Stacks Project

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CHAPTER 1

Introduction

1.1. Overview

Besides the book by Laumon and Moret-Bailly, see [**LMB00**], and the work (in progress) by Fulton et al, we think there is a place for an open source textbook on algebraic stacks and the algebraic geometry that is needed to define them. The Stacks Project attempts to do this by building the foundations starting with commutative algebra and proceeding via the theory of schemes and algebraic spaces to a comprehensive foundation for the theory of algebraic stacks.

We expect this material to be read online as a key feature are the hyperlinks giving quick access to internal references spread over many different pages. If you use an embedded pdf or dvi viewer in your browser, the cross file links should work.

This project is a collaborative effort and we encourage you to help out. Please email any typos or errors you find while reading or any suggestions, additional material, or examples you have to stacks.project@gmail.com. You can download a tarball containing all source files, extract, run make, and use a dvi or pdf viewer locally. Please feel free to edit the LaTeX files and email your improvements.

1.2. Attribution

The scope of this work is such that it is a daunting task to attribute correctly and succinctly all of those mathematicians whose work has led to the development of the theory we try to explain here. We hope eventually to generate enough community interest to find contributors willing to write sections with historical remarks for each and every chapter.

Those who contributed to this work are listed on the title page of the book version of this work and online. Here we would like to name a selection of major contributions:

(1) Jarod Alper wrote Guide to Literature.
(2) Bhargav Bhatt wrote the initial version of Étale Morphisms of Schemes.
(3) Bhargav Bhatt wrote the initial version of More on Algebra, Section 15.56.
(4) Kiran Kedlaya contributed the initial writeup of Descent, Section 34.4.
(5) The initial versions of
    (a) Algebra, Section 10.26,
    (b) Injectives, Section 19.2, and
    (c) the chapter Fields

    are from The CRing Project, courtesy of Akhil Mathew et al.
(6) Alex Perry wrote the material on projective modules, Mittag-Leffler modules, including the proof of Algebra, Theorem 10.90.5.
(7) Alex Perry wrote Formal Deformation Theory.

(8) Thibaut Pugin, Zachary Maddock and Min Lee took course notes which formed the basis for Étale Cohomology.
(9) David Rydh has contributed many helpful comments, pointed out several mistakes, helped out in an essential way with the material on residual gerbes, and was the originator for the material in More on Groupoids in Spaces, Sections 60.8 and 60.11.
(10) The material in the chapter Pro-étale Cohomology is taken from a paper by Bhargav Bhatt and Peter Scholze.

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CHAPTER 2

Conventions

2.1. Comments

The philosophy behind the conventions used in writing these documents is to choose those conventions that work.

2.2. Set theory

We use Zermelo-Fraenkel set theory with the axiom of choice. See [**Kun83**]. We do not use universes (different from SGA4). We do not stress set-theoretic issues, but we make sure everything is correct (of course) and so we do not ignore them either.

2.3. Categories

A category $\mathcal{C}$ consists of a set of objects and, for each pair of objects, a set of morphisms between them. In other words, it is what is called a "small" category in other texts. We will use "big" categories (categories whose objects form a proper class) as well, but only those that are listed in Categories, Remark 4.2.2.

2.4. Algebra

In these notes a ring is a commutative ring with a 1. Hence the category of rings has an initial object $\mathbf{Z}$ and a final object $\{0\}$ (this is the unique ring where $1 = 0$). Modules are assumed unitary. See [**Eis95**].

2.5. Notation

The natural integers are elements of $\mathbf{N} = \{1, 2, 3, \ldots\}$. The integers are elements of $\mathbf{Z} = \{\ldots, -2, -1, 0, 1, 2, \ldots\}$. The field of rational numbers is denoted $\mathbf{Q}$. The field of real numbers is denoted $\mathbf{R}$. The field of complex numbers is denoted $\mathbf{C}$.

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CHAPTER 3

Set Theory

3.1. Introduction

We need some set theory every now and then. We use Zermelo-Fraenkel set theory with the axiom of choice (ZFC) as described in [**Kun83**] and [**Jec02**].

3.2. Everything is a set

Most mathematicians think of set theory as providing the basic foundations for mathematics. So how does this really work? For example, how do we translate the sentence "X is a scheme" into set theory? Well, we just unravel the definitions: A scheme is a locally ringed space such that every point has an open neighbourhood which is an affine scheme. A locally ringed space is a ringed space such that every stalk of the structure sheaf is a local ring. A ringed space is a pair $(X, \mathcal{O}_X)$ consisting of a topological space X and a sheaf of rings $\mathcal{O}_X$ on it. A topological space is a pair (X, τ) consisting of a set X and a set of subsets $\tau \subset \mathcal{P}(X)$ satisfying the axioms of a topology. And so on and so forth.

So how, given a set S would we recognize whether it is a scheme? The first thing we look for is whether the set S is an ordered pair. This is defined (see [**Jec02**], page 7) as saying that S has the form $(a, b) := \{\{a\}, \{a, b\}\}$ for some sets a, b. If this is the case, then we would take a look to see whether a is an ordered pair (c, d). If so we would check whether $d \subset \mathcal{P}(c)$, and if so whether d forms the collection of sets for a topology on the set c. And so on and so forth.

So even though it would take a considerable amount of work to write a complete formula $\phi_{scheme}(x)$ with one free variable x in set theory that expresses the notion "x is a scheme", it is possible to do so. The same thing should be true for any mathematical object.

3.3. Classes

Informally we use the notion of a *class*. Given a formula $\phi(x, p_1, \ldots, p_n)$, we call

$$C = \{x : \phi(x, p_1, \ldots, p_n)\}$$

a *class*. A class is easier to manipulate than the formula that defines it, but it is not strictly speaking a mathematical object. For example, if R is a ring, then we may consider the class of all R-modules (since after all we may translate the sentence "M is an R-module" into a formula in set theory, which then defines a class). A *proper class* is a class which is not a set.

In this way we may consider the category of R-modules, which is a "big" category—in other words, it has a proper class of objects. Similarly, we may consider the "big" category of schemes, the "big" category of rings, etc.

3.4. Ordinals

A set T is *transitive* if $x \in T$ implies $x \subset T$. A set α is an *ordinal* if it is transitive and well-ordered by $\in$. In this case, we define $\alpha + 1 = \alpha \cup \{\alpha\}$, which is another ordinal called the *successor* of α. An ordinal α is called a *successor ordinal* if there exists an ordinal β such that $\alpha = \beta + 1$. The smallest ordinal is $\emptyset$ which is also denoted 0. If α is not 0, and not a successor ordinal, then α is called a *limit ordinal* and we have

$$\alpha = \bigcup_{\gamma \in \alpha} \gamma.$$

The first limit ordinal is ω and it is also the first infinite ordinal. The collection of all ordinals is a proper class. It is well-ordered by $\in$ in the following sense: any nonempty set (or even class) of ordinals has a least element. Given a set A of ordinals, we define the *supremum* of A to be $\sup_{\alpha \in A} \alpha = \bigcup_{\alpha \in A} \alpha$. It is the least ordinal bigger or equal to all $\alpha \in A$. Given any well ordered set $(S, \geq)$, there is a unique ordinal α such that $(S, \geq) \cong (\alpha, \in)$; this is called the *order type* of the well ordered set.

3.5. The hierarchy of sets

We define, by transfinite induction, $V_0 = \emptyset$, $V_{\alpha+1} = P(V_\alpha)$ (power set), and for a limit ordinal α,

$$V_\alpha = \bigcup_{\beta < \alpha} V_\beta.$$

Note that each V_α is a transitive set.

Lemma 3.5.1. *Every set is an element of V_α for some ordinal α.*

Proof. See [**Jec02**, Lemma 6.3]. □

In [**Kun83**, Chapter III] it is explained that this lemma is equivalent to the axiom of foundation. The *rank* of a set S is the least ordinal α such that $S \in V_\alpha$.

3.6. Cardinality

The *cardinality* of a set A is the least ordinal α such that there exists a bijection between A and α. We sometimes use the notation $\alpha = |A|$ to indicate this. We say an ordinal α is a *cardinal* if and only if it occurs as the cardinality of some set A—in other words, if $\alpha = |A|$. We use the greek letters κ, λ for cardinals. The first infinite cardinal is ω, and in this context it is denoted $\aleph_0$. A set is *countable* if its cardinality is $\leq \aleph_0$. If α is an ordinal, then we denote α^+ the least cardinal $> \alpha$. You can use this to define $\aleph_1 = \aleph_0^+$, $\aleph_2 = \aleph_1^+$, etc, and in fact you can define $\aleph_\alpha$ for any ordinal α by transfinite induction.

The *addition* of cardinals κ, λ is denoted $\kappa \oplus \lambda$; it is the cardinality of $\kappa \amalg \lambda$. The *multiplication* of cardinals κ, λ is denoted $\kappa \otimes \lambda$; it is the cardinality of $\kappa \times \lambda$. It is uninteresting since if κ and λ are infinite cardinals, then $\kappa \otimes \lambda = \max(\kappa, \lambda)$. The *exponentiation* of cardinals κ, λ is denoted κ^λ; it is the cardinality of the set of (set) maps from λ to κ. Given any set K of cardinals, the *supremum* of K is $\sup_{\kappa \in K} \kappa = \bigcup_{\kappa \in K} \kappa$, which is also a cardinal.

3.7. Cofinality

A *cofinal subset* S of a partially ordered set T is a subset $S \subset T$ such that $\forall t \in T \exists s \in S(t \leq s)$. Note that a subset of a well-ordered set is a well-ordered set (with induced ordering). Given an ordinal α, the *cofinality* $\mathrm{cf}(\alpha)$ of α is the least ordinal β which occurs as the order type of some cofinal subset of α. The cofinality of an ordinal is always a cardinal (this is clear from the definition). Hence alternatively we can define the cofinality of α as the least cardinality of a cofinal subset of α.

Lemma 3.7.1. *Suppose that $T = \mathrm{colim}_{\alpha<\beta} T_\alpha$ is a colimit of sets indexed by ordinals less than a given ordinal β. Suppose that $\varphi : S \to T$ is a map of sets. Then φ lifts to a map into T_α for some $\alpha < \beta$ provided that β is not a limit of ordinals indexed by S, in other words, if β is an ordinal with $cf(\beta) > |S|$.*

Proof. For each element $s \in S$ pick a $\alpha_s < \beta$ and an element $t_s \in T_{\alpha_s}$ which maps to $\varphi(s)$ in T. By assumption $\alpha = \sup_{s\in S} \alpha_s$ is strictly smaller than β. Hence the map $\varphi_\alpha : S \to T_\alpha$ which assigns to s the image of t_s in T_α is a solution. □

The following is essentially Grothendieck's argument for the existence of ordinals with arbitrarily large cofinality which he used to prove the existence of enough injectives in certain abelian categories, see [**Gro57**].

Proposition 3.7.2. *Let κ be a cardinal. Then there exists an ordinal whose cofinality is bigger than κ.*

Proof. If κ is finite, then $\omega = \mathrm{cf}(\omega)$ works. Let us thus assume that κ is infinite. Consider the smallest ordinal α whose cardinality is strictly greater than κ. We claim that $\mathrm{cf}(\alpha) > \kappa$. Note that α is a limit ordinal, since if $\alpha = \beta + 1$, then $|\alpha| = |\beta|$ (because α and β are infinite) and this contradicts the minimality of α. (Of course α is also a cardinal, but we do not need this.) To get a contradiction suppose $S \subset \alpha$ is a cofinal subset with $|S| \leq \kappa$. For $\beta \in S$, i.e., $\beta < \alpha$, we have $|\beta| \leq \kappa$ by minimality of α. As α is a limit ordinal and S cofinal in α we obtain $\alpha = \bigcup_{\beta\in S} \beta$. Hence $|\alpha| \leq |S| \otimes \kappa \leq \kappa \otimes \kappa \leq \kappa$ which is a contradiction with our choice of α. □

3.8. Reflection principle

Some of this material is in the chapter of [**Kun83**] called "Easy consistency proofs".

Let $\phi(x_1, \ldots, x_n)$ be a formula of set theory. Let us use the convention that this notation implies that all the free variables in ϕ occur among $x_1, \ldots, x_n$. Let M be a set. The formula $\phi^M(x_1, \ldots, x_n)$ is the formula obtained from $\phi(x_1, \ldots, x_n)$ by replacing all the $\forall x$ and $\exists x$ by $\forall x \in M$ and $\exists x \in M$, respectively. So the formula $\phi(x_1, x_2) = \exists x(x \in x_1 \wedge x \in x_2)$ is turned into $\phi^M(x_1, x_2) = \exists x \in M(x \in x_1 \wedge x \in x_2)$. The formula ϕ^M is called the *relativization of ϕ to M*.

Theorem 3.8.1. *Suppose given $\phi_1(x_1, \ldots, x_n), \ldots, \phi_m(x_1, \ldots, x_n)$ a* **finite** *collection of formulas of set theory. Let M_0 be a set. There exists a set M such that $M_0 \subset M$ and $\forall x_1, \ldots, x_n \in M$, we have*

$$\forall i = 1, \ldots, m,\ \phi_i^M(x_1, \ldots, x_n) \Leftrightarrow \forall i = 1, \ldots, m,\ \phi_i(x_1, \ldots, x_n).$$

In fact we may take $M = V_\alpha$ for some limit ordinal α.

Proof. See [**Jec02**, Theorem 12.14] or [**Kun83**, Theorem 7.4]. □

We view this theorem as saying the following: Given any $x_1, \ldots, x_n \in M$ the formulas hold with the bound variables ranging through all sets if and only if they hold for the bound variables ranging through elements of V_α. This theorem is a meta-theorem because it deals with the formulas of set theory directly. It actually says that given the finite list of formulas $\phi_1, \ldots, \phi_m$ with at most free variables $x_1, \ldots, x_n$ the sentence

$$\forall M_0\ \exists M,\ M_0 \subset M\ \forall x_1, \ldots, x_n \in M$$
$$\phi_1(x_1, \ldots, x_n) \wedge \ldots \wedge \phi_m(x_1, \ldots, x_n) \leftrightarrow \phi_1^M(x_1, \ldots, x_n) \wedge \ldots \wedge \phi_m^M(x_1, \ldots, x_n)$$

is provable in ZFC. In other words, whenever we actually write down a finite list of formulas ϕ_i, we get a theorem.

It is somewhat hard to use this theorem in "ordinary mathematics" since the meaning of the formulas $\phi_i^M(x_1, \ldots, x_n)$ is not so clear! Instead, we will use the idea of the proof of the reflection principle to prove the existence results we need directly.

3.9. Constructing categories of schemes

We will discuss how to apply this to produce, given an initial set of schemes, a "small" category of schemes closed under a list of natural operations. Before we do so, we introduce the size of a scheme. Given a scheme S we define

$$\text{size}(S) = \max(\aleph_0, \kappa_1, \kappa_2),$$

where we define the cardinal numbers κ_1 and κ_2 as follows:

(1) We let κ_1 be the cardinality of the set of affine opens of S.
(2) We let κ_2 be the supremum of all the cardinalities of all $\Gamma(U, \mathcal{O}_S)$ for all $U \subset S$ affine open.

Lemma 3.9.1. *For every cardinal κ, there exists a set A such that every element of A is a scheme and such that for every scheme S with $Size(S) \leq \kappa$, there is an element $X \in A$ such that $X \cong S$ (isomorphism of schemes).*

Proof. Omitted. Hint: think about how any scheme is isomorphic to a scheme obtained by glueing affines. □

We denote *Bound* the function which to each cardinal κ associates

$$Bound(\kappa) = \max\{\kappa^{\aleph_0}, \kappa^+\}. \tag{3.9.1.1}$$

We could make this function grow much more rapidly, e.g., we could set $Bound(\kappa) = \kappa^\kappa$, and the result below would still hold. For any ordinal α, we denote Sch_α the full subcategory of category of schemes whose objects are elements of V_α. Here is the result we are going to prove.

Lemma 3.9.2. *With notations size, Bound and Sch_α as above. Let S_0 be a set of schemes. There exists a limit ordinal α with the following properties:*

(1) *We have $S_0 \subset V_\alpha$; in other words, $S_0 \subset \text{Ob}(Sch_\alpha)$.*
(2) *For any $S \in \text{Ob}(Sch_\alpha)$ and any scheme T with $size(T) \leq Bound(size(S))$, there exists a scheme $S' \in \text{Ob}(Sch_\alpha)$ such that $T \cong S'$.*
(3) *For any countable[1] diagram category $\mathcal{I}$ and any functor $F : \mathcal{I} \to Sch_\alpha$, the limit $\lim_{\mathcal{I}} F$ exists in Sch_α if and only if it exists in Sch and moreover, in this case, the natural morphism between them is an isomorphism.*

[1]Both the set of objects and the morphism sets are countable. In fact you can prove the lemma with $\aleph_0$ replaced by any cardinal whatsoever in (3) and (4).

(4) *For any countable diagram category $\mathcal{I}$ and any functor $F : \mathcal{I} \to Sch_\alpha$, the colimit $\operatorname{colim}_\mathcal{I} F$ exists in Sch_α if and only if it exists in Sch and moreover, in this case, the natural morphism between them is an isomorphism.*

Proof. We define, by transfinite induction, a function f which associates to every ordinal an ordinal as follows. Let $f(0) = 0$. Given $f(\alpha)$, we define $f(\alpha + 1)$ to be the least ordinal β such that the following hold:

(1) We have $\alpha + 1 \leq \beta$ and $f(\alpha) \leq \beta$.
(2) For any $S \in \mathrm{Ob}(Sch_{f(\alpha)})$ and any scheme T with $\mathrm{size}(T) \leq Bound(\mathrm{size}(S))$, there exists a scheme $S' \in \mathrm{Ob}(Sch_\beta)$ such that $T \cong S'$.
(3) For any countable diagram category $\mathcal{I}$ and any functor $F : \mathcal{I} \to Sch_{f(\alpha)}$, if the limit $\lim_\mathcal{I} F$ or the colimit $\operatorname{colim}_\mathcal{I} F$ exists in Sch, then it is isomorphic to a scheme in Sch_β.

To see β exists, we argue as follows. Since $\mathrm{Ob}(Sch_{f(\alpha)})$ is a set, we see that $\kappa = \sup_{S \in \mathrm{Ob}(Sch_{f(\alpha)})} Bound(\mathrm{size}(S))$ exists and is a cardinal. Let A be a set of schemes obtained starting with κ as in Lemma 3.9.1. There is a set $CountCat$ of countable categories such that any countable category is isomorphic to an element of $CountCat$. Hence in (3) above we may assume that $\mathcal{I}$ is an element in $CountCat$. This means that the pairs $(\mathcal{I}, F)$ in (3) range over a set. Thus, there exists a set B whose elements are schemes such that for every $(\mathcal{I}, F)$ as in (3), if the limit or colimit exists, then it is isomorphic to an element in B. Hence, if we pick any β such that $A \cup B \subset V_\beta$ and $\beta > \max\{\alpha + 1, f(\alpha)\}$, then (1)–(3) hold. Since every nonempty collection of ordinals has a least element, we see that $f(\alpha + 1)$ is well defined. Finally, if α is a limit ordinal, then we set $f(\alpha) = \sup_{\alpha' < \alpha} f(\alpha')$.

Pick β_0 such that $S_0 \subset V_{\beta_0}$. By construction $f(\beta) \geq \beta$ and we see that also $S_0 \subset V_{f(\beta_0)}$. Moreover, as f is nondecreasing, we see $S_0 \subset V_{f(\beta)}$ is true for any $\beta \geq \beta_0$. Next, choose any ordinal $\beta_1 > \beta_0$ with cofinality $\mathrm{cf}(\beta_1) > \omega = \aleph_0$. This is possible since the cofinality of ordinals gets arbitrarily large, see Proposition 3.7.2. We claim that $\alpha = f(\beta_1)$ is a solution to the problem posed in the lemma.

The first property of the lemma holds by our choice of $\beta_1 > \beta_0$ above.

Since β_1 is a limit ordinal (as its cofinality is infinite), we get $f(\beta_1) = \sup_{\beta < \beta_1} f(\beta)$. Hence $\{f(\beta) \mid \beta < \beta_1\} \subset f(\beta_1)$ is a cofinal subset. Hence we see that

$$V_\alpha = V_{f(\beta_1)} = \bigcup\nolimits_{\beta < \beta_1} V_{f(\beta)}.$$

Now, let $S \in \mathrm{Ob}(Sch_\alpha)$. We define $\beta(S)$ to be the least ordinal β such that $S \in \mathrm{Ob}(Sch_{f(\beta)})$. By the above we see that always $\beta(S) < \beta_1$. Since $\mathrm{Ob}(Sch_{f(\beta+1)}) \subset \mathrm{Ob}(Sch_\alpha)$, we see by construction of f above that the second property of the lemma is satisfied.

Suppose that $\{S_1, S_2, \ldots\} \subset \mathrm{Ob}(Sch_\alpha)$ is a countable collection. Consider the function $\omega \to \beta_1$, $n \mapsto \beta(S_n)$. Since the cofinality of β_1 is $> \omega$, the image of this function cannot be a cofinal subset. Hence there exists a $\beta < \beta_1$ such that $\{S_1, S_2, \ldots\} \subset \mathrm{Ob}(Sch_{f(\beta)})$. It follows that any functor $F : \mathcal{I} \to Sch_\alpha$ factors through one of the subcategories $Sch_{f(\beta)}$. Thus, if there exists a scheme X that is the colimit or limit of the diagram F, then, by construction of f, we see X is isomorphic to an object of $Sch_{f(\beta+1)}$ which is a subcategory of Sch_α. This proves the last two assertions of the lemma. □

Remark 3.9.3. The lemma above can also be proved using the reflection principle. However, one has to be careful. Namely, suppose the sentence $\phi_{scheme}(X)$ expresses the property "X is a scheme", then what does the formula $\phi_{scheme}^{V_\alpha}(X)$ mean? It is true that the reflection principle says we can find α such that for all $X \in V_\alpha$ we have $\phi_{scheme}(X) \leftrightarrow \phi_{scheme}^{V_\alpha}(X)$ but this is entirely useless. It is only by combining two such statements that something interesting happens. For example suppose $\phi_{red}(X, Y)$ expresses the property "X, Y are schemes, and Y is the reduction of X" (see Schemes, Definition 25.12.5). Suppose we apply the reflection principle to the pair of formulas $\phi_1(X, Y) = \phi_{red}(X, Y)$, $\phi_2(X) = \exists Y, \phi_1(X, Y)$. Then it is easy to see that any α produced by the reflection principle has the property that given $X \in \mathrm{Ob}(Sch_\alpha)$ the reduction of X is also an object of Sch_α (left as an exercise).

Lemma 3.9.4. *Let S be an affine scheme. Let $R = \Gamma(S, \mathcal{O}_S)$. Then the size of S is equal to* $\max\{\aleph_0, |R|\}$.

Proof. There are at most $\max\{|R|, \aleph_0\}$ affine opens of $\mathrm{Spec}(R)$. This is clear since any affine open $U \subset \mathrm{Spec}(R)$ is a finite union of principal opens $D(f_1) \cup \ldots \cup D(f_n)$ and hence the number of affine opens is at most $\sup_n |R|^n = \max\{|R|, \aleph_0\}$, see [**Kun83**, Ch. I, 10.13]. On the other hand, we see that $\Gamma(U, \mathcal{O}) \subset R_{f_1} \times \ldots \times R_{f_n}$ and hence $|\Gamma(U, \mathcal{O})| \leq \max\{\aleph_0, |R_{f_1}|, \ldots, |R_{f_n}|\}$. Thus it suffices to prove that $|R_f| \leq \max\{\aleph_0, |R|\}$ which is omitted. □

Lemma 3.9.5. *Let S be a scheme. Let $S = \bigcup_{i \in I} S_i$ be an open covering. Then* $size(S) \leq \max\{|I|, \sup_i\{size(S_i)\}\}$.

Proof. Let $U \subset S$ be any affine open. Since U is quasi-compact there exist finitely many elements $i_1, \ldots, i_n \in I$ and affine opens $U_i \subset U \cap S_i$ such that $U = U_1 \cup U_2 \cup \ldots \cup U_n$. Thus

$$|\Gamma(U, \mathcal{O}_U)| \leq |\Gamma(U_1, \mathcal{O})| \otimes \ldots \otimes |\Gamma(U_n, \mathcal{O})| \leq \sup_i\{\mathrm{size}(S_i)\}$$

Moreover, it shows that the set of affine opens of S has cardinality less than or equal to the cardinality of the set

$$\coprod_{n \in \omega} \coprod_{i_1, \ldots, i_n \in I} \{\text{affine opens of } S_{i_1}\} \times \ldots \times \{\text{affine opens of } S_{i_n}\}.$$

Each of the sets inside the disjoint union has cardinality at most $\sup_i\{\mathrm{size}(S_i)\}$. The index set has cardinality at most $\max\{|I|, \aleph_0\}$, see [**Kun83**, Ch. I, 10.13]. Hence by [**Jec02**, Lemma 5.8] the cardinality of the coproduct is at most $\max\{\aleph_0, |I|\} \otimes \sup_i\{\mathrm{size}(S_i)\}$. The lemma follows. □

Lemma 3.9.6. *Let $f : X \to S$, $g : Y \to S$ be morphisms of schemes. Then we have* $size(X \times_S Y) \leq \max\{size(X), size(Y))\}$.

Proof. Let $S = \bigcup_{k \in K} S_k$ be an affine open covering. Let $X = \bigcup_{i \in I} U_i$, $Y = \bigcup_{j \in J} V_j$ be affine open coverings with I, J of cardinality $\leq \mathrm{size}(X), \mathrm{size}(Y)$. For each $i \in I$ there exists a finite set K_i of $k \in K$ such that $f(U_i) \subset \bigcup_{k \in K_i} S_k$. For each $j \in J$ there exists a finite set K_j of $k \in K$ such that $g(V_j) \subset \bigcup_{k \in K_j} S_k$. Hence $f(X), g(Y)$ are contained in $S' = \bigcup_{k \in K'} S_k$ with $K' = \bigcup_{i \in I} K_i \cup \bigcup_{j \in J} K_j$. Note that the cardinality of K' is at most $\max\{\aleph_0, |I|, |J|\}$. Applying Lemma 3.9.5 we see that it suffices to prove that $\mathrm{size}(f^{-1}(S_k) \times_{S_k} g^{-1}(S_k)) \leq \max\{\mathrm{size}(X), \mathrm{size}(Y))\}$ for $k \in K'$. In other words, we may assume that S is affine.

Assume S affine. Let $X = \bigcup_{i\in I} U_i$, $Y = \bigcup_{j\in J} V_j$ be affine open coverings with I, J of cardinality $\leq \text{size}(X), \text{size}(Y)$. Again by Lemma 3.9.5 it suffices to prove the lemma for the products $U_i \times_S V_j$. By Lemma 3.9.4 we see that it suffices to show that

$$|A \otimes_C B| \leq \max\{\aleph_0, |A|, |B|\}.$$

We omit the proof of this inequality. □

Lemma 3.9.7. *Let S be a scheme. Let $f : X \to S$ be locally of finite type with X quasi-compact. Then size$(X) \leq$ size(S).*

Proof. We can find a finite affine open covering $X = \bigcup_{i=1,\ldots n} U_i$ such that each U_i maps into an affine open S_i of S. Thus by Lemma 3.9.5 we reduce to the case where both S and X are affine. In this case by Lemma 3.9.4 we see that it suffices to show

$$|A[x_1, \ldots, x_n]| \leq \max\{\aleph_0, |A|\}.$$

We omit the proof of this inequality. □

In Algebra, Lemma 10.102.13 we will show that if $A \to B$ is an epimorphism of rings, then $|B| \leq \max(|A|, \aleph_0)$. The analogue for schemes is the following lemma.

Lemma 3.9.8. *Let $f : X \to Y$ be a monomorphism of schemes. If at least one of the following properties holds, then size$(X) \leq$ size(Y):*

(1) *f is quasi-compact,*
(2) *f is locally of finite type,*
(3) *add more here as needed.*

However, the bound does not hold for general monomorphisms.

Proof. Let $Y = \bigcup_{j\in J} V_j$ be an affine open covering of Y with $|J| \leq \text{size}(Y)$. By Lemma 3.9.5 it suffices to bound the size of the inverse image of V_j in X. Hence we reduce to the case that Y is affine, say $Y = \text{Spec}(B)$. For any affine open $\text{Spec}(A) \subset X$ we have $|A| \leq \max(|B|, \aleph_0) = \text{size}(Y)$, see remark above and Lemma 3.9.4. Thus it suffices to show that X has at most size(Y) affine opens. This is clear if X is quasi-compact, whence case (1) holds. In case (2) the number of isomorphism classes of B-algebras A that can occur is bounded by size(B), because each A is of finite type over B, hence isomorphic to an algebra $B[x_1, \ldots, x_n]/(f_1, \ldots, f_m)$ for some n, m, and $f_j \in B[x_1, \ldots, x_n]$. However, as $X \to Y$ is a monomorphism, there is a unique morphism $\text{Spec}(A) \to X$ over $Y = \text{Spec}(B)$ if there is one, hence the number of affine opens of X is bounded by the number of these isomorphism classes.

To prove the final statement of the lemma consider the ring $B = \prod_{n\in\mathbf{N}} \mathbf{F}_2$ and set $Y = \text{Spec}(B)$. For every ultrafilter $\mathcal{U}$ on $\mathbf{N}$ we obtain a maximal ideal $\mathfrak{m}_\mathcal{U}$ with residue field $\mathbf{F}_2$; the map $B \to \mathbf{F}_2$ sends the element (x_n) to $\lim_\mathcal{U} x_n$. Details omitted. The morphism of schemes $X = \coprod_\mathcal{U} \text{Spec}(\mathbf{F}_2) \to Y$ is a monomorphism as all the points are distinct. However the cardinality of the set of affine open subschemes of X is equal to the cardinality of the set of ultrafilters on $\mathbf{N}$ which is $2^{2^{\aleph_0}}$. We conclude as $|B| = 2^{\aleph_0} < 2^{2^{\aleph_0}}$. □

Lemma 3.9.9. *Let α be an ordinal as in Lemma 3.9.2 above. The category Sch_α satisfies the following properties:*

(1) *If $X, Y, S \in \mathrm{Ob}(Sch_\alpha)$, then for any morphisms $f : X \to S$, $g : Y \to S$ the fibre product $X \times_S Y$ in Sch_α exists and is a fibre product in the category of schemes.*
(2) *Given any at most countable collection $S_1, S_2, \ldots$ of elements of $\mathrm{Ob}(Sch_\alpha)$, the coproduct $\coprod_i S_i$ exists in $\mathrm{Ob}(Sch_\alpha)$ and is a coproduct in the category of schemes.*
(3) *For any $S \in \mathrm{Ob}(Sch_\alpha)$ and any open immersion $U \to S$, there exists a $V \in \mathrm{Ob}(Sch_\alpha)$ with $V \cong U$.*
(4) *For any $S \in \mathrm{Ob}(Sch_\alpha)$ and any closed immersion $T \to S$, there exists a $S' \in \mathrm{Ob}(Sch_\alpha)$ with $S' \cong T$.*
(5) *For any $S \in \mathrm{Ob}(Sch_\alpha)$ and any finite type morphism $T \to S$, there exists a $S' \in \mathrm{Ob}(Sch_\alpha)$ with $S' \cong T$.*
(6) *Suppose S is a scheme which has an open covering $S = \bigcup_{i \in I} S_i$ such that there exists a $T \in \mathrm{Ob}(Sch_\alpha)$ with (a) $\mathrm{size}(S_i) \leq \mathrm{size}(T)^{\aleph_0}$ for all $i \in I$, and (b) $|I| \leq \mathrm{size}(T)^{\aleph_0}$. Then S is isomorphic to an object of Sch_α.*
(7) *For any $S \in \mathrm{Ob}(Sch_\alpha)$ and any morphism $f : T \to S$ locally of finite type such that T can be covered by at most $\mathrm{size}(S)^{\aleph_0}$ open affines, there exists a $S' \in \mathrm{Ob}(Sch_\alpha)$ with $S' \cong T$. For example this holds if T can be covered by at most $|\mathbf{R}| = 2^{\aleph_0} = \aleph_0^{\aleph_0}$ open affines.*
(8) *For any $S \in \mathrm{Ob}(Sch_\alpha)$ and any monomorphism $T \to S$ which is either locally of finite type or quasi-compact, there exists a $S' \in \mathrm{Ob}(Sch_\alpha)$ with $S' \cong T$.*
(9) *Suppose that $T \in \mathrm{Ob}(Sch_\alpha)$ is affine. Write $R = \Gamma(T, \mathcal{O}_T)$. Then any of the following schemes is isomorphic to a scheme in Sch_α:*
   (a) *For any ideal $I \subset R$ with completion $R^* = \lim_n R/I^n$, the scheme $\mathrm{Spec}(R^*)$.*
   (b) *For any finite type R-algebra R', the scheme $\mathrm{Spec}(R')$.*
   (c) *For any localization $S^{-1}R$, the scheme $\mathrm{Spec}(S^{-1}R)$.*
   (d) *For any prime $\mathfrak{p} \subset R$, the scheme $\mathrm{Spec}(\overline{\kappa(\mathfrak{p})})$.*
   (e) *For any subring $R' \subset R$, the scheme $\mathrm{Spec}(R')$.*
   (f) *Any scheme of finite type over a ring of cardinality at most $|R|^{\aleph_0}$.*
   (g) *And so on.*

Proof. Statements (1) and (2) follow directly from the definitions. Statement (3) follows as the size of an open subscheme U of S is clearly smaller than or equal to the size of S. Statement (4) follows from (5). Statement (5) follows from (7). Statement (6) follows as the size of S is $\leq \max\{|I|, \sup_i \mathrm{size}(S_i)\} \leq \mathrm{size}(T)^{\aleph_0}$ by Lemma 3.9.5. Statement (7) follows from (6). Namely, for any affine open $V \subset T$ we have $\mathrm{size}(V) \leq \mathrm{size}(S)$ by Lemma 3.9.7. Thus, we see that (6) applies in the situation of (7). Part (8) follows from Lemma 3.9.8.

Statement (9) is translated, via Lemma 3.9.4, into an upper bound on the cardinality of the rings R^*, $S^{-1}R$, $\overline{\kappa(\mathfrak{p})}$, R', etc. Perhaps the most interesting one is the ring R^*. As a set, it is the image of a surjective map $R^{\mathbf{N}} \to R^*$. Since $|R^{\mathbf{N}}| = |R|^{\aleph_0}$, we see that it works by our choice of $Bound(\kappa)$ being at least $\kappa^{\aleph_0}$. Phew! (The cardinality of the algebraic closure of a field is the same as the cardinality of the field, or it is $\aleph_0$.) □

Remark 3.9.10. Let R be a ring. Suppose we consider the ring $\prod_{\mathfrak{p} \in \mathrm{Spec}(R)} \kappa(\mathfrak{p})$. The cardinality of this ring is bounded by $|R|^{|R|}$, but is not bounded by $|R|^{\aleph_0}$ in

general (for example if $R = \mathbf{C}[x]$). Thus the "And so on" of Lemma 3.9.9 above should be taken with a grain of salt. Of course, if it ever becomes necessary to consider these rings in arguments pertaining to fppf/étale cohomology, then we can change the function *Bound* above into the function $\kappa \mapsto \kappa^\kappa$.

3.10. Sets with group action

Let G be a group. We denote G-*Sets* the "big" category of G-sets. For any ordinal α, we denote G-$Sets_\alpha$ the full subcategory of G-*Sets* whose objects are in V_α. As a notion for size of a G-set we take $\text{size}(S) = \max\{\aleph_0, |G|, |S|\}$ (where $|G|$ and $|S|$ are the cardinality of the underlying sets). As above we use the function $Bound(\kappa) = \kappa^{\aleph_0}$.

Lemma 3.10.1. *With notations G, G-$Sets_\alpha$, size, and Bound as above. Let S_0 be a set of G-sets. There exists a limit ordinal α with the following properties:*

1. *We have $S_0 \cup \{{}_GG\} \subset \text{Ob}(G\text{-}Sets_\alpha)$.*
2. *For any $S \in \text{Ob}(G\text{-}Sets_\alpha)$ and any G-set T with $size(T) \leq Bound(size(S))$, there exists a $S' \in \text{Ob}(G\text{-}Sets_\alpha)$ that is isomorphic to T.*
3. *For any countable diagram category $\mathcal{I}$ and any functor $F : \mathcal{I} \to G\text{-}Sets_\alpha$, the limit $\lim_\mathcal{I} F$ and colimit $\text{colim}_\mathcal{I} F$ exist in G-$Sets_\alpha$ and are the same as in G-Sets.*

Proof. Omitted. Similar to but easier than the proof of Lemma 3.9.2 above. □

Lemma 3.10.2. *Let α be an ordinal as in Lemma 3.10.1 above. The category G-$Sets_\alpha$ satisfies the following properties:*

1. *The G-set ${}_GG$ is an object of G-$Sets_\alpha$.*
2. *(Co)Products, fibre products, and pushouts exist in G-$Sets_\alpha$ and are the same as their counterparts in G-Sets.*
3. *Given an object U of G-$Sets_\alpha$, any G-stable subset $O \subset U$ is isomorphic to an object of G-$Sets_\alpha$.*

Proof. Omitted. □

3.11. Coverings of a site

Suppose that $\mathcal{C}$ is a category (as in Categories, Definition 4.2.1) and that $\text{Cov}(\mathcal{C})$ is a proper class of coverings satisfying properties (1), (2), and (3) of Sites, Definition 7.6.2. We list them here:

1. If $V \to U$ is an isomorphism, then $\{V \to U\} \in \text{Cov}(\mathcal{C})$.
2. If $\{U_i \to U\}_{i\in I} \in \text{Cov}(\mathcal{C})$ and for each i we have $\{V_{ij} \to U_i\}_{j\in J_i} \in \text{Cov}(\mathcal{C})$, then $\{V_{ij} \to U\}_{i\in I, j\in J_i} \in \text{Cov}(\mathcal{C})$.
3. If $\{U_i \to U\}_{i\in I} \in \text{Cov}(\mathcal{C})$ and $V \to U$ is a morphism of $\mathcal{C}$, then $U_i \times_U V$ exists for all i and $\{U_i \times_U V \to V\}_{i\in I} \in \text{Cov}(\mathcal{C})$.

For an ordinal α, we set $\text{Cov}(\mathcal{C})_\alpha = \text{Cov}(\mathcal{C}) \cap V_\alpha$. Given an ordinal α and a cardinal κ, we set $\text{Cov}(\mathcal{C})_{\kappa,\alpha}$ equal to the set of elements $\mathcal{U} = \{\varphi_i : U_i \to U\}_{i\in I} \in \text{Cov}(\mathcal{C})_\alpha$ such that $|I| \leq \kappa$.

We recall the following notion, see Sites, Definition 7.8.2. Two families of morphisms, $\{\varphi_i : U_i \to U\}_{i\in I}$ and $\{\psi_j : W_j \to U\}_{j\in J}$, with the same target of $\mathcal{C}$ are called *combinatorially equivalent* if there exist maps $\alpha : I \to J$ and $\beta : J \to I$ such

that $\varphi_i = \psi_{\alpha(i)}$ and $\psi_j = \varphi_{\beta(j)}$. This defines an equivalence relation on families of morphisms having a fixed target.

Lemma 3.11.1. *With notations as above. Let $Cov_0 \subset Cov(\mathcal{C})$ be a set contained in $Cov(\mathcal{C})$. There exist a cardinal κ and a limit ordinal α with the following properties:*

1. *We have $Cov_0 \subset Cov(\mathcal{C})_{\kappa,\alpha}$.*
2. *The set of coverings $Cov(\mathcal{C})_{\kappa,\alpha}$ satisfies (1), (2), and (3) of Sites, Definition 7.6.2 (see above). In other words $(\mathcal{C}, Cov(\mathcal{C})_{\kappa,\alpha})$ is a site.*
3. *Every covering in $Cov(\mathcal{C})$ is combinatorially equivalent to a covering in $Cov(\mathcal{C})_{\kappa,\alpha}$.*

Proof. To prove this, we first consider the set $\mathcal{S}$ of all sets of morphisms of $\mathcal{C}$ with fixed target. In other words, an element of $\mathcal{S}$ is a subset T of $\text{Arrows}(\mathcal{C})$ such that all elements of T have the same target. Given a family $\mathcal{U} = \{\varphi_i : U_i \to U\}_{i\in I}$ of morphisms with fixed target, we define $Supp(\mathcal{U}) = \{\varphi \in \text{Arrows}(\mathcal{C}) \mid \exists i \in I, \varphi = \varphi_i\}$. Note that two families $\mathcal{U} = \{\varphi_i : U_i \to U\}_{i\in I}$ and $\mathcal{V} = \{V_j \to V\}_{j\in J}$ are combinatorially equivalent if and only if $Supp(\mathcal{U}) = Supp(\mathcal{V})$. Next, we define $\mathcal{S}_\tau \subset \mathcal{S}$ to be the subset $\mathcal{S}_\tau = \{T \in \mathcal{S} \mid \exists\, \mathcal{U} \in \text{Cov}(\mathcal{C})\ T = Supp(\mathcal{U})\}$. For every element $T \in \mathcal{S}_\tau$, set $\beta(T)$ to equal the least ordinal β such that there exists a $\mathcal{U} \in \text{Cov}(\mathcal{C})_\beta$ such that $T = \text{Supp}(\mathcal{U})$. Finally, set $\beta_0 = \sup_{T\in S_\tau} \beta(T)$. At this point it follows that every $\mathcal{U} \in \text{Cov}(\mathcal{C})$ is combinatorially equivalent to some element of $\text{Cov}(\mathcal{C})_{\beta_0}$.

Let κ be the maximum of $\aleph_0$, the cardinality $|\text{Arrows}(\mathcal{C})|$,

$$\sup_{\{U_i\to U\}_{i\in I}\in \text{Cov}(\mathcal{C})_{\beta_0}} |I|, \quad \text{and} \quad \sup_{\{U_i\to U\}_{i\in I}\in \text{Cov}_0} |I|.$$

Since κ is an infinite cardinal, we have $\kappa \otimes \kappa = \kappa$. Note that obviously $\text{Cov}(\mathcal{C})_{\beta_0} = \text{Cov}(\mathcal{C})_{\kappa,\beta_0}$.

We define, by transfinite induction, a function f which associates to every ordinal an ordinal as follows. Let $f(0) = 0$. Given $f(\alpha)$, we define $f(\alpha + 1)$ to be the least ordinal β such that the following hold:

1. We have $\alpha + 1 \leq \beta$ and $f(\alpha) \leq \beta$.
2. If $\{U_i \to U\}_{i\in I} \in \text{Cov}(\mathcal{C})_{\kappa,f(\alpha)}$ and for each i we have $\{W_{ij} \to U_i\}_{j\in J_i} \in \text{Cov}(\mathcal{C})_{\kappa,f(\alpha)}$, then $\{W_{ij} \to U\}_{i\in I, j\in J_i} \in \text{Cov}(\mathcal{C})_{\kappa,\beta}$.
3. If $\{U_i \to U\}_{i\in I} \in \text{Cov}(\mathcal{C})_{\kappa,\alpha}$ and $W \to U$ is a morphism of $\mathcal{C}$, then $\{U_i \times_U W \to W\}_{i\in I} \in \text{Cov}(\mathcal{C})_{\kappa,\beta}$.

To see β exists we note that clearly the collection of all coverings $\{W_{ij} \to U\}$ and $\{U_i \times_U W \to W\}$ that occur in (2) and (3) form a set. Hence there is some ordinal β such that V_β contains all of these coverings. Moreover, the index set of the covering $\{W_{ij} \to U\}$ has cardinality $\sum_{i\in I} |J_i| \leq \kappa \otimes \kappa = \kappa$, and hence these coverings are contained in $\text{Cov}(\mathcal{C})_{\kappa,\beta}$. Since every nonempty collection of ordinals has a least element we see that $f(\alpha + 1)$ is well defined. Finally, if α is a limit ordinal, then we set $f(\alpha) = \sup_{\alpha'<\alpha} f(\alpha')$.

Pick an ordinal β_1 such that $\text{Arrows}(\mathcal{C}) \subset V_{\beta_1}$, $\text{Cov}_0 \subset V_{\beta_0}$, and $\beta_1 \geq \beta_0$. By construction $f(\beta_1) \geq \beta_1$ and we see that the same properties hold for $V_{f(\beta_1)}$. Moreover, as f is nondecreasing this remains true for any $\beta \geq \beta_1$. Next, choose any ordinal $\beta_2 > \beta_1$ with cofinality $\text{cf}(\beta_2) > \kappa$. This is possible since the cofinality of ordinals gets arbitrarily large, see Proposition 3.7.2. We claim that the pair κ, $\alpha = f(\beta_2)$ is a solution to the problem posed in the lemma.

The first and third property of the lemma holds by our choices of κ, $\beta_2 > \beta_1 > \beta_0$ above.

Since β_2 is a limit ordinal (as its cofinality is infinite) we get $f(\beta_2) = \sup_{\beta<\beta_2} f(\beta)$. Hence $\{f(\beta) \mid \beta < \beta_2\} \subset f(\beta_2)$ is a cofinal subset. Hence we see that

$$V_\alpha = V_{f(\beta_2)} = \bigcup\nolimits_{\beta<\beta_2} V_{f(\beta)}.$$

Now, let $\mathcal{U} \in \mathrm{Cov}_{\kappa,\alpha}$. We define $\beta(\mathcal{U})$ to be the least ordinal β such that $\mathcal{U} \in \mathrm{Cov}_{\kappa,f(\beta)}$. By the above we see that always $\beta(\mathcal{U}) < \beta_2$.

We have to show properties (1), (2), and (3) defining a site hold for the pair $(\mathcal{C}, \mathrm{Cov}_{\kappa,\alpha})$. The first holds because by our choice of β_2 all arrows of $\mathcal{C}$ are contained in $V_{f(\beta_2)}$. For the third, we use that given a covering $\mathcal{U} = \{U_i \to U\}_{i\in I} \in \mathrm{Cov}(\mathcal{C})_{\kappa,\alpha}$ we have $\beta(\mathcal{U}) < \beta_2$ and hence any base change of $\mathcal{U}$ is by construction of f contained in $\mathrm{Cov}(\mathcal{C})_{\kappa,f(\beta+1)}$ and hence in $\mathrm{Cov}(\mathcal{C})_{\kappa,\alpha}$.

Finally, for the second condition, suppose that $\{U_i \to U\}_{i\in I} \in \mathrm{Cov}(\mathcal{C})_{\kappa,f(\alpha)}$ and for each i we have $\mathcal{W}_i = \{W_{ij} \to U_i\}_{j\in J_i} \in \mathrm{Cov}(\mathcal{C})_{\kappa,f(\alpha)}$. Consider the function $I \to \beta_2$, $i \mapsto \beta(\mathcal{W}_i)$. Since the cofinality of β_2 is $> \kappa \geq |I|$ the image of this function cannot be a cofinal subset. Hence there exists a $\beta < \beta_1$ such that $\mathcal{W}_i \in \mathrm{Cov}_{\kappa,f(\beta)}$ for all $i \in I$. It follows that the covering $\{W_{ij} \to U\}_{i\in I, j\in J_i}$ is an element of $\mathrm{Cov}(\mathcal{C})_{\kappa,f(\beta+1)} \subset \mathrm{Cov}(\mathcal{C})_{\kappa,\alpha}$ as desired. □

Remark 3.11.2. It is likely the case that, for some limit ordinal α, the set of coverings $\mathrm{Cov}(\mathcal{C})_\alpha$ satisfies the conditions of the lemma. This is after all what an application of the reflection principle would appear to give (modulo caveats as described at the end of Section 3.8 and in Remark 3.9.3).

3.12. Abelian categories and injectives

The following lemma applies to the category of modules over a sheaf of rings on a site.

Lemma 3.12.1. *Suppose given a big category $\mathcal{A}$ (see Categories, Remark 4.2.2). Assume $\mathcal{A}$ is abelian and has enough injectives. See Homology, Definitions 12.5.1 and 12.23.4. Then for any given set of objects $\{A_s\}_{s\in S}$ of $\mathcal{A}$, there is an abelian subcategory $\mathcal{A}' \subset \mathcal{A}$ with the following properties:*

1. *$\mathrm{Ob}(\mathcal{A}')$ is a set,*
2. *$\mathrm{Ob}(\mathcal{A}')$ contains A_s for each $s \in S$,*
3. *$\mathcal{A}'$ has enough injectives, and*
4. *an object of $\mathcal{A}'$ is injective if and only if it is an injective object of $\mathcal{A}$.*

Proof. Omitted. □

3.13. Other chapters

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CHAPTER 4

Categories

4.1. Introduction

Categories were first introduced in [**EM45**]. The category of categories (which is a proper class) is a 2-category. Similarly, the category of stacks forms a 2-category. If you already know about categories, but not about 2-categories you should read Section 4.26 as an introduction to the formal definitions later on.

4.2. Definitions

We recall the definitions, partly to fix notation.

Definition 4.2.1. A *category* $\mathcal{C}$ consists of the following data:

(1) A set of objects $\mathrm{Ob}(\mathcal{C})$.
(2) For each pair $x, y \in \mathrm{Ob}(\mathcal{C})$ a set of morphisms $\mathrm{Mor}_{\mathcal{C}}(x, y)$.
(3) For each triple $x, y, z \in \mathrm{Ob}(\mathcal{C})$ a composition map $\mathrm{Mor}_{\mathcal{C}}(y, z) \times \mathrm{Mor}_{\mathcal{C}}(x, y) \to \mathrm{Mor}_{\mathcal{C}}(x, z)$, denoted $(\phi, \psi) \mapsto \phi \circ \psi$.

These data are to satisfy the following rules:

(1) For every element $x \in \mathrm{Ob}(\mathcal{C})$ there exists a morphism $\mathrm{id}_x \in \mathrm{Mor}_{\mathcal{C}}(x, x)$ such that $\mathrm{id}_x \circ \phi = \phi$ and $\psi \circ \mathrm{id}_x = \psi$ whenever these compositions make sense.
(2) Composition is associative, i.e., $(\phi \circ \psi) \circ \chi = \phi \circ (\psi \circ \chi)$ whenever these compositions make sense.

It is customary to require all the morphism sets $\mathrm{Mor}_{\mathcal{C}}(x, y)$ to be disjoint. In this way a morphism $\phi : x \to y$ has a unique *source* x and a unique *target* y. This is not strictly necessary, although care has to be taken in formulating condition (2) above if it is not the case. It is convenient and we will often assume this is the case. In this case we say that ϕ and ψ are *composable* if the source of ϕ is equal to the target of ψ, in which case $\phi \circ \psi$ is defined. An equivalent definition would be to define a category as a quintuple $(\mathrm{Ob}, \mathrm{Arrows}, s, t, \circ)$ consisting of a set of objects, a set of morphisms (arrows), source, target and composition subject to a long list of axioms. We will occasionally use this point of view.

Remark 4.2.2. Big categories. In some texts a category is allowed to have a proper class of objects. We will allow this as well in these notes but only in the following list of cases (to be updated as we go along). In particular, when we say: "Let $\mathcal{C}$ be a category" then it is understood that $\mathrm{Ob}(\mathcal{C})$ is a set.

(1) The category *Sets* of sets.
(2) The category *Ab* of abelian groups.
(3) The category *Groups* of groups.
(4) Given a group G the category G-*Sets* of sets with a left G-action.

(5) Given a ring R the category Mod_R of R-modules.
(6) Given a field k the category of vector spaces over k.
(7) The category of rings.
(8) The category of schemes.
(9) The category *Top* of topological spaces.
(10) Given a topological space X the category $PSh(X)$ of presheaves of sets over X.
(11) Given a topological space X the category $Sh(X)$ of sheaves of sets over X.
(12) Given a topological space X the category $PAb(X)$ of presheaves of abelian groups over X.
(13) Given a topological space X the category $Ab(X)$ of sheaves of abelian groups over X.
(14) Given a small category $\mathcal{C}$ the category of functors from $\mathcal{C}$ to *Sets*.
(15) Given a category $\mathcal{C}$ the category of presheaves of sets over $\mathcal{C}$.
(16) Given a site $\mathcal{C}$ the category of sheaves of sets over $\mathcal{C}$.

One of the reason to enumerate these here is to try and avoid working with something like the "collection" of "big" categories which would be like working with the collection of all classes which I think definitively is a meta-mathematical object.

Remark 4.2.3. It follows directly from the definition that any two identity morphisms of an object x of $\mathcal{A}$ are the same. Thus we may and will speak of *the* identity morphism id_x of x.

Definition 4.2.4. A morphism $\phi : x \to y$ is an *isomorphism* of the category $\mathcal{C}$ if there exists a morphism $\psi : y \to x$ such that $\phi \circ \psi = \mathrm{id}_y$ and $\psi \circ \phi = \mathrm{id}_x$.

An isomorphism ϕ is also sometimes called an *invertible* morphism, and the morphism ψ of the definition is called the *inverse* and denoted ϕ^{-1}. It is unique if it exists. Note that given an object x of a category $\mathcal{A}$ the set of invertible elements $\mathrm{Aut}_{\mathcal{A}}(x)$ of $\mathrm{Mor}_{\mathcal{A}}(x, x)$ forms a group under composition. This group is called the *automorphism* group of x in $\mathcal{A}$.

Definition 4.2.5. A *groupoid* is a category where every morphism is an isomorphism.

Example 4.2.6. A group G gives rise to a groupoid with a single object x and morphisms $\mathrm{Mor}(x, x) = G$, with the composition rule given by the group law in G. Every groupoid with a single object is of this form.

Example 4.2.7. A set C gives rise to a groupoid $\mathcal{C}$ defined as follows: As objects we take $\mathrm{Ob}(\mathcal{C}) := C$ and for morphisms we take $\mathrm{Mor}(x, y)$ empty if $x \neq y$ and equal to $\{\mathrm{id}_x\}$ if $x = y$.

Definition 4.2.8. A *functor* $F : \mathcal{A} \to \mathcal{B}$ between two categories $\mathcal{A}, \mathcal{B}$ is given by the following data:

(1) A map $F : \mathrm{Ob}(\mathcal{A}) \to \mathrm{Ob}(\mathcal{B})$.
(2) For every $x, y \in \mathrm{Ob}(\mathcal{A})$ a map $F : \mathrm{Mor}_{\mathcal{A}}(x, y) \to \mathrm{Mor}_{\mathcal{B}}(F(x), F(y))$, denoted $\phi \mapsto F(\phi)$.

These data should be compatible with composition and identity morphisms in the following manner: $F(\phi \circ \psi) = F(\phi) \circ F(\psi)$ for a composable pair (ϕ, ψ) of morphisms of $\mathcal{A}$ and $F(\mathrm{id}_x) = \mathrm{id}_{F(x)}$.

Note that every category $\mathcal{A}$ has an *identity* functor $\mathrm{id}_{\mathcal{A}}$. In addition, given a functor $G : \mathcal{B} \to \mathcal{C}$ and a functor $F : \mathcal{A} \to \mathcal{B}$ there is a *composition* functor $G \circ F : \mathcal{A} \to \mathcal{C}$ defined in an obvious manner.

Definition 4.2.9. Let $F : \mathcal{A} \to \mathcal{B}$ be a functor.

(1) We say F is *faithful* if for any objects x, y of $\mathrm{Ob}(\mathcal{A})$ the map
$$F : \mathrm{Mor}_{\mathcal{A}}(x, y) \to \mathrm{Mor}_{\mathcal{B}}(F(x), F(y))$$
is injective.
(2) If these maps are all bijective then F is called *fully faithful*.
(3) The functor F is called *essentially surjective* if for any object $y \in \mathrm{Ob}(\mathcal{B})$ there exists an object $x \in \mathrm{Ob}(\mathcal{A})$ such that $F(x)$ is isomorphic to y in $\mathcal{B}$.

Definition 4.2.10. A *subcategory* of a category $\mathcal{B}$ is a category $\mathcal{A}$ whose objects and arrows form subsets of the objects and arrows of $\mathcal{A}$ and such that source, target and composition in $\mathcal{A}$ agree with those of $\mathcal{B}$. We say $\mathcal{A}$ is a *full subcategory* of $\mathcal{B}$ if $\mathrm{Mor}_{\mathcal{A}}(x, y) = \mathrm{Mor}_{\mathcal{B}}(x, y)$ for all $x, y \in \mathrm{Ob}(\mathcal{A})$. We say $\mathcal{A}$ is a *strictly full* subcategory of $\mathcal{B}$ if it is a full subcategory and given $x \in \mathrm{Ob}(\mathcal{A})$ any object of $\mathcal{B}$ which is isomorphic to x is also in $\mathcal{A}$.

If $\mathcal{A} \subset \mathcal{B}$ is a subcategory then the identity map is a functor from $\mathcal{A}$ to $\mathcal{B}$. Furthermore a subcategory $\mathcal{A} \subset \mathcal{B}$ is full if and only if the inclusion functor is fully faithful. Note that given a category $\mathcal{B}$ the set of full subcategories of $\mathcal{B}$ is the same as the set of subsets of $\mathrm{Ob}(\mathcal{B})$.

Remark 4.2.11. Suppose that $\mathcal{A}$ is a category. A functor F from $\mathcal{A}$ to *Sets* is a mathematical object (i.e., it is a set not a class or a formula of set theory, see Sets, Section 3.2) even though the category of sets is "big". Namely, the range of F on objects will be a set $F(\mathrm{Ob}(\mathcal{A}))$ and then we may think of F as a functor between $\mathcal{A}$ and the full subcategory of the category of sets whose objects are elements of $F(\mathrm{Ob}(\mathcal{A}))$.

Example 4.2.12. A homomorphism $p : G \to H$ of groups gives rise to a functor between the associated groupoids in Example 4.2.6. It is faithful (resp. fully faithful) if and only if p is injective (resp. an isomorphism).

Example 4.2.13. Given a category $\mathcal{C}$ and an object $X \in \mathrm{Ob}(\mathcal{C})$ we define the *category of objects over* X, denoted $\mathcal{C}/X$ as follows. The objects of $\mathcal{C}/X$ are morphisms $Y \to X$ for some $Y \in \mathrm{Ob}(\mathcal{C})$. Morphisms between objects $Y \to X$ and $Y' \to X$ are morphisms $Y \to Y'$ in $\mathcal{C}$ that make the obvious diagram commute. Note that there is a functor $p_X : \mathcal{C}/X \to \mathcal{C}$ which simply forgets the morphism. Moreover given a morphism $f : X' \to X$ in $\mathcal{C}$ there is an induced functor $F : \mathcal{C}/X' \to \mathcal{C}/X$ obtained by composition with f, and $p_X \circ F = p_{X'}$.

Example 4.2.14. Given a category $\mathcal{C}$ and an object $X \in \mathrm{Ob}(\mathcal{C})$ we define the *category of objects under* X, denoted $X/\mathcal{C}$ as follows. The objects of $X/\mathcal{C}$ are morphisms $X \to Y$ for some $Y \in \mathrm{Ob}(\mathcal{C})$. Morphisms between objects $X \to Y$ and $X \to Y'$ are morphisms $Y \to Y'$ in $\mathcal{C}$ that make the obvious diagram commute. Note that there is a functor $p_X : X/\mathcal{C} \to \mathcal{C}$ which simply forgets the morphism. Moreover given a morphism $f : X' \to X$ in $\mathcal{C}$ there is an induced functor $F : X/\mathcal{C} \to X'/\mathcal{C}$ obtained by composition with f, and $p_{X'} \circ F = p_X$.

Definition 4.2.15. Let $F, G : \mathcal{A} \to \mathcal{B}$ be functors. A *natural transformation*, or a *morphism of functors* $t : F \to G$, is a collection $\{t_x\}_{x \in \mathrm{Ob}(\mathcal{A})}$ such that

(1) $t_x : F(x) \to G(x)$ is a morphism in the category $\mathcal{B}$, and
(2) for every morphism $\phi : x \to y$ of $\mathcal{A}$ the following diagram is commutative

$$\begin{array}{ccc} F(x) & \xrightarrow{t_x} & G(x) \\ {\scriptstyle F(\phi)}\downarrow & & \downarrow{\scriptstyle G(\phi)} \\ F(y) & \xrightarrow{t_y} & G(y) \end{array}$$

Sometimes we use the diagram

$$\mathcal{A} \overset{F}{\underset{G}{\Downarrow t}} \mathcal{B}$$

to indicate that t is a morphism from F to G.

Note that every functor F comes with the *identity* transformation $\mathrm{id}_F : F \to F$. In addition, given a morphism of functors $t : F \to G$ and a morphism of functors $s : E \to F$ then the *composition* $t \circ s$ is defined by the rule

$$(t \circ s)_x = t_x \circ s_x : E(x) \to G(x)$$

for $x \in \mathrm{Ob}(\mathcal{A})$. It is easy to verify that this is indeed a morphism of functors from E to G. In this way, given categories $\mathcal{A}$ and $\mathcal{B}$ we obtain a new category, namely the category of functors between $\mathcal{A}$ and $\mathcal{B}$.

Remark 4.2.16. This is one instance where the same thing does not hold if $\mathcal{A}$ is a "big" category. For example consider functors $Sets \to Sets$. As we have currently defined it such a functor is a class and not a set. In other words, it is given by a formula in set theory (with some variables equal to specified sets)! It is not a good idea to try to consider all possible formulae of set theory as part of the definition of a mathematical object. The same problem presents itself when considering sheaves on the category of schemes for example. We will come back to this point later.

Definition 4.2.17. An *equivalence of categories* $F : \mathcal{A} \to \mathcal{B}$ is a functor such that there exists a functor $G : \mathcal{B} \to \mathcal{A}$ such that the compositions $F \circ G$ and $G \circ F$ are isomorphic to the identity functors $\mathrm{id}_{\mathcal{B}}$, respectively $\mathrm{id}_{\mathcal{A}}$. In this case we say that G is a *quasi-inverse* to F.

Lemma 4.2.18. *Let $F : \mathcal{A} \to \mathcal{B}$ be a fully faithful functor. Suppose for every $X \in \mathrm{Ob}(\mathcal{B})$ given an object $j(X)$ of $\mathcal{A}$ and an isomorphism $i_X : X \to F(j(X))$. Then there is a unique functor $j : \mathcal{B} \to \mathcal{A}$ such that j extends the rule on objects, and the isomorphisms i_X define an isomorphism of functors $id_{\mathcal{B}} \to F \circ j$. Moreover, j and F are quasi-inverse equivalences of categories.*

Proof. This lemma proves itself. □

Lemma 4.2.19. *A functor is an equivalence of categories if and only if it is both fully faithful and essentially surjective.*

Proof. Let $F : \mathcal{A} \to \mathcal{B}$ be essentially surjective and fully faithful. As by convention all categories are small and as F is essentially surjective we can, using the axiom of choice, choose for every $X \in \mathrm{Ob}(\mathcal{B})$ an object $j(X)$ of $\mathcal{A}$ and an isomorphism $i_X : X \to F(j(X))$. Then we apply Lemma 4.2.18 using that F is fully faithful. □

Definition 4.2.20. Let $\mathcal{A}$, $\mathcal{B}$ be categories. We define the *product category* $\mathcal{A} \times \mathcal{B}$ to be the category with objects $\mathrm{Ob}(\mathcal{A} \times \mathcal{B}) = \mathrm{Ob}(\mathcal{A}) \times \mathrm{Ob}(\mathcal{B})$ and

$$\mathrm{Mor}_{\mathcal{A}\times\mathcal{B}}((x,y),(x',y')) := \mathrm{Mor}_{\mathcal{A}}(x,x') \times \mathrm{Mor}_{\mathcal{B}}(y,y').$$

Composition is defined componentwise.

4.3. Opposite Categories and the Yoneda Lemma

Definition 4.3.1. Given a category $\mathcal{C}$ the *opposite category* $\mathcal{C}^{opp}$ is the category with the same objects as $\mathcal{C}$ but all morphisms reversed.

In other words $\mathrm{Mor}_{\mathcal{C}^{opp}}(x,y) = \mathrm{Mor}_{\mathcal{C}}(y,x)$. Composition in $\mathcal{C}^{opp}$ is the same as in $\mathcal{C}$ except backwards: if $\phi : y \to z$ and $\psi : x \to y$ in $\mathcal{C}^{opp}$ then $\phi \circ^{opp} \psi := \psi \circ \phi$.

Definition 4.3.2. Let $\mathcal{C}$, $\mathcal{S}$ be categories. A *contravariant* functor F from $\mathcal{C}$ to $\mathcal{S}$ is a functor $\mathcal{C}^{opp} \to \mathcal{S}$.

Concretely, a contravariant functor F is given by a map $F : \mathrm{Ob}(\mathcal{C}) \to \mathrm{Ob}(\mathcal{S})$ and for every morphism $\psi : x \to y$ in $\mathcal{C}$ a morphism $F(\psi) : F(y) \to F(x)$. These should satisfy the property that, given another morphism $\phi : y \to z$, we have $F(\phi \circ \psi) = F(\psi) \circ F(\phi)$ as morphisms $F(z) \to F(x)$. (Note the reverse of order.)

Definition 4.3.3. Let $\mathcal{C}$ be a category.

(1) A *presheaf of sets on* $\mathcal{C}$ or simply a *presheaf* is a contravariant functor F from $\mathcal{C}$ to *Sets*.
(2) The category of presheaves is denoted $PSh(\mathcal{C})$.

Of course the category of presheaves is a proper class.

Example 4.3.4. Functor of points. For any $U \in \mathrm{Ob}(\mathcal{C})$ there is a contravariant functor

$$\begin{array}{rccc} h_U & : & \mathcal{C} & \longrightarrow & Sets \\ & & X & \longmapsto & \mathrm{Mor}_{\mathcal{C}}(X,U) \end{array}$$

which takes an object X to the set $\mathrm{Mor}_{\mathcal{C}}(X,U)$. In other words h_U is a presheaf. Given a morphism $f : X \to Y$ the corresponding map $h_U(f) : \mathrm{Mor}_{\mathcal{C}}(Y,U) \to \mathrm{Mor}_{\mathcal{C}}(X,U)$ takes ϕ to $\phi \circ f$. We will always denote this presheaf $h_U : \mathcal{C}^{opp} \to Sets$. It is called the *representable presheaf* associated to U. If $\mathcal{C}$ is the category of schemes this functor is sometimes referred to as the *functor of points* of U.

Note that given a morphism $\phi : U \to V$ in $\mathcal{C}$ we get a corresponding natural transformation of functors $h(\phi) : h_U \to h_V$ defined simply by composing with the morphism $U \to V$. It is trivial to see that this turns composition of morphisms in $\mathcal{C}$ into composition of transformations of functors. In other words we get a functor

$$h : \mathcal{C} \longrightarrow \mathrm{Fun}(\mathcal{C}^{opp}, Sets) = PSh(\mathcal{C})$$

Note that the target is a "big" category, see Remark 4.2.2. On the other hand, h is an actual mathematical object (i.e. a set), compare Remark 4.2.11.

Lemma 4.3.5 (Yoneda lemma)**.** *Let $U, V \in \mathrm{Ob}(\mathcal{C})$. Given any morphism of functors $s : h_U \to h_V$ there is a unique morphism $\phi : U \to V$ such that $h(\phi) = s$. In other words the functor h is fully faithful. More generally, given any contravariant functor F and any object U of $\mathcal{C}$ we have a natural bijection*

$$\mathrm{Mor}_{PSh(\mathcal{C})}(h_U, F) \longrightarrow F(U), \quad s \longmapsto s_U(id_U).$$

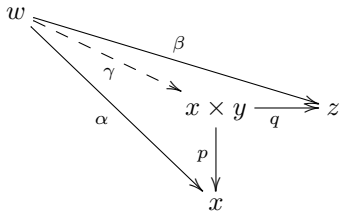
Proof. Just take $\phi = s_U(\mathrm{id}_U) \in \mathrm{Mor}_{\mathcal{C}}(U, V)$. □

Definition 4.3.6. A contravariant functor $F : \mathcal{C} \to \mathit{Sets}$ is said to be *representable* if it is isomorphic to the functor of points h_U for some object U of $\mathcal{C}$.

Choose an object U of $\mathcal{C}$ and an isomorphism $s : h_U \to F$. The Yoneda lemma guarantees that the pair (U, s) is unique up to unique isomorphism. The object U is called an object *representing* F.

4.4. Products of pairs

Definition 4.4.1. Let $x, y \in \mathrm{Ob}(\mathcal{C})$. A *product* of x and y is an object $x \times y \in \mathrm{Ob}(\mathcal{C})$ together with morphisms $p \in \mathrm{Mor}_{\mathcal{C}}(x \times y, x)$ and $q \in \mathrm{Mor}_{\mathcal{C}}(x \times y, y)$ such that the following universal property holds: for any $w \in \mathrm{Ob}(\mathcal{C})$ and morphisms $\alpha \in \mathrm{Mor}_{\mathcal{C}}(w, x)$ and $\beta \in \mathrm{Mor}_{\mathcal{C}}(w, y)$ there is a unique $\gamma \in \mathrm{Mor}_{\mathcal{C}}(w, x \times y)$ making the diagram



commute.

If a product exists it is unique up to unique isomorphism. This follows from the Yoneda lemma as the definition requires $x \times y$ to be an object of $\mathcal{C}$ such that

$$h_{x \times y}(w) = h_x(w) \times h_y(w)$$

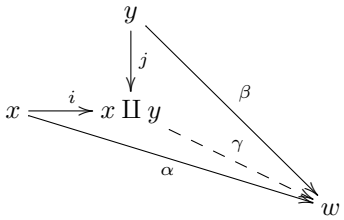
functorially in w. In other words the product $x \times y$ is an object representing the functor $w \mapsto h_x(w) \times h_y(w)$.

Definition 4.4.2. We say the category $\mathcal{C}$ *has products of pairs of objects* if a product $x \times y$ exists for any $x, y \in \mathrm{Ob}(\mathcal{C})$.

We use this terminology to distinguish this notion from the notion of "having products" or "having finite products" which usually means something else (in particular it always implies there exists a final object).

4.5. Coproducts of pairs

Definition 4.5.1. Let $x, y \in \mathrm{Ob}(\mathcal{C})$. A *coproduct*, or *amalgamated sum* of x and y is an object $x \amalg y \in \mathrm{Ob}(\mathcal{C})$ together with morphisms $i \in \mathrm{Mor}_{\mathcal{C}}(x, x \amalg y)$ and $j \in \mathrm{Mor}_{\mathcal{C}}(y, x \amalg y)$ such that the following universal property holds: for any $w \in \mathrm{Ob}(\mathcal{C})$ and morphisms $\alpha \in \mathrm{Mor}_{\mathcal{C}}(x, w)$ and $\beta \in \mathrm{Mor}_{\mathcal{C}}(y, w)$ there is a unique $\gamma \in \mathrm{Mor}_{\mathcal{C}}(x \amalg y, w)$ making the diagram



commute.

If a coproduct exists it is unique up to unique isomorphism. This follows from the Yoneda lemma (applied to the opposite category) as the definition requires $x \amalg y$ to be an object of $\mathcal{C}$ such that

$$\mathrm{Mor}_{\mathcal{C}}(x \amalg y, w) = \mathrm{Mor}_{\mathcal{C}}(x, w) \times \mathrm{Mor}_{\mathcal{C}}(y, w)$$

functorially in w.

Definition 4.5.2. We say the category $\mathcal{C}$ *has coproducts of pairs of objects* if a coproduct $x \amalg y$ exists for any $x, y \in \mathrm{Ob}(\mathcal{C})$.

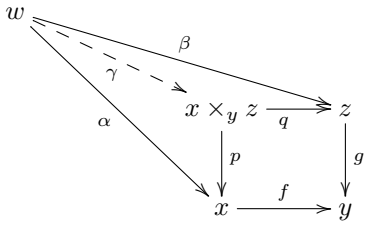
We use this terminology to distinguish this notion from the notion of "having coproducts" or "having finite coproducts" which usually means something else (in particular it always implies there exists an initial object in $\mathcal{C}$).

4.6. Fibre products

Definition 4.6.1. Let $x, y, z \in \mathrm{Ob}(\mathcal{C})$, $f \in \mathrm{Mor}_{\mathcal{C}}(x, y)$ and $g \in \mathrm{Mor}_{\mathcal{C}}(z, y)$. A *fibre product* of f and g is an object $x \times_y z \in \mathrm{Ob}(\mathcal{C})$ together with morphisms $p \in \mathrm{Mor}_{\mathcal{C}}(x \times_y z, x)$ and $q \in \mathrm{Mor}_{\mathcal{C}}(x \times_y z, z)$ making the diagram

$$\begin{array}{ccc} x \times_y z & \xrightarrow{q} & z \\ {\scriptstyle p}\downarrow & & \downarrow{\scriptstyle g} \\ x & \xrightarrow{f} & y \end{array}$$

commute, and such that the following universal property holds: for any $w \in \mathrm{Ob}(\mathcal{C})$ and morphisms $\alpha \in \mathrm{Mor}_{\mathcal{C}}(w, x)$ and $\beta \in \mathrm{Mor}_{\mathcal{C}}(w, z)$ with $f \circ \alpha = g \circ \beta$ there is a unique $\gamma \in \mathrm{Mor}_{\mathcal{C}}(w, x \times_y z)$ making the diagram



commute.

If a fibre product exists it is unique up to unique isomorphism. This follows from the Yoneda lemma as the definition requires $x \times_y z$ to be an object of $\mathcal{C}$ such that

$$h_{x \times_y z}(w) = h_x(w) \times_{h_y(w)} h_z(w)$$

functorially in w. In other words the fibre product $x \times_y z$ is an object representing the functor $w \mapsto h_x(w) \times_{h_y(w)} h_z(w)$.

Definition 4.6.2. We say a commutative diagram

$$\begin{array}{ccc} w & \longrightarrow & z \\ \downarrow & & \downarrow \\ x & \longrightarrow & y \end{array}$$

in a category is *cartesian* if w and the morphisms $w \to x$ and $w \to z$ form a fibre product of the morphisms $x \to y$ and $z \to y$.

Definition 4.6.3. We say the category $\mathcal{C}$ *has fibre products* if the fibre product exists for any $f \in \mathrm{Mor}_{\mathcal{C}}(x, z)$ and $g \in \mathrm{Mor}_{\mathcal{C}}(y, z)$.

Definition 4.6.4. A morphism $f : x \to y$ of a category $\mathcal{C}$ is said to be *representable*, if and only if for every morphism $z \to y$ in $\mathcal{C}$ the fibre product $z \times_y x$ exists.

Lemma 4.6.5. *Let $\mathcal{C}$ be a category. Let $f : x \to y$, and $g : y \to z$ be representable. Then $g \circ f : x \to z$ is representable.*

Proof. Omitted. □

Lemma 4.6.6. *Let $\mathcal{C}$ be a category. Let $f : x \to y$ be representable. Let $y' \to y$ be a morphism of $\mathcal{C}$. Then the morphism $x' := x \times_y y' \to y'$ is representable also.*

Proof. Let $z \to y'$ be a morphism. The fibre product $x' \times_{y'} z$ is supposed to represent the functor

$$\begin{aligned} w &\mapsto h_{x'}(w) \times_{h_{y'}(w)} h_z(w) \\ &= (h_x(w) \times_{h_y(w)} h_{y'}(w)) \times_{h_{y'}(w)} h_z(w) \\ &= h_x(w) \times_{h_y(w)} h_z(w) \end{aligned}$$

which is representable by assumption. □

4.7. Examples of fibre products

In this section we list examples of fibre products and we describe them.

As a really trivial first example we observe that the category of sets has fibred products and hence every morphism is representable. Namely, if $f : X \to Y$ and $g : Z \to Y$ are maps of sets then we define $X \times_Y Z$ as the subset of $X \times Z$ consisting of pairs (x, z) such that $f(x) = g(z)$. The morphisms $p : X \times_Y Z \to X$ and $q : X \times_U Z \to Z$ are the projection maps $(x, z) \mapsto x$, and $(x, z) \mapsto z$. Finally, if $\alpha : W \to X$ and $\beta : W \to Z$ are morphisms such that $f \circ \alpha = g \circ \beta$ then the map $W \to X \times Y$, $w \mapsto (\alpha(w), \beta(w))$ obviously ends up in $X \times_Y Z$ as desired.

In many categories whose objects are sets endowed with certain types of algebraic structures the fibre product of the underlying sets also provides the fibre product in the category. For example, suppose that X, Y and Z above are groups and that f, g are homomorphisms of groups. Then the set-theoretic fibre product $X \times_Y Z$ inherits the structure of a group, simply by defining the product of two pairs by the formula $(x, z) \cdot (x', z') = (xx', zz')$. Here we list those categories for which a similar reasoning works.

(1) The category *Groups* of groups.
(2) The category *G-Sets* of sets endowed with a left G-action for some fixed group G.
(3) The category of rings.
(4) The category of R-modules given a ring R.

4.8. Fibre products and representability

In this section we work out fibre products in the category of contravariant functors from a category to the category of sets. This will later be superseded during the discussion of sites, presheaves, sheaves. Of some interest is the notion of a "representable morphism" between such functors.

Lemma 4.8.1. *Let $\mathcal{C}$ be a category. Let $F, G, H : \mathcal{C}^{opp} \to Sets$ be functors. Let $a : F \to G$ and $b : H \to G$ be transformations of functors. Then the fibre product $F \times_{a,G,b} H$ in the category $Fun(\mathcal{C}^{opp}, Sets)$ exists and is given by the formula*

$$(F \times_{a,G,b} H)(X) = F(X) \times_{a_X, G(X), b_X} H(X)$$

for any object X of $\mathcal{C}$.

Proof. Omitted. □

As a special case suppose we have a morphism $a : F \to G$, an object $U \in \mathrm{Ob}(\mathcal{C})$ and an element $\xi \in G(U)$. According to the Yoneda Lemma 4.3.5 this gives a transformation $\xi : h_U \to G$. The fibre product in this case is described by the rule

$$(h_U \times_{\xi,G,a} F)(X) = \{(f, \xi') \mid f : X \to U,\ \xi' \in F(X),\ G(f)(\xi) = a_X(\xi')\}$$

If F, G are also representable, then this is the functor representing the fibre product, if it exists, see Section 4.6. The analogy with Definition 4.6.4 prompts us to define a notion of representable transformations.

Definition 4.8.2. Let $\mathcal{C}$ be a category. Let $F, G : \mathcal{C}^{opp} \to Sets$ be functors. We say a morphism $a : F \to G$ is *representable*, or that *F is relatively representable over G*, if for every $U \in \mathrm{Ob}(\mathcal{C})$ and any $\xi \in G(U)$ the functor $h_U \times_G F$ is representable.

Lemma 4.8.3. *Let $\mathcal{C}$ be a category. Let $a : F \to G$ be a morphism of contravariant functors from $\mathcal{C}$ to Sets. If a is representable, and G is a representable functor, then F is representable.*

Proof. Omitted. □

Lemma 4.8.4. *Let $\mathcal{C}$ be a category. Let $F : \mathcal{C}^{opp} \to Sets$ be a functor. Assume $\mathcal{C}$ has products of pairs of objects and fibre products. The following are equivalent:*

1. *The diagonal $F \to F \times F$ is representable.*
2. *For every U in $\mathcal{C}$, and any $\xi \in F(U)$ the map $\xi : h_U \to F$ is representable.*

Proof. Suppose the diagonal is representable, and let U, ξ be given. Consider any $V \in \mathrm{Ob}(\mathcal{C})$ and any $\xi' \in F(V)$. Note that $h_U \times h_V = h_{U \times V}$ is representable. Hence the fibre product of the maps $(\xi, \xi') : h_U \times h_V \to F \times F$ and $F \to F \times F$ is representable by assumption. This means there exists $W \in \mathrm{Ob}(\mathcal{C})$, morphisms $W \to U$, $W \to V$ and $h_W \to F$ such that

$$\begin{array}{ccc} h_W & \longrightarrow & F \\ \downarrow & & \downarrow \\ h_U \times h_V & \longrightarrow & F \times F \end{array}$$

is cartesian. We leave it to the reader to see that this implies that $h_W = h_U \times_F h_V$ as desired.

Assume (2) holds. Consider any $V \in \mathrm{Ob}(\mathcal{C})$ and any $(\xi, \xi') \in (F \times F)(V)$. We have to show that $h_V \times_{F \times F} F$ is representable. What we know is that $h_V \times_{\xi, F, \xi'} h_V$ is representable, say by W in $\mathcal{C}$ with corresponding morphisms $a, a' : W \to V$ (such that $\xi \circ a = \xi' \circ a'$). Consider $W' = W \times_{(a,a'), V \times V} V$. It is formal to show that W' represents $h_V \times_{F \times F} F$ because

$$h_{W'} = h_W \times_{h_V \times h_V} h_V = (h_V \times_{\xi, F, \xi'} h_V) \times_{h_V \times h_V} h_V = F \times_{F \times F} h_V.$$

□

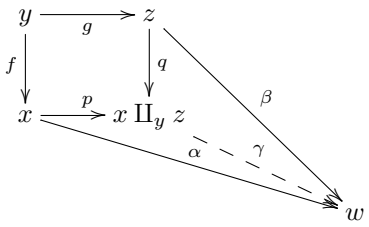
4.9. Pushouts

The dual notion to fibre products is that of pushouts.

Definition 4.9.1. Let $x, y, z \in \mathrm{Ob}(\mathcal{C})$, $f \in \mathrm{Mor}_{\mathcal{C}}(y, x)$ and $g \in \mathrm{Mor}_{\mathcal{C}}(y, z)$. A *pushout* of f and g is an object $x \amalg_y z \in \mathrm{Ob}(\mathcal{C})$ together with morphisms $p \in \mathrm{Mor}_{\mathcal{C}}(x, x \amalg_y z)$ and $q \in \mathrm{Mor}_{\mathcal{C}}(z, x \amalg_y z)$ making the diagram

$$\begin{array}{ccc} y & \xrightarrow{g} & z \\ {\scriptstyle f}\downarrow & & \downarrow{\scriptstyle q} \\ x & \xrightarrow{p} & x \amalg_y z \end{array}$$

commute, and such that the following universal property holds: For any $w \in \mathrm{Ob}(\mathcal{C})$ and morphisms $\alpha \in \mathrm{Mor}_{\mathcal{C}}(x, w)$ and $\beta \in \mathrm{Mor}_{\mathcal{C}}(z, w)$ with $\alpha \circ f = \beta \circ g$ there is a unique $\gamma \in \mathrm{Mor}_{\mathcal{C}}(x \amalg_y z, w)$ making the diagram



commute.

It is possible and straightforward to prove the uniqueness of the triple $(x \amalg_y z, p, q)$ up to unique isomorphism (if it exists) by direct arguments. Another possibility is to think of the coproduct as the product in the opposite category, thereby getting this uniqueness for free from the discussion in Section 4.6.

Definition 4.9.2. We say a commutative diagram

$$\begin{array}{ccc} y & \longrightarrow & z \\ \downarrow & & \downarrow \\ x & \longrightarrow & w \end{array}$$

in a category is *cocartesian* if w and the morphisms $x \to w$ and $z \to w$ form a pushout of the morphisms $y \to x$ and $y \to z$.

4.10. Equalizers

Definition 4.10.1. Suppose that X, Y are objects of a category $\mathcal{C}$ and that $a, b : X \to Y$ are morphisms. We say a morphism $e : Z \to X$ is an *equalizer* for the pair (a, b) if $a \circ e = b \circ e$ and if (Z, e) satisfies the following universal property: For every morphism $t : W \to X$ in $\mathcal{C}$ such that $a \circ t = b \circ t$ there exists a unique morphism $s : W \to Z$ such that $t = e \circ s$.

As in the case of the fibre product above, equalizers when they exist are unique up to unique isomorphism. There is a straightforward generalization of this definition to the case where we have more than 2 morphisms.

4.11. Coequalizers

Definition 4.11.1. Suppose that X, Y are objects of a category $\mathcal{C}$ and that $a, b : X \to Y$ are morphisms. We say a morphism $c : Y \to Z$ is a *coequalizer* for the pair (a, b) if $c \circ a = c \circ b$ and if (Z, c) satisfies the following universal property: For every morphism $t : Y \to W$ in $\mathcal{C}$ such that $t \circ a = t \circ b$ there exists a unique morphism $s : Z \to W$ such that $t = s \circ c$.

As in the case of the pushouts above, coequalizers when they exist are unique up to unique isomorphism, and this follows from the uniqueness of equalizers upon considering the opposite category. There is a straightforward generalization of this definition to the case where we have more than 2 morphisms.

4.12. Initial and final objects

Definition 4.12.1. Let $\mathcal{C}$ be a category.

(1) An object x of the category $\mathcal{C}$ is called an *initial* object if for every object y of $\mathcal{C}$ there is exactly one morphism $x \to y$.
(2) An object x of the category $\mathcal{C}$ is called a *final* object if for every object y of $\mathcal{C}$ there is exactly one morphism $y \to x$.

In the category of sets the empty set $\emptyset$ is an initial object, and in fact the only initial object. Also, any *singleton*, i.e., a set with one element, is a final object (so it is not unique).

4.13. Monomorphisms and Epimorphisms

Definition 4.13.1. Let $\mathcal{C}$ be a category and let $f : X \to Y$ be a morphism of $\mathcal{C}$.

(1) We say that f is a *monomorphism* if for every object W and every pair of morphisms $a, b : W \to X$ such that $f \circ a = f \circ b$ we have $a = b$.
(2) We say that f is an *epimorphism* if for every object W and every pair of morphisms $a, b : Y \to W$ such that $a \circ f = b \circ f$ we have $a = b$.

Example 4.13.2. In the category of sets the monomorphisms correspond to injective maps and the epimorphisms correspond to surjective maps.

Lemma 4.13.3. *Let $\mathcal{C}$ be a category, and let $f : X \to Y$ be a morphism of $\mathcal{C}$. Then*

(1) *f is a monomorphism if and only if X is the fibre product $X \times_Y X$, and*
(2) *f is an epimorphism if and only if Y is the pushout $Y \amalg_X Y$.*

Proof. Omitted. □

4.14. Limits and colimits

Let $\mathcal{C}$ be a category. A *diagram* in $\mathcal{C}$ is simply a functor $M : \mathcal{I} \to \mathcal{C}$. We say that $\mathcal{I}$ is the *index category* or that M is an $\mathcal{I}$-diagram. We will use the notation M_i to denote the image of the object i of $\mathcal{I}$. Hence for $\phi : i \to i'$ a morphism in $\mathcal{I}$ we have $M(\phi) : M_i \to M_{i'}$.

Definition 4.14.1. A *limit* of the $\mathcal{I}$-diagram M in the category $\mathcal{C}$ is given by an object $\lim_I M$ in $\mathcal{C}$ together with morphisms $p_i : \lim_I M \to M_i$ such that

(1) for $\phi : i \to i'$ a morphism in $\mathcal{I}$ we have $p_{i'} = M(\phi) \circ p_i$, and
(2) for any object W in $\mathcal{C}$ and any family of morphisms $q_i : W \to M_i$ such that for all $\phi : i \to i'$ in $\mathcal{I}$ we have $q_{i'} = M(\phi) \circ q_i$ there exists a unique morphism $q : W \to \lim_I M$ such that $q_i = p_i \circ q$ for every object i of $\mathcal{I}$.

Limits $(\lim_I M, (p_i)_{i \in \mathrm{Ob}(\mathcal{I})})$ are (if they exist) unique up to unique isomorphism by the uniqueness requirement in the definition. Products of pairs, fibred products, and equalizers are examples of limits. The limit over the empty diagram is a final object of $\mathcal{C}$. In the category of sets all limits exist. The dual notion is that of colimits.

Definition 4.14.2. A *colimit* of the $\mathcal{I}$-diagram M in the category $\mathcal{C}$ is given by an object $\mathrm{colim}_I M$ in $\mathcal{C}$ together with morphisms $s_i : M_i \to \mathrm{colim}_I M$ such that

(1) for $\phi : i \to i'$ a morphism in $\mathcal{I}$ we have $s_i = s_{i'} \circ M(\phi)$, and
(2) for any object W in $\mathcal{C}$ and any family of morphisms $t_i : M_i \to W$ such that for all $\phi : i \to i'$ in $\mathcal{I}$ we have $t_i = t_{i'} \circ M(\phi)$ there exists a unique morphism $t : \mathrm{colim}_I M \to W$ such that $t_i = t \circ s_i$ for every object i of $\mathcal{I}$.

Colimits $(\mathrm{colim}_I M, (s_i)_{i \in \mathrm{Ob}(\mathcal{I})})$ are (if they exist) unique up to unique isomorphism by the uniqueness requirement in the definition. Coproducts of pairs, pushouts, and coequalizers are examples of colimits. The colimit over an empty diagram is an initial object of $\mathcal{C}$. In the category of sets all colimits exist.

Remark 4.14.3. The index category of a (co)limit will never be allowed to have a proper class of objects. In this project it means that it cannot be one of the categories listed in Remark 4.2.2

Remark 4.14.4. We often write $\lim_i M_i$, $\mathrm{colim}_i M_i$, $\lim_{i \in \mathcal{I}} M_i$, or $\mathrm{colim}_{i \in \mathcal{I}} M_i$ instead of the versions indexed by $\mathcal{I}$. Using this notation, and using the description of limits and colimits of sets in Section 4.15 below, we can say the following. Let $M : \mathcal{I} \to \mathcal{C}$ be a diagram.

(1) The object $\lim_i M_i$ if it exists satisfies the following property
$$\mathrm{Mor}_{\mathcal{C}}(W, \lim_i M_i) = \lim_i \mathrm{Mor}_{\mathcal{C}}(W, M_i)$$
where the limit on the right takes place in the category of sets.
(2) The object $\mathrm{colim}_i M_i$ if it exists satisfies the following property
$$\mathrm{Mor}_{\mathcal{C}}(\mathrm{colim}_i M_i, W) = \lim_{i \in \mathcal{I}^{\mathrm{opp}}} \mathrm{Mor}_{\mathcal{C}}(M_i, W)$$
where on the right we have the limit over the opposite category with value in the category of sets.

By the Yoneda lemma (and its dual) this formula completely determines the limit, respectively the colimit.

As an application of the notions of limits and colimits we define products and coproducts.

Definition 4.14.5. Suppose that I is a set, and suppose given for every $i \in I$ an object M_i of the category $\mathcal{C}$. A *product* $\prod_{i \in I} M_i$ is by definition $\lim_{\mathcal{I}} M$ (if it exists) where $\mathcal{I}$ is the category having only identities as morphisms and having the elements of I as objects.

An important special case is where $I = \emptyset$ in which case the product is a final object of the category. The morphisms $p_i : \prod M_i \to M_i$ are called the *projection morphisms*.

Definition 4.14.6. Suppose that I is a set, and suppose given for every $i \in I$ an object M_i of the category $\mathcal{C}$. A *coproduct* $\coprod_{i \in I} M_i$ is by definition $\operatorname{colim}_{\mathcal{I}} M$ (if it exists) where $\mathcal{I}$ is the category having only identities as morphisms and having the elements of I as objects.

An important special case is where $I = \emptyset$ in which case the product is an initial object of the category. Note that the coproduct comes equipped with morphisms $M_i \to \coprod M_i$. These are sometimes called the *coprojections*.

Lemma 4.14.7. *Suppose that $M : \mathcal{I} \to \mathcal{C}$, and $N : \mathcal{J} \to \mathcal{C}$ are diagrams whose colimits exist. Suppose $H : \mathcal{I} \to \mathcal{J}$ is a functor, and suppose $t : M \to N \circ H$ is a transformation of functors. Then there is a unique morphism*

$$\theta : \operatorname{colim}_{\mathcal{I}} M \longrightarrow \operatorname{colim}_{\mathcal{J}} N$$

such that all the diagrams

$$\begin{array}{ccc} M_i & \longrightarrow & \operatorname{colim}_{\mathcal{I}} M \\ {\scriptstyle t_i}\downarrow & & \downarrow{\scriptstyle \theta} \\ N_{H(i)} & \longrightarrow & \operatorname{colim}_{\mathcal{J}} N \end{array}$$

commute.

Proof. Omitted. □

Lemma 4.14.8. *Suppose that $M : \mathcal{I} \to \mathcal{C}$, and $N : \mathcal{J} \to \mathcal{C}$ are diagrams whose limits exist. Suppose $H : \mathcal{I} \to \mathcal{J}$ is a functor, and suppose $t : N \circ H \to M$ is a transformation of functors. Then there is a unique morphism*

$$\theta : \lim_{\mathcal{J}} N \longrightarrow \lim_{\mathcal{I}} M$$

such that all the diagrams

$$\begin{array}{ccc} \lim_{\mathcal{J}} N & \longrightarrow & N_{H(i)} \\ \downarrow{\scriptstyle \theta} & & {\scriptstyle t_{H(i)}}\downarrow \\ \lim_{\mathcal{I}} M & \longrightarrow & M_i \end{array}$$

commute.

Proof. Omitted. □

Lemma 4.14.9. *Let $\mathcal{I}$, $\mathcal{J}$ be index categories. Let $M : \mathcal{I} \times \mathcal{J} \to \mathcal{C}$ be a functor. We have*

$$\operatorname{colim}_i \operatorname{colim}_j M_{i,j} = \operatorname{colim}_{i,j} M_{i,j} = \operatorname{colim}_j \operatorname{colim}_i M_{i,j}$$

provided all the indicated colimits exist. Similar for limits.

Proof. Omitted. □

Lemma 4.14.10. *Let $M : \mathcal{I} \to \mathcal{C}$ be a diagram. Write $I = \mathrm{Ob}(\mathcal{I})$ and $A = Arrow(\mathcal{I})$. Denote $s, t : A \to I$ the source and target maps. Suppose that $\prod_{i \in I} M_i$ and $\prod_{a \in A} M_{t(a)}$ exist. Suppose that the equalizer of*

$$\prod_{i \in I} M_i \xrightarrow[\psi]{\phi} \prod_{a \in A} M_{t(a)}$$

exists, where the morphisms are determined by their components as follows: $p_a \circ \psi = a \circ p_{s(a)}$ and $p_a \circ \phi = p_{t(a)}$. Then this equalizer is the limit of the diagram.

Proof. Omitted. □

Lemma 4.14.11. *Let $M : \mathcal{I} \to \mathcal{C}$ be a diagram. Write $I = \mathrm{Ob}(\mathcal{I})$ and $A = Arrow(\mathcal{I})$. Denote $s, t : A \to I$ the source and target maps. Suppose that $\coprod_{i \in I} M_i$ and $\coprod_{a \in A} M_{s(a)}$ exist. Suppose that the coequalizer of*

$$\coprod_{a \in A} M_{s(a)} \xrightarrow[\psi]{\phi} \coprod_{i \in I} M_i$$

exists, where the morphisms are determined by their components as follows: The component $M_{s(a)}$ maps via ψ to the component $M_{t(a)}$ via the morphism a. The component $M_{s(a)}$ maps via ϕ to the component $M_{s(a)}$ by the identity morphism. Then this coequalizer is the colimit of the diagram.

Proof. Omitted. □

4.15. Limits and colimits in the category of sets

Not only do limits and colimits exist in *Sets* but they are also easy to describe. Namely, let $M : \mathcal{I} \to Sets$, $i \mapsto M_i$ be a diagram of sets. Denote $I = \mathrm{Ob}(\mathcal{I})$. The limit is described as

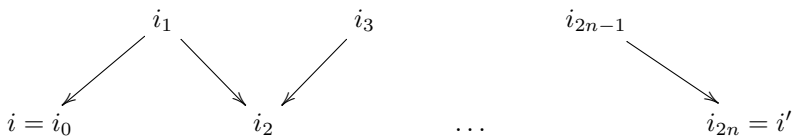
$$\lim_{\mathcal{I}} M = \{(m_i)_{i \in I} \in \prod_{i \in I} M_i \mid \forall \phi : i \to i' \text{ in } \mathcal{I}, M(\phi)(m_i) = m_{i'}\}.$$

So we think of an element of the limit as a compatible system of elements of all the sets M_i.

On the other hand, the colimit is

$$\mathrm{colim}_{\mathcal{I}} M = (\coprod_{i \in I} M_i)/\sim$$

where the equivalence relation $\sim$ is the equivalence relation generated by setting $m_i \sim m_{i'}$ if $m_i \in M_i$, $m_{i'} \in M_{i'}$ and $M(\phi)(m_i) = m_{i'}$ for some $\phi : i \to i'$. In other words, $m_i \in M_i$ and $m_{i'} \in M_{i'}$ are equivalent if there is a chain of morphisms in $\mathcal{I}$



and elements $m_{i_j} \in M_{i_j}$ mapping to each other under the maps $M_{i_{2k-1}} \to M_{i_{2k-2}}$ and $M_{i_{2k-1}} \to M_{i_{2k}}$ induced from the maps in $\mathcal{I}$ above.

This is not a very pleasant type of object to work with. But if the diagram is filtered then it is much easier to describe. We will explain this in Section 4.19.

4.16. Connected limits

A (co)limit is called connected if its index category is connected.

Definition 4.16.1. We say that a category $\mathcal{I}$ is *connected* if the equivalence relation generated by $x \sim y \Leftrightarrow \mathrm{Mor}_{\mathcal{I}}(x, y) \neq \emptyset$ has exactly one equivalence class.

Here we follow the convention of Topology, Definition 5.6.1 that connected spaces are nonempty. The following in some vague sense characterizes connected limits.

Lemma 4.16.2. *Let $\mathcal{C}$ be a category. Let X be an object of $\mathcal{C}$. Let $M : \mathcal{I} \to \mathcal{C}/X$ be a diagram in the category of objects over X. If the index category $\mathcal{I}$ is connected and the limit of M exists in $\mathcal{C}/X$, then the limit of the composition $\mathcal{I} \to \mathcal{C}/X \to \mathcal{C}$ exists and is the same.*

Proof. Let $M \to X$ be an object representing the limit in $\mathcal{C}/X$. Consider the functor

$$W \longmapsto \lim_i \mathrm{Mor}_{\mathcal{C}}(W, M_i).$$

Let (φ_i) be an element of the set on the right. Since each M_i comes equipped with a morphism $s_i : M_i \to X$ we get morphisms $f_i = s_i \circ \varphi_i : W \to X$. But as $\mathcal{I}$ is connected we see that all f_i are equal. Since $\mathcal{I}$ is nonempty there is at least one f_i. Hence this common value $W \to X$ defines the structure of an object of W in $\mathcal{C}/X$ and (φ_i) defines is an element of $\lim_i \mathrm{Mor}_{\mathcal{C}/X}(W, M_i)$. Thus we obtain a unique morphism $\phi : W \to M$ such that φ_i is the composition of ϕ with $M \to M_i$ as desired. □

Lemma 4.16.3. *Let $\mathcal{C}$ be a category. Let X be an object of $\mathcal{C}$. Let $M : \mathcal{I} \to X/\mathcal{C}$ be a diagram in the category of objects under X. If the index category $\mathcal{I}$ is connected and the colimit of M exists in $X/\mathcal{C}$, then the colimit of the composition $\mathcal{I} \to X/\mathcal{C} \to \mathcal{C}$ exists and is the same.*

Proof. Omitted. Hint: This lemma is dual to Lemma 4.16.2. □

4.17. Cofinal and initial categories

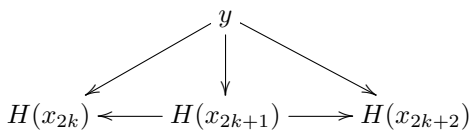
In the literature sometimes the word "final" is used instead of cofinal in the following definition.

Definition 4.17.1. Let $H : \mathcal{I} \to \mathcal{J}$ be a functor between categories. We say $\mathcal{I}$ *is cofinal in* $\mathcal{J}$ or that H is *cofinal* if

(1) for all $y \in \mathrm{Ob}(\mathcal{J})$ there exists a $x \in \mathrm{Ob}(\mathcal{I})$ and a morphism $y \to H(x)$, and
(2) given $y \in \mathrm{Ob}(\mathcal{J})$, $x, x' \in \mathrm{Ob}(\mathcal{I})$ and morphisms $y \to H(x)$ and $y \to H(x')$ there exists a sequence of morphisms

$$x = x_0 \leftarrow x_1 \to x_2 \leftarrow x_3 \to \ldots \to x_{2n} = x'$$

in $\mathcal{I}$ and morphisms $y \to H(x_i)$ in $\mathcal{J}$ such that the diagrams



commute for $k = 0, \ldots, n-1$.

Lemma 4.17.2. *Let $H : \mathcal{I} \to \mathcal{J}$ be a functor of categories. Assume $\mathcal{I}$ is cofinal in $\mathcal{J}$. Then for every diagram $M : \mathcal{J} \to \mathcal{C}$ we have a canonical isomorphism*

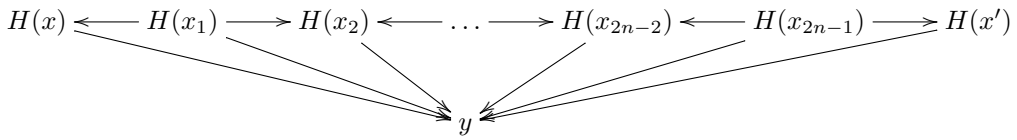
$$\operatorname{colim}_{\mathcal{I}} M \circ H = \operatorname{colim}_{\mathcal{J}} M$$

if either side exists.

Proof. Omitted. □

Definition 4.17.3. Let $H : \mathcal{I} \to \mathcal{J}$ be a functor between categories. We say $\mathcal{I}$ *is initial in* $\mathcal{J}$ or that H is *initial* if

(1) for all $y \in \mathrm{Ob}(\mathcal{J})$ there exists a $x \in \mathrm{Ob}(\mathcal{I})$ and a morphism $H(x) \to y$,
(2) for any $y \in \mathrm{Ob}(\mathcal{J})$, $x, x' \in \mathrm{Ob}(\mathcal{I}')$ and morphisms $H(x) \to y$, $H(x') \to y$ in $\mathcal{J}$ there exists $n \geq 0$ and a commutative diagram



where all the horizontal morphisms come from morphisms in $\mathcal{I}$, and the vertical arrows in $\mathcal{J}$.

This is just the dual notion to "cofinal" functors.

Lemma 4.17.4. *Let $H : \mathcal{I} \to \mathcal{J}$ be a functor of categories. Assume $\mathcal{I}$ is initial in $\mathcal{J}$. Then for every diagram $M : \mathcal{J} \to \mathcal{C}$ the limit $\lim_{\mathcal{J}} M$ exists if and only if $\lim_{\mathcal{I}} M$ exists and if so these limits agree.*

Proof. Omitted. □

Lemma 4.17.5. *Let $F : \mathcal{I} \to \mathcal{I}'$ be a functor. Assume*

(1) *the fibre categories (see Definition 4.30.2) of $\mathcal{I}$ over $\mathcal{I}'$ are all connected, and*
(2) *for every morphism $\alpha' : x' \to y'$ in $\mathcal{I}'$ there exist a morphism $\alpha : x \to y$ in $\mathcal{I}$ such that $F(\alpha) = \alpha'$.*

Then for every diagram $M : \mathcal{I}' \to \mathcal{C}$ the colimit $\operatorname{colim}_{\mathcal{I}} M \circ F$ exists if and only if $\operatorname{colim}_{\mathcal{I}'} M$ exists and if so these colimits agree.

Proof. One can prove this by showing that $\mathcal{I}$ is cofinal in $\mathcal{I}'$ and applying Lemma 4.17.2. But we can also prove it directly as follows. It suffices to show that for any object T of $\mathcal{C}$ we have

$$\lim_{\mathcal{I}} \operatorname{Mor}_{\mathcal{C}}(M_{F(i)}, T) = \lim_{\mathcal{I}'} \operatorname{Mor}_{\mathcal{C}}(M_{i'}, T)$$

If $(g_{i'})_{i' \in \mathrm{Ob}(\mathcal{I}')}$ is an element of the right hand side, then setting $f_i = g_{F(i)}$ we obtain an element $(f_i)_{i \in \mathrm{Ob}(\mathcal{I})}$ of the left hand side. Conversely, let $(f_i)_{i \in \mathrm{Ob}(\mathcal{I})}$ be an element of the left hand side. Note that on each (connected) fibre category $\mathcal{I}_{i'}$ the functor $M \circ F$ is constant with value $M_{i'}$. Hence the morphisms f_i for $i \in \mathrm{Ob}(\mathcal{I})$ with $F(i) = i'$ are all the same and determine a well defined morphism $g_{i'} : M_{i'} \to T$. By assumption (2) the collection $(g_{i'})_{i' \in \mathrm{Ob}(\mathcal{I}')}$ defines an element of the right hand side. □

4.18. Finite limits and colimits

A *finite* (co)limit is a (co)limit whose diagram category is finite, i.e., the diagram category has finitely many objects and finitely many morphisms. A (co)limit is called *nonempty* if the index category is nonempty. A (co)limit is called *connected* if the index category is connected, see Definition 4.16.1. It turns out that there are "enough" finite diagram categories.

Lemma 4.18.1. *Let $\mathcal{I}$ be a category with*

(1) $\mathrm{Ob}(\mathcal{I})$ *is finite, and*
(2) *there exist finitely many morphisms $f_1, \ldots, f_m \in \mathrm{Arrows}(\mathcal{I})$ such that every morphism of $\mathcal{I}$ is a composition $f_{j_1} \circ f_{j_2} \circ \ldots \circ f_{j_k}$.*

Then there exists a functor $F : \mathcal{J} \to \mathcal{I}$ such that

(a) *$\mathcal{J}$ is a finite category, and*
(b) *for any diagram $M : \mathcal{I} \to \mathcal{C}$ the (co)limit of M over $\mathcal{I}$ exists if and only if the (co)limit of $M \circ F$ over $\mathcal{J}$ exists and in this case the (co)limits are canonically isomorphic.*

Moreover, $\mathcal{J}$ is connected (resp. nonempty) if and only if $\mathcal{I}$ is so.

Proof. Say $\mathrm{Ob}(\mathcal{I}) = \{x_1, \ldots, x_n\}$. Denote $s, t : \{1, \ldots, m\} \to \{1, \ldots, n\}$ the functions such that $f_j : x_{s(j)} \to x_{t(j)}$. We set $\mathrm{Ob}(\mathcal{J}) = \{y_1, \ldots, y_n, z_1, \ldots, z_n\}$ Besides the identity morphisms we introduce morphisms $g_j : y_{s(j)} \to z_{t(j)}$, $j = 1, \ldots, m$ and morphisms $h_i : y_i \to z_i$, $i = 1, \ldots, n$. Since all of the nonidentity morphisms in $\mathcal{J}$ go from an y to a z there are no compositions to define and no associativity to check. Set $F(y_i) = F(z_i) = x_i$. Set $F(g_j) = f_j$ and $F(h_i) = \mathrm{id}_{x_i}$. It is clear that F is a functor. It is clear that $\mathcal{J}$ is finite. It is clear that $\mathcal{J}$ is connected, resp. nonempty if and only if $\mathcal{I}$ is so.

Let $M : \mathcal{I} \to \mathcal{C}$ be a diagram. Consider an object W of $\mathcal{C}$ and morphisms $q_i : W \to M(x_i)$ as in Definition 4.14.1. Then by taking $q_i : W \to M(F(y_i)) = M(F(z_i)) = M(x_i)$ we obtain a family of maps as in Definition 4.14.1 for the diagram $M \circ F$. Conversely, suppose we are given maps $qy_i : W \to M(F(y_i))$ and $qz_i : W \to M(F(z_i))$ as in Definition 4.14.1 for the diagram $M \circ F$. Since

$$M(F(h_i)) = \mathrm{id} : M(F(y_i)) = M(x_i) \longrightarrow M(x_i) = M(F(z_i))$$

we conclude that $qy_i = qz_i$ for all i. Set q_i equal to this common value. The compatibility of $q_{s(j)} = qy_{s(j)}$ and $q_{t(j)} = qz_{t(j)}$ with the morphism $M(f_j)$ guarantees that the family q_i is compatible with all morphisms in $\mathcal{I}$ as by assumption every such morphism is a composition of the morphisms f_j. Thus we have found a canonical bijection

$$\lim_{B \in \mathrm{Ob}(\mathcal{J})} \mathrm{Mor}_{\mathcal{C}}(W, M(F(B))) = \lim_{A \in \mathrm{Ob}(\mathcal{I})} \mathrm{Mor}_{\mathcal{C}}(W, M(A))$$

which implies the statement on limits in the lemma. The statement on colimits is proved in the same way (proof omitted). □

Lemma 4.18.2. *Let $\mathcal{C}$ be a category. The following are equivalent:*

(1) *Connected finite limits exist in $\mathcal{C}$.*
(2) *Equalizers and fibre products exist in $\mathcal{C}$.*

Proof. Since equalizers and fibre products are finite connected limits we see that (1) implies (2). For the converse, let $\mathcal{I}$ be a finite connected diagram category. Let

$F : \mathcal{J} \to \mathcal{I}$ be the functor of diagram categories constructed in the proof of Lemma 4.18.1. Then we see that we may replace $\mathcal{I}$ by $\mathcal{J}$. The result is that we may assume that $\mathrm{Ob}(\mathcal{I}) = \{x_1, \ldots, x_n\} \amalg \{y_1, \ldots, y_m\}$ with $n, m \geq 1$ such that all nonidentity morphisms in $\mathcal{I}$ are morphisms $f : x_i \to y_j$ for some i and j.

Suppose that $n > 1$. Since $\mathcal{I}$ is connected there exist indices i_1, i_2 and j_0 and morphisms $a : x_{i_1} \to y_{j_0}$ and $b : x_{i_2} \to y_{j_0}$. Consider the category

$$\mathcal{I}' = \{x\} \amalg \{x_1, \ldots, \hat{x}_{i_1}, \ldots, \hat{x}_{i_2}, \ldots x_n\} \amalg \{y_1, \ldots, y_m\}$$

with

$$\mathrm{Mor}_{\mathcal{I}'}(x, y_j) = \mathrm{Mor}_{\mathcal{I}}(x_{i_1}, y_j) \amalg \mathrm{Mor}_{\mathcal{I}}(x_{i_2}, y_j)$$

and all other morphism sets the same as in $\mathcal{I}$. For any functor $M : \mathcal{I} \to \mathcal{C}$ we can construct a functor $M' : \mathcal{I}' \to \mathcal{C}$ by setting

$$M'(x) = M(x_{i_1}) \times_{M(a), M(y_j), M(b)} M(x_{i_2})$$

and for a morphism $f' : x \to y_j$ corresponding to, say, $f : x_{i_1} \to y_j$ we set $M'(f) = M(f) \circ \mathrm{pr}_1$. Then the functor M has a limit if and only if the functor M' has a limit (proof omitted). Hence by induction we reduce to the case $n = 1$.

If $n = 1$, then the limit of any $M : \mathcal{I} \to \mathcal{C}$ is the successive equalizer of pairs of maps $x_1 \to y_j$ hence exists by assumption. □

Lemma 4.18.3. *Let $\mathcal{C}$ be a category. The following are equivalent:*

(1) *Nonempty finite limits exist in $\mathcal{C}$.*
(2) *Products of pairs and equalizers exist in $\mathcal{C}$.*
(3) *Products of pairs and fibre products exist in $\mathcal{C}$.*

Proof. Since products of pairs, fibre products, and equalizers are limits with nonempty index categories we see that (1) implies both (2) and (3). Assume (2). Then finite nonempty products and equalizers exist. Hence by Lemma 4.14.10 we see that finite nonempty limits exist, i.e., (1) holds. Assume (3). If $a, b : A \to B$ are morphisms of $\mathcal{C}$, then the equalizer of a, b is

$$(A \times_{a,B,b} A) \times_{(pr_1, pr_2), A \times A, \Delta} A.$$

Thus (3) implies (2), and the lemma is proved. □

Lemma 4.18.4. *Let $\mathcal{C}$ be a category. The following are equivalent:*

(1) *Finite limits exist in $\mathcal{C}$.*
(2) *Finite products and equalizers exist.*
(3) *The category has a final object and fibred products exist.*

Proof. Since products of pairs, fibre products, equalizers, and final objects are limits over finite index categories we see that (1) implies both (2) and (3). By Lemma 4.14.10 above we see that (2) implies (1). Assume (3). Note that the product $A \times A$ is the fibre product over the final object. If $a, b : A \to B$ are morphisms of $\mathcal{C}$, then the equalizer of a, b is

$$(A \times_{a,B,b} A) \times_{(pr_1, pr_2), A \times A, \Delta} A.$$

Thus (3) implies (2) and the lemma is proved. □

Lemma 4.18.5. *Let $\mathcal{C}$ be a category. The following are equivalent:*

(1) *Connected finite colimits exist in $\mathcal{C}$.*
(2) *Coequalizers and pushouts exist in $\mathcal{C}$.*

Proof. Omitted. Hint: This is dual to Lemma 4.18.2. □

Lemma 4.18.6. *Let $\mathcal{C}$ be a category. The following are equivalent:*

(1) *Nonempty finite colimits exist in $\mathcal{C}$.*
(2) *Coproducts of pairs and coequalizers exist in $\mathcal{C}$.*
(3) *Coproducts of pairs and pushouts exist in $\mathcal{C}$.*

Proof. Omitted. Hint: This is the dual of Lemma 4.18.3. □

Lemma 4.18.7. *Let $\mathcal{C}$ be a category. The following are equivalent:*

(1) *finite colimits exist in $\mathcal{C}$,*
(2) *finite coproducts and coequalizers exist in $\mathcal{C}$, and*
(3) *$\mathcal{C}$ has an initial object and pushouts exist.*

Proof. Omitted. Hint: This is dual to Lemma 4.18.4. □

4.19. Filtered colimits

Colimits are easier to compute or describe when they are over a filtered diagram. Here is the definition.

Definition 4.19.1. We say that a diagram $M : \mathcal{I} \to \mathcal{C}$ is *directed*, or *filtered* if the following conditions hold:

(1) the category $\mathcal{I}$ has at least one object,
(2) for every pair of objects x, y of $\mathcal{I}$ there exists an object z and morphisms $x \to z$, $y \to z$, and
(3) for every pair of objects x, y of $\mathcal{I}$ and every pair of morphisms $a, b : x \to y$ of $\mathcal{I}$ there exists a morphism $c : y \to z$ of $\mathcal{I}$ such that $M(c \circ a) = M(c \circ b)$ as morphisms in $\mathcal{C}$.

We say that an index category $\mathcal{I}$ is *directed*, or *filtered* if $\mathrm{id} : \mathcal{I} \to \mathcal{I}$ is filtered (in other words you erase the M in part (3) above.)

We observe that any diagram with filtered index category is filtered, and this is how filtered colimits usually come about. In fact, if $M : \mathcal{I} \to \mathcal{C}$ is a filtered diagram, then we can factor M as $\mathcal{I} \to \mathcal{I}' \to \mathcal{C}$ where $\mathcal{I}'$ is a filtered index category[1] such that $\mathrm{colim}_{\mathcal{I}} M$ exists if and only if $\mathrm{colim}_{\mathcal{I}'} M'$ exists in which case the colimits are canonically isomorphic.

Suppose that $M : \mathcal{I} \to \mathit{Sets}$ is a filtered diagram. In this case we may describe the equivalence relation in the formula

$$\mathrm{colim}_{\mathcal{I}} M = (\coprod\nolimits_{i \in I} M_i)/\sim$$

simply as follows

$$m_i \sim m_{i'} \Leftrightarrow \exists i'', \phi : i \to i'', \phi' : i' \to i'', M(\phi)(m_i) = M(\phi')(m_{i'}).$$

In other words, two elements are equal in the colimit if and only if they "eventually become equal".

[1] Namely, let $\mathcal{I}'$ have the same objects as $\mathcal{I}$ but where $\mathrm{Mor}_{\mathcal{I}'}(x, y)$ is the quotient of $\mathrm{Mor}_{\mathcal{I}}(x, y)$ by the equivalence relation which identifies $a, b : x \to y$ if $M(a) = M(b)$.

Lemma 4.19.2. *Let $\mathcal{I}$ and $\mathcal{J}$ be index categories. Assume that $\mathcal{I}$ is filtered and $\mathcal{J}$ is finite. Let $M : \mathcal{I} \times \mathcal{J} \to Sets$, $(i, j) \mapsto M_{i,j}$ be a diagram of diagrams of sets. In this case*

$$\operatorname{colim}_i \lim_j M_{i,j} = \lim_j \operatorname{colim}_i M_{i,j}.$$

In particular, colimits over $\mathcal{I}$ commute with finite products, fibre products, and equalizers of sets.

Proof. Omitted. In fact, it is a fun exercise to prove that a category is filtered if and only if colimits over the category commute with finite limits (into the category of sets). □

We give a counter example to the lemma in the case where $\mathcal{J}$ is infinite. Namely, let $\mathcal{I}$ consist of $\mathbf{N} = \{1, 2, 3, \ldots\}$ with a unique morphism $i \to i'$ whenever $i \leq i'$. Let $\mathcal{J}$ consist of the discrete category $\mathbf{N} = \{1, 2, 3, \ldots\}$ (only morphisms are identities). Let $M_{i,j} = \{1, 2, \ldots, i\}$ with obvious inclusion maps $M_{i,j} \to M_{i',j}$ when $i \leq i'$. In this case $\operatorname{colim}_i M_{i,j} = \mathbf{N}$ and hence

$$\lim_j \operatorname{colim}_i M_{i,j} = \prod\nolimits_j \mathbf{N} = \mathbf{N}^{\mathbf{N}}$$

On the other hand $\lim_j M_{i,j} = \prod_j M_{i,j}$ and hence

$$\operatorname{colim}_i \lim_j M_{i,j} = \bigcup\nolimits_i \{1, 2, \ldots, i\}^{\mathbf{N}}$$

which is smaller than the other limit.

It turns out we sometimes need a more finegrained control over the possible conditions one can impose on index categories. Thus we add some lemmas on the possible things one can require.

Lemma 4.19.3. *Let $\mathcal{I}$ be an index category, i.e., a category. Assume that for every pair of objects x, y of $\mathcal{I}$ there exists an object z and morphisms $x \to z$ and $y \to z$. Then colimits of diagrams of sets over $\mathcal{I}$ commute with nonempty products.*

Proof. Let M and N be diagrams of sets over $\mathcal{I}$. To prove the lemma we have to show that the canonical map

$$\operatorname{colim}(M_i \times N_i) \longrightarrow \operatorname{colim} M_i \times \operatorname{colim} N_i$$

is an isomorphism. If $\mathcal{I}$ is empty, then this is true because the colimit of sets over the empty category is the empty set. If $\mathcal{I}$ is nonempty, then we construct a map $\operatorname{colim} M_i \times \operatorname{colim} N_i \to \operatorname{colim} M_i$ as follows. Suppose that $m \in M_i$ and $n \in N_j$ give rise to elements s and t of the respective colimits. By assumption we can find $a : i \to k$ and $b : j \to k$ in $\mathcal{I}$. Then $(M(a)(m), N(b)(n))$ is an element of $M_k \times N_k$ and we map (s, t) to the corresponding element of $\operatorname{colim} M_i \times N_i$. We omit the verification that this map is well defined and that it is an inverse of the map displayed above. □

Lemma 4.19.4. *Let $\mathcal{I}$ be an index category, i.e., a category. Assume that for every pair of objects x, y of $\mathcal{I}$ there exists an object z and morphisms $x \to z$ and $y \to z$. Let $M : \mathcal{I} \to Ab$ be a diagram of abelian groups over $\mathcal{I}$. Then the set underlying $\operatorname{colim}_i M_i$ is the colimit of M viewed as a diagram of sets over $\mathcal{I}$.*

Proof. In this proof all colimits are taken in the category of sets. By Lemma 4.19.3 we have $\operatorname{colim} M_i \times \operatorname{colim} M_i = \operatorname{colim}(M_i \times M_i)$ hence we can use the maps $+ : M_i \times M_i \to M_i$ to define an addition map on $\operatorname{colim} M_i$. A straightforward argument, which we omit, shows that the set $\operatorname{colim} M_i$ with this addition is the colimit in the category of abelian groups. □

Lemma 4.19.5. *Let $\mathcal{I}$ be an index category, i.e., a category. Assume that for every solid diagram*

$$\begin{array}{ccc} x & \longrightarrow & y \\ \downarrow & & \vdots \\ z & \dashrightarrow & w \end{array}$$

in $\mathcal{I}$ there exists an object w and dotted arrows making the diagram commute. Then $\mathcal{I}$ is a (possibly empty) disjoint union of categories satisfying the condition above and the condition of Lemma 4.19.3.

Proof. If $\mathcal{I}$ is the empty category, then the lemma is true. Otherwise, we define a relation on objects of $\mathcal{I}$ by saying that $x \sim y$ if there exists a z and morphisms $x \to z$ and $y \to z$. This is an equivalence relation by the assumption of the lemma. Hence $\mathrm{Ob}(\mathcal{I})$ is a disjoint union of equivalence classes. Let $\mathcal{I}_j$ be the full subcategories corresponding to these equivalence classes. Then $\mathcal{I} = \coprod \mathcal{I}_j$ as desired. □

Lemma 4.19.6. *Let $\mathcal{I}$ be an index category, i.e., a category. Assume that for every solid diagram*

$$\begin{array}{ccc} x & \longrightarrow & y \\ \downarrow & & \vdots \\ z & \dashrightarrow & w \end{array}$$

in $\mathcal{I}$ there exists an object w and dotted arrows making the diagram commute. Then an injective morphism $M \to N$ of diagrams of sets (resp. abelian groups) over $\mathcal{I}$ gives rise to an injective map $\operatorname{colim} M_i \to \operatorname{colim} N_i$ of sets (resp. abelian groups).

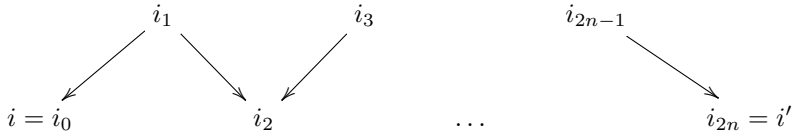
Proof. We first show that it suffices to prove the lemma for the case of a diagram of sets. Namely, by Lemma 4.19.5 we can write $\mathcal{I} = \coprod \mathcal{I}_j$ where each $\mathcal{I}_j$ satisfies the condition of the lemma as well as the condition of Lemma 4.19.3. Thus, if M is a diagram of abelian groups over $\mathcal{I}$, then

$$\operatorname{colim}_{\mathcal{I}} M = \bigoplus\nolimits_j \operatorname{colim}_{\mathcal{I}_j} M|_{\mathcal{I}_j}$$

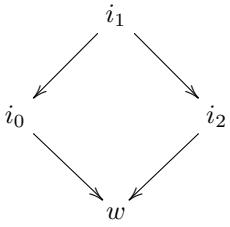
It follows that it suffices to prove the result for the categories $\mathcal{I}_j$. Howeover, colimits of abelian groups over these categories are computed by the colimits of the underlying sets (Lemma 4.19.4) hence we reduce to the case of an injective map of diagrams of sets.

Here we say that $M \to N$ is injective if all the maps $M_i \to N_i$ are injective. In fact, we will identify M_i with the image of $M_i \to N_i$, i.e., we will think of M_i as a subset of N_i. We will use the description of the colimits given in Section 4.15 without further mention. Let $s, s' \in \operatorname{colim} M_i$ map to the same element of $\operatorname{colim} N_i$. Say s comes from an element m of M_i and s' comes from an element m' of $M_{i'}$.

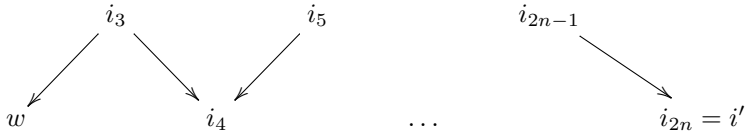
Then we can find a sequence $i = i_0, i_1, \ldots, i_n = i'$ of objects of $\mathcal{I}$ and morphisms



and elements $n_{i_j} \in N_{i_j}$ mapping to each other under the maps $N_{i_{2k-1}} \to N_{i_{2k-2}}$ and $N_{i_{2k-1}} \to N_{i_{2k}}$ induced from the maps in $\mathcal{I}$ above with $n_{i_0} = m$ and $n_{i_{2n}} = m'$. We will prove by induction on n that this implies $s = s'$. The base case $n = 0$ is trivial. Assume $n \geq 1$. Using the assumption on $\mathcal{I}$ we find a commutative diagram



We conclude that m and n_{i_2} map to the same element of N_w because both are the image of the element n_{i_1}. In particular, this element is an element $m'' \in M_w$ which gives rise to the same element as s in $\operatorname{colim} M_i$. Then we find the chain



and the elements n_{i_j} for $j \geq 3$ which has a smaller length than the chain we started with. This proves the induction step and the proof of the lemma is complete. □

Lemma 4.19.7. *Let $\mathcal{I}$ be an index category, i.e., a category. Assume*

(1) *for every pair of morphisms $a : w \to x$ and $b : w \to y$ in $\mathcal{I}$ there exists an object z and morphisms $c : x \to z$ and $d : y \to z$ such that $c \circ a = d \circ b$, and*
(2) *for every pair of morphisms $a, b : x \to y$ there exists a morphism $c : y \to z$ such that $c \circ a = c \circ b$.*

Then $\mathcal{I}$ is a (possibly empty) union of disjoint filtered index categories $\mathcal{I}_j$.

Proof. If $\mathcal{I}$ is the empty category, then the lemma is true. Otherwise, we define a relation on objects of $\mathcal{I}$ by saying that $x \sim y$ if there exists a z and morphisms $x \to z$ and $y \to z$. This is an equivalence relation by the first assumption of the lemma. Hence $\mathrm{Ob}(\mathcal{I})$ is a disjoint union of equivalence classes. Let $\mathcal{I}_j$ be the full subcategories corresponding to these equivalence classes. The rest is clear from the definitions. □

Lemma 4.19.8. *Let $\mathcal{I}$ be an index category satisfying the hypotheses of Lemma 4.19.7 above. Then colimits over $\mathcal{I}$ commute with fibre products and equalizers in sets (and more generally with finite connected limits).*

Proof. By Lemma 4.19.7 we may write $\mathcal{I} = \coprod \mathcal{I}_j$ with each $\mathcal{I}_j$ filtered. By Lemma 4.19.2 we see that colimits of $\mathcal{I}_j$ commute with equalizers and fibred products. Thus it suffices to show that equalizers and fibre products commute with coproducts in the category of sets (including empty coproducts). In other words, given a set J and sets A_j, B_j, C_j and set maps $A_j \to B_j$, $C_j \to B_j$ for $j \in J$ we have to show that

$$(\coprod\nolimits_{j\in J} A_j) \times_{(\coprod_{j\in J} B_j)} (\coprod\nolimits_{j\in J} C_j) = \coprod\nolimits_{j\in J} A_j \times_{B_j} C_j$$

and given $a_j, a'_j : A_j \to B_j$ that

$$\text{Equalizer}(\coprod\nolimits_{j\in J} a_j, \coprod\nolimits_{j\in J} a'_j) = \coprod\nolimits_{j\in J} \text{Equalizer}(a_j, a'_j)$$

This is true even if $J = \emptyset$. Details omitted. □

4.20. Cofiltered limits

Limits are easier to compute or describe when they are over a cofiltered diagram. Here is the definition.

Definition 4.20.1. We say that a diagram $M : \mathcal{I} \to \mathcal{C}$ is *codirected* or *cofiltered* if the following conditions hold:

(1) the category $\mathcal{I}$ has at least one object,
(2) for every pair of objects x, y of $\mathcal{I}$ there exists an object z and morphisms $z \to x$, $z \to y$, and
(3) for every pair of objects x, y of $\mathcal{I}$ and every pair of morphisms $a, b : x \to y$ of $\mathcal{I}$ there exists a morphism $c : w \to x$ of $\mathcal{I}$ such that $M(a \circ c) = M(b \circ c)$ as morphisms in $\mathcal{C}$.

We say that an index category $\mathcal{I}$ is *codirected*, or *cofiltered* if $\text{id} : \mathcal{I} \to \mathcal{I}$ is cofiltered (in other words you erase the M in part (3) above.)

We observe that any diagram with cofiltered index category is cofiltered, and this is how this situation usually occurs.

As an example of why cofiltered limits of sets are "easier" than general ones, we mention the fact that a cofiltered diagram of finite nonempty sets has nonempty limit (Lemma 4.21.5). This result does not hold for a general limit of finite nonempty sets.

4.21. Limits and colimits over partially ordered sets

A special case of diagrams is given by systems over partially ordered sets.

Definition 4.21.1. Let $(I, \geq)$ be a partially ordered set. Let $\mathcal{C}$ be a category.

(1) A *system over I in $\mathcal{C}$*, sometimes called a *inductive system over I in $\mathcal{C}$* is given by objects M_i of $\mathcal{C}$ and for every $i \leq i'$ a morphism $f_{ii'} : M_i \to M_{i'}$ such that $f_{ii} = \text{id}$ and such that $f_{ii''} = f_{i'i''} \circ f_{ii'}$ whenever $i \leq i' \leq i''$.
(2) An *inverse system over I in $\mathcal{C}$*, sometimes called a *projective system over I in $\mathcal{C}$* is given by objects M_i of $\mathcal{C}$ and for every $i \geq i'$ a morphism $f_{ii'} : M_i \to M_{i'}$ such that $f_{ii} = \text{id}$ and such that $f_{ii''} = f_{i'i''} \circ f_{ii'}$ whenever $i \geq i' \geq i''$. (Note reversal of inequalities.)

We will say $(M_i, f_{ii'})$ is a (inverse) system over I to denote this. The maps $f_{ii'}$ are sometimes called the *transition maps*.

In other words a system over I is just a diagram $M : \mathcal{I} \to \mathcal{C}$ where $\mathcal{I}$ is the category with objects I and a unique arrow $i \to i'$ if and only $i \leq i'$. And an inverse system is a diagram $M : \mathcal{I}^{opp} \to \mathcal{C}$. From this point of view we could take (co)limits of any (inverse) system over I. However, it is customary to take *only colimits of systems over I* and *only limits of inverse systems over I*. More precisely: Given a system $(M_i, f_{ii'})$ over I the colimit of the system $(M_i, f_{ii'})$ is defined as

$$\operatorname{colim}_{i \in I} M_i = \operatorname{colim}_{\mathcal{I}} M,$$

i.e., as the colimit of the corresponding diagram. Given a inverse system $(M_i, f_{ii'})$ over I the limit of the inverse system $(M_i, f_{ii'})$ is defined as

$$\lim_{i \in I} M_i = \lim_{\mathcal{I}^{opp}} M,$$

i.e., as the limit of the corresponding diagram.

Definition 4.21.2. With notation as above. We say the system (resp. inverse system) $(M_i, f_{ii'})$ is a *directed system* (resp. *directed inverse system*) if the partially ordered set I is *directed*: I is nonempty and for all $i_1, i_2 \in I$ there exists $i \in I$ such that $i_1 \leq i$ and $i_2 \leq i$.

In this case the colimit is sometimes (unfortunately) called the "direct limit". We will not use this last terminology. It turns out that diagrams over a filtered category are no more general than directed systems in the following sense.

Lemma 4.21.3. *Let $\mathcal{I}$ be a filtered index category. There exists a directed partially ordered set $(I, \geq)$ and a system $(x_i, \varphi_{ii'})$ over I in $\mathcal{I}$ with the following properties:*

(1) *For every category $\mathcal{C}$ and every diagram $M : \mathcal{I} \to \mathcal{C}$ with values in $\mathcal{C}$, denote $(M(x_i), M(\varphi_{ii'}))$ the corresponding system over I. If $\operatorname{colim}_{i \in I} M(x_i)$ exists then so does $\operatorname{colim}_{\mathcal{I}} M$ and the transformation*

$$\theta : \operatorname{colim}_{i \in I} M(x_i) \longrightarrow \operatorname{colim}_{\mathcal{I}} M$$

of Lemma 4.14.7 is an isomorphism.

(2) *For every category $\mathcal{C}$ and every diagram $M : \mathcal{I}^{opp} \to \mathcal{C}$ in $\mathcal{C}$, denote $(M(x_i), M(\varphi_{ii'}))$ the corresponding inverse system over I. If $\lim_{i \in I} M(x_i)$ exists then so does $\lim_{\mathcal{I}} M$ and the transformation*

$$\theta : \lim_{\mathcal{I}^{opp}} M \longrightarrow \lim_{i \in I} M(x_i)$$

of Lemma 4.14.8 is an isomorphism.

Proof. As mentioned in the beginning of the section, we may view partially ordered sets as categories and systems as functors. Throughout the proof, we will freely shift between these two points of view. We prove the first statement by constructing a category $\mathcal{I}_0$, corresponding to a directed set, and a cofinal functor $M_0 \colon \mathcal{I}_0 \to \mathcal{I}$. Then, by Lemma 4.17.2, the colimit of a diagram $M \colon \mathcal{I} \to \mathcal{C}$ coincides with the colimit of the diagram $M \circ M_0 \colon \mathcal{I}_0 \to \mathcal{C}$, from which the statement follows. The second stament is dual to the first and may be proved by interpreting a limit in $\mathcal{C}$ as a colimit in $\mathcal{C}^{opp}$. We omit the details.

A category $\mathcal{F}$ is called *finitely generated* if there exists a finite set F of arrows in $\mathcal{F}$, such that each arrow in $\mathcal{F}$ may be obtained by composing arrows from F. In particular, this implies that $\mathcal{F}$ has finitely many objects. We start the proof by reducing to the case when $\mathcal{I}$ has the property that every finitely generated

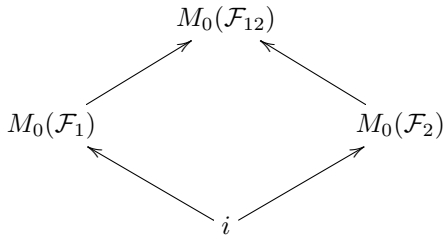
subcategory of $\mathcal{I}$ may be extended to a finitely generated subcategory with a unique final object.

Let ω denote the directed set of finite ordinals, which we view as a filtered category. It is easy to verify that the product category $\mathcal{I}\times\omega$ is also filtered, and the projection $\Pi\colon \mathcal{I}\times\omega\to\mathcal{I}$ is cofinal.

Now let $\mathcal{F}$ be any finitely generated subcategory of $\mathcal{I}\times\omega$. By using the axioms of a filtered category and a simple induction argument on a finite set of generators of $\mathcal{F}$, we may construct a cocone $(\{f_i\}, i_\infty)$ in $\mathcal{I}$ for the diagram $\mathcal{F}\to\mathcal{I}$. That is, a morphism $f_i\colon i\to i_\infty$ for every object i in $\mathcal{F}$ such that for each arrow $f\colon i\to i'$ in $\mathcal{F}$ we have $f_i = f\circ f_{i'}$. We also choose i_∞ such that it is not contained in $\mathcal{F}$. This is possible since we may always post-compose the arrows f_i with an arrow which is the identity on the $\mathcal{I}$-component and strictly increasing on the ω-component. Now let $\mathcal{F}^+$ denote the category consisting of all objects and arrows in $\mathcal{F}$ together with the object i_∞, the identity arrow id_{i_∞} and the arrows f_i. Since there are no arrows from i_∞ in $\mathcal{F}^+$ to any object of $\mathcal{F}$, the arrow set in $\mathcal{F}^+$ is closed under composition, so $\mathcal{F}^+$ is indeed a category. By construction, it is a finitely generated subcategory of $\mathcal{I}$ which has i_∞ as unique final object. Since, by Lemma 4.17.2, the colimit of any diagram $M\colon\mathcal{I}\to\mathcal{C}$ coincides with the colimit of $M\circ\Pi$, this gives the desired reduction.

The set of all finitely generated subcategories of $\mathcal{I}$ with a unique final object is naturally ordered by inclusion. We take $\mathcal{I}_0$ to be the category corresponding to this set. We also have a functor $M_0\colon\mathcal{I}_0\to\mathcal{I}$, which takes an arrow $\mathcal{F}\subset\mathcal{F}'$ in $\mathcal{I}_0$ to the unique map from the final object of $\mathcal{F}$ to the final object of $\mathcal{F}'$. Given any two finitely generated subcategories of $\mathcal{I}$, the category generated by these two categories is also finitely generated. By our assumption on $\mathcal{I}$, it is also contained in a finitely generated subcategory of $\mathcal{I}$ with a unique final object. This shows that $\mathcal{I}_0$ is directed.

Finally, we verify that M_0 is cofinal. Since any object of $\mathcal{I}$ is the final object in the subcategory consisting of only that object and its identity arrow, the functor M_0 is surjective on objects. In particular, Condition (1) of Definition 4.17.1 is satisfied. Given an object i of $\mathcal{I}$, $\mathcal{F}_1, \mathcal{F}_2$ in $\mathcal{I}_0$ and maps $\varphi_1 : i\to M_0(\mathcal{F}_1)$ and $\varphi_2 : i\to M_0(\mathcal{F}_2)$ in $\mathcal{I}$, we can take $\mathcal{F}_{12}$ to be a finitely generated category with a unique final object containg $\mathcal{F}_1$, $\mathcal{F}_2$ and the morphisms φ_1, φ_2. The resulting diagram commutes



since it lives in the category $\mathcal{F}_{12}$ and $M_0(\mathcal{F}_{12})$ is final in this category. Hence also Condition (2) is satisfied, which concludes the proof. □

Remark 4.21.4. Note that a finite directed set $(I, \geq)$ always has a greatest object i_∞. Hence any colimit of a system $(M_i, f_{ii'})$ over such a set is trivial in the sence

that the colimit equals M_{i_∞}. In contrast, a colimit indexed by a finite filtered category need not be trival. For instace, let $\mathcal{I}$ be the category with a single object i and a single non-trivial morphisms e satisfying $e = e \circ e$. The colimit of a diagram $M : \mathcal{I} \to Sets$ is the image of the idempotent $M(e)$. This illustrates that something like the trick of passing to $\mathcal{I} \times \omega$ in the proof of Lemma 4.21.3 is essential.

Lemma 4.21.5. *If $S : \mathcal{I} \to Sets$ is a cofiltered diagram of sets and all the S_i are finite nonempty, then $\lim_i S_i$ is nonempty. In other words, the limit of a directed inverse system of finite nonempty sets is nonempty.*

Proof. The two statements are equivalent by Lemma 4.21.3. Let I be a directed partially ordered set and let $(S_i)_{i\in I}$ be an inverse system of finite nonempty sets over I. Let us say that a *subsystem* T is a family $T = (T_i)_{i\in I}$ of nonempty subsets $T_i \subset S_i$ such that $T_{i'}$ is mapped into T_i by the transition map $S_{i'} \to S_i$ for all $i' \geq i$. Denote $\mathcal{T}$ the set of subsystems. We order $\mathcal{T}$ by inclusion. Suppose T_α, $\alpha \in A$ is a totally ordered family of elements of $\mathcal{T}$. Say $T_\alpha = (T_{\alpha,i})_{i\in I}$. Then we can find a lower bound $T = (T_i)_{i\in I}$ by setting $T_i = \bigcap_{\alpha\in A} T_{\alpha,i}$ which is manifestly a finite nonempty subset of S_i as all the $T_{\alpha,i}$ are nonempty and as the T_α form a totally ordered family. Thus we may apply Zorn's lemma to see that $\mathcal{T}$ has minimal elements.

Let's analyze what a minimal element $T \in \mathcal{T}$ looks like. First observe that the maps $T_{i'} \to T_i$ are all surjective. Namely, as I is a directed partially ordered set and T_i is finite, the intersection $T_i' = \bigcap_{i'\geq i} \mathrm{Im}(T_{i'} \to T_i)$ is nonempty. Thus $T' = (T_i')$ is a subsystem contained in T and by minimality $T' = T$. Finally, we claim that T_i is a singleton for each i. Namely, if $x \in T_i$, then we can define $T_{i'}' = (T_{i'} \to T_i)^{-1}(\{x\})$ for $i' \geq i$ and $T_j' = T_j$ if $j \not\geq i$. This is another subsystem as we've seen above that the transition maps of the subsystem T are surjective. By minimality we see that $T = T'$ which indeed implies that T_i is a singleton. This holds for every $i \in I$, hence we see that $T_i = \{x_i\}$ for some $x_i \in S_i$ with $x_{i'} \mapsto x_i$ under the map $S_{i'} \to S_i$ for every $i' \geq i$. In other words, $(x_i) \in \lim S_i$ and the lemma is proved. □

4.22. Essentially constant systems

Let $M : \mathcal{I} \to \mathcal{C}$ be a diagram in a category $\mathcal{C}$. Assume the index category $\mathcal{I}$ is filtered. In this case there are three successively stronger notions which pick out an object X of $\mathcal{C}$. The first is just

$$X = \mathrm{colim}_{i\in\mathcal{I}} M_i.$$

Then X comes equipped with projection morphisms $M_i \to X$. A stronger condition would be to require that X is the colimit and that there exists an $i \in \mathcal{I}$ and a morphism $X \to M_i$ such that the composition $X \to M_i \to X$ is id_X. A stronger condition is the following.

Definition 4.22.1. Let $M : \mathcal{I} \to \mathcal{C}$ be a diagram in a category $\mathcal{C}$.

(1) Assume the index category $\mathcal{I}$ is filtered. We say M *is essentially constant* with *value* X if $X = \mathrm{colim}_i M_i$ and there exists an $i \in \mathcal{I}$ and a morphism $X \to M_i$ such that
   (a) $X \to M_i \to X$ is id_X, and
   (b) for all j there exist k and morphisms $i \to k$ and $j \to k$ such that the morphism $M_j \to M_k$ equals the composition $M_j \to X \to M_i \to M_k$.

(2) Assume the index category $\mathcal{I}$ is cofiltered. We say M is *essentially constant* with *value* X if $X = \lim_i M_i$ and there exists an $i \in \mathcal{I}$ and a morphism $M_i \to X$ such that
   (a) $X \to M_i \to X$ is id_X, and
   (b) for all j there exist k and morphisms $k \to i$ and $k \to j$ such that the morphism $M_k \to M_j$ equals the composition $M_k \to M_i \to X \to M_j$.

Which of the two versions is meant will be clear from context. If there is any confusion we will distinguish between these by saying that the first version means M is essentially constant as an *ind-object*, and in the second case we will say it is essentially constant as an *pro-object*. This terminology is further explained in Remarks 4.22.3 and 4.22.4. In fact we will often use the terminology "essentially constant system" which formally speaking is only defined for systems over directed partially ordered sets.

Definition 4.22.2. Let $\mathcal{C}$ be a category. A directed system $(M_i, f_{ii'})$ is an *essentially constant system* if M viewed as a functor $I \to \mathcal{C}$ defines an essentially constant diagram. A directed inverse system $(M_i, f_{ii'})$ is an *essentially constant inverse system* if M viewed as a functor $I^{opp} \to \mathcal{C}$ defines an essentially constant inverse diagram.

If $(M_i, f_{ii'})$ is an essentially constant system and the morphisms $f_{ii'}$ are monomorphisms, then for all $i \leq i'$ sufficiently large the morphisms $f_{ii'}$ are isomorphisms. In general this need not be the case however. An example is the system

$$\mathbf{Z}^2 \to \mathbf{Z}^2 \to \mathbf{Z}^2 \to \ldots$$

with maps given by $(a, b) \mapsto (a + b, 0)$. This system is essentially constant with value $\mathbf{Z}$. A non-example is to let $M = \bigoplus_{n \geq 0} \mathbf{Z}$ and to let $S : M \to M$ be the shift operator $(a_0, a_1, \ldots) \mapsto (a_1, a_2, \ldots)$. In this case the system $M \to M \to M \to \ldots$ with transition maps S has colimit 0, and a map $0 \to M$ but the system is not essentially constant.

Remark 4.22.3. Let $\mathcal{C}$ be a category. There exists a big category Ind-$\mathcal{C}$ of *ind-objects of* $\mathcal{C}$. Namely, if $F : \mathcal{I} \to \mathcal{C}$ and $G : \mathcal{J} \to \mathcal{C}$ are filtered diagrams in $\mathcal{C}$, then we can define

$$\mathrm{Mor}_{\text{Ind-}\mathcal{C}}(F, G) = \lim_i \mathrm{colim}_j \mathrm{Mor}_{\mathcal{C}}(F(i), G(j)).$$

There is a canonical functor $\mathcal{C} \to$ Ind-$\mathcal{C}$ which maps X to the *constant system* on X. This is a fully faithful embedding. In this language one sees that a diagram F is essentially constant if and only F is isomorphic to a constant system. If we ever need this material, then we will formulate this into a lemma and prove it here.

Remark 4.22.4. Let $\mathcal{C}$ be a category. There exists a big category Pro-$\mathcal{C}$ of *pro-objects* of $\mathcal{C}$. Namely, if $F : \mathcal{I} \to \mathcal{C}$ and $G : \mathcal{J} \to \mathcal{C}$ are cofiltered diagrams in $\mathcal{C}$, then we can define

$$\mathrm{Mor}_{\text{Pro-}\mathcal{C}}(F, G) = \lim_j \mathrm{colim}_i \mathrm{Mor}_{\mathcal{C}}(F(i), G(j)).$$

There is a canonical functor $\mathcal{C} \to$ Pro-$\mathcal{C}$ which maps X to the *constant system* on X. This is a fully faithful embedding. In this language one sees that a diagram F is essentially constant if and only F is isomorphic to a constant system. If we ever need this material, then we will formulate this into a lemma and prove it here.

Lemma 4.22.5. *Let $\mathcal{C}$ be a category. Let $M : \mathcal{I} \to \mathcal{C}$ be a diagram with filtered (resp. cofiltered) index category $\mathcal{I}$. Let $F : \mathcal{C} \to \mathcal{D}$ be a functor. If M is essentially constant as an ind-object (resp. pro-object), then so is $F \circ M : \mathcal{I} \to \mathcal{D}$.*

Proof. If X is a value for M, then it follows immediately from the definition that $F(X)$ is a value for $F \circ M$. □

Lemma 4.22.6. *Let $\mathcal{C}$ be a category. Let $M : \mathcal{I} \to \mathcal{C}$ be a diagram with filtered index category $\mathcal{I}$. The following are equivalent*

(1) *M is an essentially constant ind-object, and*
(2) *$X = \operatorname{colim}_i M_i$ exists and for any W in $\mathcal{C}$ the map*

$$\operatorname{colim}_i \operatorname{Mor}_{\mathcal{C}}(W, M_i) \longrightarrow \operatorname{Mor}_{\mathcal{C}}(W, X)$$

is bijective.

Proof. Assume (2) holds. Then $\mathrm{id}_X \in \operatorname{Mor}_{\mathcal{C}}(X, X)$ comes from a morphism $X \to M_i$ for some i, i.e., $X \to M_i \to X$ is the identity. Then both maps

$$\operatorname{Mor}_{\mathcal{C}}(W, X) \longrightarrow \operatorname{colim}_i \operatorname{Mor}_{\mathcal{C}}(W, M_i) \longrightarrow \operatorname{Mor}_{\mathcal{C}}(W, X)$$

are bijective for all W where the first one is induced by the morphism $X \to M_i$ we found above, and the composition is the identity. This means that the composition

$$\operatorname{colim}_i \operatorname{Mor}_{\mathcal{C}}(W, M_i) \longrightarrow \operatorname{Mor}_{\mathcal{C}}(W, X) \longrightarrow \operatorname{colim}_i \operatorname{Mor}_{\mathcal{C}}(W, M_i)$$

is the identity too. Setting $W = M_j$ and starting with id_{M_j} in the colimit, we see that $M_j \to X \to M_i \to M_k$ is equal to $M_j \to M_k$ for some k large enough. This proves (1) holds. The proof of (1) $\Rightarrow$ (2) is omitted. □

Lemma 4.22.7. *Let $\mathcal{C}$ be a category. Let $M : \mathcal{I} \to \mathcal{C}$ be a diagram with cofiltered index category $\mathcal{I}$. The following are equivalent*

(1) *M is an essentially constant pro-object, and*
(2) *$X = \lim_i M_i$ exists and for any W in $\mathcal{C}$ the map*

$$\operatorname{colim}_i \operatorname{Mor}_{\mathcal{C}}(M_i, W) \longrightarrow \operatorname{Mor}_{\mathcal{C}}(X, W)$$

is bijective.

Proof. Assume (2) holds. Then $\mathrm{id}_X \in \operatorname{Mor}_{\mathcal{C}}(X, X)$ comes from a morphism $M_i \to X$ for some i, i.e., $X \to M_i \to X$ is the identity. Then both maps

$$\operatorname{Mor}_{\mathcal{C}}(X, W) \longrightarrow \operatorname{colim}_i \operatorname{Mor}_{\mathcal{C}}(M_i, W) \longrightarrow \operatorname{Mor}_{\mathcal{C}}(X, W)$$

are bijective for all W where the first one is induced by the morphism $M_i \to X$ we found above, and the composition is the identity. This means that the composition

$$\operatorname{colim}_i \operatorname{Mor}_{\mathcal{C}}(M_i, W) \longrightarrow \operatorname{Mor}_{\mathcal{C}}(X, W) \longrightarrow \operatorname{colim}_i \operatorname{Mor}_{\mathcal{C}}(M_i, W)$$

is the identity too. Setting $W = M_j$ and starting with id_{M_j} in the colimit, we see that $M_k \to M_i \to X \to M_j$ is equal to $M_k \to M_j$ for some k large enough. This proves (1) holds. The proof of (1) $\Rightarrow$ (2) is omitted. □

4.23. Exact functors

Definition 4.23.1. Let $F : \mathcal{A} \to \mathcal{B}$ be a functor.

(1) Suppose all finite limits exist in $\mathcal{A}$. We say F is *left exact* if it commutes with all finite limits.
(2) Suppose all finite colimits exist in $\mathcal{A}$. We say F is *right exact* if it commutes with all finite colimits.
(3) We say F is *exact* if it is both left and right exact.

Lemma 4.23.2. *Let $F : \mathcal{A} \to \mathcal{B}$ be a functor. Suppose all finite limits exist in $\mathcal{A}$, see Lemma 4.18.4. The following are equivalent:*

(1) *F is left exact,*
(2) *F commutes with finite products and equalizers, and*
(3) *F transforms a final object of $\mathcal{A}$ into a final object of $\mathcal{B}$, and commutes with fibre products.*

Proof. Lemma 4.14.10 shows that (2) implies (1). Suppose (3) holds. The fibre product over the final object is the product. If $a, b : A \to B$ are morphisms of $\mathcal{A}$, then the equalizer of a, b is

$$(A \times_{a,B,b} A) \times_{(pr_1,pr_2),A\times A,\Delta} A.$$

Thus (3) implies (2). Finally (1) implies (3) because the empty limit is a final object, and fibre products are limits. □

4.24. Adjoint functors

Definition 4.24.1. Let $\mathcal{C}$, $\mathcal{D}$ be categories. Let $u : \mathcal{C} \to \mathcal{D}$ and $v : \mathcal{D} \to \mathcal{C}$ be functors. We say that u is a *left adjoint* of v, or that v is a *right adjoint* to u if there are bijections

$$\mathrm{Mor}_{\mathcal{D}}(u(X), Y) \longrightarrow \mathrm{Mor}_{\mathcal{C}}(X, v(Y))$$

functorial in $X \in \mathrm{Ob}(\mathcal{C})$, and $Y \in \mathrm{Ob}(\mathcal{D})$.

In other words, this means that there is a *given* isomorphism of functors $\mathcal{C}^{opp} \times \mathcal{D} \to \mathit{Sets}$ from $\mathrm{Mor}_{\mathcal{D}}(u(-), -)$ to $\mathrm{Mor}_{\mathcal{C}}(-, v(-))$. For any object X of $\mathcal{C}$ we obtain a morphism $X \to v(u(X))$ corresponding to $\mathrm{id}_{u(X)}$. Similarly, for any object Y of $\mathcal{D}$ we obtain a morphism $u(v(Y)) \to Y$ corresponding to $\mathrm{id}_{v(Y)}$. These maps are called the *adjunction maps*. The adjunction maps are functorial in X and Y, hence we obtain morphisms of functors $\mathrm{id}_{\mathcal{C}} \to v \circ u$ and $v \circ u \to \mathrm{id}_{\mathcal{D}}$. Moreover, if $\alpha : u(X) \to Y$ and $\beta : X \to v(Y)$ are morphisms, then the following are equivalent

(1) α and β correspond to each other via the bijection of the definition,
(2) β is the composition $X \to v(u(X)) \xrightarrow{v(\beta)} v(Y)$, and
(3) α is the composition $u(X) \xrightarrow{u(\alpha)} u(v(Y)) \to Y$.

In this way one can reformulate the notion of adjoint functors in terms of adjunction maps.

Lemma 4.24.2. *Let u be a left adjoint to v as in Definition 4.24.1. Then*

(1) *u is fully faithful $\Leftrightarrow$ $id \cong v \circ u$.*
(2) *v is fully faithful $\Leftrightarrow$ $u \circ v \cong id$.*

Proof. Assume u is fully faithful. We have to show the adjunction map $X \to v(u(X))$ is an isomorphism. Let $X' \to v(u(X))$ be any morphism. By adjointness this corresponds to a morphism $u(X') \to u(X)$. By fully faithfulness of u this corresponds to a morphism $X' \to X$. Thus we see that $X \to v(u(X))$ defines a bijection $\mathrm{Mor}(X', X) \to \mathrm{Mor}(X', v(u(X)))$. Hence it is an isomorphism. Conversely, if $\mathrm{id} \cong v \circ u$ then u has to be fully faithful, as v defines an inverse on morphism sets.

Part (2) is dual to part (1). □

Lemma 4.24.3. *Let u be a left adjoint to v as in Definition 4.24.1.*

(1) *Suppose that $M : \mathcal{I} \to \mathcal{C}$ is a diagram, and suppose that $\mathrm{colim}_{\mathcal{I}} M$ exists in $\mathcal{C}$. Then $u(\mathrm{colim}_{\mathcal{I}} M) = \mathrm{colim}_{\mathcal{I}} u \circ M$. In other words, u commutes with (representable) colimits.*
(2) *Suppose that $M : \mathcal{I} \to \mathcal{D}$ is a diagram, and suppose that $\lim_{\mathcal{I}} M$ exists in $\mathcal{D}$. Then $v(\lim_{\mathcal{I}} M) = \lim_{\mathcal{I}} v \circ M$. In other words v commutes with representable limits.*

Proof. A morphism from a colimit into an object is the same as a compatible system of morphisms from the constituents of the limit into the object, see Remark 4.14.4. So

$$\begin{aligned} \mathrm{Mor}_{\mathcal{D}}(u(\mathrm{colim}_{i\in\mathcal{I}} M_i), Y) &= \mathrm{Mor}_{\mathcal{C}}(\mathrm{colim}_{i\in\mathcal{I}} M_i, v(Y)) \\ &= \lim_{i\in\mathcal{I}^{opp}} \mathrm{Mor}_{\mathcal{C}}(M_i, v(Y)) \\ &= \lim_{i\in\mathcal{I}^{opp}} \mathrm{Mor}_{\mathcal{D}}(u(M_i), Y) \end{aligned}$$

proves that $u(\mathrm{colim}_{i\in\mathcal{I}} M_i)$ is the colimit we are looking for. A similar argument works for the other statement. □

Lemma 4.24.4. *Let u be a left adjoint of v as in Definition 4.24.1.*

(1) *If $\mathcal{C}$ has finite colimits, then u is right exact.*
(2) *If $\mathcal{D}$ has finite limits, then v is left exact.*

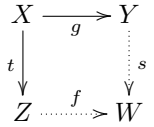
Proof. Obvious from the definitions and Lemma 4.24.3. □

4.25. Localization in categories

The basic idea of this section is given a category $\mathcal{C}$ and a set of arrows to construct a functor $F : \mathcal{C} \to S^{-1}\mathcal{C}$ such that all elements of S become invertible in $S^{-1}\mathcal{C}$ and such that F is universal among all functors with this property. References for this section are [**GZ67**, Chapter I, Section 2] and [**Ver96**, Chapter II, Section 2].

Definition 4.25.1. Let $\mathcal{C}$ be a category. A set of arrows S of $\mathcal{C}$ is called a *left multiplicative system* if it has the following properties:

LMS1 The identity of every object of $\mathcal{C}$ is in S and the composition of two composable elements of S is in S.
LMS2 Every solid diagram



with $t \in S$ can be completed to a commutative dotted square with $s \in S$.

LMS3 For every pair of morphisms $f, g : X \to Y$ and $t \in S$ with target X such that $f \circ t = g \circ t$ there exists a $s \in S$ with source Y such that $s \circ f = s \circ g$.

A set of arrows S of $\mathcal{C}$ is called a *right multiplicative system* if it has the following properties:

RMS1 The identity of every object of $\mathcal{C}$ is in S and the composition of two composable elements of S is in S.

RMS2 Every solid diagram

$$\begin{array}{ccc} X & \overset{g}{\cdots\!\!>} & Y \\ {\scriptstyle t}\vdots & & \downarrow{\scriptstyle s} \\ Z & \xrightarrow{f} & W \end{array}$$

with $s \in S$ can be completed to a commutative dotted square with $t \in S$.

RMS3 For every pair of morphisms $f, g : X \to Y$ and $s \in S$ with source Y such that $s \circ f = s \circ g$ there exists a $t \in S$ with target X such that $f \circ t = g \circ t$.

A set of arrows S of $\mathcal{C}$ is called a *multiplicative system* if it is both a left multiplicative system and a right multiplicative system. In other words, this means that MS1, MS2, MS3 hold, where MS1 = LMS1 = RMS1, MS2 = LMS2 + RMS2, and MS3 = LMS3 + RMS3.

These conditions are useful to construct the categories $S^{-1}\mathcal{C}$ as follows.

Left calculus of fractions. Let $\mathcal{C}$ be a category and let S be a left multiplicative system. We define a new category $S^{-1}\mathcal{C}$ as follows (we verify this works in the proof of Lemma 4.25.2):

(1) We set $\mathrm{Ob}(S^{-1}\mathcal{C}) = \mathrm{Ob}(\mathcal{C})$.

(2) Morphisms $X \to Y$ of $S^{-1}\mathcal{C}$ are given by pairs $(f : X \to Y', s : Y \to Y')$ with $s \in S$ up to equivalence. (Think of this as $s^{-1}f : X \to Y$.)

(3) Two pairs $(f_1 : X \to Y_1, s_1 : Y \to Y_1)$ and $(f_2 : X \to Y_2, s_2 : Y \to Y_2)$ are said to be equivalent if there exists a third pair $(f_3 : X \to Y_3, s_3 : Y \to Y_3)$ and morphisms $u : Y_1 \to Y_3$ and $v : Y_2 \to Y_3$ of $\mathcal{C}$ fitting into the commutative diagram

$$\begin{array}{ccccc} & & Y_1 & & \\ & {\scriptstyle f_1}\nearrow & \downarrow{\scriptstyle u} & \nwarrow{\scriptstyle s_1} & \\ X & \xrightarrow{f_3} & Y_3 & \xleftarrow{s_3} & Y \\ & {\scriptstyle f_2}\searrow & \uparrow{\scriptstyle v} & \swarrow{\scriptstyle s_2} & \\ & & Y_2 & & \end{array}$$

(4) The composition of the equivalence classes of the pairs $(f : X \to Y', s : Y \to Y')$ and $(g : Y \to Z', t : Z \to Z')$ is defined as the equivalence class of a pair $(h \circ f : X \to Z'', u \circ t : Z \to Z'')$ where h and $u \in S$ are chosen to fit into a commutative diagram

$$\begin{array}{ccc} Y & \xrightarrow{g} & Z' \\ {\scriptstyle s}\downarrow & & \downarrow{\scriptstyle u} \\ Y' & \xrightarrow{h} & Z'' \end{array}$$

which exists by assumption.

Lemma 4.25.2. *Let $\mathcal{C}$ be a category and let S be a left multiplicative system.*

(1) *The relation on pairs defined above is an equivalence relation.*
(2) *The composition rule given above is well defined on equivalence classes.*
(3) *Composition is associative and hence $S^{-1}\mathcal{C}$ is a category.*

Proof. Proof of (1). Let us say two pairs $p_1 = (f_1 : X \to Y_1, s_1 : Y \to Y_1)$ and $p_2 = (f_2 : X \to Y_2, s_2 : Y \to Y_2)$ are elementary equivalent if there exists a morphism $a : Y_1 \to Y_2$ of $\mathcal{C}$ such that $a \circ f_1 = f_2$ and $a \circ s_1 = s_2$. Diagram:

$$\begin{array}{ccccc} X & \xrightarrow[f_1]{} & Y_1 & \xleftarrow[s_1]{} & Y \\ \| & & \downarrow{\scriptstyle a} & & \| \\ X & \xrightarrow[f_2]{} & Y_2 & \xleftarrow[s_2]{} & Y \end{array}$$

Let us denote this property by saying $p_1 E p_2$. Note that pEp and $aEb, bEc \Rightarrow aEc$. Part (1) claims that the relation $p \sim p' \Leftrightarrow \exists q : pEq \wedge p'Eq$ is an equivalence relation. A simple formal argument, using the properties of E above shows that it suffices to prove $p_3 E p_1, p_3 E p_2 \Rightarrow p_1 \sim p_2$. Thus suppose that we are given a commutative diagram

$$\begin{array}{ccccc} & & Y_1 & & \\ & \nearrow^{f_1} & \uparrow{\scriptstyle a_{31}} & \nwarrow^{s_1} & \\ X & \xrightarrow[f_3]{} & Y_3 & \xleftarrow[s_3]{} & Y \\ & \searrow_{f_2} & \downarrow{\scriptstyle a_{32}} & \swarrow_{s_2} & \\ & & Y_2 & & \end{array}$$

with $s_i \in S$. First we apply LMS2 to get a commutative diagram

$$\begin{array}{ccc} Y & \xrightarrow[s_3]{} & Y_3 \\ {\scriptstyle s_1}\downarrow & & \vdots{\scriptstyle s_{34}} \\ Y_1 & \overset{a_{14}}{\dashrightarrow} & Y_4 \end{array}$$

with $s_{34} \in S$. Then we have $s_{34} \circ s_3 = a_{14} \circ a_{31} \circ s_3$. Hence by LMS3 there exists a morphism $s_{44} : Y_4 \to Y_4'$, $s_{44} \in S$ such that $s_{44} \circ s_{34} = s_{44} \circ a_{14} \circ a_{31}$. Hence after replacing Y_4 by Y_4', a_{14} by $s_{44} \circ a_{14}$, and s_{34} by $s_{44} \circ s_{34}$ we may assume that $s_{34} = a_{14} \circ a_{31}$. Next, we apply LMS2 to get a commutative diagram

$$\begin{array}{ccc} Y_3 & \xrightarrow[s_{34}]{} & Y_4 \\ {\scriptstyle a_{32}}\downarrow & & \vdots{\scriptstyle s_{45}} \\ Y_2 & \overset{a_{25}}{\dashrightarrow} & Y_5 \end{array}$$

with $s_{45} \in S$. Thus we obtain a pair $p_5 = (s_{45} \circ s_{34} \circ f_3 : X \to Y_5, s_{45} \circ s_{34} \circ s_3 : Y \to Y_5)$ and the morphisms $s_{45} \circ a_{14} : Y_1 \to Y_5$ and $a_{25} : Y_2 \to Y_5$ show that indeed $p_1 E p_5$ and $p_2 E p_5$ as desired.

Proof of (2). Let $p = (f : X \to Y', s : Y \to Y')$ and $q = (g : Y \to Z', t : Z \to Z')$ be pairs as in the definition of composition above. To compose we have to choose

a diagram

$$\begin{array}{ccc} Y & \xrightarrow{g} & Z' \\ {\scriptstyle s}\downarrow & & \downarrow{\scriptstyle u_2} \\ Y' & \xrightarrow{h_2} & Z_2 \end{array}$$

We first show that the equivalence class of the pair $r_2 = (h_2 \circ f : X \to Z_2, u_2 \circ t : Z \to Z_2)$ is independent of the choice of (Z_2, h_2, u_2). Namely, suppose that (Z_3, h_3, u_3) is another choice with corresponding composition $r_3 = (h_3 \circ f : X \to Z_3, u_3 \circ t : Z \to Z_3)$. Then by LMS2 we can choose a diagram

$$\begin{array}{ccc} Z' & \xrightarrow[u_3]{} & Z_3 \\ {\scriptstyle u_2}\downarrow & & \downarrow{\scriptstyle u_{34}} \\ Z_2 & \xrightarrow{h_{24}} & Z_4 \end{array}$$

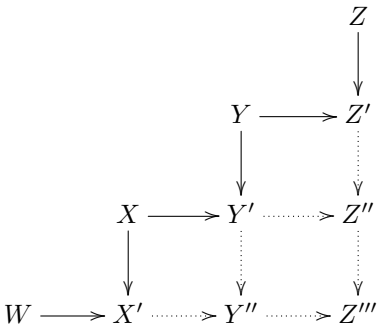
with $u_{34} \in S$. Hence we obtain a pair $r_4 = (h_{24} \circ h_2 \circ f : X \to Z_4, u_{34} \circ u_3 \circ t : Z \to Z_4)$ and the morphisms $h_{24} : Z_2 \to Z_4$ and $u_{34} : Z_3 \to Z_4$ show that we have $r_2 E r_4$ and $r_3 E r_4$ as desired. Thus it now makes sense to define $p \circ q$ as the equivalence class of all possible pairs r obtained as above.

To finish the proof of (2) we have to show that given pairs p_1, p_2, q such that $p_1 E p_2$ then $p_1 \circ q = p_2 \circ q$ and $q \circ p_1 = q \circ p_2$ whenever the compositions make sense. To do this, write $p_1 = (f_1 : X \to Y_1, s_1 : Y \to Y_1)$ and $p_2 = (f_2 : X \to Y_2, s_2 : Y \to Y_2)$ and let $a : Y_1 \to Y_2$ be a morphism of $\mathcal{C}$ such that $f_2 = a \circ f_1$ and $s_2 = a \circ s_1$. First assume that $q = (g : Y \to Z', t : Z \to Z')$. In this case choose a commutative diagram as the one on the left

$$\begin{array}{ccc} Y & \xrightarrow{g} & Z' \\ {\scriptstyle s_2}\downarrow & & \downarrow{\scriptstyle u} \\ Y_2 & \xrightarrow{h} & Z'' \end{array} \quad \Rightarrow \quad \begin{array}{ccc} Y & \xrightarrow{g} & Z' \\ {\scriptstyle s_1}\downarrow & & \downarrow{\scriptstyle u} \\ Y_1 & \xrightarrow{h \circ a} & Z'' \end{array}$$

which implies the diagram on the right is commutative as well. Using these diagrams we see that both compositions are the equivalence class of $(h \circ a \circ f_1 : X \to Z'', u \circ t : Z \to Z'')$. Thus $p_1 \circ q = p_2 \circ q$. The proof of the other case, in which we have to show $q \circ p_1 = q \circ p_2$, is omitted.

Proof of (3). We have to prove associativity of composition. Consider a solid diagram



which gives rise to three composable pairs. Using LMS2 we can choose the dotted arrows making the squares commutative and such that the vertical arrows are in S. Then it is clear that the composition of the three pairs is the equivalence class of the pair $(W \to Z''', Z \to Z''')$ gotten by composing the horizontal arrows on the bottom row and the vertical arrows on the right column. □

We can "write any finite collection of morphisms with the same target as fractions with common denominator".

Lemma 4.25.3. *Let $\mathcal{C}$ be a category and let S be a left multiplicative system of morphisms of $\mathcal{C}$. Given any finite collection $g_i : X_i \to Y$ of morphisms of $S^{-1}\mathcal{C}$ we can find an element $s : Y \to Y'$ of S and $f_i : X_i \to Y'$ such that g_i is the equivalence class of the pair $(f_i : X_i \to Y', s : Y \to Y')$.*

Proof. For each i choose a representative $(X_i \to Y_i, s_i : Y \to Y_i)$. The lemma follows if we can find a morphism $s : Y \to Y'$ in S such that for each i there is a morphism $a_i : Y_i \to Y'$ with $a_i \circ s_i = s$. If we have two indices $i = 1, 2$, then we can do this by completing the square

$$\begin{array}{ccc} Y & \xrightarrow[s_2]{} & Y_2 \\ {\scriptstyle s_1}\downarrow & & \downarrow{\scriptstyle t_2} \\ Y_1 & \xrightarrow{a_1} & Y' \end{array}$$

with $t_2 \in S$ as is possible by Definition 4.25.1. Then $s = t_2 \circ s_2 \in S$ works. If we have $n > 2$ morphisms, then we use the above trick to reduce to the case of $n - 1$ morphisms, and we win by induction. □

There is an easy characterization of equality of morphisms if they have the same denominator.

Lemma 4.25.4. *Let $\mathcal{C}$ be a category and let S be a left multiplicative system of morphisms of $\mathcal{C}$. Let $A, B : X \to Y$ be morphisms of $S^{-1}\mathcal{C}$ which are the equivalence classes of $(f : X \to Y', s : Y \to Y')$ and $(g : X \to Y', s : Y \to Y')$. Then $A = B$ if and only if there exists a morphism $a : Y' \to Y''$ with $a \circ s \in S$ and such that $a \circ f = a \circ g$.*

Proof. The equality of A and B means that there exists a commutative diagram

$$\begin{array}{ccccc} & & Y' & & \\ & {\scriptstyle f}\nearrow & \downarrow{\scriptstyle u} & \nwarrow{\scriptstyle s} & \\ X & \xrightarrow{h} & Z & \xleftarrow{t} & Y \\ & {\scriptstyle g}\searrow & \uparrow{\scriptstyle v} & \swarrow{\scriptstyle s} & \\ & & Y' & & \end{array}$$

with $t \in S$. In particular $u \circ s = v \circ s$. Hence by LMS3 there exists a $s' : Z \to Y''$ in S such that $s' \circ u = s' \circ v$. Setting a equal to this common value does the job. □

Remark 4.25.5. Let $\mathcal{C}$ be a category. Let S be a left multiplicative system. Given an object Y of $\mathcal{C}$ we denote Y/S the category whose objects are $s : Y \to Y'$ with

$s \in S$ and whose morphisms are commutative diagrams

$$\begin{array}{ccc} & Y & \\ {\scriptstyle s}\swarrow & & \searrow{\scriptstyle t} \\ Y' & \xrightarrow{a} & Y'' \end{array}$$

where $a : Y' \to Y''$ is arbitrary. We claim that the category Y/S is filtered (see Definition 4.19.1). Namely, LMS1 implies that $\mathrm{id}_Y : Y \to Y$ is in Y/S hence Y/S is nonempty. LMS2 implies that given $s_1 : Y \to Y_1$ and $s_2 : Y \to Y_2$ we can find a diagram

$$\begin{array}{ccc} Y & \xrightarrow[s_2]{} & Y_2 \\ {\scriptstyle s_1}\downarrow & & \downarrow{\scriptstyle t} \\ Y_1 & \xrightarrow{a} & Y_3 \end{array}$$

with $t \in S$. Hence $s_1 : Y \to Y_1$ and $s_2 : Y \to Y_2$ both map to $t \circ s_2 : Y \to Y_3$ in Y/S. Finally, given two morphisms a, b from $s_1 : Y \to Y_1$ to $s_2 : Y \to Y_2$ in Y/S we see that $a \circ s_1 = b \circ s_1$ hence by LMS3 there exists a $t : Y_2 \to Y_3$ such that $t \circ a = t \circ b$. Now the combined results of Lemmas 4.25.3 and 4.25.4 tell us that

$$\mathrm{Mor}_{S^{-1}\mathcal{C}}(X, Y) = \mathrm{colim}_{(s:Y \to Y') \in Y/S} \mathrm{Mor}_{\mathcal{C}}(X, Y') \tag{4.25.5.1}$$

This formula expressing morphisms in $S^{-1}\mathcal{C}$ as a filtered colimit of morphisms in $\mathcal{C}$ is occasionally useful.

Lemma 4.25.6. *Let $\mathcal{C}$ be a category and let S be a left multiplicative system of morphisms of $\mathcal{C}$.*

1. *The rules $X \mapsto X$ and $(f : X \to Y) \mapsto (f : X \to Y, id_Y : Y \to Y)$ define a functor $Q : \mathcal{C} \to S^{-1}\mathcal{C}$.*
2. *For any $s \in S$ the morphism $Q(s)$ is an isomorphism in $S^{-1}\mathcal{C}$.*
3. *If $G : \mathcal{C} \to \mathcal{D}$ is any functor such that $G(s)$ is invertible for every $s \in S$, then there exists a unique functor $H : S^{-1}\mathcal{C} \to \mathcal{D}$ such that $H \circ Q = G$.*

Proof. Parts (1) and (2) are clear. To see (3) just set $H(X) = G(X)$ and set $H((f : X \to Y', s : Y \to Y')) = G(s)^{-1} \circ G(f)$. Details omitted. □

Lemma 4.25.7. *Let $\mathcal{C}$ be a category and let S be a left multiplicative system of morphisms of $\mathcal{C}$. The localization functor $Q : \mathcal{C} \to S^{-1}\mathcal{C}$ commutes with finite colimits.*

Proof. This is clear from (4.25.5.1), Remark 4.14.4, and Lemma 4.14.9. □

Lemma 4.25.8. *Let $\mathcal{C}$ be a category. Let S be a left multiplicative system. If $f : X \to Y$, $f' : X' \to Y'$ are two morphisms of $\mathcal{C}$ and if*

$$\begin{array}{ccc} Q(X) & \xrightarrow[a]{} & Q(X') \\ {\scriptstyle Q(f)}\downarrow & & \downarrow{\scriptstyle Q(f')} \\ Q(Y) & \xrightarrow{b} & Q(Y') \end{array}$$

is a commutative diagram in $S^{-1}\mathcal{C}$, then there exists a morphism $f'' : X'' \to Y''$ in $\mathcal{C}$ and a commutative diagram

$$\begin{array}{ccccc} X & \xrightarrow[g]{} & X'' & \xleftarrow[s]{} & X' \\ {\scriptstyle f}\downarrow & & \downarrow{\scriptstyle f''} & & \downarrow{\scriptstyle f'} \\ Y & \xrightarrow{h} & Y'' & \xleftarrow{t} & Y' \end{array}$$

in $\mathcal{C}$ with $s, t \in S$ and $a = s^{-1}g$, $b = t^{-1}h$.

Proof. We choose maps and objects in the following way: First write $a = s^{-1}g$ for some $s : X' \to X''$ in S and $g : X \to X''$. By LMS2 we can find $t : Y' \to Y''$ in S and $f'' : X'' \to Y''$ such that

$$\begin{array}{ccccc} X & \xrightarrow[g]{} & X'' & \xleftarrow[s]{} & X' \\ {\scriptstyle f}\downarrow & & \downarrow{\scriptstyle f''} & & \downarrow{\scriptstyle f'} \\ Y & & Y'' & \xleftarrow{t} & Y' \end{array}$$

commutes. Now in this diagram we are going to repeatedly change our choice of

$$X'' \xrightarrow{f''} Y'' \xleftarrow{t} Y'$$

by postcomposing both t and f'' by a morphism $d : Y'' \to Y'''$ with the property that $d \circ t \in S$. According to Remark 4.25.5 we may after such a replacement assume that there exists a morphism $h : Y \to Y''$ such that $b = t^{-1}h$. At this point we have everything as in the lemma except that we don't know that the left square of the diagram commutes. However, we do know that $Q(f''g) = Q(hf)$ in $S^{-1}\mathcal{D}$ because the right square commutes, the outer square commutes in $S^{-1}\mathcal{D}$ by assumption, and because $Q(s), Q(t)$ are isomorphisms. Hence using Lemma 4.25.4 we can find a morphism $d : Y'' \to Y'''$ in S (!) such that $df''g = dhf$. Hence we make one more replacement of the kind described above and we win. □

Right calculus of fractions. Let $\mathcal{C}$ be a category and let S be a right multiplicative system. We define a new category $S^{-1}\mathcal{C}$ as follows (we verify this works in the proof of Lemma 4.25.9):

(1) We set $\mathrm{Ob}(S^{-1}\mathcal{C}) = \mathrm{Ob}(\mathcal{C})$.
(2) Morphisms $X \to Y$ of $S^{-1}\mathcal{C}$ are given by pairs $(f : X' \to Y, s : X' \to X)$ with $s \in S$ up to equivalence. (Think of this as $fs^{-1} : X \to Y$.)
(3) Two pairs $(f_1 : X_1 \to Y, s_1 : X_1 \to X)$ and $(f_2 : X_2 \to Y, s_2 : X_2 \to X)$ are said to be equivalent if there exists a third pair $(f_3 : X_3 \to Y, s_3 : X_3 \to X)$ and morphisms $u : X_3 \to X_1$ and $v : X_3 \to X_2$ of $\mathcal{C}$ fitting into the commutative diagram

$$\begin{array}{ccccc} & & X_1 & & \\ & {\scriptstyle s_1}\swarrow & \uparrow{\scriptstyle u} & \searrow{\scriptstyle f_1} & \\ X & \xleftarrow{s_3} & X_3 & \xrightarrow{f_3} & Y \\ & {\scriptstyle s_2}\nwarrow & \downarrow{\scriptstyle v} & \nearrow{\scriptstyle f_2} & \\ & & X_2 & & \end{array}$$

(4) The composition of the equivalence classes of the pairs $(f : X' \to Y, s : X' \to X)$ and $(g : Y' \to Z, t : Y' \to Y)$ is defined as the equivalence class of a pair $(g \circ h : X'' \to Z, s \circ u : X'' \to X)$ where h and $u \in S$ are chosen to fit into a commutative diagram

$$\begin{array}{ccc} X'' & \xrightarrow{h} & Y' \\ {\scriptstyle u}\downarrow & & \downarrow{\scriptstyle t} \\ X' & \xrightarrow{f} & Y \end{array}$$

which exists by assumption.

Lemma 4.25.9. *Let $\mathcal{C}$ be a category and let S be a right multiplicative system.*

(1) *The relation on pairs defined above is an equivalence relation.*
(2) *The composition rule given above is well defined on equivalence classes.*
(3) *Composition is associative and hence $S^{-1}\mathcal{C}$ is a category.*

Proof. This lemma is dual to Lemma 4.25.2. It follows formally from that lemma by replacing $\mathcal{C}$ by its opposite category in which S is a left multiplicative system. □

We can "write any finite collection of morphisms with the same source as fractions with common denominator".

Lemma 4.25.10. *Let $\mathcal{C}$ be a category and let S be a right multiplicative system of morphisms of $\mathcal{C}$. Given any finite collection $g_i : X \to Y_i$ of morphisms of $S^{-1}\mathcal{C}$ we can find an element $s : X' \to X$ of S and $f_i : X' \to Y_i$ such that g_i is the equivalence class of the pair $(f_i : X' \to Y_i, s : X' \to X)$.*

Proof. This lemma is the dual of Lemma 4.25.3 and follows formally from that lemma by replacing all categories in sight by their opposites. □

There is an easy characterization of equality of morphisms if they have the same denominator.

Lemma 4.25.11. *Let $\mathcal{C}$ be a category and let S be a right multiplicative system of morphisms of $\mathcal{C}$. Let $A, B : X \to Y$ be morphisms of $S^{-1}\mathcal{C}$ which are the equivalence classes of $(f : X' \to Y, s : X' \to X)$ and $(g : X' \to Y, s : X' \to X)$. Then $A = B$ if and only if there exists a morphism $a : X'' \to X'$ with $s \circ a \in S$ and such that $f \circ a = g \circ a$.*

Proof. This is dual to Lemma 4.25.4. □

Remark 4.25.12. Let $\mathcal{C}$ be a category. Let S be a right multiplicative system. Given an object X of $\mathcal{C}$ we denote S/X the category whose objects are $s : X' \to X$ with $s \in S$ and whose morphisms are commutative diagrams

$$\begin{array}{ccc} X' & \xrightarrow{a} & X'' \\ & {\scriptstyle s}\searrow \quad \swarrow{\scriptstyle t} & \\ & X & \end{array}$$

where $a : X' \to X''$ is arbitrary. The category S/X is cofiltered (see Definition 4.20.1). (This is dual to the corresponding statement in Remark 4.25.5.) Now the combined results of Lemmas 4.25.10 and 4.25.11 tell us that

$$\mathrm{Mor}_{S^{-1}\mathcal{C}}(X, Y) = \mathrm{colim}_{(s:X' \to X) \in (S/X)^{opp}} \mathrm{Mor}_{\mathcal{C}}(X', Y) \tag{4.25.12.1}$$

This formula expressing morphisms in $S^{-1}\mathcal{C}$ as a filtered colimit of morphisms in $\mathcal{C}$ is occasionally useful.

Lemma 4.25.13. *Let $\mathcal{C}$ be a category and let S be a right multiplicative system of morphisms of $\mathcal{C}$.*

1. *The rules $X \mapsto X$ and $(f : X \to Y) \mapsto (f : X \to Y, id_X : X \to X)$ define a functor $Q : \mathcal{C} \to S^{-1}\mathcal{C}$.*
2. *For any $s \in S$ the morphism $Q(s)$ is an isomorphism in $S^{-1}\mathcal{C}$.*
3. *If $G : \mathcal{C} \to \mathcal{D}$ is any functor such that $G(s)$ is invertible for every $s \in S$, then there exists a unique functor $H : S^{-1}\mathcal{C} \to \mathcal{D}$ such that $H \circ Q = G$.*

Proof. This lemma is the dual of Lemma 4.25.6 and follows formally from that lemma by replacing all categories in sight by their opposites. □

Lemma 4.25.14. *Let $\mathcal{C}$ be a category and let S be a right multiplicative system of morphisms of $\mathcal{C}$. The localization functor $Q : \mathcal{C} \to S^{-1}\mathcal{C}$ commutes with finite limits.*

Proof. This is clear from (4.25.12.1), Remark 4.14.4, and Lemma 4.14.9. □

Lemma 4.25.15. *Let $\mathcal{C}$ be a category. Let S be a right multiplicative system. If $f : X \to Y$, $f' : X' \to Y'$ are two morphisms of $\mathcal{C}$ and if*

$$\begin{array}{ccc} Q(X) & \xrightarrow{a} & Q(X') \\ {\scriptstyle Q(f)}\downarrow & & \downarrow{\scriptstyle Q(f')} \\ Q(Y) & \xrightarrow{b} & Q(Y') \end{array}$$

is a commutative diagram in $S^{-1}\mathcal{C}$, then there exists a morphism $f'' : X'' \to Y''$ in $\mathcal{C}$ and a commutative diagram

$$\begin{array}{ccccc} X & \xleftarrow{s} & X'' & \xrightarrow{g} & X' \\ {\scriptstyle f}\downarrow & & \downarrow{\scriptstyle f''} & & \downarrow{\scriptstyle f'} \\ Y & \xleftarrow{t} & Y'' & \xrightarrow{h} & Y' \end{array}$$

in $\mathcal{C}$ with $s, t \in S$ and $a = gs^{-1}$, $b = ht^{-1}$.

Proof. This lemma is dual to Lemma 4.25.8 but we can also prove it directly as follows. We choose maps and objects in the following way: First write $b = ht^{-1}$ for some $t : Y'' \to Y$ in S and $h : Y'' \to Y'$. By RMS2 we can find $s : X'' \to X$ in S and $f'' : X'' \to Y''$ such that

$$\begin{array}{ccccc} X & \xleftarrow{s} & X'' & & X' \\ {\scriptstyle f}\downarrow & & \downarrow{\scriptstyle f''} & & \downarrow{\scriptstyle f'} \\ Y & \xleftarrow{t} & Y'' & \xrightarrow{h} & Y' \end{array}$$

commutes. Now in this diagram we are going to repeatedly change our choice of

$$X \xleftarrow{s} X'' \xrightarrow{f''} Y''$$

by precomposing both s and f'' by a morphism $d : X''' \to X''$ with the property that $s \circ d \in S$. According to Remark 4.25.12 we may after such a replacement assume that there exists a morphism $g : X'' \to X'$ such that $a = gs^{-1}$. At this

point we have everything as in the lemma except that we don't know that the right square of the diagram commutes. However, we do know that $Q(f'g) = Q(hf'')$ in $S^{-1}\mathcal{D}$ because the left square commutes, the outer square commutes in $S^{-1}\mathcal{D}$ by assumption, and because $Q(s), Q(t)$ are isomorphisms. Hence using Lemma 4.25.11 we can find a morphism $d : X''' \to X''$ in S (!) such that $f'gd = hf''d$. Hence we make one more replacement of the kind described above and we win. □

Multiplicative systems and two sided calculus of fractions. If S is a multiplicative system then left and right calculus of fractions given canonically isomorphic categories.

Lemma 4.25.16. *Let $\mathcal{C}$ be a category and let S be a multiplicative system. The category of left fractions and the category of right fractions $S^{-1}\mathcal{C}$ are canonically isomorphic.*

Proof. Denote $\mathcal{C}_{left}$, $\mathcal{C}_{right}$ the two categories of fractions. By the universal properties of Lemmas 4.25.6 and 4.25.13 we obtain functors $\mathcal{C}_{left} \to \mathcal{C}_{right}$ and $\mathcal{C}_{right} \to \mathcal{C}_{left}$. By the uniqueness of these functors they are each others inverse. □

Definition 4.25.17. Let $\mathcal{C}$ be a category and let S be a multiplicative system. We say S is *saturated* if, in addition to MS1, MS2, MS3 we also have

MS4 Given three composable morphisms f, g, h, if $fg, gh \in S$, then $g \in S$.

Note that a saturated multiplicative system contains all isomorphisms. Moreover, if f, g, h are composable morphisms in a category and fg, gh are isomorphisms, then g is an isomorphism (because then g has both a left and a right inverse, hence is invertible).

Lemma 4.25.18. *Let $\mathcal{C}$ be a category and let S be a multiplicative system. Denote $Q : \mathcal{C} \to S^{-1}\mathcal{C}$ the localization functor. The set*

$$\hat{S} = \{f \in Arrows(\mathcal{C}) \mid Q(f) \text{ is an isomorphism}\}$$

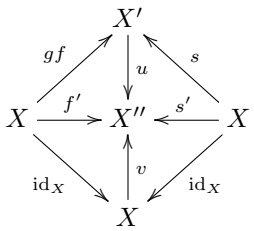
is equal to

$$S' = \{f \in Arrows(\mathcal{C}) \mid \text{there exist } g, h \text{ such that } gf, fh \in S\}$$

and is the smallest saturated multiplicative system containing S. In particular, if S is saturated, then $\hat{S} = S$.

Proof. It is clear that $S \subset S' \subset \hat{S}$ because elements of S' map to morphisms in $S^{-1}\mathcal{C}$ which have both left and right inverses. Note that S' satisfies MS4, and that $\hat{S}$ satisfies MS1. Next, we prove that $S' = \hat{S}$.

Let $f \in \hat{S}$. Let $s^{-1}g = ht^{-1}$ be the inverse morphism in $S^{-1}\mathcal{C}$. (We may use both left fractions and right fractions to describe morphisms in $S^{-1}\mathcal{C}$, see Lemma 4.25.16.) The relation $\mathrm{id}_X = s^{-1}gf$ in $S^{-1}\mathcal{C}$ means there exists a commutative diagram



for some morphisms f', u, v and $s' \in S$. Hence $ugf = s' \in S$. Similarly, using that $\mathrm{id}_Y = fht^{-1}$ one proves that $fhw \in S$ for some w. We conclude that $f \in S'$. Thus $S' = \hat{S}$. Provided we prove that $S' = \hat{S}$ is a multiplicative system it is now clear that this implies that $S' = \hat{S}$ is the smallest saturated system containing S.

Our remarks above take care of MS1 and MS4, so to finish the proof of the lemma we have to show that LMS2, RMS2, LMS3, RMS3 hold for $\hat{S}$. Let us check that LMS2 holds for $\hat{S}$. Suppose we have a solid diagram

$$\begin{array}{ccc} X & \xrightarrow{g} & Y \\ {\scriptstyle t}\downarrow & & \vdots\,{\scriptstyle s} \\ Z & \overset{f}{\dashrightarrow} & W \end{array}$$

with $t \in \hat{S}$. Pick a morphism $a : Z \to Z'$ such that $at \in S$. Then we can use LMS2 for S to find a commutative diagram

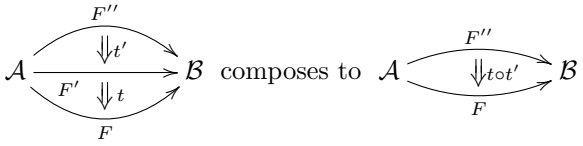
$$\begin{array}{ccc} X & \xrightarrow{g} & Y \\ {\scriptstyle t}\downarrow & & \\ Z & & \downarrow{\scriptstyle s} \\ {\scriptstyle a}\downarrow & & \\ Z' & \xrightarrow{f'} & W \end{array}$$

and setting $f = f' \circ a$ we win. The proof of RMS2 is dual to this. Finally, suppose given a pair of morphisms $f, g : X \to Y$ and $t \in \hat{S}$ with target X such that $ft = gt$. Then we pick a morphism b such that $tb \in S$. Then $ftb = gtb$ which implies by LMS3 for S that there exists an $s \in S$ with source Y such that $sf = sg$ as desired. The proof of RMS3 is dual to this. □

4.26. Formal properties

In this section we discuss some formal properties of the 2-category of categories. This will lead us to the definition of a (strict) 2-category later.

Let us denote $\mathrm{Ob}(Cat)$ the class of all categories. For every pair of categories $\mathcal{A}, \mathcal{B} \in \mathrm{Ob}(Cat)$ we have the "small" category of functors $\mathrm{Fun}(\mathcal{A}, \mathcal{B})$. Composition of transformation of functors such as



is called *vertical* composition. We will use the usual symbol $\circ$ for this. Next, we will define *horizontal* composition. In order to do this we explain a bit more of the structure at hand.

Namely for every triple of categories $\mathcal{A}$, $\mathcal{B}$, and $\mathcal{C}$ there is a composition law

$$\circ : \mathrm{Ob}(\mathrm{Fun}(\mathcal{B}, \mathcal{C})) \times \mathrm{Ob}(\mathrm{Fun}(\mathcal{A}, \mathcal{B})) \longrightarrow \mathrm{Ob}(\mathrm{Fun}(\mathcal{A}, \mathcal{C}))$$

coming from composition of functors. This composition law is associative, and identity functors act as units. In other words – forgetting about transformations of functors – we see that Cat forms a category. How does this structure interact with the morphisms between functors?

Well, given $t : F \to F'$ a transformation of functors $F, F' : \mathcal{A} \to \mathcal{B}$ and a functor $G : \mathcal{B} \to \mathcal{C}$ we can define a transformation of functors $G \circ F \to G \circ F'$. We will denote this transformation ${}_Gt$. It is given by the formula $({}_Gt)_x = G(t_x) : G(F(x)) \to G(F'(x))$ for all $x \in \mathcal{A}$. In this way composition with G becomes a functor

$$\mathrm{Fun}(\mathcal{A}, \mathcal{B}) \longrightarrow \mathrm{Fun}(\mathcal{A}, \mathcal{C}).$$

To see this you just have to check that ${}_G(\mathrm{id}_F) = \mathrm{id}_{G \circ F}$ and that ${}_G(t_1 \circ t_2) = {}_Gt_1 \circ {}_Gt_2$. Of course we also have that ${}_{\mathrm{id}_{\mathcal{A}}}t = t$.

Similarly, given $s : G \to G'$ a transformation of functors $G, G' : \mathcal{B} \to \mathcal{C}$ and $F : \mathcal{A} \to \mathcal{B}$ a functor we can define s_F to be the transformation of functors $G \circ F \to G' \circ F$ given by $(s_F)_x = s_{F(x)} : G(F(x)) \to G'(F(x))$ for all $x \in \mathcal{A}$. In this way composition with F becomes a functor

$$\mathrm{Fun}(\mathcal{B}, \mathcal{C}) \longrightarrow \mathrm{Fun}(\mathcal{A}, \mathcal{C}).$$

To see this you just have to check that $(\mathrm{id}_G)_F = \mathrm{id}_{G \circ F}$ and that $(s_1 \circ s_2)_F = s_{1,F} \circ s_{2,F}$. Of course we also have that $s_{\mathrm{id}_{\mathcal{B}}} = s$.

These constructions satisfy the additional properties

$${}_{G_1}({}_{G_2}t) = {}_{G_1 \circ G_2}t, \ (s_{F_1})_{F_2} = s_{F_1 \circ F_2}, \text{ and } {}_H(s_F) = ({}_Hs)_F$$

whenever these make sense. Finally, given functors $F, F' : \mathcal{A} \to \mathcal{B}$, and $G, G' : \mathcal{B} \to \mathcal{C}$ and transformations $t : F \to F'$, and $s : G \to G'$ the following diagram is commutative

$$\begin{array}{ccc} G \circ F & \xrightarrow{{}_Gt} & G \circ F' \\ \downarrow{\scriptstyle s_F} & & \downarrow{\scriptstyle s_{F'}} \\ G' \circ F & \xrightarrow[{}_{G'}t]{} & G' \circ F' \end{array}$$

in other words ${}_{G'}t \circ s_F = s_{F'} \circ {}_Gt$. To prove this we just consider what happens on any object $x \in \mathrm{Ob}(\mathcal{A})$:

$$\begin{array}{ccc} G(F(x)) & \xrightarrow{G(t_x)} & G(F'(x)) \\ \downarrow{\scriptstyle s_{F(x)}} & & \downarrow{\scriptstyle s_{F'(x)}} \\ G'(F(x)) & \xrightarrow[G'(t_x)]{} & G'(F'(x)) \end{array}$$

which is commutative because s is a transformation of functors. This compatibility relation allows us to define horizontal composition.

Definition 4.26.1. Given a diagram as in the left hand side of:

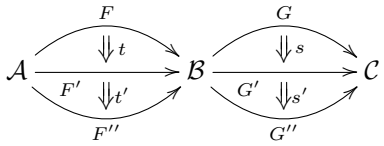
$$\mathcal{A} \underset{F'}{\overset{F}{\Downarrow t}} \mathcal{B} \underset{G'}{\overset{G}{\Downarrow s}} \mathcal{C} \quad \text{gives} \quad \mathcal{A} \underset{G' \circ F'}{\overset{G \circ F}{\Downarrow s * t}} \mathcal{C}$$

we define the *horizontal* composition $s \star t$ to be the transformation of functors ${}_{G'}t \circ s_F = s_{F'} \circ {}_Gt$.

Now we see that we may recover our previously constructed transformations ${}_Gt$ and s_F as ${}_Gt = \mathrm{id}_G \star t$ and $s_F = s \star \mathrm{id}_F$. Furthermore, all of the rules we found above are consequences of the properties stated in the lemma that follows.

Lemma 4.26.2. *The horizontal and vertical compositions have the following properties*

(1) $\circ$ *and* $\star$ *are associative,*
(2) *the identity transformations* id_F *are units for* $\circ$*,*
(3) *the identity transformations of the identity functors* $id_{id_\mathcal{A}}$ *are units for* $\star$ *and* $\circ$*, and*
(4) *given a diagram*



we have $(s' \circ s) \star (t' \circ t) = (s' \star t') \circ (s \star t)$.

Proof. The last statement turns using our previous notation into the following equation

$$s'_{F''} \circ {}_{G'}t' \circ s_{F'} \circ {}_Gt = (s' \circ s)_{F''} \circ {}_G(t' \circ t).$$

According to our result above applied to the middle composition we may rewrite the left hand side as $s'_{F''} \circ s_{F''} \circ {}_Gt' \circ {}_Gt$ which is easily shown to be equal to the right hand side. □

Another way of formulating condition (4) of the lemma is that composition of functors and horizontal composition of transformation of functors gives rise to a functor

$$(\circ, \star) : \mathrm{Fun}(\mathcal{B}, \mathcal{C}) \times \mathrm{Fun}(\mathcal{A}, \mathcal{B}) \longrightarrow \mathrm{Fun}(\mathcal{A}, \mathcal{C})$$

whose source is the product category, see Definition 4.2.20.

4.27. 2-categories

We will give a definition of (strict) 2-categories as they appear in the setting of stacks. Before you read this take a look at Section 4.26 and Example 4.28.2. Basically, you take this example and you write out all the rules satisfied by the objects, 1-morphisms and 2-morphisms in that example.

Definition 4.27.1. A (strict) 2-*category* $\mathcal{C}$ consists of the following data

(1) A set of objects $\mathrm{Ob}(\mathcal{C})$.
(2) For each pair $x, y \in \mathrm{Ob}(\mathcal{C})$ a category $\mathrm{Mor}_\mathcal{C}(x, y)$. The objects of $\mathrm{Mor}_\mathcal{C}(x, y)$ will be called 1-*morphisms* and denoted $F : x \to y$. The morphisms between these 1-morphisms will be called 2-*morphisms* and denoted $t : F' \to F$. The composition of 2-morphisms in $\mathrm{Mor}_\mathcal{C}(x, y)$ will be called *vertical* composition and will be denoted $t \circ t'$ for $t : F' \to F$ and $t' : F'' \to F'$.

(3) For each triple $x, y, z \in \mathrm{Ob}(\mathcal{C})$ a functor

$$(\circ, \star) : \mathrm{Mor}_{\mathcal{C}}(y, z) \times \mathrm{Mor}_{\mathcal{C}}(x, y) \longrightarrow \mathrm{Mor}_{\mathcal{C}}(x, z).$$

The image of the pair of 1-morphisms (F, G) on the left hand side will be called the *composition* of F and G, and denoted $F \circ G$. The image of the pair of 2-morphisms (t, s) will be called the *horizontal* composition and denoted $t \star s$.

These data are to satisfy the following rules:

(1) The set of objects together with the set of 1-morphisms endowed with composition of 1-morphisms forms a category.
(2) Horizontal composition of 2-morphisms is associative.
(3) The identity 2-morphism $\mathrm{id}_{\mathrm{id}_x}$ of the identity 1-morphism id_x is a unit for horizontal composition.

This is obviously not a very pleasant type of object to work with. On the other hand, there are lots of examples where it is quite clear how you work with it. The only example we have so far is that of the 2-category whose objects are a given collection of categories, 1-morphisms are functors between these categories, and 2-morphisms are natural transformations of functors, see Section 4.26. As far as this text is concerned all 2-categories will be sub 2-categories of this example. Here is what it means to be a sub 2-category.

Definition 4.27.2. Let $\mathcal{C}$ be a 2-category. A *sub 2-category* $\mathcal{C}'$ of $\mathcal{C}$, is given by a subset $\mathrm{Ob}(\mathcal{C}')$ of $\mathrm{Ob}(\mathcal{C})$ and sub categories $\mathrm{Mor}_{\mathcal{C}'}(x, y)$ of the categories $\mathrm{Mor}_{\mathcal{C}}(x, y)$ for all $x, y \in \mathrm{Ob}(\mathcal{C}')$ such that these, together with the operations $\circ$ (composition 1-morphisms), $\circ$ (vertical composition 2-morphisms), and $\star$ (horizontal composition) form a 2-category.

Remark 4.27.3. Big 2-categories. In many texts a 2-category is allowed to have a class of objects (but hopefully a "class of classes" is not allowed). We will allow these "big" 2-categories as well, but only in the following list of cases (to be updated as we go along):

(1) The 2-category of categories *Cat*.
(2) The $(2, 1)$-category of categories *Cat*.
(3) The 2-category of groupoids *Groupoids*.
(4) The $(2, 1)$-category of groupoids *Groupoids*.
(5) The 2-category of fibred categories over a fixed category.
(6) The $(2, 1)$-category of fibred categories over a fixed category.

Note that in each case the class of objects of the 2-category $\mathcal{C}$ is a proper class, but for all objects $x, y \in \mathrm{Ob}(C)$ the category $\mathrm{Mor}_{\mathcal{C}}(x, y)$ is "small" (according to our conventions).

The notion of equivalence of categories that we defined in Section 4.2 extends to the more general setting of 2-categories as follows.

Definition 4.27.4. Two objects x, y of a 2-category are *equivalent* if there exist 1-morphisms $F : x \to y$ and $G : y \to x$ such that $F \circ G$ is 2-isomorphic to id_y and $G \circ F$ is 2-isomorphic to id_x.

Sometimes we need to say what it means to have a functor from a category into a 2-category.

Definition 4.27.5. Let $\mathcal{A}$ be a category and let $\mathcal{C}$ be a 2-category.

(1) A *functor* from an ordinary category into a 2-category will ignore the 2-morphisms unless mentioned otherwise. In other words, it will be a "usual" functor into the category formed out of 2-category by forgetting all the 2-morphisms.
(2) A *weak functor*, or a *pseudo functor* φ from $\mathcal{A}$ into the 2-category $\mathcal{C}$ is given by the following data
   (a) a map $\varphi : \mathrm{Ob}(\mathcal{A}) \to \mathrm{Ob}(\mathcal{C})$,
   (b) for every pair $x, y \in \mathrm{Ob}(\mathcal{A})$, and every morphism $f : x \to y$ a 1-morphism $\varphi(f) : \varphi(x) \to \varphi(y)$,
   (c) for every $x \in \mathrm{Ob}(A)$ a 2-morphism $\alpha_x : \mathrm{id}_{\varphi(x)} \to \varphi(\mathrm{id}_x)$, and
   (d) for every pair of composable morphisms $f : x \to y$, $g : y \to z$ of $\mathcal{A}$ a 2-morphism $\alpha_{g,f} : \varphi(g \circ f) \to \varphi(g) \circ \varphi(f)$.

   These data are subject to the following conditions:
   (a) the 2-morphisms α_x and $\alpha_{g,f}$ are all isomorphisms,
   (b) for any morphism $f : x \to y$ in $\mathcal{A}$ we have $\alpha_{\mathrm{id}_y,f} = \alpha_y \star \mathrm{id}_{\varphi(f)}$:

$$\varphi(x) \underset{\varphi(f)}{\overset{\varphi(f)}{\Downarrow \mathrm{id}_{\varphi(f)}}} \varphi(y) \underset{\varphi(\mathrm{id}_y)}{\overset{\mathrm{id}_y}{\Downarrow \alpha_y}} \varphi(y) \;=\; \varphi(x) \underset{\varphi(\mathrm{id}_y)\circ\varphi(f)}{\overset{\varphi(f)}{\Downarrow \alpha_{\mathrm{id}_y,f}}} \varphi(y)$$

   (c) for any morphism $f : x \to y$ in $\mathcal{A}$ we have $\alpha_{f,\mathrm{id}_x} = \mathrm{id}_{\varphi(f)} \star \alpha_x$,
   (d) for any triple of composable morphisms $f : w \to x$, $g : x \to y$, and $h : y \to z$ of $\mathcal{A}$ we have

$$(\mathrm{id}_{\varphi(h)} \star \alpha_{g,f}) \circ \alpha_{h,g\circ f} = (\alpha_{h,g} \star \mathrm{id}_{\varphi(f)}) \circ \alpha_{h\circ g,f}$$

   in other words the following diagram with objects 1-morphisms and arrows 2-morphisms commutes

$$\begin{array}{ccc} \varphi(h\circ g\circ f) & \xrightarrow[\alpha_{h\circ g,f}]{} & \varphi(h\circ g)\circ\varphi(f) \\ \Big\downarrow{\scriptstyle \alpha_{h,g\circ f}} & & \Big\downarrow{\scriptstyle \alpha_{h,g}\star\mathrm{id}_{\varphi(f)}} \\ \varphi(h)\circ\varphi(g\circ f) & \xrightarrow{\mathrm{id}_{\varphi(h)}\star\alpha_{g,f}} & \varphi(h)\circ\varphi(g)\circ\varphi(f) \end{array}$$

Again this is not a very workable notion, but it does sometimes come up. There is a theorem that says that any pseudo-functor is isomorphic to a functor. Finally, there are the notions of *functor between* 2*-categories*, and *pseudo functor between* 2*-categories*. This last notion leads us into 3-category territory. We would like to avoid having to define this at almost any cost!

4.28. (2, 1)-categories

Some 2-categories have the property that all 2-morphisms are isomorphisms. These will play an important role in the following, and they are easier to work with.

Definition 4.28.1. A (strict) $(2,1)$-*category* is a 2-category in which all 2-morphisms are isomorphisms.

Example 4.28.2. The 2-category *Cat*, see Remark 4.27.3, can be turned into a $(2,1)$-category by only allowing isomorphisms of functors as 2-morphisms.

In fact, more generally any 2-category $\mathcal{C}$ produces a $(2,1)$-category by considering the sub 2-category $\mathcal{C}'$ with the same objects and 1-morphisms but whose 2-morphisms are the invertible 2-morphisms of $\mathcal{C}$. In this situation we will say "*let $\mathcal{C}'$ be the $(2,1)$-category associated to $\mathcal{C}$*" or similar. For example, the $(2,1)$-category of groupoids means the 2-category whose objects are groupoids, whose 1-morphisms are functors and whose 2-morphisms are isomorphisms of functors. Except that this is a bad example as a transformation between functors between groupoids is automatically an isomorphism!

Remark 4.28.3. Thus there are variants of the construction of Example 4.28.2 above where we look at the 2-category of groupoids, or categories fibred in groupoids over a fixed category, or stacks. And so on.

4.29. 2-fibre products

In this section we introduce 2-fibre products. Suppose that $\mathcal{C}$ is a 2-category. We say that a diagram

$$\begin{array}{ccc} w & \longrightarrow & y \\ \downarrow & & \downarrow \\ x & \longrightarrow & z \end{array}$$

2-commutes if the two 1-morphisms $w \to y \to z$ and $w \to x \to z$ are 2-isomorphic. In a 2-category it is more natural to ask for 2-commutativity of diagrams than for actually commuting diagrams. (Indeed, some may say that we should not work with strict 2-categories at all, and in a "weak" 2-category the notion of a commutative diagram of 1-morphisms does not even make sense.) Correspondingly the notion of a fibre product has to be adjusted.

Let $\mathcal{C}$ be a 2-category. Let $x, y, z \in \mathrm{Ob}(\mathcal{C})$ and $f \in \mathrm{Mor}_{\mathcal{C}}(x, z)$ and $g \in \mathrm{Mor}_{\mathcal{C}}(y, z)$. In order to define the 2-fibre product of f and g we are going to look at 2-commutative diagrams

$$\begin{array}{ccc} w & \xrightarrow[a]{} & x \\ {\scriptstyle b}\downarrow & & \downarrow{\scriptstyle f} \\ y & \xrightarrow[g]{} & z. \end{array}$$

Now in the case of categories, the fibre product is a final object in the category of such diagrams. Correspondingly a 2-fibre product is a final object in a 2-category (see definition below). The 2-*category of* 2-*commutative diagrams* is the 2-category defined as follows:

(1) Objects are quadruples (w, a, b, ϕ) as above where ϕ is an invertible 2-morphism $\phi : f \circ a \to g \circ b$,
(2) 1-morphisms from (w', a', b', ϕ') to (w, a, b, ϕ) are given by $(k : w' \to w, \alpha : a' \to a \circ k, \beta : b' \to b \circ k)$ such that

$$\begin{array}{ccc} f \circ a' & \xrightarrow[\mathrm{id}_f \star \alpha]{} & f \circ a \circ k \\ {\scriptstyle \phi'}\downarrow & & \downarrow{\scriptstyle \phi \star \mathrm{id}_k} \\ f \circ b' & \xrightarrow[\mathrm{id}_f \star \beta]{} & f \circ b \circ k \end{array}$$

is commutative,

(3) given a second 1-morphism $(k', \alpha', \beta') : (w'', a'', b'', \phi'') \to (w', \alpha', \beta', \phi')$ the composition of 1-morphisms is given by the rule

$$(k, \alpha, \beta) \circ (k', \alpha', \beta') = (k \circ k', (\alpha \star \mathrm{id}_{k'}) \circ \alpha', (\beta \star \mathrm{id}_{k'}) \circ \beta'),$$

(4) a 2-morphism between 1-morphisms (k_i, α_i, β_i), $i = 1, 2$ with the same is given by a 2-morphism $\delta : k_1 \to k_2$ such that

$$\begin{array}{ccc} a' & \xrightarrow[\alpha_1]{} & a \circ k_1 \\ & {\scriptstyle \alpha_2} \searrow & \downarrow {\scriptstyle \mathrm{id}_a \star \delta} \\ & & a \circ k_2 \end{array} \qquad \begin{array}{ccc} b \circ k_1 & \xleftarrow[\beta_1]{} & b' \\ {\scriptstyle \mathrm{id}_b \star \delta} \downarrow & \swarrow {\scriptstyle \beta_2} & \\ b \circ k_2 & & \end{array}$$

commute,

(5) vertical composition of 2-morphisms is given by vertical composition of the morphisms δ in $\mathcal{C}$, and

(6) horizontal composition of the diagram

$$(w'', a'', b'', \phi'') \underset{(k_2', \alpha_2', \beta_2')}{\overset{(k_1', \alpha_1', \beta_1')}{\Downarrow \delta'}} (w', a', b', \phi') \underset{(k_2, \alpha_2, \beta_2)}{\overset{(k_1, \alpha_1, \beta_1)}{\Downarrow \delta}} (w, a, b, \phi)$$

is given by the diagram

$$(w'', a'', b'', \phi'') \underset{(k_2 \circ k_2', (\alpha_2 \star \mathrm{id}_{k_2'}) \circ \alpha_2', (\beta_2 \star \mathrm{id}_{k_2'}) \circ \beta_2')}{\overset{(k_1 \circ k_1', (\alpha_1 \star \mathrm{id}_{k_1'}) \circ \alpha_1', (\beta_1 \star \mathrm{id}_{k_1'}) \circ \beta_1')}{\Downarrow \delta \star \delta'}} (w, a, b, \phi)$$

Note that if $\mathcal{C}$ is actually a $(2,1)$-category, the morphisms α and β in (2) above are automatically also isomorphisms[2]. In addition the 2-category of 2-commutative diagrams is also a $(2,1)$-category if $\mathcal{C}$ is a $(2,1)$-category.

Definition 4.29.1. A *final object* of a $(2,1)$-category $\mathcal{C}$ is an object x such that

(1) for every $y \in \mathrm{Ob}(\mathcal{C})$ there is a morphism $y \to x$, and
(2) every two morphisms $y \to x$ are isomorphic by a unique 2-morphism.

Likely, in the more general case of 2-categories there are different flavours of final objects. We do not want to get into this and hence we only define 2-fibre products in the $(2,1)$-case.

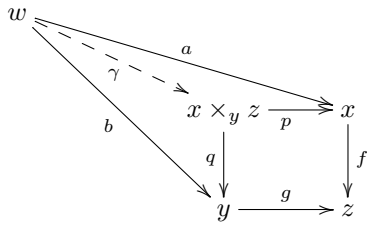
Definition 4.29.2. Let $\mathcal{C}$ be a $(2,1)$-category. Let $x, y, z \in \mathrm{Ob}(\mathcal{C})$ and $f \in \mathrm{Mor}_{\mathcal{C}}(x, z)$ and $g \in \mathrm{Mor}_{\mathcal{C}}(y, z)$. A *2-fibre product of f and g* is a final object in the category of 2-commutative diagrams described above. If a 2-fibre product exists we will denote it $x \times_z y \in \mathrm{Ob}(\mathcal{C})$, and denote the required morphisms $p \in \mathrm{Mor}_{\mathcal{C}}(x \times_z y, x)$ and $q \in \mathrm{Mor}_{\mathcal{C}}(x \times_z y, y)$ making the diagram

$$\begin{array}{ccc} x \times_y z & \xrightarrow{p} & x \\ {\scriptstyle q} \downarrow & & \downarrow {\scriptstyle f} \\ y & \xrightarrow{g} & z \end{array}$$

[2]In fact it seems in the 2-category case that one could define another 2-category of 2-commutative diagrams where the direction of the arrows α, β is reversed, or even where the direction of only one of them is reversed. This is why we restrict to $(2,1)$-categories later on.

2-commute and we will denote the given invertible 2-morphism exhibiting this by $\psi : f \circ p \to g \circ q$.

Thus the following universal property holds: for any $w \in \mathrm{Ob}(\mathcal{C})$ and morphisms $a \in \mathrm{Mor}_{\mathcal{C}}(w, x)$ and $b \in \mathrm{Mor}_{\mathcal{C}}(w, y)$ with a given 2-isomorphism $\phi : f \circ a \to g \circ b$ there is a $\gamma \in \mathrm{Mor}_{\mathcal{C}}(w, x \times_z y)$ making the diagram



2-commute such that for suitable choices of $a \to p \circ \gamma$ and $b \to q \circ \gamma$ the diagram

$$\begin{array}{ccc} f \circ a & \longrightarrow & f \circ p \circ \gamma \\ \downarrow{\scriptstyle \phi} & & \downarrow{\scriptstyle \psi \star \mathrm{id}_\gamma} \\ g \circ b & \longrightarrow & g \circ q \circ \gamma \end{array}$$

commutes. Moreover γ is unique up to isomorphism. Of course the exact properties are finer than this. All of the cases of 2-fibre products that we will need later on come from the following example of 2-fibre products in the 2-category of categories.

Example 4.29.3. Let $\mathcal{A}$, $\mathcal{B}$, and $\mathcal{C}$ be categories. Let $F : \mathcal{A} \to \mathcal{C}$ and $G : \mathcal{B} \to \mathcal{C}$ be functors. We define a category $\mathcal{A} \times_{\mathcal{C}} \mathcal{B}$ as follows:

(1) an object of $\mathcal{A} \times_{\mathcal{C}} \mathcal{B}$ is a triple (A, B, f), where $A \in \mathrm{Ob}(\mathcal{A})$, $B \in \mathrm{Ob}(\mathcal{B})$, and $f : F(A) \to G(B)$ is an isomorphism in $\mathcal{C}$,
(2) a morphism $(A, B, f) \to (A', B', f')$ is given by a pair (a, b), where $a : A \to A'$ is a morphism in $\mathcal{A}$, and $b : B \to B'$ is a morphism in $\mathcal{B}$ such that the diagram

$$\begin{array}{ccc} F(A) & \xrightarrow{f} & G(B) \\ \downarrow{\scriptstyle F(a)} & & \downarrow{\scriptstyle G(b)} \\ F(A') & \xrightarrow{f'} & G(B') \end{array}$$

is commutative.

Moreover, we define functors $p : \mathcal{A} \times_{\mathcal{C}} \mathcal{B} \to \mathcal{A}$ and $q : \mathcal{A} \times_{\mathcal{C}} \mathcal{B} \to \mathcal{B}$ by setting

$$p(A, B, f) = A, \quad q(A, B, f) = B,$$

in other words, these are the forgetful functors. We define a transformation of functors $\psi : F \circ p \to G \circ q$. On the object $\xi = (A, B, f)$ it is given by $\psi_\xi = f : F(p(\xi)) = F(A) \to G(B) = G(q(\xi))$.

Lemma 4.29.4. *In the $(2, 1)$-category of categories 2-fibre products exist and are given by the construction of Example 4.29.3.*

Proof. Let us check the universal property: let $\mathcal{W}$ be a category, let $a : \mathcal{W} \to \mathcal{A}$ and $b : \mathcal{W} \to \mathcal{B}$ be functors, and let $t : F \circ a \to G \circ b$ be an isomorphism of functors.

Consider the functor $\gamma : \mathcal{W} \to \mathcal{A} \times_{\mathcal{C}} \mathcal{B}$ given by $W \mapsto (a(W), b(W), t_W)$. (Check this is a functor omitted.) Moreover, consider $\alpha : a \to p \circ \gamma$ and $\beta : b \to q \circ \gamma$ obtained from the identities $p \circ \gamma = a$ and $q \circ \gamma = b$. Then it is clear that (γ, α, β) is a morphism from (W, a, b, t) to $(\mathcal{A} \times_{\mathcal{C}} \mathcal{B}, p, q, \psi)$.

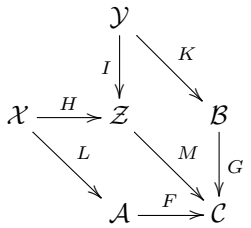
Let $(k, \alpha', \beta') : (W, a, b, t) \to (\mathcal{A} \times_{\mathcal{C}} \mathcal{B}, p, q, \psi)$ be a second such morphism. For an object W of $\mathcal{W}$ let us write $k(W) = (a_k(W), b_k(W), t_{k,W})$. Hence $p(k(W)) = a_k(W)$ and so on. The map α' corresponds to functorial maps $\alpha' : a(W) \to a_k(W)$. Since we are working in the $(2,1)$-category of categories, in fact each of the maps $a(W) \to a_k(W)$ is an isomorphism. We can use these (and their counterparts $b(W) \to b_k(W)$) to get isomorphisms

$$\delta_W : \gamma(W) = (a(W), b(W), t_W) \longrightarrow (a_k(W), b_k(W), t_{k,W}) = k(W).$$

It is straightforward to show that δ defines a 2-isomorphism between γ and k in the 2-category of 2-commutative diagrams as desired. □

Remark 4.29.5. Let $\mathcal{A}$, $\mathcal{B}$, and $\mathcal{C}$ be categories. Let $F : \mathcal{A} \to \mathcal{C}$ and $G : \mathcal{B} \to \mathcal{C}$ be functors. Another, slightly more symmetrical, construction of a 2-fibre product $\mathcal{A} \times_{\mathcal{C}} \mathcal{B}$ is as follows. An object is a quintuple (A, B, C, a, b) where A, B, C are objects of $\mathcal{A}, \mathcal{B}, \mathcal{C}$ and where $a : F(A) \to C$ and $b : G(B) \to C$ are isomorphisms. A morphism $(A, B, C, a, b) \to (A', B', C', a', b')$ is given by a triple of morphisms $A \to A', B \to B', C \to C'$ compatible with the morphisms a, b, a', b'. We can prove directly that this leads to a 2-fibre product. However, it is easier to observe that the functor $(A, B, C, a, b) \mapsto (A, B, b^{-1} \circ a)$ gives an equivalence from the category of quintuples to the category constructed in Example 4.29.3.

Lemma 4.29.6. *Let*



be a 2-commutative diagram of categories. A choice of isomorphisms $\alpha : G \circ K \to M \circ I$ and $\beta : M \circ H \to F \circ L$ determines a morphism

$$\mathcal{X} \times_{\mathcal{Z}} \mathcal{Y} \longrightarrow \mathcal{A} \times_{\mathcal{C}} \mathcal{B}$$

of 2-fibre products associated to this situation.

Proof. Just use the functor

$$(X, Y, \phi) \longmapsto (L(X), K(Y), \alpha_Y^{-1} \circ M(\phi) \circ \beta_X^{-1})$$

on objects and

$$(a, b) \longmapsto (L(a), K(b))$$

on morphisms. □

Lemma 4.29.7. *Assumptions as in Lemma 4.29.6.*

(1) *If K and L are faithful then the morphism $\mathcal{Y} \times_{\mathcal{Z}} \mathcal{X} \to \mathcal{B} \times_{\mathcal{C}} \mathcal{A}$ is faithful.*
(2) *If K and L are fully faithful and M is faithful then the morphism $\mathcal{Y} \times_{\mathcal{Z}} \mathcal{X} \to \mathcal{B} \times_{\mathcal{C}} \mathcal{A}$ is fully faithful.*

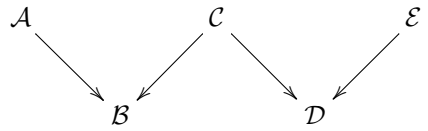
(3) *If K and L are equivalences and M is fully faithful then the morphism $\mathcal{Y} \times_{\mathcal{Z}} \mathcal{X} \to \mathcal{B} \times_{\mathcal{C}} \mathcal{A}$ is an equivalence.*

Proof. Let (X, Y, ϕ) and (X', Y', ϕ') be objects of $\mathcal{X} \times_{\mathcal{Z}} \mathcal{Y}$. Set $Z = H(X)$ and identify it with $I(Y)$ via ϕ. Also, identify $M(Z)$ with $F(L(X))$ via α_X and identify $M(Z)$ with $G(K(Y))$ via β_Y. Similarly for $Z' = H(X')$ and $M(Z')$. The map on morphisms is the map

$$\begin{array}{c} \mathrm{Mor}_{\mathcal{X}}(X, X') \times_{\mathrm{Mor}_{\mathcal{Z}}(Z,Z')} \mathrm{Mor}_{\mathcal{Y}}(Y, Y') \\ \downarrow \\ \mathrm{Mor}_{\mathcal{A}}(L(X), L(X')) \times_{\mathrm{Mor}_{\mathcal{C}}(M(Z),M(Z'))} \mathrm{Mor}_{\mathcal{B}}(K(Y), K(Y')) \end{array}$$

Hence parts (1) and (2) follow. Moreover, if K and L are equivalences and M is fully faithful, then any object (A, B, ϕ) is in the essential image for the following reasons: Pick X, Y such that $L(X) \cong A$ and $K(Y) \cong B$. Then the fully faithfulness of M guarantees that we can find an isomorphism $H(X) \cong I(Y)$. Some details omitted. □

Lemma 4.29.8. *Let*



be a diagram of categories and functors. Then there is a canonical isomorphism

$$(\mathcal{A} \times_{\mathcal{B}} \mathcal{C}) \times_{\mathcal{D}} \mathcal{E} \cong \mathcal{A} \times_{\mathcal{B}} (\mathcal{C} \times_{\mathcal{D}} \mathcal{E})$$

of categories.

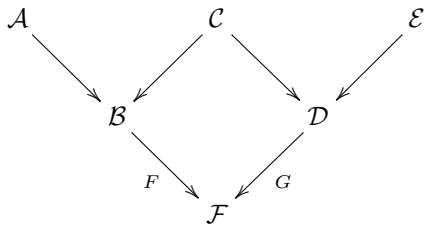
Proof. Just use the functor

$$((A, C, \phi), E, \psi) \longmapsto (A, (C, E, \psi), \phi)$$

if you know what I mean. □

Henceforth we do not write the parentheses when dealing with fibred products of more than 2 categories.

Lemma 4.29.9. *Let*



be a commutative diagram of categories and functors. Then there is a canonical functor

$$pr_{02} : \mathcal{A} \times_{\mathcal{B}} \mathcal{C} \times_{\mathcal{D}} \mathcal{E} \longrightarrow \mathcal{A} \times_{\mathcal{F}} \mathcal{E}$$

of categories.

Proof. If we write $\mathcal{A} \times_{\mathcal{B}} \mathcal{C} \times_{\mathcal{D}} \mathcal{E}$ as $(\mathcal{A} \times_{\mathcal{B}} \mathcal{C}) \times_{\mathcal{D}} \mathcal{E}$ then we can just use the functor

$$((A, C, \phi), E, \psi) \longmapsto (A, E, G(\psi) \circ F(\phi))$$

if you know what I mean. □

Lemma 4.29.10. *Let*

$$\mathcal{A} \to \mathcal{B} \leftarrow \mathcal{C} \leftarrow \mathcal{D}$$

be a diagram of categories and functors. Then there is a canonical isomorphism

$$\mathcal{A} \times_{\mathcal{B}} \mathcal{C} \times_{\mathcal{C}} \mathcal{D} \cong \mathcal{A} \times_{\mathcal{C}} \mathcal{D}$$

of categories.

Proof. Omitted. □

We claim that this means you can work with these 2-fibre products just like with ordinary fibre products. Here are some further lemmas that actually come up later.

Lemma 4.29.11. *Let*

$$\begin{array}{ccc} \mathcal{C}_3 & \longrightarrow & \mathcal{S} \\ \downarrow & & \downarrow \Delta \\ \mathcal{C}_1 \times \mathcal{C}_2 & \xrightarrow{G_1 \times G_2} & \mathcal{S} \times \mathcal{S} \end{array}$$

be a 2-fibre product of categories. Then there is a canonical isomorphism $\mathcal{C}_3 \cong \mathcal{C}_1 \times_{G_1, \mathcal{S}, G_2} \mathcal{C}_2$.

Proof. We may assume that $\mathcal{C}_3$ is the category $(\mathcal{C}_1 \times \mathcal{C}_2) \times_{\mathcal{S} \times \mathcal{S}} \mathcal{S}$ constructed in Example 4.29.3. Hence an object is a triple $((X_1, X_2), S, \phi)$ where $\phi = (\phi_1, \phi_2) : (G_1(X_1), G_2(X_2)) \to (S, S)$ is an isomorphism. Thus we can associate to this the triple $(X_1, X_2, \phi_2 \circ \phi_1^{-1})$. Conversely, if (X_1, X_2, ψ) is an object of $\mathcal{C}_1 \times_{G_1, \mathcal{S}, G_2} \mathcal{C}_2$, then we can associate to this the triple $((X_1, X_2), G_1(X_1), (\mathrm{id}_{G_1(X_1)}, \psi))$. We claim these constructions given mutually inverse functors. We omit describing how to deal with morphisms and show they are mutually inverse. □

Lemma 4.29.12. *Let*

$$\begin{array}{ccc} \mathcal{C}' & \longrightarrow & \mathcal{S} \\ \downarrow & & \downarrow \Delta \\ \mathcal{C} & \xrightarrow{G_1 \times G_2} & \mathcal{S} \times \mathcal{S} \end{array}$$

be a 2-fibre product of categories. Then there is a canonical isomorphism

$$\mathcal{C}' \cong (\mathcal{C} \times_{G_1, \mathcal{S}, G_2} \mathcal{C}) \times_{(p,q), \mathcal{C} \times \mathcal{C}, \Delta} \mathcal{C}.$$

Proof. An object of the right hand side is given by $((C_1, C_2, \phi), C_3, \psi)$ where $\phi : G_1(C_1) \to G_2(C_2)$ is an isomorphism and $\psi = (\psi_1, \psi_2) : (C_1, C_2) \to (C_3, C_3)$ is an isomorphism. Hence we can associate to this the triple $(C_3, G_1(C_1), (G_1(\psi_1^{-1}), \varphi^{-1} \circ G_2(\psi_2^{-1})))$ which is an object of $\mathcal{C}'$. Details omitted. □

Lemma 4.29.13. *Let* $\mathcal{A} \to \mathcal{C}$, $\mathcal{B} \to \mathcal{C}$ *and* $\mathcal{C} \to \mathcal{D}$ *be functors between categories. Then the diagram*

$$\begin{array}{ccc} \mathcal{A} \times_{\mathcal{C}} \mathcal{B} & \longrightarrow & \mathcal{A} \times_{\mathcal{D}} \mathcal{B} \\ \downarrow & & \downarrow \\ \mathcal{C} & \xrightarrow{\Delta_{\mathcal{C}/\mathcal{D}}} & \mathcal{C} \times_{\mathcal{D}} \mathcal{C} \end{array}$$

is a 2-fibre product diagram.

Proof. Omitted. □

Lemma 4.29.14. *Let*

$$\begin{array}{ccc} \mathcal{U} & \longrightarrow & \mathcal{V} \\ \downarrow & & \downarrow \\ \mathcal{X} & \longrightarrow & \mathcal{Y} \end{array}$$

be a 2-fibre product. Then the diagram

$$\begin{array}{ccc} \mathcal{U} & \longrightarrow & \mathcal{U} \times_{\mathcal{V}} \mathcal{U} \\ \downarrow & & \downarrow \\ \mathcal{X} & \longrightarrow & \mathcal{X} \times_{\mathcal{Y}} \mathcal{X} \end{array}$$

is 2-cartesian.

Proof. This is a purely 2-category theoretic statement, valid in any $(2,1)$-category with 2-fibre products. Explicitly, it follows from the following chain of equivalences:

$$\begin{aligned} \mathcal{X} \times_{(\mathcal{X} \times_{\mathcal{Y}} \mathcal{X})} (\mathcal{U} \times_{\mathcal{V}} \mathcal{U}) &= \mathcal{X} \times_{(\mathcal{X} \times_{\mathcal{Y}} \mathcal{X})} ((\mathcal{X} \times_{\mathcal{Y}} \mathcal{V}) \times_{\mathcal{V}} (\mathcal{X} \times_{\mathcal{Y}} \mathcal{V})) \\ &= \mathcal{X} \times_{(\mathcal{X} \times_{\mathcal{Y}} \mathcal{X})} (\mathcal{X} \times_{\mathcal{Y}} \mathcal{X} \times_{\mathcal{Y}} \mathcal{V}) \\ &= \mathcal{X} \times_{\mathcal{Y}} \mathcal{V} = \mathcal{U} \end{aligned}$$

see Lemmas 4.29.8 and 4.29.10. □

4.30. Categories over categories

In this section we have a functor $p : \mathcal{S} \to \mathcal{C}$. We think of $\mathcal{S}$ as being on top and of $\mathcal{C}$ as being at the bottom. To make sure that everybody knows what we are talking about we define the 2-category of categories over $\mathcal{C}$.

Definition 4.30.1. Let $\mathcal{C}$ be a category. The 2-*category of categories over* $\mathcal{C}$ is the sub 2-category of *Cat* defined as follows:

(1) Its objects will be functors $p : \mathcal{S} \to \mathcal{C}$.
(2) Its 1-morphisms $(\mathcal{S}, p) \to (\mathcal{S}', p')$ will be functors $G : \mathcal{S} \to \mathcal{S}'$ such that $p' \circ G = p$.
(3) Its 2-morphisms $t : G \to H$ for $G, H : (\mathcal{S}, p) \to (\mathcal{S}', p')$ will be morphisms of functors such that $p'(t_x) = \mathrm{id}_{p(x)}$ for all $x \in \mathrm{Ob}(\mathcal{S})$.

In this situation we will denote

$$\mathrm{Mor}_{Cat/\mathcal{C}}(\mathcal{S}, \mathcal{S}')$$

the category of 1-morphisms between $(\mathcal{S}, p)$ and $(\mathcal{S}', p')$

Since we have defined this as a sub 2-category of *Cat* we do not have to check any of the axioms. Rather we just have to check things such as "vertical composition of 2-morphisms over $\mathcal{C}$ gives another 2-morphism over $\mathcal{C}$". This is clear.

Analogously to the fibre of a map of spaces, we have the notion of a fibre category, and some notions of lifting associated to this situation.

Definition 4.30.2. Let $\mathcal{C}$ be a category. Let $p : \mathcal{S} \to \mathcal{C}$ be a category over $\mathcal{C}$.

(1) The *fibre category* over an object $U \in \mathrm{Ob}(\mathcal{C})$ is the category $\mathcal{S}_U$ with objects
$$\mathrm{Ob}(\mathcal{S}_U) = \{x \in \mathrm{Ob}(\mathcal{S}) : p(x) = U\}$$
and morphisms
$$\mathrm{Mor}_{\mathcal{S}_U}(x, y) = \{\phi \in \mathrm{Mor}_{\mathcal{S}}(x, y) : p(\phi) = \mathrm{id}_U\}.$$
(2) A *lift* of an object $U \in \mathrm{Ob}(\mathcal{C})$ is an object $x \in \mathrm{Ob}(\mathcal{S})$ such that $p(x) = U$, i.e., $x \in \mathrm{Ob}(\mathcal{S}_U)$. We will also sometime say that x *lies over* U.
(3) Similarly, a *lift* of a morphism $f : V \to U$ in $\mathcal{C}$ is a morphism $\phi : y \to x$ in $\mathcal{S}$ such that $p(\phi) = f$. We sometimes say that ϕ *lies over* f.

There are some observations we could make here. For example if $F : (\mathcal{S}, p) \to (\mathcal{S}', p')$ is a 1-morphism of categories over $\mathcal{C}$, then F induces functors of fibre categories $F : \mathcal{S}_U \to \mathcal{S}'_U$. Similarly for 2-morphisms.

Here is the obligatory lemma describing the 2-fibre product in the $(2, 1)$-category of categories over $\mathcal{C}$.

Lemma 4.30.3. *Let $\mathcal{C}$ be a category. The $(2, 1)$-category of categories over $\mathcal{C}$ has 2-fibre products. Suppose that $F : \mathcal{X} \to \mathcal{S}$ and $G : \mathcal{Y} \to \mathcal{S}$ are morphisms of categories over $\mathcal{C}$. An explicit 2-fibre product $\mathcal{X} \times_{\mathcal{S}} \mathcal{Y}$ is given by the following description*

(1) *an object of $\mathcal{X} \times_{\mathcal{S}} \mathcal{Y}$ is a quadruple (U, x, y, f), where $U \in \mathrm{Ob}(\mathcal{C})$, $x \in \mathrm{Ob}(\mathcal{X}_U)$, $y \in \mathrm{Ob}(\mathcal{Y}_U)$, and $f : F(x) \to G(y)$ is an isomorphism in $\mathcal{S}_U$,*
(2) *a morphism $(U, x, y, f) \to (U', x', y', f')$ is given by a pair (a, b), where $a : x \to x'$ is a morphism in $\mathcal{X}$, and $b : y \to y'$ is a morphism in $\mathcal{Y}$ such that*
   (a) *a and b induced the same morphism $U \to U'$, and*
   (b) *the diagram*
$$\begin{array}{ccc} F(x) & \xrightarrow{f} & G(y) \\ \downarrow{\scriptstyle F(a)} & & \downarrow{\scriptstyle G(b)} \\ F(x') & \xrightarrow{f'} & G(y') \end{array}$$
   is commutative.

The functors $p : \mathcal{X} \times_{\mathcal{S}} \mathcal{Y} \to \mathcal{X}$ and $q : \mathcal{X} \times_{\mathcal{S}} \mathcal{Y} \to \mathcal{Y}$ are the forgetful functors in this case. The transformation $\psi : F \circ p \to G \circ q$ is given on the object $\xi = (U, x, y, f)$ by $\psi_\xi = f : F(p(\xi)) = F(x) \to G(y) = G(q(\xi))$.

Proof. Let us check the universal property: let $p_W : \mathcal{W} \to \mathcal{C}$ be a category over $\mathcal{C}$, let $X : \mathcal{W} \to \mathcal{X}$ and $Y : \mathcal{W} \to \mathcal{Y}$ be functors over $\mathcal{C}$, and let $t : F \circ X \to G \circ Y$ be an isomorphism of functors over $\mathcal{C}$. The desired functor $\gamma : \mathcal{W} \to \mathcal{X} \times_{\mathcal{S}} \mathcal{Y}$ is given by $W \mapsto (p_W(W), X(W), Y(W), t_W)$. Details omitted; compare with Lemma 4.29.4. □

Lemma 4.30.4. *Let $\mathcal{C}$ be a category. Let $f : \mathcal{X} \to \mathcal{S}$ and $g : \mathcal{Y} \to \mathcal{S}$ be morphisms of categories over $\mathcal{C}$. For any object U of $\mathcal{C}$ we have the following identity of fibre categories*
$$(\mathcal{X} \times_{\mathcal{S}} \mathcal{Y})_U = \mathcal{X}_U \times_{\mathcal{S}_U} \mathcal{Y}_U$$

Proof. Omitted. □

4.31. Fibred categories

A very brief discussion of fibred categories is warranted.

Let $p : \mathcal{S} \to \mathcal{C}$ be a category over $\mathcal{C}$. Given an object $x \in \mathcal{S}$ with $p(x) = U$, and given a morphism $f : V \to U$, we can try to take some kind of "fibre product $V \times_U x$" (or a *base change* of x via $V \to U$). Namely, a morphism from an object $z \in \mathcal{S}$ into "$V \times_U x$" should be given by a pair (φ, g), where $\varphi : z \to x$, $g : p(z) \to V$ such that $p(\varphi) = f \circ g$. Pictorially:

$$\begin{array}{ccccc} z & \longrightarrow & ? & \longrightarrow & x \\ \downarrow{\scriptstyle p} & & \downarrow{\scriptstyle p} & & \downarrow{\scriptstyle p} \\ p(Z) & \longrightarrow & V & \xrightarrow{f} & U \end{array}$$

If such a morphism $V \times_U x \to x$ exists then it is called a strongly cartesian morphism.

Definition 4.31.1. Let $\mathcal{C}$ be a category. Let $p : \mathcal{S} \to \mathcal{C}$ be a category over $\mathcal{C}$. A *strongly cartesian morphism*, or more precisely a *strongly $\mathcal{C}$-cartesian morphism* is a morphism $\varphi : y \to x$ of $\mathcal{S}$ such that for every $z \in \mathrm{Ob}(\mathcal{S})$ the map

$$\mathrm{Mor}_{\mathcal{S}}(z, y) \longrightarrow \mathrm{Mor}_{\mathcal{S}}(z, x) \times_{\mathrm{Mor}_{\mathcal{C}}(p(z), p(x))} \mathrm{Mor}_{\mathcal{C}}(p(z), p(y)),$$

given by $\psi \longmapsto (\varphi \circ \psi, p(\psi))$ is bijective.

Note that by the Yoneda Lemma 4.3.5, given $x \in \mathrm{Ob}(\mathcal{S})$ lying over $U \in \mathrm{Ob}(\mathcal{C})$ and the morphism $f : V \to U$ of $\mathcal{C}$, if there is a strongly cartesian morphism $\varphi : y \to x$ with $p(\varphi) = f$, then (y, φ) is unique up to unique isomorphism. This is clear from the definition above, as the functor

$$z \longmapsto \mathrm{Mor}_{\mathcal{S}}(z, x) \times_{\mathrm{Mor}_{\mathcal{C}}(p(z), U)} \mathrm{Mor}_{\mathcal{C}}(p(z), V)$$

only depends on the data $(x, U, f : V \to U)$. Hence we will sometimes use $V \times_U x \to x$ or $f^*x \to x$ to denote a strongly cartesian morphism which is a lift of f.

Lemma 4.31.2. *Let $\mathcal{C}$ be a category. Let $p : \mathcal{S} \to \mathcal{C}$ be a category over $\mathcal{C}$.*

1. *The composition of two strongly cartesian morphisms is strongly cartesian.*
2. *Any isomorphism of $\mathcal{S}$ is strongly cartesian.*
3. *Any strongly cartesian morphism φ such that $p(\varphi)$ is an isomorphism, is an isomorphism.*

Proof. Proof of (1). Let $\varphi : y \to x$ and $\psi : z \to y$ be strongly cartesian. Let t be an arbitrary object of $\mathcal{S}$. Then we have

$$\begin{aligned} &\mathrm{Mor}_{\mathcal{S}}(t, z) \\ &= \mathrm{Mor}_{\mathcal{S}}(t, y) \times_{\mathrm{Mor}_{\mathcal{C}}(p(t), p(y))} \mathrm{Mor}_{\mathcal{C}}(p(t), p(z)) \\ &= \mathrm{Mor}_{\mathcal{S}}(t, x) \times_{\mathrm{Mor}_{\mathcal{C}}(p(t), p(x))} \mathrm{Mor}_{\mathcal{C}}(p(t), p(y)) \times_{\mathrm{Mor}_{\mathcal{C}}(p(t), p(y))} \mathrm{Mor}_{\mathcal{C}}(p(t), p(z)) \\ &= \mathrm{Mor}_{\mathcal{S}}(t, x) \times_{\mathrm{Mor}_{\mathcal{C}}(p(t), p(x))} \mathrm{Mor}_{\mathcal{C}}(p(t), p(z)) \end{aligned}$$

hence $z \to x$ is strongly cartesian.

Proof of (2). Let $y \to x$ be an isomorphism. Then $p(y) \to p(x)$ is an isomorphism too. Hence $\mathrm{Mor}_{\mathcal{C}}(p(z), p(y)) \to \mathrm{Mor}_{\mathcal{C}}(p(z), p(x))$ is a bijection. Hence $\mathrm{Mor}_{\mathcal{S}}(z, x) \times_{\mathrm{Mor}_{\mathcal{C}}(p(z), p(x))} \mathrm{Mor}_{\mathcal{C}}(p(z), p(y))$ is bijective to $\mathrm{Mor}_{\mathcal{S}}(z, x)$. Hence the

displayed map of Definition 4.31.1 is a bijection as $y \to x$ is an isomorphism, and we conclude that $y \to x$ is strongly cartesian.

Proof of (3). Assume $\varphi : y \to x$ is strongly cartesian with $p(\varphi) : p(y) \to p(x)$ an isomorphism. Applying the definition with $z = x$ shows that $(\mathrm{id}_x, p(\varphi)^{-1})$ comes from a unique morphism $\chi : x \to y$. We omit the verification that χ is the inverse of φ. □

Lemma 4.31.3. *Let $F : \mathcal{A} \to \mathcal{B}$ and $G : \mathcal{B} \to \mathcal{C}$ be composable functors between categories. Let $x \to y$ be a morphism of $\mathcal{A}$. If $x \to y$ is strongly $\mathcal{B}$-cartesian and $F(x) \to F(y)$ is strongly $\mathcal{A}$-cartesian, then $x \to y$ is strongly $\mathcal{A}$-cartesian.*

Proof. This follows directly from the definition. □

Lemma 4.31.4. *Let $\mathcal{C}$ be a category. Let $p : \mathcal{S} \to \mathcal{C}$ be a category over $\mathcal{C}$. Let $x \to y$ and $z \to y$ be morphisms of $\mathcal{S}$. Assume*

1. *$x \to y$ is strongly cartesian,*
2. *$p(x) \times_{p(y)} p(z)$ exists, and*
3. *there exists a strongly cartesian morphism $a : w \to z$ in $\mathcal{S}$ with $p(w) = p(x) \times_{p(y)} p(z)$ and $p(a) = pr_2 : p(x) \times_{p(y)} p(z) \to p(z)$.*

Then the fibre product $x \times_y z$ exists and is isomorphic to w.

Proof. Since $x \to y$ is strongly cartesian there exists a unique morphism $b : w \to x$ such that $p(b) = \mathrm{pr}_1$. To see that w is the fibre product we compute

$$\begin{aligned}
&\mathrm{Mor}_{\mathcal{S}}(t, w) \\
&= \mathrm{Mor}_{\mathcal{S}}(t, z) \times_{\mathrm{Mor}_{\mathcal{C}}(p(t), p(z))} \mathrm{Mor}_{\mathcal{C}}(p(t), p(w)) \\
&= \mathrm{Mor}_{\mathcal{S}}(t, z) \times_{\mathrm{Mor}_{\mathcal{C}}(p(t), p(z))} (\mathrm{Mor}_{\mathcal{C}}(p(t), p(x)) \times_{\mathrm{Mor}_{\mathcal{C}}(p(t), p(y))} \mathrm{Mor}_{\mathcal{C}}(p(t), p(z))) \\
&= \mathrm{Mor}_{\mathcal{S}}(t, z) \times_{\mathrm{Mor}_{\mathcal{C}}(p(t), p(y))} \mathrm{Mor}_{\mathcal{C}}(p(t), p(x)) \\
&= \mathrm{Mor}_{\mathcal{S}}(t, z) \times_{\mathrm{Mor}_{\mathcal{S}}(t, y)} \mathrm{Mor}_{\mathcal{S}}(t, y) \times_{\mathrm{Mor}_{\mathcal{C}}(p(t), p(y))} \mathrm{Mor}_{\mathcal{C}}(p(t), p(x)) \\
&= \mathrm{Mor}_{\mathcal{S}}(t, z) \times_{\mathrm{Mor}_{\mathcal{S}}(t, y)} \mathrm{Mor}_{\mathcal{S}}(t, x)
\end{aligned}$$

as desired. The first equality holds because $a : w \to z$ is strongly cartesian and the last equality holds because $x \to y$ is strongly cartesian. □

Definition 4.31.5. Let $\mathcal{C}$ be a category. Let $p : \mathcal{S} \to \mathcal{C}$ be a category over $\mathcal{C}$. We say $\mathcal{S}$ is a *fibred category over $\mathcal{C}$* if given any $x \in \mathrm{Ob}(\mathcal{S})$ lying over $U \in \mathrm{Ob}(\mathcal{C})$ and any morphism $f : V \to U$ of $\mathcal{C}$, there exists a strongly cartesian morphism $f^*x \to x$ lying over f.

Assume $p : \mathcal{S} \to \mathcal{C}$ is a fibred category. For every $f : V \to U$ and $x \in \mathrm{Ob}(\mathcal{S}_U)$ as in the definition we may choose a strongly cartesian morphism $f^*x \to x$ lying over f. By the axiom of choice we may choose $f^*x \to x$ for all $f : V \to U = p(x)$ simultaneously. We claim that for every morphism $\phi : x \to x'$ in $\mathcal{S}_U$ and $f : V \to U$ there is a unique morphism $f^*\phi : f^*x \to f^*x'$ in $\mathcal{S}_V$ such that

$$\begin{array}{ccc}
f^*x & \xrightarrow[f^*\phi]{} & f^*x' \\
\downarrow & & \downarrow \\
x & \xrightarrow{\phi} & x'
\end{array}$$

commutes. Namely, the arrow exists and is unique because $f^*x' \to x'$ is strongly cartesian. The uniqueness of this arrow guarantees that f^* (now also defined on morphisms) is a functor $f^* : \mathcal{S}_U \to \mathcal{S}_V$.

Definition 4.31.6. Assume $p : \mathcal{S} \to \mathcal{C}$ is a fibred category.

(1) A *choice of pullbacks*[3] for $p : \mathcal{S} \to \mathcal{C}$ is given by a choice of a strongly cartesian morphism $f^*x \to x$ lying over f for any morphism $f : V \to U$ of $\mathcal{C}$ and any $x \in \mathrm{Ob}(\mathcal{S}_U)$.
(2) Given a choice of pullbacks, for any morphism $f : V \to U$ of $\mathcal{C}$ the functor $f^* : \mathcal{S}_U \to \mathcal{S}_V$ described above is called a *pullback functor* (associated to the choices $f^*x \to x$ made above).

Of course we may always assume our choice of pullbacks has the property that $\mathrm{id}_U^* x = x$, although in practice this is a useless property without imposing further assumptions on the pullbacks.

Lemma 4.31.7. *Assume $p : \mathcal{S} \to \mathcal{C}$ is a fibred category. Assume given a choice of pullbacks for $p : \mathcal{S} \to \mathcal{C}$.*

(1) *For any pair of composable morphisms $f : V \to U$, $g : W \to V$ there is a unique isomorphism*

$$\alpha_{g,f} : (f \circ g)^* \longrightarrow g^* \circ f^*$$

as functors $\mathcal{S}_U \to \mathcal{S}_W$ such that for every $y \in \mathrm{Ob}(\mathcal{S}_U)$ the following diagram commutes

$$\begin{array}{ccc} g^*f^*y & \longrightarrow & f^*y \\ {\scriptstyle (\alpha_{g,f})_y}\uparrow & & \downarrow \\ (f \circ g)^*y & \longrightarrow & y \end{array}$$

(2) *If $f = id_U$, then there is a canonical isomorphism $\alpha_U : id \to (id_U)^*$ as functors $\mathcal{S}_U \to \mathcal{S}_U$.*
(3) *The quadruple $(U \mapsto \mathcal{S}_U, f \mapsto f^*, \alpha_{g,f}, \alpha_U)$ defines a pseudo functor from $\mathcal{C}^{opp}$ to the $(2,1)$-category of categories, see Definition 4.27.5.*

Proof. In fact, it is clear that the commutative diagram of part (1) uniquely determines the morphism $(\alpha_{g,f})_y$ in the fibre category $\mathcal{S}_W$. It is an isomorphism since both the morphism $(f \circ g)^*y \to y$ and the morphism and the composition $g^*f^*y \to f^*y \to Y$ are strongly cartesian morphisms lifting $f \circ g$ (see discussion following Definition 4.31.1 and Lemma 4.31.2). In the same way, since $\mathrm{id}_x : x \to x$ is clearly strongly cartesian over id_U (with $U = p(x)$) we see that there exists an isomorphism $(\alpha_U)_x : x \to (\mathrm{id}_U)^*x$. (Of course we could have assumed beforehand that $f^*x = x$ whenever f is an identity morphism, but it is better for the sake of generality not to assume this.) We omit the verification that $\alpha_{g,f}$ and α_U so obtained are transformations of functors. We also omit the verification of (3). □

[3]This is probably nonstandard terminology. In some texts this is called a "cleavage" but it conjures up the wrong image. Maybe a "cleaving" would be a better word. A related notion is that of a "splitting", but in many texts a "splitting" means a choice of pullbacks such that $g^*f^* = (f \circ g)^*$ for any composable pair of morphisms. Compare also with Definition 4.34.2.

Lemma 4.31.8. *Let $\mathcal{C}$ be a category. Let $\mathcal{S}_1$, $\mathcal{S}_2$ be categories over $\mathcal{C}$. Suppose that $\mathcal{S}_1$ and $\mathcal{S}_2$ are equivalent as categories over $\mathcal{C}$. Then $\mathcal{S}_1$ is fibred over $\mathcal{C}$ if and only if $\mathcal{S}_2$ is fibred over $\mathcal{C}$.*

Proof. Denote $p_i : \mathcal{S}_i \to \mathcal{C}$ the given functors. Let $F : \mathcal{S}_1 \to \mathcal{S}_2$, $G : \mathcal{S}_2 \to \mathcal{S}_1$ be functors over $\mathcal{C}$, and let $i : F \circ G \to \mathrm{id}_{\mathcal{S}_2}$, $j : G \circ F \to \mathrm{id}_{\mathcal{S}_1}$ be isomorphisms of functors over $\mathcal{C}$. We claim that in this case F maps strongly cartesian morphisms to strongly cartesian morphisms. Namely, suppose that $\varphi : y \to x$ is strongly cartesian in $\mathcal{S}_1$. Set $f : V \to U$ equal to $p_1(\varphi)$. Suppose that $z' \in \mathrm{Ob}(\mathcal{S}_2)$, with $W = p_2(z')$, and we are given $g : W \to V$ and $\psi' : z' \to F(x)$ such that $p_2(\psi') = f \circ g$. Then

$$\psi = j \circ G(\psi') : G(z') \to G(F(x)) \to x$$

is a morphism in $\mathcal{S}_1$ with $p_1(\psi) = f \circ g$. Hence by assumption there exists a unique morphism $\xi : G(z') \to y$ lying over g such that $\psi = \varphi \circ \xi$. This in turn gives a morphism

$$\xi' = F(\xi) \circ i^{-1} : z' \to F(G(z')) \to F(y)$$

lying over g with $\psi' = F(\varphi) \circ \xi'$. We omit the verification that ξ' is unique. □

The conclusion from Lemma 4.31.8 is that equivalences map strongly cartesian morphisms to strongly cartesian morphisms. But this may not be the case for an arbitrary functor between fibred categories over $\mathcal{C}$. Hence we define the 2-category of fibred categories as follows.

Definition 4.31.9. Let $\mathcal{C}$ be a category. The 2-*category of fibred categories over* $\mathcal{C}$ is the sub 2-category of the 2-category of categories over $\mathcal{C}$ (see Definition 4.30.1) defined as follows:

(1) Its objects will be fibred categories $p : \mathcal{S} \to \mathcal{C}$.
(2) Its 1-morphisms $(\mathcal{S}, p) \to (\mathcal{S}', p')$ will be functors $G : \mathcal{S} \to \mathcal{S}'$ such that $p' \circ G = p$ and such that G maps strongly cartesian morphisms to strongly cartesian morphisms.
(3) Its 2-morphisms $t : G \to H$ for $G, H : (\mathcal{S}, p) \to (\mathcal{S}', p')$ will be morphisms of functors such that $p'(t_x) = \mathrm{id}_{p(x)}$ for all $x \in \mathrm{Ob}(\mathcal{S})$.

In this situation we will denote

$$\mathrm{Mor}_{Fib/\mathcal{C}}(\mathcal{S}, \mathcal{S}')$$

the category of 1-morphisms between $(\mathcal{S}, p)$ and $(\mathcal{S}', p')$

Note the condition on 1-morphisms. Note also that this is a true 2-category and not a $(2,1)$-category. Hence when taking 2-fibre products we first pass to the associated $(2,1)$-category.

Lemma 4.31.10. *Let $\mathcal{C}$ be a category. The $(2,1)$-category of fibred categories over $\mathcal{C}$ has 2-fibre products, and they are described as in Lemma 4.30.3.*

Proof. Basically what one has to show here is that given $F : \mathcal{X} \to \mathcal{S}$ and $G : \mathcal{Y} \to \mathcal{S}$ morphisms of fibred categories over $\mathcal{C}$, then the category $\mathcal{X} \times_{\mathcal{S}} \mathcal{Y}$ described in Lemma 4.30.3 is fibred. Let us show that $\mathcal{X} \times_{\mathcal{S}} \mathcal{Y}$ has plenty of strongly cartesian morphisms. Namely, suppose we have (U, x, y, ϕ) an object of $\mathcal{X} \times_{\mathcal{S}} \mathcal{Y}$. And suppose $f : V \to U$ is a morphism in $\mathcal{C}$. Choose strongly cartesian morphisms $a : f^*x \to x$ in $\mathcal{X}$ lying over f and $b : f^*y \to y$ in $\mathcal{Y}$ lying over f. By assumption $F(a)$ and $G(b)$ are strongly cartesian. Since $\phi : F(x) \to G(y)$ is an isomorphism, by the uniqueness of strongly cartesian morphisms we find a unique isomorphism $f^*\phi$:

$F(f^*x) \to G(f^*y)$ such that $G(b) \circ f^*\phi = \phi \circ F(a)$. In other words $(G(a), G(b)) : (V, f^*x, f^*y, f^*\phi) \to (U, x, y, \phi)$ is a morphism in $\mathcal{X} \times_{\mathcal{S}} \mathcal{Y}$. We omit the verification that this is a strongly cartesian morphism (and that these are in fact the only strongly cartesian morphisms). □

Lemma 4.31.11. *Let $\mathcal{C}$ be a category. Let $U \in \mathrm{Ob}(\mathcal{C})$. If $p : \mathcal{S} \to \mathcal{C}$ is a fibred category and p factors through $p' : \mathcal{S} \to \mathcal{C}/U$ then $p' : \mathcal{S} \to \mathcal{C}/U$ is a fibred category.*

Proof. Suppose that $\varphi : x' \to x$ is strongly cartesian with respect to p. We claim that φ is strongly cartesian with respect to p' also. Set $g = p'(\varphi)$, so that $g : V'/U \to V/U$ for some morphisms $f : V \to U$ and $f' : V' \to U$. Let $z \in \mathrm{Ob}(\mathcal{S})$. Set $p'(z) = (W \to U)$. To show that φ is strongly cartesian for p' we have to show

$$\mathrm{Mor}_{\mathcal{S}}(z, x') \longrightarrow \mathrm{Mor}_{\mathcal{S}}(z, x) \times_{\mathrm{Mor}_{\mathcal{C}/U}(W/U, V/U)} \mathrm{Mor}_{\mathcal{C}/U}(W/U, V'/U),$$

given by $\psi' \longmapsto (\varphi \circ \psi', p'(\psi'))$ is bijective. Suppose given an element (ψ, h) of the right hand side, then in particular $g \circ h = p(\psi)$, and by the condition that φ is strongly cartesian we get a unique morphism $\psi' : z \to x'$ with $\psi = \varphi \circ \psi'$ and $p(\psi') = h$. OK, and now $p'(\psi') : W/U \to V/U$ is a morphism whose corresponding map $W \to V$ is h, hence equal to h as a morphism in $\mathcal{C}/U$. Thus ψ' is a unique morphism $z \to x'$ which maps to the given pair (ψ, h). This proves the claim.

Finally, suppose given $g : V'/U \to V/U$ and x with $p'(x) = V/U$. Since $p : \mathcal{S} \to \mathcal{C}$ is a fibred category we see there exists a strongly cartesian morphism $\varphi : x' \to x$ with $p(\varphi) = g$. By the same argument as above it follows that $p'(\varphi) = g : V'/U \to V/U$. And as seen above the morphism φ is strongly cartesian. Thus the conditions of Definition 4.31.5 are satisfied and we win. □

Lemma 4.31.12. *Let $\mathcal{A} \to \mathcal{B} \to \mathcal{C}$ be functors between categories. If $\mathcal{A}$ is fibred over $\mathcal{B}$ and $\mathcal{B}$ is fibred over $\mathcal{C}$, then $\mathcal{A}$ is fibred over $\mathcal{C}$.*

Proof. This follows from the definitions and Lemma 4.31.3. □

Lemma 4.31.13. *Let $p : \mathcal{S} \to \mathcal{C}$ be a fibred category. Let $x \to y$ and $z \to y$ be morphisms of $\mathcal{S}$ with $x \to y$ strongly cartesian. If $p(x) \times_{p(y)} p(z)$ exists, then $x \times_y z$ exists, $p(x \times_y z) = p(x) \times_{p(y)} p(z)$, and $x \times_y z \to z$ is strongly cartesian.*

Proof. Pick a strongly cartesian morphism $\mathrm{pr}_2^* z \to z$ lying over $\mathrm{pr}_2 : p(x) \times_{p(y)} p(z) \to p(z)$. Then $\mathrm{pr}_2^* z = x \times_y z$ by Lemma 4.31.4. □

Lemma 4.31.14. *Let $\mathcal{C}$ be a category. Let $F : \mathcal{X} \to \mathcal{Y}$ be a 1-morphism of fibred categories over $\mathcal{C}$. There exist 1-morphisms of fibred categories over $\mathcal{C}$*

$$\mathcal{X} \underset{w}{\overset{u}{\rightleftarrows}} \mathcal{X}' \xrightarrow{v} \mathcal{Y}$$

such that $F = v \circ u$ and such that

(1) *$u : \mathcal{X} \to \mathcal{X}'$ is fully faithful,*
(2) *w is left adjoint to u, and*
(3) *$v : \mathcal{X}' \to \mathcal{Y}$ is a fibred category.*

Proof. Denote $p : \mathcal{X} \to \mathcal{C}$ and $q : \mathcal{Y} \to \mathcal{C}$ the structure functors. We construct $\mathcal{X}'$ explicitly as follows. An object of $\mathcal{X}'$ is a quadruple (U, x, y, f) where $x \in \mathrm{Ob}(\mathcal{X}_U)$, $y \in \mathrm{Ob}(\mathcal{Y}_U)$ and $f : y \to F(x)$ is a morphism in $\mathcal{Y}_U$. A morphism

$(a, b) : (U, x, y, f) \to (U', x', y', f')$ is given by $a : x \to x'$ and $b : y \to y'$ with $p(a) = q(b) : U \to U'$ and such that $f' \circ b = F(a) \circ f$.

Let us make a choice of pullbacks for both p and q and let us use the same notation to indicate them. Let (U, x, y, f) be an object and let $h : V \to U$ be a morphism. Consider the morphism $c : (V, h^*x, h^*y, h^*f) \to (U, x, y, f)$ coming from the given strongly cartesian maps $h^*x \to x$ and $h^*y \to y$. We claim c is strongly cartesian in $\mathcal{X}'$ over $\mathcal{C}$. Namely, suppose we are given an object (W, x', y', f') of $\mathcal{X}'$, a morphism $(a, b) : (W, x', y', f') \to (U, x, y, f)$ lying over $W \to U$, and a factorization $W \to V \to U$ of $W \to U$ through h. As $h^*x \to X$ and $h^*y \to y$ are strongly cartesian we obtain morphisms $a' : x' \to h^*x$ and $b' : y' \to h^*y$ lying over the given morphism $W \to V$. Consider the diagram

$$\begin{array}{ccccc} y' & \longrightarrow & h^*y & \longrightarrow & y \\ {\scriptstyle f'}\downarrow & & {\scriptstyle h^*f}\downarrow & & {\scriptstyle f}\downarrow \\ F(x') & \longrightarrow & F(h^*x) & \longrightarrow & F(x) \end{array}$$

The outer rectangle and the right square commute. Since F is a 1-morphism of fibred categories the morphism $F(h^*x) \to F(x)$ is strongly cartesian. Hence the left square commutes by the universal property of strongly cartesian morphisms. This proves that $\mathcal{X}'$ is fibred over $\mathcal{C}$.

The functor $u : \mathcal{X} \to \mathcal{X}'$ is given by $x \mapsto (p(x), x, F(x), \mathrm{id})$. This is fully faithful. The functor $\mathcal{X}' \to \mathcal{Y}$ is given by $(U, x, y, f) \mapsto y$. The functor $w : \mathcal{X}' \to \mathcal{X}$ is given by $(U, x, y, f) \mapsto x$. Each of these functors is a 1-morphism of fibred categories over $\mathcal{C}$ by our description of strongly cartesian morphisms of $\mathcal{X}'$ over $\mathcal{C}$. Adjointness of w and u means that

$$\mathrm{Mor}_{\mathcal{X}}(x, x') = \mathrm{Mor}_{\mathcal{X}'}((U, x, y, f), (p(x'), x', F(x'), \mathrm{id})),$$

which follows immediately from the definitions.

Finally, we have to show that $\mathcal{X}' \to \mathcal{Y}$ is a fibred category. Let $c : y' \to y$ be a morphism in $\mathcal{Y}$ and let (U, x, y, f) be an object of $\mathcal{X}'$ lying over y. Set $V = q(y')$ and let $h = q(c) : V \to U$. Let $a : h^*x \to x$ and $b : h^*y \to y$ be the strongly cartesian morphisms covering h. Since F is a 1-morphism of fibred categories we may identify $h^*F(x) = F(h^*x)$ with strongly cartesian morphism $F(a) : F(h^*x) \to F(x)$. By the universal property of $b : h^*y \to y$ there is a morphism $c' : y' \to h^*y$ in $\mathcal{Y}_V$ such that $c = b \circ c'$. We claim that

$$(a, c) : (V, h^*x, y', h^*f \circ b') \longrightarrow (U, x, y, f)$$

is strongly cartesian in $\mathcal{X}'$ over $\mathcal{Y}$. To see this let (W, x_1, y_1, f_1) be an object of $\mathcal{X}'$, let $(a_1, b_1) : (W, x_1, y_1, f_1) \to (U, x, y, f)$ be a morphism and let $b_1 = c \circ c_1$ for some morphism $c_1 : y_1 \to y'$. Then

$$(a_1', c_1) : (W, x_1, y_1, f_1) \longrightarrow (V, h^*x, y', h^*f \circ b')$$

(where $a_1' : x_1 \to h^*x$ is the unique morphism lying over the given morphism $p(a_1) = q(b_1) : W \to V$ such that $a_1 = a \circ a_1'$) is the desired morphism. $\square$

4.32. Inertia

Given a fibred categories $p : \mathcal{S} \to \mathcal{C}$ and $p' : \mathcal{S}' \to \mathcal{C}$ over a category $\mathcal{C}$ and a 1-morphism $F : \mathcal{S} \to \mathcal{S}'$ we have the diagonal morphism

$$\Delta = \Delta_{\mathcal{S}/\mathcal{S}'} : \mathcal{S} \longrightarrow \mathcal{S} \times_{\mathcal{S}'} \mathcal{S}$$

in the $(2,1)$-category of fibred categories over $\mathcal{C}$.

Lemma 4.32.1. *Let $\mathcal{C}$ be a category. Let $p : \mathcal{S} \to \mathcal{C}$ and $p' : \mathcal{S}' \to \mathcal{C}$ be fibred categories. Let $F : \mathcal{S} \to \mathcal{S}'$ be a 1-morphism of fibred categories over $\mathcal{C}$. Consider the category $\mathcal{I}_{\mathcal{S}/\mathcal{S}'}$ over $\mathcal{C}$ whose*

1. *objects are pairs (x, α) where $x \in \mathrm{Ob}(\mathcal{S})$ and $\alpha : x \to x$ is an automorphism with $F(\alpha) = id$,*
2. *morphisms $(x, \alpha) \to (y, \beta)$ are given by morphisms $\phi : x \to y$ such that*

$$\begin{array}{ccc} x & \xrightarrow{\phi} & y \\ {\scriptstyle\alpha}\downarrow & & \downarrow{\scriptstyle\beta} \\ x & \xrightarrow{\phi} & y \end{array}$$

   commutes, and
3. *the functor $\mathcal{I}_{\mathcal{S}/\mathcal{S}'} \to \mathcal{C}$ is given by $(x, \alpha) \mapsto p(x)$.*

Then

1. *there is an equivalence*

$$\mathcal{I}_{\mathcal{S}/\mathcal{S}'} \longrightarrow \mathcal{S} \times_{\Delta,(\mathcal{S}\times_{\mathcal{S}'}\mathcal{S}),\Delta} \mathcal{S}$$

   in the $(2,1)$-category of categories over $\mathcal{C}$, and
2. *$\mathcal{I}_{\mathcal{S}/\mathcal{S}'}$ is a fibred category over $\mathcal{C}$.*

Proof. Note that (2) follows from (1) by Lemma 4.31.10. Thus it suffices to prove (1). We will use without further mention the construction of the 2-fibre product from Lemma 4.31.10. In particular an object of $\mathcal{S} \times_{\Delta,(\mathcal{S}\times_{\mathcal{S}'}\mathcal{S}),\Delta} \mathcal{S}$ is a triple $(x, y, (\iota, \kappa))$ where x and y are objects of $\mathcal{S}$, and $(\iota, \kappa) : (x, x, \mathrm{id}_{F(x)}) \to (y, y, \mathrm{id}_{F(y)})$ is an isomorphism in $\mathcal{S} \times_{\mathcal{S}'} \mathcal{S}$. This just means that $\iota, \kappa : x \to y$ are isomorphisms and that $F(\iota) = F(\kappa)$. Consider the functor

$$I_{\mathcal{S}/\mathcal{S}'} \longrightarrow \mathcal{S} \times_{\Delta,(\mathcal{S}\times_{\mathcal{S}'}\mathcal{S}),\Delta} \mathcal{S}$$

which to an object (x, α) of the left hand side assigns the object $(x, x, (\alpha, \mathrm{id}_x))$ of the right hand side and to a morphism ϕ of the left hand side assigns the morphism (ϕ, ϕ) of the right hand side. We claim that a quasi-inverse to that morphism is given by the functor

$$\mathcal{S} \times_{\Delta,(\mathcal{S}\times_{\mathcal{S}'}\mathcal{S}),\Delta} \mathcal{S} \longrightarrow I_{\mathcal{S}/\mathcal{S}'}$$

which to an object $(x, y, (\iota, \kappa))$ of the left hand side assigns the object $(x, \kappa^{-1} \circ \iota)$ of the right hand side and to a morphism $(\phi, \phi') : (x, y, (\iota, \kappa)) \to (z, w, (\lambda, \mu))$ of the left hand side assigns the morphism ϕ. Indeed, the endo-functor of $I_{\mathcal{S}/\mathcal{S}'}$ induced by composing the two functors above is the identity on the nose, and the endo-functor induced on $\mathcal{S} \times_{\Delta,(\mathcal{S}\times_{\mathcal{S}'}\mathcal{S}),\Delta} \mathcal{S}$ is isomorphic to the identity via the natural isomorphism

$$(\iota^{-1} \circ \kappa, \kappa \circ \iota^{-1} \circ \kappa) : (x, x, (\kappa^{-1} \circ \iota, \mathrm{id}_x)) \longrightarrow (x, y, (\iota, \kappa)).$$

Some details omitted. □

Definition 4.32.2. Let $\mathcal{C}$ be a category.

(1) Let $F : \mathcal{S} \to \mathcal{S}'$ be a 1-morphism of fibred categories over $\mathcal{C}$. The *relative inertia of $\mathcal{S}$ over $\mathcal{S}'$* is the fibred category $\mathcal{I}_{\mathcal{S}/\mathcal{S}'} \to \mathcal{C}$ of Lemma 4.32.1.
(2) By the *inertia fibred category $\mathcal{I}_\mathcal{S}$ of $\mathcal{S}$* we mean $\mathcal{I}_\mathcal{S} = \mathcal{I}_{\mathcal{S}/\mathcal{C}}$.

Note that there are canonical 1-morphisms

$$\mathcal{I}_{\mathcal{S}/\mathcal{S}'} \longrightarrow \mathcal{S} \quad \text{and} \quad \mathcal{I}_\mathcal{S} \longrightarrow \mathcal{S} \tag{4.32.2.1}$$

of fibred categories over $\mathcal{C}$. In terms of the description of Lemma 4.32.1 these simply map the object (x, α) to the object x and the morphism $\phi : (x, \alpha) \to (y, \beta)$ to the morphism $\phi : x \to y$. There is also a *neutral section*

$$e : \mathcal{S} \to \mathcal{I}_{\mathcal{S}/\mathcal{S}'} \quad \text{and} \quad e : \mathcal{S} \to \mathcal{I}_\mathcal{S} \tag{4.32.2.2}$$

defined by the rules $x \mapsto (x, \mathrm{id}_x)$ and $(\phi : x \to y) \mapsto \phi$. This is a right inverse to (4.32.2.1). Given a 2-commutative square

$$\begin{array}{ccc} \mathcal{S}_1 & \xrightarrow[G]{} & \mathcal{S}_2 \\ {\scriptstyle F_1}\downarrow & & \downarrow{\scriptstyle F_2} \\ \mathcal{S}_1' & \xrightarrow{G'} & \mathcal{S}_2' \end{array}$$

there is a *functoriality map*

$$\mathcal{I}_{\mathcal{S}_1/\mathcal{S}_1'} \longrightarrow \mathcal{I}_{\mathcal{S}_2/\mathcal{S}_2'} \quad \text{and} \quad \mathcal{I}_{\mathcal{S}_1} \longrightarrow \mathcal{I}_{\mathcal{S}_2} \tag{4.32.2.3}$$

defined by the rules $(x, \alpha) \mapsto (G(x), G(\alpha))$ and $\phi \mapsto G(\phi)$. In particular there is always a comparison map

$$\mathcal{I}_{\mathcal{S}/\mathcal{S}'} \longrightarrow \mathcal{I}_\mathcal{S} \tag{4.32.2.4}$$

and all the maps above are compatible with this.

Lemma 4.32.3. *Let $F : \mathcal{S} \to \mathcal{S}'$ be a 1-morphism of categories fibred over a category $\mathcal{C}$. Then the diagram*

$$\begin{array}{ccc} \mathcal{I}_{\mathcal{S}/\mathcal{S}'} & \xrightarrow[(4.32.2.4)]{} & \mathcal{I}_\mathcal{S} \\ {\scriptstyle F\circ(4.32.2.1)}\downarrow & & \downarrow{\scriptstyle (4.32.2.3)} \\ \mathcal{S}' & \xrightarrow{e} & \mathcal{I}_{\mathcal{S}'} \end{array}$$

is a 2-fibre product.

Proof. Omitted. □

4.33. Categories fibred in groupoids

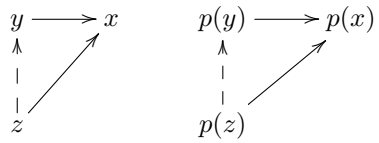
In this section we explain how to think about categories in groupoids and we see how they are basically the same as functors with values in the $(2, 1)$-category of groupoids.

Definition 4.33.1. We say that $\mathcal{S}$ is *fibred in groupoids* over $\mathcal{C}$ if the following two conditions hold:

(1) For every morphism $f : V \to U$ in $\mathcal{C}$ and every lift x of U there is a lift $\phi : y \to x$ of f with target x.

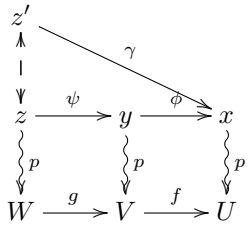
(2) For every pair of morphisms $\phi : y \to x$ and $\psi : z \to x$ and any morphism $f : p(z) \to p(y)$ such that $p(\phi) \circ f = p(\psi)$ there exists a unique lift $\chi : z \to y$ of f such that $\phi \circ \chi = \psi$.

Condition (2) phrased differently says that applying the functor p gives a bijection between the sets of dotted arrows in the following commutative diagram below:



Another way to think about the second condition is the following. Suppose that $g : W \to V$ and $f : V \to U$ are morphisms in $\mathcal{C}$. Let $x \in \mathrm{Ob}(\mathcal{S}_U)$. By the first condition we can lift f to $\phi : y \to x$ and then we can lift g to $\psi : z \to y$. Instead of doing this two step process we can directly lift $g \circ f$ to $\gamma : z' \to x$. This gives the solid arrows in the diagram

(4.33.1.1)



where the squiggly arrows represent not morphisms but the functor p. Applying the second condition to the arrows $\phi \circ \psi$, γ and id_W we conclude that there is a unique morphism $\chi : z \to z'$ in $\mathcal{S}_W$ such that $\gamma \circ \chi = \phi \circ \psi$. Similarly there is a unique morphism $z' \to z$. The uniqueness implies that the morphisms $z' \to z$ and $z \to z'$ are mutually inverse, in other words isomorphisms.

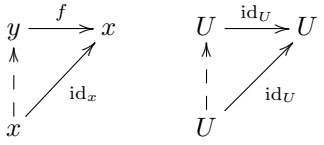
It should be clear from this discussion that a category fibred in groupoids is very closely related to a fibred category. Here is the result.

Lemma 4.33.2. *Let $p : \mathcal{S} \to \mathcal{C}$ be a functor. The following are equivalent*

(1) *$p : \mathcal{S} \to \mathcal{C}$ is a category fibred in groupoids, and*
(2) *all fibre categories are groupoids and $\mathcal{S}$ is a fibred category over $\mathcal{C}$.*

Moreover, in this case every morphism of $\mathcal{S}$ is strongly cartesian. In addition, given $f^*x \to x$ lying over f for all $f : V \to U = p(x)$ the data $(U \mapsto \mathcal{S}_U, f \mapsto f^*, \alpha_{f,g}, \alpha_U)$ constructed in Lemma 4.31.7 defines a pseudo functor from $\mathcal{C}^{opp}$ in to the $(2,1)$-category of groupoids.

Proof. Assume $p : \mathcal{S} \to \mathcal{C}$ is fibred in groupoids. To show all fibre categories $\mathcal{S}_U$ for $U \in \mathrm{Ob}(\mathcal{C})$ are groupoids, we must exhibit for every $f : y \to x$ in $\mathcal{S}_U$ an inverse morphism. The diagram on the left (in $\mathcal{S}_U$) is mapped by p to the diagram on the right:



Since only id_U makes the diagram on the right commute, there is a unique $g : x \to y$ making the diagram on the left commute, so $fg = \mathrm{id}_x$. By a similar argument there is a unique $h : y \to x$ so that $gh = \mathrm{id}_y$. Then $fgh = f : y \to x$. We have $fg = \mathrm{id}_x$, so $h = f$. Condition (2) of Definition 4.33.1 says exactly that every morphism of $\mathcal{S}$ is strongly cartesian. Hence condition (1) of Definition 4.33.1 implies that $\mathcal{S}$ is a fibred category over $\mathcal{C}$.

Conversely, assume all fibre categories are groupoids and $\mathcal{S}$ is a fibred category over $\mathcal{C}$. We have to check conditions (1) and (2) of Definition 4.33.1. The first condition follows trivially. Let $\phi : y \to x$, $\psi : z \to x$ and $f : p(z) \to p(y)$ such that $p(\phi) \circ f = p(\psi)$ be as in condition (2) of Definition 4.33.1. Write $U = p(x)$, $V = p(y)$, $W = p(z)$, $p(\phi) = g : V \to U$, $p(\psi) = h : W \to U$. Choose a strongly cartesian $g^*x \to x$ lying over g. Then we get a morphism $i : y \to g^*x$ in $\mathcal{S}_V$, which is therefore an isomorphism. We also get a morphism $j : z \to g^*x$ corresponding to the pair (ψ, f) as $g^*x \to x$ is strongly cartesian. Then one checks that $\chi = i^{-1} \circ j$ is a solution.

We have seen in the proof of (1) $\Rightarrow$ (2) that every morphism of $\mathcal{S}$ is strongly cartesian. The final statement follows directly from Lemma 4.31.7. □

Lemma 4.33.3. *Let $\mathcal{C}$ be a category. Let $p : \mathcal{S} \to \mathcal{C}$ be a fibred category. Let $\mathcal{S}'$ be the subcategory of $\mathcal{S}$ defined as follows*

(1) $\mathrm{Ob}(\mathcal{S}') = \mathrm{Ob}(\mathcal{S})$, *and*
(2) *for $x, y \in \mathrm{Ob}(\mathcal{S}')$ the set of morphisms between x and y in $\mathcal{S}'$ is the set of of strongly cartesian morphisms between x and y in $\mathcal{S}$.*

Let $p' : \mathcal{S}' \to \mathcal{C}$ be the restriction of p to $\mathcal{S}'$. Then $p' : \mathcal{S}' \to \mathcal{C}$ is fibred in groupoids.

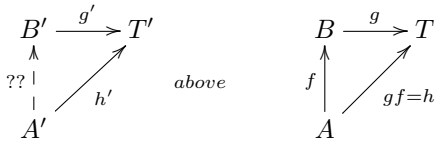
Proof. Note that the construction makes sense since by Lemma 4.31.2 the identity morphism of any object of $\mathcal{S}$ is strongly cartesian, and the composition of strongly cartesian morphisms is strongly cartesian. The first lifting property of Definition 4.33.1 follows from the condition that in a fibred category given any morphism $f : V \to U$ and x lying over U there exists a strongly cartesian morphism $\varphi : y \to x$ lying over f. Let us check the second lifting property of Definition 4.33.1 for the category $p' : \mathcal{S}' \to \mathcal{C}$ over $\mathcal{C}$. To do this we argue as in the discussion following Definition 4.33.1. Thus in Diagram 4.33.1.1 the morphisms ϕ, ψ and γ are strongly cartesian morphisms of $\mathcal{S}$. Hence γ and $\phi \circ \psi$ are strongly cartesian morphisms of $\mathcal{S}$ lying over the same arrow of $\mathcal{C}$ and having the same target in $\mathcal{S}$. By the discussion following Definition 4.31.1 this means these two arrows are isomorphic as desired (here we use also that any isomorphism in $\mathcal{S}$ is strongly cartesian, by Lemma 4.31.2 again). □

Example 4.33.4. A homomorphism of groups $p : G \to H$ gives rise to a functor $p : \mathcal{S} \to \mathcal{C}$ as in Example 4.2.12. This functor $p : \mathcal{S} \to \mathcal{C}$ is fibred in groupoids if and only if p is surjective. The fibre category $\mathcal{S}_U$ over the (unique) object $U \in \mathrm{Ob}(\mathcal{C})$ is the category associated to the kernel of p as in Example 4.2.6.

Given $p : \mathcal{S} \to \mathcal{C}$, we can ask: if the fibre category $\mathcal{S}_U$ is a groupoid for all $U \in \mathrm{Ob}(\mathcal{C})$, must $\mathcal{S}$ be fibred in groupoids over $\mathcal{C}$? We can see the answer is no as follows. Start with a category fibred in groupoids $p : \mathcal{S} \to \mathcal{C}$. Altering the morphisms in $\mathcal{S}$ which do not map to the identity morphism on some object does not alter the categories $\mathcal{S}_U$. Hence we can violate the existence and uniqueness conditions on lifts.

One example is the functor from Example 4.33.4 when $G \to H$ is not surjective. Here is another example.

Example 4.33.5. Let $\mathrm{Ob}(\mathcal{C}) = \{A, B, T\}$ and $\mathrm{Mor}_{\mathcal{C}}(A, B) = \{f\}$, $\mathrm{Mor}_{\mathcal{C}}(B, T) = \{g\}$, $\mathrm{Mor}_{\mathcal{C}}(A, T) = \{h\} = \{gf\}$, plus the identity morphism for each object. See the diagram below for a picture of this category. Now let $\mathrm{Ob}(\mathcal{S}) = \{A', B', T'\}$ and $\mathrm{Mor}_{\mathcal{S}}(A', B') = \emptyset$, $\mathrm{Mor}_{\mathcal{S}}(B', T') = \{g'\}$, $\mathrm{Mor}_{\mathcal{S}}(A', T') = \{h'\}$, plus the identity morphisms. The functor $p : \mathcal{S} \to \mathcal{C}$ is obvious. Then for every $U \in \mathrm{Ob}(\mathcal{C})$, $\mathcal{S}_U$ is the category with one object and the identity morphism on that object, so a groupoid, but the morphism $f : A \to B$ cannot be lifted. Similarly, if we declare $\mathrm{Mor}_{\mathcal{S}}(A', B') = \{f'_1, f'_2\}$ and $\mathrm{Mor}_{\mathcal{S}}(A', T') = \{h'\} = \{g'f'_1\} = \{g'f'_2\}$, then the fibre categories are the same and $f : A \to B$ in the diagram below has two lifts.



Later we would like to make assertions such as "any category fibred in groupoids over $\mathcal{C}$ is equivalent to a split one", or "any category fibred in groupoids whose fibre categories are setlike is equivalent to a category fibred in sets". The notion of equivalence depends on the 2-category we are working with.

Definition 4.33.6. Let $\mathcal{C}$ be a category. The 2-*category of categories fibred in groupoids over* $\mathcal{C}$ is the sub 2-category of the 2-category of fibred categories over $\mathcal{C}$ (see Definition 4.31.9) defined as follows:

(1) Its objects will be categories $p : \mathcal{S} \to \mathcal{C}$ fibred in groupoids.
(2) Its 1-morphisms $(\mathcal{S}, p) \to (\mathcal{S}', p')$ will be functors $G : \mathcal{S} \to \mathcal{S}'$ such that $p' \circ G = p$ (since every morphism is strongly cartesian G automatically preserves them).
(3) Its 2-morphisms $t : G \to H$ for $G, H : (\mathcal{S}, p) \to (\mathcal{S}', p')$ will be morphisms of functors such that $p'(t_x) = \mathrm{id}_{p(x)}$ for all $x \in \mathrm{Ob}(\mathcal{S})$.

Note that every 2-morphism is automatically an isomorphism! Hence this is actually a $(2, 1)$-category and not just a 2-category. Here is the obligatory lemma on 2-fibre products.

Lemma 4.33.7. *Let $\mathcal{C}$ be a category. The 2-category of categories fibred in groupoids over $\mathcal{C}$ has 2-fibre products, and they are described as in Lemma 4.30.3.*

Proof. By Lemma 4.31.10 the fibre product as described in Lemma 4.30.3 is a fibred category. Hence it suffices to prove that the fibre categories are groupoids, see Lemma 4.33.2. By Lemma 4.30.4 it is enough to show that the 2-fibre product of groupoids is a groupoid, which is clear (from the construction in Lemma 4.29.4 for example). □

Lemma 4.33.8. *Let $p : \mathcal{S} \to \mathcal{C}$ and $p' : \mathcal{S}' \to \mathcal{C}$ be categories fibred in groupoids, and suppose that $G : \mathcal{S} \to \mathcal{S}'$ is a functor over $\mathcal{C}$.*

(1) *Then G is faithful (resp. fully faithful, resp. an equivalence) if and only if for each $U \in \mathrm{Ob}(\mathcal{C})$ the induced functor $G_U : \mathcal{S}_U \to \mathcal{S}'_U$ is faithful (resp. fully faithful, resp. an equivalence).*

(2) *If G is an equivalence, then G is an equivalence in the 2-category of categories fibred in groupoids over $\mathcal{C}$.*

Proof. Let x, y be objects of $\mathcal{S}$ lying over the same object U. Consider the commutative diagram

$$\begin{array}{ccc} \mathrm{Mor}_{\mathcal{S}}(x,y) & \xrightarrow{\quad G \quad} & \mathrm{Mor}_{\mathcal{S}'}(G(x),G(y)) \\ & {\scriptstyle p}\searrow \quad \swarrow{\scriptstyle p'} & \\ & \mathrm{Mor}_{\mathcal{C}}(U,U) & \end{array}$$

From this diagram it is clear that if G is faithful (resp. fully faithful) then so is each G_U.

Suppose G is an equivalence. For every object x' of $\mathcal{S}'$ there exists an object x of $\mathcal{S}$ such that $G(x)$ is isomorphic to x'. Suppose that x' lies over U' and x lies over U. Then there is an isomorphism $f : U' \to U$ in $\mathcal{C}$, namely, p' applied to the isomorphism $x' \to G(x)$. By the axioms of a category fibred in groupoids there exists an arrow $f^*x \to x$ of $\mathcal{S}$ lying over f. Hence there exists an isomorphism $\alpha : x' \to G(f^*x)$ such that $p'(\alpha) = \mathrm{id}_{U'}$ (this time by the axioms for $\mathcal{S}'$). All in all we conclude that for every object x' of $\mathcal{S}'$ we can choose a pair $(o_{x'}, \alpha_{x'})$ consisting of an object $o_{x'}$ of $\mathcal{S}$ and an isomorphism $\alpha_{x'} : x' \to G(o_{x'})$ with $p(\alpha_{x'}) = \mathrm{id}_{p'(x')}$. From this point on we proceed as usual (see proof of Lemma 4.2.19) to produce an inverse functor $F : \mathcal{S}' \to \mathcal{S}$, by taking $x' \mapsto o_{x'}$ and $\varphi' : x' \to y'$ to the unique arrow $\varphi_{\varphi'} : o_{x'} \to o_{y'}$ with $\alpha_{x'}^{-1} \circ G(\varphi_{\varphi'}) \circ \alpha_{y'} = \varphi'$. With these choices F is a functor over $\mathcal{C}$. We omit the verification that $G \circ F$ and $F \circ G$ are 2-isomorphic (in the 2-category of categories fibred in groupoids over $\mathcal{C}$).

Suppose that G_U is faithful (resp. fully faithful) for all $U \in \mathrm{Ob}(\mathcal{C})$. To show that G is faithful (resp. fully faithful) we have to show for any objects $x, y \in \mathrm{Ob}(\mathcal{S})$ that G induces an injection (resp. bijection) between $\mathrm{Mor}_{\mathcal{S}}(x,y)$ and $\mathrm{Mor}_{\mathcal{S}'}(G(x),G(y))$. Set $U = p(x)$ and $V = p(y)$. It suffices to prove that G induces an injection (resp. bijection) between morphism $x \to y$ lying over f to morphisms $G(x) \to G(y)$ lying over f for any morphism $f : U \to V$. Now fix $f : U \to V$. Denote $f^*y \to y$ a pullback. Then also $G(f^*y) \to G(y)$ is a pullback. The set of morphisms from x to y lying over f is bijective to the set of morphisms between x and f^*y lying over id_U. (By the second axiom of a category fibred in groupoids.) Similarly the set of morphisms from $G(x)$ to $G(y)$ lying over f is bijective to the set of morphisms between $G(x)$ and $G(f^*y)$ lying over id_U. Hence the fact that G_U is faithful (resp. fully faithful) gives the desired result.

Finally suppose for all G_U is an equivalence for all U, so it is fully faithful and essentially surjective. We have seen this implies G is fully faithful, and thus to prove it is an equivalence we have to prove that it is essentially surjective. This is clear, for if $z' \in \mathrm{Ob}(\mathcal{S}')$ then $z' \in \mathrm{Ob}(\mathcal{S}'_U)$ where $U = p'(z')$. Since G_U is essentially surjective we know that z' is isomorphic, in $\mathcal{S}'_U$, to an object of the form $G_U(z)$ for some $z \in \mathrm{Ob}(\mathcal{S}_U)$. But morphisms in $\mathcal{S}'_U$ are morphisms in $\mathcal{S}'$ and hence z' is isomorphic to $G(z)$ in $\mathcal{S}'$. □

Lemma 4.33.9. *Let $\mathcal{C}$ be a category. Let $p : \mathcal{S} \to \mathcal{C}$ and $p' : \mathcal{S}' \to \mathcal{C}$ be categories fibred in groupoids. Let $G : \mathcal{S} \to \mathcal{S}'$ be a functor over $\mathcal{C}$. Then G is fully faithful if*

and only if the diagonal

$$\Delta_G : \mathcal{S} \longrightarrow \mathcal{S} \times_{G,\mathcal{S}',G} \mathcal{S}$$

is an equivalence.

Proof. By Lemma 4.33.8 it suffices to look at fibre categories over an object U of $\mathcal{C}$. An object of the right hand side is a triple (x, x', α) where $\alpha : G(x) \to G(x')$ is a morphism in $\mathcal{S}'_U$. The functor Δ_G maps the object x of $\mathcal{S}_U$ to the triple $(x, x, \mathrm{id}_{G(x)})$. Note that (x, x', α) is in the essential image of Δ_G if and only if $\alpha = G(\beta)$ for some morphism $\beta : x \to x'$ in $\mathcal{S}_U$ (details omitted). Hence in order for Δ_G to be an equivalence, every α has to be the image of a morphism $\beta : x \to x'$, and also every two distinct morphisms $\beta, \beta' : x \to x'$ have to given distinct morphisms $G(\beta), G(\beta')$. This proves one direction of the lemma. We omit the proof of the other direction. □

Lemma 4.33.10. *Let $\mathcal{C}$ be a category. Let $\mathcal{S}_i$, $i = 1, 2, 3, 4$ be categories fibred in groupoids over $\mathcal{C}$. Suppose that $\varphi : \mathcal{S}_1 \to \mathcal{S}_2$ and $\psi : \mathcal{S}_3 \to \mathcal{S}_4$ are equivalences over $\mathcal{C}$. Then*

$$\mathrm{Mor}_{Cat/\mathcal{C}}(\mathcal{S}_2, \mathcal{S}_3) \longrightarrow \mathrm{Mor}_{Cat/\mathcal{C}}(\mathcal{S}_1, \mathcal{S}_4), \quad \alpha \longmapsto \psi \circ \alpha \circ \varphi$$

is an equivalence of categories.

Proof. This is a generality and holds in any 2-category. □

Lemma 4.33.11. *Let $\mathcal{C}$ be a category. If $p : \mathcal{S} \to \mathcal{C}$ is fibred in groupoids, then so is the inertia fibred category $\mathcal{I}_\mathcal{S} \to \mathcal{C}$.*

Proof. Clear from the construction in Lemma 4.32.1 or by using (from the same lemma) that $I_\mathcal{S} \to \mathcal{S} \times_{\Delta, \mathcal{S} \times_\mathcal{C} \mathcal{S}, \Delta} \mathcal{S}$ is an equivalence and appealing to Lemma 4.33.7. □

Lemma 4.33.12. *Let $\mathcal{C}$ be a category. Let $U \in \mathrm{Ob}(\mathcal{C})$. If $p : \mathcal{S} \to \mathcal{C}$ is a category fibred in groupoids and p factors through $p' : \mathcal{S} \to \mathcal{C}/U$ then $p' : \mathcal{S} \to \mathcal{C}/U$ is fibred in groupoids.*

Proof. We have already seen in Lemma 4.31.11 that p' is a fibred category. Hence it suffices to prove the fibre categories are groupoids, see Lemma 4.33.2. For $V \in \mathrm{Ob}(\mathcal{C})$ we have

$$\mathcal{S}_V = \coprod\nolimits_{f : V \to U} \mathcal{S}_{(f : V \to U)}$$

where the left hand side is the fibre category of p and the right hand side is the disjoint union of the fibre categories of p'. Hence the result. □

Lemma 4.33.13. *Let $\mathcal{A} \to \mathcal{B} \to \mathcal{C}$ be functors between categories. If $\mathcal{A}$ is fibred in groupoids over $\mathcal{B}$ and $\mathcal{B}$ is fibred in groupoids over $\mathcal{C}$, then $\mathcal{A}$ is fibred in groupoids over $\mathcal{C}$.*

Proof. One can prove this directly from the definition. However, we will argue using the criterion of Lemma 4.33.2. By Lemma 4.31.12 we see that $\mathcal{A}$ is fibred over $\mathcal{C}$. To finish the proof we show that the fibre category $\mathcal{A}_U$ is a groupoid for U in $\mathcal{C}$. Namely, if $x \to y$ is a morphism of $\mathcal{A}_U$, then its image in $\mathcal{B}$ is an isomorphism as $\mathcal{B}_U$ is a groupoid. But then $x \to y$ is an isomorphism, for example by Lemma 4.31.2 and the fact that every morphism of $\mathcal{A}$ is strongly $\mathcal{B}$-cartesian (see Lemma 4.33.2). □

Lemma 4.33.14. *Let $p : \mathcal{S} \to \mathcal{C}$ be a category fibred in groupoids. Let $x \to y$ and $z \to y$ be morphisms of $\mathcal{S}$. If $p(x) \times_{p(y)} p(z)$ exists, then $x \times_y z$ exists and $p(x \times_y z) = p(x) \times_{p(y)} p(z)$.*

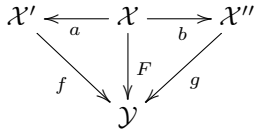
Proof. Follows from Lemma 4.31.13. □

Lemma 4.33.15. *Let $\mathcal{C}$ be a category. Let $F : \mathcal{X} \to \mathcal{Y}$ be a 1-morphism of categories fibred in groupoids over $\mathcal{C}$. There exists a factorization $\mathcal{X} \to \mathcal{X}' \to \mathcal{Y}$ by 1-morphisms of categories fibred in groupoids over $\mathcal{C}$ such that $\mathcal{X} \to \mathcal{X}'$ is an equivalence over $\mathcal{C}$ and such that $\mathcal{X}'$ is a category fibred in groupoids over $\mathcal{Y}$.*

Proof. Denote $p : \mathcal{X} \to \mathcal{C}$ and $q : \mathcal{Y} \to \mathcal{C}$ the structure functors. We construct $\mathcal{X}'$ explicitly as follows. An object of $\mathcal{X}'$ is a quadruple (U, x, y, f) where $x \in \mathrm{Ob}(\mathcal{X}_U)$, $y \in \mathrm{Ob}(\mathcal{Y}_U)$ and $f : F(x) \to y$ is an isomorphism in $\mathcal{Y}_U$. A morphism $(a, b) : (U, x, y, f) \to (U', x', y', f')$ is given by $a : x \to x'$ and $b : y \to y'$ with $p(a) = q(b)$ and such that $f' \circ F(a) = b \circ f$. In other words $\mathcal{X}' = \mathcal{X} \times_{F,\mathcal{Y},\mathrm{id}} \mathcal{Y}$ with the construction of the 2-fibre product from Lemma 4.30.3. By Lemma 4.33.7 we see that $\mathcal{X}'$ is a category fibred in groupoids over $\mathcal{C}$ and that $\mathcal{X}' \to \mathcal{Y}$ is a morphism of categories over $\mathcal{C}$. As functor $\mathcal{X} \to \mathcal{X}'$ we take $x \mapsto (p(x), x, F(x), \mathrm{id}_{F(x)})$ on objects and $(a : x \to x') \mapsto (a, F(a))$ on morphisms. It is clear that the composition $\mathcal{X} \to \mathcal{X}' \to \mathcal{Y}$ equals F. We omit the verification that $\mathcal{X} \to \mathcal{X}'$ is an equivalence of fibred categories over $\mathcal{C}$.

Finally, we have to show that $\mathcal{X}' \to \mathcal{Y}$ is a category fibred in groupoids. Let $b : y' \to y$ be a morphism in $\mathcal{Y}$ and let (U, x, y, f) be an object of $\mathcal{X}'$ lying over y. Because $\mathcal{X}$ is fibred in groupoids over $\mathcal{C}$ we can find a morphism $a : x' \to x$ lying over $U' = q(y') \to q(y) = U$. Since $\mathcal{Y}$ is fibred in groupoids over $\mathcal{C}$ and since both $F(x') \to F(x)$ and $y' \to y$ lie over the same morphism $U' \to U$ we can find $f' : F(x') \to y'$ lying over $\mathrm{id}_{U'}$ such that $f \circ F(a) = b \circ f'$. Hence we obtain $(a, b) : (U', x', y', f') \to (U, x, y, f)$. This verifies the first condition (1) of Definition 4.33.1. To see (2) let $(a, b) : (U', x', y', f') \to (U, x, y, f)$ and $(a', b') : (U'', x'', y'', f'') \to (U, x, y, f)$ be morphisms of $\mathcal{X}'$ and let $b'' : y' \to y''$ be a morphism of $\mathcal{Y}$ such that $b' \circ b'' = b$. We have to show that there exists a unique morphism $a'' : x' \to x''$ such that $f'' \circ F(a'') = b'' \circ f'$ and such that $(a', b') \circ (a'', b'') = (a, b)$. Because $\mathcal{X}$ is fibred in groupoids we know there exists a unique morphism $a'' : x' \to x''$ such that $a' \circ a'' = a$ and $p(a'') = q(b'')$. Because $\mathcal{Y}$ is fibred in groupoids we see that $F(a'')$ is the unique morphism $F(x') \to F(x'')$ such that $F(a') \circ F(a'') = F(a)$ and $q(F(a'')) = q(b'')$. The relation $f'' \circ F(a'') = b'' \circ f'$ follows from this and the given relations $f \circ F(a) = b \circ f'$ and $f \circ F(a') = b' \circ f''$. □

Lemma 4.33.16. *Let $\mathcal{C}$ be a category. Let $F : \mathcal{X} \to \mathcal{Y}$ be a 1-morphism of categories fibred in groupoids over $\mathcal{C}$. Assume we have a 2-commutative diagram*



where a and b are equivalences of categories over $\mathcal{C}$ and f and g are categories fibred in groupoids. Then there exists an equivalence $h : \mathcal{X}'' \to \mathcal{X}'$ of categories over $\mathcal{Y}$ such that $h \circ b$ is 2-isomorphic to a as 1-morphisms of categories over $\mathcal{C}$.

If the diagram above actually commutes, then we can arrange it so that $h \circ b$ is 2-isomorphic to a as 1-morphisms of categories over $\mathcal{Y}$.

Proof. We will show that both $\mathcal{X}'$ and $\mathcal{X}''$ over $\mathcal{Y}$ are equivalent to the category fibred in groupoids $\mathcal{X} \times_{F,\mathcal{Y},\mathrm{id}} \mathcal{Y}$ over $\mathcal{Y}$, see proof of Lemma 4.33.15. Choose a quasi-inverse $b^{-1} : \mathcal{X}'' \to \mathcal{X}$ in the 2-category of categories over $\mathcal{C}$. Since the right triangle of the diagram is 2-commutative we see that

$$\begin{array}{ccc} \mathcal{X} & \xleftarrow[b^{-1}]{} & \mathcal{X}'' \\ {\scriptstyle F}\downarrow & & \downarrow{\scriptstyle g} \\ \mathcal{Y} & \longleftarrow & \mathcal{Y} \end{array}$$

is 2-commutative. Hence we obtain a 1-morphism $c : \mathcal{X}'' \to \mathcal{X} \times_{F,\mathcal{Y},\mathrm{id}} \mathcal{Y}$ by the universal property of the 2-fibre product. Moreover c is a morphism of categories over $\mathcal{Y}$ (!) and an equivalence (by the assumption that b is an equivalence, see Lemma 4.29.7). Hence c is an equivalence in the 2-category of categories fibred in groupoids over $\mathcal{Y}$ by Lemma 4.33.8.

We still have to construct a 2-isomorphism between $c \circ b$ and the functor $d : \mathcal{X} \to \mathcal{X} \times_{F,\mathcal{Y},\mathrm{id}} \mathcal{Y}$, $x \mapsto (p(x), x, F(x), \mathrm{id}_{F(x)})$ constructed in the proof of Lemma 4.33.15. Let $\alpha : F \to g \circ b$ and $\beta : b^{-1} \circ b \to \mathrm{id}$ be 2-isomorphisms between 1-morphisms of categories over $\mathcal{C}$. Note that $c \circ b$ is given by the rule

$$x \mapsto (p(x), b^{-1}(b(x)), g(b(x)), \alpha_x \circ F(\beta_x))$$

on objects. Then we see that

$$(\beta_x, \alpha_x) : (p(x), x, F(x), \mathrm{id}_{F(x)}) \longrightarrow (p(x), b^{-1}(b(x)), g(b(x)), \alpha_x \circ F(\beta_x))$$

is a functorial isomorphism which gives our 2-morphism $d \to b \circ c$. Finally, if the diagram commutes then α_x is the identity for all x and we see that this 2-morphism is a 2-morphism in the 2-category of categories over $\mathcal{Y}$. □

4.34. Presheaves of categories

In this section we compare the notion of fibred categories with the closely related notion of a "presheaf of categories". The basic construction is explained in the following example.

Example 4.34.1. Let $\mathcal{C}$ be a category. Suppose that $F : \mathcal{C}^{opp} \to Cat$ is a functor to the 2-category of categories, see Definition 4.27.5. For $f : V \to U$ in $\mathcal{C}$ we will suggestively write $F(f) = f^*$ for the functor from $F(U)$ to $F(V)$. From this we can construct a fibred category $\mathcal{S}_F$ over $\mathcal{C}$ as follows. Define

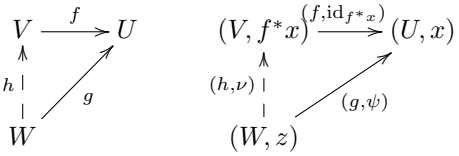
$$\mathrm{Ob}(\mathcal{S}_F) = \{(U, x) \mid U \in \mathrm{Ob}(\mathcal{C}), x \in \mathrm{Ob}(F(U))\}.$$

For $(U, x), (V, y) \in \mathrm{Ob}(\mathcal{S}_F)$ we define

$$\begin{aligned} \mathrm{Mor}_{\mathcal{S}_F}((V, y), (U, x)) &= \{(f, \phi) \mid f \in \mathrm{Mor}_{\mathcal{C}}(V, U), \phi \in \mathrm{Mor}_{F(V)}(y, f^*x)\} \\ &= \coprod\nolimits_{f \in \mathrm{Mor}_{\mathcal{C}}(V,U)} \mathrm{Mor}_{F(V)}(y, f^*x) \end{aligned}$$

In order to define composition we use that $g^* \circ f^* = (f \circ g)^*$ for a pair of composable morphisms of $\mathcal{C}$ (by definition of a functor into a 2-category). Namely, we define the composition of $\psi : z \to g^*y$ and $\phi : y \to f^*x$ to be $g^*(\phi) \circ \psi$. The functor $p_F : \mathcal{S}_F \to \mathcal{C}$ is given by the rule $(U, x) \mapsto U$. Let us check that this is indeed

a fibred category. Given $f : V \to U$ in $\mathcal{C}$ and (U, x) a lift of U, then we claim $(f, \mathrm{id}_{f^*x}) : (V, f^*x) \to (U, x)$ is a strongly cartesian lift of f. We have to show a h in the diagram on the left determines (h, ν) on the right:



Just take $\nu = \psi$ which works because $f \circ h = g$ and hence $g^*x = h^*f^*x$. Moreover, this is the only lift making the diagram (on the right) commute.

Definition 4.34.2. Let $\mathcal{C}$ be a category. Suppose that $F : \mathcal{C}^{opp} \to Cat$ is a functor to the 2-category of categories. We will write $p_F : \mathcal{S}_F \to \mathcal{C}$ for the fibred category constructed in Example 4.34.1. A *split fibred category* is a fibred category isomorphic (!) over $\mathcal{C}$ to one of these categories $\mathcal{S}_F$.

Lemma 4.34.3. *Let $\mathcal{C}$ be a category. Let $\mathcal{S}$ be a fibred category over $\mathcal{C}$. Then $\mathcal{S}$ is split if and only if for some choice of pullbacks (see Definition 4.31.6) the pullback functors $(f \circ g)^*$ and $g^* \circ f^*$ are equal.*

Proof. This is immediate from the definitions. □

Lemma 4.34.4. *Let $p : \mathcal{S} \to \mathcal{C}$ be a fibred category. There exists a functor $F : \mathcal{C} \to Cat$ such that $\mathcal{S}$ is equivalent to $\mathcal{S}_F$ in the 2-category of fibred categories over $\mathcal{C}$. In other words, every fibred category is equivalent to a split one.*

Proof. Let us make a choice of pullbacks (see Definition 4.31.6). By Lemma 4.31.7 we get pullback functors f^* for every morphism f of $\mathcal{C}$.

We construct a new category $\mathcal{S}'$ as follows. The objects of $\mathcal{S}'$ are pairs (x, f) consisting of a morphism $f : V \to U$ of $\mathcal{C}$ and an object x of $\mathcal{S}$ over U, i.e., $x \in \mathrm{Ob}(\mathcal{S}_U)$. The functor $p' : \mathcal{S}' \to \mathcal{C}$ will map the pair (x, f) to the source of the morphism f, in other words $p'(x, f : V \to U) = V$. A morphism $\varphi : (x_1, f_1 : V_1 \to U_1) \to (x_2, f_2 : V_2 \to U_2)$ is given by a pair (φ, g) consisting of a morphism $g : V_1 \to V_2$ and a morphism $\varphi : f_1^*x_1 \to f_2^*x_2$ with $p(\varphi) = g$. It is no problem to define the composition law: $(\varphi, g) \circ (\psi, h) = (\varphi \circ \psi, g \circ h)$ for any pair of composable morphisms. There is a natural functor $\mathcal{S} \to \mathcal{S}'$ which simply maps x over U to the pair (x, id_x).

At this point we need to check that p' makes $\mathcal{S}'$ into a fibred category over $\mathcal{C}$, and we need to check that $\mathcal{S} \to \mathcal{S}'$ is an equivalence of categories over $\mathcal{C}$ which maps strongly cartesian morphisms to strongly cartesian morphisms. We omit the verifications.

Finally, we can define pullback functors on $\mathcal{S}'$ by setting $g^*(x, f) = (x, f \circ g)$ on objects if $g : V' \to V$ and $f : V \to U$. On morphisms $(\varphi, \mathrm{id}_V) : (x_1, f_1) \to (x_2, f_2)$ between morphisms in $\mathcal{S}'_V$ we set $g^*(\varphi, \mathrm{id}_V) = (g^*\varphi, \mathrm{id}_{V'})$ where we use the unique identifications $g^*f_i^*x_i = (f_i \circ g)^*x_i$ from Lemma 4.31.7 to think of $g^*\varphi$ as a morphism from $(f_1 \circ g)^*x_1$ to $(f_2 \circ g)^*x_2$. Clearly, these pullback functors g^* have the property that $g_1^* \circ g_2^* = (g_2 \circ g_1)^*$, in other words $\mathcal{S}'$ is split as desired. □

4.35. Presheaves of groupoids

In this section we compare the notion of categories fibred in groupoids with the closely related notion of a "presheaf of groupoids". The basic construction is explained in the following example.

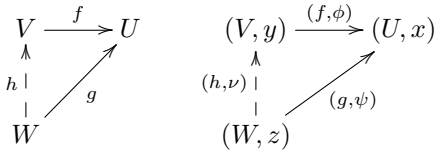
Example 4.35.1. This example is the analogue of Example 4.34.1, for "presheaves of groupoids" instead of "presheaves of categories". The output will be a category fibred in groupoids instead of a fibred category. Suppose that $F : \mathcal{C}^{opp} \to \text{Groupoids}$ is a functor to the category of groupoids, see Definition 4.27.5. For $f : V \to U$ in $\mathcal{C}$ we will suggestively write $F(f) = f^*$ for the functor from $F(U)$ to $F(V)$. We construct a category $\mathcal{S}_F$ fibred in groupoids over $\mathcal{C}$ as follows. Define

$$\text{Ob}(\mathcal{S}_F) = \{(U, x) \mid U \in \text{Ob}(\mathcal{C}), x \in \text{Ob}(F(U))\}.$$

For $(U, x), (V, y) \in \text{Ob}(\mathcal{S}_F)$ we define

$$\begin{aligned}\text{Mor}_{\mathcal{S}_F}((V, y), (U, x)) &= \{(f, \phi) \mid f \in \text{Mor}_{\mathcal{C}}(V, U), \phi \in \text{Mor}_{F(V)}(y, f^*x)\} \\ &= \coprod\nolimits_{f \in \text{Mor}_{\mathcal{C}}(V,U)} \text{Mor}_{F(V)}(y, f^*x)\end{aligned}$$

In order to define composition we use that $g^* \circ f^* = (f \circ g)^*$ for a pair of composable morphisms of $\mathcal{C}$ (by definition of a functor into a 2-category). Namely, we define the composition of $\psi : z \to g^*y$ and $\phi : y \to f^*x$ to be $g^*(\phi) \circ \psi$. The functor $p_F : \mathcal{S}_F \to \mathcal{C}$ is given by the rule $(U, x) \mapsto U$. The condition that $F(U)$ is a groupoid for every U guarantees that $\mathcal{S}_F$ is fibred in groupoids over $\mathcal{C}$, as we have already seen in Example 4.34.1 that $\mathcal{S}_F$ is a fibred category, see Lemma 4.33.2. But we can also prove conditions (1), (2) of Definition 4.33.1 directly as follows: (1) Lifts of morphisms exist since given $f : V \to U$ in $\mathcal{C}$ and (U, x) an object of $\mathcal{S}_F$ over U, then $(f, \text{id}_{f^*x}) : (V, f^*x) \to (U, x)$ is a lift of f. (2) Suppose given solid diagrams as follows



Then for the dotted arrows we have $\nu = (h^*\phi)^{-1} \circ \psi$ so given h there exists a ν which is unique by uniqueness of inverses.

Definition 4.35.2. Let $\mathcal{C}$ be a category. Suppose that $F : \mathcal{C}^{opp} \to \textit{Groupoids}$ is a functor to the 2-category of groupoids. We will write $p_F : \mathcal{S}_F \to \mathcal{C}$ for the category fibred in groupoids constructed in Example 4.35.1. A *split category fibred in groupoids* is a category fibred in groupoids isomorphic (!) over $\mathcal{C}$ to one of these categories $\mathcal{S}_F$.

Lemma 4.35.3. *Let $p : \mathcal{S} \to \mathcal{C}$ be a category fibred in groupoids. There exists a functor $F : \mathcal{C} \to \text{Groupoids}$ such that $\mathcal{S}$ is equivalent to $\mathcal{S}_F$ over $\mathcal{C}$. In other words, every category fibred in groupoids is equivalent to a split one.*

Proof. Make a choice of pullbacks (see Definition 4.31.6). By Lemmas 4.31.7 and 4.33.2 we get pullback functors f^* for every morphism f of $\mathcal{C}$.

We construct a new category $\mathcal{S}'$ as follows. The objects of $\mathcal{S}'$ are pairs (x, f) consisting of a morphism $f : V \to U$ of $\mathcal{C}$ and an object x of $\mathcal{S}$ over U, i.e.,

$x \in \mathrm{Ob}(\mathcal{S}_U)$. The functor $p' : \mathcal{S}' \to \mathcal{C}$ will map the pair (x, f) to the source of the morphism f, in other words $p'(x, f : V \to U) = V$. A morphism $\varphi : (x_1, f_1 : V_1 \to U_1) \to (x_2, f_2 : V_2 \to U_2)$ is given by a pair (φ, g) consisting of a morphism $g : V_1 \to V_2$ and a morphism $\varphi : f_1^* x_1 \to f_2^* x_2$ with $p(\varphi) = g$. It is no problem to define the composition law: $(\varphi, g) \circ (\psi, h) = (\varphi \circ \psi, g \circ h)$ for any pair of composable morphisms. There is a natural functor $\mathcal{S} \to \mathcal{S}'$ which simply maps x over U to the pair (x, id_x).

At this point we need to check that p' makes $\mathcal{S}'$ into a category fibred in groupoids over $\mathcal{C}$, and we need to check that $\mathcal{S} \to \mathcal{S}'$ is an equivalence of categories over $\mathcal{C}$. We omit the verifications.

Finally, we can define pullback functors on $\mathcal{S}'$ by setting $g^*(x, f) = (x, f \circ g)$ on objects if $g : V' \to V$ and $f : V \to U$. On morphisms $(\varphi, \mathrm{id}_V) : (x_1, f_1) \to (x_2, f_2)$ between morphisms in $\mathcal{S}'_V$ we set $g^*(\varphi, \mathrm{id}_V) = (g^*\varphi, \mathrm{id}_{V'})$ where we use the unique identifications $g^* f_i^* x_i = (f_i \circ g)^* x_i$ from Lemma 4.33.2 to think of $g^*\varphi$ as a morphism from $(f_1 \circ g)^* x_1$ to $(f_2 \circ g)^* x_2$. Clearly, these pullback functors g^* have the property that $g_1^* \circ g_2^* = (g_2 \circ g_1)^*$, in other words $\mathcal{S}'$ is split as desired. □

We will see an alternative proof of this lemma in Section 4.39.

4.36. Categories fibred in sets

Definition 4.36.1. A category is called *discrete* if the only morphisms are the identity morphisms.

A discrete category has only one interesting piece of information: its set of objects. Thus we sometime confuse discrete categories with sets.

Definition 4.36.2. Let $\mathcal{C}$ be a category. A *category fibred in sets*, or a *category fibred in discrete categories* is a category fibred in groupoids all of whose fibre categories are discrete.

We want to clarify the relationship between categories fibred in sets and presheaves (see Definition 4.3.3). To do this it makes sense to first make the following definition.

Definition 4.36.3. Let $\mathcal{C}$ be a category. The 2-*category of categories fibred in sets over* $\mathcal{C}$ is the sub 2-category of the category of categories fibred in groupoids over $\mathcal{C}$ (see Definition 4.33.6) defined as follows:

(1) Its objects will be categories $p : \mathcal{S} \to \mathcal{C}$ fibred in sets.
(2) Its 1-morphisms $(\mathcal{S}, p) \to (\mathcal{S}', p')$ will be functors $G : \mathcal{S} \to \mathcal{S}'$ such that $p' \circ G = p$ (since every morphism is strongly cartesian G automatically preserves them).
(3) Its 2-morphisms $t : G \to H$ for $G, H : (\mathcal{S}, p) \to (\mathcal{S}', p')$ will be morphisms of functors such that $p'(t_x) = \mathrm{id}_{p(x)}$ for all $x \in \mathrm{Ob}(\mathcal{S})$.

Note that every 2-morphism is automatically an isomorphism. Hence this 2-category is actually a $(2, 1)$-category. Here is the obligatory lemma on the existence of 2-fibre products.

Lemma 4.36.4. *Let $\mathcal{C}$ be a category. The 2-category of categories fibred in sets over $\mathcal{C}$ has 2-fibre products. More precisely, the 2-fibre product described in Lemma 4.30.3 returns a category fibred in sets if one starts out with such.*

Proof. Omitted. □

Example 4.36.5. This example is the analogue of Examples 4.34.1 and 4.35.1 for presheaves instead of "presheaves of categories". The output will be a category fibred in sets instead of a fibred category. Suppose that $F : \mathcal{C}^{opp} \to \mathit{Sets}$ is a presheaf. For $f : V \to U$ in $\mathcal{C}$ we will suggestively write $F(f) = f^* : F(U) \to F(V)$. We construct a category $\mathcal{S}_F$ fibred in sets over $\mathcal{C}$ as follows. Define

$$\mathrm{Ob}(\mathcal{S}_F) = \{(U, x) \mid U \in \mathrm{Ob}(\mathcal{C}), x \in \mathrm{Ob}(F(U))\}.$$

For $(U, x), (V, y) \in \mathrm{Ob}(\mathcal{S}_F)$ we define

$$\mathrm{Mor}_{\mathcal{S}_F}((V, y), (U, x)) = \{f \in \mathrm{Mor}_{\mathcal{C}}(V, U) \mid f^*x = y\}$$

Composition is inherited from composition in $\mathcal{C}$ which works as $g^* \circ f^* = (f \circ g)^*$ for a pair of composable morphisms of $\mathcal{C}$. The functor $p_F : \mathcal{S}_F \to \mathcal{C}$ is given by the rule $(U, x) \mapsto U$. As every fibre category $\mathcal{S}_{F,U}$ is discrete with underlying set $F(U)$ and we have already see in Example 4.35.1 that $\mathcal{S}_F$ is a category fibred in groupoids, we conclude that $\mathcal{S}_F$ is fibred in sets.

Lemma 4.36.6. *Let $\mathcal{C}$ be a category. The only 2-morphisms between categories fibred in sets are identities. In other words, the 2-category of categories fibred in sets is a category. Moreover, there is an equivalence of categories*

$$\left\{\begin{array}{c}\textit{the category of presheaves}\\ \textit{of sets over } \mathcal{C}\end{array}\right\} \leftrightarrow \left\{\begin{array}{c}\textit{the category of categories}\\ \textit{fibred in sets over } \mathcal{C}\end{array}\right\}$$

The functor from left to right is the construction $F \to \mathcal{S}_F$ discussed in Example 4.36.5. The functor from right to left assigns to $p : \mathcal{S} \to \mathcal{C}$ the presheaf of objects $U \mapsto \mathrm{Ob}(\mathcal{S}_U)$.

Proof. The first assertion is clear, as the only morphisms in the fibre categories are identities.

Suppose that $p : \mathcal{S} \to \mathcal{C}$ is fibred in sets. Let $f : V \to U$ be a morphism in $\mathcal{C}$ and let $x \in \mathrm{Ob}(\mathcal{S}_U)$. Then there is exactly one choice for the object f^*x. Thus we see that $(f \circ g)^*x = g^*(f^*x)$ for f, g as in Lemma 4.33.2. It follows that we may think of the assignments $U \mapsto \mathrm{Ob}(\mathcal{S}_U)$ and $f \mapsto f^*$ as a presheaf on $\mathcal{C}$. □

Here is an important example of a category fibred in sets.

Example 4.36.7. Let $\mathcal{C}$ be a category. Let $X \in \mathrm{Ob}(\mathcal{C})$. Consider the representable presheaf $h_X = \mathrm{Mor}_{\mathcal{C}}(-, X)$ (see Example 4.3.4). On the other hand, consider the category $p : \mathcal{C}/X \to \mathcal{C}$ from Example 4.2.13. The fibre category $(\mathcal{C}/X)_U$ has as objects morphisms $h : U \to X$, and only identities as morphisms. Hence we see that under the correspondence of Lemma 4.36.6 we have

$$h_X \longleftrightarrow \mathcal{C}/X.$$

In other words, the category $\mathcal{C}/X$ is canonically equivalent to the category $\mathcal{S}_{h_X}$ associated to h_X in Example 4.36.5.

For this reason it is tempting to define a "representable" object in the 2-category of categories fibred in groupoids to be a category fibred in sets whose associated presheaf is representable. However, this is would not be a good definition for use since we prefer to have a notion which is invariant under equivalences. To make this precise we study exactly which categories fibred in groupoids are equivalent to categories fibred in sets.

4.37. Categories fibred in setoids

Definition 4.37.1. Let us call a category a *setoid*[4] if it is a groupoid where every object has exactly one automorphism: the identity.

If C is a set with an equivalence relation $\sim$, then we can make a setoid $\mathcal{C}$ as follows: $\mathrm{Ob}(\mathcal{C}) = C$ and $\mathrm{Mor}_{\mathcal{C}}(x, y) = \emptyset$ unless $x \sim y$ in which case we set $\mathrm{Mor}_{\mathcal{C}}(x, y) = \{1\}$. Transitivity of $\sim$ means that we can compose morphisms. Conversely any setoid category defines an equivalence relation on its objects (isomorphism) such that you recover the category (up to unique isomorphism – not equivalence) from the procedure just described.

Discrete categories are setoids. For any setoid $\mathcal{C}$ there is a canonical procedure to make a discrete category equivalent to it, namely one replaces $\mathrm{Ob}(\mathcal{C})$ by the set of isomorphism classes (and adds identity morphisms). In terms of sets endowed with an equivalence relation this corresponds to taking the quotient by the equivalence relation.

Definition 4.37.2. Let $\mathcal{C}$ be a category. A *category fibred in setoids* is a category fibred in groupoids all of whose fibre categories are setoids.

Below we will clarify the relationship between categories fibred in setoids and categories fibred in sets.

Definition 4.37.3. Let $\mathcal{C}$ be a category. The 2-*category of categories fibred in setoids over* $\mathcal{C}$ is the sub 2-category of the category of categories fibred in groupoids over $\mathcal{C}$ (see Definition 4.33.6) defined as follows:

(1) Its objects will be categories $p : \mathcal{S} \to \mathcal{C}$ fibred in setoids.
(2) Its 1-morphisms $(\mathcal{S}, p) \to (\mathcal{S}', p')$ will be functors $G : \mathcal{S} \to \mathcal{S}'$ such that $p' \circ G = p$ (since every morphism is strongly cartesian G automatically preserves them).
(3) Its 2-morphisms $t : G \to H$ for $G, H : (\mathcal{S}, p) \to (\mathcal{S}', p')$ will be morphisms of functors such that $p'(t_x) = \mathrm{id}_{p(x)}$ for all $x \in \mathrm{Ob}(\mathcal{S})$.

Note that every 2-morphism is automatically an isomorphism. Hence this 2-category is actually a $(2, 1)$-category.
Here is the obligatory lemma on the existence of 2-fibre products.

Lemma 4.37.4. *Let $\mathcal{C}$ be a category. The 2-category of categories fibred in setoids over $\mathcal{C}$ has 2-fibre products. More precisely, the 2-fibre product described in Lemma 4.30.3 returns a category fibred in setoids if one starts out with such.*

Proof. Omitted. □

Lemma 4.37.5. *Let $\mathcal{C}$ be a category. Let $\mathcal{S}$ be a category over $\mathcal{C}$.*

(1) *If $\mathcal{S} \to \mathcal{S}'$ is an equivalence over $\mathcal{C}$ with $\mathcal{S}'$ fibred in sets over $\mathcal{C}$, then*
   (a) *$\mathcal{S}$ is fibred in setoids over $\mathcal{C}$, and*
   (b) *for each $U \in \mathrm{Ob}(\mathcal{C})$ the map $\mathrm{Ob}(\mathcal{S}_U) \to \mathrm{Ob}(\mathcal{S}'_U)$ identifies the target as the set of isomorphism classes of the source.*
(2) *If $p : \mathcal{S} \to \mathcal{C}$ is a category fibred in setoids, then there exists a category fibred in sets $p' : \mathcal{S}' \to \mathcal{C}$ and an equivalence $can : \mathcal{S} \to \mathcal{S}'$ over $\mathcal{C}$.*

[4] A set on steroids!?

Proof. Let us prove (2). An object of the category $\mathcal{S}'$ will be a pair (U, ξ), where $U \in \mathrm{Ob}(\mathcal{C})$ and ξ is an isomorphism class of objects of $\mathcal{S}_U$. A morphism $(U, \xi) \to (V, \psi)$ is given by a morphism $x \to y$, where $x \in \xi$ and $y \in \psi$. Here we identify two morphisms $x \to y$ and $x' \to y'$ if they induce the same morphism $U \to V$, and if for some choices of isomorphisms $x \to x'$ in $\mathcal{S}_U$ and $y \to y'$ in $\mathcal{S}_V$ the compositions $x \to x' \to y'$ and $x \to y \to y'$ agree. By construction there are surjective maps on objects and morphisms from $\mathcal{S} \to \mathcal{S}'$. We define composition of morphisms in $\mathcal{S}'$ to be the unique law that turns $\mathcal{S} \to \mathcal{S}'$ into a functor. Some details omitted. □

Thus categories fibred in setoids are exactly the categories fibred in groupoids which are equivalent to categories fibred in sets. Moreover, an equivalence of categories fibred in sets is an isomorphism by Lemma 4.36.6.

Lemma 4.37.6. *Let $\mathcal{C}$ be a category. The construction of Lemma 4.37.5 part (2) gives a functor*

$$F : \left\{ \begin{array}{c} \text{the 2-category of categories} \\ \text{fibred in setoids over } \mathcal{C} \end{array} \right\} \longrightarrow \left\{ \begin{array}{c} \text{the category of categories} \\ \text{fibred in sets over } \mathcal{C} \end{array} \right\}$$

(see Definition 4.27.5). This functor is an equivalence in the following sense:

1. *for any two 1-morphisms $f, g : \mathcal{S}_1 \to \mathcal{S}_2$ with $F(f) = F(g)$ there exists a unique 2-isomorphism $f \to g$,*
2. *for any morphism $h : F(\mathcal{S}_1) \to F(\mathcal{S}_2)$ there exists a 1-morphism $f : \mathcal{S}_1 \to \mathcal{S}_2$ with $F(f) = h$, and*
3. *any category fibred in sets $\mathcal{S}$ is equal to $F(\mathcal{S})$.*

In particular, defining $F_i \in PSh(\mathcal{C})$ by the rule $F_i(U) = \mathrm{Ob}(\mathcal{S}_{i,U})/\cong$, we have

$$\mathrm{Mor}_{Cat/\mathcal{C}}(\mathcal{S}_1, \mathcal{S}_2) \Big/ 2\text{-isomorphism} = \mathrm{Mor}_{PSh(\mathcal{C})}(F_1, F_2)$$

More precisely, given any map $\phi : F_1 \to F_2$ there exists a 1-morphism $f : \mathcal{S}_1 \to \mathcal{S}_2$ which induces ϕ on isomorphism classes of objects and which is unique up to unique 2-isomorphism.

Proof. By Lemma 4.36.6 the target of F is a category hence the assertion makes sense. The construction of Lemma 4.37.5 part (2) assigns to $\mathcal{S}$ the category fibred in sets whose value over U is the set of isomorphism classes in $\mathcal{S}_U$. Hence it is clear that it defines a functor as indicated. Let $f, g : \mathcal{S}_1 \to \mathcal{S}_2$ with $F(f) = F(g)$ be as in (1). For each object U of $\mathcal{C}$ and each object x of $\mathcal{S}_{1,U}$ we see that $f(x) \cong g(x)$ by assumption. As $\mathcal{S}_2$ is fibred in setoids there exists a unique isomorphism $t_x : f(x) \to g(x)$ in $\mathcal{S}_{2,U}$. Clearly the rule $x \mapsto t_x$ gives the desired 2-isomorphism $f \to g$. We omit the proofs of (2) and (3). To see the final assertion use Lemma 4.36.6 to see that the right hand side is equal to $\mathrm{Mor}_{Cat(\mathcal{C})}(F(\mathcal{S}_1), F(\mathcal{S}_2))$ and apply (1) and (2) above. □

Here is another characterization of categories fibred in setoids among all categories fibred in groupoids.

Lemma 4.37.7. *Let $\mathcal{C}$ be a category. Let $p : \mathcal{S} \to \mathcal{C}$ be a category fibred in groupoids. The following are equivalent:*

1. *$p : \mathcal{S} \to \mathcal{C}$ be a category fibred in setoids, and*
2. *the canonical 1-morphism $\mathcal{I}_{\mathcal{S}} \to \mathcal{S}$, see (4.32.2.1), is an equivalence (of categories over $\mathcal{C}$).*

Proof. Assume (2). The category $\mathcal{I}_\mathcal{S}$ has objects (x, α) where $x \in \mathcal{S}$, say with $p(x) = U$, and $\alpha : x \to x$ is a morphism in $\mathcal{S}_U$. Hence if $\mathcal{I}_\mathcal{S} \to \mathcal{S}$ is an equivalence over $\mathcal{C}$ then every pair of objects (x, α), (x, α') are isomorphic in the fibre category of $\mathcal{I}_\mathcal{S}$ over U. Looking at the definition of morphisms in $\mathcal{I}_\mathcal{S}$ we conclude that α, α' are conjugate in the group of automorphisms of x. Hence taking $\alpha' = \mathrm{id}_x$ we conclude that every automorphism of x is equal to the identity. Since $\mathcal{S} \to \mathcal{C}$ is fibred in groupoids this implies that $\mathcal{S} \to \mathcal{C}$ is fibred in setoids. We omit the proof of (1) $\Rightarrow$ (2). □

Lemma 4.37.8. *Let $\mathcal{C}$ be a category. The construction of Lemma 4.37.6 which associates to a category fibred in setoids a presheaf is compatible with products, in the sense that the presheaf associated to a 2-fibre product $\mathcal{X} \times_\mathcal{Y} \mathcal{Z}$ is the fibre product of the presheaves associated to $\mathcal{X}, \mathcal{Y}, \mathcal{Z}$.*

Proof. Let $U \in \mathrm{Ob}(\mathcal{C})$. The lemma just says that

$$\mathrm{Ob}((\mathcal{X} \times_\mathcal{Y} \mathcal{Z})_U)/\cong \quad \text{equals} \quad \mathrm{Ob}(\mathcal{X}_U)/\cong \times_{\mathrm{Ob}(\mathcal{Y}_U)/\cong} \mathrm{Ob}(\mathcal{Z}_U)/\cong$$

the proof of which we omit. (But note that this would not be true in general if the category $\mathcal{Y}_U$ is not a setoid.) □

4.38. Representable categories fibred in groupoids

Here is our definition of a representable category fibred in groupoids. As promised this is invariant under equivalences.

Definition 4.38.1. Let $\mathcal{C}$ be a category. A category fibred in groupoids $p : \mathcal{S} \to \mathcal{C}$ is called *representable* if there exists an object X of $\mathcal{C}$ and an equivalence $j : \mathcal{S} \to \mathcal{C}/X$ (in the 2-category of groupoids over $\mathcal{C}$).

The usual abuse of notation is to say that X *represents* $\mathcal{S}$ and not mention the equivalence j. We spell out what this entails.

Lemma 4.38.2. *Let $\mathcal{C}$ be a category. Let $p : \mathcal{S} \to \mathcal{C}$ be a category fibred in groupoids.*

1. *$\mathcal{S}$ is representable if and only if the following conditions are satisfied:*
   (a) *$\mathcal{S}$ is fibred in setoids, and*
   (b) *the presheaf $U \mapsto \mathrm{Ob}(\mathcal{S}_U)/\cong$ is representable.*
2. *If $\mathcal{S}$ is representable the pair (X, j), where j is the equivalence $j : \mathcal{S} \to \mathcal{C}/X$ is uniquely determined up to isomorphism.*

Proof. The first assertion follows immediately from Lemma 4.37.5. For the second, suppose that $j' : \mathcal{S} \to \mathcal{C}/X'$ is a second such pair. Choose a 1-morphism $t' : \mathcal{C}/X' \to \mathcal{S}$ such that $j' \circ t' \cong \mathrm{id}_{\mathcal{C}/X'}$ and $t' \circ j' \cong \mathrm{id}_\mathcal{S}$. Then $j \circ t' : \mathcal{C}/X' \to \mathcal{C}/X$ is an equivalence. Hence it is an isomorphism, see Lemma 4.36.6. Hence by the Yoneda Lemma 4.3.5 (via Example 4.36.7 for example) it is given by an isomorphism $X' \to X$. □

Lemma 4.38.3. *Let $\mathcal{C}$ be a category. Let $\mathcal{X}$, $\mathcal{Y}$ be categories fibred in groupoids over $\mathcal{C}$. Assume that $\mathcal{X}$, $\mathcal{Y}$ are representable by objects X, Y of $\mathcal{C}$. Then*

$$\mathrm{Mor}_{Cat/\mathcal{C}}(\mathcal{X}, \mathcal{Y}) \Big/ 2\text{-isomorphism} = \mathrm{Mor}_\mathcal{C}(X, Y)$$

More precisely, given $\phi : X \to Y$ there exists a 1-morphism $f : \mathcal{X} \to \mathcal{Y}$ which induces ϕ on isomorphism classes of objects and which is unique up to unique 2-isomorphism.

Proof. By Example 4.36.7 we have $\mathcal{C}/X = \mathcal{S}_{h_X}$ and $\mathcal{C}/Y = \mathcal{S}_{h_Y}$. By Lemma 4.37.6 we have

$$\mathrm{Mor}_{Cat/\mathcal{C}}(\mathcal{X}, \mathcal{Y}) \Big/ 2\text{-isomorphism} = \mathrm{Mor}_{PSh(\mathcal{C})}(h_X, h_Y)$$

By the Yoneda Lemma 4.3.5 we have $\mathrm{Mor}_{PSh(\mathcal{C})}(h_X, h_Y) = \mathrm{Mor}_{\mathcal{C}}(X, Y)$. □

4.39. Representable 1-morphisms

Let $\mathcal{C}$ be a category. In this section we explain what it means for a 1-morphism between categories fibred in groupoids over $\mathcal{C}$ to be representable. Note that the 2-category of categories fibred in groupoids over $\mathcal{C}$ is a "full" sub 2-category of the 2-category of categories over $\mathcal{C}$ (see Definition 4.33.6). Hence if $\mathcal{S}$, $\mathcal{S}'$ are fibred in groupoids over $\mathcal{C}$ then

$$\mathrm{Mor}_{Cat/\mathcal{C}}(\mathcal{S}, \mathcal{S}')$$

denotes the category of 1-morphisms in this 2-category (see Definition 4.30.1). These are all groupoids, see remarks following Definition 4.33.6. Here is the 2-category analogue of the Yoneda lemma.

Lemma 4.39.1 (2-Yoneda lemma)**.** *Let $\mathcal{S} \to \mathcal{C}$ be fibred in groupoids. Let $U \in \mathrm{Ob}(\mathcal{C})$. The functor*

$$\mathrm{Mor}_{Cat/\mathcal{C}}(\mathcal{C}/U, \mathcal{S}) \longrightarrow \mathcal{S}_U$$

given by $G \mapsto G(id_U)$ is an equivalence.

Proof. Make a choice of pullbacks for $\mathcal{S}$ (see Definition 4.31.6). We define a functor

$$\mathcal{S}_U \longrightarrow \mathrm{Mor}_{Cat/\mathcal{C}}(\mathcal{C}/U, \mathcal{S})$$

as follows. Given $x \in \mathrm{Ob}(\mathcal{S}_U)$ the associated functor is

(1) on objects: $(f : V \to U) \mapsto f^*x$, and
(2) on morphisms: the arrow $(g : V'/U \to V/U)$ maps to the composition

$$(f \circ g)^*x \xrightarrow{(\alpha_{g,f})_x} g^*f^*x \to f^*x$$

where $\alpha_{g,f}$ is as in Lemma 4.33.2.

We omit the verification that this is an inverse to the functor of the lemma. □

Remark 4.39.2. We can use the 2-Yoneda lemma to give an alternative proof of Lemma 4.35.3. Let $p : \mathcal{S} \to \mathcal{C}$ be a category fibred in groupoids. We define a contravariant functor F from $\mathcal{C}$ to the category of groupoids as follows: for $U \in \mathrm{Ob}(\mathcal{C})$ let

$$F(U) = \mathrm{Mor}_{Cat/\mathcal{C}}(\mathcal{C}/U, \mathcal{S}).$$

If $f : U \to V$ the induced functor $\mathcal{C}/U \to \mathcal{C}/V$ induces the morphism $F(f) : F(V) \to F(U)$. Clearly F is a functor. Let $\mathcal{S}'$ be the associated category fibred in groupoids from Example 4.35.1. There is an obvious functor $G : \mathcal{S}' \to \mathcal{S}$ over $\mathcal{C}$ given by taking the pair (U, x), where $U \in \mathrm{Ob}(\mathcal{C})$ and $x \in F(U)$, to $x(\mathrm{id}_U) \in \mathcal{S}$. Now Lemma 4.39.1 implies that for each U,

$$G_U : \mathcal{S}'_U = F(U) = \mathrm{Mor}_{Cat/\mathcal{C}}(\mathcal{C}/U, \mathcal{S}) \to \mathcal{S}_U$$

is an equivalence, and thus G equivalence between $\mathcal{S}$ and $\mathcal{S}'$ by Lemma 4.33.8.

Let $\mathcal{C}$ be a category. Let $\mathcal{X}$, $\mathcal{Y}$ be categories fibred in groupoids over $\mathcal{C}$. Let $U \in \mathrm{Ob}(\mathcal{C})$. Let $F : \mathcal{X} \to \mathcal{Y}$ and $G : \mathcal{C}/U \to \mathcal{Y}$ be 1-morphisms of categories fibred in groupoids over $\mathcal{C}$. We want to describe the 2-fibre product

$$\begin{array}{ccc} (\mathcal{C}/U) \times_{\mathcal{Y}} \mathcal{X} & \longrightarrow & \mathcal{X} \\ \downarrow & & \downarrow F \\ \mathcal{C}/U & \xrightarrow{G} & \mathcal{Y} \end{array}$$

Let $y = G(\mathrm{id}_U) \in \mathcal{Y}_U$. Make a choice of pullbacks for $\mathcal{Y}$ (see Definition 4.31.6). Then G is isomorphic to the functor $(f : V \to U) \mapsto f^*y$, see Lemma 4.39.1 and its proof. We may think of an object of $(\mathcal{C}/U) \times_{\mathcal{Y}} \mathcal{X}$ as a quadruple $(V, f : V \to U, x, \phi)$, see Lemma 4.30.3. Using the description of G above we may think of ϕ as an isomorphism $\phi : f^*y \to F(x)$ in $\mathcal{Y}_V$.

Lemma 4.39.3. *In the situation above the fibre category of $(\mathcal{C}/U) \times_{\mathcal{Y}} \mathcal{X}$ over an object $f : V \to U$ of $\mathcal{C}/U$ is the category described as follows:*

1. *objects are pairs (x, ϕ), where $x \in \mathrm{Ob}(\mathcal{X}_V)$, and $\phi : f^*y \to F(x)$ is a morphism in $\mathcal{Y}_V$,*
2. *the set of morphisms between (x, ϕ) and (x', ϕ') is the set of morphisms $\psi : x \to x'$ in $\mathcal{X}_V$ such that $F(\psi) = \phi' \circ \phi^{-1}$.*

Proof. See discussion above. □

Lemma 4.39.4. *Let $\mathcal{C}$ be a category. Let $\mathcal{X}$, $\mathcal{Y}$ be categories fibred in groupoids over $\mathcal{C}$. Let $F : \mathcal{X} \to \mathcal{Y}$ be a 1-morphism. Let $G : \mathcal{C}/U \to \mathcal{Y}$ be a 1-morphism. Then*

$$(\mathcal{C}/U) \times_{\mathcal{Y}} \mathcal{X} \longrightarrow \mathcal{C}/U$$

is a category fibred in groupoids.

Proof. We have already seen in Lemma 4.33.7 that the composition

$$(\mathcal{C}/U) \times_{\mathcal{Y}} \mathcal{X} \longrightarrow \mathcal{C}/U \longrightarrow \mathcal{C}$$

is a category fibred in groupoids. Then the lemma follows from Lemma 4.33.12. □

Definition 4.39.5. Let $\mathcal{C}$ be a category. Let $\mathcal{X}$, $\mathcal{Y}$ be categories fibred in groupoids over $\mathcal{C}$. Let $F : \mathcal{X} \to \mathcal{Y}$ be a 1-morphism. We say F is *representable*, or that $\mathcal{X}$ *is relatively representable over* $\mathcal{Y}$, if for every $U \in \mathrm{Ob}(\mathcal{C})$ and any $G : \mathcal{C}/U \to \mathcal{Y}$ the category fibred in groupoids

$$(\mathcal{C}/U) \times_{\mathcal{Y}} \mathcal{X} \longrightarrow \mathcal{C}/U$$

is representable over $\mathcal{C}/U$.

Lemma 4.39.6. *Let $\mathcal{C}$ be a category. Let $\mathcal{X}$, $\mathcal{Y}$ be categories fibred in groupoids over $\mathcal{C}$. Let $F : \mathcal{X} \to \mathcal{Y}$ be a 1-morphism. If F is representable then every one of the functors*

$$F_U : \mathcal{X}_U \longrightarrow \mathcal{Y}_U$$

between fibre categories is faithful.

Proof. Clear from the description of fibre categories in Lemma 4.39.3 and the characterization of representable fibred categories in Lemma 4.38.2. □

Lemma 4.39.7. *Let $\mathcal{C}$ be a category. Let $\mathcal{X}$, $\mathcal{Y}$ be categories fibred in groupoids over $\mathcal{C}$. Let $F : \mathcal{X} \to \mathcal{Y}$ be a 1-morphism. Make a choice of pullbacks for $\mathcal{Y}$. Assume*

(1) *each functor $F_U : \mathcal{X}_U \longrightarrow \mathcal{Y}_U$ between fibre categories is faithful, and*
(2) *for each U and each $y \in \mathcal{Y}_U$ the presheaf*

$$(f : V \to U) \longmapsto \{(x, \phi) \mid x \in \mathcal{X}_V, \phi : f^*y \to F(x)\}/\cong$$

is a representable presheaf on $\mathcal{C}/U$.

Then F is representable.

Proof. Clear from the description of fibre categories in Lemma 4.39.3 and the characterization of representable fibred categories in Lemma 4.38.2. □

Before we state the next lemma we point out that the 2-category of categories fibred in groupoids is a $(2,1)$-category, and hence we know what it means to say that it has a final object (see Definition 4.29.1). And it has a final object namely $\mathrm{id} : \mathcal{C} \to \mathcal{C}$. Thus we define 2-*products* of categories fibred in groupoids over $\mathcal{C}$ as the 2-fibred products

$$\mathcal{X} \times \mathcal{Y} := \mathcal{X} \times_{\mathcal{C}} \mathcal{Y}.$$

With this definition in place the following lemma makes sense.

Lemma 4.39.8. *Let $\mathcal{C}$ be a category. Let $\mathcal{S} \to \mathcal{C}$ be a category fibred in groupoids. Assume $\mathcal{C}$ has products of pairs of objects and fibre products. The following are equivalent:*

(1) *The diagonal $\mathcal{S} \to \mathcal{S} \times \mathcal{S}$ is representable.*
(2) *For every U in $\mathcal{C}$, any $G : \mathcal{C}/U \to \mathcal{S}$ is representable.*

Proof. Suppose the diagonal is representable, and let U, G be given. Consider any $V \in \mathrm{Ob}(\mathcal{C})$ and any $G' : \mathcal{C}/V \to \mathcal{S}$. Note that $\mathcal{C}/U \times \mathcal{C}/V = \mathcal{C}/U \times V$ is representable. Hence the fibre product

$$\begin{array}{ccc} (\mathcal{C}/U \times V) \times_{(\mathcal{S}\times\mathcal{S})} \mathcal{S} & \longrightarrow & \mathcal{S} \\ \downarrow & & \downarrow \\ \mathcal{C}/U \times V & \xrightarrow{(G,G')} & \mathcal{S} \times \mathcal{S} \end{array}$$

is representable by assumption. This means there exists $W \to U \times V$ in $\mathcal{C}$, such that

$$\begin{array}{ccc} \mathcal{C}/W & \longrightarrow & \mathcal{S} \\ \downarrow & & \downarrow \\ \mathcal{C}/U \times \mathcal{C}/V & \longrightarrow & \mathcal{S} \times \mathcal{S} \end{array}$$

is cartesian. This implies that $\mathcal{C}/W \cong \mathcal{C}/U \times_{\mathcal{S}} \mathcal{C}/V$ (see Lemma 4.29.11) as desired.

Assume (2) holds. Consider any $V \in \mathrm{Ob}(\mathcal{C})$ and any $(G, G') : \mathcal{C}/V \to \mathcal{S} \times \mathcal{S}$. We have to show that $\mathcal{C}/V \times_{\mathcal{S}\times\mathcal{S}} \mathcal{S}$ is representable. What we know is that $\mathcal{C}/V \times_{G,\mathcal{S},G'} \mathcal{C}/V$ is representable, say by $a : W \to V$ in $\mathcal{C}/V$. The equivalence

$$\mathcal{C}/W \to \mathcal{C}/V \times_{G,\mathcal{S},G'} \mathcal{C}/V$$

followed by the second projection to $\mathcal{C}/V$ gives a second morphism $a' : W \to V$. Consider $W' = W \times_{(a,a'),V\times V} V$. There exists an equivalence

$$\mathcal{C}/W' \cong \mathcal{C}/V \times_{\mathcal{S}\times\mathcal{S}} \mathcal{S}$$

namely

$$\begin{aligned} \mathcal{C}/W' &\cong \mathcal{C}/W \times_{(\mathcal{C}/V \times \mathcal{C}/V)} \mathcal{C}/V \\ &\cong \big(\mathcal{C}/V \times_{(G,\mathcal{S},G')} \mathcal{C}/V\big) \times_{(\mathcal{C}/V \times \mathcal{C}/V)} \mathcal{C}/V \\ &\cong \mathcal{C}/V \times_{(\mathcal{S} \times \mathcal{S})} \mathcal{S} \end{aligned}$$

(for the last isomorphism see Lemma 4.29.12) which proves the lemma. □

Biographical notes: Parts of this have been taken from Vistoli's notes [**Vis**].

4.40. Other chapters

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CHAPTER 5

Topology

5.1. Introduction

Basic topology will be explained in this document. A reference is [**Eng77**].

5.2. Basic notions

The following notions are considered basic and will not be defined, and or proved. This does not mean they are all necessarily easy or well known.

(1) X is a *topological space*,
(2) $x \in X$ is a *point*,
(3) $x \in X$ is a *closed point*,
(4) $E \subset X$ is a *dense* set,
(5) $f : X_1 \to X_2$ is *continuous*,
(6) a continuous map of spaces $f : X \to Y$ is *open* if $f(U)$ is open in Y for $U \subset X$ open,
(7) a continuous map of spaces $f : X \to Y$ is *closed* if $f(Z)$ is closed in Y for $Z \subset X$ closed,
(8) a *neighbourhood of* $x \in X$ is any subset $E \subset X$ which contains an open subset that contains x,
(9) the *induced topology* on a subset $E \subset X$,
(10) $\mathcal{U} : U = \bigcup_{i \in I} U_i$ is an *open covering of* U (note: we allow any U_i to be empty and we even allow, in case U is empty, the empty set for I),
(11) the open covering $\mathcal{V}$ is a *refinement* of the open covering $\mathcal{U}$ (if $\mathcal{V} : V = \bigcup_{j \in J} V_j$ and $\mathcal{U} : U = \bigcup_{i \in I} U_i$ this means each V_j is completely contained in one of the U_i),
(12) $\{E_i\}_{i \in I}$ *is a fundamental system of neighbourhoods of* x *in* X,
(13) a topological space X is called *Hausdorff* or *separated* if and only if for every distinct pair of points $x, y \in X$ there exist disjoint opens $U, V \subset X$ such that $x \in U$, $y \in V$,
(14) the *product* of two topological spaces,
(15) the *fibre product* $X \times_Y Z$ of a pair of continuous maps $f : X \to Y$ and $g : Z \to Y$,
(16) etc.

5.3. Hausdorff spaces

The category of topological spaces has finite products.

Lemma 5.3.1. *Let X be a topological space. The following are equivalent*

(1) *X is Hausdorff,*
(2) *the diagonal $\Delta(X) \subset X \times X$ is closed.*

Proof. Omitted. □

Lemma 5.3.2. *Let $f : X \to Y$ be a continuous map of topological spaces. If Y is Hausdorff, then the graph of f is closed in $X \times Y$.*

Proof. The graph is the inverse image of the diagonal under the map $X \times Y \to Y \times Y$. Thus the lemma follows from Lemma 5.3.1. □

Lemma 5.3.3. *Let $f : X \to Y$ be a continuous map of topological spaces. Let $s : Y \to X$ be a continuous map such that $f \circ s = id_Y$. If X is Hausdorff, then $s(Y)$ is closed.*

Proof. This follows from Lemma 5.3.1 as $s(Y) = \{x \in X \mid x = s(f(x))\}$. □

Lemma 5.3.4. *Let $X \to Z$ and $Y \to Z$ be continuous maps of topological spaces. If Z is Hausdorff, then $X \times_Z Y$ is closed in $X \times Y$.*

Proof. This follows from Lemma 5.3.1 as $X \times_Z Y$ is the inverse image of $\Delta(Z)$ under $X \times Y \to Z \times Z$. □

5.4. Bases

Basic material on bases for topological spaces.

Definition 5.4.1. Let X be a topological space. A collection of subsets $\mathcal{B}$ of X is called a *base for the topology on X* or a *basis for the topology on X* if the following conditions hold:

(1) Every element $B \in \mathcal{B}$ is open in X.
(2) For every open $U \subset X$ and every $x \in U$, there exists an element $B \in \mathcal{B}$ such that $x \in B \subset U$.

Lemma 5.4.2. *Let X be a topological space. Let $\mathcal{B}$ be a basis for the topology on X. Let $\mathcal{U} : U = \bigcup_i U_i$ be an open covering of $U \subset X$. There exists an open covering $U = \bigcup V_j$ which is a refinement of $\mathcal{U}$ such that each V_j is an element of the basis $\mathcal{B}$.*

Proof. Omitted. □

Definition 5.4.3. Let X be a topological space. A collection of subsets $\mathcal{B}$ of X is called a *subbase for the topology on X* or a *subbasis for the topology on X* if the finite intersections of elements of $\mathcal{B}$ forms a basis for the topology on X.

In particular every element of $\mathcal{B}$ is open.

Lemma 5.4.4. *Let X be a set. Given any collection $\mathcal{B}$ of subsets of X there is a unique topology on X such that $\mathcal{B}$ is a subbase for this topology.*

Proof. Omitted. □

5.5. Submersive maps

If X is a topological space and $E \subset X$ is a subset, then we usually endow E with the *induced topology*.

Lemma 5.5.1. *Let X be a topological space. Let Y be a set and let $f : Y \to X$ be an injective map of sets. The induced topology on Y is the topology characterized by each of the following statements*

(1) *it is the weakest topology on Y such that f is continuous,*
(2) *the open subsets of Y are $f^{-1}(U)$ for $U \subset X$ open,*
(3) *the closed subsets of Y are the sets $f^{-1}(Z)$ for $Z \subset X$ closed.*

Proof. Omitted. □

Dually, if X is a topological space and $X \to Y$ is a surjection of sets, then Y can be endowed with the *quotient topology*.

Lemma 5.5.2. *Let X be a topological space. Let Y be a set and let $f : X \to Y$ be a surjective map of sets. The quotient topology on Y is the topology characterized by each of the following statements*

(1) *it is the strongest toplogy on Y such that f is continuous,*
(2) *a subset V of Y is open if and only if $f^{-1}(V)$ is open,*
(3) *a subset Z of Y is closed if and only if $f^{-1}(Z)$ is closed.*

Proof. Omitted. □

Let $f : X \to Y$ be a continuous map of topological spaces. In this case we obtain a factorization $X \to f(Y) \to Y$ of maps of sets. We can endow $f(Y)$ with the quotient topology coming from the surjection $Y \to f(Y)$ or with the induced topology coming from the injection $f(Y) \to X$. The map

$$(f(Y), \text{quotient topology}) \longrightarrow (f(Y), \text{induced topology})$$

is continuous.

Definition 5.5.3. Let $f : X \to Y$ be a continuous map of topological spaces.

(1) We say f is a *strict map of topological spaces* if the induced topology and the quotient topology on $f(Y)$ agree (see discussion above).
(2) We say f is *submersive*[1] if f is surjective and strict.

Thus a continuous map $f : X \to Y$ is submersive if f is surjection and for any $T \subset Y$ we have T is open or closed if and only if $f^{-1}(T)$ is so. In other words, Y has the quotient topology relative to the surjection $X \to Y$.

Lemma 5.5.4. *Let $f : X \to Y$ be surjective, open, continuous map of topological spaces. Let $T \subset Y$ be a subset. Then*

(1) *$f^{-1}(\overline{T}) = \overline{f^{-1}(T)}$,*
(2) *$T \subset Y$ is closed if and only $f^{-1}(T)$ is closed,*
(3) *$T \subset Y$ is open if and only $f^{-1}(T)$ is open, and*
(4) *$T \subset Y$ is locally closed if and only $f^{-1}(T)$ is locally closed.*

In particular we see that f is submersive.

Proof. It is clear that $\overline{f^{-1}(T)} \subset f^{-1}(\overline{T})$. If $x \in X$, and $x \notin \overline{f^{-1}(T)}$, then there exists an open neighbourhood $x \in U \subset X$ with $U \cap f^{-1}(T) = \emptyset$. Since f is open we see that $f(U)$ is an open neighbourhood of $f(x)$ not meeting T. Hence $x \notin f^{-1}(\overline{T})$. This proves (1). Part (2) is an easy consequences of this. Part (3) is obvious from the fact that f is open. For (4), if $f^{-1}(T)$ is locally closed, then $f^{-1}(T) \subset \overline{f^{-1}(T)} = f^{-1}(\overline{T})$ is open, and hence by (3) applied to the map $f^{-1}(\overline{T}) \to \overline{T}$ we see that T is open in $\overline{T}$, i.e., T is locally closed. □

[1]This is very different from the notion of a submersion between differential manifolds! It is probably a good idea to use "strict and surjective" in stead of "submersive".

5.6. Connected components

Definition 5.6.1. Let X be a topological space.

(1) We say X is *connected* if X is not empty and whenever $X = T_1 \coprod T_2$ with $T_i \subset X$ open and closed, then either $T_1 = \emptyset$ or $T_2 = \emptyset$.
(2) We say $T \subset X$ is a *connected component* of X if T is a maximal connected subset of X.

The empty space is not connected.

Lemma 5.6.2. *Let $f : X \to Y$ be a continuous map of topological spaces. If $E \subset X$ is a connected subset, then $f(E) \subset Y$ is connected as well.*

Proof. Omitted. □

Lemma 5.6.3. *Let X be a topological space. If $T \subset X$ is connected, then so is its closure. Each point of X is contained in a connected component. Connected components are always closed, but not necessarily open.*

Proof. Let $\overline{T}$ be the closure of the connected subset T. Suppose $\overline{T} = T_1 \coprod T_2$ with $T_i \subset \overline{T}$ open and closed. Then $T = (T \cap T_1) \coprod (T \cap T_2)$. Hence T equals one of the two, say $T = T_1 \cap T$. Thus clearly $\overline{T} \subset T_1$ as desired.

Pick a point $x \in X$. Consider the set A of connected subsets $x \in T_\alpha \subset X$. Note that A is nonempty since $\{x\} \in A$. There is a partial ordering on A coming from inclusion: $\alpha \leq \alpha' \Leftrightarrow T_\alpha \subset T_{\alpha'}$. Choose a maximal totally ordered subset $A' \subset A$, and let $T = \bigcup_{\alpha \in A'} T_\alpha$. We claim that T is connected. Namely, suppose that $T = T_1 \coprod T_2$ is a disjoint union of two open and closed subsets of T. For each $\alpha \in A'$ we have either $T_\alpha \subset T_1$ or $T_\alpha \subset T_2$, by connectedness of T_α. Suppose that for some $\alpha_0 \in A'$ we have $T_{\alpha_0} \not\subset T_1$ (say, if not we're done anyway). Then, since A' is totally ordered we see immediately that $T_\alpha \subset T_2$ for all $\alpha \in A'$. Hence $T = T_2$.

To get an example where connected components are not open, just take an infinite product $\prod_{n \in \mathbf{N}} \{0, 1\}$ with the product topology. This is a totally disconnected space so connected components are singletons, which are not open. □

Lemma 5.6.4. *Let $f : X \to Y$ be a continuous map of topological spaces. Assume that*

(1) *all fibres of f are connected, and*
(2) *a set $T \subset Y$ is closed if and only if $f^{-1}(T)$ is closed.*

Then f induces a bijection between the sets of connected components of X and Y.

Proof. Let $T \subset Y$ be a connected component. Note that T is closed, see Lemma 5.6.3. The lemma follows if we show that $p^{-1}(T)$ is connected because any connected subset of X maps into a connected component of Y by Lemma 5.6.2. Suppose that $p^{-1}(T) = Z_1 \coprod Z_2$ with Z_1, Z_2 closed. For any $t \in T$ we see that $p^{-1}(\{t\}) = Z_1 \cap p^{-1}(\{t\}) \coprod Z_2 \cap p^{-1}(\{t\})$. By (1) we see $p^{-1}(\{t\})$ is connected we conclude that either $p^{-1}(\{t\}) \subset Z_1$ or $p^{-1}(\{t\}) \subset Z_2$. In other words $T = T_1 \coprod T_2$ with $p^{-1}(T_i) = Z_i$. By (2) we conclude that T_i is closed in Y. Hence either $T_1 = \emptyset$ or $T_2 = \emptyset$ as desired. □

Lemma 5.6.5. *Let $f : X \to Y$ be a continuous map of topological spaces. Assume that (a) f is open, (b) all fibres of f are connected. Then f induces a bijection between the sets of connected components of X and Y.*

Proof. This is a special case of Lemma 5.6.4. □

Lemma 5.6.6. *Let $f : X \to Y$ be a continuous map of nonempty topological spaces. Assume that (a) Y is connected, (b) f is open and closed, and (c) there is a point $y \in Y$ such that the fiber $f^{-1}(y)$ is a finite set. Then X has at most $|f^{-1}(y)|$ connected components. Hence any connected component T of X is open and closed, and $p(T)$ is a nonempty open and closed subset of Y, which is therefore equal to Y.*

Proof. If the topological space X has at least N connected components for some $N \in \mathbb{N}$, we find by induction a decomposition $X = X_1 \amalg \cdots \amalg X_N$ of X as a disjoint union of N nonempty open and closed subsets $X_1, \ldots, X_N$ of X. As f is open and closed, each $f(X_i)$ is a nonempty open and closed subset of Y and is hence equal to Y. In particular the intersection $X_i \cap p^{-1}(y)$ is nonempty for each $1 \leq i \leq N$. Hence $p^{-1}(y)$ has at least N elements. □

Definition 5.6.7. A topological space is *totally disconnected* if the connected components are all singletons.

A discrete space is totally disconnected. A totally disconnected space need not be discrete, for example $\mathbf{Q} \subset \mathbf{R}$ is totally disconnected but not discrete.

Lemma 5.6.8. *Let X be a topological space. Let $\pi_0(X)$ be the set of connected components of X. Let $X \to \pi_0(X)$ be the map which sends $x \in X$ to the connected component of X passing through x. Endow $\pi_0(X)$ with the quotient topology. Then $\pi_0(X)$ is a totally disconnected space and any continuous map $X \to Y$ from X to a totally disconnected space Y factors through $\pi_0(X)$.*

Proof. By Lemma 5.6.4 the connected components of $\pi_0(X)$ are the singletons. We omit the proof of the second statement. □

Definition 5.6.9. A topological space X is called *locally connected* if every point $x \in X$ has a fundamental system of connected neighbourhoods.

Lemma 5.6.10. *Let X be a topological space. If X is locally connected, then*

(1) *any open subset of X is locally connected, and*
(2) *the connected components of X are open.*

So also the connected components of open subsets of X are open. In particular, every point has a fundamental system of open connected neighbourhoods.

Proof. Omitted. □

5.7. Irreducible components

Definition 5.7.1. Let X be a topological space.

(1) We say X is *irreducible*, if X is not empty, and whenever $X = Z_1 \cup Z_2$ with Z_i closed, we have $X = Z_1$ or $X = Z_2$.
(2) We say $Z \subset X$ is an *irreducible component* of X if Z is a maximal irreducible subset of X.

An irreducible space is obviously connected.

Lemma 5.7.2. *Let $f : X \to Y$ be a continuous map of topological spaces. If $E \subset X$ is an irreducible subset, then $f(E) \subset Y$ is irreducible as well.*

Proof. Suppose $f(E)$ is the union of $Z_1 \cap f(E)$ and $Z_2 \cap f(E)$, for two distinct closed subsets Z_1 and Z_2 of Y; this is equal to the intersection $(Z_1 \cup Z_2) \cap f(E)$, so $f(E)$ is then contained in the union $Z_1 \cup Z_2$. For the irreducibility of $f(E)$ it suffices to show that it is contained in either Z_1 or Z_2. The relation $f(E) \subseteq Z_1 \cup Z_2$ shows that $f^{-1}(f(E)) \subseteq f^{-1}(Z_1 \cup Z_2)$; as the right-hand side is clearly equal to $f^{-1}(Z_1) \cup f^{-1}(Z_2)$ and since $E \subseteq f^{-1}(f(E))$, it follows that $E \subseteq f^{-1}(Z_1) \cup f^{-1}(Z_2)$, from which one concludes by the irreducibility of E that $E \subseteq f^{-1}(Z_1)$ or $E \subseteq f^{-1}(Z_2)$. Hence one sees that either $f(E) \subseteq f(f^{-1}(Z_1)) \subseteq Z_1$ or $f(E) \subseteq Z_2$. □

Lemma 5.7.3. *Let X be a topological space. If $T \subset X$ is irreducible so is its closure in X. Any irreducible component of X is closed. Every point of X is contained in some irreducible component of X.*

Proof. Let $\overline{T}$ be the closure of the irreducible subset T. If $\overline{T} = Z_1 \cup Z_2$ with $Z_i \subset \overline{T}$ closed, then $T = (T \cap Z_1) \cup (T \cap Z_2)$ and hence T equals one of the two, say $T = Z_1 \cap T$. Thus clearly $\overline{T} \subset Z_1$ as desired.

Pick a point $x \in X$. Consider the set A of irreducible subsets $x \in T_\alpha \subset X$. Note that A is nonempty since $\{x\} \in A$. There is a partial ordening on A coming from inclusion: $\alpha \leq \alpha' \Leftrightarrow T_\alpha \subset T_{\alpha'}$. Choose a maximal totally ordered subset $A' \subset A$, and let $T = \bigcup_{\alpha \in A'} T_\alpha$. We claim that T is irreducible. Namely, suppose that $T = Z_1 \cup Z_2$ is a union of two closed subsets of T. For each $\alpha \in A'$ we have either $T_\alpha \subset Z_1$ or $T_\alpha \subset Z_2$, by irreducibility of T_α. Suppose that for some $\alpha_0 \in A'$ we have $T_{\alpha_0} \not\subset Z_1$ (say, if not we're done anyway). Then, since A' is totally ordered we see immediately that $T_\alpha \subset Z_2$ for all $\alpha \in A'$. Hence $T = Z_2$. □

A singleton is irreducible. Thus if $x \in X$ is a point then the closure $\overline{\{x\}}$ is an irreducible closed subset of X.

Definition 5.7.4. Let X be a topological space.

(1) Let $Z \subset X$ be an irreducible closed subset. A *generic point* of Z is a point $\xi \in Z$ such that $Z = \overline{\{\xi\}}$.
(2) The space X is called *Kolmogorov*, if for every $x, x' \in X$, $x \neq x'$ there exists a closed subset of X which contains exactly one of the two points.
(3) The space X is called *sober* if every irreducible closed subset has a unique generic point.

A space X is Kolmogorov if for $x_1, x_2 \in X$ we have $x_1 = x_2$ if and only if $\overline{\{x_1\}} = \overline{\{x_2\}}$. Hence we see that a sober topological space is Kolmogorov.

Lemma 5.7.5. *Let X be a topological space. If X has an open covering $X = \bigcup X_i$ with X_i sober (resp. Kolmogorov), then X is sober (resp. Kolmogorov).*

Proof. Omitted. □

Example 5.7.6. Recall that a topological space X is Hausdorff iff for every distinct pair of points $x, y \in X$ there exist disjoint opens $U, V \subset X$ such that $x \in U$, $y \in V$. In this case X is irreducible if and only if X is a singleton. Similarly, any subset of X is irreducible if and only if it is a singleton. Hence a Hausdorff space is sober.

Lemma 5.7.7. *Let $f : X \to Y$ be a continuous map of topological spaces. Assume that (a) Y is irreducible, (b) f is open, and (c) there exists a dense collection of points $y \in Y$ such that $f^{-1}(y)$ is irreducible. Then X is irreducible.*

Proof. Suppose $Y = Z_1 \cup Z_2$ with Z_i closed. Consider the open sets $U_1 = Z_1 \setminus Z_2 = Y \setminus Z_2$ and $U_2 = Z_2 \setminus Z_1 = Y \setminus Z_2$. To get a contradiction assume that U_1 and U_2 are both nonempty. By (b) we see that $f(U_i)$ is open. By (a) we have X irreducible and hence $f(U_1) \cap f(U_2) \neq \emptyset$. By (c) there is a point y which corresponds to a point of this intersection such that the fibre $X_y = f^{-1}(y)$ is irreducible. Then $X_y \cap U_1$ and $X_y \cap U_2$ are nonempty disjoint open subsets of X_y which is a contradiction. □

Lemma 5.7.8. *Let $f : X \to Y$ be a continuous map of topological spaces. Assume that (a) f is open, and (b) for every $y \in Y$ the fibre $f^{-1}(y)$ is irreducible. Then f induces a bijection between irreducible components.*

Proof. We point out that assumption (b) implies that f is surjective (see Definition 5.7.1). Let $T \subset Y$ be an irreducible component. Note that T is closed, see Lemma 5.7.3. The lemma follows if we show that $p^{-1}(T)$ is irreducible because any irreducible subset of X maps into an irreducible component of Y by Lemma 5.7.2. Note that $p^{-1}(T) \to T$ satisfies the assumptions of Lemma 5.7.7. Hence we win. □

5.8. Noetherian topological spaces

Definition 5.8.1. A topological space is called *Noetherian* if the descending chain condition holds for closed subsets of X. A topological space is called *locally Noetherian* if every point has a neighbourhood which is Noetherian.

Lemma 5.8.2. *Let X be a Noetherian topological space.*

(1) *Any subset of X with the induced topology is Noetherian.*
(2) *The space X has finitely many irreducible components.*
(3) *Each irreducible component of X contains a nonempty open of X.*

Proof. Let $T \subset X$ be a subset of X. Let $T_1 \supset T_2 \supset \ldots$ be a descending chain of closed subsets of T. Write $T_i = T \cap Z_i$ with $Z_i \subset X$ closed. Consider the descending chain of closed subsets $Z_1 \supset Z_1 \cap Z_2 \supset Z_1 \cap Z_2 \cap Z_3 \ldots$ This stabilizes by assumption and hence the original sequence of T_i stabilizes. Thus T is Noetherian.

Let A be the set of closed subsets of X which do not have finitely many irreducible components. Assume that A is not empty to arrive at a contradiction. The set A is partially ordered by inclusion: $\alpha \leq \alpha' \Leftrightarrow Z_\alpha \subset Z_{\alpha'}$. By the descending chain condition we may find a smallest element of A, say Z. As Z is not a finite union of irreducible components, it is not irreducible. Hence we can write $Z = Z' \cup Z''$ and both are strictly smaller closed subsets. By construction $Z' = \bigcup Z'_i$ and $Z'' = \bigcup Z''_j$ are finite unions of their irreducible components. Hence $Z = \bigcup Z'_i \cup \bigcup Z''_j$ is a finite union of irreducible closed subsets. After removing redundant members of this expression, this will be the decomposition of Z into its irreducible components, a contradiction.

Let $Z \subset X$ be an irreducible component of X. Let $Z_1, \ldots, Z_n$ be the other irreducible components of X. Consider $U = Z \setminus (Z_1 \cup \ldots \cup Z_n)$. This is not empty since otherwise the irreducible space Z would be contained in one of the other Z_i. Because $X = Z \cup Z_1 \cup \ldots Z_n$ (see Lemma 5.7.3), also $U = X \setminus (Z_1 \cup \ldots \cup Z_n)$ and hence open in X. Thus Z contains a nonempty open of X. □

Lemma 5.8.3. *Let $f : X \to Y$ be a continuous map of topological spaces.*

(1) *If X is Noetherian, then $f(X)$ is Noetherian.*

(2) *If X is locally Noetherian and f open, then $f(X)$ is locally Noetherian.*

Proof. In case (1), suppose that $Z_1 \supset Z_2 \supset Z_2 \supset \ldots$ is a descending chain of closed subsets of $f(X)$ (as usual with the induced topology as a subset of Y). Then $f^{-1}(Z_1) \supset f^{-1}(Z_2) \supset f^{-1}(Z_3) \supset \ldots$ is a descending chain of closed subsets of X. Hence this chain stabilizes. Since $f(f^{-1}(Z_i)) = Z_i$ we conclude that $Z_1 \supset Z_2 \supset Z_2 \supset \ldots$ stabilizes also. In case (2), let $y \in f(X)$. Choose $x \in X$ with $f(x) = y$. By assumption there exists a neighbourhood $E \subset X$ of x which is Noetherian. Then $f(E) \subset f(X)$ is a neighbourhood which is Noetherian by part (1). □

Lemma 5.8.4. *Let X be a topological space. Let $X_i \subset X$, $i = 1, \ldots, n$ be a finite collection of subsets. If each X_i is Noetherian (with the induced topology), then $\bigcup_{i=1,\ldots,n} X_i$ is Noetherian (with the induced topology).*

Proof. Omitted. □

Example 5.8.5. Any Noetherian topological space has a closed point (combine Lemmas 5.11.8 and 5.11.13). Let $X = \{1, 2, 3, \ldots\}$. Define a topology on X with opens $\emptyset$, $\{1, 2, \ldots, n\}$, $n \geq 1$ and X. Thus X is a locally Noetherian topological space, without any closed points. This space cannot be the underlying topological space of a locally Noetherian scheme, see Properties, Lemma 27.5.8.

Lemma 5.8.6. *Let X be a locally Noetherian topological space. Then X is locally connected.*

Proof. Let $x \in X$. Let E be a neighbourhood of x. We have to find a connected neighbourhood of x contained in E. By assumption there exists a neighbourhood E' of x which is Noetherian. Then $E \cap E'$ is Noetherian, see Lemma 5.8.2. Let $E \cap E' = Y_1 \cup \ldots \cup Y_n$ be the decomposition into irreducible components, see Lemma 5.8.2. Let $E'' = \bigcup_{x \in Y_i} Y_i$. This is a connected subset of $E \cap E'$ containing x. It contains the open $E \cap E' \setminus (\bigcup_{x \notin Y_i} Y_i)$ of $E \cap E'$ and hence it is a neighbourhood of x in X. This proves the lemma. □

5.9. Krull dimension

Definition 5.9.1. Let X be a topological space.

(1) A *chain of irreducible closed subsets* of X is a sequence $Z_0 \subset Z_1 \subset \ldots \subset Z_n \subset X$ with Z_i closed irreducible and $Z_i \neq Z_{i+1}$ for $i = 0, \ldots, n-1$.
(2) The *length* of a chain $Z_0 \subset Z_1 \subset \ldots \subset Z_n \subset X$ of irreducible closed subsets of X is the integer n.
(3) The *dimension* or more precisely the *Krull dimension* $\dim(X)$ of X is the element of $\{-\infty, 0, 1, 2, 3, \ldots, \infty\}$ defined by the formula:

$$\dim(X) = \sup\{\text{lengths of chains of irreducible closed subsets}\}$$

Thus $\dim(X) = -\infty$ if and only if X is the empty space.
(4) Let $x \in X$. The *Krull dimension of X at x* is defined as

$$\dim_x(X) = \min\{\dim(U), x \in U \subset X \text{ open}\}$$

the minimum of $\dim(U)$ where U runs over the open neighbourhoods of x in X.

Note that if $U' \subset U \subset X$ are open then $\dim(U') \leq \dim(U)$. Hence if $\dim_x(X) = d$ then x has a fundamental system of open neighbourhoods U with $\dim(U) = \dim_x(X)$.

Example 5.9.2. The Krull dimension of the usual Euclidean space $\mathbf{R}^n$ is 0.

Example 5.9.3. Let $X = \{s, \eta\}$ with open sets given by $\{\emptyset, \{\eta\}, \{s, \eta\}\}$. In this case a maximal chain of irreducible closed subsets is $\{s\} \subset \{s, \eta\}$. Hence $\dim(X) = 1$. It is easy to generalize this example to get a $(n+1)$-element topological space of Krull dimension n.

Definition 5.9.4. Let X be a topological space. We say that X is *equidimensional* if every irreducible component of X has the same dimension.

5.10. Codimension and catenary spaces

We only define the codimension of irreducible closed subsets.

Definition 5.10.1. Let X be a topological space. Let $Y \subset X$ be an irreducible closed subset. The *codimension* of Y in X is the supremum of the lengths e of chains

$$Y = Y_0 \subset Y_1 \subset \ldots \subset Y_e \subset X$$

of irreducible closed subsets in X starting with Y. We will denote this $\operatorname{codim}(Y, X)$.

The codimension is an element of $\{0, 1, 2, \ldots\} \cup \{\infty\}$. If $\operatorname{codim}(Y, X) < \infty$, then every chain can be extended to a maximal chain (but these do not all have to have the same length).

Lemma 5.10.2. *Let X be a topological space. Let $Y \subset X$ be an irreducible closed subset. Let $U \subset X$ be an open subset such that $Y \cap U$ is nonempty. Then*

$$\operatorname{codim}(Y, X) = \operatorname{codim}(Y \cap U, U)$$

Proof. The rule $T \mapsto \overline{T}$ defines a bijective inclusion preserving map between the closed irreducible subsets of U and the closed irreducible subsets of X which meet U. Using this the lemma easily follows. Details omitted. □

Example 5.10.3. Let $X = [0, 1]$ be the unit interval with the following topology: The sets $[0, 1]$, $(1 - 1/n, 1]$ for $n \in \mathbf{N}$, and $\emptyset$ are open. So the closed sets are $\emptyset$, $\{0\}$, $[0, 1 - 1/n]$ for $n > 1$ and $[0, 1]$. This is clearly a Noetherian topological space. But the irreducible closed subset $Y = \{0\}$ has infinite codimension $\operatorname{codim}(Y, X) = \infty$. To see this we just remark that all the closed sets $[0, 1 - 1/n]$ are irreducible.

Definition 5.10.4. Let X be a topological space. We say X is *catenary* if for every pair of irreducible closed subsets $T \subset T'$ we have $\operatorname{codim}(T, T') < \infty$ and every maximal chain of irreducible closed subsets

$$T = T_0 \subset T_1 \subset \ldots \subset T_e = T'$$

has the same length (equal to the codimension).

Lemma 5.10.5. *Let X be a topological space. The following are equivalent:*

1. *X is catenary,*
2. *X has an open covering by catenary spaces.*

Moreover, in this case any locally closed subspace of X is catenary.

Proof. Suppose that X is catenary and that $U \subset X$ is an open subset. The rule $T \mapsto \overline{T}$ defines a bijective inclusion preserving map between the closed irreducible subsets of U and the closed irreducible subsets of X which meet U. Using this the lemma easily follows. Details omitted. □

Lemma 5.10.6. *Let X be a topological space. The following are equivalent:*

(1) *X is catenary, and*
(2) *for pair of irreducible closed subsets $Y \subset Y'$ we have $codim(Y, Y') < \infty$ and for every triple $Y \subset Y' \subset Y''$ of irreducible closed subsets we have*
$$codim(Y, Y'') = codim(Y, Y') + codim(Y', Y'').$$

Proof. Omitted. □

5.11. Quasi-compact spaces and maps

The phrase "compact" will be reserved for Hausdorff topological spaces. And many spaces occurring in algebraic geometry are not Hausdorff.

Definition 5.11.1. Quasi-compactness.

(1) We say that a topological space X is *quasi-compact* if every open covering of X has a finite refinement.
(2) We say that a continuous map $f : X \to Y$ is *quasi-compact* if the inverse image $f^{-1}(V)$ of every quasi-compact open $V \subset Y$ is quasi-compact.
(3) We say a subset $Z \subset X$ is *retrocompact* if the inclusion map $Z \to X$ is quasi-compact.

In many texts on topology a space is called *compact* if it is quasi-compact and Hausdorff; and in other texts the Hausdorff condition is omitted. To avoid confusion in algebraic geometry we use the term quasi-compact. Note that the notion of quasi-compactness of a map is very different from the notion of a "proper map" in topology, since there one requires the inverse image of any (quasi-)compact subset of the target to be (quasi-)compact, whereas in the definition above we only consider quasi-compact *open* sets.

Lemma 5.11.2. *A composition of quasi-compact maps is quasi-compact.*

Proof. Omitted. □

Lemma 5.11.3. *A closed subset of a quasi-compact topological space is quasi-compact.*

Proof. Let $E \subset X$ be a closed subset of the quasi-compact space X. Let $E = \bigcup V_j$ be an open covering. Choose $U_j \subset X$ open such that $V_j = E \cap U_j$. Then $X = (X \setminus E) \cup \bigcup U_j$ is an open covering of X. Hence $X = (X \setminus E) \cup U_{j_1} \cup \ldots \cup U_{j_n}$ for some n and indices j_i. Thus $E = V_{j_1} \cup \ldots \cup V_{j_n}$ as desired. □

Lemma 5.11.4. *Let X be a Hausdorff topological space.*

(1) *If $E \subset X$ is quasi-compact, then it is closed.*
(2) *If $E_1, E_2 \subset X$ are disjoint quasi-compact subsets then there exists opens $E_i \subset U_i$ with $U_1 \cap U_2 = \emptyset$.*

Proof. Proof of (1). Let $x \in X$, $x \notin E$. For every $e \in E$ we can find disjoint opens V_e and U_e with $e \in V_e$ and $x \in U_e$. Since $E \subset \bigcup V_e$ we can find finitely many $e_1, \ldots, e_n$ such that $E \subset V_{e_1} \cup \ldots \cup V_{e_n}$. Then $U = U_{e_1} \cap \ldots \cap U_{e_n}$ is an open neighbourhood of x which avoids $V_{e_1} \cup \cdots \cup V_{e_n}$. In particular it avoids E. Thus E is closed.

Proof of (2). In the proof of (1) we have seen that given $x \in E_1$ we can find an open neighbourhood $x \in U_x$ and an open $E_2 \subset V_x$ such that $U_x \cap V_x = \emptyset$. Because

E_1 is quasi-compact we can find a finite number $x_i \in E_1$ such that $E_1 \subset U = U_{x_1} \cup \ldots \cup U_{x_n}$. We take $V = V_{x_1} \cap \ldots \cap V_{x_n}$ to finish the proof. □

Lemma 5.11.5. *Let X be a quasi-compact Hausdorff space. Let $E \subset X$. The following are equivalent: (a) E is closed in X, (b) E is quasi-compact.*

Proof. The implication (a) ⇒ (b) is Lemma 5.11.3. The implication (b) ⇒ (a) is Lemma 5.11.4. □

The following is really a reformulation of the quasi-compact property.

Lemma 5.11.6. *Let X be a quasi-compact topological space. If $\{Z_\alpha\}_{\alpha \in A}$ is a collection of closed subsets such that the intersection of each finite subcollection is nonempty, then $\bigcap_{\alpha \in A} Z_\alpha$ is nonempty.*

Proof. Omitted. □

Lemma 5.11.7. *Let $f : X \to Y$ be a continuous map of topological spaces.*

(1) *If X is quasi-compact, then $f(X)$ is quasi-compact.*
(2) *If f is quasi-compact, then $f(X)$ is retrocompact.*

Proof. If $f(X) = \bigcup V_i$ is an open covering, then $X = \bigcup f^{-1}(V_i)$ is an open covering. Hence if X is quasi-compact then $X = f^{-1}(V_{i_1}) \cup \ldots \cup f^{-1}(V_{i_n})$ for some $i_1, \ldots, i_n \in I$ and hence $f(X) = V_{i_1} \cup \ldots \cup V_{i_n}$. This proves (1). Assume f is quasi-compact, and let $V \subset Y$ be quasi-compact open. Then $f^{-1}(V)$ is quasi-compact, hence by (1) we see that $f(f^{-1}(V)) = f(X) \cap V$ is quasi-compact. Hence $f(X)$ is retrocompact. □

Lemma 5.11.8. *Let X be a topological space. Assume that*

(1) *X is nonempty,*
(2) *X is quasi-compact, and*
(3) *X is Kolmogorov.*

Then X has a closed point.

Proof. Consider the set

$$\mathcal{T} = \{Z \subset X \mid Z = \overline{\{x\}} \text{ for some } x \in X\}$$

of all closures of singletons in X. It is nonempty since X is nonempty. Make $\mathcal{T}$ into a partially ordered set using the relation of inclusion. Suppose Z_α, $\alpha \in A$ is a totally ordered subset of $\mathcal{T}$. By Lemma 5.11.6 we see that $\bigcap_{\alpha \in A} Z_\alpha \neq \emptyset$. Hence there exists some $x \in \bigcap_{\alpha \in A} Z_\alpha$ and we see that $Z = \overline{\{x\}} \in \mathcal{T}$ is a lower bound for the family. By Zorn's lemma there exists a minimal element $Z \in \mathcal{T}$. As X is Kolmogorov we conclude that $Z = \{x\}$ for some x and $x \in X$ is a closed point. □

Lemma 5.11.9. *Let X be a quasi-compact Kolmogorov space. Then the set X_0 of closed points of X is quasi-compact.*

Proof. Let $X_0 = \bigcup U_{i,0}$ be an open covering. Write $U_{i,0} = X_0 \cap U_i$ for some open $U_i \subset X$. Consider the complement Z of $\bigcup U_i$. This is a closed subset of X, hence quasi-compact (Lemma 5.11.3) and Kolmogorov. By Lemma 5.11.8 if Z is nonempty it would have a closed point which contradicts the fact that $X_0 \subset \bigcup U_i$. Hence $Z = \emptyset$ and $X = \bigcup U_i$. Since X is quasi-compact this covering has a finite subcover and we conclude. □

Lemma 5.11.10. *Let X be a topological space. Assume*

(1) *X is quasi-compact,*
(2) *X has a basis for the topology consisting of quasi-compact opens, and*
(3) *the intersection of two quasi-compact opens is quasi-compact.*

For any $x \in X$ the connected component of X containing x is the intersection of all open and closed subsets of X containing x.

Proof. Let T be the connected component containing x. Let $S = \bigcap_{\alpha \in A} Z_\alpha$ be the intersection of all open and closed subsets Z_α of X containing x. Note that S is closed in X. Note that any finite intersection of Z_α's is a Z_α. Because T is connected and $x \in T$ we have $T \subset S$. It suffices to show that S is connected. If not, then there exists a disjoint union decomposition $S = B \coprod C$ with B and C open and closed in S. In particular, B and C are closed in X, and so quasi-compact by Lemma 5.11.3 and assumption (1). By assumption (2) there exist quasi-compact opens $U, V \subset X$ with $B = S \cap U$ and $C = S \cap V$ (details omitted). Then $U \cap V \cap S = \emptyset$. Hence $\bigcap_\alpha U \cap V \cap Z_\alpha = \emptyset$. By assumption (3) the intersection $U \cap V$ is quasi-compact. By Lemma 5.11.6 for some $\alpha \in A$ we have $U \cap V \cap Z_\alpha = \emptyset$. Hence $Z_\alpha = U \cap Z_\alpha \coprod V \cap Z_\alpha$ is a decomposition into two open pieces, hence $U \cap Z_\alpha$ and $V \cap Z_\alpha$ are open and closed in X. Thus, if $x \in B$ say, then we see that $S \subset U \cap Z_\alpha$ and we conclude that $C = \emptyset$. □

Lemma 5.11.11. *Let X be a topological space. Assume X is quasi-compact and Hausdorff. For any $x \in X$ the connected component of X containing x is the intersection of all open and closed subsets of X containing x.*

Proof. Let T be the connected component containing x. Let $S = \bigcap_{\alpha \in A} Z_\alpha$ be the intersection of all open and closed subsets Z_α of X containing x. Note that S is closed in X. Note that any finite intersection of Z_α's is a Z_α. Because T is connected and $x \in T$ we have $T \subset S$. It suffices to show that S is connected. If not, then there exists a disjoint union decomposition $S = B \coprod C$ with B and C open and closed in S. In particular, B and C are closed in X, and so quasi-compact by Lemma 5.11.3. By Lemma 5.11.4 there exist disjoint opens $U, V \subset X$ with $B \subset U$ and $C \subset V$. Then $X \setminus U \cup V$ is closed in X hence quasi-compact (Lemma 5.11.3). It follows that $(X \setminus U \cup V) \cap Z_\alpha = \emptyset$ for some α by Lemma 5.11.6. In other words, $Z_\alpha \subset U \cup V$. Thus $Z_\alpha = Z_\alpha \cap V \amalg Z_\alpha \cap U$ is a decomposition into two open pieces, hence $U \cap Z_\alpha$ and $V \cap Z_\alpha$ are open and closed in X. Thus, if $x \in B$ say, then we see that $S \subset U \cap Z_\alpha$ and we conclude that $C = \emptyset$. □

Lemma 5.11.12. *Let X be a topological space. Assume*

(1) *X is quasi-compact,*
(2) *X has a basis for the topology consisting of quasi-compact opens, and*
(3) *the intersection of two quasi-compact opens is quasi-compact.*

For a subset $T \subset X$ the following are equivalent:

(a) *T is an intersection of open and closed subsets of X, and*
(b) *T is closed in X and is a union of connected components of X.*

Proof. It is clear that (a) implies (b). Assume (b). Let $x \in X$, $x \notin T$. Let $x \in C \subset X$ be the connected component of X containing x. By Lemma 5.11.10 we see that $C = \bigcap V_\alpha$ is the intersection of all open and closed subsets V_α of X which contain C. In particular, any pairwise intersection $V_\alpha \cap V_\beta$ occurs as a V_α. As T is

a union of connected components of X we see that $C \cap T = \emptyset$. Hence $T \cap \bigcap V_\alpha = \emptyset$. Since T is quasi-compact as a closed subset of a quasi-compact space (see Lemma 5.11.3) we deduce that $T \cap V_\alpha = \emptyset$ for some α, see Lemma 5.11.6. For this α we see that $U_\alpha = X \setminus V_\alpha$ is an open and closed subset of X which contains T and not x. The lemma follows. □

Lemma 5.11.13. *Let X be a Noetherian topological space.*

(1) *The space X is quasi-compact.*
(2) *Any subset of X is retrocompact.*

Proof. Suppose $X = \bigcup U_i$ is an open covering of X indexed by the set I which does not have a refinement by a finite open covering. Choose $i_1, i_2, \ldots$ elements of I inductively in the following way: If $X \neq U_{i_1} \cup \ldots \cup U_{i_n}$ then choose i_{n+1} such that $U_{i_{n+1}}$ is not contained in $U_{i_1} \cup \ldots \cup U_{i_n}$. Thus we see that $X \supset (X \setminus U_{i_1}) \supset (X \setminus U_{i_1} \cup U_{i_2}) \supset \ldots$ is a strictly decreasing infinite sequence of closed subsets. This contradicts the fact that X is Noetherian. This proves the first assertion. The second assertion is now clear since every subset of X is Noetherian by Lemma 5.8.2. □

Lemma 5.11.14. *A quasi-compact locally Noetherian space is Noetherian.*

Proof. The conditions imply immediately that X has a finite covering by Noetherian subsets, and hence is Noetherian by Lemma 5.8.4. □

Lemma 5.11.15 (Alexander subbase theorem)**.** *Let X be a topological space. Let $\mathcal{B}$ be a subbase for X. If every covering of X by elements of $\mathcal{B}$ has a finite refinement, then X is quasi-compact.*

Proof. Assume there is an open covering of X which does not have a finite refinement. Using Zorn's lemma we can choose a maximal open covering $X = \bigcup_{i \in I} U_i$ which does not have a finite refinement (details omitted). In other words, if $U \subset X$ is any open which does not occur as one of the U_i, then the covering $X = U \cup \bigcup_{i \in I} U_i$ does have a finite refinement. Let $I' \subset I$ be the set of indices such that $U_i \in \mathcal{B}$. Then $\bigcup_{i \in I'} U_i \neq X$, since otherwise we would get a finite refinement covering X by our assumption on $\mathcal{B}$. Pick $x \in X$, $x \notin \bigcup_{i \in I'} U_i$. Pick $i \in I$ with $x \in U_i$. Pick $V_1, \ldots, V_n \in \mathcal{B}$ such that $x \in V_1 \cap \ldots \cap V_n \subset U_i$. This is possible as $\mathcal{B}$ is a subbasis for X. Note that V_j does not occur as a U_i. By maximality of the chosen covering we see that for each j there exist $i_{j,1}, \ldots, i_{j,n_j} \in I$ such that $X = V_j \cup U_{i_{j,1}} \cup \ldots \cup U_{i_{j,n_j}}$. Since $V_1 \cap \ldots \cap V_n \subset U_i$ we conclude that $X = U_i \cup \bigcup U_{i_{j,l}}$ a contradiction. □

5.12. Locally quasi-compact spaces

Recall that a neighbourhood of a point need not be open.

Definition 5.12.1. A topological space X is called *locally quasi-compact*[2] if every point has a fundamental system of quasi-compact neighbourhoods.

The term *locally compact space* in the literature often refers to a space as in the following lemma.

[2]This may not be standard notation. Alternative notions used in the literature are: (1) Every point has some quasi-compact neighbourhood, and (2) Every point has a closed quasi-compact neighbourhood. A scheme has the property that every point has a fundamental system of open quasi-compact neighbourhoods.

Lemma 5.12.2. *A Hausdorff space is locally quasi-compact if and only if every point has a quasi-compact neighbourhood.*

Proof. Let X be a Hausdorff space. Let $x \in X$ and let $x \in E \subset X$ be a quasi-compact neighbourhood. Then E is closed by Lemma 5.11.4. Suppose that $x \in U \subset X$ is an open neighbourhood of x. Then $Z = E \setminus U$ is a closed subset of E not containing x. Hence we can find a pair of disjoint open subsets $W, V \subset E$ of E such that $x \in V$ and $Z \subset W$, see Lemma 5.11.4. It follows that $\overline{V} \subset E$ is a closed neighbourhood of x contained in $E \cap U$. Also $\overline{V}$ is quasi-compact as a closed subset of E (Lemma 5.11.3). In this way we obtain a fundamental system of quasi-compact neighbourhoods of x. □

Lemma 5.12.3. *Let X be a Hausdorff and quasi-compact space. Let $X = \bigcup_{i \in I} U_i$ be an open covering. Then there exists an open covering $X = \bigcup_{i \in I} V_i$ such that $\overline{V_i} \subset U_i$ for all i.*

Proof. Let $x \in X$. Choose an $i(x) \in I$ such that $x \in U_{i(x)}$. Since $X \setminus U_{i(x)}$ and $\{x\}$ are disjoint closed subsets of X, by Lemmas 5.11.3 and 5.11.4 there exists an open neighbourhood U_x of x whose closure is disjoint from $X \setminus U_{i(x)}$. Thus $\overline{U_x} \subset U_{i(x)}$. Since X is quasi-compact, there is a finite list of points $x_1, \ldots, x_m$ such that $X = U_{x_1} \cup \ldots \cup U_{x_m}$. Setting $V_i = \bigcup_{i=i(x_j)} U_{x_j}$ the proof is finished. □

Lemma 5.12.4. *Let X be a Hausdorff and quasi-compact space. Let $X = \bigcup_{i \in I} U_i$ be an open covering. Suppose given an integer $p \geq 0$ and for every $(p+1)$-tuple $i_0, \ldots, i_p$ of I an open covering $U_{i_0} \cap \ldots \cap U_{i_p} = \bigcup W_{i_0 \ldots i_p, k}$. Then there exists an open covering $X = \bigcup_{j \in J} V_j$ and a map $\alpha : J \to I$ such that $\overline{V_j} \subset U_{\alpha(j)}$ and such that each $V_{j_0} \cap \ldots \cap V_{j_p}$ is contained in $W_{\alpha(j_0) \ldots \alpha(j_p), k}$ for some k.*

Proof. Since X is quasi-compact, there is a reduction to the case where I is finite (details omitted). We prove the result for I finite by induction on p. The base case $p = 0$ is immediate by taking a covering as in Lemma 5.12.3 refining the open covering $X = \bigcup W_{i_0, k}$.

Induction step. Assume the lemma proven for $p-1$. For all p-tuples $i'_0, \ldots, i'_{p-1}$ of I let $U_{i'_0} \cap \ldots \cap U_{i'_{p-1}} = \bigcup W_{i'_0 \ldots i'_{p-1}, k}$ be a common refinement of the coverings $U_{i_0} \cap \ldots \cap U_{i_p} = \bigcup W_{i_0 \ldots i_p, k}$ for those $(p+1)$-tuples such that $\{i'_0, \ldots, i'_{p-1}\} = \{i_0, \ldots, i_p\}$ (equality of sets). (There are finitely many of these as I is finite.) By induction there exists a solution for these opens, say $X = \bigcup V_j$ and $\alpha : J \to I$. At this point the covering $X = \bigcup_{j \in J} V_j$ and α satisfies $\overline{V_j} \subset U_{\alpha(j)}$ and each $V_{j_0} \cap \ldots \cap V_{j_p}$ is contained in $W_{\alpha(j_0) \ldots \alpha(j_p), k}$ for some k if there is a repetition in $\alpha(j_0), \ldots, \alpha(j_p)$. Of course, we may and do assume that J is finite.

Fix $i_0, \ldots, i_p \in I$ pairwise distinct. Consider $(p+1)$-tuples $j_0, \ldots, j_p \in J$ with $i_0 = \alpha(j_0), \ldots, i_p = \alpha(j_p)$ such that $V_{j_0} \cap \ldots \cap V_{j_p}$ is **not** contained in $W_{\alpha(j_0) \ldots \alpha(j_p), k}$ for any k. Let N be the number of such $(p+1)$-tuples. We will show how to decrease N. Since

$$\overline{V_{j_0}} \cap \ldots \cap \overline{V_{j_p}} \subset U_{i_0} \cap \ldots \cap U_{i_p} = \bigcup W_{i_0 \ldots i_p, k}$$

we find a finite set $K = \{k_1, \ldots, k_t\}$ such that the LHS is contained in $\bigcup_{k \in K} W_{i_0 \ldots i_p, k}$. Then we consider the open covering

$$V_{j_0} = (V_{j_0} \setminus \overline{V_{j_1}} \cup \ldots \cup \overline{V_{j_p}}) \cup (\bigcup\nolimits_{k \in K} V_{j_0} \cap W_{i_0 \ldots i_p, k})$$

The first open on the RHS intersects $V_{j_1} \cap \ldots \cap V_{j_p}$ in the empty set and the other opens $V_{j_0,k}$ of the RHS satisfy $V_{j_0,k} \cap V_{j_1} \ldots \cap V_{j_p} \subset W_{\alpha(j_0)\ldots\alpha(j_p),k}$. Set $J' = J \amalg K$. For $j \in J$ set $V'_j = V_j$ if $j \neq j_0$ and set $V'_{j_0} = V_{j_0} \setminus \overline{V_{j_1}} \cup \ldots \cup \overline{V_{j_p}}$. For $k \in K$ set $V'_k = V_{j_0,k}$. Finally, the map $\alpha' : J' \to I$ is given by α on J and maps every element of K to i_0. A simple check shows that N has decreased by one under this replacement. Repeating this procedure N times we arrive at the situation where $N = 0$.

To finish the proof we argue by induction on the number M of $(p+1)$-tuples $i_0, \ldots, i_p \in I$ with pairwise distinct entries for which there exists a $(p+1)$-tuple $j_0, \ldots, j_p \in J$ with $i_0 = \alpha(j_0), \ldots, i_p = \alpha(j_p)$ such that $V_{j_0} \cap \ldots \cap V_{j_p}$ is **not** contained in $W_{\alpha(j_0)\ldots\alpha(j_p),k}$ for any k. To do this, we claim that the operation performed in the previous paragraph does not increase M. This follows formally from the fact that the map $\alpha' : J' \to I$ factors through a map $\beta : J' \to J$ such that $V'_{j'} \subset V_{\beta(j')}$. □

Lemma 5.12.5. *Let X be a Hausdorff and locally quasi-compact space. Let $Z \subset X$ be a quasi-compact (hence closed) subset. Suppose given an integer $p \geq 0$, a set I, for every $i \in I$ an open $U_i \subset X$, and for every $(p+1)$-tuple $i_0, \ldots, i_p$ of I an open $W_{i_0\ldots i_p} \subset U_{i_0} \cap \ldots \cap U_{i_p}$ such that*

(1) *$Z \subset \bigcup U_i$, and*
(2) *for every $i_0, \ldots, i_p$ we have $W_{i_0\ldots i_p} \cap Z = U_{i_0} \cap \ldots \cap U_{i_p} \cap Z$.*

Then there exist opens V_i of X such that we have $Z \subset \bigcup V_i$, for all i we have $\overline{V_i} \subset U_i$, and we have $V_{i_0} \cap \ldots \cap V_{i_p} \subset W_{i_0\ldots i_p}$ for all $(p+1)$-tuples $i_0, \ldots, i_p$.

Proof. Since Z is quasi-compact, there is a reduction to the case where I is finite (details omitted). Because X is locally quasi-compact and Z is quasi-compact, we can find a neighbourhood $Z \subset E$ which is quasi-compact, i.e., E is quasi-compact and contains an open neighbourhood of Z in X. If we prove the result after replacing X by E, then the result follows. Hence we may assume X is quasi-compact.

We prove the result in case I is finite and X is quasi-compact by induction on p. The base case is $p = 0$. In this case we have $X = (X \setminus Z) \cup \bigcup W_i$. By Lemma 5.12.3 we can find a covering $X = V \cup \bigcup V_i$ by opens $V_i \subset W_i$ and $V \subset X \setminus Z$ with $\overline{V_i} \subset W_i$ for all i. Then we see that we obtain a solution of the problem posed by the lemma.

Induction step. Assume the lemma proven for $p-1$. Set $W_{j_0\ldots j_{p-1}}$ equal to the intersection of all $W_{i_0\ldots i_p}$ with $\{j_0, \ldots, j_{p-1}\} = \{i_0, \ldots, i_p\}$ (equality of sets). By induction there exists a solution for these opens, say $V_i \subset U_i$. It follows from our choice of $W_{j_0\ldots j_{p-1}}$ that we have $V_{i_0} \cap \ldots \cap V_{i_p} \subset W_{i_0\ldots i_p}$ for all $(p+1)$-tuples $i_0, \ldots, i_p$ where $i_a = i_b$ for some $0 \leq a < b \leq p$. Thus we only need to modify our choice of V_i if $V_{i_0} \cap \ldots \cap V_{i_p} \not\subset W_{i_0\ldots i_p}$ for some $(p+1)$-tuple $i_0, \ldots, i_p$ with pairwise distinct elements. In this case we have

$$T = \overline{V_{i_0} \cap \ldots \cap V_{i_p} \setminus W_{i_0\ldots i_p}} \subset \overline{V_{i_0}} \cap \ldots \cap \overline{V_{i_p}} \setminus W_{i_0\ldots i_p}$$

is a closed subset of X contained in $U_{i_0} \cap \ldots \cap U_{i_p}$ not meeting Z. Hence we can replace V_{i_0} by $V_{i_0} \setminus T$ to "fix" the problem. After repeating this finitely many times for each of the problem tuples, the lemma is proven. □

5.13. Limits of spaces

The category of topological spaces has products. Namely, if I is a set and for $i \in I$ we are given a topological space X_i then we endow $\prod_{i\in I} X_i$ with the *product topology*. As a basis for the topology we use sets of the form $\prod U_i$ where $U_i \subset X_i$ is open and $U_i = X_i$ for almost all i.

The category of topological spaces has equalizers. Namely, if $a, b : X \to Y$ are morphisms of topological spaces, then the equalizer of a and b is the subset $\{x \in X \mid a(x) = b(x)\} \subset X$ endowed with the induced topology.

Lemma 5.13.1. *The category of topological spaces has limits.*

Proof. This follows from the discussion above and Categories, Lemma 4.14.10. □

Theorem 5.13.2 (Tychonov)**.** *A product of quasi-compact spaces is quasi-compact.*

Proof. Let I be a set and for $i \in I$ let X_i be a quasi-compact topological space. Set $X = \prod X_i$. Let $\mathcal{B}$ be the set of subsets of X of the form $U_i \times \prod_{i'\in I, i'\neq i} X_{i'}$ where $U_i \subset X_i$ is open. By construction this family is a subbasis for the topology on X. By Lemma 5.11.15 it suffices to show that any covering $X = \bigcup_{j\in J} B_j$ by elements B_j of $\mathcal{B}$ has a finite refinement. We can decompose $J = \coprod J_i$ so that if $j \in J_i$, then $B_j = U_j \times \prod_{i'\neq i} X_{i'}$ with $U_j \subset X_i$ open. If $X_i = \bigcup_{j\in J_i} U_j$, then there is a finite refinement and we conclude that $X = \bigcup_{j\in J} B_j$ has a finite refinement. If this is not the case, the for every i we can choose an point $x_i \in X_i$ which is not in $\bigcup_{j\in J_i} U_j$. But then the point $x = (x_i)_{i\in I}$ is an element of X not contained in $\bigcup_{j\in J} B_j$, a contradiction. □

The following lemma does not hold if one drops the assumption that the spaces X_i are Hausdorff, see Examples, Section 78.2.

Lemma 5.13.3. *Let $\mathcal{I}$ be a category and let $i \mapsto X_i$ be a diagram over $\mathcal{I}$ in the category of topological spaces. If each X_i is quasi-compact and Hausdorff, then* $\lim X_i$ *is quasi-compact.*

Proof. Recall that $\lim X_i$ is a subspace of $\prod X_i$. By Theorem 5.13.2 this product is quasi-compact. Hence it suffices to show that $\lim X_i$ is a closed subspace of $\prod X_i$ (Lemma 5.11.3). If $\varphi : j \to k$ is a morphism of $\mathcal{I}$, then let $\Gamma_\varphi \subset X_j \times X_k$ denote the graph of the corresponding continuous map $X_j \to X_k$. By Lemma 5.3.2 this graph is closed. It is clear that $\lim X_i$ is the intersection of the closed subsets

$$\Gamma_\varphi \times \prod\nolimits_{l\neq j,k} X_l \subset \prod X_i$$

Thus the result follows. □

5.14. Constructible sets

Definition 5.14.1. Let X be a topological space. Let $E \subset X$ be a subset of X.

(1) We say E is *constructible*[3] in X if E is a finite union of subsets of the form $U \cap V^c$ where $U, V \subset X$ are open and retrocompact in X.
(2) We say E is *locally constructible* in X if there exists an open covering $X = \bigcup V_i$ such that each $E \cap V_i$ is constructible in V_i.

[3]In the second edition of EGA I [**GD71**] this was called a "globally constructible" set and a the terminology "constructible" was used for what we call a locally constructible set.

Lemma 5.14.2. *The collection of constructible sets is closed under finite intersections, finite unions and complements.*

Proof. Note that if U_1, U_2 are open and retrocompact in X then so is $U_1 \cup U_2$ because the union of two quasi-compact subsets of X is quasi-compact. It is also true that $U_1 \cap U_2$ is retrocompact. Namely, suppose $U \subset X$ is quasi-compact open, then $U_2 \cap U$ is quasi-compact because U_2 is retrocompact in X, and then we conclude $U_1 \cap (U_2 \cap U)$ is quasi-compact because U_1 is retrocompact in X. From this it is formal to show that the complement of a constructible set is constructible, that finite unions of constructibles are constructible, and that finite intersections of constructibles are constructible. □

Lemma 5.14.3. *Let $f : X \to Y$ be a continuous map of topological spaces. If the inverse image of every retrocompact open subset of Y is retrocompact in X, then inverse images of constructible sets are constructible.*

Proof. This is true because $f^{-1}(U \cap V^c) = f^{-1}(U) \cap f^{-1}(V)^c$, combined with the definition of constructible sets. □

Lemma 5.14.4. *Let $U \subset X$ be open. For a constructible set $E \subset X$ the intersection $E \cap U$ is constructible in U.*

Proof. Suppose that $V \subset X$ is retrocompact open in X. It suffices to show that $V \cap U$ is retrocompact in U by Lemma 5.14.3. To show this let $W \subset U$ be open and quasi-compact. Then W is open and quasi-compact in X. Hence $V \cap W = V \cap U \cap W$ is quasi-compact as V is retrocompact in X. □

Lemma 5.14.5. *Let $U \subset X$ be a retrocompact open. Let $E \subset U$. If E is constructible in U, then E is constructible in X.*

Proof. Suppose that $V, W \subset U$ are retrocompact open in U. Then V, W are retrocompact open in X (Lemma 5.11.2). Hence $V \cap (U \setminus W) = V \cap (X \setminus W)$ is constructible in X. We conclude since every constructible subset of U is a finite union of subsets of the form $V \cap (U \setminus W)$. □

Lemma 5.14.6. *Let X be a topological space. Let $E \subset X$ be a subset. Let $X = V_1 \cup \ldots \cup V_m$ be a finite covering by retrocompact opens. Then E is constructible in X if and only if $E \cap V_j$ is constructible in V_j for each $j = 1, \ldots, m$.*

Proof. If E is constructible in X, then by Lemma 5.14.4 we see that $E \cap V_j$ is constructible in V_j for all j. Conversely, suppose that $E \cap V_j$ is constructible in V_j for each $j = 1, \ldots, m$. Then $E = \bigcup E \cap V_j$ is a finite union of constructible sets by Lemma 5.14.5 and hence constructible. □

Lemma 5.14.7. *Let X be a topological space. Let $Z \subset X$ be a closed subset such that $X \setminus Z$ is quasi-compact. Then for a constructible set $E \subset X$ the intersection $E \cap Z$ is constructible in Z.*

Proof. Suppose that $V \subset X$ is retrocompact open in X. It suffices to show that $V \cap Z$ is retrocompact in Z by Lemma 5.14.3. To show this let $W \subset Z$ be open and quasi-compact. The subset $W' = W \cup (X \setminus Z)$ is quasi-compact, open, and $W = Z \cap W'$. Hence $V \cap Z \cap W = V \cap Z \cap W'$ is a closed subset of the quasi-compact open $V \cap W'$ as V is retrocompact in X. Thus $V \cap Z \cap W$ is quasi-compact by Lemma 5.11.3. □

Lemma 5.14.8. *Let X be a topological space. Let $T \subset X$ be a subset. Suppose*
(1) *T is retrocompact in X,*
(2) *quasi-compact opens form a basis for the topology on X.*
Then for a constructible set $E \subset X$ the intersection $E \cap T$ is constructible in T.

Proof. Suppose that $V \subset X$ is retrocompact open in X. It suffices to show that $V \cap T$ is retrocompact in T by Lemma 5.14.3. To show this let $W \subset T$ be open and quasi-compact. By assumption (2) we can find a quasi-compact open $W' \subset X$ such that $W = T \cap W'$ (details omitted). Hence $V \cap T \cap W = V \cap T \cap W'$ is the intersection of T with the quasi-compact open $V \cap W'$ as V is retrocompact in X. Thus $V \cap T \cap W$ is quasi-compact. □

Lemma 5.14.9. *Let $Z \subset X$ be a closed subset whose complement is retrocompact open. Let $E \subset Z$. If E is constructible in Z, then E is constructible in X.*

Proof. Suppose that $V \subset Z$ is retrocompact open in Z. Consider the open subset $\tilde{V} = V \cup (X \setminus Z)$ of X. Let $W \subset X$ be quasi-compact open. Then

$$W \cap \tilde{V} = (V \cap W) \cup ((X \setminus Z) \cap W).$$

The first part is quasi-compact as $V \cap W = V \cap (Z \cap W)$ and $(Z \cap W)$ is quasi-compact open in Z (Lemma 5.11.3) and V is retrocompact in Z. The second part is quasi-compact as $(X \setminus Z)$ is retrocompact in X. In this way we see that $\tilde{V}$ is retrocompact in X. Thus if $V_1, V_2 \subset Z$ are retrocompact open, then

$$V_1 \cap (Z \setminus V_2) = \tilde{V}_1 \cap (X \setminus \tilde{V}_2)$$

is constructible in X. We conclude since every constructible subset of Z is a finite union of subsets of the form $V_1 \cap (Z \setminus V_2)$. □

Lemma 5.14.10. *Let X be a topological space. Every constructible subset of X is retrocompact.*

Proof. Let $E = \bigcup_{i=1,\ldots,n} U_i \cap V_i^c$ with U_i, V_i retrocompact open in X. Let $W \subset X$ be quasi-compact open. Then $E \cap W = \bigcup_{i=1,\ldots,n} U_i \cap V_i^c \cap W$. Thus it suffices to show that $U \cap V^c \cap W$ is quasi-compact if U, V are retrocompact open and W is quasi-compact open. This is true because $U \cap V^c \cap W$ is a closed subset of the quasi-compact $U \cap W$ so Lemma 5.11.3 applies. □

Question: Does the following lemma also hold if we assume X is a quasi-compact topological space? Compare with Lemma 5.14.7.

Lemma 5.14.11. *Let X be a topological space. Assume X has a basis consisting of quasi-compact opens. For E, E' constructible in X, the intersection $E \cap E'$ is constructible in E.*

Proof. Combine Lemmas 5.14.8 and 5.14.10. □

Lemma 5.14.12. *Let X be a topological space. Assume X has a basis consisting of quasi-compact opens. Let E be constructible in X and $F \subset E$ constructible in E. Then F is constructible in X.*

Proof. Observe that any retrocompact subset T of X has a basis for the topology consisting of quasi-compact opens. In particular this holds for any constructible subset (Lemma 5.14.10). Write $E = E_1 \cup \ldots \cup E_n$ with $E_i = U_i \cap V_i^c$ where $U_i, V_i \subset X$ are retrocompact open. Note that $E_i = E \cap E_i$ is constructible in E by

Lemma 5.14.11. Hence $F \cap E_i$ is constructible in E_i by Lemma 5.14.11. Thus it suffices to prove the lemma in case $E = U \cap V^c$ where $U, V \subset X$ are retrocompact open. In this case the inclusion $E \subset X$ is a composition

$$E = U \cap V^c \to U \to X$$

Then we can apply Lemma 5.14.9 to the first inclusion and Lemma 5.14.5 to the second. □

Lemma 5.14.13. *Let X be a topological space which has a basis for the topology consisting of quasi-compact opens. Let $E \subset X$ be a subset. Let $X = E_1 \cup \ldots \cup E_m$ be a finite covering by constructible subsets. Then E is constructible in X if and only if $E \cap E_j$ is constructible in E_j for each $j = 1, \ldots, m$.*

Proof. Combine Lemmas 5.14.11 and 5.14.12. □

Lemma 5.14.14. *Let X be a topological space. Suppose that $Z \subset X$ is irreducible. Let $E \subset X$ be a finite union of locally closed subsets (e.g. E is constructible). The following are equivalent*

(1) *The intersection $E \cap Z$ contains an open dense subset of Z.*
(2) *The intersection $E \cap Z$ is dense in Z.*

If Z has a generic point ξ, then this is also equivalent to

(3) *We have $\xi \in E$.*

Proof. Write $E = \bigcup U_i \cap Z_i$ as the finite union of intersections of open sets U_i and closed sets Z_i. Suppose that $E \cap Z$ is dense in Z. Note that the closure of $E \cap Z$ is the union of the closures of the intersections $U_i \cap Z_i \cap Z$. As Z is irreducible we conclude that the closure of $U_i \cap Z_i \cap Z$ is Z for some i. Fix such an i. It follows that $Z \subset Z_i$ since otherwise the closed subset $Z \cap Z_i$ of Z would not be dense in Z. Then $U_i \cap Z_i \cap Z = U_i \cap Z$ is an open nonempty subset of Z. Because Z is irreducible, it is open dense. Hence $E \cap Z$ contains an open dense subset of Z. The converse is obvious.

Suppose that $\xi \in Z$ is a generic point. Of course if (1) $\Leftrightarrow$ (2) holds, then $\xi \in E$. Conversely, if $\xi \in E$, then $\xi \in U_i \cap Z_i$ for some $i = i_0$. Clearly this implies $Z \subset Z_{i_0}$ and hence $U_{i_0} \cap Z_{i_0} \cap Z = U_{i_0} \cap Z$ is an open not empty subset of Z. We conclude as before. □

5.15. Constructible sets and Noetherian spaces

Lemma 5.15.1. *Let X be a Noetherian topological space. Constructible sets in X are finite unions of locally closed subsets of X.*

Proof. This follows immediately from Lemma 5.11.13. □

Lemma 5.15.2. *Let $f : X \to Y$ be a continuous map of Noetherian topological spaces. If $E \subset Y$ is constructible in Y, then $f^{-1}(E)$ is constructible in X.*

Proof. Follows immediately from Lemma 5.15.1 and the definition of a continuous map. □

Lemma 5.15.3. *Let X be a Noetherian topological space. Let $E \subset X$ be a subset. The following are equivalent*

(1) *E is constructible in X, and*

(2) *for every irreducible closed $Z \subset X$ the intersection $E \cap Z$ either contains a nonempty open of Z or is not dense in Z.*

Proof. Assume E is constructible and $Z \subset X$ irreducible closed. Then $E \cap Z$ is constructible in Z by Lemma 5.15.2. Hence $E \cap Z$ is a finite union of nonempty locally closed subsets T_i of Z. Clearly if none of the T_i is open in Z, then $E \cap Z$ is not dense in Z. In this way we see that (1) implies (2).

Conversely, assume (2) holds. Consider the set $\mathcal{S}$ of closed subsets Y of X such that $E \cap Y$ is not constructible in Y. If $\mathcal{S} \neq \emptyset$, then it has a smallest element Y as X is Noetherian. Let $Y = Y_1 \cup \ldots \cup Y_r$ be the decomposition of Y into its irreducible components, see Lemma 5.8.2. If $r > 1$, then each $Y_i \cap E$ is constructible in Y_i and hence a finite union of locally closed subsets of Y_i. Thus $E \cap Y$ is a finite union of locally closed subsets of Y too and we conclude that $E \cap Y$ is constructible in Y by Lemma 5.15.1. This is a contradiction and so $r = 1$. If $r = 1$, then Y is irreducible, and by assumption (2) we see that $E \cap Y$ either (a) contains an open V of Y or (b) is not dense in Y. In case (a) we see, by minimality of Y, that $E \cap (Y \setminus V)$ is a finite union of locally closed subsets of $Y \setminus V$. Thus $E \cap Y$ is a finite union of locally closed subsets of Y and is constructible by Lemma 5.15.1. This is a contradiction and so we must be in case (b). In case (b) we see that $E \cap Y = E \cap Y'$ for some proper closed subset $Y' \subset Y$. By minimality of Y we see that $E \cap Y'$ is a finite union of locally closed subsets of Y' and we see that $E \cap Y' = E \cap Y$ is a finite union of locally closed subsets of Y and is constructible by Lemma 5.15.1. This contradiction finishes the proof of the lemma. □

Lemma 5.15.4. *Let X be a Noetherian topological space. Let $x \in X$. Let $E \subset X$ be constructible in X. The following are equivalent*

(1) *E is a neighbourhood of x, and*
(2) *for every irreducible closed subset Y of X which contains x the intersection $E \cap Y$ is dense in Y.*

Proof. It is clear that (1) implies (2). Assume (2). Consider the set $\mathcal{S}$ of closed subsets Y of X containing x such that $E \cap Y$ is not a neighbourhood of x in Y. If $\mathcal{S} \neq \emptyset$, then it has a smallest element Y as X is Noetherian. Let $Y = Y_1 \cup \ldots \cup Y_r$ be the decomposition of Y into its irreducible components, see Lemma 5.8.2. If $r > 1$, then each $Y_i \cap E$ is a neighbourhood of x in Y_i by minimality of Y. Thus $E \cap Y$ is a neighbourhood of x in Y. This is a contradiction and so $r = 1$. If $r = 1$, then Y is irreducible, and by assumption (2) we see that $E \cap Y$ is dense in Y. Thus $E \cap Y$ contains an open V of Y, see Lemma 5.15.3. If $x \in V$ then $E \cap Y$ is a neighbourhood of x in Y which is a contradiction. If $x \notin V$, then $Y' = Y \setminus V$ is a proper closed subset of Y containing x. By minimality of Y we see that $E \cap Y'$ contains an open neighbourhood $V' \subset Y'$ of x in Y'. But then $V' \cup V$ is an open neighbourhood of x in Y contained in E, a contradiction. This contradiction finishes the proof of the lemma. □

Lemma 5.15.5. *Let X be a Noetherian topological space. Let $E \subset X$ be a subset. The following are equivalent*

(1) *E is open in X, and*
(2) *for every irreducible closed subset Y of X the intersection $E \cap Y$ is either empty or contains a nonempty open of Y.*

Proof. This follows formally from Lemmas 5.15.3 and 5.15.4. □

5.16. Characterizing proper maps

We include a section discussing the notion of a proper map in usual topology. It turns out that in topology, the notion of being proper is the same as the notion of being universally closed, in the sense that any base change is a closed morphism (not just taking products with spaces). The reason for doing this is that in algebraic geometry we use this notion of universal closedness as the basis for our definition of properness.

Lemma 5.16.1 (Tube lemma)**.** *Let X and Y be topological spaces. Let $A \subset X$ and $B \subset Y$ be quasi-compact subsets. Let $A \times B \subset W \subset X \times Y$ with W open in $X \times Y$. Then there exists opens $A \subset U \subset X$ and $B \subset V \subset Y$ such that $U \times V \subset W$.*

Proof. For every $a \in A$ and $b \in B$ there exist opens $U_{(a,b)}$ of X and $V_{(a,b)}$ of Y such that $(a, b) \in U_{(a,b)} \times V_{(a,b)} \subset W$. Fix b and we see there exist a finite number $a_1, \ldots, a_n$ such that $A \subset U_{(a_1,b)} \cup \ldots \cup U_{(a_n,b)}$. Hence $A \times \{b\} \subset (U_{(a_1,b)} \cup \ldots \cup U_{(a_n,b)}) \times (V_{(a_1,b)} \cap \ldots \cap V_{(a_n,b)}) \subset W$. Thus for every $b \in B$ there exists opens $U_b \subset X$ and $V_b \subset Y$ such that $A \times \{b\} \subset U_b \times V_b \subset W$. As above there exist a finite number $b_1, \ldots, b_m$ such that $B \subset V_{b_1} \cup \ldots \cup V_{b_m}$. Then we win because $A \times B \subset (U_{b_1} \cap \ldots \cap U_{b_m}) \times (V_{b_1} \cup \ldots \cup V_{b_m})$. □

The notation in the following definition may be slightly different from what you are used to.

Definition 5.16.2. Let $f : X \to Y$ be a continuous map between topological spaces.

(1) We say that the map f is *closed* iff the image of every closed subset is closed.
(2) We say that the map f is *proper*[4] iff the map $Z \times X \to Z \times Y$ is closed for any topological space Z.
(3) We say that the map f is *quasi-proper* iff the inverse image $f^{-1}(V)$ of every quasi-compact subset $V \subset Y$ is quasi-compact.
(4) We say that f is *universally closed* iff the map $f' : Z \times_Y X \to Z$ is closed for any map $g : Z \to Y$.

The following lemma is useful later.

Lemma 5.16.3. *A topological space X is quasi-compact if and only if the projection map $Z \times X \to Z$ is closed for any topological space Z.*

Proof. (See also remark below.) If X is not quasi-compact, there exists an open covering $X = \bigcup_{i \in I} U_i$ such that no finite number of U_i cover X. Let Z be the subset of the power set $\mathcal{P}(I)$ of I consisting of I and all nonempty finite subsets of I. Define a topology on Z with as a basis for the topology the following sets:

(1) All subsets of $Z \setminus \{I\}$.
(2) The empty set.
(3) For every finite subset K of I the set $U_K := \{J \subset I \mid J \in Z,\ K \subset J\})$.

It is left to the reader to verify this is the basis for a topology. Consider the subset of $Z \times X$ defined by the formula

$$M = \{(J, x) \mid J \in Z,\ x \in \bigcap\nolimits_{i \in J} U_i^c)\}$$

[4]This is the terminology used in [**Bou71**]. Usually this is what is called "universally closed" in the literature. Thus our notion of proper does not involve any separation conditions.

If $(J, x) \notin M$, then $x \in U_i$ for some $i \in J$. Hence $U_{\{i\}} \times U_i \subset Z \times X$ is an open subset containing (J, x) and not intersecting M. Hence M is closed. The projection of M to Z is $Z - \{I\}$ which is not closed. Hence $Z \times X \to Z$ is not closed.

Assume X is quasi-compact. Let Z be a topological space. Let $M \subset Z \times X$ be closed. Let $z \in Z$ be a point which is not in $\mathrm{pr}_1(M)$. By the Tube Lemma 5.16.1 there exists an open $U \subset Z$ such that $U \times X$ is contained in the complement of M. Hence $\mathrm{pr}_1(M)$ is closed. □

Remark 5.16.4. Lemma 5.16.3 is a combination of [**Bou71**, I, p. 75, Lemme 1] and [**Bou71**, I, p. 76, Corrolaire 1].

Theorem 5.16.5. *Let $f : X \to Y$ be a continuous map between topological spaces. The following condition is equivalent.*

1. *The map f is quasi-proper and closed.*
2. *The map f is proper.*
3. *The map f is universally closed.*
4. *The map f is closed and $f^{-1}(y)$ is quasi-compact for any $y \in Y$.*

Proof. (See also the remark below.) If the map f satisfies (1), it automatically satisfies (4) because any single point is quasi-compact.

Assume map f satisfies (4). We will prove it is universally closed, i.e., (3) holds. Let $g : Z \to Y$ be a continuous map of topological spaces and consider the diagram

$$\begin{array}{ccc} Z \times_Y X & \xrightarrow{g'} & X \\ {\scriptstyle f'}\downarrow & & \downarrow{\scriptstyle f} \\ Z & \xrightarrow{g} & Y \end{array}$$

During the proof we will use that $Z \times_Y X \to Z \times X$ is a homeomorphism onto its image, i.e., that we may identify $Z \times_Y X$ with the corresponding subset of $Z \times X$ with the induced topology. The image of $f' : X \times_Y Z \to Z$ is $\mathrm{Im}(f') = \{z : g(z) \in f(X)\}$. Because $f(X)$ is closed, we see that $\mathrm{Im}(f')$ is a closed subspace of Z. Consider a closed subset $P \subset X \times_Y Z$. Let $z \in Z$, $z \notin f'(P)$. If $z \notin \mathrm{Im}(f')$, then $Z \setminus \mathrm{Im}(f')$ is an open neighbourhood which avoids $f'(P)$. If z is in $\mathrm{Im}(f')$ then $(f')^{-1}\{z\} = \{z\} \times f^{-1}\{g(z)\}$ and $f^{-1}\{g(z)\}$ is quasi-compact by assumption. Because P is a closed subset of $Z \times_Y X$, we have a closed P' of $Z \times X$ such that $P = P' \cap Z \times_Y X$. Since $(f')^{-1}\{z\}$ is a subset of $P^c = P'^c \cup (Z \times_Y X)^c$, and since $(f')^{-1}\{z\}$ is disjoint from $(Z \times_Y X)^c$ we see that $(f')^{-1}\{z\}$ is contained in P'^c. We may apply the Tube Lemma 5.16.1 to $(f')^{-1}\{z\} = \{z\} \times f^{-1}\{g(z)\} \subset (P')^c \subset Z \times X$. This gives $V \times U$ containing $(f')^{-1}\{z\}$ where U and V are open sets in X and Z respectively and $U \times V$ has empty intersection with P'. Hence z is contained in V and V has empty intersection with the image of P. As a result, the map f is universally closed.

The implication (3) $\Rightarrow$ (2) is trivial. Namely, given any topological space Z consider the projection morphism $g : Z \times Y \to Y$. Then it is easy to see that f' is the map $Z \times X \to Z \times Y$, in other words that $(Z \times Y) \times_Y X = Z \times X$. (This identification is a purely categorical property having nothing to do with topological spaces per se.)

Assume f satisfies (2). We will prove it satisfies (1). Note that f is closed as f can be identified with the map $\{pt\} \times X \to \{pt\} \times Y$ which is assumed closed. Choose any quasi-compact subset $K \subset Y$. Let Z be any topological space. Because

$Z \times X \to Z \times Y$ is closed we see the map $Z \times f^{-1}(K) \to Z \times K$ is closed (if T is closed in $Z \times f^{-1}(K)$, write $T = Z \times f^{-1}(K) \cap T'$ for some closed $T' \subset Z \times X$). Because K is quasi-compact, $K \times Z \to Z$ is closed by Lemma 5.16.3. Hence the composition $Z \times f^{-1}(K) \to Z \times K \to Z$ is closed and therefore $f^{-1}(K)$ must be quasi-compact by Lemma 5.16.3 again. □

Remark 5.16.6. Here are some references to the literature. In [**Bou71**, I, p. 75, Theorem 1] you can find: (2) ⇔ (4). In [**Bou71**, I, p. 77, Proposition 6] you can find: (2) ⇒ (1). Of course, trivially we have (1) ⇒ (4). Thus (1), (2) and (4) are equivalent. Fan Zhou claimed and proved that (3) and (4) are equivalent; let me know if you find a reference in the literature.

Lemma 5.16.7. *Let $f : X \to Y$ be a continuous map of topological spaces. If X is quasi-compact and Y is Hausdorff, then f is proper.*

Proof. Since every point of Y is closed, we see from Lemma 5.11.3 that the closed subset $f^{-1}(y)$ of X is quasi-compact for all $y \in Y$. Thus, by Theorem 5.16.5 it suffices to show that f is closed. If $E \subset X$ is closed, then it is quasi-compact (Lemma 5.11.3), hence $f(E) \subset Y$ is quasi-compact (Lemma 5.11.7), hence $f(E)$ is closed in Y (Lemma 5.11.4). □

Lemma 5.16.8. *Let $f : X \to Y$ be a continuous map of topological spaces. If f is bijective, X is quasi-compact, and Y is Hausdorff, then f is a homeomorphism.*

Proof. This follows immediately from Lemma 5.16.7 which tells us that f is closed, i.e., f^{-1} is continuous. □

5.17. Jacobson spaces

Definition 5.17.1. Let X be a topological space. Let X_0 be the set of closed points of X. We say that X is *Jacobson* if every closed subset $Z \subset X$ is the closure of $Z \cap X_0$.

Let X be a Jacobson space and let X_0 be the set of closed points of X with the induced topology. Clearly, the definition implies that the morphism $X_0 \to X$ induces a bijection between the closed subsets of X_0 and the closed subsets of X. Thus many properties of X are inherited by X_0. For example, the Krull dimensions of X and X_0 are the same.

Lemma 5.17.2. *Let X be a topological space. Let X_0 be the set of closed points of X. Suppose that for every irreducible closed subset $Z \subset X$ the intersection $X_0 \cap Z$ is dense in Z. Then X is Jacobson.*

Proof. Let $Z \subset X$ be closed. According to Lemma 5.7.3 we have $Z = \bigcup Z_i$ with Z_i irreducible and closed. Thus is $X_0 \cap Z_i$ is dense in each Z_i, then $X_0 \cap Z$ is dense in Z. □

Lemma 5.17.3. *Let X be a sober, Noetherian topological space. If X is not Jacobson, then there exists a non-closed point $\xi \in X$ such that $\{\xi\}$ is locally closed.*

Proof. Assume X is sober, Noetherian and not Jacobson. By Lemma 5.17.2 there exists an irreducible closed subset $Z \subset X$ which is not the closure of its closed points. Since X is Noetherian we may assume Z is minimal with this property. Let $\xi \in Z$ be the unique generic point (here we use X is sober). Note that the closed

points are dense in $\overline{\{z\}}$ for any $z \in Z$, $z \neq \xi$ by minimality of Z. Hence the closure of the set of closed points of Z is a closed subset containing all $z \in Z$, $z \neq \xi$. Hence $\{\xi\}$ is locally closed as desired. □

Lemma 5.17.4. *Let X be a topological space. Let $X = \bigcup U_i$ be an open covering. Then X is Jacobson if and only if each U_i is Jacobson. Moreover, in this case $X_0 = \bigcup U_{i,0}$.*

Proof. Let X be a topological space. Let X_0 be the set of closed points of X. Let $U_{i,0}$ be the set of closed points of U_i. Then $X_0 \cap U_i \subset U_{i,0}$ but equality may not hold in general.

First, assume that each U_i is Jacobson. We claim that in this case $X_0 \cap U_i = U_{i,0}$. Namely, suppose that $x \in U_{i,0}$, i.e., x is closed in U_i. Let $\overline{\{x\}}$ be the closure in X. Consider $\overline{\{x\}} \cap U_j$. If $x \notin U_j$, then $\overline{\{x\}} \cap U_j = \emptyset$. If $x \in U_j$, then $U_i \cap U_j \subset U_j$ is an open subset of U_j containing x. Let $T' = U_j \setminus U_i \cap U_j$ and $T = \{x\} \coprod T'$. Then T, T' are closed subsets of U_j and T contains x. As U_j is Jacobson we see that the closed points of U_j are dense in T. Because $T = \{x\} \coprod T'$ this can only be the case if x is closed in U_j. Hence $\overline{\{x\}} \cap U_j = \{x\}$. We conclude that $\overline{\{x\}} = \{x\}$ as desired.

Let $Z \subset X$ be a closed subset (still assuming each U_i is Jacobson). Since now we know that $X_0 \cap Z \cap U_i = U_{i,0} \cap Z$ are dense in $Z \cap U_i$ it follows immediately that $X_0 \cap Z$ is dense in Z.

Conversely, assume that X is Jacobson. Let $Z \subset U_i$ be closed. Then $X_0 \cap \overline{Z}$ is dense in $\overline{Z}$. Hence also $X_0 \cap Z$ is dense in Z, because $\overline{Z} \setminus Z$ is closed. As $X_0 \cap U_i \subset U_{i,0}$ we see that $U_{i,0} \cap Z$ is dense in Z. Thus U_i is Jacobson as desired. □

Lemma 5.17.5. *Let X be Jacobson. The following types of subsets $T \subset X$ are Jacobson:*

1. *Open subspaces.*
2. *Closed subspaces.*
3. *Locally closed subspaces.*
4. *Finite unions of locally closed subspaces.*
5. *Constructible sets.*
6. *Any subset $T \subset X$ which locally on X is a finite union of locally closed subsets.*

In each of these cases closed points of T are closed in X.

Proof. Let X_0 be the set of closed points of X. For any subset $T \subset X$ we let $(*)$ denote the property:

$(*)$ For every closed subset $Z \subset T$ the set $Z \cap X_0$ is dense in Z.

Note that always $X_0 \cap T \subset T_0$. Hence property $(*)$ implies that T is Jacobson. In addition it clearly implies that every closed point of T is closed in X.

Let $U \subset X$ be an open subset. Suppose $Z \subset U$ is closed. Then $X_0 \cap \overline{Z}$ is dense in $\overline{Z}$. Hence $X_0 \cap Z$ is dense in Z, because $\overline{Z} \setminus Z$ is closed. Thus $(*)$ holds.

Let $Z \subset X$ be a closed subset. Since closed subsets of Z are the same as closed subsets of X contained in Z property $(*)$ is immediate.

Let $T \subset X$ be locally closed. Write $T = U \cap Z$ for some open $U \subset X$ and some closed $Z \subset X$. Note that closed subsets of T are the same thing as closed subsets

of U which happen to be contained in Z. Hence $(*)$ holds for T because we proved it for U above.

Suppose $T_i \subset X$, $i = 1, \ldots, n$ are locally closed subsets. Let $T = T_1 \cup \ldots \cup T_n$. Suppose $Z \subset T$ is closed. Then $Z_i = Z \cap T_i$ is closed in T_i. By $(*)$ for T_i we see that $Z_i \cap X_0$ is dense in Z_i. Clearly this implies that $X_0 \cap Z$ is dense in Z, and property $(*)$ holds for T.

The case of constructible subsets is subsumed in the case of finite unions of locally closed subsets, see Definition 5.14.1.

The condition of the last assertion means that there exists an open covering $X = \bigcup U_i$ such that each $T \cap U_i$ is a finite union of locally closed subsets of U_i. We conclude that T is Jacobson by Lemma 5.17.4 and the case of a finite union of locally closed subsets dealt with above. It is formal to deduce $(*)$ for T from $(*)$ for all the inclusions $T \cap U_i \subset U_i$ and the assertions $X_0 = \bigcup U_{i,0}$ and $T_0 = \bigcup (T \cap U_i)_0$ from Lemma 5.17.4. □

Lemma 5.17.6. *A finite Kolmogorov Jacobson space is discrete.*

Proof. By induction on the number of points. The lemma holds if the space is empty. If X is a non-empty finite Kolmogorov space, choose a closed point $x \in X$, see Lemma 5.11.8. Then $U = X \setminus \{x\}$ is a finite Jacobson space, see Lemma 5.17.5. By induction U is a finite discrete space, hence all its points are closed. By Lemma 5.17.5 all the points of U are also closed in X and we win. □

Lemma 5.17.7. *Suppose X is a Jacobson topological space. Let X_0 be the set of closed points of X. There is a bijective, inclusion preserving correspondence*

$$\{\textit{constructible subsets of } X\} \leftrightarrow \{\textit{constructible subsets of } X_0\}$$

given by $E \mapsto E \cap X_0$. This correspondence preserves the subset of retrocompact open subsets, as well as complements of these.

Proof. Obvious from Lemma 5.17.5 above. □

Lemma 5.17.8. *Suppose X is a Jacobson topological space. Let X_0 be the set of closed points of X. There is a bijective, inclusion preserving correspondence*

$$\{\textit{finite unions loc. closed subsets of } X\} \leftrightarrow \{\textit{finite unions loc. closed subsets of } X_0\}$$

given by $E \mapsto E \cap X_0$. This correspondence preserves the subsets of locally closed, of open and of closed subsets.

Proof. Obvious from Lemma 5.17.5 above. □

5.18. Specialization

Definition 5.18.1. Let X be a topological space.

(1) If $x, x' \in X$ then we say x is a *specialization* of x', or x' is a *generalization* of x if $x \in \overline{\{x'\}}$. Notation: $x' \rightsquigarrow x$.
(2) A subset $T \subset X$ is *stable under specialization* if for all $x' \in T$ and every specialization $x' \rightsquigarrow x$ we have $x \in T$.
(3) A subset $T \subset X$ is *stable under generalization* if for all $x \in T$ and every generalization x' of x we have $x' \in T$.

Lemma 5.18.2. *Let X be a topological space.*

(1) *Any closed subset of X is stable under specialization.*
(2) *Any open subset of X is stable under generalization.*
(3) *A subset $T \subset X$ is stable under specialization if and only if the complement T^c is stable under generalization.*

Proof. Omitted. □

Definition 5.18.3. Let $f : X \to Y$ be a continuous map of topological spaces.

(1) We say that *specializations lift along f* or that f is *specializing* if given $y' \rightsquigarrow y$ in Y and any $x' \in X$ with $f(x') = y'$ there exists a specialization $x' \rightsquigarrow x$ of x' in X such that $f(x) = y$.
(2) We say that *generalizations lift along f* or that f is *generalizing* if given $y' \rightsquigarrow y$ in Y and any $x \in X$ with $f(x) = y$ there exists a generalization $x' \rightsquigarrow x$ of x in X such that $f(x') = y'$.

Lemma 5.18.4. *Suppose $f : X \to Y$ and $g : Y \to Z$ are continuous maps of topological spaces. If specializations lift along both f and g then specializations lift along $g \circ f$. Similarly for "generalizations lift along".*

Proof. Omitted. □

Lemma 5.18.5. *Let $f : X \to Y$ be a continuous map of topological spaces.*

(1) *If specializations lift along f, and if $T \subset X$ is stable under specialization, then $f(T) \subset Y$ is stable under specialization.*
(2) *If generalizations lift along f, and if $T \subset X$ is stable under generalization, then $f(T) \subset Y$ is stable under generalization.*

Proof. Omitted. □

Lemma 5.18.6. *Let $f : X \to Y$ be a continuous map of topological spaces.*

(1) *If f is closed then specializations lift along f.*
(2) *If f is open, X is a Noetherian topological space, each irreducible closed subset of X has a generic point, and Y is Kolmogorov then generalizations lift along f.*

Proof. Assume f is closed. Let $y' \rightsquigarrow y$ in Y and any $x' \in X$ with $f(x') = y'$ be given. Consider the closed subset $T = \overline{\{x'\}}$ of X. Then $f(T) \subset Y$ is a closed subset, and $y' \in f(T)$. Hence also $y \in f(T)$. Hence $y = f(x)$ with $x \in T$, i.e., $x' \rightsquigarrow x$.

Assume f is open, X Noetherian, every irreducible closed subset of X has a generic point, and Y is Kolmogorov. Let $y' \rightsquigarrow y$ in Y and any $x \in X$ with $f(x) = y$ be given. Consider $T = f^{-1}(\{y'\}) \subset X$. Take an open neighbourhood $x \in U \subset X$ of x. Then $f(U) \subset Y$ is open and $y \in f(U)$. Hence also $y' \in f(U)$. In other words, $T \cap U \neq \emptyset$. This proves that $x \in \overline{T}$. Since X is Noetherian, T is Noetherian (Lemma 5.8.2). Hence it has a decomposition $T = T_1 \cup \ldots \cup T_n$ into irreducible components. Then correspondingly $\overline{T} = \overline{T_1} \cup \ldots \cup \overline{T_n}$. By the above $x \in \overline{T_i}$ for some i. By assumption there exists a generic point $x' \in \overline{T_i}$, and we see that $x' \rightsquigarrow x$. As $x' \in \overline{T}$ we see that $f(x') \in \overline{\{y'\}}$. Note that $f(\overline{T_i}) = f(\overline{\{x'\}}) \subset \overline{\{f(x')\}}$. If $f(x') \neq y'$, then since Y is Kolmogorov $f(x')$ is not a generic point of the irreducible closed subset $\overline{\{y'\}}$ and the inclusion $\overline{\{f(x')\}} \subset \overline{\{y'\}}$ is strict, i.e., $y' \notin f(\overline{T_i})$. This contradicts the fact that $f(T_i) = \{y'\}$. Hence $f(x') = y'$ and we win. □

Lemma 5.18.7. *Suppose that $s, t : R \to U$ and $\pi : U \to X$ are continuous maps of topological spaces such that*

(1) *π is open,*
(2) *U is sober,*
(3) *s, t have finite fibres,*
(4) *generalizations lift along s, t,*
(5) *$(t, s)(R) \subset U \times U$ is an equivalence relation on U and X is the quotient of U by this equivalence relation (as a set).*

Then X is Kolmogorov.

Proof. Properties (3) and (5) imply that a point x corresponds to an finite equivalence class $\{u_1, \ldots, u_n\} \subset U$ of the equivalence relation. Suppose that $x' \in X$ is a second point corresponding to the equivalence class $\{u'_1, \ldots, u'_m\} \subset U$. Suppose that $u_i \rightsquigarrow u'_j$ for some i, j. Then for any $r' \in R$ with $s(r') = u'_j$ by (4) we can find $r \rightsquigarrow r'$ with $s(r) = u_i$. Hence $t(t) \rightsquigarrow t(r')$. Since $\{u'_1, \ldots, u'_m\} = t(s^{-1}(\{u'_j\}))$ we conclude that every element of $\{u'_1, \ldots, u'_m\}$ is the specialization of an element of $\{u_1, \ldots, u_n\}$. Thus $\overline{\{u_1\}} \cup \ldots \cup \overline{\{u_n\}}$ is a union of equivalence classes, hence of the form $\pi^{-1}(Z)$ for some subset $Z \subset X$. By (1) we see that Z is closed in X and in fact $Z = \overline{\{x\}}$ because $\pi(\overline{\{u_i\}}) \subset \overline{\{x\}}$ for each i. In other words, $x \rightsquigarrow x'$ if and only if some lift of x in U specializes to some lift of x' in U, if and only if every lift of x' in U is a specialization of some lift of x in U.

Suppose that both $x \rightsquigarrow x'$ and $x' \rightsquigarrow x$. Say x corresponds to $\{u_1, \ldots, u_n\}$ and x' corresponds to $\{u'_1, \ldots, u'_m\}$ as above. Then, by the results of the preceding paragraph, we can find a sequence

$$\ldots \rightsquigarrow u'_{j_3} \rightsquigarrow u_{i_3} \rightsquigarrow u'_{j_2} \rightsquigarrow u_{i_2} \rightsquigarrow u'_{j_1} \rightsquigarrow u_{i_1}$$

which must repeat, hence by (2) we conclude that $\{u_1, \ldots, u_n\} = \{u'_1, \ldots, u'_m\}$, i.e., $x = x'$. Thus X is Kolmogorov. □

Lemma 5.18.8. *Let $f : X \to Y$ be a morphism of topological spaces. Suppose that Y is a sober topological space, and f is surjective. If either specializations or generalizations lift along f, then* $\dim(X) \geq \dim(Y)$.

Proof. Assume specializations lift along f. Let $Z_0 \subset Z_1 \subset \ldots Z_e \subset Y$ be a chain of irreducible closed subsets of X. Let $\xi_e \in X$ be a point mapping to the generic point of Z_e. By assumption there exists a specialization $\xi_e \rightsquigarrow \xi_{e-1}$ in X such that ξ_{e-1} maps to the generic point of Z_{e-1}. Continuing in this manner we find a sequence of specializations

$$\xi_e \rightsquigarrow \xi_{e-1} \rightsquigarrow \ldots \rightsquigarrow \xi_0$$

with ξ_i mapping to the generic point of Z_i. This clearly implies the sequence of irreducible closed subsets

$$\overline{\{\xi_0\}} \subset \overline{\{\xi_1\}} \subset \ldots \overline{\{\xi_e\}}$$

is a chain of length e in X. The case when generalizations lift along f is similar. □

Lemma 5.18.9. *Let X be a Noetherian sober topological space. Let $E \subset X$ be a subset of X.*

(1) *If E is constructible and stable under specialization, then E is closed.*
(2) *If E is constructible and stable under generalization, then E is open.*

Proof. Let E be constructible and stable under generalization. Let $Y \subset X$ be an irreducible closed subset with generic point $\xi \in Y$. If $E \cap Y$ is nonempty, then it contains ξ (by stability under generalization) and hence is dense in Y, hence it contains a nonempty open of Y, see Lemma 5.15.3. Thus E is open by Lemma 5.15.5. This proves (2). To prove (1) apply (2) to the complement of E in X. □

5.19. Dimension functions

It scarcely makes sense to consider dimension functions unless the space considered is sober (Definition 5.7.4). Thus the definition below can be improved by considering the sober topological space associated to X. Since the underlying topological space of a scheme is sober we do not bother with this improvement.

Definition 5.19.1. Let X be a topological space.

(1) Let $x, y \in X$, $x \neq y$. Suppose $x \rightsquigarrow y$, that is y is a specialization of x. We say y is an *immediate specialization* of x if there is no $z \in X \setminus \{x, y\}$ with $x \rightsquigarrow z$ and $z \rightsquigarrow y$.
(2) A map $\delta : X \to \mathbf{Z}$ is called a *dimension function*[5] if
   (a) whenever $x \rightsquigarrow y$ and $x \neq y$ we have $\delta(x) > \delta(y)$, and
   (b) for every immediate specialization $x \rightsquigarrow y$ in X we have $\delta(x) = \delta(y) + 1$.

It is clear that if δ is a dimension function, then so is $\delta + t$ for any $t \in \mathbf{Z}$. Here is a fun lemma.

Lemma 5.19.2. *Let X be a topological space. If X is sober and has a dimension function, then X is catenary. Moreover, for any $x \rightsquigarrow y$ we have*

$$\delta(x) - \delta(y) = \mathit{codim}\left(\overline{\{y\}}, \overline{\{x\}}\right).$$

Proof. Suppose $Y \subset Y' \subset X$ are irreducible closed subsets. Let $\xi \in Y$, $\xi' \in Y'$ be their generic points. Then we see immediately from the definitions that $\mathrm{codim}(Y, Y') \leq \delta(\xi) - \delta(\xi') < \infty$. In fact the first inequality is an equality. Namely, suppose

$$Y = Y_0 \subset Y_1 \subset \ldots \subset Y_e = Y'$$

is any maximal chain of irreducible closed subsets. Let $\xi_i \in Y_i$ denote the generic point. Then we see that $\xi_i \rightsquigarrow \xi_{i+1}$ is an immediate specialization. Hence we see that $e = \delta(\xi) - \delta(\xi')$ as desired. This also proves the last statement of the lemma. □

Lemma 5.19.3. *Let X be a topological space. Let δ, δ' be two dimension functions on X. If X is locally Noetherian and sober then $\delta - \delta'$ is locally constant on X.*

Proof. Let $x \in X$ be a point. We will show that $\delta - \delta'$ is constant in a neighbourhood of x. We may replace X by an open neighbourhood of x in X which is Noetherian. Hence we may assume X is Noetherian and sober. Let $Z_1, \ldots, Z_r$ be the irreducible components of X passing through x. (There are finitely many as X is Noetherian, see Lemma 5.8.2.) Let $\xi_i \in Z_i$ be the generic point. Note $Z_1 \cup \ldots \cup Z_r$ is a neighbourhood of x in X (not necessarily closed). We claim that $\delta - \delta'$ is constant on $Z_1 \cup \ldots \cup Z_r$. Namely, if $y \in Z_i$, then

$$\delta(x) - \delta(y) = \delta(x) - \delta(\xi_i) + \delta(\xi_i) - \delta(y) = -\mathrm{codim}(\overline{\{x\}}, Z_i) + \mathrm{codim}(\overline{\{y\}}, Z_i)$$

[5]This is likely nonstandard notation. This notion is usually introduced only for (locally) Noetherian schemes, in which case condition (a) is implied by (b).

by Lemma 5.19.2. Similarly for δ'. Whence the result. □

Lemma 5.19.4. *Let X be locally Noetherian, sober and catenary. Then any point has an open neighbourhood $U \subset X$ which has a dimension function.*

Proof. We will use repeatedly that an open subspace of a catenary space is catenary, see Lemma 5.10.5 and that a Noetherian topological space has finitely many irreducible components, see Lemma 5.8.2. In the proof of Lemma 5.19.3 we saw how to construct such a function. Namely, we first replace X by a Noetherian open neighbourhood of x. Next, we let $Z_1, \ldots, Z_r \subset X$ be the irreducible components of X. Let

$$Z_i \cap Z_j = \bigcup Z_{ijk}$$

be the decomposition into irreducible components. We replace X by

$$X \setminus \left(\bigcup\nolimits_{x \notin Z_i} Z_i \cup \bigcup\nolimits_{x \notin Z_{ijk}} Z_{ijk} \right)$$

so that we may assume $x \in Z_i$ for all i and $x \in Z_{ijk}$ for all i, j, k. For $y \in X$ choose any i such that $y \in Z_i$ and set

$$\delta(y) = -\operatorname{codim}(\overline{\{x\}}, Z_i) + \operatorname{codim}(\overline{\{y\}}, Z_i).$$

We claim this is a dimension function. First we show that it is well defined, i.e., independent of the choice of i. Namely, suppose that $y \in Z_{ijk}$ for some i, j, k. Then we have (using Lemma 5.10.6)

$$\begin{aligned} \delta(y) &= -\operatorname{codim}(\overline{\{x\}}, Z_i) + \operatorname{codim}(\overline{\{y\}}, Z_i) \\ &= -\operatorname{codim}(\overline{\{x\}}, Z_{ijk}) - \operatorname{codim}(Z_{ijk}, Z_i) + \operatorname{codim}(\overline{\{y\}}, Z_{ijk}) + \operatorname{codim}(Z_{ijk}, Z_i) \\ &= -\operatorname{codim}(\overline{\{x\}}, Z_{ijk}) + \operatorname{codim}(\overline{\{y\}}, Z_{ijk}) \end{aligned}$$

which is symmetric in i and j. We omit the proof that it is a dimension function. □

Remark 5.19.5. Combining Lemmas 5.19.3 and 5.19.4 we see that on a catenary, locally Noetherian, sober topological space the obstruction to having a dimension function is an element of $H^1(X, \mathbf{Z})$.

5.20. Nowhere dense sets

Definition 5.20.1. Let X be a topological space.

(1) Given a subset $T \subset X$ the *interior* of T is the largest open subset of X contained in T.
(2) A subset $T \subset X$ is called *nowhere dense* if the closure of T has empty interior.

Lemma 5.20.2. *Let X be a topological space. The union of a finite number of nowhere dense sets is a nowhere dense set.*

Proof. Omitted. □

Lemma 5.20.3. *Let X be a topological space. Let $U \subset X$ be an open. Let $T \subset U$ be a subset. If T is nowhere dense in U, then T is nowhere dense in X.*

Proof. Assume T is nowhere dense in U. Suppose that $x \in X$ is an interior point of the closure $\overline{T}$ of T in X. Say $x \in V \subset \overline{T}$ with $V \subset X$ open in X. Note that $\overline{T} \cap U$ is the closure of T in U. Hence the interior of $\overline{T} \cap U$ being empty implies $V \cap U = \emptyset$. Thus x cannot be in the closure of U, a fortiori cannot be in the closure of T, a contradiction. □

Lemma 5.20.4. *Let X be a topological space. Let $X = \bigcup U_i$ be an open covering. Let $T \subset X$ be a subset. If $T \cap U_i$ is nowhere dense in U_i for all i, then T is nowhere dense in X.*

Proof. Omitted. (Hint: closure commutes with intersecting with opens.) □

Lemma 5.20.5. *Let $f : X \to Y$ be a continuous map of topological spaces. Let $T \subset X$ be a subset. If f identifies X with a closed subset of Y and T is nowhere dense in X, then also $f(T)$ is nowhere dense in Y.*

Proof. Omitted. □

Lemma 5.20.6. *Let $f : X \to Y$ be a continuous map of topological spaces. Let $T \subset Y$ be a subset. If f is open and T is a closed nowhere dense subset of Y, then also $f^{-1}(T)$ is a closed nowhere dense subset of X. If f is surjective and open, then T is closed nowhere dense if and only if $f^{-1}(T)$ is closed nowhere dense.*

Proof. Omitted. (Hint: In the first case the interior of $f^{-1}(T)$ maps into the interior of T, and in the second case the interior of $f^{-1}(T)$ maps onto the interior of T.) □

5.21. Profinite spaces

Here is the definition.

Definition 5.21.1. A topological space is *profinite* if it is homeomorphic to a limit of a diagram of finite discrete spaces.

This is not the most convenient characterization of a profinite space.

Lemma 5.21.2. *Let X be a topological space. The following are equivalent*

1. *X is a profinite space, and*
2. *X is Hausdorff, quasi-compact, and totally disconnected.*

If this is true, then X is a cofiltered limit of finite discrete spaces.

Proof. Assume (1). Choose a diagram $i \mapsto X_i$ of finite discrete spaces such that $X = \lim X_i$. As each X_i is Hausdorff and quasi-compact we find that X is quasi-compact by Lemma 5.13.3. If $x, x' \in X$ are distinct points, then x and x' map to distinct points in some X_i. Hence x and x' have disjoint open neighbourhoods, i.e., X is Hausdorff. In exactly the same way we see that X is totally disconnected.

Assume (2). Let $\mathcal{I}$ be the set of finite disjoint union decompositions $X = \coprod_{i \in I} U_i$ with U_i open (and closed). For each $I \in \mathcal{I}$ there is a continuous map $X \to I$ sending a point of U_i to i. We define a partial ordering: $I \leq I'$ for $I, I' \in \mathcal{I}$ if and only if the covering corresponding to I' refines the covering corresponding to I. In this case we obtain a canonical map $I' \to I$. In other words we obtain an inverse system of finite discrete spaces over $\mathcal{I}$. The maps $X \to I$ fit together and we obtain a continuous map

$$X \longrightarrow \lim_{I \in \mathcal{I}} I$$

We claim this map is a homeomorphism, which finishes the proof. (The final assertion follows too as the partially ordered set $\mathcal{I}$ is directed: given two disjoint union decompositions of X we can find a third refining either.) Namely, the map is injective as X is totally disconnected and hence $\{x\}$ is the intersection of all open and closed subsets of X containing x (Lemma 5.11.11), the map is surjective by Lemma 5.11.6. By Lemma 5.16.8 the map is a homeomorphism. □

Lemma 5.21.3. *Let X be a profinite space. Every open covering of X has a refinement by a finite covering $X = \coprod U_i$ with U_i open and closed.*

Proof. Write $X = \lim X_i$ as a limit of an inverse system of finite discrete spaces over a directed partially ordered set I (Lemma 5.21.2). Denote $f_i : X \to X_i$ the projection. For every point $x = (x_i) \in X$ a fundamental system of open neighbourhoods is the collection $f_i^{-1}(\{x_i\})$. Thus, as X is quasi-compact, we may assume we have an open covering

$$X = f_{i_1}^{-1}(x_{i_1}) \cup \ldots \cup f_{i_n}^{-1}(\{x_{i_n}\})$$

Choose $i \in I$ with $i \geq i_j$ for $j = 1, \ldots, n$ (this is possible as I is a directed partially ordered set). Then we see that the covering

$$X = \coprod\nolimits_{t \in X_i} f_i^{-1}(\{t\})$$

refines the given covering and is of the desired form. □

Lemma 5.21.4. *Let X be a topological space. If X is quasi-compact and every connected component of X is the intersection of the open and closed subsets containing it, then $\pi_0(X)$ is a profinite space.*

Proof. We will use Lemma 5.21.2 to prove this. Since $\pi_0(X)$ is the image of a quasi-compact space it is quasi-compact (Lemma 5.11.7). It is totally disconnected by construction (Lemma 5.6.8). Let $C, D \subset X$ be distinct connected components of X. Write $C = \bigcap U_\alpha$ as the intersection of the open and closed subsets of X containing C. Any finite intersection of U_α's is another. Since $\bigcap U_\alpha \cap D = \emptyset$ we conclude that $U_\alpha \cap D = \emptyset$ for some α (use Lemmas 5.6.3, 5.11.3 and 5.11.6) Since U_α is open and closed, it is the union of the connected components it contains, i.e., U_α is the inverse image of some open and closed subset $V_\alpha \subset \pi_0(X)$. This proves that the points corresponding to C and D are contained in disjoint open subsets, i.e., $\pi_0(X)$ is Hausdorff. □

5.22. Spectral spaces

The material in this section is taken from [**Hoc69**] and [**Hoc67**]. In his thesis Hochster proves (among other things) that the spectral spaces are exactly the topological spaces that occur as the spectrum of a ring.

Definition 5.22.1. A topological space X is called *spectral* if it is sober, quasi-compact, the intersection of two quasi-compact opens is quasi-compact, and the collection of quasi-compact opens forms a basis for the topology. A continuous map $f : X \to Y$ of spectral spaces is called *spectral* if the inverse image of a quasi-compact open is quasi-compact.

In other words a continuous map of spectral space is spectral if and only if it is quasi-compact (Definition 5.11.1).

Let X be a spectral space. The *constructible topology* on X is the topology which has as a subbase of opens the sets U and U^c where U is a quasi-compact open of X. Note that since X is spectral an open $U \subset X$ is retrocompact if and only if U is quasi-compact. Hence the constructible topology can also be characterized as the coarsest topology such that every constructible subset of X is both open and closed. Since the collection of quasi-compact opens is a basis for the topology on X we see that the constructible topology is stronger than the given topology on X.

Lemma 5.22.2. *Let X be a spectral space. The constructible topology is Hausdorff and quasi-compact.*

Proof. Since the collection of all quasi-compact opens forms a basis for the topology on X, it is clear that X is Hausdorff in the constructible topology.

Let $\mathcal{B}$ be the collection of subsets $B \subset X$ with B either quasi-compact open or closed with quasi-compact complement. If $B \in \mathcal{B}$ then $B^c \in \mathcal{B}$. It suffices to show every covering covering $X = \bigcup_{i \in I} B_i$ with $B_i \in \mathcal{B}$ has a finite refinement, see Lemma 5.11.15. Taking complements we see that we have to show that any family $\{B_i\}_{i \in I}$ of elements of $\mathcal{B}$ such that $B_{i_n} \cap \ldots \cap B_{i_n} \neq \emptyset$ for all n and all $i_1, \ldots, i_n \in I$ has a common point of intersection. We may and do assume $B_i \neq B_{i'}$ for $i \neq i'$.

To get a contradiction assume $\{B_i\}_{i \in I}$ is a maximal family of elements of $\mathcal{B}$ having the finite intersection property but empty intersection. An application of Zorn's lemma shows that we may assume our family is maximal (details omitted). Let $I' \subset I$ be those indices such that B_i is closed and set $Z = \bigcap_{i \in I'} B_i$. This is a closed subset of X. If Z is reducible, then we can write $Z = Z' \cup Z''$ as a union of two closed subsets, neither equal to Z. This means in particular that we can find a quasi-compact open $U' \subset X$ meeting Z' but not Z''. Similarly, we can find a quasi-compact open $U'' \subset X$ meeting Z'' but not Z'. Set $B' = X \setminus U'$ and $B'' = X \setminus U''$. Note that $Z'' \subset B'$ and $Z' \subset B''$. If there exist a finite number of indices $i_1, \ldots, i_n \in I$ such that $B' \cap B_{i_1} \cap \ldots \cap B_{i_n} = \emptyset$ as well as a finite number of indices $j_1, \ldots, j_m \in I$ such that $B'' \cap B_{j_1} \cap \ldots \cap B_{j_m} = \emptyset$ then we find that $Z \cap B_{i_1} \cap \ldots \cap B_{i_n} \cap B_{j_1} \cap \ldots \cap B_{j_m} = \emptyset$. However, the set $B_{i_1} \cap \ldots \cap B_{i_n} \cap B_{j_1} \cap \ldots \cap B_{j_m}$ is quasi-compact hence we would find a finite number of indices $i'_1, \ldots, i'_l \in I'$ with $B_{i_1} \cap \ldots \cap B_{i_n} \cap B_{j_1} \cap \ldots \cap B_{j_m} \cap B_{i'_1} \cap \ldots \cap B_{i'_l} = \emptyset$ a contradiction. Thus we see that we may add either B' or B'' to the given family contradicting maximality. We conclude that Z is irreducible. However, this leads to a contradiction as well, as now every nonempty (by the same argument as above) open $Z \cap B_i$ for $i \in I \setminus I'$ contains the unique generic point of Z. This contradiction proves the lemma. □

Lemma 5.22.3. *Let X be a spectral space. Let $E \subset X$ be closed in the constructible topology (for example constructible or closed). Then E with the induced topology is a spectral space.*

Proof. Let $Z \subset E$ be a closed irreducible subset. Let η be the generic point of the closure $\overline{Z}$ of Z in X. To prove that E is sober, we show that $\eta \in E$. If not, then since E is closed in the constructible topology, there exists a constructible subset $F \subset X$ such that $\eta \in F$ and $F \cap E = \emptyset$. By Lemma 5.14.14 this implies $F \cap \overline{Z}$ contains a nonempty open subset of $\overline{Z}$. But this is impossible as $\overline{Z}$ is the closure of Z and $Z \cap F = \emptyset$.

Since E is closed in the constructible topology, it is quasi-compact in the constructible topology (Lemmas 5.11.3 and 5.22.2). Hence a fortiori it is quasi-compact

in the topology coming from X. If $U \subset X$ is a quasi-compact open, then $E \cap U$ is closed in the constructible topology, hence quasi-compact (as seen above). It follows that the quasi-compact open subsets of E are the intersections $E \cap U$ with U quasi-compact open in X. These form a basis for the topology. Finally, given two $U, U' \subset X$ quasi-compact opens, the intersection $(E \cap U) \cap (E \cap U') = E \cap (U \cap U')$ and $U \cap U'$ is quasi-compact as X is spectral. This finishes the proof. □

Lemma 5.22.4. *Let X be a spectral space. Let $E \subset X$ be a constructible subset.*

(1) *If $x \in \overline{E}$, then x is the specialization of a point of E.*
(2) *If E is stable under specialization, then E is closed.*
(3) *If E is stable under generalization, then E is open.*

Proof. Proof of (1). Let $x \in \overline{E}$. Let $\{U_i\}$ be the set of quasi-compact open neighbourhoods of x. A finite intersection of the U_i is another one. The intersection $U_i \cap E$ is nonempty for all i. Since the subsets $U_i \cap E$ are closed in the constructible topology we see that $\bigcap(U_i \cap E)$ is nonempty by Lemma 5.22.2 and Lemma 5.11.6. Since X is a sober space and $\{U_i\}$ is a fundamental system of open neighbourhoods of x, we see that $\bigcap U_i$ is the set of generalizations of x. Thus x is a specialization of a point of E.

Part (2) is immediate from (1).

Proof of (3). Assume E is stable under generalization. The complement of E is constructible (Lemma 5.14.2) and closed under specialization (Lemma 5.18.2). Hence the complement is closed by (2), i.e., E is open. □

Lemma 5.22.5. *Let X be a spectral space. Let $x, y \in X$. Then either there exists a third point specializing to both x and y, or there exist disjoint open neighbourhoods containing x and y.*

Proof. Let $\{U_i\}$ be the set of quasi-compact open neighbourhoods of x. A finite intersection of the U_i is another one. Let $\{V_j\}$ be the set of quasi-compact open neighbourhoods of y. A finite intersection of the V_j is another one. If $U_i \cap V_j$ is empty for some i, j we are done. If not, then the intersection $U_i \cap V_j$ is nonempty for all i. The sets $U_i \cap V_j$ are closed in the constructible topology on X. By Lemma 5.22.2 we see that $\bigcap(U_i \cap V_j)$ is nonempty (Lemma 5.11.6). Since X is a sober space and $\{U_i\}$ is a fundamental system of open neighbourhoods of x, we see that $\bigcap U_i$ is the set of generalizations of x. Similarly, $\bigcap V_j$ is the set of generalizations of y. Thus any element of $\bigcap(U_i \cap V_j)$ specializes to both x and y. □

Lemma 5.22.6. *Let X be a spectral space. The following are equivalent*

(1) *X is profinite,*
(2) *X is Hausdorff,*
(3) *X is totally disconnected,*
(4) *every quasi-compact open is closed,*
(5) *there are no nontrivial specializations between points,*
(6) *every point of X is closed,*
(7) *every point of X is the generic point of an irreducible component of X,*
(8) *add more here.*

Proof. The implication (1) $\Rightarrow$ (2) is trivial. If every quasi-compact open is closed, then X is Hausforff, so (4) $\Rightarrow$ (2).

It is clear that (4), (5), and (6) are equivalent since X is sober. It follows from Lemma 5.22.5 that this implies X is Hausdorff.

If X is totally disconnected, then every point is closed. So (3) implies (6).

Thus every condition implies that X is Hausdorff. Conversely, if X is Hausdorff, then every quasi-compact open is also closed (Lemma 5.11.4). This implies that X is totally disconnected. Hence it is profinite, by Lemma 5.21.2. This also implies (4), (5), and (6) hold. □

Lemma 5.22.7. *If X is a spectral space, then $\pi_0(X)$ is a profinite space.*

Proof. Combine Lemmas 5.11.10 and 5.21.4. □

Lemma 5.22.8. *The product of two spectral spaces is spectral.*

Proof. Let X, Y be spectral spaces. Denote $p : X \times Y \to X$ and $q : X \times Y \to Y$ the projections. Let $Z \subset X \times Y$ be a closed irreducible subset. Then $p(Z) \subset X$ is irreducible and $q(Z) \subset Y$ is irreducible. Let $x \in X$ be the generic point of the closure of $p(X)$ and let $y \in Y$ be the generic point of the closure of $q(Y)$. If $(x, y) \notin Z$, then there exist opens $x \in U \subset X$, $y \in V \subset Y$ such that $Z \cap U \times V = \emptyset$. Hence Z is contained in $(X \setminus U) \times Y \cup X \times (Y \setminus V)$. Since Z is irreducible, we see that either $Z \subset (X \setminus U) \times Y$ or $Z \subset X \times (Y \setminus V)$. In the first case $p(Z) \subset (X \setminus U)$ and in the second case $q(Z) \subset (Y \setminus V)$. Both cases are absurd as x is in the closure of $p(Z)$ and y is in the closure of $q(Z)$. Thus we conclude that $(x, y) \in Z$, which means that (x, y) is the generic point for Z.

A basis of the topology of $X \times Y$ are the opens of the form $U \times V$ with $U \subset X$ and $V \subset Y$ quasi-compact open (here we use that X and Y are spectral). Then $U \times V$ is quasi-compact as the product of quasi-compact spaces is quasi-compact. Moreover, any quasi-compact open of $X \times Y$ is a finite union of such quasi-compact rectangles $U \times V$. It follows that the intersection of two such is again quasi-compact (since X and Y are spectral). This concludes the proof. □

Lemma 5.22.9. *Let $f : X \to Y$ be a continuous map of topological spaces. if*

(1) *X and Y are spectral,*
(2) *f is spectral and bijective, and*
(3) *generalizations (resp. specializations) lift along f.*

Then f is a homeomorphism.

Proof. Since f is spectral it defines a continuous map between X and Y in the constructible topology. By Lemmas 5.22.2 and 5.16.8 it follows that $X \to Y$ is a homeomorphism in the constructible topology. Let $U \subset X$ be quasi-compact open. Then $f(U)$ is constructible in Y. Let $y \in Y$ specialize to a point in $f(U)$. By the last assumption we see that $f^{-1}(y)$ specializes to a point of U. Hence $y \in U$. Thus $y \in f(U)$. It follows that $f(U)$ is open, see Lemma 5.22.4. Whence f is a homeomorphism. To prove the lemma in case specializations lift along f one shows instead that $f(Z)$ is closed if $X \setminus Z$ is a quasi-compact open of X. □

Lemma 5.22.10. *The inverse limit of a directed inverse system of finite sober topological spaces is a spectral topological space.*

Proof. Let I be a directed partially ordered set. Let X_i be an inverse system of finite sober spaces over I. Let $X = \lim X_i$ which exists by Lemma 5.13.1. Because I is directed we can describe X as follows. As a set $X = \lim X_i$. Denote $p_i : X \to X_i$ the projection. A basis for the topology is given by the opens $p_i^{-1}(U_i)$ for $U_i \subset X_i$ open. Note that $p_i^{-1}(U_i)$ is quasi-compact as a profinite topological space (Lemma 5.21.2). Since an open covering of $p_i^{-1}(U_i)$ is in particular an open covering in the profinite topology, we conclude that $p_i^{-1}(U_i)$ is quasi-compact. Given $U_i \subset X_i$ and $U_j \subset X_j$, then $p_i^{-1}(U_i) \cap p_j^{-1}(U_j) = p_k^{-1}(U_k)$ for some $k \geq i, j$ and open $U_k \subset X_k$. Finally, if $Z \subset X$ is irreducible and closed, then $p_i(Z) \subset X_i$ is irreducible and therefore has a unique generic point ξ_i (because X_i is a finite sober topological space). Then $\xi = \lim \xi_i$ is a generic point of Z (it is a point of Z as Z is closed). This finishes the proof. □

Lemma 5.22.11. *Let W be the topological space with two points one closed the other not. A topological space is spectral if and only if it is homeomorphic to a closed subspace of a product of copies of W.*

Proof. Write $W = \{0, 1\}$ where 0 is a specialization of 1 but not vice versa. Let I be a set. The space $\prod_{i \in I} W$ is spectral by Lemma 5.22.10. Thus we see that a closed subspace of $\prod_{i \in I} W$ is a spectral space by Lemma 5.22.3.

For the converse, let X be a spectral space. Let $U \subset X$ be a quasi-compact open. Consider the continuous map

$$f_U : X \longrightarrow W$$

which maps every point in U to 1 and every point in $X \setminus U$ to 0. Taking the product of these maps we obtain a continuous map

$$f = \prod f_U : X \longrightarrow \prod\nolimits_U W$$

If $x', x \in X$ are distinct, then since X is sober either x' is not a specialization of x or conversely. In either case (as the quasi-compact opens form a basis for the topology of X) there exists a quasi-compact open $U \subset X$ such that $f_U(x') \neq f_U(x)$. Thus f is injective. Let $Y = f(X)$ endowed with the induced topology. By construction the map $f : X \to Y$ is spectral. Let $y' \rightsquigarrow y$ be a specialization in Y and say $f(x') = y'$ and $f(x) = y$. Arguing as above we see that $x' \rightsquigarrow x$, since otherwise there is a U such that $x \in U$ and $x' \notin U$, which would imply $f_U(x') \not\rightsquigarrow f_U(x)$. We conclude that $f : X \to Y$ is a homeomorphism by Lemma 5.22.9. □

Lemma 5.22.12. *A topological space is spectral if and only if it is a directed inverse limit of finite sober topological spaces.*

Proof. One direction is given by Lemma 5.22.10. For the converse, assume X is spectral. Then we may assume $X \subset \prod_{i \in I} W$ is a closed subset where $W = \{0, 1\}$ as in the proof of Lemma 5.22.11. We can write

$$\prod\nolimits_{i \in I} W = \lim\nolimits_{J \subset I \text{ finite}} \prod\nolimits_{j \in J} W$$

as a cofiltered limit. For each J, let $X_J \subset \prod_{j \in J} W$ be the image of X. Then we see that $X = \lim X_J$ as sets because X is closed in the product. A formal argument (omitted) on limits shows that $X = \lim X_J$ as topological spaces. □

5.23. Stone-Čech compactification

The Stone-Čech compactification of a topological space X is a map $X \to \beta(X)$ from X to a Hausdorff quasi-compact space $\beta(X)$ which is universal for such maps. We prove this exists by a standard argument using the following simple lemma.

Lemma 5.23.1. *Let $f : X \to Y$ be a continuous map of topological spaces. Assume that $f(X)$ is dense in Y and that Y is Hausdorff. Then the cardinality of Y is at most the cardinality of $P(P(X))$ where P is the power set operation.*

Proof. Let $S = f(X) \subset Y$. Let $\mathcal{D}$ be the set of all closed domains of Y, i.e., subsets $D \subset Y$ which equal the closure of its interior. Note that the closure of an open subset of Y is a closed domain. For $y \in Y$ consider the set

$$I_y = \{T \subset S \mid \text{ there exists } D \in \mathcal{D} \text{ with } T = S \cap D \text{ and } y \in D\}$$

Since S is dense in Y for every closed domain D we see that $S \cap D$ is dense in D. Hence, if $D \cap S = D' \cap S$ for $D, D' \in \mathcal{D}$, then $D = D'$. Thus $I_y = I_{y'}$ implies that $y = y'$ because the Hausdorff condition assures us that we can find a closed domain containing y but not y'. The result follows. □

Let X be a topological space. Let κ be the cardinality of $P(P(X))$ as in the lemma above. There is a set I of isomorphism classes of continuous maps $f : X \to Y$ which has dense image and where Y is Hausdorff and quasi-compact. For $i \in I$ choose a representative $f_i : X \to Y_i$. Consider the map

$$\prod f_i : X \longrightarrow \prod\nolimits_{i \in I} Y_i$$

and denote $\beta(X)$ the closure of the image. Since each Y_i is Hausdorff, so is $\beta(X)$. Since each Y_i is quasi-compact, so is $\beta(X)$ (use Theorem 5.13.2 and Lemma 5.11.3).

Let us show the canonical map $X \to \beta(X)$ satisfies the universal property with respect to maps to Hausdorff, quasi-compact spaces. Namely, let $f : X \to Y$ be such a morphism. Let $Z \subset Y$ be the closure of $f(X)$. By Lemma 5.23.1 the cardinality of Z is at most κ. Thus $X \to Z$ is isomorphic to one of the maps $f_i : X \to Y_i$, say $f_{i_0} : X \to Y_{i_0}$. Thus f factors as $X \to \beta(X) \to \prod Y_i \to Y_{i_0} \cong Z \to Y$ as desired.

Lemma 5.23.2. *Let X be a Hausdorff, locally quasi-compact space. There exists a map $X \to X^*$ which identifies X as an open subspace of a quasi-compact Hausdorff space X^* such that $X^* \setminus X$ is a singleton (one point compactification). In particular, the map $X \to \beta(X)$ identifies X with an open subspace of $\beta(X)$.*

Proof. Set $X^* = X \amalg \{\infty\}$. We declare a subset V of X^* to be open if either $V \subset X$ is open in X, or $\infty \in V$ and $U = V \cap X$ is an open of X such that $X \setminus U$ is quasi-compact. We omit the verification that this defines a topology. It is clear that $X \to X^*$ identifies X with an open subspace of X.

Since X is locally quasi-compact, every point $x \in X$ has a quasi-compact neighbourhood $x \in E \subset X$. Then E is closed (Lemma 5.11.3) and $V = (X \setminus E) \amalg \{\infty\}$ is an open neighbourhood of ∞ disjoint from the interior of E. Thus X^* is Hausdorff.

Let $X^* = \bigcup V_i$ be an open covering. Then for some i, say i_0, we have $\infty \in V_{i_0}$. By construction $Z = X^* \setminus V_{i_0}$ is quasi-compact. Hence the covering $Z \subset \bigcup_{i \neq i_0} Z \cap V_i$ has a finite refinement which implies that the given covering of X^* has a finite refinement. Thus X^* is quasi-compact.

The map $X \to X^*$ factors as $X \to \beta(X) \to X^*$ by the universal property of the Stone-Čech compactification. Let $\varphi : \beta(X) \to X^*$ be this factorization. Then $X \to \varphi^{-1}(X)$ is a section to $\varphi^{-1}(X) \to X$ hence has closed image (Lemma 5.3.3). Since the image of $X \to \beta(X)$ is dense we conclude that $X = \varphi^{-1}(X)$. □

5.24. Extremally disconnected spaces

The material in this section is taken from [**Gle58**] (with a slight modification as in [**Rai59**]). In Gleason's paper it is shown that in the category of quasi-compact Hausdorff spaces, the "projective objects" are exactly the extremally disconnected spaces.

Definition 5.24.1. A topological space X is called *extremally disconnected* if the closure of every open subset of X is open.

If X is Hausdorff and extremally disconnected, then X is totally disconnected (this isn't true in general). If X is quasi-compact, Hausdorff, and extremally disconnected, then X is profinite by Lemma 5.21.2, but the converse does not holds in general. Namely, Gleason shows that in an extremally disconnected Hausdorff space X a convergent sequence $x_1, x_2, x_3, \ldots$ is eventually constant. Hence for example the p-adic integers $\mathbf{Z}_p = \lim \mathbf{Z}/p^n\mathbf{Z}$ is a profinite space which is not extremally disconnected.

Lemma 5.24.2. *Let $f : X \to Y$ be a continuous map of topological spaces. Assume f is surjective and $f(E) \neq Y$ for all proper closed subsets $E \subset X$. Then for $U \subset X$ open the subset $f(U)$ is contained in the closure of $Y \setminus f(X \setminus U)$.*

Proof. Pick $y \in f(U)$ and let $V \subset Y$ be any open neighbourhood of y. We will show that V intersects $Y \setminus f(X \setminus U)$. Note that $W = U \cap f^{-1}(V)$ is a nonempty open subset of X, hence $f(X \setminus W) \neq Y$. Take $y' \in Y$, $y' \notin f(X \setminus W)$. It is elementary to show that $y' \in V$ and $y' \in Y \setminus f(X \setminus U)$. □

Lemma 5.24.3. *Let X be an extremally disconnected space. If $U, V \subset X$ are disjoint open subsets, then $\overline{U}$ and $\overline{V}$ are disjoint too.*

Proof. By assumption $\overline{U}$ is open, hence $V \cap \overline{U}$ is open and disjoint from U, hence empty because $\overline{U}$ is the intersection of all the closed subsets of X containing U. This means the open $\overline{V} \cap \overline{U}$ avoids V hence is empty by the same argument. □

Lemma 5.24.4. *Let $f : X \to Y$ be a continuous map of Hausforff quasi-compact topological spaces. If Y is extremally disconnected, f is surjective, and $f(Z) \neq Y$ for every proper closed subset Z of X, then f is a homeomorphism.*

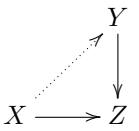
Proof. By Lemma 5.16.8 it suffices to show that f is injective. Suppose that $x, x' \in X$ are distinct points with $y = f(x) = f(x')$. Choose disjoint open neighbourhoods $U, U' \subset X$ of x, x'. Observe that f is closed (Lemma 5.16.7) hence $T = f(X \setminus U)$ and $T' = f(X \setminus U')$ are closed in Y. Since X is the union of $X \setminus U$ and $X \setminus U'$ we see that $Y = T \cup T'$. By Lemma 5.24.2 we see that y is contained in the closure of $Y \setminus T$ and the closure of $Y \setminus T'$. On the other hand, by Lemma 5.24.3, this intersection is empty. In this way we obtain the desired contradiction. □

Lemma 5.24.5. *Let $f : X \to Y$ be a continuous surjective map of Hausforff quasi-compact topological spaces. There exists a quasi-compact subset $E \subset X$ such that $f(E) = Y$ but $f(E') \neq Y$ for all proper closed subsets $E' \subset E$.*

Proof. We will use without further mention that the quasi-compact subsets of X are exactly the closed subsets (Lemma 5.11.5). Consider the collection $\mathcal{E}$ of all quasi-compact subsets $E \subset X$ with $f(E) = Y$ ordered by inclusion. We will use Zorn's lemma to show that $\mathcal{E}$ has a minimal element. To do this it suffices to show that given a totally ordered family E_λ of elements of $\mathcal{E}$ the intersection $\bigcap E_\lambda$ is an element of $\mathcal{E}$. It is quasi-compact as it is closed. For every $y \in Y$ the sets $E_\lambda \cap f^{-1}(\{y\})$ are nonempty and closed, hence the intersection $\bigcap E_\lambda \cap f^{-1}(\{y\}) = \bigcap (E_\lambda \cap f^{-1}(\{y\}))$ is nonempty by Lemma 5.11.6. This finishes the proof. □

Proposition 5.24.6. *Let X be a Hausdorff, quasi-compact topological space. The following are equivalent*

1. *X is extremally disconnected,*
2. *for any surjective continuous map $f : Y \to X$ with Y Hausdorff quasi-compact there exists a continuous section, and*
3. *for any solid commutative diagram*



of continuous maps of quasi-compact Hausdorff spaces with $Y \to Z$ surjective, there is a dotted arrow in the category of topological spaces making the diagram commute.

Proof. It is clear that (3) implies (2). On the other hand, if (2) holds and $X \to Z$ and $Y \to Z$ are as in (3), then (2) assures there is a section to the projection $X \times_Z Y \to X$ which implies a suitable dotted arrow exists (details omitted). Thus (3) is equivalent to (2).

Assume X is extremally disconnected and let $f : Y \to X$ be as in (2). By Lemma 5.24.5 there exists a quasi-compact subset $E \subset Y$ such that $f(E) = X$ but $f(E') \neq X$ for all proper closed subsets $E' \subset E$. By Lemma 5.24.4 we find that $f|_E : E \to X$ is a homeomorphism, the inverse of which gives the desired section.

Assume (2). Let $U \subset X$ be open with complement Z. Consider the continuous surjection $f : \overline{U} \amalg Z \to X$. Let σ be a section. Then $\overline{U} = \sigma^{-1}(\overline{U})$ is open. Thus X is extremally disconnected. □

Lemma 5.24.7. *Let $f : X \to X$ be a continuous selfmap of a Hausdorff topological space. If f is not id_X, then there exists a proper closed subset $E \subset X$ such that $X = E \cup f(E)$.*

Proof. Pick $p \in X$ with $f(p) \neq p$. Choose disjoint open neighbourhoods $p \in U$, $f(p) \in V$ and set $E = X \setminus U \cap f^{-1}(V)$. □

Example 5.24.8. We can use Proposition 5.24.6 to see that the Stone-Čech compactification $\beta(X)$ of a discrete space X is extremally disconnected. Namely, let $f : Y \to \beta(X)$ be a continuous surjection where Y is quasi-compact and Hausdorff. Then we can lift the map $X \to \beta(X)$ to a continuous (!) map $X \to Y$ as X is discrete. By the universal property of the Stone-Čech compactification we see that we obtain a factorization $X \to \beta(X) \to Y$. Since $\beta(X) \to Y \to \beta(X)$ equals the identity on the dense subset X we conclude that we get a section. In particular,

we conclude that the Stone-Čech compactification of a discrete space is totally disconnected, whence profinite (see discussion following Definition 5.24.1 and Lemma 5.21.2).

Using the supply of extremally disconnected spaces given by Example 5.24.8 we can prove that every quasi-compact Hausdorff space has a "projective cover" in the category of quasi-compact Hausdorff spaces.

Lemma 5.24.9. *Let X be a quasi-compact Hausdorff space. There exists a continuous surjection $X' \to X$ with X' quasi-compact, Hausdorff, and extremally disconnected. If we require that every proper closed subset of X' does not map onto X, then X' is unique up to isomorphism.*

Proof. Let $Y = X$ but endowed with the discrete topology. Let $X' = \beta(Y)$. The continuous map $Y \to X$ factors as $Y \to X' \to X$. This proves the first statement of the lemma by Example 5.24.8.

By Lemma 5.24.5 we can find a quasi-compact subset $E \subset X'$ such that no proper closed subset of E surjects onto X. Because X' is extremally disconnected there exists a continuous map $f : X' \to E$ over X (Proposition 5.24.6). Composing f with the map $E \to X'$ gives a continuous selfmap $f|_E : E \to E$. This map has to be id_E as otherwise Lemma 5.24.7 shows that E isn't minimal. Thus the id_E factors through the extremally disconnected space X'. A formal, categorical argument (using the characterization of Proposition 5.24.6 shows that E is extremally disconnected.

To prove uniqueness, suppose we have a second $X'' \to X$ minimal cover. By the lifting property proven in Proposition 5.24.6 we can find a continuous map $g : X' \to X''$ over X. Observe that g is a closed map (Lemma 5.16.7). Hence $g(X') \subset X''$ is a closed subset surjecting onto X and we conclude $g(X') = X''$ by minimality of X''. On the other hand, if $E \subset X'$ is a proper closed subset, then $g(E) \neq X''$ as E does not map onto X by minimality of X'. By Lemma 5.24.4 we see that g is an isomorphism. □

Remark 5.24.10. Let X be a quasi-compact Hausdorff space. Let κ be an infinite cardinal bigger or equal than the cardinality of X. Then the cardinality of the minimal quasi-compact, Hausdorff, extremally disconnected cover $X' \to X$ (Lemma 5.24.9) is at most 2^{2^κ}. Namely, choose a subset $S \subset X'$ mapping bijectively to X. By minimality of X' the set S is dense in X'. Thus $|X'| \leq 2^{2^\kappa}$ by Lemma 5.23.1.

5.25. Miscellany

The following lemma applies to the underlying topological space associated to a quasi-separated scheme.

Lemma 5.25.1. *Let X be a topological space which*

(1) *has a basis of the topology consisting of quasi-compact opens, and*
(2) *has the property that the intersection of any two quasi-compact opens is quasi-compact.*

Then

(1) *X is locally quasi-compact,*
(2) *a quasi-compact open $U \subset X$ is retrocompact,*

(3) *any quasi-compact open $U \subset X$ has a cofinal system of open coverings $\mathcal{U} : U = \bigcup_{j \in J} U_j$ with J finite and all U_j and $U_j \cap U_{j'}$ quasi-compact,*
(4) *add more here.*

Proof. Omitted. □

Definition 5.25.2. Let X be a topological space. We say $x \in X$ is an *isolated point* of X if $\{x\}$ is open in X.

5.26. Partitions and stratifications

Stratifications can be defined in many different ways. We welcome comments on the choice of definitions in this section.

Definition 5.26.1. Let X be a topological space. A *partition* of X is a decomposition $X = \coprod X_i$ into locally closed subsets X_i. The X_i are called the *parts* of the partition. Given two partitions of X we say one *refines* the other if the parts of one are unions of parts of the other.

Thus we can say that X has a partition into connected components and a partition into irreducible components and that the partition into irreducible components refines the partition into connected components.

Definition 5.26.2. Let X be a topological space. A *good stratification* of X is a partition $X = \coprod X_i$ such that for all $i, j \in I$ we have

$$X_i \cap \overline{X_j} \neq \emptyset \Rightarrow X_i \subset \overline{X_j}.$$

Given a good stratification $X = \coprod_{i \in I} X_i$ we obtain a partial ordering on I by setting $i \leq j$ if and only if $X_i \subset \overline{X_j}$. Then we see that

$$\overline{X_j} = \bigcup_{i \leq j} X_i$$

However, what often happens in algebraic geometry is that one just has that the left hand side is a subset of the right hand side in the formulat above. This leads to the following definition.

Definition 5.26.3. Let X be a topological space. A *stratification* of X is given by a partition $X = \coprod_{i \in I} X_i$ and a partial ordering on I such that for each $j \in I$ we have

$$\overline{X_j} \subset \bigcup_{i \leq j} X_i$$

The parts X_i are called the *strata* of the stratification.

We often impose additional conditions on the stratification. For example, we say a stratification is *locally finite* if every point has a neighbourhood which meets only finitely many strata.

Remark 5.26.4. Given a locally finite stratification $X = \coprod X_i$ of a topological space X, we obtain a family of closed subsets $Z_i = \bigcup_{j \leq i} X_j$ of X indexed by I such that

$$Z_i \cap Z_j = \bigcup_{k \leq i,j} Z_k$$

Conversely, given closed subsets $Z_i \subset X$ indexed by a partially ordered set I such that $X = \bigcup Z_i$, such that every point has a neighbourhood meeting only finitely many Z_i, and such that the displayed formula holds, then we obtain a locally finite stratification of X by setting $X_i = Z_i \setminus \bigcup_{j < i} Z_j$.

Lemma 5.26.5. *Let X be a topological space. Let $X = \coprod X_i$ be a finite partition of X. Then there exists a finite stratification of X refining it.*

Proof. Let $T_i = \overline{X_i}$ and $\Delta_i = T_i \setminus X_i$. Let S be the set of all intersections of T_i and Δ_i. (For example $T_1 \cap T_2 \cap \Delta_4$ is an element of S.) Then $S = \{Z_s\}$ is a finite collection of closed subsets of X such that $Z_s \cap Z_{s'} \in S$ for all $s, s' \in S$. Define a partial ordering on S by inclusion. Then set $Y_s = Z_s \setminus \bigcup_{s'<s} Z_{s'}$ to get the desired stratification. □

Lemma 5.26.6. *Let X be a topological space. Suppose $X = T_1 \cup \ldots \cup T_n$ is written as a union of constructible subsets. There exists a finite stratification $X = \coprod X_i$ with each X_i constructible such that each T_k is a union of strata.*

Proof. By definition of constructible subsets, we can write each T_i as a finite union of $U \cap V^c$ with $U, V \subset X$ retrocompact open. Hence we may assume that $T_i = U_i \cap V_i^c$ with $U_i, V_i \subset X$ retrocompact open. Let S be the finite set of closed subsets of X consisting of $\emptyset, X, U_i^c, V_i^c$ and finite intersections of these. Write $S = \{Z_s\}$. If $s \in S$, then Z_s is constructible (Lemma 5.14.2). Moreover, $Z_s \cap Z_{s'} \in S$ for all $s, s' \in S$. Define a partial ordering on S by inclusion. Then set $Y_s = Z_s \setminus \bigcup_{s'<s} Z_{s'}$ to get the desired stratification. □

Lemma 5.26.7. *Let X be a Noetherian topological space. Any finite partition of X can be refined by a finite good stratification.*

Proof. Let $X = \coprod X_i$ be a finite partition of X. Let Z be an irreducible component of X. Since $X = \bigcup \overline{X_i}$ with finite index set, there is an i such that $Z \subset \overline{X_i}$. Since X_i is locally closed this implies that $Z \cap X_i$ contains an open of Z. Thus $Z \cap X_i$ contains an open U of X (Lemma 5.8.2). Write $X_i = U \amalg X_i^1 \amalg X_i^2$ with $X_i^1 = (X_i \setminus U) \cap \overline{U}$ and $X_i^2 = (X_i \setminus U) \cap \overline{U}^c$. For $i' \neq i$ we set $X_{i'}^1 = X_{i'} \cap \overline{U}$ and $X_{i'}^2 = X_{i'} \cap \overline{U}^c$. Then

$$X \setminus U = \coprod X_l^k$$

is a partition such that $\overline{U} \setminus U = \bigcup X_l^1$. Note that $X \setminus U$ is closed and strictly smaller than X. By Noetherian induction we can refine this partition by a finite good stratification $X \setminus U = \coprod_{\alpha \in A} T_\alpha$. Then $X = U \amalg \coprod_{\alpha \in A} T_\alpha$ is a finite good stratification of X refining the partition we started with. □

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CHAPTER 6

Sheaves on Spaces

6.1. Introduction

Basic properties of sheaves on topological spaces will be explained in this document. A reference is [**God73**].

This will be superseded by the discussion of sheaves over sites later in the documents. But perhaps it makes sense to briefly define some of the notions here.

6.2. Basic notions

The following notions are considered basic and will not be defined, and or proved. This does not mean they are all necessarily easy or well known.

(1) Let X be a topological space. The phrase: "Let $U = \bigcup_{i \in I} U_i$ be an open covering" means the following: I is a set, and for each $i \in I$ we are given an open subset $U_i \subset X$. Furthermore U is the union of the U_i. It is allowed to have $I = \emptyset$ in which case there are no U_i and $U = \emptyset$. It is also allowed, in case $I \neq \emptyset$ to have any or all of the U_i be empty.
(2) etc, etc.

6.3. Presheaves

Definition 6.3.1. Let X be a topological space.

(1) A *presheaf $\mathcal{F}$ of sets on X* is a rule which assigns to each open $U \subset X$ a set $\mathcal{F}(U)$ and to each inclusion $V \subset U$ a map $\rho^U_V : \mathcal{F}(U) \to \mathcal{F}(V)$ such that $\rho^U_U = \mathrm{id}_{\mathcal{F}(U)}$ and whenever $W \subset V \subset U$ we have $\rho^U_W = \rho^V_W \circ \rho^U_V$.
(2) A *morphism $\varphi : \mathcal{F} \to \mathcal{G}$ of presheaves of sets on X* is a rule which assigns to each open $U \subset X$ a map of sets $\varphi : \mathcal{F}(U) \to \mathcal{G}(U)$ compatible with restriction maps, i.e., whenever $V \subset U \subset X$ are open the diagram

$$\begin{array}{ccc} \mathcal{F}(U) & \xrightarrow{\varphi} & \mathcal{G}(U) \\ \downarrow{\scriptstyle \rho^U_V} & & \downarrow{\scriptstyle \rho^U_V} \\ \mathcal{F}(V) & \xrightarrow{\varphi} & \mathcal{G}(V) \end{array}$$

commutes.
(3) The category of presheaves of sets on X will be denoted $PSh(X)$.

The elements of the set $\mathcal{F}(U)$ are called the *sections* of $\mathcal{F}$ over U. For every $V \subset U$ the map $\rho^U_V : \mathcal{F}(U) \to \mathcal{F}(V)$ is called the *restriction map*. We will use the notation $s|_V := \rho^U_V(s)$ if $s \in \mathcal{F}(U)$. This notation is consistent with the notion of restriction of functions from topology because if $W \subset V \subset U$ and s is a section of $\mathcal{F}$ over

U then $s|_W = (s|_V)|_W$ by the property of the restriction maps expressed in the definition above.

Another notation that is often used is to indicate sections over an open U by the symbol $\Gamma(U, -)$ or by $H^0(U, -)$. In other words, the following equalities are tautological

$$\Gamma(U, \mathcal{F}) = \mathcal{F}(U) = H^0(U, \mathcal{F}).$$

In this chapter we will not use this notation, but in others we will.

Definition 6.3.2. Let X be a topological space. Let A be a set. The *constant presheaf with value* A is the presheaf that assigns the set A to every open $U \subset X$, and such that all restriction mappings are id_A.

6.4. Abelian presheaves

In this section we briefly point out some features of the category of presheaves that allow one to define presheaves of abelian groups.

Example 6.4.1. Let X be a topological space X. Consider a rule $\mathcal{F}$ that associates to every open subset a singleton set. Since every set has a unique map into a singleton set, there exist unique restriction maps ρ^U_V. The resulting structure is a presheaf of sets. It is a final object in the category of presheaves of sets, by the property of singleton sets mentioned above. Hence it is also unique up to unique isomorphism. We will sometimes write $*$ for this presheaf.

Lemma 6.4.2. *Let X be a topological space. The category of presheaves of sets on X has products (see Categories, Definition 4.14.5). Moreover, the set of sections of the product $\mathcal{F} \times \mathcal{G}$ over an open U is the product of the sets of sections of $\mathcal{F}$ and $\mathcal{G}$ over U.*

Proof. Namely, suppose $\mathcal{F}$ and $\mathcal{G}$ are presheaves of sets on the topological space X. Consider the rule $U \mapsto \mathcal{F}(U) \times \mathcal{G}(U)$, denoted $\mathcal{F} \times \mathcal{G}$. If $V \subset U \subset X$ are open then define the restriction mapping

$$(\mathcal{F} \times \mathcal{G})(U) \longrightarrow (\mathcal{F} \times \mathcal{G})(V)$$

by mapping $(s, t) \mapsto (s|_V, t|_V)$. Then it is immediately clear that $\mathcal{F} \times \mathcal{G}$ is a presheaf. Also, there are projection maps $p : \mathcal{F} \times \mathcal{G} \to \mathcal{F}$ and $q : \mathcal{F} \times \mathcal{G} \to \mathcal{G}$. We leave it to the reader to show that for any third presheaf $\mathcal{H}$ we have $\mathrm{Mor}(\mathcal{H}, \mathcal{F} \times \mathcal{G}) = \mathrm{Mor}(\mathcal{H}, \mathcal{F}) \times \mathrm{Mor}(\mathcal{H}, \mathcal{G})$. □

Recall that if $(A, + : A \times A \to A, - : A \to A, 0 \in A)$ is an abelian group, then the zero and the negation maps are uniquely determined by the addition law. In other words, it makes sense to say "let $(A, +)$ be an abelian group".

Lemma 6.4.3. *Let X be a topological space. Let $\mathcal{F}$ be a presheaf of sets. Consider the following types of structure on $\mathcal{F}$:*

1. *For every open U the structure of an abelian group on $\mathcal{F}(U)$ such that all restriction maps are abelian group homomorphisms.*
2. *A map of presheaves $+ : \mathcal{F} \times \mathcal{F} \to \mathcal{F}$, a map of presheaves $- : \mathcal{F} \to \mathcal{F}$ and a map $0 : * \to \mathcal{F}$ (see Example 6.4.1) satisfying all the axioms of $+, -, 0$ in a usual abelian group.*

(3) *A map of presheaves* $+ : \mathcal{F} \times \mathcal{F} \to \mathcal{F}$*, a map of presheaves* $- : \mathcal{F} \to \mathcal{F}$ *and a map* $0 : * \to \mathcal{F}$ *such that for each open* $U \subset X$ *the quadruple* $(\mathcal{F}(U), +, -, 0)$ *is an abelian group,*
(4) *A map of presheaves* $+ : \mathcal{F} \times \mathcal{F} \to \mathcal{F}$ *such that for every open* $U \subset X$ *the map* $+ : \mathcal{F}(U) \times \mathcal{F}(U) \to \mathcal{F}(U)$ *defines the structure of an abelian group.*

There are natural bijections between the collections of types of data (1) - (4) above.

Proof. Omitted. □

The lemma says that to give an abelian group object $\mathcal{F}$ in the category of presheaves is the same as giving a presheaf of sets $\mathcal{F}$ such that all the sets $\mathcal{F}(U)$ are endowed with the structure of an abelian group and such that all the restriction mappings are group homomorphisms. For most algebra structures we will take this approach to (pre)sheaves of such objects, i.e., we will define a (pre)sheaf of such objects to be a (pre)sheaf $\mathcal{F}$ of sets all of whose sets of sections $\mathcal{F}(U)$ are endowed with this structure compatibly with the restriction mappings.

Definition 6.4.4. Let X be a topological space.

(1) A *presheaf of abelian groups on* X or an *abelian presheaf over* X is a presheaf of sets $\mathcal{F}$ such that for each open $U \subset X$ the set $\mathcal{F}(U)$ is endowed with the structure of an abelian group, and such that all restriction maps ρ_V^U are homomorphisms of abelian groups, see Lemma 6.4.3 above.
(2) A *morphism of abelian presheaves over* X $\varphi : \mathcal{F} \to \mathcal{G}$ is a morphism of presheaves of sets which induces a homomorphism of abelian groups $\mathcal{F}(U) \to \mathcal{G}(U)$ for every open $U \subset X$.
(3) The category of presheaves of abelian groups on X is denoted $PAb(X)$.

Example 6.4.5. Let X be a topological space. For each $x \in X$ suppose given an abelian group M_x. For $U \subset X$ open we set

$$\mathcal{F}(U) = \bigoplus\nolimits_{x \in U} M_x.$$

We denote a typical element in this abelian group by $\sum_{i=1}^n m_{x_i}$, where $x_i \in U$ and $m_{x_i} \in M_{x_i}$. (Of course we may always choose our representation such that $x_1, \ldots, x_n$ are pairwise distinct.) We define for $V \subset U \subset X$ open a restriction mapping $\mathcal{F}(U) \to \mathcal{F}(V)$ by mapping an element $s = \sum_{i=1}^n m_{x_i}$ to the element $s|_V = \sum_{x_i \in V} m_{x_i}$. We leave it to the reader to verify that this is a presheaf of abelian groups.

6.5. Presheaves of algebraic structures

Let us clarify the definition of presheaves of algebraic structures. Suppose that $\mathcal{C}$ is a category and that $F : \mathcal{C} \to Sets$ is a faithful functor. Typically F is a "forgetful" functor. For an object $M \in \mathrm{Ob}(\mathcal{C})$ we often call $F(M)$ the *underlying set* of the object M. If $M \to M'$ is a morphism in $\mathcal{C}$ we call $F(M) \to F(M')$ the *underlying map of sets*. In fact, we will often not distinguish between an object and its underlying set, and similarly for morphisms. So we will say a map of sets $F(M) \to F(M')$ is a *morphism of algebraic structures*, if it is equal to $F(f)$ for some morphism $f : M \to M'$ in $\mathcal{C}$.

In analogy with Definition 6.4.4 above a "presheaf of objects of $\mathcal{C}$" could be defined by the following data:

(1) a presheaf of sets $\mathcal{F}$, and
(2) for every open $U \subset X$ a choice of an object $A(U) \in \mathrm{Ob}(\mathcal{C})$

subject to the following conditions (using the phraseology above)

(1) for every open $U \subset X$ the set $\mathcal{F}(U)$ is the underlying set of $A(U)$, and
(2) for every $V \subset U \subset X$ open the map of sets $\rho_V^U : \mathcal{F}(U) \to \mathcal{F}(V)$ is a morphism of algebraic structures.

In other words, for every $V \subset U$ open in X the restriction mappings ρ_V^U is the image $F(\alpha_V^U)$ for some unique morphism $\alpha_V^U : A(U) \to A(V)$ in the category $\mathcal{C}$. The uniqueness is forced by the condition that F is faithful; it also implies that $\alpha_W^U = \alpha_W^V \circ \alpha_V^U$ whenever $W \subset V \subset U$ are open in X. The system $(A(-), \alpha_V^U)$ is what we will define as a presheaf with values in $\mathcal{C}$ on X, compare Sites, Definition 7.2.2. We recover our presheaf of sets $(\mathcal{F}, \rho_V^U)$ via the rules $\mathcal{F}(U) = F(A(U))$ and $\rho_V^U = F(\alpha_V^U)$.

Definition 6.5.1. Let X be a topological space. Let $\mathcal{C}$ be a category.

(1) A *presheaf $\mathcal{F}$ on X with values in $\mathcal{C}$* is given by a rule which assigns to every open $U \subset X$ an object $\mathcal{F}(U)$ of $\mathcal{C}$ and to each inclusion $V \subset U$ a morphism $\rho_V^U : \mathcal{F}(U) \to \mathcal{F}(V)$ in $\mathcal{C}$ such that whenever $W \subset V \subset U$ we have $\rho_W^U = \rho_W^V \circ \rho_V^U$.
(2) A *morphism $\varphi : \mathcal{F} \to \mathcal{G}$ of presheaves with value in $\mathcal{C}$* is given by a morphism $\varphi : \mathcal{F}(U) \to \mathcal{G}(U)$ in $\mathcal{C}$ compatible with restriction morphisms.

Definition 6.5.2. Let X be a topological space. Let $\mathcal{C}$ be a category. Let $F : \mathcal{C} \to \mathit{Sets}$ be a faithful functor. Let $\mathcal{F}$ be a presheaf on X with values in $\mathcal{C}$. The presheaf of sets $U \mapsto F(\mathcal{F}(U))$ is called the *underlying presheaf of sets of $\mathcal{F}$*.

It is customary to use the same letter $\mathcal{F}$ to denote the underlying presheaf of sets, and this makes sense according to our discussion preceding Definition 6.5.1. In particular, the phrase "let $s \in \mathcal{F}(U)$" or "let s be a section of $\mathcal{F}$ over U" signifies that $s \in F(\mathcal{F}(U))$.

This notation and these definitions apply in particular to: *Presheaves of (not necessarily abelian) groups, rings, modules over a fixed ring, vector spaces over a fixed field,* etc and *morphisms between these.*

6.6. Presheaves of modules

Suppose that $\mathcal{O}$ is a presheaf of rings on X. We would like to define the notion of a presheaf of $\mathcal{O}$-modules over X. In analogy with Definition 6.4.4 we are tempted to define this as a sheaf of sets $\mathcal{F}$ such that for every open $U \subset X$ the set $\mathcal{F}(U)$ is endowed with the structure of an $\mathcal{O}(U)$-module compatible with restriction mappings (of $\mathcal{F}$ and $\mathcal{O}$). However, it is customary (and equivalent) to define it as in the following definition.

Definition 6.6.1. Let X be a topological space, and let $\mathcal{O}$ be a presheaf of rings on X.

(1) A *presheaf of $\mathcal{O}$-modules* is given by an abelian presheaf $\mathcal{F}$ together with a map of presheaves of sets

$$\mathcal{O} \times \mathcal{F} \longrightarrow \mathcal{F}$$

such that for every open $U \subset X$ the map $\mathcal{O}(U) \times \mathcal{F}(U) \to \mathcal{F}(U)$ defines the structure of an $\mathcal{O}(U)$-module structure on the abelian group $\mathcal{F}(U)$.

(2) A *morphism* $\varphi : \mathcal{F} \to \mathcal{G}$ *of presheaves of* $\mathcal{O}$*-modules* is a morphism of abelian presheaves $\varphi : \mathcal{F} \to \mathcal{G}$ such that the diagram

$$\begin{array}{ccc} \mathcal{O} \times \mathcal{F} & \longrightarrow & \mathcal{F} \\ {\scriptstyle \mathrm{id} \times \varphi} \downarrow & & \downarrow {\scriptstyle \varphi} \\ \mathcal{O} \times \mathcal{G} & \longrightarrow & \mathcal{G} \end{array}$$

commutes.
(3) The set of $\mathcal{O}$-module morphisms as above is denoted $\mathrm{Hom}_{\mathcal{O}}(\mathcal{F}, \mathcal{G})$.
(4) The category of presheaves of $\mathcal{O}$-modules is denoted $PMod(\mathcal{O})$.

Suppose that $\mathcal{O}_1 \to \mathcal{O}_2$ is a morphism of presheaves of rings on X. In this case, if $\mathcal{F}$ is a presheaf of $\mathcal{O}_2$-modules then we can think of $\mathcal{F}$ as a presheaf of $\mathcal{O}_1$-modules by using the composition

$$\mathcal{O}_1 \times \mathcal{F} \to \mathcal{O}_2 \times \mathcal{F} \to \mathcal{F}.$$

We sometimes denote this by $\mathcal{F}_{\mathcal{O}_1}$ to indicate the restriction of rings. We call this the *restriction of* $\mathcal{F}$. We obtain the restriction functor

$$PMod(\mathcal{O}_2) \longrightarrow PMod(\mathcal{O}_1)$$

On the other hand, given a presheaf of $\mathcal{O}_1$-modules $\mathcal{G}$ we can construct a presheaf of $\mathcal{O}_2$-modules $\mathcal{O}_2 \otimes_{p,\mathcal{O}_1} \mathcal{G}$ by the rule

$$\left(\mathcal{O}_2 \otimes_{p,\mathcal{O}_1} \mathcal{G}\right)(U) = \mathcal{O}_2(U) \otimes_{\mathcal{O}_1(U)} \mathcal{G}(U)$$

The index p stands for "presheaf" and not "point". This presheaf is called the tensor product presheaf. We obtain the *change of rings* functor

$$PMod(\mathcal{O}_1) \longrightarrow PMod(\mathcal{O}_2)$$

Lemma 6.6.2. *With X, $\mathcal{O}_1$, $\mathcal{O}_2$, $\mathcal{F}$ and $\mathcal{G}$ as above there exists a canonical bijection*

$$\mathrm{Hom}_{\mathcal{O}_1}(\mathcal{G}, \mathcal{F}_{\mathcal{O}_1}) = \mathrm{Hom}_{\mathcal{O}_2}(\mathcal{O}_2 \otimes_{p,\mathcal{O}_1} \mathcal{G}, \mathcal{F})$$

In other words, the restriction and change of rings functors are adjoint to each other.

Proof. This follows from the fact that for a ring map $A \to B$ the restriction functor and the change of ring functor are adjoint to each other. □

6.7. Sheaves

In this section we explain the sheaf condition.

Definition 6.7.1. Let X be a topological space.

(1) A *sheaf* $\mathcal{F}$ *of sets on* X is a presheaf of sets which satisfies the following additional property: Given any open covering $U = \bigcup_{i \in I} U_i$ and any collection of sections $s_i \in \mathcal{F}(U_i)$, $i \in I$ such that $\forall i, j \in I$

$$s_i|_{U_i \cap U_j} = s_j|_{U_i \cap U_j}$$

there exists a unique section $s \in \mathcal{F}(U)$ such that $s_i = s|_{U_i}$ for all $i \in I$.
(2) A *morphism of sheaves of sets* is simply a morphism of presheaves of sets.
(3) The category of sheaves of sets on X is denoted $Sh(X)$.

Remark 6.7.2. There is always a bit of confusion as to whether it is necessary to say something about the set of sections of a sheaf over the empty set $\emptyset \subset X$. It is necessary, and we already did if you read the definition right. Namely, note that the empty set is covered by the empty open covering, and hence the "collection of section s_i" from the definition above actually form an element of the empty product which is the final object of the category the sheaf has values in. In other words, if you read the definition right you automatically deduce that $\mathcal{F}(\emptyset) = $ *a final object*, which in the case of a sheaf of sets is a singleton. If you do not like this argument, then you can just require that $\mathcal{F}(\emptyset) = \{*\}$.

In particular, this condition will then ensure that if $U, V \subset X$ are open and *disjoint* then

$$\mathcal{F}(U \cup V) = \mathcal{F}(U) \times \mathcal{F}(V).$$

(Because the fibre product over a final object is a product.)

Example 6.7.3. Let X, Y be topological spaces. Consider the rule $\mathcal{F}$ wich associates to the open $U \subset X$ the set

$$\mathcal{F}(U) = \{f : U \to Y \mid f \text{ is continuous}\}$$

with the obvious restriction mappings. We claim that $\mathcal{F}$ is a sheaf. To see this suppose that $U = \bigcup_{i \in I} U_i$ is an open covering, and $f_i \in \mathcal{F}(U_i)$, $i \in I$ with $f_i|_{U_i \cap U_j} = f_j|_{U_i \cap U_j}$ for all $i, j \in I$. In this case define $f : U \to Y$ by setting $f(u)$ equal to the value of $f_i(u)$ for any $i \in I$ such that $u \in U_i$. This is well defined by assumption. Moreover, $f : U \to Y$ is a map such that its restriction to U_i agrees with the continuous map U_i. Hence clearly f is continuous!

We can use the result of the example to define constant sheaves. Namely, suppose that A is a set. Endow A with the discrete topology. Let $U \subset X$ be an open subset. Then we have

$$\{f : U \to A \mid f \text{ continuous}\} = \{f : U \to A \mid f \text{ locally constant}\}.$$

Thus the rule which assigns to an open all locally constant maps into A is a sheaf.

Definition 6.7.4. Let X be a topological space. Let A be a set. The *constant sheaf with value* A denoted $\underline{A}$, or $\underline{A}_X$ is the sheaf that assigns to an open $U \subset X$ the set of all locally constant maps $U \to A$ with restriction mappings given by restrictions of functions.

Example 6.7.5. Let X be a topological space. Let $(A_x)_{x \in X}$ be a family of sets A_x indexed by points $x \in X$. We are going to construct a sheaf of sets Π from this data. For $U \subset X$ open set

$$\Pi(U) = \prod_{x \in U} A_x.$$

For $V \subset U \subset X$ open define a restriction mapping by the following rule: An element $s = (a_x)_{x \in U} \in \Pi(U)$ restricts to $s|_V = (a_x)_{x \in V}$. It is obvious that this defines a presheaf of sets. We claim this is a sheaf. Namely, let $U = \bigcup U_i$ be an open covering. Suppose that $s_i \in \Pi(U_i)$ are such that s_i and s_j agree over $U_i \cap U_j$. Write $s_i = (a_{i,x})_{x \in U_i}$. The compatibility condition implies that $a_{i,x} = a_{j,x}$ in the set A_x whenever $x \in U_i \cap U_j$. Hence there exists a unique element $s = (a_x)_{x \in U}$ in $\Pi(U) = \prod_{x \in U} A_x$ with the property that $a_x = a_{i,x}$ whenever $x \in U_i$ for some i. Of course this element s has the property that $s|_{U_i} = s_i$ for all i.

Example 6.7.6. Let X be a topological space. Suppose for each $x \in X$ we are given an abelian group M_x. Consider the presheaf $\mathcal{F} : U \mapsto \bigoplus_{x \in U} M_x$ defined in Example 6.4.5. This is not a sheaf in general. For example, if X is an infinite set with the discrete topology, then the sheaf condition would imply that $\mathcal{F}(X) = \prod_{x \in X} \mathcal{F}(\{x\})$ but by definition we have $\mathcal{F}(X) = \bigoplus_{x \in X} M_x = \bigoplus_{x \in X} \mathcal{F}(\{x\})$. And an infinite direct sum is in general different from an infinite direct product.

However, if X is a topological space such that every open of X is quasi-compact, then $\mathcal{F}$ *is* a sheaf. This is left as an exercise to the reader.

6.8. Abelian sheaves

Definition 6.8.1. Let X be a topological space.

(1) An *abelian sheaf on X* or *sheaf of abelian groups on X* is an abelian presheaf on X such that the underlying presheaf of sets is a sheaf.
(2) The category of sheaves of abelian groups is denoted $Ab(X)$.

Let X be a topological space. In the case of an abelian presheaf $\mathcal{F}$ the sheaf condition with regards to an open covering $U = \bigcup U_i$ is often expressed by saying that the complex of abelian groups

$$0 \to \mathcal{F}(U) \to \prod\nolimits_i \mathcal{F}(U_i) \to \prod\nolimits_{(i_0,i_1)} \mathcal{F}(U_{i_0} \cap U_{i_1})$$

is exact. The first map is the usual one, whereas the second maps the element $(s_i)_{i \in I}$ to the element

$$(s_{i_0}|_{U_{i_0} \cap U_{i_1}} - s_{i_1}|_{U_{i_0} \cap U_{i_1}})_{(i_0,i_1)} \in \prod\nolimits_{(i_0,i_1)} \mathcal{F}(U_{i_0} \cap U_{i_1})$$

6.9. Sheaves of algebraic structures

Let us clarify the definition of sheaves of certain types of structures. First, let us reformulate the sheaf condition. Namely, suppose that $\mathcal{F}$ is a presheaf of sets on the topological space X. The sheaf condition can be reformulated as follows. Let $U = \bigcup_{i \in I} U_i$ be an open covering. Consider the diagram

$$\mathcal{F}(U) \longrightarrow \prod\nolimits_{i \in I} \mathcal{F}(U_i) \rightrightarrows \prod\nolimits_{(i_0,i_1) \in I \times I} \mathcal{F}(U_{i_0} \cap U_{i_1})$$

Here the left map is defined by the rule $s \mapsto \prod_{i \in I} s|_{U_i}$. The two maps on the right are the maps

$$\prod\nolimits_i s_i \mapsto \prod\nolimits_{(i_0,i_1)} s_{i_0}|_{U_{i_0} \cap U_{i_1}} \text{ resp. } \prod\nolimits_i s_i \mapsto \prod\nolimits_{(i_0,i_1)} s_{i_1}|_{U_{i_0} \cap U_{i_1}}.$$

The sheaf condition exactly says that the left arrow is the equalizer of the right two. This generalizes immediately to the case of presheaves with values in a category as long as the category has products.

Definition 6.9.1. Let X be a topological space. Let $\mathcal{C}$ be a category with products. A presheaf $\mathcal{F}$ with values in $\mathcal{C}$ on X is a *sheaf* if for every open covering the diagram

$$\mathcal{F}(U) \longrightarrow \prod\nolimits_{i \in I} \mathcal{F}(U_i) \rightrightarrows \prod\nolimits_{(i_0,i_1) \in I \times I} \mathcal{F}(U_{i_0} \cap U_{i_1})$$

is an equalizer diagram in the category $\mathcal{C}$.

Suppose that $\mathcal{C}$ is a category and that $F : \mathcal{C} \to Sets$ is a faithful functor. A good example to keep in mind is the case where $\mathcal{C}$ is the category of abelian groups and F is the forgetful functor. Consider a presheaf $\mathcal{F}$ with values in $\mathcal{C}$ on X. We would like to reformulate the condition above in terms of the underlying presheaf of sets (Definition 6.5.2). Note that the underlying presheaf of sets is a sheaf of sets if and only if all the diagrams

$$F(\mathcal{F}(U)) \longrightarrow \prod_{i \in I} F(\mathcal{F}(U_i)) \rightrightarrows \prod_{(i_0,i_1) \in I \times I} F(\mathcal{F}(U_{i_0} \cap U_{i_1}))$$

of sets – after applying the forgetful functor F – are equalizer diagrams! Thus we would like $\mathcal{C}$ to have products and equalizers and we would like F to commute with them. This is equivalent to the condition that $\mathcal{C}$ has limits and that F commutes with them, see Categories, Lemma 4.14.10. But this is not yet good enough (see Example 6.9.4); we also need F to *reflect isomorphisms*. This property means that given a morphism $f : A \to A'$ in $\mathcal{C}$, then f is an isomorphism if (and only if) $F(f)$ is a bijection.

Lemma 6.9.2. *Suppose the category $\mathcal{C}$ and the functor $F : \mathcal{C} \to Sets$ have the following properties:*

(1) *F is faithful,*
(2) *$\mathcal{C}$ has limits and F commutes with them, and*
(3) *the functor F reflects isomorphisms.*

Let X be a topological space. Let $\mathcal{F}$ be a presheaf with values in $\mathcal{C}$. Then $\mathcal{F}$ is a sheaf if and only if the underlying presheaf of sets is a sheaf.

Proof. Assume that $\mathcal{F}$ is a sheaf. Then $\mathcal{F}(U)$ is the equalizer of the diagram above and by assumption we see $F(\mathcal{F}(U))$ is the equalizer of the corresponding diagram of sets. Hence $F(\mathcal{F})$ is a sheaf of sets.

Assume that $F(\mathcal{F})$ is a sheaf. Let $E \in \mathrm{Ob}(\mathcal{C})$ be the equalizer of the two parallel arrows in Definition 6.9.1. We get a canonical morphism $\mathcal{F}(U) \to E$, simply because $\mathcal{F}$ is a presheaf. By assumption, the induced map $F(\mathcal{F}(U)) \to F(E)$ is an isomorphism, because $F(E)$ is the equalizer of the corresponding diagram of sets. Hence we see $\mathcal{F}(U) \to E$ is an isomorphism by condition (3) of the lemma. □

The lemma in particular applies to *sheaves of groups, rings, algebras over a fixed ring, modules over a fixed ring, vector spaces over a fixed field,* etc. In other words, these are presheaves of groups, rings, modules over a fixed ring, vector spaces over a fixed field, etc such that the underlying presheaf of sets is a sheaf.

Example 6.9.3. Let X be a topological space. For each open $U \subset X$ consider the $\mathbf{R}$-algebra $\mathcal{C}^0(U) = \{f : U \to \mathbf{R} \mid f \text{ is continuous}\}$. There are obvious restriction mappings that turn this into a presheaf of $\mathbf{R}$-algebras over X. By Example 6.7.3 it is a sheaf of sets. Hence by the Lemma 6.9.2 it is a sheaf of $\mathbf{R}$-algebras over X.

Example 6.9.4. Consider the category of topological spaces *Top*. There is a natural faithful functor $Top \to Sets$ which commutes with products and equalizers. But it does not reflect isomorphisms. And, in fact it turns out that the analogue of Lemma 6.9.2 is wrong. Namely, suppose $X = \mathbf{N}$ with the discrete topology. Let A_i, for $i \in \mathbf{N}$ be a discrete topological space. For any subset $U \subset \mathbf{N}$ define $\mathcal{F}(U) = \prod_{i \in U} A_i$ with the discrete topology. Then this is a presheaf of topological spaces whose underlying presheaf of sets is a sheaf, see Example 6.7.5. However, if each A_i

has at least two elements, then this is not a sheaf of topological spaces according to Definition 6.9.1. The reader may check that putting the *product topology* on each $\mathcal{F}(U) = \prod_{i \in U} A_i$ does lead to a sheaf of topological spaces over X.

6.10. Sheaves of modules

Definition 6.10.1. Let X be a topological space. Let $\mathcal{O}$ be a sheaf of rings on X.

(1) A *sheaf of $\mathcal{O}$-modules* is a presheaf of $\mathcal{O}$-modules $\mathcal{F}$, see Definition 6.6.1, such that the underlying presheaf of abelian groups $\mathcal{F}$ is a sheaf.
(2) A *morphism of sheaves of $\mathcal{O}$-modules* is a morphism of presheaves of $\mathcal{O}$-modules.
(3) Given sheaves of $\mathcal{O}$-modules $\mathcal{F}$ and $\mathcal{G}$ we denote $\mathrm{Hom}_{\mathcal{O}}(\mathcal{F}, \mathcal{G})$ the set of morphism of sheaves of $\mathcal{O}$-modules.
(4) The category of sheaves of $\mathcal{O}$-modules is denoted $\mathit{Mod}(\mathcal{O})$.

This definition kind of makes sense even if $\mathcal{O}$ is just a presheaf of rings, although we do not know any examples where this is useful, and we will avoid using the terminology "sheaves of $\mathcal{O}$-modules" in case $\mathcal{O}$ is not a sheaf of rings.

6.11. Stalks

Let X be a topological space. Let $x \in X$ be a point. Let $\mathcal{F}$ be a presheaf of sets on X. The *stalk of $\mathcal{F}$ at x* is the set

$$\mathcal{F}_x = \mathrm{colim}_{x \in U} \mathcal{F}(U)$$

where the colimit is over the set of open neighbourhoods U of x in X. The set of open neighbourhoods is (partially) ordered by (reverse) inclusion: We say $U \geq U' \Leftrightarrow U \subset U'$. The transition maps in the system are given by the restriction maps of $\mathcal{F}$. See Categories, Section 4.21 for notation and terminology regarding (co)limits over systems. Note that the colimit is a directed colimit. Thus it is easy to describe $\mathcal{F}_x$. Namely,

$$\mathcal{F}_x = \{(U, s) \mid x \in U, s \in \mathcal{F}(U)\}/\sim$$

with equivalence relation given by $(U, s) \sim (U', s')$ if and only if there exists an open $U'' \subset U \cap U'$ with $x \in U''$ and $s|_{U''} = s'|_{U''}$. By abuse of notation we will often denote (U, s), s_x, or even s the corresponding element in $\mathcal{F}_x$. Also we will say $s = s'$ in $\mathcal{F}_x$ for two local sections of $\mathcal{F}$ defined in an open neighbourhood of x to denote that they have the same image in $\mathcal{F}_x$.

An obvious consequence of this definition is that for any open $U \subset X$ there is a canonical map

$$\mathcal{F}(U) \longrightarrow \prod\nolimits_{x \in U} \mathcal{F}_x$$

defined by $s \mapsto \prod_{x \in U}(U, s)$. Think about it!

Lemma 6.11.1. *Let $\mathcal{F}$ be a sheaf of sets on the topological space X. For every open $U \subset X$ the map*

$$\mathcal{F}(U) \longrightarrow \prod\nolimits_{x \in U} \mathcal{F}_x$$

is injective.

Proof. Suppose that $s, s' \in \mathcal{F}(U)$ map to the same element in every stalk $\mathcal{F}_x$ for all $x \in U$. This means that for every $x \in U$, there exists an open $V^x \subset U$, $x \in V^x$ such that $s|_{V^x} = s'|_{V^x}$. But then $U = \bigcup_{x \in U} V^x$ is an open covering. Thus by the uniqueness in the sheaf condition we see that $s = s'$. □

Definition 6.11.2. Let X be a topological space. A presheaf of sets $\mathcal{F}$ on X is *separated* if for every open $U \subset X$ the map $\mathcal{F}(U) \to \prod_{x \in U} \mathcal{F}_x$ is injective.

Another observation is that the construction of the stalk $\mathcal{F}_x$ is functorial in the presheaf $\mathcal{F}$. In other words, it gives a functor

$$PSh(X) \longrightarrow Sets,\ \mathcal{F} \longmapsto \mathcal{F}_x.$$

This functor is called the *stalk functor*. Namely, if $\varphi : \mathcal{F} \to \mathcal{G}$ is a morphism of presheaves, then we define $\varphi_x : \mathcal{F}_x \to \mathcal{G}_x$ by the rule $(U, s) \mapsto (U, \varphi(s))$. To see that this works we have to check that if $(U, s) = (U', s')$ in $\mathcal{F}_x$ then also $(U, \varphi(s)) = (U', \varphi(s'))$ in $\mathcal{G}_x$. This is clear since φ is compatible with the restriction mappings.

Example 6.11.3. Let X be a topological space. Let A be a set. Denote temporarily A_p the constant presheaf with value A (p for presheaf – not for point). There is a canonical map of presheaves $A_p \to \underline{A}$ into the constant sheaf with value A. For every point we have canonical bijections $A = (A_p)_x = \underline{A}_x$, where the second map is induced by functoriality from the map $A_p \to \underline{A}$.

Example 6.11.4. Suppose $X = \mathbf{R}^n$ with the Euclidean topology. Consider the presheaf of $\mathcal{C}^\infty$ functions on X, denoted $\mathcal{C}^\infty_{\mathbf{R}^n}$. In other words, $\mathcal{C}^\infty_{\mathbf{R}^n}(U)$ is the set of $\mathcal{C}^\infty$-functions $f : U \to \mathbf{R}$. As in Example 6.7.3 it is easy to show that this is a sheaf. In fact it is a sheaf of $\mathbf{R}$-vector spaces.

Next, let $x \in X = \mathbf{R}^n$ be a point. How do we think of an element in the stalk $\mathcal{C}^\infty_{\mathbf{R}^n, x}$? Such an element is given by a $\mathcal{C}^\infty$-function f whose domain contains x. And a pair of such functions f, g determine the same element of the stalk if they agree in a neighbourhood of x. In other words, an element if $\mathcal{C}^\infty_{\mathbf{R}^n, x}$ is the same thing as what is sometimes called a *germ of a $\mathcal{C}^\infty$-function at x*.

Example 6.11.5. Let X be a topological space. Let A_x be a set for each $x \in X$. Consider the sheaf $\mathcal{F} : U \mapsto \prod_{x \in U} A_x$ of Example 6.7.5. We would just like to point out here that the stalk $\mathcal{F}_x$ of $\mathcal{F}$ at x is in general *not* equal to the set A_x. Of course there is a map $\mathcal{F}_x \to A_x$, but that is in general the best you can say. For example, suppose $x = \lim x_n$ with $x_n \neq x_m$ for all $n \neq m$ and suppose that $A_y = \{0, 1\}$ for all $y \in X$. Then $\mathcal{F}_x$ maps onto the (infinite) set of tails of sequences of 0s and 1s. Namely, every open neighbourhood of x contains almost all of the x_n. On the other hand, if every neighbourhood of x contains a point y such that $A_y = \emptyset$, then $\mathcal{F}_x = \emptyset$.

6.12. Stalks of abelian presheaves

We first deal with the case of abelian groups as a model for the general case.

Lemma 6.12.1. *Let X be a topological space. Let $\mathcal{F}$ be a presheaf of abelian groups on X. There exists a unique structure of an abelian group on $\mathcal{F}_x$ such that for every $U \subset X$ open, $x \in U$ the map $\mathcal{F}(U) \to \mathcal{F}_x$ is a group homomorphism. Moreover,*

$$\mathcal{F}_x = \operatorname{colim}_{x \in U} \mathcal{F}(U)$$

holds in the category of abelian groups.

Proof. We define addition of a pair of elements (U, s) and (V, t) as the pair $(U \cap V, s|_{U\cap V} + t|_{U\cap V})$. The rest is easy to check. □

What is crucial in the proof above is that the partially ordered set of open neighbourhoods is a directed system (compare Categories, Definition 4.21.2). Namely, the coproduct of two abelian groups A, B is the direct sum $A \oplus B$, whereas the coproduct in the category of sets is the disjoint union $A \coprod B$, showing that colimits in the category of abelian groups do not agree with colimits in the category of sets in general.

6.13. Stalks of presheaves of algebraic structures

The proof of Lemma 6.12.1 will work for any type of algebraic structure such that directed colimits commute with the forgetful functor.

Lemma 6.13.1. *Let $\mathcal{C}$ be a category. Let $F : \mathcal{C} \to Sets$ be a functor. Assume that*

(1) *F is faithful, and*
(2) *directed colimits exist in $\mathcal{C}$ and F commutes with them.*

Let X be a topological space. Let $x \in X$. Let $\mathcal{F}$ be a presheaf with values in $\mathcal{C}$. Then

$$\mathcal{F}_x = \operatorname{colim}_{x\in U} \mathcal{F}(U)$$

exists in $\mathcal{C}$. Its underlying set is equal to the stalk of the underlying presheaf of sets of $\mathcal{F}$. Furthermore, the construction $\mathcal{F} \mapsto \mathcal{F}_x$ is a functor from the category of presheaves with values in $\mathcal{C}$ to $\mathcal{C}$.

Proof. Omitted. □

By the very definition, all the morphisms $\mathcal{F}(U) \to \mathcal{F}_x$ are morphisms in the category $\mathcal{C}$ which (after applying the forgetful functor F) turn into the corresponding maps for the underlying sheaf of sets. As usual we will not distinguish between the morphism in $\mathcal{C}$ and the underlying map of sets, which is permitted since F is faithful.

This lemma applies in particular to: *Presheaves of (not necessarily abelian) groups, rings, modules over a fixed ring, vector spaces over a fixed field.*

6.14. Stalks of presheaves of modules

Lemma 6.14.1. *Let X be a topological space. Let $\mathcal{O}$ be a presheaf of rings on X Let $\mathcal{F}$ be a presheaf $\mathcal{O}$-modules. Let $x \in X$. The canonical map $\mathcal{O}_x \times \mathcal{F}_x \to \mathcal{F}_x$ coming from the multiplication map $\mathcal{O} \times \mathcal{F} \to \mathcal{F}$ defines a $\mathcal{O}_x$-module structure on the abelian group $\mathcal{F}_x$.*

Proof. Omitted. □

Lemma 6.14.2. *Let X be a topological space. Let $\mathcal{O} \to \mathcal{O}'$ be a morphism of presheaves of rings on X Let $\mathcal{F}$ be a presheaf $\mathcal{O}$-modules. Let $x \in X$. We have*

$$\mathcal{F}_x \otimes_{\mathcal{O}_x} \mathcal{O}'_x = (\mathcal{F} \otimes_{p,\mathcal{O}} \mathcal{O}')_x$$

as $\mathcal{O}'_x$-modules.

Proof. Omitted. □

6.15. Algebraic structures

In this section we mildly formalize the notions we have encountered in the sections above.

Definition 6.15.1. A *type of algebraic structure* is given by a category $\mathcal{C}$ and a functor $F : \mathcal{C} \to Sets$ with the following properties

(1) F is faithful,
(2) $\mathcal{C}$ has limits and F commutes with limits,
(3) $\mathcal{C}$ has filtered colimits and F commutes with them, and
(4) F reflects isomorphisms.

We make this definition to point out the properties we will use in a number of arguments below. But we will not actually study this notion in any great detail, since we are prohibited from studying "big" categories by convention, except for those listed in Categories, Remark 4.2.2. Among those the following have the required properties.

Lemma 6.15.2. *The following categories, endowed with the obvious forgetful functor, define types of algebraic structures:*

(1) *The category of pointed sets.*
(2) *The category of abelian groups.*
(3) *The category of groups.*
(4) *The category of monoids.*
(5) *The category of rings.*
(6) *The category of R-modules for a fixed ring R.*
(7) *The category of Lie algebras over a fixed field.*

Proof. Omitted. □

From now on we will think of a (pre)sheaf of algebraic structures and their stalks, in terms of the underlying (pre)sheaf of sets. This is allowable by Lemmas 6.9.2 and 6.13.1.

In the rest of this section we point out some results on algebraic structures that will be useful in the future.

Lemma 6.15.3. *Let $(\mathcal{C}, F)$ be a type of algebraic structure.*

(1) *$\mathcal{C}$ has a final object 0 and $F(0) = \{*\}$.*
(2) *$\mathcal{C}$ has products and $F(\prod A_i) = \prod F(A_i)$.*
(3) *$\mathcal{C}$ has fibre products and $F(A \times_B C) = F(A) \times_{F(B)} F(C)$.*
(4) *$\mathcal{C}$ has equalizers, and if $E \to A$ is the equalizer of $a, b : A \to B$, then $F(E) \to F(A)$ is the equalizer of $F(a), F(b) : F(A) \to F(B)$.*
(5) *$A \to B$ is a monomorphism if and only if $F(A) \to F(B)$ is injective.*
(6) *if $F(a) : F(A) \to F(B)$ is surjective, then a is an epimorphism.*
(7) *given $A_1 \to A_2 \to A_3 \to \ldots$, then $\operatorname{colim} A_i$ exists and $F(\operatorname{colim} A_i) = \operatorname{colim} F(A_i)$, and more generally for any filtered colimit.*

Proof. Omitted. The only interesting statement is (5) which follows because $A \to B$ is a monomorphism if and only if $A \to A \times_B A$ is an isomorphism, and then applying the fact that F reflects isomorphisms. □

Lemma 6.15.4. *Let $(\mathcal{C}, F)$ be a type of algebraic structure. Suppose that $A, B, C \in \mathrm{Ob}(\mathcal{C})$. Let $f : A \to B$ and $g : C \to B$ be morphisms of $\mathcal{C}$. If $F(g)$ is injective, and $Im(F(f)) \subset Im(F(g))$, then f factors as $f = g \circ t$ for some morphism $t : A \to C$.*

Proof. Consider $A \times_B C$. The assumptions imply that $F(A \times_B C) = F(A) \times_{F(B)} F(C) = F(A)$. Hence $A = A \times_B C$ because F reflects isomorphisms. The result follows. □

Example 6.15.5. The lemma will be applied often to the following situation. Suppose that we have a diagram

$$\begin{array}{ccc} A & \longrightarrow & B \\ & & \downarrow \\ C & \longrightarrow & D \end{array}$$

in $\mathcal{C}$. Suppose $C \to D$ is injective on underlying sets, and suppose that the composition $A \to B \to D$ has image on underlying sets in the image of $C \to D$. Then we get a commutative diagram

$$\begin{array}{ccc} A & \longrightarrow & B \\ \downarrow & & \downarrow \\ C & \longrightarrow & D \end{array}$$

in $\mathcal{C}$.

Example 6.15.6. Let $F : \mathcal{C} \to \mathit{Sets}$ be a type of algebraic structures. Let X be a topological space. Suppose that for every $x \in X$ we are given an object $A_x \in \mathrm{ob}(\mathcal{C})$. Consider the presheaf Π with values in $\mathcal{C}$ on X defined by the rule $\Pi(U) = \prod_{x \in U} A_x$ (with obvious restriction mappings). Note that the associated presheaf of sets $U \mapsto F(\Pi(U)) = \prod_{x \in U} F(A_x)$ is a sheaf by Example 6.7.5. Hence Π is a sheaf of algebraic structures of type $(\mathcal{C}, F)$. This gives many examples of sheaves of abelian groups, groups, rings, etc.

6.16. Exactness and points

In any category we have the notion of epimorphism, monomorphism, isomorphism, etc.

Lemma 6.16.1. *Let X be a topological space. Let $\varphi : \mathcal{F} \to \mathcal{G}$ be a morphism of sheaves of sets on X.*

1. *The map φ is a monomorphism in the category of sheaves if and only if for all $x \in X$ the map $\varphi_x : \mathcal{F}_x \to \mathcal{G}_x$ is injective.*
2. *The map φ is an epimorphism in the category of sheaves if and only if for all $x \in X$ the map $\varphi_x : \mathcal{F}_x \to \mathcal{G}_x$ is surjective.*
3. *The map φ is a isomorphism in the category of sheaves if and only if for all $x \in X$ the map $\varphi_x : \mathcal{F}_x \to \mathcal{G}_x$ is bijective.*

Proof. Omitted. □

It follows that in the category of sheaves of sets the notions epimorphism and monomorphism can be described as follows.

Definition 6.16.2. Let X be a topological space.

(1) A presheaf $\mathcal{F}$ is called a *subpresheaf* of a presheaf $\mathcal{G}$ if $\mathcal{F}(U) \subset \mathcal{G}(U)$ for all open $U \subset X$ such that the restriction maps of $\mathcal{G}$ induce the restriction maps of $\mathcal{F}$. If $\mathcal{F}$ and $\mathcal{G}$ are sheaves, then $\mathcal{F}$ is called a *subsheaf* of $\mathcal{G}$. We sometimes indicate this by the notation $\mathcal{F} \subset \mathcal{G}$.
(2) A morphism of presheaves of sets $\varphi : \mathcal{F} \to \mathcal{G}$ on X is called *injective* if and only if $\mathcal{F}(U) \to \mathcal{G}(U)$ is injective for all U open in X.
(3) A morphism of presheaves of sets $\varphi : \mathcal{F} \to \mathcal{G}$ on X is called *surjective* if and only if $\mathcal{F}(U) \to \mathcal{G}(U)$ is surjective for all U open in X.
(4) A morphism of sheaves of sets $\varphi : \mathcal{F} \to \mathcal{G}$ on X is called *injective* if and only if $\mathcal{F}(U) \to \mathcal{G}(U)$ is injective for all U open in X.
(5) A morphism of sheaves of sets $\varphi : \mathcal{F} \to \mathcal{G}$ on X is called *surjective* if and only if for every open U of X and every section s of $\mathcal{G}(U)$ there exists an open covering $U = \bigcup U_i$ such that $s|_{U_i}$ is in the image of $\mathcal{F}(U_i) \to \mathcal{G}(U)$ for all i.

Lemma 6.16.3. *Let X be a topological space.*

(1) *Epimorphisms (resp. monomorphisms) in the category of presheaves are exactly the surjective (resp. injective) maps of presheaves.*
(2) *Epimorphisms (resp. monomorphisms) in the category of sheaves are exactly the surjective (resp. injective) maps of sheaves, and are exactly those maps with are surjective (resp. injective) on all the stalks.*
(3) *The sheafification of a surjective (resp. injective) morphism of presheaves of sets is surjective (resp. injective).*

Proof. Omitted. □

Lemma 6.16.4. *let X be a topological space. Let $(\mathcal{C}, F)$ be a type of algebraic structure. Suppose that $\mathcal{F}$, $\mathcal{G}$ are sheaves on X with values in $\mathcal{C}$. Let $\varphi : \mathcal{F} \to \mathcal{G}$ be a map of the underlying sheaves of sets. If for all points $x \in X$ the map $\mathcal{F}_x \to \mathcal{G}_x$ is a morphism of algebraic structures, then φ is a morphism of sheaves of algebraic structures.*

Proof. Let U be an open subset of X. Consider the diagram of (underlying) sets

$$\begin{array}{ccc} \mathcal{F}(U) & \longrightarrow & \prod_{x \in U} \mathcal{F}_x \\ \downarrow & & \downarrow \\ \mathcal{G}(U) & \longrightarrow & \prod_{x \in U} \mathcal{G}_x \end{array}$$

By assumption, and previous results, all but the left vertical arrow are morphisms of algebraic structures. In addition the bottom horizontal arrow is injective, see Lemma 6.11.1. Hence we conclude by Lemma 6.15.4, see also Example 6.15.5 □

Short exact sequences of abelian sheaves, etc will be discussed in the chapter on sheaves of modules. See Modules, Section 17.3.

6.17. Sheafification

In this section we explain how to get the sheafification of a presheaf on a topological space. We will use stalks to describe the sheafification in this case. This is different from the general procedure described in Sites, Section 7.10, and perhaps somewhat easier to understand.

The basic construction is the following. Let $\mathcal{F}$ be a presheaf of sets $\mathcal{F}$ on a topological space X. For every open $U \subset X$ we define

$$\mathcal{F}^\#(U) = \{(s_u) \in \prod_{u \in U} \mathcal{F}_u \text{ such that } (*)\}$$

where $(*)$ is the property:

(*) For every $u \in U$, there exists an open neighbourhood $u \in V \subset U$, and a section $\sigma \in \mathcal{F}(V)$ such that for all $v \in V$ we have $s_v = (V, \sigma)$ in $\mathcal{F}_v$.

Note that $(*)$ is a condition for each $u \in U$, and that given $u \in U$ the truth of this condition depends only on the values s_v for v in any open neighbourhood of u. Thus it is clear that, if $V \subset U \subset X$ are open, the projection maps

$$\prod_{u \in U} \mathcal{F}_u \longrightarrow \prod_{v \in V} \mathcal{F}_v$$

maps elements of $\mathcal{F}^\#(U)$ into $\mathcal{F}^\#(V)$. In other words, we get the structure of a presheaf of sets on $\mathcal{F}^\#$.

Furthermore, the map $\mathcal{F}(U) \to \prod_{u \in U} \mathcal{F}_u$ described in Section 6.11 clearly has image in $\mathcal{F}^\#(U)$. In addition, if $V \subset U \subset X$ are open then we have the following commutative diagram

$$\begin{array}{ccccc} \mathcal{F}(U) & \longrightarrow & \mathcal{F}^\#(U) & \longrightarrow & \prod_{u \in U} \mathcal{F}_u \\ \downarrow & & \downarrow & & \downarrow \\ \mathcal{F}(V) & \longrightarrow & \mathcal{F}^\#(V) & \longrightarrow & \prod_{v \in V} \mathcal{F}_v \end{array}$$

where the vertical maps are induced from the restriction mappings. Thus we see that there is a canonical morphism of presheaves $\mathcal{F} \to \mathcal{F}^\#$.

In Example 6.7.5 we saw that the rule $\Pi(\mathcal{F}) : U \mapsto \prod_{u \in U} \mathcal{F}_u$ is a sheaf, with obvious restriction mappings. And by construction $\mathcal{F}^\#$ is a subpresheaf of this. In other words, we have morphisms of presheaves

$$\mathcal{F} \to \mathcal{F}^\# \to \Pi(\mathcal{F}).$$

In addition the rule that associates to $\mathcal{F}$ the sequence above is clearly functorial in the presheaf $\mathcal{F}$. This notation will be used in the proofs of the lemmas below.

Lemma 6.17.1. *The presheaf $\mathcal{F}^\#$ is a sheaf.*

Proof. It is probably better for the reader to find their own explanation of this than to read the proof here. In fact the lemma is true for the same reason as why the presheaf of continuous function is a sheaf, see Example 6.7.3 (and this analogy can be made precise using the "espace étalé").

Anyway, let $U = \bigcup U_i$ be an open covering. Suppose that $s_i = (s_{i,u})_{u \in U_i} \in \mathcal{F}^\#(U_i)$ such that s_i and s_j agree over $U_i \cap U_j$. Because $\Pi(\mathcal{F})$ is a sheaf, we find an element $s = (s_u)_{u \in U}$ in $\prod_{u \in U} \mathcal{F}_u$ restricting to s_i on U_i. We have to check property $(*)$. Pick $u \in U$. Then $u \in U_i$ for some i. Hence by $(*)$ for s_i, there exists a V open, $u \in V \subset U_i$ and a $\sigma \in \mathcal{F}(V)$ such that $s_{i,v} = (V, \sigma)$ in $\mathcal{F}_v$ for all $v \in V$. Since $s_{i,v} = s_v$ we get $(*)$ for s. $\square$

Lemma 6.17.2. *Let X be a topological space. Let $\mathcal{F}$ be a presheaf of sets on X. Let $x \in X$. Then $\mathcal{F}_x = \mathcal{F}^\#_x$.*

Proof. The map $\mathcal{F}_x \to \mathcal{F}_x^\#$ is injective, since already the map $\mathcal{F}_x \to \Pi(\mathcal{F})_x$ is injective. Namely, there is a canonical map $\Pi(\mathcal{F})_x \to \mathcal{F}_x$ which is a left inverse to the map $\mathcal{F}_x \to \Pi(\mathcal{F})_x$, see Example 6.11.5. To show that it is surjective, suppose that $\overline{s} \in \mathcal{F}_x^\#$. We can find an open neighbourhood U of x such that $\overline{s}$ is the equivalence class of (U, s) with $s \in \mathcal{F}^\#(U)$. By definition, this means there exists an open neighbourhood $V \subset U$ of x and a section $\sigma \in \mathcal{F}(V)$ such that $s|_V$ is the image of σ in $\Pi(\mathcal{F})(V)$. Clearly the class of (V, σ) defines an element of $\mathcal{F}_x$ mapping to $\overline{s}$. □

Lemma 6.17.3. *Let $\mathcal{F}$ be a presheaf of sets on X. Any map $\mathcal{F} \to \mathcal{G}$ into a sheaf of sets factors uniquely as $\mathcal{F} \to \mathcal{F}^\# \to \mathcal{G}$.*

Proof. Clearly, there is a commutative diagram

$$\begin{array}{ccccc} \mathcal{F} & \longrightarrow & \mathcal{F}^\# & \longrightarrow & \Pi(\mathcal{F}) \\ \downarrow & & \downarrow & & \downarrow \\ \mathcal{G} & \longrightarrow & \mathcal{G}^\# & \longrightarrow & \Pi(\mathcal{G}) \end{array}$$

So it suffices to prove that $\mathcal{G} = \mathcal{G}^\#$. To see this it suffices to prove, for every point $x \in X$ the map $\mathcal{G}_x \to \mathcal{G}_x^\#$ is bijective, by Lemma 6.16.1. And this is Lemma 6.17.2 above. □

This lemma really says that there is an adjoint pair of functors: $i : Sh(X) \to PSh(X)$ (inclusion) and $\# : PSh(X) \to Sh(X)$ (sheafification). The formula is that

$$\mathrm{Mor}_{PSh(X)}(\mathcal{F}, i(\mathcal{G})) = \mathrm{Mor}_{Sh(X)}(\mathcal{F}^\#, \mathcal{G})$$

which says that sheafification is a left adjoint of the inclusion functor. See Categories, Section 4.24.

Example 6.17.4. See Example 6.11.3 for notation. The map $A_p \to \underline{A}$ induces a map $A_p^\# \to \underline{A}$. It is easy to see that this is an isomorphism. In words: The sheafification of the constant presheaf with value A is the constant sheaf with value A.

Lemma 6.17.5. *Let X be a topological space. A presheaf $\mathcal{F}$ is separated (see Definition 6.11.2) if and only if the canonical map $\mathcal{F} \to \mathcal{F}^\#$ is injective.*

Proof. This is clear from the construction of $\mathcal{F}^\#$ in this section. □

6.18. Sheafification of abelian presheaves

The following strange looking lemma is likely unnecessary, but very convenient to deal with sheafification of presheaves of algebraic structures.

Lemma 6.18.1. *Let X be a topological space. Let $\mathcal{F}$ be a presheaf of sets on X. Let $U \subset X$ be open. There is a canonical fibre product diagram*

$$\begin{array}{ccc} \mathcal{F}^\#(U) & \longrightarrow & \Pi(\mathcal{F})(U) \\ \downarrow & & \downarrow \\ \prod_{x \in U} \mathcal{F}_x & \longrightarrow & \prod_{x \in U} \Pi(\mathcal{F})_x \end{array}$$

where the maps are the following:

(1) *The left vertical map has components* $\mathcal{F}^\#(U) \to \mathcal{F}^\#_x = \mathcal{F}_x$ *where the equality is Lemma 6.17.2.*
(2) *The top horizontal map comes from the map of presheaves* $\mathcal{F} \to \Pi(\mathcal{F})$ *described in Section 6.17.*
(3) *The right vertical map has obvious component maps* $\Pi(\mathcal{F})(U) \to \Pi(\mathcal{F})_x$.
(4) *The bottom horizontal map has components* $\mathcal{F}_x \to \Pi(\mathcal{F})_x$ *which come from the map of presheaves* $\mathcal{F} \to \Pi(\mathcal{F})$ *described in Section 6.17.*

Proof. It is clear that the diagram commutes. We have to show it is a fibre product diagram. The bottom horizontal arrow is injective since all the maps $\mathcal{F}_x \to \Pi(\mathcal{F})_x$ are injective (see beginning proof of Lemma 6.17.2). A section $s \in \Pi(\mathcal{F})(U)$ is in $\mathcal{F}^\#$ if and only if $(*)$ holds. But $(*)$ says that around every point the section s comes from a section of $\mathcal{F}$. By definition of the stalk functors, this is equivalent to saying that the value of s in every stalk $\Pi(\mathcal{F})_x$ comes from an element of the stalk $\mathcal{F}_x$. Hence the lemma. □

Lemma 6.18.2. *Let X be a topological space. Let $\mathcal{F}$ be an abelian presheaf on X. Then there exists a unique structure of abelian sheaf on $\mathcal{F}^\#$ such that $\mathcal{F} \to \mathcal{F}^\#$ is a morphism of abelian presheaves. Moreover, the following adjointness property holds*

$$\mathrm{Mor}_{PAb(X)}(\mathcal{F}, i(\mathcal{G})) = \mathrm{Mor}_{Ab(X)}(\mathcal{F}^\#, \mathcal{G}).$$

Proof. Recall the sheaf of sets $\Pi(\mathcal{F})$ defined in Section 6.17. All the stalks $\mathcal{F}_x$ are abelian groups, see Lemma 6.12.1. Hence $\Pi(\mathcal{F})$ is a sheaf of abelian groups by Example 6.15.6. Also, it is clear that the map $\mathcal{F} \to \Pi(\mathcal{F})$ is a morphism of abelian presheaves. If we show that condition $(*)$ of Section 6.17 defines a subgroup of $\Pi(\mathcal{F})(U)$ for all open subsets $U \subset X$, then $\mathcal{F}^\#$ canonically inherits the structure of abelian sheaf. This is quite easy to do by hand, and we leave it to the reader to find a good simple argument. The argument we use here, which generalizes to presheaves of algebraic structures is the following: Lemma 6.18.1 show that $\mathcal{F}^\#(U)$ is the fibre product of a diagram of abelian groups. Thus $\mathcal{F}^\#$ is an abelian subgroup as desired.

Note that at this point $\mathcal{F}^\#_x$ is an abelian group by Lemma 6.12.1 and that $\mathcal{F}_x \to \mathcal{F}^\#_x$ is a bijection (Lemma 6.17.2) and a homomorphism of abelian groups. Hence $\mathcal{F}_x \to \mathcal{F}^\#_x$ is an isomorphism of abelian groups. This will be used below without further mention.

To prove the adjointness property we use the adjointness property of sheafification of presheaves of sets. For example if $\psi : \mathcal{F} \to i(\mathcal{G})$ is morphism of presheaves then we obtain a morphism of sheaves $\psi' : \mathcal{F}^\# \to \mathcal{G}$. What we have to do is to check that this is a morphism of abelian sheaves. We may do this for example by noting that it is true on stalks, by Lemma 6.17.2, and then using Lemma 6.16.4 above. □

6.19. Sheafification of presheaves of algebraic structures

Lemma 6.19.1. *Let X be a topological space. Let $(\mathcal{C}, F)$ be a type of algebraic structure. Let $\mathcal{F}$ be a presheaf with values in $\mathcal{C}$ on X. Then there exists a sheaf $\mathcal{F}^\#$ with values in $\mathcal{C}$ and a morphism $\mathcal{F} \to \mathcal{F}^\#$ of presheaves with values in $\mathcal{C}$ with the following properties:*

(1) *The map $\mathcal{F} \to \mathcal{F}^\#$ identifies the underlying sheaf of sets of $\mathcal{F}^\#$ with the sheafification of the underlying presheaf of sets of $\mathcal{F}$.*

(2) *For any morphism $\mathcal{F} \to \mathcal{G}$, where $\mathcal{G}$ is a sheaf with values in $\mathcal{C}$ there exists a unique factorization $\mathcal{F} \to \mathcal{F}^\# \to \mathcal{G}$.*

Proof. The proof is the same as the proof of Lemma 6.18.2, with repeated application of Lemma 6.15.4 (see also Example 6.15.5). The main idea however, is to define $\mathcal{F}^\#(U)$ as the fibre product in $\mathcal{C}$ of the diagram

$$\begin{array}{ccc} & & \Pi(\mathcal{F})(U) \\ & & \downarrow \\ \prod_{x \in U} \mathcal{F}_x & \longrightarrow & \prod_{x \in U} \Pi(\mathcal{F})_x \end{array}$$

compare Lemma 6.18.1. □

6.20. Sheafification of presheaves of modules

Lemma 6.20.1. *Let X be a topological space. Let $\mathcal{O}$ be a presheaf of rings on X Let $\mathcal{F}$ be a presheaf $\mathcal{O}$-modules. Let $\mathcal{O}^\#$ be the sheafification of $\mathcal{O}$. Let $\mathcal{F}^\#$ be the sheafification of $\mathcal{F}$ as a presheaf of abelian groups. There exists a map of sheaves of sets*

$$\mathcal{O}^\# \times \mathcal{F}^\# \longrightarrow \mathcal{F}^\#$$

which makes the diagram

$$\begin{array}{ccc} \mathcal{O} \times \mathcal{F} & \longrightarrow & \mathcal{F} \\ \downarrow & & \downarrow \\ \mathcal{O}^\# \times \mathcal{F}^\# & \longrightarrow & \mathcal{F}^\# \end{array}$$

commute and which makes $\mathcal{F}^\#$ into a sheaf of $\mathcal{O}^\#$-modules. In addition, if $\mathcal{G}$ is a sheaf of $\mathcal{O}^\#$-modules, then any morphism of presheaves of $\mathcal{O}$-modules $\mathcal{F} \to \mathcal{G}$ (into the restriction of $\mathcal{G}$ to a $\mathcal{O}$-module) factors uniquely as $\mathcal{F} \to \mathcal{F}^\# \to \mathcal{G}$ where $\mathcal{F}^\# \to \mathcal{G}$ is a morphism of $\mathcal{O}^\#$-modules.

Proof. Omitted. □

This actually means that the functor $i : \mathit{Mod}(\mathcal{O}^\#) \to \mathit{PMod}(\mathcal{O})$ (combining restriction and including sheaves into presheaves) and the sheafification functor of the lemma $^\# : \mathit{PMod}(\mathcal{O}) \to \mathit{Mod}(\mathcal{O}^\#)$ are adjoint. In a formula

$$\mathrm{Mor}_{\mathit{PMod}(\mathcal{O})}(\mathcal{F}, i\mathcal{G}) = \mathrm{Mor}_{\mathit{Mod}(\mathcal{O}^\#)}(\mathcal{F}^\#, \mathcal{G})$$

Let X be a topological space. Let $\mathcal{O}_1 \to \mathcal{O}_2$ be a morphism of sheaves of rings on X. In Section 6.6 we defined a restriction functor and a change of rings functor on presheaves of modules associated to this situation.

If $\mathcal{F}$ is a sheaf of $\mathcal{O}_2$-modules then the restriction $\mathcal{F}_{\mathcal{O}_1}$ of $\mathcal{F}$ is clearly a sheaf of $\mathcal{O}_1$-modules. We obtain the restriction functor

$$\mathit{Mod}(\mathcal{O}_2) \longrightarrow \mathit{Mod}(\mathcal{O}_1)$$

On the other hand, given a sheaf of $\mathcal{O}_1$-modules $\mathcal{G}$ the presheaf of $\mathcal{O}_2$-modules $\mathcal{O}_2 \otimes_{p,\mathcal{O}_1} \mathcal{G}$ is in general not a sheaf. Hence we define the *tensor product sheaf* $\mathcal{O}_2 \otimes_{\mathcal{O}_1} \mathcal{G}$ by the formula

$$\mathcal{O}_2 \otimes_{\mathcal{O}_1} \mathcal{G} = (\mathcal{O}_2 \otimes_{p,\mathcal{O}_1} \mathcal{G})^\#$$

as the sheafification of our construction for presheaves. We obtain the *change of rings* functor

$$Mod(\mathcal{O}_1) \longrightarrow Mod(\mathcal{O}_2)$$

Lemma 6.20.2. *With X, $\mathcal{O}_1$, $\mathcal{O}_2$, $\mathcal{F}$ and $\mathcal{G}$ as above there exists a canonical bijection*

$$\mathrm{Hom}_{\mathcal{O}_1}(\mathcal{G}, \mathcal{F}_{\mathcal{O}_1}) = \mathrm{Hom}_{\mathcal{O}_2}(\mathcal{O}_2 \otimes_{\mathcal{O}_1} \mathcal{G}, \mathcal{F})$$

In other words, the restriction and change of rings functors are adjoint to each other.

Proof. This follows from Lemma 6.6.2 and the fact that $\mathrm{Hom}_{\mathcal{O}_2}(\mathcal{O}_2 \otimes_{\mathcal{O}_1} \mathcal{G}, \mathcal{F}) = \mathrm{Hom}_{\mathcal{O}_2}(\mathcal{O}_2 \otimes_{p,\mathcal{O}_1} \mathcal{G}, \mathcal{F})$ because $\mathcal{F}$ is a sheaf. □

Lemma 6.20.3. *Let X be a topological space. Let $\mathcal{O} \to \mathcal{O}'$ be a morphism of sheaves of rings on X. Let $\mathcal{F}$ be a sheaf $\mathcal{O}$-modules. Let $x \in X$. We have*

$$\mathcal{F}_x \otimes_{\mathcal{O}_x} \mathcal{O}'_x = (\mathcal{F} \otimes_{\mathcal{O}} \mathcal{O}')_x$$

as $\mathcal{O}'_x$-modules.

Proof. Follows directly from Lemma 6.14.2 and the fact that taking stalks commutes with sheafification. □

6.21. Continuous maps and sheaves

Let $f : X \to Y$ be a continuous map of topological spaces. We will define the pushforward and pullback functors for presheaves and sheaves.

Let $\mathcal{F}$ be a presheaf of sets on X. We define the *pushforward* of $\mathcal{F}$ by the rule

$$f_*\mathcal{F}(V) = \mathcal{F}(f^{-1}(V))$$

for any open $V \subset Y$. Given $V_1 \subset V_2 \subset Y$ open the restriction map is given by the commutativity of the diagram

$$\begin{array}{ccc} f_*\mathcal{F}(V_2) & = & \mathcal{F}(f^{-1}(V_2)) \\ \downarrow & & \downarrow \text{restriction for } \mathcal{F} \\ f_*\mathcal{F}(V_1) & = & \mathcal{F}(f^{-1}(V_1)) \end{array}$$

It is clear that this defines a presheaf of sets. The construction is clearly functorial in the presheaf $\mathcal{F}$ and hence we obtain a functor

$$f_* : PSh(X) \longrightarrow PSh(Y).$$

Lemma 6.21.1. *Let $f : X \to Y$ be a continuous map. Let $\mathcal{F}$ be a sheaf of sets on X. Then $f_*\mathcal{F}$ is a sheaf on Y.*

Proof. This immediately follows from the fact that if $V = \bigcup V_j$ is an open covering in Y, then $f^{-1}(V) = \bigcup f^{-1}(V_j)$ is an open covering in X. □

As a consequence we obtain a functor

$$f_* : Sh(X) \longrightarrow Sh(Y).$$

This is compatible with composition in the following strong sense.

Lemma 6.21.2. *Let $f : X \to Y$ and $g : Y \to Z$ be continuous maps of topological spaces. The functors $(g \circ f)_*$ and $g_* \circ f_*$ are equal (on both presheaves and sheaves of sets).*

Proof. This is because $(g \circ f)_*\mathcal{F}(W) = \mathcal{F}((g \circ f)^{-1}W)$ and $(g_* \circ f_*)\mathcal{F}(W) = \mathcal{F}(f^{-1}g^{-1}W)$ and $(g \circ f)^{-1}W = f^{-1}g^{-1}W$. □

Let $\mathcal{G}$ be a presheaf of sets on Y. The *pullback presheaf* $f_p\mathcal{G}$ of a given presheaf $\mathcal{G}$ is defined as the left adjoint of the pushforward f_* on presheaves. In other words it should be a presheaf $f_p\mathcal{G}$ on X such that

$$\mathrm{Mor}_{PSh(X)}(f_p\mathcal{G}, \mathcal{F}) = \mathrm{Mor}_{PSh(Y)}(\mathcal{G}, f_*\mathcal{F}).$$

By the Yoneda lemma this determines the pullback uniquely. It turns out that it actually exists.

Lemma 6.21.3. *Let $f : X \to Y$ be a continuous map. There exists a functor $f_p : PSh(Y) \to PSh(X)$ which is left adjoint to f_*. For a presheaf $\mathcal{G}$ it is determined by the rule*

$$f_p\mathcal{G}(U) = \mathrm{colim}_{f(U) \subset V} \mathcal{G}(V)$$

where the colimit is over the collection of open neighbourhoods V of $f(U)$ in Y. The colimits are over directed partially ordered sets. (The restriction mappings of $f_p\mathcal{G}$ are explained in the proof.)

Proof. The colimit is over the partially ordered set consisting of open subset $V \subset Y$ which contain $f(U)$ with ordering by reverse inclusion. This is a directed partially ordered set, since if V, V' are in it then so is $V \cap V'$. Furthermore, if $U_1 \subset U_2$, then every open neighbourhood of $f(U_2)$ is an open neighbourhood of $f(U_1)$. Hence the system defining $f_p\mathcal{G}(U_2)$ is a subsystem of the one defining $f_p\mathcal{G}(U_1)$ and we obtain a restriction map (for example by applying the generalities in Categories, Lemma 4.14.7).

Note that the construction of the colimit is clearly functorial in $\mathcal{G}$, and similarly for the restriction mappings. Hence we have defined f_p as a functor.

A small useful remark is that there exists a canonical map $\mathcal{G}(U) \to f_p\mathcal{G}(f^{-1}(U))$, because the system of open neighbourhoods of $f(f^{-1}(U))$ contains the element U. This is compatible with restriction mappings. In other words, there is a canonical map $i_{\mathcal{G}} : \mathcal{G} \to f_*f_p\mathcal{G}$.

Let $\mathcal{F}$ be a presheaf of sets on X. Suppose that $\psi : f_p\mathcal{G} \to \mathcal{F}$ is a map of presheaves of sets. The corresponding map $\mathcal{G} \to f_*\mathcal{F}$ is the map $f_*\psi \circ i_{\mathcal{G}} : \mathcal{G} \to f_*f_p\mathcal{G} \to f_*\mathcal{F}$.

Another small useful remark is that there exists a canonical map $c_{\mathcal{F}} : f_pf_*\mathcal{F} \to \mathcal{F}$. Namely, let $U \subset X$ open. For every open neighbourhood $V \supset f(U)$ in Y there exists a map $f_*\mathcal{F}(V) = \mathcal{F}(f^{-1}(V)) \to \mathcal{F}(U)$, namely the restriction map on $\mathcal{F}$. And this is certainly compatible wrt restriction mappings between values of $\mathcal{F}$ on f^{-1} of varying opens containing $f(U)$. Thus we obtain a canonical map $f_pf_*\mathcal{F}(U) \to \mathcal{F}(U)$. Another trivial verification show that these maps are compatible with restrictions and define a map $c_{\mathcal{F}}$ of presheaves of sets.

Suppose that $\varphi : \mathcal{G} \to f_*\mathcal{F}$ is a map of presheaves of sets. Consider $f_p\varphi : f_p\mathcal{G} \to f_pf_*\mathcal{F}$. Postcomposing with $c_{\mathcal{F}}$ gives the desired map $c_{\mathcal{F}} \circ f_p\varphi : f_p\mathcal{G} \to \mathcal{F}$. We omit the verification that this construction is inverse to the construction in the other direction given above. □

Lemma 6.21.4. *Let $f : X \to Y$ be a continuous map. Let $x \in X$. Let $\mathcal{G}$ be a presheaf of sets on Y. There is a canonical bijection of stalks $(f_p\mathcal{G})_x = \mathcal{G}_{f(x)}$.*

Proof. This you can see as follows

$$\begin{aligned}(f_p\mathcal{G})_x &= \operatorname{colim}_{x\in U} f_p\mathcal{G}(U) \\ &= \operatorname{colim}_{x\in U} \operatorname{colim}_{f(U)\subset V} \mathcal{G}(V) \\ &= \operatorname{colim}_{f(x)\in V} \mathcal{G}(V) \\ &= \mathcal{G}_{f(x)}\end{aligned}$$

Here we have used Categories, Lemma 4.14.9, and the fact that any V open in Y containing $f(x)$ occurs in the third description above. Details omitted. □

Let $\mathcal{G}$ be a sheaf of sets on Y. The *pullback sheaf* $f^{-1}\mathcal{G}$ is defined by the formula

$$f^{-1}\mathcal{G} = (f_p\mathcal{G})^\#.$$

Sheafification is a left adjoint to the inclusion of sheaves in presheaves, and f_p is a left adjoint to f_* on presheaves. As a formal consequence we obtain that f^{-1} is a left adjoint of pushforward on sheaves. In other words,

$$\operatorname{Mor}_{Sh(X)}(f^{-1}\mathcal{G}, \mathcal{F}) = \operatorname{Mor}_{Sh(Y)}(\mathcal{G}, f_*\mathcal{F}).$$

The formal argument is given in the setting of abelian sheaves in the next section.

Lemma 6.21.5. *Let $x \in X$. Let $\mathcal{G}$ be a sheaf of sets on Y. There is a canonical bijection of stalks $(f^{-1}\mathcal{G})_x = \mathcal{G}_{f(x)}$.*

Proof. This is a combination of Lemmas 6.17.2 and 6.21.4. □

Lemma 6.21.6. *Let $f : X \to Y$ and $g : Y \to Z$ be continuous maps of topological spaces. The functors $(g \circ f)^{-1}$ and $f^{-1} \circ g^{-1}$ are canonically isomorphic. Similarly $(g \circ f)_p \cong f_p \circ g_p$ on presheaves.*

Proof. To see this use that adjoint functors are unique up to unique isomorphism, and Lemma 6.21.2. □

Definition 6.21.7. Let $f : X \to Y$ be a continuous map. Let $\mathcal{F}$ be a sheaf of sets on X and let $\mathcal{G}$ be a sheaf of sets on Y. An *f-map* $\xi : \mathcal{G} \to \mathcal{F}$ is a collection of maps $\xi_V : \mathcal{G}(V) \to \mathcal{F}(f^{-1}(V))$ indexed by open subsets $V \subset Y$ such that

$$\begin{array}{ccc} \mathcal{G}(V) & \xrightarrow[\xi_V]{} & \mathcal{F}(f^{-1}V) \\ {\scriptstyle \text{restriction of } \mathcal{G}}\downarrow & & \downarrow{\scriptstyle \text{restriction of } \mathcal{F}} \\ \mathcal{G}(V') & \xrightarrow{\xi_{V'}} & \mathcal{F}(f^{-1}V') \end{array}$$

commutes for all $V' \subset V \subset Y$ open.

Lemma 6.21.8. *Let $f : X \to Y$ be a continuous map. Let $\mathcal{F}$ be a sheaf of sets on X and let $\mathcal{G}$ be a sheaf of sets on Y. There are canonical bijections between the following three sets:*

1. *The set of maps $\mathcal{G} \to f_*\mathcal{F}$.*
2. *The set of maps $f^{-1}\mathcal{G} \to \mathcal{F}$.*
3. *The set of f-maps $\xi : \mathcal{G} \to \mathcal{F}$.*

Proof. We leave the easy verification to the reader. □

It is sometimes convenient to think about f-maps instead of maps between sheaves either on X or on Y. We define composition of f-maps as follows.

Definition 6.21.9. Suppose that $f : X \to Y$ and $g : Y \to Z$ are continuous maps of topological spaces. Suppose that $\mathcal{F}$ is a sheaf on X, $\mathcal{G}$ is a sheaf on Y, and $\mathcal{H}$ is a sheaf on Z. Let $\varphi : \mathcal{G} \to \mathcal{F}$ be an f-map. Let $\psi : \mathcal{H} \to \mathcal{G}$ be an g-map. The *composition of φ and ψ* is the $(g \circ f)$-map $\varphi \circ \psi$ defined by the commutativity of the diagrams

$$\begin{array}{ccc} \mathcal{H}(W) & \xrightarrow[(\varphi\circ\psi)_W]{} & \mathcal{F}(f^{-1}g^{-1}W) \\ & {\scriptstyle \psi_W} \searrow \quad \nearrow {\scriptstyle \varphi_{g^{-1}W}} & \\ & \mathcal{G}(g^{-1}W) & \end{array}$$

We leave it to the reader to verify that this works. Another way to think about this is to think of $\varphi \circ \psi$ as the composition

$$\mathcal{H} \xrightarrow{\psi} g_*\mathcal{G} \xrightarrow{g_*\varphi} g_*f_*\mathcal{F} = (g \circ f)_*\mathcal{F}$$

Now, doesn't it seem that thinking about f-maps is somehow easier?

Finally, given a continuous map $f : X \to Y$, and an f-map $\varphi : \mathcal{G} \to \mathcal{F}$ there is a natural map on stalks

$$\varphi_x : \mathcal{G}_{f(x)} \longrightarrow \mathcal{F}_x$$

for all $x \in X$. The image of a representative (V, s) of an element in $\mathcal{G}_{f(x)}$ is mapped to the element in $\mathcal{F}_x$ with representative $(f^{-1}V, \varphi_V(s))$. We leave it to the reader to see that this is well defined. Another way to state it is that it is the unique map such that all diagrams

$$\begin{array}{ccc} \mathcal{F}(f^{-1}V) & \longrightarrow & \mathcal{F}_x \\ {\scriptstyle \varphi_V}\uparrow & & \uparrow{\scriptstyle \varphi_x} \\ \mathcal{G}(V) & \longrightarrow & \mathcal{G}_{f(x)} \end{array}$$

(for $x \in V \subset Y$ open) commute.

Lemma 6.21.10. *Suppose that $f : X \to Y$ and $g : Y \to Z$ are continuous maps of topological spaces. Suppose that $\mathcal{F}$ is a sheaf on X, $\mathcal{G}$ is a sheaf on Y, and $\mathcal{H}$ is a sheaf on Z. Let $\varphi : \mathcal{G} \to \mathcal{F}$ be an f-map. Let $\psi : \mathcal{H} \to \mathcal{G}$ be an g-map. Let $x \in X$ be a point. The map on stalks $(\varphi \circ \psi)_x : \mathcal{H}_{g(f(x))} \to \mathcal{F}_x$ is the composition*

$$\mathcal{H}_{g(f(x))} \xrightarrow{\psi_{f(x)}} \mathcal{G}_{f(x)} \xrightarrow{\varphi_x} \mathcal{F}_x$$

Proof. Immediate from Definition 6.21.9 and the definition of the map on stalks above. □

6.22. Continuous maps and abelian sheaves

Let $f : X \to Y$ be a continuous map. We claim there are functors

$$\begin{aligned} f_* : PAb(X) &\longrightarrow PAb(Y) \\ f_* : Ab(X) &\longrightarrow Ab(Y) \\ f_p : PAb(Y) &\longrightarrow PAb(X) \\ f^{-1} : Ab(Y) &\longrightarrow Ab(X) \end{aligned}$$

with similar properties to their counterparts in Section 6.21. To see this we argue in the following way.

Each of the functors will be constructed in the same way as the corresponding functor in Section 6.21. This works because all the colimits in that section are directed colimits (but we will work through it below).

First off, given an abelian presheaf $\mathcal{F}$ on X and an abelian presheaf $\mathcal{G}$ on Y we define

$$\begin{aligned} f_*\mathcal{F}(V) &= \mathcal{F}(f^{-1}(V)) \\ f_p\mathcal{G}(U) &= \text{colim}_{f(U)\subset V}\, \mathcal{G}(V) \end{aligned}$$

as abelian groups. The restriction mappings are the same as the restriction mappings for presheaves of sets (and they are all homomorphisms of abelian groups).

The assignments $\mathcal{F} \mapsto f_*\mathcal{F}$ and $\mathcal{G} \to f_p\mathcal{G}$ are functors on the categories of presheaves of abelian groups. This is clear, as (for example) a map of abelian presheaves $\mathcal{G}_1 \to \mathcal{G}_2$ gives rise to a map of directed systems $\{\mathcal{G}_1(V)\}_{f(U)\subset V} \to \{\mathcal{G}_2(V)\}_{f(U)\subset V}$ all of whose maps are homomorphisms and hence gives rise to a homomorphism of abelian groups $f_p\mathcal{G}_1(U) \to f_p\mathcal{G}_2(U)$.

The functors f_* and f_p are adjoint on the category of presheaves of abelian groups, i.e., we have

$$\text{Mor}_{PAb(X)}(f_p\mathcal{G}, \mathcal{F}) = \text{Mor}_{PAb(Y)}(\mathcal{G}, f_*\mathcal{F}).$$

To prove this, note that the map $i_{\mathcal{G}} : \mathcal{G} \to f_*f_p\mathcal{G}$ from the proof of Lemma 6.21.3 is a map of abelian presheaves. Hence if $\psi : f_p\mathcal{G} \to \mathcal{F}$ is a map of abelian presheaves, then the corresponding map $\mathcal{G} \to f_*\mathcal{F}$ is the map $f_*\psi \circ i_{\mathcal{G}} : \mathcal{G} \to f_*f_p\mathcal{G} \to f_*\mathcal{F}$ is also a map of abelian presheaves. For the other direction we point out that the map $c_{\mathcal{F}} : f_pf_*\mathcal{F} \to \mathcal{F}$ from the proof of Lemma 6.21.3 is a map of abelian presheaves as well (since it is made out of restriction mappings of $\mathcal{F}$ which are all homomorphisms). Hence given a map of abelian presheaves $\varphi : \mathcal{G} \to f_*\mathcal{F}$ the map $c_{\mathcal{F}} \circ f_p\varphi : f_p\mathcal{G} \to \mathcal{F}$ is a map of abelian presheaves as well. Since these constructions $\psi \mapsto f_*\psi$ and $\varphi \mapsto c_{\mathcal{F}} \circ f_p\varphi$ are inverse to each other as constructions on maps of presheaves of sets we see they are also inverse to each other on maps of abelian presheaves.

If $\mathcal{F}$ is an abelian sheaf on Y, then $f_*\mathcal{F}$ is an abelian sheaf on X. This is true because of the definition of an abelian sheaf and because this is true for sheaves of sets, see Lemma 6.21.1. This defines the functor f_* on the category of abelian sheaves.

We define $f^{-1}\mathcal{G} = (f_p\mathcal{G})^\#$ as before. Adjointness of f_* and f^{-1} follows formally as in the case of presheaves of sets. Here is the argument:

$$\begin{aligned} \text{Mor}_{Ab(X)}(f^{-1}\mathcal{G}, \mathcal{F}) &= \text{Mor}_{PAb(X)}(f_p\mathcal{G}, \mathcal{F}) \\ &= \text{Mor}_{PAb(Y)}(\mathcal{G}, f_*\mathcal{F}) \\ &= \text{Mor}_{Ab(Y)}(\mathcal{G}, f_*\mathcal{F}) \end{aligned}$$

Lemma 6.22.1. *Let $f : X \to Y$ be a continuous map.*

1. *Let $\mathcal{G}$ be an abelian presheaf on Y. Let $x \in X$. The bijection $\mathcal{G}_{f(x)} \to (f_p\mathcal{G})_x$ of Lemma 6.21.4 is an isomorphism of abelian groups.*
2. *Let $\mathcal{G}$ be an abelian sheaf on Y. Let $x \in X$. The bijection $\mathcal{G}_{f(x)} \to (f^{-1}\mathcal{G})_x$ of Lemma 6.21.5 is an isomorphism of abelian groups.*

Proof. Omitted. □

Given a continuous map $f : X \to Y$ and sheaves of abelian groups $\mathcal{F}$ on X, $\mathcal{G}$ on Y, the notion of an *f-map $\mathcal{G} \to \mathcal{F}$ of sheaves of abelian groups* makes sense. We can just define it exactly as in Definition 6.21.7 (replacing maps of sets with homomorphisms of abelian groups) or we can simply say that it is the same as a map of abelian sheaves $\mathcal{G} \to f_*\mathcal{F}$. We will use this notion freely in the following. The group of f-maps between $\mathcal{G}$ and $\mathcal{F}$ will be in canonical bijection with the groups $\mathrm{Mor}_{Ab(X)}(f^{-1}\mathcal{G}, \mathcal{F})$ and $\mathrm{Mor}_{Ab(Y)}(\mathcal{G}, f_*\mathcal{F})$.

Composition of f-maps is defined in exactly the same manner as in the case of f-maps of sheaves of sets. In addition, given an f-map $\mathcal{G} \to \mathcal{F}$ as above, the induced maps on stalks

$$\varphi_x : \mathcal{G}_{f(x)} \longrightarrow \mathcal{F}_x$$

are abelian group homomorphisms.

6.23. Continuous maps and sheaves of algebraic structures

Let $(\mathcal{C}, F)$ be a type of algebraic structure. For a topological space X let us introduce the notation:

(1) $PSh(X, \mathcal{C})$ will be the category of presheaves with values in $\mathcal{C}$.
(2) $Sh(X, \mathcal{C})$ will be the category of sheaves with values in $\mathcal{C}$.

Let $f : X \to Y$ be a continuous map of topological spaces. The same arguments as in the previous section show there are functors

$$\begin{aligned} f_* : PSh(X, \mathcal{C}) &\longrightarrow PSh(Y, \mathcal{C}) \\ f_* : Sh(X, \mathcal{C}) &\longrightarrow Sh(Y, \mathcal{C}) \\ f_p : PSh(Y, \mathcal{C}) &\longrightarrow PSh(X, \mathcal{C}) \\ f^{-1} : Sh(Y, \mathcal{C}) &\longrightarrow Sh(X, \mathcal{C}) \end{aligned}$$

constructed in the same manner and with the same properties as the functors constructed for abelian (pre)sheaves. In particular there are commutative diagrams

$$\begin{array}{ccc} PSh(X, \mathcal{C}) & \xrightarrow{f_*} & PSh(Y, \mathcal{C}) \\ \downarrow F & & \downarrow F \\ PSh(X) & \xrightarrow{f_*} & PSh(Y) \end{array} \qquad \begin{array}{ccc} Sh(X, \mathcal{C}) & \xrightarrow{f_*} & Sh(Y, \mathcal{C}) \\ \downarrow F & & \downarrow F \\ Sh(X) & \xrightarrow{f_*} & Sh(Y) \end{array}$$

$$\begin{array}{ccc} PSh(Y, \mathcal{C}) & \xrightarrow{f_p} & PSh(X, \mathcal{C}) \\ \downarrow F & & \downarrow F \\ PSh(Y) & \xrightarrow{f_p} & PSh(X) \end{array} \qquad \begin{array}{ccc} Sh(Y, \mathcal{C}) & \xrightarrow{f^{-1}} & Sh(X, \mathcal{C}) \\ \downarrow F & & \downarrow F \\ Sh(Y) & \xrightarrow{f^{-1}} & Sh(X) \end{array}$$

The main formulas to keep in mind are the following

$$\begin{aligned} f_*\mathcal{F}(V) &= \mathcal{F}(f^{-1}(V)) \\ f_p\mathcal{G}(U) &= \operatorname{colim}_{f(U)\subset V} \mathcal{G}(V) \\ f^{-1}\mathcal{G} &= (f_p\mathcal{G})^\# \\ (f_p\mathcal{G})_x &= \mathcal{G}_{f(x)} \\ (f^{-1}\mathcal{G})_x &= \mathcal{G}_{f(x)} \end{aligned}$$

Each of these formulas has the property that they hold in the category $\mathcal{C}$ and that upon taking underlying sets we get the corresponding formula for presheaves of sets. In addition we have the adjointness properties

$$\begin{aligned} \operatorname{Mor}_{PSh(X,\mathcal{C})}(f_p\mathcal{G}, \mathcal{F}) &= \operatorname{Mor}_{PSh(Y,\mathcal{C})}(\mathcal{G}, f_*\mathcal{F}) \\ \operatorname{Mor}_{Sh(X,\mathcal{C})}(f^{-1}\mathcal{G}, \mathcal{F}) &= \operatorname{Mor}_{Sh(Y,\mathcal{C})}(\mathcal{G}, f_*\mathcal{F}). \end{aligned}$$

To prove these, the main step is to construct the maps

$$i_{\mathcal{G}} : \mathcal{G} \longrightarrow f_*f_p\mathcal{G}$$

and

$$c_{\mathcal{F}} : f_pf_*\mathcal{F} \longrightarrow \mathcal{F}$$

which occur in the proof of Lemma 6.21.3 as morphisms of presheaves with values in $\mathcal{C}$. This may be safely left to the reader since the constructions are exactly the same as in the case of presheaves of sets.

Given a continuous map $f : X \to Y$ and sheaves of algebraic structures $\mathcal{F}$ on X, $\mathcal{G}$ on Y, the notion of an *f-map $\mathcal{G} \to \mathcal{F}$ of sheaves of algebraic structures* makes sense. We can just define it exactly as in Definition 6.21.7 (replacing maps of sets with morphisms in $\mathcal{C}$) or we can simply say that it is the same as a map of sheaves of algebraic structures $\mathcal{G} \to f_*\mathcal{F}$. We will use this notion freely in the following. The set of f-maps between $\mathcal{G}$ and $\mathcal{F}$ will be in canonical bijection with the sets $\operatorname{Mor}_{Sh(X,\mathcal{C})}(f^{-1}\mathcal{G}, \mathcal{F})$ and $\operatorname{Mor}_{Sh(Y,\mathcal{C})}(\mathcal{G}, f_*\mathcal{F})$.

Composition of f-maps is defined in exactly the same manner as in the case of f-maps of sheaves of sets. In addition, given an f-map $\mathcal{G} \to \mathcal{F}$ as above, the induced maps on stalks

$$\varphi_x : \mathcal{G}_{f(x)} \longrightarrow \mathcal{F}_x$$

are homomorphisms of algebraic structures.

Lemma 6.23.1. *Let $f : X \to Y$ be a continuous map of topological spaces. Suppose given sheaves of algebraic structures $\mathcal{F}$ on X, $\mathcal{G}$ on Y. Let $\varphi : \mathcal{G} \to \mathcal{F}$ be an f-map of underlying sheaves of sets. If for every $V \subset Y$ open the map of sets $\varphi_V : \mathcal{G}(V) \to \mathcal{F}(f^{-1}V)$ is the effect of a morphism in $\mathcal{C}$ on underlying sets, then φ comes from a unique f-morphism between sheaves of algebraic structures.*

Proof. Omitted. □

6.24. Continuous maps and sheaves of modules

The case of sheaves of modules is more complicated. The reason is that the natural setting for defining the pullback and pushforward functors, is the setting of ringed spaces, which we will define below. First we state a few obvious lemmas.

Lemma 6.24.1. *Let $f : X \to Y$ be a continuous map of topological spaces. Let $\mathcal{O}$ be a presheaf of rings on X. Let $\mathcal{F}$ be a presheaf of $\mathcal{O}$-modules. There is a natural map of underlying presheaves of sets*

$$f_*\mathcal{O} \times f_*\mathcal{F} \longrightarrow f_*\mathcal{F}$$

which turns $f_*\mathcal{F}$ into a presheaf of $f_*\mathcal{O}$-modules. This construction is functorial in $\mathcal{F}$.

Proof. Let $V \subset Y$ is open. We define the map of the lemma to be the map

$$f_*\mathcal{O}(V) \times f_*\mathcal{F}(V) = \mathcal{O}(f^{-1}V) \times \mathcal{F}(f^{-1}V) \to \mathcal{F}(f^{-1}V) = f_*\mathcal{F}(V).$$

Here the arrow in the middle is the multiplication map on X. We leave it to the reader to see this is compatible with restriction mappings and defines a structure of $f_*\mathcal{O}$-module on $f_*\mathcal{F}$. □

Lemma 6.24.2. *Let $f : X \to Y$ be a continuous map of topological spaces. Let $\mathcal{O}$ be a presheaf of rings on Y. Let $\mathcal{G}$ be a presheaf of $\mathcal{O}$-modules. There is a natural map of underlying presheaves of sets*

$$f_p\mathcal{O} \times f_p\mathcal{G} \longrightarrow f_p\mathcal{G}$$

which turns $f_p\mathcal{G}$ into a presheaf of $f_p\mathcal{O}$-modules. This construction is functorial in $\mathcal{G}$.

Proof. Let $U \subset X$ is open. We define the map of the lemma to be the map

$$\begin{aligned} f_p\mathcal{O}(U) \times f_p\mathcal{G}(U) &= \operatorname{colim}_{f(U)\subset V} \mathcal{O}(V) \times \operatorname{colim}_{f(U)\subset V} \mathcal{G}(V) \\ &= \operatorname{colim}_{f(U)\subset V} (\mathcal{O}(V) \times \mathcal{G}(V)) \\ &\to \operatorname{colim}_{f(U)\subset V} \mathcal{G}(V) \\ &= f_p\mathcal{G}(U). \end{aligned}$$

Here the arrow in the middle is the multiplication map on Y. The second equality holds because directed colimits commute with finite limits, see Categories, Lemma 4.19.2. We leave it to the reader to see this is compatible with restriction mappings and defines a structure of $f_p\mathcal{O}$-module on $f_p\mathcal{G}$. □

Let $f : X \to Y$ be a continuous map. Let $\mathcal{O}_X$ be a presheaf of rings on X and let $\mathcal{O}_Y$ be a presheaf of rings on Y. So at the moment we have defined functors

$$\begin{aligned} f_* : PMod(\mathcal{O}_X) &\longrightarrow PMod(f_*\mathcal{O}_X) \\ f_p : PMod(\mathcal{O}_Y) &\longrightarrow PMod(f_p\mathcal{O}_Y) \end{aligned}$$

These satisfy some compatibilities as follows.

Lemma 6.24.3. *Let $f : X \to Y$ be a continuous map of topological spaces. Let $\mathcal{O}$ be a presheaf of rings on Y. Let $\mathcal{G}$ be a presheaf of $\mathcal{O}$-modules. Let $\mathcal{F}$ be a presheaf of $f_p\mathcal{O}$-modules. Then*

$$\operatorname{Mor}_{PMod(f_p\mathcal{O})}(f_p\mathcal{G}, \mathcal{F}) = \operatorname{Mor}_{PMod(\mathcal{O})}(\mathcal{G}, f_*\mathcal{F}).$$

Here we use Lemmas 6.24.2 and 6.24.1, and we think of $f_*\mathcal{F}$ as an $\mathcal{O}$-module via the map $i_{\mathcal{O}} : \mathcal{O} \to f_*f_p\mathcal{O}$ (defined first in the proof of Lemma 6.21.3).

Proof. Note that we have

$$\mathrm{Mor}_{PAb(X)}(f_p\mathcal{G}, \mathcal{F}) = \mathrm{Mor}_{PAb(Y)}(\mathcal{G}, f_*\mathcal{F}).$$

according to Section 6.22. So what we have to prove is that under this correspondence, the subsets of module maps correspond. In addition, the correspondence is determined by the rule

$$\Big(\psi : f_p\mathcal{G} \to \mathcal{F}\Big) \longmapsto \Big(f_*\psi \circ i_{\mathcal{G}} : \mathcal{G} \to f_*f_p\mathcal{G} \to f_*\mathcal{F}\Big)$$

Hence, using the functoriality of the pushforward we see that it suffices to prove that the map $i_{\mathcal{G}} : \mathcal{G} \to f_*f_p\mathcal{G}$ is compatible with module structure, which we leave to the reader. □

Lemma 6.24.4. *Let $f : X \to Y$ be a continuous map of topological spaces. Let $\mathcal{O}$ be a presheaf of rings on X. Let $\mathcal{F}$ be a presheaf of $\mathcal{O}$-modules. Let $\mathcal{G}$ be a presheaf of $f_*\mathcal{O}$-modules. Then*

$$\mathrm{Mor}_{PMod(\mathcal{O})}(\mathcal{O} \otimes_{p,f_pf_*\mathcal{O}} f_p\mathcal{G}, \mathcal{F}) = \mathrm{Mor}_{PMod(f_*\mathcal{O})}(\mathcal{G}, f_*\mathcal{F}).$$

Here we use Lemmas 6.24.2 and 6.24.1, and we use the map $c_{\mathcal{O}} : f_pf_*\mathcal{O} \to \mathcal{O}$ in the definition of the tensor product.

Proof. This follows from the equalities

$$\begin{aligned} \mathrm{Mor}_{PMod(\mathcal{O})}(\mathcal{O} \otimes_{p,f_pf_*\mathcal{O}} f_p\mathcal{G}, \mathcal{F}) &= \mathrm{Mor}_{PMod(f_pf_*\mathcal{O})}(f_p\mathcal{G}, \mathcal{F}_{f_pf_*\mathcal{O}}) \\ &= \mathrm{Mor}_{PMod(f_*\mathcal{O})}(\mathcal{G}, f_*\mathcal{F}). \end{aligned}$$

which is a combination of Lemmas 6.6.2 and 6.24.3. □

Lemma 6.24.5. *Let $f : X \to Y$ be a continuous map of topological spaces. Let $\mathcal{O}$ be a sheaf of rings on X. Let $\mathcal{F}$ be a sheaf of $\mathcal{O}$-modules. The pushforward $f_*\mathcal{F}$, as defined in Lemma 6.24.1 is a sheaf of $f_*\mathcal{O}$-modules.*

Proof. Obvious from the definition and Lemma 6.21.1. □

Lemma 6.24.6. *Let $f : X \to Y$ be a continuous map of topological spaces. Let $\mathcal{O}$ be a sheaf of rings on Y. Let $\mathcal{G}$ be a sheaf of $\mathcal{O}$-modules. There is a natural map of underlying presheaves of sets*

$$f^{-1}\mathcal{O} \times f^{-1}\mathcal{G} \longrightarrow f^{-1}\mathcal{G}$$

which turns $f^{-1}\mathcal{G}$ into a sheaf of $f^{-1}\mathcal{O}$-modules.

Proof. Recall that f^{-1} is defined as the composition of the functor f_p and sheafification. Thus the lemma is a combination of Lemma 6.24.2 and Lemma 6.20.1. □

Let $f : X \to Y$ be a continuous map. Let $\mathcal{O}_X$ be a sheaf of rings on X and let $\mathcal{O}_Y$ be a sheaf of rings on Y. So now we have defined functors

$$\begin{aligned} f_* : \mathit{Mod}(\mathcal{O}_X) &\longrightarrow \mathit{Mod}(f_*\mathcal{O}_X) \\ f^{-1} : \mathit{Mod}(\mathcal{O}_Y) &\longrightarrow \mathit{Mod}(f^{-1}\mathcal{O}_Y) \end{aligned}$$

These satisfy some compatibilities as follows.

Lemma 6.24.7. *Let $f : X \to Y$ be a continuous map of topological spaces. Let $\mathcal{O}$ be a sheaf of rings on Y. Let $\mathcal{G}$ be a sheaf of $\mathcal{O}$-modules. Let $\mathcal{F}$ be a sheaf of $f^{-1}\mathcal{O}$-modules. Then*

$$\mathrm{Mor}_{Mod(f^{-1}\mathcal{O})}(f^{-1}\mathcal{G}, \mathcal{F}) = \mathrm{Mor}_{Mod(\mathcal{O})}(\mathcal{G}, f_*\mathcal{F}).$$

Here we use Lemmas 6.24.6 and 6.24.5, and we think of $f_*\mathcal{F}$ as an $\mathcal{O}$-module by restriction via $\mathcal{O} \to f_*f^{-1}\mathcal{O}$.

Proof. Argue by the equalities

$$\begin{aligned} \mathrm{Mor}_{Mod(f^{-1}\mathcal{O})}(f^{-1}\mathcal{G}, \mathcal{F}) &= \mathrm{Mor}_{Mod(f_p\mathcal{O})}(f_p\mathcal{G}, \mathcal{F}) \\ &= \mathrm{Mor}_{Mod(\mathcal{O})}(\mathcal{G}, f_*\mathcal{F}). \end{aligned}$$

where the second is Lemmas 6.24.3 and the first is by Lemma 6.20.1. □

Lemma 6.24.8. *Let $f : X \to Y$ be a continuous map of topological spaces. Let $\mathcal{O}$ be a sheaf of rings on X. Let $\mathcal{F}$ be a sheaf of $\mathcal{O}$-modules. Let $\mathcal{G}$ be a sheaf of $f_*\mathcal{O}$-modules. Then*

$$\mathrm{Mor}_{Mod(\mathcal{O})}(\mathcal{O} \otimes_{f^{-1}f_*\mathcal{O}} f^{-1}\mathcal{G}, \mathcal{F}) = \mathrm{Mor}_{Mod(f_*\mathcal{O})}(\mathcal{G}, f_*\mathcal{F}).$$

Here we use Lemmas 6.24.6 and 6.24.5, and we use the canonical map $f^{-1}f_*\mathcal{O} \to \mathcal{O}$ in the definition of the tensor product.

Proof. This follows from the equalities

$$\begin{aligned} \mathrm{Mor}_{Mod(\mathcal{O})}(\mathcal{O} \otimes_{f^{-1}f_*\mathcal{O}} f^{-1}\mathcal{G}, \mathcal{F}) &= \mathrm{Mor}_{Mod(f^{-1}f_*\mathcal{O})}(f^{-1}\mathcal{G}, \mathcal{F}_{f^{-1}f_*\mathcal{O}}) \\ &= \mathrm{Mor}_{Mod(f_*\mathcal{O})}(\mathcal{G}, f_*\mathcal{F}). \end{aligned}$$

which are a combination of Lemma 6.20.2 and 6.24.7. □

Let $f : X \to Y$ be a continuous map. Let $\mathcal{O}_X$ be a (pre)sheaf of rings on X and let $\mathcal{O}_Y$ be a (pre)sheaf of rings on Y. So at the moment we have defined functors

$$\begin{aligned} f_* : PMod(\mathcal{O}_X) &\longrightarrow PMod(f_*\mathcal{O}_X) \\ f_* : Mod(\mathcal{O}_X) &\longrightarrow Mod(f_*\mathcal{O}_X) \\ f_p : PMod(\mathcal{O}_Y) &\longrightarrow PMod(f_p\mathcal{O}_Y) \\ f^{-1} : Mod(\mathcal{O}_Y) &\longrightarrow Mod(f^{-1}\mathcal{O}_Y) \end{aligned}$$

Clearly, usually the pair of functors (f_*, f^{-1}) on sheaves of modules are not adjoint, because their target categories do not match. Namely, as we saw above, it works only if by some miracle the sheaves of rings $\mathcal{O}_X, \mathcal{O}_Y$ satisfy the relations $\mathcal{O}_X = f^{-1}\mathcal{O}_Y$ and $\mathcal{O}_Y = f_*\mathcal{O}_X$. This is almost never true in practice. We interrupt the discussion to define the correct notion of morphism for which a suitable adjoint pair of functors on sheaves of modules exists.

6.25. Ringed spaces

Let X be a topological space and let $\mathcal{O}_X$ be a sheaf of rings on X. We are supposed to think of the sheaf of rings $\mathcal{O}_X$ as a sheaf of functions on X. And if $f : X \to Y$ is a "suitable" map, then by composition a function on Y turns into a function on X. Thus there should be a natural f-map from $\mathcal{O}_Y$ to $\mathcal{O}_X$ See Definition 6.21.7, and the remarks in previous sections for terminology. For a precise example, see Example 6.25.2 below. Here is the relevant abstract definition.

Definition 6.25.1. A *ringed space* is a pair $(X, \mathcal{O}_X)$ consisting of a topological space X and a sheaf of rings $\mathcal{O}_X$ on X. A *morphism of ringed spaces* $(X, \mathcal{O}_X) \to (Y, \mathcal{O}_Y)$ is a pair consisting of a continuous map $f : X \to Y$ and an f-map of sheaves of rings $f^\sharp : \mathcal{O}_Y \to \mathcal{O}_X$.

Example 6.25.2. Let $f : X \to Y$ be a continuous map of topological spaces. Consider the sheaves of continuous real valued functions $\mathcal{C}^0_X$ on X and $\mathcal{C}^0_Y$ on Y, see Example 6.9.3. We claim that there is a natural f-map $f^\sharp : \mathcal{C}^0_Y \to \mathcal{C}^0_X$ associated to f. Namely, we simply define it by the rule

$$\begin{array}{ccc} \mathcal{C}^0_Y(V) & \longrightarrow & \mathcal{C}^0_X(f^{-1}V) \\ h & \longmapsto & h \circ f \end{array}$$

Strictly speaking we should write $f^\sharp(h) = h \circ f|_{f^{-1}(V)}$. It is clear that this is a family of maps as in Definition 6.21.7 and compatible with the $\mathbf{R}$-algebra structures. Hence it is an f-map of sheaves of $\mathbf{R}$-algebras, see Lemma 6.23.1.

Of course there are lots of other situations where there is a canonical morphism of ringed spaces associated to a geometrical type of morphism. For example, if M, N are $\mathcal{C}^\infty$-manifolds and $f : M \to N$ is a infinitely differentiable map, then f induces a canonical morphism of ringed spaces $(M, \mathcal{C}^\infty_M) \to (N, \mathcal{C}^\infty_N)$. The construction (which is identical to the above) is left to the reader.

It may not be completely obvious how to compose morphisms of ringed spaces hence we spell it out here.

Definition 6.25.3. Let $(f, f^\sharp) : (X, \mathcal{O}_X) \to (Y, \mathcal{O}_Y)$ and $(g, g^\sharp) : (Y, \mathcal{O}_Y) \to (Z, \mathcal{O}_Z)$ be morphisms of ringed spaces. Then we define the *composition of morphisms of ringed spaces* by the rule

$$(g, g^\sharp) \circ (f, f^\sharp) = (g \circ f, f^\sharp \circ g^\sharp).$$

Here we use composition of f-maps defined in Definition 6.21.9.

6.26. Morphisms of ringed spaces and modules

We have now introduced enough notation so that we are able to define the pullback and pushforward of modules along a morphism of ringed spaces.

Definition 6.26.1. Let $(f, f^\sharp) : (X, \mathcal{O}_X) \to (Y, \mathcal{O}_Y)$ be a morphism of ringed spaces.

(1) Let $\mathcal{F}$ be a sheaf of $\mathcal{O}_X$-modules. We define the *pushforward* of $\mathcal{F}$ as the sheaf of $\mathcal{O}_Y$-modules which as a sheaf of abelian groups equals $f_*\mathcal{F}$ and with module structure given by the restriction via $f^\sharp : \mathcal{O}_Y \to f_*\mathcal{O}_X$ of the module structure given in Lemma 6.24.5.

(2) Let $\mathcal{G}$ be a sheaf of $\mathcal{O}_Y$-modules. We define the *pullback* $f^*\mathcal{G}$ to be the sheaf of $\mathcal{O}_X$-modules defined by the formula

$$f^*\mathcal{G} = \mathcal{O}_X \otimes_{f^{-1}\mathcal{O}_Y} f^{-1}\mathcal{G}$$

where the ring map $f^{-1}\mathcal{O}_Y \to \mathcal{O}_X$ is the map corresponding to $f^\sharp$, and where the module structure is given by Lemma 6.24.6.

Thus we have defined functors

$$\begin{aligned} f_* : Mod(\mathcal{O}_X) &\longrightarrow Mod(\mathcal{O}_Y) \\ f^* : Mod(\mathcal{O}_Y) &\longrightarrow Mod(\mathcal{O}_X) \end{aligned}$$

The final result on these functors is that they are indeed adjoint as expected.

Lemma 6.26.2. *Let $(f, f^\sharp) : (X, \mathcal{O}_X) \to (Y, \mathcal{O}_Y)$ be a morphism of ringed spaces. Let $\mathcal{F}$ be a sheaf of $\mathcal{O}_X$-modules. Let $\mathcal{G}$ be a sheaf of $\mathcal{O}_Y$-modules. There is a canonical bijection*

$$\mathrm{Hom}_{\mathcal{O}_X}(f^*\mathcal{G}, \mathcal{F}) = \mathrm{Hom}_{\mathcal{O}_Y}(\mathcal{G}, f_*\mathcal{F}).$$

In other words: the functor f^* is the left adjoint to f_*.

Proof. This follows from the work we did before:

$$\begin{aligned} \mathrm{Hom}_{\mathcal{O}_X}(f^*\mathcal{G}, \mathcal{F}) &= \mathrm{Mor}_{Mod(\mathcal{O}_X)}(\mathcal{O}_X \otimes_{f^{-1}\mathcal{O}_Y} f^{-1}\mathcal{G}, \mathcal{F}) \\ &= \mathrm{Mor}_{Mod(f^{-1}\mathcal{O}_Y)}(f^{-1}\mathcal{G}, \mathcal{F}_{f^{-1}\mathcal{O}_Y}) \\ &= \mathrm{Hom}_{\mathcal{O}_Y}(\mathcal{G}, f_*\mathcal{F}). \end{aligned}$$

Here we use Lemmas 6.20.2 and 6.24.7. □

Lemma 6.26.3. *Let $f : X \to Y$ and $g : Y \to Z$ be morphisms of ringed spaces. The functors $(g \circ f)_*$ and $g_* \circ f_*$ are equal. There is a canonical isomorphism of functors $(g \circ f)^* \cong f^* \circ g^*$.*

Proof. The result on pushforwards is a consequence of Lemma 6.21.2 and our definitions. The result on pullbacks follows from this by the same argument as in the proof of Lemma 6.21.6. □

Given a morphism of ringed spaces $(f, f^\sharp) : (X, \mathcal{O}_X) \to (Y, \mathcal{O}_Y)$, and a sheaf of $\mathcal{O}_X$-modules $\mathcal{F}$, a sheaf of $\mathcal{O}_Y$-modules $\mathcal{G}$ on Y, the notion of an *f-map $\varphi : \mathcal{G} \to \mathcal{F}$ of sheaves of modules* makes sense. We can just define it as an f-map $\varphi : \mathcal{G} \to \mathcal{F}$ of abelian sheaves such that for all open $V \subset Y$ the map

$$\mathcal{G}(V) \longrightarrow \mathcal{F}(f^{-1}V)$$

is an $\mathcal{O}_Y(V)$-module map. Here we think of $\mathcal{F}(f^{-1}V)$ as an $\mathcal{O}_Y(V)$-module via the map $f_V^\sharp : \mathcal{O}_Y(V) \to \mathcal{O}_X(f^{-1}V)$. The set of f-maps between $\mathcal{G}$ and $\mathcal{F}$ will be in canonical bijection with the sets $\mathrm{Mor}_{Mod(\mathcal{O}_X)}(f^*\mathcal{G}, \mathcal{F})$ and $\mathrm{Mor}_{Mod(\mathcal{O}_Y)}(\mathcal{G}, f_*\mathcal{F})$. See above.

Composition of f-maps is defined in exactly the same manner as in the case of f-maps of sheaves of sets. In addition, given an f-map $\mathcal{G} \to \mathcal{F}$ as above, and $x \in X$ the induced map on stalks

$$\varphi_x : \mathcal{G}_{f(x)} \longrightarrow \mathcal{F}_x$$

is an $\mathcal{O}_{Y,f(x)}$-module map where the $\mathcal{O}_{Y,f(x)}$-module structure on $\mathcal{F}_x$ comes from the $\mathcal{O}_{X,x}$-module structure via the map $f_x^\sharp : \mathcal{O}_{Y,f(x)} \to \mathcal{O}_{X,x}$. Here is a related lemma.

Lemma 6.26.4. *Let $(f, f^\sharp) : (X, \mathcal{O}_X) \to (Y, \mathcal{O}_Y)$ be a morphism of ringed spaces. Let $\mathcal{G}$ be a sheaf of $\mathcal{O}_Y$-modules. Let $x \in X$. Then*

$$f^*\mathcal{G}_x = \mathcal{F}_{f(x)} \otimes_{\mathcal{O}_{Y,f(x)}, f_x^\sharp} \mathcal{O}_{X,x}$$

as $\mathcal{O}_{X,x}$-modules.

Proof. This follows from Lemma 6.20.3 and the identification of the stalks of pullback sheaves at x with the corresponding stalks at $f(x)$. See the formulae in Section 6.23 for example. □

6.27. Skyscraper sheaves and stalks

Definition 6.27.1. Let X be a topological space.

(1) Let $x \in X$ be a point. Denote $i_x : \{x\} \to X$ the inclusion map. Let A be a set and think of A as a sheaf on the one point space $\{x\}$. We call $i_{x,*}A$ the *skyscraper sheaf at x with value A*.
(2) If in (1) above A is an abelian group then we think of $i_{x,*}A$ as a sheaf of abelian groups on X.
(3) If in (1) above A is an algebraic structure then we think of $i_{x,*}A$ as a sheaf of algebraic structures.
(4) If $(X, \mathcal{O}_X)$ is a ringed space, then we think of $i_x : \{x\} \to X$ as a morphism of ringed spaces $(\{x\}, \mathcal{O}_{X,x}) \to (X, \mathcal{O}_X)$ and if A is a $\mathcal{O}_{X,x}$-module, then we think of $i_{x,*}A$ as a sheaf of $\mathcal{O}_X$-modules.
(5) We say a sheaf of sets $\mathcal{F}$ is a *skyscraper sheaf* if there exists an point x of X and a set A such that $\mathcal{F} \cong i_{x,*}A$.
(6) We say a sheaf of abelian groups $\mathcal{F}$ is a *skyscraper sheaf* if there exists an point x of X and an abelian group A such that $\mathcal{F} \cong i_{x,*}A$ as sheaves of abelian groups.
(7) We say a sheaf of algebraic structures $\mathcal{F}$ is a *skyscraper sheaf* if there exists an point x of X and an algebraic structure A such that $\mathcal{F} \cong i_{x,*}A$ as sheaves of algebraic structures.
(8) If $(X, \mathcal{O}_X)$ is a ringed space and $\mathcal{F}$ is a sheaf of $\mathcal{O}_X$-modules, then we say $\mathcal{F}$ is a *skyscraper sheaf* if there exists a point $x \in X$ and a $\mathcal{O}_{X,x}$-module A such that $\mathcal{F} \cong i_{x,*}A$ as sheaves of $\mathcal{O}_X$-modules.

Lemma 6.27.2. *Let X be a topological space, $x \in X$ a point, and A a set. For any point $x' \in X$ the stalk of the skyscraper sheaf at x with value A at x' is*

$$(i_{x,*}A)_{x'} = \begin{cases} A & \text{if} \quad x' \in \overline{\{x\}} \\ \{*\} & \text{if} \quad x' \notin \overline{\{x\}} \end{cases}$$

A similar description holds for the case of abelian groups, algebraic structures and sheaves of modules.

Proof. Omitted. □

Lemma 6.27.3. *Let X be a topological space, and let $x \in X$ a point. The functors $\mathcal{F} \mapsto \mathcal{F}_x$ and $A \mapsto i_{x,*}A$ are adjoint. In a formula*

$$\mathrm{Mor}_{Sets}(\mathcal{F}_x, A) = \mathrm{Mor}_{Sh(X)}(\mathcal{F}, i_{x,*}A).$$

A similar statement holds for the case of abelian groups, algebraic structures. In the case of sheaves of modules we have

$$\mathrm{Hom}_{\mathcal{O}_{X,x}}(\mathcal{F}_x, A) = \mathrm{Hom}_{\mathcal{O}_X}(\mathcal{F}, i_{x,*}A).$$

Proof. Omitted. Hint: The stalk functor can be seen as the pullback functor for the morphism $i_x : \{x\} \to X$. Then the adjointness follows from adjointness of i_x^{-1} and $i_{x,*}$ (resp. i_x^* and $i_{x,*}$ in the case of sheaves of modules). □

6.28. Limits and colimits of presheaves

Let X be a topological space. Let $\mathcal{I} \to PSh(X)$, $i \mapsto \mathcal{F}_i$ be a diagram.

(1) Both $\lim_i \mathcal{F}_i$ and $\operatorname{colim}_i \mathcal{F}_i$ exist.
(2) For any open $U \subset X$ we have
$$(\lim_i \mathcal{F}_i)(U) = \lim_i \mathcal{F}_i(U)$$
and
$$(\operatorname{colim}_i \mathcal{F}_i)(U) = \operatorname{colim}_i \mathcal{F}_i(U).$$
(3) Let $x \in X$ be a point. In general the stalk of $\lim_i \mathcal{F}_i$ at x is not equal to the limit of the stalks. But if the diagram category is finite then it is the case. In other words, the stalk functor is left exact (see Categories, Definition 4.23.1).
(4) Let $x \in X$. We always have
$$(\operatorname{colim}_i \mathcal{F}_i)_x = \operatorname{colim}_i \mathcal{F}_{i,x}.$$

The proofs are all easy.

6.29. Limits and colimits of sheaves

Let X be a topological space. Let $\mathcal{I} \to Sh(X)$, $i \mapsto \mathcal{F}_i$ be a diagram.

(1) Both $\lim_i \mathcal{F}_i$ and $\operatorname{colim}_i \mathcal{F}_i$ exist.
(2) The inclusion functor $i : Sh(X) \to PSh(X)$ commutes with limits. In other words, we may compute the limit in the category of sheaves as the limit in the category of presheaves. In particular, for any open $U \subset X$ we have
$$(\lim_i \mathcal{F}_i)(U) = \lim_i \mathcal{F}_i(U).$$
(3) The inclusion functor $i : Sh(X) \to PSh(X)$ does not commute with colimits in general (not even with finite colimits – think surjections). The colimit is computed as the sheafification of the colimit in the category of presheaves:
$$\operatorname{colim}_i \mathcal{F}_i = \Big(U \mapsto \operatorname{colim}_i \mathcal{F}_i(U)\Big)^{\#}.$$
(4) Let $x \in X$ be a point. In general the stalk of $\lim_i \mathcal{F}_i$ at x is not equal to the limit of the stalks. But if the diagram category is finite then it is the case. In other words, the stalk functor is left exact.
(5) Let $x \in X$. We always have
$$(\operatorname{colim}_i \mathcal{F}_i)_x = \operatorname{colim}_i \mathcal{F}_{i,x}.$$
(6) The sheafification functor $^{\#} : PSh(X) \to Sh(X)$ commutes with all colimits, and with finite limits. But it does not commute with all limits.

The proofs are all easy. Here is an example of what is true for directed colimits of sheaves.

Lemma 6.29.1. *Let X be a topological space. Let I be a directed partially ordered set. Let $(\mathcal{F}_i, \varphi_{ii'})$ be a system of sheaves of sets over I, see Categories, Section 4.21. Let $U \subset X$ be an open subset. Consider the canonical map*
$$\Psi : \operatorname{colim}_i \mathcal{F}_i(U) \longrightarrow (\operatorname{colim}_i \mathcal{F}_i)(U)$$
(1) *If all the transition maps are injective then Ψ is injective for any open U.*

(2) *If U is quasi-compact, then Ψ is injective.*
(3) *If U is quasi-compact and all the transition maps are injective then Ψ is an isomorphism.*
(4) *If U has a cofinal system of open coverings $\mathcal{U} : U = \bigcup_{j\in J} U_j$ with J finite and $U_j \cap U_{j'}$ quasi-compact for all $j, j' \in J$, then Ψ is bijective.*

Proof. Assume all the transition maps are injective. In this case the presheaf $\mathcal{F}' : V \mapsto \operatorname{colim}_i \mathcal{F}_i(V)$ is separated (see Definition 6.11.2). By the discussion above we have $(\mathcal{F}')^\# = \operatorname{colim}_i \mathcal{F}_i$. By Lemma 6.17.5 we see that $\mathcal{F}' \to (\mathcal{F}')^\#$ is injective. This proves (1).

Assume U is quasi-compact. Suppose that $s \in \mathcal{F}_i(U)$ and $s' \in \mathcal{F}_{i'}(U)$ give rise to elements on the left hand side which have the same image under Ψ. Since U is quasi-compact this means there exists a finite open covering $U = \bigcup_{j=1,\ldots,m} U_j$ and for each j an index $i_j \in I$, $i_j \geq i$, $i_j \geq i'$ such that $\varphi_{ii_j}(s) = \varphi_{i'i_j}(s')$. Let $i'' \in I$ be $\geq$ than all of the i_j. We conclude that $\varphi_{ii''}(s)$ and $\varphi_{i'i''}(s)$ agree on the opens U_j for all j and hence that $\varphi_{ii''}(s) = \varphi_{i'i''}(s)$. This proves (2).

Assume U is quasi-compact and all transition maps injective. Let s be an element of the target of Ψ. Since U is quasi-compact there exists a finite open covering $U = \bigcup_{j=1,\ldots,m} U_j$, for each j an index $i_j \in I$ and $s_j \in \mathcal{F}_{i_j}(U_j)$ such that $s|_{U_j}$ comes from s_j for all j. Pick $i \in I$ which is $\geq$ than all of the i_j. By (1) the sections $\varphi_{i_j i}(s_j)$ agree over the overlaps $U_j \cap U_{j'}$. Hence they glue to a section $s' \in \mathcal{F}_i(U)$ which maps to s under Ψ. This proves (3).

Assume the hypothesis of (4). Let s be an element of the target of Ψ. By assumption there exists a finite open covering $U = \bigcup_{j=1,\ldots,m} U_j$, with $U_j \cap U_{j'}$ quasi-compact for all $j, j' \in J$ and for each j an index $i_j \in I$ and $s_j \in \mathcal{F}_{i_j}(U_j)$ such that $s|_{U_j}$ is the image of s_j for all j. Since $U_j \cap U_{j'}$ is quasi-compact we can apply (2) and we see that there exists an $i_{jj'} \in I$, $i_{jj'} \geq i_j$, $i_{jj'} \geq i_{j'}$ such that $\varphi_{i_j i_{jj'}}(s_j)$ and $\varphi_{i_{j'} i_{jj'}}(s_{j'})$ agree over $U_j \cap U_{j'}$. Choose an index $i \in I$ wich is bigger or equal than all the $i_{jj'}$. Then we see that the sections $\varphi_{i_j i}(s_j)$ of $\mathcal{F}_i$ glue to a section of $\mathcal{F}_i$ over U. This section is mapped to the element s as desired. □

Example 6.29.2. Let $X = \{s_1, s_2, \xi_1, \xi_2, \xi_3, \ldots\}$ as a set. Declare a subset $U \subset X$ to be open if $s_1 \in U$ or $s_2 \in U$ implies U contains all of the ξ_i. Let $U_n = \{\xi_n, \xi_{n+1}, \ldots\}$, and let $j_n : U_n \to X$ be the inclusion map. Set $\mathcal{F}_n = j_{n,*}\underline{\mathbf{Z}}$. There are transition maps $\mathcal{F}_n \to \mathcal{F}_{n+1}$. Let $\mathcal{F} = \operatorname{colim} \mathcal{F}_n$. Note that $\mathcal{F}_{n,\xi_m} = 0$ if $m < n$ because $\{\xi_m\}$ is an open subset of X which misses U_n. Hence we see that $\mathcal{F}_{\xi_n} = 0$ for all n. On the other hand the stalk $\mathcal{F}_{s_i}$, $i = 1, 2$ is the colimit

$$M = \operatorname{colim}_n \prod\nolimits_{m\geq n} \mathbf{Z}$$

which is not zero. We conclude that the sheaf $\mathcal{F}$ is the direct sum of the skyscraper sheaves with value M at the closed points s_1 and s_2. Hence $\Gamma(X, \mathcal{F}) = M \oplus M$. On the other hand, the reader can verify that $\operatorname{colim}_n \Gamma(X, \mathcal{F}_n) = M$. Hence some condition is necessary in part (4) of Lemma 6.29.1 above.

6.30. Bases and sheaves

Sometimes there exists a basis for the topology consisting of opens that are easier to work with than general opens. For convenience we give here some definitions and simple lemmas in order to facilitate working with (pre)sheaves in such a situation.

Definition 6.30.1. Let X be a topological space. Let $\mathcal{B}$ be a basis for the topology on X.

(1) A *presheaf $\mathcal{F}$ of sets on $\mathcal{B}$* is a rule which assigns to each $U \in \mathcal{B}$ a set $\mathcal{F}(U)$ and to each inclusion $V \subset U$ of elements of $\mathcal{B}$ a map $\rho_V^U : \mathcal{F}(U) \to \mathcal{F}(V)$ such that whenever $W \subset V \subset U$ in $\mathcal{B}$ we have $\rho_W^U = \rho_W^V \circ \rho_V^U$.
(2) A *morphism $\varphi : \mathcal{F} \to \mathcal{G}$ of presheaves of sets on $\mathcal{B}$* is a rule which assigns to each element $U \in \mathcal{B}$ a map of sets $\varphi : \mathcal{F}(U) \to \mathcal{G}(U)$ compatible with restriction maps.

As in the case of usual presheaves we use the terminology of sections, restrictions of sections, etc. In particular, we may define the *stalk* of $\mathcal{F}$ at a point $x \in X$ by the colimit

$$\mathcal{F}_x = \operatorname{colim}_{U \in \mathcal{B}, x \in U} \mathcal{F}(U).$$

As in the case of the stalk of a presheaf on X this limit is directed. The reason is that the collection of $U \in \mathcal{B}$, $x \in U$ is a fundamental system of open neighbourhoods of x.

It is easy to make examples to show that the notion of a presheaf on X is very different from the notion of a presheaf on a basis for the topology on X. This does not happen in the case of sheaves. A much more useful notion therefore, is the following.

Definition 6.30.2. Let X be a topological space. Let $\mathcal{B}$ be a basis for the topology on X.

(1) A *sheaf $\mathcal{F}$ of sets on $\mathcal{B}$* is a presheaf of sets on $\mathcal{B}$ which satisfies the following additional property: Given any $U \in \mathcal{B}$, and any covering $U = \bigcup_{i \in I} U_i$ with $U_i \in \mathcal{B}$, and any coverings $U_i \cap U_j = \bigcup_{k \in I_{ij}} U_{ijk}$ with $U_{ijk} \in \mathcal{B}$ the sheaf condition holds:
   (∗∗) For any collection of sections $s_i \in \mathcal{F}(U_i)$, $i \in I$ such that $\forall i, j \in I$, $\forall k \in I_{ij}$
   $$s_i|_{U_{ijk}} = s_j|_{U_{ijk}}$$
   there exists a unique section $s \in \mathcal{F}(U)$ such that $s_i = s|_{U_i}$ for all $i \in I$.
(2) A *morphism of sheaves of sets on $\mathcal{B}$* is simply a morphism of presheaves of sets.

First we explain that it suffices to check the sheaf condition (∗∗) on a cofinal system of coverings. In the situation of the definition, suppose $U \in \mathcal{B}$. Let us temporarily denote $\operatorname{Cov}_{\mathcal{B}}(U)$ the set of all coverings of U by elements of $\mathcal{B}$. Note that $\operatorname{Cov}_{\mathcal{B}}(U)$ is partially ordered by refinement. A subset $C \subset \operatorname{Cov}_{\mathcal{B}}(U)$ is a cofinal system, if for every $\mathcal{U} \in \operatorname{Cov}_{\mathcal{B}}(U)$ there exists a covering $\mathcal{V} \in C$ which refines $\mathcal{U}$.

Lemma 6.30.3. *With notation as above. For each $U \in \mathcal{B}$, let $C(U) \subset Cov_{\mathcal{B}}(U)$ be a cofinal system. For each $U \in \mathcal{B}$, and each $\mathcal{U} : U = \bigcup U_i$ in $C(U)$, let coverings $\mathcal{U}_{ij} : U_i \cap U_j = \bigcup U_{ijk}$, $U_{ijk} \in \mathcal{B}$ be given. Let $\mathcal{F}$ be a presheaf of sets on $\mathcal{B}$. The following are equivalent*

(1) *The presheaf $\mathcal{F}$ is a sheaf on $\mathcal{B}$.*
(2) *For every $U \in \mathcal{B}$ and every covering $\mathcal{U} : U = \bigcup U_i$ in $C(U)$ the sheaf condition (∗∗) holds (for the given coverings $\mathcal{U}_{ij}$).*

Proof. We have to show that (2) implies (1). Suppose that $U \in \mathcal{B}$, and that $\mathcal{U} : U = \bigcup_{i \in I} U_i$ is an arbitrary covering by elements of $\mathcal{B}$. Because the system $C(U)$ is cofinal we can find an element $\mathcal{V} : U = \bigcup_{j \in J} V_j$ in $C(U)$ which refines $\mathcal{U}$. This means there exists a map $\alpha : J \to I$ such that $V_j \subset U_{\alpha(i)}$.

Note that if $s, s' \in \mathcal{F}(U)$ are sections such that $s|_{U_i} = s'|_{U_i}$, then

$$s|_{V_j} = (s|_{U_{\alpha(j)}})|_{V_j} = (s'|_{U_{\alpha(j)}})|_{V_j} = s'|_{V_j}$$

for all j. Hence by the uniqueness in $(**)$ for the covering $\mathcal{V}$ we conclude that $s = s'$. Thus we have proved the uniqueness part of $(**)$ for our arbitrary covering $\mathcal{U}$.

Suppose furthermore that $U_i \cap U_{i'} = \bigcup_{k \in I_{ii'}} U_{ii'k}$ are arbitrary coverings by $U_{ii'k} \in \mathcal{B}$. Let us try to prove the existence part of $(**)$ for the system $(\mathcal{U}, \mathcal{U}_{ij})$. Thus let $s_i \in \mathcal{F}(U_i)$ and suppose we have

$$s_i|_{U_{ijk}} = s_{i'}|_{U_{ii'k}}$$

for all i, i', k. Set $t_j = s_{\alpha(i)}|_{V_j}$, where $\mathcal{V}$ and α are as above.

There is one small kink in the argument here. Namely, let $\mathcal{V}_{jj'} : V_j \cap V_{j'} = \bigcup_{l \in J_{jj'}} V_{jj'l}$ be the covering given to us by the statement of the lemma. It is not a priori clear that

$$t_j|_{V_{jj'l}} = t_{j'}|_{V_{jj'l}}$$

for all j, j', l. To see this, note that we do have

$$t_j|_W = t_{j'}|_W \text{ for all } W \in \mathcal{B}, W \subset V_{jj'l} \cap U_{\alpha(j)\alpha(j')k}$$

for all $k \in I_{\alpha(j)\alpha(j')}$, by our assumption on the family of elements s_i. And since $V_j \cap V_{j'} \subset U_{\alpha(j)} \cap U_{\alpha(j')}$ we see that $t_j|_{V_{jj'l}}$ and $t_{j'}|_{V_{jj'l}}$ agree on the members of a covering of $V_{jj'l}$ by elements of $\mathcal{B}$. Hence by the uniqueness part proved above we finally deduce the desired equality of $t_j|_{V_{jj'l}}$ and $t_{j'}|_{V_{jj'l}}$. Then we get the existence of an element $t \in \mathcal{F}(U)$ by property $(**)$ for $(\mathcal{V}, \mathcal{V}_{jj'})$.

Again there is a small snag. We know that t restricts to t_j on V_j but we do not yet know that t restricts to s_i on U_i. To conclude this note that the sets $U_i \cap V_j$, $j \in J$ cover U_i. Hence also the sets $U_{i\alpha(j)k} \cap V_j$, $j \in J$, $k \in I_{i\alpha(j)}$ cover U_i. We leave it to the reader to see that t and s_i restrict to the same section of $\mathcal{F}$ on any $W \in \mathcal{B}$ which is contained in one of the open sets $U_{i\alpha(j)k} \cap V_j$, $j \in J$, $k \in I_{i\alpha(j)}$. Hence by the uniqueness part seen above we win. □

Lemma 6.30.4. *Let X be a topological space. Let $\mathcal{B}$ be a basis for the topology on X. Assume that for every pair $U, U' \in \mathcal{B}$ we have $U \cap U' \in \mathcal{B}$. For each $U \in \mathcal{B}$, let $C(U) \subset Cov_{\mathcal{B}}(U)$ be a cofinal system. Let $\mathcal{F}$ be a presheaf of sets on $\mathcal{B}$. The following are equivalent*

1. *The presheaf $\mathcal{F}$ is a sheaf on $\mathcal{B}$.*
2. *For every $U \in \mathcal{B}$ and every covering $\mathcal{U} : U = \bigcup U_i$ in $C(U)$ and for every family of sections $s_i \in \mathcal{F}(U_i)$ such that $s_i|_{U_i \cap U_j} = s_j|_{U_i \cap U_j}$ there exists a unique section $s \in \mathcal{F}(U)$ which restricts to s_i on U_i.*

Proof. This is a reformulation of Lemma 6.30.3 above in the special case where the coverings $\mathcal{U}_{ij}$ each consist of a single element. But also this case is much easier and is an easy exercise to do directly. □

Lemma 6.30.5. *Let X be a topological space. Let $\mathcal{B}$ be a basis for the topology on X. Let $U \in \mathcal{B}$. Let $\mathcal{F}$ be a sheaf of sets on $\mathcal{B}$. The map*

$$\mathcal{F}(U) \to \prod\nolimits_{x \in U} \mathcal{F}_x$$

identifies $\mathcal{F}(U)$ with the elements $(s_x)_{x \in U}$ with the property

(∗) *For any $x \in U$ there exists a $V \in \mathcal{B}$, $x \in V$ and a section $\sigma \in \mathcal{F}(V)$ such that for all $y \in V$ we have $s_y = (V, \sigma)$ in $\mathcal{F}_y$.*

Proof. First note that the map $\mathcal{F}(U) \to \prod_{x \in U} \mathcal{F}_x$ is injective by the uniqueness in the sheaf condition of Definition 6.30.2. Let (s_x) be any element on the right hand side which satisfies (∗). Clearly this means we can find a covering $U = \bigcup U_i$, $U_i \in \mathcal{B}$ such that $(s_x)_{x \in U_i}$ comes from certain $\sigma_i \in \mathcal{F}(U_i)$. For every $y \in U_i \cap U_j$ the sections σ_i and σ_j agree in the stalk $\mathcal{F}_y$. Hence there exists an element $V_{ijy} \in \mathcal{B}$, $y \in V_{ijy}$ such that $\sigma_i|_{V_{ijy}} = \sigma_j|_{V_{ijy}}$. Thus the sheaf condition (∗∗) of Definition 6.30.2 applies to the system of σ_i and we obtain a section $s \in \mathcal{F}(U)$ with the desired property. □

Let X be a topological space. Let $\mathcal{B}$ be a basis for the topology on X. There is a natural restriction functor from the category of sheaves of sets on X to the category of sheaves of sets on $\mathcal{B}$. It turns out that this is an equivalence of categories. In down to earth terms this means the following.

Lemma 6.30.6. *Let X be a topological space. Let $\mathcal{B}$ be a basis for the topology on X. Let $\mathcal{F}$ be a sheaf of sets on $\mathcal{B}$. There exists a unique sheaf of sets $\mathcal{F}^{ext}$ on X such that $\mathcal{F}^{ext}(U) = \mathcal{F}(U)$ for all $U \in \mathcal{B}$ compatibly with the restriction mappings.*

Proof. We first construct a presheaf $\mathcal{F}^{ext}$ with the desired property. Namely, for an arbitrary open $U \subset X$ we define $\mathcal{F}^{ext}(U)$ as the set of elements $(s_x)_{x \in U}$ such that (∗) of Lemma 6.30.5 holds. It is clear that there are restriction mappings that turn $\mathcal{F}^{ext}$ into a presheaf of sets. Also, by Lemma 6.30.5 we see that $\mathcal{F}(U) = \mathcal{F}^{ext}(U)$ whenever U is an element of the basis $\mathcal{B}$. To see $\mathcal{F}^{ext}$ is a sheaf one may argue as in the proof of Lemma 6.17.1. □

Note that we have

$$\mathcal{F}_x = \mathcal{F}^{ext}_x$$

in the situation of the lemma. This is so because the collection of elements of $\mathcal{B}$ containing x forms a fundamental system of open neighbourhoods of x.

Lemma 6.30.7. *Let X be a topological space. Let $\mathcal{B}$ be a basis for the topology on X. Denote $Sh(\mathcal{B})$ the category of sheaves on $\mathcal{B}$. There is an equivalence of categories*

$$Sh(X) \longrightarrow Sh(\mathcal{B})$$

which assigns to a sheaf on X its restriction to the members of $\mathcal{B}$.

Proof. The inverse functor in given in Lemma 6.30.6 above. Checking the obvious functorialities is left to the reader. □

This ends the discussion of sheaves of sets on a basis $\mathcal{B}$. Let $(\mathcal{C}, F)$ be a type of algebraic structure. At the end of this section we would like to point out that the constructions above work for sheaves with values in $\mathcal{C}$. Let us briefly define the relevant notions.

Definition 6.30.8. Let X be a topological space. Let $\mathcal{B}$ be a basis for the topology on X. Let $(\mathcal{C}, F)$ be a type of algebraic structure.

(1) A *presheaf $\mathcal{F}$ with values in $\mathcal{C}$ on $\mathcal{B}$* is a rule which assigns to each $U \in \mathcal{B}$ an object $\mathcal{F}(U)$ of $\mathcal{C}$ and to each inclusion $V \subset U$ of elements of $\mathcal{B}$ a morphism $\rho_V^U : \mathcal{F}(U) \to \mathcal{F}(V)$ in $\mathcal{C}$ such that whenever $W \subset V \subset U$ in $\mathcal{B}$ we have $\rho_W^U = \rho_W^V \circ \rho_V^U$.
(2) A *morphism $\varphi : \mathcal{F} \to \mathcal{G}$ of presheaves with values in $\mathcal{C}$ on $\mathcal{B}$* is a rule which assigns to each element $U \in \mathcal{B}$ a morphism of algebraic structures $\varphi : \mathcal{F}(U) \to \mathcal{G}(U)$ compatible with restriction maps.
(3) Given a presheaf $\mathcal{F}$ with values in $\mathcal{C}$ on $\mathcal{B}$ we say that $U \mapsto F(\mathcal{F}(U))$ is the underlying presheaf of sets.
(4) A *sheaf $\mathcal{F}$ with values in $\mathcal{C}$ on $\mathcal{B}$* is a presheaf with values in $\mathcal{C}$ on $\mathcal{B}$ whose underlying presheaf of sets is a sheaf.

At this point we can define the *stalk* at $x \in X$ of a presheaf with values in $\mathcal{C}$ on $\mathcal{B}$ as the directed colimit

$$\mathcal{F}_x = \operatorname{colim}_{U \in \mathcal{B}, x \in U} \mathcal{F}(U).$$

It exists as an object of $\mathcal{C}$ because of our assumptions on $\mathcal{C}$. Also, we see that the underlying set of $\mathcal{F}_x$ is the stalk of the underlying presheaf of sets on $\mathcal{B}$.

Note that Lemmas 6.30.3, 6.30.4 and 6.30.5 refer to the sheaf property which we have defined in terms of the associated presheaf of sets. Hence they generalize without change to the notion of a presheaf with values in $\mathcal{C}$. The analogue of Lemma 6.30.6 need some care. Here it is.

Lemma 6.30.9. *Let X be a topological space. Let $(\mathcal{C}, F)$ be a type of algebraic structure. Let $\mathcal{B}$ be a basis for the topology on X. Let $\mathcal{F}$ be a sheaf with values in $\mathcal{C}$ on $\mathcal{B}$. There exists a unique sheaf $\mathcal{F}^{ext}$ with values in $\mathcal{C}$ on X such that $\mathcal{F}^{ext}(U) = \mathcal{F}(U)$ for all $U \in \mathcal{B}$ compatibly with the restriction mappings.*

Proof. By the conditions imposed on the pair $(\mathcal{C}, F)$ it suffices to come up with a presheaf $\mathcal{F}^{ext}$ which does the correct thing on the level of underlying presheaves of sets. Thus our first task is to construct a suitable object $\mathcal{F}^{ext}(U)$ for all open $U \subset X$. We could do this by imitating Lemma 6.18.1 in the setting of presheaves on $\mathcal{B}$. However, a slightly different method (but basically equivalent) is the following: Define it as the directed colimit

$$\mathcal{F}^{ext}(U) := \operatorname{colim}_{\mathcal{U}} FIB(\mathcal{U})$$

over all coverings $\mathcal{U} : U = \bigcup_{i \in I} U_i$ by $U_i \in \mathcal{B}$ of the fibre product

$$\begin{array}{ccc} FIB(\mathcal{U}) & \longrightarrow & \prod_{x \in U} \mathcal{F}_x \\ \downarrow & & \downarrow \\ \prod_{i \in I} \mathcal{F}(U_i) & \longrightarrow & \prod_{i \in I} \prod_{x \in U_i} \mathcal{F}_x \end{array}$$

By the usual arguments, see Lemma 6.15.4 and Example 6.15.5 it suffices to show that this construction on underlying sets is the same as the definition using $(**)$ above. Details left to the reader. □

Note that we have

$$\mathcal{F}_x = \mathcal{F}_x^{ext}$$

as objects in $\mathcal{C}$ in the situation of the lemma. This is so because the collection of elements of $\mathcal{B}$ containing x forms a fundamental system of open neighbourhoods of x.

Lemma 6.30.10. *Let X be a topological space. Let $\mathcal{B}$ be a basis for the topology on X. Let $(\mathcal{C}, F)$ be a type of algebraic structure. Denote $Sh(\mathcal{B}, \mathcal{C})$ the category of sheaves with values in $\mathcal{C}$ on $\mathcal{B}$. There is an equivalence of categories*

$$Sh(X, \mathcal{C}) \longrightarrow Sh(\mathcal{B}, \mathcal{C})$$

which assigns to a sheaf on X its restriction to the members of $\mathcal{B}$.

Proof. The inverse functor in given in Lemma 6.30.9 above. Checking the obvious functorialities is left to the reader. □

Finally we come to the case of (pre)sheaves of modules on a basis. We will use the easy fact that the category of presheaves of sets on a basis has products and that they are described by taking products of values on elements of the bases.

Definition 6.30.11. Let X be a topological space. Let $\mathcal{B}$ be a basis for the topology on X. Let $\mathcal{O}$ be a presheaf of rings on $\mathcal{B}$.

(1) A *presheaf of $\mathcal{O}$-modules $\mathcal{F}$ on $\mathcal{B}$* is a presheaf of abelian groups on $\mathcal{B}$ together with a morphism of presheaves of sets $\mathcal{O} \times \mathcal{F} \to \mathcal{F}$ such that for all $U \in \mathcal{B}$ the map $\mathcal{O}(U) \times \mathcal{F}(U) \to \mathcal{F}(U)$ turns the group $\mathcal{F}(U)$ into an $\mathcal{O}(U)$-module.
(2) A *morphism $\varphi : \mathcal{F} \to \mathcal{G}$ of presheaves of $\mathcal{O}$-modules on $\mathcal{B}$* is a morphism of abelian presheaves on $\mathcal{B}$ which induces an $\mathcal{O}(U)$-module homomorphism $\mathcal{F}(U) \to \mathcal{G}(U)$ for every $U \in \mathcal{B}$.
(3) Suppose that $\mathcal{O}$ is a sheaf of rings on $\mathcal{B}$. A *sheaf $\mathcal{F}$ of $\mathcal{O}$-modules on $\mathcal{B}$* is a presheaf of $\mathcal{O}$-modules on $\mathcal{B}$ whose underlying presheaf of abelian groups is a sheaf.

We can define the *stalk* at $x \in X$ of a presheaf of $\mathcal{O}$-modules on $\mathcal{B}$ as the directed colimit

$$\mathcal{F}_x = \operatorname{colim}_{U \in \mathcal{B}, x \in U} \mathcal{F}(U).$$

It is a $\mathcal{O}_x$-module.

Note that Lemmas 6.30.3, 6.30.4 and 6.30.5 refer to the sheaf property which we have defined in terms of the associated presheaf of sets. Hence they generalize without change to the notion of a presheaf of $\mathcal{O}$-modules. The analogue of Lemma 6.30.6 is as follows.

Lemma 6.30.12. *Let X be a topological space. Let $\mathcal{O}$ be a sheaf of rings on $\mathcal{B}$. Let $\mathcal{B}$ be a basis for the topology on X. Let $\mathcal{F}$ be a sheaf with values in $\mathcal{C}$ on $\mathcal{B}$. Let $\mathcal{O}^{ext}$ be the sheaf of rings on X extending $\mathcal{O}$ and let $\mathcal{F}^{ext}$ be the abelian sheaf on X extending $\mathcal{F}$, see Lemma 6.30.9. There exists a canonical map*

$$\mathcal{O}^{ext} \times \mathcal{F}^{ext} \longrightarrow \mathcal{F}^{ext}$$

which agrees with the given map over elements of $\mathcal{B}$ and which endows $\mathcal{F}^{ext}$ with the structure of an $\mathcal{O}^{ext}$-module.

Proof. It suffices to construct the multiplication map on the level of presheaves of sets. Perhaps the easiest way to see this is to prove directly that if $(f_x)_{x\in U}$, $f_x \in \mathcal{O}_x$ and $(m_x)_{x\in U}$, $m_x \in \mathcal{F}_x$ satisfy $(*)$, then the element $(f_x m_x)_{x\in U}$ also satisfies $(*)$. Then we get the desired result, because in the proof of Lemma 6.30.6 we construct the extension in terms of families of elements of stalks satisfying $(*)$. □

Note that we have

$$\mathcal{F}_x = \mathcal{F}_x^{ext}$$

as $\mathcal{O}_x$-modules in the situation of the lemma. This is so because the collection of elements of $\mathcal{B}$ containing x forms a fundamental system of open neighbourhoods of x, or simply because it is true on the underlying sets.

Lemma 6.30.13. *Let X be a topological space. Let $\mathcal{B}$ be a basis for the topology on X. Let $\mathcal{O}$ be a sheaf of rings on X. Denote $Mod(\mathcal{O}|_{\mathcal{B}})$ the category of sheaves of $\mathcal{O}|_{\mathcal{B}}$-modules on $\mathcal{B}$. There is an equivalence of categories*

$$Mod(\mathcal{O}) \longrightarrow Mod(\mathcal{O}|_{\mathcal{B}})$$

which assigns to a sheaf of $\mathcal{O}$-modules on X its restriction to the members of $\mathcal{B}$.

Proof. The inverse functor in given in Lemma 6.30.12 above. Checking the obvious functorialities is left to the reader. □

Finally, we address the question of the relationship of this with continuous maps. This is now very easy thanks to the work above. First we do the case where there is a basis on the target given.

Lemma 6.30.14. *Let $f : X \to Y$ be a continuous map of topological spaces. Let $(\mathcal{C}, F)$ be a type of algebraic structures. Let $\mathcal{F}$ be a sheaf with values in $\mathcal{C}$ on X. Let $\mathcal{G}$ be a sheaf with values in $\mathcal{C}$ on Y. Let $\mathcal{B}$ be a basis for the topology on Y. Suppose given for every $V \in \mathcal{B}$ a morphism*

$$\varphi_V : \mathcal{G}(V) \longrightarrow \mathcal{F}(f^{-1}V)$$

of $\mathcal{C}$ compatible with restriction mappings. Then there is a unique f-map (see Definition 6.21.7 and discussion of f-maps in Section 6.23) $\varphi : \mathcal{G} \to \mathcal{F}$ recovering φ_V for $V \in \mathcal{B}$.

Proof. This is trivial because the collection of maps amounts to a morphism between the restrictions of $\mathcal{G}$ and $f_*\mathcal{F}$ to $\mathcal{B}$. By Lemma 6.30.10 this is the same as giving a morphism from $\mathcal{G}$ to $f_*\mathcal{F}$, which by Lemma 6.21.8 is the same as an f-map. See also Lemma 6.23.1 and the discussion preceding it for how to deal with the case of sheaves of algebraic structures. □

Here is the analogue for ringed spaces.

Lemma 6.30.15. *Let $(f, f^\sharp) : (X, \mathcal{O}_X) \to (Y, \mathcal{O}_Y)$ be a morphism of ringed spaces. Let $\mathcal{F}$ be a sheaf of $\mathcal{O}_X$-modules. Let $\mathcal{G}$ be a sheaf of $\mathcal{O}_Y$-modules. Let $\mathcal{B}$ be a basis for the topology on Y. Suppose given for every $V \in \mathcal{B}$ a $\mathcal{O}_Y(V)$-module map*

$$\varphi_V : \mathcal{G}(V) \longrightarrow \mathcal{F}(f^{-1}V)$$

(where $\mathcal{F}(f^{-1}V)$ has a module structure using $f_V^\sharp : \mathcal{O}_Y(V) \to \mathcal{O}_X(f^{-1}V)$) compatible with restriction mappings. Then there is a unique f-map (see discussion of f-maps in Section 6.26) $\varphi : \mathcal{G} \to \mathcal{F}$ recovering φ_V for $V \in \mathcal{B}$.

Proof. Same as the proof of the corresponding lemma for sheaves of algebraic structures above. □

Lemma 6.30.16. *Let $f : X \to Y$ be a continuous map of topological spaces. Let $(\mathcal{C}, F)$ be a type of algebraic structures. Let $\mathcal{F}$ be a sheaf with values in $\mathcal{C}$ on X. Let $\mathcal{G}$ be a sheaf with values in $\mathcal{C}$ on Y. Let $\mathcal{B}_Y$ be a basis for the topology on Y. Let $\mathcal{B}_X$ be a basis for the topology on X. Suppose given for every $V \in \mathcal{B}_Y$, and $U \in \mathcal{B}_X$ such that $f(U) \subset V$ a morphism*

$$\varphi_V^U : \mathcal{G}(V) \longrightarrow \mathcal{F}(U)$$

of $\mathcal{C}$ compatible with restriction mappings. Then there is a unique f-map (see Definition 6.21.7 and the discussion of f-maps in Section 6.23) $\varphi : \mathcal{G} \to \mathcal{F}$ recovering φ_V^U as the composition

$$\mathcal{G}(V) \xrightarrow{\varphi_V} \mathcal{F}(f^{-1}(V)) \xrightarrow{restr.} \mathcal{F}(U)$$

for every pair (U, V) as above.

Proof. Let us first proves this for sheaves of sets. Fix $V \subset Y$ open. Pick $s \in \mathcal{G}(V)$. We are going to construct an element $\varphi_V(s) \in \mathcal{F}(f^{-1}V)$. We can define a value $\varphi(s)_x$ in the stalk $\mathcal{F}_x$ for every $x \in f^{-1}V$ by picking a $U \in \mathcal{B}_X$ with $x \in U \subset f^{-1}V$ and setting $\varphi(s)_x$ equal to the equivalence class of $(U, \varphi_V^U(s))$ in the stalk. Clearly, the family $(\varphi(s)_x)_{x \in f^{-1}V}$ satisfies condition $(*)$ because the maps φ_V^U for varying U are compatible with restrictions in the sheaf $\mathcal{F}$. Thus, by the proof of Lemma 6.30.6 we see that $(\varphi(s)_x)_{x \in f^{-1}V}$ corresponds to a unique element $\varphi_V(s)$ of $\mathcal{F}(f^{-1}V)$. Thus we have defined a set map $\varphi_V : \mathcal{G}(V) \to \mathcal{F}(f^{-1}V)$. The compatibility between φ_V and φ_V^U follows from Lemma 6.30.5.

We leave it to the reader to show that the construction of φ_V is compatible with restriction mappings as we vary $v \in \mathcal{B}_Y$. Thus we may apply Lemma 6.30.14 above to "glue" them to the desired f-map.

Finally, we note that the map of sheaves of sets so constructed satisfies the property that the map on stalks

$$\mathcal{G}_{f(x)} \longrightarrow \mathcal{F}_x$$

is the colimit of the system of maps φ_V^U as $V \in \mathcal{B}_Y$ varies over those elements that contain $f(x)$ and $U \in \mathcal{B}_X$ varies over those elements that contain x. In particular, if $\mathcal{G}$ and $\mathcal{F}$ are the underlying sheaves of sets of sheaves of algebraic structures, then we see that the maps on stalks is a morphism of algebraic structures. Hence we conclude that the associated map of sheaves of underlying sets $f^{-1}\mathcal{G} \to \mathcal{F}$ satisfies the assumptions of Lemma 6.23.1. We conclude that $f^{-1}\mathcal{G} \to \mathcal{F}$ is a morphism of sheaves with values in $\mathcal{C}$. And by adjointness this means that φ is an f-map of sheaves of algebraic structures. □

Lemma 6.30.17. *Let $(f, f^\sharp) : (X, \mathcal{O}_X) \to (Y, \mathcal{O}_Y)$ be a morphism of ringed spaces. Let $\mathcal{F}$ be a sheaf of $\mathcal{O}_X$-modules. Let $\mathcal{G}$ be a sheaf of $\mathcal{O}_Y$-modules. Let $\mathcal{B}_Y$ be a basis for the topology on Y. Let $\mathcal{B}_X$ be a basis for the topology on X. Suppose given for every $V \in \mathcal{B}_Y$, and $U \in \mathcal{B}_X$ such that $f(U) \subset V$ a $\mathcal{O}_Y(V)$-module map*

$$\varphi_V^U : \mathcal{G}(V) \longrightarrow \mathcal{F}(U)$$

compatible with restriction mappings. Here the $\mathcal{O}_Y(V)$-module structure on $\mathcal{F}(U)$ comes from the $\mathcal{O}_X(U)$-module structure via the map $f_V^\sharp : \mathcal{O}_Y(V) \to \mathcal{O}_X(f^{-1}V) \to$

$\mathcal{O}_X(U)$. Then there is a unique f-map of sheaves of modules (see Definition 6.21.7 and the discussion of f-maps in Section 6.26) $\varphi : \mathcal{G} \to \mathcal{F}$ recovering φ_V^U as the composition

$$\mathcal{G}(V) \xrightarrow{\varphi_V} \mathcal{F}(f^{-1}(V)) \xrightarrow{restrc.} \mathcal{F}(U)$$

for every pair (U, V) as above.

Proof. Similar to the above and omitted. □

6.31. Open immersions and (pre)sheaves

Let X be a topological space. Let $j : U \to X$ be the inclusion of an open subset U into X. In Section 6.21 we have defined functors j_* and j^{-1} such that j_* is right adjoint to j^{-1}. It turns out that for an open immersion there is a left adjoint for j^{-1}, which we will denote $j_!$. First we point out that j^{-1} has a particularly simple description in the case of an open immersion.

Lemma 6.31.1. *Let X be a topological space. Let $j : U \to X$ be the inclusion of an open subset U into X.*

1. *Let $\mathcal{G}$ be a presheaf of sets on X. The presheaf $j_p\mathcal{G}$ (see Section 6.21) is given by the rule $V \mapsto \mathcal{G}(V)$ for $V \subset U$ open.*
2. *Let $\mathcal{G}$ be a sheaf of sets on X. The sheaf $j^{-1}\mathcal{G}$ is given by the rule $V \mapsto \mathcal{G}(V)$ for $V \subset U$ open.*
3. *For any point $u \in U$ and any sheaf $\mathcal{G}$ on X we have a canonical identification of stalks*
$$j^{-1}\mathcal{G}_u = (\mathcal{G}|_U)_u = \mathcal{G}_u.$$
4. *On the category of presheaves of U we have $j_p j_* = id$.*
5. *On the category of sheaves of U we have $j^{-1}j_* = id$.*

The same description holds for (pre)sheaves of abelian groups, (pre)sheaves of algebraic structures, and (pre)sheaves of modules.

Proof. The colimit in the definition of $j_p\mathcal{G}(V)$ is over collection of all $W \subset X$ open such that $V \subset W$ ordered by reverse inclusion. Hence this has a largest element, namely V. This proves (1). And (2) follows because the assignment $V \mapsto \mathcal{G}(V)$ for $V \subset U$ open is clearly a sheaf if $\mathcal{G}$ is a sheaf. Assertion (3) follows from (2) since the collection of open neighbourhoods of u which are contained in U is cofinal in the collection of all open neighbourhoods of u in X. Parts (4) and (5) follow by computing $j^{-1}j_*\mathcal{F}(V) = j_*\mathcal{F}(V) = \mathcal{F}(V)$.

The exact same arguments work for (pre)sheaves of abelian groups and (pre)sheaves of algebraic structures. □

Definition 6.31.2. Let X be a topological space. Let $j : U \to X$ be the inclusion of an open subset.

1. Let $\mathcal{G}$ be a presheaf of sets, abelian groups or algebraic structures on X. The presheaf $j_p\mathcal{G}$ described in Lemma 6.31.1 is called the *restriction of $\mathcal{G}$ to U* and denoted $\mathcal{G}|_U$.
2. Let $\mathcal{G}$ be a sheaf of sets on X, abelian groups or algebraic structures on X. The sheaf $j^{-1}\mathcal{G}$ is called the *restriction of $\mathcal{G}$ to U* and denoted $\mathcal{G}|_U$.
3. If $(X, \mathcal{O})$ is a ringed space, then the pair $(U, \mathcal{O}|_U)$ is called the *open subspace of $(X, \mathcal{O})$ associated to U*.

(4) If $\mathcal{G}$ is a presheaf of $\mathcal{O}$-modules then $\mathcal{G}|_U$ together with the multiplication map $\mathcal{O}|_U \times \mathcal{G}|_U \to \mathcal{G}|_U$ (see Lemma 6.24.6) is called the *restriction of $\mathcal{G}$ to U*.

We leave a definition of the restriction of presheaves of modules to the reader. Ok, so in this section we will discuss a left adjoint to the restriction functor. Here is the definition in the case of (pre)sheaves of sets.

Definition 6.31.3. Let X be a topological space. Let $j : U \to X$ be the inclusion of an open subset.

(1) Let $\mathcal{F}$ be a presheaf of sets on U. We define the *extension of $\mathcal{F}$ by the empty set* $j_{p!}\mathcal{F}$ to be the presheaf of sets on X defined by the rule
$$j_{p!}\mathcal{F}(V) = \begin{cases} \emptyset & \text{if} \quad V \not\subset U \\ \mathcal{F}(V) & \text{if} \quad V \subset U \end{cases}$$
with obvious restriction mappings.
(2) Let $\mathcal{F}$ be a sheaf of sets on U. We define the *extension of $\mathcal{F}$ by the empty set* $j_!\mathcal{F}$ to be the sheafification of the presheaf $j_{p!}\mathcal{F}$.

Lemma 6.31.4. *Let X be a topological space. Let $j : U \to X$ be the inclusion of an open subset.*

(1) *The functor $j_{p!}$ is a left adjoint to the restriction functor j_p (see Lemma 6.31.1).*
(2) *The functor $j_!$ is a left adjoint to restriction, in a formula*
$$\mathrm{Mor}_{Sh(X)}(j_!\mathcal{F}, \mathcal{G}) = \mathrm{Mor}_{Sh(U)}(\mathcal{F}, j^{-1}\mathcal{G}) = \mathrm{Mor}_{Sh(U)}(\mathcal{F}, \mathcal{G}|_U)$$
bifunctorially in $\mathcal{F}$ and $\mathcal{G}$.
(3) *Let $\mathcal{F}$ be a sheaf of sets on U. The stalks of the sheaf $j_!\mathcal{F}$ are described as follows*
$$j_!\mathcal{F}_x = \begin{cases} \emptyset & \text{if} \quad x \notin U \\ \mathcal{F}_x & \text{if} \quad x \in U \end{cases}$$
(4) *On the category of presheaves of U we have $j_p j_{p!} = id$.*
(5) *On the category of sheaves of U we have $j^{-1}j_! = id$.*

Proof. To map $j_{p!}\mathcal{F}$ into $\mathcal{G}$ it is enough to map $\mathcal{F}(V) \to \mathcal{G}(V)$ whenever $V \subset U$ compatibly with restriction mappings. And by Lemma 6.31.1 the same description holds for maps $\mathcal{F} \to \mathcal{G}|_U$. The adjointness of $j_!$ and restriction follows from this and the properties of sheafification. The identification of stalks is obvious from the definition of the extension by the empty set and the definition of a stalk. Statements (4) and (5) follow by computing the value of the sheaf on any open of U. □

Note that if $\mathcal{F}$ is a sheaf of abelian groups on U, then in general $j_!\mathcal{F}$ as defined above, is not a sheaf of abelian groups, for example because some of its stalks are empty (hence not abelian groups for sure). Thus we need to modify the definition of $j_!$ depending on the type of sheaves we consider. The reason for choosing the empty set in the definition of the extension by the empty set, is that it is the initial object in the category of sets. Thus in the case of abelian groups we use 0 (and more generally for sheaves with values in any abelian category).

Definition 6.31.5. Let X be a topological space. Let $j : U \to X$ be the inclusion of an open subset.

(1) Let $\mathcal{F}$ be an abelian presheaf on U. We define the *extension* $j_{p!}\mathcal{F}$ *of* $\mathcal{F}$ *by* 0 to be the abelian presheaf on X defined by the rule
$$j_{p!}\mathcal{F}(V) = \begin{cases} 0 & \text{if} \quad V \not\subset U \\ \mathcal{F}(V) & \text{if} \quad V \subset U \end{cases}$$
with obvious restriction mappings.
(2) Let $\mathcal{F}$ be an abelian sheaf on U. We define the *extension* $j_!\mathcal{F}$ *of* $\mathcal{F}$ *by* 0 to be the sheafification of the abelian presheaf $j_{p!}\mathcal{F}$.
(3) Let $\mathcal{C}$ be a category having an initial object e. Let $\mathcal{F}$ be a presheaf on U with values in $\mathcal{C}$. We define the *extension* $j_{p!}\mathcal{F}$ *of* $\mathcal{F}$ *by* e to be the presheaf on X with values in $\mathcal{C}$ defined by the rule
$$j_{p!}\mathcal{F}(V) = \begin{cases} e & \text{if} \quad V \not\subset U \\ \mathcal{F}(V) & \text{if} \quad V \subset U \end{cases}$$
with obvious restriction mappings.
(4) Let $(\mathcal{C}, F)$ be a type of algebraic structure such that $\mathcal{C}$ has an initial object e. Let $\mathcal{F}$ be a sheaf of algebraic structures on U (of the give type). We define the *extension* $j_!\mathcal{F}$ *of* $\mathcal{F}$ *by* e to be the sheafification of the presheaf $j_{p!}\mathcal{F}$ defined above.
(5) Let $\mathcal{O}$ be a presheaf of rings on X. Let $\mathcal{F}$ be a presheaf of $\mathcal{O}|_U$-modules. In this case we define the *extension by* 0 to be the presheaf of $\mathcal{O}$-modules which is equal to $j_{p!}\mathcal{F}$ as an abelian presheaf endowed with the multiplication map $\mathcal{O} \times j_{p!}\mathcal{F} \to j_{p!}\mathcal{F}$.
(6) Let $\mathcal{O}$ be a sheaf of rings on X. Let $\mathcal{F}$ be a sheaf of $\mathcal{O}|_U$-modules. In this case we define the *extension by* 0 to be the $\mathcal{O}$-module which is equal to $j_!\mathcal{F}$ as an abelian sheaf endowed with the multiplication map $\mathcal{O} \times j_!\mathcal{F} \to j_!\mathcal{F}$.

It is true that one can define $j_!$ in the setting of sheaves of algebraic structures (see below). However, it depends on the type of algebraic structures involved what the resulting object is. For example, if $\mathcal{O}$ is a sheaf of rings on U, then $j_{!,rings}\mathcal{O} \neq j_{!,abelian}\mathcal{O}$ since the initial object in the category of rings is $\mathbf{Z}$ and the initial object in the category of abelian groups is 0. In particular the functor $j_!$ *does not commute with taking underlying sheaves of sets*, in contrast to what we have seen so far! We separate out the case of (pre)sheaves of abelian groups, (pre)sheaves of algebraic structures and (pre)sheaves of modules as usual.

Lemma 6.31.6. *Let X be a topological space. Let $j : U \to X$ be the inclusion of an open subset. Consider the functors of restriction and extension by* 0 *for abelian (pre)sheaves.*

(1) *The functor $j_{p!}$ is a left adjoint to the restriction functor j_p (see Lemma 6.31.1).*
(2) *The functor $j_!$ is a left adjoint to restriction, in a formula*
$$\mathrm{Mor}_{Ab(X)}(j_!\mathcal{F}, \mathcal{G}) = \mathrm{Mor}_{Ab(U)}(\mathcal{F}, j^{-1}\mathcal{G}) = \mathrm{Mor}_{Ab(U)}(\mathcal{F}, \mathcal{G}|_U)$$
bifunctorially in $\mathcal{F}$ and $\mathcal{G}$.
(3) *Let $\mathcal{F}$ be an abelian sheaf on U. The stalks of the sheaf $j_!\mathcal{F}$ are described as follows*
$$j_!\mathcal{F}_x = \begin{cases} 0 & \textit{if} \quad x \notin U \\ \mathcal{F}_x & \textit{if} \quad x \in U \end{cases}$$
(4) *On the category of abelian presheaves of U we have $j_p j_{p!} = id$.*

(5) *On the category of abelian sheaves of U we have $j^{-1}j_! = id$.*

Proof. Omitted. □

Lemma 6.31.7. *Let X be a topological space. Let $j : U \to X$ be the inclusion of an open subset. Let $(\mathcal{C}, F)$ be a type of algebraic structure such that $\mathcal{C}$ has an initial object e. Consider the functors of restriction and extension by e for (pre)sheaves of algebraic structure defined above.*

(1) *The functor $j_{p!}$ is a left adjoint to the restriction functor j_p (see Lemma 6.31.1).*
(2) *The functor $j_!$ is a left adjoint to restriction, in a formula*
$$\mathrm{Mor}_{Sh(X,\mathcal{C})}(j_!\mathcal{F}, \mathcal{G}) = \mathrm{Mor}_{Sh(U,\mathcal{C})}(\mathcal{F}, j^{-1}\mathcal{G}) = \mathrm{Mor}_{Sh(U,\mathcal{C})}(\mathcal{F}, \mathcal{G}|_U)$$
bifunctorially in $\mathcal{F}$ and $\mathcal{G}$.
(3) *Let $\mathcal{F}$ be a sheaf on U. The stalks of the sheaf $j_!\mathcal{F}$ are described as follows*
$$j_!\mathcal{F}_x = \begin{cases} e & \text{if} \quad x \notin U \\ \mathcal{F}_x & \text{if} \quad x \in U \end{cases}$$
(4) *On the category of presheaves of algebraic structures on U we have $j_p j_{p!} = id$.*
(5) *On the category of sheaves of algebraic structures on U we have $j^{-1}j_! = id$.*

Proof. Omitted. □

Lemma 6.31.8. *Let $(X, \mathcal{O})$ be a ringed space. Let $j : (U, \mathcal{O}|_U) \to (X, \mathcal{O})$ be an open subspace. Consider the functors of restriction and extension by 0 for (pre)sheaves of modules defined above.*

(1) *The functor $j_{p!}$ is a left adjoint to restriction, in a formula*
$$\mathrm{Mor}_{PMod(\mathcal{O})}(j_{p!}\mathcal{F}, \mathcal{G}) = \mathrm{Mor}_{PMod(\mathcal{O}|_U)}(\mathcal{F}, \mathcal{G}|_U)$$
bifunctorially in $\mathcal{F}$ and $\mathcal{G}$.
(2) *The functor $j_!$ is a left adjoint to restriction, in a formula*
$$\mathrm{Mor}_{Mod(\mathcal{O})}(j_!\mathcal{F}, \mathcal{G}) = \mathrm{Mor}_{Mod(\mathcal{O}|_U)}(\mathcal{F}, \mathcal{G}|_U)$$
bifunctorially in $\mathcal{F}$ and $\mathcal{G}$.
(3) *Let $\mathcal{F}$ be a sheaf of $\mathcal{O}$-modules on U. The stalks of the sheaf $j_!\mathcal{F}$ are described as follows*
$$j_!\mathcal{F}_x = \begin{cases} 0 & \text{if} \quad x \notin U \\ \mathcal{F}_x & \text{if} \quad x \in U \end{cases}$$
(4) *On the category of sheaves of $\mathcal{O}|_U$-modules on U we have $j^{-1}j_! = id$.*

Proof. Omitted. □

Note that by the lemmas above, both the functors j_* and $j_!$ are fully faithful embeddings of the category of sheaves on U into the category of sheaves on X. It is only true for the functor $j_!$ that one can easily describe the essential image of this functor.

Lemma 6.31.9. *Let X be a topological space. Let $j : U \to X$ be the inclusion of an open subset. The functor*
$$j_! : Sh(U) \longrightarrow Sh(X)$$

is fully faithful. Its essential image consists exactly of those sheaves $\mathcal{G}$ such that $\mathcal{G}_x = \emptyset$ for all $x \in X \setminus U$.

Proof. Fully faithfulness follows formally from $j^{-1}j_! = \mathrm{id}$. We have seen that any sheaf in the image of the functor has the property on the stalks mentioned in the lemma. Conversely, suppose that $\mathcal{G}$ has the indicated property. Then it is easy to check that

$$j_! j^{-1}\mathcal{G} \to \mathcal{G}$$

is an isomorphism on all stalks and hence an isomorphism. □

Lemma 6.31.10. *Let X be a topological space. Let $j : U \to X$ be the inclusion of an open subset. The functor*

$$j_! : Ab(U) \longrightarrow Ab(X)$$

is fully faithful. Its essential image consists exactly of those sheaves $\mathcal{G}$ such that $\mathcal{G}_x = 0$ for all $x \in X \setminus U$.

Proof. Omitted. □

Lemma 6.31.11. *Let X be a topological space. Let $j : U \to X$ be the inclusion of an open subset. Let $(\mathcal{C}, F)$ be a type of algebraic structure such that $\mathcal{C}$ has an initial object e. The functor*

$$j_! : Sh(U, \mathcal{C}) \longrightarrow Sh(X, \mathcal{C})$$

is fully faithful. Its essential image consists exactly of those sheaves $\mathcal{G}$ such that $\mathcal{G}_x = e$ for all $x \in X \setminus U$.

Proof. Omitted. □

Lemma 6.31.12. *Let $(X, \mathcal{O})$ be a ringed space. Let $j : (U, \mathcal{O}|_U) \to (X, \mathcal{O})$ be an open subspace. The functor*

$$j_! : Mod(\mathcal{O}|_U) \longrightarrow Mod(\mathcal{O})$$

is fully faithful. Its essential image consists exactly of those sheaves $\mathcal{G}$ such that $\mathcal{G}_x = 0$ for all $x \in X \setminus U$.

Proof. Omitted. □

Remark 6.31.13. Let $j : U \to X$ be an open immersion of topological spaces as above. Let $x \in X$, $x \notin U$. Let $\mathcal{F}$ be a sheaf of sets on U. Then $\mathcal{F}_x = \emptyset$ by Lemma 6.31.4. Hence $j_!$ does not transform a final object of $Sh(U)$ into a final object of $Sh(X)$ unless $U = X$. According to our conventions in Categories, Section 4.23 this means that the functor $j_!$ is not left exact as a functor between the categories of sheaves of sets. It will be shown later that $j_!$ on abelian sheaves is exact, see Modules, Lemma 17.3.4.

6.32. Closed immersions and (pre)sheaves

Let X be a topological space. Let $i : Z \to X$ be the inclusion of a closed subset Z into X. In Section 6.21 we have defined functors i_* and i^{-1} such that i_* is right adjoint to i^{-1}.

Lemma 6.32.1. *Let X be a topological space. Let $i : Z \to X$ be the inclusion of a closed subset Z into X. Let $\mathcal{F}$ be a sheaf of sets on Z. The stalks of $i_*\mathcal{F}$ are described as follows*

$$i_*\mathcal{F}_x = \begin{cases} \{*\} & \text{if} \quad x \notin Z \\ \mathcal{F}_x & \text{if} \quad x \in Z \end{cases}$$

where $\{*\}$ denotes a singleton set. Moreover, $i^{-1}i_* = id$ on the category of sheaves of sets on Z. Moreover, the same holds for abelian sheaves on Z, resp. sheaves of algebraic structures on Z where $\{*\}$ has to be replaced by 0, resp. a final object of the category of algebraic structures.

Proof. If $x \notin Z$, then there exist arbitrarily small open neighbourhoods U of x which do not meet Z. Because $\mathcal{F}$ is a sheaf we have $\mathcal{F}(i^{-1}(U)) = \{*\}$ for any such U, see Remark 6.7.2. This proves the first case. The second case comes from the fact that for $z \in Z$ any open neighbourhood of z is of the form $Z \cap U$ for some open U of X. For the statement that $i^{-1}i_* = \text{id}$ consider the canonical map $i^{-1}i_*\mathcal{F} \to \mathcal{F}$. This is an isomorphism on stalks (see above) and hence an isomorphism.

For sheaves of abelian groups, and sheaves of algebraic structures you argue in the same manner. □

Lemma 6.32.2. *Let X be a topological space. Let $i : Z \to X$ be the inclusion of a closed subset. The functor*

$$i_* : Sh(Z) \longrightarrow Sh(X)$$

is fully faithful. Its essential image consists exactly of those sheaves $\mathcal{G}$ such that $\mathcal{G}_x = \{*\}$ for all $x \in X \setminus Z$.

Proof. Fully faithfulness follows formally from $i^{-1}i_* = \text{id}$. We have seen that any sheaf in the image of the functor has the property on the stalks mentioned in the lemma. Conversely, suppose that $\mathcal{G}$ has the indicated property. Then it is easy to check that

$$\mathcal{G} \to i_*i^{-1}\mathcal{G}$$

is an isomorphism on all stalks and hence an isomorphism. □

Lemma 6.32.3. *Let X be a topological space. Let $i : Z \to X$ be the inclusion of a closed subset. The functor*

$$i_* : Ab(Z) \longrightarrow Ab(X)$$

is fully faithful. Its essential image consists exactly of those sheaves $\mathcal{G}$ such that $\mathcal{G}_x = 0$ for all $x \in X \setminus Z$.

Proof. Omitted. □

Lemma 6.32.4. *Let X be a topological space. Let $i : Z \to X$ be the inclusion of a closed subset. Let $(\mathcal{C}, F)$ be a type of algebraic structure with final object 0. The functor*

$$i_* : Sh(Z, \mathcal{C}) \longrightarrow Sh(X, \mathcal{C})$$

is fully faithful. Its essential image consists exactly of those sheaves $\mathcal{G}$ such that $\mathcal{G}_x = 0$ for all $x \in X \setminus Z$.

Proof. Omitted. □

Remark 6.32.5. Let $i : Z \to X$ be a closed immersion of topological spaces as above. Let $x \in X$, $x \notin Z$. Let $\mathcal{F}$ be a sheaf of sets on Z. Then $(i_*\mathcal{F})_x = \{*\}$ by Lemma 6.32.1. Hence if $\mathcal{F} = *\amalg *$, where $*$ is the singleton sheaf, then $i_*\mathcal{F}_x = \{*\} \neq i_*(*)_x \amalg i_*(*)_x$ because the latter is a two point set. According to our conventions in Categories, Section 4.23 this means that the functor i_* is not right exact as a functor between the categories of sheaves of sets. In particular, it cannot have a right adjoint, see Categories, Lemma 4.24.4.

On the other hand, we will see later (see Modules, Lemma 17.6.3) that i_* on abelian sheaves is exact, and does have a right adjoint, namely the functor that associates to an abelian sheaf on X the sheaf of sections supported in Z.

Remark 6.32.6. We have not discussed the relationship between closed immersions and ringed spaces. This is because the notion of a closed immersion of ringed spaces is best discussed in the setting of quasi-coherent sheaves, see Modules, Section 17.13.

6.33. Glueing sheaves

In this section we glue sheaves defined on the members of a covering of X. We first deal with maps.

Lemma 6.33.1. *Let X be a topological space. Let $X = \bigcup U_i$ be an open covering. Let $\mathcal{F}$, $\mathcal{G}$ be sheaves of sets on X. Given a collection*

$$\varphi_i : \mathcal{F}|_{U_i} \longrightarrow \mathcal{G}|_{U_i}$$

of maps of sheaves such that for all $i, j \in I$ the maps φ_i, φ_j restrict to the same map $\mathcal{F}|_{U_i \cap U_j} \to \mathcal{G}|_{U_i \cap U_j}$ then there exists a unique map of sheaves

$$\varphi : \mathcal{F} \longrightarrow \mathcal{G}$$

whose restriction to each U_i agrees with φ_i.

Proof. Omitted. □

The previous lemma implies that given two sheaves $\mathcal{F}$, $\mathcal{G}$ on the topological space X the rule

$$U \longmapsto \mathrm{Mor}_{Sh(U)}(\mathcal{F}|_U, \mathcal{G}|_U)$$

defines a sheaf. This is a kind of *internal hom sheaf*. It is seldom used in the setting of sheaves of sets, and more usually in the setting of sheaves of modules, see Modules, Section 17.19.

Let X be a topological space. Let $X = \bigcup_{i \in I} U_i$ be an open covering. For each $i \in I$ let $\mathcal{F}_i$ be a sheaf of sets on U_i. For each pair $i, j \in I$, let

$$\varphi_{ij} : \mathcal{F}_i|_{U_i \cap U_j} \longrightarrow \mathcal{F}_j|_{U_i \cap U_j}$$

be an isomorphism of sheaves of sets. Assume in addition that for every triple of indices $i, j, k \in I$ the following diagram is commutative

$$\begin{array}{ccc} \mathcal{F}_i|_{U_i \cap U_j \cap U_k} & \xrightarrow{\varphi_{ik}} & \mathcal{F}_k|_{U_i \cap U_j \cap U_k} \\ & {\scriptstyle \varphi_{ij}} \searrow \quad \nearrow {\scriptstyle \varphi_{jk}} & \\ & \mathcal{F}_j|_{U_i \cap U_j \cap U_k} & \end{array}$$

We will call such a collection of data $(\mathcal{F}_i, \varphi_{ij})$ a *glueing data for sheaves of sets with respect to the covering* $X = \bigcup U_i$.

Lemma 6.33.2. *Let X be a topological space. Let $X = \bigcup_{i \in I} U_i$ be an open covering. Given any glueing data $(\mathcal{F}_i, \varphi_{ij})$ for sheaves of sets with respect to the covering $X = \bigcup U_i$ there exists a sheaf of sets $\mathcal{F}$ on X together with isomorphisms*

$$\varphi_i : \mathcal{F}|_{U_i} \to \mathcal{F}_i$$

such that the diagrams

$$\begin{array}{ccc} \mathcal{F}|_{U_i \cap U_j} & \xrightarrow[\varphi_i]{} & \mathcal{F}_i|_{U_i \cap U_j} \\ {\scriptstyle id}\downarrow & & \downarrow{\scriptstyle \varphi_{ij}} \\ \mathcal{F}|_{U_i \cap U_j} & \xrightarrow{\varphi_j} & \mathcal{F}_j|_{U_i \cap U_j} \end{array}$$

are commutative.

Proof. Actually we can write a formula for the set of sections of $\mathcal{F}$ over an open $W \subset X$. Namely, we define

$$\mathcal{F}(W) = \{(s_i)_{i \in I} \mid s_i \in \mathcal{F}_i(W \cap U_i), \varphi_{ij}(s_i|_{W \cap U_i \cap U_j}) = s_j|_{W \cap U_i \cap U_j}\}.$$

Restriction mappings for $W' \subset W$ are defined by the restricting each of the s_i to $W' \cap U_i$. The sheaf condition for $\mathcal{F}$ follows immediately from the sheaf condition for each of the $\mathcal{F}_i$.

We still have to prove that $\mathcal{F}|_{U_i}$ maps isomorphically to $\mathcal{F}_i$. Let $W \subset U_i$. In this case the condition in the definition of $\mathcal{F}(W)$ implies that $s_j = \varphi_{ij}(s_i|_{W \cap U_j})$. And the commutativity of the diagrams in the definition of a glueing data assures that we may start with *any* section $s \in \mathcal{F}_i(W)$ and obtain a compatible collection by setting $s_i = s$ and $s_j = \varphi_{ij}(s_i|_{W \cap U_j})$. Thus the lemma follows. □

Lemma 6.33.3. *Let X be a topological space. Let $X = \bigcup U_i$ be an open covering. Let $(\mathcal{F}_i, \varphi_{ij})$ be a glueing data of sheaves of abelian groups, resp. sheaves of algebraic structures, resp. sheaves of $\mathcal{O}$-modules for some sheaf of rings $\mathcal{O}$ on X. Then the construction in the proof of Lemma 6.33.2 above leads to a sheaf of abelian groups, resp. sheaf of algebraic structures, resp. sheaf of $\mathcal{O}$-modules.*

Proof. This is true because in the construction the set of sections $\mathcal{F}(W)$ over an open W is given as the equalizer of the maps

$$\prod_{i \in I} \mathcal{F}_i(W \cap U_i) \rightrightarrows \prod_{i,j \in I} \mathcal{F}_i(W \cap U_i \cap U_j)$$

And in each of the cases envisioned this equalizer gives an object in the relevant category whose underlying set is the object considered in the cited lemma. □

Lemma 6.33.4. *Let X be a topological space. Let $X = \bigcup_{i \in I} U_i$ be an open covering. The functor which associates to a sheaf of sets $\mathcal{F}$ the following collection of glueing data*

$$(\mathcal{F}|_{U_i}, (\mathcal{F}|_{U_i})|_{U_i \cap U_j} \to (\mathcal{F}|_{U_j})|_{U_i \cap U_j})$$

with respect to the covering $X = \bigcup U_i$ defines an equivalence of categories between $Sh(X)$ and the category of glueing data. A similar statement holds for abelian sheaves, resp. sheaves of algebraic structures, resp. sheaves of $\mathcal{O}$-modules.

Proof. The functor is fully faithful by Lemma 6.33.1 and essentially surjective (via an explicitly given quasi-inverse functor) by Lemma 6.33.2. □

This lemma means that if the sheaf $\mathcal{F}$ was constructed from the glueing data $(\mathcal{F}_i, \varphi_{ij})$ and if $\mathcal{G}$ is a sheaf on X, then a morphism $f : \mathcal{F} \to \mathcal{G}$ is given by a collection of morphisms of sheaves

$$f_i : \mathcal{F}_i \longrightarrow \mathcal{G}|_{U_i}$$

compatible with the glueing maps φ_{ij}. Similarly, to give a morphism of sheaves $g : \mathcal{G} \to \mathcal{F}$ is the same as giving a collection of morphisms of sheaves

$$g_i : \mathcal{G}|_{U_i} \longrightarrow \mathcal{F}_i$$

compatible with the glueing maps φ_{ij}.

6.34. Other chapters

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CHAPTER 7

Sites and Sheaves

7.1. Introduction

The notion of a site was introduced by Grothendieck to be able to study sheaves in the étale topology of schemes. The basic reference for this notion is perhaps [**AGV71**]. Our notion of a site differs from that in [**AGV71**]; what we call a site is called a category endowed with a pretopology in [**AGV71**, Exposé II, Définition 1.3]. The reason we do this is that in algebraic geometry it is often convenient to work with a given class of coverings, for example when defining when a property of schemes is local in a given topology, see Descent, Section 34.11. Our exposition will closely follow [**Art62**]. We will not use universes.

7.2. Presheaves

Let $\mathcal{C}$ be a category. A *presheaf of sets* is a contravariant functor $\mathcal{F}$ from $\mathcal{C}$ to *Sets* (see Categories, Remark 4.2.11). So for every object U of $\mathcal{C}$ we have a set $\mathcal{F}(U)$. The elements of this set are called the *sections* of $\mathcal{F}$ over U. For every morphism $f : V \to U$ the map $\mathcal{F}(f) : \mathcal{F}(U) \to \mathcal{F}(V)$ is called the *restriction map* and is often denoted $f^* : \mathcal{F}(U) \to \mathcal{F}(V)$. Another way of expressing this is to say that $f^*(s)$ is the *pullback* of s via f. Functoriality means that $g^*f^*(s) = (f \circ g)^*(s)$. Sometimes we use the notation $s|_V := f^*(s)$. This notation is consistent with the notion of restriction of functions from topology because if $W \to V \to U$ are morphisms in $\mathcal{C}$ and s is a section of $\mathcal{F}$ over U then $s|_W = (s|_V)|_W$ by the functorial nature of $\mathcal{F}$. Of course we have to be careful since it may very well happen that there is more than one morphism $V \to U$ and it is certainly not going to be the case that the corresponding pullback maps are equal.

Definition 7.2.1. A *presheaf of sets* on $\mathcal{C}$ is a contravariant functor from $\mathcal{C}$ to *Sets*. *Morphisms of presheaves* are transformations of functors. The category of presheaves of sets is denoted $PSh(\mathcal{C})$.

Note that for any object U of $\mathcal{C}$ the functor of points h_U, see Categories, Example 4.3.4 is a presheaf. These are called the *representable presheaves*. These presheaves have the pleasing property that for any presheaf $\mathcal{F}$ we have

$$\text{Mor}_{PSh(\mathcal{C})}(h_U, \mathcal{F}) = \mathcal{F}(U). \tag{7.2.1.1}$$

This is the Yoneda lemma (Categories, Lemma 4.3.5).

Similarly, we can define the notion of a presheaf of abelian groups, rings, etc. More generally we may define a presheaf with values in a category.

Definition 7.2.2. Let $\mathcal{C}$, $\mathcal{A}$ be categories. A *presheaf* $\mathcal{F}$ on $\mathcal{C}$ with values in $\mathcal{A}$ is a contravariant functor from $\mathcal{C}$ to $\mathcal{A}$, i.e., $\mathcal{F} : \mathcal{C}^\circ \to \mathcal{A}$. A *morphism* of presheaves $\mathcal{F} \to \mathcal{G}$ on $\mathcal{C}$ with values in $\mathcal{A}$ is a transformation of functors from $\mathcal{F}$ to $\mathcal{G}$.

These form the objects and morphisms of the category of presheaves on $\mathcal{C}$ with values in $\mathcal{A}$.

Remark 7.2.3. As already pointed out we may consider the category presheaves with values in any of the "big" categories listed in Categories, Remark 4.2.2. These will be "big" categories as well and they will be listed in the above mentioned remark as we go along.

7.3. Injective and surjective maps of presheaves

Definition 7.3.1. Let $\mathcal{C}$ be a category, and let $\varphi : \mathcal{F} \to \mathcal{G}$ be a map of presheaves of sets.

(1) We say that φ is *injective* if for every object U of $\mathcal{C}$ we have $\alpha : \mathcal{F}(U) \to \mathcal{G}(U)$ is injective.
(2) We say that φ is *surjective* if for every object U of $\mathcal{C}$ we have $\alpha : \mathcal{F}(U) \to \mathcal{G}(U)$ is surjective.

Lemma 7.3.2. *The injective (resp. surjective) maps defined above are exactly the monomorphisms (resp. epimorphisms) of $PSh(\mathcal{C})$. A map is an isomorphism if and only if it is both injective and surjective.*

Proof. Omitted. □

Definition 7.3.3. We say $\mathcal{F}$ is a *subpresheaf* of $\mathcal{G}$ if for every object $U \in \mathrm{Ob}(\mathcal{C})$ the set $\mathcal{F}(U)$ is a subset of $\mathcal{G}(U)$, compatibly with the restriction mappings.

In other words, the inclusion maps $\mathcal{F}(U) \to \mathcal{G}(U)$ glue together to give an (injective) morphism of presheaves $\mathcal{F} \to \mathcal{G}$.

Lemma 7.3.4. *Let $\mathcal{C}$ be a category. Suppose that $\varphi : \mathcal{F} \to \mathcal{G}$ is a morphism of presheaves of sets on $\mathcal{C}$. There exists a unique subpresheaf $\mathcal{G}' \subset \mathcal{G}$ such that φ factors as $\mathcal{F} \to \mathcal{G}' \to \mathcal{G}$ and such that the first map is surjective.*

Proof. Omitted. □

Definition 7.3.5. Notation as in Lemma 7.3.4. We say that $\mathcal{G}'$ is the *image of* φ.

7.4. Limits and colimits of presheaves

Let $\mathcal{C}$ be a category. Limits and colimits exist in the category $PSh(\mathcal{C})$. In addition, for any $U \in \mathrm{ob}(\mathcal{C})$ the functor

$$PSh(\mathcal{C}) \longrightarrow Sets, \quad \mathcal{F} \longmapsto \mathcal{F}(U)$$

commutes with limits and colimits. Perhaps the easiest way to prove these statement is the following. Given a diagram $\mathcal{F} : \mathcal{I} \to PSh(\mathcal{C})$ define presheaves

$$\mathcal{F}_{\lim} : U \longmapsto \lim_{i \in \mathcal{I}} \mathcal{F}_i(U) \text{ and } \mathcal{F}_{\mathrm{colim}} : U \longmapsto \mathrm{colim}_{i \in \mathcal{I}} \mathcal{F}_i(U)$$

There are clearly projection maps $\mathcal{F}_{\lim} \to \mathcal{F}_i$ and canonical maps $\mathcal{F}_i \to \mathcal{F}_{\mathrm{colim}}$. These maps satisfy the requirements of the maps of a limit (reps. colimit) of Categories, Definition 4.14.1 (resp. Categories, Definition 4.14.2). Finally, if $(\mathcal{G}, q_i : \mathcal{G} \to \mathcal{F}_i)$ is another system (as in the definition of a limit), then we get for every U a system of maps $\mathcal{G}(U) \to \mathcal{F}_i(U)$ with suitable functoriality requirements. And thus a unique map $\mathcal{G}(U) \to \mathcal{F}_{\lim}(U)$. It is easy to verify these are compatible as we vary U and arise from the desired map $\mathcal{G} \to \mathcal{F}_{\lim}$. A similar argument works in the case of the colimit.

7.5. Functoriality of categories of presheaves

Let $u : \mathcal{C} \to \mathcal{D}$ be a functor between categories. In this case we denote

$$u^p : PSh(\mathcal{D}) \longrightarrow PSh(\mathcal{C})$$

the functor that associates to $\mathcal{G}$ on $\mathcal{D}$ the presheaf $u^p\mathcal{G} = \mathcal{G} \circ u$. Note that by the previous section this functor commutes with all limits.

For $V \in \mathrm{ob}(\mathcal{D})$ let $\mathcal{I}_V^u$ denote the category with

$$\begin{aligned} \mathrm{Ob}(\mathcal{I}_V^u) &= \{(U, \phi) \mid U \in \mathrm{Ob}(\mathcal{C}), \phi : V \to u(U)\} \\ \mathrm{Mor}_{\mathcal{I}_V^u}((U, \phi), (U', \phi')) &= \{f : U \to U' \text{ in } \mathcal{C} \mid u(f) \circ \phi = \phi'\} \end{aligned} \tag{7.5.0.1}$$

We sometimes drop the subscript ^{u} from the notation and we simply write $\mathcal{I}_V$. We will use these categories to define a left adjoint to the functor u^p. Before we do so we prove a few technical lemmas.

Lemma 7.5.1. *Let $u : \mathcal{C} \to \mathcal{D}$ be a functor between categories. Suppose that $\mathcal{C}$ has fibre products and equalizers, and that u commutes with them. Then the categories $(\mathcal{I}_V)^{opp}$ satisfy the hypotheses of Categories, Lemma 4.19.7.*

Proof. There are two conditions to check.

First, suppose we are given three objects $\phi : V \to u(U)$, $\phi' : V \to u(U')$, and $\phi'' : V \to u(U'')$ and morphisms $a : U' \to U$, $b : U'' \to U$ such that $u(a) \circ \phi = \phi'$ and $u(b) \circ \phi = \phi''$. We have to show there exists another object $\phi''' : V \to u(U''')$ and morphisms $c : U''' \to U'$ and $d : U''' \to U''$ such that $u(c) \circ \phi = \phi'''$, $u(d) \circ \phi = \phi'''$ and $a \circ c = b \circ d$. We take $U''' = U' \times_U U''$ with c and d the projection morphisms. This works as u commutes with fibre products; we omit the verification.

Second, suppose we are given two objects $\phi : V \to u(U)$ and $\phi' : V \to u(U')$ and morphisms $a, b : (U, \phi) \to (U', \phi')$. We have to find a morphism $c : (U'', \phi'') \to (U, \phi)$ which equalizes a and b. Let $c : U'' \to U$ be the equalizer of a and b in the category $\mathcal{C}$. As u commutes with equalizers and since $u(a) \circ \phi = u(b) \circ \phi = \phi'$ we obtain a morphism $\phi'' : V \to u(U'')$. □

Lemma 7.5.2. *Let $u : \mathcal{C} \to \mathcal{D}$ be a functor between categories. Assume*

1. *the category $\mathcal{C}$ has a final object X and $u(X)$ is a final object of $\mathcal{D}$, and*
2. *the category $\mathcal{C}$ has fibre products and u commutes with them.*

Then the index categories $(\mathcal{I}_V^u)^{opp}$ are filtered (see Categories, Definition 4.19.1).

Proof. The assumptions imply that the assumptions of Lemma 7.5.1 are satisfied (see the discussion in Categories, Section 4.18). By Categories, Lemma 4.19.7 we see that $\mathcal{I}_V$ is a (possibly empty) disjoint union of directed categories. Hence it suffices to show that $\mathcal{I}_V$ is connected.

First, we show that $\mathcal{I}_V$ is nonempty. Namely, let X be the final object of $\mathcal{C}$, which exists by assumption. Let $V \to u(X)$ be the morphism coming from the fact that $u(X)$ is final in $\mathcal{D}$ by assumption. This gives an object of $\mathcal{I}_V$.

Second, we show that $\mathcal{I}_V$ is connected. Let $\phi_1 : V \to u(U_1)$ and $\phi_2 : V \to u(U_2)$ be in $\mathrm{Ob}(\mathcal{I}_V)$. By assumption $U_1 \times U_2$ exists and $u(U_1 \times U_2) = u(U_1) \times u(U_2)$. Consider the morphism $\phi : V \to u(U_1 \times U_2)$ corresponding to (ϕ_1, ϕ_2) by the universal property of products. Clearly the object $\phi : V \to u(U_1 \times U_2)$ maps to both $\phi_1 : V \to u(U_1)$ and $\phi_2 : V \to u(U_2)$. □

Given $g : V' \to V$ in $\mathcal{D}$ we get a functor $\overline{g} : \mathcal{I}_V \to \mathcal{I}_{V'}$ by setting $\overline{g}(U, \phi) = (U, \phi \circ g)$ on objects. Given a presheaf $\mathcal{F}$ on $\mathcal{C}$ we obtain a functor

$$\mathcal{F}_V : \mathcal{I}_V^{opp} \longrightarrow Sets, \quad (U, \phi) \longmapsto \mathcal{F}(U).$$

In other words, $\mathcal{F}_V$ is a presheaf of sets on $\mathcal{I}_V$. Note that we have $\mathcal{F}_{V'} \circ \overline{g} = \mathcal{F}_V$. We define

$$u_p\mathcal{F}(V) := \operatorname{colim}_{\mathcal{I}_V^{opp}} \mathcal{F}_V$$

As a colimit we obtain for each $(U, \phi) \in \mathrm{Ob}(\mathcal{I}_V)$ a canonical map $\mathcal{F}(U) \xrightarrow{c(\phi)} u_p\mathcal{F}(V)$. For $g : V' \to V$ as above there is a canonical restriction map $g^* : u_p\mathcal{F}(V) \to u_p\mathcal{F}(V')$ compatible with $\mathcal{F}_{V'} \circ \overline{g} = \mathcal{F}_V$ by Categories, Lemma 4.14.7. It is the unique map so that for all $(U, \phi) \in \mathrm{Ob}(\mathcal{I}_V)$ the diagram

$$\begin{array}{ccc} \mathcal{F}(U) & \xrightarrow{c(\phi)} & u_p\mathcal{F}(V) \\ {\scriptstyle \mathrm{id}}\downarrow & & \downarrow{\scriptstyle g^*} \\ \mathcal{F}(U) & \xrightarrow{c(\phi \circ g)} & u_p\mathcal{F}(V') \end{array}$$

commutes. The uniqueness of these maps implies that we obtain a presheaf. This presheaf will be denoted $u_p\mathcal{F}$.

Lemma 7.5.3. *There is a canonical map $\mathcal{F}(U) \to u_p\mathcal{F}(u(U))$, which is compatible with restriction maps (on $\mathcal{F}$ and on $u_p\mathcal{F}$).*

Proof. This is just the map $c(\mathrm{id}_{u(U)})$ introduced above. □

Note that any map of presheaves $\mathcal{F} \to \mathcal{F}'$ gives rise to compatible systems of maps between functors $\mathcal{F}_Y \to \mathcal{F}'_Y$, and hence to a map of presheaves $u_p\mathcal{F} \to u_p\mathcal{F}'$. In other words, we have defined a functor

$$u_p : PSh(\mathcal{C}) \longrightarrow PSh(\mathcal{D})$$

Lemma 7.5.4. *The functor u_p is a left adjoint to the functor u^p. In other words the formula*

$$\mathrm{Mor}_{PSh(\mathcal{C})}(\mathcal{F}, u^p\mathcal{G}) = \mathrm{Mor}_{PSh(\mathcal{D})}(u_p\mathcal{F}, \mathcal{G})$$

holds bifunctorially in $\mathcal{F}$ and $\mathcal{G}$.

Proof. Let $\mathcal{G}$ be a presheaf on $\mathcal{D}$ and let $\mathcal{F}$ be a presheaf on $\mathcal{C}$. We will show that the displayed formula holds by constructing maps either way. We will leave it to the reader to verify they are each others inverse.

Given a map $\alpha : u_p\mathcal{F} \to \mathcal{G}$ we get $u^p\alpha : u^pu_p\mathcal{F} \to u^p\mathcal{G}$. Lemma 7.5.3 says that there is a map $\mathcal{F} \to u^pu_p\mathcal{F}$. The composition of the two gives the desired map. (The good thing about this construction is that it is clearly functorial in everything in sight.)

Conversely, given a map $\beta : \mathcal{F} \to u^p\mathcal{G}$ we get a map $u_p\beta : u_p\mathcal{F} \to u_pu^p\mathcal{G}$. We claim that the functor $u^p\mathcal{G}_Y$ on $\mathcal{I}_Y$ has a canonical map to the constant functor with value $\mathcal{G}(Y)$. Namely, for every object (X, ϕ) of $\mathcal{I}_Y$, the value of $u^p\mathcal{G}_Y$ on this object is $\mathcal{G}(u(X))$ which maps to $\mathcal{G}(Y)$ by $\mathcal{G}(\phi) = \phi^*$. This is a transformation of functors because $\mathcal{G}$ is a functor itself. This leads to a map $u_pu^p\mathcal{G}(Y) \to \mathcal{G}(Y)$. Another trivial verification shows that this is functorial in Y leading to a map of presheaves $u_pu^p\mathcal{G} \to \mathcal{G}$. The composition $u_p\mathcal{F} \to u_pu^p\mathcal{G} \to \mathcal{G}$ is the desired map. □

Remark 7.5.5. Suppose that $\mathcal{A}$ is a category such that any diagram $\mathcal{I}_Y \to \mathcal{A}$ has a colimit in $\mathcal{A}$. In this case it is clear that there are functors u^p and u_p, defined in exactly the same way as above, on the categories of presheaves with values in $\mathcal{A}$. Moreover, the adjointness of the pair u^p and u_p continues to hold in this setting.

Lemma 7.5.6. *Let $u : \mathcal{C} \to \mathcal{D}$ be a functor between categories. For any object U of $\mathcal{C}$ we have $u_p h_U = h_{u(U)}$.*

Proof. By adjointness of u_p and u^p we have

$$\mathrm{Mor}_{PSh(\mathcal{D})}(u_p h_U, \mathcal{G}) = \mathrm{Mor}_{PSh(\mathcal{C})}(h_U, u^p\mathcal{G}) = u^p\mathcal{G}(U) = \mathcal{G}(u(U))$$

and hence by Yoneda's lemma we see that $u_p h_U = h_{u(U)}$ as presheaves. □

7.6. Sites

Our notion of a site uses the following type of structures.

Definition 7.6.1. Let $\mathcal{C}$ be a category, see Conventions, Section 2.3. A *family of morphisms with fixed target* in $\mathcal{C}$ is given by an object $U \in \mathrm{Ob}(\mathcal{C})$, a set I and for each $i \in I$ a morphism $U_i \to U$ of $\mathcal{C}$ with target U. We use the notation $\{U_i \to U\}_{i\in I}$ to indicate this.

It can happen that the set I is empty! This notation is meant to suggest an open covering as in topology.

Definition 7.6.2. A *site*[1] is given by a category $\mathcal{C}$ and a set $\mathrm{Cov}(\mathcal{C})$ of families of morphisms with fixed target $\{U_i \to U\}_{i\in I}$, called *coverings of* $\mathcal{C}$, satisfying the following axioms

(1) If $V \to U$ is an isomorphism then $\{V \to U\} \in \mathrm{Cov}(\mathcal{C})$.
(2) If $\{U_i \to U\}_{i\in I} \in \mathrm{Cov}(\mathcal{C})$ and for each i we have $\{V_{ij} \to U_i\}_{j\in J_i} \in \mathrm{Cov}(\mathcal{C})$, then $\{V_{ij} \to U\}_{i\in I, j\in J_i} \in \mathrm{Cov}(\mathcal{C})$.
(3) If $\{U_i \to U\}_{i\in I} \in \mathrm{Cov}(\mathcal{C})$ and $V \to U$ is a morphism of $\mathcal{C}$ then $U_i \times_U V$ exists for all i and $\{U_i \times_U V \to V\}_{i\in I} \in \mathrm{Cov}(\mathcal{C})$.

Remark 7.6.3. (On set theoretic issues – skip on a first reading.) The main reason for introducing sites is to study the category of sheaves on a site, because it is the generalization of the category of sheaves on a topological space that has been so important in algebraic geometry. In order to avoid thinking about things like "classes of classes" and so on, we will not allow sites to be "big" categories, in contrast to what we do for categories and 2-categories.

Suppose that $\mathcal{C}$ is a category and that $\mathrm{Cov}(\mathcal{C})$ is a proper class of coverings satisfying (1), (2) and (3) above. We will not allow this as a site either, mainly because we are going to take limits over coverings. However, there are several natural ways to replace $\mathrm{Cov}(\mathcal{C})$ by a set of coverings or a slightly different structure that give rise to the same category of sheaves. For example:

(1) In Sets, Section 3.11 we show how to pick a suitable set of coverings that gives the same category of sheaves.
(2) Another thing we can do is to take the associated topology (see Definition 7.46.2). The resulting topology on $\mathcal{C}$ has the same category of sheaves. Two topologies have the same categories of sheaves if and only if they

[1] This notation differs from that of [**AGV71**], as explained in the introduction.

are equal, see Theorem 7.48.2. A topology on a category is given by a choice of sieves on objects. The collection of all possible sieves and even all possible topologies on $\mathcal{C}$ is a set.

(3) We could also slightly modify the notion of a site, see Remark 7.46.4 below, and end up with a canonical set of coverings which is contained in the powerset of the set of arrows of $\mathcal{C}$.

Each of these solutions has some minor drawback. For the first, one has to check that constructions later on do not depend on the choice of the set of coverings. For the second, one has to learn about topologies and redo many of the arguments for sites. For the third, see the last sentence of Remark 7.46.4.

Our approach will be to work with sites as in Definition 7.6.2 above. Given a category $\mathcal{C}$ with a proper class of coverings as above, we will replace this by a set of coverings producing a site using Sets, Lemma 3.11.1. It is shown in Lemma 7.8.6 below that the resulting category of sheaves (the topos) is independent of this choice. We leave it to the reader to use one of the other two strategies to deal with these issues if he/she so desires.

Example 7.6.4. Let X be a topological space. Let X_{Zar} be the category whose objects consist of all the open sets U in X and whose morphisms are just the inclusion maps. That is, there is at most one morphism between any two objects in X_{Zar}. Now define $\{U_i \to U\}_{i\in I} \in \mathrm{Cov}(X_{Zar})$ if and only if $\bigcup U_i = U$. Conditions (1) and (2) above are clear, and (3) is also clear once we realize that in X_{Zar} we have $U \times V = U \cap V$. Note that in particular the empty set has to be an element of X_{Zar} since otherwise this would not work in general. Furthermore, it is equally important, as we will see later, to allow the *empty covering of the empty set as a covering*! We turn X_{Zar} into a site by choosing a suitable set of coverings $\mathrm{Cov}(X_{Zar})_{\kappa,\alpha}$ as in Sets, Lemma 3.11.1. Presheaves and sheaves (as defined below) on the site X_{Zar} agree exactly with the usual notion of a presheaves and sheaves on a topological space, as defined in Sheaves, Section 6.1.

Example 7.6.5. Let G be a group. Consider the category G-*Sets* whose objects are sets X with a left G-action, with G-equivariant maps as the morphisms. An important example is ${}_GG$ which is the G-set whose underlying set is G and action given by left multiplication. This category has fiber products, see Categories, Section 4.7. We declare $\{\varphi_i : U_i \to U\}_{i\in I}$ to be a covering if $\bigcup_{i\in I} \varphi_i(U_i) = U$. This gives a class of coverings on G-*Sets* which is easily see to satisfy conditions (1), (2), and (3) of Definition 7.6.2. The result is not a site since both the collection of objects of the underlying category and the collection of coverings form a proper class. We first replace by G-*Sets* by a full subcategory G-$Sets_\alpha$ as in Sets, Lemma 3.10.1. After this the site $(G\text{-}Sets_\alpha, \mathrm{Cov}_{\kappa,\alpha'}(G\text{-}Sets_\alpha))$ gotten by suitably restricting the collection of coverings as in Sets, Lemma 3.11.1 will be denoted $\mathcal{T}_G$.

As a special case, if the group G is countable, then we can let $\mathcal{T}_G$ be the category of countable G-sets and coverings those jointly surjective families of morphisms $\{\varphi_i : U_i \to U\}_{i\in I}$ such that I is countable.

Example 7.6.6. Let $\mathcal{C}$ be a category. There is a canonical way to turn this into a site where $\{\mathrm{id}_U : U \to U\}$ are the coverings. Sheaves on this site are the presheaves on $\mathcal{C}$. This corresponding topology is called the *chaotic* or *indiscrete topology*.

7.7. Sheaves

Let $\mathcal{C}$ be a site. Before we introduce the notion of a sheaf with values in a category we explain what it means for a presheaf of sets to be a sheaf. Let $\mathcal{F}$ be a presheaf of sets on $\mathcal{C}$ and let $\{U_i \to U\}_{i\in I}$ be an element of $\mathrm{Cov}(\mathcal{C})$. By assumption all the fibre products $U_i \times_U U_j$ exist in $\mathcal{C}$. There are two natural maps

$$\prod_{i\in I} \mathcal{F}(U_i) \xrightarrow[\mathrm{pr}_1^*]{\mathrm{pr}_0^*} \prod_{(i_0,i_1)\in I\times I} \mathcal{F}(U_{i_0} \times_U U_{i_1})$$

which we will denote pr_i^*, $i = 0, 1$ as indicated in the displayed equation. Namely, an element of the left hand side corresponds to a family $(s_i)_{i\in I}$, where each s_i is a section of $\mathcal{F}$ over U_i. For each pair $(i_0, i_1) \in I \times I$ we have the projection morphisms

$$\mathrm{pr}_{i_0}^{(i_0,i_1)} : U_{i_0} \times_U U_{i_1} \longrightarrow U_{i_0} \text{ and } \mathrm{pr}_{i_1}^{(i_0,i_1)} : U_{i_0} \times_U U_{i_1} \longrightarrow U_{i_1}.$$

Thus we may pull back either the section s_{i_0} via the first of these maps or the section s_{i_1} via the second. Explicitly the maps we referred to above are

$$\mathrm{pr}_0^* : (s_i)_{i\in I} \longmapsto \left(\mathrm{pr}_{i_0}^{(i_0,i_1),*}(s_{i_0})\right)_{(i_0,i_1)\in I\times I}$$

and

$$\mathrm{pr}_1^* : (s_i)_{i\in I} \longmapsto \left(\mathrm{pr}_{i_1}^{(i_0,i_1),*}(s_{i_1})\right)_{(i_0,i_1)\in I\times I}.$$

Finally consider the natural map

$$\mathcal{F}(U) \longrightarrow \prod\nolimits_{i\in I} \mathcal{F}(U_i), \quad s \longmapsto (s|_{U_i})_{i\in I}$$

where we have used the notation $s|_{U_i}$ to indicate the pullback of s via the map $U_i \to U$. It is clear from the functorial natural of $\mathcal{F}$ and the commutativity of the fibre product diagrams that $\mathrm{pr}_0^*((s|_{U_i})_{i\in I}) = \mathrm{pr}_1^*((s|_{U_i})_{i\in I})$.

Definition 7.7.1. Let $\mathcal{C}$ be a site, and let $\mathcal{F}$ be a presheaf of sets on $\mathcal{C}$. We say $\mathcal{F}$ is a *sheaf* if for every covering $\{U_i \to U\}_{i\in I} \in \mathrm{Cov}(\mathcal{C})$ the diagram

$$\mathcal{F}(U) \longrightarrow \prod_{i\in I} \mathcal{F}(U_i) \xrightarrow[\mathrm{pr}_1^*]{\mathrm{pr}_0^*} \prod_{(i_0,i_1)\in I\times I} \mathcal{F}(U_{i_0} \times_U U_{i_1}) \tag{7.7.1.1}$$

represents the first arrow as the equalizer of pr_0^* and pr_1^*.

Loosely speaking this means that given sections $s_i \in \mathcal{F}(U_i)$ such that

$$s_i|_{U_i\times_U U_j} = s_j|_{U_i\times_U U_j}$$

in $\mathcal{F}(U_i \times_U U_j)$ for all pairs $(i, j) \in I \times I$ then there exists a unique $s \in \mathcal{F}(U)$ such that $s_i = s|_{U_i}$.

Remark 7.7.2. If the covering $\{U_i \to U\}_{i\in I}$ is the empty family (this means that $I = \emptyset$), then the sheaf condition signifies that $\mathcal{F}(U) = \{*\}$ is a singleton set. This is true because in (7.7.1.1) the second and third sets are empty products in the category of sets, which are final objects in the category of sets, hence singletons.

Example 7.7.3. Let X be a topological space. Let X_{Zar} be the site constructed in Example 7.6.4. The notion of a sheaf on X_{Zar} coincides with the notion of a sheaf on X introduced in Sheaves, Definition 6.7.1.

Example 7.7.4. Let X be a topological space. Let us consider the site X'_{Zar} which is the same as the site X_{Zar} of Example 7.6.4 except that we disallow the empty covering of the empty set. In other words, we do allow the covering $\{\emptyset \to \emptyset\}$ but we do not allow the covering whose index set is empty. It is easy to show that this still defines a site. However, we claim that the sheaves on X'_{Zar} are different from the sheaves on X_{Zar}. For example, as an extreme case consider the situation where $X = \{p\}$ is a singleton. Then the objects of X'_{Zar} are $\emptyset, X$ and the coverings are $\{\{\emptyset \to \emptyset\}, \{X \to X\}\}$. Clearly, a sheaf on this is given by any choice of a set $\mathcal{F}(\emptyset)$ and any choice of a set $\mathcal{F}(X)$, together with any restriction map $\mathcal{F}(X) \to \mathcal{F}(\emptyset)$. Thus sheaves on X'_{Zar} are the same as usual sheaves on the two point space $\{\eta, p\}$ with open sets $\{\emptyset, \{\eta\}, \{p, \eta\}\}$. In general sheaves on X'_{Zar} are the same as sheaves on the space $X \amalg \{\eta\}$, with opens given by the empty set and any set of the form $U \cup \{\eta\}$ for $U \subset X$ open.

Definition 7.7.5. The category $Sh(\mathcal{C})$ of sheaves of sets is the full subcategory of the category $PSh(\mathcal{C})$ whose objects are the sheaves of sets.

Let $\mathcal{A}$ be a category. If products indexed by I, and $I \times I$ exist in $\mathcal{A}$ for any I that occurs as an index set for covering families then Definition 7.7.1 above makes sense, and defines a notion of a sheaf on $\mathcal{C}$ with values in $\mathcal{A}$. Note that the diagram in $\mathcal{A}$

$$\mathcal{F}(U) \longrightarrow \prod_{i \in I} \mathcal{F}(U_i) \underset{\mathrm{pr}_1^*}{\overset{\mathrm{pr}_0^*}{\rightrightarrows}} \prod_{(i_0, i_1) \in I \times I} \mathcal{F}(U_{i_0} \times_U U_{i_1})$$

is an equalizer diagram if and only if for every object X of $\mathcal{A}$ the diagram of sets

$$\mathrm{Mor}_{\mathcal{A}}(X, \mathcal{F}(U)) \longrightarrow \prod \mathrm{Mor}_{\mathcal{A}}(X, \mathcal{F}(U_i)) \underset{\mathrm{pr}_1^*}{\overset{\mathrm{pr}_0^*}{\rightrightarrows}} \prod \mathrm{Mor}_{\mathcal{A}}(X, \mathcal{F}(U_{i_0} \times_U U_{i_1}))$$

is an equalizer diagram.

Suppose $\mathcal{A}$ is arbitrary. Let $\mathcal{F}$ be a presheaf with values in $\mathcal{A}$. Choose any object $X \in \mathrm{Ob}(\mathcal{A})$. Then we get a presheaf of sets $\mathcal{F}_X$ defined by the rule

$$\mathcal{F}_X(U) = \mathrm{Mor}_{\mathcal{A}}(X, \mathcal{F}(U)).$$

From the above it follows that a good definition is obtained by requiring all the presheaves $\mathcal{F}_X$ to be sheaves of sets.

Definition 7.7.6. Let $\mathcal{C}$ be a site, let $\mathcal{A}$ be a category and let $\mathcal{F}$ be a presheaf on $\mathcal{C}$ with values in $\mathcal{A}$. We say that $\mathcal{F}$ is a *sheaf* if for all objects X of $\mathcal{A}$ the presheaf of sets $\mathcal{F}_X$ (defined above) is a sheaf.

7.8. Families of morphisms with fixed target

This section is meant to introduce some notions regarding families of morphisms with the same target.

Definition 7.8.1. Let $\mathcal{C}$ be a category. Let $\mathcal{U} = \{U_i \to U\}_{i \in I}$ be a family of morphisms of $\mathcal{C}$ with fixed target. Let $\mathcal{V} = \{V_j \to V\}_{j \in J}$ be another.

(1) A *morphism of families of maps with fixed target of $\mathcal{C}$ from $\mathcal{U}$ to $\mathcal{V}$*, or simply a *morphism from $\mathcal{U}$ to $\mathcal{V}$* is given by a morphism $U \to V$, a map

of sets $\alpha : I \to J$ and for each $i \in I$ a morphism $U_i \to V_{\alpha(i)}$ such that the diagram

$$\begin{array}{ccc} U_i & \longrightarrow & V_{\alpha(i)} \\ \downarrow & & \downarrow \\ U & \longrightarrow & V \end{array}$$

is commutative.

(2) In the special case that $U = V$ and $U \to V$ is the identity we call $\mathcal{U}$ a *refinement* of the family $\mathcal{V}$.

A trivial but important remark is that if $\mathcal{V} = \{V_j \to V\}_{j \in J}$ is the *empty family of maps*, i.e., if $J = \emptyset$, then no family $\mathcal{U} = \{U_i \to V\}_{j \in I}$ with $I \neq \emptyset$ can refine $\mathcal{V}$!

Definition 7.8.2. Let $\mathcal{C}$ be a category. Let $\mathcal{U} = \{\varphi_i : U_i \to U\}_{i \in I}$, and $\mathcal{V} = \{\psi_j : V_j \to U\}_{j \in J}$ be two families of morphisms with fixed target.

(1) We say $\mathcal{U}$ and $\mathcal{V}$ are *combinatorially equivalent* if there exist maps $\alpha : I \to J$ and $\beta : J \to I$ such that $\varphi_i = \psi_{\alpha(i)}$ and $\psi_j = \varphi_{\beta(j)}$.

(2) We say $\mathcal{U}$ and $\mathcal{V}$ are *tautologically equivalent* if there exist maps $\alpha : I \to J$ and $\beta : J \to I$ and for all $i \in I$ and $j \in J$ commutative diagrams

$$\begin{array}{ccc} U_i & \longrightarrow & V_{\alpha(i)} \\ & \searrow \quad \swarrow & \\ & U & \end{array} \qquad \begin{array}{ccc} V_j & \longrightarrow & U_{\beta(j)} \\ & \searrow \quad \swarrow & \\ & U & \end{array}$$

with isomorphisms as horizontal arrows.

Lemma 7.8.3. *Let $\mathcal{C}$ be a category. Let $\mathcal{U} = \{\varphi_i : U_i \to U\}_{i \in I}$, and $\mathcal{V} = \{\psi_j : V_j \to U\}_{j \in J}$ be two families of morphisms with the same fixed target.*

(1) *If $\mathcal{U}$ and $\mathcal{V}$ are combinatorially equivalent then they are tautologically equivalent.*

(2) *If $\mathcal{U}$ and $\mathcal{V}$ are tautologically equivalent then $\mathcal{U}$ is a refinement of $\mathcal{V}$ and $\mathcal{V}$ is a refinement of $\mathcal{U}$.*

(3) *The relation "being combinatorially equivalent" is an equivalence relation on all families of morphisms with fixed target.*

(4) *The relation "being tautologically equivalent" is an equivalence relation on all families of morphisms with fixed target.*

(5) *The relation "$\mathcal{U}$ refines $\mathcal{V}$ and $\mathcal{V}$ refines $\mathcal{U}$" is an equivalence relation on all families of morphisms with fixed target.*

Proof. Omitted. □

In the following lemma, given a category $\mathcal{C}$, a presheaf $\mathcal{F}$ on $\mathcal{C}$, a family $\mathcal{U} = \{U_i \to U\}_{i \in I}$ such that all fibre products $U_i \times_U U_{i'}$ exist, we say that *the sheaf condition for $\mathcal{F}$ with respect to $\mathcal{U}$* holds if the diagram (7.7.1.1) is an equalizer diagram.

Lemma 7.8.4. *Let $\mathcal{C}$ be a category. Let $\mathcal{U} = \{\varphi_i : U_i \to U\}_{i \in I}$, and $\mathcal{V} = \{\psi_j : V_j \to U\}_{j \in J}$ be two families of morphisms with the same fixed target. Assume that the fibre products $U_i \times_U U_{i'}$ and $V_j \times_U V_{j'}$ exist. If $\mathcal{U}$ and $\mathcal{V}$ are tautologically equivalent, then for any presheaf $\mathcal{F}$ on $\mathcal{C}$ the sheaf condition for $\mathcal{F}$ with respect to $\mathcal{U}$ is equivalent to the sheaf condition for $\mathcal{F}$ with respect to $\mathcal{V}$.*

Proof. First, note that if $\varphi : A \to B$ is an isomorphism in the category $\mathcal{C}$, then $\varphi^* : \mathcal{F}(B) \to \mathcal{F}(A)$ is an isomorphism. Let $\beta : J \to I$ be a map and let $\psi_j : V_j \to U_{\beta(j)}$ be isomorphisms over U which are assumed to exist by hypothesis. Let us show that the sheaf condition for $\mathcal{V}$ implies the sheaf condition for $\mathcal{U}$. Suppose given sections $s_i \in \mathcal{F}(U_i)$ such that

$$s_i|_{U_i \times_U U_{i'}} = s_{i'}|_{U_i \times_U U_{i'}}$$

in $\mathcal{F}(U_i \times_U U_{i'})$ for all pairs $(i, i') \in I \times I$. Then we can define $s_j = \psi_j^* s_{\beta(j)}$. For any pair $(j, j') \in J \times J'$ the morphism $\psi_j \times_{\mathrm{id}_U} \psi_{j'} : V_j \times_U V_{j'} \to U_{\beta(j)} \times_U U_{\beta(j')}$ is an isomorphism as well. Hence by transport of structure we see that

$$s_j|_{V_j \times_U V_{j'}} = s_{j'}|_{V_j \times_U V_{j'}}$$

as well. The sheaf condition w.r.t. $\mathcal{V}$ implies there exists a unique s such that $s|_{V_j} = s_j$ for all $j \in J$. By the first remark of the proof this implies that $s|_{U_i} = s_i$ for all $i \in \mathrm{Im}(\beta)$ as well. Suppose that $i \in I$, $i \notin \mathrm{Im}(\beta)$. For such an i we have isomorphisms $U_i \to V_{\alpha(i)} \to U_{\beta(\alpha(i))}$ over U. This gives a morphism $U_i \to U_i \times_U U_{\beta(\alpha(i))}$ which is a section of the projection. Because s_i and $s_{\beta(\alpha(i))}$ restrict to the same element on the fibre product we conclude that $s_{\beta(\alpha(i))}$ pulls back to s_i via $U_i \to U_{\beta(\alpha(i))}$. Thus we see that also $s_i = s|_{U_i}$ as desired. □

Lemma 7.8.5. *Let $\mathcal{C}$ be a category. Let Cov_i, $i = 1, 2$ be two sets of families of morphisms with fixed target which each define the structure of a site on $\mathcal{C}$.*

1. *If every $\mathcal{U} \in Cov_1$ is tautologically equivalent to some $\mathcal{V} \in Cov_2$, then $Sh(\mathcal{C}, Cov_2) \subset Sh(\mathcal{C}, Cov_1)$. If also, every $\mathcal{U} \in Cov_2$ is tautologically equivalent to some $\mathcal{V} \in Cov_1$ then the category of sheaves are equal.*
2. *Suppose that for each $\mathcal{U} \in Cov_1$ there exists a $\mathcal{V} \in Cov_2$ such that $\mathcal{V}$ refines $\mathcal{U}$. In this case $Sh(\mathcal{C}, Cov_2) \subset Sh(\mathcal{C}, Cov_1)$. If also for every $\mathcal{U} \in Cov_2$ there exists a $\mathcal{V} \in Cov_1$ such that $\mathcal{V}$ refines $\mathcal{U}$, then the categories of sheaves are equal.*

Proof. Part (1) follows directly from Lemma 7.8.4 and the definitions.

We advise the reader to **skip the proof of (2)** on a first reading. Let $\mathcal{F}$ be a sheaf of sets for the site $(\mathcal{C}, \mathrm{Cov}_2)$. Let $\mathcal{U} \in \mathrm{Cov}_1$, say $\mathcal{U} = \{U_i \to U\}_{i \in I}$. Choose a refinement $\mathcal{V} \in \mathrm{Cov}_2$ of $\mathcal{U}$, say $\mathcal{V} = \{V_j \to U\}_{j \in J}$ and refinement given by $\alpha : J \to I$ and $f_j : V_j \to U_{\alpha(j)}$.

First let $s, s' \in \mathcal{F}(U)$. If for all $i \in I$ we have $s|_{U_i} = s'|_{U_i}$, then we also have $s|_{V_j} = s'|_{V_j}$ for all $j \in J$. This implies that $s = s'$ by the sheaf condition for $\mathcal{F}$ with respect to Cov_2. Hence we see that the unicity in the sheaf condition for $\mathcal{F}$ and the site $(\mathcal{C}, \mathrm{Cov}_1)$ holds.

Next, suppose given $s_i \in \mathcal{F}(U_i)$ such that $s_i|_{U_i \times_U U_{i'}} = s_{i'}|_{U_i \times_U U_{i'}}$ for all $i, i' \in I$. Set $s_j = f_j^*(s_{\alpha(j)}) \in \mathcal{F}(V_j)$. Since the morphisms f_j are morphisms over U we obtain induced morphisms $f_{jj'} : V_j \times_U V_{j'} \to U_{\alpha(i)} \times_U U_{\alpha(i')}$ compatible with the $f_j, f_{j'}$ via the projection maps. It follows that

$$s_j|_{V_j \times_U V_{j'}} = f_{jj'}^*(s_{\alpha(j)}|_{U_{\alpha(j)} \times_U U_{\alpha(j')}}) = f_{jj'}^*(s_{\alpha(j')}|_{U_{\alpha(j)} \times_U U_{\alpha(j')}}) = s_{j'}|_{V_j \times_U V_{j'}}$$

for all $j, j' \in J$. Hence, by the sheaf condition for $\mathcal{F}$ with respect to Cov_2, we get a section $s \in \mathcal{F}(U)$ which restricts to s_j on each V_j. We are done if we show s restricts to s_{i_0} on U_{i_0} for any $i_0 \in I$. For each $i_0 \in I$ the family $\mathcal{U}' = \{U_i \times_U U_{i_0} \to U_{i_0}\}_{i \in I}$ is an element of Cov_1 by the axioms of a site. Also, the family $\mathcal{V}' = \{V_j \times_U U_{i_0} \to$

$U_{i_0}\}_{j\in J}$ is an element of Cov_2. Then $\mathcal{V}'$ refines $\mathcal{U}'$ via $\alpha : J \to I$ and the maps $f'_j = f_j \times \mathrm{id}_{U_{i_0}}$. The element s_{i_0} restricts to $s_i|_{U_i \times_U U_{i_0}}$ on the members of the covering $\mathcal{U}'$ and hence via $(f'_j)^*$ to the elements $s_j|_{V_j \times_U U_{i_0}}$ on the members of the covering $\mathcal{V}'$. By construction of s this is the same as the family of restrictions of $s|_{U_{i_0}}$ to the members of the covering $\mathcal{V}'$. Hence by the sheaf condition for $\mathcal{F}$ with respect to Cov_2 we see that $s|_{U_{i_0}} = s_{i_0}$ as desired. □

Lemma 7.8.6. *Let $\mathcal{C}$ be a category. Let $Cov(\mathcal{C})$ be a proper class of coverings satisfying conditions (1), (2) and (3) of Definition 7.6.2. Let $Cov_1, Cov_2 \subset Cov(\mathcal{C})$ be two subsets of $Cov(\mathcal{C})$ which endow $\mathcal{C}$ with the structure of a site. If every covering $\mathcal{U} \in Cov(\mathcal{C})$ is combinatorially equivalent to a covering in Cov_1 and combinatorially equivalent to a covering in Cov_2, then $Sh(\mathcal{C}, Cov_1) = Sh(\mathcal{C}, Cov_2)$.*

Proof. This is clear from Lemmas 7.8.5 and 7.8.3 above as the hypothesis implies that every covering $\mathcal{U} \in \mathrm{Cov}_1 \subset \mathrm{Cov}(\mathcal{C})$ is combinatorially equivalent to an element of Cov_2, and similarly with the roles of Cov_1 and Cov_2 reversed. □

7.9. The example of G-sets

As an example, consider the site $\mathcal{T}_G$ of Example 7.6.5. We will describe the category of sheaves on $\mathcal{T}_G$. The answer will turn out to be independent of the choices made in defining $\mathcal{T}_G$. In fact, during the proof we will need only the following properties of the site $\mathcal{T}_G$:

(a) $\mathcal{T}_G$ is a full subcategory of G-*Sets*,
(b) $\mathcal{T}_G$ contains the G-set ${}_GG$,
(c) $\mathcal{T}_G$ has fibre products and they are the same as in G-*Sets*,
(d) given $U \in \mathrm{Ob}(\mathcal{T}_G)$ and a G-invariant subset $O \subset U$, there exists an object of $\mathcal{T}_G$ isomorphic to O, and
(e) any surjective family of maps $\{U_i \to U\}_{i\in I}$, with $U, U_i \in \mathrm{Ob}(\mathcal{T}_G)$ is combinatorially equivalent to a covering of $\mathcal{T}_G$.

These properties hold by Sets, Lemmas 3.10.2 and 3.11.1.

Remark that the map

$$\mathrm{Hom}_G({}_GG, {}_GG) \longrightarrow G^{opp}, \varphi \longmapsto \varphi(1)$$

is an isomorphism of groups. The inverse map sends $g \in G$ to the map $R_g : s \mapsto sg$ (i.e. right multiplication). Note that $R_{g_1g_2} = R_{g_2} \circ R_{g_1}$ so the opposite is necessary.

This implies that for every presheaf $\mathcal{F}$ on $\mathcal{T}_G$ the value $\mathcal{F}({}_GG)$ inherits the structure of a G-set as follows: $g \cdot s$ for $g \in G$ and $s \in \mathcal{F}({}_GG)$ defined by $\mathcal{F}(R_g)(s)$. This is a left action because

$$(g_1g_2) \cdot s = \mathcal{F}(R_{g_1g_2})(s) = \mathcal{F}(R_{g_2} \circ R_{g_1})(s) = \mathcal{F}(R_{g_1})(\mathcal{F}(R_{g_2})(s)) = g_1 \cdot (g_2 \cdot s).$$

Here we've used that $\mathcal{F}$ is contravariant. Note that if $\mathcal{F} \to \mathcal{G}$ is a morphism of presheaves of sets on $\mathcal{T}_G$ then we get a map $\mathcal{F}({}_GG) \to \mathcal{G}({}_GG)$ which is compatible with the G-actions we have just defined. All in all we have constructed a functor

$$PSh(\mathcal{T}_G) \longrightarrow G\text{-}Sets, \quad \mathcal{F} \longmapsto \mathcal{F}({}_GG).$$

We leave it to the reader to verify that this construction has the pleasing property that the representable presheaf h_U is mapped to something canonically isomorphic to U. In a formula $h_U({}_GG) = \mathrm{Hom}_G({}_GG, U) \cong U$.

Suppose that S is a G-set. We define a presheaf $\mathcal{F}_S$ by the formula[2]

$$\mathcal{F}_S(U) = \mathrm{Mor}_{G\text{-}Sets}(U, S).$$

This is clearly a presheaf. On the other hand, suppose that $\{U_i \to U\}_{i\in I}$ is a covering in $\mathcal{T}_G$. This implies that $\coprod_i U_i \to U$ is surjective. Thus it is clear that the map

$$\mathcal{F}_S(U) = \mathrm{Mor}_{G\text{-}Sets}(U, S) \longrightarrow \prod \mathcal{F}_S(U_i) = \prod \mathrm{Mor}_{G\text{-}Sets}(U_i, S)$$

is injective. And, given a family of G-equivariant maps $s_i : U_i \to S$, such that all the diagrams

$$\begin{array}{ccc} U_i \times_U U_j & \longrightarrow & U_j \\ \downarrow & & \downarrow{\scriptstyle s_j} \\ U_i & \xrightarrow{s_i} & S \end{array}$$

commute, there is a unique G-equivariant map $s : U \to S$ such that s_i is the composition $U_i \to U \to S$. Namely, we just define $s(u) = s_i(u_i)$ where $i \in I$ is any index such that there exists some $u_i \in U_i$ mapping to u under the map $U_i \to U$. The commutativity of the diagrams above implies exactly that this construction is well defined. All in all we have constructed a functor

$$G\text{-}Sets \longrightarrow Sh(\mathcal{T}_G), \quad S \longmapsto \mathcal{F}_S.$$

We now have the following diagram of categories and functors

$$\begin{array}{ccc} PSh(\mathcal{T}_G) & \xrightarrow{\mathcal{F}\mapsto\mathcal{F}({}_GG)} & G\text{-}Sets \\ & \nwarrow \quad \swarrow_{S\mapsto\mathcal{F}_S} & \\ & Sh(\mathcal{T}_G) & \end{array}$$

It is immediate from the definitions that $\mathcal{F}_S({}_GG) = \mathrm{Mor}_G({}_GG, S) = S$, the last equality by evaluation at 1. This almost proves the following.

Proposition 7.9.1. *The functors $\mathcal{F} \mapsto \mathcal{F}({}_GG)$ and $S \mapsto \mathcal{F}_S$ define quasi-inverse equivalences between $Sh(\mathcal{T}_G)$ and G-Sets.*

Proof. We have already seen that composing the functors one way around is isomorphic to the identity functor. In the other direction, for any sheaf $\mathcal{H}$ there is a natural map of sheaves

$$can : \mathcal{H} \longrightarrow \mathcal{F}_{\mathcal{H}({}_GG)}.$$

Namely, for any object U of $\mathcal{T}_G$ we let can_U be the map

$$\begin{array}{ccc} \mathcal{H}(U) & \longrightarrow & \mathcal{F}_{\mathcal{H}({}_GG)}(U) = \mathrm{Mor}_G(U, \mathcal{H}({}_GG)) \\ s & \longmapsto & (u \mapsto \alpha_u^* s). \end{array}$$

Here $\alpha_u : {}_GG \to U$ is the map $\alpha_u(g) = gu$ and $\alpha_u^* : \mathcal{H}(U) \to \mathcal{H}({}_GG)$ is the pullback map. A trivial but confusing verification shows that this is indeed a map of presheaves. We have to show that *can* is an isomorphism. We do this by showing can_U is an isomorphism for all $U \in \mathrm{ob}(\mathcal{T}_G)$. We leave the (important but easy) case that $U = {}_GG$ to the reader. A general object U of $\mathcal{T}_G$ is a disjoint union of G-orbits: $U = \coprod_{i\in I} O_i$. The family of maps $\{O_i \to U\}_{i\in I}$ is tautologically equivalent

[2]It may appear this is the representable presheaf defined by S. This may not be the case because S may not be an object of $\mathcal{T}_G$ which was chosen to be a sufficiently large set of G-sets.

to a covering in $\mathcal{T}_G$ (by the properties of $\mathcal{T}_G$ listed at the beginning of this section). Hence by Lemma 7.8.4 the sheaf $\mathcal{H}$ satisfies the sheaf property with respect to $\{O_i \to U\}_{i\in I}$. The sheaf property for this covering implies $\mathcal{H}(U) = \prod_i \mathcal{H}(O_i)$. Hence it suffices to show that can_U is an isomorphism when U consists of a single G-orbit. Let $u \in U$ and let $H \subset G$ be its stabilizer. Clearly, $\mathrm{Mor}_G(U, \mathcal{H}({}_GG)) = \mathcal{H}({}_GG)^H$ equals the subset of H-invariant elements. On the other hand consider the covering $\{{}_GG \to U\}$ given by $g \mapsto gu$ (again it is just combinatorially equivalent to some covering of $\mathcal{T}_G$, and again this doesn't matter). Note that the fibre product $({}_GG) \times_U ({}_GG)$ is equal to $\{(g, gh), g \in G, h \in H\} \cong \prod_{h\in H} {}_GG$. Hence the sheaf property for this covering reads as

$$\mathcal{H}(U) \longrightarrow \mathcal{H}({}_GG) \underset{\mathrm{pr}_1^*}{\overset{\mathrm{pr}_0^*}{\rightrightarrows}} \prod_{h\in H} \mathcal{H}({}_GG).$$

The two maps pr_i^* into the factor $\mathcal{H}({}_GG)$ differ by multiplication by h. Now the result follows from this and the fact that can is an isomorphism for $U = {}_GG$. $\square$

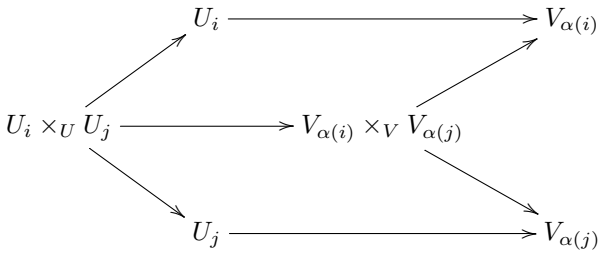
7.10. Sheafification

In order to define the sheafification we study the zeroth Cech cohomology group of a covering and its functoriality properties.

Let $\mathcal{F}$ be a presheaf of sets on $\mathcal{C}$, and let $\mathcal{U} = \{U_i \to U\}_{i\in I}$ be a covering of $\mathcal{C}$. Let us use the notation $\mathcal{F}(\mathcal{U})$ to indicate the equalizer

$$H^0(\mathcal{U}, \mathcal{F}) = \{(s_i)_{i\in I} \in \prod\nolimits_i \mathcal{F}(U_i) \mid s_i|_{U_i\times_U U_j} = s_j|_{U_i\times_U U_j} \ \forall i, j \in I\}.$$

As we will see later, this is the zeroth Cech cohomology of $\mathcal{F}$ over U with respect to the covering $\mathcal{U}$. A small remark is that we can define $H^0(\mathcal{U}, \mathcal{F})$ as soon as all the morphisms $U_i \to U$ are representable, i.e., $\mathcal{U}$ need not be a covering of the site. There is a canonical map $\mathcal{F}(U) \to H^0(\mathcal{U}, \mathcal{F})$. It is clear that a morphism of coverings $\mathcal{U} \to \mathcal{V}$ induces commutative diagrams



This in turn produces a map $H^0(\mathcal{V}, \mathcal{F}) \to H^0(\mathcal{U}, \mathcal{F})$, compatible with the map $\mathcal{F}(V) \to \mathcal{F}(U)$.

By construction, a presheaf $\mathcal{F}$ is a sheaf if and only if for every covering $\mathcal{U}$ of $\mathcal{C}$ the natural map $\mathcal{F}(U) \to H^0(\mathcal{U}, \mathcal{F})$ is bijective. We will use this notion to prove the following simple lemma about limits of sheaves.

Lemma 7.10.1. *Let $\mathcal{F} : \mathcal{I} \to Sh(\mathcal{C})$ be a diagram. Then $\lim_{\mathcal{I}} \mathcal{F}$ exists and is equal to the limit in the category of presheaves.*

Proof. Let $\lim_i \mathcal{F}_i$ be the limit as a presheaf. We will show that this is a sheaf and then it will trivially follow that it is a limit in the category of sheaves. To prove the sheaf property, let $\mathcal{V} = \{V_j \to V\}_{j\in J}$ be a covering. Let $(s_j)_{j\in J}$ be an element of $H^0(\mathcal{V}, \lim_i \mathcal{F}_i)$. Using the projection maps we get elements $(s_{j,i})_{j\in J}$ in $H^0(\mathcal{V}, \mathcal{F}_i)$. By the sheaf property for $\mathcal{F}_i$ we see that there is a unique $s_i \in \mathcal{F}_i(V)$ such that $s_{j,i} = s_i|_{V_j}$. Let $\phi : i \to i'$ be a morphism of the index category. We would like to show that $\mathcal{F}(\phi) : \mathcal{F}_i \to \mathcal{F}_{i'}$ maps s_i to $s_{i'}$. We know this is true for the sections $s_{i,j}$ and $s_{i',j}$ for all j and hence by the sheaf property for $\mathcal{F}_{i'}$ this is true. At this point we have an element $s = (s_i)_{i\in \mathrm{Ob}(\mathcal{I})}$ of $(\lim_i \mathcal{F}_i)(V)$. We leave it to the reader to see this element has the required property that $s_j = s|_{V_j}$. □

Example 7.10.2. A particular example is the limit over the empty diagram. This gives the final object in the category of (pre)sheaves. It is the sheaf that associates to each object U of $\mathcal{C}$ a singleton set, with unique restriction mappings. We often denote this sheaf by $*$.

Let $\mathcal{J}_U$ be the category of all coverings of U. In other words, the objects of $\mathcal{J}_U$ are the coverings of U in $\mathcal{C}$, and the morphisms are the refinements. By our conventions on sites this is indeed a category, i.e., the collection of objects and morphisms forms a set. Note that $\mathrm{Ob}(\mathcal{J}_U)$ is not empty since $\{\mathrm{id}_U\}$ is an object of it. According to the remarks above the construction $\mathcal{U} \mapsto H^0(\mathcal{U}, \mathcal{F})$ is a contravariant functor on $\mathcal{J}_U$. We define

$$\mathcal{F}^+(U) = \mathrm{colim}_{\mathcal{J}_U^{opp}} H^0(\mathcal{U}, \mathcal{F})$$

See Categories, Section 4.14 for a discussion of limits and colimits. We point out that later we will see that $\mathcal{F}^+(U)$ is the zeroth Cech cohomology of $\mathcal{F}$ over U.

Before we say more about the structure of the colimit, we turn the collection of sets $\mathcal{F}^+(U)$, $U \in \mathrm{Ob}(\mathcal{C})$ into a presheaf. Namely, let $V \to U$ be a morphism of $\mathcal{C}$. By the axioms of a site there is a functor[3]

$$\mathcal{J}_U \longrightarrow \mathcal{J}_V, \quad \{U_i \to U\} \longmapsto \{U_i \times_U V \to V\}.$$

Note that the projection maps furnish a functorial morphism of coverings $\{U_i \times_U V \to V\} \to \{U_i \to U\}$ and hence, by the construction above, a functorial map of sets $H^0(\{U_i \to U\}, \mathcal{F}) \to H^0(\{U_i \times_U V \to V\}, \mathcal{F})$. In other words, there is a transformation of functors from $H^0(-, \mathcal{F}) : \mathcal{J}_U \to \mathit{Sets}$ to the composition $\mathcal{J}_U \to \mathcal{J}_V \xrightarrow{H^0(-,\mathcal{F})} \mathit{Sets}$. Hence by generalities of colimits we obtain a canonical map $\mathcal{F}^+(U) \to \mathcal{F}^+(V)$. In terms of the description of the set $\mathcal{F}^+(U)$ above, it just takes the element associated with $s = (s_i) \in H^0(\{U_i \to U\}, \mathcal{F})$ to the element associated with $(s_i|_{V\times_U U_i}) \in H^0(\{U_i \times_U V \to V\}, \mathcal{F})$.

Lemma 7.10.3. *The constructions above define a presheaf $\mathcal{F}^+$ together with a canonical map of presheaves $\mathcal{F} \to \mathcal{F}^+$.*

Proof. All we have to do is to show that given morphisms $W \to V \to U$ the composition $\mathcal{F}^+(U) \to \mathcal{F}^+(V) \to \mathcal{F}^+(W)$ equals the map $\mathcal{F}^+(U) \to \mathcal{F}^+(W)$. This can be shown directly by verifying that, given a covering $\{U_i \to U\}$ and $s = (s_i) \in H^0(\{U_i \to U\}, \mathcal{F})$, we have canonically $W \times_U U_i \cong W \times_V (V \times_U U_i)$, and $s_i|_{W\times_U U_i}$ corresponds to $(s_i|_{V\times_U U_i})|_{W\times_V (V\times_U U_i)}$ via this isomorphism. □

[3]This construction actually involves a choice of the fibre products $U_i \times_U V$ and hence the axiom of choice. The resulting map does not depend on the choices made, see below.

More indirectly, the result of Lemma 7.10.6 shows that we may pullback an element s as above via any morphism from any covering of W to $\{U_i \to U\}$ and we will always end up with the same element in $\mathcal{F}^+(W)$.

Lemma 7.10.4. *The association $\mathcal{F} \mapsto (\mathcal{F} \to \mathcal{F}^+)$ is a functor.*

Proof. Instead of proving this we state exactly what needs to be proven. Let $\mathcal{F} \to \mathcal{G}$ be a map of presheaves. Prove the commutativity of:

$$\begin{array}{ccc} \mathcal{F} & \longrightarrow & \mathcal{F}^+ \\ \downarrow & & \downarrow \\ \mathcal{G} & \longrightarrow & \mathcal{G}^+ \end{array}$$

□

The next two lemmas imply that the colimits above are colimits over a directed partially ordered set.

Lemma 7.10.5. *Given a pair of coverings $\{U_i \to U\}$ and $\{V_j \to U\}$ of a given object U of the site $\mathcal{C}$, there exists a covering which is a common refinement.*

Proof. Since $\mathcal{C}$ is a site we have that for every i the family $\{V_j \times_U U_i \to U_i\}_j$ is a covering. And, then another axiom implies that $\{V_j \times_U U_i \to U\}_{i,j}$ is a covering of U. Clearly this covering refines both given coverings. □

Lemma 7.10.6. *Any two morphisms $f, g : \mathcal{U} \to \mathcal{V}$ of coverings inducing the same morphism $U \to V$ induce the same map $H^0(\mathcal{V}, \mathcal{F}) \to H^0(\mathcal{U}, \mathcal{F})$.*

Proof. Let $\mathcal{U} = \{U_i \to U\}_{i \in I}$ and $\mathcal{V} = \{V_j \to V\}_{j \in J}$. The morphism f consists of a map $U \to V$, a map $\alpha \colon I \to J$ and maps $f_i \colon U_i \to V_{\alpha(i)}$. Likewise, g determines a map $\beta \colon I \to J$ and maps $g_i \colon U_i \to V_{\beta(i)}$. As f and g induce the same map $U \to V$, the diagram

$$\begin{array}{ccccc} & & V_{\alpha(i)} & & \\ & \overset{f_i}{\nearrow} & & \searrow & \\ U_i & & & & V \\ & \underset{g_i}{\searrow} & & \nearrow & \\ & & V_{\beta(i)} & & \end{array}$$

is commutative for every $i \in I$. Hence f and g factor through the fibre product

$$\begin{array}{ccccc} & & V_{\alpha(i)} & & \\ & \overset{f_i}{\nearrow} & \uparrow{\scriptstyle \mathrm{pr}_1} & & \\ U_i & \xrightarrow{\varphi} & V_{\alpha(i)} \times_V V_{\beta(i)} & & \\ & \underset{g_i}{\searrow} & \downarrow{\scriptstyle \mathrm{pr}_2} & & \\ & & V_{\beta(i)}. & & \end{array}$$

Now let $s = (s_j)_j \in H^0(\mathcal{V}, \mathcal{F})$. Then for all $i \in I$:

$$(f^*s)_i = f_i^*(s_{\alpha(i)}) = \varphi^* \mathrm{pr}_1^*(s_{\alpha(i)}) = \varphi^* \mathrm{pr}_2^*(s_{\beta(i)}) = g_i^*(s_{\beta(i)}) = (g^*s)_i,$$

where the middle equality is given by the definition of $H^0(\mathcal{V}, \mathcal{F})$. This shows that the maps $H^0(\mathcal{V}, \mathcal{F}) \to H^0(\mathcal{U}, \mathcal{F})$ induced by f and g are equal. □

Remark 7.10.7. In particular this lemma shows that if $\mathcal{U}$ is a refinement of $\mathcal{V}$, and if $\mathcal{V}$ is a refinement of $\mathcal{U}$, then there is a canonical identification $H^0(\mathcal{U}, \mathcal{F}) = H^0(\mathcal{V}, \mathcal{F})$.

From these two lemmas, and the fact that $\mathcal{J}_U$ is nonempty, it follows that the diagram $H^0(-, \mathcal{F}) : \mathcal{J}_U^{opp} \to Sets$ is filtered, see Categories, Definition 4.19.1. Hence, by Categories, Section 4.19 the colimit $\mathcal{F}^+(U)$ may be described in the following straightforward manner. Namely, every element in the set $\mathcal{F}^+(U)$ arises from an element $s \in H^0(\mathcal{U}, \mathcal{F})$ for some covering $\mathcal{U}$ of U. Given a second element $s' \in H^0(\mathcal{U}', \mathcal{F})$ then s and s' determine the same element of the colimit if and only if there exists a covering $\mathcal{V}$ of U and refinements $f : \mathcal{V} \to \mathcal{U}$ and $f' : \mathcal{V} \to \mathcal{U}'$ such that $f^*s = (f')^*s'$ in $H^0(\mathcal{V}, \mathcal{F})$. Since the trivial covering $\{\mathrm{id}_U\}$ is an object of $\mathcal{J}_U$ we get a canonical map $\mathcal{F}(U) \to \mathcal{F}^+(U)$.

Lemma 7.10.8. *The map $\theta : \mathcal{F} \to \mathcal{F}^+$ has the following property: For every object U of $\mathcal{C}$ and every section $s \in \mathcal{F}^+(U)$ there exists a covering $\{U_i \to U\}$ such that $s|_{U_i}$ is in the image of $\theta : \mathcal{F}(U_i) \to \mathcal{F}^+(U_i)$.*

Proof. Namely, let $\{U_i \to U\}$ be a covering such that s arises from the element $(s_i) \in H^0(\{U_i \to U\}, \mathcal{F})$. According to Lemma 7.10.6 we may consider the covering $\{U_i \to U_i\}$ and the (obvious) morphism of coverings $\{U_i \to U_i\} \to \{U_i \to U\}$ to compute the pullback of s to an element of $\mathcal{F}^+(U_i)$. And indeed, using this covering we get exactly $\theta(s_i)$ for the restriction of s to U_i. □

Definition 7.10.9. We say that a presheaf of sets $\mathcal{F}$ on a site $\mathcal{C}$ is *separated* if, for all coverings of $\{U_i \to U\}$, the map $\mathcal{F}(U) \to \prod \mathcal{F}(U_i)$ is injective.

Theorem 7.10.10. *With $\mathcal{F}$ as above*

1. *The presheaf $\mathcal{F}^+$ is separated.*
2. *If $\mathcal{F}$ is separated, then $\mathcal{F}^+$ is a sheaf and the map of presheaves $\mathcal{F} \to \mathcal{F}^+$ is injective.*
3. *If $\mathcal{F}$ is a sheaf, then $\mathcal{F} \to \mathcal{F}^+$ is an isomorphism.*
4. *The presheaf $\mathcal{F}^{++}$ is always a sheaf.*

Proof. Proof of (1). Suppose that $s, s' \in \mathcal{F}^+(U)$ and suppose that there exists some covering $\{U_i \to U\}$ such that $s|_{U_i} = s'|_{U_i}$ for all i. We now have three coverings of U: the covering $\{U_i \to U\}$ above, a covering $\mathcal{U}$ for s as in Lemma 7.10.8, and a similar covering $\mathcal{U}'$ for s'. By Lemma 7.10.5, we can find a common refinement, say $\{W_j \to U\}$. This means we have $s_j, s'_j \in \mathcal{F}(W_j)$ such that $s|_{W_j} = \theta(s_j)$, similarly for $s'|_{W_j}$, and such that $\theta(s_j) = \theta(s'_j)$. This last equality means that there exists some covering $\{W_{jk} \to W_j\}$ such that $s_j|_{W_{jk}} = s'_j|_{W_{jk}}$. Then since $\{W_{jk} \to U\}$ is a covering we see that s, s' map to the same element of $H^0(\{W_{jk} \to U\}, \mathcal{F})$ as desired.

Proof of (2). It is clear that $\mathcal{F} \to \mathcal{F}^+$ is injective because all the maps $\mathcal{F}(U) \to H^0(\mathcal{U}, \mathcal{F})$ are injective. It is also clear that, if $\mathcal{U} \to \mathcal{U}'$ is a refinement, then $H^0(\mathcal{U}', \mathcal{F}) \to H^0(\mathcal{U}, \mathcal{F})$ is injective. Now, suppose that $\{U_i \to U\}$ is a covering, and let (s_i) be a family of elements of $\mathcal{F}^+(U_i)$ satisfying the sheaf condition $s_i|_{U_i \times_U U_j} = s_j|_{U_i \times_U U_j}$ for all $i, j \in I$. Choose coverings (as in Lemma 7.10.8) $\{U_{ij} \to U_i\}$ such

that $s_i|_{U_{ij}}$ is the image of the (unique) element $s_{ij} \in \mathcal{F}(U_{ij})$. The sheaf condition implies that s_{ij} and $s_{i'j'}$ agree over $U_{ij} \times_U U_{i'j'}$ because it maps to $U_i \times_U U_{i'}$ and we have the equality there. Hence $(s_{ij}) \in H^0(\{U_{ij} \to U\}, \mathcal{F})$ gives rise to an element $s \in \mathcal{F}^+(U)$. We leave it to the reader to verify that $s|_{U_i} = s_i$.

Proof of (3). This is immediate from the definitions because the sheaf property says exactly that every map $\mathcal{F} \to H^0(\mathcal{U}, \mathcal{F})$ is bijective (for every covering $\mathcal{U}$ of U).

Statement (4) is now obvious. □

Definition 7.10.11. Let $\mathcal{C}$ be a site and let $\mathcal{F}$ be a presheaf of sets on $\mathcal{C}$. The sheaf $\mathcal{F}^\# := \mathcal{F}^{++}$ together with the canonical map $\mathcal{F} \to \mathcal{F}^\#$ is called the *sheaf associated to* $\mathcal{F}$.

Proposition 7.10.12. *The canonical map $\mathcal{F} \to \mathcal{F}^\#$ has the following universal property: For any map $\mathcal{F} \to \mathcal{G}$, where $\mathcal{G}$ is a sheaf of sets, there is a unique map $\mathcal{F}^\# \to \mathcal{G}$ such that $\mathcal{F} \to \mathcal{F}^\# \to \mathcal{G}$ equals the given map.*

Proof. By Lemma 7.10.4 we get a commutative diagram

$$\begin{array}{ccccc} \mathcal{F} & \longrightarrow & \mathcal{F}^+ & \longrightarrow & \mathcal{F}^{++} \\ \downarrow & & \downarrow & & \downarrow \\ \mathcal{G} & \longrightarrow & \mathcal{G}^+ & \longrightarrow & \mathcal{G}^{++} \end{array}$$

and by Theorem 7.10.10 the lower horizontal maps are isomorphisms. The uniqueness follows from Lemma 7.10.8 which says that every section of $\mathcal{F}^\#$ locally comes from sections of $\mathcal{F}$. □

It is clear from this result that the functor $\mathcal{F} \mapsto (\mathcal{F} \to \mathcal{F}^\#)$ is unique up to unique isomorphism of functors. Actually, let us temporarily denote $i : Sh(\mathcal{C}) \to PSh(\mathcal{C})$ the functor of inclusion. The result above actually says that

$$\mathrm{Mor}_{PSh(\mathcal{C})}(\mathcal{F}, i(\mathcal{G})) = \mathrm{Mor}_{Sh(\mathcal{C})}(\mathcal{F}^\#, \mathcal{G}).$$

In other words, the functor of sheafification is the left adjoint to the inclusion functor i. We finish this section with a couple of lemmas.

Lemma 7.10.13. *Let $\mathcal{F} : \mathcal{I} \to Sh(\mathcal{C})$ be a diagram. Then $\mathrm{colim}_{\mathcal{I}} \mathcal{F}$ exists and is the sheafification of the colimit in the category of presheaves.*

Proof. Since the sheafification functor is a left adjoint it commutes with all colimits, see Categories, Lemma 4.24.3. Hence, since $PSh(\mathcal{C})$ has colimits, we deduce that $Sh(\mathcal{C})$ has colimits (which are the sheafifications of the colimits in presheaves). □

Lemma 7.10.14. *The functor $PSh(\mathcal{C}) \to Sh(\mathcal{C})$, $\mathcal{F} \mapsto \mathcal{F}^\#$ is exact.*

Proof. Since it is a left adjoint it is right exact, see Categories, Lemma 4.24.4. On the other hand, by Lemmas 7.10.5 and Lemma 7.10.6 the colimits in the construction of $\mathcal{F}^+$ are really over the directed partially ordered set $\mathrm{Ob}(\mathcal{J}_U)$ where $\mathcal{U} \geq \mathcal{U}'$ if and only if $\mathcal{U}$ is a refinement of $\mathcal{U}'$. Hence by Categories, Lemma 4.19.2 we see that $\mathcal{F} \to \mathcal{F}^+$ commutes with finite limits (as a functor from presheaves to presheaves). Then we conclude using Lemma 7.10.1. □

Lemma 7.10.15. *Let $\mathcal{C}$ be a site. Let $\mathcal{F}$ be a presheaf of sets on $\mathcal{C}$. Denote $\theta^2 : \mathcal{F} \to \mathcal{F}^\#$ the canonical map of $\mathcal{F}$ into its sheafification. Let U be an object of $\mathcal{C}$. Let $s \in \mathcal{F}^\#(U)$. There exists a covering $\{U_i \to U\}$ and sections $s_i \in \mathcal{F}(U_i)$ such that*

(1) *$s|_{U_i} = \theta^2(s_i)$, and*
(2) *for every i, j there exists a covering $\{U_{ijk} \to U_i \times_U U_j\}$ of $\mathcal{C}$ such that the pullback of s_i and s_j to each U_{ijk} agree.*

Conversely, given any covering $\{U_i \to U\}$, elements $s_i \in \mathcal{F}(U_i)$ such that (2) holds, then there exists a unique section $s \in \mathcal{F}^\#(U)$ such that (1) holds.

Proof. Omitted. □

7.11. Quasi-compact objects and colimits

To be able to use the same language as in the case of topological spaces we introduce the following terminology.

Definition 7.11.1. Let $\mathcal{C}$ be a site. An object U of $\mathcal{C}$ is *quasi-compact* if every covering of U in $\mathcal{C}$ can be refined by a finite covering.

The following lemma is the analogue of Sheaves, Lemma 6.29.1 for sites.

Lemma 7.11.2. *Let $\mathcal{C}$ be a site. Let $\mathcal{I} \to Sh(\mathcal{C})$, $i \mapsto \mathcal{F}_i$ be a filtered diagram of sheaves of sets. Let $U \in \mathrm{Ob}(\mathcal{C})$. Consider the canonical map*

$$\Psi : \mathrm{colim}_i \mathcal{F}_i(U) \longrightarrow (\mathrm{colim}_i \mathcal{F}_i)(U)$$

With the terminology introduced above:

(1) *If all the transition maps are injective then Ψ is injective for any U.*
(2) *If U is quasi-compact, then Ψ is injective.*
(3) *If U is quasi-compact and all the transition maps are injective then Ψ is an isomorphism.*
(4) *If U has a cofinal system of coverings $\{U_j \to U\}_{j \in J}$ with J finite and $U_j \times_U U_{j'}$ quasi-compact for all $j, j' \in J$, then Ψ is bijective.*

Proof. Assume all the transition maps are injective. In this case the presheaf $\mathcal{F}' : V \mapsto \mathrm{colim}_i \mathcal{F}_i(V)$ is separated (see Definition 7.10.9). By Lemma 7.10.13 we have $(\mathcal{F}')^\# = \mathrm{colim}_i \mathcal{F}_i$. By Theorem 7.10.10 we see that $\mathcal{F}' \to (\mathcal{F}')^\#$ is injective. This proves (1).

Assume U is quasi-compact. Suppose that $s \in \mathcal{F}_i(U)$ and $s' \in \mathcal{F}_{i'}(U)$ give rise to elements on the left hand side which have the same image under Ψ. Since U is quasi-compact this means there exists a finite covering $\{U_j \to U\}_{j=1,\ldots,m}$ and for each j an index $i_j \in I$, $i_j \geq i$, $i_j \geq i'$ such that $\varphi_{ii_j}(s) = \varphi_{i'i_j}(s')$. Let $i'' \in I$ be $\geq$ than all of the i_j. We conclude that $\varphi_{ii''}(s)$ and $\varphi_{i'i''}(s)$ agree on U_j for all j and hence that $\varphi_{ii''}(s) = \varphi_{i'i''}(s)$. This proves (2).

Assume U is quasi-compact and all transition maps injective. Let s be an element of the target of Ψ. Since U is quasi-compact there exists a finite covering $\{U_j \to U\}_{j=1,\ldots,m}$, for each j an index $i_j \in I$ and $s_j \in \mathcal{F}_{i_j}(U_j)$ such that $s|_{U_j}$ comes from s_j for all j. Pick $i \in I$ which is $\geq$ than all of the i_j. By (1) the sections $\varphi_{i_j i}(s_j)$ agree over $U_j \times_U U_{j'}$. Hence they glue to a section $s' \in \mathcal{F}_i(U)$ which maps to s under Ψ. This proves (3).

Assume the hypothesis of (4). Let s be an element of the target of Ψ. By assumption there exists a finite covering $\{U_j \to U\}_{j=1,\ldots,m} U_j$, with $U_j \times_U U_{j'}$ quasi-compact for all $j, j' \in J$ and for each j an index $i_j \in I$ and $s_j \in \mathcal{F}_{i_j}(U_j)$ such that $s|_{U_j}$ is the image of s_j for all j. Since $U_j \times_U U_{j'}$ is quasi-compact we can apply (2) and we see that there exists an $i_{jj'} \in I$, $i_{jj'} \geq i_j$, $i_{jj'} \geq i_{j'}$ such that $\varphi_{i_j i_{jj'}}(s_j)$ and $\varphi_{i_{j'} i_{jj'}}(s_{j'})$ agree over $U_j \times_U U_{j'}$. Choose an index $i \in I$ wich is bigger or equal than all the $i_{jj'}$. Then we see that the sections $\varphi_{i_j i}(s_j)$ of $\mathcal{F}_i$ glue to a section of $\mathcal{F}_i$ over U. This section is mapped to the element s as desired. □

We need an analogue of the above result in the case that the site is also the colimit of sites. Namely, suppose that we have

(1) a directed index set I,
(2) for $i \in I$ a site $\mathcal{C}_i$,
(3) for $i' \geq i$ a morphisms of sites $f_{i'i} : \mathcal{C}_{i'} \to \mathcal{C}_i$ given by a continuous functor $u_{i'i} : \mathcal{C}_i \to \mathcal{C}_{i'}$

such that $f_{i''i} = f_{i'i} \circ f_{i''i'}$. For simplicity we assume that every covering in $\mathcal{C}_i$ has a finite index set. Then we can define

(1) a category $\mathcal{C} = \operatorname{colim} \mathcal{C}_i$ using the functors $u_{i'i}$, and
(2) a set of families of morphisms with fixed target $\mathrm{Cov}(\mathcal{C})$ defined as the union of the images of $\mathrm{Cov}(\mathcal{C}_i)$ by the coprojections $u_i : \mathcal{C}_i \to \mathcal{C}$.

Lemma 7.11.3. *In the situation above the category $\mathcal{C}$ endowed with $Cov(\mathcal{C})$ is a site and the functors u_i define morphisms of sites $f_i : \mathcal{C} \to \mathcal{C}_i$.*

Proof. Let $\{U_i^j \to U_i\}$ be a covering of $\mathcal{C}_i$ for some i. Let's write $U^j \to U$, resp. $U_{i'}^j \to U_{i'}$ instead of $u_i(U_i^j) \to u_i(U_i)$, resp. $u_{i'i}(U_i^j) \to u_{i'i}(U)$. We will use similar notation for other coverings, objects, morphisms that will occur in the proof.

Let's prove that if $V \to U$ is a morphism of $\mathcal{C}$, then $U^j \times_U V$ exists. Namely, $V \to U$ comes from $V_{i'} \to U_{i'}$ for some $i' \geq i$. Then $U_{i'}^j \times_{U_{i'}} V_{i'}$ exists. Since each $u_{i''i'}$ is continuous we see that

$$u_{i''i'}(U_{i'}^j \times_{U_{i'}} V_{i'}) = U_{i''}^j \times_{U_{i''}} V_{i''}$$

for all $i'' \geq i'$. It follows by a formal argument that $U^j \times_U V$ exists and is the image of $U_{i'}^j \times_{U_{i'}} V_{i'}$ by the functor $u_{i'}$. It also follows that $\{U^j \times_U V \to V\}$ is a covering in $\mathcal{C}$. In this way we see that axiom (3) of Definition 7.6.2 holds.

To verify axiom (2) of Definition 7.6.2 let $\{U^j \to U\}_{j \in J}$ be a covering of $\mathcal{C}$ and for each j let $\{U^{jk} \to U^j\}$ be a covering of $\mathcal{C}$. Then we can find an i and a covering $\{U_i^j \to U_i\}_{j \in J}$ of $\mathcal{C}_i$ whose image by u_i is $\{U^j \to U\}$. Since J is **finite** we may choose an $i' \geq i$ and coverings $\{U_{i'}^{jk} \to U_{i'}^j\}$ of $\mathcal{C}_{i'}$ whose image by $u_{i'}$ gives $\{U^{jk} \to U^j\}$. Then we conclude that $\{U^{jk} \to U\}$ is a covering of $\mathcal{C}$.

We omit the proof of axiom (1) of Definition 7.6.2.

It follows from the arguments above that the functors u_i are continuous. To finish the proof we have to show that $f_i^{-1} := u_{i,s}$ is an exact functor $Sh(\mathcal{C}_i) \to Sh(\mathcal{C})$. In fact it suffices to show that f_i^{-1} is left exact, because it is right exact as a left adjoint (Categories, Lemma 4.24.4). Before we do this we describe this functor more explicity for a particular index $0 \in I$.

Let $\mathcal{F}_0$ be a sheaf on $\mathcal{C}_0$ and for $i \geq 0$ we set $\mathcal{F}_i = f_{i0}^{-1}\mathcal{F}_0$. Let $i \geq 0$ be an index and let U_i be an object of $\mathcal{C}_i$. We claim that

$$f_0^{-1}\mathcal{F}_0(U) = \operatorname{colim}_{i' \geq i} \mathcal{F}_{i'}(U_{i'}) \tag{7.11.3.1}$$

where U and $U_{i'}$ are the images of U_i in $\mathcal{C}$ and $\mathcal{C}_{i'}$ as above.

Proof of the claim. A straightforward argument with colimits (similar to the above) shows that the sheaf $f_0^{-1}\mathcal{F}_0$ is the sheaf associated to the presheaf $\mathcal{G}$ given by the right hand side of (7.11.3.1). We omit the proof. Thus to finish the proof we have to show that the right hand side satisfies the sheaf condition. However, this is clear because any covering $\{U^j \to U\}$ in $\mathcal{C}$ comes from a finite level, say $\mathcal{U}_{i'} = \{U_{i'}^j \to U_{i'}\}$ for some $i' \geq i$. Then we have

$$H^0(\mathcal{U}, \mathcal{G}) = \operatorname{colim}_{i'' \geq i'} H^0(\mathcal{U}_{i''}, \mathcal{F}_{i''}) = \operatorname{colim}_{i'' \geq i'} \mathcal{F}_{i'}(U_{i'}) = \mathcal{G}(U)$$

as desired. The first equality holds because filtered colimits commute with finite limits (Categories, Lemma 4.19.2).

Assume the claim holds. Then f_0^{-1} commutes with finite limits because finite limits of sheaves are computed in the category of presheaves (Lemma 7.10.1) and filtered colimits commute with finite limits. □

We continue the discussion. Assume given

(1) a sheaf $\mathcal{F}_i$ on $\mathcal{C}_i$ for all $i \in I$,
(2) for $i' \geq i$ a map $\varphi_{i'i} : f_{i'i}^{-1}\mathcal{F}_i \to \mathcal{F}_{i'}$ of sheaves on $\mathcal{C}_{i'}$

such that $\varphi_{i''i} = \varphi_{i''i'} \circ f_{i''i'}^{-1}\varphi_{i'i}$ whenever $i'' \geq i' \geq i$.

Lemma 7.11.4. *In the situation discussed above set $\mathcal{F} = \operatorname{colim} f_i^{-1}\mathcal{F}_i$. Let $0 \in I$, $X_0 \in Ob(\mathcal{C}_0)$, set $X = u_0(X_0)$ and $X_i = u_{i0}(X_0)$ for $i \geq 0$. Then*

$$\operatorname{colim}_{i \geq 0} \mathcal{F}_i(X_i) = \mathcal{F}(X)$$

Proof. We may and do replace I by $\{i \in I, i \geq 0\}$. A formal argument shows that

$$\operatorname{colim}_{i \in I} \mathcal{F}_i(X_i) = \operatorname{colim}_{j \in I} \operatorname{colim}_{i \geq j} f_{ij}^{-1}\mathcal{F}_j(X_i)$$

By (7.11.3.1) we see that the inner colimit is equal to $f_j^{-1}\mathcal{F}_j(X)$ hence we conclude by Lemma 7.11.3 applied with $\mathcal{B} = \mathrm{Ob}(\mathcal{C})$ and $\mathrm{Cov} = \mathrm{Cov}(\mathcal{C})$. □

7.12. Injective and surjective maps of sheaves

Definition 7.12.1. Let $\mathcal{C}$ be a site, and let $\varphi : \mathcal{F} \to \mathcal{G}$ be a map of sheaves of sets.

(1) We say that φ is *injective* if for every object U of $\mathcal{C}$ the map $\varphi : \mathcal{F}(U) \to \mathcal{G}(U)$ is injective.
(2) We say that φ is *surjective* if for every object U of $\mathcal{C}$ and every section $s \in \mathcal{G}(U)$ there exists a covering $\{U_i \to U\}$ such that for all i the restriction $s|_{U_i}$ is in the image of $\varphi : \mathcal{F}(U_i) \to \mathcal{G}(U_i)$.

Lemma 7.12.2. *The injective (resp. surjective) maps defined above are exactly the monomorphisms (resp. epimorphisms) of the category $Sh(\mathcal{C})$. A map of sheaves is an isomorphism if and only if it is both injective and surjective.*

Proof. Omitted. □

Lemma 7.12.3. *Let $\mathcal{C}$ be a site. Let $\mathcal{F} \to \mathcal{G}$ be a surjection of sheaves of sets. Then the diagram*

$$\mathcal{F} \times_{\mathcal{G}} \mathcal{F} \rightrightarrows \mathcal{F} \longrightarrow \mathcal{G}$$

represents $\mathcal{G}$ as a coequalizer.

Proof. Let $\mathcal{H}$ be a sheaf of sets and let $\varphi : \mathcal{F} \to \mathcal{H}$ be a map of sheaves equalizing the two maps $\mathcal{F} \times_{\mathcal{G}} \mathcal{F} \to \mathcal{F}$. Let $\mathcal{G}' \subset \mathcal{G}$ be the presheaf image of the map $\mathcal{F} \to \mathcal{G}$. As the product $\mathcal{F} \times_{\mathcal{G}} \mathcal{F}$ may be computed in the category of presheaves we see that it is equal to the presheaf product $\mathcal{F} \times_{\mathcal{G}'} \mathcal{F}$. Hence φ induces a unique map of presheaves $\psi' : \mathcal{G}' \to \mathcal{H}$. Since $\mathcal{G}$ is the sheafification of $\mathcal{G}'$ by Lemma 7.12.2 we conclude that ψ' extends uniquely to a map of sheaves $\psi : \mathcal{G} \to \mathcal{H}$. We omit the verification that φ is equal to the composition of ψ and the given map. □

7.13. Representable sheaves

Let $\mathcal{C}$ be a category. The canonical topology is the finest topology such that all representable presheaves are sheaves (it is formally defined in Definition 7.45.12 but we will not need this). This topology is not always the topology associated to the structure of a site on $\mathcal{C}$. We will give a collection of coverings that generates this topology in case $\mathcal{C}$ has fibered products. First we give the following general definition.

Definition 7.13.1. Let $\mathcal{C}$ be a category. We say that a family $\{U_i \to U\}$ is an *effective epimorphism* if all the morphisms $U_i \to U$ are representable (see Categories, Definition 4.6.4), and for any $X \in \mathrm{Ob}(\mathcal{C})$ the sequence

$$\mathrm{Mor}_{\mathcal{C}}(U, X) \longrightarrow \mathrm{Mor}_{\mathcal{C}}(U_i, X) \rightrightarrows \mathrm{Mor}_{\mathcal{C}}(U_i \times_U U_j, X)$$

is an equalizer diagram. We say that a family $\{U_i \to U\}$ is a *universal effective epimorphism* if for any morphism $V \to U$ the base change $\{U_i \times_U V \to V\}$ is an effective epimorphism.

The class of families which are universal effective epimorphisms satisfies the axioms of Definition 7.6.2. If $\mathcal{C}$ has fibre products, then the associated topology is the canonical topology. (In this case, to get a site argue as in Sets, Lemma 3.11.1.)

Conversely, suppose that $\mathcal{C}$ is a site such that all representable presheaves are sheaves. Then clearly, all coverings are universal effective epimorphisms. Thus the following definition is the "correct" one in the setting of sites.

Definition 7.13.2. We say that the topology on a site $\mathcal{C}$ is *weaker than the canonical topology*, or that the topology is *subcanonical* if all the coverings of $\mathcal{C}$ are universal effective epimorphisms.

A representable sheaf is a representable presheaf which is also a sheaf. Since it is perhaps better to avoid this terminology when the topology is not subcanonical, we only define it formally in that case.

Definition 7.13.3. Let $\mathcal{C}$ be a site whose topology is subcanonical. The Yoneda embedding h (see Categories, Section 4.3) presents $\mathcal{C}$ as a full subcategory of the category of sheaves of $\mathcal{C}$. In this case we call sheaves of the form h_U with $U \in \mathrm{Ob}(\mathcal{C})$ *representable sheaves* on $\mathcal{C}$. Notation: Sometimes, the representable sheaf h_U associated to U is denoted $\underline{U}$.

Note that we have in the situation of the definition

$$\mathrm{Mor}_{Sh(\mathcal{C})}(h_U, \mathcal{F}) = \mathcal{F}(U)$$

for every sheaf $\mathcal{F}$, since it holds for presheaves, see (7.2.1.1). In general the presheaves h_U are not sheaves and to get a sheaf you have to sheafify them. In this case we still have

$$\mathrm{Mor}_{Sh(\mathcal{C})}(h_U^\#, \mathcal{F}) = \mathrm{Mor}_{PSh(\mathcal{C})}(h_U, \mathcal{F}) = \mathcal{F}(U) \tag{7.13.3.1}$$

for every sheaf $\mathcal{F}$. Namely, the first equality holds by the adjointness property of $\#$ and the second is (7.2.1.1).

Lemma 7.13.4. *Let $\mathcal{C}$ be a site. If $\{U_i \to U\}_{i\in I}$ is a covering of the site $\mathcal{C}$, then the morphism of presheaves of sets*

$$\coprod\nolimits_{i\in I} h_{U_i} \to h_U$$

becomes surjective after sheafification.

Proof. By Lemma 7.12.2 above we have to show that $\coprod_{i\in I} h_{U_i}^\# \to h_U^\#$ is an epimorphism. Let $\mathcal{F}$ be a sheaf of sets. A morphism $h_U^\# \to \mathcal{F}$ corresponds to a section $s \in \mathcal{F}(U)$. Hence the injectivity of $\mathrm{Mor}(h_U^\#, \mathcal{F}) \to \prod_i \mathrm{Mor}(h_{U_i}^\#, \mathcal{F})$ follows directly from the sheaf property of $\mathcal{F}$. □

The next lemma says, in the case the topology is weaker than the canonical topology, that every sheaf is made up out of representable sheaves in a way.

Lemma 7.13.5. *Let $\mathcal{C}$ be a site. Let $E \subset \mathrm{Ob}(\mathcal{C})$ be a subset such that every object of $\mathcal{C}$ has a covering by elements of E. Let $\mathcal{F}$ be a sheaf of sets. There exists a diagram of sheaves of sets*

$$\mathcal{F}_1 \rightrightarrows \mathcal{F}_0 \longrightarrow \mathcal{F}$$

which represents $\mathcal{F}$ as a coequalizer, such that $\mathcal{F}_i$, $i = 0, 1$ are coproducts of sheaves of the form $h_U^\#$ with $U \in E$.

Proof. First we show there is an epimorphism $\mathcal{F}_0 \to \mathcal{F}$ of the desired type. Namely, just take

$$\mathcal{F}_0 = \coprod\nolimits_{U\in E, s\in\mathcal{F}(U)} (h_U)^\# \longrightarrow \mathcal{F}$$

Here the arrow restricted to the component corresponding to (U, s) maps the element $\mathrm{id}_U \in h_U^\#(U)$ to the section $s \in \mathcal{F}(U)$. This is an epimorphism according to Lemma 7.12.2 and our condition on E. To construct $\mathcal{F}_1$ first set $\mathcal{G} = \mathcal{F}_0 \times_{\mathcal{F}} \mathcal{F}_0$ and then construct an epimorphism $\mathcal{F}_1 \to \mathcal{G}$ as above. See Lemma 7.12.3. □

7.14. Continuous functors

Definition 7.14.1. Let $\mathcal{C}$ and $\mathcal{D}$ be sites. A functor $u : \mathcal{C} \to \mathcal{D}$ is called *continuous* if for every $\{V_i \to V\}_{i\in I} \in \mathrm{Cov}(\mathcal{C})$ we have the following

(1) $\{u(V_i) \to u(V)\}_{i\in I}$ is in $\mathrm{Cov}(\mathcal{D})$, and
(2) for any morphism $T \to V$ in $\mathcal{C}$ the morphism $u(T \times_V V_i) \to u(T) \times_{u(V)} u(V_i)$ is an isomorphism.

Recall that given a functor u as above, and a presheaf of sets $\mathcal{F}$ on $\mathcal{D}$ we have defined $u^p\mathcal{F}$ to be simply the presheaf $\mathcal{F} \circ u$, in other words

$$u^p\mathcal{F}(V) = \mathcal{F}(u(V))$$

for every object V of $\mathcal{C}$.

Lemma 7.14.2. *Let $\mathcal{C}$ and $\mathcal{D}$ be sites. Let $u : \mathcal{C} \to \mathcal{D}$ be a continuous functor. If $\mathcal{F}$ is a sheaf on $\mathcal{D}$ then $u^p\mathcal{F}$ is a sheaf as well.*

Proof. Let $\{V_i \to V\}$ be a covering. By assumption $\{u(V_i) \to u(V)\}$ is a covering in $\mathcal{D}$ and $u(V_i \times_V V_j) = u(V_i) \times_{u(V)} u(V_j)$. Hence the sheaf condition for $u^p\mathcal{F}$ and the covering $\{V_i \to V\}$ is precisely the same as the sheaf condition for $\mathcal{F}$ and the covering $\{u(V_i) \to u(V)\}$. □

In order to avoid confusion we sometimes denote

$$u^s : Sh(\mathcal{D}) \longrightarrow Sh(\mathcal{C})$$

the functor u^p restricted to the subcategory of sheaves of sets.

Lemma 7.14.3. *In the situation of Lemma 7.14.2. The functor $u_s : \mathcal{G} \mapsto (u_p\mathcal{G})^\#$ is a left adjoint to u^s.*

Proof. Follows directly from Lemma 7.5.4 and Proposition 7.10.12. □

Here is a technical lemma.

Lemma 7.14.4. *In the situation of Lemma 7.14.2. For any presheaf $\mathcal{G}$ on $\mathcal{C}$ we have $(u_p\mathcal{G})^\# = (u_p(\mathcal{G}^\#))^\#$.*

Proof. For any sheaf $\mathcal{F}$ on $\mathcal{D}$ we have

$$\begin{aligned}
\mathrm{Mor}_{Sh(\mathcal{D})}(u_s(\mathcal{G}^\#), \mathcal{F}) &= \mathrm{Mor}_{Sh(\mathcal{C})}(\mathcal{G}^\#, u^s\mathcal{F}) \\
&= \mathrm{Mor}_{PSh(\mathcal{C})}(\mathcal{G}^\#, u^p\mathcal{F}) \\
&= \mathrm{Mor}_{PSh(\mathcal{C})}(\mathcal{G}, u^p\mathcal{F}) \\
&= \mathrm{Mor}_{PSh(\mathcal{D})}(u_p\mathcal{G}, \mathcal{F}) \\
&= \mathrm{Mor}_{Sh(\mathcal{D})}((u_p\mathcal{G})^\#, \mathcal{F})
\end{aligned}$$

and the result follows from the Yoneda lemma. □

Lemma 7.14.5. *Let $u : \mathcal{C} \to \mathcal{D}$ be a continuous functor between sites. For any object U of $\mathcal{C}$ we have $u_s h_U^\# = h_{u(U)}^\#$.*

Proof. Follows from Lemmas 7.5.6 and 7.14.4. □

Remark 7.14.6. (Skip on first reading.) Let $\mathcal{C}$ and $\mathcal{D}$ be sites. Let us use the definition of tautologically equivalent families of maps, see Definition 7.8.2 to (slightly) weaken the conditions defining continuity. Let $u : \mathcal{C} \to \mathcal{D}$ be a functor. Let us call u *quasi-continuous* if for every $\mathcal{V} = \{V_i \to V\}_{i \in I} \in \mathrm{Cov}(\mathcal{C})$ we have the following

(1') the family of maps $\{u(V_i) \to u(V)\}_{i \in I}$ is tautologically equivalent to an element of $\mathrm{Cov}(\mathcal{D})$, and
(2) for any morphism $T \to V$ in $\mathcal{C}$ the morphism $u(T \times_V V_i) \to u(T) \times_{u(V)} u(V_i)$ is an isomorphism.

We are going to see that Lemmas 7.14.2 and 7.14.3 hold in case u is quasi-continuous as well.

We first remark that the morphisms $u(V_i) \to u(V)$ are representable, since they are isomorphic to representable morphisms (by the first condition). In particular, the family $u(\mathcal{V}) = \{u(V_i) \to u(V)\}_{i \in I}$ gives rise to a zeroth Cech cohomology group $H^0(u(\mathcal{V}), \mathcal{F})$ for any presheaf $\mathcal{F}$ on $\mathcal{D}$. Let $\mathcal{U} = \{U_j \to u(V)\}_{j \in J}$ be an element of $\mathrm{Cov}(\mathcal{D})$ tautologically equivalent to $\{u(V_i) \to u(V)\}_{i \in I}$. Note that $u(\mathcal{V})$ is a refinement of $\mathcal{U}$ and vice versa. Hence by Remark 7.10.7 we see that $H^0(u(\mathcal{V}), \mathcal{F}) = H^0(\mathcal{U}, \mathcal{F})$. In particular, if $\mathcal{F}$ is a sheaf, then $\mathcal{F}(u(V)) = H^0(u(\mathcal{V}), \mathcal{F})$ because of the sheaf property expressed in terms of zeroth Cech cohomology groups. We conclude that $u^p\mathcal{F}$ is a sheaf if $\mathcal{F}$ is a sheaf, since $H^0(\mathcal{V}, u^p\mathcal{F}) = H^0(u(\mathcal{V}), \mathcal{F})$ which we just observed is equal to $\mathcal{F}(u(V)) = u^p\mathcal{F}(V)$. Thus Lemma 7.14.2 holds. Lemma 7.14.3 follows immediately.

7.15. Morphisms of sites

Definition 7.15.1. Let $\mathcal{C}$ and $\mathcal{D}$ be sites. A *morphism of sites* $f : \mathcal{D} \to \mathcal{C}$ is given by a continuous functor $u : \mathcal{C} \to \mathcal{D}$ such that the functor u_s is exact.

Notice how the functor u goes in the direction *opposite* the morphism f. If $f \leftrightarrow u$ is a morphism of sites then we use the notation $f^{-1} = u_s$ and $f_* = u^s$. The functor f^{-1} is called the *pullback functor* and the functor f_* is called the *pushforward functor*. As in topology we have the following adjointness property

$$\mathrm{Mor}_{Sh(\mathcal{D})}(f^{-1}\mathcal{G}, \mathcal{F}) = \mathrm{Mor}_{Sh(\mathcal{C})}(\mathcal{G}, f_*\mathcal{F})$$

The motivation for this definition comes from the following example.

Example 7.15.2. Let $f : X \to Y$ be a continuous map of topological spaces. Recall that we have sites X_{Zar} and Y_{Zar}, see Example 7.6.4. Consider the functor $u : Y_{Zar} \to X_{Zar}$, $V \mapsto f^{-1}(V)$. This functor is clearly continuous because inverse images of open coverings are open coverings. (Actually, this depends on how you chose sets of coverings for X_{Zar} and Y_{Zar}. But in any case the functor is quasi-continuous, see Remark 7.14.6.) It is easy to verify that the functor u^s equals the usual pushforward functor f_* from topology. Hence, since u_s is an adjoint and since the usual topological pullback functor f^{-1} is an adjoint as well, we get a canonical isomorphism $f^{-1} \cong u_s$. Since f^{-1} is exact we deduce that u_s is exact. Hence u defines a morphism of sites $f : X_{Zar} \to Y_{Zar}$, which we may denote f as well since we've already seen the functors u_s, u^s agree with their usual notions anyway.

Lemma 7.15.3. *Let $\mathcal{C}_i$, $i = 1, 2, 3$ be sites. Let $u : \mathcal{C}_2 \to \mathcal{C}_1$ and $v : \mathcal{C}_3 \to \mathcal{C}_2$ be continuous functors which induce morphisms of sites. Then the functor $u \circ v : \mathcal{C}_3 \to \mathcal{C}_1$ is continuous and defines a morphism of sites $\mathcal{C}_1 \to \mathcal{C}_3$.*

Proof. It is immediate from the definitions that $u \circ v$ is a continuous functor. In addition, we clearly have $(u \circ v)^p = v^p \circ u^p$, and hence $(u \circ v)^s = v^s \circ u^s$. Hence functors $(u \circ v)_s$ and $u_s \circ v_s$ are both left adjoints of $(u \circ v)^s$. Therefore $(u \circ v)_s \cong u_s \circ v_s$ and we conclude that $(u \circ v)_s$ is exact as a composition of exact functors. □

Definition 7.15.4. Let $\mathcal{C}_i$, $i = 1, 2, 3$ be sites. Let $f : \mathcal{C}_1 \to \mathcal{C}_2$ and $g : \mathcal{C}_2 \to \mathcal{C}_3$ be morphisms of sites given by continuous functors $u : \mathcal{C}_2 \to \mathcal{C}_1$ and $v : \mathcal{C}_3 \to \mathcal{C}_2$. The *composition* $g \circ f$ is the morphism of sites corresponding to the functor $u \circ v$.

In this situation we have $(g \circ f)_* = g_* \circ f_*$ and $(g \circ f)^{-1} = f^{-1} \circ g^{-1}$ (see proof of Lemma 7.15.3).

Lemma 7.15.5. *Let $\mathcal{C}$ and $\mathcal{D}$ be sites. Let $u : \mathcal{C} \to \mathcal{D}$ be continuous. Assume all the categories $(\mathcal{I}^u_V)^{opp}$ of Section 7.5 are filtered. Then u defines a morphism of sites $\mathcal{D} \to \mathcal{C}$, in other words u_s is exact.*

Proof. Since u_s is the left adjoint of u^s we see that u_s is right exact, see Categories, Lemma 4.24.4. Hence it suffices to show that u_s is left exact. In other words we have to show that u_s commutes with finite limits. Because the categories $\mathcal{I}_Y^{opp}$ are filtered we see that u_p commutes with finite limits, see Categories, Lemma 4.19.2 (this also uses the description of limits in *PSh*, see Section 7.4). And since sheafification commutes with finite limits as well (Lemma 7.10.14) we conclude because $u_s = \# \circ u_p$. □

Proposition 7.15.6. *Let $\mathcal{C}$ and $\mathcal{D}$ be sites. Let $u : \mathcal{C} \to \mathcal{D}$ be continuous. Assume furthermore the following:*

(1) *the category $\mathcal{C}$ has a final object X and $u(X)$ is a final object of $\mathcal{D}$, and*
(2) *the category $\mathcal{C}$ has fibre products and u commutes with them.*

Then u defines a morphism of sites $\mathcal{D} \to \mathcal{C}$, in other words u_s is exact.

Proof. This follows from Lemmas 7.5.2 and 7.15.5. □

Remark 7.15.7. The conditions of Proposition 7.15.6 above are equivalent to saying that u is left exact, i.e., commutes with finite limits. See Categories, Lemmas 4.18.4 and 4.23.2. It seems more natural to phrase it in terms of final objects and fibre products since this seems to have more geometric meaning in the examples.

Lemma 7.18.4 will provide another way to prove a continuous functor gives rise to a morphism of sites.

Remark 7.15.8. (Skip on first reading.) Let $\mathcal{C}$ and $\mathcal{D}$ be sites. Analogously to Definition 7.15.1 we say that a *quasi-morphism of sites* $f : \mathcal{D} \to \mathcal{C}$ is given by a quasi-continuous functor $u : \mathcal{C} \to \mathcal{D}$ (see Remark 7.14.6) such that u_s is exact. The analogue of Proposition 7.15.6 in this setting is obtained by replacing the word "continuous" by the word "quasi-continuous", and replacing the word "morphism" by "quasi-morphism". The proof is literally the same.

In Definition 7.15.1 the condition that u_s be exact cannot be omitted. For example, the conclusion of the following lemma need not hold if one only assumes that u is continuous.

Lemma 7.15.9. *Let $f : \mathcal{D} \to \mathcal{C}$ be a morphism of sites given by the functor $u : \mathcal{C} \to \mathcal{D}$. Given any object V of $\mathcal{D}$ there exists a covering $\{V_j \to V\}$ such that for every j there exists a morphism $V_j \to u(U_j)$ for some object U_j of $\mathcal{C}$.*

Proof. Since $f^{-1} = u_s$ is exact we have $f^{-1}* = *$ where $*$ denotes the final object of the category of sheaves (Example 7.10.2). Since $f^{-1}* = u_s*$ is the sheafification of u_p* we see there exists a covering $\{V_j \to V\}$ such that $(u_p*)(V_j)$ is nonempty. Since $(u_p*)(V_j)$ is a colimit over the category $\mathcal{I}^u_{V_j}$ whose objects are morphisms $V_j \to u(U)$ the lemma follows. □

7.16. Topoi

Here is a definition of a topos which is suitable for our purposes. Namely, a topos is the category of sheaves on a site. In order to specify a topos you just specify the site. The real difference between a topos and a site lies in the definition of morphisms. Namely, it turns out that there are lots of morphisms of topoi which do not come from morphisms of the underlying sites.

Definition 7.16.1 (Topoi)**.** A *topos* is the category $Sh(\mathcal{C})$ of sheaves on a site $\mathcal{C}$.

(1) Let $\mathcal{C}$, $\mathcal{D}$ be sites. A *morphism of topoi* f from $Sh(\mathcal{D})$ to $Sh(\mathcal{C})$ is given by a pair of functors $f_* : Sh(\mathcal{D}) \to Sh(\mathcal{C})$ and $f^{-1} : Sh(\mathcal{C}) \to Sh(\mathcal{D})$ such that
   (a) we have
$$\mathrm{Mor}_{Sh(\mathcal{D})}(f^{-1}\mathcal{G}, \mathcal{F}) = \mathrm{Mor}_{Sh(\mathcal{C})}(\mathcal{G}, f_*\mathcal{F})$$
   bifunctorially, and
   (b) the functor f^{-1} commutes with finite limits, i.e., is left exact.

(2) Let $\mathcal{C}$, $\mathcal{D}$, $\mathcal{E}$ be sites. Given morphisms of topoi $f : Sh(\mathcal{D}) \to Sh(\mathcal{C})$ and $g : Sh(\mathcal{E}) \to Sh(\mathcal{D})$ the *composition* $f \circ g$ is the morphism of topoi defined by the functors $(f \circ g)_* = f_* \circ g_*$ and $(f \circ g)^{-1} = g^{-1} \circ f^{-1}$.

Suppose that $\alpha : \mathcal{S}_1 \to \mathcal{S}_2$ is an equivalence of (possibly "big") categories. If $\mathcal{S}_1$, $\mathcal{S}_2$ are topoi, then setting $f_* = \alpha$ and f^{-1} equal to a quasi-inverse of α gives a morphism $f : \mathcal{S}_1 \to \mathcal{S}_2$ of topoi. Moreover this morphism is an equivalence in the 2-category of topoi (see Section 7.35). Thus it makes sense to say "$\mathcal{S}$ is a topos" if $\mathcal{S}$ is equivalent to the category of sheaves on a site (and not necessarily equal to the category of sheaves on a site). We will occasionally use this abuse of notation.

Two examples of topoi. The *empty topos* is topos of sheaves on the site $\mathcal{C}$, where $\mathcal{C}$ has a single object $\emptyset$ and a single morphism $\mathrm{id}_\emptyset$ and a single covering, namely the empty covering of $\emptyset$. We will sometimes write $\emptyset$ for this site. This is a site and every sheaf on $\mathcal{C}$ assigns a singleton to $\emptyset$. Thus $Sh(\emptyset)$ is equivalent to the category having a single object and a single morphism. The *punctual topos* is the topos of sheaves on the site $\mathcal{C}$ which has a single object pt and one morphism id_{pt} and whose only covering is the covering $\{\mathrm{id}_{pt}\}$. We will simply write pt for this site. It is clear that the category of sheaves = the category of presheaves = the category of sets. In a formula $Sh(pt) = Sets$.

Let $\mathcal{C}$ and $\mathcal{D}$ be sites. Let $f : Sh(\mathcal{D}) \to Sh(\mathcal{C})$ be a morphism of topoi. Note that f_* commutes with all limits and that f^{-1} commutes with all colimits, see Categories, Lemma 4.24.3. In particular, the condition on f^{-1} in the definition above guarantees that f^{-1} is exact. Morphisms of topoi are often constructed using either Lemma 7.20.1 or the following lemma.

Lemma 7.16.2. *Given a morphism of sites $f : \mathcal{D} \to \mathcal{C}$ corresponding to the functor $u : \mathcal{C} \to \mathcal{D}$ the pair of functors $(f^{-1} = u_s, f_* = u^s)$ is a morphism of topoi.*

Proof. This is obvious from Definition 7.15.1. □

Remark 7.16.3. There are many sites that give rise to the topos $Sh(pt)$. A useful example is the following. Suppose that S is a set (of sets) which contains at least one nonempty element. Let $\mathcal{S}$ be the category whose objects are elements of S and whose morphisms are arbitrary set maps. Assume that $\mathcal{S}$ has fibre products. For example this will be the case if $S = \mathcal{P}(\text{infinite set})$ is the power set of any infinite

set (exercise in set theory). Make $\mathcal{S}$ into a site by declaring surjective families of maps to be coverings (and choose a suitable sufficiently large set of covering families as in Sets, Section 3.11). We claim that $Sh(\mathcal{S})$ is equivalent to the category of sets.

We first prove this in case S contains $e \in S$ which is a singleton. In this case, there is an equivalence of topoi $i : Sh(pt) \to Sh(\mathcal{S})$ given by the functors

$$i^{-1}\mathcal{F} = \mathcal{F}(e), \quad i_*E = (U \mapsto \mathrm{Mor}_{Sets}(U, E)) \tag{7.16.3.1}$$

Namely, suppose that $\mathcal{F}$ is a sheaf on $\mathcal{S}$. For any $U \in \mathrm{Ob}(\mathcal{S}) = S$ we can find a covering $\{\varphi_u : e \to U\}_{u \in U}$, where φ_u maps the unique element of e to $u \in U$. The sheaf condition implies in this case that $\mathcal{F}(U) = \prod_{u \in U} \mathcal{F}(e)$. In other words $\mathcal{F}(U) = \mathrm{Mor}_{Sets}(U, \mathcal{F}(e))$. Moreover, this rule is compatible with restriction mappings. Hence the functor

$$i_* : Sets = Sh(pt) \longrightarrow Sh(\mathcal{S}), \quad E \longmapsto (U \mapsto \mathrm{Mor}_{Sets}(U, E))$$

is an equivalence of categories, and its inverse is the functor i^{-1} given above.

If $\mathcal{S}$ does not contain a singleton, then the functor i_* as defined above still makes sense. To show that it is still an equivalence in this case, choose any nonempty $\tilde{e} \in S$ and a map $\varphi : \tilde{e} \to \tilde{e}$ whose image is a singleton. For any sheaf $\mathcal{F}$ set

$$\mathcal{F}(e) := \mathrm{Im}(\mathcal{F}(\varphi) : \mathcal{F}(\tilde{e}) \longrightarrow \mathcal{F}(\tilde{e}))$$

and show that this is a quasi-inverse to i_*. Details omitted.

Remark 7.16.4. (Set theoretical issues related to morphisms of topoi. Skip on a first reading.) A morphism of topoi as defined above is not a set but a class. In other words it is given by a mathematical formula rather than a mathematical object. Although we may contemplate the collection of all morphisms between two given topoi, it is not a good idea to introduce it as a mathematical object. On the other hand, suppose $\mathcal{C}$ and $\mathcal{D}$ are given sites. Consider a functor $\Phi : \mathcal{C} \to Sh(\mathcal{D})$. Such a thing is a set, in other words, it is a mathematical object. We may, in succession, ask the following questions on Φ.

(1) Is it true, given a sheaf $\mathcal{F}$ on $\mathcal{D}$, that the rule $U \mapsto \mathrm{Mor}_{Sh(\mathcal{D})}(\Phi(U), \mathcal{F})$ defines a sheaf on $\mathcal{C}$? If so, this defines a functor $\Phi_* : Sh(\mathcal{D}) \to Sh(\mathcal{C})$.
(2) Is it true that Φ_* has a left adjoint? If so, write Φ^{-1} for this left adjoint.
(3) Is it true that Φ^{-1} is exact?

If the last question still has the answer "yes", then we obtain a morphism of topoi (Φ_*, Φ^{-1}). Moreover, given any morphism of topoi (f_*, f^{-1}) we may set $\Phi(U) = f^{-1}(h_U^\#)$ and obtain a functor Φ as above with $f_* \cong \Phi_*$ and $f^{-1} \cong \Phi^{-1}$ (compatible with adjoint property). The upshot is that by working with the collection of Φ instead of morphisms of topoi, we (a) replaced the notion of a morphism of topoi by a mathematical object, and (b) the collection of Φ forms a class (and not a collection of classes). Of course, more can be said, for example one can work out more precisely the significance of conditions (2) and (3) above; we do this in the case of points of topoi in Section 7.31.

Remark 7.16.5. (Skip on first reading.) Let $\mathcal{C}$ and $\mathcal{D}$ be sites. A quasi-morphism of sites $f : \mathcal{D} \to \mathcal{C}$ (see Remark 7.15.8) gives rise to a morphism of topoi f from $Sh(\mathcal{D})$ to $Sh(\mathcal{C})$ exactly as in Lemma 7.16.2.

7.17. G-sets and morphisms

Let $\varphi : G \to H$ be a homomorphism of groups. Choose (suitable) sites $\mathcal{T}_G$ and $\mathcal{T}_H$ as in Example 7.6.5 and Section 7.9. Let $u : \mathcal{T}_H \to \mathcal{T}_G$ be the functor which assigns to a H-set U the G-set U_φ which has the same underlying set but G action defined by $g \cdot u = \varphi(g)u$. It is clear that u commutes with finite limits and is continuous[4]. Applying Proposition 7.15.6 and Lemma 7.16.2 we obtain a morphism of topoi

$$f : Sh(\mathcal{T}_G) \longrightarrow Sh(\mathcal{T}_H)$$

associated with φ. Using Proposition 7.9.1 we see that we get a pair of adjoint functors

$$f_* : G\text{-}Sets \longrightarrow H\text{-}Sets, \quad f^{-1} : H\text{-}Sets \longrightarrow G\text{-}Sets.$$

Let's work out what are these functors in this case.

We first work out a formula for f_*. Recall that given a G-set S the corresponding sheaf $\mathcal{F}_S$ on $\mathcal{T}_G$ is given by the rule $\mathcal{F}_S(U) = \mathrm{Mor}_G(U, S)$. And on the other hand, given a sheaf $\mathcal{G}$ on $\mathcal{T}_H$ the corresponding H-set is given by the rule $\mathcal{G}({}_H H)$. Hence we see that

$$f_*S = \mathrm{Mor}_{G\text{-}Sets}(({}_H H)_\varphi, S).$$

If we work this out a little bit more then we get

$$f_*S = \{a : H \to S \mid a(gh) = ga(h)\}$$

with left H-action given by $(h \cdot a)(h') = a(h'h)$ for any element $a \in f_*S$.

Next, we explicitly compute f^{-1}. Note that since the topology on $\mathcal{T}_G$ and $\mathcal{T}_H$ is subcanonical, all representable presheaves are sheaves. Moreover, given an object V of $\mathcal{T}_H$ we see that $f^{-1}h_V$ is equal to $h_{u(V)}$ (see Lemma 7.14.5). Hence we see that $f^{-1}S = S_\varphi$ for representable sheaves. Since every sheaf on $\mathcal{T}_H$ is a coproduct of representable sheaves we conclude that this is true in general. Hence we see that for any H-set T we have

$$f^{-1}T = T_\varphi.$$

The adjunction between f^{-1} and f_* is evidenced by the formula

$$\mathrm{Mor}_{G\text{-}Sets}(T_\varphi, S) = \mathrm{Mor}_{H\text{-}Sets}(T, f_*S)$$

with f_*S as above. This can be proved directly. Moreover, it is then clear that (f^{-1}, f_*) form an adjoint pair and that f^{-1} is exact. So alternatively to the above the morphism of topoi $f : G\text{-}Sets \to H\text{-}Sets$ can be defined directly in this manner.

7.18. More functoriality of presheaves

In this section we revisit the material of Section 7.5. Let $u : \mathcal{C} \to \mathcal{D}$ be a functor between categories. Recall that

$$u^p : PSh(\mathcal{D}) \longrightarrow PSh(\mathcal{C})$$

is the functor that associates to $\mathcal{G}$ on $\mathcal{D}$ the presheaf $u^p\mathcal{G} = \mathcal{G} \circ u$. It turns out that this functor not only has a left adjoint (namely u_p) but also a right adjoint.

Namely, for any $V \in \mathrm{Ob}(\mathcal{D})$ we define a category ${}_V\mathcal{I} = {}^u_V\mathcal{I}$. Its objects are pairs $(U, \psi : u(U) \to V)$. Note that the arrow is in the opposite direction from the arrow

[4]Set theoretical remark: First choose $\mathcal{T}_H$. Then choose $\mathcal{T}_G$ to contain $u(\mathcal{T}_H)$ and such that every covering in $\mathcal{T}_H$ corresponds to a covering in $\mathcal{T}_G$. This is possible by Sets, Lemmas 3.10.1, 3.10.2 and 3.11.1.

we used in defining the category $\mathcal{I}_V^u$ in Section 7.5. A morphism $(U, \psi) \to (U', \psi')$ is given by a morphism $\alpha : U \to U'$ such that $\psi = \psi' \circ u(\alpha)$. In addition, given any presheaf of sets $\mathcal{F}$ on $\mathcal{C}$ we introduce the functor ${}_V\mathcal{F} : {}_V\mathcal{I}^{opp} \to Sets$, which is defined by the rule ${}_V\mathcal{F}(U, \psi) = \mathcal{F}(U)$. We define

$$ {}_pu(\mathcal{F})(V) := \lim_{{}_V\mathcal{I}^{opp}} {}_V\mathcal{F} $$

As a limit there are projection maps $c(\psi) : {}_pu(\mathcal{F})(V) \to \mathcal{F}(U)$ for every object (U, ψ) of ${}_V\mathcal{I}$. In fact,

$$ {}_pu(\mathcal{F})(V) = \left\{ \begin{array}{c} \text{collections } s_{(U,\psi)} \in \mathcal{F}(U) \\ \forall \beta : (U_1, \psi_1) \to (U_2, \psi_2) \text{ in } {}_V\mathcal{I} \\ \text{we have } \beta^* s_{(U_2,\psi_2)} = s_{(U_1,\psi_1)} \end{array} \right\} $$

where the correspondence is given by $s \mapsto s_{(U,\psi)} = c(\psi)(s)$. We leave it to the reader to define the restriction mappings ${}_pu(\mathcal{F})(V) \to {}_pu(\mathcal{F})(V')$ associated to any morphism $V' \to V$ of $\mathcal{D}$. The resulting presheaf will be denoted ${}_pu\mathcal{F}$.

Lemma 7.18.1. *There is a canonical map ${}_pu\mathcal{F}(u(U)) \to \mathcal{F}(U)$, which is compatible with restriction maps.*

Proof. This is just the projection map $c(\mathrm{id}_{u(U)})$ above. □

Note that any map of presheaves $\mathcal{F} \to \mathcal{F}'$ gives rise to compatible systems of maps between functors ${}_V\mathcal{F} \to {}_V\mathcal{F}'$, and hence to a map of presheaves ${}_pu\mathcal{F} \to {}_pu\mathcal{F}'$. In other words, we have defined a functor

$$ {}_pu : PSh(\mathcal{C}) \longrightarrow PSh(\mathcal{D}) $$

Lemma 7.18.2. *The functor ${}_pu$ is a right adjoint to the functor u^p. In other words the formula*

$$ \mathrm{Mor}_{PSh(\mathcal{C})}(u^p\mathcal{G}, \mathcal{F}) = \mathrm{Mor}_{PSh(\mathcal{D})}(\mathcal{G}, {}_pu\mathcal{F}) $$

holds bifunctorially in $\mathcal{F}$ and $\mathcal{G}$.

Proof. This is proved in exactly the same way as the proof of Lemma 7.5.4. We note that the map $u^p{}_pu\mathcal{F} \to \mathcal{F}$ from Lemma 7.18.1 is the map that is used to go from the right to the left.

Alternately, think of a presheaf of sets $\mathcal{F}$ on $\mathcal{C}$ as a presheaf $\mathcal{F}'$ on $\mathcal{C}^{opp}$ with values in $Sets^{opp}$, and similarly on $\mathcal{D}$. Check that $({}_pu\mathcal{F})' = u_p(\mathcal{F}')$, and that $(u^p\mathcal{G})' = u^p(\mathcal{G}')$. By Remark 7.5.5 we have the adjointness of u_p and u^p for presheaves with values in $Sets^{opp}$. The result then follows formally from this. □

Thus given a functor $u : \mathcal{C} \to \mathcal{D}$ of categories we obtain a sequence of functors

$$ u_p, u^p, {}_pu $$

between categories of presheaves where in each consequtive pair the first is left adjoint to the second.

Lemma 7.18.3. *Let $u : \mathcal{C} \to \mathcal{D}$ and $v : \mathcal{D} \to \mathcal{C}$ be functors of categories. Assume that v is right adjoint to u. Then we have*

1. *$u^p h_V = h_{v(V)}$ for any V in $\mathcal{D}$,*
2. *the category $\mathcal{I}_U^v$ has an initial object,*
3. *the category ${}_V^u\mathcal{I}$ has a final object,*
4. *${}_pu = v^p$, and*
5. *$u^p = v_p$.*

Proof. Proof of (1). Let V be an object of $\mathcal{D}$. We have $u^p h_V = h_{v(V)}$ because $u^p h_V(U) = \mathrm{Mor}_{\mathcal{D}}(u(U), V) = \mathrm{Mor}_{\mathcal{C}}(U, v(V))$ by assumption.

Proof of (2). Let U be an object of $\mathcal{C}$. Let $\eta : U \to v(u(U))$ be the map adjoint to the map $\mathrm{id} : u(U) \to u(U)$. Then we claim $(u(U), \eta)$ is an initial object of $\mathcal{I}_U^v$. Namely, given an object $(V, \phi : U \to v(V))$ of $\mathcal{I}_U^v$ the morphism ϕ is adjoint to a map $\psi : u(U) \to V$ which then defines a morphism $(u(U), \eta) \to (V, \phi)$.

Proof of (3). Let V be an object of $\mathcal{D}$. Let $\xi : u(v(V)) \to V$ be the map adjoint to the map $\mathrm{id} : v(V) \to v(V)$. Then we calim $(v(V), \xi)$ is a final object of ${}_V^u\mathcal{I}$. Namely, given an object $(U, \psi : u(U) \to V)$ of ${}_V^u\mathcal{I}$ the morphism ψ is adjoint to a map $\phi : U \to v(V)$ which then defines a morphism $(U, \psi) \to (v(V), \xi)$.

Hence for any presheaf $\mathcal{F}$ on $\mathcal{C}$ we have

$$\begin{aligned} v^p\mathcal{F}(V) &= \mathcal{F}(v(V)) \\ &= \mathrm{Mor}_{PSh(\mathcal{C})}(h_{v(V)}, \mathcal{F}) \\ &= \mathrm{Mor}_{PSh(\mathcal{C})}(u^p h_V, \mathcal{F}) \\ &= \mathrm{Mor}_{PSh(\mathcal{D})}(h_V, {}_pu\mathcal{F}) \\ &= {}_pu\mathcal{F}(V) \end{aligned}$$

which proves part (2). Part (3) follows by the uniqueness of adjoint functors. □

Lemma 7.18.4. *A continuous functor of sites which has a continuous left adjoint defines a morphism of sites.*

Proof. Let $u : \mathcal{C} \to \mathcal{D}$ be a continuous functor of sites. Let $w : \mathcal{D} \to \mathcal{C}$ be a continuous left adjoint. Then $u_p = w^p$ by Lemma 7.18.3. Hence $u_s = w^s$ has a left adjoint, namely w_s (Lemma 7.14.3). Thus u_s has both a right and a left adjoint, whence is exact (Categories, Lemma 4.24.4). □

7.19. Cocontinuous functors

There is another way to construct morphisms of topoi. This involves using cocontinuous functors between sites defined as follows.

Definition 7.19.1. Let $\mathcal{C}$ and $\mathcal{D}$ be sites. Let $u : \mathcal{C} \to \mathcal{D}$ be a functor. The functor u is called *cocontinuous* if for every $U \in \mathrm{Ob}(\mathcal{C})$ and every covering $\{V_j \to u(U)\}_{j \in J}$ of $\mathcal{D}$ there exists a covering $\{U_i \to U\}_{i \in I}$ of $\mathcal{C}$ such that the family of maps $\{u(U_i) \to u(U)\}_{i \in I}$ refines the covering $\{V_j \to u(U)\}_{j \in J}$.

Note that $\{u(U_i) \to u(U)\}_{i \in I}$ is in general *not* a covering of the site $\mathcal{D}$.

Lemma 7.19.2. *Let $\mathcal{C}$ and $\mathcal{D}$ be sites. Let $u : \mathcal{C} \to \mathcal{D}$ be cocontinuous. Let $\mathcal{F}$ be a sheaf on $\mathcal{C}$. Then ${}_pu\mathcal{F}$ is a sheaf on $\mathcal{D}$, which we will denote ${}_su\mathcal{F}$.*

Proof. Let $\{V_j \to V\}_{j \in J}$ be a covering of the site $\mathcal{D}$. We have to show that

$$ {}_pu\mathcal{F}(V) \longrightarrow \prod {}_pu\mathcal{F}(V_j) \rightrightarrows \prod {}_pu\mathcal{F}(V_j \times_V V_{j'}) $$

is an equalizer diagram. Since ${}_pu$ is right adjoint to u^p we have

$$ {}_pu\mathcal{F}(V) = \mathrm{Mor}_{PSh(\mathcal{D})}(h_V, {}_pu\mathcal{F}) = \mathrm{Mor}_{PSh(\mathcal{C})}(u^p h_V, \mathcal{F}) = \mathrm{Mor}_{Sh(\mathcal{C})}((u^p h_V)^\#, \mathcal{F}) $$

Hence it suffices to show that

$$ \coprod u^p h_{V_j \times_V V_{j'}} \rightrightarrows \coprod u^p h_{V_j} \longrightarrow u^p h_V \tag{7.19.2.1} $$

becomes a coequalizer diagram after sheafification. (Recall that a coproduct in the category of sheaves is the sheafification of the coproduct in the category of presheaves, see Lemma 7.10.13.)

We first show that the second arrow of (7.19.2.1) becomes surjective after sheafification. To do this we use Lemma 7.12.2. Thus it suffices to show a section s of $u^p h_V$ over U lifts to a section of $\coprod u^p h_{V_j}$ on the members of a covering of U. Note that s is a morphism $s : u(U) \to V$. Then $\{V_j \times_{V,s} u(U) \to u(U)\}$ is a covering of $\mathcal{D}$. Hence, as u is cocontinuous, there is a covering $\{U_i \to U\}$ such that $\{u(U_i) \to u(U)\}$ refines $\{V_j \times_{V,s} u(U) \to u(U)\}$. This means that each restriction $s|_{U_i} : u(U_i) \to V$ factors through a morphism $s_i : u(U_i) \to V_j$ for some j, i.e., $s|_{U_i}$ is in the image of $u^p h_{V_j}(U_i) \to u^p h_V(U_i)$ as desired.

Let $s, s' \in (\coprod u^p h_{V_j})^\#(U)$ map to the same element of $(u^p h_V)^\#(U)$. To finish the proof of the lemma we show that after replacing U by the members of a covering that s, s' are the image of the same section of $\coprod u^p h_{V_j \times_V V_{j'}}$ by the two maps of (7.19.2.1). We may first replace U by the members of a covering and assume that $s \in u^p h_{V_j}(U)$ and $s' \in u^p h_{V_{j'}}(U)$. A second such replacement guarantees that s and s' have the same image in $u^p h_V(U)$ instead of in the sheafification. Hence $s : u(U) \to V_j$ and $s' : u(U) \to V_{j'}$ are morphisms of $\mathcal{D}$ such that

$$\begin{array}{ccc} u(U) & \xrightarrow[s']{} & V_{j'} \\ {\scriptstyle s}\downarrow & & \downarrow \\ V_j & \longrightarrow & V \end{array}$$

is commutative. Thus we obtain $t = (s, s') : u(U) \to V_j \times_V V_{j'}$, i.e., a section $t \in u^p h_{V_j \times_V V_{j'}}(U)$ which maps to s, s' as desired. □

Lemma 7.19.3. *Let $\mathcal{C}$ and $\mathcal{D}$ be sites. Let $u : \mathcal{C} \to \mathcal{D}$ be cocontinuous. The functor $Sh(\mathcal{D}) \to Sh(\mathcal{C})$, $\mathcal{G} \mapsto (u^p \mathcal{G})^\#$ is a left adjoint to the functor ${}_s u$ introduced in Lemma 7.19.2 above. Moreover, it is exact.*

Proof. Let us prove the adjointness property as follows

$$\begin{aligned} \mathrm{Mor}_{Sh(\mathcal{C})}((u^p\mathcal{G})^\#, \mathcal{F}) &= \mathrm{Mor}_{PSh(\mathcal{C})}(u^p\mathcal{G}, \mathcal{F}) \\ &= \mathrm{Mor}_{PSh(\mathcal{D})}(\mathcal{G}, {}_p u\mathcal{F}) \\ &= \mathrm{Mor}_{Sh(\mathcal{D})}(\mathcal{G}, {}_s u\mathcal{F}). \end{aligned}$$

Thus it is a left adjoint and hence right exact, see Categories, Lemma 4.24.4. We have seen that sheafification is left exact, see Lemma 7.10.14. Moreover, the inclusion $i : Sh(\mathcal{D}) \to PSh(\mathcal{D})$ is left exact by Lemma 7.10.1. Finally, the functor u^p is left exact because it is a right adjoint (namely to u_p). Thus the functor is the composition $^\# \circ u^p \circ i$ of left exact functors, hence left exact. □

We finish this section with a technical lemma.

Lemma 7.19.4. *In the situation of Lemma 7.19.3. For any presheaf $\mathcal{G}$ on $\mathcal{D}$ we have $(u^p\mathcal{G})^\# = (u^p(\mathcal{G}^\#))^\#$.*

Proof. For any sheaf $\mathcal{F}$ on $\mathcal{C}$ we have

$$\begin{aligned}
\mathrm{Mor}_{Sh(\mathcal{C})}((u^p(\mathcal{G}^\#))^\#, \mathcal{F}) &= \mathrm{Mor}_{Sh(\mathcal{D})}(\mathcal{G}^\#, {}_su\mathcal{F}) \\
&= \mathrm{Mor}_{Sh(\mathcal{D})}(\mathcal{G}^\#, {}_pu\mathcal{F}) \\
&= \mathrm{Mor}_{PSh(\mathcal{D})}(\mathcal{G}, {}_pu\mathcal{F}) \\
&= \mathrm{Mor}_{PSh(\mathcal{C})}(u^p\mathcal{G}, \mathcal{F}) \\
&= \mathrm{Mor}_{Sh(\mathcal{C})}((u^p\mathcal{G})^\#, \mathcal{F})
\end{aligned}$$

and the result follows from the Yoneda lemma. □

Remark 7.19.5. Let $u : \mathcal{C} \to \mathcal{D}$ be a functor between categories. Given morphisms $g : u(U) \to V$ and $f : W \to V$ in $\mathcal{D}$ we can define consider the functor

$$\mathcal{C}^{opp} \longrightarrow Sets, \quad T \longmapsto \mathrm{Mor}_{\mathcal{C}}(T, U) \times_{\mathrm{Mor}_{\mathcal{D}}(u(T),V)} \mathrm{Mor}_{\mathcal{D}}(u(T), W)$$

If this functor is representable, denote $U \times_{g,V,f} W$ the corresponding object of $\mathcal{C}$. Assume that $\mathcal{C}$ and $\mathcal{D}$ are sites. Consider the property P: for every covering $\{f_j : V_j \to V\}$ of $\mathcal{D}$ and any morphism $g : u(U) \to V$ we have

(1) $U \times_{g,V,f_i} V_i$ exists for all i, and
(2) $\{U \times_{g,V,f_i} V_i \to U\}$ is a covering of $\mathcal{C}$.

Please note the similarity with the definition of continuous functors. If u has P then u is cocontinuous (details omitted). Many of the cocontinuous functors we will encounter satisfy P.

7.20. Cocontinuous functors and morphisms of topoi

It is clear from the above that a cocontinuous functor u gives a morphism of topoi in the same direction as u. Thus this is in the opposite direction from the morphism of topoi associated (under certain conditions) to a continuous u as in Definition 7.15.1, Proposition 7.15.6, and Lemma 7.16.2.

Lemma 7.20.1. *Let $\mathcal{C}$ and $\mathcal{D}$ be sites. Let $u : \mathcal{C} \to \mathcal{D}$ be cocontinuous. The functors $g_* = {}_su$ and $g^{-1} = (u^p\ )^\#$ define a morphism of topoi g from $Sh(\mathcal{C})$ to $Sh(\mathcal{D})$.*

Proof. This is exactly the content of Lemma 7.19.3. □

Lemma 7.20.2. *Let $u : \mathcal{C} \to \mathcal{D}$, and $v : \mathcal{D} \to \mathcal{E}$ be cocontinuous functors. Then $v \circ u$ is cocontinuous and we have $h = g \circ f$ where $f : Sh(\mathcal{C}) \to Sh(\mathcal{D})$, resp. $g : Sh(\mathcal{D}) \to Sh(\mathcal{E})$, resp. $h : Sh(\mathcal{C}) \to Sh(\mathcal{E})$ is the morphism of topoi associated to u, resp. v, resp. $v \circ u$.*

Proof. Let $U \in \mathrm{Ob}(\mathcal{C})$. Let $\{E_i \to v(u(U))\}$ be a covering of U in $\mathcal{E}$. By assumption there exists a covering $\{D_j \to u(U)\}$ in $\mathcal{D}$ such that $\{v(D_j) \to v(u(U))\}$ refines $\{E_i \to v(u(U))\}$. Also by assumption there exists a covering $\{C_l \to U\}$ in $\mathcal{C}$ such that $\{u(C_l) \to u(U)\}$ refines $\{D_j \to u(U)\}$. Then it is true that $\{v(u(C_l)) \to v(u(U))\}$ refines the covering $\{E_i \to v(u(U))\}$. This proves that $v \circ u$ is cocontinuous. To prove the last assertion it suffices to show that ${}_sv \circ {}_su = {}_s(v \circ u)$. It suffices to prove that ${}_pv \circ {}_pu = {}_p(v \circ u)$, see Lemma 7.19.2. Since ${}_pu$, resp. ${}_pv$, resp. ${}_p(v \circ u)$ is right adjoint to u^p, resp. v^p, resp. $(v \circ u)^p$ it suffices to prove that $u^p \circ v^p = (v \circ u)^p$. And this is direct from the definitions. □

Example 7.20.3. Let X be a topological space. Let $j : U \to X$ be the inclusion of an open subspace. Recall that we have sites X_{Zar} and U_{Zar}, see Example 7.6.4. Recall that we have the functor $u : X_{Zar} \to U_{Zar}$ associated to j which is continuous and gives rise to a morphism of sites $U_{Zar} \to X_{Zar}$, see Example 7.15.2. This also gives a morphism of topoi (j_*, j^{-1}). Next, consider the functor $v : U_{Zar} \to X_{Zar}$, $V \mapsto v(V) = V$ (just the same open but now thought of as an object of X_{Zar}). This functor is cocontinuous. Namely, if $v(V) = \bigcup_{j\in J} W_j$ is an open covering in X, then each W_j must be a subset of U and hence is of the form $v(V_j)$, and trivially $V = \bigcup_{j\in J} V_j$ is an open covering in U. We conclude by Lemma 7.20.1 above that there is a morphism of topoi associated to v

$$Sh(U) \longrightarrow Sh(X)$$

given by ${}_sv$ and $(v^p\ )^\#$. We claim that actually $(v^p\ )^\# = j^{-1}$ and that ${}_sv = j_*$, in other words, that this is the same morphism of topoi as the one given above. Perhaps the easiest way to see this is to realize that for any sheaf $\mathcal{G}$ on X we have $v^p\mathcal{G}(V) = \mathcal{G}(V)$ which according to Sheaves, Lemma 6.31.1 is a description of $j^{-1}\mathcal{G}$ (and hence sheafification is superfluous in this case). The equality of ${}_sv$ and j_* follows by uniqueness of adjoint functors (but may also be computed directly).

Example 7.20.4. This example is a slight generalization of Example 7.20.3. Let $f : X \to Y$ be a continuous map of topological spaces. Assume that f is open. Recall that we have sites X_{Zar} and Y_{Zar}, see Example 7.6.4. Recall that we have the functor $u : Y_{Zar} \to X_{Zar}$ associated to f which is continuous and gives rise to a morphism of sites $X_{Zar} \to Y_{Zar}$, see Example 7.15.2. This also gives a morphism of topoi (f_*, f^{-1}). Next, consider the functor $v : X_{Zar} \to Y_{Zar}$, $U \mapsto v(U) = f(U)$. This functor is cocontinuous. Namely, if $f(U) = \bigcup_{j\in J} V_j$ is an open covering in Y, then setting $U_j = f^{-1}(V_j) \cap U$ we get an open covering $U = \bigcup U_j$ such that $f(U) = \bigcup f(U_j)$ is a refinement of $f(U) = \bigcup V_j$. We conclude by Lemma 7.20.1 above that there is a morphism of topoi associated to v

$$Sh(X) \longrightarrow Sh(Y)$$

given by ${}_sv$ and $(v^p\ )^\#$. We claim that actually $(v^p\ )^\# = f^{-1}$ and that ${}_sv = f_*$, in other words, that this is the same morphism of topoi as the one given above. For any sheaf $\mathcal{G}$ on Y we have $v^p\mathcal{G}(U) = \mathcal{G}(f(U))$. On the other hand, we may compute $u_p\mathcal{G}(U) = \operatorname{colim}_{f(U)\subset V} \mathcal{G}(V) = \mathcal{G}(f(U))$ because clearly $(f(U), U \subset f^{-1}(f(U)))$ is an initial object of the category $\mathcal{I}^u_U$ of Section 7.5. Hence $u_p = v^p$ and we conclude $f^{-1} = u_s = (v^p\ )^\#$. The equality of ${}_sv$ and f_* follows by uniqueness of adjoint functors (but may also be computed directly).

In the first Example 7.20.3 the functor v is also continuous. But in the second Example 7.20.4 it is generally not continuous because condition (2) of Definition 7.14.1 may fail. Hence the following lemma applies to the first example, but not to the second.

Lemma 7.20.5. *Let $\mathcal{C}$ and $\mathcal{D}$ be sites. Let $u : \mathcal{C} \to \mathcal{D}$ be a functor. Assume that*

(a) *u is cocontinuous, and*
(b) *u is continuous.*

Let $g : Sh(\mathcal{C}) \to Sh(\mathcal{D})$ be the associated morphism of topoi. Then

(1) *sheafification in the formula $g^{-1} = (u^p\ )^\#$ is unnecessary, in other words $g^{-1}(\mathcal{G})(U) = \mathcal{G}(u(U))$,*

(2) g^{-1} *has a left adjoint* $g_! = (u_p\ )^\#$, *and*
(3) g^{-1} *commutes with arbitrary limits and colimits.*

Proof. By Lemma 7.14.2 for any sheaf $\mathcal{G}$ on $\mathcal{D}$ the presheaf $u^p\mathcal{G}$ is a sheaf on $\mathcal{C}$. And then we see the adjointness by the following string of equalities

$$\begin{aligned} \mathrm{Mor}_{Sh(\mathcal{C})}(\mathcal{F}, g^{-1}\mathcal{G}) &= \mathrm{Mor}_{PSh(\mathcal{C})}(\mathcal{F}, u^p\mathcal{G}) \\ &= \mathrm{Mor}_{PSh(\mathcal{D})}(u_p\mathcal{F}, \mathcal{G}) \\ &= \mathrm{Mor}_{Sh(\mathcal{D})}(g_!\mathcal{F}, \mathcal{G}) \end{aligned}$$

The statement on limits and colimits follows from the discussion in Categories, Section 4.24. □

In the situation of Lemma 7.20.5 above we see that we have a sequence of adjoint functors

$$g_!,\ g^{-1},\ g_*.$$

The functor $g_!$ is *not* exact in general, because it does not transform a final object of $Sh(\mathcal{C})$ into a final object of $Sh(\mathcal{D})$ in general. See Sheaves, Remark 6.31.13. On the other hand, in the topological setting of Example 7.20.3 the functor $j_!$ is exact on abelian sheaves, see Modules, Lemma 17.3.4. The following lemma gives the generalization to the case of sites.

Lemma 7.20.6. *Let $\mathcal{C}$ and $\mathcal{D}$ be sites. Let $u : \mathcal{C} \to \mathcal{D}$ be a functor. Assume that*

(a) *u is cocontinuous,*
(b) *u is continuous, and*
(c) *fibre products and equalizers exist in $\mathcal{C}$ and u commutes with them.*

In this case the functor $g_!$ above commutes with fibre products and equalizers (and more generally with finite connected limits).

Proof. Assume (a), (b), and (c). We have $g_! = (u_p\ )^\#$. Recall (Lemma 7.10.1) that limits of sheaves are equal to the corresponding limits as presheaves. And sheafification commutes with finite limits (Lemma 7.10.14). Thus it suffices to show that u_p commutes with fibre products and equalizers. To do this it suffices that colimits over the categories $(\mathcal{I}^u_V)^{opp}$ of Section 7.5 commute with fibre products and equalizers. This follows from Lemma 7.5.1 and Categories, Lemma 4.19.8. □

The following lemma deals with a case that is even more like the morphism associated to an open immersion of topological spaces.

Lemma 7.20.7. *Let $\mathcal{C}$ and $\mathcal{D}$ be sites. Let $u : \mathcal{C} \to \mathcal{D}$ be a functor. Assume that*

(a) *u is cocontinuous,*
(b) *u is continuous, and*
(c) *u is fully faithful.*

For $g_!, g^{-1}, g_*$ as above the canonical maps $\mathcal{F} \to g^{-1}g_!\mathcal{F}$ and $g^{-1}g_*\mathcal{F} \to \mathcal{F}$ are isomorphisms for all sheaves $\mathcal{F}$ on $\mathcal{C}$.

Proof. Let X be an object of $\mathcal{C}$. In Lemmas 7.19.2 and 7.20.5 we have seen that sheafification is not necessary for the functors $g^{-1} = (u^p\ )^\#$ and $g_* = ({}_pu\ )^\#$. We may compute $(g^{-1}g_*\mathcal{F})(X) = g_*\mathcal{F}(u(X)) = \lim \mathcal{F}(Y)$. Here the limit is over the category of pairs $(Y, u(Y) \to u(X))$ where the morphisms $u(Y) \to u(X)$ are not required to be of the form $u(\alpha)$ with α a morphism of $\mathcal{C}$. By assumption (c) we see

that they automatically come from morphisms of $\mathcal{C}$ and we deduce that the limit is the value on $(X, u(\mathrm{id}_X))$, i.e., $\mathcal{F}(X)$. This proves that $g^{-1}g_*\mathcal{F} = \mathcal{F}$.

On the other hand, $(g^{-1}g_!\mathcal{F})(X) = g_!\mathcal{F}(u(X)) = (u_p\mathcal{F})^\#(u(X))$, and $u_p\mathcal{F}(u(X)) = \operatorname{colim} \mathcal{F}(Y)$. Here the colimit is over the category of pairs $(Y, u(X) \to u(Y))$ where the morphisms $u(X) \to u(Y)$ are not required to be of the form $u(\alpha)$ with α a morphism of $\mathcal{C}$. By assumption (c) we see that they automatically come from morphisms of $\mathcal{C}$ and we deduce that the colimit is the value on $(X, u(\mathrm{id}_X))$, i.e., $\mathcal{F}(X)$. Thus for every $X \in \mathrm{Ob}(\mathcal{C})$ we have $u_p\mathcal{F}(u(X)) = \mathcal{F}(X)$. Since u is cocontinuous and continuous any covering of $u(X)$ in $\mathcal{D}$ can be refined by a covering (!) $\{u(X_i) \to u(X)\}$ of $\mathcal{D}$ where $\{X_i \to X\}$ is a covering in $\mathcal{C}$. This implies that $(u_p\mathcal{F})^+(u(X)) = \mathcal{F}(X)$ also, since in the colimit defining the value of $(u_p\mathcal{F})^+$ on $u(X)$ we may restrict to the cofinal system of coverings $\{u(X_i) \to u(X)\}$ as above. Hence we see that $(u_p\mathcal{F})^+(u(X)) = \mathcal{F}(X)$ for all objects X of $\mathcal{C}$ as well. Repeating this argument one more time gives the equality $(u_p\mathcal{F})^\#(u(X)) = \mathcal{F}(X)$ for all objects X of $\mathcal{C}$. This produces the desired equality $g^{-1}g_!\mathcal{F} = \mathcal{F}$. □

Finally, here is a case that does not have any corresponding topological example. We will use this lemma to see what happens when we enlarge a "partial universe" of schemes keeping the same topology. In the situation of the lemma, the morphism of topoi $g : Sh(\mathcal{C}) \to Sh(\mathcal{D})$ identifies $Sh(\mathcal{C})$ as a subtopos of $Sh(\mathcal{D})$ (Section 7.42) and moreover, the given embedding has a retraction.

Lemma 7.20.8. *Let $\mathcal{C}$ and $\mathcal{D}$ be sites. Let $u : \mathcal{C} \to \mathcal{D}$ be a functor. Assume that*

(a) *u is cocontinuous,*
(b) *u is continuous,*
(c) *u is fully faithful,*
(d) *fibre products exist in $\mathcal{C}$ and u commutes with them, and*
(e) *there exist final objects $e_\mathcal{C} \in \mathrm{Ob}(\mathcal{C})$, $e_\mathcal{D} \in \mathrm{Ob}(\mathcal{D})$ such that $u(e_\mathcal{C}) = e_\mathcal{D}$.*

Let $g_!, g^{-1}, g_*$ be as above. Then, u defines a morphism of sites $f : \mathcal{D} \to \mathcal{C}$ with $f_* = g^{-1}$, $f^{-1} = g_!$. The composition

$$Sh(\mathcal{C}) \xrightarrow{g} Sh(\mathcal{D}) \xrightarrow{f} Sh(\mathcal{C})$$

is isomorphic to the identity morphism of the topos $Sh(\mathcal{C})$. Moreover, the functor f^{-1} is fully faithful.

Proof. By assumption the functor u satisfies the hypotheses of Proposition 7.15.6. Hence u defines a morphism of sites and hence a morphism of topoi f as in Lemma 7.16.2. The formulas $f_* = g^{-1}$ and $f^{-1} = g_!$ are clear from the lemma cited and Lemma 7.20.5. We have $f_* \circ g_* = g^{-1} \circ g_* \cong \mathrm{id}$, and $g^{-1} \circ f^{-1} = g^{-1} \circ g_! \cong \mathrm{id}$ by Lemma 7.20.7.

We still have to show that f^{-1} is fully faithful. Let $\mathcal{F}, \mathcal{G} \in \mathrm{Ob}(Sh(\mathcal{C}))$. We have to show that the map

$$\mathrm{Mor}_{Sh(\mathcal{C})}(\mathcal{F}, \mathcal{G}) \longrightarrow \mathrm{Mor}_{Sh(\mathcal{D})}(f^{-1}\mathcal{F}, f^{-1}\mathcal{G})$$

is bijective. But the right hand side is equal to

$$\begin{aligned} \mathrm{Mor}_{Sh(\mathcal{D})}(f^{-1}\mathcal{F}, f^{-1}\mathcal{G}) &= \mathrm{Mor}_{Sh(\mathcal{C})}(f_*f^{-1}\mathcal{F}, \mathcal{G}) \\ &= \mathrm{Mor}_{Sh(\mathcal{C})}(g^{-1}f^{-1}\mathcal{F}, \mathcal{G}) \\ &= \mathrm{Mor}_{Sh(\mathcal{C})}(\mathcal{F}, \mathcal{G}) \end{aligned}$$

(the first equality by adjunction) which proves what we want. □

Example 7.20.9. Let X be a topological space. Let $i : Z \to X$ be the inclusion of a subset (with induced topology). Consider the functor $u : X_{Zar} \to Z_{Zar}$, $U \mapsto u(U) = Z \cap U$. At first glance it may appear that this functor is cocontinuous as well. After all, since Z has the induced topology, shouldn't any covering of $U \cap Z$ it come from a covering of U in X? Not so! Namely, what if $U \cap Z = \emptyset$? In that case, the empty covering is a covering of $U \cap Z$, and the empty covering can only be refined by the empty covering. Thus we conclude that u cocontinuous $\Rightarrow$ every nonempty open U of X has nonempty intersection with Z. But this is not sufficient. For example, if $X = \mathbf{R}$ the real number line with the usual topology, and $Z = \mathbf{R} \setminus \{0\}$, then there is an open covering of Z, namely $Z = \{x < 0\} \cup \bigcup_n \{1/n < x\}$ which cannot be refined by the restriction of any open covering of X.

7.21. Cocontinuous functors which have a right adjoint

It may happen that a cocontinuous functor u has a right adjoint v. In this case it is often the case that v is continuous, and if so, then it defines a morphism of topoi (which is the same as the one defined by u).

Lemma 7.21.1. *Let $\mathcal{C}$ and $\mathcal{D}$ be sites. Let $u : \mathcal{C} \to \mathcal{D}$, and $v : \mathcal{D} \to \mathcal{C}$ be functors. Assume that u is cocontinuous, and that v is a right adjoint to u. Let $g : Sh(\mathcal{C}) \to Sh(\mathcal{D})$ be the morphism of topoi associated to u, see Lemma 7.20.1. Then $g_*\mathcal{F}$ is equal to the presheaf $v^p\mathcal{F}$, in other words, $(g_*\mathcal{F})(V) = \mathcal{F}(v(V))$.*

Proof. We have $u^p h_V = h_{v(V)}$ by Lemma 7.18.3. By Lemma 7.19.4 this implies that $g^{-1}(h_V^\#) = (u^p h_V^\#)^\# = (u^p h_V)^\# = h_{v(V)}^\#$. Hence for any sheaf $\mathcal{F}$ on $\mathcal{C}$ we have

$$\begin{aligned}(g_*\mathcal{F})(V) &= \mathrm{Mor}_{Sh(\mathcal{D})}(h_V^\#, g_*\mathcal{F}) \\ &= \mathrm{Mor}_{Sh(\mathcal{C})}(g^{-1}(h_V^\#), \mathcal{F}) \\ &= \mathrm{Mor}_{Sh(\mathcal{C})}(h_{v(V)}^\#, \mathcal{F}) \\ &= \mathcal{F}(v(V))\end{aligned}$$

which proves the lemma. □

In the situation of Lemma 7.21.1 we see that v^p transforms sheaves into sheaves. Hence we can define $v^s = v^p$ restricted to sheaves. Just as in Lemma 7.14.3 we see that $v_s : \mathcal{G} \mapsto (v_p\mathcal{G})^\#$ is a left adjoint to v^s. On the other hand, we have $v^s = g_*$ and g^{-1} is a left adjoint of g_* as well. We conclude that $g^{-1} = v_s$ is exact.

Lemma 7.21.2. *In the situation of Lemma 7.21.1. We have $g_* = v^s = v^p$ and $g^{-1} = v_s = (v_p\ )^\#$. If v is continuous then v defines a morphism of sites f from $\mathcal{C}$ to $\mathcal{D}$ whose associated morphism of topoi is equal to the morphism g associated to the cocontinuous functor u. In other words, a continuous functor which has a cocontinuous left adjoint defines a morphism of sites.*

Proof. Clear from the discussion above the lemma and Definitions 7.15.1 and Lemma 7.16.2. □

7.22. Cocontinuous functors which have a left adjoint

It may happen that a cocontinuous functor u has a left adjoint w.

Lemma 7.22.1. *Let $\mathcal{C}$ and $\mathcal{D}$ be sites. Let $g : Sh(\mathcal{C}) \to Sh(\mathcal{D})$ be the morphism of topoi associated to a continuous and cocontinuous functor $u : \mathcal{C} \to \mathcal{D}$, see Lemmas 7.20.1 and 7.20.5.*

1. *If $w : \mathcal{D} \to \mathcal{C}$ is a left adjoint to u, then*
   (a) *$g_!\mathcal{F}$ is the sheaf associated to the presheaf $w^p\mathcal{F}$, and*
   (b) *$g_!$ is exact.*
2. *if w is a continuous left adjoint, then $g_!$ has a left adjoint.*
3. *If w is a cocontinuous left adjoint, then $g_! = h^{-1}$ and $g^{-1} = h_*$ where $h : Sh(\mathcal{D}) \to Sh(\mathcal{C})$ is the morphism of topoi associated to w.*

Proof. Recall that $g_!\mathcal{F}$ is the sheafification of $u_p\mathcal{F}$. Hence (1)(a) follows from the fact that $u_p = w^p$ by Lemma 7.18.3.

To see (1)(b) note that $g_!$ commutes with all colimits as $g_!$ is a left adjoint (Categories, Lemma 4.24.3). Let $i \mapsto \mathcal{F}_i$ be a finite diagram in $Sh(\mathcal{C})$. Then $\lim \mathcal{F}_i$ is computed in the category of presheaves (Lemma 7.10.1). Since w^p is a right adjoint (Lemma 7.5.4) we see that $w^p \lim \mathcal{F}_i = \lim w^p\mathcal{F}_i$. Since sheafification is exact (Lemma 7.10.14) we conclude by (1)(a).

Assume w is continuous. Then $g_! = (w^p\ )^\# = w^s$ but sheafification isn't necessary and one has the left adjoint w_s, see Lemmas 7.14.2 and 7.14.3.

Assume w is cocontinuous. The equality $g_! = h^{-1}$ follows from (1)(a) and the definitions. The equality $g^{-1} = h_*$ follows from the equality $g_! = h^{-1}$ and uniqueness of adjoint functor. Alternatively one can deduce it from Lemma 7.21.1. □

7.23. Existence of lower shriek

In this section we discuss some cases of morphisms of topoi f for which f^{-1} has a left adjoint $f_!$.

Lemma 7.23.1. *Let $\mathcal{C}$, $\mathcal{D}$ be two sites. Let $f : Sh(\mathcal{D}) \to Sh(\mathcal{C})$ be a morphism of topoi. Let $E \subset \mathrm{Ob}(\mathcal{D})$ be a subset such that*

1. *for $V \in E$ there exists a sheaf $\mathcal{G}$ on $\mathcal{C}$ such that $f^{-1}\mathcal{F}(V) = \mathrm{Mor}_{Sh(\mathcal{C})}(\mathcal{G}, \mathcal{F})$ functorially for $\mathcal{F}$ in $Sh(\mathcal{C})$,*
2. *every object of $\mathcal{D}$ has a covering by objects of E.*

Then f^{-1} has a left adjoint $f_!$.

Proof. By the Yoneda lemma (Categories, Lemma 4.3.5) the sheaf $\mathcal{G}_V$ corresponding to $V \in E$ is defined up to unique isomorphism by the formula $f^{-1}\mathcal{F}(V) = \mathrm{Mor}_{Sh(\mathcal{C})}(\mathcal{G}_V, \mathcal{F})$. Recall that $f^{-1}\mathcal{F}(V) = \mathrm{Mor}_{Sh(\mathcal{D})}(h_V^\#, f^{-1}\mathcal{F})$. Denote $i_V : h_V^\# \to f^{-1}\mathcal{G}_V$ the map corresponding to id in $\mathrm{Mor}(\mathcal{G}_V, \mathcal{G}_V)$. Functoriality in (1) implies that the bijection is given by

$$\mathrm{Mor}_{Sh(\mathcal{C})}(\mathcal{G}_V, \mathcal{F}) \to \mathrm{Mor}_{Sh(\mathcal{D})}(h_V^\#, f^{-1}\mathcal{F}), \quad \varphi \mapsto f^{-1}\varphi \circ i_V$$

For any $V_1, V_2 \in E$ there is a canonical map

$$\mathrm{Mor}_{Sh(\mathcal{D})}(h_{V_2}^\#, h_{V_1}^\#) \to \mathrm{Hom}_{Sh(\mathcal{C})}(\mathcal{G}_{V_2}, \mathcal{G}_{V_1}), \quad \varphi \mapsto f_!(\varphi)$$

which is characterized by $f^{-1}(f_!(\varphi)) \circ i_{V_2} = i_{V_1} \circ \varphi$. Note that $\varphi \mapsto f_!(\varphi)$ is compatible with composition; this can be seen directly from the characterization.

Hence $h_V^\# \mapsto \mathcal{G}_V$ and $\varphi \mapsto f_!\varphi$ is a functor from the full subcategory of $Sh(\mathcal{D})$ whose objects are the $h_V^\#$.

Let J be a set and let $J \to E$, $j \mapsto V_j$ be a map. Then we have a functorial bijection

$$\mathrm{Mor}_{Sh(\mathcal{C})}(\coprod \mathcal{G}_{V_j}, \mathcal{F}) \longrightarrow \mathrm{Mor}_{Sh(\mathcal{D})}(\coprod h_{V_j}^\#, f^{-1}\mathcal{F})$$

using the product of the bijections above. Hence we can extend the functor $f_!$ to the full subcategory of $Sh(\mathcal{D})$ whose objects are coproducts of $h_V^\#$ with $V \in E$.

Given an arbitrary sheaf $\mathcal{H}$ on $\mathcal{D}$ we choose an coequalizer diagram

$$\mathcal{H}_1 \rightrightarrows \mathcal{H}_0 \longrightarrow \mathcal{H}$$

where $\mathcal{H}_i = \coprod h_{V_{i,j}}^\#$ is a coproduct with $V_{i,j} \in E$. This is possible by assumption (2), see Lemma 7.13.5 (for those worried about set theoretical issues, note that the construction given in Lemma 7.13.5 is canonical). Define $f_!(\mathcal{H})$ to be the sheaf on $\mathcal{C}$ which makes

$$f_!\mathcal{H}_1 \rightrightarrows f_!\mathcal{H}_0 \longrightarrow f_!\mathcal{H}$$

Then

$$\begin{aligned}
\mathrm{Mor}(f_!\mathcal{H}, \mathcal{F}) &= \text{Equalizer}(\ \mathrm{Mor}(f_!\mathcal{H}_0, \mathcal{F}) \rightrightarrows \mathrm{Mor}(f_!\mathcal{H}_1, \mathcal{F})\ ) \\
&= \text{Equalizer}(\ \mathrm{Mor}(\mathcal{H}_0, f^{-1}\mathcal{F}) \rightrightarrows \mathrm{Mor}(\mathcal{H}_1, f^{-1}\mathcal{F})\ ) \\
&= \mathrm{Hom}(\mathcal{H}, f^{-1}\mathcal{F})
\end{aligned}$$

Hence we see that we can extend $f_!$ to the whole category of sheaves on $\mathcal{D}$. □

7.24. Localization

Let $\mathcal{C}$ be a site. Let $U \in \mathrm{Ob}(\mathcal{C})$. See Categories, Example 4.2.13 for the definition of the category $\mathcal{C}/U$ of objects over U. We turn $\mathcal{C}/U$ into a site by declaring a family of morphisms $\{V_j \to V\}$ of objects over U to be a covering of $\mathcal{C}/U$ if and only if it is a covering in $\mathcal{C}$. Consider the forgetful functor

$$j_U : \mathcal{C}/U \longrightarrow \mathcal{C}.$$

This is clearly cocontinuous and continuous. Hence by the results of the previous sections we obtain a morphism of topoi

$$j_U : Sh(\mathcal{C}/U) \longrightarrow Sh(\mathcal{C})$$

given by j_U^{-1} and j_{U*}, as well as a functor $j_{U!}$.

Definition 7.24.1. Let $\mathcal{C}$ be a site. Let $U \in \mathrm{Ob}(\mathcal{C})$.

1. The site $\mathcal{C}/U$ is called the *localization of the site $\mathcal{C}$ at the object U*.
2. The morphism of topoi $j_U : Sh(\mathcal{C}/U) \to Sh(\mathcal{C})$ is called the *localization morphism*.
3. The functor j_{U*} is called the *direct image functor*.
4. For a sheaf $\mathcal{F}$ on $\mathcal{C}$ the sheaf $j_U^{-1}\mathcal{F}$ is called the *restriction of $\mathcal{F}$ to $\mathcal{C}/U$*.
5. For a sheaf $\mathcal{G}$ on $\mathcal{C}/U$ the sheaf $j_{U!}\mathcal{G}$ is called the *extension of $\mathcal{G}$ by the empty set*.

The restriction $j_U^{-1}\mathcal{F}$ is the sheaf defined by the rule $j_U^{-1}\mathcal{F}(X/U) = \mathcal{F}(X)$ as expected. The extension by the empty set also has a very easy description in this case; here it is.

Lemma 7.24.2. *Let $\mathcal{C}$ be a site. Let $U \in \mathrm{Ob}(\mathcal{C})$. Let $\mathcal{G}$ be a presheaf on $\mathcal{C}/U$. Then $j_{U!}(\mathcal{G}^\#)$ is the sheaf associated to the presheaf*

$$V \longmapsto \coprod\nolimits_{\varphi \in \mathrm{Mor}_{\mathcal{C}}(V,U)} \mathcal{G}(V \xrightarrow{\varphi} U)$$

with obvious restriction mappings.

Proof. By Lemma 7.20.5 we have $j_{U!}(\mathcal{G}^\#) = ((j_U)_p\mathcal{G}^\#)^\#$. By Lemma 7.14.4 this is equal to $((j_U)_p\mathcal{G})^\#$. Hence it suffices to prove that $(j_U)_p$ is given by the formula above for any presheaf $\mathcal{G}$ on $\mathcal{C}/U$. OK, and by the definition in Section 7.5 we have

$$(j_U)_p\mathcal{G}(V) = \mathrm{colim}_{(W/U, V \to W)}\, \mathcal{G}(W)$$

Now it is clear that the category of pairs $(W/U, V \to W)$ has an object $O_\varphi = (\varphi : V \to U, \mathrm{id} : V \to V)$ for every $\varphi : V \to U$, and moreover for any object there is a unique morphism from one of the O_φ into it. The result follows. □

Lemma 7.24.3. *Let $\mathcal{C}$ be a site. Let $U \in \mathrm{Ob}(\mathcal{C})$. Let X/U be an object of $\mathcal{C}/U$. Then we have $j_{U!}(h^\#_{X/U}) = h^\#_X$.*

Proof. Denote $p : X \to U$ the structure morphism of X. By Lemma 7.24.2 we see $j_{U!}(h^\#_{X/U})$ is the sheaf associated to the presheaf

$$V \longmapsto \coprod\nolimits_{\varphi \in \mathrm{Mor}_{\mathcal{C}}(V,U)} \{\psi : V \to X \mid p \circ \psi = \varphi\}$$

This is clearly the same thing as $\mathrm{Mor}_{\mathcal{C}}(V, X)$. Hence the lemma follows. □

We have $j_{U!}(*) = h^\#_U$ by either of the two lemmas above. Hence for every sheaf $\mathcal{G}$ over $\mathcal{C}/U$ there is a canonical map of sheaves $j_{U!}\mathcal{G} \to h^\#_U$. This characterizes sheaves in the essential image of $j_{U!}$.

Lemma 7.24.4. *Let $\mathcal{C}$ be a site. Let $U \in \mathrm{Ob}(\mathcal{C})$. The functor $j_{U!}$ gives an equivalence of categories*

$$Sh(\mathcal{C}/U) \longrightarrow Sh(\mathcal{C})/h^\#_U$$

Proof. We explain how to get a functor from $Sh(\mathcal{C})/h^\#_U$ to $Sh(\mathcal{C}/U)$. Suppose that $\varphi : \mathcal{F} \to h^\#_U$ is given. For any object $a : X \to U$ of $\mathcal{C}/U$ we consider the set $\mathcal{F}_\varphi(X \to U)$ of elements $s \in \mathcal{F}(X)$ which under φ map to the image of $a \in \mathrm{Mor}_{\mathcal{C}}(X, U) = h_U(X)$ in $h^\#_U(X)$. It is easy to see that $(X \to U) \mapsto \mathcal{F}_\varphi(X \to U)$ is a sheaf on $\mathcal{C}/U$. The verification that $(\mathcal{F}, \varphi) \mapsto \mathcal{F}_\varphi$ is an inverse to the functor $j_{U!}$ is omitted. □

The lemma says the functor $j_{U!}$ is the composition

$$Sh(\mathcal{C}/U) \to Sh(\mathcal{C})/h^\#_U \to Sh(\mathcal{C})$$

where the first arrow is an equivalence.

Lemma 7.24.5. *Let $\mathcal{C}$ be a site. Let $U \in \mathrm{Ob}(\mathcal{C})$. The functor $j_{U!}$ commutes with with fibre products and equalizers (and more generally finite connected limits). In particular, if $\mathcal{F} \subset \mathcal{F}'$ in $Sh(\mathcal{C}/U)$, then $j_{U!}\mathcal{F} \subset j_{U!}\mathcal{F}'$.*

Proof. This follows from the fact that an isomorphism of categories commutes with all limits and the functor $Sh(\mathcal{C})/h^\#_U \to Sh(\mathcal{C})$ commutes with fibre products and equalizers. Alternatively, one can prove this directly using the description of $j_{U!}$ in Lemma 7.24.2 using that sheafification is exact. (Also, in case $\mathcal{C}$ has fibre products and equalizers, the result follows from Lemma 7.20.6.) □

Lemma 7.24.6. *Let $\mathcal{C}$ be a site. Let $U \in \mathrm{Ob}(\mathcal{C})$. For any sheaf $\mathcal{F}$ on $\mathcal{C}$ we have $j_{U!}j_U^{-1}\mathcal{F} = \mathcal{F} \times h_U^\#$.*

Proof. This is clear from the description of $j_{U!}$ in Lemma 7.24.2. □

Lemma 7.24.7. *Let $\mathcal{C}$ be a site. Let $f : V \to U$ be a morphism of $\mathcal{C}$. Then there exists a commutative diagram*

$$\begin{array}{ccc} \mathcal{C}/V & \xrightarrow{\quad j \quad} & \mathcal{C}/U \\ & {\scriptstyle j_V}\searrow \quad \swarrow{\scriptstyle j_U} & \\ & \mathcal{C} & \end{array}$$

of cocontinuous functors. Here $j : \mathcal{C}/V \to \mathcal{C}/U$, $(a : W \to V) \mapsto (f \circ a : W \to U)$ is identified with the functor $j_{V/U} : (\mathcal{C}/U)/(V/U) \to \mathcal{C}/U$ via the identification $(\mathcal{C}/U)/(V/U) = \mathcal{C}/V$. Moreover we have $j_{V!} = j_{U!} \circ j_!$, $j_V^{-1} = j^{-1} \circ j_U^{-1}$, and $j_{V*} = j_{U*} \circ j_*$.

Proof. The commutativity of the diagram is immediate. The agreement of j with $j_{V/U}$ follows from the definitions. By Lemma 7.20.2 we see that the following diagram of morphisms of topoi

$$\begin{array}{ccc} Sh(\mathcal{C}/V) & \xrightarrow{\quad j \quad} & Sh(\mathcal{C}/U) \\ & {\scriptstyle j_V}\searrow \quad \swarrow{\scriptstyle j_U} & \\ & Sh(\mathcal{C}) & \end{array} \tag{7.24.7.1}$$

is commutative. This proves that $j_V^{-1} = j^{-1} \circ j_U^{-1}$ and $j_{V*} = j_{U*} \circ j_*$. The equality $j_{V!} = j_{U!} \circ j_!$ follows formally from adjointness properties. □

Lemma 7.24.8. *Notation $\mathcal{C}$, $f : V \to U$, j_U, j_V, and j as in Lemma 7.24.7. Via the identifications $Sh(\mathcal{C}/V) = Sh(\mathcal{C})/h_V^\#$ and $Sh(\mathcal{C}/U) = Sh(\mathcal{C})/h_U^\#$ of Lemma 7.24.4 the functor j^{-1} has the following description*

$$j^{-1}(\mathcal{H} \xrightarrow{\varphi} h_U^\#) = (\mathcal{H} \times_{\varphi, h_U^\#, f} h_V^\# \to h_V^\#).$$

Proof. Suppose that $\varphi : \mathcal{H} \to h_U^\#$ is an object of $Sh(\mathcal{C})/h_U^\#$. By the proof of Lemma 7.24.4 this corresponds to the sheaf $\mathcal{H}_\varphi$ on $\mathcal{C}/U$ defined by the rule

$$(a : W \to U) \longmapsto \{s \in \mathcal{H}(W) \mid \varphi(s) = a\}$$

on $\mathcal{C}/U$. The pullback $j^{-1}\mathcal{H}_\varphi$ to $\mathcal{C}/V$ is given by the rule

$$(a : W \to V) \longmapsto \{s \in \mathcal{H}(W) \mid \varphi(s) = f \circ a\}$$

by the description of $j^{-1} = j_{U/V}^{-1}$ as the restriction of $\mathcal{H}_\varphi$ to $\mathcal{C}/V$. On the other hand, applying the rule to the object

$$\mathcal{H}' = \mathcal{H} \times_{\varphi, h_U^\#, f} h_V^\# \xrightarrow{\quad \varphi' \quad} h_V^\#$$

of $Sh(\mathcal{C})/h_V^\#$ we get $\mathcal{H}'_{\varphi'}$ given by

$$\begin{aligned} (a : W \to V) \longmapsto & \{s' \in \mathcal{H}'(W) \mid \varphi'(s') = a\} \\ = & \{(s, a') \in \mathcal{H}(W) \times h_V^\#(W) \mid a' = a \text{ and } \varphi(s) = f \circ a'\} \end{aligned}$$

which is exactly the same rule as the one describing $j^{-1}\mathcal{H}_\varphi$ above. □

Remark 7.24.9. Localization and presheaves. Let $\mathcal{C}$ be a category. Let U be an object of $\mathcal{C}$. Strictly speaking the functors j_U^{-1}, j_{U*} and $j_{U!}$ have not been defined for presheaves. But of course, we can think of a presheaf as a sheaf for the chaotic topology on $\mathcal{C}$ (see Example 7.6.6). Hence we also obtain a functor

$$j_U^{-1} : PSh(\mathcal{C}) \longrightarrow PSh(\mathcal{C}/U)$$

and functors

$$j_{U*}, j_{U!} : PSh(\mathcal{C}/U) \longrightarrow PSh(\mathcal{C})$$

which are right, left adjoint to j_U^{-1}. By Lemma 7.24.2 we see that $j_{U!}\mathcal{G}$ is the presheaf

$$V \longmapsto \coprod\nolimits_{\varphi \in \mathrm{Mor}_{\mathcal{C}}(V,U)} \mathcal{G}(V \xrightarrow{\varphi} U)$$

In addition the functor $j_{U!}$ commutes with fibre products and equalizers.

Remark 7.24.10. Let $\mathcal{C}$ be a site. Let $U \to V$ be a morphism of $\mathcal{C}$. The cocontinuous functors $\mathcal{C}/U \to \mathcal{C}$ and $j : \mathcal{C}/U \to \mathcal{C}/V$ (Lemma 7.24.7) satisfy property P of Remark 7.19.5. For example, if we have objects (X/U), (W/V), a morphism $g : j(X/U) \to (W/V)$, and a covering $\{f_i : (W_i/V) \to (W/V)\}$ then $(X \times_W W_i/U)$ is an avatar of $(X/U) \times_{g,(W/V),f_i} (W_i/V)$ and the family $\{(X \times_W W_i/U) \to (X/U)\}$ is a covering of $\mathcal{C}/U$.

7.25. Glueing sheaves

This section is the analogue of Sheaves, Section 6.33.

Lemma 7.25.1. *Let $\mathcal{C}$ be a site. Let $\{U_i \to U\}$ be a covering of $\mathcal{C}$. Let $\mathcal{F}$, $\mathcal{G}$ be sheaves on $\mathcal{C}$. Given a collection*

$$\varphi_i : \mathcal{F}|_{\mathcal{C}/U_i} \longrightarrow \mathcal{G}|_{\mathcal{C}/U_i}$$

of maps of sheaves such that for all $i, j \in I$ the maps φ_i, φ_j restrict to the same map $\mathcal{F}|_{\mathcal{C}/U_i \times_U U_j} \to \mathcal{G}|_{\mathcal{C}/U_i \times_U U_j}$ then there exists a unique map of sheaves

$$\varphi : \mathcal{F}|_{\mathcal{C}/U} \longrightarrow \mathcal{G}|_{\mathcal{C}/U}$$

whose restriction to each $\mathcal{C}/U_i$ agrees with φ_i.

Proof. Omitted. Note that the restrictions are always those of Lemma 7.24.7. □

The previous lemma implies that given two sheaves $\mathcal{F}$, $\mathcal{G}$ on a site $\mathcal{C}$ the rule

$$U \longmapsto \mathrm{Mor}_{Sh(\mathcal{C}/U)}(\mathcal{F}|_{\mathcal{C}/U}, \mathcal{G}|_{\mathcal{C}/U})$$

defines a sheaf. This is a kind of *internal hom sheaf*. It is seldom used in the setting of sheaves of sets, and more usually in the setting of sheaves of modules, see Modules on Sites, Section 18.27.

Let $\mathcal{C}$ be a site. Let $\{U_i \to U\}_{i \in I}$ be a covering of $\mathcal{C}$. For each $i \in I$ let $\mathcal{F}_i$ be a sheaf of sets on $\mathcal{C}/U_i$. For each pair $i, j \in I$, let

$$\varphi_{ij} : \mathcal{F}_i|_{\mathcal{C}/U_i \times_U U_j} \longrightarrow \mathcal{F}_j|_{\mathcal{C}/U_i \times_U U_j}$$

be an isomorphism of sheaves of sets. Assume in addition that for every triple of indices $i, j, k \in I$ the following diagram is commutative

$$\begin{array}{ccc} \mathcal{F}_i|_{\mathcal{C}/U_i \times_U U_j \times_U U_k} & \xrightarrow{\varphi_{ik}} & \mathcal{F}_k|_{\mathcal{C}/U_i \times_U U_j \times_U U_k} \\ & {\scriptstyle \varphi_{ij}} \searrow \quad \nearrow {\scriptstyle \varphi_{jk}} & \\ & \mathcal{F}_j|_{\mathcal{C}/U_i \times_U U_j \times_U U_k} & \end{array}$$

We will call such a collection of data $(\mathcal{F}_i, \varphi_{ij})$ a *glueing data for sheaves of sets with respect to the covering* $\{U_i \to U\}_{i \in I}$.

Lemma 7.25.2. *Let $\mathcal{C}$ be a site. Let $\{U_i \to U\}_{i \in I}$ be a covering of $\mathcal{C}$. Given any glueing data $(\mathcal{F}_i, \varphi_{ij})$ for sheaves of sets with respect to the covering $\{U_i \to U\}_{i \in I}$ there exists a sheaf of sets $\mathcal{F}$ on $\mathcal{C}/U$ together with isomorphisms*

$$\varphi_i : \mathcal{F}|_{\mathcal{C}/U_i} \to \mathcal{F}_i$$

such that the diagrams

$$\begin{array}{ccc} \mathcal{F}|_{\mathcal{C}/U_i \times_U U_j} & \xrightarrow{\varphi_i} & \mathcal{F}_i|_{\mathcal{C}/U_i \times_U U_j} \\ {\scriptstyle id}\downarrow & & \downarrow{\scriptstyle \varphi_{ij}} \\ \mathcal{F}|_{\mathcal{C}/U_i \times_U U_j} & \xrightarrow{\varphi_j} & \mathcal{F}_j|_{\mathcal{C}/U_i \times_U U_j} \end{array}$$

are commutative.

Proof. Let us describe how to construct the sheaf $\mathcal{F}$ on $\mathcal{C}/U$. Let $a : V \to U$ be an object of $\mathcal{C}/U$. Then

$$\mathcal{F}(V/U) = \{(s_i)_{i \in I} \in \prod_{i \in I} \mathcal{F}_i(U_i \times_U V/U_i) \mid \varphi_{ij}(s_i|_{U_i \times_U U_j \times_U V}) = s_j|_{U_i \times_U U_j \times_U V}\}$$

We omit the construction of the restriction mappings. We omit the verification that this is a sheaf. We omit the construction of the isomorphisms φ_i, and we omit proving the commutativity of the diagrams of the lemma. □

Let $\mathcal{C}$ be a site. Let $\{U_i \to U\}_{i \in I}$ be a covering of $\mathcal{C}$. Let $\mathcal{F}$ be a sheaf on $\mathcal{C}/U$. Associated to $\mathcal{F}$ we have its *canonical glueing data* given by the restrictions $\mathcal{F}|_{\mathcal{C}/U_i}$ and the canonical isomorphisms

$$\left(\mathcal{F}|_{\mathcal{C}/U_i}\right)|_{\mathcal{C}/U_i \times_U U_j} = \left(\mathcal{F}|_{\mathcal{C}/U_j}\right)|_{\mathcal{C}/U_i \times_U U_j}$$

coming from the fact that the composition of the functors $\mathcal{C}/U_i \times_U U_j \to \mathcal{C}/U_i \to \mathcal{C}/U$ and $\mathcal{C}/U_i \times_U U_j \to \mathcal{C}/U_j \to \mathcal{C}/U$ are equal.

Lemma 7.25.3. *Let $\mathcal{C}$ be a site. Let $\{U_i \to U\}_{i \in I}$ be a covering of $\mathcal{C}$. The category $Sh(\mathcal{C}/U)$ is equivalent to the category of glueing data via the functor that associates to $\mathcal{F}$ on $\mathcal{C}/U$ the canonical glueing data.*

Proof. In Lemma 7.25.1 we saw that the functor is fully faithful, and in Lemma 7.25.2 we proved that it is essentially surjective (by explicitly constructing a quasi-inverse functor). □

7.26. More localization

In this section we prove a few lemmas on localization where we impose some additional hypotheses on the site on or the object we are localizing at.

Lemma 7.26.1. *Let $\mathcal{C}$ be a site. Let $U \in \mathrm{Ob}(\mathcal{C})$. If the topology on $\mathcal{C}$ is subcanonical, see Definition 7.13.2, and if $\mathcal{G}$ is a sheaf on $\mathcal{C}/U$, then*

$$j_{U!}(\mathcal{G})(V) = \coprod\nolimits_{\varphi \in \mathrm{Mor}_{\mathcal{C}}(V,U)} \mathcal{G}(V \xrightarrow{\varphi} U),$$

in other words sheafification is not necessary in Lemma 7.24.2.

Proof. Let $\mathcal{V} = \{V_i \to V\}_{i \in I}$ be a covering of V in the site $\mathcal{C}$. We are going to check the sheaf condition for the presheaf $\mathcal{H}$ of Lemma 7.24.2 directly. Let $(s_i, \varphi_i)_{i \in I} \in \prod_i \mathcal{H}(V_i)$, This means $\varphi_i : V_i \to U$ is a morphism in $\mathcal{C}$, and $s_i \in \mathcal{G}(V_i \xrightarrow{\varphi_i} U)$. The restriction of the pair (s_i, φ_i) to $V_i \times_V V_j$ is the pair $(s_i|_{V_i \times_V V_j/U}, \mathrm{pr}_1 \circ \varphi_i)$, and likewise the restriction of the pair (s_j, φ_j) to $V_i \times_V V_j$ is the pair $(s_j|_{V_i \times_V V_j/U}, \mathrm{pr}_2 \circ \varphi_j)$. Hence, if the family (s_i, φ_i) lies in $\check{H}^0(\mathcal{V}, \mathcal{H})$, then we see that $\mathrm{pr}_1 \circ \varphi_i = \mathrm{pr}_2 \circ \varphi_j$. The condition that the topology on $\mathcal{C}$ is weaker than the canonical topology then implies that there exists a unique morphism $\varphi : V \to U$ such that φ_i is the composition of $V_i \to V$ with φ. At this point the sheaf condition for $\mathcal{G}$ guarantees that the sections s_i glue to a unique section $s \in \mathcal{G}(V \xrightarrow{\varphi} U)$. Hence $(s, \varphi) \in \mathcal{H}(V)$ as desired. □

Lemma 7.26.2. *Let $\mathcal{C}$ be a site. Let $U \in \mathrm{Ob}(\mathcal{C})$. Assume $\mathcal{C}$ has products of pairs of objects. Then*

1. *the functor j_U has a continuous right adjoint, namely the functor $v(X) = X \times U/U$,*
2. *the functor v defines a morphism of sites $\mathcal{C}/U \to \mathcal{C}$ whose associated morphism of topoi equals $j_U : Sh(\mathcal{C}/U) \to Sh(\mathcal{C})$, and*
3. *we have $j_{U*}\mathcal{F}(X) = \mathcal{F}(X \times U/U)$.*

Proof. The functor v being right adjoint to j_U means that given Y/U and X we have

$$\mathrm{Mor}_{\mathcal{C}}(Y, X) = \mathrm{Mor}_{\mathcal{C}/U}(Y/U, X \times U/U)$$

which is clear. To check that v is continuous let $\{X_i \to X\}$ be a covering of $\mathcal{C}$. By the third axiom of a site (Definition 7.6.2) we see that

$$\{X_i \times_X (X \times U) \to X \times_X (X \times U)\} = \{X_i \times U \to X \times U\}$$

is a covering of $\mathcal{C}$ also. Hence v is continuous. The other statements of the lemma follow from Lemmas 7.21.1 and 7.21.2. □

Lemma 7.26.3. *Let $\mathcal{C}$ be a site. Let $U \to V$ be a morphism of $\mathcal{C}$. Assume $\mathcal{C}$ has fibre products. Let j be as in Lemma 7.24.7. Then*

1. *the functor $j : \mathcal{C}/U \to \mathcal{C}/V$ has a continuous right adjoint, namely the functor $v : (X/V) \mapsto (X \times_V U/U)$,*
2. *the functor v defines a morphism of sites $\mathcal{C}/U \to \mathcal{C}/V$ whose associated morphism of topoi equals j, and*
3. *we have $j_*\mathcal{F}(X/U) = \mathcal{F}(X \times_V U/U)$.*

Proof. Follows from Lemma 7.26.2 since j may be viewed as a localization functor by Lemma 7.24.7. □

A fundamental property of an open immersion is that the restriction of the push-forward and the restriction of the extension by the empty set produces back the original sheaf. This is not always true for the functors associated to j_U above. It is true when U is a "subobject of the final object".

Lemma 7.26.4. *Let $\mathcal{C}$ be a site. Let $U \in \mathrm{Ob}(\mathcal{C})$. Assume that every X in $\mathcal{C}$ has at most one morphism to U. Let $\mathcal{F}$ be a sheaf on $\mathcal{C}/U$. The canonical maps $\mathcal{F} \to j_U^{-1} j_{U!}\mathcal{F}$ and $j_U^{-1} j_{U*}\mathcal{F} \to \mathcal{F}$ are isomorphisms.*

Proof. If $\mathcal{C}$ has fibre products, then this is a special case of Lemma 7.20.7. In general we have the following direct proof.

Let X/U be an object over U. In Lemmas 7.19.2 and 7.20.5 we have seen that sheafification is not necessary for the functors $j_U^{-1} = (u^p\ )^\#$ and $j_{U*} = ({}_p u)^\#$. We may compute $(j_U^{-1} j_{U*}\mathcal{F})(X/U) = j_{U*}\mathcal{F}(X) = \lim \mathcal{F}(Y/U)$. Here the limit is over the category of pairs $(Y/U, Y \to X)$ where the morphisms $Y \to X$ are not required to be over U. By our assumption however we see that they are automatically morphisms over U and we deduce that the limit is the value on id_X, i.e., $\mathcal{F}(X/U)$. This proves that $j_U^{-1} j_{U*}\mathcal{F} = \mathcal{F}$.

On the other hand, $(j_U^{-1} j_{U!}\mathcal{F})(X/U) = j_{U!}\mathcal{F}(X) = (u_p\mathcal{F})^\#(X)$, and $u_p\mathcal{F}(X) = \mathrm{colim}\, \mathcal{F}(Y/U)$. Here the colimit is over the category of pairs $(Y/U, X \to Y)$ where the morphisms $X \to Y$ are not required to be over U. By our assumption however we see that they are automatically morphisms over U and we deduce that the colimit is the value on id_X, i.e., $\mathcal{F}(X/U)$. This shows that the sheafification is not necessary (since any object over X is automatically in a unique way an object over U) and the result follows. □

7.27. Localization and morphisms

The following lemma is important in order to understand relation between localization and morphisms of sites and topoi.

Lemma 7.27.1. *Let $f : \mathcal{C} \to \mathcal{D}$ be a morphism of sites corresponding to the continuous functor $u : \mathcal{D} \to \mathcal{C}$. Let $V \in \mathrm{Ob}(\mathcal{D})$ and set $U = u(V)$. Then the functor $u' : \mathcal{D}/V \to \mathcal{C}/U$, $V'/V \mapsto u(V')/U$ determines a morphism of sites $f' : \mathcal{C}/U \to \mathcal{D}/V$. The morphism f' fits into a commutative diagram of topoi*

$$\begin{array}{ccc} Sh(\mathcal{C}/U) & \xrightarrow{j_U} & Sh(\mathcal{C}) \\ {\scriptstyle f'}\downarrow & & \downarrow{\scriptstyle f} \\ Sh(\mathcal{D}/V) & \xrightarrow{j_V} & Sh(\mathcal{D}). \end{array}$$

Using the identifications $Sh(\mathcal{C}/U) = Sh(\mathcal{C})/h_U^\#$ and $Sh(\mathcal{D}/V) = Sh(\mathcal{D})/h_V^\#$ of Lemma 7.24.4 the functor $(f')^{-1}$ is described by the rule

$$(f')^{-1}(\mathcal{H} \xrightarrow{\varphi} h_V^\#) = (f^{-1}\mathcal{H} \xrightarrow{f^{-1}\varphi} h_U^\#).$$

Finally, we have $f'_* j_U^{-1} = j_V^{-1} f_*$.

Proof. It is clear that u' is continuous, and hence we get functors $f'_* = (u')^s = (u')^p$ (see Sections 7.5 and 7.14) and an adjoint $(f')^{-1} = (u')_s = ((u')_p\ )^\#$. The assertion $f'_* j_U^{-1} = j_V^{-1} f_*$ follows as

$$(j_V^{-1} f_*\mathcal{F})(V'/V) = f_*\mathcal{F}(V') = \mathcal{F}(u(V')) = (j_U^{-1}\mathcal{F})(u(V')/U) = (f'_* j_U^{-1}\mathcal{F})(V'/V)$$

which holds even for presheaves. What isn't clear a priori is that $(f')^{-1}$ is exact, that the diagram commutes, and that the description of $(f')^{-1}$ holds.

Let $\mathcal{H}$ be a sheaf on $\mathcal{D}/V$. Let us compute $j_{U!}(f')^{-1}\mathcal{H}$. We have

$$\begin{aligned} j_{U!}(f')^{-1}\mathcal{H} &= ((j_U)_p(u'_p\mathcal{H})^\#)^\# \\ &= ((j_U)_p u'_p\mathcal{H})^\# \\ &= (u_p(j_V)_p\mathcal{H})^\# \\ &= f^{-1}j_{V!}\mathcal{H} \end{aligned}$$

The first equality by unwinding the definitions. The second equality by Lemma 7.14.4. The third equality because $u \circ j_V = j_U \circ u'$. The fourth equality by Lemma 7.14.4 again. All of the equalities above are isomorphisms of functors, and hence we may interpret this as saying that the following diagram of categories and functors is commutative

$$\begin{array}{ccccc} Sh(\mathcal{C}/U) & \xrightarrow[j_{U!}]{} & Sh(\mathcal{C})/h_U^\# & \longrightarrow & Sh(\mathcal{C}) \\ {\scriptstyle (f')^{-1}}\uparrow & & {\scriptstyle f^{-1}}\uparrow & & {\scriptstyle f^{-1}}\uparrow \\ Sh(\mathcal{D}/V) & \xrightarrow{j_{V!}} & Sh(\mathcal{D})/h_V^\# & \longrightarrow & Sh(\mathcal{D}) \end{array}$$

The middle arrow makes sense as $f^{-1}h_V^\# = (h_{u(V)})^\# = h_U^\#$, see Lemma 7.14.5. In particular this proves the description of $(f')^{-1}$ given in the statement of the lemma. Since by Lemma 7.24.4 the left horizontal arrows are equivalences and since f^{-1} is exact by assumption we conclude that $(f')^{-1} = u'_s$ is exact. Namely, because it is a left adjoint it is already right exact (Categories, Lemma 4.24.3). Hence we only need to show that it transforms a final object into a final object and commutes with fibre products (Categories, Lemma 4.23.2). Both are clear for the induced functor $f^{-1} : Sh(\mathcal{D})/h_V^\# \to Sh(\mathcal{C})/h_U^\#$. This proves that f' is a morphism of sites.

We still have to verify that $(f')^{-1}j_V^{-1} = j_U^{-1}f^{-1}$. To see this use the formula above and the description in Lemma 7.24.6. Namely, combined these give, for any sheaf $\mathcal{G}$ on $\mathcal{D}$, that

$$j_{U!}(f')^{-1}j_V^{-1}\mathcal{G} = f^{-1}j_{V!}j_V^{-1}\mathcal{G} = f^{-1}(\mathcal{G} \times h_V^\#) = f^{-1}\mathcal{G} \times h_U^\# = j_{U!}j_U^{-1}f^{-1}\mathcal{G}.$$

Since the functor $j_{U!}$ induces an equivalence $Sh(\mathcal{C}/U) \to Sh(\mathcal{C})/h_U^\#$ we conclude. □

The following lemma is a special case of the more general Lemma 7.27.1 above.

Lemma 7.27.2. *Let $\mathcal{C}$, $\mathcal{D}$ be sites. Let $u : \mathcal{D} \to \mathcal{C}$ be a functor. Let $V \in \mathrm{Ob}(\mathcal{D})$. Set $U = u(V)$. Assume that*

(1) *$\mathcal{C}$ and $\mathcal{D}$ have all finite limits,*
(2) *u is continuous, and*
(3) *u commutes with finite limits.*

There exists a commutative diagram of morphisms of sites

$$\begin{array}{ccc} \mathcal{C}/U & \xrightarrow[j_U]{} & \mathcal{C} \\ {\scriptstyle f'}\downarrow & & \downarrow{\scriptstyle f} \\ \mathcal{D}/V & \xrightarrow{j_V} & \mathcal{D} \end{array}$$

where the right vertical arrow corresponds to u, the left vertical arrow corresponds to the functor $u' : \mathcal{D}/V \to \mathcal{C}/U$, $V'/V \mapsto u(V')/u(V)$ and the horizontal arrows correspond to the functors $\mathcal{C} \to \mathcal{C}/U$, $X \mapsto X \times U$ and $\mathcal{D} \to \mathcal{D}/V$, $Y \mapsto Y \times V$ as in Lemma 7.26.2. Moreover, the associated diagram of morphisms of topoi is equal to the diagram of Lemma 7.27.1. In particular we have $f'_* j_U^{-1} = j_V^{-1} f_*$.

Proof. Note that u satisfies the assumptions of Proposition 7.15.6 and hence induces a morphism of sites $f : \mathcal{C} \to \mathcal{D}$ by that proposition. It is clear that u induces a functor u' as indicated. It is clear that this functor also satisfies the assumptions of Proposition 7.15.6. Hence we get a morphism of sites $f' : \mathcal{C}/U \to \mathcal{D}/V$. The diagram commutes by our definition of composition of morphisms of sites (see Definition 7.15.4) and because

$$u(Y \times V) = u(Y) \times u(V) = u(Y) \times U$$

which shows that the diagram of categories and functors opposite to the diagram of the lemma commutes. □

At this point we can localize a site, we know how to relocalize, and we can localize a morphism of sites at an object of the site downstairs. If we combine these then we get the following kind of diagram.

Lemma 7.27.3. *Let $f : \mathcal{C} \to \mathcal{D}$ be a morphism of sites corresponding to the continuous functor $u : \mathcal{C} \to \mathcal{D}$. Let $V \in \mathrm{Ob}(\mathcal{D})$, $U \in \mathrm{Ob}(\mathcal{C})$ and $c : U \to u(V)$ a morphism of $\mathcal{C}$. There exists a commutative diagram of topoi*

$$\begin{array}{ccc} Sh(\mathcal{C}/U) & \xrightarrow[j_U]{} & Sh(\mathcal{C}) \\ {\scriptstyle f_c}\downarrow & & \downarrow{\scriptstyle f} \\ Sh(\mathcal{D}/V) & \xrightarrow{j_V} & Sh(\mathcal{D}). \end{array}$$

We have $f_c = f' \circ j_{U/u(V)}$ where $f' : Sh(\mathcal{C}/u(V)) \to Sh(\mathcal{D}/V)$ is as in Lemma 7.27.1 and $j_{U/u(V)} : Sh(\mathcal{C}/U) \to Sh(\mathcal{C}/u(V))$ is as in Lemma 7.24.7. Using the identifications $Sh(\mathcal{C}/U) = Sh(\mathcal{C})/h_U^\#$ and $Sh(\mathcal{D}/V) = Sh(\mathcal{D})/h_V^\#$ of Lemma 7.24.4 the functor $(f_c)^{-1}$ is described by the rule

$$(f_c)^{-1}(\mathcal{H} \xrightarrow{\varphi} h_V^\#) = (f^{-1}\mathcal{H} \times_{f^{-1}\varphi, h^\#_{u(V)}, c} h_U^\# \to h_U^\#).$$

Finally, given any morphisms $b : V' \to V$, $a : U' \to U$ and $c' : U' \to u(V')$ such that

$$\begin{array}{ccc} U' & \xrightarrow[c']{} & u(V') \\ {\scriptstyle a}\downarrow & & \downarrow{\scriptstyle u(b)} \\ U & \xrightarrow{c} & u(V) \end{array}$$

commutes, then the diagram

$$\begin{array}{ccc} Sh(\mathcal{C}/U') & \xrightarrow[j_{U'/U}]{} & Sh(\mathcal{C}/U) \\ {\scriptstyle f_{c'}}\downarrow & & \downarrow{\scriptstyle f_c} \\ Sh(\mathcal{D}/V') & \xrightarrow{j_{V'/V}} & Sh(\mathcal{D}/V). \end{array}$$

commutes.

Proof. This lemma proves itself, and is more a collection of things we know at this stage of the development of theory. For example the commutativity of the first square follows from the commutativity of Diagram (7.24.7.1) and the commutativity of the diagram in Lemma 7.27.1. The description of f_c^{-1} follows on combining Lemma 7.24.8 with Lemma 7.27.1. The commutativity of the last square then follows from the equality

$$f^{-1}\mathcal{H} \times_{h^\#_{u(V)},c} h^\#_U \times_{h^\#_U} h^\#_{U'} = f^{-1}(\mathcal{H} \times_{h^\#_V} h^\#_{V'}) \times_{h^\#_{u(V'),c'}} h^\#_{U'}$$

which is formal using that $f^{-1}h^\#_V = h^\#_{u(V)}$ and $f^{-1}h^\#_{V'} = h^\#_{u(V')}$, see Lemma 7.14.5. □

In the following lemma we find another kind of functoriality of localization, in case the morphism of topoi comes from a cocontinuous functor. This is a kind of diagram which is different from the diagram in Lemma 7.27.1, and in particular, in general the equality $f'_* j_U^{-1} = j_V^{-1} f_*$ seen in Lemma 7.27.1 does not hold in the situation of the following lemma.

Lemma 7.27.4. *Let $\mathcal{C}$, $\mathcal{D}$ be sites. Let $u : \mathcal{C} \to \mathcal{D}$ be a cocontinuous functor. Let U be an object of $\mathcal{C}$, and set $V = u(U)$. We have a commutative diagram*

$$\begin{array}{ccc} \mathcal{C}/U & \xrightarrow{j_U} & \mathcal{C} \\ {\scriptstyle u'}\downarrow & & \downarrow{\scriptstyle u} \\ \mathcal{D}/V & \xrightarrow{j_V} & \mathcal{D} \end{array}$$

where the left vertical arrow is $u' : \mathcal{C}/U \to \mathcal{D}/V$, $U'/U \mapsto V'/V$. Then u' is cocontinuous also and we get a commutative diagram of topoi

$$\begin{array}{ccc} Sh(\mathcal{C}/U) & \xrightarrow{j_U} & Sh(\mathcal{C}) \\ {\scriptstyle f'}\downarrow & & \downarrow{\scriptstyle f} \\ Sh(\mathcal{D}/V) & \xrightarrow{j_V} & Sh(\mathcal{D}) \end{array}$$

where f (resp. f') corresponds to u (resp. u').

Proof. The commutativity of the first diagram is clear. It implies the commutativity of the second diagram provided we show that u' is cocontinuous.

Let U'/U be an object of $\mathcal{C}/U$. Let $\{V_j/V \to u(U')/V\}_{j\in J}$ be a covering of $u(U')/V$ in $\mathcal{D}/V$. Since u is cocontinuous there exists a covering $\{U'_i \to U'\}_{i\in I}$ such that the family $\{u(U'_i) \to u(U')\}$ refines the covering $\{V_j \to u(U')\}$ in $\mathcal{D}$. In other words, there exists a map of index sets $\alpha : I \to J$ and morphisms $\phi_i : u(U'_i) \to V_{\alpha(i)}$ over U'. Think of U'_i as an object over U via the composition $U'_i \to U' \to U$. Then $\{U'_i/U \to U'/U\}$ is a covering of $\mathcal{C}/U$ such that $\{u(U'_i)/V \to u(U')/V\}$ refines $\{V_j/V \to u(U')/V\}$ (use the same α and the same maps ϕ_i). Hence $u' : \mathcal{C}/U \to \mathcal{D}/V$ is cocontinuous. □

7.28. Morphisms of topoi

In this section we show that any morphism of topoi is equivalent to a morphism of topoi which comes from a morphism of sites.

Lemma 7.28.1. *Let $\mathcal{C}$, $\mathcal{D}$ be sites. Let $u : \mathcal{C} \to \mathcal{D}$ be a functor. Assume that*

(1) *u is cocontinuous,*
(2) *u is continuous,*
(3) *given $a, b : U' \to U$ in $\mathcal{C}$ such that $u(a) = u(b)$, then there exists a covering $\{f_i : U'_i \to U'\}$ in $\mathcal{C}$ such that $a \circ f_i = b \circ f_i$,*
(4) *given $U', U \in \mathrm{Ob}(\mathcal{C})$ and a morphism $c : u(U') \to u(U)$ in $\mathcal{D}$ there exists a covering $\{f_i : U'_i \to U'\}$ in $\mathcal{C}$ and morphisms $c_i : U'_i \to U$ such that $u(c_i) = c \circ u(f_i)$, and*
(5) *given $V \in \mathrm{Ob}(\mathcal{D})$ there exists a covering of V in $\mathcal{D}$ of the form $\{u(U_i) \to V\}_{i \in I}$.*

Then the morphism of topoi

$$g : Sh(\mathcal{C}) \longrightarrow Sh(\mathcal{D})$$

associated to the cocontinuous functor u by Lemma 7.20.1 is an equivalence.

Proof. Assume u satisfies properties (1) – (5). We will show that the adjunction mappings

$$\mathcal{G} \longrightarrow g_* g^{-1} \mathcal{G} \quad \text{and} \quad g^{-1} g_* \mathcal{F} \longrightarrow \mathcal{F}$$

are isomorphisms.

Note that Lemma 7.20.5 applies and we have $g^{-1}\mathcal{G}(U) = \mathcal{G}(u(U))$ for any sheaf $\mathcal{G}$ on $\mathcal{D}$. Next, let $\mathcal{F}$ be a sheaf on $\mathcal{C}$, and let V be an object of $\mathcal{D}$. By definition we have $g_*\mathcal{F}(V) = \lim_{u(U) \to V} \mathcal{F}(U)$. Hence

$$g^{-1} g_* \mathcal{F}(U) = \lim_{U', u(U') \to u(U)} \mathcal{F}(U')$$

where the morphisms $\psi : u(U') \to u(U)$ need not be of the form $u(\alpha)$. The category of such pairs (U', ψ) has a final object, namely (U, id), which gives rise to the map from the limit into $\mathcal{F}(U)$. Let $(s_{(U',\psi)})$ be an element of the limit. We want to show that $s_{(U',\psi)}$ is uniquely determined by the value $s_{(U,\mathrm{id})} \in \mathcal{F}(U)$. By property (4) given any (U', ψ) there exists a covering $\{U'_i \to U'\}$ such that the compositions $u(U'_i) \to u(U') \to u(U)$ are of the form $u(c_i)$ for some $c_i : U'_i \to U$ in $\mathcal{C}$. Hence

$$s_{(U',\psi)}|_{U'_i} = c_i^*(s_{U,\mathrm{id}}).$$

Since $\mathcal{F}$ is a sheaf it follows that indeed $s_{(U',\psi)}$ is determined by $s_{(U,\mathrm{id})}$. This proves uniqueness. For existence, assume given any $s \in \mathcal{F}(U)$, $\psi : u(U') \to u(U)$, $\{f_i : U'_i \to U'\}$ and $c_i : U'_i \to U$ such that $\psi \circ u(f_i) = u(c_i)$ as above. We claim there exists a (unique) element $s_{(U',\psi)} \in \mathcal{F}(U')$ such that

$$s_{(U',\psi)}|_{U'_i} = c_i^*(s).$$

Namely, a priori it is not clear the elements $c_i^*(s)|_{U'_i \times_{U'} U'_j}$ and $c_j^*(s)|_{U'_i \times_{U'} U'_j}$ agree, since the diagram

$$\begin{array}{ccc} U'_i \times_{U'} U'_j & \xrightarrow{\mathrm{pr}_2} & U'_j \\ {\scriptstyle \mathrm{pr}_1}\downarrow & & \downarrow{\scriptstyle c_j} \\ U'_i & \xrightarrow{c_i} & U \end{array}$$

need not commute. But condition (3) of the lemma guarantees that there exist coverings $\{f_{ijk} : U'_{ijk} \to U'_i \times_{U'} U'_j\}_{k \in K_{ij}}$ such that $c_i \circ \mathrm{pr}_1 \circ f_{ijk} = c_j \circ \mathrm{pr}_2 \circ f_{ijk}$. Hence

$$f_{ijk}^* \left(c_i^* s|_{U'_i \times_{U'} U'_j} \right) = f_{ijk}^* \left(c_j^* s|_{U'_i \times_{U'} U'_j} \right)$$

Hence $c_i^*(s)|_{U_i' \times_{U'} U_j'} = c_j^*(s)|_{U_i' \times_{U'} U_j'}$ by the sheaf condition for $\mathcal{F}$ and hence the existence of $s_{U',\psi}$ also by the sheaf condition for $\mathcal{F}$. The uniqueness guarantees that the collection $(s_{U',\psi})$ so obtained is an element of the limit with $s_{(U,\psi)} = s$. This proves that $g^{-1}g_*\mathcal{F} \to \mathcal{F}$ is an isomorphism.

Let $\mathcal{G}$ be a sheaf on $\mathcal{D}$. Let V be an object of $\mathcal{D}$. Then we see that

$$g_*g^{-1}\mathcal{G}(V) = \lim_{U,\psi:u(U)\to V} \mathcal{G}(u(U))$$

By the preceding paragraph we see that the value of the sheaf $g_*g^{-1}\mathcal{G}$ on an object V of the form $V = u(U)$ is equal to $\mathcal{G}(u(U))$. (Formally, this holds because we have $g^{-1}g_*g^{-1} \cong g^{-1}$, and the description of g^{-1} given at the beginning of the proof; informally just by comparing limits here and above.) Hence the adjunction mapping $\mathcal{G} \to g_*g^{-1}\mathcal{G}$ has the property that it is a bijection on sections over any object of the form $u(U)$. Since by axiom (5) there exists a covering of V by objects of the form $u(U)$ we see easily that the adjunction map is an isomorphism. □

It will be convenient to give cocontinuous functors as in Lemma 7.28.1 a name.

Definition 7.28.2. Let $\mathcal{C}$, $\mathcal{D}$ be sites. A *special cocontinuous functor u from $\mathcal{C}$ to $\mathcal{D}$* is a cocontinuous functor $u : \mathcal{C} \to \mathcal{D}$ satisfying the assumptions and conclusions of Lemma 7.28.1.

Lemma 7.28.3. *Let $\mathcal{C}$, $\mathcal{D}$ be sites. Let $u : \mathcal{C} \to \mathcal{D}$ be a special cocontinuous functor. For every object U of $\mathcal{C}$ we have a commutative diagram*

$$\begin{array}{ccc} \mathcal{C}/U & \xrightarrow{j_U} & \mathcal{C} \\ \downarrow & & \downarrow u \\ \mathcal{D}/u(U) & \xrightarrow{j_{u(U)}} & \mathcal{D} \end{array}$$

as in Lemma 7.27.4. The left vertical arrow is a special cocontinuous functor. Hence in the commutative diagram of topoi

$$\begin{array}{ccc} Sh(\mathcal{C}/U) & \xrightarrow{j_U} & Sh(\mathcal{C}) \\ \downarrow & & \downarrow u \\ Sh(\mathcal{D}/u(U)) & \xrightarrow{j_{u(U)}} & Sh(\mathcal{D}) \end{array}$$

the vertical arrows are equivalences.

Proof. We have seen the existence and commutativity of the diagrams in Lemma 7.27.4. We have to check hypotheses (1) – (5) of Lemma 7.28.1 for the induced functor $u : \mathcal{C}/U \to \mathcal{D}/u(U)$. This is completely mechanical.

Property (1). This is Lemma 7.27.4.

Property (2). Let $\{U_i'/U' \to U'/U\}_{i\in I}$ be a covering of U'/U in $\mathcal{C}/U$. Because u is continuous we see that $\{u(U_i')/u(U') \to u(U')/u(U)\}_{i\in I}$ is a covering of $u(U')/u(U)$ in $\mathcal{D}/u(U)$. Hence (2) holds for $u : \mathcal{C}/U \to \mathcal{D}/u(U)$.

Property (3). Let $a, b : U''/U \to U'/U$ in $\mathcal{C}/U$ be morphisms such that $u(a) = u(b)$ in $\mathcal{D}/u(U)$. Because u satisfies (3) we see there exists a covering $\{f_i : U_i'' \to U''\}$ in $\mathcal{C}$ such that $a \circ f_i = b \circ f_i$. This gives a covering $\{f_i : U_i''/U \to U''/U\}$ in $\mathcal{C}/U$ such that $a \circ f_i = b \circ f_i$. Hence (3) holds for $u : \mathcal{C}/U \to \mathcal{D}/u(U)$.

Property (4). Let $U''/U, U'/U \in \mathrm{Ob}(\mathcal{C}/U)$ and a morphism $c : u(U'')/u(U) \to u(U')/u(U)$ in $\mathcal{D}/u(U)$ be given. Because u satisfies property (4) there exists a covering $\{f_i : U_i'' \to U''\}$ in $\mathcal{C}$ and morphisms $c_i : U_i'' \to U'$ such that $u(c_i) = c \circ u(f_i)$. We think of U_i'' as an object over U via the composition $U_i'' \to U'' \to U$. It may not be true that c_i is a morphism over U! But since $u(c_i)$ is a morphism over $u(U)$ we may apply property (3) for u and find coverings $\{f_{ik} : U_{ik}'' \to U_i''\}$ such that $c_{ik} = c_i \circ f_{ik} : U_{ik}'' \to U'$ are morphisms over U. Hence $\{f_i \circ f_{ik} : U_{ik}''/U \to U''/U\}$ is a covering in $\mathcal{C}/U$ such that $u(c_{ik}) = c \circ u(f_{ik})$. Hence (4) holds for $u : \mathcal{C}/U \to \mathcal{D}/u(U)$.

Property (5). Let $h : V \to u(U)$ be an object of $\mathcal{D}/u(U)$. Because u satisfies property (5) there exists a covering $\{c_i : u(U_i) \to V\}$ in $\mathcal{D}$. By property (3) we can find coverings $\{f_{ij} : U_{ij} \to U_i\}$ and morphisms $c_{ij} : U_{ij} \to U$ such that $u(c_{ij}) = h \circ c_i \circ u(f_{ij})$. Hence $\{u(U_{ij})/u(U) \to V/u(U)\}$ is a covering in $\mathcal{D}/u(U)$ of the desired shape and we conclude that (5) holds for $u : \mathcal{C}/U \to \mathcal{D}/u(U)$. □

Lemma 7.28.4. *Let $\mathcal{C}$ be a site. Let $\mathcal{C}' \subset Sh(\mathcal{C})$ be a full subcategory (with a set of objects) such that*

(1) *$h_U^\# \in \mathrm{Ob}(\mathcal{C}')$ for all $U \in \mathrm{Ob}(\mathcal{C})$, and*
(2) *$\mathcal{C}'$ is preserved under fibre products in $Sh(\mathcal{C})$.*

Declare a covering of $\mathcal{C}'$ to be any family $\{\mathcal{F}_i \to \mathcal{F}\}_{i \in I}$ of maps such that $\coprod_{i \in I} \mathcal{F}_i \to \mathcal{F}$ is a surjective map of sheaves. Then

(1) *$\mathcal{C}'$ is a site (after choosing a set of coverings, see Sets, Lemma 3.11.1),*
(2) *representable presheaves on $\mathcal{C}'$ are sheaves (i.e., the topology on $\mathcal{C}'$ is subcanonical, see Definition 7.13.2),*
(3) *the functor $v : \mathcal{C} \to \mathcal{C}'$, $U \mapsto h_U^\#$ is a special cocontinuous functor, hence induces an equivalence $g : Sh(\mathcal{C}) \to Sh(\mathcal{C}')$,*
(4) *for any $\mathcal{F} \in \mathrm{Ob}(\mathcal{C}')$ we have $g^{-1} h_{\mathcal{F}} = \mathcal{F}$, and*
(5) *for any $U \in \mathrm{Ob}(\mathcal{C})$ we have $g_* h_U^\# = h_{v(U)} = h_{h_U^\#}$.*

Proof. Warning: Some of the statements above may look be a bit confusing at first; this is because objects of $\mathcal{C}'$ can also be viewed as sheaves on $\mathcal{C}$! We omit the proof that the coverings of $\mathcal{C}'$ as described in the lemma satisfy the conditions of Definition 7.6.2.

Suppose that $\{\mathcal{F}_i \to \mathcal{F}\}$ is a surjective family of morphisms of sheaves. Let $\mathcal{G}$ be another sheaf. Part (2) of the lemma says that the equalizer of

$$\mathrm{Mor}_{Sh(\mathcal{C})}(\coprod_{i \in I} \mathcal{F}_i, \mathcal{G}) \rightrightarrows \mathrm{Mor}_{Sh(\mathcal{C})}(\coprod_{(i_0, i_1) \in I \times I} \mathcal{F}_{i_0} \times_{\mathcal{F}} \mathcal{F}_{i_1}, \mathcal{G})$$

is $\mathrm{Mor}_{Sh(\mathcal{C})}(\mathcal{F}, \mathcal{G})$. This is clear (for example use Lemma 7.12.2).

To prove (3) we have to check conditions (1) – (5) of Lemma 7.28.1. The fact that v is cocontinuous is equivalent to the description of surjective maps of sheaves in Lemma 7.12.2. The functor v is continuous because $U \mapsto h_U^\#$ commutes with fibre products, and transforms coverings into coverings (see Lemma 7.10.14, and Lemma 7.13.4). Properties (3), (4) of Lemma 7.28.1 are statements about morphisms $f : h_U^\# \to h_{U'}^\#$. Such a morphism is the same thing as an element of $h_{U'}^\#(U)$. Hence (3) and (4) are immediate from the construction of the sheafification. Property (5) of Lemma 7.28.1 is Lemma 7.13.5. Denote $g : Sh(\mathcal{C}) \to Sh(\mathcal{C}')$ the equivalence of topoi associated with v by Lemma 7.28.1.

Let $\mathcal{F}$ be as in part (4) of the lemma. For any $U \in \mathrm{Ob}(\mathcal{C})$ we have

$$g^{-1}h_{\mathcal{F}}(U) = h_{\mathcal{F}}(v(U)) = \mathrm{Mor}_{Sh(\mathcal{C})}(h_U^\#, \mathcal{F}) = \mathcal{F}(U)$$

The first equality by Lemma 7.20.5. Thus part (4) holds.

Let $\mathcal{F} \in \mathrm{Ob}(\mathcal{C}')$. Let $U \in \mathrm{Ob}(\mathcal{C})$. Then

$$\begin{aligned} g_*h_U^\#(\mathcal{F}) &= \mathrm{Mor}_{Sh(\mathcal{C}')}(h_{\mathcal{F}}, g_*h_U^\#) \\ &= \mathrm{Mor}_{Sh(\mathcal{C})}(g^{-1}h_{\mathcal{F}}, h_U^\#) \\ &= \mathrm{Mor}_{Sh(\mathcal{C})}(\mathcal{F}, h_U^\#) \\ &= \mathrm{Mor}_{\mathcal{C}'}(\mathcal{F}, h_U^\#) \end{aligned}$$

as desired (where the third equality was shown above). □

Using this we can massage any topos to live over a site having all finite limits.

Lemma 7.28.5. *Let $Sh(\mathcal{C})$ be a topos. Let $\{\mathcal{F}_i\}_{i\in I}$ be a set of sheaves on $\mathcal{C}$. There exists an equivalence of topoi $g : Sh(\mathcal{C}) \to Sh(\mathcal{C}')$ induced by a special cocontinuous functor $u : \mathcal{C} \to \mathcal{C}'$ of sites such that*

1. *$\mathcal{C}'$ has a subcanonical topology,*
2. *a family $\{V_j \to V\}$ of morphisms of $\mathcal{C}'$ is (combinatorially equivalent to) a covering of $\mathcal{C}'$ if and only if $\coprod h_{V_i} \to h_V$ is surjective,*
3. *$\mathcal{C}'$ has fibre products and a final object (i.e., $\mathcal{C}'$ has all finite limits),*
4. *every subsheaf of a representable sheaf on $\mathcal{C}'$ is representable, and*
5. *each $g_*\mathcal{F}_i$ is a representable sheaf.*

Proof. Consider the full subcategory $\mathcal{C}_1 \subset Sh(\mathcal{C})$ consisting of all $h_U^\#$ for all $U \in \mathrm{Ob}(\mathcal{C})$, the given sheaves $\mathcal{F}_i$ and the final sheaf $*$ (see Example 7.10.2). We are going to inductively define full subcategories

$$\mathcal{C}_1 \subset \mathcal{C}_2 \subset \mathcal{C}_2 \subset \ldots \subset Sh(\mathcal{C})$$

Namely, given $\mathcal{C}_n$ let $\mathcal{C}_{n+1}$ be the full subcategory consisting of all fibre products and subsheaves of objects of $\mathcal{C}_n$. (Note that $\mathcal{C}_{n+1}$ has a set of objects.) Set $\mathcal{C}' = \bigcup_{n\geq 1} \mathcal{C}_n$. A covering in $\mathcal{C}'$ is any family $\{\mathcal{G}_j \to \mathcal{G}\}_{j\in J}$ of morphisms of objects of $\mathcal{C}'$ such that $\coprod \mathcal{G}_j \to \mathcal{G}$ is surjective as a map of sheaves on $\mathcal{C}$. The functor $v : \mathcal{C} \to \mathcal{C}'$ is given by $U \mapsto h_U^\#$. Apply Lemma 7.28.4. □

Here is the goal of the current section.

Lemma 7.28.6. *Let $\mathcal{C}$, $\mathcal{D}$ be sites. Let $f : Sh(\mathcal{C}) \to Sh(\mathcal{D})$ be a morphism of topoi. Then there exists a site $\mathcal{C}'$ and a diagram of functors*

$$\mathcal{C} \xrightarrow[v]{} \mathcal{C}' \xleftarrow[u]{} \mathcal{D}$$

such that

1. *the functor v is a special cocontinuous functor,*
2. *the functor u commutes with fibre products, is continuous and defines a morphism of sites $\mathcal{C}' \to \mathcal{D}'$, and*
3. *the morphism of topoi f agrees with the composition of morphisms of topoi*

$$Sh(\mathcal{C}) \longrightarrow Sh(\mathcal{C}') \longrightarrow Sh(\mathcal{D})$$

where the first arrow comes from v via Lemma 7.28.1 and the second arrow from u via Lemma 7.16.2.

Proof. Consider the full subcategory $\mathcal{C}_1 \subset Sh(\mathcal{C})$ consisting of all $h_U^\#$ and all $f^{-1}h_V^\#$ for all $U \in \mathrm{Ob}(\mathcal{C})$ and all $V \in \mathrm{Ob}(\mathcal{D})$. Let $\mathcal{C}_{n+1}$ be a full subcategory consisting of all fibre products of objects of $\mathcal{C}_n$. Set $\mathcal{C}' = \bigcup_{n\geq 1} \mathcal{C}_n$. A covering in $\mathcal{C}'$ is any family $\{\mathcal{F}_i \to \mathcal{F}\}_{i\in I}$ such that $\coprod_{i\in I} \mathcal{F}_i \to \mathcal{F}$ is surjective as a map of sheaves on $\mathcal{C}$. The functor $v : \mathcal{C} \to \mathcal{C}'$ is given by $U \mapsto h_U^\#$. The functor $u : \mathcal{D} \to \mathcal{C}'$ is given by $V \mapsto f^{-1}h_V^\#$.

Part (1) follows from Lemma 7.28.4.

Proof of (2) and (3) of the lemma. The functor u commutes with fibre products as both $V \mapsto h_V^\#$ and f^{-1} do. Moreover, since f^{-1} is exact and commutes with arbitrary colimits we see that it transforms a covering into a surjective family of morphisms of sheaves. Hence u is continuous. To see that it defines a morphism of sites we still have to see that u_s is exact. In order to do this we will show that $g^{-1} \circ u_s = f^{-1}$. Namely, then since g^{-1} is an equivalence and f^{-1} is exact we will conclude. Because g^{-1} is adjoint to g_*, and u_s is adjoint to u^s, and f^{-1} is adjoint to f_* it also suffices to prove that $u^s \circ g_* = f_*$. Let U be an object of $\mathcal{C}$ and let V be an object of $\mathcal{D}$. Then

$$\begin{aligned}(u^s g_* h_U^\#)(V) &= g_* h_U^\#(f^{-1}h_V^\#) \\ &= \mathrm{Mor}_{Sh(\mathcal{C})}(f^{-1}h_V^\#, h_U^\#) \\ &= \mathrm{Mor}_{Sh(\mathcal{D})}(h_V^\#, f_* h_U^\#) \\ &= f_* h_U^\#(V)\end{aligned}$$

The first equality because $u^s = u^p$. The second equality by Lemma 7.28.4 (5). The third equality by adjointness of f_* and f^{-1} and the final equality by properties of sheafification and the Yoneda lemma. We omit the verification that these identities are functorial in U and V. Hence we see that we have $u^s \circ g_* = f_*$ for sheaves of the form $h_U^\#$. This implies that $u^s \circ g_* = f_*$ and we win (some details omitted). □

Remark 7.28.7. Notation and assumptions as in Lemma 7.28.6. If the site $\mathcal{D}$ has a final object and fibre products then the functor $u : \mathcal{D} \to \mathcal{C}'$ satisfies all the assumptions of Proposition 7.15.6. Namely, in addition to the properties mentioned in the lemma u also transforms the final object of $\mathcal{D}$ into the final object of $\mathcal{C}'$. This is clear from the construction of u. Hence, if we first apply Lemmas 7.28.5 to $\mathcal{D}$ and then Lemma 7.28.6 to the resulting morphism of topoi $Sh(\mathcal{C}) \to Sh(\mathcal{D}')$ we obtain the following statement: Any morphism of topoi $f : Sh(\mathcal{C}) \to Sh(\mathcal{D})$ fits into a commutative diagram

$$\begin{array}{ccc} Sh(\mathcal{C}) & \xrightarrow{f} & Sh(\mathcal{D}) \\ \downarrow{\scriptstyle g} & & \downarrow{\scriptstyle e} \\ Sh(\mathcal{C}') & \xrightarrow{f'} & Sh(\mathcal{D}') \end{array}$$

where the following properties hold:

(1) the morphisms e and g are equivalences given by special cocontinuous functors $\mathcal{C} \to \mathcal{C}'$ and $\mathcal{D} \to \mathcal{D}'$,
(2) the sites $\mathcal{C}'$ and $\mathcal{D}'$ have fibre products, final objects and have subcanonical topologies,

(3) the morphism $f' : \mathcal{C}' \to \mathcal{D}'$ comes from a morphism of sites corresponding to a functor $u : \mathcal{D}' \to \mathcal{C}'$ to which Proposition 7.15.6 applies, and
(4) given any set of sheaves $\mathcal{F}_i$ (resp. $\mathcal{G}_j$) on $\mathcal{C}$ (resp. $\mathcal{D}$) we may assume each of these is a representable sheaf on $\mathcal{C}'$ (resp. $\mathcal{D}'$).

It is often useful to replace $\mathcal{C}$ and $\mathcal{D}$ by $\mathcal{C}'$ and $\mathcal{D}'$.

Remark 7.28.8. Notation and assumptions as in Lemma 7.28.6. Suppose that in addition the original morphism of topoi $Sh(\mathcal{C}) \to Sh(\mathcal{D})$ is an equivalence. Then the construction in the proof of Lemma 7.28.6 gives two functors

$$\mathcal{C} \to \mathcal{C}' \leftarrow \mathcal{D}$$

which are both special continuous functors. Hence in this case we can actually factor the morphism of topoi as a composition

$$Sh(\mathcal{C}) \to Sh(\mathcal{C}') = Sh(\mathcal{D}') \leftarrow Sh(\mathcal{D})$$

as in Remark 7.28.7, but with the middle morphism an identity.

7.29. Localization of topoi

We repeat some of the material on localization to the apparently more general case of topoi. In reality this is not more general since we may always enlarge the underlying sites to assume that we are localizing at objects of the site.

Lemma 7.29.1. *Let $\mathcal{C}$ be a site. Let $\mathcal{F}$ be a sheaf on $\mathcal{C}$. Then the category $Sh(\mathcal{C})/\mathcal{F}$ is a topos. There is a canonical morphism of topoi*

$$j_{\mathcal{F}} : Sh(\mathcal{C})/\mathcal{F} \longrightarrow Sh(\mathcal{C})$$

which is a localization as in Section 7.24 such that

(1) *the functor $j_{\mathcal{F}}^{-1}$ is the functor $\mathcal{H} \mapsto \mathcal{H} \times \mathcal{F}/\mathcal{F}$, and*
(2) *the functor $j_{\mathcal{F}!}$ is the forgetful functor $\mathcal{G}/\mathcal{F} \mapsto \mathcal{G}$.*

Proof. Apply Lemma 7.28.5. This means we may assume $\mathcal{C}$ is a site with subcanonical topology, and $\mathcal{F} = h_U = h_U^\#$ for some $U \in \mathrm{Ob}(\mathcal{C})$. Hence the material of Section 7.24 applies. In particular, there is an equivalence $Sh(\mathcal{C}/U) = Sh(\mathcal{C})/h_U^\#$ such that the composition

$$Sh(\mathcal{C}/U) \to Sh(\mathcal{C})/h_U^\# \to Sh(\mathcal{C})$$

is equal to $j_{U!}$, see Lemma 7.24.4. Denote $a : Sh(\mathcal{C})/h_U^\# \to Sh(\mathcal{C}/U)$ the inverse functor, so $j_{\mathcal{F}!} = j_{U!} \circ a$, $j_{\mathcal{F}}^{-1} = j_U^{-1} \circ a$ and $j_{\mathcal{F},*} = j_{U,*} \circ a$. The description of $j_{\mathcal{F}!}$ follows from the above. The description of $j_{\mathcal{F}}^{-1}$ follows from Lemma 7.24.6. □

Remark 7.29.2. In the situation of Lemma 7.29.1 we can also describe the functor $j_{\mathcal{F},*}$. It is the functor which associates to $\varphi : \mathcal{G} \to \mathcal{F}$ the sheaf

$$U \longmapsto \{\alpha : \mathcal{F}|_U \to \mathcal{G}|_U \text{ such that } \alpha \text{ is a right inverse to } \varphi|_U\}$$

In order to prove that this works the introduction of $\mathcal{H}om$-sheaves is desirable, hence we postpone this to a later time.

Lemma 7.29.3. *Let $\mathcal{C}$ be a site. Let $\mathcal{F}$ be a sheaf on $\mathcal{C}$. Let $\mathcal{C}/\mathcal{F}$ be the category of pairs (U, s) where $U \in \mathrm{Ob}(\mathcal{C})$ and $s \in \mathcal{F}(U)$. Let a covering in $\mathcal{C}/\mathcal{F}$ be a family $\{(U_i, s_i) \to (U, s)\}$ such that $\{U_i \to U\}$ is a covering of $\mathcal{C}$. Then $j : \mathcal{C}/\mathcal{F} \to \mathcal{C}$ is a continuous and cocontinuous functor of sites which induces a morphism of topi*

$j : Sh(\mathcal{C}/\mathcal{F}) \to Sh(\mathcal{C})$. In fact, there is an equivalence $Sh(\mathcal{C}/\mathcal{F}) = Sh(\mathcal{C})/\mathcal{F}$ which turns j into $j_{\mathcal{F}}$.

Proof. We omit the verification that $\mathcal{C}/\mathcal{F}$ is a site and that j is continuous and cocontinuous. By Lemma 7.20.5 there exists a morphism of topoi j as indicated, with $j^{-1}\mathcal{G}(U, s) = \mathcal{G}(U)$, and there is a left adjoint $j_!$ to j^{-1}. A morphism $\varphi : * \to g^{-1}\mathcal{G}$ on $\mathcal{C}/\mathcal{F}$ is the same thing as a rule which assigns to every pair (U, s) a section $\varphi(s) \in \mathcal{G}(U)$ compatible with restriction maps. Hence this is the same thing as a morphism $\varphi : \mathcal{F} \to \mathcal{G}$ over $\mathcal{C}$. We conclude that $j_!* = \mathcal{F}$. In particular, for every $\mathcal{H} \in Sh(\mathcal{C}/\mathcal{F})$ there is a canonical map

$$j_!\mathcal{H} \to j_!* = \mathcal{F}$$

i.e., we obtain a functor $j'_! : Sh(\mathcal{C}/\mathcal{F}) \to Sh(\mathcal{C})/\mathcal{F}$. An inverse to this functor is the rule which assigns to an object $\varphi : \mathcal{G} \to \mathcal{F}$ of $Sh(\mathcal{C})/\mathcal{F}$ the sheaf

$$a(\mathcal{G}/\mathcal{F}) : (U, s) \longmapsto \{t \in \mathcal{G}(U) \mid \varphi(t) = s\}$$

We omit the verification that $a(\mathcal{G}/\mathcal{F})$ is a sheaf and that a is inverse to $j'_!$. □

Definition 7.29.4. Let $\mathcal{C}$ be a site. Let $\mathcal{F}$ be a sheaf on $\mathcal{C}$.

(1) The topos $Sh(\mathcal{C})/\mathcal{F}$ is called the *localization of the topos $Sh(\mathcal{C})$ at $\mathcal{F}$*.
(2) The morphism of topoi $j_{\mathcal{F}} : Sh(\mathcal{C})/\mathcal{F} \to Sh(\mathcal{C})$ of Lemma 7.29.1 is called the *localization morphism*.

We are going to show that whenever the sheaf $\mathcal{F}$ is equal to $h_U^\#$ for some object U of the site, then the localization of the topos is equal to the category of sheaves on the localization of the site at U. Moreover, we are going to check that any functorialities are compatible with this identification.

Lemma 7.29.5. *Let $\mathcal{C}$ be a site. Let $\mathcal{F} = h_U^\#$ for some object U of $\mathcal{C}$. Then $j_{\mathcal{F}} : Sh(\mathcal{C})/\mathcal{F} \to Sh(\mathcal{C})$ constructed in Lemma 7.29.1 agrees with the morphism of topoi $j_U : Sh(\mathcal{C}/U) \to Sh(\mathcal{C})$ constructed in Section 7.24 via the identification $Sh(\mathcal{C}/U) = Sh(\mathcal{C})/h_U^\#$ of Lemma 7.24.4.*

Proof. We have seen in Lemma 7.24.4 that the composition $Sh(\mathcal{C}/U) \to Sh(\mathcal{C})/h_U^\# \to Sh(\mathcal{C})$ is $j_{U!}$. The functor $Sh(\mathcal{C})/h_U^\# \to Sh(\mathcal{C})$ is $j_{\mathcal{F}!}$ by Lemma 7.29.1. Hence $j_{\mathcal{F}!} = j_{U!}$ via the identification. So $j_{\mathcal{F}}^{-1} = j_U^{-1}$ (by adjointness) and so $j_{\mathcal{F},*} = j_{U,*}$ (by adjointness again). □

Lemma 7.29.6. *Let $\mathcal{C}$ be a site. If $s : \mathcal{G} \to \mathcal{F}$ is a morphism of sheaves on $\mathcal{C}$ then there exists a natural commutative diagram of morphisms of topoi*

$$\begin{array}{ccc} Sh(\mathcal{C})/\mathcal{G} & \xrightarrow{\quad j \quad} & Sh(\mathcal{C})/\mathcal{F} \\ & {\scriptstyle j_{\mathcal{G}}} \searrow \quad \swarrow {\scriptstyle j_{\mathcal{F}}} & \\ & Sh(\mathcal{C}) & \end{array}$$

where $j = j_{\mathcal{G}/\mathcal{F}}$ is the localization of the topos $Sh(\mathcal{C})/\mathcal{F}$ at the object $\mathcal{G}/\mathcal{F}$. In particular we have

$$j^{-1}(\mathcal{H} \to \mathcal{F}) = (\mathcal{H} \times_{\mathcal{F}} \mathcal{G} \to \mathcal{G}).$$

and

$$j_!(\mathcal{E} \xrightarrow{e} \mathcal{F}) = (\mathcal{E} \xrightarrow{s \circ e} \mathcal{G}).$$

Proof. The description of j^{-1} and $j_!$ comes from the description of those functors in Lemma 7.29.1. The equality of functors $j_{\mathcal{G}!} = j_{\mathcal{F}!} \circ j_!$ is clear from the description of these functors (as forgetful functors). By adjointness we also obtain the equalities $j_{\mathcal{G}}^{-1} = j^{-1} \circ j_{\mathcal{F}}^{-1}$, and $j_{\mathcal{G},*} = j_{\mathcal{F},*} \circ j_*$. □

Lemma 7.29.7. *Assume $\mathcal{C}$ and $s : \mathcal{G} \to \mathcal{F}$ are as in Lemma 7.29.6. If $\mathcal{G} = h_V^\#$ and $\mathcal{F} = h_U^\#$ and $s : \mathcal{G} \to \mathcal{F}$ comes from a morphism $V \to U$ of $\mathcal{C}$ then the diagram in Lemma 7.29.6 is identified with diagram (7.24.7.1) via the identifications $Sh(\mathcal{C}/V) = Sh(\mathcal{C})/h_V^\#$ and $Sh(\mathcal{C}/U) = Sh(\mathcal{C})/h_U^\#$ of Lemma 7.24.4.*

Proof. This is true because the descriptions of j^{-1} agree. See Lemma 7.24.8 and Lemma 7.29.6. □

7.30. Localization and morphisms of topoi

This section is the analogue of Section 7.27 for morphisms of topoi.

Lemma 7.30.1. *Let $f : Sh(\mathcal{C}) \to Sh(\mathcal{D})$ be a morphism of topoi. Let $\mathcal{G}$ be a sheaf on $\mathcal{D}$. Set $\mathcal{F} = f^{-1}\mathcal{G}$. Then there exists a commutative diagram of topoi*

$$\begin{array}{ccc} Sh(\mathcal{C})/\mathcal{F} & \xrightarrow[j_{\mathcal{F}}]{} & Sh(\mathcal{C}) \\ {\scriptstyle f'}\downarrow & & \downarrow{\scriptstyle f} \\ Sh(\mathcal{D})/\mathcal{G} & \xrightarrow{j_{\mathcal{G}}} & Sh(\mathcal{D}). \end{array}$$

The morphism f' is characterized by the property that

$$(f')^{-1}(\mathcal{H} \xrightarrow{\varphi} \mathcal{G}) = (f^{-1}\mathcal{H} \xrightarrow{f^{-1}\varphi} \mathcal{F})$$

and we have $f'_* j_{\mathcal{F}}^{-1} = j_{\mathcal{G}}^{-1} f_*$.

Proof. Since the statement is about topoi and does not refer to the underlying sites we may change sites at will. Hence by the discussion in Remark 7.28.7 we may assume that f is given by a continuous functor $u : \mathcal{D} \to \mathcal{C}$ satisfying the assumptions of Proposition 7.15.6 between sites having all finite limits and subcanonical topologies, and such that $\mathcal{G} = h_V$ for some object V of $\mathcal{D}$. Then $\mathcal{F} = f^{-1}h_V = h_{u(V)}$ by Lemma 7.14.5. By Lemma 7.27.1 we obtain a commutative diagram of morphisms of topoi

$$\begin{array}{ccc} Sh(\mathcal{C}/U) & \xrightarrow[j_U]{} & Sh(\mathcal{C}) \\ {\scriptstyle f'}\downarrow & & \downarrow{\scriptstyle f} \\ Sh(\mathcal{D}/V) & \xrightarrow{j_V} & Sh(\mathcal{D}), \end{array}$$

and we have $f'_* j_U^{-1} = j_V^{-1} f_*$. By Lemma 7.29.5 we may identify $j_{\mathcal{F}}$ and j_U and $j_{\mathcal{G}}$ and j_V. The description of $(f')^{-1}$ is given in Lemma 7.27.1. □

Lemma 7.30.2. *Let $f : \mathcal{C} \to \mathcal{D}$ be a morphism of sites given by the continuous functor $u : \mathcal{D} \to \mathcal{C}$. Let V be an object of $\mathcal{D}$. Set $U = u(V)$. Set $\mathcal{G} = h_V^\#$, and $\mathcal{F} = h_U^\# = f^{-1}h_V^\#$ (see Lemma 7.14.5). Then the diagram of morphisms of topoi of Lemma 7.30.1 agrees with the diagram of morphisms of topoi of Lemma 7.27.1 via the identifications $j_{\mathcal{F}} = j_U$ and $j_{\mathcal{G}} = j_V$ of Lemma 7.29.5.*

Proof. This is not a complete triviality as the choice of morphism of sites giving rise to f made in the proof of Lemma 7.30.1 may be different from the morphisms of sites given to us in the lemma. But in both cases the functor $(f')^{-1}$ is described by the same rule. Hence they agree and the associated morphism of topoi is the same. Some details omitted. □

Lemma 7.30.3. *Let $f : Sh(\mathcal{C}) \to Sh(\mathcal{D})$ be a morphism of topoi. Let $\mathcal{G} \in Sh(\mathcal{D})$, $\mathcal{F} \in Sh(\mathcal{C})$ and $s : \mathcal{F} \to f^{-1}\mathcal{G}$ a morphism of sheaves. There exists a commutative diagram of topoi*

$$\begin{array}{ccc} Sh(\mathcal{C})/\mathcal{F} & \xrightarrow[j_{\mathcal{F}}]{} & Sh(\mathcal{C}) \\ {\scriptstyle f_s}\downarrow & & \downarrow{\scriptstyle f} \\ Sh(\mathcal{D})/\mathcal{G} & \xrightarrow{j_{\mathcal{G}}} & Sh(\mathcal{D}). \end{array}$$

We have $f_s = f' \circ j_{\mathcal{F}/f^{-1}\mathcal{G}}$ where $f' : Sh(\mathcal{C})/f^{-1}\mathcal{G} \to Sh(\mathcal{D})/\mathcal{F}$ is as in Lemma 7.30.1 and $j_{\mathcal{F}/f^{-1}\mathcal{G}} : Sh(\mathcal{C})/\mathcal{F} \to Sh(\mathcal{C})/f^{-1}\mathcal{G}$ is as in Lemma 7.29.6. The functor $(f_s)^{-1}$ is described by the rule

$$(f_s)^{-1}(\mathcal{H} \xrightarrow{\varphi} \mathcal{G}) = (f^{-1}\mathcal{H} \times_{f^{-1}\varphi, f^{-1}\mathcal{G}, s} \mathcal{F} \to \mathcal{F}).$$

Finally, given any morphisms $b : \mathcal{G}' \to \mathcal{G}$, $a : \mathcal{F}' \to \mathcal{F}$ and $s' : \mathcal{F}' \to f^{-1}\mathcal{G}'$ such that

$$\begin{array}{ccc} \mathcal{F}' & \xrightarrow[s']{} & f^{-1}\mathcal{G}' \\ {\scriptstyle a}\downarrow & & \downarrow{\scriptstyle f^{-1}b} \\ \mathcal{F} & \xrightarrow{s} & f^{-1}\mathcal{G} \end{array}$$

commutes, then the diagram

$$\begin{array}{ccc} Sh(\mathcal{C})/\mathcal{F}' & \xrightarrow[j_{\mathcal{F}'/\mathcal{F}}]{} & Sh(\mathcal{C})/\mathcal{F} \\ {\scriptstyle f_{s'}}\downarrow & & \downarrow{\scriptstyle f_s} \\ Sh(\mathcal{D})/\mathcal{G}' & \xrightarrow{j_{\mathcal{G}'/\mathcal{G}}} & Sh(\mathcal{D})/\mathcal{G}. \end{array}$$

commutes.

Proof. The commutativity of the first square follows from the commutativity of the diagram in Lemma 7.29.6 and the commutativity of the diagram in Lemma 7.30.1. The description of f_s^{-1} follows on combining the descriptions of $(f')^{-1}$ in Lemma 7.30.1 with the description of $(j_{\mathcal{F}/f^{-1}\mathcal{G}})^{-1}$ in Lemma 7.29.6. The commutativity of the last square then follows from the equality

$$f^{-1}\mathcal{H} \times_{f^{-1}\mathcal{G}, s} \mathcal{F} \times_{\mathcal{F}} \mathcal{F}' = f^{-1}(\mathcal{H} \times_{\mathcal{G}} \mathcal{G}') \times_{\mathcal{G}', s'} \mathcal{F}'$$

which is formal. □

Lemma 7.30.4. *Let $f : \mathcal{C} \to \mathcal{D}$ be a morphism of sites given by the continuous functor $u : \mathcal{D} \to \mathcal{C}$. Let V be an object of $\mathcal{D}$. Let $c : U \to u(V)$ be a morphism. Set $\mathcal{G} = h_V^\#$ and $\mathcal{F} = h_U^\# = f^{-1}h_V^\#$. Let $s : \mathcal{F} \to f^{-1}\mathcal{G}$ be the map induced by c. Then the diagram of morphisms of topoi of Lemma 7.27.3 agrees with the diagram of morphisms of topoi of Lemma 7.30.3 via the identifications $j_{\mathcal{F}} = j_U$ and $j_{\mathcal{G}} = j_V$ of Lemma 7.29.5.*

Proof. This follows on combining Lemmas 7.29.7 and 7.30.2. □

7.31. Points

Definition 7.31.1. Let $\mathcal{C}$ be a site. A *point of the topos* $Sh(\mathcal{C})$ is a morphism of topoi p from $Sh(pt)$ to $Sh(\mathcal{C})$.

We will define a point of a site in terms of a functor $u : \mathcal{C} \to Sets$. It will turn out later that u will define a morphism of sites which gives rise to a point of the topos associated to $\mathcal{C}$, see Lemma 7.31.8.

Let $\mathcal{C}$ be a site. Let $p = u$ be a functor $u : \mathcal{C} \to Sets$. This curious language is introduced because it seems funny to talk about neighbourhoods of functors; so we think of a "point" p as a geometric thing which is given by a categorical datum, namely the functor u. The fact that p is actually equal to u does not matter. A *neighbourhood* of p is a pair (U, x) with $U \in \mathrm{Ob}(\mathcal{C})$ and $x \in u(U)$. A *morphism of neighbourhoods* $(V, y) \to (U, x)$ is given by a morphism $\alpha : V \to U$ of $\mathcal{C}$ such that $u(\alpha)(y) = x$. Note that the category of neighbourhoods isn't a "big" category.

We define the *stalk* of a presheaf $\mathcal{F}$ at p as

$$\mathcal{F}_p = \mathrm{colim}_{\{(U,x)\}^{opp}} \mathcal{F}(U). \tag{7.31.1.1}$$

The colimit is over the opposite of the category of neighbourhoods of p. In other words, an element of $\mathcal{F}_p$ is given by a triple (U, x, s), where (U, x) is a neighbourhood of p and $s \in \mathcal{F}(U)$. Equality of triples is the equivalence relation generated by $(U, x, s) \sim (V, y, \alpha^* s)$ when α is as above.

Note that if $\varphi : \mathcal{F} \to \mathcal{G}$ is a morphism of presheaves of sets, then we get a canonical map of stalks $\varphi_p : \mathcal{F}_p \to \mathcal{G}_p$. Thus we obtain a *stalk functor*

$$PSh(\mathcal{C}) \longrightarrow Sets, \quad \mathcal{F} \longmapsto \mathcal{F}_p.$$

We have defined the stalk functor using any functor $p = u : \mathcal{C} \to Sets$. No conditions are necessary for the definition to work[5]. On the other hand, it is probably better not to use this notion unless p actually is a point (see definition below), since in general the stalk functor does not have good properties.

Definition 7.31.2. Let $\mathcal{C}$ be a site. A *point* p *of the site* $\mathcal{C}$ is given by a functor $u : \mathcal{C} \to Sets$ such that

(1) For every covering $\{U_i \to U\}$ of $\mathcal{C}$ the map $\coprod u(U_i) \to u(U)$ is surjective.
(2) For every covering $\{U_i \to U\}$ of $\mathcal{C}$ and every morphism $V \to U$ the maps $u(U_i \times_U V) \to u(U_i) \times_{u(U)} u(V)$ are bijective.
(3) The stalk functor $Sh(\mathcal{C}) \to Sets$, $\mathcal{F} \to \mathcal{F}_p$ is left exact.

The conditions should be familiar since they are modeled after those of Definitions 7.14.1 and 7.15.1. Note that (3) implies that $*_p = \{*\}$, see Example 7.10.2. Hence $u(U) \neq \emptyset$ for at least some U (because the empty colimit produces the empty set). We will show below (Lemma 7.31.7) that this does give rise to a point of the topos $Sh(\mathcal{C})$. Before we do so, we prove some lemmas for general functors u.

Lemma 7.31.3. *Let $\mathcal{C}$ be a site. Let $p = u : \mathcal{C} \to Sets$ be a functor. There are functorial isomorphisms $(h_U)_p = u(U)$ for $U \in \mathrm{Ob}(\mathcal{C})$.*

[5] One should try to avoid the case where $u(U) = \emptyset$ for all U.

Proof. An element of $(h_U)_p$ is given by a triple (V, y, f), where $V \in \mathrm{Ob}(\mathcal{C})$, $y \in u(V)$ and $f \in h_U(V) = \mathrm{Mor}_{\mathcal{C}}(V, U)$. Two such (V, y, f), (V', y', f') determine the same object if there exists a morphism $\phi : V \to V'$ such that $u(\phi)(x) = x'$ and $f' \circ \phi = f$, and in general you have to take chains of identities like this to get the correct equivalence relation. In any case, every (V, y, f) is equivalent to the element $(U, u(f)(y), \mathrm{id}_U)$. If ϕ exists as above, then the triples (V, y, f), (V', y', f') determine the same triple $(U, u(f)(y), \mathrm{id}_U) = (U, u(f')(y'), \mathrm{id}_U)$. This proves that the map $u(U) \to (h_U)_p$, $x \mapsto$ class of (U, x, id_U) is bijective. □

Let $\mathcal{C}$ be a site. Let $p = u : \mathcal{C} \to \mathit{Sets}$ be a functor. In analogy with the constructions in Section 7.5 given a set E we define a presheaf u^pE by the rule

$$U \longmapsto u^pE(U) = \mathrm{Mor}_{\mathit{Sets}}(u(U), E) = \mathrm{Map}(u(U), E). \tag{7.31.3.1}$$

This defines a functor $u^p : \mathit{Sets} \to \mathit{PSh}(\mathcal{C})$, $E \mapsto u^pE$.

Lemma 7.31.4. *For any functor $u : \mathcal{C} \to \mathit{Sets}$. The functor u^p is a right adjoint to the stalk functor on presheaves.*

Proof. Let $\mathcal{F}$ be a presheaf on $\mathcal{C}$. Let E be a set. A morphism $\mathcal{F} \to u^pE$ is given by a compatible system of maps $\mathcal{F}(U) \to \mathrm{Map}(u(U), E)$, i.e., a compatible system of maps $\mathcal{F}(U) \times u(U) \to E$. And by definition of $\mathcal{F}_p$ a map $\mathcal{F}_p \to E$ is given by a rule associating with each triple (U, x, σ) an element in E such that equivalent triples map to the same element, see discussion surrounding Equation (7.31.1.1). This also means a compatible system of maps $\mathcal{F}(U) \times u(U) \to E$. □

In analogy with Section 7.14 we have the following lemma.

Lemma 7.31.5. *Let $\mathcal{C}$ be a site. Let $p = u : \mathcal{C} \to \mathit{Sets}$ be a functor. Suppose that for every covering $\{U_i \to U\}$ of $\mathcal{C}$*

1. *the map $\coprod u(U_i) \to u(U)$ is surjective and*
2. *the maps $u(U_i \times_U U_j) \to u(U_i) \times_{u(U)} u(U_j)$ are surjective.*

Then we have

1. *the presheaf u^pE is a sheaf for all sets E, denote it u^sE,*
2. *the stalk functor $\mathit{Sh}(\mathcal{C}) \to \mathit{Sets}$ and the functor $u^s : \mathit{Sets} \to \mathit{Sh}(\mathcal{C})$ are adjoint, and*
3. *we have $\mathcal{F}_p = \mathcal{F}_p^\#$ for every presheaf of sets $\mathcal{F}$.*

Proof. The first assertion is immediate from the definition of a sheaf, assumptions (1) and (2), and the definition of u^pE. The second is a restatement of the adjointness of u^p and the stalk functor (but now restricted to sheaves). The third assertion follows as, for any set E, we have

$$\mathrm{Map}(\mathcal{F}_p, E) = \mathrm{Mor}_{\mathit{PSh}(\mathcal{C})}(\mathcal{F}, u^pE) = \mathrm{Mor}_{\mathit{Sh}(\mathcal{C})}(\mathcal{F}^\#, u^sE) = \mathrm{Map}(\mathcal{F}_p^\#, E)$$

by the adjointness property of sheafification. □

In particular Lemma 7.31.5 holds when $p = u$ is a point. In this case we think of the sheaf u^sE as the "skyscraper" sheaf with value E at p.

Definition 7.31.6. Let p be a point of the site $\mathcal{C}$ given by the functor u. For a set E we define $p_*E = u^sE$ the sheaf described in Lemma 7.31.5 above. We sometimes call this a *skyscraper sheaf*.

In particular we have the following adjointness property of skyscraper sheaves and stalks:

$$\mathrm{Mor}_{Sh(\mathcal{C})}(\mathcal{F}, p_*E) = \mathrm{Map}(\mathcal{F}_p, E)$$

This motivates the notation $p^{-1}\mathcal{F} = \mathcal{F}_p$ which we will sometimes use.

Lemma 7.31.7. *Let $\mathcal{C}$ be a site.*

1. *Let p be a point of the site $\mathcal{C}$. Then the pair of functors (p_*, p^{-1}) introduced above define a morphism of topoi $Sh(pt) \to Sh(\mathcal{C})$.*
2. *Let $p = (p_*, p^{-1})$ be a point of the topos $Sh(\mathcal{C})$. Then the functor $u : U \mapsto p^{-1}(h_U^\#)$ gives rise to a point p' of the site $\mathcal{C}$ whose associated morphism of topoi $(p'_*, (p')^{-1})$ is equal to p.*

Proof. Proof of (1). By the above the functors p_* and p^{-1} are adjoint. The functor p^{-1} is required to be exact by Definition 7.31.2. Hence the conditions imposed in Definition 7.16.1 are all satisfied and we see that (1) holds.

Proof of (2). Let $\{U_i \to U\}$ be a covering of $\mathcal{C}$. Then $\coprod (h_{U_i})^\# \to h_U^\#$ is surjective, see Lemma 7.13.4. Since p^{-1} is exact (by definition of a morphism of topoi) we conclude that $\coprod u(U_i) \to u(U)$ is surjective. This proves part (1) of Definition 7.31.2. Sheafification is exact, see Lemma 7.10.14. Hence if $U \times_V W$ exists in $\mathcal{C}$, then

$$h^\#_{U \times_V W} = h^\#_U \times_{h^\#_V} h^\#_W$$

and we see that $u(U \times_V W) = u(U) \times_{u(V)} u(W)$ since p^{-1} is exact. This proves part (2) of Definition 7.31.2. Let $p' = u$, and let $\mathcal{F}_{p'}$ be the stalk functor defined by Equation (7.31.1.1) using u. There is a canonical comparison map $c : \mathcal{F}_{p'} \to \mathcal{F}_p = p^{-1}\mathcal{F}$. Namely, given a triple (U, x, σ) representing an element ξ of $\mathcal{F}_{p'}$ we think of σ as a map $\sigma : h_U^\# \to \mathcal{F}$ and we can set $c(\xi) = p^{-1}(\sigma)(x)$ since $x \in u(U) = p^{-1}(h_U^\#)$. By Lemma 7.31.3 we see that $(h_U)_{p'} = u(U)$. Since conditions (1) and (2) of Definition 7.31.2 hold for p' we also have $(h_U^\#)_{p'} = (h_U)_{p'}$ by Lemma 7.31.5. Hence we have

$$(h_U^\#)_{p'} = (h_U)_{p'} = u(U) = p^{-1}(h_U^\#)$$

We claim this bijection equals the comparison map $c : (h_U^\#)_{p'} \to p^{-1}(h_U^\#)$ (verification omitted). Any sheaf on $\mathcal{C}$ is a coequalizer of maps of coproducts of sheaves of the form $h_U^\#$, see Lemma 7.13.5. The stalk functor $\mathcal{F} \mapsto \mathcal{F}_{p'}$ and the functor p^{-1} commute with arbitrary colimits (as they are both left adjoints). We conclude c is an isomorphism for every sheaf $\mathcal{F}$. Thus the stalk functor $\mathcal{F} \mapsto \mathcal{F}_{p'}$ is isomorphic to p^{-1} and we in particular see that it is exact. This proves condition (3) of Definition 7.31.2 holds and p' is a point. The final assertion has already been shown above, since we saw that $p^{-1} = (p')^{-1}$. □

Actually a point always corresponds to a morphism of sites as we show in the following lemma.

Lemma 7.31.8. *Let $\mathcal{C}$ be a site. Let p be a point of $\mathcal{C}$ given by $u : \mathcal{C} \to Sets$. Let S_0 be an infinite set such that $u(U) \subset S_0$ for all $U \in \mathrm{Ob}(\mathcal{C})$. Let $\mathcal{S}$ be the site constructed out of the powerset $S = \mathcal{P}(S_0)$ in Remark 7.16.3. Then*

1. *there is an equivalence $i : Sh(pt) \to Sh(\mathcal{S})$,*
2. *the functor $u : \mathcal{C} \to \mathcal{S}$ induces a morphism of sites $f : \mathcal{S} \to \mathcal{C}$, and*

(3) *the composition*

$$Sh(pt) \to Sh(\mathcal{S}) \to Sh(\mathcal{C})$$

is the morphism of topoi (p_*, p^{-1}) *of Lemma 7.31.7.*

Proof. Part (1) we saw in Remark 7.16.3. Moreover, recall that the equivalence associates to the set E the sheaf i_*E on $\mathcal{S}$ defined by the rule $V \mapsto \mathrm{Mor}_{Sets}(V, E)$. Part (2) is clear from the definition of a point of $\mathcal{C}$ (Definition 7.31.2) and the definition of a morphism of sites (Definition 7.15.1). Finally, consider f_*i_*E. By construction we have

$$f_*i_*E(U) = i_*E(u(U)) = \mathrm{Mor}_{Sets}(u(U), E)$$

which is equal to $p_*E(U)$, see Equation (7.31.3.1). This proves (3). □

Contrary to what happens in the topological case it is not always true that the stalk of the skyscraper sheaf with value E is E. Here is what is true in general.

Lemma 7.31.9. *Let* $\mathcal{C}$ *be a site. Let* $p : Sh(pt) \to Sh(\mathcal{C})$ *be a point of the topos associated to* $\mathcal{C}$*. For any set* E *there are canonical maps*

$$E \longrightarrow (p_*E)_p \longrightarrow E$$

whose composition is id_E*.*

Proof. There is always an adjunction map $(p_*E)_p = p^{-1}p_*E \to E$. This map is an isomorphism when $E = \{*\}$ because p_* and p^{-1} are both left exact, hence transform the final object into the final object. Hence given $e \in E$ we can consider the map $i_e : \{*\} \to E$ which gives

$$\begin{array}{ccc} p^{-1}p_*\{*\} & \xrightarrow{p^{-1}p_*i_e} & p^{-1}p_*E \\ \cong \downarrow & & \downarrow \\ \{*\} & \xrightarrow{i_e} & E \end{array}$$

whence the map $E \to (p_*E)_p = p^{-1}p_*E$ as desired. □

Lemma 7.31.10. *Let* $\mathcal{C}$ *be a site. Let* $p : Sh(pt) \to Sh(\mathcal{C})$ *be a point of the topos associated to* $\mathcal{C}$*. The functor* $p_* : Sets \to Sh(\mathcal{C})$ *has the following properties: It commutes with arbitrary limits, it is left exact, it is faithful, it transforms surjections into surjections, it commutes with coequalizers, it reflects injections, it reflects surjections, and it reflects isomorphisms.*

Proof. Because p_* is a right adjoint it commutes with arbitrary limits and it is left exact. The fact that $p^{-1}p_*E \to E$ is a canonically split surjection implies that p_* is faithful, reflects injections, reflects surjections, and reflects isomorphisms. By Lemma 7.31.7 we may assume that p comes from a point $u : \mathcal{C} \to Sets$ of the underlying site $\mathcal{C}$. In this case the sheaf p_*E is given by

$$p_*E(U) = \mathrm{Mor}_{Sets}(u(U), E)$$

see Equation (7.31.3.1) and Definition 7.31.6. It follows immediately from this formula that p_* transforms surjections into surjections and coequalizers into coequalizers. □

7.32. Constructing points

In this section we give criteria for when a functor from a site to the category of sets defines a point of that site.

Lemma 7.32.1. *Let $\mathcal{C}$ be a site. Assume that $\mathcal{C}$ has a final object X and fibred products. Let $p = u : \mathcal{C} \to Sets$ be a functor such that*

(1) *$u(X)$ is a singleton set, and*
(2) *for every pair of morphisms $U \to W$ and $V \to W$ with the same target the map $u(U \times_W V) \to u(U) \times_{u(W)} u(V)$ is bijective.*

Then the opposite of the category of neighbourhoods of p is filtered. Moreover, the stalk functor $Sh(\mathcal{C}) \to Sets$, $\mathcal{F} \to \mathcal{F}_p$ commutes with finite limits.

Proof. This is analogous to the proof of Lemma 7.5.2 above. The assumptions on $\mathcal{C}$ imply that $\mathcal{C}$ has finite limits. See Categories, Lemma 4.18.4. Assumption (1) implies that the category of neighbourhoods is nonempty. Suppose (U, x) and (V, y) are neighbourhoods. Then $u(U \times V) = u(U \times_X V) = u(U) \times_{u(X)} u(V) = u(U) \times u(V)$ by (2). Hence there exists a neighbourhood $(U \times_X V, z)$ mapping to both (U, x) and (V, y). Let $a, b : (V, y) \to (U, x)$ be two morphisms in the category of neighbourhoods. Let W be the equalizer of $a, b : V \to U$. As in the proof of Categories, Lemma 4.18.4 we may write W in terms of fibre products:

$$W = (V \times_{a,U,b} V) \times_{(pr_1,pr_2),V\times V,\Delta} V$$

The bijectivity in (2) guarantees there exists an element $z \in u(W)$ which maps to $((y, y), y)$. Then $(W, z) \to (V, y)$ equalizes a, b as desired.

Let $\mathcal{I} \to Sh(\mathcal{C})$, $i \mapsto \mathcal{F}_i$ be a finite diagram of sheaves. We have to show that the stalk of the limit of this system agrees with the limit of the stalks. Let $\mathcal{F}$ be the limit of the system as a *presheaf*. According to Lemma 7.10.1 this is a sheaf and it is the limit in the category of sheaves. Hence we have to show that $\mathcal{F}_p = \lim_{\mathcal{I}} \mathcal{F}_{i,p}$. Recall also that $\mathcal{F}$ has a simple description, see Section 7.4. Thus we have to show that

$$\lim_i \operatorname{colim}_{\{(U,x)\}^{opp}} \mathcal{F}_i(U) = \operatorname{colim}_{\{(U,x)\}^{opp}} \lim_i \mathcal{F}_i(U).$$

This holds, by Categories, Lemma 4.19.2, because we just showed the opposite of the category of neighbourhoods is filtered. □

Proposition 7.32.2. *Let $\mathcal{C}$ be a site. Assume that finite limits exist in $\mathcal{C}$. (I.e., $\mathcal{C}$ has fibre products, and a final object.) A point p of such a site $\mathcal{C}$ is given by a functor $u : \mathcal{C} \to Sets$ such that*

(1) *u commutes with finite limits, and*
(2) *if $\{U_i \to U\}$ is a covering, then $\coprod_i u(U_i) \to u(U)$ is surjective.*

Proof. Suppose first that p is a point (Definition 7.31.2) given by a functor u. Condition (2) is satisfied directly from the definition of a point. By Lemma 7.31.3 we have $(h_U)_p = u(U)$. By Lemma 7.31.5 we have $(h_U^\#)_p = (h_U)_p$. Thus we see that u is equal to the composition of functors

$$\mathcal{C} \xrightarrow{h} PSh(\mathcal{C}) \xrightarrow{\#} Sh(\mathcal{C}) \xrightarrow{()_p} Sets$$

Each of these functors is left exact, and hence we see u satisfies (1).

Conversely, suppose that u satisfies (1) and (2). In this case we immediately see that u satisfies the first two conditions of Definition 7.31.2. And its stalk functor

is exact, because it is a left adjoint by Lemma 7.31.5 and it commutes with finite limits by Lemma 7.32.1. □

Remark 7.32.3. In fact, let $\mathcal{C}$ be a site. Assume $\mathcal{C}$ has a final object X and fibre products. Let $p = u : \mathcal{C} \to Sets$ be a functor such that

(1) $u(X) = \{*\}$ a singleton, and
(2) for every pair of morphisms $U \to W$ and $V \to W$ with the same target the map $u(U \times_W V) \to u(U) \times_{u(W)} u(V)$ is surjective.
(3) for every covering $\{U_i \to U\}$ the map $\coprod u(U_i) \to u(U)$ is surjective.

Then, in general, p is **not** a point of $\mathcal{C}$. An example is the category $\mathcal{C}$ with two objects $\{U, X\}$ and exactly one non-identity arrow, namely $U \to X$. We endow $\mathcal{C}$ with the trivial topology, i.e., the only coverings are $\{U \to U\}$ and $\{X \to X\}$. A sheaf $\mathcal{F}$ is the same thing as a presheaf and consists of a triple $(A, B, A \to B)$: namely $A = \mathcal{F}(X)$, $B = \mathcal{F}(U)$ and $A \to B$ is the restriction mapping corresponding to $U \to X$. Note that $U \times_X U = U$ so fibre products exist. Consider the functor $u = p$ with $u(X) = \{*\}$ and $u(U) = \{*_1, *_2\}$. This satisfies (1), (2), and (3), but the corresponding stalk functor (7.31.1.1) is the functor

$$(A, B, A \to B) \longmapsto B \coprod\nolimits_A B$$

which isn't exact. Namely, consider $(\emptyset, \{1\}, \emptyset \to \{1\}) \to (\{1\}, \{1\}, \{1\} \to \{1\})$ which is an injective map of sheaves, but is transformed into the noninjective map of sets

$$\{1\} \coprod \{1\} \longrightarrow \{1\} \coprod\nolimits_{\{1\}} \{1\}$$

by the stalk functor.

Example 7.32.4. Let X be a topological space. Let X_{Zar} be the site of Example 7.6.4. Let $x \in X$ be a point. Consider the functor

$$u : X_{Zar} \longrightarrow Sets, \quad U \mapsto \begin{cases} \emptyset & \text{if} \quad x \notin U \\ \{*\} & \text{if} \quad x \in U \end{cases}$$

This functor commutes with product and fibred products, and turns coverings into surjective families of maps. Hence we obtain a point p of the site X_{Zar}. It is immediately verified that the stalk functor agrees with the stalk at x defined in Sheaves, Section 6.11.

Example 7.32.5. Let X be a topological space. What are the points of the topos $Sh(X)$? To see this, let X_{Zar} be the site of Example 7.6.4. By Lemma 7.31.7 a point of $Sh(X)$ corresponds to a point of this site. Let p be a point of the site X_{Zar} given by the functor $u : X_{Zar} \to Sets$. We are going to use the characterization of such a u in Proposition 7.32.2. This implies immediately that $u(\emptyset) = \emptyset$ and $u(U \cap V) = u(U) \times u(V)$. In particular we have $u(U) = u(U) \times u(U)$ via the diagonal map which implies that $u(U)$ is either a singleton or empty. Moreover, if $U = \bigcup U_i$ is an open covering then

$$u(U) = \emptyset \Rightarrow \forall i,\ u(U_i) = \emptyset \quad \text{and} \quad u(U) \neq \emptyset \Rightarrow \exists i,\ u(U_i) \neq \emptyset.$$

We conclude that there is a unique largest open $W \subset X$ with $u(W) = \emptyset$, namely the union of all the opens U with $u(U) = \emptyset$. Let $Z = X \setminus W$. If $Z = Z_1 \cup Z_2$ with $Z_i \subset Z$ closed, then $W = (X \setminus Z_1) \cap (X \setminus Z_2)$ so $\emptyset = u(W) = u(X \setminus Z_1) \times u(X \setminus Z_2)$ and we conclude that $u(X \setminus Z_1) = \emptyset$ or that $u(X \setminus Z_2) = \emptyset$. This means that

$X \setminus Z_1 = W$ or that $X \setminus Z_2 = W$. In other words, Z is irreducible. Now we see that u is described by the rule

$$u : X_{Zar} \longrightarrow Sets, \quad U \mapsto \begin{cases} \emptyset & \text{if} \quad Z \cap U = \emptyset \\ \{*\} & \text{if} \quad Z \cap U \neq \emptyset \end{cases}$$

Note that for any irreducible closed $Z \subset X$ this functor satisfies assumptions (1), (2) of Proposition 7.32.2 and hence defines a point. In other words we see that points of the site X_{Zar} are in one-to-one correspondence with irreducible closed subsets of X. In particular, if X is a sober topological space, then points of X_{Zar} and points of X are in one to one correspondence, see Example 7.32.4.

Example 7.32.6. Consider the site $\mathcal{T}_G$ described in Example 7.6.5 and Section 7.9. The forgetful functor $u : \mathcal{T}_G \to Sets$ commutes with products and fibred products and turns coverings into surjective families. Hence it defines a point of $\mathcal{T}_G$. We identify $Sh(\mathcal{T}_G)$ and G-$Sets$. The stalk functor

$$p^{-1} : Sh(\mathcal{T}_G) = G\text{-}Sets \longrightarrow Sets$$

is the forgetful functor. The pushforward p_* is the functor

$$Sets \longrightarrow Sh(\mathcal{T}_G) = G\text{-}Sets$$

which maps a set S to the G-set $\mathrm{Map}(G, S)$ with action $g \cdot \psi = \psi \circ R_g$ where R_g is right multiplication. In particular we have $p^{-1}p_*S = \mathrm{Map}(G, S)$ as a set and the maps $S \to \mathrm{Map}(G, S) \to S$ of Lemma 7.31.9 are the obvious ones.

Example 7.32.7. Let $\mathcal{C}$ be a category endowed with the chaotic topology (Example 7.6.6). For every object U_0 of $\mathcal{C}$ the functor $u : U \mapsto \mathrm{Mor}_{\mathcal{C}}(U_0, U)$ defines a point p of $\mathcal{C}$. Namely, conditions (1) and (2) of Definition 7.31.2 are immediate as the only coverings are given by identity maps. Condition (2) holds because $\mathcal{F}_p = \mathcal{F}(U_0)$ and since the topology is discrete taking sections over U_0 is an exact functor.

7.33. Points and morphisms of topoi

In this section we make a few remarks about points and morphisms of topoi.

Lemma 7.33.1. *Let $f : \mathcal{D} \to \mathcal{C}$ be a morphism of sites given by a continuous functor $u : \mathcal{C} \to \mathcal{D}$. Let p be a point of $\mathcal{D}$ given by the functor $v : \mathcal{D} \to Sets$, see Definition 7.31.2. Then the functor $v \circ u : \mathcal{C} \to Sets$ defines a point q of $\mathcal{C}$ and moreover there is a canonical identification*

$$(f^{-1}\mathcal{F})_p = \mathcal{F}_q$$

for any sheaf $\mathcal{F}$ on $\mathcal{C}$.

First proof Lemma 7.33.1. Note that since u is continuous and since v defines a point, it is immediate that $v \circ u$ satisfies conditions (1) and (2) of Definition 7.31.2. Let us prove the displayed equality. Let $\mathcal{F}$ be a sheaf on $\mathcal{C}$. Then

$$\mathcal{F}_q = \mathrm{colim}_{(U,x)} \mathcal{F}(U)$$

where the colimit is over objects U in $\mathcal{C}$ and elements $x \in v(u(U))$. Similarly, we have

$$\begin{aligned}(f^{-1}\mathcal{F})_p &= (u_p\mathcal{F})_p \\ &= \operatorname{colim}_{(V,x)} \operatorname{colim}_{U,\phi:V\to u(U)} \mathcal{F}(U) \\ &= \operatorname{colim}_{(V,x,U,\phi:V\to u(U))} \mathcal{F}(U) \\ &= \operatorname{colim}_{(U,x)} \mathcal{F}(U) \\ &= \mathcal{F}_q\end{aligned}$$

Explanation: The first equality holds because $f^{-1}\mathcal{F} = (u_p\mathcal{F})^\#$ and because $\mathcal{G}_p = \mathcal{G}_p^\#$ for any presheaf $\mathcal{G}$, see Lemma 7.31.5. The second equality holds by the definition of u_p. In the third equality we simply combine colimits. To see the fourth equality we apply Categories, Lemma 4.17.5 to the functor F of diagram categories defined by the rule $F((V, x, U, \phi : V \to u(U))) = (U, v(\phi)(x))$. The lemma applies, because F has a right inverse, namely $(U, x) \mapsto (u(U), x, U, \mathrm{id} : u(U) \to u(U))$ and because there is always a morphism

$$(V, x, U, \phi : V \to u(U)) \longrightarrow (u(U), v(\phi)(x), U, \mathrm{id} : u(U) \to u(U))$$

in the fibre category over (U, x) which shows the fibre categories are connected. The fifth equality is clear. Hence now we see that q also satisfies condition (3) of Definition 7.31.2 because it is a composition of exact functors. This finishes the proof. □

Second proof Lemma 7.33.1. By Lemma 7.31.8 we may factor (p_*, p^{-1}) as

$$Sh(pt) \xrightarrow{i} Sh(\mathcal{S}) \xrightarrow{h} Sh(\mathcal{D})$$

where the second morphism of topoi comes from a morphism of sites $h : \mathcal{S} \to \mathcal{D}$ induced by the functor $v : \mathcal{D} \to \mathcal{S}$ (which makes sense as $\mathcal{S} \subset Sets$ is a full subcategory containing every object in the image of v). By Lemma 7.15.3 the composition $v \circ u : \mathcal{C} \to \mathcal{S}$ defines a morphism of sites $g : \mathcal{S} \to \mathcal{C}$. In particular, the functor $v \circ u : \mathcal{C} \to \mathcal{S}$ is continuous which by the definition of the coverings in $\mathcal{S}$, see Remark 7.16.3, means that $v \circ u$ satisfies conditions (1) and (2) of Definition 7.31.2. On the other hand, we see that

$$g_*i_*E(U) = i_*E(v(u(U)) = \mathrm{Mor}_{Sets}(v(u(U)), E)$$

by the construction of i in Remark 7.16.3. Note that this is the same as the formula for which is equal to $(v \circ u)^p E$, see Equation (7.31.3.1). By Lemma 7.31.5 the functor $g_*i_* = (v \circ u)^p = (v \circ u)^s$ is right adjoint to the stalk functor $\mathcal{F} \mapsto \mathcal{F}_q$. Hence we see that the stalk functor q^{-1} is canonically isomorphic to $i^{-1} \circ g^{-1}$. Hence it is exact and we conclude that q is a point. Finally, as we have $g = f \circ h$ by construction we see that $q^{-1} = i^{-1} \circ h^{-1} \circ f^{-1} = p^{-1} \circ f^{-1}$, i.e., we have the displayed formula of the lemma. □

Lemma 7.33.2. *Let $f : Sh(\mathcal{D}) \to Sh(\mathcal{C})$ be a morphism of topoi. Let $p : Sh(pt) \to Sh(\mathcal{D})$ be a point. Then $q = f \circ p$ is a point of the topos $Sh(\mathcal{C})$ and we have a canonical identification*

$$(f^{-1}\mathcal{F})_p = \mathcal{F}_q$$

for any sheaf $\mathcal{F}$ on $\mathcal{C}$.

Proof. This is immediate from the definitions and the fact that we can compose morphisms of topoi. □

7.34. Localization and points

In this section we show that points of a localization $\mathcal{C}/U$ are constructed in a simple manner from the points of $\mathcal{C}$.

Lemma 7.34.1. *Let $\mathcal{C}$ be a site. Let p be a point of $\mathcal{C}$ given by $u : \mathcal{C} \to Sets$. Let U be an object of $\mathcal{C}$ and let $x \in u(U)$. The functor*

$$v : \mathcal{C}/U \longrightarrow Sets, \quad (\varphi : V \to U) \longmapsto \{y \in u(V) \mid u(\varphi)(y) = x\}$$

defines a point q of the site $\mathcal{C}/U$ such that the diagram

$$\begin{array}{ccc} & & Sh(pt) \\ & {\scriptstyle q}\swarrow & \downarrow{\scriptstyle p} \\ Sh(\mathcal{C}/U) & \xrightarrow{j_U} & Sh(\mathcal{C}) \end{array}$$

commutes. In other words $\mathcal{F}_p = (j_U^{-1}\mathcal{F})_q$ for any sheaf on $\mathcal{C}$.

Proof. Choose S and $\mathcal{S}$ as in Lemma 7.31.8. We may identify $Sh(pt) = Sh(\mathcal{S})$ as in that lemma, and we may write $p = f : Sh(\mathcal{S}) \to Sh(\mathcal{C})$ for the morphism of topoi induced by u. By Lemma 7.27.1 we get a commutative diagram of topoi

$$\begin{array}{ccc} Sh(\mathcal{S}/u(U)) & \xrightarrow[j_{u(U)}]{} & Sh(\mathcal{S}) \\ {\scriptstyle p'}\downarrow & & \downarrow{\scriptstyle p} \\ Sh(\mathcal{C}/U) & \xrightarrow{j_U} & Sh(\mathcal{C}), \end{array}$$

where p' is given by the functor $u' : \mathcal{C}/U \to \mathcal{S}/u(U)$, $V/U \mapsto u(V)/u(U)$. Consider the functor $j_x : \mathcal{S} \cong \mathcal{S}/x$ obtained by assigning to a set E the set E endowed with the constant map $E \to u(U)$ with value x. Then j_x is a fully faithful cocontinuous functor which has a continuous right adjoint $v_x : (\psi : E \to u(U)) \mapsto \psi^{-1}(\{x\})$. Note that $j_U \circ j_x = \mathrm{id}_{\mathcal{S}}$, and $v_x \circ u' = v$. These observations imply that we have the following commutative diagram of topoi

$$\begin{array}{ccccc} Sh(\mathcal{S}) & & & & \\ & {\scriptstyle a}\searrow & & & \\ & & Sh(\mathcal{S}/u(U)) & \xrightarrow[j_{u(U)}]{} & Sh(\mathcal{S}) \\ & {\scriptstyle q}\swarrow & {\scriptstyle p'}\downarrow & & \downarrow{\scriptstyle p} \\ & & Sh(\mathcal{C}/U) & \xrightarrow{j_U} & Sh(\mathcal{C}) \end{array} \qquad p$$

Namely:

(1) The morphism $a : Sh(\mathcal{S}) \to Sh(\mathcal{S}/u(U))$ is the morphism of topoi associated to the cocontinuous functor j_x, which equals the morphism associated to the continuous functor v_x, see Lemma 7.20.1 and Section 7.21.
(2) The composition $p \circ j_{u(U)} \circ a = p$ since $j_{u(U)} \circ j_x = \mathrm{id}_{\mathcal{S}}$.

(3) The composition $p' \circ a$ gives a morphism of topoi. Moreover, it is the morphism of topoi associated to the continuous functor $v_x \circ u' = v$. Hence v does indeed define a point q of $\mathcal{C}/U$ which fits into the diagram above by construction.

This ends the proof of the lemma. □

Lemma 7.34.2. *Let $\mathcal{C}$, p, u, U be as in Lemma 7.34.1. The construction of Lemma 7.34.1 gives a one to one correspondence between points q of $\mathcal{C}/U$ lying over p and elements x of $u(U)$.*

Proof. Let q be a point of $\mathcal{C}/U$ given by the functor $v : \mathcal{C}/U \to Sets$ such that $j_U \circ q = p$ as morphisms of topoi. Recall that $u(V) = p^{-1}(h_V^\#)$ for any object V of $\mathcal{C}$, see Lemma 7.31.7. Similarly $v(V/U) = q^{-1}(h_{V/U}^\#)$ for any object V/U of $\mathcal{C}/U$. Consider the following two diagrams

$$\begin{array}{ccc} \mathrm{Mor}_{\mathcal{C}/U}(W/U, V/U) & \longrightarrow & \mathrm{Mor}_{\mathcal{C}}(W, V) \\ \downarrow & & \downarrow \\ \mathrm{Mor}_{\mathcal{C}/U}(W/U, U/U) & \longrightarrow & \mathrm{Mor}_{\mathcal{C}}(W, U) \end{array} \qquad \begin{array}{ccc} h_{V/U}^\# & \longrightarrow & j_U^{-1}(h_V^\#) \\ \downarrow & & \downarrow \\ h_{U/U}^\# & \longrightarrow & j_U^{-1}(h_U^\#) \end{array}$$

The right hand diagram is the sheafification of the diagram of presheaves on $\mathcal{C}/U$ which maps W/U to the left hand diagram of sets. (There is a small technical point to make here, namely, that we have $(j_U^{-1}h_V)^\# = j_U^{-1}(h_V^\#)$ and similarly for h_U, see Lemma 7.19.4.) Note that the left hand diagram of sets is cartesian. Since sheafification is exact (Lemma 7.10.14) we conclude that the right hand diagram is cartesian.

Apply the exact functor q^{-1} to the right hand diagram to get a cartesian diagram

$$\begin{array}{ccc} v(V/U) & \longrightarrow & u(V) \\ \downarrow & & \downarrow \\ v(U/U) & \longrightarrow & u(U) \end{array}$$

of sets. Here we have used that $q^{-1} \circ j^{-1} = p^{-1}$. Since U/U is a final object of $\mathcal{C}/U$ we see that $v(U/U)$ is a singleton. Hence the image of $v(U/U)$ in $u(U)$ is an element x, and the top horizontal map gives a bijection $v(V/U) \to \{y \in u(V) \mid y \mapsto x \text{ in } u(U)\}$ as desired. □

Lemma 7.34.3. *Let $\mathcal{C}$ be a site. Let p be a point of $\mathcal{C}$ given by $u : \mathcal{C} \to Sets$. Let U be an object of $\mathcal{C}$. For any sheaf $\mathcal{G}$ on $\mathcal{C}/U$ we have*

$$(j_{U!}\mathcal{G})_p = \coprod\nolimits_q \mathcal{G}_q$$

where the coproduct is over the points q of $\mathcal{C}/U$ associated to elements $x \in u(U)$ as in Lemma 7.34.1.

Proof. We use the description of $j_{U!}\mathcal{G}$ as the sheaf associated to the presheaf $V \mapsto \coprod_{\varphi \in \mathrm{Mor}_{\mathcal{C}}(V,U)} \mathcal{G}(V/_\varphi U)$ of Lemma 7.24.2. Also, the stalk of $j_{U!}\mathcal{G}$ at p is equal to the stalk of this presheaf, see Lemma 7.31.5. Hence we see that

$$(j_{U!}\mathcal{G})_p = \mathrm{colim}_{(V,y)} \coprod\nolimits_{\varphi : V \to U} \mathcal{G}(V/_\varphi U)$$

To each element (V, y, φ, s) of this colimit, we can assign $x = u(\varphi)(y) \in u(U)$. Hence we obtain

$$(j_{U!}\mathcal{G})_p = \coprod\nolimits_{x \in u(U)} \operatorname{colim}_{(\varphi: V \to U, y),\ u(\varphi)(y) = x} \mathcal{G}(V/_\varphi U).$$

This is equal to the expression of the lemma by our construction of the points q. □

Remark 7.34.4. Warning: The result of Lemma 7.34.3 has no analogue for $j_{U,*}$.

7.35. 2-morphisms of topoi

This is a brief section concerning the notion of a 2-morphism of topoi.

Definition 7.35.1. Let $f, g : Sh(\mathcal{C}) \to Sh(\mathcal{D})$ be two morphisms of topoi. A *2-morphism from f to g* is given by a transformation of functors $t : f_* \to g_*$.

Pictorially we sometimes represent t as follows:

$$Sh(\mathcal{C}) \underset{g}{\overset{f}{\Downarrow t}} Sh(\mathcal{D})$$

Note that since f^{-1} is adjoint to f_* and g^{-1} is adjoint to g_* we see that t induces also a transformation of functors $t : g^{-1} \to f^{-1}$ (usually denoted by the same symbol) uniquely characterized by the condition that the diagram

$$\begin{array}{ccc} \operatorname{Mor}_{Sh(\mathcal{C})}(\mathcal{G}, f_*\mathcal{F}) & = & \operatorname{Mor}_{Sh(\mathcal{C})}(f^{-1}\mathcal{G}, \mathcal{F}) \\ {\scriptstyle t\circ -}\downarrow & & \downarrow{\scriptstyle -\circ t} \\ \operatorname{Mor}_{Sh(\mathcal{C})}(\mathcal{G}, g_*\mathcal{F}) & = & \operatorname{Mor}_{Sh(\mathcal{C})}(g^{-1}\mathcal{G}, \mathcal{F}) \end{array}$$

commutes. Because of set theoretic difficulties (see Remark 7.16.4) we do not obtain a 2-category of topoi. But we can still define horizontal and vertical composition and show that the axioms of a strict 2-category listed in Categories, Section 4.27 hold. Namely, vertical composition of 2-morphisms is clear (just compose transformations of functors), composition of 1-morphisms has been defined in Definition 7.16.1, and horizontal composition of

$$Sh(\mathcal{C}) \underset{g}{\overset{f}{\Downarrow t}} Sh(\mathcal{D}) \underset{g'}{\overset{f'}{\Downarrow s}} Sh(\mathcal{E})$$

is defined by the transformation of functors $s \star t$ introduced in Categories, Definition 4.26.1. Explicitly, $s \star t$ is given by

$$f'_*f_*\mathcal{F} \xrightarrow{f'_*t} f'_*g_*\mathcal{F} \xrightarrow{s} g'_*g_*\mathcal{F} \quad \text{or} \quad f'_*f_*\mathcal{F} \xrightarrow{s} g'_*f_*\mathcal{F} \xrightarrow{g'_*t} g'_*g_*\mathcal{F}$$

(these maps are equal). Since these definitions agree with the ones in Categories, Section 4.26 it follows from Categories, Lemma 4.26.2 that the axioms of a strict 2-category hold with these definitions.

7.36. Morphisms between points

Lemma 7.36.1. *Let $\mathcal{C}$ be a site. Let $u, u' : \mathcal{C} \to Sets$ be two functors, and let $t : u' \to u$ be a transformation of functors. Then we obtain a canonical transformation of stalk functors $t_{stalk} : \mathcal{F}_{p'} \to \mathcal{F}_p$ which agrees with t via the identifications of Lemma 7.31.3.*

Proof. Omitted. □

Definition 7.36.2. Let $\mathcal{C}$ be a site. Let p, p' be points of $\mathcal{C}$ given by functors $u, u' : \mathcal{C} \to Sets$. A *morphism* $f : p \to p'$ is given by a transformation of functors

$$f_u : u' \to u.$$

Note how the transformation of functors goes the other way. This makes sense, as we will see later, by thinking of the morphism f as a kind of 2-arrow pictorially as follows:

$$Sets = Sh(pt) \xrightarrow[\Downarrow f]{p,\ p'} Sh(\mathcal{C})$$

Namely, we will see later that f_u induces a canonical transformation of functors $p_* \to p'_*$ between the skyscraper sheaf constructions.

This is a fairly important notion, and deserves a more complete treatment here. List of desiderata

(1) Describe the automorphisms of the point of $\mathcal{T}_G$ described in Example 7.32.6.
(2) Describe $\mathrm{Mor}(p, p')$ in terms of $\mathrm{Mor}(p_*, p'_*)$.
(3) Specialization of points in topological spaces. Show that if $x' \in \overline{\{x\}}$ in the topological space X, then there is a morphism $p \to p'$, where p (resp. p') is the point of X_{Zar} associated to x (resp. x').

7.37. Sites with enough points

Definition 7.37.1. Let $\mathcal{C}$ be a site.

(1) A family of points $\{p_i\}_{i \in I}$ is called *conservative* if for every map of sheaves $\phi : \mathcal{F} \to \mathcal{G}$ which is an isomorphism on all the fibres $\mathcal{F}_{p_i} \to \mathcal{G}_{p_i}$ is an isomorphism.
(2) We say that $\mathcal{C}$ *has enough points* if there exists a conservative family of points.

It turns out that you can then check "exactness" at the stalks.

Lemma 7.37.2. *Let $\mathcal{C}$ be a site and let $\{p_i\}_{i \in I}$ be a conservative family of points. Then*

(1) *Given any map of sheaves $\varphi : \mathcal{F} \to \mathcal{G}$ we have $\forall i, \varphi_{p_i}$ injective implies φ injective.*
(2) *Given any map of sheaves $\varphi : \mathcal{F} \to \mathcal{G}$ we have $\forall i, \varphi_{p_i}$ surjective implies φ surjective.*
(3) *Given any pair of maps of sheaves $\varphi_1, \varphi_2 : \mathcal{F} \to \mathcal{G}$ we have $\forall i, \varphi_{1,p_i} = \varphi_{2,p_i}$ implies $\varphi_1 = \varphi_2$.*

(4) *Given a finite diagram $\mathcal{G} : \mathcal{J} \to Sh(\mathcal{C})$, a sheaf $\mathcal{F}$ and morphisms $q_j : \mathcal{F} \to \mathcal{G}_j$ then $(\mathcal{F}, q_j)$ is a limit of the diagram if and only if for each i the stalk $(\mathcal{F}_{p_i}, (q_j)_{p_i})$ is one.*
(5) *Given a finite diagram $\mathcal{F} : \mathcal{J} \to Sh(\mathcal{C})$, a sheaf $\mathcal{G}$ and morphisms $e_j : \mathcal{F}_j \to \mathcal{G}$ then $(\mathcal{G}, e_j)$ is a colimit of the diagram if and only if for each i the stalk $(\mathcal{G}_{p_i}, (e_j)_{p_i})$ is one.*

Proof. We will use over and over again that all the stalk functors commute with any finite limits and colimits and hence with products, fibred products, etc. We will also use that injective maps are the monomorphisms and the surjective maps are the epimorphisms. A map of sheaves $\varphi : \mathcal{F} \to \mathcal{G}$ is injective if and only if $\mathcal{F} \to \mathcal{F} \times_{\mathcal{G}} \mathcal{F}$ is an isomorphism. Hence (1). Similarly, $\varphi : \mathcal{F} \to \mathcal{G}$ is surjective if and only if $\mathcal{G} \amalg_{\mathcal{F}} \mathcal{G} \to \mathcal{G}$ is an isomorphism. Hence (2). The maps $a, b : \mathcal{F} \to \mathcal{G}$ are equal if and only if $\mathcal{F} \times_{a,\mathcal{G},b} \mathcal{F} \to \mathcal{F} \times \mathcal{F}$ is an isomorphism. Hence (3). The assertions (4) and (5) follow immediately from the definitions and the remarks at the start of this proof. □

Lemma 7.37.3. *Let $\mathcal{C}$ be a site and let $\{(p_i, u_i)\}_{i \in I}$ be a family of points. The family is conservative if and only if for every sheaf $\mathcal{F}$ and every $U \in \mathrm{Ob}(\mathcal{C})$ and every pair of distinct sections $s, s' \in \mathcal{F}(U)$, $s \neq s'$ there exists an i and $x \in u_i(U)$ such that the triples (U, x, s) and (U, x, s') define distinct elements of $\mathcal{F}_{p_i}$.*

Proof. Suppose that the family is conservative and that $\mathcal{F}$, U, and s, s' are as in the lemma. The sections s, s' define maps $a, a' : (h_U)^\# \to \mathcal{F}$ which are distinct. Hence, by Lemma 7.37.2 there is an i such that $a_{p_i} \neq a'_{p_i}$. Recall that $(h_U)^\#_{p_i} = u_i(U)$, by Lemmas 7.31.3 and 7.31.5. Hence there exists an $x \in u_i(U)$ such that $a_{p_i}(x) \neq a'_{p_i}(x)$ in $\mathcal{F}_{p_i}$. Unwinding the definitions you see that (U, x, s) and (U, x, s') are as in the statement of the lemma.

To prove the converse, assume the condition on the existence of points of the lemma. Let $\phi : \mathcal{F} \to \mathcal{G}$ be a map of sheaves which is an isomorphism at all the stalks. We have to show that ϕ is both injective and surjective, see Lemma 7.12.2. Injectivity is an immediate consequence of the assumption. To show surjectivity we have to show that $\mathcal{G} \amalg_{\mathcal{F}} \mathcal{G} \to \mathcal{G}$ is an isomorphism (Categories, Lemma 4.13.3). Since this map is clearly surjective, it suffices to check injectivity which follows as $\mathcal{G} \amalg_{\mathcal{F}} \mathcal{G} \to \mathcal{G}$ is injective on all stalks by assumption. □

In the following lemma the points $q_{i,x}$ are exactly all the points of $\mathcal{C}/U$ lying over the point p_i according to Lemma 7.34.2.

Lemma 7.37.4. *Let $\mathcal{C}$ be a site. Let U be an object of $\mathcal{C}$. let $\{(p_i, u_i)\}_{i \in I}$ be a family of points of $\mathcal{C}$. For $x \in u_i(U)$ let $q_{i,x}$ be the point of $\mathcal{C}/U$ constructed in Lemma 7.34.1. If $\{p_i\}$ is a conservative family of points, then $\{q_{i,x}\}_{i \in I, x \in u_i(U)}$ is a conservative family of points of $\mathcal{C}/U$. In particular, if $\mathcal{C}$ has enough points, then so does every localization $\mathcal{C}/U$.*

Proof. We know that $j_{U!}$ induces an equivalence $j_{U!} : Sh(\mathcal{C}/U) \to Sh(\mathcal{C})/h_U^\#$, see Lemma 7.24.4. Moreover, we know that $(j_{U!}\mathcal{G})_{p_i} = \coprod_x \mathcal{G}_{q_{i,x}}$, see Lemma 7.34.3. Hence the result follows formally. □

The following lemma tells us we can check the existence of points locally on the site.

Lemma 7.37.5. *Let $\mathcal{C}$ be a site. Let $\{U_i\}_{i\in I}$ be a family of objects of $\mathcal{C}$. Assume*

(1) *$\coprod h^\#_{U_i} \to *$ is a surjective map of sheaves, and*
(2) *each localization $\mathcal{C}/U_i$ has enough points.*

Then $\mathcal{C}$ has enough points.

Proof. For each $i \in I$ let $\{p_j\}_{j\in J_i}$ be a conservative family of points of $\mathcal{C}/U_i$. For $j \in J_i$ denote $q_j : Sh(pt) \to Sh(\mathcal{C})$ the composition of p_j with the localization morphism $Sh(\mathcal{C}/U_i) \to Sh(\mathcal{C})$. Then q_j is a point, see Lemma 7.33.2. We claim that the family of points $\{q_j\}_{j\in\coprod J_i}$ is conservative. Namely, let $\mathcal{F} \to \mathcal{G}$ be a map of sheaves on $\mathcal{C}$ such that $\mathcal{F}_{q_j} \to \mathcal{G}_{q_j}$ is an isomorphism for all $j \in \coprod J_i$. Let W be an object of $\mathcal{C}$. By assumption (1) there exists a covering $\{W_a \to W\}$ and morphisms $W_a \to U_{i(a)}$. Since $(\mathcal{F}|_{\mathcal{C}/U_{i(a)}})_{p_j} = \mathcal{F}_{q_j}$ and $(\mathcal{G}|_{\mathcal{C}/U_{i(a)}})_{p_j} = \mathcal{G}_{q_j}$ by Lemma 7.33.2 we see that $\mathcal{F}|_{U_{i(a)}} \to \mathcal{G}|_{U_{i(a)}}$ is an isomorphism since the family of points $\{p_j\}_{j\in J_{i(a)}}$ is conservative. Hence $\mathcal{F}(W_a) \to \mathcal{G}(W_a)$ is bijective for each a. Similarly $\mathcal{F}(W_a \times_W W_b) \to \mathcal{G}(W_a \times_W W_b)$ is bijective for each a, b. By the sheaf condition this shows that $\mathcal{F}(W) \to \mathcal{G}(W)$ is bijective, i.e., $\mathcal{F} \to \mathcal{G}$ is an isomorphism. □

7.38. Criterion for existence of points

This section corresponds to Deligne's appendix to [**AGV71**, Exposé VI]. In fact it is almost literally the same.

Let $\mathcal{C}$ be a site. Suppose that $(I, \geq)$ is a directed partially ordered set, and that $(U_i, f_{ii'})$ is an inverse system over I, see Categories, Definition 4.21.1. Given the data $(I, \geq, U_i, f_{ii'})$ we define

$$u : \mathcal{C} \longrightarrow Sets, \quad u(V) = \operatorname{colim}_i \operatorname{Mor}_{\mathcal{C}}(U_i, V)$$

Let $\mathcal{F} \mapsto \mathcal{F}_p$ be the stalk functor associated to u as in Section 7.31. It is direct from the definition that actually

$$\mathcal{F}_p = \operatorname{colim}_i \mathcal{F}(U_i)$$

in this special case. Note that u commutes with all finite limits (I mean those that are representable in $\mathcal{C}$) because each of the functors $V \mapsto \operatorname{Mor}_{\mathcal{C}}(U_i, V)$ do, see Categories, Lemma 4.19.2.

We say that a system $(I, \geq, U_i, f_{ii'})$ is a *refinement* of $(J, \geq, V_j, g_{jj'})$ if $J \subset I$, the ordening on J induced from that of I and $V_j = U_j$, $g_{jj'} = f_{jj'}$ (in words, the inverse system over J is induced by that over I). Let u be the functor associated to $(I, \geq, U_i, f_{ii'})$ and let u' be the functor associated to $(J, \geq, V_j, g_{jj'})$. This induces a transformation of functors

$$u' \longrightarrow u$$

simply because the colimits for u' are over a subsystem of the systems in the colimits for u. In particular we get an associated transformation of stalk functors $\mathcal{F}_{p'} \to \mathcal{F}_p$, see Lemma 7.36.1.

Lemma 7.38.1. *Let $\mathcal{C}$ be a site. Let $(J, \geq, V_j, g_{jj'})$ be a system as above with associated pair of functors (u', p'). Let $\mathcal{F}$ be a sheaf on $\mathcal{C}$. Let $s, s' \in \mathcal{F}_{p'}$ be distinct elements. Let $\{W_k \to W\}$ be a finite covering of $\mathcal{C}$. Let $f \in u(W)$. There exists a refinement $(I, \geq, U_i, f_{ii'})$ of $(J, \geq, V_j, g_{jj'})$ such that s, s' map to distinct elements of $\mathcal{F}_p$ and that the image of f in $u'(W)$ is in the image of one of the $u'(W_k)$.*

Proof. There exists a $j_0 \in J$ such that f is defined by $f' : V_{j_0} \to W$. For $j \geq j_0$ we set $V_{j,k} = V_j \times_{f' \circ f_{jj_0}, W} W_k$. Then $\{V_{j,k} \to V_j\}$ is a finite covering in the site $\mathcal{C}$. Hence $\mathcal{F}(V_j) \subset \prod_k \mathcal{F}(V_{j,k})$. By Categories, Lemma 4.19.2 once again we see that

$$\mathcal{F}_{p'} = \operatorname{colim}_j \mathcal{F}(V_j) \longrightarrow \prod\nolimits_k \operatorname{colim}_j \mathcal{F}(V_{j,k})$$

is injective. Hence there exists a k such that s and s' have distinct image in $\operatorname{colim}_j \mathcal{F}(V_{j,k})$. Let $J_0 = \{j \in J, j \geq j_0\}$ and $I = J \amalg J_0$. We order I so that no element of the second summand is smaller than any element of the first, but otherwise using the ordering on J. If $j \in I$ is in the first summand then we use V_j and if $j \in I$ is in the second summand then we use $V_{j,k}$. We omit the definition of the transition maps of the inverse system. By the above it follows that s, s' have distinct image in $\mathcal{F}_p$. Moreover, the restriction of f' to $V_{j,k}$ factors through W_k by construction. □

Lemma 7.38.2. *Let $\mathcal{C}$ be a site. Let $(J, \geq, V_j, g_{jj'})$ be a system as above with associated pair of functors (u', p'). Let $\mathcal{F}$ be a sheaf on $\mathcal{C}$. Let $s, s' \in \mathcal{F}_{p'}$ be distinct elements. There exists a refinement $(I, \geq, U_i, f_{ii'})$ of $(J, \geq, V_j, g_{jj'})$ such that s, s' map to distinct elements of $\mathcal{F}_p$ and such that for every finite covering $\{W_k \to W\}$ of the site $\mathcal{C}$, and any $f \in u'(W)$ the image of f in $u(W)$ is in the image of one of the $u(W_k)$.*

Proof. Let E be the set of pairs $(\{W_k \to W\}, f \in u'(W))$. Consider pairs $(E' \subset E, (I, \geq, U_i, f_{ii'}))$ such that

(1) $(I, \geq, U_i, g_{ii'})$ is a refinement of $(J, \geq, V_j, g_{jj'})$,
(2) s, s' map to distinct elements of $\mathcal{F}_p$, and
(3) for every pair $(\{W_k \to W\}, f \in u'(W)) \in E'$ we have that the image of f in $u(W)$ is in the image of one of the $u(W_k)$.

We order such pairs by inclusion in the first factor and by refinement in the second. Denote $\mathcal{S}$ the class of all pairs $(E' \subset E, (I, \geq, U_i, f_{ii'}))$ as above. We claim that the hypothesis of Zorn's lemma holds for $\mathcal{S}$. Namely, suppose that $(E'_a, (I_a, \geq, U_i, f_{ii'}))_{a \in A}$ is a totally ordered subset of $\mathcal{S}$. Then we can define $E' = \bigcup_{a \in A} E'_a$ and we can set $I = \bigcup_{a \in A} I_a$. We claim that the corresponding pair $(E', (I, \geq, U_i, f_{ii'}))$ is an element of $\mathcal{S}$. Conditions (1) and (3) are clear. For condition (2) you note that

$$u = \operatorname{colim}_{a \in A} u_a \text{ and correspondingly } \mathcal{F}_p = \operatorname{colim}_{a \in A} \mathcal{F}_{p_a}$$

The distinctness of the images of s, s' in this stalk follows from the description of a directed colimit of sets, see Categories, Section 4.19. We will simply write $(E', (I, \ldots)) = \bigcup_{a \in A} (E'_a, (I_a, \ldots))$ in this situation.

OK, so Zorn's Lemma would apply if $\mathcal{S}$ was a set, and this would, combined with Lemma 7.38.1 above easily prove the lemma. It doesn't since $\mathcal{S}$ is a class. In order to circumvent this we choose a well ordering on E. For $e \in E$ set $E'_e = \{e' \in E \mid e' \leq e\}$. By transfinite induction we construct pairs $(E'_e, (I_e, \ldots)) \in \mathcal{S}$ such that $e_1 \leq e_2 \Rightarrow (E'_{e_1}, (I_{e_1}, \ldots)) \leq (E'_{e_2}, (I_{e_2}, \ldots))$. Let $e \in E$, say $e = (\{W_k \to W\}, f \in u'(W))$. If e has a predecessor $e-1$, then we let $(I_e, \ldots)$ be a refinement of $(I_{e-1}, \ldots)$ as in Lemma 7.38.1 with respect to the system $e = (\{W_k \to W\}, f \in u'(W))$. If e does not have a predecessor, then we let $(I_e, \ldots)$ be a refinement of $\bigcup_{e' < e} (I_{e'}, \ldots)$ with respect to the system $e = (\{W_k \to W\}, f \in u'(W))$. Finally, the union $\bigcup_{e \in E} I_e$ will be a solution to the problem posed in the lemma. □

Proposition 7.38.3. *Let $\mathcal{C}$ be a site. Assume that*

(1) *finite limits exist in $\mathcal{C}$, and*
(2) *every covering $\{U_i \to U\}_{i \in I}$ has a refinement by a finite covering of $\mathcal{C}$.*

Then $\mathcal{C}$ has enough points.

Proof. We have to show that given any sheaf $\mathcal{F}$ on $\mathcal{C}$, any $U \in \mathrm{Ob}(\mathcal{C})$, and any distinct sections $s, s' \in \mathcal{F}(U)$, there exists a point p such that s, s' have distinct image in $\mathcal{F}_p$. See Lemma 7.37.3. Consider the system $(J, \geq, V_j, g_{jj'})$ with $J = \{1\}$, $V_1 = U$, $g_{11} = \mathrm{id}_U$. Apply Lemma 7.38.2. By the result of that lemma we get a system $(I, \geq, U_i, f_{ii'})$ refining our system such that $s_p \neq s'_p$ and such that moreover for every finite covering $\{W_k \to W\}$ of the site $\mathcal{C}$ the map $\coprod_k u(W_k) \to u(W)$ is surjective. Since every covering of $\mathcal{C}$ can be refined by a finite covering we conclude that $\coprod_k u(W_k) \to u(W)$ is surjective for *any* covering $\{W_k \to W\}$ of the site $\mathcal{C}$. This implies that $u = p$ is a point, see Proposition 7.32.2 (and the discussion at the beginning of this section which guarantees that u commutes with finite limits). □

7.39. Weakly contractible objects

A *weakly contractible* object of a site is one that satisfies the equivalent conditions of the following lemma.

Lemma 7.39.1. *Let $\mathcal{C}$ be a site. Let U be an object of $\mathcal{C}$. The following conditions are equivalent*

(1) *For every covering $\{U_i \to U\}$ there exists a map of sheaves $h_U^\# \to \coprod h_{U_i}^\#$ inverse to the sheafification of $\coprod h_{U_i} \to h_U$.*
(2) *For every surjection of sheaves of sets $\mathcal{F} \to \mathcal{G}$ the map $\mathcal{F}(U) \to \mathcal{G}(U)$ is surjective.*

Proof. Assume (1) and let $\mathcal{F} \to \mathcal{G}$ be a surjective map of sheaves of sets. For $s \in \mathcal{G}(U)$ there exists a covering $\{U_i \to U\}$ and $t_i \in \mathcal{F}(U_i)$ mapping to $s|_{U_i}$, see Definition 7.12.1. Think of t_i as a map $t_i : h_{U_i}^\# \to \mathcal{F}$ via (7.13.3.1). Then precomposing $\coprod t_i : \coprod h_{U_i}^\# \to \mathcal{F}$ with the map $h_U^\# \to \coprod h_{U_i}^\#$ we get from (1) we obtain a section $t \in \mathcal{F}(U)$ mapping to s. Thus (2) holds.

Assume (2) holds. Let $\{U_i \to U\}$ be a covering. Then $\coprod h_{U_i}^\# \to h_U^\#$ is surjective (Lemma 7.13.4). Hence by (2) there exists a section s of $\coprod h_{U_i}^\#$ mapping to the section id_U of $h_U^\#$. This section corresponds to a map $h_U^\# \to \coprod h_{U_i}^\#$ inverse to the sheafification of $\coprod h_{U_i} \to h_U$ which proves (1). □

Definition 7.39.2. Let $\mathcal{C}$ be a site.

(1) We say an object U of $\mathcal{C}$ is *weakly contractible* if the equivalent conditions of Lemma 7.39.1 hold.
(2) We say a site has *enough weakly contractible objects* if every object U of $\mathcal{C}$ has a covering $\{U_i \to U\}$ with U_i weakly contractible for all i.
(3) More generally, if P is a property of objects of $\mathcal{C}$ we say that $\mathcal{C}$ has *enough P objects* if every object U of $\mathcal{C}$ has a covering $\{U_i \to U\}$ such that U_i has P for all i.

The small étale site of $\mathbf{A}^1_{\mathbf{C}}$ does not have any weakly contractible objects. On the other hand, the small pro-étale site of any scheme has enough contractible objects.

7.40. Exactness properties of pushforward

Let f be a morphism of topoi. The functor f_* in general is only left exact. There are many additional conditions one can impose on this functor to single out particular classes of morphisms of topoi. We collect them here and note some of the logical dependencies. Some parts of the following lemma are purely category theoretical (i.e., they do not depend on having a morphism of topoi, just having a pair of adjoint functors is enough).

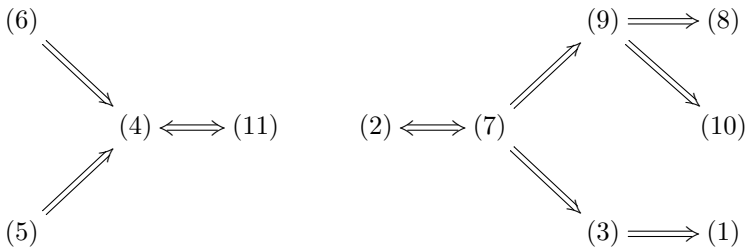
Lemma 7.40.1. *Let $f : Sh(\mathcal{C}) \to Sh(\mathcal{D})$ be a morphism of topoi. Consider the following properties (on sheaves of sets):*

(1) *f_* is faithful,*
(2) *f_* is fully faithful,*
(3) *$f^{-1}f_*\mathcal{F} \to \mathcal{F}$ is surjective for all $\mathcal{F}$ in $Sh(\mathcal{C})$,*
(4) *f_* transforms surjections into surjections,*
(5) *f_* commutes with coequalizers,*
(6) *f_* commutes with pushouts,*
(7) *$f^{-1}f_*\mathcal{F} \to \mathcal{F}$ is an isomorphism for all $\mathcal{F}$ in $Sh(\mathcal{C})$,*
(8) *f_* reflects injections,*
(9) *f_* reflects surjections,*
(10) *f_* reflects bijections, and*
(11) *for any surjection $\mathcal{F} \to f^{-1}\mathcal{G}$ there exists a surjection $\mathcal{G}' \to \mathcal{G}$ such that $f^{-1}\mathcal{G}' \to f^{-1}\mathcal{G}$ factors through $\mathcal{F} \to f^{-1}\mathcal{G}$.*

Then we have the following implications

(a) *(2) $\Rightarrow$ (1),*
(b) *(3) $\Rightarrow$ (1),*
(c) *(7) $\Rightarrow$ (1), (2), (3), (8), (9), (10).*
(d) *(3) $\Leftrightarrow$ (9),*
(e) *(6) $\Rightarrow$ (4) and (5) $\Rightarrow$ (4),*
(f) *(4) $\Leftrightarrow$ (11),*
(g) *(9) $\Rightarrow$ (8), (10), and*
(h) *(2) $\Leftrightarrow$ (7).*

Picture



Proof. Proof of (a): This is immediate from the definitions.

Proof of (b). Suppose that $a, b : \mathcal{F} \to \mathcal{F}'$ are maps of sheaves on $\mathcal{C}$. If $f_*a = f_*b$, then $f^{-1}f_*a = f^{-1}f_*b$. Consider the commutative diagram

$$\begin{array}{ccc} \mathcal{F} & \rightrightarrows & \mathcal{F}' \\ \uparrow & & \uparrow \\ f^{-1}f_*\mathcal{F} & \rightrightarrows & f^{-1}f_*\mathcal{F}' \end{array}$$

If the bottom two arrows are equal and the vertical arrows are surjective then the top two arrows are equal. Hence (b) follows.

Proof of (c). Suppose that $a : \mathcal{F} \to \mathcal{F}'$ is a map of sheaves on $\mathcal{C}$. Consider the commutative diagram

$$\begin{array}{ccc} \mathcal{F} & \longrightarrow & \mathcal{F}' \\ \uparrow & & \uparrow \\ f^{-1}f_*\mathcal{F} & \longrightarrow & f^{-1}f_*\mathcal{F}' \end{array}$$

If (7) holds, then the vertical arrows are isomorphisms. Hence if f_*a is injective (resp. surjective, resp. bijective) then the bottom arrow is injective (resp. surjective, resp. bijective) and hence the top arrow is injective (resp. surjective, resp. bijective). Thus we see that (7) implies (8), (9), (10). It is clear that (7) implies (3). The implications (7) $\Rightarrow$ (2), (1) follow from (a) and (h) which we will see below.

Proof of (d). Assume (3). Suppose that $a : \mathcal{F} \to \mathcal{F}'$ is a map of sheaves on $\mathcal{C}$ such that f_*a is surjective. As f^{-1} is exact this implies that $f^{-1}f_*a : f^{-1}f_*\mathcal{F} \to f^{-1}f_*\mathcal{F}'$ is surjective. Combined with (3) this implies that a is surjective. This means that (9) holds. Assume (9). Let $\mathcal{F}$ be a sheaf on $\mathcal{C}$. We have to show that the map $f^{-1}f_*\mathcal{F} \to \mathcal{F}$ is surjective. It suffices to show that $f_*f^{-1}f_*\mathcal{F} \to f_*\mathcal{F}$ is surjective. And this is true because there is a canonical map $f_*\mathcal{F} \to f_*f^{-1}f_*\mathcal{F}$ which is a one-sided inverse.

Proof of (e). We use Categories, Lemma 4.13.3 without further mention. If $\mathcal{F} \to \mathcal{F}'$ is surjective then $\mathcal{F}' \amalg_{\mathcal{F}} \mathcal{F}' \to \mathcal{F}'$ is an isomorphism. Hence (6) implies that

$$f_*\mathcal{F}' \amalg_{f_*\mathcal{F}} f_*\mathcal{F}' = f_*(\mathcal{F}' \amalg_{\mathcal{F}} \mathcal{F}') \longrightarrow f_*\mathcal{F}'$$

is an isomorphism also. And this in turn implies that $f_*\mathcal{F} \to f_*\mathcal{F}'$ is surjective. Hence we see that (6) implies (4). If $\mathcal{F} \to \mathcal{F}'$ is surjective then $\mathcal{F}'$ is the coequalizer of the two projections $\mathcal{F} \times_{\mathcal{F}'} \mathcal{F} \to \mathcal{F}$ by Lemma 7.12.3. Hence if (5) holds, then $f_*\mathcal{F}'$ is the coequalizer of the two projections

$$f_*(\mathcal{F} \times_{\mathcal{F}'} \mathcal{F}) = f_*\mathcal{F} \times_{f_*\mathcal{F}'} f_*\mathcal{F} \longrightarrow f_*\mathcal{F}$$

which clearly means that $f_*\mathcal{F} \to f_*\mathcal{F}'$ is surjective. Hence (5) implies (4) as well.

Proof of (f). Assume (4). Let $\mathcal{F} \to f^{-1}\mathcal{G}$ be a surjective map of sheaves on $\mathcal{C}$. By (4) we see that $f_*\mathcal{F} \to f_*f^{-1}\mathcal{G}$ is surjective. Let $\mathcal{G}'$ be the fibre product

$$\begin{array}{ccc} f_*\mathcal{F} & \longrightarrow & f_*f^{-1}\mathcal{G} \\ \uparrow & & \uparrow \\ \mathcal{G}' & \longrightarrow & \mathcal{G} \end{array}$$

so that $\mathcal{G}' \to \mathcal{G}$ is surjective also. Consider the commutative diagram

$$\begin{array}{ccc} \mathcal{F} & \longrightarrow & f^{-1}\mathcal{G} \\ \uparrow & & \uparrow \\ f^{-1}f_*\mathcal{F} & \longrightarrow & f^{-1}f_*f^{-1}\mathcal{G} \\ \uparrow & & \uparrow \\ f^{-1}\mathcal{G}' & \longrightarrow & f^{-1}\mathcal{G} \end{array}$$

and we see the required result. Conversely, assume (11). Let $a : \mathcal{F} \to \mathcal{F}'$ be surjective map of sheaves on $\mathcal{C}$. Consider the fibre product diagram

$$\begin{array}{ccc} \mathcal{F} & \longrightarrow & \mathcal{F}' \\ \uparrow & & \uparrow \\ \mathcal{F}'' & \longrightarrow & f^{-1}f_*\mathcal{F}' \end{array}$$

Because the lower horizontal arrow is surjective and by (11) we can find a surjection $\gamma : \mathcal{G}' \to f_*\mathcal{F}'$ such that $f^{-1}\gamma$ factors through $\mathcal{F}'' \to f^{-1}f_*\mathcal{F}'$:

$$\begin{array}{ccccc} & & \mathcal{F} & \longrightarrow & \mathcal{F}' \\ & & \uparrow & & \uparrow \\ f^{-1}\mathcal{G}' & \longrightarrow & \mathcal{F}'' & \longrightarrow & f^{-1}f_*\mathcal{F}' \end{array}$$

Pushing this down using f_* we get a commutative diagram

$$\begin{array}{ccccc} & & f_*\mathcal{F} & \longrightarrow & f_*\mathcal{F}' \\ & & \uparrow & & \uparrow \\ f_*f^{-1}\mathcal{G}' & \longrightarrow & f_*\mathcal{F}'' & \longrightarrow & f_*f^{-1}f_*\mathcal{F}' \\ \uparrow & & & & \uparrow \\ \mathcal{G}' & & \longrightarrow & & f_*\mathcal{F}' \end{array}$$

which proves that (4) holds.

Proof of (g). Assume (9). We use Categories, Lemma 4.13.3 without further mention. Let $a : \mathcal{F} \to \mathcal{F}'$ be a map of sheaves on $\mathcal{C}$ such that f_*a is injective. This means that $f_*\mathcal{F} \to f_*\mathcal{F} \times_{f_*\mathcal{F}'} f_*\mathcal{F} = f_*(\mathcal{F} \times_{\mathcal{F}'} \mathcal{F})$ is an isomorphism. Thus by (9) we see that $\mathcal{F} \to \mathcal{F} \times_{\mathcal{F}'} \mathcal{F}$ is surjective, i.e., an isomorphism. Thus a is injective, i.e., (8) holds. Since (10) is trivially equivalent to (8) + (9) we are done with (g).

Proof of (h). This is Categories, Lemma 4.24.2. □

Here is a condition on a morphism of sites which guarantees that the functor f_* transforms surjective maps into surjective maps.

Lemma 7.40.2. *Let $f : \mathcal{D} \to \mathcal{C}$ be a morphism of sites associated to the continuous functor $u : \mathcal{C} \to \mathcal{D}$. Assume that for any object U of $\mathcal{C}$ and any covering $\{V_j \to u(U)\}$ in $\mathcal{D}$ there exists a covering $\{U_i \to U\}$ in $\mathcal{C}$ such that the map of sheaves*

$$\coprod h^\#_{u(U_i)} \to h^\#_{u(U)}$$

factors through the map of sheaves

$$\coprod h^\#_{V_j} \to h^\#_{u(U)}.$$

Then f_* transforms surjective maps of sheaves into surjective maps of sheaves.

Proof. Let $a : \mathcal{F} \to \mathcal{G}$ be a surjective map of sheaves on $\mathcal{D}$. Let U be an object of $\mathcal{C}$ and let $s \in f_*\mathcal{G}(U) = \mathcal{G}(u(U))$. By assumption there exists a covering $\{V_j \to u(U)\}$ and sections $s_j \in \mathcal{F}(V_j)$ with $a(s_j) = s|_{V_j}$. Now we may think of the sections s, s_j and a as giving a commutative diagram of maps of sheaves

$$\begin{array}{ccc} \coprod h^\#_{V_j} & \xrightarrow[\coprod s_j]{} & \mathcal{F} \\ \downarrow & & \downarrow a \\ h^\#_{u(U)} & \xrightarrow{s} & \mathcal{G} \end{array}$$

By assumption there exists a covering $\{U_i \to U\}$ such that we can enlarge the commutative diagram above as follows

$$\begin{array}{ccccc} & & \coprod h^\#_{V_j} & \xrightarrow[\coprod s_j]{} & \mathcal{F} \\ & \nearrow & \downarrow & & \downarrow a \\ \coprod h^\#_{u(U_i)} & \longrightarrow & h^\#_{u(U)} & \xrightarrow{s} & \mathcal{G} \end{array}$$

Because $\mathcal{F}$ is a sheaf the map from the left lower corner to the right upper corner corresponds to a family of sections $s_i \in \mathcal{F}(u(U_i))$, i.e., sections $s_i \in f_*\mathcal{F}(U_i)$. The commutativity of the diagram implies that $a(s_i)$ is equal to the restriction of s to U_i. In other words we have shown that f_*a is a surjective map of sheaves. □

Example 7.40.3. Assume $f : \mathcal{D} \to \mathcal{C}$ satisfies the assumptions of Lemma 7.40.2. Then it is in general not the case that f_* commutes with coequalizers or pushouts. Namely, suppose that f is the morphism of sites associated to the morphism of topological spaces $X = \{1, 2\} \to Y = \{*\}$ (see Example 7.15.2), where Y is a singleton space, and $X = \{1, 2\}$ is a discrete space with two points. A sheaf $\mathcal{F}$ on X is given by a pair (A_1, A_2) of sets. Then $f_*\mathcal{F}$ corresponds to the set $A_1 \times A_2$. Hence if $a = (a_1, a_2), b = (b_1, b_2) : (A_1, A_2) \to (B_1, B_2)$ are maps of sheaves on X, then the coequalizer of a, b is (C_1, C_2) where C_i is the coequalizer of a_i, b_i, and the coequalizer of f_*a, f_*b is the coequalizer of

$$a_1 \times a_2, b_1 \times b_2 : A_1 \times A_2 \longrightarrow B_1 \times B_2$$

which is in general different from $C_1 \times C_2$. Namely, if $A_2 = \emptyset$ then $A_1 \times A_2 = \emptyset$, and hence the coequalizer of the displayed arrows is $B_1 \times B_2$, but in general $C_1 \neq B_1$. A similar example works for pushouts.

The following lemma gives a criterion for when a morphism of sites has a functor f_* which reflects injections and surjections. Note that this also implies that f_* is faithful and that the map $f^{-1}f_*\mathcal{F} \to \mathcal{F}$ is always surjective.

Lemma 7.40.4. *Let $f : \mathcal{D} \to \mathcal{C}$ be a morphism of sites given by the functor $u : \mathcal{C} \to \mathcal{D}$. Assume that for every object V of $\mathcal{D}$ there exist objects U_i of $\mathcal{C}$ and morphisms $u(U_i) \to V$ such that $\{u(U_i) \to V\}$ is a covering of $\mathcal{D}$. In this case the functor $f_* : Sh(\mathcal{D}) \to Sh(\mathcal{C})$ reflects injections and surjections.*

Proof. Let $\alpha : \mathcal{F} \to \mathcal{G}$ be maps of sheaves on $\mathcal{D}$. By assumption for every object V of $\mathcal{D}$ we get $\mathcal{F}(V) \subset \prod \mathcal{F}(u(U_i)) = \prod f_*\mathcal{F}(U_i)$ by the sheaf condition for some $U_i \in \mathrm{Ob}(\mathcal{C})$ and similarly for $\mathcal{G}$. Hence it is clear that if $f_*\alpha$ is injective, then α is injective. In other words f_* reflects injections.

Suppose that $f_*\alpha$ is surjective. Then for $V, U_i, u(U_i) \to V$ as above and a section $s \in \mathcal{G}(V)$, there exist coverings $\{U_{ij} \to U_i\}$ such that $s|_{u(U_{ij})}$ is in the image of $\mathcal{F}(u(U_{ij}))$. Since $\{u(U_{ij}) \to V\}$ is a covering (as u is continuous and by the axioms of a site) we conclude that s is locally in the image. Thus α is surjective. In other words f_* reflects surjections. □

Example 7.40.5. We construct a morphism $f : \mathcal{D} \to \mathcal{C}$ satisfying the assumptions of Lemma 7.40.4. Namely, let $\varphi : G \to H$ be a morphism of finite groups. Consider the sites $\mathcal{D} = \mathcal{T}_G$ and $\mathcal{C} = \mathcal{T}_H$ of countable G-sets and H-sets and coverings countable families of jointly surjective maps (Example 7.6.5). Let $u : \mathcal{T}_H \to \mathcal{T}_G$ be the functor described in Section 7.17 and $f : \mathcal{T}_G \to \mathcal{T}_H$ the corresponding morphism of sites. If φ is injective, then every countable G-set is, as a G-set, the quotient of a countable H-set (this fails if φ isn't injective). Thus f satisfies the hypothesis of Lemma 7.40.4. If the sheaf $\mathcal{F}$ on $\mathcal{T}_G$ corresponds to the G-set S, then the canonical map

$$f^{-1}f_*\mathcal{F} \longrightarrow \mathcal{F}$$

corresponds to the map

$$\mathrm{Map}_G(H, S) \longrightarrow S, \quad a \longmapsto a(1_H)$$

If φ is injective but not surjective, then this map is surjective (as it should according to Lemma 7.40.4) but not injective in general (for example take $G = \{1\}$, $H = \{1, \sigma\}$, and $S = \{1, 2\}$). Moreover, the functor f_* does not commute with coequalizers or pushouts (for $G = \{1\}$ and $H = \{1, \sigma\}$).

7.41. Almost cocontinuous functors

Let $\mathcal{C}$ be a site. The category $PSh(\mathcal{C})$ has an initial object, namely the presheaf which assigns the empty set to each object of $\mathcal{C}$. Let us denote this presheaf by $\emptyset$. It follows from the properties of sheafification that the sheafification $\emptyset^\#$ of $\emptyset$ is an initial object of the category $Sh(\mathcal{C})$ of sheaves on $\mathcal{C}$.

Definition 7.41.1. Let $\mathcal{C}$ be a site. We say an object U of $\mathcal{C}$ is *sheaf theoretically empty* if $\emptyset^\# \to h_U^\#$ is an isomorphism of sheaves.

The following lemma makes this notion more explicit.

Lemma 7.41.2. *Let $\mathcal{C}$ be a site. Let U be an object of $\mathcal{C}$. The following are equivalent:*

1. *U is sheaf theoretically empty,*
2. *$\mathcal{F}(U)$ is a singleton for each sheaf $\mathcal{F}$,*
3. *$\emptyset^\#(U)$ is a singleton,*
4. *$\emptyset^\#(U)$ is nonempty, and*
5. *the empty family is a covering of U in $\mathcal{C}$.*

Moreover, if U is sheaf theoretically empty, then for any morphism $U' \to U$ of $\mathcal{C}$ the object U' is sheaf theoretically empty.

Proof. For any sheaf $\mathcal{F}$ we have $\mathcal{F}(U) = \mathrm{Mor}_{Sh(\mathcal{C})}(h_U^\#, \mathcal{F})$. Hence, we see that (1) and (2) are equivalent. It is clear that (2) implies (3) implies (4). If every covering of U is given by a nonempty family, then $\emptyset^+(U)$ is empty by definition of the plus construction. Note that $\emptyset^+ = \emptyset^\#$ as $\emptyset$ is a separated presheaf, see Theorem 7.10.10. Thus we see that (4) implies (5). If (5) holds, then $\mathcal{F}(U)$ is a singleton for every sheaf $\mathcal{F}$ by the sheaf condition for $\mathcal{F}$, see Remark 7.7.2. Thus (5) implies (2) and (1) – (5) are equivalent. The final assertion of the lemma follows from Axiom (3) of Definition 7.6.2 applied the empty covering of U. □

Definition 7.41.3. Let $\mathcal{C}$, $\mathcal{D}$ be sites. Let $u : \mathcal{C} \to \mathcal{D}$ be a functor. We say u is *almost cocontinuous* if for every object U of $\mathcal{C}$ and every covering $\{V_j \to u(U)\}_{j\in J}$ there exists a covering $\{U_i \to U\}_{i\in I}$ in $\mathcal{C}$ such that for each i in I we have at least one of the following two conditions

(1) $u(U_i)$ is sheaf theoretically empty, or
(2) the morphism $u(U_i) \to u(U)$ factors through V_j for some $j \in J$.

The motivation for this definition comes from a closed immersion $i : Z \to X$ of topological spaces. As discussed in Example 7.20.9 the continuous functor $X_{Zar} \to Z_{Zar}$, $U \mapsto Z \cap U$ is not cocontinuous. But it is almost cocontinuous in the sense defined above. We know that i_* while not exact on sheaves of sets, is exact on sheaves of abelian groups, see Sheaves, Remark 6.32.5. And this holds in general for continuous and almost cocontinuous functors.

Lemma 7.41.4. *Let $\mathcal{C}$, $\mathcal{D}$ be sites. Let $u : \mathcal{C} \to \mathcal{D}$ be a functor. Assume that u is continuous and almost cocontinuous. Let $\mathcal{G}$ be a presheaf on $\mathcal{D}$ such that $\mathcal{G}(V)$ is a singleton whenever V is sheaf theoretically empty. Then $(u^p\mathcal{G})^\# = u^p(\mathcal{G}^\#)$.*

Proof. Let $U \in \mathrm{Ob}(\mathcal{C})$. We have to show that $(u^p\mathcal{G})^\#(U) = u^p(\mathcal{G}^\#)(U)$. It suffices to show that $(u^p\mathcal{G})^+(U) = u^p(\mathcal{G}^+)(U)$ since $\mathcal{G}^+$ is another presheaf for which the assumption of the lemma holds. We have

$$u^p(\mathcal{G}^+)(U) = \mathcal{G}^+(u(U)) = \mathrm{colim}_{\mathcal{V}}\, \check{H}^0(\mathcal{V}, \mathcal{G})$$

where the colimit is over the coverings $\mathcal{V}$ of $u(U)$ in $\mathcal{D}$. On the other hand, we see that

$$u^p(\mathcal{G})^+(U) = \mathrm{colim}_{\mathcal{U}}\, \check{H}^0(u(\mathcal{U}), \mathcal{G})$$

where the colimit is over the category of coverings $\mathcal{U} = \{U_i \to U\}_{i\in I}$ of U in $\mathcal{C}$ and $u(\mathcal{U}) = \{u(U_i) \to u(U)\}_{i\in I}$. The condition that u is continuous means that each $u(\mathcal{U})$ is a covering. Write $I = I_1 \amalg I_2$, where

$$I_2 = \{i \in I \mid u(U_i) \text{ is sheaf theoretically empty}\}$$

Then $u(\mathcal{U})' = \{u(U_i) \to u(U)\}_{i\in I_1}$ is still a covering of because each of the other pieces can be covered by the empty family and hence can be dropped by Axiom (2) of Definition 7.6.2. Moreover, $\check{H}^0(u(\mathcal{U}), \mathcal{G}) = \check{H}^0(u(\mathcal{U})', \mathcal{G})$ by our assumption on $\mathcal{G}$. Finally, the condition that u is almost cocontinuous implies that for every covering $\mathcal{V}$ of $u(U)$ there exists a covering $\mathcal{U}$ of U such that $u(\mathcal{U})'$ refines $\mathcal{V}$. It follows that the two colimits displayed above have the same value as desired. □

Lemma 7.41.5. *Let $\mathcal{C}$, $\mathcal{D}$ be sites. Let $u : \mathcal{C} \to \mathcal{D}$ be a functor. Assume that u is continuous and almost cocontinuous. Then $u^s = u^p : Sh(\mathcal{D}) \to Sh(\mathcal{C})$ commutes with pushouts and coequalizers (and more generally finite connected colimits).*

Proof. Let $\mathcal{I}$ be a finite connected index category. Let $\mathcal{I} \to Sh(\mathcal{D})$, $i \mapsto \mathcal{G}_i$ by a diagram. We know that the colimit of this diagram is the sheafification of the colimit in the category of presheaves, see Lemma 7.10.13. Denote $\operatorname{colim}^{Psh}$ the colimit in the category of presheaves. Since $\mathcal{I}$ is finite and connected we see that $\operatorname{colim}_i^{Psh} \mathcal{G}_i$ is a presheaf satisfying the assumptions of Lemma 7.41.4 (because a finite connected colimit of singleton sets is a singleton). Hence that lemma gives

$$\begin{aligned} u^s(\operatorname{colim}_i \mathcal{G}_i) &= u^s((\operatorname{colim}_i^{Psh} \mathcal{G}_i)^\#) \\ &= (u^p(\operatorname{colim}_i^{Psh} \mathcal{G}_i))^\# \\ &= (\operatorname{colim}_i^{PSh} u^p(\mathcal{G}_i))^\# \\ &= \operatorname{colim}_i u^s(\mathcal{G}_i) \end{aligned}$$

as desired. □

Lemma 7.41.6. *Let $f : \mathcal{D} \to \mathcal{C}$ be a morphism of sites associated to the continuous functor $u : \mathcal{C} \to \mathcal{D}$. If u is almost cocontinuous then f_* commutes with pushouts and coequalizers (and more generally finite connected colimits).*

Proof. This is a special case of Lemma 7.41.5. □

7.42. Subtopoi

Here is the definition.

Definition 7.42.1. Let $\mathcal{C}$ and $\mathcal{D}$ be sites. A morphism of topoi $f : Sh(\mathcal{D}) \to Sh(\mathcal{C})$ is callend an *embedding* if f_* is fully faithfull.

According to Lemma 7.40.1 this is equivalent to asking the adjunction map $f^{-1}f_*\mathcal{F} \to \mathcal{F}$ to be an isomorphism for every sheaf $\mathcal{F}$ on $\mathcal{D}$.

Definition 7.42.2. Let $\mathcal{C}$ be a site. A strictly full subcategory $E \subset Sh(\mathcal{C})$ is a *subtopos* if there exists an embedding of topoi $f : Sh(\mathcal{D}) \to Sh(\mathcal{C})$ such that E is equal to the essential image of the functor f_*.

The subtopoi constructed in the following lemma will be dubbed "open" in the definition later on.

Lemma 7.42.3. *Let $\mathcal{C}$ be a site. Let $\mathcal{F}$ be a sheaf on $\mathcal{C}$. The following are equivalent*

1. *$\mathcal{F}$ is a subobject of the final object of $\mathcal{C}$, and*
2. *the topos $Sh(\mathcal{C})/\mathcal{F}$ is a subtopos of $Sh(\mathcal{C})$.*

Proof. We have seen in Lemma 7.29.1 that $Sh(\mathcal{C})/\mathcal{F}$ is a topos. In fact, we recall the proof. First we apply Lemma 7.28.5 to see that we may assume $\mathcal{C}$ is a site with a subcanonical topology, fibre products, a final object X, and an object U with $\mathcal{F} = h_U$. The proof of Lemma 7.29.1 shows that the morphism of topoi $j_{\mathcal{F}} : Sh(\mathcal{C})/\mathcal{F} \to Sh(\mathcal{C})$ is equal (modulo certain identifications) to the localization morphism $j_U : Sh(\mathcal{C}/U) \to Sh(\mathcal{C})$.

Assume (2). This means that $j_U^{-1}j_{U,*}\mathcal{G} \to \mathcal{G}$ is an isomorphism for all sheaves $\mathcal{G}$ on $\mathcal{C}/U$. For any object Z/U of $\mathcal{C}/U$ we have

$$(j_{U,*}h_{Z/U})(U) = \operatorname{Mor}_{\mathcal{C}/U}(U \times_X U/U, Z/U)$$

by Lemma 7.26.2. Setting $\mathcal{G} = h_{Z/U}$ in the equality above we obtain

$$\operatorname{Mor}_{\mathcal{C}/U}(U \times_X U/U, Z/U) = \operatorname{Mor}_{\mathcal{C}/U}(U, Z/U)$$

for all Z/U. By Yoneda's lemma (Categories, Lemma 4.3.5) this implies $U \times_X U = U$. By Categories, Lemma 4.13.3 $U \to X$ is a monomorphism, in other words (1) holds.

Assume (1). Then $j_U^{-1} j_{U,*} = \text{id}$ by Lemma 7.26.4. □

Definition 7.42.4. Let $\mathcal{C}$ be a site. A strictly full subcategory $E \subset Sh(\mathcal{C})$ is an *open subtopos* if there exists a subsheaf $\mathcal{F}$ of the final object of $Sh(\mathcal{C})$ such that E is the subtopos $Sh(\mathcal{C})/\mathcal{F}$ described in Lemma 7.42.3.

This means there is a bijection between the collection of open subtopoi of $Sh(\mathcal{C})$ and the set of subobjects of the final object of $Sh(\mathcal{C})$. Given an open subtopos there is a "closed" complement.

Lemma 7.42.5. *Let $\mathcal{C}$ be a site. Let $\mathcal{F}$ be a subsheaf of the final object $*$ of $Sh(\mathcal{C})$. The full subcategory of sheaves $\mathcal{G}$ such that $\mathcal{F} \times \mathcal{G} \to \mathcal{F}$ is an isomorphism is a subtopos of $Sh(\mathcal{C})$.*

Proof. We apply Lemma 7.28.5 to see that we may assume $\mathcal{C}$ is a site with the properties listed in that lemma. In particular $\mathcal{C}$ has a final object X (so that $* = h_X$) and an object U with $\mathcal{F} = h_U$.

Let $\mathcal{D} = \mathcal{C}$ as a category but a covering is a family $\{V_j \to V\}$ of morphisms such that $\{V_i \to V\} \cup \{U \times_X V \to V\}$ is a covering. By our choice of $\mathcal{C}$ this means exactly that

$$h_{U \times_X V} \coprod \coprod h_{V_i} \longrightarrow h_V$$

is surjective. We claim that $\mathcal{D}$ is a site, i.e., the coverings satisfy the conditions (1), (2), (3) of Definition 7.6.2. Condition (1) holds. For condition (2) suppose that $\{V_i \to V\}$ and $\{V_{ij} \to V_i\}$ are coverings of $\mathcal{D}$. Then the composition

$$\coprod \left(h_{U \times_X V_i} \coprod \coprod h_{V_{ij}} \right) \longrightarrow h_{U \times_X V} \coprod \coprod h_{V_i} \longrightarrow h_V$$

is surjective. Since each of the morphisms $U \times_X V_i \to V$ factors through $U \times_X V$ we see that

$$h_{U \times_X V} \coprod \coprod h_{V_{ij}} \longrightarrow h_V$$

is surjective, i.e., $\{V_{ij} \to V\}$ is a covering of V in $\mathcal{D}$. Condition (3) follows similarly as a base change of a surjective map of sheaves is surjective.

Note that the (identity) functor $u : \mathcal{C} \to \mathcal{D}$ is continuous and commutes with fibre products and final objects. Hence we obtain a morphism $f : \mathcal{D} \to \mathcal{C}$ of sites (Proposition 7.15.6). Observe that f_* is the identity functor on underlying presheaves, hence fully faithful. To finish the proof we have to show that the essential image of f_* is the full subcategory $E \subset Sh(\mathcal{C})$ singled out in the lemma. To do this, note that $\mathcal{G} \in \text{Ob}(Sh(\mathcal{C}))$ is in E if and only if $\mathcal{G}(U \times_X V)$ is a singleton for all objects V of $\mathcal{C}$. Thus such a sheaf satisfies the sheaf property for all coverings of $\mathcal{D}$ (argument omitted). Conversely, if $\mathcal{G}$ satisfies the sheaf property for all coverings of $\mathcal{D}$, then $\mathcal{G}(U \times_X V)$ is a singleton, as in $\mathcal{D}$ the object $U \times_X V$ is covered by the empty covering. □

Definition 7.42.6. Let $\mathcal{C}$ be a site. A strictly full subcategory $E \subset Sh(\mathcal{C})$ is an *closed subtopos* if there exists a subsheaf $\mathcal{F}$ of the final object of $Sh(\mathcal{C})$ such that E is the subtopos described in Lemma 7.42.5.

All right, and now we can define what it means to have a closed immersion and an open immersion of topoi.

Definition 7.42.7. Let $f : Sh(\mathcal{D}) \to Sh(\mathcal{C})$ be a morphism of topoi.

(1) We say f is an *open immersion* if f is an embedding and the essential image of f_* is an open subtopos.
(2) We say f is a *closed immersion* if f is an embedding and the essential image of f_* is a closed subtopos.

Lemma 7.42.8. *Let $i : Sh(\mathcal{D}) \to Sh(\mathcal{C})$ be a closed immersion of topoi. Then i_* is fully faithful, transforms surjections into surjections, commutes with coequalizers, commutes with pushouts, reflects injections, reflects surjections, and reflects bijections.*

Proof. Let $\mathcal{F}$ be a subsheaf of the final object $*$ of $Sh(\mathcal{C})$ and let $E \subset Sh(\mathcal{C})$ be the full subcategory consisting of those $\mathcal{G}$ such that $\mathcal{F} \times \mathcal{G} \to \mathcal{F}$ is an isomorphism. By Lemma 7.42.5 the functor i_* is isomorphic to the inclusion functor $\iota : E \to Sh(\mathcal{C})$.

Let $j_{\mathcal{F}} : Sh(\mathcal{C})/\mathcal{F} \to Sh(\mathcal{C})$ be the localization functor (Lemma 7.29.1). Note that E can also be described as the collection of sheaves $\mathcal{G}$ such that $j_{\mathcal{F}}^{-1}\mathcal{G} = *$.

Let $a, b : \mathcal{G}_1 \to \mathcal{G}_2$ be two morphism of E. To prove ι commutes with coequalizers it suffices to show that the coequalizer of a, b in $Sh(\mathcal{C})$ lies in E. This is clear because the coequalizer of two morphisms $* \to *$ is $*$ and because $j_{\mathcal{F}}^{-1}$ is exact. Similarly for pushouts.

Thus i_* satisfies properties (5), (6), and (7) of Lemma 7.40.1 and hence the morphism i satisfies all properties mentioned in that lemma, in particular the ones mentioned in this lemma. □

7.43. Sheaves of algebraic structures

In Sheaves, Section 6.15 we introduced a type of algebraic structure to be a pair $(\mathcal{A}, s)$, where $\mathcal{A}$ is a category, and $s : \mathcal{A} \to Sets$ is a functor such that

(1) s is faithful,
(2) $\mathcal{A}$ has limits and s commutes with limits,
(3) $\mathcal{A}$ has filtered colimits and s commutes with them, and
(4) s reflects isomorphisms.

For such a type of algebraic structure we saw that a presheaf $\mathcal{F}$ with values in $\mathcal{A}$ on a space X is a sheaf if and only if the associated presheaf of sets is a sheaf. Moreover, we worked out the notion of stalk, and given a continuous map $f : X \to Y$ we defined adjoint functors pushforward and pullback on sheaves of algebraic structures which agrees with pushforward and pullback on the underlying sheaves of sets. In addition extending a sheaf of algebraic structures from a basis to all opens of a space, works as expected.

Part of this material still works in the setting of sites and sheaves. Let $(\mathcal{A}, s)$ be a type of algebraic structure. Let $\mathcal{C}$ be a site. Let us denote $PSh(\mathcal{C}, \mathcal{A})$, resp. $Sh(\mathcal{C}, \mathcal{A})$ the category of presheaves, resp. sheaves with values in $\mathcal{A}$ on $\mathcal{C}$.

(α) A presheaf with values in $\mathcal{A}$ is a sheaf if and only if its underlying presheaf of sets is a sheaf. See the proof of Sheaves, Lemma 6.9.2.

(β) Given a presheaf $\mathcal{F}$ with values in $\mathcal{A}$ the presheaf $\mathcal{F}^\# = (\mathcal{F}^+)^+$ is a sheaf. This is true since the colimits in the sheafification process are filtered, and even colimits over directed partially ordered sets (see Section 7.10, especially the proof of Lemma 7.10.14) and since s commutes with filtered colimits.

(γ) We get the following commutative diagram

$$\begin{array}{ccc} Sh(\mathcal{C}, \mathcal{A}) & \underset{\#}{\rightleftarrows} & PSh(\mathcal{C}, \mathcal{A}) \\ \downarrow s & & \downarrow s \\ Sh(\mathcal{C}) & \rightleftarrows & PSh(\mathcal{C}) \end{array}$$

(δ) We have $\mathcal{F} = \mathcal{F}^\#$ if and only if $\mathcal{F}$ is a sheaf of algebraic structures.

(ϵ) The functor $^\#$ is adjoint to the inclusion functor:

$$\mathrm{Mor}_{PSh(\mathcal{C},\mathcal{A})}(\mathcal{G}, \mathcal{F}) = \mathrm{Mor}_{Sh(\mathcal{C},\mathcal{A})}(\mathcal{G}^\#, \mathcal{F})$$

The proof is the same as the proof of Proposition 7.10.12.

(ζ) The functor $\mathcal{F} \mapsto \mathcal{F}^\#$ is left exact. The proof is the same as the proof of Lemma 7.10.14.

Definition 7.43.1. Let $f : \mathcal{D} \to \mathcal{C}$ be a morphism of sites given by a functor $u : \mathcal{C} \to \mathcal{D}$. We define the *pushforward* functor for presheaves of algebraic structures by the rule $u^p\mathcal{F}(U) = \mathcal{F}(uU)$, and for sheaves of algebraic structures by the same rule, namely $f_*\mathcal{F}(U) = \mathcal{F}(uU)$.

The problem comes with trying the define the pullback. The reason is that the colimits defining the functor u_p in Section 7.5 may not be filtered. Thus the axioms above are not enough in general to define the pullback of a (pre)sheaf of algebraic structures. Nonetheless, in almost all cases the following lemma is sufficient to define pushforward, and pullback of (pre)sheaves of algebraic structures.

Lemma 7.43.2. *Suppose the functor $u : \mathcal{C} \to \mathcal{D}$ satisfies the hypotheses of Proposition 7.15.6, and hence gives rise to a morphism of sites $f : \mathcal{D} \to \mathcal{C}$. In this case the pullback functor f^{-1} (resp. u^p) and the pushforward functor f_* (resp. u_p) extend to an adjoint pair of functors on the categories of sheaves (resp. presheaves) of algebraic structures. Moreover, these functors commute with taking the underlying sheaf (resp. presheaf) of sets.*

Proof. We have defined $f_* = u^p$ above. In the course of the proof of Proposition 7.15.6 we saw that all the colimits used to define u_p are filtered under the assumptions of the proposition. Hence we conclude from the definition of a type of algebraic structure that we may define u_p by exactly the same colimits as a functor on presheaves of algebraic structures. Adjointness of u_p and u^p is proved in exactly the same way as the proof of Lemma 7.5.4. The discussion of sheafification of presheaves of algebraic structures above then implies that we may define $f^{-1}(\mathcal{F}) = (u_p\mathcal{F})^\#$. □

We briefly discuss a method for dealing with pullback and pushforward for a general morphism of sites, and more generally for any morphism of topoi.

Let $\mathcal{C}$ be a site. In the case $\mathcal{A} = Ab$, we may think of an abelian (pre)sheaf on $\mathcal{C}$ as a quadruple $(\mathcal{F}, +, 0, i)$. Here the data are

(D1) $\mathcal{F}$ is a sheaf of sets,
(D2) $+ : \mathcal{F} \times \mathcal{F} \to \mathcal{F}$ is a morphism of sheaves of sets,
(D3) $0 : * \to \mathcal{F}$ is a morphism from the singleton sheaf (see Example 7.10.2) to $\mathcal{F}$, and
(D4) $i : \mathcal{F} \to \mathcal{F}$ is a morphism of sheaves of sets.

These data have to satisfy the following axioms

(A1) $+$ is associative and commutative,
(A2) 0 is a unit for $+$, and
(A3) $+ \circ (1, i) = 0 \circ (\mathcal{F} \to *)$.

Compare Sheaves, Lemma 6.4.3. Let $f : \mathcal{D} \to \mathcal{C}$ be a morphism of sites. Note that since f^{-1} is exact we have $f^{-1} * = *$ and $f^{-1}(\mathcal{F} \times \mathcal{F}) = f^{-1}\mathcal{F} \times f^{-1}\mathcal{F}$. Thus we can define $f^{-1}\mathcal{F}$ simply as the quadruple $(f^{-1}\mathcal{F}, f^{-1}+, f^{-1}0, f^{-1}i)$. The axioms are going to be preserved because f^{-1} is a functor which commutes with finite limits. Finally it is not hard to check that f_* and f^{-1} are adjoint as usual.

In [**AGV71**] this method is used. They introduce something called an "*espèce the structure algébrique ≪définie par limites projectives finie≫*". For such an espèce you can use the method described above to define a pair of adjoint functors f^{-1} and f_* as above. This clearly works for most algebraic structures that one encounters in practice. Instead of formalizing this construction we simply list those algebraic structures for which this method works (to be verified case by case). In fact, this method works for any morphism of topoi.

Proposition 7.43.3. *Let $\mathcal{C}$, $\mathcal{D}$ be sites. Let $f = (f^{-1}, f_*)$ be a morphism of topoi from $Sh(\mathcal{D}) \to Sh(\mathcal{C})$. The method introduced above gives rise to an adjoint pair of functors (f^{-1}, f_*) on sheaves of algebraic structures compatible with taking the underlying sheaves of sets for the following types of algebraic structures:*

(1) *pointed sets,*
(2) *abelian groups,*
(3) *groups,*
(4) *monoids,*
(5) *rings,*
(6) *modules over a fixed ring, and*
(7) *lie algebras over a fixed field.*

Moreover, in each of these cases the results above labeled (α), (β), (γ), (δ), (ϵ), and (ζ) hold.

Proof. The final statement of the proposition holds simply since each of the listed categories, endowed with the obvious forgetful functor, is indeed a type of algebraic structure in the sense explained at the beginning of this section. See Sheaves, Lemma 6.15.2.

Proof of (2). We think of a sheaf of abelian groups as a quadruple $(\mathcal{F}, +, 0, i)$ as explained in the discussion preceding the proposition. If $(\mathcal{F}, +, 0, i)$ lives on $\mathcal{C}$, then its pullback is defined as $(f^{-1}\mathcal{F}, f^{-1}+, f^{-1}0, f^{-1}i)$. If $(\mathcal{G}, +, 0, i)$ lives on $\mathcal{D}$, then its pushforward is defined as $(f_*\mathcal{G}, f_*+, f_*0, f_*i)$. This works because $f_*(\mathcal{G} \times \mathcal{G}) = f_*\mathcal{G} \times f_*\mathcal{G}$. Adjointness follows from adjointness of the set based functors, since

$$\mathrm{Mor}_{Ab(\mathcal{C})}((\mathcal{F}_1, +, 0, i), (\mathcal{F}_2, +, 0, i)) = \left\{ \begin{array}{c} \varphi \in \mathrm{Mor}_{Sh(\mathcal{C})}(\mathcal{F}_1, \mathcal{F}_2) \\ \varphi \text{ is compatible with } +, 0, i \end{array} \right\}$$

Details left to the reader.

This method also works for sheaves of rings by thinking of a sheaf of rings (with unit) as a sixtuple $(\mathcal{O}, +, 0, i, \cdot, 1)$ satisfying a list of axioms that you can find in any elementary algebra book.

A sheaf of pointed sets is a pair $(\mathcal{F}, p)$, where $\mathcal{F}$ is a sheaf of sets, and $p : * \to \mathcal{F}$ is a map of sheaves of sets.

A sheaf of groups is given by a quadruple $(\mathcal{F}, \cdot, 1, i)$ with suitable axioms.

A sheaf of monoids is given by a pair $(\mathcal{F}, \cdot)$ with suitable axiom.

Let R be a ring. An sheaf of R-modules is given by a quintuple $(\mathcal{F}, +, 0, i, \{\lambda_r\}_{r \in R})$, where the quadruple $(\mathcal{F}, +, 0, i)$ is a sheaf of abelian groups as above, and $\lambda_r : \mathcal{F} \to \mathcal{F}$ is a family of morphisms of sheaves of sets such that $\lambda_r \circ 0 = 0$, $\lambda_r \circ + = + \circ (\lambda_r, \lambda_r)$, $\lambda_{r+r'} = + \circ \lambda_r \times \lambda_{r'} \circ (\mathrm{id}, \mathrm{id})$, $\lambda_{rr'} = \lambda_r \circ \lambda_{r'}$, $\lambda_1 = \mathrm{id}$, $\lambda_0 = 0 \circ (\mathcal{F} \to *)$. □

We will discuss the category of sheaves of modules over a sheaf of rings in Modules on Sites, Section 18.10.

Remark 7.43.4. Let $\mathcal{C}$, $\mathcal{D}$ be sites. Let $u : \mathcal{D} \to \mathcal{C}$ be a continuous functor which gives rise to a morphism of sites $\mathcal{C} \to \mathcal{D}$. Note that even in the case of abelian groups we have not defined a pullback functor for presheaves of abelian groups. Since all colimits are representable in the category of abelian groups, we certainly may define a functor u_p^{ab} on abelian presheaves by the same colimits as we have used to define u_p on presheaves of sets. It will also be the case that u_p^{ab} is adjoint to u^p on the categories of abelian presheaves. However, it will not always be the case that u_p^{ab} agrees with u_p on the underlying presheaves of sets.

7.44. Pullback maps

It sometimes happens that a site $\mathcal{C}$ does not have a final object. In this case we define the global section functor as follows.

Definition 7.44.1. The *global sections* of a presheaf of sets $\mathcal{F}$ over a site $\mathcal{C}$ is the set

$$\Gamma(\mathcal{C}, \mathcal{F}) = \mathrm{Mor}_{PSh(\mathcal{C})}(*, \mathcal{F})$$

where $*$ is the final object in the category of presheaves on $\mathcal{C}$, i.e., the presheaf which associates to every object a singleton.

Of course the same definition applies to sheaves as well. Here is one way to compute global sections.

Lemma 7.44.2. *Let $\mathcal{C}$ be a site. Let $a, b : V \to U$ be objects of $\mathcal{C}$ such that*

$$h_V^\# \rightrightarrows h_U^\# \longrightarrow *$$

is a coequalizer in $Sh(\mathcal{C})$. Then $\Gamma(\mathcal{C}, \mathcal{F})$ is the equalizer of $a^*, b^* : \mathcal{F}(U) \to \mathcal{F}(V)$.

Proof. Since $\mathrm{Mor}_{Sh(\mathcal{C})}(h_U^\#, \mathcal{F}) = \mathcal{F}(U)$ this is clear from the definitions. □

Now, let $f : Sh(\mathcal{D}) \to Sh(\mathcal{C})$ be a morphism of topoi. Then for any sheaf $\mathcal{G}$ on $\mathcal{D}$ there is a pullback map

$$f^{-1} : \Gamma(\mathcal{D}, \mathcal{F}) \longrightarrow \Gamma(\mathcal{C}, f^{-1}\mathcal{F})$$

Namely, as f^{-1} is exact it transforms $*$ into $*$. We can generalize this a bit by considering a pair of sheaves $\mathcal{F}, \mathcal{G}$ on $\mathcal{C}, \mathcal{D}$ together with a map $f^{-1}\mathcal{F} \to \mathcal{G}$. Then we compose to get a map

$$\Gamma(\mathcal{D}, \mathcal{F}) \longrightarrow \Gamma(\mathcal{C}, \mathcal{G})$$

A slightly more general construction which occurs frequently in nature is the following. Suppose that we have a commutative diagram of morphisms of topoi

$$\begin{array}{ccc} Sh(\mathcal{D}) & \xrightarrow{\quad f \quad} & Sh(\mathcal{C}) \\ & {\scriptstyle h}\searrow \quad \swarrow{\scriptstyle g} & \\ & Sh(\mathcal{B}) & \end{array}$$

Next, suppose that we have a sheaf $\mathcal{F}$ on $\mathcal{D}$. Then there is a *pullback map*

$$f^{-1} : g_*\mathcal{F} \longrightarrow h_* f^{-1}\mathcal{F}$$

Namely, it is just the map coming from the identification $h_* f^{-1}\mathcal{F} = g_* f_* f^{-1}\mathcal{F}$ together with the canonical map $\mathcal{F} \to f_* f^{-1}\mathcal{F}$ pushed down to $\mathcal{B}$. Again, if we have a pair of sheaves $\mathcal{F}, \mathcal{G}$ on $\mathcal{C}, \mathcal{D}$ together with a map $f^{-1}\mathcal{F} \to \mathcal{G}$, then we compose to get a map

$$g_*\mathcal{F} \longrightarrow h_*\mathcal{G}$$

Restricting to sections over an object of $\mathcal{B}$ one recovers the pullback map on global sections in many cases, see (insert future reference here). A seemingly more general situation is where we have a commutative diagram of topoi

$$\begin{array}{ccc} Sh(\mathcal{D}) & \xrightarrow{f} & Sh(\mathcal{C}) \\ {\scriptstyle h}\downarrow & & \downarrow{\scriptstyle g} \\ Sh(\mathcal{B}) & \xrightarrow{e} & Sh(\mathcal{A}) \end{array}$$

and a sheaf $\mathcal{G}$ on $\mathcal{C}$. Then there is a map $e^{-1}g_*\mathcal{G} \to h_* f^{-1}\mathcal{G}$. Namely, this map is adjoint to a map $g_*\mathcal{G} \to e_* h_* f^{-1}\mathcal{G} = (e \circ h)_* f^{-1}\mathcal{G}$ which is the pullback map just described.

7.45. Topologies

In this section we define what a topology on a category is as defined in [**AGV71**]. One can develop all of the machinery of sheaves and topoi in this language. A modern exposition of this material can be found in [**KS06**]. However, the case of most interest for algebraic geometry is the topology defined by a site on its underlying category. Thus we strongly suggest the first time reader **skip this section and all other sections of this chapter**!

Definition 7.45.1. Let $\mathcal{C}$ be a category. Let $U \in \mathrm{Ob}(\mathcal{C})$. A *sieve* S *on* U is a subpresheaf $S \subset h_U$.

In other words, a sieve on U picks out for each object $T \in \mathrm{Ob}(\mathcal{C})$ a subset $S(T)$ of the set of all morphisms $T \to U$. In fact, the only condition on the collection of subsets $S(T) \subset h_U(T) = \mathrm{Mor}_{\mathcal{C}}(T, U)$ is the following rule

$$\left.\begin{array}{c} (\alpha : T \to U) \in S(T) \\ g : T' \to T \end{array}\right\} \Rightarrow (\alpha \circ g : T' \to U) \in S(T') \tag{7.45.1.1}$$

A good mental picture to keep in mind is to think of the map $S \to h_U$ as a "morphism from S to U".

Lemma 7.45.2. *Let $\mathcal{C}$ be a category. Let $U \in \mathrm{Ob}(\mathcal{C})$.*

1. *The collection of sieves on U is a set.*
2. *Inclusion defines a partial ordering on this set.*
3. *Unions and intersections of sieves are sieves.*
4. *Given a family of morphisms $\{U_i \to U\}_{i \in I}$ of $\mathcal{C}$ with target U there exists a unique smallest sieve S on U such that each $U_i \to U$ belongs to $S(U_i)$.*
5. *The sieve $S = h_U$ is the maximal sieve.*
6. *The empty subpresheaf is the minimal sieve.*

Proof. By our definition of subpresheaf, the collection of all subpresheaves of a presheaf $\mathcal{F}$ is a subset of $\prod_{U \in \mathrm{Ob}(\mathcal{C})} \mathcal{P}(\mathcal{F}(U))$. And this is a set. (Here $\mathcal{P}(A)$ denotes the powerset of A.) Hence the collection of sieves on U is a set.

The partial ordering is defined by: $S \leq S'$ if and only if $S(T) \subset S'(T)$ for all $T \to U$. Notation: $S \subset S'$.

Given a collection of sieves S_i, $i \in I$ on U we can define $\bigcup S_i$ as the sieve with values $(\bigcup S_i)(T) = \bigcup S_i(T)$ for all $T \in \mathrm{Ob}(\mathcal{C})$. We define the intersection $\bigcap S_i$ in the same way.

Given $\{U_i \to U\}_{i \in I}$ as in the statement, consider the morphisms of presheaves $h_{U_i} \to h_U$. We simply define S as the union of the images (Definition 7.3.5) of these maps of presheaves.

The last two statements of the lemma are obvious. □

Definition 7.45.3. Let $\mathcal{C}$ be a category. Given a family of morphisms $\{f_i : U_i \to U\}_{i \in I}$ of $\mathcal{C}$ with target U we say the sieve S on U described in Lemma 7.45.2 part (4) is the *sieve on U generated by the morphisms f_i*.

Definition 7.45.4. Let $\mathcal{C}$ be a category. Let $f : V \to U$ be a morphism of $\mathcal{C}$. Let $S \subset h_U$ be a sieve. We define the *pullback of S by f* to be the sieve $S \times_U V$ of V defined by the rule

$$(\alpha : T \to V) \in (S \times_U V)(T) \Leftrightarrow (f \circ \alpha : T \to U) \in S(T)$$

We leave it to the reader to see that this is indeed a sieve (hint: use Equation 7.45.1.1). We also sometimes call $S \times_U V$ the *base change* of S by $f : V \to U$.

Lemma 7.45.5. *Let $\mathcal{C}$ be a category. Let $U \in \mathrm{Ob}(\mathcal{C})$. Let S be a sieve on U. If $f : V \to U$ is in S, then $S \times_U V = h_V$ is maximal.*

Proof. Trivial from the definitions. □

Definition 7.45.6. Let $\mathcal{C}$ be a category. A *topology on $\mathcal{C}$* is given by the following datum:

For every $U \in \mathrm{Ob}(\mathcal{C})$ a subset $J(U)$ of the set of all sieves on U.

These sets $J(U)$ have to satisfy the following conditions

1. For every morphism $f : V \to U$ in $\mathcal{C}$, and every element $S \in J(U)$ the pullback $S \times_U V$ is an element of $J(V)$.
2. If S and S' are sieves on $U \in \mathrm{Ob}(\mathcal{C})$, if $S \in J(U)$, and if for all $f \in S(V)$ the pullback $S' \times_U V$ belongs to $J(V)$, then S' belongs to $J(U)$.

(3) For every $U \in \mathrm{Ob}(\mathcal{C})$ the maximal sieve $S = h_U$ belongs to $J(U)$.

In this case, the sieves belonging to $J(U)$ are called the *covering sieves*.

Lemma 7.45.7. *Let $\mathcal{C}$ be a category. Let J be a topology on $\mathcal{C}$. Let $U \in \mathrm{Ob}(\mathcal{C})$.*

(1) *Finite intersections of elements of $J(U)$ are in $J(U)$.*
(2) *If $S \in J(U)$ and $S' \supset S$, then $S' \in J(U)$.*

Proof. Let $S, S' \in J(U)$. Consider $S'' = S \cap S'$. For every $V \to U$ in $S(U)$ we have

$$S' \times_U V = S'' \times_U V$$

simply because $V \to U$ already is in S. Hence by the second axiom of the definition we see that $S'' \in J(U)$.

Let $S \in J(U)$ and $S' \supset S$. For every $V \to U$ in $S(U)$ we have $S' \times_U V = h_V$ by Lemma 7.45.5. Thus $S' \times_U V \in J(V)$ by the third axiom. Hence $S' \in J(U)$ by the second axiom. □

Definition 7.45.8. Let $\mathcal{C}$ be a category. Let J, J' be two topologies on $\mathcal{C}$. We say that J is *finer* than J' if and only if for every object U of $\mathcal{C}$ we have $J'(U) \subset J(U)$.

In other words, any covering sieve of J' is a covering sieve of J. There exists a finest topology on $\mathcal{C}$, namely that topology where any sieve is a covering sieve. This is called the *discrete topology* of $\mathcal{C}$. There also exists a coarsest topology. Namely, the topology where $J(U) = \{h_U\}$ for all objects U. This is called the *chaotic* or *indiscrete topology*.

Lemma 7.45.9. *Let $\mathcal{C}$ be a category. Let $\{J_i\}_{i \in I}$ be a set of topologies.*

(1) *The rule $J(U) = \bigcap J_i(U)$ defines a topology on $\mathcal{C}$.*
(2) *There is a coarsest topology finer than all of the topologies J_i.*

Proof. The first part is direct from the definitions. The second follows by taking the intersection of all topologies finer than all of the J_i. □

At this point we can define without any motivation what a sheaf is.

Definition 7.45.10. Let $\mathcal{C}$ be a category endowed with a topology J. Let $\mathcal{F}$ be a presheaf of sets on $\mathcal{C}$. We say that $\mathcal{F}$ is a *sheaf* on $\mathcal{C}$ if for every $U \in \mathrm{Ob}(\mathcal{C})$ and for every covering sieve S of U the canonical map

$$\mathrm{Mor}_{PSh(\mathcal{C})}(h_U, \mathcal{F}) \longrightarrow \mathrm{Mor}_{PSh(\mathcal{C})}(S, \mathcal{F})$$

is bijective.

Recall that the left hand side of the displayed formula equals $\mathcal{F}(U)$. In other words, $\mathcal{F}$ is a sheaf if and only if a section of $\mathcal{F}$ over U is the same thing as a compatible collection of sections $s_{T,\alpha} \in \mathcal{F}(T)$ parametrized by $(\alpha : T \to U) \in S(T)$, and this for every covering sieve S on U.

Lemma 7.45.11. *Let $\mathcal{C}$ be a category. Let $\{\mathcal{F}_i\}_{i \in I}$ be a collection of presheaves of sets on $\mathcal{C}$. For each $U \in \mathrm{Ob}(\mathcal{C})$ denote $J(U)$ the set of sieves S with the following property: For every morphism $V \to U$, the maps*

$$\mathrm{Mor}_{PSh(\mathcal{C})}(h_V, \mathcal{F}_i) \longrightarrow \mathrm{Mor}_{PSh(\mathcal{C})}(S \times_U V, \mathcal{F}_i)$$

are bijective for all $i \in I$. Then J defines a topology on $\mathcal{C}$. This topology is the finest topology in which all of the $\mathcal{F}_i$ are sheaves.

Proof. If we show that J is a topology, then the last statement of the lemma immediately follows. The first and second axioms of a topology are immediately verified. Thus, assume that we have an object U, and sieves S, S' of U such that $S \in J(U)$, and for all $V \to U$ in $S(V)$ we have $S' \times_U V \in J(V)$. We have to show that $S' \in J(U)$. In other words, we have to show that for any $f : W \to U$, the maps

$$\mathcal{F}_i(W) = \mathrm{Mor}_{PSh(\mathcal{C})}(h_W, \mathcal{F}_i) \longrightarrow \mathrm{Mor}_{PSh(\mathcal{C})}(S' \times_U W, \mathcal{F}_i)$$

are bijective for all $i \in I$. Pick an element $i \in I$ and pick an element $\varphi \in \mathrm{Mor}_{PSh(\mathcal{C})}(S' \times_U W, \mathcal{F}_i)$. We will construct a section $s \in \mathcal{F}_i(W)$ mapping to φ.

Suppose $\alpha : V \to W$ is an element of $S \times_U W$. According to the definition of pullbacks we see that the composition $f \circ \alpha : V \to W \to U$ is in S. Hence $S' \times_U V$ is in $J(W)$ by assumption on the pair of sieves S, S'. Now we have a commutative diagram of presheaves

$$\begin{array}{ccc} S' \times_U V & \longrightarrow & h_V \\ \downarrow & & \downarrow \\ S' \times_U W & \longrightarrow & h_W \end{array}$$

The restriction of φ to $S' \times_U V$ corresponds to an element $s_{V,\alpha} \in \mathcal{F}_i(V)$. This we see from the definition of J, and because $S' \times_U V$ is in $J(W)$. We leave it to the reader to check that the rule $(V, \alpha) \mapsto s_{V,\alpha}$ defines an element $\psi \in \mathrm{Mor}_{PSh(\mathcal{C})}(S \times_U W, \mathcal{F}_i)$. Since $S \in J(U)$ we see immediately from the definition of J that ψ corresponds to an element s of $\mathcal{F}_i(W)$.

We leave it to the reader to verify that the construction $\varphi \mapsto s$ is inverse to the natural map displayed above. □

Definition 7.45.12. Let $\mathcal{C}$ be a category. The finest topology on $\mathcal{C}$ such that all representable presheaves are sheaves, see Lemma 7.45.11, is called the *canonical topology* of $\mathcal{C}$.

7.46. The topology defined by a site

Suppose that $\mathcal{C}$ is a category, and suppose that $\mathrm{Cov}_1(\mathcal{C})$ and $\mathrm{Cov}_2(\mathcal{C})$ are sets of coverings that define the structure of a site on $\mathcal{C}$. In this situation it can happen that the categories of sheaves (of sets) for $\mathrm{Cov}_1(\mathcal{C})$ and $\mathrm{Cov}_2(\mathcal{C})$ are the same, see for example Lemma 7.8.5.

It turns out that the category of sheaves on $\mathcal{C}$ with respect to some topology J determines and is determined by the topology J. This is a nontrivial statement which we will address later, see Theorem 7.48.2.

Accepting this for the moment it makes sense to study the topology determined by a site.

Lemma 7.46.1. *Let $\mathcal{C}$ be a site with coverings $Cov(\mathcal{C})$. For every object U of $\mathcal{C}$, let $J(U)$ denote the set of sieves S on U with the following property: there exists a covering $\{f_i : U_i \to U\}_{i \in I} \in Cov(\mathcal{C})$ so that the sieve S' generated by the f_i (see Definition 7.45.3) is contained in S.*

1. *This J is a topology on $\mathcal{C}$.*
2. *A presheaf $\mathcal{F}$ is a sheaf for this topology (see Definition 7.45.10) if and only if it is a sheaf on the site (see Definition 7.7.1).*

Proof. To prove the first assertion we just note that axioms (1), (2) and (3) of the definition of a site (Definition 7.6.2) directly imply the axioms (3), (2) and (1) of the definition of a topology (Definition 7.45.6). As an example we prove J has property (2). Namely, let U be an object of $\mathcal{C}$, let S, S' be sieves on U such that $S \in J(U)$, and such that for every $V \to U$ in $S(V)$ we have $S' \times_U V \in J(V)$. By definition of $J(U)$ we can find a covering $\{f_i : U_i \to U\}$ of the site such that S the image of $h_{U_i} \to h_U$ is contained in S. Since each $S' \times_U U_i$ is in $J(U_i)$ we see that there are coverings $\{U_{ij} \to U_i\}$ of the site such that $h_{U_{ij}} \to h_{U_i}$ is contained in $S' \times_U U_i$. By definition of the base change this means that $h_{U_{ij}} \to h_U$ is contained in the subpresheaf $S' \subset h_U$. By axiom (2) for sites we see that $\{U_{ij} \to U\}$ is a covering of U and we conclude that $S' \in J(U)$ by definition of J.

Let $\mathcal{F}$ be a presheaf. Suppose that $\mathcal{F}$ is a sheaf in the topology J. We will show that $\mathcal{F}$ is a sheaf on the site as well. Let $\{f_i : U_i \to U\}_{i \in I}$ be a covering of the site. Let $s_i \in \mathcal{F}(U_i)$ be a family of sections such that $s_i|_{U_i \times_U U_j} = s_j|_{U_i \times_U U_j}$ for all i, j. We have to show that there exists a unique section $s \in \mathcal{F}(U)$ restricting back to the s_i on the U_i. Let $S \subset h_U$ be the sieve generated by the f_i. Note that $S \in J(U)$ by definition. In stead of constructing s, by the sheaf condition in the topology, it suffices to construct an element

$$\varphi \in \mathrm{Mor}_{PSh(\mathcal{C})}(S, \mathcal{F}).$$

Take $\alpha \in S(T)$ for some object $T \in \mathcal{U}$. This means exactly that $\alpha : T \to U$ is a morphism which factors through f_i for some $i \in I$ (and maybe more than 1). Pick such an index i and a factorization $\alpha = f_i \circ \alpha_i$. Define $\varphi(\alpha) = \alpha_i^* s_i$. If i', $\alpha = f_i \circ \alpha'_{i'}$ is a second choice, then $\alpha_i^* s_i = (\alpha'_{i'})^* s_{i'}$ exactly because of our condition $s_i|_{U_i \times_U U_j} = s_j|_{U_i \times_U U_j}$ for all i, j. Thus $\varphi(\alpha)$ is well defined. We leave it to the reader to verify that φ, which in turn determines s is correct in the sense that s restricts back to s_i.

Let $\mathcal{F}$ be a presheaf. Suppose that $\mathcal{F}$ is a sheaf on the site $(\mathcal{C}, \mathrm{Cov}(\mathcal{C}))$. We will show that $\mathcal{F}$ is a sheaf for the topology J as well. Let U be an object of $\mathcal{C}$. Let S be a covering sieve on U with respect to the topology J. Let

$$\varphi \in \mathrm{Mor}_{PSh(\mathcal{C})}(S, \mathcal{F}).$$

We have to show there is a unique element in $\mathcal{F}(U) = \mathrm{Mor}_{PSh(\mathcal{C})}(h_U, \mathcal{F})$ which restricts back to φ. By definition there exists a covering $\{f_i : U_i \to U\}_{i \in I} \in \mathrm{Cov}(\mathcal{C})$ such that $f_i : U_i \in U$ belongs to $S(U_i)$. Hence we can set $s_i = \varphi(f_i) \in \mathcal{F}(U_i)$. Then it is a pleasant exercise to see that $s_i|_{U_i \times_U U_j} = s_j|_{U_i \times_U U_j}$ for all i, j. Thus we obtain the desired section s by the sheaf condition for $\mathcal{F}$ on the site $(\mathcal{C}, \mathrm{Cov}(\mathcal{C}))$. Details left to the reader. □

Definition 7.46.2. Let $\mathcal{C}$ be a site with coverings $\mathrm{Cov}(\mathcal{C})$. The *topology associated to* $\mathcal{C}$ is the topology J constructed in Lemma 7.46.1 above.

Let $\mathcal{C}$ be a category. Let $\mathrm{Cov}_1(\mathcal{C})$ and $\mathrm{Cov}_2(\mathcal{C})$ be two coverings defining the structure of a site on $\mathcal{C}$. It may very well happen that the topologies defined by these are the same. If this happens then we say $\mathrm{Cov}_1(\mathcal{C})$ and $\mathrm{Cov}_2(\mathcal{C})$ *define the same topology* on $\mathcal{C}$. And if this happens then the categories of sheaves are the same, by Lemma 7.46.1.

It is usually the case that we only care about the topology defined by a collection of coverings, and we view the possibility of choosing different sets of coverings as a tool to study the topology.

Remark 7.46.3. Enlarging the class of coverings. Clearly, if $\mathrm{Cov}(\mathcal{C})$ defines the structure of a site on $\mathcal{C}$ then we may add to $\mathcal{C}$ any set of families of morphisms with fixed target tautologically equivalent (see Definition 7.8.2) to elements of $\mathrm{Cov}(\mathcal{C})$ without changing the topology.

Remark 7.46.4. Shrinking the class of coverings. Let $\mathcal{C}$ be a site. Consider the power set $\mathcal{S} = P(\mathrm{Arrow}(\mathcal{C}))$ (power set) of the set of morphisms, i.e., the set of all sets of morphisms. Let $\mathcal{S}_\tau \subset \mathcal{S}$ be the subset consisting of those $T \in \mathcal{S}$ such that (a) all $\varphi \in T$ have the same target, (b) the collection $\{\varphi\}_{\varphi \in T}$ is tautologically equivalent (see Definition 7.8.2) to some covering in $\mathrm{Cov}(\mathcal{C})$. Clearly, considering the elements of $\mathcal{S}_\tau$ as the coverings, we do not get exactly the notion of a site as defined in Definition 7.6.2. The structure $(\mathcal{C}, \mathcal{S}_\tau)$ we get satisfies slightly modified conditions. The modified conditions are:

(0') $\mathrm{Cov}(\mathcal{C}) \subset P(\mathrm{Arrow}(\mathcal{C}))$,
(1') If $V \to U$ is an isomorphism then $\{V \to U\} \in \mathrm{Cov}(\mathcal{C})$.
(2') If $\{U_i \to U\}_{i \in I} \in \mathrm{Cov}(\mathcal{C})$ and for each i we have $\{V_{ij} \to U_i\}_{j \in J_i} \in \mathrm{Cov}(\mathcal{C})$, then $\{V_{ij} \to U\}_{i \in I, j \in J_i}$ is tautologically equivalent to an element of $\mathrm{Cov}(\mathcal{C})$.
(3') If $\{U_i \to U\}_{i \in I} \in \mathrm{Cov}(\mathcal{C})$ and $V \to U$ is a morphism of $\mathcal{C}$ then $U_i \times_U V$ exists for all i and $\{U_i \times_U V \to V\}_{i \in I}$ is tautologically equivalent to an element of $\mathrm{Cov}(\mathcal{C})$.

And it is easy to verify that, given a structure satisfying (0') – (3') above, then after suitably enlarging $\mathrm{Cov}(\mathcal{C})$ (compare Sets, Section 3.11) we get a site. Obviously there is little difference between this notion and the actual notion of a site, at least from the point of view of the topology. There are two benefits: because of condition (0') above the coverings automatically form a set, and because of (0') the totality of all structures of this type forms a set as well. The price you pay for this is that you have to keep writing "tautologically equivalent" everywhere.

7.47. Sheafification in a topology

In this section we explain the analogue of the sheafification construction in a topology.

Let $\mathcal{C}$ be a category. Let J be a topology on $\mathcal{C}$. Let $\mathcal{F}$ be a presheaf of sets. For every $U \in \mathrm{Ob}(\mathcal{C})$ we define

$$L\mathcal{F}(U) = \mathrm{colim}_{S \in J(U)^{opp}} \mathrm{Mor}_{PSh(\mathcal{C})}(S, \mathcal{F})$$

as a colimit. Here we think of $J(U)$ as a partially ordered set, ordered by inclusion, see Lemma 7.45.2. The transition maps in the system are defined as follows. If $S \subset S'$ are in $J(U)$, then $S \to S'$ is a morphism of presheaves. Hence there is a natural restriction mapping

$$\mathrm{Mor}_{PSh(\mathcal{C})}(S, \mathcal{F}) \longrightarrow \mathrm{Mor}_{PSh(\mathcal{C})}(S', \mathcal{F}).$$

Thus we see that $S \mapsto \mathrm{Mor}_{PSh(\mathcal{C})}(S, \mathcal{F})$ is a directed system as in Categories, Definition 4.21.1 provided we reverse the ordering on $J(U)$ (which is what the

superscript ^{opp} is supposed to indicate). In particular, since $h_U \in J(U)$ there is a canonical map

$$\ell : \mathcal{F}(U) \longrightarrow L\mathcal{F}(U)$$

coming from the identification $\mathcal{F}(U) = \mathrm{Mor}_{PSh(\mathcal{C})}(h_U, \mathcal{F})$. In addition, the colimit defining $L\mathcal{F}(U)$ is directed since for any pair of covering sieves S, S' on U the sieve $S \cap S'$ is a covering sieve too, see Lemma 7.45.2.

Let $f : V \to U$ be a morphism in $\mathcal{C}$. Let $S \in J(U)$. There is a commutative diagram

$$\begin{array}{ccc} S \times_U V & \longrightarrow & h_V \\ \downarrow & & \downarrow \\ S & \longrightarrow & h_U \end{array}$$

We can use the left vertical map to get canonical restriction maps

$$\mathrm{Mor}_{PSh(\mathcal{C})}(S, \mathcal{F}) \to \mathrm{Mor}_{PSh(\mathcal{C})}(S \times_U V, \mathcal{F}).$$

Base change $S \mapsto S \times_U V$ induces an order preserving map $J(U) \to J(V)$. And the restriction maps define a transformation of functors as in Categories, Lemma categories-lemma-functorial-colimit. Hence we get a natural restriction map

$$L\mathcal{F}(U) \longrightarrow L\mathcal{F}(V).$$

Lemma 7.47.1. *In the situation above.*

(1) *The assignment $U \mapsto L\mathcal{F}(U)$ combined with the restriction mappings defined above is a presheaf.*
(2) *The maps ℓ glue to give a morphism of presheaves $\ell : \mathcal{F} \to L\mathcal{F}$.*
(3) *The rule $\mathcal{F} \mapsto (\mathcal{F} \xrightarrow{\ell} L\mathcal{F})$ is a functor.*
(4) *If $\mathcal{F}$ is a subpresheaf of $\mathcal{G}$, then $L\mathcal{F}$ is a subpresheaf of $L\mathcal{G}$.*
(5) *The map $\ell : \mathcal{F} \to L\mathcal{F}$ has the following property: For every section $s \in L\mathcal{F}(U)$ there exists a covering sieve S on U and an element $\varphi \in \mathrm{Mor}_{PSh(\mathcal{C})}(S, \mathcal{F})$ such that $\ell(\varphi)$ equals the restriction of s to S.*

Proof. Omitted. □

Definition 7.47.2. Let $\mathcal{C}$ be a category. Let J be a topology on $\mathcal{C}$. We say that a presheaf of sets $\mathcal{F}$ is *separated* if for every object U and every covering sieve S on U the canonical map $\mathcal{F}(U) \to \mathrm{Mor}_{PSh(\mathcal{C})}(S, \mathcal{F})$ is injective.

Theorem 7.47.3. *Let $\mathcal{C}$ be a category. Let J be a topology on $\mathcal{C}$. Let $\mathcal{F}$ be a presheaf of sets.*

(1) *The presheaf $L\mathcal{F}$ is separated.*
(2) *If $\mathcal{F}$ is separated, then $L\mathcal{F}$ is a sheaf and the map of presheaves $\mathcal{F} \to L\mathcal{F}$ is injective.*
(3) *If $\mathcal{F}$ is a sheaf, then $\mathcal{F} \to L\mathcal{F}$ is an isomorphism.*
(4) *The presheaf $LL\mathcal{F}$ is always a sheaf.*

Proof. Part (3) is trivial from the definition of L and the definition of a sheaf (Definition 7.45.10). Part (4) follows formally from the others.

We sketch the proof of (1). Suppose S is a covering sieve of the object U. Suppose that $\varphi_i \in L\mathcal{F}(U)$, $i = 1, 2$ map to the same element in $\mathrm{Mor}_{PSh(\mathcal{C})}(S, L\mathcal{F})$. We may find a single covering sieve S' on U such that both φ_i are represented by elements

$\varphi_i \in \mathrm{Mor}_{PSh(\mathcal{C})}(S', \mathcal{F})$. We may assume that $S' = S$ by replacing both S and S' by $S' \cap S$ which is also a covering sieve, see Lemma 7.45.2. Suppose $V \in \mathrm{Ob}(\mathcal{C})$, and $\alpha : V \to U$ in $S(V)$. Then we have $S \times_U V = h_V$, see Lemma 7.45.5. Thus the restrictions of φ_i via $V \to U$ correspond to sections $s_{i,V,\alpha}$ of $\mathcal{F}$ over V. The assumption is that there exist a covering sieve $S_{V,\alpha}$ of V such that $s_{i,V,\alpha}$ restrict to the same element of $\mathrm{Mor}_{PSh(\mathcal{C})}(S_{V,\alpha}, \mathcal{F})$. Consider the sieve S'' on U defined by the rule

$$\begin{aligned} (f : T \to U) \in S''(T) \quad \Leftrightarrow \quad & \exists\ V,\ \alpha : V \to U,\ \alpha \in S(V), \\ & \exists\ g : T \to V,\ g \in S_{V,\alpha}(T), \\ & f = \alpha \circ g \end{aligned} \tag{7.47.3.1}$$

By axiom (2) of a topology we see that S'' is a covering sieve on U. By construction we see that φ_1 and φ_2 restrict to the same element of $\mathrm{Mor}_{PSh(\mathcal{C})}(S'', L\mathcal{F})$ as desired.

We sketch the proof of (2). Assume that $\mathcal{F}$ is a separated presheaf of sets on $\mathcal{C}$ with respect to the topology J. Let S be a covering sieve of the object U of $\mathcal{C}$. Suppose that $\varphi \in \mathrm{Mor}_{\mathcal{C}}(S, L\mathcal{F})$. We have to find an element $s \in L\mathcal{F}(U)$ restricting to φ. Suppose $V \in \mathrm{Ob}(\mathcal{C})$, and $\alpha : V \to U$ in $S(V)$. The value $\varphi(\alpha) \in L\mathcal{F}(V)$ is given by a covering sieve $S_{V,\alpha}$ of V and a morphism of presheaves $\varphi_{V,\alpha} : S_{V,\alpha} \to \mathcal{F}$. As in the proof above, define a covering sieve S'' on U by Equation (7.47.3.1). We define

$$\varphi'' : S'' \longrightarrow \mathcal{F}$$

by the following simple rule: For every $f : T \to U$, $f \in S''(T)$ choose V, α, g as in Equation (7.47.3.1). Then set

$$\varphi''(f) = \varphi_{V,\alpha}(g).$$

We claim this is independent of the choice of V, α, g. Consider a second such choiceV', α', g'. The restrictions of $\varphi_{V,\alpha}$ and $\varphi_{V',\alpha'}$ to the intersection of the following covering sieves on T

$$(S_{V,\alpha} \times_{V,g} T) \cap (S_{V',\alpha'} \times_{V',g'} T)$$

agree. Namely, these restrictions both correspond to the restriction of φ to T (via f) and the desired equality follows because $\mathcal{F}$ is separated. Denote the common restriction ψ. The independence of choice follows because $\varphi_{V,\alpha}(g) = \psi(\mathrm{id}_T) = \varphi_{V',\alpha'}(g')$. OK, so now φ'' gives an element $s \in L\mathcal{F}(U)$. We leave it to the reader to check that s restricts to φ. □

Definition 7.47.4. Let $\mathcal{C}$ be a category endowed with a topology J. Let $\mathcal{F}$ be a presheaf of sets on $\mathcal{C}$. The sheaf $\mathcal{F}^\# := LL\mathcal{F}$ together with the canonical map $\mathcal{F} \to \mathcal{F}^\#$ is called the *sheaf associated to* $\mathcal{F}$.

Proposition 7.47.5. *Let $\mathcal{C}$ be a category endowed with a topology. Let $\mathcal{F}$ be a presheaf of sets on $\mathcal{C}$. The canonical map $\mathcal{F} \to \mathcal{F}^\#$ has the following universal property: For any map $\mathcal{F} \to \mathcal{G}$, where $\mathcal{G}$ is a sheaf of sets, there is a unique map $\mathcal{F}^\# \to \mathcal{G}$ such that $\mathcal{F} \to \mathcal{F}^\# \to \mathcal{G}$ equals the given map.*

Proof. Same as the proof of Proposition 7.10.12. □

7.48. Topologies and sheaves

Lemma 7.48.1. *Let $\mathcal{C}$ be a category endowed with a topology J. Let U be an object of $\mathcal{C}$. Let S be a sieve on U. The following are equivalent*

(1) *The sieve S is a covering sieve.*
(2) *The sheafification $S^\# \to h_U^\#$ of the map $S \to h_U$ is an isomorphism.*

Proof. First we make a couple of general remarks. We will use that $S^\# = LLS$, and $h_U^\# = LLh_U$. In particular, by Lemma 7.47.1, we see that $S^\# \to h_U^\#$ is injective. Note that $\mathrm{id}_U \in h_U(U)$. Hence it gives rise to sections of Lh_U and $h_U^\# = LLh_U$ over U which we will also denote id_U.

Suppose S is a covering sieve. It clearly suffices to find a morphism $h_U \to S^\#$ such that the composition $h_U \to h_U^\#$ is the canonical map. To find such a map it suffices to find a section $s \in S^\#(U)$ wich restricts to id_U. But since S is a covering sieve, the element $\mathrm{id}_S \in \mathrm{Mor}_{PSh(\mathcal{C})}(S, S)$ gives rise to a section of LS over U which restricts to id_U in Lh_U. Hence we win.

Suppose that $S^\# \to h_U^\#$ is an isomorphism. Let $1 \in S^\#(U)$ be the element corresponding to id_U in $h_U^\#(U)$. Because $S^\# = LLS$ there exists a covering sieve S' on U such that 1 comes from a

$$\varphi \in \mathrm{Mor}_{PSh(\mathcal{C})}(S', LS).$$

This in turn means that for every $\alpha : V \to U$, $\alpha \in S'(V)$ there exists a covering sieve $S_{V,\alpha}$ on V such that $\varphi(\mathrm{id}_V)$ corresponds to a morphism of presheaves $S_{V,\alpha} \to S$. In other words $S_{V,\alpha}$ is contained in $S \times_U V$. By the second axiom of a topology we see that S is a covering sieve. □

Theorem 7.48.2. *Let $\mathcal{C}$ be a category. Let J, J' be topologies on $\mathcal{C}$. The following are equivalent*

(1) *$J = J'$,*
(2) *sheaves for the topology J are the same as sheaves for the topology J'.*

Proof. It is a tautology that if $J = J'$ then the notions of sheaves are the same. Conversely, Lemma 7.48.1 characterizes covering sieves in terms of the sheafification functor. But the sheafification functor $PSh(\mathcal{C}) \to Sh(\mathcal{C}, J)$ is the right adjoint of the inclusion functor $Sh(\mathcal{C}, J) \to PSh(\mathcal{C})$. Hence if the subcategories $Sh(\mathcal{C}, J)$ and $Sh(\mathcal{C}, J')$ are the same, then the sheafification functors are the same and hence the collections of covering sieves are the same. □

Lemma 7.48.3. *Assumption and notation as in Theorem 7.48.2. Then $J \subset J'$ if and only if every sheaf for the topology J' is a sheaf for the topology J.*

Proof. One direction is clear. For the other direction suppose that $Sh(\mathcal{C}, J') \subset Sh(\mathcal{C}, J)$. By formal nonsense this implies that if $\mathcal{F}$ is a presheaf of sets, and $\mathcal{F} \to \mathcal{F}^\#$, resp. $\mathcal{F} \to \mathcal{F}^{\#,\prime}$ is the sheafification wrt J, resp. J' then there is a canonical map $\mathcal{F}^\# \to \mathcal{F}^{\#,\prime}$ such that $\mathcal{F} \to \mathcal{F}^\# \to \mathcal{F}^{\#,\prime}$ equals the canonical map $\mathcal{F} \to \mathcal{F}^{\#,\prime}$. Of course, $\mathcal{F}^\# \to \mathcal{F}^{\#,\prime}$ identifies the second sheaf as the sheafification of the first with respect to the topology J'. Apply this to the map $S \to h_U$ of

Lemma 7.48.1. We get a commutative diagram

$$\begin{array}{ccccc} S & \longrightarrow & S^\# & \longrightarrow & S^{\#,\prime} \\ \downarrow & & \downarrow & & \downarrow \\ h_U & \longrightarrow & h_U^\# & \longrightarrow & h_U^{\#,\prime} \end{array}$$

And clearly, if S is a covering sieve for the topology J then the middle vertical map is an isomorphism (by the lemma) and we conclude that the right vertical map is an isomorphism as it is the sheafification of the one in the middle wrt J'. By the lemma again we conclude that S is a covering sieve for J' as well. □

7.49. Topologies and continuous functors

Explain how a continuous functor gives an adjoint pair of functors on sheaves.

7.50. Points and topologies

Recall from Section 7.31 that given a functor $p = u : \mathcal{C} \to Sets$ we can define a stalk functor

$$PSh(\mathcal{C}) \longrightarrow Sets, \mathcal{F} \longmapsto \mathcal{F}_p.$$

Definition 7.50.1. Let $\mathcal{C}$ be a category. Let J be a topology on $\mathcal{C}$. A *point* p of the topology is given by a functor $u : \mathcal{C} \to Sets$ such that

(1) For every covering sieve S on U the map $S_p \to (h_U)_p$ is surjective.
(2) The stalk functor $Sh(\mathcal{C}) \to Sets$, $\mathcal{F} \to \mathcal{F}_p$ is exact.

7.51. Other chapters

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CHAPTER 8

Stacks

8.1. Introduction

In this very short chapter we introduce stacks, and stacks in groupoids. See [**DM69**], and [**Vis**].

8.2. Presheaves of morphisms associated to fibred categories

Let $\mathcal{C}$ be a category. Let $p : \mathcal{S} \to \mathcal{C}$ be a fibred category, see Categories, Section 4.31. Suppose that $x, y \in \mathrm{Ob}(\mathcal{S}_U)$ are objects in the fibre category over U. We are going to define a functor

$$Mor(x, y) : (\mathcal{C}/U)^{opp} \longrightarrow Sets.$$

In other words this will be a presheaf on $\mathcal{C}/U$, see Sites, Definition 7.2.2. Make a choice of pullbacks as in Categories, Definition 4.31.6. Then, for $f : V \to U$ we set

$$Mor(x, y)(f : V \to U) = \mathrm{Mor}_{\mathcal{S}_V}(f^*x, f^*y).$$

Let $f' : V' \to U$ be a second object of $\mathcal{C}/U$. We also have to define the restriction map corresponding to a morphism $g : V'/U \to V/U$ in $\mathcal{C}/U$, in other words $g : V' \to V$ and $f' = f \circ g$. This will be a map

$$\mathrm{Mor}_{\mathcal{S}_V}(f^*x, f^*y) \longrightarrow \mathrm{Mor}_{\mathcal{S}_{V'}}(f'^*x, f'^*y), \quad \phi \longmapsto \phi|_{V'}$$

This map will basically be g^*, except that this transforms an element ϕ of the left hand side into an element $g^*\phi$ of $\mathrm{Mor}_{\mathcal{S}_{V'}}(g^*f^*x, g^*f^*y)$. At this point we use the transformation $\alpha_{g,f}$ of Categories, Lemma 4.31.7. In a formula, the restriction map is described by

$$\phi|_{V'} = (\alpha_{g,f})_y^{-1} \circ g^*\phi \circ (\alpha_{g,f})_x.$$

Of course, nobody thinks of this restriction map in this way. We will only do this once in order to verify the following lemma.

Lemma 8.2.1. *This actually does give a presheaf.*

Proof. Let $g : V'/U \to V/U$ be as above and similarly $g' : V''/U \to V'/U$ be morphisms in $\mathcal{C}/U$. So $f' = f \circ g$ and $f'' = f' \circ g' = f \circ g \circ g'$. Let $\phi \in \mathrm{Mor}_{\mathcal{S}_V}(f^*x, f^*y)$. Then we have

$$\begin{aligned}
& (\alpha_{g\circ g',f})_y^{-1} \circ (g \circ g')^*\phi \circ (\alpha_{g\circ g',f})_x \\
= \; & (\alpha_{g\circ g',f})_y^{-1} \circ (\alpha_{g',g})_{f^*y}^{-1} \circ (g')^*g^*\phi \circ (\alpha_{g',g})_{f^*x} \circ (\alpha_{g\circ g',f})_x \\
= \; & (\alpha_{g',f'})_y^{-1} \circ (g')^*(\alpha_{g,f})_y^{-1} \circ (g')^*g^*\phi \circ (g')^*(\alpha_{g,f})_x \circ (\alpha_{g',f'})_x \\
= \; & (\alpha_{g',f'})_y^{-1} \circ (g')^*\Big((\alpha_{g,f})_y^{-1} \circ g^*\phi \circ (\alpha_{g,f})_x\Big) \circ (\alpha_{g',f'})_x
\end{aligned}$$

which is what we want, namely $\phi|_{V''} = (\phi|_{V'})|_{V''}$. The first equality holds because $\alpha_{g',g}$ is a transformation of functors, and hence

$$\begin{array}{ccc} (g \circ g')^* f^* x & \xrightarrow[(g\circ g')^*\phi]{} & (g \circ g')^* f^* y \\ {\scriptstyle (\alpha_{g',g})_{f^*x}}\downarrow & & \downarrow{\scriptstyle (\alpha_{g',g})_{f^*y}} \\ (g')^* g^* f^* x & \xrightarrow{(g')^* g^* \phi} & (g')^* g^* f^* y \end{array}$$

commutes. The second equality holds because of property (d) of a pseudo functor since $f' = f \circ g$ (see Categories, Definition 4.27.5). The last equality follows from the fact that $(g')^*$ is a functor. □

From now on we often omit mentioning the transformations $\alpha_{g,f}$ and we simply identify the functors $g^* \circ f^*$ and $(f \circ g)^*$. In particular, given $g : V'/U \to V/U$ the restriction mappings for the presheaf $Mor(x, y)$ will sometimes be denoted $\phi \mapsto g^*\phi$. We formalize the construction in a definition.

Definition 8.2.2. Let $\mathcal{C}$ be a category. Let $p : \mathcal{S} \to \mathcal{C}$ be a fibred category, see Categories, Section 4.31. Given an object U of $\mathcal{C}$ and objects x, y of the fibre category, the *presheaf of morphisms from x to y* is the presheaf

$$(f : V \to U) \longmapsto \mathrm{Mor}_{\mathcal{S}_V}(f^*x, f^*y)$$

described above. It is denoted $Mor(x, y)$. The subpresheaf $Isom(x, y)$ whose values over V is the set of isomorphisms $f^*x \to f^*y$ in the fibre category $\mathcal{S}_V$ is called the *presheaf of isomorphisms from x to y*.

If $\mathcal{S}$ is fibred in groupoids then of course $Isom(x, y) = Mor(x, y)$, and it is customary to use the *Isom* notation.

Lemma 8.2.3. *Let $F : \mathcal{S}_1 \to \mathcal{S}_2$ be a 1-morphism of fibred categories over the category $\mathcal{C}$. Let $U \in \mathrm{Ob}(\mathcal{C})$ and $x, y \in \mathrm{Ob}(\mathcal{S}_U)$. Then F defines a canonical morphism of presheaves*

$$Mor_{\mathcal{S}_1}(x, y) \longrightarrow Mor_{\mathcal{S}_2}(F(x), F(y))$$

on $\mathcal{C}/U$.

Proof. By Categories, Definition 4.31.9 the functor F maps strongly cartesian morphisms to strongly cartesian morphisms. Hence if $f : V \to U$ is a morphism in $\mathcal{C}$, then there are canonical isomorphisms $\alpha_V : f^*F(x) \to F(f^*x)$, $\beta_V : f^*F(y) \to F(f^*y)$ such that $f^*F(x) \to F(f^*x) \to F(x)$ is the canonical morphism $f^*F(x) \to F(x)$, and similarly for β_V. Thus we may define

$$\begin{array}{ccc} Mor_{\mathcal{S}_1}(x, y)(f : V \to U) & = & \mathrm{Mor}_{\mathcal{S}_{1,V}}(f^*x, f^*y) \\ & & \downarrow \\ Mor_{\mathcal{S}_2}(F(x), F(y))(f : V \to U) & = & \mathrm{Mor}_{\mathcal{S}_{2,V}}(f^*F(x), f^*F(y)) \end{array}$$

by $\phi \mapsto \beta_V^{-1} \circ F(\phi) \circ \alpha_V$. We omit the verification that this is compatible with the restriction mappings. □

Remark 8.2.4. Suppose that $p : \mathcal{S} \to \mathcal{C}$ is fibred in groupoids. In this case we can prove Lemma 8.2.1 using Categories, Lemma 4.34.4 which says that $\mathcal{S} \to \mathcal{C}$ is equivalent to the category associated to a contravariant functor $F : \mathcal{C} \to \mathrm{Groupoids}$.

In the case of the fibred category associated to F we have $g^* \circ f^* = (f \circ g)^*$ on the nose and there is no need to use the maps $\alpha_{g,f}$. In this case the lemma is (even more) trivial. Of course then one uses that the $Mor(x, y)$ presheaf is unchanged when passing to an equivalent fibred category which follows from Lemma 8.2.3.

Lemma 8.2.5. *Let $\mathcal{C}$ be a category. Let $p : \mathcal{S} \to \mathcal{C}$ be a fibred category, see Categories, Section 4.31. Let $U \in \mathrm{Ob}(\mathcal{C})$ and let $x, y \in \mathrm{Ob}(\mathcal{S}_U)$. Denote $x, y : \mathcal{C}/U \to \mathcal{S}$ also the corresponding 1-morphisms, see Categories, Lemma 4.39.1. Then*

1. *the 2-fibre product $\mathcal{S} \times_{\mathcal{S} \times \mathcal{S}, (x,y)} \mathcal{C}/U$ is fibred in setoids over $\mathcal{C}/U$, and*
2. *$Isom(x, y)$ is the presheaf of sets corresponding to this category fibred in setoids, see Categories, Lemma 4.37.6.*

Proof. Omitted. Hint: Objects of the 2-fibre product are $(a : V \to U, z, a : V \to U, (\alpha, \beta))$ where $\alpha : z \to a^*x$ and $\beta : z \to a^*y$ are isomorphisms in $\mathcal{S}_V$. Thus the relationship with $Isom(x, y)$ comes by assigning to such an object the isomorphism $\beta \circ \alpha^{-1}$. □

8.3. Descent data in fibred categories

In this section we define the notion of a descent datum in the abstract setting of a fibred category. Before we do so we point out that this is completely analogous to descent data for quasi-coherent sheaves (Descent, Section 34.2) and descent data for schemes over schemes (Descent, Section 34.30).

We will use the convention where the projection maps $\mathrm{pr}_i : X \times \ldots \times X \to X$ are labeled starting with $i = 0$. Hence we have $\mathrm{pr}_0, \mathrm{pr}_1 : X \times X \to X$, $\mathrm{pr}_0, \mathrm{pr}_1, \mathrm{pr}_2 : X \times X \times X \to X$, etc.

Definition 8.3.1. Let $\mathcal{C}$ be a category. Let $p : \mathcal{S} \to \mathcal{C}$ be a fibred category. Make a choice of pullbacks as in Categories, Definition 4.31.6. Let $\mathcal{U} = \{f_i : U_i \to U\}_{i \in I}$ be a family of morphisms of $\mathcal{C}$. Assume all the fibre products $U_i \times_U U_j$, and $U_i \times_U U_j \times_U U_k$ exist.

1. A *descent datum (X_i, φ_{ij}) in $\mathcal{S}$ relative to the family $\{f_i : U_i \to U\}$* is given by an object X_i of $\mathcal{S}_{U_i}$ for each $i \in I$, an isomorphism $\varphi_{ij} : \mathrm{pr}_0^* X_i \to \mathrm{pr}_1^* X_j$ in $\mathcal{S}_{U_i \times_U U_j}$ for each pair $(i, j) \in I^2$ such that for every triple of indices $(i, j, k) \in I^3$ the diagram
$$\begin{array}{ccc} \mathrm{pr}_0^* X_i & \xrightarrow{\mathrm{pr}_{02}^* \varphi_{ik}} & \mathrm{pr}_2^* X_k \\ & {\scriptstyle \mathrm{pr}_{01}^* \varphi_{ij}} \searrow \quad \nearrow {\scriptstyle \mathrm{pr}_{12}^* \varphi_{jk}} & \\ & \mathrm{pr}_1^* X_j & \end{array}$$
in the category $\mathcal{S}_{U_i \times_U U_j \times_U U_k}$ commutes. This is called the *cocycle condition*.
2. A *morphism $\psi : (X_i, \varphi_{ij}) \to (X'_i, \varphi'_{ij})$ of descent data* is given by a family $\psi = (\psi_i)_{i \in I}$ of morphisms $\psi_i : X_i \to X'_i$ in $\mathcal{S}_{U_i}$ such that all the diagrams
$$\begin{array}{ccc} \mathrm{pr}_0^* X_i & \xrightarrow{\varphi_{ij}} & \mathrm{pr}_1^* X_j \\ {\scriptstyle \mathrm{pr}_0^* \psi_i} \downarrow & & \downarrow {\scriptstyle \mathrm{pr}_1^* \psi_j} \\ \mathrm{pr}_0^* X'_i & \xrightarrow{\varphi'_{ij}} & \mathrm{pr}_1^* X'_j \end{array}$$
in the categories $\mathcal{S}_{U_i \times_U U_j}$ commute.

(3) The category of descent data relative to $\mathcal{U}$ is denoted $DD(\mathcal{U})$.

The fibre products $U_i \times_U U_j$ and $U_i \times_U U_j \times_U U_k$ will exist if each of the morphisms $f_i : U_i \to U$ is *representable*, see Categories, Definition 4.6.4. Recall that in a site one of the conditions for a covering $\{U_i \to U\}$ is that each of the morphisms is representable, see Sites, Definition 7.6.2 part (3). In fact the main interest in the definition above is where $\mathcal{C}$ is a site and $\{U_i \to U\}$ is a covering of $\mathcal{C}$. However, a descent datum is just an abstract gadget that can be defined as above. This is useful: for example, given a fibred category over $\mathcal{C}$ one can look at the collection of families with respect to which descent data are effective, and try to use these as the family of coverings for a site.

Remarks 8.3.2. Two remarks on Definition 8.3.1 are in order. Let $p : \mathcal{S} \to \mathcal{C}$ be a fibred category. Let $\{f_i : U_i \to U\}_{i \in I}$, and (X_i, φ_{ij}) be as in Definition 8.3.1.

(1) There is a diagonal morphism $\Delta : U_i \to U_i \times_U U_i$. We can pull back φ_{ii} via this morphism to get an automorphism $\Delta^*\varphi_{ii} \in \mathrm{Aut}_{U_i}(x_i)$. On pulling back the cocycle condition for the triple (i, i, i) by $\Delta_{123} : U_i \to U_i \times_U U_i \times_U U_i$ we deduce that $\Delta^*\varphi_{ii} \circ \Delta^*\varphi_{ii} = \Delta^*\varphi_{ii}$; thus $\Delta^*\varphi_{ii} = \mathrm{id}_{x_i}$.
(2) There is a morphism $\Delta_{13} : U_i \times_U U_j \to U_i \times_U U_j \times_U U_i$ and we can pull back the cocycle condition for the triple (i, j, i) to get the identity $(\sigma^*\varphi_{ji}) \circ \varphi_{ij} = \mathrm{id}_{\mathrm{pr}_0^* x_i}$, where $\sigma : U_i \times_U U_j \to U_j \times_U U_i$ is the switching morphism.

Lemma 8.3.3. *(Pullback of descent data.) Let $\mathcal{C}$ be a category. Let $p : \mathcal{S} \to \mathcal{C}$ be a fibred category. Make a choice pullbacks as in Categories, Definition 4.31.6. Let $\mathcal{U} = \{f_i : U_i \to U\}_{i \in I}$, and $\mathcal{V} = \{V_j \to V\}_{j \in J}$ be a families of morphisms of $\mathcal{C}$ with fixed target. Assume all the fibre products $U_i \times_U U_{i'}$, $U_i \times_U U_{i'} \times_U U_{i''}$, $V_j \times_V V_{j'}$, and $V_j \times_V V_{j'} \times_V V_{j''}$ exist. Let $\alpha : I \to J$, $h : U \to V$ and $g_i : U_i \to V_{\alpha(i)}$ be a morphism of families of maps with fixed target, see Sites, Definition 7.8.1.*

(1) *Let $(Y_j, \varphi_{jj'})$ be a descent datum relative to the family $\{V_j \to V\}$. The system*
$$\left(g_i^* Y_{\alpha(i)}, (g_i \times g_{i'})^* \varphi_{\alpha(i)\alpha(i')}\right)$$
is a descent datum relative to $\mathcal{U}$.
(2) *This construction defines a functor between descent data relative to $\mathcal{V}$ and descent data relative to $\mathcal{U}$.*
(3) *Given a second $\alpha' : I \to J$, $h' : U \to V$ and $g_i' : U_i \to V_{\alpha'(i)}$ morphism of families of maps with fixed target, then if $h = h'$ the two resulting functors between descent data are canonically isomorphic.*

Proof. Omitted. □

Definition 8.3.4. With $\mathcal{U} = \{U_i \to U\}_{i \in I}$, $\mathcal{V} = \{V_j \to V\}_{j \in J}$, $\alpha : I \to J$, $h : U \to V$, and $g_i : U_i \to V_{\alpha(i)}$ as in Lemma 8.3.3 the functor
$$(Y_j, \varphi_{jj'}) \longmapsto (g_i^* Y_{\alpha(i)}, (g_i \times g_{i'})^* \varphi_{\alpha(i)\alpha(i')})$$
constructed in that lemma is called the *pullback functor* on descent data.

Given $h : U \to V$, if there exists a morphism $\tilde{h} : \mathcal{U} \to \mathcal{V}$ covering h then $\tilde{h}^*$ is independent of the choice of $\tilde{h}$ as we saw in Lemma 8.3.3. Hence we will sometimes simply write h^* to indicate the pullback functor.

Definition 8.3.5. Let $\mathcal{C}$ be a category. Let $p : \mathcal{S} \to \mathcal{C}$ be a fibred category. Make a choice of pullbacks as in Categories, Definition 4.31.6. Let $\mathcal{U} = \{f_i : U_i \to U\}_{i \in I}$ be a family of morphisms with target U. Assume all the fibre products $U_i \times_U U_j$ and $U_i \times_U U_j \times_U U_k$ exist.

(1) Given an object X of $\mathcal{S}_U$ the *trivial descent datum* is the descent datum (X, id_X) with respect to the family $\{\mathrm{id}_U : U \to U\}$.
(2) Given an object X of $\mathcal{S}_U$ we have a *canonical descent datum* on the family of objects f_i^*X by pulling back the trivial descent datum (X, id_X) via the obvious map $\{f_i : U_i \to U\} \to \{\mathrm{id}_U : U \to U\}$. We denote this descent datum (f_i^*X, can).
(3) A descent datum (X_i, φ_{ij}) relative to $\{f_i : U_i \to U\}$ is called *effective* if there exists an object X of $\mathcal{S}_U$ such that (X_i, φ_{ij}) is isomorphic to (f_i^*X, can).

Note that the rule that associates to $X \in \mathcal{S}_U$ its canonical descent datum relative to $\mathcal{U}$ defines a functor

$$\mathcal{S}_U \longrightarrow DD(\mathcal{U}).$$

A descent datum is effective if and only if it is in the essential image of this functor. Let us make explicit the canonical descent datum as follows.

Lemma 8.3.6. *In the situation of Definition 8.3.5 part (2) the maps $can_{ij} : pr_0^*f_i^*X \to pr_1^*f_j^*X$ are equal to $(\alpha_{pr_1,f_j})_X \circ (\alpha_{pr_0,f_i})_X^{-1}$ where $\alpha_{\cdot,\cdot}$ is as in Categories, Lemma 4.31.7 and where we use the equality $f_i \circ pr_0 = f_j \circ pr_1$ as maps $U_i \times_U U_j \to U$.*

Proof. Omitted. □

8.4. Stacks

Here is the definition of a stack. It mixes the notion of a fibred category with the notion of descent.

Definition 8.4.1. Let $\mathcal{C}$ be a site. A *stack* over $\mathcal{C}$ is a category $p : \mathcal{S} \to \mathcal{C}$ over $\mathcal{C}$ which satisfies the following conditions:

(1) $p : \mathcal{S} \to \mathcal{C}$ is a fibred category, see Categories, Definition 4.31.5,
(2) for any $U \in \mathrm{Ob}(\mathcal{C})$ and any $x, y \in \mathcal{S}_U$ the presheaf $Mor(x, y)$ (see Definition 8.2.2) is a sheaf on the site $\mathcal{C}/U$, and
(3) for any covering $\mathcal{U} = \{f_i : U_i \to U\}_{i \in I}$ of the site $\mathcal{C}$, any descent datum in $\mathcal{S}$ relative to $\mathcal{U}$ is effective.

We find the formulation above the most convenient way to think about a stack. Namely, given a category over $\mathcal{C}$ in order to verify that it is a stack you proceed to check properties (1), (2) and (3) in that order. Certainly properties (2) and (3) do not make sense if the category isn't fibred. Without (2) we cannot prove that the descent in (3) is unique up to unique isomorphism and functorial.

The following lemma provides an alternative definition.

Lemma 8.4.2. *Let $\mathcal{C}$ be a site. Let $p : \mathcal{S} \to \mathcal{C}$ be a fibred category over $\mathcal{C}$. The following are equivalent*

(1) *$\mathcal{S}$ is a stack over $\mathcal{C}$, and*

(2) *for any covering* $\mathcal{U} = \{f_i : U_i \to U\}_{i \in I}$ *of the site* $\mathcal{C}$ *the functor*

$$\mathcal{S}_U \longrightarrow DD(\mathcal{U})$$

which associates to an object its canonical descent datum is an equivalence.

Proof. Omitted. □

Lemma 8.4.3. *Let* $p : \mathcal{S} \to \mathcal{C}$ *be a stack over the site* $\mathcal{C}$*. Let* $\mathcal{S}'$ *be a subcategory of* $\mathcal{S}$*. Assume*

(1) *if* $\varphi : y \to x$ *is a strongly cartesian morphism of* $\mathcal{S}$ *and* x *is an object of* $\mathcal{S}'$*, then* y *is isomorphic to an object of* $\mathcal{S}'$*,*
(2) $\mathcal{S}'$ *is a full subcategory of* $\mathcal{S}$*, and*
(3) *if* $\{f_i : U_i \to U\}$ *is a covering of* $\mathcal{C}$*, and* x *an object of* $\mathcal{S}$ *over* U *such that* $f_i^* x$ *is isomorphic to an object of* $\mathcal{S}'$ *for each* i*, then* x *is isomorphic to an object of* $\mathcal{S}'$*.*

Then $\mathcal{S}' \to \mathcal{C}$ *is a stack.*

Proof. Omitted. Hints: The first condition guarantees that $\mathcal{S}'$ is a fibred category. The second condition guarantees that the *Isom*-presheaves of $\mathcal{S}'$ are sheaves (as they are identical to their counter parts in $\mathcal{S}$). The third condition guarantees that the descent condition holds in $\mathcal{S}'$ as we can first descend in $\mathcal{S}$ and then (3) implies the resulting object is isomorphic to an object of $\mathcal{S}'$. □

Lemma 8.4.4. *Let* $\mathcal{C}$ *be a site. Let* $\mathcal{S}_1$*,* $\mathcal{S}_2$ *be categories over* $\mathcal{C}$*. Suppose that* $\mathcal{S}_1$ *and* $\mathcal{S}_2$ *are equivalent as categories over* $\mathcal{C}$*. Then* $\mathcal{S}_1$ *is a stack over* $\mathcal{C}$ *if and only if* $\mathcal{S}_2$ *is a stack over* $\mathcal{C}$*.*

Proof. Let $F : \mathcal{S}_1 \to \mathcal{S}_2$, $G : \mathcal{S}_2 \to \mathcal{S}_1$ be functors over $\mathcal{C}$, and let $i : F \circ G \to \mathrm{id}_{\mathcal{S}_2}$, $j : G \circ F \to \mathrm{id}_{\mathcal{S}_1}$ be isomorphisms of functors over $\mathcal{C}$. By Categories, Lemma 4.31.8 we see that $\mathcal{S}_1$ is fibred if and only if $\mathcal{S}_2$ is fibred over $\mathcal{C}$. Hence we may assume that both $\mathcal{S}_1$ and $\mathcal{S}_2$ are fibred. Moreover, the proof of Categories, Lemma 4.31.8 shows that F and G map strongly cartesian morphisms to strongly cartesian morphisms, i.e., F and G are 1-morphisms of fibred categories over $\mathcal{C}$. This means that given $U \in \mathrm{Ob}(\mathcal{C})$, and $x, y \in \mathcal{S}_{1,U}$ then the presheaves

$$Mor_{\mathcal{S}_1}(x, y), Mor_{\mathcal{S}_1}(F(x), F(y)) : (\mathcal{C}/U)^{opp} \longrightarrow Sets.$$

are identified, see Lemma 8.2.3. Hence the first is a sheaf if and only if the second is a sheaf. Finally, we have to show that if every descent datum in $\mathcal{S}_1$ is effective, then so is every descent datum in $\mathcal{S}_2$. To do this, let $(X_i, \varphi_{ii'})$ be a descent datum in $\mathcal{S}_2$ relative the covering $\{U_i \to U\}$ of the site $\mathcal{C}$. Then $(G(X_i), G(\varphi_{ii'}))$ is a descent datum in $\mathcal{S}_1$ relative the covering $\{U_i \to U\}$. Let X be an object of $\mathcal{S}_{1,U}$ such that the descent datum $(f_i^* X, can)$ is isomorphic to $(G(X_i), G(\varphi_{ii'}))$. Then $F(X)$ is an object of $\mathcal{S}_{2,U}$ such that the descent datum $(f_i^* F(X), can)$ is isomorphic to $(F(G(X_i)), F(G(\varphi_{ii'})))$ which in turn is isomorphic to the original descent datum $(X_i, \varphi_{ii'})$ using i. □

The 2-category of stacks over $\mathcal{C}$ is defined as follows.

Definition 8.4.5. Let $\mathcal{C}$ be a site. The 2*-category of stacks over* $\mathcal{C}$ is the sub 2-category of the 2-category of fibred categories over $\mathcal{C}$ (see Categories, Definition 4.31.9) defined as follows:

(1) Its objects will be stacks $p : \mathcal{S} \to \mathcal{C}$.

(2) Its 1-morphisms $(\mathcal{S}, p) \to (\mathcal{S}', p')$ will be functors $G : \mathcal{S} \to \mathcal{S}'$ such that $p' \circ G = p$ and such that G maps strongly cartesian morphisms to strongly cartesian morphisms.
(3) Its 2-morphisms $t : G \to H$ for $G, H : (\mathcal{S}, p) \to (\mathcal{S}', p')$ will be morphisms of functors such that $p'(t_x) = \mathrm{id}_{p(x)}$ for all $x \in \mathrm{Ob}(\mathcal{S})$.

Lemma 8.4.6. *Let $\mathcal{C}$ be a site. The $(2,1)$-category of stacks over $\mathcal{C}$ has 2-fibre products, and they are described as in Categories, Lemma 4.30.3.*

Proof. Let $f : \mathcal{X} \to \mathcal{S}$ and $g : \mathcal{Y} \to \mathcal{S}$ be 1-morphisms of stacks over $\mathcal{C}$ as defined above. The category $\mathcal{X} \times_{\mathcal{S}} \mathcal{Y}$ described in Categories, Lemma 4.30.3 is a fibred category according to Categories, Lemma 4.31.10. (This is where we use that f and g preserve strongly cartesian morphisms.) It remains to show that the morphism presheaves are sheaves and that descent relative to coverings of $\mathcal{C}$ is effective.

Recall that an object of $\mathcal{X} \times_{\mathcal{S}} \mathcal{Y}$ is given by a quadruple (U, x, y, ϕ). It lies over the object U of $\mathcal{C}$. Next, let (U, x', y', ϕ') be second object lying over U. Recall that $\phi : f(x) \to g(y)$, and $\phi' : f(x') \to g(y')$ are isomorphisms in the category $\mathcal{S}_U$. Let us use these isomorphisms to identify $z = f(x) = g(y)$ and $z' = f(x') = g(y')$. With this identifications it is clear that

$$Mor((U, x, y, \phi), (U, x', y', \phi')) = Mor(x, x') \times_{Mor(z,z')} Mor(y, y')$$

as presheaves. However, as the fibred product in the category of presheaves preserves sheaves (Sites, Lemma 7.10.1) we see that this is a sheaf.

Let $\mathcal{U} = \{f_i : U_i \to U\}_{i \in I}$ be a covering of the site $\mathcal{C}$. Let (X_i, χ_{ij}) be a descent datum in $\mathcal{X} \times_{\mathcal{S}} \mathcal{Y}$ relative to $\mathcal{U}$. Write $X_i = (U_i, x_i, y_i, \phi_i)$ as above. Write $\chi_{ij} = (\varphi_{ij}, \psi_{ij})$ as in the definition of the category $\mathcal{X} \times_{\mathcal{S}} \mathcal{Y}$ (see Categories, Lemma 4.30.3). It is clear that (x_i, φ_{ij}) is a descent datum in $\mathcal{X}$ and that (y_i, ψ_{ij}) is a descent datum in $\mathcal{Y}$. Since $\mathcal{X}$ and $\mathcal{Y}$ are stacks these descent data are effective. Thus we get $x \in \mathrm{Ob}(\mathcal{X}_U)$, and $y \in \mathrm{Ob}(\mathcal{Y}_U)$ with $x_i = x|_{U_i}$, and $y_i = y|_{U_i}$ compatibly with descent data. Set $z = f(x)$ and $z' = g(y)$ which are both objects of $\mathcal{S}_U$. The morphisms ϕ_i are elements of $Isom(z, z')(U_i)$ with the property that $\phi_i|_{U_i \times_U U_j} = \phi_j|_{U_i \times_U U_j}$. Hence by the sheaf property of $Isom(z, z')$ we obtain an isomorphism $\phi : z = f(x) \to z' = g(y)$. We omit the verification that the canonical descent datum associated to the object (U, x, y, ϕ) of $(\mathcal{X} \times_{\mathcal{S}} \mathcal{Y})_U$ is isomorphic to the descent datum we started with. □

Lemma 8.4.7. *Let $\mathcal{C}$ be a site. Let $\mathcal{S}_1$, $\mathcal{S}_2$ be stacks over $\mathcal{C}$. Let $F : \mathcal{S}_1 \to \mathcal{S}_2$ be a 1-morphism. Then the following are equivalent*

(1) *F is fully faithful,*
(2) *for every $U \in \mathrm{Ob}(\mathcal{C})$ and for every $x, y \in \mathrm{Ob}(\mathcal{S}_{1,U})$ the map*

$$F : Mor_{\mathcal{S}_1}(x, y) \longrightarrow Mor_{\mathcal{S}_2}(x, y)$$

is an isomorphism of sheaves on $\mathcal{C}/U$.

Proof. Omitted. □

Lemma 8.4.8. *Let $\mathcal{C}$ be a site. Let $\mathcal{S}_1$, $\mathcal{S}_2$ be stacks over $\mathcal{C}$. Let $F : \mathcal{S}_1 \to \mathcal{S}_2$ be a 1-morphism which is fully faithful. Then the following are equivalent*

(1) *F is an equivalence,*

(2) *for every* $U \in \mathrm{Ob}(\mathcal{C})$ *and for every* $x \in \mathrm{Ob}(\mathcal{S}_{2,U})$ *there exists a covering* $\{f_i : U_i \to U\}$ *such that* $f_i^* x$ *is in the essential image of the functor* $F : \mathcal{S}_{1,U_i} \to \mathcal{S}_{2,U_i}$.

Proof. The implication (1) $\Rightarrow$ (2) is immediate. To see that (2) implies (1) we have to show that every x as in (2) is in the essential image of the functor F. To do this choose a covering as in (2), $x_i \in \mathrm{Ob}(\mathcal{S}_{1,U_i})$, and isomorphisms $\varphi_i : F(x_i) \to f_i^* x$. Then we get a descent datum for $\mathcal{S}_1$ relative to $\{f_i : U_i \to U\}$ by taking

$$\varphi_{ij} : x_i|_{U_i \times_U U_j} \longrightarrow x_j|_{U_i \times_U U_j}$$

the arrow such that $F(\varphi_{ij}) = \varphi_j^{-1} \circ \varphi_i$. This descent datum is effective by the axioms of a stack, and hence we obtain an object x_1 of $\mathcal{S}_1$ over U. We omit the verification that $F(x_1)$ is isomorphic to x over U. □

Remark 8.4.9. (Cutting down a "big" stack to get a stack.) Let $\mathcal{C}$ be a site. Suppose that $p : \mathcal{S} \to \mathcal{C}$ is functor from a "big" category to $\mathcal{C}$, i.e., suppose that the collection of objects of $\mathcal{S}$ forms a proper class. Finally, suppose that $p : \mathcal{S} \to \mathcal{C}$ satisfies conditions (1), (2), (3) of Definition 8.4.1. In general there is no way to replace $p : \mathcal{S} \to \mathcal{C}$ by a equivalent category such that we obtain a stack. The reason is that it can happen that a fibre categories $\mathcal{S}_U$ may have a proper class of isomorphism classes of objects. On the other hand, suppose that

(4) for every $U \in \mathrm{Ob}(\mathcal{C})$ there exists a set $S_U \subset \mathrm{Ob}(\mathcal{S}_U)$ such that every object of $\mathcal{S}_U$ is isomorphic in $\mathcal{S}_U$ to an element of S_U.

In this case we can find a full subcategory $\mathcal{S}_{small}$ of $\mathcal{S}$ such that, setting $p_{small} = p|_{\mathcal{S}_{small}}$, we have

(a) the functor $p_{small} : \mathcal{S}_{small} \to \mathcal{C}$ defines a stack, and
(b) the inclusion $\mathcal{S}_{small} \to \mathcal{S}$ is fully faithful and essentially surjective.

(Hint: For every $U \in \mathrm{Ob}(\mathcal{C})$ let $\alpha(U)$ denote the smallest ordinal such that $\mathrm{Ob}(\mathcal{S}_U) \cap V_{\alpha(U)}$ surjects onto the set of isomorphism classes of $\mathcal{S}_U$, and set $\alpha = \sup_{U \in \mathrm{Ob}(\mathcal{C})} \alpha(U)$. Then take $\mathrm{Ob}(\mathcal{S}_{small}) = \mathrm{Ob}(\mathcal{S}) \cap V_\alpha$. For notation used see Sets, Section 3.5.)

8.5. Stacks in groupoids

Among stacks those which are fibred in groupoids are somewhat easier to comprehend. We redefine them as follows.

Definition 8.5.1. A *stack in groupoids* over a site $\mathcal{C}$ is a category $p : \mathcal{S} \to \mathcal{C}$ over $\mathcal{C}$ such that

(1) $p : \mathcal{S} \to \mathcal{C}$ is fibred in groupoids over $\mathcal{C}$ (see Categories, Definition 4.33.1),
(2) for all $U \in \mathrm{Ob}(\mathcal{C})$, for all $x, y \in \mathrm{Ob}(\mathcal{S}_U)$ the presheaf $Isom(x, y)$ is a sheaf on the site $\mathcal{C}/U$, and
(3) for all coverings $\mathcal{U} = \{U_i \to U\}$ in $\mathcal{C}$, all descent data (x_i, ϕ_{ij}) for $\mathcal{U}$ are effective.

Usually the hardest part to check is the third condition. Here is the lemma comparing this with the notion of a stack.

Lemma 8.5.2. *Let* $\mathcal{C}$ *be a site. Let* $p : \mathcal{S} \to \mathcal{C}$ *be a category over* $\mathcal{C}$. *The following are equivalent*

(1) $\mathcal{S}$ *is a stack in groupoids over* $\mathcal{C}$,

(2) $\mathcal{S}$ *is a stack over* $\mathcal{C}$ *and all fibre categories are groupoids, and*
(3) $\mathcal{S}$ *is fibred in groupoids over* $\mathcal{C}$ *and is a stack over* $\mathcal{C}$*.*

Proof. Omitted, but see Categories, Lemma 4.33.2. □

Lemma 8.5.3. *Let* $\mathcal{C}$ *be a site. Let* $p : \mathcal{S} \to \mathcal{C}$ *be a stack. Let* $p' : \mathcal{S}' \to \mathcal{C}$ *be the category fibred in groupoids associated to* $\mathcal{S}$ *constructed in Categories, Lemma 4.33.3. Then* $p' : \mathcal{S}' \to \mathcal{C}$ *is a stack in groupoids.*

Proof. Recall that the morphisms in $\mathcal{S}'$ are exactly the strongly cartesian morphisms of $\mathcal{S}$, and that any isomorphism of $\mathcal{S}$ is such a morphism. Hence descent data in $\mathcal{S}'$ are exactly the same thing as descent data in $\mathcal{S}$. Now apply Lemma 8.4.2. Some details omitted. □

Lemma 8.5.4. *Let* $\mathcal{C}$ *be a site. Let* $\mathcal{S}_1$*,* $\mathcal{S}_2$ *be categories over* $\mathcal{C}$*. Suppose that* $\mathcal{S}_1$ *and* $\mathcal{S}_2$ *are equivalent as categories over* $\mathcal{C}$*. Then* $\mathcal{S}_1$ *is a stack in groupoids over* $\mathcal{C}$ *if and only if* $\mathcal{S}_2$ *is a stack in groupoids over* $\mathcal{C}$*.*

Proof. Follows by combining Lemmas 8.5.2 and 8.4.4. □

The 2-category of stacks in groupoids over $\mathcal{C}$ is defined as follows.

Definition 8.5.5. Let $\mathcal{C}$ be a site. The 2-*category of stacks in groupoids over* $\mathcal{C}$ is the sub 2-category of the 2-category of stacks over $\mathcal{C}$ (see Definition 8.4.5) defined as follows:

(1) Its objects will be stacks in groupoids $p : \mathcal{S} \to \mathcal{C}$.
(2) Its 1-morphisms $(\mathcal{S}, p) \to (\mathcal{S}', p')$ will be functors $G : \mathcal{S} \to \mathcal{S}'$ such that $p' \circ G = p$. (Since every morphism is strongly cartesian every functor preserves them.)
(3) Its 2-morphisms $t : G \to H$ for $G, H : (\mathcal{S}, p) \to (\mathcal{S}', p')$ will be morphisms of functors such that $p'(t_x) = \mathrm{id}_{p(x)}$ for all $x \in \mathrm{Ob}(\mathcal{S})$.

Note that any 2-morphism is automatically an isomorphism, so that in fact the 2-category of stacks in groupoids over $\mathcal{C}$ is a (strict) $(2, 1)$-category.

Lemma 8.5.6. *Let* $\mathcal{C}$ *be a category. The* 2*-category of stacks in groupoids over* $\mathcal{C}$ *has 2-fibre products, and they are described as in Categories, Lemma 4.30.3.*

Proof. This is clear from Categories, Lemma 4.33.7 and Lemmas 8.5.2 and 8.4.6. □

8.6. Stacks in setoids

This is just a brief section saying that a stack in sets is the same thing as a sheaf of sets. Please consult Categories, Section 4.37 for notation.

Definition 8.6.1. Let $\mathcal{C}$ be a site.

(1) A *stack in setoids* over $\mathcal{C}$ is a stack over $\mathcal{C}$ all of whose fibre categories are setoids.
(2) A *stack in sets*, or a *stack in discrete categories* is a stack over $\mathcal{C}$ all of whose fibre categories are discrete.

From the discussion in Section 8.5 this is the same thing as a stack in groupoids whose fibre categories are setoids (resp. discrete). Moreover, it is also the same thing as a category fibred in setoids (resp. sets) which is a stack.

Lemma 8.6.2. *Let $\mathcal{C}$ be a site. Under the equivalence*

$$\left\{\begin{array}{c}\text{the category of presheaves}\\ \text{of sets over } \mathcal{C}\end{array}\right\} \leftrightarrow \left\{\begin{array}{c}\text{the category of categories}\\ \text{fibred in sets over } \mathcal{C}\end{array}\right\}$$

of Categories, Lemma 4.36.6 the stacks in sets correspond precisely to the sheaves.

Proof. Omitted. Hint: Show that effectivity of descent corresponds exactly to the sheaf condition. □

Lemma 8.6.3. *Let $\mathcal{C}$ be a site. Let $\mathcal{S}$ be a category fibred in setoids over $\mathcal{C}$. Then $\mathcal{S}$ is a stack in setoids if and only if the unique equivalent category $\mathcal{S}'$ fibred in sets (see Categories, Lemma 4.37.5) is a stack in sets. In other words, if and only if the presheaf*

$$U \longmapsto \mathrm{Ob}(\mathcal{S}_U)/\cong$$

is a sheaf.

Proof. Omitted. □

Lemma 8.6.4. *Let $\mathcal{C}$ be a site. Let $\mathcal{S}_1$, $\mathcal{S}_2$ be categories over $\mathcal{C}$. Suppose that $\mathcal{S}_1$ and $\mathcal{S}_2$ are equivalent as categories over $\mathcal{C}$. Then $\mathcal{S}_1$ is a stack in setoids over $\mathcal{C}$ if and only if $\mathcal{S}_2$ is a stack in setoids over $\mathcal{C}$.*

Proof. By Categories, Lemma 4.37.5 we see that a category $\mathcal{S}$ over $\mathcal{C}$ is fibred in setoids over $\mathcal{C}$ if and only if it is equivalent over $\mathcal{C}$ to a category fibred in sets. Hence we see that $\mathcal{S}_1$ is fibred in setoids over $\mathcal{C}$ if and only if $\mathcal{S}_2$ is fibred in setoids over $\mathcal{C}$. Hence now the lemma follows from Lemma 8.6.3. □

The 2-category of stacks in setoids over $\mathcal{C}$ is defined as follows.

Definition 8.6.5. Let $\mathcal{C}$ be a site. The 2-*category of stacks in setoids over* $\mathcal{C}$ is the sub 2-category of the 2-category of stacks over $\mathcal{C}$ (see Definition 8.4.5) defined as follows:

(1) Its objects will be stacks in setoids $p : \mathcal{S} \to \mathcal{C}$.
(2) Its 1-morphisms $(\mathcal{S}, p) \to (\mathcal{S}', p')$ will be functors $G : \mathcal{S} \to \mathcal{S}'$ such that $p' \circ G = p$. (Since every morphism is strongly cartesian every functor preserves them.)
(3) Its 2-morphisms $t : G \to H$ for $G, H : (\mathcal{S}, p) \to (\mathcal{S}', p')$ will be morphisms of functors such that $p'(t_x) = \mathrm{id}_{p(x)}$ for all $x \in \mathrm{Ob}(\mathcal{S})$.

Note that any 2-morphism is automatically an isomorphism, so that in fact the 2-category of stacks in setoids over $\mathcal{C}$ is a (strict) $(2,1)$-category.

Lemma 8.6.6. *Let $\mathcal{C}$ be a site. The 2-category of stacks in setoids over $\mathcal{C}$ has 2-fibre products, and they are described as in Categories, Lemma 4.30.3.*

Proof. This is clear from Categories, Lemmas 4.33.7 and 4.37.4 and Lemmas 8.5.2 and 8.4.6. □

Lemma 8.6.7. *Let $\mathcal{C}$ be a site. Let $\mathcal{S}, \mathcal{T}$ be stacks in groupoids over $\mathcal{C}$ and let $\mathcal{R}$ be a stack in setoids over $\mathcal{C}$. Let $f : \mathcal{T} \to \mathcal{S}$ and $g : \mathcal{R} \to \mathcal{S}$ be 1-morphisms. If f is faithful, then the 2-fibre product*

$$\mathcal{T} \times_{f,\mathcal{S},g} \mathcal{R}$$

is a stack in setoids over $\mathcal{C}$.

Proof. Immediate from the explicit description of the 2-fibre product in Categories, Lemma 4.30.3. □

Lemma 8.6.8. *Let $\mathcal{C}$ be a site. Let $\mathcal{S}$ be a stack in groupoids over $\mathcal{C}$ and let $\mathcal{S}_i$, $i = 1, 2$ be stacks in setoids over $\mathcal{C}$. Let $f_i : \mathcal{S}_i \to \mathcal{S}$ be 1-morphisms. Then the 2-fibre product*

$$\mathcal{S}_1 \times_{f_1, \mathcal{S}, f_2} \mathcal{S}_2$$

is a stack in setoids over $\mathcal{C}$.

Proof. This is a special case of Lemma 8.6.7 as f_2 is faithful. □

Lemma 8.6.9. *Let $\mathcal{C}$ be a site. Let*

$$\begin{array}{ccc} \mathcal{T}_2 & \longrightarrow & \mathcal{T}_1 \\ {\scriptstyle G'}\downarrow & & \downarrow{\scriptstyle G} \\ \mathcal{S}_2 & \xrightarrow{F} & \mathcal{S}_1 \end{array}$$

be a 2-cartesian diagram of stacks in groupoids over $\mathcal{C}$. Assume

1. *for every $U \in \mathrm{Ob}(\mathcal{C})$ and $x \in \mathrm{Ob}((\mathcal{S}_1)_U)$ there exists a covering $\{U_i \to U\}$ such that $x|_{U_i}$ is in the essential image of $F : (\mathcal{S}_2)_{U_i} \to (\mathcal{S}_1)_{U_i}$, and*
2. *G' is faithful,*

then G is faithful.

Proof. We may assume that $\mathcal{T}_2$ is the category $\mathcal{S}_2 \times_{\mathcal{S}_1} \mathcal{T}_1$ described in Categories, Lemma 4.30.3. By Categories, Lemma 4.33.8 the faithfulness of G, G' can be checked on fibre categories. Suppose that y, y' are objects of $\mathcal{T}_1$ over the object U of $\mathcal{C}$. Let $\alpha, \beta : y \to y'$ be morphisms of $(\mathcal{T}_1)_U$ such that $G(\alpha) = G(\beta)$. Our object is to show that $\alpha = \beta$. Considering instead $\gamma = \alpha^{-1} \circ \beta$ we see that $G(\gamma) = \mathrm{id}_{G(y)}$ and we have to show that $\gamma = \mathrm{id}_y$. By assumption we can find a covering $\{U_i \to U\}$ such that $G(y)|_{U_i}$ is in the essential image of $F : (\mathcal{S}_2)_{U_i} \to (\mathcal{S}_1)_{U_i}$. Since it suffices to show that $\gamma|_{U_i} = \mathrm{id}$ for each i, we may therefore assume that we have $f : F(x) \to G(y)$ for some object x of $\mathcal{S}_2$ over U and morphisms f of $(\mathcal{S}_1)_U$. In this case we get a morphism

$$(1, \gamma) : (U, x, y, f) \longrightarrow (U, x, y, f)$$

in the fibre category of $\mathcal{S}_2 \times_{\mathcal{S}_1} \mathcal{T}_1$ over U whose image under G' in $\mathcal{S}_1$ is id_x. As G' is faithful we conclude that $\gamma = \mathrm{id}_y$ and we win. □

Lemma 8.6.10. *Let $\mathcal{C}$ be a site. Let*

$$\begin{array}{ccc} \mathcal{T}_2 & \longrightarrow & \mathcal{T}_1 \\ \downarrow & & \downarrow{\scriptstyle G} \\ \mathcal{S}_2 & \xrightarrow{F} & \mathcal{S}_1 \end{array}$$

be a 2-cartesian diagram of stacks in groupoids over $\mathcal{C}$. If

1. *$F : \mathcal{S}_2 \to \mathcal{S}_1$ is fully faithful,*
2. *for every $U \in \mathrm{Ob}(\mathcal{C})$ and $x \in \mathrm{Ob}((\mathcal{S}_1)_U)$ there exists a covering $\{U_i \to U\}$ such that $x|_{U_i}$ is in the essential image of $F : (\mathcal{S}_2)_{U_i} \to (\mathcal{S}_1)_{U_i}$, and*
3. *$\mathcal{T}_2$ is a stack in setoids.*

then $\mathcal{T}_1$ is a stack in setoids.

Proof. We may assume that $\mathcal{T}_2$ is the category $\mathcal{S}_2 \times_{\mathcal{S}_1} \mathcal{T}_1$ described in Categories, Lemma 4.30.3. Pick $U \in \mathrm{Ob}(\mathcal{C})$ and $y \in \mathrm{Ob}((\mathcal{T}_1)_U)$. We have to show that the sheaf $Aut(y)$ on $\mathcal{C}/U$ is trivial. To to this we may replace U by the members of a covering of U. Hence by assumption (2) we may assume that there exists an object $x \in \mathrm{Ob}((\mathcal{S}_2)_U)$ and an isomorphism $f : F(x) \to G(y)$. Then $y' = (U, x, y, f)$ is an object of $\mathcal{T}_2$ over U which is mapped to y under the projection $\mathcal{T}_2 \to \mathcal{T}_1$. Because F is fully faithful by (1) the map $Aut(y') \to Aut(y)$ is surjective, use the explicit description of morphisms in $\mathcal{T}_2$ in Categories, Lemma 4.30.3. Since by (3) the sheaf $Aut(y')$ is trivial we get the result of the lemma. □

8.7. The inertia stack

Let $p : \mathcal{S} \to \mathcal{C}$ and $p' : \mathcal{S}' \to \mathcal{C}$ be fibred categories over the category $\mathcal{C}$. Let $F : \mathcal{S} \to \mathcal{S}'$ be a 1-morphism of fibred categories over $\mathcal{C}$. Recall that we have defined in Categories, Definition 4.32.2 an *relative inertia fibred category* $\mathcal{I}_{\mathcal{S}/\mathcal{S}'} \to \mathcal{C}$ as the category whose objects are pairs (x, α) where $x \in \mathrm{Ob}(\mathcal{S})$ and $\alpha : x \to x$ with $F(\alpha) = \mathrm{id}_{F(x)}$. There is also an absolute version, namely the *inertia* $\mathcal{I}_{\mathcal{S}}$ of $\mathcal{S}$. These inertia categories are actually stacks over $\mathcal{C}$ provided that $\mathcal{S}$ and $\mathcal{S}'$ are stacks.

Lemma 8.7.1. *Let $\mathcal{C}$ be a site. Let $p : \mathcal{S} \to \mathcal{C}$ and $p' : \mathcal{S}' \to \mathcal{C}$ be stacks over the site $\mathcal{C}$. Let $F : \mathcal{S} \to \mathcal{S}'$ be a 1-morphism of stacks over $\mathcal{C}$.*

1. *The inertia $\mathcal{I}_{\mathcal{S}/\mathcal{S}'}$ and $\mathcal{I}_{\mathcal{S}}$ are stacks over $\mathcal{C}$.*
2. *If $\mathcal{S}, \mathcal{S}'$ are stacks in groupoids over $\mathcal{S}$, then so are $\mathcal{I}_{\mathcal{S}/\mathcal{S}'}$ and $\mathcal{I}_{\mathcal{S}}$.*
3. *If $\mathcal{S}, \mathcal{S}'$ are stacks in setoids over $\mathcal{S}$, then so are $\mathcal{I}_{\mathcal{S}/\mathcal{S}'}$ and $\mathcal{I}_{\mathcal{S}}$.*

Proof. The first three assertions follow from Lemmas 8.4.6, 8.5.6, and 8.6.6 and the equivalence in Categories, Lemma 4.32.1 part (1). □

Lemma 8.7.2. *Let $\mathcal{C}$ be a site. If $\mathcal{S}$ is a stack in groupoids, then the canonical 1-morphism $\mathcal{I}_{\mathcal{S}} \to \mathcal{S}$ is an equivalence if and only if $\mathcal{S}$ is a stack in setoids.*

Proof. Follows directly from Categories, Lemma 4.37.7. □

8.8. Stackification of fibred categories

Here is the result.

Lemma 8.8.1. *Let $\mathcal{C}$ be a site. Let $p : \mathcal{S} \to \mathcal{C}$ be a fibred category over $\mathcal{C}$. There exists a stack $p' : \mathcal{S}' \to \mathcal{C}$ and a 1-morphism $G : \mathcal{S} \to \mathcal{S}'$ of fibred categories over $\mathcal{C}$ (see Categories, Definition 4.31.9) such that*

1. *for every $U \in \mathrm{Ob}(\mathcal{C})$, and any $x, y \in \mathrm{Ob}(\mathcal{S}_U)$ the map*
$$Mor(x, y) \longrightarrow Mor(G(x), G(y))$$
induced by G identifies the right hand side with the sheafification of the left hand side, and
2. *for every $U \in \mathrm{Ob}(\mathcal{C})$, and any $x' \in \mathrm{Ob}(\mathcal{S}'_U)$ there exists a covering $\{U_i \to U\}_{i \in I}$ such that for every $i \in I$ the object $x'|_{U_i}$ is in the essential image of the functor $G : \mathcal{S}_U \to \mathcal{S}'_U$.*

Moreover the stack $\mathcal{S}'$ is determined up to unique 2-isomorphism by these conditions.

Proof by naive method. In this proof method we proceed in stages:

First, given x lying over U and any object y of $\mathcal{S}$, we say that two morphisms $a, b : x \to y$ of $\mathcal{S}$ lying over the same arrow of $\mathcal{C}$ are *locally equal* if there exists a covering $\{f_i : U_i \to U\}$ of $\mathcal{C}$ such that the compositions

$$f_i^* x \to x \xrightarrow{a} y, \quad f_i^* x \to x \xrightarrow{b} y$$

are equal. This gives an equivalence relation $\sim$ on arrows of $\mathcal{S}$. If $b \sim b'$ then $a \circ b \circ c \sim a \circ b' \circ c$ (verification omitted). Hence we can quotient out by this equivalence relation to obtain a new category $\mathcal{S}^1$ over $\mathcal{C}$ together with a morphism $G^1 : \mathcal{S} \to \mathcal{S}^1$.

One checks that G^1 preserves strongly cartesian morphisms and that $\mathcal{S}^1$ is a fibred category over $\mathcal{C}$. Checks omitted. Thus we reduce to the case where locally equal morphisms are equal.

Next, we add morphisms as follows. Given x lying over U and any object y of lying over V a *locally defined morphism from x to y* is given by

(1) a morphism $f : U \to V$,
(2) a covering $\{f_i : U_i \to U\}$ of U, and
(3) morphisms $a_i : f_i^* x \to Y$ with $p(a_i) = h \circ f_i$

with the property that the compositions

$$(f_i \times f_j)^* x \to f_i^* x \xrightarrow{a_i} y, \quad (f_i \times f_j)^* x \to f_j^* x \xrightarrow{a_j} y$$

are equal. Note that a usual morphism $a : x \to y$ gives a locally defined morphism $(p(a) : U \to V, \{\mathrm{id}_U\}, a)$. We say two locally defined morphisms $(f, \{f_i : U_i \to U\}, a_i)$ and $(g, \{g_j : U_i \to U\}, b_j)$ are *equal* if $f = g$ and the compositions

$$(f_i \times g_j)^* x \to f_i^* x \xrightarrow{a_i} y, \quad (f_i \times g_j)^* x \to g_j^* x \xrightarrow{b_j} y$$

are equal (this is the right condition since we are in the situation where locally equal morphisms are equal). To compose locally defined morphisms $(f, \{f_i : U_i \to U\}, a_i)$ from x to y and $(g, \{g_j : V_j \to V\}, b_j)$ from y to z lying over W, just take $g \circ f : U \to W$, the covering $\{U_i \times_V V_j \to U\}$, and as maps the compositions

$$x|_{U_i \times_V V_j} \xrightarrow{\mathrm{pr}_0^* a_i} y|_{V_j} \xrightarrow{b_j} z$$

We omit the verification that this is a locally defined morphism.

One checks that $\mathcal{S}^2$ with the same objects as $\mathcal{S}$ and with locally defined morphisms as morphisms is a category over $\mathcal{C}$, that there is a functor $G^2 : \mathcal{S} \to \mathcal{S}^2$ over $\mathcal{C}$, that this functor preserves strongly cartesian objects, and that $\mathcal{S}^2$ is a fibred category over $\mathcal{C}$. Checks omitted. This reduces one to the case where the morphism presheaves of $\mathcal{S}$ are all sheaves, by checking that the effect of using locally defined morphisms is to take the sheafification of the (separated) morphisms presheaves.

Finally, in the case where the morphism presheaves are all sheaves we have to add objects in order to make sure descent conditions are effective in the end result. The simplest way to do this is to consider the category $\mathcal{S}'$ whose objects are pairs $(\mathcal{U}, \xi)$ where $\mathcal{U} = \{U_i \to U\}$ is a covering of $\mathcal{C}$ and $\xi = (X_i, \varphi_{ii'})$ is a descent datum relative $\mathcal{U}$. Suppose given two such data $(\mathcal{U}, \xi) = (\{f_i : U_i \to U\}, x_i, \varphi_{ii'})$ and $(\mathcal{V}, \eta) = (\{g_j : V_j \to V\}, y_j, \psi_{jj'})$. We define

$$\mathrm{Mor}_{\mathcal{S}'}((\mathcal{U}, \xi), (\mathcal{V}, \eta))$$

as the set of (f, a_{ij}), where $f : U \to V$ and

$$a_{ij} : x_i|_{U_i \times_V V_j} \longrightarrow y_j$$

are morphisms of $\mathcal{S}$ lying over $U_i \times_V V_j \to V_j$. These have to satisfy the following condition: for any $i, i' \in I$ and $j, j' \in J$ set $W = (U_i \times_U U_{i'}) \times_V (V_j \times_V V_{j'})$. Then

$$\begin{array}{ccc} x_i|_W & \xrightarrow[a_{ij}|_W]{} & y_j|_W \\ {\scriptstyle \varphi_{ii'}|_W}\downarrow & & \downarrow{\scriptstyle \psi_{jj'}|_W} \\ x_{i'}|_W & \xrightarrow{a_{i'j'}|_W} & y_{j'}|_W \end{array}$$

commutes. At this point you have to verify the following things:

(1) there is a well defined composition on morphisms as above,
(2) this turns $\mathcal{S}'$ into a category over $\mathcal{C}$,
(3) there is a functor $G : \mathcal{S} \to \mathcal{S}'$ over $\mathcal{C}$,
(4) for x, y objects of $\mathcal{S}$ we have $\mathrm{Mor}_{\mathcal{S}}(x, y) = \mathrm{Mor}_{\mathcal{S}'}(G(x), G(y))$,
(5) any object of $\mathcal{S}'$ locally comes from an object of $\mathcal{S}$, i.e., part (2) of the lemma holds,
(6) G preserves strongly cartesian morphisms,
(7) $\mathcal{S}'$ is a fibred category over $\mathcal{C}$, and
(8) $\mathcal{S}'$ is a stack over $\mathcal{C}$.

This is all not hard but there is a lot of it. Details omitted. □

Less naive proof. Here is a less naive proof. By Categories, Lemma 4.34.4 there exists an equivalence of fibred categories $\mathcal{S} \to \mathcal{S}'$ where $\mathcal{S}'$ is a split fibred category, i.e., one in which the pullback functors compose on the nose. Obviously the lemma for $\mathcal{S}'$ implies the lemma for $\mathcal{S}$. Hence we may think of $\mathcal{S}$ as a presheaf in categories.

Consider the 2-category *Cat* temporarily as a category by forgetting about 2-morphisms. Let us think of a category as a quintuple $(\mathrm{Ob}, \mathrm{Arrows}, s, t, \circ)$ as in Categories, Section 4.2. Consider the forgetful functor

$$forget : Cat \to Sets, \quad (\mathrm{Ob}, \mathrm{Arrows}, s, t, \circ) \mapsto \mathrm{Ob} \coprod \mathrm{Arrows}.$$

Then $forget$ is faithful, *Cat* has limits and $forget$ commutes with them, *Cat* has directed colimits and $forget$ commutes with them, and $forget$ reflects isomorphisms. Hence, according to the first part of Sites, Section 7.43 we can sheafify presheaves with values in *Cat*, and the result commutes with $forget$. Applying this to $\mathcal{S}$ we obtain a sheafification $\mathcal{S}^\#$ which has a sheaf of objects and a sheaf of morphisms both of which are the sheafifications of the corresponding presheaves for $\mathcal{S}$. In this case it is quite easy to see that the map $\mathcal{S} \to \mathcal{S}^\#$ has the properties (1) and (2) of the lemma.

However, the category $\mathcal{S}^\#$ may not yet be a stack since, although the presheaf of objects is a sheaf, the descent condition may not yet be satisfied. To remedy this we have to add more objects. But the argument above does reduce us to the case where $\mathcal{S} = \mathcal{S}_F$ for some sheaf(!) $F : \mathcal{C}^{opp} \to Cat$ of categories. In this case consider the functor $F' : \mathcal{C}^{opp} \to Cat$ defined by

(1) The set $\mathrm{Ob}(F'(U))$ is the set of pairs $(\mathcal{U}, \xi)$ where $\mathcal{U} = \{U_i \to U\}$ is a covering of U and $\xi = (x_i, \varphi_{ii'})$ is a descent datum relative to $\mathcal{U}$.

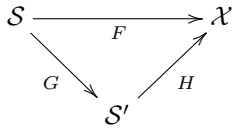
(2) A morphism in $F'(U)$ from $(\mathcal{U}, \xi)$ to $(\mathcal{V}, \eta)$ is an element of
$$\operatorname{colim} \operatorname{Mor}_{DD(\mathcal{W})}(a^*\xi, b^*\eta)$$
where the colimit is over all common refinements $a : \mathcal{W} \to \mathcal{U}$, $b : \mathcal{W} \to \mathcal{V}$. This colimit is filtered (verification omitted). Hence composition of morphisms in $F(U)$ is defined by finding a common refinement and composing in $DD(\mathcal{W})$.

(3) Given $h : V \to U$ and an object $(\mathcal{U}, \xi)$ of $F'(U)$ we set $F'(h)(\mathcal{U}, \xi)$ equal to $(V \times_U \mathcal{U}, \mathrm{pr}_1^*\xi)$. More precisely, if $\mathcal{U} = \{U_i \to U\}$ and $\xi = (x_i, \varphi_{ii'})$, then $V \times_U \mathcal{U} = \{V \times_U U_i \to V\}$ which comes with a canonical morphism $\mathrm{pr}_1 : V \times_U \mathcal{U} \to \mathcal{U}$ and $\mathrm{pr}_1^*\xi$ is the pullback of ξ with respect to this morphism (see Definition 8.3.4).

(4) Given $h : V \to U$, objects $(\mathcal{U}, \xi)$ and $(\mathcal{V}, \eta)$ and a morphism between them, represented by $a : \mathcal{W} \to \mathcal{U}$, $b : \mathcal{W} \to \mathcal{V}$, and $\alpha : a^*\xi \to b^*\eta$, then $F'(h)(\alpha)$ is represented by $a' : V \times_U \mathcal{W} \to V \times_U \mathcal{U}$, $b' : V \times_U \mathcal{W} \to V \times_U \mathcal{V}$, and the pullback α' of the morphism α via the map $V \times_U \mathcal{W} \to \mathcal{W}$. This works since pullbacks in $\mathcal{S}_F$ commute on the nose.

There is a map $F \to F'$ given by associating to an object x of $F(U)$ the object $(\{U \to U\}, (x, triv))$ of $F'(U)$. At this point you have to check that the corresponding functor $\mathcal{S}_F \to \mathcal{S}_{F'}$ has properties (1) and (2) of the lemma, and finally that $\mathcal{S}_{F'}$ is a stack. Details omitted. □

Lemma 8.8.2. *Let $\mathcal{C}$ be a site. Let $p : \mathcal{S} \to \mathcal{C}$ be a fibred category over $\mathcal{C}$. Let $p' : \mathcal{S}' \to \mathcal{C}$ and $G : \mathcal{S} \to \mathcal{S}'$ the stack and 1-morphism constructed in Lemma 8.8.1. This construction has the following universal property: Given a stack $q : \mathcal{X} \to \mathcal{C}$ and a 1-morphism $F : \mathcal{S} \to \mathcal{S}'$ of fibred categories over $\mathcal{C}$ there exists a 1-morphism $H : \mathcal{S}' \to \mathcal{X}$ such that the diagram*



is 2-commutative.

Proof. Omitted. Hint: Suppose that $x' \in \mathrm{Ob}(\mathcal{S}'_U)$. By the result of Lemma 8.8.1 there exists a covering $\{U_i \to U\}_{i \in I}$ such that $x'|_{U_i} = G(x_i)$ for some $x_i \in \mathrm{Ob}(\mathcal{S}_{U_i})$. Moreover, there exist coverings $\{U_{ijk} \to U_i \times_U U_j\}$ and isomorphisms $\alpha_{ijk} : x_i|_{U_{ijk}} \to x_j|_{U_{ijk}}$ with $G(\alpha_{ijk}) = \mathrm{id}_{x'|_{U_{ijk}}}$. Set $y_i = F(x_i)$. Then you can check that
$$F(\alpha_{ijk}) : y_i|_{U_{ijk}} \to y_j|_{U_{ijk}}$$
agree on overlaps and therefore (as $\mathcal{X}$ is a stack) define a morphism $\beta_{ij} : y_i|_{U_i \times_U U_j} \to y_j|_{U_i \times_U U_j}$. Next, you check that the β_{ij} define a descent datum. Since $\mathcal{X}$ is a stack these descent data are effective and we find an object y of $\mathcal{X}_U$ agreeing with $G(x_i)$ over U_i. The hint is to set $H(x') = y$. □

Lemma 8.8.3. *Notation and assumptions as in Lemma 8.8.2. There is a canonical equivalence of categories*
$$\operatorname{Mor}_{Fib/\mathcal{C}}(\mathcal{S}, \mathcal{X}) = \operatorname{Mor}_{Stacks/\mathcal{C}}(\mathcal{S}', \mathcal{X})$$
given by the constructions in the proof of the aforementioned lemma.

Proof. Omitted. □

Lemma 8.8.4. *Let $\mathcal{C}$ be a site. Let $f : \mathcal{X} \to \mathcal{Y}$ and $g : \mathcal{Z} \to \mathcal{Y}$ be morphisms of fibred categories over $\mathcal{C}$. In this case the stackification of the 2-fibre product is the 2-fibre product of the stackifications.*

Proof. Let us denote $\mathcal{X}', \mathcal{Y}', \mathcal{Z}'$ the stackifications and $\mathcal{W}$ the stackification of $\mathcal{X} \times_{\mathcal{Y}} \mathcal{Z}$. By construction of 2-fibre products there is a canonical 1-morphism $\mathcal{X} \times_{\mathcal{Y}} \mathcal{Z} \to \mathcal{X}' \times_{\mathcal{Y}'} \mathcal{Z}'$. As the second 2-fibre product is a stack (see Lemma 8.4.6) this 1-morphism induces a 1-morphism $h : \mathcal{W} \to \mathcal{X}' \times_{\mathcal{Y}'} \mathcal{Z}'$ by the universal property of stackification, see Lemma 8.8.2. Now h is a morphism of stacks, and we may check that it is an equivalence using Lemmas 8.4.7 and 8.4.8.

Thus we first prove that h induces isomorphisms of Mor-sheaves. Let ξ, ξ' be objects of $\mathcal{W}$ over $U \in \mathrm{Ob}(\mathcal{C})$. We want to show that

$$h : Mor(\xi, \xi') \longrightarrow Mor(h(\xi), h(\xi'))$$

is an isomorphism. To do this we may work locally on U (see Sites, Section 7.25). Hence by construction of $\mathcal{W}$ (see Lemma 8.8.1) we may assume that ξ, ξ' actually come from objects (x, y, α) and (x', y', α') of $\mathcal{X} \times_{\mathcal{Y}} \mathcal{Z}$ over U. By the same lemma once more we see that in this case $Mor(\xi, \xi')$ is the sheafification of

$$V/U \longmapsto \mathrm{Mor}_{\mathcal{X}_V}(x|_V, x'|_V) \times_{\mathrm{Mor}_{\mathcal{Z}_V}(f(x)|_V, f(x')|_V)} \mathrm{Mor}_{\mathcal{Y}_V}(y|_V, y'|_V)$$

and that $Mor(h(\xi), h(\xi'))$ is equal to the fibre product

$$Mor(i(x), i(x')) \times_{Mor(k(f(x)), k(f(x')))} Mor(j(x), j(x'))$$

where $i : \mathcal{X} \to \mathcal{X}'$, $j : \mathcal{Y} \to \mathcal{Y}'$, and $k : \mathcal{Z} \to \mathcal{Z}'$ are the canonical functors. Thus the first displayed map of this paragraph is an isomorphism as sheafification is exact (and hence the sheafification of a fibre product of presheaves is the fibre product of the sheafifications).

Finally, we have to check that any object of $\mathcal{X}' \times_{\mathcal{Y}'} \mathcal{Z}'$ over U is locally on U in the essential image of h. Write such an object as a triple (x', y', α). Then x' locally comes from an object of $\mathcal{X}$, y' locally comes from an object of $\mathcal{Y}$, and having made suitable replacements for x', y' the morphism α of $\mathcal{Z}'_U$ locally comes from a morphism of $\mathcal{Z}$. In other words, we have shown that any object of $\mathcal{X}' \times_{\mathcal{Y}'} \mathcal{Z}'$ over U is locally on U in the essential image of $\mathcal{X} \times_{\mathcal{Y}} \mathcal{Z} \to \mathcal{X}' \times_{\mathcal{Y}'} \mathcal{Z}'$, hence a fortiori it is locally in the essential image of h. □

Lemma 8.8.5. *Let $\mathcal{C}$ be a site. Let $\mathcal{X}$ be a fibred category over $\mathcal{C}$. The stackification of the inertia fibred category $\mathcal{I}_{\mathcal{X}}$ is inertia of the stackification of $\mathcal{X}$.*

Proof. This follows from the fact that stackification is compatible with 2-fibre products by Lemma 8.8.4 and the fact that there is a formula for the inertia in terms of 2-fibre products of categories over $\mathcal{C}$, see Categories, Lemma 4.32.1. □

8.9. Stackification of categories fibred in groupoids

Here is the result.

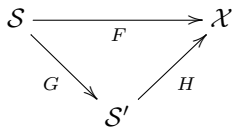
Lemma 8.9.1. *Let $\mathcal{C}$ be a site. Let $p : \mathcal{S} \to \mathcal{C}$ be a category fibred in groupoids over $\mathcal{C}$. There exists a stack in groupoids $p' : \mathcal{S}' \to \mathcal{C}$ and a 1-morphism $G : \mathcal{S} \to \mathcal{S}'$ of categories fibred in groupoids over $\mathcal{C}$ (see Categories, Definition 4.33.6) such that*

(1) *for every* $U \in \mathrm{Ob}(\mathcal{C})$*, and any* $x, y \in \mathrm{Ob}(\mathcal{S}_U)$ *the map*
$$Mor(x, y) \longrightarrow Mor(G(x), G(y))$$
induced by G *identifies the right hand side with the sheafification of the left hand side, and*
(2) *for every* $U \in \mathrm{Ob}(\mathcal{C})$*, and any* $x' \in \mathrm{Ob}(\mathcal{S}'_U)$ *there exists a covering* $\{U_i \to U\}_{i \in I}$ *such that for every* $i \in I$ *the object* $x'|_{U_i}$ *is in the essential image of the functor* $G : \mathcal{S}_{U_i} \to \mathcal{S}'_{U_i}$.

Moreover the stack in groupoids $\mathcal{S}'$ *is determined up to unique* 2*-isomorphism by these conditions.*

Proof. Apply Lemma 8.8.1. The result will be a stack in groupoids by applying Lemma 8.5.2. □

Lemma 8.9.2. *Let* $\mathcal{C}$ *be a site. Let* $p : \mathcal{S} \to \mathcal{C}$ *be a category fibred in groupoids over* $\mathcal{C}$*. Let* $p' : \mathcal{S}' \to \mathcal{C}$ *and* $G : \mathcal{S} \to \mathcal{S}'$ *the stack in groupoids and* 1*-morphism constructed in Lemma 8.9.1. This construction has the following universal property: Given a stack in groupoids* $q : \mathcal{X} \to \mathcal{C}$ *and a* 1*-morphism* $F : \mathcal{S} \to \mathcal{X}$ *of categories over* $\mathcal{C}$ *there exists a* 1*-morphism* $H : \mathcal{S}' \to \mathcal{X}$ *such that the diagram*



is 2*-commutative.*

Proof. This is a special case of Lemma 8.8.2. □

Lemma 8.9.3. *Let* $\mathcal{C}$ *be a site. Let* $f : \mathcal{X} \to \mathcal{Y}$ *and* $g : \mathcal{Y} \to \mathcal{Z}$ *be morphisms categories fibred in groupoids over* $\mathcal{C}$*. In this case the stackification of the* 2*-fibre product is the* 2*-fibre product of the stackifications.*

Proof. This is a special case of Lemma 8.8.4. □

8.10. Inherited topologies

It turns out that a fibred category over a site inherits a canonical topology from the underlying site.

Lemma 8.10.1. *Let* $\mathcal{C}$ *be a site. Let* $p : \mathcal{S} \to \mathcal{C}$ *be a fibred category. Let* $Cov(\mathcal{S})$ *be the set of families* $\{x_i \to x\}_{i \in I}$ *of morphisms in* $\mathcal{S}$ *with fixed target such that (a) each* $x_i \to x$ *is strongly cartesian, and (b)* $\{p(x_i) \to p(x)\}_{i \in I}$ *is a covering of* $\mathcal{C}$*. Then* $(\mathcal{S}, Cov(\mathcal{S}))$ *is a site.*

Proof. We have to check the three conditions of Sites, Definition 7.6.2.

(1) If $x \to y$ is an isomorphism of $\mathcal{S}$, then it is strongly cartesian by Categories, Lemma 4.31.2 and $p(x) \to p(y)$ is an isomorphism of $\mathcal{C}$. Thus $\{p(x) \to p(y)\}$ is a covering of $\mathcal{C}$ whence $\{x \to y\} \in \mathrm{Cov}(\mathcal{S})$.
(2) If $\{x_i \to x\}_{i \in I} \in \mathrm{Cov}(\mathcal{S})$ and for each i we have $\{y_{ij} \to x_i\}_{j \in J_i} \in \mathrm{Cov}(\mathcal{S})$, then each composition $p(y_{ij}) \to p(x)$ is strongly cartesian by Categories, Lemma 4.31.2 and $\{p(y_{ij}) \to p(x)\}_{i \in I, j \in J_i} \in \mathrm{Cov}(\mathcal{C})$. Hence also $\{y_{ij} \to x\}_{i \in I, j \in J_i} \in \mathrm{Cov}(\mathcal{S})$.

(3) Suppose $\{x_i \to x\}_{i\in I} \in \mathrm{Cov}(\mathcal{S})$ and $y \to x$ is a morphism of $\mathcal{S}$. As $\{p(x_i) \to p(x)\}$ is a covering of $\mathcal{C}$ we see that $p(x_i) \times_{p(x)} p(y)$ exists. Hence Categories, Lemma 4.31.13 implies that $x_i \times_x y$ exists, that $p(x_i \times_x y) = p(x_i) \times_{p(x)} p(y)$, and that $x_i \times_x y \to y$ is strongly cartesian. Since also $\{p(x_i) \times_{p(x)} p(y) \to p(y)\}_{i\in I} \in \mathrm{Cov}(\mathcal{C})$ we conclude that $\{x_i \times_x y \to y\}_{i\in I} \in \mathrm{Cov}(\mathcal{S})$

This finishes the proof. □

Note that if $p : \mathcal{S} \to \mathcal{C}$ is fibred in groupoids, then the coverings of the site $\mathcal{S}$ in Lemma 8.10.1 are characterized by

$$\{x_i \to x\} \in \mathrm{Cov}(\mathcal{S}) \Leftrightarrow \{p(x_i) \to p(x)\} \in \mathrm{Cov}(\mathcal{C})$$

because every morphism of $\mathcal{S}$ is strongly cartesian.

Definition 8.10.2. Let $\mathcal{C}$ be a site. Let $p : \mathcal{S} \to \mathcal{C}$ be a fibred category. We say $(\mathcal{S}, \mathrm{Cov}(\mathcal{S}))$ as in Lemma 8.10.1 is the *structure of site on $\mathcal{S}$ inherited from $\mathcal{C}$*. We sometimes indicate this by saying that *$\mathcal{S}$ is endowed with the topology inherited from $\mathcal{C}$*.

In particular we obtain a topos of sheaves $Sh(\mathcal{S})$ in this situation. It turns out that this topos is functorial with respect to 1-morphisms of fibred categories.

Lemma 8.10.3. *Let $\mathcal{C}$ be a site. Let $F : \mathcal{X} \to \mathcal{Y}$ be a 1-morphism of fibred categories over $\mathcal{C}$. Then F is a continuous and cocontinuous functor between the structure of sites inherited from $\mathcal{C}$. Hence F induces a morphism of topoi $f : Sh(\mathcal{X}) \to Sh(\mathcal{Y})$ with $f_* = {}_sF = {}_pF$ and $f^{-1} = F^s = F^p$. In particular $f^{-1}(\mathcal{G})(x) = \mathcal{G}(F(x))$ for a sheaf $\mathcal{G}$ on $\mathcal{Y}$ and object x of $\mathcal{X}$.*

Proof. We first prove that F is continuous. Let $\{x_i \to x\}_{i\in I}$ be a covering of $\mathcal{X}$. By Categories, Definition 4.31.9 the functor F transforms strongly cartesian morphisms into strongly cartesian morphisms, hence $\{F(x_i) \to F(x)\}_{i\in I}$ is a covering of $\mathcal{Y}$. This proves part (1) of Sites, Definition 7.14.1. Moreover, let $x' \to x$ be a morphism of $\mathcal{X}$. By Categories, Lemma 4.31.13 the fibre product $x_i \times_x x'$ exists and $x_i \times_x x' \to x'$ is strongly cartesian. Hence $F(x_i \times_x x') \to F(x')$ is strongly cartesian. By Categories, Lemma 4.31.13 applied to $\mathcal{Y}$ this means that $F(x_i \times_x x') = F(x_i) \times_{F(x)} F(x')$. This proves part (2) of Sites, Definition 7.14.1 and we conclude that F is continuous.

Next we prove that F is cocontinuous. Let $x \in \mathrm{Ob}(\mathcal{X})$ and let $\{y_i \to F(x)\}_{i\in I}$ be a covering in $\mathcal{Y}$. Denote $\{U_i \to U\}_{i\in I}$ the corresponding covering of $\mathcal{C}$. For each i choose a strongly cartesian morphism $x_i \to x$ in $\mathcal{X}$ lying over $U_i \to U$. Then $F(x_i) \to F(x)$ and $y_i \to F(x)$ are both a strongly cartesian morphisms in $\mathcal{Y}$ lying over $U_i \to U$. Hence there exists a unique isomorphism $F(x_i) \to y_i$ in $\mathcal{Y}_{U_i}$ compatible with the maps to $F(x)$. Thus $\{x_i \to x\}_{i\in I}$ is a covering of $\mathcal{X}$ such that $\{F(x_i) \to F(x)\}_{i\in I}$ is isomorphic to $\{y_i \to F(x)\}_{i\in I}$. Hence F is cocontinuous, see Sites, Definition 7.19.1.

The final assertion follows from the first two, see Sites, Lemmas 7.20.1, 7.19.2, and 7.20.5. □

Lemma 8.10.4. *Let $\mathcal{C}$ be a site. Let $p : \mathcal{X} \to \mathcal{C}$ and $q : \mathcal{Y} \to \mathcal{C}$ be stacks in groupoids. Let $F : \mathcal{X} \to \mathcal{Y}$ be a 1-morphism of categories over $\mathcal{C}$. If F turns $\mathcal{X}$ into*

a category fibred in groupoids over $\mathcal{Y}$, then $\mathcal{X}$ is a stack in groupoids over $\mathcal{Y}$ (with topology inherited from $\mathcal{C}$).

Proof. Let us prove descent for objects. Let $\{y_i \to y\}$ be a covering of $\mathcal{Y}$. Let (x_i, φ_{ij}) be a descent datum in $\mathcal{X}$ with respect to this covering. Then (x_i, φ_{ij}) is also a descent datum with respect to the covering $\{q(y_i) \to q(y)\}$ of $\mathcal{C}$. As $\mathcal{X}$ is a stack in groupoids we obtain an object x over $q(y)$ and isomorphisms $\psi_i : x|_{q(y_i)} \to x_i$ over $q(y_i)$ compatible with the φ_{ij}, i.e., such that

$$\varphi_{ij} = \psi_j|_{q(y_i)\times_{q(y)} q(y_j)} \circ \psi_i^{-1}|_{q(y_i)\times_{q(y)} q(y_j)}.$$

Consider the sheaf $I = Isom_{\mathcal{Y}}(F(x), y)$ on $\mathcal{C}/p(x)$. Note that $s_i = F(\psi_i) \in I(q(x_i))$ because $F(x_i) = y_i$. Because $F(\varphi_{ij}) = \mathrm{id}$ (as we started with a descent datum over $\{y_i \to y\}$) the displayed formula shows that $s_i|_{q(y_i)\times_{q(y)} q(y_j)} = s_j|_{q(y_i)\times_{q(y)} q(y_j)}$. Hence the local sections s_i glue to $s : F(x) \to y$. As F is fibred in groupoids we see that x is isomorphic to an object x' with $F(x') = y$. We omit the verification that x' in the fibre category of $\mathcal{X}$ over y is a solution to the problem of descent posed by the descent datum (x_i, φ_{ij}). We also omit the proof of the sheaf property of the *Isom*-presheaves of $\mathcal{X}/\mathcal{Y}$. □

Lemma 8.10.5. *Let $\mathcal{C}$ be a site. Let $p : \mathcal{X} \to \mathcal{C}$ be a stack. Endow $\mathcal{X}$ with the topology inherited from $\mathcal{C}$ and let $q : \mathcal{Y} \to \mathcal{X}$ be a stack. Then $\mathcal{Y}$ is a stack over $\mathcal{C}$. If p and q define stacks in groupoids, then $\mathcal{Y}$ is a stack in groupoids over $\mathcal{C}$.*

Proof. We check the three conditions in Definition 8.4.1 to prove that $\mathcal{Y}$ is a stack over $\mathcal{C}$. By Categories, Lemma 4.31.12 we find that $\mathcal{Y}$ is a fibred category over $\mathcal{C}$. Thus condition (1) holds.

Let U be an object of $\mathcal{C}$ and let y_1, y_2 be objects of $\mathcal{Y}$ over U. Denote $x_i = q(y_i)$ in $\mathcal{X}$. Consider the map of presheaves

$$q : Mor_{\mathcal{Y}/\mathcal{C}}(y_1, y_2) \longrightarrow Mor_{\mathcal{X}/\mathcal{C}}(x_1, x_2)$$

on $\mathcal{C}/U$, see Lemma 8.2.3. Let $\{U_i \to U\}$ be a covering and let φ_i be a section of the presheaf on the left over U_i such that φ_i and φ_j restrict to the same section over $U_i \times_U U_j$. We have to find a morphism $\varphi : x_1 \to x_2$ restricting to φ_i. Note that $q(\varphi_i) = \psi|_{U_i}$ for some morphism $\psi : x_1 \to x_2$ over U because the second presheaf is a sheaf (by assumption). Let $y_{12} \to y_2$ be the stronly $\mathcal{X}$-cartesian morphism of $\mathcal{Y}$ lying over ψ. Then φ_i corresponds to a morphism $\varphi_i' : y_1|_{U_i} \to y_{12}|_{U_i}$ over $x_1|_{U_i}$. In other words, φ_i' now define local sections of the presheaf

$$Mor_{\mathcal{Y}/\mathcal{X}}(y_1, y_{12})$$

over the members of the covering $\{x_1|_{U_i} \to x_1\}$. By assumption these glue to a unique morphism $y_1 \to y_{12}$ which composed with the given morphism $y_{12} \to y_2$ produces the desired morphism $y_1 \to y_2$.

Finally, we show that descent data are effective. Let $\{f_i : U_i \to U\}$ be a covering of $\mathcal{C}$ and let (y_i, φ_{ij}) be a descent datum relative to this covering (Definition 8.3.1). Setting $x_i = q(y_i)$ and $\psi_{ij} = q(\varphi_{ij})$ we obtain a descent datum (x_i, ψ_{ij}) for the covering in $\mathcal{X}$. By assumption on $\mathcal{X}$ we may assume $x_i = x|_{U_i}$ and the ψ_{ij} equal to the canonical descent datum (Definition 8.3.5). In this case $\{x|_{U_i} \to x\}$ is a covering and we can view (y_i, φ_{ij}) as a descent datum relative to this covering. By our assumption that $\mathcal{Y}$ is a stack over $\mathcal{C}$ we see that it is effective which finishes the proof of condition (3).

The final assertion follows because $\mathcal{Y}$ is a stack over $\mathcal{C}$ and is fibred in groupoids by Categories, Lemma 4.33.13. □

8.11. Gerbes

Gerbes are a special kind of stacks in groupoids.

Definition 8.11.1. A *gerbe* over a site $\mathcal{C}$ is a category $p : \mathcal{S} \to \mathcal{C}$ over $\mathcal{C}$ such that

(1) $p : \mathcal{S} \to \mathcal{C}$ is a stack in groupoids over $\mathcal{C}$ (see Definition 8.5.1),
(2) for $U \in \mathrm{Ob}(\mathcal{C})$ there exists a covering $\{U_i \to U\}$ in $\mathcal{C}$ such that $\mathcal{S}_{U_i}$ is nonempty, and
(3) for $U \in \mathrm{Ob}(\mathcal{C})$ and $x, y \in \mathrm{Ob}(\mathcal{S}_U)$ there exists a covering $\{U_i \to U\}$ in $\mathcal{C}$ such that $x|_{U_i} \cong y|_{U_i}$ in $\mathcal{S}_{U_i}$.

In other words, a gerbe is a stack in groupoids such that any two objects are locally isomorphic and such that objects exist locally.

Lemma 8.11.2. *Let $\mathcal{C}$ be a site. Let $\mathcal{S}_1$, $\mathcal{S}_2$ be categories over $\mathcal{C}$. Suppose that $\mathcal{S}_1$ and $\mathcal{S}_2$ are equivalent as categories over $\mathcal{C}$. Then $\mathcal{S}_1$ is a gerbe over $\mathcal{C}$ if and only if $\mathcal{S}_2$ is a gerbe over $\mathcal{C}$.*

Proof. Assume $\mathcal{S}_1$ is a gerbe over $\mathcal{C}$. By Lemma 8.5.4 we see $\mathcal{S}_2$ is a stack in groupoids over $\mathcal{C}$. Let $F : \mathcal{S}_1 \to \mathcal{S}_2$, $G : \mathcal{S}_2 \to \mathcal{S}_1$ be equivalences of categories over $\mathcal{C}$. Given $U \in \mathrm{Ob}(\mathcal{C})$ we see that there exists a covering $\{U_i \to U\}$ such that $(\mathcal{S}_1)_{U_i}$ is nonempty. Applying F we see that $(\mathcal{S}_2)_{U_i}$ is nonempty. Given $U \in \mathrm{Ob}(\mathcal{C})$ and $x, y \in \mathrm{Ob}((\mathcal{S}_2)_U)$ there exists a covering $\{U_i \to U\}$ in $\mathcal{C}$ such that $G(x)|_{U_i} \cong G(y)|_{U_i}$ in $(\mathcal{S}_1)_{U_i}$. By Categories, Lemma 4.33.8 this implies $x|_{U_i} \cong y|_{U_i}$ in $(\mathcal{S}_2)_{U_i}$. □

We want to generalize the definition of gerbes a bit. Namely, let $F : \mathcal{X} \to \mathcal{Y}$ be a 1-morphism of stacks in groupoids over a site $\mathcal{C}$. We want to say what it means for $\mathcal{X}$ to be a gerbe over $\mathcal{Y}$. By Section 8.10 the category $\mathcal{Y}$ inherits the structure of a site from $\mathcal{C}$. A naive guess is: Just require that $\mathcal{X} \to \mathcal{Y}$ *is* a gerbe in the sense above. Except the notion so obtained is not invariants under replacing $\mathcal{X}$ by an equivalent stack in groupoids over $\mathcal{C}$; this is even the case for the property of being fibred in groupoids over $\mathcal{Y}$. However, it turns out that we can replace $\mathcal{X}$ by an equivalent stack in groupoids over $\mathcal{Y}$ which is fibred in groupoids over $\mathcal{Y}$, and then the property of being a gerbe over $\mathcal{Y}$ is independent of this choice. Here is the precise formulation.

Lemma 8.11.3. *Let $\mathcal{C}$ be a site. Let $p : \mathcal{X} \to \mathcal{C}$ and $q : \mathcal{Y} \to \mathcal{C}$ be stacks in groupoids. Let $F : \mathcal{X} \to \mathcal{Y}$ be a 1-morphism of categories over $\mathcal{C}$. The following are equivalent*

(1) *For some (equivalently any) factorization $F = F' \circ a$ where $a : \mathcal{X} \to \mathcal{X}'$ is an equivalence of categories over $\mathcal{C}$ and F' is fibred in groupoids, the map $F' : \mathcal{X}' \to \mathcal{Y}$ is a gerbe (with the topology on $\mathcal{Y}$ inherited from $\mathcal{C}$).*
(2) *The following two conditions are satisfied*
   (a) *for $y \in \mathrm{Ob}(\mathcal{Y})$ lying over $U \in \mathrm{Ob}(\mathcal{C})$ there exists a covering $\{U_i \to U\}$ in $\mathcal{C}$ and objects x_i of $\mathcal{X}$ over U_i such that $F(x_i) \cong y|_{U_i}$ in $\mathcal{Y}_{U_i}$, and*
   (b) *for $U \in \mathrm{Ob}(\mathcal{C})$, $x, x' \in \mathrm{Ob}(\mathcal{X}_U)$, and $b : F(x) \to F(x')$ in $\mathcal{Y}_U$ there exists a covering $\{U_i \to U\}$ in $\mathcal{C}$ and morphisms $a_i : x|_{U_i} \to x'|_{U_i}$ in $\mathcal{X}_{U_i}$ with $F(a_i) = b|_{U_i}$.*

Proof. By Categories, Lemma 4.33.15 there exists a factorization $F = F' \circ a$ where $a : \mathcal{X} \to \mathcal{X}'$ is an equivalence of categories over $\mathcal{C}$ and F' is fibred in groupoids. By Categories, Lemma 4.33.16 given any two such factorizations $F = F' \circ a = F'' \circ b$ we have that $\mathcal{X}'$ is equivalent to $\mathcal{X}''$ as categories over $\mathcal{Y}$. Hence Lemma 8.11.2 guarantees that the condition (1) is independent of the choice of the factorization. Moreover, this means that we may assume $\mathcal{X}' = \mathcal{X} \times_{F,\mathcal{Y},\mathrm{id}} \mathcal{Y}$ as in the proof of Categories, Lemma 4.33.15

Let us prove that (a) and (b) imply that $\mathcal{X}' \to \mathcal{Y}$ is a gerbe. First of all, by Lemma 8.10.4 we see that $\mathcal{X}' \to \mathcal{Y}$ is a stack in groupoids. Next, let y be an object of $\mathcal{Y}$ lying over $U \in \mathrm{Ob}(\mathcal{C})$. By (a) we can find a covering $\{U_i \to U\}$ in $\mathcal{C}$ and objects x_i of $\mathcal{X}$ over U_i and isomorphisms $f_i : F(x_i) \to y|_{U_i}$ in $\mathcal{Y}_{U_i}$. Then $(U_i, x_i, y|_{U_i}, f_i)$ are objects of $\mathcal{X}'_{U_i}$, i.e., the second condition of Definition 8.11.1 holds. Finally, let (U, x, y, f) and (U, x', y, f') be objects of $\mathcal{X}'$ lying over the same object $y \in \mathrm{Ob}(\mathcal{Y})$. Set $b = (f')^{-1} \circ f$. By condition (b) we can find a covering $\{U_i \to U\}$ and isomorphisms $a_i : x|_{U_i} \to x'|_{U_i}$ in $\mathcal{X}_{U_i}$ with $F(a_i) = b|_{U_i}$. Then

$$(a_i, \mathrm{id}) : (U, x, y, f)|_{U_i} \to (U, x', y, f')|_{U_i}$$

is a morphism in $\mathcal{X}'_{U_i}$ as desired. This proves that (2) implies (1).

To prove that (1) implies (2) one reads the arguments in the preceding paragraph backwards. Details omitted. □

Definition 8.11.4. Let $\mathcal{C}$ be a site. Let $\mathcal{X}$ and $\mathcal{Y}$ be stacks in groupoids over $\mathcal{C}$. Let $F : \mathcal{X} \to \mathcal{Y}$ be a 1-morphism of categories over $\mathcal{C}$. We say $\mathcal{X}$ is a *gerbe over* $\mathcal{Y}$ if the equivalent conditions of Lemma 8.11.3 are satisfied.

This definition does not conflict with Definition 8.11.1 when $\mathcal{Y} = \mathcal{C}$ because in this case we may take $\mathcal{X}' = \mathcal{X}$ in part (1) of Lemma 8.11.3. Note that conditions (2)(a) and (2)(b) of Lemma 8.11.3 are quite close in spirit to conditions (2) and (3) of Definition 8.11.1. Namely, (2)(a) says that the map of presheaves of isomorphism classes of objects becomes a surjection after sheafification. Moreover, (2)(b) says that

$$Isom_{\mathcal{X}}(x, x') \longrightarrow Isom_{\mathcal{Y}}(F(x), F(x'))$$

is a surjection of sheaves on $\mathcal{C}/U$ for any U and $x, x' \in \mathrm{Ob}(\mathcal{X}_U)$.

Lemma 8.11.5. *Let $\mathcal{C}$ be a site. Let*

$$\begin{array}{ccc} \mathcal{X}' & \xrightarrow{G'} & \mathcal{X} \\ {\scriptstyle F'}\downarrow & & \downarrow{\scriptstyle F} \\ \mathcal{Y}' & \xrightarrow{G} & \mathcal{Y} \end{array}$$

be a 2-fibre product of stacks in groupoids over $\mathcal{C}$. If $\mathcal{X}$ is a gerbe over $\mathcal{Y}$, then $\mathcal{X}'$ is a gerbe over $\mathcal{Y}'$.

Proof. By the uniqueness property of a 2-fibre product may assume that $\mathcal{X}' = \mathcal{Y}' \times_{\mathcal{Y}} \mathcal{X}$ as in Categories, Lemma 4.30.3. Let us prove properties (2)(a) and (2)(b) of Lemma 8.11.3 for $\mathcal{Y}' \times_{\mathcal{Y}} \mathcal{X} \to \mathcal{Y}'$.

Let y' be an object of $\mathcal{Y}'$ lying over the object U of $\mathcal{C}$. By assumption there exists a covering $\{U_i \to U\}$ of U and objects $x_i \in \mathcal{X}_{U_i}$ with isomorphisms $\alpha_i : G(y')|_{U_i} \to F(x_i)$. Then $(U_i, y'|_{U_i}, x_i, \alpha_i)$ is an object of $\mathcal{Y}' \times_{\mathcal{Y}} \mathcal{X}$ over U_i whose image in $\mathcal{Y}'$ is $y'|_{U_i}$. Thus (2)(a) holds.

Let $U \in \mathrm{Ob}(\mathcal{C})$, let x'_1, x'_2 be objects of $\mathcal{Y}' \times_{\mathcal{Y}} \mathcal{X}$ over U, and let $b' : F'(x'_1) \to F'(x'_2)$ be a morphism in $\mathcal{Y}'_U$. Write $x'_i = (U, y'_i, x_i, \alpha_i)$. Note that $F'(x'_i) = x_i$ and $G'(x'_i) = y'_i$. By assumption there exists a covering $\{U_i \to U\}$ in $\mathcal{C}$ and morphisms $a_i : x_1|_{U_i} \to x_2|_{U_i}$ in $\mathcal{X}_{U_i}$ with $F(a_i) = G(b')|_{U_i}$. Then $(b'|_{U_i}, a_i)$ is a morphism $x'_1|_{U_i} \to x'_2|_{U_i}$ as required in (2)(b). □

Lemma 8.11.6. *Let $\mathcal{C}$ be a site. Let $F : \mathcal{X} \to \mathcal{Y}$ and $G : \mathcal{Y} \to \mathcal{Z}$ be 1-morphisms of stacks in groupoids over $\mathcal{C}$. If $\mathcal{X}$ is a gerbe over $\mathcal{Y}$ and $\mathcal{Y}$ is a gerbe over $\mathcal{Z}$, then $\mathcal{X}$ is a gerbe over $\mathcal{Z}$.*

Proof. Let us prove properties (2)(a) and (2)(b) of Lemma 8.11.3 for $\mathcal{X} \to \mathcal{Z}$.

Let z be an object of $\mathcal{Z}$ lying over the object U of $\mathcal{C}$. By assumption on G there exists a covering $\{U_i \to U\}$ of U and objects $y_i \in \mathcal{Y}_{U_i}$ such that $G(y_i) \cong z|_{U_i}$. By assumption on F there exist coverings $\{U_{ij} \to U_i\}$ and objects $x_{ij} \in \mathcal{X}_{U_{ij}}$ such that $F(x_{ij}) \cong y_i|_{U_{ij}}$. Then $\{U_{ij} \to U\}$ is a covering of $\mathcal{C}$ and $(G \circ F)(x_{ij}) \cong z|_{U_{ij}}$. Thus (2)(a) holds.

Let $U \in \mathrm{Ob}(\mathcal{C})$, let x_1, x_2 be objects of $\mathcal{X}$ over U, and let $c : (G \circ F)(x_1) \to (G \circ F)(x_2)$ be a morphism in $\mathcal{Z}_U$. By assumption on G there exists a covering $\{U_i \to U\}$ of U and morphisms $b_i : F(x_1)|_{U_i} \to F(x_2)|_{U_i}$ in $\mathcal{Y}_{U_i}$ such that $G(b_i) = c|_{U_i}$. By assumption on F there exist coverings $\{U_{ij} \to U_i\}$ and morphisms $a_{ij} : x_1|_{U_{ij}} \to x_2|_{U_{ij}}$ in $\mathcal{X}_{U_{ij}}$ such that $F(a_{ij}) = b_i|_{U_{ij}}$. Then $\{U_{ij} \to U\}$ is a covering of $\mathcal{C}$ and $(G \circ F)(a_{ij}) = c|_{U_{ij}}$ as required in (2)(b). □

Lemma 8.11.7. *Let $\mathcal{C}$ be a site. Let*

$$\begin{array}{ccc} \mathcal{X}' & \xrightarrow{G'} & \mathcal{X} \\ {\scriptstyle F'}\downarrow & & \downarrow{\scriptstyle F} \\ \mathcal{Y}' & \xrightarrow{G} & \mathcal{Y} \end{array}$$

be a 2-cartesian diagram of stacks in groupoids over $\mathcal{C}$. If for every $U \in \mathrm{Ob}(\mathcal{C})$ and $x \in \mathrm{Ob}(\mathcal{Y}_U)$ there exists a covering $\{U_i \to U\}$ such that $x|_{U_i}$ is in the essential image of $G : \mathcal{Y}'_{U_i} \to \mathcal{Y}_{U_i}$ and $\mathcal{X}'$ is a gerbe over $\mathcal{Y}'$, then $\mathcal{X}$ is a gerbe over $\mathcal{Y}$.

Proof. By the uniqueness property of a 2-fibre product may assume that $\mathcal{X}' = \mathcal{Y}' \times_{\mathcal{Y}} \mathcal{X}$ as in Categories, Lemma 4.30.3. Let us prove properties (2)(a) and (2)(b) of Lemma 8.11.3 for $\mathcal{X} \to \mathcal{Y}$.

Let y be an object of $\mathcal{Y}$ lying over the object U of $\mathcal{C}$. By assumption there exists a covering $\{U_i \to U\}$ of U and objects $y'_i \in \mathcal{Y}'_{U_i}$ with $G(y'_i) \cong y|_{U_i}$. By (2)(a) for $\mathcal{X}' \to \mathcal{Y}'$ there exist coverings $\{U_{ij} \to U_i\}$ and objects x'_{ij} of $\mathcal{X}'$ over U_{ij} with $F'(x'_{ij})$ isomorphic to the restriction of y'_i to U_{ij}. Then $\{U_{ij} \to U\}$ is a covering of $\mathcal{C}$ and $G'(x'_{ij})$ are objects of $\mathcal{X}$ over U_{ij} whose images in $\mathcal{Y}$ are isomorphic to the restrictions $y|_{U_{ij}}$. This proves (2)(a) for $\mathcal{X} \to \mathcal{Y}$.

Let $U \in \mathrm{Ob}(\mathcal{C})$, let x_1, x_2 be objects of $\mathcal{X}$ over U, and let $b : F(x_1) \to F(x_2)$ be a morphism in $\mathcal{Y}_U$. By assumption we may choose a covering $\{U_i \to U\}$ and objects y'_{1i}, y'_{2i} of $\mathcal{Y}'$ over U_i such that there exist isomorphisms $\alpha_{1i} : G(y'_{1i}) \cong F(x_1)|_{U_i}$ and $\alpha_{2i} : G(y'_{2i}) \cong F(x_2)|_{U_i}$. Then we get objects $x'_{1i} = (U_i, y'_{1i}, x_1|_{U_i}, \alpha_{1i})$ and $x'_{2i} = (U_i, y'_{1i}, x_1|_{U_i}, \alpha_{2i})$ of $\mathcal{X}'$ over U_i. The restriction $b|_{U_i}$ is a morphism $F'(x'_{1i}) \to F'(x'_{2i})$. By (2)(b) for $\mathcal{X}' \to \mathcal{Y}'$ there exist coverings $\{U_{ij} \to U_i\}$ and morphisms $a'_{ij} : x'_{1i}|_{U_{ij}} \to x'_{2i}|_{U_{ij}}$ such that $F'(a'_{ij}) = b|_{U_{ij}}$. Unwinding the definition of

morphisms in $\mathcal{Y}' \times_{\mathcal{Y}} \mathcal{X}$ we see that $G'(a'_{ij}) : x_1|_{U_{ij}} \to x_2|_{U_{ij}}$ are the morphism we're looking for, i.e., (2)(b) holds for $\mathcal{X} \to \mathcal{Y}$. □

8.12. Functoriality for stacks

In this section we study what happens if we want to change the base site of a stack. This section can be skipped on a first reading.

Let $u : \mathcal{C} \to \mathcal{D}$ be a functor between categories. Let $p : \mathcal{S} \to \mathcal{D}$ be a category over $\mathcal{D}$. In this situation we denote $u^p\mathcal{S}$ the category over $\mathcal{C}$ defined as follows

(1) An object of $u^p\mathcal{S}$ is a pair (U, y) consisting of an object U of $\mathcal{C}$ and an object y of $\mathcal{S}_{u(U)}$.
(2) A morphism $(a, \beta) : (U, y) \to (U', y')$ is given by a morphism $a : U \to U'$ of $\mathcal{C}$ and a morphism $\beta : y \to y'$ of $\mathcal{S}$ such that $p(\beta) = u(a)$.

Note that with these definitions the fibre category of $u^p\mathcal{S}$ over U is equal to the fibre category of $\mathcal{S}$ over $u(U)$.

Lemma 8.12.1. *In the situation above, if $\mathcal{S}$ is a fibred category over $\mathcal{D}$ then $u^p\mathcal{S}$ is a fibred category over $\mathcal{C}$.*

Proof. Please take a look at the discussion surrounding Categories, Definitions 4.31.1 and 4.31.5 before reading this proof. Let $(a, \beta) : (U, y) \to (U', y')$ be a morphism of $u^p\mathcal{S}$. We claim that (a, β) is strongly cartesian if and only if β is strongly cartesian. First, assume β is strongly cartesian. Consider any second morphism $(a_1, \beta_1) : (U_1, y_1) \to (U', y')$ of $u^p\mathcal{S}$. Then

$$
\begin{aligned}
&\mathrm{Mor}_{u^p\mathcal{S}}((U_1, y_1), (U, y)) \\
&= \mathrm{Mor}_{\mathcal{C}}(U_1, U) \times_{\mathrm{Mor}_{\mathcal{D}}(u(U_1), u(U))} \mathrm{Mor}_{\mathcal{S}}(y_1, y) \\
&= \mathrm{Mor}_{\mathcal{C}}(U_1, U) \times_{\mathrm{Mor}_{\mathcal{D}}(u(U_1), u(U))} \mathrm{Mor}_{\mathcal{S}}(y_1, y') \times_{\mathrm{Mor}_{\mathcal{D}}(u(U_1), u(U'))} \mathrm{Mor}_{\mathcal{D}}(u(U_1), u(U)) \\
&= \mathrm{Mor}_{\mathcal{S}}(y_1, y') \times_{\mathrm{Mor}_{\mathcal{D}}(u(U_1), u(U'))} \mathrm{Mor}_{\mathcal{C}}(U_1, U) \\
&= \mathrm{Mor}_{u^p\mathcal{S}}((U_1, y_1), (U', y')) \times_{\mathrm{Mor}_{\mathcal{C}}(U_1, U')} \mathrm{Mor}_{\mathcal{C}}(U_1, U)
\end{aligned}
$$

the second equality as β is strongly cartesian. Hence we see that indeed (a, β) is strongly cartesian. Conversely, suppose that (a, β) is strongly cartesian. Choose a strongly cartesian morphism $\beta' : y'' \to y'$ in $\mathcal{S}$ with $p(\beta') = u(a)$. Then bot $(a, \beta) : (U, y) \to (U, y')$ and $(a, \beta') : (U, y'') \to (U, y)$ are strongly cartesian and lift a. Hence, by the uniqueness of strongly cartesian morphisms (see discussion in Categories, Section 4.31) there exists an isomorphism $\iota : y \to y''$ in $\mathcal{S}_{u(U)}$ such that $\beta = \beta' \circ \iota$, which implies that β is strongly cartesian in $\mathcal{S}$ by Categories, Lemma 4.31.2.

Finally, we have to show that given (U', y') and $U \to U'$ we can find a strongly cartesian morphism $(U, y) \to (U', y')$ in $u^p\mathcal{S}$ lifting the morphism $U \to U'$. This follows from the above as by assumption we can find a strongly cartesian morphism $y \to y'$ lifting the morphism $u(U) \to u(U')$. □

Lemma 8.12.2. *Let $u : \mathcal{C} \to \mathcal{D}$ be a continuous functor of sites. Let $p : \mathcal{S} \to \mathcal{D}$ be a stack over $\mathcal{D}$. Then $u^p\mathcal{S}$ is a stack over $\mathcal{S}$.*

Proof. We have seen in Lemma 8.12.1 that $u^p\mathcal{S}$ is a fibred category over $\mathcal{S}$. Moreover, in the proof of that lemma we have seen that a morphism (a, β) of $u^p\mathcal{S}$ is

strongly cartesian if and only β is strongly cartesian in $\mathcal{S}$. Hence, given a morphism $a : U \to U'$ of $\mathcal{C}$, not only do we have the equalities $(u^p\mathcal{S})_U = \mathcal{S}_U$ and $(u^p\mathcal{S})_{U'} = \mathcal{S}_{U'}$, but via these equalities the pullback functors agree; in a formula $a^*(U', y') = (U, u(a)^* y')$.

Having said this, let $\mathcal{U} = \{U_i \to U\}$ be a covering of $\mathcal{C}$. As u is continuous we see that $\mathcal{V} = \{u(U_i) \to u(U)\}$ is a covering of $\mathcal{D}$, and that $u(U_i \times_U U_j) = u(U_i) \times_{u(U)} u(U_j)$ and similarly for the triple fibre products $U_i \times_U U_j \times_U U_k$. As we have the identifications of fibre categories and pullbacks we see that descend data relative to $\mathcal{U}$ are identical to descend data relative to $\mathcal{V}$. Since by assumption we have effective descent in $\mathcal{S}$ we conclude the same holds for $u^p\mathcal{S}$. □

Lemma 8.12.3. *Let $u : \mathcal{C} \to \mathcal{D}$ be a continuous functor of sites. Let $p : \mathcal{S} \to \mathcal{D}$ be a stack in groupoids over $\mathcal{D}$. Then $u^p\mathcal{S}$ is a stack in groupoids over $\mathcal{S}$.*

Proof. This follows immediately from Lemma 8.12.2 and the fact that all fibre categories are groupoids. □

Definition 8.12.4. Let $f : \mathcal{D} \to \mathcal{C}$ be a morphism of sites given by the continuous functor $u : \mathcal{C} \to \mathcal{D}$. Let $\mathcal{S}$ be a fibred category over $\mathcal{D}$. In this setting we write $f_*\mathcal{S}$ for the fibred category $u^p\mathcal{S}$ defined above. We say that $f_*\mathcal{S}$ is the *pushforward of $\mathcal{S}$ along f*.

By the results above we know that $f_*\mathcal{S}$ is a stack (in groupoids) if $\mathcal{S}$ is a stack (in groupoids). It is harder to define the pullback of a stack (and we'll need additional assumptions for our particular construction – feel free to write up and submit a more general construction). We do this in several steps.

Let $u : \mathcal{C} \to \mathcal{D}$ be a functor between categories. Let $p : \mathcal{S} \to \mathcal{C}$ be a category over $\mathcal{C}$. In this setting we define a category $u_{pp}\mathcal{S}$ as follows:

(1) An object of $u_{pp}\mathcal{S}$ is a triple $(U, \phi : V \to u(U), x)$ where $U \in \mathrm{Ob}(\mathcal{C})$, the map $\phi : V \to u(U)$ is a morphism in $\mathcal{D}$, and $x \in \mathrm{Ob}(\mathcal{S}_U)$.
(2) A morphism
$$(U_1, \phi_1 : V_1 \to u(U_1), x_1) \longrightarrow (U_2, \phi_2 : V_2 \to u(U_2), x_2)$$
of $u_{pp}\mathcal{S}$ is given by a (a, b, α) where $a : U_1 \to U_2$ is a morphism of $\mathcal{C}$, $b : V_1 \to V_2$ is a morphism of $\mathcal{D}$, and $\alpha : x_1 \to x_2$ is morphism of $\mathcal{S}$, such that $p(\alpha) = a$ and the diagram
$$\begin{array}{ccc} V_1 & \xrightarrow{b} & V_2 \\ {\scriptstyle \phi_1}\downarrow & & \downarrow{\scriptstyle \phi_2} \\ u(U_1) & \xrightarrow{u(a)} & u(U_2) \end{array}$$
commutes in $\mathcal{D}$.

We think of $u_{pp}\mathcal{S}$ as a category over $\mathcal{D}$ via
$$p_{pp} : u_{pp}\mathcal{S} \longrightarrow \mathcal{D}, \quad (U, \phi : V \to u(U), x) \longmapsto V.$$
The fibre category of $u_{pp}\mathcal{S}$ over an object V of $\mathcal{D}$ does not have a simple description. Moreover, it is in general not the case that $u_{pp}\mathcal{S}$ is a fibred category over $\mathcal{D}$ if $\mathcal{S}$ is a fibred category over $\mathcal{C}$.

Lemma 8.12.5. *In the situation above assume*

(1) $p : \mathcal{S} \to \mathcal{C}$ *is a fibred category,*
(2) $\mathcal{C}$ *has nonempty finite limits, and*
(3) $u : \mathcal{C} \to \mathcal{D}$ *commutes with nonempty finite limits.*

Consider the set $R \subset Arrows(u_{pp}\mathcal{S})$ *of morphisms of the form*

$$(a, id_V, \alpha) : (U', \phi' : V \to u(U'), x') \longrightarrow (U, \phi : V \to u(U), x)$$

with α *strongly cartesian. Then* R *is a right multiplicative system.*

Proof. According to Categories, Definition 4.25.1 we have to check RMS1, RMS2, RMS3. Condition RMS1 holds as a composition of strongly cartesian morphisms is strongly cartesian, see Categories, Lemma 4.31.2.

To check RMS2 suppose we have a morphism

$$(a, b, \alpha) : (U_1, \phi_1 : V_1 \to u(U_1), x_1) \longrightarrow (U, \phi : V \to u(U), x)$$

of $u_{pp}\mathcal{S}$ and a morphism

$$(c, \mathrm{id}_V, \gamma) : (U', \phi' : V \to u(U'), x') \longrightarrow (U, \phi : V \to u(U), x)$$

with γ strongly cartesian from R. In this situation set $U_1' = U_1 \times_U U'$, and denote $a' : U_1' \to U'$ and $c' : U_1' \to U_1$ the projections. As $u(U_1') = u(U_1) \times_{u(U)} u(U')$ we see that $\phi_1' = (\phi_1, \phi') : V_1 \to u(U_1')$ is a morphism in $\mathcal{D}$. Let $\gamma_1 : x_1' \to x_1$ be a strongly cartesian morphism of $\mathcal{S}$ with $p(\gamma_1) = \phi_1'$ (which exists because $\mathcal{S}$ is a fibred category over $\mathcal{C}$). Then as $\gamma : x' \to x$ is strongly cartesian there exists a unique morphism $\alpha' : x_1' \to x'$ with $p(\alpha') = a'$. At this point we see that

$$(a', b, \alpha') : (U_1, \phi_1 : V_1 \to u(U_1'), x_1') \longrightarrow (U, \phi : V \to u(U'), x')$$

is a morphism and that

$$(c', \mathrm{id}_{V_1}, \gamma_1) : (U_1', \phi_1' : V_1 \to u(U_1'), x_1') \longrightarrow (U_1, \phi : V_1 \to u(U_1), x_1)$$

is an element of R which form a solution of the existence problem posed by RMS2.

Finally, suppose that

$$(a, b, \alpha), (a', b', \alpha') : (U_1, \phi_1 : V_1 \to u(U_1), x_1) \longrightarrow (U, \phi : V \to u(U), x)$$

are two morphisms of $u_{pp}\mathcal{S}$ and suppose that

$$(c, \mathrm{id}_V, \gamma) : (U, \phi : V \to u(U), x) \longrightarrow (U', \phi : V \to u(U'), x')$$

is an element of R which equalizes the morphisms (a, b, α) and (a', b', α'). This implies in particular that $b = b'$. Let $d : U_2 \to U_1$ be the equalizer of a, a' which exists (see Categories, Lemma 4.18.3). Moreover, $u(d) : u(U_2) \to u(U_1)$ is the equalizer of $u(a), u(a')$ hence (as $b = b'$) there is a morphism $\phi_2 : V_1 \to u(U_2)$ such that $\phi_1 = u(d) \circ \phi_1$. Let $\delta : x_2 \to x_1$ be a strongly cartesian morphism of $\mathcal{S}$ with $p(\delta) = u(d)$. Now we claim that $\alpha \circ \delta = \alpha' \circ \delta$. This is true because γ is strongly cartesian, $\gamma \circ \alpha \circ \delta = \gamma \circ \alpha' \circ \delta$, and $p(\alpha \circ \delta) = p(\alpha' \circ \delta)$. Hence the arrow

$$(d, \mathrm{id}_{V_1}, \delta) : (U_2, \phi_2 : V_1 \to u(U_2), x_2) \longrightarrow (U_1, \phi_1 : V_1 \to u(U_1), x_1)$$

is an element of R and equalizes (a, b, α) and (a', b', α'). Hence R satisfies RMS3 as well. □

Lemma 8.12.6. *With notation and assumptions as in Lemma 8.12.5. Set* $u_p\mathcal{S} = R^{-1}u_{pp}\mathcal{S}$*, see Categories, Section 4.25. Then* $u_p\mathcal{S}$ *is a fibred category over* $\mathcal{D}$*.*

Proof. We use the description of $u_p\mathcal{S}$ given just above Categories, Lemma 4.25.9. Note that the functor $p_{pp} : u_{pp}\mathcal{S} \to \mathcal{D}$ transforms every element of R to an identity morphism. Hence by Categories, Lemma 4.25.13 we obtain a canonical functor $p_p : u_p\mathcal{S} \to \mathcal{D}$ extending the given functor. This is how we think of $u_p\mathcal{S}$ as a category over $\mathcal{D}$.

First we want to characterize the $\mathcal{D}$-strongly cartesian morphisms in $u_p\mathcal{S}$. A morphism $f : X \to Y$ of $u_p\mathcal{S}$ is the equivalence class of a pair $(f' : X' \to Y, r : X' \to X)$ with $r \in R$. In fact, in $u_p\mathcal{S}$ we have $f = (f', 1) \circ (r, 1)^{-1}$ with obvious notation. Note that an isomorphism is always strongly cartesian, as are compositions of strongly cartesian morphisms, see Categories, Lemma 4.31.2. Hence f is strongly cartesian if and only if $(f', 1)$ is so. Thus the following claim completely characterizes strongly cartesian morphisms. Claim: A morphism

$$(a, b, \alpha) : X_1 = (U_1, \phi_1 : V_1 \to u(U_1), x_1) \longrightarrow (U_2, \phi_2 : V_2 \to u(U_2), x_2) = X_2$$

of $u_{pp}\mathcal{S}$ has image $f = ((a, b, \alpha), 1)$ strongly cartesian in $u_p\mathcal{S}$ if and only if α is a strongly cartesian morphism of $\mathcal{S}$.

Assume α strongly cartesian. Let $X = (U, \phi : V \to u(U), x)$ be another object, and let $f_2 : X \to X_2$ be a morphism of $u_p\mathcal{S}$ such that $p_p(f_2) = b \circ b_1$ for some $b_1 : U \to U_1$. To show that f is strongly cartesian we have to show that there exists a unique morphism $f_1 : X \to X_1$ in $u_p\mathcal{S}$ such that $p_p(f_1) = b_1$ and $f_2 = f \circ f_1$ in $u_p\mathcal{S}$. Write $f_2 = (f_2' : X' \to X_2, r : X' \to X)$. Again we can write $f_2 = (f_2', 1) \circ (r, 1)^{-1}$ in $u_p\mathcal{S}$. Since $(r, 1)$ is an isomorphism whose image in $\mathcal{D}$ is an identity we see that finding a morphism $f_1 : X \to X_1$ with the required properties is the same thing as finding a morphism $f_1' : X' \to X_1$ in $u_p\mathcal{S}$ with $p(f_1') = b_1$ and $f_2' = f \circ f_1'$. Hence we may assume that f_2 is of the form $f_2 = ((a_2, b_2, \alpha_2), 1)$ with $b_2 = b \circ b_1$. Here is a picture

$$\begin{array}{ccc} & & (U_1, V_1 \to u(U_1), x_1) \\ & & \downarrow{\scriptstyle (a,b,\alpha)} \\ (U, V \to u(U), x) & \xrightarrow{(a_2,b_2,\alpha_2)} & (U_2, V_2 \to u(U_2), x_2) \end{array}$$

Now it is clear how to construct the morphism f_1. Namely, set $U' = U \times_{U_2} U_1$ with projections $c : U' \to U$ and $a_1 : U' \to U_1$. Pick a strongly cartesian morphism $\gamma : x' \to x$ lifting the morphism c. Since $b_2 = b \circ b_1$, and since $u(U') = u(U) \times_{u(U_2)} u(U_1)$ we see that $\phi' = (\phi, \phi_1 \circ b_1) : V \to u(U')$. Since α is strongly cartesian, and $a \circ a_1 = a_2 \circ c = p(\alpha_2 \circ \gamma)$ there exists a morphism $\alpha_1 : x' \to x_1$ lifting a_1 such that $\alpha \circ \alpha_1 = \alpha_2 \circ \gamma$. Set $X' = (U', \phi' : V \to u(U'), x')$. Thus we see that

$$f_1 = ((a_1, b_1, \alpha_1) : X' \to X_1, (c, \mathrm{id}_V, \gamma) : X' \to X) : X \longrightarrow X_1$$

works, in fact the diagram

$$\begin{array}{ccc} (U', \phi' : V \to u(U'), x') & \xrightarrow{(a_1,b_1,\alpha_1)} & (U_1, V_1 \to u(U_1), x_1) \\ \downarrow{\scriptstyle (c,\mathrm{id}_V,\gamma)} & & \downarrow{\scriptstyle (a,b,\alpha)} \\ (U, V \to u(U), x) & \xrightarrow{(a_2,b_2,\alpha_2)} & (U_2, V_2 \to u(U_2), x_2) \end{array}$$

is commutative by construction. This proves existence.

Next we prove uniqueness, still in the special case $f = ((a, b, \alpha), 1)$ and $f_2 = ((a_2, b_2, \alpha_2), 1)$. We strongly advise the reader to skip this part. Suppose that

$g_1, g_1' : X \to X_1$ are two morphisms of $u_p\mathcal{S}$ such that $p_p(g_1) = p_p(g_1') = b_1$ and $f_2 = f \circ g_1 = f \circ g_1'$. Our goal is to show that $g_1 = g_1'$. By Categories, Lemma 4.25.10 we may represent g_1 and g_1' as the equivalence classes of $(f_1 : X' \to X_1, r : X' \to X)$ and $(f_1' : X' \to X_1, r : X' \to X)$ for some $r \in R$. By Categories, Lemma 4.25.11 we see that $f_2 = f \circ g_1 = f \circ g_1'$ means that there exists a morphism $r' : X'' \to X'$ in $u_{pp}\mathcal{S}$ such that $r' \circ r \in R$ and

$$(a, b, \alpha) \circ f_1 \circ r' = (a, b, \alpha) \circ f_1' \circ r' = (a_2, b_2, \alpha_2) \circ r'$$

in $u_{pp}\mathcal{S}$. Note that now g_1 is represented by $(f_1 \circ r', r \circ r')$ and similarly for g_1'. Hence we may assume that

$$(a, b, \alpha) \circ f_1 = (a, b, \alpha) \circ f_1' = (a_2, b_2, \alpha_2).$$

Write $r = (c, \mathrm{id}_V, \gamma) : (U', \phi' : V \to u(U'), x')$, $f_1 = (a_1, b_1, \alpha_1)$, and $f_1' = (a_1', b_1, \alpha_1')$. Here we have used the condition that $p_p(g_1) = p_p(g_1')$. The equalities above are now equivalent to $a \circ a_1 = a \circ a_1' = a_2 \circ c$ and $\alpha \circ \alpha_1 = \alpha \circ \alpha_1' = \alpha_2 \circ \gamma$. It need not be the case that $a_1 = a_1'$ in this situation. Thus we have to precompose by one more morphism from R. Namely, let $U'' = \mathrm{Eq}(a_1, a_1')$ be the equalizer of a_1 and a_1' which is a subobject of U'. Denote $c' : U'' \to U'$ the canonical monomorphism. Because of the relations among the morphisms above we see that $V \to u(U')$ maps into $u(U'') = u(\mathrm{Eq}(a_1, a_1')) = \mathrm{Eq}(u(a_1), u(a_1'))$. Hence we get a new object $(U'', \phi'' : V \to u(U''), x'')$, where $\gamma' : x'' \to x'$ is a strongly cartesian morphism lifting γ. Then we see that we may precompose f_1 and f_1' with the element $(c', \mathrm{id}_V, \gamma')$ of R. After doing this, i.e., replacing $(U', \phi' : V \to u(U'), x')$ with $(U'', \phi'' : V \to u(U''), x'')$, we get back to the previous situation where in addition we now have that $a_1 = a_1'$. In this case it follows formally from the fact that α is strongly cartesian (!) that $\alpha_1 = \alpha_1'$. This shows that $g_1 = g_1'$ as desired.

We omit the proof of the fact that for any strongly cartesian morphism of $u_p\mathcal{S}$ of the form $((a, b, \alpha), 1)$ the morphism α is strongly cartesian in $\mathcal{S}$. (We do not need the characterization of strongly cartesian morphisms in the rest of the proof, although we do use it later in this section.)

Let $(U, \phi : V \to u(U), x)$ be an object of $u_p\mathcal{S}$. Let $b : V' \to V$ be a morphism of $\mathcal{D}$. Then the morphism

$$(\mathrm{id}_U, b, \mathrm{id}_x) : (U, \phi \circ b : V' \to u(U), x) \longrightarrow (U, \phi : V \to u(U), x)$$

is strongly cartesian by the result of the preceding paragraphs and we win. □

Lemma 8.12.7. *With notation and assumptions as in Lemma 8.12.6. If $\mathcal{S}$ is fibred in groupoids, then $u_p\mathcal{S}$ is fibred in groupoids.*

Proof. By Lemma 8.12.6 we know that $u_p\mathcal{S}$ is a fibred category. Let $f : X \to Y$ be a morphism of $u_p\mathcal{S}$ with $p_p(f) = \mathrm{id}_V$. We are done if we can show that f is invertible, see Categories, Lemma 4.33.2. Write f as the equivalence class of a pair $((a, b, \alpha), r)$ with $r \in R$. Then $p_p(r) = \mathrm{id}_V$, hence $p_{pp}((a, b, \alpha)) = \mathrm{id}_V$. Hence $b = \mathrm{id}_V$. But any morphism of $\mathcal{S}$ is strongly cartesian, see Categories, Lemma 4.33.2 hence we see that $(a, b, \alpha) \in R$ is invertible in $u_p\mathcal{S}$ as desired. □

Lemma 8.12.8. *Let $u : \mathcal{C} \to \mathcal{D}$ be a functor. Let $p : \mathcal{S} \to \mathcal{C}$ and $q : \mathcal{T} \to \mathcal{D}$ be categories over $\mathcal{C}$ and $\mathcal{D}$. Assume that*

(1) *$p : \mathcal{S} \to \mathcal{C}$ is a fibred category,*
(2) *$q : \mathcal{T} \to \mathcal{D}$ is a fibred category,*

(3) $\mathcal{C}$ *has nonempty finite limits, and*
(4) $u : \mathcal{C} \to \mathcal{D}$ *commutes with nonempty finite limits.*

Then we have a canonical equivalence of categories

$$\mathrm{Mor}_{Fib/\mathcal{C}}(\mathcal{S}, u^p\mathcal{T}) = \mathrm{Mor}_{Fib/\mathcal{D}}(u_p\mathcal{S}, \mathcal{T})$$

of morphism categories.

Proof. In this proof we use the notation x/U to denote an object x of $\mathcal{S}$ which lies over U in $\mathcal{C}$. Similarly y/V denotes an object y of $\mathcal{T}$ which lies over V in $\mathcal{D}$. In the same vein $\alpha/a : x/U \to x'/U'$ denotes the morphism $\alpha : x \to x'$ with image $a : U \to U'$ in $\mathcal{C}$.

Let $G : u_p\mathcal{S} \to \mathcal{T}$ be a 1-morphism of fibred categories over $\mathcal{D}$. Denote $G' : u_{pp}\mathcal{S} \to \mathcal{T}$ the composition of G with the canonical (localization) functor $u_{pp}\mathcal{S} \to u_p\mathcal{S}$. Then consider the functor $H : \mathcal{S} \to u^p\mathcal{T}$ given by

$$H(x/U) = (U, G'(U, \mathrm{id}_{u(U)} : u(U) \to u(U), x))$$

on objects and by

$$H((\alpha, a) : x/U \to x'/U') = G'(a, u(a), \alpha)$$

on morphisms. Since G transforms strongly cartesian morphisms into strongly cartesian morphisms, we see that if α is strongly cartesian, then $H(\alpha)$ is strongly cartesian. Namely, we've seen in the proof of Lemma 8.12.6 that in this case the map $(a, u(a), \alpha)$ becomes strongly cartesian in $u_p\mathcal{S}$. Clearly this construction is functorial in G and we obtain a functor

$$A : \mathrm{Mor}_{Fib/\mathcal{D}}(u_p\mathcal{S}, \mathcal{T}) \longrightarrow \mathrm{Mor}_{Fib/\mathcal{C}}(\mathcal{S}, u^p\mathcal{T})$$

Conversely, let $H : \mathcal{S} \to u^p\mathcal{T}$ be a 1-morphism of fibred categories. Recall that an object of $u^p\mathcal{T}$ is a pair (U, y) with $y \in \mathrm{Ob}(\mathcal{T}_{u(U)})$. We denote $\mathrm{pr} : u^p\mathcal{T} \to \mathcal{T}$ the functor $(U, y) \mapsto y$. In this case we define a functor $G' : u_{pp}\mathcal{S} \to \mathcal{T}$ by the rules

$$G'(U, \phi : V \to u(U), x) = \phi^*\mathrm{pr}(H(x))$$

on objects and we let

$$G'((a, b, \alpha) : (U, \phi : V \to u(U), x) \to (U', \phi' : V' \to u(U'), x')) = \beta$$

be the unique morphism $\beta : \phi^*\mathrm{pr}(H(x)) \to (\phi')^*\mathrm{pr}(H(x'))$ such that $q(\beta) = b$ and the diagram

$$\begin{array}{ccc} \phi^*\mathrm{pr}(H(x)) & \xrightarrow[\beta]{} & (\phi')^*\mathrm{pr}(H(x')) \\ \downarrow & & \downarrow \\ \mathrm{pr}(H(x)) & \xrightarrow{\mathrm{pr}(H(a,\alpha))} & \mathrm{pr}(H(x')) \end{array}$$

Such a morphism exists and is unique because $\mathcal{T}$ is a fibred category.

We check that $G'(r)$ is an isomorphism if $r \in R$. Namely, if

$$(a, \mathrm{id}_V, \alpha) : (U', \phi' : V \to u(U'), x') \longrightarrow (U, \phi : V \to u(U), x)$$

with α strongly cartesian is an element of the right multiplicative system R of Lemma 8.12.5 then $H(\alpha)$ is strongly cartesian, and $\mathrm{pr}(H(\alpha))$ is strongly cartesian, see proof of Lemma 8.12.1. Hence in this case the morphism β has $q(\beta) = \mathrm{id}_V$ and is strongly cartesian. Hence β is an isomorphism by Categories, Lemma 4.31.2. Thus by Categories, Lemma 4.25.13 we obtain a canonical extension $G : u_p\mathcal{S} \to \mathcal{T}$.

Next, let us prove that G transforms strongly cartesian morphisms into strongly cartesian morphisms. Suppose that $f : X \to Y$ is a strongly cartesian. By the characterization of strongly cartesian morphisms in $u_p\mathcal{S}$ we can write f as $((a, b, \alpha) : X' \to Y, r : X' \to Y)$ where $r \in R$ and α strongly cartesian in $\mathcal{S}$. By the above it suffices to show that $G'(a, b\alpha)$ is strongly cartesian. As before the condition that α is strongly cartesian implies that $\text{pr}(H(a, \alpha)) : \text{pr}(H(x)) \to \text{pr}(H(x'))$ is strongly cartesian in $\mathcal{T}$. Since in the commutative square above now all arrows except possibly β is strongly cartesian it follows that also β is strongly cartesian as desired. Clearly the construction $H \mapsto G$ is functorial in H and we obtain a functor

$$B : \text{Mor}_{Fib/\mathcal{C}}(\mathcal{S}, u^p\mathcal{T}) \longrightarrow \text{Mor}_{Fib/\mathcal{D}}(u_p\mathcal{S}, \mathcal{T})$$

To finish the proof of the lemma we have to show that the functors A and B are mutually quasi-inverse. We omit the verifications. □

Definition 8.12.9. Let $f : \mathcal{D} \to \mathcal{C}$ be a morphism of sites given by a continuous functor $u : \mathcal{C} \to \mathcal{D}$ satisfying the hypotheses and conclusions of Sites, Proposition 7.15.6. Let $\mathcal{S}$ be a stack over $\mathcal{C}$. In this setting we write $f^{-1}\mathcal{S}$ for the stackification of the fibred category $u_p\mathcal{S}$ over $\mathcal{D}$ constructed above. We say that $f^{-1}\mathcal{S}$ is the *pullback of $\mathcal{S}$ along f*.

Of course, if $\mathcal{S}$ is a stack in groupoids, then $f^{-1}\mathcal{S}$ is a stack in groupoids by Lemmas 8.9.1 and 8.12.7.

Lemma 8.12.10. *Let $f : \mathcal{D} \to \mathcal{C}$ be a morphism of sites given by a continuous functor $u : \mathcal{C} \to \mathcal{D}$ satisfying the hypotheses and conclusions of Sites, Proposition 7.15.6. Let $p : \mathcal{S} \to \mathcal{C}$ and $q : \mathcal{T} \to \mathcal{D}$ be stacks. Then we have a canonical equivalence of categories*

$$\text{Mor}_{Stacks/\mathcal{C}}(\mathcal{S}, f_*\mathcal{T}) = \text{Mor}_{Stacks/\mathcal{D}}(f^{-1}\mathcal{S}, \mathcal{T})$$

of morphism categories.

Proof. For $i = 1, 2$ an i-morphism of stacks is the same thing as a i-morphism of fibred categories, see Definition 8.4.5. By Lemma 8.12.8 we have already

$$\text{Mor}_{Fib/\mathcal{C}}(\mathcal{S}, u^p\mathcal{T}) = \text{Mor}_{Fib/\mathcal{D}}(u_p\mathcal{S}, \mathcal{T})$$

Hence the result follows from Lemma 8.8.3 as $u^p\mathcal{T} = f_*\mathcal{T}$ and $f^{-1}\mathcal{S}$ is the stackification of $u_p\mathcal{S}$. □

Lemma 8.12.11. *Let $f : \mathcal{D} \to \mathcal{C}$ be a morphism of sites given by a continuous functor $u : \mathcal{C} \to \mathcal{D}$ satisfying the hypotheses and conclusions of Sites, Proposition 7.15.6. Let $\mathcal{S} \to \mathcal{C}$ be a fibred category, and let $\mathcal{S} \to \mathcal{S}'$ be the stackification of $\mathcal{S}$. Then $f^{-1}\mathcal{S}'$ is the stackification of $u_p\mathcal{S}$.*

Proof. Omitted. Hint: This is the analogue of Sites, Lemma 7.14.4. □

The following lemma tells us that the 2-category of stacks over Sch_{fppf} is a "full 2-sub category" of the 2-category of stacks over Sch'_{fppf} provided that Sch'_{fppf} contains Sch_{fppf} (see Topologies, Section 33.10).

Lemma 8.12.12. *Let $\mathcal{C}$ and $\mathcal{D}$ be sites. Let $u : \mathcal{C} \to \mathcal{D}$ be a functor satisfying the assumptions of Sites, Lemma 7.20.8. Let $f : \mathcal{D} \to \mathcal{C}$ be the corresponding morphism of sites. Then*

(1) *for every stack $p : \mathcal{S} \to \mathcal{C}$ the canonical functor $\mathcal{S} \to f_* f^{-1}\mathcal{S}$ is an equivalence of stacks,*
(2) *given stacks $\mathcal{S}, \mathcal{S}'$ over $\mathcal{C}$ the construction f^{-1} induces an equivalence*
$$\mathrm{Mor}_{Stacks/\mathcal{C}}(\mathcal{S}, \mathcal{S}') \longrightarrow \mathrm{Mor}_{Stacks/\mathcal{D}}(f^{-1}\mathcal{S}, f^{-1}\mathcal{S}')$$
of morphism categories.

Proof. Note that by Lemma 8.12.10 we have an equivalence of categories
$$\mathrm{Mor}_{Stacks/\mathcal{D}}(f^{-1}\mathcal{S}, f^{-1}\mathcal{S}') = \mathrm{Mor}_{Stacks/\mathcal{C}}(\mathcal{S}, f_* f^{-1}\mathcal{S}')$$
Hence (2) follows from (1).

To prove (1) we are going to use Lemma 8.4.8. This lemma tells us that we have to show that $can : \mathcal{S} \to f_* f^{-1}\mathcal{S}$ is fully faithful and that all objects of $f_* f^{-1}\mathcal{S}$ are locally in the essential image.

We quickly describe the functor can, see proof of Lemma 8.12.8. To do this we introduce the functor $c'' : \mathcal{S} \to u_{pp}\mathcal{S}$ defined by $c''(x/U) = (U, \mathrm{id} : u(U) \to u(U), x)$, and $c''(\alpha/a) = (a, u(a), \alpha)$. We set $c' : \mathcal{S} \to u_p\mathcal{S}$ equal to the composition of c'' and the canonical functor $u_{pp}\mathcal{S} \to u_p\mathcal{S}$. We set $c : \mathcal{S} \to f^{-1}\mathcal{S}$ equal to the composition of c' and the canonical functor $u_p\mathcal{S} \to f^{-1}\mathcal{S}$. Then $can : \mathcal{S} \to f_* f^{-1}\mathcal{S}$ is the functor which to x/U associates the pair $(U, c(x))$ and to α/a the morphism $(a, c(\alpha))$.

Fully faithfulness. To prove this we are going to use Lemma 8.4.7. Let $U \in \mathrm{Ob}(\mathcal{C})$. Let $x, y \in \mathcal{S}_U$. First off, as u is fully faithful, we have
$$\mathrm{Mor}_{(f_* f^{-1}\mathcal{S})_U}(can(x), can(y)) = \mathrm{Mor}_{(f^{-1}\mathcal{S})_{u(U)}}(c(x), c(y))$$
directly from the definition of f_*. Similar holds after pulling back to any U'/U. Because $f^{-1}\mathcal{S}$ is the stackification of $u_p\mathcal{S}$, and since u is continuous and cocontinuous the presheaf
$$U'/U \longmapsto \mathrm{Mor}_{(f^{-1}\mathcal{S})_{u(U')}}(c(x|_{U'}), c(y|_{U'}))$$
is the sheafification of the presheaf
$$U'/U \longmapsto \mathrm{Mor}_{(u_p\mathcal{S})_{u(U')}}(c'(x|_{U'}), c'(y|_{U'}))$$
Hence to finish the proof of fully faithfulness it suffices to show that for any U and x, y the map
$$\mathrm{Mor}_{\mathcal{S}_U}(x, y) \longrightarrow \mathrm{Mor}_{(u_p\mathcal{S})_U}(c'(x), c'(y))$$
is bijective. A morphism $f : x \to y$ in $u_p\mathcal{S}$ over $u(U)$ is given by an equivalence class of diagrams
$$\begin{array}{ccc} (U', \phi : u(U) \to u(U'), x') & \xrightarrow[(a,b,\alpha)]{} & (U, \mathrm{id} : u(U) \to u(U), y) \\ \downarrow{\scriptstyle (c, \mathrm{id}_{u(U)}, \gamma)} & & \\ (U, \mathrm{id} : u(U) \to u(U), x) & & \end{array}$$
with γ strongly cartesian and $b = \mathrm{id}_{u(U)}$. But since u is fully faithful we can write $\phi = u(c')$ for some morphism $c' : U \to U'$ and then we see that $a \circ c' = \mathrm{id}_U$ and $c \circ c' = \mathrm{id}_{U'}$. Because γ is strongly cartesian we can find a morphism $\gamma' : x \to x'$ lifting c' such that $\gamma \circ \gamma' = \mathrm{id}_x$. By definition of the equivalence classes defining morphisms in $u_p\mathcal{S}$ it follows that the morphism
$$(U, \mathrm{id} : u(U) \to u(U), x) \xrightarrow[(\mathrm{id}, \mathrm{id}, \alpha \circ \gamma')]{} (U, \mathrm{id} : u(U) \to u(U), y)$$

of $u_{pp}\mathcal{S}$ induces the morphism f in $u_p\mathcal{S}$. This proves that the map is surjective. We omit the proof that it is injective.

Finally, we have to show that any object of $f_*f^{-1}\mathcal{S}$ locally comes from an object of $\mathcal{S}$. This is clear from the constructions (details omitted). □

8.13. Stacks and localization

Let $\mathcal{C}$ be a site. Let U be an object of $\mathcal{C}$. We want to understand stacks over $\mathcal{C}/U$ as stacks over $\mathcal{C}$ together with a morphism towards U. The following lemma is the reason why this is easier to do when the presheaf h_U is a sheaf.

Lemma 8.13.1. *Let $\mathcal{C}$ be a site. Let $U \in \mathrm{Ob}(\mathcal{C})$. Then $j_U : \mathcal{C}/U \to \mathcal{C}$ is a stack over $\mathcal{C}$ if and only if h_U is a sheaf.*

Proof. Combine Lemma 8.6.3 with Categories, Example 4.36.7. □

Assume that $\mathcal{C}$ is a site, and U is an object of $\mathcal{C}$ whose associated representable presheaf is a sheaf. We denote $j : \mathcal{C}/U \to \mathcal{C}$ the localization functor.

Construction A. Let $p : \mathcal{S} \to \mathcal{C}/U$ be a stack over the site $\mathcal{C}/U$. We define a stack $j_!p : j_!\mathcal{S} \to \mathcal{C}$ as follows:

(1) As a category $j_!\mathcal{S} = \mathcal{S}$, and
(2) the functor $j_!p : j_!\mathcal{S} \to \mathcal{C}$ is just the composition $j \circ p$.

We omit the verification that this is a stack (hint: Use that h_U is a sheaf to glue morphisms to U). There is a canonical functor

$$j_!\mathcal{S} \longrightarrow \mathcal{C}/U$$

namely the functor p which is a 1-morphism of stacks over $\mathcal{C}$.

Construction B. Let $q : \mathcal{T} \to \mathcal{C}$ be a stack over $\mathcal{C}$ which is endowed with a morphism of stacks $p : \mathcal{T} \to \mathcal{C}/U$ over $\mathcal{C}$. In this case it is automatically the case that $p : \mathcal{T} \to \mathcal{C}/U$ is a stack over $\mathcal{C}/U$.

Lemma 8.13.2. *Assume that $\mathcal{C}$ is a site, and U is an object of $\mathcal{C}$ whose associated representable presheaf is a sheaf. Constructions A and B above define mutually inverse (!) functors of 2-categories*

$$\left\{\begin{array}{c}\text{2-category of}\\ \text{stacks over } \mathcal{C}/U\end{array}\right\} \leftrightarrow \left\{\begin{array}{c}\text{2-category of pairs } (\mathcal{T}, p) \text{ consisting}\\ \text{of a stack } \mathcal{T} \text{ over } \mathcal{C} \text{ and a morphism}\\ p : \mathcal{T} \to \mathcal{C}/U \text{ of stacks over } \mathcal{C}\end{array}\right\}$$

Proof. This is clear. □

8.14. Other chapters

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CHAPTER 9

Fields

9.1. Introduction

In this chapter, we shall discuss the theory of fields. Recall that a *field* is a ring in which all nonzero elements are invertible. Equivalently, the only two ideals of a field are (0) and (1) since any nonzero element is a unit. Consequently fields will be the simplest cases of much of the theory developed later.

The theory of field extensions has a different feel from standard commutative algebra since, for instance, any morphism of fields is injective. Nonetheless, it turns out that questions involving rings can often be reduced to questions about fields. For instance, any domain can be embedded in a field (its quotient field), and any *local ring* (that is, a ring with a unique maximal ideal; we have not defined this term yet) has associated to it its residue field (that is, its quotient by the maximal ideal). A knowledge of field extensions will thus be useful.

9.2. Basic definitions

Because we have placed this chapter before the chapter discussing commutative algebra we need to introduce some of the basic definitions here before we discuss these in greater detail in the algebra chapters.

Definition 9.2.1. An *field* is a nonzero ring where every nonzero element is invertible. Given a field a *subfield* is a subring that is itself a field.

For a field k, we write k^* for the subset $k \setminus \{0\}$. This generalizes the usual notation R^* that refers to the group of invertible elements in a ring R.

Definition 9.2.2. A *domain* or an *integral domain* is a nonzero ring where 0 is the only zerodivisor.

9.3. Examples of fields

To get started, let us begin by providing several examples of fields. The reader should recall that if R is a ring and $I \subset R$ an ideal, then R/I is a field precisely when I is a maximal ideal.

Example 9.3.1 (Rational numbers)**.** The rational numbers form a field. It is called the *field of rational numbers* and denoted $\mathbf{Q}$.

Example 9.3.2 (Prime fields)**.** If p is a prime number, then $\mathbf{Z}/(p)$ is a field, denoted $\mathbf{F}_p$. Indeed, (p) is a maximal ideal in $\mathbf{Z}$. Thus, fields may be finite: $\mathbf{F}_p$ contains p elements.

Example 9.3.3. In a principal ideal domain, an ideal generated by an irreducible element is maximal. Now, if k is a field, then the polynomial ring $k[x]$ is a principal ideal domain. It follows that if $P \in k[x]$ is an irreducible polynomial (that is, a nonconstant polynomial that does not admit a factorization into terms of smaller degrees), then $k[x]/(P)$ is a field. It contains a copy of k in a natural way. This is a very general way of constructing fields. For instance, the complex numbers $\mathbf{C}$ can be constructed as $\mathbf{R}[x]/(x^2+1)$.

Example 9.3.4 (Quotient fields)**.** Recall that, given a domain A, there is an imbedding $A \to K(A)$ into a field $K(A)$ constructed from A in exactly the same manner that $\mathbf{Q}$ is constructed from $\mathbf{Z}$. Formally the elements of $K(A)$ are (equivalence classes of) fractions a/b, $a, b \in A$, $b \neq 0$. As usual $a/b = a'/b'$ if and only if $ab' = ba'$. This is called the *quotient field* or *field of fractions*. The quotient field has the following universal property: given an injective ring map $\varphi : A \to K$ to a field K, there is a unique map $\psi : K(A) \to K$ making

$$\begin{array}{ccc} K(A) & \xrightarrow{\psi} & K \\ \uparrow & \nearrow_{\varphi} & \\ A & & \end{array}$$

commute. Indeed, it is clear how to define such a map: we set $\psi(a/b) = \varphi(a)\varphi(b)^{-1}$ where injectivity of φ assures that $\varphi(b) \neq 0$ if $b \neq 0$.

Example 9.3.5 (Field of rational functions)**.** If k is a field, then we can consider the field $k(x)$ of rational functions over k. This is the quotient field of the polynomial ring $k[x]$. In other words, it is the set of quotients F/G for $F, G \in k[x]$, $G \neq 0$ with the obvious equivalence relation.

Example 9.3.6. Let X be a Riemann surface. Let $\mathbf{C}(X)$ denote the set of meromorphic functions on X. Then $\mathbf{C}(X)$ is a ring under multiplication and addition of functions. It turns out that in fact $\mathbf{C}(X)$ is a field. Namely, if a nonzero function $f(z)$ is meromorphic, so is $1/f(z)$. For example, let S^2 be the Riemann sphere; then we know from complex analysis that the ring of meromorphic functions $\mathbf{C}(S^2)$ is the field of rational functions $\mathbf{C}(z)$.

9.4. Vector spaces

One reason fields are so nice is that the theory of modules over fields (i.e. vector spaces), is very simple.

Lemma 9.4.1. *If k is a field, then every k-module is free.*

Proof. Indeed, by linear algebra we know that a k-module (i.e. vector space) V has a *basis* $\mathcal{B} \subset V$, which defines an isomorphism from the free vector space on $\mathcal{B}$ to V. □

Lemma 9.4.2. *Every exact sequence of modules over a field splits.*

Proof. This follows from Lemma 9.4.1 as every vector space is a projective module. □

This is another reason why much of the theory in future chapters will not say very much about fields, since modules behave in such a simple manner. Note

that Lemma 9.4.2 is a statement about the *category* of k-modules (for k a field), because the notion of exactness is inherently arrow-theoretic, i.e., makes use of purely categorical notions, and can in fact be phrased within a so-called *abelian category*.

Henceforth, since the study of modules over a field is linear algebra, and since the ideal theory of fields is not very interesting, we shall study what this chapter is really about: *extensions* of fields.

9.5. The characteristic of a field

In the category of rings, there is an *initial object* $\mathbf{Z}$: any ring R has a map from $\mathbf{Z}$ into it in precisely one way. For fields, there is no such initial object. Nonetheless, there is a family of objects such that every field can be mapped into in exactly one way by exactly one of them, and in no way by the others.

Let F be a field. Think of F as a ring to get a ring map $f : \mathbf{Z} \to F$. The image of this ring map is a domain (as a subring of a field) hence the kernel of f is a prime ideal in $\mathbf{Z}$. Hence the kernel of f is either (0) or (p) for some prime number p.

In the first case we see that f is injective, and in this case we think of $\mathbf{Z}$ as a subring of F. Moreover, since every nonzero element of F is invertible we see that it makes sense to talk about $p/q \in F$ for $p, q \in \mathbf{Z}$ with $q \neq 0$. Hence in this case we may and we do think of $\mathbf{Q}$ as a subring of F. One can easily see that this is the smallest subfield of F in this case.

In the second case, i.e., when $\mathrm{Ker}(f) = (p)$ we see that $\mathbf{Z}/(p) = \mathbf{F}_p$ is a subring of F. Clearly it is the smallest subfield of F.

Arguing in this way we see that every field contains a smallest subfield which is either $\mathbf{Q}$ or finite equal to $\mathbf{F}_p$ for some prime number p.

Definition 9.5.1. The *characteristic* of a field F is 0 if $\mathbf{Z} \subset F$, or is a prime p if $p = 0$ in F. The *prime subfield of* F is the smallest subfield of F which is either $\mathbf{Q} \subset F$ if the characteristic is zero, or $\mathbf{F}_p \subset F$ if the characteristic is $p > 0$.

It is easy to see that if $E \subset F$ is a subfield, then the characteristic of E is the same as the characteristic of F.

Example 9.5.2. The characteristic of $\mathbf{F}_p$ is p, and that of $\mathbf{Q}$ is 0.

9.6. Field extensions

In general, though, we are interested not so much in fields by themselves but in field *extensions*. This is perhaps analogous to studying not rings but *algebras* over a fixed ring. The nice thing for fields is that the notion of a "field over another field" just recovers the notion of a field extension, by the next result.

Proposition 9.6.1. *If F is a field and R is a nonzero ring, then any ring homomorphism $\varphi : F \to R$ is injective.*

Proof. Indeed, let $a \in \mathrm{Ker}(\varphi)$ be a nonzero element. Then we have $\varphi(1) = \varphi(a^{-1}a) = \varphi(a^{-1})\varphi(a) = 0$. Thus $1 = \varphi(1) = 0$ and R is the zero ring. □

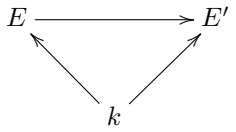
Definition 9.6.2. If F is a field contained in a field E, then E is said to be a *field extension* of F. We shall write E/F to indicate that E is an extension of F.

So if F, F' are fields, and $F \to F'$ is any ring-homomorphism, we see by Lemma 9.6.1 that it is injective, and F' can be regarded as an extension of F, by a slight abuse of language. Alternatively, a field extension of F is just an F-algebra that happens to be a field. This is completely different than the situation for general rings, since a ring homomorphism is not necessarily injective.

Let k be a field. There is a *category* of field extensions of k. An object of this category is an extension E/k, that is a (necessarily injective) morphism of fields

$$k \to E,$$

while a morphism between extensions E/k and E'/k is a k-algebra morphism $E \to E'$; alternatively, it is a commutative diagram



The set of morphisms from $E \to E'$ in the category of extensions of k will be denoted by $\mathrm{Mor}_k(E, E')$.

Definition 9.6.3. A *tower* of fields $E_n/E_{n-1}/\ldots/E_0$ consists of a sequence of extensions of fields E_n/E_{n-1}, E_{n-1}/E_{n-2}, ..., E_1/E_0.

Let us give a few examples of field extensions.

Example 9.6.4. Let k be a field, and $P \in k[x]$ an irreducible polynomial. We have seen that $k[x]/(P)$ is a field (Example 9.3.3). Since it is also a k-algebra in the obvious way, it is an extension of k.

Example 9.6.5. If X is a Riemann surface, then the field of meromorphic functions $\mathbf{C}(X)$ (Example 9.3.6) is an extension field of $\mathbf{C}$, because any element of $\mathbf{C}$ induces a meromorphic — indeed, holomorphic — constant function on X.

Let F/k be a field extension. Let $S \subset F$ be any subset. Then there is a *smallest* subextension of F (that is, a subfield of F containing k) that contains S. To see this, consider the family of subfields of F containing S and k, and take their intersection; one checks that this is a field. By a standard argument one shows, in fact, that this is the set of elements of F that can be obtained via a finite number of elementary algebraic operations (addition, multiplication, subtraction, and division) involving elements of k and S.

Definition 9.6.6. If F/k is an extension and $S \subset F$, we write $k(S)$ for the smallest subextension of F containing S. We will say that S *generates* the extension $k(S)/k$.

For instance, $\mathbf{C}$ is generated by i over $\mathbf{R}$.

Exercise 9.6.7. Show that $\mathbf{C}$ does not have a countable set of generators over $\mathbf{Q}$.

Let us now classify extensions generated by one element.

Lemma 9.6.8 (Classification of simple extensions)**.** *If a field extension F/k is generated by one element, then it is F is k-isomorphic either to the rational function field $k(t)/k$ or to one of the extensions $k[t]/(P)$ for $P \in k[t]$ irreducible.*

We will see that many of the most important cases of field extensions are generated by one element, so this is actually useful.

Proof. Let $\alpha \in F$ be such that $F = k(\alpha)$; by assumption, such an α exists. There is a morphism of rings

$$k[t] \to F$$

sending the indeterminate t to α. The image is a domain, so the kernel is a prime ideal. Thus, it is either (0) or (P) for $P \in k[t]$ irreducible.

If the kernel is (P) for $P \in k[t]$ irreducible, then the map factors through $k[t]/(P)$, and induces a morphism of fields $k[t]/(P) \to F$. Since the image contains α, we see easily that the map is surjective, hence an isomorphism. In this case, $k[t]/(P) \simeq F$.

If the kernel is trivial, then we have an injection $k[t] \to F$. One may thus define a morphism of the quotient field $k(t)$ into F; given a quotient $R(t)/Q(t)$ with $R(t), Q(t) \in k[t]$, we map this to $R(\alpha)/Q(\alpha)$. The hypothesis that $k[t] \to F$ is injective implies that $Q(\alpha) \neq 0$ unless Q is the zero polynomial. The quotient field of $k[t]$ is the rational function field $k(t)$, so we get a morphism $k(t) \to F$ whose image contains α. It is thus surjective, hence an isomorphism. □

9.7. Finite extensions

If F/E is a field extension, then evidently F is also a vector space over E (the scalar action is just multiplication in F).

Definition 9.7.1. Let F/E be an extension of fields. The dimension of F considered as an E-vector space is called the *degree* of the extension and is denoted $[F : E]$. If $[F : E] < \infty$ then F is said to be a *finite* extension of E.

Example 9.7.2. The field $\mathbf{C}$ is a two dimensional vector space over $\mathbf{R}$ with basis $1, i$. Thus $\mathbf{C}$ is a finite extension of $\mathbf{R}$ of degree 2.

Lemma 9.7.3. *Let $K/E/F$ be a tower of algebraic field extensions. If K is finite over F, then K is finite over E.*

Proof. Direct from the definition. □

Let us now consider the degree in the most important special example, that given by Lemma 9.6.8, in the next two examples.

Example 9.7.4 (Degree of a rational function field)**.** If k is any field, then the rational function field $k(t)$ is *not* a finite extension. For example the elements $\{t^n, n \in \mathbf{Z}\}$ are linearly independent over k.

In fact, if k is uncountable, then $k(t)$ is *uncountably* dimensional as a k-vector space. To show this, we claim that the family of elements $\{1/(t-\alpha), \alpha \in k\} \subset k(t)$ is linearly independent over k. A nontrivial relation between them would lead to a contradiction: for instance, if one works over $\mathbf{C}$, then this follows because $\frac{1}{t-\alpha}$, when considered as a meromorphic function on $\mathbf{C}$, has a pole at α and nowhere else. Consequently any sum $\sum c_i \frac{1}{t-\alpha_i}$ for the $c_i \in k^*$, and $\alpha_i \in k$ distinct, would have poles at each of the α_i. In particular, it could not be zero.

Amusingly, this leads to a quick proof of the Hilbert Nullstellensatz over the complex numbers. For a slightly more general result, see Algebra, Theorem 10.32.11.

Example 9.7.5 (Degree of a simple algebraic extension)**.** Consider a monogenic field extension E/k of the form discussed in Example 9.3.5. In other words, $E =$

$k[t]/(P)$ for $P \in k[t]$ an irreducible polynomial. Then the degree $[E : k]$ is just the degree the degree $d = \deg(P)$ of the polynomial P. Indeed, say

$$P = a_d t^d + a_1 t^{d-1} + \cdots + a_0. \tag{9.7.5.1}$$

with $a_d \neq 0$. Then the images of $1, t, \ldots, t^{d-1}$ in $k[t]/(P)$ are linearly independent over k, because any relation involving them would have degree strictly smaller than that of P, and P is the element of smallest degree in the ideal (P).

Conversely, the set $S = \{1, t, \ldots, t^{d-1}\}$ (or more properly their images) spans $k[t]/(P)$ as a vector space. Indeed, we have by (9.7.5.1) that $a_d t^d$ lies in the span of S. Since a_d is invertible, we see that t^d is in the span of S. Similarly, the relation $tP(t) = 0$ shows that the image of t^{d+1} lies in the span of $\{1, t, \ldots, t^d\}$ — by what was just shown, thus in the span of S. Working upward inductively, we find that the image of t^n for $n \geq d$ lies in the span of S.

This confirms the observation that $[\mathbf{C} : \mathbf{R}] = 2$, for instance. More generally, if k is a field, and $\alpha \in k$ is not a square, then the irreducible polynomial $x^2 - \alpha \in k[x]$ allows one to construct an extension $k[x]/(x^2 - \alpha)$ of degree two. We shall write this as $k(\sqrt{\alpha})$. Such extensions will be called *quadratic,* for obvious reasons.

The basic fact about the degree is that it is *multiplicative in towers.*

Lemma 9.7.6 (Multiplicativity)**.** *Suppose given a tower of fields $F/E/k$. Then*

$$[F : k] = [F : E][E : k]$$

Proof. Let $\alpha_1, \ldots, \alpha_n \in F$ be an E-basis for F. Let $\beta_1, \ldots, \beta_m \in E$ be a k-basis for E. Then the claim is that the set of products $\{\alpha_i \beta_j, 1 \leq i \leq n, 1 \leq j \leq m\}$ is a k-basis for F. Indeed, let us check first that they span F over k.

By assumption, the $\{\alpha_i\}$ span F over E. So if $f \in F$, there are $a_i \in E$ with

$$f = \sum\nolimits_i a_i \alpha_i,$$

and, for each i, we can write $a_i = \sum b_{ij} \beta_j$ for some $b_{ij} \in k$. Putting these together, we find

$$f = \sum\nolimits_{i,j} b_{ij} \alpha_i \beta_j,$$

proving that the $\{\alpha_i \beta_j\}$ span F over k.

Suppose now that there existed a nontrivial relation

$$\sum\nolimits_{i,j} c_{ij} \alpha_i \beta_j = 0$$

for the $c_{ij} \in k$. In that case, we would have

$$\sum\nolimits_i \alpha_i \left(\sum\nolimits_j c_{ij} \beta_j \right) = 0,$$

and the inner terms lie in E as the β_j do. Now E-linear independence of the $\{\alpha_i\}$ shows that the inner sums are all zero. Then k-linear independence of the $\{\beta_j\}$ shows that the c_{ij} all vanish. □

We sidetrack to a slightly tangential definition.

Definition 9.7.7. A field K is said to be a *number field* if it has characteristic 0 and the extension $\mathbf{Q} \subset K$ is finite.

Number fields are the basic objects in algebraic number theory. We shall see later that, for the analog of the integers $\mathbf{Z}$ in a number field, something kind of like unique factorization still holds (though strict unique factorization generally does not!).

9.8. Algebraic extensions

An important class of extensions are those where every element generates a finite extension.

Definition 9.8.1. Consider a field extension F/E. An element $\alpha \in F$ is said to be *algebraic* over E if α is the root of some nonzero polynomial with coefficients in E. If all elements of F are algebraic then F is said to be an *algebraic extension* of E.

By Lemma 9.6.8, the subextension $E(\alpha)$ is isomorphic either to the rational function field $E(t)$ or to a quotient ring $E[t]/(P)$ for $P \in E[t]$ an irreducible polynomial. In the latter case, α is algebraic over E (in fact, the proof of Lemma 9.6.8 shows that we can pick P such that α is a root of P); in the former case, it is not.

Example 9.8.2. The field $\mathbf{C}$ is algebraic over $\mathbf{R}$. Namely, if $\alpha = a + ib$ in $\mathbf{C}$, then $\alpha^2 - 2a\alpha + a^2 + b^2 = 0$ is a polynomial equation for α over $\mathbf{R}$.

Example 9.8.3. Let X be a compact Riemann surface, and let $f \in \mathbf{C}(X) - \mathbf{C}$ any nonconstant meromorphic function on X (see Example 9.3.6). Then it is known that $\mathbf{C}(X)$ is algebraic over the subextension $\mathbf{C}(f)$ generated by f. We shall not prove this.

Lemma 9.8.4. *Let $K/E/F$ be a tower of field extensions.*

(1) *If $\alpha \in K$ is algebraic over F, then α is algebraic over E.*
(2) *if K is algebraic over F, then K is algebraic over E.*

Proof. This is immediate from the definitions. □

We now show that there is a deep connection between finiteness and being algebraic.

Lemma 9.8.5. *A finite extension is algebraic. In fact, an extension E/k is algebraic if and only if every subextension $k(\alpha)/k$ generated by some $\alpha \in E$ is finite.*

In general, it is very false that an algebraic extension is finite.

Proof. Let E/k be finite, say of degree n. Choose $\alpha \in E$. Then the elements $\{1, \alpha, \ldots, \alpha^n\}$ are linearly dependent over E, or we would necessarily have $[E : k] > n$. A relation of linear dependence now gives the desired polynomial that α must satisfy.

For the last assertion, note that a monogenic extension $k(\alpha)/k$ is finite if and only α is algebraic over k, by Examples 9.7.4 and 9.7.5. So if E/k is algebraic, then each $k(\alpha)/k$, $\alpha \in E$, is a finite extension, and conversely. □

We can extract a lemma of the last proof (really of Examples 9.7.4 and 9.7.5): a monogenic extension is finite if and only if it is algebraic. We shall use this observation in the next result.

Lemma 9.8.6. *Let k be a field, and let $\alpha_1, \alpha_2, \ldots, \alpha_n$ be elements of some extension field such that each α_i is finite over k. Then the extension $k(\alpha_1, \ldots, \alpha_n)/k$ is finite. That is, a finitely generated algebraic extension is finite.*

Proof. Indeed, each extension $k(\alpha_1, \ldots, \alpha_{i+1})/k(\alpha_1, \ldots, \alpha_i)$ is generated by one element and algebraic, hence finite. By multiplicativity of degree (Lemma 9.7.6) we obtain the result. □

The set of complex numbers that are algebraic over $\mathbf{Q}$ are simply called the *algebraic numbers.* For instance, $\sqrt{2}$ is algebraic, i is algebraic, but π is not. It is a basic fact that the algebraic numbers form a field, although it is not obvious how to prove this from the definition that a number is algebraic precisely when it satisfies a nonzero polynomial equation with rational coefficients (e.g. by polynomial equations).

Lemma 9.8.7. *Let E/k be a field extension. Then the elements of E algebraic over k form a subextension of E/k.*

Proof. Let $\alpha, \beta \in E$ be algebraic over k. Then $k(\alpha, \beta)/k$ is a finite extension by Lemma 9.8.6. It follows that $k(\alpha + \beta) \subset k(\alpha, \beta)$ is a finite extension, which implies that $\alpha + \beta$ is algebraic by Lemma 9.8.5. Similarly for the difference, product and quotient of α and β. □

Many nice properties of field extensions, like those of rings, will have the property that they will be preserved by towers and composita.

Lemma 9.8.8. *Let E/k and F/E be algebraic extensions of fields. Then F/k is an algebraic extension of fields.*

Proof. Choose $\alpha \in F$. Then α is algebraic over E. The key observation is that α is algebraic over a finitely generated subextension of k. That is, there is a finite set $S \subset E$ such that α is algebraic over $k(S)$: this is clear because being algebraic means that a certain polynomial in $E[x]$ that α satisfies exists, and as S we can take the coefficients of this polynomial. It follows that α is algebraic over $k(S)$. In particular, the extension $k(S, \alpha)/k(S)$ is finite. Since S is a finite set, and $k(S)/k$ is algebraic, Lemma 9.8.6 shows that $k(S)/k$ is finite. Using multiplicativity (Lemma 9.7.6) we find that $k(S, \alpha)/k$ is finite, so α is algebraic over k. □

The method of proof in the previous argument — that being algebraic over E was a property that *descended* to a finitely generated subextension of E — is an idea that recurs throughout algebra. It often allows one to reduce general commutative algebra questions to the Noetherian case for example.

Lemma 9.8.9. *Let E/F be an algebraic extension of fields. Then the cardinality $|E|$ of E is at most* $\max(\aleph_0, |F|)$.

Proof. Let S be the set of nonconstant polynomials with coefficients in F. For every $P \in S$ the set of roots $r(P, E) = \{\alpha \in E \mid P(\alpha) = 0\}$ is finite (details omitted). Moreover, the fact that E is algebraic over F implies that $E = \bigcup_{P \in S} r(P, E)$. It is clear that S has cardinality bounded by $\max(\aleph_0, |F|)$ because the cardinality of a finite product of copies of F has cardinality at most $\max(\aleph_0, |F|)$. Thus so does E. □

9.9. Minimal polynomials

Let E/k be a field extension, and let $\alpha \in E$ be algebraic over k. Then α satisfies a (nontrivial) polynomial equation in $k[x]$. Consider the set of polynomials $P \in k[x]$ such that $P(\alpha) = 0$; by hypothesis, this set does not just contain the zero

polynomial. It is easy to see that this set is an *ideal*. Indeed, it is the kernel of the map

$$k[x] \to E, \quad x \mapsto \alpha$$

Since $k[x]$ is a PID, there is a *generator* $P \in k[x]$ of this ideal. If we assume P monic, without loss of generality, then P is uniquely determined.

Definition 9.9.1. The polynomial P above is called the *minimal polynomial* of α over k.

The minimal polynomial has the following characterization: it is the monic polynomial, of smallest degree, that annihilates α. Any nonconstant multiple of P will have larger degree, and only multiples of P can annihilate α. This explains the name *minimal*.

Clearly the minimal polynomial is *irreducible*. This is equivalent to the assertion that the ideal in $k[x]$ consisting of polynomials annihilating α is prime. This follows from the fact that the map $k[x] \to E, x \mapsto \alpha$ is a map into a domain (even a field), so the kernel is a prime ideal.

Lemma 9.9.2. *The degree of the minimal polynomial is* $[k(\alpha) : k]$.

Proof. This is just a restatement of the argument in Lemma 9.6.8: the observation is that if P is the minimal polynomial of α, then the map

$$k[x]/(P) \to k(\alpha), \quad x \mapsto \alpha$$

is an isomorphism as in the aforementioned proof, and we have counted the degree of such an extension (see Example 9.7.5). □

So the observation of the above proof is that if $\alpha \in E$ is algebraic, then $k(\alpha) \subset E$ is isomorphic to $k[x]/(P)$.

9.10. Algebraic closure

The "fundamental theorem of algebra" states that $\mathbf{C}$ is algebraically closed. A beautiful proof of this result uses Liouville's theorem in complex analysis, we shall give another proof (see Lemma 9.20.1).

Definition 9.10.1. A field F is said to be *algebraically closed* if every algebraic extension E/F is trivial, i.e., $E = F$.

This may not be the definition in every text. Here is the lemma comparing it with the other one.

Lemma 9.10.2. *Let F be a field. The following are equivalent*

(1) *F is algebraically closed,*
(2) *every irreducible polynomial over F is linear,*
(3) *every nonconstant polynomial over F has a root,*
(4) *every nonconstant polynomial over F is a product of linear factors.*

Proof. If F is algebraically closed, then every irreducible polynomial is linear. Namely, if there exists an irreducible polynomial of degree > 1, then this generates a nontrivial finite (hence algebraic) field extension, see Example 9.7.5. Thus (1) implies (2). If every irreducible polynomial is linear, then every irreducible polynomial has a root, whence every nonconstant polynomial has a root. Thus (2) implies (3).

Assume every nonconstant polynomial has a root. Let $P \in F[x]$ be nonconstant. If $P(\alpha) = 0$ with $\alpha \in F$, then we see that $P = (x - \alpha)Q$ for some $Q \in F[x]$ (by division with remainder). Thus we can argue by induction on the degree that any nonconstant polynomial can be written as a product $c\prod(x - \alpha_i)$.

Finally, suppose that every nonconstant polynomial over F is a product of linear factors. Let E/F be an algebraic extension. Then all the simple subextensions $F(\alpha)/F$ of E are necessarily trivial (because the only irreducible polynomials are linear by assumption). Thus $E = F$. We see that (4) implies (1) and we are done. □

Now we want to define a "universal" algebraic extension of a field. Actually, we should be careful: the algebraic closure is *not* a universal object. That is, the algebraic closure is not unique up to *unique* isomorphism: it is only unique up to isomorphism. But still, it will be very handy, if not functorial.

Definition 9.10.3. Let F be a field. We say F is *algebraically closed* if every algebraic extension E/F is trivial, i.e., $E = F$. An *algebraic closure* of F is a field $\overline{F}$ containing F such that:

(1) $\overline{F}$ is algebraic over F.
(2) $\overline{F}$ is algebraically closed.

If F is algebraically closed, then F is its own algebraic closure. We now prove the basic existence result.

Theorem 9.10.4. *Every field has an algebraic closure.*

The proof will mostly be a red herring to the rest of the chapter. However, we will want to know that it is *possible* to embed a field inside an algebraically closed field, and we will often assume it done.

Proof. Let F be a field. By Lemma 9.8.9 the cardinality of an algebraic extension of F is bounded by $\max(\aleph_0, |F|)$. Choose a set S containing F with $|S| > \max(\aleph_0, |F|)$. Let's consider triples (E, σ_E, μ_E) where

(1) E is a set with $F \subset E \subset S$, and
(2) $\sigma_E : E \times E \to E$ and $\mu_E : E \times E \to E$ are maps of sets such that (E, σ_E, μ_E) defines the structure of a field extension of F (in particular $\sigma_E(a, b) = a +_F b$ for $a, b \in F$ and similarly for μ_E), and
(3) $F \subset E$ is an algebraic field extension.

The collection of all triples (E, σ_E, μ_E) forms a set I. For $i \in I$ we will denote $E_i = (E_i, \sigma_i, \mu_i)$ the corresponding field extension fo F. We define a partial ordering on I by declaring $i \leq i'$ if and only if $E_i \subset E_{i'}$ (this makes sense as E_i and $E_{i'}$ are subsets of the same set S) and we have $\sigma_i = \sigma_{i'}|_{E_i \times E_i}$ and $\mu_i = \mu_{i'}|_{E_i \times E_i}$, in other words, $E_{i'}$ is a field extension of E_i.

Let $T \subset I$ be a totally ordered subset. Then it is clear that $E_T = \bigcup_{i \in T} E_i$ with induced maps $\sigma_T = \bigcup \sigma_i$ and $\mu_T = \bigcup \mu_i$ is another element of I. In other words every totally order subset of I has a upper bound in I. By Zorn's lemma there exists a maximal element (E, σ_E, μ_E) in I. We claim that E is an algebraic closure. Since by definition of I the extension E/F is algebraic, it suffices to show that E is algebraically closed.

To see this we argue by contradiction. Namely, suppose that E is not algebraically closed. Then there exists an irreducible polynomial P over E of degree > 1, see Lemma 9.10.2. By Lemma 9.8.5 we obtain a nontrivial finite extension $E' = E[x]/(P)$. Observe that E'/F is algebraic by Lemma 9.8.8. Thus the cardinality of E' is $\leq \max(\aleph_0, |F|)$. By elementary set theory we can extend the given injection $E \subset S$ to an injection $E' \to S$. In other words, we may think of E' as an element of our set I contradicting the maximality of E. This contradiction completes the proof. □

Lemma 9.10.5. *Let F be a field. Let $\overline{F}$ be an algebraic closure of F. Let M/F be an algebraic extension. Then there is a morphism of F-extensions $M \to \overline{F}$.*

Proof. Consider the set I of pairs (E, φ) where $F \subset E \subset M$ is a subextension and $\varphi : E \to \overline{F}$ is a morphism of F-extensions. We partially order the set I by declaring $(E, \varphi) \leq (E', \varphi')$ if and only if $E \subset E'$ and $\varphi'|_E = \varphi$. If $T = \{(E_t, \varphi_t)\} \subset I$ is a totally ordered subset, then $\bigcup \varphi_t : \bigcup E_t \to \overline{F}$ is an element of I. Thus every totally ordered subset of I has an upper bound. By Zorn's lemma there exists a maximal element (E, φ) in I. We claim that $E = M$, which will finish the proof. If not, then pick $\alpha \in M$, $\alpha \notin E$. The α is algebraic over E, see Lemma 9.8.4. Let P be the minimal polynomial of α over E. Let P^φ be the image of P by φ in $\overline{F}[x]$. Since $\overline{F}$ is algebraically closed there is a root β of P^φ in $\overline{F}$. Then we can extend φ to $\varphi' : E(\alpha) = E[x]/(P) \to \overline{F}$ by mapping x to β. This contradicts the maximality of (E, φ) as desired. □

Lemma 9.10.6. *Any two algebraic closures of a field are isomorphic.*

Proof. Let F be a field. If M and $\overline{F}$ are algebraic closures of F, then there exists a morphism of F-extensions $\varphi : M \to \overline{F}$ by Lemma 9.10.5. Now the image $\varphi(M)$ is algebraically closed. On the other hand, the extension $\varphi(M) \subset \overline{F}$ is algebraic by Lemma 9.8.4. Thus $\varphi(M) = \overline{F}$. □

9.11. Relatively prime polynomials

Let K be an algebraically closed field. Then the ring $K[x]$ has a very simple ideal structure as we saw in Lemma 9.10.2. In particular, every polynomial $P \in K[x]$ can be written as

$$P = c(x - \alpha_1) \ldots (x - \alpha_n),$$

where c is the constant term and the $\alpha_1, \ldots, \alpha_n \in k$ are the roots of P (counted with multiplicity). Clearly, he only irreducible polynomials in $K[x]$ are the linear polynomials $c(x - \alpha)$, $c, \alpha \in K$ (and $c \neq 0$).

Definition 9.11.1. If k is any field, we say that two polynomials in $k[x]$ are *relatively prime* if they generate the unit ideal in $k[x]$.

Continuing the discussion above, if K is an algebraically closed field, two polynomials in $K[x]$ are relatively prime if and only if they have no common roots. This follows because the maximal ideals of $K[x]$ are of the form $(x - \alpha)$, $\alpha \in K$. So if $F, G \in K[x]$ have no common root, then (F, G) cannot be contained in any $(x - \alpha)$ (as then they would have a common root at α).

If k is *not* algebraically closed, then this still gives information about when two polynomials in $k[x]$ generate the unit ideal.

Lemma 9.11.2. *Two polynomials in $k[x]$ are relatively prime precisely when they have no common roots in an algebraic closure $\overline{k}$ of k.*

Proof. The claim is that any two polynomials P, Q generate (1) in $k[x]$ if and only if they generate (1) in $\overline{k}[x]$. This is a piece of linear algebra: a system of linear equations with coefficients in k has a solution if and only if it has a solution in any extension of k. Consequently, we can reduce to the case of an algebraically closed field, in which case the result is clear from what we have already proved. □

9.12. Separable extensions

In characteristic p something funny happens with irreducible polynomials over fields. We explain this in the following lemma.

Lemma 9.12.1. *Let F be a field. Let $P \in F[x]$ be an irreducible polynomial over F. Let $P' = dP/dx$ be the derivative of P with respect to x. Then one of the following two cases happens*

(1) *P and P' are relatively prime, or*
(2) *P' is the zero polynomial.*

Then second case can only happen if F has characteristic $p > 0$. In this case $P(x) = Q(x^q)$ where $q = p^f$ is a power of p and $Q \in F[x]$ is an irreducible polynomial such that Q and Q' are relatively prime.

Proof. Note that P' has degree $< \deg(P)$. Hence if P and P' are not relatively prime, then $(P, P') = (R)$ where R is a polynomial of degree $< \deg(P)$ contradicting the irreducibily of P. This proves we have the dichotomy between (1) and (2).

Assume we are in case (2) and $P = a_d x^d + \ldots + a_0$. Then $P' = d a_d x^{d-1} + \ldots + a_1$. In characteristic 0 we see that this forces $a_d, \ldots, a_1 = 0$ which would mean P is constant a contradiction. Thus we conclude that the characteristic p is positive. In this case the condition $P' = 0$ forces $a_i = 0$ whenever $p \not| i$. In other words, $P(x) = P_1(x^p)$ for some nonconstant polynomial P_1. Clearly, P_1 is irreducible as well. By induction on the degree we see that $P_1(x) = Q(x^q)$ as in the statement of the lemma, hence $P(x) = Q(x^{pq})$ and the lemma is proved. □

Definition 9.12.2. Let F be a field. Let K/F be an extension of fields.

(1) We say an irreducible polynomial P over F is *separable* if it is relatively prime to its derivative.
(2) Given $\alpha \in K$ algebraic over F we say α is *separable* over F if its minimal polynomial is separable over F.
(3) If K is an algebraic extension of F, we say K is *separable*[1] over F if every element of K is separable over F.

By Lemma 9.12.1 in characteristic 0 every irreducible polynomial is separable, every algebraic element in an extension is separable, and every algebraic extension is separable.

Lemma 9.12.3. *Let $K/E/F$ be a tower of algebraic field extensions.*

(1) *If $\alpha \in K$ is separable over F, then α is separable over E.*
(2) *if K is separable over F, then K is separable over E.*

[1]For nonalgebraic extensions this definition does not make sense and is not the correct one.

Proof. We will use Lemma 9.12.1 without further mention. Let P be the minimal polynomial of α over F. Let Q be the minimal polynomial of α over E. Then Q divides P in the polynomial ring $E[x]$, say $P = QR$. Then $P' = Q'R + QR'$. Thus if $Q' = 0$, then Q divides P and P' hence $P' = 0$ by the lemma. This proves (1). Part (2) follows immediately from (1) and the definitions. □

Lemma 9.12.4. *Let F be a field. An irreducible polynomial P over F is separable if and only if P has pairwise distinct roots in an algebraic closure of F.*

Proof. Suppose that $\alpha \in F$ is a root of both P and P'. Then $P = (x - \alpha)Q$ for some polynomial Q. Taking derivatives we obtain $P' = Q + (x - \alpha)Q'$. Thus α is a root of Q. Hence we see that if P and P' have a common root, then P does not have pairwise distinct roots. Conversely, if P has a repeated root, i.e., $(x-\alpha)^2$ divides P, then α is a root of both P and P'. Combined with Lemma 9.11.2 this proves the lemma. □

Lemma 9.12.5. *Let F be a field and let $\overline{F}$ be an algebraic closure of F. Let $p > 0$ be the characteristic of F. Let P be a polynomial over F. Then the set of roots of P and $P(x^p)$ in $\overline{F}$ have the same cardinality (not counting multiplicity).*

Proof. Clearly, α is a root of $P(x^p)$ if and only if α^p is a root of P. In other words, the roots of $P(x^p)$ are the roots of $x^p - \beta$, where β is a root of P. Thus it suffices to show that the map $\overline{F} \to \overline{F}$, $\alpha \mapsto \alpha^p$ is bijective. It is surjective, as $\overline{F}$ is algebraically closed which means that every element has a pth root. It is injective because $\alpha^p = \beta^p$ implies $(\alpha - \beta)^p = 0$ because the characteristic is p. And of course in a field $x^p = 0$ implies $x = 0$. □

Let F be a field and let P be an irreducible polynomial over F. Then we know that $P = Q(x^q)$ for some separable irreducible polynomial Q (Lemma 9.12.1) where q is a power of the characteristic p (and if the characteristic is zero, then $q = 1$[2] and $Q = P$). By Lemma 9.12.5 the number of roots of P and Q in any algebraic closure of F is the same. By Lemma 9.12.4 this number is equal to the degree of Q.

Definition 9.12.6. Let F be a field. Let P be an irreducible polynomial over F. The *separable degree* of P is the cardinality of the set of roots of P in any algebraic closure of F (see discussion above). Notation $\deg_s(P)$.

The separable degree of P always divides the degree and the quotient is a power of the characteristic. If the characteristic is zero, then $\deg_s(P) = \deg(P)$.

Situation 9.12.7. Here F be a field and K/F is a finite extension generated by elements $\alpha_1, \ldots, \alpha_n \in K$. We set $K_0 = F$ and

$$K_i = F(\alpha_1, \ldots, \alpha_i)$$

to obtain a tower of finite extensions $K = K_r/K_{r-1}/\ldots/K_0 = F$. Denote P_i the minimal polynomial of α_i over K_{i-1}. Finally, we fix an algebraic closure $\overline{F}$ of F.

Let F, K, α_i, and $\overline{F}$ be as in Situation 9.12.7. Suppose that $\varphi : K \to \overline{F}$ is a morphism of extensions of F. Then we obtain maps $\varphi_i : K_i \to \overline{F}$. In particular, we can take the image of $P_i \in K_{i-1}[x]$ by φ_{i-1} to get a polynomial $P_i^\varphi \in \overline{F}[x]$.

[2] A good convention for this chapter is to set $0^0 = 1$.

Lemma 9.12.8. *In Situation 9.12.7 the correspondence*

$$\mathrm{Mor}_F(K, \overline{F}) \longrightarrow \{(\beta_1, \ldots, \beta_n) \text{ as below}\}, \quad \varphi \longmapsto (\varphi(\alpha_1), \ldots, \varphi(\alpha_n))$$

is a bijection. Here the right hand side is the set of n-tuples $(\beta_1, \ldots, \beta_n)$ of elements of $\overline{F}$ such that β_i is a root of P_i^φ.

Proof. Let $(\beta_1, \ldots, \beta_n)$ be an element of the right hand side. We construct a map of fields corresponding to it by induction. Namely, we set $\varphi_0 : K_0 \to \overline{F}$ equal to the given map $K_0 = F \subset \overline{F}$. Having constructed $\varphi_{i-1} : K_{i-1} \to \overline{F}$ we observe that $K_i = K_{i-1}[x]/(P_i)$. Hence we can set φ_i equal to the unique map $K_i \to \overline{F}$ inducing φ_{i-1} on K_{i-1} and mapping x to β_i. This works precisely as β_i is a root of P_i^φ. Uniqueness implies that the two constructions are mutually inverse. □

Lemma 9.12.9. *In Situation 9.12.7 we have* $|\mathrm{Mor}_F(K, \overline{F})| = \prod_{i=1}^n \deg_s(P_i)$.

Proof. This follows immediately from Lemma 9.12.8. Observe that a key ingredient we are tacitly using here is the well-definedness of the separable degree of an irreducible polynomial which was observed just prior to Definition 9.12.6. □

We now use the result above to characterize separable field extensions.

Lemma 9.12.10. *Assumptions and notation as in Situation 9.12.7. If each P_i is separable, i.e., each α_i is separable over K_{i-1}, then*

$$|\mathrm{Mor}_F(K, \overline{F})| = [K : F]$$

and the field extension K/F is separable. If one of the α_i is not separable over K_{i-1}, then $|\mathrm{Mor}_F(K, \overline{F})| < [K : F]$.

Proof. If α_i is separable over K_{i-1} then $\deg_s(P_i) = \deg(P_i) = [K_i : K_{i-1}]$ (last equality by Lemma 9.9.2). By multiplicativity (Lemma 9.7.6) we have

$$[K : F] = \prod [K_i : K_{i-1}] = \prod \deg(P_i) = \prod \deg_s(P_i) = |\mathrm{Mor}_F(K, \overline{F})|$$

where the last equality is Lemma 9.12.9. By the exact same argument we get the strict inequality $|\mathrm{Mor}_F(K, \overline{F})| < [K : F]$ if one of the α_i is not separable over K_{i-1}.

Finally, assume again that each α_i is separable over K_{i-1}. Let $\gamma = \gamma_1 \in K$ be arbitrary. Then we can find additional elements $\gamma_2, \ldots, \gamma_m$ such that $K = F(\gamma_1, \ldots, \gamma_m)$ (for example we could take $\gamma_2 = \alpha_1, \ldots, \gamma_{n+1} = \alpha_n$). Then we see by the last part of the lemma (already proven above) that if γ is not separable over F we would have the strict inequality $|\mathrm{Mor}_F(K, \overline{F})| < [K : F]$ contradicting the very first part of the lemma (already prove above as well). □

Lemma 9.12.11. *Let K/F be a finite extension of fields. Let $\overline{F}$ be an algebraic closure of F. Then we have*

$$|\mathrm{Mor}_F(K, \overline{F})| \leq [K : F]$$

with equality if and only if K is separable over F.

Proof. This is a corollary of Lemma 9.12.10. Namely, since K/F is finite we can find finitely many elements $\alpha_1, \ldots, \alpha_n \in K$ generating K over F (for example we can choose the α_i to be a basis of K over F). If K/F is separable, then each α_i is separable over $F(\alpha_1, \ldots, \alpha_{i-1})$ by Lemma 9.12.3 and we get equality by Lemma 9.12.10. On the other hand, if we have equality, then no matter how we choose $\alpha_1, \ldots, \alpha_n$ we get that α_1 is separable over F by Lemma 9.12.10. Since we can

start the sequence with an arbitrary element of K it follows that K is separable over F. □

Lemma 9.12.12. *Let E/k and F/E be separable algebraic extensions of fields. Then F/k is a separable extension of fields.*

Proof. Choose $\alpha \in F$. Then α is separable algebraic over E. Let $P = x^d + \sum_{i<d} a_i x^i$ be the minimal polynomial of α over E. Each a_i is separable algebraic over k. Consider the tower of fields

$$k \subset k(a_0) \subset k(a_0, a_1) \subset \ldots \subset k(a_0, \ldots, a_{d-1}) \subset k(a_0, \ldots, a_{d-1}, \alpha)$$

Because a_i is separable algebraic over k it is separable algebraic over $k(a_0, \ldots, a_{i-1})$ by Lemma 9.12.3. Finally, α is separable algebraic over $k(a_0, \ldots, a_{d-1})$ because it is a root of P which is irreducible (as it is irreducible over the possibly bigger field E) and separable (as it is separable over E). Thus $k(a_0, \ldots, a_{d-1}, \alpha)$ is separable over k by Lemma 9.12.10 and we conclude that α is separable over k as desired. □

Lemma 9.12.13. *Let E/k be a field extension. Then the elements of E separable over k form a subextension of E/k.*

Proof. Let $\alpha, \beta \in E$ be separable over k. Then β is separable over $k(\alpha)$ by Lemma 9.12.3. Thus we can apply Lemma 9.12.3 to $k(\alpha, \beta)$ to see that $k(\alpha, \beta)$ is separable over k. □

9.13. Purely inseparable extensions

Purely inseparable extensions are the opposite of the separable extensions defined in the previous section. These extensions only show up in positive characteristic.

Definition 9.13.1. Let F be a field of characteristic $p > 0$. Let K/F be an extension.

(1) An element $\alpha \in K$ is *purely inseparable* over F if there exists a power q of p such that $\alpha^q \in F$.
(2) The extension K/F is said to be *purely inseparable* if and only if every element of K is purely inseparable over F.

Observe that a purely inseparable extension is necessarily algebraic. Let F be a field of characteristic $p > 0$. An example of a purely inseparable extension is gotten by adjoining the pth root of an element $t \in F$ which does not yet have one. Namely, the lemma below shows that $P = x^p - t$ is irreducible, and hence

$$K = F[x]/(P) = F[t^{1/p}]$$

is a field. And K is purely inseparable over F because every element

$$a_0 + a_1 t^{1/p} + \ldots + a_{p-1} t^{p-1/p}, a_i \in F$$

has pth power equal to

$$(a_0 + a_1 t^{1/p} + \ldots + a_{p-1} t^{p-1/p})^p = a_0^p + a_1^p t + \ldots + a_{p-1}^p t^{p-1} \in F$$

This situation occurs for the field $\mathbf{F}_p(t)$ of rational functions over $\mathbf{F}_p$.

Lemma 9.13.2. *Let p be a prime number. Let F be a field of characteristic p. Let $t \in F$ be an element which does not have a pth root in F. Then the polynomial $x^p - t$ is irreducible over F.*

Proof. To see this, suppose that we have a factorization $x^p - t = fg$. Taking derivatives we get $f'g + fg' = 0$. Note that neither $f' = 0$ nor $g' = 0$ as the degrees of f and g are smaller than p. Moreover, $\deg(f') < \deg(f)$ and $\deg(g') < \deg(g)$. We conclude that f and g have a factor in common. Thus if $x^p - t$ is reducible, then it is of the form $x^p - t = cf^n$ for some irreducible f, $c \in F^*$, and $n > 1$. Since p is a prime number this implies $n = p$ and f linear, which would imply $x^p - t$ has a root in F. Contradiction. □

We will see that taking pth roots is a very important operation in characteristic p.

Lemma 9.13.3. *Let E/k and F/E be purely inseparable extensions of fields. Then F/k is a purely inseparable extension of fields.*

Proof. Say the characteristic of k is p. Choose $\alpha \in F$. Then $\alpha^q \in E$ for some p-power q. Whereupon $(\alpha^q)^{q'} \in k$ for some p-power q'. Hence $\alpha^{qq'} \in k$. □

Lemma 9.13.4. *Let E/k be a field extension. Then the elements of E purely-inseparable over k form a subextension of E/k.*

Proof. Let p be the characteristic of k. Let $\alpha, \beta \in E$ be purely inseparable over k. Say $\alpha^q \in k$ and $\beta^{q'} \in k$ for some p-powers q, q'. If q'' is a p-power, then $(\alpha + \beta)^{q''} = \alpha^{q''} + \beta^{q''}$. Hence if $q'' \geq q, q'$, then we conclude that $\alpha + \beta$ is purely inseparable over k. Similarly for the difference, product and quotient of α and β. □

Lemma 9.13.5. *Let E/F be a finite purely inseparable field extension of characteristic $p > 0$. Then there exists a sequence of elements $\alpha_1, \ldots, \alpha_n \in E$ such that we obtain a tower of fields*

$$E = F(\alpha_1, \ldots, \alpha_n) \supset F(\alpha_1, \ldots, \alpha_{n-1}) \supset \ldots \supset F(\alpha_1) \supset F$$

such that each intermediate extension is of degree p and comes from adjoining a pth root. Namely, $\alpha_i^p \in F(\alpha_1, \ldots, \alpha_{i-1})$ is an element which does not have a pth root in $F(\alpha_1, \ldots, \alpha_{i-1})$ for $i = 1, \ldots, n$.

Proof. By induction on the degree of E/F. If the degree of the extension is 1 then the result is clear (with $n = 0$). If not, then choose $\alpha \in E$, $\alpha \notin F$. Say $\alpha^{p^r} \in F$ for some $r > 0$. Pick r minimal and replace α by $\alpha^{p^{r-1}}$. Then $\alpha \notin F$, but $\alpha^p \in F$. Then $t = \alpha^p$ is not a pth power in F (because that would imply $\alpha \in F$, see Lemma 9.12.5 or its proof). Thus $F \subset F(\alpha)$ is a subextension of degree p (Lemma 9.13.2). By induction we find $\alpha_1, \ldots, \alpha_n \in E$ generating $E/F(\alpha)$ satisfying the conclusions of the lemma. The sequence $\alpha, \alpha_1, \ldots, \alpha_n$ does the job for the extension E/F. □

Lemma 9.13.6. *Let E/F be an algebraic field extension. There exists a unique subextension $F \subset E_{sep} \subset E$ such that E_{sep}/F is separable and E/E_{sep} is purely inseparable.*

Proof. If the characteristic is zero we set $E_{sep} = E$. Assume the characteristic if $p > 0$. Let E_{sep} be the set of elements of E which are separable over F. This is a subextension by Lemma 9.12.13 and of course E_{sep} is separable over F. Given an α in E there exists a p-power q such that α^q is separable over F. Namely, q is that power of p such that the minimal polynomial of α is of the form $P(x^q)$ with P separable algebraic, see Lemma 9.12.1. Hence E/E_{sep} is purely inseparable. Uniqueness is clear. □

Definition 9.13.7. Let E/F be an algebraic field extension. Let E_{sep} be the subextension found in Lemma 9.13.6.

(1) The integer $[E_{sep} : F]$ is called the *separable degree* of the extension. Notation $[E : F]_s$.
(2) The integer $[E : E_{sep}]$ is called the *inseparable degree*, or the *degree of inseparability* of the extension. Notation $[E : F]_i$.

Of course in characteristic 0 we have $[E : F] = [E : F]_s$ and $[E : F]_i = 1$. By multipliciativity (Lemma 9.7.6) we have

$$[E : F] = [E : F]_s[E : F]_i$$

even in case some of these degrees are infinite. In fact, the separable degree and the inseparable degree are multiplicative too (see Lemma 9.13.9).

Lemma 9.13.8. *Let K/F be a finite extension. Let $\overline{F}$ be an algebraic closure of F. Then $[K : F]_s = |\operatorname{Mor}_F(K, \overline{F})|$.*

Proof. We first prove this when K/F is purely inseparable. Namely, we claim that in this case there is a unique map $K \to \overline{F}$. This can be seen by choosing a sequence of elements $\alpha_1, \ldots, \alpha_n \in K$ as in Lemma 9.13.5. The irreducible polynmial of α_i over $F(\alpha_1, \ldots, \alpha_{i-1})$ is $x^p - \alpha_i^p$. Applying Lemma 9.12.9 we see that $|\operatorname{Mor}_F(K, \overline{F})| = 1$. On the other hand, $[K : F]_s = 1$ in this case hence the equality holds.

Let's return to a general finite extension K/F. In this case choose $F \subset K_s \subset K$ as in Lemma 9.13.6. By Lemma 9.12.11 we have $|\operatorname{Mor}_F(K_s, \overline{F})| = [K_s : F] = [K : F]_s$. On the other hand, every field map $\sigma' : K_s \to \overline{F}$ extends to a unique field map $\sigma : K \to \overline{F}$ by the result of the previous paragraph. In other words $|\operatorname{Mor}_F(K, \overline{F})| = |\operatorname{Mor}_F(K_s, \overline{F})|$ and the proof is done. □

Lemma 9.13.9 (Multiplicativity)**.** *Suppose given a tower of algebraic field extensions $K/E/F$. Then*

$$[K : F]_s = [K : E]_s[E : F]_s \quad \text{and} \quad [K : F]_i = [K : E]_i[E : F]_i$$

Proof. We first prove this in case K is finite over F. Since we have multiplicativity for the usual degree (by Lemma 9.7.6) it suffices to prove one of the two formulas. By Lemma 9.13.8 we have $[K : F]_s = |\operatorname{Mor}_F(K, \overline{F})|$. By the same lemma, given any $\sigma \in \operatorname{Mor}_F(E, \overline{F})$ the number of extensions of σ to a map $\tau : K \to \overline{F}$ is $[K : E]_s$. Namely, via $E \cong \sigma(E) \subset \overline{F}$ we can view $\overline{F}$ as an algebraic closure of E. Combined with the fact that there are $[E : F]_s = |\operatorname{Mor}_F(E, \overline{F})|$ choices for σ we obtain the result.

If the extensions are infinite one can write K as the union of all finite subextension $F \subset K' \subset K$. For each K' we set $E' = E \cap K'$. Then we have the formulas of the lemma for $K'/E'/F$ by the first paragraph. Since $[K : F]_s = \sup\{[K' : F]_s\}$ and similarly for the other degrees (some details omitted) we obtain the result in general. □

9.14. Normal extensions

Let $P \in F[x]$ be a nonconstant polynomial over a field F. We say P *splits completely into linear factors over F* or *splits completely over F* if there exist $c \in F^*$, $n \geq 1$,

$\alpha_1, \ldots, \alpha_n \in F$ such that

$$P = c(x - \alpha_1) \ldots (x - \alpha_n)$$

in $F[x]$. Normal extensions are defined as follows.

Definition 9.14.1. Let E/F be an algebraic field extension. We say E is *normal* over F if for all $\alpha \in E$ the minimal polynomial P of α over F splits completely into linear factors over E.

As in the case of separable extensions, it takes a bit of work to establish the basic properties of this notion.

Lemma 9.14.2. *Let $K/E/F$ be a tower of algebraic field extensions. If K is normal over F, then K is normal over E.*

Proof. Let $\alpha \in K$. Let P be the minimal polynomial of α over F. Let Q be the minimal polynomial of α over E. Then Q divides P in the polynomial ring $E[x]$, say $P = QR$. Hence, if P splits completely over K, then so does Q. □

Lemma 9.14.3. *Let F be a field. Let M/F be an algebraic extension. Let $F \subset E_i \subset M$, $i \in I$ be subextensions with E_i/F normal. Then $\bigcap E_i$ is normal over F.*

Proof. Direct from the definitions. □

Lemma 9.14.4. *Let E/F be an algebraic extension of fields. Let $\overline{F}$ be an algebraic closure of F. The following are equivalent*

1. *E is normal over F, and*
2. *for every pair $\sigma, \sigma' \in \mathrm{Mor}_F(E, \overline{F})$ we have $\sigma(E) = \sigma'(E)$.*

Proof. Let $\mathcal{P}$ be the set of all minimal polynomials over F of all elements of E. Set

$$T = \{\beta \in \overline{F} \mid P(\beta) = 0 \text{ for some } P \in \mathcal{P}\}$$

It is clear that if E is normal over F, then $\sigma(E) = T$ for all $\sigma \in \mathrm{Mor}_F(E, \overline{F})$. Thus we see that (1) implies (2).

Conversely, assume (2). Pick $\beta \in T$. We can find a corresponding $\alpha \in E$ whose minimal polynomial $P \in \mathcal{P}$ annihilates β. Because $F(\alpha) = F[x]/(P)$ we can find an element $\sigma_0 \in \mathrm{Mor}_F(F(\alpha), \overline{F})$ mapping α to β. By Lemma 9.10.5 we can extend σ_0 to a $\sigma \in \mathrm{Mor}_F(E, \overline{F})$. Whence we see that β is in the common image of all embeddings $\sigma : E \to \overline{F}$. It follows that $\sigma(E) = T$ for any σ. Fix a σ. Now let $P \in \mathcal{P}$. Then we can write

$$P = (x - \beta_1) \ldots (x - \beta_n)$$

for some n and $\beta_i \in \overline{F}$ by Lemma 9.10.2. Observe that $\beta_i \in T$. Thus $\beta_i = \sigma(\alpha_i)$ for some $\alpha_i \in E$. Thus $P = (x - \alpha_1) \ldots (x - \alpha_n)$ splits completely over E. This finishes the proof. □

Definition 9.14.5. Let E/F be an extension of fields. Then $\mathrm{Aut}(E/F)$ or $\mathrm{Aut}_F(E)$ denotes automorphisms group of E as an object of the category of F-extensions. Elements of $\mathrm{Aut}(E/F)$ are called *automorphisms of E over F* or *automorphisms of E/F*.

Here is a characterization of normal extensions in terms of automorphisms.

Lemma 9.14.6. *Let E/F be a finite extension. We have*

$$|Aut(E/F)| \leq [E : F]_s$$

with equality if and only if E is normal over F.

Proof. Choose an algebraic closure $\overline{F}$ of F. Recall that $[E : F] = |\operatorname{Mor}_F(E, \overline{F})|$. Pick an element $\sigma_0 \in \operatorname{Mor}_F(E, \overline{F})$. Then the map

$$\operatorname{Aut}(E/F) \longrightarrow \operatorname{Mor}_F(E, \overline{F}), \quad \tau \longmapsto \sigma_0 \circ \tau$$

is injective. Thus the inequality. If equality holds, then every $\sigma \in \operatorname{Mor}_F(E, \overline{F})$ is gotten by precomposing σ_0 by an automorphism. Hence $\sigma(E) = \sigma_0(E)$. Thus E is normal over F by Lemma 9.14.4.

Conversely, assume that E/F is normal. Then by Lemma 9.14.4 we have $\sigma(E) = \sigma_0(E)$ for all $\sigma \in \operatorname{Mor}_F(E, \overline{F})$. Thus we get an automorphism of E over F by setting $\tau = \sigma_0^{-1} \circ \sigma$. Whence the map displayed above is surjective. □

9.15. Splitting fields

The following lemma is a useful tool for constructing normal field extensions.

Lemma 9.15.1. *Let F be a field. Let $P \in F[x]$ be a nonconstant polynomial. There exists a smallest field extension E/F such that P splits completely over E. Moreover, the field extension E/F is normal and unique up to (nonunique) isomorphism.*

Proof. Choose an algebraic closure $\overline{F}$. Then we can write $P = c(x-\beta_1)\ldots(x-\beta_n)$ in $\overline{F}[x]$, see Lemma 9.10.2. Note that $c \in F^*$. Set $E = F(\beta_1, \ldots, \beta_n)$. Then it is clear that E is minimal with the requirement that P splits completely over E.

Next, let E' be another minimal field extension of F such that P splits completely over E'. Write $P = c(x - \alpha_1)\ldots(x - \alpha_n)$ with $c \in F$ and $\alpha_i \in E'$. Again it follows from minimality that $E' = F(\alpha_1, \ldots, \alpha_n)$. Moreover, if we pick any $\sigma : E' \to \overline{F}$ (Lemma 9.10.5) then we immediately see thart $\sigma(\alpha_i) = \beta_{\tau(i)}$ for some permutation $\tau : \{1, \ldots, n\} \to \{1, \ldots, n\}$. Thus $\sigma(E') = E$. This implies that E' is a normal extension of F by Lemma 9.14.4 and that $E \cong E'$ as extensions of F thereby finishing the proof. □

Definition 9.15.2. Let F be a field. Let $P \in F[x]$ be a nonconstant polynomial. The field extension E/F constructed in Lemma 9.15.1 is called the *splitting field of P over F*.

The field constructed in the next lemma is sometimes called the *normal closure* of E over F.

Lemma 9.15.3. *Let E/F be a finite extension of fields. There exists a unique (up to nonunique isomorphism) smallest finite extension K/E such that K is normal over F.*

Proof. Choose generators $\alpha_1, \ldots, \alpha_n$ of E over F. Let $P_1, \ldots, P_n$ be the minimal polynomials of $\alpha_1, \ldots, \alpha_n$ over F. Set $P = P_1 \ldots P_n$. Observe that $(x-\alpha_1)\ldots(x-\alpha_n)$ divides P, since each $(x - \alpha_i)$ divides P_i. Say $P = (x - \alpha_1)\ldots(x - \alpha_n)Q$. Let K/E be the splitting field of P over E. We claim that K is the splitting field of P over F as well (which implies that K is normal over F). This is clear because

K/E is generated by the roots of Q over E and E is generated by the roots of $(x-\alpha_1)\ldots(x-\alpha_n)$ over F, hence K is generated by the roots of P over F.

Uniqueness. Suppose that K'/E is a second smallest extension such that K'/F is normal. Choose an algebraic closure $\overline{F}$ and an embedding $\sigma_0 : E \to \overline{F}$. By Lemma 9.10.5 we can extend σ_0 to $\sigma : K \to \overline{F}$ and $\sigma' : K' \to \overline{F}$. By Lemma 9.14.3 we see that $\sigma(K) \cap \sigma'(K')$ is normal over F. By minimality we conclude that $\sigma(K) = \sigma(K')$. Thus $\sigma \circ (\sigma')^{-1} : K' \to K$ gives an isomorphism of extensions of E. □

9.16. Roots of unity

Let F be a field. For an integer $n \geq 1$ we set

$$\mu_n(F) = \{\zeta \in F \mid \zeta^n = 1\}$$

This is called the *group of nth roots of unity* or *nth roots of* 1. It is an abelian group under multiplication with neutral element given by 1. Observe that in a field the number of roots of a polynomial of degree d is always at most d. Hence we see that $|\mu_n(F)| \leq n$ as it is defined by a polynomial equation of degree n. Of course every element of $\mu_n(F)$ has order dividing n. Moreover, the subgroups

$$\mu_d(F) \subset \mu_n(F), \quad d|n$$

each have at most d elements. This implies that $\mu_n(F)$ is cyclic.

Lemma 9.16.1. *Let A be an abelian group of exponent dividing n such that $\{x \in A \mid dx = 0\}$ has cardinality at most d for all $d|n$. Then A is cyclic of order dividing n.*

Proof. The conditions imply that $|A| \leq n$, in particular A is finite. The structure of finite abelian groups shows that $A = \mathbf{Z}/e_1\mathbf{Z} \oplus \ldots \oplus \mathbf{Z}/e_r\mathbf{Z}$ for some integers $1 < e_1|e_2|\ldots|e_r$. This would imply that $\{x \in A \mid e_x = 0\}$ has cardinality e_1^r. Hence $r = 1$. □

Applying this to the field $\mathbf{F}_p$ we obtain the celebrated result that the group $(\mathbf{Z}/p\mathbf{Z})^*$ is a cyclic group. More about this in the section on finite fields.

One more observation is often useful: If F has characteristic $p > 0$, then $\mu_{p^n}(F) = \{1\}$. This is true because raising to the pth power is an injective map on fields of characteristic p as we have seen in the proof of Lemma 9.12.5. (Of course, it also follows from the statement of that lemma itself.)

9.17. Finite fields

Let F be a finite field. It is clear that F has positive characteristic as we cannot have an injection $\mathbf{Q} \to F$. Say the characteristic of F is p. The extension $\mathbf{F}_p \subset F$ is finite. Hence we see that F has $q = p^f$ elements for some $f \geq 1$.

Let us think about the group of units F^*. This is a finite abelian group, so it has some exponent e. Then $F^* = \mu_e(F)$ and we see from the discussion in Section 9.16 that F^* is a cyclic group of order $q-1$. (A posteriori it follows that $e = q-1$ as well.) In particular, if $\alpha \in F^*$ is a generator then it clearly is true that

$$F = \mathbf{F}_p(\alpha)$$

In other words, the extension $F/\mathbf{F}_p$ is generated by a single element. Of course, the same thing is true for any extension of finite fields E/F (because E is already generated by a single element over the prime field).

9.18. Primitive elements

Let E/F be a finite extension of fields. An element $\alpha \in E$ is called a *primitive element of E over F* if $E = F(\alpha)$.

Lemma 9.18.1 (Primitive element)**.** *Let E/F be a finite extension of fields. The following are equivalent*

(1) *there exists a prmitive element for E over F, and*
(2) *there are finitely many subextensions $E/K/F$.*

Moreover, (1) and (2) hold if E/F is separable.

Proof. Let $\alpha \in E$ be a primitive element. Let P be the minimal polynomial of α over F. Let $E \subset M$ be a splitting field for P over E, so that $P(x) = (x-\alpha)(x-\alpha_2)\ldots(x-\alpha_n)$ over M. For ease of notation we set $\alpha_1 = \alpha$. Next, let $E/K/F$ be a subextension. Let Q be the minimal polynomial of α over K. Observe that $\deg(Q) = [E : K]$. Writing $Q = x^d + \sum_{i<d} a_i x^i$ we claim that K is equal to $L = F(a_0, \ldots, a_{d-1})$. Indeed α has degree d over L and $L \subset K$. Hence $[E : L] = [E : K]$ and it follows that $[K : L] = 1$, i.e., $K = L$. Thus it suffices to show there are at most finitely many possibilities for the polynomial Q. This is clear because we have a factorization $P = QR$ in $K[x]$ in particular in $E[x]$. Since we have unique factorization in $E[x]$ there are at most finitely many monic factors of P in $E[x]$.

If F is a finite field (equivalently E is a finite field), then E/F has a primitive element by the discussion in Section 9.17. Next, assume F is infinite and there are at most finitely many proper subfields $E/K/F$. List them, say $K_1, \ldots, K_N$. Then each $K_i \subset E$ is a proper sub F-vector space. As F is infinite we can find a vector $\alpha \in E$ with $\alpha \notin K_i$ for all i (a finite union of proper subvector spaces is never a subvector space; details omitted). Then α is a primitive element for E over F.

Having established the equivalence of (1) and (2) we now turn to the final statement of the lemma. Choose an algebraic closure $\overline{F}$ of F. Enumerate the elements $\sigma_1, \ldots, \sigma_n \in \mathrm{Mor}_F(E, \overline{F})$. Since E/F is separable we have $n = [E : F]$ by Lemma 9.12.11. Note that if $i \neq j$, then

$$V_{ij} = \mathrm{Ker}(\sigma_i - \sigma_j : E \longrightarrow \overline{F})$$

is not equal to E. Hence arguing as in the preceding paragraph we can find $\alpha \in E$ with $\alpha \notin V_{ij}$ for all $i \neq j$. It follows that $|\mathrm{Mor}_F(F(\alpha), \overline{F})| \geq n$. On the other hand $[F(\alpha) : F] \leq [E : F]$. Hence equality by Lemma 9.12.11 and we conclude that $E = F(\alpha)$. □

9.19. Galois theory

Here is the definition.

Definition 9.19.1. A field extension E/F is called *Galois* if it is algebraic, separable, and normal.

It turns out that a finite extension is Galois if and only if it has the "correct" number of automorphisms.

Lemma 9.19.2. *Let E/F be a finite extension of fields. Then E is Galois over F if and only if $|Aut(E/F)| = [E : F]$.*

Proof. Assume $|\mathrm{Aut}(E/F)| = [E : F]$. By Lemma 9.14.6 this implies that E/F is separable and normal, hence Galois. Conversely, if E/F is separable then $[E : F] = [E : F]_s$ and if E/F is in addition normal, then Lemma 9.14.6 implies that $|\mathrm{Aut}(E/F)| = [E : F]$. □

Motivated by the lemma above we introduce the Galois group as follows.

Definition 9.19.3. If E/F is a Galois extension, then the group $\mathrm{Aut}(E/F)$ is called the *Galois group* and it is denoted $\mathrm{Gal}(E/F)$.

It turns out that if L/K is an infinite Galois extension, then one should think of the Galois group as a topological group. We will return to this later (insert future reference here). In this chapter we mainly restrict ourselves to finite Galois extensions.

Lemma 9.19.4. *Let $K/E/F$ be a tower of algebraic field extensions. If K is Galois over F, then K is Galois over E.*

Proof. Combine Lemmas 9.14.2 and 9.12.3. □

Let G be a group acting on a field K (by field automorphisms). We will often use the notation

$$K^G = \{x \in K \mid \sigma(x) = x \ \forall \sigma \in G\}$$

and we will call this the *fixed field* for the action of G on K.

Lemma 9.19.5. *Let K be a field. Let G be a finite group acting faithfully on K. Then the extension K/K^G is Galois, we have $[K : K^G] = |G|$, and the Galois group of the extension is G.*

Proof. Given $\alpha \in K$ consider the orbit $G \cdot \alpha \subset K$ of α under the group action. Consider the polynomial

$$P = \prod\nolimits_{\beta \in G\cdot\alpha} (x - \beta) \in K[x]$$

The key to the whole lemma is that this polynomial is invariant under the action of G and hence has coefficients in K^G Namely, for $\sigma \in G$ we have

$$P^\sigma = \prod\nolimits_{\beta \in G\cdot\alpha} (x - \tau(\beta)) = \prod\nolimits_{\beta \in G\cdot\alpha} (x - \beta) = P$$

because the map $\beta \mapsto \tau(\beta)$ is a permutation of the orbit $G \cdot \alpha$. Thus $P \in K^G$. Since also $P(\alpha) = 0$ as α is an element of its orbit we conclude that the extension K/K^G is algebraic. Moreover, the minimal polynomial Q of α over K^G divides the polynomial P just constructed. Hence Q is separable (by Lemma 9.12.4 for example) and we conclude that K/K^G is separable. Thus K/K^G is Galois. To finish the proof it suffices to show that $[K : K^G] = |G|$ since then G will be the Galois group by Lemma 9.19.2.

Pick finitely many elements $\alpha_i \in K$, $i = 1, \ldots, n$ such that $\sigma(\alpha_i) = \alpha_i$ for $i = 1, \ldots, n$ implies σ is the neutral element of G. Set

$$L = K^G(\{\sigma(\alpha_i); 1 \leq i \leq n, \sigma \in G\}) \subset K$$

and observe that the action of G on K induces an action of G on L. We will show that L has degree $|G|$ over K^G. This will finish the proof, since if $L \subset K$ is proper,

then we can add an element $\alpha \in K$, $\alpha \notin L$ to our list of elements $\alpha_1, \ldots, \alpha_n$ without increasing L which is absurd. This reduces us to the case that K/K^G is finite which is treated in the next paragraph.

Assume K/K^G is finite. By Lemma 9.18.1 we can find $\alpha \in K$ such that $K = K^G(\alpha)$. By the construction in the first paragraph of this proof we see that α has degree at most $|G|$ over K. However, the degree cannot be less than $|G|$ as G acts faithfully on $K^G(\alpha) = L$ by construction and the inequality of Lemma 9.14.6. □

Theorem 9.19.6 (Fundamental theorem of Galois theory)**.** *Let L/K be a finite Galois extension with Galois group G. Then we have $K = L^G$ and the map*

$$\{\textit{subgroups of } G\} \longrightarrow \{\textit{subextensions } K \subset M \subset L\}, \quad H \longmapsto L^H$$

is a bijection whose inverse maps M to $Gal(L/M)$.

Proof. By Lemma 9.19.4 given a subextension $L/M/K$ the extension L/M is Galois. Of course L/M is also finite (Lemma 9.7.3). Thus $|\mathrm{Gal}(L/M)| = [L : M]$ by Lemma 9.19.2. Conversely, if $H \subset G$ is a finite subgroup, then $[L : L^H] = |H|$ by Lemma 9.19.5. It follows formally from these two observations that we obtain a bijective correspondence as in the theorem. □

9.20. The complex numbers

The fundamental theorem of algebra states that the complex numbers is an algebraically closed field. In this section we discuss this briefly.

The first remark we'd like to make is that you need to use a little bit of input from calculus in order to prove this. We will use the intuitively clear fact that every odd degree polynomial over the reals has a real root. Namely, let $P(x) = a_{2k+1}x^{2k+1} + \ldots + a_0 \in \mathbf{R}[x]$ for some $k \geq 0$ and $a_{2k+1} \neq 0$. We may and do assume $a_{2k+1} > 0$. Then for $x \in \mathbf{R}$ very large (positive) we see that $P(x) > 0$ as the term $a_{2k+1}x^{2k+1}$ dominates all the other terms. Similarly, if $x \ll 0$, then $P(x) < 0$ by the same reason (and this is where we use that the degree is odd). Hence by the intermediate value theorem there is an $x \in \mathbf{R}$ with $P(x) = 0$.

A conclusion we can draw from the above is the $\mathbf{R}$ has no nontrivial odd degree field extensions, as elements of such extensions would have odd degree minimal polynomials.

Next, let $K/\mathbf{R}$ be a finite Galois extension with Galois group G. Let $P \subset G$ be a 2-sylow subgroup. Then $K^P/\mathbf{R}$ is an odd degree extension, hence by the above $K^P = K$, which in turn implies $G = P$. (All of these arguments rely on Galois theory of course.) Thus G is a 2-group. If G is nontrivial, then we see that $\mathbf{C} \subset K$ as $\mathbf{C}$ is (up to isomorphism) the only degree degree 2 extension of $\mathbf{R}$. If G has more than 2 elements we would obtain a quadratic extension of $\mathbf{C}$. This is absurd as every complex number has a square root.

The conclusion: $\mathbf{C}$ is algebraically closed. Namely, if not then we'd get a nontrivial finite extension $\mathbf{C} \subset K$ which we could assume normal (hence Galois) over $\mathbf{R}$ by Lemma 9.15.3. But we've seen above that then $K = \mathbf{C}$.

Lemma 9.20.1 (Fundamental theorem of algebra)**.** *The field $\mathbf{C}$ is algebraically closed.*

Proof. See discussion above. □

9.21. Kummer extensions

Let K be a field. Let $n \geq 2$ be an integer such that K contains a primitive nth root of 1. Let $a \in K$. Let L be an extension of K obtained by adjoining a root b of the equation $x^n = a$. Then L/K is Galois. If $G = \mathrm{Gal}(L/K)$ is the Galois group, then the map

$$G \longrightarrow \mu_n(K), \quad \sigma \longmapsto \sigma(b)/b$$

is an injective homomorphism of groups. In particular, G is cyclic of order dividing n as a subgroup of the cyclic group $\mu_n(K)$. Kummer theory gives a converse.

Lemma 9.21.1 (Kummer extensions)**.** *Let $K \subset L$ be a Galois extension of fields whose Galois group is $\mathbf{Z}/n\mathbf{Z}$. Assume moreover that the characteristic of K is prime to n and that K contains a primitive nth root of 1. Then $L = K[z]$ with $z^n \in K$.*

Proof. Omitted. □

9.22. Artin-Schreier extensions

Let K be a field of characteristic $p > 0$. Let $a \in K$. Let L be an extension of K obtained by adjoining a root b of the equation $x^p - x = a$. Then L/K is Galois. If $G = \mathrm{Gal}(L/K)$ is the Galois group, then the map

$$G \longrightarrow \mathbf{Z}/p\mathbf{Z}, \quad \sigma \longmapsto \sigma(b) - b$$

is an injective homomorphism of groups. In particular, G is cyclic of order dividing p as a subgroup of $\mathbf{Z}/p\mathbf{Z}$. The theory of Artin-Schreier extensions gives a converse.

Lemma 9.22.1 (Artin-Schreier extensions)**.** *Let $K \subset L$ be a Galois extension of fields of characteristic $p > 0$ with Galois group $\mathbf{Z}/p\mathbf{Z}$. Then $L = K[z]$ with $z^p - z \in K$.*

Proof. Omitted. □

9.23. Transcendence

We recall the standard definitions.

Definition 9.23.1. Let $k \subset K$ be a field extension.

(1) A collection of elements $\{x_i\}_{i \in I}$ of K is called *algebraically independent* over k if the map
$$k[X_i; i \in I] \longrightarrow K$$
which maps X_i to x_i is injective.
(2) The field of fractions of a polynomial ring $k[x_i; i \in I]$ is denoted $k(x_i; i \in I)$.
(3) A *purely transcendental extension* of k is any field extension $k \subset K$ isomorphic to the field of fractions of a polynomial ring over k.
(4) A *transcendence basis* of K/k is a collection of elements $\{x_i\}_{i \in I}$ which are algebraically independent over k and such that the extension $k(x_i; i \in I) \subset K$ is algebraic.

Example 9.23.2. The field $\mathbf{Q}(\pi)$ is purely transcendental because π isn't the root of an nonzero polynomial with rational coefficients. In particular, $\mathbf{Q}(\pi) \cong \mathbf{Q}(x)$.

Lemma 9.23.3. *Let E/F be a field extension. A transcendence basis of K over k exists. Any two transcendence bases have the same cardinality.*

Proof. Let A be an algebraically independent subset of E. Let G be a subset of E containing A that generates E/F. We claim we can find a transcendence basis B such that $A \subset B \subset G$. To prove this consider the collection of algebraically independent subsets $\mathcal{B}$ whose members are subsets of G that contain A. Define a partial ordering on $\mathcal{B}$ using inclusion. Then $\mathcal{B}$ contains at least one element A. The union of the elements of a totally ordered subset T of $\mathcal{B}$ is an algebraically independent subset of E over F since any algebraic dependence relation would have occurred in one of the elements of T (since polynomials only involve finitely many variables). The union also contains A and is contained in G. By Zorn's lemma, there is a maximal element $B \in \mathcal{B}$. Now we claim E is algebraic over EFB). This is because if it wasn't then there would be an element $f \in G$ transcendental over $F(B)$ since $E(G) = F$. Then $B \cup \{f\}$ wold be algebraically independent contradicting the maximality of B. Thus B is our transcendence basis.

Let B and B' be two transcendence bases. Without loss of generality, we can assume that $|B'| \leq |B|$. Now we divide the proof into two cases: the first case is that B is an infinite set. Then for each $\alpha \in B'$, there is a finite set B_α such that α is algebraic over $E(B_\alpha)$ since any algebraic dependence relation only uses finitely many indeterminates. Then we define $B^* = \bigcup_{\alpha \in B'} B_\alpha$. By construction, $B^* \subset B$, but we claim that in fact the two sets are equal. To see this, suppose that they are not equal, say there is an element $\beta \in B \setminus B^*$. We know β is algebraic over $E(B')$ which is algebraic over $E(B^*)$. Therefore β is algebraic over $E(B^*)$, a contradiction. So $|B| \leq |\bigcup_{\alpha \in B'} B_\alpha|$. Now if B' is finite, then so is B so we can assume B' is infinite; this means

$$|B| \leq |\bigcup\nolimits_{\alpha \in B'} B_\alpha| = |B'|$$

because each B_α is finite and B' is infinite. Therefore in the infinite case, $|B| = |B'|$.

Now we need to look at the case where B is finite. In this case, B' is also finite, so suppose $B = \{\alpha_1, \ldots, \alpha_n\}$ and $B' = \{\beta_1, \ldots, \beta_m\}$ with $m \leq n$. We perform induction on m: if $m = 0$ then E/F is algebraic so $B = \emptyset$ so $n = 0$. If $m > 0$, there is an irreducible polynomial $f \in E[x, y_1, \ldots, y_n]$ such that $f(\beta_1, \alpha_1, \ldots, \alpha_n) = 0$ and such that x occurs in f. Since β_1 is not algebraic over F, f must involve some y_i so without loss of generality, assume f uses y_1. Let $B^* = \{\beta_1, \alpha_2, \ldots, \alpha_n\}$. We claim that B^* is a basis for E/F. To prove this claim, we see that we have a tower of algebraic extensions

$$E/F(B^*, \alpha_1)/F(B^*)$$

since α_1 is algebraic over $F(B^*)$. Now we claim that B^* (counting multiplicity of elements) is algebraically independent over E because if it weren't, then there would be an irreducible $g \in E[x, y_2, \ldots, y_n]$ such that $g(\beta_1, \alpha_2, \ldots, \alpha_n) = 0$ which must involve x making β_1 algebraic over $E(\alpha_2, \ldots, \alpha_n)$ which would make α_1 algebraic over $E(\alpha_2, \ldots, \alpha_n)$ which is impossible. So this means that $\{\alpha_2, \ldots, \alpha_n\}$ and $\{\beta_2, \ldots, \beta_m\}$ are bases for E over $F(\beta_1)$ which means by induction, $m = n$. □

Definition 9.23.4. Let $k \subset K$ be a field extension. The *transcendence degree* of K over k is the cardinality of a transcendence basis of K over k. It is denoted $\mathrm{trdeg}_k(K)$.

Lemma 9.23.5. *Let $k \subset K \subset L$ be field extensions. Then*

$$trdeg_k(L) = trdeg_K(L) + trdeg_k(K).$$

Proof. Choose a transcendence basis $A \subset K$ of K over k. Choose a transcendence basis $B \subset L$ of L over K. Then it is straightforward to see that $A \cup B$ is a transcendence basis of L over k. □

Example 9.23.6. Consider the field extension $\mathbf{Q}(e, \pi)$ formed by adjoining the numbers e and π. This field extension has transcendence degree at least 1 since both e and π are transcendental over the rationals. However, this field extension might have transcendence degree 2 if e and π are algebraically independent. Whether or not this is true is unknown and whence the problem of determining $trdeg(\mathbf{Q}(e, \pi))$ is open.

Example 9.23.7. Let F be a field and $E = F(t)$. Then $\{t\}$ is a transcendence basis since $E = F(t)$. However, $\{t^2\}$ is also a transcendence basis since $F(t)/F(t^2)$ is algebraic. This illustrates that while we can always decompose an extension E/F into an algebraic extension E/F' and a purely transcendental extension F'/F, this decomposition is not unique and depends on choice of transcendence basis.

Example 9.23.8. Let X be a compact Riemann surface. Then the function field $\mathbf{C}(X)$ (see Example 9.3.6) has transcendence degree one over $\mathbf{C}$. In fact, *any* finitely generated extension of $\mathbf{C}$ of transcendence degree one arises from a Riemann surface. There is even an equivalence of categories between the category of compact Riemann surfaces and (non-constant) holomorphic maps and the opposite category of finitely generated extensions of $\mathbf{C}$ and morphisms of $\mathbf{C}$-algebras. See [**For91**].

There is an algebraic version of the above statement as well. Given an (irreducible) algebraic curve in projective space over an algebraically closed field k (e.g. the complex numbers), one can consider its "field of rational functions:" basically, functions that look like quotients of polynomials, where the denominator does not identically vanish on the curve. There is a similar anti-equivalence of categories (insert future reference here) between smooth projective curves and non-constant morphisms of curves and finitely generated extensions of k of transcendence degree one. See [**Har77**].

Definition 9.23.9. Let $k \subset K$ be a field extension.

1. The *algebraic closure of k in K* is the subfield k' of K consisting of elements of K which are algebraic over k.
2. We say k is *algebraically closed in K* if every element of K which is algebraic over k is contained in k.

Lemma 9.23.10. *Let $k \subset K$ be a finitely generated field extension. The algebraic closure of k in K is finite over k.*

Proof. Let $x_1, \ldots, x_r \in K$ be a transcendence basis for K over k. Then $n = [K : k(x_1, \ldots, x_r)] < \infty$. Suppose that $k \subset k' \subset K$ with k'/k finite. In this case $[k'(x_1, \ldots, x_r) : k(x_1, \ldots, x_r)] = [k' : k] < \infty$. Hence

$$[k' : k] = [k'(x_1, \ldots, x_r) : k(x_1, \ldots, x_r)] < [K : k(x_1, \ldots, x_r)] = n.$$

In other words, the degrees of finite subextensions are bounded and the lemma follows. □

9.24. Linearly disjoint extensions

Let k be a field, K and L field extensions of k. Suppose also that K and L are embedded in some larger field Ω.

Definition 9.24.1. Consider a diagram

$$\begin{array}{ccc} L & \longrightarrow & \Omega \\ \uparrow & & \uparrow \\ k & \longrightarrow & K \end{array} \tag{9.24.1.1}$$

of field extensions. The *compositum of K and L in Ω* written KL is the smallest subfield of Ω containing both L and K.

It is clear that KL is generated by the set $K \cup L$ over k, generated by the set K over L, and generated by the set L over K.

Warning: The (isomorphism class of the) composition depends on the choice of the embeddings of K and L into Ω. For example consider the number fields $K = \mathbf{Q}(2^{1/8}) \subset \mathbf{R}$ and $L = \mathbf{Q}(2^{1/12}) \subset \mathbf{R}$. The compositum inside $\mathbf{R}$ is the field $\mathbf{Q}(2^{1/24})$ of degree 24 over Q. However, if we embed $K = Q[x]/(x^8 - 2)$ into $\mathbf{C}$ by mapping x to $2^{1/8}e^{2\pi i/8}$, then the compositum $Q(2^{1/12}, 2^{1/8}e^{2\pi i/8})$ contains $i = e^{2\pi i/4}$ and has degree 48 over $\mathbf{Q}$ (we omit showing the degree is 48, but the existence of i certainly proves the two composita are not isomorphic).

Definition 9.24.2. Consider a diagram of fields as in (9.24.1.1). We say K and L are said to be *linearly disjoint over k in Ω* if the map

$$K \otimes_k L \longrightarrow KL, \quad \sum x_i \otimes y_i \longmapsto \sum x_i y_i$$

is injective.

The following lemma does not seem to fit anywhere else.

Lemma 9.24.3. *Let E/F be a normal algebraic field extension. There exist subextensions $E/E_{sep}/F'$ and $E/E_{insep}/F$ such that*

1. *$F \subset E_{sep}$ is Galois and $E_{sep} \subset E$ is purely inseparable,*
2. *$F \subset E_{insep}$ is purely inseparable and $E_{insep} \subset E$ is Galois,*
3. *$E = E_{sep} \otimes_F E_{insep}$.*

Proof. We found the subfield E_{sep} in Lemma 9.13.6. We set $E_{insep} = E^{\mathrm{Aut}(E/F)}$. Details omitted. □

9.25. Review

In this section we give a quick review of what has transpired above.

Let $k \subset K$ be a field extension. Let $\alpha \in K$. Then we have the following possibilities:

1. The element α is transcendental over k.
2. The element α is algebraic over k. Denote $P(T) \in k[T]$ its *minimal polynomial*. This is a monic polynomial $P(T) = T^d + a_1T^{d-1} + \ldots + a_d$ with coefficients in k. It is irreducible and $P(\alpha) = 0$. These properties uniquely determine P, and the integer d is called the *degree of α over k*. There are two subcases:

(a) The polynomial $\mathrm{d}P/\mathrm{d}T$ is not identically zero. This is equivalent to the condition that $P(T) = \prod_{i=1,\ldots,d}(T - \alpha_i)$ for pairwise distinct elements $\alpha_1, \ldots, \alpha_d$ in the algebraic closure of k. In this case we say that α is *separable* over k.
(b) The $\mathrm{d}P/\mathrm{d}T$ is identically zero. In this case the characteristic p of k is > 0, and P is actually a polynomial in T^p. Clearly there exists a largest power $q = p^e$ such that P is a polynomial in T^q. Then the element α^q is separable over k.

Definition 9.25.1. Algebraic field extensions.

(1) A field extension $k \subset K$ is called *algebraic* if every element of K is algebraic over k.
(2) An algebraic extension $k \subset k'$ is called *separable* if every $\alpha \in k'$ is separable over k.
(3) An algebraic extension $k \subset k'$ is called *purely inseparable* if the characteristic of k is $p > 0$ and for every element $\alpha \in k'$ there exists a power q of p such that $\alpha^q \in k$.
(4) An algebraic extension $k \subset k'$ is called *normal* if for every $\alpha \in k'$ the minimal polynomial $P(T) \in k[T]$ of α over k splits completely into linear factors over k'.
(5) An algebraic extension $k \subset k'$ is called *Galois* if it is separable and normal.

The following lemma does not seem to fit anywhere else.

Lemma 9.25.2. *Let K be a field of characteristic $p > 0$. Let $K \subset L$ be a separable algebraic extension. Let $\alpha \in L$.*

(1) *If the coefficients of the minimal polynomial of α over K are pth powers in K then α is a pth power in L.*
(2) *More generally, if $P \in K[T]$ is a polynomial such that (a) α is a root of P, (b) P has pairwise distinct roots in an algebraic closure, and (c) all coefficients of P are pth powers, then α is a pth power in L.*

Proof. It follows from the definitions that (2) implies (1). Assume P is as in (2). Write $P(T) = \sum_{i=0}^{d} a_i T^{d-i}$ and $a_i = b_i^p$. The polynomial $Q(T) = \sum_{i=0}^{d} b_i T^{d-i}$ has distinct roots in an algebraic closure as well, because the roots of Q are the pth roots of the roots of P. If α is not a pth power, then $T^p - \alpha$ is an irreducible polynomial over L (Lemma 9.13.2). Moreover Q and $T^p - \alpha$ have a root in common in an algebraic closure $\overline{L}$. Thus Q and $T^p - \alpha$ are not relatively prime, which implies $T^p - \alpha | Q$ in $L[T]$. This contradicts the fact that the roots of Q are pairwise distinct. □

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CHAPTER 10

Commutative Algebra

10.1. Introduction

Basic commutative algebra will be explained in this document. A reference is [**Mat70**].

10.2. Conventions

A ring is commutative with 1. The zero ring is a ring. In fact it is the only ring that does not have a prime ideal. The Kronecker symbol δ_{ij} will be used. If $R \to S$ is a ring map and $\mathfrak{q}$ a prime of S, then we use the notation "$\mathfrak{p} = R \cap \mathfrak{q}$" to indicate the prime which is the inverse image of $\mathfrak{q}$ under $R \to S$ even if R is not a subring of S and even if $R \to S$ is not injective.

10.3. Basic notions

The following notions are considered basic and will not be defined, and or proved. This does not mean they are all necessarily easy or well known.

(1) R is a *ring*,
(2) $x \in R$ is *nilpotent*,
(3) $x \in R$ is a *zerodivisor*,
(4) $x \in R$ is a *unit*,
(5) $e \in R$ is an *idempotent*,
(6) an idempotent $e \in R$ is called *trivial* if $e = 1$ or $e = 0$,
(7) $\varphi : R_1 \to R_2$ is a *ring homomorphism*,
(8) $\varphi : R_1 \to R_2$ is *of finite presentation*, or R_2 *is a finitely presented* R_1*-algebra*, see Definition 10.6.1,
(9) $\varphi : R_1 \to R_2$ is *of finite type*, or R_2 *is a finite type* R_1*-algebra*, see Definition 10.6.1,
(10) $\varphi : R_1 \to R_2$ is *finite*, or R_2 *is a finite* R_1*-algebra*,
(11) R is a *(integral) domain*,
(12) R is *reduced*,
(13) R is *Noetherian*,
(14) R is a *principal ideal domain* or a *PID*,
(15) R is a *Euclidean domain*,
(16) R is a *unique factorization domain* or a *UFD*,
(17) R is a *discrete valuation ring* or a *dvr*,
(18) K is a *field*,
(19) $K \subset L$ is a *field extension*,
(20) $K \subset L$ is an *algebraic field extension*,
(21) $\{t_i\}_{i \in I}$ is a *transcendence basis* for L over K,
(22) the *transcendence degree* $\text{trdeg}(L/K)$ of L over K,

(23) the field k is *algebraically closed*,
(24) if $K \subset L$ is algebraic, and $K \subset k$ an extension with k algebraically closed, then there exists a map ring map $L \to k$ extending the map on K,
(25) $I \subset R$ is an *ideal*,
(26) $I \subset R$ is *radical*,
(27) if I is an ideal then we have its *radical* $\sqrt{I}$,
(28) $I \subset R$ is *nilpotent* means that $I^n = 0$ for some $n \in \mathbf{N}$,
(29) $I \subset R$ is *locally nilpotent* means that every element of I is nilpotent,
(30) $\mathfrak{p} \subset R$ is a *prime ideal*,
(31) if $\mathfrak{p} \subset R$ is prime and if $I, J \subset R$ are ideal, and if $IJ \subset \mathfrak{p}$, then $I \subset \mathfrak{p}$ or $J \subset \mathfrak{p}$.
(32) $\mathfrak{m} \subset R$ is a *maximal ideal*,
(33) any nonzero ring has a maximal ideal,
(34) the *Jacobson radical* of R is $\text{rad}(R) = \bigcap_{\mathfrak{m} \subset R} \mathfrak{m}$ the intersection of all the maximal ideals of R,
(35) the ideal (T) *generated* by a subset $T \subset R$,
(36) the *quotient ring* R/I,
(37) an ideal I in the ring R is prime if and only if R/I is a domain,
(38) an ideal I in the ring R is maximal if and only if the ring R/I is a field,
(39) if $\varphi : R_1 \to R_2$ is a ring homomorphism, and if $I \subset R_2$ is an ideal, then $\varphi^{-1}(I)$ is an ideal of R_1,
(40) if $\varphi : R_1 \to R_2$ is a ring homomorphism, and if $I \subset R_1$ is an ideal, then $\varphi(I) \cdot R_2$ (sometimes denoted $I \cdot R_2$, or IR_2) is the ideal of R_2 generated by $\varphi(I)$,
(41) if $\varphi : R_1 \to R_2$ is a ring homomorphism, and if $\mathfrak{p} \subset R_2$ is a prime ideal, then $\varphi^{-1}(\mathfrak{p})$ is a prime ideal of R_1,
(42) M is an *R-module*,
(43) for $m \in M$ the *annihilator* $I = \{f \in R \mid fm = 0\}$ of m in R,
(44) $N \subset M$ is an *R-submodule*,
(45) M is an *Noetherian R-module*,
(46) M is a *finite R-module*,
(47) M is a *finitely generated R-module*,
(48) M is a *finitely presented R-module*,
(49) M is a *free R-module*,
(50) if $0 \to K \to L \to M \to 0$ is a short exact sequence of R-modules and K, M are free, then L is free,
(51) if $N \subset M \subset L$ are R-modules, then $L/M = (L/N)/(M/N)$,
(52) S is a *multiplicative subset of R*,
(53) the *localization* $R \to S^{-1}R$ of R,
(54) if R is a ring and S is a multiplicative subset of R then $S^{-1}R$ is the zero ring if and only if S contains 0,
(55) if R is a ring and if the multiplicative subset S consists completely of nonzerodivisors, then $R \to S^{-1}R$ is injective,
(56) if $\varphi : R_1 \to R_2$ is a ring homomorphism, and S is a multiplicative subsets of R_1, then $\varphi(S)$ is a multiplicative subset of R_2,
(57) if S, S' are multiplicative subsets of R, and if SS' denotes the set of products $SS' = \{r \in R \mid \exists s \in S, \exists s' \in S', r = ss'\}$ then SS' is a multiplicative subset of R,

(58) if S, S' are multiplicative subsets of R, and if $\overline{S}$ denotes the image of S in $(S')^{-1}R$, then $(SS')^{-1}R = \overline{S}^{-1}((S')^{-1}R)$,
(59) the *localization* $S^{-1}M$ of the R-module M,
(60) the functor $M \mapsto S^{-1}M$ preserves injective maps, surjective maps, and exactness,
(61) if S, S' are multiplicative subsets of R, and if M is an R-module, then $(SS')^{-1}M = S^{-1}((S')^{-1}M)$,
(62) if R is a ring, I and ideal of R and S a multiplicative subset of R, then $S^{-1}I$ is an ideal of $S^{-1}R$, and we have $S^{-1}R/S^{-1}I = \overline{S}^{-1}(R/I)$, where $\overline{S}$ is the image of S in R/I,
(63) if R is a ring, and S a multiplicative subset of R, then any ideal I' of $S^{-1}R$ is of the form $S^{-1}I$, where one can take I to be the inverse image of I' in R,
(64) if R is a ring, M an R-module, and S a multiplicative subset of R, then any submodule N' of $S^{-1}M$ is of the form $S^{-1}N$ for some submodule $N \subset M$, where one can take N to be the inverse image of N' in M,
(65) if $S = \{1, f, f^2, \ldots\}$ then $R_f = S^{-1}R$ and $M_f = S^{-1}M$,
(66) if $S = R \setminus \mathfrak{p} = \{x \in R \mid x \notin \mathfrak{p}\}$ for some prime ideal $\mathfrak{p}$, then it is customary to denote $R_\mathfrak{p} = S^{-1}R$ and $M_\mathfrak{p} = S^{-1}M$,
(67) a *local ring* is a ring with exactly one maximal ideal,
(68) a *semi-local ring* is a ring with finitely many maximal ideals,
(69) if $\mathfrak{p}$ is a prime in R, then $R_\mathfrak{p}$ is a local ring with maximal ideal $\mathfrak{p}R_\mathfrak{p}$,
(70) the *residue field*, denoted $\kappa(\mathfrak{p})$, of the prime $\mathfrak{p}$ in the ring R is the quotient $R_\mathfrak{p}/\mathfrak{p}R_\mathfrak{p} = (R \setminus \mathfrak{p})^{-1}R/\mathfrak{p}$,
(71) given R and M_1, M_2 the *tensor product* $M_1 \otimes_R M_2$,
(72) etc.

10.4. Snake lemma

The snake lemma and its variants are discussed in the setting of abelian categories in Homology, Section 12.5.

Lemma 10.4.1. *Suppose given a commutative diagram*

$$\begin{array}{ccccccc} & & X & \longrightarrow & Y & \longrightarrow & Z \longrightarrow 0 \\ & & \downarrow{\scriptstyle \alpha} & & \downarrow{\scriptstyle \beta} & & \downarrow{\scriptstyle \gamma} \\ 0 & \longrightarrow & U & \longrightarrow & V & \longrightarrow & W \end{array}$$

of abelian groups with exact rows, then there is a canonical exact sequence

$$Ker(\alpha) \to Ker(\beta) \to Ker(\gamma) \to Coker(\alpha) \to Coker(\beta) \to Coker(\gamma)$$

Moreover, if $X \to Y$ is injective, then the first map is injective, and if $V \to W$ is surjective, then the last map is surjective.

Proof. The map $\partial : \mathrm{Ker}(\gamma) \to \mathrm{Coker}(\alpha)$ is defined as follows. Take $z \in \mathrm{Ker}(\gamma)$. Choose $y \in Y$ mapping to z. Then $\beta(y) \in V$ maps to zero in W. Hence $\beta(y)$ is the image of some $u \in U$. Set $\partial z = \overline{u}$ the class of u in the cokernel of α. Proof of exactness is omitted. □

10.5. Finite modules and finitely presented modules

Just some basic notation and lemmas.

Definition 10.5.1. Let R be a ring. Let M be an R-module

(1) We say M is a *finite R-module*, or a *finitely generated R-module* if there exist $n \in \mathbf{N}$ and $x_1, \ldots, x_n \in M$ such that every element of M is a R-linear combination of the x_i. Equivalently, this means there exists a surjection $R^{\oplus n} \to M$ for some $n \in \mathbf{N}$.
(2) We say M is a *finitely presented R-module* or an *R-module of finite presentation* if there exist integers $n, m \in \mathbf{N}$ and an exact sequence

$$R^{\oplus m} \longrightarrow R^{\oplus n} \longrightarrow M \longrightarrow 0$$

Informally this means that M is finitely generated and that the module of relations among these generators is finitely generated as well. A choice of an exact sequence as in the definition is called a *presentation* of M.

Lemma 10.5.2. *Let R be a ring. Let $\alpha : R^{\oplus n} \to M$ and $\beta : N \to M$ be module maps. If $Im(\alpha) \subset Im(\beta)$, then there exists an R-module map $\gamma : R^{\oplus n} \to N$ such that $\alpha = \beta \circ \gamma$.*

Proof. Let $e_i = (0, \ldots, 0, 1, 0, \ldots, 0)$ be the ith basis vector of $R^{\oplus n}$. Let $x_i \in N$ be an element with $\alpha(e_i) = \beta(x_i)$ which exists by assumption. Set $\gamma(a_1, \ldots, a_n) = \sum a_i x_i$. By construction $\alpha = \beta \circ \gamma$. □

Lemma 10.5.3. *Let R be a ring. Let*

$$0 \to M_1 \to M_2 \to M_3 \to 0$$

be a short exact sequence of R-modules.

(1) *If M_1 and M_3 are finite R-modules, then M_2 is a finite R-module.*
(2) *If M_1 and M_3 are finitely presented R-modules, then M_2 is a finitely presented R-module.*
(3) *If M_2 is a finite R-module, then M_3 is a finite R-module.*
(4) *If M_2 is a finitely presented R-module and M_1 is a finite R-module, then M_3 is a finitely presented R-module.*
(5) *If M_3 is a finitely presented R-module and M_2 is a finite R-module, then M_1 is a finite R-module.*

Proof. Proof of (1). If $x_1, \ldots, x_n$ are generators of M_1 and $y_1, \ldots, y_m \in M_2$ are elements whose images in M_3 are generators of M_3, then $x_1, \ldots, x_n, y_1, \ldots, y_m$ generate M_2.

Part (3) is immediate from the definition.

Proof of (5). Assume M_3 is finitely presented and M_2 finite. Choose a presentation

$$R^{\oplus m} \to R^{\oplus n} \to M_3 \to 0$$

By Lemma 10.5.2 there exists a map $R^{\oplus n} \to M_2$ such that the solid diagram

$$\begin{array}{ccccccccc} & & R^{\oplus m} & \longrightarrow & R^{\oplus n} & \longrightarrow & M_3 & \longrightarrow & 0 \\ & & \vdots & & \downarrow & & \downarrow{\scriptstyle \mathrm{id}} & & \\ 0 & \longrightarrow & M_1 & \longrightarrow & M_2 & \longrightarrow & M_3 & \longrightarrow & 0 \end{array}$$

commutes. This produces the dotted arrow. By the snake lemma (Lemma 10.4.1) we see that we get an isomorphism

$$\mathrm{Coker}(R^{\oplus m} \to M_1) \cong \mathrm{Coker}(R^{\oplus n} \to M_2)$$

In particular we conclude that $\mathrm{Coker}(R^{\oplus m} \to M_1)$ is a finite R-module. Since $\mathrm{Im}(R^{\oplus m} \to M_1)$ is finite by (3), we see that M_1 is finite by part (1).

Proof of (4). Assume M_2 is finitely presented and M_1 is finite. Choose a presentation $R^{\oplus m} \to R^{\oplus n} \to M_2 \to 0$. Choose a surjection $R^{\oplus k} \to M_1$. By Lemma 10.5.2 there exists a factorization $R^{\oplus k} \to R^{\oplus n} \to M_1$ of this map. Then $R^{\oplus k+m} \to R^{\oplus n} \to M_3 \to 0$ is a presentation.

Proof of (2). Assume that M_1 and M_3 are finitely presented. The argument in the proof of part (1) produces a commutative diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & R^{\oplus n} & \longrightarrow & R^{\oplus n+m} & \longrightarrow & R^{\oplus m} & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & M_1 & \longrightarrow & M_2 & \longrightarrow & M_3 & \longrightarrow & 0 \end{array}$$

with surjective vertical arrows. By the snake lemma we obtain a short exact sequence

$$0 \to \mathrm{Ker}(R^{\oplus n} \to M_1) \to \mathrm{Ker}(R^{\oplus n+m} \to M_2) \to \mathrm{Ker}(R^{\oplus m} \to M_3) \to 0$$

By part (5) we see that the outer two modules are finite. Hence the middle one is finite too. By (4) we see that M_2 is of finite presentation. □

Lemma 10.5.4. *Let R be a ring, and let M be a finite R-module. There exists a filtration by R-submodules*

$$0 = M_0 \subset M_1 \subset \ldots \subset M_n = M$$

such that each quotient M_i/M_{i-1} is isomorphic to R/I_i for some ideal I_i of R.

Proof. By induction on the number of generators of M. Let $x_1, \ldots, x_r \in M$ be a minimal number of generators. Let $M' = Rx_1 \subset M$. Then M/M' has $r-1$ generators and the induction hypothesis applies. And clearly $M' \cong R/I_1$ with $I_1 = \{f \in R \mid fx_1 = 0\}$. □

Lemma 10.5.5. *Let $R \to S$ be a ring map. Let M be an S-module. If M is finite as an R-module, then M is finite as an S-module.*

Proof. In fact, any R-generating set of M is also an S-generating set of M, since the R-module structure is induced by the image of R in S. □

10.6. Ring maps of finite type and of finite presentation

Definition 10.6.1. Let $R \to S$ be a ring map.

(1) We say $R \to S$ is of *finite type*, or that *S is a finite type R-algebra* if there exists an $n \in \mathbf{N}$ and an surjection of R-algebras $R[x_1, \ldots, x_n] \to S$.
(2) We say $R \to S$ is of *finite presentation* if there exist integers $n, m \in \mathbf{N}$ and polynomials $f_1, \ldots, f_m \in R[x_1, \ldots, x_n]$ and an isomorphism of R-algebras $R[x_1, \ldots, x_n]/(f_1, \ldots, f_m) \cong S$.

Informally this means that S is finitely generated as an R-algebra and that the ideal of relations among the generators is finitely generated. A choice of a surjection $R[x_1, \ldots, x_n] \to S$ as in the definition is sometimes called a *presentation* of S.

Lemma 10.6.2. *The notions finite type and finite presentation have the following permanence properties.*

(1) *A composition of ring maps of finite type is of finite type.*
(2) *A composition of ring maps of finite presentation is of finite presentation.*
(3) *Given $R \to S' \to S$ with $R \to S$ of finite type, then $S' \to S$ is of finite type.*
(4) *Given $R \to S' \to S$, with $R \to S$ of finite presentation, and $R \to S'$ of finite type, then $S' \to S$ is of finite presentation.*

Proof. We only prove the last assertion. Write $S = R[x_1, \ldots, x_n]/(f_1, \ldots, f_m)$ and $S' = R[y_1, \ldots, y_a]/I$. Say that the class $\bar{y}_i$ of y_i maps to h_i mod $(f_1, \ldots, f_m)$ in S. Then it is clear that $S' = S[x_1, \ldots, x_n]/(f_1, \ldots, f_m, h_1 - \bar{y}_1, \ldots, h_m - \bar{y}_m)$. □

Lemma 10.6.3. *Let $R \to S$ be a ring map of finite presentation. For any surjection $\alpha : R[x_1, \ldots, x_n] \to S$ the kernel of α is a finitely generated ideal in $R[x_1, \ldots, x_n]$.*

Proof. Write $S = R[y_1, \ldots, y_m]/(f_1, \ldots, f_k)$. Choose $g_i \in R[y_1, \ldots, y_m]$ which are lifts of $\alpha(x_i)$. Then we see that $S = R[x_i, y_j]/(f_j, x_i - g_i)$. Choose $h_j \in R[x_1, \ldots, x_n]$ such that $\alpha(h_j)$ corresponds to y_j mod $(f_1, \ldots, f_k)$. Consider the map $\psi : R[x_i, y_j] \to R[x_i]$, $x_i \mapsto x_i$, $y_j \mapsto h_j$. Then the kernel of α is the image of $(f_j, x_i - g_i)$ under ψ and we win. □

Lemma 10.6.4. *Let $R \to S$ be a ring map. Let M be an S-module. Assume $R \to S$ is of finite type and M is finitely presented as an R-module. Then M is finitely presented as an S-module.*

Proof. This is similar to the proof of part (4) of Lemma 10.6.2. We may assume $S = R[x_1, \ldots, x_n]/J$. Choose $y_1, \ldots, y_m \in M$ which generate M as an R-module and choose relations $\sum a_{ij} y_j = 0$, $i = 1, \ldots, t$ which generate the kernel of $R^{\oplus n} \to M$. For any $i = 1, \ldots, n$ and $j = 1, \ldots, m$ write

$$x_i y_j = \sum a_{ijk} y_k$$

for some $a_{ijk} \in R$. Consider the S-module N generated by $y_1, \ldots, y_m$ subject to the relations $\sum a_{ij} y_j = 0$, $i = 1, \ldots, t$ and $x_i y_j = \sum a_{ijk} y_k$, $i = 1, \ldots, n$ and $j = 1, \ldots, m$. Then N has a presentation

$$S^{\oplus nm+t} \longrightarrow S^{\oplus m} \longrightarrow N \longrightarrow 0$$

By construction there is a surjective map $\varphi : N \to M$. To finish the proof we show φ is injective. Suppose $z = \sum b_j y_j \in N$ for some $b_j \in S$. We may think of b_j as a polynomial in $x_1, \ldots, x_n$ with coefficients in R. By applying the relations of the form $x_i y_j = \sum a_{ijk} y_k$ we can inductively lower the degree of the polynomials. Hence we see that $z = \sum c_j y_j$ for some $c_j \in R$. Hence if $\varphi(z) = 0$ then the vector $(c_1, \ldots, c_m)$ is an R-linear combination of the vectors $(a_{i1}, \ldots, a_{im})$ and we conclude that $z = 0$ as desired. □

10.7. Finite ring maps

Definition 10.7.1. Let $\varphi : R \to S$ be a ring map. We say $\varphi : R \to S$ is *finite* if S is finite as an R-module.

Lemma 10.7.2. *Let $R \to S$ be a finite ring map. Let M be an S-module. Then M is finite as an R-module if and only if M is finite as an S-module.*

Proof. One of the implications follows from Lemma 10.5.5. To see the other assume that M is finite as an S-module. Pick $x_1, \ldots, x_n \in S$ which generate S as an R-module. Pick $y_1, \ldots, y_m \in M$ which generate M as an S-module. Then $x_j y_j$ generate M as an R-module. □

Lemma 10.7.3. *Suppose that $R \to S$ and $S \to T$ are finite ring maps. Then $R \to T$ is finite.*

Proof. If t_i generate T as an S-module and s_j generate S as an R-module, then $t_i s_j$ generate T as an R-module. (Also follows from Lemma 10.7.2.) □

Lemma 10.7.4. *Let $R \to S$ be a finite and finitely presented ring map. Let M be an S-module. Then M is finitely presented as an R-module if and only if M is finitely presented as an S-module.*

Proof. One of the implications follows from Lemma 10.6.4. To see the other assume that M is finitely presented as an S-module. Pick a presentation

$$S^{\oplus m} \longrightarrow S^{\oplus n} \longrightarrow M \longrightarrow 0$$

As S is finite as an R-module, the kernel of $S^{\oplus n} \to M$ is a finite R-module. Thus from Lemma 10.5.3 we see that it suffices to prove that S is finitely presented as an R-module.

Pick $x_1, \ldots, x_n \in S$ which generate S as an R-module. Write $x_i^2 = \sum a_{ik} x_k$ for some $a_{ik} \in R$. Let $J = \mathrm{Ker}(R[X_1, \ldots, X_n] \to S)$ where $R[X_1, \ldots, X_n] \to S$ is the R-algebra map determined by $X_i \mapsto x_i$. Let $h_i = X_i^2 - \sum a_{ik} X_k$ which is an element of J. Let $I = (h_1, \ldots, h_n)$ so that $I \subset J$. Note that

$$S' = R[X_1, ..., X_n]/I$$

is free as an R-module with basis the I-congruence classes of $X^E = X_1^{e_1} \ldots X_n^{e_n}$ where $E = (e_1, \ldots, e_n)$ is a multi-index with $e_i \in \{0, 1\}$. (Details omitted.) By Lemma 10.6.3 there exist finitely many $g_1, \ldots, g_m \in J$ such that $J = (g_1, \ldots, g_m)$. Denote g_j' the image of g_j in S'. Then $S = S'/(g_1', \ldots, g_m')$. Hence as an R-module we have

$$S = S' / \sum\nolimits_{j=1,\ldots,m} \sum\nolimits_{E=(e_1,\ldots,e_n),\ e_i \in \{0,1\}} R X^E g_j'$$

which is a finitely presented R-module. □

10.8. Colimits

Some of the material in this section overlaps with the general discussion on colimits in Categories, Sections 4.14 – 4.21.

Definition 10.8.1. A *partially ordered set* is a set I together with a relation $\leq$ which is associative (if $i \leq j$ and $j \leq k$ then $i \leq k$) and reflexive ($i \leq i$ for all $i \in I$). A *directed set* $(I, \leq)$ is a partially ordered set $(I, \leq)$ such that I is not empty and such that $\forall i, j \in I$, there exists $k \in I$ with $i \leq k, j \leq k$.

It is customary to drop the $\leq$ from the notation when talking about a partially ordered set. This is the same as the notion defined in Categories, Section 4.21.

Definition 10.8.2. Let $(I, \leq)$ be a partially ordered set. A *system* (M_i, μ_{ij}) *of R-modules over I* consists of a family of R-modules $\{M_i\}_{i \in I}$ indexed by I and a family of R-module maps $\{\mu_{ij} : M_i \to M_j\}_{i \leq j}$ such that for all $i \leq j \leq k$

$$\mu_{ii} = \mathrm{id}_{M_i} \tag{10.8.2.1}$$

$$\mu_{ik} = \mu_{jk} \circ \mu_{ij} \tag{10.8.2.2}$$

We say (M_i, μ_{ij}) is a *directed system* if I is a directed set.

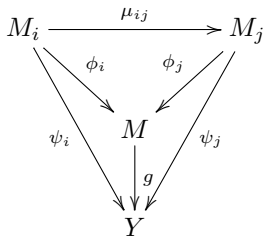
This is the same as the notion defined in Categories, Definition 4.21.1 and Section 4.21. We refer to Categories, Definition 4.14.2 for the definition of a colimit of a diagram/system in any category.

Lemma 10.8.3. *Let (M_i, μ_{ij}) be a system of R-modules over the partially ordered set I. The colimit of the system (M_i, μ_{ij}) is the quotient R-module $(\bigoplus_{i \in I} M_i)/Q$ where Q is the R-submodule generated by all elements*

$$\iota_i(x_i) - \iota_j(\mu_{ij}(x_i))$$

where $\iota_i : M_i \to \bigoplus_{i \in I} M_i$ is the natural inclusion. We denote the colimit $M = \mathrm{colim}_i M_i$. We denote $\pi : \bigoplus_{i \in I} M_i \to M$ the projection map and $\phi_i = \pi \circ \iota_i : M_i \to M$.

Proof. This lemma is a special case of Categories, Lemma 4.14.11 but we will also prove it directly in this case. Namely, note that $\phi_i = \phi_j \circ \mu_{ij}$ in the above construction. To show the pair (M, ϕ_i) is the colimit we have to show it satisfies the universal property: for any other such pair (Y, ψ_i) with $\psi_i : M_i \to Y$, $\psi_i = \psi_j \circ \mu_{ij}$, there is a unique R-module homomorphism $g : M \to Y$ such that the following diagram commutes:



And this is clear because we can define g by taking the map ψ_i on the summand M_i in the direct sum $\bigoplus M_i$. □

Lemma 10.8.4. *Let (M_i, μ_{ij}) be a system of R-modules over the partially ordered set I. Assume that I is directed. The colimit of the system (M_i, μ_{ij}) is canonically isomorphic to the module M defined as follows:*

(1) *as a set let*

$$M = \left(\coprod\nolimits_{i \in I} M_i\right) / \sim$$

where for $m \in M_i$ and $m' \in M_{i'}$ we have

$$m \sim m' \Leftrightarrow \mu_{ij}(m) = \mu_{i'j}(m') \text{ for some } j \geq i, i'$$

(2) *as an abelian group for $m \in M_i$ and $m' \in M_{i'}$ we define the sum of the classes of m and m' in M to be the class of $\mu_{ij}(m) + \mu_{i'j}(m')$ where $j \in I$ is any index with $i \leq j$ and $i' \leq j$, and*
(3) *as an R-module define for $m \in M_i$ and $x \in R$ the product of x and the class of m in M to be the class of xm in M.*

The canonical maps $\phi_i : M_i \to M$ are induced by the canonical maps $M_i \to \coprod_{i \in I} M_i$.

Proof. Omitted. Compare with Categories, Section 4.19. □

Lemma 10.8.5. *Let (M_i, μ_{ij}) be a directed system. Let $M = \operatorname{colim} M_i$ with $\mu_i : M_i \to M$, then $\mu_i(x_i) = 0$ for $x_i \in M_i$ if and only if there exists $j \geq i$ such that $\mu_{ij}(x_i) = 0$.*

Proof. This is clear from the description of the directed colimit in Lemma 10.8.4. □

Example 10.8.6. Consider the partially ordered set $I = \{a, b, c\}$ with $a < b$ and $a < c$ and no other strict inequalities. A system $(M_a, M_b, M_c, \mu_{ab}, \mu_{ac})$ over I consists of three R-modules M_a, M_b, M_c and two R-module homomorphisms $\mu_{ab} : M_a \to M_b$ and $\mu_{ac} : M_a \to M_c$. The colimit of the system is just

$$M := \operatorname{colim}_{i \in I} M_i = \operatorname{Coker}(M_a \to M_b \oplus M_c)$$

where the map is $\mu_{ab} \oplus -\mu_{ac}$. Thus the kernel of the canonical map $M_a \to M$ is $\operatorname{Ker}(\mu_{ab}) + \operatorname{Ker}(\mu_{ac})$. And the kernel of the canonical map $M_b \to M$ is the image of $\operatorname{Ker}(\mu_{ac})$ under the map μ_{ab}. Hence clearly the result of Lemma 10.8.5 is false for general systems.

Definition 10.8.7. Let (M_i, μ_{ij}), (N_i, ν_{ij}) be systems of R-modules over the same partially ordered set I. A *homomorphism of systems* Φ from (M_i, μ_{ij}) to (N_i, ν_{ij}) is by definition a family of R-module homomorphisms $\phi_i : M_i \to N_i$ such that $\phi_j \circ \mu_{ij} = \nu_{ij} \circ \phi_i$ for all $i \leq j$.

This is the same notion as a transformation of functors between the associated diagrams $M : I \to \operatorname{Mod}_R$ and $N : I \to \operatorname{Mod}_R$, in the language of categories. The following lemma is a special case of Categories, Lemma 4.14.7.

Lemma 10.8.8. *Let (M_i, μ_{ij}), (N_i, ν_{ij}) be systems of R-modules over the same partially ordered set. A morphism of systems $\Phi = (\phi_i)$ from (M_i, μ_{ij}) to (N_i, ν_{ij}) induces a unique homomorphism*

$$\operatorname{colim} \phi_i : \operatorname{colim} M_i \longrightarrow \operatorname{colim} N_i$$

such that

$$\begin{array}{ccc} M_i & \longrightarrow & \operatorname{colim} M_i \\ {\scriptstyle \phi_i}\downarrow & & \downarrow{\scriptstyle \operatorname{colim} \phi_i} \\ N_i & \longrightarrow & \operatorname{colim} N_i \end{array}$$

commutes for all $i \in I$.

Proof. Write $M = \operatorname{colim} M_i$ and $N = \operatorname{colim} N_i$ and $\phi = \operatorname{colim} \phi_i$ (as yet to be constructed). We will use the explicit description of M and N in Lemma 10.8.3

without further mention. The condition of the lemma is equivalent to the condition that

$$\begin{array}{ccc} \bigoplus_{i\in I} M_i & \longrightarrow & M \\ {\scriptstyle \bigoplus \phi_i}\downarrow & & \downarrow{\scriptstyle \phi} \\ \bigoplus_{i\in I} N_i & \longrightarrow & N \end{array}$$

commutes. Hence it is clear that if ϕ exists, then it is unique. To see that ϕ exists, it suffices to show that the kernel of the upper horizontal arrow is mapped by $\bigoplus \phi_i$ to the kernel of the lower horizontal arrow. To see this, let $j \leq k$ and $x_j \in M_j$. Then

$$(\bigoplus \phi_i)(x_j - \mu_{jk}(x_j)) = \phi_j(x_j) - \phi_k(\mu_{jk}(x_j)) = \phi_j(x_j) - \nu_{jk}(\phi_i(x_j))$$

which is in the kernel of the lower horizontal arrow as required. □

Lemma 10.8.9. *Let I be a directed partially ordered set. Let (L_i, λ_{ij}), (M_i, μ_{ij}), and (N_i, ν_{ij}) be systems of R-modules over I. Let $\varphi_i : L_i \to M_i$ and $\psi_i : M_i \to N_i$ be morphisms of systems over I. Assume that for all $i \in I$ the sequence of R-modules*

$$L_i \xrightarrow{\varphi_i} M_i \xrightarrow{\psi_i} N_i$$

is a complex with homology H_i. Then the R-modules H_i form a system over I, the sequence of R-modules

$$\operatorname{colim}_i L_i \xrightarrow{\varphi} \operatorname{colim}_i M_i \xrightarrow{\psi} \operatorname{colim}_i N_i$$

is a complex as well, and denoting H its homology we have

$$H = \operatorname{colim}_i H_i.$$

Proof. We are going to repeatedly use the description of colimits over I as in Lemma 10.8.4 without further mention. Let $h \in H$. Since $H = \operatorname{Ker}(\varphi)/\operatorname{Im}(\psi)$ we see that h is the class mod $\operatorname{Im}(\psi)$ of an element $[m]$ in $\operatorname{Ker}(\psi) \subset \operatorname{colim}_i M_i$. Choose an i such that $[m]$ comes from an element $m \in M_i$. Choose a $j \geq i$ such that $\nu_{ij}(\psi_i(m)) = 0$ which is possible since $[m] \in \operatorname{Ker}(\psi)$. After replacing i by j and m by $\mu_{ij}(m)$ we see that we may assume $m \in \operatorname{Ker}(\psi_i)$. This shows that the map $\operatorname{colim}_i H_i \to H$ is surjective.

Suppose that $h_i \in H_i$ has image zero on H. Since $H_i = \operatorname{Ker}(\psi_i)/\operatorname{Im}(\varphi_i)$ we may represent h_i by an element $m \in \operatorname{Ker}(\psi_i) \subset M_i$. The assumption on the vanishing of h_i in H means that the class of m in $\operatorname{colim}_i M_i$ lies in the image of φ. Hence there exists an $j \geq i$ and a $l \in L_j$ such that $\varphi_j(l) = \mu_{ij}(m)$. Clearly this shows that the image of h_i in H_j is zero. This proves the injectivity of $\operatorname{colim}_i H_i \to H$. □

Example 10.8.10. Taking colimits is not exact in general. Consider the partially ordered set $I = \{a, b, c\}$ with $a < b$ and $a < c$ and no other strict inequalities, as in Example 10.8.6. Consider the map of systems $(0, \mathbf{Z}, \mathbf{Z}, 0, 0) \to (\mathbf{Z}, \mathbf{Z}, \mathbf{Z}, 1, 1)$. From the description of the colimit in Example 10.8.6 we see that the associated map of colimits is not injective, even though the map of systems is injective on each object. Hence the result of Lemma 10.8.9 is false for general systems.

Lemma 10.8.11. *Let $\mathcal{I}$ be an index category satisfying the assumptions of Categories, Lemma 4.19.7. Then taking colimits of diagrams of abelian groups over $\mathcal{I}$ is exact (i.e., the analogue of Lemma 10.8.9 holds in this situation).*

Proof. By Categories, Lemma 4.19.7 we may write $\mathcal{I} = \coprod_{j \in J} \mathcal{I}_j$ with each $\mathcal{I}_j$ a filtered category, and J possibly empty. By Categories, Lemma 4.21.3 taking colimits over the index categories $\mathcal{I}_j$ is the same as taking the colimit over some directed partially ordered set. Hence Lemma 10.8.9 applies to these colimits. This reduces the problem to showing that coproducts in the category of R-modules over the set J are exact. In other words, exact sequences $L_j \to M_j \to N_j$ of R modules we have to show that

$$\bigoplus\nolimits_{j \in J} L_j \longrightarrow \bigoplus\nolimits_{j \in J} M_j \longrightarrow \bigoplus\nolimits_{j \in J} N_j$$

is exact. This can be verified by hand, and holds even if J is empty. □

For purposes of reference, we define what it means to have a relation between elements of a module.

Definition 10.8.12. Let R be a ring. Let M be an R-module. Let $n \geq 0$ and $x_i \in M$ for $i = 1, \ldots, n$. A *relation* between $x_1, \ldots, x_n$ in M is a sequence of elements $f_1, \ldots, f_n \in R$ such that $\sum_{i=1,\ldots,n} f_i x_i = 0$.

Lemma 10.8.13. *Let R be a ring and let M be an R-module. Then M is the colimit of a directed system (M_i, μ_{ij}) of R-modules with all M_i finitely presented R-modules.*

Proof. Consider any finite subset $S \subset M$ and any finite collection of relations E among the elements of S. So each $s \in S$ corresponds to $x_s \in M$ and each $e \in E$ consists of a vector of elements $f_{e,s} \in R$ such that $\sum f_{e,s} x_s = 0$. Let $M_{S,E}$ be the cokernel of the map

$$R^{\#E} \longrightarrow R^{\#S}, \quad (g_e)_{e \in E} \longmapsto (\sum g_e f_{e,s})_{s \in S}.$$

There are canonical maps $M_{S,E} \to M$. If $S \subset S'$ and if the elements of E correspond, via this map, to relations in E', then there is an obvious map $M_{S,E} \to M_{S',E'}$ commuting with the maps to M. Let I be the set of pairs (S, E) with ordering by inclusion as above. It is clear that the colimit of this directed system is M. □

10.9. Localization

Definition 10.9.1. Let R be a ring, S a subset of R. We say S is a *multiplicative subset of R* is $1 \in S$ and S is closed under multiplication, i.e., $s, s' \in S \Rightarrow ss' \in S$.

Given a ring A and a multiplicative subset S, we define a relation on $A \times S$ as follows:

$$(x, s) \sim (y, t) \iff \exists u \in S, \text{ such that } (xt - ys)u = 0$$

It is easily checked that this is an equivalence relation. Let x/s (or $\frac{x}{s}$) be the equivalence class of (x, s) and $S^{-1}A$ be the set of all equivalence classes. Define addition and multiplication in $S^{-1}A$ as follows:

$$x/s + y/t = (xt + ys)/st \tag{10.9.1.1}$$

$$x/s \cdot y/t = xy/st \tag{10.9.1.2}$$

One can check that $S^{-1}A$ becomes a ring under these operations.

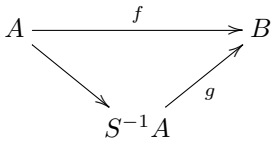
Definition 10.9.2. This ring is called the *localization of A with respect to S*.

We have a natural ring map from A to its localization $S^{-1}A$,

$$A \longrightarrow S^{-1}A, \quad x \longmapsto x/1$$

which is sometimes called the *localization map*. In general the localization map is not injective, unless S contains no zerodivisors. For, if $x/1 = 0$, then there is a $u \in S$ such that $xu = 0$ in A and hence $x = 0$ since there are no zerodivisors in S. The localization of a ring has the following universal property.

Proposition 10.9.3. *Let $f : A \to B$ be a ring map that sends every element in S to a unit of B. Then there is a unique homomorphism $g : S^{-1}A \to B$ such that the following diagram commutes.*



Proof. Existence. We define a map g as follows. For $x/s \in S^{-1}A$, let $g(x/s) = f(x)f(s)^{-1} \in B$. It is easily checked from the definition that this is a well-defined ring map. And it is also clear that this makes the diagram commutative.

Uniqueness. We now show that if $g' : S^{-1}A \to B$ satisfies $g'(x/1) = f(x)$, then $g = g'$. Hence $f(s) = g'(s/1)$ for $s \in S$ by the commutativity of the diagram. But then $g'(1/s)f(s) = 1$ in B, which implies that $g'(1/s) = f(s)^{-1}$ and hence $g'(x/s) = g'(x/1)g'(1/s) = f(x)f(s)^{-1} = g(x/s)$. □

Lemma 10.9.4. *The localization $S^{-1}A$ is the zero ring if and only if $0 \in S$.*

Proof. If $0 \in S$, any pair $(a, s) \sim (0, 1)$ by definition. If $0 \notin S$, then clearly $1/1 \neq 0/1$ in $S^{-1}A$. □

Lemma 10.9.5. *Let R be a ring. Let $S \subset R$ be a multiplicative subset. The category of $S^{-1}R$-modules is equivalent to the category of R-modules N with the property that every $s \in S$ acts as an automorphism on N.*

Proof. The functor which defines the equivalence associates to an $S^{-1}R$-module M the same module but now viewed as an R-module via the localization map $R \to S^{-1}R$. Conversely, if N is an R-module, such that every $s \in S$ acts via an automorphism s_N, then we can think of N as an $S^{-1}R$-module by letting x/s act via $x_N \circ s_N^{-1}$. We omit the verification that these two functors are quasi-inverse to each other. □

The notion of localization of a ring can be generalized to the localization of a module. Let A be a ring, S a multiplicative subset of A and M an A-module. We define a relation on $M \times S$ as follows

$$(m, s) \sim (n, t) \iff \exists u \in S, \text{ such that } (mt - ns)u = 0$$

This is clearly an equivalence relation. Denote by m/s (or $\frac{m}{s}$) be the equivalence class of (m, s) and $S^{-1}M$ be the set of all equivalence classes. Define the addition and scalar multiplication as follows

$$m/s + n/t = (mt + ns)/st \tag{10.9.5.1}$$

$$m/s \cdot n/t = mn/st \tag{10.9.5.2}$$

It is clear that this makes $S^{-1}M$ an $S^{-1}A$ module.

Definition 10.9.6. The $S^{-1}A$-module $S^{-1}M$ is called the *localization* of M at S.

Note that there is an A-module map $M \to S^{-1}M$, $m \mapsto m/1$ which is sometimes called the *localization map*. It satisfies the following universal property.

Lemma 10.9.7. *Let R be a ring. Let $S \subset R$ a multiplicative subset. Let M, N be R-modules. Assume all the elements of S act as automorphisms on N. Then the canonical map*

$$\mathrm{Hom}_R(S^{-1}M, N) \longrightarrow \mathrm{Hom}_R(M, N)$$

induced by the localization map, is an isomorphism.

Proof. It is clear that the map is well-defined and R-linear. Injectivity: Let $\alpha \in \mathrm{Hom}_R(S^{-1}M, N)$ and take an arbitrary element $m/s \in S^{-1}M$. Then, since $s \cdot \alpha(m/s) = \alpha(m/1)$, we have $\alpha(m/s) = s^{-1}(\alpha(m/1))$, so α is completely determined by what it does on the image of M in $S^{-1}M$. Surjectivity: Let $\beta : M \to N$ be a given R-linear map. We need to show that it can be "extended" to $S^{-1}M$. Define a map of sets

$$M \times S \to N, \quad (m, s) \mapsto s^{-1}(m)$$

Clearly, this map respects the equivalence relation from above, so it descends to a well-defined map $\alpha : S^{-1}M \to N$. It remains to show that this map is R-linear, so take $r, r' \in R$ as well as $s, s' \in S$ and $m, m' \in M$. Then

$$\begin{aligned}
\alpha(r \cdot m/s + r' \cdot m'/s') &= \alpha((r \cdot s\prime \cdot m + r\prime \cdot s \cdot m')/(ss')) \\
&= (ss')^{-1}(\beta(r \cdot s\prime \cdot m + r\prime \cdot s \cdot m') \\
&= (ss')^{-1}(r \cdot s'\beta(m) + r' \cdot s\beta(m') \\
&= r\alpha(m/s) + r'\alpha(m'/s')
\end{aligned}$$

and we win. □

Example 10.9.8. Let A be a ring and let M be an A-module. Here are some important examples of localizations.

(1) Given $\mathfrak{p}$ a prime ideal of A consider $S = A \setminus \mathfrak{p}$. It is immediately checked that S is a multiplicative set. In this case we denote $A_\mathfrak{p}$ and $M_\mathfrak{p}$ the localization of A and M with respect to S respectively. These are called the *localization of A, resp. M at $\mathfrak{p}$*.
(2) Let $f \in A$. Consider $S = \{1, f, f^2, \cdots\}$. This is clearly a multiplicative subset of A. In this case we denote A_f (resp. M_f) the localization $S^{-1}A$ (resp. $S^{-1}M$). This is called the *localization of A, resp. M with respect to f*. Note that $A_f = 0$ if and only if f is nilpotent in A.
(3) Let $S = \{f \in A \mid f \text{ is not a zerodivisor in } A\}$. This is a multiplicative subset of A. In this case the ring $Q(A) = S^{-1}A$ is called either the *total quotient ring*, or the *total ring of fractions* of A.

Lemma 10.9.9. *Let R be a ring. Let $S \subset R$ be a multiplicative subset. Let M be an R-module. Then*

$$S^{-1}M = \mathrm{colim}_{f \in S} M_f$$

where the partial ordering on S is given by $f \geq f' \Leftrightarrow f = f'f''$ for some $f' \in R$ in which case the map $M_{f'} \to M_f$ is given by $m/(f')^e \mapsto m(f'')^e/f^e$.

Proof. Omitted. Hint: Use the universal property of Lemma 10.9.7. □

In the following paragraph, let A denote a ring, and M, N denote modules over A.

If S and S' are multiplicative sets of A, then it is clear that

$$SS' = \{ss' : s \in S,\ s' \in S'\}$$

is also a multiplicative set of A. Then the following holds.

Proposition 10.9.10. *Let $\overline{S}$ be the image of S in $S'^{-1}A$, then $(SS')^{-1}A$ is isomorphic to $\overline{S}^{-1}(S'^{-1}A)$.*

Proof. The map sending $x \in A$ to $x/1 \in (SS'^{-1})A$ induces a map sending $x/s \in S'^{-1}A$ to $x/s \in (SS'^{-1})A$, by universal property. The image of the elements in $\overline{S}$ are invertible in $(SS'^{-1})A$. By the universal property we get a map $f : \overline{S}^{-1}(S'^{-1}A) \to (SS'^{-1})A$ which maps $(x/t')/(s/s')$ to $(x/t') \cdot (s/s')^{-1}$.

On the other hand, the map from A to $\overline{S}^{-1}(S'^{-1}A)$ sending $x \in A$ to $(x/1)/(1/1)$ also induces a map $g : (SS'^{-1})A \to \overline{S}^{-1}(S'^{-1}A)$ which sends x/ss' to $(x/s')/(s/1)$, by the universal property again. It is immediately checked that f and g are inverse to each other, hence they are both isomorphisms. □

For the module M we have

Proposition 10.9.11. *View $S'^{-1}M$ as an A-module, then $S^{-1}(S'^{-1}M)$ is isomorphic to $(SS')^{-1}M$.*

Proof. Note that given a A-module M, we have not proved any universal property for $S^{-1}M$. Hence we cannot reason as in the preceding proof; we have to construct the isomorphism explicitly.

We define the maps as follows

$$f : S^{-1}(S'^{-1}M) \longrightarrow (SS')^{-1}M, \quad \frac{x/s'}{s} \mapsto x/ss'$$

$$g : (SS')^{-1}M \longrightarrow S^{-1}(S'^{-1}M), \quad x/t \mapsto \frac{x/s'}{s} \text{ for some } s \in S, s' \in S', \text{ and } t = ss'$$

We have to check that these homomorphisms are well-defined, that is, independent the choice of the fraction. This is easily checked and it is also straightforward to show that they are inverse to each other. □

If $u : M \to N$ is an A homomorphism, then the localization indeed induces a well-defined $S^{-1}A$ homomorphism $S^{-1}u : S^{-1}M \to S^{-1}N$ which sends x/s to $u(x)/s$. It is immediately checked that this construction is functorial, so that S^{-1} is actually a functor from the category of A-modules to the category of $S^{-1}A$-modules. Moreover this functor is exact, as we show in the following proposition.

Proposition 10.9.12. *Let $L \xrightarrow{u} M \xrightarrow{v} N$ is an exact sequence of R modules. Then $S^{-1}L \to S^{-1}M \to S^{-1}N$ is also exact.*

Proof. First it is clear that $S^{-1}L \to S^{-1}M \to S^{-1}N$ is a complex since localization is a functor. Next suppose that x/s maps to zero in $S^{-1}N$ for some $x/s \in S^{-1}M$. Then by definition there is a $t \in S$ such that $v(xt) = v(x)t = 0$ in M, which means $xt \in \mathrm{Ker}(v)$. By the exactness of $L \to M \to N$ we have $xt = u(y)$ for some y in L. Then x/s is the image of y/st. This proves the exactness. □

Lemma 10.9.13. *Localization respects quotients, i.e. if N is a submodule of M, then $S^{-1}(M/N) \simeq (S^{-1}M)/(S^{-1}N)$.*

Proof. From the exact sequence

$$0 \longrightarrow N \longrightarrow M \longrightarrow M/N \longrightarrow 0$$

we have

$$0 \longrightarrow S^{-1}N \longrightarrow S^{-1}M \longrightarrow S^{-1}(M/N) \longrightarrow 0$$

The corollary then follows. □

If, in the preceding Corollary, we take $N = I$ and $M = A$ for an ideal I of A, we see that $S^{-1}A/S^{-1}I \simeq S^{-1}(A/I)$ as A-modules. The next proposition shows that they are isomorphic as rings.

Proposition 10.9.14. *Let I be an ideal of A, S a multiplicative set of A. Then $S^{-1}I$ is an ideal of $S^{-1}A$ and $\overline{S}^{-1}(A/I)$ is isomorphic to $S^{-1}A/S^{-1}I$, where $\overline{S}$ is the image of S in A/I.*

Proof. The fact that $S^{-1}I$ is an ideal is clear since I itself is an ideal. Define

$$f : S^{-1}A \longrightarrow \overline{S}^{-1}(A/I), \quad x/s \mapsto \overline{x}/\overline{s}$$

where $\overline{x}$ and $\overline{s}$ are the images of x and s in A/I. We shall keep similar notations in this proof. This map is well-defined by the universal property of $S^{-1}A$, and $S^{-1}I$ is contained in the kernel of it, therefore it induces a map

$$\overline{f} : S^{-1}A/S^{-1}I \longrightarrow \overline{S}^{-1}(A/I), \quad \overline{x/s} \mapsto \overline{x}/\overline{s}$$

On the other hand, the map $A \to S^{-1}A/S^{-1}I$ sending x to $\overline{x/1}$ induces a map $A/I \to S^{-1}A/S^{-1}I$ sending $\overline{x}$ to $\overline{x/1}$. The image of $\overline{S}$ is invertible in $S^{-1}A/S^{-1}I$, thus induces a map

$$g : \overline{S}^{-1}(A/I) \longrightarrow S^{-1}A/S^{-1}I, \quad \frac{\overline{x}}{\overline{s}} \mapsto \overline{x/s}$$

by the universal property. It is then clear that $\overline{f}$ and g are inverse to each other, hence are both isomorphisms. □

We now consider how submodules behave in localization.

Lemma 10.9.15. *Any submodule N' of $S^{-1}M$ is of the form $S^{-1}N$ for some $N \subset M$. Indeed one can take N to be the inverse image of N' in M.*

Proof. Let N be the inverse image of N' in M. Then one can see that $S^{-1}N \supset N'$. To show they are equal, take x/s in $S^{-1}N$, where $s \in S$ and $x \in N$. This yields that $x/1 \in N'$. Since N' is an $S^{-1}R$-submodule we have $x/s = x/1 \cdot 1/s \in N'$. This finishes the proof. □

Taking $M = A$ and $N = I$ an ideal of A, we have the following corollary, which can be viewed as a converse of the first part of Proposition 10.9.14.

Lemma 10.9.16. *Each ideal I' of $S^{-1}A$ takes the form $S^{-1}I$, where one can take I to be the inverse image of I' in A.*

Proof. Immediate from Lemma 10.9.15. □

The next lemma concerns the spectrum and localization. FIXME: This should be moved later in the manuscript.

Lemma 10.9.17. *Let S be a multiplicative set of A. Then the map*

$$f : \mathrm{Spec}(S^{-1}A) \longrightarrow \mathrm{Spec}(A)$$

induced by the canonical ring map $A \to S^{-1}A$ is a homeomorphism onto its image and $Im(f) = \{\mathfrak{p} \in \mathrm{Spec}(R) : \mathfrak{p} \cap S = \emptyset\}$.

Proof. Denote the localization map by $\varphi : A \to S^{-1}A$. We first show that $\mathrm{Im}(f) = \{\mathfrak{p} \in \mathrm{Spec}(R) : \mathfrak{p} \cap S = \emptyset\}$. For any ideal $\mathfrak{q}$ of $S^{-1}A$, $\varphi^{-1}(\mathfrak{q}) \cap S = 0$. Otherwise if $x \neq 0 \in \varphi^{-1}(\mathfrak{q}) \cap S$, then $x/1 \in \mathfrak{q}$. But $x \in S$, hence $x/1$ is invertible in $S^{-1}A$ which is impossible since $\mathfrak{q}$ is a prime ideal. For any prime ideal $\mathfrak{p}$ in A which does not meet S, $S^{-1}\mathfrak{p}$ is an ideal in $S^{-1}A$. Moreover it is a prime ideal. This is because $S^{-1}A/S^{-1}\mathfrak{p}$ is isomorphic to $\overline{S}^{-1}(A/\mathfrak{p})$ and the localization of an integral domain is again an integral domain.

We still have to show that this map is open, i.e. we have to show that the image of a standard open set is again open. For any $x/s \in S^{-1}A$, we claim that the image of $D(x/s)$ is $D(x) \cap \mathrm{Im}(f)$. First if $x/s \notin S^{-1}\mathfrak{p}$ for some prime ideal $\mathfrak{p}$ of A, then $x \notin \mathfrak{p}$. Conversely, if $x \notin \mathfrak{p}$ and $\mathfrak{p}$ does not meet S, then $x/s \notin S^{-1}\mathfrak{p}$. This is due to that fact that $\mathfrak{p} \cap S = \emptyset$.

Thus f is indeed an homeomorphism onto its image. □

10.10. Internal Hom

If R is a ring, and M, N are R-modules, then

$$\mathrm{Hom}_R(M, N) = \{\varphi : M \to N\}$$

is the set of R-linear maps from M to N. This set comes with the structure of an abelian group by setting $(\varphi+\psi)(m) = \varphi(m)+\psi(m)$, as usual. In fact, $\mathrm{Hom}_R(M, N)$ is also an R-module via the rule $(x\varphi)(m) = x\varphi(m) = \varphi(xm)$.

Given maps $a : M \to M'$ and $b : N \to N'$ of R-modules, we can pre-compose and post-compose homomorphisms by a and b. This leads to the following commutative diagram

$$\begin{array}{ccc} \mathrm{Hom}_R(M', N) & \xrightarrow[b\circ -]{} & \mathrm{Hom}_R(M', N') \\ {\scriptstyle -\circ a}\downarrow & & \downarrow{\scriptstyle -\circ a} \\ \mathrm{Hom}_R(M, N') & \xrightarrow{b\circ -} & \mathrm{Hom}_R(M, N) \end{array}$$

In fact, the maps in this diagram are R-module maps. Thus Hom_R defines an additive functor

$$\mathrm{Mod}_R^{opp} \times \mathrm{Mod}_R \longrightarrow \mathrm{Mod}_R, \quad (M, N) \longmapsto \mathrm{Hom}_R(M, N)$$

Lemma 10.10.1. *Exactness and Hom_R. Let R be a ring.*

1. *Let $M_1 \to M_2 \to M_3 \to 0$ be a complex of R-modules. Then $M_1 \to M_2 \to M_3 \to 0$ is exact if and only if $0 \to \mathrm{Hom}_R(M_3, N) \to \mathrm{Hom}_R(M_2, N) \to \mathrm{Hom}_R(M_1, N)$ is exact for all R-modules N.*
2. *Let $0 \to M_1 \to M_2 \to M_3$ be a complex of R-modules. Then $0 \to M_1 \to M_2 \to M_3$ is exact if and only if $0 \to \mathrm{Hom}_R(N, M_1) \to \mathrm{Hom}_R(N, M_2) \to \mathrm{Hom}_R(N, M_1)$ is exact for all R-modules N.*

Proof. Omitted. □

Lemma 10.10.2. *Let R be a ring. Let M be a finitely presented R-module. Let N be an R-module. Let $f \in R$. Then* $\mathrm{Hom}_R(M,N)_f \cong \mathrm{Hom}_{R_f}(M_f, N_f)$.

Proof. Choose a presentation

$$\bigoplus\nolimits_{j=1,\ldots,m} R \longrightarrow \bigoplus\nolimits_{i=1,\ldots,n} R \to M \to 0.$$

By Lemma 10.10.1 this gives an exact sequence

$$0 \to \mathrm{Hom}_R(M,N) \to \bigoplus\nolimits_{i=1,\ldots,n} N \longrightarrow \bigoplus\nolimits_{j=1,\ldots,m} N.$$

Inverting f we get an exact sequence

$$0 \to (\mathrm{Hom}_R(M,N))_f \to \bigoplus\nolimits_{i=1,\ldots,n} N_f \longrightarrow \bigoplus\nolimits_{j=1,\ldots,m} N_f$$

and the result follows since M_f sits in an exact sequence

$$\bigoplus\nolimits_{j=1,\ldots,m} R_f \longrightarrow \bigoplus\nolimits_{i=1,\ldots,n} R_f \to M_f \to 0$$

which induces (by Lemma 10.10.1) the exact sequence

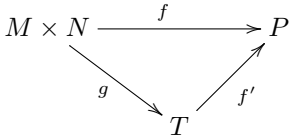
$$0 \to \mathrm{Hom}_{R_f}(M_f, N_f) \to \bigoplus\nolimits_{i=1,\ldots,n} N_f \longrightarrow \bigoplus\nolimits_{j=1,\ldots,m} N_f$$

which is the same as the one above. □

10.11. Tensor products

Definition 10.11.1. Let R be a ring, M, N, P be three R-modules. A mapping $f : M \times N \to P$ (where $M \times N$ is viewed only as Cartesian product of two R-modules) is said to be *R-bilinear* if for each $x \in M$ the mapping $y \mapsto f(x,y)$ of N into P is R-linear, and for each $y \in N$ the mapping $x \mapsto f(x,y)$ is also R-linear.

Lemma 10.11.2. *Let M, N be R-modules. Then there exists a pair (T, g) where T is an R-module, and $g : M \times N \to T$ an R-bilinear mapping, with the following universal property: For any R-module P and any R-bilinear mapping $f : M \times N \to P$, there exists a unique R-linear mapping $\tilde{f} : T \to P$ such that $f = \tilde{f} \circ g$. In other words, the following diagram commutes:*



Moreover, if (T, g) and (T', g') are two pairs with this property, then there exists a unique isomorphism $j : T \to T'$ such that $j \circ g = g'$.

The R-module T which satisfies the above universal property is called the *tensor product* of R-modules M and N, denoted as $M \otimes_R N$.

Proof. We first prove the existence of such R-module T. Let M, N be R-modules. Let T be the quotient module P/Q, where P is the free R-module $R^{M \times N}$ and Q is

the R-module generated by all elements of the following types: $(x \in M, y \in N)$

$$(x + x', y) - (x, y) - (x', y) \tag{10.11.2.1}$$

$$(x, y + y') - (x, y) - (x, y') \tag{10.11.2.2}$$

$$(ax, y) - a \cdot (x, y) \tag{10.11.2.3}$$

$$(x, ay) - a \cdot (x, y) \tag{10.11.2.4}$$

Let $\pi : M \times N \to T$ denote the natural map. This map is R-bilinear, as implied by the above relations when we check the bilinearity conditions. Denote the image $\pi(x, y) = x \otimes y$, then these elements generate T. Now let $f : M \times N \to P$ be an R-bilinear map, then we can define $f' : T \to P$ by extending the mapping $f'(x \otimes y) = f(x, y)$. Clearly $f = f' \circ \pi$. Moreover, f' is uniquely determined by the value on the generating sets $\{x \otimes y : x \in M, y \in N\}$. Suppose there is another pair (T', g') satisfying the same properties. Then there is a unique $j : T \to T'$ and also $j' : T' \to T$ such that $g' = j \circ g$, $g = j' \circ g'$. But then both the maps $(j \circ j') \circ g$ and g satisfies the universal properties, so by uniqueness they are equal, and hence $j' \circ j$ is identity on T. Similarly $(j' \circ j) \circ g' = g'$ and $j \circ j'$ is identity on T'. So j is an isomorphism. □

Lemma 10.11.3. *Let M, N, P be R-modules, then the bilinear maps*

$$(x, y) \mapsto y \otimes x \tag{10.11.3.1}$$

$$(x + y, z) \mapsto x \otimes z + y \otimes z \tag{10.11.3.2}$$

$$(r, x) \mapsto rx \tag{10.11.3.3}$$

induce unique isomorphisms

$$M \otimes_R N \to N \otimes_R M, \tag{10.11.3.4}$$

$$(M \oplus N) \otimes_R P \to (M \otimes_R P) \oplus (N \otimes_R P), \tag{10.11.3.5}$$

$$R \otimes_R M \to M \tag{10.11.3.6}$$

Proof. Omitted. □

We may generalize the tensor product of two R-modules to finitely many R-modules, and set up a correspondence between the multi-tensor product with multilinear mappings. Using almost the same construction one can prove that:

Lemma 10.11.4. *Let $M_1, \ldots, M_r$ be R-modules. Then there exists a pair (T, g) consisting of an R-module T and an R-multilinear mapping $g : M_1 \times \ldots \times M_r \to T$ with the universal property: For any R-multilinear mapping $f : M_1 \times \ldots \times M_r \to P$ there exists a unique R-module homomorphism $f' : T \to P$ such that $f' \circ g = f$. Such a module T is unique up to unique isomorphism. We denote it $M_1 \otimes_R \ldots \otimes_R M_r$ and we denote the universal multilinear map $(m_1, \ldots, m_r \mapsto m_1 \otimes \ldots \otimes m_r$.*

Proof. Omitted. □

Lemma 10.11.5. *The homomorphisms*

$$(M \otimes_R N) \otimes_R P \to M \otimes_R N \otimes_R P \to M \otimes_R (N \otimes_R P) \tag{10.11.5.1}$$

such that $f((x \otimes y) \otimes z) = x \otimes y \otimes z$ and $g(x \otimes y \otimes z) = x \otimes (y \otimes z)$, $x \in M, y \in N, z \in P$ are well-defined and are isomorphisms.

Proof. We shall prove f is well-defined and is an isomorphism, and this proof carries analogously to g. Fix any $z \in P$, then the mapping $(x, y) \mapsto x \otimes y \otimes z$, $x \in M, y \in N$, is R-bilinear in x and y, and hence induces homomorphism $f_z : M \otimes N \to M \otimes N \otimes P$ $f_z(x \otimes y) = x \otimes y \otimes z$. Then consider $(M \otimes N) \times P \to M \otimes N \otimes P$ given by $(w, z) \mapsto f_z(w)$. The map is R-bilinear and thus induces $f : (M \otimes_R N) \otimes_R P \to M \otimes_R N \otimes_R P$ and $f((x \otimes y) \otimes z) = x \otimes y \otimes z$. To construct the inverse, we note that the map $\pi : M \times N \times P \to (M \otimes N) \otimes P$ is R-trilinear. Therefore, it induces an R-linear map $h : M \otimes N \otimes P \to (M \otimes N) \otimes P$ which agrees with the universal property. Here we see that $h(x \otimes y \otimes z) = (x \otimes y) \otimes z$. From the explicit expression of f and h, $f \circ h$ and $h \circ f$ are identity maps of $M \otimes N \otimes P$ and $(M \otimes N) \otimes P$ respectively, hence f is our desired isomorphism. □

Doing induction we see that this extends to multi-tensor products. Combined with Lemma 10.11.3 we see that the tensor product operation on the category of R-modules is associative, commutative and distributive.

Definition 10.11.6. An abelian group N is called an (A, B)-*bimodule* if it is both an A-module and a B-module, and the actions $A \to End(M)$ and $B \to End(M)$ are compatible in the sense that $(ax)b = a(xb)$ for all $a \in A, b \in B, x \in N$. Usually we denote it as ${}_AN_B$.

Lemma 10.11.7. *For A-module M, B-module P and (A, B)-bimodule N, the modules $(M \otimes_A N) \otimes_B P$ and $M \otimes_A (N \otimes_B P)$ can both be given (A, B)-bimodule structure, and moreover*

$$(M \otimes_A N) \otimes_B P \cong M \otimes_A (N \otimes_B P). \tag{10.11.7.1}$$

Proof. A priori $M \otimes_A N$ is an A-module, but we can give it a B-module structure by letting

$$(x \otimes y)b = x \otimes yb, \quad x \in M, y \in N, b \in B$$

Thus $M \otimes_A N$ becomes an (A, B)-bimodule. Similarly for $N \otimes_B P$, and thus for $(M \otimes_A N) \otimes_B P$ and $M \otimes_A (N \otimes_B P)$. By Lemma 10.11.5, these two modules are isomorphic as both as A-module and B-module via the same mapping. □

Lemma 10.11.8. *For any three R-modules M, N, P,*

$$\mathrm{Hom}_R(M \otimes_R N, P) \cong \mathrm{Hom}_R(M, \mathrm{Hom}_R(N, P)) \tag{10.11.8.1}$$

Proof. An R-linear map $\hat{f} \in \mathrm{Hom}_R(M \otimes_R N, P)$ corresponds to an R-bilinear map $f : M \times N \to P$. For each $x \in M$ the mapping $y \mapsto f(x, y)$ is R-linear by the universal property. Thus f corresponds to a map $\phi_f : M \to \mathrm{Hom}_R(N, P)$. This map is R-linear since

$$\phi_f(ax + y)(z) = f(ax + y, z) = af(x, z) + f(y, z) = (a\phi_f(x) + \phi_f(y))(z),$$

for all $y \in N$ and for all $a \in R$, $x, z \in M$. Conversely, any $f \in \mathrm{Hom}_R(M, \mathrm{Hom}_R(N, P))$ defines an R-bilinear map $M \times N \to P$, namely $(x, y) \mapsto f(x)(y)$. So this is a natural one-to-one correspondence between the two modules. □

Lemma 10.11.9 (Tensor products commute with colimits)**.** *Let (M_i, μ_{ij}) be a system over the partially ordered set I. Let N be an R-module. Then*

$$\mathrm{colim}(M_i \otimes N) \cong (\mathrm{colim}\, M_i) \otimes N. \tag{10.11.9.1}$$

Moreover, the isomorphism is induced by the homomorphisms $\mu_i \otimes 1 : M_i \otimes N \to M \otimes N$ where $M = \mathrm{colim}_i M_i$ with natural maps $\mu_i : M_i \to M$.

Proof. First proof. The functor $M' \mapsto M' \otimes_R N$ is left adjoint to the functor $N' \mapsto \mathrm{Hom}_R(N, N')$ by Lemma 10.11.8. Thus $M' \mapsto M' \otimes_R N$ commutes with all colimits, see Categories, Lemma 4.24.3.

Second direct proof. Let $P = \mathrm{colim}(M_i \otimes N)$, $M = \mathrm{colim}\, M_i$. Then for all $i \leq j$, the following diagram commutes:

$$\begin{array}{ccc} M_i \otimes N & \xrightarrow[\mu_i \otimes 1]{} & M \otimes N \\ {\scriptstyle \mu_{ij}\otimes 1}\downarrow & & \downarrow{\scriptstyle \mathrm{id}} \\ M_j \otimes N & \xrightarrow{\mu_j \otimes 1} & M \otimes N \end{array}$$

By Lemma 10.8.8, these maps induce a unique homomorphism $\psi : P \to M \otimes N$, with $\lambda_i : M_i \otimes N \to P$ given by $\lambda_i = \pi \circ (\iota_i \otimes 1)$.

To construct the inverse map, for each $i \in I$, there is the canonical R-bilinear mapping $g_i : M_i \times N \to M_i \otimes N$. This induces a unique mapping $\widehat{\phi} : M \times N \to P$ such that $\widehat{\phi} \circ (\mu_i \times 1) = \lambda_i \circ g_i$. It is R-bilinear. Thus it induces an R-linear mapping $\phi : M \otimes N \to P$. From the commutative diagram below:

$$\begin{array}{ccccccc} M_i \times N & \xrightarrow{g_i} & M_i \otimes N & \xrightarrow[\mathrm{id}]{} & M_i \otimes N & & \\ \downarrow{\scriptstyle \mu_i \times \mathrm{id}} & & \downarrow{\scriptstyle \lambda_i} & & \downarrow{\scriptstyle \mu_i \otimes \mathrm{id}} & \searrow^{\lambda_i} & \\ M \times N & \xrightarrow{\widehat{\phi}} & P & \xrightarrow{\psi} & M \otimes N & \xrightarrow{\phi} & P \end{array}$$

we see that $\psi \circ \widehat{\phi} = g$, the canonical R-bilinear mapping $g : M \times N \to M \otimes N$. So $\psi \circ \phi$ is identity on $M \otimes N$. From the right-hand square and triangle, $\phi \circ \psi$ is also identity on P. □

Lemma 10.11.10. *Let*

$$M_1 \xrightarrow{f} M_2 \xrightarrow{g} M_3 \to 0 \tag{10.11.10.1}$$

be an exact sequence of R-modules and homomorphisms, and let N be any R-module. Then the sequence

$$M_1 \otimes N \xrightarrow{f\otimes 1} M_2 \otimes N \xrightarrow{g\otimes 1} M_3 \otimes N \to 0 \tag{10.11.10.2}$$

is exact. In other words, the functor $- \otimes_R N$ is right exact, in the sense that tensoring each term in the original right exact sequence preserves the exactness.

Proof. We apply the functor $\mathrm{Hom}(-, \mathrm{Hom}(N, P))$ to the first exact sequence. We obtain

$$0 \to \mathrm{Hom}(M_3, \mathrm{Hom}(N, P)) \to \mathrm{Hom}(M_2, \mathrm{Hom}(N, P)) \to \mathrm{Hom}(M_1, \mathrm{Hom}(N, P))$$

By Lemma 10.11.8, we have

$$0 \to \mathrm{Hom}(M_3 \otimes N, P) \to \mathrm{Hom}(M_2 \otimes N, P) \to \mathrm{Hom}(M_1 \otimes N, P)$$

Using the pullback property again, we arrive at the desired exact sequence. □

Remark 10.11.11. However, tensor product does NOT preserve exact sequences in general. In other words, if $M_1 \to M_2 \to M_3$ is exact, then it is not necessarily true that $M_1 \otimes N \to M_2 \otimes N \to M_3 \otimes N$ is exact for arbitrary R-module N.

Example 10.11.12. Consider the injective map $2 : \mathbf{Z} \to \mathbf{Z}$ viewed as a map of $\mathbf{Z}$-modules. Let $N = \mathbf{Z}/2$. Then the induced map $\mathbf{Z} \otimes \mathbf{Z}/2 \to \mathbf{Z} \otimes \mathbf{Z}/2$ is NOT injective. This is because for $x \otimes y \in \mathbf{Z} \otimes \mathbf{Z}/2$,

$$(2 \otimes 1)(x \otimes y) = 2x \otimes y = x \otimes 2y = x \otimes 0 = 0$$

Therefore the induced map is the zero map while $\mathbf{Z} \otimes N \neq 0$.

Remark 10.11.13. For R-modules N, if the functor $- \otimes_R N$ is exact, i.e. tensoring with N preserves all exact sequences, then N is said to be *flat* R-module. We will discuss this later.

Lemma 10.11.14. *Let R be a ring. Let M and N be R-modules.*

(1) *If N and M are finite, then so is $M \otimes_R N$.*
(2) *If N and M are finitely presented, then so is $M \otimes_R N$.*

Proof. Suppose M is finite. Then choose a presentation $0 \to K \to R^{\otimes n} \to M \to 0$. This gives an exact sequence $K \otimes_R N \to N^{\oplus n} \to M \otimes_R N \to 0$ by Lemma 10.11.10. We conclude that if N is finite too then $M \otimes_R N$ is a quotient of a finite module, hence finite, see Lemma 10.5.3. Similarly, if both N and M are finitely presented, then we see that K is finite and that $M \otimes_R N$ is a quotient of the finitely presented module $N^{\oplus n}$ by a finite module, namely $K \otimes_R N$, and hence finitely presented, see Lemma 10.5.3. □

Lemma 10.11.15. *Let M be an R-module. Then the $S^{-1}R$ modules $S^{-1}M$ and $S^{-1}R \otimes_R M$ are canonically isomorphic, and the unique isomorphism $f : S^{-1}R \otimes_R M \to S^{-1}M$ is given by*

$$f((a/s) \otimes m) = am/s, \forall a \in R, m \in M, s \in S \tag{10.11.15.1}$$

Proof. Obviously, the map $f' : S^{-1}R \times M \to S^{-1}M$ given by $f((am, s)) = am/s$ is bilinear, and thus by the universal property, this map induces a unique $S^{-1}R$-module homomorphism $f : S^{-1}R \otimes_R M \to S^{-1}M$ as in the statement of the lemma. Actually every element in $S^{-1}M$ is of the form m/s, $m \in M, s \in S$ and every element in $S^{-1}R \otimes_R M$ is of the form $1/s \otimes m$. To see the latter fact, write an element in $S^{-1}R \otimes_R M$ as

$$\sum_k \frac{a_k}{s_k} \otimes m_k = \sum_k \frac{a_k t_k}{s} \otimes m_k = \frac{1}{s} \otimes \sum_k a_k t_k m_k = \frac{1}{s} \otimes m$$

Where $m = \sum_k a_k t_k m_k$. Then it is obvious that f is surjective, and if $f(\frac{1}{s} \otimes m) = m/s = 0$ then there exists $t' \in S$ with $tm = 0$ in M. Then we have

$$\frac{1}{s} \otimes m = \frac{1}{st} \otimes tm = \frac{1}{st} \otimes 0 = 0$$

Therefore f is injective. □

Lemma 10.11.16. *Let M, N be R-modules, then there is a canonical $S^{-1}R$-module isomorphism $f : S^{-1}M \otimes_{S^{-1}R} S^{-1}N \to S^{-1}(M \otimes_R N)$, given by*

$$f((m/s) \otimes (n/t) = (m \otimes n)/st$$

Proof. We may use Lemma 10.11.7 and Lemma 10.11.15 repeatedly to see that these two $S^{-1}R$-modules are isomorphic, noting that $S^{-1}R$ is an $(R, S^{-1}R)$-bimodule:

$$S^{-1}(M \otimes_R N) \cong S^{-1}R \otimes_R (M \otimes_R N) \tag{10.11.16.1}$$

$$\cong S^{-1}M \otimes_R N \tag{10.11.16.2}$$

$$\cong (S^{-1}M \otimes_{S^{-1}R} S^{-1}R) \otimes_R N \tag{10.11.16.3}$$

$$\cong S^{-1}M \otimes_{S^{-1}R} (S^{-1}R \otimes_R N) \tag{10.11.16.4}$$

$$\cong S^{-1}M \otimes_{S^{-1}R} S^{-1}N \tag{10.11.16.5}$$

This isomorphism is easily seen to be the one stated in the lemma. □

10.12. Tensor algebra

Let R be a ring. Let M be an R-module. We define the *tensor algebra of M over R* to be the noncommutative R-algebra

$$\mathrm{T}(M) = \mathrm{T}_R(M) = \bigoplus_{n \geq 0} \mathrm{T}^n(M)$$

with $\mathrm{T}^0(M) = R$, $\mathrm{T}^1(M) = M$, $\mathrm{T}^2(M) = M \otimes_R M$, $\mathrm{T}^3(M) = M \otimes_R M \otimes_R M$, and so on. Multiplication is defined by the rule that on pure tensors we have

$$(x_1 \otimes x_2 \otimes \ldots \otimes x_n) \cdot (y_1 \otimes y_2 \otimes \ldots \otimes y_m) = x_1 \otimes x_2 \otimes \ldots \otimes x_n \otimes y_1 \otimes y_2 \otimes \ldots \otimes y_m$$

and we extend this by linearity.

We define the *exterior algebra $\wedge(M)$ of M over R* to be the quotient of $\mathrm{T}(M)$ by the two sided ideal generated by the elements $x \otimes x \in \mathrm{T}^2(M)$. The image of a pure tensor $x_1 \otimes \ldots \otimes x_n$ in $\wedge^n(M)$ is denoted $x_1 \wedge \ldots \wedge x_n$. These elements generate $\wedge^n(M)$, they are R-linear in each x_i and they are zero when two of the x_i are equal (i.e., alternating). The multiplication on $\wedge(M)$ is graded commutative, i.e., $x \wedge y = -y \wedge x$.

An example of this is when $M = Rx_1 \oplus \ldots \oplus Rx_n$ is a finite free module. In this case $\wedge(M)$ is free over R with basis the elements

$$x_{i_1} \wedge \ldots \wedge x_{i_r}$$

with $0 \leq r \leq n$ and $1 \leq i_1 < i_2 < \ldots < i_r \leq n$.

We define the *symmetric algebra $\mathrm{Sym}(M)$ of M over R* to be the quotient of $\mathrm{T}(M)$ by the two sided ideal generated by the elements $x \otimes y - y \otimes x \in \mathrm{T}^2(M)$. The image of a pure tensor $x_1 \otimes \ldots \otimes x_n$ in $\mathrm{Sym}^n(M)$ is denoted just $x_1 \ldots x_n$. These elements generate $\mathrm{Sym}^n(M)$, these are R-linear in each x_i and $x_1 \ldots x_n = x'_1 \ldots x'_n$ if the sequence of elements $x_1, \ldots, x_n$ is a permutation of the sequence $x'_1, \ldots, x'_n$. Thus we see that $\mathrm{Sym}(M)$ is commutative.

An example of this is when $M = Rx_1 \oplus \ldots \oplus Rx_n$ is a finite free module. In this case $\mathrm{Sym}(M) = R[x_1, \ldots, x_n]$ is a polynomial algebra.

Lemma 10.12.1. *Let R be a ring. Let M be an R-module. If M is a free R-module, so is each symmetric and exterior power.*

Proof. Omitted, but see above for the finite free case. □

Lemma 10.12.2. *Let R be a ring. Let $M_2 \to M_1 \to M \to 0$ be an exact sequence of R-modules. There are exact sequences*

$$M_2 \otimes_R Sym^{n-1}(M_1) \to Sym^n(M_1) \to Sym^n(M) \to 0$$

and similarly

$$M_2 \otimes_R \wedge^{n-1}(M_1) \to \wedge^n(M_1) \to \wedge^n(M) \to 0$$

Proof. Omitted. □

Lemma 10.12.3. *Let R be a ring. Let M be an R-module. Let x_i, $i \in I$ be a given system of generators of M as an R-module. Let $n \geq 2$. There exists a canonical exact sequence*

$$\bigoplus_{1 \leq j_1 < j_2 \leq n} \bigoplus_{i_1, i_2 \in I} T^{n-2}(M) \oplus \bigoplus_{1 \leq j_1 < j_2 \leq n} \bigoplus_{i \in I} T^{n-2}(M) \to T^n(M) \to \wedge^n(M) \to 0$$

where the pure tensor $m_1 \otimes \ldots \otimes m_{n-2}$ in the first summand maps to

$$m_1 \otimes \ldots \otimes x_{i_1} \otimes \ldots \otimes x_{i_2} \otimes \ldots \otimes m_{n-2} + m_1 \otimes \ldots \otimes x_{i_2} \otimes \ldots \otimes x_{i_1} \otimes \ldots \otimes m_{n-2}$$

and $m_1 \otimes \ldots \otimes m_{n-2}$ in the second summand maps to

$$m_1 \otimes \ldots \otimes x_i \otimes \ldots \otimes x_i \otimes \ldots \otimes m_{n-2}$$

There is also a canonical exact sequence

$$\bigoplus_{1 \leq j_1 < j_2 \leq n} \bigoplus_{i_1, i_2 \in I} T^{n-2}(M) \to T^n(M) \to Sym^n(M) \to 0$$

where the pure tensor $m_1 \otimes \ldots \otimes m_{n-2}$ maps to

$$m_1 \otimes \ldots \otimes x_{i_1} \otimes \ldots \otimes x_{i_2} \otimes \ldots \otimes m_{n-2} - m_1 \otimes \ldots \otimes x_{i_2} \otimes \ldots \otimes x_{i_1} \otimes \ldots \otimes m_{n-2}$$

Proof. Omitted. □

Lemma 10.12.4. *Let R be a ring. Let M_i be a directed system of R-modules. Then* $\operatorname{colim}_i T(M) = T(\operatorname{colim}_i M_i)$ *and similarly for the symmetric and exterior algebras.*

Proof. Omitted. □

10.13. Base change

We formally introduce base change in algebra as follows.

Definition 10.13.1. Let $\varphi : R \to S$ be a ring map. Let M be an S-module. Let $R \to R'$ be any ring map. The *base change* of φ by $R \to R'$ is the ring map $R' \to S \otimes_R R'$. In this situation we often write $S' = S \otimes_R R'$. The *base change* of the S-module M is the S'-module $M \otimes_R R'$.

If $S = R[x_i]/(f_j)$ for some collection of variables x_i, $i \in I$ and some collection of polynomials $f_j \in R[x_i]$, $j \in J$, then $S \otimes_R R' = R'[x_i]/(f'_j)$, where $f'_j \in R'[x_i]$ is the image of f_j under the map $R[x_i] \to R'[x_i]$ induced by $R \to R'$. This simple remark is the key to understanding base change.

Lemma 10.13.2. *Let $R \to S$ be a ring map. Let M be an S-module. Let $R \to R'$ be a ring map and let $S' = S \otimes_R R'$ and $M' = M \otimes_R R'$ be the base changes.*

1. *If M is a finite S-module, then the base change M' is a finite S'-module.*
2. *If M is an S-module finite presentation, then the base change M' is an S'-module of finite presentation.*

(3) *If $R \to S$ is of finite type, then the base change $R' \to S'$ is of finite type.*
(4) *If $R \to S$ is of finite presentation, then the base change $R' \to S'$ is of finite presentation.*

Proof. Proof of (1). Take a surjective, R-linear map $R^{\oplus n} \to M \to 0$. By Lemma 10.11.3 and 10.11.10 the result after tensoring with R' is a surjection $R'^{\oplus n} \to M' \to 0$, so M' is a finitely generated R'-module. Proof of (2). Take a presentation $R^{\oplus m} \to R^{\oplus n} \to M \to 0$. By Lemma 10.11.3 and 10.11.10 the result after tensoring with R' gives a finite presentation $R'^{\oplus m} \to R'^{\oplus n} \to M' \to 0$, of the R'-module M'. Proof of (3). This follows by the remark preceding the lemma as we can take I to be finite by assumption. Proof of (4). This follows by the remark preceding the lemma as we can take I and J to be finite by assumption. □

Let $\varphi : R \to S$ be a ring map. Given a S-module N we obtain an R-module N_R by the rule $r \cdot n = \varphi(r)n$. This is sometimes called the *restriction* of N to R.

Lemma 10.13.3. *Let $R \to S$ be a ring map. The functors $Mod_S \to Mod_R$, $N \mapsto N_R$ (restriction) and $Mod_R \to Mod_S$, $M \mapsto M \otimes_R S$ (base change) are adjoint functors. In a formula*

$$\mathrm{Hom}_R(M, N_R) = \mathrm{Hom}_S(M \otimes_R S, N)$$

Proof. If $\alpha : M \to N_R$ is an R-module map, then we define $\alpha' : M \otimes_R S \to N$ by the rule $\alpha'(m \otimes s) = s\alpha(m)$. If $\beta : M \otimes_R S \to N$ is an S-module map, we define $\beta' : M \to N_R$ by the rule $\beta'(m) = \beta(m \otimes 1)$. We omit the verification that these constructions are mutually inverse. □

The lemma above tells us that restriction has a left adjoint, namely base change. It also has a right adjoint.

Lemma 10.13.4. *Let $R \to S$ be a ring map. The functors $Mod_S \to Mod_R$, $N \mapsto N_R$ (restriction) and $Mod_R \to Mod_S$, $M \mapsto \mathrm{Hom}_R(S, M)$ are adjoint functors. In a formula*

$$\mathrm{Hom}_R(N_R, M) = \mathrm{Hom}_S(N, \mathrm{Hom}_R(S, M))$$

Proof. If $\alpha : N_R \to M$ is an R-module map, then we define $\alpha' : N \to \mathrm{Hom}_R(S, M)$ by the rule $\alpha'(n) = (s \mapsto \alpha(sn))$. If $\beta : N \to \mathrm{Hom}_R(S, M)$ is an S-module map, we define $\beta' : N_R \to M$ by the rule $\beta'(n) = \beta(n)(1)$. We omit the verification that these constructions are mutually inverse. □

Lemma 10.13.5. *Let $R \to S$ be a ring map. Given S-modules M, N and an R-module P we have*

$$\mathrm{Hom}_R(M \otimes_S N, P) = \mathrm{Hom}_S(M, \mathrm{Hom}_R(N, P))$$

Proof. This can be proved directly, but it is also a consequence of Lemmas 10.13.4 and 10.11.8. Namely, we have

$$\begin{aligned} \mathrm{Hom}_R(M \otimes_S N, P) &= \mathrm{Hom}_S(M \otimes_S N, \mathrm{Hom}_R(S, P)) \\ &= \mathrm{Hom}_S(M, \mathrm{Hom}_S(N, \mathrm{Hom}_R(S, P))) \\ &= \mathrm{Hom}_S(M, \mathrm{Hom}_R(N, P)) \end{aligned}$$

as desired. □

10.14. Miscellany

The proofs in this section should not refer to any results except those from the section on basic notions, Section 10.3.

Lemma 10.14.1. *Let $R \to R'$ be a ring map and let $I \subset R$ be a locally nilpotent ideal. Then IR' is a locally nilpotent ideal of R'.*

Proof. This follows from the fact that if $x, y \in R'$ are nilpotent, then $x + y$ is nilpotent too. Namely, if $x^n = 0$ and $y^m = 0$, then $(x + y)^{n+m-1} = 0$. □

Lemma 10.14.2. *Let R be a ring, I and J two ideals and $\mathfrak{p}$ a prime ideal containing the product IJ. Then $\mathfrak{p}$ contains I or J.*

Proof. Assume the contrary and take $x \in I \setminus \mathfrak{p}$ and $y \in I \setminus \mathfrak{p}$. Their product is an element of $IJ \subseteq \mathfrak{p}$, which contradicts the assumption that $\mathfrak{p}$ was prime. □

Lemma 10.14.3 (Prime avoidance)**.** *Let R be a ring. Let $I_i \subset R$, $i = 1, \ldots, r$, and $J \subset R$ be ideals. Assume*

1. *$J \not\subset I_i$ for $i = 1, \ldots, r$, and*
2. *all but two of I_i are prime ideals.*

Then there exists an $x \in J$, $x \notin I_i$ for all i.

Proof. The result is true for $r = 1$. If $r = 2$, then let $x, y \in J$ with $x \notin I_1$ and $y \notin I_2$. We are done unless $x \in I_2$ and $y \in I_1$. Then the element $x + y$ cannot be in I_1 (since that would mean $x + y - y \in I_1$) and it also cannot be in I_2.

For $r \geq 3$, assume the result holds for $r - 1$. After renumbering we may assume that I_r is prime. We may also assume there are no inclusions among the I_i. Pick $x \in J$, $x \notin I_i$ for all $i = 1, \ldots, r - 1$. If $x \notin I_r$ we are done. So assume $x \in I_r$. If $JI_1 \ldots I_{r-1} \subset I_r$ then $J \subset I_r$ (by Lemma 10.14.2) a contradiction. Pick $y \in JI_1 \ldots I_{r-1}$, $y \notin I_r$. Then $x + y$ works. □

Lemma 10.14.4 (Chinese remainder)**.** *Let R be a ring.*

1. *If $I_1, \ldots, I_r$ are ideals such that $I_a + I_b = R$ when $a \neq b$, then $I_1 \cap \ldots \cap I_r = I_1 I_2 \ldots I_r$ and $R/(I_1 I_2 \ldots I_r) \cong R/I_1 \times \ldots \times R/I_r$.*
2. *If $\mathfrak{m}_1, \ldots, \mathfrak{m}_r$ are pairwise distinct maximal ideals then $\mathfrak{m}_a + \mathfrak{m}_b = R$ for $a \neq b$ and the above applies.*

Proof. Let us first prove $I_1 \cap \ldots \cap I_r = I_1 \ldots I_r$ as this will also imply the injectivity of the induced ring homomorphism $R/(I_1 \ldots I_r) \to R/I_1 \times \ldots \times R/I_r$. The inclusion $I_1 \cap \ldots \cap I_r \supseteq I_1 \ldots I_r$ is always fulfilled since ideals are closed under multiplication with arbitrary ring elements. To prove the other inclusion, we claim that the ideals

$$I_1 \ldots \hat{I}_i \ldots I_r, \quad i = 1, \ldots, r$$

generate the ring R. We prove this by induction on r. It holds when $r = 2$. If $r > 2$, then we see that R is the sum of the ideals $I_1 \ldots \hat{I}_i \ldots I_{r-1}$, $i = 1, \ldots, r - 1$. Hence I_r is the sum of the ideals $I_1 \ldots \hat{I}_i \ldots I_r$, $i = 1, \ldots, r - 1$. Applying the same argument with the reverse ordering on the ideals we see that I_1 is the sum of the ideals $I_1 \ldots \hat{I}_i \ldots I_r$, $i = 2, \ldots, r$. Since $R = I_1 + I_r$ by assumption we see that R is the sum of the ideals displayed above. Therefore we can find elements $a_i \in I_1 \ldots \hat{I}_i \ldots I_r$ such that their sum is one. Multiplying this equation by an element of $I_1 \cap \ldots \cap I_r$ gives the other inclusion. It remains to show that the

canonical map $R/(I_1 \ldots I_r) \to R/I_1 \times \ldots \times R/I_r$ is surjective. For this, consider its action on the equation $1 = \sum_{i=1}^r a_i$ we derived above. On the one hand, a ring morphism sends 1 to 1 and on the other hand, the image of any a_i is zero in R/I_j for $j \neq i$. Therefore, the image of a_i in R/I_i is the identity. So given any element $(\bar{b_1}, \ldots, \bar{b_r}) \in R/I_1 \times \ldots \times R/I_r$, the element $\sum_{i=1}^r a_i \cdot b_i$ is an inverse image in R.

To see (2), by the very definition of being distinct maximal ideals, we have $\mathfrak{m}_a + \mathfrak{m}_b = R$ for $a \neq b$ and so the above applies. □

Lemma 10.14.5. *Let R be a ring. Let $n \geq m$. Let $A = (a_{ij})$ be an $n \times m$ matrix with coefficients in R. Let $J \subset R$ be the ideal generated by the $m \times m$ minors of A.*

1. *For any $f \in J$ there exists a $m \times n$ matrix B such that $BA = f1_{m \times m}$.*
2. *If $f \in R$ and $BA = f1_{m \times m}$ for some $m \times m$ matrix B, then $f^m \in J$.*

Proof. For $I \subset \{1, \ldots, n\}$ with $|I| = m$ denote E_I the $m \times n$ matrix of the projection

$$R^{\oplus n} = \bigoplus\nolimits_{i \in \{1,\ldots,n\}} R \longrightarrow \bigoplus\nolimits_{i \in I} R$$

and set $A_I = E_I A$, i.e., A_I is the $m \times m$ matrix whose rows are the rows of A with indices in I. Let B_I be the adjugate (transpose of cofactor) matrix to A_I, i.e., such that $A_I B_I = B_I A_I = \det(A_I)$. If $f \in J$ then we can write $f = \sum c_I \det(A_I)$ for some $c_I \in R$. Set $B = \sum c_I B_I E_I$ to see that (1) holds.

If $f1_{m \times m} = BA$ then by the Cauchy-Binet formula we have $f^m = \sum b_I \det(A_I)$ where b_I is the determinant of the $m \times m$ matrix whose columns are the columns of B with indices in I. □

Lemma 10.14.6. *Let R be a ring. Let $n \geq m$. Let $A = (a_{ij})$ be an $n \times m$ matrix with coefficients in R, written in block form as*

$$A = \begin{pmatrix} A_1 \\ A_2 \end{pmatrix}$$

where A_1 has size $m \times m$. Let B be the adjugate (transpose of cofactor) matrix to A_1. Then

$$AB = \begin{pmatrix} f1_{m \times m} \\ C \end{pmatrix}$$

where $f = \det(A_1)$ and c_{ij} is (up to sign) the determinant of the $m \times m$ minor of A corresponding to the rows $1, \ldots, \hat{j}, \ldots, m, i$.

Proof. Since the adjugate has the property $A_1 B = B A_1 = f$ the first block of the expression for AB is correct. Note that

$$c_{ij} = \sum\nolimits_k a_{ik} b_{kj} = \sum (-1)^{j+k} a_{ik} \det(A_1^{jk})$$

where A_1^{ij} means A_1 with the jth row and kth column removed. This last expression is the row expansion of the determinant of the matrix in the statement of the lemma. □

10.15. Cayley-Hamilton

Lemma 10.15.1. *Let R be a ring. Let $A = (a_{ij})$ be an $n \times n$ matrix with coefficients in R. Let $P(x) \in R[x]$ be the characteristic polynomial of A (defined as $\det(xid_{n\times n} - A)$). Then $P(A) = 0$ in $Mat(n \times n, R)$.*

Proof. We reduce the question to the well-known Cayley-Hamilton theorem from linear algebra in several steps:

(1) If $\phi : S \to R$ is a ring morphism and b_{ij} are inverse images of the a_{ij} under this map, then it suffices to show the statement for S and (b_{ij}) since ϕ is a ring morphism.
(2) If $\psi : R \hookrightarrow S$ is an injective ring morphism, it clearly suffices to show the result for S and the a_{ij} considered as elements of S.
(3) Thus we may first reduce to the case $R = \mathbb{Z}[X_{ij}]$, $a_{ij} = X_{ij}$ of a polynomial ring and then further to the case $R = \mathbb{Q}(X_{ij})$ where we may finally apply Cayley-Hamilton.

□

Lemma 10.15.2. *Let R be a ring. Let M be a finite R-module. Let $\varphi : M \to M$ be an endomorphism. Then there exists a monic polynomial $P \in R[T]$ such that $P(\varphi) = 0$ as an endomorphism of M.*

Proof. Choose a surjective R-module map $R^{\oplus n} \to M$, given by $(a_1, \ldots, a_n) \mapsto \sum a_i x_i$ for some generators $x_i \in M$. Choose $(a_{i1}, \ldots, a_{in}) \in R^{\oplus n}$ such that $\varphi(x_i) = \sum a_{ij} x_j$. In other words the diagram

$$\begin{array}{ccc} R^{\oplus n} & \longrightarrow & M \\ {\scriptstyle A}\downarrow & & \downarrow{\scriptstyle \varphi} \\ R^{\oplus n} & \longrightarrow & M \end{array}$$

is commutative where $A = (a_{ij})$. By Lemma 10.15.1 there exists a monic polynomial P such that $P(A) = 0$. Then it follows that $P(\varphi) = 0$. □

Lemma 10.15.3. *Let R be a ring. Let $I \subset R$ be an ideal. Let M be a finite R-module. Let $\varphi : M \to M$ be an endomorphism such that $\varphi(M) \subset IM$. Then there exists a monic polynomial $P = t^n + a_1 t^{n-1} + \ldots + a_n \in R[T]$ such that $a_j \in I^j$ and $P(\varphi) = 0$ as an endomorphism of M.*

Proof. Choose a surjective R-module map $R^{\oplus n} \to M$, given by $(a_1, \ldots, a_n) \mapsto \sum a_i x_i$ for some generators $x_i \in M$. Choose $(a_{i1}, \ldots, a_{in}) \in I^{\oplus n}$ such that $\varphi(x_i) = \sum a_{ij} x_j$. In other words the diagram

$$\begin{array}{ccc} R^{\oplus n} & \longrightarrow & M \\ {\scriptstyle A}\downarrow & & \downarrow{\scriptstyle \varphi} \\ I^{\oplus n} & \longrightarrow & M \end{array}$$

is commutative where $A = (a_{ij})$. By Lemma 10.15.1 the polynomial $P(t) = \det(t\mathrm{id}_{n\times n} - A)$ has all the desired properties. □

As a fun example application we prove the following surprising lemma.

Lemma 10.15.4. *Let R be a ring. Let M be a finite R-module. Let $\varphi : M \to M$ be a surjective R-module map. Then φ is an isomorphism.*

First proof. Write $R' = R[x]$ and think of M as a finite R'-module with x acting via φ. Set $I = (x) \subset R'$. By our assumption that φ is surjective we have $IM = M$. Hence we may apply Lemma 10.15.3 to M as an R' module, the ideal I and the endomorphism id_M. We conclude that $(1 + a_1 + \ldots + a_n)\mathrm{id}_M = 0$ with $a_j \in I$. Write $a_j = b_j(x)x$ for some $b_j(x) \in R[x]$. Translating back into φ we see that $\mathrm{id}_M = -(\sum_{j=1,\ldots,n} b_j(\varphi)\varphi$ and hence φ is invertible. □

Second proof. We perform induction on the number of generators of M over R. If M is generated by one element, then $M \cong R/I$ for some ideal $I \subset R$. In this case we may replace R by R/I so that $M = R$. In this case $\varphi : R \to R$ is given by multiplication on M by an element $r \in R$. The surjectivity of φ forces r invertible, since φ must hit 1, which implies that φ is invertible.

Now assume that we have proven the lemma in the case of modules generated by $n - 1$ elements, and are examining a module M generated by n elements. Let A mean the ring $R[t]$, and regard the module M as an A-module by letting t act via φ; since M is finite over R, it is finite over $R[t]$ as well, and since we're trying to prove φ injective, a set-theoretic property, we might as well prove the endomorphism $t : M \to M$ over A injective. We have reduced our problem to the case our endomorphism is multiplication by an element of the ground ring. Let $M' \subset M$ denote the sub-A-module generated by the first $n-1$ of the generators of M, and consider the diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & M' & \longrightarrow & M & \longrightarrow & M/M' & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle \varphi|_{M'}} & & \downarrow{\scriptstyle \varphi} & & \downarrow{\scriptstyle \varphi \bmod M'} & & \\ 0 & \longrightarrow & M' & \longrightarrow & M & \longrightarrow & M/M' & \longrightarrow & 0, \end{array}$$

where the restriction of φ to M' and the map induced by φ on the quotient M/M' are well-defined since φ is multiplication by an element in the base, and M' and M/M' are A-modules in their own right. By the case $n = 1$ the map $M/M' \to M/M'$ is an isomorphism. A diagram chase implies that $\varphi|_{M'}$ is surjective hence by induction $\varphi|_{M'}$ is an isomorphism. This forces the middle column to be an isomorphism by the snake lemma. □

10.16. The spectrum of a ring

We arbitrarily decide that the spectrum of a ring as a topological space is part of the algebra chapter, whereas an affine scheme is part of the chapter on schemes.

Definition 10.16.1. Let R be a ring.

1. The *spectrum* of R is the set of prime ideals of R. It is usually denoted $\mathrm{Spec}(R)$.
2. Given a subset $T \subset R$ we let $V(T) \subset \mathrm{Spec}(R)$ be the set of primes containing T, i.e., $V(T) = \{\mathfrak{p} \in \mathrm{Spec}(R) \mid \forall f \in T, f \in \mathfrak{p}\}$.
3. Given an element $f \in R$ we let $D(f) \subset \mathrm{Spec}(R)$ be the set of primes not containing f.

Lemma 10.16.2. *Let R be a ring.*

(1) *The spectrum of a ring R is empty if and only if R is the zero ring.*
(2) *Every nonzero ring has a maximal ideal.*
(3) *Every nonzero ring has a minimal prime ideal.*
(4) *Given an ideal $I \subset R$ and a prime ideal $I \subset \mathfrak{p}$ there exists a prime $I \subset \mathfrak{q} \subset \mathfrak{p}$ such that $\mathfrak{q}$ is minimal over I.*
(5) *If $T \subset R$, and if (T) is the ideal generated by T in R, then $V((T)) = V(T)$.*
(6) *If I is an ideal and $\sqrt{I}$ is its radical, see basic notion (27), then $V(I) = V(\sqrt{I})$.*
(7) *Given an ideal I of R we have $\sqrt{I} = \bigcap_{I \subset \mathfrak{p}} \mathfrak{p}$.*
(8) *If I is an ideal then $V(I) = \emptyset$ if and only if I is the unit ideal.*
(9) *If I, J are ideals of R then $V(I) \cup V(J) = V(I \cap J)$.*
(10) *If $(I_a)_{a \in A}$ is a set of ideals of R then $\cap_{a \in A} V(I_a) = V(\cup_{a \in A} I_a)$.*
(11) *If $f \in R$, then $D(f) \coprod V(f) = \mathrm{Spec}(R)$.*
(12) *If $f \in R$ then $D(f) = \emptyset$ if and only if f is nilpotent.*
(13) *If $f = uf'$ for some unit $u \in R$, then $D(f) = D(f')$.*
(14) *If $I \subset R$ is an ideal, and $\mathfrak{p}$ is a prime of R with $\mathfrak{p} \notin V(I)$, then there exists an $f \in R$ such that $\mathfrak{p} \in D(f)$, and $D(f) \cap V(I) = \emptyset$.*
(15) *If $f, g \in R$, then $D(fg) = D(f) \cap D(g)$.*
(16) *If $f_i \in R$ for $i \in I$, then $\bigcup_{i \in I} D(f_i)$ is the complement of $V(\{f_i\}_{i \in I})$ in $\mathrm{Spec}(R)$.*
(17) *If $f \in R$ and $D(f) = \mathrm{Spec}(R)$, then f is a unit.*

Proof.

(1) This is a direct consequence of (2) and (3).
(2) Let $\mathfrak{A}$ be the set of all proper ideals of R. This set is ordered by inclusion and is non-empty, since $(0) \in \mathfrak{A}$ is a proper ideal. Let A be a totally ordered subset of R. $\bigcup_{I \in A} I$ is in fact an ideal. Since $1 \notin I$ for all $I \in A$, the union does not contain 1 and thus is proper. Hence $\bigcup_{I \in A} I$ is in $\mathfrak{A}$ and is an upper bound for the set A. Thus by Zorn's lemma $\mathfrak{A}$ has a maximal element, which is the sought-after maximal ideal.
(3) Since R is nonzero, it contains a maximal ideal which is a prime ideal. Thus the set $\mathfrak{A}$ of all prime ideals of R is nonempty. $\mathfrak{A}$ is ordered by reverse-inclusion. Let A be a totally ordered subset of $\mathfrak{A}$. It's pretty clear that $J = \bigcap_{I \in A} I$ is in fact an ideal. Not so clear, however, is that it is prime. Let $xy \in J$. Then $xy \in I$ for all $I \in A$. Now let $B = \{I \in A | y \in I\}$. Let $K = \bigcap_{I \in B} I$. Since A is totally ordered, either $K = J$ (and we're done, since then $y \in J$) or $K \supset J$ and for all $I \in A$ such that I is properly contained in K, we have $y \notin I$. But that means that for all those $I, x \in I$, since they are prime. Hence $x \in J$. In either case, J is prime as desired. Hence by Zorn's lemma we get a maximal element which in this case is a minimal prime ideal.
(4) This is the same exact argument as (3) except you only consider prime ideals contained in $\mathfrak{p}$ and containing I.
(5) (T) is the smallest ideal containing T. Hence if $T \subset I$, some ideal, then $(T) \subset I$ as well. Hence if $I \in V(T)$, then $I \in V((T))$ as well. The other inclusion is obvious.
(6) Since $I \subset \sqrt{I}, V(\sqrt{I}) \subset V(I)$. Now let $\mathfrak{p} \in V(I)$. Let $x \in \sqrt{I}$. Then $x^n \in I$ for some n. Hence $x^n \in \mathfrak{p}$. But since $\mathfrak{p}$ is prime, a boring induction argument gets you that $x \in \mathfrak{p}$. Hence $\sqrt{I} \subset \mathfrak{p}$ and $\mathfrak{p} \in V(\sqrt{I})$.

(7) Let $f \in R \setminus \sqrt{I}$. Then $f^n \notin I$ for all n. Hence $S = \{1, f, f^2, \ldots\}$ is a multiplicative subset, not containing 0. Take a prime ideal $\bar{\mathfrak{p}} \subset S^{-1}R$ containing $S^{-1}I$. Then the pull-back $\mathfrak{p}$ in R of $\bar{\mathfrak{p}}$ is a prime ideal containing I that does not intersect S. This shows that $\bigcap_{I \subset \mathfrak{p}} \mathfrak{p} \subset \sqrt{I}$. Now if $a \in \sqrt{I}$, then $a^n \in I$ for some n. Hence if $I \subset \mathfrak{p}$, then $a^n \in \mathfrak{p}$. But since $\mathfrak{p}$ is prime, we have $a \in \mathfrak{p}$. Thus the equality is shown.
(8) I is not the unit ideal iff I is contained in some maximal ideal (to see this, apply (2) to the ring R/I) which is therefore prime.
(9) If $\mathfrak{p} \in V(I) \cup V(J)$, then $I \subset \mathfrak{p}$ or $J \subset \mathfrak{p}$ which means that $I \cap J \subset \mathfrak{p}$. Now if $I \cap J \subset \mathfrak{p}$, then $IJ \subset \mathfrak{p}$ and hence either I of J is in $\mathfrak{p}$, since $\mathfrak{p}$ is prime.
(10) $\mathfrak{p} \in \bigcap_{a \in A} V(I_a) \Leftrightarrow I_a \subset \mathfrak{p}, \forall a \in A \Leftrightarrow \mathfrak{p} \in V(\cup_{a \in A} I_a)$
(11) If $\mathfrak{p}$ is a prime ideal and $f \in R$, then either $f \in \mathfrak{p}$ or $f \notin \mathfrak{p}$ (strictly) which is what the disjoint union says.
(12) If $a \in R$ is nilpotent, then $a^n = 0$ for some n. Hence $a^n \in \mathfrak{p}$ for any prime ideal. Thus $a \in \mathfrak{p}$ as can be shown by induction and $D(f) = \emptyset$. Now, as shown in (7), if $a \in R$ is not nilpotent, then there is a prime ideal that does not contain it.
(13) $f \in \mathfrak{p} \Leftrightarrow uf \in \mathfrak{p}$, since u is invertible.
(14) If $\mathfrak{p} \notin V(I)$, then $\exists f \in I \setminus \mathfrak{p}$. Then $f \notin \mathfrak{p}$ so $\mathfrak{p} \in D(f)$. Also if $\mathfrak{q} \in D(f)$, then $f \notin \mathfrak{q}$ and thus I is not contained in $\mathfrak{q}$. Thus $D(f) \cap V(I) = \emptyset$.
(15) If $fg \in \mathfrak{p}$, then $f \in \mathfrak{p}$ or $g \in \mathfrak{p}$. Hence if $f \notin \mathfrak{p}$ and $g \notin \mathfrak{p}$, then $fg \notin \mathfrak{p}$. Since $\mathfrak{p}$ is an ideal, if $fg \notin \mathfrak{p}$, then $f \notin \mathfrak{p}$ and $g \notin \mathfrak{p}$.
(16) $\mathfrak{p} \in \bigcup_{i \in I} D(f_i) \Leftrightarrow \exists i \in I, f_i \notin \mathfrak{p} \Leftrightarrow \mathfrak{p} \in \operatorname{Spec}(R) \setminus V(\{f_i\}_{i \in I})$
(17) If $D(f) = \operatorname{Spec}(R)$, then $V(f) = \emptyset$ and hence $fR = R$, so f is a unit.

□

The lemma implies that the subsets $V(T)$ from Definition 10.16.1 form the closed subsets of a topology on $\operatorname{Spec}(R)$. And it also shows that the sets $D(f)$ are open and form a basis for this topology.

Definition 10.16.3. Let R be a ring. The topology on $\operatorname{Spec}(R)$ whose closed sets are the sets $V(T)$ is called the *Zariski* topology. The open subsets $D(f)$ are called the *standard opens* of $\operatorname{Spec}(R)$.

It should be clear from context whether we consider $\operatorname{Spec}(R)$ just as a set or as a topological space.

Lemma 10.16.4. *Suppose that $\varphi : R \to R'$ is a ring homomorphism. The induced map*

$$\operatorname{Spec}(\varphi) : \operatorname{Spec}(R') \longrightarrow \operatorname{Spec}(R), \quad \mathfrak{p}' \longmapsto \varphi^{-1}(\mathfrak{p}')$$

is continuous for the Zariski topologies. In fact, for any element $f \in R$ we have $\operatorname{Spec}(\varphi)^{-1}(D(f)) = D(\varphi(f))$.

Proof. It is basic notion (41) that $\mathfrak{p} := \varphi^{-1}(\mathfrak{p}')$ is indeed a prime ideal of R. The last assertion of the lemma follows directly from the definitions, and implies the first. □

If $\varphi' : R' \to R''$ is a second ring homomorphism then the composition

$$\operatorname{Spec}(R') \longrightarrow \operatorname{Spec}(R') \longrightarrow \operatorname{Spec}(R'')$$

equals $\operatorname{Spec}(\varphi' \circ \varphi)$. In other words, Spec is a contravariant functor from the category of rings to the category of topological spaces.

Lemma 10.16.5. *Let R be a ring. Let $S \subset R$ be a multiplicative subset. The map $R \to S^{-1}R$ induces via the functoriality of* Spec *a homeomorphism*

$$\mathrm{Spec}(S^{-1}R) \longrightarrow \{\mathfrak{p} \in \mathrm{Spec}(R) \mid S \cap \mathfrak{p} = \emptyset\}$$

where the topology on the right hand side is that induced from the Zariski topology on $\mathrm{Spec}(R)$*. The inverse map is given by $\mathfrak{p} \mapsto S^{-1}\mathfrak{p}$.*

Proof. Denote the right hand side of the arrow of the lemma by D. Choose a prime $\mathfrak{p}' \subset S^{-1}R$ and let $\mathfrak{p}$ the inverse image of $\mathfrak{p}'$ in R. Since $\mathfrak{p}'$ does not contain 1 we see that $\mathfrak{p}$ does not contain any element of S. Hence $\mathfrak{p} \in D$ and we see that the image is contained in D. Let $\mathfrak{p} \in D$. By assumption the image $\overline{S}$ does not contain 0. By basic notion (54) $\overline{S}^{-1}(R/\mathfrak{p})$ is not the zero ring. By basic notion (62) we see $S^{-1}R/S^{-1}\mathfrak{p} = \overline{S}^{-1}(R/\mathfrak{p})$ is a domain, and hence $S^{-1}\mathfrak{p}$ is a prime. The equality of rings also shows that the inverse image of $S^{-1}\mathfrak{p}$ in R is equal to $\mathfrak{p}$, because $R/\mathfrak{p} \to \overline{S}^{-1}(R/\mathfrak{p})$ is injective by basic notion (55). This proves that the map $\mathrm{Spec}(S^{-1}R) \to \mathrm{Spec}(R)$ is bijective onto D with inverse as given. It is continuous by Lemma 10.16.4. Finally, let $D(g) \subset \mathrm{Spec}(S^{-1}R)$ be a standard open. Write $g = h/s$ for some $h \in R$ and $s \in S$. Since g and $h/1$ differ by a unit we have $D(g) = D(h/1)$ in $\mathrm{Spec}(S^{-1}R)$. Hence by Lemma 10.16.4 and the bijectivity above the image of $D(g) = D(h/1)$ is $D \cap D(h)$. This proves the map is open as well. □

Lemma 10.16.6. *Let R be a ring. Let $f \in R$. The map $R \to R_f$ induces via the functoriality of* Spec *a homeomorphism*

$$\mathrm{Spec}(R_f) \longrightarrow D(f) \subset \mathrm{Spec}(R).$$

The inverse is given by $\mathfrak{p} \mapsto \mathfrak{p} \cdot R_f$.

Proof. This is a special case of Lemma 10.16.5. □

It is not the case that every "affine open" of a spectrum is a standard open. See Example 10.25.4.

Lemma 10.16.7. *Let R be a ring. Let $I \subset R$ be an ideal. The map $R \to R/I$ induces via the functoriality of* Spec *a homeorphism*

$$\mathrm{Spec}(R/I) \longrightarrow V(I) \subset \mathrm{Spec}(R).$$

The inverse is given by $\mathfrak{p} \mapsto \mathfrak{p}/I$.

Proof. It is immediate that the image is contained in $V(I)$. On the other hand, if $\mathfrak{p} \in V(I)$ then $\mathfrak{p} \supset I$ and we may consider the ideal $\mathfrak{p}/I \subset R/I$. Using basic notion (51) we see that $(R/I)/(\mathfrak{p}/I) = R/\mathfrak{p}$ is a domain and hence $\mathfrak{p}/I$ is a prime ideal. From this it is immediately clear that the image of $D(f + I)$ is $D(f) \cap V(I)$, and hence the map is a homeomorphism. □

Remark 10.16.8. A fundamental commutative diagram associated to $\varphi : R \to S$, $\mathfrak{q} \subset S$ and $\mathfrak{p} = \varphi^{-1}(\mathfrak{q})$ is the following

$$\begin{array}{ccccccc}
\kappa(\mathfrak{q}) = S_\mathfrak{q}/\mathfrak{q}S_\mathfrak{q} & \longleftarrow & S_\mathfrak{q} & \longleftarrow & S & \longrightarrow & S/\mathfrak{q} \longrightarrow \kappa(\mathfrak{q}) = \text{f.f.}(S/\mathfrak{q}) \\
\uparrow & & \uparrow & & \uparrow & & \uparrow \\
\kappa(\mathfrak{p}) \otimes_R S = S_\mathfrak{p}/\mathfrak{p}S_\mathfrak{p} & \longleftarrow & S_\mathfrak{p} & \longleftarrow & S & \longrightarrow & S/\mathfrak{p}S \longrightarrow (R \setminus \mathfrak{p})^{-1}S/\mathfrak{p}S \\
\uparrow & & \uparrow & & \uparrow & & \uparrow \\
\kappa(\mathfrak{p}) = R_\mathfrak{p}/\mathfrak{p}R_\mathfrak{p} & \longleftarrow & R_\mathfrak{p} & \longleftarrow & R & \longrightarrow & R/\mathfrak{p} \longrightarrow \kappa(\mathfrak{p}) = \text{f.f.}(R/\mathfrak{p})
\end{array}$$

In this diagram the arrows in the outer left and outer right columns are identical. The horizontal maps induce on the associated spectrums always a homeomorphism onto the image. The lower two rows of the diagram make sense without assuming $\mathfrak{q}$ exists. The lower squares induce fibre squares of topological spaces. This diagram shows that $\mathfrak{p}$ is in the image of the map on Spec if and only if $S \otimes_R \kappa(\mathfrak{p})$ is not the zero ring.

Lemma 10.16.9. *Let $\varphi : R \to S$ be a ring map. Let $\mathfrak{p}$ be a prime of R. The following are equivalent*

(1) *$\mathfrak{p}$ is in the image of* $\mathrm{Spec}(S) \to \mathrm{Spec}(R)$,
(2) $S \otimes_R \kappa(\mathfrak{p}) \neq 0$,
(3) $S_\mathfrak{p}/\mathfrak{p}S_\mathfrak{p} \neq 0$,
(4) $(S/\mathfrak{p}S)_\mathfrak{p} \neq 0$, *and*
(5) $\mathfrak{p} = \varphi^{-1}(\mathfrak{p}S)$.

Proof. We have already seen the equivalence of the first two in Remark 10.16.8. The others are just reformulations of this. □

Lemma 10.16.10. *Let R be a ring. The space* $\mathrm{Spec}(R)$ *is quasi-compact.*

Proof. It suffices to prove that any covering of $\mathrm{Spec}(R)$ by standard opens can be refined by a finite covering. Thus suppose that $\mathrm{Spec}(R) = \cup D(f_i)$ for a set of elements $\{f_i\}_{i \in I}$ of R. This means that $\cap V(f_i) = \emptyset$. According to Lemma 10.16.2 this means that $V(\{f_i\}) = \emptyset$. According to the same lemma this means that the ideal generated by the f_i is the unit ideal of R. This means that we can write 1 as a *finite* sum: $1 = \sum_{i \in J} r_i f_i$ with $J \subset I$ finite. And then it follows that $\mathrm{Spec}(R) = \cup_{i \in J} D(f_i)$. □

Lemma 10.16.11. *Let R be a ring. The topology on $X = \mathrm{Spec}(R)$ has the following properties:*

(1) *X is quasi-compact,*
(2) *X has a basis for the topology consisting of quasi-compact opens, and*
(3) *the intersection of any two quasi-compact opens is quasi-compact.*

Proof. The spectrum of a ring is quasi-compact, see Lemma 10.16.10. It has a basis for the topology consisting of the standard opens $D(f) = \mathrm{Spec}(R_f)$ (Lemma 10.16.6) which are quasi-compact by the first remark. The intersection of two standard opens is quasi-compact as $D(f) \cap D(g) = D(fg)$. Given any two quasi-compact opens $U, V \subset X$ we may write $U = D(f_1) \cup \ldots \cup D(f_n)$ and $V = D(g_1) \cup \ldots \cup D(g_m)$. Then $U \cap V = \bigcup D(f_i g_j)$ which is quasi-compact. □

10.17. Local rings

Local rings are the bread and butter of algebraic geometry.

Definition 10.17.1. A *local ring* is a ring with exactly one maximal ideal. The maximal ideal is often denoted $\mathfrak{m}_R$ in this case. We often say "let $(R, \mathfrak{m}, \kappa)$ be a local ring" to indicate that R is local, $\mathfrak{m}$ is its unique maximal ideal and $\kappa = R/\mathfrak{m}$ is its residue field. A *local homomorphism of local rings* is a ring map $\varphi : R \to S$ such that R and S are local rings and such that $\varphi(\mathfrak{m}_R) \subset \mathfrak{m}_S$. If it is given that R and S are local rings, then the phrase "*local ring map* $\varphi : R \to S$" means that φ is a local homomorphism of local rings.

A field is a local ring. Any ring map between fields is a local homomorphism of local rings.

Lemma 10.17.2. *Let R be a ring. The following are equivalent:*

1. *R is a local ring,*
2. *$\mathrm{Spec}(R)$ has exactly one closed point,*
3. *R has a maximal ideal $\mathfrak{m}$ and every element of $R \setminus \mathfrak{m}$ is a unit, and*
4. *R is not the zero ring and for every $x \in R$ either x or $1 - x$ is invertible or both.*

Proof. Let R be a ring, and $\mathfrak{m}$ a maximal ideal. If $x \in R \setminus \mathfrak{m}$, and x is not a unit then there is a maximal ideal $\mathfrak{m}'$ containing x. Hence R has at least two maximal ideals. Conversely, if $\mathfrak{m}'$ is another maximal ideal, then choose $x \in \mathfrak{m}'$, $x \notin \mathfrak{m}$. Clearly x is not a unit. This proves the equivalence of (1) and (3). The equivalence (1) and (2) is tautological. If R is local then (4) holds since x is either in $\mathfrak{m}$ or not. If (4) holds, and $\mathfrak{m}$, $\mathfrak{m}'$ are distinct maximal ideals then we may choose $x \in R$ such that $x \bmod \mathfrak{m}' = 0$ and $x \bmod \mathfrak{m} = 1$ by the Chinese remainder theorem (Lemma 10.14.4). This element x is not invertible and neither is $1 - x$ which is a contradiction. Thus (4) and (1) are equivalent. □

The localization $R_\mathfrak{p}$ of a ring R at a prime $\mathfrak{p}$ is a local ring with maximal ideal $\mathfrak{p}R_\mathfrak{p}$. Namely, the quotient $R_\mathfrak{p}/\mathfrak{p}R_\mathfrak{p}$ is the fraction field of the domain $R/\mathfrak{p}$ and every element of $R_\mathfrak{p}$ which is not contained in $\mathfrak{p}R_\mathfrak{p}$ is invertible.

Lemma 10.17.3. *Let $\varphi : R \to S$ be a ring map. Assume R and S are local rings. The following are equivalent:*

1. *φ is a local ring map,*
2. *$\varphi(\mathfrak{m}_R) \subset \mathfrak{m}_S$, and*
3. *$\varphi^{-1}(\mathfrak{m}_S) = \mathfrak{m}_R$.*
4. *For any $x \in R$, if $\varphi(x)$ is invertible in S, then x is invertible in R.*

Proof. Conditions (1) and (2) are equivalent by definition. If (3) holds then (2) holds. Conversely, if (2) holds, then $\varphi^{-1}(\mathfrak{m}_S)$ is a prime ideal containing the maximal ideal $\mathfrak{m}_R$, hence $\varphi^{-1}(\mathfrak{m}_S) = \mathfrak{m}_R$. Finally, (4) is the contrapositive of (2) by Lemma 10.17.2. □

Let $\varphi : R \to S$ be a ring map. Let $\mathfrak{q} \subset S$ be a prime and set $\mathfrak{p} = \varphi^{-1}(\mathfrak{q})$. Then the induced ring map $R_\mathfrak{p} \to S_\mathfrak{q}$ is a local ring map.

10.18. Nakayama's lemma

We quote from [**Mat70**]: "This simple but important lemma is due to T. Nakayama, G. Azumaya and W. Krull. Priority is obscure, and although it is usually called the Lemma of Nakayama, late Prof. Nakayama did not like the name."

We recall that the radical $\text{rad}(R)$ of a ring R is the intersection of all maximal ideals of R. If R is local then $\text{rad}(R)$ is the maximal ideal of R.

Lemma 10.18.1 (Nakayama's lemma)**.** *Let R be a ring, let M be an R-module, and let $I \subset R$ be an ideal.*

1. *If $IM = M$ and M is finite, then there exists a $f \in 1 + I$ such that $fM = 0$.*
2. *If $IM = M$, M is finite, and $I \subset \text{rad}(R)$, then $M = 0$.*
3. *If $N, N' \subset M$, $M = N + IN'$, and N' is finite, then there exists a $f \in 1+I$ such that $M_f = N_f$.*
4. *If $N, N' \subset M$, $M = N + IN'$, N' is finite, and $I \subset \text{rad}(R)$, then $M = N$.*
5. *If $N \to M$ is a module map, $N/IN \to M/IM$ is surjective, and M is finite, then there exists a $f \in 1 + I$ such that $N_f \to M_f$ is surjective.*
6. *If $N \to M$ is a module map, $N/IN \to M/IM$ is surjective, M is finite, and $I \subset \text{rad}(R)$, then $N \to M$ is surjective.*
7. *If $x_1, \ldots, x_n \in M$ generate M/IM and M is finite, then there exists an $f \in 1 + I$ such that $x_1, \ldots, x_n$ generate M_f over R_f.*
8. *If $x_1, \ldots, x_n \in M$ generate M/IM, M is finite, and $I \subset \text{rad}(R)$, then M is generated by $x_1, \ldots, x_n$.*
9. *If $IM = M$, I is nilpotent, then $M = 0$.*
10. *If $N, N' \subset M$, $M = N + IN'$, and I is nilpotent then $M = N$.*
11. *If $N \to M$ is a module map, I is nilpotent, and $N/IN \to M/IM$ is surjective, then $N \to M$ is surjective.*
12. *If $\{x_\alpha\}_{\alpha \in A}$ is a set of elements of M which generate M/IM and I is nilpotent, then M is generated by the x_α.*

Proof. Proof of (1). Choose generators $y_1, \ldots, y_m$ of M over R. For each i we can write $y_i = \sum z_{ij} y_j$ with $z_{ij} \in I$. In other words $\sum_j (\delta_{ij} - z_{ij}) y_j = 0$. Let f be the determinant f of the $m \times m$ matrix $A = (\delta_{ij} - z_{ij})$. Note that $f \in 1 + I$. By Lemma 10.14.5 there exists an $m \times m$ matrix B such that $BA = f 1_{m \times m}$. Writing out we see that $f y_j = \sum_{i,j} b_{hi} a_{ij} y_j = 0$ for every j. This implies that f annihilates M.

By Lemma 10.16.2 an element of $1 + \text{rad}(R)$ is invertible element of R. Hence we see that (1) implies (2). We obtain (3) by applying (1) to M/N which is finite as N' is finite. We obtain (4) by applying (2) to M/N which is finite as N' is finite. We obtain (5) by applying (3) to M and the submodules $\text{Im}(N \to M)$ and M. We obtain (6) by applying (4) to M and the submodules $\text{Im}(N \to M)$ and M. We obtain (7) by applying (5) to the map $R^{\oplus n} \to M$, $(a_1, \ldots, a_n) \mapsto a_1 x_1 + \ldots + a_n x_n$. We obtain (8) by applying (6) to the map $R^{\oplus n} \to M$, $(a_1, \ldots, a_n) \mapsto a_1 x_1 + \ldots + a_n x_n$.

Part (9) holds because if $M = IM$ then $M = I^n M$ for all $n \geq 0$ and I being nilpotent means $I^n = 0$ for some $n \gg 0$. Parts (10), (11), and (12) follow from (9) by the arguments used above. $\square$

10.19. Open and closed subsets of spectra

It turns out that open and closed subsets of a spectrum correspond to idempotents of the ring.

Lemma 10.19.1. *Let R be a ring. Let $e \in R$ be an idempotent. In this case*

$$\mathrm{Spec}(R) = D(e) \coprod D(1-e).$$

Proof. Note that an idempotent e of a domain is either 1 or 0. Hence we see that

$$\begin{aligned} D(e) &= \{\mathfrak{p} \in \mathrm{Spec}(R) \mid e \notin \mathfrak{p}\} \\ &= \{\mathfrak{p} \in \mathrm{Spec}(R) \mid e \neq 0 \text{ in } \kappa(\mathfrak{p})\} \\ &= \{\mathfrak{p} \in \mathrm{Spec}(R) \mid e = 1 \text{ in } \kappa(\mathfrak{p})\} \end{aligned}$$

Similarly we have

$$\begin{aligned} D(1-e) &= \{\mathfrak{p} \in \mathrm{Spec}(R) \mid 1-e \notin \mathfrak{p}\} \\ &= \{\mathfrak{p} \in \mathrm{Spec}(R) \mid e \neq 1 \text{ in } \kappa(\mathfrak{p})\} \\ &= \{\mathfrak{p} \in \mathrm{Spec}(R) \mid e = 0 \text{ in } \kappa(\mathfrak{p})\} \end{aligned}$$

Since the image of e in any residue field is either 1 or 0 we deduce that $D(e)$ and $D(1-e)$ cover all of $\mathrm{Spec}(R)$. □

Lemma 10.19.2. *Let R_1 and R_2 be rings. Let $R = R_1 \times R_2$. The maps $R \to R_1$, $(x, y) \mapsto x$ and $R \to R_2$, $(x, y) \mapsto y$ induce continuous maps $\mathrm{Spec}(R_1) \to \mathrm{Spec}(R)$ and $\mathrm{Spec}(R_2) \to \mathrm{Spec}(R)$. The induced map*

$$\mathrm{Spec}(R_1) \coprod \mathrm{Spec}(R_2) \longrightarrow \mathrm{Spec}(R)$$

is a homeomorphism. In other words, the spectrum of $R = R_1 \times R_2$ is the disjoint union of the spectrum of R_1 and the spectrum of R_2.

Proof. Write $1 = e_1 + e_2$ with $e_1 = (1, 0)$ and $e_2 = (0, 1)$. Note that e_1 and $e_2 = 1 - e_1$ are idempotents. We leave it to the reader to show that $R_1 = R_{e_1}$ is the localization of R at e_1. Similarly for e_2. Thus the statement of the lemma follows from Lemma 10.19.1 combined with Lemma 10.16.6. □

We reprove the following lemma later after introducing a glueing lemma for functions. See Section 10.21.

Lemma 10.19.3. *Let R be a ring. For each $U \subset \mathrm{Spec}(R)$ which is open and closed there exists a unique idempotent $e \in R$ such that $U = D(e)$. This induces a 1-1 correspondence between open and closed subsets $U \subset \mathrm{Spec}(R)$ and idempotents $e \in R$.*

First proof of Lemma 10.19.3. Let $U \subset \mathrm{Spec}(R)$ be open and closed. Since U is closed it is quasi-compact by Lemma 10.16.10, and similarly for its complement. Write $U = \bigcup_{i=1}^n D(f_i)$ as a finite union of standard opens. Similarly, write $\mathrm{Spec}(R) \setminus U = \bigcup_{j=1}^m D(g_j)$ as a finite union of standard opens. Since $\emptyset = D(f_i) \cap D(g_j) = D(f_i g_j)$ we see that $f_i g_j$ is nilpotent by Lemma 10.16.2. Let $I = (f_1, \ldots, f_n) \subset R$ and let $J = (g_1, \ldots, g_m) \subset R$. Note that $V(J)$ equals U, that $V(I)$ equals the complement of U, so $\mathrm{Spec}(R) = V(I) \coprod V(J)$. By the remark on nilpotency above, we see that $(IJ)^N = (0)$ for some sufficiently large integer N. Since $\bigcup D(f_i) \cup \bigcup D(g_j) = \mathrm{Spec}(R)$ we see that $I + J = R$, see Lemma 10.16.2. By raising this equation to the $2N$th power we conclude that $I^N + J^N = R$. Write $1 = x + y$ with

$x \in I^N$ and $y \in J^N$. Then $1 = (x+y)^2 = x^2 + y^2$ because $I^N J^N = (0)$. Then $z = x - x^2 \in I^N \cap J^N$. Thus $zx = 0$ and $z^2 = 0$. Hence $(x-z) - (x-z)^2 = x - x^2 - z = 0$. In other words, $e = x - z$ is an idempotent contained in $I^N \subset I$, and the idempotent $e' = 1 - e = y + z$ is contained in $J^N \subset J$. This shows that the idempotent e maps to 1 in every residue field $\kappa(\mathfrak{p})$ for $\mathfrak{p} \in V(J)$ and that e maps to 0 in $\kappa(\mathfrak{p})$ for every $\mathfrak{p} \in V(I)$.

To see uniqueness suppose that e_1, e_2 are distinct idempotents in R. We have to show there exists a prime $\mathfrak{p}$ such that $e_1 \in \mathfrak{p}$ and $e_2 \notin \mathfrak{p}$, or conversely. Write $e_i' = 1 - e_i$. If $e_1 \neq e_2$, then $0 \neq e_1 - e_2 = e_1(e_2 + e_2') - (e_1 + e_1')e_2 = e_1 e_2' - e_1' e_2$. Hence either the idempotent $e_1 e_2' \neq 0$ or $e_1' e_2 \neq 0$. An idempotent is not nilpotent, and hence we find a prime $\mathfrak{p}$ such that either $e_1 e_2' \notin \mathfrak{p}$ or $e_1' e_2 \notin \mathfrak{p}$, by Lemma 10.16.2. It is easy to see this gives the desired prime. □

Lemma 10.19.4. *Let R be a nonzero ring. Then* $\mathrm{Spec}(R)$ *is connected if and only if R has no nontrivial idempotents.*

Proof. Obvious from Lemma 10.19.3. □

Lemma 10.19.5. *Let R be a ring. Let I be a finitely generated ideal. Assume that $I = I^2$. Then $V(I)$ is open and closed in* $\mathrm{Spec}(R)$*, and $R/I \cong R_e$ for some idempotent $e \in R$.*

Proof. By Nakayama's Lemma 10.18.1 there exists an element $f = 1 + i$, $i \in I$ in R such that $fI = 0$. It follows that $V(I) = D(f)$ by a simple argument. Also, $0 = fi = i + i^2$, and hence $f^2 = 1 + i + i + i^2 = 1 + i = f$, so f is an idempotent. Consider the canonical map $R \to R_f$. It is surjective since $x/f^n = x/f = xf/f^2 = xf/f = x/1$ in R_f. Any element of I is in the kernel since $fI = 0$. If $x \mapsto 0$ in R_f, then $f^n x = 0$ for some $n > 0$ and hence $(1+i)x = 0$ hence $x \in I$. □

10.20. Connected components of spectra

Connected components of spectra are not as easy to understand as one may think at first. This is because we are used to the topology of locally connected spaces, but the spectrum of a ring is in general not locally connected.

Lemma 10.20.1. *Let R be a ring. Let $T \subset \mathrm{Spec}(R)$ be a subset of the spectrum. The following are equivalent*

1. *T is closed and is a union of connected components of* $\mathrm{Spec}(R)$*,*
2. *T is an intersection of open and closed subsets of* $\mathrm{Spec}(R)$*, and*
3. *$T = V(I)$ where $I \subset R$ is an ideal generated by idempotents.*

Moreover, the ideal in (3) if it exists is unique.

Proof. By Lemma 10.16.11 and Topology, Lemma 5.11.12 we see that (1) and (2) are equivalent. Assume (2) and write $T = \bigcap U_\alpha$ with $U_\alpha \subset \mathrm{Spec}(R)$ open and closed. Then $U_\alpha = D(e_\alpha)$ for some idempotent $e_\alpha \in A$ by Lemma 10.19.3. Then setting $I = (1 - e_\alpha)$ we see that $T = V(I)$, i.e., (3) holds. Finally, assume (3). Write $T = V(I)$ and $I = (e_\alpha)$ for some collection of idempotents e_α. Then it is clear that $T = \bigcap V(e_\alpha) = \bigcap D(1 - e_\alpha)$.

Suppose that I is an ideal generated by idempotents. Let $e \in R$ be an idempotent such that $V(I) \subset V(e)$. Then by Lemma 10.16.2 we see that $e^n \in I$ for some $n \geq 1$. As e is an idempotent this means that $e \in I$. Hence we see that I is generated

by exactly those idempotents e such that $T \subset V(e)$. In other words, the ideal I is completely determined by the closed subset T which proves uniqueness. □

Lemma 10.20.2. *Let R be a ring. A connected component of $\mathrm{Spec}(R)$ is of the form $V(I)$, where I is an ideal generated by idempotents such that every idempotent of R either maps to 0 or 1 in R/I.*

Proof. Let $\mathfrak{p}$ be a prime of R. By Lemma 10.16.11 we have see that the hypotheses of Topology, Lemma 5.11.10 are satisfied for the topological space $\mathrm{Spec}(R)$. Hence the connected component of $\mathfrak{p}$ in $\mathrm{Spec}(R)$ is the intersection of open and closed subsets of $\mathrm{Spec}(R)$ containing $\mathfrak{p}$. Hence it equals $V(I)$ where I is generated by the idempotents $e \in R$ such that e maps to 0 in $\kappa(\mathfrak{p})$, see Lemma 10.19.3. Any idempotent e which is not in this collection clearly maps to 1 in R/I. □

10.21. Glueing functions

In this section we show that given an open covering

$$\mathrm{Spec}(R) = \bigcup\nolimits_{i=1}^{n} D(f_i)$$

by standard opens, and given an element $h_i \in R_{f_i}$ for each i such that $h_i = h_j$ as elements of $R_{f_i f_j}$ then there exists a unique $h \in R$ such that the image of h in R_{f_i} is h_i. This result can be interpreted in two ways:

(1) The rule $D(f) \mapsto R_f$ is a sheaf of rings on the standard opens, see Sheaves, Section 6.30.
(2) If we think of elements of R_f as the "algebraic" or "regular" functions on $D(f)$, then these glue as would continuous, resp. differentiable functions on a topological, resp. differentiable manifold.

At the end of this section we use this result to reprove the lemma describing open and closed subsets in terms of idempotents.

Lemma 10.21.1. *Let R be a ring, and let $f_1, f_2, \ldots f_n \in R$ generate the unit ideal in R. Then the following sequence is exact:*

$$0 \longrightarrow R \longrightarrow \bigoplus\nolimits_i R_{f_i} \longrightarrow \bigoplus\nolimits_{i,j} R_{f_i f_j}$$

where the maps $\alpha : R \longrightarrow \bigoplus_i R_{f_i}$ and $\beta : \bigoplus_i R_{f_i} \longrightarrow \bigoplus_{i,j} R_{f_i f_j}$ are defined as

$$\alpha(x) = \left(\frac{x}{1}, \ldots, \frac{x}{1}\right) \text{ and } \beta\left(\frac{x_1}{f_1}, \ldots, \frac{x_n}{f_n}\right) = \left(\frac{x_i}{f_i} - \frac{x_j}{f_j} \text{ in } R_{f_i f_j}\right).$$

Proof. We first show that α is injective, and then that the image of α equals the kernel of β. Assume there exists $x \in R$ such that $\alpha(x) = (0, \ldots, 0)$. Then $\frac{x}{1} = 0$ in R_{f_i} for all i. This means, for all i, there exists a number n_i such that

$$f_i^{n_i} x = 0$$

Since the f_i generate R, we can pick a_i so

$$1 = \sum\nolimits_{i=1}^{n} a_i f_i$$

Then for all $M \geq \sum n_i$, we have

$$1^M = \left(\sum a_i f_i\right)^M,$$

where each term has a factor of at least $f_i^{n_i}$ for some i. Therefore,

$$x = 1x = 1^M x = \left(\sum a_i f_i\right)^M x = 0.$$

Thus, if $\alpha(x) = 0$, $x = 0$ and α is injective. We check that the image of α equals the kernel of β. First, note that for $x \in R$,

$$\beta(\alpha(x)) = \beta\left(\frac{x}{1}, \ldots, \frac{x}{1}\right) = (\frac{x}{1} - \frac{x}{1} \; in \; R_{f_i f_j}) = 0.$$

Therefore, the image of α is in the kernel of β, and it remains only to verify that if

$$\beta\left(\frac{x_1}{f_1}, \ldots, \frac{x_n}{f_n}\right) = 0,$$

then there exists $x \in R$ so that for all i,

$$\frac{x}{1} = \frac{x_i}{f_i}$$

Assume we have $x_1, \ldots, x_n$ such that

$$\beta\left(\frac{x_1}{f_1}, \ldots, \frac{x_n}{f_n}\right) = 0.$$

Then, for all pairs i, j, there exists an n_{ij} such that

$$f_i^{n_{ij}} f_j^{n_{ij}} (f_j x_i - f_i x_j) = 0$$

Choosing N so $N \geq n_{ij}$ for all i, j, we see that

$$f_i^N f_j^N (f_j x_i - f_i x_j) = 0$$

Define elements $\widetilde{x_i}$ and $\widetilde{f_i}$ as follows:

$$\widetilde{f_i} = f_i^{N+1}, \qquad \widetilde{x_i} = f_i^N x_i.$$

Notice that

$$\frac{\widetilde{x_i}}{\widetilde{f_i}} = \frac{x_i}{f_i}.$$

Also, we can use this to rewrite the above equation to get the following equality, for all i, j,

$$\widetilde{f_j}\widetilde{x_i} = \widetilde{f_i}\widetilde{x_j}.$$

Since $f_1, \ldots, f_n$ generate R, we clearly have that $\widetilde{f_1}, \ldots, \widetilde{f_n}$ also generate R. Therefore, there exist $a_1, \ldots, a_n$ in R so that

$$1 = \sum_{i=1}^n a_i \widetilde{f_i}$$

Therefore, we finally conclude that for all i,

$$\frac{x_i}{f_i} = \frac{\widetilde{x_i}}{\widetilde{f_i}} = \sum_{j=1}^n \frac{a_j \widetilde{f_j} \widetilde{x_i}}{\widetilde{f_i}} = \sum_{j=1}^n \frac{a_j \widetilde{f_i} \widetilde{x_j}}{\widetilde{f_i}} = \frac{\sum_{j=1}^n a_j \widetilde{x_j}}{1}.$$

Thus, we have

$$\alpha\left(\sum_{j=1}^n a_j \widetilde{x_j}\right) = \left(\frac{x_1}{f_1}, \ldots, \frac{x_n}{f_n}\right),$$

as required. There the sequence is exact. □

Lemma 10.21.2. *Let R be a ring. Let $f_1, \ldots, f_n$ be elements of R generating the unit ideal. Let M be an R-module. The sequence*

$$0 \to M \xrightarrow{\alpha} \bigoplus\nolimits_{i=1}^{n} M_{f_i} \xrightarrow{\beta} \bigoplus\nolimits_{i,j=1}^{n} M_{f_i f_j}$$

is exact, where $\alpha(m) = (m/1, \ldots, m/1)$ and $\beta(m_1/f_1^{e_1}, \ldots, m_n/f_n^{e_n}) = (m_i/f_i^{e_i} - m_j/f_j^{e_j})_{(i,j)}$.

Proof. The same as the proof of Lemma 10.21.1. □

Second proof of Lemma 10.19.3. Having assured ourselves (Lemma 10.21.1) that for generators $f_1, \ldots, f_n$ for the unit ideal of a ring R the sequence

$$0 \to R \to \bigoplus\nolimits_{i=1}^{n} R_{f_i} \to \bigoplus\nolimits_{i,j} R_{f_i f_j}$$

is exact, we now provide an alternate proof of the surjectivity of the map from idempotents e of R to open and closed subsets of $\mathrm{Spec}(R)$ presented in Lemma 10.19.3. Let $U \subset \mathrm{Spec}(R)$ be open and closed, and W be its complement. We can write U and V as unions of standard opens such that $U = \bigcup_{i=1}^{n} D(f_i)$ and $W = \bigcup_{j=1}^{m} D(g_j)$. Since $\mathrm{Spec}(R) = \bigcup D(f_i) \cup \bigcup D(g_j)$, we observe that the collection $\{f_i; g_j\}$ must generate the unit ideal in R by Lemma 10.16.2. So the following sequence is exact.

(10.21.2.1)

$$0 \to R \xrightarrow{\alpha} \bigoplus\nolimits_{i=1}^{n} R_{f_i} \oplus \bigoplus\nolimits_{j=1}^{m} R_{g_j} \to \bigoplus\nolimits_{i_1,i_2} R_{f_{i_1} f_{i_2}} \oplus \bigoplus\nolimits_{i,j} R_{f_i g_j} \oplus \bigoplus\nolimits_{j_1,j_2} R_{g_{j_1} g_{j_2}}$$

However, notice that for any pair i, j, $D(f_i) \cap D(g_j) = \emptyset$ since $D(f_i) \subset U$ and $D(g_j) \subset W$). From part (15) of Lemma 10.16.2 we recall that $D(f_i g_j) = D(f_i) \cap D(g_j) = \emptyset$. Therefore by Lemma 10.16.5 $\mathrm{Spec}(R_{f_i g_j}) = D(f_i g_j) = \emptyset$, implying that $R_{f_i g_j}$ is the zero ring for each pair i, j by part (3) of Lemma 10.16.2. Consider the element $(1, \ldots, 1, 0, \ldots, 0) \in \bigoplus_{i=1}^{n} R_{f_i} \oplus \bigoplus_{j=1}^{m} R_{g_j}$ whose coordinates are 1 in each R_{f_i} and 0 in each R_{g_j}. This is sent to 0 under the map

$$\beta : \bigoplus\nolimits_{i=1}^{n} R_{f_i} \oplus \bigoplus\nolimits_{j=1}^{m} R_{g_j} \to \bigoplus\nolimits_{i_1,i_2} R_{f_{i_1} f_{i_2}} \oplus \bigoplus\nolimits_{j_1,j_2} R_{g_{j_1} g_{j_2}}$$

so by the exactness of the sequence (10.21.2.1), there must be some element of R whose image under α is $(1, \ldots, 1, 0, \ldots, 0)$. Call it e. We see that $\alpha(e^2) = \alpha(e)^2 = (1, \ldots, 1, 0, \ldots, 0) = \alpha(e)$. Since α is injective, $e = e^2$ in R and e is an idempotent of R. We claim that $U = D(e)$. Notice that for arbitrary j, the map $R \to R_{g_j}$ maps e to 0. Therefore there must be some positive integer k_j such that $g_j^{k_j}(e - 0) = 0$ in R. Multiplying by e as necessary, we see that $(g_j e)^{k_j} = 0$, so $g_j e$ is nilpotent in R. By Lemma 10.16.2 $D(g_j) \cap D(e) = D(g_j e) = \emptyset$. So since $V = \bigcup D(g_j)$, $D(e) \cap V = \emptyset$ and $D(e) \subset U$. Furthermore, for arbitrary i, the map $R \to R_{f_i}$ maps e to 1, so there must be some l_i such that $f_i^{l_i}(e - 1) = 0$ in R. Hence $f_i^{l_i} e = f_i^{l_i}$. Suppose $\mathfrak{p} \in \mathrm{Spec}(R)$ contains e, then $\mathfrak{p}$ contains $f_i^{l_i} e = f_i^{l_i}$, and since $\mathfrak{p}$ is prime, $f_i \in \mathfrak{p}$. So $V(e) \subset V(f_i)$, implying that $D(f_i) \subset D(e)$. Therefore $U = \bigcup D(f_i) \subset D(e)$, and $U = D(e)$. Therefore any open and closed subset of $\mathrm{Spec}(R)$ is the standard open of an idempotent as desired. □

The following we have already seen above, but we state it explicitly here for convenience.

Lemma 10.21.3. *Let R be a ring. If $\mathrm{Spec}(R) = U \amalg V$ with both U and V open then $R \cong R_1 \times R_2$ with $U \cong \mathrm{Spec}(R_1)$ and $V \cong \mathrm{Spec}(R_2)$ via the maps in Lemma 10.19.2. Moreover, both R_1 and R_2 are localizations as well as quotients of the ring R.*

Proof. By Lemma 10.19.3 we have $U = D(e)$ and $V = D(1-e)$ for some idempotent e. By Lemma 10.21.1 we see that $R \cong R_e \times R_{1-e}$ (since clearly $R_{e(1-e)} = 0$ so the glueing condition is trivial; of course it is trivial to prove the product decomposition directly in this case). The lemma follows. □

Lemma 10.21.4. *Let R be a ring. Let $f_1, \ldots, f_n \in R$. Let M be an R-module. Then $M \to \bigoplus M_{f_i}$ is injective if and only if*

$$M \longrightarrow \bigoplus\nolimits_{i=1,\ldots,n} M, \quad m \longmapsto (f_1 m, \ldots, f_n m)$$

is injective.

Proof. The map $M \to \bigoplus M_{f_i}$ is injective if and only if for all $m \in M$ and $e_1, \ldots, e_n \geq 1$ such that $f_i^{e_i} m = 0$, $i = 1, \ldots, n$ we have $m = 0$. This clearly implies the displayed map is injective. Conversely, suppose the displayed map is injective and $m \in M$ and $e_1, \ldots, e_n \geq 1$ are such that $f_i^{e_i} m = 0$, $i = 1, \ldots, n$. If $e_i = 1$ for all i, then we immediately conclude that $m = 0$ from the injectivity of the displayed map. Next, we prove this holds for any such data by induction on $e = \sum e_i$. The base case is $e = n$, and we have just dealt with this. If some $e_i > 1$, then set $m' = f_i m$. By induction we see that $m' = 0$. Hence we see that $f_i m = 0$, i.e., we may take $e_i = 1$ which decreases e and we win. □

10.22. More glueing results

In this section we put a number of standard results of the form: if something is true for all members of a standard open covering then it is true. In fact, it often suffices to check things on the level of local rings as in the following lemma.

Lemma 10.22.1. *Let R be a ring.*

(1) *For an element x of an R-module M the following are equivalent*
   (a) *$x = 0$,*
   (b) *x maps to zero in $M_\mathfrak{p}$ for all $\mathfrak{p} \in \mathrm{Spec}(R)$,*
   (c) *x maps to zero in $M_\mathfrak{m}$ for all maximal ideals $\mathfrak{m}$ of R.*

   In other words, the map $M \to \prod_\mathfrak{m} M_\mathfrak{m}$ is injective.
(2) *Given an R-module M the following are equivalent*
   (a) *M is zero,*
   (b) *$M_\mathfrak{p}$ is zero for all $\mathfrak{p} \in \mathrm{Spec}(R)$,*
   (c) *$M_\mathfrak{m}$ is zero for all maximal ideals $\mathfrak{m}$ of R.*
(3) *Given a complex $M_1 \to M_2 \to M_3$ of R-modules the following are equivalent*
   (a) *$M_1 \to M_2 \to M_3$ is exact,*
   (b) *for every prime $\mathfrak{p}$ of R the localization $M_{1,\mathfrak{p}} \to M_{2,\mathfrak{p}} \to M_{3,\mathfrak{p}}$ is exact,*
   (c) *for every maximal ideal $\mathfrak{m}$ of R the localization $M_{1,\mathfrak{m}} \to M_{2,\mathfrak{m}} \to M_{3,\mathfrak{m}}$ is exact.*
(4) *Given a map $f : M \to M'$ of R-modules the following are equivalent*
   (a) *f is injective,*
   (b) *$f_\mathfrak{p} : M_\mathfrak{p} \to M'_\mathfrak{p}$ is injective for all primes $\mathfrak{p}$ of R,*

(c) $f_{\mathfrak{m}} : M_{\mathfrak{m}} \to M'_{\mathfrak{m}}$ *is injective for all maximal ideals* $\mathfrak{m}$ *of* R.

(5) *Given a map* $f : M \to M'$ *of* R*-modules the following are equivalent*
   (a) f *is surjective,*
   (b) $f_{\mathfrak{p}} : M_{\mathfrak{p}} \to M'_{\mathfrak{p}}$ *is surjective for all primes* $\mathfrak{p}$ *of* R,
   (c) $f_{\mathfrak{m}} : M_{\mathfrak{m}} \to M'_{\mathfrak{m}}$ *is surjective for all maximal ideals* $\mathfrak{m}$ *of* R.

(6) *Given a map* $f : M \to M'$ *of* R*-modules the following are equivalent*
   (a) f *is bijective,*
   (b) $f_{\mathfrak{p}} : M_{\mathfrak{p}} \to M'_{\mathfrak{p}}$ *is bijective for all primes* $\mathfrak{p}$ *of* R,
   (c) $f_{\mathfrak{m}} : M_{\mathfrak{m}} \to M'_{\mathfrak{m}}$ *is bijective for all maximal ideals* $\mathfrak{m}$ *of* R.

Proof. Let $x \in M$ as in (1). Let $I = \{f \in R \mid fx = 0\}$. It is easy to see that I is an ideal (it is the annihilator of x). Condition (1)(c) means that for all maximal ideals $\mathfrak{m}$ there exists an $f \in R \setminus \mathfrak{m}$ such that $fx = 0$. In other words, $V(I)$ does not contain a closed point. By Lemma 10.16.2 we see I is the unit ideal. Hence x is zero, i.e., (1)(a) holds. This proves (1).

Part (2) follows by applying (1) to all elements of M simultaneously.

Proof of (3). Let H be the homology of the sequence, i.e., $H = \mathrm{Ker}(M_2 \to M_3)/\mathrm{Im}(M_1 \to M_2)$. By Proposition 10.9.12 we have that $H_{\mathfrak{p}}$ is the homology of the sequence $M_{1,\mathfrak{p}} \to M_{2,\mathfrak{p}} \to M_{3,\mathfrak{p}}$. Hence (3) is a consequence of (2).

Parts (4) and (5) are special cases of (3). Part (6) follows formally on combining (4) and (5). □

Lemma 10.22.2. *Let* R *be a ring. Let* M *be an* R*-module. Let* S *be an* R*-algebra. Suppose that* $f_1, \ldots, f_n$ *is a finite list of elements of* R *such that* $\bigcup D(f_i) = \mathrm{Spec}(R)$ *in other words* $(f_1, \ldots, f_n) = R$.

(1) *If each* $M_{f_i} = 0$ *then* $M = 0$.
(2) *If each* M_{f_i} *is a finite* R_{f_i}*-module, then* M *is a finite* R*-module.*
(3) *If each* M_{f_i} *is a finitely presented* R_{f_i}*-module, then* M *is a finitely presented* R*-module.*
(4) *Let* $M \to N$ *be a map of* R*-modules. If* $M_{f_i} \to N_{f_i}$ *is an isomorphism for each* i *then* $M \to N$ *is an isomorphism.*
(5) *Let* $0 \to M'' \to M \to M' \to 0$ *be a complex of* R*-module. If* $0 \to M''_{f_i} \to M_{f_i} \to M'_{f_i} \to 0$ *is exact for each* i*, then* $0 \to M'' \to M \to M' \to 0$ *is exact.*
(6) *If each* R_{f_i} *is Noetherian, then* R *is Noetherian.*
(7) *If each* S_{f_i} *is a finite type* R*-algebra, so is* S.
(8) *If each* S_{f_i} *is of finite presentation over* R*, so is* S.

Proof. We prove each of the parts in turn.

(1) By Proposition 10.9.10 this implies $M_{\mathfrak{p}} = 0$ for all $\mathfrak{p} \in \mathrm{Spec}(R)$, so we conclude by Lemma 10.22.1.
(2) For each i take a finite generating set X_i of M_{f_i}. Without loss of generality, we may assume that the elements of X_i are in the image of the localization map $M \to M_{f_i}$, so we take a finite set Y_i of preimages of the elements of X_i in M. Let Y be the union of these sets. This is still a finite set. Consider the obvious R-linear map $R^Y \to M$ sending the basis element e_y to y. By assumption this map is surjective after localizing at an arbitrary prime ideal $\mathfrak{p}$ of R, so it surjective by Lemma 10.22.1 and M is finitely generated.

(3) By (2) we have a short exact sequence

$$0 \to K \to R^n \to M \to 0$$

Since localization is an exact functor and M_{f_i} is finitely presented we see that K_{f_i} is finitely generated for all $1 \leq i \leq n$ by Lemma 10.5.3. By (2) this implies that K is a finite R-module and therefore M is finitely presented.

(4) By Proposition 10.9.10 the assumption implies that the induced morphism on localizations at all prime ideals is an isomorphism, so we conclude by Lemma 10.22.1.

(5) By Proposition 10.9.10 the assumption implies that the induced sequence of localizations at all prime ideals is short exact, so we conclude by Lemma 10.22.1.

(6) We will show that every ideal of R has a finite generating set: For this, let $I \subseteq R$ be an arbitrary ideal. By Proposition 10.9.12 each $I_{f_i} \subseteq R_{f_i}$ is an ideal. These are all finitely generated by assumption, so we conclude by (2).

(7) For each i take a finite generating set X_i of S_{f_i}. Without loss of generality, we may assume that the elements of X_i are in the image of the localization map $S \to S_{f_i}$, so we take a finite set Y_i of preimages of the elements of X_i in S. Let Y be the union of these sets. This is still a finite set. Consider the algebra homomorphism $R[X_y]_{y \in Y} \to S$ induced by Y. Since it is an algebra homomorphism, the image T is an R-submodule of the R-module S, so we can consider the quotient module S/T. By assumption, this is zero if we localize at the f_i, so it is zero by item 1. and therefore S is an R-algebra of finite type.

(8) By the previous item, there exists a surjective R-algebra homomorphism $R[X_1, ..., X_n] \to S$. Let K be the kernel of this map. This is an ideal in $R[X_1, ..X_n]$, finitely generated in each localization at f_i. Since the f_i generate the unit ideal in R, they also generate the unit ideal in $R[X_1, ..., X_n]$, so an application of (2) finishes the proof.

□

Lemma 10.22.3. *Let $R \to S$ be a ring map. Suppose that $g_1, \ldots, g_m$ is a finite list of elements of S such that $\bigcup D(g_j) = \mathrm{Spec}(S)$ in other words $(g_1, \ldots, g_m) = S$.*

(1) *If each S_{g_i} is of finite type over R, then S is of finite type over R.*
(2) *If each S_{g_i} is of finite presentation over R, then S is of finite presentation over R.*

Proof. Omitted. □

The following lemma is better stated and proved in the more general context of flat descent. However, it makes sense to state it here since it fits well with the above.

Lemma 10.22.4. *Let R be a ring. Let $f_1, \ldots, f_n \in R$ be elements which generate the unit ideal in R. Suppose we are given the following data:*

(1) *For each i an R_{f_i}-module M_i.*
(2) *For each pair i, j an $R_{f_i f_j}$-module isomorphism $\psi_{ij} : (M_i)_{f_j} \to (M_j)_{f_i}$.*

which satisfy the "cocycle condition" that all the diagrams

$$\begin{array}{ccc} (M_i)_{f_jf_k} & \xrightarrow{\psi_{ik}} & (M_k)_{f_if_j} \\ & {\scriptstyle\psi_{ij}}\searrow \quad \nearrow{\scriptstyle\psi_{jk}} & \\ & (M_j)_{f_if_k} & \end{array}$$

commute (for all triples i, j, k). Given this data define

$$M = Ker\Big(\bigoplus\nolimits_{1\leq i\leq n} M_i \longrightarrow \bigoplus\nolimits_{1\leq i,j\leq n} (M_i)_{f_i}\Big)$$

where $(m_1, \ldots, m_n)$ maps to the element whose (i, j)th entry is $m_i/1 - \psi_{ji}(m_j/1)$. Then the natural map $M \to M_i$ identifies M_i with M_{f_i}. Moreover $\psi_{ij}(m/1) = m/1$ for all $m \in M$ (with obvious notation).

Proof. Omitted. □

10.23. Total rings of fractions

We can apply the glueing results above to prove something about total rings of fractions $Q(R)$. Namely, Lemma 10.23.2 below.

Lemma 10.23.1. *Let R be a ring. Let $S \subset R$ be a multiplicative subset consisting of nonzerodivisors. Then $Q(R) \cong Q(S^{-1}R)$. In particular $Q(R) \cong Q(Q(R))$.*

Proof. If $x \in S^{-1}R$ is a nonzerodivisor, and $x = r/f$ for some $r \in R$, $f \in S$, then r is a nonzerodivisor in R. Whence the lemma. □

Lemma 10.23.2. *Let R be a ring. Assume that R has finitely many minimal primes $\mathfrak{q}_1, \ldots, \mathfrak{q}_t$, and that $\mathfrak{q}_1 \cup \ldots \cup \mathfrak{q}_t$ is the set of zerodivisors of R. Then the total ring of fractions $Q(R)$ (Example 10.9.8) is equal to $R_{\mathfrak{q}_1} \times \ldots \times R_{\mathfrak{q}_t}$.*

Proof. There are natural maps $Q(R) \to R_{\mathfrak{q}_i}$ since any nonzerodivisor is contained in $R \setminus \mathfrak{q}_i$. Hence a natural map $Q(R) \to R_{\mathfrak{q}_1} \times \ldots \times R_{\mathfrak{q}_t}$. For any nonminimal prime $\mathfrak{p} \subset R$ we see that $\mathfrak{p} \not\subset \mathfrak{q}_1 \cup \ldots \cup \mathfrak{q}_t$ by Lemma 10.14.3. Hence $\mathrm{Spec}(Q(R)) = \{\mathfrak{q}_1, \ldots, \mathfrak{q}_t\}$ (as subsets of $\mathrm{Spec}(R)$, see Lemma 10.16.5). Therefore $\mathrm{Spec}(Q(R))$ is a finite discrete set and it follows that $Q(R) = A_1 \times \ldots \times A_t$ with $\mathrm{Spec}(A_i) = \{q_i\}$, see Lemma 10.21.3. Moreover A_i is a local ring, which is a localization of R. Hence $A_i \cong R_{\mathfrak{q}_i}$. □

10.24. Irreducible components of spectra

We show that irreducible components of the spectrum of a ring correspond to the minimal primes in the ring.

Lemma 10.24.1. *Let R be a ring.*

1. *For a prime $\mathfrak{p} \subset R$ the closure of $\{\mathfrak{p}\}$ in the Zariski topology is $V(\mathfrak{p})$. In a formula $\overline{\{\mathfrak{p}\}} = V(\mathfrak{p})$.*
2. *The irreducible closed subsets of $\mathrm{Spec}(R)$ are exactly the subsets $V(\mathfrak{p})$, with $\mathfrak{p} \subset R$ a prime.*
3. *The irreducible components (see Topology, Definition 5.7.1) of $\mathrm{Spec}(R)$ are exactly the subsets $V(\mathfrak{p})$, with $\mathfrak{p} \subset R$ a minimal prime.*

Proof. Note that if $\mathfrak{p} \in V(I)$, then $I \subset \mathfrak{p}$. Hence, clearly $\overline{\{\mathfrak{p}\}} = V(\mathfrak{p})$. In particular $V(\mathfrak{p})$ is the closure of a singleton and hence irreducible. The second assertion implies the third. To show the second, let $V(I) \subset \mathrm{Spec}(R)$ with I a radical ideal. If I is not prime, then choose $a, b \in R$, $a, b \notin I$ with $ab \in I$. In this case $V(I, a) \cup V(I, b) = V(I)$, but neither $V(I, b) = V(I)$ nor $V(I, a) = V(I)$, by Lemma 10.16.2. Hence $V(I)$ is not irreducible. □

In other words, this lemma shows that every irreducible closed subset of $\mathrm{Spec}(R)$ is of the form $V(\mathfrak{p})$ for some prime $\mathfrak{p}$. Since $V(\mathfrak{p}) = \overline{\{\mathfrak{p}\}}$ we see that each irreducible closed subset has a unique generic point, see Topology, Definition 5.7.4. In particular, $\mathrm{Spec}(R)$ is a sober topological space. We record this fact in the following lemma.

Lemma 10.24.2. *The spectrum of a ring is a spectral space, see Topology, Definition 5.22.1.*

Proof. Formally this follows from Lemma 10.24.1 and Lemma 10.16.11. See also discussion above. □

Lemma 10.24.3. *Let R be a ring. Let $\mathfrak{p} \subset R$ be a prime.*

1. *the set of irreducible closed subsets of $\mathrm{Spec}(R)$ passing through $\mathfrak{p}$ is in one-to-one correspondence with primes $\mathfrak{q} \subset R_\mathfrak{p}$.*
2. *The set of irreducible components of $\mathrm{Spec}(R)$ passing through $\mathfrak{p}$ is in one-to-one correspondence with minimal primes $\mathfrak{q} \subset R_\mathfrak{p}$.*

Proof. Omitted. □

Lemma 10.24.4. *Let $\mathfrak{p}$ be a minimal prime of a ring R. Every element of the maximal ideal of $R_\mathfrak{p}$ is nilpotent. If R is reduced then $R_\mathfrak{p}$ is a field.*

Proof. If some element x of $\mathfrak{p}R_\mathfrak{p}$ is not nilpotent, then $D(x) \neq \emptyset$, see Lemma 10.16.2. This contradicts the minimality of $\mathfrak{p}$. If R is reduced, then $\mathfrak{p}R_\mathfrak{p} = 0$ and hence it is a field. □

Lemma 10.24.5. *Let R be a ring. Let $\mathfrak{p}$ be a minimal prime of R. Let $W \subset \mathrm{Spec}(R)$ be a quasi-compact open not containing the point $\mathfrak{p}$. Then there exists an $f \in R$, $f \notin \mathfrak{p}$ such that $D(f) \cap W = \emptyset$.*

Proof. Since W is quasi-compact we may write it as a finite union of standard affine opens $D(g_i)$, $i = 1, \ldots, n$. Since $\mathfrak{p} \notin W$ we have $g_i \in \mathfrak{p}$ for all i. By Lemma 10.24.4 each g_i is nilpotent in $R_\mathfrak{p}$. Hence we can find an $f \in R$, $f \notin \mathfrak{p}$ such that for all i we have $fg_i^{n_i} = 0$ for some $n_i > 0$. Then $D(f)$ works. □

Lemma 10.24.6. *Let R be a ring. Let $X = \mathrm{Spec}(R)$ as a topological space. The following are equivalent*

1. *X is profinite,*
2. *X is Hausdorff,*
3. *X is totally disconnected.*
4. *every quasi-compact open of X is closed,*
5. *there are no nontrivial inclusions between its prime ideals,*
6. *every prime ideal is a maximal ideal,*
7. *every prime ideal is minimal,*
8. *every standard open $D(f) \subset X$ is closed, and*

(9) *add more here.*

Proof. First proof. It is clear that (5), (6), and (7) are equivalent. It is clear that (4) and (8) are equivalent as every quasi-compact open is a finite union of standard opens. The implication (7) $\Rightarrow$ (4) follows from Lemma 10.24.5. Assume (4) holds. Let $\mathfrak{p}, \mathfrak{p}'$ be distinct primes of R. Choose an $f \in \mathfrak{p}'$, $f \notin \mathfrak{p}$ (if needed switch $\mathfrak{p}$ with $\mathfrak{p}'$). Then $\mathfrak{p}' \notin D(f)$ and $\mathfrak{p} \in D(f)$. By (4) the open $D(f)$ is also closed. Hence $\mathfrak{p}$ and $\mathfrak{p}'$ are in disjoint open neighbourhoods whose union is X. Thus X is Hausdorff and totally disconnected. Thus (4) $\Rightarrow$ (2) and (3). If (3) holds then there cannot be any specializations between points of $\mathrm{Spec}(R)$ and we see that (5) holds. If X is Hausdorff then every point is closed, so (2) implies (6). Thus (2), (3), (4), (5), (6), (7) and (8) are equivalent. Any profinite space is Hausdorff, so (1) implies (2). If X satisfies (2) and (3), then X (being quasi-compact by Lemma 10.16.10) is profinite by Topology, Lemma 5.21.2.

Second proof. Besides the equivalence of (4) and (8) this follows from Lemma 10.24.2 and purely topological facts, see Topology, Lemma 5.22.6. $\square$

Lemma 10.24.7. *Let R be a reduced ring. Then R is a subring of a product of fields. In fact, $R \subset \prod_{\mathfrak{p}\ minimal} R_{\mathfrak{p}}$ is such an embedding.*

Proof. This is clear from Lemma 10.24.4 and the fact that $\bigcap_{\mathfrak{p}} \mathfrak{p} = (0)$ in a reduced ring, see Lemma 10.16.2. $\square$

10.25. Examples of spectra of rings

In this section we put some examples of spectra.

Example 10.25.1. In this example we describe $X = \mathrm{Spec}(\mathbf{Z}[x]/(x^2-4))$. Let $\mathfrak{p}$ be an arbitrary prime in X. Let $\phi : \mathbf{Z} \to \mathbf{Z}[x]/(x^2-4)$ be the natural ring map. Then, $\phi^{-1}(\mathfrak{p})$ is a prime in $\mathbf{Z}$. If $\phi^{-1}(\mathfrak{p}) = (2)$, then since $\mathfrak{p}$ contains 2, it corresponds to a prime ideal in $\mathbf{Z}[x]/(x^2-4,2) \cong (\mathbf{Z}/2\mathbf{Z})[x]/(x^2)$ via the map $\mathbf{Z}[x]/(x^2-4) \to \mathbf{Z}[x]/(x^2-4,2)$. Any prime in $(\mathbf{Z}/2\mathbf{Z})[x]/(x^2)$ corresponds to a prime in $(\mathbf{Z}/2\mathbf{Z})[x]$ containing (x^2). Such primes will then contain x. Since $(\mathbf{Z}/2\mathbf{Z}) \cong (\mathbf{Z}/2\mathbf{Z})[x]/(x)$ is a field, (x) is a maximal ideal. Since any prime contains (x) and (x) is maximal, the ring contains only one prime (x). Thus, in this case, $\mathfrak{p} = (2, x)$. Now, if $\phi^{-1}(\mathfrak{p}) = (q)$ for $q > 2$, then since $\mathfrak{p}$ contains q, it corresponds to a prime ideal in $\mathbf{Z}[x]/(x^2-4,q) \cong (\mathbf{Z}/q\mathbf{Z})[x]/(x^2-4)$ via the map $\mathbf{Z}[x]/(x^2-4) \to \mathbf{Z}[x]/(x^2-4,q)$. Any prime in $(\mathbf{Z}/q\mathbf{Z})[x]/(x^2-4)$ corresponds to a prime in $(\mathbf{Z}/q\mathbf{Z})[x]$ containing $(x^2-4) = (x-2)(x+2)$. Hence, these primes must contain either $x-2$ or $x+2$. Since $(\mathbf{Z}/q\mathbf{Z})[x]$ is a PID, all nonzero primes are maximal, and so there are precisely 2 primes in $(\mathbf{Z}/q\mathbf{Z})[x]$ containing $(x-2)(x+2)$, namely $(x-2)$ and $(x+2)$. In conclusion, there exist two primes $(q, x-2)$ and $(q, x+2)$ since $2 \neq -2 \in \mathbf{Z}/(q)$. Finally, we treat the case where $\phi^{-1}(\mathfrak{p}) = (0)$. Notice that $\mathfrak{p}$ corresponds to a prime ideal in $\mathbf{Z}[x]$ that contains $(x^2-4) = (x-2)(x+2)$. Hence, $\mathfrak{p}$ contains either $(x-2)$ or $(x+2)$. Hence, $\mathfrak{p}$ corresponds to a prime in $\mathbf{Z}[x]/(x-2)$ or one in $\mathbf{Z}[x]/(x+2)$ that intersects $\mathbf{Z}$ only at 0, by assumption. Since $\mathbf{Z}[x]/(x-2) \cong \mathbf{Z}$ and $\mathbf{Z}[x]/(x+2) \cong \mathbf{Z}$, this means that $\mathfrak{p}$ must correspond to 0 in one of these rings. Thus, $\mathfrak{p} = (x-2)$ or $\mathfrak{p} = (x+2)$ in the original ring.

Example 10.25.2. In this example we describe $X = \mathrm{Spec}(\mathbf{Z}[x])$. Fix $\mathfrak{p} \in X$. Let $\phi : \mathbf{Z} \to \mathbf{Z}[x]$ and notice that $\phi^{-1}(\mathfrak{p}) \in \mathrm{Spec}(\mathbf{Z})$. If $\phi^{-1}(\mathfrak{p}) = (q)$ for q a prime

number $q > 0$, then $\mathfrak{p}$ corresponds to a prime in $(\mathbf{Z}/(q))[x]$, which must be generated by a polynomial that is irreducible in $(\mathbf{Z}/(q))[x]$. If we choose a representative of this polynomial with minimal degree, then it will also be irreducible in $\mathbf{Z}[x]$. Hence, in this case $\mathfrak{p} = (q, f_q)$ where f_q is an irreducible polynomial in $\mathbf{Z}[x]$ that is irreducible when viewed in $(\mathbf{Z}/(q)[x])$. Now, assume that $\phi^{-1}(\mathfrak{p}) = (0)$. In this case, $\mathfrak{p}$ must be generated by nonconstant polynomials which, since $\mathfrak{p}$ is prime, may be assumed to be irreducible in $\mathbf{Z}[x]$. By Gauss' lemma, these polynomials are also irreducible in $\mathbf{Q}[x]$. Since $\mathbf{Q}[x]$ is a Euclidean domain, if there are at least two distinct irreducibles f, g generating $\mathfrak{p}$, then $1 = af + bg$ for $a, b \in \mathbf{Q}[x]$. Multiplying through by a common denominator, we see that $m = \bar{a}f + \bar{b}g$ for $\bar{a}, \bar{b} \in \mathbf{Z}[x]$ and nonzero $m \in \mathbf{Z}$. This is a contradiction. Hence, $\mathfrak{p}$ is generated by one irreducible polynomial in $\mathbf{Z}[x]$.

Example 10.25.3. In this example we describe $X = \mathrm{Spec}(k[x, y])$ when k is an arbitrary field. Clearly (0) is prime, and any principal ideal generated by an irreducible polynomial will also be a prime since $k[x, y]$ is a unique factorization domain. Now assume $\mathfrak{p}$ is an element of X that is not principal. Since $k[x, y]$ is a Noetherian UFD, the prime ideal $\mathfrak{p}$ can be generated by a finite number of irreducible polynomials $(f_1, \ldots, f_n)$. Now, I claim that if f, g are irreducible polynomials in $k[x, y]$ that are not associates, then $(f, g) \cap k[x] \neq 0$. To do this, it is enough to show that f and g are relatively prime when viewed in $k(x)[y]$. In this case, $k(x)[y]$ is a Euclidean domain, so by applying the Euclidean algorithm and clearing denominators, we obtain $p = af + bg$ for $p, a, b \in k[x]$. Thus, assume this is not the case, that is, that some nonunit $h \in k(x)[y]$ divides both f and g. Then, by Gauss's lemma, for some $a, b \in k(x)$ we have $ah|f$ and $bh|g$ for $ah, bh \in k[x]$ since $\mathrm{Frac}(k[x]) = k(x)$. By irreducibility, $ah = f$ and $bh = g$ (since $h \notin k(x)$). So, back in $k(x)[y]$, f, g are associates, as $\frac{a}{b}g = f$. Since $k(x) = \mathrm{Frac}(k[x])$, we can write $g = \frac{r}{s}f$ for elements $r, s \in k[x]$ sharing no common factors. This implies that $sg = rf$ in $k[x, y]$ and so s must divide f since $k[x, y]$ is a UFD. Hence, $s = 1$ or $s = f$. If $s = f$, then $r = g$, implying $f, g \in k[x]$ and thus must be units in $k(x)$ and relatively prime in $k(x)[y]$, contradicting our hypothesis. If $s = 1$, then $g = rf$, another contradiction. Thus, we must have f, g relatively prime in $k(x)[y]$, a Euclidean domain. Thus, we have reduced to the case $\mathfrak{p}$ contains some irreducible polynomial $p \in k[x] \subseteq k[x, y]$. By the above, $\mathfrak{p}$ corresponds to a prime in the ring $k[x, y]/(p) = k(\alpha)[y]$, where α is an element algebraic over k with minimum polynomial p. This is a PID, and so any prime ideal corresponds to (0) or an irreducible polynomial in $k(\alpha)[y]$. Thus, $\mathfrak{p}$ is of the form (p) or (p, f) where f is a polynomial in $k[x, y]$ that is irreducible in the quotient $k[x, y]/(p)$.

Example 10.25.4. Consider the ring

$$R = \{f \in \mathbf{Q}[z] \text{ with } f(0) = f(1)\}.$$

Consider the map

$$\varphi : \mathbf{Q}[A, B] \to R$$

defined by $\varphi(A) = z^2 - z$ and $\varphi(B) = z^3 - z^2$. It is easily checked that $(A^3 - B^2 + AB) \subseteq \ker(\varphi)$ and that $A^3 - B^2 + AB$ is irreducible. Assume that φ is surjective; then since R is an integral domain (it is a subring of an integral domain), $\ker(\phi)$ must be a prime ideal of $\mathbf{Q}[A, B]$. The prime ideals which contain $(A^3 - B^2 + AB)$ are $(A^3 - B^2 + AB)$ itself and any maximal ideal (f, g) with $f, g \in \mathbf{Q}[A, B]$ such that f is irreducible mod g. But R is not a field, so the kernel must be $(A^3 - B^2 + AB)$;

hence φ gives an isomorphism $R \to \mathbf{Q}[A,B]/(A^3 - B^2 + AB)$.

To see that φ is surjective, we must express any $f \in R$ as a $\mathbf{Q}$-coefficient polynomial in $A(z) = z^2 - z$ and $B(z) = z^3 - z^2$. Note the relation $zA(z) = B(z)$. Let $a = f(0) = f(1)$. Then $z(z-1)$ must divide $f(z) - a$, so we can write $f(z) = z(z-1)g(z) + a = A(z)g(z) + a$. If $\deg(g) < 2$, then $h(z) = c_1 z + c_0$ and $f(z) = A(z)(c_1 z + c_0) + a = c_1 B(z) + c_0 A(z) + a$, so we are done. If $\deg(g) \geq 2$, then by the polynomial division algorithm, we can write $g(z) = A(z)h(z) + b_1 z + b_0$ ($\deg(h) \leq \deg(g) - 2$), so $f(z) = A(z)^2 h(z) + b_1 B(z) + b_0 A(z)$. Applying division to $h(z)$ and iterating, we obtain an expression for $f(z)$ as a polynomial in $A(z)$ and $B(z)$; hence φ is surjective.

Now let $a \in \mathbf{Q}$, $a \neq 0, \frac{1}{2}, 1$ and consider

$$R_a = \{f \in \mathbf{Q}[z, \frac{1}{z-a}] \text{ with } f(0) = f(1)\}.$$

This is a finitely generated $\mathbf{Q}$-algebra as well: it is easy to check that the functions $z^2 - z$, $z^3 - z$, and $\frac{a^2-a}{z-a} + z$ generate R_a as an $\mathbf{Q}$-algebra. We have the following inclusions:

$$R \subset R_a \subset \mathbf{Q}[z, \frac{1}{z-a}], \qquad R \subset \mathbf{Q}[z] \subset \mathbf{Q}[z, \frac{1}{z-a}].$$

Recall (Lemma 10.16.5) that for a ring T and a multiplicative subset $S \subset T$, the ring map $T \to S^{-1}T$ induces a map on spectra $\mathrm{Spec}(S^{-1}T) \to \mathrm{Spec}(T)$ which is a homeomorphism onto the subset

$$\{\mathfrak{p} \in \mathrm{Spec}(T) \mid S \cap \mathfrak{p} = \emptyset\} \subseteq \mathrm{Spec}(T).$$

When $S = \{1, f, f^2, \ldots\}$ for some $f \in T$, this is the open set $D(f) \subset T$. We now verify a corresponding property for the ring map $R \to R_a$: we will show that the map $\theta : \mathrm{Spec}(R_a) \to \mathrm{Spec}(R)$ induced by inclusion $R \subset R_a$ is a homeomorphism onto an open subset of $\mathrm{Spec}(R)$ by verifying that θ is an injective local homeomorphism. We do so with respect to an open cover of $\mathrm{Spec}(R_a)$ by two distinguished opens, as we now describe. For any $r \in \mathbf{Q}$, let $\mathrm{ev}_r : R \to \mathbf{Q}$ be the homomorphism given by evaluation at r. Note that for $r = 0$ and $r = 1 - a$, this can be extended to a homomorphism $\mathrm{ev}'_r : R_a \to \mathbf{Q}$ (the latter because $\frac{1}{z-a}$ is well-defined at $z = 1-a$, since $a \neq \frac{1}{2}$). However, ev_a does not extend to R_a. Write $\mathfrak{m}_r = \ker(\mathrm{ev}_r)$; it is easy to check that

$$\mathfrak{m}_0 = (z^2 - z, z^3 - z),$$
$$\mathfrak{m}_a = ((z-1+a)(z-a), (z^2 - 1 + a)(z-a)), \text{ and}$$
$$\mathfrak{m}_{1-a} = ((z-1+a)(z-a), (z-1+a)(z^2 - a)).$$

(To do so, note that the right-hand sides are clearly contained in the left-hand sides. Then check that the right-hand sides are maximal ideals by writing the generators in terms of A and B, and viewing R as $\mathbf{Q}[A,B]/(A^3 - B^2 + AB)$.) Note that $\mathfrak{m}_a$ is not in the image of θ: we have $(z^2-1+a)(z-a) - (z-1+a)(z-a) = (z^2-z)(z-a)$ is in $\mathfrak{m}_a$, so $z^2 - z = \frac{(z^2-z)(z-a)}{z-a}$ is in $\mathfrak{m}_a R_a$. Hence no ideal I of R_a can satisfy $I \cap R = \mathfrak{m}_\mathfrak{a}$, as such an I would have to contain $z^2 - z$, which is in R but not in $\mathfrak{m}_a$. The distinguished open set $D((z-1+a)(z-a)) \subset \mathrm{Spec}(R)$ is equal to the complement of the closed set $\{\mathfrak{m}_a, \mathfrak{m}_{1-a}\}$. Then check that $R_{(z-1+a)(z-a)} = (R_a)_{(z-1+a)(z-a)}$; calling this localized ring R', then, it follows that the map $R \to R'$ factors as $R \to R_a \to R'$. By Lemma 10.16.5, then, these maps express $\mathrm{Spec}(R') \subseteq \mathrm{Spec}(R_a)$ and $\mathrm{Spec}(R') \subseteq \mathrm{Spec}(R)$ as open subsets; hence $\theta : \mathrm{Spec}(R_a) \to \mathrm{Spec}(R)$, when

restricted to $D((z-1+a)(z-a))$, is a homeomorphism onto an open subset. Similarly, θ restricted to $D((z^2+z+2a-2)(z-a)) \subseteq \operatorname{Spec}(R_a)$ is a homeomorphism onto the open subset $D((z^2+z+2a-2)(z-a)) \subseteq \operatorname{Spec}(R)$. Depending on whether $z^2+z+2a-2$ is irreducible or not over $\mathbf{Q}$, this former distinguished open set has complement equal to one or two closed points along with the closed point $\mathfrak{m}_a$. Furthermore, the ideal in R_a generated by the elements $(z^2+z+2a-a)(z-a)$ and $(z-1+a)(z-a)$ is all of R_a, so these two distinguished open sets cover $\operatorname{Spec}(R_a)$. Hence in order to show that θ is a homeomorphism onto $\operatorname{Spec}(R)-\{\mathfrak{m}_a\}$, it suffices to show that these one or two points can never equal $\mathfrak{m}_{1-a}$. And this is indeed the case, since $1-a$ is a root of $z^2+z+2a-2$ if and only of $a=0$ or $a=1$, both of which do not occur.

Despite this homeomorphism which mimics the behavior of a localization at an element of R, while $\mathbf{Q}[z, \frac{1}{z-a}]$ is the localization of $\mathbf{Q}[z]$ at the maximal ideal $(z-a)$, the ring R_a is *not* a localization of R: Any localization $S^{-1}R$ results in more units than the original ring R. The units of R are $\mathbf{Q}^\times$, the units of $\mathbf{Q}$. If $\frac{f}{(z-a)^k}$ is a unit in R_a ($f \in R$ and $k \geq 0$ an integer), then we have

$$\frac{f}{(z-a)^k} \cdot \frac{g}{(z-a)^\ell} = 1$$

for some $g \in R$ and some integer $\ell \geq 0$. Since R is an integral domain, this is equivalent to

$$fg = (z-a)^{k+\ell}.$$

But $(z-a)^{k+\ell}$ is only an element of R if $k=\ell=0$; hence f, g are units in R as well. Hence R_a has no more units than R does, and thus cannot be a localization of R.

We used the fact that $a \neq 0, 1$ to ensure that $\frac{1}{z-a}$ makes sense at $z=0,1$. We used the fact that $a \neq 1/2$ in a few places: (1) In order to be able to talk about the kernel of ev_{1-a} on R_a, which ensures that $\mathfrak{m}_{1-a}$ is a point of R_a (i.e., that R_a is missing just one point of R). (2) At the end in order to conclude that $(z-a)^{k+\ell}$ can only be in R for $k=\ell=0$; indeed, if $a=1/2$, then this is in R as long as $k+\ell$ is even. Hence there would indeed be more units in R_a than in R, and R_a could possibly be a localization of R.

10.26. A meta-observation about prime ideals

This section is taken from the CRing project. Let R be a ring and let $S \subset R$ be a multiplicative subset. A consequence of Lemma 10.16.5 is that an ideal $I \subset R$ maximal with respect to the property of not intersecting S is prime. The reason is that $I = R \cap \mathfrak{m}$ for some maximal ideal $\mathfrak{m}$ of the ring $S^{-1}R$. It turns out that for many properties of ideals, the maximal ones are prime. A general method of seeing this was developed in [**LR08**]. In this section, we digress to explain this phenomenon.

Let R be a ring. If I is an ideal of R and $a \in R$, we define

$$(I : a) = \{x \in R \mid xa \in I\}.$$

More generally, if $J \subset R$ is an ideal, we define

$$(I : J) = \{x \in R \mid xJ \subset I\}.$$

Lemma 10.26.1. *Let R be a ring. For a principal ideal $J \subset R$, and for any ideal $I \subset J$ we have $I = J(I : J)$.*

Proof. Say $J = (a)$. Then $(I : J) = (I : a)$. Since $I \subset J$ we see that any $y \in I$ is of the form $y = xa$ for some $x \in (I : a)$. Hence $I \subset J(I : J)$. Conversely, if $x \in (I : a)$, then $xJ = (xa) \subset I$, which proves the other inclusion. □

Let $\mathcal{F}$ be a collection of ideals of R. We are interested in conditions that will guarantee that the maximal elements in the complement of $\mathcal{F}$ are prime.

Definition 10.26.2. Let R be a ring. Let $\mathcal{F}$ be a set of ideals of R. We say $\mathcal{F}$ is an *Oka family* if $R \in \mathcal{F}$ and whenever $I \subset R$ is an ideal and $(I : a), (I, a) \in \mathcal{F}$ for some $a \in R$, then $I \in \mathcal{F}$.

Let us give some examples of Oka families. The first example is the basic example discussed in the introduction to this section.

Example 10.26.3. Let R be a ring and let S be a multiplicative subset of R. We claim that $\mathcal{F} = \{I \subset R \mid I \cap S \neq \emptyset\}$ is an Oka family. Namely, suppose that $(I : a), (I, a) \in \mathcal{F}$ for some $a \in R$. Then pick $s \in (I, a) \cap S$ and $s' \in (I : a) \cap S$. Then $ss' \in I \cap S$ and hence $I \in \mathcal{F}$. Thus $\mathcal{F}$ is an Oka family.

Example 10.26.4. Let R be a ring, $I \subset R$ an ideal, and $a \in R$. If $(I : a)$ is generated by $a_1, \ldots, a_n$ and (I, a) is generated by $a, b_1, \ldots, b_m$ with $b_1, \ldots, b_m \in I$, then I is generated by $aa_1, \ldots, aa_n, b_1, \ldots, b_m$. To see this, note that if $x \in I$, then $x \in (I, a)$ is a linear combination of $a, b_1, \ldots, b_m$, but the coefficient of a must lie in $(I : a)$. As a result, we deduce that the family of finitely generated ideals is an Oka family.

Example 10.26.5. Let us show that the family of principal ideals of a ring R is an Oka family. Indeed, suppose $I \subset R$ is an ideal, $a \in R$, and (I, a) and $(I : a)$ are principal. Note that $(I : a) = (I : (I, a))$. Setting $J = (I, a)$, we find that J is principal and $(I : J)$ is too. By Lemma 10.26.1 we have $I = J(I : J)$. Thus we find in our situation that since $J = (I, a)$ and $(I : J)$ are principal, I is principal.

Example 10.26.6. Let R be a ring. Let κ be an infinite cardinal. The family of ideals which can be generated by at most κ elements is an Oka family. The argument is analogous to the argument in Example 10.26.4 and is omitted.

Proposition 10.26.7. *If $\mathcal{F}$ is an Oka family of ideals, then any maximal element of the complement of $\mathcal{F}$ is prime.*

Proof. Suppose $I \notin \mathcal{F}$ is maximal with respect to not being in $\mathcal{F}$ but I is not prime. Note that $I \neq R$ because $R \in \mathcal{F}$. Since I is not prime we can find $a, b \in R - I$ with $ab \in I$. it follows that $(I, a) \neq I$ and $(I : a)$ contains $b \notin I$ so also $(I : a) \neq I$. Thus $(I : a), (I, a)$ both strictly contain I, so they must belong to $\mathcal{F}$. By the Oka condition, we have $I \in \mathcal{F}$, a contradiction. □

At this point we are able to turn most of the examples above into a lemma about prime ideals in a ring.

Lemma 10.26.8. *Let R be a ring. Let S be a multiplicative subset of R. An ideal $I \subset R$ which is maximal with respect to the property that $I \cap S = \emptyset$ is prime.*

Proof. This is the example discussed in the introduction to this section. For an alternative proof, combine Example 10.26.3 with Proposition 10.26.7. □

Lemma 10.26.9. *Let R be a ring.*

(1) *An ideal $I \subset R$ maximal with respect to not being finitely generated is prime.*
(2) *If every prime ideal of R is finitely generated, then every ideal of R is finitely generated*[1].

Proof. The first assertion is an immediate consequence of Example 10.26.4 and Proposition 10.26.7. For the second, suppose that there exists an ideal $I \subset R$ which is not finitely generated. The union of a totally ordered chain $\{I_\alpha\}$ of ideals that are not finitely generated is not finitely generated; indeed, if $I = \bigcup I_\alpha$ were generated by $a_1, \ldots, a_n$, then all the generators would belong to some I_α and would consequently generate it. By Zorn's lemma, there is an ideal maximal with respect to being not finitely generated. By the first part this ideal is prime. □

Lemma 10.26.10. *Let R be a ring.*

(1) *An ideal $I \subset R$ maximal with respect to not being principal is prime.*
(2) *If every prime ideal of R is principal, then every ideal of R is principal.*

Proof. The first part follows from Example 10.26.5 and Proposition 10.26.7. For the second, suppose that there exists an ideal $I \subset R$ which is not principal. The union of a totally ordered chain $\{I_\alpha\}$ of ideals that not principal is not principal; indeed, if $I = \bigcup I_\alpha$ were generated by a, then a would belong to some I_α and a would generate it. By Zorn's lemma, there is an ideal maximal with respect to not being principal. This ideal is necessarily prime by the first part. □

Lemma 10.26.11. *Let R be a ring.*

(1) *An ideal maximal among the ideals which do not contain a nonzerodivisor is prime.*
(2) *If every nonzero prime ideal in R contains a nonzerodivisor, then R is a domain.*

Proof. Consider the set S of nonzerodivisors. It is a multiplicative subset of R. Hence any ideal maximal with respect to not intersecting S is prime, see Lemma 10.26.8. Thus, if every nonzero prime ideal contains a nonzerodivisor, then (0) is prime, i.e., R is a domain. □

Remark 10.26.12. Let R be a ring. Let κ be an infinite cardinal. By applying Example 10.26.6 and Proposition 10.26.7 we see that any ideal maximal with respect to the property of not being generated by κ elements is prime. This result is not so useful because there exists a ring for which every prime ideal of R can be generated by $\aleph_0$ elements, but some ideal cannot. Namely, let k be a field, let T be a set whose cardinality is greater than $\aleph_0$ and let

$$R = k[\{x_n\}_{n\geq 1}, \{z_{t,n}\}_{t\in T, n\geq 0}]/(x_n^2, z_{t,n}^2, x_n z_{t,n} - z_{t,n-1})$$

This is a local ring with unique prime ideal $\mathfrak{m} = (x_n)$. But the ideal $(z_{t,n})$ cannot be generated by countably many elements.

[1] Later we will say that R is Noetherian.

10.27. Images of ring maps of finite presentation

In this section we prove some results on the topology of maps $\mathrm{Spec}(S) \to \mathrm{Spec}(R)$ induced by ring maps $R \to S$, mainly Chevalley's Theorem. In order to do this we will use the notions of constructible sets, quasi-compact sets, retrocompact sets, and so on which are defined in Topology, Section 5.11.

Lemma 10.27.1. *Let $U \subset \mathrm{Spec}(R)$ be open. The following are equivalent:*

1. *U is retrocompact in $\mathrm{Spec}(R)$,*
2. *U is quasi-compact,*
3. *U is a finite union of standard opens, and*
4. *there exists a finitely generated ideal $I \subset R$ such that $X \setminus V(I) = U$.*

Proof. We have (1) $\Rightarrow$ (2) because $\mathrm{Spec}(R)$ is quasi-compact, see Lemma 10.16.10. We have (2) $\Rightarrow$ (3) because standard opens form a basis for the topology. Proof of (3) $\Rightarrow$ (1). Let $U = \bigcup_{i=1\ldots n} D(f_i)$. To show that U is retrocompact in $\mathrm{Spec}(R)$ it suffices to show that $U \cap V$ is quasi-compact for any quasi-compact open V of $\mathrm{Spec}(R)$. Write $V = \bigcup_{j=1\ldots m} D(g_j)$ which is possible by (2) $\Rightarrow$ (3). Each standard open is homeomorphic to the spectrum of a ring and hence quasi-compact, see Lemmas 10.16.6 and 10.16.10. Thus $U \cap V = (\bigcup_{i=1\ldots n} D(f_i)) \cap (\bigcup_{j=1\ldots m} D(g_j)) = \bigcup_{i,j} D(f_i g_j)$ is a finite union of quasi-compact opens hence quasi-compact. To finish the proof note that (4) is equivalent to (3) by Lemma 10.16.2. □

Lemma 10.27.2. *Let $\varphi : R \to S$ be a ring map. The induced continuous map $f : \mathrm{Spec}(S) \to \mathrm{Spec}(R)$ is quasi-compact. For any constructible set $E \subset \mathrm{Spec}(R)$ the inverse image $f^{-1}(E)$ is constructible in $\mathrm{Spec}(S)$.*

Proof. We first show that the inverse image of any quasi-compact open $U \subset \mathrm{Spec}(R)$ is quasi-compact. By Lemma 10.27.1 we may write U as a finite open of standard opens. Thus by Lemma 10.16.4 we see that $f^{-1}(U)$ is a finite union of standard opens. Hence $f^{-1}(U)$ is quasi-compact by Lemma 10.27.1 again. The second assertion now follows from Topology, Lemma 5.14.3. □

Lemma 10.27.3. *Let R be a ring and let $T \subset \mathrm{Spec}(R)$ be constructible. Then there exists a ring map $R \to S$ of finite presentation such that T is the image of $\mathrm{Spec}(S)$ in $\mathrm{Spec}(R)$.*

Proof. Let $T \subset \mathrm{Spec}(R)$ be constructible. The spectrum of a finite product of rings is the disjoint union of the spectra, see Lemma 10.19.2. Hence if $T = T_1 \cup T_2$ and the result holds for T_1 and T_2, then the result holds for T. In particular we may assume that $T = U \cap V^c$, where $U, V \subset \mathrm{Spec}(R)$ are retrocompact open. By Lemma 10.27.1 we may write $T = (\bigcup D(f_i)) \cap (\bigcup D(g_j))^c = \bigcup \big(D(f_i) \cap V(g_1, \ldots, g_m)\big)$. In fact we may assume that $T = D(f) \cap V(g_1, \ldots, g_m)$ (by the argument on unions above). In this case T is the image of the map $R \to (R/(g_1, \ldots, g_m))_f$, see Lemmas 10.16.6 and 10.16.7. □

Lemma 10.27.4. *Let R be a ring. Let f be an element of R. Let $S = R_f$. Then the image of a constructible subset of $\mathrm{Spec}(S)$ is constructible in $\mathrm{Spec}(R)$.*

Proof. We repeatedly use Lemma 10.27.1 without mention. Let U, V be quasi-compact open in $\mathrm{Spec}(S)$. We will show that the image of $U \cap V^c$ is constructible. Under the identification $\mathrm{Spec}(S) = D(f)$ of Lemma 10.16.6 the sets U, V correspond

to quasi-compact opens U', V' of $\mathrm{Spec}(R)$. Hence it suffices to show that $U' \cap (V')^c$ is constructible in $\mathrm{Spec}(R)$ which is clear. □

Lemma 10.27.5. *Let R be a ring. Let I be a finitely generated ideal of R. Let $S = R/I$. Then the image of a constructible of $\mathrm{Spec}(S)$ is constructible in $\mathrm{Spec}(R)$.*

Proof. If $I = (f_1, \ldots, f_m)$, then we see that $V(I)$ is the complement of $\bigcup D(f_i)$, see Lemma 10.16.2. Hence it is constructible, by Lemma 10.27.1. Denote the map $R \to S$ by $f \mapsto \overline{f}$. We have to show that if $\overline{U}, \overline{V}$ are retrocompact opens of $\mathrm{Spec}(S)$, then the image of $\overline{U} \cap \overline{V}^c$ in $\mathrm{Spec}(R)$ is constructible. By Lemma 10.27.1 we may write $\overline{U} = \bigcup D(\overline{g_i})$. Setting $U = \bigcup D(g_i)$ we see $\overline{U}$ has image $U \cap V(I)$ which is constructible in $\mathrm{Spec}(R)$. Similarly the image of $\overline{V}$ equals $V \cap V(I)$ for some retrocompact open V of $\mathrm{Spec}(R)$. Hence the image of $\overline{U} \cap \overline{V}^c$ equals $U \cap V(I) \cap V^c$ as desired. □

Lemma 10.27.6. *Let R be a ring. The map $\mathrm{Spec}(R[x]) \to \mathrm{Spec}(R)$ is open, and the image of any standard open is a quasi-compact open.*

Proof. It suffices to show that the image of a standard open $D(f)$, $f \in R[x]$ is quasi-compact open. The image of $D(f)$ is the image of $\mathrm{Spec}(R[x]_f) \to \mathrm{Spec}(R)$. Let $\mathfrak{p} \subset R$ be a prime ideal. Let $\overline{f}$ be the image of f in $\kappa(\mathfrak{p})[x]$. Recall, see Lemma 10.16.9, that $\mathfrak{p}$ is in the image if and only if $R[x]_f \otimes_R \kappa(\mathfrak{p}) = \kappa(\mathfrak{p})[x]_{\overline{f}}$ is not the zero ring. This is exactly the condition that f does not map to zero in $\kappa(\mathfrak{p})[x]$, in other words, that some coefficient of f is not in $\mathfrak{p}$. Hence we see: if $f = a_d x^d + \ldots a_0$, then the image of $D(f)$ is $D(a_d) \cup \ldots \cup D(a_0)$. □

We prove a property of characteristic polynomials which will be used below.

Lemma 10.27.7. *Let $R \to A$ be a ring homomorphism. Assume $A \cong R^{\oplus n}$ as an R-module. Let $f \in A$. The multiplication map $m_f : A \to A$ is R-linear and hence has a characteristic polynomial $P(T) = T^n + r_{n-1}T^{n-1} + \cdots + r_0 \in R[T]$. For any prime $\mathfrak{p} \in \mathrm{Spec}(R)$, f acts nilpotently on $A \otimes_R \kappa(\mathfrak{p})$ if and only if $\mathfrak{p} \in V(r_0, \ldots, r_{n-1})$.*

Proof. This follows quite easily once we prove that the characteristic polynomial $\bar{P}(T) \in \kappa(\mathfrak{p})[T]$ of the multiplication map $m_{\bar{f}} : A \otimes_R \kappa(\mathfrak{p}) \to A \otimes_R \kappa(\mathfrak{p})$ which multiplies elements of $A \otimes_R \kappa(\mathfrak{p})$ by $\bar{f}$, the image of f viewed in $\kappa(\mathfrak{p})$, is just the image of $P(T)$ in $\kappa(\mathfrak{p})[T]$. Let (a_{ij}) be the matrix of the map m_f with entries in R, using a basis $e_1, \ldots, e_n$ of A as an R-module. Then, $A \otimes_R \kappa(\mathfrak{p}) \cong (R \otimes_R \kappa(\mathfrak{p}))^{\oplus n} = \kappa(\mathfrak{p})^n$, which is an n-dimensional vector space over $\kappa(\mathfrak{p})$ with basis $e_1 \otimes 1, \ldots, e_n \otimes 1$. The image $\bar{f} = f \otimes 1$, and so the multiplication map $m_{\bar{f}}$ has matrix $(a_{ij} \otimes 1)$. Thus, the characteristic polynomial is precisely the image of $P(T)$.

From linear algebra, we know that a linear transformation acts nilpotently on an n-dimensional vector space if and only if the characteristic polynomial is T^n (since the characteristic polynomial divides some power of the minimal polynomial). Hence, f acts nilpotently on $A \otimes_R \kappa(\mathfrak{p})$ if and only if $\bar{P}(T) = T^n$. This occurs if and only if $r_i \in \mathfrak{p}$ for all $0 \leq i \leq n-1$, that is when $\mathfrak{p} \in V(r_0, \ldots, r_{n-1})$. □

Lemma 10.27.8. *Let R be a ring. Let $f, g \in R[x]$ be polynomials. Assume the leading coefficient of g is a unit of R. There exists elements $r_i \in R$, $i = 1 \ldots, n$ such that the image of $D(f) \cap V(g)$ in $\mathrm{Spec}(R)$ is $\bigcup_{i=1,\ldots,n} D(r_i)$.*

Proof. Write $g = ux^d + a_{d-1}x^{d-1} + \ldots + a_0$, where d is the degree of g, and hence $u \in R^*$. Consider the ring $A = R[x]/(g)$. It is, as an R-module, finite free with basis the images of $1, x, \ldots, x^{d-1}$. Consider multiplication by (the image of) f on A. This is an R-module map. Hence we can let $P(T) \in R[T]$ be the characteristic polynomial of this map. Write $P(T) = T^d + r_{d-1}T^{d-1} + \ldots + r_0$. We claim that $r_0, \ldots, r_{d-1}$ have the desired property. We will use below the property of characteristic polynomials that

$$\mathfrak{p} \in V(r_0, \ldots, r_{d-1}) \Leftrightarrow \text{multiplication by } f \text{ is nilpotent on } A \otimes_R \kappa(\mathfrak{p}).$$

This was proved in Lemma 10.27.7.

Suppose $\mathfrak{q} \in D(f) \cap V(g)$, and let $\mathfrak{p} = \mathfrak{q} \cap R$. Then there is a nonzero map $A \otimes_R \kappa(\mathfrak{p}) \to \kappa(\mathfrak{q})$ which is compatible with multiplication by f. And f acts as a unit on $\kappa(\mathfrak{q})$. Thus we conclude $\mathfrak{p} \notin V(r_0, \ldots, r_{d-1})$.

On the other hand, suppose that $r_i \notin \mathfrak{p}$ for some prime $\mathfrak{p}$ of R and some $0 \leq i \leq d-1$. Then multiplication by f is not nilpotent on the algebra $A \otimes_R \kappa(\mathfrak{p})$. Hence there exists a maximal ideal $\overline{\mathfrak{q}} \subset A \otimes_R \kappa(\mathfrak{p})$ not containing the image of f. The inverse image of $\overline{\mathfrak{q}}$ in $R[x]$ is an element of $D(f) \cap V(g)$ mapping to $\mathfrak{p}$. □

Theorem 10.27.9 (Chevalley's Theorem)**.** *Suppose that $R \to S$ is of finite presentation. The image of a constructible subset of* $\mathrm{Spec}(S)$ *in* $\mathrm{Spec}(R)$ *is constructible.*

Proof. Write $S = R[x_1, \ldots, x_n]/(f_1, \ldots, f_m)$. We may factor $R \to S$ as $R \to R[x_1] \to R[x_1, x_2] \to \ldots \to R[x_1, \ldots, x_{n-1}] \to S$. Hence we may assume that $S = R[x]/(f_1, \ldots, f_m)$. In this case we factor the map as $R \to R[x] \to S$, and by Lemma 10.27.5 we reduce to the case $S = R[x]$. By Lemma 10.27.1 suffices to show that if $T = (\bigcup_{i=1\ldots n} D(f_i)) \cap V(g_1, \ldots, g_m)$ for $f_i, g_j \in R[x]$ then the image in $\mathrm{Spec}(R)$ is constructible. Since finite unions of constructible sets are constructible, it suffices to deal with the case $n = 1$, i.e., when $T = D(f) \cap V(g_1, \ldots, g_m)$.

Note that if $c \in R$, then we have

$$\mathrm{Spec}(R) = V(c) \coprod D(c) = \mathrm{Spec}(R/(c)) \coprod \mathrm{Spec}(R_c)),$$

and correspondingly $\mathrm{Spec}(R[x]) = V(c) \coprod D(c) = \mathrm{Spec}(R/(c)[x]) \coprod \mathrm{Spec}(R_c[x]))$. The intersection of $T = D(f) \cap V(g_1, \ldots, g_m)$ with each part still has the same shape, with f, g_i replaced by their images in $R/(c)[x]$, respectively $R_c[x]$. Note that the image of T in $\mathrm{Spec}(R)$ is the union of the image of $T \cap V(c)$ and $T \cap D(c)$. Using Lemmas 10.27.4 and 10.27.5 it suffices to prove the images of both parts are constructible in $\mathrm{Spec}(R/(c))$, respectively $\mathrm{Spec}(R_c)$.

Let us assume we have $T = D(f) \cap V(g_1, \ldots, g_m)$ as above, with $\deg(g_1) \leq \deg(g_2) \leq \ldots \leq \deg(g_m)$. We are going to use descending induction on m, and on the degrees of the g_i. Let $d = \deg(g_1)$, i.e., $g_1 = cx^{d_1} + l.o.t$ with $c \in R$ not zero. Cutting R up into the pieces $R/(c)$ and R_c we either lower the degree of g_1 (and this is covered by induction) or we reduce to the case where c is invertible. If c is invertible, and $m > 1$, then write $g_2 = c'x^{d_2} + l.o.t.$ In this case consider $g_2' = g_2 - (c'/c)x^{d_2-d_1}g_1$. Since the ideals $(g_1, g_2, \ldots, g_m)$ and $(g_1, g_2', g_3, \ldots, g_m)$ are equal we see that $T = D(f) \cap V(g_1, g_2', g_3 \ldots, g_m)$. But here the degree of g_2' is strictly less than the degree of g_2 and hence this case is covered by induction.

The bases case for the induction above are the cases (a) $T = D(f) \cap V(g)$ where the leading coefficient of g is invertible, and (b) $T = D(f)$. These two cases are dealt with in Lemmas 10.27.8 and 10.27.6. □

10.28. More on images

In this section we collect a few additional lemmas concerning the image on Spec for ring maps. See also Section 10.38 for example.

Lemma 10.28.1. *Let $R \subset S$ be an inclusion of domains. Assume that $R \to S$ is of finite type. There exists a nonzero $f \in R$, and a nonzero $g \in S$ such that $R_f \to S_{fg}$ is of finite presentation.*

Proof. By induction on the number of generators of S over R.

Suppose that S is generated by a single element over R. Then $S = R[x]/\mathfrak{q}$ for some prime ideal $\mathfrak{q} \subset R[x]$. If $\mathfrak{q} = (0)$ there is nothing to prove. If $\mathfrak{q} \neq (0)$, then let $g \in \mathfrak{q}$ be an element with minimal degree in x. Since $K[x] = f.f.(R)[x]$ is a PID we see that g is irreducible over K and that $f.f.(S) = K[x]/(g)$. Write $g = a_d x^d + \ldots + a_0$ with $a_i \in R$ and $a_d \neq 0$. After inverting a_d in R we may assume that g is monic. Hence we see that $R \to R[x]/(g) \to S$ with the last map surjective. But $R[x]/(g) = R \oplus Rx \oplus \ldots \oplus Rx^{d-1}$ maps injectively into $f.f.(S) = K[x]/(g) = K \oplus Kx \oplus \ldots \oplus Kx^{d-1}$. Thus $S \cong R[x]/(g)$ is finitely presented.

Suppose that S is generated by $n > 1$ elements over R. Say $x_1, \ldots, x_n \in S$ generate S. Denote $S' \subset S$ the subring generated by $x_1, \ldots, x_{n-1}$. By induction hypothesis we see that there exist $f \in R$ and $g \in S'$ nonzero such that $R_f \to S'_{fg}$ is of finite presentation. Next we apply the induction hypothesis to $S'_{fg} \to S_{fg}$ to see that there exist $f' \in S'_{fg}$ and $g' \in S_{fg}$ such that $S'_{fgf'} \to S_{fgf'g'}$ is of finite presentation. We leave it to the reader to conclude. □

Lemma 10.28.2. *Let $R \to S$ be a finite type ring map. Denote $X = \mathrm{Spec}(R)$ and $Y = \mathrm{Spec}(S)$. Write $f : Y \to X$ the induced map of spectra. Let $E \subset Y = \mathrm{Spec}(S)$ be a constructible set. If a point $\xi \in X$ is in $f(E)$, then $\overline{\{\xi\}} \cap f(E)$ contains an open dense subset of $\overline{\{\xi\}}$.*

Proof. Let $\xi \in X$ be a point of $f(E)$. Choose a point $\eta \in E$ mapping to ξ. Let $\mathfrak{p} \subset R$ be the prime corresponding to ξ and let $\mathfrak{q} \subset S$ be the prime corresponding to η. Consider the diagram

$$\begin{array}{ccccccc} \eta & \longrightarrow & E \cap Y' & \longrightarrow & Y' = \mathrm{Spec}(S/\mathfrak{q}) & \longrightarrow & Y \\ \downarrow & & \downarrow & & \downarrow & & \downarrow \\ \xi & \longrightarrow & f(E) \cap X' & \longrightarrow & X' = \mathrm{Spec}(R/\mathfrak{p}) & \longrightarrow & X \end{array}$$

By Lemma 10.27.2 the set $E \cap Y'$ is constructible in Y'. It follows that we may replace X by X' and Y by Y'. Hence we may assume that $R \subset S$ is an inclusion of domains, ξ is the generic point of X, and η is the generic point of Y. By Lemma 10.28.1 combined with Chevalley's theorem (Theorem 10.27.9) we see that there exist dense opens $U \subset X$, $V \subset Y$ such that $f(V) \subset U$ and such that $f : V \to U$ maps constructible sets to constructible sets. Note that $E \cap V$ is constructible in V, see Topology, Lemma 5.14.4. Hence $f(E \cap V)$ is constructible in U and

contains ξ. By Topology, Lemma 5.14.14 we see that $f(E \cap V)$ contains a dense open $U' \subset U$. □

At the end of this section we present a few more results on images of maps on Spectra that have nothing to do with constructible sets.

Lemma 10.28.3. *Let $\varphi : R \to S$ be a ring map. The following are equivalent:*

1. *The map $\mathrm{Spec}(S) \to \mathrm{Spec}(R)$ is surjective.*
2. *For any radical ideal $I \subset R$ the inverse image of IS in R is equal to I.*
3. *For every prime $\mathfrak{p}$ of R the inverse image of $\mathfrak{p}S$ in R is $\mathfrak{p}$.*

In this case the same is true after any base change: Given a ring map $R \to R'$ the ring map $R' \to R' \otimes_R S$ has the equivalent properties (1), (2), (3) also.

Proof. The implication (2) $\Rightarrow$ (3) is immediate. If $I \subset R$ is a radical ideal, then Lemma 10.16.2 guarantees that $I = \bigcap_{I \subset \mathfrak{p}} \mathfrak{p}$. Hence (3) $\Rightarrow$ (2). By Lemma 10.16.9 we have $\mathfrak{p} = \varphi^{-1}(\mathfrak{p}S)$ if and only if $\mathfrak{p}$ is in the image. Hence (1) $\Leftrightarrow$ (3). Thus (1), (2), and (3) are equivalent.

Assume (1) holds. Let $R \to R'$ be a ring map. Let $\mathfrak{p}' \subset R'$ be a prime ideal lying over the prime $\mathfrak{p}$ of R. To see that $\mathfrak{p}'$ is in the image of $\mathrm{Spec}(R' \otimes_R S) \to \mathrm{Spec}(R')$ we have to show that $(R' \otimes_R S) \otimes_{R'} \kappa(\mathfrak{p}')$ is not zero, see Lemma 10.16.9. But we have

$$(R' \otimes_R S) \otimes_{R'} \kappa(\mathfrak{p}') = S \otimes_R \kappa(\mathfrak{p}) \otimes_{\kappa(\mathfrak{p})} \kappa(\mathfrak{p}')$$

which is not zero as $S \otimes_R \kappa(\mathfrak{p})$ is not zero by assumption and $\kappa(\mathfrak{p}) \to \kappa(\mathfrak{p}')$ is an extension of fields. □

Lemma 10.28.4. *Let R be a domain. Let $\varphi : R \to S$ be a ring map. The following are equivalent:*

1. *The ring map $R \to S$ is injective.*
2. *The image $\mathrm{Spec}(S) \to \mathrm{Spec}(R)$ contains a dense set of points.*
3. *There exists a prime ideal $\mathfrak{q} \subset S$ whose inverse image in R is (0).*

Proof. Let K be the field of fractions of the domain R. Assume that $R \to S$ is injective. Since localization is exact we see that $K \to S \otimes_R K$ is injective. Hence there is a prime mapping to (0) by Lemma 10.16.9.

Note that (0) is dense in $\mathrm{Spec}(R)$, so that the last condition implies the second.

Suppose the second condition holds. Let $f \in R$, $f \neq 0$. As R is a domain we see that $V(f)$ is a proper closed subset of R. By assumption there exists a prime $\mathfrak{q}$ of S such that $\varphi(f) \notin \mathfrak{q}$. Hence $\varphi(f) \neq 0$. Hence $R \to S$ is injective. □

Lemma 10.28.5. *Let $R \subset S$ be an injective ring map. Then $\mathrm{Spec}(S) \to \mathrm{Spec}(R)$ hits all the minimal primes of $\mathrm{Spec}(R)$.*

Proof. Let $\mathfrak{p} \subset R$ be a minimal prime. In this case $R_\mathfrak{p}$ has a unique prime ideal. Hence it suffices to show that $S_\mathfrak{p}$ is not zero. And this follows from the fact that localization is exact, see Proposition 10.9.12. □

Lemma 10.28.6. *Let $R \to S$ be a ring map. The following are equivalent:*

1. *The kernel of $R \to S$ consists of nilpotent elements.*
2. *The minimal primes of R are in the image of $\mathrm{Spec}(S) \to \mathrm{Spec}(R)$.*
3. *The image of $\mathrm{Spec}(S) \to \mathrm{Spec}(R)$ is dense in $\mathrm{Spec}(R)$.*

Proof. Let $I = \mathrm{Ker}(R \to S)$. Note that $\sqrt{(0)} = \bigcap_{\mathfrak{q} \subset S} \mathfrak{q}$, see Lemma 10.16.2. Hence $\sqrt{I} = \bigcap_{\mathfrak{q} \subset S} R \cap \mathfrak{q}$. Thus $V(I) = V(\sqrt{I})$ is the closure of the image of $\mathrm{Spec}(S) \to \mathrm{Spec}(R)$. This shows that (1) is equivalent to (3). It is clear that (2) implies (3). Finally, assume (1). We may replace R by R/I and S by S/IS without affecting the topology of the spectra and the map. Hence the implication (1) $\Rightarrow$ (2) follows from Lemma 10.28.5. □

10.29. Noetherian rings

A ring R is *Noetherian* if any ideal of R is finitely generated. This is clearly equivalent to the ascending chain condition for ideals of R. By Lemma 10.26.9 it suffices to check that every prime ideal of R is finitely generated.

Lemma 10.29.1. *Any finitely generated ring over a Noetherian ring is Noetherian. Any localization of a Noetherian ring is Noetherian.*

Proof. The statement on localizations follows from the fact that any ideal $J \subset S^{-1}R$ is of the form $I \cdot S^{-1}R$. Any quotient R/I of a Noetherian ring R is Noetherian because any ideal $\overline{J} \subset R/I$ is of the form J/I for some ideal $I \subset J \subset R$. Thus it suffices to show that if R is Noetherian so is $R[X]$. Suppose $J_1 \subset J_2 \subset \ldots$ is an ascending chain of ideals in $R[X]$. Consider the ideals $I_{i,d}$ defined as the ideal of elements of R which occur as leading coefficients of degree d polynomials in J_i. Clearly $I_{i,d} \subset I_{i',d'}$ whenever $i \leq i'$ and $d \leq d'$. By the ascending chain condition in R there are at most finitely many distinct ideals among all of the $I_{i,d}$. (Hint: Any infinite set of elements of $\mathbf{N} \times \mathbf{N}$ contains an increasing infinite sequence.) Take i_0 so large that $I_{i,d} = I_{i_0,d}$ for all $i \geq i_0$ and all d. Suppose $f \in J_i$ for some $i \geq i_0$. By induction on the degree $d = \deg(f)$ we show that $f \in J_{i_0}$. Namely, there exists a $g \in J_{i_0}$ whose degree is d and which has the same leading coefficient as f. By induction $f - g \in J_{i_0}$ and we win. □

Lemma 10.29.2. *If R is a Noetherian ring, then so is the formal power series ring $R[[x_1, \ldots, x_n]]$.*

Proof. Since $R[[x_1, \ldots, x_{n+1}]] \cong R[[x_1, \ldots, x_n]][[x_{n+1}]]$ it suffices to prove the statement that $R[[x]]$ is Noetherian if R is Noetherian. Let $I \subset R[[x]]$ be a ideal. We have to show that I is a finitely generated ideal. For each integer d denote $I_d = \{a \in R \mid ax^d + \text{h.o.t.} \in I\}$. Then we see that $I_0 \subset I_1 \subset \ldots$ stabilizes as R is Noetherian. Choose d_0 such that $I_{d_0} = I_{d_0+1} = \ldots$. For each $d \leq d_0$ choose elements $f_{d,j} \in I \cap (x^d)$, $j = 1, \ldots, n_d$ such that if we write $f_{d,j} = a_{d,j}x^d + \text{h.o.t}$ then $I_d = (a_{d,j})$. Denote $I' = (\{f_{d,j}\}_{d=0,\ldots,d_0, j=1,\ldots,n_d})$. Then it is clear that $I' \subset I$. Pick $f \in I$. First we may choose $c_{d,i} \in R$ such that

$$f - \sum c_{d,i} f_{d,i} \in (x^{d_0+1}) \cap I.$$

Next, we can choose $c_{i,1} \in R$, $i = 1, \ldots, n_{d_0}$ such that

$$f - \sum c_{d,i} f_{d,i} - \sum c_{i,1} x f_{d_0,i} \in (x^{d_0+2}) \cap I.$$

Next, we can choose $c_{i,2} \in R$, $i = 1, \ldots, n_{d_0}$ such that

$$f - \sum c_{d,i} f_{d,i} - \sum c_{i,1} x f_{d_0,i} - \sum c_{i,2} x^2 f_{d_0,i} \in (x^{d_0+3}) \cap I.$$

And so on. In the end we see that

$$f = \sum c_{d,i} f_{d,i} + \sum\nolimits_i (\sum\nolimits_e c_{i,e} x^e) f_{d_0,i}$$

is contained in I' as desired. □

The following lemma, although easy, is useful because finite type $\mathbf{Z}$-algebras come up quite often in a technique called "absolute Noetherian reduction".

Lemma 10.29.3. *Any finite type algebra over a field is Noetherian. Any finite type algebra over $\mathbf{Z}$ is Noetherian.*

Proof. This is immediate from Lemma 10.29.1 and the fact that fields are Noetherian rings and that $\mathbf{Z}$ is Noetherian ring (because it is a principal ideal domain). □

Lemma 10.29.4. *Let R be a Noetherian ring.*

(1) *Any finite R-module is of finite presentation.*
(2) *Any finite type R-algebra is of finite presentation over R.*

Proof. Let M be a finite R-module. By Lemma 10.5.4 we can find a finite filtration of M whose successive quotients are of the form R/I. Since any ideal is finitely generated, each of the quotients R/I is finitely presented. Hence M is finitely presented by Lemma 10.5.3. This proves (1). To see (2) note that any ideal of $R[x_1, \ldots, x_n]$ is finitely generated by Lemma 10.29.1. □

Lemma 10.29.5. *If R is a Noetherian ring then* $\mathrm{Spec}(R)$ *is a Noetherian topological space, see Topology, Definition 5.8.1.*

Proof. This is because any closed subset of $\mathrm{Spec}(R)$ is uniquely of the form $V(I)$ with I a radical ideal, see Lemma 10.16.2. And this correspondence is inclusion reversing. Thus the result follows from the definitions. □

Lemma 10.29.6. *If R is a Noetherian ring then* $\mathrm{Spec}(R)$ *has finitely many irreducible components. In other words R has finitely many minimal primes.*

Proof. By Lemma 10.29.5 and Topology, Lemma 5.8.2 we see there are finitely many irreducible components. By Lemma 10.24.1 these correspond to minimal primes of R. □

Lemma 10.29.7. *Let k be a field and let R be a Noetherian k-algebra. If $k \subset K$ is a finitely generated field extension the $K \otimes_k R$ is Noetherian.*

Proof. Write $K = S^{-1}B$ where B is a finite type k-algebra, and $S \subset B$ is a multiplicative subset. Then we have $K \otimes_k R = S^{-1}(B \otimes_k R)$. Hence $K \otimes_k R$ is a localization of the finite type R-algebra $B \otimes_k R$ which is Noetherian by Lemma 10.29.1. □

Here is fun lemma that is sometimes useful.

Lemma 10.29.8. *Any surjective endomorphism of a Noetherian ring is an isomorphism.*

Proof. If $f : R \to R$ were such an endomorphism but not injective, then

$$\mathrm{Ker}(f) \subset \mathrm{Ker}(f \circ f) \subset \mathrm{Ker}(f \circ f \circ f) \subset \ldots$$

would be a strictly increasing chain of ideals. □

10.30. Curiosity

Lemma 10.21.3 explains what happens if $V(I)$ is open for some ideal $I \subset R$. But what if $\mathrm{Spec}(S^{-1}R)$ is closed in $\mathrm{Spec}(R)$? The next two lemmas give a partial answer. For more information see Section 10.103.

Lemma 10.30.1. *Let R be a ring. Let $S \subset R$ be a multiplicative subset. Assume the image of the map $\mathrm{Spec}(S^{-1}R) \to \mathrm{Spec}(R)$ is closed. Then $S^{-1}R \cong R/I$ for some ideal $I \subset R$.*

Proof. Let $I = \mathrm{Ker}(R \to S^{-1}R)$ so that $V(I)$ contains the image. Say the image is the closed subset $V(I') \subset \mathrm{Spec}(R)$ for some ideal $I' \subset R$. So $V(I') \subset V(I)$. For $f \in I'$ we see that $f/1 \in S^{-1}R$ is contained in every prime ideal. Hence f^n maps to zero in $S^{-1}R$ for some $n \geq 1$ (Lemma 10.16.2). Hence $V(I') = V(I)$. Then this implies every $g \in S$ is invertible mod I. Hence we get ring maps $R/I \to S^{-1}R$ and $S^{-1}R \to R/I$. The first map is injective by choice of I. The second is the map $S^{-1}R \to S^{-1}(R/I) = R/I$ which has kernel $S^{-1}I$ because localization is exact. Since $S^{-1}I = 0$ we see also the second map is injective. Hence $S^{-1}R \cong R/I$. □

Lemma 10.30.2. *Let R be a ring. Let $S \subset R$ be a multiplicative subset. Assume the image of the map $\mathrm{Spec}(S^{-1}R) \to \mathrm{Spec}(R)$ is closed. If R is Noetherian, or $\mathrm{Spec}(R)$ is a Noetherian topological space, or S is finitely generated as a monoid, then $R \cong S^{-1}R \times R'$ for some ring R'.*

Proof. By Lemma 10.30.1 we have $S^{-1}R \cong R/I$ for some ideal $I \subset R$. By Lemma 10.21.3 it suffices to show that $V(I)$ is open. If R is Noetherian then $\mathrm{Spec}(R)$ is a Noetherian topological space, see Lemma 10.29.5. If $\mathrm{Spec}(R)$ is a Noetherian topological space, then the complement $\mathrm{Spec}(R)\backslash V(I)$ is quasi-compact, see Topology, Lemma 5.11.13. Hence there exist finitely many $f_1, \ldots, f_n \in I$ such that $V(I) = V(f_1, \ldots, f_n)$. Since each f_i maps to zero in $S^{-1}R$ there exists a $g \in S$ such that $gf_i = 0$ for $i = 1, \ldots, n$. Hence $D(g) = V(I)$ as desired. In case S is finitely generated as a monoid, say S is generated by $g_1, \ldots, g_m$, then $S^{-1}R \cong R_{g_1 \ldots g_m}$ and we conclude that $V(I) = D(g_1 \ldots g_m)$. □

10.31. Hilbert Nullstellensatz

Theorem 10.31.1 (Hilbert Nullstellensatz)**.** *Let k be a field.*

1. *For any maximal ideal $\mathfrak{m} \subset k[x_1, \ldots, x_n]$ the field extension $k \subset \kappa(\mathfrak{m})$ is finite.*
2. *Any radical ideal $I \subset k[x_1, \ldots, x_n]$ is the intersection of maximal ideals containing it.*

The same is true in any finite type k-algebra.

Proof. It is enough to prove part (1) of the theorem for the case of a polynomial algebra $k[x_1, \ldots, x_n]$, because any finitely generated k-algebra is a quotient of such a polynomial algebra. We prove this by induction on n. The case $n = 0$ is clear. Suppose that $\mathfrak{m}$ is a maximal ideal in $k[x_1, \ldots, x_n]$. Let $\mathfrak{p} \subset k[x_n]$ be the intersection of $\mathfrak{m}$ with $k[x_n]$.

If $\mathfrak{p} \neq (0)$, then $\mathfrak{p}$ is maximal and generated by an irreducible monic polynomial P (because of the Euclidean algorithm in $k[x_n]$). Then $k' = k[x_n]/\mathfrak{p}$ is a finite field extension of k and contained in $\kappa(\mathfrak{m})$. In this case we get a surjection

$$k'[x_1, \ldots, x_{n-1}] \to k'[x_1, \ldots, x_n] = k' \otimes_k k[x_1, \ldots, x_n] \longrightarrow \kappa(\mathfrak{m})$$

and hence we see that $\kappa(\mathfrak{m})$ is a finite extension of k' by induction hypothesis. Thus $\kappa(\mathfrak{m})$ is finite over k as well.

If $\mathfrak{p} = (0)$ we consider the ring extension $k[x_n] \subset k[x_1, \ldots, x_n]/\mathfrak{m}$. This is a finitely generated ring extension, hence of finite presentation by Lemmas 10.29.3 and 10.29.4. Thus the image of $\mathrm{Spec}(k[x_1, \ldots, x_n]/\mathfrak{m})$ in $\mathrm{Spec}(k[x_n])$ is constructible by Theorem 10.27.9. Since the image contains (0) we conclude that it contains a standard open $D(f)$ for some $f \in k[x_n]$ nonzero. Since clearly $D(f)$ is infinite we get a contradiction with the assumption that $k[x_1, \ldots, x_n]/\mathfrak{m}$ is a field (and hence has a spectrum consisting of one point).

To prove part (2) let $I \subset R$ be radical, with R of finite type over k. Let $f \in R$, $f \notin I$. Pick a maximal ideal $\mathfrak{m}'$ in the nonzero ring $R_f/IR_f = (R/I)_f$. Let $\mathfrak{m} \subset R$ be the inverse image of $\mathfrak{m}'$ in R. We see that $I \subset \mathfrak{m}$ and $f \notin \mathfrak{m}$. If we show that $\mathfrak{m}$ is a maximal ideal of R, then we are done. We clearly have

$$k \subset R/\mathfrak{m} \subset \kappa(\mathfrak{m}').$$

By part (1) the field extension $k \subset \kappa(\mathfrak{m}')$ is finite. By elementary field theory we conclude that $R/\mathfrak{m}$ is a field. □

Lemma 10.31.2. *Let R be a ring. Let K be a field. If $R \subset K$ and K is of finite type over R, then there exists a $f \in R$ such that R_f is a field, and $R_f \subset K$ is a finite field extension.*

Proof. By Lemma 10.28.2 there exist a nonempty open $U \subset \mathrm{Spec}(R)$ contained in the image $\{(0)\}$ of $\mathrm{Spec}(K) \to \mathrm{Spec}(R)$. Choose $f \in R$, $f \neq 0$ such that $D(f) \subset U$, i.e., $D(f) = \{(0)\}$. Then R_f is a domain whose spectrum has exactly one point and R_f is a field. Then K is a finitely generated algebra over the field R_f and hence a finite field extension of R_f by the Hilbert Nullstellensatz (Theorem 10.31.1). □

10.32. Jacobson rings

Let R be a ring. The closed points of $\mathrm{Spec}(R)$ are the maximal ideals of R. Often rings which occur naturally in algebraic geometry have lots of maximal ideals. For example finite type algebras over a field or over $\mathbf{Z}$. We will show that these are examples of Jacobson rings.

Definition 10.32.1. Let R be a ring. We say that R is a *Jacobson ring* if every radical ideal I is the intersection of the maximal ideals containing it.

Lemma 10.32.2. *Any algebra of finite type over a field is Jacobson.*

Proof. This follows from Theorem 10.31.1 and Definition 10.32.1. □

Lemma 10.32.3. *Let R be a ring. If every prime ideal of R is the intersection of the maximal ideals containing it, then R is Jacobson.*

Proof. This is immediately clear from the fact that every radical ideal $I \subset R$ is the intersection of the primes containing it. See Lemma 10.16.2. □

Lemma 10.32.4. *A ring R is Jacobson if and only if $\mathrm{Spec}(R)$ is Jacobson, see Topology, Definition 5.17.1.*

Proof. Suppose R is Jacobson. Let $Z \subset \mathrm{Spec}(R)$ be a closed subset. We have to show that the set of closed points in Z is dense in Z. Let $U \subset \mathrm{Spec}(R)$ be an open such that $U \cap Z$ is nonempty. We have to show $Z \cap U$ contains a closed point of $\mathrm{Spec}(R)$. We may assume $U = D(f)$ as standard opens form a basis for the topology on $\mathrm{Spec}(R)$. According to Lemma 10.16.2 we may assume that $Z = V(I)$, where I is a radical ideal. We see also that $f \notin I$. By assumption, there exists a maximal ideal $\mathfrak{m} \subset R$ such that $I \subset \mathfrak{m}$ but $f \notin \mathfrak{m}$. Hence $\mathfrak{m} \in D(f) \cap V(I) = U \cap Z$ as desired.

Conversely, suppose that $\mathrm{Spec}(R)$ is Jacobson. Let $I \subset R$ be a radical ideal. Let $J = \cap_{I \subset \mathfrak{m}} \mathfrak{m}$ be the intersection of the maximal ideals containing I. Clearly J is radical, $V(J) \subset V(I)$, and $V(J)$ is the smallest closed subset of $V(I)$ containing all the closed points of $V(I)$. By assumption we see that $V(J) = V(I)$. But Lemma 10.16.2 shows there is a bijection between Zariski closed sets and radical ideals, hence $I = J$ as desired. □

Lemma 10.32.5. *Let R be a ring. If R is not Jacobson there exist a prime $\mathfrak{p} \subset R$, an element $f \in R$ such that the following hold*

(1) *$\mathfrak{p}$ is not a maximal ideal,*
(2) *$f \notin \mathfrak{p}$,*
(3) *$V(\mathfrak{p}) \cap D(f) = \{\mathfrak{p}\}$, and*
(4) *$(R/\mathfrak{p})_f$ is a field.*

On the other hand, if R is Jacobson, then for any pair $(\mathfrak{p}, f)$ such that (1) and (2) hold the set $V(\mathfrak{p}) \cap D(f)$ is infinite.

Proof. Assume R is not Jacobson. By Lemma 10.32.4 this means there exists an closed subset $T \subset \mathrm{Spec}(R)$ whose set $T_0 \subset T$ of closed points is not dense in T. Choose an $f \in R$ such that $T_0 \subset V(f)$ but $T \not\subset V(f)$. Note that $T \cap D(f)$ is homeomorphic to $\mathrm{Spec}((R/I)_f)$ if $T = V(I)$, see Lemmas 10.16.7 and 10.16.6. As any ring has a maximal ideal (Lemma 10.16.2) we can choose a closed point t of space $T \cap D(f)$. Then t corresponds to a prime ideal $\mathfrak{p} \subset R$ which is not maximal (as $t \notin T_0$). Thus (1) holds. By construction $f \notin \mathfrak{p}$, hence (2). As t is a closed point of $T \cap D(f)$ we see that $V(\mathfrak{p}) \cap D(f) = \{\mathfrak{p}\}$, i.e., (3) holds. Hence we conclude that $(R/\mathfrak{p})_f$ is a domain whose spectrum has one point, hence (4) holds (for example combine Lemmas 10.17.2 and 10.24.4).

Conversely, suppose that R is Jacobson and $(\mathfrak{p}, f)$ satisfy (1) and (2). If $V(\mathfrak{p}) \cap V(f) = \{\mathfrak{p}, \mathfrak{q}_1, \ldots, \mathfrak{q}_t\}$ then $\mathfrak{p} \neq \mathfrak{q}_i$ implies there exists an element $g \in R$ such that $g \notin \mathfrak{p}$ but $g \in \mathfrak{q}_i$ for all i. Hence $V(\mathfrak{p}) \cap D(fg) = \{\mathfrak{p}\}$ which is impossible since each locally closed subset of $\mathrm{Spec}(R)$ contains at least one closed point as $\mathrm{Spec}(R)$ is a Jacobson topological space. □

Lemma 10.32.6. *The ring $\mathbf{Z}$ is a Jacobson ring. More generally, let R be a ring such that*

(1) *R is a domain,*
(2) *R is Noetherian,*
(3) *any nonzero prime ideal is a maximal ideal, and*
(4) *R has infinitely many maximal ideals.*

Then R is a Jacobson ring.

Proof. Let R satisfy (1), (2), (3) and (4). The statement means that $(0) = \bigcap_{\mathfrak{m} \subset R} \mathfrak{m}$. Since R has infinitely many maximal ideals it suffices to show that any nonzero $x \in R$ is contained in at most finitely many maximal ideals, in other words that $V(x)$ is finite. By Lemma 10.16.7 we see that $V(x)$ is homeomorphic to $\mathrm{Spec}(R/xR)$. By assumption (3) every prime of R/xR is minimal and hence corresponds to an irreducible component of $\mathrm{Spec}(R)$ (Lemma 10.24.1). As R/xR is Noetherian, the topological space $\mathrm{Spec}(R/xR)$ is Noetherian (Lemma 10.29.5) and has finitely many irreducible components (Topology, Lemma 5.8.2). Thus $V(x)$ is finite as desired. □

Example 10.32.7. Let A be an infinite set. For each $\alpha \in A$, let k_α be a field. We claim that $R = \prod_{\alpha \in A} k_\alpha$ is Jacobson. First, note that any element $f \in R$ has the form $f = ue$, with $u \in R$ a unit and $e \in R$ an idempotent (left to the reader). Hence $D(f) = D(e)$, and $R_f = R_e = R/(1 - e)$ is a quotient of R. Actually, any ring with this property is Jacobson. Namely, say $\mathfrak{p} \subset R$ is a prime ideal and $f \in R$, $f \notin \mathfrak{p}$. We have to find a maximal ideal $\mathfrak{m}$ of R such that $\mathfrak{p} \subset \mathfrak{m}$ and $f \notin \mathfrak{m}$. Because R_f is a quotient of R we see that any maximal ideal of R_f corresponds to a maximal ideal of R not containing f. Hence the result follows by choosing a maximal ideal of R_f containing $\mathfrak{p}R_f$.

Example 10.32.8. A domain R with finitely many maximal ideals $\mathfrak{m}_i$, $i = 1, \ldots, n$ is not a Jacobson ring, except when it is a field. Namely, in this case (0) is not the intersection of the maximal ideals $(0) \neq \mathfrak{m}_1 \cap \mathfrak{m}_2 \cap \ldots \cap \mathfrak{m}_n \supset \mathfrak{m}_1 \cdot \mathfrak{m}_2 \cdot \ldots \cdot \mathfrak{m}_n \neq 0$. In particular a discrete valuation ring, or any local ring with at least two prime ideals is not a Jacobson ring.

Lemma 10.32.9. *Let $R \to S$ be a ring map. Let $\mathfrak{m} \subset R$ be a maximal ideal. Let $\mathfrak{q} \subset S$ be a prime ideal lying over $\mathfrak{m}$ such that $\kappa(\mathfrak{m}) \subset \kappa(\mathfrak{q})$ is an algebraic field extension. Then $\mathfrak{q}$ is a maximal ideal of S.*

Proof. Consider the diagram

$$\begin{array}{ccccc} S & \longrightarrow & S/\mathfrak{q} & \longrightarrow & \kappa(\mathfrak{q}) \\ \uparrow & & \uparrow & & \\ R & \longrightarrow & R/\mathfrak{m} & & \end{array}$$

We see that $\kappa(\mathfrak{m}) \subset S/\mathfrak{q} \subset \kappa(\mathfrak{q})$. Because the field extension $\kappa(\mathfrak{m}) \subset \kappa(\mathfrak{q})$ is algebraic, any ring between $\kappa(\mathfrak{m})$ and $\kappa(\mathfrak{q})$ is a field (by elementary field theory). Thus $S/\mathfrak{q}$ is a field, and a posteriori equal to $\kappa(\mathfrak{q})$. □

Lemma 10.32.10. *Suppose that k is a field and suppose that V is a nonzero vector space over k. Assume the dimension of V (which is a cardinal number) is smaller than the cardinality of k. Then for any linear operator $T : V \to V$ there exists some monic polynomial $P(t) \in k[t]$ such that $P(T)$ is not invertible.*

Proof. If not then V inherits the structure of a vector space over the field $k(t)$. But the dimension of $k(t)$ over k is at least the cardinality of k for example due to the fact that the elements $\frac{1}{t-\lambda}$ are k-linearly independent. □

Here is another version of Hilbert's Nullstellensatz.

Theorem 10.32.11. *Let k be a field. Let S be a k-algebra generated over k by the elements $\{x_i\}_{i\in I}$. Assume the cardinality of I is smaller than the cardinality of k. Then*

1. *for all maximal ideals $\mathfrak{m} \subset S$ the field extension $k \subset \kappa(\mathfrak{m})$ is algebraic, and*
2. *S is a Jacobson ring.*

Proof. If I is finite then the result follows from the Hilbert Nullstellensatz, Theorem 10.31.1. In the rest of the proof we assume I is infinite. It suffices to prove the result for $\mathfrak{m} \subset k[\{x_i\}_{i\in I}]$ maximal in the polynomial ring on variables x_i, since S is a quotient of this. As I is infinite the set of monomials $x_{i_1}^{e_1} \ldots x_{i_r}^{e_r}$, $i_1, \ldots, i_r \in I$ and $e_1, \ldots, e_r \geq 0$ has cardinality at most equal to the cardinality of I. Because the cardinality of $I \times \ldots \times I$ is the cardinality of I, and also the cardinality of $\bigcup_{n\geq 0} I^n$ has the same cardinality. (If I is finite, then this is not true and in that case this proof only works if k is uncountable.)

To arrive at a contradiction pick $T \in \kappa(\mathfrak{m})$ transcendental over k. Note that the k-linear map $T : \kappa(\mathfrak{m}) \to \kappa(\mathfrak{m})$ given by multiplication by T has the property that $P(T)$ is invertible for all monic polynomials $P(t) \in k[t]$. Also, $\kappa(\mathfrak{m})$ has dimension at most the cardinality of I over k since it is a quotient of the vector space $k[\{x_i\}_{i\in I}]$ over k (whose dimension is $\#I$ as we saw above). This is impossible by Lemma 10.32.10.

To show that S is Jacobson we argue as follows. If not then there exists a prime $\mathfrak{q} \subset S$ and an element $f \in S$, $f \notin \mathfrak{q}$ such that $\mathfrak{q}$ is not maximal and $(S/\mathfrak{q})_f$ is a field, see Lemma 10.32.5. But note that $(S/\mathfrak{q})_f$ is generated by at most $\#I + 1$ elements. Hence the field extension $k \subset (R/\mathfrak{q})_f$ is algebraic (by the first part of the proof). This implies that $\kappa(\mathfrak{q})$ is an algebraic extension of k hence $\mathfrak{q}$ is maximal by Lemma 10.32.9. This contradiction finishes the proof. □

Lemma 10.32.12. *Let k be a field. Let S be a k-algebra. For any field extension $k \subset K$ whose cardinality is larger than the cardinality of S we have*

1. *for every maximal ideal $\mathfrak{m}$ of S_K the field $\kappa(\mathfrak{m})$ is algebraic over K, and*
2. *S_K is a Jacobson ring.*

Proof. Choose $k \subset K$ such that the cardinality of K is greater than the cardinality of S. Since the elements of S generate the K-algebra S_K we see that Theorem 10.32.11 applies. □

Example 10.32.13. The trick in the proof of Theorem 10.32.11 really does not work if k is a countable field and I is countable too. Let k be a countable field. Let x be a variable, and let $k(x)$ be the field of rational functions in x. Consider the polynomial algebra $R = k[x, \{x_f\}_{f\in k[x]-\{0\}}]$. Let $I = (\{fx_f - 1\}_{f\in k[x]-\{0\}})$. Note that I is a proper ideal in R. Choose a maximal ideal $I \subset \mathfrak{m}$. Then $k \subset R/\mathfrak{m}$ is isomorphic to $k(x)$, and is not algebraic over k.

Lemma 10.32.14. *Let R be a Jacobson ring. Let $f \in R$. The ring R_f is Jacobson and maximal ideals of R_f correspond to maximal ideals of R.*

Proof. By Topology, Lemma 5.17.5 we see that $D(f) = \mathrm{Spec}(R_f)$ is Jacobson and that closed points of $D(f)$ correspond to closed points in $\mathrm{Spec}(R)$ which happen to lie in $D(f)$. Thus we win by Lemma 10.32.4. □

Example 10.32.15. Here is a simple example that shows Lemma 10.32.14 to be false if R is not Jacobson. Consider the ring $R = \mathbf{Z}_{(2)}$, i.e., the localization of $\mathbf{Z}$ at the prime (2). The localization of R at the element 2 is isomorphic to $\mathbf{Q}$, in a formula: $R_2 \cong \mathbf{Q}$. Clearly the map $R \to R_2$ maps the closed point of $\mathrm{Spec}(\mathbf{Q})$ to the generic point of $\mathrm{Spec}(R)$.

Example 10.32.16. Here is a simple example that shows Lemma 10.32.14 is false if R is Jacobson but we localize at infinitely many elements. Namely, let $R = \mathbf{Z}$ and consider the localization $(R \setminus \{0\})^{-1}R \cong \mathbf{Q}$ of R at the set of all nonzero elements. Clearly the map $\mathbf{Z} \to \mathbf{Q}$ maps the closed point of $\mathrm{Spec}(\mathbf{Q})$ to the generic point of $\mathrm{Spec}(\mathbf{Z})$.

Lemma 10.32.17. *Let R be a Jacobson ring. Let $I \subset R$ be an ideal. The ring R/I is Jacobson and maximal ideals of R/I correspond to maximal ideals of R.*

Proof. The proof is the same as the proof of Lemma 10.32.14. □

Proposition 10.32.18. *Let R be a Jacobson ring. Let $R \to S$ be a ring map of finite type. Then*

1. *The ring S is Jacobson.*
2. *The map $\mathrm{Spec}(S) \to \mathrm{Spec}(R)$ transforms closed points to closed points.*
3. *For $\mathfrak{m}' \subset S$ maximal lying over $\mathfrak{m} \subset R$ the field extension $\kappa(\mathfrak{m}) \subset \kappa(\mathfrak{m}')$ is finite.*

Proof. Let $A \to B \to C$ be finite type ring maps. Suppose $\mathrm{Spec}(C) \to \mathrm{Spec}(B)$ and $\mathrm{Spec}(B) \to \mathrm{Spec}(A)$ map closed points to closed points, and induce finite residue field extensions on residue fields at closed points. Then so does $\mathrm{Spec}(C) \to \mathrm{Spec}(A)$. Thus it is clear that if we factor $R \to S$ as $R \to S' \to S$ for some finite type R-algebra S', then it suffices to prove the lemma for $R \to S'$ and then $S' \to S$. Writing $S = R[x_1, \ldots, x_n]/I$ we see that it suffices to prove the lemma in the cases $S = R[x]$ and $S = R/I$. The case $S = R/I$ is Lemma 10.32.17.

The case $S = R[x]$. Take an irreducible closed subset $Z \subset \mathrm{Spec}(R[x])$. In other words $Z = V(\mathfrak{q})$ for some prime $\mathfrak{q} \subset R[x]$. Set $\mathfrak{p} = \mathfrak{q} \cap R$. Let $U \subset \mathrm{Spec}(R[x])$ be open such that $U \cap Z \neq \emptyset$. We have to find a closed point in $U \cap Z$. In fact, we will find

(∗) a closed point y of $U \cap Z$ which maps to a closed point x of $\mathrm{Spec}(R)$ such that additionally $\kappa(x) \subset \kappa(y)$ is finite.

To do this we may assume $U = D(f)$ for some $f \in R[x]$. In this case $U \cap V(\mathfrak{q}) \neq \emptyset$ means $f \notin \mathfrak{q}$. Consider the diagram

$$\begin{array}{ccc} R[x] & \longrightarrow & R/\mathfrak{p}[x] \\ \uparrow & & \uparrow \\ R & \longrightarrow & R/\mathfrak{p} \end{array}$$

It suffices to solve the problem on the right hand side of this diagram. Thus we see we may assume R is Jacobson, a domain and $\mathfrak{p} = (0)$.

In case $\mathfrak{q} = (0)$, write $f = a_d x^d + \ldots + a_0$. We see that not all a_i are zero. Take any maximal ideal $\mathfrak{m}$ of R such that $a_i \notin \mathfrak{m}$ for some i (here we use R is Jacobson). Next, choose a maximal ideal $\overline{\mathfrak{m}}' \subset (R/\mathfrak{m})[x]$ not containing the image of f (possible because $\kappa(\mathfrak{m})[x]$ is Jacobson). Then the inverse image $\mathfrak{m}' \subset R[x]$ defines a closed

point of $U \cap Z$ and maps to $\mathfrak{m}$. Also, by construction $\kappa(\mathfrak{m}) \subset \kappa(\mathfrak{m}')$ is finite. Thus we have shown $(*)$ in this case.

In case $\mathfrak{q} \neq (0)$, let K be the fraction field of R. Write $\mathfrak{q}K[x] = (g)$ for some irreducible $g \in K[x]$. Clearing denominators, we may assume that $g \in R[x]$, and hence in $\mathfrak{q}$. Write $g = b_e x^e + \ldots + b_0$, $b_i \in R$ with $b_e \neq 0$. The maps $R \to R_{b_e}$ and $R[x] \to R[x]_{b_e}$ satisfies the conclusion of the lemma, by Lemma 10.32.14 and moreover induce isomorphisms on residue fields. Hence, in order to prove $(*)$, we may replace R by R_{b_e} and assume that g is monic. In this case we see that $R[x]/\mathfrak{q}$ is a quotient of the finite free R-module $R[x]/(g) = R \oplus Rx \oplus \ldots \oplus Rx^{e-1}$. But on the other hand we have $R[x]/(g) \subset K[x]/(g) = K[x]/\mathfrak{q}K[x]$. Hence $\mathfrak{q} = (g)$, and $Z = V(\mathfrak{q}) = V(g)$. At this point, by Lemma 10.27.8 the image of $D(f) \cap V(g)$ in $\mathrm{Spec}(R)$ is $D(r_1) \cup \ldots \cup D(r_d)$ for some $r_i \in R$ (of course it is nonempty). Take any maximal ideal $\mathfrak{m} \subset R$ in this image (possible because R is Jacobson) and take any prime $\mathfrak{m}' \subset R[x]$ corresponding to a point of $D(f) \cap V(g)$ lying over $\mathfrak{m}$. Note that the residue field extension $\kappa(\mathfrak{m}) \subset \kappa(\mathfrak{m}')$ is finite (because $g \in \mathfrak{m}'$). By Lemma 10.32.9 we see that $\mathfrak{m}'$ is a closed point. This proves $(*)$ in this case.

At this point we are done. Namely, $(*)$ implies that $\mathrm{Spec}(R[x])$ is Jacobson (via Lemma 10.32.4). Also, if Z is a singleton closed set, then $(*)$ implies that $Z = \{\mathfrak{m}'\}$ with $\mathfrak{m}'$ lying over a maximal ideal $\mathfrak{m} \subset R$ such that $\kappa(\mathfrak{m}) \subset \kappa(\mathfrak{m}')$ is finite. □

Lemma 10.32.19. *Any finite type algebra over* $\mathbf{Z}$ *is Jacobson.*

Proof. Combine Lemma 10.32.6 and Proposition 10.32.18. □

Lemma 10.32.20. *Let $R \to S$ be a finite type ring map of Jacobson rings. Denote $X = \mathrm{Spec}(R)$ and $Y = \mathrm{Spec}(S)$. Write $f : Y \to X$ the induced map of spectra. Let $E \subset Y = \mathrm{Spec}(S)$ be a constructible set. Denote with a subscript $_0$ the set of closed points of a topological space.*

1. *We have $f(E)_0 = f(E_0) = X_0 \cap f(E)$.*
2. *A point $\xi \in X$ is in $f(E)$ if and only if $\overline{\{\xi\}} \cap f(E_0)$ is dense in $\overline{\{\xi\}}$.*

Proof. We have a commutative diagram of continuous maps

$$\begin{array}{ccc} E & \longrightarrow & Y \\ \downarrow & & \downarrow \\ f(E) & \longrightarrow & X \end{array}$$

Suppose $x \in f(E)$ is closed in $f(E)$. Then $f^{-1}(\{x\}) \cap E$ is closed in E. Hence $f^{-1}(\{x\}) \cap E$ is constructible, nonempty in Y. By Topology, Lemma 5.17.5, the intersection $Y_0 \cap f^{-1}(\{x\}) \cap E$ is not empty. Thus there exists $y \in Y_0$ mapping to x. Since clearly $y \in E_0$ we see that $x \in f(E_0)$. This proves that $f(E)_0 \subset f(E_0)$. Proposition 10.32.18 implies that $f(E_0) \subset X_0 \cap f(E)$. The inclusion $X_0 \cap f(E) \subset f(E)_0$ is trivial. This proves the first assertion.

Suppose that $\xi \in f(E)$. According to Lemma 10.28.2 the set $f(E) \cap \overline{\{\xi\}}$ contains a dense open subset of $\overline{\{\xi\}}$. Since X is Jacobson we conclude that $f(E) \cap \overline{\{\xi\}}$ contains a dense set of closed points, see Topology, Lemma 5.17.5. We conclude by part (1) of the lemma.

On the other hand, suppose that $\overline{\{\xi\}} \cap f(E_0)$ is dense in $\overline{\{\xi\}}$. By Lemma 10.27.3 there exists a ring map $S \to S'$ of finite presentation such that E is the image of

$Y' := \mathrm{Spec}(S') \to Y$. Then E_0 is the image of Y'_0 by the first part of the lemma applied to the ring map $S \to S'$. Thus we may assume that $E = Y$ by replacing S by S'. Suppose ξ corresponds to $\mathfrak{p} \subset R$. Consider the diagram

$$\begin{array}{ccc} S & \longrightarrow & S/\mathfrak{p}S \\ \uparrow & & \uparrow \\ R & \longrightarrow & R/\mathfrak{p} \end{array}$$

This diagram and the density of $f(Y_0) \cap V(\mathfrak{p})$ in $V(\mathfrak{p})$ shows that the morphism $R/\mathfrak{p} \to S/\mathfrak{p}S$ satisfies condition (2) of Lemma 10.28.4. Hence we conclude there exists a prime $\overline{\mathfrak{q}} \subset S/\mathfrak{p}S$ mapping to (0). In other words the inverse image $\mathfrak{q}$ of $\overline{\mathfrak{q}}$ in S maps to $\mathfrak{p}$ as desired. □

The conclusion of the lemma above is that we can read off the image of f from the set of closed points of the image. This is a little nicer in case the map is of finite presentation because then we know that images of constructibles are constructible. Before we state it we introduce some notation. Denote $\mathrm{Constr}(X)$ the set of constructible Let $R \to S$ be a ring map. Denote $X = \mathrm{Spec}(R)$ and $Y = \mathrm{Spec}(S)$. Write $f : Y \to X$ the induced map of spectra. Denote with a subscript $_0$ the set of closed points of a topological space.

Lemma 10.32.21. *With notation as above. Assume that R is a Noetherian Jacobson ring. Further assume $R \to S$ is of finite type. There is a commutative diagram*

$$\begin{array}{ccc} \mathit{Constr}(Y) & \xrightarrow{E \mapsto E_0} & \mathit{Constr}(Y_0) \\ \downarrow{\scriptstyle E \mapsto f(E)} & & \downarrow{\scriptstyle E \mapsto f(E)} \\ \mathit{Constr}(X) & \xrightarrow{E \mapsto E_0} & \mathit{Constr}(X_0) \end{array}$$

where the horizontal arrows are the bijections from Topology, Lemma 5.17.7.

Proof. Since $R \to S$ is of finite type, it is of finite presentation, see Lemma 10.29.4. Thus the image of a constructible set in X is constructible in Y by Chevalley's theorem (Theorem 10.27.9). Combined with Lemma 10.32.20 the lemma follows. □

To illustrate the use of Jacobson rings, we give the following two examples.

Example 10.32.22. Let k be a field. The space $\mathrm{Spec}(k[x,y]/(xy))$ has two irreducible components: namely the x-axis and the y-axis. As a generalization, let

$$R = k[x_{11}, x_{12}, x_{21}, x_{22}, y_{11}, y_{12}, y_{21}, y_{22}]/\mathfrak{a},$$

where $\mathfrak{a}$ is the ideal in $k[x_{11}, x_{12}, x_{21}, x_{22}, y_{11}, y_{12}, y_{21}, y_{22}]$ generated by the entries of the 2×2 product matrix

$$\begin{pmatrix} x_{11} & x_{12} \\ x_{21} & x_{22} \end{pmatrix} \begin{pmatrix} y_{11} & y_{12} \\ y_{21} & y_{22} \end{pmatrix}.$$

In this example we will describe $\mathrm{Spec}(R)$.

To prove the statement about $\mathrm{Spec}(k[x,y]/(xy))$ we argue as follows. If $\mathfrak{p} \subset k[x,y]$ is any ideal containing xy, then either x or y would be contained in $\mathfrak{p}$. Hence the minimal such prime ideals are just (x) and (y). In case k is algebraically closed,

the max-Spec of these components can then be visualized as the point sets of y- and x-axis.

For the generalization, note that we may identify the closed points of the spectrum of $k[x_{11}, x_{12}, x_{21}, x_{22}, y_{11}, y_{12}, y_{21}, y_{22}])$ with the space of matrices

$$\left\{(X,Y) \in \mathrm{Mat}(2,k) \times \mathrm{Mat}(2,k) \mid X = \begin{pmatrix} x_{11} & x_{12} \\ x_{21} & x_{22} \end{pmatrix}, Y = \begin{pmatrix} y_{11} & y_{12} \\ y_{21} & y_{22} \end{pmatrix}\right\}$$

at least if k is algebraically closed. Now define a group action of $\mathrm{GL}(2,k) \times \mathrm{GL}(2,k) \times \mathrm{GL}(2,k)$ on the space of matrices $\{(X,Y)\}$ by

$$(g_1, g_2, g_3) \times (X,Y) \mapsto ((g_1 X g_2^{-1}, g_2 Y g_3^{-1})).$$

Here, also observe that the algebraic set

$$\mathrm{GL}(2,k) \times \mathrm{GL}(2,k) \times \mathrm{GL}(2,k) \subset \mathrm{Mat}(2,k) \times \mathrm{Mat}(2,k) \times \mathrm{Mat}(2,k)$$

is irreducible since it is the max spectrum of the domain

$$k[x_{11}, x_{12}, \ldots, z_{21}, z_{22}, (x_{11}x_{22} - x_{12}x_{21})^{-1}, (y_{11}y_{22} - y_{12}y_{21})^{-1}, (z_{11}z_{22} - z_{12}z_{21})^{-1}].$$

Since the image of irreducible an algebraic set is still irreducible, it suffices to classify the orbits of the set $\{(X,Y) \in \mathrm{Mat}(2,k) \times \mathrm{Mat}(2,k) | XY = 0\}$ and take their closures. From standard linear algebra, we are reduced to the following three cases:

(1) $\exists (g_1, g_2)$ such that $g_1 X g_2^{-1} = I_{2\times 2}$. Then Y is necessarily 0, which as an algebraic set is invariant under the group action. It follows that this orbit is contained in the irreducible algebraic set defined by the prime ideal $(y_{11}, y_{12}, y_{21}, y_{22})$. Taking the closure, we see that $(y_{11}, y_{12}, y_{21}, y_{22})$ is actually a component.

(2) $\exists (g_1, g_2)$ such that

$$g_1 X g_2^{-1} = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}.$$

This case occurs if and only if X is a rank 1 matrix, and furthermore, Y is killed by such an X if and only if

$$x_{11}y_{11} + x_{12}y_{21} = 0; \quad x_{11}y_{12} + x_{12}y_{22} = 0;$$
$$x_{21}y_{11} + x_{22}y_{21} = 0; \quad x_{21}y_{12} + x_{22}y_{22} = 0.$$

Fix a rank 1 X, such non zero Y's satisfying the above equations form an irreducible algebraic set for the following reason($Y = 0$ is contained the previous case): $0 = g_1 X g_2^{-1} g_2 Y$ implies that

$$g_2 Y = \begin{pmatrix} 0 & 0 \\ y'_{21} & y'_{22} \end{pmatrix}.$$

With a further $\mathrm{GL}(2,k)$-action on the right by g_3, $g_2 Y$ can be brought into

$$g_2 Y g_3^{-1} = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix},$$

and thus such Y's form an irreducible algebraic set isomorphic to the image of $\mathrm{GL}(2,k)$ under this action. Finally, notice that the "rank 1" condition for X's forms an open dense subset of the irreducible algebraic set $\det X = x_{11}x_{22} - x_{12}x_{21} = 0$. It now follows that all the five equations define an irreducible component $(x_{11}y_{11} + x_{12}y_{21}, x_{11}y_{12} + x_{12}y_{22}, x_{21}y_{11} +$

$x_{22}y_{21}, x_{21}y_{12} + x_{22}y_{22}, x_{11}x_{22} - x_{12}x_{21}$) in the open subset of the space of pairs of nonzero matrices. It can be shown that the pair of equations $\det X = 0$, $\det Y = 0$ cuts $\mathrm{Spec}(R)$ in an irreducible component with the above locus an open dense subset.

(3) $\exists(g_1, g_2)$ such that $g_1 X g_2^{-1} = 0$, or equivalently, $X = 0$. Then Y can be arbitrary and this component is thus defined by $(x_{11}, x_{12}, x_{21}, x_{22})$.

Example 10.32.23. For another example, consider $R = k[\{t_{ij}\}_{i,j=1}^n]/\mathfrak{a}$, where $\mathfrak{a}$ is the ideal generated by the entries of the product matrix $T^2 - T$, $T = (t_{ij})$. From linear algebra, we know that under the $GL(n,k)$-action defined by $g, T \mapsto gTg^{-1}$, T is classified by the its rank and each T is conjugate to some $\mathrm{diag}(1, \ldots, 1, 0, \ldots, 0)$, which has r 1's and $n-r$ 0's. Thus each orbit of such a $\mathrm{diag}(1, \ldots, 1, 0, \ldots, 0)$ under the group action forms an irreducible component and every idempotent matrix is contained in one such orbit. Next we will show that any two different orbits are necessarily disjoint. For this purpose we only need to cook up polynomial functions that take different values on different orbits. In characteristic 0 cases, such a function can be taken to be $f(t_{ij}) = trace(T) = \sum_{i=1}^n t_{ii}$. In positive characteristic cases, things are slightly more tricky since we might have $trace(T) = 0$ even if $T \neq 0$. For instance, $char = 3$

$$trace \begin{pmatrix} 1 & & \\ & 1 & \\ & & 1 \end{pmatrix} = 3 = 0$$

Anyway, these components can be separated using other functions. For instance, in the characteristic 3 case, $tr(\wedge^3 T)$ takes value 1 on the components corresponding to $diag(1,1,1)$ and 0 on other components.

10.33. Finite and integral ring extensions

Trivial lemmas concerning finite and integral ring maps. We recall the definition.

Definition 10.33.1. Let $\varphi : R \to S$ be a ring map.

(1) An element $s \in S$ is *integral over* R if there exists a monic polynomial $P(x) \in R[x]$ such that $P^\varphi(s) = 0$, where $P^\varphi(x) \in S[x]$ is the image of P under $\varphi : R[x] \to S[x]$.

(2) The ring map φ is *integral* if every $s \in S$ is integral over R.

Lemma 10.33.2. *Let $\varphi : R \to S$ be a ring map. Let $y \in S$. If there exists a finite R-submodule M of S such that $1 \in M$ and $yM \subset M$, then y is integral over R.*

Proof. Let $x_1 = 1 \in M$ and $x_i \in M$, $i = 2, \ldots, n$ be a finite set of elements generating M as an R-module. Write $yx_i = \sum \varphi(a_{ij})x_j$ for some $a_{ij} \in R$. Let $P(T) \in R[T]$ be the characteristic polynomial of the $n \times n$ matrix $A = (a_{ij})$. By Lemma 10.15.1 we see $P(A) = 0$. By construction the map $\pi : R^n \to M$, $(a_1, \ldots, a_n) \mapsto \sum \varphi(a_i)x_i$ commutes with $A : R^n \to R^n$ and multiplication by y. In a formula $\pi(Av) = y\pi(v)$. Thus $P(y) = P(y)\cdot 1 = P(y)\cdot x_1 = P(y)\cdot\pi((1,0,\ldots,0)) = \pi(P(A)(1,0,\ldots,0)) = 0$. □

Lemma 10.33.3. *A finite ring extension is integral.*

Proof. Let $R \to S$ be finite. Let $y \in S$. Apply Lemma 10.33.2 to $M = S$ to see that y is integral over R. □

Lemma 10.33.4. *Let $\varphi : R \to S$ be a ring map. Let $s_1, \ldots, s_n$ be a finite set of elements of S. In this case s_i is integral over R for all $i = 1, \ldots, n$ if and only if there exists an R-subalgebra $S' \subset S$ finite over R containing all of the s_i.*

Proof. If each s_i is integral, then the subalgebra generated by $\varphi(R)$ and the s_i is finite over R. Namely, if s_i satisfies a monic equation of degree d_i over R, then this subalgebra is generated as an R-module by the elements $s_1^{e_1} \ldots s_n^{e_n}$ with $0 \leq e_i \leq d_i - 1$. Conversely, suppose given a finite R-subalgebra S' containing all the s_i. Then all of the s_i are integral by Lemma 10.33.3. □

Lemma 10.33.5. *Let $R \to S$ be a ring map. The following are equivalent*

1. *$R \to S$ is finite,*
2. *$R \to S$ is integral and of finite type, and*
3. *there exist $x_1, \ldots, x_n \in S$ which generate S as an algebra over R such that each x_i is integral over R.*

Proof. Clear from Lemma 10.33.4. □

Lemma 10.33.6. *Suppose that $R \to S$ and $S \to T$ are integral ring maps. Then $R \to T$ is integral.*

Proof. Let $t \in T$. Let $P(x) \in S[x]$ be a monic polynomial such that $P(t) = 0$. Apply Lemma 10.33.4 to the finite set of coefficients of P. Hence t is integral over some subalgebra $S' \subset S$ finite over R. Apply Lemma 10.33.4 again to find a subalgebra $T' \subset T$ finite over S' and containing t. Lemma 10.7.3 applied to $R \to S' \to T'$ shows that T' is finite over R. The integrality of t over R now follows from Lemma 10.33.3. □

Lemma 10.33.7. *Let $R \to S$ be a ring homomorphism. The set*

$$S' = \{s \in S \mid s \text{ is integral over } R\}$$

is an R-subalgebra of S.

Proof. This is clear from Lemmas 10.33.4 and 10.33.3. □

Definition 10.33.8. Let $R \to S$ be a ring map. The ring $S' \subset S$ of elements integral over R, see Lemma 10.33.7, is called the *integral closure* of R in S. If $R \subset S$ we say that R is *integrally closed* in S if $R = S'$.

In particular, we see that $R \to S$ is integral if and only if the integral closure of R in S is all of S.

Lemma 10.33.9. *Integral closure commutes with localization: If $A \to B$ is a ring map, and $S \subset A$ is a multiplicative subset, then the integral closure of $S^{-1}A$ in $S^{-1}B$ is $S^{-1}B'$, where $B' \subset B$ is the integral closure of A in B.*

Proof. Since localization is exact we see that $S^{-1}B' \subset S^{-1}B$. Suppose $x \in B'$ and $f \in S$. Then $x^d + \sum_{i=1,\ldots,d} a_i x^{d-i} = 0$ in B for some $a_i \in A$. Hence also

$$(x/f)^d + \sum\nolimits_{i=1,\ldots,d} a_i/f^i (x/f)^{d-i} = 0$$

in $S^{-1}B$. In this way we see that $S^{-1}B'$ is contained in the integral closure of $S^{-1}A$ in $S^{-1}B$. Conversely, suppose that $x/f \in S^{-1}B$ is integral over $S^{-1}A$. Then we have

$$(x/f)^d + \sum\nolimits_{i=1,\ldots,d} (a_i/f_i)(x/f)^{d-i} = 0$$

in $S^{-1}B$ for some $a_i \in A$ and $f_i \in S$. This means that

$$(f'f_1 \ldots f_d x)^d + \sum\nolimits_{i=1,\ldots,d} f^i (f')^i f_1^i \ldots f_i^{i-1} \ldots f_d^i a_i (f'f_1 \ldots f_d x)^{d-i} = 0$$

for a suitable $f' \in S$. Hence $f'f_1 \ldots f_d x \in B'$ and thus $x/f \in S^{-1}B'$ as desired. □

Lemma 10.33.10. *Let $\varphi : R \to S$ be a ring map. Let $x \in S$. The following are equivalent:*

(1) *x is integral over R, and*
(2) *for every $\mathfrak{p} \in R$ the element $x \in S_\mathfrak{p}$ is integral over $R_\mathfrak{p}$.*

Proof. It is clear that (1) implies (2). Assume (1). Consider the R-algebra $S' \subset S$ generated by $\varphi(R)$ and x. Let $\mathfrak{p}$ be a prime ideal of R. Then we know that $x^d + \sum_{i=1,\ldots,d} \varphi(a_i) x^{d-i} = 0$ in $S_\mathfrak{p}$ for some $a_i \in R_\mathfrak{p}$. Hence we see, by looking at which denominators occur, that for some $f \in R$, $f \notin \mathfrak{p}$ we have $a_i \in R_f$ and $x^d + \sum_{i=1,\ldots,d} \varphi(a_i) x^{d-i} = 0$ in S_f. This implies that S'_f is finite over R_f. Since $\mathfrak{p}$ was arbitrary and $\mathrm{Spec}(R)$ is quasi-compact (Lemma 10.16.10) we can find finitely many elements $f_1, \ldots, f_n \in R$ which generate the unit ideal of R such that S'_{f_i} is finite over R. Hence we conclude from Lemma 10.22.2 that S' is finite over R. Hence x is integral over R by Lemma 10.33.4. □

Lemma 10.33.11. *Let $R \to S$ and $R \to R'$ be ring maps. Set $S' = R' \otimes_R S$.*

(1) *If $R \to S$ is integral so is $R' \to S'$.*
(2) *If $R \to S$ is finite so is $R' \to S'$.*

Proof. We prove (1). Let $s_i \in S$ be generators for S over R. Each of these satisfies a monic polynomial equation P_i over R. Hence the elements $1 \otimes s_i \in S'$ generate S' over R' and satisfy the corresponding polynomial P_i' over R'. Since these elements generate S' over R' we see that S' is integral over R'. Proof of (2) omitted. □

Lemma 10.33.12. *Let $R \to S$ be a ring map. Let $f_1, \ldots, f_n \in R$ generate the unit ideal.*

(1) *If each $R_{f_i} \to S_{f_i}$ is integral, so is $R \to S$.*
(2) *If each $R_{f_i} \to S_{f_i}$ is finite, so is $R \to S$.*

Proof. Proof of (1). Let $s \in S$. Consider the ideal $I \subset R[x]$ of polynomials P such that $P(s) = 0$. Let $J \subset R$ denote the ideal (!) of leading coefficients of elements of I. By assumption and clearing denominators we see that $f_i^{n_i} \in J$ for all i and certain $n_i \geq 0$. Hence J contains 1 and we see s is integral over R. Proof of (2) omitted. □

Lemma 10.33.13. *Let $A \to B \to C$ be ring maps.*

(1) *If $A \to C$ is integral so is $B \to C$.*
(2) *If $A \to C$ is finite so is $B \to C$.*

Proof. Omitted. □

Lemma 10.33.14. *Let $A \to B \to C$ be ring maps. Let B' be the integral closure of A in B, let C' be the integral closure of B' in C. Then C' is the integral closure of A in C.*

Proof. Omitted. □

Lemma 10.33.15. *Suppose that $R \to S$ is an integral ring extension with $R \subset S$. Then $\varphi : \mathrm{Spec}(S) \to \mathrm{Spec}(R)$ is surjective.*

Proof. Let $\mathfrak{p} \subset R$ be a prime ideal. We have to show $\mathfrak{p}S_{\mathfrak{p}} \neq S_{\mathfrak{p}}$, see Lemma 10.16.9. The localization $R_{\mathfrak{p}} \to S_{\mathfrak{p}}$ is injective (as localization is exact) and integral by Lemma 10.33.9 or 10.33.11. Hence we may replace R, S by $R_{\mathfrak{p}}$, $S_{\mathfrak{p}}$ and we may assume R is local with maximal ideal $\mathfrak{m}$ and it suffices to show that $\mathfrak{m}S \neq S$. Suppose $1 = \sum f_i s_i$ with $f_i \in \mathfrak{m}$ and $s_i \in S$ in order to get a contradiction. Let $R \subset S' \subset S$ be such that $R \to S'$ is finite and $s_i \in S'$, see Lemma 10.33.4. The equation $1 = \sum f_i s_i$ implies that the finite R-module S' satisfies $S' = \mathfrak{m}S'$. Hence by Nakayama's Lemma 10.18.1 we see $S' = 0$. Contradiction. □

Lemma 10.33.16. *Let R be a ring. Let K be a field. If $R \subset K$ and K is integral over R, then R is a field and K is an algebraic extension. If $R \subset K$ and K is finite over R, then R is a field and K is a finite algebraic extension.*

Proof. Assume that $R \subset K$ is integral. By Lemma 10.33.15 we see that $\mathrm{Spec}(R)$ has 1 point. Since clearly R is a domain we see that $R = R_{(0)}$ is a field. The other assertions are immediate from this. □

Lemma 10.33.17. *Let k be a field. Let S be a k-algebra over k.*

1. *If S is a domain and finite dimensional over k, then S is a field.*
2. *If S is integral over k and a domain, then S is a field.*
3. *If S is integral over k then every prime of S is a maximal ideal (see Lemma 10.24.6 for more consequences).*

Proof. The statement on primes follows from the statement "integral + domain $\Rightarrow$ field". Let S integral over k and assume S is a domain, Take $s \in S$. By Lemma 10.33.4 we may find a finite dimensional k-subalgebra $k \subset S' \subset S$. containing s. Hence S is a field if we can prove the first statement. Assume S finite dimensional over k and a domain. Pick $s \in S$. Since S is a domain the multiplication map $s : S \to S$ is surjective by dimension reasons. Hence there exists an element $s_1 \in S$ such that $ss_1 = 1$. So S is a field. □

Lemma 10.33.18. *Suppose $R \to S$ is integral. Let $\mathfrak{q}, \mathfrak{q}' \in \mathrm{Spec}(S)$ be distinct primes having the same image in $\mathrm{Spec}(R)$. Then neither $\mathfrak{q} \subset \mathfrak{q}'$ nor $\mathfrak{q}' \subset \mathfrak{q}$.*

Proof. Let $\mathfrak{p} \subset R$ be the image. By Remark 10.16.8 the primes $\mathfrak{q}, \mathfrak{q}'$ correspond to ideals in $S \otimes_R \kappa(\mathfrak{p})$. Thus the lemma follows from Lemma 10.33.17. □

Lemma 10.33.19. *Suppose $R \to S$ is finite. Then the fibres of $\mathrm{Spec}(S) \to \mathrm{Spec}(R)$ are finite.*

Proof. By the discussion in Remark 10.16.8 the fibres are the spectra of the rings $S \otimes_R \kappa(\mathfrak{p})$. As $R \to S$ is finite, these fibre rings are finite over $\kappa(\mathfrak{p})$ hence Noetherian by Lemma 10.29.1. By Lemma 10.33.18 every prime of $S \otimes_R \kappa(\mathfrak{p})$ is a minimal prime. Hence by Lemma 10.29.6 there are at most finitely many. □

Lemma 10.33.20. *Let $R \to S$ be a ring map such that S is integral over R. Let $\mathfrak{p} \subset \mathfrak{p}' \subset R$ be primes. Let $\mathfrak{q}$ be a prime of S mapping to $\mathfrak{p}$. Then there exists a prime $\mathfrak{q}'$ with $\mathfrak{q} \subset \mathfrak{q}'$ mapping to $\mathfrak{p}'$.*

Proof. We may replace R by $R/\mathfrak{p}$ and S by $S/\mathfrak{q}$. This reduces us to the situation of having an integral extension of domains $R \subset S$ and a prime $\mathfrak{p}' \subset R$. By Lemma 10.33.15 we win. □

The property expressed in the lemma above is called the "going up property" for the ring map $R \to S$, see Definition 10.38.1.

Lemma 10.33.21. *Let R be a ring. Let $x, y \in R$ be nonzerodivisors. Let $R[x/y] \subset R_{xy}$ be the R-subalgebra generated by x/y, and similarly for the subalgebras $R[y/x]$ and $R[x/y, y/x]$. If R is integrally closed in R_x or R_y, then the sequence*

$$0 \to R \xrightarrow{(-1,1)} R[x/y] \oplus R[y/x] \xrightarrow{(1,1)} R[x/y, y/x] \to 0$$

is a short exact sequence of R-modules.

Proof. Since $x/y \cdot y/x = 1$ it is clear that the map $R[x/y] \oplus R[y/x] \to R[x/y, y/x]$ is surjective. Let $\alpha \in R[x/y] \cap R[y/x]$. To show exactness in the middle we have to prove that $\alpha \in R$. By assumption we may write

$$\alpha = a_0 + a_1 x/y + \ldots + a_n (x/y)^n = b_0 + b_1 y/x + \ldots + b_m (y/x)^m$$

for some $n, m \geq 0$ and $a_i, b_j \in R$. Pick some $N > \max(n, m)$. Consider the finite R-submodule M of R_{xy} generated by the elements

$$(x/y)^N, (x/y)^{N-1}, \ldots, x/y, 1, y/x, \ldots, (y/x)^{N-1}, (y/x)^N$$

We claim that $\alpha M \subset M$. Namely, it is clear that $(x/y)^i(b_0 + b_1 y/x + \ldots + b_m (y/x)^m) \in M$ for $0 \leq i \leq N$ and that $(y/x)^i(a_0 + a_1 x/y + \ldots + a_n (x/y)^n) \in M$ for $0 \leq i \leq N$. Hence α is integral over R by Lemma 10.33.2. Note that $\alpha \in R_x$, so if R is integrally closed in R_x then $\alpha \in R$ as desired. □

10.34. Normal rings

We first introduce the notion of a normal domain, and then we introduce the (very general) notion of a normal ring.

Definition 10.34.1. A domain R is called *normal* if it is integrally closed in its field of fractions.

Lemma 10.34.2. *Let $R \to S$ be a ring map. If S is a normal domain, then the integral closure of R in S is a normal domain.*

Proof. Omitted. □

The following notion is occasionally useful when studying normality.

Definition 10.34.3. Let R be a domain.

(1) An element g of the fraction field of R is called *almost integral over R* if there exists an element $r \in R$, $r \neq 0$ such that $rg^n \in R$ for all $n \geq 0$.
(2) The domain R is called *completely normal* if every almost integral element of the fraction field of R is contained in R.

The following lemma shows that a Noetherian domain is normal if and only if it is completely normal.

Lemma 10.34.4. *Let R be a domain with fraction field K. If $u, v \in K$ are almost integral over R, then so are $u+v$ and uv. Any element $g \in K$ which is integral over R is almost integral over R. If R is Noetherian then the converse holds as well.*

Proof. If $ru^n \in R$ for all $n \geq 0$ and $v^n r' \in R$ for all $n \geq 0$, then $(uv)^n rr'$ and $(u+v)^n rr'$ are in R for all $n \geq 0$. Hence the first assertion. Suppose $g \in K$ is integral over R. In this case there exists an $d > 0$ such that the ring $R[g]$ is generated by $1, g, \ldots, g^d$ as an R-module. Let $r \in R$ be a common denominator of the elements $1, g, \ldots, g^d \in K$. It is follows that $rR[g] \subset R$, and hence g is almost integral over R.

Suppose R is Noetherian and $g \in K$ is almost integral over R. Let $r \in R$, $r \neq 0$ be as in the definition. Then $R[g] \subset \frac{1}{r}R$ as an R-module. Since R is Noetherian this implies that $R[g]$ is finite over R. Hence g is integral over R, see Lemma 10.33.3. □

Lemma 10.34.5. *Any localization of a normal domain is normal.*

Proof. Let R be a normal domain, and let $S \subset R$ be a multiplicative subset. Suppose g is an element of the fraction field of R which is integral over $S^{-1}R$. Let $P = x^d + \sum_{j<d} a_j x^j$ be a polynomial with $a_i \in S^{-1}R$ such that $P(g) = 0$. Choose $s \in S$ such that $sa_i \in R$ for all i. Then sg satisfies the monic polynomial $x^d + \sum_{j<d} s^{d-j} a_j x^j$ which has coefficients $s^{d-j}a_j$ in R. Hence $sg \in R$ because R is normal. Hence $g \in S^{-1}R$. □

Lemma 10.34.6. *A principal ideal domain is normal.*

Proof. Let R be a principal ideal domain. Let $g = a/b$ be an element of the fraction field of R integral over R. Because R is a principal ideal domain we may divide out a common factor of a and b and assume $(a, b) = R$. In this case, any equation $(a/b)^n + r_{n-1}(a/b)^{n-1} + \ldots + r_0 = 0$ with $r_i \in R$ would imply $a^n \in (b)$. This contradicts $(a, b) = R$ unless b is a unit in R. □

Lemma 10.34.7. *Let R be a domain with fraction field K. Suppose $f = \sum \alpha_i x^i$ is an element of $K[x]$.*

1. *If f is integral over $R[x]$ then all α_i are integral over $R[x]$, and*
2. *If f is almost integral over $R[x]$ then all α_i are almost integral over $R[x]$.*

Proof. We first prove the second statement. Write $f = \alpha_d x^d + \ldots + \alpha_r x^r$ with $\alpha_r \neq 0$. By assumption there exists $h = b_d x^d + \ldots + b_s x^s \in R[x]$, $b_s \neq 0$ such that $f^n h \in R[x]$ for all $n \geq 0$. This implies that $b_s \alpha_r^n \in R$ for all $n \geq 0$. Hence α_r is almost integral over R. Since the set of almost integral elements form a subring we deduce that $f - \alpha_r x^r = \alpha_d x^d + \ldots + \alpha_{r-1} x^{r-1}$ is almost integral over $R[x]$. By induction on $d - r$ we win.

In order to prove the first statement we will use absolute Noetherian reduction. Namely, write $\alpha_i = a_i/b_i$ and let $P(t) = t^d + \sum_{j<d} f_j t^j$ be a polynomial with coefficients $f_j \in R[x]$ such that $P(g) = 0$. Let $f_j = \sum f_{ji} x^i$. Consider the subring $R_0 \subset R$ generated by the finite list of elements a_i, b_i, f_{ji} of R. It is a domain; let K_0 be its field of fractions. Since R_0 is a finite type $\mathbf{Z}$-algebra it is Noetherian, see Lemma 10.29.3. It is still the case that $g \in K_0[x]$ is integral over $R_0[x]$, because all the identities in R among the elements a_i, b_i, f_{ji} also hold in R_0. By Lemma 10.34.4 the element g is almost integral over $R_0[x]$. By the first part of the lemma, the elements α_i are almost integral over R_0. And since R_0 is Noetherian, the are integral over R_0, see Lemma 10.34.4. Of course, then they are integral over R. □

Lemma 10.34.8. *Let R be a normal domain. Then $R[x]$ is a normal domain.*

Proof. The result is true if R is a field K because $K[x]$ is a euclidean domain and hence a principal ideal domain and hence normal by Lemma 10.34.6. Let g be an element of the fraction field of $R[x]$ which is integral over $R[x]$. Because g is integral over $K[x]$ where K is the fraction field of R we may write $g = \alpha_d x^d + \alpha_{d-1}x^{d-1} + \ldots + \alpha_0$ with $\alpha_i \in K$. By Lemma 10.34.7 the elements α_i are integral over R and hence are in R. □

Lemma 10.34.9. *Let R be a domain. The following are equivalent:*

(1) *The domain R is a normal domain,*
(2) *for every prime $\mathfrak{p} \subset R$ the local ring $R_\mathfrak{p}$ is a normal domain, and*
(3) *for every maximal ideal $\mathfrak{m}$ the ring $R_\mathfrak{m}$ is a normal domain.*

Proof. This follows easily from the fact that for any domain R we have

$$R = \bigcap\nolimits_\mathfrak{m} R_\mathfrak{m}$$

inside the fraction field of R. Namely, if g is an element of the right hand side then the ideal $I = \{x \in R \mid xg \in R\}$ is not contained in any maximal ideal $\mathfrak{m}$, whence $I = R$. □

Lemma 10.34.9 shows that the following definition is compatible with Definition 10.34.1. (It is the definition from EGA – see [**DG67**, IV, 5.13.5 and 0, 4.1.4].)

Definition 10.34.10. A ring R is called *normal* if for every prime $\mathfrak{p} \subset R$ the localization $R_\mathfrak{p}$ is a normal domain (see Definition 10.34.1).

Note that a normal ring is a reduced ring, as R is a subring of the product of its localizations at all primes (see for example Lemma 10.22.1).

Lemma 10.34.11. *A normal ring is integrally closed in its total ring of fractions.*

Proof. Let R be a normal ring. Let $x \in Q(R)$ be an element of the total ring of fractions of R integral over R. Set $I = \{f \in R, fx \in R\}$. Let $\mathfrak{p} \subset R$ be a prime. As $R \subset R_\mathfrak{p}$ is flat we see that $R_\mathfrak{p} \subset Q(R) \otimes R_\mathfrak{p}$. As $R_\mathfrak{p}$ is a normal domain we see that $x \otimes 1$ is an element of $R_\mathfrak{p}$. Hence we can find $a, f \in R$, $f \notin \mathfrak{p}$ such that $x \otimes 1 = a \otimes 1/f$. This means that $fx - a$ maps to zero in $Q(R) \otimes_R R_\mathfrak{p} = Q(R)_\mathfrak{p}$, which in turn means that there exists an $f' \in R$, $f' \notin \mathfrak{p}$ such that $f'fx = f'a$ in R. In other words, $ff' \in I$. Thus I is an ideal which isn't contained in any of the prime ideals of R, i.e., $I = R$ and $x \in R$. □

Lemma 10.34.12. *A localization of a normal ring is a normal ring.*

Proof. Omitted. □

Lemma 10.34.13. *Let R be a normal ring. Then $R[x]$ is a normal ring.*

Proof. Let $\mathfrak{q}$ be a prime of $R[x]$. Set $\mathfrak{p} = R \cap \mathfrak{q}$. Then we see that $R_\mathfrak{p}[x]$ is a normal domain by Lemma 10.34.8. Hence $(R[x])_\mathfrak{q}$ is a normal domain by Lemma 10.34.5. □

Lemma 10.34.14. *Let R be a ring. Assume R is reduced and has finitely many minimal primes. Then the following are equivalent:*

(1) *R is a normal ring,*
(2) *R is integrally closed in its total ring of fractions, and*
(3) *R is a finite product of normal domains.*

Proof. Let $\mathfrak{q}_1, \ldots, \mathfrak{q}_t$ be the minimal primes of R. By Lemma 10.23.2 we have $Q(R) = R_{\mathfrak{q}_1} \times \ldots \times R_{\mathfrak{q}_t}$, and by Lemma 10.24.4 each factor is a field. Denote $e_i = (0, \ldots, 0, 1, 0, \ldots, 0)$ the ith idempotent of $Q(R)$.

If R is integrally closed in $Q(R)$, then it contains in particular the idempotents e_i, and we see that R is a product of t domains (see Sections 10.20 and 10.21). Hence it is clear that R is a finite product of normal domains.

If R is normal, then it is clear that $e_i \in R_{\mathfrak{p}}$ for every prime ideal $\mathfrak{p}$ of R. Hence we see that R contains the elements e_i (see proof of Lemma 10.34.9). We conclude that R is a product of t domains as before. Each of these t domains is normal by Lemma 10.34.9 and the assumption that R is a normal ring. Hence it follows that R is a finite product of normal domains.

We omit the verification that (3) implies (1) and (2). □

Lemma 10.34.15. *Let $(R_i, \varphi_{ii'})$ be a directed system (Categories, Definition 10.8.2) of rings. If each R_i is a normal ring so is $R = \operatorname{colim}_i R_i$.*

Proof. Let $\mathfrak{p} \subset R$ be a prime ideal. Set $\mathfrak{p}_i = R_i \cap \mathfrak{p}$ (usual abuse of notation). Then we see that $R_{\mathfrak{p}} = \operatorname{colim}_i (R_i)_{\mathfrak{p}_i}$. Since each $(R_i)_{\mathfrak{p}_i}$ is a normal domain we reduce to proving the statement of the lemma for normal domains. If $a, b \in R$ and a/b satisfies a monic polynomial $P(T) \in R[T]$, then we can find a (sufficiently large) $i \in I$ such that a, b, P all come from objects a_i, b_i, P_i over R_i. Since R_i is normal we see $a_i/b_i \in R_i$ and hence also $a/b \in R$. □

10.35. Going down for integral over normal

We first play around a little bit with the notion of elements integral over an ideal, and then we prove the theorem referred to in the section title.

Definition 10.35.1. Let $\varphi : R \to S$ be a ring map. Let $I \subset R$ be an ideal. We say an element $g \in S$ is *integral over I* if there exists a monic polynomial $P = x^d + \sum_{j<d} a_j x^j$ with coefficients $a_j \in I^{d-j}$ such that $P^\varphi(g) = 0$ in S.

This is mostly used when $\varphi = \mathrm{id}_R : R \to R$. In this case the set I' of elements integral over I is called the *integral closure of I*. We will see that I' is an ideal of R (and of course $I \subset I'$).

Lemma 10.35.2. *Let $\varphi : R \to S$ be a ring map. Let $I \subset R$ be an ideal. Let $A = \sum I^n t^n \subset R[t]$ be the subring of the polynomial ring generated by $R \oplus It \subset R[t]$. An element $s \in S$ is integral over I if and only if the element $st \in S[t]$ is integral over A.*

Proof. Suppose st is integral over A. Let $P = x^d + \sum_{j<d} a_j x^j$ be a monic polynomial with coefficients in A such that $P^\varphi(st) = 0$. Let $a'_j \in A$ be the degree $d - j$ part of a_i, in other words $a'_j = a''_j t^{d-j}$ with $a''_j \in I^{d-j}$. For degree reasons we still have $(st)^d + \sum_{j<d} \varphi(a''_j) t^{d-j} (st)^j = 0$. Hence we see that s is integral over I.

Suppose that s is integral over I. Say $P = x^d + \sum_{j<d} a_j x^j$ with $a_j \in I^{d-j}$. The we immediately find a polynomial $Q = x^d + \sum_{j<d} (a_j t^{d-j}) x^j$ with coefficients in A which proves that st is integral over A. □

Lemma 10.35.3. *Let $\varphi : R \to S$ be a ring map. Let $I \subset R$ be an ideal. The set of elements of S which are integral over I form a R-submodule of S. Furthermore, if $s \in S$ is integral over R, and s' is integral over I, then ss' is integral over I.*

Proof. Closure under addition is clear from the characterization of Lemma 10.35.2. Any element $s \in S$ which is integral over R corresponds to the degree 0 element s of $S[x]$ which is integral over A (because $R \subset A$). Hence we see that multiplication by s on $S[x]$ preserves the property of being integral over A, by Lemma 10.33.7. □

Lemma 10.35.4. *Suppose $\varphi : R \to S$ is integral. Suppose $I \subset R$ is an ideal. Then every element of IS is integral over I.*

Proof. Immediate from Lemma 10.35.3. □

Lemma 10.35.5. *Let R be a domain with field of fractions K. Let $n, m \in \mathbf{N}$ and $a_0, \ldots, a_{n-1}, b_0, \ldots, b_{m-1} \in R$. If the polynomial $x^n + a_{n-1}x^{n-1} + \ldots + a_0$ divides the polynomial $x^m + b_{m-1}x^{m-1} + \ldots + b_0$ in $K[x]$ then*

1. *$a_0, \ldots, a_{n-1}$ are integral over the subring of R generated by $b_0, \ldots, b_{m-1}$, and*
2. *each a_i lies in $\sqrt{(b_0, \ldots, b_m)}$.*

Proof. Let $K \supset R$ be the fraction field of R. Let $L \supset K$ be a field extension such that we can write $x^m + b_{m-1}x^{m-1} + \ldots + b_0 = \prod_{i=1}^m (x - \beta_i)$ with $\beta_i \in L$. Each β_i is integral over the subring generated by $b_0, \ldots, b_{m-1}$. Since each a_i is a homogeneous polynomial in $\beta_1, \ldots, \beta_m$ we deduce the same for the a_i.

Choose $c_0, \ldots, c_{m-n-1} \in K$ such that

$$x^m + b_{m-1}x^{m-1} + \ldots + b_0 = (x^n + a_{n-1}x^{n-1} + \ldots + a_0)(x^{m-n} + c_{m-n-1}x^{m-n-1} + \ldots + c_0).$$

By the first part we see that the elements c_i are integral over R. Let R' be the sub R-algebra of K generated by $c_0, \ldots, c_{m-n-1}$. By Lemmas 10.33.15 and 10.28.3 we see that $R \cap \sqrt{(b_0, \ldots, b_m)R'} = \sqrt{(b_0, \ldots, b_m)}$. Thus we may replace R by R' and assume $c_i \in R$. Dividing out the radical $\sqrt{(b_0, \ldots, b_m)}$ we get a reduced ring $\overline{R}$. We have to show that the images $\overline{a_i} \in \overline{R}$ are zero. And in $\overline{R}[x]$ we have the relation

$$x^m = x^m + \overline{b_{m-1}}x^{m-1} + \ldots + \overline{b_0} = (x^n + \overline{a_{n-1}}x^{n-1} + \ldots + \overline{a_0})(x^{m-n} + \overline{c_{m-n-1}}x^{m-n-1} + \ldots + \overline{c_0}).$$

It is easy to see that this implies $\overline{a_i} = 0$ for all i. For example one can see this by localizing at all the minimal primes, see Lemma 10.24.7. □

Lemma 10.35.6. *Let $R \subset S$ be an inclusion of domains. Assume R is normal. Let $g \in S$ be integral over R. Then the minimal polynomial of g has coefficients in R.*

Proof. Let $P = x^m + b_{m-1}x^{m-1} + \ldots + b_0$ be a polynomial with coefficients in R such that $P(g) = 0$. Let $Q = x^n + a_{n-1}x^{n-1} + \ldots + a_0$ be the minimal polynomial for g over the fraction field K of R. Then Q divides P in $K[x]$. By Lemma 10.35.5 we see the a_i are integral over R. Since R is normal this means they are in R. □

Proposition 10.35.7. *Let $R \subset S$ be an inclusion of domains. Assume R is normal and S integral over R. Let $\mathfrak{p} \subset \mathfrak{p}' \subset R$ be primes. Let $\mathfrak{q}'$ be a prime of S with $\mathfrak{p}' = R \cap \mathfrak{q}'$. Then there exists a prime $\mathfrak{q}$ with $\mathfrak{q} \subset \mathfrak{q}'$ such that $\mathfrak{p} = R \cap \mathfrak{q}$. In other words: the going down property holds for $R \to S$, see Definition 10.38.1.*

Proof. Let $\mathfrak{p}$, $\mathfrak{p}'$ and $\mathfrak{q}'$ be as in the statement. We have to show there is a prime $\mathfrak{q}$, $\mathfrak{q} \subset \mathfrak{q}'$ such that $R \cap \mathfrak{q} = \mathfrak{p}$. This is the same as finding a prime of $S_{\mathfrak{q}'}$ mapping to $\mathfrak{p}$. According to Lemma 10.16.9 we have to show that $\mathfrak{p}S_{\mathfrak{q}'} \cap R = \mathfrak{p}$. Pick $z \in \mathfrak{p}S_{\mathfrak{q}'} \cap R$. We may write $z = y/g$ with $y \in \mathfrak{p}S$ and $g \in S$, $g \notin \mathfrak{q}'$. Written differently we have $zg = y$.

By Lemma 10.35.4 there exists a monic polynomial $P = x^m + b_{m-1}x^{m-1} + \ldots + b_0$ with $b_i \in \mathfrak{p}$ such that $P(y) = 0$.

By Lemma 10.35.6 the minimal polynomial of g over K has coefficients in R. Write it as $Q = x^n + a_{n-1}x^{n-1} + \ldots + a_0$. Note that not all a_i, $i = n-1, \ldots, 0$ are in $\mathfrak{p}$ since that would imply $g^n = \sum_{j<n} a_j g^j \in \mathfrak{p}S \subset \mathfrak{p}'S \subset \mathfrak{q}'$ which is a contradiction.

Since $y = zg$ we see immediately from the above that $Q' = x^n + za_{n-1}x^{n-1} + \ldots + z^n a_0$ is the minimal polynomial for y. Hence Q' divides P and by Lemma 10.35.5 we see that $z^j a_{n-j} \in \sqrt{(b_0, \ldots, b_{m-1})} \subset \mathfrak{p}$, $j = 1, \ldots, n$. Because not all a_i, $i = n-1, \ldots, 0$ are in $\mathfrak{p}$ we conclude $z \in \mathfrak{p}$ as desired. □

10.36. Flat modules and flat ring maps

One often used result is that if $M = \operatorname{colim}_{i \in \mathcal{I}} M_i$ is a colimit of R-modules and if N is an R-module then

$$M \otimes N = \operatorname{colim}_{i \in \mathcal{I}} M_i \otimes_R N,$$

see Lemma 10.11.9. This property is usually expressed by saying that $\otimes$ *commutes with colimits*. Another often used result is that if $0 \to N_1 \to N_2 \to N_3 \to 0$ is an exact sequence and if M is any R-module, then

$$M \otimes_R N_1 \to M \otimes_R N_2 \to M \otimes_R N_3 \to 0$$

is still exact, see Lemma 10.11.10. Both of these properties tell us that the functor $N \mapsto M \otimes_R N$ *is right exact*. See Categories, Section 4.23 and Homology, Section 12.7. An R-module M is flat if $N \mapsto N \otimes_R M$ is also left exact, i.e., if it is exact. Here is the precise definition.

Definition 10.36.1. Let R be a ring.

1. An R-module M is called *flat* if whenever $N_1 \to N_2 \to N_3$ is an exact sequence of R-modules the sequence $M \otimes_R N_1 \to M \otimes_R N_1 \to M \otimes_R N_1$ is exact as well.
2. An R-module M is called *faithfully flat* if the complex of R-modules $N_1 \to N_2 \to N_3$ is exact if and only if the sequence $M \otimes_R N_1 \to M \otimes_R N_1 \to M \otimes_R N_1$ is exact.
3. A ring map $R \to S$ is called *flat* if S is flat as an R-module.
4. A ring map $R \to S$ is called *faithfully flat* if S is faithfully flat as an R-module.

Lemma 10.36.2. *Let R be a ring. Let $\{M_i, \varphi_{ii'}\}$ be a directed system of flat R-modules. Then $\operatorname{colim}_i M_i$ is a flat R-module.*

Proof. This follows as $\otimes$ commutes with colimits and because directed colimits are exact, see Lemma 10.8.9. □

Lemma 10.36.3. *A composition of (faithfully) flat ring maps is (faithfully) flat. If $R \to R'$ is flat, and M' is a flat R'-module, then M' is a flat R-module.*

Proof. Omitted. □

Lemma 10.36.4. *Let M be an R-module. The following are equivalent:*

1. *M is flat over R.*
2. *for every injection of R-modules $N \subset N'$ the map $N \otimes_R M \to N' \otimes_R M$ is injective.*
3. *for every ideal $I \subset R$ the map $I \otimes_R M \to R \otimes_R M = M$ is injective.*
4. *for every finitely generated ideal $I \subset R$ the map $I \otimes_R M \to R \otimes_R M = M$ is injective.*

Proof. The implications (1) implies (2) implies (3) implies (4) are all trivial. Thus we prove (4) implies (1). Suppose that $N_1 \to N_2 \to N_3$ is exact. Let $K = \mathrm{Ker}(N_2 \to N_3)$ and $Q = \mathrm{Im}(N_2 \to N_3)$. Then we get maps

$$N_1 \otimes_R M \to K \otimes_R M \to N_2 \otimes_R M \to Q \otimes_R M \to N_3 \otimes_R M$$

Observe that the first and third arrows are surjective. Thus if we show that the second and fourth arrows are injective, then we are done. Hence it suffices to show that $- \otimes_R M$ transforms injective R-module maps into injective R-module maps.

Assume $K \to M$ is an injective R-module map and let $x \in \mathrm{Ker}(K \otimes_R M \to N \otimes_R M)$. We have to show that x is zero. It is clear that the module $N = \bigcup N_i$ is the union of its finitely generated R-submodules N_i. Set $K_i = K \cap N_i$. For some i our x comes from an element $x_i \in K_i \otimes_R M$. After increasing i the element x_i maps to zero in $M_i \otimes_R M$ (because tensor product commutes with colimits). Thus we may assume N is a finite R-module.

Assume N is a finite R-module. Write $N = R^{\oplus n}/L$ and $K = L'/L$. It suffices to prove that $L \otimes_R M \to M^{\oplus n}$ and $L' \otimes_R M \to M^{\oplus n}$ are injective. Namely, if so, then we see that $K \otimes_R M = L' \otimes_R M / L \otimes_R M \to M^{\oplus n}/L \otimes_R M$ is injective too.

Thus it suffices to show that $L \otimes_R M \to M^{oplusn}$ is injective when $L \subset R^{\oplus n}$ is an R-submodule. We do this by induction on n. The base case $n = 1$ we handle below. For the induction step assume $n > 1$ and set $L' = L \cap R \oplus 0^{\oplus n-1}$. Then $L'' = L/L'$ is a submodule of $R^{\oplus n-1}$. We obtain a diagram

$$\begin{array}{ccccccccc} & & L' \otimes_R M & \longrightarrow & L \otimes_R M & \longrightarrow & L'' \otimes_R M & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & M & \longrightarrow & M^{\oplus n} & \longrightarrow & M^{\oplus n-1} & \longrightarrow & 0 \end{array}$$

By induction hypothesis and the base case the left and right vertical arrows are injective. The rows are exact. It follows that the middle vertical arrow is injective too.

The base case of the induction above is when $L \subset R$ is an ideal. In other words, we have to show that $I \otimes_R M \to M$ is injective for any ideal I of R. We know this is true when I is finitely generated. However, $I = \bigcup I_\alpha$ is the union of its finitely generated subideals I_α. In other words, $I = \mathrm{colim}\, I_\alpha$. Since $\otimes$ commutes with colimits we see that $I \otimes_R M = \mathrm{colim}\, I_\alpha \otimes_R M$ and since all the morphisms $I_\alpha \otimes_R M \to M$ are injective by assumption, the same is true for $I \otimes_R M \to M$. □

Lemma 10.36.5. *Let $\{R_i, \varphi_{ii'}\}$ be a system of rings of the directed partially ordered set I. Let $R = \mathrm{colim}_i R_i$. Let M be an R-module such that M is flat as an R_i-module for all i. Then M is flat as an R-module.*

Proof. Let $\mathfrak{a} \subset R$ be a finitely generated ideal. By Lemma 10.36.4 it suffices to show that $\mathfrak{a} \otimes_R M \to M$ is injective. We can find an $i \in I$ and a finitely generated ideal $\mathfrak{a}' \subset R_i$ such that $\mathfrak{a} = \mathfrak{a}'R$. Then $\mathfrak{a} = \operatorname{colim}_{i' \geq i} \mathfrak{a}' R_{i'}$. Hence the map $\mathfrak{a} \otimes_R M \to M$ is the colimit of the maps

$$\mathfrak{a}' R_{i'} \otimes_{R_{i'}} M \longrightarrow M$$

which are all injective by assumption. Since $\otimes$ commutes with colimits and since colimits over I are exact by Lemma 10.8.9 we win. □

Lemma 10.36.6. *Suppose that M is flat over R, and that $R \to R'$ is a ring map. Then $M \otimes_R R'$ is flat over R'.*

Proof. For any R'-module N we have a canonical isomorphism $N \otimes_{R'} (R' \otimes_R M) = N \otimes_R M$. Hence the exactness of $- \otimes_{R'} (R' \otimes_R M)$ follows from the exactness of $- \otimes_R M$. □

Lemma 10.36.7. *Let $R \to R'$ be a faithfully flat ring map. Let M be a module over R, and set $M' = R' \otimes_R M$. Then M is flat over R if and only if M' is flat over R'.*

Proof. By Lemma 10.36.6 we see that if M is flat then M' is flat. For the converse, suppose that M' is flat. Let $N_1 \to N_2 \to N_3$ be an exact sequence of R-modules. We want to show that $N_1 \otimes_R M \to N_2 \otimes_R M \to N_3 \otimes_R M$ is exact. We know that $N_1 \otimes_R R' \to N_2 \otimes_R R' \to N_3 \otimes_R R'$ is exact, because $R \to R'$ is flat. Flatness of M' implies that $N_1 \otimes_R R' \otimes_{R'} M' \to N_2 \otimes_R R' \otimes_{R'} M' \to N_3 \otimes_R R' \otimes_{R'} M'$ is exact. We may write this as $N_1 \otimes_R M \otimes_R R' \to N_2 \otimes_R M \otimes_R R' \to N_3 \otimes_R M \otimes_R R'$. Finally, faithful flatness implies that $N_1 \otimes_R M \to N_2 \otimes_R M \to N_3 \otimes_R M$ is exact. □

Lemma 10.36.8. *Let R be a ring. Let $S \to S'$ be a faithfully flat map of R-algebras. Let M be a module over S, and set $M' = S' \otimes_S M$. Then M is flat over R if and only if M' is flat over R.*

Proof. Let $N \to N'$ be an injection of R-modules. By the faithful flatness of $S \to S'$ we have

$$\operatorname{Ker}(N \otimes_R M \to N' \otimes_R M) \otimes_S S' = \operatorname{Ker}(N \otimes_R M' \to N' \otimes_R M')$$

Hence the equivalence of the lemma follows from the second characterization of flatness in Lemma 10.36.4. □

Lemma 10.36.9. *Let $R \to S$ be a ring map. Let M be an S-module. If M is flat as an R-module and faithfully flat as an S-module, then $R \to S$ is flat.*

Proof. Let $N_1 \to N_2 \to N_3$ be an exact sequence of R-modules. By assumption $N_1 \otimes_R M \to N_2 \otimes_R M \to N_3 \otimes_R M$ is exact. We may write this as

$$N_1 \otimes_R S \otimes_S M \to N_2 \otimes_R S \otimes_S M \to N_3 \otimes_R S \otimes_S M.$$

By faithful flatness of M over S we conclude that $N_1 \otimes_R S \to N_2 \otimes_R S \to N_3 \otimes_R S$ is exact. Hence $R \to S$ is flat. □

Let R be a ring. Let M be an R-module. Let $\sum f_i x_i = 0$ be a relation in M. We say the relation $\sum f_i x_i$ is *trivial* if there exist an integer $m \geq 0$, elements $y_j \in M$, $j = 1, \ldots, m$, and elements $a_{ij} \in R$, $i = 1, \ldots, n$, $j = 1, \ldots, m$ such that

$$x_i = \sum\nolimits_j a_{ij} y_j, \forall i, \quad \text{and} \quad 0 = \sum\nolimits_i f_i a_{ij}, \forall j.$$

Lemma 10.36.10 (Equational criterion of flatness)**.** *A module M over R is flat if and only if every relation in M is trivial.*

Proof. Assume M is flat and let $\sum f_i x_i = 0$ be a relation. Let $I = (f_1, \ldots, f_n)$, and let $K = \mathrm{Ker}(R^n \to I)$. So we have the short exact sequence $0 \to K \to R^n \to I \to 0$. Then $\sum f_i \otimes x_i$ is an element of $I \otimes_R M$ which maps to zero in $R \otimes_R M = M$. By flatness $\sum f_i \otimes x_i$ is zero in $I \otimes_R M$. Thus there exists an element of $K \otimes_R M$ mapping to $\sum e_i \otimes m_i \in R^n \otimes_R M$. Write this element as $\sum k_j \otimes y_j$ and then write the image of k_j in R^n as $\sum a_{ij} e_i$ to get the result.

Assume every relation is trivial, let I be a finitely generated ideal, and let $x = \sum f_i \otimes x_i$ be an element of $I \otimes_R M$ mapping to zero in $R \otimes_R M = M$. This just means exactly that $\sum h_i x_i$ is a relation in M. And the fact that it is trivial implies easily that x is zero, because

$$x = \sum f_i \otimes x_i = \sum f_i \otimes \left(\sum a_{ij} y_j \right) = \sum \left(\sum f_i a_{ij} \right) \otimes y_j = 0$$

□

Lemma 10.36.11. *Suppose that R is a ring, $0 \to M'' \to M' \to M \to 0$ a short exact sequence, and N an R-module. If M is flat then $N \otimes_R M'' \to N \otimes_R M'$ is injective, i.e., the sequence*

$$0 \to N \otimes_R M'' \to N \otimes_R M' \to N \otimes_R M \to 0$$

is a short exact sequence.

Proof. Let $R^{(I)} \to N$ be a surjection from a free module onto N with kernel K. The result follows by a simple diagram chase from the following diagram

$$\begin{array}{ccccccccc}
 & & 0 & & 0 & & 0 & & \\
 & & \uparrow & & \uparrow & & \uparrow & & \\
 & & M'' \otimes_R N & \to & M' \otimes_R N & \to & M \otimes_R N & \to & 0 \\
 & & \uparrow & & \uparrow & & \uparrow & & \\
0 & \to & (M'')^{(I)} & \to & (M')^{(I)} & \to & M^{(I)} & \to & 0 \\
 & & \uparrow & & \uparrow & & \uparrow & & \\
 & & M'' \otimes_R N & \to & M' \otimes_R N & \to & M \otimes_R N & \to & 0 \\
 & & & & & & \uparrow & & \\
 & & & & & & 0 & &
\end{array}$$

with exact rows and columns. The middle row is exact because tensoring with the free module $R^{(I)}$ is exact. □

Lemma 10.36.12. *Suppose that $0 \to M' \to M \to M'' \to 0$ is a short exact sequence of R-modules. If M' and M'' are flat so is M. If M and M'' are flat so is M'.*

Proof. We will use the criterion that a module N is flat if for every ideal $I \subset R$ the map $N \otimes_R I \to N$ is injective, see Lemma 10.36.4. Consider an ideal $I \subset R$. Consider the diagram

$$\begin{array}{ccccccccc}
0 & \to & M' & \to & M & \to & M'' & \to & 0 \\
 & & \uparrow & & \uparrow & & \uparrow & & \\
 & & M' \otimes_R I & \to & M \otimes_R I & \to & M'' \otimes_R I & \to & 0
\end{array}$$

with exact rows. This immediately proves the first assertion. The second follows because if M'' is flat then the lower left horizontal arrow is injective by Lemma 10.36.11. □

Lemma 10.36.13. *Let R be a ring. Let M be an R-module. The following are equivalent*

(1) *M is faithfully flat, and*
(2) *M is flat and for all R-module homomorphisms $\alpha : N \to N'$ we have $\alpha = 0$ if and only if $\alpha \otimes id_M = 0$.*

Proof. If M is faithfully flat, then $0 \to \mathrm{Ker}(\alpha) \to N \to 0$ is exact if and only if the same holds after tensoring with M. This proves (1) implies (2). For the other, assume (2). Let $N_1 \to N_2 \to N_3$ be a complex, and assume the complex $N_1 \otimes_R M \to N_2 \otimes_R M \to N_3 \otimes_R M$ is exact. Take $x \in \mathrm{Ker}(N_2 \to N_3)$, and consider the map $\alpha : R \to N_2/\mathrm{Im}(N_1)$, $r \mapsto rx + \mathrm{Im}(N_1)$. By the exactness of the complex $- \otimes_R M$ we see that $\alpha \otimes \mathrm{id}_M$ is zero. By assumption we get that α is zero. Hence x is in the image of $N_1 \to N_2$. □

Lemma 10.36.14. *Let M be a flat R-module. The following are equivalent:*

(1) *M is faithfully flat,*
(2) *for all $\mathfrak{p} \in \mathrm{Spec}(R)$ the tensor product $M \otimes_R \kappa(\mathfrak{p})$ is nonzero, and*
(3) *for all maximal ideals $\mathfrak{m}$ of R the tensor product $M \otimes_R \kappa(\mathfrak{m}) = M/\mathfrak{m}M$ is nonzero.*

Proof. Assume M faithfully flat. Since $R \to \kappa(\mathfrak{p})$ is not zero we deduce that $M \to M \otimes_R \kappa(\mathfrak{p})$ is not zero, see Lemma 10.36.13.

Conversely assume that M is flat and that $M/\mathfrak{m}M$ is never zero. Suppose that $N_1 \to N_2 \to N_3$ is a complex and suppose that $N_1 \otimes_R M \to N_2 \otimes_R M \to N_3 \otimes_R M$ is exact. Let H be the cohomology of the complex, so $H = \mathrm{Ker}(N_2 \to N_3)/\mathrm{Im}(N_1 \to N_2)$. By flatness we see that $H \otimes_R M = 0$. Take $x \in H$ and let $I = \{f \in R \mid fx = 0\}$ be its annihilator. Since $R/I \subset H$ we get $M/IM \subset H \otimes_R M = 0$ by flatness of M. If $I \neq R$ we may choose a maximal ideal $I \subset \mathfrak{m} \subset R$. This immediately gives a contradiction. □

Lemma 10.36.15. *Let $R \to S$ be a flat ring map. The following are equivalent:*

(1) *$R \to S$ is faithfully flat,*
(2) *the induced map on* Spec *is surjective, and*
(3) *any closed point $x \in \mathrm{Spec}(R)$ is in the image of the map $\mathrm{Spec}(S) \to \mathrm{Spec}(R)$.*

Proof. This follows quickly from Lemma 10.36.14, because we saw in Remark 10.16.8 that $\mathfrak{p}$ is in the image if and only if the ring $S \otimes_R \kappa(\mathfrak{p})$ is nonzero. □

Lemma 10.36.16. *A flat local ring homomorphism of local rings is faithfully flat.*

Proof. Immediate from Lemma 10.36.15. □

Lemma 10.36.17. *Let $R \to S$ be flat. Let $\mathfrak{p} \subset \mathfrak{p}'$ be primes of R. Let $\mathfrak{q}' \subset S$ be a prime of S mapping to $\mathfrak{p}'$. Then there exists a prime $\mathfrak{q} \subset \mathfrak{q}'$ mapping to $\mathfrak{p}$.*

Proof. Namely, consider the flat local ring map $R_{\mathfrak{p}'} \to S_{\mathfrak{q}'}$. By Lemma 10.36.16 this is faithfully flat. By Lemma 10.36.15 there is a prime mapping to $\mathfrak{p}R_{\mathfrak{p}'}$. The inverse image of this prime in S does the job. □

The property of $R \to S$ described in the lemma is called the "going down property". See Definition 10.38.1.

Lemma 10.36.18. *If $R \to S$ is a faithfully flat ring map then for every R-module M the map $M \to S \otimes_R M$, $x \mapsto 1 \otimes x$ is injective.*

Proof. This is true because the base change $S \otimes_R M \to S \otimes_R S \otimes_R M$ by the faithfully flat ring map $R \to S$ is injective: It has a section, namely $s \otimes s' \otimes m \mapsto ss' \otimes m$. □

We finish with some remarks on flatness and localization.

Lemma 10.36.19. *Let R be a ring. Let $S \subset R$ be a multiplicative subset.*

(1) *The localization $S^{-1}R$ is a flat R-algebra.*
(2) *If M is a $S^{-1}R$-module, then M is a flat R-module if and only if M is a flat $S^{-1}R$-module.*
(3) *Suppose M is an R-module. Then M is a flat R-module if and only if $M_{\mathfrak{p}}$ is a flat $R_{\mathfrak{p}}$-module for all primes $\mathfrak{p}$ of R.*
(4) *Suppose M is an R-module. Then M is a flat R-module if and only if $M_{\mathfrak{m}}$ is a flat $R_{\mathfrak{m}}$-module for all maximal ideals $\mathfrak{m}$ of R.*
(5) *Suppose $R \to A$ is a ring map, M is an A-module, and $g_1, \ldots, g_m \in A$ are elements generating the unit ideal of A. Then M is flat over R if and only if each localization M_{g_i} is flat over R.*
(6) *Suppose $R \to A$ is a ring map, and M is an A-module. Then M is a flat R-module if and only if the localization $M_{\mathfrak{q}}$ is a flat $R_{\mathfrak{p}}$-module (with $\mathfrak{p}$ the prime of R lying under $\mathfrak{q}$) for all primes $\mathfrak{q}$ of A.*
(7) *Suppose $R \to A$ is a ring map, and M is an A-module. Then M is a flat R-module if and only if the localization $M_{\mathfrak{m}}$ is a flat $R_{\mathfrak{p}}$-module (with $\mathfrak{p} = R \cap \mathfrak{m}$) for all maximal ideals $\mathfrak{m}$ of A.*

Proof. Let us prove the last statement of the lemma. In the proof we will use repeatedly that localization is exact and commutes with tensor product, see Sections 10.9 and 10.11.

Suppose $R \to A$ is a ring map, and M is an A-module. Assume that $M_{\mathfrak{m}}$ is a flat $R_{\mathfrak{p}}$-module for all maximal ideals $\mathfrak{m}$ of A (with $\mathfrak{p} = R \cap \mathfrak{m}$). Let $I \subset R$ be an ideal. We have to show the map $I \otimes_R M \to M$ is injective. We can think of this as a map of A-modules. By assumption the localization $(I \otimes_R M)_{\mathfrak{m}} \to M_{\mathfrak{m}}$ is injective because $(I \otimes_R M)_{\mathfrak{m}} = I_{\mathfrak{p}} \otimes_{R_{\mathfrak{p}}} M_{\mathfrak{m}}$. Hence the kernel of $I \otimes_R M \to M$ is zero by Lemma 10.22.1. Hence M is flat over R.

Conversely, assume M is flat over R. Pick a prime $\mathfrak{q}$ of A lying over the prime $\mathfrak{p}$ of R. Suppose that $I \subset R_{\mathfrak{p}}$ is an ideal. We have to show that $I \otimes_{R_{\mathfrak{p}}} M_{\mathfrak{q}} \to M_{\mathfrak{q}}$ is injective. We can write $I = J_{\mathfrak{p}}$ for some ideal $J \subset R$. Then the map $I \otimes_{R_{\mathfrak{p}}} M_{\mathfrak{q}} \to M_{\mathfrak{q}}$ is just the localization (at $\mathfrak{q}$) of the map $J \otimes_R M \to M$ which is injective. Since localization is exact we see that $M_{\mathfrak{q}}$ is a flat $R_{\mathfrak{p}}$-module.

This proves (7) and (6). The other statements follow in a straightforward way from the last statement (proofs omitted). □

Lemma 10.36.20. *Let R be a ring. Let $\{S_i, \varphi_{ii'}\}$ be a directed system of faithfully flat R-algebras. Then $S = \operatorname{colim}_i S_i$ is a faithfully flat R-algebra.*

Proof. By Lemma 10.36.2 we see that S is flat. Let $\mathfrak{m} \subset R$ be a maximal ideal. By Lemma 10.36.15 none of the rings $S_i/\mathfrak{m}S_i$ is zero. Hence $S/\mathfrak{m}S = \operatorname{colim} S_i/\mathfrak{m}S_i$ is nonzero either as 1 is not equal to zero. Thus the image of $\operatorname{Spec}(S) \to \operatorname{Spec}(R)$ contains $\mathfrak{m}$ and we see that $R \to S$ is faithfully flat by Lemma 10.36.15. □

10.37. Supports and annihilators

Some very basic definitions and lemmas.

Definition 10.37.1. Let R be a ring and let M be an R-module. The *support of* M is the set

$$\operatorname{Supp}(M) = \{\mathfrak{p} \in \operatorname{Spec}(R) \mid M_\mathfrak{p} \neq 0\}$$

Lemma 10.37.2. *Let R be a ring. Let M be an R-module. Then*

$$M = (0) \Leftrightarrow \mathit{Supp}(M) = \emptyset.$$

Proof. Actually, Lemma 10.22.1 even shows that $\operatorname{Supp}(M)$ always contains a maximal ideal if M is not zero. □

Definition 10.37.3. Let R be a ring. Let M be an R-module.

(1) Given an element $m \in M$ the *annihilator of* m is the ideal

$$\operatorname{Ann}_R(m) = \operatorname{Ann}(m) = \{f \in R \mid fm = 0\}.$$

(2) The *annihilator of* M is the ideal

$$\operatorname{Ann}_R(M) = \operatorname{Ann}(M) = \{f \in R \mid fm = 0 \ \forall m \in M\}.$$

Lemma 10.37.4. *Let $R \to S$ be a flat ring map. Let M be an R-module and $m \in M$. Then $\mathit{Ann}_R(m)S = \mathit{Ann}_S(m \otimes 1)$. If M is a finite R-module, then $\mathit{Ann}_R(M)S = \mathit{Ann}_S(M \otimes_R S)$.*

Proof. Set $I = \operatorname{Ann}_R(m)$. By definition there is an exact sequence $0 \to I \to R \to M$ where the map $R \to M$ sends f to fm. Using flatness we obtain an exact sequence $0 \to I \otimes_R S \to S \to M \otimes_R S$ which proves the first assertion. If $m_1, \ldots, m_n$ is a set of generators of M then $\operatorname{Ann}_R(M) = \bigcap \operatorname{Ann}_R(m_i)$. Similarly $\operatorname{Ann}_S(M \otimes_R S) = \bigcap \operatorname{Ann}_S(m_i \otimes 1)$. Set $I_i = \operatorname{Ann}_R(m_i)$. Then it suffices to show that $\bigcap_{i=1,\ldots,n}(I_i S) = (\bigcap_{i=1,\ldots,n} I_i)S$. This follows by tensoring the injective map $R/(\bigcap_{i=1,\ldots,n} I_i) \to \bigoplus_{i=1,\ldots,n} R/I_i$ with S. □

Lemma 10.37.5. *Let R be a ring and let M be an R-module. If M is finite, then $\mathit{Supp}(M)$ is closed. More precisely, if $I = \mathit{Ann}(M)$ is the annihilator of M, then $V(I) = \mathit{Supp}(M)$.*

Proof. We will show that $V(I) = \operatorname{Supp}(M)$.

Suppose $\mathfrak{p} \in \operatorname{Supp}(M)$. Then $M_\mathfrak{p} \neq 0$. Hence by Nakayama's Lemma 10.18.1 we have $M \otimes_R \kappa(\mathfrak{p}) \neq 0$. Hence $I \subset \mathfrak{p}$.

Conversely, suppose that $\mathfrak{p} \notin \operatorname{Supp}(M)$. Then $M_\mathfrak{p} = 0$. Let $x_1, \ldots, x_r \in M$ be generators. By Lemma 10.9.9 there exists an $f \in R$, $f \notin \mathfrak{p}$ such that $x_i/1 = 0$ in M_f. Hence $f^{n_i} x_i = 0$ for some $n_i \geq 1$. Hence $f^n M = 0$ for $n = \max\{n_i\}$ as desired. □

Lemma 10.37.6. *Let R be a ring, let M be an R-module, and let $x \in M$. Then $\mathfrak{p} \in V(\mathit{Ann}(m))$ if and only if x does not map to zero in $M_\mathfrak{p}$.*

Proof. We may replace M by $Rm \subset M$. Then (1) $\text{Ann}(m) = \text{Ann}(M)$ and (2) x does not map to zero in $M_\mathfrak{p}$ if and only if $\mathfrak{p} \in \text{Supp}(M)$. The result now follows from Lemma 10.37.5. □

Lemma 10.37.7. *Let R be a ring and let M be an R-module. If M is a finitely presented R-module, then Supp(M) is a closed subset of* $\text{Spec}(R)$ *whose complement is quasi-compact.*

Proof. Choose a presentation

$$R^{\oplus m} \longrightarrow R^{\oplus n} \longrightarrow M \to 0$$

Let $A \in \text{Mat}(n \times m, R)$ be the matrix of the first map. By Nakayama's Lemma 10.18.1 we see that

$$M_\mathfrak{p} \neq 0 \Leftrightarrow M \otimes \kappa(\mathfrak{p}) \neq 0 \Leftrightarrow \text{rank}(A \bmod \mathfrak{p}) < n.$$

Hence, if I is the ideal of R generated by the $n \times n$ minors of A, then $\text{Supp}(M) = V(I)$. Since I is finitely generated, say $I = (f_1, \ldots, f_t)$, we see that $\text{Spec}(R) \setminus V(I)$ is a finite union of the standard opens $D(f_i)$, hence quasi-compact. □

Lemma 10.37.8. *Let R be a ring and let M be an R-module.*

1. *If M is finite then the support of M/IM is Supp$(M) \cap V(I)$.*
2. *If $N \subset M$, then Supp$(N) \subset$ Supp(M).*
3. *If Q is a quotient module of M then Supp$(Q) \subset$ Supp(M).*
4. *If $0 \to N \to M \to Q \to 0$ is a short exact sequence then Supp$(M) =$ Supp$(Q) \cup$ Supp(N).*

Proof. The functors $M \mapsto M_\mathfrak{p}$ are exact. This immediately implies all but the first assertion. For the first assertion we need to show that $M_\mathfrak{p} \neq 0$ and $I \subset \mathfrak{p}$ implies $(M/IM)_\mathfrak{p} = M_\mathfrak{p}/IM_\mathfrak{p} \neq 0$. This follows from Nakayama's Lemma 10.18.1. □

10.38. Going up and going down

Suppose $\mathfrak{p}$, $\mathfrak{p}'$ are primes of the ring R. Let $X = \text{Spec}(R)$ with the Zariski topology. Denote $x \in X$ the point corresponding to $\mathfrak{p}$ and $x' \in X$ the point corresponding to $\mathfrak{p}'$. Then we have:

$$x' \rightsquigarrow x \Leftrightarrow \mathfrak{p}' \subset \mathfrak{p}.$$

In words: x is a specialization of x' if and only if $\mathfrak{p}' \subset \mathfrak{p}$. See Topology, Section 5.18 for terminology and notation.

Definition 10.38.1. Let $\varphi : R \to S$ be a ring map.

1. We say a $\varphi : R \to S$ satisfies *going up* if given primes $\mathfrak{p} \subset \mathfrak{p}'$ in R and a prime $\mathfrak{q}$ in S lying over $\mathfrak{p}$ there exists a prime $\mathfrak{q}'$ of S such that (a) $\mathfrak{q} \subset \mathfrak{q}'$, and (b) $\mathfrak{q}'$ lies over $\mathfrak{p}'$.
2. We say a $\varphi : R \to S$ satisfies *going down* if given primes $\mathfrak{p} \subset \mathfrak{p}'$ in R and a prime $\mathfrak{q}'$ in S lying over $\mathfrak{p}'$ there exists a prime $\mathfrak{q}$ of S such that (a) $\mathfrak{q} \subset \mathfrak{q}'$, and (b) $\mathfrak{q}$ lies over $\mathfrak{p}$.

So far we have see the following cases of this:

1. An integral ring map satisfies going up, see Lemma 10.33.20.
2. As a special case finite ring maps satisfy going up.
3. As a special case quotient maps $R \to R/I$ satisfy going up.
4. A flat ring map satisfies going down, see Lemma 10.36.17

(5) As a special case any localization satisfies going down.
(6) An extension $R \subset S$ of domains, with R normal and S integral over R satisfies going down, see Proposition 10.35.7.

Here is another case where going down holds.

Lemma 10.38.2. *Let $R \to S$ be a ring map. If the induced map $\varphi : \mathrm{Spec}(S) \to \mathrm{Spec}(R)$ is open, then $R \to S$ satisfies going down.*

Proof. Suppose that $\mathfrak{p} \subset \mathfrak{p}' \subset R$ and $\mathfrak{q}' \subset S$ lies over $\mathfrak{p}'$. As φ is open, for every $g \in S$, $g \notin \mathfrak{q}'$ we see that $\mathfrak{p}$ is in the image of $D(g) \subset \mathrm{Spec}(S)$. In other words $S_g \otimes_R \kappa(\mathfrak{p})$ is not zero. Since $S_{\mathfrak{q}'}$ is the directed colimit of these S_g this implies that $S_{\mathfrak{q}'} \otimes_R \kappa(\mathfrak{p})$ is not zero, see Lemmas 10.9.9 and 10.11.9. Hence $\mathfrak{p}$ is in the image of $\mathrm{Spec}(S_{\mathfrak{q}'}) \to \mathrm{Spec}(R)$ as desired. □

Lemma 10.38.3. *Let $R \to S$ be a ring map.*

(1) *$R \to S$ satisfies going down if and only if generalizations lift along the map $\mathrm{Spec}(S) \to \mathrm{Spec}(R)$, see Topology, Definition 5.18.3.*
(2) *$R \to S$ satisfies going up if and only if specializations lift along the map $\mathrm{Spec}(S) \to \mathrm{Spec}(R)$, see Topology, Definition 5.18.3.*

Proof. Omitted. □

Lemma 10.38.4. *Suppose $R \to S$ and $S \to T$ are ring maps satisfying going down. Then so does $R \to T$. Similarly for going up.*

Proof. According to Lemma 10.38.3 this follows from Topology, Lemma 5.18.4 □

Lemma 10.38.5. *Let $R \to S$ be a ring map. Let $T \subset \mathrm{Spec}(R)$ be the image of $\mathrm{Spec}(S)$. If T is stable under specialization, then T is closed.*

Proof. We give two proofs.

First proof. Let $\mathfrak{p} \subset R$ be a prime ideal such that the corresponding point of $\mathrm{Spec}(R)$ is in the closure of T. This means that for ever $f \in R$, $f \notin \mathfrak{p}$ we have $D(f) \cap T \neq \emptyset$. Note that $D(f) \cap T$ is the image of $\mathrm{Spec}(S_f)$ in $\mathrm{Spec}(R)$. Hence we conclude that $S_f \neq 0$. In other words, $1 \neq 0$ in the ring S_f. Since $S_{\mathfrak{p}}$ is the directed limit of the rings S_f we conclude that $1 \neq 0$ in $S_{\mathfrak{p}}$. In other words, $S_{\mathfrak{p}} \neq 0$ and considering the image of $\mathrm{Spec}(S_{\mathfrak{p}}) \to \mathrm{Spec}(S) \to \mathrm{Spec}(R)$ we see there exists a $\mathfrak{p}' \in T$ with $\mathfrak{p}' \subset \mathfrak{p}$. As we assumed T closed under specialization we conclude $\mathfrak{p}$ is a point of T as desired.

Second proof. Let $I = \mathrm{Ker}(R \to S)$. We may replace R by R/I. In this case the ring map $R \to S$ is injective. By Lemma 10.28.5 all the minimal primes of R are contained in the image T. Hence if T is stable under specialization then it contains all primes. □

Lemma 10.38.6. *Let $R \to S$ be a ring map. The following are equivalent:*

(1) *Going up holds for $R \to S$, and*
(2) *the map $\mathrm{Spec}(S) \to \mathrm{Spec}(R)$ is closed.*

Proof. It is a general fact that specializations lift along a closed map of topological spaces, see Topology, Lemma 5.18.6. Hence the second condition implies the first.

Assume that going up holds for $R \to S$. Let $V(I) \subset \mathrm{Spec}(S)$ be a closed set. We want to show that the image of $V(I)$ in $\mathrm{Spec}(R)$ is closed. The ring map

$S \to S/I$ obviously satisfies going up. Hence $R \to S \to S/I$ satisfies going up, by Lemma 10.38.4. Replacing S by S/I it suffices to show the image T of $\mathrm{Spec}(S)$ in $\mathrm{Spec}(R)$ is closed. By Topology, Lemmas 5.18.2 and 5.18.5 this image is stable under specialization. Thus the result follows from Lemma 10.38.5. □

Lemma 10.38.7. *Let R be a ring. Let $E \subset \mathrm{Spec}(R)$ be a constructible subset.*

(1) *If E is stable under specialization, then E is closed.*
(2) *If E is stable under generalization, then E is open.*

Proof. First proof. The first assertion follows from Lemma 10.38.5 combined with Lemma 10.27.3. The second follows because the complement of a constructible set is constructible (see Topology, Lemma 5.14.2), the first part of the lemma and Topology, Lemma 5.18.2.

Second proof. Since $\mathrm{Spec}(R)$ is a spectral space by Lemma 10.24.2 this is a special case of Topology, Lemma 5.22.4. □

Proposition 10.38.8. *Let $R \to S$ be flat and of finite presentation. Then $\mathrm{Spec}(R) \to \mathrm{Spec}(S)$ is open. More generally this holds for any ring map $R \to S$ of finite presentation which satisfies going down.*

Proof. Assume that $R \to S$ has finite presentation and satisfies going down. It suffices to prove that the image of a standard open $D(f)$ is open. Since $S \to S_f$ satisfies going down as well, we see that $R \to S_f$ satisfies going down. Thus after replacing S by S_f we see it suffices to prove the image is open. By Chevalley's theorem (Theorem 10.27.9) the image is a constructible set E. And E is stable under generalization because $R \to S$ satisfies going down, see Topology, Lemmas 5.18.2 and 5.18.5. Hence E is open by Lemma 10.38.7. □

Lemma 10.38.9. *Let k be a field, and let R, S be k-algebras. Let $S' \subset S$ be a sub k-algebra, and let $f \in S' \otimes_k R$. In the commutative diagram*

$$\begin{array}{ccc} \mathrm{Spec}((S \otimes_k R)_f) & \longrightarrow & \mathrm{Spec}((S' \otimes_k R)_f) \\ & \searrow \quad \swarrow & \\ & \mathrm{Spec}(R) & \end{array}$$

the images of the diagonal arrows are the same.

Proof. Let $\mathfrak{p} \subset R$ be in the image of the south-west arrow. This means (Lemma 10.16.9) that

$$(S' \otimes_k R)_f \otimes_R \kappa(\mathfrak{p}) = (S' \otimes_k \kappa(\mathfrak{p}))_f$$

is not the zero ring, i.e., $S' \otimes_k \kappa(\mathfrak{p})$ is not the zero ring and the image of f in it is not nilpotent. The ring map $S' \otimes_k \kappa(\mathfrak{p}) \to S \otimes_k \kappa(\mathfrak{p})$ is injective. Hence also $S \otimes_k \kappa(\mathfrak{p})$ is not the zero ring and the image of f in it is not nilpotent. Hence $(S \otimes_k R)_f \otimes_R \kappa(\mathfrak{p})$ is not the zero ring. Thus (Lemma 10.16.9) we see that $\mathfrak{p}$ is in the image of the south-east arrow as desired. □

Lemma 10.38.10. *Let k be a field. Let R and S be k-algebras. The map $\mathrm{Spec}(S \otimes_k R) \to \mathrm{Spec}(R)$ is open.*

Proof. Let $f \in R \otimes_k S$. It suffices to prove that the image of the standard open $D(f)$ is open. Let $S' \subset S$ be a finite type k-subalgebra such that $f \in S' \otimes_k R$. The map $R \to S' \otimes_k R$ is flat and of finite presentation, hence the image U of $\mathrm{Spec}((S' \otimes_k R)_f) \to \mathrm{Spec}(R)$ is open by Proposition 10.38.8. By Lemma 10.38.9 this is also the image of $D(f)$ and we win. □

Here is a tricky lemma that is sometimes useful.

Lemma 10.38.11. *Let $R \to S$ be a ring map. Let $\mathfrak{p} \subset R$ be a prime. Assume that*

1. *there exists a unique prime $\mathfrak{q} \subset S$ lying over $\mathfrak{p}$, and*
2. *either*
   (a) *going up holds for $R \to S$, or*
   (b) *going down holds for $R \to S$ and there is at most one prime of S above every prime of R.*

Then $S_\mathfrak{p} = S_\mathfrak{q}$.

Proof. Consider any prime $\mathfrak{q}' \subset S$ which corresponds to a point of $\mathrm{Spec}(S_\mathfrak{p})$. This means that $\mathfrak{p}' = R \cap \mathfrak{q}'$ is contained in $\mathfrak{p}$. Here is a picture

$$\begin{array}{ccccc} \mathfrak{q}' & \text{———} & ? & \text{———} & S \\ | & & | & & | \\ \mathfrak{p}' & \text{———} & \mathfrak{p} & \text{———} & R \end{array}$$

Assume (1) and (2)(a). By going up there exists a prime $\mathfrak{q}'' \subset S$ with $\mathfrak{q}' \subset \mathfrak{q}''$ and $\mathfrak{q}''$ lying over $\mathfrak{p}$. By the uniqueness of $\mathfrak{q}$ we conclude that $\mathfrak{q}'' = \mathfrak{q}$. In other words $\mathfrak{q}'$ defines a point of $\mathrm{Spec}(S_\mathfrak{q})$.

Assume (1) and (2)(b). By going down there exists a prime $\mathfrak{q}'' \subset \mathfrak{q}$ lying over $\mathfrak{p}'$. By the uniqueness of primes lying over $\mathfrak{p}'$ we see that $\mathfrak{q}' = \mathfrak{q}''$. In other words $\mathfrak{q}'$ defines a point of $\mathrm{Spec}(S_\mathfrak{q})$.

In both cases we conclude that the map $\mathrm{Spec}(S_\mathfrak{q}) \to \mathrm{Spec}(S_\mathfrak{p})$ is bijective. Clearly this means all the elements of $S - \mathfrak{q}$ are all invertible in $S_\mathfrak{p}$, in other words $S_\mathfrak{p} = S_\mathfrak{q}$. □

The following lemma is a generalization of going down for flat ring maps.

Lemma 10.38.12. *Let $R \to S$ be a ring map. Let N be a finite S-module flat over R. Endow $\mathrm{Supp}(N) \subset \mathrm{Spec}(S)$ with the induced topology. Then generalizations lift along $\mathrm{Supp}(N) \to \mathrm{Spec}(R)$.*

Proof. The meaning of the statement is as follows. Let $\mathfrak{p} \subset \mathfrak{p}' \subset R$ be primes. Let $\mathfrak{q}' \subset S$ be a prime $\mathfrak{q}' \in \mathrm{Supp}(N)$ Then there exists a prime $\mathfrak{q} \subset \mathfrak{q}'$, $\mathfrak{q} \in \mathrm{Supp}(N)$ lying over $\mathfrak{p}$. As N is flat over R we see that $N_{\mathfrak{q}'}$ is flat over $R_{\mathfrak{p}'}$, see Lemma 10.36.19. As $N_{\mathfrak{q}'}$ is finite over $S_{\mathfrak{q}'}$ and not zero since $\mathfrak{q}' \in \mathrm{Supp}(N)$ we see that $N_{\mathfrak{q}'} \otimes_{S_{\mathfrak{q}'}} \kappa(\mathfrak{q}')$ is nonzero by Nakayama's Lemma 10.18.1. Thus $N_{\mathfrak{q}'} \otimes_{R_{\mathfrak{p}'}} \kappa(\mathfrak{p}')$ is also not zero. We conclude from Lemma 10.36.14 that $N_{\mathfrak{q}'} \otimes_{R_{\mathfrak{p}'}} \kappa(\mathfrak{p})$ is nonzero. Let $J \subset S_{\mathfrak{q}'} \otimes_{R_{\mathfrak{p}'}} \kappa(\mathfrak{p})$ be the annihilator of the finite nonzero module $N_{\mathfrak{q}'} \otimes_{R_{\mathfrak{p}'}} \kappa(\mathfrak{p})$. Since J is a proper ideal we can choose a prime $\mathfrak{q} \subset S$ which corresponds to a prime of $S_{\mathfrak{q}'} \otimes_{R_{\mathfrak{p}'}} \kappa(\mathfrak{p})/J$. This prime is in the support of N, lies over $\mathfrak{p}$, and is contained in $\mathfrak{q}'$ as desired. □

10.39. Separable extensions

In this section we talk about separability for nonalgebraic field extensions. This is closely related to the concept of geometrically reduced algebras, see Definition 10.40.1.

Definition 10.39.1. Let $k \subset K$ be a field extension.

(1) We say K is *separably generated over* k if there exists a transcendence basis $\{x_i; i \in I\}$ of K/k such that the extension $k(x_i; i \in I) \subset K$ is a separable algebraic extension.
(2) We say K is *separable over* k if for every subextension $k \subset K' \subset K$ with K' finitely generated over k, the extension $k \subset K'$ is separably generated.

With this awkward definition it is not clear that a separably generated field extension is itself separable. It will turn out that this is the case, see Lemma 10.41.2.

Lemma 10.39.2. *Let $k \subset K$ be a separable field extension. For any subextension $k \subset K' \subset K$ the field extension $k \subset K'$ is separable.*

Proof. This is direct from the definition. □

Lemma 10.39.3. *Let $k \subset K$ be a separably generated, and finitely generated field extension. Set $r = trdeg_k(K)$. Then there exist elements $x_1, \ldots, x_{r+1}$ of K such that*

(1) *$x_1, \ldots, x_r$ is a transcendence basis of K over k,*
(2) *$K = k(x_1, \ldots, x_{r+1})$, and*
(3) *x_{r+1} is separable over $k(x_1, \ldots, x_r)$.*

Proof. Combine the definition with Fields, Lemma 9.18.1. □

Lemma 10.39.4. *Let $k \subset K$ be a finitely generated field extension. There exists a diagram*

$$\begin{array}{ccc} K & \longrightarrow & K' \\ \uparrow & & \uparrow \\ k & \longrightarrow & k' \end{array}$$

where $k \subset k'$, $K \subset K'$ are finite purely inseparable field extensions such that $k' \subset K'$ is a separably generated field extension.

Proof. This lemma is only interesting when the characteristic of k is $p > 0$. Choose $x_1, \ldots, x_r$ a transcendence basis of K over k. As K is finitely generated over k the extension $k(x_1, \ldots, x_r) \subset K$ is finite. Let $k(x_1, \ldots, x_r) \subset K_{sep} \subset K$ be the subextension found in Fields, Lemma 9.13.6. If $K = K_{sep}$ then we are done. We will use induction on $d = [K : K_{sep}]$.

Assume that $d > 1$. Choose a $\beta \in K$ with $\alpha = \beta^p \in K_{sep}$ and $\beta \notin K_{sep}$. Let $P = T^d + a_1 T^{d-1} + \ldots + a_d$ be the minimal polynomial of α over $k(x_1, \ldots, x_r)$. Let $k \subset k'$ be a finite purely inseparable extension obtained by adjoining pth roots such that each a_i is a pth power in $k'(x_1^{1/p}, \ldots, x_r^{1/p})$. Such an extension exists; details

omitted. Let L be a field fitting into the diagram

$$\begin{array}{ccc} K & \longrightarrow & L \\ \uparrow & & \uparrow \\ k(x_1, \ldots, x_r) & \longrightarrow & k'(x_1^{1/p}, \ldots, x_r^{1/p}) \end{array}$$

We may and do assume L is the compositum of K and $k'(x_1^{1/p}, \ldots, x_r^{1/p})$. Let $k'(x_1^{1/p}, \ldots, x_r^{1/p}) \subset L_{sep} \subset L$ be the subextension found in Fields, Lemma 9.13.6. Then L_{sep} is the compositum of K_{sep} and $k'(x_1^{1/p}, \ldots, x_r^{1/p})$. The element $\alpha \in L_{sep}$ is a zero of the polynomial P all of whose coefficients are pth powers in $k'(x_1^{1/p}, \ldots, x_r^{1/p})$ and whose roots are pairwise distinct. By Fields, Lemma 9.25.2 we see that $\alpha = (\alpha')^p$ for some $\alpha' \in L_{sep}$. Clearly, this means that β maps to $\alpha' \in L_{sep}$. In other words, we get the tower of fields

$$\begin{array}{ccc} K & \longrightarrow & L \\ \uparrow & & \uparrow \\ K_{sep}(\beta) & \longrightarrow & L_{sep} \\ \uparrow & & \| \\ K_{sep} & \longrightarrow & L_{sep} \\ \uparrow & & \uparrow \\ k(x_1, \ldots, x_r) & \longrightarrow & k'(x_1^{1/p}, \ldots, x_r^{1/p}) \\ \uparrow & & \uparrow \\ k & \longrightarrow & k' \end{array}$$

Thus this construction leads to a new situation with $[L : L_{sep}] < [K : K_{sep}]$. By induction we can find $k' \subset k''$ and $L \subset L'$ as in the lemma for the extension $k' \subset L$. Then the extensions $k \subset k''$ and $K \subset L''$ work for the extension $k \subset K$. This proves the lemma. □

10.40. Geometrically reduced algebras

The main result on geometrically reduced algebras is Lemma 10.41.3. We suggest the reader skip to the lemma after reading the definition.

Definition 10.40.1. Let k be a field. Let S be a k-algebra. We say S is *geometrically reduced over* k if for every field extension $k \subset K$ the K-algebra $K \otimes_k S$ is reduced.

Let k be a field and let S be a reduced k algebra. To check that S is geometrically reduced it will suffice to check that $\overline{k} \otimes_k S$ is reduced (where $\overline{k}$ denotes the algebraic closure of k). In fact it is enough to check this for finite purely inseparable field extensions $k \subset k'$. See Lemma 10.41.3.

Lemma 10.40.2. *Elementary properties of geometrically reducedness. Let k be a field. Let S be a k-algebra.*

(1) *If S is geometrically reduced over k so is every k-subalgebra.*
(2) *If all finitely generated k-subalgebras of S are geometrically reduced, then S is geometrically reduced.*
(3) *A directed colimit of geometrically reduced k-algebras is geometrically reduced.*
(4) *If S is geometrically reduced over k, then any localization of S is geometrically reduced over k.*

Proof. Omitted. The second and third property follow from the fact that tensor product commutes with colimits. □

Lemma 10.40.3. *Let k be a field. If R is geometrically reduced over k, and $S \subset R$ is a multiplicative subset, then the localization $S^{-1}R$ is geometrically reduced over k. If R is geometrically reduced over k, then $R[x]$ is geometrically reduced over k.*

Proof. Omitted. Hints: A localization of a reduced ring is reduced, and localization commutes with tensor products. □

In the proofs of the following lemmas we will repeatedly use the following observation: Suppose that $R' \subset R$ and $S' \subset S$ are inclusions of k-algebras. Then the map $R' \otimes_k S' \to R \otimes_k S$ is injective.

Lemma 10.40.4. *Let k be a field. Let R, S be k-algebras.*

(1) *If $R \otimes_k S$ is nonreduced, then there exist finitely generated subalgebras $R' \subset R$, $S' \subset S$ such that $R' \otimes_k S'$ is not reduced.*
(2) *If $R \otimes_k S$ contains a nonzero zerodivisor, then there exist finitely generated subalgebras $R' \subset R$, $S' \subset S$ such that $R' \otimes_k S'$ contains a nonzero zerodivisor.*
(3) *If $R \otimes_k S$ contains a nontrivial idempotent, then there exist finitely generated subalgebras $R' \subset R$, $S' \subset S$ such that $R' \otimes_k S'$ contains a nontrivial idempotent.*

Proof. Suppose $z \in R \otimes_k S$ is nilpotent. We may write $z = \sum_{i=1,\ldots,n} x_i \otimes y_i$. Thus we may take R' the k-subalgebra generated by the x_i and S' the k-subalgebra generated by the y_i. The second and third statements are proved in the same way. □

Lemma 10.40.5. *Let k be a field. Let S be a geometrically reduced k-algebra. Let R be any reduced k-algebra. Then $R \otimes_k S$ is reduced.*

Proof. By Lemma 10.40.4 we may assume that R is of finite type over k. Then R, as a reduced Noetherian ring, embeds into a finite product of fields (see Lemmas 10.23.2, 10.29.6, and 10.24.4). Hence we may assume R is a finite product of fields. In this case the reducedness follows from the definition. □

Lemma 10.40.6. *Let k be a field. Let S be a reduced k-algebra. Let $k \subset K$ be either a separable field extension, or a separably generated field extension. Then $K \otimes_k S$ is reduced.*

Proof. Assume $k \subset K$ is separable. By Lemma 10.40.4 we may assume that S is of finite type over k and K is finitely generated over k. Then S embeds into a finite product of fields, namely its total ring of fractions (see Lemmas 10.24.4 and 10.23.2). Hence we may actually assume that S is a domain. We choose $x_1, \ldots, x_{r+1} \in K$ as

in Lemma 10.39.3. Let $P \in k(x_1, \ldots, x_r)[T]$ be the minimal polynomial of x_{r+1}. It is a separable polynomial. It is easy to see that $k[x_1, \ldots, x_r] \otimes_k S = S[x_1, \ldots, x_r]$ is a domain. This implies $k(x_1, \ldots, x_r) \otimes_k S$ is a domain as it is a localization of $S[x_1, \ldots, x_r]$. The ring extension $k(x_1, \ldots, x_r) \otimes_k S \subset K \otimes_k S$ is generated by a single element x_{r+1} with a single equation, namely P. Hence $K \otimes_k S$ embeds into $f.f.(k(x_1, \ldots, x_n) \otimes_k S)[T]/(P)$. Since P is separable this is a finite product of fields and we win.

At this point we do not yet know that a separably generated field extension is separable, so we have to prove the lemma in this case also. To do this suppose that $\{x_i\}_{i \in I}$ is a separating transcendence basis for K over k. For any finite set of elements $\lambda_j \in K$ there exists a finite subset $T \subset I$ such that $k(\{x_i\}_{i \in T}) \subset k(\{x_i\}_{i \in T} \cup \{\lambda_j\})$ is finite separable. Hence we see that K is a directed colimit of finitely generated and separably generated extensions of k. Thus the argument of the preceding paragraph applies to this case as well. □

Lemma 10.40.7. *Let k be a field and let S be a k-algebra. Assume that S is reduced and that $S_\mathfrak{p}$ is geometrically reduced for every minimal prime $\mathfrak{p}$ of S. Then S is geometrically reduced.*

Proof. Since S is reduced the map $S \to \prod_{\mathfrak{p} \text{ minimal}} S_\mathfrak{p}$ is injective, see Lemma 10.24.7. If $k \subset K$ is a field extension, then the maps

$$S \otimes_k K \to (\prod S_\mathfrak{p}) \otimes_k K \to \prod S_\mathfrak{p} \otimes_k K$$

are injective: the first as $k \to K$ is flat and the second by inspection because K is a free k-module. As $S_\mathfrak{p}$ is geometrically reduced the ring on the right is reduced. Thus we see that $S \otimes_k K$ is reduced as a subring of a reduced ring. □

10.41. Separable extensions, continued

In this section we continue the discussion started in Section 10.39. Let p be a prime number and let k be a field of characteristic p. In this case we write $k^{1/p}$ for the extension of k gotten by adjoining pth roots of all the elements of k to k. (In other words it is the subfield of an algebraic closure of k generated by the pth roots of elements of k.)

Lemma 10.41.1. *Let k be a field of characteristic $p > 0$. Let $k \subset K$ be a field extension. The following are equivalent:*

1. *K is separable over k,*
2. *the ring $K \otimes_k k^{1/p}$ is reduced, and*
3. *K is geometrically reduced over k.*

Proof. The implication (1) ⇒ (3) follows from Lemma 10.40.6. The implication (3) ⇒ (2) is immediate.

Assume (2). Let $k \subset L \subset K$ be a subextension such that L is a finitely generated field extension of k. We have to show that we can find a separating transcendence basis of L. The assumption implies that $L \otimes_k k^{1/p}$ is reduced. Let $x_1, \ldots, x_r$ be a transcendence basis of L over k such that the degree of inseparability of the finite extension $k(x_1, \ldots, x_r) \subset L$ is minimal. If L is separable over $k(x_1, \ldots, x_r)$ then we win. Assume this is not the case to get a contradiction. Then there exists an element $\alpha \in L$ which is not separable over $k(x_1, \ldots, x_r)$. Let $P(T) \in k(x_1, \ldots, x_r)[T]$ be

the minimal polynomial of α over $k(x_1, \ldots, x_r)$. After replacing α by $f\alpha$ for some nonzero $f \in k[x_1, \ldots, x_r]$ we may and do assume that P lies in $k[x_1, \ldots, x_r, T]$. Because α is not separable P is a polynomial in T^p, see Fields, Lemma 9.12.1. Let dp be the degree of P as a polynomial in T. Since P is the minimal polynomial of α the monomials

$$x_1^{e_1} \ldots x_r^{e_r} \alpha^e$$

for $e < dp$ are linearly independent over k in L. We claim that $\mathrm{d}P/\mathrm{d}x_i$ is not identically zero for at least one i. Namely, if this was not the case, then P is actually a polynomial in $x_1^p, \ldots, x_r^p, T^p$. In that case we can consider $P^{1/p} \in k^{1/p}[x_1, \ldots, x_r, T]$. This would map to $P^{1/p}(x_1, \ldots, x_r, \alpha)$ which is a nilpotent element of $k^{1/p} \otimes_k L$ and hence zero. On the other hand, $P^{1/p}(x_1, \ldots, x_r, \alpha)$ is a $k^{1/p}$-linear combination the monomials listed above, hence nonzero in $k^{1/p} \otimes_k L$. This is a contradiction which proves our claim. Thus, after renumbering, we may assume that $\mathrm{d}P/\mathrm{d}x_1$ is not zero. Then we see that x_1 is separably algebraic over $k(x_2, \ldots, x_r, \alpha)$, and that $x_2, \ldots, x_r, \alpha$ is a transcendence basis of L over k. This means that the degree of inseparability of the finite extension $k(x_2, \ldots, x_r, \alpha) \subset L$ is less than the degree of inseparability of the finite extension $k(x_1, \ldots, x_r) \subset L$, which is a contradiction. □

Lemma 10.41.2. *A separably generated field extension is separable.*

Proof. Combine Lemma 10.40.6 with Lemma 10.41.1. □

In the following lemma we will use the notion of the perfect closure which is defined in Definition 10.42.5.

Lemma 10.41.3. *Let k be a field. Let S be a k-algebra. The following are equivalent:*

1. *$k' \otimes_k S$ is reduced for every finite purely inseparable extension k' of k,*
2. *$k^{1/p} \otimes_k S$ is reduced,*
3. *$k^{perf} \otimes_k S$ is reduced, where k^{perf} is the perfect closure of k,*
4. *$\overline{k} \otimes_k S$ is reduced, where $\overline{k}$ is the algebraic closure of k, and*
5. *S is geometrically reduced over k.*

Proof. Note that any finite purely inseparable extension $k \subset k'$ embeds in k^{perf}. Moreover, $k^{1/p}$ embeds into k^{perf} which embeds into $\overline{k}$. Thus it is clear that (5) $\Rightarrow$ (4) $\Rightarrow$ (3) $\Rightarrow$ (2) and that (3) $\Rightarrow$ (1).

We prove that (1) $\Rightarrow$ (5). Assume $k' \otimes_k S$ is reduced for every finite purely inseparable extension k' of k. Let $k \subset K$ be an extension of fields. We have to show that $K \otimes_k S$ is reduced. By Lemma 10.40.4 we reduce to the case where $k \subset K$ is a finitely generated field extension. Choose a diagram

$$\begin{array}{ccc} K & \longrightarrow & K' \\ \uparrow & & \uparrow \\ k & \longrightarrow & k' \end{array}$$

as in Lemma 10.39.4. By assumption $k' \otimes_k S$ is reduced. By Lemma 10.40.6 it follows that $K' \otimes_k S$ is reduced. Hence we conclude that $K \otimes_k S$ is reduced as desired.

Finally we prove that (2) $\Rightarrow$ (5). Assume $k^{1/p} \otimes_k S$ is reduced. Then S is reduced. Moreover, for each localization $S_{\mathfrak{p}}$ at a minimal prime $\mathfrak{p}$, the ring $k^{1/p} \otimes_k S_{\mathfrak{p}}$ is a localization of $k^{1/p} \otimes_k S$ hence is reduced. But $S_{\mathfrak{p}}$ is a field by Lemma 10.24.4, hence $S_{\mathfrak{p}}$ is geometrically reduced by Lemma 10.41.1. It follows from Lemma 10.40.7 that S is geometrically reduced. □

10.42. Perfect fields

Here is the definition.

Definition 10.42.1. Let k be a field. We say k is *perfect* if every field extension of k is separable over k.

Lemma 10.42.2. *A field k is perfect if and only if it is a field of characteristic 0 or a field of characteristic $p > 0$ such that every element has a pth root.*

Proof. The characteristic zero case is clear. Assume the characteristic of k is $p > 0$. If k is perfect, then all the field extensions where we adjoin a pth root of an element of k have to be trivial, hence every element of k has a pth root. Conversely if every element has a pth root, then $k = k^{1/p}$ and every field extension of k is separable by Lemma 10.41.1. □

Lemma 10.42.3. *Let $k \subset K$ be a finitely generated field extension. There exists a diagram*

$$\begin{array}{ccc} K & \longrightarrow & K' \\ \uparrow & & \uparrow \\ k & \longrightarrow & k' \end{array}$$

where $k \subset k'$, $K \subset K'$ are finite purely inseparable field extensions such that $k' \subset K'$ is a separable field extension. In this situation we can assume that $K' = k'K$ is the compositum, and also that $K' = (k' \otimes_k K)_{red}$.

Proof. By Lemma 10.39.4 we can find such a diagram with $k' \subset K'$ separably generated. By Lemma 10.41.2 this implies that K' is separable over k'. The compositum $k'K$ is a subextension of $k' \subset K'$ and hence $k' \subset k'K$ is separable by Lemma 10.39.2. The ring $(k' \otimes_k K)_{red}$ is a domain as for some $n \gg 0$ the map $x \mapsto x^{p^n}$ maps it into K. Hence it is a field by Lemma 10.33.17. Thus $(k' \otimes_k K)_{red} \to K'$ maps it isomorphically onto $k'K$. □

Lemma 10.42.4. *For every field k there exists a purely inseparable extension $k \subset k'$ such that k' is perfect. The field extension $k \subset k'$ is unique up to unique isomorphism.*

Proof. If the characteristic of k is zero, then $k' = k$ is the unique choice. Assume the characteristic of k is $p > 0$. For every $n > 0$ there exists a unique algebraic extension $k \subset k^{1/p^n}$ such that (a) every element $\lambda \in k$ has a p^nth root in k^{1/p^n} and (b) for every element $\mu \in k^{1/p^n}$ we have $\mu^{p^n} \in k$. Namely, consider the ring map $k \to k^{1/p^n} = k$, $x \mapsto x^{p^n}$. This is injective and satisfies (a) and (b). It is clear that $k^{1/p^n} \subset k^{1/p^{n+1}}$ as extensions of k via the map $y \mapsto y^p$. Then we can take $k' = \bigcup k^{1/p^n}$. Some details omitted. □

Definition 10.42.5. Let k be a field. The field extension $k \subset k'$ of Lemma 10.42.4 is called the *perfect closure* of k. Notation $k \subset k^{perf}$.

Note that if $k \subset k'$ is any algebraic purely inseparable extension, then $k' \subset k^{perf}$. Namely, $(k')^{perf}$ is isomorphic to k^{perf} by the uniqueness of Lemma 10.42.4.

Lemma 10.42.6. *Let k be a perfect field. Any reduced k algebra is geometrically reduced over k. Let R, S be k-algebras. Assume both R and S are reduced. Then the k-algebra $R \otimes_k S$ is reduced.*

Proof. The first statement follows from Lemma 10.41.3. For the second statement use the first statement and Lemma 10.40.5. □

10.43. Geometrically irreducible algebras

Let $k \subset k'$ be an algebraic purely inseparable field extension. Then for any k-algebra R the ring map $R \to k' \otimes_k R$ induces a homeomorphism of spectra. The reason for this is the slightly more general Lemma 10.43.2 below.

Lemma 10.43.1. *Let p be a prime number. Let $n, m > 0$ be two integers. There exists an integer a such that $(x+y)^{p^a}, p^a(x+y) \in \mathbf{Z}[x^{p^n}, p^n x, y^{p^m}, p^m y]$.*

Proof. This is clear for $p^a(x+y)$ as soon as $a \geq n, m$. In fact, pick $a \gg n, m$. Write

$$(x+y)^{p^a} = \sum\nolimits_{i,j \geq 0, i+j=p^a} \binom{p^a}{i,j} x^i y^j$$

The condition $(x+y)^{p^a} \in \mathbf{Z}[x^{p^n}, p^n x, y^{p^m}, p^m y]$ holds if

$$p^{nr+mr'} \mid \binom{p^a}{i,j}$$

where $i = qp^n + r$ with $r \in \{0, \ldots, p^n - 1\}$ and $j = q'p^m + r'$ with $r' \in \{0, \ldots, p^m - 1\}$. If $r = r' = 0$ then the divisibility holds. If $r \neq 0$, then we write

$$\binom{p^a}{i,j} = \frac{p^a}{i} \binom{p^a - 1}{i-1, j}$$

Since $r \neq 0$ the rational number p^a/i has p-adic valuation at least $a-(n-1)$ (because i is not divisible by p^n). Thus $\binom{p^a}{i,j}$ is divisible by p^{a-n+1} in this case. Similarly, we see that if $r' \neq 0$, then $\binom{p^a}{i,j}$ is divisible by p^{a-m+1}. Picking $a = np^n + mp^m + n + m$ will work. □

Lemma 10.43.2. *Let $\varphi : R \to S$ be a ring map. If*

1. *for any $x \in S$ there exists $n > 0$ such that x^n is in the image of φ, and*
2. *for any $x \in Ker(\varphi)$ there exists $n > 0$ such that $x^n = 0$,*

then φ induces a homeomorphism on spectra. Given a prime number p such that

(a) *S is generated as an R-algebra by elements x such that there exists an $n > 0$ with $x^{p^n} \in \varphi(R)$ and $p^n x \in \varphi(R)$, and*
(b) *the kernel of φ is generated by nilpotent elements,*

then (1) and (2) hold, and for any ring map $R \to R'$ the ring map $R' \to R' \otimes_R S$ also satisfies (a), (b), (1), and (2) and in particular induces a homeomorphism on spectra.

Proof. Assume (1) and (2). Let $\mathfrak{q}, \mathfrak{q}'$ be primes of S lying over the same prime ideal $\mathfrak{p}$ of R. Suppose $x \in S$ with $x \in \mathfrak{q}$, $x \notin \mathfrak{q}'$. Then $x^n \in \mathfrak{q}$ and $x^n \notin \mathfrak{q}'$ for all $n > 0$. If $x^n = \varphi(y)$ with $y \in R$ for some $n > 0$ then

$$x^n \in \mathfrak{q} \Rightarrow y \in \mathfrak{p} \Rightarrow x^n \in \mathfrak{q}'$$

which is a contradiction. Hence there does not exist an x as above and we conclude that $\mathfrak{q} = \mathfrak{q}'$, i.e., the map on spectra is injective. By assumption (2) the kernel $I = \mathrm{Ker}(\varphi)$ is contained in every prime, hence $\mathrm{Spec}(R) = \mathrm{Spec}(R/I)$ as topological spaces. As the induced map $R/I \to S$ is integral by assumption (1) Lemma 10.33.15 shows that $\mathrm{Spec}(S) \to \mathrm{Spec}(R/I)$ is surjective. Combining the above we see that $\mathrm{Spec}(S) \to \mathrm{Spec}(R)$ is bijective. If $x \in S$ is arbitrary, and we pick $y \in R$ such that $\varphi(y) = x^n$ for some $n > 0$, then we see that the open $D(x) \subset \mathrm{Spec}(S)$ corresponds to the open $D(y) \subset \mathrm{Spec}(R)$ via the bijection above. Hence we see that the map $\mathrm{Spec}(S) \to \mathrm{Spec}(R)$ is a homeomorphism.

Assume (a) and (b). Note that (b) is equivalent to (2). We claim that for any $x \in S$ there exists an integer $n > 0$ such that $x^{p^n}, p^n x \in \varphi(R)$. By assumption (a) it suffices to show that the set of elements with this property forms a R-sub algebra. Suppose $x, y \in S$ and $n, m > 0$ such that $x^{p^n}, y^{p^m}, p^n x, p^m y \in \varphi(R)$. Then $(xy)^{p^{n+m}}, p^{n+m} xy \in \varphi(R)$ and we see that xy satisfies the condition. The condition holds for $x + y$ by Lemma 10.43.1 and the claim is proved. In particular it follows from the first part of the lemma that $\mathrm{Spec}(S) \to \mathrm{Spec}(R)$ is a homeomorphism. In particular it is surjective which is a property preserved under any base change, see Lemma 10.28.3. Therefore for any $R \to R'$ the kernel of the ring map $R' \to R' \otimes_R S$ consists of nilpotent elements, see Lemma 10.28.6, in other words (b) holds for $R' \to R' \otimes_R S$. Finally, it is clear that (a) is preserved under base change which finishes the proof. □

Lemma 10.43.3. *Let $R \to S$ be a ring map. Assume*

(a) $\mathrm{Spec}(R)$ *is irreducible,*
(b) $R \to S$ *is flat,*
(c) $R \to S$ *is of finite presentation,*
(d) *the fibre rings $S \otimes_R \kappa(\mathfrak{p})$ have irreducible spectra for a dense collection of primes $\mathfrak{p}$ of R.*

Then $\mathrm{Spec}(S)$ *is irreducible. This is true more generally with (b) + (c) replaced by "the map* $\mathrm{Spec}(S) \to \mathrm{Spec}(R)$ *is open".*

Proof. The assumptions (b) and (c) imply that the map on spectra is open, see Proposition 10.38.8. Hence the lemma follows from Topology, Lemma 5.7.7. □

Lemma 10.43.4. *Let k be a separably algebraically closed field. Let R, S be k-algebras. If R, S have a unique minimal prime, so does $R \otimes_k S$.*

Proof. Let $k \subset \overline{k}$ be a perfect closure, see Definition 10.42.5. By assumption $\overline{k}$ is algebraically closed. The ring maps $R \to R \otimes_k \overline{k}$ and $S \to S \otimes_k \overline{k}$ and $R \otimes_k S \to (R \otimes_k S) \otimes_k \overline{k} = (R \otimes_k \overline{k}) \otimes_{\overline{k}} (S \otimes_k \overline{k})$ satisfy the assumptions of Lemma 10.43.2. Hence we may assume k is algebraically closed.

We may replace R and S by their reductions. Hence we may assume that R and S are domains. By Lemma 10.42.6 we see that $R \otimes_k S$ is reduced. Hence its spectrum is reducible if and only if it contains a nonzero zerodivisor. By Lemma 10.40.4 we reduce to the case where R and S are domains of finite type over k algebraically closed.

Note that the ring map $R \to R \otimes_k S$ is of finite presentation and flat. Moreover, for every maximal ideal $\mathfrak{m}$ of R we have $(R \otimes_k S) \otimes_R R/\mathfrak{m} \cong S$ because $k \cong R/\mathfrak{m}$ by the Hilbert Nullstellensatz Theorem 10.31.1. Moreover, the set of maximal ideals is

dense in the spectrum of R since $\mathrm{Spec}(R)$ is Jacobson, see Lemma 10.32.2. Hence we see that Lemma 10.43.3 applies to the ring map $R \to R \otimes_k S$ and we conclude that the spectrum of $R \otimes_k S$ is irreducible as desired. □

Lemma 10.43.5. *Let k be a field. Let R be a k-algebra. The following are equivalent*

(1) *for every field extension $k \subset k'$ the spectrum of $R \otimes_k k'$ is irreducible, and*
(2) *for every finite separable field extension $k \subset k'$ the spectrum of $R \otimes_k k'$ is irreducible.*

Proof. Let $k \subset k^{perf}$ be a perfect closure of k, see Definition 10.42.5. By Lemma 10.43.2 we may replace R by $(R \otimes_k k^{perf})_{reduction}$ and k by k^{perf} (some details omitted). Hence we may assume that R is geometrically reduced over k.

Assume R is geometrically reduced over k. For any extension of fields $k \subset k'$ we see irreducibility of the spectrum of $R \otimes_k k'$ is equivalent to $R \otimes_k k'$ being a domain. Assume (2). Let $k \subset \overline{k}$ be a separable algebraic closure of k. Using Lemma 10.40.4 we see that (2) is equivalent to $R \otimes_k \overline{k}$ being a domain. For any field extension $k \subset k'$, there exists a field extension $\overline{k} \subset \overline{k}'$ with $k' \subset \overline{k}'$. By Lemma 10.43.4 we see that $R \otimes_k \overline{k}'$ is a domain. If $R \otimes_k k'$ is not a domain, then also $R \otimes_k \overline{k}'$ is not a domain, contradiction. □

Definition 10.43.6. Let k be a field. Let S be a k-algebra. We say S is *geometrically irreducible over k* if for every field extension $k \subset k'$ the spectrum of $S \otimes_k k'$ is irreducible[2].

By Lemma 10.43.5 it suffices to check this for finite separable field extensions $k \subset k'$.

Lemma 10.43.7. *Let k be a field. Let R be a k-algebra. If k is separably algebraically closed then R is geometrically irreducible over k if and only if the spectrum of R is irreducible.*

Proof. Immediate from the remark following Definition 10.43.6. □

Lemma 10.43.8. *Let k be a field. Let S be a k-algebra.*

(1) *If S is geometrically irreducible over k so is every k-subalgebra.*
(2) *If all finitely generated k-subalgebras of S are geometrically irreducible, then S is geometrically irreducible.*
(3) *A directed colimit of geometrically irreducible k-algebras is geometrically irreducible.*

Proof. Let $S' \subset S$ be a subalgebra. Then for any extension $k \subset k'$ the ring map $S' \otimes_k k' \to S \otimes_k k'$ is injective also. Hence (1) follows from Lemma 10.28.5 (and the fact that the image of an irreducible space under a continuous map is irreducible). The second and third property follow from the fact that tensor product commutes with colimits. □

Lemma 10.43.9. *Let k be a field. Let S be a geometrically irreducible k-algebra. Let R be any k-algebra. The map*

$$\mathrm{Spec}(R \otimes_k S) \longrightarrow \mathrm{Spec}(R)$$

induces a bijection on irreducible components.

[2] An irreducible space is nonempty.

Proof. Recall that irreducible components correspond to minimal primes (Lemma 10.24.1). As $R \to R \otimes_k S$ is flat we see by going down (Lemma 10.36.17) that any minimal prime of $R \otimes_k S$ lies over a minimal prime of R. Conversely, if $\mathfrak{p} \subset R$ is a (minimal) prime then

$$R \otimes_k S/\mathfrak{p}(R \otimes_k S) = (R/\mathfrak{p}) \otimes_k S \subset f.f.(R/\mathfrak{p}) \otimes_k S$$

by flatness of $R \to R \otimes_k S$. The ring $f.f.(R/\mathfrak{p}) \otimes_k S$ has irreducible spectrum by assumption. It follows that $R \otimes_k S/\mathfrak{p}(R \otimes_k S)$ has a single minimal prime (Lemma 10.28.5). In other words, the inverse image of the irreducible set $V(\mathfrak{p})$ is irreducible. Hence the lemma follows. □

Let us make some remarks on the notion of geometrically irreducible field extensions.

Lemma 10.43.10. *Let $k \subset K$ be a field extension. If k is algebraically closed in K, then K is geometrically irreducible over k.*

Proof. Let $k \subset k'$ be a finite separable extension, say generated by $\alpha \in k'$ over k (see Fields, Lemma 9.18.1). Let $P = T^d + a_1 T^{d-1} + \ldots + a_d \in k[T]$ be the minimal polynomial of α. Then $K \otimes_k k' \cong K[T]/(P)$. The only way the spectrum of $K[T]/(P)$ can be reducible is if P is reducible in $K[T]$. Say $P = P_1 P_2$ is a nontrivial factorization of P into monic polynomials. Let $b_1, \ldots, b_t \in K$ be the coefficients of P_1. Then we see that b_i is algebraic over k by Lemma 10.35.5. Hence the lemma follows. □

Lemma 10.43.11. *Let $k \subset K$ be a field extension. Consider the subextension $k \subset k' \subset K$ such that $k \subset k'$ is separable algebraic and $k' \subset K$ maximal with this property. Then K is geometrically irreducible over k'. If K/k is a finitely generated field extension, then $[k' : k] < \infty$.*

Proof. Let $k'' \subset K$ be the algebraic closure of k in K. By Lemma 10.43.10 we see that K is geometrically irreducible over k''. Since $k' \subset k''$ is purely inseparable we see from Lemma 10.43.2 that also the extension $k' \subset K$ is geometrically irreducible (some details omitted). If $k \subset K$ is finitely generated, then k' is finite over k by Fields, Lemma 9.23.10. □

Lemma 10.43.12. *Let $k \subset K$ be an extension of fields. Let $k \subset \overline{k}$ be a separable algebraic closure. Then $Gal(\overline{k}/k)$ acts transitively on the primes of $\overline{k} \otimes_k K$.*

Proof. Let $k \subset k' \subset K$ be the subextension found in Lemma 10.43.11. Note that as $k \subset \overline{k}$ is integral all the prime ideals of $\overline{k} \otimes_k K$ and $\overline{k} \otimes_k k'$ are maximal, see Lemma 10.33.18. In particular the residue field of any prime ideal of $\overline{k} \otimes_k k'$ is isomorphic to $\overline{k}$. Hence the prime ideals of $\overline{k} \otimes_k k'$ correspond one to one to elements of $\mathrm{Hom}_k(k', \overline{k})$ with $\sigma \in \mathrm{Hom}_k(k', \overline{k})$ corresponding to the kernel $\mathfrak{p}_\sigma$ of $1 \otimes \sigma : \overline{k} \otimes_k k' \to \overline{k}$. In particular $\mathrm{Gal}(\overline{k}/k)$ acts transitively on this set. Finally, since K is geometrically irreducible over k' we see that there is a unique prime of $\overline{k} \otimes_k K$ lying over each $\mathfrak{p}_\sigma$ since the set of these primes is the set of primes in the ring

$$(\overline{k} \otimes_k K) \otimes_{(\overline{k} \otimes_k k'), 1 \otimes \sigma} \kappa(\mathfrak{p}_\sigma) = \overline{k} \otimes_{\overline{k}} (K \otimes_{k'} \overline{k}) = K \otimes_{k', \sigma} \overline{k}$$

Thus the lemma holds. □

10.44. Geometrically connected algebras

Lemma 10.44.1. *Let k be a separably algebraically closed field. Let R, S be k-algebras. If $\mathrm{Spec}(R)$, and $\mathrm{Spec}(S)$ are connected, then so is $\mathrm{Spec}(R \otimes_k S)$.*

Proof. Recall that $\mathrm{Spec}(R)$ is connected if and only if R has no nontrivial idempotents, see Lemma 10.19.4. Hence, by Lemma 10.40.4 we may assume R and S are of finite type over k. In this case R and S are Noetherian, and have finitely many minimal primes, see Lemma 10.29.6. Thus we may argue by induction on $n + m$ where n, resp. m is the number of irreducible components of $\mathrm{Spec}(R)$, resp. $\mathrm{Spec}(S)$. Of course the case where either n or m is zero is trivial. If $n = m = 1$, i.e., $\mathrm{Spec}(R)$ and $\mathrm{Spec}(S)$ both have one irreducible component, then the result holds by Lemma 10.43.4. Suppose that $n > 1$. Let $\mathfrak{p} \subset R$ be a minimal prime corresponding to the irreducible closed subset $T \subset \mathrm{Spec}(R)$. Let $I \subset R$ be such that $T' = V(I) \subset \mathrm{Spec}(R)$ is the closure of the complement of T. Note that this means that $T' = \mathrm{Spec}(R/I)$ (Lemma 10.16.7) has $n - 1$ irreducible components. Then $T \cup T' = \mathrm{Spec}(R)$, and $T \cap T' = V(\mathfrak{p} + I) = \mathrm{Spec}(R/(\mathfrak{p} + I))$ is not empty as $\mathrm{Spec}(R)$ is assumed connected. The inverse image of T in $\mathrm{Spec}(R \otimes_k S)$ is $\mathrm{Spec}(R/\mathfrak{p} \otimes_k S)$, and the inverse of T' in $\mathrm{Spec}(R \otimes_k S)$ is $\mathrm{Spec}(R/I \otimes_k S)$. By induction these are both connected. The inverse image of $T \cap T'$ is $\mathrm{Spec}(R/(\mathfrak{p} + I) \otimes_k S)$ which is nonempty. Hence $\mathrm{Spec}(R \otimes_k S)$ is connected. □

Lemma 10.44.2. *Let k be a field. Let R be a k-algebra. The following are equivalent*

1. *for every field extension $k \subset k'$ the spectrum of $R \otimes_k k'$ is connected, and*
2. *for every finite separable field extension $k \subset k'$ the spectrum of $R \otimes_k k'$ is connected.*

Proof. For any extension of fields $k \subset k'$ the connectivity of the spectrum of $R \otimes_k k'$ is equivalent to $R \otimes_k k'$ having no nontrivial idempotents, see Lemma 10.19.4. Assume (2). Let $k \subset \overline{k}$ be a separable algebraic closure of k. Using Lemma 10.40.4 we see that (2) is equivalent to $R \otimes_k \overline{k}$ having no nontrivial idempotents. For any field extension $k \subset k'$, there exists a field extension $\overline{k} \subset \overline{k}'$ with $k' \subset \overline{k}'$. By Lemma 10.44.1 we see that $R \otimes_k \overline{k}'$ has no nontrivial idempotents. If $R \otimes_k k'$ has a nontrivial idempotent, then also $R \otimes_k \overline{k}'$, contradiction. □

Definition 10.44.3. Let k be a field. Let S be a k-algebra. We say S is *geometrically connected over k* if for every field extension $k \subset k'$ the spectrum of $R \otimes_k k'$ is connected.

By Lemma 10.44.2 it suffices to check this for finite separable field extensions $k \subset k'$.

Lemma 10.44.4. *Let k be a field. Let R be a k-algebra. If k is separably algebraically closed then R is geometrically connected over k if and only if the spectrum of R is connected.*

Proof. Immediate from the remark following Definition 10.44.3. □

Lemma 10.44.5. *Let k be a field. Let S be a k-algebra.*

1. *If S is geometrically connected over k so is every k-subalgebra.*
2. *If all finitely generated k-subalgebras of S are geometrically connected, then S is geometrically connected.*

(3) *A directed colimit of geometrically irreducible k-algebras is geometrically connected.*

Proof. This follows from the characterization of connectedness in terms of the nonexistence of nontrivial idempotents. The second and third property follow from the fact that tensor product commutes with colimits. □

The following lemma will be superseded by the more general Varieties, Lemma 32.5.4.

Lemma 10.44.6. *Let k be a field. Let S be a geometrically connected k-algebra. Let R be any k-algebra. The map*

$$R \longrightarrow R \otimes_k S$$

induces a bijection on idempotents, and the map

$$\mathrm{Spec}(R \otimes_k S) \longrightarrow \mathrm{Spec}(R)$$

induces a bijection on connected components.

Proof. The second assertion follows from the first combined with Lemma 10.20.2. By Lemmas 10.44.5 and 10.40.4 we may assume that R and S are of finite type over k. Then we see that also $R \otimes_k S$ is of finite type over k. Note that in this case all the rings are Noetherian and hence their spectra have finitely many connected components (since they have finitely many irreducible components, see Lemma 10.29.6). In particular, all connected components in question are open! Hence via Lemma 10.21.3 we see that the first statement of the lemma in this case is equivalent to the second. Let's prove this. As the algebra S is geometrically connected and nonzero we see that all fibres of $X = \mathrm{Spec}(R \otimes_k S) \to \mathrm{Spec}(R) = Y$ are connected and nonempty. Also, as $R \to R \otimes_k S$ is flat of finite presentation the map $X \to Y$ is open (Proposition 10.38.8). Topology, Lemma 5.6.5 shows that $X \to Y$ induces bijection on connected components. □

10.45. Geometrically integral algebras

Definition 10.45.1. Let k be a field. Let S be a k-algebra. We say S is *geometrically integral over k* if for every field extension $k \subset k'$ the ring of $S \otimes_k k'$ is a domain.

Any question about geometrically integral algebras can be translated in a question about geometrically reduced and irreducible algebras.

Lemma 10.45.2. *Let k be a field. Let S be a k-algebra. In this case S is geometrically integral over k if and only if S is geometrically irreducible as well as geometrically reduced over k.*

Proof. Omitted. □

Lemma 10.45.3. *Let k be a field. Let S be a geometrically integral k-algebra. Let R be a k-algebra and an integral domain. Then $R \otimes_k S$ is an integral domain.*

Proof. By Lemma 10.40.5 the ring $R \otimes_k S$ is reduced and by Lemma 10.43.9 the ring $R \otimes_k S$ is irreducible (the spectrum has just one irreducible component), so $R \otimes_k S$ is an integral domain. □

10.46. Valuation rings

Here are some definitions.

Definition 10.46.1. Valuation rings.

(1) Let K be a field. Let A, B be local rings contained in K. We say that B *dominates* A if $A \subset B$ and $\mathfrak{m}_A = A \cap \mathfrak{m}_B$.
(2) Let A be a ring. We say A is a *valuation ring* if A is a local domain, not a field, and if A is maximal for the relation of domination among local rings contained in the fraction field of A.
(3) Let A be a valuation ring with fraction field K. If $R \subset K$ is a subring of K, then we say A is *centered* on R if $R \subset A$.

Lemma 10.46.2. *Let K be a field. Let $A \subset K$ be a local subring which is not a field. Then there exists a valuation ring with fraction field K dominating A.*

Proof. During this proof, and during this proof only, the phrase "local ring" will mean "local ring, not a field". We consider the collection of local subrings of K as a partially ordered set using the relation of domination. Suppose that $\{A_i\}_{i \in I}$ is a totally ordered collection of local subrings of K. Then $B = \bigcup A_i$ is a local subring which dominates all of the A_i. Hence by Zorn's Lemma, it suffices to show that if $A \subset K$ is a local ring whose fraction field is not K, then there exists a local ring $B \subset K$, $B \neq A$ dominating A.

Pick $t \in K$ which is not in the fraction field of A. If t is transcendental over A, then $A[t] \subset K$ and hence $A[t]_{(t,\mathfrak{m})} \subset K$ is a local ring dominating A. Suppose t is algebraic over A. Then for some $a \in A$ the element at is integral over A. In this case the subring $A' \subset K$ generated by A and ta is finite over A. By Lemma 10.33.15 there exists a prime ideal $\mathfrak{m}' \subset A'$ lying over $\mathfrak{m}$. Then $A'_{\mathfrak{m}'}$ clearly dominates A and we win. □

Lemma 10.46.3. *Let A be a valuation ring with maximal ideal $\mathfrak{m}$ and fraction field K. Let $x \in K$. Then either $x \in A$ or $x^{-1} \in A$ or both.*

Proof. Assume that x is not in A. Let A' denote the subring of K generated by A and x. Since A is a valuation ring we see that there is no prime of A' lying over $\mathfrak{m}$. Hence we can write $1 = \sum_{i=0}^{d} t_i x^i$ with $t_i \in \mathfrak{m}$. This implies that $(1 - t_0)(x^{-1})^d - \sum t_i (x^{-1})^{d-i} = 0$. In particular we see that x^{-1} is integral over A. Thus the subring A'' of K generated by A and x^{-1} is finite over A and we see there exists a prime ideal $\mathfrak{m}'' \subset A''$ lying over $\mathfrak{m}$ by Lemma 10.33.15. Since A is a valuation ring we conclude that $A = (A'')_{\mathfrak{m}''}$ and hence $x^{-1} \in A$. □

Lemma 10.46.4. *Let $A \subset K$ be a subring of a field K such that for all $x \in K$ either $x \in A$ or $x^{-1} \in A$ or both. Then either $A = K$ or A is a valuation ring with fraction field K.*

Proof. If A is not K, then A is not a field and there is a nonzero maximal ideal $\mathfrak{m}$. If $\mathfrak{m}'$ is a second maximal ideal, then choose $x, y \in A$ with $x \in \mathfrak{m}$, $y \notin \mathfrak{m}$, $x \notin \mathfrak{m}'$, and $y \in \mathfrak{m}'$ (see Lemma 10.14.3). Then neither $x/y \in A$ nor $y/x \in A$ contradicting the assumption of the lemma. Thus we see that A is a local ring. Suppose that A' is a local ring contained in K which dominates A. Let $x \in A'$. We have to show that $x \in A$. If not, then $x^{-1} \in A$, and of course $x^{-1} \in \mathfrak{m}_A$. But then $x^{-1} \in \mathfrak{m}_{A'}$ which contradicts $x \in A'$. □

Lemma 10.46.5. *Let $K \subset L$ be an extension of fields. If $B \subset L$ is a valuation ring, then $A = K \cap B$ is either a field or a valuation ring.*

Proof. Omitted. Hint: Combine Lemmas 10.46.3 and 10.46.4. □

Lemma 10.46.6. *Let A be a valuation ring. For any prime ideal $\mathfrak{p} \subset A$ the quotient $A/\mathfrak{p}$ is either a field or a valuation ring. The same is true for the localization $A_\mathfrak{p}$.*

Proof. Follows immediately from the characterization of valuation rings given in Lemma 10.46.5. □

Lemma 10.46.7. *Let A' be a valuation ring with residue field K. Let A be a valuation ring with fraction field K. Then $C = \{\lambda \in A' \mid \lambda \bmod \mathfrak{m}_{A'} \in A\}$ is a valuation ring.*

Proof. Note that $\mathfrak{m}_{A'} \subset C$ and $C/\mathfrak{m}_{A'} = A$. In particular, the fraction field of C is equal to the fraction field of A'. We will use the criterion of Lemma 10.46.4 to prove the lemma. Let x be an element of the fraction field of C. By the lemma we may assume $x \in A'$. If $x \in \mathfrak{m}_{A'}$, then we see $x \in C$. If not, then x is a unit of A' and we also have $x^{-1} \in A'$. Hence either x or x^{-1} maps to an element of A by the lemma again. □

Lemma 10.46.8. *Let A be a valuation ring. Then A is a normal domain.*

Proof. Suppose x is in the field of fractions of A and integral over A, say $x^d + \sum_{i<d} a_i x^i = 0$. By Lemma 10.46.4 either $x \in A$ (and we're done) or $x^{-1} \in A$. In the second case we see that $x = -\sum a_i x^{i-d} \in A$ as well. □

Lemma 10.46.9. *Let A be a normal domain with fraction field K. For every $x \in K$, $x \notin A$ there exists a valuation ring $A \subset V \subset K$ with fraction field K such that $x \notin V$. In other words, A is the intersection of all valuation rings in K containing A.*

Proof. Suppose $x \in K$, $x \notin A$. Consider $B = A[x^{-1}]$. Then $x \notin B$. Namely, if $x = a_0 + a_1 x^{-1} + \ldots + a_d x^{-d}$ then $x^{d+1} - a_0 x^d - \ldots - a_d = 0$ and x is integral over A in contradiction with the fact that A is normal. Thus x^{-1} is not a unit in B. Thus $V(x^{-1}) \subset \mathrm{Spec}(B)$ is not empty (Lemma 10.16.2), and we can choose a prime $\mathfrak{p} \subset B$ with $x^{-1} \in \mathfrak{p}$. Choose a valuation ring $V \subset K$ dominating $B_\mathfrak{p}$ (Lemma 10.46.2). Then $x \notin V$ as $x^{-1} \in \mathfrak{m}_V$. □

An *totally ordered abelian group* is a pair $(\Gamma, \geq)$ consisting of an abelian group Γ endowed with a total ordering $\geq$ such that $\gamma \geq \gamma' \Rightarrow \gamma + \gamma'' \geq \gamma' + \gamma''$ for all $\gamma, \gamma', \gamma'' \in \Gamma$.

Lemma 10.46.10. *Let A be a valuation ring with field of fractions K. Set $\Gamma = K^*/A^*$ (with group law written additively). For $\gamma, \gamma' \in \Gamma$ define $\gamma \geq \gamma'$ if and only if $\gamma - \gamma'$ is in the image of $A - \{0\} \to \Gamma$. Then $(\Gamma, \geq)$ is a totally ordered abelian group.*

Proof. Omitted, but follows easily from Lemma 10.46.3. □

Definition 10.46.11. Let A be a valuation ring.

(1) The totally ordered abelian group $(\Gamma, \geq)$ is called the *value group* of the valuation ring A.

(2) The map $v : A - \{0\} \to \Gamma$ and also $v : K^* \to \Gamma$ is called the *valuation* associated to A.
(3) The valuation ring A is called a *discrete valuation ring* if $\Gamma \cong \mathbf{Z}$.

Note that if $\Gamma \cong \mathbf{Z}$ then there is a unique such isomorphism such that $1 \geq 0$. If the isomorphism is chosen in this way, then the ordering becomes the usual ordering of the integers.

Lemma 10.46.12. *Let A be a valuation ring. The valuation $v : A - \{0\} \to \Gamma_{\geq 0}$ has the following properties:*

(1) $v(a) = 0 \Leftrightarrow a \in A^*$,
(2) $v(ab) = v(a) + v(b)$,
(3) $v(a + b) \geq min(v(a), v(b))$.

Proof. Omitted. □

Lemma 10.46.13. *Let A be a ring. The following are equivalent*

(1) *A is a valuation ring,*
(2) *A is a local domain and every finitely generated ideal of A is principal.*

Proof. Assume A is a valuation ring and let $f_1, \ldots, f_n \in A$. Choose i such that $v(f_i)$ is minimal among $v(f_j)$. Then $(f_i) = (f_1, \ldots, f_n)$. Conversely, assume A is a local domain and every finitely generated ideal of A is principal. Pick $f, g \in A$ and write $(f, g) = (h)$. Then $f = ah$ and $g = bh$ and $h = cf + dg$ for some $a, b, c, d \in A$. Thus $ac + bd = 1$ and we see that either a or b is a unit, i.e., either g/f or f/g is an element of A. This shows A is a valuation ring by Lemma 10.46.4. □

Lemma 10.46.14. *Let $(\Gamma, \geq)$ be a totally ordered abelian group. Let K be a field. Let $v : K^* \to \Gamma$ be a homomorphism of abelian groups. Then*

$$A = \{x \in K \mid x = 0 \text{ or } v(x) \geq 0\}$$

is a valuation ring with value group $Im(v) \subset \Gamma$, with maximal ideal

$$\mathfrak{m} = \{x \in K \mid x = 0 \text{ or } v(x) > 0\}$$

and with group of units

$$A^* = \{x \in K^* \mid v(x) = 0\}.$$

Proof. Omitted. □

Let $(\Gamma, \geq)$ be a totally ordered abelian group. An *ideal of* Γ is a subset $I \subset \Gamma$ such that all elements of I are ≥ 0 and $\gamma \in I$, $\gamma' \geq \gamma$ implies $\gamma' \in I$. We say that such an ideal is *prime* if $\gamma + \gamma' \in I, \gamma, \gamma' \geq 0 \Rightarrow \gamma \in I$ or $\gamma' \in I$.

Lemma 10.46.15. *Let A be a valuation ring. Ideals in A correspond $1 - 1$ with ideals of Γ. This bijection is inclusion preserving, and maps prime ideals to prime ideals.*

Proof. Omitted. □

Lemma 10.46.16. *A valuation ring is Noetherian if and only if it is a discrete valuation ring.*

Proof. Suppose A is a discrete valuation ring with valuation $v : A \setminus \{0\} \to \mathbf{Z}$ normalized so that $\text{Im}(v) \subset \mathbf{Z}_{\geq 0}$. By Lemma 10.46.15 the ideals of A are the subsets $I_n = \{0\} \cup v^{-1}(\mathbf{Z}_{\geq n})$. It is clear that any element $x \in A$ with $v(x) = n$ generates I_n. Hence A is a PID so certainly Noetherian.

Suppose A is a Noetherian valuation ring with value group Γ. By Lemma 10.46.15 we see the ascending chain condition holds for ideals in Γ. In particular, by considering the subsets $\gamma + \Gamma_{\geq 0}$ with $\gamma > 0$ we see there exists a smallest element γ_0 which is bigger than 0. Let $\gamma \in \Gamma$ be an element $\gamma > 0$. Consider the sequence of elements γ, $\gamma - \gamma_0$, $\gamma - 2\gamma_0$, etc. By the ascending chain condition these cannot all be > 0. Let $\gamma - n\gamma_0$ be the last one ≥ 0. By minimality of γ_0 we see that $0 = \gamma - n\gamma_0$. Hence Γ is a cyclic group as desired. □

10.47. More Noetherian rings

Lemma 10.47.1. *Let R be a Noetherian ring. Any finite R-module is of finite presentation. Any submodule of a finite R-module is finite. The ascending chain condition holds for R-submodules of a finite R-module.*

Proof. We first show that any submodule N of a finite R-module M is finite. We do this by induction on the number of generators of M. If this number is 1, then $N = J/I \subset M = R/I$ for some ideals $I \subset J \subset R$. Thus the definition of Noetherian implies the result. If the number of generators of M is greater than 1, then we can find a short exact sequence $0 \to M' \to M \to M'' \to 0$ where M' and M'' have fewer generators. Note that setting $N' = M' \cap N$ and $N'' = \text{Im}(N \to M'')$ gives a similar short exact sequence for N. Hence the result follows from the induction hypothesis since the number of generators of N is at most the number of generators of N' plus the number of generators of N''.

To show that M is finitely presented just apply the previous result to the kernel of a presentation $R^n \to M$.

It is well known and easy to prove that the ascending chain condition for R-submodules of M is equivalent to the condition that every submodule of M is a finite R-module. We omit the proof. □

Definition 10.47.2. Let R be a ring. Let $I \subset R$ be an ideal. We say I is *locally nilpotent* if for every $x \in I$ there exists an $n \in \mathbf{N}$ such that $x^n = 0$. We say I is *nilpotent* if there exists an $n \in \mathbf{N}$ such that $I^n = 0$.

Lemma 10.47.3. *Let R be a Noetherian ring. Let I, J be ideals of R. Suppose $J \subset \sqrt{I}$. Then $J^n \subset I$ for some n. In particular, in a Noetherian ring the notions of "locally nilpotent ideal" and "nilpotent ideal" coincide.*

Proof. Say $J = (f_1, \ldots, f_s)$. By assumption $f_i^{d_i} \in I$. Take $n = d_1 + d_2 + \ldots + d_s + 1$. □

Lemma 10.47.4 (Artin-Rees)**.** *Suppose that R is Noetherian, $I \subset R$ an ideal. Let $N \subset M$ be finite R-modules. There exists a constant $c > 0$ such that $I^n M \cap N = I^{n-c}(I^c M \cap N)$.*

Proof. Consider the ring $S = R \oplus I \oplus I^2 \oplus \ldots = \bigoplus_{n \geq 0} I^n$. Convention: $I^0 = R$. Multiplication maps $I^n \times I^m$ into I^{n+m} by multiplication in R. Note that if $I = (f_1, \ldots, f_t)$ then S is a quotient of the Noetherian ring $R[X_1, \ldots, X_t]$. The

map just sends the monomial $X_1^{e_1} \ldots X_t^{e_t}$ to $f_1^{e_1} \ldots f_t^{e_t}$. Thus S is Noetherian. Similarly, consider the module $M \oplus IM \oplus I^2M \oplus \ldots = \bigoplus_{n \geq 0} I^n M$. This is a finitely generated S-module. Namely, if $x_1, \ldots, x_r$ generate M over R, then they also generate $\bigoplus_{n \geq 0} I^n M$ over S. Next, consider the submodule $\bigoplus_{n \geq 0} I^n M \cap N$. This is an S-submodule, as is easily verified. By Lemma 10.47.1 it is finitely generated as an S-module, say by $\xi_j \in \bigoplus_{n \geq 0} I^n M \cap N$, $j = 1, \ldots, s$. We may assume by decomposing each ξ_j into its homogeneous pieces that each $\xi_j \in I^{d_j} M \cap N$ for some d_j. Set $c = \max\{d_j\}$. Then for all $n \geq c$ every element in $I^n M \cap N$ is of the form $\sum h_j \xi_j$ with $h_j \in I^{n-d_j}$. The lemma now follows from this and the trivial observation that $I^{n-d_j}(I^{d_j} M \cap N) \subset I^{n-c}(I^c M \cap N)$. □

Lemma 10.47.5. *Suppose that $0 \to K \to M \xrightarrow{f} N$ is an exact sequence of finitely generated modules over a Noetherian ring R. Let $I \subset R$ be an ideal. Then there exists a c such that*

$$f^{-1}(I^n N) = K + I^{n-c} f^{-1}(I^c N) \quad \text{and} \quad f(M) \cap I^n N \subset f(I^{n-c} M)$$

for all $n \geq c$.

Proof. Apply Lemma 10.47.4 to $\mathrm{Im}(f) \subset N$ and note that $f : I^{n-c} M \to I^{n-c} f(M)$ is surjective. □

Lemma 10.47.6. *Let R be a Noetherian local ring. Let $I \subset R$ be a proper ideal. Let M be a finite R-module. Then $\bigcap_{n \geq 0} I^n M = 0$.*

Proof. Let $N = \bigcap_{n \geq 0} I^n M$. Then $N = I^n M \cap N$ for all $n \geq 0$. By the Artin-Rees Lemma 10.47.4 we see that $N = I^n M \cap N \subset IN$ for some suitably large n. By Nakayama's Lemma 10.18.1 we see that $N = 0$. □

Lemma 10.47.7. *Let R be a Noetherian ring. Let $I \subset R$ be an ideal. Let M be a finite R-module. Let $N = \bigcap_n I^n M$.*

1. *For every prime $\mathfrak{p}$, $I \subset \mathfrak{p}$ there exists a $f \in R$, $f \notin \mathfrak{p}$ such that $N_f = 0$.*
2. *If $I \subset rad(R)$ is contained in the Jacobson radical of R, then $N = 0$.*

Proof. Proof of (1). Let $x_1, \ldots, x_n$ be generators for the module N, see Lemma 10.47.1. For every prime $\mathfrak{p}$, $I \subset \mathfrak{p}$ we see that the image of N in the localization $M_\mathfrak{p}$ is zero, by Lemma 10.47.6. Hence we can find $g_i \in R$, $g_i \notin \mathfrak{p}$ such that x_i maps to zero in N_{g_i}. Thus $N_{g_1 g_2 \ldots g_n} = 0$.

Part (2) follows from (1) and Lemma 10.22.1. □

Remark 10.47.8. Lemma 10.47.6 in particular implies that $\bigcap_n I^n = (0)$ when $I \subset R$ is a non-unit ideal in a Noetherian local ring R. More generally, let R be a Noetherian ring and $I \subset R$ an ideal. Suppose that $f \in \bigcap_{n \in \mathbf{N}} I^n$. Then Lemma 10.47.7 says that for every prime ideal $I \subset \mathfrak{p}$ there exists a $g \in R$, $g \notin \mathfrak{p}$ such that f maps to zero in R_g. In algebraic geometry we express this by saying that "f is zero in an open neighbourhood of the closed set $V(I)$ of $\mathrm{Spec}(R)$".

Lemma 10.47.9 (Artin-Tate)**.** *Let R be a Noetherian ring. Let S be a finitely generated R-algebra. If $T \subset S$ is an R-subalgebra such that S is finitely generated as a T-module, then T is a finite type over R.*

Proof. Choose elements $x_1, \ldots, x_n \in S$ which generate S as an R-algebra. Choose $y_1, \ldots, y_m$ in S which generate S as a T-module. Thus there exist $a_{ij} \in T$ such that $x_i = \sum a_{ij} y_j$. There also exist $b_{ijk} \in T$ such that $y_i y_j = \sum b_{ijk} y_k$. Let $T' \subset T$ be the sub R-algebra generated by a_{ij} and b_{ijk}. This is a finitely generated R-algebra, hence Noetherian. Consider the algebra

$$S' = T'[Y_1, \ldots, Y_m]/(Y_i Y_j - \sum b_{ijk} Y_k).$$

Note that S' is finite over T', namely as a T'-module it is generated by the classes of $1, Y_1, \ldots, Y_m$. Consider the T'-algebra homomorphism $S' \to S$ which maps Y_i to y_i. Because $a_{ij} \in T'$ we see that x_j is in the image of this map. Thus $S' \to S$ is surjective. Therefore S is finite over T' as well. Since T' is Noetherian and we conclude that $T \subset S$ is finite over T' and we win. □

10.48. Length

Definition 10.48.1. Let R be a ring. For any R-module M we define the *length* of M over R by the formula

$$\text{length}_R(M) = \sup\{n \mid \exists\ 0 = M_0 \subset M_1 \subset \ldots \subset M_n = M,\ M_i \neq M_{i+1}\}.$$

In other words it is the supremum of the lengths of chains of submodules. There is an obvious notion of when a chain of submodules is a refinement of another. This gives a partial ordering on the collection of all chains of submodules, with the smallest chain having the shape $0 = M_0 \subset M_1 = M$ if M is not zero. We note the obvious fact that if the length of M is finite, then every chain can be refined to a maximal chain. But it is not as obvious that all maximal chains have the same length (as we will see later).

Lemma 10.48.2. *Let R be a ring. Let M be an R-module. If $\text{length}_R(M) < \infty$ then M is a finite R-module.*

Proof. Omitted. □

Lemma 10.48.3. *If $0 \to M' \to M \to M'' \to 0$ is a short exact sequence of modules over R then the length of M is the sum of the lengths of M' and M''.*

Proof. Given filtrations of M' and M'' of lengths n', n'' it is easy to make a corresponding filtration of M of length $n' + n''$. Thus we see that $\text{length}_R M \geq \text{length}_R M' + \text{length}_R M''$. Conversely, given a filtration $M_0 \subset M_1 \subset \ldots \subset M_n$ of M consider the induced filtrations $M_i' = M_i \cap M'$ and $M_i'' = \text{Im}(M_i \to M'')$. Let n' (resp. n'') be the number of steps in the filtration $\{M_i'\}$ (resp. $\{M_i''\}$). If $M_i' = M_{i+1}'$ and $M_i'' = M_{i+1}''$ then $M_i = M_{i+1}$. Hence we conclude that $n' + n'' \geq n$. Combined with the earlier result we win. □

Lemma 10.48.4. *Let R be a local ring with maximal ideal $\mathfrak{m}$. Let M be an R-module.*

1. *If M is a finite module and $\mathfrak{m}^n M \neq 0$ for all $n \geq 0$, then $\text{length}_R(M) = \infty$.*
2. *If M has finite length then $\mathfrak{m}^n M = 0$ for some n.*

Proof. Assume M is finite and $\mathfrak{m}^n M \neq 0$ for all $n \geq 0$. By Nakayama's Lemma 10.18.1 all the steps in the filtration $0 \subset \mathfrak{m}^n M \subset \mathfrak{m}^{n-1} M \subset \ldots \subset \mathfrak{m} M \subset M$ are distinct. Hence the length is infinite, i.e., (1) holds. Combine (1) and Lemma 10.48.2 to see (2). □

Lemma 10.48.5. *Let $R \to S$ be a ring map. Let M be an S-module. We always have $\text{length}_R(M) \geq \text{length}_S(M)$. If $R \to S$ is surjective then equality holds.*

Proof. A filtration of M by S-submodules gives rise a filtration of M by R-submodules. This proves the inequality. And if $R \to S$ is surjective, then any R-submodule of M is automatically a S-submodule. Hence equality in this case. □

Lemma 10.48.6. *Let R be a ring with maximal ideal $\mathfrak{m}$. Suppose that M is an R-module with $\mathfrak{m}M = 0$. Then the length of M as an R-module agrees with the dimension of M as a $R/\mathfrak{m}$ vector space. The length is finite if and only if M is a finite R-module.*

Proof. The first part is a special case of Lemma 10.48.5. Thus the length is finite if and only if M has a finite basis as a $R/\mathfrak{m}$-vector space if and only if M has a finite set of generators as an R-module. □

Lemma 10.48.7. *Let R be a ring. Let M be an R-module. Let $S \subset R$ be a multiplicative subset. Then $\text{length}_R(M) \geq \text{length}_{S^{-1}R}(S^{-1}M)$.*

Proof. Any submodule $N' \subset S^{-1}M$ is of the form $S^{-1}N$ for some R-submodule $N \subset M$, by Lemma 10.9.15. The lemma follows. □

Lemma 10.48.8. *Let R be a ring with finitely generated maximal ideal $\mathfrak{m}$. (For example R Noetherian.) Suppose that M is a finite R-module with $\mathfrak{m}^n M = 0$ for some n. Then $\text{length}_R(M) < \infty$.*

Proof. Consider the filtration $0 = \mathfrak{m}^n M \subset \mathfrak{m}^{n-1}M \subset \ldots \subset \mathfrak{m}M \subset M$. All of the subquotients are finitely generated R-modules to which Lemma 10.48.6 applies. We conclude by additivity, see Lemma 10.48.3. □

Definition 10.48.9. Let R be a ring. Let M be an R-module. We say M is *simple* if $M \neq 0$ and every submodule of M is either equal to M or to 0.

Lemma 10.48.10. *Let R be a ring. Let M be an R-module. The following are equivalent:*

1. *M is simple,*
2. *$\text{length}_R(M) = 1$, and*
3. *$M \cong R/\mathfrak{m}$ for some maximal ideal $\mathfrak{m} \subset R$.*

Proof. Let $\mathfrak{m}$ be a maximal ideal of R. By Lemma 10.48.6 the module $R/\mathfrak{m}$ has length 1. The equivalence of the first two assertions is tautological. Suppose that M is simple. Choose $x \in M$, $x \neq 0$. As M is simple we have $M = R \cdot x$. Let $I \subset R$ be the annihilator of x, i.e., $I = \{f \in R \mid fx = 0\}$. The map $R/I \to M$, $f \bmod I \mapsto fx$ is an isomorphism, hence R/I is a simple R-module. Since $R/I \neq 0$ we see $I \neq R$. Let $I \subset \mathfrak{m}$ be a maximal ideal containing I. If $I \neq \mathfrak{m}$, then $\mathfrak{m}/I \subset R/I$ is a nontrivial submodule contradicting the simplicity of R/I. Hence we see $I = \mathfrak{m}$ as desired. □

Lemma 10.48.11. *Let R be a ring. Let M be a finite length R-module. Let $\ell = \text{length}_R(M)$. Choose any maximal chain of submodules*

$$0 = M_0 \subset M_1 \subset M_2 \subset \ldots \subset M_n = M$$

with $M_i \neq M_{i-1}$, $i = 1, \ldots, n$. Then

1. *$n = \ell$,*

(2) *each M_i/M_{i-1} is simple,*
(3) *each M_i/M_{i-1} is of the form $R/\mathfrak{m}_i$ for some maximal ideal $\mathfrak{m}_i$,*
(4) *given a maximal ideal $\mathfrak{m} \subset R$ we have*

$$\#\{i \mid \mathfrak{m}_i = \mathfrak{m}\} = length_{R_\mathfrak{m}}(M_\mathfrak{m}).$$

Proof. If M_i/M_{i-1} is not simple then we can refine the filtration and the filtration is not maximal. Thus we see that M_i/M_{i-1} is simple. By Lemma 10.48.10 the modules M_i/M_{i-1} have length 1 and are of the form $R/\mathfrak{m}_i$ for some maximal ideals $\mathfrak{m}_i$. By additivity of length, Lemma 10.48.3, we see $n = \ell$. Since localization is exact, we see that

$$0 = (M_0)_\mathfrak{m} \subset (M_1)_\mathfrak{m} \subset (M_2)_\mathfrak{m} \subset \ldots \subset (M_n)_\mathfrak{m} = M_\mathfrak{m}$$

is a filtration of $M_\mathfrak{m}$ with successive quotients $(M_i/M_{i-1})_\mathfrak{m}$. Thus the last statement follows directly from the fact that given maximal ideals $\mathfrak{m}$, $\mathfrak{m}'$ of R we have

$$(R/\mathfrak{m}')_\mathfrak{m} \cong \begin{cases} 0 & \text{if } \mathfrak{m} \neq \mathfrak{m}', \\ R_\mathfrak{m}/\mathfrak{m}R_\mathfrak{m} & \text{if } \mathfrak{m} = \mathfrak{m}' \end{cases}$$

This we leave to the reader. □

Lemma 10.48.12. *Let A be a local ring with maximal ideal $\mathfrak{m}$. Let B be a semi-local ring with maximal ideals $\mathfrak{m}_i$, $i = 1, \ldots, n$. Suppose that $A \to B$ is a homomorphism such that each $\mathfrak{m}_i$ lies over $\mathfrak{m}$ and such that*

$$[\kappa(\mathfrak{m}_i) : \kappa(\mathfrak{m})] < \infty.$$

Let M be a B-module of finite length. Then

$$length_A(M) = \sum\nolimits_{i=1,\ldots,n} [\kappa(\mathfrak{m}_i) : \kappa(\mathfrak{m})] length_{B_{\mathfrak{m}_i}}(M_{\mathfrak{m}_i}),$$

in particular $length_A(M) < \infty$.

Proof. Choose a maximal chain

$$0 = M_0 \subset M_1 \subset M_2 \subset \ldots \subset M_n = M$$

by B-submodules as in Lemma 10.48.11. Then each quotient M_i/M_{i-1} is isomorphic to $\kappa(\mathfrak{m}_{j(i)})$ for some $j(i) \in \{1, \ldots, n\}$. Moreover $\text{length}_A(\kappa(\mathfrak{m}_i)) = [\kappa(\mathfrak{m}_i) : \kappa(\mathfrak{m})]$ by Lemma 10.48.6. The lemma follows by additivity of lengths (Lemma 10.48.3). □

Lemma 10.48.13. *Let $A \to B$ be a flat local homomorphism of local rings. Then for any A-module M we have*

$$length_A(M) length_B(B/\mathfrak{m}_A B) = length_B(M \otimes_A B).$$

In particular, if $length_B(B/\mathfrak{m}_A B) < \infty$ then M has finite length if and only if $M \otimes_A B$ has finite length.

Proof. The ring map $A \to B$ is faithfully flat by Lemma 10.36.16. Hence if $0 = M_0 \subset M_1 \subset \ldots \subset M_n = M$ is a chain of length n in M, then the corresponding chain $0 = M_0 \otimes_A B \subset M_1 \otimes_A B \subset \ldots \subset M_n \otimes_A B = M \otimes_A B$ has length n also. This proves $\text{length}_A(M) = \infty \Rightarrow \text{length}_B(M \otimes_A B) = \infty$. Next, assume $\text{length}_A(M) < \infty$. In this case we see that M has a filtration of length $\ell = \text{length}_A(M)$ whose quotients are $A/\mathfrak{m}_A$. Arguing as above we see that $M \otimes_A B$ has a filtration of length ℓ whose quotients are isomorphic to $B \otimes_A A/\mathfrak{m}_A = B/\mathfrak{m}_A B$. Thus the lemma follows. □

Lemma 10.48.14. *Let $A \to B \to C$ be flat local homomorphisms of local rings. Let M be an A-module of finite length. Then*

$$length_B(B/\mathfrak{m}_A B) length_C(C/\mathfrak{m}_B C) = length_C(C/\mathfrak{m}_A C)$$

Proof. Follows from Lemma 10.48.13 applied to the ring map $B \to C$ and the B-module $M = B/\mathfrak{m}_A B$ □

10.49. Artinian rings

Artinian rings, and especially local Artinian rings, play an important role in algebraic geometry, for example in deformation theory.

Definition 10.49.1. A ring R is *Artinian* if it satisfies the descending chain condition for ideals.

Lemma 10.49.2. *Suppose R is a finite dimensional algebra over a field. Then R is Artinian.*

Proof. The descending chain condition for ideals obviously holds. □

Lemma 10.49.3. *If R is Artinian then R has only finitely many maximal ideals.*

Proof. Suppose that $\mathfrak{m}_i$, $i = 1, 2, 3, \ldots$ are maximal ideals. Then $\mathfrak{m}_1 \supset \mathfrak{m}_1 \cap \mathfrak{m}_2 \supset \mathfrak{m}_1 \cap \mathfrak{m}_2 \cap \mathfrak{m}_3 \supset \ldots$ is an infinite descending sequence (because by the Chinese remainder theorem all the maps $R \to \oplus_{i=1}^n R/\mathfrak{m}_i$ are surjective). □

Lemma 10.49.4. *Let R be Artinian. The radical $rad(R)$ of R is a nilpotent ideal.*

Proof. Denote the radical I. Note that $I \supset I^2 \supset I^3 \supset \ldots$ is a descending sequence. Thus $I^n = I^{n+1}$ for some n. Set $J = \{x \in R \mid xI^n = 0\}$. We have to show $J = R$. If not, choose an ideal $J' \neq J$, $J \subset J'$ minimal (possible by the Artinian property). Then $J' = J + Rx$ for some $x \in R$. By NAK, Lemma 10.18.1, we have $IJ' \subset J$. Hence $xI^{n+1} \subset xI \cdot I^n \subset J \cdot I^n = 0$. Since $I^{n+1} = I^n$ we conclude $x \in J$. Contradiction. □

Lemma 10.49.5. *Let R be a ring. Let $I \subset R$ be a locally nilpotent ideal. Then $R \to R/I$ induces a bijection on idempotents.*

First proof of Lemma 10.49.5. As I is locally nilpotent it is contained in every prime ideal. Hence $\mathrm{Spec}(R/I) = V(I) = \mathrm{Spec}(R)$. Hence the lemma follows from Lemma 10.19.3. □

Second proof of Lemma 10.49.5. First assume I is nilpotent. Suppose $\overline{e} \in R/I$ is an idempotent. We have to lift $\overline{e}$ to an idempotent of R. Choose a lift $e \in R$ such that $x = e^2 - e \in I^k$ for some $k \geq 1$. Let $e' = e - (2e-1)x = 3e^2 - 2e^3$, which is another lift of $\overline{e}$. Then

$$(e')^2 - e' = (4e^2 - 4e - 3)(e^2 - e)^2 \in I^{2k}$$

by a simple computation. Hence e' is an idempotent in R/I^{2k}. By successively improving the approximation as above we reach a stage where $I^k = 0$, and we win.

Next, suppose I is locally nilpotent. Let $\overline{e} \in R/I$ be an idempotent. Let $f \in R$ be any element lifting $\overline{e}$. Denote $R' \subset R$ the $\mathbf{Z}$-subalgebra of R generated by f. Denote $I' = R' \cap I$. Since R' is Noetherian, see Lemma 10.29.3 we see that I' is nilpotent, see Lemma 10.47.3. On the other hand we have $R'/I' \subset R/I$ and hence

the image $\overline{f} \in R'/I'$ of f is an idempotent. Thus by the first part of the proof we see that we can find an idempotent $e \in R'$ which is a lift of $\overline{f}$. Then $e \in R$ is also a lift of $\overline{e}$ in R/I. □

Lemma 10.49.6. *Let A be a possibly noncommutative algebra. Let $e \in A$ be an element such that $x = e^2 - e$ is nilpotent. Then there exists an idempotent of the form $e' = e + x(\sum a_{i,j} e^i x^j) \in A$ with $a_{i,j} \in \mathbf{Z}$.*

Proof. Consider the ring $R_n = \mathbf{Z}[e]/((e^2 - e)^n)$. It is clear that if we can prove the result for each R_n then the lemma follows. In R_n consider the ideal $I = (e^2 - e)$ and apply Lemma 10.49.5. □

Lemma 10.49.7. *Any ring with finitely many maximal ideals and locally nilpotent radical is the product of its localizations at its maximal ideals. Also, all primes are maximal.*

Proof. Let R be a ring with finitely many maximal ideals $\mathfrak{m}_1, \ldots, \mathfrak{m}_n$. Let $I = \bigcap_{i=1}^n \mathfrak{m}_i$ be the radical of R. Assume I is locally nilpotent. Let $\mathfrak{p}$ be a prime ideal of R. Since every prime contains every nilpotent element of R we see $\mathfrak{p} \supset \mathfrak{m}_1 \cap \ldots \cap \mathfrak{m}_n$. Since $\mathfrak{m}_1 \cap \ldots \cap \mathfrak{m}_n \supset \mathfrak{m}_1 \ldots \mathfrak{m}_n$ we conclude $\mathfrak{p} \supset \mathfrak{m}_1 \ldots \mathfrak{m}_n$. Hence $\mathfrak{p} \supset \mathfrak{m}_i$ for some i, and so $\mathfrak{p} = \mathfrak{m}_i$. By the Chinese remainder theorem (Lemma 10.14.4) we have $R/I \cong \bigoplus R/\mathfrak{m}_i$ which is a product of fields. Hence by Lemma 10.49.5 there are idempotents e_i, $i = 1, \ldots, n$ with $e_i \bmod \mathfrak{m}_j = \delta_{ij}$. Hence $R = \prod Re_i$, and each Re_i is a ring with exactly one maximal ideal. □

Lemma 10.49.8. *A ring R is Artinian if and only if it has finite length as a module over itself. Any such ring R is both Artinian and Noetherian, any prime ideal of R is a maximal ideal, and R is equal to the (finite) product of its localizations at its maximal ideals.*

Proof. If R has finite length over itself then it satisfies both the ascending chain condition and the descending chain condition for ideals. Hence it is both Noetherian and Artinian. Any Artinian ring is equal to product of its localizations at maximal ideals by Lemmas 10.49.3, 10.49.4, and 10.49.7.

Suppose that R is Artinian. We will show R has finite length over itself. It suffices to exhibit a chain of submodules whose successive quotients have finite length. By what we said above we may assume that R is local, with maximal ideal $\mathfrak{m}$. By Lemma 10.49.4 we have $\mathfrak{m}^n = 0$ for some n. Consider the sequence $0 = \mathfrak{m}^n \subset \mathfrak{m}^{n-1} \subset \ldots \subset \mathfrak{m} \subset R$. By Lemma 10.48.6 the length of each subquotient $\mathfrak{m}^j/\mathfrak{m}^{j+1}$ is the dimension of this as a vector space over $\kappa(\mathfrak{m})$. This has to be finite since otherwise we would have an infinite descending chain of subvector spaces which would correspond to an infinite descending chain of ideals in R. □

10.50. Homomorphisms essentially of finite type

Some simple remarks on localizations of finite type ring maps.

Definition 10.50.1. Let $R \to S$ be a ring map.

(1) We say that $R \to S$ is *essentially of finite type* if S is the localization of an R-algebra of finite type.
(2) We say that $R \to S$ is *essentially of finite presentation* if S is the localization of an R-algebra of finite presentation.

Lemma 10.50.2. *The class of ring maps which are essentially of finite type is preserved under composition. Similarly for essentially of finite presentation.*

Proof. Omitted. □

Lemma 10.50.3. *Let $R \to S$ be a ring map. Assume S is an Artinian local ring with maximal ideal $\mathfrak{m}$. Then*

1. *$R \to S$ is finite if and only if $R \to S/\mathfrak{m}$ is finite,*
2. *$R \to S$ is of finite type if and only if $R \to S/\mathfrak{m}$ is of finite type.*
3. *$R \to S$ is essentially of finite type if and only if the composition $R \to S/\mathfrak{m}$ is essentially of finite type.*

Proof. If $R \to S$ is finite, then $R \to S/\mathfrak{m}$ is finite by Lemma 10.7.3. Conversely, assume $R \to S/\mathfrak{m}$ is finite. As S has finite length over itself (Lemma 10.49.8) we can choose a filtration

$$0 \subset I_1 \subset \ldots \subset I_n = S$$

by ideals such that $I_i/I_{i-1} \cong S/\mathfrak{m}$ as S-modules. Thus S has a filtration by R-submodules I_i such that each successive quotient is a finite R-module. Thus S is a finite S-module by Lemma 10.5.3.

If $R \to S$ is of finite type, then $R \to S/\mathfrak{m}$ is of finite type by Lemma 10.6.2. Conversely, assume that $R \to S/\mathfrak{m}$ is of finite type. Choose $f_1, \ldots, f_n \in S$ which map to generators of $S/\mathfrak{m}$. Then $A = R[x_1, \ldots, x_n] \to S$, $x_i \mapsto f_i$ is a ring map such that $A \to S/\mathfrak{m}$ is surjective (in particular finite). Hence $A \to S$ is finite by part (1) and we see that $R \to S$ is of finite type by Lemma 10.6.2.

If $R \to S$ is essentially of finite type, then $R \to S/\mathfrak{m}$ is essentially of finite type by Lemma 10.50.2. Conversely, assume that $R \to S/\mathfrak{m}$ is essentially of finite type. Suppose $S/\mathfrak{m}$ is the localization of $R[x_1, \ldots, x_n]/I$. Choose $f_1, \ldots, f_n \in S$ whose congruence classes modulo $\mathfrak{m}$ correspond to the congruence classes of $x_1, \ldots, x_n$ modulo I. Consider the map $R[x_1, \ldots, x_n] \to S$, $x_i \mapsto f_i$ with kernel J. Set $A = R[x_1, \ldots, x_n]/J \subset S$ and $\mathfrak{p} = A \cap \mathfrak{m}$. Note that $A/\mathfrak{p} \subset S/\mathfrak{m}$ is equal to the image of $R[x_1, \ldots, x_n]/I$ in $S/\mathfrak{m}$. Hence $\kappa(\mathfrak{p}) = S/\mathfrak{m}$. Thus $A_\mathfrak{p} \to S$ is finite by part (1). We conclude that S is essentially of finite type by Lemma 10.50.2. □

10.51. K-groups

Let R be a ring. We will introduce two abelian groups associated to R. The first of the two is denoted $K_0'(R)$ and has the following properties:

1. For every finite R-module M there is given an element $[M]$ in $K_0'(R)$,
2. for every short exact sequence $0 \to M' \to M \to M'' \to 0$ we have the relation $[M] = [M'] + [M'']$,
3. the group $K_0'(R)$ is generated by the elements $[M]$, and
4. all relations in $K_0'(R)$ are $\mathbf{Z}$-linear combinations of the relations coming from exact sequences as above.

The actual construction is a bit more annoying since one has to take care that the collection of all finitely generated R-modules is a proper class. However, this problem can be overcome by taking as set of generators of the group $K_0'(R)$ the elements $[R^n/K]$ where n ranges over all integers and K ranges over all submodules $K \subset R^n$. The generators of for the subgroup of relations imposed on these elements will be the relations coming from short exact sequences whose terms are of the form

R^n/K. The element $[M]$ is defined by choosing n and K such that $M \cong R^n/K$ and putting $[M] = [R^n/K]$. Details left to the reader.

Lemma 10.51.1. *If R is an Artinian local ring then the length function defines a natural abelian group homomorphism $\text{length}_R : K'_0(R) \to \mathbf{Z}$.*

Proof. The length of any finite R-module is finite, because it is the quotient of R^n which has finite length by Lemma 10.49.8. And the length function is additive, see Lemma 10.48.3. □

The second of the two is denoted $K_0(R)$ and has the following properties:

(1) For every finite projective R-module M there is given an element $[M]$ in $K_0(R)$,
(2) for every short exact sequence $0 \to M' \to M \to M'' \to 0$ of finite projective R-modules we have the relation $[M] = [M'] + [M'']$,
(3) the group $K_0(R)$ is generated by the elements $[M]$, and
(4) all relations in $K_0(R)$ are $\mathbf{Z}$-linear combinations of the relations coming from exact sequences as above.

The construction of this group is done as above.

We note that there is an obvious map $K_0(R) \to K'_0(R)$ which is not an isomorphism in general.

Example 10.51.2. Note that if $R = k$ is a field then we clearly have $K_0(k) = K'_0(k) \cong \mathbf{Z}$ with the isomorphism given by the dimension function (which is also the length function).

Example 10.51.3. Let k be a field. Then $K_0(k[x]) = K'_0(k[x]) = \mathbf{Z}$.

Since $R = k[x]$ is a principal ideal domain, any finite projective R-module is free. In a short exact sequence of modules

$$0 \to M' \to M \to M'' \to 0$$

we have $\text{rank}(M) = \text{rank}(M') + \text{rank}(M'')$, which gives $K_0(k[x]) = \mathbf{Z}$.

As for K'_0, the structure theorem for modules of a PID says that any finitely generated R-module is of the form $M = R^r \times R/(d_1) \times \cdots \times R/(d_k)$. Consider the short exact sequence

$$0 \to (d_i) \to R \to R/(d_i) \to 0$$

Since the ideal (d_i) is isomorphic to R as a module (it is free with generator d_i), in $K'_0(R)$ we have $[(d_i)] = [R]$. Then $[R/(d_i)] = [(d_i)] - [R] = 0$. From this it follows that any torsion part "disappears" in K'_0. Again the rank of the free part determines that $K'_0(k[x]) = \mathbf{Z}$, and the canonical homomorphism from K_0 to K'_0 is an isomorphism.

Example 10.51.4. Let k be a field. Let $R = \{f \in k[x] \mid f(0) = f(1)\}$, compare Example 10.25.4. In this case $K_0(R) \cong k^* \oplus \mathbf{Z}$, but $K'_0(R) = \mathbf{Z}$.

Lemma 10.51.5. *Let $R = R_1 \times R_2$. Then $K_0(R) = K_0(R_1) \times K_0(R_2)$ and $K'_0(R) = K'_0(R_1) \times K'_0(R_2)$*

Proof. Omitted. □

Lemma 10.51.6. *Let R be an Artinian local ring. The map $\text{length}_R : K'_0(R) \to \mathbf{Z}$ of Lemma 10.51.1 is an isomorphism.*

Proof. Omitted. □

Lemma 10.51.7. *Let R be a local ring. Every finite projective R-module is finite free. The map $rank_R : K_0(R) \to \mathbf{Z}$ defined by $[M] \to rank_R(M)$ is well defined and an isomorphism.*

Proof. Let P be a finite projective R-module. The n generators of P give a surjection $R^n \to P$, and since P is projective it follows that $R^n \cong P \oplus Q$ for some projective module Q.

If $\mathfrak{m} \subset R$ is the maximal ideal, then $P/\mathfrak{m}$ and $Q/\mathfrak{m}$ are $R/\mathfrak{m}$-vector spaces, with $P/\mathfrak{m} \oplus Q/\mathfrak{m} \cong (R/\mathfrak{m})^n$. Say that $\dim P = p$, $\dim Q = q$, so $p + q = n$.

Choose elements $a_1, \ldots, a_p$ in P and $b_1, \ldots, b_q$ in Q lying above bases for $P/\mathfrak{m}$ and $Q/\mathfrak{m}$. The homomorphism $R^n \to P \oplus Q \cong R^n$ given by $(r_1, \ldots, r_n) \mapsto r_1a_1 + \ldots + r_pa_p + r_{p+1}b_1 + \ldots + r_nb_q$ is a matrix A which is invertible over $R/\mathfrak{m}$. Let B be a matrix over R lying over the inverse of A in $R/\mathfrak{m}$. $AB = I + M$, where M is a matrix whose entries all lie in $\mathfrak{m}$. Thus $\det AB = 1 + x$, for $x \in \mathfrak{m}$, so AB is invertible, so A is invertible.

The homomorphism $R^p \to P$ given by $(r_1, \ldots, r_p) \mapsto r_1a_1 + \ldots + r_pa_p$ inherits injectivity and surjectivity from A. Hence, $P \cong R^p$.

Next we show that the rank of a finite projective module over R is well defined: if $P \cong R^\alpha \cong R^\beta$, then $\alpha = \beta$. This is immediate in the vector space case, and so it is true in the general module case as well, by dividing out the maximal ideal on both sides. If $0 \to R^\alpha \to R^\beta \to R^\gamma \to 0$ is exact, the sequence splits, so $R^\beta \cong R^\alpha \oplus R^\gamma$, so $\beta = \alpha + \gamma$.

So far we have seen that the map $\text{rank}_R : K_0(R) \to \mathbf{Z}$ is a well-defined homomorphism. It is surjective because $\text{rank}_R[R] = 1$. It is injective because the element of $K_0(R)$ with rank $\pm\alpha$ is uniquely $\pm[R^\alpha]$. □

Lemma 10.51.8. *Let R be a local Artinian ring. There is a commutative diagram*

$$\begin{array}{ccc} K_0(R) & \longrightarrow & K_0'(R) \\ {\scriptstyle rank_R}\downarrow & & \downarrow{\scriptstyle length_R} \\ \mathbf{Z} & \xrightarrow{length_R(R)} & \mathbf{Z} \end{array}$$

where the vertical maps are isomorphisms by Lemmas 10.51.6 and 10.51.7.

Proof. By induction on the rank of M. Suppose $[M] \in K_0(R)$. Then M is a finite projective R-module over a local ring, so M is free; $M \cong R^n$ for some n. The claim is that $\text{rank}(M)\text{length}_R(R) = \text{length}_R(M)$, or equivalently that $n\text{length}_R(R) = \text{length}_R(R^n)$ for all $n \geq 1$. When $n = 1$, this is clearly true. Suppose that $(n-1)\text{length}_R(R) = \text{length}_R(R^{n-1})$. Then since there is a split short exact sequence

$$0 \to R \to R^n \to R^{n-1} \to 0$$

by Lemma 10.48.3 we have

$$\begin{aligned} \text{length}_R(R^n) &= \text{length}_R(R) + \text{length}_R(R^{n-1}) \\ &= \text{length}_R(R) + (n-1)\text{length}_R(R) \\ &= n\text{length}_R(R) \end{aligned}$$

as desired. □

10.52. Graded rings

A *graded ring* will be for us a ring S endowed with a direct sum decomposition $S = \bigoplus_{d \geq 0} S_d$ such that $S_d \cdot S_e \subset S_{d+e}$. Note that we do not allow nonzero elements in negative degrees. The *irrelevant ideal* is the ideal $S_+ = \bigoplus_{d>0} S_d$. A *graded module* will be an S-module M endowed with a direct sum decomposition $M = \bigoplus_{n \in \mathbf{Z}} M_n$ such that $S_d \cdot M_e \subset M_{d+e}$. Note that for modules we do allow nonzero elements in negative degrees. We think of S as a graded S-module by setting $S_{-k} = (0)$ for $k > 0$. An element x (resp. f) of M (resp. S) is called *homogeneous* if $x \in M_d$ (resp. $f \in S_d$) for some d. A *map of graded S-modules* is a map of S-modules $\varphi : M \to M'$ such that $\varphi(M_d) \subset M'_d$. We do not allow maps to shift degrees. Let us denote $\mathrm{GrHom}_0(M, N)$ the S_0-module of homomorphisms of graded modules from M to N.

At this point there are the notions of graded ideal, graded quotient ring, graded submodule, graded quotient module, graded tensor product, etc. We leave it to the reader to find the relevant definitions, and lemmas. For example: A short exact sequence of graded modules is short exact in every degree.

Given a graded ring S, a graded S-module M and $n \in \mathbf{Z}$ we denote $M(n)$ the graded S-module with $M(n)_d = M_{n+d}$. This is called the *twist of M by n*. In particular we get modules $S(n)$, $n \in \mathbf{Z}$ which will play an important role in the study of projective schemes. There are some obvious functorial isomorphisms such as $(M \oplus N)(n) = M(n) \oplus N(n)$, $(M \otimes_S N)(n) = M \otimes_S N(n) = M(n) \otimes_S N$. In addition we can define a graded S-module structure on the S_0-module

$$\mathrm{GrHom}(M, N) = \bigoplus\nolimits_{n \in \mathbf{Z}} \mathrm{GrHom}_n(M, N), \quad \mathrm{GrHom}_n(M, N) = \mathrm{GrHom}_0(M, N(n)).$$

We omit the definition of the multiplication.

Let S be a graded ring. Let $d \geq 1$ be an integer. We set $S^{(d)} = \bigoplus_{n \geq 0} S_{nd}$. We think of $S^{(d)}$ as a graded ring with degree n summand $(S^{(d)})_n = S_{nd}$. Given a graded S-module M we can similarly consider $M^{(d)} = \bigoplus_{n \in \mathbf{Z}} M_{nd}$ which is a graded $S^{(d)}$-module.

Lemma 10.52.1. *Let $R \to S$ be a homomorphism of graded rings. Let $S' \subset S$ be the integral closure of R in S. Then*

$$S' = \bigoplus\nolimits_{d \geq 0} S' \cap S_d,$$

i.e., S' is a graded R-subalgebra of S.

Proof. We have to show the following: If $s = s_n + s_{n+1} + \ldots + s_m \in S'$, then each homogeneous part $s_j \in S'$. We will prove this by induction on $m - n$ over all homomorphisms $R \to S$ of graded rings. First note that it is immediate that s_0 is integral over R_0 (hence over R) as there is a ring map $S \to S_0$ compatible with the ring map $R \to R_0$. Thus, after replacing s by $s - s_0$, we may assume $n > 0$. Consider the extension of graded rings $R[t, t^{-1}] \to S[t, t^{-1}]$ where t has degree 0. There is a commutative diagram

$$\begin{array}{ccc} S[t, t^{-1}] & \xrightarrow[s \mapsto t^{\deg(s)} s]{} & S[t, t^{-1}] \\ \uparrow & & \uparrow \\ R[t, t^{-1}] & \xrightarrow{r \mapsto r^{\deg(r)}} & R[t, t^{-1}] \end{array}$$

where the horizontal maps are ring automorphisms. Hence the integral closure C of $S[t, t^{-1}]$ over $R[t, t^{-1}]$ maps into itself. Thus we see that

$$t^m(s_n + s_{n+1} + \ldots + s_m) - (t^n s_n + t^{n+1}s_{n+1} + \ldots + t^m s_m) \in C$$

which implies by induction hypothesis that each $(t^m - t^i)s_i \in C$ for $i = n, \ldots, m-1$. Note that for any ring A and $m > i \geq n > 0$ we have $A[t, t^{-1}]/(t^m - t^i - 1) \cong A[t]/(t^m - t^i - 1) \supset A$ because $t(t^{m-1} - t^{i-1}) = 1$ in $A[t]/(t^m - t^i - 1)$. Since $t^m - t^i$ maps to 1 we see the image of s_i in the ring $S[t]/(t^m - t^i - 1)$ is integral over $R[t]/(t^m - t^i - 1)$ for $i = n, \ldots, m-1$. Since $R \to R[t]/(t^m - t^i - 1)$ is finite we see that s_i is integral over R by transitivity, see Lemma 10.33.6. Finally, we also conclude that $s_m = s - \sum_{i=n,\ldots,m-1} s_i$ is integral over R. □

10.53. Proj of a graded ring

Let S be a graded ring. A *homogeneous ideal* is simply an ideal $I \subset S$ which is also a graded submodule of S. Equivalently, it is an ideal generated by homogeneous elements. Equivalently, if $f \in I$ and

$$f = f_0 + f_1 + \ldots + f_n$$

is the decomposition of f into homogeneous parts in S then $f_i \in I$ for each i. To check that a homogeneous ideal $\mathfrak{p}$ is prime it suffices to check that if $ab \in \mathfrak{p}$ with a, b homogeneous then either $a \in \mathfrak{p}$ or $b \in \mathfrak{p}$.

Definition 10.53.1. Let S be a graded ring. We define $\text{Proj}(S)$ to be the set of homogeneous, prime ideals $\mathfrak{p}$ of S such that $S_+ \not\subset \mathfrak{p}$. As $\text{Proj}(S)$ is a subset of $\text{Spec}(S)$ and we endow it with the induced topology. The topological space $\text{Proj}(S)$ is called the *homogeneous spectrum* of the graded ring S.

Note that by construction there is a continuous map

$$\text{Proj}(S) \longrightarrow \text{Spec}(S_0)$$

Let $S = \oplus_{d \geq 0} S_d$ be a graded ring. Let $f \in S_d$ and assume that $d \geq 1$. We define $S_{(f)}$ to be the subring of S_f consisting of elements of the form r/f^n with r homogeneous and $\deg(r) = nd$. If M is a graded S-module, then we define the $S_{(f)}$-module $M_{(f)}$ as the sub module of M_f consisting of elements of the form x/f^n with x homogeneous of degree nd.

Lemma 10.53.2. *Let S be a $\mathbf{Z}$-graded ring. Let $f \in S_d$, $d > 0$ and assume f is invertible in S. The set $G \subset \text{Spec}(S)$ of $\mathbf{Z}$-graded primes of S (with induced topology) maps homeomorphically to $\text{Spec}(S_0)$.*

Proof. First we show that the map is a bijection by constructing an inverse. Namely, if $\mathfrak{p}_0$ is a prime of S_0, then $\mathfrak{p}_0 S$ is a $\mathbf{Z}$-graded ideal of S such that $\mathfrak{p}_0 S \cap S_0 = \mathfrak{p}_0$. And if $ab \in \mathfrak{p}_0 S$ with a, b homogeneous, then $a^d b^d / f^{\deg(a)+\deg(b)} \in \mathfrak{p}_0$. Thus either $a^d/f^{\deg(a)} \in \mathfrak{p}_0$ or $b^d/f^{\deg(b)} \in \mathfrak{p}_0$, in other words either $a^d \in \mathfrak{p}_0 S$ or $b^d \in \mathfrak{p}_0 S$. It follows that $\sqrt{\mathfrak{p}_0 S}$ is a $\mathbf{Z}$-graded prime ideal of S whose intersection with S_0 is $\mathfrak{p}_0$.

To show that the map is a homeomorphism we show that the image of $G \cap D(g)$ is open. If $g = \sum g_i$ with $g_i \in S_i$, then by the above $G \cap D(g)$ maps onto the set $\bigcup D(g_i^d/f^i)$ which is open. □

For $f \in S$ homogeneous of degree > 0 we define

$$D_+(f) = \{\mathfrak{p} \in \text{Proj}(S) \mid f \notin \mathfrak{p}\}.$$

Finally, for a homogeneous ideal $I \subset S$ we define

$$V_+(I) = \{\mathfrak{p} \in \text{Proj}(S) \mid I \subset \mathfrak{p}\}.$$

We will use more generally the notation $V_+(E)$ for any set E of homogeneous elements $E \subset S$.

Lemma 10.53.3 (Topology on Proj)**.** *Let $S = \oplus_{d \geq 0} S_d$ be a graded ring.*

1. *The sets $D_+(f)$ are open in Proj(S).*
2. *We have $D_+(ff') = D_+(f) \cap D_+(f')$.*
3. *Let $g = g_0 + \ldots + g_m$ be an element of S with $g_i \in S_i$. Then*
$$D(g) \cap \mathit{Proj}(S) = (D(g_0) \cap \mathit{Proj}(S)) \cup \bigcup\nolimits_{i \geq 1} D_+(g_i).$$
4. *Let $g_0 \in S_0$ be a homogeneous element of degree 0. Then*
$$D(g_0) \cap \mathit{Proj}(S) = \bigcup\nolimits_{f \in S_d,\ d \geq 1} D_+(g_0 f).$$
5. *The open sets $D_+(f)$ form a basis for the topology of Proj(S).*
6. *Let $f \in S$ be homogeneous of positive degree. The ring S_f has a natural $\mathbf{Z}$-grading. The ring maps $S \to S_f \leftarrow S_{(f)}$ induce homeomorphisms*
$$D_+(f) \leftarrow \{\mathbf{Z}\text{-graded primes of } S_f\} \to \text{Spec}(S_{(f)}).$$
7. *There exists an S such that Proj(S) is not quasi-compact.*
8. *The sets $V_+(I)$ are closed.*
9. *Any closed subset $T \subset \mathit{Proj}(S)$ is of the form $V_+(I)$ for some homogeneous ideal $I \subset S$.*
10. *For any graded ideal $I \subset S$ we have $V_+(I) = \emptyset$ if and only if $S_+ \subset \sqrt{I}$.*

Proof. Since $D_+(f) = \text{Proj}(S) \cap D(f)$, these sets are open. Similarly the sets $V_+(I) = \text{Proj}(S) \cap V(E)$ are closed.

Suppose that $T \subset \text{Proj}(S)$ is closed. Then we can write $T = \text{Proj}(S) \cap V(J)$ for some ideal $J \subset S$. By definition of a homogeneous ideal if $g \in J$, $g = g_0 + \ldots + g_m$ with $g_d \in S_d$ then $g_d \in \mathfrak{p}$ for all $\mathfrak{p} \in T$. Thus, letting $I \subset S$ be the ideal generated by the homogeneous parts of the elements of J we have $T = V_+(I)$.

The formula for $\text{Proj}(S) \cap D(g)$, with $g \in S$ is direct from the definitions. Consider the formula for $\text{Proj}(S) \cap D(g_0)$. The inclusion of the right hand side in the left hand side is obvious. For the other inclusion, suppose $g_0 \notin \mathfrak{p}$ with $\mathfrak{p} \in \text{Proj}(S)$. If all $g_0 f \in \mathfrak{p}$ for all homogeneous f of positive degree, then we see that $S_+ \subset \mathfrak{p}$ which is a contradiction. This gives the other inclusion.

The collection of opens $D(g) \cap \text{Proj}(S)$ forms a basis for the topology since the standard opens $D(g) \subset \text{Spec}(S)$ form a basis for the topology on $\text{Spec}(S)$. By the formulas above we can express $D(g) \cap \text{proj}(S)$ as a union of opens $D_+(f)$. Hence the collection of opens $D_+(f)$ forms a basis for the topology also.

First we note that $D_+(f)$ may be identified with a subset (with induced topology) of $D(f) = \text{Spec}(S_f)$ via Lemma 10.16.6. Note that the ring S_f has a $\mathbf{Z}$-grading. The homogeneous elements are of the form r/f^n with $r \in S$ homogeneous and have degree $\deg(r/f^n) = \deg(r) - n\deg(f)$. The subset $D_+(f)$ corresponds exactly to

those prime ideals $\mathfrak{p} \subset S_f$ which are $\mathbf{Z}$-graded ideals (i.e., generated by homogeneous elements). Hence we have to show that the set of $\mathbf{Z}$-graded prime ideals of S_f maps homeomorphically to $\mathrm{Spec}(S_{(f)})$. This follows from Lemma 10.53.2.

Let $S = \mathbf{Z}[X_1, X_2, X_3, \ldots]$ with grading such that each X_i has degree 1. Then it is easy to see that

$$\mathrm{Proj}(S) = \bigcup\nolimits_{i=1}^{\infty} D_+(X_i)$$

does not have a finite refinement.

Let $I \subset S$ be a graded ideal. If $\sqrt{I} \supset S_+$ then $V_+(I) = \emptyset$ since every prime $\mathfrak{p} \in \mathrm{Proj}(S)$ does not contain S_+ by definition. Conversely, suppose that $S_+ \not\subset \sqrt{I}$. Then we can find an element $f \in S_+$ such that f is not nilpotent modulo I. Clearly this means that one of the homogeneous parts of f is not nilpotent modulo I, in other words we may (and do) assume that f is homogeneous. This implies that $IS_f \neq 0$, in other words that $(S/I)_f$ is not zero. Hence $(S/I)_{(f)} \neq 0$ since it is a ring which maps into $(S/I)_f$. Pick a prime $\mathfrak{q} \subset (S/I)_{(f)}$. This corresponds to a graded prime of S/I, not containing the irrelevant ideal $(S/I)_+$. And this in turn corresponds to a graded prime ideal $\mathfrak{p}$ of S, containing I but not containing S_+ as desired. □

Example 10.53.4. Let R be a ring. If $S = R[X]$ with $\deg(X) = 1$, then the natural map $\mathrm{Proj}(S) \to \mathrm{Spec}(R)$ is a bijection and in fact a homeomorphism. Namely, suppose $\mathfrak{p} \in \mathrm{Proj}(S)$. Since $S_+ \not\subset \mathfrak{p}$ we see that $X \notin \mathfrak{p}$. Thus if $aX^n \in \mathfrak{p}$ with $a \in R$ and $n > 0$, then $a \in \mathfrak{p}$. It follows that $\mathfrak{p} = \mathfrak{p}_0 S$ with $\mathfrak{p}_0 = \mathfrak{p} \cap R$.

If $\mathfrak{p} \in \mathrm{Proj}(S)$, then we define $S_{(\mathfrak{p})}$ to be the ring whose elements are fractions r/f where $r, f \in S$ are homogeneous elements of the same degree such that $f \notin \mathfrak{p}$. As usual we say $r/f = r'/f'$ if and only if there exists some $f'' \in S$ homogeneous, $f'' \notin \mathfrak{p}$ such that $f''(rf' - r'f) = 0$. Given a graded S-module M we let $M_{(\mathfrak{p})}$ be the $S_{(\mathfrak{p})}$-module whose elements are fractions x/f with $x \in M$ and $f \in S$ homogeneous of the same degree such that $f \notin \mathfrak{p}$. We say $x/f = x'/f'$ if and only if there exists some $f'' \in S$ homogeneous, $f'' \notin \mathfrak{p}$ such that $f''(xf' - x'f) = 0$.

Lemma 10.53.5. *Let S be a graded ring. Let M be a graded S-module. Let $\mathfrak{p}$ be an element of $\mathrm{Proj}(S)$. Let $f \in S$ be a homogeneous element of positive degree such that $f \notin \mathfrak{p}$, i.e., $\mathfrak{p} \in D_+(f)$. Let $\mathfrak{p}' \subset S_{(f)}$ be the element of $\mathrm{Spec}(S_{(f)})$ corresponding to $\mathfrak{p}$ as in Lemma 10.53.3. Then $S_{(\mathfrak{p})} = (S_{(f)})_{\mathfrak{p}'}$ and compatibly $M_{(\mathfrak{p})} = (M_{(f)})_{\mathfrak{p}'}$.*

Proof. We define a map $\psi : M_{(\mathfrak{p})} \to (M_{(f)})_{\mathfrak{p}'}$. Let $x/g \in M_{(\mathfrak{p})}$. We set $\psi(x/g) = (x^{\deg(f)}/f^{\deg(x)})/(g^{\deg(f)}/f^{\deg(g)})$. This makes sense since $\deg(x) = \deg(g)$, and since $g^{\deg(f)}/f^{\deg(g)} \notin \mathfrak{p}'$. We omit the verification that ψ is well defined, a module map and an isomorphism. □

Here is a graded variant of Lemma 10.14.3.

Lemma 10.53.6. *Suppose S is a graded ring, $\mathfrak{p}_i$, $i = 1, \ldots, r$ homogeneous prime ideals and $I \subset S_+$ a graded ideal. Assume $I \not\subset \mathfrak{p}_i$ for all i. Then there exists a homogeneous element $x \in I$ of positive degree such that $x \notin \mathfrak{p}_i$ for all i.*

Proof. We may assume there are no inclusions among the $\mathfrak{p}_i$. The result is true for $r = 1$. Suppose the result holds for $r - 1$. Pick $x \in I$ homogeneous of positive degree such that $x \notin \mathfrak{p}_i$ for all $i = 1, \ldots, r-1$. If $x \notin \mathfrak{p}_r$ we are done. So assume

$x \in \mathfrak{p}_r$. If $I\mathfrak{p}_1 \ldots \mathfrak{p}_{r-1} \subset \mathfrak{p}_r$ then $I \subset \mathfrak{p}_r$ a contradiction. Pick $y \in I\mathfrak{p}_1 \ldots \mathfrak{p}_{r-1}$ homogeneous and $y \notin \mathfrak{p}_r$. Then $x^{\deg(y)} + y^{\deg(x)}$ works. □

Lemma 10.53.7. *Let S be a graded ring. Let $\mathfrak{p} \subset S$ be a prime. Let $\mathfrak{q}$ be the homogeneous ideal of S generated by the homogeneous elements of $\mathfrak{p}$. Then $\mathfrak{q}$ is a prime ideal of S.*

Proof. Suppose $f, g \in S$ are such that $fg \in \mathfrak{q}$. Let f_d (resp. g_e) be the homogeneous part of f (resp. g) of degree d (resp. e). Assume d, e are maxima such that $f_d \neq 0$ and $g_e \neq 0$. By assumption we can write $fg = \sum a_i f_i$ with $f_i \in \mathfrak{p}$ homogeneous. Say $\deg(f_i) = d_i$. Then $f_d g_e = \sum a_i' f_i$ with a_i' to homogeneous par of degree $d + e - d_i$ of a_i (or 0 if $d + e - d_i < 0$). Hence $f_d \in \mathfrak{p}$ or $g_e \in \mathfrak{p}$. Hence $f_d \in \mathfrak{q}$ or $g_e \in \mathfrak{q}$. In the first case replace f by $f - f_d$, in the second case replace g by $g - g_e$. Then still $fg \in \mathfrak{q}$ but the discrete invariant $d + e$ has been decreased. Thus we may continue in this fashion until either f or g is zero. This clearly shows that $fg \in \mathfrak{q}$ implies either $f \in \mathfrak{q}$ or $g \in \mathfrak{q}$ as desired. □

Lemma 10.53.8. *Let S be a graded ring.*

1. *Any minimal prime of S is a homogeneous ideal of S.*
2. *Given a homogeneous ideal $I \subset S$ any minimal prime over I is homogeneous.*

Proof. The first assertion holds because the prime $\mathfrak{q}$ constructed in Lemma 10.53.7 satisfies $\mathfrak{q} \subset \mathfrak{p}$. The second because we may consider S/I and apply the first part. □

Lemma 10.53.9. *Let R be a ring. Let S be a graded R-algebra. Assume that S is of finite type over R. Then for every homogeneous $f \in S_+$ the ring $S_{(f)}$ is of finite type over R.*

Proof. Choose $f_1, \ldots, f_n \in S$ which generate S as an R-algebra. We may assume that each f_i is homogeneous (by decomposing each f_i into its homogeneous components). An element of $S_{(f)}$ is a sum of the form

$$\sum\nolimits_{e \deg(f) = \sum e_i \deg(f_i)} \lambda_{e_1 \ldots e_n} f_1^{e_1} \ldots f_n^{e_n} / f^e$$

with $\lambda_{e_1 \ldots e_n} \in R$. Thus $S_{(f)}$ is generated as an R-algebra by the $f_1^{e_1} \ldots f_n^{e_n}/f^e$ with the property that $e \deg(f) = \sum e_i \deg(f_i)$. If $e_i \geq \deg(f)$ then we can write this as

$$f_1^{e_1} \ldots f_n^{e_n}/f^e = f_i^{\deg(f)}/f^{\deg(f_i)} \cdot f_1^{e_1} \ldots f_i^{e_i - \deg(f)} \ldots f_n^{e_n}/f^{e - \deg(f_i)}$$

Thus we only need the elements $f_i^{\deg(f)}/f^{\deg(f_i)}$ as well as the elements $f_1^{e_1} \ldots f_n^{e_n}/f^e$ with $e \deg(f) = \sum e_i \deg(f_i)$ and $e_i \leq \deg(f)$. This is a finite list and we win. □

Lemma 10.53.10. *Let R be a ring. Let R' be a finite type R-algebra, and let M be a finite R'-module. There exists a graded R-algebra S, a graded S-module N and an element $f \in S$ homogeneous of degree 1 such that*

1. *$R' \cong S_{(f)}$ and $M \cong N_{(f)}$ (as modules),*
2. *$S_0 = R$ and S is generated by finitely many elements of degree 1 over R, and*
3. *N is a finite S-module.*

Proof. We may write $R' = R[x_1, \ldots, x_n]/I$ for some ideal I. For an element $g \in R[x_1, \ldots, x_n]$ denote $\tilde{g} \in R[x_0, \ldots, x_n]$ the element homogeneous of minimal degree such that $g = \tilde{g}(1, x_1, \ldots, x_n)$. Let $\tilde{I} \subset R[X_0, \ldots, X_n]$ generated by all elements $\tilde{g}$, $g \in I$. Set $S = R[X_0, \ldots, X_n]/\tilde{I}$ and denote f the image of X_0 in S. By construction we have an isomorphism

$$S_{(f)} \longrightarrow R', \quad X_i/X_0 \longmapsto x_i.$$

To do the same thing with the module M we choose a presentation

$$M = (R')^{\oplus r} / \sum\nolimits_{j \in J} R' k_j$$

with $k_j = (k_{1j}, \ldots, k_{rj})$. Let $d_{ij} = \deg(\tilde{k}_{ij})$. Set $d_j = \max\{d_{ij}\}$. Set $K_{ij} = X_0^{d_j - d_{ij}} \tilde{k}_{ij}$ which is homogeneous of degree d_j. With this notation we set

$$N = \mathrm{Coker}\Big(\bigoplus\nolimits_{j \in J} S(-d_j) \xrightarrow{(K_{ij})} S^{\oplus r} \Big)$$

which works. Some details omitted. □

10.54. Blow up algebras

In this section we make some elementary observations about blowing up.

Definition 10.54.1. Let R be a ring. Let $I \subset R$ be an ideal.

(1) The *blowup algebra*, or the *Rees algebra*, associated to the pair (R, I) is the graded R-algebra
$$\mathrm{Bl}_I(R) = \bigoplus\nolimits_{n \geq 0} I^n = R \oplus I \oplus I^2 \oplus \ldots$$
where the summand I^n is placed in degree n.

(2) Let $a \in I$ be an element. Denote $a^{(1)}$ the element a seen as an element of degree 1 in the Rees algebra. Then the *affine blowup algebra* $R[\frac{I}{a}]$ is the algebra $(\mathrm{Bl}_I(R))_{(a^{(1)})}$ constructed in Section 10.53.

In other words, an element of $R[\frac{I}{a}]$ is represented by an expression of the form x/a^n with $x \in I^n$. Two representatives x/a^n and y/a^m define the same element if and only if $a^k(a^m x - a^n y) = 0$ for some $k \geq 0$.

Lemma 10.54.2. *Let R be a ring, $I \subset R$ an ideal, and $a \in I$. The image of a in the blowup algebra $R' = R[\frac{I}{a}]$ is a nonzerodivisor and $IR' = aR'$.*

Proof. Immediate from the description of $R[\frac{I}{a}]$ above. □

Lemma 10.54.3. *Let R be a ring, $I \subset R$ an ideal, and $a \in I$. Set $R' = R[\frac{I}{a}]$. If $f \in R$ is such that $V(f) = V(I)$, then f maps to a nonzerodivisor in R' and $R'_f = R'_a = R_f$.*

Proof. We will use the results of Lemma 10.54.2 without further mention. The assumption $V(f) = V(I)$ implies $V(fR') = V(IR') = V(aR')$. Hence $a^n = fb$ and $f^m = ac$ for some $b, c \in R'$. The lemma follows. □

Lemma 10.54.4. *If R is a domain then every (affine) blowup algebra of R is a domain.*

Proof. Omitted. □

Lemma 10.54.5. *If R is reduced then every (affine) blowup algebra of R is reduced.*

Proof. Omitted. □

Lemma 10.54.6. *Let R be a ring. Let $I \subset R$ be an ideal. Let $a \in I$. If a is not contained in any minimal prime of R, then $\mathrm{Spec}(R[\frac{I}{a}]) \to \mathrm{Spec}(R)$ has dense image.*

Proof. If $a^k x = 0$ for $x \in R$, then x is contained in all the minimal primes of R and hence nilpotent, see Lemma 10.16.2. Thus the kernel of $R \to R[\frac{I}{a}]$ consists of nilpotent elements. Hence the result follows from Lemma 10.28.6. □

Lemma 10.54.7. *Let $(R, \mathfrak{m})$ be a local domain with fraction field K. Let $R \subset A \subset K$ be a valuation ring which dominates R. Then*

$$A = \operatorname{colim} R[\tfrac{I}{a}]$$

is a directed colimit of affine blowups $R \to R[\frac{I}{a}]$ with the following two properties

(1) *$a \in I \subset \mathfrak{m}$,*
(2) *I is finitely generated, and*
(3) *the fibre ring of $R \to R[\frac{I}{a}]$ at $\mathfrak{m}$ is not zero.*

Proof. Consider a finite subset $E \subset A$. Say $E = \{e_1, \ldots, e_n\}$. Choose a nonzero $a \in R$ such that we can write $e_i = f_i/a$ for all $i = 1, \ldots, n$. Set $I = (f_1, \ldots, f_n, a)$. We claim that $R[\frac{I}{a}] \subset A$. This is clear as an element of $R[\frac{I}{a}]$ can be represented as a polynomial in the elements e_i. The lemma follows immediately from this observation. □

10.55. Noetherian graded rings

A bit of theory on Noetherian graded rings including some material on Hilbert polynomials.

Lemma 10.55.1. *Let S be a graded ring. A set of homogeneous elements $f_i \in S_+$ generates S as an algebra over S_0 if and only if they generate S_+ as an ideal of S.*

Proof. If the f_i generate S as an algebra over S_0 then every element in S_+ is a polynomial without constant term in the f_i and hence S_+ is generated by the f_i as an ideal. Conversely, suppose that $S_+ = \sum Sf_i$. We will prove that any element f of S can be written as a polynomial in the f_i with coefficients in S_0. It suffices to do this for homogeneous elements. Say f has degree d. Then we may perform induction on d. The case $d = 0$ is immediate. If $d > 0$ then $f \in S_+$ hence we can write $f = \sum g_i f_i$ for some $g_i \in S$. As S is graded we can replace g_i by its homogeneous component of degree $d - \deg(f_i)$. By induction we see that each g_i is a polynomial in the f_i and we win. □

Lemma 10.55.2. *A graded ring S is Noetherian if and only if S_0 is Noetherian and S_+ is finitely generated as an ideal of S.*

Proof. It is clear that if S is Noetherian then $S_0 = S/S_+$ is Noetherian and S_+ is finitely generated. Conversely, assume S_0 is Noetherian and S_+ finitely generated as an ideal of S. Pick generators $S_+ = (f_1, \ldots, f_n)$. By decomposing the f_i into homogeneous pieces we may assume each f_i is homogeneous. By Lemma 10.55.1 we see that $S_0[X_1, \ldots X_n] \to S$ sending X_i to f_i is surjective. Thus S is Noetherian by Lemma 10.29.1. □

Definition 10.55.3. Let A be an abelian group. We say that a function $f : n \mapsto f(n) \in A$ defined for all sufficient large integers n is a *numerical polynomial* if there exists $r \geq 0$, elements $a_0, \ldots, a_r \in A$ such that

$$f(n) = \sum\nolimits_{i=0}^{r} \binom{n}{i} a_i$$

for all $n \gg 0$.

The reason for using the binomial coefficients is the elementary fact that any polynomial $P \in \mathbf{Q}[T]$ all of whose values at integer points are integers, is equal to a sum $P(T) = \sum a_i \binom{T}{i}$ with $a_i \in \mathbf{Z}$. Note that in particular the expressions $\binom{T+1}{i+1}$ are of this form.

Lemma 10.55.4. *If $A \to A'$ is a homomorphism of abelian groups and if $f : n \mapsto f(n) \in A$ is a numerical polynomial, then so is the composition.*

Proof. This is immediate from the definitions. □

Lemma 10.55.5. *Suppose that $f : n \mapsto f(n) \in A$ is defined for all n sufficiently large and suppose that $n \mapsto f(n) - f(n-1)$ is a numerical polynomial. Then f is a numerical polynomial.*

Proof. Let $f(n) - f(n-1) = \sum_{i=0}^{r} \binom{n}{i} a_i$ for all $n \gg 0$. Set $g(n) = f(n) - \sum_{i=0}^{r} \binom{n+1}{i+1} a_i$. Then $g(n) - g(n-1) = 0$ for all $n \gg 0$. Hence g is eventually constant, say equal to a_{-1}. We leave it to the reader to show that $a_{-1} + \sum_{i=0}^{r} \binom{n+1}{i+1} a_i$ has the required shape (see remark above the lemma). □

Lemma 10.55.6. *If M is a finitely generated graded S-module, and if S is finitely generated over S_0, then each M_n is a finite S_0-module.*

Proof. Suppose the generators of M are m_i and the generators of S are f_i. By taking homogeneous components we may assume that the m_i and the f_i are homogeneous and we may assume $f_i \in S_+$. In this case it is clear that each M_n is generated over S_0 by the "monomials" $\prod f_i^{e_i} m_j$ whose degree is n. □

Proposition 10.55.7. *Suppose that S is a Noetherian graded ring and M a finite graded S-module. Consider the function*

$$\mathbf{Z} \longrightarrow K_0'(S_0), \quad n \longmapsto [M_n]$$

see Lemma 10.55.6. If S_+ is generated by elements of degree 1, then this function is a numerical polynomial.

Proof. We prove this by induction on the minimal number of generators of S_1. If this number is 0, then $M_n = 0$ for all $n \gg 0$ and the result holds. To prove the induction step, let $x \in S_1$ be one of a minimal set of generators, such that the induction hypothesis applies to the graded ring $S/(x)$.

First we show the result holds if x is nilpotent on M. This we do by induction on the minimal integer r such that $x^r M = 0$. If $r = 1$, then M is a module over S/xS and the result holds (by the other induction hypothesis). If $r > 1$, then we can find a short exact sequence $0 \to M' \to M \to M'' \to 0$ such that the integers r', r'' are strictly smaller than r. Thus we know the result for M'' and M'. Hence we get the result for M because of the relation $[M_d] = [M'_d] + [M''_d]$ in $K_0'(R)$.

If x is not nilpotent on M, let $M' \subset M$ be the largest submodule on which x is nilpotent. Consider the exact sequence $0 \to M' \to M \to M/M' \to 0$ we see again it suffices to prove the result for M/M'. In other words we may assume that multiplication by x is injective.

Let $\overline{M} = M/xM$. Note that the map $x : M \to M$ is *not* a map of graded S-modules, since it does not map M_d into M_d. Namely, for each d we have the following short exact sequence

$$0 \to M_d \xrightarrow{x} M_{d+1} \to \overline{M}_{d+1} \to 0$$

This proves that $[M_{d+1}] - [M_d] = [\overline{M}_{d+1}]$. Hence we win by Lemma 10.55.5. □

Remark 10.55.8. If S is still Noetherian but S is not generated in degree 1, then the function associated to a graded S-module is a periodic polynomial (i.e., it is a numerical polynomial on the congruence classes of integers modulo n for some n).

Example 10.55.9. Suppose that $S = k[X_1, \ldots, X_d]$. By Example 10.51.2 we may identify $K_0(k) = K_0'(k) = \mathbf{Z}$. Hence any finitely generated graded $k[X_1, \ldots, X_d]$-module gives rise to a numerical polynomial $n \mapsto \dim_k(M_n)$.

Lemma 10.55.10. *Let k be a field. Suppose that $I \subset k[X_1, \ldots, X_d]$ is a nonzero graded ideal. Let $M = k[X_1, \ldots, X_d]/I$. Then the numerical polynomial $n \mapsto \dim_k(M_n)$ (see Example 10.55.9) has degree $< d - 1$ (or is zero if $d = 1$).*

Proof. The numerical polynomial associated to the graded module $k[X_1, \ldots, X_n]$ is $n \mapsto \binom{n-1+d}{d-1}$. For any nonzero homogeneous $f \in I$ of degree e and any degree $n >> e$ we have $I_n \supset f \cdot k[X_1, \ldots, X_d]_{n-e}$ and hence $\dim_k(I_n) \geq \binom{n-e-1+d}{d-1}$. Hence $\dim_k(M_n) \leq \binom{n-1+d}{d-1} - \binom{n-e-1+d}{d-1}$. We win because the last expression has degree $< d - 1$ (or is zero if $d = 1$). □

10.56. Noetherian local rings

In all of this section $(R, \mathfrak{m}, \kappa)$ is a Noetherian local ring. We develop some theory on Hilbert functions of modules in this section. Let M be a finite R-module. We define the *Hilbert function* of M to be the function

$$\varphi_M : n \longmapsto \text{length}_R(\mathfrak{m}^n M/\mathfrak{m}^{n+1} M)$$

defined for all integers $n \geq 0$. Another important invariant is the function

$$\chi_M : n \longmapsto \text{length}_R(M/\mathfrak{m}^{n+1} M)$$

defined for all integers $n \geq 0$. Note that we have by Lemma 10.48.3 that

$$\chi_M(n) = \sum\nolimits_{i=0}^{n} \varphi_M(i).$$

There is a variant of this construction which uses an ideal of definition.

Definition 10.56.1. Let $(R, \mathfrak{m})$ be a local Noetherian ring. An ideal $I \subset R$ such that $\sqrt{I} = \mathfrak{m}$ is called *an ideal of definition of R.*

Let $I \subset R$ be an ideal of definition. Because R is Noetherian this means that $\mathfrak{m}^r \subset I$ for some r, see Lemma 10.47.3. Hence any finite R-module annihilated by a power of I has a finite length, see Lemma 10.48.8. Thus it makes sense to define

$$\varphi_{I,M}(n) = \text{length}_R(I^n M/I^{n+1} M) \quad \text{and} \quad \chi_{I,M}(n) = \text{length}_R(M/I^{n+1} M)$$

for all $n \geq 0$. Again we have that

$$\chi_{I,M}(n) = \sum\nolimits_{i=0}^{n} \varphi_{I,M}(i).$$

Lemma 10.56.2. *Suppose that $M' \subset M$ are finite R-modules with finite length quotient. Then there exists a constants c_1, c_2 such that for all $n \geq c_2$ we have*

$$c_1 + \chi_{I,M'}(n - c_2) \leq \chi_{I,M}(n) \leq c_1 + \chi_{I,M'}(n)$$

Proof. Since M/M' has finite length there is a $c_2 \geq 0$ such that $I^{c_2}M \subset M'$. Let $c_1 = \text{length}_R(M/M')$. For $n \geq c_2$ we have

$$\begin{aligned} \chi_{I,M}(n) &= \text{length}_R(M/I^{n+1}M) \\ &= c_1 + \text{length}_R(M'/I^{n+1}M) \\ &\leq c_1 + \text{length}_R(M'/I^{n+1}M') \\ &= c_1 + \chi_{I,M'}(n) \end{aligned}$$

On the other hand, since $I^{c_2}M \subset M'$, we have $I^nM \subset I^{n-c_2}M'$ for $n \geq c_2$. Thus for $n \geq c_2$ we get

$$\begin{aligned} \chi_{I,M}(n) &= \text{length}_R(M/I^{n+1}M) \\ &= c_1 + \text{length}_R(M'/I^{n+1}M) \\ &\geq c_1 + \text{length}_R(M'/I^{n+1-c_2}M') \\ &= c_1 + \chi_{I,M'}(n - c_2) \end{aligned}$$

which finishes the proof. □

Lemma 10.56.3. *Suppose that $0 \to M' \to M \to M'' \to 0$ is a short exact sequence of finite R-modules. Then there exists a submodule $N \subset M'$ with finite colength l and $c \geq 0$ such that*

$$\chi_{I,M}(n) = \chi_{I,M''}(n) + \chi_{I,N}(n - c) + l$$

and

$$\varphi_{I,M}(n) = \varphi_{I,M''}(n) + \varphi_{I,N}(n - c)$$

for all $n \geq c$.

Proof. Note that $M/I^nM \to M''/I^nM''$ is surjective with kernel $M'/M' \cap I^nM$. By the Artin-Rees Lemma 10.47.4 there exists a constant c such that $M' \cap I^nM = I^{n-c}(M' \cap I^cM)$. Denote $N = M' \cap I^cM$. Note that $I^cM' \subset N \subset M'$. Hence $\text{length}_R(M'/M' \cap I^nM) = \text{length}_R(M'/N) + \text{length}_R(N/I^{n-c}N)$ for $n \geq c$. From the short exact sequence

$$0 \to M'/M' \cap I^nM \to M/I^nM \to M''/I^nM'' \to 0$$

and additivity of lengths (Lemma 10.48.3) we obtain the equality

$$\chi_{I,M}(n-1) = \chi_{I,M''}(n-1) + \chi_{I,N}(n-c-1) + \text{length}_R(M'/N)$$

for $n \geq c$. we have $\varphi_{I,M}(n) = \chi_{I,M}(n) - \chi_{I,M}(n-1)$ and similarly for the modules M'' and N. Hence we get $\varphi_{I,M}(n) = \varphi_{I,M''}(n) + \varphi_{I,N}(n-c)$ for $n \geq c$. □

Lemma 10.56.4. *Suppose that I, I' are two ideals of definition for the Noetherian local ring R. Let M be a finite R-module. There exists a constant a such that $\chi_{I,M}(n) \leq \chi_{I',M}(an)$ for $n \geq 1$.*

Proof. There exists an integer c such that $(I')^c \subset I$. Hence we get a surjection $M/(I')^{c(n+1)}M \to M/I^{n+1}M$. Whence the result with $a = c + 1$. □

Proposition 10.56.5. *Let R be a Noetherian local ring. Let M be a finite R-module. Let $I \subset R$ be an ideal of definition. The Hilbert function $\varphi_{I,M}$ and the function $\chi_{I,M}$ are numerical polynomials.*

Proof. Consider the graded ring $S = R/I \oplus I/I^2 \oplus I^2/I^3 \oplus \ldots = \bigoplus_{d\geq 0} I^d/I^{d+1}$. Consider the graded S-module $N = M/IM \oplus IM/I^2M \oplus \ldots = \bigoplus_{d\geq 0} I^dM/I^{d+1}M$. This pair (S, N) satisfies the hypotheses of Proposition 10.55.7. Hence the result follows from that proposition and Lemma 10.51.1. □

Definition 10.56.6. Let R be a Noetherian local ring. Let M be a finite R-module. The *Hilbert polynomial* of M over R is the element $P(t) \in \mathbf{Q}[t]$ such that $P(n) = \varphi_M(n)$ for $n \gg 0$.

By Proposition 10.56.5 we see that the Hilbert polynomial exists.

Lemma 10.56.7. *Let R be a Noetherian local ring. Let M be a finite R-module.*

1. *The degree of the numerical polynomial $\varphi_{I,M}$ is independent of the ideal of definition I.*
2. *The degree of the numerical polynomial $\chi_{I,M}$ is independent of the ideal of definition I.*

Proof. Part (2) follows immediately from Lemma 10.56.4. Part (1) follows from (2) because $\varphi_{I,M}(n) = \chi_{I,M}(n) - \chi_{I,M}(n-1)$ for $n \geq 1$. □

Definition 10.56.8. Let R be a local Noetherian ring and M a finite R-module. We denote $d(M)$ the element of $\{-\infty, 0, 1, 2, \ldots\}$ defined as follows:

1. If $M = 0$ we set $d(M) = -\infty$,
2. if $M \neq 0$ then $d(M)$ is the degree of the numerical polynomial χ_M.

If $\mathfrak{m}^n M \neq 0$ for all n, then we see that $d(M)$ is the degree $+1$ of the Hilbert polynomial of M.

Lemma 10.56.9. *Let R be a Noetherian local ring. Let $I \subset R$ be an ideal of definition. Let M be a finite R-module which does not have finite length. If $M' \subset M$ is a submodule with finite colength, then $\chi_{I,M} - \chi_{I,M'}$ is a polynomial of degree $<$ degree of either polynomial.*

Proof. Follows from Lemma 10.56.2 by elementary calculus. □

Lemma 10.56.10. *Let R be a Noetherian local ring. Let $I \subset R$ be an ideal of definition. Let $0 \to M' \to M \to M'' \to 0$ be a short exact sequence of finite R-modules. Then*

1. *if M' does not have finite length, then $\chi_{I,M} - \chi_{I,M''} - \chi_{I,M'}$ is a numerical polynomial of degree $<$ the degree of $\chi_{I,M'}$,*
2. *$\max\{\deg(\chi_{I,M'}), \deg(\chi_{I,M''})\} = \deg(\chi_{I,M})$, and*
3. *$\max\{d(M'), d(M'')\} = d(M)$,*

Proof. We first prove (1). Let $N \subset M'$ be as in Lemma 10.56.3. By Lemma 10.56.9 the numerical polynomial $\chi_{I,M'} - \chi_{I,N}$ has degree $<$ the common degree of $\chi_{I,M'}$ and $\chi_{I,N}$. By Lemma 10.56.3 the difference

$$\chi_{I,M}(n) - \chi_{I,M''}(n) - \chi_{I,N}(n - c)$$

is constant for $n \gg 0$. By elementary calculus the difference $\chi_{I,N}(n) - \chi_{I,N}(n-c)$ has degree $<$ the degree of $\chi_{I,N}$ which is bigger than zero (see above). Putting everything together we obtain (1).

Note that the leading coefficients of $\chi_{I,M'}$ and $\chi_{I,M''}$ are nonnegative. Thus the degree of $\chi_{I,M'} + \chi_{I,M''}$ is equal to the maximum of the degrees. Thus if M' does not have finite length, then (2) follows from (1). If M' does have finite length, then $I^nM \to I^nM''$ is an isomorphism for all $n \gg 0$ by Artin-Rees (Lemma 10.47.4). Thus $M/I^nM \to M''/I^nM''$ is a surjection with kernel M' for $n \gg 0$ and we see that $\chi_{I,M}(n) - \chi_{I,M''}(n) = \text{length}(M')$ for all $n \gg 0$. Thus (2) holds in this case also.

Proof of (3). This follows from (2) except if one of M, M', or M'' is zero. We omit the proof in these special cases. □

10.57. Dimension

Definition 10.57.1. The *Krull dimension* of the ring R is the Krull dimension of the topological space $\text{Spec}(R)$, see Topology, Definition 5.9.1. In other words it is the supremum of the integers $n \geq 0$ such that there exists a chain of prime ideals of length n:

$$\mathfrak{p}_0 \subset \mathfrak{p}_1 \subset \ldots \subset \mathfrak{p}_n, \quad \mathfrak{p}_i \neq \mathfrak{p}_{i+1}.$$

Definition 10.57.2. The *height* of a prime ideal $\mathfrak{p}$ of a ring R is the dimension of the local ring $R_\mathfrak{p}$.

Lemma 10.57.3. *The Krull dimension of R is the supremum of the heights of its (maximal) primes.*

Proof. This is so because we can always add a maximal ideal at the end of a chain of prime ideals. □

Lemma 10.57.4. *A Noetherian ring of dimension 0 is Artinian. Conversely, any Artinian ring is Noetherian of dimension zero.*

Proof. By Lemma 10.29.5 the space $\text{Spec}(R)$ is Noetherian. By Topology, Lemma 5.8.2 we see that $\text{Spec}(R)$ has finitely many irreducible components, say $\text{Spec}(R) = Z_1 \cup \ldots Z_r$. According to Lemma 10.24.1, each $Z_i = V(\mathfrak{p}_i)$ with $\mathfrak{p}_i$ a minimal ideal. Since the dimension is 0 these $\mathfrak{p}_i$ are also maximal. Thus $\text{Spec}(R)$ is the discrete topological space with elements $\mathfrak{p}_i$. All elements f of the radical $I = \cap \mathfrak{p}_i$ are nilpotent since otherwise R_f would not be the zero ring and we would have another prime. Since I is finitely generated we conclude that I is nilpotent, Lemma 10.47.3. By Lemma 10.49.7 R is the product of its local rings. By Lemma 10.48.8 each of these has finite length over R. Hence we conclude that R is Artinian by Lemma 10.49.8.

If R is Artinian then by Lemma 10.49.8 it is Noetherian. All of its primes are maximal by a combination of Lemmas 10.49.3, 10.49.4 and 10.49.7. □

In the following we will use the invariant $d(-)$ defined in Definition 10.56.8. Here is a warm up lemma.

Lemma 10.57.5. *Let R be a Noetherian local ring. Then* $\dim(R) = 0 \Leftrightarrow d(R) = 0$.

Proof. This is because $d(R) = 0$ if and only if R has finite length as an R-module. See Lemma 10.49.8. □

Proposition 10.57.6. *Let R be a ring. The following are equivalent:*

(1) *R is Artinian,*
(2) *R is Noetherian and $\dim(R) = 0$,*
(3) *R has finite length as a module over itself,*
(4) *R is a finite product of Artinian local rings,*
(5) *R is Noetherian and $\mathrm{Spec}(R)$ is a finite discrete topological space,*
(6) *R is a finite product of Noetherian local rings of dimension 0,*
(7) *R is a finite product of Noetherian local rings R_i with $d(R_i) = 0$,*
(8) *R is a finite product of Noetherian local rings R_i whose maximal ideals are nilpotent,*
(9) *R is Noetherian, has finitely many maximal ideals and its radical ideal is nilpotent, and*
(10) *R is Noetherian and there are no strict inclusions among its primes.*

Proof. This is a combination of Lemmas 10.49.7, 10.49.8, 10.57.4, and 10.57.5. □

Lemma 10.57.7. *Let R be a local Noetherian ring. The following are equivalent:*

(1) *$\dim(R) = 1$,*
(2) *$d(R) = 1$,*
(3) *there exists an $x \in \mathfrak{m}$, x not nilpotent such that $V(x) = \{\mathfrak{m}\}$,*
(4) *there exists an $x \in \mathfrak{m}$, x not nilpotent such that $\mathfrak{m} = \sqrt{(x)}$, and*
(5) *there exists an ideal of definition generated by 1 element, and no ideal of definition is generated by 0 elements.*

Proof. First, assume that $\dim(R) = 1$. Let $\mathfrak{p}_i$ be the minimal primes of R. Because the dimension is 1 the only other prime of R is $\mathfrak{m}$. According to Lemma 10.29.6 there are finitely many. Hence we can find $x \in \mathfrak{m}$, $x \notin \mathfrak{p}_i$, see Lemma 10.14.3. Thus the only prime containing x is $\mathfrak{m}$ and hence (3).

If (3) then $\mathfrak{m} = \sqrt{(x)}$ by Lemma 10.16.2, and hence (4). The converse is clear as well. The equivalence of (4) and (5) follows from directly the definitions.

Assume (5). Let $I = (x)$ be an ideal of definition. Note that I^n/I^{n+1} is a quotient of R/I via multiplication by x^n and hence $\mathrm{length}_R(I^n/I^{n+1})$ is bounded. Thus $d(R) = 0$ or $d(R) = 1$, but $d(R) = 0$ is excluded by the assumption that 0 is not an ideal of definition.

Assume (2). To get a contradiction, assume there exist primes $\mathfrak{p} \subset \mathfrak{q} \subset \mathfrak{m}$, with both inclusions strict. Pick some ideal of definition $I \subset R$. We will repeatedly use Lemma 10.56.10. First of all it implies, via the exact sequence $0 \to \mathfrak{p} \to R \to R/\mathfrak{p} \to 0$, that $d(R/\mathfrak{p}) \leq 1$. But it clearly cannot be zero. Pick $x \in \mathfrak{q}$, $x \notin \mathfrak{p}$. Consider the short exact sequence

$$0 \to R/\mathfrak{p} \to R/\mathfrak{p} \to R/(xR + \mathfrak{p}) \to 0.$$

This implies that $\chi_{I,R/\mathfrak{p}} - \chi_{I,R/\mathfrak{p}} - \chi_{I,R/(xR+\mathfrak{p})} = -\chi_{I,R/(xR+\mathfrak{p})}$ has degree < 1. In other words, $d(R/(xR + \mathfrak{p}) = 0$, and hence $\dim(R/(xR + \mathfrak{p})) = 0$, by Lemma 10.57.5. But $R/(xR + \mathfrak{p})$ has the distinct primes $\mathfrak{q}/(xR + \mathfrak{p})$ and $\mathfrak{m}/(xR + \mathfrak{p})$ which gives the desired contradiction. □

Proposition 10.57.8. *Let R be a local Noetherian ring. The following are equivalent:*

(1) *$\dim(R) = d$,*

(2) $d(R) = d$,
(3) *there exists an ideal of definition generated by d elements, and no ideal of definition is generated by fewer than d elements.*

Proof. This proof is really just the same as the proof of Lemma 10.57.7. We will prove the proposition by induction on d. By Lemmas 10.57.5 and 10.57.7 we may assume that $d > 1$. Denote the minimal number of generators for an ideal of definition of R by $d'(R)$. We will prove that the inequalities $\dim(R) \geq d'(R) \geq d(R) \geq \dim(R)$, and hence they are all equal.

First, assume that $\dim(R) = d$. Let $\mathfrak{p}_i$ be the minimal primes of R. According to Lemma 10.29.6 there are finitely many. Hence we can find $x \in \mathfrak{m}$, $x \notin \mathfrak{p}_i$, see Lemma 10.14.3. Note that every maximal chain of primes starts with some $\mathfrak{p}_i$, hence the dimension of R/xR is at most $d - 1$. By induction there are $x_2, \ldots, x_d$ which generate an ideal of definition in R/xR. Hence R has an ideal of definition generated by (at most) d elements.

Assume $d'(R) = d$. Let $I = (x_1, \ldots, x_d)$ be an ideal of definition. Note that I^n/I^{n+1} is a quotient of a direct sum of $\binom{d+n-1}{d-1}$ copies R/I via multiplication by all degree n monomials in $x_1, \ldots, x_n$. Hence $\text{length}_R(I^n/I^{n+1})$ is bounded by a polynomial of degree $d - 1$. Thus $d(R) \leq d$.

Assume $d(R) = d$. Consider a chain of primes $\mathfrak{p} \subset \mathfrak{q} \subset \mathfrak{q}_2 \subset \ldots \subset \mathfrak{p}_e = \mathfrak{m}$, with all inclusions strict, and $e \geq 2$. Pick some ideal of definition $I \subset R$. We will repeatedly use Lemma 10.56.10. First of all it implies, via the exact sequence $0 \to \mathfrak{p} \to R \to R/\mathfrak{p} \to 0$, that $d(R/\mathfrak{p}) \leq d$. But it clearly cannot be zero. Pick $x \in \mathfrak{q}$, $x \notin \mathfrak{p}$. Consider the short exact sequence

$$0 \to R/\mathfrak{p} \to R/\mathfrak{p} \to R/(xR + \mathfrak{p}) \to 0.$$

This implies that $\chi_{I,R/\mathfrak{p}} - \chi_{I,R/\mathfrak{p}} - \chi_{I,R/(xR+\mathfrak{p})} = -\chi_{I,R/(xR+\mathfrak{p})}$ has degree $< d$. In other words, $d(R/(xR + \mathfrak{p})) \leq d - 1$, and hence $\dim(R/(xR + \mathfrak{p})) \leq d - 1$, by induction. Now $R/(xR+\mathfrak{p})$ has the chain of prime ideals $\mathfrak{q}/(xR+\mathfrak{p}) \subset \mathfrak{q}_2/(xR+\mathfrak{p}) \subset \ldots \subset \mathfrak{q}_e/(xR + \mathfrak{p})$ which gives $e - 1 \leq d - 1$. Since we started with an arbitrary chain of primes this proves that $\dim(R) \leq d(R)$.

Reading back the reader will see we proved the circular inequalities as desired. □

Let $(R, \mathfrak{m})$ be a Noetherian local ring. From the above it is clear that $\mathfrak{m}$ cannot be generated by fewer than $\dim(R)$ variables. By Nakayama's Lemma 10.18.1 the minimal number of generators of $\mathfrak{m}$ equals $\dim_{\kappa(\mathfrak{m})} \mathfrak{m}/\mathfrak{m}^2$. Hence we have the following fundamental inequality

$$\dim(R) \leq \dim_{\kappa(\mathfrak{m})} \mathfrak{m}/\mathfrak{m}^2.$$

It turns out that the rings where equality holds have a lot of good properties. They are called regular local rings.

Definition 10.57.9. Let $(R, \mathfrak{m})$ be a Noetherian local ring of dimension d.

(1) A *system of parameters of R* is a sequence of elements $x_1, \ldots, x_d \in \mathfrak{m}$ which generates an ideal of definition of R,
(2) if there exist $x_1, \ldots, x_d \in \mathfrak{m}$ such that $\mathfrak{m} = (x_1, \ldots, x_d)$ then we call R a *regular local ring* and $x_1, \ldots, x_d$ a *regular system of parameters*.

The following two lemmas are clear from the proofs of the lemmas and proposition above, but we spell them out so we have convenient references.

Lemma 10.57.10. *Let R be a Noetherian ring.*

(1) *Let $x \in R$, $\mathfrak{p}, \mathfrak{q} \in \operatorname{Spec}(R)$. Suppose that $\mathfrak{p} \subset (\mathfrak{p}, x) \subset \mathfrak{q}$ and $\mathfrak{q}$ minimal over $(\mathfrak{p}, x)$. Then there is no prime strictly between $\mathfrak{p}$ and $\mathfrak{q}$.*
(2) *If $x \in R$ and $x \in \mathfrak{p}$ is minimal over (x) then the height of $\mathfrak{p}$ is 0 or 1.*

Proof. Consider the situation of the first assertion. The primes containing $\mathfrak{p}$ and contained in $\mathfrak{q}$ correspond to primes of $R_\mathfrak{q}/\mathfrak{p}R_\mathfrak{q}$, and the primes containing x correspond to the ones containing the image of x. Thus we may assume R is a Noetherian local domain, $\mathfrak{p} = (0)$ and $\mathfrak{q}$ maximal. Now since $\sqrt{(x)}$ is the intersection of the prime ideals containing it, and since $\mathfrak{q}$ is the only prime containing x by minimality, we see that $\sqrt{(x)} = \mathfrak{q}$. Hence Lemma 10.57.7 applies. The second assertion follows from the first. □

Lemma 10.57.11. *Suppose that R is a Noetherian local ring and $x \in \mathfrak{m}$ an element of its maximal ideal. Then $\dim R \leq \dim R/xR + 1$. If x is not contained in any of the minimal primes of R then equality holds. (For example if x is a nonzerodivisor.)*

Proof. If $x_1, \ldots, x_{\dim R/xR} \in R$ map to elements of R/xR which generate an ideal of definition for R/xR, then $x, x_1, \ldots, x_{\dim R/xR}$ generate an ideal of definition for R. Hence the inequality by Proposition 10.57.8. On the other hand, if x is not contained in any minimal prime of R, then the chains of primes in R/xR all give rise to chains in R which are at least one step away from being maximal. □

Lemma 10.57.12. *Let $(R, \mathfrak{m})$ be a Noetherian local ring. Suppose $x_1, \ldots, x_d \in \mathfrak{m}$ generate an ideal of definition and $d = \dim(R)$. Then $\dim(R/(x_1, \ldots, x_i)) = d - i$ for all $i = 1, \ldots, d$.*

Proof. Follows either from the proof of Proposition 10.57.8, or by using induction on d and Lemma 10.57.11. □

10.58. Applications of dimension theory

We can use the results on dimension to prove certain rings are Jacobson.

Lemma 10.58.1. *Let R be a Noetherian local domain of dimension ≥ 2. A nonempty open subset $U \subset \operatorname{Spec}(R)$ is infinite.*

Proof. To get a contradiction, assume that $U \subset \operatorname{Spec}(R)$ is finite. In this case $(0) \in U$ and $\{(0)\}$ is an open subset of U (because the complement of $\{(0)\}$ is the union of the closures of the other points). Thus we may assume $U = \{(0)\}$. Let $\mathfrak{m} \subset R$ be the maximal ideal. We can find an $x \in \mathfrak{m}$, $x \neq 0$ such that $V(x) \cup U = \operatorname{Spec}(R)$. In other words we see that $D(x) = \{(0)\}$. In particular we see that $\dim(R/xR) = \dim(R) - 1 \geq 1$, see Lemma 10.57.11. Let $\overline{y}_2, \ldots, \overline{y}_{\dim(R)} \in R/xR$ generate an ideal of definition of R/xR, see Proposition 10.57.8. Choose lifts $y_2, \ldots, y_{\dim(R)} \in R$, so that $x, y_2, \ldots, y_{\dim(R)}$ generate an ideal of definition in R. This implies that $\dim(R/(y_2)) = \dim(R) - 1$ and $\dim(R/(y_2, x)) = \dim(R) - 2$, see Lemma 10.57.12. Hence there exists a prime $\mathfrak{p}$ containing y_2 but not x. This contradicts the fact that $D(x) = \{(0)\}$. □

Lemma 10.58.2. *Noetherian Jacobson rings.*

(1) *Any Noetherian domain R of dimension 1 with infinitely many primes is Jacobson.*
(2) *Any Noetherian ring such that every prime $\mathfrak{p}$ is either maximal or contained in infinitely many prime ideals is Jacobson.*

Proof. Part (1) is a reformulation of Lemma 10.32.6.

Let R be a Noetherian ring such that every non-maximal prime $\mathfrak{p}$ is contained in infinitely many prime ideals. Assume $\mathrm{Spec}(R)$ is not Jacobson to get a contradiction. By Lemmas 10.24.1 and 10.29.5 we see that $\mathrm{Spec}(R)$ is a sober, Noetherian topological space. By Topology, Lemma 5.17.3 we see that there exists a non-maximal ideal $\mathfrak{p} \subset R$ such that $\{\mathfrak{p}\}$ is a locally closed subset of $\mathrm{Spec}(R)$. In other words, $\mathfrak{p}$ is not maximal and $\{\mathfrak{p}\}$ is an open subset of $V(\mathfrak{p})$. Consider a prime $\mathfrak{q} \subset R$ with $\mathfrak{p} \subset \mathfrak{q}$. Recall that the topology on the spectrum of $(R/\mathfrak{p})_\mathfrak{q} = R_\mathfrak{q}/\mathfrak{p}R_\mathfrak{q}$ is induced from that of $\mathrm{Spec}(R)$, see Lemmas 10.16.5 and 10.16.7. Hence we see that $\{(0)\}$ is a locally closed subset of $\mathrm{Spec}((R/\mathfrak{p})_\mathfrak{q})$. By Lemma 10.58.1 we conclude that $\dim((R/\mathfrak{p})_\mathfrak{q}) = 1$. Since this holds for every $\mathfrak{q} \supset \mathfrak{p}$ we conclude that $\dim(R/\mathfrak{p}) = 1$. At this point we use the assumption that $\mathfrak{p}$ is contained in infinitely many primes to see that $\mathrm{Spec}(R/\mathfrak{p})$ is infinite. Hence by part (1) of the lemma we see that $V(\mathfrak{p}) \cong \mathrm{Spec}(R/\mathfrak{p})$ is the closure of its closed points. This is the desired contradiction since it means that $\{\mathfrak{p}\} \subset V(\mathfrak{p})$ cannot be open. □

10.59. Support and dimension of modules

Lemma 10.59.1. *Let R be a Noetherian ring, and let M be a finite R-module. There exists a filtration by R-submodules*

$$0 = M_0 \subset M_1 \subset \ldots \subset M_n = M$$

such that each quotient M_i/M_{i-1} is isomorphic to $R/\mathfrak{p}_i$ for some prime ideal $\mathfrak{p}_i$ of R.

Proof. By Lemma 10.5.4 it suffices to do the case $M = R/I$ for some ideal I. Consider the set S of ideals J such that the lemma does not hold for the module R/J, and order it by inclusion. To arrive at a contradiction, assume that S is not empty. Because R is Noetherian, S has a maximal element J. By definition of S, the ideal J cannot be prime. Pick $a, b \in R$ such that $ab \in J$, but neither $a \in J$ nor $b \in J$. Consider the filtration $0 \subset aR/(J \cap aR) \subset R/J$. Note that $aR/(J \cap aR)$ is a quotient of $R/(J + bR)$ and the second quotient equals $R/(aR + J)$. Hence by maximality of J, each of these has a filtration as above and hence so does R/J. Contradiction. □

Lemma 10.59.2. *Let R, M, M_i, $\mathfrak{p}_i$ as in Lemma 10.59.1. All of the primes $\mathfrak{p}_i$ are in the support of M.*

Proof. Since localization is exact, we see that $(R/\mathfrak{p}_i)_{\mathfrak{p}_i}$ is a subquotient of $M_{\mathfrak{p}_i}$. Hence $M_{\mathfrak{p}_i}$ is not zero. □

Lemma 10.59.3. *Suppose that R is a Noetherian local ring with maximal ideal $\mathfrak{m}$. Let M be a finite R-module. Then $\mathrm{Supp}(M) = \{\mathfrak{m}\}$ if and only if M has finite length over R.*

Proof. Assume that $\mathrm{Supp}(M) = \{\mathfrak{m}\}$. It suffices to show that all the primes $\mathfrak{p}_i$ in the filtration of Lemma 10.59.1 are the maximal ideal. This is clear by Lemma 10.59.2.

Suppose that M has finite length over R. Then $\mathfrak{m}^n M = 0$ by Lemma 10.48.4. Since some element of $\mathfrak{m}$ maps to a unit in $R_\mathfrak{p}$ for any prime $\mathfrak{p} \neq \mathfrak{m}$ in R we see $M_\mathfrak{p} = 0$. □

Lemma 10.59.4. *Let R be a Noetherian ring. Let $I \subset R$ be an ideal. Let M be a finite R-module. Then $I^n M = 0$ for some $n \geq 0$ if and only if $Supp(M) \subset V(I)$.*

Proof. It is clear that $I^n M = 0$ for some $n \geq 0$ implies $\mathrm{Supp}(M) \subset V(I)$. Suppose that $\mathrm{Supp}(M) \subset V(I)$. Choose a filtration $0 = M_0 \subset M_1 \subset \ldots \subset M_n = M$ as in Lemma 10.59.1. Each of the primes $\mathfrak{p}_i$ is contained in $V(I)$ by Lemma 10.59.2. Hence $I \subset \mathfrak{p}_i$ and I annihilates M_i/M_{i-1}. Hence I^n annihilates M. □

Lemma 10.59.5. *Let R, M, M_i, $\mathfrak{p}_i$ as in Lemma 10.59.1. The minimal elements of the set $\{\mathfrak{p}_i\}$ are the minimal elements of $Supp(M)$, and the number of times a minimal prime $\mathfrak{p}$ occurs is*

$$\#\{i \mid \mathfrak{p}_i = \mathfrak{p}\} = length_{R_\mathfrak{p}} M_\mathfrak{p}.$$

Proof. We have already seen $\{\mathfrak{p}_i\} \subset \mathrm{Supp}(M)$, in Lemma 10.59.2. Let $\mathfrak{p} \in \mathrm{Supp}(M)$ be minimal. The support of $M_\mathfrak{p}$ is the set consisting of the maximal ideal $\mathfrak{p}R_\mathfrak{p}$. Hence by Lemma 10.59.3 the length of $M_\mathfrak{p}$ is finite and > 0. Next we note that $M_\mathfrak{p}$ has a filtration with subquotients $(R/\mathfrak{p}_i)_\mathfrak{p} = R_\mathfrak{p}/\mathfrak{p}_i R_\mathfrak{p}$ These are zero if $\mathfrak{p}_i \not\subset \mathfrak{p}$ and equal to $\kappa(\mathfrak{p})$ if $\mathfrak{p}_i \subset \mathfrak{p}$ because by minimality of $\mathfrak{p}$ we have $\mathfrak{p}_i = \mathfrak{p}$ in this case. The result follows since $\kappa(\mathfrak{p})$ has length 1. □

Lemma 10.59.6. *Let R be a Noetherian local ring. Let M be a finite R-module. Then $d(M) = \dim(Supp(M))$.*

Proof. Let $M_i, \mathfrak{p}_i$ be as in Lemma 10.59.1. By Lemma 10.56.10 we obtain the equality $d(M) = \max\{d(R/\mathfrak{p}_i)\}$. By Proposition 10.57.8 we have $d(R/\mathfrak{p}_i) = \dim(R/\mathfrak{p}_i)$. Trivially $\dim(R/\mathfrak{p}_i) = \dim V(\mathfrak{p}_i)$. Since all minimal primes of $\mathrm{Supp}(M)$ occur among the $\mathfrak{p}_i$ we win. □

10.60. Associated primes

Here is the standard definition. For non-Noetherian rings and non-finite modules it may be more appropriate to use the definition in Section 10.63.

Definition 10.60.1. Let R be a ring. Let M be an R-module. A prime $\mathfrak{p}$ of R is *associated* to M if there exists an element $m \in M$ whose annihilator is $\mathfrak{p}$. The set of all such primes is denoted $\mathrm{Ass}_R(M)$ or $\mathrm{Ass}(M)$.

Lemma 10.60.2. *Let R be a ring. Let M be an R-module. Then $Ass(M) \subset Supp(M)$.*

Proof. If $m \in M$ has annihilator $\mathfrak{p}$, then in particular no element of $R \backslash \mathfrak{p}$ annihilates m. Hence m is a nonzero element of $M_\mathfrak{p}$, i.e., $\mathfrak{p} \in \mathrm{Supp}(M)$. □

Lemma 10.60.3. *Let R be a ring. Let $0 \to M' \to M \to M'' \to 0$ be a short exact sequence of R-modules. Then $Ass(M') \subset Ass(M)$ and $Ass(M) \subset Ass(M') \cup Ass(M'')$.*

Proof. Omitted. □

Lemma 10.60.4. *Let R be a ring, and M an R-module. Suppose there exists a filtration by R-submodules*

$$0 = M_0 \subset M_1 \subset \ldots \subset M_n = M$$

such that each quotient M_i/M_{i-1} is isomorphic to $R/\mathfrak{p}_i$ for some prime ideal $\mathfrak{p}_i$ of R. Then $Ass(M) \subset \{\mathfrak{p}_1, \ldots, \mathfrak{p}_n\}$.

Proof. By induction on the length n of the filtration $\{M_i\}$. Pick $m \in M$ whose annihilator is a prime $\mathfrak{p}$. If $m \in M_{n-1}$ we are done by induction. If not, then m maps to a nonzero element of $M/M_{n-1} \cong R/\mathfrak{p}_n$. Hence we have $\mathfrak{p} \subset \mathfrak{p}_n$. If equality does not hold, then we can find $f \in \mathfrak{p}_n$, $f \notin \mathfrak{p}$. In this case the annihilator of fm is still $\mathfrak{p}$ and $fm \in M_{n-1}$. Thus we win by induction. □

Lemma 10.60.5. *Let R be a Noetherian ring. Let M be a finite R-module. Then $Ass(M)$ is finite.*

Proof. Immediate from Lemma 10.60.4 and Lemma 10.59.1. □

Proposition 10.60.6. *Let R be a Noetherian ring. Let M be a finite R-module. The following sets of primes are the same:*

1. *The minimal primes in the support of M.*
2. *The minimal primes in $Ass(M)$.*
3. *For any filtration $0 = M_0 \subset M_1 \subset \ldots \subset M_{n-1} \subset M_n = M$ with $M_i/M_{i-1} \cong R/\mathfrak{p}_i$ the minimal primes of the set $\{\mathfrak{p}_i\}$.*

Proof. Part of this we saw in Lemma 10.59.5. It suffices to prove that if $\mathfrak{p}$ is a minimal element of the set $\{\mathfrak{p}_i\}$ then it is the annihilator of an element of M. Let i be minimal such that $\mathfrak{p} = \mathfrak{p}_i$. Pick $m \in M_i$, $m \notin M_{i-1}$. The annihilator of m is contained in $\mathfrak{p}_i = \mathfrak{p}$ and contains $\mathfrak{p}_1\mathfrak{p}_2 \ldots \mathfrak{p}_i$. By our choice of i we have $\mathfrak{p}_1\mathfrak{p}_2 \ldots \mathfrak{p}_{i-1} \not\subset \mathfrak{p}_i$. Pick $f \in \mathfrak{p}_1\mathfrak{p}_2 \ldots \mathfrak{p}_{i-1}$, $f \notin \mathfrak{p}$. Then fm has annihilator $\mathfrak{p}$. □

Lemma 10.60.7. *Let R be a Noetherian ring. Let M be an R-module. Then*

$$M = (0) \Leftrightarrow Ass(M) = \emptyset.$$

Proof. If $M = (0)$, then $\mathrm{Ass}(M) = \emptyset$ by definition. If $M \neq 0$, pick any nonzero finitely generated submodule $M' \subset M$, for example a submodule generated by a single nonzero element. By Lemma 10.37.2 we see that $\mathrm{Supp}(M')$ is nonempty. By Proposition 10.60.6 this implies that $\mathrm{Ass}(M')$ is nonempty. By Lemma 10.60.3 this implies $\mathrm{Ass}(M) \neq \emptyset$. □

Lemma 10.60.8. *Let R be a Noetherian ring. Let M be an R-module. Any $\mathfrak{p} \in Supp(M)$ which is minimal among the elements of $Supp(M)$ is an element of $Ass(M)$.*

Proof. If M is a finite R-module, then this is a consequence of Proposition 10.60.6. In general write $M = \bigcup M_\lambda$ as the union of its finite submodules, and use that $\mathrm{Supp}(M) = \bigcup \mathrm{Supp}(M_\lambda)$ and $\mathrm{Ass}(M) = \bigcup \mathrm{Ass}(M_\lambda)$. □

Lemma 10.60.9. *Let R be a Noetherian ring. Let M be an R-module. The union $\bigcup_{\mathfrak{q} \in Ass(M)} \mathfrak{q}$ is the set of elements of R which are zerodivisors on M.*

Proof. Any element in any associated prime clearly is a zerodivisor on M. Conversely, suppose $x \in R$ is a zerodivisor on M. Consider the submodule $N = \{m \in M \mid xm = 0\}$. Since N is not zero it has an associated prime $\mathfrak{q}$ by Lemma 10.60.7. Then $x \in \mathfrak{q}$ and $\mathfrak{q}$ is an associated prime of M by Lemma 10.60.3. □

Lemma 10.60.10. *Let $\varphi : R \to S$ be a ring map. Let M be an S-module. Then* $\mathrm{Spec}(\varphi)(Ass_S(M)) \subset Ass_R(M)$.

Proof. If $\mathfrak{q} \in \mathrm{Ass}_S(M)$, then there exists an m in M such that the annihilator of m in S is $\mathfrak{q}$. Then the annihilator of m in R is $\mathfrak{q} \cap R$. □

Remark 10.60.11. Let $\varphi : R \to S$ be a ring map. Let M be an S-module. Then it is not always the case that $\mathrm{Spec}(\varphi)(\mathrm{Ass}_S(M)) \supset \mathrm{Ass}_R(M)$. For example, consider the ring map $R = k \to S = k[x_1, x_2, x_3, \ldots]/(x_i^2)$ and $M = S$. Then $\mathrm{Ass}_R(M)$ is not empty, but $\mathrm{Ass}_S(S)$ is empty.

Lemma 10.60.12. *Let $\varphi : R \to S$ be a ring map. Let M be an S-module. If S is Noetherian, then* $\mathrm{Spec}(\varphi)(Ass_S(M)) = Ass_R(M)$.

Proof. We have already seen in Lemma 10.60.10 that $\mathrm{Spec}(\varphi)(\mathrm{Ass}_S(M)) \subset \mathrm{Ass}_R(M)$. For the converse, choose a prime $\mathfrak{p} \in \mathrm{Ass}_R(M)$. Let $m \in M$ be an element such that the annihilator of x in R is $\mathfrak{p}$. Let $I = \{g \in S \mid gm = 0\}$ be the annihilator of m in S. Then $R/\mathfrak{p} \subset S/I$ is injective, hence there exists a prime $\mathfrak{q} \subset S$ lying over $\mathfrak{p}$, see Lemma 10.28.5. By Proposition 10.60.6 we see that $\mathfrak{q}$ is an associated prime of S/I, hence an associated prime of M by Lemma 10.60.3 and we win. □

Lemma 10.60.13. *Let R be a ring. Let I be an ideal. Let M be an R/I-module. Via the canonical injection* $\mathrm{Spec}(R/I) \to \mathrm{Spec}(R)$ *we have* $Ass_{R/I}(M) = Ass_R(M)$.

Proof. Omitted. □

Lemma 10.60.14. *Let R be a ring. Let M be an R-module. Let $\mathfrak{p} \subset R$ be a prime.*

1. *If $\mathfrak{p} \in Ass(M)$ then $\mathfrak{p}R_\mathfrak{p} \in Ass(M_\mathfrak{p})$.*
2. *If $\mathfrak{p}$ is finitely generated then the converse holds as well.*

Proof. If $\mathfrak{p} \in \mathrm{Ass}(M)$ there exists an element $m \in M$ whose annihilator is $\mathfrak{p}$. As localization is exact (Proposition 10.9.12) we see that the annihilator of $m/1$ in $M_\mathfrak{p}$ is $\mathfrak{p}R_\mathfrak{p}$ hence (1) holds. Assume $\mathfrak{p}R_\mathfrak{p} \in \mathrm{Ass}(M_\mathfrak{p})$ and $\mathfrak{p} = (f_1, \ldots, f_n)$. Let m/g be an element of $M_\mathfrak{p}$ whose annihilator is $\mathfrak{p}R_\mathfrak{p}$. This implies that the annihilator of m is contained in $\mathfrak{p}$. As $f_i m/g = 0$ in $M_\mathfrak{p}$ we see there exists a $g_i \in R$, $g_i \notin \mathfrak{p}$ such that $g_i f_i m = 0$ in M. Combined we see the annihilator of $g_1 \ldots g_n m$ is $\mathfrak{p}$. Hence $\mathfrak{p} \in \mathrm{Ass}(M)$. □

Lemma 10.60.15. *Let R be a ring. Let M be an R-module. Let $S \subset R$ be a multiplicative subset. Via the canonical injection* $\mathrm{Spec}(S^{-1}R) \to \mathrm{Spec}(R)$ *we have*

1. $Ass_R(S^{-1}M) = Ass_{S^{-1}R}(S^{-1}M)$,
2. $Ass_R(M) \cap \mathrm{Spec}(S^{-1}R) \subset Ass_R(S^{-1}M)$, *and*
3. *if R is Noetherian this inclusion is an equality.*

Proof. The first equality follows, since if $m \in S^{-1}M$, then the annihilator of m in R is the intersection of the annihilator of m in $S^{-1}R$ with R. The displayed inclusion and equality in the Noetherian case follows from Lemma 10.60.14 since for $\mathfrak{p} \in R$, $S \cap \mathfrak{p} = \emptyset$ we have $M_\mathfrak{p} = (S^{-1}M)_{S^{-1}\mathfrak{p}}$. □

Lemma 10.60.16. *Let R be a ring. Let M be an R-module. Let $S \subset R$ be a multiplicative subset. Assume that every $s \in S$ is a nonzerodivisor on M. Then*

$$Ass_R(M) = Ass_R(S^{-1}M).$$

Proof. As $M \subset S^{-1}M$ by assumption we get the inclusion $\text{Ass}(M) = \text{Ass}(S^{-1}M)$ from Lemma 10.60.3. Conversely, suppose that $n/s \in S^{-1}M$ is an element whose annihilator is a prime ideal $\mathfrak{p}$. Then the annihilator of $n \in M$ is also $\mathfrak{p}$. □

Lemma 10.60.17. *Let R be a Noetherian local ring with maximal ideal $\mathfrak{m}$. Let $I \subset \mathfrak{m}$ be an ideal. Let M be a finite R-module. The following are equivalent:*

1. *There exists an $x \in I$ which is not a zerodivisor on M.*
2. *We have $I \not\subset \mathfrak{q}$ for all $\mathfrak{q} \in Ass(M)$.*

Proof. If there exists a nonzerodivisor x in I, then x clearly cannot be in any associated prime of M. Conversely, suppose $I \not\subset \mathfrak{q}$ for all $\mathfrak{q} \in \text{Ass}(M)$. In this case we can choose $x \in I$, $x \notin \mathfrak{q}$ for all $\mathfrak{q} \in \text{Ass}(M)$ by Lemmas 10.60.5 and 10.14.3. By Lemma 10.60.9 the element x is not a zerodivisor on M. □

Lemma 10.60.18. *Let R be a ring. Let M be an R-module. If R is Noetherian the map*

$$M \longrightarrow \prod\nolimits_{\mathfrak{p} \in Ass(M)} M_{\mathfrak{p}}$$

is injective.

Proof. Let $x \in M$ be an element of the kernel of the map. Then if $\mathfrak{p}$ is an associated prime of $Rx \subset M$ we see on the one hand that $\mathfrak{p} \in \text{Ass}(M)$ (Lemma 10.60.3) and on the other hand that $(Rx)_{\mathfrak{p}} \subset M_{\mathfrak{p}}$ is not zero. This contradiction shows that $\text{Ass}(Rx) = \emptyset$. Hence $Rx = 0$ by Lemma 10.60.7. □

10.61. Symbolic powers

We only make the following definition in the case of a Noetherian ring although the formula itself makes sense in general.

Definition 10.61.1. Let R be a Noetherian ring. Let $\mathfrak{p}$ be a prime ideal. For $n \geq 0$ the nth *symbolic power* of $\mathfrak{p}$ is the ideal $\mathfrak{p}^{(n)} = \text{Ker}(R \to R_{\mathfrak{p}}/\mathfrak{p}^n R_{\mathfrak{p}})$.

Note that $\mathfrak{p}^n \subset \mathfrak{p}^{(n)}$ but equality does not always hold.

Lemma 10.61.2. *Let R be a Noetherian ring. Let $\mathfrak{p}$ be a prime ideal. Let $n > 0$. Then $Ass(R/\mathfrak{p}^{(n)}) = \{\mathfrak{p}\}$.*

Proof. If $\mathfrak{q}$ is an associated prime of $R/\mathfrak{p}^{(n)}$ then clearly $\mathfrak{p} \subset \mathfrak{q}$. On the other hand, any element $x \in R$, $x \notin \mathfrak{p}$ is a nonzerodivisor on $R/\mathfrak{p}^{(n)}$. Namely, if $y \in R$ and $xy \in \mathfrak{p}^{(n)} = R \cap \mathfrak{p}^n R_{\mathfrak{p}}$ then $y \in \mathfrak{p}^n R_{\mathfrak{p}}$, hence $y \in \mathfrak{p}^{(n)}$. Hence the lemma follows. □

10.62. Relative assassin

Discussion of relative assassins. Let $R \to S$ be a ring map. Let N be an S-module. In this situation we can introduce the following sets of primes $\mathfrak{q}$ of S:

- A with $\mathfrak{p} = R \cap \mathfrak{q}$ we have that $\mathfrak{q} \in \text{Ass}_S(N \otimes_R \kappa(\mathfrak{p}))$,
- A' with $\mathfrak{p} = R \cap \mathfrak{q}$ we have that $\mathfrak{q}$ is in the image of $\text{Ass}_{S \otimes \kappa(\mathfrak{p})}(N \otimes_R \kappa(\mathfrak{p}))$ under the canonical map $\text{Spec}(S \otimes_R \kappa(\mathfrak{p})) \to \text{Spec}(S)$,
- A_{fin} with $\mathfrak{p} = R \cap \mathfrak{q}$ we have that $\mathfrak{q} \in \text{Ass}_S(N/\mathfrak{p}N)$,

A'_{fin} for some prime $\mathfrak{p}' \subset R$ we have $\mathfrak{q} \in \mathrm{Ass}_S(N/\mathfrak{p}'N)$,
B for some R-module M we have $\mathfrak{q} \in \mathrm{Ass}_S(N \otimes_R M)$, and
B_{fin} for some finite R-module M we have $\mathfrak{q} \in \mathrm{Ass}_S(N \otimes_R M)$.

Let us determine some of the relations between theses sets.

Lemma 10.62.1. *Let $R \to S$ be a ring map. Let N be an S-module. Let A, A', A_{fin}, B, and B_{fin} be the subsets of* $\mathrm{Spec}(S)$ *introduced above.*

(1) *We always have $A = A'$.*
(2) *We always have $A_{fin} \subset A$, $B_{fin} \subset B$, $A_{fin} \subset A'_{fin} \subset B_{fin}$ and $A \subset B$.*
(3) *If S is Noetherian, then $A = A_{fin}$ and $B = B_{fin}$.*
(4) *If N is flat over R, then $A = A_{fin} = A'_{fin}$ and $B = B_{fin}$.*
(5) *If R is Noetherian and N is flat over R, then all of the sets are equal, i.e., $A = A' = A_{fin} = A'_{fin} = B = B_{fin}$.*

Proof. Some of the arguments in the proof will be repeated in the proofs of later lemmas which are more precise than this one (because they deal with a given module M or a given prime $\mathfrak{p}$ and not with the collection of all of them).

Proof of (1). Let $\mathfrak{p}$ be a prime of R. Then we have

$$\mathrm{Ass}_S(N \otimes_R \kappa(\mathfrak{p})) = \mathrm{Ass}_{S/\mathfrak{p}S}(N \otimes_R \kappa(\mathfrak{p})) = \mathrm{Ass}_{S \otimes_R \kappa(\mathfrak{p})}(N \otimes_R \kappa(\mathfrak{p}))$$

the first equality by Lemma 10.60.13 and the second by Lemma 10.60.15 part (1). This prove that $A = A'$. The inclusion $A_{fin} \subset A'_{fin}$ is clear.

Proof of (2). Each of the inclusions is immediate from the definitions except perhaps $A_{fin} \subset A$ which follows from Lemma 10.60.15 and the fact that we require $\mathfrak{p} = R \cap \mathfrak{q}$ in the formulation of A_{fin}.

Proof of (3). The equality $A = A_{fin}$ follows from Lemma 10.60.15 part (3) if S is Noetherian. Let $\mathfrak{q} = (g_1, \ldots, g_m)$ be a finitely generated prime ideal of S. Say $z \in N \otimes_R M$ is an element whose annihilator is $\mathfrak{q}$. We may pick a finite submodule $M' \subset M$ such that z is the image of $z' \in N \otimes_R M'$. Then $\mathrm{Ann}_S(z') \subset \mathfrak{q} = \mathrm{Ann}_S(z)$. Since $N \otimes_R -$ commutes with colimits and since M is the directed colimit of finite R-modules we can find $M' \subset M'' \subset M$ such that the image $z'' \in N \otimes_R M''$ is annihilated by $g_1, \ldots, g_m$. Hence $\mathrm{Ann}_S(z'') = \mathfrak{q}$. This proves that $B = B_{fin}$ if S is Noetherian.

Proof of (4). If N is flat, then the functor $N \otimes_R -$ is exact. In particular, if $M' \subset M$, then $N \otimes_R M' \subset N \otimes_R M$. Hence if $z \in N \otimes_R M$ is an element whose annihilator $\mathfrak{q} = \mathrm{Ann}_S(z)$ is a prime, then we can pick any finite R-submodule $M' \subset M$ such that $z \in N \otimes_R M'$ and we see that the annihilator of z as an element of $N \otimes_R M'$ is equal to $\mathfrak{q}$. Hence $B = B_{fin}$. Let $\mathfrak{p}'$ be a prime of R and let $\mathfrak{q}$ be a prime of S which is an associated prime of $N/\mathfrak{p}'N$. This implies that $\mathfrak{p}'S \subset \mathfrak{q}$. As N is flat over R we see that $N/\mathfrak{p}'N$ is flat over the integral domain $R/\mathfrak{p}'$. Hence every nonzero element of $R/\mathfrak{p}'$ is a nonzerodivisor on $N/\mathfrak{p}'$. Hence none of these elements can map to an element of $\mathfrak{q}$ and we conclude that $\mathfrak{p}' = R \cap \mathfrak{q}$. Hence $A_{fin} = A'_{fin}$. Finally, by Lemma 10.60.16 we see that $\mathrm{Ass}_S(N/\mathfrak{p}'N) = \mathrm{Ass}_S(N \otimes_R \kappa(\mathfrak{p}'))$, i.e., $A'_{fin} = A$.

Proof of (5). We only need to prove $A'_{fin} = B_{fin}$ as the other equalities have been proved in (4). To see this let M be a finite R-module. By Lemma 10.59.1 there exists a filtration by R-submodules

$$0 = M_0 \subset M_1 \subset \ldots \subset M_n = M$$

such that each quotient M_i/M_{i-1} is isomorphic to $R/\mathfrak{p}_i$ for some prime ideal $\mathfrak{p}_i$ of R. Since N is flat we obtain a filtration by S-submodules

$$0 = N \otimes_R M_0 \subset N \otimes_R M_1 \subset \ldots \subset N \otimes_R M_n = N \otimes_R M$$

such that each subquotient is isomorphic to $N/\mathfrak{p}_i N$. By Lemma 10.60.3 we conclude that $\text{Ass}_S(N \otimes_R M) \subset \bigcup \text{Ass}_S(N/\mathfrak{p}_i N)$. Hence we see that $B_{fin} \subset A'_{fin}$. Since the other inclusion is part of (2) we win. □

We define the relative assassin of N over S/R to be the set $A = A'$ above. As a motivation we point out that it depends only on the fibre modules $N \otimes_R \kappa(\mathfrak{p})$ over the fibre rings. As in the case of the assassin of a module we warn the reader that this notion makes most sense when the fibre rings $S \otimes_R \kappa(\mathfrak{p})$ are Noetherian, for example if $R \to S$ is of finite type.

Definition 10.62.2. Let $R \to S$ be a ring map. Let N be an S-module. The *relative assassin of N over S/R* is the set

$$\text{Ass}_{S/R}(N) = \{\mathfrak{q} \subset S \mid \mathfrak{q} \in \text{Ass}_S(N \otimes_R \kappa(\mathfrak{p})) \text{ with } \mathfrak{p} = R \cap \mathfrak{q}\}.$$

This is the set named A in Lemma 10.62.1.

The spirit of the next few results is that they are about the relative assassin, even though this may not be apparent.

Lemma 10.62.3. *Let $R \to S$ be a ring map. Let M be an R-module, and let N be an S-module. If N is flat as R-module, then*

$$Ass_S(M \otimes_R N) \supset \bigcup\nolimits_{\mathfrak{p} \in Ass_R(M)} Ass_S(N/\mathfrak{p}N)$$

and if R is Noetherian then we have equality.

Proof. If $\mathfrak{p} \in \text{Ass}_R(M)$ then there exists an injection $R/\mathfrak{p} \to M$. As N is flat over R we obtain an injection $R/\mathfrak{p} \otimes_R N \to M \otimes_R M$. Since $R/\mathfrak{p} \otimes_R N = N/\mathfrak{p}N$ we conclude that $\text{Ass}_S(N/\mathfrak{p}N) \subset \text{Ass}_S(M \otimes_R N)$, see Lemma 10.60.3. Hence the right hand side is contained in the left hand side.

Write $M = \bigcup M_\lambda$ as the union of its finitely generated R-submodules. Then also $N \otimes_R M = \bigcup N \otimes_R M_\lambda$ (as N is R-flat). By definition of associated primes we see that $\text{Ass}_S(N \otimes_R M) = \bigcup \text{Ass}_S(N \otimes_R M_\lambda)$ and $\text{Ass}_R(M) = \bigcup \text{Ass}(M_\lambda)$. Hence we may assume M is finitely generated.

Let $\mathfrak{q} \in \text{Ass}_S(M \otimes_R N)$, and assume R is Noetherian and M is a finite R-module. To finish the proof we have to show that $\mathfrak{q}$ is an element of the right hand side. First we observe that $\mathfrak{q}S_\mathfrak{q} \in \text{Ass}_{S_\mathfrak{q}}((M \otimes_R N)_\mathfrak{q})$, see Lemma 10.60.14. Let $\mathfrak{p}$ be the corresponding prime of R. Note that

$$(M \otimes_R N)_\mathfrak{q} = M \otimes_R N_\mathfrak{q} = M_\mathfrak{p} \otimes_{R_\mathfrak{p}} N_\mathfrak{q}$$

If $\mathfrak{p}R_\mathfrak{p} \notin \text{Ass}_{R_\mathfrak{p}}(M_\mathfrak{p})$ then there exists an element $x \in \mathfrak{p}R_\mathfrak{p}$ which is a nonzerodivisor in $M_\mathfrak{p}$ (see Lemma 10.60.17). Since $N_\mathfrak{q}$ is flat over $R_\mathfrak{p}$ we see that the image of x in $\mathfrak{q}S_\mathfrak{q}$ is a nonzerodivisor on $(M \otimes_R N)_\mathfrak{q}$. This is a contradiction with the assumption that $\mathfrak{q}S_\mathfrak{q} \in \text{Ass}_S((M \otimes_R N)_\mathfrak{q})$. Hence we conclude that $\mathfrak{p}$ is one of the associated primes of M.

Continuing the argument we choose a filtration

$$0 = M_0 \subset M_1 \subset \ldots \subset M_n = M$$

such that each quotient M_i/M_{i-1} is isomorphic to $R/\mathfrak{p}_i$ for some prime ideal $\mathfrak{p}_i$ of R, see Lemma 10.59.1. (By Lemma 10.60.4 we have $\mathfrak{p}_i = \mathfrak{p}$ for at least one i.) This gives a filtration

$$0 = M_0 \otimes_R N \subset M_1 \otimes_R N \subset \ldots \subset M_n \otimes_R N = M \otimes_R N$$

with subquotients isomorphic to $N/\mathfrak{p}_i N$. If $\mathfrak{p}_i \neq \mathfrak{p}$ then $\mathfrak{q}$ cannot be associated to the module $N/\mathfrak{p}_i N$ by the result of the preceding paragraph (as $\mathrm{Ass}_R(R/\mathfrak{p}_i) = \{\mathfrak{p}_i\}$). Hence we conclude that $\mathfrak{q}$ is associated to $N/\mathfrak{p}N$ as desired. □

Lemma 10.62.4. *Let $R \to S$ be a ring map. Let N be an S-module. Assume N is flat as an R-module and R is a domain with fraction field K. Then*

$$Ass_S(N) = Ass_S(N \otimes_R K) = Ass_{S \otimes_R K}(N \otimes_R K)$$

via the canonical inclusion $\mathrm{Spec}(S \otimes_R K) \subset \mathrm{Spec}(S)$.

Proof. Note that $S \otimes_R K = (R \setminus \{0\})^{-1}S$ and $N \otimes_R K = (R \setminus \{0\})^{-1}N$. For any nonzero $x \in R$ multiplication by x on N is injective as N is flat over R. Hence the lemma follows from Lemma 10.60.16 combined with Lemma 10.60.15 part (1). □

Lemma 10.62.5. *Let $R \to S$ be a ring map. Let M be an R-module, and let N be an S-module. Assume N is flat as R-module. Then*

$$Ass_S(M \otimes_R N) \supset \bigcup\nolimits_{\mathfrak{p} \in Ass_R(M)} Ass_{S \otimes_R \kappa(\mathfrak{p})}(N \otimes_R \kappa(\mathfrak{p}))$$

where we use Remark 10.16.8 to think of the spectra of fibre rings as subsets of $\mathrm{Spec}(S)$. *If R is Noetherian then this inclusion is an equality.*

Proof. This is equivalent to Lemma 10.62.3 by Lemmas 10.60.13, 10.36.6, and 10.62.4. □

Remark 10.62.6. Let $R \to S$ be a ring map. Let N be an S-module. Let $\mathfrak{p}$ be a prime of R. Then

$$\mathrm{Ass}_S(N \otimes_R \kappa(\mathfrak{p})) = \mathrm{Ass}_{S/\mathfrak{p}S}(N \otimes_R \kappa(\mathfrak{p})) = \mathrm{Ass}_{S \otimes_R \kappa(\mathfrak{p})}(N \otimes_R \kappa(\mathfrak{p})).$$

The first equality by Lemma 10.60.13 and the second by Lemma 10.60.15 part (1).

10.63. Weakly associated primes

This is a variant on the notion of an associated prime that is useful for non-Noetherian ring and non-finite modules.

Definition 10.63.1. Let R be a ring. Let M be an R-module. A prime $\mathfrak{p}$ of R is *weakly associated* to M if there exists an element $m \in M$ such that $\mathfrak{p}$ is minimal among the prime ideals containing the annihilator $\mathrm{Ann}(m) = \{f \in R \mid fm = 0\}$. The set of all such primes is denoted $\mathrm{WeakAss}_R(M)$ or $\mathrm{WeakAss}(M)$.

Thus an associated prime is a weakly associated prime. Here is a characterization in terms of the localization at the prime.

Lemma 10.63.2. *Let R be a ring. Let M be an R-module. Let $\mathfrak{p}$ be a prime of R. The following are equivalent:*

1. *$\mathfrak{p}$ is weakly associated to M,*
2. *$\mathfrak{p}R_\mathfrak{p}$ is weakly associated to $M_\mathfrak{p}$, and*
3. *$M_\mathfrak{p}$ contains an element whose annihilator has radical equal to $\mathfrak{p}R_\mathfrak{p}$.*

Proof. Assume (1). Then there exists an element $m \in M$ such that $\mathfrak{p}$ is minimal among the primes containing the annihilator $I = \{x \in R \mid xm = 0\}$ of m. As localization is exact, the annihilator of m in $M_\mathfrak{p}$ is $I_\mathfrak{p}$. Hence $\mathfrak{p}R_\mathfrak{p}$ is a minimal prime of $R_\mathfrak{p}$ containing the annihilator $I_\mathfrak{p}$ of m in $M_\mathfrak{p}$. This implies (2) holds, and also (3) as it implies that $\sqrt{I_\mathfrak{p}} = \mathfrak{p}R_\mathfrak{p}$.

Applying the implication (1) $\Rightarrow$ (3) to $M_\mathfrak{p}$ over $R_\mathfrak{p}$ we see that (2) $\Rightarrow$ (3).

Finally, assume (3). This means there exists an element $m/f \in M_\mathfrak{p}$ whose annihilator has radical equal to $\mathfrak{p}R_\mathfrak{p}$. Then the annihilator $I = \{x \in R \mid xm = 0\}$ of m in M is such that $\sqrt{I_\mathfrak{p}} = \mathfrak{p}R_\mathfrak{p}$. Clearly this means that $\mathfrak{p}$ contains I and is minimal among the primes containing I, i.e., (1) holds. □

Lemma 10.63.3. *Let R be a ring. Let $0 \to M' \to M \to M'' \to 0$ be a short exact sequence of R-modules. Then $\mathit{WeakAss}(M') \subset \mathit{WeakAss}(M)$ and $\mathit{WeakAss}(M) \subset \mathit{WeakAss}(M') \cup \mathit{WeakAss}(M'')$.*

Proof. We will use the characterization of weakly associated primes of Lemma 10.63.2. Let $\mathfrak{p}$ be a prime of R. As localization is exact we obtain the short exact sequence $0 \to M'_\mathfrak{p} \to M_\mathfrak{p} \to M''_\mathfrak{p} \to 0$. Suppose that $m \in M_\mathfrak{p}$ is an element whose annihilator has radical $\mathfrak{p}R_\mathfrak{p}$. Then either the image $\overline{m}$ of m in $M''_\mathfrak{p}$ is zero and $m \in M'_\mathfrak{p}$, or the annihilator of $\overline{m}$ is $\mathfrak{p}R_\mathfrak{p}$. This proves that $\mathrm{WeakAss}(M) \subset \mathrm{WeakAss}(M') \cup \mathrm{WeakAss}(M'')$. The inclusion $\mathrm{WeakAss}(M') \subset \mathrm{WeakAss}(M)$ is immediate from the definitions. □

Lemma 10.63.4. *Let R be a ring. Let M be an R-module. Then*

$$M = (0) \Leftrightarrow \mathit{WeakAss}(M) = \emptyset$$

Proof. If $M = (0)$ then $\mathrm{WeakAss}(M) = \emptyset$ by definition. Conversely, suppose that $M \neq 0$. Pick a nonzero element $m \in M$. Write $I = \{x \in R \mid xm = 0\}$ the annihilator of m. Then $R/I \subset M$. Hence $\mathrm{WeakAss}(R/I) \subset \mathrm{WeakAss}(M)$ by Lemma 10.63.3. But as $I \neq R$ we have $V(I) = \mathrm{Spec}(R/I)$ contains a minimal prime, see Lemmas 10.16.2 and 10.16.7, and we win. □

Lemma 10.63.5. *Let R be a ring. Let M be an R-module. Then*

$$\mathit{Ass}(M) \subset \mathit{WeakAss}(M) \subset \mathit{Supp}(M).$$

Proof. The first inclusion is immediate from the definitions. If $\mathfrak{p} \in \mathrm{WeakAss}(M)$, then by Lemma 10.63.2 we have $M_\mathfrak{p} \neq 0$, hence $\mathfrak{p} \in \mathrm{Supp}(M)$. □

Lemma 10.63.6. *Let R be a ring. Let M be an R-module. The union $\bigcup_{\mathfrak{q} \in \mathit{WeakAss}(M)} \mathfrak{q}$ is the set elements of R which are zerodivisors on M.*

Proof. Suppose $f \in \mathfrak{q} \in \mathrm{WeakAss}(M)$. Then there exists an element $m \in M$ such that $\mathfrak{q}$ is minimal over $I = \{x \in R \mid xm = 0\}$. Hence there exists a $g \in R$, $g \notin \mathfrak{p}$ and $n > 0$ such that $f^n gm = 0$. Note that $gm \neq 0$ as $g \notin I$. If we take n minimal as above, then $f(f^{n-1}gm) = 0$ and $f^{n-1}gm \neq 0$, so f is a zerodivisor on M. Conversely, suppose $f \in R$ is a zerodivisor on M. Consider the submodule $N = \{m \in M \mid fm = 0\}$. Since N is not zero it has a weakly associated prime $\mathfrak{q}$ by Lemma 10.63.4. Clearly $f \in \mathfrak{q}$ and by Lemma 10.63.3 $\mathfrak{q}$ is an associated prime of M. □

Lemma 10.63.7. *Let R be a ring. Let M be an R-module. Any $\mathfrak{p} \in \mathit{Supp}(M)$ which is minimal among the elements of $\mathit{Supp}(M)$ is an element of $\mathit{WeakAss}(M)$.*

Proof. Note that $\text{Supp}(M_\mathfrak{p}) = \{\mathfrak{p}R_\mathfrak{p}\}$ in $\text{Spec}(R_\mathfrak{p})$. In particular $M_\mathfrak{p}$ is nonzero, and hence $\text{WeakAss}(M_\mathfrak{p}) \neq \emptyset$ by Lemma 10.63.4. Since $\text{WeakAss}(M_\mathfrak{p}) \subset \text{Supp}(M_\mathfrak{p})$ by Lemma 10.63.5 we conclude that $\text{WeakAss}(M_\mathfrak{p}) = \{\mathfrak{p}R_\mathfrak{p}\}$, whence $\mathfrak{p} \in \text{WeakAss}(M)$ by Lemma 10.63.2. □

Lemma 10.63.8. *Let R be a ring. Let M be an R-module. Let $\mathfrak{p}$ be a prime ideal of R which is finitely generated. Then*

$$\mathfrak{p} \in \mathit{Ass}(M) \Leftrightarrow \mathfrak{p} \in \mathit{WeakAss}(M).$$

In particular, if R is Noetherian, then $\mathit{Ass}(M) = \mathit{WeakAss}(M)$.

Proof. Write $\mathfrak{p} = (g_1, \ldots, g_n)$ for some $g_i \in R$. It is enough the prove the implication "$\Leftarrow$" as the other implication holds in general, see Lemma 10.63.5. Assume $\mathfrak{p} \in \text{WeakAss}(M)$. By Lemma 10.63.2 there exists an element $m \in M_\mathfrak{p}$ such that $I = \{x \in R_\mathfrak{p} \mid xm = 0\}$ has radical $\mathfrak{p}R_\mathfrak{p}$. Hence for each i there exists a smallest $e_i > 0$ such that $g_i^{e_i} m = 0$ in $M_\mathfrak{p}$. If $e_i > 1$ for some i, then we can replace m by $g_i^{e_i - 1} m \neq 0$ and decrease $\sum e_i$. Hence we may assume that the annihilator of $m \in M_\mathfrak{p}$ is $(g_1, \ldots, g_n)R_\mathfrak{p} = \mathfrak{p}R_\mathfrak{p}$. By Lemma 10.60.14 we see that $\mathfrak{p} \in \text{Ass}(M)$. □

Remark 10.63.9. Let $\varphi : R \to S$ be a ring map. Let M be an S-module. Then it is not always the case that $\text{Spec}(\varphi)(\text{WeakAss}_S(M)) \subset \text{WeakAss}_R(M)$ contrary to the case of associated primes (see Lemma 10.60.10). An example is to consider the ring map

$$R = k[x_1, x_2, x_3, \ldots] \to S = k[x_1, x_2, x_3, \ldots, y_1, y_2, y_3, \ldots]/(x_1 y_1, x_2 y_2, x_3 y_3, \ldots)$$

and $M = S$. In this case $\mathfrak{q} = \sum x_i S$ is a minimal prime of S, hence a weakly associated prime of $M = S$ (see Lemma 10.63.7). But on the other hand, for any nonzero element of S the annihilator in R is finitely generated, and hence does not have radical equal to $R \cap \mathfrak{q} = (x_1, x_2, x_3, \ldots)$ (details omitted).

Lemma 10.63.10. *Let $\varphi : R \to S$ be a ring map. Let M be an S-module. Then we have $\text{Spec}(\varphi)(\mathit{WeakAss}_S(M)) \supset \mathit{WeakAss}_R(M)$.*

Proof. Let $\mathfrak{p}$ be an element of $\text{WeakAss}_R(M)$. Then there exists an $m \in M_\mathfrak{p}$ whose annihilator $I = \{x \in R_\mathfrak{p} \mid xm = 0\}$ has radical $\mathfrak{p}R_\mathfrak{p}$. Consider the radical $J = \{x \in S_\mathfrak{p} \mid xm = 0\}$ of m in $S_\mathfrak{p}$. As $IS_\mathfrak{p} \subset J$ we see that any minimal prime $\mathfrak{q} \subset S_\mathfrak{p}$ over J lies over $\mathfrak{p}$. Moreover such a $\mathfrak{q}$ corresponds to a weakly associated prime of M for example by Lemma 10.63.2. □

Remark 10.63.11. Let $\varphi : R \to S$ be a ring map. Let M be an S-module. Denote $f : \text{Spec}(S) \to \text{Spec}(R)$ the associated map on spectra. Then we have

$$f(\text{Ass}_S(M)) \subset \text{Ass}_R(M) \subset \text{WeakAss}_R(M) \subset f(\text{WeakAss}_S(M))$$

see Lemmas 10.60.10, 10.63.10, and 10.63.5. In general all of the inclusions may be strict, see Remarks 10.60.11 and 10.63.9. If S is Noetherian, then all the inclusions are equalities as the outer two are equal by Lemma 10.63.8.

Lemma 10.63.12. *Let $\varphi : R \to S$ be a ring map. Let M be an S-module. Denote $f : \text{Spec}(S) \to \text{Spec}(R)$ the associated map on spectra. If φ is a finite ring map, then*

$$\mathit{WeakAss}_R(M) = f(\mathit{WeakAss}_S(M)).$$

Proof. One of the inclusions has already been proved, see Remark 10.63.11. To prove the other assume $\mathfrak{q} \in \text{WeakAss}_S(M)$ and let $\mathfrak{p}$ be the corresponding prime of R. Let $m \in M$ be an element such that $\mathfrak{q}$ is a minimal prime over $J = \{g \in S \mid gm = 0\}$. Thus the radical of $JS_\mathfrak{q}$ is $\mathfrak{q}S_\mathfrak{q}$. As $R \to S$ is finite there are finitely many primes $\mathfrak{q} = \mathfrak{q}_1, \mathfrak{q}_2, \ldots, \mathfrak{q}_l$ over $\mathfrak{p}$, see Lemma 10.33.19. Pick $x \in \mathfrak{q}$ with $x \notin \mathfrak{q}_i$ for $i > 1$, see Lemma 10.14.3. By the above there exists an element $y \in S$, $y \notin \mathfrak{q}$ and an integer $t > 0$ such that $yx^t m = 0$. Thus the element $ym \in M$ is annihilated by x^t, hence ym maps to zero in $M_{\mathfrak{q}_i}$, $i = 2, \ldots, l$. To be sure, ym does not map to zero in $S_\mathfrak{q}$.

The ring $S_\mathfrak{p}$ is semi-local with maximal ideals $\mathfrak{q}_i S_\mathfrak{p}$ by going up for finite ring maps, see Lemma 10.33.20. If $f \in \mathfrak{p}R_\mathfrak{p}$ then some power of f ends up in $JS_\mathfrak{q}$ hence for some $n > 0$ we see that $f^t ym$ maps to zero in $M_\mathfrak{q}$. As ym vanishes at the other maximal ideals of $S_\mathfrak{p}$ we conclude that $f^t ym$ is zero in $M_\mathfrak{p}$, see Lemma 10.22.1. In this way we see that $\mathfrak{p}$ is a minimal prime over the annihilator of ym in R and we win. □

Lemma 10.63.13. *Let R be a ring. Let I be an ideal. Let M be an R/I-module. Via the canonical injection $\text{Spec}(R/I) \to \text{Spec}(R)$ we have $\mathit{WeakAss}_{R/I}(M) = \mathit{WeakAss}_R(M)$.*

Proof. Omitted. □

Lemma 10.63.14. *Let R be a ring. Let M be an R-module. Let $S \subset R$ be a multiplicative subset. Via the canonical injection $\text{Spec}(S^{-1}R) \to \text{Spec}(R)$ we have $\mathit{WeakAss}_R(S^{-1}M) = \mathit{WeakAss}_{S^{-1}R}(S^{-1}M)$ and*

$$\mathit{WeakAss}(M) \cap \text{Spec}(S^{-1}R) = \mathit{WeakAss}(S^{-1}M).$$

Proof. Suppose that $m \in S^{-1}M$. Let $I = \{x \in R \mid xm = 0\}$ and $I' = \{x' \in S^{-1}R \mid x'm = 0\}$. Then $I' = S^{-1}I$ and $I \cap S = \emptyset$ unless $I = R$ (verifications omitted). Thus primes in $S^{-1}R$ minimal over I' correspond bijectively to primes in R minimal over I and avoiding S. This proves the equality $\text{WeakAss}_R(S^{-1}M) = \text{WeakAss}_{S^{-1}R}(S^{-1}M)$. The second equality follows from Lemma 10.60.14 since for $\mathfrak{p} \in R$, $S \cap \mathfrak{p} = \emptyset$ we have $M_\mathfrak{p} = (S^{-1}M)_{S^{-1}\mathfrak{p}}$. □

Lemma 10.63.15. *Let R be a ring. Let M be an R-module. Let $S \subset R$ be a multiplicative subset. Assume that every $s \in S$ is a nonzerodivisor on M. Then*

$$\mathit{WeakAss}(M) = \mathit{WeakAss}(S^{-1}M).$$

Proof. As $M \subset S^{-1}M$ by assumption we obtain $\text{WeakAss}(M) \subset \text{WeakAss}(S^{-1}M)$ from Lemma 10.63.3. Conversely, suppose that $n/s \in S^{-1}M$ is an element with annihilator I and $\mathfrak{p}$ a prime which is minimal over I. Then the annihilator of $n \in M$ is I and $\mathfrak{p}$ is a prime minimal over I. □

Lemma 10.63.16. *Let R be a ring. Let M be an R-module. The map*

$$M \longrightarrow \prod\nolimits_{\mathfrak{p} \in \mathit{WeakAss}(M)} M_\mathfrak{p}$$

is injective.

Proof. Let $x \in M$ be an element of the kernel of the map. Set $N = Rx \subset M$. If $\mathfrak{p}$ is a weakly associated prime of N we see on the one hand that $\mathfrak{p} \in \text{WeakAss}(M)$ (Lemma 10.63.3) and on the other hand that $N_\mathfrak{p} \subset M_\mathfrak{p}$ is not zero.

This contradiction shows that $\text{WeakAss}(N) = \emptyset$. Hence $N = 0$, i.e., $x = 0$ by Lemma 10.63.4. □

Lemma 10.63.17. *Let $R \to S$ be a ring map. Let N be an S-module. Assume N is flat as an R-module and R is a domain with fraction field K. Then*

$$\mathit{WeakAss}_S(N) = \mathit{WeakAss}_{S \otimes_R K}(N \otimes_R K)$$

via the canonical inclusion $\text{Spec}(S \otimes_R K) \subset \text{Spec}(S)$.

Proof. Note that $S \otimes_R K = (R \setminus \{0\})^{-1}S$ and $N \otimes_R K = (R \setminus \{0\})^{-1}N$. For any nonzero $x \in R$ multiplication by x on N is injective as N is flat over R. Hence the lemma follows from Lemma 10.63.15. □

10.64. Embedded primes

Here is the definition.

Definition 10.64.1. Let R be a ring. Let M be an R-module.

(1) The associated primes of M which are not minimal among the associated primes of M are called the *embedded associated primes* of M.
(2) The *embedded primes of R* are the embedded associated primes of R as an R-module.

Here is a way to get rid of these.

Lemma 10.64.2. *Let R be a Noetherian ring. Let M be a finite R-module. Consider the set of R-submodules*

$$\{K \subset M \mid \mathit{Supp}(K) \text{ nowhere dense in } \mathit{Supp}(M)\}.$$

This set has a maximal element K and the quotient $M' = M/K$ has the following properties

(1) $\mathit{Supp}(M) = \mathit{Supp}(M')$,
(2) *M' has no embedded associated primes,*
(3) *for any $f \in R$ which is contained in all embedded associated primes of M we have $M_f \cong M'_f$.*

Proof. Let $\mathfrak{q}_1, \ldots, \mathfrak{q}_t$ denote the minimal primes in the support of M. Let $\mathfrak{p}_1, \ldots, \mathfrak{p}_s$ denote the embedded associated primes of M. Then $\text{Ass}(M) = \{\mathfrak{q}_j, \mathfrak{p}_i\}$. There are finitely many of these, see Lemma 10.60.5. Set $I = \prod_{i=1,\ldots,s} \mathfrak{p}_i$. Then $I \not\subset \mathfrak{q}_j$ for any j. Hence by Lemma 10.14.3 we can find an $f \in I$ such that $f \notin \mathfrak{q}_j$ for all $j = 1, \ldots, t$. Set $M' = \text{Im}(M \to M_f)$. This implies that $M_f \cong M'_f$. Since $M' \subset M_f$ we see that $\text{Ass}(M') \subset \text{Ass}(M_f) = \{\mathfrak{q}_j\}$. Thus M' has no embedded associated primes.

Moreover, the support of $K = \text{Ker}(M \to M')$ is contained in $V(\mathfrak{p}_1) \cup \ldots \cup V(\mathfrak{p}_s)$, because $\text{Ass}(K) \subset \text{Ass}(M)$ (see Lemma 10.60.3) and $\text{Ass}(K)$ contains none of the $\mathfrak{q}_i$ by construction. Clearly, K is in fact the largest submodule of M whose support is contained in $V(\mathfrak{p}_1) \cup \ldots \cup V(\mathfrak{p}_t)$. This implies that K is the maximal element of the set displayed in the lemma. □

Lemma 10.64.3. *Let R be a Noetherian ring. Let M be a finite R-module. For any $f \in R$ we have $(M')_f = (M_f)'$ where $M \to M'$ and $M_f \to (M_f)'$ are the quotients constructed in Lemma 10.64.2.*

Proof. Omitted. □

Lemma 10.64.4. *Let R be a Noetherian ring. Let M be a finite R-module without embedded associated primes. Let $I = \{x \in R \mid xM = 0\}$. Then the ring R/I has no embedded primes.*

Proof. We may replace R by R/I. Hence we may assume every nonzero element of R acts nontrivially on M. By Lemma 10.37.5 this implies that $\mathrm{Spec}(R)$ equals the support of M. Suppose that $\mathfrak{p}$ is an embedded prime of R. Let $x \in R$ be an element whose annihilator is $\mathfrak{p}$. Consider the nonzero module $N = xM \subset M$. It is annihilated by $\mathfrak{p}$. Hence any associated prime $\mathfrak{q}$ of N contains $\mathfrak{p}$ and is also an associated prime of M. Then $\mathfrak{q}$ would be an embedded associated prime of M which contradicts the assumption of the lemma. □

10.65. Regular sequences and depth

There is a characterization of depth in terms of Ext-groups that we will discuss in Section 10.67. Here we develop some basic properties of regular sequences and we prove the inequality between depth and dimension.

Definition 10.65.1. Let R be a ring. Let M be an R-module. A sequence of elements $f_1, \ldots, f_r$ of R is called *M-regular* if the following conditions hold:

(1) f_i is a nonzerodivisor on $M/(f_1, \ldots, f_{i-1})M$ for each $i = 1, \ldots, r$, and
(2) the module $M/(f_1, \ldots, f_r)M$ is not zero.

If I is an ideal of R and $f_1, \ldots, f_r \in I$ then we call $f_1, \ldots, f_r$ a *M-regular sequence in I*. If $M = R$, we call $f_1, \ldots, f_r$ simply a *regular sequence* (in I).

Please pay attention to the fact that the definition depends on the order of the elements $f_1, \ldots, f_r$ (see examples below). Some papers/books drop the requirement that the module $M/(f_1, \ldots, f_r)M$ is nonzero. This has the advantage that being a regular sequence is preserved under localization. However, we will use this definition mainly to define the depth of a module in case R is local; in that case the f_i are required to be in the maximal ideal – a condition which is not preserved under going from R to a localization $R_{\mathfrak{p}}$.

Example 10.65.2. Let k be a field. In the ring $k[x, y, z]$ the sequence $x, y(1 - x), z(1 - x)$ is regular but the sequence $y(1 - x), z(1 - x), x$ is not.

Example 10.65.3. Let k be a field. Consider the ring $k[x, y, w_0, w_1, w_2, \ldots]/I$ where I is generated by yw_i, $i = 0, 1, 2, \ldots$ and $w_i - xw_{i+1}$, $i = 0, 1, 2, \ldots$. The sequence x, y is regular, but y is a zerodivisor. Moreover you can localize at the maximal ideal (x, y, w_i) and still get an example.

Definition 10.65.4. Let R be a ring, and $I \subset R$ an ideal. Let M be an R-module. The *I-depth* of M is the supremum in $\{-\infty, 0, 1, 2, \ldots, \infty\}$ of the lengths of M-regular sequences in I. We denote it $\mathrm{depth}_I(M)$. If $(R, \mathfrak{m})$ is local we call $\mathrm{depth}_{\mathfrak{m}}(M)$ simply the *depth* of M.

Elucidation: By Definition 10.65.1 the empty sequence is not a regular sequence on the zero module. Hence $\mathrm{depth}_I(0) = -\infty$ and in fact the zero module is the only module whose I-depth is $-\infty$ (in references using the modified definition of regular sequences — see discussion above — the depth of the zero module is $+\infty$).

A module M has I-depth 0 if and only if M is nonzero and I does not contain an M-regular element.

Example 10.65.2 shows depth does not behave well even if the ring is Noetherian, and Example 10.65.3 shows that it does not behave well if the ring is local but non-Noetherian. We will see later depth behaves well if the ring is local Noetherian. The following two lemmas are an indication of this.

Lemma 10.65.5. *Let R be a local Noetherian ring. Let M be a finite R-module. Let $x_1, \ldots, x_c$ be an M-regular sequence. Then any permutation of the x_i is a regular sequence as well.*

Proof. First we do the case $c = 2$. Consider $K \subset M$ the kernel of $x_2 : M \to M$. For any $z \in K$ we know that $z = x_1 z'$ for some $z' \in M$ because x_2 is a nonzerodivisor on M/x_1M. Because x_1 is a nonzerodivisor on M we see that $x_2 z' = 0$ as well. Hence $x_1 : K \to K$ is surjective. Thus $K = 0$ by Nakayama's Lemma 10.18.1. Next, consider multiplication by x_1 on M/x_2M. If $z \in M$ maps to an element $\overline{z} \in M/x_2M$ in the kernel of this map, then $x_1 z = x_2 y$ for some $y \in M$. But then since x_1, x_2 is a regular sequence we see that $y = x_1 y'$ for some $y' \in M$. Hence $x_1(z - x_2 y') = 0$ and hence $z = x_2 y'$ and hence $\overline{z} = 0$ as desired.

For the general case, observe that any permutation is a composition of transpositions of adjacent indices. Hence it suffices to prove that

$$x_1, \ldots, x_{i-2}, x_i, x_{i-1}, x_{i+1}, \ldots, x_c$$

is an M-regular sequence. This follows from the case we just did applied to the module $M/(x_1, \ldots, x_{i-2})$ and the length 2 regular sequence x_{i-1}, x_i. □

Lemma 10.65.6. *Let R be a Noetherian local ring. Let M be a finite R-module. Then* $\dim(\mathit{Supp}(M)) \geq \mathit{depth}(M)$.

Proof. By Lemma 10.59.6 it suffices to prove that if $f \in \mathfrak{m}$ is a nonzerodivisor on M, then $d(M/fM) \leq d(M) - 1$. The existence of f shows that M does not have finite length. Consider the exact sequence

$$0 \to M \xrightarrow{f} M \to M/fM \to 0$$

and apply Lemma 10.56.10. It shows that $d(M/fM) < d(M)$. □

Here are a few more results on depth.

Lemma 10.65.7. *Let R, S be local rings. Let $R \to S$ be a flat local ring homomorphism. Let $x_1, \ldots, x_r$ be a sequence in R. Let M be an R-module. The following are equivalent*

1. *$x_1, \ldots, x_r$ is an M-regular sequence in R, and*
2. *the images of $x_1, \ldots, x_r$ in S form a $M \otimes_R S$-regular sequence.*

Proof. This is so because $R \to S$ is faithfully flat by Lemma 10.36.16. □

Lemma 10.65.8. *Let R be a Noetherian ring. Let M be a finite R-module. Let $\mathfrak{p}$ be a prime. Let $x_1, \ldots, x_r$ be a sequence in R whose image in $R_\mathfrak{p}$ forms an $M_\mathfrak{p}$-regular sequence. Then there exists a $g \in R$, $g \notin \mathfrak{p}$ such that the image of $x_1, \ldots, x_r$ in R_g forms an M_g-regular sequence.*

Proof. Set

$$K_i = \mathrm{Ker}\left(x_i : M/(x_1, \ldots, x_{i-1})M \to M/(x_1, \ldots, x_{i-1})M\right).$$

This is a finite R-module whose localization at $\mathfrak{p}$ is zero by assumption. Hence there exists a $g \in R$, $g \notin \mathfrak{p}$ such that $(K_i)_g = 0$ for all $i = 1, \ldots, r$. This g works. □

Lemma 10.65.9. *Let A be a ring. Let I be an ideal generated by a regular sequence $f_1, \ldots, f_n$ in A. Let $g_1, \ldots, g_m \in A$ be elements whose images $\overline{g}_1, \ldots, \overline{g}_m$ form a regular sequence in A/I. Then $f_1, \ldots, f_n, g_1, \ldots, g_m$ is a regular sequence in A.*

Proof. This follows immediately from the definitions. □

Lemma 10.65.10. *Let R be a ring. Let M be an R-module. Let $f_1, \ldots, f_r \in R$ be M-regula. Then for $e_1, \ldots, e_r > 0$ the sequence $f_1^{e_1}, \ldots, f_r^{e_r}$ is$M$-regular too.*

Proof. We will show that $f_1^e, f_2, \ldots, f_r$ is an M-regular sequence by induction on e. The case $e = 1$ is trivial. Since f_1 is a nonzerodivisor we have a short exact sequence

$$0 \to M/f_1M \xrightarrow{f_1^{e-1}} M/f_1^eM \to M/f_1^{e-1}M \to 0$$

By induction the elements $f_2, \ldots, f_r$ are M/f_1M and $M/f_1^{e-1}M$-regular sequences. It follows from the snake lemma that they are also M/f_1^eM-regular sequences. □

Lemma 10.65.11. *Let R be a ring. Let $f_1, \ldots, f_r \in R$ which do not generate the unit ideal. The following are equivalent:*

1. *any permutation of $f_1, \ldots, f_r$ is a regular sequence,*
2. *any subsequence of $f_1, \ldots, f_r$ (in the given order) is a regular sequence, and*
3. *$f_1x_1, \ldots, f_rx_r$ is a regular sequence in the polynomial ring $R[x_1, \ldots, x_r]$.*

Proof. It is clear that (1) implies (2). We prove (2) implies (1) by induction on r. The case $r = 1$ is trivial. The case $r = 2$ says that if $a, b \in R$ are a regular sequence and b is a nonzerodivisor, then b, a is a regular sequence. This is clear because the kernel of $a : R/(b) \to R/(b)$ is isomorphic to the kernel of $b : R/(a) \to R/(a)$ if both a and b are nonzerodivisors. The case $r > 2$. Assume (2) holds and say we want to prove $f_{\sigma(1)}, \ldots, f_{\sigma(r)}$ is a regular sequence for some permutation σ. We already know that $f_{\sigma(1)}, \ldots, f_{\sigma(r-1)}$ is a regular sequence by induction. Hence it suffices to show that f_s where $s = \sigma(r)$ is a nonzerodivisor modulo $f_1, \ldots, \hat{f}_s, \ldots, f_r$. If $s = r$ we are done. If $s < r$, then note that f_s and f_r are both nonzerodivisors in the ring $R/(f_1, \ldots, \hat{f}_s, \ldots, f_{r-1})$ (by induction hypothesis again). Since we know f_s, f_r is a regular sequence in that ring we conclude by the case of sequence of length 2 that f_r, f_s is too.

Note that $R[x_1, \ldots, x_r]/(f_1x_1, \ldots, f_ix_i)$ as an R-module is a direct sum of the modules

$$R/I_E \cdot x_1^{e_1} \ldots x_r^{e_r}$$

indexed by multi-indices $E = (e_1, \ldots, e_r)$ where I_E is the ideal generated by $f_j^{e_j}$ for $1 \leq j \leq i$ with $e_j > 0$. Hence $f_{i+1}x_i$ is a nonzerodivisor on this if and only if the maps f_{i+1} is a nonzerodivisor on R/I_E for all E. Thus it is clear that (3) implies (2) by taking $e_1, \ldots, e_i \in \{0, 1\}$. Conversely, if (2) holds, then any sequence of the form $f_1^{e_1}, \ldots, f_i^{e_i}, f_{i+1}$ (but omitting those powers with zero exponent) is a regular sequence by Lemma 10.65.10, i.e., f_{i+1} is a nonzerodivisor on R/I_E. □

10.66. Quasi-regular sequences

There is a notion of regular sequence which is slightly weaker than that of a regular sequence and easier to use. Let R be a ring and let $f_1, \ldots, f_c \in R$. Set $J = (f_1, \ldots, f_c)$. Let M be an R-module. Then there is a canonical map

$$M/JM \otimes_{R/J} R/J[X_1, \ldots, X_c] \longrightarrow \bigoplus\nolimits_{n \geq 0} J^n M/J^{n+1}M \tag{10.66.0.1}$$

of graded $R/J[X_1, \ldots, X_c]$-modules defined by the rule

$$\overline{m} \otimes X_1^{e_1} \ldots X_c^{e_c} \longmapsto f_1^{e_1} \ldots f_c^{e_c} m \bmod J^{e_1 + \ldots + e_c + 1} M.$$

Note that (10.66.0.1) is always surjective.

Definition 10.66.1. Let R be a ring. Let M be an R-module. A sequence of elements $f_1, \ldots, f_c$ of R is called *M-quasi-regular* if (10.66.0.1) is an isomorphism. If $M = R$, we call $f_1, \ldots, f_c$ simply a *quasi-regular sequence*.

So if $f_1, \ldots, f_c$ is a quasi-regular sequence, then

$$R/J[X_1, \ldots, X_c] = \bigoplus\nolimits_{n \geq 0} J^n/J^{n+1}$$

where $J = (f_1, \ldots, f_c)$. It is clear that being a quasi-regular sequence is independent of the order of $f_1, \ldots, f_c$.

Lemma 10.66.2. *Let R be a ring.*

1. *A regular sequence $f_1, \ldots, f_c$ of R is a quasi-regular sequence.*
2. *Suppose that M is an R-module and that $f_1, \ldots, f_c$ is an M-regular sequence. Then $f_1, \ldots, f_c$ is an M-quasi-regular sequence.*

Proof. Set $J = (f_1, \ldots, f_c)$. We prove the first assertion by induction on c. We have to show that given any relation $\sum_{|I|=n} a_I f^I \in J^{n+1}$ with $a_I \in R$ we actually have $a_I \in J$ for all multi-indices I. Since any element of J^{n+1} is of the form $\sum_{|I|=n} b_I f^I$ with $b_I \in J$ we may assume, after replacing a_I by $a_I - b_I$, the relation reads $\sum_{|I|=n} a_I f^I = 0$. We can rewrite this as

$$\sum\nolimits_{e=0}^{n} \left(\sum\nolimits_{|I'|=n-e} a_{I',e} f^{I'} \right) f_c^e = 0$$

Here and below the "primed" multi-indices I' are required to be of the form $I' = (i_1, \ldots, i_{d-1}, 0)$. We will show by descending induction on $l \in \{0, \ldots, n\}$ that if we have a relation

$$\sum\nolimits_{e=0}^{l} \left(\sum\nolimits_{|I'|=n-e} a_{I',e} f^{I'} \right) f_c^e = 0$$

then $a_{I',e} \in J$ for all I', e. Namely, set $J' = (f_1, \ldots, f_{c-1})$. We observe that $\sum_{|I'|=n-l} a_{I',l} f^{I'}$ is mapped into J' by f_c^l and hence (because f_c is not a zerodivisor on R/J') it is in J'. By induction hypotheses (for the induction on c), we see that $a_{I',l} \in J'$. This allows us to rewrite the term $(\sum_{|I'|=n-l} a_{I',l} f^{I'}) f_c^l$ in the form $(\sum_{|I'|=n-l+1} f_c b_{I',l-1} f^{I'}) f_c^{l-1}$. This gives a new relation of the form

$$\sum\nolimits_{|I'|=n-l+1} (a_{I',l-1} + f_c b_{I',l-1}) f^{I'}) f_c^{l-1} + \sum\nolimits_{e=0}^{l-2} \left(\sum\nolimits_{|I'|=n-e} a_{I',e} f^{I'} \right) f_c^e = 0$$

Now by the induction hypothesis (on l this time) we see that all $a_{I',l-1} + f_c b_{I',l-1} \in J$ and all $a_{I',e} \in J$ for $e \leq l-2$. This, combined with $a_{I',l} \in J' \subset J$ seen above, finishes the proof of the induction step.

The second assertion means that given any formal expression $F = \sum_{|I|=n} m_I X^I$, $m_I \in M$ with $\sum m_I f^I \in J^{n+1}M$, then all the coefficients m_I are in J. This is proved in exactly the same way as we prove the corresponding result for the first assertion above. □

Lemma 10.66.3. *Let $R \to R'$ be a flat ring map. Let M be an R-module. Suppose that $f_1, \ldots, f_r \in R$ form an M-quasi-regular sequence. Then the images of $f_1, \ldots, f_r$ in R' form a $M \otimes_R R'$-quasi-regular sequence.*

Proof. Set $J = (f_1, \ldots, f_r)$, $J' = JR'$ and $M' = M \otimes_R R'$. Because $R \to S$ is flat the sequences $0 \to J^{n+1}M \to J^nM \to J^nM/J^{n+1}M \to 0$ remain exact on tensoring with S. Hence $(J')^nM'/(J')^{n+1}M' = J^nM/J^{n+1}M \otimes_R R'$. Thus the isomorphism $J^n/J^{n+1} \otimes_R M \to J^nM/J^{n+1}M$ gives rise to the corresponding isomorphism for M'. □

Lemma 10.66.4. *Let R be a Noetherian ring. Let M be a finite R-module. Let $\mathfrak{p}$ be a prime. Let $x_1, \ldots, x_c$ be a sequence in R whose image in $R_\mathfrak{p}$ forms an $M_\mathfrak{p}$-quasi-regular sequence. Then there exists a $g \in R$, $g \notin \mathfrak{p}$ such that the image of $x_1, \ldots, x_c$ in R_g forms an M_g-quasi-regular sequence.*

Proof. Consider the kernel K of the map (10.66.0.1). As $M/JM \otimes_{R/J} R/J[X_1, \ldots, X_c]$ is a finite $R/J[X_1, \ldots, X_c]$-module and as $R/J[X_1, \ldots, X_c]$ is Noetherian, we see that K is also a finite $R/J[X_1, \ldots, X_c]$-module. Pick homogeneous generators $k_1, \ldots, k_t \in K$. By assumption for each $i = 1, \ldots, t$ there exists a $g_i \in R$, $g_i \notin \mathfrak{p}$ such that $g_i k_i = 0$. Hence $g = g_1 \ldots g_t$ works. □

Lemma 10.66.5. *Let R be a ring. Let M be an R-module. Let $f_1, \ldots, f_c \in R$ be an M-quasi-regular sequence. For any i the sequence $\overline{f}_{i+1}, \ldots, \overline{f}_c$ of $\overline{R} = R/(f_1, \ldots, f_i)$ is an $\overline{M} = M/(f_1, \ldots, f_i)M$-quasi-regular sequence.*

Proof. It suffices to prove this for $i = 1$. Set $\overline{J} = (\overline{f}_2, \ldots, \overline{f}_c) \subset \overline{R}$. Then

$$\begin{aligned} \overline{J}^n\overline{M}/\overline{J}^{n+1}\overline{M} &= (J^nM + f_1M)/(J^{n+1}M + f_1M) \\ &= J^nM/(J^{n+1}M + J^nM \cap f_1M). \end{aligned}$$

Thus, in order to prove the lemma it suffices to show that $J^{n+1}M + J^nM \cap f_1M = J^{n+1}M + f_1J^{n-1}M$ because that will show that $\bigoplus_{n\geq 0} \overline{J}^n\overline{M}/\overline{J}^{n+1}\overline{M}$ is the quotient of $\bigoplus_{n\geq 0} J^nM/J^{n+1} \cong M/JM[X_1, \ldots, X_c]$ by X_1. Actually, we have $J^nM \cap f_1M = f_1J^{n-1}M$. Namely, if $m \notin J^{n-1}M$, then $f_1m \notin J^nM$ because $\bigoplus J^nM/J^{n+1}M$ is the polynomial algebra $M/J[X_1, \ldots, X_c]$ by assumption. □

Lemma 10.66.6. *Let $(R, \mathfrak{m})$ be a local Noetherian ring. Let M be a nonzero finite R-module. Let $f_1, \ldots, f_c \in \mathfrak{m}$ be an M-quasi-regular sequence. Then $f_1, \ldots, f_c$ is an M-regular sequence.*

Proof. Set $J = (f_1, \ldots, f_c)$. Let us show that f_1 is a nonzerodivisor on M. Suppose $x \in M$ is not zero. By the Artin-Rees lemma there exists an integer r such that $x \in J^rM$ but $x \notin J^{r+1}M$, see Lemma 10.47.6. Then $f_1x \in J^{r+1}M$ is an element whose

class in $J^{r+1}M/J^{r+2}M$ is nonzero by the assumed structure of $\bigoplus J^nM/J^{n+1}M$. Whence $f_1x \neq 0$.

Now we can finish the proof by induction on c using Lemma 10.66.5. □

Remark 10.66.7 (Koszul regular sequences)**.** In the paper [**Kab71**] the author introduces two more regularity conditions for sequences $x_1, \ldots, x_r$ of elements of a ring R. Namely, we say the sequence is *Koszul-regular* if $H_i(K_\bullet(R, x_\bullet)) = 0$ for $i \geq 1$ where $K_\bullet(R, x_\bullet)$ is the Koszul complex. The sequence is called *H_1-regular* if $H_1(K_\bullet(R, x_\bullet)) = 0$. If R is a local ring (possibly nonnoetherian) and the sequence consists of elements of the maximal ideal, then one has the implications regular $\Rightarrow$ Koszul-regular $\Rightarrow$ H_1-regular $\Rightarrow$ quasi-regular. By examples the author shows that these implications cannot be reversed in general. We introduce these notions in more detail in More on Algebra, Section 15.20.

Remark 10.66.8. Let k be a field. Consider the ring

$$A = k[x, y, w, z_0, z_1, z_2, \ldots]/(y^2z_0 - wx, z_0 - yz_1, z_1 - yz_2, \ldots)$$

In this ring x is a nonzerodivisor and the image of y in A/xA gives a quasi-regular sequence. But it is not true that x, y is a quasi-regular sequence in A because $(x, y)/(x, y)^2$ isn't free of rank two over $A/(x, y)$ due to the fact that $wx = 0$ in $(x, y)/(x, y)^2$ but w isn't zero in $A/(x, y)$. Hence the analogue of Lemma 10.65.9 does not hold for quasi-regular sequences.

Lemma 10.66.9. *Let R be a ring. Let $J = (f_1, \ldots, f_r)$ be an ideal of R. Let M be an R-module. Set $\overline{R} = R/\bigcap_{n\geq 0} J^n$, $\overline{M} = M/\bigcap_{n\geq 0} J^nM$, and denote $\overline{f}_i$ the image of f_i in $\overline{R}$. Then $f_1, \ldots, f_r$ is M-quasi-regular if and only if $\overline{f}_1, \ldots, \overline{f}_r$ is $\overline{M}$-quasi-regular.*

Proof. This is true because $J^nM/J^{n+1}M \cong \overline{J}^n\overline{M}/\overline{J}^{n+1}\overline{M}$. □

10.67. Ext groups and depth

In this section we do a tiny bit of homological algebra, in order to establish some fundamental properties of depth over Noetherian local rings.

Lemma 10.67.1. *Let R be a ring. Let M be an R-module.*

(1) *There exists an exact complex*

$$\ldots \to F_2 \to F_1 \to F_0 \to M \to 0.$$

with F_i free R-modules.

(2) *If R is Noetherian and M finite over R, then we can choose the complex such that F_i is finite free. In other words, we can find an exact complex*

$$\ldots \to R^{\oplus n_2} \to R^{\oplus n_1} \to R^{\oplus n_0} \to M \to 0.$$

Proof. Let us explain only the Noetherian case. As a first step choose a surjection $R^{n_0} \to M$. Then having constructed an exact complex of length e we simply choose a surjection $R^{n_{e+1}} \to \mathrm{Ker}(R^{n_e} \to R^{n_{e-1}})$ which is possible because R is Noetherian. □

Definition 10.67.2. Let R be a ring. Let M be an R-module.

(1) A (left) *resolution* $F_\bullet \to M$ of M is an exact complex
$$\ldots \to F_2 \to F_1 \to F_0 \to M \to 0$$
of R-modules.
(2) A *resolution of M by free R-modules* is a resolution $F_\bullet \to M$ where each F_i is a free R-module.
(3) A *resolution of M by finite free R-modules* is a resolution $F_\bullet \to M$ where each F_i is a finite free R-module.

We often use the notation $F_\bullet$ to denote a complex of R-modules
$$\ldots \to F_i \to F_{i-1} \to \ldots$$
In this case we often use d_i or $d_{F,i}$ to denote the map $F_i \to F_{i-1}$. In this section we are always going to assume that F_0 is the last nonzero term in the complex. The *ith homology group of the complex* $F_\bullet$ is the group $H_i = \mathrm{Ker}(d_{F,i})/\mathrm{Im}(d_{F,i+1})$. A *map of complexes* $\alpha : F_\bullet \to G_\bullet$ is given by maps $\alpha_i : F_i \to G_i$ such that $\alpha_{i-1} \circ d_{F,i} = d_{G,i-1} \circ \alpha_i$. Such a map induces a map on homology $H_i(\alpha) : H_i(F_\bullet) \to H_i(G_\bullet)$. If $\alpha, \beta : F_\bullet \to G_\bullet$ are maps of complexes, then a *homotopy* between α and β is given by a collection of maps $h_i : F_i \to G_{i+1}$ such that $\alpha_i - \beta_i = d_{G,i+1} \circ h_i + h_{i-1} \circ d_{F,i}$.

We will use a very similar notation regarding complexes of the form $F^\bullet$ which look like
$$\ldots \to F^i \xrightarrow{d^i} F^{i+1} \to \ldots$$
There are maps of complexes, homotopies, etc. In this case we set $H^i(F^\bullet) = \mathrm{Ker}(d^i)/\mathrm{Im}(d^{i-1})$ and we call it the *ith cohomology group*.

Lemma 10.67.3. *Any two homotopic maps of complexes induce the same maps on (co)homology groups.*

Proof. Omitted. □

Lemma 10.67.4. *Let R be a ring. Let $M \to N$ be a map of R-modules. Let $F_\bullet \to M$ be a resolution by free R-modules and let $N_\bullet \to N$ be an arbitrary resolution. Then*

(1) *there exists a map of complexes $F_\bullet \to N_\bullet$ inducing the given map*
$$M = \mathit{Coker}(F_1 \to F_0) \to \mathit{Coker}(N_1 \to N_0) = N$$
(2) *two maps $\alpha, \beta : F_\bullet \to N_\bullet$ inducing the same map $M \to N$ are homotopic.*

Proof. Proof of (1). Because F_0 is free we can find a map $F_0 \to N_0$ lifting the map $F_0 \to M \to N$. We obtain an induced map $F_1 \to F_0 \to N_0$ wich ends up in the image of $N_1 \to N_0$. Since F_1 is free we may lift this to a map $F_1 \to N_1$. This in turn induces a map $F_2 \to F_1 \to N_1$ which maps to zero into N_0. Since $N_\bullet$ is exact we see that the image of this map is contained in the image of $N_2 \to N_1$. Hence we may lift to get a map $F_2 \to N_2$. Repeat.

Proof of (2). To show that α, β are homotopic it suffices to show the difference $\gamma = \alpha - \beta$ is homotopic to zero. Note that the image of $\gamma_0 : F_0 \to N_0$ is contained in the image of $N_1 \to N_0$. Hence we may lift γ_0 to a map $h_0 : F_0 \to N_1$. Consider the map $\gamma_1' = \gamma_1 - h_0 \circ d_{F,1}$. By our choice of h_0 we see that the image of γ_1' is contained in the kernel of $N_1 \to N_0$. Since $N_\bullet$ is exact we may lift γ_1' to a map $h_1 : F_1 \to N_2$. At this point we have $\gamma_1 = h_0 \circ d_{F,1} + d_{N,2} \circ h_1$. Repeat. □

At this point we are ready to define the groups $\mathrm{Ext}^i_R(M, N)$. Namely, choose a resolution $F_\bullet$ of M by free R-modules, see Lemma 10.67.1. Consider the (cohomological) complex

$$\mathrm{Hom}_R(F_\bullet, N) : \mathrm{Hom}_R(F_0, N) \to \mathrm{Hom}_R(F_1, N) \to \mathrm{Hom}_R(F_2, N) \to \ldots$$

We define $\mathrm{Ext}^i_R(M, N)$ to be the ith cohomology group of this complex.[3] The following lemma explains in what sense this is well defined.

Lemma 10.67.5. *Let R be a ring. Let M_1, M_2, N be R-modules. Suppose that $F_\bullet$ is a free resolution of the module M_1, and $G_\bullet$ is a free resolution of the module M_2. Let $\varphi : M_1 \to M_2$ be a module map. Let $\alpha : F_\bullet \to G_\bullet$ be a map of complexes inducing φ on $M_1 = Coker(d_{F,1}) \to M_2 = Coker(d_{G,1})$, see Lemma 10.67.4. Then the induced maps*

$$H^i(\alpha) : H^i(\mathrm{Hom}_R(F_\bullet, N)) \longrightarrow H^i(\mathrm{Hom}_R(G_\bullet, N))$$

are independent of the choice of α. If φ is an isomorphism, so are all the maps $H^i(\alpha)$. If $M_1 = M_2$, $F_\bullet = G_\bullet$, and φ is the identity, so are all the maps $H_i(\alpha)$.

Proof. Another map $\beta : F_\bullet \to G_\bullet$ inducing φ is homotopic to α by Lemma 10.67.4. Hence the maps $\mathrm{Hom}_R(F_\bullet, N) \to \mathrm{Hom}_R(G_\bullet, N)$ are homotopic. Hence the independence result follows from Lemma 10.67.3.

Suppose that φ is an isomorphism. Let $\psi : M_2 \to M_1$ be an inverse. Choose $\beta : G_\bullet \to F_\bullet$ be a map inducing $\psi : M_2 = \mathrm{Coker}(d_{G,1}) \to M_1 = \mathrm{Coker}(d_{F,1})$, see Lemma 10.67.4. Ok, and now consider the map $H^i(\alpha) \circ H^i(\beta) = H^i(\alpha \circ \beta)$. By the above the map $H^i(\alpha \circ \beta)$ is the *same* as the map $H^i(\mathrm{id}_{G_\bullet}) = \mathrm{id}$. Similarly for the composition $H^i(\beta) \circ H^i(\alpha)$. Hence $H^i(\alpha)$ and $H^i(\beta)$ are inverses of each other. □

Lemma 10.67.6. *Let R be a ring. Let M be an R-module. Let $0 \to N' \to N \to N'' \to 0$ be a short exact sequence. Then we get a long exact sequence*

$$\begin{aligned} 0 \to \mathrm{Hom}_R(M, N') \to \mathrm{Hom}_R(M, N) \to \mathrm{Hom}_R(M, N'') \\ \to Ext^1_R(M, N') \to Ext^1_R(M, N) \to Ext^1_R(M, N'') \to \ldots \end{aligned}$$

Proof. Pick a free resolution $F_\bullet \to M$. Since each of the F_i are free we see that we get a short exact sequence of complexes

$$0 \to \mathrm{Hom}_R(F_\bullet, N') \to \mathrm{Hom}_R(F_\bullet, N) \to \mathrm{Hom}_R(F_\bullet, N'') \to 0$$

Thus we get the long exact sequence from the snake lemma applied to this. □

Lemma 10.67.7. *Let R be a ring. Let N be an R-module. Let $0 \to M' \to M \to M'' \to 0$ be a short exact sequence. Then we get a long exact sequence*

$$\begin{aligned} 0 \to \mathrm{Hom}_R(M'', N) \to \mathrm{Hom}_R(M, N) \to \mathrm{Hom}_R(M', N) \\ \to Ext^1_R(M'', N) \to Ext^1_R(M, N) \to Ext^1_R(M', N) \to \ldots \end{aligned}$$

Proof. Pick sets of generators $\{m'_{i'}\}_{i' \in I'}$ and $\{m''_{i''}\}_{i'' \in I''}$ of M' and M''. For each $i'' \in I''$ choose a lift $\tilde{m}''_{i''} \in M$ of the element $m''_{i''} \in M''$. Set $F' = \bigoplus_{i' \in I'} R$, $F'' = \bigoplus_{i'' \in I''} R$ and $F = F' \oplus F''$. Mapping the generators of these free modules

[3] At this point it would perhaps be more appropriate to say "an" in stead of "the" Ext-group.

to the corresponding chosen generators gives surjective R-module maps $F' \to M'$, $F'' \to M''$, and $F \to M$. We obtain a map of short exact sequences

$$\begin{array}{ccccccccc} 0 & \to & M' & \to & M & \to & M'' & \to & 0 \\ & & \uparrow & & \uparrow & & \uparrow & & \\ 0 & \to & F' & \to & F & \to & F'' & \to & 0 \end{array}$$

By the snake lemma we see that the sequence of kernels $0 \to K' \to K \to K'' \to 0$ is short exact sequence of R-modules. Hence we can continue this process indefinitively. In other words we obtain a short exact sequence of resolutions fitting into the diagram

$$\begin{array}{ccccccccc} 0 & \to & M' & \to & M & \to & M'' & \to & 0 \\ & & \uparrow & & \uparrow & & \uparrow & & \\ 0 & \to & F'_\bullet & \to & F_\bullet & \to & F''_\bullet & \to & 0 \end{array}$$

Because each of the sequences $0 \to F'_n \to F_n \to F''_n \to 0$ is split exact (by construction) we obtain a short exact sequence of complexes

$$0 \to \mathrm{Hom}_R(F''_\bullet, N) \to \mathrm{Hom}_R(F_\bullet, N) \to \mathrm{Hom}_R(F'_\bullet, N) \to 0$$

by applying the $\mathrm{Hom}_R(-, N)$ functor. Thus we get the long exact sequence from the snake lemma applied to this. □

Lemma 10.67.8. *Let R be a ring. Let M, N be R-modules. Any $x \in R$ such that either $xN = 0$, or $xM = 0$ annihilates each of the modules $Ext^i_R(M, N)$.*

Proof. Pick a free resolution $F_\bullet$ of M. Since $\mathrm{Ext}^i_R(M, N)$ is defined as the cohomology of the complex $\mathrm{Hom}_R(F_\bullet, N)$ the lemma is clear when $xN = 0$. If $xM = 0$, then we see that multiplication by x on $F_\bullet$ lifts the zero map on M. Hence by Lemma 10.67.5 we see that it induces the same map on Ext groups as the zero map. □

Lemma 10.67.9. *Let R be a Noetherian ring. Let M, N be finite R-modules. Then $Ext^i_R(M, N)$ is a finite R-module for all i.*

Proof. This holds because $\mathrm{Ext}^i_R(M, N)$ is computed as the cohomology groups of a complex $\mathrm{Hom}_R(F_\bullet, N)$ with each F_n a finite free R-module. □

Lemma 10.67.10. *Let R be a Noetherian local ring with maximal ideal $\mathfrak{m}$. Let M be a nonzero finite R-module. Then $depth_R(M)$ is equal to the smallest integer i such that $Ext^i_R(R/\mathfrak{m}, M)$ is nonzero.*

Proof. Let $\delta(M)$ denote the depth of M and let $i(M)$ denote the smallest integer i such that $\mathrm{Ext}^i_R(R/\mathfrak{m}, M)$ is nonzero. We will see in a moment that $i(M) < \infty$. By Lemma 10.60.17 we have $\delta(M) = 0$ if and only if $i(M) = 0$, because $\mathfrak{m} \in \mathrm{Ass}(M)$ exactly means that $i(M) = 0$. Hence if $\delta(M)$ or $i(M)$ is > 0, then we may choose $x \in \mathfrak{m}$ such that (a) x is a nonzerodivisor on M, and (b) $\mathrm{depth}(M/xM) = \delta(M) - 1$. Consider the long exact sequence of Ext-groups associated to the short exact sequence $0 \to M \to M \to M/xM \to 0$ by Lemma 10.67.6:

$$\begin{aligned} 0 \to \mathrm{Hom}_R(\kappa, M) \to \mathrm{Hom}_R(\kappa, M) \to \mathrm{Hom}_R(\kappa, M/xM) \\ \to \mathrm{Ext}^1_R(\kappa, M) \to \mathrm{Ext}^1_R(\kappa, M) \to \mathrm{Ext}^1_R(\kappa, M/xM) \to \ldots \end{aligned}$$

Since $x \in \mathfrak{m}$ all the maps $\mathrm{Ext}^i_R(\kappa, M) \to \mathrm{Ext}^i_R(\kappa, M)$ are zero, see Lemma 10.67.8. Thus it is clear that $i(M/xM) = i(M) - 1$. Induction, e.g., on $\dim(\mathrm{Support}(M))$, finishes the proof. □

Lemma 10.67.11. *Let R be a local Noetherian ring. Let $0 \to N' \to N \to N'' \to 0$ be a short exact sequence of nonzero finite R-modules.*

(1) $\mathit{depth}(N'') \geq \min\{\mathit{depth}(N), \mathit{depth}(N') - 1\}$
(2) $\mathit{depth}(N') \geq \min\{\mathit{depth}(N), \mathit{depth}(N'') + 1\}$

Proof. This is easy using the results above. Hint: Use the characterization of depth using the Ext groups $\mathrm{Ext}^i(\kappa, N)$, see Lemma 10.67.10, and use the long exact cohomology sequence

$$\begin{aligned} 0 \to \mathrm{Hom}_R(\kappa, N') \to \mathrm{Hom}_R(\kappa, N) \to \mathrm{Hom}_R(\kappa, N'') \\ \to \mathrm{Ext}^1_R(\kappa, N') \to \mathrm{Ext}^1_R(\kappa, N) \to \mathrm{Ext}^1_R(\kappa, N'') \to \ldots \end{aligned}$$

from Lemma 10.67.6. □

Lemma 10.67.12. *Let R be a local Noetherian ring. Let M be a finite R-module. If $x \in \mathfrak{m}$ is a nonzerodivisor on M, then $\mathit{depth}(M/xM) = \mathit{depth}(M) - 1$.*

Proof. By the short exact sequence $0 \to M \to M \to M/xM \to 0$ and Lemma 10.67.11 we see that the depth drops by at most 1. On the other hand, if $x_1, \ldots, x_\delta \in \mathfrak{m}$ is a regular sequence for M/xM, then $x, x_1, \ldots, x_r$ is a regular sequence for M. Hence we see that the depth drops by at least 1. □

10.68. Functorialities for Ext

In this section we briefly discuss the functoriality of Ext with respect to change of ring, etc. Here is a list of items to work out.

(1) Given $R \to R'$, an R-module M and an R'-module N' the R-module $\mathrm{Ext}^i_R(M, N')$ has a natural R'-module structure. Moreover, there is a canonical R'-linear map $\mathrm{Ext}^i_{R'}(M \otimes_R R', N') \to \mathrm{Ext}^i_R(M, N')$.
(2) Given $R \to R'$ and R-modules M, N there is a natural R-module map $\mathrm{Ext}^i_R(M, N) \to \mathrm{Ext}^i_R(M, N \otimes_R R')$.

Lemma 10.68.1. *Given a flat ring map $R \to R'$, an R-module M, and an R'-module N' the natural maps*

$$\mathit{Ext}^i_{R'}(M \otimes_R R', N') \to \mathit{Ext}^i_R(M, N')$$

are isomorphisms.

Proof. Choose a free resolution $F_\bullet$ of M. Since $R \to R'$ is flat we see that $F_\bullet \otimes_R R'$ is a free resolution of $M \otimes_R R'$ over R'. The statement is that the map

$$\mathrm{Hom}_{R'}(F_\bullet \otimes_R R', N') \to \mathrm{Hom}_R(F_\bullet, N')$$

induces an isomorphism on homology group, which is true because it is an isomorphism of complexes by Lemma 10.13.3. □

10.69. An application of Ext groups

This section should briefly discuss the relationship between $\mathrm{Ext}^1_R(Q, N)$ and extensions (see Homology, Section 12.6). Omitted.

Lemma 10.69.1. *Let R be a Noetherian local ring with maximal ideal $\mathfrak{m}$. Let $N \to M$ be a homomorphism of finite R-modules. Suppose that there exists arbitrarily large n such that $N/\mathfrak{m}^n N \to M/\mathfrak{m}^n M$ is a split injection. Then $N \to M$ is a split injection.*

Proof. Assume $\varphi : N \to M$ satisfies the assumptions of the lemma. Note that this implies that $\mathrm{Ker}(\varphi) \subset \mathfrak{m}^n N$ for arbitrarily large n. Hence by Lemma 10.47.6 we see that φ is injection. Let $Q = M/N$ so that we have a short exact sequence

$$0 \to N \to M \to Q \to 0.$$

Let

$$F_2 \xrightarrow{d_2} F_1 \xrightarrow{d_1} F_0 \to Q \to 0$$

be a finite free resolution of Q. We can choose a map $\alpha : F_0 \to M$ lifting the map $F_0 \to Q$. This induces a map $\beta : F_1 \to N$ such that $\beta \circ d_2 = 0$. The extension above is split if and only if there exists a map $\gamma : F_0 \to N$ such that $\beta = \gamma \circ d_1$. In other words, the class of β in $\mathrm{Ext}^1_R(Q, N)$ is the obstruction to splitting the short exact sequence above.

Suppose n is a large integer such that $N/\mathfrak{m}^n N \to M/\mathfrak{m}^n M$ is a split injection. This implies

$$0 \to N/\mathfrak{m}^n N \to M/\mathfrak{m}^n M \to Q/\mathfrak{m}^n Q \to 0.$$

is still short exact. Also, the sequence

$$F_1/\mathfrak{m}^n F_1 \xrightarrow{d_1} F_0/\mathfrak{m}^n F_0 \to Q/\mathfrak{m}^n Q \to 0$$

is still exact. Arguing as above we see that the map $\overline{\beta} : F_1/\mathfrak{m}^n F_1 \to N/\mathfrak{m}^n N$ induced by β is equal to $\gamma_n \circ d_1$ for some map $\overline{\gamma_n} : F_0/\mathfrak{m}^n F_0 \to N/\mathfrak{m}^n$. Since F_0 is free we can lift $\overline{\gamma_n}$ to a map $\gamma_n : F_0 \to N$ and then we see that $\beta - \gamma_n \circ d_1$ is a map from F_1 into $\mathfrak{m}^n N$. In other words we conclude that

$$\beta \in \mathrm{Im}\Big(\mathrm{Hom}_R(F_0, N) \to \mathrm{Hom}_R(F_1, N) \Big) + \mathfrak{m}^n \mathrm{Hom}_R(F_1, N).$$

for this n.

Since we have this property for arbitrarily large n by assumption we conclude (by Lemma 10.47.4) that β is actually in the image of the map $\mathrm{Hom}_R(F_0, N) \to \mathrm{Hom}_R(F_1, N)$ as desired. □

10.70. Tor groups and flatness

In this section we use some of the homological algebra developed in the previous section to explain what Tor groups are. Namely, suppose that R is a ring and that M, N are two R-modules. Choose a resolution $F_\bullet$ of M by free R-modules. See Lemma 10.67.1. Consider the homological complex

$$F_\bullet \otimes_R N : \ldots \to F_2 \otimes_R N \to F_1 \otimes_R N \to F_0 \otimes_R N$$

We define $\mathrm{Tor}_i^R(M, N)$ to be the ith homology group of this complex. The following lemma explains in what sense this is well defined.

Lemma 10.70.1. *Let R be a ring. Let M_1, M_2, N be R-modules. Suppose that $F_\bullet$ is a free resolution of the module M_1 and that $G_\bullet$ is a free resolution of the module M_2. Let $\varphi : M_1 \to M_2$ be a module map. Let $\alpha : F_\bullet \to G_\bullet$ be a map of complexes inducing φ on $M_1 = Coker(d_{F,1}) \to M_2 = Coker(d_{G,1})$, see Lemma 10.67.4. Then the induced maps*

$$H_i(\alpha) : H_i(F_\bullet \otimes_R N) \longrightarrow H_i(G_\bullet \otimes_R N)$$

are independent of the choice of α. If φ is an isomorphism, so are all the maps $H_i(\alpha)$. If $M_1 = M_2$, $F_\bullet = G_\bullet$, and φ is the identity, so are all the maps $H_i(\alpha)$.

Proof. The proof of this lemma is identical to the proof of Lemma 10.67.5. □

Not only does this lemma imply that the Tor modules are well defined, but it also provides for the functoriality of the constructions $(M, N) \mapsto \mathrm{Tor}_i^R(M, N)$ in the first variable. Of course the functoriality in the second variable is evident. We leave it to the reader to see that each of the Tor_i^R is in fact a functor

$$\mathrm{Mod}_R \times \mathrm{Mod}_R \to \mathrm{Mod}_R.$$

Here Mod_R denotes the category of R-modules, and for the definition of the product category see Categories, Definition 4.2.20. Namely, given morphisms of R-modules $M_1 \to M_2$ and $N_1 \to N_2$ we get a commutative diagram

$$\begin{array}{ccc} \mathrm{Tor}_i^R(M_1, N_1) & \longrightarrow & \mathrm{Tor}_i^R(M_1, N_2) \\ \downarrow & & \downarrow \\ \mathrm{Tor}_i^R(M_2, N_1) & \longrightarrow & \mathrm{Tor}_i^R(M_2, N_2) \end{array}$$

Lemma 10.70.2. *Let R be a ring and let M be an R-module. Suppose that $0 \to N' \to N \to N'' \to 0$ is a short exact sequence of R-modules. There exists a long exact sequence*

$$\begin{array}{c} M \otimes_R N' \to M \otimes_R N \to M \otimes_R N'' \to 0 \\ Tor_1^R(M, N') \to Tor_1^R(M, N) \to Tor_1^R(M, N'') \to \end{array}$$

Proof. The proof of this is the same as the proof of Lemma 10.67.6. □

Consider a homological double complex of R-modules

$$\begin{array}{ccccccc} \ldots & \xrightarrow{d} & A_{2,0} & \xrightarrow{d} & A_{1,0} & \xrightarrow{d} & A_{0,0} \\ & & \uparrow{\scriptstyle \delta} & & \uparrow{\scriptstyle \delta} & & \uparrow{\scriptstyle \delta} \\ \ldots & \xrightarrow{d} & A_{2,1} & \xrightarrow{d} & A_{1,1} & \xrightarrow{d} & A_{0,1} \\ & & \uparrow{\scriptstyle \delta} & & \uparrow{\scriptstyle \delta} & & \uparrow{\scriptstyle \delta} \\ \ldots & \xrightarrow{d} & A_{2,2} & \xrightarrow{d} & A_{1,2} & \xrightarrow{d} & A_{0,2} \\ & & \uparrow{\scriptstyle \delta} & & \uparrow{\scriptstyle \delta} & & \uparrow{\scriptstyle \delta} \\ & & \ldots & & \ldots & & \ldots \end{array}$$

This means that $d_{i,j} : A_{i,j} \to A_{i-1,j}$ and $\delta_{i,j} : A_{i,j} \to A_{i,j-1}$ have the following properties

(1) Any composition of two $d_{i,j}$ is zero. In other words the rows of the double complex are complexes.
(2) Any composition of two $\delta_{i,j}$ is zero. In other words the columns of the double complex are complexes.
(3) For any pair (i, j) we have $\delta_{i-1,j} \circ d_{i,j} = d_{i,j-1} \circ \delta_{i,j}$. In other words, all the squares commute.

The correct thing to do is to associate a spectral sequence to any such double complex. However, for the moment we can get away with doing something slightly easier.

Namely, for the purposes of this section only, given a double complex $(A_{\bullet,\bullet}, d, \delta)$ set $R(A)_j = \mathrm{Coker}(A_{1,j} \to A_{0,j})$ and $U(A)_i = \mathrm{Coker}(A_{i,1} \to A_{i,0})$. (The letters R and U are meant to suggest Right and Up.) We endow $R(A)_\bullet$ with the structure of a complex using the maps δ. Similarly we endow $U(A)_\bullet$ with the structure of a complex using the maps d. In other words we obtain the following huge commutative diagram

$$\begin{array}{ccccccccc}
\ldots & \xrightarrow{d} & U(A)_2 & \xrightarrow{d} & U(A)_1 & \xrightarrow{d} & U(A)_0 & & \\
 & & \uparrow & & \uparrow & & \uparrow & & \\
\ldots & \xrightarrow{d} & A_{2,0} & \xrightarrow{d} & A_{1,0} & \xrightarrow{d} & A_{0,0} & \longrightarrow & R(A)_0 \\
 & & \uparrow{\scriptstyle\delta} & & \uparrow{\scriptstyle\delta} & & \uparrow{\scriptstyle\delta} & & \uparrow{\scriptstyle\delta} \\
\ldots & \xrightarrow{d} & A_{2,1} & \xrightarrow{d} & A_{1,1} & \xrightarrow{d} & A_{0,1} & \longrightarrow & R(A)_1 \\
 & & \uparrow{\scriptstyle\delta} & & \uparrow{\scriptstyle\delta} & & \uparrow{\scriptstyle\delta} & & \uparrow{\scriptstyle\delta} \\
\ldots & \xrightarrow{d} & A_{2,2} & \xrightarrow{d} & A_{1,2} & \xrightarrow{d} & A_{0,2} & \longrightarrow & R(A)_2 \\
 & & \uparrow{\scriptstyle\delta} & & \uparrow{\scriptstyle\delta} & & \uparrow{\scriptstyle\delta} & & \uparrow{\scriptstyle\delta} \\
 & & \ldots & & \ldots & & \ldots & & \ldots
\end{array}$$

(This is no longer a double complex of course.) It is clear what a morphism $\Phi : (A_{\bullet,\bullet}, d, \delta) \to (B_{\bullet,\bullet}, d, \delta)$ of double complexes is, and it is clear that this induces morphisms of complexes $R(\Phi) : R(A)_\bullet \to R(B)_\bullet$ and $U(\Phi) : U(A)_\bullet \to U(B)_\bullet$.

Lemma 10.70.3. *Let $(A_{\bullet,\bullet}, d, \delta)$ be a double complex such that*

(1) *Each row $A_{\bullet,j}$ is a resolution of $R(A)_j$.*
(2) *Each column $A_{i,\bullet}$ is a resolution of $U(A)_i$.*

Then there are canonical isomorphisms

$$H_i(R(A)_\bullet) \cong H_i(U(A)_\bullet).$$

The isomorphisms are functorial with respect to morphisms of double complexes with the properties above.

Proof. We will show that $H_i(R(A)_\bullet))$ and $H_i(U(A)_\bullet)$ are canonically isomorphic to a third group. Namely

$$\mathbf{H}_i(A) := \frac{\{(a_{i,0}, a_{i-1,1}, \ldots, a_{0,i}) \mid d(a_{i,0}) = \delta(a_{i-1,1}), \ldots, d(a_{1,i-1}) = \delta(a_{0,i})\}}{\{d(a_{i+1,0}) - \delta(a_{i,1}), d(a_{i,1}) - \delta(a_{i-1,2}), \ldots, d(a_{1,i}) - \delta(a_{0,i+1})\}}$$

Here we use the notational convention that $a_{i,j}$ denotes an element of $A_{i,j}$. In other words, an element of $\mathbf{H}_i$ is represented by a zig-zag, represented as follows for $i = 2$

$$\begin{array}{ccccc}
a_{2,0} & \longmapsto & d(a_{2,0}) = \delta(a_{1,1}) & & \\
 & & \uparrow & & \\
 & & a_{1,1} & \longmapsto & d(a_{1,1}) = \delta(a_{0,2}) \\
 & & & & \uparrow \\
 & & & & a_{0,2}
\end{array}$$

Naturally, we divide out by "trivial" zig-zags, namely the submodule generated by elements of the form $(0, \ldots, 0, -\delta(a_{t+1,t-i}), d(a_{t+1,t-i}), 0, \ldots, 0)$. Note that there are canonical homomorphisms

$$\mathbf{H}_i(A) \to H_i(R(A)_\bullet), \quad (a_{i,0}, a_{i-1,1}, \ldots, a_{0,i}) \mapsto \text{class of image of } a_{0,i}$$

and

$$\mathbf{H}_i(A) \to H_i(U(A)_\bullet), \quad (a_{i,0}, a_{i-1,1}, \ldots, a_{0,i}) \mapsto \text{class of image of } a_{i,0}$$

First we show that these maps are surjective. Suppose that $\overline{r} \in H_i(R(A)_\bullet)$. Let $r \in R(A)_i$ be a cocycle representing the class of $\overline{r}$. Let $a_{0,i} \in A_{0,i}$ be an element which maps to r. Because $\delta(r) = 0$, we see that $\delta(a_{0,i})$ is in the image of d. Hence there exists an element $a_{1,i-1} \in A_{1,i-1}$ such that $d(a_{1,i-1}) = \delta(a_{0,i})$. This in turn implies that $\delta(a_{1,i-1})$ is in the kernel of d (because $d(\delta(a_{1,i-1})) = \delta(d(a_{1,i-1})) = \delta(\delta(a_{0,i})) = 0$. By exactness of the rows we find an element $a_{2,i-2}$ such that $d(a_{2,i-2}) = \delta(a_{1,i-1})$. And so on until a full zig-zag is found. Of course surjectivity of $\mathbf{H}_i \to H_i(U(A))$ is shown similarly.

To prove injectivity we argue in exactly the same way. Namely, suppose we are given a zig-zag $(a_{i,0}, a_{i-1,1}, \ldots, a_{0,i})$ which maps to zero in $H_i(R(A)_\bullet)$. This means that $a_{0,i}$ maps to an element of $\text{Coker}(A_{i,1} \to A_{i,0})$ which is in the image of $\delta : \text{Coker}(A_{i+1,1} \to A_{i+1,0}) \to \text{Coker}(A_{i,1} \to A_{i,0})$. In other words, $a_{0,i}$ is in the image of $\delta \oplus d : A_{0,i+1} \oplus A_{1,i} \to A_{0,i}$. From the definition of trivial zig-zags we see that we may modify our zig-zag by a trivial one and assume that $a_{0,i} = 0$. This immediately implies that $d(a_{1,i-1}) = 0$. As the rows are exact this implies that $a_{1,i-1}$ is in the image of $d : A_{2,i-1} \to A_{1,i-1}$. Thus we may modify our zig-zag once again by a trivial zig-zag and assume that our zig-zag looks like $(a_{i,0}, a_{i-1,1}, \ldots, a_{2,i-2}, 0, 0)$. Continuing like this we obtain the desired injectivity.

If $\Phi : (A_{\bullet,\bullet}, d, \delta) \to (B_{\bullet,\bullet}, d, \delta)$ is a morphism of double complexes both of which satisfy the conditions of the lemma, then we clearly obtain a commutative diagram

$$\begin{array}{ccccc} H_i(U(A)_\bullet) & \longleftarrow & \mathbf{H}_i(A) & \longrightarrow & H_i(R(A)_\bullet) \\ \downarrow & & \downarrow & & \downarrow \\ H_i(U(B)_\bullet) & \longleftarrow & \mathbf{H}_i(B) & \longrightarrow & H_i(R(B)_\bullet) \end{array}$$

This proves the functoriality. □

Remark 10.70.4. The isomorphism constructed above is the "correct" one only up to signs. A good part of homological algebra is concerned with choosing signs for various maps and showing commutativity of diagrams with intervention of suitable signs. For the moment we will simply use the isomorphism as given in the proof above, and worry about signs later.

Lemma 10.70.5. *Let R be a ring. For any $i \geq 0$ the functors $Mod_R \times Mod_R \to Mod_R$, $(M, N) \mapsto Tor_i^R(M, N)$ and $(M, N) \mapsto Tor_i^R(N, M)$ are canonically isomorphic.*

Proof. Let $F_\bullet$ be a free resolution of the module M and let $G_\bullet$ be a free resolution of the module N. Consider the double complex $(A_{i,j}, d, \delta)$ defined as follows:

(1) set $A_{i,j} = F_i \otimes_R G_j$,
(2) set $d_{i,j} : F_i \otimes_R G_j \to F_{i-1} \otimes G_j$ equal to $d_{F,i} \otimes \text{id}$, and

(3) set $\delta_{i,j} : F_i \otimes_R G_j \to F_i \otimes G_{j-1}$ equal to $\mathrm{id} \otimes d_{G,j}$.

This double complex is usually simply denoted $F_\bullet \otimes_R G_\bullet$.

Since each G_j is free, and hence flat we see that each row of the double complex is exact except in homological degree 0. Since each F_i is free and hence flat we see that each column of the double complex is exact except in homological degree 0. Hence the double complex satisfies the conditions of Lemma 10.70.3.

To see what the lemma says we compute $R(A)_\bullet$ and $U(A)_\bullet$. Namely,

$$\begin{aligned} R(A)_i &= \mathrm{Coker}(A_{1,i} \to A_{0,i}) \\ &= \mathrm{Coker}(F_1 \otimes_R G_i \to F_0 \otimes_R G_i) \\ &= \mathrm{Coker}(F_1 \to F_0) \otimes_R G_i \\ &= M \otimes_R G_i \end{aligned}$$

In fact these isomorphisms are compatible with the differentials δ and we see that $R(A)_\bullet = M \otimes_R G_\bullet$ as homological complexes. In exactly the same way we see that $U(A)_\bullet = F_\bullet \otimes_R N$. We get

$$\begin{aligned} \mathrm{Tor}_i^R(M, N) &= H_i(F_\bullet \otimes_R N) \\ &= H_i(U(A)_\bullet) \\ &= H_i(R(A)_\bullet) \\ &= H_i(M \otimes_R G_\bullet) \\ &= H_i(G_\bullet \otimes_R M) \\ &= \mathrm{Tor}_i^R(N, M) \end{aligned}$$

Here the third equality is Lemma 10.70.3, and the fifth equality uses the isomorphism $V \otimes W = W \otimes V$ of the tensor product.

Functoriality. Suppose that we have R-modules M_ν, N_ν, $\nu = 1, 2$. Let $\varphi : M_1 \to M_2$ and $\psi : N_1 \to N_2$ be morphisms of R-modules. Suppose that we have free resolutions $F_{\nu,\bullet}$ for M_ν and free resolutions $G_{\nu,\bullet}$ for N_ν. By Lemma 10.67.4 we may choose maps of complexes $\alpha : F_{1,\bullet} \to F_{2,\bullet}$ and $\beta : G_{1,\bullet} \to G_{2,\bullet}$ compatible with φ and ψ. We claim that the pair (α, β) induces a morphism of double complexes

$$\alpha \otimes \beta : F_{1,\bullet} \otimes_R G_{1,\bullet} \longrightarrow F_{2,\bullet} \otimes_R G_{2,\bullet}$$

This is really a very straightforward check using the rule that $F_{1,i} \otimes_R G_{1,j} \to F_{2,i} \otimes_R G_{2,j}$ is given by $\alpha_i \otimes \beta_j$ where α_i, resp. β_j is the degree i, resp. j component of α, resp. β. The reader also readily verifies that the induced maps $R(F_{1,\bullet} \otimes_R G_{1,\bullet})_\bullet \to R(F_{2,\bullet} \otimes_R G_{2,\bullet})_\bullet$ agrees with the map $M_1 \otimes_R G_{1,\bullet} \to M_2 \otimes_R G_{2,\bullet}$ induced by $\varphi \otimes \beta$. Similarly for the map induced on the $U(-)_\bullet$ complexes. Thus the statement on functoriality follows from the statement on functoriality in Lemma 10.70.3. □

Remark 10.70.6. An interesting case occurs when $M = N$ in the above. In this case we get a canonical map $\mathrm{Tor}_i^R(M, M) \to \mathrm{Tor}_i^R(M, M)$. Note that this map is not the identity, because even when $i = 0$ this map is not the identity! For example, if V is a vector space of dimension n over a field, then the switch map $V \otimes_k V \to V \otimes_k V$ has $(n^2 + n)/2$ eigenvalues $+1$ and $(n^2 - n)/2$ eigenvalues -1. In characteristic 2 it is not even diagonalizable. Note that even changing the sign of the map will not get rid of this.

Lemma 10.70.7. *Let R be a ring. Let M be an R-module. The following are equivalent:*

(1) *The module M is flat over R.*
(2) *For all $i > 0$ the functor $Tor_i^R(M, -)$ is zero.*
(3) *The functor $Tor_1^R(M, -)$ is zero.*
(4) *For all ideals $I \subset R$ we have $Tor_1^R(M, R/I) = 0$.*
(5) *For all finitely generated ideals $I \subset R$ we have $Tor_1^R(M, R/I) = 0$.*

Proof. Suppose M is flat. Let N be an R-module. Let $F_\bullet$ be a free resolution of N. Then $F_\bullet \otimes_R M$ is a resolution of $N \otimes_R M$, by flatness of M. Hence all higher Tor groups vanish.

It now suffices to show that the last condition implies that M is flat. Let $I \subset R$ be an ideal. Consider the short exact sequence $0 \to I \to R \to R/I \to 0$. Apply Lemma 10.70.2. We get an exact sequence

$$\mathrm{Tor}_1^R(M, R/I) \to M \otimes_R I \to M \otimes_R R \to M \otimes_R R/I \to 0$$

Since obviously $M \otimes_R R = M$ we conclude that the last hypothesis implies that $M \otimes_R I \to M$ is injective for every finitely generated ideal I. Thus M is flat by Lemma 10.36.4. □

Remark 10.70.8. The proof of Lemma 10.70.7 actually shows that

$$\mathrm{Tor}_1^R(M, R/I) = \mathrm{Ker}(I \otimes_R M \to M).$$

10.71. Functorialities for Tor

In this section we briefly discuss the functoriality of Tor with respect to change of ring, etc. Here is a list of items to work out.

(1) Given a ring map $R \to R'$, an R-module M and an R'-module N' the R-modules $\mathrm{Tor}_i^R(M, N')$ have a natural R'-module structure.
(2) Given a ring map $R \to R'$ and R-modules M, N there is a natural R-module map $\mathrm{Tor}_i^R(M, N) \to \mathrm{Tor}_i^{R'}(M \otimes_R R', N \otimes_R R')$.
(3) Given a ring map $R \to R'$ an R-module M and an R'-module N' there exists a natural R'-module map $\mathrm{Tor}_i^R(M, N') \to \mathrm{Tor}_i^{R'}(M \otimes_R R', N')$.

Lemma 10.71.1. *Given a flat ring map $R \to R'$ and R-modules M, N the natural R-module map $Tor_i^R(M, N) \otimes_R R' \to Tor_i^{R'}(M \otimes_R R', N \otimes_R R')$ is an isomorphism for all i.*

Proof. Omitted. This is true because a free resolution $F_\bullet$ of M over R stays exact when tensoring with R' over R and hence $(F_\bullet \otimes_R N) \otimes_R R'$ computes the Tor groups over R'. □

10.72. Projective modules

Some lemmas on projective modules.

Definition 10.72.1. Let R be a ring. An R-module P is *projective* if and only if the functor $\mathrm{Hom}_R(P, -) : \mathrm{Mod}_R \to \mathrm{Mod}_R$ is an exact functor.

The functor $\mathrm{Hom}_R(M, -)$ is left exact for any R-module M, see Lemma 10.10.1. Hence the condition for P to be projective really signifies that given a surjection of R-modules $N \to N'$ the map $\mathrm{Hom}_R(P, N) \to \mathrm{Hom}_R(P, N')$ is surjective.

Lemma 10.72.2. *Let R be a ring. Let P be an R-module. The following are equivalent*

(1) *P is projective,*
(2) *P is a direct summand of a free R-module, and*
(3) *$Ext^1_R(P, M) = 0$ for every R-module M.*

Proof. Assume P is projective. Choose a surjection $\pi : F \to P$ where F is a free R-module. As P is projective there exists a $i \in \mathrm{Hom}_R(P, F)$ such that $i \circ \pi = \mathrm{id}_P$. In other words $F \cong \mathrm{Ker}(\pi) \oplus i(P)$ and we see that P is a direct summand of F.

Conversely, assume that $P \oplus Q = F$ is a free R-module. Note that the free module $F = \bigoplus_{i \in I} R$ is projective as $\mathrm{Hom}_R(F, M) = \prod_{i \in I} M$ and the functor $M \mapsto \prod_{i \in I} M$ is exact. Then $\mathrm{Hom}_R(F, -) = \mathrm{Hom}_R(P, -) \times \mathrm{Hom}_R(Q, -)$ as functors, hence both P and Q are projective.

Assume $P \oplus Q = F$ is a free R-module. Then we have a free resolution $F_\bullet$ of the form

$$\ldots F \xrightarrow{a} F \xrightarrow{b} F \to P \to 0$$

where the maps a, b alternate and are equal to the projector onto P and Q. Hence the complex $\mathrm{Hom}_R(F_\bullet, M)$ is split exact in degrees ≥ 1, whence we see the vanishing in (3).

Assume $\mathrm{Ext}^1_R(P, M) = 0$ for every R-module M. Pick a free resolution $F_\bullet \to P$. Set $M = \mathrm{Im}(F_1 \to F_0) = \mathrm{Ker}(F_0 \to M)$. Consider the element $\xi \in \mathrm{Ext}^1_R(P, M)$ given by the class of the quotient map $\pi : F_1 \to M$. Since ξ is zero there exists a map $s : F_0 \to M$ such that $\pi = s \circ (F_1 \to F_0)$. Clearly, this means that

$$F_0 = \mathrm{Ker}(s) \oplus \mathrm{Ker}(F_0 \to P) = P \oplus \mathrm{Ker}(F_0 \to P)$$

and we win. □

Lemma 10.72.3. *A direct sum of projective modules is projective.*

Proof. This is true by the characterization of projectives as direct summands of free modules in Lemma 10.72.2. □

Lemma 10.72.4. *Let R be a ring. Let $I \subset R$ be a nilpotent ideal. Let $\overline{P}$ be a projective R-module. Then there exists a projective R-module P such that $P/IP \cong \overline{P}$.*

Proof. We can choose a set A and a direct sum decomposition $\bigoplus_{\alpha \in A} R/I = \overline{P} \oplus \overline{K}$ for some R/I-module $\overline{K}$. Write $F = \bigoplus_{\alpha \in A} R$ for the free R-module on A. Choose a lift $p : F \to F$ of the projector $\overline{p}$ associated to the direct summand $\overline{P}$ of $\bigoplus_{\alpha \in A} R/I$. Note that $p^2 - p \in \mathrm{End}_R(F)$ is a nilpotent endomorphism of F (as I is nilpotent and the matrix entries of $p^2 - p$ are in I; more precisely, if $I^n = 0$, then $(p^2 - p)^n = 0$). Hence by Lemma 10.49.6 we can modify our choice of p and assume that p is a projector. Set $P = \mathrm{Im}(p)$. □

Lemma 10.72.5. *Let R be a ring. Let $I \subset R$ be an ideal. Let M be an R-module. Assume*

(1) *I is nilpotent,*
(2) *M/IM is a projective R/I-module,*
(3) *M is a flat R-module.*

Then M is a projective R-module.

Proof. By Lemma 10.72.4 we can find a projective R-module P and an isomorphism $P/IP \to M/IM$. We are going to show that M is isomorphic to P which will finish the proof. Because P is projective we can lift the map $P \to P/IP \to M/IM$ to an R-module map $P \to M$ which is an isomorphism modulo I. By Nakayama's Lemma 10.18.1 the map $P \to M$ is surjective. It remains to show that $P \to M$ is injective. Since $I^n = 0$ for some n, we can use the filtrations

$$0 = I^n M \subset I^{n-1}M \subset \ldots \subset IM \subset M$$
$$0 = I^n P \subset I^{n-1}P \subset \ldots \subset IP \subset P$$

to see that it suffices to show that the induced maps $I^a P/I^{a+1}P \to I^a M/I^{a+1}M$ are injective. Since both P and M are flat R-modules we can identify this with the map

$$I^a/I^{a+1} \otimes_{R/I} P/IP \longrightarrow I^a/I^{a+1} \otimes_{R/I} M/IM$$

induced by $P \to M$. Since we chose $P \to M$ such that the induced map $P/IP \to M/IM$ is an isomorphism, we win. □

10.73. Finite projective modules

Definition 10.73.1. Let R be a ring and M an R-module. We say that M is *locally free* if we can cover $\mathrm{Spec}(R)$ by standard opens $D(f_i)$, $i \in I$ such that M_{f_i} is a free R_{f_i}-module for all $i \in I$. We say that M is *finite locally free* if each M_{f_i} is finite free.

Note that a finite locally free R-module is automatically finitely presented by Lemma 10.22.2.

Lemma 10.73.2. *Let R be a ring and let M be an R-module. The following are equivalent*

1. *M is finitely presented and R-flat,*
2. *M is finite projective,*
3. *M is a direct summand of a finite free R-module,*
4. *M is finitely presented and for all $\mathfrak{p} \in \mathrm{Spec}(R)$ the localization $M_\mathfrak{p}$ is free,*
5. *M is finitely presented and for all maximal ideals $\mathfrak{m} \subset R$ the localization $M_\mathfrak{m}$ is free,*
6. *M is finite and locally free,*
7. *M is finite locally free, and*
8. *M is finite, for every prime $\mathfrak{p}$ the module $M_\mathfrak{p}$ is free, and the function*

$$\rho_M : \mathrm{Spec}(R) \to \mathbf{Z}, \quad \mathfrak{p} \longmapsto \dim_{\kappa(\mathfrak{p})} M \otimes_R \kappa(\mathfrak{p})$$

is locally constant in the Zariski topology.

Proof. First suppose M is finite projective, i.e., (2) holds. Take a surjection $R^n \to M$ and let K be the kernel. Since M is projective, $0 \to K \to R^n \to M \to 0$ splits. Hence (2) $\Rightarrow$ (3). The implication (3) $\Rightarrow$ (2) follows from the fact that a direct summand of a projective is projective, see Lemma 10.72.2.

Assume (3), so we can write $K \oplus M \cong R^{\oplus n}$. So K is a direct summand of R^n and thus finitely generated. This shows $M = R^{\oplus n}/K$ is finitely presented. In other words, (3) $\Rightarrow$ (1).

Assume M is finitely presented and flat, i.e., (1) holds. We will prove that (7) holds. Pick any prime $\mathfrak{p}$ and $x_1, \ldots, x_r \in M$ which map to a basis of $M \otimes_R \kappa(\mathfrak{p})$.

By Nakayama's Lemma 10.18.1 these elements generate M_g for some $g \in R$, $g \notin \mathfrak{p}$. The corresponding surjection $\varphi : R_g^{\oplus r} \to M_g^{\oplus r}$ has the following two properties: (a) $\mathrm{Ker}(\varphi)$ is a finite R_g-module (see Lemma 10.5.3) and (b) $\mathrm{Ker}(\varphi) \otimes \kappa(\mathfrak{p}) = 0$ by flatness of M_g over R_g (see Lemma 10.36.11). Hence by Nakayama's lemma again there exists a $g' \in R_g$ such that $\mathrm{Ker}(\varphi)_{g'} = 0$. In other words, $M_{gg'}$ is free.

A finite locally free module is a finite module, see Lemma 10.22.2, hence (7) $\Rightarrow$ (6). It is clear that (6) $\Rightarrow$ (7) and that (7) $\Rightarrow$ (8).

A finite locally free module is a finitely presented module, see Lemma 10.22.2, hence (7) $\Rightarrow$ (4). Of course (4) implies (5). Since we may check flatness locally (see Lemma 10.36.19) we conclude that (5) implies (1). At this point we have

$$\begin{array}{ccccccc} (2) & \Longleftrightarrow & (3) & \Longrightarrow & (1) & \Longrightarrow & (7) \Longleftrightarrow (6) \\ & & & & \Uparrow & & \Downarrow \quad \searrow \\ & & & & (5) & \Longleftarrow & (4) \qquad (8) \end{array}$$

Suppose that M satisfies (1), (4), (5), (6), and (7). We will prove that (3) holds. It suffices to show that M is projective. We have to show that $\mathrm{Hom}_R(M, -)$ is exact. Let $0 \to N'' \to N \to N' \to 0$ be a short exact sequence of R-module. We have to show that $0 \to \mathrm{Hom}_R(M, N'') \to \mathrm{Hom}_R(M, N) \to \mathrm{Hom}_R(M, N') \to 0$ is exact. As M is finite locally free there exist a covering $\mathrm{Spec}(R) = \bigcup D(f_i)$ such that M_{f_i} is finite free. By Lemma 10.10.2 we see that

$$0 \to \mathrm{Hom}_R(M, N'')_{f_i} \to \mathrm{Hom}_R(M, N)_{f_i} \to \mathrm{Hom}_R(M, N')_{f_i} \to 0$$

is equal to $0 \to \mathrm{Hom}_{R_{f_i}}(M_{f_i}, N''_{f_i}) \to \mathrm{Hom}_{R_{f_i}}(M_{f_i}, N_{f_i}) \to \mathrm{Hom}_{R_{f_i}}(M_{f_i}, N'_{f_i}) \to 0$ which is exact as M_{f_i} is free and as the localization $0 \to N''_{f_i} \to N_{f_i} \to N'_{f_i} \to 0$ is exact (as localization is exact). Whence we see that $0 \to \mathrm{Hom}_R(M, N'') \to \mathrm{Hom}_R(M, N) \to \mathrm{Hom}_R(M, N') \to 0$ is exact by Lemma 10.22.2.

Finally, assume that (8) holds. Pick a maximal ideal $\mathfrak{m} \subset R$. Pick $x_1, \ldots, x_r \in M$ which map to a $\kappa(\mathfrak{m})$-basis of $M \otimes_R \kappa(\mathfrak{m}) = M/\mathfrak{m}M$. In particular $\rho_M(\mathfrak{m}) = r$. By Nakayama's Lemma 10.18.1 there exists an $f \in R$, $f \notin \mathfrak{m}$ such that $x_1, \ldots, x_r$ generate M_f over R_f. By the assumption that ρ_M is locally constant there exists a $g \in R$, $g \notin \mathfrak{m}$ such that ρ_M is constant equal to r on $D(g)$. We claim that

$$\Psi : R_{fg}^{\oplus r} \longrightarrow M_{fg}, \quad (a_1, \ldots, a_r) \longmapsto \sum a_i x_i$$

is an isomorphism. This claim will show that M is finite locally free, i.e., that (7) holds. To see the claim it suffices to show that the induced map on localizations $\Psi_{\mathfrak{p}} : R_{\mathfrak{p}}^{\oplus r} \to M_{\mathfrak{p}}$ is an isomorphism for all $\mathfrak{p} \in D(fg)$, see Lemma 10.22.1. By our choice of f the map $\Psi_{\mathfrak{p}}$ is surjective. By assumption (8) we have $M_{\mathfrak{p}} \cong R_{\mathfrak{p}}^{\oplus \rho_M(\mathfrak{p})}$ and by our choice of g we have $\rho_M(\mathfrak{p}) = r$. Hence $\Psi_{\mathfrak{p}}$ determines a surjection $R_{\mathfrak{p}}^{\oplus r} \to M_{\mathfrak{p}} \cong R_{\mathfrak{p}}^{\oplus r}$ whence is an isomorphism by Lemma 10.15.4. (Of course this last fact follows from a simple matrix argument also.) $\square$

Remark 10.73.3. It is not true that a finite R-module which is R-flat is automatically projective. A counter example is where $R = \mathcal{C}^\infty(\mathbf{R})$ is the ring of infinitely differentiable functions on $\mathbf{R}$, and $M = R_{\mathfrak{m}} = R/I$ where $\mathfrak{m} = \{f \in R \mid f(0) = 0\}$ and $I = \{f \in R \mid \exists \epsilon, \epsilon > 0 : f(x) = 0 \ \forall x, |x| < \epsilon\}$.

Lemma 10.73.4. *(Warning: see Remark 10.73.3.) Suppose R is a local ring, and M is a finite flat R-module. Then M is finite free.*

Proof. Follows from the equational criterion of flatness, see Lemma 10.36.10. Namely, suppose that $x_1, \ldots, x_r \in M$ map to a basis of $M/\mathfrak{m}M$. By Nakayama's Lemma 10.18.1 these elements generate M. We want to show there is no relation among the x_i. Instead, we will show by induction on n that if $x_1, \ldots, x_n \in M$ are linearly independent in the vector space $M/\mathfrak{m}M$ then they are independent over R.

The base case of the induction is where we have $x \in M$, $x \notin \mathfrak{m}M$ and a relation $fx = 0$. By the equational criterion there exist $y_j \in M$ and $a_j \in R$ such that $x = \sum a_j y_j$ and $fa_j = 0$ for all j. Since $x \notin \mathfrak{m}M$ we see that at least one a_j is a unit and hence $f = 0$.

Suppose that $\sum f_i x_i$ is a relation among $x_1, \ldots, x_n$. By our choice of x_i we have $f_i \in \mathfrak{m}$. According to the equational criterion of flatness there exist $a_{ij} \in R$ and $y_j \in M$ such that $x_i = \sum a_{ij} y_j$ and $\sum f_i a_{ij} = 0$. Since $x_n \notin \mathfrak{m}M$ we see that $a_{nj} \notin \mathfrak{m}$ for at least one j. Since $\sum f_i a_{ij} = 0$ we get $f_n = \sum_{i=1}^{n-1} (-a_{ij}/a_{nj}) f_i$. The relation $\sum f_i x_i = 0$ now can be rewritten as $\sum_{i=1}^{n-1} f_i(x_i + (-a_{ij}/a_{nj})x_n) = 0$. Note that the elements $x_i + (-a_{ij}/a_{nj})x_n$ map to $n-1$ linearly independent elements of $M/\mathfrak{m}M$. By induction assumption we get that all the f_i, $i \leq n-1$ have to be zero, and also $f_n = \sum_{i=1}^{n-1} (-a_{ij}/a_{nj}) f_i$. This proves the induction step. □

Lemma 10.73.5. *Let $R \to S$ be a flat local homomorphism of local rings. Let M be a finite R-module. Then M is finite projective over R if and only if $M \otimes_R S$ is finite projective over S.*

Proof. Suppose that $M \otimes_R S$ is finite projective over S. By Lemma 10.73.2 it is finite free. Pick $x_1, \ldots, x_r \in M$ whose residue classes generate $M/\mathfrak{m}_R M$. Clearly we see that $x_1 \otimes 1, \ldots, x_r \otimes 1$ are a basis for $M \otimes_R S$. This implies that the map $R^{\oplus r} \to M, (a_i) \mapsto \sum a_i x_i$ becomes an isomorphism after tensoring with S. By faithful flatness of $R \to S$, see Lemma 10.36.16 we see that it is an isomorphism. □

Lemma 10.73.6. *Let R be a semi-local ring. Let M be a finite locally free module. If M has constant rank, then M is free. In particular, if R has connected spectrum, then M is free.*

Proof. Omitted. Hints: First show that $M/\mathfrak{m}_i M$ has the same dimension d for all maximal ideal $\mathfrak{m}_1, \ldots, \mathfrak{m}_n$ of R using the spectrum is connected. Next, show that there exist elements $x_1, \ldots, x_d \in M$ which form a basis for each $M/\mathfrak{m}_i M$ by the Chinese remainder theorem. Finally show that $x_1, \ldots, x_d$ is a basis for M. □

Here is a technical lemma that is used in the chapter on groupoids.

Lemma 10.73.7. *Let R be a local ring with maximal ideal $\mathfrak{m}$ and infinite residue field. Let $R \to S$ be a ring map. Let M be an S-module and let $N \subset M$ be an R-submodule. Assume*

1. *S is semi-local and $\mathfrak{m}S$ is contained in the radical of S,*
2. *M is a finite free S-module, and*
3. *N generates M as an S-module.*

Then N contains an S-basis of M.

Proof. Assume M is free of rank n. Let $I = \text{rad}(S)$. By Nakayama's Lemma 10.18.1 a sequence of elements $m_1, \ldots, m_n$ is a basis for M if and only if $\overline{m}_i \in M/IM$ generate M/IM. Hence we may replace M by M/IM, N by $N/(N \cap IM)$, R by $R/\mathfrak{m}$, and S by S/IS. In this case we see that S is a finite product of fields $S = k_1 \times \ldots \times k_r$ and $M = k_1^{\oplus n} \times \ldots \times k_r^{\oplus n}$. The fact that $N \subset M$ generates M as an S-module means that there exist $x_j \in N$ such that a linear combination $\sum a_j x_j$ with $a_j \in S$ has a nonzero component in each factor $k_i^{\oplus n}$. Because $R = k$ is an infinite field, this means that also some linear combination $y = \sum c_j x_j$ with $c_j \in k$ has a nonzero component in each factor. Hence $y \in N$ generates a free direct summand $Sy \subset M$. By induction on n the result holds for M/Sy and the submodule $\overline{N} = N/(N \cap Sy)$. In other words there exist $\overline{y}_2, \ldots, \overline{y}_n$ in $\overline{N}$ which (freely) generate M/Sy. Then $y, y_2, \ldots, y_n$ (freely) generate M and we win. □

10.74. Open loci defined by module maps

The set of primes where a given module map is surjective, or an isomorphism is sometimes open. In the case of finite projective modules we can look at the rank of the map.

Lemma 10.74.1. *Let R be a ring. Let $\varphi : M \to N$ be a map of R-modules with N a finite R-module. Then we have the equality*

$$\begin{aligned} U &= \{\mathfrak{p} \subset R \mid \varphi_\mathfrak{p} : M_\mathfrak{p} \to N_\mathfrak{p} \text{ is surjective}\} \\ &= \{\mathfrak{p} \subset R \mid \varphi \otimes \kappa(\mathfrak{p}) : M \otimes \kappa(\mathfrak{p}) \to N \otimes \kappa(\mathfrak{p}) \text{ is surjective}\} \end{aligned}$$

and U is an open subset of $\text{Spec}(R)$.

Proof. This is a reformulation of Nakayama's lemma, see Lemma 10.18.1. Details omitted. □

Lemma 10.74.2. *Let R be a ring. Let $\varphi : M \to N$ be a map of finitely presented R-modules. Then*

$$U = \{\mathfrak{p} \subset R \mid \varphi_\mathfrak{p} : M_\mathfrak{p} \to N_\mathfrak{p} \text{ is an isomorphism}\}$$

is an open subset of $\text{Spec}(R)$.

Proof. Let $\mathfrak{p} \in U$. Pick a presentation $N = R^{\oplus n}/\sum_{j=1,\ldots,m} Rk_j$. Denote e_i the image in N of the ith basis vector of $R^{\oplus n}$. For each $i \in \{1, \ldots, n\}$ choose an element $m_i \in M_\mathfrak{p}$ such that $\varphi(m_i) = f_i e_i$ for some $f_i \in R$, $f_i \notin \mathfrak{p}$. This is possible as $\varphi_\mathfrak{p}$ is an isomorphism. Set $f = f_1 \ldots f_n$ and let $\psi : R_f^{\oplus n} \to M$ be the map which maps the ith basis vector to m_i/f_i. Note that $\varphi_f \circ \psi$ is the localization at f of the given map $R^{\oplus n} \to N$. As $\varphi_\mathfrak{p}$ is an isomorphism we see that $\psi(k_j)$ is an element of M which maps to zero in $M_\mathfrak{p}$. Hence we see that there exist $g_j \in R$, $g_j \notin \mathfrak{p}$ such that $g_j \psi(k_j) = 0$. Setting $g = g_1 \ldots g_m$, we see that ψ_g factors through N_{fg} to give a map $N_{fg} \to M_{fg}$. By construction this map is inverse to φ_{fg}. Hence φ_{fg} is an isomorphism, which implies that $D(fg) \subset U$ as desired. □

Lemma 10.74.3. *Let R be a ring. Let $\varphi : P_1 \to P_2$ be a map of finite projective modules. Then*

1. *The set U of primes $\mathfrak{p} \in \text{Spec}(R)$ such that $\varphi \otimes \kappa(\mathfrak{p})$ is injective is open.*
2. *For any $f \in R$ such that $D(f) \subset U$ the module $\text{Coker}(\varphi)_f$ is finite projective over R_f.*

(3) *The set V of primes $\mathfrak{p} \in \mathrm{Spec}(R)$ such that $\varphi \otimes \kappa(\mathfrak{p})$ is an isomorphism is open.*
(4) *For any $f \in R$ such that $D(f) \subset V$ the map $\varphi : P_{1,f} \to P_{2,f}$ is an isomorphism of modules over R_f.*

Proof. To prove the sets U and V are open we may work locally on $\mathrm{Spec}(R)$. Thus we may replace R by a suitable localization and assume that $P_1 = R^{n_1}$ and $P_2 = R^{n_2}$, see Lemma 10.73.2. In this case injectivity of $\varphi \otimes \kappa(\mathfrak{p})$ is equivalent to some $n_1 \times n_1$ minor f of the matrix of φ being invertible in $\kappa(\mathfrak{p})$. Thus $D(f) \subset U$. Similarly for V, but in that case with the added assumption that $m = n$ (and hence f is just the determinant of the map).

Now suppose $D(f) \subset U$. By Lemma 10.73.2 it suffices to prove that $\mathrm{Coker}(\varphi)$ is finite projective locally on $D(f)$. Thus, as we saw above, we may assume that $P_1 = R^{n_1}$ and $P_2 = R^{n_2}$ and that some minor of the matrix of φ is invertible in R. If the minor in question corresponds to the first n_1 basis vectors of R^{n_2}, then using the last $n_2 - n_1$ basis vectors we get a map $R^{n_2-n_1} \to R^{n_2} \to \mathrm{Coker}(\varphi)$ which is easily seen to be an isomorphism. If $D(f) \subset V$ the argument is even easier. □

10.75. Faithfully flat descent for projectivity of modules

In the next few sections we prove, following Raynaud and Gruson [**GR71**], that the projectivity of modules descends along faithfully flat ring maps. The idea of the proof is to use dévissage à la Kaplansky [**Kap58**] to reduce to the case of countably generated modules. Given a well-behaved filtration of a module M, dévissage allows us to express M as a direct sum of successive quotients of the filtering submodules (see Section 10.79). Using this technique, we prove that a projective module is a direct sum of countably generated modules (Theorem 10.79.5). To prove descent of projectivity for countably generated modules, we introduce a "Mittag-Leffler" condition on modules, prove that a countably generated module is projective if and only if it is flat and Mittag-Leffler (Theorem 10.88.3), and then show that the property of being a flat Mittag-Leffler module descends (Lemma 10.90.1). Finally, given an arbitrary module M whose base change by a faithfully flat ring map is projective, we filter M by submodules whose successive quotients are countably generated projective modules, and then by dévissage conclude M is a direct sum of projectives, hence projective itself (Theorem 10.90.5).

We note that there is an error in the proof of faithfully flat descent of projectivity in [**GR71**]. There, descent of projectivity along faithfully flat ring maps is deduced from descent of projectivity along a more general type of ring map ([**GR71**, Example 3.1.4(1) of Part II]). However, the proof of descent along this more general type of map is incorrect, as explained in [**Gru73**]. Patching this hole in the proof of faithfully flat descent of projectivity comes down to proving that the property of being a flat Mittag-Leffler module descends along faithfully flat ring maps. We do this in Lemma 10.90.1.

10.76. Characterizing flatness

In this section we discuss criteria for flatness. The main result in this section is Lazard's theorem (Theorem 10.76.4 below), which says that a flat module is the colimit of a directed system of free finite modules. We remind the reader of the

"equational criterion for flatness", see Lemma 10.36.10. It turns out that this can be massaged into a seemingly much stronger property.

Lemma 10.76.1. *Let M be an R-module. The following are equivalent:*

(1) *M is flat.*
(2) *If $f : R^n \to M$ is a module map and $x \in \ker(f)$, then there are module maps $h : R^n \to R^m$ and $g : R^m \to M$ such that $f = g \circ h$ and $x \in \ker(h)$.*
(3) *Suppose $f : R^n \to M$ is a module map, $N \subset \ker(f)$ any submodule, and $h : R^n \to R^m$ a map such that $N \subset \ker(h)$ and f factors through h. Then given any $x \in \ker(f)$ we can find a map $h' : R^n \to R^{m'}$ such that $N + Rx \subset \ker(h')$ and f factors through h'.*
(4) *If $f : R^n \to M$ is a module map and $N \subset \ker(f)$ is a finitely generated submodule, then there are module maps $h : R^n \to R^m$ and $g : R^m \to M$ such that $f = g \circ h$ and $N \subset \ker(h)$.*

Proof. That (1) is equivalent to (2) is just a reformulation of the equational criterion for flatness. To show (2) implies (3), let $g : R^m \to M$ be the map such that f factors as $f = g \circ h$. By (2) find $h'' : R^m \to R^{m'}$ such that h'' kills $h(x)$ and $g : R^m \to M$ factors through h''. Then taking $h' = h'' \circ h$ works. (3) implies (4) by induction on the number of generators of $N \subset \ker(f)$ in (4). Clearly (4) implies (2). □

Lemma 10.76.2. *Let M be an R-module. Then M is flat if and only if the following condition holds: if P is a finitely presented R-module and $f : P \to M$ a module map, then there is a free finite R-module F and module maps $h : P \to F$ and $g : F \to M$ such that $f = g \circ h$.*

Proof. This is just a reformulation of condition (4) from Lemma 10.76.1. □

Lemma 10.76.3. *Let M be an R-module. Then M is flat if and only if the following condition holds: for every finitely presented R-module P, if $N \to M$ is a surjective R-module map, then the induced map $\mathrm{Hom}_R(P, N) \to \mathrm{Hom}_R(P, M)$ is surjective.*

Proof. First suppose M is flat. We must show that if P is finitely presented, then given a map $f : P \to M$, it factors through the map $N \to M$. By Lemma 10.76.2 the map f factors through a map $F \to M$ where F is free and finite. Since F is free, this map factors through $N \to M$. Thus f factors through $N \to M$.

Conversely, suppose the condition of the lemma holds. Let $f : P \to M$ be a map from a finitely presented module P. Choose a free module N with a surjection $N \to M$ onto M. Then f factors through $N \to M$, and since P is finitely generated, f factors through a free finite submodule of N. Thus M satisfies the condition of Lemma 10.76.2, hence is flat. □

Theorem 10.76.4 (Lazard's theorem)**.** *Let M be an R-module. Then M is flat if and only if it is the colimit of a directed system of free finite R-modules.*

Proof. A colimit of a directed system of flat modules is flat, as taking directed colimits is exact and commutes with tensor product. Hence if M is the colimit of a directed system of free finite modules then M is flat.

For the converse, first recall that any module M can be written as the colimit of a directed system of finitely presented modules, in the following way. Choose a

surjection $f : R^I \to M$ for some set I, and let K be the kernel. Let E be the set of ordered pairs (J, N) where J is a finite subset of I and N is a finitely generated submodule of $R^J \cap K$. Then E is made into a directed partially ordered set by defining $(J, N) \leq (J', N')$ if and only if $J \subset J'$ and $N \subset N'$. Define $M_e = R^J/N$ for $e = (J, N)$, and define $f_{ee'} : M_e \to M_{e'}$ to be the natural map for $e \leq e'$. Then $(M_e, f_{ee'})$ is a directed system and the natural maps $f_e : M_e \to M$ induce an isomorphism $\operatorname{colim}_{e \in E} M_e \xrightarrow{\cong} M$.

Now suppose M is flat. Let $I = M \times \mathbf{Z}$, write (x_i) for the canonical basis of R^I, and take in the above discussion $f : R^I \to M$ to be the map sending x_i to the projection of i onto M. To prove the theorem it suffices to show that the $e \in E$ such that M_e is free form a cofinal subset of E. So let $e = (J, N) \in E$ be arbitrary. By Lemma 10.76.2 there is a free finite module F and maps $h : R^J/N \to F$ and $g : F \to M$ such that the natural map $f_e : R^J/N \to M$ factors as $R^J/N \xrightarrow{h} F \xrightarrow{g} M$. We are going to realize F as $M_{e'}$ for some $e' \geq e$.

Let $\{b_1, \ldots, b_n\}$ be a finite basis of F. Choose n distinct elements $i_1, \ldots, i_n \in I$ such that $i_\ell \notin J$ for all ℓ, and such that the image of x_{i_ℓ} under $f : R^I \to M$ equals the image of b_ℓ under $g : F \to M$. This is possible by our choice of I. Now let $J' = J \cup \{i_1, \ldots, i_n\}$, and define $R^{J'} \to F$ by $x_i \mapsto h(x_i)$ for $i \in J$ and $x_{i_\ell} \mapsto b_\ell$ for $\ell = 1, \ldots, n$. Let $N' = \ker(R^{J'} \to F)$. Observe:

(1) $R^{J'} \to F$ factors $f : R^I \to M$, hence $N' \subset K = \ker(f)$;
(2) $R^{J'} \to F$ is a surjection onto a free finite module, hence it splits and so N' is finitely generated;
(3) $J \subset J'$ and $N \subset N'$.

By (1) and (2) $e' = (J', N')$ is in E, by (3) $e' \geq e$, and by construction $M_{e'} = R^{J'}/N' \cong F$ is free. □

10.77. Universally injective module maps

Next we discuss universally injective module maps, which are in a sense complementary to flat modules (see Lemma 10.77.5). We follow Lazard's thesis [**Laz67**]; also see [**Lam99**].

Definition 10.77.1. Let $f : M \to N$ be a map of R-modules. Then f is called *universally injective* if for every R-module Q, the map $f \otimes_R \mathrm{id}_Q : M \otimes_R Q \to N \otimes_R Q$ is injective. A sequence $0 \to M_1 \to M_2 \to M_3 \to 0$ of R-modules is called *universally exact* if it is exact and $M_1 \to M_2$ is universally injective.

Example 10.77.2. Examples of universally exact maps.

(1) A split short exact sequence is universally exact since tensoring commutes with taking direct sums.
(2) The colimit of a directed system of universally exact sequences is universally exact. This follows from the fact that taking directed colimits is exact and that tensoring commutes with taking colimits. In particular the colimit of a directed system of split exact sequences is universally exact. We will see below that, conversely, any universally exact sequence arises in this way.

Next we give a list of criteria for a short exact sequence to be universally exact. They are analogues of criteria for flatness given above. Parts (3)-(6) below correspond,

respectively, to the criteria for flatness given in Lemmas 10.36.10, 10.76.1, 10.76.3, and Theorem 10.76.4.

Theorem 10.77.3. *Let*

$$0 \to M_1 \xrightarrow{f_1} M_2 \xrightarrow{f_2} M_3 \to 0$$

be an exact sequence of R-modules. The following are equivalent:

(1) *The sequence $0 \to M_1 \to M_2 \to M_3 \to 0$ is universally exact.*
(2) *For every finitely presented R-module Q, the sequence*

$$0 \to M_1 \otimes_R Q \to M_2 \otimes_R Q \to M_3 \otimes_R Q \to 0$$

is exact.
(3) *Given elements $x_i \in M_1$ $(i = 1, \ldots, n)$, $y_j \in M_2$ $(j = 1, \ldots, m)$, and $a_{ij} \in R$ $(i = 1, \ldots, n, j = 1, \ldots, m)$ such that for all i*

$$f_1(x_i) = \sum\nolimits_j a_{ij} y_j,$$

there exists $z_j \in M_1$ $(j = 1, \ldots, m)$ such that for all i,

$$x_i = \sum\nolimits_j a_{ij} z_j.$$

(4) *Given a commutative diagram of R-module maps*

$$\begin{array}{ccc} R^n & \longrightarrow & R^m \\ \downarrow & & \downarrow \\ M_1 & \xrightarrow{f_1} & M_2 \end{array}$$

where m and n are integers, there exists a map $R^m \to M_1$ making the top triangle commute.
(5) *For every finitely presented R-module P, the R-module map $\mathrm{Hom}_R(P, M_2) \to \mathrm{Hom}_R(P, M_3)$ is surjective.*
(6) *The sequence $0 \to M_1 \to M_2 \to M_3 \to 0$ is the colimit of a directed system of split exact sequences of the form*

$$0 \to M_1 \to M_{2,i} \to M_{3,i} \to 0$$

where the $M_{3,i}$ are finitely presented.

Proof. Obviously (1) implies (2).

Next we show (2) implies (3). Let $f_1(x_i) = \sum_j a_{ij} y_j$ be relations as in (3). Let (f_j) be a basis for R^m, (e_i) a basis for R^n, and $R^m \to R^n$ the map given by $f_j \mapsto \sum_i a_{ij} e_i$. Let Q be the cokernel of $R^m \to R^n$. Then tensoring $R^m \to R^n \to Q \to 0$ by the map $f_1 : M_1 \to M_2$, we get a commutative diagram

$$\begin{array}{ccccccc} M_1^{\oplus m} & \longrightarrow & M_1^{\oplus n} & \longrightarrow & M_1 \otimes_R Q & \longrightarrow & 0 \\ \downarrow & & \downarrow & & \downarrow & & \\ M_2^{\oplus m} & \longrightarrow & M_2^{\oplus n} & \longrightarrow & M_2 \otimes_R Q & \longrightarrow & 0 \end{array}$$

where $M_1^{\oplus m} \to M_1^{\oplus n}$ is given by

$$(z_1, \ldots, z_m) \mapsto (\textstyle\sum_j a_{1j} z_j, \ldots, \sum_j a_{nj} z_j),$$

and $M_2^{\oplus m} \to M_2^{\oplus n}$ is given similarly. We want to show $x = (x_1, \ldots, x_n) \in M_1^{\oplus n}$ is in the image of $M_1^{\oplus m} \to M_1^{\oplus n}$. By (2) the map $M_1 \otimes Q \to M_2 \otimes Q$ is injective, hence by exactness of the top row it is enough to show x maps to 0 in $M_2 \otimes Q$, and so by exactness of the bottom row it is enough to show the image of x in $M_2^{\oplus n}$ is in the image of $M_2^{\oplus m} \to M_2^{\oplus n}$. This is true by assumption.

Condition (4) is just a translation of (3) into diagram form.

Next we show (4) implies (5). Let $\varphi : P \to M_3$ be a map from a finitely presented R-module P. We must show that φ lifts to a map $P \to M_2$. Choose a presentation of P,

$$R^n \xrightarrow{g_1} R^m \xrightarrow{g_2} P \to 0.$$

Using freeness of R^n and R^m, we can construct $h_2 : R^m \to M_2$ and then $h_1 : R^n \to M_1$ such that the following diagram commutes

$$\begin{array}{ccccccccc} & & R^n & \xrightarrow{g_1} & R^m & \xrightarrow{g_2} & P & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle h_1} & & \downarrow{\scriptstyle h_2} & & \downarrow{\scriptstyle \varphi} & & \\ 0 & \longrightarrow & M_1 & \xrightarrow{f_1} & M_2 & \xrightarrow{f_2} & M_3 & \longrightarrow & 0. \end{array}$$

By (4) there is a map $k_1 : R^m \to M_1$ such that $k_1 \circ g_1 = h_1$. Now define $h_2' : R^m \to M_2$ by $h_2' = h_2 - f_1 \circ k_1$. Then

$$h_2' \circ g_1 = h_2 \circ g_1 - f_1 \circ k_1 \circ g_1 = h_2 \circ g_1 - f_1 \circ h_1 = 0.$$

Hence by passing to the quotient h_2' defines a map $\varphi' : P \to M_2$ such that $\varphi' \circ g_2 = h_2'$. In a diagram, we have

$$\begin{array}{ccc} R^m & \xrightarrow{g_2} & P \\ {\scriptstyle h_2'}\downarrow & \swarrow{\scriptstyle \varphi'} & \downarrow{\scriptstyle \varphi} \\ M_2 & \xrightarrow{f_2} & M_3. \end{array}$$

where the top triangle commutes. We claim that φ' is the desired lift, i.e. that $f_2 \circ \varphi' = \varphi$. From the definitions we have

$$f_2 \circ \varphi' \circ g_2 = f_2 \circ h_2' = f_2 \circ h_2 - f_2 \circ f_1 \circ k_1 = f_2 \circ h_2 = \varphi \circ g_2.$$

Since g_2 is surjective, this finishes the proof.

Now we show (5) implies (6). Write M_3 as the colimit of a directed system of finitely presented modules $M_{3,i}$. Let $M_{2,i}$ be the fiber product of $M_{3,i}$ and M_2 over M_3—by definition this is the submodule of $M_2 \times M_{3,i}$ consisting of elements whose two projections onto M_3 are equal. Let $M_{1,i}$ be the kernel of the projection $M_{2,i} \to M_{3,i}$. Then we have a directed system of exact sequences

$$0 \to M_{1,i} \to M_{2,i} \to M_{3,i} \to 0,$$

and for each i a map of exact sequences

$$\begin{array}{ccccccccc} 0 & \longrightarrow & M_{1,i} & \longrightarrow & M_{2,i} & \longrightarrow & M_{3,i} & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & M_1 & \longrightarrow & M_2 & \longrightarrow & M_3 & \longrightarrow & 0 \end{array}$$

compatible with the directed system. From the definition of the fiber product $M_{2,i}$, it follows that the map $M_{1,i} \to M_1$ is an isomorphism. By (5) there is a map

$M_{3,i} \to M_2$ lifting $M_{3,i} \to M_3$, and by the universal property of the fiber product this gives rise to a section of $M_{2,i} \to M_{3,i}$. Hence the sequences

$$0 \to M_{1,i} \to M_{2,i} \to M_{3,i} \to 0$$

split. Passing to the colimit, we have a commutative diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \operatorname{colim} M_{1,i} & \longrightarrow & \operatorname{colim} M_{2,i} & \longrightarrow & \operatorname{colim} M_{3,i} & \longrightarrow & 0 \\ & & \downarrow \cong & & \downarrow & & \downarrow \cong & & \\ 0 & \longrightarrow & M_1 & \longrightarrow & M_2 & \longrightarrow & M_3 & \longrightarrow & 0 \end{array}$$

with exact rows and outer vertical maps isomorphisms. Hence $\operatorname{colim} M_{2,i} \to M_2$ is also an isomorphism and (6) holds.

Condition (6) implies (1) by Example 10.77.2 (2). □

The previous theorem shows that a universally exact sequence is always a colimit of split short exact sequences. If the cokernel of a universally injective map is finitely presented, then in fact the map itself splits:

Lemma 10.77.4. *Let*

$$0 \to M_1 \to M_2 \to M_3 \to 0$$

be an exact sequence of R-modules. Suppose M_3 is of finite presentation. Then

$$0 \to M_1 \to M_2 \to M_3 \to 0$$

is universally exact if and only if it is split.

Proof. A split sequence is always universally exact. Conversely, if the sequence is universally exact, then by Theorem 10.77.3 (5) applied to $P = M_3$, the map $M_2 \to M_3$ admits a section. □

The following lemma shows how universally injective maps are complementary to flat modules.

Lemma 10.77.5. *Let M be an R-module. Then M is flat if and only if any exact sequence of R-modules*

$$0 \to M_1 \to M_2 \to M \to 0$$

is universally exact.

Proof. This follows from Lemma 10.76.3 and Theorem 10.77.3 (5). □

Example 10.77.6. Non-split and non-flat universally exact sequences.

(1) In spite of Lemma 10.77.4, it is possible to have a short exact sequence of R-modules

$$0 \to M_1 \to M_2 \to M_3 \to 0$$

that is universally exact but non-split. For instance, take $R = \mathbf{Z}$, let $M_1 = \bigoplus_{n=1}^{\infty} \mathbf{Z}$, let $M_2 = \prod_{n=1}^{\infty} \mathbf{Z}$, and let M_3 be the cokernel of the inclusion $M_1 \to M_2$. Then M_1, M_2, M_3 are all flat since they are torsion-free, so by Lemma 10.77.5,

$$0 \to M_1 \to M_2 \to M_3 \to 0$$

is universally exact. However there can be no section $s : M_3 \to M_2$. In fact, if x is the image of $(2, 2^2, 2^3, \ldots) \in M_2$ in M_3, then any module map

$s : M_3 \to M_2$ must kill x. This is because $x \in 2^n M_3$ for any $n \geq 1$, hence $s(x)$ is divisible by 2^n for all $n \geq 1$ and so must be 0.

(2) In spite of Lemma 10.77.5, it is possible to have a short exact sequence of R-modules
$$0 \to M_1 \to M_2 \to M_3 \to 0$$
that is universally exact but with M_1, M_2, M_3 all non-flat. In fact if M is any non-flat module, just take the split exact sequence
$$0 \to M \to M \oplus M \to M \to 0.$$
For instance over $R = \mathbf{Z}$, take M to be any torsion module.

(3) Taking the direct sum of an exact sequence as in (1) with one as in (2), we get a short exact sequence of R-modules
$$0 \to M_1 \to M_2 \to M_3 \to 0$$
that is universally exact, non-split, and such that M_1, M_2, M_3 are all non-flat.

We end this section with a simple observation.

Lemma 10.77.7. *Let $0 \to M_1 \to M_2 \to M_3 \to 0$ be a universally exact sequence of R-modules, and suppose M_2 is flat. Then M_1 and M_3 are flat.*

Proof. Let $0 \to N \to N' \to N'' \to 0$ be a short exact sequence of R-modules. Consider the commutative diagram
$$\begin{array}{ccccc} M_1 \otimes_R N & \longrightarrow & M_2 \otimes_R N & \longrightarrow & M_3 \otimes_R N \\ \downarrow & & \downarrow & & \downarrow \\ M_1 \otimes_R N' & \longrightarrow & M_2 \otimes_R N' & \longrightarrow & M_3 \otimes_R N' \\ \downarrow & & \downarrow & & \downarrow \\ M_1 \otimes_R N'' & \longrightarrow & M_2 \otimes_R N'' & \longrightarrow & M_3 \otimes_R N'' \end{array}$$
(we have dropped the 0's on the boundary). By assumption the rows give short exact sequences and the arrow $M_2 \otimes N \to M_2 \otimes N'$ is injective. Clearly this implies that $M_1 \otimes N \to M_1 \otimes N'$ is injective and we see that M_1 is flat. In particular the left and middle columns give rise to short exact sequences. It follows from a diagram chase that the arrow $M_3 \otimes N \to M_3 \otimes N'$ is injective. Hence M_3 is flat. □

Lemma 10.77.8. *Let R be a ring. Let $M \to M'$ be a universally injective R-module map. Then for any R-module N the map $M \otimes_R N \to M' \otimes_R N$ is universally injective.*

Proof. Omitted. □

Lemma 10.77.9. *Let R be a ring. A composition of universally injective R-module maps is universally injective.*

Proof. Omitted. □

Lemma 10.77.10. *Let R be a ring. Let $M \to M'$ and $M \to M''$ be R-module maps. If $M \to M''$ is universally injective, then $M \to M'$ is universally injective.*

Proof. Omitted. □

Lemma 10.77.11. *Let $R \to S$ be a faithfully flat ring map. Then $R \to S$ is universally injective as a map of R-modules. In particular $R \cap IS = I$ for any ideal $I \subset R$.*

Proof. Let N be an R-module. We have to show that $N \to N \otimes_R S$ is injective. As S is faithfully flat as an R-module, it suffices to prove this after tensoring with S. Hence it suffices to show that $N \otimes_R S \to N \otimes_R S \otimes_R S$, $n \otimes s \mapsto N \otimes 1 \otimes s$ is injective. This is true because there is a section, namely, $n \otimes s \otimes s' \mapsto n \otimes ss'$. □

Lemma 10.77.12. *Let $R \to S$ be a ring map. Let $M \to M'$ be a map of S-modules. The following are equivalent*

1. *$M \to M'$ is universally injective as a map of R-modules,*
2. *for each prime $\mathfrak{q}$ of S the map $M_\mathfrak{q} \to M'_\mathfrak{q}$ is universally injective as a map of R-modules,*
3. *for each maximal ideal $\mathfrak{m}$ of S the map $M_\mathfrak{m} \to M'_\mathfrak{m}$ is universally injective as a map of R-modules,*
4. *for each prime $\mathfrak{q}$ of S the map $M_\mathfrak{q} \to M'_\mathfrak{q}$ is universally injective as a map of $R_\mathfrak{p}$-modules, where $\mathfrak{p}$ is the inverse image of $\mathfrak{q}$ in R, and*
5. *for each maximal ideal $\mathfrak{m}$ of S the map $M_\mathfrak{m} \to M'_\mathfrak{m}$ is universally injective as a map of $R_\mathfrak{p}$-modules, where $\mathfrak{p}$ is the inverse image of $\mathfrak{m}$ in R.*

Proof. Let N be an R-module. Let $\mathfrak{q}$ be a prime of S lying over the prime $\mathfrak{p}$ of R. Then we have

$$(M \otimes_R N)_\mathfrak{q} = M_\mathfrak{q} \otimes_R N = M_\mathfrak{q} \otimes_{R_\mathfrak{p}} N_\mathfrak{p}.$$

Moreover, the same thing holds for M' and localization is exact. Also, if N is an $R_\mathfrak{p}$-module, then $N_\mathfrak{p} = N$. Using this the equivalences can be proved in a straightforward manner.

For example, suppose that (5) holds. Let $K = \mathrm{Ker}(M \otimes_R N \to M' \otimes_R N)$. By the remarks above we see that $K_\mathfrak{m} = 0$ for each maximal ideal $\mathfrak{m}$ of S. Hence $K = 0$ by Lemma 10.22.1. Thus (1) holds. Conversely, suppose that (1) holds. Take any $\mathfrak{q} \subset S$ lying over $\mathfrak{p} \subset R$. Take any module N over $R_\mathfrak{p}$. Then by assumption $\mathrm{Ker}(M \otimes_R N \to M' \otimes_R N) = 0$. Hence by the formulae above and the fact that $N = N_\mathfrak{p}$ we see that $\mathrm{Ker}(M_\mathfrak{q} \otimes_{R_\mathfrak{p}} N \to M'_\mathfrak{q} \otimes_{R_\mathfrak{p}} N) = 0$. In other words (4) holds. Of course (4) $\Rightarrow$ (5) is immediate. Hence (1), (4) and (5) are all equivalent. We omit the proof of the other equivalences. □

Lemma 10.77.13. *Let $\varphi : A \to B$ be a ring map. Let $S \subset A$ and $S' \subset B$ be multiplicative subsets such that $\varphi(S) \subset S'$. Let $M \to M'$ be a map of B-modules.*

1. *If $M \to M'$ is universally injective as a map of A-modules, then $(S')^{-1}M \to (S')^{-1}M'$ is universally injective as a map of A-modules and as a map of $S^{-1}A$-modules.*
2. *If M and M' are $(S')^{-1}B$-modules, then $M \to M'$ is universally injective as a map of A-modules if and only if it is universally injective as a map of $S^{-1}A$-modules.*

Proof. You can prove this using Lemma 10.77.12 but you can also prove it directly as follows. Assume $M \to M'$ is A-universally injective. Let Q be an A-module. Then $Q \otimes_A M \to Q \otimes_A M'$ is injective. Since localization is exact we see that $(S')^{-1}(Q \otimes_A M) \to (S')^{-1}(Q \otimes_A M')$ is injective. As $(S')^{-1}(Q \otimes_A M) = Q \otimes_A (S')^{-1}M$ and similarly for M' we see that $Q \otimes_A (S')^{-1}M \to Q \otimes_A (S')^{-1}M'$

is injective, hence $(S')^{-1}M \to (S')^{-1}M'$ is universally injective as a map of A-modules. This proves the first part of (1). To see (2) we can use the following two facts: (a) if Q is an $S^{-1}A$-module, then $Q \otimes_A S^{-1}A = Q$, i.e., tensoring with Q over A is the same thing as tensoring with Q over $S^{-1}A$, (b) if M is any A-module on which the elements of S are invertible, then $M \otimes_A Q = M \otimes_{S^{-1}A} S^{-1}Q$. Part (2) follows from this immediately. □

10.78. Descent for finite projective modules

In this section we give an elementary proof of the fact that the property of being a *finite* projective module descends along faithfully flat ring maps. The proof does not apply when we drop the finiteness condition. However, the method is indicative of the one we shall use to prove descent for the property of being a *countably generated* projective module—see the comments at the end of this section.

Lemma 10.78.1. *Let M be an R-module. Then M is finite projective if and only if M is finitely presented and flat.*

Proof. This is part of Lemma 10.73.2. However, at this point we can give a more elegant proof of the implication (1) ⇒ (2) of that lemma as follows. If M is finitely presented and flat, then take a surjection $R^n \to M$. By Lemma 10.76.3 applied to $P = M$, the map $R^n \to M$ admits a section. So M is a direct summand of a free module and hence projective. □

Here are some properties of modules that descend.

Lemma 10.78.2. *Let $R \to S$ be a faithfully flat ring map. Let M be an R-module. Then*

1. *if the S-module $M \otimes_R S$ is of finite type, then M is of finite type,*
2. *if the S-module $M \otimes_R S$ is of finite presentation, then M is of finite presentation,*
3. *if the S-module $M \otimes_R S$ is flat, then M is flat, and*
4. *add more here as needed.*

Proof. Assume $M \otimes_R S$ is of finite type. Let $y_1, \ldots, y_m$ be generators of $M \otimes_R S$ over S. Write $y_j = \sum x_i \otimes f_i$ for some $x_1, \ldots, x_n \in M$. Then we see that the map $\varphi : R^{\oplus n} \to M$ has the property that $\varphi \otimes \mathrm{id}_S : S^{\oplus n} \to M \otimes_R S$ is surjective. Since $R \to S$ is faithfully flat we see that φ is surjective, and M is finitely generated.

Assume $M \otimes_R S$ is of finite presentation. By (1) we see that M is of finite type. Choose a surjection $R^{\oplus n} \to M$ and denote K the kernel. As $R \to S$ is flat we see that $K \otimes_R S$ is the kernel of the base change $S^{\oplus n} \to M \otimes_R S$. As $M \otimes_R S$ is of finite presentation we conclude that $K \otimes_R S$ is of finite type. Hence by (1) we see that K is of finite type and hence M is of finite presentation.

Part (3) is Lemma 10.36.7. □

Proposition 10.78.3. *Let $R \to S$ be a faithfully flat ring map. Let M be an R-module. If the S-module $M \otimes_R S$ is finite projective, then M is finite projective.*

Proof. Follows from Lemmas 10.78.1 and 10.78.2. □

The next few sections are about removing the finiteness assumption by using dévissage to reduce to the countably generated case. In the countably generated case, the strategy is to find a characterization of countably generated projective

modules analogous to Lemma 10.78.1, and then to prove directly that this characterization descends. We do this by introducing the notion of a Mittag-Leffler module and proving that if a module M is countably generated, then it is projective if and only if it is flat and Mittag-Leffler (Theorem 10.88.3). When M is finitely generated, this statement reduces to Lemma 10.78.1 (since, according to Example 10.86.1 (1), a finitely generated module is Mittag-Leffler if and only if it is finitely presented).

10.79. Transfinite dévissage of modules

In this section we introduce a dévissage technique for decomposing a module into a direct sum. The main result is that a projective module is a direct sum of countably generated modules (Theorem 10.79.5 below). We follow [**Kol96**].

Definition 10.79.1. Let M be an R-module. A *direct sum dévissage* of M is a family of submodules $(M_\alpha)_{\alpha\in S}$, indexed by an ordinal S and increasing (with respect to inclusion), such that:

(0) $M_0 = 0$;
(1) $M = \bigcup_\alpha M_\alpha$;
(2) if $\alpha \in S$ is a limit ordinal, then $M_\alpha = \bigcup_{\beta<\alpha} M_\beta$;
(3) if $\alpha + 1 \in S$, then M_α is a direct summand of $M_{\alpha+1}$.

If moreover

(4) $M_{\alpha+1}/M_\alpha$ is countably generated for $\alpha + 1 \in S$,

then $(M_\alpha)_{\alpha\in S}$ is called a *Kaplansky dévissage* of M.

The terminology is justified by the following lemma.

Lemma 10.79.2. *Let M be an R-module. If $(M_\alpha)_{\alpha\in S}$ is a direct sum dévissage of M, then $M \cong \bigoplus_{\alpha+1\in S} M_{\alpha+1}/M_\alpha$.*

Proof. By property (3) of a direct sum dévissage, there is an inclusion $M_{\alpha+1}/M_\alpha \to M$ for each $\alpha \in S$. Consider the map

$$f : \bigoplus_{\alpha+1\in S} M_{\alpha+1}/M_\alpha \to M$$

given by the sum of these inclusions. Transfinite induction on S shows that the image contains M_α for every $\alpha \in S$: for $\alpha = 0$ this is true by (0); if $\alpha + 1$ is a successor ordinal then it is clearly true; and if α is a limit ordinal and it is true for $\beta < \alpha$, then it is true for α by (2). Hence f is surjective by (1).

Transfinite induction on S also shows that for every $\beta \in S$ the restriction

$$f_\beta : \bigoplus\nolimits_{\alpha+1\leq\beta} M_{\alpha+1}/M_\alpha \longrightarrow M$$

of f is injective: For $\beta = 0$ it is true. If it is true for all $\beta' < \beta$, then let x be in the kernel and write $x = (x_{\alpha+1})_{\alpha+1\leq\beta}$ in terms of its components $x_{\alpha+1} \in M_{\alpha+1}/M_\alpha$. By property (3) both $(x_{\alpha+1})_{\alpha+1<\beta}$ and $x_{\beta+1}$ map to 0. Hence $x_{\beta+1} = 0$ and, by the assumption that the restriction $f_{\beta'}$ is injective for all $\beta' < \beta$, also $x_{\alpha+1} = 0$ for every $\alpha + 1 < \beta$. So $x = 0$ and f_β is injective, which finishes the induction. We conclude that f is injective since f_β is for each $\beta \in S$. □

Lemma 10.79.3. *Let M be an R-module. Then M is a direct sum of countably generated R-modules if and only if it admits a Kaplansky dévissage.*

Proof. The lemma takes care of the "if" direction. Conversely, suppose $M = \bigoplus_{i\in I} N_i$ where each N_i is a countably generated R-module. Well-order I so that we can think of it as an ordinal. Then setting $M_i = \bigoplus_{j<i} N_j$ gives a Kaplansky dévissage $(M_i)_{i\in I}$ of M. □

Theorem 10.79.4. *Suppose M is a direct sum of countably generated R-modules. If P is a direct summand of M, then P is also a direct sum of countably generated R-modules.*

Proof. Write $M = P \oplus Q$. We are going to construct a Kaplansky dévissage $(M_\alpha)_{\alpha\in S}$ of M which, in addition to the defining properties (0)-(4), satisfies:

(5) Each M_α is a direct summand of M;
(6) $M_\alpha = P_\alpha \oplus Q_\alpha$, where $P_\alpha = P \cap M_\alpha$ and $Q = Q \cap M_\alpha$.

(Note: if properties (0)-(2) hold, then in fact property (3) is equivalent to property (5).)

To see how this implies the theorem, it is enough to show that $(P_\alpha)_{\alpha\in S}$ forms a Kaplansky dévissage of P. Properties (0), (1), and (2) are clear. By (5) and (6) for (M_α), each P_α is a direct summand of M. Since $P_\alpha \subset P_{\alpha+1}$, this implies P_α is a direct summand of $P_{\alpha+1}$; hence (3) holds for (P_α). For (4), note that

$$M_{\alpha+1}/M_\alpha \cong P_{\alpha+1}/P_\alpha \oplus Q_{\alpha+1}/Q_\alpha,$$

so $P_{\alpha+1}/P_\alpha$ is countably generated because this is true of $M_{\alpha+1}/M_\alpha$.

It remains to construct the M_α. Write $M = \bigoplus_{i\in I} N_i$ where each N_i is a countably generated R-module. Choose a well-ordering of I. By transfinite induction we are going to define an increasing family of submodules M_α of M, one for each ordinal α, such that M_α is a direct sum of some subset of the N_i.

For $\alpha = 0$ let $M_0 = 0$. If α is a limit ordinal and M_β has been defined for all $\beta < \alpha$, then define $M_\alpha = \bigcup_{\beta<\alpha} M_\beta$. Since each M_β for $\beta < \alpha$ is a direct sum of a subset of the N_i, the same will be true of M_α. If $\alpha + 1$ is a successor ordinal and M_α has been defined, then define $M_{\alpha+1}$ as follows. If $M_\alpha = M$, then let $M_{\alpha+1} = M$. If not, choose the smallest $j \in I$ such that N_j is not contained in M_α. We will construct an infinite matrix $(x_{mn}), m, n = 1, 2, 3, \ldots$ such that:

(1) N_j is contained in the submodule of M generated by the entries x_{mn};
(2) if we write any entry $x_{k\ell}$ in terms of its P- and Q-components, $x_{k\ell} = y_{k\ell} + z_{k\ell}$, then the matrix (x_{mn}) contains a set of generators for each N_i for which $y_{k\ell}$ or $z_{k\ell}$ has nonzero component.

Then we define $M_{\alpha+1}$ to be the submodule of M generated by M_α and all x_{mn}; by property (2) of the matrix (x_{mn}), $M_{\alpha+1}$ will be a direct sum of some subset of the N_i. To construct the matrix (x_{mn}), let $x_{11}, x_{12}, x_{13}, \ldots$ be a countable set of generators for N_j. Then if $x_{11} = y_{11} + z_{11}$ is the decomposition into P- and Q-components, let $x_{21}, x_{22}, x_{23}, \ldots$ be a countable set of generators for the sum of the N_i for which y_{11} or z_{11} have nonzero component. Repeat this process on x_{12} to get elements $x_{31}, x_{32}, \ldots$, the third row of our matrix. Repeat on x_{21} to get the fourth row, on x_{13} to get the fifth, and so on, going down along successive

anti-diagonals as indicated below:

$$\begin{pmatrix} x_{11} & x_{12} & x_{13} & x_{14} & \cdots \\ & \swarrow & \swarrow & \swarrow & \\ x_{21} & x_{22} & x_{23} & \cdots & \\ & \swarrow & \swarrow & & \\ x_{31} & x_{32} & \cdots & & \\ & \swarrow & & & \\ x_{41} & \cdots & & & \\ \cdots & & & & \end{pmatrix}.$$

Transfinite induction on I (using the fact that we constructed $M_{\alpha+1}$ to contain N_j for the smallest j such that N_j is not contained in M_α) shows that for each $i \in I$, N_i is contained in some M_α. Thus, there is some large enough ordinal S satisfying: for each $i \in I$ there is $\alpha \in S$ such that N_i is contained in M_α. This means $(M_\alpha)_{\alpha \in S}$ satisfies property (1) of a Kaplansky dévissage of M. The family $(M_\alpha)_{\alpha \in S}$ moreover satisfies the other defining properties, and also (5) and (6) above: properties (0), (2), (4), and (6) are clear by construction; property (5) is true because each M_α is by construction a direct sum of some N_i; and (3) is implied by (5) and the fact that $M_\alpha \subset M_{\alpha+1}$. □

As a corollary we get the result for projective modules stated at the beginning of the section.

Theorem 10.79.5. *If P is a projective R-module, then P is a direct sum of countably generated projective R-modules.*

Proof. A module is projective if and only if it is a direct summand of a free module, so this follows from Theorem 10.79.4. □

10.80. Projective modules over a local ring

In this section we prove a very cute result: a projective module M over a local ring is free (Theorem 10.80.4 below). Note that with the additional assumption that M is finite, this result is Lemma 10.73.4. In general we have:

Lemma 10.80.1. *Let R be a ring. Then every projective R-module is free if and only if every countably generated projective R-module is free.*

Proof. Follows immediately from Theorem 10.79.5. □

Here is a criterion for a countably generated module to be free.

Lemma 10.80.2. *Let M be a countably generated R-module. Suppose any direct summand N of M satisfies: any element of N is contained in a free direct summand of N. Then M is free.*

Proof. Let $x_1, x_2, \ldots$ be a countable set of generators for M. By the assumption on M, we can construct by induction free R-modules $F_1, F_2, \ldots$ such that for every positive integer n, $\bigoplus_{i=1}^n F_i$ is a direct summand of M and contains $x_1, \ldots, x_n$. Then $M = \bigoplus_{i=1}^\infty F_i$. □

Lemma 10.80.3. *Let P be a projective module over a local ring R. Then any element of P is contained in a free direct summand of P.*

Proof. Since P is projective it is a direct summand of some free R-module F, say $F = P \oplus Q$. Let $x \in P$ be the element that we wish to show is contained in a free direct summand of P. Let B be a basis of F such that the number of basis elements needed in the expression of x is minimal, say $x = \sum_{i=1}^n a_i e_i$ for some $e_i \in B$ and $a_i \in R$. Then no a_j can be expressed as a linear combination of the other a_i; for if $a_j = \sum_{i \neq j} a_i b_i$ for some $b_i \in R$, then replacing e_i by $e_i + b_i e_j$ for $i \neq j$ and leaving unchanged the other elements of B, we get a new basis for F in terms of which x has a shorter expression.

Let $e_i = y_i + z_i, y_i \in P, z_i \in Q$ be the decomposition of e_i into its P- and Q-components. Write $y_i = \sum_{j=1}^n b_{ij} e_j + t_i$, where t_i is a linear combination of elements in B other than $e_1, \ldots, e_n$. To finish the proof it suffices to show that the matrix (b_{ij}) is invertible. For then the map $F \to F$ sending $e_i \mapsto y_i$ for $i = 1, \ldots, n$ and fixing $B \setminus \{e_1, \ldots, e_n\}$ is an isomorphism, so that $y_1, \ldots, y_n$ together with $B \setminus \{e_1, \ldots, e_n\}$ form a basis for F. Then the submodule N spanned by $y_1, \ldots, y_n$ is a free submodule of P; N is a direct summand of P since $N \subset P$ and both N and P are direct summands of F; and $x \in N$ since $x \in P$ implies $x = \sum_{i=1}^n a_i e_i = \sum_{i=1}^n a_i y_i$.

Now we prove that (b_{ij}) is invertible. Plugging $y_i = \sum_{j=1}^n b_{ij} e_j + t_i$ into $\sum_{i=1}^n a_i e_i = \sum_{i=1}^n a_i y_i$ and equating the coefficients of e_j gives $a_j = \sum_{i=1}^n a_i b_{ij}$. But as noted above, our choice of B guarantees that no a_j can be written as a linear combination of the other a_i. Thus b_{ij} is a non-unit for $i \neq j$, and $1 - b_{ii}$ is a non-unit—so in particular b_{ii} is a unit—for all i. But a matrix over a local ring having units along the diagonal and non-units elsewhere is invertible, as its determinant is a unit. □

Theorem 10.80.4. *If P is a projective module over a local ring R, then P is free.*

Proof. Follows from Lemmas 10.80.1, 10.80.2, and 10.80.3. □

10.81. Mittag-Leffler systems

The purpose of this section is to define Mittag-Leffler systems and why it is a useful property.

In the following, I will be a directed partially ordered set, see Categories, Definition 4.21.2. Let $(A_i, \varphi_{ji} : A_j \to A_i)$ be an inverse system of sets or of modules indexed by I, see Categories, Definition 4.21.2. This is a directed inverse system as we assumed I directed. For each $i \in I$, the images $\varphi_{ji}(A_j) \subset A_i$ for $j \geq i$ form a decreasing family. Let $A_i' = \bigcap_{j \geq i} \varphi_{ji}(A_j)$. Then $\varphi_{ji}(A_j') \subset A_i'$ for $j \geq i$, hence by restricting we get a directed inverse system $(A_i', \varphi_{ji}|_{A_j'})$. From the construction of the limit of an inverse system in the category of sets or modules, we have $\lim A_i = \lim A_i'$. The Mittag-Leffler condition on (A_i, φ_{ji}) is that A_i' equals $\varphi_{ji}(A_j)$ for some $j \geq i$ (and hence equals $\varphi_{ki}(A_k)$ for all $k \geq j$):

Definition 10.81.1. Let (A_i, φ_{ji}) be a directed inverse system of sets over I. Then we say (A_i, φ_{ji}) is *Mittag-Leffler inverse system* if for each $i \in I$, the decreasing family $\varphi_{ji}(A_j) \subset A_i$ for $j \geq i$ stabilizes. Explicitly, this means that for each $i \in I$, there exists $j \geq i$ such that for $k \geq j$ we have $\varphi_{ki}(A_k) = \varphi_{ji}(A_j)$. If (A_i, φ_{ji}) is a directed inverse system of modules over a ring R, we say that it is Mittag-Leffler if the underlying inverse system of sets is Mittag-Leffler.

Example 10.81.2. If (A_i, φ_{ji}) is a directed inverse system of sets or of modules and the maps φ_{ji} are surjective, then clearly the system is Mittag-Leffler. Conversely, suppose (A_i, φ_{ji}) is Mittag-Leffler. Let $A'_i \subset A_i$ be the stable image of $\varphi_{ji}(A_j)$ for $j \geq i$. Then $\varphi_{ji}|_{A'_j} : A'_j \to A'_i$ is surjective for $j \geq i$ and $\lim A_i = \lim A'_i$. Hence the limit of the Mittag-Leffler system (A_i, φ_{ji}) can also be written as the limit of a directed inverse system over I with surjective maps.

Lemma 10.81.3. *Let (A_i, φ_{ji}) be a directed inverse system over I. Suppose I is countable. If (A_i, φ_{ji}) is Mittag-Leffler and the A_i are nonempty, then $\lim A_i$ is nonempty.*

Proof. Let $i_1, i_2, i_3, \ldots$ be an enumeration of the elements of I. Define inductively a sequence of elements $j_n \in I$ for $n = 1, 2, 3, \ldots$ by the conditions: $j_1 = i_1$, and $j_n \geq i_n$ and $j_n \geq j_m$ for $m < n$. Then the sequence j_n is increasing and forms a cofinal subset of I. Hence we may assume $I = \{1, 2, 3, \ldots\}$. So by Example 10.81.2 we are reduced to showing that the limit of an inverse system of nonempty sets with surjective maps indexed by the positive integers is nonempty. This is obvious. □

The Mittag-Leffler condition will be important for us because of the following exactness property.

Lemma 10.81.4. *Let*

$$0 \to A_i \xrightarrow{f_i} B_i \xrightarrow{g_i} C_i \to 0$$

be an exact sequence of directed inverse systems of abelian groups over I. Suppose I is countable. If (A_i) is Mittag-Leffler, then

$$0 \to \lim A_i \to \lim B_i \to \lim C_i \to 0$$

is exact.

Proof. Taking limits of directed inverse systems is left exact, hence we only need to prove surjectivity of $\lim B_i \to \lim C_i$. So let $(c_i) \in \lim C_i$. For each $i \in I$, let $E_i = g_i^{-1}(c_i)$, which is nonempty since $g_i : B_i \to C_i$ is surjective. The system of maps $\varphi_{ji} : B_j \to B_i$ for (B_i) restrict to maps $E_j \to E_i$ which make (E_i) into an inverse system of nonempty sets. It is enough to show that (E_i) is Mittag-Leffler. For then Lemma 10.81.3 would show $\lim E_i$ is nonempty, and taking any element of $\lim E_i$ would give an element of $\lim B_i$ mapping to (c_i).

By the injection $f_i : A_i \to B_i$ we will regard A_i as a subset of B_i. Since (A_i) is Mittag-Leffler, if $i \in I$ then there exists $j \geq i$ such that $\varphi_{ki}(A_k) = \varphi_{ji}(A_j)$ for $k \geq j$. We claim that also $\varphi_{ki}(E_k) = \varphi_{ji}(E_j)$ for $k \geq j$. Always $\varphi_{ki}(E_k) \subset \varphi_{ji}(E_j)$ for $k \geq j$. For the reverse inclusion let $e_j \in E_j$, and we need to find $x_k \in E_k$ such that $\varphi_{ki}(x_k) = \varphi_{ji}(e_j)$. Let $e'_k \in E_k$ be any element, and set $e'_j = \varphi_{kj}(e'_k)$. Then $g_j(e_j - e'_j) = c_j - c_j = 0$, hence $e_j - e'_j = a_j \in A_j$. Since $\varphi_{ki}(A_k) = \varphi_{ji}(A_j)$, there exists $a_k \in A_k$ such that $\varphi_{ki}(a_k) = \varphi_{ji}(a_j)$. Hence

$$\varphi_{ki}(e'_k + a_k) = \varphi_{ji}(e'_j) + \varphi_{ji}(a_j) = \varphi_{ji}(e_j),$$

so we can take $x_k = e'_k + a_k$. □

10.82. Inverse systems

In many papers (and in this section) the term *inverse system* is used to indicate an inverse system over the partially ordered set $(\mathbf{N}, \geq)$. We briefly discuss such systems in this section. This material will be discussed more broadly in Homology, Section 12.26. Suppose we are given a ring R and a sequence of R-modules

$$M_1 \xleftarrow{\varphi_2} M_2 \xleftarrow{\varphi_3} M_3 \leftarrow \ldots$$

with maps as indicated. By composing successive maps we obtain maps $\varphi_{ii'} : M_i \to M_{i'}$ whenever $i \geq i'$ such that moreover $\varphi_{ii''} = \varphi_{i'i''} \circ \varphi_{ii'}$ whenever $i \geq i' \geq i''$. Conversely, given the system of maps $\varphi_{ii'}$ we can set $\varphi_i = \varphi_{i(i-1)}$ and recover the maps displayed above. In this case

$$\lim M_i = \{(x_i) \in \prod M_i \mid \varphi_i(x_i) = x_{i-1},\ i = 2, 3, \ldots\}$$

compare with Categories, Section 4.15. As explained in Homology, Section 12.26 this is actually a limit in the category of R-modules, as defined in Categories, Section 4.14.

Lemma 10.82.1. *Let R be a ring. Let $0 \to K_i \to L_i \to M_i \to 0$ be short exact sequences of R-modules, $i \geq 1$ which fit into maps of short exact sequences*

$$\begin{array}{ccccccccc} 0 & \longrightarrow & K_i & \longrightarrow & L_i & \longrightarrow & M_i & \longrightarrow & 0 \\ & & \uparrow & & \uparrow & & \uparrow & & \\ 0 & \longrightarrow & K_{i+1} & \longrightarrow & L_{i+1} & \longrightarrow & M_{i+1} & \longrightarrow & 0 \end{array}$$

If for every i there exists a $c = c(i) \geq i$ such that $Im(K_c \to K_i) = Im(K_j \to K_i)$ for all $j \geq c$, then the sequence

$$0 \to \lim K_i \to \lim L_i \to \lim M_i \to 0$$

is exact.

Proof. This is a special case of the more general Lemma 10.81.4. □

10.83. Mittag-Leffler modules

A Mittag-Leffler module is (very roughly) a module which can be written as a directed limit whose dual is a Mittag-Leffler system. To be able to give a precise definition we need to do a bit of work.

Definition 10.83.1. Let (M_i, f_{ij}) be a directed system of R-modules. We say that (M_i, f_{ij}) is a *Mittag-Leffler directed system of modules* if each M_i is an R-module of finite presentation and if for every R-module N, the inverse system

$$(\mathrm{Hom}_R(M_i, N), \mathrm{Hom}_R(f_{ij}, N))$$

is Mittag-Leffler.

We are going to characterize those R-modules that are colimits of Mittag-Leffler directed systems of modules.

Definition 10.83.2. Let $f : M \to N$ and $g : M \to M'$ be maps of R-modules. Then we say g *dominates* f if for any R-module Q, we have $\ker(f \otimes_R \mathrm{id}_Q) \subset \ker(g \otimes_R \mathrm{id}_Q)$.

It is enough to check this condition for finitely presented modules.

Lemma 10.83.3. *Let $f : M \to N$ and $g : M \to M'$ be maps of R-modules. Then g dominates f if and only if for any finitely presented R-module Q, we have $\ker(f \otimes_R id_Q) \subset \ker(g \otimes_R id_Q)$.*

Proof. Suppose $\ker(f \otimes_R \mathrm{id}_Q) \subset \ker(g \otimes_R \mathrm{id}_Q)$ for all finitely presented modules Q. If Q is an arbitrary module, write $Q = \mathrm{colim}_{i \in I} Q_i$ as a colimit of a directed system of finitely presented modules Q_i. Then $\ker(f \otimes_R \mathrm{id}_{Q_i}) \subset \ker(g \otimes_R \mathrm{id}_{Q_i})$ for all i. Since taking directed colimits is exact and commutes with tensor product, it follows that $\ker(f \otimes_R \mathrm{id}_Q) \subset \ker(g \otimes_R \mathrm{id}_Q)$. □

The above definition of domination is related to the usual notion of domination of maps as follows.

Lemma 10.83.4. *Let $f : M \to N$ and $g : M \to M'$ be maps of R-modules. Suppose $\mathrm{Coker}(f)$ is of finite presentation. Then g dominates f if and only if g factors through f, i.e. there exists a module map $h : N \to M'$ such that $g = h \circ f$.*

Proof. Consider the pushout of f and g,

$$\begin{array}{ccc} M & \xrightarrow{f} & N \\ {\scriptstyle g}\downarrow & & \downarrow{\scriptstyle g'} \\ M' & \xrightarrow{f'} & N' \end{array}$$

where N' is $M' \oplus N$ modulo the submodule consisting of elements $(g(x), -f(x))$ for $x \in M$. We are going to show that the two conditions we wish to prove equivalent are each equivalent to f' being universally injective.

From the definition of N' we have a short exact sequence

$$0 \to \ker(f) \cap \ker(g) \to \ker(f) \to \ker(f') \to 0.$$

Since tensoring commutes with taking pushouts, we have such a short exact sequence

$$0 \to \ker(f \otimes \mathrm{id}_Q) \cap \ker(g \otimes \mathrm{id}_Q) \to \ker(f \otimes \mathrm{id}_Q) \to \ker(f' \otimes \mathrm{id}_Q) \to 0$$

for every R-module Q. So f' is universally injective if and only if $\ker(f \otimes \mathrm{id}_Q) \subset \ker(g \otimes \mathrm{id}_Q)$ for every Q, if and only if g dominates f.

On the other hand, from the definition of the pushout it follows that $\mathrm{Coker}(f') = \mathrm{Coker}(f)$, so $\mathrm{Coker}(f')$ is of finite presentation. Then by Lemma 10.77.4, f' is universally injective if and only if

$$0 \to M' \xrightarrow{f'} N' \to \mathrm{Coker}(f') \to 0$$

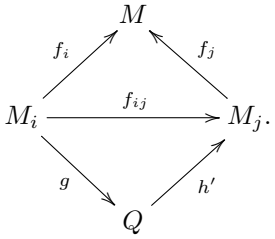
splits. This is the case if and only if there is a map $h' : N' \to M'$ such that $h' \circ f' = \mathrm{id}_{M'}$. From the universal property of the pushout, the existence of such an h' is equivalent to g factoring through f. □

Proposition 10.83.5. *Let M be an R-module. Let (M_i, f_{ij}) be a directed system of finitely presented R-modules, indexed by I, such that $M = \mathrm{colim}\, M_i$. Let $f_i : M_i \to M$ be the canonical map. The following are equivalent:*

(1) *For every finitely presented R-module P and module map $f : P \to M$, there exists a finitely presented R-module Q and a module map $g : P \to Q$ such that g and f dominate each other, i.e., $\ker(f \otimes_R id_N) = \ker(g \otimes_R id_N)$ for every R-module N.*

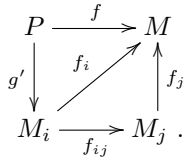
(2) *For each $i \in I$, there exists $j \geq i$ such that $f_{ij} : M_i \to M_j$ dominates $f_i : M_i \to M$.*
(3) *For each $i \in I$, there exists $j \geq i$ such that $f_{ij} : M_i \to M_j$ factors through $f_{ik} : M_i \to M_k$ for all $k \geq i$.*
(4) *For every R-module N, the inverse system $(\mathrm{Hom}_R(M_i, N), \mathrm{Hom}_R(f_{ij}, N))$ is Mittag-Leffler.*
(5) *For $N = \prod_{s \in I} M_s$, the inverse system $(\mathrm{Hom}_R(M_i, N), \mathrm{Hom}_R(f_{ij}, N))$ is Mittag-Leffler.*

Proof. First we prove the equivalence of (1) and (2). Suppose (1) holds and let $i \in I$. Corresponding to the map $f_i : M_i \to M$, we can choose $g : M_i \to Q$ as in (1). Since M_i and Q are of finite presentation, so is $\mathrm{Coker}(g)$. Then by Lemma 10.83.4, $f_i : M_i \to M$ factors through $g : M_i \to Q$, say $f_i = h \circ g$ for some $h : Q \to M$. Then since Q is finitely presented, h factors through $M_j \to M$ for some $j \geq i$, say $h = f_j \circ h'$ for some $h' : Q \to M_j$. In total we have a commutative diagram



Thus f_{ij} dominates g. But g dominates f_i, so f_{ij} dominates f_i.

Conversely, suppose (2) holds. Let P be of finite presentation and $f : P \to M$ a module map. Then f factors through $f_i : M_i \to M$ for some $i \in I$, say $f = f_i \circ g'$ for some $g' : P \to M_i$. Choose by (2) a $j \geq i$ such that f_{ij} dominates f_i. We have a commutative diagram



From the diagram and the fact that f_{ij} dominates f_i, we find that f and $f_{ij} \circ g'$ dominate each other. Hence taking $g = f_{ij} \circ g' : P \to M_j$ works.

Next we prove (2) is equivalent to (3). Let $i \in I$. It is always true that f_i dominates f_{ik} for $k \geq i$, since f_i factors through f_{ik}. If (2) holds, choose $j \geq i$ such that f_{ij} dominates f_i. Then since domination is a transitive relation, f_{ij} dominates f_{ik} for $k \geq i$. All M_i are of finite presentation, so $\mathrm{Coker}(f_{ik})$ is of finite presentation for $k \geq i$. By Lemma 10.83.4, f_{ij} factors through f_{ik} for all $k \geq i$. Thus (2) implies (3). On the other hand, if (3) holds then for any R-module N, $f_{ij} \otimes_R \mathrm{id}_N$ factors through $f_{ik} \otimes_R \mathrm{id}_N$ for $k \geq i$. So $\ker(f_{ik} \otimes_R \mathrm{id}_N) \subset \ker(f_{ij} \otimes_R \mathrm{id}_N)$ for $k \geq i$. But $\ker(f_i \otimes_R \mathrm{id}_N : M_i \otimes_R N \to M \otimes_R N)$ is the union of $\ker(f_{ik} \otimes_R \mathrm{id}_N)$ for $k \geq i$. Thus $\ker(f_i \otimes_R \mathrm{id}_N) \subset \ker(f_{ij} \otimes_R \mathrm{id}_N)$ for any R-module N, which by definition means f_{ij} dominates f_i.

It is trivial that (3) implies (4) implies (5). We show (5) implies (3). Let $N = \prod_{s\in I} M_s$. If (5) holds, then given $i \in I$ choose $j \geq i$ such that

$$\mathrm{Im}(\mathrm{Hom}(M_j, N) \to \mathrm{Hom}(M_i, N)) = \mathrm{Im}(\mathrm{Hom}(M_k, N) \to \mathrm{Hom}(M_i, N))$$

for all $k \geq j$. Passing the product over $s \in I$ outside of the Hom's and looking at the maps on each component of the product, this says

$$\mathrm{Im}(\mathrm{Hom}(M_j, M_s) \to \mathrm{Hom}(M_i, M_s)) = \mathrm{Im}(\mathrm{Hom}(M_k, M_s) \to \mathrm{Hom}(M_i, M_s))$$

for all $k \geq j$ and $s \in I$. Taking $s = j$ we have

$$\mathrm{Im}(\mathrm{Hom}(M_j, M_j) \to \mathrm{Hom}(M_i, M_j)) = \mathrm{Im}(\mathrm{Hom}(M_k, M_j) \to \mathrm{Hom}(M_i, M_j))$$

for all $k \geq j$. Since f_{ij} is the image of $\mathrm{id} \in \mathrm{Hom}(M_j, M_j)$ under $\mathrm{Hom}(M_j, M_j) \to \mathrm{Hom}(M_i, M_j)$, this shows that for any $k \geq j$ there is $h \in \mathrm{Hom}(M_k, M_j)$ such that $f_{ij} = h \circ f_{ik}$. If $j \geq k$ then we can take $h = f_{kj}$. Hence (3) holds. □

Definition 10.83.6. Let M be an R-module. We say that M is *Mittag-Leffler* if the equivalent conditions of Proposition 10.83.5 hold.

In particular a finitely presented module is Mittag-Leffler.

Remark 10.83.7. Let M be a flat R-module. By Lazard's theorem (Theorem 10.76.4) we can write $M = \mathrm{colim}\, M_i$ as the colimit of a directed system (M_i, f_{ij}) where the M_i are free finite R-modules. For M to be Mittag-Leffler, it is enough for the inverse system of duals $(\mathrm{Hom}_R(M_i, R), \mathrm{Hom}_R(f_{ij}, R))$ to be Mittag-Leffler. This follows from criterion (4) of Proposition 10.83.5 and the fact that for a free finite R-module F, there is a functorial isomorphism $\mathrm{Hom}_R(F, R) \otimes_R N \cong \mathrm{Hom}_R(F, N)$ for any R-module N.

Lemma 10.83.8. *If R is a ring and M, N are Mittag-Leffler modules over R, then $M \otimes_R N$ is a Mittag-Leffler module.*

Proof. Write $M = \mathrm{colim}_{i\in I} M_i$ and $N = \mathrm{colim}_{j\in J} N_j$ as directed colimits of finitely presented R-modules. Denote $f_{ii'} : M_i \to M_{i'}$ and $g_{jj'} : N_j \to N_{j'}$ the transition maps. Then $M_i \otimes_R N_j$ is a finitely presented R-module (see Lemma 10.11.14), and $M \otimes_R N = \mathrm{colim}_{(i,j)\in I\times J} M_i \otimes_R M_j$. Pick $(i, j) \in I \times J$. By the definition of a Mittag-Leffler module we have Proposition 10.83.5 (3) for both systems. In other words there exist $i' \geq i$ and $j' \geq j$ such that for every choice of $i'' \geq i$ and $j'' \geq j$ there exist maps $a : M_{i''} \to M_{i'}$ and $b : M_{j''} \to M_{j'}$ such that $f_{ii'} = a \circ f_{ii''}$ and $g_{jj'} = b \circ g_{jj''}$. Then it is clear that $a \otimes b : M_{i''} \otimes_R N_{j''} \to M_{i'} \otimes_R N_{j'}$ serves the same purpose for the system $(M_i \otimes_R N_j, f_{ii'} \otimes g_{jj'})$. Thus by the characterization Proposition 10.83.5 (3) we conclude that $M \otimes_R N$ is Mittag-Leffler. □

Lemma 10.83.9. *Let R be a ring and M an R-module. Then M is Mittag-Leffler if and only if for every finite free R-module F and module map $f : F \to M$, there exists a finitely presented R-module Q and a module map $g : F \to Q$ such that g and f dominate each other, i.e., $\ker(f \otimes_R id_N) = \ker(g \otimes_R id_N)$ for every R-module N.*

Proof. Since the condition is clear weaker than condition (1) of Proposition 10.83.5 we see that a Mittag-Leffler module satisfies the condition. Conversely, suppose that M satisfies the condition and that $f : P \to M$ is an R-module map from a finitely presented R-module P into M. Choose a surjection $F \to P$ where F is a finite free R-module. By assumption we can find a map $F \to Q$ where Q is a finitely presented

R-module such that $F \to Q$ and $F \to M$ dominate each other. In particular, the kernel of $F \to Q$ contains the kernel of $F \to P$, hence we obtain an R-module map $g : P \to Q$ such that $F \to Q$ is equal to the composition $F \to P \to Q$. Let N be any R-module and consider the commutative diagram

$$\begin{array}{ccc} F \otimes_R N & \longrightarrow & Q \otimes_R N \\ \downarrow & \nearrow & \\ P \otimes_R N & \longrightarrow & M \otimes_R N \end{array}$$

By assumption the kernels of $F \otimes_R N \to Q \otimes_R N$ and $F \otimes_R N \to M \otimes_R N$ are equal. Hence, as $F \otimes_R N \to P \otimes_R N$ is surjective, also the kernels of $P \otimes_R N \to Q \otimes_R N$ and $P \otimes_R N \to M \otimes_R N$ are equal. □

Lemma 10.83.10. *Let $R \to S$ be a finite and finitely presented ring map. Let M be an S-module. If M is a Mittag-Leffler module over S then M is a Mittag-Leffler module over R.*

Proof. Assume M is a Mittag-Leffler module over S. Write $M = \operatorname{colim} M_i$ as a directed colimit of finitely presented S-modules M_i. As M is Mittag-Leffler over S there exists for each i an index $j \geq i$ such that for all $k \geq j$ there is a factorization $f_{ij} = h \circ f_{ik}$ (where h depends on i, the choice of j and k). Note that by Lemma 10.7.4 the modules M_i are also finitely presented as R-modules. Moreover, all the maps f_{ij}, f_{ik}, h are maps of R-modules. Thus we see that the system (M_i, f_{ij}) satisfies the same condition when viewed as a system of R-modules. Thus M is Mittag-Leffler as an R-module. □

Lemma 10.83.11. *Let R be a ring. Let $S = R/I$ for some finitely generated ideal I. Let M be an S-module. Then M is a Mittag-Leffler module over R if and only if M is a Mittag-Leffler module over S.*

Proof. One implication follows from Lemma 10.83.10. To prove the other, assume M is Mittag-Leffler as an R-module. Write $M = \operatorname{colim} M_i$ as a directed colimit of finitely presented S-modules. As I is finitely generated, the ring S is finite and finitely presented as an R-algebra, hence the modules M_i are finitely presented as R-modules, see Lemma 10.7.4. Next, let N be any S-module. Note that for each i we have $\operatorname{Hom}_R(M_i, N) = \operatorname{Hom}_S(M_i, N)$ as $R \to S$ is surjective. Hence the condition that the inverse system $(\operatorname{Hom}_R(M_i, N))_i$ satisfies Mittag-Leffler, implies that the system $(\operatorname{Hom}_S(M_i, N))_i$ satisfies Mittag-Leffler. Thus M is Mittag-Leffler over S by definition. □

Remark 10.83.12. Let $R \to S$ be a finite and finitely presented ring map. Let M be an S-module which is Mittag-Leffler as an R-module. Then it is in general not the case that if M is Mittag-Leffler as an S-module. For example suppose that S is the ring of dual numbers over R, i.e., $S = R \oplus R\epsilon$ with $\epsilon^2 = 0$. Then an S-module consists of an R-module M endowed with a square zero R-linear endomorphism $\epsilon : M \to M$. Now suppose that M_0 is an R-module which is not Mittag-Leffler. Choose a presentation $F_1 \xrightarrow{u} F_0 \to M_0 \to 0$ with F_1 and F_0 free R-modules. Set $M = F_1 \oplus F_0$ with

$$\epsilon = \begin{pmatrix} 0 & 0 \\ u & 0 \end{pmatrix} : M \longrightarrow M.$$

Then $M/\epsilon M \cong F_1 \oplus M_0$ is not Mittag-Leffler over $R = S/\epsilon S$, hence not Mittag-Leffler over S (see Lemma 10.83.11). On the other hand, $M/\epsilon M = M \otimes_S S/\epsilon S$ which would be Mittag-Leffler over S if M was, see Lemma 10.83.8.

10.84. Interchanging direct products with tensor

Let M be an R-module and let $(Q_\alpha)_{\alpha \in A}$ be a family of R-modules. Then there is a canonical map $M \otimes_R \left(\prod_{\alpha \in A} Q_\alpha\right) \to \prod_{\alpha \in A}(M \otimes_R Q_\alpha)$ given on pure tensors by $x \otimes (q_\alpha) \mapsto (x \otimes q_\alpha)$. This map is not necessarily injective or surjective, as the following example shows.

Example 10.84.1. Take $R = \mathbf{Z}$, $M = \mathbf{Q}$, and consider the family $Q_n = \mathbf{Z}/n$ for $n \geq 1$. Then $\prod_n (M \otimes Q_n) = 0$. However there is an injection $\mathbf{Q} \to M \otimes (\prod_n Q_n)$ obtained by tensoring the injection $\mathbf{Z} \to \prod_n Q_n$ by M, so $M \otimes (\prod_n Q_n)$ is nonzero. Thus $M \otimes (\prod_n Q_n) \to \prod_n (M \otimes Q_n)$ is not injective.

On the other hand, take again $R = \mathbf{Z}$, $M = \mathbf{Q}$, and let $Q_n = \mathbf{Z}$ for $n \geq 1$. The image of $M \otimes (\prod_n Q_n) \to \prod_n (M \otimes Q_n) = \prod_n M$ consists precisely of sequences of the form $(a_n/m)_{n \geq 1}$ with $a_n \in \mathbf{Z}$ and m some nonzero integer. Hence the map is not surjective.

We determine below the precise conditions needed on M for the map $M \otimes_R (\prod_\alpha Q_\alpha) \to \prod_\alpha (M \otimes_R Q_\alpha)$ to be surjective, bijective, or injective for all choices of $(Q_\alpha)_{\alpha \in A}$. This is relevant because the modules for which it is injective turn out to be exactly Mittag-Leffler modules (Proposition 10.84.5). In what follows, if M is an R-module and A a set, we write M^A for the product $\prod_{\alpha \in A} M$.

Proposition 10.84.2. *Let M be an R-module. The following are equivalent:*

1. *M is finitely generated.*
2. *For every family $(Q_\alpha)_{\alpha \in A}$ of R-modules, the canonical map $M \otimes_R (\prod_\alpha Q_\alpha) \to \prod_\alpha (M \otimes_R Q_\alpha)$ is surjective.*
3. *For every R-module Q and every set A, the canonical map $M \otimes_R Q^A \to (M \otimes_R Q)^A$ is surjective.*
4. *For every set A, the canonical map $M \otimes_R R^A \to M^A$ is surjective.*

Proof. First we prove (1) implies (2). Choose a surjection $R^n \to M$ and consider the commutative diagram

$$\begin{array}{ccc} R^n \otimes_R (\prod_\alpha Q_\alpha) & \xrightarrow{\cong} & \prod_\alpha (R^n \otimes_R Q_\alpha) \\ \downarrow & & \downarrow \\ M \otimes_R (\prod_\alpha Q_\alpha) & \longrightarrow & \prod_\alpha (M \otimes_R Q_\alpha). \end{array}$$

The top arrow is an isomorphism and the vertical arrows are surjections. We conclude that the bottom arrow is a surjection.

Obviously (2) implies (3) implies (4), so it remains to prove (4) implies (1). In fact for (1) to hold it suffices that the element $d = (x)_{x \in M}$ of M^M is in the image of the map $f : M \otimes_R R^M \to M^M$. In this case $d = \sum_{i=1}^n f(x_i \otimes a_i)$ for some $x_i \in M$ and $a_i \in R^M$. If for $x \in M$ we write $p_x : M^M \to M$ for the projection onto the x-th factor, then

$$x = p_x(d) = \sum\nolimits_{i=1}^n p_x(f(x_i \otimes a_i)) = \sum\nolimits_{i=1}^n p_x(a_i) x_i.$$

Thus $x_1, \ldots, x_n$ generate M. □

Proposition 10.84.3. *Let M be an R-module. The following are equivalent:*

1. *M is finitely presented.*
2. *For every family $(Q_\alpha)_{\alpha \in A}$ of R-modules, the canonical map $M \otimes_R (\prod_\alpha Q_\alpha) \to \prod_\alpha (M \otimes_R Q_\alpha)$ is bijective.*
3. *For every R-module Q and every set A, the canonical map $M \otimes_R Q^A \to (M \otimes_R Q)^A$ is bijective.*
4. *For every set A, the canonical map $M \otimes_R R^A \to M^A$ is bijective.*

Proof. First we prove (1) implies (2). Choose a presentation $R^m \to R^n \to M$ and consider the commutative diagram

$$\begin{array}{ccccccc} R^m \otimes_R (\prod_\alpha Q_\alpha) & \longrightarrow & R^m \otimes_R (\prod_\alpha Q_\alpha) & \longrightarrow & M \otimes_R (\prod_\alpha Q_\alpha) & \longrightarrow & 0 \\ \downarrow \cong & & \downarrow \cong & & \downarrow & & \\ \prod_\alpha (R^m \otimes_R Q_\alpha) & \longrightarrow & \prod_\alpha (R^n \otimes_R Q_\alpha) & \longrightarrow & \prod_\alpha (M \otimes_R Q_\alpha) & \longrightarrow & 0. \end{array}$$

The first two vertical arrows are isomorphisms and the rows are exact. This implies that the map $M \otimes_R (\prod_\alpha Q_\alpha) \to \prod_\alpha (M \otimes_R Q_\alpha)$ is surjective and, by a diagram chase, also injective. Hence (2) holds.

Obviously (2) implies (3) implies (4), so it remains to prove (4) implies (1). From Proposition 10.84.2, if (4) holds we already know that M is finitely generated. So we can choose a surjection $F \to M$ where F is free and finite. Let K be the kernel. We must show K is finitely generated. For any set A, we have a commutative diagram

$$\begin{array}{ccccccccc} & & K \otimes_R R^A & \longrightarrow & F \otimes_R R^A & \longrightarrow & M \otimes_R R^A & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle f_3} & & \downarrow{\scriptstyle f_2} \cong & & \downarrow{\scriptstyle f_1} \cong & & \\ 0 & \longrightarrow & K^A & \longrightarrow & F^A & \longrightarrow & M^A & \longrightarrow & 0. \end{array}$$

The map f_1 is an isomorphism by assumption, the map f_2 is a isomorphism since F is free and finite, and the rows are exact. A diagram chase shows that f_3 is surjective, hence by Proposition 10.84.2 we get that K is finitely generated. □

We need the following lemma for the next proposition.

Lemma 10.84.4. *Let M be an R-module, P a finitely presented R-module, and $f : P \to M$ a map. Let Q be an R-module and suppose $x \in \ker(P \otimes Q \to M \otimes Q)$. Then there exists a finitely presented R-module P' and a map $f' : P \to P'$ such that f factors through f' and $x \in \ker(P \otimes Q \to P' \otimes Q)$.*

Proof. Write M as a colimit $M = \operatorname{colim}_{i \in I} M_i$ of a directed system of finitely presented modules M_i. Since P is finitely presented, the map $f : P \to M$ factors through $M_j \to M$ for some $j \in I$. Upon tensoring by Q we have a commutative diagram

$$\begin{array}{ccc} & M_j \otimes Q & \\ \nearrow & & \searrow \\ P \otimes Q & \longrightarrow & M \otimes Q. \end{array}$$

The image y of x in $M_j \otimes Q$ is in the kernel of $M_j \otimes Q \to M \otimes Q$. Since $M \otimes Q = \operatorname{colim}_{i\in I}(M_i \otimes Q)$, this means y maps to 0 in $M_{j'} \otimes Q$ for some $j' \geq j$. Thus we may take $P' = M_{j'}$ and f' to be the composite $P \to M_j \to M_{j'}$. □

Proposition 10.84.5. *Let M be an R-module. The following are equivalent:*

1. *M is Mittag-Leffler.*
2. *For every family $(Q_\alpha)_{\alpha\in A}$ of R-modules, the canonical map $M \otimes_R (\prod_\alpha Q_\alpha) \to \prod_\alpha (M \otimes_R Q_\alpha)$ is injective.*

Proof. First we prove (1) implies (2). Suppose M is Mittag-Leffler and let x be in the kernel of $M \otimes_R (\prod_\alpha Q_\alpha) \to \prod_\alpha (M \otimes_R Q_\alpha)$. Write M as a colimit $M = \operatorname{colim}_{i\in I} M_i$ of a directed system of finitely presented modules M_i. Then $M \otimes_R (\prod_\alpha Q_\alpha)$ is the colimit of $M_i \otimes_R (\prod_\alpha Q_\alpha)$. So x is the image of an element $x_i \in M_i \otimes_R (\prod_\alpha Q_\alpha)$. We must show that x_i maps to 0 in $M_j \otimes_R (\prod_\alpha Q_\alpha)$ for some $j \geq i$. Since M is Mittag-Leffler, we may choose $j \geq i$ such that $M_i \to M_j$ and $M_i \to M$ dominate each other. Then consider the commutative diagram

$$\begin{array}{ccc}
M \otimes_R (\prod_\alpha Q_\alpha) & \longrightarrow & \prod_\alpha (M \otimes_R Q_\alpha) \\
\uparrow & & \uparrow \\
M_i \otimes_R (\prod_\alpha Q_\alpha) & \xrightarrow{\cong} & \prod_\alpha (M_i \otimes_R Q_\alpha) \\
\downarrow & & \downarrow \\
M_j \otimes_R (\prod_\alpha Q_\alpha) & \xrightarrow{\cong} & \prod_\alpha (M_j \otimes_R Q_\alpha)
\end{array}$$

whose bottom two horizontal maps are isomorphisms, according to Proposition 10.84.3. Since x_i maps to 0 in $\prod_\alpha (M \otimes_R Q_\alpha)$, its image in $\prod_\alpha (M_i \otimes_R Q_\alpha)$ is in the kernel of the map $\prod_\alpha (M_i \otimes_R Q_\alpha) \to \prod_\alpha (M \otimes_R Q_\alpha)$. But this kernel equals the kernel of $\prod_\alpha (M_i \otimes_R Q_\alpha) \to \prod_\alpha (M_j \otimes_R Q_\alpha)$ according to the choice of j. Thus x_i maps to 0 in $\prod_\alpha (M_j \otimes_R Q_\alpha)$ and hence to 0 in $M_j \otimes_R (\prod_\alpha Q_\alpha)$.

Now suppose (2) holds. We prove M satisfies formulation (1) of being Mittag-Leffler from Proposition 10.83.5. Let $f : P \to M$ be a map from a finitely presented module P to M. Choose a set B of representatives of the isomorphism classes of finitely presented R-modules. Let A be the set of pairs (Q, x) where $Q \in B$ and $x \in \ker(P \otimes Q \to M \otimes Q)$. For $\alpha = (Q, x) \in A$, we write Q_α for Q and x_α for x. Consider the commutative diagram

$$\begin{array}{ccc}
M \otimes_R (\prod_\alpha Q_\alpha) & \longrightarrow & \prod_\alpha (M \otimes_R Q_\alpha) \\
\uparrow & & \uparrow \\
P \otimes_R (\prod_\alpha Q_\alpha) & \xrightarrow{\cong} & \prod_\alpha (P \otimes_R Q_\alpha)
\end{array}$$

The top arrow is an injection by assumption, and the bottom arrow is an isomorphism by Proposition 10.84.3. Let $x \in P \otimes_R (\prod_\alpha Q_\alpha)$ be the element corresponding to $(x_\alpha) \in \prod_\alpha (P \otimes_R Q_\alpha)$ under this isomorphism. Then $x \in \ker(P \otimes_R (\prod_\alpha Q_\alpha) \to M \otimes_R (\prod_\alpha Q_\alpha))$ since the top arrow in the diagram is injective. By Lemma 10.84.4, we get a finitely presented module P' and a map $f' : P \to P'$ such that $f : P \to M$ factors through f' and $x \in \ker(P \otimes_R (\prod_\alpha Q_\alpha) \to P' \otimes_R (\prod_\alpha Q_\alpha))$. We have a

commutative diagram

$$\begin{array}{ccc} P' \otimes_R (\prod_\alpha Q_\alpha) & \xrightarrow{\cong} & \prod_\alpha (P' \otimes_R Q_\alpha) \\ \uparrow & & \uparrow \\ P \otimes_R (\prod_\alpha Q_\alpha) & \xrightarrow{\cong} & \prod_\alpha (P \otimes_R Q_\alpha) \end{array}$$

where both the top and bottom arrows are isomorphisms by Proposition 10.84.3. Thus since x is in the kernel of the left vertical map, (x_α) is in the kernel of the right vertical map. This means $x_\alpha \in \ker(P \otimes_R Q_\alpha \to P' \otimes_R Q_\alpha)$ for every $\alpha \in A$. By the definition of A this means $\ker(P \otimes_R Q \to P' \otimes_R Q) \supset \ker(P \otimes_R Q \to M \otimes_R Q)$ for all finitely presented Q and, since $f : P \to M$ factors through $f' : P \to P'$, actually equality holds. By Lemma 10.83.3, f and f' dominate each other. □

Lemma 10.84.6. *Let $0 \to M_1 \to M_2 \to M_3 \to 0$ be a universally exact sequence of R-modules. Then:*

(1) *If M_2 is Mittag-Leffler, then M_1 is Mittag-Leffler.*
(2) *If M_1 and M_3 are Mittag-Leffler, then M_2 is Mittag-Leffler.*

Proof. For any family $(Q_\alpha)_{\alpha \in A}$ of R-modules we have a commutative diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & M_1 \otimes_R (\prod_\alpha Q_\alpha) & \longrightarrow & M_2 \otimes_R (\prod_\alpha Q_\alpha) & \longrightarrow & M_3 \otimes_R (\prod_\alpha Q_\alpha) & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & \prod_\alpha (M_1 \otimes Q_\alpha) & \longrightarrow & \prod_\alpha (M_2 \otimes Q_\alpha) & \longrightarrow & \prod_\alpha (M_3 \otimes Q_\alpha) & \longrightarrow & 0 \end{array}$$

with exact rows. Thus (1) and (2) follow from Proposition 10.84.5. □

Lemma 10.84.7. *If $M = \bigoplus_{i \in I} M_i$ is a direct sum of R-modules, then M is Mittag-Leffler if and only if each M_i is Mittag-Leffler.*

Proof. The "only if" direction follows from Lemma 10.84.6 (1) and the fact that a split short exact sequence is universally exact. For the converse, first note that if I is finite then this follows from Lemma 10.84.6 (2). For general I, if all M_i are Mittag-Leffler then we prove the same of M by verifying condition (1) of Proposition 10.83.5. Let $f : P \to M$ be a map from a finitely presented module P. Then f factors as $P \xrightarrow{f'} \bigoplus_{i' \in I'} M_{i'} \hookrightarrow \bigoplus_{i \in I} M_i$ for some finite subset I' of I. By the finite case $\bigoplus_{i' \in I'} M_{i'}$ is Mittag-Leffler and hence there exists a finitely presented module Q and a map $g : P \to Q$ such that g and f' dominate each other. Then also g and f dominate each other. □

Lemma 10.84.8. *Let $R \to S$ be a ring map. Let M be an S-module. If S is Mittag-Leffler as an R-module, and M is flat and Mittag-Leffler as an S-module, then M is Mittag-Leffler as an R-module.*

Proof. We deduce this from the characterization of Proposition 10.84.5. Namely, suppose that Q_α is a family of R-modules. Consider the composition

$$\begin{array}{c} M \otimes_R \prod_\alpha Q_\alpha = M \otimes_S S \otimes_R \prod_\alpha Q_\alpha \\ \downarrow \\ M \otimes_S \prod_\alpha (S \otimes_R Q_\alpha) \\ \downarrow \\ \prod_\alpha (M \otimes_S \otimes_R Q_\alpha) = \prod_\alpha (M \otimes_R Q_\alpha) \end{array}$$

The first arrows is injective as M is flat over S and S is Mittag-Leffler over R and the second arrow is injective as M is Mittag-Leffler over S. Hence M is Mittag-Leffler over R. □

10.85. Coherent rings

We use the discussion on interchanging $\prod$ and $\otimes$ to determine for which rings products of flat modules are flat. It turns out that these are the so-called coherent rings. You may be more familiar with the notion of a coherent $\mathcal{O}_X$-module on a ringed space, see Modules, Section 17.12.

Definition 10.85.1. Let R be a ring. Let M be an R-module.

(1) We say M is a *coherent module* if it is finitely generated and every finitely generated submodule of M is finitely presented over R.
(2) We say R is a *coherent ring* if it is coherent as a module over itself.

Thus a ring is coherent if and only if every finitely generated ideal is finitely presented as a module. The category of coherent modules is abelian.

Lemma 10.85.2. *Let R be a ring.*

(1) *A finite type submodule of a coherent module is coherent.*
(2) *Let $\varphi : N \to M$ be a homomorphism from a finite module to a coherent module. Then $Ker(\varphi)$ is finite.*
(3) *Let $\varphi : N \to M$ be a homomorphism of coherent modules. Then $Ker(\varphi)$ and $Coker(\varphi)$ are coherent modules.*
(4) *The category of coherent modules is an abelian subcategory of Mod_R.*
(5) *Given a short exact sequence of R-modules $0 \to M_1 \to M_2 \to M_3 \to 0$ if two out of three are coherent so is the third.*

Proof. The first statement is immediate from the definition. During the rest of the proof we will use the results of Lemma 10.5.3 without further mention.

Let $\varphi : N \to M$ satisfy the assumptions of (2). Suppose that N is generated by $x_1, \ldots, x_n$. By Definition 10.85.1 the kernel K of the induced map $\bigoplus_{i=1}^n R \to M$, $e_i \mapsto \varphi(x_i)$ is of finite type. Hence $\mathrm{Ker}(\varphi)$ which is the image of the composition $K \to \bigoplus_{i=1}^n R \to N$ is of finite type. This proves (2).

Let $\varphi : N \to M$ satisfy the assumptions of (3). By (2) the kernel of φ is of finite type and hence by (1) it is coherent.

With the same hypotheses let us show that $\mathrm{Coker}(\varphi)$ is coherent. Since M is finite so is $\mathrm{Coker}(\varphi)$. Let $\overline{x}_i \in \mathrm{Coker}(\varphi)$. We have to show that the kernel of the

associated morphism $\overline{\Psi} : \bigoplus_{i=1}^n R \to \mathrm{Coker}(\varphi)$ is finite. Choose $x_i \in M$ lifting $\overline{x}_i$. Thus $\overline{\Psi}$ lifts to $\Psi : \bigoplus_{i=1}^n R \to M$. Consider the following diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \mathrm{Ker}(\Psi) & \longrightarrow & \bigoplus_{i=1}^n R & \longrightarrow & M & \longrightarrow & 0 \\ & & \downarrow & & \| & & \downarrow & & \\ 0 & \longrightarrow & \mathrm{Ker}(\overline{\Psi}) & \longrightarrow & \bigoplus_{i=1}^n R & \longrightarrow & \mathrm{Coker}(\varphi) & \longrightarrow & 0 \end{array}$$

By the snake lemma we get a short exact sequence $0 \to \mathrm{Ker}(\Psi) \to \mathrm{Ker}(\overline{\Psi}) \to \mathrm{Im}(\varphi) \to 0$. Hence we see that $\mathrm{Ker}(\overline{\Psi})$ is finite.

Statement (4) follows from (3).

Let $0 \to M_1 \to M_2 \to M_3 \to 0$ be a short exact sequence of R-modules. It suffices to prove that if M_1 and M_3 are coherent so is M_2. By Lemma 10.5.3 we see that M_2 is finite. Let $x_1, \ldots, x_n$ be finitely many elements of M_2. We have to show that the module of relations K between them is finite. Consider the following commutative diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & 0 & \longrightarrow & \bigoplus_{i=1}^n R & \longrightarrow & \bigoplus_{i=1}^n R & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & M_1 & \longrightarrow & M_2 & \longrightarrow & M_3 & \longrightarrow & 0 \end{array}$$

with obvious notation. By the snake lemma we get an exact sequence $0 \to K \to K_3 \to M_1$ where K_3 is the module of relations among the images of the x_i in M_3. Since M_3 is coherent we see that K_3 is a finite module. Since M_1 is coherent we see that the image I of $K_3 \to M_1$ is coherent. Hence K is the kernel of the map $K_3 \to I$ between a finite module and a coherent module and hence finite by (2). □

Lemma 10.85.3. *Let R be a ring. If R is coherent, then a module is coherent if and only if it is finitely presented.*

Proof. It is clear that a coherent module is finitely presented (over any ring). Conversely, if R is coherent, then $R^{\oplus n}$ is coherent and so is the cokernel of any map $R^{\oplus m} \to R^{\oplus n}$, see Lemma 10.85.2. □

Lemma 10.85.4. *A Noetherian ring is a coherent ring.*

Proof. By Lemma 10.29.4 any finite R-module is finitely presented. In particular any ideal of R is finitely presented. □

Proposition 10.85.5. *Let R be a ring. The following are equivalent*

(1) *R is coherent,*
(2) *any product of flat R-modules is flat, and*
(3) *for every set A the module R^A is flat.*

Proof. Assume R coherent, and let Q_α, $\alpha \in A$ be a set of flat R-modules. We have to show that $I \otimes_R \prod_\alpha Q_\alpha \to \prod Q_\alpha$ is injective for every finitely generated ideal I of R, see Lemma 10.36.4. Since R is coherent I is an R-module of finite presentation. Hence $I \otimes_R \prod_\alpha Q_\alpha = \prod I \otimes_R Q_\alpha$ by Proposition 10.84.3. The desired injectivity follows as $I \otimes_R Q_\alpha \to Q_\alpha$ is injective by flatness of Q_α.

The implication (2) $\Rightarrow$ (3) is trivial.

Assume that the R-module R^A is flat for every set A. Let I be a finitely generated ideal in R. Then $I \otimes_R R^A \to R^A$ is injective by assumption. By Proposition 10.84.2 and the finiteness of I the image is equal to I^A. Hence $I \otimes_R R^A = I^A$ for every set A and we conclude that I is finitely presented by Proposition 10.84.3. □

10.86. Examples and non-examples of Mittag-Leffler modules

We end this section with some examples and non-examples of Mittag-Leffler modules.

Example 10.86.1. Mittag-Leffler modules.

(1) Any finitely presented module is Mittag-Leffler. This follows, for instance, from Proposition 10.83.5 (1). In general, it is true that a finitely generated module is Mittag-Leffler if and only it is finitely presented. This follows from Propositions 10.84.2, 10.84.3, and 10.84.5.
(2) A free module is Mittag-Leffler since it satisfies condition (1) of Proposition 10.83.5.
(3) By the previous example together with Lemma 10.84.7, projective modules are Mittag-Leffler.

We also want to add to our list of examples power series rings over a Noetherian ring R. This will be a consequence the following lemma.

Lemma 10.86.2. *Let M be a flat R-module. Suppose the following condition holds: if F is a free finite R-module and $x \in F \otimes_R M$, then there exists a smallest submodule F' of F such that $x \in F' \otimes_R M$. Then M is Mittag-Leffler.*

Proof. By Theorem 10.76.4 we can write M as the colimit $M = \operatorname{colim}_{i \in I} M_i$ of a directed system (M_i, f_{ij}) of free finite R-modules. By Remark 10.83.7, it suffices to show that the inverse system $(\mathrm{Hom}_R(M_i, R), \mathrm{Hom}_R(f_{ij}, R))$ is Mittag-Leffler. In other words, fix $i \in I$ and for $j \geq i$ let Q_j be the image of $\mathrm{Hom}_R(M_j, R) \to \mathrm{Hom}_R(M_i, R)$; we must show that the Q_j stabilize.

Since M_i is free and finite, we can make the identification $\mathrm{Hom}_R(M_i, M_j) = \mathrm{Hom}_R(M_i, R) \otimes_R M_j$ for all j. Using the fact that the M_j are free, it follows that for $j \geq i$, Q_j is the smallest submodule of $\mathrm{Hom}_R(M_i, R)$ such that $f_{ij} \in Q_j \otimes_R M_j$. Under the identification $\mathrm{Hom}_R(M_i, M) = \mathrm{Hom}_R(M_i, R) \otimes_R M$, the canonical map $f_i : M_i \to M$ is in $\mathrm{Hom}_R(M_i, R) \otimes_R M$. By the assumption on M, there exists a smallest submodule Q of $\mathrm{Hom}_R(M_i, R)$ such that $f_i \in Q \otimes_R M$. We are going to show that the Q_j stabilize to Q.

For $j \geq i$ we have a commutative diagram

$$\begin{array}{ccc} Q_j \otimes_R M_j & \longrightarrow & \mathrm{Hom}_R(M_i, R) \otimes_R M_j \\ \downarrow & & \downarrow \\ Q_j \otimes_R M & \longrightarrow & \mathrm{Hom}_R(M_i, R) \otimes_R M. \end{array}$$

Since $f_{ij} \in Q_j \otimes_R M_j$ maps to $f_i \in \mathrm{Hom}_R(M_i, R) \otimes_R M$, it follows that $f_i \in Q_j \otimes_R M$. Hence, by the choice of Q, we have $Q \subset Q_j$ for all $j \geq i$.

Since the Q_j are decreasing and $Q \subset Q_j$ for all $j \geq i$, to show that the Q_j stabilize to Q it suffices to find a $j \geq i$ such that $Q_j \subset Q$. As an element of

$$\mathrm{Hom}_R(M_i, R) \otimes_R M = \operatorname{colim}_{j \in J}(\mathrm{Hom}_R(M_i, R) \otimes_R M_j),$$

f_i is the colimit of f_{ij} for $j \geq i$, and f_i also lies in the submodule

$$\text{colim}_{j \in J}(Q \otimes_R M_j) \subset \text{colim}_{j \in J}(\text{Hom}_R(M_i, R) \otimes_R M_j).$$

It follows that for some $j \geq i$, f_{ij} lies in $Q \otimes_R M_j$. Since Q_j is the smallest submodule of $\text{Hom}_R(M_i, R)$ with $f_{ij} \in Q_j \otimes_R M_j$, we conclude $Q_j \subset Q$. □

Lemma 10.86.3. *Let R be a Noetherian ring and A a set. Then $M = R^A$ is a flat and Mittag-Leffler R-module.*

Proof. Combining Lemma 10.85.4 and Proposition 10.85.5 we see that M is flat over R. We show that M satisfies the condition of Lemma 10.86.2. Let F be a free finite R-module. If F' is any submodule of F then it is finitely presented since R is Noetherian. So by Proposition 10.84.3 we have a commutative diagram

$$\begin{array}{ccc} F' \otimes_R M & \longrightarrow & F \otimes_R M \\ \downarrow \cong & & \downarrow \cong \\ (F')^A & \longrightarrow & F^A \end{array}$$

by which we can identify the map $F' \otimes_R M \to F \otimes_R M$ with $(F')^A \to F^A$. Hence if $x \in F \otimes_R M$ corresponds to $(x_\alpha) \in F^A$, then the submodule of F' of F generated by the x_α is the smallest submodule of F such that $x \in F' \otimes_R M$. □

Lemma 10.86.4. *Let R be a Noetherian ring and n a positive integer. Then the R-module $M = R[[t_1, \ldots, t_n]]$ is flat and Mittag-Leffler.*

Proof. As an R-module, we have $M = R^A$ for a (countable) set A. Hence this lemma is a special case of Lemma 10.86.3. □

Example 10.86.5. Non Mittag-Leffler modules.

(1) By Example 10.84.1 and Proposition 10.84.5, $\mathbf{Q}$ is not a Mittag-Leffler $\mathbf{Z}$-module.
(2) We prove below (Theorem 10.88.3) that for a flat and countably generated module, projectivity is equivalent to being Mittag-Leffler. Thus any flat, countably generated, non-projective module M is an example of a non-Mittag-Leffler module. For such an example, see Remark 10.73.3.
(3) Let k be a field. Let $R = k[[x]]$. The R-module $M = \prod_{n \in \mathbf{N}} R/(x^n)$ is not Mittag-Leffler. Namely, consider the element $\xi = (\xi_1, \xi_2, \xi_3, \ldots)$ defined by $\xi_{2^m} = x^{2^{m-1}}$ and $\xi_n = 0$ else, so

$$\xi = (0, x, 0, x^2, 0, 0, 0, x^4, 0, 0, 0, 0, 0, 0, 0, x^8, \ldots)$$

Then the annihilator of ξ in $M/x^{2^m}M$ is generated $x^{2^{m-1}}$ for $m \gg 0$. But if M was Mittag-Leffler, then there would exist a finite R-module Q and an element $\xi' \in Q$ such that the annihilator of ξ' in Q/x^lQ agrees with the annihilator of ξ in M/x^lM for all $l \geq 1$, see Proposition 10.83.5 (1). Now you can prove there exists an integer $a \geq 0$ such that the annihilator of ξ' in Q/x^lQ is generated by either x^a or x^{l-a} for all $l \gg 0$ (depending on whether $\xi' \in Q$ is torsion or not). The combination of the above would give for all $l = 2^m >> 0$ the equality $a = l/2$ or $l - a = l/2$ which is nonsensical.

(4) The same argument shows that (x)-adic completion of $\bigoplus_{n\in\mathbf{N}} R/(x^n)$ is not Mittag-Leffler over $R = k[[x]]$ (hint: ξ is actually an element of this completion).
(5) Let $R = k[a,b]/(a^2, ab, b^2)$. Let S be the finitely presented R-algebra with presentation $S = R[t]/(at - b)$. Then as an R-module S is countably generated and indecomposable (details omitted). On the other hand, R is Artinian local, hence complete local, hence a henselian local ring, see Lemma 10.143.10. If S was Mittag-Leffler as an R-module, then it would be a direct sum of finite R-modules by Lemma 10.143.32. Thus we conclude that S is not Mittag-Leffler as an R-module.

10.87. Countably generated Mittag-Leffler modules

It turns out that countably generated Mittag-Leffler modules have a particularly simple structure.

Lemma 10.87.1. *Let M be an R-module. Write $M = \operatorname{colim}_{i\in I} M_i$ where (M_i, f_{ij}) is a directed system of finitely presented R-modules. If M is Mittag-Leffler and countably generated, then there is a directed countable subset $I' \subset I$ such that $M \cong \operatorname{colim}_{i\in I'} M_i$.*

Proof. Let $x_1, x_2, \ldots$ be a countable set of generators for M. For each x_n choose $i \in I$ such that x_n is in the image of the canonical map $f_i : M_i \to M$; let $I'_0 \subset I$ be the set of all these i. Now since M is Mittag-Leffler, for each $i \in I'_0$ we can choose $j \in I$ such that $j \geq i$ and $f_{ij} : M_i \to M_j$ factors through $f_{ik} : M_i \to M_k$ for all $k \geq i$ (condition (3) of Proposition 10.83.5); let I'_1 be the union of I'_0 with all of these j. Since I'_1 is a countable, we can enlarge it to a countable directed set $I'_2 \subset I$. Now we can apply the same procedure to I'_2 as we did to I'_0 to get a new countable set $I'_3 \subset I$. Then we enlarge I'_3 to a countable directed set I'_4. Continuing in this way—adding in a j as in Proposition 10.83.5 (3) for each $i \in I'_\ell$ if ℓ is odd and enlarging I'_ℓ to a directed set if ℓ is even—we get a sequence of subsets $I'_\ell \subset I$ for $\ell \geq 0$. The union $I' = \bigcup I'_\ell$ satisfies:

(1) I' is countable and directed;
(2) each x_n is in the image of $f_i : M_i \to M$ for some $i \in I'$;
(3) if $i \in I'$, then there is $j \in I'$ such that $j \geq i$ and $f_{ij} : M_i \to M_j$ factors through $f_{ik} : M_i \to M_k$ for all $k \in I$ with $k \geq i$. In particular $\ker(f_{ik}) \subset \ker(f_{ij})$ for $k \geq i$.

We claim that the canonical map $\operatorname{colim}_{i\in I'} M_i \to \operatorname{colim}_{i\in I} M_i = M$ is an isomorphism. By (2) it is surjective. For injectivity, suppose $x \in \operatorname{colim}_{i\in I'} M_i$ maps to 0 in $\operatorname{colim}_{i\in I} M_i$. Representing x by an element $\tilde{x} \in M_i$ for some $i \in I'$, this means that $f_{ik}(\tilde{x}) = 0$ for some $k \in I, k \geq i$. But then by (3) there is $j \in I', j \geq i$, such that $f_{ij}(\tilde{x}) = 0$. Hence $x = 0$ in $\operatorname{colim}_{i\in I'} M_i$. □

Lemma 10.87.1 implies that a countably generated Mittag-Leffler module M over R is the colimit of a system

$$M_1 \to M_2 \to M_3 \to M_4 \to \ldots$$

with each M_n a finitely presented R-module. To see this argue as in the proof of Lemma 10.81.3 to see that a countable directed partially ordered set has a cofinal subset isomorphic to $(\mathbf{N}, \geq)$. Suppose $R = k[x_1, x_2, x_3, \ldots]$ and $M = R/(x_i)$. Then M is finitely generated but not finitely presented, hence not Mittag-Leffler (see

Example 10.86.1 part (1)). But of course you can write $M = \operatorname{colim}_n M_n$ by taking $M_n = R/(x_1, \ldots, x_n)$, hence the condition that you can write M as such a limit does not imply that M is Mittag-Leffler.

Lemma 10.87.2. *Let R be a ring. Let M be an R-module. Assume M is Mittag-Leffler and countably generated. For any R-module map $f : P \to M$ with P finitely generated there exists an endomorphism $\alpha : M \to M$ such that*

(1) *$\alpha : M \to M$ factors through a finitely presented R-module, and*
(2) *$\alpha \circ f = f$.*

Proof. Write $M = \operatorname{colim}_{i \in I} M_i$ as a directed colimit of finitely presented R-modules with I countable, see Lemma 10.87.1. The transition maps are denoted f_{ij} and we use $f_i : M_i \to M$ to note the canonical maps into M. Set $N = \prod_{s \in I} M_s$. Denote

$$M_i^* = \operatorname{Hom}_R(M_i, N) = \prod_{s \in S} \operatorname{Hom}_R(M_i, M_s)$$

so that (M_i^*) is an inverse system of R-modules over I. Note that $\operatorname{Hom}_R(M, N) = \lim M_i^*$. As M is Mittag-Leffler, we find for every $i \in I$ an index $k(i) \geq i$ such that

$$E_i := \bigcap\nolimits_{i' \geq i} \operatorname{Im}(M_{i'}^* \to M_i^*) = \operatorname{Im}(M_{k(i)}^* \to M_i^*)$$

Choose $i \in I$ such that $\operatorname{Im}(P \to M) \subset \operatorname{Im}(M_i \to M)$. This is possible as P is finitely generated. Set $k = k(i)$ for this i. Let $x = (0, \ldots, 0, \operatorname{id}_{M_k}, 0, \ldots, 0) \in M_k^*$ and note that this maps to $y = (0, \ldots, 0, f_{ik}, 0, \ldots, 0) \in M_i^*$. By our choice of k we see that $y \in E_i$. By Example 10.81.2 the transition maps $E_{i'} \to E_i$ are surjective for each $i' \geq i$ and $\lim E_i = \lim M_i^* = \operatorname{Hom}_R(M, N)$. Hence Lemma 10.81.3 guarantees there exists an element $z \in \operatorname{Hom}_R(M, N)$ which maps to y in $E_i \subset M_i^*$. Let z_k be the kth component of z. Then $z_k : M \to M_k$ is a homomorphism such that

$$\begin{array}{ccc} M & \xrightarrow{z_k} & M_k \\ {\scriptstyle f_i}\uparrow & \nearrow_{f_{ik}} & \\ M_i & & \end{array}$$

commutes. Let $\alpha : M \to M$ be the composition $f_k \circ z_k : M \to M_k \to M$. Then α factors through a finitely presented module by construction and $\alpha \circ f_i = f_i$. Since the image of f is contained in the image of f_i this also implies that $\alpha \circ f = f$. $\square$

We will see later (see Lemma 10.143.32) that Lemma 10.87.2 means that a countably generated Mittag-Leffler module over a henselian local ring is a direct sum of finitely presented modules.

10.88. Characterizing projective modules

The goal of this section is to prove that a module is projective if and only if it is flat, Mittag-Leffler, and a direct sum of countably generated modules (Theorem 10.88.3 below).

Lemma 10.88.1. *Let M be an R-module. If M is flat, Mittag-Leffler, and countably generated, then M is projective.*

Proof. By Lazard's theorem (Theorem 10.76.4), we can write $M = \operatorname{colim}_{i \in I} M_i$ for a directed system of finite free R-modules (M_i, f_{ij}) indexed by a set I. By Lemma 10.87.1, we may assume I is countable. Now let

$$0 \to N_1 \to N_2 \to N_3 \to 0$$

be an exact sequence of R-modules. We must show that applying $\operatorname{Hom}_R(M, -)$ preserves exactness. Since M_i is finite free,

$$0 \to \operatorname{Hom}_R(M_i, N_1) \to \operatorname{Hom}_R(M_i, N_2) \to \operatorname{Hom}_R(M_i, N_3) \to 0$$

is exact for each i. Since M is Mittag-Leffler, $(\operatorname{Hom}_R(M_i, N_1))$ is a Mittag-Leffler inverse system. So by Lemma 10.81.4,

$$0 \to \lim_{i \in I} \operatorname{Hom}_R(M_i, N_1) \to \lim_{i \in I} \operatorname{Hom}_R(M_i, N_2) \to \lim_{i \in I} \operatorname{Hom}_R(M_i, N_3) \to 0$$

is exact. But for any R-module N there is a functorial isomorphism $\operatorname{Hom}_R(M, N) \cong \lim_{i \in I} \operatorname{Hom}_R(M_i, N)$, so

$$0 \to \operatorname{Hom}_R(M, N_1) \to \operatorname{Hom}_R(M, N_2) \to \operatorname{Hom}_R(M, N_3) \to 0$$

is exact. □

Remark 10.88.2. Lemma 10.88.1 does not hold without the countable generation assumption. For example, the $\mathbf{Z}$-module $M = \mathbf{Z}[[x]]$ is flat and Mittag-Leffler but not projective. It is Mittag-Leffler by Lemma 10.86.4. Subgroups of free abelian groups are free, hence a projective $\mathbf{Z}$-module is in fact free and so are its submodules. Thus to show M is not projective it suffices to produce a non-free submodule. Fix a prime p and consider the submodule N consisting of power series $f(x) = \sum a_i x^i$ such that for every integer $m \geq 1$, p^m divides a_i for all but finitely many i. Then $\sum a_i p^i x^i$ is in N for all $a_i \in \mathbf{Z}$, so N is uncountable. Thus if N were free it would have uncountable rank and the dimension of N/pN over $\mathbf{Z}/p$ would be uncountable. This is not true as the elements $x^i \in N/pN$ for $i \geq 0$ span N/pN.

Theorem 10.88.3. *Let M be an R-module. Then M is projective if and only it satisfies:*

1. *M is flat,*
2. *M is Mittag-Leffler,*
3. *M is a direct sum of countably generated R-modules.*

Proof. First suppose M is projective. Then M is a direct summand of a free module, so M is flat and Mittag-Leffler since these properties pass to direct summands. By Kaplansky's theorem (Theorem 10.79.5), M satisfies (3).

Conversely, suppose M satisfies (1)-(3). Since being flat and Mittag-Leffler passes to direct summands, M is a direct sum of flat, Mittag-Leffler, countably generated R-modules. Lemma 10.88.1 implies M is a direct sum of projective modules. Hence M is projective. □

Lemma 10.88.4. *Let $f : M \to N$ be universally injective map of R-modules. Suppose M is a direct sum of countably generated R-modules, and suppose N is flat and Mittag-Leffler. Then M is projective.*

Proof. By Lemmas 10.77.7 and 10.84.6, M is flat and Mittag-Leffler, so the conclusion follows from Theorem 10.88.3. □

Lemma 10.88.5. *Let R be a Noetherian ring and let M be a R-module. Suppose M is a direct sum of countably generated R-modules, and suppose there is a universally injective map $M \to R[[t_1, \ldots, t_n]]$ for some n. Then M is projective.*

Proof. Follows from Lemmas 10.88.4 and 10.86.4. □

10.89. Ascending properties of modules

All of the properties of a module in Theorem 10.88.3 ascend along arbitrary ring maps:

Lemma 10.89.1. *Let $R \to S$ be a ring map. Let M be an R-module. Then:*

1. *If M is flat, then the S-module $M \otimes_R S$ is flat.*
2. *If M is Mittag-Leffler, then the S-module $M \otimes_R S$ is Mittag-Leffler.*
3. *If M is a direct sum of countably generated R-modules, then the S-module $M \otimes_R S$ is a direct sum of countably generated S-modules.*
4. *If M is projective, then the S-module $M \otimes_R S$ is projective.*

Proof. All are obvious except (2). For this, use formulation (3) of being Mittag-Leffler from Proposition 10.83.5 and the fact that tensoring commutes with taking colimits. □

10.90. Descending properties of modules

We address the faithfully flat descent of the properties from Theorem 10.88.3 that characterize projectivity. In the presence of flatness, the property of being a Mittag-Leffler module descends:

Lemma 10.90.1. *Let $R \to S$ be a faithfully flat ring map. Let M be an R-module. If the S-module $M \otimes_R S$ is flat and Mittag-Leffler, then M is flat and Mittag-Leffler.*

Proof. By Lemma 10.78.2, flatness descends, so M is flat. Thus by Lazard's theorem (Theorem 10.76.4) we can write $M = \operatorname{colim}_{i \in I} M_i$ where (M_i, f_{ij}) is a directed system of free finite R-modules. According to Remark 10.83.7, to prove M is Mittag-Leffler it is enough to show that $(\operatorname{Hom}_R(M_i, R))$ is a Mittag-Leffler inverse system.

Since tensoring commutes with colimits, $M \otimes_R S = \operatorname{colim}(M_i \otimes_R S)$. Since $M \otimes_R S$ is Mittag-Leffler this means $(\operatorname{Hom}_S(M_i \otimes_R S, S))$ is a Mittag-Leffler inverse system. So for every $i \in I$, the family $\operatorname{Im}(\operatorname{Hom}_S(M_j \otimes_R S, S) \to \operatorname{Hom}_S(M_i \otimes_R S, S))$ for $j \geq i$ stabilizes. Because M_i is free and finite there is a functorial isomorphism $\operatorname{Hom}_S(M_i \otimes_R S, S) \cong \operatorname{Hom}_R(M_i, R) \otimes_R S$, and because $R \to S$ is faithfully flat, tensoring by S commutes with taking the image of a module map. Thus we find that for every $i \in I$, the family $\operatorname{Im}(\operatorname{Hom}_R(M_j, R) \to \operatorname{Hom}_R(M_i, R)) \otimes_R S$ for $j \geq i$ stabilizes. But if N is an R-module and $N' \subset N$ a submodule such that $N' \otimes_R S = N \otimes_R S$, then $N' = N$ by faithful flatness of S. We conclude that for every $i \in I$, the family $\operatorname{Im}(\operatorname{Hom}_R(M_j, R) \to \operatorname{Hom}_R(M_i, R))$ for $j \geq i$ stabilizes. So M is Mittag-Leffler. □

At this point the faithfully flat descent of countably generated projective modules follows easily.

Lemma 10.90.2. *Let $R \to S$ be a faithfully flat ring map. Let M be an R-module. If the S-module $M \otimes_R S$ is countably generated and projective, then M is countably generated and projective.*

Proof. Follows from Lemma 10.90.1, the fact that countable generation descends, and Theorem 10.88.3. □

All that remains is to use dévissage to reduce descent of projectivity in the general case to the countably generated case. First, two simple lemmas.

Lemma 10.90.3. *Let $R \to S$ be a ring map, let M be an R-module, and let Q be a countably generated S-submodule of $M \otimes_R S$. Then there exists a countably generated R-submodule P of M such that $Im(P \otimes_R S \to M \otimes_R S)$ contains Q.*

Proof. Let $y_1, y_2, \ldots$ be generators for Q and write $y_j = \sum_k x_{jk} \otimes s_{jk}$ for some $x_{jk} \in M$ and $s_{jk} \in S$. Then take P be the submodule of M generated by the x_{jk}. □

Lemma 10.90.4. *Let $R \to S$ be a ring map, and let M be an R-module. Suppose $M \otimes_R S = \bigoplus_{i \in I} Q_i$ is a direct sum of countably generated S-modules Q_i. If N is a countably generated submodule of M, then there is a countably generated submodule N' of M such that $N' \supset N$ and $Im(N' \otimes_R S \to M \otimes_R S) = \bigoplus_{i \in I'} Q_i$ for some subset $I' \subset I$.*

Proof. Let $N'_0 = N$. We construct by induction an increasing sequence of countably generated submodules $N'_\ell \subset M$ for $\ell = 0, 1, 2, \ldots$ such that: if I'_ℓ is the set of $i \in I$ such that the projection of $\mathrm{Im}(N'_\ell \otimes_R S \to M \otimes_R S)$ onto Q_i is nonzero, then $\mathrm{Im}(N'_{\ell+1} \otimes_R S \to M \otimes_R S)$ contains Q_i for all $i \in I'_\ell$. To construct $N'_{\ell+1}$ from N'_ℓ, let Q be the sum of (the countably many) Q_i for $i \in I'_\ell$, choose P as in Lemma 10.90.3, and then let $N'_{\ell+1} = N'_\ell + P$. Having constructed the N'_ℓ, just take $N' = \bigcup_\ell N'_\ell$ and $I' = \bigcup_\ell I'_\ell$. □

Theorem 10.90.5. *Let $R \to S$ be a faithfully flat ring map. Let M be an R-module. If the S-module $M \otimes_R S$ is projective, then M is projective.*

Proof. We are going to construct a Kaplansky dévissage of M to show that it is a direct sum of projective modules and hence projective. By Theorem 10.79.5 we can write $M \otimes_R S = \bigoplus_{i \in I} Q_i$ as a direct sum of countably generated S-modules Q_i. Choose a well-ordering on M. By transfinite induction we are going to define an increasing family of submodules M_α of M, one for each ordinal α, such that $M_\alpha \otimes_R S$ is a direct sum of some subset of the Q_i.

For $\alpha = 0$ let $M_0 = 0$. If α is a limit ordinal and M_β has been defined for all $\beta < \alpha$, then define $M_\beta = \bigcup_{\beta<\alpha} M_\beta$. Since each $M_\beta \otimes_R S$ for $\beta < \alpha$ is a direct sum of a subset of the Q_i, the same will be true of $M_\alpha \otimes_R S$. If $\alpha+1$ is a successor ordinal and M_α has been defined, then define $M_{\alpha+1}$ as follows. If $M_\alpha = M$, then let $M_{\alpha+1} = M$. Otherwise choose the smallest $x \in M$ (with respect to the fixed well-ordering) such that $x \notin M_\alpha$. Since S is flat over R, $(M/M_\alpha) \otimes_R S = M \otimes_R S / M_\alpha \otimes_R S$, so since $M_\alpha \otimes_R S$ is a direct sum of some Q_i, the same is true of $(M/M_\alpha) \otimes_R S$. By Lemma 10.90.4, we can find a countably generated R-submodule P of M/M_α containing the image of x in M/M_α and such that $P \otimes_R S$ (which equals $\mathrm{Im}(P \otimes_R S \to M \otimes_R S)$ since S is flat over R) is a direct sum of some Q_i. Since $M \otimes_R S = \bigoplus_{i \in I} Q_i$ is projective and projectivity passes to direct summands, $P \otimes_R S$ is also projective. Thus by Lemma 10.90.2, P is projective. Finally we define $M_{\alpha+1}$ to be the preimage of P in M, so that $M_{\alpha+1}/M_\alpha = P$ is countably generated and projective. In particular M_α is a direct summand of $M_{\alpha+1}$ since projectivity of $M_{\alpha+1}/M_\alpha$ implies the sequence $0 \to M_\alpha \to M_{\alpha+1} \to M_{\alpha+1}/M_\alpha \to 0$ splits.

Transfinite induction on M (using the fact that we constructed $M_{\alpha+1}$ to contain the smallest $x \in M$ not contained in M_α) shows that each $x \in M$ is contained in some M_α. Thus, there is some large enough ordinal S satisfying: for each $x \in M$ there is $\alpha \in S$ such that $x \in M_\alpha$. This means $(M_\alpha)_{\alpha \in S}$ satisfies property (1) of a Kaplansky dévissage of M. The other properties are clear by construction. We conclude $M = \bigoplus_{\alpha+1 \in S} M_{\alpha+1}/M_\alpha$. Since each $M_{\alpha+1}/M_\alpha$ is projective by construction, M is projective. □

10.91. Completion

Suppose that R is a ring and I is an ideal. We define the *completion of R with respect to I* to be the limit

$$R^\wedge = \lim_n R/I^n.$$

An element of $R^\wedge$ is simply given by a sequence of elements $f_n \in R/I^n$ such that $f_n \equiv f_{n+1} \bmod I^n$ for all n. Similarly, if M is an R-module then we define the *completion of M with respect to I* to be the limit

$$M^\wedge = \lim_n M/I^n M.$$

An element of $M^\wedge$ is simply given by a sequence of elements $m_n \in M/I^n M$ such that $m_n \equiv m_{n+1} \bmod I^n M$ for all n. From this description it is clear that there are always canonical maps

$$M \longrightarrow M^\wedge, \quad \text{and} \quad M \otimes_R R^\wedge \longrightarrow M^\wedge.$$

Moreover, given a map $\varphi : M \to N$ of modules we get an induced map $\varphi^\wedge : M^\wedge \to N^\wedge$ on completions making the diagram

$$\begin{array}{ccc} M & \longrightarrow & N \\ \downarrow & & \downarrow \\ M^\wedge & \longrightarrow & N^\wedge \end{array}$$

commute. In general completion is not an exact functor, see Examples, Section 78.6. Here are some initial positive results.

Lemma 10.91.1. *Let R be a ring. Let $I \subset R$ be an ideal. Let $\varphi : M \to N$ be a map of R-modules.*

1. *If $M/IM \to N/IN$ is surjective, then $M^\wedge \to N^\wedge$ is surjective.*
2. *If $M \to N$ is surjective, then $M^\wedge \to N^\wedge$ is surjective.*
3. *If $0 \to K \to M \to N \to 0$ is a short exact sequence of R-modules and N is flat, then $0 \to K^\wedge \to M^\wedge \to N^\wedge \to 0$ is a short exact sequence.*
4. *The map $M \otimes_R R^\wedge \to M^\wedge$ is surjective for any finite R-module M.*

Proof. Assume $M/IM \to N/IN$ is surjective. Then the map $M/I^n M \to N/I^n N$ is surjective for each $n \geq 1$ by Nakayama's lemma (Lemma 10.18.1). Set $K_n = \{x \in M \mid \varphi(x) \in I^n N\}$. Thus we get short exact sequences

$$0 \to K_n/I^n M \to M/I^n M \to N/I^n N \to 0$$

We claim that the canonical map $K_{n+1}/I^{n+1}M \to K_n/I^n M$ is surjective. Namely, if $x \in K_n$ write $\varphi(x) = \sum z_j n_j$ with $z_j \in I^n$, $n_j \in N$. By assumption we can write $n_j = \varphi(m_j) + \sum z_{jk} n_{jk}$ with $m_j \in M$, $z_{jk} \in I$ and $n_{jk} \in N$. Hence

$$\varphi(x - \sum z_j m_j) = \sum z_j z_{jk} n_{jk}.$$

This means that $x' = x - \sum z_j m_j \in K_{n+1}$ maps to x which proves the claim. Now we may apply Lemma 10.82.1 to the inverse system of short exact sequences above to see (1). Part (2) is a special case of (1). If the assumptions of (3) hold, then for each n the sequence

$$0 \to K/I^nK \to M/I^nM \to N/I^nN \to 0$$

is short exact by Lemma 10.36.11. Hence we can directly apply Lemma 10.82.1 to conclude (3) is true. To see (4) choose generators $x_i \in M$, $i = 1, \ldots, n$. Then the map $R^{\oplus n} \to M$, $(a_1, \ldots, a_n) \mapsto \sum a_i x_i$ is surjective. Hence by (2) we see $(R^\wedge)^{\oplus n} \to M^\wedge$, $(a_1, \ldots, a_n) \mapsto \sum a_i x_i$ is surjective. Assertion (4) follows from this. □

Lemma 10.91.2. *Suppose R is Noetherian.*

(1) *If $N \to M$ is an injective map of finite R-modules, then the map on completions $N^\wedge \to M^\wedge$ is injective.*
(2) *If M is a finite R-module, then $M^\wedge = M \otimes_R R^\wedge$.*

Proof. For the first statement, by the Artin-Rees Lemma 10.47.4, we have a constant c such that $I^nM \cap N$ equals $I^{n-c}(I^cM \cap N) \subset I^{n-c}N$. Thus if $(n_i) \in N^\wedge$ maps to zero in $M^\wedge$, then each n_i maps to zero in $N/I^{i-c}N$. And hence $n_{i-c} = 0$. Thus $N^\wedge \to M^\wedge$ is injective.

For the second statement let $0 \to K \to R^t \to M \to 0$ be the presentation of M corresponding to the generators $x_1, \ldots, x_t$ of M. By Lemma 10.91.1 $(R^t)^\wedge \to M^\wedge$ is surjective, and for any finitely generated R-module the canonical map $M \otimes_R R^\wedge \to M^\wedge$ is surjective. Hence to prove the second statement it suffices to prove the kernel of $(R^t)^\wedge \to M^\wedge$ is exactly $K^\wedge$.

Let $(x_n) \in (R^t)^\wedge$ be in the kernel. Note that each x_n is in the image of the map $K/I^nK \to (R/I^n)^t$. Choose c such that $(I^n)^t \cap K \subset I^{n-c}K$, which is possible by Artin-Rees (Lemma 10.47.4). For each $n \geq 0$ choose $y_n \in K/I^{n+c}K$ mapping to x_{n+c}, and set $z_n = y_n \bmod I^nK$. The elements z_n satisfy $z_{n+1} - z_n \bmod I^nK = y_{n+1} - y_n \bmod I^nK$, and $y_{n+1} - y_n \in I^{n+c}R^t$ by construction. Hence $z_{n+1} = z_n \bmod I^nK$ by the choice of c above. In other words $(z_n) \in K^\wedge$ maps to (x_n) as desired. □

Lemma 10.91.3. *Let R be a Noetherian ring. Let $I \subset R$ be an ideal.*

(1) *The ring map $R \to R^\wedge$ is flat.*
(2) *The functor $M \mapsto M^\wedge$ is exact on the category of finitely generated R-modules.*

Proof. Consider $I \otimes_R R^\wedge \to R \otimes_R R^\wedge = R^\wedge$. According to Lemma 10.91.2 this is identified with $I^\wedge \to R^\wedge$ and $I^\wedge \to R^\wedge$ is injective. Part (1) follows from Lemma 10.36.4. Part (2) follows from part (1) and Lemma 10.91.2 part (2). □

Lemma 10.91.4. *Let R be a Noetherian local ring. Let $\mathfrak{m} \subset R$ be the maximal ideal. Let $I \subset \mathfrak{m}$ be an ideal. The ring map $R \to R^\wedge$ is faithfully flat. In particular the completion with respect to $\mathfrak{m}$, namely $\lim_n R/\mathfrak{m}^n$ is faithfully flat.*

Proof. By Lemma 10.91.3 it is flat. The composition $R \to R^\wedge \to R/\mathfrak{m}$ where the last map is the projection map $R^\wedge \to R/I$ combined with $R/I \to R/\mathfrak{m}$ shows that $\mathfrak{m}$ is in the image of $\mathrm{Spec}(R^\wedge) \to \mathrm{Spec}(R)$. Hence the map is faithfully flat by Lemma 10.36.14. □

Definition 10.91.5. Let R be a ring. Let $I \subset R$ be an ideal. Let M be an R-module. We say M is *I-adically complete* if the map

$$M \longrightarrow M^\wedge = \lim_n M/I^nM$$

is an isomorphism[4]. We say R is *I-adically complete* if R is complete as an R-module.

It is not true that the completion of an R-module M with respect to I is I-adically complete. For an example see Examples, Section 78.4. Here is a lemma from an unpublished note of Lenstra and de Smit.

Lemma 10.91.6. *Let R be a ring. Let $I \subset R$ be an ideal. Let M be an R-module. Denote $K_n = Ker(M^\wedge \to M/I^nM)$. Then $M^\wedge$ is I-adically complete if and only if K_n is equal to $I^nM^\wedge$ for all $n \geq 1$.*

Proof. The module $I^nM^\wedge$ is contained in K_n. Thus for each $n \geq 1$ there is a canonical exact sequence

$$0 \to K_n/I^nM^\wedge \to M^\wedge/I^nM^\wedge \to M/I^nM \to 0.$$

As $I^nM^\wedge$ maps onto $I^nM/I^{n+1}M$ we see that $K_{n+1} + I^nM^\wedge = K_n$. Thus the inverse system $\{K_n/I^nM^\wedge\}_{n\geq 1}$ has surjective transition maps. By Lemma 10.82.1 we see that there is a short exact sequence

$$0 \to \lim_n K_n/I^nM^\wedge \to (M^\wedge)^\wedge \to M^\wedge \to 0$$

Hence $M^\wedge$ is complete if and only if $K_n/I^nM^\wedge = 0$ for all $n \geq 1$. □

Lemma 10.91.7. *Let R be a ring. Let I be a finitely generated ideal of R. For any R-module M the completion $M^\wedge$ is complete. In particular $R^\wedge$ is complete.*

Proof. Let $K_n = \mathrm{Ker}(M^\wedge \to M/I^nM)$. By Lemma 10.91.6 we have to show that $K_n = I^nM^\wedge$. Write $I = (f_1, \ldots, f_t)$. Let $z \in K_n$. Write $z = (\overline{z}_m)$ with $\overline{z}_m \in M/I^mM$. Choose $z_m \in M$ mapping to $\overline{z}_m$ in M/I^mM. Then $z_{m+1} = z_m \bmod I^m$. Write $z_{n+1} = z_n + \delta_n$, $z_{n+2} = z_{n+1} + \delta_{n+1}$, etc. Then $\delta_m \in I^mM$. Thus the infinite sum

$$z = z_n + \delta_n + \delta_{n+1} + \delta_{n+2} + \ldots$$

converges in $M^\wedge$. For $m \geq n$ we have $\delta_m \in I^mM$ hence we can write

$$\delta_m = \sum\nolimits_{j_1+\ldots+j_t=n} f_1^{j_1} \ldots f_t^{j_t} \alpha_{J,m}$$

with $\alpha_{J,m} \in I^{m-n}M$. Our assumption $z \in K_n$ means $z_n \in I^nM$ hence we can also write

$$z_n = \sum\nolimits_{j_1+\ldots+j_t=n} f_1^{j_1} \ldots f_t^{j_t} \alpha_J$$

with $\alpha_J \in M$. Then we can set

$$z_J = \alpha_J + \alpha_{J,n} + \alpha_{J,n+1} + \alpha_{J,n+2} + \ldots$$

as an element of $M^\wedge$. By construction $z = \sum_J f_1^{j_1} \ldots f_t^{j_t} z_J$. Hence z is an element of $I^nM^\wedge$ as desired. □

Lemma 10.91.8. *Let R be a Noetherian ring. Let I be an ideal of R. Let M be an R-module. Then the completion $M^\wedge$ of M with respect to I is I-adically complete.*

[4]This includes the condition that $\bigcap I^nM = (0)$.

Proof. This is a special case of Lemma 10.91.7 because I is a finitely generated ideal. □

Lemma 10.91.9. *Let R be a ring. Let $I \subset R$ be an ideal. Assume*

(1) *R/I is a Noetherian ring,*
(2) *I is finitely generated.*

Then $R^\wedge$ is a Noetherian ring complete with respect to $IR^\wedge$.

Proof. By Lemma 10.91.7 we see that $R^\wedge$ is I-adically complete. Hence it is also $IR^\wedge$-adically complete. Since $R^\wedge/IR^\wedge = R/I$ is Noetherian we see that after replacing R by $R^\wedge$ we may in addition to assumptions (1) and (2) assume that also R is I-adically complete.

Let $f_1, \ldots, f_t$ be generators of I. Then there is a surjection of rings $R/I[T_1, \ldots, T_t] \to \bigoplus I^n/I^{n+1}$ mapping T_i to the element $\overline{f}_i \in I/I^2$. Hence $\bigoplus I^n/I^{n+1}$ is a Noetherian ring. Let $J \subset R$ be an ideal. Consider the ideal

$$\bigoplus J \cap I^n/J \cap I^{n+1} \subset \bigoplus I^n/I^{n+1}.$$

Let $\overline{g}_1, \ldots, \overline{g}_m$ be generators of this ideal. We may choose $\overline{g}_j$ to be a homogeneous element of degree d_j and we may pick $g_j \in J \cap I^{d_j}$ mapping to $\overline{g}_j \in J \cap I^{d_j}/J \cap I^{d_j+1}$. We claim that $g_1, \ldots, g_m$ generate J.

Let $x \in J \cap I^n$. There exist $a_j \in I^{\max(0, n-d_j)}$ such that $x - \sum a_j g_j \in J \cap I^{n+1}$. The reason is that $J \cap I^n/J \cap I^{n+1}$ is equal to $\sum \overline{g}_j I^{n-d_j}/I^{n-d_j+1}$ by our choice of $g_1, \ldots, g_m$. Hence starting with $x \in J$ we can find a sequence of vectors $(a_{1,n}, \ldots, a_{m,n})_{n \geq 0}$ with $a_{j,n} \in I^{\max(0,n-d_j)}$ such that

$$x = \sum\nolimits_{n=0,\ldots,N} \sum\nolimits_{j=1,\ldots,m} a_{j,n} g_j \bmod I^{N+1}$$

Setting $A_j = \sum_{n \geq 0} a_{j,n}$ we see that $x = \sum A_j g_j$ as R is complete. Hence J is finitely generated and we win. □

Lemma 10.91.10. *Let R be a Noetherian ring. Let I be an ideal of R. The completion $R^\wedge$ of R with respect to I is Noetherian.*

Proof. This is a direct consequence of Lemma 10.91.9. It can also be seen directly as follows. Choose generators $f_1, \ldots, f_n$ of I. Consider the map

$$R[[x_1, \ldots, x_n]] \longrightarrow R^\wedge, \quad x_i \longmapsto f_i.$$

This is a well defined and surjective ring map (details omitted). Since $R[[x_1, \ldots, x_n]]$ is Noetherian (see Lemma 10.29.2) we win. □

Lemma 10.91.11. *Let R be a ring. Let $I \subset R$ be an ideal. Then*

(1) *any element of $1 + I$ maps to an invertible element of $R^\wedge$,*
(2) *any element of $1 + IR^\wedge$ is invertible in $R^\wedge$, and*
(3) *the ideal $IR^\wedge$ is contained in the radical of $R^\wedge$.*

Proof. Let $x \in IR^\wedge$. Let u_n be the image of $1 - x + x^2 - x^3 + \ldots + (-x)^n$ in R/I^{n+1}. Note that (u_n) defines an element u of $R^\wedge$. By construction the element $(1 + x)u_n$ in R/I^n is 1. Hence we see that u is the inverse of $1 + x$ in $R^\wedge$. This proves (2) which implies (1). Let $\mathfrak{m} \subset R^\wedge$ be a maximal ideal and set $M = R^\wedge/\mathfrak{m}$. If $IR^\wedge \not\subset \mathfrak{m}$ then $IM = M$. By Nakayama's Lemma 10.18.1 there exists an $f \in IR^\wedge$ such that $(1 + f)M = 0$. This is a contradiction with (2). □

Lemma 10.91.12. *Let A be a ring. Let $I = (f_1, \ldots, f_r)$ be a finitely generated ideal. If $M \to \lim M/f_i^n M$ is surjective for each i, then $M \to \lim M/I^n M$ is surjective.*

Proof. Note that $\lim M/I^n M = \lim M/(f_1^n, \ldots, f_r^n)M$ as $I^n \supset (f_1^n, \ldots, f_r^n) \supset I^{rn}$. An element ξ of $\lim M/(f_1^n, \ldots, f_r^n)M$ can be symbolically written as

$$\xi = \sum\nolimits_{n \geq 0} \sum\nolimits_i f_i^n x_{n,i}$$

with $x_{n,i} \in M$. If $M \to \lim M/f_i^n M$ is surjective, then there is an $x_i \in M$ mapping to $\sum x_{n,i} f_i^n$ in $\lim M/f_i^n M$. Then $x = \sum x_i$ maps to ξ in $\lim M/I^n M$. □

Lemma 10.91.13. *Let A be a ring. Let $I \subset J \subset A$ be ideals. If M is J-adically complete and I is finitely generated, then M is I-adically complete.*

Proof. Assume M is J-adically complete and I is finitely generated. We have $\bigcap I^n M = 0$ because $\bigcap J^n M = 0$. By Lemma 10.91.12 it suffices to prove the surjectivity of $M \to \lim M/I^n M$ in case I is generated by a single element. Say $I = (f)$. Let $x_n \in M$ with $x_{n+1} - x_n \in f^n M$. We have to show there exists an $x \in M$ such that $x_n - x \in f^n M$ for all n. As $x_{n+1} - x_n \in J^n M$ and as M is J-adically complete, there exists an element $x \in M$ such that $x_n - x \in J^n M$. Replacing x_n by $x_n - x$ we may assume that $x_n \in J^n M$. To finish the proof we will show that this implies $x_n \in I^n M$. Namely, write $x_n - x_{n+1} = f^n z_n$. Then

$$x_n = f^n(z_n + f z_{n+1} + f^2 z_{n+2} + \ldots)$$

The sum $z_n + f z_{n+1} + f^2 z_{n+2} + \ldots$ converges in M as $f^c \in J^c$. The sum $f^n(z_n + f z_{n+1} + f^2 z_{n+2} + \ldots)$ converges in M to x_n because the partial sums equal $x_n - x_{n+c}$ and $x_{n+c} \in J^{n+c} M$. □

Lemma 10.91.14. *Let R be a ring. Let I, J be ideals of R. Assume there exist integers $c, d > 0$ such that $I^c \subset J$ and $J^d \subset I$. Then completion with respect to I agrees with completion with respect to J for any R-module. In particular an R-module M is I-adically complete if and only if it is J-adically complete.*

Proof. Consider the system of maps $M/I^n M \to M/J^{\lfloor n/d \rfloor} M$ and the system of maps $M/J^m M \to M/I^{\lfloor m/c \rfloor} M$ to get mutually inverse maps between the completions. □

Lemma 10.91.15. *Let R be a ring. Let I be an ideal of R. Let M be an I-adically complete R-module, and let $K \subset M$ be an R-submodule. The following are equivalent*

1. *$K = \bigcap (K + I^n M)$ and*
2. *M/K is I-adically complete.*

Proof. Set $N = M/K$. By Lemma 10.91.1 the map $M = M^\wedge \to N^\wedge$ is surjective. Hence $N \to N^\wedge$ is surjective. It is easy to see that the kernel of $N \to N^\wedge$ is the module $\bigcap (K + I^n M)/K$. □

Lemma 10.91.16. *Let R be a ring. Let I be an ideal of R. Let M be an R-module. If (a) R is I-adically complete, (b) M is a finite R-module, and (c) $\bigcap I^n M = (0)$, then M is I-adically complete.*

Proof. By Lemma 10.91.1 the map $M = M \otimes_R R = M \otimes_R R^\wedge \to M^\wedge$ is surjective. The kernel of this map is $\bigcap I^n M$ hence zero by assumption. Hence $M \cong M^\wedge$ and M is complete. □

Lemma 10.91.17. *Let R be a ring. Let $I \subset R$ be an ideal. Let M be an R-module. Assume*

(1) *R is I-adically complete,*
(2) *$\bigcap_{n \geq 1} I^n M = (0)$, and*
(3) *M/IM is a finite R/I-module.*

Then M is a finite R-module.

Proof. Let $x_1, \ldots, x_n \in M$ be elements whose images in M/IM generate M/IM as a R/I-module. Denote $M' \subset M$ the R-submodule generated by $x_1, \ldots, x_n$. By Lemma 10.91.1 the map $(M')^\wedge \to M^\wedge$ is surjective. Since $\bigcap I^n M = 0$ we see in particular that $\bigcap I^n M' = (0)$. Hence by Lemma 10.91.16 we see that M' is complete, and we conclude that $M' \to M^\wedge$ is surjective. Finally, the kernel of $M \to M^\wedge$ is zero since it is equal to $\bigcap I^n M = (0)$. Hence we conclude that $M \cong M' \cong M^\wedge$ is finitely generated. □

Suppose $R \to S$ is a local homomorphism of local rings $(R, \mathfrak{m})$ and $(S, \mathfrak{n})$. Let $S^\wedge$ be the completion of S with respect to $\mathfrak{n}$. In general $S^\wedge$ is not the $\mathfrak{m}$-adic completion of S. If $\mathfrak{n}^t \subset \mathfrak{m}S$ for some $t \geq 1$ then we do have $S^\wedge = \lim S/\mathfrak{m}^n S$ by Lemma 10.91.14. In some cases this even implies that $S^\wedge$ is finite over $R^\wedge$.

Lemma 10.91.18. *Let $R \to S$ be a local homomorphism of local rings $(R, \mathfrak{m})$ and $(S, \mathfrak{n})$. Let $R^\wedge$, resp. $S^\wedge$ be the completion of R, resp. S with respect to $\mathfrak{m}$, resp. $\mathfrak{n}$. If $\mathfrak{m}$ and $\mathfrak{n}$ are finitely generated and $\dim_{\kappa(\mathfrak{m})} S/\mathfrak{m}S < \infty$, then*

(1) *$S^\wedge$ is equal to the $\mathfrak{m}$-adic completion of S, and*
(2) *$S^\wedge$ is a finite $R^\wedge$-module.*

Proof. We have $\mathfrak{m}S \subset \mathfrak{n}$ because $R \to S$ is a local ring map. The assumption $\dim_{\kappa(\mathfrak{m})} S/\mathfrak{m}S < \infty$ implies that $S/\mathfrak{m}S$ is an Artinian ring, see Lemma 10.49.2. Hence has dimension 0, see Lemma 10.57.4, hence $\mathfrak{n} = \sqrt{\mathfrak{m}S}$. This and the fact that $\mathfrak{n}$ is finitely generated implies that $\mathfrak{n}^t \subset \mathfrak{m}S$ for some $t \geq 1$. By Lemma 10.91.14 we see that $S^\wedge$ can be identified with the $\mathfrak{m}$-adic completion of S. As $\mathfrak{m}$ is finitely generated we see from Lemma 10.91.7 that $S^\wedge$ and $R^\wedge$ are $\mathfrak{m}$-adically complete. At this point we may apply Lemma 10.91.17 to $S^\wedge$ as an $R^\wedge$-module to conclude. □

Lemma 10.91.19. *Let R be a Noetherian ring. Let $R \to S$ be a finite ring map. Let $\mathfrak{p} \subset R$ be a prime and let $\mathfrak{q}_1, \ldots, \mathfrak{q}_m$ be the primes of S lying over $\mathfrak{p}$ (Lemma 10.33.19). Then*

$$R_\mathfrak{p}^\wedge \otimes_R S = S_{\mathfrak{q}_1}^\wedge \times \ldots \times S_{\mathfrak{q}_m}^\wedge$$

where the local rings $R_\mathfrak{p}$ and $S_{\mathfrak{q}_i}$ are completed with respect to their maximal ideals.

Proof. We may replace R by the localization $R_\mathfrak{p}$ and S by $S_\mathfrak{p} = S \otimes_R R_\mathfrak{p}$. Hence we may assume that R is a local Noetherian ring and that $\mathfrak{p} = \mathfrak{m}$ is its maximal ideal. The $\mathfrak{q}_i S_{\mathfrak{q}_i}$-adic completion $S_{\mathfrak{q}_i}^\wedge$ is equal to the $\mathfrak{m}$-adic completion by Lemma 10.91.18. For every $n \geq 1$ prime ideals of $S/\mathfrak{m}^n S$ are in 1-to-1 correspondence with the maximal ideals $\mathfrak{q}_1, \ldots, \mathfrak{q}_m$ of S (by going up for S over R, see Lemma 10.33.20). Hence $S/\mathfrak{m}^n S = \prod S_{\mathfrak{q}_i}/\mathfrak{m}^n S_{\mathfrak{q}_i}$ by Lemma 10.49.8 (using for example Proposition

10.57.6 to see that $S/\mathfrak{m}^n S$ is Artinian). Hence the $\mathfrak{m}$-adic completion $S^\wedge$ of S is equal to $\prod S^\wedge_{\mathfrak{q}_i}$. Finally, we have $R^\wedge \otimes_R S = S^\wedge$ by Lemma 10.91.2. □

Lemma 10.91.20. *Let R be a ring. Let $I \subset R$ be an ideal. Let $0 \to K \to P \to M \to 0$ be a short exact sequence of R-modules. If M is flat over R and M/IM is a projective R/I-module, then the sequence of I-adic completions*

$$0 \to K^\wedge \to P^\wedge \to M^\wedge \to 0$$

is a split exact sequence.

Proof. As M is flat, each of the sequences

$$0 \to K/I^n K \to P/I^n P \to M/I^n M \to 0$$

is short exact, see Lemma 10.36.11 and the sequence $0 \to K^\wedge \to P^\wedge \to M^\wedge \to 0$ is a short exact sequence, see Lemma 10.91.1. It suffices to show that we can find splittings $s_n : M/I^n M \to P/I^n P$ such that $s_{n+1} \bmod I^n = s_n$. We will construct these s_n by induction on n. Pick any splitting s_1, which exists as M/IM is a projective R/I-module. Assume given s_n for some $n > 0$. Set $P_{n+1} = \{x \in P \mid x \bmod I^n P \in \mathrm{Im}(s_n)\}$. The map $\pi : P_{n+1}/I^{n+1}P_{n+1} \to M/I^{n+1}M$ is surjective (details omitted). As $M/I^{n+1}M$ is projective as a R/I^{n+1}-module by Lemma 10.72.5 we may choose a section $t : M/I^{n+1}M \to P_{n+1}/I^{n+1}P_{n+1}$ of π. Setting s_{n+1} equal to the composition of t with the canonical map $P_{n+1}/I^{n+1}P_{n+1} \to P/I^{n+1}P$ works. □

10.92. Taking limits of modules

In this section we discuss what happens when we take a limit of modules.

Lemma 10.92.1. *Let A be a ring. Let $I \subset A$ be an ideal. Let (M_n) be an inverse system of A-modules. Set $M = \lim M_n$. If $M_n = M_{n+1}/I^n M_{n+1}$ and I is finitely generated then $M/I^n M = M_n$ and M is I-adically complete.*

Proof. As $M_{n+1} \to M_n$ is surjective, the map $M \to M_1$ is surjective. Pick $x_t \in M$, $t \in T$ mapping to generators of M_1. This gives a map $\bigoplus_{t\in T} A \to M$. Note that the images of x_t in M_n generate M_n for all n too. Consider the exact sequences

$$0 \to K_n \to \bigoplus\nolimits_{t\in T} A/I^n \to M_n \to 0$$

We claim the map $K_{n+1} \to K_n$ is surjective. Namely, if $y \in K_n$ choose a lift $y' \in \bigoplus_{t\in T} A/I^{n+1}$. Then y' maps to an element of $I^n M_{n+1}$ by our assumption $M_n = M_{n+1}/I^n M_{n+1}$. Hence we can modify our choice of y' by an element of $\bigoplus_{t\in T} I^n/I^{n+1}$ so that y' maps to zero in M_{n+1}. Then $y' \in K_{n+1}$ maps to y. Hence (K_n) is a sequence of modules with surjective transition maps and we obtain an exact sequence

$$0 \to \lim K_n \to \left(\bigoplus\nolimits_{t\in T} A\right)^\wedge \to M \to 0$$

by Lemma 10.82.1. Fix an integer m. As I is finitely generated, the completion with respect to I is complete (Lemma 10.91.7) which implies that $(\bigoplus_{t\in T} A)^\wedge/I^m(\bigoplus_{t\in T} A)^\wedge = \bigoplus_{t\in T} A/I^m$ (Lemma 10.91.6). We obtain a short exact sequence

$$(\lim K_n)/I^m(\lim K_n) \to \bigoplus\nolimits_{t\in T} A/I^m \to M/I^m M \to 0$$

Since $\lim K_n \to K_m$ is surjective we conclude that $M/I^m M = M_m$. It follows in particular that M is I-adically complete. □

10.93. Criteria for flatness

In this section we prove some important technical lemmas in the Noetherian case. We will (partially) generalize these to the non-Noetherian case in Section 10.122.

Lemma 10.93.1. *Suppose that $R \to S$ is a local homomorphism of Noetherian local rings. Denote $\mathfrak{m}$ the maximal ideal of R. Let $u : M \to N$ be a map of finite S-modules. Assume N flat over R. If $\overline{u} : M/\mathfrak{m}M \to N/\mathfrak{m}N$ is injective then u is injective. In this case $N/u(M)$ is flat over R.*

Proof. First we claim that $u_n : M/\mathfrak{m}^n M \to N/\mathfrak{m}^n N$ is injective for all $n \geq 1$. We proceed by induction, the base case given by assumption. By our assumption that N is flat over R we have a short exact sequence $0 \to N \otimes_R \mathfrak{m}^n/\mathfrak{m}^{n+1} \to N/\mathfrak{m}^{n+1}N \to N/\mathfrak{m}^n N \to 0$. Also, $N \otimes_R \mathfrak{m}^n/\mathfrak{m}^{n+1} = N/\mathfrak{m}N \otimes_{R/\mathfrak{m}} \mathfrak{m}^n/\mathfrak{m}^{n+1}$. We have a similar exact sequence $M \otimes_R \mathfrak{m}^n/\mathfrak{m}^{n+1} \to M/\mathfrak{m}^{n+1}M \to M/\mathfrak{m}^n M \to 0$ for M except we do not have the zero on the left. We also have $M \otimes_R \mathfrak{m}^n/\mathfrak{m}^{n+1} = M/\mathfrak{m}M \otimes_{R/\mathfrak{m}} \mathfrak{m}^n/\mathfrak{m}^{n+1}$. Thus the map u_{n+1} is injective as both u_n and the map $\overline{u} \otimes \mathrm{id}_{\mathfrak{m}^n/\mathfrak{m}^{n+1}}$ are.

Note that $\lim_n M/\mathfrak{m}^n M$ is the completion of the module M with respect to the ideal $I = \mathfrak{m}S$, and similarly for N. Since M and N are finite S-modules we have $M^\wedge = M \otimes S^\wedge$ and similarly for N, see Lemma 10.91.2. We conclude that $u \otimes 1 : M \otimes S^\wedge \to N \otimes S^\wedge$ is injective. Since $S^\wedge$ is faithfully flat over S, see Lemma 10.91.4, we conclude that u is injective, see Lemma 10.36.4.

Finally, we have to prove that $I \otimes_R N/u(M) \to N/u(M)$ is injective for every ideal $I \subset R$. Consider the diagram

$$\begin{array}{ccccccccc} & & 0 & & 0 & & 0 & & \\ & & \uparrow & & \uparrow & & \uparrow & & \\ & & M/IM & \to & N/IN & \to & N/(IN + u(M)) & \to & 0 \\ & & \uparrow & & \uparrow & & \uparrow & & \\ 0 & \to & M & \to & N & \to & N/u(M) & \to & 0 \\ & & \uparrow & & \uparrow & & \uparrow & & \\ & & M \otimes_R I & \to & N \otimes_R I & \to & N/u(M) \otimes_R I & \to & 0 \end{array}$$

The arrow $N \otimes_R I \to N$ is injective. Chasing through the diagram we see that it suffices to prove that M/IM injects into N/IN. Note that M/IM and N/IN are modules over the Noetherian ring S/IS, N/IN is flat over R/I and $u \bmod I : M/IM \to N/IN$ is injective module $\mathfrak{m}$ we may apply the result above to $u \bmod I$, and we win. □

Lemma 10.93.2. *Suppose that $R \to S$ is a flat and local ring homomorphism of Noetherian local rings. Denote $\mathfrak{m}$ the maximal ideal of R. Suppose $f \in S$ is a nonzerodivisor in $S/\mathfrak{m}S$. Then S/fS is flat over R, and f is a nonzerodivisor in S.*

Proof. Follows directly from Lemma 10.93.1. □

Lemma 10.93.3. *Suppose that $R \to S$ is a flat and local ring homomorphism of Noetherian local rings. Denote $\mathfrak{m}$ the maximal ideal of R. Suppose $f_1, \ldots, f_c$ is a sequence of elements of S such that the images $\overline{f}_1, \ldots, \overline{f}_c$ form a regular sequence in $S/\mathfrak{m}S$. Then $f_1, \ldots, f_c$ is a regular sequence in S and each of the quotients $S/(f_1, \ldots, f_i)$ is flat over R.*

Proof. Induction and Lemma 10.93.2. □

Lemma 10.93.4. *Let $R \to S$ be a local homomorphism of Noetherian local rings. Let $\mathfrak{m}$ be the maximal ideal of R. Let M be a finite S-modules. Suppose that (a) $M/\mathfrak{m}M$ is a free $S/\mathfrak{m}S$-module, and (b) M is flat over R. Then M is free and S is flat over R.*

Proof. Let $\overline{x}_1, \ldots, \overline{x}_n$ be a basis for the free module $M/\mathfrak{m}M$. Choose $x_1, \ldots, x_n \in M$ with x_i mapping to $\overline{x}_i$. Let $u : S^{\oplus n} \to M$ be the map which maps the ith standard basis vector to x_i. By Lemma 10.93.1 we see that u is injective. On the other hand, by Nakayama's Lemma 10.18.1 the map is surjective. The lemma follows. □

Lemma 10.93.5. *Let $R \to S$ be a local homomorphism of local Noetherian rings. Let $\mathfrak{m}$ be the maximal ideal of R. Let $0 \to F_e \to F_{e-1} \to \ldots \to F_0$ be a finite complex of finite S-modules. Assume that each F_i is R-flat, and that the complex $0 \to F_e/\mathfrak{m}F_e \to F_{e-1}/\mathfrak{m}F_{e-1} \to \ldots \to F_0/\mathfrak{m}F_0$ is exact. Then $0 \to F_e \to F_{e-1} \to \ldots \to F_0$ is exact, and moreover the module Coker$(F_1 \to F_0)$ is R-flat.*

Proof. By induction on e. If $e = 1$, then this is exactly Lemma 10.93.1. If $e > 1$, we see by Lemma 10.93.1 that $F_e \to F_{e-1}$ is injective and that $C = \text{Coker}(F_e \to F_{e-1})$ is a finite S-module flat over R. Hence we can apply the induction hypothesis to the complex $0 \to C \to F_{e-2} \to \ldots \to F_0$. We deduce that $C \to F_{e-2}$ is injective and the exactness of the complex follows, as well as the flatness of the cokernel of $F_1 \to F_0$. □

In the rest of this section we prove two versions of what is called the "*local criterion of flatness*". Note also the interesting Lemma 10.122.1 below.

Lemma 10.93.6. *Let R be a local ring with maximal ideal $\mathfrak{m}$ and residue field $\kappa = R/\mathfrak{m}$. Let M be an R-module. If $Tor_1^R(\kappa, M) = 0$, then for every finite length R-module N we have $Tor_1^R(N, M) = 0$.*

Proof. By descending induction on the length of N. If the length of N is 1, then $N \cong \kappa$ and we are done. If the length of N is more than 1, then we can fit N into a short exact sequence $0 \to N' \to N \to N'' \to 0$ where N', N'' are finite length R-modules of smaller length. The vanishing of $\text{Tor}_1^R(N, M)$ follows from the vanishing of $\text{Tor}_1^R(N', M)$ and $\text{Tor}_1^R(N'', M)$ (induction hypothesis) and the long exact sequence of Tor groups, see Lemma 10.70.2. □

Lemma 10.93.7 (Local criterion for flatness)**.** *Let $R \to S$ be a local homomorphism of local Noetherian rings. Let $\mathfrak{m}$ be the maximal ideal of R, and let $\kappa = R/\mathfrak{m}$. Let M be a finite S-module. If $Tor_1^R(\kappa, M) = 0$, then M is flat over R.*

Proof. Let $I \subset R$ be an ideal. By Lemma 10.36.4 it suffices to show that $I \otimes_R M \to M$ is injective. By Remark 10.70.8 we see that this kernel is equal to $\text{Tor}_1^R(M, R/I)$. By Lemma 10.93.6 we see that $J \otimes_R M \to M$ is injective for all ideals of finite colength.

Choose $n >> 0$ and consider the following short exact sequence

$$0 \to I \cap \mathfrak{m}^n \to I \oplus \mathfrak{m}^n \to I + \mathfrak{m}^n \to 0$$

This is a sub sequence of the short exact sequence $0 \to R \to R^{\oplus 2} \to R \to 0$. Thus we get the diagram

$$\begin{array}{ccccc} (I \cap \mathfrak{m}^n) \otimes_R M & \longrightarrow & I \otimes_R M \oplus \mathfrak{m}^n \otimes_R M & \longrightarrow & (I + \mathfrak{m}^n) \otimes_R M \\ \downarrow & & \downarrow & & \downarrow \\ M & \longrightarrow & M \oplus M & \longrightarrow & M \end{array}$$

Note that $I + \mathfrak{m}^n$ and $\mathfrak{m}^n$ are ideals of finite colength. Thus a diagram chase shows that $\mathrm{Ker}((I \cap \mathfrak{m}^n) \otimes_R M \to M) \to \mathrm{Ker}(I \otimes_R M \to M)$ is surjective. We conclude in particular that $K = \mathrm{Ker}(I \otimes_R M \to M)$ is contained in the image of $(I \cap \mathfrak{m}^n) \otimes_R M$ in $I \otimes_R M$. By Artin-Rees, Lemma 10.47.4 we see that K is contained in $\mathfrak{m}^{n-c}(I \otimes_R M)$ for some $c > 0$ and all $n >> 0$. Since $I \otimes_R M$ is a finite S-module (!) and since S is Noetherian, we see that this implies $K = 0$. Namely, the above implies K maps to zero in the $\mathfrak{m}S$-adic completion of $I \otimes_R M$. But the map from S to its $\mathfrak{m}S$-adic completion is faithfully flat by Lemma 10.91.4. Hence $K = 0$, as desired. □

In the following we often encounter the conditions "M/IM is flat over R/I and $\mathrm{Tor}_1^R(R/I, M) = 0$". The following lemma gives some consequences of these conditions (it is a generalization of Lemma 10.93.6).

Lemma 10.93.8. *Let R be a ring. Let $I \subset R$ be an ideal. Let M be an R-module. If M/IM is flat over R/I and $Tor_1^R(R/I, M) = 0$ then*

1. *M/I^nM is flat over R/I^n for all $n \geq 1$, and*
2. *for any module N which is annihilated by I^m for some $m \geq 0$ we have $Tor_1^R(N, M) = 0$.*

In particular, if I is nilpotent, then M is flat over R.

Proof. Assume M/IM is flat over R/I and $\mathrm{Tor}_1^R(R/I, M) = 0$. Let N be an R/I-module. Choose a short exact sequence

$$0 \to K \to \bigoplus\nolimits_{i \in I} R/I \to N \to 0$$

By the long exact sequence of Tor and the vanishing of $\mathrm{Tor}_1^R(R/I, M)$ we get

$$0 \to \mathrm{Tor}_1^R(N, M) \to K \otimes_R M \to (\bigoplus\nolimits_{i \in I} R/I) \otimes_R M \to N \otimes_R M \to 0$$

But since K, $\bigoplus_{i \in I} R/I$, and N are all annihilated by I we see that

$$K \otimes_R M = K \otimes_{R/I} M/IM,$$
$$(\bigoplus\nolimits_{i \in I} R/I) \otimes_R M = (\bigoplus\nolimits_{i \in I} R/I) \otimes_{R/I} M/IM,$$
$$N \otimes_R M = N \otimes_{R/I} M/IM.$$

As M/IM is flat over R/I we conclude that

$$0 \to K \otimes_{R/I} M/IM \to (\bigoplus\nolimits_{i \in I} R/I) \otimes_{R/I} M/IM \to N \otimes_{R/} M/IM \to 0$$

is exact. Combining this with the above we conclude that $\mathrm{Tor}_1^R(N, M) = 0$ for any R-module N annihilated by I.

In particular, if we apply this to the module I/I^2, then we conclude that the sequence

$$0 \to I^2 \otimes_R M \to I \otimes_R M \to I/I^2 \otimes_R M \to 0$$

is short exact. This implies that $I^2 \otimes_R M \to M$ is injective and it implies that $I/I^2 \otimes_{R/I} M/IM = IM/I^2M$.

Let us prove that M/I^2M is flat over R/I^2. Let $I^2 \subset J$ be an ideal. We have to show that $J/I^2 \otimes_{R/I^2} M/I^2M \to M/I^2M$ is injective, see Lemma 10.36.4. As M/IM is flat over R/I we know that the map $(I+J)/I \otimes_{R/I} M/IM \to M/IM$ is injective. The sequence

$$(I \cap J)/I^2 \otimes_{R/I^2} M/I^2M \to J/I^2 \otimes_{R/I^2} M/I^2M \to (I+J)/I \otimes_{R/I} M/IM \to 0$$

is exact, as you get it by tensoring the exact sequence $0 \to (I \cap J) \to J \to (I+J)/I \to 0$ by M/I^2M. Hence suffices to prove the injectivity of the map $(I \cap J)/I^2 \otimes_{R/I} M/IM \to IM/I^2M$. However, the map $(I \cap J)/I^2 \to I/I^2$ is injective and as M/IM is flat over R/I the map $(I \cap J)/I^2 \otimes_{R/I} M/IM \to I/I^2 \otimes_{R/I} M/IM$ is injective. Since we have previously seen that $I/I^2 \otimes_{R/I} M/IM = IM/I^2M$ we obtain the desired injectivity.

Hence we have proven that the assumptions imply: (a) $\mathrm{Tor}_1^R(N, M) = 0$ for all N annihilated by I, (b) $I^2 \otimes_R M \to M$ is injective, and (c) M/I^2M is flat over R/I^2. Thus we can continue by induction to get the same results for I^n for all $n \geq 1$. □

Lemma 10.93.9 (Variant of the local criterion)**.** *Let $R \to S$ be a local homomorphism of Noetherian local rings. Let $I \neq R$ be an ideal in R. Let M be a finite S-module. If $Tor_1^R(M, R/I) = 0$ and M/IM is flat over R/I, then M is flat over R.*

Proof. First proof: By Lemma 10.93.8 we see that $\mathrm{Tor}_1^R(\kappa, M)$ is zero where κ is the residue field of R. Hence we see that M is flat over R by Lemma 10.93.7.

Second proof: Let $\mathfrak{m}$ be the maximal ideal of R. We will show that $\mathfrak{m} \otimes_R M \to M$ is injective, and then apply Lemma 10.93.7. Suppose that $\sum f_i \otimes x_i \in \mathfrak{m} \otimes_R M$ and that $\sum f_i x_i = 0$ in M. By the equational criterion for flatness Lemma 10.36.10 applied to M/IM over R/I we see there exist $\overline{a}_{ij} \in R/I$ and $\overline{y}_j \in M/IM$ such that $x_i \bmod IM = \sum_j \overline{a}_{ij}\overline{y}_j$ and $0 = \sum_i (f_i \bmod I)\overline{a}_{ij}$. Let $a_{ij} \in R$ be a lift of $\overline{a}_{ij}$ and similarly let $y_j \in M$ be a lift of $\overline{y}_j$. Then we see that

$$\begin{aligned} \sum f_i \otimes x_i &= \sum f_i \otimes x_i + \sum f_i a_{ij} \otimes y_j - \sum f_i \otimes a_{ij} y_j \\ &= \sum f_i \otimes (x_i - \sum a_{ij} y_j) + \sum (\sum f_i a_{ij}) \otimes y_j \end{aligned}$$

Since $x_i - \sum a_{ij} y_j \in IM$ and $\sum f_i a_{ij} \in I$ we see that there exists an element in $I \otimes_R M$ which maps to our given element $\sum f_i \otimes x_i$ in $\mathfrak{m} \otimes_R M$. But $I \otimes_R M \to M$ is injective by assumption (see Remark 10.70.8) and we win. □

In particular, in the situation of Lemma 10.93.9, suppose that $I = (x)$ is generated by a single element x which is a nonzerodivisor in R. Then $\mathrm{Tor}_1^R(M, R/(x)) = (0)$ if and only if x is a nonzerodivisor on M.

Lemma 10.93.10. *Let $R \to S$ be a ring map. Let $I \subset R$ be an ideal. Let M be an S-module. Assume*

1. *R is a Noetherian ring,*
2. *S is a Noetherian ring,*
3. *M is a finite S-module, and*
4. *for each $n \geq 1$ the module M/I^nM is flat over R/I^n.*

Then for every $\mathfrak{q} \in V(IS)$ the localization $M_\mathfrak{q}$ is flat over R. In particular, if S is local and IS is contained in its maximal ideal, then M is flat over R.

Proof. We are going to use Lemma 10.93.9. By assumption M/IM is flat over R/I. Hence it suffices to check that $\mathrm{Tor}_1^R(M, R/I)$ is zero on localization at $\mathfrak{q}$. By Remark 10.70.8 this Tor group is equal to $K = \mathrm{Ker}(I \otimes_R M \to M)$. We know for each $n \geq 1$ that the kernel $\mathrm{Ker}(I/I^n \otimes_{R/I^n} M/I^nM \to M/I^nM)$ is zero. Since there is a module map $I/I^n \otimes_{R/I^n} M/I^nM \to (I \otimes_R M)/I^{n-1}(I \otimes_R M)$ we conclude that $K \subset I^{n-1}(I \otimes_R M)$ for each n. By the Artin-Rees lemma, and more precisely Lemma 10.47.7 we conclude that $K_\mathfrak{q} = 0$, as desired. □

Lemma 10.93.11. *Let $R \to R' \to R''$ be ring maps. Let M be an R-module. Suppose that $M \otimes_R R'$ is flat over R'. Then the natural map $Tor_1^R(M, R') \otimes_{R'} R'' \to Tor_1^R(M, R'')$ is onto.*

Proof. Let $F_\bullet$ be a free resolution of M over R. The complex $F_2 \otimes_R R' \to F_1 \otimes_R R' \to F_0 \otimes_R R'$ computes $\mathrm{Tor}_1^R(M, R')$. The complex $F_2 \otimes_R R'' \to F_1 \otimes_R R'' \to F_0 \otimes_R R''$ computes $\mathrm{Tor}_1^R(M, R'')$. Note that $F_i \otimes_R R' \otimes_{R'} R'' = F_i \otimes_R R''$. Let $K' = \mathrm{Ker}(F_1 \otimes_R R' \to F_0 \otimes_R R')$ and similarly $K'' = \mathrm{Ker}(F_1 \otimes_R R'' \to F_0 \otimes_R R'')$. Thus we have an exact sequence

$$0 \to K' \to F_1 \otimes_R R' \to F_0 \otimes_R R' \to M \otimes_R R' \to 0.$$

By the assumption that $M \otimes_R R'$ is flat over R', the sequence $0 \to K' \otimes_{R'} R'' \to F_1 \otimes_R R'' \to F_0 \otimes_R R'' \to M \otimes_R R'' \to 0$ is still exact. This means that $K'' = K' \otimes_{R'} R''$. Since $\mathrm{Tor}_1^R(M, R')$ is a quotient of K' and $\mathrm{Tor}_1^R(M, R'')$ is a quotient of K'' we win. □

Lemma 10.93.12. *Let $R \to R'$ be a ring map. Let $I \subset R$ be an ideal and $I' = IR'$. Let M be an R-module and set $M' = M \otimes_R R'$. The natural map $Tor_1^R(R'/I', M) \to Tor_1^{R'}(R'/I', M')$ is surjective.*

Proof. Let $F_2 \to F_1 \to F_0 \to M \to 0$ be a free resolution of M over R. Set $F_i' = F_i \otimes_R R'$. The sequence $F_2' \to F_1' \to F_0' \to M' \to 0$ may no longer be exact at F_1'. A free resolution of M' over R' therefore looks like

$$F_2' \oplus F_2'' \to F_1' \to F_0' \to M' \to 0$$

for a suitable free module F_2'' over R'. Next, note that $F_i \otimes_R R'/I' = F_i'/IF_i' = F_i'/I'F_i'$. So the complex $F_2'/I'F_2' \to F_1'/I'F_1' \to F_0'/I'F_0'$ computes $\mathrm{Tor}_1^R(M, R'/I')$. On the other hand $F_i' \otimes_{R'} R'/I' = F_i'/I'F_i'$ and similarly for F_2''. Thus the complex $F_2'/I'F_2' \oplus F_2''/I'F_2'' \to F_1'/I'F_1' \to F_0'/I'F_0'$ computes $\mathrm{Tor}_1^{R'}(M', R'/I')$. Since the vertical map on complexes

$$\begin{array}{ccccc} F_2'/I'F_2' & \longrightarrow & F_1'/I'F_1' & \longrightarrow & F_0'/I'F_0' \\ \downarrow & & \downarrow & & \downarrow \\ F_2'/I'F_2' \oplus F_2''/I'F_2'' & \longrightarrow & F_1'/I'F_1' & \longrightarrow & F_0'/I'F_0' \end{array}$$

clearly induces a surjection on cohomology we win. □

Lemma 10.93.13. *Let*

$$\begin{array}{ccc} S & \longrightarrow & S' \\ \uparrow & & \uparrow \\ R & \longrightarrow & R' \end{array}$$

be a commutative diagram of local homomorphisms of local Noetherian rings. Let $I \subset R$ be an ideal. Let M be an S-module. Denote $I' = IR'$ and $M' = M \otimes_S S'$. Assume that

(1) *S' is a localization of the tensor product $S \otimes_R R'$,*
(2) *M/IM is flat over R/I,*
(3) *$Tor_1^R(M, R/I) \to Tor_1^{R'}(M', R'/I')$ is zero.*

Then M' is flat over R'.

Proof. Since S' is a localization of $S \otimes_R R'$ we see that M' is a localization of $M \otimes_R R'$. Note that by Lemma 10.36.6 the module $M/IM \otimes_{R/I} R'/I' = M \otimes_R R'/I'(M \otimes_R R')$ is flat over R'/I'. Hence also $M'/I'M'$ is flat over R'/I' as the localization of a flat module is flat. By Lemma 10.93.9 it suffices to show that $\mathrm{Tor}_1^{R'}(M', R'/I')$ is zero. Since M' is a localization of $M \otimes_R R'$, the last assumption implies that it suffices to show that $\mathrm{Tor}_1^R(M, R/I) \otimes_R R' \to \mathrm{Tor}_1^{R'}(M \otimes_R R', R'/I')$ is surjective.

By Lemma 10.93.12 we see that $\mathrm{Tor}_1^R(M, R'/I') \to \mathrm{Tor}_1^{R'}(M \otimes_R R', R'/I')$ is surjective. So now it suffices to show that $\mathrm{Tor}_1^R(M, R/I) \otimes_R R' \to \mathrm{Tor}_1^R(M, R'/I')$ is surjective. This follows from Lemma 10.93.11 by looking at the ring maps $R \to R/I \to R'/I'$ and the module M. □

Please compare the lemma below to Lemma 10.95.8 (the case of a nilpotent ideal) and Lemma 10.122.8 (the case of finitely presented algebras).

Lemma 10.93.14 (Critère de platitude par fibres; Noetherian case)**.** *Let R, S, S' be Noetherian local rings and let $R \to S \to S'$ be local ring homomorphisms. Let $\mathfrak{m} \subset R$ be the maximal ideal. Let M be an S'-module. Assume*

(1) *The module M is finite over S'.*
(2) *The module M is not zero.*
(3) *The module $M/\mathfrak{m}M$ is a flat $S/\mathfrak{m}S$-module.*
(4) *The module M is a flat R-module.*

Then S is flat over R and M is a flat S-module.

Proof. Set $I = \mathfrak{m}S \subset S$. Then we see that M/IM is a flat S/I-module because of (3). Since $\mathfrak{m} \otimes_R S' \to I \otimes_S S'$ is surjective we see that also $\mathfrak{m} \otimes_R M \to I \otimes_S M$ is surjective. Consider

$$\mathfrak{m} \otimes_R M \to I \otimes_S M \to M.$$

As M is flat over R the composition is injective and so both arrows are injective. In particular $\mathrm{Tor}_1^S(S/I, M) = 0$ see Remark 10.70.8. By Lemma 10.93.9 we conclude that M is flat over S. Note that since $M/\mathfrak{m}_{S'}M$ is not zero by Nakayama's Lemma 10.18.1 we see that actually M is faithfully flat over S by Lemma 10.36.14 (since it forces $M/\mathfrak{m}_S M \neq 0$).

Consider the exact sequence $0 \to \mathfrak{m} \to R \to \kappa \to 0$. This gives an exact sequence $0 \to \mathrm{Tor}_1^R(\kappa, S) \to \mathfrak{m} \otimes_R S \to I \to 0$. Since M is flat over S this gives an exact

sequence $0 \to \mathrm{Tor}_1^R(\kappa, S) \otimes_S M \to \mathfrak{m} \otimes_R M \to I \otimes_S M \to 0$. By the above this implies that $\mathrm{Tor}_1^R(\kappa, S) \otimes_S M = 0$. Since M is faithfully flat over S this implies that $\mathrm{Tor}_1^R(\kappa, S) = 0$ and we conclude that S is flat over R by Lemma 10.93.7. □

10.94. Base change and flatness

Some lemmas which deal with what happens with flatness when doing a base change.

Lemma 10.94.1. *Let*

$$\begin{array}{ccc} S & \longrightarrow & S' \\ \uparrow & & \uparrow \\ R & \longrightarrow & R' \end{array}$$

be a commutative diagram of local homomorphisms of local rings. Assume that S' is a localization of the tensor product $S \otimes_R R'$. Let M be an S-module and set $M' = S' \otimes_S M$.

(1) *If M is flat over R then M' is flat over R'.*
(2) *If M' is flat over R' and $R \to R'$ is flat then M is flat over R.*

In particular we have

(3) *If S is flat over R then S' is flat over R'.*
(4) *If $R' \to S'$ and $R \to R'$ are flat then S is flat over R.*

Proof. Proof of (1). If M is flat over R, then $M \otimes_R R'$ is flat over R' by Lemma 10.36.6. If $W \subset S \otimes_R R'$ is the multiplicative subset such that $W^{-1}(S \otimes_R R') = S'$ then $M' = W^{-1}(M \otimes_R R')$. Hence M' is flat over R' as the localization of a flat module, see Lemma 10.36.19 part (5). This proves (1) and in particular, we see that (3) holds.

Proof of (2). Suppose that M' is flat over R' and $R \to R'$ is flat. By (3) applied to the diagram reflected in the northwest diagonal we see that $S \to S'$ is flat. Thus $S \to S'$ is faithfully flat by Lemma 10.36.16. We are going to use the criterion of Lemma 10.36.4 (3) to show that M is flat. Let $I \subset R$ be an ideal. If $I \otimes_R M \to M$ has a kernel, so does $(I \otimes_R M) \otimes_S S' \to M \otimes_S S' = M'$. Note that $I \otimes_R R' = IR'$ as $R \to R'$ is flat, and that

$$(I \otimes_R M) \otimes_S S' = (I \otimes_R R') \otimes_{R'} (M \otimes_S S') = IR' \otimes_{R'} M'.$$

From flatness of M' over R' we conclude that this maps injectively into M'. This concludes the proof of (2), and hence (4) is true as well. □

10.95. Flatness criteria over Artinian rings

We discuss some flatness criteria for modules over Artinian rings. Note that an Artinian local ring has a nilpotent maximal ideal so that the following two lemmas apply to Artinian local rings.

Lemma 10.95.1. *Let $(R, \mathfrak{m})$ be a local ring with nilpotent maximal ideal $\mathfrak{m}$. Let M be a flat R-module. If A is a set and $x_\alpha \in M$, $\alpha \in A$ is a collection of elements of M, then the following are equivalent:*

(1) *$\{\overline{x}_\alpha\}_{\alpha \in A}$ forms a basis for the vector space $M/\mathfrak{m}M$ over $R/\mathfrak{m}$, and*
(2) *$\{x_\alpha\}_{\alpha \in A}$ forms a basis for M over R.*

Proof. The implication (2) $\Rightarrow$ (1) is immediate. We will prove the other implication by using induction on n to show that $\{x_\alpha\}_{\alpha\in A}$ forms a basis for $M/\mathfrak{m}^n M$ over $R/\mathfrak{m}^n$. The case $n = 1$ holds by assumption (1). Assume the statement holds for some $n \geq 1$. By Nakayama's Lemma 10.18.1 the elements x_α generate M, in particular $M/\mathfrak{m}^{n+1}M$. The exact sequence $0 \to \mathfrak{m}^n/\mathfrak{m}^{n+1} \to R/\mathfrak{m}^{n+1} \to R/\mathfrak{m}^n \to 0$ gives on tensoring with M the exact sequence

$$0 \to \mathfrak{m}^n M/\mathfrak{m}^{n+1}M \to M/\mathfrak{m}^{n+1}M \to M/\mathfrak{m}^n M \to 0$$

Here we are using that M is flat. Moreover, we have $\mathfrak{m}^n M/\mathfrak{m}^{n+1}M = M/\mathfrak{m}M \otimes_{R/\mathfrak{m}} \mathfrak{m}^n/\mathfrak{m}^{n+1}$ by flatness of M again. Now suppose that $\sum f_\alpha x_\alpha = 0$ in $M/\mathfrak{m}^{n+1}M$. Then by induction hypothesis $f_\alpha \in \mathfrak{m}^n$ for each α. By the short exact sequence above we then conclude that $\sum \overline{f}_\alpha \otimes \overline{x}_\alpha$ is zero in $\mathfrak{m}^n/\mathfrak{m}^{n+1} \otimes_{R/\mathfrak{m}} M/\mathfrak{m}M$. Since $\overline{x}_\alpha$ forms a basis we conclude that each of the congruence classes $\overline{f}_\alpha \in \mathfrak{m}^n/\mathfrak{m}^{n+1}$ is zero and we win. □

Lemma 10.95.2. *Let R be a local ring with nilpotent maximal ideal. Let M be an R-module. The following are equivalent*

(1) *M is flat over R,*
(2) *M is a free R-module, and*
(3) *M is a projective R-module.*

Proof. Since any projective module is flat (as a direct summand of a free module) and every free module is projective, it suffices to prove that a flat module is free. Let M be a flat module. Let A be a set and let $x_\alpha \in M$, $\alpha \in A$ be elements such that $\overline{x_\alpha} \in M/\mathfrak{m}M$ forms a basis over the residue field of R. By Lemma 10.95.1 the x_α are a basis for M over R and we win. □

Lemma 10.95.3. *Let R be a ring. Let $I \subset R$ be an ideal. Let M be an R-module. Let A be a set and let $x_\alpha \in M$, $\alpha \in A$ be a collection of elements of M. Assume*

(1) *I is nilpotent,*
(2) *$\{\overline{x}_\alpha\}_{\alpha\in A}$ forms a basis for M/IM over R/I, and*
(3) *$Tor_1^R(R/I, M) = 0$.*

Then M is free on $\{x_\alpha\}_{\alpha\in A}$ over R.

Proof. Let R, I, M, $\{x_\alpha\}_{\alpha\in A}$ be as in the lemma and satisfy assumptions (1), (2), and (3). By Nakayama's Lemma 10.18.1 the elements x_α generate M over R. The assumption $\mathrm{Tor}_1^R(R/I, M) = 0$ implies that we have a short exact sequence

$$0 \to I \otimes_R M \to M \to M/IM \to 0.$$

Let $\sum f_\alpha x_\alpha = 0$ be a relation in M. By choice of x_α we see that $f_\alpha \in I$. Hence we conclude that $\sum f_\alpha \otimes x_\alpha = 0$ in $I \otimes_R M$. The map $I \otimes_R M \to I/I^2 \otimes_{R/I} M/IM$ and the fact that $\{x_\alpha\}_{\alpha\in A}$ forms a basis for M/IM implies that $f_\alpha \in I^2$! Hence we conclude that there are no relations among the images of the x_α in M/I^2M. In other words, we see that M/I^2M is free with basis the images of the x_α. Using the map $I \otimes_R M \to I/I^3 \otimes_{R/I^2} M/I^2M$ we then conclude that $f_\alpha \in I^3$! And so on. Since $I^n = 0$ for some n by assumption (1) we win. □

Lemma 10.95.4. *Let $\varphi : R \to R'$ be a ring map. Let $I \subset R$ be an ideal. Let M be an R-module. Assume*

(1) *M/IM is flat over R/I, and*

(2) $R' \otimes_R M$ *is flat over* R'.

Set $I_2 = \varphi^{-1}(\varphi(I^2)R')$. *Then* M/I_2M *is flat over* R/I_2.

Proof. We may replace R, M, and R' by R/I_2, M/I_2M, and $R'/\varphi(I)^2R'$. Then $I^2 = 0$ and φ is injective. By Lemma 10.93.8 and the fact that $I^2 = 0$ it suffices to prove that $\mathrm{Tor}_1^R(R/I, M) = K = \mathrm{Ker}(I \otimes_R M \to M)$ is zero. Set $M' = M \otimes_R R'$ and $I' = IR'$. By assumption the map $I' \otimes_{R'} M' \to M'$ is injective. Hence K maps to zero in

$$I' \otimes_{R'} M' = I' \otimes_R M = I' \otimes_{R/I} M/IM.$$

Then $I \to I'$ is an injective map of R/I-modules. Since M/IM is flat over R/I the map

$$I \otimes_{R/I} M/IM \longrightarrow I' \otimes_{R/I} M/IM$$

is injective. This implies that K is zero in $I \otimes_R M = I \otimes_{R/I} M/IM$ as desired. □

Lemma 10.95.5. *Let* $\varphi : R \to R'$ *be a ring map. Let* $I \subset R$ *be an ideal. Let* M *be an* R*-module. Assume*

(1) I *is nilpotent,*
(2) $R \to R'$ *is injective,*
(3) M/IM *is flat over* R/I*, and*
(4) $R' \otimes_R M$ *is flat over* R'.

Then M *is flat over* R.

Proof. Define inductively $I_1 = I$ and $I_{n+1} = \varphi^{-1}(\varphi(I_n)^2R')$ for $n \geq 1$. Note that by Lemma 10.95.4 we find that M/I_nM is flat over R/I_n for each $n \geq 1$. It is clear that $\varphi(I_n) \subset \varphi(I)^{2^n}R'$. Since I is nilpotent we see that $\varphi(I_n) = 0$ for some n. As φ is injective we conclude that $I_n = 0$ for some n and we win. □

Here is the local Artinian version of the local criterion for flatness.

Lemma 10.95.6. *Let* R *be an Artinian local ring. Let* M *be an* R*-module. Let* $I \subset R$ *be a proper ideal. The following are equivalent*

(1) M *is flat over* R*, and*
(2) M/IM *is flat over* R/I *and* $Tor_1^R(R/I, M) = 0$.

Proof. The implication (1) $\Rightarrow$ (2) follows immediately from the definitions. Assume M/IM is flat over R/I and $\mathrm{Tor}_1^R(R/I, M) = 0$. By Lemma 10.95.2 this implies that M/IM is free over R/I. Pick a set A and elements $x_\alpha \in M$ such that the images in M/IM form a basis. By Lemma 10.95.3 we conclude that M is free and in particular flat. □

It turns out that flatness descends along injective homomorphism whose source is an Artinian ring.

Lemma 10.95.7. *Let* $R \to S$ *be a ring map. Let* M *be an* R*-module. Assume*

(1) R *is Artinian*
(2) $R \to S$ *is injective, and*
(3) $M \otimes_R S$ *is a flat* S*-module.*

Then M *is a flat* R*-module.*

Proof. First proof: Let $I \subset R$ be the radical of R. Then I is nilpotent and M/IM is flat over R/I as R/I is a product of fields, see Section 10.49. Hence M is flat by an application of Lemma 10.95.5.

Second proof: By Lemma 10.49.8 we may write $R = \prod R_i$ as a finite product of local Artinian rings. This induces similar product decompositions for both R and S. Hence we reduce to the case where R is local Artinian (details omitted).

Assume that $R \to S$, M are as in the lemma satisfying (1), (2), and (3) and in addition that R is local with maximal ideal $\mathfrak{m}$. Let A be a set and $x_\alpha \in A$ be elements such that $\overline{x}_\alpha$ forms a basis for $M/\mathfrak{m}M$ over $R/\mathfrak{m}$. By Nakayama's Lemma 10.18.1 we see that the elements x_α generate M as an R-module. Set $N = S \otimes_R M$ and $I = \mathfrak{m}S$. Then $\{1 \otimes x_\alpha\}_{\alpha \in A}$ is a family of elements of N which form a basis for N/IN. Moreover, since N is flat over S we have $\mathrm{Tor}_1^S(S/I, N) = 0$. Thus we conclude from Lemma 10.95.3 that N is free on $\{1 \otimes x_\alpha\}_{\alpha \in A}$. The injectivity of $R \to S$ then guarantees that there cannot be a nontrivial relation among the x_α with coefficients in R. □

Please compare the lemma below to Lemma 10.93.14 (the case of Noetherian local rings) and Lemma 10.122.8 (the case of finitely presented algebras).

Lemma 10.95.8 (Critère de platitude par fibres: Nilpotent case)**.** *Let*

$$\begin{array}{ccc} S & \longrightarrow & S' \\ & \nwarrow \quad \nearrow & \\ & R & \end{array}$$

be a commutative diagram in the category of rings. Let $I \subset R$ be a nilpotent ideal and M an S'-module. Assume

(1) *The module M/IM is a flat S/IS-module.*
(2) *The module M is a flat R-module.*

Then M is a flat S-module and $S_\mathfrak{q}$ is flat over R for every $\mathfrak{q} \subset S$ such that $M \otimes_S \kappa(\mathfrak{q})$ is nonzero.

Proof. As M is flat over R tensoring with the short exact sequence $0 \to I \to R \to R/I \to 0$ gives a short exact sequence

$$0 \to I \otimes_R M \to M \to M/IM \to 0.$$

Note that $I \otimes_R M \to IS \otimes_S M$ is surjective. Combined with the above this means both maps in

$$I \otimes_R M \to IS \otimes_S M \to M$$

are injective. Hence $\mathrm{Tor}_1^S(IS, M) = 0$ (see Remark 10.70.8) and we conclude that M is a flat S-module by Lemma 10.93.8. To finish we need to show that $S_\mathfrak{q}$ is flat over R for any prime $\mathfrak{q} \subset S$ such that $M \otimes_S \kappa(\mathfrak{q})$ is nonzero. This follows from Lemma 10.36.14 and 10.36.9. □

10.96. What makes a complex exact?

Some of this material can be found in the paper [**BE73**] by Buchsbaum and Eisenbud.

Situation 10.96.1. Here R is a ring, and we have a complex

$$0 \to R^{n_e} \xrightarrow{\varphi_e} R^{n_{e-1}} \xrightarrow{\varphi_{e-1}} \ldots \xrightarrow{\varphi_{i+1}} R^{n_i} \xrightarrow{\varphi_i} R^{n_{i-1}} \xrightarrow{\varphi_{i-1}} \ldots \xrightarrow{\varphi_1} R^{n_0}$$

In other words we require $\varphi_i \circ \varphi_{i+1} = 0$ for $i = 1, \ldots, e-1$.

Lemma 10.96.2. *In Situation 10.96.1. Suppose R is a local ring with maximal ideal $\mathfrak{m}$. Suppose that for some i, $e \leq i \leq 1$ some matrix coefficient of the map φ_i is invertible. Then the complex $0 \to R^{n_e} \to R^{n_{e-1}} \to \ldots \to R^{n_0}$ is isomorphic to the direct sum of a complex $0 \to R^{n_e} \to \ldots \to R^{n_i - 1} \to R^{n_{i-1}-1} \to \ldots \to R^{n_0}$ and the complex $0 \to 0 \to \ldots \to R \to R \to 0 \to \ldots \to 0$ where the map $R \to R$ is the identity map.*

Proof. The assumption means, after a change of basis of R^{n_i} and $R^{n_{i-1}}$ that the first basis vector of R^{n_i} is mapped via φ_i to the first basis vector of $R^{n_{i-1}}$. Let e_j denote the jth basis vector of R^{n_i} and f_k the kth basis vector of $R^{n_{i-1}}$. Write $\varphi_i(e_j) = \sum a_{jk} f_k$. So $a_{1k} = 0$ unless $k = 1$ and $a_{11} = 1$. Change basis on R^{n_i} again by setting $e'_j = e_j - a_{j1}e_1$ for $j > 1$. After this change of coordinates we have $a_{j1} = 0$ for $j > 1$. Note the image of $R^{n_{i+1}} \to R^{n_i}$ is contained in the subspace spanned by e_j, $j > 1$. Note also that $R^{n_{i-1}} \to R^{n_{i-2}}$ has to annihilate f_1 since it is in the image. These conditions and the shape of the matrix (a_{jk}) for φ_i imply the lemma. □

Let us say that an acyclic complex of the form $\ldots \to 0 \to R \to R \to 0 \to \ldots$ is *trivial*. The lemma above clearly says that any finite complex of finite free modules over a local ring is up to direct sums with trivial complexes the same as a complex all of whose maps have all matrix coefficients in the maximal ideal.

Lemma 10.96.3. *In Situation 10.96.1. Let R be a Artinian local ring. Suppose that $0 \to R^{n_e} \to R^{n_{e-1}} \to \ldots \to R^{n_0}$ is an exact complex. Then the complex is isomorphic to a direct sum of trivial complexes.*

Proof. By induction on the integer $\sum n_i$. Clearly $\mathrm{Ass}(R) = \{\mathfrak{m}\}$. Pick $x \in R$, $x \neq 0$, $\mathfrak{m}x = 0$. Pick a basis vector $e_i \in R^{n_e}$. Since xe_i is not mapped to zero by exactness of the complex we deduce that some matrix coefficient of the map $R^{n_e} \to R^{n_{e-1}}$ is not in $\mathfrak{m}$. Lemma 10.96.2 then allows us to decrease $\sum n_i$. □

Below we define the rank of a map of finite free modules. This is just one possible definition of rank. It is just the definition that works in this section; there are others that may be more convenient in other settings.

Definition 10.96.4. Let R be a ring. Suppose that $\varphi : R^m \to R^n$ is a map of finite free modules.

(1) The *rank* of φ is the maximal r such that $\wedge^r\varphi : \wedge^r R^m \to \wedge^r R^n$ is nonzero.
(2) We let $I(\varphi) \subset R$ be the ideal generated by the $r \times r$ minors of the matrix of φ, where r is the rank as defined above.

Lemma 10.96.5. *In Situation 10.96.1, suppose the complex is isomorphic to a direct sum of trivial complexes. Then we have*

(1) *the maps φ_i have rank $r_i = n_i - n_{i+1} + \ldots + (-1)^{e-i-1}n_{e-1} + (-1)^{e-i}n_e$,*
(2) *for all i, $1 \leq i \leq e$ we have $\mathrm{rank}(\varphi_{i+1}) + \mathrm{rank}(\varphi_i) = n_i$,*
(3) *each $I(\varphi_i) = R$.*

Proof. We may assume the complex is the direct sum of trivial complexes. Then for each i we can split the standard basis elements of R^{n_i} into those that map to a basis element of $R^{n_{i-1}}$ and those that are mapped to zero (and these are mapped onto by basis elements of $R^{n_{i+1}}$). Using descending induction starting with $i = e$ it is easy to prove that there are r_{i+1}-basis elements of R^{n_i} which are mapped to zero and r_i which are mapped to basis elements of $R^{n_{i-1}}$. From this the result follows. □

Lemma 10.96.6. *Let R be a local Noetherian ring. Suppose that $\varphi : R^m \to R^n$ is a map of finite free modules. The following are equivalent*

1. *φ is injective.*
2. *the rank of φ is m and either $I(\varphi) = R$ or it contains a nonzerodivisor.*

Proof. If any matrix coefficient of φ is not in $\mathfrak{m}$, then we apply Lemma 10.96.2 to write φ as the sum of $1 : R \to R$ and a map $\varphi' : R^{m-1} \to R^{n-1}$. It is easy to see that the lemma for φ' implies the lemma for φ. Thus we may assume from the outset that all the matrix coefficients of φ are in $\mathfrak{m}$.

Suppose φ is injective. We may assume $m > 0$. Let $\mathfrak{q} \in \mathrm{Ass}(R)$. Let $x \in R$ be an element whose annihilator is $\mathfrak{q}$. Note that φ induces a injective map $xR^m \to xR^n$ which is isomorphic to the map $\varphi_\mathfrak{q} : (R/\mathfrak{q})^m \to (R/\mathfrak{q})^n$ induced by φ. Since $R/\mathfrak{q}$ is a domain we deduce immediately by localizing to its fraction field that the rank of $\varphi_\mathfrak{q}$ is m and that $I(\varphi_\mathfrak{q})$ is not the zero ideal. Hence we conclude by Lemma 10.60.17.

Conversely, assume that the rank of φ is m and that $I(\varphi)$ contains a nonzerodivisor x. The rank being m implies $n \geq m$. By Lemma 10.14.5 we can find a map $\psi : R^n \to R^m$ such that $\psi \circ \varphi = x\mathrm{id}_{R^m}$. Thus φ is injective. □

Lemma 10.96.7. *In Situation 10.96.1. Suppose R is a local Noetherian ring with maximal ideal $\mathfrak{m}$. Assume $\mathfrak{m} \in Ass(R)$, in other words R has depth 0. Suppose that the complex is exact. In this case the complex is isomorphic to a direct sum of trivial complexes.*

Proof. The proof is the same as in Lemma 10.96.3, except using Lemma 10.96.6 to guarantee that $I(\varphi_e) = R$, and hence some matrix coefficient of φ_e is not in $\mathfrak{m}$. □

Lemma 10.96.8. *In Situation 10.96.1, suppose R is a local Noetherian ring, and suppose that the complex is exact. Let x be an element of the maximal ideal which is a nonzerodivisor. The complex $0 \to (R/xR)^{n_e} \to \ldots \to (R/xR)^{n_1}$ is still exact.*

Proof. Follows easily from the snake lemma. □

Lemma 10.96.9 (Acyclicity lemma)**.** *Let R be a local Noetherian ring. Let $0 \to M_e \to M_{e-1} \to \ldots \to M_0$ be a complex of finite R-modules. Assume $depth(M_i) \geq i$. Let i be the largest index such that the complex is not exact at M_i. If $i > 0$ then $Ker(M_i \to M_{i-1})/Im(M_{i+1} \to M_i)$ has depth ≥ 1.*

Proof. Let $H = \mathrm{Ker}(M_i \to M_{i-1})/\mathrm{Im}(M_{i+1} \to M_i)$ be the cohomology group in question. We may break the complex into short exact sequences $0 \to M_e \to M_{e-1} \to K_{e-2} \to 0$, $0 \to K_j \to M_j \to K_{j-1} \to 0$, for $i + 2 \leq j \leq e - 2$, $0 \to K_{i+1} \to M_{i+1} \to B_i \to 0$, $0 \to K_i \to M_i \to M_{i-1}$, and $0 \to B_i \to K_i \to H \to 0$. We proceed up through these complexes to prove the statements about depths, repeatedly using Lemma 10.67.11. First of all, since $\mathrm{depth}(M_e) \geq e$, and

$\text{depth}(M_{e-1}) \geq e-1$ we deduce that $\text{depth}(K_{e-2}) \geq e-1$. At this point the sequences $0 \to K_j \to M_j \to K_{j-1} \to 0$ for $i+2 \leq j \leq e-2$ imply similarly that $\text{depth}(K_{j-1}) \geq j$ for $i+2 \leq j \leq e-2$. The sequence $0 \to K_{i+1} \to M_{i+1} \to B_i \to 0$ then shows that $\text{depth}(B_i) \geq i+1$. The sequence $0 \to K_i \to M_i \to M_{i-1}$ shows that $\text{depth}(K_i) \geq 1$ since M_i has depth $\geq i \geq 1$ by assumption. The sequence $0 \to B_i \to K_i \to H \to 0$ then implies the result. □

Proposition 10.96.10. *In Situation 10.96.1, suppose R is a local Noetherian ring. The complex is exact if and only if for all i, $1 \leq i \leq e$ the following two conditions are satisfied:*

(1) *we have $\text{rank}(\varphi_{i+1}) + \text{rank}(\varphi_i) = n_i$, and*
(2) *$I(\varphi_i) = R$, or $I(\varphi_i)$ contains a regular sequence of length i.*

Proof. This proof is very similar to the proof of Lemma 10.96.6. As in the proof of Lemma 10.96.6 we may assume that all matrix entries of each φ_i are elements of the maximal ideal. We may also assume that $e \geq 1$.

Assume the complex is exact. Let $q \in \text{Ass}(R)$. (There is at least one such prime.) Note that the ring $R_{\mathfrak{q}}$ has depth 0. We apply Lemmas 10.96.7 and 10.96.5 to the localized complex over $R_{\mathfrak{q}}$. all of the ideals $I(\varphi_i)_{\mathfrak{q}}$, $e \geq i \geq 1$ are equal to $R_{\mathfrak{q}}$. Thus none of the ideals $I(\varphi_i)$ is contained in $\mathfrak{q}$. This implies that $I(\varphi_e)I(\varphi_{e-1})\ldots I(\varphi_1)$ is not contained in any of the associated primes of R. By Lemma 10.14.3 we may choose $x \in I(\varphi_e)I(\varphi_{e-1})\ldots I(\varphi_1)$, $x \notin \mathfrak{q}$ for all $q \in \text{Ass}(R)$. According to Lemma 10.96.8 the complex $0 \to (R/xR)^{n_e} \to \ldots \to (R/xR)^{n_1}$ is exact. By induction on e all the ideals $I(\varphi_i)/xR$ have a regular sequence of length $i-1$. This proves that $I(\varphi_i)$ contains a regular sequence of length i.

Assume the two conditions on the ranks of φ_i and the ideals $I(\varphi_i)$ is satisfied. Note that $I(\varphi_i) \subset \mathfrak{m}$ for all i because of what was said in the first paragraph of the proof. Hence the assumption in particular implies that $\text{depth}(R) \geq e$. By induction on the dimension of R we may assume the complex is exact when localized at any nonmaximal prime of R. Thus $\text{Ker}(\varphi_i)/\text{Im}(\varphi_{i+1})$ has support $\{\mathfrak{m}\}$ and hence (if nonzero) depth 0. By Lemma 10.96.9 we see that the complex is exact. □

10.97. Auslander-Buchsbaum

The following result can be found in [**AB57**].

Proposition 10.97.1. *Let R be a Noetherian local ring. Let M be a nonzero finite R-module which has finite projective dimension $\text{pd}_R(M)$. Then we have*

$$\text{depth}(R) = \text{pd}_R(M) + \text{depth}(M)$$

Proof. We prove this by induction on $\text{depth}(M)$. The most interesting case is the case $\text{depth}(M) = 0$. In this case, let

$$0 \to R^{n_e} \to R^{n_{e-1}} \to \ldots \to R^{n_0} \to M \to 0$$

be a minimal finite free resolution, so $e = \text{pd}_R(M)$. By Lemma 10.96.2 we may assume all matrix coefficients of the maps in the complex are contained in the maximal ideal of R. Then on the one hand, by Proposition 10.96.10 we see that

$\text{depth}(R) \geq e$. On the other hand, breaking the long exact sequence into short exact sequences

$$\begin{aligned} 0 \to R^{n_e} \to R^{n_{e-1}} \to K_{e-2} \to 0, \\ 0 \to K_{e-2} \to R^{n_{e-2}} \to K_{e-3} \to 0, \\ \ldots, \\ 0 \to K_0 \to R^{n_0} \to M \to 0 \end{aligned}$$

we see, using Lemma 10.67.11, that

$$\begin{aligned} \text{depth}(K_{e-2}) \geq \text{depth}(R) - 1, \\ \text{depth}(K_{e-3}) \geq \text{depth}(R) - 2, \\ \ldots, \\ \text{depth}(K_0) \geq \text{depth}(R) - (e-1), \\ \text{depth}(M) \geq \text{depth}(R) - e \end{aligned}$$

and since $\text{depth}(M) = 0$ we conclude $\text{depth}(R) \leq e$. This finishes the proof of the case $\text{depth}(M) = 0$.

Induction step. If $\text{depth}(M) > 0$, then we pick $x \in \mathfrak{m}$ which is a nonzero divisor on both M and R. This is possible, because either $\text{pd}_R(M) > 0$ and $\text{depth}(R) > 0$ by the aforementioned Proposition 10.96.10 or $\text{pd}_R(M) = 0$ in which case M is finite free hence also $\text{depth}(R) = \text{depth}(M) > 0$. Thus $\text{depth}(R \oplus M) > 0$ by Lemma 10.67.11 (for example) and we can find an $x \in \mathfrak{m}$ which is a nonzerodivisor on both R and M. Let

$$0 \to R^{n_e} \to R^{n_{e-1}} \to \ldots \to R^{n_0} \to M \to 0$$

be a minimal resolution as above. An application of the snake lemma shows that

$$0 \to (R/xR)^{n_e} \to (R/xR)^{n_{e-1}} \to \ldots \to (R/xR)^{n_0} \to M/xM \to 0$$

is a minimal resolution too. Thus $\text{pd}_R(M) = \text{pd}_{R/xR}(M/xM)$. By Lemma 10.67.12 we have $\text{depth}(R/xR) = \text{depth}(R) - 1$ and $\text{depth}(M/xM) = \text{depth}(M) - 1$. Till now depths have all been depths as R modules, but we observe that $\text{depth}_R(M/xM) = \text{depth}_{R/xR}(M/xM)$ and similarly for R/xR. By induction hypothesis we see that the Auslander-Buchsbaum formula holds for M/xM over R/xR. Since the depths of both R/xR and M/xM have decreased by one and the projective dimension has not changed we conclude. □

10.98. Cohen-Macaulay modules

Here we just do a minimal amount of work to show that Cohen-Macaulay modules have good properties. We postpone using Ext groups to establish the connection with duality and so on.

Definition 10.98.1. Let R be a Noetherian local ring. Let M be a finite R-module. We say M is *Cohen-Macaulay* if $\dim(\text{Support}(M)) = \text{depth}(M)$.

Let R be a local Noetherian ring. Let M be a Cohen-Macaulay module, and let $f_1, \ldots, f_d$ be an M-regular sequence with $d = \dim(\text{Support}(M))$. We say that $g \in \mathfrak{m}$ is *good with respect to* $(M, f_1, \ldots, f_d)$ if for all $i = 0, 1, \ldots, d-1$ we have $\dim(\text{Support}(M) \cap V(g, f_1, \ldots, f_i)) = d - i - 1$. This is equivalent to the condition that $\dim((\text{Support}(M/(f_1, \ldots, f_i)M) \cap V(g) = d - i - 1$ for $i = 0, 1, \ldots, d-1$.

Lemma 10.98.2. *Notation and assumptions as above. If g is good with respect to $(M, f_1, \ldots, f_d)$, then (a) g is a nonzerodivisor on M, and (b) M/gM is Cohen-Macaulay with maximal regular sequence $f_1, \ldots, f_{d-1}$.*

Proof. We prove the lemma by induction on d. If $d = 0$, then M is finite and there is no case to which the lemma applies. If $d = 1$, then we have to show that $g : M \to M$ is injective. The kernel K has support $\{\mathfrak{m}\}$ because by assumption $\dim \mathrm{Supp}(M) \cap V(g) = 0$. Hence K has finite length. Hence $f_1 : K \to K$ injective implies the length of the image is the length of K, and hence $f_1 K = K$, which by Nakayama's Lemma 10.18.1 implies $K = 0$. Also, $\dim \mathrm{Supp}(M/gM) = 0$ and so M/gM is Cohen-Macaulay of depth 0.

For $d > 1$ we essentially argue in the same way. Let $K \subset M$ be the kernel of multiplication by g. As above $f_1 : K \to K$ cannot be surjective if $K \neq 0$ Consider the commutative diagram

$$\begin{array}{ccccccccc} 0 & \to & M & \xrightarrow{f_1} & M & \to & M/f_1M & \to & 0 \\ & & \downarrow g & & \downarrow g & & \downarrow g & & \\ 0 & \to & M & \xrightarrow{f_1} & M & \to & M/f_1M & \to & 0 \end{array}$$

This shows that the kernel K_1 of $g : M/f_1M \to M/f_1M$ cannot be zero if K is not zero. But g is good for $(M/f_1M, f_2, \ldots, f_d)$, as is easy seen from the definition. We conclude that $K_1 = 0$, and so $K = 0$. From the snake lemma we see that $0 \to M/gM \to M/gM \to M/(f_1, g)M \to 0$ is exact. By induction, we have that $M/(g, f_1)M$ is Cohen-Macaulay with regular sequence $f_2, \ldots, f_{d-1}$. Thus M/gM is Cohen-Macaulay with regular sequence $f_1, \ldots, f_{d-1}$. □

Lemma 10.98.3. *Let R be a Noetherian local ring. Let M be a Cohen-Macaulay module over R. Suppose $g \in \mathfrak{m}$ is such that $\dim(\mathit{Supp}(M) \cap V(g)) = \dim(\mathit{Supp}(M)) - 1$. Then (a) g is a nonzerodivisor on M, and (b) M/gM is Cohen-Macaulay of depth one less.*

Proof. Choose a M-regular sequence $f_1, \ldots, f_d$ with $d = \dim(\mathrm{Supp}(M))$. If g is is good with respect to $(M, f_1, \ldots, f_d)$ we win by Lemma 10.98.2. In particular the lemma holds if $d = 1$. (The case $d = 0$ does not occur.) Assume $d > 1$. Choose an element $h \in R$ such that (a) h is good with respect to $(M, f_1, \ldots, f_d)$, and (b) $\dim(\mathrm{Supp}(M) \cap V(h, g) = d-2$. To see h exists, let $\{\mathfrak{q}_i\}$ be the (finite) set of minimal primes of the closed sets $\mathrm{Supp}(M)$, $\mathrm{Supp}(M) \cap V(f_1, \ldots, f_i)$, $i = 1, \ldots, d-1$, and $\mathrm{Supp}(M) \cap V(g)$. None of these $\mathfrak{q}_i$ is equal to $\mathfrak{m}$ and hence we may find $h \in \mathfrak{m}$, $h \notin \mathfrak{q}_i$ by Lemma 10.14.3. It is clear that h satisfies (a) and (b). At this point we may apply Lemma 10.98.2 to conclude that M/hM is Cohen-Macaulay. By (b) we see that the pair $(M/hM, g)$ satisfies the induction hypothesis. Hence $M/(h, g)M$ is Cohen-Macaulay, and $g : M/hM \to M/hM$ is injective. From this it follows easily that $g : M \to M$ is injective, by a snake lemma argument. This in its turn implies that $h : M/gM \to M/gM$ is injective. Combined with the fact that $M/(g, h)M$ is Cohen-Macaulay this finishes the proof. □

Proposition 10.98.4. *Let R be a Noetherian local ring, with maximal ideal $\mathfrak{m}$. Let M be a Cohen-Macaulay module over R whose support has dimension d. Suppose that $g_1, \ldots, g_c$ are elements of $\mathfrak{m}$ such that $\dim(\mathit{Supp}(M/(g_1, \ldots, g_c)M)) = d - c$. Then $g_1, \ldots, g_c$ is an M-regular sequence, and can be extended to a maximal M-regular sequence.*

Proof. Let $Z = \mathrm{Supp}(M) \subset \mathrm{Spec}(R)$. By Lemma 10.57.11 in the chain $Z \supset Z \cap V(g_1) \supset \ldots \supset Z \cap V(g_1, \ldots, g_c)$ each step decreases the dimension at most by 1. Hence by assumption each step decreases the dimension by exactly 1 each time. Thus we may successively apply Lemma 10.98.3 to the modules $M/(g_1, \ldots, g_i)$ and the element g_{i+1}.

To extend $g_1, \ldots, g_c$ by one element if $c < d$ we simply choose an element $g_{c+1} \in \mathfrak{m}$ which is not in any of the finitely many minimal primes of $Z \cap V(g_1, \ldots, g_c)$, using Lemma 10.14.3. □

10.99. Cohen-Macaulay rings

Definition 10.99.1. A Noetherian local ring R is called *Cohen-Macaulay* if it is Cohen-Macaulay as a module over itself.

Note that this is equivalent to requiring the existence of a R-regular sequence $x_1, \ldots, x_d$ of the maximal ideal such that $R/(x_1, \ldots, x_d)$ has dimension 0. We will usually just say "regular sequence" and not "R-regular sequence".

Lemma 10.99.2. *Let R be a Noetherian local Cohen-Macaulay ring with maximal ideal $\mathfrak{m}$. Let $x_1, \ldots, x_c \in \mathfrak{m}$ be elements. Then*

$$x_1, \ldots, x_c \text{ is a regular sequence } \Leftrightarrow \dim(R/(x_1, \ldots, x_c)) = \dim(R) - c$$

If so $x_1, \ldots, x_c$ can be extended to a regular sequence of length $\dim(R)$ and each quotient $R/(x_1, \ldots, x_i)$ is a Cohen-Macaulay ring of dimension $\dim(R) - i$.

Proof. This is a reformulation of Proposition 10.98.4 in the case where the module is equal to R. □

Lemma 10.99.3. *Let R be Noetherian local. Suppose R is Cohen-Macaulay of dimension d. Any maximal chain of ideals $\mathfrak{p}_0 \subset \mathfrak{p}_1 \subset \ldots \subset \mathfrak{p}_n$ has length $n = d$.*

Proof. Choose an element $x \in \mathfrak{p}_1$, with x not in any of the minimal primes of R, and in particular $x \notin \mathfrak{p}_0$. (See Lemma 10.14.3.) Then $\dim(R/xR) < \dim(R)$ and R/xR is Cohen-Macaulay by Proposition 10.98.4. By induction the chain $\mathfrak{p}_1/xR \subset \ldots \subset \mathfrak{p}_n/xR$ has length $d - 1$. □

Lemma 10.99.4. *Suppose R is a Noetherian local Cohen-Macaulay ring of dimension d. For any prime $\mathfrak{p} \subset R$ we have*

$$\dim(R) = \dim(R_\mathfrak{p}) + \dim(R/\mathfrak{p}).$$

Proof. This is immediate from the result on maximal sequences above, by looking at maximal sequences which have $\mathfrak{p}$ in them. □

Lemma 10.99.5. *Suppose R is a Cohen-Macaulay local ring. For any prime $\mathfrak{p} \subset R$ the ring $R_\mathfrak{p}$ is Cohen-Macaulay as well.*

Proof. Suppose that $\dim(R) = d$ and that $\dim(R/\mathfrak{p}) = d - c$. We may choose $f_1, \ldots, f_c \in \mathfrak{p}$ such that $\dim V(f_1, \ldots, f_i) = d - i$, using Lemma 10.14.3 at each step to avoid minimal primes of $V(f_1, \ldots, f_{i-1})$. Then $\mathfrak{p}$ is minimal over $(f_1, \ldots, f_c)$ and hence the support of $R_\mathfrak{p}/(f_1, \ldots, f_c)R_\mathfrak{p}$ consists of the maximal ideal of $R_\mathfrak{p}$. In other words $R_\mathfrak{p}$ has an ideal of definition generated by c elements, and has dimension c by Lemma 10.99.4. □

Definition 10.99.6. A Noetherian ring R is called *Cohen-Macaulay* if all its local rings are Cohen-Macaulay.

Lemma 10.99.7. *Suppose R is a Cohen-Macaulay ring. Any polynomial algebra over R is Cohen-Macaulay.*

Proof. By induction on the number of variables it suffices to prove that $R[x]$ is Cohen-Macaulay if R is. Let $\mathfrak{q} \subset R[x]$ be a prime, and let $\mathfrak{p}$ be its image. Let $f_1, \ldots, f_d$ be a regular sequence in the maximal ideal of $R_\mathfrak{p}$ of length $d = \dim(R_\mathfrak{p})$. Note that since $R[x]$ is flat over R the localization $R[x]_\mathfrak{q}$ is flat over $R_\mathfrak{p}$. Hence, by Lemma 10.65.7, the sequence $f_1, \ldots, f_d$ is a regular sequence of length d in $R[x]_\mathfrak{q}$. The quotient $R[x]_\mathfrak{q}/(f_1, \ldots, f_d)$ is a localization of $(R_\mathfrak{p}/(f_1, \ldots, f_d))[x]$ at a prime $\overline{\mathfrak{q}}$. It is clear that either $\overline{\mathfrak{q}}$ contains a monic polynomial f in $(R_\mathfrak{p}/(f_1, \ldots, f_d))[x]$, or $\overline{\mathfrak{q}}$ equals the kernel of $(R_\mathfrak{p}/(f_1, \ldots, f_d))[x] \to \kappa(\mathfrak{p})[x]$. In the first case the monic polynomial f is a nonzerodivisor in $(R_\mathfrak{p}/(f_1, \ldots, f_d))[x]$ and hence in $R[x]_\mathfrak{q}/(f_1, \ldots, f_d)$, and $x_1, \ldots, x_d, f$ is a regular sequence in $R[x]_\mathfrak{q}$ such that $\dim(R[x]_\mathfrak{q}/(x_1, \ldots, x_d, f)) = 0$. In the second case it is already the case that $\dim R[x]_\mathfrak{q}/(f_1, \ldots, f_d) = 0$. □

Lemma 10.99.8. *Let R be a Noetherian local Cohen-Macaulay ring of dimension d. Let $0 \to K \to R^{\oplus n} \to M \to 0$ be an exact sequence of R-modules with $K \neq 0$. Then either $\text{depth}(K) > \text{depth}(M)$ or $\text{depth}(K) = \text{depth}(M) = d$.*

Proof. If $\text{depth}(M) = 0$ the lemma is clear. Let $x \in \mathfrak{m}$ be a nonzerodivisor on M and on R. Then x is a nonzerodivisor on M and on K and it follows by an easy diagram chase that $0 \to K/xK \to (R/xR)^n \to M/xM \to 0$ is exact. Thus the result follows from the result for K/xK over R/xR which has smaller dimension. □

Definition 10.99.9. Let R be a Noetherian local Cohen-Macaulay ring. A finite module M over R is called a *maximal Cohen-Macaulay* module if $\text{depth}(M) = \dim(R)$.

Lemma 10.99.10. *Let R be a local Noetherian Cohen-Macaulay ring of dimension d. Let M be a finite R module of depth e. There exists an exact complex*

$$0 \to K \to F_{d-e-1} \to \ldots \to F_0 \to M \to 0$$

with each F_i finite free and K maximal Cohen-Macaulay.

Proof. Immediate from the definition and Lemma 10.99.8. □

Lemma 10.99.11. *Let $\varphi : A \to B$ be a map of local rings. Assume that B is Noetherian and Cohen-Macaulay and that $\mathfrak{m}_B = \sqrt{\varphi(\mathfrak{m}_A)B}$. Then there exists a sequence of elements $f_1, \ldots, f_{\dim(B)}$ in A such that $\varphi(f_1), \ldots, \varphi(f_{\dim(B)})$ is a regular sequence in B.*

Proof. By induction on $\dim(B)$ it suffices to prove: If $\dim(B) \geq 1$, then we can find an element f of A which maps to a nonzerodivisor in B. By Lemma 10.99.2 it suffices to find $f \in A$ whose image in B is not contained in any of the finitely many minimal primes $\mathfrak{q}_1, \ldots, \mathfrak{q}_r$ of B. By the assumption that $\mathfrak{m}_B = \sqrt{\varphi(\mathfrak{m}_A)B}$ we see that $\mathfrak{m}_A \not\subset \varphi^-(\mathfrak{q}_i)$. Hence we can find f by Lemma 10.14.3. □

10.100. Catenary rings

Definition 10.100.1. A ring R is said to be *catenary* if for any pair of prime ideals $\mathfrak{p} \subset \mathfrak{q}$, all maximal chains of primes $\mathfrak{p} = \mathfrak{p}_0 \subset \mathfrak{p}_1 \subset \ldots \subset \mathfrak{p}_e = \mathfrak{q}$ have the same (finite) length.

Lemma 10.100.2. *A ring R is catenary if and only if the topological space* $\mathrm{Spec}(R)$ *is catenary (see Topology, Definition 5.10.4).*

Proof. Immediate from the definition and the characterization of irreducible closed subsets. □

Lemma 10.100.3. *Any localization of a catenary ring is catenary.*

Proof. Omitted. □

Lemma 10.100.4. *Any quotient of a catenary ring is catenary.*

Proof. Omitted. □

In general it is not the case that a finitely generated R-algebra is catenary if R is. Thus we make the following definition.

Definition 10.100.5. A ring R is said to be *universally catenary* if R is Noetherian and every R algebra of finite type is catenary.

By Lemma 10.100.4 this just means that R is Noetherian and that each polynomial algebra $R[x_1, \ldots, x_n]$ is catenary.

Lemma 10.100.6. *A Cohen-Macaulay ring is universally catenary.*

Proof. Since a polynomial algebra over R is Cohen-Macaulay, by Lemma 10.99.7, it suffices to show that a Cohen-Macaulay ring is catenary. Let R be Cohen-Macaulay and $\mathfrak{p} \subset \mathfrak{q}$ primes of R. By definition $R_\mathfrak{q}$ and $R_\mathfrak{p}$ are Cohen-Macaulay. Take a maximal chain of primes $\mathfrak{p} = \mathfrak{p}_0 \subset \mathfrak{p}_1 \subset \ldots \subset \mathfrak{p}_n = \mathfrak{q}$. Next choose a maximal chain of primes $\mathfrak{q}_0 \subset \mathfrak{q}_1 \subset \ldots \subset \mathfrak{q}_m = \mathfrak{p}$. By Lemma 10.99.3 we have $n + m = \dim(R_\mathfrak{q})$. And we have $m = \dim(R_\mathfrak{p})$ by the same lemma. Hence $n = \dim(R_\mathfrak{q}) - \dim(R_\mathfrak{p})$ is independent of choices. □

10.101. Regular local rings

It is not that easy to show that all prime localizations of a regular local ring are regular. In fact, quite a bit of the material developed so far is geared towards a proof of this fact. See Proposition 10.105.5, and trace back the references.

Lemma 10.101.1. *Let R be a regular local ring with maximal ideal $\mathfrak{m}$. The graded ring $\bigoplus \mathfrak{m}^n/\mathfrak{m}^{n+1}$ is isomorphic to the graded polynomial algebra $\kappa(\mathfrak{m})[X_1, \ldots, X_d]$.*

Proof. Let $x_1, \ldots, x_d$ be a minimal set of generators for the maximal ideal $\mathfrak{m}$. Write $\kappa = \kappa(\mathfrak{m})$. There is a surjection $\kappa[X_1, \ldots, X_d] \to \bigoplus \mathfrak{m}^n/\mathfrak{m}^{n+1}$, which maps the class of x_i in $\mathfrak{m}/\mathfrak{m}^2$ to X_i. Since $d(R) = d$ we know that the numerical polynomial $n \mapsto \dim_\kappa \mathfrak{m}^n/\mathfrak{m}^{n+1}$ has degree d. By Lemma 10.55.10 we conclude that the surjection $\kappa[X_1, \ldots, X_d] \to \bigoplus \mathfrak{m}^n/\mathfrak{m}^{n+1}$ is an isomorphism. □

Lemma 10.101.2. *Any regular local ring is a domain.*

Proof. We will use that $\bigcap \mathfrak{m}^n = 0$ by Lemma 10.47.6. Let $f, g \in R$ such that $fg = 0$. Suppose that $f \in \mathfrak{m}^a$ and $g \in \mathfrak{m}^b$, with a, b maximal. Since $fg = 0 \in \mathfrak{m}^{a+b+1}$ we see from the result of Lemma 10.101.1 that either $f \in \mathfrak{m}^{a+1}$ or $g \in \mathfrak{m}^{b+1}$. Contradiction. □

Lemma 10.101.3. *Let R be a regular local ring and let $x_1, \ldots, x_d$ be a minimal set of generators for the maximal ideal $\mathfrak{m}$. Then $x_1, \ldots, x_d$ is a regular sequence, and each $R/(x_1, \ldots, x_c)$ is a regular local ring of dimension $d - c$. In particular R is Cohen-Macaulay.*

Proof. Note that R/x_1R is a Noetherian local ring of dimension $\geq d-1$ by Lemma 10.57.11 with $x_2, \ldots, x_d$ generating the maximal ideal. Hence it is a regular local ring by definition. Since R is a domain by Lemma 10.101.2 x_1 is a nonzerodivisor. □

Lemma 10.101.4. *Let R be a regular local ring. Let $I \subset R$ be an ideal such that R/I is a regular local ring as well. Then there exists a minimal set of generators $x_1, \ldots, x_d$ for the maximal $\mathfrak{m}$ of R such that $I = (x_1, \ldots, x_c)$ for some $0 \leq c \leq d$.*

Proof. Say $\dim(R) = d$ and $\dim(R/I) = d - c$. Denote $\overline{\mathfrak{m}} = \mathfrak{m}/I$ the maximal ideal of R/I. Let $\kappa = R/\mathfrak{m}$. We have $\dim_\kappa(I/\mathfrak{m}^2) = \dim_\kappa(\mathfrak{m}/\mathfrak{m}^2) - \dim(\overline{\mathfrak{m}}/\overline{\mathfrak{m}}^2) = d - (d - c) = c$ by the definition of a regular local ring. Hence we can choose $x_1, \ldots, x_c \in I$ whose images in $\mathfrak{m}/\mathfrak{m}^2$ are linearly independent, and supplement with $x_{c+1}, \ldots, x_d$ to get a minimal system of generators. □

Lemma 10.101.5. *Let R be a Noetherian local ring. Let $x \in \mathfrak{m}$. Let M be a finite R-module such that x is a nonzerodivisor on M and M/xM is free over R/xR. Then M is free over R.*

Proof. Let $m_1, \ldots, m_r$ be elements of M which map to a R/xR-basis of M/xM. By Nakayama's Lemma 10.18.1 $m_1, \ldots, m_r$ generate M. If $\sum a_i m_i = 0$ is a relation, then $a_i \in xR$ for all i. Hence $a_i = b_i x$ for some $b_i \in R$. Hence the kernel K of $R^r \to M$ satisfies $xK = K$ and hence is zero by Nakayama's lemma. □

Lemma 10.101.6. *Let R be a regular local ring. Any maximal Cohen-Macaulay module over R is free.*

Proof. Let M be a maximal Cohen-Macaulay module over R. Let $x \in \mathfrak{m}$ be part of a regular sequence generating $\mathfrak{m}$. Then x is a nonzerodivisor on M by Proposition 10.98.4, and M/xM is a maximal Cohen-Macaulay module over R/xR. By induction on $\dim(R)$ we see that M/xM is free. We win by Lemma 10.101.5. □

Lemma 10.101.7. *Suppose R is a Noetherian local ring. Let $x \in \mathfrak{m}$ be a nonzerodivisor such that R/xR is a regular local ring. Then R is a regular local ring. More generally, if $x_1, \ldots, x_r$ is a regular sequence in R such that $R/(x_1, \ldots, x_r)$ is a regular local ring, then R is a regular local ring.*

Proof. This is true because x together with the lifts of a system of minimal generators of the maximal ideal of R/xR will give $\dim(R)$ generators of $\mathfrak{m}$. Use Lemma 10.57.11. The last statement follows from the first and induction. □

Lemma 10.101.8. *Let $(R_i, \varphi_{ii'})$ be a directed system of local rings whose transition maps are local ring maps. If each R_i is a regular local ring and $R = \operatorname{colim} R_i$ is Noetherian, then R is a regular local ring.*

Proof. Let $\mathfrak{m} \subset R$ be the maximal ideal; it is the colimit of the maximal ideal $\mathfrak{m}_i \subset R_i$. We prove the lemma by induction on $d = \dim \mathfrak{m}/\mathfrak{m}^2$. If $d = 0$, then $R = R/\mathfrak{m}$ is a field and R is a regular local ring. If $d > 0$ pick an $x \in \mathfrak{m}$, $x \notin \mathfrak{m}^2$. For some i we can find an $x_i \in \mathfrak{m}_i$ mapping to x. Note that $R/xR = \operatorname{colim}_{i' \geq i} R_{i'}/x_i R_{i'}$ is a Noetherian local ring. By Lemma 10.101.3 we see that $R_{i'}/x_i R_{i'}$ is a regular local ring. Hence by induction we see that R/xR is a regular local ring. Since each R_i is a domain (Lemma 10.101.1) we see that R is a domain. Hence x is a nonzerodivisor and we conclude that R is a regular local ring by Lemma 10.101.7. □

10.102. Epimorphisms of rings

In any category there is a notion of an *epimorphism*. Some of this material is taken from [**Laz69**] and [**Maz68**].

Lemma 10.102.1. *Let $R \to S$ be a ring map. The following are equivalent*

(1) *$R \to S$ is an epimorphism,*
(2) *the two ring maps $S \to S \otimes_R S$ are equal,*
(3) *either of the ring maps $S \to S \otimes_R S$ is an isomorphism, and*
(4) *the ring map $S \otimes_R S \to S$ is an isomorphism.*

Proof. Omitted. □

Lemma 10.102.2. *The composition of two epimorphisms of rings is an epimorphism.*

Proof. Omitted. Hint: This is true in any category. □

Lemma 10.102.3. *If $R \to S$ is an epimorphism of rings and $R \to R'$ is any ring map, then $R' \to R' \otimes_R S$ is an epimorphism.*

Proof. Omitted. Hint: True in any category with pushouts. □

Lemma 10.102.4. *If $A \to B \to C$ are ring maps and $A \to C$ is an epimorphism, so is $B \to C$.*

Proof. Omitted. Hint: This is true in any category. □

This means in particular, that if $R \to S$ is an epimorphism with image $\overline{R} \subset S$, then $\overline{R} \to S$ is an epimorphism. Hence while proving results for epimorphisms we may often assume the map is injective. The following lemma means in particular that every localization is an epimorphism.

Lemma 10.102.5. *Let $R \to S$ be a ring map. The following are equivalent:*

(1) *$R \to S$ is an epimorphism, and*
(2) *$R_\mathfrak{p} \to S_\mathfrak{p}$ is an epimorphism for each prime $\mathfrak{p}$ of R.*

Proof. Since $S_\mathfrak{p} = R_\mathfrak{p} \otimes_R S$ (see Lemma 10.11.15) we see that (1) implies (2) by Lemma 10.102.3. Conversely, assume that (2) holds. Let $a, b : S \to A$ be two ring maps from S to a ring A equalizing the map $R \to S$. By assumption we see that for every prime $\mathfrak{p}$ of R the induced maps $a_\mathfrak{p}, b_\mathfrak{p} : S_\mathfrak{p} \to A_\mathfrak{p}$ are the same. Hence $a = b$ as $A \subset \prod_\mathfrak{p} A_\mathfrak{p}$, see Lemma 10.22.1. □

Lemma 10.102.6. *Let $R \to S$ be a ring map. The following are equivalent*

(1) *$R \to S$ is an epimorphism and finite, and*
(2) *$R \to S$ is surjective.*

Proof. (This lemma seems to have been reproved many times in the literature, and has many different proofs.) It is clear that a surjective ring map is an epimorphism. Suppose that $R \to S$ is a finite ring map such that $S \otimes_R S \to S$ is an isomorphism. Our goal is to show that $R \to S$ is surjective. Assume S/R is not zero. The exact sequence $R \to S \to S/R \to 0$ leads to an exact sequence

$$R \otimes_R S \to S \otimes_R S \to S/R \otimes_R S \to 0.$$

Our assumption implies that the first arrow is an isomorphism, hence we conclude that $S/R \otimes_R S = 0$. Hence also $S/R \otimes_R S/R = 0$. By Lemma 10.5.4 there exists a surjection of R-modules $S/R \to R/I$ for some proper ideal $I \subset R$. Hence there exists a surjection $S/R \otimes_R S/R \to R/I \otimes_R R/I = R/I \neq 0$, contradiction. □

Lemma 10.102.7. *A faithfully flat epimorphism is an isomorphism.*

Proof. This is clear from Lemma 10.102.1 part (3) as the map $S \to S \otimes_R S$ is the map $R \to S$ tensored with S. □

Lemma 10.102.8. *If $k \to S$ is an epimorphism and k is a field, then $S = k$ or $S = 0$.*

Proof. This is clear from the result of Lemma 10.102.7 (as any nonzero algebra over k is faithfully flat), or by arguing directly that $R \to R \otimes_k R$ cannot be surjective unless $\dim_k(R) \leq 1$. □

Lemma 10.102.9. *Let $R \to S$ be an epimorphism of rings. Then* $\mathrm{Spec}(S) \to \mathrm{Spec}(R)$ *is injective.*

Proof. Let $\mathfrak{p}$ be a prime of R. The fibre of the map is the spectrum of the fibre ring $S \otimes_R \kappa(\mathfrak{p})$. By Lemma 10.102.3 the map $\kappa(\mathfrak{p}) \to S \otimes_R \kappa(\mathfrak{p})$ is an epimorphism, and hence by Lemma 10.102.8 there is either one point or no points lying over $\mathfrak{p}$. □

Lemma 10.102.10. *Let R be a ring. Let M, N be R-modules. Let $\{x_i\}_{i \in I}$ be a set of generators of M. Let $\{y_j\}_{j \in J}$ be a set of generators of N. Let $\{m_j\}_{j \in J}$ be a family of elements of M with $m_j = 0$ for all but finitely many j. Then*

$$\sum\nolimits_{j \in J} m_j \otimes y_j = 0 \text{ in } M \otimes_R N$$

is equivalent to the following: There exist $a_{i,j} \in R$ with $a_{i,j} = 0$ for all but finitely many pairs (i, j) such that

$$m_j = \sum\nolimits_{i \in I} a_{i,j} x_i \quad \text{for all } j \in J,$$
$$0 = \sum\nolimits_{j \in J} a_{i,j} y_j \quad \text{for all } i \in I.$$

Proof. The sufficiency is immediate. Suppose that $\sum_{j \in J} m_j \otimes y_j = 0$. Consider the short exact sequence

$$0 \to K \to \bigoplus\nolimits_{j \in J} R \to N \to 0$$

where the jth basis vector of $\bigoplus_{j \in J} R$ maps to y_j. Tensor this with M to get the exact sequence

$$K \otimes_R M \to \bigoplus\nolimits_{j \in J} M \to N \otimes_R M \to 0.$$

The assumption implies that there exist elements $k_i \in K$ such that $\sum k_i \otimes x_i$ maps to the element $(m_j)_{j \in J}$ of the middle. Writing $k_i = (a_{i,j})_{j \in J}$ and we obtain what we want. □

Lemma 10.102.11. *Let $\varphi : R \to S$ be a ring map. Let $g \in S$. The following are equivalent:*

(1) *$g \otimes 1 = 1 \otimes g$ in $S \otimes_R S$, and*
(2) *there exist $n \geq 0$ and elements $y_i, z_j \in S$ and $x_{i,j} \in R$ for $1 \leq i, j \leq n$ such that*
   (a) *$g = \sum_{i,j \leq n} x_{i,j} y_i z_j$,*
   (b) *for each j we have $\sum x_{i,j} y_i \in \varphi(R)$, and*
   (c) *for each i we have $\sum x_{i,j} z_j \in \varphi(R)$.*

Proof. It is clear that (2) implies (1). Conversely, suppose that $g \otimes 1 = 1 \otimes g$. Choose generators $\{s_i\}_{i \in I}$ of S as an R-module with $0, 1 \in I$ and $s_0 = 1$ and $s_1 = g$. Apply Lemma 10.102.10 to the relation $g \otimes s_0 + (-1) \otimes s_1 = 0$. We see that there exist $a_{i,j} \in R$ such that $g = \sum_i a_{i,0} s_i$, $-1 = \sum_i a_{i,1} s_i$, and for $j \neq 0, 1$ we have $0 = \sum_i a_{i,j} s_i$, and moreover for all i we have $\sum_j a_{i,j} s_j = 0$. Then we have

$$\sum\nolimits_{i,j \neq 0} a_{i,j} s_i s_j = -g + a_{0,0}$$

and for each $j \neq 0$ we have $\sum_{i \neq 0} a_{i,j} s_i \in R$. This proves that $-g + a_{0,0}$ can be written as in (2). It follows that g can be written as in (2). Details omitted. Hint: Show that the set of elements of S which have an expression as in (2) form an R-subalgebra of S. □

Remark 10.102.12. Let $R \to S$ be a ring map. Sometimes the set of elements $g \in S$ such that $g \otimes 1 = 1 \otimes g$ is called the *epicenter* of S. It is an R-algebra. By the construction of Lemma 10.102.11 we get for each g in the epicenter a matrix factorization

$$(g) = YXZ$$

with $X \in \mathrm{Mat}(n \times n, R)$, $Y \in \mathrm{Mat}(1 \times n, S)$, and $Z \in \mathrm{Mat}(n \times 1, S)$. Namely, let $x_{i,j}, y_i, z_j$ be as in part (2) of the lemma. Set $X = (x_{i,j})$, let y be the row vector whose entries are the y_i and let z be the column vector whose entries are the z_j. With this notation conditions (b) and (c) of Lemma 10.102.11 mean exactly that $YX \in \mathrm{Mat}(1 \times n, R)$, $XZ \in \mathrm{Mat}(n \times 1, R)$. It turns out to be very convenient to consider the triple of matrices (X, YX, XZ). Given $n \in \mathbf{N}$ and a triple (P, U, V) we say that (P, U, V) is a *n-triple associated to g* if there exists a matrix factorization as above such that $P = X$, $U = YX$ and $V = XZ$.

Lemma 10.102.13. *Let $R \to S$ be an epimorphism of rings. Then the cardinality of S is at most the cardinality of R. In a formula: $|S| \leq |R|$.*

Proof. The condition that $R \to S$ is an epimorphism means that each $g \in S$ satisfies $g \otimes 1 = 1 \otimes g$, see Lemma 10.102.1. We are going to use the notation introduced in Remark 10.102.12. Suppose that $g, g' \in S$ and suppose that (P, U, V) is an n-triple which is associated to both g and g'. Then we claim that $g = g'$. Namely, write $(P, U, V) = (X, YX, XZ)$ for a matrix factorization $(g) = YXZ$ of g and write $(P, U, V) = (X', Y'X', X'Z')$ for a matrix factorization $(g') = Y'X'Z'$ of g'. Then we see that

$$(g) = YXZ = UZ = Y'X'Z = Y'PZ = Y'XZ = Y'V = Y'X'Z' = (g')$$

and hence $g = g'$. This implies that the cardinality of S is bounded by the number of possible triples, which has cardinality at most $\sup_{n \in \mathbf{N}} |R|^n$. If R is infinite then this is at most $|R|$, see [**Kun83**, Ch. I, 10.13].

If R is a finite ring then the argument above only proves that S is at worst countable. In fact in this case R is Artinian and the map $R \to S$ is surjective. We omit the proof of this case. $\square$

Lemma 10.102.14. *Let $R \to S$ be an epimorphism of rings. Let N_1, N_2 be S-modules. Then $\mathrm{Hom}_S(N_1, N_2) = \mathrm{Hom}_R(N_1, N_2)$. In other words, the restriction functor $Mod_S \to Mod_R$ is fully faithful.*

Proof. Let $\varphi : N_1 \to N_2$ be an R-linear map. For any $x \in N_1$ consider the map $S \otimes_R S \to N_2$ defined by the rule $g \otimes g' \mapsto g\varphi(g'x)$. Since both maps $S \to S \otimes_R S$ are isomorphisms (Lemma 10.102.1), we conclude that $g\varphi(g'x) = gg'\varphi(x) = \varphi(gg'x)$. Thus φ is S-linear. $\square$

10.103. Pure ideals

The material in this section is discussed in many papers, see for example [**Laz67**], [**Bko70**], and [**DM83**].

Definition 10.103.1. Let R be a ring. We say that $I \subset R$ is *pure* if the quotient ring R/I is flat over R.

Lemma 10.103.2. *Let R be a ring. Let $I \subset R$ be an ideal. The following are equivalent:*

1. *I is pure,*
2. *for every ideal $J \subset R$ we have $J \cap I = IJ$,*
3. *for every finitely generated ideal $J \subset R$ we have $J \cap I = JI$,*
4. *for every $x \in R$ we have $(x) \cap I = xI$,*
5. *for every $x \in I$ we have $x = yx$ for some $y \in I$,*
6. *for every $x_1, \ldots, x_n \in I$ there exists a $y \in I$ such that $x_i = yx_i$ for all $i = 1, \ldots, n$,*
7. *for every prime $\mathfrak{p}$ of R we have $IR_\mathfrak{p} = 0$ or $IR_\mathfrak{p} = R_\mathfrak{p}$,*
8. *$Supp(I) = \mathrm{Spec}(R) \setminus V(I)$,*
9. *I is the kernel of the map $R \to (1 + I)^{-1}R$,*
10. *$R/I \cong S^{-1}R$ as R-algebras for some multiplicative subset S of R, and*
11. *$R/I \cong (1 + I)^{-1}R$ as R-algebras.*

Proof. For any ideal J of R we have the short exact sequence $0 \to J \to R \to R/J \to 0$. Tensoring with R/I we get an exact sequence $J \otimes_R R/I \to R/I \to R/I + J \to 0$ and $J \otimes_R R/I = R/JI$. Thus the equivalence of (1), (2), and (3) follows from Lemma 10.36.4. Moreover, these imply (4).

The implication (4) $\Rightarrow$ (5) is trivial. Assume (5) and let $x_1, \ldots, x_n \in I$. Choose $y_i \in I$ such that $x_i = y_i x_i$. Let $y \in I$ be the element such that $1 - y = \prod_{i=1,\ldots,n}(1 - y_i)$. Then $x_i = yx_i$ for all $i = 1, \ldots, n$. Hence (6) holds, and it follows that (5) $\Leftrightarrow$ (6).

Assume (5). Let $x \in I$. Then $x = yx$ for some $y \in I$. Hence $x(1 - y) = 0$, which shows that x maps to zero in $(1 + I)^{-1}R$. Of course the kernel of the map $R \to (1+I)^{-1}R$ is always contained in I. Hence we see that (5) implies (9). Assume (9). Then for any $x \in I$ we see that $x(1 - y) = 0$ for some $y \in I$. In other words, $x = yx$. We conclude that (5) is equivalent to (9).

Assume (5). Let $\mathfrak{p}$ be a prime of R. If $\mathfrak{p} \notin V(I)$, then $IR_\mathfrak{p} = R_\mathfrak{p}$. If $\mathfrak{p} \in V(I)$, in other words, if $I \subset \mathfrak{p}$, then $x \in I$ implies $x(1 - y) = 0$ for some $y \in I$, implies x maps to zero in $R_\mathfrak{p}$, i.e., $IR_\mathfrak{p} = 0$. Thus we see that (7) holds.

Assume (7). Then $(R/I)_\mathfrak{p}$ is either 0 or $R_\mathfrak{p}$ for any prime $\mathfrak{p}$ of R. Hence by Lemma 10.36.19 we see that (1) holds. At this point we see that all of (1) – (7) and (9) are equivalent.

As $IR_\mathfrak{p} = I_\mathfrak{p}$ we see that (7) implies (8). Finally, if (8) holds, then this means exactly that $I_\mathfrak{p}$ is the zero module if and only if $\mathfrak{p} \in V(I)$, which is clearly saying that (7) holds. Now (1) – (9) are equivalent.

Assume (1) – (9) hold. Then $R/I \subset (1+I)^{-1}R$ by (9) and the map $R/I \to (1+I)^{-1}R$ is also surjective by the description of localizations at primes afforded by (7). Hence (11) holds.

The implication (11) $\Rightarrow$ (10) is trivial. And (10) implies that (1) holds because a localization of R is flat over R, see Lemma 10.36.19. □

Lemma 10.103.3. *Let R be a ring. If $I, J \subset R$ are pure ideals, then $V(I) = V(J)$ implies $I = J$.*

Proof. For example, by property (7) of Lemma 10.103.2 we see that $I = \mathrm{Ker}(R \to \prod_{\mathfrak{p} \in V(I)} R_\mathfrak{p})$ can be recovered from the closed subset associated to it. □

Lemma 10.103.4. *Let R be a ring. The rule $I \mapsto V(I)$ determines a bijection*

$$\{I \subset R \text{ pure}\} \leftrightarrow \{Z \subset \mathrm{Spec}(R) \text{ closed and closed under generalizations}\}$$

Proof. Let I be a pure ideal. Then since $R \to R/I$ is flat, by going up generalizations lift along the map $\mathrm{Spec}(R/I) \to \mathrm{Spec}(R)$. Hence $V(I)$ is closed under generalizations. This shows that the map is well defined. By Lemma 10.103.3 the map is injective. Suppose that $Z \subset \mathrm{Spec}(R)$ is closed and closed under generalizations. Let $J \subset R$ be the radical ideal such that $Z = V(J)$. Let $I = \{x \in R : x \in xJ\}$. Note that I is an ideal. We claim that I is pure and that $V(I) = V(J)$. If the claim is true then the map of the lemma is surjective and the lemma holds.

Note that $I \subset J$, so that $V(J) \subset V(I)$. Let $I \subset \mathfrak{p}$ be a prime. Consider the multiplicative subset $S = (R \setminus \mathfrak{p})(1+J)$. By definition of I and $I \subset \mathfrak{p}$ we see that $0 \notin S$. Hence we can find a prime $\mathfrak{q}$ of R which is disjoint from S, see Lemmas 10.9.4 and 10.16.5. Hence $\mathfrak{q} \subset \mathfrak{p}$ and $\mathfrak{q} \cap (1+J) = \emptyset$. This implies that $\mathfrak{q} + J$ is a proper ideal of R. Let $\mathfrak{m}$ be a maximal ideal containing $\mathfrak{q} + J$. Then we get $\mathfrak{m} \in V(J)$ and hence $\mathfrak{q} \in V(J) = Z$ as Z was assumed to be closed under generalization. This in turn implies $\mathfrak{p} \in V(J)$ as $\mathfrak{q} \subset \mathfrak{p}$. Thus we see that $V(I) = V(J)$.

Finally, since $V(I) = V(J)$ (and J radical) we see that $J = \sqrt{I}$. Pick $x \in I$, so that $x = xy$ for some $y \in J$ by definition. Then $x = xy = xy^2 = \ldots = xy^n$. Since $y^n \in I$ for some $n > 0$ we conclude that property (5) of Lemma 10.103.2 holds and we see that I is indeed pure. □

Lemma 10.103.5. *Let R be a ring. Let $I \subset R$ be an ideal. The following are equivalent*

(1) *I is pure and finitely generated,*
(2) *I is generated by an idempotent,*
(3) *I is pure and $V(I)$ is open, and*
(4) *R/I is a projective R-module.*

Proof. If (1) holds, then $I = I \cap I = I^2$ by Lemma 10.103.2. Hence I is generated by an idempotent by Lemma 10.19.5. Thus (1) $\Rightarrow$ (2). If (2) holds, then $I = (e)$ and $R = (1-e) \oplus (e)$ as an R-module hence R/I is flat and I is pure and $V(I) = D(1-e)$ is open. Thus (2) $\Rightarrow$ (1) + (3). Finally, assume (3). Then $V(I)$ is open and closed, hence $V(I) = D(1-e)$ for some idempotent e of R, see Lemma 10.19.3. The ideal $J = (e)$ is a pure ideal such that $V(J) = V(I)$ hence $I = J$ by Lemma 10.103.3. In this way we see that (3) $\Rightarrow$ (2). By Lemma 10.73.2 we see that (4) is equivalent to the assertion that I is pure and R/I finitely presented. Moreover, R/I is finitely presented if and only if I is finitely generated, see Lemma 10.5.3. Hence (4) is equivalent to (1). □

We can use the above to characterize those rings for which every finite flat module is finitely presented.

Lemma 10.103.6. *Let R be a ring. The following are equivalent:*

(1) *every $Z \subset \mathrm{Spec}(R)$ which is closed and closed under generalizations is also open, and*
(2) *any finite flat R-module is finite locally free.*

Proof. If any finite flat R-module is finite locally free then the support of R/I where I is a pure ideal is open. Hence the implication (2) $\Rightarrow$ (1) follows from Lemma 10.103.3.

For the converse assume that R satisfies (1). Let M be a finite flat R-module. The support $Z = \mathrm{Supp}(M)$ of M is closed, see Lemma 10.37.5. On the other hand, if $\mathfrak{p} \subset \mathfrak{p}'$, then by Lemma 10.73.4 the module $M_{\mathfrak{p}'}$ is free, and $M_{\mathfrak{p}} = M_{\mathfrak{p}'} \otimes_{R_{\mathfrak{p}'}} R_{\mathfrak{p}}$ Hence $\mathfrak{p}' \in \mathrm{Supp}(M) \Rightarrow \mathfrak{p} \in \mathrm{Supp}(M)$, in other words, the support is closed under generalization. As R satisfies (1) we see that the support of M is open and closed. Suppose that M is generated by r elements $m_1, \ldots, m_r$. The modules $\wedge^i(M)$, $i = 1, \ldots, r$ are finite flat R-modules also, because $\wedge^i(M)_{\mathfrak{p}} = \wedge^i(M_{\mathfrak{p}})$ is free over $R_{\mathfrak{p}}$. Note that $\mathrm{Supp}(\wedge^{i+1}(M)) \subset \mathrm{Supp}(\wedge^i(M))$. Thus we see that there exists a decomposition

$$\mathrm{Spec}(R) = U_0 \coprod U_1 \coprod \ldots \coprod U_r$$

by open and closed subsets such that the support of $\wedge^i(M)$ is $U_r \cup \ldots \cup U_i$ for all $i = 0, \ldots, r$. Let $\mathfrak{p}$ be a prime of R, and say $\mathfrak{p} \in U_i$. Note that $\wedge^i(M) \otimes_R \kappa(\mathfrak{p}) = \wedge^i(M \otimes_R \kappa(\mathfrak{p}))$. Hence, after possibly renumbering $m_1, \ldots, m_r$ we may assume that $m_1, \ldots, m_i$ generate $M \otimes_R \kappa(\mathfrak{p})$. By Nakayama's Lemma 10.18.1 we get a surjection

$$R_f^{\oplus i} \longrightarrow M_f, \quad (a_1, \ldots, a_i) \longmapsto \sum a_i m_i$$

for some $f \in R$, $f \notin \mathfrak{p}$. We may also assume that $D(f) \subset U_i$. This means that $\wedge^i(M_f) = \wedge^i(M)_f$ is a flat R_f module whose support is all of $\mathrm{Spec}(R_f)$. By the above it is generated by a single element, namely $m_1 \wedge \ldots \wedge m_i$. Hence $\wedge^i(M)_f \cong R_f/J$ for some pure ideal $J \subset R_f$ with $V(J) = \mathrm{Spec}(R_f)$. Clearly this means that $J = (0)$, see Lemma 10.103.3. Thus $m_1 \wedge \ldots \wedge m_i$ is a basis for $\wedge^i(M_f)$ and it follows that the displayed map is injective as well as surjective. This proves that M is finite locally free as desired. □

10.104. Rings of finite global dimension

The following lemma is often used to compare different projective resolutions of a given module.

Lemma 10.104.1 (Schanuel's lemma)**.** *Let R be a ring. Let M be an R-module. Suppose that $0 \to K \to P_1 \to M \to 0$ and $0 \to L \to P_2 \to M \to 0$ are two short exact sequences, with P_i projective. Then $K \oplus P_2 \cong L \oplus P_1$.*

Proof. Consider the module N defined by the short exact sequence $0 \to N \to P_1 \oplus P_2 \to M \to 0$, where the last map is the sum of the two maps $P_i \to M$. It is easy to see that the projection $N \to P_1$ is surjective with kernel L, and that $N \to P_2$ is surjective with kernel K. Since P_i are projective we have $N \cong K \oplus P_2 \cong L \oplus P_1$. □

Definition 10.104.2. Let R be a ring. Let M be an R-module. We say M has *finite projective dimension* if it has a finite length resolution by projective R-modules. The minimal length of such a resolution is called the *projective dimension* of M.

It is clear that the projective dimension of M is 0 if and only if M is a projective module. The following lemma explains to what extend the projective dimension is independent of the choice of a projective resolution.

Lemma 10.104.3. *Let R be a ring. Suppose that M is an R-module of projective dimension d. Suppose that $F_e \to F_{e-1} \to \ldots \to F_0 \to M \to 0$ is exact with F_i projective and $e \geq d - 1$. Then the kernel of $F_e \to F_{e-1}$ is projective (or the kernel of $F_0 \to M$ is projective in case $e = 0$).*

Proof. We prove this by induction on d. If $d = 0$, then M is projective. In this case there is a splitting $F_0 = \mathrm{Ker}(F_0 \to M) \oplus M$, and hence $\mathrm{Ker}(F_0 \to M)$ is projective. This finishes the proof if $e = 0$, and if $e > 0$, then replacing M by $\mathrm{Ker}(F_0 \to M)$ we decrease e.

Next assume $d > 0$. Let $0 \to P_d \to P_{d-1} \to \ldots \to P_0 \to M \to 0$ be a minimal length finite resolution with P_i projective. According to Schanuel's Lemma 10.104.1 we have $P_0 \oplus \mathrm{Ker}(F_0 \to M) \cong F_0 \oplus \mathrm{Ker}(P_0 \to M)$. This proves the case $d = 1$, $e = 0$, because then the right hand side is $F_0 \oplus P_1$ which is projective. Hence now we may assume $e > 0$. The module $F_0 \oplus \mathrm{Ker}(P_0 \to M)$ has the finite projective resolution $0 \to P_d \oplus F_0 \to P_{d-1} \oplus F_0 \to \ldots \to P_1 \oplus F_0 \to \mathrm{Ker}(P_0 \to M) \oplus F_0 \to 0$ of length $d - 1$. By induction on d we see that the kernel of $F_e \oplus P_0 \to F_{e-1} \oplus P_0$ is projective. This implies the lemma. □

Lemma 10.104.4. *Let R be a ring. Let M be an R-module. Let $n \geq 0$. The following are equivalent*

1. *M has projective dimension $\leq n$,*
2. *$Ext^i_R(M, N) = 0$ for all R-modules N and all $i \geq n + 1$, and*
3. *$Ext^{n+1}_R(M, N) = 0$ for all R-modules N.*

Proof. Assume (1). Choose a free resolution $F_\bullet \to M$ of M. Denote $d_e : F_e \to F_{e-1}$. By Lemma 10.104.3 we see that $P_e = \mathrm{Ker}(d_e)$ is projective for $e \geq n - 1$. This implies that $F_e \cong P_e \oplus P_{e-1}$ for $e \geq n$ where d_e maps the summand P_{e-1} isomorphically to P_{e-1} in F_{e-1}. Hence, for any R-module N the complex $\mathrm{Hom}_R(F_\bullet, N)$ is split exact in degrees $\geq n + 1$. Whence (2) holds. The implication (2) $\Rightarrow$ (3) is trivial.

Assume (3) holds. If $n = 0$ then M is projective by Lemma 10.72.2 and we see that (1) holds. If $n > 0$ choose a free R-module F and a surjection $F \to M$ with kernel K. By Lemma 10.67.7 and the vanishing of $\mathrm{Ext}^i_R(F, N)$ for all $i > 0$ by part (1)

we see that $\mathrm{Ext}^n_R(K, N) = 0$ for all R-modules N. Hence by induction we see that N has projective dimension $\leq n-1$. Then M has projective dimension $\leq n$ as any finite projective resolution of N gives a projective resolution of length one more for M by adding F to the front. □

Lemma 10.104.5. *Let R be a ring. Let $0 \to M' \to M \to M'' \to 0$ be a short exact sequence of R-modules.*

1. *If M has projective dimension $\leq n$ and M'' has projective dimension $\leq n+1$, then M' has projective dimension $\leq n$.*
2. *If M' and M'' have projective dimension $\leq n$ then M has projective dimension $\leq n$.*
3. *If M' has projective dimension $\leq n$ and M has projective dimension $\leq n+1$ then M'' has projective dimension $\leq n+1$.*

Proof. Combine the characterization of projective dimension in Lemma 10.104.4 with the long exact sequence of ext groups in Lemma 10.67.7. □

Definition 10.104.6. Let R be a ring. The ring R is said to have *finite global dimension* if there exists an integer n such that every R-module has a resolution by projective R-modules of length at most n. The minimal such n is then called the *global dimension* of R.

The argument in the proof of the following lemma can be found in the paper [**Aus55**] by Auslander.

Lemma 10.104.7. *Let R be a ring. The following are equivalent*

1. *R has finite global dimension $\leq n$,*
2. *every finite R-module has projective dimension $\leq n$, and*
3. *every cyclic R-module R/I has projective dimension $\leq n$.*

Proof. It is clear that (1) $\Rightarrow$ (3). Assume (3). Since every finite R-module has a finite filtration by cyclic modules, see Lemma 10.5.4 we see that (2) follows by Lemma 10.104.5.

Assume (2). Let M be an arbitrary R-module. Choose a set $E \subset M$ of generators of M. Choose a well ordering on E. For $e \in E$ denote M_e the submodule of M generated by the elements $e' \in E$ with $e' \leq e$. Then $M = \bigcup_{e \in E} M_e$. Note that for each $e \in E$ the quotient

$$M_e / \bigcup\nolimits_{e' < e} M_{e'}$$

is either zero or generated by one element, hence has projective dimension $\leq n$. To finish the proof we claim that any time we have a well ordered set E and a module $M = \bigcup_{e \in E} M_e$ such that the quotients $M_e / \bigcup_{e' < e} M_{e'}$ have projective dimension $\leq n$, then M has projective dimension $\leq n$.

We may prove this statement by induction on n. If $n = 0$, then we will show, by transfinite induction that M is projective. Namely, for each $e \in E$ we may choose a splitting $M_e = \bigcup_{e' < e} M_{e'} \oplus P_e$ because $P_e = M_e / \bigcup_{e' < e} M_{e'}$ is projective. Hence it follows that $M = \bigoplus_{e \in E} P_e$ and we conclude that M is projective, see Lemma 10.72.3.

If $n > 0$, then for $e \in E$ we denote F_e the free R-module on the set of elements of M_e. Then we have a system of short exact sequences

$$0 \to K_e \to F_e \to M_e \to 0$$

over the well ordered set E. Note that the transition maps $F_{e'} \to F_e$ and $K_{e'} \to K_e$ are injective too. Set $F = \bigcup F_e$ and $K = \bigcup K_e$. Then

$$0 \to K_e/\bigcup_{e'<e} K_{e'} \to F_e/\bigcup_{e'<e} F_{e'} \to M_e/\bigcup_{e'<e} M_{e'} \to 0$$

is a short exact sequence of R-modules too and $F_e/\bigcup_{e'<e} F_{e'}$ is the free R-module on the set of elements in M_e which are not contained in $\bigcup_{e'<e} M_{e'}$. Hence by Lemma 10.104.5 we see that the projective dimension of $K_e/\bigcup_{e'<e} K_{e'}$ is at most $n-1$. By induction we conclude that K has projective dimension at most $n-1$. Whence M has projective dimension at most n and we win. □

Lemma 10.104.8. *Let R be a ring. Let M be an R-module. Let $S \subset R$ be a multiplicative subset.*

1. *If M has projective dimension $\leq n$, then $S^{-1}M$ has projective dimension $\leq n$ over $S^{-1}R$.*
2. *If R has finite global dimension $\leq n$, then $S^{-1}R$ has finite global dimension $\leq n$.*

Proof. Let $0 \to P_n \to P_{n-1} \to \ldots \to P_0 \to M \to 0$ be a projective resolution. As localization is exact, see Proposition 10.9.12, and as each $S^{-1}P_i$ is a projective $S^{-1}R$-module, see Lemma 10.89.1, we see that $0 \to S^{-1}P_n \to \ldots \to S^{-1}P_0 \to S^{-1}M \to 0$ is a projective resolution of $S^{-1}M$. This proves (1). Let M' be an $S^{-1}R$-module. Note that $M' = S^{-1}M'$. Hence we see that (2) follows from (1). □

10.105. Regular rings and global dimension

We can use the material on rings of finite global dimension to give another characterization of regular local rings.

Proposition 10.105.1. *Let R be a regular local ring of dimension d. Every finite R-module M of depth e has a finite free resolution*

$$0 \to F_{d-e} \to \ldots \to F_0 \to M \to 0.$$

In particular a regular local ring has global dimension $\leq d$.

Proof. This is clear in view of Lemma 10.101.6 and Lemma 10.99.10. □

Lemma 10.105.2. *Let R be a Noetherian ring. Then R has finite global dimension if and only if there exists an integer n such that for all maximal ideals $\mathfrak{m}$ of R the ring $R_\mathfrak{m}$ has global dimension $\leq n$.*

Proof. We saw, Lemma 10.104.8 that if R has finite global dimension n, then all the localizations $R_\mathfrak{m}$ have finite global dimension at most n. Conversely, suppose that all the $R_\mathfrak{m}$ have global dimension n. Let M be a finite R-module. Let $0 \to K_n \to F_{n-1} \to \ldots \to F_0 \to M \to 0$. be a resolution with F_i finite free. Then K_n is a finite R-module. According to Lemma 10.104.3 and the assumption all the modules $K_n \otimes_R R_\mathfrak{m}$ are projective. Hence by Lemma 10.73.2 the module K_n is finite projective. □

Lemma 10.105.3. *Suppose that R is a Noetherian local ring with maximal ideal $\mathfrak{m}$ and residue field κ. In this case the projective dimension of κ is $\geq \dim_\kappa \mathfrak{m}/\mathfrak{m}^2$.*

Proof. Let $x_1, \ldots x_n$ be elements of $\mathfrak{m}$ whose images in $\mathfrak{m}/\mathfrak{m}^2$ form a basis. Consider the *Koszul complex* on $x_1, \ldots, x_n$. This is the complex

$$0 \to \wedge^n R^n \to \wedge^{n-1} R^n \to \wedge^{n-2} R^n \to \ldots \to \wedge^i R^n \to \ldots \to R^n \to R$$

with maps given by

$$e_{j_1} \wedge \ldots \wedge e_{j_i} \longmapsto \sum_{a=1}^{i} (-1)^{i+1} x_{j_a} e_{j_1} \wedge \ldots \wedge \hat{e}_{j_a} \wedge \ldots \wedge e_{j_i}$$

It is easy to see that this is a complex $K_\bullet(R, x_\bullet)$. Note that the cokernel of the last map of $K_\bullet(R, x_\bullet)$ is clearly κ.

Now, let $F_\bullet \to \kappa$ by any finite resolution by finite free R-modules. By Lemma 10.96.2 we may assume all the maps in the complex $F_\bullet$ have to property that $\mathrm{Im}(F_i \to F_{i-1}) \subset \mathfrak{m} F_{i-1}$, because removing a trivial summand from the resolution can at worst shorten the resolution. By Lemma 10.67.4 we can find a map of complexes $\alpha : K_\bullet(R, x_\bullet) \to F_\bullet$ inducing the identity on κ. We will prove by induction that the maps $\alpha_i : \wedge^i R^n = K_i(R, x_\bullet) \to F_i$ have the property that $\alpha_i \otimes \kappa : \wedge^i \kappa^n \to F_i \otimes \kappa$ are injective. This will prove the lemma since it clearly shows that $F_n \neq 0$.

The result is clear for $i = 0$ because the composition $R \xrightarrow{\alpha_0} F_0 \to \kappa$ is nonzero. Note that F_0 must have rank 1 since otherwise the map $F_1 \to F_0$ whose cokernel is a single copy of κ cannot have image contained in $\mathfrak{m} F_0$. For α_1 we use that $x_1, \ldots, x_n$ is a minimal set of generators for $\mathfrak{m}$. Namely, we saw above that $F_0 = R$ and $F_1 \to F_0 = R$ has image $\mathfrak{m}$. We have a commutative diagram

$$\begin{array}{ccccccc} R^n & = & K_1(R, x_\bullet) & \to & K_0(R, x_\bullet) & = & R \\ & & \downarrow & & \downarrow & & \downarrow \\ & & F_1 & \to & F_0 & = & R \end{array}$$

where the rightmost vertical arrow is given by multiplication by a unit. Hence we see that the image of the composition $R^n \to F_1 \to F_0 = R$ is also equal to $\mathfrak{m}$. Thus the map $R^n \otimes \kappa \to F_1 \otimes \kappa$ has to be injective since $\dim_\kappa(\mathfrak{m}/\mathfrak{m}^2) = n$.

Suppose the injectivity of $\alpha_j \otimes \kappa$ has been proved for all $j \leq i - 1$. Consider the commutative diagram

$$\begin{array}{ccccccc} \wedge^i R^n & = & K_i(R, x_\bullet) & \to & K_{i-1}(R, x_\bullet) & = & \wedge^{i-1} R^n \\ & & \downarrow & & \downarrow & & \\ & & F_i & \to & F_{i-1} & & \end{array}$$

We know that $\wedge^{i-1}\kappa^n \to F_{i-1} \otimes \kappa$ is injective. This proves that $\wedge^{i-1}\kappa^n \otimes_\kappa \mathfrak{m}/\mathfrak{m}^2 \to F_{i-1} \otimes \mathfrak{m}/\mathfrak{m}^2$ is injective. Also, by our choice of the complex, F_i maps into $\mathfrak{m} F_{i-1}$, and similarly for the Koszul complex. Hence we get a commutative diagram

$$\begin{array}{ccc} \wedge^i \kappa^n & \to & \wedge^{i-1}\kappa^n \otimes \mathfrak{m}/\mathfrak{m}^n \\ \downarrow & & \downarrow \\ F_i \otimes \kappa & \to & F_{i-1} \otimes \mathfrak{m}/\mathfrak{m}^2 \end{array}$$

At this point it suffices to verify the map $\wedge^i \kappa^n \to \wedge^{i-1}\kappa^n \otimes \mathfrak{m}/\mathfrak{m}^n$ is injective, which can be done by hand. □

Lemma 10.105.4. *Let R be a Noetherian local ring. Suppose that the residue field κ has finite projective dimension n over R. In this case* $\dim(R) \geq n$.

Proof. Let $F_\bullet$ be a finite resolution of κ by finite free R-modules. By Lemma 10.96.2 we may assume all the maps in the complex $F_\bullet$ have to property that $\mathrm{Im}(F_i \to F_{i-1}) \subset \mathfrak{m}F_{i-1}$, because removing a trivial summand from the resolution can at worst shorten the resolution. Say $F_n \neq 0$ and $F_i = 0$ for $i > n$, so that the projective dimension of κ is n. By Proposition 10.96.10 we see that $\mathrm{depth}(I(\varphi_n)) \geq n$ since $I(\varphi_n)$ cannot equal R by our choice of the complex. Thus by Lemma 10.65.6 also $\dim(R) \geq n$. □

Proposition 10.105.5. *A Noetherian local ring whose residue field has finite projective dimension is a regular local ring. In particular a Noetherian local ring of finite global dimension is a regular local ring.*

Proof. By Lemmas 10.105.3 and 10.105.4 we see that $\dim(R) \geq \dim_\kappa(\mathfrak{m}/\mathfrak{m}^2)$. Thus the result follows immediately from Definition 10.57.9. □

By Propositions 10.105.5 and 10.105.1 we see that a Noetherian local ring is a regular local ring if and only if it has finite global dimension. Furthermore, any localization $R_\mathfrak{p}$ has finite global dimension, see Lemma 10.104.8, and hence is a regular local ring. Thus it now makes sense to make the following definition, because it does not conflict with the earlier definition of a regular local ring.

Definition 10.105.6. A Noetherian ring R is said to be *regular* if all the localizations $R_\mathfrak{p}$ are regular local rings.

Note that this is not the same as asking R to have finite global dimension, even assuming R is Noetherian. This is because there is an example of a regular Noetherian ring which does not have finite global dimension, namely because it does not have finite dimension.

Lemma 10.105.7. *Let R be a Noetherian ring. The following are equivalent:*

1. *R has finite global dimension n,*
2. *there exists an integer n such that all the localizations $R_\mathfrak{m}$ at maximal ideals are regular of dimension $\leq n$ with equality for at least one $\mathfrak{m}$, and*
3. *there exists an integer n such that all the localizations $R_\mathfrak{p}$ at prime ideals are regular of dimension $\leq n$ with equality for at least one $\mathfrak{p}$.*

Proof. This is a reformulation of Lemma 10.105.2 in view of the discussion surrounding Definition 10.105.6. See especially Propositions 10.105.1 and 10.105.5. □

Lemma 10.105.8. *Let $R \to S$ be a local homomorphism of local Noetherian rings. Assume that $R \to S$ is flat and that S is regular. Then R is regular.*

Proof. Let $\mathfrak{m} \subset R$ be the maximal ideal and let $\kappa = R/\mathfrak{m}$ be the residue field. Let $d = \dim S$. Choose any resolution $F_\bullet \to \kappa$ with each F_i a finite free R-module. Set $K_d = \mathrm{Ker}(F_{d-1} \to F_{d-2})$. By flatness of $R \to S$ the complex $0 \to K_d \otimes_R S \to F_{d-1} \otimes_R S \to \ldots \to F_0 \otimes_R S \to \kappa \otimes_R S \to 0$ is still exact. Because the global dimension of S is d, see Proposition 10.105.1, we see that $K_d \otimes_R S$ is a finite free S-module (see also Lemma 10.104.3). By Lemma 10.73.5 we see that K_d is a finite free R-module. Hence κ has finite projective dimension and R is regular by Proposition 10.105.5. □

10.106. Homomorphisms and dimension

This section contains a collection of easy results relating dimensions of rings when there are maps between them.

Lemma 10.106.1. *Suppose $R \to S$ is a ring map satisfying either going up, see Definition 10.38.1, or going down see Definition 10.38.1. Assume in addition that* $\mathrm{Spec}(S) \to \mathrm{Spec}(R)$ *is surjective. Then* $\dim(R) \leq \dim(S)$.

Proof. Assume going up. Take any chain $\mathfrak{p}_0 \subset \mathfrak{p}_1 \subset \ldots \subset \mathfrak{p}_e$ of prime ideals in R. By surjectivity we may choose a prime $\mathfrak{q}_0$ mapping to $\mathfrak{p}_0$. By going up we may extend this to a chain of length e of primes $\mathfrak{q}_i$ lying over $\mathfrak{p}_i$. Thus $\dim(S) \geq \dim(R)$. The case of going down is exactly the same. See also Topology, Lemma 5.18.8 for a purely topological version. □

Lemma 10.106.2. *Suppose that $R \to S$ is a ring map with the going up property, see Definition 10.38.1. If $\mathfrak{q} \subset S$ is a maximal ideal. Then the inverse image of $\mathfrak{q}$ in R is a maximal ideal too.*

Proof. Trivial. □

Lemma 10.106.3. *Suppose that $R \to S$ is a ring map such that S is integral over R. Then* $\dim(R) \geq \dim(S)$*, and every closed point of* $\mathrm{Spec}(S)$ *maps to a closed point of* $\mathrm{Spec}(R)$.

Proof. Immediate from Lemmas 10.33.18 and 10.106.2 and the definitions. □

Lemma 10.106.4. *Suppose $R \subset S$ and S integral over R. Then* $\dim(R) = \dim(S)$.

Proof. This is a combination of Lemmas 10.33.20, 10.33.15, 10.106.1, and 10.106.3. □

Definition 10.106.5. Suppose that $R \to S$ is a ring map. Let $\mathfrak{q} \subset S$ be a prime lying over the prime $\mathfrak{p}$ of R. The *local ring of the fibre at* $\mathfrak{q}$ is the local ring

$$S_\mathfrak{q}/\mathfrak{p}S_\mathfrak{q} = (S/\mathfrak{p}S)_\mathfrak{q} = (S \otimes_R \kappa(\mathfrak{p}))_\mathfrak{q}$$

Lemma 10.106.6. *Let $R \to S$ be a homomorphism of Noetherian rings. Let $\mathfrak{q} \subset S$ be a prime lying over the prime $\mathfrak{p}$. Then*

$$\dim(S_\mathfrak{q}) \leq \dim(R_\mathfrak{p}) + \dim(S_\mathfrak{q}/\mathfrak{p}S_\mathfrak{q}).$$

Proof. We use the characterization of dimension of Proposition 10.57.8. Let $x_1, \ldots, x_d$ be elements of $\mathfrak{p}$ generating an ideal of definition of $R_\mathfrak{p}$ with $d = \dim(R_\mathfrak{p})$. Let $y_1, \ldots, y_e$ be elements of $\mathfrak{q}$ generating an ideal of definition of $S_\mathfrak{q}/\mathfrak{p}S_\mathfrak{q}$ with $e = \dim(S_\mathfrak{q}/\mathfrak{p}S_\mathfrak{q})$. It is clear that $S_\mathfrak{q}/(x_1, \ldots, x_d, y_1, \ldots, y_e)$ has a nilpotent maximal ideal. Hence $x_1, \ldots, x_d, y_1, \ldots, y_e$ generate an ideal of definition if $S_\mathfrak{q}$. □

Lemma 10.106.7. *Let $R \to S$ be a homomorphism of Noetherian rings. Let $\mathfrak{q} \subset S$ be a prime lying over the prime $\mathfrak{p}$. Assume the going down property holds for $R \to S$ (for example if $R \to S$ is flat, see Lemma 10.36.17). Then*

$$\dim(S_\mathfrak{q}) = \dim(R_\mathfrak{p}) + \dim(S_\mathfrak{q}/\mathfrak{p}S_\mathfrak{q}).$$

Proof. By Lemma 10.106.6 we have an inequality $\dim(S_\mathfrak{q}) \leq \dim(R_\mathfrak{p}) + \dim(S_\mathfrak{q}/\mathfrak{p}S_\mathfrak{q})$. To get equality, choose a chain of primes $\mathfrak{p}S \subset \mathfrak{q}_0 \subset \mathfrak{q}_1 \subset \ldots \subset \mathfrak{q}_d = \mathfrak{q}$ with $d = \dim(S_\mathfrak{q}/\mathfrak{p}S_\mathfrak{q})$. On the other hand, choose a chain of primes $\mathfrak{p}_0 \subset \mathfrak{p}_1 \subset \ldots \subset \mathfrak{p}_e = \mathfrak{p}$ with $e = \dim(R_\mathfrak{p})$. By the going down theorem we may choose $\mathfrak{q}_{-1} \subset \mathfrak{q}_0$ lying over

$\mathfrak{p}_{e-1}$. And then we may choose $\mathfrak{q}_{-2} \subset \mathfrak{q}_{e-1}$ lying over $\mathfrak{p}_{e-2}$. Inductively we keep going until we get a chain $\mathfrak{q}_{-e} \subset \ldots \subset \mathfrak{q}_d$ of length $e + d$. □

Lemma 10.106.8. *Let $R \to S$ be a local homomorphism of local Noetherian rings. Assume*

(1) *R is regular,*
(2) *$S/\mathfrak{m}_R S$ is regular, and*
(3) *$R \to S$ is flat.*

Then S is regular.

Proof. By Lemma 10.106.7 we have $\dim(S) = \dim(R) + \dim(S/\mathfrak{m}_R S)$. Pick generators $x_1, \ldots, x_d \in \mathfrak{m}_R$ with $d = \dim(R)$, and pick $y_1, \ldots, y_e \in \mathfrak{m}_S$ which generate the maximal ideal of $S/\mathfrak{m}_R S$ with $e = \dim(S/\mathfrak{m}_R S)$. Then we see that $x_1, \ldots, x_d, y_1, \ldots, y_e$ are elements which generate the maximal ideal of S and $e+d = \dim(S)$. □

The lemma below will later be used to show that rings of finite type over a field are Cohen-Macaulay if and only if they are quasi-finite flat over a polynomial ring. It is a partial converse to Lemma 10.122.1.

Lemma 10.106.9. *Let $R \to S$ be a local homomorphism of Noetherian local rings. Assume R Cohen-Macaulay. If S is finite flat over R, or if S is flat over R and $\dim(S) \leq \dim(R)$, then S is Cohen-Macaulay and $\dim(R) = \dim(S)$.*

Proof. Let $x_1, \ldots, x_d \in \mathfrak{m}_R$ be a regular sequence of length $d = \dim(R)$. By Lemma 10.65.7 this maps to a regular sequence in S. Hence S is Cohen-Macaulay if $\dim(S) \leq d$. This is true if S is finite flat over R by Lemma 10.106.4. And in the second case we assumed it. □

10.107. The dimension formula

Recall the definitions of catenary (Definition 10.100.1) and universally catenary (Definition 10.100.5).

Lemma 10.107.1. *Let $R \to S$ be a ring map. Let $\mathfrak{q}$ be a prime of S lying over the prime $\mathfrak{p}$ of R. Assume that*

(1) *R is Noetherian,*
(2) *$R \to S$ is of finite type,*
(3) *R, S are domains, and*
(4) *$R \subset S$.*

Then we have

$$\textit{height}(\mathfrak{q}) \leq \textit{height}(\mathfrak{p}) + \textit{trdeg}_R(S) - \textit{trdeg}_{\kappa(\mathfrak{p})}\kappa(\mathfrak{q})$$

with equality if R is universally catenary.

Proof. Suppose that $R \subset S' \subset S$ is a finitely generated R-subalgebra of S. In this case set $\mathfrak{q}' = S' \cap \mathfrak{q}$. The lemma for the ring maps $R \to S'$ and $S' \to S$ implies the lemma for $R \to S$ by additivity of transcendence degree in towers of fields. Hence we can use induction on the number of generators of S over R and reduce to the case where S is generated by one element over R.

Case I: $S = R[x]$ is a polynomial algebra over R. In this case we have $\text{trdeg}_R(S) = 1$. Also $R \to S$ is flat and hence

$$\dim(S_\mathfrak{q}) = \dim(R_\mathfrak{p}) + \dim(S_\mathfrak{q}/\mathfrak{p}S_\mathfrak{q})$$

see Lemma 10.106.7. Let $\mathfrak{r} = \mathfrak{m}S$. Then $\text{trdeg}_{\kappa(\mathfrak{p})}\kappa(\mathfrak{q}) = 1$ is equivalent to $\mathfrak{q} = \mathfrak{r}$, and implies that $\dim(S_\mathfrak{q}/\mathfrak{p}S_\mathfrak{q}) = 0$. In the same vein $\text{trdeg}_{\kappa(\mathfrak{p})}\kappa(\mathfrak{q}) = 0$ is equivalent to having a strict inclusion $\mathfrak{q} \subset \mathfrak{r}$, which implies that $\dim(S_\mathfrak{q}/\mathfrak{p}S_\mathfrak{q}) = 1$. Thus we are done with case I with equality in every instance.

Case II: $S = R[x]/\mathfrak{n}$ with $\mathfrak{n} \neq 0$. In this case we have $\text{trdeg}_R(S) = 0$. Denote $\mathfrak{q}' \subset R[x]$ the prime corresponding to $\mathfrak{q}$. Thus we have

$$S_\mathfrak{q} = (R[x])_{\mathfrak{q}'}/\mathfrak{n}(R[x])_{\mathfrak{q}'}$$

By the previous case we have $\dim((R[x])_{\mathfrak{q}'}) = \dim(R_\mathfrak{p}) + 1 - \text{trdeg}_{\kappa(\mathfrak{p})}\kappa(\mathfrak{q})$. Since $\mathfrak{n} \neq 0$ we see that the dimension of $S_\mathfrak{q}$ decreases by at least one, see Lemma 10.57.11, which proves the inequality of the lemma. To see the equality in case R is universally catenary note that $\mathfrak{n} \subset R[x]$ is a height one prime as it corresponds to a nonzero prime in $f.f.(R)[x]$. Hence any maximal chain of primes in $R[x]_{\mathfrak{q}'}/\mathfrak{n}$ corresponds to a maximal chain of primes with length 1 greater between $\mathfrak{q}'$ and (0) in $R[x]$. If R is universally catenary these all have the same length equal to the height of $\mathfrak{q}'$. This proves that $\dim(R[x]_{\mathfrak{q}'}/\mathfrak{n}) = \dim(R[x]_{\mathfrak{q}'}) - 1$ as desired. □

The following lemma says that generically finite maps tend to be quasi-finite in codimension 1.

Lemma 10.107.2. *Let $A \to B$ be a ring map. Assume*

(1) *$A \subset B$ is an extension of domains.*
(2) *A is Noetherian,*
(3) *$A \to B$ is of finite type, and*
(4) *the extension $f.f.(A) \subset f.f.(B)$ is finite.*

Let $\mathfrak{p} \subset A$ be a prime such that $\dim(A_\mathfrak{p}) = 1$. Then there are at most finitely many primes of B lying over $\mathfrak{p}$.

Proof. By the dimension formula (Lemma 10.107.1) for any prime $\mathfrak{q}$ lying over $\mathfrak{p}$ we have

$$\dim(B_\mathfrak{q}) \leq \dim(A_\mathfrak{p}) - \text{trdeg}_{\kappa(\mathfrak{p})}\kappa(\mathfrak{q}).$$

As the domain $B_\mathfrak{q}$ has at least 2 prime ideals we see that $\dim(B_\mathfrak{q}) \geq 1$. We conclude that $\dim(B_\mathfrak{q}) = 1$ and that the extension $\kappa(\mathfrak{p}) \subset \kappa(\mathfrak{q})$ is algebraic. Hence $\mathfrak{q}$ defines a closed point of its fibre $\text{Spec}(B \otimes_A \kappa(\mathfrak{p}))$, see Lemma 10.32.9. Since $B \otimes_A \kappa(\mathfrak{p})$ is a Noetherian ring the fibre $\text{Spec}(B \otimes_A \kappa(\mathfrak{p}))$ is a Noetherian topological space, see Lemma 10.29.5. A Noetherian topological space consisting of closed points is finite, see for example Topology, Lemma 5.8.2. □

10.108. Dimension of finite type algebras over fields

In this section we compute the dimension of a polynomial ring over a field. We also prove that the dimension of a finite type domain over a field is the dimension of its local rings at maximal ideals. We will establish the connection with the transcendence degree over the ground field in Section 10.110.

Lemma 10.108.1. *Let $\mathfrak{m}$ be a maximal ideal in $k[x_1, \ldots, x_n]$. The ideal $\mathfrak{m}$ is generated by n elements. The dimension of $k[x_1, \ldots, x_n]_{\mathfrak{m}}$ is n. Hence $k[x_1, \ldots, x_n]_{\mathfrak{m}}$ is a regular local ring of dimension n.*

Proof. By the Hilbert Nullstellensatz (Theorem 10.31.1) we know the residue field $\kappa = \kappa(\mathfrak{m})$ is a finite extension of k. Denote $\alpha_i \in \kappa$ the image of x_i. Denote $\kappa_i = k(\alpha_1, \ldots, \alpha_i) \subset \kappa$, $i = 1, \ldots, n$ and $\kappa_0 = k$. Note that $\kappa_i = k[\alpha_1, \ldots, \alpha_i]$ by field theory. Define inductively elements $f_i \in \mathfrak{m} \cap k[x_1, \ldots, x_i]$ as follows: Let $P_i(T) \in \kappa_{i-1}[T]$ be the monic minimal polynomial of α_i over κ_{i-1}. Let $Q_i(T) \in k[x_1, \ldots, x_{i-1}][T]$ be a monic lift of $P_i(T)$ (of the same degree). Set $f_i = Q_i(x_i)$. Note that if $d_i = \deg_T(P_i) = \deg_T(Q_i) = \deg_{x_i}(f_i)$ then $d_1 d_2 \ldots d_i = [\kappa_i : k]$ by elementary field theory.

We claim that for all $i = 0, 1, \ldots, n$ there is an isomorphism

$$\psi_i : k[x_1, \ldots, x_i]/(f_1, \ldots, f_i) \cong \kappa_i.$$

By construction the composition $k[x_1, \ldots, x_i] \to k[x_1, \ldots, x_n] \to \kappa$ is surjective onto κ_i and $f_1, \ldots, f_i$ are in the kernel. This gives a surjective homomorphism. We prove ψ_i is injective by induction. It is clear for $i = 0$. Given the statement for i we prove it for $i + 1$. The ring extension $k[x_1, \ldots, x_i]/(f_1, \ldots, f_i) \to k[x_1, \ldots, x_{i+1}]/(f_1, \ldots, f_{i+1})$ is generated by 1 element over a field and one irreducible equation. By elementary field theory $k[x_1, \ldots, x_{i+1}]/(f_1, \ldots, f_{i+1})$ is a field, and hence ψ_i is injective.

This implies that $\mathfrak{m} = (f_1, \ldots, f_n)$. Moreover, we also conclude that

$$k[x_1, \ldots, x_n]/(f_1, \ldots, f_i) \cong \kappa_i[x_{i+1}, \ldots, x_n].$$

Hence $(f_1, \ldots, f_i)$ is a prime ideal. Thus

$$(0) \subset (f_1) \subset (f_1, f_2) \subset \ldots \subset (f_1, \ldots, f_n) = \mathfrak{m}$$

is a chain of primes of length n. The lemma follows. □

Proposition 10.108.2. *A polynomial algebra in n variables over a field is a regular ring. It has global dimension n. All localizations at maximal ideals are regular local rings of dimension n.*

Proof. By Lemma 10.108.1 all localizations $k[x_1, \ldots, x_n]_{\mathfrak{m}}$ at maximal ideals are regular local rings of dimension n. Hence we conclude by Lemma 10.105.7. □

Lemma 10.108.3. *Let k be a field. Let $\mathfrak{p} \subset \mathfrak{q} \subset k[x_1, \ldots, x_n]$ be a pair of primes. Any maximal chain of primes between $\mathfrak{p}$ and $\mathfrak{q}$ has length height($\mathfrak{q}$) − height($\mathfrak{p}$).*

Proof. By Proposition 10.108.2 any local ring of $k[x_1, \ldots, x_n]$ is regular. Hence all local rings are Cohen-Macaulay, see Lemma 10.101.3. The local rings at maximal ideals have dimension n hence every maximal chain of primes in $k[x_1, \ldots, x_n]$ has length n, see Lemma 10.99.3. Hence every maximal chain of primes between (0) and $\mathfrak{p}$ has length height($\mathfrak{p}$), see Lemma 10.99.4 for example. Putting these together leads to the assertion of the lemma. □

Lemma 10.108.4. *Let k be a field. Let S be a finite type k-algebra which is an integral domain. Then $\dim(S) = \dim(S_{\mathfrak{m}})$ for any maximal ideal $\mathfrak{m}$ of S. In words: every maximal chain of primes has length equal to the dimension of S.*

Proof. Write $S = k[x_1, \ldots, x_n]/\mathfrak{p}$. By Proposition 10.108.2 and Lemma 10.108.3 all the maximal chains of primes in S (which necessarily end with a maximal ideal) have length $n - \text{height}(\mathfrak{p})$. Thus this number is the dimension of S and of $S_\mathfrak{m}$ for any maximal ideal $\mathfrak{m}$ of S. □

Recall that we defined the dimension $\dim_x(X)$ of a topological space X at a point x in Topology, Definition 5.9.1.

Lemma 10.108.5. *Let k be a field. Let S be a finite type k-algebra. Let $X = \text{Spec}(S)$. Let $\mathfrak{p} \subset S$ be a prime ideal and let $x \in X$ be the associated point. The following numbers are equal*

1. $\dim_x(X)$,
2. $\max \dim(Z)$ *where the maximum is over those irreducible components Z of X passing through x, and*
3. $\min \dim(S_\mathfrak{m})$ *where the minimum is over maximal ideals $\mathfrak{m}$ with $\mathfrak{p} \subset \mathfrak{m}$.*

Proof. Let $X = \bigcup_{i \in I} Z_i$ be the decomposition of X into its irreducible components. There are finitely many of them (see Lemmas 10.29.3 and 10.29.5). Let $I' = \{i \mid x \in Z_i\}$, and let $T = \bigcup_{x \notin I'} Z_i$. Then $U = X \setminus T$ is an open subset of X containing the point x. The number (2) is $\max_{i \in I'} \dim(Z_i)$. For any open $W \subset U$, with $x \in W$ the irreducible components of W are the irreducible sets $W_i = Z_i \cap W$ for $i \in I'$. Note that each W_i, $i \in I'$ contains a closed point because X is Jacobson, see Section 10.32. By Lemma 10.108.4 we see that $\dim(W_i) = \dim(Z_i)$ for any $i \in I'$. Hence $\dim(W)$ is equal to the number (2). This proves that (1) = (2).

Let $\mathfrak{m} \supset \mathfrak{p}$ be any maximal ideal containing $\mathfrak{p}$. Let $x_0 \in X$ be the corresponding point. First of all, x_0 is contained in all the irreducible components Z_i, $i \in I'$. Let $\mathfrak{q}_i$ denote the minimal primes of S corresponding to the irreducible components Z_i. For each i such that $x_0 \in Z_i$ (which is equivalent to $\mathfrak{m} \supset \mathfrak{q}_i$) we have a surjection

$$S_\mathfrak{m} \longrightarrow S_\mathfrak{m}/\mathfrak{q}_i S_\mathfrak{m} = (S/\mathfrak{q}_i)_\mathfrak{m}$$

Moreover, the primes $\mathfrak{q}_i S_\mathfrak{m}$ so obtained exhaust the minimal primes of the Noetherian local ring $S_\mathfrak{m}$, see Lemma 10.24.3. We conclude, using Lemma 10.108.4, that the dimension of $S_\mathfrak{m}$ is the maximum of the dimensions of the Z_i passing through x_0. To finish the proof of the lemma it suffices to show that we can choose x_0 such that $x_0 \in Z_i \Rightarrow i \in I'$. Because S is Jacobson (as we saw above) it is enough to show that $V(\mathfrak{p}) \setminus T$ (with T as above) is nonempty. And this is clear since it contains the point x (i.e. $\mathfrak{p}$). □

Lemma 10.108.6. *Let k be a field. Let S be a finite type k-algebra. Let $X = \text{Spec}(S)$. Let $\mathfrak{m} \subset S$ be a maximal ideal and let $x \in X$ be the associated closed point. Then $\dim_x(X) = \dim(S_\mathfrak{m})$.*

Proof. This is a special case of Lemma 10.108.5. □

Lemma 10.108.7. *Let k be a field. Let S be a finite type k algebra. Assume that S is Cohen-Macaulay. Then $\text{Spec}(S) = \coprod T_d$ is a finite disjoint union of open and closed subsets T_d with T_d equidimensional (see Topology, Definition 5.9.4) of dimension d. Equivalently, S is a product of rings S_d, $d = 0, \ldots, \dim(S)$ such that every maximal ideal $\mathfrak{m}$ of S_d has height d.*

Proof. The equivalence of the two statements follows from Lemma 10.21.3. Let $\mathfrak{m} \subset S$ be a maximal ideal. Every maximal chain of primes in $S_\mathfrak{m}$ has the same length equal to $\dim(S_\mathfrak{m})$, see Lemma 10.99.3. Hence, the dimension of the irreducible components passing through the point corresponding to $\mathfrak{m}$ all have dimension equal to $\dim(S_\mathfrak{m})$, see Lemma 10.108.4. Since $\mathrm{Spec}(S)$ is a Jacobson topological space the intersection of any two irreducible components of it contains a closed point if nonempty, see Lemmas 10.32.2 and 10.32.4. Thus we have shown that any two irreducible components that meet have the same dimension. The lemma follows easily from this, and the fact that $\mathrm{Spec}(S)$ has a finite number of irreducible components (see Lemmas 10.29.3 and 10.29.5). □

10.109. Noether normalization

In this section we prove variants of the Noether normalization lemma. The key ingredient we will use is contained in the following two lemmas.

Lemma 10.109.1. *Let $n \in \mathbf{N}$. Let N be a finite nonempty set of multi-indices $\nu = (\nu_1, \ldots, \nu_n)$. Given $e = (e_1, \ldots, e_n)$ we set $e \cdot \nu = \sum e_i \nu_i$. Then for $e_1 \gg e_2 \gg \ldots \gg e_{n-1} \gg e_n$ we have: If $\nu, \nu' \in N$ then*

$$(e \cdot \nu = e \cdot \nu') \Leftrightarrow (\nu = \nu')$$

Proof. Say $N = \{\nu_j\}$ with $\nu_j = (\nu_{j1}, \ldots, \nu_{jn})$. Let $A_i = \max_j \nu_{ji} - \min_j \nu_{ji}$. If for each i we have $e_{i-1} > A_i e_i + A_{i+1} e_{i+1} + \ldots + A_n e_n$ then the lemma holds. For suppose that $e \cdot (\nu - \nu') = 0$. Then for $n \geq 2$,

$$e_1(\nu_1 - \nu'_1) = \sum\nolimits_{i=2}^{n} e_i(\nu'_i - \nu_i).$$

We may assume that $(\nu_1 - \nu'_1) \geq 0$. If $(\nu_1 - \nu'_1) > 0$, then

$$e_1(\nu_1 - \nu'_1) \geq e_1 > A_2 e_2 + \ldots + A_n e_n \geq \sum\nolimits_{i=2}^{n} e_i |\nu'_i - \nu_i| \geq \sum\nolimits_{i=2}^{n} e_i(\nu'_i - \nu_i).$$

This contradiction implies that $\nu'_1 = \nu_1$. By induction, $\nu'_i = \nu_i$ for $2 \leq i \leq n$. □

Lemma 10.109.2. *Let R be a ring. Let $g \in R[x_1, \ldots, x_n]$ be an element which is nonconstant, i.e., $g \notin R$. For $e_1 \gg e_2 \gg \ldots \gg e_{n-1} \gg e_n = 1$ the polynomial*

$$g(x_1 + x_n^{e_1}, x_2 + x_n^{e_2}, \ldots, x_{n-1} + x_n^{e_{n-1}}, x_n) = a x_n^d + \text{lower order terms in } x_n$$

where $d > 0$ and $a \in R$ is one of the nonzero coefficients of g.

Proof. Write $g = \sum_{\nu \in N} a_\nu x^\nu$ with $a_\nu \in R$ not zero. Here N is a finite set of multi-indices as in Lemma 10.109.1 and $x^\nu = x_1^{\nu_1} \ldots x_n^{\nu_n}$. Note that the leading term in

$$(x_1 + x_n^{e_1})^{\nu_1} \ldots (x_{n-1} + x_n^{e_{n-1}})^{\nu_{n-1}} x_n^{\nu_n} \text{ is } x_n^{e_1\nu_1 + \ldots + e_{n-1}\nu_{n-1} + \nu_n}.$$

Hence the lemma follows from Lemma 10.109.1 which guarantees that there is exactly one nonzero term $a_\nu x^\nu$ of g which gives rise to the leading term of $g(x_1 + x_n^{e_1}, x_2 + x_n^{e_2}, \ldots, x_{n-1} + x_n^{e_{n-1}}, x_n)$, i.e., $a = a_\nu$ for the unique $\nu \in N$ such that $e \cdot \nu$ is maximal. □

Lemma 10.109.3. *Let k be a field. Let $S = k[x_1, \ldots, x_n]/I$ for some ideal I. If $I \neq 0$, then there exist $y_1, \ldots, y_{n-1} \in k[x_1, \ldots, x_n]$ such that S is finite over $k[y_1, \ldots, y_{n-1}]$. Moreover we may choose y_i to be in the $\mathbf{Z}$-subalgebra of $k[x_1, \ldots, x_n]$ generated by $x_1, \ldots, x_n$.*

Proof. Pick $f \in I$, $f \neq 0$. It suffices to show the lemma for $k[x_1, \ldots, x_n]/(f)$ since S is a quotient of that ring. We will take $y_i = x_i - x_n^{e_i}$, $i = 1, \ldots, n-1$ for suitable integers e_i. When does this work? It suffices to show that $\overline{x_n} \in k[x_1, \ldots, x_n]/(f)$ is integral over the ring $k[y_1, \ldots, y_{n-1}]$. The equation for $\overline{x_n}$ over this ring is

$$f(y_1 + x_n^{e_1}, \ldots, y_{n-1} + x_n^{e_{n-1}}, x_n) = 0.$$

Hence we are done if we can show there exists integers e_i such that the leading coefficient w.r.t. x_n of the equation above is a nonzero element of k. This can be achieved for example by choosing $e_1 \gg e_2 \gg \ldots \gg e_{n-1}$, see Lemma 10.109.2. □

Lemma 10.109.4. *Let k be a field. Let $S = k[x_1, \ldots, x_n]/I$ for some ideal I. There exist $r \geq 0$, and $y_1, \ldots, y_r \in k[x_1, \ldots, x_n]$ such that (a) the map $k[y_1, \ldots, y_r] \to S$ is injective, and (b) the map $k[y_1, \ldots, y_r] \to S$ is finite. In this case the integer r is the dimension of S. Moreover we may choose y_i to be in the $\mathbf{Z}$-subalgebra of $k[x_1, \ldots, x_n]$ generated by $x_1, \ldots, x_n$.*

Proof. By induction on n, with $n = 0$ being trivial. If $I = 0$, then take $r = n$ and $y_i = x_i$. If $I \neq 0$, then choose $y_1, \ldots, y_{n-1}$ as in Lemma 10.109.3. Let $S' \subset S$ be the subring generated by the images of the y_i. By induction we can choose r and $z_1, \ldots, z_r \in k[y_1, \ldots, y_{n-1}]$ such that (a), (b) hold for $k[z_1, \ldots, z_r] \to S'$. Since $S' \to S$ is injective and finite we see (a), (b) hold for $k[z_1, \ldots, z_r] \to S$. The last assertion follows from Lemma 10.106.4. □

Lemma 10.109.5. *Let k be a field. Let S be a finite type k algebra and denote $X = \mathrm{Spec}(S)$. Let $\mathfrak{q}$ be a prime of S, and let $x \in X$ be the corresponding point. There exists a $g \in S$, $g \notin \mathfrak{q}$ such that $\dim(S_g) = \dim_x(X) =: d$ and such that there exists a finite injective map $k[y_1, \ldots, y_d] \to S_g$.*

Proof. Note that by definition $\dim_x(X)$ is the minimum of the dimensions of S_g for $g \in S$, $g \notin \mathfrak{q}$, i.e., the minimum is attained. Thus the lemma follows from Lemma 10.109.4. □

Lemma 10.109.6. *Let k be a field. Let $\mathfrak{q} \subset k[x_1, \ldots, x_n]$ be a prime ideal. Set $r = trdeg_k\ \kappa(\mathfrak{q})$. Then there exists a finite ring map $\varphi : k[y_1, \ldots, y_n] \to k[x_1, \ldots, x_n]$ such that $\varphi^{-1}(\mathfrak{q}) = (y_{r+1}, \ldots, y_n)$.*

Proof. By induction on n. The case $n = 0$ is clear. Assume $n > 0$. If $r = n$, then $\mathfrak{q} = (0)$ and the result is clear. Choose a nonzero $f \in \mathfrak{q}$. Of course f is nonconstant. After applying an automorphism of the form

$$k[x_1, \ldots, x_n] \longrightarrow k[x_1, \ldots, x_n], \quad x_n \mapsto x_n, \quad x_i \mapsto x_i + x_n^{e_i}\ (i < n)$$

we may assume that f is monic in x_n over $k[x_1, \ldots, x_n]$, see Lemma 10.109.2. Hence the ring map

$$k[y_1, \ldots, y_n] \longrightarrow k[x_1, \ldots, x_n], \quad y_n \mapsto f, \quad y_i \mapsto x_i\ (i < n)$$

is finite. Moreover $y_n \in \mathfrak{q} \cap k[y_1, \ldots, y_n]$ by construction. Thus $\mathfrak{q} \cap k[y_1, \ldots, y_n] = \mathfrak{p}k[y_1, \ldots, y_n] + (y_n)$ where $\mathfrak{p} \subset k[y_1, \ldots, y_{n-1}]$ is a prime ideal. Note that $\kappa(\mathfrak{p}) \subset \kappa(\mathfrak{q})$ is finite, and hence $r = \mathrm{trdeg}_k\ \kappa(\mathfrak{p})$. Apply the induction hypothesis to the pair $(k[y_1, \ldots, y_{n-1}], \mathfrak{p})$ and we obtain a finite ring map $k[z_1, \ldots, z_{n-1}] \to k[y_1, \ldots, y_{n-1}]$ such that $\mathfrak{p} \cap k[z_1, \ldots, z_{n-1}] = (z_{r+1}, \ldots, z_{n-1})$. We extend the ring

map $k[z_1, \ldots, z_{n-1}] \to k[y_1, \ldots, y_{n-1}]$ to a ring map $k[z_1, \ldots, z_n] \to k[y_1, \ldots, y_n]$ by mapping z_n to y_n. The composition of the ring maps

$$k[z_1, \ldots, z_n] \to k[y_1, \ldots, y_n] \to k[x_1, \ldots, x_n]$$

solves the problem. □

Lemma 10.109.7. *Let $R \to S$ be an injective finite type map of domains. Then there exists an integer d and factorization*

$$R \to R[y_1, \ldots, y_d] \to S' \to S$$

by injective maps such that S' is finite over $R[y_1, \ldots, y_d]$ and such that $S'_f \cong S_f$ for some nonzero $f \in R$.

Proof. Pick $x_1, \ldots, x_n \in S$ which generate S over R. Let $K = f.f.(R)$ and $S_K = S \otimes_R K$. By Lemma 10.109.4 we can find $y_1, \ldots, y_d \in S$ such that $K[y_1, \ldots, y_d] \to S_K$ is a finite injective map. Note that $y_i \in S$ because we may pick the y_j in the $\mathbf{Z}$-algebra generated by $x_1, \ldots, x_n$. As a finite ring map is integral (see Lemma 10.33.3) we can find monic $P_i \in K[y_1, \ldots, y_d][T]$ such that $P_i(x_i) = 0$ in S_K. Let $f \in R$ be a nonzero element such that $fP_i \in R[y_1, \ldots, y_d][T]$ for all i. Set $x'_i = fx_i$ and let $S' \subset S$ be the subalgebra generated by $y_1, \ldots, y_d$ and $x'_1, \ldots, x'_n$. Note that x'_i is integral over $R[y_1, \ldots, y_d]$ as we have $Q_i(x'_i) = 0$ where $Q_i = f^{\deg_T(P_i)} P_i(T/f)$ which is a monic polynomial in T with coefficients in $R[y_1, \ldots, y_d]$ by our choice of f. Hence $R[y_1, \ldots, y_n] \subset S'$ is finite by Lemma 10.33.5. By construction $S'_f \cong S_f$ and we win. □

10.110. Dimension of finite type algebras over fields, reprise

This section is a continuation of Section 10.108. In this section we establish the connection between dimension and transcendence degree over the ground field for finite type domains over a field.

Lemma 10.110.1. *Let k be a field. Let S be a finite type k algebra which is an integral domain. Let $K = f.f.(S)$ be the field of fractions of S. Let $r = trdeg(K/k)$ be the transcendence degree of K over k. Then $\dim(S) = r$. Moreover, the local ring of S at every maximal ideal has dimension r.*

Proof. We may write $S = k[x_1, \ldots, x_n]/\mathfrak{p}$. By Lemma 10.108.3 all local rings of S at maximal ideals have the same dimension. Apply Lemma 10.109.4. We get a finite injective ring map

$$k[y_1, \ldots, y_d] \to S$$

with $d = \dim(S)$. Clearly, $k(y_1, \ldots, y_d) \subset K$ is a finite extension and we win. □

Lemma 10.110.2. *Let k be a field. Let S be a finite type k-algebra. Let $\mathfrak{q} \subset \mathfrak{q}' \subset S$ be distinct prime ideals. Then $trdeg_k\ \kappa(\mathfrak{q}') < trdeg_k\ \kappa(\mathfrak{q})$.*

Proof. By Lemma 10.110.1 we have $\dim V(\mathfrak{q}) = \mathrm{trdeg}_k\ \kappa(\mathfrak{q})$ and similarly for $\mathfrak{q}'$. Hence the result follows as the strict inclusion $V(\mathfrak{q}') \subset V(\mathfrak{q})$ implies a strict inequality of dimensions. □

The following lemma generalizes Lemma 10.108.6.

Lemma 10.110.3. *Let k be a field. Let S be a finite type k algebra. Let $X = \mathrm{Spec}(S)$. Let $\mathfrak{p} \subset S$ be a prime ideal, and let $x \in X$ be the corresponding point. Then we have*

$$\dim_x(X) = \dim(S_\mathfrak{p}) + \mathit{trdeg}_k\ \kappa(\mathfrak{p}).$$

Proof. By Lemma 10.110.1 we know that $r = \mathrm{trdeg}_k\ \kappa(\mathfrak{p})$ is equal to the dimension of $V(\mathfrak{p})$. Pick any maximal chain of primes $\mathfrak{p} \subset \mathfrak{p}_1 \subset \ldots \subset \mathfrak{p}_r$ starting with $\mathfrak{p}$ in S. This has length r by Lemma 10.108.4. Let $\mathfrak{q}_i$, $i \in I'$ be the minimal primes of S which are contained in $\mathfrak{p}$. These correspond $1-1$ to minimal primes in $S_\mathfrak{p}$ via the rule $\mathfrak{q}_i \mapsto \mathfrak{q}_i S_\mathfrak{p}$. By Lemma 10.108.5 we know that $\dim_x(X)$ is equal to the maximum of the dimensions of the rings $S/\mathfrak{q}_i$. For each i pick a maximal chain of primes $\mathfrak{q}_i \subset \mathfrak{p}_1 \subset \ldots \subset \mathfrak{p}_{s(i)} = \mathfrak{p}$. Then $\dim(S_\mathfrak{p}) = \max_{i \in I'} s(i)$. Now, each chain

$$\mathfrak{q}_i \subset \mathfrak{p}_1 \subset \ldots \subset \mathfrak{p}_{s(i)} = \mathfrak{p} \subset \mathfrak{p}_1 \subset \ldots \subset \mathfrak{p}_r$$

is a maximal chain in $S/\mathfrak{q}_i$, and by what was said before we have $\dim_x(X) = \max_{i \in I'} r + s(i)$. The lemma follows. □

The following lemma says that the codimension of one finite type Spec in another is the difference of heights.

Lemma 10.110.4. *Let k be a field. Let $S' \to S$ be a surjection of finite type k algebras. Let $\mathfrak{p} \subset S$ be a prime ideal, and let $\mathfrak{p}'$ be the corresponding prime ideal of S'. Let $X = \mathrm{Spec}(S)$, resp. $X' = \mathrm{Spec}(S')$, and let $x \in X$, resp. $x' \in X'$ be the point corresponding to $\mathfrak{p}$, resp. $\mathfrak{p}'$. Then*

$$\dim_{x'} X' - \dim_x X = \mathit{height}(\mathfrak{p}') - \mathit{height}(\mathfrak{p}).$$

Proof. Immediate from Lemma 10.110.3. □

Lemma 10.110.5. *Let k be a field. Let S be a finite type k-algebra. Let $k \subset K$ be a field extension. Then $\dim(S) = \dim(K \otimes_k S)$.*

Proof. By Lemma 10.109.4 there exists a finite injective map $k[y_1, \ldots, y_d] \to S$ with $d = \dim(S)$. Since K is flat over k we also get a finite injective map $K[y_1, \ldots, y_d] \to K \otimes_k S$. The result follows from Lemma 10.106.4. □

Lemma 10.110.6. *Let k be a field. Let S be a finite type k-algebra. Set $X = \mathrm{Spec}(S)$. Let $k \subset K$ be a field extension. Set $S_K = K \otimes_k S$, and $X_K = \mathrm{Spec}(S_K)$. Let $\mathfrak{q} \subset S$ be a prime corresponding to $x \in X$ and let $\mathfrak{q}_K \subset S_K$ be a prime corresponding to $x_K \in X_K$ lying over $\mathfrak{q}$. Then $\dim_x X = \dim_{x_K} X_K$.*

Proof. Choose a presentation $S = k[x_1, \ldots, x_n]/I$. This gives a presentation $K \otimes_k S = K[x_1, \ldots, x_n]/(K \otimes_k I)$. Let $\mathfrak{q}'_K \subset K[x_1, \ldots, x_n]$, resp. $\mathfrak{q}' \subset k[x_1, \ldots, x_n]$ be the corresponding primes. Consider the following commutative diagram of Noetherian local rings

$$\begin{array}{ccc} K[x_1, \ldots, x_n]_{\mathfrak{q}'_K} & \longrightarrow & (K \otimes_k S)_{\mathfrak{q}_K} \\ \uparrow & & \uparrow \\ k[x_1, \ldots, x_n]_{\mathfrak{q}'} & \longrightarrow & S_\mathfrak{q} \end{array}$$

Both vertical arrows are flat because they are localizations of the flat ring maps $S \to S_K$ and $k[x_1, \ldots, x_n] \to K[x_1, \ldots, x_n]$. Moreover, the vertical arrows have the same fibre rings. Hence, we see from Lemma 10.106.7 that $\mathrm{height}(\mathfrak{q}') - \mathrm{height}(\mathfrak{q}) = \mathrm{height}(\mathfrak{q}'_K) - \mathrm{height}(\mathfrak{q}_K)$. Denote $x' \in X' = \mathrm{Spec}(k[x_1, \ldots, x_n])$ and $x'_K \in X'_K =$

$\mathrm{Spec}(K[x_1, \ldots, x_n])$ the points corresponding to $\mathfrak{q}'$ and $\mathfrak{q}'_K$. By Lemma 10.110.4 and what we showed above we have

$$\begin{aligned} n - \dim_x X &= \dim_{x'} X' - \dim_x X \\ &= \mathrm{height}(\mathfrak{q}') - \mathrm{height}(\mathfrak{q}) \\ &= \mathrm{height}(\mathfrak{q}'_K) - \mathrm{height}(\mathfrak{q}_K) \\ &= \dim_{x'_K} X'_K - \dim_{x_K} X_K \\ &= n - \dim_{x_K} X_K \end{aligned}$$

and the lemma follows. □

10.111. Dimension of graded algebras over a field

Here is a basic result.

Lemma 10.111.1. *Let k be a field. Let S be a finitely generated graded algebra over k. Assume $S_0 = k$. Let $P(T) \in \mathbf{Q}[T]$ be the polynomial such that $\dim(S_d) = P(d)$ for all $d \gg 0$. See Proposition 10.55.7. Then*

1. *The irrelevant ideal S_+ is a maximal ideal $\mathfrak{m}$.*
2. *Any minimal prime of S is a homogeneous ideal and is contained in $S_+ = \mathfrak{m}$.*
3. *We have $\dim(S) = \deg(P) + 1 = \dim_x \mathrm{Spec}(S)$ (with the convention that $\deg(0) = -1$) where x is the point corresponding to the maximal ideal $S_+ = \mathfrak{m}$.*
4. *The Hilbert function of the local ring $R = S_{\mathfrak{m}}$ is equal to the Hilbert function of S.*

Proof. The first statement is obvious. The second follows from Lemma 10.53.8. The equality $\dim(S) = \dim_x \mathrm{Spec}(S)$ follows from the fact that every irreducible component passes through x according to (2). Hence we may compute this dimension as the dimension of the local ring $R = S_{\mathfrak{m}}$ with $\mathfrak{m} = S_+$ by Lemma 10.108.6. Since $\mathfrak{m}^d/\mathfrak{m}^{d+1} \cong \mathfrak{m}^d R/\mathfrak{m}^{d+1} R$ we see that the Hilbert function of the local ring R is equal to the Hilbert function of S, which is (4). We conclude the last equality of (3) by Proposition 10.57.8. □

10.112. Generic flatness

Basically this says that a finite type algebra over a domain becomes flat after inverting a single element of the domain. There are several versions of this result (in increasing order of strength).

Lemma 10.112.1. *Let $R \to S$ be a ring map. Let M be an S-module. Assume*

1. *R is Noetherian,*
2. *R is a domain,*
3. *$R \to S$ is of finite type, and*
4. *M is a finite type S-module.*

Then there exists a nonzero $f \in R$ such that M_f is a free R_f-module.

Proof. Let K be the fraction field of R. Set $S_K = K \otimes_R S$. This is an algebra of finite type over K. We will argue by induction on $d = \dim(S_K)$ (which is finite for example by Noether normalization, see Section 10.109). Fix $d \geq 0$. Assume we know that the lemma holds in all cases where $\dim(S_K) < d$.

Suppose given $R \to S$ and M as in the lemma with $\dim(S_K) = d$. By Lemma 10.59.1 there exists a filtration $0 \subset M_1 \subset M_2 \subset \ldots \subset M_n = M$ so that M_i/M_{i-1} is isomorphic to $S/\mathfrak{q}$ for some prime $\mathfrak{q}$ of S. Note that $\dim((S/\mathfrak{q})_K) \leq \dim(S_K)$. Also, note that an extension of free modules is free (see basic notion 50). Thus we may assume $M = S$ and that S is a domain of finite type over R.

If $R \to S$ has a nontrivial kernel, then take a nonzero $f \in R$ in this kernel. In this case $S_f = 0$ and the lemma holds. (This is really the case $d = -1$ and the start of the induction.) Hence we may assume that $R \to S$ is a finite type extension of Noetherian domains.

Apply Lemma 10.109.7 and replace R by R_f (with f as in the lemma) to get a factorization

$$R \subset R[y_1, \ldots, y_d] \subset S$$

where the second extension is finite. Note that $f.f.(R[y_1, \ldots, y_d]) \subset f.f.(S)$ is a finite extension of fields. Choose $z_1, \ldots, z_r \in S$ which form a basis for $f.f.(S)$ over $f.f.(R[y_1, \ldots, y_d])$. This gives a short exact sequence

$$0 \to R[y_1, \ldots, y_d]^{\oplus r} \xrightarrow{(z_1, \ldots, z_r)} S \to N \to 0$$

By construction N is a finite $R[y_1, \ldots, y_d]$-module whose support does not contain the generic point (0) of $\mathrm{Spec}(R[y_1, \ldots, y_d])$. By Lemma 10.37.5 there exists a nonzero $g \in R[y_1, \ldots, y_d]$ such that g annihilates N, so we may view N as a finite module over $S' = R[y_1, \ldots, y_d]/(g)$. Since $\dim(S'_K) < d$ by induction there exists a nonzero $f \in R$ such that N_f is a free R_f-module. Since $(R[y_1, \ldots, y_d])_f \cong R_f[y_1, \ldots, y_d]$ is free also we conclude by the already mentioned fact that an extension of free modules is free. □

Lemma 10.112.2. *Let $R \to S$ be a ring map. Let M be an S-module. Assume*

1. *R is a domain,*
2. *$R \to S$ is of finite presentation, and*
3. *M is an S-module of finite presentation.*

Then there exists a nonzero $f \in R$ such that M_f is a free R_f-module.

Proof. Write $S = R[x_1, \ldots, x_n]/(g_1, \ldots, g_m)$. For $g \in R[x_1, \ldots, x_n]$ denote $\overline{g}$ its image in S. We may write $M = S^{\oplus t}/\sum Sn_i$ for some $n_i \in S^{\oplus t}$. Write $n_i = (\overline{g}_{i1}, \ldots, \overline{g}_{it})$ for some $g_{ij} \in R[x_1, \ldots, x_n]$. Let $R_0 \subset R$ be the subring generated by all the coefficients of all the elements $g_i, g_{ij} \in R[x_1, \ldots, x_n]$. Define $S_0 = R_0[x_1, \ldots, x_n]/(g_1, \ldots, g_m)$. Define $M_0 = S_0^{\oplus t}/\sum S_0 n_i$. Then R_0 is a domain of finite type over $\mathbf{Z}$ and hence Noetherian (see Lemma 10.29.1). Moreover via the injection $R_0 \to R$ we have $S \cong R \otimes_{R_0} S_0$ and $M \cong R \otimes_{R_0} M_0$. Applying Lemma 10.112.1 we obtain a nonzero $f \in R_0$ such that $(M_0)_f$ is a free $(R_0)_f$-module. Hence $M_f = R_f \otimes_{(R_0)_f} (M_0)_f$ is a free R_f-module. □

Lemma 10.112.3. *Let $R \to S$ be a ring map. Let M be an S-module. Assume*

1. *R is a domain,*
2. *$R \to S$ is of finite type, and*
3. *M is a finite type S-module.*

Then there exists a nonzero $f \in R$ such that

(a) *M_f and S_f are free as R_f-modules, and*

(b) *S_f is a finitely presented R_f-algebra and M_f is a finitely presented S_f-module.*

Proof. We first prove the lemma for $S = R[x_1, \ldots, x_n]$, and then we deduce the result in general.

Assume $S = R[x_1, \ldots, x_n]$. Choose elements $m_1, \ldots, m_t$ which generate M. This gives a short exact sequence

$$0 \to N \to S^{\oplus t} \xrightarrow{(m_1, \ldots, m_t)} M \to 0.$$

Denote K the fraction field of R. Denote $S_K = K \otimes_R S = K[x_1, \ldots, x_n]$, and similarly $N_K = K \otimes_R N$, $M_K = K \otimes_R M$. As $R \to K$ is flat the sequence remains flat after tensoring with K. As $S_K = K[x_1, \ldots, x_n]$ is a Noetherian ring (see Lemma 10.29.1) we can find finitely many elements $n'_1, \ldots, n'_s \in N_K$ which generate it. Choose $n_1, \ldots, n_r \in N$ such that $n'_i = \sum a_{ij} n_j$ for some $a_{ij} \in K$. Set

$$M' = S^{\oplus t} / \sum\nolimits_{i=1,\ldots,r} Sn_i$$

By construction M' is a finitely presented S-module, and there is a surjection $M' \to M$ which induces an isomorphism $M'_K \cong M_K$. We may apply Lemma 10.112.2 to $R \to S$ and M' and we find an $f \in R$ such that M'_f is a free R_f-module. Thus $M'_f \to M_f$ is a surjection of modules over the domain R_f where the source is a free module and which becomes an isomorphism upon tensoring with K. Thus it is injective as $M'_f \subset M'_K$ as it is free over the domain R_f. Hence $M'_f \to M_f$ is an isomorphism and the result is proved.

For the general case, choose a surjection $R[x_1, \ldots, x_n] \to S$. Think of both S and M as finite modules over $R[x_1, \ldots, x_n]$. By the special case proved above there exists a nonzero $f \in R$ such that both S_f and M_f are free as R_f-modules and finitely presented as $R_f[x_1, \ldots, x_n]$-modules. Clearly this implies that S_f is a finitely presented R_f-algebra and that M_f is a finitely presented S_f-module. □

Let $R \to S$ be a ring map. Let M be an S-module. Consider the following condition on an element $f \in R$:

$$\begin{cases} S_f & \text{is of finite presentation over } R_f \\ M_f & \text{is of finite presentation as } S_f\text{-module} \\ S_f, M_f & \text{are free as } R_f\text{-modules} \end{cases} \tag{10.112.3.1}$$

We define

$$U(R \to S, M) = \bigcup\nolimits_{f \in R \text{ with (10.112.3.1)}} D(f) \tag{10.112.3.2}$$

which is an open subset of $\mathrm{Spec}(R)$.

Lemma 10.112.4. *Let $R \to S$ be a ring map. Let $0 \to M_1 \to M_2 \to M_3 \to 0$ be a short exact sequence of S-modules. Then*

$$U(R \to S, M_1) \cap U(R \to S, M_3) \subset U(R \to S, M_2).$$

Proof. Let $u \in U(R \to S, M_1) \cap U(R \to S, M_3)$. Choose $f_1, f_3 \in R$ such that $u \in D(f_1)$, $u \in D(f_3)$ and such that (10.112.3.1) holds for f_1 and M_1 and for f_3 and M_3. Then set $f = f_1 f_3$. Then $u \in D(f)$ and (10.112.3.1) holds for f and both M_1 and M_3. An extension of free modules is free, and an extension of finitely presented modules is finitely presented (Lemma 10.5.3). Hence we see that (10.112.3.1) holds for f and M_2. Thus $u \in U(R \to S, M_2)$ and we win. □

Lemma 10.112.5. *Let $R \to S$ be a ring map. Let M be an S-module. Let $f \in R$. Using the identification* $\mathrm{Spec}(R_f) = D(f)$ *we have $U(R_f \to S_f, M_f) = D(f) \cap U(R \to S, M)$.*

Proof. Suppose that $u \in U(R_f \to S_f, M_f)$. Then there exists an element $g \in R_f$ such that $u \in D(g)$ and such that (10.112.3.1) holds for the pair $((R_f)_g \to (S_f)_g, (M_f)_g)$. Write $g = a/f^n$ for some $a \in R$. Set $h = af$. Then $R_h = (R_f)_g$, $S_h = (S_f)_g$, and $M_h = (M_f)_g$. Moreover $u \in D(h)$. Hence $u \in U(R \to S, M)$. Conversely, suppose that $u \in D(f) \cap U(R \to S, M)$. Then there exists an element $g \in R$ such that $u \in D(g)$ and such that (10.112.3.1) holds for the pair $(R_g \to S_g, M_g)$. Then it is clear that (10.112.3.1) also holds for the pair $(R_{fg} \to S_{fg}, M_{fg}) = ((R_f)_g \to (S_f)_g, (M_f)_g)$. Hence $u \in U(R_f \to S_f, M_f)$ and we win. □

Lemma 10.112.6. *Let $R \to S$ be a ring map. Let M be an S-module. Let $U \subset \mathrm{Spec}(R)$ be a dense open. Assume there is a covering $U = \bigcup_{i \in I} D(f_i)$ of opens such that $U(R_{f_i} \to S_{f_i}, M_{f_i})$ is dense in $D(f_i)$ for each $i \in I$. Then $U(R \to S, M)$ is dense in* $\mathrm{Spec}(R)$.

Proof. In view of Lemma 10.112.5 this is a purely topological statement. Namely, by that lemma we see that $U(R \to S, M) \cap D(f_i)$ is dense in $D(f_i)$ for each $i \in I$. By Topology, Lemma 5.20.4 we see that $U(R \to S, M) \cap U$ is dense in U. Since U is dense in $\mathrm{Spec}(R)$ we conclude that $U(R \to S, M)$ is dense in $\mathrm{Spec}(R)$. □

Lemma 10.112.7. *Let $R \to S$ be a ring map. Let M be an S-module. Assume*

1. *$R \to S$ is of finite type,*
2. *M is a finite S-module, and*
3. *R is reduced.*

Then there exists a subset $U \subset \mathrm{Spec}(R)$ such that

1. *U is open and dense in* $\mathrm{Spec}(R)$,
2. *for every $u \in U$ there exists an $f \in R$ such that $u \in D(f) \subset U$ and such that we have*
   (a) *M_f and S_f are free over R_f,*
   (b) *S_f is a finitely presented R_f-algebra, and*
   (c) *M_f is a finitely presented S_f-module.*

Proof. Note that the lemma is equivalent to the statement that the open $U(R \to S, M)$, see Equation (10.112.3.2), is dense in $\mathrm{Spec}(R)$. We first prove the lemma for $S = R[x_1, \ldots, x_n]$, and then we deduce the result in general.

Proof of the case $S = R[x_1, \ldots, x_n]$ and M any finite module over S. Note that in this case $S_f = R_f[x_1, \ldots, x_n]$ is free and of finite presentation over R_f, so we do not have to worry about the conditions regarding S, only those that concern M. We will use induction on n.

There exists a finite filtration

$$0 \subset M_1 \subset M_2 \subset \ldots \subset M_t = M$$

such that $M_i/M_{i-1} \cong S/J_i$ for some ideal $J_i \subset S$, see Lemma 10.5.4. Since a finite intersection of dense opens is dense open, we see from Lemma 10.112.4 that it suffices to prove the lemma for each of the modules R/J_i. Hence we may assume that $M = S/J$ for some ideal J of $S = R[x_1, \ldots, x_n]$.

Let $I \subset R$ be the ideal generated by the coefficients of elements of J. Let $U_1 = \mathrm{Spec}(R) \setminus V(I)$ and let

$$U_2 = \mathrm{Spec}(R) \setminus \overline{U_1}.$$

Then it is clear that $U = U_1 \cup U_2$ is dense in $\mathrm{Spec}(R)$. Let $f \in R$ be an element such that either (a) $D(f) \subset U_1$ or (b) $D(f) \subset U_2$. If for any such f the lemma holds for the pair $(R_f \to R_f[x_1, \ldots, x_n], M_f)$ then by Lemma 10.112.6 we see that $U(R \to S, M)$ is dense in $\mathrm{Spec}(R)$. Hence we may assume either (a) $I = R$, or (b) $V(I) = \mathrm{Spec}(R)$.

In case (b) we actually have $I = 0$ as R is reduced! Hence $J = 0$ and $M = S$ and the lemma holds in this case.

In case (a) we have to do a little bit more work. Note that every element of I is actually the coefficient of a monomial of an element of J, because the set of coefficients of elements of J forms an ideal (details omitted). Hence we find an element

$$g = \sum\nolimits_{K \in E} a_K x^K \in J$$

where E is a finite set of multi-indices $K = (k_1, \ldots, k_n)$ with at least one coefficient a_{K_0} a unit in R. Actually we can find one which has a coefficient equal to 1 as $1 \in I$ in case (a). Let $m = \#\{K \in E \mid a_K \text{ is not a unit}\}$. Note that $0 \leq m \leq \#E - 1$. We will argue by induction on m.

The case $m = 0$. In this case all the coefficients a_K, $K \in E$ of g are units and $E \neq \emptyset$. If $E = \{K_0\}$ is a singleton and $K_0 = (0, \ldots, 0)$, then g is a unit and $J = S$ so the result holds for sure. (This happens in particular when $n = 0$ and it provides the base case of the induction on n.) If not $E = \{(0, \ldots, 0)\}$, then at least one K is not equal to $(0, \ldots, 0)$, i.e., $g \notin R$. At this point we employ the usual trick of Noether normalization. Namely, we consider

$$G(y_1, \ldots, y_n) = g(y_1 + y_n^{e_1}, y_2 + y_n^{e_2}, \ldots, y_{n-1} + y_n^{e_{n-1}}, y_n)$$

with $0 \ll e_{n-1} \ll e_{n-2} \ll \ldots \ll e_1$. By Lemma 10.109.2 it follows that $G(y_1, \ldots, y_n)$ as a polynomial in y_n looks like

$$a_K y_n^{k_n + \sum_{i=1,\ldots,n-1} e_i k_i} + \text{lower order terms in } y_n$$

As a_K is a unit we conclude that $M = R[x_1, \ldots, x_n]/J$ is finite over $R[y_1, \ldots, y_{n-1}]$. Hence $U(R \to R[x_1, \ldots, x_n], M) = U(R \to R[y_1, \ldots, y_{n-1}], M)$ and we win by induction on n.

The case $m > 0$. Pick a multi-index $K \in E$ such that a_K is not a unit. As before set $U_1 = \mathrm{Spec}(R_{a_K}) = \mathrm{Spec}(R) \setminus V(a_K)$ and set

$$U_2 = \mathrm{Spec}(R) \setminus \overline{U_1}.$$

Then it is clear that $U = U_1 \cup U_2$ is dense in $\mathrm{Spec}(R)$. Let $f \in R$ be an element such that either (a) $D(f) \subset U_1$ or (b) $D(f) \subset U_2$. If for any such f the lemma holds for the pair $(R_f \to R_f[x_1, \ldots, x_n], M_f)$ then by Lemma 10.112.6 we see that $U(R \to S, M)$ is dense in $\mathrm{Spec}(R)$. Hence we may assume either (a) $a_K R = R$, or (b) $V(a_K) = \mathrm{Spec}(R)$. In case (a) the number m drops, as a_K has turned into a unit. In case (b), since R is reduced, we conclude that $a_K = 0$. Hence the set E decreases so the number m drops as well. In both cases we win by induction on m.

At this point we have proven the lemma in case $S = R[x_1, \ldots, x_n]$. Assume that $(R \to S, M)$ is an arbitrary pair satisfying the conditions of the lemma. Choose

a surjection $R[x_1, \ldots, x_n] \to S$. Observe that, with the notation introduced in (10.112.3.2), we have

$$U(R \to S, M) = U(R \to R[x_1, \ldots, x_n], S) \cap U(R \to R[x_1, \ldots, x_n], S)$$

Hence as we've just finished proving the right two opens are dense also the open on the left is dense. □

10.113. Around Krull-Akizuki

One application of Krull-Akizuki is to show that there are plenty of discrete valuation rings. More generally in this section we show how to construct discrete valuation rings dominating Noetherian local rings.

First we show how to dominate a Noetherian local domain by a 1-dimensional Noetherian local domain by blowing up the maximal ideal.

Lemma 10.113.1. *Let R be a local Noetherian domain with fraction field K. Assume R is not a field. Then there exist $R \subset R' \subset K$ with*

1. *R' local Noetherian of dimension 1,*
2. *$R \to R'$ a local ring map, i.e., R' dominates R, and*
3. *$R \to R'$ essentially of finite type.*

Proof. Choose any valuation ring $A \subset K$ dominating R (which exist by Lemma 10.46.2). Denote v the corresponding valuation. Let $x_1, \ldots, x_r$ be a minimal set of generators of the maximal ideal $\mathfrak{m}$ of R. We may and do assume that $v(x_r) = \min\{v(x_1), \ldots, v(x_r)\}$. Consider the ring

$$S = R[x_1/x_r, x_2/x_r, \ldots, x_{r-1}/x_r] \subset K.$$

Note that $\mathfrak{m}S = x_rS$ is a principal ideal. Note that $S \subset A$ and that $v(x_r) > 0$, hence we see that $x_rS \neq S$. Choose a minimal prime $\mathfrak{q}$ over x_rS. Then $\text{height}(\mathfrak{q}) = 1$ by Lemma 10.57.10 and $\mathfrak{q}$ lies over $\mathfrak{m}$. Hence we see that $R' = S_{\mathfrak{q}}$ is a solution. □

The spectrum of the ring R' in the following lemma is really the blow up of $\text{Spec}(R)$ in the maximal ideal of R (at least if case R is reduced).

Lemma 10.113.2. *Let R be a local ring with maximal ideal $\mathfrak{m}$. Assume R is Noetherian, dimension 1 and that $\dim(\mathfrak{m}/\mathfrak{m}^2) > 1$. Then there exists a ring map $R \to R'$ such that*

1. *$R \to R'$ is finite,*
2. *$R \to R'$ is not an isomorphism, and*
3. *for every $f \in \mathfrak{m}$ the map $R_f \to R'_f$ is an isomorphism.*

Proof. If $\mathfrak{m}$ is an associated prime of R then we can take $R' = R/I$ with $I = \{x \in R \mid \mathfrak{m}x = 0\}$. Hence we may assume that $\text{depth}(R) = 1$. In other words, we may assume R is Cohen-Macaulay.

Denote $\kappa = R/\mathfrak{m}$ the residue field of R. Consider the graded κ-algebra $S = \bigoplus_{d \geq 0} \mathfrak{m}^d/\mathfrak{m}^{d+1}$. This is a Noetherian ring, and hence has finitely many minimal primes $\mathfrak{q}_j$. Since the dimension of R is 1 we know the Hilbert function of R is eventually constant, see Proposition 10.57.8. Hence there exists an integer $d_0 \geq 0$ and an integer $r > 0$ such that $\dim_\kappa(\mathfrak{m}^n/\mathfrak{m}^{n+1}) = r$ for all $d \geq d_0$. By Lemma 10.111.1 we have $\dim(S) = 1$ and each $\mathfrak{q}_i$ is a homogeneous prime ideal. Note that $\dim(S) = 1$ implies none of the $\mathfrak{q}_i$ is equal to S_+. Hence by Lemma 10.53.6 we

may choose $f \in S_+$ homogeneous not contained in any $\mathfrak{q}_j$. Then $\dim(S/fS) = 0$. This implies that $\dim_\kappa(S/fS) < \infty$, see for example Lemma 10.109.4. Hence we see that $fS_n = S_{n+\deg(f)}$ for all $n \gg 0$.

Set $d = \deg(f)$. Choose $x \in \mathfrak{m}^d$ which maps to f. Note that since $fS_n = S_{n+d}$ for $n \gg 0$ we have $x\mathfrak{m}^n = \mathfrak{m}^{n+d}$ by Nakayama's Lemma 10.18.1 for $n \gg 0$. Hence $\mathfrak{m} = \sqrt{(x)}$. Since R is Cohen-Macaulay this implies that x is a nonzerodivisor. Choose generators $x_1, \ldots, x_t$ of $\mathfrak{m}^d$ as an ideal of R. Set

$$R' = R[x_1/x, x_2/x, \ldots, x_t/x] \subset R[1/x].$$

Note that since x is a nonzerodivisor we have $R \subset R'$. Since $\mathfrak{m} = \sqrt{(x)}$ we see that x is invertible in R_g for any $g \in \mathfrak{m}$, whence (3).

We claim that R' is finite over R. Namely, choose n such that $x\mathfrak{m}^{nd} = \mathfrak{m}^{(n+1)d}$. Then we can write $x_i^{n+1} = xf_i(x_1, \ldots, x_t)$ with $f_i \in R[X_1, \ldots, X_t]$ homogeneous of degree n. Hence we see that

$$(x_i/x)^{n+1} = f_i(x_1/x, \ldots, x_t/x)$$

in R' with the right hand side of degree $\leq n$. Hence any element in R' can be expressed as a sum of $(x_1/x)^{i_1} \ldots (x_t/x)^{i_t}$ with $i_j \leq n$. This proves that R' is finite over R.

Finally we show that $R \neq R'$. We argue by contradiction. Suppose $R' = R$. This means that $x_1/x, \ldots, x_t/x \in R$ for all i. In other words this means that $\mathfrak{m}^d = (x)$. Choose $y_1, \ldots, y_s \in \mathfrak{m}$ a minimal generating set. The assumption of the lemma implies $s \geq 2$. For some $i_1, \ldots, i_s \geq 0$, $\sum i_j = d$ we have $x = uy_1^{i_1} \ldots y_s^{i_s}$ for some unit u in R. We may assume $i_1 > 0$. Then

$$y_1^{i_1 - 1} y_2^{i_2 + 1} y_3^{i_3} \ldots y_s^{i_s} \in \mathfrak{m}^d$$

is a multiple of x hence a multiple of $y_1^{i_1} \ldots y_s^{i_s}$. Hence we see that $y_2/y_1 \in R$. This is a contradiction with the minimality of $y_1, \ldots, y_s$. □

Example 10.113.3. Consider the Noetherian local ring

$$R = k[[x, y]]/(y^2)$$

It has dimension 1 and it is Cohen-Macaulay. The result of applying the procedure of Lemma 10.113.2 to R is the extension

$$k[[x, y]]/(y^2) \subset k[[x, z]]/(z^2), \quad y \mapsto xz$$

in other words it is gotten by adjoining y/x to R. The effect of repeating the construction $n > 1$ times is to adjoin the element y/x^n.

Example 10.113.4. Let k be a field of characteristic $p > 0$ such that k has infinite degree over its subfield k^p of pth powers. For example $k = \mathbf{F}_p(t_1, t_2, t_3, \ldots)$. Consider the ring

$$A = \left\{ \sum a_i x^i \in k[[x]] \text{ such that } [k^p(a_0, a_1, a_2, \ldots) : k^p] < \infty \right\}$$

Then A is a discrete valuation ring and its completion is $A^\wedge = k[[x]]$. Note that the field extension $f.f.(A) \subset f.f.(k[[x]])$ is infinite purely inseparable. Choose any $f \in k[[x]]$, $f \notin A$. Let $R = A[f] \subset k[[x]]$. Then R is a Noetherian local domain of dimension 1 whose completion $R^\wedge$ is nonreduced (think!).

Remark 10.113.5. Suppose that R is a 1-dimensional semi-local Noetherian domain. If there is a maximal ideal $\mathfrak{m} \subset R$ such that $R_\mathfrak{m}$ is not regular, then we may apply the procedure of (the proof of) Lemma 10.113.2 to $(R, \mathfrak{m})$ to get a finite ring extension $R \subset R_1$. (Note that $\mathrm{Spec}(R_1) \to \mathrm{Spec}(R)$ is the blow up of $\mathrm{Spec}(R)$ in the ideal $\mathfrak{m}$.) Of course R_1 is a 1-dimensional semi-local Noetherian domain with the same fraction field as R. If R_1 is not a regular semi-local ring, then we may repeat the construction to get $R_1 \subset R_2$. Thus we get a sequence

$$R \subset R_1 \subset R_2 \subset R_3 \subset \ldots$$

of finite ring extensions which may stop if R_n is regular for some n. Resolution of singularities would be the claim that eventually R_n is indeed regular. In reality this is not the case. Namely, there exists a characteristic 0 Noetherian local domain A of dimension 1 whose completion is nonreduced, see [**FR70**, Proposition 3.1] or our Examples, Section 78.10. For an example in characteristic $p > 0$ see Example 10.113.4. Since the construction of blowing up commutes with completion it is easy to see the sequence never stabilizes. See [**Ben73**] for a discussion (mostly in positive characteristic). On the other hand, if the completion of R in all of its maximal ideals is reduced, then the procedure stops (insert future reference here).

Lemma 10.113.6. *Let A be a ring. The following are equivalent.*

1. *The ring A is a discrete valuation ring.*
2. *The ring A is a valuation ring and Noetherian.*
3. *The ring A is a regular local ring of dimension 1.*
4. *The ring A is a Noetherian local domain with maximal ideal $\mathfrak{m}$ generated by a single nonzero element.*
5. *The ring A is a Noetherian local normal domain of dimension 1.*

In this case if π is a generator of the maximal ideal of A, then every element of A can be uniquely written as $u\pi^n$, where $u \in A$ is a unit.

Proof. The equivalence of (1) and (2) is Lemma 10.46.16. Moreover, in the proof of Lemma 10.46.16 we saw that if A is a discrete valuation ring, then A is a PID, hence (3). Note that a regular local ring is a domain (see Lemma 10.101.2). Using this the equivalence of (3) and (4) follows from dimension theory, see Section 10.57.

Assume (3) and let π be a generator of the maximal ideal $\mathfrak{m}$. For all $n \geq 0$ we have $\dim_{A/\mathfrak{m}} \mathfrak{m}^n/\mathfrak{m}^{n+1} = 1$ because it is generated by π^n (and it cannot be zero). In particular $\mathfrak{m}^n = (\pi^n)$ and the graded ring $\bigoplus \mathfrak{m}^n/\mathfrak{m}^{n+1}$ is isomorphic to the polynomial ring $A/\mathfrak{m}[T]$. For $x \in A \setminus \{0\}$ define $v(x) = \max\{n \mid x \in \mathfrak{m}^n\}$. In other words $x = u\pi^{v(x)}$ with $u \in A^*$. By the remarks above we have $v(xy) = v(x) + v(y)$ for all $x, y \in A \setminus \{0\}$. We extend this to the field of fractions K of A by setting $v(a/b) = v(a) - v(b)$ (well defined by multiplicativity shown above). Then it is clear that A is the set of elements of K which have valuation ≥ 0. Hence we see that A is a valuation ring by Lemma 10.46.14.

A valuation ring is a normal domain by Lemma 10.46.8. Hence we see that the equivalent conditions (1) – (3) imply (5). Assume (5). Suppose that $\mathfrak{m}$ cannot be generated by 1 element to get a contradiction. Then Lemma 10.113.2 implies there is a finite ring map $A \to A'$ which is an isomorphism after inverting any nonzero element of $\mathfrak{m}$ but not an isomorphism. In particular $A' \subset f.f.(A)$. Since $A \to A'$ is finite it is integral (see Lemma 10.33.3). Since A is normal we get $A = A'$ a contradiction. $\square$

Definition 10.113.7. Let A be a discrete valuation ring. A *uniformizer* is an element $\pi \in A$ which generates the maximal ideal of A.

By Lemma 10.113.6 any two uniformizers of a discrete valuation ring are associates.

Lemma 10.113.8. *Let R be a domain with fraction field K. Let M be an R-submodule of $K^{\oplus r}$. Assume R is local Noetherian of dimension 1. For any nonzero $x \in R$ we have $\text{length}_R(R/xR) < \infty$ and*

$$\text{length}_R(M/xM) \leq r \cdot \text{length}_R(R/xR).$$

Proof. If x is a unit then the result is true. Hence we may assume $x \in \mathfrak{m}$ the maximal ideal of R. Since x is not zero and R is a domain we have $\dim(R/xR) = 0$, and hence R/xR has finite length. Consider $M \subset K^{\oplus r}$ as in the lemma. We may assume that the elements of M generate $K^{\oplus r}$ as a K-vector space after replacing $K^{\oplus r}$ by a smaller subspace if necessary.

Suppose first that M is a finite R-module. In that case we can clear denominators and assume $M \subset R^{\oplus r}$. Since M generates $K^{\oplus r}$ as a vectors space we see that $R^{\oplus r}/M$ has finite length. In particular there exists an integer $c \geq 0$ such that $x^c R^{\oplus r} \subset M$. Note that $M \supset xM \supset x^2M \supset \ldots$ is a sequence of modules with successive quotients each isomorphic to M/xM. Hence we see that

$$n\text{length}_R(M/xM) = \text{length}_R(M/x^nM).$$

The same argument for $M = R^{\oplus r}$ shows that

$$n\text{length}_R(R^{\oplus r}/xR^{\oplus r}) = \text{length}_R(R^{\oplus r}/x^nR^{\oplus r}).$$

By our choice of c above we see that x^nM is sandwiched between $x^nR^{\oplus r}$ and $x^{n+c}R^{\oplus r}$. This easily gives that

$$r(n+c)\text{length}_R(R/xR) \geq n\text{length}_R(M/xM) \geq r(n-c)\text{length}_R(R/xR)$$

Hence in the finite case we actually get the result of the lemma with equality.

Suppose now that M is not finite. Suppose that the length of M/xM is $\geq k$ for some natural number k. Then we can find

$$0 \subset N_0 \subset N_1 \subset N_2 \subset \ldots N_k \subset M/xM$$

with $N_i \neq N_{i+1}$ for $i = 0, \ldots k-1$. Choose an element $m_i \in M$ whose congruence class mod xM falls into N_i but not into N_{i-1} for $i = 1, \ldots, k$. Consider the finite R-module $M' = Rm_1 + \ldots + Rm_k \subset M$. Let $N_i' \subset M'/xM'$ be the inverse image of N_i. It is clear that $N_i' \neq N_{i+1}'$ by our choice of m_i. Hence we see that $\text{length}_R(M'/xM') \geq k$. By the finite case we conclude $k \leq r\text{length}_R(R/xR)$ as desired. □

Here is a first application.

Lemma 10.113.9. *Let $R \to S$ be a homomorphism of domains inducing an injection of fraction fields $K \subset L$. If R is Noetherian local of dimension 1 and $[L : K] < \infty$ then*

1. *each prime ideal $\mathfrak{n}_i$ of S lying over the maximal ideal $\mathfrak{m}$ of R is maximal,*
2. *there are finitely many of these, and*
3. *$[\kappa(\mathfrak{n}_i) : \kappa(\mathfrak{m})] < \infty$ for each i.*

Proof. Pick $x \in \mathfrak{m}$ nonzero. Apply Lemma 10.113.8 to the submodule $S \subset L \cong K^{\oplus n}$ where $n = [L : K]$. Thus the ring S/xS has finite length over R. It follows that $S/\mathfrak{m}S$ has finite length over $\kappa(\mathfrak{m})$. In other words, $\dim_{\kappa(\mathfrak{m})} S/\mathfrak{m}S$ is finite (Lemma 10.48.6). Thus $S/\mathfrak{m}S$ is Artinian (Lemma 10.49.2). The structural results on Artinian rings implies parts (1) and (2), see for example Lemma 10.49.8. Part (3) is implied by the finiteness esthablished above. □

Lemma 10.113.10. *Let R be a domain with fraction field K. Let M be an R-submodule of $K^{\oplus r}$. Assume R is Noetherian of dimension 1. For any nonzero $x \in R$ we have $\text{length}_R(M/xM) < \infty$.*

Proof. Since R has dimension 1 we see that x is contained in finitely many primes $\mathfrak{m}_i$, $i = 1, \ldots, n$, each maximal. Since R is Noetherian we see that R/xR is Artinian, see Proposition 10.57.6. Hence R/xR is a quotient of $\prod R/\mathfrak{m}_i^{e_i}$ for certain e_i because that $\mathfrak{m}_1^{e_1} \ldots \mathfrak{m}_n^{e_n} \subset (x)$ for suitably large e_i as R/xR is Artinian (see Section 10.49). Hence M/xM similarly decomposes as a product $\prod (M/xM)_{\mathfrak{m}_i} = \prod M/(\mathfrak{m}_i^{e_i}, x)M$ of its localizations at the $\mathfrak{m}_i$. By Lemma 10.113.8 applied to $M_{\mathfrak{m}_i}$ over $R_{\mathfrak{m}_i}$ we see each $M_{\mathfrak{m}_i}/xM_{\mathfrak{m}_i} = (M/xM)_{\mathfrak{m}_i}$ has finite length over $R_{\mathfrak{m}_i}$. It easily follows that M/xM has finite length over R. □

Lemma 10.113.11 (Krull-Akizuki)**.** *Let R be a domain with fraction field K. Let $K \subset L$ be a finite extension of fields. Assume R is Noetherian and* $\dim(R) = 1$*. In this case any ring A with $R \subset A \subset L$ is Noetherian.*

Proof. To begin we may assume that L is the fraction field of A by replacing L by the fraction field of A if necessary. Let $I \subset A$ be an ideal. Clearly I generates L as a K-vector space. Hence we see that $I \cap R \neq (0)$. Pick any nonzero $x \in I \cap R$. Then we get $I/xA \subset A/xA$. By Lemma 10.113.10 the R-module A/xA has finite length as an R-module. Hence I/xA has finite length as an R-module. Hence I is finitely generated as an ideal in A. □

Lemma 10.113.12. *Let R be a Noetherian local domain with fraction field K. Assume that R is not a field. Let $K \subset L$ be a finitely generated field extension. Then there exists discrete valuation ring A with fraction field L which dominates R.*

Proof. If L is not finite over K choose a transcendence basis $x_1, \ldots, x_r$ of L over K and replace R by $R[x_1, \ldots, x_r]$ localized at the maximal ideal generated by $\mathfrak{m}_R$ and $x_1, \ldots, x_r$. Thus we may assume $K \subset L$ finite.

By Lemma 10.113.1 we may assume $\dim(R) = 1$.

Let $A \subset L$ be the integral closure of R in L. By Lemma 10.113.11 this is Noetherian. By Lemma 10.33.15 there is a prime ideal $\mathfrak{q} \subset A$ lying over the maximal ideal of R. By Lemma 10.113.6 the ring $A_{\mathfrak{q}}$ is a discrete valuation ring dominating R as desired. □

10.114. Factorization

Here are some notions and relations between them that are typically taught in a first year course on algebra at the undergraduate level.

Definition 10.114.1. Let R be a domain.

(1) Elements $x, y \in R$ are called *associates* if there exists a unit $u \in R^*$ such that $x = uy$.
(2) An element $x \in R$ is called *irreducible* if it is nonzero, not a unit and whenever $x = yz$, $y, z \in R$, then y is either a unit or an associate of x.
(3) An element $x \in R$ is called *prime* if the ideal generated by x is a prime ideal.

Lemma 10.114.2. *Let R be a domain. Let $x, y \in R$. Then x, y are associates if and only if $(x) = (y)$.*

Proof. Omitted. □

Lemma 10.114.3. *Let R be a domain. Consider the following conditions:*

(1) *The ring R satisfies the ascending chain condition for principal ideals.*
(2) *Every nonzero, nonunit element $a \in R$ has a factorization $a = b_1 \ldots b_k$ with each b_i an irreducible element of R.*

Then (1) implies (2).

Proof. Omitted. □

Definition 10.114.4. A *unique factorization domain*, abbreviated *UFD*, is a domain R such that if $x \in R$ is a nonzero, nonunit, then x has a factorization into irreducibles, and if

$$x = a_1 \ldots a_m = b_1 \ldots b_n$$

are factorizations into irreducibles then $n = m$ and there exists a permutation $\sigma : \{1, \ldots, n\} \to \{1, \ldots, n\}$ such that a_i and $b_{\sigma(i)}$ are associates.

Lemma 10.114.5. *Let R be a domain. Assume every nonzero, nonunit factors into irreducibles. Then R is a UFD if and only if every irreducible element is prime.*

Proof. Omitted. □

Definition 10.114.6. A *principal ideal domain*, abbreviated *PID*, is a domain R such that every ideal is a principal ideal.

Lemma 10.114.7. *A principal ideal domain is a unique factorization domain.*

Proof. Omitted. □

Definition 10.114.8. A *Dedekind domain* is a domain R such that every nonzero ideal $I \subset R$ can be written as a product

$$I = \mathfrak{p}_1 \ldots \mathfrak{p}_r$$

of nonzero prime ideals uniquely up to permutation of the $\mathfrak{p}_i$.

Lemma 10.114.9. *Let A be a ring. Let I and J be nonzero ideals of A such that $IJ = (f)$ for some nonzerodivisor $f \in A$. Then I and J are finitely generated ideals and finitely locally free of rank 1 as A-modules.*

Proof. It suffices to show that I and J are finite locally free A-modules of rank 1, see Lemma 10.73.2. To do this, write $f = \sum_{i=1,\ldots,n} x_i y_i$ with $x_i \in I$ and $y_i \in J$. We can also write $x_i y_i = a_i f$ for some $a_i \in A$. Since f is a nonzerodivisor we see that $\sum a_i = 1$. Thus it suffices to show that each I_{a_i} and J_{a_i} is free of rank 1 over A_{a_i}. After replacing A by A_{a_i} we conclude that $f = xy$ for some $x \in I$ and $y \in J$. Note that both x and y are nonzerodivisors. We claim that $I = (x)$ and $J = (y)$

which finishes the proof. Namely, if $x' \in I$, then $x'y = af = axy$ for some $a \in A$. Hence $x' = ax$ and we win. □

Lemma 10.114.10. *Let R be a ring. The following are equivalent*

(1) *R is a Dedekind domain,*
(2) *R is a Noetherian domain, and for every maximal ideal $\mathfrak{m}$ the local ring $R_\mathfrak{m}$ is a discrete valuation ring, and*
(3) *R is a Noetherian, normal domain, and $\dim(R) \leq 1$.*

Proof. Assume (1). The argument is nontrivial because we did not assume that R was Noetherian in our definition of a Dedekind domain. Let $\mathfrak{p} \subset R$ be a prime ideal. Observe that $\mathfrak{p} \neq \mathfrak{p}^2$ by uniqueness of the factorizations in the definition. Pick $x \in \mathfrak{p}$ with $x \notin \mathfrak{p}^2$. Let $y \in \mathfrak{p}$ be a second element (for example $y = 0$). Write $(x, y) = \mathfrak{p}_1 \ldots \mathfrak{p}_r$. Since $(x, y) \subset \mathfrak{p}$ at least one of the primes $\mathfrak{p}_i$ is contained in $\mathfrak{p}$. But as $x \notin \mathfrak{p}^2$ there is at most one. Thus exactly one of $\mathfrak{p}_1, \ldots, \mathfrak{p}_r$ is contained in $\mathfrak{p}$, say $\mathfrak{p}_1 \subset \mathfrak{p}$. We conclude that $(x, y)R_\mathfrak{p} = \mathfrak{p}_1 R_\mathfrak{p}$ is prime for every choice of y. We claim that $(x)R_\mathfrak{p} = \mathfrak{p}R_\mathfrak{p}$. Namely, pick $y \in \mathfrak{p}$. By the above applied with y^2 we see that $(x, y^2)R_\mathfrak{p}$ is prime. Hence $y \in (x, y^2)R_\mathfrak{p}$, i.e., $y = ax + by^2$ in $R_\mathfrak{p}$. Thus $(1 - by)y = ax \in (x)R_\mathfrak{p}$, i.e., $y \in (x)R_\mathfrak{p}$ as desired.

Writing $(x) = \mathfrak{p}_1 \ldots \mathfrak{p}_r$ anew with $\mathfrak{p}_1 \subset \mathfrak{p}$ we conlude that $\mathfrak{p}_1 R_\mathfrak{p} = \mathfrak{p}R_\mathfrak{p}$, i.e., $\mathfrak{p}_1 = \mathfrak{p}$. Moreover, $\mathfrak{p}_1 = \mathfrak{p}$ is a finitely generated ideal of R by Lemma 10.114.9. We conclude that R is Noetherian by Lemma 10.26.9. Moreover, it follows that $R_\mathfrak{m}$ is a discrete valuation ring for every prime ideal $\mathfrak{p}$, see Lemma 10.113.6.

The equivalence of (2) and (3) follows from Lemmas 10.34.9 and 10.113.6. Assume (2) and (3) are satisfied. Let $I \subset R$ be an ideal. We will construct a factorization of I. If I is prime, then there is nothing to prove. If not, pick $I \subset \mathfrak{p}$ with $\mathfrak{p} \subset R$ maximal. Let $J = \{x \in R \mid x\mathfrak{p} \subset I\}$. We claim $J\mathfrak{p} = I$. It suffices to check this after localization at the maximal ideals $\mathfrak{m}$ of A (the formation of J commutes with localization and we use Lemma 10.22.1). Then either $\mathfrak{p}R_\mathfrak{m} = R_\mathfrak{m}$ and the result is clear, or $\mathfrak{p}R_\mathfrak{m} = \mathfrak{m}R_\mathfrak{m}$. In the last case $\mathfrak{p}R_\mathfrak{m} = (\pi)$ and the case where $\mathfrak{p}$ is principal is immediate. By Noetherian induction the ideal J has a factorization and we obtain the desired factorization of I. We omit the proof of uniqueness of the factorization. □

The following is a variant of the Krull-Akizuki lemma.

Lemma 10.114.11. *Let A be a Noetherian domain of dimension 1 with fraction field K. Let $K \subset L$ be a finite extension. Let B be the integral closure of A in L. Then B is a Dedekind domain and $\mathrm{Spec}(B) \to \mathrm{Spec}(A)$ is surjective, has finite fibres, and induces finite residue field extensions.*

Proof. By Krull-Akizuki (Lemma 10.113.11) the ring B is Noetherian. By Lemma 10.106.4 $\dim(B) = 1$. Thus B is a Dedekind domain by Lemma 10.114.10. Surjectivity of the map on spectra follows from Lemma 10.33.15. The last two statements follow from Lemma 10.113.9. □

10.115. Orders of vanishing

Lemma 10.115.1. *Let R be a semi-local Noetherian ring of dimension 1. If $a, b \in R$ are nonzerodivisors then*

$$length_R(R/(ab)) = length_R(R/(a)) + length_R(R/(b))$$

and these lengths are finite.

Proof. We saw the finiteness in Lemma 10.113.10. Additivity holds since there is a short exact sequence $0 \to R/(a) \to R/(ab) \to R/(b) \to 0$ where the first map is given by multiplication by b. (Use length is additive, see Lemma 10.48.3.) □

Definition 10.115.2. Suppose that K is a field, and $R \subset K$ is a local[5] Noetherian subring of dimension 1 with fraction field K. In this case we define the *order of vanishing along* R

$$\text{ord}_R : K^* \longrightarrow \mathbf{Z}$$

by the rule

$$\text{ord}_R(x) = \text{length}_R(R/(x))$$

if $x \in R$ and we set $\text{ord}_R(x/y) = \text{ord}_R(x) - \text{ord}_R(y)$ for $x, y \in R$ both nonzero.

We can use the order of vanishing to compare lattices in a vector space. Here is the definition.

Definition 10.115.3. Let R be a Noetherian local domain of dimension 1 with fraction field K. Let V be a finite dimensional K-vector space. A *lattice in* V is a finite R-submodule $M \subset V$ such that $V = K \otimes_R M$.

The condition $V = K \otimes_R M$ signifies that M contains a basis for the vector space K. We remark that in many places in the literature the notion of a lattice may be defined only in case the ring R is a discrete valuation ring. If R is a discrete valuation ring then any lattice is a free R-module, and this may not be the case in general.

Lemma 10.115.4. *Let R be a Noetherian local domain of dimension 1 with fraction field K. Let V be a finite dimensional K-vector space.*

(1) *If M is a lattice in V and $M \subset M' \subset V$ is an R-submodule of V containing M then the following are equivalent*
   (a) *M' is a lattice,*
   (b) *$\text{length}_R(M'/M)$ is finite, and*
   (c) *M' is finitely generated.*

(2) *If M is a lattice in V and $M' \subset M$ is an R-submodule of M then M' is a lattice if and only if $\text{length}_R(M/M')$ is finite.*

(3) *If M, M' are lattices in V, then so are $M \cap M'$ and $M + M'$.*

(4) *If $M \subset M' \subset M'' \subset V$ are lattices in V then*

$$\text{length}_R(M''/M) = \text{length}_R(M'/M) + \text{length}_R(M''/M').$$

(5) *If M, M', N, N' are lattices in V and $N \subset M \cap M'$, $M + M' \subset N'$, then we have*

$$\begin{aligned} & \text{length}_R(M/M \cap M') - \text{length}_R(M'/M \cap M') \\ = \ & \text{length}_R(M/N) - \text{length}_R(M'/N) \\ = \ & \text{length}_R(M + M'/M') - \text{length}_R(M + M'/M) \\ = \ & \text{length}_R(N'/M') - \text{length}_R(N'/M) \end{aligned}$$

[5] We could also define this when R is only semi-local but this is probably never really what you want!

Proof. Proof of (1). Assume (1)(a). Say $y_1, \ldots, y_m$ generate M'. Then each $y_i = x_i/f_i$ for some $x_i \in M$ and nonzero $f_i \in R$. Hence we see that $f_1 \ldots f_m M' \subset M$. Since R is Noetherian local of dimension 1 we see that $\mathfrak{m}^n \subset (f_1 \ldots f_m)$ for some n (for example combine Lemmas 10.57.11 and Proposition 10.57.6 or combine Lemmas 10.113.8 and 10.48.4). In other words $\mathfrak{m}^n M' \subset M$ for some n Hence $\text{length}(M'/M) < \infty$ by Lemma 10.48.8, in other words (1)(b) holds. Assume (1)(b). Then M'/M is a finite R-module (see Lemma 10.48.2). Hence M' is a finite R-module as an extension of finite R-modules. Hence (1)(c). The implication (1)(c) $\Rightarrow$ (1)(a) follows from the remark following Definition 10.115.3.

Proof of (2). Suppose M is a lattice in V and $M' \subset M$ is an R-submodule. We have seen in (1) that if M' is a lattice, then $\text{length}_R(M/M') < \infty$. Conversely, assume that $\text{length}_R(M/M') < \infty$. Then M' is finitely generated as R is Noetherian and for some n we have $\mathfrak{m}^n M \subset M'$ (Lemma 10.48.4). Hence it follows that M' contains a basis for V, and M' is a lattice.

Proof of (3). Assume M, M' are lattices in V. Since R is Noetherian the submodule $M \cap M'$ of M is finite. As M is a lattice we can find $x_1, \ldots, x_n \in M$ which form a K-basis for V. Because M' is a lattice we can write $x_i = y_i/f_i$ with $y_i \in M'$ and $f_i \in R$. Hence $f_i x_i \in M \cap M'$. Hence $M \cap M'$ is a lattice also. The fact that $M + M'$ is a lattice follows from part (1).

Part (4) follows from additivity of lengths (Lemma 10.48.3) and the exact sequence

$$0 \to M'/M \to M''/M \to M''/M' \to 0$$

Part (5) follows from repeatedly applying part (4). □

Definition 10.115.5. Let R be a Noetherian local domain of dimension 1 with fraction field K. Let V be a finite dimensional K-vector space. Let M, M' be two lattices in V. The *distance between M and M'* is the integer

$$d(M, M') = \text{length}_R(M/M \cap M') - \text{length}_R(M'/M \cap M')$$

of Lemma 10.115.4 part (5).

In particular, if $M' \subset M$, then $d(M, M') = \text{length}_R(M/M')$.

Lemma 10.115.6. *Let R be a Noetherian local domain of dimension 1 with fraction field K. Let V be a finite dimensional K-vector space. This distance function has the property that*

$$d(M, M'') = d(M, M') + d(M', M'')$$

whenever given three lattices M, M', M'' of V. In particular we have $d(M, M') = -d(M', M)$.

Proof. Omitted. □

Lemma 10.115.7. *Let R be a Noetherian local domain of dimension 1 with fraction field K. Let V be a finite dimensional K-vector space. Let $\varphi : V \to V$ be a K-linear isomorphism. For any lattice $M \subset V$ we have*

$$d(M, \varphi(M)) = ord_R(\det(\varphi))$$

Proof. We can see that the integer $d(M, \varphi(M))$ does not depend on the lattice M as follows. Suppose that M' is a second such lattice. Then we see that

$$\begin{aligned} d(M, \varphi(M)) &= d(M, M') + d(M', \varphi(M)) \\ &= d(M, M') + d(\varphi(M'), \varphi(M)) + d(M', \varphi(M')) \end{aligned}$$

Since φ is an isomorphism we see that $d(\varphi(M'), \varphi(M)) = d(M', M) = -d(M, M')$, and hence $d(M, \varphi(M)) = d(M', \varphi(M'))$. Moreover, both sides of the equation (of the lemma) are additive in φ, i.e.,

$$\mathrm{ord}_R(\det(\varphi \circ \psi)) = \mathrm{ord}_R(\det(\varphi)) + \mathrm{ord}_R(\det(\psi))$$

and also

$$\begin{aligned} d(M, \varphi(\psi((M))) &= d(M, \psi(M)) + d(\psi(M), \varphi(\psi(M))) \\ &= d(M, \psi(M)) + d(M, \varphi(M)) \end{aligned}$$

by the independence shown above. Hence it suffices to prove the lemma for generators of $\mathrm{GL}(V)$. Choose an isomorphism $K^{\oplus n} \cong V$. Then $\mathrm{GL}(V) = \mathrm{GL}_n(K)$ is generated by elementary matrices E. The result is clear for E equal to the identity matrix. If $E = E_{ij}(\lambda)$ with $i \neq j$, $\lambda \in K$, $\lambda \neq 0$, for example

$$E_{12}(\lambda) = \begin{pmatrix} 1 & \lambda & \ldots \\ 0 & 1 & \ldots \\ \ldots & \ldots & \ldots \end{pmatrix}$$

then with respect to a different basis we get $E_{12}(1)$. The result is clear for $E = E_{12}(1)$ by taking as lattice $R^{\oplus n} \subset K^{\oplus n}$. Finally, if $E = E_i(a)$, with $a \in K^*$ for example

$$E_1(a) = \begin{pmatrix} a & 0 & \ldots \\ 0 & 1 & \ldots \\ \ldots & \ldots & \ldots \end{pmatrix}$$

then $E_1(a)(R^{\oplus b}) = aR \oplus R^{\oplus n-1}$ and it is clear that $d(R^{\oplus n}, aR \oplus R^{\oplus n-1}) = \mathrm{ord}_R(a)$ as desired. □

Lemma 10.115.8. *Let $A \to B$ be a ring map. Assume*

(1) *A is a Noetherian local domain of dimension 1,*
(2) *$A \subset B$ is a finite extension of domains.*

Let $K = f.f.(A)$ and $L = f.f.(B)$ so that L is a finite field extension of K. Let $y \in L^*$ and $x = Nm_{L/K}(y)$. In this situation B is semi-local. Let $\mathfrak{m}_i$, $i = 1, \ldots, n$ be the maximal ideals of B. Then

$$ord_A(x) = \sum\nolimits_i [\kappa(\mathfrak{m}_i) : \kappa(\mathfrak{m}_A)] ord_{B_{\mathfrak{m}_i}}(y)$$

where ord is defined as in Definition 10.115.2.

Proof. The ring B is semi-local by Lemma 10.107.2. Write $y = b/b'$ for some $b, b' \in B$. By the additivity of ord and multiplicativity of Nm it suffices to prove the lemma for $y = b$ or $y = b'$. In other words we may assume $y \in B$. In this case the left hand side of the formula is

$$\sum [\kappa(\mathfrak{m}_i) : \kappa(\mathfrak{m}_A)] \mathrm{length}_{B_{\mathfrak{m}_i}}((B/yB)_{\mathfrak{m}_i})$$

By Lemma 10.48.12 this is equal to $\mathrm{length}_A(B/yB)$. By Lemma 10.115.7 we have

$$\mathrm{length}_A(B/yB) = d(B, yB) = \mathrm{ord}_A(\det_K(L \xrightarrow{y} L)).$$

Since $x = \mathrm{Nm}_{L/K}(y) = \det_K(L \xrightarrow{y} L)$ by definition the lemma is proved. □

We can extend some of the results above to reduced 1-dimensional Noetherian local rings which are not domains by the following lemma.

Lemma 10.115.9. *Let $(R, \mathfrak{m})$ be a reduced Noetherian local ring of dimension 1 and let $x \in \mathfrak{m}$ be a nonzerodivisor. Let $\mathfrak{q}_1, \ldots, \mathfrak{q}_r$ be the minimal primes of R. Then*

$$length_R(R/(x)) = \sum\nolimits_i ord_{R/\mathfrak{q}_i}(x)$$

Proof. Note that $R_i = R/\mathfrak{q}_i$ is a Noetherian 1-dimensional local domain. Denote $K_i = f.f.(R_i)$. If x is a unit in R, then both sides are zero. Hence we may assume $x \in \mathfrak{m}$. Consider the map $\Psi : R \to \prod R_i$. As R is reduced this map is injective, see Lemma 10.16.2. By Lemma 10.23.2 we have $Q(R) = \prod K_i$. Hence the finite R-module $\mathrm{Coker}(\Psi)$ is annihilated by a nonzerodivisor $y \in R$, hence has support $\{\mathfrak{m}\}$, is annihilated by some power of x and has finite length over R, see Lemma 10.59.3. Consider the short exact sequence

$$0 \to R \to \prod R_i \to \mathrm{Coker}(\Psi) \to 0$$

Applying multiplication by x^n to this for $n \gg 0$ we obtain from the snake lemma

$$0 \to \mathrm{Coker}(\Psi) \to R/x^nR \to \prod R_i/x^nR_i \to \mathrm{Coker}(\Psi) \to 0$$

Thus we see that

$$\mathrm{length}_R(R/x^nR) = \mathrm{length}_R(\prod R_i/x^nR_i) = \sum \mathrm{length}_R(R_i/x^nR_i)$$

by Lemma 10.48.3. By Lemma 10.48.5 we have $\mathrm{length}_R(R_i/x^nR_i) = \mathrm{length}_{R_i}(R_i/x^nR_i)$. Now the result follows from the additivity of Lemma 10.115.1 and the definition of the order of vanishing along R_i. □

10.116. Quasi-finite maps

Consider a ring map $R \to S$ of finite type. A map $\mathrm{Spec}(S) \to \mathrm{Spec}(R)$ is quasi-finite at a point if that point is isolated in its fibre. This means that the fibre is zero dimensional at that point. In this section we study the basic properties of this important but technical notion. More advanced material can be found in the next section.

Lemma 10.116.1. *Let k be a field. Let S be a finite type k algebra. Let $\mathfrak{q}$ be a prime of S. The following are equivalent:*

1. *$\mathfrak{q}$ is an isolated point of $\mathrm{Spec}(S)$,*
2. *$S_\mathfrak{q}$ is finite over k,*
3. *there exists a $g \in S$, $g \notin \mathfrak{q}$ such that $D(g) = \{\mathfrak{q}\}$,*
4. *$\dim_\mathfrak{q} \mathrm{Spec}(S) = 0$,*
5. *$\mathfrak{q}$ is a closed point of $\mathrm{Spec}(S)$ and $\dim(S_\mathfrak{q}) = 0$, and*
6. *the field extension $k \subset \kappa(\mathfrak{q})$ is finite and $\dim(S_\mathfrak{q}) = 0$.*

In this case $S = S_\mathfrak{q} \times S'$ for some finite type k-algebra S'. Also, the element g as in (3) has the property $S_\mathfrak{q} = S_g$.

Proof. Suppose $\mathfrak{q}$ is an isolated point of $\mathrm{Spec}(S)$, i.e., $\{\mathfrak{q}\}$ is open in $\mathrm{Spec}(S)$. Because $\mathrm{Spec}(S)$ is a Jacobson space (see Lemmas 10.32.2 and 10.32.4) we see that $\mathfrak{q}$ is a closed point. Hence $\{\mathfrak{q}\}$ is open and closed in $\mathrm{Spec}(S)$. By Lemmas 10.19.3 and 10.21.3 we may write $S = S_1 \times S_2$ with $\mathfrak{q}$ corresponding to the only point

$\mathrm{Spec}(S_1)$. Hence $S_1 = S_\mathfrak{q}$ is a zero dimensional ring of finite type over k. Hence it is finite over k for example by Lemma 10.109.4. We have proved (1) implies (2).

Suppose $S_\mathfrak{q}$ is finite over k. Then $S_\mathfrak{q}$ is Artinian local, see Lemma 10.49.2. So $\mathrm{Spec}(S_\mathfrak{q}) = \{\mathfrak{q}S_\mathfrak{q}\}$ by Lemma 10.49.8. Consider the exact sequence $0 \to K \to S \to S_\mathfrak{q} \to Q \to 0$. It is clear that $K_\mathfrak{q} = Q_\mathfrak{q} = 0$. Also, K is a finite S-module as S is Noetherian and Q is a finite S-modules since $S_\mathfrak{q}$ is finite over k. Hence there exists $g \in S$, $g \notin \mathfrak{q}$ such that $K_g = Q_g = 0$. Thus $S_\mathfrak{q} = S_g$ and $D(g) = \{\mathfrak{q}\}$. We have proved that (2) implies (3).

Suppose $D(g) = \{\mathfrak{q}\}$. Since $D(g)$ is open by construction of the topology on $\mathrm{Spec}(S)$ we see that $\mathfrak{q}$ is an isolated point of $\mathrm{Spec}(S)$. We have proved that (3) implies (1). In other words (1), (2) and (3) are equivalent.

Assume $\dim_\mathfrak{q} \mathrm{Spec}(S) = 0$. This means that there is some open neighbourhood of $\mathfrak{q}$ in $\mathrm{Spec}(S)$ which has dimension zero. Then there is an open neighbourhood of the form $D(g)$ which has dimension zero. Since S_g is Noetherian we conclude that S_g is Artinian and $D(g) = \mathrm{Spec}(S_g)$ is a finite discrete set, see Proposition 10.57.6. Thus $\mathfrak{q}$ is an isolated point of $D(g)$ and, by the equivalence of (1) and (2) above applied to $\mathfrak{q}S_g \subset S_g$, we see that $S_\mathfrak{q} = (S_g)_{\mathfrak{q}S_g}$ is finite over k. Hence (4) implies (2). It is clear that (1) implies (4). Thus (1) – (4) are all equivalent.

Lemma 10.108.6 gives the implication (5) $\Rightarrow$ (4). The implication (4) $\Rightarrow$ (6) follows from Lemma 10.110.3. The implication (6) $\Rightarrow$ (5) follows from Lemma 10.32.9. At this point we know (1) – (6) are equivalent.

The two statements at the end of the lemma we saw during the course of the proof of the equivalence of (1), (2) and (3) above. □

Lemma 10.116.2. *Let $R \to S$ be a ring map of finite type. Let $\mathfrak{q} \subset S$ be a prime lying over $\mathfrak{p} \subset R$. Let $F = \mathrm{Spec}(S \otimes_R \kappa(\mathfrak{p}))$ be the fibre of $\mathrm{Spec}(S) \to \mathrm{Spec}(R)$, see Remark 10.16.8. Denote $\overline{\mathfrak{q}} \in F$ the point corresponding to $\mathfrak{q}$. The following are equivalent*

1. *$\overline{\mathfrak{q}}$ is an isolated point of F,*
2. *$S_\mathfrak{q}/\mathfrak{p}S_\mathfrak{q}$ is finite over $\kappa(\mathfrak{p})$,*
3. *there exists a $g \in S$, $g \notin \mathfrak{q}$ such that the only prime of $D(g)$ mapping to $\mathfrak{p}$ is $\mathfrak{q}$,*
4. *$\dim_{\overline{\mathfrak{q}}}(F) = 0$,*
5. *$\overline{\mathfrak{q}}$ is a closed point of F and $\dim(S_\mathfrak{q}/\mathfrak{p}S_\mathfrak{q}) = 0$, and*
6. *the field extension $\kappa(\mathfrak{p}) \subset \kappa(\mathfrak{q})$ is finite and $\dim(S_\mathfrak{q}/\mathfrak{p}S_\mathfrak{q}) = 0$.*

Proof. Note that $S_\mathfrak{q}/\mathfrak{p}S_\mathfrak{q} = (S \otimes_R \kappa(\mathfrak{p}))_{\overline{\mathfrak{q}}}$. Moreover $S \otimes_R \kappa(\mathfrak{p})$ is of finite type over $\kappa(\mathfrak{p})$. The conditions correspond exactly to the conditions of Lemma 10.116.1 for the $\kappa(\mathfrak{p})$-algebra $S \otimes_R \kappa(\mathfrak{p})$ and the prime $\overline{\mathfrak{q}}$, hence they are equivalent. □

Definition 10.116.3. Let $R \to S$ be a finite type ring map. Let $\mathfrak{q} \subset S$ be a prime.

1. If the equivalent conditions of Lemma 10.116.2 are satisfied then we say $R \to S$ is *quasi-finite at* $\mathfrak{q}$.
2. We say a ring map $A \to B$ is *quasi-finite* if it is of finite type and quasi-finite at all primes of B.

Lemma 10.116.4. *Let $R \to S$ be a finite type ring map. Then $R \to S$ is quasi-finite if and only if for all primes $\mathfrak{p} \subset R$ the fibre $S \otimes_R \kappa(\mathfrak{p})$ is finite over $\kappa(\mathfrak{p})$.*

Proof. If the fibres are finite then the map is clearly quasi-finite. For the converse, note that $S \otimes_R \kappa(\mathfrak{p})$ is a $\kappa(\mathfrak{p})$-algebra of finite type over k of dimension 0. Hence it is finite over k for example by Lemma 10.109.4. □

Lemma 10.116.5. *Let $R \to S$ be a finite type ring map. Let $\mathfrak{q} \subset S$ be a prime lying over $\mathfrak{p} \subset R$. Let $f \in R$, $f \notin \mathfrak{p}$ and $g \in S$, $g \notin \mathfrak{q}$. Then $R \to S$ is quasi-finite at $\mathfrak{q}$ if and only if $R_f \to S_{fg}$ is quasi-finite at $\mathfrak{q}S_{fg}$.*

Proof. The fibre of $\mathrm{Spec}(S_{fg}) \to \mathrm{Spec}(R_f)$ is homeomorphic to an open subset of the fibre of $\mathrm{Spec}(S) \to \mathrm{Spec}(R)$. Hence the lemma follows from part (1) of the equivalent conditions of Lemma 10.116.2. □

Lemma 10.116.6. *Let*

$$\begin{array}{ccc} S & \longrightarrow & S' \\ \uparrow & & \uparrow \\ R & \longrightarrow & R' \end{array} \qquad \begin{array}{ccc} \mathfrak{q} & \longrightarrow & \mathfrak{q}' \\ \downarrow & & \downarrow \\ \mathfrak{p} & \longrightarrow & \mathfrak{p}' \end{array}$$

be a commutative diagram of rings with primes as indicated. Assume $R \to S$ of finite type, and $S \otimes_R R' \to S'$ surjective. If $R \to S$ is quasi-finite at $\mathfrak{q}$, then $R' \to S'$ is quasi-finite at $\mathfrak{q}'$.

Proof. Write $S \otimes_R \kappa(\mathfrak{p}) = S_1 \times S_2$ with S_1 finite over $\kappa(\mathfrak{p})$ and such that $\mathfrak{q}$ corresponds to a point of S_1 as in Lemma 10.116.1. Because $S \otimes_R R' \to S'$ surjective the canonical map $(S \otimes_R \kappa(\mathfrak{p})) \otimes_{\kappa(\mathfrak{p})} \kappa(\mathfrak{p}') \to S' \otimes_{R'} \kappa(\mathfrak{p}')$ is surjective. Let S_i' be the image of $S_i \otimes_{\kappa(\mathfrak{p})} \kappa(\mathfrak{p}')$ in $S' \otimes_{R'} \kappa(\mathfrak{p}')$. Then $S' \otimes_{R'} \kappa(\mathfrak{p}') = S_1' \times S_2'$ and S_1' is finite over $\kappa(\mathfrak{p}')$. The map $S' \otimes_{R'} \kappa(\mathfrak{p}') \to \kappa(\mathfrak{q}')$ factors through S_1' (i.e. it annihilates the factor S_2') because the map $S \otimes_R \kappa(\mathfrak{p}) \to \kappa(\mathfrak{q})$ factors through S_1 (i.e. it annihilates the factor S_2). Thus $\mathfrak{q}'$ corresponds to a point of $\mathrm{Spec}(S_1')$ in the disjoint union decomposition of the fibre: $\mathrm{Spec}(S' \otimes_{R'} \kappa(\mathfrak{p}')) = \mathrm{Spec}(S_1') \amalg \mathrm{Spec}(S_1')$. (See Lemma 10.19.2.) Since S_1' is finite over a field, it is Artinian ring, and hence $\mathrm{Spec}(S_1')$ is a finite discrete set. (See Proposition 10.57.6.) We conclude $\mathfrak{q}'$ is isolated in its fibre as desired. □

Lemma 10.116.7. *A composition of quasi-finite ring maps is quasi-finite.*

Proof. Suppose $A \to B$ and $B \to C$ are quasi-finite ring maps. By Lemma 10.6.2 we see that $A \to C$ is of finite type. Let $\mathfrak{r} \subset C$ be a prime of C lying over $\mathfrak{q} \subset B$ and $\mathfrak{p} \subset A$. Since $A \to B$ and $B \to C$ are quasi-finite at $\mathfrak{q}$ and $\mathfrak{r}$ respectively, then there exist $b \in B$ and $c \in C$ such that $\mathfrak{q}$ is the only prime of $D(b)$ which maps to $\mathfrak{p}$ and similarly $\mathfrak{r}$ is the only prime of $D(c)$ which maps to $\mathfrak{q}$. If $c' \in C$ is the image of $b \in B$, then $\mathfrak{r}$ is the only prime of $D(cc')$ which maps to $\mathfrak{p}$. Therefore $A \to C$ is quasi-finite at $\mathfrak{r}$. □

Lemma 10.116.8. *Let $R \to S$ be a ring map of finite type. Let $R \to R'$ be any ring map. Set $S' = R' \otimes_R S$.*

1. *The set $\{\mathfrak{q}' \mid R' \to S' \text{ quasi-finite at } \mathfrak{q}'\}$ is the inverse image of the corresponding set of* $\mathrm{Spec}(S)$ *under the canonical map* $\mathrm{Spec}(S') \to \mathrm{Spec}(S)$.
2. *If* $\mathrm{Spec}(R') \to \mathrm{Spec}(R)$ *is surjective, then $R \to S$ is quasi-finite if and only if $R' \to S'$ is quasi-finite.*
3. *Any base change of a quasi-finite ring map is quasi-finite.*

Proof. Let $\mathfrak{p}' \subset R'$ be a prime lying over $\mathfrak{p} \subset R$. Then the fibre ring $S' \otimes_{R'} \kappa(\mathfrak{p}')$ is the base change of the fibre ring $S \otimes_R \kappa(\mathfrak{p})$ by the field extension $\kappa(\mathfrak{p}) \to \kappa(\mathfrak{p}')$. Hence the first assertion follows from the invariance of dimension under field extension (Lemma 10.110.6) and Lemma 10.116.1. The stability of quasi-finite maps under base change follows from this and the stability of finite type property under base change. The second assertion follows since the assumption implies that given a prime $\mathfrak{q} \subset S$ we can find a prime $\mathfrak{q}' \subset S'$ lying over it. □

The following lemma is not quite about quasi-finite ring maps, but it does not seem to fit anywhere else so well.

Lemma 10.116.9. *Let $R \to S$ be a ring map of finite type. Let $\mathfrak{p} \subset R$ be a minimal prime. Assume that there are at most finitely many primes of S lying over $\mathfrak{p}$. Then there exists a $g \in R$, $g \notin \mathfrak{p}$ such that the ring map $R_g \to S_g$ is finite.*

Proof. Let $x_1, \ldots, x_n$ be generators of S over R. Since $\mathfrak{p}$ is a minimal prime we have that $\mathfrak{p}R_\mathfrak{p}$ is a locally nilpotent ideal, see Lemma 10.24.4. Hence $\mathfrak{p}S_\mathfrak{p}$ is a locally nilpotent ideal, see Lemma 10.14.1. By assumption the finite type $\kappa(\mathfrak{p})$-algebra $S_\mathfrak{p}/\mathfrak{p}S_\mathfrak{p}$ has finitely many primes. Hence (for example by Lemma 10.109.4) $\kappa(\mathfrak{p}) \to S_\mathfrak{p}/\mathfrak{p}S_\mathfrak{p}$ is a finite ring map. Thus we may find monic polynomials $P_i \in R_\mathfrak{p}[X]$ such that $P_i(x_i)$ maps to zero in $S_\mathfrak{p}/\mathfrak{p}S_\mathfrak{p}$. By what we said above there exist $e_i \geq 1$ such that $P(x_i)^{e_i} = 0$ in $S_\mathfrak{p}$. Let $g_1 \in R$, $g_1 \notin \mathfrak{p}$ be an element such that $P_i \in R[1/g_1]$ for all i. Next, let $g_2 \in R$, $g_2 \notin \mathfrak{p}$ be an element such that $P(x_i) = 0$ in $S_{g_1 g_2}$. Setting $g = g_1 g_2$ we win. □

10.117. Zariski's Main Theorem

In this section our aim is to prove the algebraic version of Zariski's Main theorem. This theorem will be the basis of many further developments in the theory of schemes and morphisms of schemes later in the project.

Let $R \to S$ be a ring map of finite type. Our goal in this section is to show that the set of points of $\mathrm{Spec}(S)$ where the map is quasi-finite is *open* (Theorem 10.117.13). In fact, it will turn out that there exists a finite ring map $R \to S'$ such that in some sense the quasi-finite locus of S/R is open in $\mathrm{Spec}(S')$ (but we will not prove this in the algebra chapter since we do not develop the language of schemes here – for the case where $R \to S$ is quasi-finite see Lemma 10.117.15). These statements are somewhat tricky to prove and we do it by a long list of lemmas concerning integral and finite extensions of rings. This material may be found in [**Ray70**], and [**Pes66**]. We also found notes by Thierry Coquand helpful.

Lemma 10.117.1. *Let $\varphi : R \to S$ be a ring map. Suppose $t \in S$ satisfies the relation $\varphi(a_0) + \varphi(a_1)t + \ldots + \varphi(a_n)t^n = 0$. Then $\varphi(a_n)t$ is integral over R.*

Proof. Namely, multiply the equation $\varphi(a_0) + \varphi(a_1)t + \ldots + \varphi(a_n)t^n = 0$ with $\varphi(a_n)^{n-1}$ and write it as $\varphi(a_0 a_n^{n-1}) + \varphi(a_1 a_n^{n-2})(\varphi(a_n)t) + \ldots + (\varphi(a_n)t)^n = 0$. □

The following lemma is in some sense the key lemma in this section.

Lemma 10.117.2. *Let R be a ring. Let $\varphi : R[x] \to S$ be a ring map. Let $t \in S$. Assume that (a) t is integral over $R[x]$, and (b) there exists a monic $p \in R[x]$ such that $t\varphi(p) \in Im(\varphi)$. Then there exists a $q \in R[x]$ such that $t - \varphi(q)$ is integral over R.*

Proof. Write $t\varphi(p) = \varphi(r)$ for some $r \in R[x]$. Using euclidean division, write $r = qp + r'$ with $q, r' \in R[x]$ and $\deg(r') < \deg(p)$. We may replace t by $t - \varphi(q)$ which is still integral over $R[x]$, so that we obtain $t\varphi(p) = \varphi(r')$. In the ring S_t we may write this as $\varphi(p) - (1/t)\varphi(r') = 0$. This implies that $\varphi(x)$ gives an element of the localization S_t which is integral over $\varphi(R)[1/t] \subset S_t$. On the other hand, t is integral over the subring $\varphi(R)[\varphi(x)] \subset S$. Combined we conclude that t is integral over the subring $\varphi(R)[1/t] \subset S_t$, see Lemma 10.33.6. In other words there exists an equation of the form $t^d + \sum_{i<d}(\varphi(r_i)/t^{n_i})t^i = 0$ in S_t with $r_i \in R$. This means that $t^{d+N} + \sum_{i<d}\varphi(r_i)t^{i+N-n_i} = 0$ in S for some N large enough. In other words t is integral over R. □

Lemma 10.117.3. *Let R be a ring and let $\varphi : R[x] \to S$ be a ring map. Let $t \in S$. If t is integral over $R[x]$, then there exists an $\ell \geq 0$ such that for every $a \in R$ the element $\varphi(a)^\ell t$ is integral over $\varphi_a : R[y] \to S$, defined by $y \mapsto \varphi(ax)$ and $r \mapsto \varphi(r)$ for $r \in R$.*

Proof. Say $t^d + \sum_{i<d}\varphi(f_i)t^i = 0$ with $f_i \in R[x]$. Let ℓ be the maximum degree in x of all the f_i. Multiply the equation by $\varphi(a)^\ell$ to get $\varphi(a)^\ell t^d + \sum_{i<d}\varphi(a^\ell f_i)t^i = 0$. Note that each $\varphi(a^\ell f_i)$ is in the image of φ_a. The result follows from Lemma 10.117.1. □

Lemma 10.117.4. *Let R be a ring. Let $\varphi : R[x] \to S$ be a ring map. Let $t \in S$. Assume t is integral over $R[x]$. Let $p \in R[x]$, $p = a_0 + a_1x + \ldots + a_kx^k$ such that $t\varphi(p) \in Im(\varphi)$. Then there exists a $q \in R[x]$ and $n \geq 0$ such that $\varphi(a_k)^n t - \varphi(q)$ is integral over R.*

Proof. By Lemma 10.117.3 there exists an $\ell \geq 0$ such that the element $\varphi(a_k)^\ell t$ is integral over the map $\varphi' : R[y] \to S$, $\varphi'(y) = \varphi(a_kx)$ and $\varphi'(r) = \varphi(r)$, for $r \in R$. The polynomial $p' = a_k^{k-1}a_0 + a_k^{k-2}a_1y + \ldots + y^k$ is monic and $t\varphi'(p') = \varphi(a_k^{k-1})t\varphi(p) \in \mathrm{Im}(\varphi)$. By definition of φ' this implies there exists a $n \geq k-1$ such that $\varphi(a_k^n)t\varphi'(p') \in \mathrm{Im}(\varphi')$. If also $n \geq \ell$, then $\varphi(a_k)^n t$ is still integral over $R[y]$. By Lemma 10.117.2 we see that $\varphi(a_k)^n t - \varphi'(q)$ is integral over R for some $q \in R[y]$. Again by the simple relationship between φ' and φ this implies the lemma. □

Situation 10.117.5. Let R be a ring. Let $\varphi : R[x] \to S$ be finite. Let

$$J = \{g \in S \mid gS \subset \mathrm{Im}(\varphi)\}$$

be the "conductor ideal" of φ. Assume $\varphi(R) \subset S$ integrally closed in S.

Lemma 10.117.6. *In Situation 10.117.5. Suppose $u \in S$, $a_0, \ldots, a_k \in R$, $u\varphi(a_0 + a_1x + \ldots + a_kx^k) \in J$. Then there exists an $m \geq 0$ such that $u\varphi(a_k)^m \in J$.*

Proof. Assume that S is generated by $t_1, \ldots, t_n$ as an $R[x]$-module. In this case $J = \{g \in S \mid gt_i \in \mathrm{Im}(\varphi) \text{ for all } i\}$. Note that each element ut_i is integral over $R[x]$, see Lemma 10.33.3. We have $\varphi(a_0+a_1x+\ldots+a_kx^k)ut_i \in \mathrm{Im}(\varphi)$. By Lemma 10.117.4, for each i there exists an integer n_i and an element $q_i \in R[x]$ such that $\varphi(a_k^{n_i})ut_i - \varphi(q_i)$ is integral over R. By assumption this element is in $\varphi(R)$ and hence $\varphi(a_k^{n_i})ut_i \in \mathrm{Im}(\varphi)$. It follows that $m = \max\{n_1, \ldots, n_n\}$ works. □

Lemma 10.117.7. *In Situation 10.117.5. Suppose $u \in S$, $a_0, \ldots, a_k \in R$, $u\varphi(a_0 + a_1x + \ldots + a_kx^k) \in \sqrt{J}$. Then $u\varphi(a_i) \in \sqrt{J}$ for all i.*

Proof. Under the assumptions of the lemma we have $u^n\varphi(a_0+a_1x+\ldots+a_kx^k)^n \in J$ for some $n \geq 1$. By Lemma 10.117.6 we deduce $u^n\varphi(a_k^{nm}) \in J$ for some $m \geq 1$. Thus $u\varphi(a_k) \in \sqrt{J}$, and so $u\varphi(a_0 + a_1x + \ldots + a_kx^k) - u\varphi(a_k) = u\varphi(a_0 + a_1x + \ldots + a_{k-1}x^{k-1}) \in \sqrt{J}$. We win by induction on k. □

This lemma suggests the following definition.

Definition 10.117.8. Given an inclusion of rings $R \subset S$ and an element $x \in S$ we say that x is *strongly transcendental over* R if whenever $u(a_0+a_1x+\ldots+a_kx^k) = 0$ with $u \in S$ and $a_i \in R$, then we have $ua_i = 0$ for all i.

Note that if S is a domain then this is the same as saying that x as an element of the fraction field of S is transcendental over the fraction field of R.

Lemma 10.117.9. *Suppose $R \subset S$ is an inclusion of reduced rings and suppose that $x \in S$ is strongly transcendental over R. Let $\mathfrak{q} \subset S$ be a minimal prime and let $\mathfrak{p} = R \cap \mathfrak{q}$. Then the image of x in $S/\mathfrak{q}$ is strongly transcendental over the subring $R/\mathfrak{p}$.*

Proof. Suppose $u(a_0 + a_1x + \ldots + a_kx^k) \in \mathfrak{q}$. By Lemma 10.24.4 the local ring $S_\mathfrak{q}$ is a field, and hence $u(a_0 + a_1x + \ldots + a_kx^k)$ is zero in $S_\mathfrak{q}$. Thus $uu'(a_0 + a_1x + \ldots + a_kx^k) = 0$ for some $u' \in S$, $u' \notin \mathfrak{q}$. Since x is strongly transcendental over R we get $uu'a_i = 0$ for all i. This in turn implies that $ua_i \in \mathfrak{q}$. □

Lemma 10.117.10. *Suppose $R \subset S$ is an inclusion of domains and let $x \in S$. Assume x is (strongly) transcendental over R and that S is finite over $R[x]$. Then $R \to S$ is not quasi-finite at any prime of S.*

Proof. As a first case, assume that R is normal, see Definition 10.34.10. By Lemma 10.34.13 we see that $R[x]$ is normal. Take a prime $\mathfrak{q} \subset S$, and set $\mathfrak{p} = R \cap \mathfrak{q}$. Assume that the extension $\kappa(\mathfrak{p}) \subset \kappa(\mathfrak{q})$ is finite. This would be the case if $R \to S$ is quasi-finite at $\mathfrak{q}$. Let $\mathfrak{r} = R[x] \cap \mathfrak{q}$. Then since $\kappa(\mathfrak{p}) \subset \kappa(\mathfrak{r}) \subset \kappa(\mathfrak{q})$ we see that the extension $\kappa(\mathfrak{p}) \subset \kappa(\mathfrak{r})$ is finite too. Thus the inclusion $\mathfrak{r} \supset \mathfrak{p}R[x]$ is strict. By going down for $R[x] \subset S$, see Proposition 10.35.7, we find a prime $\mathfrak{q}' \subset \mathfrak{q}$, lying over the prime $\mathfrak{p}R[x]$. Hence the fibre $\mathrm{Spec}(S \otimes_R \kappa(\mathfrak{p}))$ contains a point not equal to $\mathfrak{q}$, namely $\mathfrak{q}'$, whose closure contains $\mathfrak{q}$ and hence $\mathfrak{q}$ is not isolated in its fibre.

If R is not normal, let $R \subset R' \subset K$ be the integral closure R' of R in its field of fractions K. Let $S \subset S' \subset L$ be the subring S' of the field of fractions L of S generated by R' and S. Note that by construction the map $S \otimes_R R' \to S'$ is surjective. This implies that $R'[x] \subset S'$ is finite. Also, the map $S \subset S'$ induces a surjection on Spec, see Lemma 10.33.15. We conclude by Lemma 10.116.6 and the normal case we just discussed. □

Lemma 10.117.11. *Suppose $R \subset S$ is an inclusion of reduced rings. Assume $x \in S$ be strongly transcendental over R, and S finite over $R[x]$. Then $R \to S$ is not quasi-finite at any prime of S.*

Proof. Let $\mathfrak{q} \subset S$ be any prime. Choose a minimal prime $\mathfrak{q}' \subset \mathfrak{q}$. According to Lemmas 10.117.9 and 10.117.10 the extension $R/(R \cap \mathfrak{q}') \subset S/\mathfrak{q}'$ is not quasi-finite at the prime corresponding to $\mathfrak{q}$. By Lemma 10.116.6 the extension $R \to S$ is not quasi-finite at $\mathfrak{q}$. □

Lemma 10.117.12. *Let R be a ring. Let $S = R[x]/I$. Let $\mathfrak{q} \subset S$ be a prime. Assume $R \to S$ is quasi-finite at $\mathfrak{q}$. Let $S' \subset S$ be the integral closure of R in S. Then there exists an element $g \in S'$, $g \notin \mathfrak{q}$ such that $S'_g \cong S_g$.*

Proof. Let $\mathfrak{p}$ be the image of $\mathfrak{q}$ in $\mathrm{Spec}(R)$. The assumption that $R \to S$ is quasi-finite at $\mathfrak{q}$ implies there exists an $f \in I$, $f = a_n x^n + \ldots + a_0$ such that some $a_i \notin \mathfrak{p}$. In particular there exists a relation $b_m x^m + \ldots + b_0 = 0$ with $b_j \in S'$, $j = 0, \ldots, m$ and $b_j \notin \mathfrak{q} \cap S'$ for some j. We prove the lemma by induction on m.

The case $b_m \in \mathfrak{q}$. In this case we have $b_m x \in S'$ by Lemma 10.117.1. Set $b'_{m-1} = b_m x + b_{m-1}$. Then

$$b'_{m-1} x^{m-1} + b_{m-2} x^{m-2} + \ldots + b_0 = 0$$

Since b'_{m-1} is congruent to b_{m-1} modulo $S' \cap \mathfrak{q}$ we see that it is still the case that one of $b'_{m-1}, b_{m-2}, \ldots, b_0$ is not in $S' \cap \mathfrak{q}$. Thus we win by induction on m.

The case $b_m \notin \mathfrak{q}$. In this case x is integral over S'_{b_m}, in fact $b_m x \in S'$ by Lemma 10.117.1. Hence the injective map $S'_{b_m} \to S_{b_m}$ is also surjective, i.e., an isomorphism as desired. □

Theorem 10.117.13 (Zariski's Main Theorem)**.** *Let R be a ring. Let $R \to S$ be a finite type R-algebra. Let $S' \subset S$ be the integral closure of R in S. Let $\mathfrak{q} \subset S$ be a prime of S. If $R \to S$ is quasi-finite at $\mathfrak{q}$ then there exists a $g \in S'$, $g \notin \mathfrak{q}$ such that $S'_g \cong S_g$.*

Proof. There exist finitely many elements $x_1, \ldots, x_n \in S$ such that S is finite over the R-sub algebra generated by $x_1, \ldots, x_n$. (For example generators of S over R.) We prove the proposition by induction on the minimal such number n.

The case $n = 0$ is trivial, because in this case $S' = S$, see Lemma 10.33.3.

The case $n = 1$. We may and do replace R by its integral closure in S, in particular this means that $R \subset S$. Consider the map $\varphi : R[x] \to S$, $x \mapsto x_1$. (We will see that φ is not injective below.) By assumption φ is finite. Hence we are in Situation 10.117.5. Let $J \subset S$ be the "conductor ideal" defined in Situation 10.117.5. Consider the diagram

$$\begin{array}{ccccccc} R[x] & \longrightarrow & S & \longrightarrow & S/\sqrt{J} & \longleftarrow & R/(R \cap \sqrt{J})[x] \\ & \nwarrow & \uparrow & & \uparrow & \nearrow & \\ & & R & \longrightarrow & R/(R \cap \sqrt{J}) & & \end{array}$$

According to Lemma 10.117.7 the image of x in the quotient $S/\sqrt{J}$ is strongly transcendental over $R/(R \cap \sqrt{J})$. Hence by Lemma 10.117.11 the ring map $R/(R \cap \sqrt{J}) \to S/\sqrt{J}$ is not quasi-finite at any prime of $S/\sqrt{J}$. By Lemma 10.116.6 we deduce that $\mathfrak{q}$ does not lie in $V(J) \subset \mathrm{Spec}(S)$. Thus there exists an element $s \in J$, $s \notin \mathfrak{q}$. By definition of J we may write $s = \varphi(f)$ for some polynomial $f \in R[x]$. Now let $I = \mathrm{Ker}(R[x] \to S)$. Since $\varphi(f) \in J$ we get $(R[x]/I)_f \cong S_{\varphi(f)}$. Also $s \notin \mathfrak{q}$ means that $f \notin \varphi^{-1}(\mathfrak{q})$. Thus $\varphi^{-1}(\mathfrak{q})$ is a prime of $R[x]/I$ at which $R \to R[x]/I$ is quasi-finite, see Lemma 10.116.5. Let $C \subset R[x]/I$ be the integral closure of R. By Lemma 10.117.12 there exists an element $h \in C$, $h \notin \varphi^{-1}(\mathfrak{q})$ such that $C_h \cong (R[x]/I)_h$. We conclude that $(R[x]/I)_{fh} = S_{\varphi(fh)}$ is isomorphic to a principal localization $C_{h'}$ of C for some $h' \in C$, $h' \notin \varphi^{-1}(\mathfrak{q})$. Since $\varphi(C) \subset S'$ we get $g = \varphi(h') \in S'$, $g \notin \mathfrak{q}$ and

moreover the injective map $S'_g \to S_g$ is also surjective because by our choice of h' the map $C_{h'} \to S_g$ is surjective.

The case $n > 1$. Consider the subring $R' \subset S$ which is the integral closure of $R[x_1, \ldots, x_{n-1}]$ in S. By Lemma 10.116.6 the extension S/R' is quasi-finite at $\mathfrak{q}$. Also, note that S is finite over $R'[x_n]$. By the case $n = 1$ above, there exists a $g' \in R'$, $g' \notin \mathfrak{q}$ such that $(R')_{g'} \cong S_{g'}$. At this point we cannot apply induction to $R \to R'$ since R' may not be finite type over R. Since S is finitely generated over R we deduce in particular that $(R')_{g'}$ is finitely generated over R. Say the elements g', and $y_1/(g')^{n_1}, \ldots, y_N/(g')^{n_N}$ with $y_i \in R'$ generate $(R')_{g'}$ over R. Let R'' be the R-sub algebra of R' generated by $x_1, \ldots, x_{n-1}, y_1, \ldots, y_N, g'$. This has the property $(R'')_{g'} \cong S_{g'}$. Surjectivity because of how we chose y_i, injectivity because $R'' \subset R'$, and localization is exact. Note that R'' is finite over $R[x_1, \ldots, x_{n-1}]$ because of our choice of R', see Lemma 10.33.4. Let $\mathfrak{q}'' = R'' \cap \mathfrak{q}$. Since $(R'')_{\mathfrak{q}''} = S_{\mathfrak{q}}$ we see that $R \to R''$ is quasi-finite at $\mathfrak{q}''$, see Lemma 10.116.2. We apply our induction hypothesis to $R \to R''$, $\mathfrak{q}''$ and $x_1, \ldots, x_{n-1} \in R''$ and we find a subring $R''' \subset R''$ which is integral over R and an element $g'' \in R'''$, $g'' \notin \mathfrak{q}''$ such that $(R''')_{g''} \cong (R'')_{g''}$. Write the image of g' in $(R'')_{g''}$ as $g'''/(g'')^n$ for some $g''' \in R'''$. Set $g = g''g''' \in R'''$. Then it is clear that $g \notin \mathfrak{q}$ and $(R''')_g \cong S_g$. Since by construction we have $R''' \subset S'$ we also have $S'_g \cong S_g$ as desired. □

Lemma 10.117.14. *Let $R \to S$ be a finite type ring map. The set of points $\mathfrak{q}$ of $\operatorname{Spec}(S)$ at which S/R is quasi-finite is open in $\operatorname{Spec}(S)$.*

Proof. Let $\mathfrak{q} \subset S$ be a point at which the ring map is quasi-finite. By Theorem 10.117.13 there exists an integral ring extension $R \to S'$, $S' \subset S$ and an element $g \in S'$, $g \notin \mathfrak{q}$ such that $S'_g \cong S_g$. Since S and hence S_g are of finite type over R we may find finitely many elements $y_1, \ldots, y_N$ of S' such that $S''_g \cong S$ where $S'' \subset S'$ is the sub R-algebra generated by $g, y_1, \ldots, y_N$. Since S'' is finite over R (see Lemma 10.33.4) we see that S'' is quasi-finite over R (see Lemma 10.116.4). It is easy to see that this implies that S''_g is quasi-finite over R, for example because the property of being quasi-finite at a prime depends only on the local ring at the prime. Thus we see that S_g is quasi-finite over R. By the same token this implies that $R \to S$ is quasi-finite at every prime of S which lies in $D(g)$. □

Lemma 10.117.15. *Let $R \to S$ be a finite type ring map. Suppose that S is quasi-finite over R. Let $S' \subset S$ be the integral closure of R in S. Then*

1. *$\operatorname{Spec}(S) \to \operatorname{Spec}(S')$ is a homeomorphism onto an open subset,*
2. *if $g \in S'$ and $D(g)$ is contained in the image of the map, then $S'_g \cong S_g$, and*
3. *there exists a finite R-algebra $S'' \subset S'$ such that (1) and (2) hold for the ring map $S'' \to S$.*

Proof. Because S/R is quasi-finite we may apply Theorem 10.117.13 to each point $\mathfrak{q}$ of $\operatorname{Spec}(S)$. Since $\operatorname{Spec}(S)$ is quasi-compact, see Lemma 10.16.10, we may choose a finite number of $g_i \in S'$, $i = 1, \ldots, n$ such that $S'_{g_i} = S_{g_i}$, and such that $g_1, \ldots, g_n$ generate the unit ideal in S (in other words the standard opens of $\operatorname{Spec}(S)$ associated to $g_1, \ldots, g_n$ cover all of $\operatorname{Spec}(S)$).

Suppose that $D(g) \subset \operatorname{Spec}(S')$ is contained in the image. Then $D(g) \subset \bigcup D(g_i)$. In other words, $g_1, \ldots, g_n$ generate the unit ideal of S'_g. Note that $S'_{gg_i} \cong S_{gg_i}$ by our choice of g_i. Hence $S'_g \cong S_g$ by Lemma 10.22.2.

We construct a finite algebra $S'' \subset S'$ as in (3). To do this note that each $S'_{g_i} \cong S_{g_i}$ is a finite type R-algebra. For each i pick some elements $y_{ij} \in S'$ such that each S'_{g_i} is generated as R-algebra by $1/g_i$ and the elements y_{ij}. Then set S'' equal to the sub R-algebra of S' generated by all g_i and all the y_{ij}. Details omitted. □

10.118. Applications of Zariski's Main Theorem

Here is an immediate application characterizing the finite maps of 1-dimensional semi-local rings among the quasi-finite ones as those where equality always holds in the formula of Lemma 10.115.8.

Lemma 10.118.1. *Let $A \subset B$ be an extension of domains. Assume*

(1) *A is a local Noetherian ring of dimension 1,*
(2) *$A \to B$ is of finite type, and*
(3) *the extension $K = f.f.(A) \subset L = f.f.(B)$ is a finite field extension.*

Then B is semi-local. Let $x \in \mathfrak{m}_A$, $x \neq 0$. Let $\mathfrak{m}_i$, $i = 1, \ldots, n$ be the maximal ideals of B. Then

$$[L : K] ord_A(x) \geq \sum\nolimits_i [\kappa(\mathfrak{m}_i) : \kappa(\mathfrak{m}_A)] ord_{B_{\mathfrak{m}_i}}(x)$$

where ord is defined as in Definition 10.115.2. We have equality if and only if $A \to B$ is finite.

Proof. The ring B is semi-local by Lemma 10.107.2. Let B' be the integral closure of A in B. By Lemma 10.117.15 we can find a finite A-subalgebra $C \subset B'$ such that on setting $\mathfrak{n}_i = C \cap \mathfrak{m}_i$ we have $C_{\mathfrak{n}_i} \cong B_{\mathfrak{m}_i}$ and the primes $\mathfrak{n}_1, \ldots, \mathfrak{n}_n$ are pairwise distinct. The ring C is semi-local by Lemma 10.107.2. Let $\mathfrak{p}_j$, $j = 1, \ldots, m$ be the other maximal ideals of C (the "missing points"). By Lemma 10.115.8 we have

$$\text{ord}_A(x^{[L:K]}) = \sum\nolimits_i [\kappa(\mathfrak{n}_i) : \kappa(\mathfrak{m}_A)] \text{ord}_{C_{\mathfrak{n}_i}}(x) + \sum\nolimits_j [\kappa(\mathfrak{p}_j) : \kappa(\mathfrak{m}_A)] \text{ord}_{C_{\mathfrak{p}_j}}(x)$$

hence the inequality follows. In case of equality we conclude that $m = 0$ (no "missing points"). Hence $C \subset B$ is an inclusion of semi-local rings inducing a bijection on maximal ideals and an isomorphism on all localizations at maximal ideals. So if $b \in B$, then $I = \{x \in C \mid xb \in C\}$ is an ideal of C which is not contained in any of the maximal ideals of C, and hence $I = C$, hence $b \in C$. Thus $B = C$ and B is finite over A. □

Here is a more standard application of Zariski's main theorem to the structure of local homomorphisms of local rings.

Lemma 10.118.2. *Let $(R, \mathfrak{m}_R) \to (S, \mathfrak{m}_S)$ be a local homomorphism of local rings. Assume*

(1) *$R \to S$ is essentially of finite type,*
(2) *$\kappa(\mathfrak{m}_R) \subset \kappa(\mathfrak{m}_S)$ is finite, and*
(3) *$\dim(S/\mathfrak{m}_R S) = 0$.*

Then S is the localization of a finite R-algebra.

Proof. Let S' be a finite type R-algebra such that $S = S'_{\mathfrak{q}'}$ for some prime $\mathfrak{q}'$ of S'. By Definition 10.116.3 we see that $R \to S'$ is quasi-finite at $\mathfrak{q}'$. After replacing S' by $S'_{g'}$ for some $g' \in S'$, $g' \notin \mathfrak{q}'$ we may assume that $R \to S'$ is quasi-finite, see Lemma 10.117.14. Then by Lemma 10.117.15 there exists a finite R-algebra S''

and elements $g' \in S'$, $g' \notin \mathfrak{q}'$ and $g'' \in S''$ such that $S'_{g'} \cong S''_{g''}$ as R-algebras. This proves the lemma. □

Lemma 10.118.3. *Let $R \to S$ be a ring map, $\mathfrak{q}$ a prime of S lying over $\mathfrak{p}$ in R. If*

(1) *R is Noetherian,*
(2) *$R \to S$ is of finite type, and*
(3) *$R \to S$ is quasi-finite at $\mathfrak{q}$,*

then $R^\wedge_\mathfrak{p} \otimes_R S = S^\wedge_\mathfrak{q} \times B$ for some $R^\wedge_\mathfrak{p}$-algebra B.

Proof. There exists a finite R-algebra $S' \subset S$ and an element $g \in S'$, $g \notin \mathfrak{q}' = S' \cap \mathfrak{q}$ such that $S'_g = S_g$ and in particular $S'_{\mathfrak{q}'} = S_\mathfrak{q}$, see Lemma 10.117.15. We have

$$R^\wedge_\mathfrak{p} \otimes_R S' = (S'_{\mathfrak{q}'})^\wedge \times B'$$

by Lemma 10.91.19. Note that we have a commutative diagram

$$\begin{array}{ccc} R^\wedge_\mathfrak{p} \otimes_R S & \longrightarrow & S^\wedge_\mathfrak{q} \\ \uparrow & & \uparrow \\ R^\wedge_\mathfrak{p} \otimes_R S' & \longrightarrow & (S'_{\mathfrak{q}'})^\wedge \end{array}$$

where the right vertical is an isomorphism and the lower horizontal arrow is the projection map of the product decomposition above. The lemma follows. □

10.119. Dimension of fibres

We study the behaviour of dimensions of fibres, using Zariski's main theorem. Recall that we defined the dimension $\dim_x(X)$ of a topological space X at a point x in Topology, Definition 5.9.1.

Definition 10.119.1. Suppose that $R \to S$ is of finite type, and let $\mathfrak{q} \subset S$ be a prime lying over a prime $\mathfrak{p}$ of R. We define the *relative dimension of S/R at $\mathfrak{q}$*, denoted $\dim_\mathfrak{q}(S/R)$, to be the dimension of $\mathrm{Spec}(S \otimes_R \kappa(\mathfrak{p}))$ at the point corresponding to $\mathfrak{q}$. We let $\dim(S/R)$ be the supremum of $\dim_\mathfrak{q}(S/R)$ over all $\mathfrak{q}$. This is called the *relative dimension of S/R*.

In particular, $R \to S$ is quasi-finite at $\mathfrak{q}$ if and only if $\dim_\mathfrak{q}(S/R) = 0$. The following lemma is more or less a reformulation of Zariski's Main Theorem.

Lemma 10.119.2. *Let $R \to S$ be a finite type ring map. Let $\mathfrak{q} \subset S$ be a prime. Suppose that $\dim_\mathfrak{q}(S/R) = n$. There exists a $g \in S$, $g \notin \mathfrak{q}$ such that S_g is quasi-finite over a polynomial algebra $R[t_1, \ldots, t_n]$.*

Proof. The ring $\overline{S} = S \otimes_R \kappa(\mathfrak{p})$ is of finite type over $\kappa(\mathfrak{p})$. Let $\overline{\mathfrak{q}}$ be the prime of $\overline{S}$ corresponding to $\mathfrak{q}$. By definition of the dimension of a topological space at a point there exists an open $U \subset \mathrm{Spec}(\overline{S})$ with $\overline{q} \in U$ and $\dim(U) = n$. Since the topology on $\mathrm{Spec}(\overline{S})$ is induced from the topology on $\mathrm{Spec}(S)$ (see Remark 10.16.8), we can find a $g \in S$, $g \notin \mathfrak{q}$ with image $\overline{g} \in \overline{S}$ such that $D(\overline{g}) \subset U$. Thus after replacing S by S_g we see that $\dim(\overline{S}) = n$.

Next, choose generators $x_1, \ldots, x_N$ for S as an R-algebra. By Lemma 10.109.4 there exist elements $y_1, \ldots, y_n$ in the $\mathbf{Z}$-subalgebra of S generated by $x_1, \ldots, x_N$ such that the map $R[t_1, \ldots, t_n] \to S$, $t_i \mapsto y_i$ has the property that $\kappa(\mathfrak{p})[t_1 \ldots, t_n] \to \overline{S}$ is finite. In particular, S is quasi-finite over $R[t_1, \ldots, t_n]$ at $\mathfrak{q}$. Hence, by Lemma

10.117.14 we may replace S by S_g for some $g \in S$, $g \notin \mathfrak{q}$ such that $R[t_1, \ldots, t_n] \to S$ is quasi-finite. □

Lemma 10.119.3. *Let $R \to S$ be a ring map. Let $\mathfrak{q} \subset S$ be a prime lying over the prime $\mathfrak{p}$ of R. Assume*

(1) *$R \to S$ is of finite type,*
(2) *$\dim_{\mathfrak{q}}(S/R) = n$, and*
(3) *$trdeg_{\kappa(\mathfrak{p})}\kappa(\mathfrak{q}) = r$.*

Then there exist $f \in R$, $f \notin \mathfrak{p}$, $g \in S$, $g \notin \mathfrak{q}$ and a quasi-finite ring map

$$\varphi : R_f[x_1, \ldots, x_n] \longrightarrow S_g$$

such that $\varphi^{-1}(\mathfrak{q}S_g) = (\mathfrak{p}, x_{r+1}, \ldots, x_n)R_f[x_{r+1}, \ldots, x_n]$

Proof. After replacing S by a principal localization we may assume there exists a quasi-finite ring map $\varphi : R[t_1, \ldots, t_n] \to S$, see Lemma 10.119.2. Set $\mathfrak{q}' = \varphi^{-1}(\mathfrak{q})$. Let $\overline{\mathfrak{q}}' \subset \kappa(\mathfrak{p})[t_1, \ldots, t_n]$ be the prime corresponding to $\mathfrak{q}'$. By Lemma 10.109.6 there exists a finite ring map $\kappa(\mathfrak{p})[x_1, \ldots, x_n] \to \kappa(\mathfrak{p})[t_1, \ldots, t_n]$ such that the inverse image of $\overline{\mathfrak{q}}'$ is $(x_{r+1}, \ldots, x_n)$. Let $\overline{h}_i \in \kappa(\mathfrak{p})[t_1, \ldots, t_n]$ be the image of x_i. We can find an element $f \in R$, $f \notin \mathfrak{p}$ and $h_i \in R_f[t_1, \ldots, t_n]$ which map to $\overline{h}_i$ in $\kappa(\mathfrak{p})[t_1, \ldots, t_n]$. Then the ring map

$$R_f[x_1, \ldots, x_n] \longrightarrow R_f[t_1, \ldots, t_n]$$

becomes finite after tensoring with $\kappa(\mathfrak{p})$. In particular, $R_f[t_1, \ldots, t_n]$ is quasi-finite over $R_f[x_1, \ldots, x_n]$ at the prime $\mathfrak{q}'R_f[t_1, \ldots, t_n]$. Hence, by Lemma 10.117.14 there exists a $g \in R_f[t_1, \ldots, t_n]$, $g \notin \mathfrak{q}'R_f[t_1, \ldots, t_n]$ such that $R_f[x_1, \ldots, x_n] \to R_f[t_1, \ldots, t_n, 1/g]$ is quasi-finite. Thus we see that the composition

$$R_f[x_1, \ldots, x_n] \longrightarrow R_f[t_1, \ldots, t_n, 1/g] \longrightarrow S_{\varphi(g)}$$

is quasi-finite and we win. □

Lemma 10.119.4. *Let $R \to S$ be a finite type ring map. Let $\mathfrak{q} \subset S$ be a prime lying over $\mathfrak{p} \subset R$. If $R \to S$ is quasi-finite at $\mathfrak{q}$, then $\dim(S_{\mathfrak{q}}) \leq \dim(R_{\mathfrak{p}})$.*

Proof. If $R_{\mathfrak{p}}$ is Noetherian (and hence $S_{\mathfrak{q}}$ Noetherian since it is essentially of finite type over $R_{\mathfrak{p}}$) then this follows immediately from Lemma 10.106.6 and the definitions. In the general case we can use Zariski's Main Theorem 10.117.13 to write $S_{\mathfrak{q}} = S'_{\mathfrak{q}'}$ for some ring S' integral over $R_{\mathfrak{p}}$. Thus the result follows from Lemma 10.106.3. □

Lemma 10.119.5. *Let k be a field. Let S be a finite type k-algebra. Suppose there is a quasi-finite k-algebra map $k[t_1, \ldots, t_n] \subset S$. Then $\dim(S) \leq n$.*

Proof. By Lemma 10.108.1 the dimension of any local ring of $k[t_1, \ldots, t_n]$ is at most n. Thus the result follows from Lemma 10.119.4. □

Lemma 10.119.6. *Let $R \to S$ be a finite type ring map. Let $\mathfrak{q} \subset S$ be a prime. Suppose that $\dim_{\mathfrak{q}}(S/R) = n$. There exists an open neighbourhood V of $\mathfrak{q}$ in $\mathrm{Spec}(S)$ such that $\dim_{\mathfrak{q}'}(S/R) \leq n$ for all $\mathfrak{q}' \in V$.*

Proof. By Lemma 10.119.2 we see that we may assume that S is quasi-finite over a polynomial algebra $R[t_1, \ldots, t_n]$. Considering the fibres, we reduce to Lemma 10.119.5. □

In other words, the lemma says that the set of points where the fibre has dimension $\leq n$ is open in $\mathrm{Spec}(S)$. The next lemma says that formation of this open commutes with base change. If the ring map is of finite presentation then this set is quasi-compact open (see below).

Lemma 10.119.7. *Let $R \to S$ be a finite type ring map. Let $R \to R'$ be any ring map. Set $S' = R' \otimes_R S$ and denote $f : \mathrm{Spec}(S') \to \mathrm{Spec}(S)$ the associated map on spectra. Let $n \geq 0$. The inverse image $f^{-1}(\{\mathfrak{q} \in \mathrm{Spec}(S) \mid \dim_{\mathfrak{q}}(S/R) \leq n\})$ is equal to $\{\mathfrak{q}' \in \mathrm{Spec}(S') \mid \dim_{\mathfrak{q}'}(S'/R') \leq n\}$.*

Proof. The condition is formulated in terms of dimensions of fibre rings which are of finite type over a field. Combined with Lemma 10.110.6 this yields the lemma. □

Lemma 10.119.8. *Let $R \to S$ be a ring homomorphism of finite presentation. Let $n \geq 0$. The set*

$$V_n = \{\mathfrak{q} \in \mathrm{Spec}(S) \mid \dim_{\mathfrak{q}}(S/R) \leq n\}$$

is a quasi-compact open subset of $\mathrm{Spec}(S)$.

Proof. It is open by Lemma 10.119.6. Let $S = R[x_1, \ldots, x_n]/(f_1, \ldots, f_m)$ be a presentation of S. Let R_0 be the $\mathbf{Z}$-subalgebra of R generated by the coefficients of the polynomials f_i. Let $S_0 = R_0[x_1, \ldots, x_n]/(f_1, \ldots, f_m)$. Then $S = R \otimes_{R_0} S_0$. By Lemma 10.119.7 V_n is the inverse image of an open $V_{0,n}$ under the quasi-compact continuous map $\mathrm{Spec}(S) \to \mathrm{Spec}(S_0)$. Since S_0 is Noetherian we see that $V_{0,n}$ is quasi-compact. □

Lemma 10.119.9. *Let R be a valuation ring with residue field k and field of fractions K. Let S be a domain containing R such that S is of finite type over R. If $S \otimes_R k$ is not the zero ring then*

$$\dim(S \otimes_R k) = \dim(S \otimes_R K)$$

In fact, $\mathrm{Spec}(S \otimes_R k)$ *is equidimensional.*

Proof. It suffices to show that $\dim_{\mathfrak{q}}(S/k)$ is equal to $\dim(S \otimes_R K)$ for every prime $\mathfrak{q}$ of S containing $\mathfrak{m}_R S$. Pick such a prime. By Lemma 10.119.6 the inequality $\dim_{\mathfrak{q}}(S/k) \geq \dim(S \otimes_R K)$ holds. Set $n = \dim_{\mathfrak{q}}(S/k)$. By Lemma 10.119.2 after replacing S by S_g for some $g \in S$, $g \notin \mathfrak{q}$ there exists a quasi-finite ring map $R[t_1, \ldots, t_n] \to S$. If $\dim(S \otimes_R K) < n$, then $K[t_1, \ldots, t_n] \to S \otimes_R K$ has a nonzero kernel. Say $f = \sum a_I t_1^{i_1} \ldots t_n^{i_n}$. After dividing f by a nonzero coefficient of f with minimal valuation, we may assume $f \in R[t_1, \ldots, t_n]$ and some a_I does not map to zero in k. Hence the ring map $k[t_1, \ldots, t_n] \to S \otimes_R k$ has a nonzero kernel which implies that $\dim(S \otimes_R k) < n$. Contradiction. □

10.120. Algebras and modules of finite presentation

In this section we discuss some standard results where the key feature is that the assumption involves a finite type or finite presentation assumption.

Lemma 10.120.1. *Let $R \to S$ be a ring map. Let $R \to R'$ be a faithfully flat ring map. Set $S' = R' \otimes_R S$. Then $R \to S$ is of finite type if and only if $R' \to S'$ is of finite type.*

Proof. It is clear that if $R \to S$ is of finite type then $R' \to S'$ is of finite type. Assume that $R' \to S'$ is of finite type. Say $y_1, \ldots, y_m$ generate S' over R'. Write $y_j = \sum_i a_{ij} \otimes x_{ji}$ for some $a_{ij} \in R'$ and $x_{ji} \in S$. Let $A \subset S$ be the R-subalgebra generated by the x_{ij}. By flatness we have $A' := R' \otimes_R A \subset S'$, and by construction $y_j \in A'$. Hence $A' = S'$. By faithful flatness $A = S$. □

Lemma 10.120.2. *Let $R \to S$ be a ring map. Let $R \to R'$ be a faithfully flat ring map. Set $S' = R' \otimes_R S$. Then $R \to S$ is of finite presentation if and only if $R' \to S'$ is of finite presentation.*

Proof. It is clear that if $R \to S$ is of finite presentation then $R' \to S'$ is of finite presentation. Assume that $R' \to S'$ is of finite presentation. By Lemma 10.120.1 we see that $R \to S$ is of finite type. Write $S = R[x_1, \ldots, x_n]/I$. By flatness $S' = R'[x_1, \ldots, x_n]/R' \otimes I$. Say $g_1, \ldots, g_m$ generate $R' \otimes I$ over $R'[x_1, \ldots, x_n]$. Write $g_j = \sum_i a_{ij} \otimes f_{ji}$ for some $a_{ij} \in R'$ and $f_{ji} \in I$. Let $J \subset I$ be the ideal generated by the f_{ij}. By flatness we have $R' \otimes_R J \subset R' \otimes_R I$, and both are ideals over $R'[x_1, \ldots, x_n]$. By construction $g_j \in R' \otimes_R J$. Hence $R' \otimes_R J = R' \otimes_R I$. By faithful flatness $J = I$. □

Lemma 10.120.3. *Let R be a ring. Let $I \subset R$ be an ideal. Let $S \subset R$ be a multiplicative subset. Set $R' = S^{-1}(R/I) = S^{-1}R/S^{-1}I$.*

1. *For any finite R'-module M' there exists a finite R-module M such that $S^{-1}(M/IM) \cong M'$.*
2. *For any finitely presented R'-module M' there exists a finitely presented R-module M such that $S^{-1}(M/IM) \cong M'$.*

Proof. Proof of (1). Choose a short exact sequence $0 \to K' \to (R')^{\oplus n} \to M' \to 0$. Let $K \subset R^{\oplus n}$ be the inverse image of K' under the map $R^{\oplus n} \to (R')^{\oplus n}$. Then $M = R^{\oplus n}/K$ works.

Proof of (2). Choose a presentation $(R')^{\oplus m} \to (R')^{\oplus n} \to M' \to 0$. Suppose that the first map is given by the matrix $A' = (a'_{ij})$ and the second map is determined by generators $x'_i \in M'$, $i = 1, \ldots, n$. As $R' = S^{-1}(R/I)$ we can choose $s \in S$ and a matrix $A = (a_{ij})$ with coefficients in R such that $a'_{ij} = a_{ij}/s \bmod S^{-1}I$. Let M be the finitely presented R-module with presentation $R^{\oplus m} \to R^{\oplus n} \to M \to 0$ where the first map is given by the matrix A and the second map is determined by generators $x_i \in M$, $i = 1, \ldots, n$. Then the map $M \to M'$, $x_i \mapsto x'_i$ induces an isomorphism $S^{-1}(M/IM) \cong M'$. □

Lemma 10.120.4. *Let R be a ring. Let $S \subset R$ be a multiplicative subset. Let M be an R-module.*

1. *If $S^{-1}M$ is a finite $S^{-1}R$-module then there exists a finite R-module M' and a map $M' \to M$ which induces an isomorphism $S^{-1}M' \to S^{-1}M$.*
2. *If $S^{-1}M$ is a finitely presented $S^{-1}R$-module then there exists an R-module M' of finite presentation and a map $M' \to M$ which induces an isomorphism $S^{-1}M' \to S^{-1}M$.*

Proof. Proof of (1). Let $x_1, \ldots, x_n \in M$ be elements which generate $S^{-1}M$ as an $S^{-1}R$-module. Let M' be the R-submodule of M generated by $x_1, \ldots, x_n$.

Proof of (2). Let $x_1, \ldots, x_n \in M$ be elements which generate $S^{-1}M$ as an $S^{-1}R$-module. Let $K = \mathrm{Ker}(R^{\oplus n} \to M)$ where the map is given by the rule $(a_1, \ldots, a_n) \mapsto$

$\sum a_i x_i$. By Lemma 10.5.3 we see that $S^{-1}K$ is a finite $S^{-1}R$-module. By (1) we can find a finite type submodule $K' \subset K$ with $S^{-1}K' = S^{-1}K$. Take $M' = \mathrm{Coker}(K' \to R^{\oplus n})$. □

Lemma 10.120.5. *Let R be a ring. Let $\mathfrak{p} \subset R$ be a prime ideal. Let M be an R-module.*

(1) *If $M_\mathfrak{p}$ is a finite $R_\mathfrak{p}$-module then there exists a finite R-module M' and a map $M' \to M$ which induces an isomorphism $M'_\mathfrak{p} \to M_\mathfrak{p}$.*
(2) *If $M_\mathfrak{p}$ is a finitely presented $R_\mathfrak{p}$-module then there exists an R-module M' of finite presentation and a map $M' \to M$ which induces an isomorphism $M'_\mathfrak{p} \to M_\mathfrak{p}$.*

Proof. This is a special case of Lemma 10.120.4 □

Lemma 10.120.6. *Let $\varphi : R \to S$ be a ring map. Let $\mathfrak{q} \subset S$ be a prime lying over $\mathfrak{p} \subset R$. Assume*

(1) *S is of finite presentation over R,*
(2) *φ induces an isomorphism $R_\mathfrak{p} \cong S_\mathfrak{q}$.*

Then there exist $f \in R$, $f \notin \mathfrak{p}$ and an R_f-algebra C such that $S_f \cong R_f \times C$ as R_f-algebras.

Proof. Write $S = R[x_1, \ldots, x_n]/(g_1, \ldots, g_m)$. Let $a_i \in R_\mathfrak{p}$ be an element mapping to the image of x_i in $S_\mathfrak{q}$. Write $a_i = b_i/f$ for some $f \in R$, $f \notin \mathfrak{p}$. After replacing R by R_f and x_i by $x_i - a_i$ we may assume that $S = R[x_1, \ldots, x_n]/(g_1, \ldots, g_m)$ such that x_i maps to zero in $S_\mathfrak{q}$. Then if c_j denotes the constant term of g_i we conclude that c_i maps to zero in $R_\mathfrak{p}$. After another replacement of R we may assume that the constant coefficients c_j of the g_j are zero. Thus we obtain an R-algebra map $S \to R$, $x_i \mapsto 0$ whose kernel is the ideal $(x_1, \ldots, x_n)$.

Note that $\mathfrak{q} = \mathfrak{p}S + (x_1, \ldots, x_n)$. Write $g_j = \sum a_{ji} x_i + h.o.t.$. Since $S_\mathfrak{q} = R_\mathfrak{p}$ we have $\mathfrak{p} \otimes \kappa(\mathfrak{p}) = \mathfrak{q} \otimes \kappa(\mathfrak{q})$. It follows that $m \times n$ matrix $A = (a_{ij})$ defines a surjective map $\kappa(\mathfrak{p})^{\oplus m} \to \kappa(\mathfrak{p})^{\oplus n}$. Thus after inverting some element of R not in $\mathfrak{p}$ we may assume there are $b_{ij} \in R$ such that $\sum b_{ij} g_j = x_i + h.o.t.$. We conclude that $(x_1, \ldots, x_n) = (x_1, \ldots, x_n)^2$ in S. It follows from Lemma 10.19.5 that $(x_1, \ldots, x_n)$ is generated by an idempotent e. Setting $C = eS$ finishes the proof. □

Lemma 10.120.7. *Let R be a ring. Let S, S' be of finite presentation over R. Let $\mathfrak{q} \subset S$ and $\mathfrak{q}' \subset S'$ be primes. If $S_\mathfrak{q} \cong S_{\mathfrak{q}'}$ as R-algebras, then there exist $g \in S$, $g \notin \mathfrak{q}$ and $g' \in S'$, $g' \notin \mathfrak{q}'$ such that $S_g \cong S'_{g'}$ as R-algebras.*

Proof. Let $\psi : S_\mathfrak{q} \to S_{\mathfrak{q}'}$ be the isomorphism of the hypothesis of the lemma. Write $S = R[x_1, \ldots, x_n]/(f_1, \ldots, f_r)$ and $S' = R[y_1, \ldots, y_m]/J$. For each $i = 1, \ldots, n$ choose a fraction h_i/g_i with $h_i, g_i \in R[y_1, \ldots, y_m]$ and $g_i \bmod J$ not in $\mathfrak{q}'$ which represents the image of x_i under ψ. After replacing S' by $S'_{g_1 \ldots g_n}$ and $R[y_1, \ldots, y_m, y_{m+1}]$ (mapping y_{m+1} to $1/(g_1 \ldots g_n)$) we may assume that $\psi(x_i)$ is the image of some $h_i \in R[y_1, \ldots, y_m]$. Consider the elements $f_j(h_1, \ldots, h_n) \in R[y_1, \ldots, y_m]$. Since ψ kills each f_j we see that there exists a $g \in R[y_1, \ldots, y_m]$, $g \bmod J \notin \mathfrak{q}'$ such that $g f_j(h_1, \ldots, h_n) \in J$ for each $j = 1, \ldots, r$. After replacing S' by S'_g and $R[y_1, \ldots, y_m, y_{m+1}]$ as before we may assume that $f_j(h_1, \ldots, h_n) \in J$. Thus we obtain a ring map $S \to S'$, $x_i \mapsto h_i$ which induces ψ on local rings. By Lemma 10.6.2 the map $S' \to S$ is of finite presentation. By Lemma 10.120.6 we

may assume that $S = S' \times C$. Thus localizing S at the idempotent corresponding to the factor C we obtain the result. □

Lemma 10.120.8. *Let R be a ring. Let $I \subset R$ be a locally nilpotent ideal. Let $S \to S'$ be an R-algebra map such that $S \to S'/IS'$ is surjective and such that S' is of finite type over R. Then $S \to S'$ is surjective.*

Proof. Write $S' = R[x_1, \ldots, x_m]/K$ for some ideal K. By assumption there exist $g_j = x_j + \sum \delta_{j,J} x^J \in R[x_1, \ldots, x_n]$ with $\delta_{j,J} \in I$ and with $g_j \bmod K \in \mathrm{Im}(S \to S')$. Hence it suffices to show that $g_1, \ldots, g_m$ generate $R[x_1, \ldots, x_n]$. Let $R_0 \subset R$ be a finitely generated $\mathbf{Z}$-subalgebra of R containing at least the $\delta_{j,J}$. Then $R_0 \cap I$ is a nilpotent ideal (by Lemma 10.47.3). It follows that $R_0[x_1, \ldots, x_n]$ is generated by $g_1, \ldots, g_m$ (because $x_j \mapsto g_j$ defines an automorphism of $R_0[x_1, \ldots, x_m]$; details omitted). Since R is the union of the subrings R_0 we win. □

Lemma 10.120.9. *Let R be a ring. Let $I \subset R$ be an ideal. Let $S \to S'$ be an R-algebra map. Let $IS \subset \mathfrak{q} \subset S$ be a prime ideal. Assume that*

1. *$S \to S'$ is surjective,*
2. *$S_\mathfrak{q}/IS_\mathfrak{q} \to S'_\mathfrak{q}/IS'_\mathfrak{q}$ is an isomorphism,*
3. *S is of finite type over R,*
4. *S' of finite presentation over R, and*
5. *$S'_\mathfrak{q}$ is flat over R.*

Then $S_g \to S'_g$ is an isomorphism for some $g \in S$, $g \notin \mathfrak{q}$.

Proof. Let $J = \mathrm{Ker}(S \to S')$. By Lemma 10.6.2 J is a finitely generated ideal. Since $S'_\mathfrak{q}$ is flat over R we see that $J_\mathfrak{q}/IJ_\mathfrak{q} \subset S_\mathfrak{q}/IS_\mathfrak{q}$ (apply Lemma 10.36.11 to $0 \to J \to S \to S' \to 0$). By assumption (2) we see that $J_\mathfrak{q}/IJ_\mathfrak{q}$ is zero. By Nakayama's lemma (Lemma 10.18.1) we see that there exists a $g \in S$, $g \notin \mathfrak{q}$ such that $J_g = 0$. Hence $S_g \cong S'_g$ as desired. □

Lemma 10.120.10. *Let R be a ring. Let $I \subset R$ be an ideal. Let $S \to S'$ be an R-algebra map. Assume that*

1. *I is locally nilpotent,*
2. *$S/IS \to S'/IS'$ is an isomorphism,*
3. *S is of finite type over R,*
4. *S' of finite presentation over R, and*
5. *S' is flat over R.*

Then $S \to S'$ is an isomorphism.

Proof. By Lemma 10.120.8 the map $S \to S'$ is surjective. As I is locally nilpotent, so are the ideals IS and IS' (Lemma 10.14.1). Hence every prime ideal $\mathfrak{q}$ of S contains IS and (trivially) $S_\mathfrak{q}/IS_\mathfrak{q} \cong S'_\mathfrak{q}/IS'_\mathfrak{q}$. Thus Lemma 10.120.9 applies and we see that $S_\mathfrak{q} \to S'_\mathfrak{q}$ is an isomorphism for every prime $\mathfrak{q} \subset S$. It follows that $S \to S'$ is injective for example by Lemma 10.22.1. □

10.121. Colimits and maps of finite presentation

In this section we prove some preliminary lemmas which will eventually help us prove result using absolute Noetherian reduction. we begin discussing how we will think about colimits in this section.

Let $(\Lambda, \geq)$ a partially ordered set. A system of rings over Λ is given by a ring R_λ for every $\lambda \in \Lambda$, and a morphism $R_\lambda \to R_\mu$ whenever $\lambda \leq \mu$. These morphisms have to satisfy the rule that $R_\lambda \to R_\mu \to R_\nu$ is equal to the map $R_\lambda \to R_\nu$ for all $\lambda \leq \mu \leq \nu$. See Categories, Section 4.21. We will often assume that $(I, \leq)$ is *directed*, which means that Λ is nonempty and given $\lambda, \mu \in \Lambda$ there exists a $\nu \in \Lambda$ with $\lambda \leq \nu$ and $\mu \leq \nu$. Recall that the colimit $\operatorname{colim}_\lambda R_\lambda$ is sometimes called a "direct limit" in this case (but we will not use this terminology).

Lemma 10.121.1. *Let $R \to A$ be a ring map. There exists a directed system A_λ of R-algebras of finite presentation such that $A = \operatorname{colim}_\lambda A_\lambda$. If A is of finite type over R we may arrange it so that all the transition maps are surjective.*

Proof. Compare with the proof of Lemma 10.8.13. Consider any finite subset $S \subset A$, and any finite collection of polynomial relations E among the elements of S. So each $s \in S$ corresponds to $x_s \in A$ and each $e \in E$ consists of a polynomial $f_e \in R[X_s; s \in S]$ such that $f_e(x_s) = 0$. Let $A_{S,E} = R[X_s; s \in S]/(f_e; e \in E)$ which is a finitely presented R-algebra. There are canonical maps $A_{S,E} \to A$. If $S \subset S'$ and if the elements of E correspond, via the map $R[X_s; s \in S] \to R[X_s; s \in S']$, to a subset of E', then there is an obvious map $A_{S,E} \to A_{S',E'}$ commuting with the maps to A. Thus, setting Λ equal the set of pairs (S, E) with ordering by inclusion as above, we get a directed partially ordered set. It is clear that the colimit of this directed system is A.

For the last statement, suppose $A = R[x_1, \ldots, x_n]/I$. In this case, consider the subset $\Lambda' \subset \Lambda$ consisting of those systems (S, E) above with $S = \{x_1, \ldots, x_n\}$. It is easy to see that still $A = \operatorname{colim}_{\lambda' \in \Lambda'} A_{\lambda'}$. Moreover, the transition maps are clearly surjective. □

It turns out that we can characterize ring maps of finite presentation as follows. This in some sense says that the algebras of finite presentation are the "compact" objects in the category of R-algebras.

Lemma 10.121.2. *Let $\varphi : R \to S$ be a ring map. Then φ is of finite presentation if and only if for every directed system A_λ of R-algebras we have*

$$\operatorname{colim}_\lambda \operatorname{Hom}_R(S, A_\lambda) = \operatorname{Hom}_R(S, \operatorname{colim}_\lambda A_\lambda)$$

Proof. Suppose $S = R[x_1, \ldots, x_n]/(f_1, \ldots, f_m)$. If $\chi : S \to \operatorname{colim} A_\lambda$ is a map, then each x_i maps to some element in the image of some A_{λ_i}. We may pick $\mu \geq \lambda_i$, $i = 1, \ldots, n$ and assume $\chi(x_i)$ is the image of $y_i \in A_\mu$ for $i = 1, \ldots, n$. Consider $z_j = f_j(y_1, \ldots, y_n) \in A_\mu$. Since χ is a homomorphism the image of z_j in $\operatorname{colim}_\lambda A_\lambda$ is zero. Hence there exists a $\mu_j \geq \mu$ such that z_j maps to zero in A_{μ_j}. Pick $\nu \geq \mu_j$, $j = 1, \ldots, m$. Then the images of $z_1, \ldots, z_m$ are zero in A_ν. This exactly means that the y_i map to elements $y_i' \in A_\nu$ which satisfy the relations $f_j(y_1', \ldots, y_n') = 0$. Thus we obtain a ring map $S \to A_\nu$ as desired.

Conversely, suppose the displayed formula holds always. By Lemma 10.121.1 we may write $S = \operatorname{colim}_\lambda S_\lambda$ with S_λ of finite presentation over R. Then the identity map factors as

$$S \to S_\lambda \to S$$

for some λ. Hence we see that S is finitely generated over R (because S_λ is). Thus we may choose the system such that all transition maps are surjective. In this case a factorization of the identity as above can only exist if $S = S_\lambda$. □

But more is true. Namely, given $R = \operatorname{colim}_\lambda R_\lambda$ we see that the category of finitely presented R-modules is equivalent to the limit of the category of finitely presented R_λ-modules. Similarly for the categories of finitely presented R-algebras.

Lemma 10.121.3. *Let A be a ring and let M, N be A-modules. Suppose that $R = \operatorname{colim}_{i \in I} R_i$ is a directed colimit of A-algebras.*

1. *If M is a finite A-module, and $u, u' : M \to N$ are A-module maps such that $u \otimes 1 = u' \otimes 1 : M \otimes_A R \to N \otimes_A R$ then for some i we have $u \otimes 1 = u' \otimes 1 : M \otimes_A R_i \to N \otimes_A R_i$.*
2. *If N is a finite A-module and $u : M \to N$ is an A-module map such that $u \otimes 1 : M \otimes_A R \to N \otimes_A R$ is surjective, then for some i the map $u \otimes 1 : M \otimes_A R_i \to N \otimes_A R_i$ is surjective.*
3. *If N is a finitely presented A-module, and $v : N \otimes_A R \to M \otimes_A R$ is an R-module map, then there exists an i and an R_i-module map $v_i : N \otimes_A R_i \to M \otimes_A R_i$ such that $v = v_i \otimes 1$.*
4. *If M is a finite A-module, N is a finitely presented A-module, and $u : M \to N$ is an R-module map such that $u \otimes 1 : M \otimes_A R \to N \otimes_A R$ is an isomorphism, then for some i the map $u \otimes 1 : M \otimes_A R_i \to N \otimes_A R_i$ is an isomorphism.*

Proof. To prove (1) assume u is as in (1) and let $x_1, \ldots, x_m \in M$ be generators. Since $N \otimes_A R = \operatorname{colim}_i N \otimes_A R_i$ we may pick an $i \in I$ such that $u(x_j) \otimes 1 = u'(x_j) \otimes 1$ in $M \otimes_A R_i$, $j = 1, \ldots, m$. For such an i we have $u \otimes 1 = u' \otimes 1 : M \otimes_A R_i \to N \otimes_A R_i$.

To prove (2) assume $u \otimes 1$ surjective and let $y_1, \ldots, y_m \in N$ be generators. Since $N \otimes_A R = \operatorname{colim}_i N \otimes_A R_i$ we may pick an $i \in I$ and $z_j \in M \otimes_A R_i$, $j = 1, \ldots, m$ whose images in $N \otimes_A R$ equal $y_j \otimes 1$. For such an i the map $u \otimes 1 : M \otimes_A R_i \to N \otimes_A R_i$ is surjective.

To prove (3) let $y_1, \ldots, y_m \in N$ be generators. Let $K = \operatorname{Ker}(A^{\oplus m} \to N)$ where the map is given by the rule $(a_1, \ldots, a_m) \mapsto \sum a_j x_j$. Let $k_1, \ldots, k_t$ be generators for K. Say $k_s = (k_{s1}, \ldots, k_{sm})$. Since $M \otimes_A R = \operatorname{colim}_i M \otimes_A R_i$ we may pick an $i \in I$ and $z_j \in M \otimes_A R_i$, $j = 1, \ldots, m$ whose images in $M \otimes_A R$ equal $v(y_j \otimes 1)$. We want to use the z_j to define the map $v_i : N \otimes_A R_i \to M \otimes_A R_i$. Since $K \otimes_A R_i \to R_i^{\oplus m} \to N \otimes_A R_i \to 0$ is a presentation, it suffices to check that $\xi_s = \sum_j k_{sj} z_j$ is zero in $M \otimes_A R_i$ for each $s = 1, \ldots, t$. This may not be the case, but since the image of ξ_s in $M \otimes_A R$ is zero we see that it will be the case after increasing i a bit.

To prove (4) assume $u \otimes 1$ is an isomorphism, that M is finite, and that N is finitely presented. Let $v : N \otimes_A R \to M \otimes_A R$ be an inverse to $u \otimes 1$. Apply part (3) to get a map $v_i : N \otimes_A R_i \to M \otimes_A R_i$ for some i. Apply part (1) to see that, after increasing i we have $v_i \circ (u \otimes 1) = \mathrm{id}_{M \otimes_R R_i}$ and $(u \otimes 1) \circ v_i = \mathrm{id}_{N \otimes_R R_i}$. □

Lemma 10.121.4. *Suppose that $R = \operatorname{colim}_{i \in I} R_i$ is a directed colimit of rings. Then the category of finitely presented R-modules is the colimit of the categories of finitely presented R_λ-modules. More precisely*

1. *Given a finitely presented R-module M there exists a $\lambda \in \Lambda$ and a finitely presented R_λ-module M_λ such that $M \cong M_\lambda \otimes_{R_\lambda} R$.*
2. *Given a $\lambda \in \Lambda$, finitely presented R_λ-modules M_λ, N_λ, and an R-module map $\varphi : M_\lambda \otimes_{R_\lambda} R \to N_\lambda \otimes_{R_\lambda} R$, then there exists a $\mu \geq \lambda$ and an R_μ-module map $\varphi_\mu : M_\lambda \otimes_{R_\lambda} R_\mu \to N_\lambda \otimes_{R_\lambda} R_\mu$ such that $\varphi = \varphi_\mu \otimes 1_R$.*

(3) *Given a $\lambda \in \Lambda$, finitely presented R_λ-modules M_λ, N_λ, and R-module maps $\varphi_\lambda, \psi_\lambda : M_\lambda \to N_\lambda$ such that $\varphi \otimes 1_R = \psi \otimes 1_R$, then $\varphi \otimes 1_{R_\mu} = \psi \otimes 1_{R_\mu}$ for some $\mu \geq \lambda$.*

Proof. To prove (1) choose a presentation $R^{\oplus m} \to R^{\oplus n} \to M \to 0$. Suppose that the first map is given by the matrix $A = (a_{ij})$. We can choose a $\lambda \in \Lambda$ and a matrix $A_\lambda = (a_{\lambda,ij})$ with coefficients in R_λ which maps to A in R. Then we simply let M_λ be the R_λ-module with presentation $R_\lambda^{\oplus m} \to R_\lambda^{\oplus n} \to M_\lambda \to 0$ where the first arrow is given by A_λ.

Parts (3) and (4) follow from Lemma 10.121.3. □

Lemma 10.121.5. *Let A be a ring and let B, C be A-algebras. Suppose that $R = \operatorname{colim}_{i \in I} R_i$ is a directed colimit of A-algebras.*

(1) *If B is a finite type A-algebra, and $u, u' : B \to C$ are A-algebra maps such that $u \otimes 1 = u' \otimes 1 : B \otimes_A R \to C \otimes_A R$ then for some i we have $u \otimes 1 = u' \otimes 1 : B \otimes_A R_i \to C \otimes_A R_i$.*
(2) *If C is a finite type A-algebra and $u : B \to C$ is an A-algebra map such that $u \otimes 1 : B \otimes_A R \to C \otimes_A R$ is surjective, then for some i the map $u \otimes 1 : B \otimes_A R_i \to C \otimes_A R_i$ is surjective.*
(3) *If C is of finite presentation over A and $v : C \otimes_A R \to B \otimes_A R$ is an R-algebra map, then there exists an i and an R_i-algebra map $v_i : C \otimes_A R_i \to B \otimes_A R_i$ such that $v = v_i \otimes 1$.*
(4) *If B is a finite type A-algebra, C is a finitely presented A-algebra, and $u \otimes 1 : B \otimes_A R \to C \otimes_A R$ is an isomorphism, then for some i the map $u \otimes 1 : B \otimes_A R_i \to C \otimes_A R_i$ is an isomorphism.*

Proof. To prove (1) assume u is as in (1) and let $x_1, \ldots, x_m \in B$ be generators. Since $B \otimes_A R = \operatorname{colim}_i B \otimes_A R_i$ we may pick an $i \in I$ such that $u(x_j) \otimes 1 = u'(x_j) \otimes 1$ in $B \otimes_A R_i$, $j = 1, \ldots, m$. For such an i we have $u \otimes 1 = u' \otimes 1 : B \otimes_A R_i \to C \otimes_A R_i$.

To prove (2) assume $u \otimes 1$ surjective and let $y_1, \ldots, y_m \in C$ be generators. Since $B \otimes_A R = \operatorname{colim}_i B \otimes_A R_i$ we may pick an $i \in I$ and $z_j \in B \otimes_A R_i$, $j = 1, \ldots, m$ whose images in $C \otimes_A R$ equal $y_j \otimes 1$. For such an i the map $u \otimes 1 : B \otimes_A R_i \to C \otimes_A R_i$ is surjective.

To prove (3) let $c_1, \ldots, c_m \in C$ be generators. Let $K = \operatorname{Ker}(A[x_1, \ldots, x_m] \to N)$ where the map is given by the rule $x_j \mapsto \sum c_j$. Let $f_1, \ldots, f_t$ be generators for K as an ideal in $A[x_1, \ldots, x_m]$. We think of $f_j = f_j(x_1, \ldots, x_m)$ as a polynomial. Since $B \otimes_A R = \operatorname{colim}_i B \otimes_A R_i$ we may pick an $i \in I$ and $z_j \in B \otimes_A R_i$, $j = 1, \ldots, m$ whose images in $B \otimes_A R$ equal $v(c_j \otimes 1)$. We want to use the z_j to define a map $v_i : C \otimes_A R_i \to B \otimes_A R_i$. Since $K \otimes_A R_i \to R_i[x_1, \ldots, x_m] \to C \otimes_A R_i \to 0$ is a presentation, it suffices to check that $\xi_s = f_j(z_1, \ldots, z_m)$ is zero in $B \otimes_A R_i$ for each $s = 1, \ldots, t$. This may not be the case, but since the image of ξ_s in $B \otimes_A R$ is zero we see that it will be the case after increasing i a bit.

To prove (4) assume $u \otimes 1$ is an isomorphism, that B is a finite type A-algebra, and that C is a finitely presented A-algebra. Let $v : B \otimes_A R \to C \otimes_A R$ be an inverse to $u \otimes 1$. let $v_i : C \otimes_A R_i \to B \otimes_A R_i$ be as in part (3). Apply part (1) to see that, after increasing i we have $v_i \circ (u \otimes 1) = \mathrm{id}_{B \otimes_R R_i}$ and $(u \otimes 1) \circ v_i = \mathrm{id}_{C \otimes_R R_i}$. □

Lemma 10.121.6. *Suppose that $R = \operatorname{colim}_{i \in I} R_i$ is a directed colimit of rings. Then the category of finitely presented R-algebras is the colimit of the categories of finitely presented R_λ-algebras. More precisely*

1. *Given a finitely presented R-algebra A there exists a $\lambda \in \Lambda$ and a finitely presented R_λ-algebra A_λ such that $A \cong A_\lambda \otimes_{R_\lambda} R$.*
2. *Given a $\lambda \in \Lambda$, finitely presented R_λ-algebras A_λ, B_λ, and an R-algebra map $\varphi : A_\lambda \otimes_{R_\lambda} R \to B_\lambda \otimes_{R_\lambda} R$, then there exists a $\mu \geq \lambda$ and an R_μ-algebra map $\varphi_\mu : A_\lambda \otimes_{R_\lambda} R_\mu \to B_\lambda \otimes_{R_\lambda} R_\mu$ such that $\varphi = \varphi_\mu \otimes 1_R$.*
3. *Given a $\lambda \in \Lambda$, finitely presented R_λ-algebras A_λ, B_λ, and R-algebra maps $\varphi_\lambda, \psi_\lambda : A_\lambda \to B_\lambda$ such that $\varphi \otimes 1_R = \psi \otimes 1_R$, then $\varphi \otimes 1_{R_\mu} = \psi \otimes 1_{R_\mu}$ for some $\mu \geq \lambda$.*

Proof. To prove (1) choose a presentation $A = R[x_1, \ldots, x_n]/(f_1, \ldots, f_m)$. We can choose a $\lambda \in \Lambda$ and elements $f_{\lambda,j} \in R_\lambda[x_1, \ldots, x_n]$ mapping to $f_j \in R[x_1, \ldots, x_n]$. Then we simply let $A_\lambda = R_\lambda[x_1, \ldots, x_n]/(f_{\lambda,1}, \ldots, f_{\lambda,m})$.

Parts (3) and (4) follow from Lemma 10.121.5. □

Lemma 10.121.7. *Suppose $R \to S$ is a local homomorphism of local rings. There exists a directed set $(\Lambda, \leq)$, and a system of local homomorphisms $R_\lambda \to S_\lambda$ of local rings such that*

1. *The colimit of the system $R_\lambda \to S_\lambda$ is equal to $R \to S$.*
2. *Each R_λ is essentially of finite type over $\mathbf{Z}$.*
3. *Each S_λ is essentially of finite type over R_λ.*

Proof. Denote $\varphi : R \to S$ the ring map. Let $\mathfrak{m} \subset R$ be the maximal ideal of R and let $\mathfrak{n} \subset S$ be the maximal ideal of S. Let

$$\Lambda = \{(A, B) \mid A \subset R, B \subset S, \#A < \infty, \#B < \infty, \varphi(A) \subset B\}.$$

As partial ordering we take the inclusion relation. For each $\lambda = (A, B) \in \Lambda$ we let R'_λ be the sub $\mathbf{Z}$-algebra generated by $a \in A$, and we let S'_λ be the sub $\mathbf{Z}$-algebra generated by b, $b \in B$. Let R_λ be the localization of R'_λ at the prime ideal $R'_\lambda \cap \mathfrak{m}$ and let S_λ be the localization of S'_λ at the prime ideal $S'_\lambda \cap \mathfrak{n}$. In a picture

$$\begin{array}{ccccccc} B & \longrightarrow & S'_\lambda & \longrightarrow & S_\lambda & \longrightarrow & S \\ \uparrow & & \uparrow & & \uparrow & & \uparrow \\ A & \longrightarrow & R'_\lambda & \longrightarrow & R_\lambda & \longrightarrow & R \end{array}.$$

The transition maps are clear. We leave the proofs of the other assertions to the reader. □

Lemma 10.121.8. *Suppose $R \to S$ is a local homomorphism of local rings. Assume that S is essentially of finite type over R. Then there exists a directed set $(\Lambda, \leq)$, and a system of local homomorphisms $R_\lambda \to S_\lambda$ of local rings such that*

1. *The colimit of the system $R_\lambda \to S_\lambda$ is equal to $R \to S$.*
2. *Each R_λ is essentially of finite type over $\mathbf{Z}$.*
3. *Each S_λ is essentially of finite type over R_λ.*
4. *For each $\lambda \leq \mu$ the map $S_\lambda \otimes_{R_\lambda} R_\mu \to S_\mu$ presents S_μ as the localization of a quotient of $S_\lambda \otimes_{R_\lambda} R_\mu$.*

Proof. Denote $\varphi : R \to S$ the ring map. Let $\mathfrak{m} \subset R$ be the maximal ideal of R and let $\mathfrak{n} \subset S$ be the maximal ideal of S. Let $x_1, \ldots, x_n \in S$ be elements such that S is a localization of the sub R-algebra of S generated by $x_1, \ldots, x_n$. In other words, S is a quotient of a localization of the polynomial ring $R[x_1, \ldots, x_n]$.

Let $\Lambda = \{A \subset R \mid \#A < \infty\}$ be the set of finite subsets of R. As partial ordering we take the inclusion relation. For each $\lambda = A \in \Lambda$ we let R'_λ be the sub $\mathbf{Z}$-algebra generated by $a \in A$, and we let S'_λ be the sub $\mathbf{Z}$-algebra generated by $\varphi(a)$, $a \in A$ and the elements $x_1, \ldots, x_n$. Let R_λ be the localization of R'_λ at the prime ideal $R'_\lambda \cap \mathfrak{m}$ and let S_λ be the localization of S'_λ at the prime ideal $S'_\lambda \cap \mathfrak{n}$. In a picture

$$\begin{array}{ccccccc} \varphi(A) \coprod \{x_i\} & \longrightarrow & S'_\lambda & \longrightarrow & S_\lambda & \longrightarrow & S \\ \uparrow & & \uparrow & & \uparrow & & \uparrow \\ A & \longrightarrow & R'_\lambda & \longrightarrow & R_\lambda & \longrightarrow & R \end{array}$$

It is clear that if $A \subset B$ corresponds to $\lambda \leq \mu$ in Λ, then there are canonical maps $R_\lambda \to R_\mu$, and $S_\lambda \to S_\mu$ and we obtain a system over the directed set Λ.

The assertion that $R = \operatorname{colim} R_\lambda$ is clear because all the maps $R_\lambda \to R$ are injective and any element of R eventually is in the image. The same argument works for $S = \operatorname{colim} S_\lambda$. Assertions (2), (3) are true by construction. The final assertion holds because clearly the maps $S'_\lambda \otimes_{R'_\lambda} R'_\mu \to S'_\mu$ are surjective. □

Lemma 10.121.9. *Suppose $R \to S$ is a local homomorphism of local rings. Assume that S is essentially of finite presentation over R. Then there exists a directed set $(\Lambda, \leq)$, and a system of local homomorphism $R_\lambda \to S_\lambda$ of local rings such that*

1. *The colimit of the system $R_\lambda \to S_\lambda$ is equal to $R \to S$.*
2. *Each R_λ is essentially of finite type over $\mathbf{Z}$.*
3. *Each S_λ is essentially of finite type over R_λ.*
4. *For each $\lambda \leq \mu$ the map $S_\lambda \otimes_{R_\lambda} R_\mu \to S_\mu$ presents S_μ as the localization of $S_\lambda \otimes_{R_\lambda} R_\mu$ at a prime ideal.*

Proof. By assumption we may choose an isomorphism $\Phi : (R[x_1, \ldots, x_n]/I)_{\mathfrak{q}} \to S$ where $I \subset R[x_1, \ldots, x_n]$ is a finitely generated ideal, and $\mathfrak{q} \subset R[x_1, \ldots, x_n]/I$ is a prime. (Note that $R \cap \mathfrak{q}$ is equal to the maximal ideal $\mathfrak{m}$ of R.) We also choose generators $f_1, \ldots, f_m \in I$ for the ideal I. Write R in any way as a colimit $R = \operatorname{colim} R_\lambda$ over a directed set $(\Lambda, \leq)$, with each R_λ local and essentially of finite type over $\mathbf{Z}$. There exists some $\lambda_0 \in \Lambda$ such that f_j is the image of some $f_{j,\lambda_0} \in R_{\lambda_0}[x_1, \ldots, x_n]$. For all $\lambda \geq \lambda_0$ denote $f_{j,\lambda} \in R_\lambda[x_1, \ldots, x_n]$ the image of f_{j,λ_0}. Thus we obtain a system of ring maps

$$R_\lambda[x_1, \ldots, x_n]/(f_{1,\lambda}, \ldots, f_{n,\lambda}) \to R[x_1, \ldots, x_n]/(f_1, \ldots, f_n) \to S$$

Set $\mathfrak{q}_\lambda$ the inverse image of $\mathfrak{q}$. Set $S_\lambda = (R_\lambda[x_1, \ldots, x_n]/(f_{1,\lambda}, \ldots, f_{n,\lambda}))_{\mathfrak{q}_\lambda}$. We leave it to the reader to see that this works. □

Remark 10.121.10. Suppose that $R \to S$ is a local homomorphism of local rings, which is essentially of finite presentation. Take any system $(\Lambda, \leq)$, $R_\lambda \to S_\lambda$ with the properties listed in Lemma 10.121.8. What may happen is that this is the "wrong" system, namely, it may happen that property (4) of Lemma 10.121.9 is not satisfied. Here is an example. Let k be a field. Consider the ring

$$R = k[[z, y_1, y_2, \ldots]]/(y_i^2 - zy_{i+1}).$$

Set $S = R/zR$. As system take $\Lambda = \mathbf{N}$ and $R_n = k[[z, y_1, \ldots, y_n]]/(\{y_i^2 - zy_{i+1}\}_{i \leq n-1})$ and $S_n = R_n/(z, y_n^2)$. All the maps $S_n \otimes_{R_n} R_{n+1} \to S_{n+1}$ are not localizations (i.e., isomorphisms in this case) since $1 \otimes y_{n+1}^2$ maps to zero. If we take instead $S_n' = R_n/zR_n$ then the maps $S_n' \otimes_{R_n} R_{n+1} \to S_{n+1}'$ are isomorphisms. The moral of this remark is that we do have to be a little careful in choosing the systems.

Lemma 10.121.11. *Suppose $R \to S$ is a local homomorphism of local rings. Assume that S is essentially of finite presentation over R. Let M be a finitely presented S-module. Then there exists a directed set $(\Lambda, \leq)$, and a system of local homomorphisms $R_\lambda \to S_\lambda$ of local rings together with S_λ-modules M_λ, such that*

1. *The colimit of the system $R_\lambda \to S_\lambda$ is equal to $R \to S$. The colimit of the system M_λ is M.*
2. *Each R_λ is essentially of finite type over $\mathbf{Z}$.*
3. *Each S_λ is essentially of finite type over R_λ.*
4. *Each M_λ is finite over S_λ.*
5. *For each $\lambda \leq \mu$ the map $S_\lambda \otimes_{R_\lambda} R_\mu \to S_\mu$ presents S_μ as the localization of $S_\lambda \otimes_{R_\lambda} R_\mu$ at a prime ideal.*
6. *For each $\lambda \leq \mu$ the map $M_\lambda \otimes_{S_\lambda} S_\mu \to M_\mu$ is an isomorphism.*

Proof. As in the proof of Lemma 10.121.9 we may first write $R = \operatorname{colim} R_\lambda$ as a directed colimit of local $\mathbf{Z}$-algebras which are essentially of finite type. Next, we may assume that for some $\lambda_1 \in \Lambda$ there exist $f_{j,\lambda_1} \in R_{\lambda_1}[x_1, \ldots, x_n]$ such that

$$S = \operatorname{colim}_{\lambda \geq \lambda_1} S_\lambda, \text{ with } S_\lambda = (R_\lambda[x_1, \ldots, x_n]/(f_{1,\lambda}, \ldots, f_{m,\lambda}))_{\mathfrak{q}_\lambda}$$

Choose a presentation

$$S^{\oplus s} \to S^{\oplus t} \to M \to 0$$

of M over S. Let $A \in \operatorname{Mat}(t \times s, S)$ be the matrix of the presentation. For some $\lambda_2 \in \Lambda$, $\lambda_2 \geq \lambda_1$ we can find a matrix $A_{\lambda_2} \in \operatorname{Mat}(t \times s, S_{\lambda_2})$ which maps to A. For all $\lambda \geq \lambda_2$ we let $M_\lambda = \operatorname{Coker}(S_\lambda^{\oplus s} \xrightarrow{A_\lambda} S_\lambda^{\oplus t})$. We leave it to the reader to see that this works. □

Lemma 10.121.12. *Suppose $R \to S$ is a ring map. Then there exists a directed set $(\Lambda, \leq)$, and a system of ring maps $R_\lambda \to S_\lambda$ such that*

1. *The colimit of the system $R_\lambda \to S_\lambda$ is equal to $R \to S$.*
2. *Each R_λ is of finite type over $\mathbf{Z}$.*
3. *Each S_λ is of finite type over R_λ.*

Proof. This is the non-local version of Lemma 10.121.7. Proof is similar and left to the reader. □

Lemma 10.121.13. *Suppose $R \to S$ is a ring map. Assume that S is of finite type over R. Then there exists a directed set $(\Lambda, \leq)$, and a system of ring maps $R_\lambda \to S_\lambda$ such that*

1. *The colimit of the system $R_\lambda \to S_\lambda$ is equal to $R \to S$.*
2. *Each R_λ is of finite type over $\mathbf{Z}$.*
3. *Each S_λ is of finite type over R_λ.*
4. *For each $\lambda \leq \mu$ the map $S_\lambda \otimes_{R_\lambda} R_\mu \to S_\mu$ presents S_μ as a quotient of $S_\lambda \otimes_{R_\lambda} R_\mu$.*

Proof. This is the non-local version of Lemma 10.121.8. Proof is similar and left to the reader. □

Lemma 10.121.14. *Suppose $R \to S$ is a ring map. Assume that S is of finite presentation over R. Then there exists a directed set $(\Lambda, \leq)$, and a system of ring maps $R_\lambda \to S_\lambda$ such that*

1. *The colimit of the system $R_\lambda \to S_\lambda$ is equal to $R \to S$.*
2. *Each R_λ is of finite type over $\mathbf{Z}$.*
3. *Each S_λ is of finite type over R_λ.*
4. *For each $\lambda \leq \mu$ the map $S_\lambda \otimes_{R_\lambda} R_\mu \to S_\mu$ is an isomorphism.*

Proof. This is the non-local version of Lemma 10.121.9. Proof is similar and left to the reader. □

Lemma 10.121.15. *Suppose $R \to S$ is a ring map. Assume that S is of finite presentation over R. Let M be a finitely presented S-module. Then there exists a directed set $(\Lambda, \leq)$, and a system of ring maps $R_\lambda \to S_\lambda$ together with S_λ-modules M_λ, such that*

1. *The colimit of the system $R_\lambda \to S_\lambda$ is equal to $R \to S$. The colimit of the system M_λ is M.*
2. *Each R_λ is of finite type over $\mathbf{Z}$.*
3. *Each S_λ is of finite type over R_λ.*
4. *Each M_λ is finite over S_λ.*
5. *For each $\lambda \leq \mu$ the map $S_\lambda \otimes_{R_\lambda} R_\mu \to S_\mu$ is an isomorphism.*
6. *For each $\lambda \leq \mu$ the map $M_\lambda \otimes_{S_\lambda} S_\mu \to S_\mu$ is an isomorphism.*

In particular, for every $\lambda \in \Lambda$ we have

$$M = M_\lambda \otimes_{S_\lambda} S = M_\lambda \otimes_{R_\lambda} R.$$

Proof. This is the non-local version of Lemma 10.121.11. Proof is similar and left to the reader. □

10.122. More flatness criteria

The following lemma is often used in algebraic geometry to show that a finite morphism from a normal surface to a smooth surface is flat. It is a partial converse to Lemma 10.106.9 because a finite local ring map certainly satisfies condition (3).

Lemma 10.122.1. *Let $R \to S$ be a local homomorphism of Noetherian local rings. Assume*

1. *R is regular,*
2. *S Cohen-Macaulay,*
3. *$\mathfrak{m}_S = \sqrt{\mathfrak{m}_R S}$, and*
4. *$\dim(R) = \dim(S)$.*

Then $R \to S$ is flat.

Proof. By induction on $\dim(R)$. The case $\dim(R) = 0$ is trivial, because then R is a field. Assume $\dim(R) > 0$. By (4) this implies that $\dim(S) > 0$. Let $\mathfrak{q}_1, \ldots, \mathfrak{q}_r$ be the minimal primes of S. Note that $\mathfrak{q}_i \neq \mathfrak{m}_S$, hence $\mathfrak{p}_i = R \cap \mathfrak{q}_i$ is not equal to $\mathfrak{m}_R$ by (3). Pick $x \in \mathfrak{m}$, $x \notin \mathfrak{m}^2$, and $x \notin \mathfrak{p}_i$, see Lemma 10.14.3. Hence we see that x is not contained in any of the minimal primes of S. Hence x is a nonzerodivisor on S by (2), see Lemma 10.99.2 and S/xS is Cohen-Macaulay with

$\dim(S/xS) = \dim(S) - 1$. By (1) and Lemma 10.101.3 the ring R/xR is regular with $\dim(R/xR) = \dim(R) - 1$. By induction we see that $R/xR \to S/xS$ is flat. Hence we conclude by Lemma 10.93.9 (see also the remarks following it). □

Lemma 10.122.2. *Let $R \to S$ be a homomorphism of Noetherian local rings. Assume that R is a regular local ring and that a regular system of parameters maps to a regular sequence in S. Then $R \to S$ is flat.*

Proof. Suppose that $x_1, \ldots, x_d$ are a system of parameters of R which map to a regular sequence in S. Note that $S/(x_1, \ldots, x_d)S$ is flat over $R/(x_1, \ldots, x_d)$ as the latter is a field. Then x_d is a nonzerodivisor in $S/(x_1, \ldots, x_{d-1})S$ hence $S/(x_1, \ldots, x_{d-1})S$ is flat over $R/(x_1, \ldots, x_{d-1})$ by the local criterion of flatness (see Lemma 10.93.9 and remarks following). Then x_{d-1} is a nonzerodivisor in $S/(x_1, \ldots, x_{d-2})S$ hence $S/(x_1, \ldots, x_{d-2})S$ is flat over $R/(x_1, \ldots, x_{d-2})$ by the local criterion of flatness (see Lemma 10.93.9 and remarks following). Continue till one reaches the conclusion that S is flat over R. □

The following lemma is the key to proving that results for finitely presented modules over finitely presented rings over a base ring follow from the corresponding results for finite modules in the Noetherian case.

Lemma 10.122.3. *Let $R \to S$, M, Λ, $R_\lambda \to S_\lambda$, M_λ be as in Lemma 10.121.11. Assume that M is flat over R. Then for some $\lambda \in \Lambda$ the module M_λ is flat over R_λ.*

Proof. Pick some $\lambda \in \Lambda$ and consider

$$\mathrm{Tor}_1^{R_\lambda}(M_\lambda, R_\lambda/\mathfrak{m}_\lambda) = \mathrm{Ker}(\mathfrak{m}_\lambda \otimes_{R_\lambda} M_\lambda \to M_\lambda).$$

See Remark 10.70.8. The right hand side shows that this is a finitely generated S_λ-module (because S_λ is Noetherian and the modules in question are finite). Let $\xi_1, \ldots, \xi_n$ be generators. Because M is flat over R we have that $0 = \mathrm{Ker}(\mathfrak{m}_\lambda R \otimes_R M \to M)$. Since $\otimes$ commutes with colimits we see there exists a $\lambda' \geq \lambda$ such that each ξ_i maps to zero in $\mathfrak{m}_\lambda R_{\lambda'} \otimes_{R_{\lambda'}} M_{\lambda'}$. Hence we see that

$$\mathrm{Tor}_1^{R_\lambda}(M_\lambda, R_\lambda/\mathfrak{m}_\lambda) \longrightarrow \mathrm{Tor}_1^{R_{\lambda'}}(M_{\lambda'}, R_{\lambda'}/\mathfrak{m}_\lambda R_{\lambda'})$$

is zero. Note that $M_\lambda \otimes_{R_\lambda} R_\lambda/\mathfrak{m}_\lambda$ is flat over $R_\lambda/\mathfrak{m}_\lambda$ because this last ring is a field. Hence we may apply Lemma 10.93.13 to get that $M_{\lambda'}$ is flat over $R_{\lambda'}$. □

Using the lemma above we can start to reprove the results of Section 10.93 in the non-Noetherian case.

Lemma 10.122.4. *Suppose that $R \to S$ is a local homomorphism of local rings. Denote $\mathfrak{m}$ the maximal ideal of R. Let $u : M \to N$ be a map of S-modules. Assume*

1. *S is essentially of finite presentation over R,*
2. *M, N are finitely presented over S,*
3. *N is flat over R, and*
4. *$\overline{u} : M/\mathfrak{m}M \to N/\mathfrak{m}N$ is injective.*

Then u is injective, and $N/u(M)$ is flat over R.

Proof. By Lemma 10.121.11 and its proof we can find a system $R_\lambda \to S_\lambda$ of local ring maps together with maps of S_λ-modules $u_\lambda : M_\lambda \to N_\lambda$ satisfying the conclusions (1) – (6) for both N and M of that lemma and such that the colimit of the maps u_λ is u. By Lemma 10.122.3 we may assume that N_λ is flat over R_λ for

all sufficiently large λ. Denote $\mathfrak{m}_\lambda \subset R_\lambda$ the maximal ideal and $\kappa_\lambda = R_\lambda/\mathfrak{m}_\lambda$, resp. $\kappa = R/\mathfrak{m}$ the residue fields.

Consider the map

$$\Psi_\lambda : M_\lambda/\mathfrak{m}_\lambda M_\lambda \otimes_{\kappa_\lambda} \kappa \longrightarrow M/\mathfrak{m}M.$$

Since $S_\lambda/\mathfrak{m}_\lambda S_\lambda$ is essentially of finite type over the field κ_λ we see that the tensor product $S_\lambda/\mathfrak{m}_\lambda S_\lambda \otimes_{\kappa_\lambda} \kappa$ is essentially of finite type over κ. Hence it is a Noetherian ring and we conclude the kernel of Ψ_λ is finitely generated. Since $M/\mathfrak{m}M$ is the colimit of the system $M_\lambda/\mathfrak{m}_\lambda M_\lambda$ and κ is the colimit of the fields κ_λ there exists a $\lambda' > \lambda$ such that the kernel of Ψ_λ is generated by the kernel of

$$\Psi_{\lambda,\lambda'} : M_\lambda/\mathfrak{m}_\lambda M_\lambda \otimes_{\kappa_\lambda} \kappa_{\lambda'} \longrightarrow M_{\lambda'}/\mathfrak{m}_{\lambda'} M_{\lambda'}.$$

By construction there exists a multiplicative subset $W \subset S_\lambda \otimes_{R_\lambda} R_{\lambda'}$ such that $S_{\lambda'} = W^{-1}(S_\lambda \otimes_{R_\lambda} R_{\lambda'})$ and

$$W^{-1}(M_\lambda/\mathfrak{m}_\lambda M_\lambda \otimes_{\kappa_\lambda} \kappa_{\lambda'}) = M_{\lambda'}/\mathfrak{m}_{\lambda'} M_{\lambda'}.$$

Now suppose that x is an element of the kernel of

$$\Psi_{\lambda'} : M_{\lambda'}/\mathfrak{m}_{\lambda'} M_{\lambda'} \otimes_{\kappa_{\lambda'}} \kappa \longrightarrow M/\mathfrak{m}M.$$

Then for some $w \in W$ we have $wx \in M_\lambda/\mathfrak{m}_\lambda M_\lambda \otimes \kappa$. Hence $wx \in \mathrm{Ker}(\Psi_\lambda)$. Hence wx is a linear combination of elements in the kernel of $\Psi_{\lambda,\lambda'}$. Hence $wx = 0$ in $M_{\lambda'}/\mathfrak{m}_{\lambda'} M_{\lambda'} \otimes_{\kappa_{\lambda'}} \kappa$, hence $x = 0$ because w is invertible in $S_{\lambda'}$. We conclude that the kernel of $\Psi_{\lambda'}$ is zero for all sufficiently large λ'!

By the result of the preceding paragraph we may assume that the kernel of Ψ_λ is zero for all λ sufficiently large, which implies that the map $M_\lambda/\mathfrak{m}_\lambda M_\lambda \to M/\mathfrak{m}M$ is injective. Combined with $\overline{u}$ being injective this formally implies that also $\overline{u_\lambda} : M_\lambda/\mathfrak{m}_\lambda M_\lambda \to N_\lambda/\mathfrak{m}_\lambda N_\lambda$ is injective. By Lemma 10.93.1 we conclude that (for all sufficiently large λ) the map u_λ is injective and that $N_\lambda/u_\lambda(M_\lambda)$ is flat over R_λ. The lemma follows. □

Lemma 10.122.5. *Suppose that $R \to S$ is a local ring homomorphism of local rings. Denote $\mathfrak{m}$ the maximal ideal of R. Suppose*

(1) *S is essentially of finite presentation over R,*
(2) *S is flat over R, and*
(3) *$f \in S$ is a nonzerodivisor in $S/\mathfrak{m}S$.*

Then S/fS is flat over R, and f is a nonzerodivisor in S.

Proof. Follows directly from Lemma 10.122.4. □

Lemma 10.122.6. *Suppose that $R \to S$ is a local ring homomorphism of local rings. Denote $\mathfrak{m}$ the maximal ideal of R. Suppose*

(1) *$R \to S$ is essentially of finite presentation,*
(2) *$R \to S$ is flat, and*
(3) *$f_1, \ldots, f_c$ is a sequence of elements of S such that the images $\overline{f}_1, \ldots, \overline{f}_c$ form a regular sequence in $S/\mathfrak{m}S$.*

Then $f_1, \ldots, f_c$ is a regular sequence in S and each of the quotients $S/(f_1, \ldots, f_i)$ is flat over R.

Proof. Induction and Lemma 10.122.5. □

Here is the version of the local criterion of flatness for the case of local ring maps which are locally of finite presentation.

Lemma 10.122.7. *Let $R \to S$ be a local homomorphism of local rings. Let $I \neq R$ be an ideal in R. Let M be an S-module. Assume*

(1) *S is essentially of finite presentation over R,*
(2) *M is of finite presentation over S,*
(3) *$Tor_1^R(M, R/I) = 0$, and*
(4) *M/IM is flat over R/I.*

Then M is flat over R.

Proof. Let Λ, $R_\lambda \to S_\lambda$, M_λ be as in Lemma 10.121.11. Denote $I_\lambda \subset R_\lambda$ the inverse image of I. In this case the system $R/I \to S/IS$, M/IM, $R_\lambda \to S_\lambda/I_\lambda S_\lambda$, and $M_\lambda/I_\lambda M_\lambda$ satisfies the conclusions of Lemma 10.121.11 as well. Hence by Lemma 10.122.3 we may assume (after shrinking the index set Λ) that $M_\lambda/I_\lambda M_\lambda$ is flat for all λ. Pick some λ and consider

$$\mathrm{Tor}_1^{R_\lambda}(M_\lambda, R_\lambda/I_\lambda) = \mathrm{Ker}(I_\lambda \otimes_{R_\lambda} M_\lambda \to M_\lambda).$$

See Remark 10.70.8. The right hand side shows that this is a finitely generated S_λ-module (because S_λ is Noetherian and the modules in question are finite). Let $\xi_1, \ldots, \xi_n$ be generators. Because $\mathrm{Tor}_R^1(M, R/I) = 0$ and since $\otimes$ commutes with colimits we see there exists a $\lambda' \geq \lambda$ such that each ξ_i maps to zero in $\mathrm{Tor}_1^{R_{\lambda'}}(M_{\lambda'}, R_{\lambda'}/I_{\lambda'})$. The composition of the maps

$$\begin{aligned}
&R_{\lambda'} \otimes_{R_\lambda} \mathrm{Tor}_1^{R_\lambda}(M_\lambda, R_\lambda/I_\lambda) \\
&\qquad\qquad \Big\downarrow \text{surjective by Lemma 10.93.11} \\
&\mathrm{Tor}_1^{R_\lambda}(M_\lambda, R_{\lambda'}/I_\lambda R_{\lambda'}) \\
&\qquad\qquad \Big\downarrow \text{surjective up to localization by Lemma 10.93.12} \\
&\mathrm{Tor}_1^{R_{\lambda'}}(M_{\lambda'}, R_{\lambda'}/I_\lambda R_{\lambda'}) \\
&\qquad\qquad \Big\downarrow \text{surjective by Lemma 10.93.11} \\
&\mathrm{Tor}_1^{R_{\lambda'}}(M_{\lambda'}, R_{\lambda'}/I_{\lambda'}).
\end{aligned}$$

is surjective up to a localization by the reasons indicated. The localization is necessary since $M_{\lambda'}$ is not equal to $M_\lambda \otimes_{R_\lambda} R_{\lambda'}$. Namely, it is equal to $M_\lambda \otimes_{S_\lambda} S_{\lambda'}$ and $S_{\lambda'}$ is the localization of $S_\lambda \otimes_{R_\lambda} R_{\lambda'}$ whence the statement up to a localization (or tensoring with $S_{\lambda'}$). Note that Lemma 10.93.11 applies to the first and third arrows because $M_\lambda/I_\lambda M_\lambda$ is flat over R_λ/I_λ and because $M_{\lambda'}/I_\lambda M_{\lambda'}$ is flat over $R_{\lambda'}/I_\lambda R_{\lambda'}$ as it is a base change of the flat module $M_\lambda/I_\lambda M_\lambda$. The composition maps the generators ξ_i to zero as we explained above. We finally conclude that $\mathrm{Tor}_1^{R_{\lambda'}}(M_{\lambda'}, R_{\lambda'}/I_{\lambda'})$ is zero. This implies that $M_{\lambda'}$ is flat over $R_{\lambda'}$ by Lemma 10.93.9. □

Please compare the lemma below to Lemma 10.93.14 (the case of Noetherian local rings) and Lemma 10.95.8 (the case of a nilpotent ideal in the base).

Lemma 10.122.8 (Critère de platitude par fibres)**.** *Let R, S, S' be local rings and let $R \to S \to S'$ be local ring homomorphisms. Let M be an S'-module. Let $\mathfrak{m} \subset R$ be the maximal ideal. Assume*

1. *The ring maps $R \to S$ and $R \to S'$ are essentially of finite presentation.*
2. *The module M is of finite presentation over S'.*
3. *The module M is not zero.*
4. *The module $M/\mathfrak{m}M$ is a flat $S/\mathfrak{m}S$-module.*
5. *The module M is a flat R-module.*

Then S is flat over R and M is a flat S-module.

Proof. As in the proof of Lemma 10.121.9 we may first write $R = \operatorname{colim} R_\lambda$ as a directed colimit of local $\mathbf{Z}$-algebras which are essentially of finite type. Denote $\mathfrak{p}_\lambda$ the maximal ideal of R. Next, we may assume that for some $\lambda_1 \in \Lambda$ there exist $f_{j,\lambda_1} \in R_{\lambda_1}[x_1, \ldots, x_n]$ such that

$$S = \operatorname{colim}_{\lambda \geq \lambda_1} S_\lambda, \text{ with } S_\lambda = (R_\lambda[x_1, \ldots, x_n]/(f_{1,\lambda}, \ldots, f_{u,\lambda}))_{\mathfrak{q}_\lambda}$$

For some $\lambda_2 \in \Lambda$, $\lambda_2 \geq \lambda_1$ there exist $g_{j,\lambda_2} \in R_{\lambda_2}[x_1, \ldots, x_n, y_1, \ldots, y_m]$ with images $\overline{g}_{j,\lambda_2} \in S_{\lambda_2}[y_1, \ldots, y_m]$ such that

$$S' = \operatorname{colim}_{\lambda \geq \lambda_2} S'_\lambda, \text{ with } S'_\lambda = (S_\lambda[y_1, \ldots, y_m]/(\overline{g}_{1,\lambda}, \ldots, \overline{g}_{v,\lambda}))_{\overline{\mathfrak{q}}'_\lambda}$$

Note that this also implies that

$$S'_\lambda = (R_\lambda[x_1, \ldots, x_n, y_1, \ldots, y_m]/(g_{1,\lambda}, \ldots, g_{v,\lambda}))_{\mathfrak{q}'_\lambda}$$

Choose a presentation

$$(S')^{\oplus s} \to (S')^{\oplus t} \to M \to 0$$

of M over S'. Let $A \in \operatorname{Mat}(t \times s, S')$ be the matrix of the presentation. For some $\lambda_3 \in \Lambda$, $\lambda_3 \geq \lambda_2$ we can find a matrix $A_{\lambda_3} \in \operatorname{Mat}(t \times s, S_{\lambda_3})$ which maps to A. For all $\lambda \geq \lambda_3$ we let $M_\lambda = \operatorname{Coker}((S'_\lambda)^{\oplus s} \xrightarrow{A_\lambda} (S'_\lambda)^{\oplus t})$.

With these choices, we have for each $\lambda_3 \leq \lambda \leq \mu$ that $S_\lambda \otimes_{R_\lambda} R_\mu \to S_\mu$ is a localization, $S'_\lambda \otimes_{S_\lambda} S_\mu \to S'_\mu$ is a localization, and the map $M_\lambda \otimes_{S'_\lambda} S_\mu \to M_\mu$ is an isomorphism. This also implies that $S'_\lambda \otimes_{R_\lambda} R_\mu \to S'_\mu$ is a localization. Thus, since M is flat over R we see by Lemma 10.122.3 that for all λ big enough the module M_λ is flat over R_λ. Moreover, note that $\mathfrak{m} = \operatorname{colim} \mathfrak{p}_\lambda$, $S/\mathfrak{m}S = \operatorname{colim} S_\lambda/\mathfrak{p}_\lambda S_\lambda$, $S'/\mathfrak{m}S' = \operatorname{colim} S'_\lambda/\mathfrak{p}_\lambda S'_\lambda$, and $M/\mathfrak{m}M = \operatorname{colim} M_\lambda/\mathfrak{p}_\lambda M_\lambda$. Also, for each $\lambda_3 \leq \lambda \leq \mu$ we see (from the properties listed above) that

$$S'_\lambda/\mathfrak{p}_\lambda S'_\lambda \otimes_{S_\lambda/\mathfrak{p}_\lambda S_\lambda} S_\mu/\mathfrak{p}_\mu S_\mu \longrightarrow S'_\mu/\mathfrak{p}_\mu S'_\mu$$

is a localization, and the map

$$M_\lambda/\mathfrak{p}_\lambda M_\lambda \otimes_{S'_\lambda/\mathfrak{p}_\lambda S'_\lambda} S_\mu/\mathfrak{p}_\mu S'_\mu \longrightarrow M_\mu/\mathfrak{p}_\mu M_\mu$$

is an isomorphism. Hence the system $(S_\lambda/\mathfrak{p}_\lambda S_\lambda \to S'_\lambda/\mathfrak{p}_\lambda S'_\lambda, M_\lambda/\mathfrak{p}_\lambda M_\lambda)$ is a system as in Lemma 10.121.11 as well. We may apply Lemma 10.122.3 again because $M/\mathfrak{m}M$ is assumed flat over $S/\mathfrak{m}S$ and we see that $M_\lambda/\mathfrak{p}_\lambda M_\lambda$ is flat over $S_\lambda/\mathfrak{p}_\lambda S_\lambda$ for all λ big enough. Thus for λ big enough the data $R_\lambda \to S_\lambda \to S'_\lambda, M_\lambda$ satisfies the hypotheses of Lemma 10.93.14. Pick such a λ. Then $S = S_\lambda \otimes_{R_\lambda} R$ is flat over R, and $M = M_\lambda \otimes_{S_\lambda} S'_\lambda$ is flat over S (since the base change of a flat module is flat). □

The following is an easy consequence of the "critère de platitude par fibres" Lemma 10.122.8. For more results of this kind see More on Flatness, Section 37.1.

Lemma 10.122.9. *Let R, S, S' be local rings and let $R \to S \to S'$ be local ring homomorphisms. Let M be an S'-module. Let $\mathfrak{m} \subset R$ be the maximal ideal. Assume*

(1) *$R \to S'$ is essentially of finite presentation,*
(2) *$R \to S$ is essentially of finite type,*
(3) *M is of finite presentation over S',*
(4) *M is not zero,*
(5) *$M/\mathfrak{m}M$ is a flat $S/\mathfrak{m}S$-module, and*
(6) *M is a flat R-module.*

Then S is essentially of finite presentation and flat over R and M is a flat S-module.

Proof. As S is essentially of finite presentation over R we can write $S = C_{\overline{\mathfrak{q}}}$ for some finite type R-algebra C. Write $C = R[x_1, \ldots, x_n]/I$. Denote $\mathfrak{q} \subset R[x_1, \ldots, x_n]$ be the prime ideal corresponding to $\overline{\mathfrak{q}}$. Then we see that $S = B/J$ where $B = R[x_1, \ldots, x_n]_{\mathfrak{q}}$ is essentially of finite presentation over R and $J = IB$. We can find $f_1, \ldots, f_k \in J$ such that the images $\overline{f}_i \in B/\mathfrak{m}B$ generate the image $\overline{J}$ of J in the Noetherian ring $B/\mathfrak{m}B$. Hence there exist finitely generated ideals $J' \subset J$ such that $B/J' \to B/J$ induces an isomorphism

$$(B/J') \otimes_R R/\mathfrak{m} \longrightarrow B/J \otimes_R R/\mathfrak{m} = S/\mathfrak{m}S.$$

For any J' as above we see that Lemma 10.122.8 applies to the ring maps

$$R \longrightarrow B/J' \longrightarrow S'$$

and the module M. Hence we conclude that B/J' is flat over R for any choice J' as above. Now, if $J' \subset J' \subset J$ are two finitely generated ideals as above, then we conclude that $B/J' \to B/J''$ is a surjective map between flat R-algebras which are essentially of finite presentation which is an isomorphism modulo $\mathfrak{m}$. Hence Lemma 10.122.4 implies that $B/J' = B/J''$, i.e., $J' = J''$. Clearly this means that J is finitely generated, i.e., S is essentially of finite presentation over R. Thus we may apply Lemma 10.122.8 to $R \to S \to S'$ and we win. □

10.123. Openness of the flat locus

Lemma 10.123.1. *Let k be a field. Let S be a finite type k-algebra. Let $f_1, \ldots, f_i$ be elements of S. Assume that S is Cohen-Macaulay and equidimensional of dimension d, and that $\dim V(f_1, \ldots, f_i) \leq d - i$. Then equality holds and $f_1, \ldots, f_i$ forms a regular sequence in $S_{\mathfrak{q}}$ for every prime $\mathfrak{q}$ of $V(f_1, \ldots, f_i)$.*

Proof. If S is Cohen-Macaulay and equidimensional of dimension d, then we have $\dim(S_{\mathfrak{m}}) = d$ for all maximal ideals $\mathfrak{m}$ of S, see Lemma 10.108.7. By Proposition 10.98.4 we see that for all maximal ideals $\mathfrak{m} \in V(f_1, \ldots, f_i)$ the sequence is a regular sequence in $S_{\mathfrak{m}}$ and the local ring $S_{\mathfrak{m}}/(f_1, \ldots, f_i)$ is Cohen-Macaulay of dimension $d - i$. This actually means that $S/(f_1, \ldots, f_i)$ is Cohen-Macaulay and equidimensional of dimension $d - i$. □

Lemma 10.123.2. *Suppose that $R \to S$ is a ring map which is finite type, flat. Let d be an integer such that all fibres $S \otimes_R \kappa(\mathfrak{p})$ are Cohen-Macaulay and equidimensional of dimension d. Let $f_1, \ldots, f_i$ be elements of S. The set*

$$\{\mathfrak{q} \in V(f_1, \ldots, f_i) \mid f_1, \ldots, f_i \text{ are a regular sequence in } S_{\mathfrak{q}}/\mathfrak{p}S_{\mathfrak{q}} \text{ where } \mathfrak{p} = R \cap \mathfrak{q}\}$$

is open in $V(f_1, \ldots, f_i)$.

Proof. Write $\overline{S} = S/(f_1, \ldots, f_i)$. Suppose $\mathfrak{q}$ is an element of the set defined in the lemma, and $\mathfrak{p}$ is the corresponding prime of R. We will use relative dimension as defined in Definition 10.119.1. First, note that $d = \dim_{\mathfrak{q}}(S/R) = \dim(S_{\mathfrak{q}}/\mathfrak{p}S_{\mathfrak{q}}) + \operatorname{trdeg}_{\kappa(\mathfrak{p})} \kappa(\mathfrak{q})$ by Lemma 10.110.3. Since $f_1, \ldots, f_i$ form a regular sequence in the Noetherian local ring $S_{\mathfrak{q}}/\mathfrak{p}S_{\mathfrak{q}}$ general dimension theory tells us that $\dim(\overline{S}_{\mathfrak{q}}/\mathfrak{p}\overline{S}_{\mathfrak{q}}) = \dim(S_{\mathfrak{q}}/\mathfrak{p}S_{\mathfrak{q}}) - i$. By the same Lemma 10.110.3 we then conclude that $\dim_{\mathfrak{q}}(\overline{S}/R) = \dim(\overline{S}_{\mathfrak{q}}/\mathfrak{p}\overline{S}_{\mathfrak{q}}) + \operatorname{trdeg}_{\kappa(\mathfrak{p})} \kappa(\mathfrak{q}) = d - i$. By Lemma 10.119.6 we have $\dim_{\mathfrak{q}'}(\overline{S}/R) \leq d - i$ for all $\mathfrak{q}' \in V(f_1, \ldots, f_i) = \operatorname{Spec}(\overline{S})$ in a neighbourhood of $\mathfrak{q}$. Thus after replacing S by S_g for some $g \in S$, $g \notin \mathfrak{q}$ we may assume that the inequality holds for all $\mathfrak{q}'$. The result follows from Lemma 10.123.1. □

Lemma 10.123.3. *Let $R \to S$ is a ring map. Consider a finite homological complex of finite free S-modules:*

$$F_\bullet : 0 \to S^{n_e} \xrightarrow{\varphi_e} S^{n_{e-1}} \xrightarrow{\varphi_{e-1}} \ldots \xrightarrow{\varphi_{i+1}} S^{n_i} \xrightarrow{\varphi_i} S^{n_{i-1}} \xrightarrow{\varphi_{i-1}} \ldots \xrightarrow{\varphi_1} S^{n_0}$$

For every prime $\mathfrak{q}$ of S consider the complex $\overline{F}_{\bullet,\mathfrak{q}} = F_{\bullet,\mathfrak{q}} \otimes_R \kappa(\mathfrak{p})$ where $\mathfrak{p}$ is inverse image of $\mathfrak{q}$ in R. Assume there exists an integer d such that $R \to S$ is finite type, flat with fibres $S \otimes_R \kappa(\mathfrak{p})$ Cohen-Macaulay of dimension d. The set

$$\{\mathfrak{q} \in \operatorname{Spec}(S) \mid \overline{F}_{\bullet,\mathfrak{q}} \text{ is exact}\}$$

is open in $\operatorname{Spec}(S)$.

Proof. Let $\mathfrak{q}$ be an element of the set defined in the lemma. We are going to use Proposition 10.96.10 to show there exists a $g \in S$, $g \notin \mathfrak{q}$ such that $D(g)$ is contained in the set defined in the lemma. In other words, we are going to show that after replacing S by S_g, the set of the lemma is all of $\operatorname{Spec}(S)$. Thus during the proof we will, finitely often, replace S by such a localization. Recall that Proposition 10.96.10 characterizes exactness of complexes in terms of ranks of the maps φ_i and the ideals $I(\varphi_i)$, in case the ring is local. We first address the rank condition. Set $r_i = n_i - n_{i+1} + \ldots + (-1)^{e-i} n_e$. Note that $r_i + r_{i+1} = n_i$ and note that r_i is the expected rank of φ_i (in the exact case).

By Lemma 10.93.5 we see that if $\overline{F}_{\bullet,\mathfrak{q}}$ is exact, then the localization $F_{\bullet,\mathfrak{q}}$ is exact. In particular the complex $F_\bullet$ becomes exact after localizing by an element $g \in S$, $g \notin \mathfrak{q}$. In this case Proposition 10.96.10 applied to all localizations of S at prime ideals implies that all $(r_i + 1) \times (r_i + 1)$-minors of φ_i are zero. Thus we see that the rank of of φ_i is at most r_i.

Let $I_i \subset S$ denote the ideal generated by the $r_i \times r_i$-minors of the matrix of φ_i. By Proposition 10.96.10 the complex $\overline{F}_{\bullet,\mathfrak{q}}$ is exact if and only if for every $1 \leq i \leq e$ we have either $(I_i)_{\mathfrak{q}} = S_{\mathfrak{q}}$ or $(I_i)_{\mathfrak{q}}$ contains a $S_{\mathfrak{q}}/\mathfrak{p}S_{\mathfrak{q}}$-regular sequence of length i. Namely, by our choice of r_i above and by the bound on the ranks of the φ_i this is the only way the conditions of Proposition 10.96.10 can be satisfied.

If $(I_i)_{\mathfrak{q}} = S_{\mathfrak{q}}$, then after localizing S at some element $g \notin \mathfrak{q}$ we may assume that $I_i = S$. Clearly, this is an open condition.

If $(I_i)_{\mathfrak{q}} \neq S_{\mathfrak{q}}$, then we have a sequence $f_1, \ldots, f_i \in (I_i)_{\mathfrak{q}}$ which form a regular sequence in $S_{\mathfrak{q}}/\mathfrak{p}S_{\mathfrak{q}}$. Note that for any prime $\mathfrak{q}' \subset S$ such that $(f_1, \ldots, f_i) \not\subset \mathfrak{q}'$ we have $(I_i)_{\mathfrak{q}'} = S_{\mathfrak{q}'}$. Thus the result follows from Lemma 10.123.2. □

Theorem 10.123.4. *Let R be a ring. Let $R \to S$ be a ring map of finite presentation. Let M be a finitely presented S-module. The set*

$$\{\mathfrak{q} \in \mathrm{Spec}(S) \mid M_{\mathfrak{q}} \text{ is flat over } R\}$$

is open in $\mathrm{Spec}(S)$.

Proof. Let $\mathfrak{q} \in \mathrm{Spec}(S)$ be a prime. Let $\mathfrak{p} \subset R$ be the inverse image of $\mathfrak{q}$ in R. Note that $M_{\mathfrak{q}}$ is flat over R if and only if it is flat over $R_{\mathfrak{p}}$. Let us assume that $M_{\mathfrak{q}}$ is flat over R. We claim that there exists a $g \in S$, $g \notin \mathfrak{q}$ such that M_g is flat over R.

We first reduce to the case where R and S are of finite type over $\mathbf{Z}$. Choose a directed partially ordered set Λ and a system $(R_\lambda \to S_\Lambda, M_\lambda)$ as in Lemma 10.121.15. Set $\mathfrak{p}_\lambda$ equal to the inverse image of $\mathfrak{p}$ in R_λ. Set $\mathfrak{q}_\lambda$ equal to the inverse image of $\mathfrak{q}$ in S_λ. Then the system

$$((R_\lambda)_{\mathfrak{p}_\lambda}, (S_\lambda)_{\mathfrak{q}_\lambda}, (M_\lambda)_{\mathfrak{q}_\lambda})$$

is a system as in Lemma 10.121.11. Hence by Lemma 10.122.3 we see that for some λ the module M_λ is flat over R_λ at the prime $\mathfrak{q}_\lambda$. Suppose we can prove our claim for the system $(R_\lambda \to S_\lambda, M_\lambda, \mathfrak{q}_\lambda)$. In other words, suppose that we can find a $g \in S_\lambda$, $g \notin \mathfrak{q}_\lambda$ such that $(M_\lambda)_g$ is flat over R_λ. By Lemma 10.121.15 we have $M = M_\lambda \otimes_{R_\lambda} R$ and hence also $M_g = (M_\lambda)_g \otimes_{R_\lambda} R$. Thus by Lemma 10.36.6 we deduce the claim for the system $(R \to S, M, \mathfrak{q})$.

At this point we may assume that R and S are of finite type over $\mathbf{Z}$. We may write S as a quotient of a polynomial ring $R[x_1, \ldots, x_n]$. Of course, we may replace S by $R[x_1, \ldots, x_n]$ and assume that S is a polynomial ring over R. In particular we see that $R \to S$ is flat and all fibres rings $S \otimes_R \kappa(\mathfrak{p})$ have global dimension n.

Choose a resolution $F_\bullet$ of M over S with each F_i finite free, see Lemma 10.67.1. Let $K_n = \mathrm{Ker}(F_{n-1} \to F_{n-2})$. Note that $(K_n)_{\mathfrak{q}}$ is flat over R, since each F_i is flat over R and by assumption on M, see Lemma 10.36.12. In addition, the sequence

$$0 \to K_n/\mathfrak{p}K_n \to F_{n-1}/\mathfrak{p}F_{n-1} \to \ldots \to F_0/\mathfrak{p}F_0 \to M/\mathfrak{p}M \to 0$$

is exact upon localizing at $\mathfrak{q}$, because of vanishing of $\mathrm{Tor}_i^{R_{\mathfrak{p}}}(\kappa(\mathfrak{p}), M_{\mathfrak{q}})$. Since the global dimension of $S_{\mathfrak{q}}/\mathfrak{p}S_{\mathfrak{q}}$ is n we conclude that $K_n/\mathfrak{p}K_n$ localized at $\mathfrak{q}$ is a finite free module over $S_{\mathfrak{q}}/\mathfrak{p}S_{\mathfrak{q}}$. By Lemma 10.93.4 $(K_n)_{\mathfrak{q}}$ is free over $S_{\mathfrak{q}}$. In particular, there exists a $g \in S$, $g \notin \mathfrak{q}$ such that $(K_n)_g$ is finite free over S_g.

By Lemma 10.123.3 there exists a further localization S_g such that the complex

$$0 \to K_n \to F_{n-1} \to \ldots \to F_0$$

is exact on *all fibres* of $R \to S$. By Lemma 10.93.5 this implies that the cokernel of $F_1 \to F_0$ is flat. This proves the theorem in the Noetherian case. □

10.124. Openness of Cohen-Macaulay loci

In this section we characterize the Cohen-Macaulay property of finite type algebras in terms of flatness. We then use this to prove the set of points where such an algebra is Cohen-Macaulay is open.

Lemma 10.124.1. *Let S be a finite type algebra over a field k. Let $\varphi : k[y_1, \ldots, y_d] \to S$ be a finite ring map. As subsets of* $\mathrm{Spec}(S)$ *we have*

$$\{\mathfrak{q} \mid S_\mathfrak{q} \text{ flat over } k[y_1, \ldots, y_d]\} = \{\mathfrak{q} \mid S_\mathfrak{q} \text{ CM and } \dim_\mathfrak{q}(S/k) = d\}$$

For notation see Definition 10.119.1.

Proof. Let $\mathfrak{q} \subset S$ be a prime. Denote $\mathfrak{p} = k[y_1, \ldots, y_d] \cap \mathfrak{q}$. Note that always $\dim(S_\mathfrak{q}) \leq \dim(k[y_1, \ldots, y_d]_\mathfrak{p})$ by Lemma 10.119.4 for example. Moreover, the field extension $\kappa(\mathfrak{p}) \subset \kappa(\mathfrak{q})$ is finite and hence $\mathrm{trdeg}_k(\kappa(\mathfrak{p})) = \mathrm{trdeg}_k(\kappa(\mathfrak{q}))$.

Let $\mathfrak{q}$ be an element of the left hand side. Then Lemma 10.106.9 applies and we conclude that $S_\mathfrak{q}$ is Cohen-Macaulay and $\dim(S_\mathfrak{q}) = \dim(k[y_1, \ldots, y_d]_\mathfrak{p})$. Combined with the equality of transcendence degrees above and Lemma 10.110.3 this implies that $\dim_\mathfrak{q}(S/k) = d$. Hence $\mathfrak{q}$ is an element of the right hand side.

Let $\mathfrak{q}$ be an element of the right hand side. By the equality of transcendence degrees above, the assumption that $\dim_\mathfrak{q}(S/k) = d$ and Lemma 10.110.3 we conclude that $\dim(S_\mathfrak{q}) = \dim(k[y_1, \ldots, y_d]_\mathfrak{p})$. Hence Lemma 10.122.1 applies and we see that $\mathfrak{q}$ is an element of the left hand side. □

Lemma 10.124.2. *Let S be a finite type algebra over a field k. The set of primes $\mathfrak{q}$ such that $S_\mathfrak{q}$ is Cohen-Macaulay is open in S.*

This lemma is a special case of Lemma 10.124.4 below, so you can skip straight to the proof of that lemma if you like.

Proof. Let $\mathfrak{q} \subset S$ be a prime such that $S_\mathfrak{q}$ is Cohen-Macaulay. We have to show there exists a $g \in S$, $g \notin \mathfrak{q}$ such that the ring S_g is Cohen-Macaulay. For any $g \in S$, $g \notin \mathfrak{q}$ we may replace S by S_g and $\mathfrak{q}$ by $\mathfrak{q}S_g$. Combining this with Lemmas 10.109.5 and 10.110.3 we may assume that there exists a finite injective ring map $k[y_1, \ldots, y_d] \to S$ with $d = \dim(S_\mathfrak{q}) + \mathrm{trdeg}_k(\kappa(\mathfrak{q}))$. Set $\mathfrak{p} = k[y_1, \ldots, y_d] \cap \mathfrak{q}$. By construction we see that $\mathfrak{q}$ is an element of the right hand side of the displayed equality of Lemma 10.124.1. Hence it is also an element of the left hand side.

By Theorem 10.123.4 we see that for some $g \in S$, $g \notin \mathfrak{q}$ the ring S_g is flat over $k[y_1, \ldots, y_d]$. Hence by the equality of Lemma 10.124.1 again we conclude that all local rings of S_g are Cohen-Macaulay as desired. □

Lemma 10.124.3. *Let k be a field. Let S be a finite type k algebra. The set of Cohen-Macaulay primes forms a dense open $U \subset \mathrm{Spec}(S)$.*

Proof. The set is open by Lemma 10.124.2. It contains all minimal primes $\mathfrak{q} \subset S$ since the local ring at a minimal prime $S_\mathfrak{q}$ has dimension zero and hence is Cohen-Macaulay. □

Lemma 10.124.4. *Let R be a ring. Let $R \to S$ be of finite presentation and flat. For any $d \geq 0$ the set*

$$\left\{ \begin{array}{c} \mathfrak{q} \in \mathrm{Spec}(S) \text{ such that setting } \mathfrak{p} = R \cap \mathfrak{q} \text{ the fibre ring} \\ S_\mathfrak{q}/\mathfrak{p}S_\mathfrak{q} \text{ is Cohen-Macaulay and } \dim_\mathfrak{q}(S/R) = d \end{array} \right\}$$

is open in $\mathrm{Spec}(S)$.

Proof. Let $\mathfrak{q}$ be an element of the set indicated, with $\mathfrak{p}$ the corresponding prime of R. We have to find a $g \in S$, $g \notin \mathfrak{q}$ such that all fibre rings of $R \to S_g$ are Cohen-Macaulay. During the course of the proof we may (finitely many times) replace S by S_g for a $g \in S$, $g \notin \mathfrak{q}$. Thus by Lemma 10.119.2 we may assume there is a quasi-finite ring map $R[t_1, \ldots, t_d] \to S$ with $d = \dim_{\mathfrak{q}}(S/R)$. Let $\mathfrak{q}' = R[t_1, \ldots, t_d] \cap \mathfrak{q}$. By Lemma 10.124.1 we see that the ring map

$$R[t_1, \ldots, t_d]_{\mathfrak{q}'}/\mathfrak{p}R[t_1, \ldots, t_d]_{\mathfrak{q}'} \longrightarrow S_{\mathfrak{q}}/\mathfrak{p}S_{\mathfrak{q}}$$

is flat. Hence by the critère de platitude par fibres Lemma 10.122.8 we see that $R[t_1, \ldots, t_d]_{\mathfrak{q}'} \to S_{\mathfrak{q}}$ is flat. Hence by Theorem 10.123.4 we see that for some $g \in S$, $g \notin \mathfrak{q}$ the ring map $R[t_1, \ldots, t_d] \to S_g$ is flat. Replacing S by S_g we see that for every prime $\mathfrak{r} \subset S$, setting $\mathfrak{r}' = R[t_1, \ldots, t_d] \cap \mathfrak{r}$ and $\mathfrak{p}' = R \cap \mathfrak{r}$ the local ring map $R[t_1, \ldots, t_d]_{\mathfrak{r}'} \to S_{\mathfrak{r}}$ is flat. Hence also the base change

$$R[t_1, \ldots, t_d]_{\mathfrak{r}'}/\mathfrak{p}'R[t_1, \ldots, t_d]_{\mathfrak{r}'} \longrightarrow S_{\mathfrak{r}}/\mathfrak{p}'S_{\mathfrak{r}}$$

is flat. Hence by Lemma 10.124.1 applied with $k = \kappa(\mathfrak{p}')$ we see $\mathfrak{r}$ is in the set of the lemma as desired. □

Lemma 10.124.5. *Let R be a ring. Let $R \to S$ be flat of finite presentation. The set of primes $\mathfrak{q}$ such that the fibre ring $S_{\mathfrak{q}} \otimes_R \kappa(\mathfrak{p})$, with $\mathfrak{p} = R \cap \mathfrak{q}$ is Cohen-Macaulay is open and dense in every fibre of* $\mathrm{Spec}(S) \to \mathrm{Spec}(R)$.

Proof. The set, call it W, is open by Lemma 10.124.4. It is dense in the fibres because the intersection of W with a fibre is the corresponding set of the fibre to which Lemma 10.124.3 applies. □

Lemma 10.124.6. *Let k be a field. Let S be a finite type k-algebra. Let $k \subset K$ be a field extension, and set $S_K = K \otimes_k S$. Let $\mathfrak{q} \subset S$ be a prime of S. Let $\mathfrak{q}_K \subset S_K$ be a prime of S_K lying over $\mathfrak{q}$. Then $S_{\mathfrak{q}}$ is Cohen-Macaulay if and only if $(S_K)_{\mathfrak{q}_K}$ is Cohen-Macaulay.*

Proof. During the course of the proof we may (finitely many times) replace S by S_g for any $g \in S$, $g \notin \mathfrak{q}$. Hence using Lemma 10.109.5 we may assume that $\dim(S) = \dim_{\mathfrak{q}}(S/k) =: d$ and find a finite injective map $k[x_1, \ldots, x_d] \to S$. Note that this also induces a finite injective map $K[x_1, \ldots, x_d] \to S_K$ by base change. By Lemma 10.110.6 we have $\dim_{\mathfrak{q}_K}(S_K/K) = d$. Set $\mathfrak{p} = k[x_1, \ldots, x_d] \cap \mathfrak{q}$ and $\mathfrak{p}_K = K[x_1, \ldots, x_d] \cap \mathfrak{q}_K$. Consider the following commutative diagram of Noetherian local rings

$$\begin{array}{ccc} S_{\mathfrak{q}} & \longrightarrow & (S_K)_{\mathfrak{q}_K} \\ \uparrow & & \uparrow \\ k[x_1, \ldots, x_d]_{\mathfrak{p}} & \longrightarrow & K[x_1, \ldots, x_d]_{\mathfrak{p}_K} \end{array}$$

By Lemma 10.124.1 we have to show that the left vertical arrow is flat if and only if the right vertical arrow is flat. Because the bottom arrow is flat this equivalence holds by Lemma 10.94.1. □

Lemma 10.124.7. *Let R be a ring. Let $R \to S$ be of finite type. Let $R \to R'$ be any ring map. Set $S' = R' \otimes_R S$. Denote $f : \mathrm{Spec}(S') \to \mathrm{Spec}(S)$ the map associated to the ring map $S \to S'$. Set W equal to the set of primes $\mathfrak{q}$ such that the fibre ring $S_{\mathfrak{q}} \otimes_R \kappa(\mathfrak{p})$, $\mathfrak{p} = R \cap \mathfrak{q}$ is Cohen-Macaulay, and let W' denote the analogue for S'/R'. Then $W' = f^{-1}(W)$.*

Proof. Trivial from Lemma 10.124.6 and the definitions. □

Lemma 10.124.8. *Let R be a ring. Let $R \to S$ be a ring map which is (a) flat, (b) of finite presentation, (c) has Cohen-Macaulay fibres. Then $S = S_0 \times \ldots \times S_n$ is a product of rings S_d such that each S_d satisfies (a), (b), (c) and has all fibres equidimensional of dimension d.*

Proof. For each integer d denote $W_d \subset \mathrm{Spec}(S)$ the set defined in Lemma 10.124.4. Clearly we have $\mathrm{Spec}(S) = \coprod W_d$, and each W_d is open by the lemma we just quoted. Hence the result follows from Lemma 10.21.3. □

10.125. Differentials

In this section we define the module of differentials of a ring map.

Definition 10.125.1. Let $\varphi : R \to S$ be a ring map and let M be an S-module. A *derivation*, or more precisely an *R-derivation* into M is a map $D : S \to M$ which is additive, annihilates elements of $\varphi(R)$, and satisfies the *Leibniz rule*: $D(ab) = aD(b) + D(a)b$.

Note that $D(ra) = rD(a)$ if $r \in R$ and $a \in S$. The set of all R-derivations forms an S-module: Given two R-derivations D, D' the sum $D + D' : S \to M$, $a \mapsto D(a) + D'(a)$ is an R-derivation, and given an R-derivation D and an element $c \in S$ the scalar multiple $cD : S \to M$, $a \mapsto cD(a)$ is an R-derivation. We denote this S-module

$$\mathrm{Der}_R(S, M).$$

Also, if $\alpha : M \to N$ is an S-module map, then the composition $\alpha \circ D$ is an R-derivation into N. In this way the assignment $M \mapsto \mathrm{Der}_R(S, M)$ is a covariant functor.

Consider the following map of free S-modules

$$\bigoplus\nolimits_{(a,b) \in S^2} S[(a,b)] \oplus \bigoplus\nolimits_{(f,g) \in S^2} S[(f,g)] \oplus \bigoplus\nolimits_{r \in R} S[r] \longrightarrow \bigoplus\nolimits_{a \in S} S[a]$$

defined by the rules

$$[(a,b)] \longmapsto [a+b] - [a] - [b], \quad [(f,g)] \longmapsto [fg] - f[g] - g[f], \quad [r] \longmapsto [\varphi(r)]$$

with obvious notation. Let $\Omega_{S/R}$ be the cokernel of this map. There is a map $\mathrm{d} : S \to \Omega_{S/R}$ which maps a to the class $\mathrm{d}a$ of $[a]$ in the cokernel. This is an R-derivation by the relations imposed on $\Omega_{S/R}$, in other words

$$\mathrm{d}(a+b) = \mathrm{d}a + \mathrm{d}b, \quad \mathrm{d}(fg) = f\mathrm{d}g + g\mathrm{d}f, \quad \mathrm{d}r = 0$$

where $a, b, f, g \in S$ and $r \in R$.

Definition 10.125.2. The pair $(\Omega_{S/R}, \mathrm{d})$ is called the *module of Kähler differentials* or the *module of differentials* of S over R.

Lemma 10.125.3. *The module of differentials of S over R has the following universal property. The map*

$$\mathrm{Hom}_S(\Omega_{S/R}, M) \longrightarrow Der_R(S, M), \quad \alpha \longmapsto \alpha \circ d$$

is an isomorphism of functors.

Proof. By definition an R-derivation is a rule which associates to each $a \in S$ an element $D(a) \in M$. Thus D gives rise to a map $[D] : \bigoplus S[a] \to M$. However, the conditions of being an R-derivation exactly mean that $[D]$ annihilates the image of the map in the displayed presentation of $\Omega_{S/R}$ above. □

Lemma 10.125.4. *Let I be a directed partially ordered set. Let $(R_i \to S_i, \varphi_{ii'})$ be a system of ring maps over I, see Categories, Section 4.21. Then we have*

$$\Omega_{S/R} = \operatorname{colim}_i \Omega_{S_i/R_i}.$$

Proof. This is clear from the presentation of $\Omega_{S/R}$ given above. □

Lemma 10.125.5. *Suppose that $R \to S$ is surjective. Then $\Omega_{S/R} = 0$.*

Proof. You can see this either because all R-derivations clearly have to be zero, or because the map in the presentation of $\Omega_{S/R}$ is surjective. □

Suppose that

$$\begin{array}{ccc} S & \xrightarrow{\varphi} & S' \\ {\scriptstyle\alpha}\uparrow & & \uparrow{\scriptstyle\beta} \\ R & \xrightarrow{\psi} & R' \end{array} \tag{10.125.5.1}$$

is a commutative diagram of rings. In this case there is a natural map of modules of differentials fitting into the commutative diagram

$$\begin{array}{ccc} \Omega_{S/R} & \longrightarrow & \Omega_{S'/R'} \\ {\scriptstyle \mathrm{d}}\uparrow & & \uparrow{\scriptstyle \mathrm{d}} \\ S & \longrightarrow & S' \end{array}$$

To construct the map just use the obvious map between the presentations for $\Omega_{S/R}$ and $\Omega_{S'/R'}$. Namely,

$$\begin{array}{ccc} \bigoplus S'[(a',b')] \oplus \bigoplus S'[(f',g')] \oplus \bigoplus S'[r'] & \longrightarrow & \bigoplus S'[a'] \\ \uparrow & & \uparrow \\ \begin{array}{r}[(a,b)] \mapsto [(\varphi(a),\varphi(b)] \\ [(f,g)] \mapsto [(\varphi(f),\varphi(g)] \\ [r] \mapsto [\psi(r)]\end{array} & & [a]\mapsto[\varphi(a)] \\ \bigoplus S[(a,b)] \oplus \bigoplus S[(f,g)] \oplus \bigoplus S[r] & \longrightarrow & \bigoplus S[a] \end{array}$$

The result is simply that $f\mathrm{d}g \in \Omega_{S/R}$ is mapped to $\varphi(f)\mathrm{d}\varphi(g)$.

Lemma 10.125.6. *In diagram (10.125.5.1), suppose that $S \to S'$ is surjective with kernel $I \subset S$. Then $\Omega_{S/R} \to \Omega_{S'/R'}$ is surjective with kernel generated as an S-module by the elements the elements $\mathrm{d}a$, where $a \in S$ is such that $\varphi(a) \in \beta(R')$. (This includes in particular the elements $\mathrm{d}(i)$, $i \in I$.)*

Proof. Consider the map of presentations above. Clearly the right vertical map of free modules is surjective. Thus the map is surjective. A diagram chase shows that the following elements generate the kernel as an S-module for sure: $i\mathrm{d}a, i \in I, a \in S$, and $\mathrm{d}a$, with $a \in S$ such that $\varphi(a) = \beta(r')$ for some $r' \in R'$. Note that $\varphi(i) = \varphi(ia) = 0 = \beta(0)$, and that $\mathrm{d}(ia) = i\mathrm{d}a + a\mathrm{d}i$. Hence $i\mathrm{d}a = \mathrm{d}(ia) - a\mathrm{d}i$ is an S-linear combination of elements of the second kind. □

Lemma 10.125.7. *Let $A \to B \to C$ be ring maps. Then there is a canonical exact sequence*

$$C \otimes_B \Omega_{B/A} \to \Omega_{C/A} \to \Omega_{C/B} \to 0$$

of C-modules.

Proof. We get a diagram (10.125.5.1) by putting $R = A$, $S = C$, $R' = B$, and $S' = C$. By Lemma 10.125.6 the map $\Omega_{C/A} \to \Omega_{C/B}$ is surjective, and the kernel is generated by the elements $\mathrm{d}(c)$, where $c \in C$ is in the image of $B \to C$. The lemma follows. □

Lemma 10.125.8. *Let $\varphi : A \to B$ be a ring map.*

(1) *If $S \subset A$ is a multiplicative subset mapping to invertible elements of B, then $\Omega_{B/A} = \Omega_{B/S^{-1}A}$.*
(2) *If $S \subset B$ is a multiplicative subset then $S^{-1}\Omega_{B/A} = \Omega_{S^{-1}B/A}$.*

Proof. To show the equality of (1) it is enough to show that any A-derivation $D : B \to M$ annihilates the elements $\varphi(s)^{-1}$. This is clear from the Leibniz rule applied to $1 = \varphi(s)\varphi(s)^{-1}$. To show (2) note that there is an obvious map $S^{-1}\Omega_{B/A} \to \Omega_{S^{-1}B/A}$. To show it is an isomorphism it is enough to show that there is a A-derivation d' of $S^{-1}B$ into $S^{-1}\Omega_{B/A}$. To define it we simply set $\mathrm{d}'(b/s) = (1/s)\mathrm{d}b - (1/s^2)b\mathrm{d}s$. Details omitted. □

Lemma 10.125.9. *In diagram (10.125.5.1), suppose that $S \to S'$ is surjective with kernel $I \subset S$, and assume that $R' = R$. Then there is a canonical exact sequence of S'-modules*

$$I/I^2 \longrightarrow \Omega_{S/R} \otimes_S S' \longrightarrow \Omega_{S'/R} \longrightarrow 0$$

The leftmost map is characterized by the rule that $f \in I$ maps to $df \otimes 1$.

Proof. The middle term is $\Omega_{S/R} \otimes_S S/I$. For $f \in I$ denote $\overline{f}$ the image of f in I/I^2. To show that the map $\overline{f} \mapsto \mathrm{d}f \otimes 1$ is well defined we just have to check that $\mathrm{d}f_1 f_2 \otimes 1 = 0$ if $f_1, f_2 \in I$. And this is clear from the Leibniz rule $\mathrm{d}f_1 f_2 \otimes 1 = (f_1 \mathrm{d}f_2 + f_2 \mathrm{d}f_1) \otimes 1 = \mathrm{d}f_2 \otimes f_1 + \mathrm{d}f_1 \otimes f_2 = 0$. A similar computation show this map is $S' = S/I$-linear.

The map $\Omega_{S/R} \otimes_S S' \to \Omega_{S'/R}$ is the canonical S'-linear map associated to the S-linear map $\Omega_{S/R} \to \Omega_{S'/R}$. It is surjective because $\Omega_{S/R} \to \Omega_{S'/R}$ is surjective by Lemma 10.125.6.

The composite of the two maps is zero because $\mathrm{d}f$ maps to zero in $\Omega_{S'/R}$ for $f \in I$. Note that exactness just says that the kernel of $\Omega_{S/R} \to \Omega_{S'/R}$ is generated as an S-submodule by the submodule $I\Omega_{S/R}$ together with the elements $\mathrm{d}f$, with $f \in I$. We know by Lemma 10.125.6 that this kernel is generated by the elements $\mathrm{d}(a)$ where $\varphi(a) = \beta(r)$ for some $r \in R$. But then $a = \alpha(r) + a - \alpha(r)$, so $\mathrm{d}(a) = \mathrm{d}(a - \alpha(r))$. And $a - \alpha(r) \in I$ since $\varphi(a - \alpha(r)) = \varphi(a) - \varphi(\alpha(r)) = \beta(r) - \beta(r) = 0$. We conclude the elements $\mathrm{d}f$ with $f \in I$ already generate the kernel as an S-module, as desired. □

Lemma 10.125.10. *In diagram (10.125.5.1), suppose that $S \to S'$ is surjective with kernel $I \subset S$, and assume that $R' = R$. Moreover, assume that there exists an R-algebra map $S' \to S$ which is a right inverse to $S' \to S$. Then the exact sequence of S'-modules of Lemma 10.125.9 turns into a short exact sequence*

$$0 \longrightarrow I/I^2 \longrightarrow \Omega_{S/R} \otimes_S S' \longrightarrow \Omega_{S'/R} \longrightarrow 0$$

which is even a split short exact sequence.

Proof. Let $\beta : S' \to S$ be the right inverse to the surjection $\alpha : S \to S'$, so $S = I \oplus \beta(S')$. Clearly we can use $\beta : \Omega_{S'/R} \to \Omega_{S/R}$, to get a right inverse to the map $\Omega_{S/R} \otimes_S S' \to \Omega_{S'/R}$. On the other hand, consider the map

$$D : S \longrightarrow I/I^2, \quad x \longmapsto x - \beta(\alpha(x))$$

It is easy to show that D is an R-derivation (omitted). Moreover $xD(s) = 0$ if $x \in I, s \in S$. Hence, by the universal property D induces a map $\tau : \Omega_{S/R} \otimes_R S' \to I/I^2$. We omit the verification that it is a left inverse to $\mathrm{d} : I/I^2 \to \Omega_{S/R} \otimes_S S'$. Hence we win. □

Lemma 10.125.11. *Let $R \to S$ be a ring map. Let $I \subset S$ be an ideal. Let $n \geq 1$ be an integer. Set $S' = S/I^{n+1}$. The map $\Omega_{S/R} \to \Omega_{S'/R}$ induces an isomorphism*

$$\Omega_{S/R} \otimes_S S/I^n \longrightarrow \Omega_{S'/R} \otimes_{S'} S/I^n.$$

Proof. This follows from Lemma 10.125.9 and the fact that $\mathrm{d}(I^{n+1}) \subset I^n \Omega_{S/R}$ by the Leibniz rule for d. □

Lemma 10.125.12. *Suppose that we have ring maps $R \to R'$ and $R \to S$. Set $S' = S \otimes_R R'$, so that we obtain a diagram (10.125.5.1). Then the canonical map defined above induces an isomorphism $\Omega_{S/R} \otimes_R R' = \Omega_{S'/R'}$.*

Proof. Let $\mathrm{d}' : S' = S \otimes_R R' \to \Omega_{S/R} \otimes_R R'$ denote the map $\mathrm{d}'(\sum a_i \otimes x_i) = \mathrm{d}(a_i) \otimes x_i$. It exists because the map $S \times R' \to \Omega_{S/R} \otimes_R R'$, $(a, x) \mapsto \mathrm{d}a \otimes_R x$ is R-bilinear. This is an R'-derivation, as can be verified by a simple computation. We will show that $(\Omega_{S/R} \otimes_R R', \mathrm{d}')$ satisfies the universal property. Let $D : S' \to M'$ be an R' derivation into an S'-module. The composition $S \to S' \to M'$ is an R-derivation, hence we get an S-linear map $\varphi_D : \Omega_{S/R} \to M'$. We may tensor this with R' and get the map $\varphi'_D : \Omega_{S/R} \otimes_R R' \to M'$, $\varphi'_D(\eta \otimes x) = x\varphi_D(\eta)$. It is clear that $D = \varphi'_D \circ \mathrm{d}'$. □

The multiplication map $S \otimes_R S \to S$ is the R-algebra map which maps $a \otimes b$ to ab in S. It is also an S-algebra map, if we think of $S \otimes_R S$ as an S-algebra via either of the maps $S \to S \otimes_R S$.

Lemma 10.125.13. *Let $R \to S$ be a ring map. Let $J = Ker(S \otimes_R S \to S)$ be the kernel of the multiplication map. There is a canonical isomorphism of S-modules $\Omega_{S/R} \to J/J^2$, $a\mathrm{d}b \mapsto a \otimes b - ab \otimes 1$.*

Proof. First we show that the rule $a\mathrm{d}b \mapsto a \otimes b - ab \otimes 1$ is well defined. In order to do this we have to show that $\mathrm{d}r$ and $a\mathrm{d}b + b\mathrm{d}a - d(ab)$ map to zero. The first because $r \otimes 1 - 1 \otimes r = 0$ by definition of the tensor product. The second because $a \otimes b - ab \otimes 1 + b \otimes a - ba \otimes 1 = (a \otimes 1 - 1 \otimes a)(1 \otimes b - b \otimes 1)$ is in J^2.

We construct a map in the other direction. We may think of $S \to S \otimes_R S$, $a \mapsto a \otimes 1$ as the base change of $R \to S$. Hence we have $\Omega_{S \otimes_R S/S} = \Omega_{S/R} \otimes_S (S \otimes_R S)$, by Lemma 10.125.12. At this point the sequence of Lemma 10.125.9 gives a map

$$J/J^2 \to \Omega_{S \otimes_R S/S} \otimes_{S \otimes_R S} S = (\Omega_{S/R} \otimes_S (S \otimes_R S)) \otimes_{S \otimes_R S} S = \Omega_{S/R}.$$

We leave it to the reader to see it is the inverse of the map above. □

Lemma 10.125.14. *If $S = R[x_1, \ldots, x_n]$, then $\Omega_{S/R}$ is a finite free S-module with basis $dx_1, \ldots, dx_n$.*

Proof. We first show that $\mathrm{d}x_1, \ldots, \mathrm{d}x_n$ generate $\Omega_{S/R}$ as an S-module. To prove this we show that $\mathrm{d}g$ can be expressed as a sum $\sum g_i \mathrm{d}x_i$ for any $g \in R[x_1, \ldots, x_n]$. We do this by induction on the (total) degree of g. It is clear if the degree of g is 0, because then $\mathrm{d}g = 0$. If the degree of g is > 0, then we may write g as $c + \sum g_i x_i$ with $c \in R$ and $\deg(g_i) < \deg(g)$. By the Leibnize rule we have $\mathrm{d}g = \sum g_i \mathrm{d}x_i + \sum x_i \mathrm{d}g_i$, and hence we win by induction.

Consider the R-derivation $\partial/\partial x_i : R[x_1, \ldots, x_n] \to R[x_1, \ldots, x_n]$. (We leave it to the reader to define this; the defining property being that $\partial/\partial x_i(x_j) = \delta_{ij}$.) By the universal property this corresponds to an S-module map $l_i : \Omega_{S/R} \to R[x_1, \ldots, x_n]$ which maps $\mathrm{d}x_i$ to 1 and $\mathrm{d}x_j$ to 0 for $j \neq i$. Thus it is clear that there are no S-linear relations among the elements $\mathrm{d}x_1, \ldots, \mathrm{d}x_n$. □

Lemma 10.125.15. *Suppose $R \to S$ is of finite presentation. Then $\Omega_{S/R}$ is a finitely presented S-module.*

Proof. Write $S = R[x_1, \ldots, x_n]/(f_1, \ldots, f_m)$. Write $I = (f_1, \ldots, f_m)$. According to Lemma 10.125.9 there is an exact sequence of S-modules

$$I/I^2 \to \Omega_{R[x_1,\ldots,x_n]/R} \otimes_{R[x_1,\ldots,x_n]} S \to \Omega_{S/R} \to 0$$

The result follows from the fact that I/I^2 is a finite S-module (generated by the images of the f_i), and that the middle term is finite free by Lemma 10.125.14. □

Lemma 10.125.16. *Suppose $R \to S$ is of finite type. Then $\Omega_{S/R}$ is finitely generated S-module.*

Proof. This is very similar to, but easier than the proof of Lemma 10.125.15. □

10.126. Finite order differential operators

In this section we introduce differential operators of finite order.

Definition 10.126.1. Let $R \to S$ be a ring map. Let M, N be S-modules. Let $k \geq 0$ be an integer. We inductively define a *differential operator $D : M \to N$ of order k* to be an R-linear map such that for all $g \in S$ the map $m \mapsto D(gm) - gD(m)$ is a differential operator of order $k - 1$. For the base case $k = 0$ we define a differential operator of order 0 to be an S-linear map.

If $D : M \to N$ is a differential operator of order k, then for all $g \in S$ the map gD is a differential operator of order k. The sum of two differential operators of order k is another. Hence the set of all these

$$\mathrm{Diff}^k(M, N) = \mathrm{Diff}^k_{S/R}(M, N)$$

is an S-module. We have

$$\mathrm{Diff}^0(M, N) \subset \mathrm{Diff}^1(M, N) \subset \mathrm{Diff}^2(M, N) \subset \ldots$$

Lemma 10.126.2. *Let $R \to S$ be a ring map. Let L, M, N be S-modules. If $D : L \to M$ and $D' : M \to N$ are differential operators of order k and k', then $D' \circ D$ is a differential operator of order $k + k'$.*

Proof. Let $g \in S$. Then the map which sends $x \in L$ to

$$D'(D(gx)) - gD'(D(x)) = D'(D(gx)) - D'(gD(x)) + D'(gD(x)) - gD'(D(x))$$

is a sum of two compositions of differential operators of lower order. Hence the lemma follows by induction on $k + k'$. □

Lemma 10.126.3. *Let $R \to S$ be a ring map. Let M be an S-module. Let $k \geq 0$. There exists an S-module $P^k_{S/R}(M)$ and a canonical isomorphism*

$$Diff^k_{S/R}(M, N) = \mathrm{Hom}_S(P^k_{S/R}(M), N)$$

functorial in the S-module N.

Proof. The existence of $P^k_{S/R}(M)$ follows from general category theoretic arguments (insert future reference here), but we will also give a construction. Set $F = \bigoplus_{m \in M} S[m]$ where $[m]$ is a symbol indicating the basis element in the summand corresponding to m. Given any differential operator $D : M \to N$ we obtain an S-linear map $L_D : F \to N$ sending $[m]$ to $D(m)$. If D has order 0 then L_D annihilates the elements

$$[m + m'] - [m] - [m'], \quad g_0[m] - [g_0 m]$$

where $g_0 \in S$ and $m, m' \in M$. If D has order 1, then L_D annihilates the elements

$$[m + m' - [m] - [m'], \quad f[m] - [fm], \quad g_0 g_1[m] - g_0[g_1 m] - g_1[g_0 m] + [g_1 g_0 m]$$

where $f \in R$, $g_0, g_1 \in S$, and $m \in M$. If D has order k, then L_D annihilates the elements $[m + m'] - [m] - [m']$, $f[m] - [fm]$, and the elements

$$g_0 g_1 \ldots g_k[m] - \sum g_0 \ldots \hat{g}_i \ldots g_k[g_i m] + \ldots + (-1)^{k+1}[g_0 \ldots g_k m]$$

Conversely, if $L : F \to N$ is an S-linear map annihilating all the elements listed in the previous sentence, then $m \mapsto L([m])$ is a differential operator of order k. Thus we see that $P^k_{S/R}(M)$ is the quotient of F by the submodule generated by these elements. □

Definition 10.126.4. Let $R \to S$ be a ring map. Let M be an S-module. The module $P^k_{S/R}(M)$ constructed in Lemma 10.126.3 is called the *module of principal parts of order k* of M.

Note that the inclusions

$$\mathrm{Diff}^0(M, N) \subset \mathrm{Diff}^1(M, N) \subset \mathrm{Diff}^2(M, N) \subset \ldots$$

correspond via Yoneda's lemma (Categories, Lemma 4.3.5) to surjections

$$\ldots \to P^2_{S/R}(M) \to P^1_{S/R}(M) \to P^0_{S/R}(M) = M$$

Example 10.126.5. Let $R \to S$ be a ring map and let N be an S-module. Observe that $\mathrm{Diff}^1(S, N) = \mathrm{Der}_R(S, N) \oplus N$. Namely, if $D : S \to S$ is a differential operator of order 1 then $\sigma_D : g \mapsto D(g) - gD(1)$ is a derivation and $D = \sigma_D + \lambda_{D(1)}$ where $\lambda_x : S \to N$ is the linear map sending g to gx. It follows that $P^1_{S/R} = \Omega_{S/R} \oplus S$ by the universal property of $\Omega_{S/R}$.

Lemma 10.126.6. *Let $R \to S$ be a ring map. Let M be an S-module. There is a canonical short exact sequence*

$$0 \to \Omega_{S/R} \otimes_S M \to P^1_{S/R}(M) \to M \to 0$$

functorial in M called the sequence of principal parts.

Proof. The map $P^1_{S/R}(M) \to M$ is given above. Let N be an S-module and let $D : M \to N$ be a differential operator of order 1. For $m \in M$ the map

$$g \longmapsto D(gm) - gD(m)$$

is an R-derivation $S \to N$ by the axioms for differential operators of order 1. Thus it corresponds to a linear map $D_m : \Omega_{S/R} \to N$ determined by the rule $adb \mapsto aD(bm) - abD(m)$ (see Lemma 10.125.3). The map

$$\Omega_{S/R} \times M \longrightarrow N, \quad (\eta, m) \longmapsto D_m(\eta)$$

is S-bilinear (details omitted) and hence determines an S-linear map

$$\sigma_D : \Omega_{S/R} \otimes_S M \to N$$

In this way we obtain a map $\mathrm{Diff}^1(M, N) \to \mathrm{Hom}_S(\Omega_{S/R} \otimes_S M, N)$, $D \mapsto \sigma_D$ functorial in N. By the Yoneda lemma this corresponds a map $\Omega_{S/R} \otimes_S M \to P^1_{S/R}(M)$. It is immediate from the construction that this map is functorial in M. The sequence

$$\Omega_{S/R} \otimes_S M \to P^1_{S/R}(M) \to M \to 0$$

is exact because for every module N the sequence

$$0 \to \mathrm{Hom}_S(M, N) \to \mathrm{Diff}^1(M, N) \to \mathrm{Hom}_S(\Omega_{S/R} \otimes_S M, N)$$

is exact by inspection.

To see that $\Omega_{S/R} \otimes_S M \to P^1_{S/R}(M)$ is injective we argue as follows. Choose an exact sequence

$$0 \to M' \to F \to M \to 0$$

with F a free S-module. This induces an exact sequence

$$0 \to \mathrm{Diff}^1(M, N) \to \mathrm{Diff}^1(F, N) \to \mathrm{Diff}^1(M', N)$$

for all N. This proves that in the commutative diagram

$$\begin{array}{ccccccccc}
0 & \longrightarrow & \Omega_{S/R} \otimes_S M' & \longrightarrow & P^1_{S/R}(M') & \longrightarrow & M' & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & \Omega_{S/R} \otimes_S F & \longrightarrow & P^1_{S/R}(F) & \longrightarrow & F & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & \Omega_{S/R} \otimes_S M & \longrightarrow & P^1_{S/R}(M) & \longrightarrow & M & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
 & & 0 & & 0 & & 0 & &
\end{array}$$

the middle column is exact. The left column is exact by right exactness of $\Omega_{S/R} \otimes_S -$. By the snake lemma (see Section 10.4) it suffices to prove exactness on the left for the free module F. Using that $P^1_{S/R}(-)$ commutes with direct sums we reduce to the case $M = S$. This case is a consequence of the discussion in Example 10.126.5. □

Remark 10.126.7. Suppose given a commutative diagram of rings

$$\begin{array}{ccc} B & \longrightarrow & B' \\ \uparrow & & \uparrow \\ A & \longrightarrow & A' \end{array}$$

a B-module M, a B'-module M', and a B-linear map $M \to M'$. Then we get a compatible system of module maps

$$\begin{array}{ccccccc} \ldots & \longrightarrow & P^2_{B'/A'}(M') & \longrightarrow & P^1_{B'/A'}(M') & \longrightarrow & P^0_{B'/A'}(M') \\ & & \uparrow & & \uparrow & & \uparrow \\ \ldots & \longrightarrow & P^2_{B/A}(M) & \longrightarrow & P^1_{B/A}(M) & \longrightarrow & P^0_{B/A}(M) \end{array}$$

These maps are compatible with further composition of maps of this type. The easiest way to see this is to use the description of the modules $P^k_{B/A}(M)$ in terms of generators and relations in the proof of Lemma 10.126.3 but it can also be seen directly from the universal property of these modules. Moreover, these maps are compatible with the short exact sequences of Lemma 10.126.6.

10.127. The naive cotangent complex

Let $R \to S$ be a ring map. Denote $R[S]$ the polynomial ring whose variables are the elements $s \in S$. Let's denote $[s] \in R[S]$ the variable corresponding to $s \in S$. Thus $R[S]$ is a free R-module on the basis elements $[s_1] \ldots [s_n]$ where $s_1, \ldots, s_n$ is an unordered sequence of elements of S. There is a canonical surjection

$$R[S] \longrightarrow S, \quad [s] \longmapsto s \tag{10.127.0.1}$$

whose kernel we denote $I \subset R[S]$. It is a simple observation that I is generated by the elements $[s][s'] - [ss']$ and $[r] - r$. According to Lemma 10.125.9 there is a canonical map

$$I/I^2 \longrightarrow \Omega_{R[S]/R} \otimes_{R[S]} S \tag{10.127.0.2}$$

whose cokernel is canonically isomorphic to $\Omega_{S/R}$. Observe that the S-module $\Omega_{R[S]/R} \otimes_{R[S]} S$ is free on the generators $\mathrm{d}[s]$.

Definition 10.127.1. Let $R \to S$ be a ring map. The *naive cotangent complex* $NL_{S/R}$ is the chain complex (10.127.0.2)

$$NL_{S/R} = \big(I/I^2 \longrightarrow \Omega_{R[S]/R} \otimes_{R[S]} S\big)$$

with I/I^2 placed in (homological) degree 1 and $\Omega_{R[S]/R} \otimes_{R[S]} S$ placed in degree 0. We will denote $H_1(L_{S/R}) = H_1(NL_{S/R})$[6] the homology in degree 1.

Before we continue let us say a few words about the actual cotangent complex (Cotangent, Section 66.3). Given a ring map $R \to S$ there exists a canonical simplicial R-algebra $P_\bullet$ whose terms are polynomial algebras and which comes equipped with a canonical homotopy equivalence

$$P_\bullet \longrightarrow S$$

[6]This module is sometimes denoted $\Gamma_{S/R}$ in the literature.

The cotangent complex $L_{S/R}$ of S over R is defined as the chain complex associated to the cosimplicial module

$$\Omega_{P_\bullet/R} \otimes_{P_\bullet} S$$

The naive cotangent complex as defined above is canonically isomorphic to the truncation $\tau_{\leq 1} L_{S/R}$ (see Homology, Section 12.13 and Cotangent, Section 66.10). In particular, it is indeed the case that $H_1(NL_{S/R}) = H_1(L_{S/R})$ so our definition is compatible with the one using the cotangent complex. Moreover, $H_0(L_{S/R}) = H_0(NL_{S/R}) = \Omega_{S/R}$ as we've seen above.

Let $R \to S$ be a ring map. A *presentation of S over R* is a surjection $\alpha : P \to S$ of R-algebras where P is a polynomial algebra (on a set of variables). Often, when S is of finite type over R we will indicate this by saying: "Let $R[x_1, \ldots, x_n] \to S$ be a presentation of S/R", or "Let $0 \to I \to R[x_1, \ldots, x_n] \to S \to 0$ be a presentation of S/R" if we want to indicate that I is the kernel of the presentation. Note that the map $R[S] \to S$ used to define the naive cotangent complex is an example of a presentation.

Note that for every presentation α we obtain a two term chain complex of S-modules

$$NL(\alpha) : I/I^2 \longrightarrow \Omega_{P/R} \otimes_P S.$$

Here the term I/I^2 is placed in degree 1 and the term $\Omega_{P/R} \otimes S$ is placed in degree 0. The class of $f \in I$ in I/I^2 is mapped to $\mathrm{d}f \otimes 1$ in $\Omega_{P/R} \otimes S$. The cokernel of this complex is canonically $\Omega_{S/R}$, see Lemma 10.125.9. We call the complex $NL(\alpha)$ the *naive cotangent complex associated to the presentation $\alpha : P \to S$ of S/R*. Note that if $P = R[S]$ with its canonical surjection onto S, then we recover $NL_{S/R}$. If $P = R[x_1, \ldots, x_n]$ then will sometimes use the notation $I/I^2 \to \bigoplus_{i=1,\ldots,n} S\mathrm{d}x_i$ to denote this complex.

Suppose we are given a commutative diagram

$$\begin{array}{ccc} S & \xrightarrow{\phi} & S' \\ \uparrow & & \uparrow \\ R & \longrightarrow & R' \end{array} \tag{10.127.1.1}$$

of rings. Let $\alpha : P \to S$ be a presentation of S over R and let $\alpha : P' \to S'$ be a presentation of S' over R'. A *morphism of presentations from $\alpha : P \to S$ to $\alpha' : P' \to S'$* is defined to be an R-algebra map

$$\varphi : P \to P'$$

such that $\phi \circ \alpha = \alpha' \circ \varphi$. Note that in this case $\varphi(I) \subset I'$, where $I = \mathrm{Ker}(\alpha)$ and $I' = \mathrm{Ker}(\alpha')$. Thus φ induces a map of S-modules $I/I^2 \to I'/(I')^2$ and by functoriality of differentials also an S-module map $\Omega_{P/R} \otimes S \to \Omega_{P'/R'} \otimes S'$. These maps are compatible with the differentials of $NL(\alpha)$ and $NL(\alpha')$ and we obtain a map of naive cotangent complexes

$$NL(\alpha) \longrightarrow NL(\alpha').$$

It is often convenient to consider the induced map $NL(\alpha) \otimes_S S' \to NL(\alpha')$.

In the special case that $P = R[S]$ and $P' = R'[S']$ the map $\phi : S \to S'$ induces a canonical ring map $\varphi : P \to P'$ by the rule $[s] \mapsto [\phi(s)]$. Hence the construction

above determines a canonical(!) maps of chain complexes

$$NL_{S/R} \longrightarrow NL_{S'/R'}, \quad \text{and} \quad NL_{S/R} \otimes_S S' \longrightarrow NL_{S'/R'}$$

associated to the diagram (10.127.1.1). Note that this construction is compatible with composition: given a commutative diagram

$$\begin{array}{ccccc} S & \xrightarrow{\phi} & S' & \xrightarrow{\phi'} & S'' \\ \uparrow & & \uparrow & & \uparrow \\ R & \longrightarrow & R' & \longrightarrow & R'' \end{array}$$

we see that the composition of

$$NL_{S/R} \longrightarrow NL_{S'/R'} \longrightarrow NL_{S'/R'}$$

is the map $NL_{S/R} \to NL_{S''/R''}$ given by the outer square.

It turns out that $NL(\alpha)$ is homotopy equivalent to $NL_{S/R}$ and that the maps constructed above are well defined up to homotopy (homotopies of maps of complexes are discussed in Homology, Section 12.12 but we also spell out the exact meaning of the statements in the lemma below in its proof).

Lemma 10.127.2. *Suppose given a diagram (10.127.1.1). Let $\alpha : P \to S$ and $\alpha' : P' \to S'$ be presentations.*

1. *There exist a morphism of presentations from α to α'.*
2. *Any two morphisms of presentations induce homotopic morphisms of complexes $NL(\alpha) \to NL(\alpha')$.*
3. *The construction is compatible with compositions of morphisms of presentations (see proof for exact statement).*
4. *If $R \to R'$ and $S \to S'$ are isomorphisms, then for any map φ of presentations from α to α' the induced map $NL(\alpha) \to NL(\alpha')$ is a homotopy equivalence and a quasi-isomorphism.*

In particular, comparing α to the canonical presentation (10.127.0.1) we conclude there is a quasi-isomorphism $NL(\alpha) \to NL_{S/R}$ well defined up to homotopy and compatible with all functorialities (up to homotopy).

Proof. To construct a morphism φ from α to α', write $P = R[x_a, a \in A]$. Choose for every a an element $f_a \in P'$ such that $\alpha'(f_a) = \phi(\alpha(x_i))$. Let $\varphi : P = R[x_a, a \in A] \to P'$ be the unique R-algebra map such that $\varphi(x_a) = f_a$. This gives the morphism.

Let φ and φ' morphisms of presentations from α to α'. Let $I = \mathrm{Ker}(\alpha)$ and $I' = \mathrm{Ker}(\alpha')$. We have to construct the diagonal map h in the diagram

$$\begin{array}{ccc} I/I^2 & \xrightarrow{\mathrm{d}} & \Omega_{P/R} \otimes_P S \\ {\scriptstyle \varphi_1}\downarrow\downarrow{\scriptstyle \varphi_1'} & \swarrow_{h} & {\scriptstyle \varphi_0}\downarrow\downarrow{\scriptstyle \varphi_0'} \\ J/J^2 & \xrightarrow{\mathrm{d}} & \Omega_{P'/R'} \otimes_{P'} S' \end{array}$$

where the vertical maps are induced by φ, φ' such that

$$\varphi_1 - \varphi_1' = h \circ d \quad \text{and} \quad \varphi_0 - \varphi_0' = d \circ h$$

Consider the map $D = \varphi - \varphi' : P \to P'$. Since both φ and φ are compatible with α and α' we conclude that $\varphi - \varphi' : P \to I'$. Also $\varphi - \varphi'$ is R-linear and

$$(\varphi - \varphi')(fg) = \varphi(f)(\varphi - \varphi')(g) + (\varphi - \varphi')(f)\varphi'(g)$$

Hence the induced map $D : P \to I'/(I')^2$ is a R-derivation. Thus we obtain a canonical map $h : \Omega_{P/R} \otimes_P S \to I'/(I')^2$ such that $D = h \circ \mathrm{d}$. A calculation (omitted) shows that h is the desired homotopy.

Suppose that we have a commutative diagram

$$\begin{array}{ccccc} S & \xrightarrow{\phi} & S' & \xrightarrow{\phi'} & S'' \\ \uparrow & & \uparrow & & \uparrow \\ R & \longrightarrow & R' & \longrightarrow & R'' \end{array}$$

with finite type vertical arrows. Suppose that

(1) $\alpha : P \to S$,
(2) $\alpha' : P' \to S'$, and
(3) $\alpha'' : P'' \to S''$

are presentations. Suppose that

(1) $\varphi : P \to P$ is a morphism of presentations from α to α' and
(2) $\varphi' : P' \to P''$ is a morphism of presentations from α' to α''.

Then it is immediate that $\varphi' \circ \varphi : P \to P''$ is a morphism of presentations from α to α'' and that the induced map $NL(\alpha) \to NL(\alpha'')$ of naive cotangent complexes is the composition of the maps $NL(\alpha) \to NL(\alpha')$ and $NL(\alpha) \to NL(\alpha')$ induced by φ and φ'.

In the simple case of complexes with 2 terms a quasi-isomorphism is just a map that induces an isomorphism on both the cokernel and the kernel of the maps between the terms. Note that homotopic maps of 2 term complexes (as explained above) define the same maps on kernel and cokernel. Hence if φ is a map from a presentation α of S over R to itself, then the induced map $NL(\alpha) \to NL(\alpha)$ is a quasi-isomorphism being homotopic to the identity by part (2). To prove (4) in full generality, consider a morphism φ' from α' to α which exists by (1). The compositions $NL(\alpha) \to NL(\alpha') \to NL(\alpha)$ and $NL(\alpha') \to NL(\alpha) \to NL(\alpha')$ are homotopic to the identity maps by (3), hence these maps are homotopy equivalences by definition. It follows formally that both maps $NL(\alpha) \to NL(\alpha')$ and $NL(\alpha') \to NL(\alpha)$ are quasi-isomorphisms. Some details omitted. □

Lemma 10.127.3. *Let $A \to B$ be a polynomial algebra. Then $NL_{B/A}$ is homotopy equivalent to the chain complex $(0 \to \Omega_{B/A})$ with $\Omega_{B/A}$ in degree 0.*

Proof. Follows from Lemma 10.127.2 and the fact that $A \to B$ is a presentation of B over A with zero kernel. □

The following lemma is part of the motivation for introducing the naive cotangent complex. The cotangent complex extends this to a genuine long exact cohomology sequence. If $B \to C$ is a local complete intersection, then one can extend the sequence with a zero on the left, see More on Algebra, Lemma 15.22.6.

Lemma 10.127.4 (Jacobi-Zariski sequence)**.** *Let $A \to B \to C$ be ring maps. Choose a presentation $\alpha : A[x_s, s \in S] \to B$ with kernel I. Choose a presentation β :*

$B[y_t, t \in T] \to C$ with kernel J. Let $\gamma : A[x_s, y_t] \to C$ be the induced presentation of C with kernel K. Then we get a canonical commutative diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \Omega_{A[x_s]/A} \otimes C & \longrightarrow & \Omega_{A[x_s,y_t]/A} \otimes C & \longrightarrow & \Omega_{B[y_t]/B} \otimes C & \longrightarrow & 0 \\ & & \uparrow & & \uparrow & & \uparrow & & \\ & & I/I^2 \otimes C & \longrightarrow & K/K^2 & \longrightarrow & J/J^2 & \longrightarrow & 0 \end{array}$$

with exact rows. We get the following exact sequence of homology groups

$$H_1(NL_{B/A} \otimes_B C) \to H_1(L_{C/A}) \to H_1(L_{C/B}) \to C \otimes_B \Omega_{B/A} \to \Omega_{C/A} \to \Omega_{C/B} \to 0$$

of C-modules extending the sequence of Lemma 10.125.7. If $Tor_1^B(\Omega_{B/A}, C) = 0$, then $H_1(NL_{B/A} \otimes_B C) = H_1(L_{B/A}) \otimes_B C$.

Proof. The precise definition of the maps is omitted. The exactness of the top row follows as the $\mathrm{d}x_s$, $\mathrm{d}y_t$ form a basis for the middle module. The map γ factors

$$A[x_s, y_t] \to B[y_t] \to C$$

with surjective first arrow and second arrow equal to β. Thus we see that $K \to J$ is surjective. Moreover, the kernel of the first displayed arrow is $IA[x_s, y_t]$. Hence $I/I^2 \otimes C$ surjects onto the kernel of $J/J^2 \to K/K^2$. Finally, we can use Lemma 10.127.2 to identify the terms as homology groups of the naive cotangent complexes. The final assertion follows as the degree 0 term of the complex $NL_{B/A}$ is a free B-module. □

Remark 10.127.5. Let $A \to B$ and $\phi : B \to C$ be ring maps. Then the composition $NL_{B/A} \to NL_{C/A} \to NL_{C/B}$ is homotopy equivalent to zero. Namely, this composition is the functoriality of the naive cotangent complex for the square

$$\begin{array}{ccc} B & \xrightarrow{\phi} & C \\ \uparrow & & \uparrow \\ A & \longrightarrow & B \end{array}$$

Write $J = \mathrm{Ker}(B[C] \to C)$. An explicit homotopy is given by the map $\Omega_{A[B]/A} \otimes_A B \to J/J^2$ which maps the basis element $\mathrm{d}[b]$ to the class of $[\phi(b)] - b$ in J/J^2.

Lemma 10.127.6. *Let $A \to B$ be a surjective ring map with kernel I. Then $NL_{B/A}$ is homotopy equivalent to the chain complex $(I/I^2 \to 0)$ with I/I^2 in degree 1. In particular $H_1(L_{B/A}) = I/I^2$.*

Proof. Follows from Lemma 10.127.2 and the fact that $A \to B$ is a presentation of B over A. □

Lemma 10.127.7. *Let $A \to B \to C$ be ring maps. Assume $A \to C$ is surjective (so also $B \to C$ is). Denote $I = Ker(A \to C)$ and $J = Ker(B \to C)$. Then the sequence*

$$I/I^2 \to J/J^2 \to \Omega_{B/A} \otimes_B B/J \to 0$$

is exact.

Proof. Follows from Lemma 10.127.4 and the description of the naive cotangent complexes $NL_{C/B}$ and $NL_{C/A}$ in Lemma 10.127.6. □

Lemma 10.127.8 (Flat base change)**.** *Let $R \to S$ be a ring map. Let $\alpha : P \to S$ be a presentation. Let $R \to R'$ be a flat ring map. Let $\alpha' : P' \otimes_R R' \to S' = S \otimes_R R'$ be the induced presentation. Then $NL(\alpha) \otimes_R R' = NL(\alpha) \otimes_S S' = NL(\alpha')$. In particular, the canonical map*

$$NL_{S/R} \otimes_R R' \longrightarrow NL_{S \otimes_R R'/R'}$$

is a homotopy equivalence if $R \to R'$ is flat.

Proof. This is true because $\mathrm{Ker}(\alpha') = R' \otimes_R \mathrm{Ker}(\alpha)$ since $R \to R'$ is flat. □

Lemma 10.127.9. *Let $R_i \to S_i$ be a system of ring maps over the directed partially ordered set I. Set $R = \mathrm{colim}\, R_i$ and $S = \mathrm{colim}\, S_i$. Then $NL_{S/R} = \mathrm{colim}\, NL_{S_i/R_i}$.*

Proof. Recall that $NL_{S/R}$ is the complex $I/I^2 \to \bigoplus_{s \in S} S\mathrm{d}[s]$ where $I \subset R[S]$ is the kernel of the canonical presentation $R[S] \to S$. Now it is clear that $R[S] = \mathrm{colim}\, R_i[S_i]$ and similarly that $I = \mathrm{colim}\, I_i$ where $I_i = \mathrm{Ker}(R_i[S_i] \to S_i)$. Hence the lemma is clear. □

Lemma 10.127.10. *If $S \subset A$ is a multiplicative subset of A, then $NL_{S^{-1}A/A}$ is homotopy equivalent to the zero complex.*

Proof. Since $A \to S^{-1}A$ is flat we see that $NL_{S^{-1}A/A} \otimes_A S^{-1}A \to NL_{S^{-1}A/S^{-1}A}$ is a homotopy equivalence by flat base change (Lemma 10.127.8). Since the source of the arrow is isomorphic to $NL_{S^{-1}A/A}$ and the target of the arrow is zero (by Lemma 10.127.6) we win. □

Lemma 10.127.11. *Let $S \subset A$ is a multiplicative subset of A. Let $S^{-1}A \to B$ be a ring map. Then $NL_{B/A} \to NL_{B/S^{-1}A}$ is an homotopy equivalence.*

Proof. Choose a presentation $\alpha : P \to B$ of B over A. Then $\beta : S^{-1}P \to B$ is a presentation of B over $S^{-1}A$. A direct computation shows that we have $NL(\alpha) = NL(\beta)$ which proves the lemma as the naive cotangent complex is well defined up to homotopy by Lemma 10.127.2. □

Lemma 10.127.12. *Let $A \to B$ be a ring map. Let $g \in B$. Suppose $\alpha : P \to B$ is a presentation with kernel I. Then a presentation of B_g over A is the map*

$$\beta : P[x] \longrightarrow B_g$$

extending α and sending x to $1/b$. The kernel J of β is generated by I and the element $fx - 1$ where $f \in P$ is an element mapped to $g \in B$ by α. In this situation we have

1. $J/J^2 = (I/I^2)_g \oplus B_g(fx - 1)$,
2. $\Omega_{P[x]/A} \otimes_{P[x]} B_g = \Omega_{P/A} \otimes_P B_g \oplus B_g dx$,
3. $NL(\beta) = NL(\alpha) \otimes_B B_g \oplus (B_g \xrightarrow{g} B_g)$

Hence the canonical map $NL_{B/A} \otimes_B B_g \to NL_{B_g/A}$ is a homotopy equivalence.

Proof. This lemma proves itself. □

Lemma 10.127.13. *Let $A \to B$ be a ring map. Let $S \subset B$ be a multiplicative subset. The canonical map $NL_{B/A} \otimes_B S^{-1}B \to NL_{S^{-1}B/A}$ is a quasi-isomorphism.*

Proof. We have $S^{-1}B = \operatorname{colim}_{g \in S} B_g$ where we think of S as a directed partially ordered set (ordering by divisibility), see Lemma 10.9.9. By Lemma 10.127.12 each of the maps $NL_{B/A} \otimes_B B_g \to NL_{B_g/A}$ are quasi-isomorphisms. The lemma follows from Lemma 10.127.9. □

Lemma 10.127.14. *Let R be a ring. Let $A_1 \to A_0$, and $B_1 \to B_0$ be two two term complexes. Suppose that there exist morphisms of complexes $\varphi : A_\bullet \to B_\bullet$ and $\psi : B_\bullet \to A_\bullet$ such that $\varphi \circ \psi$ and $\psi \circ \varphi$ are homotopic to the identity maps. Then $A_1 \oplus B_0 \cong B_1 \oplus A_0$ as R-modules.*

Proof. Choose a map $h : A_0 \to B_1$ such that

$$\mathrm{id}_{A_1} - \psi_1 \circ \varphi_1 = h \circ d_A \text{ and } \mathrm{id}_{A_0} - \psi_0 \circ \varphi_0 = d_A \circ h.$$

Similarly, choose a map $h' : B_0 \to A_1$ such that

$$\mathrm{id}_{B_1} - \varphi_1 \circ \psi_1 = h \circ d_B \text{ and } \mathrm{id}_{B_0} - \varphi_0 \circ \psi_0 = d_B \circ h.$$

A trivial computation shows that

$$\begin{pmatrix} \mathrm{id}_{A_1} & -h' \circ \psi_1 + h \circ \psi_0 \\ 0 & \mathrm{id}_{B_0} \end{pmatrix} = \begin{pmatrix} \psi_1 & h \\ -d_B & \varphi_0 \end{pmatrix} \begin{pmatrix} \varphi_1 & -h' \\ d_A & \psi_0 \end{pmatrix}$$

This shows that both matrices on the right hand side are invertible and proves the lemma. □

Lemma 10.127.15. *Let $R \to S$ be a ring map of finite type. For any presentations $\alpha : R[x_1, \ldots, x_n] \to S$, and $\beta : R[y_1, \ldots, y_m] \to S$ we have*

$$I/I^2 \oplus S^{\oplus m} \cong J/J^2 \oplus S^{\oplus n}$$

as S-modules where $I = Ker(\alpha)$ and $J = Ker(\beta)$.

Proof. See Lemmas 10.127.2 and 10.127.14. □

Lemma 10.127.16. *Let $R \to S$ be a ring map of finite type. Let $g \in S$. For any presentations $\alpha : R[x_1, \ldots, x_n] \to S$, and $\beta : R[y_1, \ldots, y_m] \to S_g$ we have*

$$(I/I^2)_g \oplus S_g^{\oplus m} \cong J/J^2 \oplus S_g^{\oplus n}$$

as S_g-modules where $I = Ker(\alpha)$ and $J = Ker(\beta)$.

Proof. By Lemma 10.127.15, we see that it suffices to prove this for a single choice of α and β. Thus we may take β the presentation of Lemma 10.127.12 and the result is clear. □

10.128. Local complete intersections

The property of being a local complete intersection is somehow an intrinsic property of a Noetherian local ring. However, for the moment we just define this property for finite type algebras over a field.

Definition 10.128.1. Let k be a field. Let S be a finite type k-algebra.

(1) We say that S is a *global complete intersection over k* if there exists a presentation $S = k[x_1, \ldots, x_n]/(f_1, \ldots, f_c)$ such that $\dim(S) = n - c$.
(2) We say that S is a *local complete intersection over k* if there exists a covering $\operatorname{Spec}(S) = \bigcup D(g_i)$ such that each of the rings S_{g_i} is a global complete intersection over k.

We will also use the convention that the zero ring is a global complete intersection over k.

Suppose S is a global complete intersection $S = k[x_1, \ldots, x_n]/(f_1, \ldots, f_c)$ as in the definition. Recall that $\dim(S) = n - c$ means that all irreducible components of $\mathrm{Spec}(S)$ have dimension $\leq n - c$. Since all maximal ideals of the polynomial ring have local rings of dimension n we conclude that all irreducible components of $\mathrm{Spec}(S)$ have dimension $\geq n - c$. See Section 10.57. In other words, $\mathrm{Spec}(S)$ is equidimensional of dimension $n - c$.

Lemma 10.128.2. *Let k be a field. Let S be a finite type k-algebra. Let $g \in S$.*

1. *If S is a global complete intersection so is S_g.*
2. *If S is a local complete intersection so is S_g.*

Proof. The second statement follows immediately from the first. For the first, say that $S = k[x_1, \ldots, x_n]/(f_1, \ldots, f_c)$ with $n - c = \dim(S)$. By the remarks above S is equidimensional of dimension $n - c$, so $\dim(S_g) = n - c$ as well (or it is the zero ring in which case the lemma is true by convention). Let $g' \in k[x_1, \ldots, x_n]$ be an element whose residue class corresponds to g. Then $S_g = k[x_1, \ldots, x_n, x_{n+1}]/(f_1, \ldots, f_c, x_{n+1}g' - 1)$ as desired. □

Lemma 10.128.3. *Let k be a field. Let S be a finite type k-algebra. If S is a local complete intersection, then S is a Cohen-Macaulay ring.*

Proof. Choose a maximal prime $\mathfrak{m}$ of S. We have to show that $S_{\mathfrak{m}}$ is Cohen-Macaulay. By assumption we may assume $S = k[x_1, \ldots, x_n]/(f_1, \ldots, f_c)$ with $\dim(S) = n - c$. Let $\mathfrak{m}' \subset k[x_1, \ldots, x_n]$ be the maximal ideal corresponding to $\mathfrak{m}$. According to Proposition 10.108.2 the local ring $k[x_1, \ldots, x_n]_{\mathfrak{m}'}$ is regular local of dimension n. In particular it is Cohen-Macaulay by Lemma 10.101.3. By dimension theory (see Section 10.57) the ring $S_{\mathfrak{m}} = k[x_1, \ldots, x_n]_{\mathfrak{m}'}/(f_1, \ldots, f_c)$ has dimension $\geq n - c$. By assumption $\dim(S_{\mathfrak{m}}) \leq n - c$. Thus we get equality. This implies that $f_1, \ldots, f_c$ is a regular sequence in $k[x_1, \ldots, x_n]_{\mathfrak{m}'}$ and that $S_{\mathfrak{m}}$ is Cohen-Macaulay, see Proposition 10.98.4. □

The following is the technical key to the rest of the material in this section. An important feature of this lemma is that we may choose any presentation for the ring S, but that condition (1) does not depend on this choice.

Lemma 10.128.4. *Let k be a field. Let S be a finite type k-algebra. Let $\mathfrak{q}$ be a prime of S. Choose any presentation $S = k[x_1, \ldots, x_n]/I$. Let $\mathfrak{q}'$ be the prime of $k[x_1, \ldots, x_n]$ corresponding to $\mathfrak{q}$. Set $c = \mathrm{height}(\mathfrak{q}') - \mathrm{height}(\mathfrak{q})$, in other words $\dim_{\mathfrak{q}}(S) = n - c$ (see Lemma 10.110.4). The following are equivalent*

1. *There exists a $g \in S$, $g \notin \mathfrak{q}$ such that S_g is a global complete intersection over k.*
2. *The ideal $I_{\mathfrak{q}'} \subset k[x_1, \ldots, x_n]_{\mathfrak{q}'}$ can be generated by c elements.*
3. *The conormal module $(I/I^2)_{\mathfrak{q}}$ can be generated by c elements over $S_{\mathfrak{q}}$.*
4. *The conormal module $(I/I^2)_{\mathfrak{q}}$ is a free $S_{\mathfrak{q}}$-module of rank c.*
5. *The ideal $I_{\mathfrak{q}'}$ can be generated by a regular sequence in the regular local ring $k[x_1, \ldots, x_n]_{\mathfrak{q}'}$.*

In this case any c elements of $I_{\mathfrak{q}'}$ which generate $I_{\mathfrak{q}'}/\mathfrak{q}'I_{\mathfrak{q}'}$ form a regular sequence in the local ring $k[x_1, \ldots, x_n]_{\mathfrak{q}'}$.

Proof. Set $R = k[x_1, \ldots, x_n]_{\mathfrak{q}'}$. This is a Cohen-Macaulay local ring of dimension height($\mathfrak{q}'$), see for example Lemma 10.128.3. Moreover, $\overline{R} = R/IR = R/I_{\mathfrak{q}'} = S_{\mathfrak{q}}$ is a quotient of dimension height($\mathfrak{q}$). Let $f_1, \ldots, f_c \in I_{\mathfrak{q}'}$ be elements which generate $(I/I^2)_{\mathfrak{q}}$. By Lemma 10.18.1 we see that $f_1, \ldots, f_c$ generate $I_{\mathfrak{q}'}$. Since the dimensions work out, we conclude by Proposition 10.98.4 that $f_1, \ldots, f_c$ is a regular sequence in R. By Lemma 10.66.2 we see that $(I/I^2)_{\mathfrak{q}}$ is free. These arguments show that (2), (3), (4) are equivalent and that they imply the last statement of the lemma, and therefore they imply (5).

If (5) holds, say $I_{\mathfrak{q}'}$ is generated by a regular sequence of length e, then $\text{height}(\mathfrak{q}) = \dim(S_{\mathfrak{q}}) = \dim(k[x_1, \ldots, x_n]_{\mathfrak{q}'}) - e = \text{height}(\mathfrak{q}') - e$ by dimension theory, see Section 10.57. We conclude that $e = c$. Thus (5) implies (2).

We continue with the notation introduced in the first paragraph. For each f_i we may find $d_i \in k[x_1, \ldots, x_n]$, $d_i \notin \mathfrak{q}'$ such that $f_i' = d_i f_i \in k[x_1, \ldots, x_n]$. Then it is still true that $I_{\mathfrak{q}'} = (f_1', \ldots, f_c')R$. Hence there exists a $g' \in k[x_1, \ldots, x_n]$, $g' \notin \mathfrak{q}'$ such that $I_{g'} = (f_1', \ldots, f_c')$. Moreover, pick $g'' \in k[x_1, \ldots, x_n]$, $g'' \notin \mathfrak{q}'$ such that $\dim(S_{g''}) = \dim_{\mathfrak{q}} \text{Spec}(S)$. By Lemma 10.110.4 this dimension is equal to $n - c$. Finally, set g equal to the image of $g'g''$ in S. Then we see that

$$S_g \cong k[x_1, \ldots, x_n, x_{n+1}]/(f_1', \ldots, f_c', x_{n+1}g'g'' - 1)$$

and by our choice of g'' this ring has dimension $n - c$. Therefore it is a global complete intersection. Thus each of (2), (3), and (4) implies (1).

Assume (1). Let $S_g \cong k[y_1, \ldots, y_m]/(f_1, \ldots, f_t)$ be a presentation of S_g as a global complete intersection. Write $J = (f_1, \ldots, f_t)$. Let $\mathfrak{q}'' \subset k[y_1, \ldots, y_m]$ be the prime corresponding to $\mathfrak{q}S_g$. Note that $t = m - \dim(S_g) = \text{height}(\mathfrak{q}) - \text{height}(\mathfrak{q}'')$, see Lemma 10.110.4 for the last equality. As seen in the proof of Lemma 10.128.3 (and also above) the elements $f_1, \ldots, f_t$ form a regular sequence in the local ring $k[y_1, \ldots, y_m]_{\mathfrak{q}''}$. By Lemma 10.66.2 we see that $(J/J^2)_{\mathfrak{q}}$ is free of rank t. By Lemma 10.127.16 we have

$$J/J^2 \oplus S_g^n \cong (I/I^2)_g \oplus S_g^m$$

Thus $(I/I^2)_{\mathfrak{q}}$ is free of rank $t + n - m = m - \dim(S_g) + n - m = n - \dim(S_g) = \text{height}(\mathfrak{q}) - \text{height}(\mathfrak{q}') = c$. Thus we obtain (4). □

The result of Lemma 10.128.4 suggests the following definition.

Definition 10.128.5. Let k be a field. Let S be a local k-algebra essentially of finite type over k. We say S is a *complete intersection (over k)* if there exists a local k-algebra R and elements $f_1, \ldots, f_c \in \mathfrak{m}_R$ such that

(1) R is essentially of finite type over k,
(2) R is a regular local ring,
(3) $f_1, \ldots, f_c$ form a regular sequence in R, and
(4) $S \cong R/(f_1, \ldots, f_c)$ as k-algebras.

By the Cohen structure theorem (see Theorem 10.147.8) any complete Noetherian local ring may be written as the quotient of some regular complete local ring. Hence we may use the definition above to define the notion of a complete intersection ring for any complete Noetherian local ring. We will discuss this in Divided Power Algebra, Section 23.8. In the meantime the following lemma shows that such a definition makes sense.

Lemma 10.128.6. *Let $A \to B \to C$ be surjective local ring homomorphisms. Assume A and B are regular local rings. The following are equivalent*

(1) *$Ker(A \to C)$ is generated by a regular sequence,*
(2) *$Ker(A \to C)$ is generated by $\dim(A) - \dim(C)$ elements,*
(3) *$Ker(B \to C)$ is generated by a regular sequence, and*
(4) *$Ker(B \to C)$ is generated by $\dim(B) - \dim(C)$ elements.*

Proof. A regular local ring is Cohen-Macaulay, see Lemma 10.101.3. Hence the equivalences (1) $\Leftrightarrow$ (2) and (3) $\Leftrightarrow$ (4), see Proposition 10.98.4. By Lemma 10.101.4 the ideal $\mathrm{Ker}(A \to B)$ can be generated by $\dim(A) - \dim(B)$ elements. Hence we see that (4) implies (2).

It remains to show that (1) implies (4). We do this by induction on $\dim(A) - \dim(B)$. The case $\dim(A) - \dim(B) = 0$ is trivial. Assume $\dim(A) > \dim(B)$. Write $I = \mathrm{Ker}(A \to C)$ and $J = \mathrm{Ker}(A \to B)$. Note that $J \subset I$. Our assumption is that the minimal number of generators of I is $\dim(A) - \dim(C)$. Let $\mathfrak{m} \subset A$ be the maximal ideal. Consider the maps

$$J/\mathfrak{m}J \to I/\mathfrak{m}I \to \mathfrak{m}/\mathfrak{m}^2$$

By Lemma 10.101.4 and its proof the composition is injective. Take any element $x \in J$ which is not zero in $J/\mathfrak{m}J$. By the above and Nakayama's lemma x is an element of a minimal set of generators of I. Hence we may replace A by A/xA and I by I/xA which decreases both $\dim(A)$ and the minimal number of generators of I by 1. Thus we win. □

Lemma 10.128.7. *Let k be a field. Let S be a local k-algebra essentially of finite type over k. The following are equivalent:*

(1) *S is a complete intersection over k,*
(2) *for any surjection $R \to S$ with R a regular local ring essentially of finite presentation over k the ideal $Ker(R \to S)$ can be generated by a regular sequence,*
(3) *for some surjection $R \to S$ with R a regular local ring essentially of finite presentation over k the ideal $Ker(R \to S)$ can be generated by $\dim(R) - \dim(S)$ elements,*
(4) *there exists a global complete intersection A over k and a prime $\mathfrak{a}$ of A such that $S \cong A_{\mathfrak{a}}$, and*
(5) *there exists a local complete intersection A over k and a prime $\mathfrak{a}$ of A such that $S \cong A_{\mathfrak{a}}$.*

Proof. It is clear that (2) implies (1) and (1) implies (3). It is also clear that (4) implies (5). Let us show that (3) implies (4). Thus we assume there exists a surjection $R \to S$ with R a regular local ring essentially of finite presentation over k such that the ideal $\mathrm{Ker}(R \to S)$ can be generated by $\dim(R) - \dim(S)$ elements. We may write $R = (k[x_1, \ldots, x_n]/J)_{\mathfrak{q}}$ for some $J \subset k[x_1, \ldots, x_n]$ and some prime $\mathfrak{q} \subset k[x_1, \ldots, x_n]$ with $J \subset \mathfrak{q}$. Let $I \subset k[x_1, \ldots, x_n]$ be the kernel of the map $k[x_1, \ldots, x_n] \to S$ so that $S \cong (k[x_1, \ldots, x_n]/I)_{\mathfrak{q}}$. By assumption $(I/J)_{\mathfrak{q}}$ is generated by $\dim(R) - \dim(S)$ elements. We conclude that $I_{\mathfrak{q}}$ can be generated by $\dim(k[x_1, \ldots, x_n]_{\mathfrak{q}}) - \dim(S)$ elements by Lemma 10.128.6. From Lemma 10.128.4 we see that for some $g \in k[x_1, \ldots, x_n]$, $g \notin \mathfrak{q}$ the algebra $(k[x_1, \ldots, x_n]/I)_g$ is a global complete intersection and S is isomorphic to a local ring of it.

To finish the proof of the lemma we have to show that (5) implies (2). Assume (5) and let $\pi : R \to S$ be a surjection with R a regular local k-algebra essentially of finite type over k. By assumption we have $S = A_{\mathfrak{a}}$ for some local complete intersection A over k. Choose a presentation $R = (k[y_1, \ldots, y_m]/J)_{\mathfrak{q}}$ with $J \subset \mathfrak{q} \subset k[y_1, \ldots, y_m]$. We may and do assume that J is the kernel of the map $k[y_1, \ldots, y_m] \to R$. Let $I \subset k[y_1, \ldots, y_m]$ be the kernel of the map $k[y_1, \ldots, y_m] \to S = A_{\mathfrak{a}}$. Then $J \subset I$ and $(I/J)_{\mathfrak{q}}$ is the kernel of the surjection $\pi : R \to S$. So $S = (k[y_1, \ldots, y_m]/I)_{\mathfrak{q}}$.

By Lemma 10.120.7 we see that there exist $g \in A$, $g \notin \mathfrak{a}$ and $g' \in k[y_1, \ldots, y_m]$, $g' \notin \mathfrak{q}$ such that $A_g \cong (k[y_1, \ldots, y_m]/I)_{g'}$. After replacing A by A_g and $k[y_1, \ldots, y_m]$ by $k[y_1, \ldots, y_{m+1}]$ we may assume that $A \cong k[y_1, \ldots, y_m]/I$. Consider the surjective maps of local rings

$$k[y_1, \ldots, y_m]_{\mathfrak{q}} \to R \to S.$$

We have to show that the kernel of $R \to S$ is generated by a regular sequence. By Lemma 10.128.4 we know that $k[y_1, \ldots, y_m]_{\mathfrak{q}} \to A_{\mathfrak{a}} = S$ has this property (as A is a local complete intersection over k). We win by Lemma 10.128.6. □

Lemma 10.128.8. *Let k be a field. Let S be a finite type k-algebra. Let $\mathfrak{q}$ be a prime of S. The following are equivalent:*

1. *The local ring $S_{\mathfrak{q}}$ is a complete intersection ring (Definition 10.128.5).*
2. *There exists a $g \in S$, $g \notin \mathfrak{q}$ such that S_g is a local complete intersection over k.*
3. *There exists a $g \in S$, $g \notin \mathfrak{q}$ such that S_g is a global complete intersection over k.*
4. *For any presentation $S = k[x_1, \ldots, x_n]/I$ with $\mathfrak{q}' \subset k[x_1, \ldots, x_n]$ corresponding to $\mathfrak{q}$ any of the equivalent conditions (1) – (5) of Lemma 10.128.4 hold.*

Proof. This is a combination of Lemmas 10.128.4 and 10.128.7 and the definitions. □

Lemma 10.128.9. *Let k be a field. Let S be a finite type k-algebra. The following are equivalent:*

1. *The ring S is a local complete intersection over k.*
2. *All local rings of S are complete intersection rings over k.*
3. *All localizations of S at maximal ideals are complete intersection rings over k.*

Proof. This follows from Lemma 10.128.8, the fact that $\mathrm{Spec}(S)$ is quasi-compact and the definitions. □

The following lemma says that being a complete intersection is preserved under change of base field (in a strong sense).

Lemma 10.128.10. *Let $k \subset K$ be a field extension. Let S be a finite type algebra over k. Let $\mathfrak{q}_K$ be a prime of $S_K = K \otimes_k S$ and let $\mathfrak{q}$ be the corresponding prime of S. Then $S_{\mathfrak{q}}$ is a complete intersection over k (Definition 10.128.5) if and only if $(S_K)_{\mathfrak{q}_K}$ is a complete intersection over K.*

Proof. Choose a presentation $S = k[x_1, \ldots, x_n]/I$. This gives a presentation $S_K = K[x_1, \ldots, x_n]/I_K$ where $I_K = K \otimes_k I$. Let $\mathfrak{q}'_K \subset K[x_1, \ldots, x_n]$, resp.

$\mathfrak{q}' \subset k[x_1, \ldots, x_n]$ be the corresponding prime. We will show that the equivalent conditions of Lemma 10.128.4 hold for the pair $(S = k[x_1, \ldots, x_n]/I, \mathfrak{q})$ if and only if they hold for the pair $(S_K = K[x_1, \ldots, x_n]/I_K, \mathfrak{q}_K)$. The lemma will follow from this (see Lemma 10.128.8).

By Lemma 10.110.6 we have $\dim_{\mathfrak{q}} S = \dim_{\mathfrak{q}_K} S_K$. Hence the integer c occurring in Lemma 10.128.4 is the same for the pair $(S = k[x_1, \ldots, x_n]/I, \mathfrak{q})$ as for the pair $(S_K = K[x_1, \ldots, x_n]/I_K, \mathfrak{q}_K)$. On the other hand we have

$$\begin{aligned} I \otimes_{k[x_1,\ldots,x_n]} \kappa(\mathfrak{q}') \otimes_{\kappa(\mathfrak{q}')} \kappa(\mathfrak{q}'_K) &= I \otimes_{k[x_1,\ldots,x_n]} \kappa(\mathfrak{q}'_K) \\ &= I \otimes_{k[x_1,\ldots,x_n]} K[x_1, \ldots, x_n] \otimes_{K[x_1,\ldots,x_n]} \kappa(\mathfrak{q}'_K) \\ &= (K \otimes_k I) \otimes_{K[x_1,\ldots,x_n]} \kappa(\mathfrak{q}'_K) \\ &= I_K \otimes_{K[x_1,\ldots,x_n]} \kappa(\mathfrak{q}'_K). \end{aligned}$$

Therefore, $\dim_{\kappa(\mathfrak{q}')} I \otimes_{k[x_1,\ldots,x_n]} \kappa(\mathfrak{q}') = \dim_{\kappa(\mathfrak{q}'_K)} I_K \otimes_{K[x_1,\ldots,x_n]} \kappa(\mathfrak{q}'_K)$. Thus it follows from Nakayama's Lemma 10.18.1 that the minimal number of generators of $I_{\mathfrak{q}'}$ is the same as the minimal number of generators of $(I_K)_{\mathfrak{q}'_K}$. Thus the lemma follows from characterization (2) of Lemma 10.128.4. □

Lemma 10.128.11. *Let $k \to K$ be a field extension. Let S be a finite type k-algebra. Then S is a local complete intersection over k if and only if $S \otimes_k K$ is a local complete intersection over K.*

Proof. This follows from a combination of Lemmas 10.128.9 and 10.128.10. But we also give a different proof here (based on the same principles).

Set $S' = S \otimes_k K$. Let $\alpha : k[x_1, \ldots, x_n] \to S$ be a presentation with kernel I. Let $\alpha' : K[x_1, \ldots, x_n] \to S'$ be the induced presentation with kernel I'.

Suppose that S is a local complete intersection. Pick a prime $\mathfrak{q} \subset S'$. Denote $\mathfrak{q}'$ the corresponding prime of $K[x_1, \ldots, x_n]$, $\mathfrak{p}$ the corresponding prime of S, and $\mathfrak{p}'$ the corresponding prime of $k[x_1, \ldots, x_n]$. Consider the following diagram of Noetherian local rings

$$\begin{array}{ccc} S'_{\mathfrak{q}} & \longleftarrow & K[x_1, \ldots, x_n]_{\mathfrak{q}'} \\ \uparrow & & \uparrow \\ S_{\mathfrak{p}} & \longleftarrow & k[x_1, \ldots, x_n]_{\mathfrak{p}'} \end{array}$$

By Lemma 10.128.4 we know that $S_{\mathfrak{p}}$ is cut out by some regular sequence $f_1, \ldots, f_e$ in $k[x_1, \ldots, x_n]_{\mathfrak{p}'}$. Since the right vertical arrow is flat we see that the images of $f_1, \ldots, f_c$ form a regular sequence in $K[x_1, \ldots, x_n]_{\mathfrak{q}'}$. Because tensoring with K over k is an exact functor we have $S'_{\mathfrak{q}} = K[x_1, \ldots, x_n]_{\mathfrak{q}'}/(f_1, \ldots, f_e)$. Hence by Lemma 10.128.4 again we see that S' is a local complete intersection in a neighbourhood of $\mathfrak{q}$. Since $\mathfrak{q}$ was arbitrary we see that S' is a local complete intersection over K.

Suppose that S' is a local complete intersection. Pick a maximal ideal $\mathfrak{m}$ of S. Let $\mathfrak{m}'$ denote the corresponding maximal ideal of $k[x_1, \ldots, x_n]$. Denote $\kappa = \kappa(\mathfrak{m})$ the residue field. By Remark 10.16.8 the primes of S' lying over $\mathfrak{m}$ correspond to primes in $K \otimes_k \kappa$. By the Hilbert-Nullstellensatz Theorem 10.31.1 we have $[\kappa : k] < \infty$. Hence $K \otimes_k \kappa$ is finite nonzero over K. Hence $K \otimes_k \kappa$ has a finite number > 0 of primes which are all maximal, each of which has a residue field finite over K (see Section 10.49). Hence there are finitely many > 0 prime ideals

$\mathfrak{n} \subset S'$ lying over $\mathfrak{m}$, each of which is maximal and has a residue field which is finite over K. Pick one, say $\mathfrak{n} \subset S'$, and let $\mathfrak{n}' \subset K[x_1, \ldots, x_n]$ denote the corresponding prime ideal of $K[x_1, \ldots, x_n]$. Note that since $V(\mathfrak{m}S')$ is finite, we see that $\mathfrak{n}$ is an isolated closed point of it, and we deduce that $\mathfrak{m}S'_{\mathfrak{n}}$ is an ideal of definition of $S'_{\mathfrak{n}}$. This implies that $\dim(S_{\mathfrak{m}}) \geq \dim(S'_{\mathfrak{n}})$, for example by Lemma 10.106.6 or by the characterization of dimension in terms of minimal number of generators of ideal of definition, see Section 10.57. (In reality the dimensions are equal but we do not need this.) Consider the corresponding diagram of Noetherian local rings

$$\begin{array}{ccc} S'_{\mathfrak{n}} & \longleftarrow & K[x_1, \ldots, x_n]_{\mathfrak{n}'} \\ \uparrow & & \uparrow \\ S_{\mathfrak{m}} & \longleftarrow & k[x_1, \ldots, x_n]_{\mathfrak{m}'} \end{array}$$

According to Lemma 10.127.8 we have $NL(\alpha) \otimes_S S' = NL(\alpha')$, in particular $I'/(I')^2 = I/I^2 \otimes_S S'$. Thus $(I/I^2)_{\mathfrak{m}} \otimes_{S_{\mathfrak{m}}} \kappa$ and $(I'/(I')^2)_{\mathfrak{n}} \otimes_{S'_{\mathfrak{n}}} \kappa(\mathfrak{n})$ have the same dimension. Since $(I'/(I')^2)_{\mathfrak{n}}$ is free of rank $n - \dim S'_{\mathfrak{n}}$ we deduce that $(I/I^2)_{\mathfrak{m}}$ can be generated by $n - \dim S'_{\mathfrak{n}} \leq n - \dim S_{\mathfrak{m}}$ elements. By Lemma 10.128.4 we see that S is a local complete intersection in a neighbourhood of $\mathfrak{m}$. Since $\mathfrak{m}$ was any maximal ideal we conclude that S is a local complete intersection. □

We end with a lemma which we will later use to prove that given ring maps $T \to A \to B$ where B is syntomic over T, and B is syntomic over A, then A is syntomic over T.

Lemma 10.128.12. *Let*

$$\begin{array}{ccc} B & \longleftarrow & S \\ \uparrow & & \uparrow \\ A & \longleftarrow & R \end{array}$$

be a commutative square of local rings. Assume

(1) *R and $\overline{S} = S/\mathfrak{m}_R S$ are regular local rings,*
(2) *$A = R/I$ and $B = S/J$ for some ideals I, J,*
(3) *$J \subset S$ and $\overline{J} = J/\mathfrak{m}_R \cap J \subset \overline{S}$ are generated by regular sequences, and*
(4) *$A \to B$ and $R \to S$ are flat.*

Then I is generated by a regular sequence.

Proof. Set $\overline{B} = B/\mathfrak{m}_R B = B/\mathfrak{m}_A B$ so that $\overline{B} = \overline{S}/\overline{J}$. Let $f_1, \ldots, f_{\overline{c}} \in J$ be elements such that $\overline{f}_1, \ldots, \overline{f}_{\overline{c}} \in \overline{J}$ form a regular sequence generating $\overline{J}$. Note that $\overline{c} = \dim(\overline{S}) - \dim(\overline{B})$, see Lemma 10.128.6. By Lemma 10.93.3 the ring $S/(f_1, \ldots, f_{\overline{c}})$ is flat over R. Hence $S/(f_1, \ldots, f_{\overline{c}}) + IS$ is flat over A. The map $S/(f_1, \ldots, f_{\overline{c}}) + IS \to B$ is therefore a surjection of finite S/IS-modules flat over A which is an isomorphism modulo $\mathfrak{m}_A$, and hence an isomorphism by Lemma 10.93.1. In other words, $J = (f_1, \ldots, f_{\overline{c}}) + IS$.

By Lemma 10.128.6 again the ideal J is generated by a regular sequence of $c = \dim(S) - \dim(B)$ elements. Hence $J/\mathfrak{m}_S J$ is a vector space of dimension c. By the description of J above there exist $g_1, \ldots, g_{c-\overline{c}} \in I$ such that J is generated by $f_1, \ldots, f_{\overline{c}}, g_1, \ldots, g_{c-\overline{c}}$ (use Nakayama's Lemma 10.18.1). Consider the ring $A' = R/(g_1, \ldots, g_{c-\overline{c}})$ and the surjection $A' \to A$. We see from the above that $B = S/(f_1, \ldots, f_{\overline{c}}, g_1, \ldots, g_{c-\overline{c}})$ is flat over A' (as $S/(f_1, \ldots, f_{\overline{c}})$ is flat over R). Hence

$A' \to B$ is injective (as it is faithfully flat, see Lemma 10.36.16). Since this map factors through A we get $A' = A$. Note that $\dim(B) = \dim(A) + \dim(\overline{B})$, and $\dim(S) = \dim(R) + \dim(\overline{S})$, see Lemma 10.106.7. Hence $c - \overline{c} = \dim(R) - \dim(A)$ by elementary algebra. Thus $I = (g_1, \ldots, g_{c-\overline{c}})$ is generated by a regular sequence according to Lemma 10.128.6. □

10.129. Syntomic morphisms

Syntomic ring maps are flat finitely presented ring maps all of whose fibers are local complete intersections. We discuss general local complete intersection ring maps in More on Algebra, Section 15.22.

Definition 10.129.1. A ring map $R \to S$ is called *syntomic*, or we say S is a *flat local complete intersection over* R if it is flat, of finite presentation, and if all of its fibre rings $S \otimes_R \kappa(\mathfrak{p})$ are local complete intersections, see Definition 10.128.1.

Clearly, an algebra over a field is syntomic over the field if and only if it is a local complete intersection. Here is a pleasing feature of this definition.

Lemma 10.129.2. *Let $R \to S$ be a ring map. Let $R \to R'$ be a faithfully flat ring map. Set $S' = R' \otimes_R S$. Then $R \to S$ is syntomic if and only if $R' \to S'$ is syntomic.*

Proof. By Lemma 10.120.2 and Lemma 10.36.7 this holds for the property of being flat and for the property of being of finite presentation. The map $\mathrm{Spec}(R') \to \mathrm{Spec}(R)$ is surjective, see Lemma 10.36.15. Thus it suffices to show given primes $\mathfrak{p}' \subset R'$ lying over $\mathfrak{p} \subset R$ that $S \otimes_R \kappa(\mathfrak{p})$ is a local complete intersection if and only if $S' \otimes_{R'} \kappa(\mathfrak{p}')$ is a local complete intersection. Note that $S' \otimes_{R'} \kappa(\mathfrak{p}') = S \otimes_R \kappa(\mathfrak{p}) \otimes_{\kappa(\mathfrak{p})} \kappa(\mathfrak{p}')$. Thus Lemma 10.128.11 applies. □

Lemma 10.129.3. *Any base change of a syntomic map is syntomic.*

Proof. This is true for being flat, for being of finite presentation, and for having local complete intersections as fibres by Lemmas 10.36.6, 10.6.2 and 10.128.11. □

Lemma 10.129.4. *Let $R \to S$ be a ring map. Suppose we have $g_1, \ldots g_m \in S$ which generate the unit ideal such that each $R \to S_{g_i}$ is syntomic. Then $R \to S$ is syntomic.*

Proof. This is true for being flat and for being of finite presentation by Lemmas 10.36.19 and 10.22.3. The property of having fibre rings which are local complete intersections is local on S by its very definition, see Definition 10.128.1. □

Definition 10.129.5. Let $R \to S$ be a ring map. We say that $R \to S$ is a *relative global complete intersection* if we are given a presentation $S = R[x_1, \ldots, x_n]/(f_1, \ldots, f_c)$ such that every nonempty fibre has dimension $n - c$.

The following lemma is occasionally useful to find global presentations.

Lemma 10.129.6. *Let S be a finitely presented R-algebra which has a presentation $S = R[x_1, \ldots, x_n]/I$ such that I/I^2 is free over S. Then S has a presentation $S = R[y_1, \ldots, y_m]/(f_1, \ldots, f_c)$ such that $(f_1, \ldots, f_c)/(f_1, \ldots, f_c)^2$ is free with basis given by the classes of $f_1, \ldots, f_c$.*

Proof. Note that I is a finitely generated ideal by Lemma 10.6.3. Let $f_1, \ldots, f_c \in I$ be elements which map to a basis of I/I^2. By Nakayama's lemma (Lemma 10.18.1) there exists a $g \in 1 + I$ such that

$$g \cdot I \subset (f_1, \ldots, f_c)$$

Hence we see that

$$S \cong R[x_1, \ldots, x_n]/(f_1, \ldots, f_c)[1/g] \cong R[x_1, \ldots, x_n, x_{n+1}]/(f_1, \ldots, f_c, gx_{n+1} - 1)$$

as desired. □

Example 10.129.7. Let $n, m \geq 1$ be integers. Consider the ring map

$$\begin{array}{rcl} R = \mathbf{Z}[a_1, \ldots, a_{n+m}] & \longrightarrow & S = \mathbf{Z}[b_1, \ldots, b_n, c_1, \ldots, c_m] \\ a_1 & \longmapsto & b_1 + c_1 \\ a_2 & \longmapsto & b_2 + b_1c_1 + c_2 \\ \ldots & \ldots & \ldots \\ a_{n+m} & \longmapsto & b_nc_m \end{array}$$

In other words, this is the unique ring map of polynomial rings as indicated such that the polynomial factorization

$$x^n + a_1x^{n-1} + \ldots + a_{n+m} = (x^n + b_1x^{n-1} + \ldots + b_n)(x^m + c_1x^{m-1} + \ldots + c_m)$$

holds. Note that S is generated by $n + m$ elements over R (namely, b_i, c_j) and that there are $n + m$ equations (namely $a_k = a_k(b_i, c_j)$). In order to show that S is a relative global complete intersection over R it suffices to prove that all fibres have dimension 0.

To prove this, let $R \to k$ be a ring map into a field k. Say a_i maps to $\alpha_i \in k$. Consider the fibre ring $S_k = k \otimes_R S$. Let $k \to K$ be a field extension. A k-algebra map of $S_k \to K$ is the same thing as finding $\beta_1, \ldots, \beta_n, \gamma_1, \ldots, \gamma_m \in K$ such that

$$x^n + \alpha_1x^{n-1} + \ldots + \alpha_{n+m} = (x^n + \beta_1x^{n-1} + \ldots + \beta_n)(x^m + \gamma_1x^{m-1} + \ldots + \gamma_m).$$

Hence we see there are at most finitely many choices of such $n+m$-tuples in K. This proves that all fibres have finitely many closed points (use Hilbert's Nullstellensatz to see they all correspond to solutions in $\overline{k}$ for example) and hence that $R \to S$ is a relative global complete intersection.

Another way to argue this is to show $\mathbf{Z}[a_1, \ldots, a_{n+m}] \to \mathbf{Z}[b_1, \ldots, b_n, c_1, \ldots, c_m]$ is actually also a *finite* ring map. Namely, by Lemma 10.35.5 each of b_i, c_j is integral over R, and hence $R \to S$ is finite by Lemma 10.33.4.

Example 10.129.8. Consider the ring map

$$\begin{array}{rcl} R = \mathbf{Z}[a_1, \ldots, a_n] & \longrightarrow & S = \mathbf{Z}[\alpha_1, \ldots, \alpha_n] \\ a_1 & \longmapsto & \alpha_1 + \ldots + \alpha_n \\ \ldots & \ldots & \ldots \\ a_n & \longmapsto & \alpha_1 \ldots \alpha_n \end{array}$$

In other words this is the unique ring map of polynomial rings as indicated such that

$$x^n + a_1x^{n-1} + \ldots + a_n = \prod\nolimits_{i=1}^{n} (x + \alpha_i)$$

holds in $\mathbf{Z}[\alpha_i, x]$. Another way to say this is that a_i maps to the ith elementary symmetric function in $\alpha_1, \ldots, \alpha_n$. Note that S is generated by n elements over R subject to n equations. Hence to show that S is a global relative complete intersection over R we have to show that the fibre rings $S \otimes_R \kappa(\mathfrak{p})$ have dimension 0. This follows as in Example 10.129.7 because the ring map $\mathbf{Z}[a_1, \ldots, a_n] \to \mathbf{Z}[\alpha_1, \ldots, \alpha_n]$ is actually *finite* since each $\alpha_i \in S$ satisfies the monic equation $x^n - a_1 x^{n-1} + \ldots + (-1)^n a_n$ over R.

Lemma 10.129.9. *Suppose that A is a ring, and $P(x) = x^n + b_1 x^{n-1} + \ldots + b_n \in A[x]$ is a monic polynomial over A. Then there exists a syntomic, finite locally free, faithfully flat ring extension $A \subset A'$ such that $P(x) = \prod_{i=1,\ldots,n}(x - \beta_i)$ for certain $\beta_i \in A'$.*

Proof. Take $A' = A \otimes_R S$, where R and S are as in Example 10.129.8, where $R \to A$ maps a_i to b_i, and let $\beta_i = -1 \otimes \alpha_i$. □

Lemma 10.129.10. *Let $S = R[x_1, \ldots, x_n]/(f_1, \ldots, f_c)$ be a relative global complete intersection over R.*

1. *For any $R \to R'$ the base change $R' \otimes_R S = R'[x_1, \ldots, x_n]/(f_1, \ldots, f_c)$ is a relative global complete intersection.*
2. *For any $g \in S$ which is the image of $h \in R[x_1, \ldots, x_n]$ the ring $S_g = R[x_1, \ldots, x_n, x_{n+1}]/(f_1, \ldots, f_c, hx_{n+1} - 1)$ is a relative global complete intersection.*
3. *If $R \to S$ factors as $R \to R_f \to S$ for some $f \in R$. Then the ring $S = R_f[x_1, \ldots, x_n]/(f_1, \ldots, f_c)$ is a relative global complete intersection over R_f.*

Proof. By Lemma 10.110.5 the fibres of a base change have the same dimension as the fibres of the original map. Moreover $R' \otimes_R R[x_1, \ldots, x_n]/(f_1, \ldots, f_c) = R'[x_1, \ldots, x_n]/(f_1, \ldots, f_c)$. Thus (1) follows. The proof of (2) is that the localization at one element can be described as $S_g \cong S[x_{n+1}]/(gx_{n+1} - 1)$. Assertion (3) follows from (1) since under the assumptions of (3) we have $R_f \otimes_R S \cong S$. □

Lemma 10.129.11. *Let R be a ring. Let $S = R[x_1, \ldots, x_n]/(f_1, \ldots, f_c)$. We will find $h \in R[x_1, \ldots, x_n]$ which maps to $g \in S$ such that*

$$S_g = R[x_1, \ldots, x_n, x_{n+1}]/(f_1, \ldots, f_c, hx_{n+1} - 1)$$

is a relative global complete intersection over R in each of the following cases:

1. *Let $I \subset R$ be an ideal. If the fibres of $\mathrm{Spec}(S/IS) \to \mathrm{Spec}(R/I)$ have dimension $n - c$, then we can find (h, g) as above such that g maps to $1 \in S/IS$.*
2. *Let $\mathfrak{p} \subset R$ be a prime. If $\dim(S \otimes_R \kappa(\mathfrak{p})) = n - c$, then we can find (h, g) as above such that g maps to a unit of $S \otimes_R \kappa(\mathfrak{p})$.*
3. *Let $\mathfrak{q} \subset S$ be a prime lying over $\mathfrak{p} \subset R$. If $\dim_{\mathfrak{q}}(S/R) = n - c$, then find (h, g) as above such that $g \notin \mathfrak{q}$.*

Proof. Ad (1). By Lemma 10.119.6 there exists an open subset $W \subset \mathrm{Spec}(S)$ containing $V(IS)$ such that all fibres of $W \to \mathrm{Spec}(R)$ have dimension $\leq n - c$. Say $W = \mathrm{Spec}(S) \setminus V(J)$. Then $V(J) \cap V(IS) = \emptyset$ hence we can find a $g \in J$ which maps to $1 \in S/IS$. Let $h \in R[x_1, \ldots, x_n]$ be any preimage of g.

Ad (2). By Lemma 10.119.6 there exists an open subset $W \subset \mathrm{Spec}(S)$ containing $\mathrm{Spec}(S \otimes_R \kappa(\mathfrak{p}))$ such that all fibres of $W \to \mathrm{Spec}(R)$ have dimension $\leq n - c$. Say $W = \mathrm{Spec}(S) \setminus V(J)$. Then $V(J \cdot S \otimes_R \kappa(\mathfrak{p})) = \emptyset$. Hence we can find a $g \in J$ which maps to a unit in $S \otimes_R \kappa(\mathfrak{p})$ (details omitted). Let $h \in R[x_1, \ldots, x_n]$ be any preimage of g.

Ad (3). By Lemma 10.119.6 there exists a $g \in S$, $g \notin \mathfrak{q}$ such that all nonempty fibres of $R \to S_g$ have dimension $\leq n - c$. Let $h \in R[x_1, \ldots, x_n]$ be any element that maps to g. □

The following lemma says we can do absolute Noetherian approximation for relative complete intersections.

Lemma 10.129.12. *Let R be a ring. Let S be a relative global complete intersection with presentation $S = R[x_1, \ldots, x_n]/(f_1, \ldots, f_c)$. There exist a finite type $\mathbf{Z}$-subalgebra $R_0 \subset R$ such that $f_i \in R_0[x_1, \ldots, x_n]$ and such that*

$$S_0 = R_0[x_1, \ldots, x_n]/(f_1, \ldots, f_c)$$

is a relative global intersection over R_0.

Proof. Let $R_0 \subset R$ be the $\mathbf{Z}$-algebra of R generated by all the coefficients of the polynomials $f_1, \ldots, f_c$. Let $S_0 = R_0[x_1, \ldots, x_n]/(f_1, \ldots, f_c)$. Clearly, $S = R \otimes_{R_0} S_0$. Pick a prime $\mathfrak{q} \subset S$ and denote $\mathfrak{p} \subset R$, $\mathfrak{q}_0 \subset S_0$, and $\mathfrak{p}_0 \subset R_0$ the primes it lies over. Because $\dim(S \otimes_R \kappa(\mathfrak{p})) = n - c$ we also have $\dim(S_0 \otimes_{R_0} \kappa(\mathfrak{p}_0)) = n - c$, see Lemma 10.110.5. By Lemma 10.119.6 there exists a $g \in S_0$, $g \notin \mathfrak{q}_0$ such that all nonempty fibres of $R_0 \to (S_0)_g$ have dimension $\leq n - c$. As $\mathfrak{q}$ was arbitrary and $\mathrm{Spec}(S)$ quasi-compact, we can find finitely many $g_1, \ldots, g_m \in S_0$ such that (a) for $j = 1, \ldots, m$ the nonempty fibres of $R_0 \to (S_0)_{g_j}$ have dimension $\leq n - c$ and (b) the image of $\mathrm{Spec}(S) \to \mathrm{Spec}(S_0)$ is contained in $D(g_1) \cup \ldots \cup D(g_m)$. In other words, the images of $g_1, \ldots, g_m$ in $S = R \otimes_{R_0} S_0$ generate the unit ideal. After increasing R_0 we may assume that $g_1, \ldots, g_m$ generate the unit ideal in S_0. By (a) the nonempty fibres of $R_0 \to S_0$ all have dimension $\leq n - c$ and we conclude. □

Lemma 10.129.13. *Let R be a ring. Let $S = R[x_1, \ldots, x_n]/(f_1, \ldots, f_c)$ be a relative global complete intersection. For every prime $\mathfrak{q}$ of S, let $\mathfrak{q}'$ denote the corresponding prime of $R[x_1, \ldots, x_n]$. Then*

1. *$f_1, \ldots, f_c$ is a regular sequence in the local ring $R[x_1, \ldots, x_n]_{\mathfrak{q}'}$,*
2. *each of the rings $R[x_1, \ldots, x_n]_{\mathfrak{q}'}/(f_1, \ldots, f_i)$ is flat over R, and*
3. *the S-module $(f_1, \ldots, f_c)/(f_1, \ldots, f_c)^2$ is free with basis given by the elements $f_i \bmod (f_1, \ldots, f_c)^2$.*

Proof. First, by Lemma 10.66.2, part (3) follows from part (1). Parts (1) and (2) immediately reduce to the Noetherian case by Lemma 10.129.12 (some minor details omitted). Assume R is Noetherian. By Lemma 10.128.4 for example we see that $f_1, \ldots, f_c$ form a regular sequence in the local ring $R[x_1, \ldots, x_n]_{\mathfrak{q}'} \otimes_R \kappa(\mathfrak{p})$. Moreover, the local ring $R[x_1, \ldots, x_n]_{\mathfrak{q}'}$ is flat over $R_{\mathfrak{p}}$. Since R, and hence $R[x_1, \ldots, x_n]_{\mathfrak{q}'}$ is Noetherian we may apply Lemma 10.93.3 to conclude. □

Lemma 10.129.14. *A relative global complete intersection is syntomic, i.e., flat.*

Proof. Let $R \to S$ be a relative global complete intersection. The fibres are global complete intersections, and S is of finite presentation over R. Thus the only thing to prove is that $R \to S$ is flat. This is true by (2) of Lemma 10.129.13. □

Lemma 10.129.15. *Let $R \to S$ be a ring map. Let $\mathfrak{q} \subset S$ be a prime lying over the prime $\mathfrak{p}$ of R. The following are equivalent:*

1. *There exists an element $g \in S$, $g \notin \mathfrak{q}$ such that $R \to S_g$ is syntomic.*
2. *There exists an element $g \in S$, $g \notin \mathfrak{q}$ such that S_g is a relative global complete intersection over R.*
3. *There exists an element $g \in S$, $g \notin \mathfrak{q}$, such that $R \to S_g$ is of finite presentation, the local ring map $R_\mathfrak{p} \to S_\mathfrak{q}$ is flat, and the local ring $S_\mathfrak{q}/\mathfrak{p}S_\mathfrak{q}$ is a complete intersection ring over $\kappa(\mathfrak{p})$ (see Definition 10.128.5).*

Proof. The implication (1) $\Rightarrow$ (3) is Lemma 10.128.8. The implication (2) $\Rightarrow$ (1) is Lemma 10.129.14. It remains to show that (3) implies (2).

Assume (3). After replacing S by S_g for some $g \in S$, $g \notin \mathfrak{q}$ we may assume S is finitely presented over R. Choose a presentation $S = R[x_1, \ldots, x_n]/I$. Let $c = n - \dim_\mathfrak{q}(S/R)$. Let $\mathfrak{q}' \subset R[x_1, \ldots, x_n]$ be the prime corresponding to $\mathfrak{q}$. Write $\kappa(\mathfrak{p}) = k$. Note that $S \otimes_R k = k[x_1, \ldots, x_n]/\overline{I}$ where $\overline{I} \subset k[x_1, \ldots, x_n]$ is the ideal generated by the image of I. Let $\overline{\mathfrak{q}}' \subset k[x_1, \ldots, x_n]$ be the prime ideal generated by the image of $\mathfrak{q}'$. By Lemma 10.128.8 we see that Lemma 10.128.4 holds for $\overline{I}$ and $\overline{\mathfrak{q}}'$. Thus the dimension of $\overline{I}_{\overline{\mathfrak{q}}'}/\overline{\mathfrak{q}}'\overline{I}_{\overline{\mathfrak{q}}'}$ over $\kappa(\overline{\mathfrak{q}}')$ is c. Pick $f_1, \ldots, f_c \in I$ mapping to a basis of this vector space. The images $\overline{f}_j \in \overline{I}$ generate $\overline{I}_{\overline{\mathfrak{q}}'}$ (by Lemma 10.128.4). Set $S' = R[x_1, \ldots, x_n]/(f_1, \ldots, f_c)$. Let J be the kernel of the surjection $S' \to S$. Since S is of finite presentation J is a finitely generated ideal (Lemma 10.6.2). Consider the short exact sequence

$$0 \to J \to S' \to S \to 0$$

As $S_\mathfrak{q}$ is flat over R we see that $J_{\mathfrak{q}'} \otimes_R k \to S'_{\mathfrak{q}'} \otimes_R k$ is injective (Lemma 10.36.11). However, by construction $S'_{\mathfrak{q}'} \otimes_R k$ maps isomorphically to $S_\mathfrak{q} \otimes_R k$. Hence we conclude that $J_{\mathfrak{q}'} \otimes_R k = J_{\mathfrak{q}'}/\mathfrak{p}J_{\mathfrak{q}'} = 0$. By Nakayama's lemma (Lemma 10.18.1) we conclude that there exists a $g \in R[x_1, \ldots, x_n]$, $g \notin \mathfrak{q}'$ such that $J_g = 0$. In other words $S'_g \cong S_g$. After further localizing we see that S' (and hence S) becomes a relative global complete intersection by Lemma 10.129.11 as desired. □

Lemma 10.129.16. *Let R be a ring. Let $S = R[x_1, \ldots, x_n]/I$ for some finitely generated ideal I. If $g \in S$ is such that S_g is syntomic over R, then $(I/I^2)_g$ is a finite projective S_g-module.*

Proof. By Lemma 10.129.15 there exist finitely many elements $g_1, \ldots, g_m \in S$ which generate the unit ideal in S_g such that each S_{gg_j} is a relative global complete intersection over R. Since it suffices to prove that $(I/I^2)_{gg_j}$ is finite projective, see Lemma 10.73.2, we may assume that S_g is a relative global complete intersection. In this case the result follows from Lemmas 10.127.16 and 10.129.13. □

Lemma 10.129.17. *Let $R \to S$, $S \to S'$ be ring maps.*

1. *If $R \to S$ and $S \to S'$ are syntomic, then $R \to S'$ is syntomic.*
2. *If $R \to S$ and $S \to S'$ are relative global complete intersections, then $R \to S'$ is a relative global complete intersection.*

Proof. Assume $R \to S$ and $S \to S'$ are syntomic. This implies that $R \to S'$ is flat by Lemma 10.36.3. It also implies that $R \to S'$ is of finite presentation by

Lemma 10.6.2. Thus it suffices to show that the fibres of $R \to S'$ are local complete intersections. Choose a prime $\mathfrak{p} \subset R$. We have a factorization

$$\kappa(\mathfrak{p}) \to S \otimes_R \kappa(\mathfrak{p}) \to S' \otimes_R \kappa(\mathfrak{p}).$$

By assumption $S \otimes_R \kappa(\mathfrak{p})$ is a local complete intersection, and by Lemma 10.129.3 we see that $S \otimes_R \kappa(\mathfrak{p})$ is syntomic over $S \otimes_R \kappa(\mathfrak{p})$. After replacing S by $S \otimes_R \kappa(\mathfrak{p})$ and S' by $S' \otimes_R \kappa(\mathfrak{p})$ we may assume that R is a field. Say $R = k$.

Choose a prime $\mathfrak{q}' \subset S'$ lying over the prime $\mathfrak{q}$ of S. Our goal is to find a $g' \in S'$, $g' \notin \mathfrak{q}'$ such that $S'_{g'}$ is a global complete intersection over k. Choose a $g \in S$, $g \notin \mathfrak{q}$ such that $S_g = k[x_1, \ldots, x_n]/(f_1, \ldots, f_c)$ is a global complete intersection over k. Since $S_g \to S'_g$ is still syntomic also, and $g \notin \mathfrak{q}'$ we may replace S by S_g and S' by S'_g and assume that $S = k[x_1, \ldots, x_n]/(f_1, \ldots, f_c)$ is a global complete intersection over k. Next we choose a $g' \in S'$, $g' \notin \mathfrak{q}'$ such that $S' = S[y_1, \ldots, y_m]/(h_1, \ldots, h_d)$ is a relative global complete intersection over S. Hence we have reduced to part (2) of the lemma.

Suppose that $R \to S$ and $S \to S'$ are relative global complete intersections. Say $S = R[x_1, \ldots, x_n]/(f_1, \ldots, f_c)$ and $S' = S[y_1, \ldots, y_m]/(h_1, \ldots, h_d)$. Then

$$S' \cong R[x_1, \ldots, x_n, y_1, \ldots, y_m]/(f_1, \ldots, f_c, h'_1, \ldots, h'_d)$$

for some lifts $h'_j \in R[x_1, \ldots, x_n, y_1, \ldots, y_m]$ of the h_j. Hence it suffices to bound the dimensions of the fibres. Thus we may yet again assume $R = k$ is a field. In this case we see that we have a ring, namely S, which is of finite type over k and equidimensional of dimension $n - c$, and a finite type ring map $S \to S'$ all of whose nonempty fibre rings are equidimensional of dimension $m - d$. Then, by Lemma 10.106.6 for example applied to localizations at maximal ideals of S', we see that $\dim(S') \leq n - c + m - d$ as desired. □

The following lemma will be improved later, see Smoothing Ring Maps, Proposition 16.4.2.

Lemma 10.129.18. *Let R be a ring and let $I \subset R$ be an ideal. Let $R/I \to \overline{S}$ be a syntomic map. Then there exists elements $\overline{g}_i \in \overline{S}$ which generate the unit ideal of $\overline{S}$ such that each $\overline{S}_{g_i} \cong S_i/IS_i$ for some relative global complete intersection S_i over R.*

Proof. By Lemma 10.129.15 we find a collection of elements $\overline{g}_i \in \overline{S}$ which generate the unit ideal of $\overline{S}$ such that each $\overline{S}_{g_i}$ is a relative global complete intersection over R/I. Hence we may assume that $\overline{S}$ is a relative global complete intersection. Write $\overline{S} = (R/I)[x_1, \ldots, x_n]/(\overline{f}_1, \ldots, \overline{f}_c)$ as in Definition 10.129.5. Choose $f_1, \ldots, f_c \in R[x_1, \ldots, x_n]$ lifting $\overline{f}_1, \ldots, \overline{f}_c$. Set $S = R[x_1, \ldots, x_n]/(f_1, \ldots, f_c)$. Note that $S/IS \cong \overline{S}$. By Lemma 10.129.11 we can find $g \in S$ mapping to 1 in $\overline{S}$ such that S_g is a relative global complete intersection over R. Since $\overline{S} \cong S_g/IS_g$ this finishes the proof. □

10.130. Smooth ring maps

Let us motivate the definition of a smooth ring map by an example. Suppose R is a ring and $S = R[x, y]/(f)$ for some nonzero $f \in R[x, y]$. In this case there is an exact sequence

$$S \to S\mathrm{d}x \oplus S\mathrm{d}y \to \Omega_{S/R} \to 0$$

where the first arrow maps 1 to $\frac{\partial f}{\partial x}\mathrm{d}x + \frac{\partial f}{\partial y}\mathrm{d}y$ see Section 10.127. We conclude that $\Omega_{S/R}$ is locally free of rank 1 if the partial derivatives of f generate the unit ideal in S. In this case S is smooth of relative dimension 1 over R. But it can happen that $\Omega_{S/R}$ is locally free of rank 2 namely if both partial derivatives of f are zero. For example if for a prime p we have $p = 0$ in R and $f = x^p + y^p$ then this happens. Here $R \to S$ is a relative global complete intersection of relative dimension 1 which is not smooth. Hence, in order to check that a ring map is smooth it is not sufficient to check whether the module of differentials is free. The correct condition is the following.

Definition 10.130.1. A ring map $R \to S$ is *smooth* if it is of finite presentation and the naive cotangent complex $NL_{S/R}$ is quasi-isomorphic to a finite projective S-module placed in degree 0.

In particular, if $R \to S$ is smooth then the module $\Omega_{S/R}$ is a finite projective S-module. Moreover, by Lemma 10.130.2 the naive cotangent complex of any presentation has the same structure. Thus, for a surjection $\alpha : R[x_1, \ldots, x_n] \to S$ with kernel I the map

$$I/I^2 \longrightarrow \Omega_{R[x_1,\ldots,x_n]/R} \otimes_{R[x_1,\ldots,x_n]} S$$

is a split injection. In other words $\bigoplus_{i=1}^n S\mathrm{d}x_i \cong I/I^2 \oplus \Omega_{S/R}$ as S-modules. This implies that I/I^2 is a finite projective S-module too!

Lemma 10.130.2. *Let $R \to S$ be a ring map of finite presentation. If for some presentation α of S over R the naive cotangent complex $NL(\alpha)$ is quasi-isomorphic to a finite projective S-module placed in degree 0, then this holds for any presentation.*

Proof. Immediate from Lemma 10.127.2. □

Lemma 10.130.3. *Let $R \to S$ be a smooth ring map. Any localization S_g is smooth over R. If $f \in R$ maps to an invertible element of S, then $R_f \to S$ is smooth.*

Proof. By Lemma 10.127.13 we see that the naive cotangent complex for S_g over R is the base change of the naive cotangent complex of S over R. The assumption is that the naive cotangent complex of S/R is $\Omega_{S/R}$ and that this is a finite projective S-module. Hence so is its base change. Thus S_g is smooth over R.

For the last assertion: A presentation of S over R_f is $R_f[x_1, \ldots, x_n]/I_f$. Since $I_f/I_f^2 = (I/I^2)_f = I/I^2$ we see that this presentation has isomorphic naive cotangent complex to the presentation of S over R. The result follows. □

Lemma 10.130.4. *Let $R \to S$ be a smooth ring map. Let $R \to R'$ be any ring map. Then the base change $R' \to S' = R' \otimes_R S$ is smooth.*

Proof. Let $\alpha : R[x_1, \ldots, x_n] \to S$ be a presentation with kernel I. Let $\alpha' : R'[x_1, \ldots, x_n] \to R' \otimes_R S$ be the induced presentation. Let $I' = \mathrm{Ker}(\alpha')$. Since $0 \to I \to R[x_1, \ldots, x_n] \to S \to 0$ is exact, the sequence $R' \otimes_R I \to R'[x_1, \ldots, x_n] \to R' \otimes_r S \to 0$ is exact. Thus $R' \otimes_R I \to I'$ is surjective. By Definition 10.130.1 there is a short exact sequence

$$0 \to I/I^2 \to \Omega_{R[x_1,\ldots,x_n]/R} \otimes_{R[x_1,\ldots,x_n]} S \to \Omega_{S/R} \to 0$$

and the S-module $\Omega_{S/R}$ is finite projective. In particular I/I^2 is a direct summand of $\Omega_{R[x_1,\ldots,x_n]/R} \otimes_{R[x_1,\ldots,x_n]} S$. Consider the commutative diagram

$$\begin{array}{ccc} R' \otimes_R (I/I^2) & \longrightarrow & R' \otimes_R (\Omega_{R[x_1,\ldots,x_n]/R} \otimes_{R[x_1,\ldots,x_n]} S) \\ \downarrow & & \downarrow \\ I'/(I')^2 & \longrightarrow & \Omega_{R'[x_1,\ldots,x_n]/R'} \otimes_{R'[x_1,\ldots,x_n]} (R' \otimes_R S) \end{array}$$

Since the right vertical map is an isomorphism we see that the left vertical map is injective and surjective by what was said above. Thus we conclude that $NL(\alpha')$ is quasi-isomorphic to $\Omega_{S'/R'} \cong S' \otimes_S \Omega_{S/R}$. And this is finite projective since it is the base change of a finite projective module. □

Lemma 10.130.5. *Let k be a field. Let S be a smooth k-algebra. Then S is a local complete intersection.*

Proof. By Lemmas 10.130.4 and 10.128.11 it suffices to prove this when k is algebraically closed. Choose a presentation $\alpha : k[x_1,\ldots,x_n] \to S$ with kernel I. Let $\mathfrak{m}$ be a maximal ideal of S, and let $\mathfrak{m}' \subset I$ be the corresponding maximal ideal of $k[x_1,\ldots,x_n]$. We will show that condition (5) of Lemma 10.128.4 holds (with $\mathfrak{m}$ instead of $\mathfrak{q}$). We may write $\mathfrak{m}' = (x_1 - a_1, \ldots, x_n - a_n)$ for some $a_i \in k$, because k is algebraically closed, see Theorem 10.31.1. By our assumption that $k \to S$ is smooth the S-module map $\mathrm{d} : I/I^2 \to \bigoplus_{i=1}^n S\mathrm{d}x_i$ is a split injection. Hence the corresponding map $I/\mathfrak{m}'I \to \bigoplus \kappa(\mathfrak{m}')\mathrm{d}x_i$ is injective. Say $\dim_{\kappa(\mathfrak{m}')}(I/\mathfrak{m}'I) = c$ and pick $f_1,\ldots,f_c \in I$ which map to a $\kappa(\mathfrak{m}')$-basis of $I/\mathfrak{m}'I$. By Nakayama's Lemma 10.18.1 we see that $f_1,\ldots,f_c$ generate $I_{\mathfrak{m}'}$ over $k[x_1,\ldots,x_n]_{\mathfrak{m}'}$. Consider the commutative diagram

$$\begin{array}{ccccc} I & \longrightarrow & I/I^2 & \longrightarrow & I/\mathfrak{m}'I \\ \downarrow & & \downarrow & & \downarrow \\ \Omega_{k[x_1,\ldots,x_n]/k} & \longrightarrow & \bigoplus S\mathrm{d}x_i & \xrightarrow{\mathrm{d}x_i \mapsto x_i - a_i} & \mathfrak{m}'/(\mathfrak{m}')^2 \end{array}$$

(proof commutativity omitted). The middle vertical map is the one defining the naive cotangent complex of α. Note that the right lower horizontal arrow induces an isomorphism $\bigoplus \kappa(\mathfrak{m}')\mathrm{d}x_i \to \mathfrak{m}'/(\mathfrak{m}')^2$. Hence our generators $f_1,\ldots,f_c$ of $I_{\mathfrak{m}'}$ map to a collection of elements in $k[x_1,\ldots,x_n]_{\mathfrak{m}'}$ whose classes in $\mathfrak{m}'/(\mathfrak{m}')^2$ are linearly independent over $\kappa(\mathfrak{m}')$. Therefore they form a regular sequence in the ring $k[x_1,\ldots,x_n]_{\mathfrak{m}'}$ by Lemma 10.101.3. This verifies condition (5) of Lemma 10.128.4 hence S_g is a global complete intersection over k for some $g \in S$, $g \notin \mathfrak{m}$. As this works for any maximal ideal of S we conclude that S is a local complete intersection over k. □

Definition 10.130.6. Let R be a ring. Given integers $n \geq c \geq 0$ and $f_1,\ldots,f_c \in R[x_1,\ldots,x_n]$ we say

$$S = R[x_1,\ldots,x_n]/(f_1,\ldots,f_c)$$

is a *standard smooth algebra over* R if the polynomial

$$g = \det \begin{pmatrix} \partial f_1/\partial x_1 & \partial f_2/\partial x_1 & \ldots & \partial f_c/\partial x_1 \\ \partial f_1/\partial x_2 & \partial f_2/\partial x_2 & \ldots & \partial f_c/\partial x_2 \\ \ldots & \ldots & \ldots & \ldots \\ \partial f_1/\partial x_c & \partial f_2/\partial x_c & \ldots & \partial f_c/\partial x_c \end{pmatrix}$$

maps to an invertible element in S.

Lemma 10.130.7. *Let $S = R[x_1, \ldots, x_n]/(f_1, \ldots, f_c) = R[x_1, \ldots, x_n]/I$ be a standard smooth algebra. Then*

1. *the ring map $R \to S$ is smooth,*
2. *the S-module $\Omega_{S/R}$ is free on $dx_{c+1}, \ldots, dx_n$,*
3. *the S-module I/I^2 is free on the classes of $f_1, \ldots, f_c$,*
4. *for any $g \in S$ the ring map $R \to S_g$ is standard smooth,*
5. *for any ring map $R \to R'$ the base change $R' \to R' \otimes_R S$ is standard smooth,*
6. *if $f \in R$ maps to an invertible element in S, then $R_f \to S$ is standard smooth, and*
7. *the ring S is a relative global complete intersection over R.*

Proof. Consider the naive cotangent complex of the given presentation

$$(f_1, \ldots, f_c)/(f_1, \ldots, f_c)^2 \longrightarrow \bigoplus\nolimits_{i=1}^{n} S\mathrm{d}x_i$$

Let us compose this map with the projection onto the first c direct summands of the direct sum. According to the definition of a standard smooth algebra the classes $f_i \bmod (f_1, \ldots, f_c)^2$ map to a basis of $\bigoplus_{i=1}^{c} S\mathrm{d}x_i$. We conclude that $(f_1, \ldots, f_c)/(f_1, \ldots, f_c)^2$ is free of rank c with a basis given by the elements $f_i \bmod (f_1, \ldots, f_c)^2$, and that the homology in degree 0, i.e., $\Omega_{S/R}$, of the naive cotangent complex is a free S-module with basis the images of $\mathrm{d}x_{c+j}$, $j = 1, \ldots, n-c$. In particular, this proves $R \to S$ is smooth.

The proofs of (4) and (6) are omitted. But see the example below and the proof of Lemma 10.129.10.

Let $\varphi : R \to R'$ be any ring map. Denote $S' = R'[x_1, \ldots, x_n]/(f_1^\varphi, \ldots, f_c^\varphi)$ where f^φ is the polynomial obtained from $f \in R[x_1, \ldots, x_n]$ by applying φ to all the coefficients. Then $S' \cong R' \otimes_R S$. Moreover, the determinant of Definition 10.130.6 for S'/R' is equal to g^φ. Its image in S' is therefore the image of g via $R[x_1, \ldots, x_n] \to S \to S'$ and hence invertible. This proves (5).

To prove (7) it suffices to show that $S \otimes_R \kappa(\mathfrak{p})$ has dimension $n - c$. By (5) it suffices to prove that any standard smooth algebra $k[x_1, \ldots, x_n]/(f_1, \ldots, f_c)$ over a field k has dimension $n - c$. We already know that $k[x_1, \ldots, x_n]/(f_1, \ldots, f_c)$ is a local complete intersection by Lemma 10.130.5. Hence, since I/I^2 is free of rank c we see that it dimension $n - c$, by Lemma 10.128.4 for example. □

Example 10.130.8. Let R be a ring. Let $f_1, \ldots, f_c \in R[x_1, \ldots, x_n]$. Let

$$h = \det \begin{pmatrix} \partial f_1/\partial x_1 & \partial f_2/\partial x_1 & \ldots & \partial f_c/\partial x_1 \\ \partial f_1/\partial x_2 & \partial f_2/\partial x_2 & \ldots & \partial f_c/\partial x_2 \\ \ldots & \ldots & \ldots & \ldots \\ \partial f_1/\partial x_c & \partial f_2/\partial x_c & \ldots & \partial f_c/\partial x_c \end{pmatrix}.$$

Set $S = R[x_1, \ldots, x_{n+1}]/(f_1, \ldots, f_c, x_{n+1}h - 1)$. This is an example of a standard smooth algebra, except that the presentation is wrong and the variables should be in the following order: $x_1, \ldots, x_c, x_{n+1}, x_{c+1}, \ldots, x_n$.

Lemma 10.130.9. *A composition of standard smooth ring maps is standard smooth.*

Proof. Suppose that $R \to S$ and $S \to S'$ are standard smooth. We choose presentations $S = R[x_1, \ldots, x_n]/(f_1, \ldots, f_c)$ and $S' = S[y_1, \ldots, y_m]/(g_1, \ldots, g_d)$. Choose elements $g'_j \in R[x_1, \ldots, x_n, y_1, \ldots, y_m]$ mapping to the g_j. In this way we see $S' = R[x_1, \ldots, x_n, y_1, \ldots, y_m]/(f_1, \ldots, f_c, g'_1, \ldots, g'_d)$. To show that S' is standard smooth it suffices to verify that the determinant

$$\det \begin{pmatrix} \partial f_1/\partial x_1 & \ldots & \partial f_c/\partial x_1 & \partial g_1/\partial x_1 & \ldots & \partial g_d/\partial x_1 \\ \ldots & \ldots & \ldots & \ldots & \ldots & \ldots \\ \partial f_1/\partial x_c & \ldots & \partial f_c/\partial x_c & \partial g_1/\partial x_c & \ldots & \partial g_d/\partial x_c \\ 0 & \ldots & 0 & \partial g_1/\partial y_1 & \ldots & \partial g_d/\partial y_1 \\ \ldots & \ldots & \ldots & \ldots & \ldots & \ldots \\ 0 & \ldots & 0 & \partial g_1/\partial y_d & \ldots & \partial g_d/\partial y_d \end{pmatrix}$$

is invertible in S'. This is clear since it is the product of the two determinants which were assumed to be invertible by hypothesis. □

Lemma 10.130.10. *Let $R \to S$ be a smooth ring map. There exists an open covering of* $\mathrm{Spec}(S)$ *by standard opens $D(g)$ such that each S_g is standard smooth over R. In particular $R \to S$ is syntomic.*

Proof. Choose a presentation $\alpha : R[x_1, \ldots, x_n] \to S$ with kernel $I = (f_1, \ldots, f_m)$. For every subset $E \subset \{1, \ldots, m\}$ consider the open subset U_E where the classes $f_e, e \in E$ freely generate the finite projective S-module I/I^2, see Lemma 10.74.3. We may cover $\mathrm{Spec}(S)$ by standard opens $D(g)$ each completely contained in one of the opens U_E. For such a g we look at the presentation

$$\beta : R[x_1, \ldots, x_n, x_{n+1}] \longrightarrow S_g$$

mapping x_{n+1} to $1/g$. Setting $J = \mathrm{Ker}(\beta)$ we use Lemma 10.127.12 to see that $J/J^2 \cong (I/I^2)_g \oplus S_g$ is free. By Lemma 10.129.6 we can find a presentation $\alpha : R[x_1, \ldots, x_n] \to S$ with kernel $I = (f_1, \ldots, f_c)$ such that I/I^2 is free on the classes of $f_1, \ldots, f_c$.

Using the presentation α obtained at the end of the previous paragraph, we more or less repeat this argument with the basis elements $\mathrm{d}x_1, \ldots, \mathrm{d}x_n$ of $\Omega_{R[x_1,\ldots,x_n]/R}$. Namely, for any subset $E \subset \{1, \ldots, n\}$ of cardinality c we may consider the open subset U_E of $\mathrm{Spec}(S)$ where the differential of $NL(\alpha)$ composed with the projection

$$S^{\oplus c} \cong I/I^2 \longrightarrow \Omega_{R[x_1,\ldots,x_n]/R} \otimes_{R[x_1,\ldots,x_n]} S \longrightarrow \bigoplus\nolimits_{i \in E} S\mathrm{d}x_i$$

is an isomorphism. Again we may find a covering of $\mathrm{Spec}(S)$ by (finitely many) standard opens $D(g)$ such that each $D(g)$ is completely contained in one of the opens U_E. By renumbering, we may assume $E = \{1, \ldots, c\}$. For a g with $D(g) \subset U_E$ we look at the presentation

$$\beta : R[x_1, \ldots, x_n, x_{n+1}] \to S_g$$

mapping x_{n+1} to $1/g$. Setting $J = \mathrm{Ker}(\beta)$ we conclude from Lemma 10.127.12 that $J = (f_1, \ldots, f_c, fx_{n+1} - 1)$ where $\alpha(f) = g$ and that the composition

$$J/J^2 \longrightarrow \Omega_{R[x_1,\ldots,x_{n+1}]/R} \otimes_{R[x_1,\ldots,x_{n+1}]} S_g \longrightarrow \bigoplus\nolimits_{i=1}^{c} S_g\mathrm{d}x_i \oplus S_g\mathrm{d}x_{n+1}$$

is an isomorphism. Reordering the coordinates as $x_1, \ldots, x_c, x_{n+1}, x_{c+1}, \ldots, x_n$ we we conclude that S_g is standard smooth over R as desired.

This finishes the proof as standard smooth algebras are syntomic (Lemmas 10.130.7 and 10.129.14) and being syntomic over R is local on S (Lemma 10.129.4). □

Definition 10.130.11. Let $R \to S$ be a ring map. Let $\mathfrak{q}$ be a prime of S. We say $R \to S$ is *smooth at* $\mathfrak{q}$ if there exists a $g \in S$, $g \notin \mathfrak{q}$ such that $R \to S_g$ is smooth.

For ring maps of finite presentation we can characterize this as follows.

Lemma 10.130.12. *Let $R \to S$ be of finite presentation. Let $\mathfrak{q}$ be a prime of S. The following are equivalent*

1. *$R \to S$ is smooth at $\mathfrak{q}$,*
2. *$H_1(L_{S/R})_\mathfrak{q} = 0$ and $\Omega_{S/R,\mathfrak{q}}$ is a projective $S_\mathfrak{q}$-module, and*
3. *$H_1(L_{S/R})_\mathfrak{q} = 0$ and $\Omega_{S/R,\mathfrak{q}}$ is a flat $S_\mathfrak{q}$-module.*

Proof. We will use without further mention that formation of the naive cotangent complex commutes with localization, see Section 10.127, especially Lemma 10.127.13. It is clear that (1) implies (2) implies (3). Assume (3) holds. Note that $\Omega_{S/R}$ is a finitely presented S-module, see Lemma 10.125.15. Hence $\Omega_{S/R,\mathfrak{q}}$ is a finite free module by Lemma 10.73.4. Writing $S_\mathfrak{q}$ as the colimit of principal localizations we see from Lemma 10.121.4 that we can find a $g \in S$, $g \notin \mathfrak{q}$ such that $(\Omega_{S/R})_g$ is finite free. Choose a presentation $\alpha : R[x_1, \ldots, x_n] \to S$ with kernel I. We may work with $NL(\alpha)$ instead of $NL_{S/R}$, see Lemma 10.127.2. The surjection

$$\Omega_{R[x_1,\ldots,x_n]/R} \otimes_R S \to \Omega_{S/R} \to 0$$

has a right inverse after inverting g because $(\Omega_{S/R})_g$ is projective. Hence the image of $\mathrm{d} : (I/I^2)_g \to \Omega_{R[x_1,\ldots,x_n]/R} \otimes_R S_g$ is a direct summand and this map has a right inverse too. We conclude that $H_1(L_{S/R})_g$ is a quotient of $(I/I^2)_g$. In particular $H_1(L_{S/R})_g$ is a finite S_g-module. Thus the vanishing of $H_1(L_{S/R})_\mathfrak{q}$ implies the vanishing of $H_1(L_{S/R})_{gg'}$ for some $g' \in S$, $g' \notin \mathfrak{q}$. Then $R \to S_{gg'}$ is smooth by definition. □

Lemma 10.130.13. *Let $R \to S$ be a ring map. Then $R \to S$ is smooth if and only if $R \to S$ is smooth at every prime $\mathfrak{q}$ of S.*

Proof. The direct implication is trivial. Suppose that $R \to S$ is smooth at every prime $\mathfrak{q}$ of S. Since $\mathrm{Spec}(S)$ is quasi-compact, see Lemma 10.16.10, there exists a finite covering $\mathrm{Spec}(S) = \bigcup D(g_i)$ such that each S_{g_i} is smooth. By Lemma 10.22.3 this implies that S is of finite presentation over R. According to Lemma 10.127.13 we see that $NL_{S/R} \otimes_S S_{g_i}$ is quasi-isomorphic to a finite projective S_{g_i}-module. By Lemma 10.73.2 this implies that $NL_{S/R}$ is quasi-isomorphic to a finite projective S-module. □

Lemma 10.130.14. *A composition of smooth ring maps is smooth.*

Proof. This follows from a combination of Lemmas 10.130.10, 10.130.9 and 10.130.13. (You can also prove this in many different ways; including easier ones.) □

Lemma 10.130.15. *Let R be a ring. Let $S = R[x_1, \ldots, x_n]/(f_1, \ldots, f_c)$ be a relative global complete intersection. Let $\mathfrak{q} \subset S$ be a prime. Then $R \to S$ is smooth*

at $\mathfrak{q}$ if and only if there exists a subset $I \subset \{1, \ldots, n\}$ of cardinality c such that the polynomial

$$g_I = \det(\partial f_j/\partial x_i)_{j=1,\ldots,c,\ i \in I}.$$

does not map to an element of $\mathfrak{q}$.

Proof. By Lemma 10.129.13 we see that the naive cotangent complex associated to the given presentation of S is the complex

$$\bigoplus\nolimits_{j=1}^{c} S \cdot f_j \longrightarrow \bigoplus\nolimits_{i=1}^{n} S \cdot \mathrm{d}x_i, \quad f_j \longmapsto \sum \frac{\partial f_j}{\partial x_i} \mathrm{d}x_i.$$

The maximal minors of the matrix giving the map are exactly the polynomials g_I.

Assume g_I maps to $g \in S$, with $g \notin \mathfrak{q}$. Then the algebra S_g is smooth over R. Namely, its naive cotangent complex is quasi-isomorphic to the complex above localized at g, see Lemma 10.127.13. And by construction it is quasi-isomorphic to a free rank $n - c$ module in degree 0.

Conversely, suppose that all g_I end up in $\mathfrak{q}$. In this case the complex above tensored with $\kappa(\mathfrak{q})$ does not have maximal rank, and hence there is no localization by an element $g \in S$, $g \notin \mathfrak{q}$ where this map becomes a split injection. By Lemma 10.127.13 again there is no such localization which is smooth over R. □

Lemma 10.130.16. *Let $R \to S$ be a ring map. Let $\mathfrak{q} \subset S$ be a prime lying over the prime $\mathfrak{p}$ of R. Assume*

1. *there exists a $g \in S$, $g \notin \mathfrak{q}$ such that $R \to S_g$ is of finite presentation,*
2. *the local ring homomorphism $R_\mathfrak{p} \to S_\mathfrak{q}$ is flat,*
3. *the fibre $S \otimes_R \kappa(\mathfrak{p})$ is smooth over $\kappa(\mathfrak{p})$ at the prime corresponding to $\mathfrak{q}$.*

Then $R \to S$ is smooth at $\mathfrak{q}$.

Proof. By Lemmas 10.129.15 and 10.130.5 we see that there exists a $g \in S$ such that S_g is a relative global complete intersection. Replacing S by S_g we may assume $S = R[x_1, \ldots, x_n]/(f_1, \ldots, f_c)$ is a relative global complete intersection. For any subset $I \subset \{1, \ldots, n\}$ of cardinality c consider the polynomial $g_I = \det(\partial f_j/\partial x_i)_{j=1,\ldots,c, i \in I}$ of Lemma 10.130.15. Note that the image $\overline{g}_I$ of g_I in the polynomial ring $\kappa(\mathfrak{p})[x_1, \ldots, x_n]$ is the determinant of the partial derivatives of the images $\overline{f}_j$ of the f_j in the ring $\kappa(\mathfrak{p})[x_1, \ldots, x_n]$. Thus the lemma follows by applying Lemma 10.130.15 both to $R \to S$ and to $\kappa(\mathfrak{p}) \to S \otimes_R \kappa(\mathfrak{p})$. □

Note that the sets U, V in the following lemma are open by definition.

Lemma 10.130.17. *Let $R \to S$ be a ring map of finite presentation. Let $R \to R'$ be a flat ring map. Denote $S' = R' \otimes_R S$ the base change. Let $U \subset \mathrm{Spec}(S)$ be the set of primes at which $R \to S$ is smooth. Let $V \subset \mathrm{Spec}(S')$ the set of primes at which $R' \to S'$ is smooth. Then V is the inverse image of U under the map $f : \mathrm{Spec}(S') \to \mathrm{Spec}(S)$.*

Proof. By Lemma 10.127.8 we see that $NL_{S/R} \otimes_S S'$ is homotopy equivalent to $NL_{S'/R'}$. This already implies that $f^{-1}(U) \subset V$.

Let $\mathfrak{q}' \subset S'$ be a prime lying over $\mathfrak{q} \subset S$. Assume $\mathfrak{q}' \in V$. We have to show that $\mathfrak{q} \in U$. Since $S \to S'$ is flat, we see that $S_\mathfrak{q} \to S'_{\mathfrak{q}'}$ is faithfully flat (Lemma 10.36.16). Thus the vanishing of $H_1(L_{S'/R'})_{\mathfrak{q}'}$ implies the vanishing of $H_1(L_{S/R})_\mathfrak{q}$. By Lemma 10.73.5 applied to the $S_\mathfrak{q}$-module $(\Omega_{S/R})_\mathfrak{q}$ and the map $S_\mathfrak{q} \to S'_{\mathfrak{q}'}$ we see that $(\Omega_{S/R})_\mathfrak{q}$ is projective. Hence $R \to S$ is smooth at $\mathfrak{q}$ by Lemma 10.130.12. □

Lemma 10.130.18. *Let $k \subset K$ be a field extension. Let S be a finite type algebra over k. Let $\mathfrak{q}_K$ be a prime of $S_K = K \otimes_k S$ and let $\mathfrak{q}$ be the corresponding prime of S. Then S is smooth over k at $\mathfrak{q}$ if and only if S_K is smooth at $\mathfrak{q}_K$ over K.*

Proof. This is a special case of Lemma 10.130.17. □

Lemma 10.130.19. *Let R be a ring and let $I \subset R$ be an ideal. Let $R/I \to \overline{S}$ be a smooth ring map. Then there exists elements $\overline{g}_i \in \overline{S}$ which generate the unit ideal of $\overline{S}$ such that each $\overline{S}_{g_i} \cong S_i/IS_i$ for some (standard) smooth ring S_i over R.*

Proof. By Lemma 10.130.10 we find a collection of elements $\overline{g}_i \in \overline{S}$ which generate the unit ideal of $\overline{S}$ such that each $\overline{S}_{g_i}$ is standard smooth over R/I. Hence we may assume that $\overline{S}$ is standard smooth over R/I. Write $\overline{S} = (R/I)[x_1, \ldots, x_n]/(\overline{f}_1, \ldots, \overline{f}_c)$ as in Definition 10.130.6. Choose $f_1, \ldots, f_c \in R[x_1, \ldots, x_n]$ lifting $\overline{f}_1, \ldots, \overline{f}_c$. Set $S = R[x_1, \ldots, x_n, x_{n+1}]/(f_1, \ldots, f_c, x_{n+1}\Delta - 1)$ where $\Delta = \det(\frac{\partial f_i}{\partial x_j})_{i,j=1,\ldots,c}$ as in Example 10.130.8. This proves the lemma. □

10.131. Formally smooth maps

In this section we define formally smooth ring maps. It will turn out that a ring map of finite presentation is formally smooth if and only if it is smooth, see Proposition 10.131.13.

Definition 10.131.1. Let $R \to S$ be a ring map. We say S is *formally smooth over R* if for every commutative solid diagram

$$\begin{array}{ccc} S & \longrightarrow & A/I \\ \uparrow & \nearrow & \uparrow \\ R & \longrightarrow & A \end{array}$$

where $I \subset A$ is an ideal of square zero, a dotted arrow exists which makes the diagram commute.

Lemma 10.131.2. *Let $R \to S$ be a formally smooth ring map. Let $R \to R'$ be any ring map. Then the base change $S' = R' \otimes_R S$ is formally smooth over R'.*

Proof. Let a solid diagram

$$\begin{array}{ccccc} S & \longrightarrow & R' \otimes_R S & \longrightarrow & A/I \\ \uparrow & & \uparrow & \nearrow\!\nearrow & \uparrow \\ R & \longrightarrow & R' & \longrightarrow & A \end{array}$$

as in Definition 10.131.1 be given. By assumption the longer dotted arrow exists. By the universal property of tensor product we obtain the shorter dotted arrow. □

Lemma 10.131.3. *A composition of formally smooth ring maps is formally smooth.*

Proof. Omitted. (Hint: This is completely formal, and follows from considering a suitable diagram.) □

Lemma 10.131.4. *A polynomial ring over R is formally smooth over R.*

Proof. Suppose we have a diagram as in Definition 10.131.1 with $S = R[x_j; j \in J]$. Then there exists a dotted arrow simply by choosing lifts $a_j \in A$ of the elements in A/I to which the elements x_j map to under the top horizontal arrow. □

Lemma 10.131.5. *Let $R \to S$ be a ring map. Let $P \to S$ be a surjective R-algebra map from a polynomial ring P onto S. Denote $J \subset P$ the kernel. Then $R \to S$ is formally smooth if and only if there exists an R-algebra map $\sigma : S \to P/J^2$ which is a right inverse to the surjection $P/J^2 \to S$.*

Proof. Assume $R \to S$ is formally smooth. Consider the commutative diagram

$$\begin{array}{ccc} S & \longrightarrow & P/J \\ \uparrow & \nearrow & \uparrow \\ R & \longrightarrow & P/J^2 \end{array}$$

By assumption the dotted arrow exists. This proves that σ exists.

Conversely, suppose we have a σ as in the lemma. Let a solid diagram

$$\begin{array}{ccc} S & \longrightarrow & A/I \\ \uparrow & \nearrow & \uparrow \\ R & \longrightarrow & A \end{array}$$

as in Definition 10.131.1 be given. Because P is formally smooth by Lemma 10.131.4, there exists an R-algebra homomorphism $\psi : P \to A$ which lifts the map $P \to S \to A/I$. Clearly $\psi(J) \subset I$ and since $I^2 = 0$ we conclude that $\psi(J^2) = 0$. Hence ψ factors as $\overline{\psi} : P/J^2 \to A$. The desired dotted arrow is the composition $\overline{\psi} \circ \sigma : S \to A$. □

Remark 10.131.6. Lemma 10.131.5 holds more generally whenever P is formally smooth over R.

Lemma 10.131.7. *Let $R \to S$ be a ring map. Let $P \to S$ be a surjective R-algebra map from a polynomial ring P onto S. Denote $J \subset P$ the kernel. Then $R \to S$ is formally smooth if and only if the sequence*

$$0 \to J/J^2 \to \Omega_{P/R} \otimes_R S \to \Omega_{S/R} \to 0$$

of Lemma 10.125.9 is a split exact sequence.

Proof. Assume S is formally smooth over R. By Lemma 10.131.5 this means there exists an R-algebra map $S \to P/J^2$ which is a left inverse to the canonical map $P/J^2 \to S$. This means that

$$P/J^2 \cong S \oplus J/J^2$$

as R-algebras. Note that the middle term of the exact sequence is $\Omega_{P/R} \otimes_P S \cong \Omega_{(P/J^2)/R} \otimes_R S$ by Lemma 10.125.11. A direct computation shows that

$$\Omega_{(S \oplus J/J^2)/R} \otimes_{(S \oplus J/J^2)} S = \Omega_{S/R} \oplus J/J^2$$

as desired.

Assume the exact sequence of the lemma is split exact. Choose a splitting $\sigma : \Omega_{S/R} \to \Omega_{P/R} \otimes_R S$. For each $\lambda \in S$ choose $x_\lambda \in P$ which maps to λ. Next, for each $\lambda \in S$ choose $f_\lambda \in J$ such that

$$\mathrm{d}f_\lambda = \mathrm{d}x_\lambda - \sigma(\mathrm{d}\lambda)$$

in the middle term of the exact sequence. We claim that $s : \lambda \mapsto x_\lambda - f_\lambda \mod J^2$ is an R-algebra homomorphism $s : S \to P/J^2$. To prove this we will repeatedly

use that if $h \in J$ and $\mathrm{d}h = 0$ in $\Omega_{P/R} \otimes_R S$, then $h \in J^2$. Let $\lambda, \mu \in S$. Then $\sigma(\mathrm{d}\lambda + \mathrm{d}\mu - \mathrm{d}(\lambda + \mu)) = 0$. This implies

$$\mathrm{d}(x_\lambda + x_\mu - x_{\lambda+\mu} - f_\lambda - f_\mu + f_{\lambda+\mu}) = 0$$

which means that $x_\lambda + x_\mu - x_{\lambda+\mu} - f_\lambda - f_\mu + f_{\lambda+\mu} \in J^2$, which in turn means that $s(\lambda) + s(\mu) = s(\lambda + \mu)$. Similarly, we have $\sigma(\lambda \mathrm{d}\mu + \mu \mathrm{d}\lambda - \mathrm{d}\lambda\mu) = 0$ which implies that

$$\mu(\mathrm{d}x_\lambda - \mathrm{d}f_\lambda) + \lambda(\mathrm{d}x_\mu - \mathrm{d}f_\mu) - \mathrm{d}x_{\lambda\mu} - \mathrm{d}f_{\lambda\mu} = 0$$

in the middle term of the exact sequence. Moreover we have

$$\mathrm{d}(x_\lambda x_\mu) = x_\lambda \mathrm{d}x_\mu + x_\mu \mathrm{d}x_\lambda = \lambda \mathrm{d}x_\mu + \mu \mathrm{d}x_\lambda$$

in the middle term again. Combined these equations mean that $x_\lambda x_\mu - x_{\lambda\mu} - \mu f_\lambda - \lambda f_\mu + f_{\lambda\mu} \in J^2$ which means that $s(\lambda)s(\mu) = s(\lambda\mu)$. If $\lambda \in R$, then $\mathrm{d}\lambda = 0$ and we see that $\mathrm{d}f_\lambda = \mathrm{d}x_\lambda$, hence $\lambda - x_\lambda + f_\lambda \in J^2$ and hence $s(\lambda) = \lambda$ as desired. At this point we can apply Lemma 10.131.5 to conclude that S/R is formally smooth. □

Proposition 10.131.8. *Let $R \to S$ be a ring map. Consider a formally smooth R-algebra P and a surjection $P \to S$ with kernel J. The following are equivalent*

1. *S is formally smooth over R,*
2. *for some $P \to S$ as above there exists a section to $P/J^2 \to S$,*
3. *for all $P \to S$ as above there exists a section to $P/J^2 \to S$,*
4. *for some $P \to S$ as above the sequence $0 \to J/J^2 \to \Omega_{P/R} \otimes S \to \Omega_{S/R} \to 0$ is split exact,*
5. *for all $P \to S$ as above the sequence $0 \to J/J^2 \to \Omega_{P/R} \otimes S \to \Omega_{S/R} \to 0$ is split exact, and*
6. *the naive cotangent complex $NL_{S/R}$ is quasi-isomorphic to a projective S-module placed in degree 0.*

Proof. It is clear that (1) implies (3) implies (2), see first part of the proof of Lemma 10.131.5. It is also true that (3) implies (5) implies (4) and that (2) implies (4), see first part of the proof of Lemma 10.131.7. Finally, Lemma 10.131.7 applied to the canonical surjection $R[S] \to S$ (10.127.0.1) shows that (1) implies (6).

Assume (4) and let's prove (6). Consider the sequence of Lemma 10.127.4 associated to the ring maps $R \to P \to S$. By the implication (1) $\Rightarrow$ (6) proved above we see that $NL_{P/R} \otimes_R S$ is quasi-isomorphic to $\Omega_{P/R} \otimes_P S$ placed in degree 0. Hence $H_1(NL_{P/R} \otimes_P S) = 0$. Since $P \to S$ is surjective we see that $NL_{S/P}$ is homotopy equivalent to J/J^2 placed in degree 1 (Lemma 10.127.6). Thus we obtain the exact sequence $0 \to H_1(L_{S/R}) \to J/J^2 \to \Omega_{P/R} \otimes_P S \to \Omega_{S/R} \to 0$. By assumption we see that $H_1(L_{S/R}) = 0$ and that $\Omega_{S/R}$ is a projective S-module. Thus (6) follows.

Finally, let's prove that (6) implies (1). The assumption means that the complex $J/J^2 \to \Omega_{P/R} \otimes S$ where $P = R[S]$ and $P \to S$ is the canonical surjection (10.127.0.1). Hence Lemma 10.131.7 shows that S is formally smooth over R. □

Lemma 10.131.9. *Let $A \to B \to C$ be ring maps. Assume $B \to C$ is formally smooth. Then the sequence*

$$0 \to \Omega_{B/A} \otimes_B C \to \Omega_{C/A} \to \Omega_{C/B} \to 0$$

of Lemma 10.125.7 is a split short exact sequence.

Proof. Follows from Proposition 10.131.8 and Lemma 10.127.4. □

Lemma 10.131.10. *Let $A \to B \to C$ be ring maps with $A \to C$ formally smooth and $B \to C$ surjective with kernel $J \subset B$. Then the exact sequence*

$$0 \to J/J^2 \to \Omega_{B/A} \otimes_B C \to \Omega_{C/A} \to 0$$

of Lemma 10.125.9 is split exact.

Proof. Follows from Proposition 10.131.8, Lemma 10.127.4, and Lemma 10.125.9. □

Lemma 10.131.11. *Let $A \to B \to C$ be ring maps. Assume $A \to C$ is surjective (so also $B \to C$ is) and $A \to B$ formally smooth. Denote $I = Ker(A \to C)$ and $J = Ker(B \to C)$. Then the sequence*

$$0 \to I/I^2 \to J/J^2 \to \Omega_{B/A} \otimes_B B/J \to 0$$

of Lemma 10.127.7 is split exact.

Proof. Since $A \to B$ is formally smooth there exists a ring map $\sigma : B \to A/I^2$ whose composition with $A \to B$ equals the quotient map $A \to A/I^2$. Then σ induces a map $J/J^2 \to I/I^2$ which is inverse to the map $I/I^2 \to J/J^2$. □

Lemma 10.131.12. *Let $R \to S$ be a ring map. Let $I \subset R$ be an ideal. Assume*

1. *$I^2 = 0$,*
2. *$R \to S$ is flat, and*
3. *$R/I \to S/IS$ is formally smooth.*

Then $R \to S$ is formally smooth.

Proof. Assume (1), (2) and (3). Let $P = R[\{x_t\}_{t \in T}] \to S$ be a surjection of R-algebras with kernel J. Thus $0 \to J \to P \to S \to 0$ is a short exact sequence of flat R-modules. This implies that $I \otimes_R S = IS$, $I \otimes_R P = IP$ and $I \otimes_R J = IJ$ as well as $J \cap IP = IJ$. We will use throughout the proof that

$$\Omega_{(S/IS)/(R/I)} = \Omega_{S/R} \otimes_S (S/IS) = \Omega_{S/R} \otimes_R R/I = \Omega_{S/R}/I\Omega_{S/R}$$

and similarly for P (see Lemma 10.125.12). By Lemma 10.131.7 the sequence

$$0 \to J/(IJ + J^2) \to \Omega_{P/R} \otimes_P S/IS \to \Omega_{S/R} \otimes_S S/IS \to 0 \tag{10.131.12.1}$$

is split exact. Of course the middle term is $\bigoplus_{t \in T} S/IS\mathrm{d}x_t$. Choose a splitting $\sigma : \Omega_{P/R} \otimes_P S/IS \to J/(IJ + J^2)$. For each $t \in T$ choose an element $f_t \in J$ which maps to $\sigma(\mathrm{d}x_t)$ in $J/(IJ + J^2)$. This determines a unique S-module map

$$\tilde{\sigma} : \Omega_{P/R} \otimes_R S = \bigoplus S\mathrm{d}x_t \longrightarrow J/J^2$$

with the property that $\tilde{\sigma}(\mathrm{d}x_t) = f_t$. As σ is a section to d the difference

$$\Delta = \mathrm{id}_{J/J^2} - \tilde{\sigma} \circ \mathrm{d}$$

is a self map $J/J^2 \to J/J^2$ whose image is contained in $(IJ + J^2)/J^2$. In particular $\Delta((IJ + J^2)/J^2) = 0$ because $I^2 = 0$. This means that Δ factors as

$$J/J^2 \to J/(IJ + J^2) \xrightarrow{\overline{\Delta}} (IJ + J^2)/J^2 \to J/J^2$$

where $\overline{\Delta}$ is a S/IS-module map. Using again that the sequence (10.131.12.1) is split, we can find a S/IS-module map $\overline{\delta} : \Omega_{P/R} \otimes_P S/IS \to (IJ + J^2)/J^2$ such that $\overline{\delta} \circ d$ is equal to $\overline{\Delta}$. In the same manner as above the map $\overline{\delta}$ determines an S-module map $\delta : \Omega_{P/R} \otimes_P S \to J/J^2$. After replacing $\tilde{\sigma}$ by $\tilde{\sigma} + \delta$ a simple

computation shows that $\Delta = 0$. In other words $\tilde{\sigma}$ is a section of $J/J^2 \to \Omega_{P/R} \otimes_P S$. By Lemma 10.131.7 we conclude that $R \to S$ is formally smooth. □

Proposition 10.131.13. *Let $R \to S$ be a ring map. The following are equivalent*

(1) *$R \to S$ is of finite presentation and formally smooth,*
(2) *$R \to S$ is smooth.*

Proof. Follows from Proposition 10.131.8 and Definition 10.130.1. (Note that $\Omega_{S/R}$ is a finitely presented S-module if $R \to S$ is of finite presentation, see Lemma 10.125.15.) □

Lemma 10.131.14. *Let $R \to S$ be a smooth ring map. Then there exists a subring $R_0 \subset R$ of finite type over $\mathbf{Z}$ and a smooth ring map $R_0 \to S_0$ such that $S \cong R \otimes_{R_0} S_0$.*

Proof. We are going to use that smooth is equivalent to finite presentation and formally smooth, see Proposition 10.131.13. Write $S = R[x_1, \ldots, x_n]/(f_1, \ldots, f_m)$ and denote $I = (f_1, \ldots, f_m)$. Choose a right inverse $\sigma : S \to R[x_1, \ldots, x_n]/I^2$ to the projection to S as in Lemma 10.131.5. Choose $h_i \in R[x_1, \ldots, x_n]$ such that $\sigma(x_i \bmod I) = h_i \bmod I^2$. The fact that σ is an R-algebra homomorphism $R[x_1, \ldots, x_n]/I \to R[x_1, \ldots, x_n]/I^2$ is equivalent to the condition that

$$f_j(h_1, \ldots, h_n) = \sum\nolimits_{j_1 j_2} a_{j_1 j_2} f_{j_1} f_{j_2}$$

for certain $a_{kl} \in R[x_1, \ldots, x_n]$. Let $R_0 \subset R$ be the subring generated over $\mathbf{Z}$ by all the coefficients of the polynomials f_j, h_i, a_{kl}. Set $S_0 = R_0[x_1, \ldots, x_n]/(f_1, \ldots, f_m)$, with $I_0 = (f_1, \ldots, f_m)$. Let $\sigma_0 : S_0 \to R_0[x_1, \ldots, x_n]/I_0^2$ defined by the rule $x_i \mapsto h_i \bmod I_0^2$; this works since the a_{lk} are defined over R_0 and satisfy the same relations. Thus by Lemma 10.131.5 the ring S_0 is formally smooth over R_0. □

Lemma 10.131.15. *Let $R \to S$ be a ring map. Let $R \to R'$ be a faithfully flat ring map. Set $S' = S \otimes_R R'$. Then $R \to S$ is formally smooth if and only if $R' \to S'$ is formally smooth.*

Proof. If $R \to S$ is formally smooth, then $R' \to S'$ is formally smooth by Lemma 10.131.2. To prove the converse, assume $R' \to S'$ is formally smooth. Note that $N \otimes_R R' = N \otimes_S S'$ for any S-module N. In particular $S \to S'$ is faithfully flat also. Choose a polynomial ring $P = R[\{x_i\}_{i \in I}]$ and a surjection of R-algebras $P \to S$ with kernel J. Note that $P' = P \otimes_R R'$ is a polynomial algebra over R'. Since $R \to R'$ is flat the kernel J' of the surjection $P' \to S'$ is $J \otimes_R R'$. Hence the split exact sequence (see Lemma 10.131.7)

$$0 \to J'/(J')^2 \to \Omega_{P'/R'} \otimes_{P'} S' \to \Omega_{S'/R'} \to 0$$

is the base change via $S \to S'$ of the corresponding sequence

$$J/J^2 \to \Omega_{P/R} \otimes_P S \to \Omega_{S/R} \to 0$$

see Lemma 10.125.9. As $S \to S'$ is faithfully flat we conclude two things: (1) this sequence (without $'$) is exact too, and (2) $\Omega_{S/R}$ is a projective S-module. Namely, $\Omega_{S'/R'}$ is projective as a direct sum of the free module $\Omega_{P'/R'} \otimes_{P'} S'$ and $\Omega_{S/R} \otimes_S S' = \Omega_{S'/R'}$ by what we said above. Thus (2) follows by descent of projectivity through faithfully flat ring maps, see Theorem 10.90.5. Hence the sequence $0 \to J/J^2 \to \Omega_{P/R} \otimes_P S \to \Omega_{S/R} \to 0$ is exact also and we win by applying Lemma 10.131.7 once more. □

It turns out that smooth ring maps satisfy the following strong lifting property.

Lemma 10.131.16. *Let $R \to S$ be a smooth ring map. Given a commutative solid diagram*

$$\begin{array}{ccc} S & \longrightarrow & A/I \\ \uparrow & \nearrow & \uparrow \\ R & \longrightarrow & A \end{array}$$

where $I \subset A$ is a locally nilpotent ideal, a dotted arrow exists which makes the diagram commute.

Proof. By Lemma 10.131.14 we can extend the diagram to a commutative diagram

$$\begin{array}{ccccc} S_0 & \longrightarrow & S & \longrightarrow & A/I \\ \uparrow & & \uparrow & \nearrow & \uparrow \\ R_0 & \longrightarrow & R & \longrightarrow & A \end{array}$$

with $R_0 \to S_0$ smooth, R_0 of finite type over $\mathbf{Z}$, and $S = S_0 \otimes_{R_0} R$. Let $x_1, \ldots, x_n \in S_0$ be generators of S_0 over R_0. Let $a_1, \ldots, a_n$ be elements of A which map to the same elements in A/I as the elements $x_1, \ldots, x_n$. Denote $A_0 \subset A$ the subring generated by the image of R_0 and the elements $a_1, \ldots, a_n$. Set $I_0 = A_0 \cap I$. Then $A_0/I_0 \subset A/I$ and $S_0 \to A/I$ maps into A_0/I_0. Thus it suffices to find the dotted arrow in the diagram

$$\begin{array}{ccc} S_0 & \longrightarrow & A_0/I_0 \\ \uparrow & \nearrow & \uparrow \\ R_0 & \longrightarrow & A_0 \end{array}$$

The ring A_0 is of finite type over $\mathbf{Z}$ by construction. Hence A_0 is Noetherian, whence I_0 is nilpotent, see Lemma 10.47.3. Say $I_0^n = 0$. By Proposition 10.131.13 we can successively lift the R_0-algebra map $S_0 \to A_0/I_0$ to $S_0 \to A_0/I_0^2$, $S_0 \to A_0/I_0^3$, ..., and finally $S_0 \to A_0/I_0^n = A_0$. □

10.132. Smoothness and differentials

Some results on differentials and smooth ring maps.

Lemma 10.132.1. *Given ring maps $A \to B \to C$ with $B \to C$ smooth, then the sequence*

$$0 \to C \otimes_B \Omega_{B/A} \to \Omega_{C/A} \to \Omega_{C/B} \to 0$$

of Lemma 10.125.7 is exact.

Proof. This follows from the more general Lemma 10.131.9 because a smooth ring map is formally smooth, see Proposition 10.131.13. But it also follows directly from Lemma 10.127.4 since $H_1(L_{C/B}) = 0$ is part of the definition of smoothness of $B \to C$. □

Lemma 10.132.2. *Let $A \to B \to C$ be ring maps with $A \to C$ smooth and $B \to C$ surjective with kernel $J \subset B$. Then the exact sequence*

$$0 \to J/J^2 \to \Omega_{B/A} \otimes_B C \to \Omega_{C/A} \to 0$$

of Lemma 10.125.9 is split exact.

Proof. This follows from the more general Lemma 10.131.10 because a smooth ring map is formally smooth, see Proposition 10.131.13. □

Lemma 10.132.3. *Let $A \to B \to C$ be ring maps. Assume $A \to C$ is surjective (so also $B \to C$ is) and $A \to B$ smooth. Denote $I = Ker(A \to C)$ and $J = Ker(B \to C)$. Then the sequence*

$$0 \to I/I^2 \to J/J^2 \to \Omega_{B/A} \otimes_B B/J \to 0$$

of Lemma 10.127.7 is exact.

Proof. This follows from the more general Lemma 10.131.11 because a smooth ring map is formally smooth, see Proposition 10.131.13. □

Lemma 10.132.4. *Let $\varphi : R \to S$ be a smooth ring map. Let $\sigma : S \to R$ be a left inverse to φ. Set $I = Ker(\sigma)$. Then*

1. *I/I^2 is a finite locally free R-module, and*
2. *if I/I^2 is free, then $S^\wedge \cong R[[t_1, \ldots, t_d]]$ as R-algebras, where $S^\wedge$ is the I-adic completion of S.*

Proof. By Lemma 10.125.10 applied to $R \to S \to R$ we see that $I/I^2 = \Omega_{S/R} \otimes_{S,\sigma} R$. Since by definition of a smooth morphism the module $\Omega_{S/R}$ is finite locally free over S we deduce that (1) holds. If I/I^2 is free, then choose $f_1, \ldots, f_d \in I$ whose images in I/I^2 form an R-basis. Consider the R-algebra map defined by

$$\Psi : R[[x_1, \ldots, x_d]] \longrightarrow S^\wedge, \quad x_i \longmapsto f_i.$$

Denote $P = R[[x_1, \ldots, x_d]]$ and $J = (x_1, \ldots, x_d) \subset P$. We write $\Psi_n : P/J^n \to S/I^n$ for the induced map of quotient rings. Note that $S/I^2 = \varphi(R) \oplus I/I^2$. Thus Ψ_2 is an isomorphism. Denote $\sigma_2 : S/I^2 \to P/J^2$ the inverse of Ψ_2. We will prove by induction on n that for all $n > 2$ there exists an inverse $\sigma_n : S/I^n \to P/J^n$ of Ψ_n. Namely, as S is formally smooth over R (by Proposition 10.131.13) we see that in the solid diagram

$$\begin{array}{ccc} S & \dashrightarrow & P/J^n \\ & \searrow^{\sigma_{n-1}} & \downarrow \\ & & P/J^{n-1} \end{array}$$

of R-algebras we can fill in the dotted arrow by some R-algebra map $\tau : S \to P/J^n$ making the diagram commute. This induces an R-algebra map $\overline{\tau} : S/I^n \to P/J^n$ which is equal to σ_{n-1} modulo J^n. By construction the map Ψ_n is surjective and now $\overline{\tau} \circ \Psi_n$ is an R-algebra endomorphism of P/J^n which maps x_i to $x_i + \delta_{i,n}$ with $\delta_{i,n} \in J^{n-1}/J^n$. It follows that Ψ_n is an isomorphism and hence it has an inverse σ_n. This proves the lemma. □

10.133. Smooth algebras over fields

Warning: The following two lemmas do not hold over nonperfect fields in general.

Lemma 10.133.1. *Let k be an algebraically closed field. Let S be a finite type k-algebra. Let $\mathfrak{m} \subset S$ be a maximal ideal. Then*

$$\dim_{\kappa(\mathfrak{m})} \Omega_{S/k} \otimes_S \kappa(\mathfrak{m}) = \dim_{\kappa(\mathfrak{m})} \mathfrak{m}/\mathfrak{m}^2.$$

Proof. Since k is algebraically closed we have $\kappa(\mathfrak{m}) = k$. We may choose a presentation $0 \to I \to k[x_1, \ldots, x_n] \to S \to 0$ such that all x_i end up in $\mathfrak{m}$. Write $I = (f_1, \ldots, f_m)$. Note that each f_i is contained in $(x_1, \ldots, x_n)$, i.e., each f_i has zero constant term. Hence we may write

$$f_j = \sum a_{ij} x_i + \text{h.o.t.}$$

By Lemma 10.125.9 there is an exact sequence

$$\bigoplus S \cdot f_j \to \bigoplus S \cdot \mathrm{d}x_i \to \Omega_{S/k} \to 0.$$

Tensoring with $\kappa(\mathfrak{m}) = k$ we get an exact sequence

$$\bigoplus k \cdot f_j \to \bigoplus k \cdot \mathrm{d}x_i \to \Omega_{S/k} \otimes \kappa(\mathfrak{m}) \to 0.$$

The matrix of the map is given by the partial derivatives of the f_j evaluated at 0. In other words by the matrix (a_{ij}). Similarly there is a short exact sequence

$$(f_1, \ldots, f_m)/(x_1 f_1, \ldots, x_n f_m) \to (x_1, \ldots, x_n)/(x_1, \ldots, x_n)^2 \to \mathfrak{m}/\mathfrak{m}^2 \to 0.$$

Note that the first map is given by expanding the f_j in terms of the x_i, i.e., by the same matrix (a_{ij}). Hence the two numbers are the same. □

Lemma 10.133.2. *Let k be an algebraically closed field. Let S be a finite type k-algebra. Let $\mathfrak{m} \subset S$ be a maximal ideal. The following are equivalent:*

1. *The ring $S_{\mathfrak{m}}$ is a regular local ring.*
2. *We have $\dim_{\kappa(\mathfrak{m})} \Omega_{S/k} \otimes_S \kappa(\mathfrak{m}) \leq \dim(S_{\mathfrak{m}})$.*
3. *We have $\dim_{\kappa(\mathfrak{m})} \Omega_{S/k} \otimes_S \kappa(\mathfrak{m}) = \dim(S_{\mathfrak{m}})$.*
4. *There exists a $g \in S$, $g \notin \mathfrak{m}$ such that S_g is smooth over k. In other words S/k is smooth at $\mathfrak{m}$.*

Proof. Note that (1), (2) and (3) are equivalent by Lemma 10.133.1 and Definition 10.105.6.

Assume that S is smooth at $\mathfrak{q}$. By Lemma 10.130.10 we see that S_g is standard smooth over k for a suitable $g \in S$, $g \notin \mathfrak{m}$. Hence by Lemma 10.130.7 we see that $\Omega_{S_g/k}$ is free of rank $\dim(S_g)$. Hence by Lemma 10.133.1 we see that $\dim(S_m) = \dim(\mathfrak{m}/\mathfrak{m}^2)$ in other words $S_{\mathfrak{m}}$ is regular.

Conversely, suppose that $S_{\mathfrak{m}}$ is regular. Let $d = \dim(S_{\mathfrak{m}}) = \dim \mathfrak{m}/\mathfrak{m}^2$. Choose a presentation $S = k[x_1, \ldots, x_n]/I$ such that x_i maps to an element of $\mathfrak{m}$ for all i. In other words, $\mathfrak{m}'' = (x_1, \ldots, x_n)$ is the corresponding maximal ideal of $k[x_1, \ldots, x_n]$. Note that we have a short exact sequence

$$I/\mathfrak{m}''I \to \mathfrak{m}''/(\mathfrak{m}'')^2 \to \mathfrak{m}/(\mathfrak{m})^2 \to 0$$

Pick $c = n - d$ elements $f_1, \ldots, f_d \in I$ such that their images in $\mathfrak{m}''/(\mathfrak{m}'')^2$ span the kernel of the map to $\mathfrak{m}/(\mathfrak{m})^2$. This is clearly possible. Denote $J = (f_1, \ldots, f_c)$. So $J \subset I$. Denote $S' = k[x_1, \ldots, x_n]/J$ so there is a surjection $S' \to S$. Denote $\mathfrak{m}' = \mathfrak{m}''S'$ the corresponding maximal ideal of S'. Hence we have

$$\begin{array}{ccccc} k[x_1, \ldots, x_n] & \longrightarrow & S' & \longrightarrow & S \\ \uparrow & & \uparrow & & \uparrow \\ \mathfrak{m}'' & \longrightarrow & \mathfrak{m}' & \longrightarrow & \mathfrak{m} \end{array}$$

By our choice of J the exact sequence

$$J/\mathfrak{m}''J \to \mathfrak{m}''/(\mathfrak{m}'')^2 \to \mathfrak{m}'/(\mathfrak{m}')^2 \to 0$$

shows that $\dim(\mathfrak{m}'/(\mathfrak{m}')^2) = d$. Since $S'_{\mathfrak{m}'}$ surjects onto $S_{\mathfrak{m}}$ we see that $\dim(S_{\mathfrak{m}'}) \geq d$. Hence by the discussion preceding Definition 10.57.9 we conclude that $S'_{\mathfrak{m}'}$ is regular of dimension d as well. Because S' was cut out by $c = n - d$ equations we conclude that there exists a $g' \in S'$, $g' \notin \mathfrak{m}'$ such that $S'_{g'}$ is a global complete intersection over k, see Lemma 10.128.4. Also the map $S'_{\mathfrak{m}'} \to S_{\mathfrak{m}}$ is a surjection of Noetherian local domains of the same dimension and hence an isomorphism. By Lemma 10.120.7 we see that $S'_{g'} \cong S_g$ for some $g \in S$, $g \notin \mathfrak{m}$ and $g' \in S'$, $g' \notin \mathfrak{m}'$. All in all we conclude that after replacing S by a principal localization we may assume that S is a global complete intersection.

At this point we may write $S = k[x_1, \ldots, x_n]/(f_1, \ldots, f_c)$ with $\dim S = n - c$. Recall that the naive cotangent complex of this algebra is given by

$$\bigoplus S \cdot f_j \to \bigoplus S \cdot \mathrm{d}x_i$$

see Lemma 10.130.15. By this same lemma in order to show that S is smooth at $\mathfrak{m}$ we have to show that one of the $c \times c$ minors g_I of the matrix "A" giving the map above does not vanish at $\mathfrak{m}$. By Lemma 10.133.1 the matrix A mod $\mathfrak{m}$ has rank c. Thus we win. □

Lemma 10.133.3. *Let k be any field. Let S be a finite type k-algebra. Let $X = \mathrm{Spec}(S)$. Let $\mathfrak{q} \subset S$ be a prime corresponding to $x \in X$. The following are equivalent:*

1. *The k-algebra S is smooth at $\mathfrak{q}$ over k.*
2. *We have $\dim_{\kappa(\mathfrak{q})} \Omega_{S/k} \otimes_S \kappa(\mathfrak{q}) \leq \dim_x X$.*
3. *We have $\dim_{\kappa(\mathfrak{q})} \Omega_{S/k} \otimes_S \kappa(\mathfrak{q}) = \dim_x X$.*

Moreover, in this case the local ring $S_{\mathfrak{q}}$ is regular.

Proof. If S is smooth at $\mathfrak{q}$ over k, then there exists a $g \in S$, $g \notin \mathfrak{q}$ such that S_g is standard smooth over k, see Lemma 10.130.10. A standard smooth algebra over k has a module of differentials which is free of rank equal to the dimension, see Lemma 10.130.7. Thus we see that (1) implies (3). To finish the proof of the lemma it suffices to show that (2) implies (1) and that it implies that $S_{\mathfrak{q}}$ is regular.

Assume (2). By Nakayama's Lemma 10.18.1 we see that $\Omega_{S/k,\mathfrak{q}}$ can be generated by $\leq \dim_x X$ elements. We may replace S by S_g for some $g \in S$, $g \notin \mathfrak{q}$ such that $\Omega_{S/k}$ is generated by at most $\dim_x X$ elements. Let $K \supset k$ be an algebraically closed field extension such that there exists a k-algebra map $\psi : \kappa(\mathfrak{q}) \to K$. Consider $S_K = K \otimes_k S$. Let $\mathfrak{m} \subset S_K$ be the maximal ideal corresponding to the surjection

$$S_K = K \otimes_k S \longrightarrow K \otimes_k \kappa(\mathfrak{q}) \xrightarrow{\mathrm{id}_K \otimes \psi} K.$$

Note that $\mathfrak{m} \cap S = \mathfrak{q}$, in other words $\mathfrak{m}$ lies over $\mathfrak{q}$. By Lemma 10.110.6 the dimension of $X_K = \mathrm{Spec}(S_K)$ at the point corresponding to $\mathfrak{m}$ is $\dim_x X$. By Lemma 10.108.6 this is equal to $\dim((S_K)_{\mathfrak{m}})$. By Lemma 10.125.12 the module of differentials of S_K over K is the base change of $\Omega_{S/k}$, hence also generated by at most $\dim_x X = \dim((S_K)_{\mathfrak{m}})$ elements. By Lemma 10.133.2 we see that S_K is smooth at $\mathfrak{m}$ over K. By Lemma 10.130.17 this implies that S is smooth at $\mathfrak{q}$ over k. This proves (1). Moreover, we know by Lemma 10.133.2 that the local ring

$(S_K)_{\mathfrak{m}}$ is regular. Since $S_{\mathfrak{q}} \to (S_K)_{\mathfrak{m}}$ is flat we conclude from Lemma 10.105.8 that $S_{\mathfrak{q}}$ is regular. □

The following lemma can be significantly generalized (in several different ways).

Lemma 10.133.4. *Let k be a field. Let R be a Noetherian local ring containing k. Assume that the residue field $\kappa = R/\mathfrak{m}$ is a finitely generated separable extension of k. Then the map*

$$d : \mathfrak{m}/\mathfrak{m}^2 \longrightarrow \Omega_{R/k} \otimes_R \kappa(\mathfrak{m})$$

is injective.

Proof. We may replace R by $R/\mathfrak{m}^2$. Hence we may assume that $\mathfrak{m}^2 = 0$. By assumption we may write $\kappa = k(\overline{x}_1, \ldots, \overline{x}_r, \overline{y})$ where $\overline{x}_1, \ldots, \overline{x}_r$ is a transcendence basis of κ over k and $\overline{y}$ is separable algebraic over $k(\overline{x}_1, \ldots, \overline{x}_r)$. Say its minimal equation is $P(\overline{y}) = 0$ with $P(T) = T^d + \sum_{i<d} a_i T^i$, with $a_i \in k(\overline{x}_1, \ldots, \overline{x}_r)$ and $P'(\overline{y}) \neq 0$. Choose any lifts $x_i \in R$ of the elements $\overline{x}_i \in \kappa$. This gives a commutative diagram

$$\begin{array}{ccc} R & \longrightarrow & \kappa \\ & \overset{\varphi}{\nwarrow} & \uparrow \\ & & k(\overline{x}_1, \ldots, \overline{x}_r) \end{array}$$

of k-algebras. We want to extend the left upwards arrow φ to a k-algebra map from κ to R. To do this choose any $y \in R$ lifting $\overline{y}$. To see that it defines a k-algebra map defined on $\kappa \cong k(\overline{x}_1, \ldots, \overline{x}_r)[T]/(P)$ all we have to show is that we may choose y such that $P^\varphi(y) = 0$. If not then we compute for $\delta \in \mathfrak{m}$ that

$$P(y + \delta) = P(y) + P'(y)\delta$$

because $\mathfrak{m}^2 = 0$. Since $P'(y)\delta = P'(\overline{y})\delta$ we see that we can adjust our choice as desired. This shows that $R \cong \kappa \oplus \mathfrak{m}$ as k-algebras! From a direct computation of $\Omega_{\kappa \oplus \mathfrak{m}/k}$ the lemma follows. □

Lemma 10.133.5. *Let k be a field. Let S be a finite type k-algebra. Let $\mathfrak{q} \subset S$ be a prime. Assume $\kappa(\mathfrak{q})$ is separable over k. The following are equivalent:*

1. *The algebra S is smooth at $\mathfrak{q}$ over k.*
2. *The ring $S_{\mathfrak{q}}$ is regular.*

Proof. Denote $R = S_{\mathfrak{q}}$ and denote its maximal by $\mathfrak{m}$ and its residue field κ. By Lemma 10.133.4 and 10.125.9 we see that there is a short exact sequence

$$0 \to \mathfrak{m}/\mathfrak{m}^2 \to \Omega_{R/k} \otimes_R \kappa \to \Omega_{\kappa/k} \to 0$$

Note that $\Omega_{R/k} = \Omega_{S/k,\mathfrak{q}}$, see Lemma 10.125.8. Moreover, since κ is separable over k we have $\dim_\kappa \Omega_{\kappa/k} = \operatorname{trdeg}_k(\kappa)$. Hence we get

$$\dim_\kappa \Omega_{R/k} \otimes_R \kappa = \dim_\kappa \mathfrak{m}/\mathfrak{m}^2 + \operatorname{trdeg}_k(\kappa) \geq \dim R + \operatorname{trdeg}_k(\kappa) = \dim_{\mathfrak{q}} S$$

(see Lemma 10.110.3 for the last equality) with equality if and only if R is regular. Thus we win by applying Lemma 10.133.3. □

Lemma 10.133.6. *Let $R \to S$ be a $\mathbf{Q}$-algebra map. Let $f \in S$ be such that $\Omega_{S/R} = S df \oplus C$ for some S-submodule C. Then*

1. *f is not nilpotent, and*
2. *if S is a Noetherian local ring, then f is a nonzerodivisor in S.*

Proof. For $a \in S$ write $\mathrm{d}(a) = \theta(a)\mathrm{d}f + c(a)$ for some $\theta(a) \in S$ and $c(a) \in C$. Consider the R-derivation $S \to S$, $a \mapsto \theta(a)$. Note that $\theta(f) = 1$.

If $f^n = 0$ with $n > 1$ minimal, then $0 = \theta(f^n) = nf^{n-1}$ contradicting the minimality of n. We conclude that f is not nilpotent.

Suppose $fa = 0$. If f is a unit then $a = 0$ and we win. Assume f is not a unit. Then $0 = \theta(fa) = f\theta(a) + a$ by the Leibniz rule and hence $a \in (f)$. By induction suppose we have shown $fa = 0 \Rightarrow a \in (f^n)$. Then writing $a = f^n b$ we get $0 = \theta(f^{n+1}b) = (n+1)f^n b + f^{n+1}\theta(b)$. Hence $a = f^n b = -f^{n+1}\theta(b)/(n+1) \in (f^{n+1})$. Since in the Noetherian local ring S we have $\bigcap(f^n) = 0$, see Lemma 10.47.6 we win. □

The following is probably quite useless in applications.

Lemma 10.133.7. *Let k be a field of characteristic 0. Let S be a finite type k-algebra. Let $\mathfrak{q} \subset S$ be a prime. The following are equivalent:*

1. *The algebra S is smooth at $\mathfrak{q}$ over k.*
2. *The $S_\mathfrak{q}$-module $\Omega_{S/k,\mathfrak{q}}$ is (finite) free.*
3. *The ring $S_\mathfrak{q}$ is regular.*

Proof. In characteristic zero any field extension is separable and hence the equivalence of (1) and (3) follows from Lemma 10.133.5. Also (1) implies (2) by definition of smooth algebras. Assume that $\Omega_{S/k,\mathfrak{q}}$ is free over $S_\mathfrak{q}$. We are going to use the notation and observations made in the proof of Lemma 10.133.5. So $R = S_\mathfrak{q}$ with maximal ideal $\mathfrak{m}$ and residue field κ. Our goal is to prove R is regular.

If $\mathfrak{m}/\mathfrak{m}^2 = 0$, then $\mathfrak{m} = 0$ and $R \cong \kappa$. Hence R is regular and we win.

If $\mathfrak{m}/\mathfrak{m}^2 \neq 0$, then choose any $f \in \mathfrak{m}$ whose image in $\mathfrak{m}/\mathfrak{m}^2$ is not zero. By Lemma 10.133.4 we see that $\mathrm{d}f$ has nonzero image in $\Omega_{R/k}/\mathfrak{m}\Omega_{R/k}$. By assumption $\Omega_{R/k} = \Omega_{S/k,\mathfrak{q}}$ is finite free and hence by Nakayama's Lemma 10.18.1 we see that $\mathrm{d}f$ generates a direct summand. We apply Lemma 10.133.6 to deduce that f is a nonzerodivisor in R. Furthermore, by Lemma 10.125.9 we get an exact sequence

$$(f)/(f^2) \to \Omega_{R/k} \otimes_R R/fR \to \Omega_{(R/fR)/k} \to 0$$

This implies that $\Omega_{(R/fR)/k}$ is finite free as well. Hence by induction we see that R/fR is a regular local ring. Since $f \in \mathfrak{m}$ was a nonzerodivisor we conclude that R is regular, see Lemma 10.101.7. □

Example 10.133.8. Lemma 10.133.7 does not hold in characteristic $p > 0$. The standard examples are the ring maps

$$\mathbf{F}_p \longrightarrow \mathbf{F}_p[x]/(x^p)$$

whose module of differentials is free but is clearly not smooth, and the ring map ($p > 2$)

$$\mathbf{F}_p(t) \to \mathbf{F}_p(t)[x, y]/(x^p + y^2 + \alpha)$$

which is not smooth at the prime $\mathfrak{q} = (y, x^p - \alpha)$ but is regular.

Using the material above we can characterize smoothness at the generic point in terms of field extensions.

Lemma 10.133.9. *Let $R \to S$ be an injective finite type ring map with R and S domains. Then $R \to S$ is smooth at $\mathfrak{q} = (0)$ if and only if $f.f.(R) \subset f.f.(S)$ is a separable extension of fields.*

Proof. Assume $R \to S$ is smooth at (0). We may replace S by S_g for some nonzero $g \in S$ and assume that $R \to S$ is smooth. Set $K = f.f.(R)$. Then $K \to S \otimes_R K$ is smooth (Lemma 10.130.4). Moreover, for any field extension $K \subset K'$ the ring map $K' \to S \otimes_R K'$ is smooth as well. Hence $S \otimes_R K'$ is a regular ring by Lemma 10.133.3, in particular reduced. It follows that $S \otimes_R K$ is a geometrically reduced over K. Hence $f.f.(S)$ is geometrically reduced over K, see Lemma 10.40.3. Hence $f.f.(S)/K$ is separable by Lemma 10.41.1.

Conversely, assume that $f.f.(R) \subset f.f.(S)$ is separable. We may assume $R \to S$ is of finite presentation, see Lemma 10.28.1. It suffices to prove that $K \to S \otimes_R K$ is smooth at (0), see Lemma 10.130.17. This follows from Lemma 10.133.5, the fact that a field is a regular ring, and the assumption that $f.f.(R) \to f.f.(S)$ is separable. □

10.134. Smooth ring maps in the Noetherian case

Definition 10.134.1. Let $\varphi : B' \to B$ be a ring map. We say φ is a *small extension* if B' and B are local Artinian rings, φ is surjective and $I = \mathrm{Ker}(\varphi)$ has length 1 as a B'-module.

Clearly this means that $I^2 = 0$ and that $I = (x)$ for some $x \in B'$ such that $\mathfrak{m}'x = 0$ where $\mathfrak{m}' \subset B'$ is the maximal ideal.

Lemma 10.134.2. *Let $R \to S$ be a ring map. Let $\mathfrak{q}$ be a prime ideal of S lying over $\mathfrak{p} \subset R$. Assume R is Noetherian and $R \to S$ of finite type. The following are equivalent:*

1. *$R \to S$ is smooth at $\mathfrak{q}$,*
2. *for every surjection of local R-algebras $(B', \mathfrak{m}') \to (B, \mathfrak{m})$ with $Ker(B' \to B)$ having square zero and every solid commutative diagram*

$$\begin{array}{ccc} S & \longrightarrow & B \\ \uparrow & \nearrow & \uparrow \\ R & \longrightarrow & B' \end{array}$$

   such that $\mathfrak{q} = S \cap \mathfrak{m}$ there exists a dotted arrow making the diagram commute,
3. *same as in (2) but with $B' \to B$ ranging over small extensions, and*
4. *same as in (2) but with $B' \to B$ ranging over small extensions such that in addition $S \to R$ induces an isomorphism $\kappa(\mathfrak{q}) \cong \kappa(\mathfrak{m})$.*

Proof. Assume (1). This means there exists a $g \in S$, $g \notin \mathfrak{q}$ such that $R \to S_g$ is smooth. By Proposition 10.131.13 we know that $R \to S_g$ is formally smooth. Note that given any diagram as in (2) the map $S \to B$ factors automatically through $S_{\mathfrak{q}}$ and a fortiori through S_g. The formal smoothness of S_g over R gives us a morphism $S_g \to B'$ fitting into a similar diagram with S_g at the upper left corner. Composing with $S \to S_g$ gives the desired arrow. In other words, we have shown that (1) implies (2).

Clearly (2) implies (3) and (3) implies (4).

Assume (4). We are going to show that (1) holds, thereby finishing the proof of the lemma. Choose a presentation $S = R[x_1, \ldots, x_n]/(f_1, \ldots, f_m)$. This is possible as

S is of finite type over R and therefore of finite presentation (see Lemma 10.29.4). Set $I = (f_1, \ldots, f_m)$. Consider the naive cotangent complex

$$\mathrm{d} : I/I^2 \longrightarrow \bigoplus\nolimits_{j=1}^{m} S\mathrm{d}x_j$$

of this presentation (see Section 10.127). It suffices to show that when we localize this complex at $\mathfrak{q}$ then the map becomes a split injection, see Lemma 10.130.12. Denote $S' = R[x_1, \ldots, x_n]/I^2$. By Lemma 10.125.11 we have

$$S \otimes_{S'} \Omega_{S'/R} = S \otimes_{R[x_1,\ldots,x_n]} \Omega_{R[x_1,\ldots,x_n]/R} = \bigoplus\nolimits_{j=1}^{m} S\mathrm{d}x_j.$$

Thus the map

$$\mathrm{d} : I/I^2 \longrightarrow S \otimes_{S'} \Omega_{S'/R}$$

is the same as the map in the naive cotangent complex above. In particular the truth of the assertion we are trying to prove depends only on the three rings $R \to S' \to S$. Let $\mathfrak{q}' \subset R[x_1, \ldots, x_n]$ be the prime ideal corresponding to $\mathfrak{q}$. Since localization commutes with taking modules of differentials (Lemma 10.125.8) we see that it suffices to show that the map

$$\mathrm{d} : I_{\mathfrak{q}'}/I_{\mathfrak{q}'}^2 \longrightarrow S_{\mathfrak{q}} \otimes_{S'_{\mathfrak{q}'}} \Omega_{S'_{\mathfrak{q}'}/R} \tag{10.134.2.1}$$

coming from $R \to S'_{\mathfrak{q}'} \to S_{\mathfrak{q}}$ is a split injection.

Let $N \in \mathbf{N}$ be an integer. Consider the ring

$$B'_N = S'_{\mathfrak{q}'}/(\mathfrak{q}')^N S'_{\mathfrak{q}'} = (S'/(\mathfrak{q}')^N S')_{\mathfrak{q}'}$$

and its quotient $B_N = B'_N/IB'_N$. Note that $B_N \cong S_{\mathfrak{q}}/\mathfrak{q}^N S_{\mathfrak{q}}$. Observe that B'_N is an Artinian local ring since it is the quotient of a local Noetherian ring by a power of its maximal ideal. Consider a filtration of the kernel I_N of $B'_N \to B_N$ by B'_N-submodules

$$0 \subset J_{N,1} \subset J_{N,2} \subset \ldots \subset J_{N,n(N)} = I_N$$

such that each successive quotient $J_{N,i}/J_{N,i-1}$ has length 1. (As B'_N is Artinian such a filtration exists.) This gives a sequence of small extensions

$$B'_N \to B'_N/J_{N,1} \to B'_N/J_{N,2} \to \ldots \to B'_N/J_{N,n(N)} = B'_N/I_N = B_N = S_{\mathfrak{q}}/\mathfrak{q}^N S_{\mathfrak{q}}$$

Applying condition (4) successively to these small extensions starting with the map $S \to B_N$ we see there exists a commutative diagram

$$\begin{array}{ccc} S & \longrightarrow & B_N \\ \uparrow & \nearrow & \uparrow \\ R & \longrightarrow & B'_N \end{array}$$

Clearly the ring map $S \to B'_N$ factors as $S \to S_{\mathfrak{q}} \to B'_N$ where $S_{\mathfrak{q}} \to B'_N$ is a local homomorphism of local rings. Moreover, since the maximal ideal of B'_N to the Nth power is zero we conclude that $S_{\mathfrak{q}} \to B'_N$ factors through $S_{\mathfrak{q}}/(\mathfrak{q})^N S_{\mathfrak{q}} = B_N$. In other words we have shown that for all $N \in \mathbf{N}$ the surjection of R-algebras $B'_N \to B_N$ has a splitting.

Consider the presentation

$$I_N \to B_N \otimes_{B'_N} \Omega_{B'_N/R} \to \Omega_{B_N/R} \to 0$$

coming from the surjection $B'_N \to B_N$ with kernel I_N (see Lemma 10.125.9). By the above the R-algebra map $B'_N \to B_N$ has a right inverse. Hence by Lemma 10.125.10 we see that the sequence above is split exact! Thus for every N the map

$$I_N \longrightarrow B_N \otimes_{B'_N} \Omega_{B'_N/R}$$

is a split injection. The rest of the proof is gotten by unwinding what this means exactly. Note that

$$I_N = I_{\mathfrak{q}'}/(I_{\mathfrak{q}'}^2 + (\mathfrak{q}')^N \cap I_{\mathfrak{q}'})$$

By Artin-Rees (Lemma 10.47.4) we find a $c \geq 0$ such that

$$S_{\mathfrak{q}}/\mathfrak{q}^{N-c}S_{\mathfrak{q}} \otimes_{S_{\mathfrak{q}}} I_N = S_{\mathfrak{q}}/\mathfrak{q}^{N-c}S_{\mathfrak{q}} \otimes_{S_{\mathfrak{q}}} I_{\mathfrak{q}'}/I_{\mathfrak{q}'}^2$$

for all $N \geq c$ (these tensor product are just a fancy way of dividing by $\mathfrak{q}^{N-c}$). We may of course assume $c \geq 1$. By Lemma 10.125.11 we see that

$$S'_{\mathfrak{q}'}/(\mathfrak{q}')^{N-c}S'_{\mathfrak{q}'} \otimes_{S'_{\mathfrak{q}'}} \Omega_{B'_N/R} = S'_{\mathfrak{q}'}/(\mathfrak{q}')^{N-c}S'_{\mathfrak{q}'} \otimes_{S'_{\mathfrak{q}'}} \Omega_{S'_{\mathfrak{q}'}/R}$$

we can further tensor this by $B_N = S_{\mathfrak{q}}/\mathfrak{q}^N$ to see that

$$S_{\mathfrak{q}}/\mathfrak{q}^{N-c}S_{\mathfrak{q}} \otimes_{S'_{\mathfrak{q}'}} \Omega_{B'_N/R} = S_{\mathfrak{q}}/\mathfrak{q}^{N-c}S_{\mathfrak{q}} \otimes_{S'_{\mathfrak{q}'}} \Omega_{S'_{\mathfrak{q}'}/R}.$$

Since a split injection remains a split injection after tensoring with anything we see that

$$S_{\mathfrak{q}}/\mathfrak{q}^{N-c}S_{\mathfrak{q}} \otimes_{S_{\mathfrak{q}}} (10.134.2.1) = S_{\mathfrak{q}}/\mathfrak{q}^{N-c}S_{\mathfrak{q}} \otimes_{S_{\mathfrak{q}}} (I_N \longrightarrow B_N \otimes_{B'_N} \Omega_{B'_N/R})$$

is a split injection for all $N \geq c$. By Lemma 10.69.1 we see that (10.134.2.1) is a split injection. This finishes the proof. □

10.135. Overview of results on smooth ring maps

Here is a list of results on smooth ring maps that we proved in the preceding sections. For more precise statements and definitions please consult the references given.

(1) A ring map $R \to S$ is smooth if it is of finite presentation and the naive cotangent complex of S/R is quasi-isomorphic to a finite projective S-module in degree 0, see Definition 10.130.1.
(2) If S is smooth over R, then $\Omega_{S/R}$ is a finite projective S-module, see discussion following Definition 10.130.1.
(3) The property of being smooth is local on S, see Lemma 10.130.13.
(4) The property of being smooth is stable under base change, see Lemma 10.130.4.
(5) The property of being smooth is stable under composition, see Lemma 10.130.14.
(6) A smooth ring map is syntomic, in particular flat, see Lemma 10.130.10.
(7) A finitely presented, flat ring map with smooth fibre rings is smooth, see Lemma 10.130.16.
(8) A finitely presented ring map $R \to S$ is smooth if and only if it is formally smooth, see Proposition 10.131.13.
(9) If $R \to S$ is a finite type ring map with R Noetherian then to check that $R \to S$ is smooth it suffices to check the lifting property of formal smoothness along small extensions of Artinian local rings, see Lemma 10.134.2.

(10) A smooth ring map $R \to S$ is the base change of a smooth ring map $R_0 \to S_0$ with R_0 of finite type over $\mathbf{Z}$, see Lemma 10.131.14.
(11) Formation of the set of points where a ring map is smooth commutes with flat base change, see Lemma 10.130.17.
(12) If S is of finite type over an algebraically closed field k, and $\mathfrak{m} \subset S$ a maximal ideal, then the following are equivalent
   (a) S is smooth over k in a neighbourhood of $\mathfrak{m}$,
   (b) $S_\mathfrak{m}$ is a regular local ring,
   (c) $\dim(S_\mathfrak{m}) = \dim_{\kappa(m)} \Omega_{S/k} \otimes_S \kappa(\mathfrak{m})$.

   see Lemma 10.133.2.
(13) If S is of finite type over a field k, and $\mathfrak{q} \subset S$ a prime ideal, then the following are equivalent
   (a) S is smooth over k in a neighbourhood of $\mathfrak{q}$,
   (b) $\dim_\mathfrak{q}(S/k) = \dim_{\kappa(\mathfrak{q})} \Omega_{S/k} \otimes_S \kappa(\mathfrak{q})$.

   see Lemma 10.133.3.
(14) If S is smooth over a field, then all its local rings are regular, see Lemma 10.133.3.
(15) If S is of finite type over a field k, $\mathfrak{q} \subset S$ a prime ideal, the field extension $k \subset \kappa(\mathfrak{q})$ is separable and $S_\mathfrak{q}$ is regular, then S is smooth over k at $\mathfrak{q}$, see Lemma 10.133.5.
(16) If S is of finite type over a field k, if k has characteristic 0, if $\mathfrak{q} \subset S$ a prime ideal, and if $\Omega_{S/k,\mathfrak{q}}$ is free, then S is smooth over k at $\mathfrak{q}$, see Lemma 10.133.7.

Some of these results were proved using the notion of a standard smooth ring map, see Definition 10.130.6. This is the analogue of what a relative global complete intersection map is for the case of syntomic morphisms. It is also the easiest way to make examples.

10.136. Étale ring maps

An étale ring map is a smooth ring map whose relative dimension is equal to zero. This is the same as the following slightly more direct definition.

Definition 10.136.1. Let $R \to S$ be a ring map. We say $R \to S$ is *étale* if it is of finite presentation and the naive cotangent complex $NL_{S/R}$ is quasi-isomorphic to zero. Given a prime $\mathfrak{q}$ of S we say that $R \to S$ is *étale at* $\mathfrak{q}$ if there exists a $g \in S$, $g \notin \mathfrak{q}$ such that $R \to S_g$ is étale.

In particular we see that $\Omega_{S/R} = 0$ if S is étale over R. If $R \to S$ is smooth, then $R \to S$ is étale if and only if $\Omega_{S/R} = 0$. From our results on smooth ring maps we automatically get a whole host of results for étale maps. We summarize these in Lemma 10.136.3 below. But before we do so we prove that *any* étale ring map is standard smooth.

Lemma 10.136.2. *Any étale ring map is standard smooth. More precisely, if $R \to S$ is étale, then there exists a presentation $S = R[x_1, \ldots, x_n]/(f_1, \ldots, f_n)$ such that the image of $\det(\partial f_j/\partial x_i)$ is invertible in S.*

Proof. Let $R \to S$ be étale. Choose a presentation $S = R[x_1, \ldots, x_n]/I$. As $R \to S$ is étale we know that

$$\mathrm{d} : I/I^2 \longrightarrow \bigoplus\nolimits_{i=1,\ldots,n} S\mathrm{d}x_i$$

is an isomorphism, in particular I/I^2 is a free S-module. Thus by Lemma 10.129.6 we may assume (after possibly changing the presentation), that $I = (f_1, \ldots, f_c)$ such that the classes $f_i \bmod I^2$ form a basis of I/I^2. It follows immediately from the fact that the displayed map above is an isomorphism that $c = n$ and that $\det(\partial f_j/\partial x_i)$ is invertible in S. □

Lemma 10.136.3. *Results on étale ring maps.*

(1) *The ring map $R \to R_f$ is étale for any ring R and any $f \in R$.*
(2) *Compositions of étale ring maps are étale.*
(3) *A base change of an étale ring map is étale.*
(4) *The property of being étale is local: Given a ring map $R \to S$ and elements $g_1, \ldots, g_m \in S$ which generate the unit ideal such that $R \to S_{g_j}$ is étale for $j = 1, \ldots, m$ then $R \to S$ is étale.*
(5) *Given $R \to S$ of finite presentation, and a flat ring map $R \to R'$, set $S' = R' \otimes_R S$. The set of primes where $R \to S'$ is étale is the inverse image via $\mathrm{Spec}(S') \to \mathrm{Spec}(S)$ of the set of primes where $R \to S$ is étale.*
(6) *An étale ring map is syntomic, in particular flat.*
(7) *If S is finite type over a field k, then S is étale over k if and only if $\Omega_{S/k} = 0$.*
(8) *Any étale ring map $R \to S$ is the base change of an étale ring map $R_0 \to S_0$ with R_0 of finite type over $\mathbf{Z}$.*
(9) *Let $A = \operatorname{colim} A_i$ be a filtered colimit of rings. Let $A \to B$ be an étale ring map. Then there exists an étale ring map $A_i \to B_i$ for some i such that $B \cong A \otimes_{A_i} B_i$.*
(10) *Let A be a ring. Let S be a multiplicative subset of A. Let $S^{-1}A \to B'$ be étale. Then there exists an étale ring map $A \to B$ such that $B' \cong S^{-1}B$.*

Proof. In each case we use the corresponding result for smooth ring maps with a small argument added to show that $\Omega_{S/R}$ is zero.

Proof of (1). The ring map $R \to R_f$ is smooth and $\Omega_{R_f/R} = 0$.

Proof of (2). The composition $A \to C$ of smooth maps $A \to B$ and $B \to C$ is smooth, see Lemma 10.130.14. By Lemma 10.125.7 we see that $\Omega_{C/A}$ is zero as both $\Omega_{C/B}$ and $\Omega_{B/A}$ are zero.

Proof of (3). Let $R \to S$ be étale and $R \to R'$ be arbitrary. Then $R' \to S' = R' \otimes_R S$ is smooth, see Lemma 10.130.4. Since $\Omega_{S'/R'} = S' \otimes_S \Omega_{S/R}$ by Lemma 10.125.12 we conclude that $\Omega_{S'/R'} = 0$. Hence $R' \to S'$ is étale.

Proof of (4). Assume the hypotheses of (4). By Lemma 10.130.13 we see that $R \to S$ is smooth. We are also given that $\Omega_{S_{g_i}/R} = (\Omega_{S/R})_{g_i} = 0$ for all i. Then $\Omega_{S/R} = 0$, see Lemma 10.22.2.

Proof of (5). The result for smooth maps is Lemma 10.130.17. In the proof of that lemma we used that $NL_{S/R} \otimes_S S'$ is homotopy equivalent to $NL_{S'/R'}$. This reduces us to showing that if M is a finitely presented S-module the set of primes $\mathfrak{q}'$ of S' such that $(M \otimes_S S')_{\mathfrak{q}'} = 0$ is the inverse image of the set of primes $\mathfrak{q}$ of S such that $M_{\mathfrak{q}} = 0$. This is true (proof omitted).

Proof of (6). Follows directly from the corresponding result for smooth ring maps (Lemma 10.130.10).

Proof of (7). Follows from Lemma 10.133.3 and the definitions.

Proof of (8). Lemma 10.131.14 gives the result for smooth ring maps. The resulting smooth ring map $R_0 \to S_0$ satisfies the hypotheses of Lemma 10.124.8, and hence we may replace S_0 by the factor of relative dimension 0 over R_0.

Proof of (9). Follows from (8) since $R_0 \to A$ will factor through A_i for some i.

Proof of (10). Follows from (9), (1), and (2) since $S^{-1}A$ is a filtered colimit of principal localizations of A. □

Next we work out in more detail what it means to be étale over a field.

Lemma 10.136.4. *Let k be a field. A ring map $k \to S$ is étale if and only if S is isomorphic as a k-algebra to a finite product of finite separable extensions of k.*

Proof. If $k \to k'$ is a finite separable field extension then we can write $k' = k(\alpha) \cong k[x]/(f)$. Here f is the minimal polynomial of the element α. Since k' is separable over k we have $\gcd(f, f') = 1$. This implies that $\mathrm{d} : k' \cdot f \to k' \cdot \mathrm{d}x$ is an isomorphism. Hence $k \to k'$ is étale.

Conversely, suppose that $k \to S$ is étale. Let $\overline{k}$ be an algebraic closure of k. Then $S \otimes_k \overline{k}$ is étale over $\overline{k}$. Suppose we have the result over $\overline{k}$. Then $S \otimes_k \overline{k}$ is reduced and hence S is reduced. Also, $S \otimes_k \overline{k}$ is finite over $\overline{k}$ and hence S is finite over k. Hence S is a finite product $S = \prod k_i$ of fields, see Lemma 10.49.2 and Proposition 10.57.6. The result over $\overline{k}$ means $S \otimes_k \overline{k}$ is isomorphic to a finite product of copies of $\overline{k}$, which implies that each $k \subset k_i$ is finite separable, see for example Lemmas 10.41.1 and 10.41.3. Thus we have reduced to the case $k = \overline{k}$. In this case Lemma 10.133.2 (combined with $\Omega_{S/k} = 0$) we see that $S_{\mathfrak{m}} \cong k$ for all maximal ideals $\mathfrak{m} \subset S$. This implies the result because S is the product of the localizations at its maximal ideals by Lemma 10.49.2 and Proposition 10.57.6 again. □

Lemma 10.136.5. *Let $R \to S$ be a ring map. Let $\mathfrak{q} \subset S$ be a prime lying over $\mathfrak{p}$ in R. If S/R is étale at $\mathfrak{q}$ then*

1. *we have $\mathfrak{p}S_{\mathfrak{q}} = \mathfrak{q}S_{\mathfrak{q}}$ is the maximal ideal of the local ring $S_{\mathfrak{q}}$, and*
2. *the field extension $\kappa(\mathfrak{p}) \subset \kappa(\mathfrak{q})$ is finite separable.*

Proof. First we may replace S by S_g for some $g \in S$, $g \notin \mathfrak{q}$ and assume that $R \to S$ is étale. Then the lemma follows from Lemma 10.136.4 by unwinding the fact that $S \otimes_R \kappa(\mathfrak{p})$ is étale over $\kappa(\mathfrak{p})$. □

Lemma 10.136.6. *An étale ring map is quasi-finite.*

Proof. Let $R \to S$ be an étale ring map. By definition $R \to S$ is of finite type. For any prime $\mathfrak{p} \subset R$ the fibre ring $S \otimes_R \kappa(\mathfrak{p})$ is étale over $\kappa(\mathfrak{p})$ and hence a finite products of fields finite separable over $\kappa(\mathfrak{p})$, in particular finite over $\kappa(\mathfrak{p})$. Thus $R \to S$ is quasi-finite by Lemma 10.116.4. □

Lemma 10.136.7. *Let $R \to S$ be a ring map. Let $\mathfrak{q}$ be a prime of S lying over a prime $\mathfrak{p}$ of R. If*

1. *$R \to S$ is of finite presentation,*
2. *$R_{\mathfrak{p}} \to S_{\mathfrak{q}}$ is flat*
3. *$\mathfrak{p}S_{\mathfrak{q}}$ is the maximal ideal of the local ring $S_{\mathfrak{q}}$, and*
4. *the field extension $\kappa(\mathfrak{p}) \subset \kappa(\mathfrak{q})$ is finite separable,*

then $R \to S$ is étale at $\mathfrak{q}$.

Proof. Apply Lemma 10.116.2 to find a $g \in S$, $g \notin \mathfrak{q}$ such that $\mathfrak{q}$ is the only prime of S_g lying over $\mathfrak{p}$. We may and do replace S by S_g. Then $S \otimes_R \kappa(\mathfrak{p})$ has a unique prime, hence is a local ring, hence is equal to $S_\mathfrak{q}/\mathfrak{p}S_\mathfrak{q} \cong \kappa(\mathfrak{q})$. By Lemma 10.130.16 there exists a $g \in S$, $g \notin \mathfrak{q}$ such that $R \to S_g$ is smooth. Replace S by S_g again we may assume that $R \to S$ is smooth. By Lemma 10.130.10 we may even assume that $R \to S$ is standard smooth, say $S = R[x_1, \ldots, x_n]/(f_1, \ldots, f_c)$. Since $S \otimes_R \kappa(\mathfrak{p}) = \kappa(\mathfrak{q})$ has dimension 0 we conclude that $n = c$, i.e., if $R \to S$ is étale. □

Lemma 10.136.8. *A ring map is étale if and only if it is flat, unramified, and of finite presentation.*

Proof. This follows by combining Lemmas 10.136.3 (flatness of étale maps), 10.136.5 (étale maps are unramified), and 10.136.7 (flat and unramified maps of finite presentation are étale). □

Here is a completely new phenomenon.

Lemma 10.136.9. *Let $R \to S$ and $R \to S'$ be étale. Then any R-algebra map $S' \to S$ is étale.*

Proof. First of all we note that $S' \to S$ is of finite presentation by Lemma 10.6.2. Let $\mathfrak{q} \subset S$ be a prime ideal lying over the primes $\mathfrak{q}' \subset S'$ and $\mathfrak{p} \subset R$. By Lemma 10.136.5 the ring map $S_\mathfrak{q}/\mathfrak{p}S_\mathfrak{q} \to S'_{\mathfrak{q}'}/\mathfrak{p}S'_{\mathfrak{q}'}$ is a map finite separable extensions of $\kappa(\mathfrak{p})$. In particular it is flat. Hence by Lemma 10.122.8 we see that $S'_{\mathfrak{q}'} \to S_\mathfrak{q}$ is flat. Thus $S' \to S$ is flat. Moreover, the above also shows that $\mathfrak{q}'S_\mathfrak{q}$ is the maximal ideal of $S_\mathfrak{q}$ and that the residue field extension of $S'_{\mathfrak{q}'} \to S_\mathfrak{q}$ is finite separable. Hence from Lemma 10.136.7 we conclude that $S' \to S$ is étale at $\mathfrak{q}$. Since being étale is local (see Lemma 10.136.3) we win. □

Lemma 10.136.10. *Let $\varphi : R \to S$ be a ring map. If $R \to S$ is surjective, flat and finitely presented then there exist an idempotent $e \in R$ such that $S = R_e$.*

Proof. Since $\mathrm{Spec}(S) \to \mathrm{Spec}(R)$ is a homeomorphism onto a closed subset (see Lemma 10.16.7) and is open (see Proposition 10.38.8) we see that the image is $D(e)$ for some idempotent $e \in R$ (see Lemma 10.19.3). Thus $R_e \to S$ induces a bijection on spectra. Now this map induces an isomorphism on all local rings for example by Lemmas 10.73.4 and 10.18.1. Then it follows that $R_e \to S$ is also injective, for example see Lemma 10.22.1. □

Lemma 10.136.11. *Let R be a ring and let $I \subset R$ be an ideal. Let $R/I \to \overline{S}$ be an étale ring map. Then there exists an étale ring map $R \to S$ such that $\overline{S} \cong S/IS$ as R/I-algebras.*

Proof. By Lemma 10.136.2 we can write $\overline{S} = (R/I)[x_1, \ldots, x_n]/(\overline{f}_1, \ldots, \overline{f}_n)$ as in Definition 10.130.6 with $\overline{\Delta} = \det(\frac{\partial \overline{f}_i}{\partial x_j})_{i,j=1,\ldots,n}$ invertible in $\overline{S}$. Just take some lifts f_i and set $S = R[x_1, \ldots, x_n, x_{n+1}]/(f_1, \ldots, f_c, x_{n+1}\Delta - 1)$ where $\Delta = \det(\frac{\partial f_i}{\partial x_j})_{i,j=1,\ldots,c}$ as in Example 10.130.8. This proves the lemma. □

Lemma 10.136.12. *Consider a commutative diagram*

$$\begin{array}{ccccccccc} 0 & \longrightarrow & J & \longrightarrow & B' & \longrightarrow & B & \longrightarrow & 0 \\ & & \uparrow & & \uparrow & & \uparrow & & \\ 0 & \longrightarrow & I & \longrightarrow & A' & \longrightarrow & A & \longrightarrow & 0 \end{array}$$

with exact rows where $B' \to B$ and $A' \to A$ are surjective ring maps whose kernels are ideals of square zero. If $A \to B$ is étale, and $J = I \otimes_A B$, then $A' \to B'$ is étale.

Proof. By Lemma 10.136.11 there exists an étale ring map $A' \to C$ such that $C/IC = B$. Then $A' \to C$ is formally smooth (by Proposition 10.131.13) hence we get an A'-algebra map $\varphi : C \to B'$. Since $A' \to C$ is flat we have $I \otimes_A B = I \otimes_A C/IC = IC$. Hence the assumption that $J = I \otimes_A B$ implies that φ induces an isomorphism $IC \to J$ and an isomorphism $C/IC \to B'/IB'$, whence φ is an isomorphism. □

Example 10.136.13. Let $n, m \geq 1$ be integers. Consider the ring map

$$\begin{array}{rcl} R = \mathbf{Z}[a_1, \ldots, a_{n+m}] & \longrightarrow & S = \mathbf{Z}[b_1, \ldots, b_n, c_1, \ldots, c_m] \\ a_1 & \longmapsto & b_1 + c_1 \\ a_2 & \longmapsto & b_2 + b_1 c_1 + c_2 \\ \ldots & \ldots & \ldots \\ a_{n+m} & \longmapsto & b_n c_m \end{array}$$

of Example 10.129.7. Write symbolically

$$S = R[b_1, \ldots, c_m]/(\{a_k(b_i, c_j) - a_k\}_{k=1,\ldots,n+m})$$

where for example $a_1(b_i, c_j) = b_1 + c_1$. The matrix of partial derivatives is

$$\begin{pmatrix} 1 & c_1 & \ldots & c_m & 0 & \ldots & 0 \\ 0 & 1 & c_1 & \ldots & c_m & \ldots & 0 \\ \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots \\ 0 & \ldots & 0 & 1 & c_1 & \ldots & c_m \\ 1 & b_1 & \ldots & b_n & 0 & \ldots & 0 \\ 0 & 1 & b_1 & \ldots & b_n & \ldots & 0 \\ \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots \\ 0 & \ldots & 0 & 1 & b_1 & \ldots & b_n \end{pmatrix}$$

The determinant Δ of this matrix is better known as the *resultant* of the polynomials $g = x^n + b_1 x^{n-1} + \ldots + b_n$ and $h = x^m + c_1 x^{m-1} + \ldots + c_m$, and the matrix above is known as the *Sylvester matrix* associated to g, h. In a formula $\Delta = \mathrm{Res}_x(g, h)$. The Sylvester matrix is the transpose of the matrix of the linear map

$$\begin{array}{rcl} S[x]_{<m} \oplus S[x]_{<n} & \longrightarrow & S[x]_{<n+m} \\ a \oplus b & \longmapsto & ag + bh \end{array}$$

Let $\mathfrak{q} \subset S$ be any prime. By the above the following are equivalent:

(1) $R \to S$ is étale at $\mathfrak{q}$,
(2) $\Delta = \mathrm{Res}_x(g, h) \notin \mathfrak{q}$,
(3) the images $\overline{g}, \overline{h} \in \kappa(\mathfrak{q})[x]$ of the polynomials g, h are relatively prime in $\kappa(\mathfrak{q})[x]$.

The equivalence of (2) and (3) holds because the image of the Sylvester matrix in $\mathrm{Mat}(n+m, \kappa(\mathfrak{q}))$ has a kernel if and only if the polynomials $\overline{g}, \overline{h}$ have a factor in common. We conclude that the ring map

$$R \longrightarrow S[\frac{1}{\Delta}] = S[\frac{1}{\mathrm{Res}_x(g,h)}]$$

is étale.

Lemma 10.136.2 tells us that it does not really make sense to define a standard étale morphism to be a standard smooth morphism of relative dimension 0. As a model for an étale morphism we take the example given by a finite separable extension $k \subset k'$ of fields. Namely, we can always find an element $\alpha \in k'$ such that $k' = k(\alpha)$ and such that the minimal polynomial $f(x) \in k[x]$ of α has derivative f' which is relatively prime to f.

Definition 10.136.14. Let R be a ring. Let $g, f \in R[x]$. Assume that f is monic and the derivative f' is invertible in the localization $R[x]_g/(f)$. In this case the ring map $R \to R[x]_g/(f)$ is said to be *standard étale*.

Lemma 10.136.15. *Let $R \to R[x]_g/(f)$ be standard étale.*

1. *The ring map $R \to R[x]_g/(f)$ is étale.*
2. *For any ring map $R \to R'$ the base change $R' \to R'[x]_g/(f)$ of the standard étale ring map $R \to R[x]_g/(f)$ is standard étale.*
3. *Any principal localization of $R[x]_g/(f)$ is standard étale over R.*
4. *A composition of standard étale maps is* **not** *standard étale in general.*

Proof. Omitted. Here is an example for (4). The ring map $\mathbf{F}_2 \to \mathbf{F}_{2^2}$ is standard étale. The ring map $\mathbf{F}_{2^2} \to \mathbf{F}_{2^2} \times \mathbf{F}_{2^2} \times \mathbf{F}_{2^2} \times \mathbf{F}_{2^2}$ is standard étale. But the ring map $\mathbf{F}_2 \to \mathbf{F}_{2^2} \times \mathbf{F}_{2^2} \times \mathbf{F}_{2^2} \times \mathbf{F}_{2^2}$ is not standard étale. □

Standard étale morphisms are a convenient way to produce étale maps. Here is an example.

Lemma 10.136.16. *Let R be a ring. Let $\mathfrak{p}$ be a prime of R. Let $\kappa(\mathfrak{p}) \subset L$ be a finite separable field extension. There exists an étale ring map $R \to R'$ together with a prime $\mathfrak{p}'$ lying over $\mathfrak{p}$ such that the field extension $\kappa(\mathfrak{p}) \subset \kappa(\mathfrak{p}')$ is isomorphic to $\kappa(\mathfrak{p}) \subset L$.*

Proof. By the theorem of the primitive element we may write $L = \kappa(\mathfrak{p})[\alpha]$. Let $\overline{f} \in \kappa(\mathfrak{p})[x]$ denote the minimal polynomial for α (in particular this is monic). After replacing α by $c\alpha$ for some $c \in R$, $c \notin \mathfrak{p}$ we may assume all the coefficients of $\overline{f}$ are in the image of $R \to \kappa(\mathfrak{p})$ (verification omitted). Thus we can find a monic polynomial $f \in R[x]$ which maps to $\overline{f}$ in $\kappa(\mathfrak{p})[x]$. Since $\kappa(\mathfrak{p}) \subset L$ is separable, we see that $\gcd(\overline{f}, \overline{f}') = 1$. Hence there is an element $\gamma \in L$ such that $\overline{f}'(\alpha)\gamma = 1$. Thus we get a R-algebra map

$$\begin{aligned} R[x, 1/f']/(f) &\longrightarrow L \\ x &\longmapsto \alpha \\ 1/f' &\longmapsto \gamma \end{aligned}$$

The left hand side is a standard étale algebra R' over R and the kernel of the ring map gives the desired prime. □

Proposition 10.136.17. *Let $R \to S$ be a ring map. Let $\mathfrak{q} \subset S$ be a prime. If $R \to S$ is étale at $\mathfrak{q}$, then there exists a $g \in S$, $g \notin \mathfrak{q}$ such that $R \to S_g$ is standard étale.*

Proof. The following proof is a little roundabout and there may be ways to shorten it.

Step 1. By Definition 10.136.1 there exists a $g \in S$, $g \notin \mathfrak{q}$ such that $R \to S_g$ is étale. Thus we may assume that S is étale over R.

Step 2. By Lemma 10.136.3 there exists an étale ring map $R_0 \to S_0$ with R_0 of finite type over $\mathbf{Z}$, and a ring map $R_0 \to R$ such that $R = R \otimes_{R_0} S_0$. Denote $\mathfrak{q}_0$ the prime of S_0 corresponding to $\mathfrak{q}$. If we show the result for $(R_0 \to S_0, \mathfrak{q}_0)$ then the result follows for $(R \to S, \mathfrak{q})$ by base change. Hence we may assume that R is Noetherian.

Step 3. Note that $R \to S$ is quasi-finite by Lemma 10.136.6. By Lemma 10.117.15 there exists a finite ring map $R \to S'$, an R-algebra map $S' \to S$, an element $g' \in S'$ such that $g' \notin \mathfrak{q}$ such that $S' \to S$ induces an isomorphism $S'_{g'} \cong S_{g'}$. (Note that of course S' is not étale over R in general.) Thus we may assume that (a) R is Noetherian, (b) $R \to S$ is finite and (c) $R \to S$ is étale at $\mathfrak{q}$ (but no longer necessarily étale at all primes).

Step 4. Let $\mathfrak{p} \subset R$ be the prime corresponding to $\mathfrak{q}$. Consider the fibre ring $S \otimes_R \kappa(\mathfrak{p})$. This is a finite algebra over $\kappa(\mathfrak{p})$. Hence it is Artinian (see Lemma 10.49.2) and so a finite product of local rings

$$S \otimes_R \kappa(\mathfrak{p}) = \prod\nolimits_{i=1}^{n} A_i$$

see Proposition 10.57.6. One of the factors, say A_1, is the local ring $S_\mathfrak{q}/\mathfrak{p}S_\mathfrak{q}$ which is isomorphic to $\kappa(\mathfrak{q})$, see Lemma 10.136.5. The other factors correspond to the other primes, say $\mathfrak{q}_2, \ldots, \mathfrak{q}_n$ of S lying over $\mathfrak{p}$.

Step 5. We may choose a nonzero element $\alpha \in \kappa(\mathfrak{q})$ which generates the finite separable field extension $\kappa(\mathfrak{p}) \subset \kappa(\mathfrak{q})$ (so even if the field extension is trivial we do not allow $\alpha = 0$). Note that for any $\lambda \in \kappa(\mathfrak{p})^*$ the element $\lambda\alpha$ also generates $\kappa(\mathfrak{q})$ over $\kappa(\mathfrak{p})$. Consider the element

$$\bar{t} = (\alpha, 0, \ldots, 0) \in \prod\nolimits_{i=1}^{n} A_i = S \otimes_R \kappa(\mathfrak{p}).$$

After possibly replacing α by $\lambda\alpha$ as above we may assume that $\bar{t}$ is the image of $t \in S$. Let $I \subset R[x]$ be the kernel of the R-algebra map $R[x] \to S$ which maps x to t. Set $S' = R[x]/I$, so $S' \subset S$. Here is a diagram

$$\begin{array}{ccccc} R[x] & \longrightarrow & S' & \longrightarrow & S \\ \uparrow & \nearrow & & \nearrow & \\ R & & & & \end{array}$$

By construction the primes $\mathfrak{q}_j$, $j \geq 2$ of S all lie over the prime $(\mathfrak{p}, x)$ of $R[x]$, whereas the prime $\mathfrak{q}$ lies over a different prime of $R[x]$ because $\alpha \neq 0$.

Step 6. Denote $\mathfrak{q}' \subset S'$ the prime of S' corresponding to $\mathfrak{q}$. By the above $\mathfrak{q}$ is the only prime of S lying over $\mathfrak{q}'$. Thus we see that $S_\mathfrak{q} = S_{\mathfrak{q}'}$, see Lemma 10.38.11 (we have going up for $S' \to S$ by Lemma 10.33.20 since $S' \to S$ is finite as $R \to S$

is finite). It follows that $S'_{\mathfrak{q}'} \to S_{\mathfrak{q}}$ is finite and injective as the localization of the finite injective ring map $S' \to S$. Consider the maps of local rings

$$R_{\mathfrak{p}} \to S'_{\mathfrak{q}'} \to S_{\mathfrak{q}}$$

The second map is finite and injective. We have $S_{\mathfrak{q}}/\mathfrak{p}S_{\mathfrak{q}} = \kappa(\mathfrak{q})$, see Lemma 10.136.5. Hence a fortiori $S_{\mathfrak{q}}/\mathfrak{q}'S_{\mathfrak{q}} = \kappa(\mathfrak{q})$. Since

$$\kappa(\mathfrak{p}) \subset \kappa(\mathfrak{q}') \subset \kappa(\mathfrak{q})$$

and since α is in the image of $\kappa(\mathfrak{q}')$ in $\kappa(\mathfrak{q})$ we conclude that $\kappa(\mathfrak{q}') = \kappa(\mathfrak{q})$. Hence by Nakayama's Lemma 10.18.1 applied to the $S'_{\mathfrak{q}'}$-module map $S'_{\mathfrak{q}'} \to S_{\mathfrak{q}}$, the map $S'_{\mathfrak{q}'} \to S_{\mathfrak{q}}$ is surjective. In other words, $S'_{\mathfrak{q}'} \cong S_{\mathfrak{q}}$.

Step 7. By Lemma 10.120.7 there exist $g \in S$, $g \notin \mathfrak{q}$ and $g' \in S'$, $g' \notin \mathfrak{q}'$ such that $S'_{g'} \cong S_g$. As R is Noetherian the ring S' is finite over R because it is an R-submodule of the finite R-module S. Hence after replacing S by S' we may assume that (a) R is Noetherian, (b) S finite over R, (c) S is étale over R at $\mathfrak{q}$, and (d) $S = R[x]/I$.

Step 8. Consider the ring $S \otimes_R \kappa(\mathfrak{p}) = \kappa(\mathfrak{p})[x]/\overline{I}$ where $\overline{I} = I \cdot \kappa(\mathfrak{p})[x]$ is the ideal generated by I in $\kappa(\mathfrak{p})[x]$. As $\kappa(\mathfrak{p})[x]$ is a PID we know that $\overline{I} = (\overline{h})$ for some monic $\overline{h} \in \kappa(\mathfrak{p})$. After replacing $\overline{h}$ by $\lambda \cdot \overline{h}$ for some $\lambda \in \kappa(\mathfrak{p})$ we may assume that $\overline{h}$ is the image of some $h \in R[x]$. (The problem is that we do not know if we may choose h monic.) Also, as in Step 4 we know that $S \otimes_R \kappa(\mathfrak{p}) = A_1 \times \ldots \times A_n$ with $A_1 = \kappa(\mathfrak{q})$ a finite separable extension of $\kappa(\mathfrak{p})$ and $A_2, \ldots, A_n$ local. This implies that

$$\overline{h} = \overline{h}_1 \overline{h}_2^{e_2} \ldots \overline{h}_n^{e_n}$$

for certain pairwise coprime irreducible monic polynomials $\overline{h}_i \in \kappa(\mathfrak{p})[x]$ and certain $e_2, \ldots, e_n \geq 1$. Here the numbering is chosen so that $A_i = \kappa(\mathfrak{p})[x]/(\overline{h}_i^{e_i})$ as $\kappa(\mathfrak{p})[x]$-algebras. Note that $\overline{h}_1$ is the minimal polynomial of $\alpha \in \kappa(\mathfrak{q})$ and hence is a separable polynomial (its derivative is prime to itself).

Step 9. Let $m \in I$ be a monic element; such an element exists because the ring extension $R \to R[x]/I$ is finite hence integral. Denote $\overline{m}$ the image in $\kappa(\mathfrak{p})[x]$. We may factor

$$\overline{m} = \overline{k}\overline{h}_1^{d_1} \overline{h}_2^{d_2} \ldots \overline{h}_n^{d_n}$$

for some $d_1 \geq 1$, $d_j \geq e_j$, $j = 2, \ldots, n$ and $\overline{k} \in \kappa(\mathfrak{p})[x]$ prime to all the $\overline{h}_i$. Set $f = m^l + h$ where $l \deg(m) > \deg(h)$, and $l \geq 2$. Then f is monic as a polynomial over R. Also, the image $\overline{f}$ of f in $\kappa(\mathfrak{p})[x]$ factors as

$$\overline{f} = \overline{h}_1 \overline{h}_2^{e_2} \ldots \overline{h}_n^{e_n} + \overline{k}^l \overline{h}_1^{ld_1} \overline{h}_2^{ld_2} \ldots \overline{h}_n^{ld_n} = \overline{h}_1 (\overline{h}_2^{e_2} \ldots \overline{h}_n^{e_n} + \overline{k}^l \overline{h}_1^{ld_1 - 1} \overline{h}_2^{ld_2} \ldots \overline{h}_n^{ld_n}) = \overline{h}_1 \overline{w}$$

with $\overline{w}$ a polynomial relatively prime to $\overline{h}_1$. Set $g = f'$ (the derivative with respect to x).

Step 10. The ring map $R[x] \to S = R[x]/I$ has the properties: (1) it maps f to zero, and (2) it maps g to an element of $S \setminus \mathfrak{q}$. The first assertion is clear since f is an element of I. For the second assertion we just have to show that g does not map to zero in $\kappa(\mathfrak{q}) = \kappa(\mathfrak{p})[x]/(\overline{h}_1)$. The image of g in $\kappa(\mathfrak{p})[x]$ is the derivative of $\overline{f}$. Thus (2) is clear because

$$\overline{g} = \frac{\mathrm{d}\overline{f}}{\mathrm{d}x} = \overline{w} \frac{\mathrm{d}\overline{h}_1}{\mathrm{d}x} + \overline{h}_1 \frac{\mathrm{d}\overline{w}}{\mathrm{d}x},$$

$\overline{w}$ is prime to $\overline{h}_1$ and $\overline{h}_1$ is separable.

Step 11. We conclude that $\varphi : R[x]/(f) \to S$ is a surjective ring map, $R[x]_g/(f)$ is étale over R (because it is standard étale, see Lemma 10.136.15) and $\varphi(g) \notin \mathfrak{q}$. Pick an element $g' \in R[x]/(f)$ such that also $\varphi(g') \notin \mathfrak{q}$ and $S_{\varphi(g')}$ is étale over R (which exists since S is étale over R at $\mathfrak{q}$). Then the ring map $R[x]_{gg'}/(f) \to S_{\varphi(g)}$ is a surjective map of étale algebras over R. Hence it is étale by Lemma 10.136.9. Hence it is a localization by Lemma 10.136.10. Thus a localization of S at an element not in $\mathfrak{q}$ is isomorphic to a localization of a standard étale algebra over R which is what we wanted to show. □

The following two lemmas say that the étale topology is coarser than the topology generated by Zariski coverings and finite flat morphisms. They should be skipped on a first reading.

Lemma 10.136.18. *Let $R \to S$ be a standard étale morphism. There exists a ring map $R \to S'$ with the following properties*

1. *$R \to S'$ is finite, finitely presented, and flat (in other words S' is finite projective as an R-module),*
2. *$\mathrm{Spec}(S') \to \mathrm{Spec}(R)$ is surjective,*
3. *for every prime $\mathfrak{q} \subset S$, lying over $\mathfrak{p} \subset R$ and every prime $\mathfrak{q}' \subset S'$ lying over $\mathfrak{p}$ there exists a $g' \in S'$, $g' \notin \mathfrak{q}'$ such that the ring map $R \to S'_{g'}$ factors through a map $\varphi : S \to S'_{g'}$ with $\varphi^{-1}(\mathfrak{q}'S'_{g'}) = \mathfrak{q}$.*

Proof. Let $S = R[x]_g/(f)$ be a presentation of S as in Definition 10.136.14. Write $f = x^n + a_1x^{n-1} + \ldots + a_n$ with $a_i \in R$. By Lemma 10.129.9 there exists a finite locally free and faithfully flat ring map $R \to S'$ such that $f = \prod(x - \alpha_i)$ for certain $\alpha_i \in S'$. Hence $R \to S'$ satisfies conditions (1), (2). Let $\mathfrak{q} \subset R[x]/(f)$ be a prime ideal with $g \notin \mathfrak{q}$ (i.e., it corresponds to a prime of S). Let $\mathfrak{p} = R \cap \mathfrak{q}$ and let $\mathfrak{q}' \subset S'$ be a prime lying over $\mathfrak{p}$. Note that there are n maps of R-algebras

$$\begin{aligned} \varphi_i : R[x]/(f) &\longrightarrow S' \\ x &\longmapsto \alpha_i \end{aligned}$$

To finish the proof we have to show that for some i we have (a) the image of $\varphi_i(g)$ in $\kappa(\mathfrak{q}')$ is not zero, and (b) $\varphi_i^{-1}(\mathfrak{q}') = \mathfrak{q}$. Because then we can just take $g' = \varphi_i(g)$, and $\varphi = \varphi_i$ for that i.

Let $\overline{f}$ denote the image of f in $\kappa(\mathfrak{p})[x]$. Note that as a point of $\mathrm{Spec}(\kappa(\mathfrak{p})[x]/(\overline{f}))$ the prime $\mathfrak{q}$ corresponds to an irreducible factor f_1 of $\overline{f}$. Moreover, $g \notin \mathfrak{q}$ means that f_1 does not divide the image $\overline{g}$ of g in $\kappa(\mathfrak{p})[x]$. Denote $\overline{\alpha}_1, \ldots, \overline{\alpha}_n$ the images of $\alpha_1, \ldots, \alpha_n$ in $\kappa(\mathfrak{q}')$. Note that the polynomial $\overline{f}$ splits completely in $\kappa(\mathfrak{q}')[x]$, namely

$$\overline{f} = \prod\nolimits_i (x - \overline{\alpha}_i)$$

Moreover $\varphi_i(g)$ reduces to $\overline{g}(\overline{\alpha}_i)$. It follows we may pick i such that $f_1(\overline{\alpha}_i) = 0$ and $\overline{g}(\overline{\alpha}_i) \neq 0$. For this i properties (a) and (b) hold. Some details omitted. □

Lemma 10.136.19. *Let $R \to S$ be a ring map. Assume that*

1. *$R \to S$ is étale, and*
2. *$\mathrm{Spec}(S) \to \mathrm{Spec}(R)$ is surjective.*

Then there exists a ring map $R \to S'$ such that

(1) $R \to S'$ *is finite, finitely presented, and flat (in other words it is finite projective as an* R*-module),*
(2) $\mathrm{Spec}(S') \to \mathrm{Spec}(R)$ *is surjective,*
(3) *for every prime* $\mathfrak{q}' \subset S'$ *there exists a* $g' \in S'$, $g' \notin \mathfrak{q}'$ *such that the ring map* $R \to S'_{g'}$ *factors as* $R \to S \to S'_{g'}$.

Proof. By Proposition 10.136.17 and the quasi-compactness of $\mathrm{Spec}(S)$ (see Lemma 10.16.10) we can find $g_1, \ldots, g_n \in S$ generating the unit ideal of S such that each $R \to S_{g_i}$ is standard étale. If we prove the lemma for the ring map $R \to \prod_{i=1,\ldots,n} S_{g_i}$ then the lemma follows for the ring map $R \to S$. Hence we may assume that $S = \prod_{i=1,\ldots,n} S_i$ is a finite product of standard étale morphisms.

For each i choose a ring map $R \to S'_i$ as in Lemma 10.136.18 adapted to the standard étale morphism $R \to S_i$. Set $S' = S'_1 \otimes_R \ldots \otimes_R S'_n$; we will use the R-algebra maps $S'_i \to S'$ without further mention below. We claim this works. Properties (1) and (2) are immediate. For property (3) suppose that $\mathfrak{q}' \subset S'$ is a prime. Denote $\mathfrak{p}$ its image in $\mathrm{Spec}(R)$. Choose $i \in \{1, \ldots, n\}$ such that $\mathfrak{p}$ is in the image of $\mathrm{Spec}(S_i) \to \mathrm{Spec}(R)$; this is possible by assumption. Set $\mathfrak{q}'_i \subset S'_i$ the image of $\mathfrak{q}'$ in the spectrum of S'_i. By construction of S'_i there exists a $g'_i \in S'_i$ such that $R \to (S'_i)_{g'_i}$ factors as $R \to S_i \to (S'_i)_{g'_i}$. Hence also $R \to S'_{g'_i}$ factors as

$$R \to S_i \to (S'_i)_{g'_i} \to S'_{g'_i}$$

as desired. □

Lemma 10.136.20. *Let* R *be a ring. Let* $f \in R[x]$ *be a monic polynomial. Let* $\mathfrak{p}$ *be a prime of* R*. Let* $f \bmod \mathfrak{p} = \overline{g}\overline{h}$ *be a factorization of the image of* f *in* $\kappa(\mathfrak{p})[x]$*. If* $\gcd(\overline{g}, \overline{h}) = 1$*, then there exist*

(1) *an étale ring map* $R \to R'$*,*
(2) *a prime* $\mathfrak{p}' \subset R'$ *lying over* $\mathfrak{p}$*, and*
(3) *a factorization* $f = gh$ *in* $R'[x]$

such that

(1) $\kappa(\mathfrak{p}) = \kappa(\mathfrak{p}')$*,*
(2) $\overline{g} = g \bmod \mathfrak{p}'$*,* $\overline{h} = h \bmod \mathfrak{p}'$*, and*
(3) *the polynomials* g, h *generate the unit ideal in* $R'[x]$*.*

Proof. Suppose $\overline{g} = \overline{b}_0 x^n + \overline{b}_1 x^{n-1} + \ldots + \overline{b}_n$, and $\overline{h} = \overline{c}_0 x^m + \overline{c}_1 x^{m-1} + \ldots + \overline{c}_m$ with $\overline{b}_0, \overline{c}_0 \in \kappa(\mathfrak{p})$ nonzero. After localizing R at some element of R not contained in $\mathfrak{p}$ we may assume $\overline{b}_0$ is the image of an invertible element $b_0 \in R$. Replacing $\overline{g}$ by $\overline{g}/b_0$ and $\overline{h}$ by $b_0\overline{h}$ we reduce to the case where $\overline{g}$, $\overline{h}$ are monic (verification omitted). Say $\overline{g} = x^n + \overline{b}_1 x^{n-1} + \ldots + \overline{b}_n$, and $\overline{h} = x^m + \overline{c}_1 x^{m-1} + \ldots + \overline{c}_m$. Write $f = x^{n+m} + a_1 x^{n-1} + \ldots + a_{n+m}$. Consider the fibre product

$$R' = R \otimes_{\mathbf{Z}[a_1,\ldots,a_{n+m}]} \mathbf{Z}[b_1, \ldots, b_n, c_1, \ldots, c_m]$$

where the map $\mathbf{Z}[a_k] \to \mathbf{Z}[b_i, c_j]$ is as in Examples 10.129.7 and 10.136.13. By construction there is an R-algebra map

$$R' = R \otimes_{\mathbf{Z}[a_1,\ldots,a_{n+m}]} \mathbf{Z}[b_1, \ldots, b_n, c_1, \ldots, c_m] \longrightarrow \kappa(\mathfrak{p})$$

which maps b_i to $\overline{b}_i$ and c_j to $\overline{c}_j$. Denote $\mathfrak{p}' \subset R'$ the kernel of this map. Since by assumption the polynomials $\overline{g}, \overline{h}$ are relatively prime we see that the element $\Delta = \mathrm{Res}_x(g, h) \in \mathbf{Z}[b_i, c_j]$ (see Example 10.136.13) does not map to zero in $\kappa(\mathfrak{p})$

under the displayed map. We conclude that $R \to R'$ is étale at $\mathfrak{p}'$. In fact a solution to the problem posed in the lemma is the ring map $R \to R'[1/\Delta]$ and the prime $\mathfrak{p}'R'[1/\Delta]$. Because $\text{Res}_x(f, g)$ is invertible in this ring the Sylvester matrix is invertible over R' and hence $1 = ag + bh$ for some $a, b \in R'[x]$ see Example 10.136.13. □

The following lemmas say roughly that after an étale extension a quasi-finite ring map becomes finite. To help interpret the results recall that the locus where a finite type ring map is quasi-finite is open (see Lemma 10.117.14) and that formation of this locus commutes with arbitrary base change (see Lemma 10.116.8).

Lemma 10.136.21. *Let $R \to S' \to S$ be ring maps. Let $\mathfrak{p} \subset R$ be a prime. Let $g \in S'$ be an element. Assume*

(1) *$R \to S'$ is integral,*
(2) *$R \to S$ is finite type,*
(3) *$S'_g \cong S_g$, and*
(4) *g invertible in $S' \otimes_R \kappa(\mathfrak{p})$.*

Then there exists a $f \in R$, $f \notin \mathfrak{p}$ such that $R_f \to S_f$ is finite.

Proof. By assumption the image T of $V(g) \subset \text{Spec}(S')$ under the morphism $\text{Spec}(S') \to \text{Spec}(R)$ does not contain $\mathfrak{p}$. By Section 10.38 especially, Lemma 10.38.6 we see T is closed. Pick $f \in R$, $f \notin \mathfrak{p}$ such that $T \cap V(f) = \emptyset$. Then we see that g becomes invertible in S'_f. Hence $S'_f \cong S_f$. Thus S_f is both of finite type and integral over R_f, hence finite. □

Lemma 10.136.22. *Let $R \to S$ be a ring map. Let $\mathfrak{q} \subset S$ be a prime lying over the prime $\mathfrak{p} \subset R$. Assume $R \to S$ finite type and quasi-finite at $\mathfrak{q}$. Then there exists*

(1) *an étale ring map $R \to R'$,*
(2) *a prime $\mathfrak{p}' \subset R'$ lying over $\mathfrak{p}$,*
(3) *a product decomposition*

$$R' \otimes_R S = A \times B$$

with the following properties

(1) *$\kappa(\mathfrak{p}) = \kappa(\mathfrak{p}')$,*
(2) *$R' \to A$ is finite,*
(3) *A has exactly one prime $\mathfrak{r}$ lying over $\mathfrak{p}'$, and*
(4) *$\mathfrak{r}$ lies over $\mathfrak{q}$.*

Proof. Let $S' \subset S$ be the integral closure of R in S. Let $\mathfrak{q}' = S' \cap \mathfrak{q}$. By Zariski's Main Theorem 10.117.13 there exists a $g \in S'$, $g \notin \mathfrak{q}'$ such that $S'_g \cong S_g$. Consider the fibre rings $F = S \otimes_R \kappa(\mathfrak{p})$ and $F' = S' \otimes_R \kappa(\mathfrak{p})$. Denote $\overline{\mathfrak{q}}'$ the prime of F' corresponding to $\mathfrak{q}'$. Since F' is integral over $\kappa(\mathfrak{p})$ we see that $\overline{\mathfrak{q}}'$ is a closed point of $\text{Spec}(F')$, see Lemma 10.33.17. Note that $\mathfrak{q}$ defines an isolated closed point $\overline{\mathfrak{q}}$ of $\text{Spec}(F)$ (see Definition 10.116.3). Since $S'_g \cong S_g$ we have $F'_g \cong F_g$, so $\overline{\mathfrak{q}}$ and $\overline{\mathfrak{q}}'$ have isomorphic open neighbourhoods in $\text{Spec}(F)$ and $\text{Spec}(F')$. We conclude the set $\{\overline{\mathfrak{q}}'\} \subset \text{Spec}(F')$ is open. Combined with closedness shown above we conclude that $\overline{\mathfrak{q}}'$ defines an isolated closed point of $\text{Spec}(F')$ as well.

An additional small remark is that under the map $\text{Spec}(F) \to \text{Spec}(F')$ the point $\overline{\mathfrak{q}}$ is the only point mapping to $\overline{\mathfrak{q}}'$. This follows from the discussion above.

By Lemma 10.21.3 we may write $F' = F'_1 \times F'_2$ with $\mathrm{Spec}(F'_1) = \{\overline{\mathfrak{q}}'\}$. Since $F' = S' \otimes_R \kappa(\mathfrak{p})$, there exists an $s' \in S'$ which maps to the element $(r, 0) \in F'_1 \times F'_2 = F'$ for some $r \in R$, $r \notin \mathfrak{p}$. In fact, what we will use about s' is that it is an element of S', not contained in $\mathfrak{q}'$, and contained in any other prime lying over $\mathfrak{p}$.

Let $f(x) \in R[x]$ be a monic polynomial such that $f(s') = 0$. Denote $\overline{f} \in \kappa(\mathfrak{p})[x]$ the image. We can factor it as $\overline{f} = x^e \overline{h}$ where $\overline{h}(0) \neq 0$. By Lemma 10.136.20 we can find an étale ring extension $R \to R'$, a prime $\mathfrak{p}'$ lying over $\mathfrak{p}$, and a factorization $f = hi$ in $R'[x]$ such that $\kappa(\mathfrak{p}) = \kappa(\mathfrak{p}')$, $x^e = h \bmod \mathfrak{p}'$, $\bar{i} = i \bmod \mathfrak{p}'$, and we can write $ah + bi = 1$ in $R'[x]$ (for suitable a, b).

Consider the elements $h(s'), i(s') \in R' \otimes_R S'$. By construction we have $h(s')i(s') = f(s') = 0$. On the other hand they generate the unit ideal since $a(s')h(s') + b(s')i(s') = 1$. Thus we see that $R' \otimes_R S'$ is the product of the localizations at these elements:

$$R' \otimes_R S' = (R' \otimes_R S')_{h(s')} \times (R' \otimes_R S')_{i(s')} = S'_1 \times S'_2$$

Moreover this product decomposition is compatible with the product decomposition we found for the fibre ring F'; this comes from our choice of s', h which guarantee that $\overline{\mathfrak{q}}'$ is the only prime of F' which does not contain the image of $h(s')$ in F'. Here we use that the fibre ring of $R' \otimes_R S'$ over R' at $\mathfrak{p}'$ is the same as F' due to the fact that $\kappa(\mathfrak{p}) = \kappa(\mathfrak{p}')$. It follows that S'_1 has exactly one prime, say $\mathfrak{r}'$, lying over $\mathfrak{p}'$ and that this prime lies over $\mathfrak{q}$. Hence the element $g \in S'$ maps to an element of S'_1 not contained in $\mathfrak{r}'$.

The base change $R' \otimes_R S$ inherits a similar product decomposition

$$R' \otimes_R S = (R' \otimes_R S)_{h(s')} \times (R' \otimes_R S)_{i(s')} = S_1 \times S_2$$

It follows from the above that S_1 has exactly one prime, say $\mathfrak{r}$, lying over $\mathfrak{p}'$ (consider the fibre ring as above), and that this prime lies over $\mathfrak{q}$.

Now we may apply Lemma 10.136.21 to the ring maps $R' \to S'_1 \to S_1$, the prime $\mathfrak{p}'$ and the element g to see that after replacing R' by a principal localization we can assume that S_1 is finite over R' as desired. □

Lemma 10.136.23. *Let $R \to S$ be a ring map. Let $\mathfrak{p} \subset R$ be a prime. Assume $R \to S$ finite type. Then there exists*

1. *an étale ring map $R \to R'$,*
2. *a prime $\mathfrak{p}' \subset R'$ lying over $\mathfrak{p}$,*
3. *a product decomposition*

$$R' \otimes_R S = A_1 \times \ldots \times A_n \times B$$

with the following properties

1. *we have $\kappa(\mathfrak{p}) = \kappa(\mathfrak{p}')$,*
2. *each A_i is finite over R',*
3. *each A_i has exactly one prime $\mathfrak{r}_i$ lying over $\mathfrak{p}'$, and*
4. *$R' \to B$ not quasi-finite at any prime lying over $\mathfrak{p}'$.*

Proof. Denote $F = S \otimes_R \kappa(\mathfrak{p})$ the fibre ring of S/R at the prime $\mathfrak{p}$. As F is of finite type over $\kappa(\mathfrak{p})$ it is Noetherian and hence $\mathrm{Spec}(F)$ has finitely many isolated closed points. If there are no isolated closed points, i.e., no primes $\mathfrak{q}$ of S over $\mathfrak{p}$

such that S/R is quasi-finite at $\mathfrak{q}$, then the lemma holds. If there exists at least one such prime $\mathfrak{q}$, then we may apply Lemma 10.136.22. This gives a diagram

$$\begin{array}{ccccc} S & \longrightarrow & R' \otimes_R S & = & A_1 \times B' \\ \uparrow & & \uparrow & \nearrow & \\ R & \longrightarrow & R' & & \end{array}$$

as in said lemma. Since the residue fields at $\mathfrak{p}$ and $\mathfrak{p}'$ are the same, the fibre rings of S/R and $(A \times B)/R'$ are the same. Hence, by induction on the number of isolated closed points of the fibre we may assume that the lemma holds for $R' \to B$ and $\mathfrak{p}'$. Thus we get an étale ring map $R' \to R''$, a prime $\mathfrak{p}'' \subset R''$ and a decomposition

$$R'' \otimes_{R'} B' = A_2 \times \ldots \times A_n \times B$$

We omit the verification that the ring map $R \to R''$, the prime $\mathfrak{p}''$ and the resulting decomposition

$$R'' \otimes_R S = (R'' \otimes_{R'} A_1) \times A_2 \times \ldots \times A_n \times B$$

is a solution to the problem posed in the lemma. □

Lemma 10.136.24. *Let $R \to S$ be a ring map. Let $\mathfrak{p} \subset R$ be a prime. Assume $R \to S$ finite type. Then there exists*

1. *an étale ring map $R \to R'$,*
2. *a prime $\mathfrak{p}' \subset R'$ lying over $\mathfrak{p}$,*
3. *a product decomposition*

$$R' \otimes_R S = A_1 \times \ldots \times A_n \times B$$

with the following properties

1. *each A_i is finite over R',*
2. *each A_i has exactly one prime $\mathfrak{r}_i$ lying over $\mathfrak{p}'$,*
3. *the finite field extensions $\kappa(\mathfrak{p}') \subset \kappa(\mathfrak{r}_i)$ are purely inseparable, and*
4. *$R' \to B$ not quasi-finite at any prime lying over $\mathfrak{p}'$.*

Proof. The strategy of the proof is to make two étale ring extensions: first we control the residue fields, then we apply Lemma 10.136.23.

Denote $F = S \otimes_R \kappa(\mathfrak{p})$ the fibre ring of S/R at the prime $\mathfrak{p}$. As in the proof of Lemma 10.136.23 there are finitely may primes, say $\mathfrak{q}_1, \ldots, \mathfrak{q}_n$ of S lying over R at which the ring map $R \to S$ is quasi-finite. Let $\kappa(\mathfrak{p}) \subset L_i \subset \kappa(\mathfrak{q}_i)$ be the subfield such that $\kappa(\mathfrak{p}) \subset L_i$ is separable, and the field extension $L_i \subset \kappa(\mathfrak{q}_i)$ is purely inseparable. Let $\kappa(\mathfrak{p}) \subset L$ be a finite Galois extension into which L_i embeds for $i = 1, \ldots, n$. By Lemma 10.136.16 we can find an étale ring extension $R \to R'$ together with a prime $\mathfrak{p}'$ lying over $\mathfrak{p}$ such that the field extension $\kappa(\mathfrak{p}) \subset \kappa(\mathfrak{p}')$ is isomorphic to $\kappa(\mathfrak{p}) \subset L$. Thus the fibre ring of $R' \otimes_R S$ at $\mathfrak{p}'$ is isomorphic to $F \otimes_{\kappa(\mathfrak{p})} L$. The primes lying over $\mathfrak{q}_i$ correspond to primes of $\kappa(\mathfrak{q}_i) \otimes_{\kappa(\mathfrak{p})} L$ which is a product of fields purely inseparable over L by our choice of L and elementary field theory. These are also the only primes over $\mathfrak{p}'$ at which $R' \to R' \otimes_R S$ is quasi-finite, by Lemma 10.116.8. Hence after replacing R by R', $\mathfrak{p}$ by $\mathfrak{p}'$, and S by $R' \otimes_R S$ we may assume that for all primes $\mathfrak{q}$ lying over $\mathfrak{p}$ for which S/R is quasi-finite the field extensions $\kappa(\mathfrak{p}) \subset \kappa(\mathfrak{q})$ are purely inseparable.

Next apply Lemma 10.136.23. The result is what we want since the field extensions do not change under this étale ring extension. □

10.137. Local homomorphisms

Lemma 10.137.1. *Let $(R, \mathfrak{m}_R) \to (S, \mathfrak{m}_S)$ be a local homomorphism of local rings. Assume S is the localization of an étale ring extension of R. Then there exists a finite, finitely presented, faithfully flat ring map $R \to S'$ such that for every maximal ideal $\mathfrak{m}'$ of S' there is a factorization*

$$R \to S \to S'_{\mathfrak{m}'}.$$

of the ring map $R \to S'_{\mathfrak{m}'}$.

Proof. Write $S = T_{\mathfrak{q}}$ for some étale R-algebra T. By Proposition 10.136.17 we may assume T is standard étale. Apply Lemma 10.136.18 to the ring map $R \to T$ to get $R \to S'$. Then in particular for every maximal ideal $\mathfrak{m}'$ of S' we get a factorization $\varphi : T \to S'_{g'}$ for some $g' \notin \mathfrak{m}'$ such that $\mathfrak{q} = \varphi^{-1}(\mathfrak{m}' S'_{g'})$. Thus φ induces the desired local ring map $S \to S'_{\mathfrak{m}'}$. □

10.138. Integral closure and smooth base change

Lemma 10.138.1. *Let R be a ring. Let $f \in R[x]$ be a monic polynomial. Let $R \to B$ be a ring map. If $h \in B[x]/(f)$ is integral over R, then the element $f'h$ can be written as $f'h = \sum_i b_i x^i$ with $b_i \in B$ integral over R.*

Proof. Say $h^e + r_1 h^{e-1} + \ldots + r_e = 0$ in the ring $B[x]/(f)$ with $r_i \in R$. There exists a finite free ring extension $B \subset B'$ such that $f = (x - \alpha_1) \ldots (x - \alpha_d)$ for some $\alpha_i \in B'$, see Lemma 10.129.9. Note that each α_i is integral over R. We may represent $h = h_0 + h_1 x + \ldots + h_{d-1} x^{d-1}$ with $h_i \in B$. Then it is a universal fact that

$$f'h \equiv \sum\nolimits_{i=1,\ldots,d} h(\alpha_i)(x - \alpha_1) \ldots \widehat{(x - \alpha_i)} \ldots (x - \alpha_d)$$

as elements of $B[x]/(f)$. You prove this by evaluating both sides at the points α_i over the ring $B_{univ} = \mathbf{Z}[\alpha_i, h_j]$ (some details omitted). By our assumption that h satisfies $h^e + r_1 h^{e-1} + \ldots + r_e = 0$ in the ring $B[x]/(f)$ we see that

$$h(\alpha_i)^e + r_1 h(\alpha_i)^{e-1} + \ldots + r_e = 0$$

in B'. Hence $h(\alpha_i)$ is integral over R. Using the formula above we see that $f'h \equiv \sum_{j=0,\ldots,d-1} b'_j x^j$ in $B'[x]/(f)$ with $b'_j \in B'$ integral over R. However, since $f'h \in B[x]/(f)$ and since $1, x, \ldots, x^{d-1}$ is a B'-basis for $B'[x]/(f)$ we see that $b'_j \in B$ as desired. □

Lemma 10.138.2. *Let $R \to S$ be an étale ring map. Let $R \to B$ be any ring map. Let $A \subset B$ be the integral closure of R in B. Let $A' \subset S \otimes_R B$ be the integral closure of S in $S \otimes_R B$. Then the canonical map $S \otimes_R A \to A'$ is an isomorphism.*

Proof. The map $S \otimes_R A \to A'$ is injective because $A \subset B$ and $R \to S$ is flat. We are going to use repeatedly that taking integral closure commutes with localization, see Lemma 10.33.9. Hence we may localize on S, by Lemma 10.22.2 (the criterion for checking whether an S-module map is an isomorphism). Thus we may assume that $S = R[x]_g/(f) = (R[x]/(f))_g$ is standard étale over R, see Proposition 10.136.17. Applying localization one more time we see that A' is $(A'')_g$ where A'' is the integral closure of $R[x]/(f)$ in $B[x]/(f)$. Suppose that $a \in A''$. It suffices to show that a is

in $S \otimes_R A$. By Lemma 10.138.1 we see that $f'a = \sum a_i x^i$ with $a_i \in A$. Since f' is invertible in $B[x]_g/(f)$ (by definition of a standard étale ring map) we conclude that $a \in S \otimes_R A$ as desired. □

Example 10.138.3. Let p be a prime number. For every $n > 0$ the ring extension

$$R = \mathbf{Z}[1/p] \subset R' = \mathbf{Z}[1/p][x]/(x^{p^n} - 1)$$

has the following property: For $d < p^n$ there exist elements $\alpha_0, \ldots, \alpha_{d-1} \in R'$ such that

$$\prod\nolimits_{0 \leq i < j < d} (\alpha_i - \alpha_j)$$

is a unit in R'. Namely, take α_i equal to the class of x^i in R'. Then we have

$$T^{p^n} - 1 = \prod\nolimits_{i=0,\ldots,p^n-1} (T - \alpha_i)$$

(for example because this is clear over $\mathbf{Q}$) and hence by taking derivatives on both sides

$$p^n \alpha_i^{p^n - 1} = (\alpha_i - \alpha_1) \ldots \widehat{(\alpha_i - \alpha_i)} \ldots (\alpha_i - \alpha_1)$$

and we see this is invertible in R'.

Lemma 10.138.4. *Let $R \to S$ be a smooth ring map. Let $R \to B$ be any ring map. Let $A \subset B$ be the integral closure of R in B. Let $A' \subset S \otimes_R B$ be the integral closure of S in $S \otimes_R B$. Then the canonical map $S \otimes_R A \to A'$ is an isomorphism.*

Proof. Arguing as in the proof of Lemma 10.138.2 we may localize on S. Hence we may assume that $R \to S$ is a standard smooth ring map, see Lemma 10.130.10. By definition of a standard smooth ring map we see that S is étale over a polynomial ring $R[x_1, \ldots, x_n]$. Since we have seen the result in the case of an étale ring extension (Lemma 10.138.2) this reduces us to the case where $S = R[x]$. Thus we have to show

$$f = \sum b_i x^i \text{ integral over } R[x] \Leftrightarrow \text{each } b_i \text{ integral over } R.$$

The implication from right to left holds because the set of elements in $B[x]$ integral over $R[x]$ is a ring (Lemma 10.33.7) and contains x.

Suppose that $f \in B[x]$ is integral over $R[x]$, and assume that $f = \sum_{i<d} b_i x^i$ has degree $< d$. Since integral closure and localization commute, it suffices to show that each b_i is integral over $R[1/2]$ and over $R[1/3]$. Hence, we can find a finite free ring extension $R \subset R'$ such that R' contains $\alpha_1, \ldots, \alpha_d$ with the property that $\prod_{i<j}(\alpha_i - \alpha_j)$ is a unit in R', see Example 10.138.3. In this case we have the universal equality

$$f = \sum_i f(\alpha_i) \frac{(x - \alpha_1) \ldots \widehat{(x - \alpha_i)} \ldots (x - \alpha_d)}{(\alpha_i - \alpha_1) \ldots \widehat{(\alpha_i - \alpha_i)} \ldots (\alpha_i - \alpha_d)}.$$

OK, and the elements $f(\alpha_i)$ are integral over R' since $(R' \otimes_R B)[x] \to R' \otimes_R B$, $h \mapsto h(\alpha_i)$ is a ring map. Hence we see that the coefficients of f in $(R' \otimes_R B)[x]$ are integral over over R'. Since R' is finite over R (hence integral over R) we see that they are integral over R also, as desired. □

10.139. Formally unramified maps

It turns out to be logically more efficient to define the notion of a formally unramified map before introducing the notion of a formally étale one.

Definition 10.139.1. Let $R \to S$ be a ring map. We say S is *formally unramified over* R if for every commutative solid diagram

$$\begin{array}{ccc} S & \longrightarrow & A/I \\ \uparrow & \nearrow & \uparrow \\ R & \longrightarrow & A \end{array}$$

where $I \subset A$ is an ideal of square zero, there exists at most one dotted arrow making the diagram commute.

Lemma 10.139.2. *Let $R \to S$ be a ring map. The following are equivalent:*

(1) *$R \to S$ is formally unramified,*
(2) *the module of differentials $\Omega_{S/R}$ is zero.*

Proof. Let $J = \mathrm{Ker}(S \otimes_R S \to S)$ be the kernel of the multiplication map. Let $A_{univ} = S \otimes_R S/J^2$. Recall that $I_{univ} = J/J^2$ is isomorphic to $\Omega_{S/R}$, see Lemma 10.125.13. Moreover, the two R-algebra maps $\sigma_1, \sigma_2 : S \to A_{univ}$, $\sigma_1(s) = s \otimes 1 \bmod J^2$, and $\sigma_2(s) = 1 \otimes s \bmod J^2$ differ by the universal derivation $\mathrm{d} : S \to \Omega_{S/R} = I_{univ}$.

Assume $R \to S$ formally unramified. Then we see that $\sigma_1 = \sigma_2$. Hence $\mathrm{d}(s) = 0$ for all $s \in S$. Hence $\Omega_{S/R} = 0$.

Assume that $\Omega_{S/R} = 0$. Let $A, I, R \to A, S \to A/I$ be a solid diagram as in Definition 10.139.1. Let $\tau_1, \tau_2 : S \to A$ be two dotted arrows making the diagram commute. Consider the R-algebra map $A_{univ} \to A$ defined by the rule $s_1 \otimes s_2 \mapsto \tau_1(s_1)\tau_2(s_2)$. We omit the verification that this is well defined. Since $A_{univ} \cong S$ as $I_{univ} = \Omega_{S/R} = 0$ we conclude that $\tau_1 = \tau_2$. □

Lemma 10.139.3. *Let $R \to S$ be a ring map. The following are equivalent:*

(1) *$R \to S$ is formally unramified,*
(2) *$R \to S_{\mathfrak{q}}$ is formally unramified for all primes $\mathfrak{q}$ of S, and*
(3) *$R_{\mathfrak{p}} \to S_{\mathfrak{q}}$ is formally unramified for all primes $\mathfrak{q}$ of S with $\mathfrak{p} = R \cap \mathfrak{q}$.*

Proof. We have seen in Lemma 10.139.2 that (1) is equivalent to $\Omega_{S/R} = 0$. Similarly, by Lemma 10.125.8 we see that (2) and (3) are equivalent to $(\Omega_{S/R})_{\mathfrak{q}} = 0$ for all $\mathfrak{q}$. Hence the equivalence follows from Lemma 10.22.1. □

Lemma 10.139.4. *Let $A \to B$ be a formally unramified ring map.*

(1) *For $S \subset A$ a multiplicative subset, $S^{-1}A \to S^{-1}B$ is formally unramified.*
(2) *For $S \subset B$ a multiplicative subset, $A \to S^{-1}B$ is formally unramified.*

Proof. Follows from Lemma 10.139.3. (You can also deduce it from Lemma 10.139.2 combined with Lemma 10.125.8.) □

Lemma 10.139.5. *Let R be a ring. Let I be a directed partially ordered set. Let $(S_i, \varphi_{ii'})$ be a system of R-algebras over I. If each $R \to S_i$ is formally unramified, then $S = \mathrm{colim}_{i \in I} S_i$ is formally unramified over R*

Proof. Consider a diagram as in Definition 10.139.1. By assumption there exists at most one R-algebra map $S_i \to A$ lifting the compositions $S_i \to S \to A/I$. Since every element of S is in the image of one of the maps $S_i \to S$ we see that there is at most one map $S \to A$ fitting into the diagram. □

10.140. Conormal modules and universal thickenings

It turns out that one can define the first infinitesimal neighbourhood not just for a closed immersion of schemes, but already for any formally unramified morphism. This is based on the following algebraic fact.

Lemma 10.140.1. *Let $R \to S$ be a formally unramified ring map. There exists a surjection of R-algebras $S' \to S$ whose kernel is an ideal of square zero with the following universal property: Given any commutative diagram*

$$\begin{array}{ccc} S & \xrightarrow{a} & A/I \\ \uparrow & & \uparrow \\ R & \xrightarrow{b} & A \end{array}$$

where $I \subset A$ is an ideal of square zero, there is a unique R-algebra map $a' : S' \to A$ such that $S' \to A \to A/I$ is equal to $S' \to S \to A$.

Proof. Choose a set of generators $z_i \in S$, $i \in I$ for S as an R-algebra. Let $P = R[\{x_i_i \in I]$ denote the polynomial ring on generators x_i, $i \in I$. Consider the R-algebra map $P \to S$ which maps x_i to z_i. Let $J = \mathrm{Ker}(P \to S)$. Consider the map

$$\mathrm{d} : J/J^2 \longrightarrow \Omega_{P/R} \otimes_P S$$

see Lemma 10.125.9. This is surjective since $\Omega_{S/R} = 0$ by assumption, see Lemma 10.139.2. Note that $\Omega_{P/R}$ is free on $\mathrm{d}x_i$, and hence the module $\Omega_{P/R} \otimes_P S$ is free over S. Thus we may choose a splitting of the surjection above and write

$$J/J^2 = K \oplus \Omega_{P/R} \otimes_P S$$

Let $J^2 \subset J' \subset J$ be the ideal of P such that J'/J^2 is the second summand in the decomposition above. Set $S' = P/J'$. We obtain a short exact sequence

$$0 \to J/J' \to S' \to S \to 0$$

and we see that $J/J' \cong K$ is a square zero ideal in S'. Hence

$$\begin{array}{ccc} S & \xrightarrow{1} & S \\ \uparrow & & \uparrow \\ R & \longrightarrow & S' \end{array}$$

is a diagram as above. In fact we claim that this is an initial object in the category of diagrams. Namely, let $(I \subset A, a, b)$ be an arbitrary diagram. We may choose an R-algebra map $\beta : P \to A$ such that

$$\begin{array}{ccccc} S & \xrightarrow{1} & S & \xrightarrow{a} & A/I \\ \uparrow & & \uparrow & & \uparrow \\ R & \longrightarrow & P & \xrightarrow{\beta} & A \\ & & & \underset{b}{\frown} & \end{array}$$

is commutative. Now it may not be the case that $\beta(J') = 0$, in other words it may not be true that β factors through $S' = P/J'$. But what is clear is that $\beta(J') \subset I$ and since $\beta(J) \subset I$ and $I^2 = 0$ we have $\beta(J^2) = 0$. Thus the "obstruction" to finding a morphism from $(J/J' \subset S', 1, R \to S')$ to $(I \subset A, a, b)$ is the corresponding S-linear map $\overline{\beta} : J'/J^2 \to I$. The choice in picking β lies in the choice of $\beta(x_i)$. A different choice of β, say β', is gotten by taking $\beta'(x_i) = \beta(x_i) + \delta_i$ with $\delta_i \in I$. In this case, for $g \in J'$, we obtain

$$\beta'(g) = \beta(g) + \sum\nolimits_i \delta_i \frac{\partial g}{\partial x_i}.$$

Since the map $\mathrm{d}|_{J'/J^2} : J'/J^2 \to \Omega_{P/R} \otimes_P S$ given by $g \mapsto \frac{\partial g}{\partial x_i} \mathrm{d}x_i$ is an isomorphism by construction, we see that there is a unique choice of $\delta_i \in I$ such that $\beta'(g) = 0$ for all $g \in J'$. (Namely, δ_i is $-\overline{\beta}(g)$ where $g \in J'/J^2$ is the unique element with $\frac{\partial g}{\partial x_j} = 1$ if $i = j$ and 0 else.) The uniqueness of the solution implies the uniqueness required in the lemma. □

In the situation of Lemma 10.140.1 the R-algebra map $S' \to S$ is unique up to unique isomorphism.

Definition 10.140.2. Let $R \to S$ be a formally unramified ring map.

(1) The *universal first order thickening* of S over R is the surjection of R-algebras $S' \to S$ of Lemma 10.140.1.
(2) The *conormal module* of $R \to S$ is the kernel I of the universal first order thickening $S' \to S$, seen as a S-module.

We often denote the conormal module $C_{S/R}$ in this situation.

Lemma 10.140.3. *Let $I \subset R$ be an ideal of a ring. The universal first order thickening of R/I over R is the surjection $R/I^2 \to R/I$. The conormal module of R/I over R is $C_{(R/I)/R} = I/I^2$.*

Proof. Omitted. □

Lemma 10.140.4. *Let $A \to B$ be a formally unramified ring map. Let $\varphi : B' \to B$ be the universal first order thickening of B over A.*

(1) *Let $S \subset A$ be a multiplicative subset. Then $S^{-1}B' \to S^{-1}B$ is the universal first order thickening of $S^{-1}B$ over $S^{-1}A$. In particular $S^{-1}C_{B/A} = C_{S^{-1}B/S^{-1}A}$.*
(2) *Let $S \subset B$ be a multiplicative subset. Then $S' = \varphi^{-1}(S)$ is a multiplicative subset in B' and $(S')^{-1}B' \to S^{-1}B$ is the universal first order thickening of $S^{-1}B$ over A. In particular $S^{-1}C_{B/A} = C_{S^{-1}B/A}$.*

Note that the lemma makes sense by Lemma 10.139.4.

Proof. With notation and assumptions as in (1). Let $(S^{-1}B)' \to S^{-1}B$ be the universal first order thickening of $S^{-1}B$ over $S^{-1}A$. Note that $S^{-1}B' \to S^{-1}B$ is a surjection of $S^{-1}A$-algebras whose kernel has square zero. Hence by definition we obtain a map $(S^{-1}B)' \to S^{-1}B'$ compatible with the maps towards $S^{-1}B$. Consider any commutative diagram

$$\begin{array}{ccccc} B & \longrightarrow & S^{-1}B & \longrightarrow & D/I \\ \uparrow & & \uparrow & & \uparrow \\ A & \longrightarrow & S^{-1}A & \longrightarrow & D \end{array}$$

where $I \subset D$ is an ideal of square zero. Since B' is the universal first order thickening of B over A we obtain an A-algebra map $B' \to D$. But it is clear that the image of S in D is mapped to invertible elements of D, and hence we obtain a compatible map $S^{-1}B' \to D$. Applying this to $D = (S^{-1}B)'$ we see that we get a map $S^{-1}B' \to (S^{-1}B)'$. We omit the verification that this map is inverse to the map described above.

With notation and assumptions as in (2). Let $(S^{-1}B)' \to S^{-1}B$ be the universal first order thickening of $S^{-1}B$ over A. Note that $(S')^{-1}B' \to S^{-1}B$ is a surjection of A-algebras whose kernel has square zero. Hence by definition we obtain a map $(S^{-1}B)' \to (S')^{-1}B'$ compatible with the maps towards $S^{-1}B$. Consider any commutative diagram

$$\begin{array}{ccccc} B & \longrightarrow & S^{-1}B & \longrightarrow & D/I \\ \uparrow & & \uparrow & & \uparrow \\ A & \longrightarrow & A & \longrightarrow & D \end{array}$$

where $I \subset D$ is an ideal of square zero. Since B' is the universal first order thickening of B over A we obtain an A-algebra map $B' \to D$. But it is clear that the image of S' in D is mapped to invertible elements of D, and hence we obtain a compatible map $(S')^{-1}B' \to D$. Applying this to $D = (S^{-1}B)'$ we see that we get a map $(S')^{-1}B' \to (S^{-1}B)'$. We omit the verification that this map is inverse to the map described above. □

Lemma 10.140.5. *Let $R \to A \to B$ be ring maps. Assume $A \to B$ formally unramified. Let $B' \to B$ be the universal first order thickening of B over A. Then B' is formally unramified over A, and the canonical map $\Omega_{A/R} \otimes_A B \to \Omega_{B'/R} \otimes_{B'} B$ is an isomorphism.*

Proof. We are going to use the construction of B' from the proof of Lemma 10.140.1 although in principle it should be possible to deduce these results formally from the definition. Namely, we choose a presentation $B = P/J$, where $P = A[x_i]$ is a polynomial ring over A. Next, we choose elements $f_i \in J$ such that $\mathrm{d}f_i = \mathrm{d}x_i \otimes 1$ in $\Omega_{P/A} \otimes_P B$. Having made these choices we have $B' = P/J'$ with $J' = (f_i) + J^2$, see proof of Lemma 10.140.1.

Consider the canonical exact sequence

$$J'/(J')^2 \to \Omega_{P/A} \otimes_P B' \to \Omega_{B'/A} \to 0$$

see Lemma 10.125.9. By construction the classes of the $f_i \in J'$ map to elements of the module $\Omega_{P/A} \otimes_P B'$ which generate it modulo J'/J^2 by construction. Since J'/J^2 is a nilpotent ideal, we see that these elements generate the module altogether (by Nakayama's Lemma 10.18.1). This proves that $\Omega_{B'/A} = 0$ and hence that B' is formally unramified over A, see Lemma 10.139.2.

Since P is a polynomial ring over A we have $\Omega_{P/R} = \Omega_{A/R} \otimes_A P \oplus \bigoplus P\mathrm{d}x_i$. We are going to use this decomposition. Consider the following exact sequence

$$J'/(J')^2 \to \Omega_{P/R} \otimes_P B' \to \Omega_{B'/R} \to 0$$

see Lemma 10.125.9. We may tensor this with B and obtain the exact sequence

$$J'/(J')^2 \otimes_{B'} B \to \Omega_{P/R} \otimes_P B \to \Omega_{B'/R} \otimes_{B'} B \to 0$$

If we remember that $J' = (f_i) + J^2$ then we see that the first arrow annihilates the submodule $J^2/(J')^2$. In terms of the direct sum decomposition $\Omega_{P/R} \otimes_P B = \Omega_{A/R} \otimes_A B \oplus \bigoplus B\mathrm{d}x_i$ given we see that the submodule $(f_i)/(J')^2 \otimes_{B'} B$ maps isomorphically onto the summand $\bigoplus B\mathrm{d}x_i$. Hence what is left of this exact sequence is an isomorphism $\Omega_{A/R} \otimes_A B \to \Omega_{B'/R} \otimes_{B'} B$ as desired. □

10.141. Formally étale maps

Definition 10.141.1. Let $R \to S$ be a ring map. We say S is *formally étale over* R if for every commutative solid diagram

$$\begin{array}{ccc} S & \longrightarrow & A/I \\ \uparrow & \nearrow & \uparrow \\ R & \longrightarrow & A \end{array}$$

where $I \subset A$ is an ideal of square zero, there exists a unique dotted arrow making the diagram commute.

Clearly a ring map is formally étale if and only if it is booth formally smooth and formally unramified.

Lemma 10.141.2. *Let $R \to S$ be a ring map of finite presentation. The following are equivalent:*

(1) *$R \to S$ is formally étale,*
(2) *$R \to S$ is étale.*

Proof. Assume that $R \to S$ is formally étale. Then $R \to S$ is smooth by Proposition 10.131.13. By Lemma 10.139.2 we have $\Omega_{S/R} = 0$. Hence $R \to S$ is étale by definition.

Assume that $R \to S$ is étale. Then $R \to S$ is formally smooth by Proposition 10.131.13. By Lemma 10.139.2 it is formally unramified. Hence $R \to S$ is formally étale. □

Lemma 10.141.3. *Let R be a ring. Let I be a directed partially ordered set. Let $(S_i, \varphi_{ii'})$ be a system of R-algebras over I. If each $R \to S_i$ is formally étale, then $S = \mathrm{colim}_{i \in I} S_i$ is formally étale over R*

Proof. Consider a diagram as in Definition 10.141.1. By assumption we get unique R-algebra maps $S_i \to A$ lifting the compositions $S_i \to S \to A/I$. Hence these are compatible with the transition maps $\varphi_{ii'}$ and define a lift $S \to A$. This proves existence. The uniqueness is clear by restricting to each S_i. □

Lemma 10.141.4. *Let R be a ring. Let $S \subset R$ be any multiplicative subset. Then the ring map $R \to S^{-1}R$ is formally étale.*

Proof. Let $I \subset A$ be an ideal of square zero. What we are saying here is that given a ring map $\varphi : R \to A$ such that $\varphi(f) \mod I$ is invertible for all $f \in S$ we have also that $\varphi(f)$ is invertible in A for all $f \in S$. This is true because A^* is the inverse image of $(A/I)^*$ under the canonical map $A \to A/I$. □

10.142. Unramified ring maps

The definition of a G-unramified ring map is the one from EGA. The definition of an unramified ring map is the one from [**Ray70**].

Definition 10.142.1. Let $R \to S$ be a ring map.

(1) We say $R \to S$ is *unramified* if $R \to S$ is of finite type and $\Omega_{S/R} = 0$.
(2) We say $R \to S$ is *G-unramified* if $R \to S$ is of finite presentation and $\Omega_{S/R} = 0$.
(3) Given a prime $\mathfrak{q}$ of S we say that S is *unramified at* $\mathfrak{q}$ if there exists a $g \in S$, $g \notin \mathfrak{q}$ such that $R \to S_g$ is unramified.
(4) Given a prime $\mathfrak{q}$ of S we say that S is *G-unramified at* $\mathfrak{q}$ if there exists a $g \in S$, $g \notin \mathfrak{q}$ such that $R \to S_g$ is G-unramified.

Of course a G-unramified map is unramified.

Lemma 10.142.2. *Let $R \to S$ be a ring map. The following are equivalent*

(1) *$R \to S$ is formally unramified and of finite type, and*
(2) *$R \to S$ is unramified.*

Moreover, also the following are equivalent

(1) *$R \to S$ is formally unramified and of finite presentation, and*
(2) *$R \to S$ is G-unramified.*

Proof. Follows from Lemma 10.139.2 and the definitions. □

Lemma 10.142.3. *Properties of unramified and G-unramified ring maps.*

(1) *The base change of an unramified ring map is unramified. The base change of a G-unramified ring map is G-unramified.*
(2) *The composition of unramified ring maps is unramified. The composition of G-unramified ring maps is G-unramified.*
(3) *Any principal localization $R \to R_f$ is G-unramified and unramified.*
(4) *If $I \subset R$ is an ideal, then $R \to R/I$ is unramified. If $I \subset R$ is a finitely generated ideal, then $R \to R/I$ is G-unramified.*
(5) *An étale ring map is G-unramified and unramified.*
(6) *If $R \to S$ is of finite type (resp. finite presentation), $\mathfrak{q} \subset S$ is a prime and $(\Omega_{S/R})_\mathfrak{q} = 0$, then $R \to S$ is unramified (resp. G-unramified) at $\mathfrak{q}$.*
(7) *If $R \to S$ is of finite type (resp. finite presentation), $\mathfrak{q} \subset S$ is a prime and $\Omega_{S/R} \otimes_S \kappa(\mathfrak{q}) = 0$, then $R \to S$ is unramified (resp. G-unramified) at $\mathfrak{q}$.*
(8) *If $R \to S$ is of finite type (resp. finite presentation), $\mathfrak{q} \subset S$ is a prime lying over $\mathfrak{p} \subset R$ and $(\Omega_{S \otimes_R \kappa(\mathfrak{p})/\kappa(\mathfrak{p})})_\mathfrak{q} = 0$, then $R \to S$ is unramified (resp. G-unramified) at $\mathfrak{q}$.*
(9) *If $R \to S$ is of finite type (resp. presentation), $\mathfrak{q} \subset S$ is a prime lying over $\mathfrak{p} \subset R$ and $(\Omega_{S \otimes_R \kappa(\mathfrak{p})/\kappa(\mathfrak{p})}) \otimes_{S \otimes_R \kappa(\mathfrak{p})} \kappa(\mathfrak{q}) = 0$, then $R \to S$ is unramified (resp. G-unramified) at $\mathfrak{q}$.*
(10) *If $R \to S$ is a ring map, $g_1, \ldots, g_m \in S$ generate the unit ideal and $R \to S_{g_j}$ is unramified (resp. G-unramified) for $j = 1, \ldots, m$, then $R \to S$ is unramified (resp. G-unramified).*
(11) *If $R \to S$ is a ring map which is unramified (resp. G-unramified) at every prime of S, then $R \to S$ is unramified (resp. G-unramified).*

(12) *If $R \to S$ is G-unramified, then there exists a finite type $\mathbf{Z}$-algebra R_0 and a G-unramified ring map $R_0 \to S_0$ and a ring map $R_0 \to R$ such that $S = R \otimes_{R_0} S_0$.*
(13) *If $R \to S$ is unramified, then there exists a finite type $\mathbf{Z}$-algebra R_0 and an unramified ring map $R_0 \to S_0$ and a ring map $R_0 \to R$ such that S is a quotient of $R \otimes_{R_0} S_0$.*

Proof. We prove each point, in order.

Ad (1). Follows from Lemmas 10.125.12 and 10.13.2.

Ad (2). Follows from Lemmas 10.125.7 and 10.13.2.

Ad (3). Follows by direct computation of $\Omega_{R_f/R}$ which we omit.

Ad (4). We have $\Omega_{(R/I)/R} = 0$, see Lemma 10.125.5, and the ring map $R \to R/I$ is of finite type. If I is a finitely generated ideal then $R \to R/I$ is of finite presentation.

Ad (5). See discussion following Definition 10.136.1.

Ad (6). In this case $\Omega_{S/R}$ is a finite S-module (see Lemma 10.125.16) and hence there exists a $g \in S$, $g \notin \mathfrak{q}$ such that $(\Omega_{S/R})_g = 0$. By Lemma 10.125.8 this means that $\Omega_{S_g/R} = 0$ and hence $R \to S_g$ is unramified as desired.

Ad (7). Use Nakayama's lemma (Lemma 10.18.1) to see that the condition is equivalent to the condition of (6).

Ad (8) & (9). These are equivalent in the same manner that (6) and (7) are equivalent. Moreover $\Omega_{S \otimes_R \kappa(\mathfrak{p})/\kappa(\mathfrak{p})} = \Omega_{S/R} \otimes_S (S \otimes_R \kappa(\mathfrak{p}))$ by Lemma 10.125.12. Hence we see that (9) is equivalent to (7) since the $\kappa(\mathfrak{q})$ vector spaces in both are canonically isomorphic.

Ad (10). Follows from from Lemmas 10.22.2 and 10.125.8.

Ad (11). Follows from (6) and (7) and the fact that the spectrum of S is quasi-compact.

Ad (12). Write $S = R[x_1, \ldots, x_n]/(g_1, \ldots, g_m)$. As $\Omega_{S/R} = 0$ we can write

$$\mathrm{d}x_i = \sum h_{ij} \mathrm{d}g_j + \sum a_{ijk} g_j \mathrm{d}x_k$$

in $\Omega_{R[x_1,\ldots,x_n]/R}$ for some $h_{ij}, a_{ijk} \in R[x_1, \ldots, x_n]$. Choose a finitely generated $\mathbf{Z}$-subalgebra $R_0 \subset R$ containing all the coefficients of the polynomials g_i, h_{ij}, a_{ijk}. Set $S_0 = R_0[x_1, \ldots, x_n]/(g_1, \ldots, g_m)$. This works.

Ad (13). Write $S = R[x_1, \ldots, x_n]/I$. As $\Omega_{S/R} = 0$ we can write

$$\mathrm{d}x_i = \sum h_{ij} \mathrm{d}g_{ij} + \sum g'_{ik} \mathrm{d}x_k$$

in $\Omega_{R[x_1,\ldots,x_n]/R}$ for some $h_{ij} \in R[x_1, \ldots, x_n]$ and $g_{ij}, g'_{ik} \in I$. Choose a finitely generated $\mathbf{Z}$-subalgebra $R_0 \subset R$ containing all the coefficients of the polynomials g_{ij}, h_{ij}, g'_{ik}. Set $S_0 = R_0[x_1, \ldots, x_n]/(g_{ij}, g'_{ik})$. This works. □

Lemma 10.142.4. *Let $R \to S$ be a ring map. If $R \to S$ is unramified, then there exists an idempotent $e \in S \otimes_R S$ such that $S \otimes_R S \to S$ is isomorphic to $S \otimes_R S \to (S \otimes_R S)_e$.*

Proof. Let $J = \mathrm{Ker}(S \otimes_R S \to S)$. By assumption $J/J^2 = 0$, see Lemma 10.125.13. Since S is of finite type over R we see that J is finitely generated, namely by $x_i \otimes 1 - 1 \otimes x_i$, where x_i generate S over R. We win by Lemma 10.19.5. □

Lemma 10.142.5. *Let $R \to S$ be a ring map. Let $\mathfrak{q} \subset S$ be a prime lying over $\mathfrak{p}$ in R. If S/R is unramified at $\mathfrak{q}$ then*

(1) *we have $\mathfrak{p}S_\mathfrak{q} = \mathfrak{q}S_\mathfrak{q}$ is the maximal ideal of the local ring $S_\mathfrak{q}$, and*
(2) *the field extension $\kappa(\mathfrak{p}) \subset \kappa(\mathfrak{q})$ is finite separable.*

Proof. We may first replace S by S_g for some $g \in S$, $g \notin \mathfrak{q}$ and assume that $R \to S$ is unramified. The base change $S \otimes_R \kappa(\mathfrak{p})$ is unramified over $\kappa(\mathfrak{p})$ by Lemma 10.142.3. By Lemma 10.133.3 it is smooth hence étale over $\kappa(\mathfrak{p})$. Hence we see that $S \otimes_R \kappa(\mathfrak{p}) = (R \setminus \mathfrak{p})^{-1}S/\mathfrak{p}S$ is a product of finite separable field extensions of $\kappa(\mathfrak{p})$ by Lemma 10.136.4. This implies the lemma. □

Lemma 10.142.6. *Let $R \to S$ be a finite type ring map. Let $\mathfrak{q}$ be a prime of S. If $R \to S$ is unramified at $\mathfrak{q}$ then $R \to S$ is quasi-finite at $\mathfrak{q}$. In particular, an unramified ring map is quasi-finite.*

Proof. An unramified ring map is of finite type. Thus it is clear that the second statement follows from the first. To see the first statement apply the characterization of Lemma 10.116.2 part (2) using Lemma 10.142.5. □

Lemma 10.142.7. *Let $R \to S$ be a ring map. Let $\mathfrak{q}$ be a prime of S lying over a prime $\mathfrak{p}$ of R. If*

(1) *$R \to S$ is of finite type,*
(2) *$\mathfrak{p}S_\mathfrak{q}$ is the maximal ideal of the local ring $S_\mathfrak{q}$, and*
(3) *the field extension $\kappa(\mathfrak{p}) \subset \kappa(\mathfrak{q})$ is finite separable,*

then $R \to S$ is unramified at $\mathfrak{q}$.

Proof. By Lemma 10.142.3 (8) it suffices to show that $\Omega_{S \otimes_R \kappa(\mathfrak{p})/\kappa(\mathfrak{p})}$ is zero when localized at $\mathfrak{q}$. Hence we may replace S by $S \otimes_R \kappa(\mathfrak{p})$ and R by $\kappa(\mathfrak{p})$. In other words, we may assume that $R = k$ is a field and S is a finite type k-algebra. In this case the hypotheses imply that $S_\mathfrak{q} \cong \kappa(\mathfrak{q})$ and hence $S = \kappa(\mathfrak{q}) \times S'$ (see Lemma 10.116.1). Hence $(\Omega_{S/k})_\mathfrak{q} = \Omega_{\kappa(\mathfrak{q})/k}$ which is zero as desired. □

Proposition 10.142.8. *Let $R \to S$ be a ring map. Let $\mathfrak{q} \subset S$ be a prime. If $R \to S$ is unramified at $\mathfrak{q}$, then there exist*

(1) *a $g \in S$, $g \notin \mathfrak{q}$,*
(2) *a standard étale ring map $R \to S'$, and*
(3) *a surjective R-algebra map $S' \to S_g$.*

Proof. This proof is the "same" as the proof of Proposition 10.136.17. The proof is a little roundabout and there may be ways to shorten it.

Step 1. By Definition 10.142.1 there exists a $g \in S$, $g \notin \mathfrak{q}$ such that $R \to S_g$ is unramified. Thus we may assume that S is unramified over R.

Step 2. By Lemma 10.142.3 there exists an unramified ring map $R_0 \to S_0$ with R_0 of finite type over $\mathbf{Z}$, and a ring map $R_0 \to R$ such that S is a quotient of $R \otimes_{R_0} S_0$. Denote $\mathfrak{q}_0$ the prime of S_0 corresponding to $\mathfrak{q}$. If we show the result for $(R_0 \to S_0, \mathfrak{q}_0)$ then the result follows for $(R \to S, \mathfrak{q})$ by base change. Hence we may assume that R is Noetherian.

Step 3. Note that $R \to S$ is quasi-finite by Lemma 10.142.6. By Lemma 10.117.15 there exists a finite ring map $R \to S'$, an R-algebra map $S' \to S$, an element $g' \in S'$ such that $g' \notin \mathfrak{q}$ such that $S' \to S$ induces an isomorphism $S'_{g'} \cong S_{g'}$. (Note that

S' may not unramified over R.) Thus we may assume that (a) R is Noetherian, (b) $R \to S$ is finite and (c) $R \to S$ is unramified at $\mathfrak{q}$ (but no longer necessarily unramified at all primes).

Step 4. Let $\mathfrak{p} \subset R$ be the prime corresponding to $\mathfrak{q}$. Consider the fibre ring $S \otimes_R \kappa(\mathfrak{p})$. This is a finite algebra over $\kappa(\mathfrak{p})$. Hence it is Artinian (see Lemma 10.49.2) and so a finite product of local rings

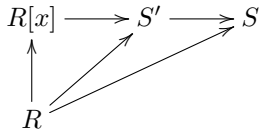
$$S \otimes_R \kappa(\mathfrak{p}) = \prod\nolimits_{i=1}^{n} A_i$$

see Proposition 10.57.6. One of the factors, say A_1, is the local ring $S_\mathfrak{q}/\mathfrak{p}S_\mathfrak{q}$ which is isomorphic to $\kappa(\mathfrak{q})$, see Lemma 10.142.5. The other factors correspond to the other primes, say $\mathfrak{q}_2, \ldots, \mathfrak{q}_n$ of S lying over $\mathfrak{p}$.

Step 5. We may choose a nonzero element $\alpha \in \kappa(\mathfrak{q})$ which generates the finite separable field extension $\kappa(\mathfrak{p}) \subset \kappa(\mathfrak{q})$ (so even if the field extension is trivial we do not allow $\alpha = 0$). Note that for any $\lambda \in \kappa(\mathfrak{p})^*$ the element $\lambda\alpha$ also generates $\kappa(\mathfrak{q})$ over $\kappa(\mathfrak{p})$. Consider the element

$$\bar{t} = (\alpha, 0, \ldots, 0) \in \prod\nolimits_{i=1}^{n} A_i = S \otimes_R \kappa(\mathfrak{p}).$$

After possibly replacing α by $\lambda\alpha$ as above we may assume that $\bar{t}$ is the image of $t \in S$. Let $I \subset R[x]$ be the kernel of the R-algebra map $R[x] \to S$ which maps x to t. Set $S' = R[x]/I$, so $S' \subset S$. Here is a diagram



By construction the primes $\mathfrak{q}_j$, $j \geq 2$ of S all lie over the prime $(\mathfrak{p}, x)$ of $R[x]$, whereas the prime $\mathfrak{q}$ lies over a different prime of $R[x]$ because $\alpha \neq 0$.

Step 6. Denote $\mathfrak{q}' \subset S'$ the prime of S' corresponding to $\mathfrak{q}$. By the above $\mathfrak{q}$ is the only prime of S lying over $\mathfrak{q}'$. Thus we see that $S_\mathfrak{q} = S_{\mathfrak{q}'}$, see Lemma 10.38.11 (we have going up for $S' \to S$ by Lemma 10.33.20 since $S' \to S$ is finite as $R \to S$ is finite). It follows that $S'_{\mathfrak{q}'} \to S_\mathfrak{q}$ is finite and injective as the localization of the finite injective ring map $S' \to S$. Consider the maps of local rings

$$R_\mathfrak{p} \to S'_{\mathfrak{q}'} \to S_\mathfrak{q}$$

The second map is finite and injective. We have $S_\mathfrak{q}/\mathfrak{p}S_\mathfrak{q} = \kappa(\mathfrak{q})$, see Lemma 10.142.5. Hence a fortiori $S_\mathfrak{q}/\mathfrak{q}'S_\mathfrak{q} = \kappa(\mathfrak{q})$. Since

$$\kappa(\mathfrak{p}) \subset \kappa(\mathfrak{q}') \subset \kappa(\mathfrak{q})$$

and since α is in the image of $\kappa(\mathfrak{q}')$ in $\kappa(\mathfrak{q})$ we conclude that $\kappa(\mathfrak{q}') = \kappa(\mathfrak{q})$. Hence by Nakayama's Lemma 10.18.1 applied to the $S'_{\mathfrak{q}'}$-module map $S'_{\mathfrak{q}'} \to S_\mathfrak{q}$, the map $S'_{\mathfrak{q}'} \to S_\mathfrak{q}$ is surjective. In other words, $S'_{\mathfrak{q}'} \cong S_\mathfrak{q}$.

Step 7. By Lemma 10.120.7 there exist $g \in S$, $g \notin \mathfrak{q}$ and $g' \in S'$, $g' \notin \mathfrak{q}'$ such that $S'_{g'} \cong S_g$. As R is Noetherian the ring S' is finite over R because it is an R-submodule of the finite R-module S. Hence after replacing S by S' we may assume that (a) R is Noetherian, (b) S finite over R, (c) S is unramified over R at $\mathfrak{q}$, and (d) $S = R[x]/I$.

Step 8. Consider the ring $S \otimes_R \kappa(\mathfrak{p}) = \kappa(\mathfrak{p})[x]/\overline{I}$ where $\overline{I} = I \cdot \kappa(\mathfrak{p})[x]$ is the ideal generated by I in $\kappa(\mathfrak{p})[x]$. As $\kappa(\mathfrak{p})[x]$ is a PID we know that $\overline{I} = (\overline{h})$ for some monic $\overline{h} \in \kappa(\mathfrak{p})$. After replacing $\overline{h}$ by $\lambda \cdot \overline{h}$ for some $\lambda \in \kappa(\mathfrak{p})$ we may assume that $\overline{h}$ is the image of some $h \in R[x]$. (The problem is that we do not know if we may choose h monic.) Also, as in Step 4 we know that $S \otimes_R \kappa(\mathfrak{p}) = A_1 \times \ldots \times A_n$ with $A_1 = \kappa(\mathfrak{q})$ a finite separable extension of $\kappa(\mathfrak{p})$ and $A_2, \ldots, A_n$ local. This implies that

$$\overline{h} = \overline{h}_1 \overline{h}_2^{e_2} \ldots \overline{h}_n^{e_n}$$

for certain pairwise coprime irreducible monic polynomials $\overline{h}_i \in \kappa(\mathfrak{p})[x]$ and certain $e_2, \ldots, e_n \geq 1$. Here the numbering is chosen so that $A_i = \kappa(\mathfrak{p})[x]/(\overline{h}_i^{e_i})$ as $\kappa(\mathfrak{p})[x]$-algebras. Note that $\overline{h}_1$ is the minimal polynomial of $\alpha \in \kappa(\mathfrak{q})$ and hence is a separable polynomial (its derivative is prime to itself).

Step 9. Let $m \in I$ be a monic element; such an element exists because the ring extension $R \to R[x]/I$ is finite hence integral. Denote $\overline{m}$ the image in $\kappa(\mathfrak{p})[x]$. We may factor

$$\overline{m} = \overline{k} \overline{h}_1^{d_1} \overline{h}_2^{d_2} \ldots \overline{h}_n^{d_n}$$

for some $d_1 \geq 1$, $d_j \geq e_j$, $j = 2, \ldots, n$ and $\overline{k} \in \kappa(\mathfrak{p})[x]$ prime to all the $\overline{h}_i$. Set $f = m^l + h$ where $l \deg(m) > \deg(h)$, and $l \geq 2$. Then f is monic as a polynomial over R. Also, the image $\overline{f}$ of f in $\kappa(\mathfrak{p})[x]$ factors as

$$\overline{f} = \overline{h}_1 \overline{h}_2^{e_2} \ldots \overline{h}_n^{e_n} + \overline{k}^l \overline{h}_1^{ld_1} \overline{h}_2^{ld_2} \ldots \overline{h}_n^{ld_n} = \overline{h}_1 (\overline{h}_2^{e_2} \ldots \overline{h}_n^{e_n} + \overline{k}^l \overline{h}_1^{ld_1 - 1} \overline{h}_2^{ld_2} \ldots \overline{h}_n^{ld_n}) = \overline{h}_1 \overline{w}$$

with $\overline{w}$ a polynomial relatively prime to $\overline{h}_1$. Set $g = f'$ (the derivative with respect to x).

Step 10. The ring map $R[x] \to S = R[x]/I$ has the properties: (1) it maps f to zero, and (2) it maps g to an element of $S \setminus \mathfrak{q}$. The first assertion is clear since f is an element of I. For the second assertion we just have to show that g does not map to zero in $\kappa(\mathfrak{q}) = \kappa(\mathfrak{p})[x]/(\overline{h}_1)$. The image of g in $\kappa(\mathfrak{p})[x]$ is the derivative of $\overline{f}$. Thus (2) is clear because

$$\overline{g} = \frac{\mathrm{d}\overline{f}}{\mathrm{d}x} = \overline{w} \frac{\mathrm{d}\overline{h}_1}{\mathrm{d}x} + \overline{h}_1 \frac{\mathrm{d}\overline{w}}{\mathrm{d}x},$$

$\overline{w}$ is prime to $\overline{h}_1$ and $\overline{h}_1$ is separable.

Step 11. We conclude that $\varphi : R[x]/(f) \to S$ is a surjective ring map, $R[x]_g/(f)$ is étale over R (because it is standard étale, see Lemma 10.136.15) and $\varphi(g) \notin \mathfrak{q}$. Thus the map $(R[x]/(f))_g \to S_{\varphi(g)}$ is the desired surjection. □

Lemma 10.142.9. *Let $R \to S$ be a ring map. Let $\mathfrak{q}$ be a prime of S lying over $\mathfrak{p} \subset R$. Assume that $R \to S$ is of finite type and unramified at $\mathfrak{q}$. Then there exist*

1. *an étale ring map $R \to R'$,*
2. *a prime $\mathfrak{p}' \subset R'$ lying over $\mathfrak{p}$.*
3. *a product decomposition*

$$R' \otimes_R S = A \times B$$

with the following properties

1. *$R' \to A$ is surjective, and*
2. *$\mathfrak{p}'A$ is a prime of A lying over $\mathfrak{p}'$ and over $\mathfrak{q}$.*

Proof. We may replace $(R \to S, \mathfrak{p}, \mathfrak{q})$ with any base change $(R' \to R' \otimes_R S, \mathfrak{p}', \mathfrak{q}')$ by a étale ring map $R \to R'$ with a prime $\mathfrak{p}'$ lying over $\mathfrak{p}$, and a choice of $\mathfrak{q}'$ lying over both $\mathfrak{q}$ and $\mathfrak{p}'$. Note also that given $R \to R'$ and $\mathfrak{p}'$ a suitable $\mathfrak{q}'$ can always be found.

The assumption that $R \to S$ is of finite type means that we may apply Lemma 10.136.24. Thus we may assume that $S = A_1 \times \ldots \times A_n \times B$, that each $R \to A_i$ is finite with exactly one prime $\mathfrak{r}_i$ lying over $\mathfrak{p}$ such that $\kappa(\mathfrak{p}) \subset \kappa(\mathfrak{r}_i)$ is purely inseparable and that $R \to B$ is not quasi-finite at any prime lying over $\mathfrak{p}$. Then clearly $\mathfrak{q} = \mathfrak{r}_i$ for some i, since an unramified morphism is quasi-finite (see Lemma 10.142.6). Say $\mathfrak{q} = \mathfrak{r}_1$. By Lemma 10.142.5 we see that $\kappa(\mathfrak{p}) \subset \kappa(\mathfrak{r}_1)$ is separable hence the trivial field extension, and that $\mathfrak{p}(A_1)_{\mathfrak{r}_1}$ is the maximal ideal. Also, by Lemma 10.38.11 (which applies to $R \to A_1$ because a finite ring map satisfies going up by Lemma 10.33.20) we have $(A_1)_{\mathfrak{r}_1} = (A_1)_{\mathfrak{p}}$. It follows from Nakayama's Lemma 10.18.1 that the map of local rings $R_{\mathfrak{p}} \to (A_1)_{\mathfrak{p}} = (A_1)_{\mathfrak{r}_1}$ is surjective. Since A_1 is finite over R we see that there exists a $f \in R$, $f \notin \mathfrak{p}$ such that $R_f \to (A_1)_f$ is surjective. After replacing R by R_f we win. □

Lemma 10.142.10. *Let $R \to S$ be a ring map. Let $\mathfrak{p}$ be a prime of R. If $R \to S$ is unramified then there exist*

(1) *an étale ring map $R \to R'$,*
(2) *a prime $\mathfrak{p}' \subset R'$ lying over $\mathfrak{p}$.*
(3) *a product decomposition*

$$R' \otimes_R S = A_1 \times \ldots \times A_n \times B$$

with the following properties

(1) *$R' \to A_i$ is surjective,*
(2) *$\mathfrak{p}' A_i$ is a prime of A_i lying over $\mathfrak{p}'$, and*
(3) *there is no prime of B lying over $\mathfrak{p}'$.*

Proof. We may apply Lemma 10.136.24. Thus, after an étale base change, we may assume that $S = A_1 \times \ldots \times A_n \times B$, that each $R \to A_i$ is finite with exactly one prime $\mathfrak{r}_i$ lying over $\mathfrak{p}$ such that $\kappa(\mathfrak{p}) \subset \kappa(\mathfrak{r}_i)$ is purely inseparable, and that $R \to B$ is not quasi-finite at any prime lying over $\mathfrak{p}$. Since $R \to S$ is quasi-finite (see Lemma 10.142.6) we see there is no prime of B lying over $\mathfrak{p}$. By Lemma 10.142.5 we see that $\kappa(\mathfrak{p}) \subset \kappa(\mathfrak{r}_i)$ is separable hence the trivial field extension, and that $\mathfrak{p}(A_i)_{\mathfrak{r}_i}$ is the maximal ideal. Also, by Lemma 10.38.11 (which applies to $R \to A_i$ because a finite ring map satisfies going up by Lemma 10.33.20) we have $(A_i)_{\mathfrak{r}_i} = (A_i)_{\mathfrak{p}}$. It follows from Nakayama's Lemma 10.18.1 that the map of local rings $R_{\mathfrak{p}} \to (A_i)_{\mathfrak{p}} = (A_i)_{\mathfrak{r}_i}$ is surjective. Since A_i is finite over R we see that there exists a $f \in R$, $f \notin \mathfrak{p}$ such that $R_f \to (A_i)_f$ is surjective. After replacing R by R_f we win. □

10.143. Henselian local rings

In this section we discuss a bit the notion of a henselian local ring. Let $(R, \mathfrak{m}, \kappa)$ be a local ring. For $a \in R$ we denote $\overline{a}$ the image of a in κ. For a polynomial $f \in R[T]$ we often denote $\overline{f}$ the image of f in $\kappa[T]$. Given a polynomial $f \in R[T]$ we denote f' the derivative of f with respect to T. Note that $\overline{f}' = \overline{f'}$.

Definition 10.143.1. Let $(R, \mathfrak{m}, \kappa)$ be a local ring.

(1) We say R is *henselian* if for every monic $f \in R[T]$ and every root $a_0 \in \kappa$ of $\overline{f}$ such that $\overline{f'}(a_0) \neq 0$ there exists an $a \in R$ such that $f(a) = 0$ and $a_0 = \overline{a}$.
(2) We say R is *strictly henselian* if R is henselian and its residue field is separably algebraically closed.

Note that the condition $\overline{f'}(a_0) \neq 0$ is equivalent to the condition that a_0 is a simple root of the polynomial $\overline{f}$. In fact, it implies that the lift $a \in R$, if it exists, is unique.

Lemma 10.143.2. *Let $(R, \mathfrak{m}, \kappa)$ be a local ring. Let $f \in R[T]$. Let $a, b \in R$ such that $f(a) = f(b) = 0$, $a = b \bmod \mathfrak{m}$, and $f'(a) \notin \mathfrak{m}$. Then $a = b$.*

Proof. Write $f(x+y) - f(x) = f'(x)y + g(x,y)y^2$ in $R[x, y]$ (this is possible as one sees by expanding $f(x + y)$; details omitted). Then we see that $0 = f(b) - f(a) = f(a + (b-a)) - f(a) = f'(a)(b-a) + c(b-a)^2$ for some $c \in R$. By assumption $f'(a)$ is a unit in R. Hence $(b - a)(1 + f'(a)^{-1}c(b - a)) = 0$. By assumption $b - a \in \mathfrak{m}$, hence $1 + f'(a)^{-1}c(b - a)$ is a unit in R. Hence $b - a = 0$ in R. □

Here is the characterization of henselian local rings.

Lemma 10.143.3. *Let $(R, \mathfrak{m}, \kappa)$ be a local ring. The following are equivalent*

(1) *R is henselian,*
(2) *for every $f \in R[T]$ and every root $a_0 \in \kappa$ of $\overline{f}$ such that $\overline{f'}(a_0) \neq 0$ there exists an $a \in R$ such that $f(a) = 0$ and $a_0 = \overline{a}$,*
(3) *for any monic $f \in R[T]$ and any factorization $\overline{f} = g_0 h_0$ with $\gcd(g_0, h_0) = 1$ there exists a factorization $f = gh$ in $R[T]$ such that $g_0 = \overline{g}$ and $h_0 = \overline{h}$,*
(4) *for any monic $f \in R[T]$ and any factorization $\overline{f} = g_0 h_0$ with $\gcd(g_0, h_0) = 1$ there exists a factorization $f = gh$ in $R[T]$ such that $g_0 = \overline{g}$ and $h_0 = \overline{h}$ and moreover $\deg_T(g) = \deg_T(g_0)$,*
(5) *for any $f \in R[T]$ and any factorization $\overline{f} = g_0 h_0$ with $\gcd(g_0, h_0) = 1$ there exists a factorization $f = gh$ in $R[T]$ such that $g_0 = \overline{g}$ and $h_0 = \overline{h}$,*
(6) *for any $f \in R[T]$ and any factorization $\overline{f} = g_0 h_0$ with $\gcd(g_0, h_0) = 1$ there exists a factorization $f = gh$ in $R[T]$ such that $g_0 = \overline{g}$ and $h_0 = \overline{h}$ and moreover $\deg_T(g) = \deg_T(g_0)$,*
(7) *for any étale ring map $R \to S$ and prime $\mathfrak{q}$ of S lying over $\mathfrak{m}$ with $\kappa = \kappa(\mathfrak{q})$ there exists a section $\tau : S \to R$ of $R \to S$,*
(8) *for any étale ring map $R \to S$ and prime $\mathfrak{q}$ of S lying over $\mathfrak{m}$ with $\kappa = \kappa(\mathfrak{q})$ there exists a section $\tau : S \to R$ of $R \to S$ with $\mathfrak{q} = \tau^{-1}(\mathfrak{m})$,*
(9) *any finite R-algebra is a product of local rings,*
(10) *any finite R-algebra is a finite product of local rings,*
(11) *any finite type R-algebra S can be written as $A \times B$ with $R \to A$ finite and $R \to B$ not quasi-finite at any prime lying over $\mathfrak{m}$,*
(12) *any finite type R-algebra S can be written as $A \times B$ with $R \to A$ finite such that each irreducible component of $\mathrm{Spec}(B \otimes_R \kappa)$ has dimension ≥ 1, and*
(13) *any quasi-finite R-algebra S can be written as $S = A \times B$ with $R \to A$ finite such that $B \otimes_R \kappa = 0$.*

Proof. Here is a list of the easier implications:

2⇒1 because in (2) we consider all polynomials and in (1) only monic ones,
5⇒3 because in (5) we consider all polynomials and in (3) only monic ones,

6⇒4 because in (6) we consider all polynomials and in (4) only monic ones,
4⇒3 is obvious,
6⇒5 is obvious,
8⇒7 is obvious,
10⇒9 is obvious,
11⇔12 by definition of being quasi-finite at a prime,
11⇒13 by definition of being quasi-finite,

Proof of 1⇒8. Assume (1). Let $R \to S$ be étale, and let $\mathfrak{q} \subset S$ be a prime ideal such that $\kappa(\mathfrak{q}) \cong \kappa$. By Proposition 10.136.17 we can find a $g \in S$, $g \notin \mathfrak{q}$ such that $R \to S_g$ is standard étale. After replacing S by S_g we may assume that $S = R[t]_g/(f)$ is standard étale. Since the prime $\mathfrak{q}$ has residue field κ it corresponds to a root a_0 of $\overline{f}$ which is not a root of $\overline{g}$. By definition of a standard étale algebra this also means that $\overline{f'}(a_0) \neq 0$. Since also f is monic by definition of a standard étale algebra again we may use that R is henselian to conclude that there exists an $a \in R$ with $a_0 = \overline{a}$ such that $f(a) = 0$. This implies that $g(a)$ is a unit of R and we obtain the desired map $\tau : S = R[t]_g/(f) \to R$ by the rule $t \mapsto a$. By construction $\tau^{-1}(\mathfrak{q}) = \mathfrak{m}$. This proves (8) holds.

Proof of 7⇒8. (This is really unimportant and should be skipped.) Assume (7) holds and assume $R \to S$ is étale. Let $\mathfrak{q}_1, \ldots, \mathfrak{q}_r$ be the other primes of S lying over $\mathfrak{m}$. Then we can find a $g \in S$, $g \notin \mathfrak{q}$ and $g \in \mathfrak{q}_i$ for $i = 1, \ldots, r$, see Lemma 10.14.3. Apply (7) to the étale ring map $R \to S_g$ and the prime $\mathfrak{q}S_g$. This gives a section $\tau_g : S_g \to R$ such that the composition $\tau : S \to S_g \to R$ has the property $\tau^{-1}(\mathfrak{q}) = \mathfrak{m}$. Minor details omitted.

Proof of 8⇒11. Assume (8) and let $R \to S$ be a finite type ring map. Apply Lemma 10.136.23. We find an étale ring map $R \to R'$ and a prime $\mathfrak{m}' \subset R'$ lying over $\mathfrak{m}$ with $\kappa = \kappa(\mathfrak{m}')$ such that $R' \otimes_R S = A' \times B'$ with A' finite over R' and B' not quasi-finite over R' at any prime lying over $\mathfrak{m}'$. Apply (8) to get a section $\tau : R' \to R$ with $\mathfrak{m} = \tau^{-1}(\mathfrak{m}')$. Then use that

$$S = (S \otimes_R R') \otimes_{R',\tau} R = (A' \times B') \otimes_{R',\tau} R = (A' \otimes_{R',\tau} R) \times (B' \otimes_{R',\tau} R)$$

which gives a decomposition as in (11).

Proof of 8⇒10. Assume (8) and let $R \to S$ be a finite ring map. Apply Lemma 10.136.23. We find an étale ring map $R \to R'$ and a prime $\mathfrak{m}' \subset R'$ lying over $\mathfrak{m}$ with $\kappa = \kappa(\mathfrak{m}')$ such that $R' \otimes_R S = A'_1 \times \ldots \times A'_n \times B'$ with A'_i finite over R' having exactly one prime over $\mathfrak{m}'$ and B' not quasi-finite over R' at any prime lying over $\mathfrak{m}'$. Apply (8) to get a section $\tau : R' \to R$ with $\mathfrak{m} = \tau^{-1}(\mathfrak{m}')$. Then we obtain

$$\begin{aligned} S &= (S \otimes_R R') \otimes_{R',\tau} R \\ &= (A'_1 \times \ldots \times A'_n \times B') \otimes_{R',\tau} R \\ &= (A'_1 \otimes_{R',\tau} R) \times \ldots \times (A'_1 \otimes_{R',\tau} R) \times (B' \otimes_{R',\tau} R) \\ &= A_1 \times \ldots \times A_n \times B \end{aligned}$$

The factor B is finite over R but $R \to B$ is not quasi-finite at any prime lying over $\mathfrak{m}$. Hence $B = 0$. The factors A_i are finite R-algebras having exactly one prime lying over $\mathfrak{m}$, hence they are local rings. This proves that S is a finite product of local rings.

Proof of 9⇒10. This holds because if S is finite over the local ring R, then it has at most finitely many maximal ideals. Namely, by going up for $R \to S$ the maximal ideals of S all lie over $\mathfrak{m}$, and $S/\mathfrak{m}S$ is Artinian hence has finitely many primes.

Proof of 10⇒1. Assume (10). Let $f \in R[T]$ be a monic polynomial and $a_0 \in \kappa$ a simple root of $\overline{f}$. Then $S = R[T]/(f)$ is a finite R-algebra. Applying (10) we get $S = A_1 \times \ldots \times A_r$ is a finite product of local R-algebras. In particular we see that $S/\mathfrak{m}S = \prod A_i/\mathfrak{m}A_i$ is the decomposition of $\kappa[T]/(\overline{f})$ as a product of local rings. This means that one of the factors, say $A_1/\mathfrak{m}A_1$ is the quotient $\kappa[T]/(\overline{f}) \to \kappa[T]/(T - a_0)$. Since A_1 is a summand of the finite free R-module S it is a finite free R-module itself. As $A_1/\mathfrak{m}A_1$ is a κ-vector space of dimension 1 we see that $A_1 \cong R$ as an R-module. Clearly this means that $R \to A_1$ is an isomorphism. Let $a \in R$ be the image of T under the map $R[T] \to S \to A_1 \to R$. Then $f(a) = 0$ and $\overline{a} = a_0$ as desired.

Proof of 13⇒1. Assume (13). Let $f \in R[T]$ be a monic polynomial and $a_0 \in \kappa$ a simple root of $\overline{f}$. Then $S_1 = R[T]/(f)$ is a finite R-algebra. Let $g \in R[T]$ be any element such that $\overline{g} = \overline{f}/(T - a_0)$. Then $S = (S_1)_g$ is a quasi-finite R-algebra such that $S \otimes_R \kappa \cong \kappa[T]_{\overline{g}}/(\overline{f}) \cong \kappa[T]/(T - a_0) \cong \kappa$. Applying (13) to S we get $S = A \times B$ with A finite over R and $B \otimes_R \kappa = 0$. In particular we see that $\kappa \cong S/\mathfrak{m}S = A/\mathfrak{m}A$. Since A is a summand of the flat R-algebra S we see that it is finite flat, hence free over R. As $A/\mathfrak{m}A$ is a κ-vector space of dimension 1 we see that $A \cong R$ as an R-module. Clearly this means that $R \to A$ is an isomorphism. Let $a \in R$ be the image of T under the map $R[T] \to S \to A \to R$. Then $f(a) = 0$ and $\overline{a} = a_0$ as desired.

Proof of 8⇒2. Assume (8). Let $f \in R[T]$ be any polynomial and let $a_0 \in \kappa$ be a simple root. Then the algebra $S = R[T]_{f'}/(f)$ is étale over R. Let $\mathfrak{q} \subset S$ be the prime generated by $\mathfrak{m}$ and $T - b$ where $b \in R$ is any element such that $\overline{b} = a_0$. Apply (8) to S and $\mathfrak{q}$ to get $\tau : S \to R$. Then the image $\tau(T) = a \in R$ works in (2).

At this point we see that (1), (2), (7), (8), (9), (10), (11), (12), (13) are all equivalent. The weakest assertion of (3), (4), (5) and (6) is (3) and the strongest is (6). Hence we still have to prove that (3) implies (1) and (1) implies (6).

Proof of 3⇒1. Assume (3). Let $f \in R[T]$ be monic and let $a_0 \in \kappa$ be a simple root of $\overline{f}$. This gives a factorization $\overline{f} = (T - a_0)h_0$ with $h_0(a_0) \neq 0$, so $\gcd(T - a_0, h_0) = 1$. Apply (3) to get a factorization $f = gh$ with $\overline{g} = T - a_0$ and $\overline{h} = h_0$. Set $S = R[T]/(f)$ which is a finite free R-algebra. We will write g, h also for the images of g and h in S. Then $gS + hS = S$ by Nakayama's Lemma 10.18.1 as the equality holds modulo $\mathfrak{m}$. Since $gh = f = 0$ in S this also implies that $gS \cap hS = 0$. Hence by the Chinese Remainder theorem we obtain $S = S/(g) \times S/(h)$. This implies that $A = S/(g)$ is a summand of a finite free R-module, hence finite free. Moreover, the rank of A is 1 as $A/\mathfrak{m}A = \kappa[T]/(T - a_0)$. Thus the map $R \to A$ is an isomorphism. Setting $a \in R$ equal to the image of T under the maps $R[T] \to S \to A \to R$ gives an element of R with $f(a) = 0$ and $\overline{a} = a_0$.

Proof of 1⇒6. Assume (1) or equivalently all of (1), (2), (7), (8), (9), (10), (11), (12), (13). Let $f \in R[T]$ be a polynomial. Suppose that $\overline{f} = g_0 h_0$ is a factorization with $\gcd(g_0, h_0) = 1$. We may and do assume that g_0 is monic. Consider $S = R[T]/(f)$. Because we have the factorization we see that the coefficients of f generate the unit

ideal in R. This implies that S has finite fibres over R, hence is quasi-finite over R. It also implies that S is flat over R by Lemma 10.93.2. Combining (13) and (10) we may write $S = A_1 \times \ldots \times A_n \times B$ where each A_i is local and finite over R, and $B \otimes_R \kappa = 0$. After reordering the factors $A_1, \ldots, A_n$ we may assume that

$$\kappa[T]/(g_0) = A_1/\mathfrak{m}A_1 \times \ldots \times A_r/\mathfrak{m}A_r,\ \kappa[T]/(h_0) = A_{r+1}/\mathfrak{m}A_{r+1} \times \ldots \times A_n/\mathfrak{m}A_n$$

as quotients of $\kappa[T]$. The finite flat R-algebra $A = A_1 \times \ldots \times A_r$ is free as an R-module, see Lemma 10.73.4. Its rank is $\deg_T(g_0)$. Let $g \in R[T]$ be the characteristic polynomial of the R-linear operator $T : A \to A$. Then g is a monic polynomial of degree $\deg_T(g) = \deg_T(g_0)$ and moreover $\overline{g} = g_0$. By Cayley-Hamilton (Lemma 10.15.1) we see that $g(T_A) = 0$ where T_A indicates the image of T in A. Hence we obtain a well defined surjective map $R[T]/(g) \to A$ which is an isomorphism by Nakayama's Lemma 10.18.1. The map $R[T] \to A$ factors through $R[T]/(f)$ by construction hence we may write $f = gh$ for some h. This finishes the proof. □

Lemma 10.143.4. *Let $(R, \mathfrak{m}, \kappa)$ be a henselian local ring.*

1. *If $R \subset S$ is a finite ring extension then S is a finite product of henselian local rings.*
2. *If $R \subset S$ is a finite local homomorphism of local rings, then S is a henselian local ring.*
3. *If $R \to S$ is a finite type ring map, and $\mathfrak{q}$ is a prime of S lying over $\mathfrak{m}$ at which $R \to S$ is quasi-finite, then $S_\mathfrak{q}$ is henselian.*
4. *If $R \to S$ is quasi-finite then $S_\mathfrak{q}$ is henselian for every prime $\mathfrak{q}$ lying over $\mathfrak{m}$.*

Proof. Part (2) implies part (1) since S as in part (1) is a finite product of its localizations at the primes lying over $\mathfrak{m}$. Part (2) follows from Lemma 10.143.3 part (10) since any finite S-algebra is also a finite R-algebra. If $R \to S$ and $\mathfrak{q}$ are as in (3), then $S_\mathfrak{q}$ is a local ring of a finite R-algebra by Lemma 10.143.3 part (11). Hence (3) follows from (1). Part (4) follows from part (3). □

Lemma 10.143.5. *A filtered colimit of henselian local rings along local homomorphisms is henselian.*

Proof. Categories, Lemma 4.21.3 says that this is really just a question about a colimit of henselian local rings over a directed partially ordered set. Let $(R_i, \varphi_{ii'})$ be such a system with each $\varphi_{ii'}$ local. Then $R = \operatorname{colim}_i R_i$ is local, and its residue field κ is $\operatorname{colim} \kappa_i$ (argument omitted). Suppose that $f \in R[T]$ is monic and that $a_0 \in \kappa$ is a simple root of $\overline{f}$. Then for some large enough i there exists an $f_i \in R_i[T]$ mapping to f and an $a_{0,i} \in \kappa_i$ mapping to a_0. Since $\overline{f_i}(a_{0,i}) \in \kappa_i$, resp. $\overline{f_i'}(a_{0,i}) \in \kappa_i$ maps to $0 = \overline{f}(a_0) \in \kappa$, resp. $0 \neq \overline{f'}(a_0) \in \kappa$ we conclude that $a_{0,i}$ is a simple root of $\overline{f_i}$. As R_i is henselian we can find $a_i \in R_i$ such that $f_i(a_i) = 0$ and $a_{0,i} = \overline{a_i}$. Then the image $a \in R$ of a_i is the desired solution. Thus R is henselian. □

Lemma 10.143.6. *Let $(R, \mathfrak{m}, \kappa)$ be a henselian local ring. Any finite type R-algebra S can be written as $S = A_1 \times \ldots \times A_n \times B$ with A_i local and finite over R and $R \to B$ not quasi-finite at any prime of B lying over $\mathfrak{m}$.*

Proof. This is a combination of parts (11) and (10) of Lemma 10.143.3. □

Lemma 10.143.7. *Let $(R, \mathfrak{m}, \kappa)$ be a strictly henselian local ring. Any finite type R-algebra S can be written as $S = A_1 \times \ldots \times A_n \times B$ with A_i local and finite over R*

and $\kappa \subset \kappa(\mathfrak{m}_{A_i})$ finite purely inseparable and $R \to B$ not quasi-finite at any prime of B lying over $\mathfrak{m}$.

Proof. First write $S = A_1 \times \ldots \times A_n \times B$ as in Lemma 10.143.6. The field extension $\kappa \subset \kappa(\mathfrak{m}_{A_i})$ is finite and κ is separably algebraically closed, hence it is finite purely inseparable. □

Lemma 10.143.8. *Let $(R, \mathfrak{m}, \kappa)$ be a henselian local ring. The category of finite étale ring extensions $R \to S$ is equivalent to the category of finite étale algebras $\kappa \to \overline{S}$ via the functor $S \mapsto S/\mathfrak{m}S$.*

Proof. Denote $\mathcal{C} \to \mathcal{D}$ the functor of categories of the statement. Suppose that $R \to S$ is finite étale. Then we may write

$$S = A_1 \times \ldots \times A_n$$

with A_i local and finite étale over S, use either Lemma 10.143.6 or Lemma 10.143.3 part (10). In particular $A_i/\mathfrak{m}A_i$ is a finite separable field extension of κ, see Lemma 10.136.5. Thus we see that every object of $\mathcal{C}$ and $\mathcal{D}$ decomposes canonically into irreducible pieces which correspond via the given functor. Next, suppose that S_1, S_2 are finite étale over R such that $\kappa_1 = S_1/\mathfrak{m}S_1$ and $\kappa_2 = S_2/\mathfrak{m}S_2$ are fields (finite separable over κ). Then $S_1 \otimes_R S_2$ is finite étale over R and we may write

$$S_1 \otimes_R S_2 = A_1 \times \ldots \times A_n$$

as before. Then we see that $\mathrm{Hom}_R(S_1, S_2)$ is identified with the set of indices $i \in \{1, \ldots, n\}$ such that $S_2 \to A_i$ is an isomorphism. To see this use that given any R-algebra map $\varphi : S_1 \to S_2$ the map $\varphi \times 1 : S_1 \otimes_R S_2 \to S_2$ is surjective, and hence is equal to projection onto one of the factors A_i. But in exactly the same way we see that $\mathrm{Hom}_\kappa(\kappa_1, \kappa_2)$ is identified with the set of indices $i \in \{1, \ldots, n\}$ such that $\kappa_2 \to A_i/\mathfrak{m}A_i$ is an isomorphism. By the discussion above these sets of indices match, and we conclude that our functor is fully faithful. Finally, let $\kappa \subset \kappa'$ be a finite separable field extension. By Lemma 10.136.16 there exists an étale ring map $R \to S$ and a prime $\mathfrak{q}$ of S lying over $\mathfrak{m}$ such that $\kappa \subset \kappa(\mathfrak{q})$ is isomorphic to the given extension. By part (1) we may write $S = A_1 \times \ldots \times A_n \times B$. Since $R \to S$ is quasi-finite we see that there exists no prime of B over $\mathfrak{m}$. Hence $S_\mathfrak{q}$ is equal to A_i for some i. Hence $R \to A_i$ is finite étale and produces the given residue field extension. Thus the functor is essentially surjective and we win. □

Lemma 10.143.9. *Let $(R, \mathfrak{m}, \kappa)$ be a strictly henselian local ring. Let $R \to S$ be an unramified ring map. Then*

$$S = A_1 \times \ldots \times A_n \times B$$

with each $R \to A_i$ surjective and no prime of B lying over $\mathfrak{m}$.

Proof. First write $S = A_1 \times \ldots \times A_n \times B$ as in Lemma 10.143.6. Now we see that $R \to A_i$ is finite unramified and A_i local. Hence the maximal ideal of A_i is $\mathfrak{m}A_i$ and its residue field $A_i/\mathfrak{m}A_i$ is a finite separable extension of κ, see Lemma 10.142.5. However, the condition that R is strictly henselian means that κ is separably algebraically closed, so $\kappa = A_i/\mathfrak{m}A_i$. By Nakayama's Lemma 10.18.1 we conclude that $R \to A_i$ is surjective as desired. □

Lemma 10.143.10. *Let $(R, \mathfrak{m}, \kappa)$ be a complete local ring, see Definition 10.147.1. Then R is henselian.*

Proof. Let $f \in R[T]$ be monic. Denote $f_n \in R/\mathfrak{m}^{n+1}[T]$ the image. Denote f_n' the derivative of f_n with respect to T. Let $a_0 \in \kappa$ be a simple root of f_0. We lift this to a solution of f over R inductively as follows: Suppose given $a_n \in R/\mathfrak{m}^{n+1}$ such that $a_n \bmod \mathfrak{m} = a_0$ and $f_n(a_n) = 0$. Pick any element $b \in R/\mathfrak{m}^{n+2}$ such that $a_n = b \bmod \mathfrak{m}^{n+1}$. Then $f_{n+1}(b) \in \mathfrak{m}^{n+1}/\mathfrak{m}^{n+2}$. Set

$$a_{n+1} = b - f_{n+1}(b)/f'_{n+1}(b)$$

(Newton's method). This makes sense as $f'_{n+1}(b) \in R_{n+1}$ is invertible by the condition on a_0. Then we compute $f_{n+1}(a_{n+1}) = f_{n+1}(b) - f_{n+1}(b) = 0$ in $R/\mathfrak{m}^{n+2}$. Since the system of elements $a_n \in R/\mathfrak{m}^{n+1}$ so constructed is compatible we get an element $a \in \lim R/\mathfrak{m}^n = R$ (here we use that R is complete). Moreover, $f(a) = 0$ since it maps to zero in each $R/\mathfrak{m}^n$. Finally $\overline{a} = a_0$ and we win. □

Lemma 10.143.11. *Let $(R, \mathfrak{m})$ be a local ring of dimension 0. Then R is henselian.*

Proof. Let $R \to S$ be a finite ring map. By Lemma 10.143.3 it suffices to show that S is a product of local rings. By Lemma 10.33.19 S has finitely many primes $\mathfrak{m}_1, \ldots, \mathfrak{m}_r$ which all lie over $\mathfrak{m}$. There are no inclusions among these primes, see Lemma 10.33.18, hence they are all maximal. Every element of $\mathfrak{m}_1 \cap \ldots \cap \mathfrak{m}_r$ is nilpotent by Lemma 10.16.2. It follows S is the product of the localizations of S at the primes $\mathfrak{m}_i$ by Lemma 10.49.7. □

Lemma 10.143.12. *Let $R \to S$ be a ring map with S henselian local. Given*

1. *an étale ring map $R \to A$,*
2. *a prime $\mathfrak{q}$ of A lying over $\mathfrak{p} = R \cap \mathfrak{m}_S$,*
3. *a $\kappa(\mathfrak{p})$-algebra map $\kappa(\mathfrak{q}) \to S/\mathfrak{m}_S$,*

then there exists a unique homomorphism of R-algebras $f : A \to S$ such that $\mathfrak{q} = f^{-1}(\mathfrak{m}_S)$.

Proof. Consider $A \otimes_R S$. This is an étale algebra over S, see Lemma 10.136.3. Moreover, the kernel

$$\mathfrak{q}' = \mathrm{Ker}(A \otimes_R S \to \kappa(\mathfrak{q}) \otimes_{\kappa(\mathfrak{p})} \kappa(\mathfrak{m}_S) \to \kappa(\mathfrak{m}_S))$$

of the map using the map given in (3) is a prime ideal lying over $\mathfrak{m}_S$ with residue field equal to the residue field of S. Hence by Lemma 10.143.3 there exists a unique splitting $\tau : A \otimes_R S \to S$ with $\tau^{-1}(\mathfrak{m}_S) = \mathfrak{q}'$. Set f equal to the composition $A \to A \otimes_R S \to S$. □

Lemma 10.143.13. *Let $R \to S$ be a ring map with S henselian local. Given*

1. *an R-algebra A which is a filtered colimit of étale R-algebras,*
2. *a prime $\mathfrak{q}$ of A lying over $\mathfrak{p} = R \cap \mathfrak{m}_S$,*
3. *a $\kappa(\mathfrak{p})$-algebra map $\kappa(\mathfrak{q}) \to S/\mathfrak{m}_S$,*

then there exists a unique homomorphism of R-algebras $f : A \to S$ such that $\mathfrak{q} = f^{-1}(\mathfrak{m}_S)$.

Proof. Write $A = \mathrm{colim}\, A_i$ as a filtered colimit of étale R-algebras. Set $\mathfrak{q}_i = A_i \cap \mathfrak{q}$. We obtain $f_i : A_i \to S$ by applying Lemma 10.143.12. Set $f = \mathrm{colim}\, f_i$. □

Lemma 10.143.14. *Let R be a ring. Let $A \to B$ be an R-algebra homomorphism. If A and B are filtered colimits of étale R-algebras, then B is a filtered colimit of étale A-algebras.*

Proof. Write $A = \operatorname{colim} A_i$ and $B = \operatorname{colim} B_j$ as filtered colimits with A_i and B_j étale over R. For each i we can find a j such that $A_i \to B$ factors through B_j, see Lemma 10.121.2. The factorization $A_i \to B_j$ is étale by Lemma 10.136.9. Since $A \to A \otimes_{A_i} B_j$ is étale (Lemma 10.136.3) it suffices to prove that $B = \operatorname{colim} A \otimes_{A_i} B_j$ where the colimit is over pairs (i, j) and factorizations $A_i \to B_j \to B$ of $A_i \to B$ (this is a directed system; details omitted). This is clear because $\operatorname{colim} A \otimes_{A_i} B_j = A \otimes_A B = B$. □

Lemma 10.143.15. *Let R be a ring. Given a commutative diagram of ring maps*

$$\begin{array}{ccc} S & \longrightarrow & K \\ \uparrow & & \uparrow \\ R & \longrightarrow & S' \end{array}$$

where S, S' are henselian local, S, S' are filtered colimits of étale R-algebras, K is a field and the arrows $S \to K$ and $S' \to K$ identify K with the residue field of both S and S'. Then there exists an unique R-algebra isomorphism $S \to S'$ compatible with the maps to K.

Proof. Follows immediately from Lemma 10.143.13. □

Lemma 10.143.16. *Let $(R, \mathfrak{m}, \kappa)$ be a local ring. There exists a local ring map $R \to R^h$ with the following properties*

1. *R^h is henselian,*
2. *R^h is a filtered colimit of étale R-algebras,*
3. *$\mathfrak{m}R^h$ is the maximal ideal of R^h, and*
4. *$\kappa = R^h/\mathfrak{m}R^h$.*

Proof. Consider the category of pairs $(S, \mathfrak{q})$ where $R \to S$ is an étale ring map, and $\mathfrak{q}$ is a prime of S lying over $\mathfrak{m}$ with $\kappa = \kappa(\mathfrak{q})$. A morphism of pairs $(S, \mathfrak{q}) \to (S', \mathfrak{q}')$ is given by an R-algebra map $\varphi : S \to S'$ such that $\varphi^{-1}(\mathfrak{q}') = \mathfrak{q}$. We set

$$R^h = \operatorname{colim}_{(S,\mathfrak{q})} S.$$

This clearly implies that R^h is canonical, since no choices were made in this construction. Moreover, property (2) is clear.

Let us show that the category of pairs is filtered, see Categories, Definition 4.19.1. The category contains the pair $(R, \mathfrak{m})$ and hence is not empty, which proves part (1) of Categories, Definition 4.19.1. Note that for any pair $(S, \mathfrak{q})$ the prime ideal $\mathfrak{q}$ is maximal, for example since $\kappa \to S/\mathfrak{q} \subset \kappa(\mathfrak{q})$ are isomorphisms. Suppose that $(S, \mathfrak{q})$ and $(S', \mathfrak{q}')$ are two objects. Set $S'' = S \otimes_R S'$ and $\mathfrak{q}'' = \mathfrak{q}S'' + \mathfrak{q}'S''$. Then $S''/\mathfrak{q}'' = S/\mathfrak{q} \otimes_R S'/\mathfrak{q}' = \kappa$ by what we said above. Moreover, $R \to S''$ is étale by Lemma 10.136.3. This proves part (2) of Categories, Definition 4.19.1. Next, suppose that $\varphi, \psi : (S, \mathfrak{q}) \to (S', \mathfrak{q}')$ are two morphisms of pairs. Consider

$$S'' = (S' \otimes_{\varphi,S,\psi} S') \otimes_{S' \otimes_R S'} S'$$

with prime ideal

$$\mathfrak{q}'' = (\mathfrak{q}' \otimes S' + S' \otimes \mathfrak{q}') \otimes S' + (S' \otimes_{\varphi,S,\psi} S') \otimes \mathfrak{q}'$$

Arguing as above (base change of étale maps is étale, composition of étale maps is étale) we see that S'' is étale over R. Moreover, the canonical map $S' \to S''$ (using the right most factor for example) equalizes φ and ψ. This proves part

(3) of Categories, Definition 4.19.1. Hence we conclude that R^h consists of triples $(S, \mathfrak{q}, f)$ with $f \in S$, and two such triples $(S, \mathfrak{q}, f)$, $(S', \mathfrak{q}', f')$ define the same element of R^h if and only if there exists a pair $(S'', \mathfrak{q}'')$ and morphisms of pairs $\varphi : (S, \mathfrak{q}) \to (S'', \mathfrak{q}'')$ and $\varphi' : (S', \mathfrak{q}') \to (S'', \mathfrak{q}'')$ such that $\varphi(f) = \varphi'(f')$.

Suppose that $x \in R^h$. Represent x by a triple $(S, \mathfrak{q}, f)$. Let $\mathfrak{q}_1, \ldots, \mathfrak{q}_r$ be the other primes of S lying over $\mathfrak{m}$. Then we can find a $g \in S$, $g \notin \mathfrak{q}$ and $g \in \mathfrak{q}_i$ for $i = 1, \ldots, r$, see Lemma 10.14.3. Consider the morphism of pairs $(S, \mathfrak{q}) \to (S_g, \mathfrak{q}S_g)$. In this way we see that we may always assume that x is given by a triple $(S, \mathfrak{q}, f)$ where $\mathfrak{q}$ is the only prime of S lying over $\mathfrak{m}$, i.e., $\sqrt{\mathfrak{m}S} = \mathfrak{q}$. But since $R \to S$ is étale, we have $\mathfrak{m}S_\mathfrak{q} = \mathfrak{q}S_\mathfrak{q}$, see Lemma 10.136.5. Hence we actually get that $\mathfrak{m}S = \mathfrak{q}$.

Suppose that $x \notin \mathfrak{m}R^h$. Represent x by a triple $(S, \mathfrak{q}, f)$ with $\mathfrak{m}S = \mathfrak{q}$. Then $f \notin \mathfrak{m}S$, i.e., $f \notin \mathfrak{q}$. Hence $(S, \mathfrak{q}) \to (S_f, \mathfrak{q}S_f)$ is a morphism of pairs such that the image of f becomes invertible. Hence x is invertible with inverse represented by the triple $(S_f, \mathfrak{q}S_f, 1/f)$. We conclude that R^h is a local ring with maximal ideal $\mathfrak{m}R^h$. The residue field is κ since we can define $R^h/\mathfrak{m}R^h \to \kappa$ by mapping a triple $(S, \mathfrak{q}, f)$ to the residue class of f module $\mathfrak{q}$.

We still have to show that R^h is henselian. Namely, suppose that $P \in R^h[T]$ is a monic polynomial and $a_0 \in \kappa$ is a simple root of the reduction $\overline{P} \in \kappa[T]$. Then we can find a pair $(S, \mathfrak{q})$ such that P is the image of a monic polynomial $Q \in S[T]$. Since $S \to R^h$ induces an isomorphism of residue fields we see that $S' = S[T]/(Q)$ has a prime ideal $\mathfrak{q}' = (\mathfrak{q}, T - a_0)$ at which $S \to S'$ is standard étale. Moreover, $\kappa = \kappa(\mathfrak{q}')$. Pick $g \in S'$, $g \notin \mathfrak{q}'$ such that $S'' = S'_g$ is étale over S. Then $(S, \mathfrak{q}) \to (S'', \mathfrak{q}'S'')$ is a morphism of pairs. Now that triple $(S'', \mathfrak{q}'S'', \text{class of } T)$ determines an element $a \in R^h$ with the properties $P(a) = 0$, and $\overline{a} = a_0$ as desired. □

Lemma 10.143.17. *Let $(R, \mathfrak{m}, \kappa)$ be a local ring. Let $\kappa \subset \kappa^{sep}$ be a separable algebraic closure. There exists a commutative diagram*

$$\begin{array}{ccccc} \kappa & \longrightarrow & \kappa & \longrightarrow & \kappa^{sep} \\ \uparrow & & \uparrow & & \uparrow \\ R & \longrightarrow & R^h & \longrightarrow & R^{sh} \end{array}$$

with the following properties

(1) *the map $R^h \to R^{sh}$ is local*
(2) *R^{sh} is strictly henselian,*
(3) *R^{sh} is a filtered colimit of étale R-algebras,*
(4) *$\mathfrak{m}R^{sh}$ is the maximal ideal of R^{sh}, and*
(5) *$\kappa^{sep} = R^{sh}/\mathfrak{m}R^{sh}$.*

Proof. This can be proved using the method followed in the proof of Lemma 10.143.16. The only difference is that, instead of pairs, one uses triples $(S, \mathfrak{q}, \alpha)$ where $R \to S$ étale, $\mathfrak{q}$ is a prime of S lying over $\mathfrak{m}$, and $\alpha : \kappa(\mathfrak{q}) \to \kappa^{sep}$ is an embedding of extensions of κ.

But we can also deduce the result directly from the result of Lemma 10.143.16. Namely, for any finite separable field sub extension $\kappa \subset \kappa' \subset \kappa^{sep}$ there exists a unique (up to unique isomorphism) finite étale local ring extension $R^h \subset R^h(\kappa')$

whose residue field extension extension reproduces the given extension, see Lemma 10.143.8. Hence we can set

$$R^{sh} = \bigcup\nolimits_{\kappa \subset \kappa' \subset \kappa^{sep}} R^h(\kappa')$$

The arrows in this system, compatible with the arrows on the level of residue fields, exist by Lemma 10.143.8. This will produce a henselian local ring by Lemma 10.143.5 since each of the rings $R^h(\kappa')$ is henselian by Lemma 10.143.4. By construction the residue field extension induced by $R^h \to R^{sh}$ is the field extension $\kappa \subset \kappa^{sep}$. We omit the proof that R^{sh} is a colimit of étale R-algebras. □

Definition 10.143.18. Let $(R, \mathfrak{m}, \kappa)$ be a local ring.

(1) The local ring map $R \to R^h$ constructed in Lemma 10.143.16 is called the *henselization* of R.
(2) Given a separable algebraic closure $\kappa \subset \kappa^{sep}$ the local ring map $R \to R^{sh}$ constructed in Lemma 10.143.17 is called the *strict henselization of R with respect to $\kappa \subset \kappa^{sep}$*.
(3) A local ring map $R \to R^{sh}$ is called a *strict henselization* of R if it is isomorphic to one of the local ring maps constructed in Lemma 10.143.17

The maps $R \to R^h \to R^{sh}$ are flat local ring homomorphisms. By Lemma 10.143.15 the R-algebras R^h and R^{sh} are well defined up to unique isomorphism by the conditions that they are henselian local, filtered colimits of étale R-algebras with residue field κ and κ^{sep}. We will discuss this and the close relationship between R and its henselization in More on Algebra, Section 15.32. In the rest of this section we prove some lemmas we discuss functoriality properties of the (strict) henselizations. This should make it clear exactly how canonical these constructions really are.

Lemma 10.143.19. *Let $R \to S$ be a local map of local rings. Let $S \to S^h$ be the henselization. Let $R \to A$ be an étale ring map and let $\mathfrak{q}$ be a prime of A lying over $\mathfrak{m}_R$ such that $R/\mathfrak{m}_R \cong \kappa(\mathfrak{q})$. Then there exists a unique morphism of rings $f : A \to S^h$ fitting into the commutative diagram*

$$\begin{array}{ccc} A & \xrightarrow{f} & S^h \\ \uparrow & & \uparrow \\ R & \longrightarrow & S \end{array}$$

such that $f^{-1}(\mathfrak{m}_{S^h}) = \mathfrak{q}$.

Proof. This is a special case of Lemma 10.143.12. □

Lemma 10.143.20. *Let $R \to S$ be a local map of local rings. Let $R \to R^h$ and $S \to S^h$ be the henselizations. There exists a unique local ring map $R^h \to S^h$ fitting into the commutative diagram*

$$\begin{array}{ccc} R^h & \xrightarrow{f} & S^h \\ \uparrow & & \uparrow \\ R & \longrightarrow & S \end{array}$$

Proof. Follows immediately from Lemma 10.143.13. □

Here is a slightly different construction of the henselization.

Lemma 10.143.21. *Let R be a ring. Let $\mathfrak{p} \subset R$ be a prime ideal. Consider the category of pairs $(S, \mathfrak{q})$ where $R \to S$ is étale and $\mathfrak{q}$ is a prime lying over $\mathfrak{p}$ such that $\kappa(\mathfrak{p}) = \kappa(\mathfrak{q})$. This category is filtered and*

$$(R_\mathfrak{p})^h = \operatorname{colim}_{(S,\mathfrak{q})} S = \operatorname{colim}_{(S,\mathfrak{q})} S_\mathfrak{q}$$

canonically.

Proof. A morphism of pairs $(S, \mathfrak{q}) \to (S', \mathfrak{q}')$ is given by an R-algebra map $\varphi : S \to S'$ such that $\varphi^{-1}(\mathfrak{q}') = \mathfrak{q}$. Let us show that the category of pairs is filtered, see Categories, Definition 4.19.1. The category contains the pair $(R, \mathfrak{p})$ and hence is not empty, which proves part (1) of Categories, Definition 4.19.1. Suppose that $(S, \mathfrak{q})$ and $(S', \mathfrak{q}')$ are two pairs. Note that $\mathfrak{q}$, resp. $\mathfrak{q}'$ correspond to primes of the fibre rings $S \otimes \kappa(\mathfrak{p})$, resp. $S' \otimes \kappa(\mathfrak{p})$ with residue fields $\kappa(\mathfrak{p})$, hence they correspond to maximal ideals of $S \otimes \kappa(\mathfrak{p})$, resp. $S' \otimes \kappa(\mathfrak{p})$. Set $S'' = S \otimes_R S'$. By the above there exists a unique prime $\mathfrak{q}'' \subset S''$ lying over $\mathfrak{q}$ and over $\mathfrak{q}'$ whose residue field is $\kappa(\mathfrak{p})$. The ring map $R \to S''$ is étale by Lemma 10.136.3. This proves part (2) of Categories, Definition 4.19.1. Next, suppose that $\varphi, \psi : (S, \mathfrak{q}) \to (S', \mathfrak{q}')$ are two morphisms of pairs. Consider

$$S'' = (S' \otimes_{\varphi,S,\psi} S') \otimes_{S' \otimes_R S'} S'$$

Arguing as above (base change of étale maps is étale, composition of étale maps is étale) we see that S'' is étale over R. The fibre ring of S'' over $\mathfrak{p}$ is

$$F'' = (F' \otimes_{\varphi,F,\psi} F') \otimes_{F' \otimes_{\kappa(\mathfrak{p})} F'} F'$$

where F', F are the fibre rings of S' and S. Since φ and ψ are morphisms of pairs the map $F' \to \kappa(\mathfrak{p})$ corresponding to $\mathfrak{p}'$ extends to a map $F'' \to \kappa(\mathfrak{p})$ and in turn corresponds to a prime ideal $\mathfrak{q}'' \subset S''$ whose residue field is $\kappa(\mathfrak{p})$. The canonical map $S' \to S''$ (using the right most factor for example) is a morphism of pairs $(S', \mathfrak{q}') \to (S'', \mathfrak{q}'')$ which equalizes φ and ψ. This proves part (3) of Categories, Definition 4.19.1. Hence we conclude that the category is filtered.

Recall that in the proof of Lemma 10.143.16 we constructed $(R_\mathfrak{p})^h$ as the corresponding colimit but starting with $R_\mathfrak{p}$ and its maximal ideal $\mathfrak{p}R_\mathfrak{p}$. Now, given any pair $(S, \mathfrak{q})$ for $(R, \mathfrak{p})$ we obtain a pair $(S_\mathfrak{p}, \mathfrak{q}S_\mathfrak{p})$ for $(R_\mathfrak{p}, \mathfrak{p}R_\mathfrak{p})$. Moreover, in this situation

$$S_\mathfrak{p} = \operatorname{colim}_{f \in R, f \notin \mathfrak{p}} S_f.$$

Hence in order to show the equalities of the lemma, it suffices to show that any pair $(S_{loc}, \mathfrak{q}_{loc})$ for $(R_\mathfrak{p}, \mathfrak{p}R_\mathfrak{p})$ is of the form $(S_\mathfrak{p}, \mathfrak{q}S_\mathfrak{p})$ for some pair $(S, \mathfrak{q})$ over $(R, \mathfrak{p})$ (some details omitted). This follows from Lemma 10.136.3. □

Lemma 10.143.22. *Let $R \to S$ be a ring map. Let $\mathfrak{q} \subset S$ be a prime lying over $\mathfrak{p} \subset R$. Let $R \to R^h$ and $S \to S^h$ be the henselizations of $R_\mathfrak{p}$ and $S_\mathfrak{q}$. The local ring map $R^h \to S^h$ of Lemma 10.143.20 identifies S^h with the henselization of $R^h \otimes_R S$ at the unique prime lying over $\mathfrak{m}^h$ and $\mathfrak{q}$.*

Proof. By Lemma 10.143.21 we see that R^h, resp. S^h are filtered colimits of étale R, resp. S-algebras. Hence we see that $R^h \otimes_R S$ is a filtered colimit of étale S-algebras A_i (Lemma 10.136.3). By Lemma 10.143.14 we see that S^h is a filtered colimit of étale $R^h \otimes_R S$-algebras. Since moreover S^h is a henselian local ring

with residue field equal to $\kappa(\mathfrak{q})$, the statement follows from the uniqueness result of Lemma 10.143.15. □

Lemma 10.143.23. *Let $R \to S$ be a ring map. Let $\mathfrak{q}$ be a prime of S lying over $\mathfrak{p}$ in R. Assume $R \to S$ is quasi-finite at $\mathfrak{q}$. The commutative diagram*

$$\begin{array}{ccc} R^h_{\mathfrak{p}} & \longrightarrow & S^h_{\mathfrak{q}} \\ \uparrow & & \uparrow \\ R_{\mathfrak{p}} & \longrightarrow & S_{\mathfrak{q}} \end{array}$$

of Lemma 10.143.20 identifies $S^h_{\mathfrak{q}}$ with the localization of $R^h_{\mathfrak{p}} \otimes_{R_{\mathfrak{p}}} S_{\mathfrak{q}}$ at the prime generated by $\mathfrak{q}$.

Proof. Note that $R^h_{\mathfrak{p}} \otimes_R S$ is quasi-finite over $R^h_{\mathfrak{p}}$ at the prime ideal corresponding to $\mathfrak{q}$, see Lemma 10.116.6. Hence the localization S' of $R^h_{\mathfrak{p}} \otimes_{R_{\mathfrak{p}}} S_{\mathfrak{q}}$ is henselian, see Lemma 10.143.4. As a localization S' is a filtered colimit of étale $R^h_{\mathfrak{p}} \otimes_{R_{\mathfrak{p}}} S_{\mathfrak{q}}$-algebras. By Lemma 10.143.22 we see that $S^h_{\mathfrak{q}}$ is the henselization of $R^h_{\mathfrak{p}} \otimes_{R_{\mathfrak{p}}} S_{\mathfrak{q}}$. Thus $S' = S^h_{\mathfrak{q}}$ by the uniqueness result of Lemma 10.143.15. □

Lemma 10.143.24. *Let R be a local ring with henselization R^h. Let $I \subset \mathfrak{m}_R$. Then R^h/IR^h is the henselization of R/I.*

Proof. This is a special case of Lemma 10.143.23. □

Lemma 10.143.25. *Let $\varphi : R \to S$ be a local map of local rings. Let $S/\mathfrak{m}_S \subset \kappa^{sep}$ be a separable algebraic closure. Let $S \to S^{sh}$ be the strict henselization of S with respect to $S/\mathfrak{m}_S \subset \kappa^{sep}$. Let $R \to A$ be an étale ring map and let $\mathfrak{q}$ be a prime of A lying over $\mathfrak{m}_R$. Given any commutative diagram*

$$\begin{array}{ccc} \kappa(\mathfrak{q}) & \xrightarrow[\phi]{} & \kappa^{sep} \\ \uparrow & & \uparrow \\ R/\mathfrak{m}_R & \xrightarrow{\varphi} & S/\mathfrak{m}_S \end{array}$$

there exists a unique morphism of rings $f : A \to S^{sh}$ fitting into the commutative diagram

$$\begin{array}{ccc} A & \xrightarrow[f]{} & S^{sh} \\ \uparrow & & \uparrow \\ R & \xrightarrow{\varphi} & S \end{array}$$

such that $f^{-1}(\mathfrak{m}_{S^h}) = \mathfrak{q}$ and the induced map $\kappa(\mathfrak{q}) \to \kappa^{sep}$ is the given one.

Proof. This is a special case of Lemma 10.143.12. □

Lemma 10.143.26. *Let $R \to S$ be a local map of local rings. Choose separable algebraic closures $R/\mathfrak{m}_R \subset \kappa_1^{sep}$ and $S/\mathfrak{m}_S \subset \kappa_2^{sep}$. Let $R \to R^{sh}$ and $S \to S^{sh}$ be*

the corresponding strict henselizations. Given any commutative diagram

$$\begin{array}{ccc} \kappa_1^{sep} & \xrightarrow{\phi} & \kappa_2^{sep} \\ \uparrow & & \uparrow \\ R/\mathfrak{m}_R & \xrightarrow{\varphi} & S/\mathfrak{m}_S \end{array}$$

There exists a unique local ring map $R^{sh} \to S^{sh}$ fitting into the commutative diagram

$$\begin{array}{ccc} R^{sh} & \xrightarrow{f} & S^{sh} \\ \uparrow & & \uparrow \\ R & \longrightarrow & S \end{array}$$

and inducing ϕ on the residue fields of R^{sh} and S^{sh}.

Proof. Follows immediately from Lemma 10.143.13. □

Lemma 10.143.27. *Let R be a ring. Let $\mathfrak{p} \subset R$ be a prime ideal. Let $\kappa(\mathfrak{p}) \subset \kappa^{sep}$ be a separable algebraic closure. Consider the category of triples $(S, \mathfrak{q}, \phi)$ where $R \to S$ is étale, $\mathfrak{q}$ is a prime lying over $\mathfrak{p}$, and $\phi : \kappa(\mathfrak{q}) \to \kappa^{sep}$ is a $\kappa(\mathfrak{p})$-algebra map. This category is filtered and*

$$(R_{\mathfrak{p}})^{sh} = \operatorname{colim}_{(S,\mathfrak{q},\phi)} S = \operatorname{colim}_{(S,\mathfrak{q},\phi)} S_{\mathfrak{q}}$$

canonically.

Proof. A morphism of triples $(S, \mathfrak{q}, \phi) \to (S', \mathfrak{q}', \phi')$ is given by an R-algebra map $\varphi : S \to S'$ such that $\varphi^{-1}(\mathfrak{q}') = \mathfrak{q}$ and such that $\phi' \circ \varphi = \phi$. Let us show that the category of pairs is filtered, see Categories, Definition 4.19.1. The category contains the triple $(R, \mathfrak{p}, \kappa(\mathfrak{p}) \subset \kappa^{sep})$ and hence is not empty, which proves part (1) of Categories, Definition 4.19.1. Suppose that $(S, \mathfrak{q}, \phi)$ and $(S', \mathfrak{q}', \phi')$ are two triples. Note that $\mathfrak{q}$, resp. $\mathfrak{q}'$ correspond to primes of the fibre rings $S \otimes \kappa(\mathfrak{p})$, resp. $S' \otimes \kappa(\mathfrak{p})$ with residue fields finite separable over $\kappa(\mathfrak{p})$ and ϕ, resp. ϕ' correspond to maps into κ^{sep}. Hence this data corresponds to $\kappa(\mathfrak{p})$-algebra maps

$$\phi : S \otimes_R \kappa(\mathfrak{p}) \longrightarrow \kappa^{sep}, \quad \phi' : S' \otimes_R \kappa(\mathfrak{p}) \longrightarrow \kappa^{sep}.$$

Set $S'' = S \otimes_R S'$. Combining the maps the above we get a unique $\kappa(\mathfrak{p})$-algebra map

$$\phi'' = \phi \otimes \phi' : S'' \otimes_R \kappa(\mathfrak{p}) \longrightarrow \kappa^{sep}$$

whose kernel corresponds to a prime $\mathfrak{q}'' \subset S''$ lying over $\mathfrak{q}$ and over $\mathfrak{q}'$, and whose residue field maps via ϕ'' to the compositum of $\phi(\kappa(\mathfrak{q}))$ and $\phi'(\kappa(\mathfrak{q}'))$ in κ^{sep}. The ring map $R \to S''$ is étale by Lemma 10.136.3. Hence $(S'', \mathfrak{q}'', \phi'')$ is a triple dominating both $(S, \mathfrak{q}, \phi)$ and $(S', \mathfrak{q}', \phi')$. This proves part (2) of Categories, Definition 4.19.1. Next, suppose that $\varphi, \psi : (S, \mathfrak{q}, \phi) \to (S', \mathfrak{q}', \phi')$ are two morphisms of pairs. Consider

$$S'' = (S' \otimes_{\varphi,S,\psi} S') \otimes_{S' \otimes_R S'} S'$$

Arguing as above (base change of étale maps is étale, composition of étale maps is étale) we see that S'' is étale over R. The fibre ring of S'' over $\mathfrak{p}$ is

$$F'' = (F' \otimes_{\varphi,F,\psi} F') \otimes_{F' \otimes_{\kappa(\mathfrak{p})} F'} F'$$

where F', F are the fibre rings of S' and S. Since φ and ψ are morphisms of triples the map $\phi' : F' \to \kappa^{sep}$ extends to a map $\phi'' : F'' \to \kappa^{sep}$ which in turn corresponds to a prime ideal $\mathfrak{q}'' \subset S''$. The canonical map $S' \to S''$ (using the right most factor for example) is a morphism of triples $(S', \mathfrak{q}', \phi') \to (S'', \mathfrak{q}'', \phi'')$ which equalizes φ and ψ. This proves part (3) of Categories, Definition 4.19.1. Hence we conclude that the category is filtered.

We still have to show that the colimit R_{colim} of the system is equal to the strict henselization of $R_\mathfrak{p}$ with respect to κ^{sep}. To see this note that the system of triples $(S, \mathfrak{q}, \phi)$ contains as a subsystem the pairs $(S, \mathfrak{q})$ of Lemma 10.143.21. Hence R_{colim} contains $R^h_\mathfrak{p}$ by the result of that lemma. Moreover, it is clear that $R^h_\mathfrak{p} \subset R_{colim}$ is a directed colimit of étale ring extensions. It follows that R_{colim} is henselian by Lemmas 10.143.4 and 10.143.5. Finally, by Lemma 10.136.16 we see that the residue field of R_{colim} is equal to κ^{sep}. Hence we conclude that R_{colim} is strictly henselian and hence equals the strict henselization of $R_\mathfrak{p}$ as desired. Some details omitted. □

Lemma 10.143.28. *Let $R \to S$ be a ring map. Let $\mathfrak{q} \subset S$ be a prime lying over $\mathfrak{p} \subset R$. Choose separable algebraic closures $\kappa(\mathfrak{p}) \subset \kappa_1^{sep}$ and $\kappa(\mathfrak{q}) \subset \kappa_2^{sep}$. Let R^{sh} and S^{sh} be the corresponding strict henselizations of $R_\mathfrak{p}$ and $S_\mathfrak{q}$. Given any commutative diagram*

$$\begin{array}{ccc} \kappa_1^{sep} & \xrightarrow{\phi} & \kappa_2^{sep} \\ \uparrow & & \uparrow \\ \kappa(\mathfrak{p}) & \xrightarrow{\varphi} & \kappa(\mathfrak{q}) \end{array}$$

The local ring map $R^{sh} \to S^{sh}$ of Lemma 10.143.26 identifies S^{sh} with the strict henselization of $R^{sh} \otimes_R S$ at a prime lying over $\mathfrak{m}^{sh}$ and $\mathfrak{q}$.

Proof. The proof is identical to the proof of Lemma 10.143.22 except that it uses Lemma 10.143.27 instead of Lemma 10.143.21. □

Lemma 10.143.29. *Let $R \to S$ be a ring map. Let $\mathfrak{q}$ be a prime of S lying over $\mathfrak{p}$ in R. Let $\kappa(\mathfrak{q}) \subset \kappa^{sep}$ be a separable algebraic closure. Assume $R \to S$ is quasi-finite at $\mathfrak{q}$. The commutative diagram*

$$\begin{array}{ccc} R^{sh}_\mathfrak{p} & \longrightarrow & S^{sh}_\mathfrak{q} \\ \uparrow & & \uparrow \\ R_\mathfrak{p} & \longrightarrow & S_\mathfrak{q} \end{array}$$

of Lemma 10.143.26 identifies $S^{sh}_\mathfrak{q}$ with a localization of $R^{sh}_\mathfrak{p} \otimes_{R_\mathfrak{p}} S_\mathfrak{q}$.

Proof. The residue field of $R^{sh}_\mathfrak{p}$ is the separable algebraic closure of $\kappa(\mathfrak{p})$ in κ^{sep}. Note that $R^{sh}_\mathfrak{p} \otimes_R S$ is quasi-finite over $R^{sh}_\mathfrak{p}$ at the prime ideal corresponding to $\mathfrak{q}$, see Lemma 10.116.6. Hence the localization S' of $R^{sh}_\mathfrak{p} \otimes_{R_\mathfrak{p}} S_\mathfrak{q}$ is henselian, see Lemma 10.143.4. Note that the residue field of S' is κ^{sep} since it contains both the separable algebraic closure of $\kappa(\mathfrak{p})$ and $\kappa(\mathfrak{q})$. Furthermore, as a localization S' is a filtered colimit of étale $R^{sh}_\mathfrak{p} \otimes_{R_\mathfrak{p}} S_\mathfrak{q}$-algebras. By Lemma 10.143.28 we see that $S^{sh}_\mathfrak{q}$ is a strict henselization of $R^{sh}_\mathfrak{p} \otimes_{R_\mathfrak{p}} S_\mathfrak{q}$. Thus $S' = S^h_\mathfrak{q}$ by the uniqueness result of Lemma 10.143.15. □

Lemma 10.143.30. *Let R be a local ring with strict henselization R^{sh}. Let $I \subset \mathfrak{m}_R$. Then R^{sh}/IR^{sh} is a strict henselization of R/I.*

Proof. This is a special case of Lemma 10.143.29. □

Lemma 10.143.31. *Let $\varphi : R \to S$ be a local homomorphism of strictly henselian local rings. Let $P_1, \ldots, P_n \in R[x_1, \ldots, x_n]$ be polynomials such that $R[x_1, \ldots, x_n]/(P_1, \ldots, P_n)$ is étale over R. Then the map*

$$R^n \longrightarrow S^n, \quad (h_1, \ldots, h_n) \longmapsto (\varphi(h_1), \ldots, \varphi(h_n))$$

induces a bijection between

$$\{(r_1, \ldots, r_n) \in R^n \mid P_i(r_1, \ldots, r_n) = 0,\ i = 1, \ldots, n\}$$

and

$$\{(s_1, \ldots, s_n) \in S^n \mid P'_i(s_1, \ldots, s_n) = 0,\ i = 1, \ldots, n\}$$

where $P'_i \in S[x_1, \ldots, x_n]$ are the images of the P_i under φ.

Proof. The first solution set is canonically isomorphic to the set

$$\mathrm{Hom}_R(R[x_1, \ldots, x_n]/(P_1, \ldots, P_n), R).$$

As R is henselian the map $R \to R/\mathfrak{m}_R$ induces a bijection between this set and the set of solutions in the residue field $R/\mathfrak{m}_R$, see Lemma 10.143.3. The same is true for S. Now since $R[x_1, \ldots, x_n]/(P_1, \ldots, P_n)$ is étale over R and $R/\mathfrak{m}_R$ is separably algebraically closed we see that $R/\mathfrak{m}_R[x_1, \ldots, x_n]/(\overline{P_1}, \ldots, \overline{P_n})$ is a finite product of copies of $R/\mathfrak{m}_R$. Hence the tensor product

$$R/\mathfrak{m}_R[x_1, \ldots, x_n]/(\overline{P_1}, \ldots, \overline{P_n}) \otimes_{R/\mathfrak{m}_R} S/\mathfrak{m}_S = S/\mathfrak{m}_S[x_1, \ldots, x_n]/(\overline{P'_1}, \ldots, \overline{P'_n})$$

is also a finite product of copies of $S/\mathfrak{m}_S$ with the same index set. This proves the lemma. □

Lemma 10.143.32. *Let R be a henselian local ring. Any countably generated Mittag-Leffler module over R is a direct sum of finitely presented R-modules.*

Proof. Let M be a countably generated and Mittag-Leffler R-module. We claim that for any element $x \in M$ there exists a direct sum decomposition $M = N \oplus K$ with $x \in N$, the module N finitely presented, and K Mittag-Leffler.

Suppose the claim is true. Choose generators $x_1, x_2, x_3, \ldots$ of M. By the claim we can inductively find direct sum decompositions

$$M = N_1 \oplus N_2 \oplus \ldots \oplus N_n \oplus K_n$$

with N_i finitely presented, $x_1, \ldots, x_n \in N_1 \oplus \ldots \oplus N_n$, and K_n Mittag-Leffler. Repeating ad infinitum we see that $M = \bigoplus N_i$.

We still have to prove the claim. Let $x \in M$. By Lemma 10.87.2 there exists an endomorphism $\alpha : M \to M$ such that α factors through a finitely presented module, and $\alpha(x) = x$. Say α factors as

$$M \xrightarrow{\pi} P \xrightarrow{i} M$$

Set $a = \pi \circ \alpha \circ i : P \to P$, so $i \circ a \circ \pi = \alpha^3$. By Lemma 10.15.2 there exists a monic polynomial $P \in R[T]$ such that $P(a) = 0$. Note that this implies formally that $\alpha^2 P(\alpha) = 0$. Hence we may think of M as a module over $R[T]/(T^2 P)$. Assume that $x \neq 0$. Then $\alpha(x) = x$ implies that $0 = \alpha^2 P(\alpha)x = P(1)x$ hence $P(1) = 0$ in

R/I where $I = \{r \in R \mid rx = 0\}$ is the annihilator of x. As $x \neq 0$ we see $I \subset \mathfrak{m}_R$, hence 1 is a root of $\overline{P} = P \bmod \mathfrak{m}_R \in R/\mathfrak{m}_R[T]$. As R is henselian we can find a factorization

$$T^2 P = (T^2 Q_1) Q_2$$

for some $Q_1, Q_2 \in R[T]$ with $Q_2 = (T-1)^e \bmod \mathfrak{m}_R R[T]$ and $Q_1(1) \neq 0 \bmod \mathfrak{m}_R$, see Lemma 10.143.3. Let $N = \mathrm{Im}(\alpha^2 Q_1(\alpha) : M \to M)$ and $K = \mathrm{Im}(Q_2(\alpha) : M \to M)$. As $T^2 Q_1$ and Q_2 generate the unit ideal of $R[T]$ we get a direct sum decomposition $M = N \oplus K$. Moreover, Q_2 acts as zero on N and $T^2 Q_1$ acts as zero on K. Note that N is a quotient of P hence is finitely generated. Also $x \in N$ because $\alpha^2 Q_1(\alpha) x = Q_1(1) x$ and $Q_1(1)$ is a unit in R. By Lemma 10.84.7 the modules N and K are Mittag-Leffler. Finally, the finitely generated module N is finitely presented as a finitely generated Mittag-Leffler module is finitely presented, see Example 10.86.1 part (1). □

10.144. Serre's criterion for normality

We introduce the following properties of Noetherian rings.

Definition 10.144.1. Let R be a Noetherian ring. Let $k \geq 0$ be an integer.

(1) We say R has property (R_k) if for every prime $\mathfrak{p}$ of height $\leq k$ the local ring $R_\mathfrak{p}$ is regular. We also say that R is *regular in codimension* $\leq k$.
(2) We say R has property (S_k) if for every prime $\mathfrak{p}$ the local ring $R_\mathfrak{p}$ has depth at least $\min\{k, \dim(R_\mathfrak{p})\}$.
(3) Let M be a finite R-module. We say M has property (S_k) if for every prime $\mathfrak{p}$ the module $M_\mathfrak{p}$ has depth at least $\min\{k, \dim(\mathrm{Supp}(M_\mathfrak{p}))\}$.

Any Noetherian ring has property (S_0) (and so does any finite module over it). Our convention that $\dim(\emptyset) = -\infty$ guarantees that the zero module has property (S_k) for all k.

Lemma 10.144.2. *Let R be a Noetherian ring. Let M be a finite R-module. The following are equivalent:*

(1) *M has no embedded associated prime, and*
(2) *M has property (S_1).*

Proof. Let $\mathfrak{p}$ be an embedded associated prime of M. Then there exists another associated prime $\mathfrak{q}$ of M such that $\mathfrak{p} \supset \mathfrak{q}$. In particular this implies that $\dim(\mathrm{Supp}(M_\mathfrak{p})) \geq 1$ (since $\mathfrak{q}$ is in the support as well). On the other hand $\mathfrak{p}R_\mathfrak{p}$ is associated to $M_\mathfrak{p}$ (Lemma 10.60.14) and hence $\mathrm{depth}(M_\mathfrak{p}) = 0$ (see Lemma 10.60.17). In other words (S_1) does not hold. Conversely, if (S_1) does not then there exists a prime $\mathfrak{p}$ such that $\dim(\mathrm{Supp}(M_\mathfrak{p})) \geq 1$ and $\mathrm{depth}(M_\mathfrak{p}) = 0$. Then we see (arguing backwards using the lemmas cited above) that $\mathfrak{p}$ is an embedded associated prime. □

Lemma 10.144.3. *Let R be a Noetherian ring. The following are equivalent:*

(1) *R is reduced, and*
(2) *R has properties (R_0) and (S_1).*

Proof. Suppose that R is reduced. Then $R_\mathfrak{p}$ is a field for every minimal prime $\mathfrak{p}$ of R, according to Lemma 10.24.4. Hence we have (R_0). Let $\mathfrak{p}$ be a prime of height ≥ 1. Then $A = R_\mathfrak{p}$ is a reduced local ring of dimension ≥ 1. Hence its maximal ideal $\mathfrak{m}$ is not an associated prime since this would mean there exists a $x \in \mathfrak{m}$ with

annihilator $\mathfrak{m}$ so $x^2 = 0$. Hence the depth of $A = R_{\mathfrak{p}}$ is at least one, by Lemma 10.60.9. This shows that (S_1) holds.

Conversely, assume that R satisfies (R_0) and (S_1). If $\mathfrak{p}$ is a minimal prime of R, then $R_{\mathfrak{p}}$ is a field by (R_0), and hence is reduced. If $\mathfrak{p}$ is not minimal, then we see that $R_{\mathfrak{p}}$ has depth ≥ 1 by (S_1) and we conclude there exists an element $t \in \mathfrak{p}R_{\mathfrak{p}}$ such that $R_{\mathfrak{p}} \to R_{\mathfrak{p}}[1/t]$ is injective. This implies that $R_{\mathfrak{p}}$ is a subring of localizations of R at primes of smaller height. Thus by induction on the height we conclude that R is reduced. □

Lemma 10.144.4 (Serre's criterion for normality)**.** *Let R be a Noetherian ring. The following are equivalent:*

(1) *R is a normal ring, and*
(2) *R has properties (R_1) and (S_2).*

Proof. Suppose that R is normal. This means by definition that R is reduced and all localizations $R_{\mathfrak{p}}$ are normal domains. In particular we see that R has (R_0) and (S_1) by Lemma 10.144.3. Hence it suffices to show that a local Noetherian normal domain R of dimension d has depth $\geq \min(2, d)$ and is regular if $d = 1$. The assertion if $d = 1$ follows from Lemma 10.113.6.

Let R be a local Noetherian normal domain with maximal ideal $\mathfrak{m}$ and dimension $d \geq 2$. Choose $x \in \mathfrak{m}$, $x \notin \mathfrak{m}^2$. If $\text{depth}(R/xR) \geq 1$ then $\text{depth}(R) \geq 2$ and we win. Assume $\text{depth}(R/xR) = 0$ to get a contradiction. This means that $\mathfrak{m}/(x)$ is an associated prime of R/xR. In other words, there exists an element $z \in R$ such that $\mathfrak{m}z \subset (x)$, but $z \notin (x)$. Consider the element z/x of the fraction field of R. Let $c \in \mathfrak{m}$ be an arbitrary nonzero element. We claim that $cz^n/x^n \in R$. Namely, $cz^n/x^n = (cz/x)z^{n-1}/x^{n-1}$. By choice of z we have $cz = c'x$ for some $c' \in R$. Note that $c' \in \mathfrak{m}$ since $x \notin \mathfrak{m}^2$. Hence $cz^n/x^n = c'z^{n-1}/x^{n-1}$ which is an element of R by induction on n. In other words, this shows that z/x is almost integral over R, see Definition 10.34.3. By Lemma 10.34.4 we see that z/x is integral over R. As R is normal we see that $z/x \in R$ which is the desired contradiction.

Suppose that R satisfies (R_1) and (S_2). By Lemma 10.144.3 we conclude that R is reduced. Hence it suffices to show that if R is a reduced local Noetherian ring of dimension d satisfying (S_2) and (R_1) then R is a normal domain. If $d = 0$, the result is clear. If $d = 1$, then the result follows from Lemma 10.113.6.

Let R be a reduced local Noetherian ring with maximal ideal $\mathfrak{m}$ and dimension d which satisfies (R_1) and (S_2). By Lemma 10.34.14 it suffices to show that R is integrally closed in its total ring of fractions. Pick $x = f/g$, with $f, g \in R$ and g a nonzerodivisor which satisfies a monic equation

$$(f/g)^n + \sum\nolimits_{i=1}^{n} a_i (f/g)^{n-i} = 0$$

with $a_i \in R$. Our goal is to show that $f \in (g) = gR$. We will prove this by induction on d. By the remarks in the previous paragraph we know this is the case when $d = 0$, and when $d = 1$, which starts the induction. Assume $d \geq 2$. Consider the short exact sequence

$$0 \to R \to R \to R/(g) \to 0.$$

By Lemma 10.67.11 this implies $\text{depth}(R/(g)) \geq 1$. Hence there exists an element $t \in \mathfrak{m}$ which is a nonzerodivisor on $R/(g)$. Hence if f has a nonzero image in $R/(g)$

then it has a nonzero image in $(R/(g))[1/t] \cong R_t/gR_t$. But by induction on the dimension the image of f is zero in R_t/gR_t (for example by localizing at all the primes of $D(t) \subset \mathrm{Spec}(R)$). Hence we win. □

Lemma 10.144.5. *A regular ring is normal.*

Proof. Let R be a regular ring. By Lemma 10.144.4 it suffices to prove that R is (R_1) and (S_2). As a regular local ring is Cohen-Macaulay, see Lemma 10.101.3, it is clear that R is (S_2). Property (R_1) is immediate. □

Lemma 10.144.6. *Let R be a Noetherian normal domain with fraction field K. Then*

(1) *for any nonzero $a \in R$ the quotient R/aR has no embedded primes, and all its associated primes have height 1*
(2)
$$R = \bigcap\nolimits_{height(\mathfrak{p})=1} R_{\mathfrak{p}}$$
(3) *For any nonzero $x \in K$ the quotient $R/(R \cap xR)$ has no embedded primes, and all its associates primes have height 1.*

Proof. By Lemma 10.144.4 we see that R has (S_2). Hence for any nonzero element $a \in R$ we see that R/aR has (S_1) (use Lemma 10.67.11 for example) Hence R/aR has no embedded primes (Lemma 10.144.2). We conclude the associated primes of R/aR are exactly the minimal primes $\mathfrak{p}$ over (a), which have height 1 as a is not zero (Lemma 10.57.10). This proves (1).

Thus, given $b \in R$ we have $b \in aR$ if and only if $b \in aR_{\mathfrak{p}}$ for every minimal prime $\mathfrak{p}$ over (a) (see Lemma 10.60.18). These primes all have height 1 as seen above so $b/a \in R$ if and only if $b/a \in R_{\mathfrak{p}}$ for all height 1 primes. Hence (2) holds.

For (3) write $x = a/b$. Let $\mathfrak{p}_1, \ldots, \mathfrak{p}_r$ be the minimal primes over (ab). These all have height 1 by the above. Then we see that $R \cap xR = \bigcap_{i=1,\ldots,r}(R \cap xR_{\mathfrak{p}_i})$ by part (2) of the lemma. Hence $R/(R \cap xR)$ is a submodule of $\bigoplus R/(R \cap xR_{\mathfrak{p}_i})$. As $R_{\mathfrak{p}_i}$ is a discrete valuation ring (by property (R_1) for the Noetherian normal domain R, see Lemma 10.144.4) we have $xR_{\mathfrak{p}_i} = \mathfrak{p}_i^{e_i} R_{\mathfrak{p}_i}$ for some $e_i \in \mathbf{Z}$. Hence the direct sum is equal to $\bigoplus_{e_i>0} R/\mathfrak{p}_i^{(e_i)}$, see Definition 10.61.1. By Lemma 10.61.2 the only associated prime of the module $R/\mathfrak{p}^{(n)}$ is $\mathfrak{p}$. Hence the set of associate primes of $R/(R \cap xR)$ is a subset of $\{\mathfrak{p}_i\}$ and there are no inclusion relations among them. This proves (3). □

10.145. Formal smoothness of fields

In this section we show that field extensions are formally smooth if and only if they are separable. However, we first prove finitely generated field extensions are separable algebraic if and only if they are formally unramified.

Lemma 10.145.1. *Let $k \subset K$ be a finitely generated field extension. The following are equivalent*

(1) *K is a finite separable field extension of k,*
(2) *$\Omega_{K/k} = 0$,*
(3) *K is formally unramified over k,*
(4) *K is unramified over k,*
(5) *K is formally étale over k,*

(6) *K is étale over k.*

Proof. The equivalence of (2) and (3) is Lemma 10.139.2. By Lemma 10.136.4 we see that (1) is equivalent to (6). Property (6) implies (5) and (4) which both in turn imply (3) (Lemmas 10.141.2, 10.142.3, and 10.142.2). Thus it suffices to show that (2) implies (1). Choose a finitely generated k-subalgebra $A \subset K$ such that K is the fraction field of the domain A. Set $S = A \setminus \{0\}$. Since $0 = \Omega_{K/k} = S^{-1}\Omega_{A/k}$ (Lemma 10.125.8) and since $\Omega_{A/k}$ is finitely generated (Lemma 10.125.16), we can replace A by a localization A_f to reduce to the case that $\Omega_{A/k} = 0$ (details omitted). Then A is unramified over k, hence K/k is finite separable for example by Lemma 10.142.5 applied with $\mathfrak{q} = (0)$. □

Lemma 10.145.2. *Let K be a field of characteristic $p > 0$. Let $a \in K$. Then $da = 0$ in $\Omega_{K/\mathbf{F}_p}$ if and only if a is a pth power.*

Proof. By Lemma 10.125.4 we see that there exists a subfield $\mathbf{F}_p \subset L \subset K$ such that $\mathbf{F}_p \subset L$ is a finitely generated field extension and such that $\mathrm{d}a$ is zero in $\Omega_{L/\mathbf{F}_p}$. Hence we may assume that K is a finitely generated field extension of $\mathbf{F}_p$.

Choose a transcendence basis $x_1, \ldots, x_r \in K$ such that K is finite separable over $\mathbf{F}_p(x_1, \ldots, x_r)$. We remark that the result holds for the purely transcendental subfield $\mathbf{F}_p(x_1, \ldots, x_r) \subset K$. Namely,

$$\Omega_{\mathbf{F}_p(x_1,\ldots,x_r)/\mathbf{F}_p} = \bigoplus\nolimits_{i=1}^{r} \mathbf{F}_p(x_1, \ldots, x_r)\mathrm{d}x_i$$

and any rational function all of whose partial derivatives are zero is a pth power. Moreover, we also have

$$\Omega_{K/\mathbf{F}_p} = \bigoplus\nolimits_{i=1}^{r} K\mathrm{d}x_i$$

since $\mathbf{F}_p(x_1, \ldots, x_r) \subset K$ is finite separable (computation omitted). Suppose $a \in K$ is an element such that $\mathrm{d}a = 0$ in the module of differentials. By our choice of x_i we see that the minimal polynomial $P(T) \in k(x_1, \ldots, x_r)[T]$ of a is separable. Write

$$P(T) = T^d + \sum\nolimits_{i=1}^{d} a_i T^{d-i}$$

and hence

$$0 = \mathrm{d}P(a) = \sum\nolimits_{i=1}^{d} a^{d-i}\mathrm{d}a_i$$

in $\Omega_{K/\mathbf{F}_p}$. By the description of $\Omega_{K/\mathbf{F}_p}$ above and the fact that P was the minimal polynomial of a, we see that this implies $\mathrm{d}a_i = 0$. Hence $a_i = b_i^p$ for each i. Therefore by Fields, Lemma 9.25.2 we see that a is a pth power. □

Lemma 10.145.3. *Let k be a field of characteristic $p > 0$. Let $a_1, \ldots, a_n \in k$ be elements such that $da_1, \ldots, da_n$ are linearly independent in $\Omega_{k/\mathbf{F}_p}$. Then the field extension $k(a_1^{1/p}, \ldots, a_n^{1/p})$ has degree p^n over k.*

Proof. By induction on n. If $n = 1$ the result is Lemma 10.145.2. For the induction step, suppose that $k(a_1^{1/p}, \ldots, a_{n-1}^{1/p})$ has degree p^{n-1} over k. We have to show that a_n does not map to a pth power in $k(a_1^{1/p}, \ldots, a_{n-1}^{1/p})$. If it does then we can write

$$\begin{aligned} a_n &= \left(\sum\nolimits_{I=(i_1,\ldots,i_{n-1}),\ 0\leq i_j\leq p-1} \lambda_I a_1^{i_1/p} \ldots a_{n-1}^{i_{n-1}/p}\right)^p \\ &= \sum\nolimits_{I=(i_1,\ldots,i_{n-1}),\ 0\leq i_j\leq p-1} \lambda_I^p a_1^{i_1} \ldots a_{n-1}^{i_{n-1}} \end{aligned}$$

Applying d we see that $\mathrm{d}a_n$ is linearly dependent on $\mathrm{d}a_i$, $i < n$. This is a contradiction. $\square$

Lemma 10.145.4. *Let k be a field of characteristic $p > 0$. The following are equivalent:*

(1) *the field extension K/k is separable (see Definition 10.39.1), and*
(2) *the map $K \otimes_k \Omega_{k/\mathbf{F}_p} \to \Omega_{K/\mathbf{F}_p}$ is injective.*

Proof. Write K as a directed colimit $K = \mathrm{colim}_i K_i$ of finitely generated field extensions $k \subset K_i$. By definition K is separable if and only if each K_i is separable over k, and by Lemma 10.125.4 we see that $K \otimes_k \Omega_{k/\mathbf{F}_p} \to \Omega_{K/\mathbf{F}_p}$ is injective if and only if each $K_i \otimes_k \Omega_{k/\mathbf{F}_p} \to \Omega_{K_i/\mathbf{F}_p}$ is injective. Hence we may assume that K/k is a finitely generated field extension.

Assume $k \subset K$ is a finitely generated field extension which is separable. Choose $x_1, \ldots, x_{r+1} \in K$ as in Lemma 10.39.3. In this case there exists an irreducible polynomial $G(X_1, \ldots, X_{r+1}) \in k[X_1, \ldots, X_{r+1}]$ such that $G(x_1, \ldots, x_{r+1}) = 0$ and such that $\partial G/\partial X_{r+1}$ is not identically zero. Moreover K is the field of fractions of the domain. $S = K[X_1, \ldots, X_{r+1}]/(G)$. Write

$$G = \sum a_I X^I, \quad X^I = X_1^{i_1} \ldots X_{r+1}^{i_{r+1}}.$$

Using the presentation of S above we see that

$$\Omega_{S/\mathbf{F}_p} = \frac{S \otimes_k \Omega_k \oplus \bigoplus_{i=1,\ldots,r+1} S\mathrm{d}X_i}{\langle \sum X^I \mathrm{d}a_I + \sum \partial G/\partial X_i \mathrm{d}X_i \rangle}$$

Since $\Omega_{K/\mathbf{F}_p}$ is the localization of the S-module $\Omega_{S/\mathbf{F}_p}$ (see Lemma 10.125.8) we conclude that

$$\Omega_{K/\mathbf{F}_p} = \frac{K \otimes_k \Omega_k \oplus \bigoplus_{i=1,\ldots,r+1} K\mathrm{d}X_i}{\langle \sum X^I \mathrm{d}a_I + \sum \partial G/\partial X_i \mathrm{d}X_i \rangle}$$

Now, since the polynomial $\partial G/\partial X_{r+1}$ is not identically zero we conclude that the map $K \otimes_k \Omega_{k/\mathbf{F}_p} \to \Omega_{S/\mathbf{F}_p}$ is injective as desired.

Assume $k \subset K$ is a finitely generated field extension and that $K \otimes_k \Omega_{k/\mathbf{F}_p} \to \Omega_{K/\mathbf{F}_p}$ is injective. (This part of the proof is the same as the argument proving Lemma 10.41.1.) Let $x_1, \ldots, x_r$ be a transcendence basis of K over k such that the degree of inseparability of the finite extension $k(x_1, \ldots, x_r) \subset K$ is minimal. If K is separable over $k(x_1, \ldots, x_r)$ then we win. Assume this is not the case to get a contradiction. Then there exists an element $\alpha \in K$ which is not separable over $k(x_1, \ldots, x_r)$. Let $P(T) \in k(x_1, \ldots, x_r)[T]$ be its minimal polynomial. Because α is not separable actually P is a polynomial in T^p. Clear denominators to get an irreducible polynomial

$$G(X_1, \ldots, X_r, T) = \sum a_{I,i} X^I T^i \in k[X_1, \ldots, X_r, T]$$

such that $G(x_1, \ldots, x_r, \alpha) = 0$ in L. Note that this means $k[X_1, \ldots, X_r, T]/(G) \subset L$. We may assume that for some pair (I_0, i_0) the coefficient $a_{I_0,i_0} = 1$. We claim that $\mathrm{d}G/\mathrm{d}X_i$ is not identically zero for at least one i. Namely, if this is not the case, then G is actually a polynomial in $X_1^p, \ldots, X_r^p, T^p$. Then this means that

$$\sum\nolimits_{(I,i) \neq (I_0,i_0)} x^I \alpha^i \mathrm{d}a_{I,i}$$

is zero in $\Omega_{K/\mathbf{F}_p}$. Note that there is no k-linear relation among the elements

$$\{x^I\alpha^i \mid a_{I,i} \neq 0 \text{ and } (I,i) \neq (I_0,i_0)\}$$

of K. Hence the assumption that $K \otimes_k \Omega_{k/\mathbf{F}_p} \to \Omega_{K/\mathbf{F}_p}$ is injective this implies that $\mathrm{d}a_{I,i} = 0$ in $\Omega_{k/\mathbf{F}_p}$ for all (I,i). By Lemma 10.145.2 we see that each $a_{I,i}$ is a pth power, which implies that G is a pth power contradicting the irreducibility of G. Thus, after renumbering, we may assume that $\mathrm{d}G/\mathrm{d}X_1$ is not zero. Then we see that x_1 is separably algebraic over $k(x_2,\ldots,x_r,\alpha)$, and that $x_2,\ldots,x_r,\alpha$ is a transcendence basis of L over k. This means that the degree of inseparability of the finite extension $k(x_2,\ldots,x_r,\alpha) \subset L$ is less than the degree of inseparability of the finite extension $k(x_1,\ldots,x_r) \subset L$, which is a contradiction. □

Lemma 10.145.5. *Let $k \subset K$ be an extension of fields. If K is formally smooth over k, then K is a separable extension of k.*

Proof. Assume K is formally smooth over k. By Lemma 10.131.9 we see that $K \otimes_k \Omega_{k/\mathbf{Z}} \to \Omega_{K/\mathbf{Z}}$ is injective. Hence K is separable over k by Lemma 10.145.4. □

Lemma 10.145.6. *Let $k \subset K$ be an extension of fields. Then K is formally smooth over k if and only if $H_1(L_{K/k}) = 0$.*

Proof. This follows from Proposition 10.131.8 and the fact that a vector spaces is free (hence projective). □

Lemma 10.145.7. *Let $k \subset K$ be an extension of fields.*

1. *If K is purely transcendental over k, then K is formally smooth over k.*
2. *If K is separable algebraic over k, then K is formally smooth over k.*
3. *If K is separable over k, then K is formally smooth over k.*

Proof. For (1) write $K = k(x_j; j \in J)$. Suppose that A is a k-algebra, and $I \subset A$ is an ideal of square zero. Let $\varphi : K \to A/I$ be a k-algebra map. Let $a_j \in A$ be an element such that $a_j \mod I = \varphi(x_j)$. Then it is easy to see that there is a unique k-algebra map $K \to A$ which maps x_j to a_j and which reduces to $\varphi \bmod I$. Hence $k \subset K$ is formally smooth.

In case (2) we see that $k \subset K$ is a colimit of étale ring extensions. An étale ring map is formally étale (Lemma 10.141.2). Hence this case follows from Lemma 10.141.3 and the trivial observation that a formally étale ring map is formally smooth.

In case (3), write $K = \operatorname{colim} K_i$ as the filtered colimit of its finitely generated sub k-extensions. By Definition 10.39.1 each K_i is separable algebraic over a purely transcendental extension of k. Hence K_i/k is formally smooth by cases (1) and (2) and Lemma 10.131.3. Thus $H_1(L_{K_i/k}) = 0$ by Lemma 10.145.6. Hence $H_1(L_{K/k}) = 0$ by Lemma 10.127.9. Hence K/k is formally smooth by Lemma 10.145.6 again. □

Lemma 10.145.8. *Let k be a field.*

1. *If the characteristic of k is zero, then any extension field of k is formally smooth over k.*
2. *If the characteristic of k is $p > 0$, then $k \subset K$ is formally smooth if and only if it is a separable field extension.*

Proof. Combine Lemmas 10.145.5 and 10.145.7. □

Here we put together all the different characterizations of separable field extensions.

Proposition 10.145.9. *Let $k \subset K$ be a field extension. If the characteristic of k is zero then*

(1) *K is separable over k,*
(2) *K is geometrically reduced over k,*
(3) *K is formally smooth over k,*
(4) *$H_1(L_{K/k}) = 0$, and*
(5) *the map $K \otimes_k \Omega_{k/\mathbf{Z}} \to \Omega_{K/\mathbf{Z}}$ is injective.*

If the characteristic of k is $p > 0$, then the following are equivalent:

(1) *K is separable over k,*
(2) *the ring $K \otimes_k k^{1/p}$ is reduced,*
(3) *K is geometrically reduced over k,*
(4) *the map $K \otimes_k \Omega_{k/\mathbf{F}_p} \to \Omega_{K/\mathbf{F}_p}$ is injective,*
(5) *$H_1(L_{K/k}) = 0$, and*
(6) *K is formally smooth over k.*

Proof. This is a combination of Lemmas 10.41.1, 10.145.8 10.145.5, and 10.145.4. □

Here is yet another characterization of finitely generated separable field extensions.

Lemma 10.145.10. *Let $k \subset K$ be a finitely generated field extension. Then K is separable over k if and only if K is the localization of a smooth k-algebra.*

Proof. Choose a finite type k-algebra R which is a domain whose fraction field is K. Lemma 10.133.9 says that $k \to R$ is smooth at (0) if and only if K/k is separable. This proves the lemma. □

Lemma 10.145.11. *Let $k \subset K$ be a field extension. Then K is a filtered colimit of global complete intersection algebras over k. If K/k is separable, then K is a filtered colimit of smooth algebras over k.*

Proof. Suppose that $E \subset K$ is a finite subset. It suffices to show that there exists a k subalgebra $A \subset K$ which contains E and which is a global complete intersection (resp. smooth) over k. The separable/smooth case follows from Lemma 10.145.10. In general let $L \subset K$ be the subfield generated by E. Pick a transcendence basis $x_1, \ldots, x_d \in L$ over k. The extension $k(x_1, \ldots, x_d) \subset L$ is finite. Say $L = k(x_1, \ldots, x_d)[y_1, \ldots, y_r]$. Pick inductively polynomials $P_i \in k(x_1, \ldots, x_d)[Y_1, \ldots, Y_r]$ such that $P_i = P_i(Y_1, \ldots, Y_i)$ is monic in Y_i over $k(x_1, \ldots, x_d)[Y_1, \ldots, Y_{i-1}]$ and maps to the minimum polynomial of y_i in $k(x_1, \ldots, x_d)[y_1, \ldots, y_{i-1}][Y_i]$. Then it is clear that $P_1, \ldots, P_r$ is a regular sequence in $k(x_1, \ldots, x_r)[Y_1, \ldots, Y_r]$ and that $L = k(x_1, \ldots, x_r)[Y_1, \ldots, Y_r]/(P_1, \ldots, P_r)$. If $h \in k[x_1, \ldots, x_d]$ is a polynomial such that $P_i \in k[x_1, \ldots, x_d, 1/h, Y_1, \ldots, Y_r]$, then we see that $P_1, \ldots, P_r$ is a regular sequence in $k[x_1, \ldots, x_d, 1/h, Y_1, \ldots, Y_r]$ and $A = k[x_1, \ldots, x_d, 1/h, Y_1, \ldots, Y_r]/(P_1, \ldots, P_r)$ is a global complete intersection. After adjusting our choice of h we may assume $E \subset A$ and we win. □

10.146. Constructing flat ring maps

The following lemma is occasionally useful.

Lemma 10.146.1. *Let $(R, \mathfrak{m}, k)$ be a local ring. Let $k \subset K$ be a field extension. There exists a local ring $(R', \mathfrak{m}', k')$, a flat local ring map $R \to R'$ such that $\mathfrak{m}' = \mathfrak{m}R'$ and such that $k \subset k'$ is isomorphic to $k \subset K$.*

Proof. Suppose that $k \subset k' = k(\alpha)$ is a monogenic extension of fields. Then k' is the residue field of a flat local extension $R \subset R'$ as in the lemma. Namely, if α is transcendental over k, then we let R' be the localization of $R[x]$ at the prime $\mathfrak{m}R[x]$. If α is algebraic with minimal polynomial $T^d + \sum \overline{\lambda}_i T^{d-i}$, then we let $R' = R[T]/(T^d + \sum \lambda_i T^{d-i})$.

Consider the collection of triples $(k', R \to R', \phi)$, where $k \subset k' \subset K$ is a subfield, $R \to R'$ is a local ring map as in the lemma, and $\phi : R' \to k'$ induces an isomorphism $R'/\mathfrak{m}R' \cong k'$ of k-extensions. These form a "big" category $\mathcal{C}$ with morphisms $(k_1, R_1, \phi_1) \to (k_2, R_2, \phi_2)$ given by ring maps $\psi : R_1 \to R_2$ such that

$$\begin{array}{ccccc} R_1 & \xrightarrow[\phi_1]{} & k_1 & \longrightarrow & K \\ {\scriptstyle\psi}\downarrow & & & & \| \\ R_2 & \xrightarrow{\phi_2} & k_2 & \longrightarrow & K \end{array}$$

commutes. This implies that $k_1 \subset k_2$.

Suppose that I is a directed partially ordered set, and $((R_i, k_i, \phi_i), \psi_{ii'})$ is a system over I, see Categories, Section 4.21. In this case we can consider

$$R' = \operatorname{colim}_{i \in I} R_i$$

This is a local ring with maximal ideal $\mathfrak{m}R'$, and residue field $k' = \bigcup_{i \in I} k_i$. Moreover, the ring map $R \to R'$ is flat as it is a colimit of flat maps (and tensor products commute with directed colimits). Hence we see that (R', k', ϕ') is an "upper bound" for the system.

An almost trivial application of Zorn's Lemma would finish the proof if $\mathcal{C}$ was a set, but it isn't. (Actually, you can make this work by finding a reasonable bound on the cardinals of the local rings occurring.) To get around this problem we choose a well ordering on K. For $x \in K$ we let $K(x)$ be the subfield of K generated by all elements of K which are $\leq x$. By transfinite induction on $x \in K$ we will produce ring maps $R \subset R(x)$ as in the lemma with residue field extension $k \subset K(x)$. Moreover, by construction we will have that $R(x)$ will contain $R(y)$ for all $y \leq x$. Namely, if x has a predecessor x', then $K(x) = K(x')[x]$ and hence we can let $R(x') \subset R(x)$ be the local ring extension constructed in the first paragraph of the proof. If x does not have a predecessor, then we first set $R'(x) = \operatorname{colim}_{x' < x} R(x')$ as in the third paragraph of the proof. The residue field of $R'(x)$ is $K'(x) = \bigcup_{x' < x} K(x')$. Since $K(x) = K'(x)[x]$ we see that we can use the construction of the first paragraph of the proof to produce $R'(x) \subset R(x)$. This finishes the proof of the lemma. □

Lemma 10.146.2. *Let $(R, \mathfrak{m}, k)$ be a local ring. If $k \subset K$ is a separable algebraic extension, then there exists a directed partially ordered set I and a system of finite étale extensions $R \subset R_i$, $i \in I$ of local rings such that $R' = \operatorname{colim} R_i$ has residue field K (as extension of k).*

Proof. Let $R \subset R'$ be the extension constructed in the proof of Lemma 10.146.1. By construction $R' = \operatorname{colim}_{\alpha \in A} R_\alpha$ where A is a well ordered set and the transition maps $R_\alpha \to R_{\alpha+1}$ are finite étale and $R_\alpha = \operatorname{colim}_{\beta < \alpha} R_\beta$ if α is not a successor. We will prove the result by transfinite induction.

Suppose the result holds for R_α, i.e., $R_\alpha = \operatorname{colim} R_i$ with R_i finite étale over R. Since $R_\alpha \to R_{\alpha+1}$ is finite étale there exists an i and a finite étale extension

$R_i \to R_{i,1}$ such that $R_{\alpha+1} = R_\alpha \otimes_{R_i} R_{i,1}$. Thus $R_{\alpha+1} = \mathrm{colim}_{i' \geq i} R_{i'} \otimes_{R_i} R_{i,1}$ and the result holds for $\alpha + 1$. Suppose α is not a successor and the result holds for R_β for all $\beta < \alpha$. Since every finite subset $E \subset R_\alpha$ is contained in R_β for some $\beta < \alpha$ and we see that E is contained in a finite étale subextension by assumption. Thus the result holds for R_α. □

Lemma 10.146.3. *Let R be a ring. Let $\mathfrak{p} \subset R$ be a prime and let $\kappa(\mathfrak{p}) \subset L$ be a finite extension of fields. Then there exists a finite free ring map $R \to S$ such that $\mathfrak{q} = \mathfrak{p}S$ is prime and $\kappa(\mathfrak{p}) \subset \kappa(\mathfrak{q})$ is isomorphic to the given extension $\kappa(\mathfrak{p}) \subset L$.*

Proof. By induction of the degree of $\kappa(\mathfrak{p}) \subset L$. If the degree is 1, then we take $R = S$. In general, if there exists a sub extension $\kappa(\mathfrak{p}) \subset L' \subset L$ then we win by induction on the degree (by first constructing $R \subset S'$ corresponding to $L'/\kappa(\mathfrak{p})$ and then construction $S' \subset S$ corresponding to L/L'). Thus we may assume that $L \supset \kappa(\mathfrak{p})$ is generated by a single element $\alpha \in L$. Let $X^d + \sum_{i<d} a_i X^i$ be the minimal polynomial of α over $\kappa(\mathfrak{p})$, so $a_i \in \kappa(\mathfrak{p})$. We may write a_i as the image of f_i/g for some $f_i, g \in R$ and $g \notin \mathfrak{p}$. After replacing α by $g\alpha$ (and correspondingly replacing a_i by $g^{d-i}a_i$) we may assume that a_i is the image of some $f_i \in R$. Then we simply take $S = R[x]/(x^d + \sum f_i x^i)$. □

10.147. The Cohen structure theorem

Here is a fundamental notion in commutative algebra.

Definition 10.147.1. Let $(R, \mathfrak{m})$ be a local ring. We say R is a *complete local ring* if the canonical map

$$R \longrightarrow \lim_n R/\mathfrak{m}^n$$

to the completion of R with respect to $\mathfrak{m}$ is an isomorphism[7].

Note that an Artinian local ring R is a complete local ring because $\mathfrak{m}_R^n = 0$ for some $n > 0$. In this section we mostly focus on Noetherian complete local rings.

Lemma 10.147.2. *Let R be a Noetherian complete local ring. Any quotient of R is also a Noetherian complete local ring. Given a finite ring map $R \to S$, then S is a product of Noetherian complete local rings.*

Proof. The ring S is Noetherian by Lemma 10.29.1. As an R-module S is complete by Lemma 10.91.2. Hence S is the product of the completions at its maximal ideals by Lemma 10.91.19. □

Lemma 10.147.3. *Let $(R, \mathfrak{m})$ be a complete local ring. If $\mathfrak{m}$ is a finitely generated ideal then R is Noetherian.*

Proof. See Lemma 10.91.9. □

Definition 10.147.4. Let $(R, \mathfrak{m})$ be a complete local ring. A subring $\Lambda \subset R$ is called a *coefficient ring* if the following conditions hold:

(1) Λ is a complete local ring with maximal ideal $\Lambda \cap \mathfrak{m}$,
(2) the residue field of Λ maps isomorphically to the residue field of R, and
(3) $\Lambda \cap \mathfrak{m} = p\Lambda$, where p is the characteristic of the residue field of R.

[7]This includes the condition that $\bigcap \mathfrak{m}^n = (0)$; in some texts this may be indicated by saying that R is complete and separated. Warning: It can happen that the completion $\lim_n R/\mathfrak{m}^n$ of a local ring is non-complete, see Examples, Lemma 78.4.1. This does not happen when $\mathfrak{m}$ is finitely generated, see Lemma 10.91.7 in which case the completion is Noetherian, see Lemma 10.91.9.

Let us make some remarks on this definition. We split the discussion into the following cases:

(1) The local ring R contains a field. This happens if either $\mathbf{Q} \subset R$, or $pR = 0$ where p is the characteristic of $R/\mathfrak{m}$. In this case a coefficient ring Λ is a field contained in R which maps isomorphically to $R/\mathfrak{m}$.
(2) The characteristic of $R/\mathfrak{m}$ is $p > 0$ but no power of p is zero in R. In this case Λ is a complete discrete valuation ring with uniformizer p and residue field $R/\mathfrak{m}$.
(3) The characteristic of $R/\mathfrak{m}$ is $p > 0$, and for some $n > 1$ we have $p^{n-1} \neq 0$, $p^n = 0$ in R. In this case Λ is an Artinian local ring whose maximal ideal is generated by p and which has residue field $R/\mathfrak{m}$.

The complete discrete valuation rings with uniformizer p above play a special role and we baptize them as follows.

Definition 10.147.5. A *Cohen ring* is a complete discrete valuation ring with uniformizer p a prime number.

Lemma 10.147.6. *Let p be a prime number. Let k be a field of characteristic p. There exists a Cohen ring Λ with $\Lambda/p\Lambda \cong k$.*

Proof. First note that the p-adic integers $\mathbf{Z}_p$ form a Cohen ring for $\mathbf{F}_p$. Let k be an arbitrary field of characteristic p. Let $\mathbf{Z}_p \to R$ be a flat local ring map such that $\mathfrak{m}_R = pR$ and $R/pR = k$, see Lemma 10.146.1. Then clearly R is a discrete valuation ring. Hence its completion is a Cohen ring for k. □

Lemma 10.147.7. *Let $p > 0$ be a prime. Let Λ be a Cohen ring with residue field of characteristic p. For every $n \geq 1$ the ring map*

$$\mathbf{Z}/p^n\mathbf{Z} \to \Lambda/p^n\Lambda$$

is formally smooth.

Proof. If $n = 1$, this follows from Proposition 10.145.9. For general n we argue by induction on n. Namely, if $\mathbf{Z}/p^n\mathbf{Z} \to \Lambda/p^n\Lambda$ is formally smooth, then we can apply Lemma 10.131.12 to the ring map $\mathbf{Z}/p^{n+1}\mathbf{Z} \to \Lambda/p^{n+1}\Lambda$ and the ideal $I = (p^n) \subset \mathbf{Z}/p^{n+1}\mathbf{Z}$. □

Theorem 10.147.8 (Cohen structure theorem)**.** *Let $(R, \mathfrak{m})$ be a complete local ring.*

(1) *R has a coefficient ring (see Definition 10.147.4),*
(2) *if $\mathfrak{m}$ is a finitely generated ideal, then R is isomorphic to a quotient*

$$\Lambda[[x_1, \ldots, x_n]]/I$$

where Λ is either a field or a Cohen ring.

Proof. Let us prove a coefficient ring exists. First we prove this in case the characteristic of the residue field κ is zero. Namely, in this case we will prove by induction on $n > 0$ that there exists a section

$$\varphi_n : \kappa \longrightarrow R/\mathfrak{m}^n$$

to the canonical map $R/\mathfrak{m}^n \to \kappa = R/\mathfrak{m}$. This is trivial for $n = 1$. If $n > 1$, let φ_{n-1} be given. The field extension $\mathbf{Q} \subset \kappa$ is formally smooth by Proposition

10.145.9. Hence we can find the dotted arrow in the following diagram

$$\begin{array}{ccc} R/\mathfrak{m}^{n-1} & \longleftarrow & R/\mathfrak{m}^n \\ {\scriptstyle \varphi_{n-1}}\uparrow & \nearrow & \uparrow \\ \kappa & \longleftarrow & \mathbf{Q} \end{array}$$

This proves the induction step. Putting these maps together

$$\lim_n \ \varphi_n : \kappa \longrightarrow R = \lim_n \ R/\mathfrak{m}^n$$

gives a map whose image is the desired coefficient ring.

Next, we prove the existence of a coefficient ring in the case where the characteristic of the residue field κ is $p > 0$. Namely, choose a Cohen ring Λ with $\kappa = \Lambda/p\Lambda$, see Lemma 10.147.6. In this case we will prove by induction on $n > 0$ that there exists a map

$$\varphi_n : \Lambda/p^n\Lambda \longrightarrow R/\mathfrak{m}^n$$

whose composition with the reduction map $R/\mathfrak{m}^n \to \kappa$ produces the given isomorphism $\Lambda/p\Lambda = \kappa$. This is trivial for $n = 1$. If $n > 1$, let φ_{n-1} be given. The ring map $\mathbf{Z}/p^n\mathbf{Z} \to \Lambda/p^n\Lambda$ is formally smooth by Lemma 10.147.7. Hence we can find the dotted arrow in the following diagram

$$\begin{array}{ccc} R/\mathfrak{m}^{n-1} & \longleftarrow & R/\mathfrak{m}^n \\ {\scriptstyle \varphi_{n-1}}\uparrow & \nearrow & \uparrow \\ \Lambda/p^n\Lambda & \longleftarrow & \mathbf{Z}/p^n\mathbf{Z} \end{array}$$

This proves the induction step. Putting these maps together

$$\lim_n \ \varphi_n : \Lambda = \lim_n \ \Lambda/p^n\Lambda \longrightarrow R = \lim_n \ R/\mathfrak{m}^n$$

gives a map whose image is the desired coefficient ring.

The final statement of the theorem is now clear. Namely, if $y_1, \ldots, y_n$ are generators of the ideal $\mathfrak{m}$, then we can use the map $\Lambda \to R$ just constructed to get a map

$$\Lambda[[x_1, \ldots, x_n]] \longrightarrow R, \quad x_i \longmapsto y_i.$$

This map is surjective on each $R/\mathfrak{m}^n$ and hence is surjective as R is complete. Some details omitted. □

Remark 10.147.9. If k is a field then the power series ring $k[[X_1, \ldots, X_d]]$ is a Noetherian complete local regular ring of dimension d. If Λ is a Cohen ring then $\Lambda[[X_1, \ldots, X_d]]$ is a complete local Noetherian regular ring of dimension $d+1$. Hence the Cohen structure theorem implies that any Noetherian complete local ring is a quotient of a regular local ring. In particular we see that a Noetherian complete local ring is universally catenary, see Lemma 10.100.6 and Lemma 10.101.3.

Lemma 10.147.10. *Let $(R, \mathfrak{m})$ be a Noetherian complete local domain. Then there exists a $R_0 \subset R$ with the following properties*

1. *R_0 is a regular complete local ring,*
2. *$R_0 \subset R$ is finite and induces an isomorphism on residue fields,*
3. *R_0 is either isomorphic to $k[[X_1, \ldots, X_d]]$ where k is a field or $\Lambda[[X_1, \ldots, X_d]]$ where Λ is a Cohen ring.*

Proof. Let Λ be a coefficient ring of R. Since R is a domain we see that either Λ is a field or Λ is a Cohen ring.

Case I: $\Lambda = k$ is a field. Let $d = \dim(R)$. Choose $x_1, \ldots, x_d \in \mathfrak{m}$ which generate an ideal of definition $I \subset R$. (See Section 10.57.) By Lemma 10.91.14 we see that R is I-adically complete as well. Consider the map $R_0 = k[[X_1, \ldots, X_d]] \to R$ which maps X_i to x_i. Note that R_0 is complete with respect to the ideal $I_0 = (X_1, \ldots, X_d)$, and that $R/I_0R \cong R/IR$ is finite over $k = R_0/I_0$ (because $\dim(R/I) = 0$, see Section 10.57.) Hence we conclude that $R_0 \to R$ is finite by Lemma 10.91.17. Since $\dim(R) = \dim(R_0)$ this implies that $R_0 \to R$ is injective (see Lemma 10.106.3), and the lemma is proved.

Case II: Λ is a Cohen ring. Let $d + 1 = \dim(R)$. Let $p > 0$ be the characteristic of the residue field k. As R is a domain we see that p is a nonzerodivisor in R. Hence $\dim(R/pR) = d$, see Lemma 10.57.11. Choose $x_1, \ldots, x_d \in R$ which generate an ideal of definition in R/pR. Then $I = (p, x_1, \ldots, x_d)$ is an ideal of definition of R. By Lemma 10.91.14 we see that R is I-adically complete as well. Consider the map $R_0 = \Lambda[[X_1, \ldots, X_d]] \to R$ which maps X_i to x_i. Note that R_0 is complete with respect to the ideal $I_0 = (p, X_1, \ldots, X_d)$, and that $R/I_0R \cong R/IR$ is finite over $k = R_0/I_0$ (because $\dim(R/I) = 0$, see Section 10.57.) Hence we conclude that $R_0 \to R$ is finite by Lemma 10.91.17. Since $\dim(R) = \dim(R_0)$ this implies that $R_0 \to R$ is injective (see Lemma 10.106.3), and the lemma is proved. □

10.148. Nagata and Japanese rings

In this section we discuss finiteness of integral closure. It turns out that this is closely related to the relationship between a local ring and its completion.

Definition 10.148.1. Let R be a a domain with field of fractions K.

(1) We say R is *N-1* if the integral closure of R in K is a finite R-module.
(2) We say R is *N-2*, or *Japanese* if for any finite extension $K \subset L$ of fields the integral closure of R in L is finite over R.

The main interest in these notions is for Noetherian rings, but here is a non-Noetherian example.

Example 10.148.2. Let k be a field. The domain $R = k[x_1, x_2, x_3, \ldots]$ is Japanese, but not Noetherian. The reason is the following. Suppose that $R \subset L$ and the field L is a finite extension of the fraction field of R. Then there exists an integer n such that L comes from a finite extension $k(x_1, \ldots, x_n) \subset L_0$ by adjoining the (transcendental) elements x_{n+1}, x_{n+2}, etc. Let S_0 be the integral closure of $k[x_1, \ldots, x_n]$ in L_0. By Proposition 10.148.31 below it is true that S_0 is finite over $k[x_1, \ldots, x_n]$. Moreover, the integral closure of R in L is $S = S_0[x_{n+1}, x_{n+2}, \ldots]$ (use Lemma 10.34.8) and hence finite over R. The same argument works for $R = \mathbf{Z}[x_1, x_2, x_3, \ldots]$.

Lemma 10.148.3. *Let R be a domain. If R is N-1 then so is any localization of R. Same for N-2.*

Proof. These statements hold because taking integral closure commutes with localization, see Lemma 10.33.9. □

Lemma 10.148.4. *Let R be a domain. Let $f_1, \ldots, f_n \in R$ generate the unit ideal. If each domain R_{f_i} is N-1 then so is R. Same for N-2.*

Proof. Assume R_{f_i} is N-2 (or N-1). Let L be a finite extension of the fraction field of R (equal to the fraction field in the N-1 case). Let S be the integral closure of R in L. By Lemma 10.33.9 we see that S_{f_i} is the integral closure of R_{f_i} in L. Hence S_{f_i} is finite over R_{f_i} by assumption. Thus S is finite over R by Lemma 10.22.2. □

Lemma 10.148.5. *Let R be a domain. Let $R \subset S$ be a quasi-finite extension of domains (for example finite). Assume R is N-2 and Noetherian. Then S is N-2.*

Proof. Let $K = f.f.(R) \subset L = f.f.(S)$. Note that this is a finite field extension (for example by Lemma 10.116.2 (2) applied to the fibre $S \otimes_R K$, and the definition of a quasi-finite ring map). Let S' be the integral closure of R in S. Then S' is contained in the integral closure of R in L which is finite over R by assumption. As R is Noetherian this implies S' is finite over R. By Lemma 10.117.15 there exist elements $g_1, \ldots, g_n \in S'$ such that $S'_{g_i} \cong S_{g_i}$ and such that $g_1, \ldots, g_n$ generate the unit ideal in S. Hence it suffices to show that S' is N-2 by Lemmas 10.148.3 and 10.148.4. Thus we have reduced to the case where S is finite over R.

Assume $R \subset S$ with hypotheses as in the lemma and moreover that S is finite over R. Let M be a finite field extension of the fraction field of S. Then M is also a finite field extension of $f.f(R)$ and we conclude that the integral closure T of R in M is finite over R. By Lemma 10.33.14 we see that T is also the integral closure of S in M and we win by Lemma 10.33.13. □

Lemma 10.148.6. *Let R be a Noetherian domain. If $R[z, z^{-1}]$ is N-1, then so is R.*

Proof. Let R' be the integral closure of R in its field of fractions K. Let S' be the integral closure of $R[z, z^{-1}]$ in its field of fractions. Clearly $R' \subset S'$. Since $K[z, z^{-1}]$ is a normal domain we see that $S' \subset K[z, z^{-1}]$. Suppose that $f_1, \ldots, f_n \in S'$ generate S' as $R[z, z^{-1}]$-module. Say $f_i = \sum a_{ij} z^j$ (finite sum), with $a_{ij} \in K$. For any $x \in R'$ we can write

$$x = \sum h_i f_i$$

with $h_i \in R[z, z^{-1}]$. Thus we see that R' is contained in the finite R-submodule $\sum Ra_{ij} \subset K$. Since R is Noetherian we conclude that R' is a finite R-module. □

Lemma 10.148.7. *Let R be a Noetherian domain, and let $R \subset S$ be a finite extension of domains. If S is N-1, then so is R. If S is N-2, then so is R.*

Proof. Omitted. (Hint: Integral closures of R in extension fields are contained in integral closures of S in extension fields.) □

Lemma 10.148.8. *Let R be a Noetherian normal domain with fraction field K. Let $K \subset L$ be a finite separable field extension. Then the integral closure of R in L is finite over R.*

Proof. Consider the trace pairing

$$L \times L \longrightarrow K, \quad (x, y) \longmapsto \langle x, y \rangle := \mathrm{Tr}_{L/K}(xy).$$

Since L/K is separable this is nondegenerate (exercise in Galois theory). Moreover, if $x \in L$ is integral over R, then $\mathrm{Tr}_{L/K}(x)$ is integral over R also, and since R is normal we see $\mathrm{Tr}_{L/K}(x) \in R$. Pick $x_1, \ldots, x_n \in L$ which are integral over R and

which form a K-basis of L. Then the integral closure $S \subset L$ is contained in the R-module

$$M = \{y \in L \mid \langle x_i, y \rangle \in R,\ i = 1, \ldots, n\}$$

By linear algebra we see that $M \cong R^{\oplus n}$ as an R-module. Hence $S \subset R^{\oplus n}$ is a finitely generated R-module as R is Noetherian. □

Example 10.148.9. Lemma 10.148.8 does not work if the ring is not Noetherian. For example consider the action of $G = \{+1, -1\}$ on $A = \mathbf{C}[x_1, x_2, x_3, \ldots]$ where -1 acts by mapping x_i to $-x_i$. The invariant ring $R = A^G$ is the $\mathbf{C}$-algebra generated by all $x_i x_j$. Hence $R \subset A$ is not finite. But R is a normal domain with fraction field $K = L^G$ the G-invariants in the fraction field L of A. And clearly A is the integral closure of R in L.

Lemma 10.148.10. *A Noetherian domain of characteristic zero is N-1 if and only if it is N-2 (i.e., Japanese).*

Proof. This is clear from Lemma 10.148.8 since every field extension in characteristic zero is separable. □

Lemma 10.148.11. *Let R be a Noetherian domain with fraction field K of characteristic $p > 0$. Then R is Japanese if and only if for every finite purely inseparable extension $K \subset L$ the integral closure of R in L is finite over R.*

Proof. Assume the integral closure of R in every finite purely inseparable field extension of K is finite. Let $K \subset L$ be any finite extension. We have to show the integral closure of R in L is finite over R. Choose a finite normal field extension $K \subset M$ containing L. As R is Noetherian it suffices to show that the integral closure of R in M is finite over R. By Fields, Lemma 9.24.3 there exists a subextension $K \subset M_{insep} \subset M$ such that M_{insep}/K is purely inseparable, and M/M_{insep} is separable. By assumption the integral closure R' of R in M_{insep} is finite over R. By Lemma 10.148.8 the integral closure R'' of R' in M is finite over R'. Then R'' is finite over R by Lemma 10.7.3. Since R'' is also the integral closure of R in M (see Lemma 10.33.14) we win. □

Lemma 10.148.12. *Let R be a Noetherian domain. If R is N-1 then $R[x]$ is N-1. If R is N-2 then $R[x]$ is N-2.*

Proof. Assume R is N-1. Let R' be the integral closure of R which is finite over R. Hence also $R'[x]$ is finite over $R[x]$. The ring $R'[x]$ is normal (see Lemma 10.34.8), hence N-1. This proves the first assertion.

For the second assertion, by Lemma 10.148.7 it suffices to show that $R'[x]$ is N-2. In other words we may and do assume that R is a normal N-2 domain. In characteristic zero we are done by Lemma 10.148.10. In characteristic $p > 0$ we have to show that the integral closure of $R[x]$ is finite in any finite purely inseparable extension of $f.f.(R[x]) = K(x) \subset L$ with $K = f.f.(R)$. Clearly there exists a finite purely inseparable field extension $K \subset L'$ and $q = p^e$ such that $L \subset L'(x^{1/q})$. As $R[x]$ is Noetherian it suffices to show that the integral closure of $R[x]$ in $L'(x^{1/q})$ is finite over $R[x]$. And this integral closure is equal to $R'[x^{1/q}]$ with $R \subset R' \subset L'$ the integral closure of R in L'. Since R is N-2 we see that R' is finite over R and hence $R'[x^{1/q}]$ is finite over $R[x]$. □

Lemma 10.148.13 (Tate)**.** *Let R be a ring. Let $x \in R$. Assume*

(1) *R is a normal Noetherian domain,*
(2) *R/xR is a Japanese domain,*
(3) *$R \cong \lim_n R/x^nR$ is complete with respect to x.*

Then R is Japanese.

Proof. We may assume $x \neq 0$ since otherwise the lemma is trivial. Let K be the fraction field of R. If the characteristic of K is zero the lemma follows from (1), see Lemma 10.148.10. Hence we may assume that the characteristic of K is $p > 0$, and we may apply Lemma 10.148.11. Thus given $K \subset L$ be a finite purely inseparable field extension we have to show that the integral closure S of R in L is finite over R.

Let q be a power of p such that $L^q \subset K$. By enlarging L if necessary we may assume there exists an element $y \in L$ such that $y^q = x$. Since $R \to S$ induces a homeomorphism of spectra (see Lemma 10.43.2) there is a unique prime ideal $\mathfrak{q} \subset S$ lying over the prime ideal $\mathfrak{p} = xR$. It is clear that

$$\mathfrak{q} = \{f \in S \mid f^q \in \mathfrak{p}\} = yS$$

since $y^q = x$. Hence $R_\mathfrak{p}$ and $S_\mathfrak{q}$ are discrete valuation rings, see Lemma 10.113.6. By Lemma 10.113.9 we see that $\kappa(\mathfrak{p}) \subset \kappa(\mathfrak{q})$ is a finite field extension. Hence the integral closure $S' \subset \kappa(\mathfrak{q})$ of R/xR is finite over R/xR by assumption (2). Since $S/yS \subset S'$ this implies that S/yS is finite over R. Note that S/y^nS has a finite filtration whose subquotients are the modules $y^iS/y^{i+1}S \cong S/yS$. Hence we see that each S/y^nS is finite over R. In particular S/xS is finite over R. Also, it is clear that $\bigcap x^nS = (0)$ since an element in the intersection has qth power contained in $\bigcap x^nR = (0)$ (Lemma 10.47.6). Thus we may apply Lemma 10.91.17 to conclude that S is finite over R, and we win. □

Lemma 10.148.14. *Let R be a ring. If R is Noetherian, a domain, and N-2, then so is $R[[x]]$.*

Proof. Apply Lemma 10.148.13 to the element $x \in R[[x]]$. □

Definition 10.148.15. Let R be a ring.

(1) We say R is *universally Japanese* if for any finite type ring map $R \to S$ with S a domain we have that S is Japanese (i.e., N-2).
(2) We say that R is a *Nagata ring* if R is Noetherian and for every prime ideal $\mathfrak{p}$ the ring $R/\mathfrak{p}$ is Japanese.

It is clear that a Noetherian universally Japanese ring is a Nagata ring. It is our goal to show that a Nagata ring is universally Japanese. This is not obvious at all, and requires some work. But first, here is a useful lemma.

Lemma 10.148.16. *Let R be a Nagata ring. Let $R \to S$ be essentially of finite type with S reduced. Then the integral closure of R in S is finite over R.*

Proof. As S is essentially of finite type over R it is Noetherian and has finitely many minimal primes $\mathfrak{q}_1, \ldots, \mathfrak{q}_m$, see Lemma 10.29.6. Since S is reduced we have $S \subset \prod S_{\mathfrak{q}_i}$ and each $S_{\mathfrak{q}_i} = K_i$ is a field, see Lemmas 10.23.2 and 10.24.4. It suffices to show that the integral closure A_i' of R in each K_i is finite over R. This is true because R is Noetherian and $A \subset \prod A_i'$. Let $\mathfrak{p}_i \subset R$ be the prime of R corresponding to $\mathfrak{q}_i$. As S is essentially of finite type over R we see that $K_i = S_{\mathfrak{q}_i} = \kappa(\mathfrak{q}_i)$ is a finitely generated field extension of $\kappa(\mathfrak{p}_i)$. Hence the algebraic closure L_i of $\kappa(\mathfrak{p}_i)$

in $\subset K_i$ is finite over $\kappa(\mathfrak{p}_i)$, see Fields, Lemma 9.23.10. It is clear that A_i' is the integral closure of $R/\mathfrak{p}_i$ in L_i, and hence we win by definition of a Nagata ring. $\square$

Lemma 10.148.17. *Let R be a ring. To check that R is universally Japanese it suffices to show: If $R \to S$ is of finite type, and S a domain then S is N-1.*

Proof. Namely, assume the condition of the lemma. Let $R \to S$ be a finite type ring map with S a domain. Let $f.f.(S) \subset L$ be a finite extension of its fraction field. Then there exists a finite ring extension $S \subset S' \subset L$ with $f.f.(S') = L$. By assumption S' is N-1, and hence the integral closure S'' of S' in L is finite over S'. Thus S'' is finite over S (Lemma 10.7.3) and S'' is the integral closure of S in L (Lemma 10.33.14). We conclude that R is universally Japanese. $\square$

Lemma 10.148.18. *If R is universally Japanese then any algebra essentially of finite type over R is universally Japanese.*

Proof. The case of an algebra of finite type over R is immediate from the definition. The general case follows on applying Lemma 10.148.3. $\square$

Lemma 10.148.19. *Let R be a Nagata ring. If $R \to S$ is a quasi-finite ring map (for example finite) then S is a Nagata ring also.*

Proof. First note that S is Noetherian as R is Noetherian and a quasi-finite ring map is of finite type. Let $\mathfrak{q} \subset S$ be a prime ideal, and set $\mathfrak{p} = R \cap \mathfrak{q}$. Then $R/\mathfrak{p} \subset S/\mathfrak{q}$ is quasi-finite and hence we conclude that $S/\mathfrak{q}$ is N-2 by Lemma 10.148.5 as desired. $\square$

Lemma 10.148.20. *A localization of a Nagata ring is a Nagata ring.*

Proof. Clear from Lemma 10.148.3. $\square$

Lemma 10.148.21. *Let R be a ring. Let $f_1, \ldots, f_n \in R$ generate the unit ideal.*

(1) *If each R_{f_i} is universally Japanese then so is R.*
(2) *If each R_{f_i} is Nagata then so is R.*

Proof. Let $\varphi : R \to S$ be a finite type ring map so that S is a domain. Then $\varphi(f_1), \ldots, \varphi(f_n)$ generate the unit ideal in S. Hence if each $S_{f_i} = S_{\varphi(f_i)}$ is N-1 then so is S, see Lemma 10.148.4. This proves (1).

If each R_{f_i} is Nagata, then each R_{f_i} is Noetherian and hence R is Noetherian, see Lemma 10.22.2. And if $\mathfrak{p} \subset R$ is a prime, then we see each $R_{f_i}/\mathfrak{p}R_{f_i} = (R/\mathfrak{p})_{f_i}$ is Japanese and hence we conclude $R/\mathfrak{p}$ is Japanese by Lemma 10.148.4. This proves (2). $\square$

Lemma 10.148.22. *A Noetherian complete local ring is a Nagata ring.*

Proof. Let R be a complete local Noetherian ring. Let $\mathfrak{p} \subset R$ be a prime. Then $R/\mathfrak{p}$ is also a complete local Noetherian ring, see Lemma 10.147.2. Hence it suffices to show that a Noetherian complete local domain R is N-2. By Lemmas 10.148.5 and 10.147.10 we reduce to the case $R = k[[X_1, \ldots, X_d]]$ where k is a field or $R = \Lambda[[X_1, \ldots, X_d]]$ where Λ is a Cohen ring.

In the case $k[[X_1, \ldots, X_d]]$ we reduce to the statement that a field is N-2 by Lemma 10.148.14. This is clear. In the case $\Lambda[[X_1, \ldots, X_d]]$ we reduce to the statement that a Cohen ring Λ is N-2. Applying Lemma 10.148.13 once more with $x = p \in \Lambda$ we reduce yet again to the case of a field. Thus we win. $\square$

Definition 10.148.23. Let $(R, \mathfrak{m})$ be a Noetherian local ring. We say R is *analytically unramified* if its completion $R^\wedge = \lim_n R/\mathfrak{m}^n$ is reduced. A prime ideal $\mathfrak{p} \subset R$ is said to be *analytically unramified* if $R/\mathfrak{p}$ is analytically unramified.

At this point we know the following are true for any Noetherian local ring R: The map $R \to R^\wedge$ is a faithfully flat local ring homomorphism (Lemma 10.91.4). The completion $R^\wedge$ is Noetherian (Lemma 10.91.9) and complete (Lemma 10.91.8). Hence the completion $R^\wedge$ is a Nagata ring (Lemma 10.148.22). Moreover, we have seen in Section 10.147 that $R^\wedge$ is a quotient of a regular local ring (Theorem 10.147.8), and hence universally catenary (Remark 10.147.9).

Lemma 10.148.24. *Let $(R, \mathfrak{m})$ be a Noetherian local ring.*

(1) *If R is analytically unramified, then R is reduced.*
(2) *If R is analytically unramified, then each minimal prime of R is analytically unramified.*
(3) *If R is reduced with minimal primes $\mathfrak{q}_1, \ldots, \mathfrak{q}_t$, and each $\mathfrak{q}_i$ is analytically unramified, then R is analytically unramified.*
(4) *If R is analytically unramified, then the integral closure of R in its total ring of fractions $Q(R)$ is finite over R.*
(5) *If R is a domain and analytically unramified, then R is N-1.*

Proof. In this proof we will use the remarks immediately following Definition 10.148.23. As $R \to R^\wedge$ is a faithfully flat local ring homomorphism it is injective and (1) follows.

Let $\mathfrak{q}$ be a minimal prime of R, and assume R is analytically unramified. Then $\mathfrak{q}$ is an associated prime of R (see Proposition 10.60.6). Hence there exists an $f \in R$ such that $\{x \in R \mid fx = 0\} = \mathfrak{q}$. Note that $(R/\mathfrak{q})^\wedge = R^\wedge/\mathfrak{q}^\wedge$, and that $\{x \in R^\wedge \mid fx = 0\} = \mathfrak{q}^\wedge$, because completion is exact (Lemma 10.91.3). If $x \in R^\wedge$ is such that $x^2 \in \mathfrak{q}^\wedge$, then $fx^2 = 0$ hence $(fx)^2 = 0$ hence $fx = 0$ hence $x \in \mathfrak{q}^\wedge$. Thus $\mathfrak{q}$ is analytically unramified and (2) holds.

Assume R is reduced with minimal primes $\mathfrak{q}_1, \ldots, \mathfrak{q}_t$, and each $\mathfrak{q}_i$ is analytically unramified. Then $R \to R/\mathfrak{q}_1 \times \ldots \times R/\mathfrak{q}_t$ is injective. Since completion is exact (see Lemma 10.91.3) we see that $R^\wedge \subset (R/\mathfrak{q}_1)^\wedge \times \ldots \times (R/\mathfrak{q}_t)^\wedge$. Hence (3) is clear.

Assume R is analytically unramified. Let $\mathfrak{p}_1, \ldots, \mathfrak{p}_s$ be the minimal primes of $R^\wedge$. Then we see that

$$Q(R^\wedge) = R_{\mathfrak{p}_1} \times \ldots \times R_{\mathfrak{p}_s}$$

with each $R_{\mathfrak{p}_i}$ a field as $R^\wedge$ is reduced (see Lemma 10.23.2). Hence the integral closure S of $R^\wedge$ in $Q(R^\wedge)$ is equal to $S = S_1 \times \ldots \times S_s$ with S_i the integral closure of $R/\mathfrak{p}_i$ in its fraction field. In particular S is finite over $R^\wedge$. Denote R' the integral closure of R in $Q(R)$. As $R \to R^\wedge$ is flat we see that $R' \otimes_R R^\wedge \subset Q(R) \otimes_R R^\wedge \subset Q(R^\wedge)$. Moreover $R' \otimes_R R^\wedge$ is integral over $R^\wedge$ (Lemma 10.33.11). Hence $R' \otimes_R R^\wedge \subset S$ is a $R^\wedge$-submodule. As $R^\wedge$ is Noetherian it is a finite $R^\wedge$-module. Thus we may find $f_1, \ldots, f_n \in R'$ such that $R' \otimes_R R^\wedge$ is generated by the elements $f_i \otimes 1$ as a $R^\wedge$-module. By faithful flatness we see that R' is generated by $f_1, \ldots, f_n$ as an R-module. This proves (4).

Part (5) is a special case of part (4). □

Lemma 10.148.25. *Let R be a Noetherian local ring. Let $\mathfrak{p} \subset R$ be a prime. Assume*

(1) $R_{\mathfrak{p}}$ *is a discrete valuation ring, and*
(2) $\mathfrak{p}$ *is analytically unramified.*

Then for any associated prime $\mathfrak{q}$ *of* $R^\wedge/\mathfrak{p}R^\wedge$ *the local ring* $(R^\wedge)_{\mathfrak{q}}$ *is a discrete valuation ring.*

Proof. Assumption (2) says that $R^\wedge/\mathfrak{p}R^\wedge$ is a reduced ring. Hence an associated prime $\mathfrak{q} \subset R^\wedge$ of $R^\wedge/\mathfrak{p}R^\wedge$ is the same thing as a minimal prime over $\mathfrak{p}R^\wedge$. In particular we see that the maximal ideal of $(R^\wedge)_{\mathfrak{q}}$ is $\mathfrak{p}(R^\wedge)_{\mathfrak{q}}$. Choose $x \in R$ such that $xR_{\mathfrak{p}} = \mathfrak{p}R_{\mathfrak{p}}$. By the above we see that $x \in (R^\wedge)_{\mathfrak{q}}$ generates the maximal ideal. As $R \to R^\wedge$ is faithfully flat we see that x is a nonzerodivisor in $(R^\wedge)_{\mathfrak{q}}$. Hence we win. □

Lemma 10.148.26. *Let* $(R, \mathfrak{m})$ *be a Noetherian local domain. Let* $x \in \mathfrak{m}$. *Assume*

(1) $x \neq 0$,
(2) R/xR *has no embedded primes, and*
(3) *for each associated prime* $\mathfrak{p} \subset R$ *of* R/xR *we have*
   (a) *the local ring* $R_{\mathfrak{p}}$ *is regular, and*
   (b) $\mathfrak{p}$ *is analytically unramified.*

Then R *is analytically unramified.*

Proof. Let $\mathfrak{p}_1, \ldots, \mathfrak{p}_t$ be the associated primes of the R-module R/xR. Since R/xR has no embedded primes we see that each $\mathfrak{p}_i$ has height 1, and is a minimal prime over (x). For each i, let $\mathfrak{q}_{i1}, \ldots, \mathfrak{q}_{is_i}$ be the associated primes of the $R^\wedge$-module $R^\wedge/\mathfrak{p}_iR^\wedge$. By Lemma 10.148.25. we see that $(R^\wedge)_{\mathfrak{q}_{ij}}$ is regular. By Lemma 10.62.3 we see that

$$\mathrm{Ass}_{R^\wedge}(R^\wedge/xR^\wedge) = \bigcup\nolimits_{\mathfrak{p}\in \mathrm{Ass}_R(R/xR)} \mathrm{Ass}_{R^\wedge}(R^\wedge/\mathfrak{p}R^\wedge) = \{\mathfrak{q}_{ij}\}.$$

Let $y \in R^\wedge$ with $y^2 = 0$. As $(R^\wedge)_{\mathfrak{q}_{ij}}$ is regular, and hence a domain (Lemma 10.101.2) we see that y maps to zero in $(R^\wedge)_{\mathfrak{q}_{ij}}$. Hence y maps to zero in $R^\wedge/xR^\wedge$ by Lemma 10.60.18. Hence $y = xy'$. Since x is a nonzerodivisor (as $R \to R^\wedge$ is flat) we see that $(y')^2 = 0$. Hence we conclude that $y \in \bigcap x^nR^\wedge = (0)$ (Lemma 10.47.6). □

Lemma 10.148.27. *Let* $(R, \mathfrak{m})$ *be a local ring. If* R *is Noetherian, a domain, and Nagata, then* R *is analytically unramified.*

Proof. By induction on $\dim(R)$. The case $\dim(R) = 0$ is trivial. Hence we assume $\dim(R) = d$ and that the lemma holds for all Noetherian Nagata domains of dimension $< d$.

Let $R \subset S$ be the integral closure of R in the field of fractions of R. By assumption S is a finite R-module. By Lemma 10.148.19 we see that S is Nagata. By Lemma 10.106.4 we see $\dim(R) = \dim(S)$. Let $\mathfrak{m}_1, \ldots, \mathfrak{m}_t$ be the maximal ideals of S. Each of these lies over the maximal ideal $\mathfrak{m}$ of R. Moreover

$$(\mathfrak{m}_1 \cap \ldots \cap \mathfrak{m}_t)^n \subset \mathfrak{m}S$$

for sufficiently large n as $S/\mathfrak{m}S$ is Artinian. By Lemma 10.91.3 $R^\wedge \to S^\wedge$ is an injective map, and by the Chinese Remainder Lemma 10.14.4 combined with Lemma 10.91.14 we have $S^\wedge = \prod S_i^\wedge$ where $S_i^\wedge$ is the completion of S with respect to the maximal ideal $\mathfrak{m}_i$. Hence it suffices to show that $S_{\mathfrak{m}_i}$ is analytically unramified. In

other words, we have reduced to the case where R is a Noetherian normal Nagata domain.

Assume R is a Noetherian, normal, local Nagata domain. Pick a nonzero $x \in \mathfrak{m}$ in the maximal ideal. We are going to apply Lemma 10.148.26. We have to check properties (1), (2), (3)(a) and (3)(b). Property (1) is clear. We have that R/xR has no embedded primes by Lemma 10.144.6. Thus property (2) holds. The same lemma also tells us each associated prime $\mathfrak{p}$ of R/xR has height 1. Hence $R_\mathfrak{p}$ is a 1-dimensional normal domain hence regular (Lemma 10.113.6). Thus (3)(a) holds. Finally (3)(b) holds by induction hypothesis, since $R/\mathfrak{p}$ is Nagata (by Lemma 10.148.19 or directly from the definition). Thus we conclude R is analytically unramified. □

Lemma 10.148.28. *Let R be a Noetherian domain. If there exists an $f \in R$ such that R_f is normal then*

$$U = \{\mathfrak{p} \in \mathrm{Spec}(R) \mid R_\mathfrak{p} \text{ is normal}\}$$

is open in $\mathrm{Spec}(R)$.

Proof. It is clear that the standard open $D(f)$ is contained in U. By Serre's criterion Lemma 10.144.4 we see that $\mathfrak{p} \notin U$ implies that for some $\mathfrak{q} \subset \mathfrak{p}$ we have either

(1) Case I: $\mathrm{depth}(R_\mathfrak{q}) < 2$ and $\dim(R_\mathfrak{q}) \geq 2$, and
(2) Case II: $R_\mathfrak{q}$ is not regular and $\dim(R_\mathfrak{q}) = 1$.

This in particular also means that $R_\mathfrak{q}$ is not normal, and hence $f \in \mathfrak{q}$. In case I we see that $\mathrm{depth}(R_\mathfrak{q}) = \mathrm{depth}(R_\mathfrak{q}/fR_\mathfrak{q}) + 1$. Hence such a prime $\mathfrak{q}$ is the same thing as an embedded associated prime of R/fR. In case II $\mathfrak{q}$ is an associated prime of R/fR of height 1. Thus there is a finite set E of such primes $\mathfrak{q}$ (see Lemma 10.60.5) and

$$\mathrm{Spec}(R) \setminus U = \bigcup\nolimits_{\mathfrak{q} \in E} V(\mathfrak{q})$$

as desired. □

Lemma 10.148.29. *Let R be a Noetherian domain. Assume*

(1) *there exists a nonzero $f \in R$ such that R_f is normal, and*
(2) *for every maximal ideal $\mathfrak{m} \subset R$ the local ring $R_\mathfrak{m}$ is N-1.*

Then R is N-1.

Proof. Set $K = f.f.(R)$. Suppose that $R \subset R' \subset K$ is a finite extension of R contained in K. Note that $R_f = R'_f$ since R_f is already normal. Hence by Lemma 10.148.28 the set of primes $\mathfrak{p}' \in \mathrm{Spec}(R')$ with $R'_{\mathfrak{p}'}$ non-normal is closed in $\mathrm{Spec}(R')$. Since $\mathrm{Spec}(R') \to \mathrm{Spec}(R)$ is closed the image of this set is closed in $\mathrm{Spec}(R)$. For such a ring R' denote $Z_{R'} \subset \mathrm{Spec}(R)$ this image.

Pick a maximal ideal $\mathfrak{m} \subset R$. Let $R_\mathfrak{m} \subset R'_\mathfrak{m}$ be the integral closure of the local ring in K. By assumption this is a finite ring extension. By Lemma 10.33.9 we can find finitely many elements $r_1, \ldots, r_n \in K$ integral over R such that $R'_\mathfrak{m}$ is generated by $r_1, \ldots, r_n$ over $R_\mathfrak{m}$. Let $R' = R[x_1, \ldots, x_n] \subset K$. With this choice it is clear that $\mathfrak{m} \notin Z_{R'}$.

As $\mathrm{Spec}(R)$ is quasi-compact, the above shows that we can find a finite collection $R \subset R'_i \subset K$ such that $\bigcap Z_{R'_i} = \emptyset$. Let R' be the subring of K generated by all of

these. It is finite over R. Also $Z_{R'} = \emptyset$. Namely, every prime $\mathfrak{p}'$ lies over a prime $\mathfrak{p}'_i$ such that $(R'_i)_{\mathfrak{p}'_i}$ is normal. This implies that $R'_{\mathfrak{p}'} = (R'_i)_{\mathfrak{p}'_i}$ is normal too. Hence R' is normal, in other words R' is the integral closure of R in K. $\square$

The following proposition says in particular that an algebra of finite type over a Nagata ring is a Nagata ring.

Proposition 10.148.30 (Nagata)**.** *Let R be a ring. The following are equivalent:*

(1) *R is a Nagata ring,*
(2) *any finite type R-algebra is Nagata, and*
(3) *R is universally Japanese and Noetherian.*

Proof. It is clear that a Noetherian universally Japanese ring is universally Nagata (i.e., condition (2) holds). Let R be a Nagata ring. We will show that any finitely generated R-algebra S is Nagata. This will prove the proposition.

Step 1. There exists a sequence of ring maps $R = R_0 \to R_1 \to R_2 \to \ldots \to R_n = S$ such that each $R_i \to R_{i+1}$ is generated by a single element. Hence by induction it suffices to prove S is Nagata if $S \cong R[x]/I$.

Step 2. Let $\mathfrak{q} \subset S$ be a prime of S, and let $\mathfrak{p} \subset R$ be the corresponding prime of R. We have to show that $S/\mathfrak{q}$ is N-2. Hence we have reduced to the proving the following: (*) Given a Nagata domain R and a monogenic extension $R \subset S$ of domains then S is N-2.

Step 3. Let R be a Nagata domain and $R \subset S$ a monogenic extension of domains. Let $R \subset R'$ be the integral closure of R in its fraction field. Let S' be the subring of $f.f.(S)$ generated by R' and S. As R' is finite over R (by the Nagata property) also S' is finite over S. Since S is Noetherian it suffices to prove that S' is N-2 (Lemma 10.148.7). Hence we have reduced to proving the following: (**) Given a normal Nagata domain R and a monogenic extension $R \subset S$ of domains then S is N-2.

Step 4: Let R be a normal Nagata domain and let $R \subset S$ be a monogenic extension of domains. Suppose the extension of fraction fields $f.f.(R) \subset f.f.(S)$ is purely transcendental. In this case $S = R[x]$. By Lemma 10.148.12 we see that S is N-2. Hence we have reduced to proving the following: (**) Given a normal Nagata domain R and a monogenic extension $R \subset S$ of domains inducing a finite extension of fraction fields then S is N-2.

Step 5. Let R be a normal Nagata domain and let $R \subset S$ be a monogenic extension of domains inducing a finite extension of fraction fields $K = f.f.(R) \subset f.f.(S) = L$. Choose an element $x \in S$ which generates S as an R-algebra. Let $L \subset M$ be a finite extension of fields. Let R' be the integral closure of R in M. Then the integral closure S' of S in M is equal to the integral closure of $R'[x]$ in M. Also $f.f.(R') = M$, and $R \subset R'$ is finite (by the Nagata property of R). This implies that R' is a Nagata ring (Lemma 10.148.19). To show that S' is finite over S is the same as showing that S' is finite over $R'[x]$. Replace R by R' and S by S' to reduce to the following statement: (***) Given a normal Nagata domain R with fraction field K, and $x \in K$, the ring $S \subset K$ generated by R and x is N-1.

Step 6. Let R be a normal Nagata domain with fraction field K. Let $x = b/a \in K$. We have to show that the ring $S \subset K$ generated by R and x is N-1. Note that

$S_a \cong R_a$ is normal. Hence by Lemma 10.148.29 it suffices to show that $S_{\mathfrak{m}}$ is N-1 for every maximal ideal $\mathfrak{m}$ of S.

With assumptions as in the preceding paragraph, pick such a maximal ideal and set $\mathfrak{n} = R \cap \mathfrak{m}$. The residue field extension $\kappa(\mathfrak{n}) \subset \kappa(\mathfrak{m})$ is finite (Theorem 10.31.1) and generated by the image of x. Hence there exists a monic polynomial $f(X) = X^d + \sum_{i=1,\ldots,d} a_i X^{d-i}$ with $f(x) \in \mathfrak{m}$. Let $K \subset K''$ be a finite extension of fields such that $f(X)$ splits completely in $K''[X]$. Let R' be the integral closure of R in K''. Let $S' \subset K'$ be the subring generated by R' and x. As R is Nagata we see R' is finite over R and Nagata (Lemma 10.148.19). Moreover, S' is finite over S. If for every maximal ideal $\mathfrak{m}'$ of S' the local ring $S'_{\mathfrak{m}'}$ is N-1, then $S'_{\mathfrak{m}}$ is N-1 by Lemma 10.148.29, which in turn implies that $S_{\mathfrak{m}}$ is N-1 by Lemma 10.148.7. After replacing R by R' and S by S', and $\mathfrak{m}$ by any of the maximal ideals $\mathfrak{m}'$ lying over $\mathfrak{m}$ we reach the situation where the polynomial f above split completely: $f(X) = \prod_{i=1,\ldots,d}(X - a_i)$ with $a_i \in R$. Since $f(x) \in \mathfrak{m}$ we see that $x - a_i \in \mathfrak{m}$ for some i. Finally, after replacing x by $x - a_i$ we may assume that $x \in \mathfrak{m}$.

To recapitulate: R is a normal Nagata domain with fraction field K, $x \in K$ and S is the subring of K generated by x and R, finally $\mathfrak{m} \subset S$ is a maximal ideal with $x \in \mathfrak{m}$. We have to show $S_{\mathfrak{m}}$ is N-1.

We will show that Lemma 10.148.26 applies to the local ring $S_{\mathfrak{m}}$ and the element x. This will imply that $S_{\mathfrak{m}}$ is analytically unramified, whereupon we see that it is N-1 by Lemma 10.148.24.

We have to check properties (1), (2), (3)(a) and (3)(b). Property (1) is trivial. Let $I = \mathrm{Ker}(R[X] \to S)$ where $X \mapsto x$. We claim that I is generated by all linear forms $aX + b$ such that $ax = b$ in K. Clearly all these linear forms are in I. If $g = a_d X^d + \ldots a_1 X + a_0 \in I$, then we see that $a_d x$ is integral over R (Lemma 10.117.1) and hence $b := a_d x \in R$ as R is normal. Then $g - (a_d X - b)X^{d-1} \in I$ and we win by induction on the degree. As a consequence we see that

$$S/xS = R[X]/(X, I) = R/J$$

where

$$J = \{b \in R \mid ax = b \text{ for some } a \in R\} = xR \cap R$$

By Lemma 10.144.6 we see that $S/xS = R/J$ has no embedded primes as an R-module, hence as an R/J-module, hence as an S/xS-module, hence as an S-module. This proves property (2). Take such an associated prime $\mathfrak{q} \subset S$ with the property $\mathfrak{q} \subset \mathfrak{m}$ (so that it is an associated prime of $S_{\mathfrak{m}}/xS_{\mathfrak{m}}$ – it does not matter for the arguments). Then $\mathfrak{q}$ is minimal over xS and hence has height 1. By the sequence of equalities above we see that $\mathfrak{p} = R \cap \mathfrak{q}$ is an associated prime of R/J, and so has height 1 (see Lemma 10.144.6). Thus $R_{\mathfrak{p}}$ is a discrete valuation ring and therefore $R_{\mathfrak{p}} \subset S_{\mathfrak{q}}$ is an equality. This shows that $S_{\mathfrak{q}}$ is regular. This proves property (3)(a). Finally, $(S/\mathfrak{q})_{\mathfrak{m}}$ is a localization of $S/\mathfrak{q}$, which is a quotient of $S/xS = R/J$. Hence $(S/\mathfrak{q})_{\mathfrak{m}}$ is a localization of a quotient of the Nagata ring R, hence Nagata (Lemmas 10.148.19 and 10.148.20) and hence analytically unramified (Lemma 10.148.27). This shows (3)(b) holds and we are done. □

Proposition 10.148.31. *The following types of rings are Nagata and in particular universally Japanese:*

(1) *fields,*

(2) *Noetherian complete local rings,*
(3) $\mathbf{Z}$*,*
(4) *Dedekind domains with fraction field of characteristic zero,*
(5) *finite type ring extensions of any of the above.*

Proof. The Noetherian complete local ring case is Lemma 10.148.22. In the other cases you just check if $R/\mathfrak{p}$ is N-2 for every prime ideal $\mathfrak{p}$ of the ring. This is clear whenever $R/\mathfrak{p}$ is a field, i.e., $\mathfrak{p}$ is maximal. Hence for the Dedekind ring case we only need to check it when $\mathfrak{p} = (0)$. But since we assume the fraction field has characteristic zero Lemma 10.148.10 kicks in. □

Example 10.148.32. A discrete valuation ring is Nagata if and only if it is N-2 (this follows immediately from the definition). The discrete valuation ring A of Example 10.113.4 is not Nagata, i.e., it is not N-2. Namely, the finite extension $A \subset R = A[f]$ is not N-1. To see this say $f = \sum a_i x^i$. For every $n \geq 1$ set $g_n = \sum_{i<n} a_i x^i \in A$. Then $h_n = (f - g_n)/x^n$ is an element of the fraction field of R and $h_n^p \in k^p[[x]] \subset A$. Hence the integral closure R' of R contains $h_1, h_2, h_3, \ldots$. Now, if R' were finite over R and hence A, then $f = x^n h_n + g_n$ would be contained in the submodule $A + x^n R'$ for all n. By Artin-Rees this would imply $f \in A$ (Lemma 10.47.6), a contradiction.

Lemma 10.148.33. *Let $(A, \mathfrak{m})$ be a Noetherian local domain which is Nagata and has fraction field of characteristic p. If $a \in A$ has a pth root in $A^\wedge$, then a is has a pth root in A.*

Proof. Consider the ring extension $A \subset B = A[x]/(x^p - a)$. If a does not have a pth root in A, then B is a domain whose completion isn't a domain. This contradicts our earlier results, as B is a Nagata (Proposition 10.148.30) and hence analytically unramified by Lemma 10.148.27. □

10.149. Ascending properties

In this section we start proving some algebraic facts concerning the "ascent" of properties of rings.

Lemma 10.149.1. *Suppose that $R \to S$ is a flat and local ring homomorphism of Noetherian local rings. Then*

$$\mathit{depth}_S(S) = \mathit{depth}_R(R) + \mathit{depth}_S(S/\mathfrak{m}_R S).$$

Proof. Denote n the right hand side of the equality of the lemma. First assume that n is zero. Then $\text{depth}(R) = 0$ (resp. $\text{depth}(S/\mathfrak{m}_R S) = 0$) which means there is a $z \in R$ (resp. $\overline{y} \in S/\mathfrak{m}_R S$) whose annihilator is $\mathfrak{m}_R$ (resp. $\mathfrak{m}_S/\mathfrak{m}_R$). Let $y \in S$ be a lift of $\overline{y}$. It follows from the fact that $R \to S$ is flat that the annihilator of z in S is $\mathfrak{m}_R S$, and hence the annihilator of zy is $\mathfrak{m}_S$. Thus $\text{depth}_S(S) = 0$ as well.

Assume $n > 0$. If $\text{depth}(S/\mathfrak{m}_R S) > 0$, then choose an $f \in \mathfrak{m}_S$ which maps to a nonzerodivisor in $S/\mathfrak{m}_R S$. According to Lemma 10.93.2 the element $f \in S$ is a nonzerodivisor and S/fS is flat over R. Hence by induction on n we have

$$\text{depth}(S/fS) = \text{depth}(R) + \text{depth}(S/(f, \mathfrak{m}_R)).$$

Since f and $\overline{f}$ are nonzerodivisors we see that $\text{depth}(S) = \text{depth}(S/fS) + 1$ and $\text{depth}(S/\mathfrak{m}_R S) = \text{depth}(S/(f, \mathfrak{m}_R)) + 1$, see Lemma 10.67.11. Hence we see that the equality holds for $R \to S$ as well.

If $n > 0$, but $\text{depth}(S/\mathfrak{m}_R S) = 0$, then choose $f \in \mathfrak{m}_R$ a nonzerodivisor. As $R \to S$ is flat it is also the case that f is a nonzerodivisor on S. By induction on n again we have

$$\text{depth}(S/fS) = \text{depth}(R/fR) + \text{depth}(S/\mathfrak{m}_R).$$

By a similar argument as above we conclude that the equality holds for $R \to S$ as well. □

Here is a more general statement involving modules, see [**DG67**, IV, Proposition 6.3.1].

Lemma 10.149.2. *We have*

$$\textit{depth}_S(M \otimes_R N) = \textit{depth}_R(M) + \textit{depth}_{S/\mathfrak{m}_R S}(N/\mathfrak{m}_R N)$$

where $R \to S$ is a local homomorphism of local Noetherian rings, M is a finite R-module, and N is a finite S-module flat over R.

Proof. Omitted, but similar to the proof of Lemma 10.149.1. □

Lemma 10.149.3. *Let $R \to S$ be a local homomorphism of local Noetherian rings. Assume*

(1) *$S/\mathfrak{m}_R S$ is Cohen-Macaulay, and*
(2) *$R \to S$ is flat.*

Then S is Cohen-Macaulay if and only if R is Cohen-Macaulay.

Proof. This follows from the definitions combined with Lemmas 10.149.1 and 10.106.7. □

Lemma 10.149.4. *Let $\varphi : R \to S$ be a ring map. Assume*

(1) *R is Noetherian,*
(2) *S is Noetherian,*
(3) *φ is flat,*
(4) *the fibre rings $S \otimes_R \kappa(\mathfrak{p})$ are Cohen-Macaulay, and*
(5) *R has property (S_k).*

Then S has property (S_k).

Proof. Let $\mathfrak{q}$ be a prime of S lying over a prime $\mathfrak{p}$ of R. By Lemma 10.149.1 we have

$$\text{depth}(S_\mathfrak{q}) = \text{depth}(S_\mathfrak{q}/\mathfrak{p}S_\mathfrak{q}) + \text{depth}(R_\mathfrak{p}).$$

On the other hand, we have

$$\dim(S_\mathfrak{q}) \leq \dim(R_\mathfrak{p}) + \dim(S_\mathfrak{q}/\mathfrak{p}S_\mathfrak{q}).$$

by Lemma 10.106.6. (Actually equality holds, by Lemma 10.106.7 but strictly speaking we do not need this.) Finally, as the fibre rings of the map are assumed Cohen-Macaulay we see that $\text{depth}(S_\mathfrak{q}/\mathfrak{p}S_\mathfrak{q}) = \dim(S_\mathfrak{q}/\mathfrak{p}S_\mathfrak{q})$. Thus the lemma follows by the following string of inequalities

$$\begin{aligned}
\text{depth}(S_\mathfrak{q}) &= \dim(S_\mathfrak{q}/\mathfrak{p}S_\mathfrak{q}) + \text{depth}(R_\mathfrak{p}) \\
&\geq \dim(S_\mathfrak{q}/\mathfrak{p}S_\mathfrak{q}) + \min(k, \dim(R_\mathfrak{p})) \\
&= \min(\dim(S_\mathfrak{q}/\mathfrak{p}S_\mathfrak{q}) + k, \dim(S_\mathfrak{q}/\mathfrak{p}S_\mathfrak{q}) + \dim(R_\mathfrak{p})) \\
&\geq \min(k, \dim(S_\mathfrak{q}))
\end{aligned}$$

as desired. □

Lemma 10.149.5. *Let $\varphi : R \to S$ be a ring map. Assume*

(1) *R is Noetherian,*
(2) *S is Noetherian*
(3) *φ is flat,*
(4) *the fibre rings $S \otimes_R \kappa(\mathfrak{p})$ are regular, and*
(5) *R has property (R_k).*

Then S has property (R_k).

Proof. Let $\mathfrak{q}$ be a prime of S lying over a prime $\mathfrak{p}$ of R. Assume that $\dim(S_\mathfrak{q}) \leq k$. Since $\dim(S_\mathfrak{q}) = \dim(R_\mathfrak{p}) + \dim(S_\mathfrak{q}/\mathfrak{p}S_\mathfrak{q})$ by Lemma 10.106.7 we see that $\dim(R_\mathfrak{p}) \leq k$. Hence $R_\mathfrak{p}$ is regular by assumption. It follows that $S_\mathfrak{q}$ is regular by Lemma 10.106.8. □

Lemma 10.149.6. *Let $\varphi : R \to S$ be a ring map. Assume*

(1) *φ is smooth,*
(2) *R is reduced.*

Then S is reduced.

Proof. First assume R is Noetherian. In this case being reduced is the same as having properties (S_1) and (R_0), see Lemma 10.144.3. Note that S is noetherian, and $R \to S$ is flat with regular fibres (see the list of results on smooth ring maps in Section 10.135). Hence we may apply Lemmas 10.149.4 and 10.149.5 and we see that S is (S_1) and (R_0), in other words reduced by Lemma 10.144.3 again.

In the general case we may find a finitely generated $\mathbf{Z}$-subalgebra $R_0 \subset R$ and a smooth ring map $R_0 \to S_0$ such that $S \cong R \otimes_{R_0} S_0$, see remark (10) in Section 10.135. Now, if $x \in S$ is an element with $x^2 = 0$, then we can enlarge R_0 and assume that x comes from an element $x_0 \in S_0$. After enlarging R_0 once more we may assume that $x_0^2 = 0$ in S_0. However, since $R_0 \subset R$ is reduced we see that S_0 is reduced and hence $x_0 = 0$ as desired. □

Lemma 10.149.7. *Let $\varphi : R \to S$ be a ring map. Assume*

(1) *φ is smooth,*
(2) *R is normal.*

Then S is normal.

Proof. First assume R is Noetherian. In this case being reduced is the same as having properties (S_2) and (R_1), see Lemma 10.144.4. Note that S is noetherian, and $R \to S$ is flat with regular fibres (see the list of results on smooth ring maps in Section 10.135). Hence we may apply Lemmas 10.149.4 and 10.149.5 and we see that S is (S_2) and (R_1), in other words reduced by Lemma 10.144.4 again.

The general case. First note that R is reduced and hence S is reduced by Lemma 10.149.6. Let $\mathfrak{q}$ be a prime of S and let $\mathfrak{p}$ be the corresponding prime of R. Note that $R_\mathfrak{p}$ is a normal domain. We have to show that $S_\mathfrak{q}$ is a normal domain. To do this we may replace R by $R_\mathfrak{p}$ and S by $S_\mathfrak{p}$. Hence we may assume that R is a normal domain.

Assume $R \to S$ smooth, and R a normal domain. We may find a finitely generated $\mathbf{Z}$-subalgebra $R_0 \subset R$ and a smooth ring map $R_0 \to S_0$ such that $S \cong R \otimes_{R_0} S_0$, see remark (10) in Section 10.135. As R_0 is a Nagata domain (see Proposition 10.148.31) we see that its integral closure R_0' is finite over R_0. Moreover, as R is

a normal domain it is clear that $R_0' \subset R$. Hence we may replace R_0 by R_0' and S_0 by $R_0' \otimes_{R_0} S_0$ and assume that R_0 is a normal Noetherian domain. By the first paragraph of the proof we conclude that S_0 is a normal ring (it need not be a domain of course). In this way we see that $R = \bigcup R_\lambda$ is the union of normal Noetherian domains and correspondingly $S = \operatorname{colim} R_\lambda \otimes_{R_0} S_0$ is the colimit of normal rings. This implies that S is a normal ring. Some details omitted. □

Lemma 10.149.8. *Let $\varphi : R \to S$ be a ring map. Assume*

(1) *φ is smooth,*
(2) *R is a regular ring.*

Then S is regular.

Proof. This follows from Lemma 10.149.5 applied for all (R_k) using Lemma 10.133.3 to see that the hypotheses are satisfied. □

10.150. Descending properties

In this section we start proving some algebraic facts concerning the "descent" of properties of rings. It turns out that it is often "easier" to descend properties than it is to ascend them. In other words, the assumption on the ring map $R \to S$ are often weaker than the assumptions in the corresponding lemma of the preceding section. However, we warn the reader that the results on descent are often useless unless the corresponding ascent can also be shown! Here is a typical result which illustrates this phenomenon.

Lemma 10.150.1. *Let $R \to S$ be a ring map. Assume that*

(1) *$R \to S$ is faithfully flat, and*
(2) *S is Noetherian.*

Then R is Noetherian.

Proof. Let $I_0 \subset I_1 \subset I_2 \subset \ldots$ be a growing sequence of ideals of R. By assumption we have $I_nS = I_{n+1}S = I_{n+2}S = \ldots$ for some n. Since $R \to S$ is flat we have $I_kS = I_k \otimes_R S$. Hence, as $R \to S$ is faithfully flat we see that $I_nS = I_{n+1}S = I_{n+2}S = \ldots$ implies that $I_n = I_{n+1} = I_{n+2} = \ldots$ as desired. □

Lemma 10.150.2. *Let $R \to S$ be a ring map. Assume that*

(1) *$R \to S$ is faithfully flat, and*
(2) *S is reduced.*

Then R is reduced.

Proof. This is clear as $R \to S$ is injective. □

Lemma 10.150.3. *Let $R \to S$ be a ring map. Assume that*

(1) *$R \to S$ is faithfully flat, and*
(2) *S is a normal ring.*

Then R is a normal ring.

Proof. Since S is reduced it follows that R is reduced. Let $\mathfrak{p}$ be a prime of R. We have to show that $R_\mathfrak{p}$ is a normal domain. Since $S_\mathfrak{p}$ is faithfully over $R_\mathfrak{p}$ too we may assume that R is local with maximal ideal $\mathfrak{m}$. Let $\mathfrak{q}$ be a prime of S lying over $\mathfrak{m}$. Then we see that $R \to S_\mathfrak{q}$ is faithfully flat (Lemma 10.36.16). Hence we may assume S is local as well. In particular S is a normal domain. Since $R \to S$ is

faithfully flat and S is a normal domain we see that R is a domain. Next, suppose that a/b is integral over R with $a, b \in R$. Then $a/b \in S$ as S is normal. Hence $a \in bS$. This means that $a : R \to R/bR$ becomes the zero map after base change to S. By faithful flatness we see that $a \in bR$, so $a/b \in R$. Hence R is normal. □

Lemma 10.150.4. *Let $R \to S$ be a ring map. Assume that*

(1) *$R \to S$ is faithfully flat, and*
(2) *S is a regular ring.*

Then R is a regular ring.

Proof. We see that R is Noetherian by Lemma 10.150.1. Let $\mathfrak{p} \subset R$ be a prime. Choose a prime $\mathfrak{q} \subset S$ lying over $\mathfrak{p}$. Then Lemma 10.105.8 applies to $R_\mathfrak{p} \to S_\mathfrak{q}$ and we conclude that $R_\mathfrak{p}$ is regular. Since $\mathfrak{p}$ was arbitrary we see R is regular. □

Lemma 10.150.5. *Let $R \to S$ be a ring map. Assume that*

(1) *$R \to S$ is faithfully flat of finite presentation, and*
(2) *S is Noetherian and has property (S_k).*

Then R is Noetherian and has property (S_k).

Proof. We have already seen that (1) and (2) imply that R is Noetherian, see Lemma 10.150.1. Let $\mathfrak{p} \subset R$ be a prime ideal. Choose a prime $\mathfrak{q} \subset S$ lying over $\mathfrak{p}$ which corresponds to a minimal prime of the fibre ring $S \otimes_R \kappa(\mathfrak{p})$. Then $A = R_\mathfrak{p} \to S_\mathfrak{q} = B$ is a flat local ring homomorphism of Noetherian local rings with $\mathfrak{m}_A B$ an ideal of definition of B. Hence $\dim(A) = \dim(B)$ (Lemma 10.106.7) and $\text{depth}(A) = \text{depth}(B)$ (Lemma 10.149.1). Hence since B has (S_k) we see that A has (S_k). □

Lemma 10.150.6. *Let $R \to S$ be a ring map. Assume that*

(1) *$R \to S$ is faithfully flat and of finite presentation, and*
(2) *S is Noetherian and has property (R_k).*

Then R is Noetherian and has property (R_k).

Proof. We have already seen that (1) and (2) imply that R is Noetherian, see Lemma 10.150.1. Let $\mathfrak{p} \subset R$ be a prime ideal and assume $\dim(R_\mathfrak{p}) \leq k$. Choose a prime $\mathfrak{q} \subset S$ lying over $\mathfrak{p}$ which corresponds to a minimal prime of the fibre ring $S \otimes_R \kappa(\mathfrak{p})$. Then $A = R_\mathfrak{p} \to S_\mathfrak{q} = B$ is a flat local ring homomorphism of Noetherian local rings with $\mathfrak{m}_A B$ an ideal of definition of B. Hence $\dim(A) = \dim(B)$ (Lemma 10.106.7). As S has (R_k) we conclude that B is a regular local ring. By Lemma 10.105.8 we conclude that A is regular. □

Lemma 10.150.7. *Let $R \to S$ be a ring map. Assume that*

(1) *$R \to S$ is smooth and surjective on spectra, and*
(2) *S is a Nagata ring.*

Then R is a Nagata ring.

Proof. Recall that a Nagata ring is the same thing as a Noetherian universally Japanese ring (Proposition 10.148.30). We have already seen that R is Noetherian in Lemma 10.150.1. Let $R \to A$ be a finite type ring map into a domain. According to Lemma 10.148.17 it suffices to check that A is N-1. It is clear that $B = A \otimes_R S$ is a finite type S-algebra and hence Nagata (Proposition 10.148.30). Since $A \to B$ is smooth (Lemma 10.130.4) we see that B is reduced (Lemma 10.149.6). Since B

is Noetherian it has only a finite number of minimal primes $\mathfrak{q}_1, \ldots, \mathfrak{q}_t$ (see Lemma 10.29.6). As $A \to B$ is flat each of these lies over $(0) \subset A$ (by going down, see Lemma 10.36.17) The total ring of fractions $Q(B)$ is the product of the $L_i = f.f.(B/\mathfrak{q}_i)$ (Lemmas 10.23.2 and 10.24.4). Moreover, the integral closure B' of B in $Q(B)$ is the product of the integral closures B'_i of the $B/\mathfrak{q}_i$ in the factors L_i (compare with Lemma 10.34.14). Since B is universally Japanese the ring extensions $B/\mathfrak{q}_i \subset B'_i$ are finite and we conclude that $B' = \prod B'_i$ is finite over B. Since $A \to B$ is flat we see that any nonzerodivisor on A maps to a nonzerodivisor on B. The corresponding map

$$Q(A) \otimes_A B = (A \setminus \{0\})^{-1}A \otimes_A B = (A \setminus \{0\})^{-1}B \to Q(B)$$

is injective (we used Lemma 10.11.15). Via this map A' maps into B'. This induces a map

$$A' \otimes_A B \longrightarrow B'$$

which is injective (by the above and the flatness of $A \to B$). Since B' is a finite B-module and B is Noetherian we see that $A' \otimes_A B$ is a finite B-module. Hence there exist finitely many elements $x_i \in A'$ such that the elements $x_i \otimes 1$ generate $A' \otimes_A B$ as a B-module. Finally, by faithful flatness of $A \to B$ we conclude that the x_i also generated A' as an A-module, and we win. □

Remark 10.150.8. The property of being "universally catenary" does not descend; not even along étale ring maps. In Examples, Section 78.11 there is a construction of a finite ring map $A \to B$ with A local Noetherian and not universally catenary, B semi-local with two maximal ideals $\mathfrak{m}$, $\mathfrak{n}$ with $B_\mathfrak{m}$ and $B_\mathfrak{n}$ regular of dimension 2 and 1 respectively, and the same residue fields as that of A. Moreover, $\mathfrak{m}_A$ generates the maximal ideal in both $B_\mathfrak{m}$ and $B_\mathfrak{n}$ (so $A \to B$ is unramified as well as finite). By Lemma 10.142.10 there exists a local étale ring map $A \to A'$ such that $B \otimes_A A' = B_1 \times B_2$ decomposes with $A' \to B_i$ surjective. This shows that A' has two minimal primes $\mathfrak{q}_i$ with $A'/\mathfrak{q}_i \cong B_i$. Since B_i is regular local (since it is étale over either $B_\mathfrak{m}$ or $B_\mathfrak{n}$) we conclude that A' is universally catenary.

10.151. Geometrically normal algebras

In this section we put some applications of ascent and descent of properties of rings.

Lemma 10.151.1. *Let k be a field. Let A be a k-algebra. The following properties of A are equivalent:*

1. *$k' \otimes_k A$ is a normal ring for every field extension $k \subset k'$,*
2. *$k' \otimes_k A$ is a normal ring for every finitely generated field extension $k \subset k'$, and*
3. *$k' \otimes_k A$ is a normal ring for every finite purely inseparable extension $k \subset k'$.*

where normal ring is as defined in Definition 10.34.10.

Proof. It is clear that (1) ⇒ (2) ⇒ (3).

Assume (2) and let $k \subset k'$ be any field extension. Then we can write $k' = \operatorname{colim}_i k_i$ as a directed colimit of finitely generated field extensions. Hence we see that $k' \otimes_k A = \operatorname{colim}_i k_i \otimes_k A$ is a directed colimit of normal rings. Thus we see that $k' \otimes_k A$ is a normal ring by Lemma 10.34.15. Hence (1) holds.

Assume (3) and let $k \subset K$ be a finitely generated field extension. By Lemma 10.42.3 we can find a diagram

$$\begin{array}{ccc} K & \longrightarrow & K' \\ \uparrow & & \uparrow \\ k & \longrightarrow & k' \end{array}$$

where $k \subset k'$, $K \subset K'$ are finite purely inseparable field extensions such that $k' \subset K'$ is separable. By Lemma 10.145.10 there exists a smooth k'-algebra B such that K' is the fraction field of B. Now we can argue as follows: Step 1: $k' \otimes_k A$ is a normal ring because we assumed (3). Step 2: $B \otimes_{k'} k' \otimes_k A$ is a normal ring as $k' \otimes_k A \to B \otimes_{k'} k' \otimes_k A$ is smooth (Lemma 10.130.4) and ascent of normality along smooth maps (Lemma 10.149.7). Step 3. $K' \otimes_{k'} k' \otimes_k A = K' \otimes_k A$ is a normal ring as it is a localization of a normal ring (Lemma 10.34.12). Step 4. Finally $K \otimes_k A$ is a normal ring by descent of normality along the faithfully flat ring map $K \otimes_k A \to K' \otimes_k A$ (Lemma 10.150.3). This proves the lemma. □

Definition 10.151.2. Let k be a field. A k-algebra R is called *geometrically normal* over k if the equivalent conditions of Lemma 10.151.1 hold.

Lemma 10.151.3. *Let k be a field. A localization of a geometrically normal k-algebra is geometrically normal.*

Proof. This is clear as being a normal ring is checked at the localizations at prime ideals. □

Lemma 10.151.4. *Let k be a field. Let A, B be k-algebras. Assume A is geometrically normal over k and B is a normal ring. Then $A \otimes_k B$ is a normal ring.*

Proof. Let $\mathfrak{r}$ be a prime ideal of $A \otimes_k B$. Denote $\mathfrak{p}$, resp. $\mathfrak{q}$ the corresponding prime of A, resp. B. Then $(A \otimes_k B)_\mathfrak{r}$ is a localization of $A_\mathfrak{p} \otimes_k B_\mathfrak{q}$. Hence it suffices to prove the result for the ring $A_\mathfrak{p} \otimes_k B_\mathfrak{q}$, see Lemma 10.34.12 and Lemma 10.151.3. Thus we may assume A and B are domains.

Assume that A and B are domains with fractions fields K and L. Note that B is the filtered colimit of its finite type normal k-sub algebras (as k is a Nagata ring, see Proposition 10.148.31, and hence the integral closure of a finite type k-sub algebra is still a finite type k-sub algebra by Proposition 10.148.30). By Lemma 10.34.15 we reduce to the case that B is of finite type over k.

Assume that A and B are domains with fractions fields K and L and B of finite type over k. In this case the ring $K \otimes_k B$ is of finite type over K, hence Noetherian (Lemma 10.29.1). In particular $K \otimes_k B$ has finitely many minimal primes (Lemma 10.29.6). Since $A \to A \otimes_k B$ is flat, this implies that $A \otimes_k B$ has finitely many minimal primes (by going down for flat ring maps – Lemma 10.36.17 – these primes all lie over $(0) \subset A$). Thus it suffices to prove that $A \otimes_k B$ is integrally closed in its total ring of fractions (Lemma 10.34.14).

We claim that $K \otimes_k B$ and $A \otimes_k L$ are both normal rings. If this is true then any element x of $Q(A \otimes_k B)$ which is integral over $A \otimes_k B$ is (by Lemma 10.34.11) contained in $K \otimes_k B \cap A \otimes_k L = A \otimes_k B$ and we're done. Since $A \otimes_K L$ is a normal ring by assumption, it suffices to prove that $K \otimes_k B$ is normal.

As A is geometrically normal over k we see K is geometrically normal over k (Lemma 10.151.3) hence K is geometrically reduced over k. Hence $K = \bigcup K_i$ is the union of finitely generated field extensions of k which are geometrically reduced (Lemma 10.40.2). Each K_i is the localization of a smooth k-algebra (Lemma 10.145.10). So $K_i \otimes_k B$ is the localization of a smooth B-algebra hence normal (Lemma 10.149.7). Thus $K \otimes_k B$ is a normal ring (Lemma 10.34.15) and we win. □

10.152. Geometrically regular algebras

Let k be a field. Let A be a Noetherian k-algebra. Let $k \subset K$ be a finitely generated field extension. Then the ring $K \otimes_k A$ is Noetherian as well, see Lemma 10.29.7. Thus the following lemma makes sense.

Lemma 10.152.1. *Let k be a field. Let A be a k-algebra. Assume A is Noetherian. The following properties of A are equivalent:*

(1) *$k' \otimes_k A$ is regular for every finitely generated field extension $k \subset k'$, and*
(2) *$k' \otimes_k A$ is regular for every finite purely inseparable extension $k \subset k'$.*

Here regular ring is as in Definition 10.105.6.

Proof. The lemma makes sense by the remarks preceding the lemma. It is clear that (1) $\Rightarrow$ (2).

Assume (2) and let $k \subset K$ be a finitely generated field extension. By Lemma 10.42.3 we can find a diagram

$$\begin{array}{ccc} K & \longrightarrow & K' \\ \uparrow & & \uparrow \\ k & \longrightarrow & k' \end{array}$$

where $k \subset k'$, $K \subset K'$ are finite purely inseparable field extensions such that $k' \subset K'$ is separable. By Lemma 10.145.10 there exists a smooth k'-algebra B such that K' is the fraction field of B. Now we can argue as follows: Step 1: $k' \otimes_k A$ is a regular ring because we assumed (2). Step 2: $B \otimes_{k'} k' \otimes_k A$ is a regular ring as $k' \otimes_k A \to B \otimes_{k'} k' \otimes_k A$ is smooth (Lemma 10.130.4) and ascent of regularity along smooth maps (Lemma 10.149.8). Step 3. $K' \otimes_{k'} k' \otimes_k A = K' \otimes_k A$ is a regular ring as it is a localization of a regular ring (immediate from the definition). Step 4. Finally $K \otimes_k A$ is a regular ring by descent of regularity along the faithfully flat ring map $K \otimes_k A \to K' \otimes_k A$ (Lemma 10.150.4). This proves the lemma. □

Definition 10.152.2. Let k be a field. Let R be a Noetherian k-algebra. The k-algebra R is called *geometrically regular* over k if the equivalent conditions of Lemma 10.152.1 hold.

It is clear from the definition that $K \otimes_k R$ is a geometrically regular algebra over K for any finitely generated field extension K of k. We will see later (More on Algebra, Proposition 15.24.1) that it suffices to check $R \otimes_k k'$ is regular whenever $k \subset k' \subset k^{1/p}$ (finite).

Lemma 10.152.3. *Let k be a field. Let $A \to B$ be a faithfully flat k-algebra map. If B is geometrically regular over k, so is A.*

Proof. Assume B is geometrically regular over k. Let $k \subset k'$ be a finite, purely inseparable extension. Then $A \otimes_k k' \to B \otimes_k k'$ is faithfully flat as a base change of $A \to B$ (by Lemmas 10.28.3 and 10.36.6) and $B \otimes_k k'$ is regular by our assumption on B over k. Then $A \otimes_k k'$ is regular by Lemma 10.150.4. □

Lemma 10.152.4. *Let k be a field. Let $A \to B$ be a smooth ring map of k-algebras. If A is geometrically regular over k, then B is geometrically regular over k.*

Proof. Let $k \subset k'$ be a finitely generated field extension. Then $A \otimes_k k' \to B \otimes_k k'$ is a smooth ring map (Lemma 10.130.4) and $A \otimes_k k'$ is regular. Hence $B \otimes_k k'$ is regular by Lemma 10.149.8. □

Lemma 10.152.5. *Let k be a field. Let A be an algebra over k. Let $k = \operatorname{colim} k_i$ be a directed colimit of subfields. If A is geometrically regular over each k_i, then A is geometrically regular over k.*

Proof. Let $k \subset k'$ be a finite purely inseparable field extension. We can get k' by adjoining finitely many variables to k and imposing finitely many polynomial relations. Hence we see that there exists an i and a finite purely inseparable field extension $k_i \subset k_i'$ such that $k_i = k \otimes_{k_i} k_i'$. Thus $A \otimes_k k' = A \otimes_{k_i} k_i'$ and the lemma is clear. □

Lemma 10.152.6. *Let $k \subset k'$ be a separable algebraic field extension. Let A be a an algebra over k'. Then A is geometrically regular over k if and only if it is geometrically regular over k'.*

Proof. Let $k \subset L$ be a finite purely inseparable field extension. Then $L' = k' \otimes_k L$ is a field (see material in Fields, Section 9.25) and $A \otimes_k L = A \otimes_{k'} L'$. Hence if A is geometrically regular over k', then A is geometrically regular over k.

Assume A is geometrically regular over k. Since k' is the filtered colimit of finite extensions of k we may assume by Lemma 10.152.5 that k'/k is finite separable. Consider the ring maps

$$k' \to A \otimes_k k' \to A.$$

Note that $A \otimes_k k'$ is geometrically regular over k' as a base change of A to k'. Note that $A \otimes_k k' \to A$ is the base change of $k' \otimes_k k' \to k'$ by the map $k' \to A$. Since k'/k is an étale extension of rings, we see that $k' \otimes_k k' \to k'$ is étale (Lemma 10.136.3). Hence A is geometrically regular over k' by Lemma 10.152.4. □

10.153. Geometrically Cohen-Macaulay algebras

This section is a bit of a misnomer, since Cohen-Macaulay algebras are automatically geometrically Cohen-Macaulay. Namely, see Lemma 10.124.6 and Lemma 10.153.2 below.

Lemma 10.153.1. *Let k be a field and let $k \subset K$ and $k \subset L$ be two field extensions such that one of them is a field extension of finite type. Then $K \otimes_k L$ is a Noetherian Cohen-Macaulay ring.*

Proof. The ring $K \otimes_k L$ is Noetherian by Lemma 10.29.7. Say K is a finite extension of the purely transcendental extension $k(t_1, \ldots, t_r)$. Then $k(t_1, \ldots, t_r) \otimes_k L \to K \otimes_k L$ is a finite free ring map. By Lemma 10.106.9 it suffices to show that $k(t_1, \ldots, t_r) \otimes_k L$ is Cohen-Macaulay. This is clear because it is a localization of

the polynomial ring $L[t_1, \ldots, t_r]$. (See for example Lemma 10.99.7 for the fact that a polynomial ring is Cohen-Macaulay.) $\square$

Lemma 10.153.2. *Let k be a field. Let S be a Noetherian k-algebra. Let $k \subset K$ be a finitely generated field extension, and set $S_K = K \otimes_k S$. Let $\mathfrak{q} \subset S$ be a prime of S. Let $\mathfrak{q}_K \subset S_K$ be a prime of S_K lying over $\mathfrak{q}$. Then $S_\mathfrak{q}$ is Cohen-Macaulay if and only if $(S_K)_{\mathfrak{q}_K}$ is Cohen-Macaulay.*

Proof. By Lemma 10.29.7 the ring S_K is Noetherian. Hence $S_\mathfrak{q} \to (S_K)_{\mathfrak{q}_K}$ is a flat local homomorphism of Noetherian local rings. Note that the fibre

$$(S_K)_{\mathfrak{q}_K}/\mathfrak{q}(S_K)_{\mathfrak{q}_K} \cong (\kappa(\mathfrak{q}) \otimes_k K)_{\mathfrak{q}'}$$

is the localization of the Cohen-Macaulay (Lemma 10.153.1) ring $\kappa(\mathfrak{q}) \otimes_k K$ at a suitable prime ideal $\mathfrak{q}'$. Hence the lemma follows from Lemma 10.149.3. $\square$

10.154. Colimits and maps of finite presentation, II

This section is a continuation of Section 10.121.

We start with an application of the openness of flatness. It says that we can approximate flat modules by flat modules which is useful.

Lemma 10.154.1. *Let $R \to S$ be a ring map. Let M be an S-module. Assume that*

1. *$R \to S$ is of finite presentation,*
2. *M is a finitely presented S-module, and*
3. *M is flat over R.*

In this case we have the following:

1. *There exists a finite type $\mathbf{Z}$-algebra R_0 and a finite type ring map $R_0 \to S_0$ and a finite S_0-module M_0 such that M_0 is flat over R_0, together with a ring maps $R_0 \to R$ and $S_0 \to S$ and an S_0-module map $M_0 \to M$ such that $S \cong R \otimes_{R_0} S_0$ and $M = S \otimes_{S_0} M_0$.*
2. *If $R = \operatorname{colim}_{\lambda \in \Lambda} R_\lambda$ is written as a directed colimit, then there exists a λ and a ring map $R_\lambda \to S_\lambda$ of finite presentation, and an S_λ-module M_λ of finite presentation such that M_λ is flat over R_λ and such that $S = R \otimes_{R_\lambda} S_\lambda$ and $M = S \otimes_{S_\lambda} M_\lambda$.*
3. *If*
$$(R \to S, M) = \operatorname{colim}_{\lambda \in \Lambda} (R_\lambda \to S_\lambda, M_\lambda)$$
is written as a directed colimit such that
   (a) *$R_\mu \otimes_{R_\lambda} S_\lambda \to S_\mu$ and $S_\mu \otimes_{S_\lambda} M_\lambda \to M_\mu$ are isomorphisms for $\mu \geq \lambda$,*
   (b) *$R_\lambda \to S_\lambda$ is of finite presentation,*
   (c) *M_λ is a finitely presented S_λ-module,*
   then for all sufficiently large λ the module M_λ is flat over R_λ.

Proof. We first write $(R \to S, M)$ as the directed colimit of a system $(R_\lambda \to S_\lambda, M_\lambda)$ as in as in Lemma 10.121.15. Let $\mathfrak{q} \subset S$ be a prime. Let $\mathfrak{p} \subset R$, $\mathfrak{q}_\lambda \subset S_\lambda$, and $\mathfrak{p}_\lambda \subset R_\lambda$ the corresponding primes. As seen in the proof of Theorem 10.123.4

$$((R_\lambda)_{\mathfrak{p}_\lambda}, (S_\lambda)_{\mathfrak{q}_\lambda}, (M_\lambda)_{\mathfrak{q}_\lambda})$$

is a system as in Lemma 10.121.11, and hence by Lemma 10.122.3 we see that for some $\lambda_\mathfrak{q} \in \Lambda$ for all $\lambda \geq \lambda_\mathfrak{q}$ the module M_λ is flat over R_λ at the prime $\mathfrak{q}_\lambda$.

By Theorem 10.123.4 we get an open subset $U_\lambda \subset \mathrm{Spec}(S_\lambda)$ such that M_λ flat over R_λ at all the primes of U_λ. Denote $V_\lambda \subset \mathrm{Spec}(S)$ the inverse image of U_λ under the map $\mathrm{Spec}(S) \to \mathrm{Spec}(S_\lambda)$. The argument above shows that for every $\mathfrak{q} \in \mathrm{Spec}(S)$ there exists a $\lambda_\mathfrak{q}$ such that $\mathfrak{q} \in V_\lambda$ for all $\lambda \geq \lambda_\mathfrak{q}$. Since $\mathrm{Spec}(S)$ is quasi-compact we see this implies there exists a single $\lambda_0 \in \Lambda$ such that $V_{\lambda_0} = \mathrm{Spec}(S)$.

The complement $\mathrm{Spec}(S_{\lambda_0}) \setminus U_{\lambda_0}$ is $V(I)$ for some ideal $I \subset S_{\lambda_0}$. As $V_{\lambda_0} = \mathrm{Spec}(S)$ we see that $IS = S$. Choose $f_1, \ldots, f_r \in I$ and $s_1, \ldots, s_n \in S$ such that $\sum f_i s_i = 1$. Since $\mathrm{colim}\, S_\lambda = S$, after increasing λ_0 we may assume there exist $s_{i,\lambda_0} \in S_{\lambda_0}$ such that $\sum f_i s_{i,\lambda_0} = 1$. Hence for this λ_0 we have $U_{\lambda_0} = \mathrm{Spec}(S_{\lambda_0})$. This proves (1).

Proof of (2). Let $(R_0 \to S_0, M_0)$ be as in (1) and suppose that $R = \mathrm{colim}\, R_\lambda$. Since R_0 is a finite type $\mathbf{Z}$ algebra, there exists a λ and a map $R_0 \to R_\lambda$ such that $R_0 \to R_\lambda \to R$ is the given map $R_0 \to R$ (see Lemma 10.121.2). Then, part (2) follows by taking $S_\lambda = R_\lambda \otimes_{R_0} S_0$ and $M_\lambda = S_\lambda \otimes_{S_0} M_0$.

Finally, we come to the proof of (3). Let $(R_\lambda \to S_\lambda, M_\lambda)$ be as in (3). Choose $(R_0 \to S_0, M_0)$ and $R_0 \to R$ as in (1). As in the proof of (2), there exists a λ_0 and a ring map $R_0 \to R_{\lambda_0}$ such that $R_0 \to R_{\lambda_0} \to R$ is the given map $R_0 \to R$. Since S_0 is of finite presentation over R_0 and since $S = \mathrm{colim}\, S_\lambda$ we see that for some $\lambda_1 \geq \lambda_0$ we get an R_0-algebra map $S_0 \to S_{\lambda_1}$ such that the composition $S_0 \to S_{\lambda_1} \to S$ is the given map $S_0 \to S$ (see Lemma 10.121.2). For all $\lambda \geq \lambda_1$ this gives maps

$$\Psi_\lambda : R_\lambda \otimes_{R_0} S_0 \longrightarrow R_\lambda \otimes_{R_{\lambda_1}} S_{\lambda_1} \cong S_\lambda$$

the last isomorphism by assumption. By construction $\mathrm{colim}_\lambda \Psi_\lambda$ is an isomorphism. Hence Ψ_λ is an isomorphism for all λ large enough by Lemma 10.121.6. In the same vein, there exists a $\lambda_2 \geq \lambda_1$ and an S_0-module map $M_0 \to M_{\lambda_2}$ such that $M_0 \to M_{\lambda_2} \to M$ is the given map $M_0 \to M$ (see Lemma 10.121.3). For $\lambda \geq \lambda_2$ there is an induced map

$$S_\lambda \otimes_{S_0} M_0 \longrightarrow S_\lambda \otimes_{S_{\lambda_2}} M_{\lambda_2} \cong M_\lambda$$

and for λ large enough this map is an isomorphism by Lemma 10.121.4. This implies (3) because M_0 is flat over R_0. □

Lemma 10.154.2. *Let $R \to A \to B$ be ring maps. Assume $A \to B$ faithfully flat of finite presentation. Then there exists a commutative diagram*

$$\begin{array}{ccccc} R & \longrightarrow & A_0 & \longrightarrow & B_0 \\ \| & & \downarrow & & \downarrow \\ R & \longrightarrow & A & \longrightarrow & B \end{array}$$

with $R \to A_0$ of finite presentation, $A_0 \to B_0$ faithfully flat of finite presentation and $B = A \otimes_{A_0} B_0$.

Proof. We first prove the lemma with R replaced $\mathbf{Z}$. By Lemma 10.154.1 there exists a diagram

$$\begin{array}{ccc} A_0 & \longrightarrow & A \\ \uparrow & & \uparrow \\ B_0 & \longrightarrow & B \end{array}$$

where A_0 is of finite type over $\mathbf{Z}$, B_0 is flat of finite presentation over A_0 such that $B = A \otimes_{A_0} B_0$. As $A_0 \to B_0$ is flat of finite presentation we see that the image of $\mathrm{Spec}(B_0) \to \mathrm{Spec}(A_0)$ is open, see Proposition 10.38.8. Hence the complement of the image is $V(I_0)$ for some ideal $I_0 \subset A_0$. As $A \to B$ is faithfully flat the map $\mathrm{Spec}(B) \to \mathrm{Spec}(A)$ is surjective, see Lemma 10.36.15. Now we use that the base change of the image is the image of the base change. Hence $I_0 A = A$. Pick a relation $\sum f_i r_i = 1$, with $r_i \in A$, $f_i \in I_0$. Then after enlarging A_0 to contain the elements r_i (and correspondingly enlarging B_0) we see that $A_0 \to B_0$ is surjective on spectra also, i.e., faithfully flat.

Thus the lemma holds in case $R = \mathbf{Z}$. In the general case, take the solution $A_0' \to B_0'$ just obtained and set $A_0 = A_0' \otimes_{\mathbf{Z}} R$, $B_0 = B_0' \otimes_{\mathbf{Z}} R$. □

Lemma 10.154.3. *Let $A = \mathrm{colim}_{i \in I} A_i$ be a directed colimit of rings. Let $0 \in I$ and $\varphi_0 : B_0 \to C_0$ a map of A_0-algebras. Assume*

(1) *$A \otimes_{A_0} B_0 \to A \otimes_{A_0} C_0$ is finite,*
(2) *C_0 is of finite type over B_0.*

Then there exists an $i \geq 0$ such that the map $A_i \otimes_{A_0} B_0 \longrightarrow A_i \otimes_{A_0} C_0$ is finite.

Proof. Let $x_1, \ldots, x_m$ be generators for C_0 over B_0. Pick monic polynomials $P_j \in A \otimes_{A_0} B_0[T]$ such that $P_j(1 \otimes x_j) = 0$ in $A \otimes_{A_0} C_0$. For some $i \geq 0$ we can find $P_{j,i} \in A_i \otimes_{A_0} B_0[T]$ mapping to P_j. Since $\otimes$ commutes with colimits we see that $P_{j,i}(1 \otimes x_j)$ is zero in $A_i \otimes_{A_0} C_0$ after possibly increasing i. Then this i works. □

Lemma 10.154.4. *Let $A = \mathrm{colim}_{i \in I} A_i$ be a directed colimit of rings. Let $0 \in I$ and $\varphi_0 : B_0 \to C_0$ a map of A_0-algebras. Assume*

(1) *$A \otimes_{A_0} B_0 \to A \otimes_{A_0} C_0$ is surjective,*
(2) *C_0 is of finite type over B_0.*

Then for some $i \geq 0$ the map $A_i \otimes_{A_0} B_0 \longrightarrow A_i \otimes_{A_0} C_0$ is surjective.

Proof. Let $x_1, \ldots, x_m$ be generators for C_0 over B_0. Pick $b_j \in A \otimes_{A_0} B_0$ mapping to $1 \otimes x_j$ in $A \otimes_{A_0} C_0$. For some $i \geq 0$ we can find $b_{j,i} \in A_i \otimes_{A_0} B_0$ mapping to b_j. Then this i works. □

Lemma 10.154.5. *Let $A = \mathrm{colim}_{i \in I} A_i$ be a directed colimit of rings. Let $0 \in I$ and $\varphi_0 : B_0 \to C_0$ a map of A_0-algebras. Assume*

(1) *$A \otimes_{A_0} B_0 \longrightarrow A \otimes_{A_0} C_0$ is étale,*
(2) *$B_0 \to C_0$ is of finite presentation.*

Then for some $i \geq 0$ the map $A_i \otimes_{A_0} B_0 \longrightarrow A_i \otimes_{A_0} C_0$ is étale.

Proof. Note that $B_0 \to C_0$ is of finite presentation, see Lemma 10.6.2. Write $C_0 = B_0[x_1, \ldots, x_n]/(f_{1,0}, \ldots, f_{m,0})$. Write $B_i = A_i \otimes_{A_0} B_0$ and $C_i = A_i \otimes_{A_0} C_0$. Note that $C_i = B_i[x_1, \ldots, x_n]/(f_{1,i}, \ldots, f_{m,i})$ where $f_{j,i}$ is the image of $f_{j,0}$ in the polynomial ring over B_i. Write $B = A \otimes_{A_0} B_0$ and $C = A \otimes_{A_0} C_0$. Note that $C = B[x_1, \ldots, x_n]/(f_1, \ldots, f_m)$ where f_j is the image of $f_{j,0}$ in the polynomial ring over B. The assumption is that the map

$$\mathrm{d} : (f_1, \ldots, f_m)/(f_1, \ldots, f_m)^2 \longrightarrow \bigoplus C \mathrm{d}x_k$$

is an isomorphism. Thus for sufficiently large i we can find elements

$$\xi_{k,i} \in (f_{1,i}, \ldots, f_{m,i})/(f_{1,i}, \ldots, f_{m,i})^2$$

with $\mathrm{d}\xi_{k,i} = \mathrm{d}x_k$ in $\bigoplus C_i \mathrm{d}x_k$. Moreover, on increasing i if necessary, we see that $\sum(\partial f_{j,i}/\partial x_k)\xi_{k,i} = f_{j,i} \bmod (f_{1,i}, \ldots, f_{m,i})^2$ since this is true in the limit. Then this i works. □

The following lemma is an application of the results above which doesn't seem to fit well anywhere else.

Lemma 10.154.6. *Let $R \to S$ be a faithfully flat ring map of finite presentation. Then there exists a commutative diagram*

$$\begin{array}{ccc} S & \longrightarrow & S' \\ & \nwarrow \quad \nearrow & \\ & R & \end{array}$$

where $R \to S'$ is quasi-finite, faithfully flat and of finite presentation.

Proof. As a first step we reduce this lemma to the case where R is of finite type over $\mathbf{Z}$. By Lemma 10.154.2 there exists a diagram

$$\begin{array}{ccc} S_0 & \longrightarrow & S \\ \uparrow & & \uparrow \\ R_0 & \longrightarrow & R \end{array}$$

where R_0 is of finite type over $\mathbf{Z}$, and S_0 is faithfully flat of finite presentation over R_0 such that $S = R \otimes_{R_0} S_0$. If we prove the lemma for the ring map $R_0 \to S_0$, then the lemma follows for $R \to S$ by base change, as the base change of a quasi-finite ring map is quasi-finite, see Lemma 10.116.8. (Of course we also use that base changes of flat maps are flat and base changes of maps of finite presentation are of finite presentation.)

Assume $R \to S$ is a faithfully flat ring map of finite presentation and that R is Noetherian (which we may assume by the preceding paragraph). Let $W \subset \mathrm{Spec}(S)$ be the open set of Lemma 10.124.4. As $R \to S$ is faithfully flat the map $\mathrm{Spec}(S) \to \mathrm{Spec}(R)$ is surjective, see Lemma 10.36.15. By Lemma 10.124.5 the map $W \to \mathrm{Spec}(R)$ is also surjective. Hence by replacing S with a product $S_{g_1} \times \ldots \times S_{g_m}$ we may assume $W = \mathrm{Spec}(S)$; here we use that $\mathrm{Spec}(R)$ is quasi-compact (Lemma 10.16.10), and that the map $\mathrm{Spec}(S) \to \mathrm{Spec}(R)$ is open (Proposition 10.38.8). Suppose that $\mathfrak{p} \subset R$ is a prime. Choose a prime $\mathfrak{q} \subset S$ lying over $\mathfrak{p}$ which corresponds to a maximal ideal of the fibre ring $S \otimes_R \kappa(\mathfrak{p})$. The Noetherian local ring $\overline{S}_\mathfrak{q} = S_\mathfrak{q}/\mathfrak{p}S_\mathfrak{q}$ is Cohen-Macaulay, say of dimension d. We may choose $f_1, \ldots, f_d$ in the maximal ideal of $S_\mathfrak{q}$ which map to a regular sequence in $\overline{S}_\mathfrak{q}$. Choose a common denominator $g \in S$, $g \notin \mathfrak{q}$ of $f_1, \ldots, f_d$, and consider the R-algebra

$$S' = S_g/(f_1, \ldots, f_d).$$

By construction there is a prime ideal $\mathfrak{q}' \subset S'$ lying over $\mathfrak{p}$ and corresponding to $\mathfrak{q}$ (via $S_g \to S'_g$). Also by construction the ring map $R \to S'$ is quasi-finite at $\mathfrak{q}$ as the local ring

$$S'_{\mathfrak{q}'}/\mathfrak{p}S'_{\mathfrak{q}'} = S_\mathfrak{q}/(f_1, \ldots, f_d) + \mathfrak{p}S_\mathfrak{q} = \overline{S}_\mathfrak{q}/(\overline{f}_1, \ldots, \overline{f}_d)$$

has dimension zero, see Lemma 10.116.2. Also by construction $R \to S'$ is of finite presentation. Finally, by Lemma 10.93.3 the local ring map $R_\mathfrak{p} \to S'_{\mathfrak{q}'}$ is flat

(this is where we use that R is Noetherian). Hence, by openness of flatness (Theorem 10.123.4), and openness of quasi-finiteness (Lemma 10.117.14) we may after replacing g by gg' for a suitable $g' \in S$, $g' \notin \mathfrak{q}$ assume that $R \to S'$ is flat and quasi-finite. The image $\mathrm{Spec}(S') \to \mathrm{Spec}(R)$ is open and contains $\mathfrak{p}$. In other words we have shown a ring S' as in the statement of the lemma exists (except possibly the faithfulness part) whose image contains any given prime. Using one more time the quasi-compactness of $\mathrm{Spec}(R)$ we see that a finite product of such rings does the job. □

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CHAPTER 11

Brauer groups

11.1. Introduction

A reference are the lectures by Serre in the Seminaire Cartan, see [**Ser55a**]. Serre in turn refers to [**Deu68**] and [**ANT44**]. We changed some of the proofs, in particular we used a fun argument of Rieffel to prove Wedderburn's theorem. Very likely this change is not an improvement and we strongly encourage the reader to read the original exposition by Serre.

11.2. Noncommutative algebras

Let k be a field. In this chapter an *algebra* A over k is a possibly noncommutative ring A together with a ring map $k \to A$ such that k maps into the center of A and such that 1 maps to an identity element of A. An *A-module* is a right A-module such that the identity of A acts as the identity.

Definition 11.2.1. Let A be a k-algebra. We say A is *finite* if $\dim_k(A) < \infty$. In this case we write $[A : k] = \dim_k(A)$.

Definition 11.2.2. A *skew field* is a possibly noncommutative ring with an identity element 1, with $1 \neq 0$, such that in which every nonzero element has a multiplicative inverse.

A skew field is a k-algebra for some k (e.g., for the prime field contained in it). We will use below that any module over a skew field is free because a maximal linearly independent set of vectors forms a basis and exists by Zorn's lemma.

Definition 11.2.3. Let A be a k-algebra. We say an A-module M is *simple* if it is nonzero and the only A-submodules are 0 and M. We say A is *simple* if the only two-sided ideals of A are 0 and A.

Definition 11.2.4. A k-algebra A is *central* if the center of A is the image of $k \to A$.

Definition 11.2.5. Given a k-algebra A we denote A^{op} the k-algebra we get by reversing the order of multiplication in A. This is called the *opposite algebra*.

11.3. Wedderburn's theorem

The following cute argument can be found in a paper of Rieffel, see [**Rie65**]. The proof could not be simpler (quote from Carl Faith's review).

Lemma 11.3.1. *Let A be a possibly noncommutative ring with 1 which contains no nontrivial two-sided ideal. Let M be a nonzero right ideal in A, and view M as a right A-module. Then A coincides with the bicommutant of M.*

Proof. Let $A' = \mathrm{End}_A(M)$, and let $A'' = \mathrm{End}_{A'}(M)$ (the bicommutant of M). Let $R : A \to A''$ be the natural homomorphism $R(a)(m) = ma$. Then R is injective, since $R(1) = \mathrm{id}_M$ and A contains no nontrivial two-sided ideal. We claim that $R(M)$ is a right ideal in A''. Namely, $R(m)a'' = R(ma'')$ for $a'' \in A''$ and m in M, because *left* multiplication of M by any element n of M represents an element of A', and so $(nm)a'' = n(ma'')$, that is, $(R(m)a'')(n) = R(ma'')(n)$ for all n in M. Finally, the product ideal AM is a two-sided ideal, and so $A = AM$. Thus $R(A) = R(A)R(M)$, so that $R(A)$ is a right ideal in A''. But $R(A)$ contains the identity element of A'', and so $R(A) = A''$. □

Lemma 11.3.2. *Let A be a k-algebra. If A is finite, then*

(1) *A has a simple module,*
(2) *any nonzero module contains a simple submodule,*
(3) *a simple module over A has finite dimension over k, and*
(4) *if M is a simple A-module, then $End_A(M)$ is a skew field.*

Proof. Of course (1) follows from (2) since A is a nonzero A-module. For (2), any submodule of minimal (finite) dimension as a k-vector space will be simple. There exists a finite dimensional one because a cyclic submodule is one. If M is simple, then $mA \subset M$ is a sub-module, hence we see (3). Any nonzero element of $\mathrm{End}_A(M)$ is an isomorphism, hence (4) holds. □

Theorem 11.3.3. *Let A be a simple finite k-algebra. Then A is a matrix algebra over a finite k-algebra K which is a skew field.*

Proof. We may choose a simple submodule $M \subset A$ and then the k-algebra $K = \mathrm{End}_A(M)$ is a skew field, see Lemma 11.3.2. By Lemma 11.3.1 we see that $A = \mathrm{End}_K(M)$. Since K is a skew field and M is finitely generated (since $\dim_k(M) < \infty$) we see that M is finite free as a left K-module. It follows immediately that $A \cong \mathrm{Mat}(n \times n, K^{op})$. □

11.4. Lemmas on algebras

Let A be a k-algebra. Let $B \subset A$ be a subalgebra. The *centralizer of B in A* is the subalgebra

$$C = \{y \in A \mid xy = yx \text{ for all } x \in B\}.$$

It is a k-algebra.

Lemma 11.4.1. *Let A, A' be k-algebras. Let $B \subset A$, $B' \subset A'$ be subalgebras with centralizers C, C'. Then the centralizer of $B \otimes_k B'$ in $A \otimes_k A'$ is $C \otimes_k C'$.*

Proof. Denote $C'' \subset A \otimes_k A'$ the centralizer of $B \otimes_k B'$. It is clear that $C \otimes_k C' \subset C''$. Conversely, every element of C'' commutes with $B \otimes 1$ hence is contained in $C \otimes_k A'$. Similarly $C'' \subset A \otimes_k C'$. Thus $C'' \subset C \otimes_k A' \cap A \otimes_k C' = C \otimes_k C'$. □

Lemma 11.4.2. *Let A be a finite simple k-algebra. Then the center k' of A is a finite field extension of k.*

Proof. Write $A = \mathrm{Mat}(n \times n, K)$ for some skew field K finite over k, see Theorem 11.3.3. By Lemma 11.4.1 the center of A is $k \otimes_k k'$ where $k' \subset K$ is the center of K. Since the center of a skew field is a field, we win. □

Lemma 11.4.3. *Let V be a k vector space. Let K be a central k-algebra which is a skew field. Let $W \subset V \otimes_k K$ be a two-sided K-sub vector space. Then W is generated as a left K-vector space by $W \cap (V \otimes 1)$.*

Proof. Let $V' \subset V$ be the k-sub vector space generated by $v \in V$ such that $v \otimes 1 \in W$. Then $V' \otimes_k K \subset W$ and we have

$$W/V' \otimes_k K \subset V/V' \otimes_k K.$$

If $\overline{v} \in V/V'$ is a nonzero vector such that $\overline{v} \otimes 1$ is contained in $W/V' \otimes_k K$, then we see that $v \otimes 1 \in W$ where $v \in V$ lifts $\overline{v}$. This contradicts our construction of V'. Hence we may replace V by V/V' and W by $W/V' \otimes_k K$ and it suffices to prove that $W \cap (V \otimes 1)$ is nonzero if W is nonzero.

To see this let $w \in W$ be a nonzero element which can be written as $W = \sum_{i=1,\ldots,n} v_i \otimes k_i$ with n minimal. We may right multiply with k_1^{-1} and assume that $k_1 = 1$. If $n = 1$, then we win because $v_1 \otimes 1 \in W$. If $n > 1$, then we see that for any $c \in K$

$$cv - vc = \sum\nolimits_{i=2,\ldots,n} v_i \otimes (ck_i - k_i c) \in W$$

and hence $ck_i - k_i c = 0$ by minimality of n. This implies that k_i is in the center of K which is k by assumption. Hence $v = (v_1 + \sum k_i v_i) \otimes 1$ contradicting the minimality of n. □

Lemma 11.4.4. *Let A be a k-algebra. Let K be a central k-algebra which is a skew field. Then any two-sided ideal $I \subset A \otimes_k K$ is of the form $J \otimes_k K$ for some two-sided ideal $J \subset A$. In particular, if A is simple, then so is $A \otimes_k K$.*

Proof. Set $J = \{a \in A \mid a \otimes 1 \in I\}$. This is a two-sided ideal of A. And $I = J \otimes_k K$ by Lemma 11.4.3. □

Lemma 11.4.5. *Let R be a possibly noncommutative ring. Let $n \geq 1$ be an integer. Let $R_n = Mat(n \times n, R)$.*

1. *The functors $M \mapsto M^{\oplus n}$ and $N \mapsto Ne_{11}$ define quasi-inverse equivalences of categories $Mod_R \leftrightarrow Mod_{R_n}$.*
2. *A two-sided ideal of R_n is of the form IR_n for some two-sided ideal I of R.*
3. *The center of R_n is equal to the center of R.*

Proof. Part (1) proves itself. If $J \subset R_n$ is a two-sided ideal, then $J = \bigoplus e_{ii} J e_{jj}$ and all of the summands $e_{ii} J e_{jj}$ are equal to each other and are a two-sided ideal I of R. This proves (2). Part (3) is clear. □

Lemma 11.4.6. *Let A be a finite simple k-algebra.*

1. *There exists exactly one simple A-module M up to isomorphism.*
2. *Any finite A-module is a direct sum of copies of a simple module.*
3. *Two finite A-modules are isomorphic if and only if they have the same dimension over k.*
4. *If $A = Mat(n \times n, K)$ with K a finite skew field extension of k, then $M = K^{\oplus n}$ is a simple A-module and $End_A(M) = K^{op}$.*
5. *If M is a simple A-module, then $L = End_A(M)$ is a skew field finite over k acting on the left on M, we have $A = End_L(M)$, and the centers of A and L agree. Also $[A : k][L : k] = \dim_k(M)^2$.*

(6) *For a finite A-module N the algebra $B = End_A(N)$ is a matrix algebra over the skew field L of (3). Moreover $End_B(N) = A$.*

Proof. By Theorem 11.3.3 we can write $A = \mathrm{Mat}(n \times n, K)$ for some finite skew field extension K of k. By Lemma 11.4.5 the category of modules over A is equivalent to the category of modules over K. Thus (1), (2), and (3) hold because every module over K is free. Part (4) holds because the equivalence transforms the K-module K to $M = K^{\oplus n}$. Using $M = K^{\oplus n}$ in (5) we see that $L = K^{op}$. The statement about the center of $L = K^{op}$ follows from Lemma 11.4.5. The statement about $\mathrm{End}_L(M)$ follows from the explicit form of M. The formula of dimensions is clear. Part (6) follows as N is isomorphic to a direct sum of copies of a simple module. □

Lemma 11.4.7. *Let A, A' be two simple k-algebras one of which is finite and central over k. Then $A \otimes_k A'$ is simple.*

Proof. Suppose that A' is finite and central over k. Write $A' = \mathrm{Mat}(n \times n, K')$, see Theorem 11.3.3. Then the center of K' is k and we conclude that $A \otimes_k K'$ is simple by Lemma 11.4.4. Hence $A \otimes_k A' = \mathrm{Mat}(n \times n, A \otimes_k K')$ is simple by Lemma 11.4.5. □

Lemma 11.4.8. *The tensor product of finite central simple algebras over k is finite, central, and simple.*

Proof. Combine Lemmas 11.4.1 and 11.4.7. □

Lemma 11.4.9. *Let A be a finite central simple algebra over k. Let $k \subset k'$ be a field extension. Then $A' = A \otimes_k k'$ is a finite central simple algebra over k'.*

Proof. Combine Lemmas 11.4.1 and 11.4.7. □

Lemma 11.4.10. *Let A be a finite central simple algebra over k. Then $A \otimes_k A^{op} \cong Mat(n \times n, k)$ where $n = [A : k]$.*

Proof. By Lemma 11.4.8 the algebra $A \otimes_k A^{op}$ is simple. Hence the map

$$A \otimes_k A^{op} \longrightarrow \mathrm{End}_k(A), \quad a \otimes a' \longmapsto (x \mapsto axa')$$

is injective. Since both sides of the arrow have the same dimension we win. □

11.5. The Brauer group of a field

Let k be a field. Consider two finite central simple algebras A and B over k. We say A and B are *similar* if there exist $n, m > 0$ such that $\mathrm{Mat}(n \times n, A) \cong \mathrm{Mat}(m \times m, B)$ as k-algebras.

Lemma 11.5.1. *Similarity.*

(1) *Similarity defines an equivalence relation on the set of isomorphism classes of finite central simple algebras over k.*
(2) *Every similarity class contains a unique (up to isomorphism) finite central skew field extension of k.*
(3) *If $A = Mat(n \times n, K)$ and $B = Mat(m \times m, K')$ for some finite central skew fields K, K' over k then A and B are similar if and only if $K \cong K'$ as k-algebras.*

Proof. Note that by Wedderburn's theorem (Theorem 11.3.3) we can always write a finite central simple algebra as a matrix algebra over a finite central skew field. Hence it suffices to prove the third assertion. To see this it suffices to show that if $A = \mathrm{Mat}(n \times n, K) \cong \mathrm{Mat}(m \times m, K') = B$ then $K \cong K'$. To see this note that for a simple module M of A we have $\mathrm{End}_A(M) = K^{op}$, see Lemma 11.4.6. Hence $A \cong B$ implies $K^{op} \cong (K')^{op}$ and we win. □

Given two finite central simple k-algebras A, B the tensor product $A \otimes_k B$ is another, see Lemma 11.4.8. Moreover if A is similar to A', then $A \otimes_k B$ is similar to $A' \otimes_k B$ because tensor products and taking matrix algebras commute. Hence tensor product defines an operation on equivalence classes of finite central simple algebras which is clearly associative and commutative. Finally, Lemma 11.4.10 shows that $A \otimes_k A^{op}$ is isomorphic to a matrix algebra, i.e., that $A \otimes_k A^{op}$ is in the similarity class of k. Thus we obtain an abelian group.

Definition 11.5.2. Let k be a field. The *Brauer group* of k is the abelian group of similarity classes of finite central simple k-algebras defined above. Notation $\mathrm{Br}(k)$.

For any map of fields $k \to k'$ we obtain a group homomorphism

$$\mathrm{Br}(k) \longrightarrow \mathrm{Br}(k'), \quad A \longmapsto A \otimes_k k'$$

see Lemma 11.4.9. In other words, $\mathrm{Br}(-)$ is a functor from the category of fields to the category of abelian groups. Observe that the Brauer group of a field is zero if and only if every finite central skew field extension $k \subset K$ is trivial.

Lemma 11.5.3. *The Brauer group of an algebraically closed field is zero.*

Proof. Let $k \subset K$ be a finite central skew field extension. For any element $x \in K$ the subring $k[x] \subset K$ is a commutative finite integral k-sub algebra, hence a field, see Algebra, Lemma 10.33.17. Since k is algebraically closed we conclude that $k[x] = k$. Since x was arbitrary we conclude $k = K$. □

Lemma 11.5.4. *Let A be a finite central simple algebra over a field k. Then $[A : k]$ is a square.*

Proof. This is true because $A \otimes_k \overline{k}$ is a matrix algebra over $\overline{k}$ by Lemma 11.5.3. □

11.6. Skolem-Noether

Theorem 11.6.1. *Let A be a finite central simple k-algebra. Let B be a simple k-algebra. Let $f, g : B \to A$ be two k-algebra homomorphisms. Then there exists an invertible element $x \in A$ such that $f(b) = xg(b)x^{-1}$ for all $b \in B$.*

Proof. Choose a simple A-module M. Set $L = \mathrm{End}_A(M)$. Then L is a skew field with center k which acts on the left on M, see Lemmas 11.3.2 and 11.4.6. Then M has two $B \otimes_k L^{op}$-module structures defined by $m \cdot_1 (b \otimes l) = lmf(b)$ and $m \cdot_2 (b \otimes l) = lmg(b)$. Since $B \otimes_k L^{op}$ is a finite simple k-algebra by Lemma 11.4.7 we see that these module structures are isomorphic by Lemma 11.4.6. Hence we find $\varphi : M \to M$ intertwining these operations. In particular φ is in the commutant of L which implies that φ is multiplication by some $x \in A$, see Lemma 11.4.6. Working out the definitions we see that x is a solution to our problem. □

Lemma 11.6.2. *Let A be a finite simple k-algebra. Any automorphism of A is inner. In particular, any automorphism of $Mat(n \times n, k)$ is inner.*

Proof. Note that A is a finite central simple algebra over the center of A which is a finite field extension of k, see Lemma 11.4.2. Hence the Skolem-Noether theorem (Theorem 11.6.1) applies. □

11.7. The centralizer theorem

Theorem 11.7.1. *Let A be a finite central simple algebra over k, and let B be a simple subalgebra of A. Then*

(1) *the centralizer C of B in A is simple,*
(2) *$[A : k] = [B : k][C : k]$, and*
(3) *the centralizer of C in A is B.*

Proof. Throughout this proof we use the results of Lemma 11.4.6 freely. Choose a simple A-module M. Set $L = \mathrm{End}_A(M)$. Then L is a skew field with center k which acts on the left on M and $A = \mathrm{End}_L(M)$. Then M is a right $B \otimes_k L^{op}$-module and $C = \mathrm{End}_{B \otimes_k L^{op}}(M)$. Since the algebra $B \otimes_k L^{op}$ is simple by Lemma 11.4.7 we see that C is simple (by Lemma 11.4.6 again).

Write $B \otimes_k L^{op} = \mathrm{Mat}(m \times m, K)$ for some skew field K finite over k. Then $C = \mathrm{Mat}(n \times n, K^{op})$ if M is isomorphic to a direct sum of n copies of the simple $B \otimes_k L^{op}$-module $K^{\oplus m}$ (the lemma again). Thus we have $\dim_k(M) = nm[K : k]$, $[B : k][L : k] = m^2[K : k]$, $[C : k] = n^2[K : k]$, and $[A : k][L : k] = \dim_k(M)^2$ (by the lemma again). We conclude that (2) holds.

Part (3) follows because of (2) applied to $C \subset A$ shows that $[B : k] = [C' : k]$ where C' is the centralizer of C in A (and the obvious fact that $B \subset C'$). □

Lemma 11.7.2. *Let A be a finite central simple algebra over k, and let B be a simple subalgebra of A. If B is a central k-algebra, then $A = B \otimes_k C$ where C is the (central simple) centralizer of B in A.*

Proof. We have $\dim_k(A) = \dim_k(B \otimes_k C)$ by Theorem 11.7.1. By Lemma 11.4.7 the tensor product is simple. Hence the natural map $B \otimes_k C \to A$ is injective hence an isomorphism. □

Lemma 11.7.3. *Let A be a finite central simple algebra over k. If $K \subset A$ is a subfield, then the following are equivalent*

(1) *$[A : k] = [K : k]^2$,*
(2) *K is its own centralizer, and*
(3) *K is a maximal commutative subring.*

Proof. Theorem 11.7.1 shows that (1) and (2) are equivalent. It is clear that (3) and (2) are equivalent. □

Lemma 11.7.4. *Let A be a finite central skew field over k. Then every maximal subfield $K \subset A$ satisfies $[A : k] = [K : k]^2$.*

Proof. Special case of Lemma 11.7.3. □

11.8. Splitting fields

Definition 11.8.1. Let A be a finite central simple k-algebra. We say a field extension $k \subset k'$ *splits* A, or k' is a *splitting field* for A if $A \otimes_k k'$ is a matrix algebra over k'.

Another way to say this is that the class of A maps to zero under the map $\mathrm{Br}(k) \to \mathrm{Br}(k')$.

Theorem 11.8.2. *Let A be a finite central simple k-algebra. Let $k \subset k'$ be a finite field extension. The following are equivalent*

(1) *k' splits A, and*
(2) *there exists a finite central simple algebra B similar to A such that $k' \subset B$ and $[B : k] = [k' : k]^2$.*

Proof. Assume (2). It suffices to show that $B \otimes_k k'$ is a matrix algebra. We know that $B \otimes_k B^{op} \cong \mathrm{End}_k(B)$. Since k' is the centralizer of k' in B^{op} by Lemma 11.7.3 we see that $B \otimes_k k'$ is the centralizer of $k \otimes k'$ in $B \otimes_k B^{op} = \mathrm{End}_k(B)$. Of course this centralizer is just $\mathrm{End}_{k'}(B)$ where we view B as a k' vector space via the embedding $k' \to B$. Thus the result.

Assume (1). This means that we have an isomorphism $A \otimes_k k' \cong \mathrm{End}_{k'}(V)$ for some k'-vector space V. Let B be the commutant of A in $\mathrm{End}_k(V)$. Note that k' sits in B. By Lemma 11.7.2 the classes of A and B add up to zero in $\mathrm{Br}(k)$. From the dimension formula in Theorem 11.7.1 we see that

$$[B : k][A : k] = \dim_k(V)^2 = [k' : k]^2 \dim_{k'}(V)^2 = [k' : k]^2[A : k].$$

Hence $[B : k] = [k' : k]^2$. Thus we have proved the result for the opposite to the Brauer class of A. However, k' splits the Brauer class of A if and only if it splits the Brauer class of the opposite algebra, so we win anyway. □

Lemma 11.8.3. *A maximal subfield of a finite central skew field K over k is a splitting field for K.*

Proof. Combine Lemma 11.7.4 with Theorem 11.8.2. □

Lemma 11.8.4. *Consider a finite central skew field K over k. Let $d^2 = [K : k]$. For any finite splitting field k' for K the degree $[k' : k]$ is divisible by d.*

Proof. By Theorem 11.8.2 there exists a finite central simple algebra B in the Brauer class of K such that $[B : k] = [k' : k]^2$. By Lemma 11.5.1 we see that $B = \mathrm{Mat}(n \times n, K)$ for some n. Then $[k' : k]^2 = n^2 d^2$ whence the result. □

Proposition 11.8.5. *Consider a finite central skew field K over k. There exists a maximal subfield $k \subset k' \subset K$ which is separable over k. In particular, every Brauer class has a finite separable spitting field.*

Proof. Since every Brauer class is represented by a finite central skew field over k, we see that the second statement follows from the first by Lemma 11.8.3.

To prove the first statement, suppose that we are given a separable subfield $k' \subset K$. Then the centralizer K' of k' in K has center k', and the problem reduces to finding a maximal subfield of K' separable over k'. Thus it suffices to prove, if $k \neq K$, that we can find an element $x \in K$, $x \notin k$ which is separable over k. This statement is clear in characteristic zero. Hence we may assume that k has characteristic $p > 0$. If the ground field k is finite then, the result is clear as well (because extensions of finite fields are always separable). Thus we may assume that k is an infinite field of positive characteristic.

To get a contradiction assume no element of K is separable over k. By the discussion in Fields, Section 9.25 this means the minimal polynomial of any $x \in K$ is of the

form $T^q - a$ where q is a power of p and $a \in k$. Since it is clear that every element of K has a minimal polynomial of degree $\leq \dim_k(K)$ we conclude that there exists a fixed p-power q such that $x^q \in k$ for all $x \in K$.

Consider the map

$$(-)^q : K \longrightarrow K$$

and write it out in terms of a k-basis $\{a_1, \ldots, a_n\}$ of K with $a_1 = 1$. So

$$(\sum x_i a_i)^q = \sum f_i(x_1, \ldots, x_n) a_i.$$

Since multiplication on A is k-bilinear we see that each f_i is a polynomial in $x_1, \ldots, x_n$ (details omitted). The choice of q above and the fact that k is infinite shows that f_i is identically zero for $i \geq 2$. Hence we see that it remains zero on extending k to its algebraic closure $\overline{k}$. But the algebra $A \otimes_k \overline{k}$ is a matrix algebra, which implies there are some elements whose qth power is not central (e.g., e_{11}). This is the desired contradiction. □

The results above allow us to characterize finite central simple algebras as follows.

Lemma 11.8.6. *Let k be a field. For a k-algebra A the following are equivalent*

1. *A is finite central simple k-algebra,*
2. *A is a finite dimensional k-vector space, k is the center of A, and A has no nontrivial two-sided ideal,*
3. *there exists $d \geq 1$ such that $A \otimes_k \bar{k} \cong Mat(d \times d, \bar{k})$,*
4. *there exists $d \geq 1$ such that $A \otimes_k k^{sep} \cong Mat(d \times d, k^{sep})$,*
5. *there exist $d \geq 1$ and a finite Galois extension $k \subset k'$ such that $A \otimes_{k'} k' \cong Mat(d \times d, k')$,*
6. *there exist $n \geq 1$ and a finite central skew field K over k such that $A \cong Mat(n \times n, K)$.*

The integer d is called the degree of A.

Proof. The equivalence of (1) and (2) is a consequence of the definitions, see Section 11.2. Assume (1). By Proposition 11.8.5 there exists a separable splitting field $k \subset k'$ for A. Of course, then a Galois closure of k'/k is a splitting field also. Thus we see that (1) implies (5). It is clear that (5) $\Rightarrow$ (4) $\Rightarrow$ (3). Assume (3). Then $A \otimes_k \overline{k}$ is a finite central simple $\overline{k}$-algebra for example by Lemma 11.4.5. This trivially implies that A is a finite central simple k-algebra. Finally, the equivalence of (1) and (6) is Wedderburn's theorem, see Theorem 11.3.3. □

11.9. Other chapters

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CHAPTER 12

Homological Algebra

12.1. Introduction

Basic homological algebra will be explained in this document. We add as needed in the other parts, since there is clearly an infinite amount of this stuff around. A reference is [**ML63**].

12.2. Basic notions

The following notions are considered basic and will not be defined, and or proved. This does not mean they are all necessarily easy or well known.

(1) Nothing yet.

12.3. Preadditive and additive categories

Here is the definition of a preadditive category.

Definition 12.3.1. A category $\mathcal{A}$ is called *preadditive* if each morphism set $\mathrm{Mor}_{\mathcal{A}}(x, y)$ is endowed with the structure of an abelian group such that the compositions

$$\mathrm{Mor}(x, y) \times \mathrm{Mor}(y, z) \longrightarrow \mathrm{Mor}(x, z)$$

are bilinear. A functor $F : \mathcal{A} \to \mathcal{B}$ of preadditive categories is called *additive* if and only if $F : \mathrm{Mor}(x, y) \to \mathrm{Mor}(F(x), F(y))$ is a homomorphism of abelian groups for all $x, y \in \mathrm{Ob}(\mathcal{A})$.

In particular for every x, y there exists at least one morphism $x \to y$, namely the zero map.

Lemma 12.3.2. *Let $\mathcal{A}$ be a preadditive category. Let x be an object of $\mathcal{A}$. The following are equivalent*

(1) *x is an initial object,*
(2) *x is a final object, and*
(3) *$id_x = 0$ in $\mathrm{Mor}_{\mathcal{A}}(x, x)$.*

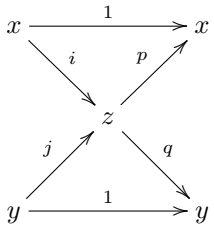
Furthermore, if such an object 0 exists, then a morphism $\alpha : x \to y$ factors through 0 if and only if $\alpha = 0$.

Proof. Omitted. □

Definition 12.3.3. In a preadditive category $\mathcal{A}$ we call *zero object*, and we denote it 0 any final and initial object as in Lemma 12.3.2 above.

Lemma 12.3.4. *Let $\mathcal{A}$ be a preadditive category. Let $x, y \in \mathrm{Ob}(\mathcal{A})$. If the product $x \times y$ exists, then so does the coproduct $x \coprod y$. If the coproduct $x \coprod y$ exists, then so does the product $x \times y$. In this case also $x \coprod y \cong x \times y$.*

Proof. Suppose that $z = x \times y$ with projections $p : z \to x$ and $q : z \to y$. Denote $i : x \to z$ the morphism corresponding to $(1, 0)$. Denote $j : y \to z$ the morphism corresponding to $(0, 1)$. Thus we have the commutative diagram



where the diagonal compositions are zero. It follows that $i \circ p + j \circ q : z \to z$ is the identity since it is a morphism which upon composing with p gives p and upon composing with q gives q. Suppose given morphisms $a : x \to w$ and $b : y \to w$. Then we can form the map $a \circ p + b \circ q : z \to w$. In this way we get a bijection $\mathrm{Mor}(z, w) = \mathrm{Mor}(x, w) \times \mathrm{Mor}(y, w)$ which show that $z = x \coprod y$.

We leave it to the reader to construct the morphisms p, q given a coproduct $x \coprod y$ instead of a product. □

Definition 12.3.5. Given a pair of objects x, y in a preadditive category $\mathcal{A}$ we call *direct sum*, and we denote it $x \oplus y$ the product $x \times y$ endowed with the morphisms i, j, p, q as in Lemma 12.3.4 above.

Remark 12.3.6. Note that the proof of Lemma 12.3.4 shows that given p and q the morphisms i, j are uniquely determined by the rules $p \circ i = \mathrm{id}_x$, $q \circ j = \mathrm{id}_y$, $p \circ j = 0$, $q \circ i = 0$. Moreover, we automatically have $i \circ p + j \circ q = \mathrm{id}_{x \oplus y}$. Similarly, given i, j the morphisms p and q are uniquely determined. Finally, given objects x, y, z and morphisms $i : x \to z$, $j : y \to z$, $p : z \to x$ and $q : z \to y$ such that $p \circ i = \mathrm{id}_x$, $q \circ j = \mathrm{id}_y$, $p \circ j = 0$, $q \circ i = 0$ and $i \circ p + j \circ q = \mathrm{id}_z$, then z is the direct sum of x and y with the four morphisms equal to i, j, p, q.

Lemma 12.3.7. *Let $\mathcal{A}$, $\mathcal{B}$ be preadditive categories. Let $F : \mathcal{A} \to \mathcal{B}$ be an additive functor. Then F transforms direct sums to direct sums and zero to zero.*

Proof. Suppose F is additive. A direct sum z of x and y is characterized by having morphisms $i : x \to z$, $j : y \to z$, $p : z \to x$ and $q : z \to y$ such that $p \circ i = \mathrm{id}_x$, $q \circ j = \mathrm{id}_y$, $p \circ j = 0$, $q \circ i = 0$ and $i \circ p + j \circ q = \mathrm{id}_z$, according to Remark 12.3.6. Clearly $F(x), F(y), F(z)$ and the morphisms $F(i), F(j), F(p), F(q)$ satisfy exactly the same relations (by additivity) and we see that $F(z)$ is a direct sum of $F(x)$ and $F(y)$. □

Definition 12.3.8. A category $\mathcal{A}$ is called *additive* if it is preadditive and finite products exist, in other words it has a zero object and direct sums.

Namely the empty product is a finite product and if it exists, then it is a final object.

Definition 12.3.9. Let $\mathcal{A}$ be a preadditive category. Let $f : x \to y$ be a morphism.

(1) A *kernel* of f is a morphism $i : z \to x$ such that (a) $f \circ i = 0$ and (b) for any $i' : z' \to x$ such that $f \circ i' = 0$ there exists a unique morphism $g : z' \to z$ such that $i' = i \circ g$.
(2) If the kernel of f exists, then we denote this $\mathrm{Ker}(f) \to x$.

(3) A *cokernel* of f is a morphism $p : y \to z$ such that (a) $p \circ f = 0$ and (b) for any $p' : y \to z'$ such that $p' \circ f = 0$ there exists a unique morphism $g : z \to z'$ such that $p' = g \circ p$.
(4) If a cokernel of f exists we denote this $y \to \text{Coker}(f)$.
(5) If a kernel of f exists, then a *coimage of* f is a cokernel for the morphism $\text{Ker}(f) \to x$.
(6) If a kernel and coimage exist then we denote this $x \to \text{Coim}(f)$.
(7) If a cokernel of f exists, then the *image of* f is a kernel of the morphism $y \to \text{Coker}(f)$.
(8) If a cokernel and image of f exist then we denote this $\text{Im}(f) \to y$.

We first relate the direct sum to kernels as follows.

Lemma 12.3.10. *Let $\mathcal{C}$ be a preadditive category. Let $x \oplus y$ with morphisms i, j, p, q as in Lemma 12.3.4 be a direct sum in $\mathcal{C}$. Then $i : x \to x \oplus y$ is a kernel of $q : x \oplus y \to y$. Dually, p is a cokernel for j.*

Proof. Let $f : z \to x \oplus y$ be a morphism such that $q \circ f = 0$. We have to show that there exists a unique morphism $g : z \to x$ such that $f = i \circ g$. Since $i \circ p + j \circ q$ is the identity on $x \oplus y$ we see that

$$f = (i \circ p + j \circ q) \circ f = i \circ p \circ f$$

and hence $g = p \circ f$ works. Uniquess holds because $p \circ i$ is the identity on x. The proof of the second statement is dual. □

Lemma 12.3.11. *Let $f : x \to y$ be a morphism in a preadditive category such that the kernel, cokernel, image and coimage all exist. Then f can be factored uniquely as $x \to Coim(f) \to Im(f) \to y$.*

Proof. There is a canonical morphism $\text{Coim}(f) \to y$ because $\text{Ker}(f) \to x \to y$ is zero. The composition $\text{Coim}(f) \to y \to \text{Coker}(f)$ is zero, because it is the unique morphism which gives rise to the morphism $x \to y \to \text{Coker}(f)$ which is zero. Hence $\text{Coim}(f) \to y$ factors uniquely through $\text{Im}(f) \to y$, which gives us the desired map. □

Example 12.3.12. Let k be a field. Consider the category of filtered vector spaces over k. (See Definition 12.16.1.) Consider the filtered vector spaces (V, F) and (W, F) with $V = W = k$ and

$$F^i V = \begin{cases} V & \text{if} \quad i < 0 \\ 0 & \text{if} \quad i \geq 0 \end{cases} \text{ and } F^i W = \begin{cases} W & \text{if} \quad i \leq 0 \\ 0 & \text{if} \quad i > 0 \end{cases}$$

The map $f : V \to W$ corresponding to id_k on the underlying vector spaces has trivial kernel and cokernel but is not an isomorphism. Note also that $\text{Coim}(f) = V$ and $\text{Im}(f) = W$. This means that the category of filtered vector spaces over k is not abelian.

12.4. Karoubian categories

Skip this section on a first reading.

Definition 12.4.1. Let $\mathcal{C}$ be a preadditive category. We say $\mathcal{C}$ is *Karoubian* if every idempotent endomorphism of an object of $\mathcal{C}$ has a kernel.

The dual notion would be that every idempotent endomorphism of an object has a cokernel. However, in view of the (dual of the) following lemma that would be an equivalent notion.

Lemma 12.4.2. *Let $\mathcal{C}$ be a preadditive category. The following are equivalent*

(1) *$\mathcal{C}$ is Karoubian,*
(2) *every idempotent endomorphism of an object of $\mathcal{C}$ has a cokernel, and*
(3) *given an idempotent endomorphism $p : z \to z$ of $\mathcal{C}$ there exists a direct sum decomposition $z = x \oplus y$ such that p corresponds to the projection onto y.*

Proof. Assume (1) and let $p : z \to z$ be as in (3). Let $x = \mathrm{Ker}(p)$ and $y = \mathrm{Ker}(1 - p)$. There are maps $x \to z$ and $y \to z$. Since $(1 - p)p = 0$ we see that $p : z \to z$ factors through y, hence we obtain a morphism $z \to y$. Similarly we obtain a morphism $z \to x$. We omit the verification that these four morphisms induce an isomorphsm $x = y \oplus z$ as in Remark 12.3.6. Thus (1) $\Rightarrow$ (3). The implication (2) $\Rightarrow$ (3) is dual. Finally, condition (3) implies (1) and (2) by Lemma 12.3.10. □

Lemma 12.4.3. *Let $\mathcal{D}$ be a preadditive category.*

(1) *If $\mathcal{D}$ has countable products and kernels of maps which have a right inverse, then $\mathcal{D}$ is Karoubian.*
(2) *If $\mathcal{D}$ has countable coproducts and cokernels of maps which have a left inverse, then $\mathcal{D}$ is Karoubian.*

Proof. Let X be an object of $\mathcal{D}$ and let $e : X \to X$ be an idempotent. The functor

$$W \longmapsto \mathrm{Ker}(\mathrm{Mor}_{\mathcal{D}}(W, X) \xrightarrow{e} \mathrm{Mor}_{\mathcal{D}}(W, X))$$

if representable if and only if e has a kernel. Note that for any abelian group A and idempotent endomorphism $e : A \to A$ we have

$$\mathrm{Ker}(e : A \to A) = \mathrm{Ker}(\Phi : \prod\nolimits_{n \in \mathbf{N}} A \to \prod\nolimits_{n \in \mathbf{N}} A)$$

where

$$\Phi(a_1, a_2, a_3, \ldots) = (ea_1 + (1 - e)a_2, ea_2 + (1 - e)a_3, \ldots)$$

Moreover, Φ has the right inverse

$$\Psi(a_1, a_2, a_3, \ldots) = (a_1, (1 - e)a_1 + ea_2, (1 - e)a_2 + ea_3, \ldots).$$

Hence (1) holds. The proof of (2) is dual (using the dual definition of a Karoubian category, namely condition (2) of Lemma 12.4.2). □

12.5. Abelian categories

An abelian category is a category satisfying just enough axioms so the snake lemma holds. An axiom (that is sometimes forgotten) is that the canonical map $\mathrm{Coim}(f) \to \mathrm{Im}(f)$ of Lemma 12.3.11 is always an isomorphism. Example 12.3.12 shows that it is necessary.

Definition 12.5.1. A category $\mathcal{A}$ is *abelian* if it is additive, if all kernels and cokernels exist, and if the natural map $\mathrm{Coim}(f) \to \mathrm{Im}(f)$ is an isomorphism for all morphisms f of $\mathcal{A}$.

Lemma 12.5.2. *Let $\mathcal{A}$ be a preadditive category. The additions on sets of morphisms make $\mathcal{A}^{opp}$ into a preadditive category. Furthermore, $\mathcal{A}$ is additive if and only if $\mathcal{A}^{opp}$ is additive, and $\mathcal{A}$ is abelian if and only if $\mathcal{A}^{opp}$ is abelian.*

Proof. Omitted. □

Definition 12.5.3. Let $f : x \to y$ be a morphism in an abelian category.

(1) We say f is *injective* if $\mathrm{Ker}(f) = 0$.
(2) We say f is *surjective* if $\mathrm{Coker}(f) = 0$.

If $x \to y$ is injective, then we say that x is a *subobject* of y and we use the notation $x \subset y$. If $x \to y$ is surjective, then we say that y is a *quotient* of x.

Lemma 12.5.4. *Let $f : x \to y$ be a morphism in an abelian category. Then*

(1) *f is injective if and only if f is a monomorphism, and*
(2) *f is surjective if and only if f is an epimorphism.*

Proof. Omitted. □

In an abelian category, if $x \subset y$ is a subobject, then we denote

$$x/y = \mathrm{Coker}(x \to y).$$

Lemma 12.5.5. *Let $\mathcal{A}$ be an abelian category. All finite limits and finite colimits exist in $\mathcal{A}$.*

Proof. To show that finite limits exist it suffices to show that finite products and equalizers exist, see Categories, Lemma 4.18.4. Finite products exist by definition and the equalizer of $a, b : x \to y$ is the kernel of $a - b$. The argument for finite colimits is similar but dual to this. □

Example 12.5.6. Let $\mathcal{A}$ be an abelian category. Pushouts and fibre products in $\mathcal{A}$ have the following simple descriptions:

(1) If $a : x \to y$, $b : z \to y$ are morphisms in $\mathcal{A}$, then we have the fibre product: $x \times_y z = \mathrm{Ker}((a, -b) : x \oplus z \to y)$.
(2) If $a : y \to x$, $b : y \to z$ are morphisms in $\mathcal{A}$, then we have the pushout: $x \amalg_y z = \mathrm{Coker}((a, -b) : y \to x \oplus z)$.

Definition 12.5.7. Let $\mathcal{A}$ be an additive category. We say a sequence of morphisms

$$\ldots \to x \to y \to z \to \ldots$$

in $\mathcal{A}$ is a *complex* if the composition of any two (drawn) arrows is zero. If $\mathcal{A}$ is abelian then we say a sequence as above is *exact at y* if $\mathrm{Im}(x \to y) = \mathrm{Ker}(y \to z)$. We say it is *exact* if it is exact at every object. A *short exact sequence* is an exact complex of the form

$$0 \to A \to B \to C \to 0.$$

In the following lemma we assume the reader knows what it means for a sequence of abelian groups to be exact.

Lemma 12.5.8. *Let $\mathcal{A}$ be an abelian category. Let $0 \to M_1 \to M_2 \to M_3 \to 0$ be a complex of $\mathcal{A}$.*

(1) *$M_1 \to M_2 \to M_3 \to 0$ is exact if and only if*

$$0 \to \mathrm{Hom}_{\mathcal{A}}(M_3, N) \to \mathrm{Hom}_{\mathcal{A}}(M_2, N) \to \mathrm{Hom}_{\mathcal{A}}(M_1, N)$$

is an exact sequence of abelian groups for all objects N of $\mathcal{A}$, and

(2) $0 \to M_1 \to M_2 \to M_3$ *is exact if and only if*

$$0 \to \mathrm{Hom}_{\mathcal{A}}(N, M_1) \to \mathrm{Hom}_{\mathcal{A}}(N, M_2) \to \mathrm{Hom}_{\mathcal{A}}(N, M_1)$$

is an exact sequence of abelian groups for all objects N *of* $\mathcal{A}$.

Proof. Omitted. Hint: See Algebra, Lemma 10.10.1. □

Definition 12.5.9. Let $\mathcal{A}$ be an abelian category. Let $i : A \to B$ and $q : B \to C$ be morphisms of $\mathcal{A}$ such that $0 \to A \to B \to C \to 0$ is a short exact sequence. We say the short exact sequence is *split* if there exist morphisms $j : C \to B$ and $p : B \to A$ such that (B, i, j, p, q) is the direct sum of A and C.

Lemma 12.5.10. *Let* $\mathcal{A}$ *be an abelian category. Let* $0 \to A \to B \to C \to 0$ *be a short exact sequence.*

(1) *Given a morphism* $s : C \to B$ *left inverse to* $B \to C$*, there exists a unique* $\pi : B \to A$ *such that* (s, π) *splits the short exact sequence as in Definition 12.5.9.*
(2) *Given a morphism* $\pi : B \to A$ *right inverse to* $A \to B$*, there exists a unique* $s : C \to B$ *such that* (s, π) *splits the short exact sequence as in Definition 12.5.9.*

Proof. Omitted. □

Lemma 12.5.11. *Let* $\mathcal{A}$ *be an abelian category. Let*

$$\begin{array}{ccc} w & \xrightarrow{f} & y \\ {\scriptstyle g}\downarrow & & \downarrow{\scriptstyle h} \\ x & \xrightarrow{k} & z \end{array}$$

be a commutative diagram.

(1) *The diagram is cartesian if and only if*

$$0 \to w \xrightarrow{(g,f)} x \oplus y \xrightarrow{(k,-h)} z$$

is exact.
(2) *The diagram is cocartesian if and only if*

$$w \xrightarrow{(g,-f)} x \oplus y \xrightarrow{(k,h)} z \to 0$$

is exact.

Proof. Let $u = (g, f) : w \to x \oplus y$ and $v = (k, -h) : x \oplus y \to z$. Let $p : x \oplus y \to x$ and $q : x \oplus y \to y$ be the canonical projections. Let $i : \mathrm{Ker}(v) \to x \oplus y$ be the canonical injection. By Example 12.5.6, the diagram is cartesian if and only if there exists an isomorphism $r : \mathrm{Ker}(v) \to w$ with $f \circ r = q \circ i$ and $g \circ r = p \circ i$. The sequence $0 \to w \xrightarrow{u} x \oplus y \xrightarrow{v} z$ is exact if and only if there exists an isomorphism $r : \mathrm{Ker}(v) \to w$ with $u \circ r = i$. But given $r : \mathrm{Ker}(v) \to w$, we have $f \circ r = q \circ i$ and $g \circ r = p \circ i$ if and only if $q \circ u \circ r = f \circ r = q \circ i$ and $p \circ u \circ r = g \circ r = p \circ i$, hence if and only if $u \circ r = i$. This proves (1), and then (2) follows by duality. □

Lemma 12.5.12. *Let $\mathcal{A}$ be an abelian category. Let*

$$\begin{array}{ccc} w & \xrightarrow{f} & y \\ {\scriptstyle g}\downarrow & & \downarrow{\scriptstyle h} \\ x & \xrightarrow{k} & z \end{array}$$

be a commutative diagram.

(1) *If the diagram is cartesian, then the morphism $Ker(f) \to Ker(k)$ induced by g is an isomorphism.*
(2) *If the diagram is cocartesian, then the morphism $Coker(f) \to Coker(k)$ induced by h is an isomorphism.*

Proof. Suppose the diagram is cartesian. Let $e : \mathrm{Ker}(f) \to \mathrm{Ker}(k)$ be induced by g. Let $i : \mathrm{Ker}(f) \to w$ and $j : \mathrm{Ker}(k) \to x$ be the canonical injections. There exists $t : \mathrm{Ker}(k) \to w$ with $f \circ t = 0$ and $g \circ t = j$. Hence, there exists $u : \mathrm{Ker}(k) \to \mathrm{Ker}(f)$ with $i \circ u = t$. It follows $g \circ i \circ u \circ e = g \circ t \circ e = j \circ e = g \circ i$ and $f \circ i \circ u \circ e = 0 = f \circ i$, hence $i \circ u \circ e = i$. Since i is a monomorphism this implies $u \circ e = \mathrm{id}_{\mathrm{Ker}(f)}$. Furthermore, we have $j \circ e \circ u = g \circ i \circ u = g \circ t = j$. Since j is a monomorphism this implies $e \circ u = \mathrm{id}_{\mathrm{Ker}(k)}$. This proves (1). Now, (2) follows by duality. □

Lemma 12.5.13. *Let $\mathcal{A}$ be an abelian category. Let*

$$\begin{array}{ccc} w & \xrightarrow{f} & y \\ {\scriptstyle g}\downarrow & & \downarrow{\scriptstyle h} \\ x & \xrightarrow{k} & z \end{array}$$

be a commutative diagram.

(1) *If the diagram is cartesian and k is an epimorphism, then the diagram is cocartesian and f is an epimorphism.*
(2) *If the diagram is cocartesian and g is a monomorphism, then the diagram is cartesian and h is a monomorphism.*

Proof. Suppose the diagram is cartesian and k is an epimorphism. Let $u = (g, -f) : w \to x \oplus y$ and let $v = (k, -h) : x \oplus y \to z$. As k is an epimorphism, v is an epimorphism, too. Therefore and by Lemma 12.5.11, the sequence $0 \to w \xrightarrow{u} x \oplus y \xrightarrow{v} z \to 0$ is exact. Thus, the diagram is cocartesian by Lemma 12.5.11. Finally, f is an epimorphism by Lemma 12.5.12 and Lemma 12.5.4. This proves (1), and (2) follows by duality. □

Lemma 12.5.14. *Let $\mathcal{A}$ be an abelian category.*

(1) *If $x \to y$ is surjective, then for every $z \to y$ the projection $x \times_y z \to z$ is surjective.*
(2) *If $x \to y$ is injective, then for every $x \to z$ the morphism $z \to z \amalg_x y$ is injective.*

Proof. Immediately from Lemma 12.5.4 and Lemma 12.5.13. □

Lemma 12.5.15. *Let $\mathcal{A}$ be an abelian category. Let $f : x \to y$ and $g : y \to z$ be morphisms with $g \circ f = 0$. Then, the following statements are equivalent:*

(1) *The sequence $x \xrightarrow{f} y \xrightarrow{g} z$ is exact.*

(2) *For every $h : w \to y$ with $g \circ h = 0$ there exist an object v, an epimorphism $k : v \to w$ and a morphism $l : v \to x$ with $h \circ k = f \circ l$.*

Proof. Let $i : \mathrm{Ker}(g) \to y$ be the canonical injection. Let $p : x \to \mathrm{Coim}(f)$ be the canonical projection. Let $j : \mathrm{Im}(f) \to \mathrm{Ker}(g)$ be the canonical injection.

Suppose (1) holds. Let $h : w \to y$ with $g \circ h = 0$. There exists $c : w \to \mathrm{Ker}(g)$ with $i \circ c = h$. Let $v = x \times_{\mathrm{Ker}(g)} w$ with canonical projections $k : v \to w$ and $l : v \to x$, so that $c \circ k = p \circ l$. Then, $h \circ k = i \circ c \circ k = i \circ j \circ p \circ l = f \circ l$. As $j \circ p$ is an epimorphism by hypothesis, k is an epimorphism by Lemma 12.5.13. This implies (2).

Suppose (2) holds. Then, $g \circ i = 0$. So, there are an object w, an epimorphism $k : w \to \mathrm{Ker}(g)$ and a morphism $l : w \to x$ with $f \circ l = i \circ k$. It follows $i \circ j \circ p \circ l = f \circ l = i \circ k$. Since i is a monomorphism we see that $j \circ p \circ l = k$ is an epimorphism. So, j is an epimorphisms and thus an isomorphism. This implies (1). □

Lemma 12.5.16. *Let $\mathcal{A}$ be an abelian category. Let*

$$\begin{array}{ccccc} x & \xrightarrow{f} & y & \xrightarrow{g} & z \\ \downarrow{\scriptstyle \alpha} & & \downarrow{\scriptstyle \beta} & & \downarrow{\scriptstyle \gamma} \\ u & \xrightarrow{k} & v & \xrightarrow{l} & w \end{array}$$

be a commutative diagram.

(1) *If the first row is exact and k is a monomorphism, then the induced sequence $Ker(\alpha) \to Ker(\beta) \to Ker(\gamma)$ is exact.*
(2) *If the second row is exact and g is an epimorphism, then the induced sequence $Coker(\alpha) \to Coker(\beta) \to Coker(\gamma)$ is exact.*

Proof. Suppose the first row is exact and k is a monomorphism. Let $a : \mathrm{Ker}(\alpha) \to \mathrm{Ker}(\beta)$ and $b : \mathrm{Ker}(\beta) \to \mathrm{Ker}(\gamma)$ be the induced morphisms. Let $h : \mathrm{Ker}(\alpha) \to x$, $i : \mathrm{Ker}(\beta) \to y$ and $j : \mathrm{Ker}(\gamma) \to z$ be the canonical injections. As j is a monomorphism we have $b \circ a = 0$. Let $c : s \to \mathrm{Ker}(\beta)$ with $b \circ c = 0$. Then, $g \circ i \circ c = j \circ b \circ c = 0$. By Lemma 12.5.15 there are an object t, an epimorphism $d : t \to s$ and a morphism $e : t \to x$ with $i \circ c \circ d = f \circ e$. Then, $k \circ \alpha \circ e = \beta \circ f \circ e = \beta \circ i \circ c \circ d = 0$. As k is a monomorphism we get $\alpha \circ e = 0$. So, there exists $m : t \to \mathrm{Ker}(\alpha)$ with $h \circ m = e$. It follows $i \circ a \circ m = f \circ h \circ m = f \circ e = i \circ c \circ d$. As i is a monomorphism we get $a \circ m = c \circ d$. Thus, Lemma 12.5.15 implies (1), and then (2) follows by duality. □

Lemma 12.5.17. *Let $\mathcal{A}$ be an abelian category. Let*

$$\begin{array}{ccccccc} & & x & \xrightarrow{f} & y & \xrightarrow{g} & z & \longrightarrow 0 \\ & & \downarrow{\scriptstyle \alpha} & & \downarrow{\scriptstyle \beta} & & \downarrow{\scriptstyle \gamma} & \\ 0 & \longrightarrow & u & \xrightarrow{k} & v & \xrightarrow{l} & w & \end{array}$$

be a commutative diagram with exact rows.

(1) *There exists a unique morphism* $\delta : Ker(\gamma) \to Coker(\alpha)$ *such that the diagram*

$$\begin{array}{ccccc} y & \xleftarrow{\pi'} & y \times_z Ker(\gamma) & \xrightarrow{\pi} & Ker(\gamma) \\ {\scriptstyle\beta}\downarrow & & & & \downarrow{\scriptstyle\delta} \\ v & \xrightarrow{\iota'} & Coker(\alpha)\amalg_u v & \xleftarrow{\iota} & Coker(\alpha) \end{array}$$

commutes, where π *and* π' *are the canonical projections and* ι *and* ι' *are the canonical coprojections.*

(2) *The induced sequence*

$$Ker(\alpha) \xrightarrow{f'} Ker(\beta) \xrightarrow{g'} Ker(\gamma) \xrightarrow{\delta} Coker(\alpha) \xrightarrow{k'} Coker(\beta) \xrightarrow{l'} Coker(\gamma)$$

is exact. If f *is injective then so is* f'*, and if* l *is surjective then so is* l'*.*

Proof. As π is an epimorphism and ι is a monomorphism by Lemma 12.5.13, uniqueness of δ is clear. Let $p = y \times_z \mathrm{Ker}(\gamma)$ and $q = \mathrm{Coker}(\alpha) \amalg_u v$. Let $h : \mathrm{Ker}(\beta) \to y$, $i : \mathrm{Ker}(\gamma) \to z$ and $j : \mathrm{Ker}(\pi) \to p$ be the canonical injections. Let $p : u \to \mathrm{Coker}(\alpha)$ be the canonical projection. Keeping in mind Lemma 12.5.13 we get a commutative diagram with exact rows

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \mathrm{Ker}(\pi) & \xrightarrow{j} & p & \xrightarrow{\pi} & \mathrm{Ker}(\gamma) & \longrightarrow & 0 \\ & & & & {\scriptstyle\pi'}\downarrow & & {\scriptstyle i}\downarrow & & \\ & & x & \xrightarrow{f} & y & \xrightarrow{g} & z & \longrightarrow & 0 \\ & & {\scriptstyle\alpha}\downarrow & & {\scriptstyle\beta}\downarrow & & {\scriptstyle\gamma}\downarrow & & \\ 0 & \longrightarrow & u & \xrightarrow{k} & v & \xrightarrow{l} & w & & \\ & & {\scriptstyle p}\downarrow & & {\scriptstyle\iota'}\downarrow & & & & \\ 0 & \longrightarrow & \mathrm{Coker}(\alpha) & \xrightarrow{\iota} & q & & & & \end{array}$$

As $l \circ \beta \circ \pi' = \gamma \circ i \circ \pi = 0$ and as the third row of the diagram above is exact, there is an $a : p \to u$ with $k \circ a = \beta \circ \pi'$. As the upper right quadrangle of the diagram above is cartesian, Lemma 12.5.12 yields an epimorphism $b : x \to \mathrm{Ker}(\pi)$ with $\pi' \circ j \circ b = f$. It follows $k \circ a \circ j \circ b = \beta \circ \pi' \circ j \circ b = \beta \circ f = k \circ \alpha$. As k is a monomorphism this implies $a \circ j \circ b = \alpha$. It follows $p \circ a \circ j \circ b = p \circ \alpha = 0$. As b is an epimorphism this implies $p \circ a \circ j = 0$. Therefore, as the top row of the diagram above is exact, there exists $\delta : \mathrm{Ker}(\gamma) \to \mathrm{Coker}(\alpha)$ with $\delta \circ \pi = p \circ a$. It follows $\iota \circ \delta \circ \pi = \iota \circ p \circ a = \iota' \circ k \circ a = \iota' \circ \beta \circ \pi'$ as desired.

As the upper right quadrangle in the diagram above is cartesian there is a $c : \mathrm{Ker}(\beta) \to p$ with $\pi' \circ c = h$ and $\pi \circ c = g'$. It follows $\iota \circ \delta \circ g' = \iota \circ \delta \circ \pi \circ c = \iota' \circ \beta \circ \pi' \circ c = \iota' \circ \beta \circ h = 0$. As ι is a monomorphism this implies $\delta \circ g' = 0$.

Next, let $d : r \to \mathrm{Ker}(\gamma)$ with $\delta \circ d = 0$. Applying Lemma 12.5.15 to the exact sequence $p \xrightarrow{\pi} \mathrm{Ker}(\gamma) \to 0$ and d yields an object s, an epimorphism $m : s \to r$ and a morphism $n : s \to p$ with $\pi \circ n = d \circ m$. As $p \circ a \circ n = \delta \circ d \circ m = 0$, applying Lemma 12.5.15 to the exact sequence $x \xrightarrow{\alpha} u \xrightarrow{p} \mathrm{Coker}(\alpha)$ and $a \circ n$ yields an object t, an epimorphism $\varepsilon : t \to s$ and a morphism $\zeta : t \to x$ with $a \circ n \circ \varepsilon = \alpha \circ \zeta$. It

holds $\beta \circ \pi' \circ n \circ \varepsilon = k \circ \alpha \circ \zeta = \beta \circ f \circ \zeta$. Let $\eta = \pi' \circ n \circ \varepsilon - f \circ \zeta : t \to y$. Then, $\beta \circ \eta = 0$. It follows that there is a $\vartheta : t \to \mathrm{Ker}(\beta)$ with $\eta = h \circ \vartheta$. It holds $i \circ g' \circ \vartheta = g \circ h \circ \vartheta = g \circ \pi' \circ n \circ \varepsilon - g \circ f \circ \zeta = i \circ \pi \circ n \circ \varepsilon = i \circ d \circ m \circ \varepsilon$. As i is a monomorphism we get $g' \circ \vartheta = d \circ m \circ \varepsilon$. Thus, as $m \circ \varepsilon$ is an epimorphism, Lemma 12.5.15 implies that $\mathrm{Ker}(\beta) \xrightarrow{g'} \mathrm{Ker}(\gamma) \xrightarrow{\delta} \mathrm{Coker}(\alpha)$ is exact. Then, the claim follows by Lemma 12.5.16 and duality. □

Lemma 12.5.18. *Let $\mathcal{A}$ be an abelian category. Let*

$$\begin{array}{ccccccccc}
x & \longrightarrow & y & \longrightarrow & z & \longrightarrow & 0 & & \\
\downarrow & \swarrow & \downarrow{\scriptstyle\alpha} & \swarrow & \downarrow{\scriptstyle\beta} & \swarrow & \downarrow{\scriptstyle\gamma} & & \\
x' & \longrightarrow & y' & \longrightarrow & z' & \longrightarrow & 0 & & \\
\downarrow & & \downarrow{\scriptstyle\alpha'} & \swarrow & \downarrow{\scriptstyle\beta'} & \swarrow & \downarrow{\scriptstyle\gamma'} & & \\
0 & \longrightarrow & u & \longrightarrow & v & \longrightarrow & w & & \\
 & \swarrow & \downarrow & \swarrow & \downarrow & \swarrow & \downarrow & & \\
0 & \longrightarrow & u' & \longrightarrow & v' & \longrightarrow & w' & &
\end{array}$$

be a commutative diagram with exact rows. Then, the induced diagram

$$\begin{array}{ccccccccccc}
Ker(\alpha) & \longrightarrow & Ker(\beta) & \longrightarrow & Ker(\gamma) & \xrightarrow{\delta} & Coker(\alpha) & \longrightarrow & Coker(\beta) & \longrightarrow & Coker(\gamma) \\
\downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow \\
Ker(\alpha') & \longrightarrow & Ker(\beta') & \longrightarrow & Ker(\gamma') & \xrightarrow{\delta'} & Coker(\alpha') & \longrightarrow & Coker(\beta') & \longrightarrow & Coker(\gamma')
\end{array}$$

commutes.

Proof. Omitted. □

Lemma 12.5.19. *Let $\mathcal{A}$ be an abelian category. Let*

$$\begin{array}{ccccccc}
w & \longrightarrow & x & \longrightarrow & y & \longrightarrow & z \\
\downarrow{\scriptstyle\alpha} & & \downarrow{\scriptstyle\beta} & & \downarrow{\scriptstyle\gamma} & & \downarrow{\scriptstyle\delta} \\
w' & \longrightarrow & x' & \longrightarrow & y' & \longrightarrow & z'
\end{array}$$

be a commutative diagram with exact rows.

(1) *If α, γ are surjective and δ is injective, then β is surjective.*
(2) *If β, δ are injective and α is surjective, then γ is injective.*

Proof. Assume α, γ are surjective and δ is injective. We may replace w' by $\mathrm{Im}(w' \to x')$, i.e., we may assume that $w' \to x'$ is injective. We may replace z by $\mathrm{Im}(y \to z)$, i.e., we may assume that $y \to z$ is surjective. Then we may apply Lemma 12.5.17 to

$$\begin{array}{ccccccc}
 & & \mathrm{Ker}(y \to z) & \longrightarrow & y & \longrightarrow & z & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & \mathrm{Ker}(y' \to z') & \longrightarrow & y' & \longrightarrow & z' & &
\end{array}$$

to conclude that $\mathrm{Ker}(y \to z) \to \mathrm{Ker}(y' \to z')$ is surjective. Finally, we apply Lemma 12.5.17 to

$$\begin{array}{ccccccc} & & w & \longrightarrow & x & \longrightarrow & \mathrm{Ker}(y \to z) \longrightarrow 0 \\ & & \downarrow & & \downarrow & & \downarrow \\ 0 & \longrightarrow & w' & \longrightarrow & x' & \longrightarrow & \mathrm{Ker}(y' \to z') \end{array}$$

to conclude that $x \to x'$ is surjective. This proves (1). The proof of (2) is dual to this. □

Lemma 12.5.20. *Let $\mathcal{A}$ be an abelian category. Let*

$$\begin{array}{ccccccccc} v & \longrightarrow & w & \longrightarrow & x & \longrightarrow & y & \longrightarrow & z \\ \downarrow{\scriptstyle \alpha} & & \downarrow{\scriptstyle \beta} & & \downarrow{\scriptstyle \gamma} & & \downarrow{\scriptstyle \delta} & & \downarrow{\scriptstyle \epsilon} \\ v' & \longrightarrow & w' & \longrightarrow & x' & \longrightarrow & y' & \longrightarrow & z' \end{array}$$

be a commutative diagram with exact rows. If β, δ are isomorphisms, ϵ is injective, and α is surjective then γ is an isomorphism.

Proof. Immediate consequence of Lemma 12.5.19. □

12.6. Extensions

Definition 12.6.1. Let $\mathcal{A}$ be an abelian category. Let $A, C \in \mathrm{Ob}(\mathcal{A})$. An *extension E of B by A* is a short exact sequence

$$0 \to A \to E \to B \to 0.$$

By abuse of language we often omit mention of the morphisms $A \to E$ and $E \to B$, although they are definitively part of the structure of an extension.

Definition 12.6.2. Let $\mathcal{A}$ be an abelian category. Let $A, B \in \mathrm{Ob}(\mathcal{A})$. The set of isomorphism classes of extensions of B by A is denoted

$$\mathrm{Ext}_{\mathcal{A}}(B, A).$$

This is called the *Ext-group*.

This definition works, because by our conventions $\mathcal{A}$ is a set, and hence $\mathrm{Ext}_{\mathcal{A}}(B, A)$ is a set. In any of the cases of "big" abelian categories listed in Categories, Remark 4.2.2. one can check by hand that $\mathrm{Ext}_{\mathcal{A}}(B, A)$ is a set as well. Also, we will see later that this is always the case when $\mathcal{A}$ has either enough projectives or enough injectives. Insert future reference here.

Actually we can turn $\mathrm{Ext}_{\mathcal{A}}(-, -)$ into a functor

$$\mathcal{A}^{opp} \times \mathcal{A} \longrightarrow \mathit{Sets}, \quad (A, B) \longmapsto \mathrm{Ext}_{\mathcal{A}}(A, B)$$

as follows:

(1) Given a morphism $B' \to B$ and an extension E of B by A we define $E' = E \times_B B'$ so that we have the following commutative diagram of short exact sequences

$$\begin{array}{ccccccccc} 0 & \longrightarrow & A & \longrightarrow & E' & \longrightarrow & B' & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & A & \longrightarrow & E & \longrightarrow & B & \longrightarrow & 0 \end{array}$$

The extension E' is called the *pullback of E via $B' \to B$*.

(2) Given a morphism $A \to A'$ and an extension E of B by A we define $E' = A' \coprod_A E$ so that we have the following commutative diagram of short exact sequences

$$\begin{array}{ccccccccc} 0 & \longrightarrow & A & \longrightarrow & E & \longrightarrow & B & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & A' & \longrightarrow & E' & \longrightarrow & B & \longrightarrow & 0 \end{array}$$

The extension E' is called the *pushout of E via $A \to A'$*.

To see that this defines a functor as indicated above there are several things to verify. First of all functoriality in the variable B requires that $(E \times_B B') \times_{B'} B'' = E \times_B B''$ which is a general property of fibre products. Dually one deals with functoriality in the variable A. Finally, given $A \to A'$ and $B' \to B$ we have to show that

$$A' \coprod\nolimits_A (E \times_B B') \cong (A' \coprod\nolimits_A E) \times_B B'$$

as extensions of B' by A'. Recall that $A' \coprod_A E$ is a quotient of $A' \oplus E$. Thus the right hand side is a quotient of $A' \oplus E \times_B B'$, and it is straightforward to see that the kernel is exactly what you need in order to get the left hand side.

Note that if E_1 and E_2 are extensions of B by A, then $E_1 \oplus E_2$ is an extension of $B \oplus B$ by $A \oplus A$. We pull back by the diagonal map $B \to B \oplus B$ and we push out by the sum map $A \oplus A \to A$ to get an extension $E_1 + E_2$ of B by A.

$$\begin{array}{ccccccccc} 0 & \longrightarrow & A \oplus A & \longrightarrow & E_1 \oplus E_2 & \longrightarrow & B \oplus B & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle \Sigma} & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & A & \longrightarrow & E' & \longrightarrow & B \oplus B & \longrightarrow & 0 \\ & & \uparrow & & \uparrow & & \uparrow{\scriptstyle \Delta} & & \\ 0 & \longrightarrow & A & \longrightarrow & E_1 + E_2 & \longrightarrow & B & \longrightarrow & 0 \end{array}$$

The extension $E_1 + E_2$ is called the *Baer sum* of the given extensions.

Lemma 12.6.3. *The construction $(E_1, E_2) \mapsto E_1 + E_2$ above defines a commutative group law on $Ext_{\mathcal{A}}(B, A)$ which is functorial in both variables.*

Proof. Omitted. □

Lemma 12.6.4. *Let $\mathcal{A}$ be an abelian category. Let $0 \to M_1 \to M_2 \to M_3 \to 0$ be a short exact sequence in $\mathcal{A}$.*

(1) *There is a canonical six term exact sequence of abelian groups*

$$\begin{array}{ccccccc} 0 \longrightarrow \mathrm{Hom}_{\mathcal{A}}(M_3, N) & \longrightarrow & \mathrm{Hom}_{\mathcal{A}}(M_2, N) & \longrightarrow & \mathrm{Hom}_{\mathcal{A}}(M_1, N) & & \\ & \searrow & & & & & \\ & & Ext_{\mathcal{A}}(M_3, N) & \longrightarrow & Ext_{\mathcal{A}}(M_2, N) & \longrightarrow & Ext_{\mathcal{A}}(M_1, N) \end{array}$$

for all objects N of $\mathcal{A}$, and

(2) *there is a canonical six term exact sequence of abelian groups*

$$\begin{array}{ccccccc} 0 & \longrightarrow & \mathrm{Hom}_{\mathcal{A}}(N, M_1) & \longrightarrow & \mathrm{Hom}_{\mathcal{A}}(N, M_2) & \longrightarrow & \mathrm{Hom}_{\mathcal{A}}(N, M_1) \\ & & & \searrow & & & \\ & & Ext_{\mathcal{A}}(N, M_1) & \longrightarrow & Ext_{\mathcal{A}}(N, M_2) & \longrightarrow & Ext_{\mathcal{A}}(N, M_1) \end{array}$$

for all objects N of $\mathcal{A}$.

Proof. Omitted. Hint: The boundary maps are defined using either the pushout or pullback of the given short exact sequence. □

12.7. Additive functors

Recall that we defined, in Categories, Definition 4.23.1 the notion of a "right exact", "left exact" and "exact" functor in the setting of a functor between categories that have finite (co)limits. Thus this applies in particular to functors between abelian categories.

Lemma 12.7.1. *Let $\mathcal{A}$ and $\mathcal{B}$ be abelian categories. Let $F : \mathcal{A} \to \mathcal{B}$ be a functor.*

(1) *If F is either left or right exact, then it is additive.*
(2) *If F is additive then it is left exact if and only if for every short exact sequence $0 \to A \to B \to C \to 0$ the sequence $0 \to F(A) \to F(B) \to F(C)$ is exact.*
(3) *If F is additive then it is right exact if and only if for every short exact sequence $0 \to A \to B \to C \to 0$ the sequence $F(A) \to F(B) \to F(C) \to 0$ is exact.*
(4) *If F is additive then it is exact if and only if for every short exact sequence $0 \to A \to B \to C \to 0$ the sequence $0 \to F(A) \to F(B) \to F(C) \to 0$ is exact.*

Proof. Let us first note that if F commutes with the empty limit or the empty colimit, then $F(0) = 0$. In particular F applied to the zero morphism is zero. We will use this below without mention.

Suppose that F is left exact, i.e., commutes with finite limits. Then $F(A \times A) = F(A) \times F(A)$ with projections $F(p)$ and $F(q)$. Hence $F(A \oplus A) = F(A) \oplus F(A)$ with all four morphisms $F(i), F(j), F(p), F(q)$ equal to their counterparts in $\mathcal{B}$ as they satisfy the same relations, see Remark 12.3.6. Then $f = F(p + q)$ is a morphism $f : F(A) \oplus F(A) \to F(A)$ such that $f \circ F(i) = F(p \circ i + q \circ i) = F(\mathrm{id}_A) = \mathrm{id}_{F(A)}$. And similarly $f \circ F(j) = \mathrm{id}_A$. We conclude that $F(p + q) = F(p) + F(q)$. For any pair of morphisms $a, b : B \to A$ the map $g = F(i \circ a + j \circ b) : F(B) \to F(A) \oplus F(A)$ is a morphism such that $F(p) \circ g = F(p \circ (i \circ a + j \circ b)) = F(a)$ and similarly $F(q) \circ g = F(b)$. Hence $g = F(i) \circ F(a) + F(j) \circ F(b)$. The sum of a and b is the composition

$$B \xrightarrow{i \circ a + j \circ b} A \oplus A \xrightarrow{p+q} A.$$

Applying F we get

$$F(B) \xrightarrow{F(i) \circ F(a) + F(j) \circ F(b)} F(A) \oplus F(A) \xrightarrow{F(p)+F(q)} A.$$

where we used the expressions for f and g obtained above. Hence F is additive.[1]

[1] I'm sure there is an infinitely slicker proof of this.

Denote $f : B \to C$ a map from B to C. Exactness of $0 \to A \to B \to C$ just means that $A = \mathrm{Ker}(f)$. Clearly the kernel of f is the equalizer of the two maps f and 0 from B to C. Hence if F commutes with limits, then $F(\mathrm{Ker}(f)) = \mathrm{Ker}(F(f))$ which exactly means that $0 \to F(A) \to F(B) \to F(C)$ is exact.

Conversely, suppose that F is additive and transforms any short exact sequence $0 \to A \to B \to C$ into an exact sequence $0 \to F(A) \to F(B) \to F(C)$. Because it is additive it commutes with direct sums and hence finite products in $\mathcal{A}$. To show it commutes with finite limits it therefore suffices to show that it commutes with equalizers. But equalizers in an abelian category are the same as the kernel of the difference map, hence it suffices to show that F commutes with taking kernels. Let $f : A \to B$ be a morphism. Factor f as $A \to I \to B$ with $f' : A \to I$ surjective and $i : I \to B$ injective. (This is possible by the definition of an abelian category.) Then it is clear that $\mathrm{Ker}(f) = \mathrm{Ker}(f')$. Also $0 \to \mathrm{Ker}(f') \to A \to I \to 0$ and $0 \to I \to B \to B/I \to 0$ are short exact. By the condition imposed on F we see that $0 \to F(\mathrm{Ker}(f')) \to F(A) \to F(I)$ and $0 \to F(I) \to F(B) \to F(B/I)$ are exact. Hence it is also the case that $F(\mathrm{Ker}(f'))$ is the kernel of the map $F(A) \to F(B)$, and we win.

The proof of (3) is similar to the proof of (2). Statement (4) is a combination of (2) and (3). □

Lemma 12.7.2. *Let $\mathcal{A}$ and $\mathcal{B}$ be abelian categories. Let $F : \mathcal{A} \to \mathcal{B}$ be an exact functor. For every pair of objects A, B of $\mathcal{A}$ the functor F induces an abelian group homomorphism*

$$Ext_{\mathcal{A}}(B, A) \longrightarrow Ext_{\mathcal{B}}(F(B), F(A))$$

which maps the extension E to $F(E)$.

Proof. Omitted. □

The following lemma is used in the proof that the category of abelian sheaves on a site is abelian, where the functor b is sheafification.

Lemma 12.7.3. *Let $a : \mathcal{A} \to \mathcal{B}$ and $b : \mathcal{B} \to \mathcal{A}$ be functors. Assume that*

1. *$\mathcal{A}$, $\mathcal{B}$ are additive categories, a, b are additive functors, and a is right adjoint to b,*
2. *$\mathcal{B}$ is abelian and b is left exact, and*
3. *$ba \cong id_{\mathcal{A}}$.*

Then $\mathcal{A}$ is abelian.

Proof. As $\mathcal{B}$ is abelian we see that all finite limits and colimits exist in $\mathcal{B}$ by Lemma 12.5.5. Since b is a left adjoint we see that b is also right exact and hence exact, see Categories, Lemma 4.24.4. Let $\varphi : B_1 \to B_2$ be a morphism of $\mathcal{B}$. In particular, if $K = \mathrm{Ker}(B_1 \to B_2)$, then K is the equalizer of 0 and φ and hence bK is the equalizer of 0 and $b\varphi$, hence bK is the kernel of $b\varphi$. Similarly, if $Q = \mathrm{Coker}(B_1 \to B_2)$, then Q is the coequalizer of 0 and φ and hence bQ is the coequalizer of 0 and $b\varphi$, hence bQ is the cokernel of $b\varphi$. Thus we see that every morphism of the form $b\varphi$ in $\mathcal{A}$ has a kernel and a cokernel. However, since $ba \cong \mathrm{id}$ we see that every morphism of $\mathcal{A}$ is of this form, and we conclude that kernels and cokernels exist in $\mathcal{A}$. In fact, the argument shows that if $\psi : A_1 \to A_2$ is a morphism then

$$\mathrm{Ker}(\psi) = b\mathrm{Ker}(a\psi), \quad \text{and} \quad \mathrm{Coker}(\psi) = b\mathrm{Coker}(a\psi).$$

Now we still have to show that $\mathrm{Coim}(\psi) = \mathrm{Im}(\psi)$. We do this as follows. First note that since $\mathcal{A}$ has kernels and cokernels it has all finite limits and colimits (see proof of Lemma 12.5.5). Hence we see by Categories, Lemma 4.24.4 that a is left exact and hence transforms kernels (=equalizers) into kernels.

$$\begin{aligned}
\mathrm{Coim}(\psi) &= \mathrm{Coker}(\mathrm{Ker}(\psi) \to A_1) && \text{by definition} \\
&= b\mathrm{Coker}(a(\mathrm{Ker}(\psi) \to A_1)) && \text{by formula above} \\
&= b\mathrm{Coker}(\mathrm{Ker}(a\psi) \to aA_1)) && a \text{ preserves kernels} \\
&= b\mathrm{Coim}(a\psi) && \text{by definition} \\
&= b\mathrm{Im}(a\psi) && \mathcal{B} \text{ is abelian} \\
&= b\mathrm{Ker}(aA_2 \to \mathrm{Coker}(a\psi)) && \text{by definition} \\
&= \mathrm{Ker}(baA_2 \to b\mathrm{Coker}(a\psi)) && b \text{ preserves kernels} \\
&= \mathrm{Ker}(A_2 \to b\mathrm{Coker}(a\psi)) && ba = \mathrm{id}_{\mathcal{A}} \\
&= \mathrm{Ker}(A_2 \to \mathrm{Coker}(\psi)) && \text{by formula above} \\
&= \mathrm{Im}(\psi) && \text{by definition}
\end{aligned}$$

Thus the lemma holds. □

12.8. Localization

In this section we note how Gabriel-Zisman localization interacts with the additive structure on a category.

Lemma 12.8.1. *Let $\mathcal{C}$ be a preadditive category. Let S be a left or right multiplicative system. There exists a canonical preadditive structure on $S^{-1}\mathcal{C}$ such that the localization functor $Q : \mathcal{C} \to S^{-1}\mathcal{C}$ is additive.*

Proof. We will prove this in the case S is a left multiplicative system. The case where S is a right multiplicative system is dual. Suppose that X, Y are objects of $\mathcal{C}$ and that $\alpha, \beta : X \to Y$ are morphisms in $S^{-1}\mathcal{C}$. According to Categories, Lemma 4.25.3 we may represent these by pairs $s^{-1}f, s^{-1}g$ with common denominator s. In this case we define $\alpha + \beta$ to be the equivalence class of $s^{-1}(f + g)$. In the rest of the proof we show that this is well defined and that composition is bilinear. Once this is done it is clear that Q is an additive functor.

Let us show construction above is well defined. An abstract way of saying this is that filtered colimits of abelian groups agree with filtered colimits of sets and to use Categories, Equation (4.25.5.1). We can work this out in a bit more detail as follows. Say $s : Y \to Y_1$ and $f, g : X \to Y_1$. Suppose we have a second representation of α, β as $(s')^{-1}f', (s')^{-1}g'$ with $s' : Y \to Y_2$ and $f', g' : X \to Y_2$. By Categories, Remark 4.25.5 we can find a morphism $s_3 : Y \to Y_3$ and morphisms $a_1 : Y_1 \to Y_3$, $a_2 : Y_2 \to Y_3$ such that $a_1 \circ s = s_3 = a_2 \circ s'$ and also $a_1 \circ f = a_2 \circ f'$ and $a_1 \circ g = a_2 \circ g'$. Hence we see that $s^{-1}(f + g)$ is equivalent to

$$\begin{aligned}
s_3^{-1}(a_1 \circ (f + g)) &= s_3^{-1}(a_1 \circ f + a_2 \circ g) \\
&= s_3^{-1}(a_2 \circ f' + a_2 \circ g') \\
&= s_3^{-1}(a_2 \circ (f' + g'))
\end{aligned}$$

which is equivalent to $(s')^{-1}(f' + g')$.

Fix $s : Y \to Y'$ and $f, g : X \to Y'$ with $\alpha = s^{-1}f$ and $\beta = s^{-1}g$ as morphisms $X \to Y$ in $S^{-1}\mathcal{C}$. To show that composition is bilinear first consider the case of a morphism $\gamma : Y \to Z$ in $S^{-1}\mathcal{C}$. Say $\gamma = t^{-1}h$ for some $h : Y \to Z'$ and $t : Z \to Z'$ in S. Using LMS2 we choose morphisms $a : Y' \to Z''$ and $t' : Z' \to Z''$ in S such that $a \circ s = t' \circ h$. Picture

$$\begin{array}{ccccc} & & & & Z \\ & & & & \downarrow{\scriptstyle t} \\ & & Y & \xrightarrow{h} & Z' \\ & & \downarrow{\scriptstyle s} & & \downarrow{\scriptstyle t} \\ X & \xrightarrow{f,g} & Y' & \xrightarrow{a} & Z'' \end{array}$$

Then $\gamma \circ \alpha = (t' \circ t)^{-1}(a \circ f)$ and $\gamma \circ \beta = (t' \circ t)^{-1}(a \circ g)$. Hence we see that $\gamma \circ (\alpha + \beta)$ is represented by $(t' \circ t)^{-1}(a \circ (f + g)) = (t' \circ t)^{-1}(a \circ f + a \circ g)$ which represents $\gamma \circ \alpha + \gamma \circ \beta$.

Finally, assume that $\delta : W \to X$ is another morphism of $S^{-1}\mathcal{C}$. Say $\delta = r^{-1}i$ for some $i : W \to X'$ and $r : X \to X'$ in S. We claim that we can find a morphism $s : Y' \to Y''$ in S and morphisms $a'', b'' : X' \to Y''$ such that the following diagram commutes

$$\begin{array}{ccccc} & & & & Y \\ & & & & \downarrow{\scriptstyle s} \\ & & X & \xrightarrow{f,g,f+g} & Y' \\ & & \downarrow{\scriptstyle s} & & \downarrow{\scriptstyle s'} \\ W & \xrightarrow{i} & X' & \xrightarrow{a'',b'',a''+b''} & Y'' \end{array}$$

Namely, using LMS2 we can first choose $s_1 : Y' \to Y_1$, $s_2 : Y' \to Y_2$ in S and $a : X' \to Y_1$, $b : X' \to Y_2$ such that $a \circ s = s_1 \circ f$ and $b \circ s = s_2 \circ f$. Then using that the category Y'/S is filtered (see Categories, Remark 4.25.5), we can find a $s' : Y' \to Y''$ and morphisms $a' : Y_1 \to Y''$, $b' : Y_2 \to Y''$ such that $s' = a' \circ s_1$ and $s' = b' \circ s_2$. Setting $a'' = a' \circ a$ and $b'' = b' \circ b$ works. At this point we see that the compositions $\alpha \circ \delta$ and $\beta \circ \delta$ are represented by $(s' \circ s)^{-1}a''$ and $(s' \circ s)^{-1}b''$. Hence $\alpha \circ \delta + \beta \circ \delta$ is represented by $(s' \circ s)^{-1}(a'' + b'')$ which by the diagram again is a representative of $(\alpha + \beta) \circ \delta$. □

Lemma 12.8.2. *Let $\mathcal{C}$ be an additive category. Let S be a left or right multiplicative system. Then $S^{-1}\mathcal{C}$ is an additive category and the localization functor $Q : \mathcal{C} \to S^{-1}\mathcal{C}$ is additive.*

Proof. By Lemma 12.8.1 we see that $S^{-1}\mathcal{C}$ is preadditive and that Q is additive. Recall that the functor Q commutes with finite colimits (resp. finite limits), see Categories, Lemmas 4.25.7 and 4.25.14. We conclude that $S^{-1}\mathcal{C}$ has a zero object and direct sums, see Lemmas 12.3.2 and 12.3.4. □

The following lemma describes the kernel (see Definition 12.9.5) of the localization functor in case we invert a multiplicative system.

Lemma 12.8.3. *Let $\mathcal{C}$ be an additive category. Let S be a multiplicative system. Let X be an object of $\mathcal{D}$. The following are equivalent*

(1) $Q(X) = 0$ *in* $S^{-1}\mathcal{C}$,
(2) *there exists* $Y \in \mathrm{Ob}(\mathcal{C})$ *such that* $0 : X \to Y$ *is an element of* S, *and*
(3) *there exists* $Z \in \mathrm{Ob}(\mathcal{C})$ *such that* $0 : Z \to X$ *is an element of* S.

Proof. If (2) holds we see that $0 = Q(0) : Q(X) \to Q(Y)$ is an isomorphism. In the additive category $S^{-1}\mathcal{C}$ this implies that $Q(X) = 0$. Hence (2) $\Rightarrow$ (1). Similarly, (3) $\Rightarrow$ (1). Suppose that $Q(X) = 0$. This implies that the morphism $f : 0 \to X$ is transformed into an isomorphism in $S^{-1}\mathcal{C}$. Hence by Categories, Lemma 4.25.18 there exists a morphism $g : Z \to 0$ such that $fg \in S$. This proves (1) $\Rightarrow$ (3). Similarly, (1) $\Rightarrow$ (2). □

Lemma 12.8.4. *Let* $\mathcal{A}$ *be an abelian category.*

(1) *If* S *is a left multiplicative system, then the category* $S^{-1}\mathcal{A}$ *has cokernels and the functor* $Q : \mathcal{A} \to S^{-1}\mathcal{A}$ *commutes with them.*
(2) *If* S *is a right multiplicative system, then the category* $S^{-1}\mathcal{A}$ *has kernels and the functor* $Q : \mathcal{A} \to S^{-1}\mathcal{A}$ *commutes with them.*
(3) *If* S *is a multiplicative system, then the category* $S^{-1}\mathcal{A}$ *is abelian and the functor* $Q : \mathcal{A} \to S^{-1}\mathcal{A}$ *is exact.*

Proof. Assume S is a left multiplicative system. Let $a : X \to Y$ be a morphism of $S^{-1}\mathcal{A}$. Then $a = s^{-1}f$ for some $s : Y \to Y'$ in S and $f : X \to Y'$. Since $Q(s)$ is an isomorphism we see that the existence of $\mathrm{Coker}(a : X \to Y)$ is equivalent to the existence of $\mathrm{Coker}(Q(f) : X \to Y')$. Since $\mathrm{Coker}(Q(f))$ is the coequalizer of 0 and $Q(f)$ we see that $\mathrm{Coker}(Q(f))$ is represented by $Q(\mathrm{Coker}(f))$ by Categories, Lemma 4.25.7. This proves (1).

Part (2) is dual to part (1).

If S is a multiplicative system, then S is both a left and a right multiplicative system. Thus we see that $S^{-1}\mathcal{A}$ has kernels and cokernels and Q commutes with kernels and cokernels. To finish the proof of (3) we have to show that Coim = Im in $S^{-1}\mathcal{A}$. Again using that any arrow in $S^{-1}\mathcal{A}$ is isomorphic to an arrow $Q(f)$ we see that the result follows from the result for $\mathcal{A}$. □

12.9. Serre subcategories

In [**Ser53**, Chapter I, Section 1] a notion of a "class" of abelian groups is defined. This notion has been extended to abelian categories by many authors (in slightly different ways). We will use the following variant which is virtually identical to Serre's original definition.

Definition 12.9.1. Let $\mathcal{A}$ be an abelian category.

(1) A *Serre subcategory* of $\mathcal{A}$ is a nonempty full subcategory $\mathcal{C}$ of $\mathcal{A}$ such that given an exact sequence
$$A \to B \to C$$
with $A, C \in \mathrm{Ob}(\mathcal{C})$, then also $B \in \mathrm{Ob}(\mathcal{C})$.
(2) A *weak Serre subcategory* of $\mathcal{A}$ is a nonempty full subcategory $\mathcal{C}$ of $\mathcal{A}$ such that given an exact sequence
$$A_0 \to A_1 \to A_2 \to A_3 \to A_4$$
with A_0, A_1, A_3, A_4 in $\mathcal{C}$, then also A_2 in $\mathcal{C}$.

In some references the second notion is called a "thick" subcategory and in other references the first notion is called a "thick" subcategory. However, it seems that the notion of a Serre subcategory is universally accepted to be the one defined above. Note that in both cases the category $\mathcal{C}$ is abelian and that the inclusion functor $\mathcal{C} \to \mathcal{A}$ is a fully faithful exact functor. Let's characterize these types of subcategories in more detail.

Lemma 12.9.2. *Let $\mathcal{A}$ be an abelian category. Let $\mathcal{C}$ be a subcategory of $\mathcal{A}$. Then $\mathcal{C}$ is a Serre subcategory if and only if the following conditions are satisfied:*

(1) $0 \in \mathrm{Ob}(\mathcal{C})$,
(2) *$\mathcal{C}$ is a strictly full subcategory of $\mathcal{A}$,*
(3) *any subobject or quotient of an object of $\mathcal{C}$ is an object of $\mathcal{C}$,*
(4) *if $A \in \mathrm{Ob}(\mathcal{A})$ is an extension of objects of $\mathcal{C}$ then also $A \in \mathrm{Ob}(\mathcal{C})$.*

Moreover, a Serre subcategory is an abelian category and the inclusion functor is exact.

Proof. Omitted. □

Lemma 12.9.3. *Let $\mathcal{A}$ be an abelian category. Let $\mathcal{C}$ be a subcategory of $\mathcal{A}$. Then $\mathcal{C}$ is a weak Serre subcategory if and only if the following conditions are satisfied:*

(1) $0 \in \mathrm{Ob}(\mathcal{C})$,
(2) *$\mathcal{C}$ is a strictly full subcategory of $\mathcal{A}$,*
(3) *kernels and cokernels in $\mathcal{A}$ of morphisms between objects of $\mathcal{C}$ are in $\mathcal{C}$,*
(4) *if $A \in \mathrm{Ob}(\mathcal{A})$ is an extension of objects of $\mathcal{C}$ then also $A \in \mathrm{Ob}(\mathcal{C})$.*

Moreover, a weak Serre subcategory is an abelian category and the inclusion functor is exact.

Proof. Omitted. □

Lemma 12.9.4. *Let $\mathcal{A}$, $\mathcal{B}$ be abelian categories. Let $F : \mathcal{A} \to \mathcal{B}$ be an exact functor. Then the full subcategory of objects C of $\mathcal{A}$ such that $F(C) = 0$ forms a Serre subcategory of $\mathcal{A}$.*

Proof. Omitted. □

Definition 12.9.5. Let $\mathcal{A}$, $\mathcal{B}$ be abelian categories. Let $F : \mathcal{A} \to \mathcal{B}$ be an exact functor. Then the full subcategory of objects C of $\mathcal{A}$ such that $F(C) = 0$ is called the *kernel of the functor F*, and is sometimes denoted $\mathrm{Ker}(F)$.

Lemma 12.9.6. *Let $\mathcal{A}$ be an abelian category. Let $\mathcal{C} \subset \mathcal{A}$ be a Serre subcategory. There exists an abelian category $\mathcal{A}/\mathcal{C}$ and an exact functor*

$$F : \mathcal{A} \longrightarrow \mathcal{A}/\mathcal{C}$$

which is essentially surjective and whose kernel is $\mathcal{C}$. The category $\mathcal{A}/\mathcal{C}$ and the functor F are characterized by the following universal property: For any exact functor $G : \mathcal{A} \to \mathcal{B}$ such that $\mathcal{C} \subset Ker(G)$ there exists a factorization $G = H \circ F$ for a unique exact functor $H : \mathcal{A}/\mathcal{C} \to \mathcal{B}$.

Proof. Consider the set of arrows of $\mathcal{A}$ defined by the following formula

$$S = \{f \in \mathrm{Arrows}(\mathcal{A}) \mid \mathrm{Ker}(f), \mathrm{Coker}(f) \in \mathrm{Ob}(\mathcal{C})\}.$$

We claim that S is a multiplicative system. To prove this we have to check MS1, MS2, MS3, see Categories, Definition 4.25.1.

It is clear that identities are elements of S. Suppose that $f : A \to B$ and $g : B \to C$ are elements of S. There are exact sequences

$$0 \to \mathrm{Ker}(f) \to \mathrm{Ker}(gf) \to \mathrm{Ker}(g) \to 0$$
$$0 \to \mathrm{Coker}(f) \to \mathrm{Coker}(gf) \to \mathrm{Coker}(g) \to 0$$

Hence it follows that $gf \in S$. This proves MS1.

Consider a solid diagram

$$\begin{array}{ccc} A & \xrightarrow{g} & B \\ {\scriptstyle t}\downarrow & & \downarrow{\scriptstyle s} \\ C & \xrightarrow{f} & C \amalg_A B \end{array}$$

with $t \in S$. Set $W = C \amalg_A B = \mathrm{Coker}((1, -1) : A \to C \oplus B)$. Then $\mathrm{Ker}(t) \to \mathrm{Ker}(s)$ is surjective and $\mathrm{Coker}(t) \to \mathrm{Coker}(s)$ is an isomorphism. Hence s is an element of S. This proves LMS2 and the proof of RMS2 is dual.

Finally, consider morphisms $f, g : B \to C$ and a morphism $s : A \to B$ in S such that $f \circ s = g \circ s$. This means that $(f - g) \circ s = 0$. In turn this means that $I = \mathrm{Im}(f - g) \subset C$ is a quotient of $\mathrm{Coker}(s)$ hence an object of $\mathcal{C}$. Thus $t : C \to C' = C/I$ is an element of S such that $t \circ (f - g) = 0$, i.e., such that $t \circ f = t \circ g$. This proves LMS3 and the proof of RMS3 is dual.

Having proved that S is a multiplicative system we set $\mathcal{A}/\mathcal{C} = S^{-1}\mathcal{A}$, and we set F equal to the localization functor Q. By Lemma 12.8.4 the category $\mathcal{A}/\mathcal{C}$ is abelian and F is exact. If X is in the kernel of $F = Q$, then by Lemma 12.8.3 we see that $0 : X \to Z$ is an element of S and hence X is an object of $\mathcal{C}$, i.e., the kernel of F is $\mathcal{C}$. Finally, if G is as in the statement of the lemma, then G turns every element of S into an isomorphism. Hence we obtain the functor $H : \mathcal{A}/\mathcal{C} \to \mathcal{B}$ from the universal property of localization, see Categories, Lemma 4.25.6. □

Lemma 12.9.7. *Let $\mathcal{A}$, $\mathcal{B}$ be abelian categories. Let $F : \mathcal{A} \to \mathcal{B}$ be an exact functor. Let $\mathcal{C} = Ker(F)$. Then the induced functor $\overline{F} : \mathcal{A}/\mathcal{C} \to \mathcal{B}$ is faithful.*

Proof. This is true because the kernel of $\overline{F}$ is zero by construction. Namely, if $f : X \to Y$ is a morphism in $\mathcal{A}/\mathcal{C}$ such that $\overline{F}(f) = 0$, then $\mathrm{Ker}(f) \to X$ and $Y \to \mathrm{Coker}(f)$ are transformed into isomorphisms by $\overline{F}$, hence are isomorphisms by the remark on the kernel of $\overline{F}$. Thus $f = 0$. □

12.10. K-groups

Definition 12.10.1. Let $\mathcal{A}$ be an abelian category. We denote $K_0(\mathcal{A})$ the *zeroth K-group of $\mathcal{A}$*. It is the abelian group constructed as follows. Take the free abelian group on the objects on $\mathcal{A}$ and for every short exact sequence $0 \to A \to B \to C \to 0$ impose the relation $[B] - [A] - [C] = 0$.

Another way to say this is that there is a presentation

$$\bigoplus_{A \to B \to C \text{ ses}} \mathbf{Z}[A \to B \to C] \longrightarrow \bigoplus_{A \in \mathrm{Ob}(\mathcal{A})} \mathbf{Z}[A] \longrightarrow K_0(\mathcal{A}) \longrightarrow 0$$

with $[A \to B \to C] \mapsto [B] - [A] - [C]$ of $K_0(\mathcal{A})$. The short exact sequence $0 \to 0 \to 0 \to 0 \to 0$ leads to the relation $[0] = 0$ in $K_0(\mathcal{A})$. There are no set-theoretical issues as all of our categories are "small" if not mentioned otherwise.

Some examples of K-groups for categories of modules over rings where computed in Algebra, Section 10.51.

Lemma 12.10.2. *Let $F : \mathcal{A} \to \mathcal{B}$ be an exact functor between abelian categories. Then F induces a homomorphism of K-groups $K_0(F) : K_0(\mathcal{A}) \to K_0(\mathcal{B})$ by simply setting $K_0(F)([A]) = [F(A)]$.*

Proof. Proves itself. □

Suppose we are given an object M of an abelian category $\mathcal{A}$ and a complex of the form

$$\ldots \longrightarrow M \xrightarrow{\varphi} M \xrightarrow{\psi} M \xrightarrow{\varphi} M \longrightarrow \ldots \tag{12.10.2.1}$$

In this situation we define

$$H^0(M, \varphi, \psi) = \mathrm{Ker}(\psi)/\mathrm{Im}(\varphi), \quad \text{and} \quad H^1(M, \varphi, \psi) = \mathrm{Ker}(\varphi)/\mathrm{Im}(\psi).$$

Lemma 12.10.3. *Let $\mathcal{A}$ be an abelian category. Let $\mathcal{C} \subset \mathcal{A}$ be a Serre subcategory and set $\mathcal{B} = \mathcal{A}/\mathcal{C}$.*

(1) *The exact functors $\mathcal{C} \to \mathcal{A}$ and $\mathcal{A} \to \mathcal{B}$ induce an exact sequence*
$$K_0(\mathcal{C}) \to K_0(\mathcal{A}) \to K_0(\mathcal{B}) \to 0$$
of K-groups, and

(2) *the kernel of $K_0(\mathcal{C}) \to K_0(\mathcal{A})$ is equal to the collection of elements of the form*
$$[H^0(M, \varphi, \psi)] - [H^1(M, \varphi, \psi)]$$
where (M, φ, ψ) is a complex as in (12.10.2.1) with the property that it becomes exact in $\mathcal{B}$; in other words that $H^0(M, \varphi, \psi)$ and $H^1(M, \varphi, \psi)$ are objects of $\mathcal{C}$.

Proof. We omit the proof of (1). The proof of (2) is in a sense completely combinatorial. First we remark that any class of the type $[H^0(M, \varphi, \psi)] - [H^1(M, \varphi, \psi)]$ is zero in $K_0(\mathcal{A})$ by the following calculation

$$\begin{aligned} 0 &= [M] - [M] \\ &= [\mathrm{Ker}(\varphi)] + [\mathrm{Im}(\varphi)] - [\mathrm{Ker}(\psi)] - [\mathrm{Im}(\psi)] \\ &= [\mathrm{Ker}(\varphi)/\mathrm{Im}(\psi)] - [\mathrm{Ker}(\psi)/\mathrm{Im}(\varphi)] \\ &= [H^1(M, \varphi, \psi)] - [H^0(M, \varphi, \psi)] \end{aligned}$$

as desired. Hence it suffices to show that any element in the kernel of $K_0(\mathcal{C}) \to K_0(\mathcal{A})$ is of this form.

Any element x in $K_0(\mathcal{C})$ can be represented as the difference $x = [P] - [Q]$ of two objects of $\mathcal{C}$ (fun exercise). Suppose that this element maps to zero in $K_0(\mathcal{A})$. This means that there exist

(1) a finite set $I = I^+ \coprod I^-$,
(2) for each $i \in I$ a short exact sequence
$$0 \to A_i \to B_i \to C_i \to 0$$
in the abelian category $\mathcal{A}$

such that

$$[P] - [Q] = \sum\nolimits_{i \in I^+} ([B_i] - [A_i] - [C_i]) - \sum\nolimits_{i \in I^-} ([B_i] - [A_i] - [C_i])$$

in the free abelian group on the objects of $\mathcal{A}$. We can rewrite this as

$$[P] + \sum\nolimits_{i \in I^+} ([A_i] + [C_i]) + \sum\nolimits_{i \in I^-} [B_i] = [Q] + \sum\nolimits_{i \in I^-} ([A_i] + [C_i]) + \sum\nolimits_{i \in I^+} [B_i].$$

Since the right and left hand side should contain the same objects of $\mathcal{A}$ counted with multiplicity, this means there should be a bijection τ between the terms which occur above. Set

$$T^+ = \{p\} \coprod \{a, c\} \times I^+ \coprod \{b\} \times I^-$$

and

$$T^- = \{q\} \coprod \{a, c\} \times I^- \coprod \{b\} \times I^+.$$

Set $T = T^+ \coprod T^- = \{p, q\} \coprod \{a, b, c\} \times I$. For $t \in T$ define

$$O(t) = \begin{cases} P & \text{if} \quad t = p \\ Q & \text{if} \quad t = q \\ A_i & \text{if} \quad t = (a, i) \\ B_i & \text{if} \quad t = (b, i) \\ C_i & \text{if} \quad t = (c, i) \end{cases}$$

Hence we can view $\tau : T^+ \to T^-$ as a bijection such that $O(t) = O(\tau(t))$ for all $t \in T^+$. Let $t_0^- = \tau(p)$ and let $t_0^+ \in T^+$ be the unique element such that $\tau(t_0^+) = q$. Consider the object

$$M^+ = \bigoplus\nolimits_{t \in T^+} O(t)$$

By using τ we see that it is equal to the object

$$M^- = \bigoplus\nolimits_{t \in T^-} O(t)$$

Consider the map

$$\varphi : M^+ \longrightarrow M^-$$

which on the summand $O(t) = A_i$ corresponding to $t = (a, i)$, $i \in I^+$ uses the map $A_i \to B_i$ into the summand $O((b, i)) = B_i$ of M^- and on the summand $O(t) = B_i$ corresponding to (b, i), $i \in I^-$ uses the map $B_i \to C_i$ into the summand $O((c, i)) = C_i$ of M^-. The map is zero on the summands corresponding to p and (c, i), $i \in I^+$. Similarly, consider the map

$$\psi : M^- \longrightarrow M^+$$

which on the summand $O(t) = A_i$ corresponding to $t = (a, i)$, $i \in I^-$ uses the map $A_i \to B_i$ into the summand $O((b, i)) = B_i$ of M^+ and on the summand $O(t) = B_i$ corresponding to (b, i), $i \in I^+$ uses the map $B_i \to C_i$ into the summand $O((c, i)) = C_i$ of M^+. The map is zero on the summands corresponding to q and (c, i), $i \in I^-$.

Note that the kernel of φ is equal to the direct sum of the summand P and the summands $O((c, i)) = C_i$, $i \in I^+$ and the subobjects A_i inside the summands $O((b, i)) = B_i$, $i \in I^-$. The image of ψ is equal to the direct sum of the summands $O((c, i)) = C_i$, $i \in I^+$ and the subobjects A_i inside the summands $O((b, i)) = B_i$, $i \in I^-$. In other words we see that

$$P \cong \mathrm{Ker}(\varphi)/\mathrm{Im}(\psi).$$

In exactly the same way we see that

$$Q \cong \mathrm{Ker}(\psi)/\mathrm{Im}(\varphi).$$

Since as we remarked above the existence of the bijection τ shows that $M^+ = M^-$ we see that the lemma follows. □

12.11. Cohomological delta-functors

Definition 12.11.1. Let $\mathcal{A}, \mathcal{B}$ be abelian categories. A *cohomological δ-functor* or simply a *δ-functor* from $\mathcal{A}$ to $\mathcal{B}$ is given by the following data:

(1) a collection $F^n : \mathcal{A} \to \mathcal{B}$, $n \geq 0$ of additive functors, and
(2) for every short exact sequence $0 \to A \to B \to C \to 0$ of $\mathcal{A}$ a collection $\delta_{A \to B \to C} : F^n(C) \to F^{n+1}(A)$, $n \geq 0$ of morphisms of $\mathcal{B}$.

These data are assumed to satisfy the following axioms

(1) for every short exact sequence as above the sequence

$$\begin{array}{ccccccc} 0 & \longrightarrow & F^0(A) & \longrightarrow & F^0(B) & \longrightarrow & F^0(C) \\ & & & \swarrow^{\delta_{A\to B\to C}} & & & \\ & & F^1(A) & \longrightarrow & F^1(B) & \longrightarrow & F^1(C) \\ & & & \swarrow^{\delta_{A\to B\to C}} & & & \\ & & F^2(A) & \longrightarrow & F^2(B) & \longrightarrow & \ldots \end{array}$$

is exact, and
(2) for every morphism $(A \to B \to C) \to (A' \to B' \to C')$ of short exact sequences of $\mathcal{A}$ the diagrams

$$\begin{array}{ccc} F^n(C) & \xrightarrow[\delta_{A\to B\to C}]{} & F^{n+1}(A) \\ \downarrow & & \downarrow \\ F^n(C') & \xrightarrow{\delta_{A'\to B'\to C'}} & F^{n+1}(A') \end{array}$$

are commutative.

Note that this in particular implies that F^0 is left exact.

Definition 12.11.2. Let $\mathcal{A}, \mathcal{B}$ be abelian categories. Let (F^n, δ_F) and (G^n, δ_G) be δ-functors from $\mathcal{A}$ to $\mathcal{B}$. A *morphism of δ-functors from F to G* is a collection of transformation of functors $t^n : F^n \to G^n$, $n \geq 0$ such that for every short exact sequence $0 \to A \to B \to C \to 0$ of $\mathcal{A}$ the diagrams

$$\begin{array}{ccc} F^n(C) & \xrightarrow[\delta_{F,A\to B\to C}]{} & F^{n+1}(A) \\ {\scriptstyle t^n}\downarrow & & \downarrow{\scriptstyle t^{n+1}} \\ G^n(C) & \xrightarrow{\delta_{G,A\to B\to C}} & G^{n+1}(A) \end{array}$$

are commutative.

Definition 12.11.3. Let $\mathcal{A}, \mathcal{B}$ be abelian categories. Let $F = (F^n, \delta_F)$ be a δ-functor from $\mathcal{A}$ to $\mathcal{B}$. We say F is a *universal δ-functor* if an only if for every

δ-functor $G = (G^n, \delta_G)$ and any morphism of functors $t : F^0 \to G^0$ there exists a unique morphism of δ-functors $\{t^n\}_{n\geq 0} : F \to G$ such that $t = t^0$.

Lemma 12.11.4. *Let $\mathcal{A}, \mathcal{B}$ be abelian categories. Let $F = (F^n, \delta_F)$ be a δ-functor from $\mathcal{A}$ to $\mathcal{B}$. Suppose that for every $n > 0$ and any $A \in \mathrm{Ob}(\mathcal{A})$ there exists an injective morphism $u : A \to B$ (depending on A and n) such that $F^n(u) : F^n(A) \to F^n(B)$ is zero. Then F is a universal δ-functor.*

Proof. Let $G = (G^n, \delta_G)$ be a δ-functor from $\mathcal{A}$ to $\mathcal{B}$ and let $t : F^0 \to G^0$ be a morphism of functors. We have to show there exists a unique morphism of δ-functors $\{t^n\}_{n\geq 0} : F \to G$ such that $t = t^0$. We construct t^n by induction on n. For $n = 0$ we set $t^0 = t$. Suppose we have already constructed a unique sequence of transformation of functors t^i for $i \leq n$ compatible with the maps δ in degrees $\leq n$.

Let $A \in \mathrm{Ob}(\mathcal{A})$. By assumption we may choose a embedding $u : A \to B$ such that $F^{n+1}(u) = 0$. Let $C = B/u(A)$. The long exact cohomology sequence for the short exact sequence $0 \to A \to B \to C \to 0$ and the δ-functor F gives that $F^{n+1}(A) = \mathrm{Coker}(F^n(B) \to F^n(C))$ by our choice of u. Since we have already defined t^n we can set

$$t^{n+1}_A : F^{n+1}(A) \to G^{n+1}(A)$$

equal to the unique map such that

$$\begin{array}{ccc} \mathrm{Coker}(F^n(B) \to F^n(C)) & \xrightarrow[t^n]{} & \mathrm{Coker}(G^n(B) \to G^n(C)) \\ \downarrow{\scriptstyle \delta_{F,A\to B\to C}} & & \downarrow{\scriptstyle \delta_{G,A\to B\to C}} \\ F^{n+1}(A) & \xrightarrow{t^{n+1}_A} & G^{n+1}(A) \end{array}$$

commutes. This is clearly uniquely determined by the requirements imposed. We omit the verification that this defines a transformation of functors. □

Lemma 12.11.5. *Let $\mathcal{A}, \mathcal{B}$ be abelian categories. Let $F : \mathcal{A} \to \mathcal{B}$ be a functor. If there exists a universal δ-functor (F^n, δ_F) from $\mathcal{A}$ to $\mathcal{B}$ with $F^0 = F$, then it is determined up to unique isomorphism of δ-functors.*

Proof. Immediate from the definitions. □

12.12. Complexes

Of course the notions of a chain complex and a cochain complex are dual and you only have to read one of the two parts of this section. So pick the one you like. (Actually, this doesn't quite work right since the conventions on numbering things are not adapted to an easy transition between chain and cochain complexes.)

A *chain complex* $A_\bullet$ in an additive category $\mathcal{A}$ is a complex

$$\ldots \to A_{n+1} \xrightarrow{d_{n+1}} A_n \xrightarrow{d_n} A_{n-1} \to \ldots$$

of $\mathcal{A}$. In other words, we are given an object A_i of $\mathcal{A}$ for all $i \in \mathbf{Z}$ and for all $i \in \mathbf{Z}$ a morphism $d_i : A_i \to A_{i-1}$ such that $d_{i-1} \circ d_i = 0$ for all i. A *morphism of chain complexes* $f : A_\bullet \to B_\bullet$ is given by a family of morphisms $f_i : A_i \to B_i$ such that

all the diagrams

$$\begin{array}{ccc} A_i & \xrightarrow[d_i]{} & A_{i-1} \\ {\scriptstyle f_i}\downarrow & & \downarrow{\scriptstyle f_{i-1}} \\ B_i & \xrightarrow{d_i} & B_{i-1} \end{array}$$

commute. The *category of chain complexes of* $\mathcal{A}$ is denoted $\mathrm{Ch}(\mathcal{A})$. The full subcategory consisting of objects of the form

$$\ldots \to A_2 \to A_1 \to A_0 \to 0 \to 0 \to \ldots$$

is denoted $\mathrm{Ch}_{\geq 0}(\mathcal{A})$. In other words, a chain complex $A_\bullet$ belongs to $\mathrm{Ch}_{\geq 0}(\mathcal{A})$ if and only if $A_i = 0$ for all $i < 0$. A *homotopy* h between a pair of morphisms of chain complexes $f, g : A_\bullet \to B_\bullet$ is is a collection of morphisms $h_i : A_i \to B_{i+1}$ such that we have

$$f_i - g_i = d_{i+1} \circ h_i + h_{i-1} \circ d_i$$

for all i. Clearly, the notions of chain complex, morphism of chain complexes, and homotopies between morphisms of chain complexes makes sense even in a preadditive category.

Lemma 12.12.1. *Let $\mathcal{A}$ be an additive category. Let $f, g : B_\bullet \to C_\bullet$ be morphisms of chain complexes. Suppose given morphisms of chain complexes $a : A_\bullet \to B_\bullet$, and $c : C_\bullet \to D_\bullet$. If $\{h_i : B_i \to C_{i+1}\}$ defines a homotopy between f and g, then $\{c_{i+1} \circ h_i \circ a_i\}$ defines a homotopy between $c \circ f \circ a$ and $c \circ g \circ a$.*

Proof. Omitted. □

In particular this means that it makes sense to define the category of chain complexes with maps up to homotopy. We'll return to this later.

Definition 12.12.2. Let $\mathcal{A}$ be an additive category. We say a morphism $a : A_\bullet \to B_\bullet$ is a *homotopy equivalence* if there exists a morphism $b : B_\bullet \to A_\bullet$ such that there exists a homotopy between $a \circ b$ and id_A and there exists a homotopy between $b \circ a$ and id_B. If there exists such a morphism between $A_\bullet$ and $B_\bullet$, then we say that $A_\bullet$ and $B_\bullet$ are *homotopy equivalent*.

In other words, two complexes are homotopy equivalent if they become isomorphic in the category of complexes up to homotopy.

Lemma 12.12.3. *Let $\mathcal{A}$ be an abelian category.*

(1) *The category of chain complexes in $\mathcal{A}$ is abelian.*
(2) *A morphism of complexes $f : A_\bullet \to B_\bullet$ is injective if and only if each $f_n : A_n \to B_n$ is injective.*
(3) *A morphism of complexes $f : A_\bullet \to B_\bullet$ is surjective if and only if each $f_n : A_n \to B_n$ is surjective.*
(4) *A sequence of chain complexes*

$$A_\bullet \xrightarrow{f} B_\bullet \xrightarrow{g} C_\bullet$$

is exact at $B_\bullet$ if and only if each sequence

$$A_i \xrightarrow{f_i} B_i \xrightarrow{g_i} C_i$$

is exact at B_i.

Proof. Omitted. □

For any $i \in \mathbf{Z}$ the ith *homology group* of a chain complex $A_\bullet$ in an abelian category is defined by the following formula

$$H_i(A_\bullet) = \mathrm{Ker}(d_i)/\mathrm{Im}(d_{i+1}).$$

If $f : A_\bullet \to B_\bullet$ is a morphism of chain complexes of $\mathcal{A}$ then we get an induced morphism $H_i(f) : H_i(A_\bullet) \to H_i(B_\bullet)$ because clearly $f_i(\mathrm{Ker}(d_i : A_i \to A_{i-1})) \subset \mathrm{Ker}(d_i : B_i \to B_{i-1})$, and similarly for $\mathrm{Im}(d_{i+1})$. Thus we obtain a functor

$$H_i : \mathrm{Ch}(\mathcal{A}) \longrightarrow \mathcal{A}.$$

Definition 12.12.4. Let $\mathcal{A}$ be an abelian category.

(1) A morphism of chain complexes $f : A_\bullet \to B_\bullet$ is called a *quasi-isomorphism* if the induced maps $H_i(f) : H_i(A_\bullet) \to H_i(B_\bullet)$ is an isomorphism for all $i \in \mathbf{Z}$.
(2) A chain complex $A_\bullet$ is called *acyclic* if all of its homology objects $H_i(A_\bullet)$ are zero.

Lemma 12.12.5. *Let $\mathcal{A}$ be an abelian category.*

(1) *If the maps $f, g : A_\bullet \to B_\bullet$ are homotopic, then the induced maps $H_i(f)$ and $H_i(g)$ are equal.*
(2) *If the map $f : A_\bullet \to B_\bullet$ is a homotopy equivalence, then f is a quasi-isomorphism.*

Proof. Omitted. □

Lemma 12.12.6. *Let $\mathcal{A}$ be an abelian category. Suppose that*

$$0 \to A_\bullet \to B_\bullet \to C_\bullet \to 0$$

is a short exact sequence of chain complexes of $\mathcal{A}$. Then there is a canonical long exact homology sequence

$$\begin{array}{ccccc} \ldots & & \ldots & & \ldots \\ & \searrow & & \searrow & \\ H_i(A_\bullet) & \longrightarrow & H_i(B_\bullet) & \longrightarrow & H_i(C_\bullet) \\ & \searrow & & \searrow & \\ H_{i-1}(A_\bullet) & \longrightarrow & H_{i-1}(B_\bullet) & \longrightarrow & H_{i-1}(C_\bullet) \\ & \searrow & & \searrow & \\ \ldots & & \ldots & & \ldots \end{array}$$

Proof. Omitted. The maps come from the Snake Lemma 12.5.17 applied to the diagrams

$$\begin{array}{ccccccc} & & A_i/\mathrm{Im}(d_{A,i+1}) & \longrightarrow & B_i/\mathrm{Im}(d_{B,i+1}) & \longrightarrow & C_i/\mathrm{Im}(d_{C,i+1}) \longrightarrow 0 \\ & & \downarrow{\scriptstyle d_{A,i}} & & \downarrow{\scriptstyle d_{B,i}} & & \downarrow{\scriptstyle d_{C,i}} \\ 0 & \longrightarrow & \mathrm{Ker}(d_{A,i-1}) & \longrightarrow & \mathrm{Ker}(d_{B,i-1}) & \longrightarrow & \mathrm{Ker}(d_{C,i-1}) \end{array}$$

□

A *cochain complex* $A^\bullet$ in an additive category $\mathcal{A}$ is a complex

$$\ldots \to A^{n-1} \xrightarrow{d^{n-1}} A^n \xrightarrow{d^n} A^{n+1} \to \ldots$$

of $\mathcal{A}$. In other words, we are given an object A^i of $\mathcal{A}$ for all $i \in \mathbf{Z}$ and for all $i \in \mathbf{Z}$ a morphism $d^i : A^i \to A^{i+1}$ such that $d^{i+1} \circ d^i = 0$ for all i. A *morphism of cochain complexes* $f : A^\bullet \to B^\bullet$ is given by a family of morphisms $f^i : A^i \to B^i$ such that all the diagrams

$$\begin{array}{ccc} A^i & \xrightarrow[d^i]{} & A^{i+1} \\ {\scriptstyle f^i}\downarrow & & \downarrow{\scriptstyle f^{i+1}} \\ B^i & \xrightarrow{d^i} & B^{i+1} \end{array}$$

commute. The *category of cochain complexes of* $\mathcal{A}$ is denoted $\mathrm{CoCh}(\mathcal{A})$. The full subcategory consisting of objects of the form

$$\ldots \to 0 \to 0 \to A^0 \to A^1 \to A^2 \to \ldots$$

is denoted $\mathrm{CoCh}_{\geq 0}(\mathcal{A})$. In other words, a cochain complex $A^\bullet$ belongs to the subcategory $\mathrm{CoCh}_{\geq 0}(\mathcal{A})$ if and only if $A^i = 0$ for all $i < 0$. A *homotopy* h between a pair of morphisms of cochain complexes $f, g : A^\bullet \to B^\bullet$ is is a collection of morphisms $h^i : A^i \to B^{i-1}$ such that we have

$$f^i - g^i = d^{i-1} \circ h^i + h^{i+1} \circ d^i$$

for all i. Clearly, the notions of cochain complex, morphism of cochain complexes, and homotopies between morphisms of cochain complexes makes sense even in a preadditive category.

Lemma 12.12.7. *Let $\mathcal{A}$ be an additive category. Let $f, g : B^\bullet \to C^\bullet$ be morphisms of cochain complexes. Suppose given morphisms of cochain complexes $a : A^\bullet \to B^\bullet$, and $c : C^\bullet \to D^\bullet$. If $\{h^i : B^i \to C^{i-1}\}$ defines a homotopy between f and g, then $\{c^{i-1} \circ h^i \circ a^i\}$ defines a homotopy between $c \circ f \circ a$ and $c \circ g \circ a$.*

Proof. Omitted. □

In particular this means that it makes sense to define the category of cochain complexes with maps up to homotopy. We'll return to this later.

Definition 12.12.8. Let $\mathcal{A}$ be an additive category. We say a morphism $a : A^\bullet \to B^\bullet$ is a *homotopy equivalence* if there exists a morphism $b : B^\bullet \to A^\bullet$ such that there exists a homotopy between $a \circ b$ and id_A and there exists a homotopy between $b \circ a$ and id_B. If there exists such a morphism between $A^\bullet$ and $B^\bullet$, then we say that $A^\bullet$ and $B^\bullet$ are *homotopy equivalent*.

In other words, two complexes are homotopy equivalent if they become isomorphic in the category of complexes up to homotopy.

Lemma 12.12.9. *Let $\mathcal{A}$ be an abelian category.*

1. *The category of cochain complexes in $\mathcal{A}$ is abelian.*
2. *A morphism of cochain complexes $f : A^\bullet \to B^\bullet$ is injective if and only if each $f^n : A^n \to B^n$ is injective.*
3. *A morphism of cochain complexes $f : A^\bullet \to B^\bullet$ is surjective if and only if each $f^n : A^n \to B^n$ is surjective.*

(4) *A sequence of cochain complexes*

$$A^\bullet \xrightarrow{f} B^\bullet \xrightarrow{g} C^\bullet$$

is exact at $B^\bullet$ if and only if each sequence

$$A^i \xrightarrow{f^i} B^i \xrightarrow{g^i} C^i$$

is exact at B^i.

Proof. Omitted. □

For any $i \in \mathbf{Z}$ the ith *cohomology group* of a cochain complex $A^\bullet$ is defined by the following formula

$$H^i(A^\bullet) = \mathrm{Ker}(d^i)/\mathrm{Im}(d^{i-1}).$$

If $f : A^\bullet \to B^\bullet$ is a morphism of cochain complexes of $\mathcal{A}$ then we get an induced morphism $H^i(f) : H^i(A^\bullet) \to H^i(B^\bullet)$ because clearly $f^i(\mathrm{Ker}(d^i : A^i \to A^{i+1})) \subset \mathrm{Ker}(d^i : B^i \to B^{i+1})$, and similarly for $\mathrm{Im}(d^{i-1})$. Thus we obtain a functor

$$H^i : \mathrm{CoCh}(\mathcal{A}) \longrightarrow \mathcal{A}.$$

Definition 12.12.10. Let $\mathcal{A}$ be an abelian category.

(1) A morphism of cochain complexes $f : A^\bullet \to B^\bullet$ of $\mathcal{A}$ is called a *quasi-isomorphism* if the induced maps $H^i(f) : H^i(A^\bullet) \to H^i(B^\bullet)$ is an isomorphism for all $i \in \mathbf{Z}$.
(2) A cochain complex $A^\bullet$ is called *acyclic* if all of its cohomology objects $H^i(A^\bullet)$ are zero.

Lemma 12.12.11. *Let $\mathcal{A}$ be an abelian category.*

(1) *If the maps $f, g : A^\bullet \to B^\bullet$ are homotopic, then the induced maps $H^i(f)$ and $H^i(g)$ are equal.*
(2) *If $f : A^\bullet \to B^\bullet$ is a homotopy equivalence, then f is a quasi-isomorphism.*

Proof. Omitted. □

Lemma 12.12.12. *Let $\mathcal{A}$ be an abelian category. Suppose that*

$$0 \to A^\bullet \to B^\bullet \to C^\bullet \to 0$$

is a short exact sequence of chain complexes of $\mathcal{A}$. Then there is a canonical long exact homology sequence

$$\begin{array}{ccccc}
\ldots & & \ldots & & \ldots \\
 & \swarrow & & \swarrow & \\
H^i(A^\bullet) & \longrightarrow & H^i(B^\bullet) & \longrightarrow & H^i(C^\bullet) \\
 & \swarrow & & \swarrow & \\
H^{i+1}(A^\bullet) & \longrightarrow & H^{i+1}(B^\bullet) & \longrightarrow & H^{i+1}(C^\bullet) \\
 & \swarrow & & \swarrow & \\
\ldots & & \ldots & & \ldots
\end{array}$$

Proof. Omitted. The maps come from the Snake Lemma 12.5.17 applied to the diagrams

$$\begin{array}{ccccccc} & & A^i/\mathrm{Im}(d_A^{i-1}) & \longrightarrow & B^i/\mathrm{Im}(d_B^{i-1}) & \longrightarrow & C^i/\mathrm{Im}(d_C^{i-1}) \longrightarrow 0 \\ & & \downarrow{\scriptstyle d_A^i} & & \downarrow{\scriptstyle d_B^i} & & \downarrow{\scriptstyle d_C^i} \\ 0 & \longrightarrow & \mathrm{Ker}(d_A^{i+1}) & \longrightarrow & \mathrm{Ker}(d_B^{i+1}) & \longrightarrow & \mathrm{Ker}(d_C^{i+1}) \end{array}$$

□

12.13. Truncation of complexes

Let $\mathcal{A}$ be an abelian category. Let $A_\bullet$ be a chain complex. There are several ways to *truncate* the complex $A_\bullet$.

(1) The *"stupid" truncation* $\sigma_{\leq n}$ is the subcomplex $\sigma_{\leq n}A_\bullet$ defined by the rule $(\sigma_{\leq n}A_\bullet)_i = 0$ if $i > n$ and $(\sigma_{\leq n}A_\bullet)_i = A_i$ if $i \leq n$. In a picture

$$\begin{array}{ccccccccccc} \sigma_{\leq n}A_\bullet & & \ldots & \longrightarrow & 0 & \longrightarrow & A_n & \longrightarrow & A_{n-1} & \longrightarrow & \ldots \\ \downarrow & & & & \downarrow & & \downarrow & & \downarrow & & \\ A_\bullet & & \ldots & \longrightarrow & A_{n+1} & \longrightarrow & A_n & \longrightarrow & A_{n-1} & \longrightarrow & \ldots \end{array}$$

Note the property $\sigma_{\leq n}A_\bullet/\sigma_{\leq n-1}A_\bullet = A_n[-n]$.

(2) The *"stupid" truncation* $\sigma_{\geq n}$ is the quotient complex $\sigma_{\geq n}A_\bullet$ defined by the rule $(\sigma_{\geq n}A_\bullet)_i = A_i$ if $i \geq n$ and $(\sigma_{\geq n}A_\bullet)_i = 0$ if $i < n$. In a picture

$$\begin{array}{ccccccccccc} A_\bullet & & \ldots & \longrightarrow & A_{n+1} & \longrightarrow & A_n & \longrightarrow & A_{n-1} & \longrightarrow & \ldots \\ \downarrow & & & & \downarrow & & \downarrow & & \downarrow & & \\ \sigma_{\geq n}A_\bullet & & \ldots & \longrightarrow & A_{n+1} & \longrightarrow & A_n & \longrightarrow & 0 & \longrightarrow & \ldots \end{array}$$

The map of complexes $\sigma_{\geq n}A_\bullet \to \sigma_{\geq n+1}A_\bullet$ is surjective with kernel $A_n[-n]$.

(3) The *canonical truncation* $\tau_{\geq n}A_\bullet$ is defined by the picture

$$\begin{array}{ccccccccccc} \tau_{\geq n}A_\bullet & & \ldots & \longrightarrow & A_{n+1} & \longrightarrow & \mathrm{Ker}(d_n) & \longrightarrow & 0 & \longrightarrow & \ldots \\ \downarrow & & & & \downarrow & & \downarrow & & \downarrow & & \\ A_\bullet & & \ldots & \longrightarrow & A_{n+1} & \longrightarrow & A_n & \longrightarrow & A_{n-1} & \longrightarrow & \ldots \end{array}$$

Note that these complexes have the property that

$$H_i(\tau_{\geq n}A_\bullet) = \begin{cases} H_i(A_\bullet) & \text{if} \quad i \geq n \\ 0 & \text{if} \quad i < n \end{cases}$$

(4) The *canonical truncation* $\tau_{\leq n}A_\bullet$ is defined by the picture

$$\begin{array}{ccccccccccc} A_\bullet & & \ldots & \longrightarrow & A_{n+1} & \longrightarrow & A_n & \longrightarrow & A_{n-1} & \longrightarrow & \ldots \\ \downarrow & & & & \downarrow & & \downarrow & & \downarrow & & \\ \tau_{\leq n}A_\bullet & & \ldots & \longrightarrow & 0 & \longrightarrow & \mathrm{Coker}(d_{n+1}) & \longrightarrow & A_{n-1} & \longrightarrow & \ldots \end{array}$$

Note that these complexes have the property that

$$H_i(\tau_{\leq n} A_\bullet) = \begin{cases} H_i(A_\bullet) & \text{if} \quad i \leq n \\ 0 & \text{if} \quad i > n \end{cases}$$

Let $\mathcal{A}$ be an abelian category. Let $A^\bullet$ be a cochain complex. There are four ways to truncate the complex $A^\bullet$.

(1) The *"stupid" truncation* $\sigma_{\geq n}$ is the subcomplex $\sigma_{\geq n} A^\bullet$ defined by the rule $(\sigma_{\geq n} A^\bullet)^i = 0$ if $i < n$ and $(\sigma_{\geq n} A^\bullet)^i = A_i$ if $i \geq n$. In a picture

$$\begin{array}{ccccccccc} \sigma_{\geq n} A^\bullet & & \ldots & \longrightarrow & 0 & \longrightarrow & A^n & \longrightarrow & A^{n+1} & \longrightarrow & \ldots \\ \downarrow & & & & \downarrow & & \downarrow & & \downarrow & & \\ A^\bullet & & \ldots & \longrightarrow & A^{n-1} & \longrightarrow & A^n & \longrightarrow & A^{n+1} & \longrightarrow & \ldots \end{array}$$

Note the property $\sigma_{\geq n} A^\bullet / \sigma_{\geq n+1} A^\bullet = A^n[-n]$.

(2) The *"stupid" truncation* $\sigma_{\leq n}$ is the quotient complex $\sigma_{\leq n} A^\bullet$ defined by the rule $(\sigma_{\geq n} A^\bullet)^i = 0$ if $i > n$ and $(\sigma_{\geq n} A^\bullet)^i = A^i$ if $i \leq n$. In a picture

$$\begin{array}{ccccccccc} A^\bullet & & \ldots & \longrightarrow & A^{n-1} & \longrightarrow & A^n & \longrightarrow & A^{n+1} & \longrightarrow & \ldots \\ \downarrow & & & & \downarrow & & \downarrow & & \downarrow & & \\ \sigma_{\leq n} A^\bullet & & \ldots & \longrightarrow & A^{n-1} & \longrightarrow & A^n & \longrightarrow & 0 & \longrightarrow & \ldots \end{array}$$

The map of complexes $\sigma_{\leq n} A^\bullet \to \sigma_{\leq n-1} A^\bullet$ is surjective with kernel $A^n[-n]$.

(3) The *canonical truncation* $\tau_{\leq n} A^\bullet$ is defined by the picture

$$\begin{array}{ccccccccc} \tau_{\leq n} A^\bullet & & \ldots & \longrightarrow & A^{n-1} & \longrightarrow & \mathrm{Ker}(d^n) & \longrightarrow & 0 & \longrightarrow & \ldots \\ \downarrow & & & & \downarrow & & \downarrow & & \downarrow & & \\ A^\bullet & & \ldots & \longrightarrow & A^{n-1} & \longrightarrow & A^n & \longrightarrow & A^{n+1} & \longrightarrow & \ldots \end{array}$$

Note that these complexes have the property that

$$H^i(\tau_{\leq n} A^\bullet) = \begin{cases} H^i(A^\bullet) & \text{if} \quad i \leq n \\ 0 & \text{if} \quad i > n \end{cases}$$

(4) The *canonical truncation* $\tau_{\geq n} A^\bullet$ is defined by the picture

$$\begin{array}{ccccccccc} A^\bullet & & \ldots & \longrightarrow & A^{n-1} & \longrightarrow & A^n & \longrightarrow & A^{n+1} & \longrightarrow & \ldots \\ \downarrow & & & & \downarrow & & \downarrow & & \downarrow & & \\ \tau_{\geq n} A^\bullet & & \ldots & \longrightarrow & 0 & \longrightarrow & \mathrm{Coker}(d^{n-1}) & \longrightarrow & A^{n+1} & \longrightarrow & \ldots \end{array}$$

Note that these complexes have the property that

$$H^i(\tau_{\leq n} A^\bullet) = \begin{cases} 0 & \text{if} \quad i < n \\ H^i(A^\bullet) & \text{if} \quad i \geq n \end{cases}$$

12.14. Homotopy and the shift functor

It is an annoying feature that signs and indices have to be part of any discussion of homological algebra[2].

[2] I am sure you think that my conventions are wrong. If so and if you feel strongly about it then drop me an email with an explanation.

Definition 12.14.1. Let $\mathcal{A}$ be an additive category. Let $A_\bullet$ be a chain complex with boundary maps $d_{A,n} : A_n \to A_{n-1}$. For any $k \in \mathbf{Z}$ we define the *k-shifted chain complex* $A[k]_\bullet$ as follows:

(1) we set $A[k]_n = A_{n+k}$, and
(2) we set $d_{A[k],n} : A[k]_n \to A[k]_{n-1}$ equal to $d_{A[k],n} = (-1)^k d_{A,n+k}$.

If $f : A_\bullet \to B_\bullet$ is a morphism of chain complexes, then we let $f[k] : A[k]_\bullet \to B[k]_\bullet$ be the morphism of chain complexes with $f[k]_n = f_{k+n}$.

Of course this means we have functors $[k] : \mathrm{Ch}(\mathcal{A}) \to \mathrm{Ch}(\mathcal{A})$ which mutually commute (on the nose, without any intervening isomorphisms of functors), such that $A[k][l]_\bullet = A[k+l]_\bullet$ and with $[0] = \mathrm{id}_{\mathrm{Ch}(\mathcal{A})}$.

Definition 12.14.2. Let $\mathcal{A}$ be an abelian category. Let $A_\bullet$ be a chain complex with boundary maps $d_{A,n} : A_n \to A_{n-1}$. For any $k \in \mathbf{Z}$ we identify $H_{i+k}(A_\bullet) \to H_i(A[k]_\bullet)$ via the identification $A_{i+k} = A[k]_i$.

This identification is functorial in $A_\bullet$. Note that since no signs are involved in this definition we actually get a compatible system of identifications of all the homology objects $H_{i-k}(A[k]_\bullet)$, which are further compatible with the identifications $A[k][l]_\bullet = A[k+l]_\bullet$ and with $[0] = \mathrm{id}_{\mathrm{Ch}(\mathcal{A})}$.

Let $\mathcal{A}$ be an additive category. Suppose that $A_\bullet$ and $B_\bullet$ are chain complexes, $a, b : A_\bullet \to B_\bullet$ are morphisms of chain complexes, and $\{h_i : A_i \to B_{i+1}\}$ is a homotopy between a and b. Recall that this means that $a_i - b_i = d_{i+1} \circ h_i + h_{i-1} \circ d_i$. What if $a = b$? Then we obtain the formula $0 = d_{i+1} \circ h_i + h_{i-1} \circ d_i$, in other words, $-d_{i+1} \circ h_i = h_{i-1} \circ d_i$. By definition above this means the collection $\{h_i\}$ above defines a morphism of chain complexes

$$A_\bullet \longrightarrow B[1]_\bullet.$$

Such a thing is the same as a morphism $A[-1]_\bullet \to B_\bullet$ by our remarks above. This proves the following lemma.

Lemma 12.14.3. *Let $\mathcal{A}$ be an additive category. Suppose that $A_\bullet$ and $B_\bullet$ are chain complexes. Given any morphism of chain complexes $a : A_\bullet \to B_\bullet$ there is a bijection between the set of homotopies from a to a and $\mathrm{Mor}_{Ch(\mathcal{A})}(A_\bullet, B[1]_\bullet)$. More generally, the set of homotopies between a and b is either empty or a principal homogeneous space under the group $\mathrm{Mor}_{Ch(\mathcal{A})}(A_\bullet, B[1]_\bullet)$.*

Proof. See above. □

Lemma 12.14.4. *Let $\mathcal{A}$ be an abelian category. Let*

$$0 \to A_\bullet \to B_\bullet \to C_\bullet \to 0$$

be a sort exact sequence of complexes. Suppose that $\{s_n : C_n \to B_n\}$ is a family of morphisms which split the short exact sequences $0 \to A_n \to B_n \to C_n \to 0$. Let $\pi_n : B_n \to A_n$ be the associated projections, see Lemma 12.5.10. Then the family of morphisms

$$\pi_{n-1} \circ d_{B,n} \circ s_n : C_n \to A_{n-1}$$

define a morphism of complexes $\delta(s) : C_\bullet \to A[-1]_\bullet$.

Proof. Denote $i : A_\bullet \to B_\bullet$ and $q : B_\bullet \to C_\bullet$ the maps of complexes in the short exact sequence. Then $i_{n-1} \circ \pi_{n-1} \circ d_{B,n} \circ s_n = d_{B,n} \circ s_n - s_{n-1} \circ d_{C,n}$. Hence $i_{n-2} \circ d_{A,n-1} \circ \pi_{n-1} \circ d_{B,n} \circ s_n = d_{B,n-1} \circ (d_{B,n} \circ s_n - s_{n-1} \circ d_{C,n}) = -d_{B,n-1} \circ s_{n-1} \circ d_{C,n}$ as desired. □

Lemma 12.14.5. *Notation and assumptions as in Lemma 12.14.4 above. The morphism of complexes $\delta(s) : C_\bullet \to A[-1]_\bullet$ induces the maps*

$$H_i(\delta(s)) : H_i(C_\bullet) \longrightarrow H_i(A[-1]_\bullet) = H_{i-1}(A_\bullet)$$

which occur in the long exact homology sequence associated to the short exact sequence of chain complexes by Lemma 12.12.6.

Proof. Omitted. □

Lemma 12.14.6. *Notation and assumptions as in Lemma 12.14.4 above. Suppose $\{s'_n : C_n \to B_n\}$ is a second choice of splittings. Write $s'_n = s_n + i_n \circ h_n$ for some unique morphisms $h_n : C_n \to A_n$. The family of maps $\{h_n : C_n \to A[-1]_{n+1}\}$ is a homotopy between the associated morphisms $\delta(s), \delta(s') : C_\bullet \to A[-1]_\bullet$.*

Proof. Omitted. □

Definition 12.14.7. Let $\mathcal{A}$ be an additive category. Let $A^\bullet$ be a cochain complex with boundary maps $d_A^n : A^n \to A^{n+1}$. For any $k \in \mathbf{Z}$ we define the *k-shifted cochain complex* $A[k]^\bullet$ as follows:

(1) we set $A[k]^n = A^{n+k}$, and
(2) we set $d^n_{A[k]} : A[k]^n \to A[k]^{n-1}$ equal to $d^n_{A[k]} = (-1)^k d_A^{n+k}$.

If $f : A^\bullet \to B^\bullet$ is a morphism of cochain complexes, then we let $f[k] : A[k]^\bullet \to B[k]^\bullet$ be the morphism of cochain complexes with $f[k]^n = f^{k+n}$.

Of course this means we have functors $[k] : \mathrm{CoCh}(\mathcal{A}) \to \mathrm{CoCh}(\mathcal{A})$ which mutually commute (on the nose, without any intervening isomorphisms of functors) and such that $A[k][l]^\bullet = A[k+l]^\bullet$ and with $[0] = \mathrm{id}_{\mathrm{CoCh}(\mathcal{A})}$.

Definition 12.14.8. Let $\mathcal{A}$ be an abelian category. Let $A^\bullet$ be a cochain complex with boundary maps $d_A^n : A^n \to A^{n+1}$. For any $k \in \mathbf{Z}$ we identify $H^{i+k}(A^\bullet) \longrightarrow H^i(A[k]^\bullet)$ via the identification $A^{i+k} = A[k]^i$.

This identification is functorial in $A^\bullet$. Note that since no signs are involved in this definition we actually get a compatible system of identifications of all the homology objects $H^{i-k}(A[k]^\bullet)$, which are further compatible with the identifications $A[k][l]^\bullet = A[k+l]^\bullet$ and with $[0] = \mathrm{id}_{\mathrm{CoCh}(\mathcal{A})}$.

Let $\mathcal{A}$ be an additive category. Suppose that $A^\bullet$ and $B^\bullet$ are cochain complexes, $a, b : A^\bullet \to B^\bullet$ are morphisms of cochain complexes, and $\{h^i : A^i \to B^{i-1}\}$ is a homotopy between a and b. Recall that this means that $a^i - b^i = d^{i-1} \circ h^i + h^{i+1} \circ d^i$. What if $a = b$? Then we obtain the formula $0 = d^{i-1} \circ h^i + h^{i+1} \circ d^i$, in other words, $-d^{i-1} \circ h^i = h^{i+1} \circ d^i$. By definition above this means the collection $\{h^i\}$ above defines a morphism of cochain complexes

$$A^\bullet \longrightarrow B[-1]^\bullet.$$

Such a thing is the same as a morphism $A[1]^\bullet \to B^\bullet$ by our remarks above. This proves the following lemma.

Lemma 12.14.9. *Let $\mathcal{A}$ be an additive category. Suppose that $A^\bullet$ and $B^\bullet$ are cochain complexes. Given any morphism of cochain complexes $a : A^\bullet \to B^\bullet$ there is a bijection between the set of homotopies from a to a and $\mathrm{Mor}_{CoCh(\mathcal{A})}(A^\bullet, B[-1]^\bullet)$. More generally, the set of homotopies between a and b is either empty or a principal homogeneous space under the group $\mathrm{Mor}_{CoCh(\mathcal{A})}(A^\bullet, B[-1]^\bullet)$.*

Proof. See above. □

Lemma 12.14.10. *Let $\mathcal{A}$ be an additive category. Let*

$$0 \to A^\bullet \to B^\bullet \to C^\bullet \to 0$$

be a complex (!) of complexes. Suppose that we are given splittings $B^n = A^n \oplus C^n$ compatible with the maps in the displayed sequence. Let $s^n : C^n \to B^n$ and $\pi^n : B^n \to A^n$ be the corresponding maps. Then the family of morphisms

$$\pi^{n+1} \circ d_B^n \circ s^n : C^n \to A^{n+1}$$

define a morphism of complexes $\delta : C^\bullet \to A[1]^\bullet$.

Proof. Denote $i : A^\bullet \to B^\bullet$ and $q : B^\bullet \to C^\bullet$ the maps of complexes in the short exact sequence. Then $i^{n+1} \circ \pi^{n+1} \circ d_B^n \circ s^n = d_B^n \circ s^n - s^{n+1} \circ d_C^n$. Hence $i^{n+2} \circ d_A^{n+1} \circ \pi^{n+1} \circ d_B^n \circ s^n = d_B^{n+1} \circ (d_B^n \circ s^n - s^{n+1} \circ d_C^n) = -d_B^{n+1} \circ s^{n+1} \circ d_C^n$ as desired. □

Lemma 12.14.11. *Notation and assumptions as in Lemma 12.14.10 above. Assume in addition that $\mathcal{A}$ is abelian. The morphism of complexes $\delta : C^\bullet \to A[1]^\bullet$ induces the maps*

$$H^i(\delta) : H^i(C^\bullet) \longrightarrow H^i(A[1]^\bullet) = H^{i+1}(A^\bullet)$$

which occur in the long exact homology sequence associated to the short exact sequence of cochain complexes by Lemma 12.12.12.

Proof. Omitted. □

Lemma 12.14.12. *Notation and assumptions as in Lemma 12.14.10. Let $\alpha : A^\bullet \to B^\bullet$, $\beta : B^\bullet \to C^\bullet$ be the given morphisms of complexes. Suppose $(s')^n : C^n \to B^n$ and $(\pi')^n : B^n \to A^n$ is a second choice of splittings. Write $(s')^n = s^n + \alpha^n \circ h^n$ and $(\pi')^n = \pi^n + g^n \circ \beta^n$ for some unique morphisms $h^n : C^n \to A^n$ and $g^n : C^n \to A^n$. Then*

1. *$g^n = -h^n$, and*
2. *the family of maps $\{g^n : C^n \to A[1]^{n-1}\}$ is a homotopy between $\delta, \delta' : C^\bullet \to A[1]^\bullet$, more precisely $(\delta')^n = \delta^n + g^{n+1} \circ d_C^n + d_{A[1]}^{n-1} \circ g^n$.*

Proof. As $(s')^n$ and $(\pi')^n$ are splittings we have $(\pi')^n \circ (s')^n = 0$. Hence

$$0 = (\pi^n + g^n \circ \beta^n) \circ (s^n + \alpha^n \circ h^n) = g^n \circ \beta^n \circ s^n + \pi^n \circ \alpha^n \circ h^n = g^n + h^n$$

which proves (1). We compute $(\delta')^n$ as follows

$$(\pi^{n+1} + g^{n+1} \circ \beta^{n+1}) \circ d_B^n \circ (s^n + \alpha^n \circ h^n) = \delta^n + g^{n+1} \circ d_C^n + d_A^n \circ h^n$$

Since $h^n = -g^n$ and since $d_{A[1]}^{n-1} = -d_A^n$ we conclude that (2) holds. □

12.15. Graded objects

We make the following definition.

Definition 12.15.1. Let $\mathcal{A}$ be an additive category. The *category of graded objects of* $\mathcal{A}$, denoted $\mathrm{Gr}(\mathcal{A})$, is the category with

(1) objects $A = (A^i)$ are families of objects A^i, $i \in \mathbf{Z}$ of objects of $\mathcal{A}$, and
(2) morphisms $f : A = (A^i) \to B = (B^i)$ are families of morphisms $f^i : A^i \to B^i$ of $\mathcal{A}$.

If $\mathcal{A}$ has countable direct sums, then we can asssociated to an object $A = (A^i)$ of $\mathrm{Gr}(\mathcal{A})$ the object

$$A = \bigoplus\nolimits_{i \in \mathbf{Z}} A^i$$

and set $k^i A = A^i$. In this case $\mathrm{Gr}(\mathcal{A})$ is equivalent to the category of pairs (A, k) consisting of an object A of $\mathcal{A}$ and a direct sum decomposition

$$A = \bigoplus\nolimits_{i \in \mathbf{Z}} k^i A$$

by direct summands indexed by $\mathbf{Z}$ and a morphism $(A, k) \to (B, k)$ of such objects is given by a morphism $\varphi : A \to B$ of $\mathcal{A}$ such that $\varphi(k^i A) \subset k^i B$ for all $i \in \mathbf{Z}$. Whenever our additive category $\mathcal{A}$ has countable direct sums we will use this equivalence without further mention.

However, with our definitions an additive or abelian category does not necessarily have all (countable) direct sums. In this case our definition still makes sense. For example, if $\mathcal{A} = \mathrm{Vect}_k$ is the category of finite dimensional vector spaces over a field k, then $\mathrm{Gr}(\mathrm{Vect}_k)$ is the category of vector spaces with a given gradation all of whose graded pieces are finite dimensional, and not the category of finite dimensional vector spaces with a given graduation.

Lemma 12.15.2. *Let $\mathcal{A}$ be an abelian category. The category of graded objects $Gr(\mathcal{A})$ is abelian.*

Proof. Omitted. □

Definition 12.15.3. Let $\mathcal{A}$ be an additive category. If $A = (A^i)$ is a graded object, then the kth *shift* $A[k]$ is the graded object with $A[k]^i = A^{k+i}$.

If A and B are graded objects of $\mathcal{A}$, then we have

$$\mathrm{Hom}_{\mathrm{Gr}(\mathcal{A})}(A, B[k]) = \mathrm{Hom}_{\mathrm{Gr}(\mathcal{A})}(A[-k], B) \tag{12.15.3.1}$$

and an element of this group is sometimes called a map of graded objects *homogeneous of degree k*.

12.16. Filtrations

A nice reference for this material is [**Del71**, Section 1]. (Note that our conventions regarding abelian categories are different.)

Definition 12.16.1. Let $\mathcal{A}$ be an abelian category.

(1) A *decreasing filtration* F on an object A is a family $(F^n A)_{n \in \mathbf{Z}}$ of subobjects of A such that

$$A \supset \ldots \supset F^n A \supset F^{n+1} A \supset \ldots \supset 0$$

(2) A *filtered object of* $\mathcal{A}$ is pair (A, F) consisting of an object A of $\mathcal{A}$ and a decreasing filtration F on A.
(3) A *morphism* $(A, F) \to (B, F)$ *of filtered objects* is given by a morphism $\varphi : A \to B$ of $\mathcal{A}$ such that $\varphi(F^i A) \subset F^i B$ for all $i \in \mathbf{Z}$.
(4) The category of filtered objects is denoted $\mathrm{Fil}(\mathcal{A})$.
(5) Given a filtered object (A, F) and a subobject $X \subset A$ the *induced filtration* on X is the filtration with $F^n X = X \cap F^n A$.
(6) Given a filtered object (A, F) and a surjection $\pi : A \to Y$ the *quotient filtration* is the filtration with $F^n Y = \pi(F^n A)$.
(7) A filtration F on an object A is said to be *finite* if there exist n, m such that $F^n A = A$ and $F^m A = 0$.
(8) The filtration on a filtered object (A, F) is said to be *separated* if $\bigcap_i F^i A = 0$ and *exhaustive* if $\bigcup F^i A = A$.

By abuse of notation we say that a morphism $f : (A, F) \to (B, F)$ of filtered objects is *injective* if $f : A \to B$ is injective in the abelian category $\mathcal{A}$. Similarly we say f is *surjective* if $f : A \to B$ is surjective in the category $\mathcal{A}$. Being injective (resp. surjective) is equivalent to being a monomorphism (resp. epimorphism) in $\mathrm{Fil}(\mathcal{A})$. By Lemma 12.16.2 this is also equivalent to having zero kernel (resp. cokernel).

Lemma 12.16.2. *Let $\mathcal{A}$ be an abelian category. The category of filtered objects $Fil(\mathcal{A})$ has the following properties:*

(1) *It is an additive category.*
(2) *It has a zero object.*
(3) *It has kernels and cokernels, images and coimages.*
(4) *In general it is not an abelian category.*

Proof. It is clear that $\mathrm{Fil}(\mathcal{A})$ is additive with direct sum given by $(A, F) \oplus (B, F) = (A \oplus B, F)$ where $F^p(A \oplus B) = F^p A \oplus F^p B$. The kernel of a morphism $f : (A, F) \to (B, F)$ of filtered objects is the injection $\mathrm{Ker}(f) \subset A$ where $\mathrm{Ker}(f)$ is endowed with the induced filtration. The cokernel of a morphism $f : A \to B$ of filtered objects is the surjection $B \to \mathrm{Coker}(f)$ where $\mathrm{Coker}(f)$ is endowed with the quotient filtration. Since all kernels and cokernels exist, so do all coimages and images. See Example 12.3.12 for the last statement. □

Definition 12.16.3. Let $\mathcal{A}$ be an abelian category. A morphism $f : A \to B$ of filtered objects of $\mathcal{A}$ is said to be *strict* if $f(F^i A) = f(A) \cap F^i B$ for all $i \in \mathbf{Z}$.

This also equivalent to requiring that $f^{-1}(F^i B) = F^i A + \mathrm{Ker}(f)$ for all $i \in \mathbf{Z}$. We characterize strict morphisms as follows.

Lemma 12.16.4. *Let $\mathcal{A}$ be an abelian category. Let $f : A \to B$ be a morphism of filtered objects of $\mathcal{A}$. The following are equivalent*

(1) *f is strict,*
(2) *the morphism $Coim(f) \to Im(f)$ of Lemma 12.3.11 is an isomorphism.*

Proof. Note that $\mathrm{Coim}(f) \to \mathrm{Im}(f)$ is an isomorphism of objects of $\mathcal{A}$, and that part (2) signifies that it is an isomorphism of filtered objects. By the description of kernels and cokernels in the proof of Lemma 12.16.2 we see that the filtration on $\mathrm{Coim}(f)$ is the quotient filtration coming from $A \to \mathrm{Coim}(f)$. Similarly, the filtration on $\mathrm{Im}(f)$ is the induced filtration coming from the injection $\mathrm{Im}(f) \to$

B. The definition of strict is exactly that the quotient filtration is the induced filtration. □

Lemma 12.16.5. *Let $\mathcal{A}$ be an abelian category. A composition of strict morphisms of filtered objects is strict.*

Proof. Let $f : A \to B$, $g : B \to C$ be strict morphisms of filtered objects. Then

$$\begin{aligned} g(f(F^pA)) &= g(f(A) \cap F^pB) \\ &\supset g(f(A)) \cap g(F^p(B)) \\ &= (g \circ f)(A) \cap (g(B) \cap F^pC) \\ &= (g \circ f)(A) \cap F^pC. \end{aligned}$$

The inclusion $g(f(F^pA)) \subset (g \circ f)(A) \cap F^pC$ is always true. □

Lemma 12.16.6. *Let $\mathcal{A}$ be an abelian category. Let $f : A \to B$ be a strict monomorphism of filtered objects. Let $g : A \to C$ be a morphism of filtered objects. Then $f \oplus g : A \to B \oplus C$ is a strict monomorphism.*

Proof. Clear from the definitions. □

Lemma 12.16.7. *Let $\mathcal{A}$ be an abelian category. Let $f : B \to A$ be a strict epimorphism of filtered objects. Let $g : C \to A$ be a morphism of filtered objects. Then $f \oplus g : B \oplus C \to A$ is a strict epimorphism.*

Proof. Clear from the definitions. □

Lemma 12.16.8. *Let $\mathcal{A}$ be an abelian category. Let (A, F), (B, F) be filtered objects. Let $u : A \to B$ be a morphism of filtered objects. If u is injective then u is strict if and only if the filtration on A is the induced filtration. If u is surjective then u is strict if and only if the filtration on B is the quotient filtration.*

Proof. This is immediate from the definition. □

The following lemma says that subobjects of a filtered object have a well defined filtration independent of a choice of writing the object as a cokernel.

Lemma 12.16.9. *Let $\mathcal{A}$ be an abelian category. Let (A, F) be a filtered object of $\mathcal{A}$. Let $X \subset Y \subset A$ be subobjects of A. On the object*

$$Y/X = Ker(A/X \to A/Y)$$

the quotient filtration coming from the induced filtration on Y and the induced filtration coming from the quotient filtration on A/X agree. Any of the morphisms $X \to Y$, $X \to A$, $Y \to A$, $Y \to A/X$, $Y \to Y/X$, $Y/X \to A/X$ are strict (with induced/quotient filtrations).

Proof. The quotient filtration Y/X is given by $F^p(Y/X) = F^pY/(X \cap F^pY) = F^pY/F^pX$ because $F^pY = Y \cap F^pA$ and $F^pX = X \cap F^pA$. The induced filtration from the injection $Y/X \to A/X$ is given by

$$\begin{aligned} F^p(Y/X) &= Y/X \cap F^p(A/X) \\ &= Y/X \cap (F^pA + X)/X \\ &= (Y \cap F^pA)/(X \cap F^pA) \\ &= F^pY/F^pX. \end{aligned}$$

Hence the first statement of the lemma. The proof of the other cases is similar. □

Lemma 12.16.10. *Let $\mathcal{A}$ be an abelian category. Let $A, B, C \in Fil(\mathcal{A})$. Let $f : A \to B$ and $g : A \to C$ be morphisms Then there exists a pushout*

$$\begin{array}{ccc} A & \xrightarrow{f} & B \\ {\scriptstyle g}\downarrow & & \downarrow{\scriptstyle g'} \\ C & \xrightarrow{f'} & C \amalg_A B \end{array}$$

in $Fil(\mathcal{A})$. If f is strict, so is f'.

Proof. Set $C \amalg_A B$ equal to $\mathrm{Coker}((1, -1) : A \to C \oplus B)$ in $\mathrm{Fil}(\mathcal{A})$. This cokernel exists, by Lemma 12.16.2. It is a pushout, see Example 12.5.6. Note that $F^p(C \times_A B)$ is the image of $F^pC \oplus F^pB$. Hence

$$(f')^{-1}(F^p(C \times_A B)) = g(f^{-1}(F^pB))) + F^pC$$

Whence the last statement. □

Lemma 12.16.11. *Let $\mathcal{A}$ be an abelian category. Let $A, B, C \in Fil(\mathcal{A})$. Let $f : B \to A$ and $g : C \to A$ be morphisms Then there exists a pushout*

$$\begin{array}{ccc} B \times_A C & \xrightarrow{f'} & B \\ {\scriptstyle g'}\downarrow & & \downarrow{\scriptstyle g} \\ C & \xrightarrow{f} & A \end{array}$$

in $Fil(\mathcal{A})$. If f is strict, so is f'.

Proof. This lemma is dual to Lemma 12.16.10. □

Let $\mathcal{A}$ be an abelian category. Let (A, F) be a filtered object of $\mathcal{A}$. We denote $\mathrm{gr}^p_F(A) = \mathrm{gr}^p(A)$ the object $F^pA/F^{p+1}A$ of $\mathcal{A}$. This defines an additive functor

$$\mathrm{gr}^p : \mathrm{Fil}(\mathcal{A}) \longrightarrow \mathcal{A}, \quad (A, F) \longmapsto \mathrm{gr}^p(A).$$

Recall that we have defined the category $\mathrm{Gr}(\mathcal{A})$ of graded objects of $\mathcal{A}$ in Section 12.15. For (A, F) in $\mathrm{Fil}(\mathcal{A})$ we may set

$$\mathrm{gr}(A) = \text{the graded object of } \mathcal{A} \text{ whose } p\text{th graded piece is } \mathrm{gr}^p(A)$$

and if $\mathcal{A}$ has countable direct sums, then we simply have

$$\mathrm{gr}(A) = \bigoplus \mathrm{gr}^p(A)$$

This defines an additive functor

$$\mathrm{gr} : \mathrm{Fil}(\mathcal{A}) \longrightarrow \mathrm{Gr}(\mathcal{A}), \quad (A, F) \longmapsto \mathrm{gr}(A).$$

Lemma 12.16.12. *Let $\mathcal{A}$ be an abelian category.*

(1) *Let A be a filtered object and $X \subset A$. Then for each p the sequence*

$$0 \to gr^p(X) \to gr^p(A) \to gr^p(A/X) \to 0$$

is exact (with induced filtration on X and quotient filtration on A/X).

(2) *Let $f : A \to B$ be a morphism of filtered objects of $\mathcal{A}$. Then for each p the sequences*

$$0 \to gr^p(Ker(f)) \to gr^p(A) \to gr^p(Coim(f)) \to 0$$

and

$$0 \to gr^p(Im(f)) \to gr^p(B) \to gr^p(Coker(f)) \to 0$$

are exact.

Proof. We have $F^{p+1}X = X \cap F^{p+1}A$, hence map $\mathrm{gr}^p(X) \to \mathrm{gr}^p(A)$ is injective. Dually the map $\mathrm{gr}^p(A) \to \mathrm{gr}^p(A/X)$ is surjective. The kernel of $F^pA/F^{p+1}A \to A/X + F^{p+1}A$ is clearly $F^{p+1}A + X \cap F^pA/F^{p+1}A = F^pX/F^{p+1}X$ hence exactness in the middle. The two short exact sequence of (2) are special cases of the short exact sequence of (1). □

Lemma 12.16.13. *Let $\mathcal{A}$ be an abelian category. Let $f : A \to B$ be a morphism of finite filtered objects of $\mathcal{A}$. The following are equivalent*

(1) *f is strict,*
(2) *the morphism $Coim(f) \to Im(f)$ is an isomorphism,*
(3) *$gr(Coim(f)) \to gr(Im(f))$ is an isomorphism,*
(4) *the sequence $gr(Ker(f)) \to gr(A) \to gr(B)$ is exact,*
(5) *the sequence $gr(A) \to gr(B) \to gr(Coker(f))$ is exact, and*
(6) *the sequence*

$$0 \to gr(Ker(f)) \to gr(A) \to gr(B) \to gr(Coker(f)) \to 0$$

is exact.

Proof. The equivalence of (1) and (2) is Lemma 12.16.4. By Lemma 12.16.12 we see that (4), (5), (6) imply (3) and that (3) implies (4), (5), (6). Hence it suffices to show that (3) implies (2). Thus we have to show that if $f : A \to B$ is an injective and surjective map of finite filtered objects which induces and isomorphism $\mathrm{gr}(A) \to \mathrm{gr}(B)$, then f induces an isomorphism of filtered objects. In other words, we have to show that $f(F^pA) = F^pB$ for all p. As the filtrations are finite we may prove this by descending induction on p. Suppose that $f(F^{p+1}A) = F^{p+1}B$. Then commutative diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & F^{p+1}A & \longrightarrow & F^pA & \longrightarrow & \mathrm{gr}^p(A) & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle f} & & \downarrow{\scriptstyle f} & & \downarrow{\scriptstyle \mathrm{gr}^p(f)} & & \\ 0 & \longrightarrow & F^{p+1}B & \longrightarrow & F^pB & \longrightarrow & \mathrm{gr}^p(B) & \longrightarrow & 0 \end{array}$$

and the five lemma imply that $f(F^pA) = F^pB$. □

Lemma 12.16.14. *Let $\mathcal{A}$ be an abelian category. Let $A \to B \to C$ be a complex of filtered objects of $\mathcal{A}$. Assume $\alpha : A \to B$ and $\beta : B \to C$ are strict morphisms of filtered objects. Then $gr(Ker(\beta)/Im(\alpha)) = Ker(gr(\beta))/Im(gr(\alpha)))$.*

Proof. This follows formally from Lemma 12.16.12 and the fact that $\mathrm{Coim}(\alpha) \cong \mathrm{Im}(\alpha)$ and $\mathrm{Coim}(\beta) \cong \mathrm{Im}(\beta)$ by Lemma 12.16.4. □

Lemma 12.16.15. *Let $\mathcal{A}$ be an abelian category. Let $A \to B \to C$ be a complex of filtered objects of $\mathcal{A}$. Assume A, B, C have finite filtrations and that $gr(A) \to gr(B) \to gr(C)$ is exact. Then*

(1) *for each $p \in \mathbf{Z}$ the sequence $gr^p(A) \to gr^p(B) \to gr^p(C)$ is exact,*
(2) *for each $p \in \mathbf{Z}$ the sequence $F^p(A) \to F^p(B) \to F^p(C)$ is exact,*
(3) *for each $p \in \mathbf{Z}$ the sequence $A/F^p(A) \to B/F^p(B) \to C/F^p(C)$ is exact,*
(4) *the maps $A \to B$ and $B \to C$ are strict, and*
(5) *$A \to B \to C$ is exact (as a sequence in $\mathcal{A}$).*

Proof. Part (1) is immediate from the definitions. We will prove (3) by induction on the length of the filtrations. If each of A, B, C has only one nonzero graded part, then (3) holds as $\mathrm{gr}(A) = A$, etc. Let n be the largest integer such that at least one of F^nA, F^nB, F^nC is nonzero. Set $A' = A/F^nA$, $B' = B/F^nB$, $C' = C/F^nC$ with induced filtrations. Note that $\mathrm{gr}(A) = F^nA \oplus \mathrm{gr}(A')$ and similarly for B and C. The induction hypothesis applies to $A' \to B' \to C'$, which implies that $A/F^p(A) \to B/F^p(B) \to C/F^p(C)$ is exact for $p \geq n$. To conclude the same for $p = n+1$, i.e., to prove that $A \to B \to C$ is exact we use the commutative diagram

$$\begin{array}{ccccccccc}
0 & \longrightarrow & F^nA & \longrightarrow & A & \longrightarrow & A' & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & F^nB & \longrightarrow & B & \longrightarrow & B' & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & F^nC & \longrightarrow & C & \longrightarrow & C' & \longrightarrow & 0
\end{array}$$

whose rows are short exact sequences of objects of $\mathcal{A}$. The proof of (2) is dual. Of course (5) follows from (2).

To prove (4) denote $f : A \to B$ and $g : B \to C$ the given morphisms. We know that $f(F^p(A)) = \mathrm{Ker}(F^p(B) \to F^p(C))$ by (2) and $f(A) = \mathrm{Ker}(g)$ by (5). Hence $f(F^p(A)) = \mathrm{Ker}(F^p(B) \to F^p(C)) = \mathrm{Ker}(g) \cap F^p(B) = f(A) \cap F^p(B)$ which proves that f is strict. The proof that g is strict is dual to this. □

12.17. Spectral sequences

A nice discussion of spectral sequences may be found in [**Eis95**]. See also [**McC01**], [**Lan02**], etc.

Definition 12.17.1. Let $\mathcal{A}$ be an abelian category.

(1) A *spectral sequence in* $\mathcal{A}$ is given by a system $(E_r, d_r)_{r \geq 1}$ where each E_r is an object of $\mathcal{A}$, each $d_r : E_r \to E_r$ is a morphism such that $d_r \circ d_r = 0$ and $E_{r+1} = \mathrm{Ker}(d_r)/\mathrm{Im}(d_r)$ for $r \geq 1$.

(2) A *morphism of spectral sequences* $f : (E_r, d_r)_{r \geq 1} \to (E'_r, d'_r)_{r \geq 1}$ is given by a family of morphisms $f_r : E_r \to E'_r$ such that $f_r \circ d_r = d'_r \circ f_r$ and such that f_{r+1} is the morphism induced by f_r via the identifications $E_{r+1} = \mathrm{Ker}(d_r)/\mathrm{Im}(d_r)$ and $E'_{r+1} = \mathrm{Ker}(d'_r)/\mathrm{Im}(d'_r)$.

We will sometimes loosen this definition somewhat and allow E_{r+1} to be an object with a given isomorphism $E_{r+1} \to \mathrm{Ker}(d_r)/\mathrm{Im}(d_r)$. In addition we sometimes have a system $(E_r, d_r)_{r \geq r_0}$ for some r_0 satisfying the properties of the definition above for indices $\geq r$. We will also call this a spectral sequence since by a simple renumbering it falls under the definition anyway. In fact, sometimes it makes sense to allow $r_0 = 0$ or even $r_0 = -1$ due to conventions in the literature.

Given a spectral sequence $(E_r, d_r)_{r \geq 1}$ we define

$$0 = B_1 \subset B_2 \subset \ldots \subset B_r \subset \ldots \subset Z_r \subset \ldots \subset Z_2 \subset Z_1 = E_1$$

by the following simple procedure. Set $B_2 = \mathrm{Im}(d_1)$ and $Z_2 = \mathrm{Ker}(d_1)$. Then it is clear that $d_2 : Z_2/B_2 \to Z_2/B_2$. Hence we can define B_3 as the unique subobject of E_1 containing B_2 such that B_3/B_2 is the image of d_2. Similarly we can define

Z_3 as the unique subobject of E_1 containing B_2 such that Z_3/B_2 is the kernel of d_2. And so on and so forth. In particular we have

$$E_r = Z_r/B_r$$

for all $r \geq 1$. I case the spectral sequence starts at $r = r_0$ then we can similarly construct B_i, Z_i as subobjects in E_{r_0}.

Definition 12.17.2. Let $\mathcal{A}$ be an abelian category. Let $(E_r, d_r)_{r\geq 1}$ be a spectral sequence.

(1) If the subobjects $Z_\infty = \bigcap Z_r$ and $B_\infty = \bigcup B_r$ of E_1 exist then we define the *limit* of the spectral sequence to be the object

$$E_\infty = Z_\infty/B_\infty.$$

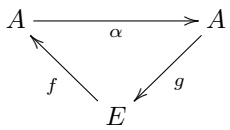
(2) We say that the spectral sequence *collapses at* E_r, or *degenerates at* E_r if the differentials $d_r, d_{r+1}, \ldots$ are all zero.

Note that if the spectral sequence collapses at E_r, then we have $E_r = E_{r+1} = \ldots = E_\infty$ (and the limit exists of course). Also, almost any abelian category we will encounter has countable sums and intersections.

12.18. Spectral sequences: exact couples

Definition 12.18.1. Let $\mathcal{A}$ be an abelian category.

(1) An *exact couple* is a datum (A, E, α, f, g) where A, E are objects of $\mathcal{A}$ and α, f, g are morphisms as in the following diagram



with the property that the kernel of each arrow is the image of its predecessor. So $\mathrm{Ker}(\alpha) = \mathrm{Im}(f)$, $\mathrm{Ker}(f) = \mathrm{Im}(g)$, and $\mathrm{Ker}(g) = \mathrm{Im}(\alpha)$.

(2) A *morphism of exact couples* $t : (A, E, \alpha, f, g) \to (A', E', \alpha', f', g')$ is given by morphisms $t_A : A \to A'$ and $t_E : E \to E'$ such that $\alpha' \circ t_A = t_A \circ \alpha$, $f' \circ t_E = t_A \circ f$, and $g' \circ t_A = t_E \circ g$.

Lemma 12.18.2. *Let $\mathcal{A}$ be an abelian category. Let (A, E, α, f, g) be an exact couple. Set*

(1) *$d = g \circ f : E \to E$ so that $d \circ d = 0$,*
(2) *$E' = Ker(d)/Im(d)$,*
(3) *$A' = Im(\alpha)$,*
(4) *$\alpha' : A' \to A'$ induced by α,*
(5) *$f' : E' \to A'$ induced by f,*
(6) *$g' : A' \to E'$ induced by "$g \circ \alpha^{-1}$".*

Then we have

(1) *$Ker(d) = f^{-1}(Ker(g)) = f^{-1}(Im(\alpha))$,*
(2) *$Im(d) = g(Im(f)) = g(Ker(\alpha))$,*
(3) *$(A', E', \alpha', f', g')$ is an exact couple.*

Proof. Omitted. □

Hence it is clear that given an exact couple (A, E, α, f, g) we get a spectral sequence by setting $E_1 = E$, $d_1 = d$, $E_2 = E'$, $d_2 = d' = g' \circ f'$, $E_3 = E''$, $d_3 = d'' = g'' \circ f''$, and so on.

Definition 12.18.3. Let $\mathcal{A}$ be an abelian category. Let (A, E, α, f, g) be an exact couple. The *spectral sequence associated to the exact couple* is the spectral sequence $(E_r, d_r)_{r \geq 1}$ with $E_1 = E$, $d_1 = d$, $E_2 = E'$, $d_2 = d' = g' \circ f'$, $E_3 = E''$, $d_3 = d'' = g'' \circ f''$, and so on.

Lemma 12.18.4. *Let $\mathcal{A}$ be an abelian category. Let (A, E, α, f, g) be an exact couple. Let $(E_r, d_r)_{r \geq 1}$ be the spectral sequence associated to the exact couple. In this case we have*

$$0 = B_1 \subset \ldots \subset B_{r+1} = g(Ker(\alpha^r)) \subset \ldots \subset Z_{r+1} = f^{-1}(Im(\alpha^r)) \subset \ldots \subset Z_1 = E$$

and the map $d_{r+1} : E_{r+1} \to E_{r+1}$ is described by the following rule: For any (test) object T of $\mathcal{A}$ and any elements $x : T \to Z_{r+1}$ and $y : T \to A$ such that $f \circ x = \alpha^r \circ y$ we have

$$d_r \circ \overline{x} = \overline{g \circ y}$$

where $\overline{x} : T \to E_{r+1}$ is the induced morphism.

Proof. Omitted. □

Note that in the situation of the lemma we obviously have

$$B_\infty = g\left(\bigcup\nolimits_r \mathrm{Ker}(\alpha^r)\right) \subset Z_\infty = f^{-1}\left(\bigcap\nolimits_r \mathrm{Im}(\alpha^r)\right)$$

provided this exist and in this case $E_\infty = Z_\infty / B_\infty$.

12.19. Spectral sequences: differential objects

Definition 12.19.1. Let $\mathcal{A}$ be an abelian category. A *differential object* of $\mathcal{A}$ is a pair (A, d) consisting of an object A of $\mathcal{A}$ endowed with a selfmap d such that $d \circ d = 0$. A *morphism of differential objects* $(A, d) \to (B, d)$ is given by a morphism $\alpha : A \to B$ such that $d \circ \alpha = \alpha \circ d$.

Lemma 12.19.2. *Let $\mathcal{A}$ be an abelian category. The category of differential objects of $\mathcal{A}$ is abelian.*

Proof. Omitted. □

Definition 12.19.3. For a differential object (A, d) we denote

$$H(A, d) = \mathrm{Ker}(d)/\mathrm{Im}(d)$$

its *homology*.

Lemma 12.19.4. *Let $\mathcal{A}$ be an abelian category. Let $0 \to (A, d) \to (B, d) \to (C, d) \to 0$ be a short exact sequence of differential objects. Then we get an exact homology sequence*

$$\ldots \to H(C, d) \to H(A, d) \to H(B, d) \to H(C, d) \to \ldots$$

Proof. Apply Lemma 12.12.12 to the short exact sequence of complexes

$$\begin{array}{ccccccccc} 0 & \to & A & \to & B & \to & C & \to & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \to & A & \to & B & \to & C & \to & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \to & A & \to & B & \to & C & \to & 0 \end{array}$$

□

We come to an important example of a spectral sequence. Let $\mathcal{A}$ be an abelian category. Let (A, d) be a differential object of $\mathcal{A}$. Let $\alpha : (A, d) \to (A, d)$ be an endomorphism of this differential object. If we assume α injective, then we get a short exact sequence

$$0 \to (A, d) \to (A, d) \to (A/\alpha A, d) \to 0$$

of differential objects. By the Lemma 12.19.4 we get an exact couple

$$\begin{array}{ccc} H(A,d) & \xrightarrow{\overline{\alpha}} & H(A,d) \\ & {\scriptstyle f}\nwarrow \quad \swarrow{\scriptstyle g} & \\ & H(A/\alpha A, d) & \end{array}$$

where g is the canonical map and f is the map defined in the snake lemma. Thus we get an associated spectral sequence! Since in this case we have $E_1 = H(A/\alpha A, d)$ we see that it makes sense to define $E_0 = A/\alpha A$ and $d_0 = d$. In other words, we start the spectral sequence with $r = 0$. According to our conventions in Section 12.17 we define a sequence of subobjects

$$0 = B_0 \subset \ldots \subset B_r \subset \ldots \subset Z_r \subset \ldots \subset Z_0 = E_0$$

with the property that $E_r = Z_r/B_r$. Namely we have for $r \geq 1$ that

(1) B_r is the image of $(\alpha^{r-1})^{-1}(dA)$ under the natural map $A \to A/\alpha A$,
(2) Z_r is the image of $d^{-1}(\alpha^r A)$ under the natural map $A \to A/\alpha A$, and
(3) $d_r : E_r \to E_r$ is given as follows: given an element $z \in Z_r$ choose an element $y \in A$ such that $d(z) = \alpha^r(y)$. Then $d_r(z+B_r+\alpha A) = y+B_r+\alpha A$.

Warning: It is not necessarily the case that $\alpha A \subset (\alpha^{r-1})^{-1}(dA)$, nor $\alpha A \subset d^{-1}(\alpha^r A)$. It is true that $(\alpha^{r-1})^{-1}(dA) \subset d^{-1}(\alpha^r A)$. We have

$$E_r = \frac{d^{-1}(\alpha^r A) + \alpha A}{(\alpha^{r-1})^{-1}(dA) + \alpha A}.$$

It is not hard to verify directly that (1) – (3) give a spectral sequence.

Definition 12.19.5. Let $\mathcal{A}$ be an abelian category. Let (A, d) be a differential object of $\mathcal{A}$. Let $\alpha : A \to A$ be an injective selfmap of A which commutes with d. The *spectral sequence associated to* (A, d, α) is the spectral sequence $(E_r, d_r)_{r\geq 0}$ described above.

12.20. Spectral sequences: filtered differential objects

Definition 12.20.1. Let $\mathcal{A}$ be an abelian category. A *filtered differential object* (K, F, d) is a filtered object (K, F) of $\mathcal{A}$ endowed with an endomorphism $d : (K, F) \to (K, F)$ whose square is zero: $d \circ d = 0$.

Let $\mathcal{A}$ be an abelian category. Let (K, F, d) be a filtered differential object of $\mathcal{A}$. Note that each $F^n K$ is a differential object by itself. Assume $\mathcal{A}$ has countable direct sums. In this case set $A = \bigoplus F^n K$ and endow it with a differential d by using d on each summand. Consider the map

$$\alpha : A \to A$$

which maps the summand $F^n A$ into the summand $F^{n-1} A$. This is clearly an injective morphism of differential modules $\alpha : (A, d) \to (A, d)$. Hence, by Definition 12.19.5 we get a spectral sequence. We will call this *the spectral sequence associated to the filtered differential object* (K, F, d).

Let us figure out the terms of this spectral sequence. First, note that $A/\alpha A = \mathrm{gr}(K)$ endowed with its differential $d = \mathrm{gr}(d)$. Hence we see that

$$E_0 = \mathrm{gr}(K), \quad d_0 = \mathrm{gr}(d).$$

Hence the homology of the graded differential object $\mathrm{gr}(K)$ is the next term:

$$E_1 = H(\mathrm{gr}(K), \mathrm{gr}(d)).$$

In addition we see that E_0 is a graded object of $\mathcal{A}$ and that d_0 is compatible with the grading. Hence clearly E_1 is a graded object as well. But it turns out that the differential d_1 does not preserve this grading; instead it shifts the degree by 1.

To work this out precisely, we define

$$Z_r^p = \frac{F^p K \cap d^{-1}(F^{p+r} K) + F^{p+1} K}{F^{p+1} K}$$

and

$$B_r^p = \frac{F^p K \cap d(F^{p-r+1} K) + F^{p+1} K}{F^{p+1} K}.$$

This notation, although quite natural, seems to be different from the notation in most places in the literature. Perhaps it does not matter, since the literature does not seem to have a consistent choice of notation either. With these choices we see that $B_r \subset E_0$, resp. $Z_r \subset E_0$ (as defined in Section 12.19) is equal to $\bigoplus_p B_r^p$, resp. $\bigoplus_p Z_r^p$. Hence if we define

$$E_r^p = Z_r^p / B_r^p$$

for $r \geq 0$ and $p \in \mathbf{Z}$, then we have $E_r = \bigoplus_p E_r^p$. We can define a differential $d_r^p : E_r^p \to E_r^{p+r}$ by the rule

$$z + F^{p+1} K \longmapsto dz + F^{p+r+1} K$$

where $z \in F^p K \cap d^{-1}(F^{p+r} K)$.

Lemma 12.20.2. *Let $\mathcal{A}$ be an abelian category. Let (K, F, d) be a filtered differential object of $\mathcal{A}$. Assume $\mathcal{A}$ has countable direct sums. The spectral sequence $(E_r, d_r)_{r \geq 0}$ associated to (K, F, d) has terms*

$$E_r = \bigoplus\nolimits_{p \in \mathbf{Z}} E_r^p, \quad d_r = \bigoplus\nolimits_{p \in \mathbf{Z}} d_r^p.$$

Furthermore, we have $E_0^p = gr^p K$, $d_0 = gr(d)$, and $E_1^p = H(gr^p(K), d)$.

Proof. Follows from the discussion above. □

Lemma 12.20.3. *Let $\mathcal{A}$ be an abelian category. Let (K, F, d) be a filtered differential object of $\mathcal{A}$. Assume $\mathcal{A}$ has countable direct sums. The spectral sequence $(E_r, d_r)_{r\geq 0}$ associated to (K, F, d) has*

$$d_1^p : E_1^p = H(gr^p(K)) \longrightarrow E_1^{p+1} = H(gr^{p+1}(K))$$

equal to the boundary map in homology associated to the short exact sequence of differential objects

$$0 \to gr^{p+1}(K) \to F^pK/F^{p+2}K \to gr^{p+1}(K) \to 0.$$

Proof. Omitted. □

Definition 12.20.4. Let $\mathcal{A}$ be an abelian category. Let (K, F, d) be a filtered differential object of $\mathcal{A}$. The *induced filtration* on $H(K, d)$ is the filtration defined by $F^pH(K, d) = \mathrm{Im}(H(F^pK, d) \to H(K, d))$.

Lemma 12.20.5. *Let $\mathcal{A}$ be an abelian category. Let (K, F, d) be a filtered differential object of $\mathcal{A}$. The associated graded $gr(H(K))$ of the cohomology of K is a graded subquotient of the graded object $E_\infty = \bigoplus E_\infty^p$.*

Proof. Recall that $E_\infty = Z_\infty/B_\infty$ by definition, with $B_\infty = \bigcup B_r$ and $Z_\infty = \bigcap Z_r$. Hence $E_\infty = \bigoplus E_\infty^p$ with $E_\infty^p = Z_\infty^p/B_\infty^p$ with $B_\infty^p = \bigcup B_r^p$ and $Z_\infty^p = \bigcap Z_r^p$. Thus

$$E_\infty^p = \frac{\bigcap_r (F^pK \cap d^{-1}(F^{p+r}K) + F^{p+1}K)}{\bigcup_r (F^pK \cap d(F^{p-r+1}K) + F^{p+1}K)}.$$

On the other hand, we have

$$\mathrm{gr}^pH(K) = \frac{\mathrm{Ker}(d) \cap F^pK + F^{p+1}K}{\mathrm{Im}(d) \cap F^pK + F^{p+1}K}$$

The result follows since

$$\mathrm{Ker}(d) \cap F^pK + F^{p+1}K \subset \bigcup\nolimits_r \left(F^pK \cap d^{-1}(F^{p+r}K) + F^{p+1}K\right) \tag{12.20.5.1}$$

and

$$\bigcap\nolimits_r \left(F^pK \cap d(F^{p-r+1}K) + F^{p+1}K\right) \subset \mathrm{Im}(d) \cap F^pK + F^{p+1}K. \tag{12.20.5.2}$$

□

Definition 12.20.6. Let $\mathcal{A}$ be an abelian category. Let (K, F, d) be a filtered differential object of $\mathcal{A}$. We say the spectral sequence associated to (K, F, d) *converges* if $\mathrm{gr}(H(K)) = E_\infty$ via Lemma 12.20.5. In this case we also say that $(E_r, d_r)_{r\geq 0}$ *abuts to* or *converges to* $H(K)$.

In the literature one finds more refined notions distinguishing between "weakly converging", "abutting" and "converging". Namely, one can require the filtration on $H(K)$ to be either "arbitrary", or "exhaustive and separated", or "exhaustive and complete" in addition to the condition that $\mathrm{gr}(H(K)) = E_\infty$. We try to avoid introducing this notation by simply adding the relevant information in the statements of the results.

Lemma 12.20.7. *Let $\mathcal{A}$ be an abelian category. Let (K, F, d) be a filtered differential object of $\mathcal{A}$. The associated spectral sequence converges if and only if for every $p \in \mathbf{Z}$ we have equality in equations (12.20.5.2) and (12.20.5.1).*

Proof. Immediate from the discussions above. □

12.21. Spectral sequences: filtered complexes

Definition 12.21.1. Let $\mathcal{A}$ be an abelian category. A *filtered complex* $K^\bullet$ *of* $\mathcal{A}$ is a complex of $\mathrm{Fil}(\mathcal{A})$ (see Definition 12.16.1).

We will denote the filtration on the objects by F. Thus F^pK^n denotes the pth step in the filtration of the nth term of the complex. Note that each $F^pK^\bullet$ is a complex of $\mathcal{A}$. Hence we could also have defined a filtered complex as a filtered object in the (abelian) category of complexes of $\mathcal{A}$. In particular $\mathrm{gr}K^\bullet$ is a graded object of the category of complexes of $\mathcal{A}$.

Let us denote d the differential of K. Forgetting the grading we can think of $\bigoplus K^n$ as a filtered differential object of $\mathcal{A}$. Hence according to Section 12.20 we obtain a spectral sequence $(E_r, d_r)_{r\geq 0}$. In this section we work out the terms of this spectral sequence, and we endow the terms of this spectral sequence with additional structure coming from the grading of K.

First we point out that $E_0^p = \mathrm{gr}^pK^\bullet$ is a complex and hence is graded. Thus E_0 is bigraded in a natural way. It is customary to use the bigrading

$$E_0 = \bigoplus\nolimits_{p,q} E_0^{p,q}, \quad E_0^{p,q} = \mathrm{gr}^pK^{p+q}$$

The idea is that $p+q$ should be thought of as the *total degree* of the (co)homology classes. Also, p is called the *filtration degree*, and q is called the *complementary degree*. The differential d_0 is compatible with this bigrading in the following way

$$d_0 = \bigoplus d_0^{p,q}, \quad d_0^{p,q} : E_0^{p,q} \to E_0^{p,q+1}.$$

Namely, d_0^p is just the differential on the complex $\mathrm{gr}^pK^\bullet$ (which occurs as gr^pE_0 just shifted a bit).

To go further we identify the objects B_r^p and Z_r^p introduced in Section 12.20 as graded objects and we work out the corresponding decompositions of the differentials. We do this in a completely straightforward manner, but again we warn the reader that our notation is not the same as notation found elsewhere. We define

$$Z_r^{p,q} = \frac{F^pK^{p+q} \cap d^{-1}(F^{p+r}K^{p+q+1}) + F^{p+1}K^{p+q}}{F^{p+1}K^{p+q}}$$

and

$$B_r^{p,q} = \frac{F^pK^{p+q} \cap d(F^{p-r+1}K^{p+q-1}) + F^{p+1}K^{p+q}}{F^{p+1}K^{p+q}}.$$

and of course $E_r^{p,q} = Z_r^{p,q}/B_r^{p,q}$. With these definitions it is completely clear that $Z_r^p = \bigoplus_q Z_r^{p,q}$, $B_r^p = \bigoplus_q B_r^{p,q}$, and $E_r^p = \bigoplus_q E_r^{p,q}$. Moreover,

$$0 \subset \ldots \subset B_r^{p,q} \subset \ldots \subset Z_r^{p,q} \subset \ldots \subset E_0^{p,q}$$

and hence it makes sense to define $Z_\infty^{p,q} = \bigcap_r Z_r^{p,q}$ and $B_\infty^{p,q} = \bigcup_r B_r^{p,q}$ and $E_\infty^{p,q} = Z_\infty^{p,q}/B_\infty^{p,q}$. Also, the map d_r^p decomposes as the direct sum of the maps

$$d_r^{p,q} : E_r^{p,q} \longrightarrow E_r^{p+r,q-r+1}, \quad z + F^{p+1}K^{p+q} \mapsto dz + F^{p+r+1}K^{p+q+1}$$

where $z \in F^pK^{p+q} \cap d^{-1}(F^{p+r}K^{p+q+1})$.

Lemma 12.21.2. *Let $\mathcal{A}$ be an abelian category. Let $(K^\bullet, F)$ be a filtered complex of $\mathcal{A}$. Assume $\mathcal{A}$ has countable direct sums. The spectral sequence $(E_r, d_r)_{r\geq 0}$ associated to $(K^\bullet, F)$ has bigraded terms*

$$E_r = \bigoplus E_r^{p,q}, \quad d_r = \bigoplus d_r^{p,q}.$$

with d_r of bidegree $(r, -r+1)$. Furthermore, we have $E_0^{p,q} = gr^p(K^{p+q})$, $d_0^{p,q} = gr^p(d^{p+q})$, and $E_1^{p,q} = H^{p+q}(gr^p(K^\bullet))$.

Proof. Follows from the discussion above. □

Lemma 12.21.3. *Let $\mathcal{A}$ be an abelian category. Let $(K^\bullet, F)$ be a filtered complex of $\mathcal{A}$. Assume $\mathcal{A}$ has countable direct sums. Let $(E_r, d_r)_{r\geq 0}$ be the spectral sequence associated to $(K^\bullet, F)$.*

(1) *The map*

$$d_1^{p,q} : E_1^{p,q} = H^{p+q}(gr^p(K^\bullet)) \longrightarrow E_1^{p+1,q} = H^{p+q+1}(gr^{p+1}(K^\bullet))$$

is equal to the boundary map in cohomology associated to the short exact sequence of complexes

$$0 \to gr^{p+1}(K^\bullet) \to F^pK^\bullet/F^{p+2}K^\bullet \to gr^{p+1}(K^\bullet) \to 0.$$

(2) *Assume that $d(F^pK) \subset F^{p+1}K$ for all $p \in \mathbf{Z}$. Then d induces the zero differential on $gr^p(K^\bullet)$ and hence $E_1^{p,q} = gr^p(K^\bullet)^{p+q}$. Furthermore, in this case*

$$d_1^{p,q} : E_1^{p,q} = gr^p(K^\bullet)^{p+q} \longrightarrow E_1^{p,q} = gr^{p+1}(K^\bullet)^{p+q+1}$$

is the morphism induced by d.

Proof. Omitted. But compare Lemma 12.20.3. □

Lemma 12.21.4. *Let $\mathcal{A}$ be an abelian category. Let $\alpha : (K^\bullet, F) \to (L^\bullet, F)$ be a morphism of filtered complexes of $\mathcal{A}$. Assume $\mathcal{A}$ has countable direct sums. Let $(E_r(K), d_r)_{r\geq 0}$, resp. $(E_r(L), d_r)_{r\geq 0}$ be the spectral sequence associated to $(K^\bullet, F)$, resp. $(L^\bullet, F)$. The morphism α induces a canonical morphism of spectral sequences $\{\alpha_r : E_r(K) \to E_r(L)\}_{r\geq 0}$ compatible with the bigradings.*

Proof. Obvious from the explicit representation of the terms of the spectral sequences. □

Definition 12.21.5. Let $\mathcal{A}$ be an abelian category. Let $(K^\bullet, F)$ be a filtered complex of $\mathcal{A}$. The *induced filtration* on $H^n(K^\bullet)$ is the filtration defined by $F^pH^n(K^\bullet) = \mathrm{Im}(H^n(F^pK^\bullet) \to H^n(K^\bullet))$.

Lemma 12.21.6. *Let $\mathcal{A}$ be an abelian category. Let $(K^\bullet, F)$ be a filtered complex of $\mathcal{A}$. The associated graded $gr(H^n(K^\bullet))$ of the cohomology of $K^\bullet$ is a graded subquotient of the graded object $\bigoplus_{p+q=n} E_\infty^{p,q}$.*

Proof. Let $q = n - p$. As in the proof of Lemma 12.20.5 we see that

$$E_\infty^{p,q} = \frac{\bigcap_r (F^pK^n \cap d^{-1}(F^{p+r}K^{n+1}) + F^{p+1}K^n)}{\bigcup_r (F^pK^n \cap d(F^{p-r+1}K^{n-1}) + F^{p+1}K^n)}.$$

On the other hand, we have

$$\mathrm{gr}^pH^n(K) = \frac{\mathrm{Ker}(d) \cap F^pK^n + F^{p+1}K^n}{\mathrm{Im}(d) \cap F^pK^n + F^{p+1}K^n} \tag{12.21.6.1}$$

The result follows since

$$\mathrm{Ker}(d) \cap F^pK^n + F^{p+1}K^n \subset \bigcup\nolimits_r \left(F^pK^n \cap d^{-1}(F^{p+r}K^{n+1}) + F^{p+1}K^n\right) \tag{12.21.6.2}$$

and

(12.21.6.3)
$$\bigcap\nolimits_r \left(F^pK^n \cap d(F^{p-r+1}K^{n-1}) + F^{p+1}K^n\right) \subset \mathrm{Im}(d) \cap F^pK^n + F^{p+1}K^n.$$

□

Definition 12.21.7. Let $\mathcal{A}$ be an abelian category. Let $(K^\bullet, F)$ be a filtered complex of $\mathcal{A}$. We say the spectral sequence associated to $(K^\bullet, F)$ *converges* if $\mathrm{gr}H^n(K^\bullet) = \bigoplus_{p+q=n} E_\infty^{p,q}$ for every $n \in \mathbf{Z}$.

This is often symbolized by the notation $E_r^{p,q} \Rightarrow H^{p+q}(K^\bullet)$. Please read the remarks following Definition 12.20.6.

Lemma 12.21.8. *Let $\mathcal{A}$ be an abelian category. Let $(K^\bullet, F)$ be a filtered complex of $\mathcal{A}$. The associated spectral sequence converges if and only if for every $p, q \in \mathbf{Z}$ we have equality in equations (12.21.6.3) and (12.21.6.2).*

Proof. Immediate from the discussions above. □

Lemma 12.21.9. *Let $\mathcal{A}$ be an abelian category. Let $(K^\bullet, F)$ be a filtered complex of $\mathcal{A}$. Assume that the filtration on each K^n is finite (see Definition 12.16.1). Then*

(1) *the filtration on each $H^n(K^\bullet)$ is finite, and*
(2) *the spectral sequence associated to $(K^\bullet, F)$ converges.*

Proof. Part (1) is clear from Equation (12.21.6.1). We will use Lemma 12.21.8 to prove part (2). Fix $p, n \in \mathbf{Z}$. Look at the left hand side of Equation (12.21.6.3). The expression is equal to the right hand side since $F^mK^{n-1} = 0$ for $m \ll 0$. Similarly, use $F^mK^{n+1} = K^{n+1}$ for $m \gg 0$ to prove equality in Equation (12.21.6.2). □

12.22. Spectral sequences: double complexes

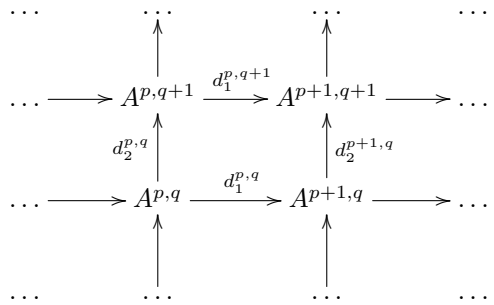
Definition 12.22.1. Let $\mathcal{A}$ be an additive category. A *double complex* in $\mathcal{A}$ is given by a system $(\{A^{p,q}, d_1^{p,q}, d_2^{p,q}\}_{p,q\in\mathbf{Z}})$, where each $A^{p,q}$ is an object of $\mathcal{A}$ and $d_1^{p,q} : A^{p,q} \to A^{p+1,q}$ and $d_2^{p,q} : A^{p,q} \to A^{p,q+1}$ are morphisms of $\mathcal{A}$ such that the following rules hold:

(1) $d_1^{p+1,q} \circ d_1^{p,q} = 0$
(2) $d_2^{p,q+1} \circ d_2^{p,q} = 0$
(3) $d_1^{p,q+1} \circ d_2^{p,q} = d_2^{p+1,q} \circ d_1^{p,q}$

for all $p, q \in \mathbf{Z}$.

This is just the cochain version of the definition. It says that each $A^{p,\bullet}$ is a cochain complex and that each $d_1^{p,\bullet}$ is a morphism of complexes $A^{p,\bullet} \to A^{p+1,\bullet}$ such that $d_1^{p+1,\bullet} \circ d_1^{p,\bullet} = 0$ as morphisms of complexes. In other words a double complex can

be seen as a complex of complexes. So in the diagram



any square commutes. Warning: In the literature one encounters a different definition where a "bicomplex" or a "double complex" has the property that the squares in the diagram anti-commute.

It is customary to denote $H_I^p(K^{\bullet,\bullet})$ the complex with terms $\mathrm{Ker}(d_1^{p,q})/\mathrm{Im}(d_1^{p-1,q})$ (varying q) and differential induced by d_2. Then $H_{II}^q(H_I^p(K^{\bullet,\bullet}))$ denotes its cohomology in degree q. It is also customary to denote $H_{II}^q(K^{\bullet,\bullet})$ the complex with terms $\mathrm{Ker}(d_2^{p,q})/\mathrm{Im}(d_2^{p,q-1})$ (varying p) and differential induced by d_1. Then $H_I^p(H_{II}^q(K^{\bullet,\bullet}))$ denotes its cohomology in degree q.

Definition 12.22.2. Let $\mathcal{A}$ be an additive category. Let $A^{\bullet,\bullet}$ be a double complex. The *associated simple complex* $sA^\bullet$, also sometimes called the *associated total complex* is given by

$$sA^n = \bigoplus\nolimits_{n=p+q} A^{p,q}$$

(if it exists) with differential

$$d_{sA}^n = \sum\nolimits_{n=p+q} (d_1^{p,q} + (-1)^p d_2^{p,q})$$

Alternatively, we sometimes write $\mathrm{Tot}(A^{\bullet,\bullet})$ to denote this complex.

If countable direct sums exist in $\mathcal{A}$ or if for each n at most finitely many $A^{p,n-p}$ are nonzero, then $sA^\bullet$ exists. Note that the definition is *not* symmetric in the indices (p, q).

There are two natural filtrations on the simple complex $sA^\bullet$ associated to the double complex $A^{\bullet,\bullet}$. Namely, we define

$$F_I^p(sA^n) = \bigoplus\nolimits_{i+j=n,\ i\geq p} A^{i,j} \quad \text{and} \quad F_{II}^p(sA^n) = \bigoplus\nolimits_{i+j=n,\ j\geq p} A^{i,j}.$$

It is immediately verified that $(sA^\bullet, F_I)$ and $(sA^\bullet, F_{II})$ are filtered complexes. By Section 12.21 we obtain two spectral sequences. It is customary to denote $({}'E_r, {}'d_r)_{r\geq 0}$ the spectral sequence associated to the filtration F_I and to denote $({}''E_r, {}''d_r)_{r\geq 0}$ the spectral sequence associated to the filtration F_{II}. Here is a description of these spectral sequences.

Lemma 12.22.3. *Let $\mathcal{A}$ be an abelian category. Let $K^{\bullet,\bullet}$ be a double complex. The spectral sequences associated to $K^{\bullet,\bullet}$ have the following terms:*

(1) ${}'E_0^{p,q} = K^{p,q}$ *with* ${}'d_0^{p,q} = (-1)^p d_2^{p,q} : K^{p,q} \to K^{p,q+1}$,
(2) ${}''E_0^{p,q} = K^{q,p}$ *with* ${}''d_0^{p,q} = d_1^{q,p} : K^{q,p} \to K^{q+1,p}$,
(3) ${}'E_1^{p,q} = H^q(K^{p,\bullet})$ *with* ${}'d_1^{p,q} = H^q(d_1^{p,\bullet})$,
(4) ${}''E_1^{p,q} = H^q(K^{\bullet,p})$ *with* ${}''d_1^{p,q} = (-1)^q H^q(d_2^{\bullet,p})$,

(5) ${}'E_2^{p,q} = H_I^p(H_{II}^q(K^{\bullet,\bullet}))$,
(6) ${}''E_2^{p,q} = H_{II}^p(H_I^q(K^{\bullet,\bullet}))$.

Proof. Omitted. □

These spectral sequences define two filtrations on $H^n(sK^\bullet)$. We will denote these F_I and F_{II}.

Definition 12.22.4. Let $\mathcal{A}$ be an abelian category. Let $K^{\bullet,\bullet}$ be a double complex. We say the spectral sequence $({}'E_r, {}'d_r)_{r\geq 0}$ *converges* if Definition 12.21.7 applies. In other words, for all n

$$\mathrm{gr}_{F_I}(H^n(sK^\bullet)) = \oplus_{p+q=n} {}'E_\infty^{p,q}$$

via the canonical comparison of Lemma 12.21.6. Similarly we say the spectral sequence $({}''E_r, {}''d_r)_{r\geq 0}$ *converges* if Definition 12.21.7 applies. In other words for all n

$$\mathrm{gr}_{F_{II}}(H^n(sK^\bullet)) = \oplus_{p+q=n} {}''E_\infty^{p,q}$$

via the canonical comparison of Lemma 12.21.6.

Same caveats as those following Definition 12.20.6.

Lemma 12.22.5 (First quadrant spectral sequence)**.** *Let $\mathcal{A}$ be an abelian category. Let $K^{\bullet,\bullet}$ be a double complex. Assume that for every $n \in \mathbf{Z}$ there are only finitely many nonzero $K^{p,q}$ with $p + q = n$. Then*

(1) *the filtrations F_I, F_{II} on each $H^n(K^\bullet)$ are finite,*
(2) *the spectral sequence $({}'E_r, {}'d_r)_{r\geq 0}$ converges, and*
(3) *the spectral sequence $({}''E_r, {}''d_r)_{r\geq 0}$ converges.*

Proof. Follows immediately from Lemma 12.21.9. □

Here is our first application of spectral sequences.

Lemma 12.22.6. *Let $\mathcal{A}$ be an abelian category. Let $K^\bullet$ be a complex. Let $A^{\bullet,\bullet}$ be a double complex. Let $\alpha^p : K^p \to A^{p,0}$ be morphisms. Assume that*

(1) *For every $n \in \mathbf{Z}$ there are only finitely many nonzero $A^{p,q}$ with $p+q = n$.*
(2) *We have $A^{p,q} = 0$ if $q < 0$.*
(3) *The morphisms α^p give rise to a morphism of complexes $\alpha : K^\bullet \to A^{\bullet,0}$.*
(4) *The complex $A^{p,\bullet}$ is exact in all degrees $q \neq 0$ and the morphism $K^p \to A^{p,0}$ induces an isomorphism $K^p \to Ker(d_2^{p,0})$.*

Then α induces a quasi-isomorphism

$$K^\bullet \longrightarrow sA^\bullet$$

of complexes. Moreover, there is a variant of this lemma involving the second variable q instead of p.

Proof. The map is simply the map given by the morphisms $K^n \to A^{n,0} \to sA^n$, which are easily seen to define a morphism of complexes. Consider the spectral sequence $({}'E_r, {}'d_r)_{r\geq 0}$ associated to the double complex $A^{\bullet,\bullet}$. By Lemma 12.22.5 this spectral sequence converges and the induced filtration on $H^n(sA^\bullet)$ is finite for each n. By Lemma 12.22.3 and assumption (4) we have ${}'E_1^{p,q} = 0$ unless $q = 0$ and ${}'E_1^{p,0} = K^p$ with differential ${}'d_1^{p,0}$ identified with d_K^p. Hence ${}'E_2^{p,0} = H^p(K^\bullet)$ and zero otherwise. This clearly implies $d_2^{p,q} = d_3^{p,q} = \ldots = 0$ for degree reasons. Hence we conclude that $H^n(sA^\bullet) = H^n(K^\bullet)$. We omit the verification

that this identification is given by the morphism of complexes $K^\bullet \to sA^\bullet$ introduced above. $\square$

Remark 12.22.7. Let $\mathcal{A}$ be an abelian category. Let $\mathcal{C} \subset \mathcal{A}$ be a weak Serre subcategory (see Definition 12.9.1). Suppose that $K^{\bullet,\bullet}$ is a double complex to which Lemma 12.22.5 applies such that for some $r \geq 0$ all the objects ${}'E_r^{p,q}$ belong to $\mathcal{C}$. We claim all the cohomology groups $H^n(sK^\bullet)$ belong to $\mathcal{C}$. Namely, the assumptions imply that the kernels and images of ${}'d_r^{p,q}$ are in $\mathcal{C}$. Whereupon we see that each ${}'E_{r+1}^{p,q}$ is in $\mathcal{C}$. By induction we see that each ${}'E_\infty^{p,q}$ is in $\mathcal{C}$. Hence each $H^n(sK^\bullet)$ has a finite filtration whose subquotients are in $\mathcal{C}$. Using that $\mathcal{C}$ is closed under extensions we conclude that $H^n(sK^\bullet)$ is in $\mathcal{C}$ as claimed.

The same result holds for the second spectral sequence associated to $K^{\bullet,\bullet}$. Similarly, if $(K^\bullet, F)$ is a filtered complex to which Lemma 12.21.9 applies and for some $r \geq 0$ all the objects $E_r^{p,q}$ belong to $\mathcal{C}$, then each $H^n(K^\bullet)$ is an object of $\mathcal{C}$.

Remark 12.22.8. Let $\mathcal{A}$ be an additive category. Let $A^{\bullet,\bullet,\bullet}$ be a triple complex. The associated total complex is the complex with terms

$$\mathrm{Tot}^n(A^{\bullet,\bullet,\bullet}) = \bigoplus\nolimits_{p+q+r=n} A^{p,q,r}$$

and differential

$$d^n_{\mathrm{Tot}(A^{\bullet,\bullet,\bullet})} = \sum\nolimits_{p+q+r=n} d_1^{p,q,r} + (-1)^p d_2^{p,q,r} + (-1)^{p+q} d_3^{p,q,r}$$

With this definition a simple calculation shows that the associated total complex is equal to

$$\mathrm{Tot}(A^{\bullet,\bullet,\bullet}) = \mathrm{Tot}(\mathrm{Tot}_{12}(A^{\bullet,\bullet,\bullet})) = \mathrm{Tot}(\mathrm{Tot}_{23}(A^{\bullet,\bullet,\bullet}))$$

In other words, we can either first combine the first two of the variables and then combine sum of those with the last, or we can first combine the last two variables and then combine the first with the sum of the last two.

Lemma 12.22.9. *Let $M^\bullet$ be a complex of abelian groups. Let*

$$\ldots \to A_2^\bullet \to A_1^\bullet \to A_0^\bullet \to M^\bullet \to 0$$

be an exact complex of complexes of abelian groups such that for all $p \in \mathbf{Z}$ the complexes

$$\ldots \to Ker(d^p_{A_2^\bullet}) \to Ker(d^p_{A_1^\bullet}) \to Ker(d^p_{A_0^\bullet}) \to Ker(d^p_{M^\bullet}) \to 0$$

are exact as well. Set $A^{p,q} = A^q_{-p}$ to obtain a double complex. Then $Tot(A^{\bullet,\bullet}) \to M^\bullet$ induced by $A_0^\bullet \to M^\bullet$ is a quasi-isomorphism.

Proof. Write $T^\bullet = \mathrm{Tot}(A^{\bullet,\bullet})$. Let $x \in \mathrm{Ker}(\mathrm{d}^0_{T^\bullet})$ represent a cohomology class ξ. Write $x = \sum_{i=n,\ldots,0} x_i$ with $x_i \in A_i^i$. Assume $n > 0$. Then x_n is in the kernel of $d^n_{A_n^\bullet}$ and maps to zero in the cohomology of $A^\bullet_{n-1}$ (because it maps to an element which is the boundary of x_{n-1} up to sign). The condition on exactness of kernels of differentials implies that the cohomology class of x_n is in the image of $H^n(A^\bullet_{n+1}) \to H^n(A^\bullet_n)$ (details omitted). Thus we can modify x by a boundary and reach the situation where x_n is a boundary. Modifying x once more we see that we may assume $x_n = 0$. By induction we see that every cohomology class ξ is represented by a cocycle $x = x_0$. Finally, the condition on exactness of kernels tells us two such cocycles x_0 and x_0' are cohomologous if and only if their image in $H^0(M^\bullet)$ are the same. $\square$

Lemma 12.22.10. *Let $M^\bullet$ be a complex of abelian groups. Let*

$$0 \to M^\bullet \to A_0^\bullet \to A_1^\bullet \to A_2^\bullet$$

be an exact complex of complexes of abelian groups such that for all $p \in \mathbf{Z}$ the complexes

$$0 \to Coker(d^p_{M^\bullet}) \to Coker(d^p_{A_0^\bullet}) \to Coker(d^p_{A_1^\bullet}) \to Coker(d^p_{A_2^\bullet}) \to \ldots$$

are exact as well. Set $A^{p,q} = A_p^q$ to obtain a double complex. Let $Tot_\pi(A^{\bullet,\bullet})$ be the product total complex associated to the double complex (see proof). Then the map $M^\bullet \to Tot_\pi(A^{\bullet,\bullet})$ induced by $M^\bullet \to A_0^\bullet$ is a quasi-isomorphism.

Proof. Abbreviating $T^\bullet = \mathrm{Tot}_\pi(A^{\bullet,\bullet})$ we define

$$T^n = \prod\nolimits_{p+q=n} A^{p,q} = \prod\nolimits_{p+q=n} A_p^q$$

As differential we use

$$\mathrm{d}((x_{p,q})) = (f_p(x_{p-1,q}) + (-1)^p \mathrm{d}_{A_p^\bullet}(x_{p,q-1}))$$

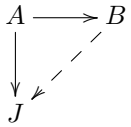
Let $x \in \mathrm{Ker}(\mathrm{d}^0_{T^\bullet})$ represent a cohomology class $\xi \in H^0(T^\bullet)$. Write $x = (x_i)$ with $x_i \in A_i^{-i}$. Note that x_0 maps to zero in $\mathrm{Coker}(A_1^{-1} \to A_1^0)$. Hence we see that $x_0 = m_0 + \mathrm{d}(y)$ for some $m_0 \in M^0$. Then $\mathrm{d}(m_0) = 0$ because $\mathrm{d}(x_0) = 0$ as x is a cocycle. Thus, replacing ξ by something in the image of $H^0(M^\bullet) \to H^0(T^\bullet)$ we may assume that x_0 is in the image of $\mathrm{d} : A_0^{-1} \to A_0^0$.

Assume $x_0 \in \mathrm{Im}(A_0^{-1} \to A_0^0)$. We claim that in this case $\xi = 0$. To prove this we find, by induction on n elements $y_1, \ldots, y_n$ with $y_i \in A_i^{-i-1}$ such that $x_0 = \mathrm{d}(y_0)$ and $x_j = f_{j-1}(y_{j-1}) + (-1)^j \mathrm{d}(y_j)$. This is clear for $n = 0$. Proof of induction step is omitted. Taking $y = (y_i)$ we find that $\mathrm{d}(y) = \xi$.

This shows that $H^0(M^\bullet) \to H^0(T^\bullet)$ is surjective. We omit the proof of injectivity. □

12.23. Injectives

Definition 12.23.1. Let $\mathcal{A}$ be an abelian category. An object $J \in \mathrm{Ob}(\mathcal{A})$ is called *injective* if for every injection $A \hookrightarrow B$ and every morphism $A \to J$ there exists a morphism $B \to J$ making the following diagram commute



Here is the obligatory characterization of injective objects.

Lemma 12.23.2. *Let $\mathcal{A}$ be an abelian category. Let I be an object of $\mathcal{A}$. The following are equivalent:*

1. *The object I is injective.*
2. *The functor $B \mapsto \mathrm{Hom}_{\mathcal{A}}(B, I)$ is exact.*
3. *Any short exact sequence*

   $$0 \to I \to A \to B \to 0$$

   in $\mathcal{A}$ is split.
4. *We have $Ext_{\mathcal{A}}(B, I) = 0$ for all $B \in \mathrm{Ob}(\mathcal{A})$.*

Proof. Omitted. □

Lemma 12.23.3. *Let $\mathcal{A}$ be an abelian category. Suppose I_ω, $\omega \in \Omega$ is a set of injective objects of $\mathcal{A}$. If $\prod_{\omega\in\Omega} I_\omega$ exists then it is injective.*

Proof. Omitted. □

Definition 12.23.4. Let $\mathcal{A}$ be an abelian category. We say $\mathcal{A}$ has *enough injectives* if every object A has an injective morphism $A \to J$ into an injective object J.

Definition 12.23.5. Let $\mathcal{A}$ be an abelian category. We say that $\mathcal{A}$ has *functorial injective embeddings* if there exists a functor

$$J : \mathcal{A} \longrightarrow \text{Arrows}(\mathcal{A})$$

such that

(1) $s \circ J = \text{id}_{\mathcal{A}}$,
(2) for any object $A \in \text{Ob}(\mathcal{A})$ the morphism $J(A)$ is injective, and
(3) for any object $A \in \text{Ob}(\mathcal{A})$ the object $t(J(A))$ is an injective object of $\mathcal{A}$.

We will denote such a functor by $A \mapsto (A \to J(A))$.

12.24. Projectives

Definition 12.24.1. Let $\mathcal{A}$ be an abelian category. An object $P \in \text{Ob}(\mathcal{A})$ is called *projective* if for every surjection $A \to B$ and every morphism $P \to B$ there exists a morphism $P \to A$ making the following diagram commute

$$\begin{array}{ccc} A & \longrightarrow & B \\ \uparrow & \nearrow & \\ P & & \end{array}$$

Here is the obligatory characterization of projective objects.

Lemma 12.24.2. *Let $\mathcal{A}$ be an abelian category. Let P be an object of $\mathcal{A}$. The following are equivalent:*

(1) *The object P is projective.*
(2) *The functor $B \mapsto \text{Hom}_{\mathcal{A}}(P, B)$ is exact.*
(3) *Any short exact sequence*

$$0 \to A \to B \to P \to 0$$

in $\mathcal{A}$ is split.
(4) *We have $Ext_{\mathcal{A}}(P, A) = 0$ for all $A \in \text{Ob}(\mathcal{A})$.*

Proof. Omitted. □

Lemma 12.24.3. *Let $\mathcal{A}$ be an abelian category. Suppose P_ω, $\omega \in \Omega$ is a set of projective objects of $\mathcal{A}$. If $\coprod_{\omega\in\Omega} P_\omega$ exists then it is projective.*

Proof. Omitted. □

Definition 12.24.4. Let $\mathcal{A}$ be an abelian category. We say $\mathcal{A}$ has *enough projectives* if every object A has an surjective morphism $P \to A$ from an projective object P onto it.

Definition 12.24.5. Let $\mathcal{A}$ be an abelian category. We say that $\mathcal{A}$ has *functorial projective surjections* if there exists a functor

$$P : \mathcal{A} \longrightarrow \text{Arrows}(\mathcal{A})$$

such that

(1) $t \circ J = \text{id}_{\mathcal{A}}$,
(2) for any object $A \in \text{Ob}(\mathcal{A})$ the morphism $P(A)$ is surjective, and
(3) for any object $A \in \text{Ob}(\mathcal{A})$ the object $s(P(A))$ is an projective object of $\mathcal{A}$.

We will denote such a functor by $A \mapsto (P(A) \to A)$.

12.25. Injectives and adjoint functors

Here are some lemmas on adjoint functors and their relationship with injectives. See also Lemma 12.7.3.

Lemma 12.25.1. *Let $\mathcal{A}$ and $\mathcal{B}$ be abelian categories. Let $u : \mathcal{A} \to \mathcal{B}$ and $v : \mathcal{B} \to \mathcal{A}$ be additive functors. Assume*

(1) *u is right adjoint to v, and*
(2) *v transforms injective maps into injective maps.*

Then u transforms injectives into injectives.

Proof. Let I be an injective object of $\mathcal{A}$. Let $\varphi : N \to M$ be an injective map in $\mathcal{B}$ and let $\alpha : N \to uI$ be a morphism. By adjointness we get a morphism $\alpha : vN \to I$ and by assumption $v\varphi : vN \to vM$ is injective. Hence as I is an injective object we get a morphism $\beta : vM \to I$ extending α. By adjointness again this corresponds to a morphism $\beta : M \to uI$ as desired. □

Remark 12.25.2. Let $\mathcal{A}$, $\mathcal{B}$, $u : \mathcal{A} \to \mathcal{B}$ and $v : \mathcal{B} \to \mathcal{A}$ be as in Lemma 12.25.1. In the presence of assumption (1) assumption (2) is equivalent to requiring that v is exact. Moreover, condition (2) is necessary. Here is an example. Let $A \to B$ be a ring map. Let $u : \text{Mod}_B \to \text{Mod}_A$ be $u(N) = N_A$ and let $v : \text{Mod}_A \to \text{Mod}_B$ be $v(M) = M \otimes_A B$. Then u is right adjoint to v, and u is exact and v is right exact, but v does not transform injective maps into injective maps in general (i.e., v is not left exact). Moreover, it is **not** the case that u transforms injective B-modules into injective A-modules. For example, if $A = \mathbf{Z}$ and $B = \mathbf{Z}/p\mathbf{Z}$, then the injective B-module $\mathbf{Z}/p\mathbf{Z}$ is not an injective $\mathbf{Z}$-module. In fact, the lemma applies to this example if and only if the ring map $A \to B$ is flat.

Lemma 12.25.3. *Let $\mathcal{A}$ and $\mathcal{B}$ be abelian categories. Let $u : \mathcal{A} \to \mathcal{B}$ and $v : \mathcal{B} \to \mathcal{A}$ be additive functors. Assume*

(1) *u is right adjoint to v,*
(2) *v transforms injective maps into injective maps,*
(3) *$\mathcal{A}$ has enough injectives, and*
(4) *$vB = 0$ implies $B = 0$ for any $B \in \text{Ob}(\mathcal{B})$.*

Then $\mathcal{B}$ has enough injectives.

Proof. Pick $B \in \text{Ob}(\mathcal{B})$. Pick an injection $vB \to I$ for I an injective object of $\mathcal{A}$. According to Lemma 12.25.1 and the assumptions the corresponding map $B \to uI$ is the injection of B into an injective object. □

Remark 12.25.4. Let $\mathcal{A}$, $\mathcal{B}$, $u : \mathcal{A} \to \mathcal{B}$ and $v : \mathcal{B} \to \mathcal{A}$ be as In Lemma 12.25.3. In the presence of conditions (1) and (2) condition (4) is equivalent to v being faithful. Moreover, condition (4) is needed. An example is to consider the case where the functors u and v are both the zero functor.

Lemma 12.25.5. *Let $\mathcal{A}$ and $\mathcal{B}$ be abelian categories. Let $u : \mathcal{A} \to \mathcal{B}$ and $v : \mathcal{B} \to \mathcal{A}$ be additive functors. Assume*

(1) *u is right adjoint to v,*
(2) *v transforms injective maps into injective maps,*
(3) *$\mathcal{A}$ has enough injectives,*
(4) *$vB = 0$ implies $B = 0$ for any $B \in \mathrm{Ob}(\mathcal{B})$, and*
(5) *$\mathcal{A}$ has functorial injective hulls.*

Then $\mathcal{B}$ has functorial injective hulls.

Proof. Let $A \mapsto (A \to J(A))$ be a functorial injective hull on $\mathcal{A}$. Then $B \mapsto (B \to uJ(vB))$ is a functorial injective hull on $\mathcal{B}$. Compare with the proof of Lemma 12.25.3. □

Lemma 12.25.6. *Let $\mathcal{A}$ and $\mathcal{B}$ be abelian categories. Let $u : \mathcal{A} \to \mathcal{B}$ be a functor. If there exists a subset $\mathcal{P} \subset \mathrm{Ob}(\mathcal{B})$ such that*

(1) *every object of $\mathcal{B}$ is a quotient of an element of $\mathcal{P}$, and*
(2) *for every $P \in \mathcal{P}$ there exists an object Q of $\mathcal{A}$ such that $\mathrm{Hom}_{\mathcal{A}}(Q, A) = \mathrm{Hom}_{\mathcal{B}}(P, u(A))$ functorially in A,*

then there exists a left adjoint v of u.

Proof. By the Yoneda lemma (Categories, Lemma 4.3.5) the object Q of $\mathcal{A}$ corresponding to P is defined up to unique isomorphism by the formula $\mathrm{Hom}_{\mathcal{A}}(Q, A) = \mathrm{Hom}_{\mathcal{B}}(P, u(A))$. Let us write $Q = v(P)$. Denote $i_P : P \to u(v(P))$ the map corresponding to $\mathrm{id}_{v(P)}$ in $\mathrm{Hom}_{\mathcal{A}}(v(P), v(P))$. Functoriality in (2) implies that the bijection is given by

$$\mathrm{Hom}_{\mathcal{A}}(v(P), A) \to \mathrm{Hom}_{\mathcal{B}}(P, u(A)), \quad \varphi \mapsto u(\varphi) \circ i_P$$

For any pair of elements $P_1, P_2 \in \mathcal{P}$ there is a canonical map

$$\mathrm{Hom}_{\mathcal{B}}(P_2, P_1) \to \mathrm{Hom}_{\mathcal{A}}(v(P_2), v(P_1)), \quad \varphi \mapsto v(\varphi)$$

which is characterized by the rule $u(v(\varphi)) \circ i_{P_2} = i_{P_1} \circ \varphi$ in $\mathrm{Hom}_{\mathcal{B}}(P_2, u(v(P_1)))$. Note that $\varphi \mapsto v(\varphi)$ is compatible with composition; this can be seen directly from the characterization. Hence $P \mapsto v(P)$ is a functor from the full subcategory of $\mathcal{B}$ whose objects are the elements of $\mathcal{P}$.

Given an arbitrary object B of $\mathcal{B}$ choose an exact sequence

$$P_2 \to P_1 \to B \to 0$$

which is possible by assumption (1). Define $v(B)$ to be the object of $\mathcal{A}$ fitting into the exact sequence

$$v(P_2) \to v(P_1) \to v(B) \to 0$$

Then

$$\begin{aligned} \mathrm{Hom}_{\mathcal{A}}(v(B), A) &= \mathrm{Ker}(\mathrm{Hom}_{\mathcal{A}}(v(P_1), A) \to \mathrm{Hom}_{\mathcal{A}}(v(P_2), A)) \\ &= \mathrm{Ker}(\mathrm{Hom}_{\mathcal{B}}(P_1, u(A)) \to \mathrm{Hom}_{\mathcal{B}}(P_2, u(A))) \\ &= \mathrm{Hom}_{\mathcal{B}}(B, u(A)) \end{aligned}$$

Hence we see that we may take $\mathcal{P} = \mathrm{Ob}(\mathcal{B})$, i.e., we see that v is everywhere defined. □

12.26. Inverse systems

Let $\mathcal{C}$ be a category. In Categories, Section 4.21 we defined the notion of an inverse system over a partially ordered set (with values in the category $\mathcal{C}$). If the partially ordered set is $\mathbf{N} = \{1, 2, 3, \ldots\}$ with the usual ordering such an inverse system over $\mathbf{N}$ is often simply called an *inverse system*. It consists quite simply of a pair $(M_i, f_{ii'})$ where each M_i, $i \in \mathbf{N}$ is an object of $\mathcal{C}$, and for each $i > i'$, $i, i' \in \mathbf{N}$ a morphism $f_{ii'} : M_i \to M_{i'}$ such that moreover $f_{i'i''} \circ f_{ii'} = f_{ii''}$ whenever this makes sense. It is clear that in fact it suffices to give the morphisms $M_2 \to M_1$, $M_3 \to M_2$, and so on. Hence an inverse system is frequently pictured as follows

$$M_1 \xleftarrow{\varphi_2} M_2 \xleftarrow{\varphi_3} M_3 \leftarrow \ldots$$

Moreover, we often omit the transition maps φ_i from the notation and we simply say "let (M_i) be an inverse system".

The collection of all inverse systems with values in $\mathcal{C}$ forms a category with the obvious notion of morphism.

Lemma 12.26.1. *Let $\mathcal{C}$ be a category.*

1. *If $\mathcal{C}$ is an additive category, then the category of inverse systems with values in $\mathcal{C}$ is an additive cateogry.*
2. *If $\mathcal{C}$ is an abelian category, then the category of inverse systems with values in $\mathcal{C}$ is an abelian cateogry. A sequence $(K_i) \to (L_i) \to (M_i)$ of inverse systems is exact if and only if each $K_i \to L_i \to N_i$ is exact.*

Proof. Omitted. □

The limit (see Categories, Section 4.21) of such an inverse system is denoted $\lim M_i$, or $\lim_i M_i$. If $\mathcal{C}$ is the category of abelian groups (or sets), then the limit always exists and in fact can be described as follows

$$\lim_i M_i = \{(x_i) \in \prod M_i \mid \varphi_i(x_i) = x_{i-1},\ i = 2, 3, \ldots\}$$

see Categories, Section 4.15. However, given a short exact sequence

$$0 \to (A_i) \to (B_i) \to (C_i) \to 0$$

of inverse systems of abelian groups it is not always the case that the associated system of limits is exact. In order to discuss this further we introduce the following notion.

Definition 12.26.2. Let $\mathcal{C}$ be an abelian category. We say the inverse system (A_i) satisfies the *Mittag-Leffler condition*, or for short is *ML*, if for every i there exists a $c = c(i) \geq i$ such that

$$\mathrm{Im}(A_k \to A_i) = \mathrm{Im}(A_c \to A_i)$$

for all $k \geq c$.

It turns out that the Mittag-Leffler condition is good enough to ensure that the lim-functor is exact, provided one works within the abelian category of abelian groups, or abelian sheaves, etc. It is shown in a paper by A. Neeman (see [**Nee02**]) that this condition is not strong enough in a general abelian category (where limits of inverse systems exist).

Lemma 12.26.3. *Let*

$$0 \to (A_i) \to (B_i) \to (C_i) \to 0$$

be a short exact sequence of inverse systems of abelian groups.

(1) *In any case the sequence*

$$0 \to \lim_i A_i \to \lim_i B_i \to \lim_i C_i$$

is exact.

(2) *If (B_i) is ML, then also (C_i) is ML.*

(3) *If (A_i) is ML, then*

$$0 \to \lim_i A_i \to \lim_i B_i \to \lim_i C_i \to 0$$

is exact.

Proof. Nice exercise. See Algebra, Lemma 10.82.1 for part (3). □

Lemma 12.26.4. *Let*

$$(A_i) \to (B_i) \to (C_i) \to (D_i)$$

be an exact sequence of inverse systems of abelian groups. If the system (A_i) is ML, then the sequence

$$\lim_i B_i \to \lim_i C_i \to \lim_i D_i$$

is exact.

Proof. Let $Z_i = \mathrm{Ker}(C_i \to D_i)$ and $I_i = \mathrm{Im}(A_i \to B_i)$. Then $\lim Z_i = \mathrm{Ker}(\lim C_i \to \lim D_i)$ and we get a short exact sequence of systems

$$0 \to (I_i) \to (B_i) \to (Z_i) \to 0$$

Moreover, by Lemma 12.26.3 we see that (I_i) has (ML), thus another application of Lemma 12.26.3 shows that $\lim B_i \to \lim Z_i$ is surjective which proves the lemma. □

The following characterization of essentially constant inverse systems shows in particular that they have ML.

Lemma 12.26.5. *Let $\mathcal{A}$ be an abelian category. Let (A_i) be an inverse system in $\mathcal{A}$ with limit $A = \lim A_i$. Then (A_i) is essentially constant (see Categories, Definition 4.22.1) if and only if there exists an i and for all $j \geq i$ a direct sum decomposition $A_j = A \oplus Z_j$ such that (a) the maps $A_{j'} \to A_j$ are compatible with the direct sum decompositions, (b) for all j there exists some $j' \geq j$ such that $Z_{j'} \to Z_j$ is zero.*

Proof. Assume (A_i) is essentially constant. Then there exists an i and a morphism $A_i \to A$ such that for all $j \geq i$ there exists a $j' \geq j$ such that $A_{j'} \to A_j$ factors as $A_{j'} \to A_i \to A \to A_j$ (the last map comes from $A = \lim A_i$). Hence setting $Z_j = \mathrm{Ker}(A_j \to A)$ for all $j \geq i$ works. Proof of the converse is omitted. □

Lemma 12.26.6. *Let*

$$0 \to (A_i) \to (B_i) \to (C_i) \to 0$$

be an exact sequence of inverse systems of abelian groups. If (A_i) has ML and (C_i) is essentially constant, then (B_i) has ML.

Proof. After renumbering we may assume that $C_i = C \oplus Z_i$ compatible with transition maps and that for all i there exists an $i' \geq i$ such that $Z_{i'} \to Z_i$ is zero, see Lemma 12.26.5. Pick i. Let $c \geq i$ by an integer such that $\mathrm{Im}(A_c \to A) = \mathrm{Im}(A_{i'} \to A_i)$ for all $i' \geq c$. Let $c' \geq c$ be an integer such that $Z_{c'} \to Z_c$ is zero. For $i' \geq c'$ consider the maps

$$\begin{array}{ccccccccc}
0 & \longrightarrow & A_{i'} & \longrightarrow & B_{i'} & \longrightarrow & C \oplus Z_{i'} & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & A_{c'} & \longrightarrow & B_{c'} & \longrightarrow & C \oplus Z_{c'} & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & A_c & \longrightarrow & B_c & \longrightarrow & C \oplus Z_c & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & A_i & \longrightarrow & B_i & \longrightarrow & C \oplus Z_i & \longrightarrow & 0
\end{array}$$

Because $Z_{c'} \to Z_c$ is zero the image $\mathrm{Im}(B_{c'} \to B_c)$ is an extension C by a subgroup $A' \subset A_c$ which contains the image of $A_{c'} \to A_c$. Hence $\mathrm{Im}(B_{c'} \to B_i)$ is an extension of C by the image of A' which is the image of $A_c \to A_i$ by our choice of c. In exactly the same way one shows that $\mathrm{Im}(B_{i'} \to B_i)$ is an extension of C by the image of $A_c \to A_i$. Hence $\mathrm{Im}(B_{c'} \to B_i) = \mathrm{Im}(B_{i'} \to B_i)$ and we win. □

The "correct" version of the following lemma is More on Algebra, Lemma 15.55.2.

Lemma 12.26.7. *Let*

$$(A_i^{-2} \to A_i^{-1} \to A_i^0 \to A_i^1)$$

be an inverse system of complexes of abelian groups and denote $A^{-2} \to A^{-1} \to A^0 \to A^1$ its limit. Denote (H_i^{-1}), (H_i^0) the inverse systems of cohomologies, and denote H^{-1}, H^0 the cohomologies of $A^{-2} \to A^{-1} \to A^0 \to A^1$. If (A_i^{-2}) and (A_i^{-1}) are ML and (H_i^{-1}) is essentially constant, then $H^0 = \lim H_i^0$.

Proof. Let $Z_i^j = \mathrm{Ker}(A_i^j \to A_i^{j+1})$ and $I_i^j = \mathrm{Im}(A_i^{j-1} \to A_i^j)$. Note that $\lim Z_i^0 = \mathrm{Ker}(\lim A_i^0 \to \lim A_i^1)$ as taking kernels commutes with limits. The systems (I_i^{-1}) and (I_i^0) have ML as quotients of the systems (A_i^{-2}) and (A_i^{-1}), see Lemma 12.26.3. Thus an exact sequence

$$0 \to (I_i^{-1}) \to (Z_i^{-1}) \to (H_i^{-1}) \to 0$$

of inverse systems where (I_i^{-1}) has ML and where (H_i^{-1}) is essentially constant by assumption. Hence (Z_i^{-1}) has ML by Lemma 12.26.6. The exact sequence

$$0 \to (Z_i^{-1}) \to (A_i^{-1}) \to (I_i^0) \to 0$$

and an application of Lemma 12.26.3 shows that $\lim A_i^{-1} \to \lim I_i^0$ is surjective. Finally, the exact sequence

$$0 \to (I_i^0) \to (Z_i^0) \to (H_i^0) \to 0$$

and Lemma 12.26.3 show that $\lim I_i^0 \to \lim Z_i^0 \to \lim H_i^0 \to 0$ is exact. Putting everything together we win. □

12.27. Exactness of products

Lemma 12.27.1. *Let I be a set. For $i \in I$ let $L_i \to M_i \to N_i$ be a complex of abelian groups. Let $H_i = Ker(M_i \to N_i)/Im(L_i \to M_i)$ be the cohomology. Then*

$$\prod L_i \to \prod M_i \to \prod N_i$$

is a complex of abelian groups with homology $\prod H_i$.

Proof. Omitted. □

12.28. Other chapters

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CHAPTER 13

Derived Categories

13.1. Introduction

We first discuss triangulated categories and localization in triangulated categories. Next, we prove that the homotopy category of complexes in an additive category is a triangulated category. Once this is done we define the derived category of an abelian category as the localization of the of homotopy category with respect to quasi-isomorphisms. A good reference is Verdier's thesis [**Ver96**].

13.2. Triangulated categories

Triangulated categories are a convenient tool to describe the type of structure inherent in the derived category of an abelian category. Some references are [**Ver96**], [**KS06**], and [**Nee01**].

13.3. The definition of a triangulated category

In this section we collect most of the definitions concerning triangulated and pre-triangulated categories.

Definition 13.3.1. Let $\mathcal{D}$ be an additive category. Let $[n] : \mathcal{D} \to \mathcal{D}$, $E \mapsto E[n]$ be a collection of additive functors indexed by $n \in \mathbf{Z}$ such that $[n] \circ [m] = [n+m]$ and $[0] = \mathrm{id}$ (equality as functors). In this situation we call *triangle* a sixtuple (X, Y, Z, f, g, h) where $X, Y, Z \in \mathrm{Ob}(\mathcal{D})$ and $f : X \to Y$, $g : Y \to Z$ and $h : Z \to X[1]$ are morphisms of $\mathcal{D}$. A *morphism of triangles* $(X, Y, Z, f, g, h) \to (X', Y', Z', f', g', h')$ is given by morphisms $a : X \to X'$, $b : Y \to Y'$ and $c : Z \to Z'$ of $\mathcal{D}$ such that $b \circ f = f' \circ a$, $c \circ g = g' \circ b$ and $a[1] \circ h = h' \circ c$.

A morphism of triangles is visualized by the following commutative diagram

$$\begin{array}{ccccccc} X & \longrightarrow & Y & \longrightarrow & Z & \longrightarrow & X[1] \\ \downarrow{\scriptstyle a} & & \downarrow{\scriptstyle b} & & \downarrow{\scriptstyle c} & & \downarrow{\scriptstyle a[1]} \\ X' & \longrightarrow & Y' & \longrightarrow & Z' & \longrightarrow & X'[1] \end{array}$$

Here is the definition of a triangulated category as given in Verdier's thesis.

Definition 13.3.2. A *triangulated category* consists of a triple $(\mathcal{D}, \{[n]\}_{n \in \mathbf{Z}}, \mathcal{T})$ where

(1) $\mathcal{D}$ is an additive category,
(2) $[n] : \mathcal{D} \to \mathcal{D}$, $E \mapsto E[n]$ be a collection of additive functors indexed by $n \in \mathbf{Z}$ such that $[n] \circ [m] = [n+m]$ and $[0] = \mathrm{id}$ (equality as functors), and
(3) $\mathcal{T}$ is a set of triangles called the *distinguished triangles*

subject to the following conditions

TR1 Any triangle isomorphic to a distinguished triangle is a distinguished triangle. Any triangle of the form $(X, X, 0, \mathrm{id}, 0, 0)$ is distinguished. For any morphism $f : X \to Y$ of $\mathcal{D}$ there exists a distinguished triangle of the form (X, Y, Z, f, g, h).

TR2 The triangle (X, Y, Z, f, g, h) is distinguished if and only if the triangle $(Y, Z, X[1], g, h, -f[1])$ is.

TR3 Given a solid commutative square

$$\begin{array}{ccccccc} X & \longrightarrow & Y & \longrightarrow & Z & \longrightarrow & X[1] \\ \downarrow{\scriptstyle a} & & \downarrow{\scriptstyle b} & & \vdots & & \downarrow{\scriptstyle a[1]} \\ X' & \longrightarrow & Y' & \longrightarrow & Z' & \longrightarrow & X'[1] \end{array}$$

whose rows are distinguished triangles there exists a morphism $c : Z \to Z'$ such that (a, b, c) is a morphism of triangles.

TR4 Given objects X, Y, Z of $\mathcal{D}$, and morphisms $f : X \to Y$, $g : Y \to Z$, and distinguished triangles (X, Y, Q_1, f, p_1, d_1), $(X, Z, Q_2, g \circ f, p_2, d_2)$, and (Y, Z, Q_3, g, p_3, d_3), there exist morphisms $a : Q_1 \to Q_2$ and $b : Q_2 \to Q_3$ such that
   (a) $(Q_1, Q_2, Q_3, a, b, p_1[1] \circ d_3)$ is a distinguished triangle,
   (b) the triple (id_X, g, a) is a morphism of triangles $(X, Y, Q_1, f, p_1, d_1) \to (X, Z, Q_2, g \circ f, p_2, d_2)$, and
   (c) the triple (f, id_Z, b) is a morphism of triangles $(X, Z, Q_2, g \circ f, p_2, d_2) \to (Y, Z, Q_3, g, p_3, d_3)$.

We will call $(\mathcal{D}, [\,], \mathcal{T})$ a *pre-triangulated category* if TR1, TR2 and TR3 hold.

The explanation of TR4 is that if you think of Q_1 as Y/X, Q_2 as Z/X and Q_3 as Z/Y, then TR4(a) expresses the isomorphism $(Z/X)/(Y/Z) \cong Z/Y$ and TR(b) and TR(c) express that we can compare the triangles $X \to Y \to Q_1 \to X[1]$ etc with morphisms of triangles. For a more precise reformulation of this idea see the proof of Lemma 13.9.2.

The sign in TR2 means that if (X, Y, Z, f, g, h) is a distinguished triangle then in the long sequence

(13.3.2.1)
$$\ldots \to Z[-1] \xrightarrow{-h[-1]} X \xrightarrow{f} Y \xrightarrow{g} Z \xrightarrow{h} X[1] \xrightarrow{-f[1]} Y[1] \xrightarrow{-g[1]} Z[1] \to \ldots$$

each four term sequence gives a distinguished triangle.

As usual we abuse notation and we simply speak of a (pre-)triangulated category $\mathcal{D}$ without explicitly introducing notation for the additional data. The notion of a pre-triangulated category is useful in finding statements equivalent to TR4.

We have the following definition of a triangulated functor.

Definition 13.3.3. Let $\mathcal{D}$, $\mathcal{D}'$ be pre-triangulated categories. An *exact functor*, or a *triangulated functor* from $\mathcal{D}$ to $\mathcal{D}'$ is a functor $F : \mathcal{D} \to \mathcal{D}'$ together with given functorial isomorphisms $\xi_X : F(X[1]) \to F(X)[1]$ such that for every distinguished triangle (X, Y, Z, f, g, h) of $\mathcal{D}$ the triangle $(F(X), F(Y), F(Z), F(f), F(g), \xi_X \circ F(h))$ is a distinguished triangle of $\mathcal{D}'$.

An exact functor is additive, see Lemma 13.4.15. When we say two triangulated categories are equivalent we mean that they are equivalent in the 2-category of triangulated categories. A 2-morphism $a : (F, \xi) \to (F', \xi')$ in this 2-category is simply a transformation of functors $a : F \to F'$ which is compatible with ξ and ξ', i.e.,

$$\begin{array}{ccc} F \circ [1] & \xrightarrow{\xi} & [1] \circ F \\ {\scriptstyle a \star 1}\downarrow & & \downarrow{\scriptstyle 1 \star a} \\ F' \circ [1] & \xrightarrow{\xi'} & [1] \circ F' \end{array}$$

commutes.

Definition 13.3.4. Let $(\mathcal{D}, [\,], \mathcal{T})$ be a pre-triangulated category. A *pre-triangulated subcategory*[1] is a pair $(\mathcal{D}', \mathcal{T}')$ such that

(1) $\mathcal{D}'$ is an additive subcategory of $\mathcal{D}$ which is preserved under $[1]$ and $[-1]$,
(2) $\mathcal{T}' \subset \mathcal{T}$ is a subset such that for every $(X, Y, Z, f, g, h) \in \mathcal{T}'$ we have $X, Y, Z \in \mathrm{Ob}(\mathcal{D}')$ and $f, g, h \in \mathrm{Arrows}(\mathcal{D}')$, and
(3) $(\mathcal{D}', [\,], \mathcal{T}')$ is a pre-triangulated category.

If $\mathcal{D}$ is a triangulated category, then we say $(\mathcal{D}', \mathcal{T}')$ is a *triangulated subcategory* if it is a pre-triangulated subcategory and $(\mathcal{D}', [\,], \mathcal{T}')$ is a triangulated category.

In this situation the inclusion functor $\mathcal{D}' \to \mathcal{D}$ is an exact functor with $\xi_X : X[1] \to X[1]$ given by the identity on $X[1]$.

We will see in Lemma 13.4.1 that for a distinguished triangle (X, Y, Z, f, g, h) in a pre-triangulated category the composition $g \circ f : X \to Z$ is zero. Thus the sequence (13.3.2.1) is a complex. A homological functor is one that turns this complex into a long exact sequence.

Definition 13.3.5. Let $\mathcal{D}$ be a pre-triangulated category. Let $\mathcal{A}$ be an abelian category. An additive functor $H : \mathcal{D} \to \mathcal{A}$ is called *homological* if for every distinguished triangle (X, Y, Z, f, g, h) the sequence

$$H(X) \to H(Y) \to H(Z)$$

is exact in the abelian category $\mathcal{A}$. An additive functor $H : \mathcal{D}^{opp} \to \mathcal{A}$ is called *cohomological* if the corresponding functor $\mathcal{D} \to \mathcal{A}^{opp}$ is homological.

If $H : \mathcal{D} \to \mathcal{A}$ is a homological functor we often write $H^n(X) = H(X[n])$ so that $H(X) = H^0(X)$. Our discussion of TR2 above implies that a distinguished triangle (X, Y, Z, f, g, h) determines a long exact sequence

$$H^{-1}(Z) \xrightarrow{h[-1]} H^0(X) \xrightarrow{f} H^0(Y) \xrightarrow{g} H^0(Z) \xrightarrow{h} H^1(X) \tag{13.3.5.1}$$

This will be called the *long exact sequence* associated to the distinguished triangle and the homological functor. As indicated we will not use any signs for the morphisms in the long exact sequence. This has the side effect that maps in the long exact sequence associated to the rotation (TR2) of a distinguished triangle differ from the maps in the sequence above by some signs.

[1] This definition may be nonstandard. If $\mathcal{D}'$ is a full subcategory then $\mathcal{T}'$ is the intersection of the set of triangles in $\mathcal{D}'$ with $\mathcal{T}$, see Lemma 13.4.14. In this case we drop $\mathcal{T}'$ from the notation.

Definition 13.3.6. Let $\mathcal{A}$ be an abelian category. Let $\mathcal{D}$ be a triangulated category. A *δ-functor from $\mathcal{A}$ to $\mathcal{D}$* is given by a functor $G : \mathcal{A} \to \mathcal{D}$ and a rule which assigns to every short exact sequence

$$0 \to A \xrightarrow{a} B \xrightarrow{b} C \to 0$$

a morphism $\delta = \delta_{A \to B \to C} : G(C) \to G(A)[1]$ such that

(1) the triangle $(G(A), G(B), G(C), G(a), G(b), \delta_{A \to B \to C})$ is a distinguished triangle of $\mathcal{D}$ for any short exact sequence as above, and
(2) for every morphism $(A \to B \to C) \to (A' \to B' \to C')$ of short exact sequences the diagram

$$\begin{array}{ccc} G(C) & \xrightarrow[\delta_{A \to B \to C}]{} & G(A)[1] \\ \downarrow & & \downarrow \\ G(C') & \xrightarrow{\delta_{A' \to B' \to C'}} & G(A')[1] \end{array}$$

is commutative.

In this situation we call $(G(A), G(B), G(C), G(a), G(b), \delta_{A \to B \to C})$ the *image of the short exact sequence under the given δ-functor*.

Note how a δ-functor comes equipped with additional structure. Strictly speaking it does not make sense to say that a given functor $\mathcal{A} \to \mathcal{D}$ is a δ-functor, but we will often do so anyway.

13.4. Elementary results on triangulated categories

Most of the results in this section are proved for pre-triangulated categories and a fortiori hold in any triangulated category.

Lemma 13.4.1. *Let $\mathcal{D}$ be a pre-triangulated category. Let (X, Y, Z, f, g, h) be a distinguished triangle. Then $g \circ f = 0$, $h \circ g = 0$ and $f[1] \circ h = 0$.*

Proof. By TR1 we know $(X, X, 0, 1, 0, 0)$ is a distinguished triangle. Apply TR3 to

$$\begin{array}{ccccccc} X & \longrightarrow & X & \longrightarrow & 0 & \longrightarrow & X[1] \\ \downarrow{\scriptstyle 1} & & \downarrow{\scriptstyle f} & & \vdots & & \downarrow{\scriptstyle 1[1]} \\ X & \xrightarrow{f} & Y & \xrightarrow{g} & Z & \xrightarrow{h} & X[1] \end{array}$$

Of course the dotted arrow is the zero map. Hence the commutativity of the diagram implies that $g \circ f = 0$. For the other cases rotate the triangle, i.e., apply TR2. □

Lemma 13.4.2. *Let $\mathcal{D}$ be a pre-triangulated category. For any object W of $\mathcal{D}$ the functor $\mathrm{Hom}_{\mathcal{D}}(W, -)$ is homological, and the functor $\mathrm{Hom}_{\mathcal{D}}(-, W)$ is cohomological.*

Proof. Consider a distinguished triangle (X, Y, Z, f, g, h). We have already seen that $g \circ f = 0$, see Lemma 13.4.1. Suppose $a : W \to Y$ is a morphism such that

$g \circ a = 0$. Then we get a commutative diagram

$$\begin{array}{ccccccc} W & \xrightarrow{1} & W & \longrightarrow & 0 & \longrightarrow & W[1] \\ \downarrow{\scriptstyle b} & & \downarrow{\scriptstyle a} & & \downarrow{\scriptstyle 0} & & \downarrow{\scriptstyle b[1]} \\ X & \longrightarrow & Y & \longrightarrow & Z & \longrightarrow & X[1] \end{array}$$

Both rows are distinguished triangles (use TR1 for the top row). Hence we can fill the dotted arrow b (first rotate using TR2, then apply TR3, and then rotate back). This proves the lemma. □

Lemma 13.4.3. *Let $\mathcal{D}$ be a pre-triangulated category. Let*

$$(a, b, c) : (X, Y, Z, f, g, h) \to (X', Y', Z', f', g', h')$$

be a morphism of distinguished triangles. If two among a, b, c are isomorphisms so is the third.

Proof. Assume that a and c are isomorphisms. For any object W of $\mathcal{D}$ write $H_W(-) = \mathrm{Hom}_{\mathcal{D}}(W, -)$. Then we get a commutative diagram of abelian groups

$$\begin{array}{ccccccccc} H_W(Z[-1]) & \longrightarrow & H_W(X) & \longrightarrow & H_W(Y) & \longrightarrow & H_W(Z) & \longrightarrow & H_W(X[1]) \\ \downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow \\ H_W(Z'[-1]) & \longrightarrow & H_W(X') & \longrightarrow & H_W(Y') & \longrightarrow & H_W(Z') & \longrightarrow & H_W(X'[1]) \end{array}$$

By assumption the right two and left two vertical arrows are bijective. As H_W is homological by Lemma 13.4.2 and the five lemma (Homology, Lemma 12.5.20) it follows that the middle vertical arrow is an isomorphism. Hence by Yoneda's lemma, see Categories, Lemma 4.3.5 we see that b is an isomorphism. This implies the other cases by rotating (using TR2). □

Remark 13.4.4. Let $\mathcal{D}$ be an additive category with translation functors $[n]$ as in Definition 13.3.1. Let us call a triangle (X, Y, Z, f, g, h) *special*[2] if for every object W of $\mathcal{D}$ the long sequence of abelian groups

$$\ldots \to \mathrm{Hom}_{\mathcal{D}}(W, X) \to \mathrm{Hom}_{\mathcal{D}}(W, Y) \to \mathrm{Hom}_{\mathcal{D}}(W, Z) \to \mathrm{Hom}_{\mathcal{D}}(W, X[1]) \to \ldots$$

is exact. The proof of Lemma 13.4.3 shows that if

$$(a, b, c) : (X, Y, Z, f, g, h) \to (X', Y', Z', f', g', h')$$

is a morphism of special triangles and if two among a, b, c are isomorphisms so is the third. There is a dual statement for *co-special* triangles, i.e., triangles which turn into long exact sequences on applying the functor $\mathrm{Hom}_{\mathcal{D}}(-, W)$. Thus distinguished triangles are special and co-special, but in general there are many more (co-)special triangles, then there are distinguished triangles.

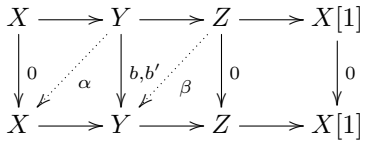
Lemma 13.4.5. *Let $\mathcal{D}$ be a pre-triangulated category. Let*

$$(0, b, 0), (0, b', 0) : (X, Y, Z, f, g, h) \to (X, Y, Z, f, g, h)$$

be endomorphisms of a distinguished triangle. Then $bb' = 0$.

[2]This is nonstandard notation.

Proof. Picture



Applying Lemma 13.4.2 we find dotted arrows α and β such that $b' = f \circ \alpha$ and $b = \beta \circ g$. Then $bb' = \beta \circ g \circ f \circ \alpha = 0$ as $g \circ f = 0$ by Lemma 13.4.1. □

Lemma 13.4.6. *Let $\mathcal{D}$ be a pre-triangulated category. Let (X, Y, Z, f, g, h) be a distinguished triangle. If*

$$\begin{array}{ccc} Z & \xrightarrow{h} & X[1] \\ {\scriptstyle c}\downarrow & & \downarrow{\scriptstyle a[1]} \\ Z & \xrightarrow{h} & X[1] \end{array}$$

is commutative and $a^2 = a$, $c^2 = c$, then there exists a morphism $b : Y \to Y$ with $b^2 = b$ such that (a, b, c) is an endomorphism of the triangle (X, Y, Z, f, g, h).

Proof. By TR3 there exists a morphism b' such that (a, b', c) is an endomorphism of (X, Y, Z, f, g, h). Then $(0, (b')^2 - b', 0)$ is also an endomorphism. By Lemma 13.4.5 we see that $(b')^2 - b'$ has square zero. Set $b = b' - (2b' - 1)((b')^2 - b') = 3(b')^2 - 2(b')^3$. A computation shows that (a, b, c) is an endomorphism and that $b^2 - b = (4(b')^2 - 4b' - 3)((b')^2 - b')^2 = 0$. □

Lemma 13.4.7. *Let $\mathcal{D}$ be a pre-triangulated category. Let $f : X \to Y$ be a morphism of $\mathcal{D}$. There exists a distinguished triangle (X, Y, Z, f, g, h) which is unique up to (nonunique) isomorphism of triangles. More precisely, given a second such distinguished triangle (X, Y, Z', f, g', h') there exists an isomorphism*

$$(1, 1, c) : (X, Y, Z, f, g, h) \longrightarrow (X, Y, Z', f, g', h')$$

Proof. Existence by TR1. Uniqueness up to isomorphism by TR3 and Lemma 13.4.3. □

Lemma 13.4.8. *Let $\mathcal{D}$ be a pre-triangulated category. Let $f : X \to Y$ be a morphism of $\mathcal{D}$. The following are equivalent*

1. *f is an isomorphism,*
2. *$(X, Y, 0, f, 0, 0)$ is a distinguished triangle, and*
3. *for any distinguished triangle (X, Y, Z, f, g, h) we have $Z = 0$.*

Proof. By TR1 the triangle $(X, X, 0, 1, 0, 0)$ is distinguished. Let (X, Y, Z, f, g, h) be a distinguished triangle. By TR3 there is a map of distinguished triangles $(1, f, 0) : (X, X, 0) \to (X, Y, Z)$. If f is an isomorphism, then $(1, f, 0)$ is an isomorphism of triangles by Lemma 13.4.3 and $Z = 0$. Conversely, if $Z = 0$, then $(1, f, 0)$ is an isomorphism of triangles as well, hence f is an isomorphism. □

Lemma 13.4.9. *Let $\mathcal{D}$ be a pre-triangulated category. Let (X, Y, Z, f, g, h) and (X', Y', Z', f', g', h') be triangles. The following are equivalent*

1. *$(X \oplus X', Y \oplus Y', Z \oplus Z', f \oplus f', g \oplus g', h \oplus h')$ is a distinguished triangle,*
2. *both (X, Y, Z, f, g, h) and (X', Y', Z', f', g', h') are distinguished triangles.*

Proof. Assume (2). By TR1 we may choose a distinguished triangle $(X \oplus X', Y \oplus Y', Q, f \oplus f', g'', h'')$. By TR3 we can find morphisms of distinguished triangles $(X, Y, Z, f, g, h) \to (X \oplus X', Y \oplus Y', Q, f \oplus f', g'', h'')$ and $(X', Y', Z', f', g', h') \to (X \oplus X', Y \oplus Y', Q, f \oplus f', g'', h'')$. Taking the direct sum of these morphisms we obtain a morphism of triangles

$$\begin{array}{c} (X \oplus X', Y \oplus Y', Z \oplus Z', f \oplus f', g \oplus g', h \oplus h') \\ \Big\downarrow{\scriptstyle (1,1,c)} \\ (X \oplus X', Y \oplus Y', Q, f \oplus f', g'', h''). \end{array}$$

In the terminology of Remark 13.4.4 this is a map of special triangles (because a direct sum of special triangles is special) and we conclude that c is an isomorphism. Thus (1) holds.

Assume (1). We will show that (X, Y, Z, f, g, h) is a distinguished triangle. First observe that (X, Y, Z, f, g, h) is a special triangle (terminology from Remark 13.4.4) as a direct summand of the distinguished hence special triangle $(X \oplus X', Y \oplus Y', Z \oplus Z', f \oplus f', g \oplus g', h \oplus h')$. Using TR1 let (X, Y, Q, f, g'', h'') be a distinguished triangle. By TR3 there exists a morphism of distinguished triangles $(X \oplus X', Y \oplus Y', Z \oplus Z', f \oplus f', g \oplus g', h \oplus h') \to (X, Y, Q, f, g'', h'')$. Composing this with the inclusion map we get a morphism of triangles

$$(1, 1, c) : (X, Y, Z, f, g, h) \longrightarrow (X, Y, Q, f, g'', h'')$$

By Remark 13.4.4 we find that c is an isomorphism and we conclude that (2) holds. □

Lemma 13.4.10. *Let $\mathcal{D}$ be a pre-triangulated category. Let (X, Y, Z, f, g, h) be a distinguished triangle.*

(1) *If $h = 0$, then there exists a right inverse $s : Z \to Y$ to g.*
(2) *For any right inverse $s : Z \to Y$ of g the map $f \oplus s : X \oplus Z \to Y$ is an isomorphism.*
(3) *For any objects X', Z' of $\mathcal{D}$ the triangle $(X', X' \oplus Z', Z', (1, 0), (0, 1), 0)$ is distinguished.*

Proof. To see (1) use that $\mathrm{Hom}_{\mathcal{D}}(Z, Y) \to \mathrm{Hom}_{\mathcal{D}}(Z, Z) \to \mathrm{Hom}_{\mathcal{D}}(Z, X[1])$ is exact by Lemma 13.4.2. By the same token, if s is as in (2), then $h = 0$ and the sequence

$$0 \to \mathrm{Hom}_{\mathcal{D}}(W, X) \to \mathrm{Hom}_{\mathcal{D}}(W, Y) \to \mathrm{Hom}_{\mathcal{D}}(W, Z) \to 0$$

is split exact (split by $s : Z \to Y$). Hence by Yoneda's lemma we see that $X \oplus Z \to Y$ is an isomorphism. The last assertion follows from TR1 and Lemma 13.4.9. □

Lemma 13.4.11. *Let $\mathcal{D}$ be a pre-triangulated category. Let $f : X \to Y$ be a morphism of $\mathcal{D}$. The following are equivalent*

(1) *f has a kernel,*
(2) *f has a cokernel,*
(3) *f is isomorphic to a map $K \oplus Z \to Z \oplus Q$ induced by id_Z.*

Proof. Any morphism isomorphic to a map of the form $X' \oplus Z \to Z \oplus Y'$ has both a kernel and a cokernel. Hence (3) $\Rightarrow$ (1), (2). Next we prove (1) $\Rightarrow$ (3). Suppose first that $f : X \to Y$ is a monomorphism, i.e., its kernel is zero. By TR1 there exists a distinguished triangle (X, Y, Z, f, g, h). By Lemma 13.4.1 the composition

$h[-1] \circ f = 0$. As f is a monomorphism we see that $h[-1] = 0$ and hence $h = 0$. Then Lemma 13.4.10 implies that $Y = X \oplus Z$, i.e., we see that (3) holds. Next, assume f has a kernel K. As $K \to X$ is a monomorphism we conclude $X = K \oplus X'$ and $f|_{X'} : X' \to Y$ is a monomorphism. Hence $Y = X' \oplus Y'$ and we win. The implication (2) $\Rightarrow$ (3) is dual to this. □

Lemma 13.4.12. *Let $\mathcal{D}$ be a pre-triangulated category. If $\mathcal{D}$ has countable products, then $\mathcal{D}$ is Karoubian. If $\mathcal{D}$ has countable coproducts, then $\mathcal{D}$ is Karoubian.*

Proof. Assume $\mathcal{D}$ has countable products. By Homology, Lemma 12.4.3 it suffices to check that morphisms which have a right inverse have kernels. Any morphism which has a right inverse is an epimorphism, hence has a kernel by Lemma 13.4.11. The second statement is dual to the first. □

The following lemma makes it slightly easier to prove that a pre-triangulated category is triangulated.

Lemma 13.4.13. *Let $\mathcal{D}$ be a pre-triangulated category. In order to prove TR4 it suffices to show that given any pair of composable morphisms $f : X \to Y$ and $g : Y \to Z$ there exist*

1. *isomorphisms $i : X' \to X$, $j : Y' \to Y$ and $k : Z' \to Z$, and then setting $f' = j^{-1}fi : X' \to Y'$ and $g' = k^{-1}gj : Y' \to Z'$ there exist*
2. *distinguished triangles $(X', Y', Q_1, f', p_1, d_1)$, $(X', Z', Q_2, g' \circ f', p_2, d_2)$ and $(Y', Z', Q_3, g', p_3, d_3)$, such that the assertion of TR4 holds.*

Proof. The replacement of X, Y, Z by X', Y', Z' is harmless by our definition of distinguished triangles and their isomorphisms. The lemma follows from the fact that the distinguished triangles $(X', Y', Q_1, f', p_1, d_1)$, $(X', Z', Q_2, g' \circ f', p_2, d_2)$ and $(Y', Z', Q_3, g', p_3, d_3)$ are unique up to isomorphism by Lemma 13.4.7. □

Lemma 13.4.14. *Let $\mathcal{D}$ be a pre-triangulated category. Assume that $\mathcal{D}'$ is an additive full subcategory of $\mathcal{D}$. The following are equivalent*

1. *there exists a set of triangles $\mathcal{T}'$ such that $(\mathcal{D}', \mathcal{T}')$ is a pre-triangulated subcategory of $\mathcal{D}$,*
2. *$\mathcal{D}'$ is preserved under $[1], [-1]$ and given any morphism $f : X \to Y$ in $\mathcal{D}'$ there exists a distinguished triangle (X, Y, Z, f, g, h) in $\mathcal{D}$ such that Z is isomorphic to an object of $\mathcal{D}'$.*

In this case $\mathcal{T}'$ is the set of distinguished triangles (X, Y, Z, f, g, h) of $\mathcal{D}$ such that $X, Y, Z \in \mathrm{Ob}(\mathcal{D}')$ and $f, g, h \in \mathit{Arrows}(\mathcal{D}')$. Finally, if $\mathcal{D}$ is a triangulated category, then (1) and (2) are also equivalent to

3. *$\mathcal{D}'$ is a triangulated subcategory.*

Proof. Omitted. □

Lemma 13.4.15. *An exact functor of pre-triangulated categories is additive.*

Proof. Let $F : \mathcal{D} \to \mathcal{D}'$ be an exact functor of pre-triangulated categories. Since $(0, 0, 0, 1_0, 1_0, 0)$ is a distinguished triangle of $\mathcal{D}$ the triangle

$$(F(0), F(0), F(0), 1_{F(0)}, 1_{F(0)}, F(0))$$

is distinguished in $\mathcal{D}'$. This implies that $1_{F(0)} \circ 1_{F(0)}$ is zero, see Lemma 13.4.1. Hence $F(0)$ is the zero object of $\mathcal{D}'$. This also implies that F applied to any zero morphism is zero (since a morphism in an additive category is zero if and only if it

factors through the zero object). Next, using that $(X, X \oplus Y, Y, (1,0), (0,1), 0)$ is a distinguished triangle, we see that $(F(X), F(X \oplus Y), F(Y), F(1,0), F(0,1), 0)$ is one too. This implies that the map $F(1,0) \oplus F(0,1) : F(X) \oplus F(Y) \to F(X \oplus Y)$ is an isomorphism, see Lemma 13.4.10. We omit the rest of the argument. □

Lemma 13.4.16. *Let $F : \mathcal{D} \to \mathcal{D}'$ be a fully faithful exact functor of pre-triangulated categories. Then a triangle (X, Y, Z, f, g, h) of $\mathcal{D}$ is distinguished if and only if $(F(X), F(Y), F(Z), F(f), F(g), F(h))$ is distinguished in $\mathcal{D}'$.*

Proof. The "only if" part is clear. Assume $(F(X), F(Y), F(Z))$ is distinguished in $\mathcal{D}'$. Pick a distinguished triangle (X, Y, Z', f, g', h') in $\mathcal{D}$. By Lemma 13.4.7 there exists an isomorphism of triangles

$$(1, 1, c') : (F(X), F(Y), F(Z)) \longrightarrow (F(X), F(Y), F(Z')).$$

Since F is fully faithful, there exists a morphism $c : Z \to Z'$ such that $F(c) = c'$. Then $(1, 1, c)$ is an isomorphism between (X, Y, Z) and (X, Y, Z'). Hence (X, Y, Z) is distinguished by TR1. □

Lemma 13.4.17. *Let $\mathcal{D}, \mathcal{D}', \mathcal{D}''$ be pre-triangulated categories. Let $F : \mathcal{D} \to \mathcal{D}'$ and $F' : \mathcal{D}' \to \mathcal{D}''$ be exact functors. Then $F' \circ F$ is an exact functor.*

Proof. Omitted. □

Lemma 13.4.18. *Let $\mathcal{D}$ be a pre-triangulated category. Let $\mathcal{A}$ be an abelian category. Let $H : \mathcal{D} \to \mathcal{A}$ be a homological functor.*

(1) *Let $\mathcal{D}'$ be a pre-triangulated category. Let $F : \mathcal{D}' \to \mathcal{D}$ be an exact functor. Then the composition $G \circ F$ is a homological functor as well.*
(2) *Let $\mathcal{A}'$ be an abelian category. Hence $G : \mathcal{A} \to \mathcal{A}'$ be an exact functor. Hence $G \circ H$ is a homological functor as well.*

Proof. Omitted. □

Lemma 13.4.19. *Let $\mathcal{D}$ be a triangulated category. Let $\mathcal{A}$ be an abelian category. Let $G : \mathcal{A} \to \mathcal{D}$ be a δ-functor.*

(1) *Let $\mathcal{D}'$ be a triangulated category. Let $F : \mathcal{D} \to \mathcal{D}'$ be an exact functor. Then the composition $F \circ G$ is a δ-functor as well.*
(2) *Let $\mathcal{A}'$ be an abelian category. Hence $H : \mathcal{A}' \to \mathcal{A}$ be an exact functor. Hence $G \circ H$ is a δ-functor as well.*

Proof. Omitted. □

Lemma 13.4.20. *Let $\mathcal{D}$ be a triangulated category. Let $\mathcal{A}$ be an abelian category. Let $G : \mathcal{A} \to \mathcal{D}$ be a δ-functor. Let $H : \mathcal{D} \to \mathcal{B}$ be a homological functor. Assume that $H^{-1}(G(A)) = 0$ for all A in $\mathcal{A}$. Then the collection*

$$\{H^n \circ G, H^n(\delta_{A \to B \to C})\}_{n \geq 0}$$

is a δ-functor from $\mathcal{A} \to \mathcal{B}$, see Homology, Definition 12.11.1.

Proof. The notation signifies the following. If $0 \to A \xrightarrow{a} B \xrightarrow{b} C \to 0$ is a short exact sequence in $\mathcal{A}$, then

$$\delta = \delta_{A \to B \to C} : G(C) \to G(A)[1]$$

is a morphism in $\mathcal{D}$ such that $(G(A), G(B), G(C), a, b, \delta)$ is a distinguished triangle, see Definition 13.3.6. Then $H^n(\delta) : H^n(G(C)) \to H^n(G(A)[1]) = H^{n+1}(G(A))$ is

clearly functorial in the short exact sequence. Finally, the long exact cohomology sequence (13.3.5.1) combined with the vanishing of $H^{-1}(G(C))$ gives a long exact sequence

$$0 \to H^0(G(A)) \to H^0(G(B)) \to H^0(G(C)) \xrightarrow{H^0(\delta)} H^1(G(A)) \to \ldots$$

in $\mathcal{B}$ as desired. □

The proof of the following result uses TR4.

Proposition 13.4.21. *Let $\mathcal{D}$ be a triangulated category. Any commutative diagram*

$$\begin{array}{ccc} X & \longrightarrow & Y \\ \downarrow & & \downarrow \\ X' & \longrightarrow & Y' \end{array}$$

can be extended to a diagram

$$\begin{array}{ccccccc} X & \longrightarrow & Y & \longrightarrow & Z & \longrightarrow & X[1] \\ \downarrow & & \downarrow & & \downarrow & & \downarrow \\ X' & \longrightarrow & Y' & \longrightarrow & Z' & \longrightarrow & X'[1] \\ \downarrow & & \downarrow & & \downarrow & & \downarrow \\ X'' & \longrightarrow & Y'' & \longrightarrow & Z'' & \longrightarrow & X''[1] \\ \downarrow & & \downarrow & & \downarrow & & \downarrow \\ X[1] & \longrightarrow & Y[1] & \longrightarrow & Z[1] & \longrightarrow & X[2] \end{array}$$

where all the squares are commutative, except for the lower right square which is anticommutative. Moreover, each of the rows and columns are distinguished triangles. Finally, the morphisms on the bottom row (resp. right column) are obtained from the morphisms of the top row (resp. left column) by applying [1].

Proof. During this proof we avoid writing the arrows in order to make the proof legible. Choose distinguished triangles (X, Y, Z), (X', Y', Z'), (X, X', X''), (Y, Y', Y''), and (X, Y', A). Note that the morphism $X \to Y'$ is both equal to the composition $X \to Y \to Y'$ and equal to the composition $X \to X' \to Y'$. Hence, we can find morphisms

(1) $a : Z \to A$ and $b : A \to Y''$, and
(2) $a' : X'' \to A$ and $b' : A \to Z'$

as in TR4. Denote $c : Y'' \to Z[1]$ the composition $Y'' \to Y[1] \to Z[1]$ and denote $c' : Z' \to X''[1]$ the composition $Z' \to X'[1] \to X''[1]$. The conclusion of our application TR4 are that

(1) (Z, A, Y'', a, b, c), (X'', A, Z', a', b', c') are distinguished triangles,
(2) $(X, Y, Z) \to (X, Y', A)$, $(X, Y', A) \to (Y, Y', Y'')$, $(X, X', X'') \to (X, Y', A)$, $(X, Y', A) \to (X', Y', Z')$ are morphisms of triangles.

First using that $(X, X', X'') \to (X, Y', A)$ and $(X, Y', A) \to (Y, Y', Y'')$. are morphisms of triangles we see the first of the commutative diagrams

$$\begin{array}{ccc} X' & \longrightarrow & Y' \\ \downarrow & & \downarrow \\ X'' & \xrightarrow{b\circ a'} & Y'' \\ \downarrow & & \downarrow \\ X[1] & \longrightarrow & Y[1] \end{array} \qquad \begin{array}{ccccc} Y & \longrightarrow & Z & \longrightarrow & X[1] \\ \downarrow & & \downarrow{\scriptstyle b'\circ a} & & \downarrow \\ Y' & \longrightarrow & Z' & \longrightarrow & X'[1] \end{array}$$

is commutative. The second is commutative too using that $(X, Y, Z) \to (X, Y', A)$ and $(X, Y', A) \to (X', Y', Z')$ are morphisms of triangles. At this point we choose a distinguished triangle (X'', Y'', Z'') starting with the map $b \circ a' : X'' \to Y''$.

Next we apply TR4 one more time to the morphisms $X'' \to A \to Y''$ and the triangles (X'', A, Z', a', b', c'), (X'', Y'', Z''), and $(A, Y'', Z[1], b, c, -a[1])$ to get morphisms $a'' : Z' \to Z''$ and $b'' : Z'' \to Z[1]$. Then $(Z', Z'', Z[1], a'', b'', -b'[1] \circ a[1])$ is a distinguished triangle, hence also $(Z, Z', Z'', -b' \circ a, a'', -b'')$ and hence also $(Z, Z', Z'', b'\circ a, a'', b'')$. Moreover, $(X'', A, Z') \to (X'', Y'', Z'')$ and $(X'', Y'', Z'') \to (A, Y'', Z[1], b, c, -a[1])$ are morphisms of triangles. At this point we have defined all the distinguished triangles and all the morphisms, and all that's left is to verify some commutativity relations.

To see that the middle square in the diagram commutes, note that the arrow $Y' \to Z'$ factors as $Y' \to A \to Z'$ because $(X, Y', A) \to (X', Y', Z')$ is a morphism of triangles. Similarly, the morphism $Y' \to Y''$ factors as $Y' \to A \to Y''$ because $(X, Y', A) \to (Y, Y', Y'')$ is a morphism of triangles. Hence the middle square commutes because the square with sides (A, Z', Z'', Y'') commutes as $(X'', A, Z') \to (X'', Y'', Z'')$ is a morphism of triangles (by TR4). The square with sides $(Y'', Z'', Y[1], Z[1])$ commutes because $(X'', Y'', Z'') \to (A, Y'', Z[1], b, c, -a[1])$ is a morphism of triangles and $c : Y'' \to Z[1]$ is the composition $Y'' \to Y[1] \to Z[1]$. The square with sides $(Z', X'[1], X''[1], Z'')$ is commutative because $(X'', A, Z') \to (X'', Y'', Z'')$ is a morphism of triangles and $c' : Z' \to X''[1]$ is the composition $Z' \to X'[1] \to X''[1]$. Finally, we have to show that the square with sides $(Z'', X''[1], Z[1], X[2])$ anticommutes. This holds because $(X'', Y'', Z'') \to (A, Y'', Z[1], b, c, -a[1])$ is a morphism of triangles and we're done. □

13.5. Localization of triangulated categories

In order to construct the derived category starting from the homotopy category of complexes, we will use a localization process.

Definition 13.5.1. Let $\mathcal{D}$ be a pre-triangulated category. We say a multiplicative system S is *compatible with the triangulated structure* if the following two conditions hold:

MS5 For $s \in S$ we have $s[n] \in S$ for all $n \in \mathbf{Z}$.

MS6 Given a solid commutative square

$$\begin{array}{ccccccc} X & \longrightarrow & Y & \longrightarrow & Z & \longrightarrow & X[1] \\ \downarrow{\scriptstyle s} & & \downarrow{\scriptstyle s'} & & \vdots & & \downarrow{\scriptstyle s[1]} \\ X' & \longrightarrow & Y' & \longrightarrow & Z' & \longrightarrow & X'[1] \end{array}$$

whose rows are distinguished triangles with $s, s' \in S$ there exists a morphism $s'' : Z \to Z'$ in S such that (s, s', s'') is a morphism of triangles.

It turns out that these axioms are not independent of the axioms defining multiplicative systems.

Lemma 13.5.2. *Let $\mathcal{D}$ be a pre-triangulated category. Let S be a set of morphisms of $\mathcal{D}$ and assume that axioms MS1, MS5, MS6 hold (see Categories, Definition 4.25.1 and Definition 13.5.1). Then MS2 holds.*

Proof. Suppose that $f : X \to Y$ is a morphism of $\mathcal{D}$ and $t : X \to X'$ an element of S. Choose a distinguished triangle (X, Y, Z, f, g, h). Next, choose a distinguished triangle $(X', Y', Z, f', g', t[1] \circ h)$ (here we use TR1 and TR2). By MS5, MS6 (and TR2 to rotate) we can find the dotted arrow in the commutative diagram

$$\begin{array}{ccccccc} X & \longrightarrow & Y & \longrightarrow & Z & \longrightarrow & X[1] \\ \downarrow{\scriptstyle t} & & \vdots{\scriptstyle s'} & & \downarrow{\scriptstyle 1} & & \downarrow{\scriptstyle t[1]} \\ X' & \longrightarrow & Y' & \longrightarrow & Z & \longrightarrow & X'[1] \end{array}$$

with moreover $s' \in S$. This proves LMS2. The proof of RMS2 is dual. □

Lemma 13.5.3. *Let $F : \mathcal{D} \to \mathcal{D}'$ be an exact functor of pre-triangulated categories. Let*

$$S = \{f \in \mathit{Arrows}(\mathcal{D}) \mid F(f) \text{ is an isomorphism}\}$$

Then S is a saturated (see Categories, Definition 4.25.17) multiplicative system compatible with the triangulated structure on $\mathcal{D}$.

Proof. We have to prove axioms MS1 – MS6, see Categories, Definitions 4.25.1 and 4.25.17 and Definition 13.5.1. MS1, MS4, and MS5 are direct from the definitions. MS6 follows from TR3 and Lemma 13.4.3. By Lemma 13.5.2 we conclude that MS2 holds. To finish the proof we have to show that MS3 holds. To do this let $f, g : X \to Y$ be morphisms of $\mathcal{D}$, and let $t : Z \to X$ be an element of S such that $f \circ t = g \circ t$. As $\mathcal{D}$ is additive this simply means that $a \circ t = 0$ with $a = f - g$. Choose a distinguished triangle (Z, X, Q, t, d, h) using TR1. Since $a \circ t = 0$ we see by Lemma 13.4.2 there exists a morphism $i : Q \to Y$ such that $i \circ d = a$. Finally, using TR1 again we can choose a triangle (Q, Y, W, i, j, k). Here is a picture

$$\begin{array}{ccccccc} Z & \xrightarrow[t]{} & X & \xrightarrow[d]{} & Q & \longrightarrow & Z[1] \\ & & \downarrow{\scriptstyle 1} & & \downarrow{\scriptstyle i} & & \\ & & X & \xrightarrow[a]{} & Y & & \\ & & & & \downarrow{\scriptstyle j} & & \\ & & & & W & & \end{array}$$

OK, and now we apply the functor F to this diagram. Since $t \in S$ we see that $F(Q) = 0$, see Lemma 13.4.8. Hence $F(j)$ is an isomorphism by the same lemma, i.e., $j \in S$. Finally, $j \circ a = j \circ i \circ d = 0$ as $j \circ i = 0$. Thus $j \circ f = j \circ g$ and we see that LMS3 holds. The proof of RMS3 is dual. □

Lemma 13.5.4. *Let $H : \mathcal{D} \to \mathcal{A}$ be a homological functor between a pre-triangulated category and an abelian category. Let*

$$S = \{f \in \mathit{Arrows}(\mathcal{D}) \mid H^i(f) \text{ is an isomorphism for all } i \in \mathbf{Z}\}$$

Then S is a saturated (see Categories, Definition 4.25.17) multiplicative system compatible with the triangulated structure on $\mathcal{D}$.

Proof. We have to prove axioms MS1 – MS6, see Categories, Definitions 4.25.1 and 4.25.17 and Definition 13.5.1. MS1, MS4, and MS5 are direct from the definitions. MS6 follows from TR3 and the long exact cohomology sequence (13.3.5.1). By Lemma 13.5.2 we conclude that MS2 holds. To finish the proof we have to show that MS3 holds. To do this let $f, g : X \to Y$ be morphisms of $\mathcal{D}$, and let $t : Z \to X$ be an element of S such that $f \circ t = g \circ t$. As $\mathcal{D}$ is additive this simply means that $a \circ t = 0$ with $a = f - g$. Choose a distinguished triangle (Z, X, Q, t, g, h) using TR1 and TR2. Since $a \circ t = 0$ we see by Lemma 13.4.2 there exists a morphism $i : Q \to Y$ such that $i \circ g = a$. Finally, using TR1 again we can choose a triangle (Q, Y, W, i, j, k). Here is a picture

$$\begin{array}{ccccccc} Z & \xrightarrow[t]{} & X & \xrightarrow[g]{} & Q & \longrightarrow & Z[1] \\ & & \downarrow{\scriptstyle 1} & & \downarrow{\scriptstyle i} & & \\ & & X & \xrightarrow[a]{} & Y & & \\ & & & & \downarrow{\scriptstyle j} & & \\ & & & & W & & \end{array}$$

OK, and now we apply the functors H^i to this diagram. Since $t \in S$ we see that $H^i(Q) = 0$ by the long exact cohomology sequence (13.3.5.1). Hence $H^i(j)$ is an isomorphism for all i by the same argument, i.e., $j \in S$. Finally, $j \circ a = j \circ i \circ g = 0$ as $j \circ i = 0$. Thus $j \circ f = j \circ g$ and we see that LMS3 holds. The proof of RMS3 is dual. □

Proposition 13.5.5. *Let $\mathcal{D}$ be a pre-triangulated category. Let S be a multiplicative system compatible with the triangulated structure. Then there exists a unique structure of a pre-triangulated category on $S^{-1}\mathcal{D}$ such that the localization functor $Q : \mathcal{D} \to S^{-1}\mathcal{D}$ is exact. Moreover, if $\mathcal{D}$ is a triangulated category, so is $S^{-1}\mathcal{D}$.*

Proof. We have seen that $S^{-1}\mathcal{D}$ is an additive category and that the localization functor Q is additive in Homology, Lemma 12.8.2, It is clear that we may define $Q(X)[n] = Q(X[n])$ since $\mathcal{S}$ is preserved under the shift functors $[n]$ by MS5. Finally, we say a triangle of $S^{-1}\mathcal{D}$ is distinguished if it is isomorphic to the image of a distinguished triangle under the localization functor Q.

Proof of TR1. The only thing to prove here is that if $a : Q(X) \to Q(Y)$ is a morphism of $S^{-1}\mathcal{D}$, then a fits into a distinguish triangle. Write $a = Q(s)^{-1} \circ Q(f)$ for some $s : Y \to Y'$ in S and $f : X \to Y'$. Choose a distinguished triangle

(X, Y', Z, f, g, h) in $\mathcal{D}$. Then we see that $(Q(X), Q(Y), Q(Z), a, Q(g) \circ Q(s), Q(h))$ is a distinguished triangle of $S^{-1}\mathcal{D}$.

Proof of TR2. This is immediate from the definitions.

Proof of TR3. Note that the existence of the dotted arrow which is required to exist may be proven after replacing the two triangles by isomorphic triangles. Hence we may assume given distinguished triangles (X, Y, Z, f, g, h) and (X', Y', Z', f', g', h') of $\mathcal{D}$ and a commutative diagram

$$\begin{array}{ccc} Q(X) & \xrightarrow[Q(f)]{} & Q(Y) \\ {\scriptstyle a}\downarrow & & \downarrow{\scriptstyle b} \\ Q(X') & \xrightarrow{Q(f')} & Q(Y') \end{array}$$

in $S^{-1}\mathcal{D}$. Now we apply Categories, Lemma 4.25.8 to find a morphism $f'' : X'' \to Y''$ in $\mathcal{D}$ and a commutative diagram

$$\begin{array}{ccccc} X & \xrightarrow[k]{} & X'' & \xleftarrow[s]{} & X' \\ {\scriptstyle f}\downarrow & & \downarrow{\scriptstyle f''} & & \downarrow{\scriptstyle f'} \\ Y & \xrightarrow{l} & Y'' & \xleftarrow{t} & Y' \end{array}$$

in $\mathcal{D}$ with $s, t \in S$ and $a = s^{-1}k$, $b = t^{-1}l$. At this point we can use TR3 for $\mathcal{D}$ and MS6 to find a commutative diagram

$$\begin{array}{ccccccc} X & \longrightarrow & Y & \longrightarrow & Z & \longrightarrow & X[1] \\ \downarrow{\scriptstyle k} & & \downarrow{\scriptstyle l} & & \downarrow{\scriptstyle m} & & \downarrow{\scriptstyle g[1]} \\ X'' & \longrightarrow & Y'' & \longrightarrow & Z'' & \longrightarrow & X''[1] \\ \uparrow{\scriptstyle s} & & \uparrow{\scriptstyle t} & & \uparrow{\scriptstyle r} & & \uparrow{\scriptstyle s[1]} \\ X' & \longrightarrow & Y' & \longrightarrow & Z' & \longrightarrow & X'[1] \end{array}$$

with $r \in S$. It follows that setting $c = Q(r)^{-1}Q(m)$ we obtain the desired morphism of triangles

$$\begin{array}{c} (Q(X), Q(Y), Q(Z), Q(f), Q(g), Q(h)) \\ \downarrow{\scriptstyle (a,b,c)} \\ (Q(X'), Q(Y'), Q(Z'), Q(f'), Q(g'), Q(h')) \end{array}$$

This proves the first statement of the lemma. If $\mathcal{D}$ is also a triangulated category, then we still have to prove TR4 in order to show that $S^{-1}\mathcal{D}$ is triangulated as well. To do this we reduce by Lemma 13.4.13 to the following statement: Given composable morphisms $a : Q(X) \to Q(Y)$ and $b : Q(Y) \to Q(Z)$ we have to produce an octahedron after possibly replacing $Q(X), Q(Y), Q(Z)$ by isomorphic objects. To do this we may first replace Y by an object such that $a = Q(f)$ for some morphism $f : X \to Y$ in $\mathcal{D}$. (More precisely, write $a = s^{-1}f$ with $s : Y \to Y'$ in S and $f : X \to Y'$. Then replace Y by Y'.) After this we similarly replace Z by an object such that $b = Q(g)$ for some morphism $g : Y \to Z$. Now we can find distinguished triangles (X, Y, Q_1, f, p_1, d_1), $(X, Z, Q_2, g \circ f, p_2, d_2)$, and

(Y, Z, Q_3, g, p_3, d_3) in $\mathcal{D}$ (by TR1), and morphisms $a : Q_1 \to Q_2$ and $b : Q_2 \to Q_3$ as in TR4. Then it is immediately verified that applying the functor Q to all these data gives a corresponding structure in $S^{-1}\mathcal{D}$ □

The universal property of the localization of a triangulated category is as follows (we formulate this for pre-triangulated categories, hence it holds a fortiori for triangulated categories).

Lemma 13.5.6. *Let $\mathcal{D}$ be a pre-triangulated category. Let S be a multiplicative system compatible with the triangulated category. Let $Q : \mathcal{D} \to S^{-1}\mathcal{D}$ be the localization functor, see Proposition 13.5.5.*

(1) *If $H : \mathcal{D} \to \mathcal{A}$ is a homological functor into an abelian category $\mathcal{A}$ such that $H(s)$ is an isomorphism for all $s \in S$, then the unique factorization $H' : S^{-1}\mathcal{D} \to \mathcal{A}$ such that $H = H' \circ Q$ (see Categories, Lemma 4.25.6) is a homological functor too.*
(2) *If $F : \mathcal{D} \to \mathcal{D}'$ is an exact functor into a pre-triangulated category $\mathcal{D}'$ such that $F(s)$ is an isomorphism for all $s \in S$, then the unique factorization $F' : S^{-1}\mathcal{D} \to \mathcal{D}'$ such that $F = F' \circ Q$ (see Categories, Lemma 4.25.6) is an exact functor too.*

Proof. This lemma proves itself. Details omitted. □

The following lemma describes the kernel (see Definition 13.6.5) of the localization functor.

Lemma 13.5.7. *Let $\mathcal{D}$ be a pre-triangulated category. Let S be a multiplicative system compatible with the triangulated structure. Let Z be an object of $\mathcal{D}$. The following are equivalent*

(1) *$Q(Z) = 0$ in $S^{-1}\mathcal{D}$,*
(2) *there exists $Z' \in \mathrm{Ob}(\mathcal{D})$ such that $0 : Z \to Z'$ is an element of S,*
(3) *there exists $Z' \in \mathrm{Ob}(\mathcal{D})$ such that $0 : Z' \to Z$ is an element of S, and*
(4) *there exists an object Z' and a distinguished triangle $(X, Y, Z \oplus Z', f, g, h)$ such that $f \in S$.*

If S is saturated, then these are also equivalent to

(4) *the morphism $0 \to Z$ is an element of S,*
(5) *the morphism $Z \to 0$ is an element of S,*
(6) *there exists a distinguished triangle (X, Y, Z, f, g, h) such that $f \in S$.*

Proof. The equivalence of (1), (2), and (3) is Homology, Lemma 12.8.3. If (2) holds, then $(Z'[-1], Z'[-1] \oplus Z, Z, (1, 0), (0, 1), 0)$ is a distinguised triangle (see Lemma 13.4.10) with "$0 \in S$". By rotating we conclude that (4) holds. If $(X, Y, Z \oplus Z', f, g, h)$ is a distinguished triangle with $f \in S$ then $Q(f)$ is an isomorphism hence $Q(Z \oplus Z') = 0$ hence $Q(Z) = 0$. Thus (1) – (4) are all equivalent.

Next, assume that S is saturated. Note that each of (4), (5), (6) implies one of the equivalent conditions (1) – (4). Suppose that $Q(Z) = 0$. Then $0 \to Z$ is a morphism of $\mathcal{D}$ which becomes an isomorphism in $S^{-1}\mathcal{D}$. According to Categories, Lemma 4.25.18 the fact that S is saturated implies that $0 \to Z$ is in S. Hence (1) $\Rightarrow$ (4). Dually (1) $\Rightarrow$ (5). Finally, if $0 \to Z$ is in S, then the triangle $(0, Z, Z, 0, \mathrm{id}_Z, 0)$ is distinguished by TR1 and TR2 and is a triangle as in (4). □

Lemma 13.5.8. *Let $\mathcal{D}$ be a triangulated category. Let S be a saturated multiplicative system in $\mathcal{D}$. Let (X, Y, Z, f, g, h) be a distinguished triangle in $\mathcal{D}$. Consider the category of morphisms of triangles*

$$\mathcal{I} = \{(s, s', s'') : (X, Y, Z, f, g, h) \to (X', Y', Z', f', g', h') \mid (s, s', s'') \in S\}$$

Then $\mathcal{I}$ is a filtered category and the functors $\mathcal{I} \to X/S$, $\mathcal{I} \to Y/S$, and $\mathcal{I} \to Z/S$ are surjective on objects.

Proof. We strongly suggest the reader skip the proof of this lemma and instead works it out on a napkin. The category $\mathcal{I}$ is nonempty as the identity provides an object. This proves the first condition of the definition of a filtered category, see Categories, Definition 4.19.1.

Note that if $s : X \to X'$ is a morphism of S, then using MS2 we can find $s' : Y \to Y'$ and $f' : X' \to Y'$ such that $f' \circ s = s' \circ f$, whereupon we can use MS6 to complete this into an object of $\mathcal{I}$. Hence certainly the surjectivity statement is correct.

Next we check condition (3) of Categories, Definition 4.19.1. Suppose $(s_1, s_1', s_1'') : (X, Y, Z) \to (X_1, Y_1, Z_1)$ and $(s_2, s_2', s_2'') : (X, Y, Z) \to (X_2, Y_2, Z_2)$ are objects of $\mathcal{I}$, and suppose $(a, b, c), (a', b', c')$ are two morphisms between them. Since $a \circ s_1 = a' \circ s_1$ there exists a morphism $s_3 : X_2 \to X_3$ such that $s_3 \circ a = s_3 \circ a'$. Using the surjectivity statement we can complete this to a morphism of triangles $(s_3, s_3', s_3'') : (X_2, Y_2, Z_2) \to (X_3, Y_3, Z_3)$ with $s_3, s_3', s_3'' \in S$. Thus $(s_3 \circ s_2, s_3' \circ s_2', s_3'' \circ s_2'') : (X, Y, Z) \to (X_3, Y_3, Z_3)$ is also an object of $\mathcal{I}$ and after composing the maps $(a, b, c), (a', b', c')$ with (s_3, s_3', s_3'') we obtain $a = a'$. By rotating we may do the same to get $b = b'$ and $c = c'$.

Finally, we check condition (2) of Categories, Definition 4.19.1. Suppose we are given two objects $(s_1, s_1', s_1'') : (X, Y, Z) \to (X_1, Y_1, Z_1)$ and $(s_2, s_2', s_2'') : (X, Y, Z) \to (X_2, Y_2, Z_2)$ of $\mathcal{I}$. Pick a morphism $s_3 : X \to X_3$ in S such that there exist morphisms $a : X_1 \to X_3$ and $a' : X_2 \to X_3$ with $s_3 = a \circ s_1$ and $s_3 = a' \circ s_2$. Because S is a saturated multiplicative system we see that $a' \in S$ (because S is the set of arrows of $\mathcal{D}$ which are turned into isomorphisms in $S^{-1}\mathcal{D}$, see Categories, Lemma 4.25.18). Hence, by the essential surjectivity above, we can find a morphism $(a, b, c) : (X_2, Y_2, Z_2) \to (X_3, Y_3, Z_3)$ with $a, b, c \in S$ such that $X \to X_3$ factors through $X \to X_1$. Replacing (X_2, Y_2, Z_2) by (X_3, Y_3, Z_3) and repeating this argument twice more, we may assume that $s_2 = a \circ s_1$, $s_2' = b \circ s_1'$, and $s_2'' = c \circ s_1''$. The problem is that it may not be the case that $b \circ f_1 = f_2 \circ a$, etc, i.e., we don't know that (a, b, c) is a morphism of triangles from (X_1, Y_1, Z_1) to (X_2, Y_2, Z_2). On the other hand, we do know that $(a, b, c) \circ (s_1, s_1', s_1'')$ is a morphism of triangles. Using MS3 one more time this means there exist morphisms $s_3 : X_2 \to X_3$, $s_3' : Y_2 \to Y_3$, and $s_3'' : Z_2 \to Z_3$ in S such that the required equalities hold after post-composing with them, e.g., $s_3' \circ b \circ f_1 = s_3' \circ f_2 \circ a$, etc. Using the essential surjectivity above once more we see that we may find a morphism of triangles $(s_4, s_4', s_4'') : (X_2, Y_2, Z_2) \to (X_4, Y_4, Z_4)$ with $s_4, s_4', s_4'' \in S$ such that s_4 factors through s_3, s_4' factors through s_3', and s_4'' factors through s_3''. We conclude that $(s_4 \circ a, s_4' \circ b, s_4'' \circ c)$ is a morphism of triangles from (X_1, Y_1, Z_1) to (X_4, Y_4, Z_4). □

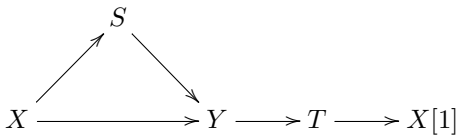
13.6. Quotients of triangulated categories

Given a triangulated category and a triangulated subcategory we can construct another triangulated category by taking the "quotient". The construction uses

a localization. This is similar to the quotient of an abelian category by a Serre subcategory, see Homology, Section 12.9. Before we do the actual construction we briefly discuss kernels of exact functors.

Definition 13.6.1. Let $\mathcal{D}$ be a pre-triangulated category. We say a full pre-triangulated subcategory $\mathcal{D}'$ of $\mathcal{D}$ is *saturated* if whenever $X \oplus Y$ is isomorphic to an object of $\mathcal{D}'$ then both X and Y are isomorphic to objects of $\mathcal{D}'$.

A saturated triangulated subcategory is sometimes called a *thick triangulated subcategory*. In some references, this is only used for strictly full triangulated subcategories (and sometimes the definition is written such that it implies strictness). There is another notion, that of an *épaisse triangulated subcategory*. The definition is that given a commutative diagram



where the second line is a distinguished triangle and S and T isomorphic to objects of $\mathcal{D}'$, then also X and Y are isomorphic to objects of $\mathcal{D}$. It turns out that this is equivalent to being saturated (this is elementary and can be found in [**Ric89a**]) and the notion of a saturated category is easier to work with.

Lemma 13.6.2. *Let $F : \mathcal{D} \to \mathcal{D}'$ be an exact functor of pre-triangulated categories. Let $\mathcal{D}''$ be the full subcategory of $\mathcal{D}$ with objects*

$$\mathrm{Ob}(\mathcal{D}'') = \{X \in \mathrm{Ob}(\mathcal{D}) \mid F(X) = 0\}$$

Then $\mathcal{D}''$ is a strictly full saturated pre-triangulated subcategory of $\mathcal{D}$. If $\mathcal{D}$ is a triangulated category, then $\mathcal{D}''$ is a triangulated subcategory.

Proof. It is clear that $\mathcal{D}''$ is preserved under $[1]$ and $[-1]$. If (X, Y, Z, f, g, h) is a distinguished triangle of $\mathcal{D}$ and $F(X) = F(Y) = 0$, then also $F(Z) = 0$ as $(F(X), F(Y), F(Z), F(f), F(g), F(h))$ is distinguished. Hence we may apply Lemma 13.4.14 to see that $\mathcal{D}''$ is a pre-triangulated subcategory (respectively a triangulated subcategory if $\mathcal{D}$ is a triangulated category). The final assertion of being saturated follows from $F(X) \oplus F(Y) = 0 \Rightarrow F(X) = F(Y) = 0$. □

Lemma 13.6.3. *Let $H : \mathcal{D} \to \mathcal{A}$ be a homological functor of a pre-triangulated category into an abelian category. Let $\mathcal{D}'$ be the full subcategory of $\mathcal{D}$ with objects*

$$\mathrm{Ob}(\mathcal{D}') = \{X \in \mathrm{Ob}(\mathcal{D}) \mid H(X[n]) = 0 \text{ for all } n \in \mathbf{Z}\}$$

Then $\mathcal{D}'$ is a strictly full saturated pre-triangulated subcategory of $\mathcal{D}$. If $\mathcal{D}$ is a triangulated category, then $\mathcal{D}'$ is a triangulated subcategory.

Proof. It is clear that $\mathcal{D}'$ is preserved under $[1]$ and $[-1]$. If (X, Y, Z, f, g, h) is a distinguished triangle of $\mathcal{D}$ and $H(X[n]) = H(Y[n]) = 0$ for all n, then also $H(Z[n]) = 0$ for all n by the long exact sequence (13.3.5.1). Hence we may apply Lemma 13.4.14 to see that $\mathcal{D}'$ is a pre-triangulated subcategory (respectively a triangulated subcategory if $\mathcal{D}$ is a triangulated category). The assertion of being

saturated follows from

$$\begin{aligned} H((X \oplus Y)[n]) = 0 &\Rightarrow H(X[n] \oplus Y[n]) = 0 \\ &\Rightarrow H(X[n]) \oplus H(Y[n]) = 0 \\ &\Rightarrow H(X[n]) = H(Y[n]) = 0 \end{aligned}$$

for all $n \in \mathbf{Z}$. □

Lemma 13.6.4. *Let $H : \mathcal{D} \to \mathcal{A}$ be a homological functor of a pre-triangulated category into an abelian category. Let $\mathcal{D}_H^+, \mathcal{D}_H^-, \mathcal{D}_H^b$ be the full subcategory of $\mathcal{D}$ with objects*

$$\begin{aligned} \mathrm{Ob}(\mathcal{D}_H^+) &= \{X \in \mathrm{Ob}(\mathcal{D}) \mid H(X[n]) = 0 \text{ for all } n \ll 0\} \\ \mathrm{Ob}(\mathcal{D}_H^-) &= \{X \in \mathrm{Ob}(\mathcal{D}) \mid H(X[n]) = 0 \text{ for all } n \gg 0\} \\ \mathrm{Ob}(\mathcal{D}_H^b) &= \{X \in \mathrm{Ob}(\mathcal{D}) \mid H(X[n]) = 0 \text{ for all } |n| \gg 0\} \end{aligned}$$

Each of these is a strictly full saturated pre-triangulated subcategory of $\mathcal{D}$. If $\mathcal{D}$ is a triangulated category, then each is a triangulated subcategory.

Proof. Let us prove this for $\mathcal{D}_H^+$. It is clear that it is preserved under [1] and [−1]. If (X, Y, Z, f, g, h) is a distinguished triangle of $\mathcal{D}$ and $H(X[n]) = H(Y[n]) = 0$ for all $n \ll 0$, then also $H(Z[n]) = 0$ for all $n \ll 0$ by the long exact sequence (13.3.5.1). Hence we may apply Lemma 13.4.14 to see that $\mathcal{D}_H^+$ is a pre-triangulated subcategory (respectively a triangulated subcategory if $\mathcal{D}$ is a triangulated category). The assertion of being saturated follows from

$$\begin{aligned} H((X \oplus Y)[n]) = 0 &\Rightarrow H(X[n] \oplus Y[n]) = 0 \\ &\Rightarrow H(X[n]) \oplus H(Y[n]) = 0 \\ &\Rightarrow H(X[n]) = H(Y[n]) = 0 \end{aligned}$$

for all $n \in \mathbf{Z}$. □

Definition 13.6.5. Let $\mathcal{D}$ be a (pre-)triangulated category.

(1) Let $F : \mathcal{D} \to \mathcal{D}'$ be an exact functor. The *kernel of F* is the strictly full saturated (pre-)triangulated subcategory described in Lemma 13.6.2.
(2) Let $H : \mathcal{D} \to \mathcal{A}$ be a homological functor. The *kernel of H* is the strictly full saturated (pre-)triangulated subcategory described in Lemma 13.6.3.

These are sometimes denoted $\mathrm{Ker}(F)$ or $\mathrm{Ker}(H)$.

The proof of the following lemma uses TR4.

Lemma 13.6.6. *Let $\mathcal{D}$ be a triangulated category. Let $\mathcal{D}' \subset \mathcal{D}$ be a full triangulated subcategory. Set*

$$S = \left\{ \begin{array}{c} f \in \mathrm{Arrows}(\mathcal{D}) \text{ such that there exists a distinguished triangle} \\ (X, Y, Z, f, g, h) \text{ of } \mathcal{D} \text{ with } Z \text{ isomorphic to an object of } \mathcal{D}' \end{array} \right\} \tag{13.6.6.1}$$

Then S is a multiplicative system compatible with the triangulated structure on $\mathcal{D}$. In this situation the following are equivalent

(1) *S is a saturated multiplicative system,*
(2) *$\mathcal{D}'$ is a saturated triangulated subcategory.*

Proof. To prove the first assertion we have to prove that MS1, MS2, MS3 and MS5, MS6 hold.

Proof of MS1. It is clear that identities are in S because $(X, X, 0, 1, 0, 0)$ is distinguished for every object X of $\mathcal{D}$ and because 0 is an object of $\mathcal{D}'$. Let $f : X \to Y$ and $g : Y \to Z$ be composable morphisms contained in S. Choose distinguished triangles (X, Y, Q_1, f, p_1, d_1), $(X, Z, Q_2, g \circ f, p_2, d_2)$, and (Y, Z, Q_3, g, p_3, d_3). By assumption we know that Q_1 and Q_3 are isomorphic to objects of $\mathcal{D}'$. By TR4 we know there exists a distinguished triangle (Q_1, Q_2, Q_3, a, b, c). Since $\mathcal{D}'$ is a triangulated subcategory we conclude that Q_2 is isomorphic to an object of $\mathcal{D}'$. Hence $g \circ f \in S$.

Proof of MS3. Let $a : X \to Y$ be a morphism and let $t : Z \to X$ be an element of S such that $a \circ t = 0$. To prove LMS3 it suffices to find an $s \in S$ such that $s \circ a = 0$, compare with the proof of Lemma 13.5.3. Choose a distinguished triangle (Z, X, Q, t, g, h) using TR1 and TR2. Since $a \circ t = 0$ we see by Lemma 13.4.2 there exists a morphism $i : Q \to Y$ such that $i \circ g = a$. Finally, using TR1 again we can choose a triangle (Q, Y, W, i, s, k). Here is a picture

$$\begin{array}{ccccccc} Z & \xrightarrow[t]{} & X & \xrightarrow[g]{} & Q & \longrightarrow & Z[1] \\ & & \downarrow{\scriptstyle 1} & & \downarrow{\scriptstyle i} & & \\ & & X & \xrightarrow[a]{} & Y & & \\ & & & & \downarrow{\scriptstyle s} & & \\ & & & & W & & \end{array}$$

Since $t \in S$ we see that Q is isomorphic to an object of $\mathcal{D}'$. Hence $s \in S$. Finally, $s \circ a = s \circ i \circ g = 0$ as $s \circ i = 0$ by Lemma 13.4.1. We conclude that LMS3 holds. The proof of RMS3 is dual.

Proof of MS5. Follows as distinguished triangles and $\mathcal{D}'$ are stable under translations

Proof of MS6. Suppose given a commutative diagram

$$\begin{array}{ccc} X & \longrightarrow & Y \\ \downarrow{\scriptstyle s} & & \downarrow{\scriptstyle s'} \\ X' & \longrightarrow & Y' \end{array}$$

with $s, s' \in S$. By Proposition 13.4.21 we can extend this to a nine square diagram. As s, s' are elements of S we see that X'', Y'' are isomorphic to objects of $\mathcal{D}'$. Since $\mathcal{D}'$ is a full triangulated subcategory we see that Z'' is also isomorphic to an object of $\mathcal{D}'$. Whence the morphism $Z \to Z'$ is an element of S. This proves MS6.

MS2 is a formal consequence of MS1, MS5, and MS6, see Lemma 13.5.2. This finishes the proof of the first assertion of the lemma.

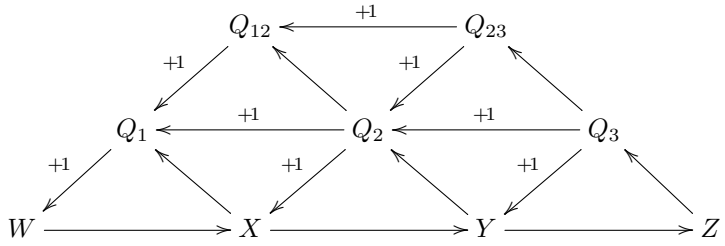
Let's assume that S is saturated. (In the following we will use rotation of distinguished triangles without further mention.) Let $X \oplus Y$ be an object isomorphic to an object of $\mathcal{D}'$. Consider the morphism $f : 0 \to X$. The composition $0 \to X \to X \oplus Y$ is an element of S as $(0, X \oplus Y, X \oplus Y, 0, 1, 0)$ is a distinguished triangle. The composition $Y[-1] \to 0 \to X$ is an element of S as $(X, X \oplus Y, Y, (1, 0), (0, 1), 0)$ is a

distinguished triangle, see Lemma 13.4.10. Hence $0 \to X$ is an element of S (as S is saturated). Thus X is isomorphic to an object of $\mathcal{D}'$ as desired.

Finally, assume $\mathcal{D}'$ is a saturated triangulated subcategory. Let

$$W \xrightarrow{h} X \xrightarrow{g} Y \xrightarrow{f} Z$$

be composable morphisms of $\mathcal{D}$ such that $fg, gh \in S$. We will build up a picture of objects as in the diagram below.



First choose distinguished triangles (W, X, Q_1), (X, Y, Q_2), (Y, Z, Q_3) (W, Y, Q_{12}), and (X, Z, Q_{23}). Denote $s : Q_2 \to Q_1[1]$ the composition $Q_2 \to X[1] \to Q_1[1]$. Denote $t : Q_3 \to Q_2[1]$ the composition $Q_3 \to Y[1] \to Q_2[1]$. By TR4 applied to the composition $W \to X \to Y$ and the composition $X \to Y \to Z$ there exist a distinguished triangles (Q_1, Q_{12}, Q_2) and (Q_2, Q_{23}, Q_3) which use the morphisms s and t. The objects Q_{12} and Q_{23} are isomorphic to objects of $\mathcal{D}'$ as $W \to Y$ and $X \to Z$ are assumed in S. Hence also $s[1]t$ is an element of S as S is closed under compositions and shifts. Note that $s[1]t = 0$ as $Y[1] \to Q_2[1] \to X[2]$ is zero, see Lemma 13.4.1. Hence $Q_3 \oplus Q_1[2]$ is isomorphic to an object of $\mathcal{D}'$, see Lemma 13.4.10. By assumption on $\mathcal{D}'$ we conclude that Q_3, Q_1 are isomorphic to objects of $\mathcal{D}'$. Looking at the distinguished triangle (Q_1, Q_{12}, Q_2) we conclude that Q_2 is also isomorphic to an object of $\mathcal{D}'$. Looking at the distinguished triangle (X, Y, Q_2) we finally conclude that $g \in S$. (It is also follows that $h, f \in S$, but we don't need this.) □

Definition 13.6.7. Let $\mathcal{D}$ be a triangulated category. Let $\mathcal{B}$ be a full triangulated subcategory. We define the *quotient category* $\mathcal{D}/\mathcal{B}$ by the formula $\mathcal{D}/\mathcal{B} = S^{-1}\mathcal{D}$, where S is the multiplicative system of $\mathcal{D}$ associated to $\mathcal{B}$ via Lemma 13.6.6. The localization functor $Q : \mathcal{D} \to \mathcal{D}/\mathcal{B}$ is called the *quotient functor* in this case.

Note that the quotient functor $Q : \mathcal{D} \to \mathcal{D}/\mathcal{B}$ is an exact functor of triangulated categories, see Proposition 13.5.5. The universal property of this construction is the following.

Lemma 13.6.8. *Let $\mathcal{D}$ be a triangulated category. Let $\mathcal{B}$ be a full triangulated subcategory of $\mathcal{D}$. Let $Q : \mathcal{D} \to \mathcal{D}/\mathcal{B}$ be the quotient functor.*

1. *If $H : \mathcal{D} \to \mathcal{A}$ is a homological functor into an abelian category $\mathcal{A}$ such that $\mathcal{B} \subset Ker(H)$ then there exists a unique factorization $H' : \mathcal{D}/\mathcal{B} \to \mathcal{A}$ such that $H = H' \circ Q$ and H' is a homological functor too.*
2. *If $F : \mathcal{D} \to \mathcal{D}'$ is an exact functor into a pre-triangulated category $\mathcal{D}'$ such that $\mathcal{B} \subset Ker(F)$ then there exists a unique factorization $F' : \mathcal{D}/\mathcal{B} \to \mathcal{D}'$ such that $F = F' \circ Q$ and F' is an exact functor too.*

Proof. This lemma follows from Lemma 13.5.6. Namely, if $f : X \to Y$ is a morphism of $\mathcal{D}$ such that for some distinguished triangle (X, Y, Z, f, g, h) the object

Z is isomorphic to an object of $\mathcal{B}$, then $H(f)$, resp. $F(f)$ is an isomorphism under the assumptions of (1), resp. (2). Details omitted. □

The kernel of the quotient functor can be described as follows.

Lemma 13.6.9. *Let $\mathcal{D}$ be a triangulated category. Let $\mathcal{B}$ be a full triangulated subcategory. The kernel of the quotient functor $Q : \mathcal{D} \to \mathcal{D}/\mathcal{B}$ is the strictly full subcategory of $\mathcal{D}$ whose objects are*

$$\mathrm{Ob}(Ker(Q)) = \left\{ \begin{array}{l} Z \in \mathrm{Ob}(\mathcal{D}) \text{ such that there exists a } Z' \in \mathrm{Ob}(\mathcal{D}) \\ \text{such that } Z \oplus Z' \text{ is isomorphic to an object of } \mathcal{B} \end{array} \right\}$$

In other words it is the smallest strictly full saturated triangulated subcategory of $\mathcal{D}$ containing $\mathcal{B}$.

Proof. First note that the kernel is automatically a strictly full triangulated subcategory containing summands of any of its objects, see Lemma 13.6.2. The description of its objects follows from the definitions and Lemma 13.5.7 part (4). □

Let $\mathcal{D}$ be a triangulated category. At this point we have constructions which induce order preserving maps between

(1) the partially ordered set of multiplicative systems S in $\mathcal{D}$ compatible with the triangulated structure, and
(2) the partially ordered set of full triangulated subcategories $\mathcal{B} \subset \mathcal{D}$.

Namely, the constructions are given by $S \mapsto \mathcal{B}(S) = \mathrm{Ker}(Q : \mathcal{D} \to S^{-1}\mathcal{D})$ and $\mathcal{B} \mapsto S(\mathcal{B})$ where $S(\mathcal{B})$ is the multiplicative set of (13.6.6.1), i.e.,

$$S(\mathcal{B}) = \left\{ \begin{array}{l} f \in \mathrm{Arrows}(\mathcal{D}) \text{ such that there exists a distinguished triangle} \\ (X, Y, Z, f, g, h) \text{ of } \mathcal{D} \text{ with } Z \text{ isomorphic to an object of } \mathcal{B} \end{array} \right\}$$

Note that it is not the case that these operations are mutually inverse.

Lemma 13.6.10. *Let $\mathcal{D}$ be a triangulated category. The operations described above have the following properties*

(1) *$S(\mathcal{B}(S))$ is the "saturation" of S, i.e., it is the smallest saturated multiplicative system in $\mathcal{D}$ containing S, and*
(2) *$\mathcal{B}(S(\mathcal{B}))$ is the "saturation" of $\mathcal{B}$, i.e., it is the smallest strictly full saturated triangulated subcategory of $\mathcal{D}$ containing $\mathcal{B}$.*

In particular, the constructions define mutually inverse maps between the (partially ordered) set of saturated multiplicative systems in $\mathcal{D}$ compatible with the triangulated structure on $\mathcal{D}$ and the (partially ordered) set of strictly full saturated triangulated subcategories of $\mathcal{D}$.

Proof. First, let's start with a full triangulated subcategory $\mathcal{B}$. Then $\mathcal{B}(S(\mathcal{B})) = \mathrm{Ker}(Q : \mathcal{D} \to \mathcal{D}/\mathcal{B})$ and hence (2) is the content of Lemma 13.6.9.

Next, suppose that S is multiplicative system in $\mathcal{D}$ compatible with the triangulation on $\mathcal{D}$. Then $\mathcal{B}(S) = \mathrm{Ker}(Q : \mathcal{D} \to S^{-1}\mathcal{D})$. Hence (using Lemma 13.4.8 in the localized category)

$$\begin{aligned} S(\mathcal{B}(S)) &= \left\{ \begin{array}{l} f \in \mathrm{Arrows}(\mathcal{D}) \text{ such that there exists a distinguished} \\ \text{triangle } (X, Y, Z, f, g, h) \text{ of } \mathcal{D} \text{ with } Q(Z) = 0 \end{array} \right\} \\ &= \{f \in \mathrm{Arrows}(\mathcal{D}) \mid Q(f) \text{ is an isomorphism}\} \\ &= \hat{S} = S' \end{aligned}$$

in the notation of Categories, Lemma 4.25.18. The final statement of that lemma finishes the proof. □

Lemma 13.6.11. *Let $H : \mathcal{D} \to \mathcal{A}$ be a homological functor from a triangulated category $\mathcal{D}$ to an abelian category $\mathcal{A}$, see Definition 13.3.5. The subcategory $Ker(H)$ of $\mathcal{D}$ is a strictly full saturated triangulated subcategory of $\mathcal{D}$ whose corresponding saturated multiplicative system (see Lemma 13.6.10) is the set*

$$S = \{f \in Arrows(\mathcal{D}) \mid H^i(f) \text{ is an isomorphism for all } i \in \mathbf{Z}\}.$$

The functor H factors through the quotient functor $Q : \mathcal{D} \to \mathcal{D}/Ker(H)$.

Proof. The category $\mathrm{Ker}(H)$ is a strictly full saturated triangulated subcategory of $\mathcal{D}$ by Lemma 13.6.3. The set S is a saturated multiplicative system compatible with the triangulated structure by Lemma 13.5.4. Recall that the multiplicative system corresponding to $\mathrm{Ker}(H)$ is the set

$$\left\{ \begin{array}{c} f \in \mathrm{Arrows}(\mathcal{D}) \text{ such that there exists a distinguished triangle} \\ (X, Y, Z, f, g, h) \text{ with } H^i(Z) = 0 \text{ for all } i \end{array} \right\}$$

By the long exact cohomology sequence, see (13.3.5.1), it is clear that f is an element of this set if and only if f is an element of S. Finally, the factorization of H through Q is a consequence of Lemma 13.6.8. □

It is clear that in the lemma above the factorization of H through $\mathcal{D}/\mathrm{Ker}(H)$ is the universal factorization. Namely, if $F : \mathcal{D} \to \mathcal{D}'$ is an exact functor of triangulated categories and if there exists a homological functor $H' : \mathcal{D}' \to \mathcal{A}$ such that $H \cong H' \circ F$, then F factors through the quotient functor $Q : \mathcal{D} \to \mathcal{D}/\mathrm{Ker}(H)$.

We put the following lemma here for lack of a better spot to put it in.

Lemma 13.6.12. *Let $\mathcal{D}$, $\mathcal{D}'$ be triangulated categories. Let $F : \mathcal{D} \to \mathcal{D}'$ and $G : \mathcal{D}' \to \mathcal{D}$ be functors. Assume that*

1. *F and G are exact functors,*
2. *F is fully faithful,*
3. *G is a right adjoint to F, and*
4. *the kernel of G is zero.*

Then F is an equivalence of categories.

Proof. Since F if fully faithful the adjunction map $\mathrm{id} \to G \circ F$ is an isomorphism. Let X be an object of $\mathcal{D}'$. Choose a distinguished triangle

$$F(G(X)) \to X \to Y \to F(G(X))[1]$$

in $\mathcal{D}'$. Applying G and using that $G(F(G(X))) = G(X)$ we find a distinguished triangle

$$G(X) \to G(X) \to G(Y) \to G(X)[1]$$

Hence $G(Y) = 0$. Thus $Y = 0$. Thus $F(G(X)) \to X$ is an isomorphism. □

13.7. The homotopy category

Let $\mathcal{A}$ be an additive category. The homotopy category $K(\mathcal{A})$ of $\mathcal{A}$ is the category of complexes of $\mathcal{A}$ with morphisms given by morphisms of complexes up to homotopy. Here is the formal definition.

Definition 13.7.1. Let $\mathcal{A}$ be an additive category.

(1) We set $\mathrm{Comp}(\mathcal{A}) = \mathrm{CoCh}(\mathcal{A})$ be the *category of (cochain) complexes*.
(2) A complex $K^\bullet$ is said to be *bounded below* if $K^n = 0$ for all $n \ll 0$.
(3) A complex $K^\bullet$ is said to be *bounded above* if $K^n = 0$ for all $n \gg 0$.
(4) A complex $K^\bullet$ is said to be *bounded* if $K^n = 0$ for all $|n| \gg 0$.
(5) We let $\mathrm{Comp}^+(\mathcal{A})$, $\mathrm{Comp}^-(\mathcal{A})$, resp. $\mathrm{Comp}^b(\mathcal{A})$ be the full subcategory of $\mathrm{Comp}(\mathcal{A})$ whose objects are the complexes which are bounded below, bounded above, resp. bounded.
(6) We let $K(\mathcal{A})$ be the category with the same objects as $\mathrm{Comp}(\mathcal{A})$ but as morphisms homotopy classes of maps of complexes (see Homology, Lemma 12.12.7).
(7) We let $K^+(\mathcal{A})$, $K^-(\mathcal{A})$, resp. $K^b(\mathcal{A})$ be the full subcategory of $K(\mathcal{A})$ whose objects are bounded below, bounded above, resp. bounded complexes of $\mathcal{A}$.

It will turn out that the categories $K(\mathcal{A})$, $K^+(\mathcal{A})$, $K^-(\mathcal{A})$, and $K^b(\mathcal{A})$ are triangulated categories. To prove this we first develop some machinery related to cones and split exact sequences.

13.8. Cones and termwise split sequences

Let $\mathcal{A}$ be an additive category, and let $K(\mathcal{A})$ denote the category of complexes of $\mathcal{A}$ with morphisms given by morphisms of complexes up to homotopy. Note that the shift functors $[n]$ on complexes, see Homology, Definition 12.14.7, give rise to functors $[n] : K(\mathcal{A}) \to K(\mathcal{A})$ such that $[n] \circ [m] = [n+m]$ and $[0] = \mathrm{id}$.

Definition 13.8.1. Let $\mathcal{A}$ be an additive category. Let $f : K^\bullet \to L^\bullet$ be a morphism of complexes of $\mathcal{A}$. The *cone* of f is the complex $C(f)^\bullet$ given by $C(f)^n = L^n \oplus K^{n+1}$ and differential

$$d^n_{C(f)} = \begin{pmatrix} d^n_L & f^{n+1} \\ 0 & -d^{n+1}_K \end{pmatrix}$$

It comes equipped with canonical morphisms of complexes $i : L^\bullet \to C(f)^\bullet$ and $p : C(f)^\bullet \to K^\bullet[1]$ induced by the obvious maps $L^n \to C(f)^n \to K^{n+1}$.

In other words $(K, L, C(f), f, i, p)$ forms a triangle:

$$K^\bullet \to L^\bullet \to C(f)^\bullet \to K^\bullet[1]$$

The formation of this triangle is functorial in the following sense.

Lemma 13.8.2. *Suppose that*

$$\begin{array}{ccc} K_1^\bullet & \xrightarrow[f_1]{} & L_1^\bullet \\ {\scriptstyle a}\downarrow & & \downarrow{\scriptstyle b} \\ K_2^\bullet & \xrightarrow{f_2} & L_2^\bullet \end{array}$$

is a diagram of morphisms of complexes which is commutative up to homotopy. Then there exists a morphism $c : C(f_1)^\bullet \to C(f_2)^\bullet$ which gives rise to a morphism of triangles $(a, b, c) : (K_1^\bullet, L_1^\bullet, C(f_1)^\bullet, f_1, i_1, p_1) \to (K_1^\bullet, L_1^\bullet, C(f_1)^\bullet, f_2, i_2, p_2)$ of $K(\mathcal{A})$.

Proof. Let $h^n : K_1^n \to L_2^{n-1}$ be a family of morphisms such that $b \circ f_1 - f_2 \circ a = d \circ h + h \circ d$. Define c^n by the matrix

$$c^n = \begin{pmatrix} b^n & h^{n+1} \\ 0 & a^{n+1} \end{pmatrix} : L_1^n \oplus K_1^{n+1} \to L_2^n \oplus K_2^{n+1}$$

A matrix computation show that c is a morphism of complexes. It is trivial that $c \circ i_1 = i_2 \circ b$, and it is trivial also to check that $p_2 \circ c = a \circ p_1$. □

Note that the morphism $c : C(f_1)^\bullet \to C(f_2)^\bullet$ constructed in the proof of Lemma 13.8.2 in general depends on the chosen homotopy h between $f_2 \circ a$ and $b \circ f_1$.

Lemma 13.8.3. *Suppose that $f : K^\bullet \to L^\bullet$ and $g : L^\bullet \to M^\bullet$ are morphisms of complexes such that $g \circ f$ is homotopic to zero. Then g factors through a morphism $C(f)^\bullet \to M^\bullet$ of $K(\mathcal{A})$.*

Proof. The assumptions say that the diagram

$$\begin{array}{ccc} K^\bullet & \xrightarrow{f} & L^\bullet \\ \downarrow & & \downarrow g \\ 0 & \longrightarrow & M^\bullet \end{array}$$

commutes up to homotopy. Since the cone on $0 \to M^\bullet$ is $M^\bullet$ the map $C(f)^\bullet \to C(0 \to M^\bullet) = M^\bullet$ of Lemma 13.8.2 is the desired map. □

Note that the morphism $C(f)^\bullet \to M^\bullet$ constructed in the proof of Lemma 13.8.3 in general depends on the chosen homotopy.

Definition 13.8.4. Let $\mathcal{A}$ be an additive category. A *termwise split injection* $\alpha : A^\bullet \to B^\bullet$ is a morphism of complexes such that each $A^n \to B^n$ is isomorphic to the inclusion of a direct summand. A *termwise split surjection* $\beta : B^\bullet \to C^\bullet$ is a morphism of complexes such that each $B^n \to C^n$ is isomorphic to the projection onto a direct summand.

Lemma 13.8.5. *Let $\mathcal{A}$ be an additive category. Let*

$$\begin{array}{ccc} A^\bullet & \xrightarrow{f} & B^\bullet \\ {\scriptstyle a}\downarrow & & \downarrow{\scriptstyle b} \\ C^\bullet & \xrightarrow{g} & D^\bullet \end{array}$$

be a diagram of morphisms of complexes commuting up to homotopy. If f is a split injection, then b is homotopic to a morphism which makes the diagram commute. If g is a split surjection, then a is homotopic to a morphism which makes the diagram commute.

Proof. Let $h^n : A^n \to D^{n-1}$ be a collection of morphisms such that $bf - ga = dh + hd$. Suppose that $\pi^n : B^n \to A^n$ are morphisms splitting the morphisms f^n. Take $b' = b - dh\pi - h\pi d$. Suppose $s^n : D^n \to C^n$ are morphisms splitting the morphisms $g^n : C^n \to D^n$. Take $a' = a + dsh + shd$. Computations omitted. □

The following lemma can be used to replace a morphism of complexes by a morphism where in each degree the map is the injection of a direct summand.

Lemma 13.8.6. *Let $\mathcal{A}$ be an additive category. Let $\alpha : K^\bullet \to L^\bullet$ be a morphism of complexes of $\mathcal{A}$. There exists a factorization*

$$K^\bullet \xrightarrow{\tilde{\alpha}} \tilde{L}^\bullet \xrightarrow{\pi} L^\bullet, \quad K^\bullet \xrightarrow{\alpha} L^\bullet$$

such that

(1) *$\tilde{\alpha}$ is a termwise split injection (see Definition 13.8.4),*
(2) *there is a map of complexes $s : L^\bullet \to \tilde{L}^\bullet$ such that $\pi \circ s = id_{L^\bullet}$ and such that $s \circ \pi$ is homotopic to $id_{\tilde{L}^\bullet}$.*

Moreover, if both $K^\bullet$ and $L^\bullet$ are in $K^+(\mathcal{A})$, $K^-(\mathcal{A})$, or $K^b(\mathcal{A})$, then so is $\tilde{L}^\bullet$.

Proof. We set

$$\tilde{L}^n = L^n \oplus K^n \oplus K^{n+1}$$

and we define

$$d^n_{\tilde{L}} = \begin{pmatrix} d^n_L & 0 & 0 \\ 0 & d^n_K & \mathrm{id}_{K^{n+1}} \\ 0 & 0 & -d^{n+1}_K \end{pmatrix}$$

In other words, $\tilde{L}^\bullet = L^\bullet \oplus C(1_{K^\bullet})$. Moreover, we set

$$\tilde{\alpha} = \begin{pmatrix} \alpha \\ \mathrm{id}_{K^n} \\ 0 \end{pmatrix}$$

which is clearly a split injection. It is also clear that it defines a morphism of complexes. We define

$$\pi = \begin{pmatrix} \mathrm{id}_{L^n} & 0 & 0 \end{pmatrix}$$

so that clearly $\pi \circ \tilde{\alpha} = \alpha$. We set

$$s = \begin{pmatrix} \mathrm{id}_{L^n} \\ 0 \\ 0 \end{pmatrix}$$

so that $\pi \circ s = \mathrm{id}_{L^\bullet}$. Finally, let $h^n : \tilde{L}^n \to \tilde{L}^{n-1}$ be the map which maps the summand K^n of $\tilde{L}^n$ via the identity morphism to the summand K^n of $\tilde{L}^{n-1}$. Then it is a trivial matter (see computations in remark below) to prove that

$$\mathrm{id}_{\tilde{L}^\bullet} - s \circ \pi = d \circ h + h \circ d$$

which finishes the proof of the lemma. □

Remark 13.8.7. To see the last displayed equality in the proof above we can argue with elements as follows. We have $s\pi(l, k, k^+) = (l, 0, 0)$. Hence the morphism of the left hand side maps (l, k, k^+) to $(0, k, k^+)$. On the other hand $h(l, k, k^+) = (0, 0, k)$ and $d(l, k, k^+) = (dl, dk + k^+, -dk^+)$. Hence $(dh + hd)(l, k, k^+) = d(0, 0, k) + h(dl, dk + k^+, -dk^+) = (0, k, -dk) + (0, 0, dk + k^+) = (0, k, k^+)$ as desired.

Lemma 13.8.8. *Let $\mathcal{A}$ be an additive category. Let $\alpha : K^\bullet \to L^\bullet$ be a morphism of complexes of $\mathcal{A}$. There exists a factorization*

$$K^\bullet \xrightarrow{i} \tilde{K}^\bullet \xrightarrow{\tilde{\alpha}} L^\bullet, \quad K^\bullet \xrightarrow{\alpha} L^\bullet$$

such that

(1) *$\tilde{\alpha}$ is a termwise split surjection (see Definition 13.8.4),*
(2) *there is a map of complexes $s : \tilde{K}^\bullet \to K^\bullet$ such that $s \circ i = id_{K^\bullet}$ and such that $i \circ s$ is homotopic to $id_{\tilde{K}^\bullet}$.*

Moreover, if both $K^\bullet$ and $L^\bullet$ are in $K^+(\mathcal{A})$, $K^-(\mathcal{A})$, or $K^b(\mathcal{A})$, then so is $\tilde{K}^\bullet$.

Proof. Dual to Lemma 13.8.6. Take

$$\tilde{K}^n = K^n \oplus L^{n-1} \oplus L^n$$

and we define

$$d^n_{\tilde{K}} = \begin{pmatrix} d^n_K & 0 & 0 \\ 0 & -d^{n-1}_L & \mathrm{id}_{L^n} \\ 0 & 0 & d^n_L \end{pmatrix}$$

in other words $\tilde{K}^\bullet = K^\bullet \oplus C(1_{L^\bullet[-1]})$. Moreover, we set

$$\tilde{\alpha} = \begin{pmatrix} \alpha & 0 & \mathrm{id}_{L^n} \end{pmatrix}$$

which is clearly a split surjection. It is also clear that it defines a morphism of complexes. We define

$$i = \begin{pmatrix} \mathrm{id}_{K^n} \\ 0 \\ 0 \end{pmatrix}$$

so that clearly $\tilde{\alpha} \circ i = \alpha$. We set

$$s = \begin{pmatrix} \mathrm{id}_{K^n} & 0 & 0 \end{pmatrix}$$

so that $s \circ i = \mathrm{id}_{K^\bullet}$. Finally, let $h^n : \tilde{K}^n \to \tilde{K}^{n-1}$ be the map which maps the summand L^{n-1} of $\tilde{K}^n$ via the identity morphism to the summand L^{n-1} of $\tilde{K}^{n-1}$. Then it is a trivial matter to prove that

$$\mathrm{id}_{\tilde{K}^\bullet} - i \circ s = d \circ h + h \circ d$$

which finishes the proof of the lemma. □

Definition 13.8.9. Let $\mathcal{A}$ be an additive category. A *termwise split sequence of complexes of $\mathcal{A}$* is a complex of complexes

$$0 \to A^\bullet \xrightarrow{\alpha} B^\bullet \xrightarrow{\beta} C^\bullet \to 0$$

together with given direct sum decompositions $B^n = A^n \oplus C^n$ compatible with α^n and β^n. We often write $s^n : C^n \to B^n$ and $\pi^n : B^n \to A^n$ for the maps induced by the direct sum decompositions. According to Homology, Lemma 12.14.10 we get an associated morphism of complexes

$$\delta : C^\bullet \longrightarrow A^\bullet[1]$$

which in degree n is the map $\pi^{n+1} \circ d^n_B \circ s^n$. In other words $(A^\bullet, B^\bullet, C^\bullet, \alpha, \beta, \delta)$ forms a triangle

$$A^\bullet \to B^\bullet \to C^\bullet \to A^\bullet[1]$$

This will be the *triangle associated to the termwise split sequence of complexes.*

Lemma 13.8.10. *Let $\mathcal{A}$ be an additive category. Let $0 \to A^\bullet \to B^\bullet \to C^\bullet \to 0$ be termwise split exact sequences as in Definition 13.8.9. Let $(\pi')^n$, $(s')^n$ be a second collection of splittings. Denote $\delta' : C^\bullet \longrightarrow A^\bullet[1]$ the morphism associated to this second set of splittings. Then*

$$(1, 1, 1) : (A^\bullet, B^\bullet, C^\bullet, \alpha, \beta, \delta) \longrightarrow (A^\bullet, B^\bullet, C^\bullet, \alpha, \beta, \delta')$$

is an isomorphism of triangles in $K(\mathcal{A})$.

Proof. The statement simply means that δ and δ' are homotopic maps of complexes. This is Homology, Lemma 12.14.12. □

Remark 13.8.11. Let $\mathcal{A}$ be an additive category. Let $0 \to A_i^\bullet \to B_i^\bullet \to C_i^\bullet \to 0$, $i = 1, 2$ be termwise split exact sequences. Suppose that $a : A_1^\bullet \to A_2^\bullet$, $b : B_1^\bullet \to B_2^\bullet$, and $c : C_1^\bullet \to C_2^\bullet$ are morphisms of complexes such that

$$\begin{array}{ccccc} A_1^\bullet & \longrightarrow & B_1^\bullet & \longrightarrow & C_1^\bullet \\ {\scriptstyle a}\downarrow & & {\scriptstyle b}\downarrow & & {\scriptstyle c}\downarrow \\ A_2^\bullet & \longrightarrow & B_2^\bullet & \longrightarrow & C_2^\bullet \end{array}$$

commutes in $K(\mathcal{A})$. In general, there does **not** exist a morphism $b' : B_1^\bullet \to B_2^\bullet$ which is homotopic to b such that the diagram above commutes in the category of complexes. Namely, consider Examples, Equation (78.48.0.1). If we could replace the middle map there by a homotopic one such that the diagram commutes, then we would have additivity of traces which we do not.

Lemma 13.8.12. *Let $\mathcal{A}$ be an additive category. Let $0 \to A_i^\bullet \to B_i^\bullet \to C_i^\bullet \to 0$, $i = 1, 2, 3$ be termwise split exact sequences of complexes. Let $b : B_1^\bullet \to B_2^\bullet$ and $b' : B_2^\bullet \to B_3^\bullet$ be morphisms of complexes such that*

$$\begin{array}{ccccc} A_1^\bullet & \longrightarrow & B_1^\bullet & \longrightarrow & C_1^\bullet \\ {\scriptstyle 0}\downarrow & & {\scriptstyle b}\downarrow & & {\scriptstyle 0}\downarrow \\ A_2^\bullet & \longrightarrow & B_2^\bullet & \longrightarrow & C_2^\bullet \end{array} \quad \text{and} \quad \begin{array}{ccccc} A_2^\bullet & \longrightarrow & B_2^\bullet & \longrightarrow & C_2^\bullet \\ {\scriptstyle 0}\downarrow & & {\scriptstyle b'}\downarrow & & {\scriptstyle 0}\downarrow \\ A_3^\bullet & \longrightarrow & B_3^\bullet & \longrightarrow & C_3^\bullet \end{array}$$

commute in $K(\mathcal{A})$. Then $b' \circ b = 0$ in $K(\mathcal{A})$.

Proof. By Lemma 13.8.5 we can replace b and b' by homotopic maps such that the right square of the left diagram commutes and the left square of the right diagram commutes. In other words, we have $\mathrm{Im}(b^n) \subset \mathrm{Im}(A_2^n \to B_2^n)$ and $\mathrm{Ker}((b')^n) \supset \mathrm{Im}(A_2^n \to B_2^n)$. Then $b \circ b' = 0$ as a map of complexes. □

Lemma 13.8.13. *Let $\mathcal{A}$ be an additive category. Let $f_1 : K_1^\bullet \to L_1^\bullet$ and $f_2 : K_2^\bullet \to L_2^\bullet$ be morphisms of complexes. Let*

$$(a, b, c) : (K_1^\bullet, L_1^\bullet, C(f_1)^\bullet, f_1, i_1, p_1) \longrightarrow (K_1^\bullet, L_1^\bullet, C(f_1)^\bullet, f_2, i_2, p_2)$$

be any morphism of triangles of $K(\mathcal{A})$. If a and b are homotopy equivalences then so is c.

Proof. Let $a^{-1} : K_2^\bullet \to K_1^\bullet$ be a morphism of complexes which is inverse to a in $K(\mathcal{A})$. Let $b^{-1} : L_2^\bullet \to L_1^\bullet$ be a morphism of complexes which is inverse to b in $K(\mathcal{A})$. Let $c' : C(f_2)^\bullet \to C(f_1)^\bullet$ be the morphism from Lemma 13.8.2 applied to $f_1 \circ a^{-1} = b^{-1} \circ f_2$. If we can show that $c \circ c'$ and $c' \circ c$ are isomorphisms in $K(\mathcal{A})$ then we win. Hence it suffices to prove the following: Given a morphism of triangles $(1, 1, c) : (K^\bullet, L^\bullet, C(f)^\bullet, f, i, p)$ in $K(\mathcal{A})$ the morphism c is an isomorphism

in $K(\mathcal{A})$. By assumption the two squares in the diagram

$$\begin{array}{ccccc} L^\bullet & \longrightarrow & C(f)^\bullet & \longrightarrow & K^\bullet[1] \\ \downarrow{\scriptstyle 1} & & \downarrow{\scriptstyle c} & & \downarrow{\scriptstyle 1} \\ L^\bullet & \longrightarrow & C(f)^\bullet & \longrightarrow & K^\bullet[1] \end{array}$$

commute up to homotopy. By construction of $C(f)^\bullet$ the rows form termwise split sequences of complexes. Thus we see that $(c-1)^2 = 0$ in $K(\mathcal{A})$ by Lemma 13.8.12. Hence c is an isomorphism in $K(\mathcal{A})$ with inverse $2 - c$. □

Hence if a and b are homotopy equivalences then the resulting morphism of triangles is an isomorphism of triangles in $K(\mathcal{A})$. It turns out that the collection of triangles of $K(\mathcal{A})$ given by cones and the collection of triangles of $K(\mathcal{A})$ given by termwise split sequences of complexes are the same up to isomorphisms, at least up to sign!

Lemma 13.8.14. *Let $\mathcal{A}$ be an additive category.*

(1) *Given a termwise split sequence of complexes $(\alpha : A^\bullet \to B^\bullet, \beta : B^\bullet \to C^\bullet, s^n, \pi^n)$ there exists a homotopy equivalence $C(\alpha)^\bullet \to C^\bullet$ such that the diagram*

$$\begin{array}{ccccccc} A^\bullet & \longrightarrow & B^\bullet & \longrightarrow & C(\alpha)^\bullet & \xrightarrow[-p]{} & A^\bullet[1] \\ \downarrow & & \downarrow & & \downarrow & & \downarrow \\ A^\bullet & \longrightarrow & B^\bullet & \longrightarrow & C^\bullet & \xrightarrow{\delta} & A^\bullet[1] \end{array}$$

defines an isomorphism of triangles in $K(\mathcal{A})$.

(2) *Given a morphism of complexes $f : K^\bullet \to L^\bullet$ there exists an isomorphism of triangles*

$$\begin{array}{ccccccc} K^\bullet & \longrightarrow & \tilde{L}^\bullet & \longrightarrow & M^\bullet & \xrightarrow[\delta]{} & K^\bullet[1] \\ \downarrow & & \downarrow & & \downarrow & & \downarrow \\ K^\bullet & \longrightarrow & L^\bullet & \longrightarrow & C(f)^\bullet & \xrightarrow{-p} & K^\bullet[1] \end{array}$$

where the upper triangle is the triangle associated to a termwise split exact sequence $K^\bullet \to \tilde{L}^\bullet \to M^\bullet$.

Proof. Proof of (1). We have $C(\alpha)^n = B^n \oplus A^{n+1}$ and we simply define $C(\alpha)^n \to C^n$ via the projection onto B^n followed by β^n. This defines a morphism of complexes because the compositions $A^{n+1} \to B^{n+1} \to C^{n+1}$ are zero. To get a homotopy inverse we take $C^\bullet \to C(\alpha)^\bullet$ given by $(s^n, -\delta^n)$ in degree n. This is a morphism of complexes because the morphism δ^n can be characterized as the unique morphism $C^n \to A^{n+1}$ such that $d \circ s^n - s^{n+1} \circ d = \alpha \circ \delta^n$, see proof of Homology, Lemma 12.14.10. The composition $C^\bullet \to C(f)^\bullet \to C^\bullet$ is the identity. The composition $C(f)^\bullet \to C^\bullet \to C(f)^\bullet$ is equal to the morphism

$$\begin{pmatrix} s^n \circ \beta^n & 0 \\ -\delta^n \circ \beta^n & 0 \end{pmatrix}$$

To see that this is homotopic to the identity map use the homotopy $h^n : C(\alpha)^n \to C(\alpha)^{n-1}$ given by the matrix

$$\begin{pmatrix} 0 & 0 \\ \pi^n & 0 \end{pmatrix} : C(\alpha)^n = B^n \oplus A^{n+1} \to B^{n-1} \oplus A^n = C(\alpha)^{n-1}$$

It is trivial to verify that

$$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} - \begin{pmatrix} s^n \\ -\delta^n \end{pmatrix} \begin{pmatrix} \beta^n & 0 \end{pmatrix} = \begin{pmatrix} d & \alpha^n \\ 0 & -d \end{pmatrix} \begin{pmatrix} 0 & 0 \\ \pi^n & 0 \end{pmatrix} + \begin{pmatrix} 0 & 0 \\ \pi^{n+1} & 0 \end{pmatrix} \begin{pmatrix} d & \alpha^{n+1} \\ 0 & -d \end{pmatrix}$$

To finish the proof of (1) we have to show that the morphisms $-p : C(\alpha)^\bullet \to A^\bullet[1]$ (see Definition 13.8.1) and $C(\alpha)^\bullet \to C^\bullet \to A^\bullet[1]$ agree up to homotopy. This is clear from the above. Namely, we can use the homotopy inverse $(s, -\delta) : C^\bullet \to C(\alpha)^\bullet$ and check instead that the two maps $C^\bullet \to A^\bullet[1]$ agree. And note that $p \circ (s, -\delta) = -\delta$ as desired.

Proof of (2). We let $\tilde{f} : K^\bullet \to \tilde{L}^\bullet$, $s : L^\bullet \to \tilde{L}^\bullet$ and $\pi : L^\bullet \to L^\bullet$ be as in Lemma 13.8.6. By Lemmas 13.8.2 and 13.8.13 the triangles $(K^\bullet, L^\bullet, C(f), i, p)$ and $(K^\bullet, \tilde{L}^\bullet, C(\tilde{f}), \tilde{i}, \tilde{p})$ are isomorphic. Note that we can compose isomorphisms of triangles. Thus we may replace $L^\bullet$ by $\tilde{L}^\bullet$ and f by $\tilde{f}$. In other words we may assume that f is a termwise split injection. In this case the result follows from part (1). □

Lemma 13.8.15. *Let $\mathcal{A}$ be an additive category. Let $A_1^\bullet \to A_2^\bullet \to \ldots \to A_n^\bullet$ be a sequence of composable morphisms of complexes. There exists a commutative diagram*

$$\begin{array}{ccccccc} A_1^\bullet & \longrightarrow & A_2^\bullet & \longrightarrow & \ldots & \longrightarrow & A_n^\bullet \\ \uparrow & & \uparrow & & & & \uparrow \\ B_1^\bullet & \longrightarrow & B_2^\bullet & \longrightarrow & \ldots & \longrightarrow & B_n^\bullet \end{array}$$

such that each morphism $B_i^\bullet \to B_{i+1}^\bullet$ is a split injection and each $B_i^\bullet \to A_i^\bullet$ is a homotopy equivalence. Moreover, if all $A_i^\bullet$ are in $K^+(\mathcal{A})$, $K^-(\mathcal{A})$, or $K^b(\mathcal{A})$, then so are the $B_i^\bullet$.

Proof. The case $n = 1$ is without content. Lemma 13.8.6 is the case $n = 2$. Suppose we have constructed the diagram except for $B_n^\bullet$. Apply Lemma 13.8.6 to the composition $B_{n-1}^\bullet \to A_{n-1}^\bullet \to A_n^\bullet$. The result is a factorization $B_{n-1}^\bullet \to B_n^\bullet \to A_n^\bullet$ as desired. □

Lemma 13.8.16. *Let $\mathcal{A}$ be an additive category. Let $(\alpha : A^\bullet \to B^\bullet, \beta : B^\bullet \to C^\bullet, s^n, \pi^n)$ be a termwise split sequence of complexes. Let $(A^\bullet, B^\bullet, C^\bullet, \alpha, \beta, \delta)$ be the associated triangle. Then the triangle $(C^\bullet[-1], A^\bullet, B^\bullet, \delta[-1], \alpha, \beta)$ is isomorphic to the triangle $(C^\bullet[-1], A^\bullet, C(\delta[-1])^\bullet, \delta[-1], i, p)$.*

Proof. We write $B^n = A^n \oplus C^n$ and we identify α^n and β^n with the natural inclusion and projection maps. By construction of δ we have

$$d_B^n = \begin{pmatrix} d_A^n & \delta^n \\ 0 & d_C^n \end{pmatrix}$$

On the other hand the cone of $\delta[-1] : C^\bullet[-1] \to A^\bullet$ is given as $C(\delta[-1])^n = A^n \oplus C^n$ with differential identical with the matrix above! Whence the lemma. □

Lemma 13.8.17. *Let $\mathcal{A}$ be an additive category. Let $f : K^\bullet \to L^\bullet$ be a morphism of complexes. The triangle $(L^\bullet, C(f)^\bullet, K^\bullet[1], i, p, f[1])$ is the triangle associated to the termwise split sequence*

$$0 \to L^\bullet \to C(f)^\bullet \to K^\bullet[1] \to 0$$

coming from the definition of the cone of f.

Proof. Immediate from the definitions. □

13.9. Distinguished triangles in the homotopy category

Since we want our boundary maps in long exact sequences of cohomology to be given by the maps in the snake lemma without signs we define distinguished triangles in the homotopy category as follows.

Definition 13.9.1. Let $\mathcal{A}$ be an additive category. A triangle (X, Y, Z, f, g, h) of $K(\mathcal{A})$ is called a *distinguished triangle of $K(\mathcal{A})$* if it is isomorphic to the triangle associated to a termwise split exact sequence of complexes, see Definition 13.8.9. Same definition for $K^+(\mathcal{A})$, $K^-(\mathcal{A})$, and $K^b(\mathcal{A})$.

Note that according to Lemma 13.8.14 a triangle of the form $(K^\bullet, L^\bullet, C(f)^\bullet, f, i, -p)$ is a distinguished triangle. This does indeed lead to a triangulated category, see Proposition 13.9.3. Before we can prove the proposition we need one more lemma in order to be able to prove TR4.

Lemma 13.9.2. *Let $\mathcal{A}$ be an additive category. Suppose that $\alpha : A^\bullet \to B^\bullet$ and $\beta : B^\bullet \to C^\bullet$ are split injections of complexes. Then there exist distinguished triangles $(A^\bullet, B^\bullet, Q_1^\bullet, \alpha, p_1, d_1)$, $(A^\bullet, C^\bullet, Q_2^\bullet, \beta \circ \alpha, p_2, d_2)$ and $(B^\bullet, C^\bullet, Q_3^\bullet, \beta, p_3, d_3)$ for which TR4 holds.*

Proof. Say $\pi_1^n : B^n \to A^n$, and $\pi_3^n : C^n \to B^n$ are the splittings. Then also $A^\bullet \to C^\bullet$ is a split injection with splittings $\pi_2^n = \pi_1^n \circ \pi_3^n$. Let us write $Q_1^\bullet$, $Q_2^\bullet$ and $Q_3^\bullet$ for the "quotient" complexes. In other words, $Q_1^n = \mathrm{Ker}(\pi_1^n)$, $Q_3^n = \mathrm{Ker}(\pi_3^n)$ and $Q_2^n = \mathrm{Ker}(\pi_2^n)$. Note that the kernels exist. Then $B^n = A^n \oplus Q_1^n$ and $C_n = B^n \oplus Q_3^n$, where we think of A^n as a subobject of B^n and so on. This implies $C^n = A^n \oplus Q_1^n \oplus Q_3^n$. Note that $\pi_2^n = \pi_1^n \circ \pi_3^n$ is zero on both Q_1^n and Q_3^n. Hence $Q_2^n = Q_1^n \oplus Q_3^n$. Consider the commutative diagram

$$\begin{array}{ccccccccc}
0 & \to & A^\bullet & \to & B^\bullet & \to & Q_1^\bullet & \to & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \to & A^\bullet & \to & C^\bullet & \to & Q_2^\bullet & \to & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \to & B^\bullet & \to & C^\bullet & \to & Q_3^\bullet & \to & 0
\end{array}$$

The rows of this diagram are termwise split exact sequences, and hence determine distinguished triangles by definition. Moreover downward arrows in the diagram above are compatible with the chosen splittings and hence define morphisms of triangles

$$(A^\bullet \to B^\bullet \to Q_1^\bullet \to A^\bullet[1]) \longrightarrow (A^\bullet \to C^\bullet \to Q_2^\bullet \to A^\bullet[1])$$

and

$$(A^\bullet \to C^\bullet \to Q_2^\bullet \to A^\bullet[1]) \longrightarrow (B^\bullet \to C^\bullet \to Q_3^\bullet \to B^\bullet[1]).$$

Note that the splittings $Q_3^n \to C^n$ of the bottom split sequence in the diagram provides a splitting for the split sequence $0 \to Q_1^\bullet \to Q_2^\bullet \to Q_3^\bullet \to 0$ upon composing

with $C^n \to Q_2^n$. It follows easily from this that the morphism $\delta : Q_3^\bullet \to Q_1^\bullet[1]$ in the corresponding distinguished triangle

$$(Q_1^\bullet \to Q_2^\bullet \to Q_3^\bullet \to Q_1^\bullet[1])$$

is equal to the composition $Q_3^\bullet \to B^\bullet[1] \to Q_1^\bullet[1]$. Hence we get a structure as in the conclusion of axiom TR4. □

Proposition 13.9.3. *Let $\mathcal{A}$ be an additive category. The category $K(\mathcal{A})$ of complexes up to homotopy with its natural translation functors and distinguished triangles as defined above is a triangulated category.*

Proof. Proof of TR1. By definition every triangle isomorphic to a distinguished one is distinguished. Also, any triangle $(A^\bullet, A^\bullet, 0, 1, 0, 0)$ is distinguished since $0 \to A^\bullet \to A^\bullet \to 0 \to 0$ is a termwise split sequence of complexes. Finally, given any morphism of complexes $f : K^\bullet \to L^\bullet$ the triangle $(K, L, C(f), f, i, -p)$ is distinguished by Lemma 13.8.14.

Proof of TR2. Let (X, Y, Z, f, g, h) be a triangle. Assume $(Y, Z, X[1], g, h, -f[1])$ is distinguished. Then there exists a termwise split sequence of complexes $A^\bullet \to B^\bullet \to C^\bullet$ such that the associated triangle $(A^\bullet, B^\bullet, C^\bullet, \alpha, \beta, \delta)$ is isomorphic to $(Y, Z, X[1], g, h, -f[1])$. Rotating back we see that (X, Y, Z, f, g, h) is isomorphic to $(C^\bullet[-1], A^\bullet, B^\bullet, -\delta[-1], \alpha, \beta)$. It follows from Lemma 13.8.16 that the triangle $(C^\bullet[-1], A^\bullet, B^\bullet, \delta[-1], \alpha, \beta)$ is isomorphic to $(C^\bullet[-1], A^\bullet, C(\delta[-1])^\bullet, \delta[-1], i, p)$. Precomposing the previous isomorphism of triangles with -1 on Y it follows that (X, Y, Z, f, g, h) is isomorphic to $(C^\bullet[-1], A^\bullet, C(\delta[-1])^\bullet, \delta[-1], i, -p)$. Hence it is distinguished by Lemma 13.8.14. On the other hand, suppose that (X, Y, Z, f, g, h) is distinguished. By Lemma 13.8.14 this means that it is isomorphic to a triangle of the form $(K^\bullet, L^\bullet, C(f), f, i, -p)$ for some morphism of complexes f. Then the rotated triangle $(Y, Z, X[1], g, h, -f[1])$ is isomorphic to $(L^\bullet, C(f), K^\bullet[1], i, -p, -f[1])$ which is isomorphic to the triangle $(L^\bullet, C(f), K^\bullet[1], i, p, f[1])$. By Lemma 13.8.17 this triangle is distinguished. Hence $(Y, Z, X[1], g, h, -f[1])$ is distinguished as desired.

Proof of TR3. Let (X, Y, Z, f, g, h) and (X', Y', Z', f', g', h') be distinguished triangles of $K(\mathcal{A})$ and let $a : X \to X'$ and $b : Y \to Y'$ be morphisms such that $f' \circ a = b \circ f$. By Lemma 13.8.14 we may assume that $(X, Y, Z, f, g, h) = (X, Y, C(f), f, i, -p)$ and $(X', Y', Z', f', g', h') = (X', Y', C(f'), f', i', -p')$. At this point we simply apply Lemma 13.8.2 to the commutative diagram given by f, f', a, b.

Proof of TR4. At this point we know that $K(\mathcal{A})$ is a pre-triangulated category. Hence we can use Lemma 13.4.13. Let $A^\bullet \to B^\bullet$ and $B^\bullet \to C^\bullet$ be composable morphisms of $K(\mathcal{A})$. By Lemma 13.8.15 we may assume that $A^\bullet \to B^\bullet$ and $B^\bullet \to C^\bullet$ are split injective morphisms. In this case the result follows from Lemma 13.9.2. □

Remark 13.9.4. Let $\mathcal{A}$ be an additive category. Exactly the same proof as the proof of Proposition 13.9.3 shows that the categories $K^+(\mathcal{A})$, $K^-(\mathcal{A})$, and $K^b(\mathcal{A})$ are triangulated categories. Namely, the cone of a morphisms between bounded (above, below) is bounded (above, below). But we prove below that these are triangulated subcategories of $K(\mathcal{A})$ which gives another proof.

Lemma 13.9.5. *Let $\mathcal{A}$ be an additive subcategory. The categories $K^+(\mathcal{A})$, $K^-(\mathcal{A})$, and $K^b(\mathcal{A})$ are full triangulated subcategories of $K(\mathcal{A})$.*

Proof. Each of the categories mentioned is a full additive subcategory. We use the criterion of Lemma 13.4.14 to show that they are triangulated subcategories. It is clear that each of the categories $K^+(\mathcal{A})$, $K^-(\mathcal{A})$, and $K^b(\mathcal{A})$ is preserved under the shift functors $[1], [-1]$. Finally, suppose that $f : A^\bullet \to B^\bullet$ is a morphism in $K^+(\mathcal{A})$, $K^-(\mathcal{A})$, or $K^b(\mathcal{A})$. Then $(A^\bullet, B^\bullet, C(f)^\bullet, f, i, -p)$ is a distinguished triangle of $K(\mathcal{A})$ with $C(f)^\bullet \in K^+(\mathcal{A})$, $K^-(\mathcal{A})$, or $K^b(\mathcal{A})$ as is clear from the construction of the cone. Thus the lemma is proved. (Alternatively, $K^\bullet \to L^\bullet$ is isomorphic to an termwise split injection of complexes in $K^+(\mathcal{A})$, $K^-(\mathcal{A})$, or $K^b(\mathcal{A})$, see Lemma 13.8.6 and then one can directly take the associated distinguished triangle.) □

Lemma 13.9.6. *Let $\mathcal{A}$, $\mathcal{B}$ be additive categories. Let $F : \mathcal{A} \to \mathcal{B}$ be an additive functor. The induced functors*

$$\begin{aligned} F : K(\mathcal{A}) &\longrightarrow K(\mathcal{B}) \\ F : K^+(\mathcal{A}) &\longrightarrow K^+(\mathcal{B}) \\ F : K^-(\mathcal{A}) &\longrightarrow K^-(\mathcal{B}) \\ F : K^b(\mathcal{A}) &\longrightarrow K^b(\mathcal{B}) \end{aligned}$$

are exact functors of triangulated categories.

Proof. Suppose $A^\bullet \to B^\bullet \to C^\bullet$ is a termwise split sequence of complexes of $\mathcal{A}$ with splittings (s^n, π^n) and associated morphism $\delta : C^\bullet \to A^\bullet[1]$, see Definition 13.8.9. Then $F(A^\bullet) \to F(B^\bullet) \to F(C^\bullet)$ is a termwise split sequence of complexes with splittings $(F(s^n), F(\pi^n))$ and associated morphism $F(\delta) : F(C^\bullet) \to F(A^\bullet)[1]$. Thus F transforms distinguished triangles into distinguished triangles. □

13.10. Derived categories

In this section we construct the derived category of an abelian category $\mathcal{A}$ by inverting the quasi-isomorphisms in $K(\mathcal{A})$. Before we do this recall that the functors $H^i : \mathrm{Comp}(\mathcal{A}) \to \mathcal{A}$ factor through $K(\mathcal{A})$, see Homology, Lemma 12.12.11. Moreover, in Homology, Definition 12.14.8 we have defined identifications $H^i(K^\bullet[n]) = H^{i+n}(K^\bullet)$. At this point it makes sense to redefine

$$H^i(K^\bullet) = H^0(K^\bullet[i])$$

in order to avoid confusion and possible sign errors.

Lemma 13.10.1. *Let $\mathcal{A}$ be an abelian category. The functor*

$$H^0 : K(\mathcal{A}) \longrightarrow \mathcal{A}$$

is homological.

Proof. Because H^0 is a functor, and by our definition of distinguished triangles it suffices to prove that given a termwise split short exact sequence of complexes $0 \to A^\bullet \to B^\bullet \to C^\bullet \to 0$ the sequence $H^0(A^\bullet) \to H^0(B^\bullet) \to H^0(C^\bullet)$ is exact. This follows from Homology, Lemma 12.12.12. □

In particular, this lemma implies that a distinguished triangle (X, Y, Z, f, g, h) in $K(\mathcal{A})$ gives rise to a long exact cohomology sequence
(13.10.1.1)

$$\ldots \longrightarrow H^i(X) \xrightarrow{H^i(f)} H^i(Y) \xrightarrow{H^i(g)} H^i(Z) \xrightarrow{H^i(h)} H^{i+1}(X) \longrightarrow \ldots$$

see (13.3.5.1). Moreover, there is a compatibility with the long exact sequence of cohomology associated to a short exact sequence of complexes (insert future reference

here). For example, if $(A^\bullet, B^\bullet, C^\bullet, \alpha, \beta, \delta)$ is the distinguished triangle associated to a termwise split exact sequence of complexes (see Definition 13.8.9), then the cohomology sequence above agrees with the one defined using the snake lemma, see Homology, Lemma 12.12.12 and for agreement of sequences, see Homology, Lemma 12.14.11.

Recall that a complex $K^\bullet$ is *acyclic* if $H^i(K^\bullet) = 0$ for all $i \in \mathbf{Z}$. Moreover, recall that a morphism of complexes $f : K^\bullet \to L^\bullet$ is a *quasi-isomorphism* if and only if $H^i(f)$ is an isomorphism for all i. See Homology, Definition 12.12.10.

Lemma 13.10.2. *Let $\mathcal{A}$ be an abelian category. The full subcategory $Ac(\mathcal{A})$ of $K(\mathcal{A})$ consisting of acyclic complexes is a strictly full saturated triangulated subcategory of $K(\mathcal{A})$. The corresponding saturated multiplicative system (see Lemma 13.6.10) of $K(\mathcal{A})$ is the set $Qis(\mathcal{A})$ of quasi-isomorphisms. In particular, the kernel of the localization functor $Q : K(\mathcal{A}) \to Qis(\mathcal{A})^{-1}K(\mathcal{A})$ is $Ac(\mathcal{A})$ and the functor H^0 factors through Q.*

Proof. We know that H^0 is a homological functor by Lemma 13.10.1. Thus this lemma is a special case of Lemma 13.6.11. □

Definition 13.10.3. Let $\mathcal{A}$ be an abelian category. Let $\mathrm{Ac}(\mathcal{A})$ and $\mathrm{Qis}(\mathcal{A})$ be as in Lemma 13.10.2. The *derived category of* $\mathcal{A}$ is the triangulated category

$$D(\mathcal{A}) = K(\mathcal{A})/\mathrm{Ac}(\mathcal{A}) = \mathrm{Qis}(\mathcal{A})^{-1}K(\mathcal{A}).$$

We denote $H^0 : D(\mathcal{A}) \to \mathcal{A}$ the unique functor whose composition with the quotient functor gives back the functor H^0 defined above. Using Lemma 13.6.4 we introduce the strictly full saturated triangulated subcategories $D^+(\mathcal{A}), D^-(\mathcal{A}), D^b(\mathcal{A})$ whose sets of objects are

$$\begin{aligned}
\mathrm{Ob}(D^+(\mathcal{A})) &= \{X \in \mathrm{Ob}(D(\mathcal{A})) \mid H^n(X) = 0 \text{ for all } n \ll 0\} \\
\mathrm{Ob}(D^-(\mathcal{A})) &= \{X \in \mathrm{Ob}(D(\mathcal{A})) \mid H^n(X) = 0 \text{ for all } n \gg 0\} \\
\mathrm{Ob}(D^b(\mathcal{A})) &= \{X \in \mathrm{Ob}(D(\mathcal{A})) \mid H^n(X) = 0 \text{ for all } |n| \gg 0\}
\end{aligned}$$

The category $D^b(\mathcal{A})$ is called the *bounded derived category* of $\mathcal{A}$.

Remark 13.10.4. In this chapter, we consistently work with "small" abelian categories (as is the convention in the Stacks project). For a "big" abelian category $\mathcal{A}$, it isn't clear that the derived category $D(\mathcal{A})$ exists, because it isn't clear that morphisms in the derived category are sets. In fact, in general they aren't, see Examples, Lemma 78.46.1. However, if $\mathcal{A}$ is a Grothendieck abelian category, and given $K^\bullet, L^\bullet$ in $K(\mathcal{A})$, then by Injectives, Theorem 19.12.6 there exists a quasi-isomorphism $L^\bullet \to I^\bullet$ to a K-injective complex $I^\bullet$ and Lemma 13.28.2 shows that

$$\mathrm{Hom}_{D(\mathcal{A})}(K^\bullet, L^\bullet) = \mathrm{Hom}_{K(\mathcal{A})}(K^\bullet, I^\bullet)$$

which is a set. Some examples of Grothendieck abelian categories are the category of modules over a ring, or more generally the category of sheaves of modules on a ringed site.

Each of the variants $D^+(\mathcal{A}), D^-(\mathcal{A}), D^b(\mathcal{A})$ can be constructed as a localization of the corresponding homotopy category. This relies on the following simple lemma.

Lemma 13.10.5. *Let $\mathcal{A}$ be an abelian category. Let $K^\bullet$ be a complex.*

(1) *If $H^n(K^\bullet) = 0$ for all $n \ll 0$, then there exists a quasi-isomorphism $K^\bullet \to L^\bullet$ with $L^\bullet$ bounded below.*

(2) *If $H^n(K^\bullet) = 0$ for all $n \gg 0$, then there exists a quasi-isomorphism $M^\bullet \to K^\bullet$ with $M^\bullet$ bounded above.*
(3) *If $H^n(K^\bullet) = 0$ for all $|n| \gg 0$, then there exists a commutative diagram of morphisms of complexes*

$$\begin{array}{ccc} K^\bullet & \longrightarrow & L^\bullet \\ \uparrow & & \uparrow \\ M^\bullet & \longrightarrow & N^\bullet \end{array}$$

where all the arrows are quasi-isomorphisms, $L^\bullet$ bounded below, $M^\bullet$ bounded above, and $N^\bullet$ a bounded complex.

Proof. Pick $a \ll 0 \ll b$ and set $M^\bullet = \tau_{\leq a}K^\bullet$, $L^\bullet = K^\bullet/\tau_{\leq b}K^\bullet$, and $N^\bullet = L^\bullet/M^\bullet$. See Homology, Section 12.13 for the truncation functors. □

To state the following lemma denote $\mathrm{Ac}^+(\mathcal{A})$, $\mathrm{Ac}^-(\mathcal{A})$, resp. $\mathrm{Ac}^b(\mathcal{A})$ the intersection of $K^+(\mathcal{A})$, $K^-(\mathcal{A})$, resp. $K^b(\mathcal{A})$ with $\mathrm{Ac}(\mathcal{A})$. Denote $\mathrm{Qis}^+(\mathcal{A})$, $\mathrm{Qis}^-(\mathcal{A})$, resp. $\mathrm{Qis}^b(\mathcal{A})$ the intersection of $K^+(\mathcal{A})$, $K^-(\mathcal{A})$, resp. $K^b(\mathcal{A})$ with $\mathrm{Qis}(\mathcal{A})$.

Lemma 13.10.6. *Let $\mathcal{A}$ be an abelian category. The subcategories $Ac^+(\mathcal{A})$, $Ac^-(\mathcal{A})$, resp. $Ac^b(\mathcal{A})$ are strictly full saturated triangulated subcategories of $K^+(\mathcal{A})$, $K^-(\mathcal{A})$, resp. $K^b(\mathcal{A})$. The corresponding saturated multiplicative systems (see Lemma 13.6.10) are the sets $Qis^+(\mathcal{A})$, $Qis^-(\mathcal{A})$, resp. $Qis^b(\mathcal{A})$.*

(1) *The kernel of the functor $K^+(\mathcal{A}) \to D^+(\mathcal{A})$ is $Ac^+(\mathcal{A})$ and this induces an equivalence of triangulated categories*
$$K^+(\mathcal{A})/Ac^+(\mathcal{A}) = Qis^+(\mathcal{A})^{-1}K^+(\mathcal{A}) \longrightarrow D^+(\mathcal{A})$$
(2) *The kernel of the functor $K^-(\mathcal{A}) \to D^-(\mathcal{A})$ is $Ac^-(\mathcal{A})$ and this induces an equivalence of triangulated categories*
$$K^-(\mathcal{A})/Ac^-(\mathcal{A}) = Qis^-(\mathcal{A})^{-1}K^-(\mathcal{A}) \longrightarrow D^-(\mathcal{A})$$
(3) *The kernel of the functor $K^b(\mathcal{A}) \to D^b(\mathcal{A})$ is $Ac^b(\mathcal{A})$ and this induces an equivalence of triangulated categories*
$$K^b(\mathcal{A})/Ac^b(\mathcal{A}) = Qis^b(\mathcal{A})^{-1}K^b(\mathcal{A}) \longrightarrow D^b(\mathcal{A})$$

Proof. The initial statements follow from Lemma 13.6.11 by considering the restriction of the homological functor H^0. The statement on kernels in (1), (2), (3) is a consequence of the definitions in each case. Each of the functors is essentially surjective by Lemma 13.10.5. To finish the proof we have to show the functors are fully faithful. We first do this for the bounded below version.

Suppose that $K^\bullet, L^\bullet$ are bounded above complexes. A morphism between these in $D(\mathcal{A})$ is of the form $s^{-1}f$ for a pair $f : K^\bullet \to (L')^\bullet$, $s : L^\bullet \to (L')^\bullet$ where s is a quasi-isomorphism. This implies that $(L')^\bullet$ has cohomology bounded below. Hence by Lemma 13.10.5 we can choose a quasi-isomorphism $s' : (L')^\bullet \to (L'')^\bullet$ with $(L'')^\bullet$ bounded below. Then the pair $(s' \circ f, s' \circ s)$ defines a morphism in $\mathrm{Qis}^+(\mathcal{A})^{-1}K^+(\mathcal{A})$. Hence the functor is "full". Finally, suppose that the pair $f : K^\bullet \to (L')^\bullet$, $s : L^\bullet \to (L')^\bullet$ defines a morphism in $\mathrm{Qis}^+(\mathcal{A})^{-1}K^+(\mathcal{A})$ which is zero in $D(\mathcal{A})$. This means that there exists a quasi-isomorphism $s' : (L')^\bullet \to (L'')^\bullet$ such that $s' \circ f = 0$. Using Lemma 13.10.5 once more we obtain a quasi-isomorphism $s'' : (L'')^\bullet \to (L''')^\bullet$ with $(L''')^\bullet$ bounded below. Thus we see that $s'' \circ s' \circ f = 0$

which implies that $s^{-1}f$ is zero in $\mathrm{Qis}^+(\mathcal{A})^{-1}K^+(\mathcal{A})$. This finishes the proof that the functor in (1) is an equivalence.

The proof of (2) is dual to the proof of (1). To prove (3) we may use the result of (2). Hence it suffices to prove that the functor $\mathrm{Qis}^b(\mathcal{A})^{-1}K^b(\mathcal{A}) \to \mathrm{Qis}^-(\mathcal{A})^{-1}K^-(\mathcal{A})$ is fully faithful. The argument given in the previous paragraph applies directly to show this where we consistently work with complexes which are already bounded above. □

13.11. The canonical delta-functor

The derived category should be the receptacle for the universal cohomology functor. In order to state the result we use the notion of a δ-functor from an abelian category into a triangulated category, see Definition 13.3.6.

Consider the functor $\mathrm{Comp}(\mathcal{A}) \to K(\mathcal{A})$. This functor is **not** a δ-functor in general. The easiest way to see this is to consider a nonsplit short exact sequence $0 \to A \to B \to C \to 0$ of objects of $\mathcal{A}$. Since $\mathrm{Hom}_{K(\mathcal{A})}(C[0], A[1]) = 0$ we see that any distinguished triangle arising from this short exact sequence would look like $(A[0], B[0], C[0], a, b, 0)$. But the existence of such a distinguished triangle in $K(\mathcal{A})$ implies that the extension is split. A contradiction.

It turns out that the functor $\mathrm{Comp}(\mathcal{A}) \to D(\mathcal{A})$ is a δ-functor. In order to see this we have to define the morphisms δ associated to a short exact sequence

$$0 \to A^\bullet \xrightarrow{a} B^\bullet \xrightarrow{b} C^\bullet \to 0$$

of complexes in the abelian category $\mathcal{A}$. Consider the cone $C(a)^\bullet$ of the morphism a. We have $C(a)^n = B^n \oplus A^{n+1}$ and we define $q^n : C(a)^n \to C^n$ via the projection to B^n followed by b^n. Hence a morphism of complexes

$$q : C(a)^\bullet \longrightarrow C^\bullet.$$

It is clear that $q \circ i = b$ where i is as in Definition 13.8.1. Note that, as $a^\bullet$ is injective in each degree, the kernel of q is identified with the cone of $\mathrm{id}_{A^\bullet}$ which is acyclic. Hence we see that q is a quasi-isomorphism. According to Lemma 13.8.14 the triangle

$$(A, B, C(a), a, i, -p)$$

is a distinguished triangle in $K(\mathcal{A})$. As the localization functor $K(\mathcal{A}) \to D(\mathcal{A})$ is exact we see that $(A, B, C(a), a, i, -p)$ is a distinguished triangle in $D(\mathcal{A})$. Since q is a quasi-isomorphism we see that q is an isomorphism in $D(\mathcal{A})$. Hence we deduce that

$$(A, B, C, a, b, -p \circ q^{-1})$$

is a distinguished triangle of $D(\mathcal{A})$. This suggests the following lemma.

Lemma 13.11.1. *Let $\mathcal{A}$ be an abelian category. The functor $Comp(\mathcal{A}) \to D(\mathcal{A})$ defined has the natural structure of a δ-functor, with*

$$\delta_{A^\bullet \to B^\bullet \to C^\bullet} = -p \circ q^{-1}$$

with p and q as explained above. The same construction turns the functors $Comp^+(\mathcal{A}) \to D^+(\mathcal{A})$, $Comp^-(\mathcal{A}) \to D^-(\mathcal{A})$, and $Comp^b(\mathcal{A}) \to D^b(\mathcal{A})$ into δ-functors.

Proof. We have already seen that this choice leads to a distinguished triangle whenever given a short exact sequence of complexes. We have to show that given a commutative diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & A^\bullet & \xrightarrow{a} & B^\bullet & \xrightarrow{b} & C^\bullet & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle f} & & \downarrow{\scriptstyle g} & & \downarrow{\scriptstyle h} & & \\ 0 & \longrightarrow & (A')^\bullet & \xrightarrow{a'} & (B')^\bullet & \xrightarrow{b'} & (C')^\bullet & \longrightarrow & 0 \end{array}$$

we get the desired commutative diagram of Definition 13.3.6 (2). By Lemma 13.8.2 the pair (f, g) induces a canonical morphism $c : C(a)^\bullet \to C(a')^\bullet$. It is a simple computation to show that $q' \circ c = h \circ q$ and $f[1] \circ p = p' \circ c$. From this the result follows directly. □

Lemma 13.11.2. *Let $\mathcal{A}$ be an abelian category. Let*

$$\begin{array}{ccccccccc} 0 & \longrightarrow & A^\bullet & \longrightarrow & B^\bullet & \longrightarrow & C^\bullet & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & D^\bullet & \longrightarrow & E^\bullet & \longrightarrow & F^\bullet & \longrightarrow & 0 \end{array}$$

be a commutative diagram of morphisms of complexes such that the rows are short exact sequences of complexes, and the vertical arrows are quasi-isomorphisms. The δ-functor of Lemma 13.11.1 above maps the to short exact sequences $0 \to A^\bullet \to B^\bullet \to C^\bullet \to 0$ and $0 \to D^\bullet \to E^\bullet \to F^\bullet \to 0$ to isomorphic distinguished triangles.

Proof. Trivial from the fact that $K(\mathcal{A}) \to D(\mathcal{A})$ transforms quasi-isomorphisms into isomorphisms and that the associated distinguished triangles are functorial. □

Lemma 13.11.3. *Let $\mathcal{A}$ be an abelian category. Let*

$$0 \longrightarrow A^\bullet \longrightarrow B^\bullet \longrightarrow C^\bullet \longrightarrow 0$$

be a short exact sequences of complexes. Assume this short exact sequence is termwise split. Let $(A^\bullet, B^\bullet, C^\bullet, \alpha, \beta, \delta)$ be the distinguished triangle of $K(\mathcal{A})$ associated to the sequence. The δ-functor of Lemma 13.11.1 above maps the short exact sequences $0 \to A^\bullet \to B^\bullet \to C^\bullet \to 0$ to a triangle isomorphic to the distinguished triangle

$$(A^\bullet, B^\bullet, C^\bullet, \alpha, \beta, \delta).$$

Proof. Follows from Lemma 13.8.14. □

Remark 13.11.4. Let $\mathcal{A}$ be an abelian category. Let $K^\bullet$ be a complex of $\mathcal{A}$. Let $a \in \mathbf{Z}$. We claim there is a canonical distinguished triangle

$$\tau_{\leq a} K^\bullet \to K^\bullet \to \tau_{\geq a+1} K^\bullet \to (\tau_{\leq a} K^\bullet)[1]$$

in $D(\mathcal{A})$. Here we have used the canonical truncation functors τ from Homology, Section 12.13. Namely, we first take the distinguished triangle associated by our δ-functor (Lemma 13.11.1) to the short exact sequence of complexes

$$0 \to \tau_{\leq a} K^\bullet \to K^\bullet \to K^\bullet / \tau_{\leq a} K^\bullet \to 0$$

Next, we use that the map $K^\bullet \to \tau_{\geq a+1} K^\bullet$ factors through a quasi-isomorphism $K^\bullet / \tau_{\leq a} K^\bullet \to \tau_{\geq a+1} K^\bullet$ by the description of cohomology groups in Homology, Section 12.13. In a similar way we obtain canonical distinguished triangles

$$\tau_{\leq a} K^\bullet \to \tau_{\leq a+1} K^\bullet \to H^{a+1}(K^\bullet)[-a-1] \to (\tau_{\leq a} K^\bullet)[1]$$

and

$$H^a(K^\bullet)[-a] \to \tau_{\geq a}K^\bullet \to \tau_{\geq a+1}K^\bullet \to H^a(K^\bullet)[-a+1]$$

Lemma 13.11.5. *Let $\mathcal{A}$ be an abelian category. Let*

$$K_0^\bullet \to K_1^\bullet \to \ldots \to K_n^\bullet$$

be maps of complexes such that

(1) $H^i(K_0^\bullet) = 0$ *for* $i > 0$,
(2) $H^{-j}(K_j^\bullet) \to H^{-j}(K_{j+1}^\bullet)$ *is zero.*

Then the composition $K_1^\bullet \to K_n^\bullet$ factors through $\tau_{\leq -n}K_n^\bullet \to K_n^\bullet$ in $D(\mathcal{A})$.

Proof. The case $n = 1$. Since $\tau_{\leq 0}K_0^\bullet = K_0^\bullet$ in $D(\mathcal{A})$ we can replace $K_0^\bullet$ by $\tau_{\leq 0}K_0^\bullet$ and $K_1^\bullet$ by $\tau_{\leq 0}K_1^\bullet$. Consider the distinguished triangle

$$\tau_{\leq -1}K_1^\bullet \to K_1^\bullet \to H^0(K_1^\bullet)[0] \to (\tau_{\leq -1}K_1^\bullet)[1]$$

(Remark 13.11.4). The composition $K_0^\bullet \to K_1^\bullet \to H^0(K_1^\bullet)[0]$ is zero as it is equal to $K_0^\bullet \to H^0(K_0^\bullet)[0] \to H^0(K_1^\bullet)[0]$ which is zero by assumption. The fact that $\mathrm{Hom}_{D(\mathcal{A})}(K_0^\bullet, -)$ is a homological functor (Lemma 13.4.2), allows us to find the desired factorization. For $n = 2$ we get a factorization $K_0^\bullet \to \tau_{\leq -1}K_1^\bullet$ by the case $n = 1$ and we can apply the case $n = 1$ to the map of complexes $\tau_{\leq -1}K_1^\bullet \to \tau_{\leq -1}K_2^\bullet$ to get a factorization $\tau_{\leq -1}K_1^\bullet \to \tau_{\leq -2}K_2^\bullet$. The general case is proved in exactly the same manner. □

13.12. Triangulated subcategories of the derived category

Let $\mathcal{A}$ be an abelian category. In this section we are going to look for strictly full saturated triangulated subcategories $\mathcal{D}' \subset D(\mathcal{A})$ and in the bounded versions.

Here is a simple construction. Let $\mathcal{B} \subset \mathcal{A}$ be a weak Serre subcategory, see Homology, Section 12.9. We let $D_{\mathcal{B}}(\mathcal{A})$ the full subcategory of $D(\mathcal{A})$ whose objects are

$$\mathrm{Ob}(D_{\mathcal{B}}(\mathcal{A})) = \{X \in \mathrm{Ob}(D(\mathcal{A})) \mid H^n(X) \text{ is an object of } \mathcal{B} \text{ for all } n\}$$

We also define $D^+_{\mathcal{B}}(\mathcal{A}) = D^+(\mathcal{A}) \cap D_{\mathcal{B}}(\mathcal{A})$ and similarly for the other bounded versions.

Lemma 13.12.1. *Let $\mathcal{A}$ be an abelian category. Let $\mathcal{B} \subset \mathcal{A}$ be a weak Serre subcategory. The category $D_{\mathcal{B}}(\mathcal{A})$ is a strictly full saturated abelian subcategory of $D(\mathcal{A})$. Similarly for the bounded versions.*

Proof. It is clear that $D_{\mathcal{B}}(\mathcal{A})$ is an additive subcategory preserved under the translation functors. If $X \oplus Y$ is in $D_{\mathcal{B}}(\mathcal{A})$, then it is clear that both X and Y are in $D_{\mathcal{B}}$ because $H^n(X \oplus Y) = H^n(X) \oplus H^n(Y)$, hence both $H^n(X)$ and $H^n(Y)$ are kernels of maps between maps of an object of $\mathcal{B}$, hence objects of $\mathcal{B}$. By Lemma 13.4.14 it therefore suffices to show that given a distinguished triangle (X, Y, Z, f, g, h) such that X and Y are in $D_{\mathcal{B}}(\mathcal{A})$ then Z is an object of $D_{\mathcal{B}}(\mathcal{A})$. The long exact cohomology sequence (13.10.1.1) and the definition of a weak Serre subcategory (see Homology, Definition 12.9.1) show that $H^n(Z)$ is an object of $\mathcal{B}$ for all n. Thus Z is an object of $D_{\mathcal{B}}(\mathcal{A})$. □

An interesting feature of the situation of the lemma is that the functor $D(\mathcal{B}) \to D(\mathcal{A})$ factors through a canonical exact functor

$$D(\mathcal{B}) \longrightarrow D_{\mathcal{B}}(\mathcal{A}) \tag{13.12.1.1}$$

After all a complex made from objects of $\mathcal{B}$ certainly gives rise to an object of $D_{\mathcal{B}}(\mathcal{A})$ and as distinguished triangles in $D_{\mathcal{B}}(\mathcal{A})$ are exactly the distinguished triangles of $D(\mathcal{A})$ whose vertices are in $D_{\mathcal{B}}(\mathcal{A})$ we see that the functor is exact since $D(\mathcal{B}) \to D(\mathcal{A})$ is exact. Similarly we obtain functors $D^+(\mathcal{B}) \longrightarrow D^+_{\mathcal{B}}(\mathcal{A})$ etc for the bounded versions. A key question in many cases is whether the displayed functor is an equivalence.

Now, suppose that $\mathcal{B}$ is a Serre subcategory of $\mathcal{A}$. In this case we have the quotient functor $\mathcal{A} \to \mathcal{A}/\mathcal{B}$, see Homology, Lemma 12.9.6. In this case $D_{\mathcal{B}}(\mathcal{A})$ is the kernel of the functor $D(\mathcal{A}) \to D(\mathcal{A}/\mathcal{B})$. Thus we obtain a canonical functor

$$D(\mathcal{A})/D_{\mathcal{B}}(\mathcal{A}) \longrightarrow D(\mathcal{A}/\mathcal{B})$$

by Lemma 13.6.8. Similarly for the bounded versions.

Lemma 13.12.2. *Let $\mathcal{A}$ be an abelian category. Let $\mathcal{B} \subset \mathcal{A}$ be a Serre subcategory. Then $D(\mathcal{A}) \to D(\mathcal{A}/\mathcal{B})$ is essentially surjective.*

Proof. We will use the description of the category $\mathcal{A}/\mathcal{B}$ in the proof of Homology, Lemma 12.9.6. Let $(X^\bullet, d^\bullet)$ be a complex of $\mathcal{A}/\mathcal{B}$. For each i we have an object X^i of $\mathcal{A}$ and $d^i = (s^i, f^i)$ where $s^i : Y^i \to X^i$ is a morphism of $\mathcal{A}$ whose kernel and cokernel are in $\mathcal{B}$ and $f^i : Y^i \to X^{i+1}$ is an arbitrary morphism of $\mathcal{A}$. Next, consider the complex

$$\ldots \to X^i \oplus Y^i \oplus Y^{i+1} \to X^{i+1} \oplus Y^{i+1} \oplus Y^{i+2} \to \ldots$$

in $\mathcal{A}$ with differential given by

$$\begin{pmatrix} 0 & f^i & s^{i+1} \\ 0 & 0 & -\mathrm{id}_{Y^{i+1}} \\ 0 & 0 & 0 \end{pmatrix}.$$

This complex becomes quasi-isomorphic to the complex $(X^\bullet, d^\bullet)$ in $\mathcal{A}/\mathcal{B}$ by the maps

$$(\mathrm{id}_{X^i}, s^i, 0) : X^i \oplus Y^i \oplus Y^{i+1} \to X^i$$

Calculation omitted. □

Lemma 13.12.3. *Let $\mathcal{A}$ be an abelian category. Let $\mathcal{B} \subset \mathcal{A}$ be a Serre subcategory. Suppose that the functor $v : \mathcal{A} \to \mathcal{A}/\mathcal{B}$ has a left adjoint $u : \mathcal{A}/\mathcal{B} \to \mathcal{A}$ such that $vu \cong id$. Then*

$$D(\mathcal{A})/D_{\mathcal{B}}(\mathcal{A}) = D(\mathcal{A}/\mathcal{B})$$

and similarly for the bounded versions.

Proof. The functor $D(v) : D(\mathcal{A}) \to D(\mathcal{A}/\mathcal{B})$ is essentially surjective by Lemma 13.12.2. For an object X of $D(\mathcal{A})$ the adjunction mapping $c_X : uvX \to X$ maps to an isomorphism in $D(\mathcal{A}/\mathcal{B})$ because $vuv \cong v$ by the assumption that $vu \cong \mathrm{id}$. Thus in a distinguished triangle (uvX, X, Z, c_X, g, h) the object Z is an object of $D_{\mathcal{B}}(\mathcal{A})$ as we see by looking at the long exact cohomology sequence. Hence c_X is an element of the multiplicative system used to define the quotient category $D(\mathcal{A})/D_{\mathcal{B}}(\mathcal{A})$. Thus $uvX \cong X$ in $D(\mathcal{A})/D_{\mathcal{B}}(\mathcal{A})$. For $X, Y \in \mathrm{Ob}(\mathcal{A})$) the map

$$\mathrm{Hom}_{D(\mathcal{A})/D_{\mathcal{B}}(\mathcal{A})}(X, Y) \longrightarrow \mathrm{Hom}_{D(\mathcal{A}/\mathcal{B})}(vX, vY)$$

is bijective because u gives an inverse (by the remarks above). □

13.13. Filtered derived categories

A reference for this section is [**Ill72**, I, Chapter V]. Let $\mathcal{A}$ be an abelian category. In this section we will define the filtered derived category $DF(\mathcal{A})$ of $\mathcal{A}$. In short, we will define it as the derived category of the exact category of objects of $\mathcal{A}$ endowed with a finite filtration. (Thus our construction is a special case of a more general construction of the derived category of an exact category, see for example [**Büh10**], [**Kel90**].) Illusie's filtered derived category is the full subcategory of ours consisting of those objects whose filtration is finite. (In our category the filtration is still finite in each degree, but may not be uniformly bounded.) The rationale for our choice is that it is not harder and it allows us to apply the discussion to the spectral sequences of Lemma 13.20.3, see also Remark 13.20.4.

We will use the notation regarding filtered objects introduced in Homology, Section 12.16. The category of filtered objects of $\mathcal{A}$ is denoted $\mathrm{Fil}(\mathcal{A})$. All filtrations will be decreasing by fiat.

Definition 13.13.1. Let $\mathcal{A}$ be an abelian category. The *category of finite filtered objects of* $\mathcal{A}$ is the category of filtered objects (A, F) of $\mathcal{A}$ whose filtration F is finite. We denote it $\mathrm{Fil}^f(\mathcal{A})$.

Thus $\mathrm{Fil}^f(\mathcal{A})$ is a full subcategory of $\mathrm{Fil}(\mathcal{A})$. For each $p \in \mathbf{Z}$ there is a functor $\mathrm{gr}^p : \mathrm{Fil}^f(\mathcal{A}) \to \mathcal{A}$. There is a functor

$$\mathrm{gr} = \bigoplus\nolimits_{p \in \mathbf{Z}} \mathrm{gr}^p : \mathrm{Fil}^f(\mathcal{A}) \to \mathrm{Gr}(\mathcal{A})$$

where $\mathrm{Gr}(\mathcal{A})$ is the category of graded objects of $\mathcal{A}$, see Homology, Definition 12.15.1. Finally, there is a functor

$$(\text{forget } F) : \mathrm{Fil}^f(\mathcal{A}) \longrightarrow \mathcal{A}$$

which associates to the filtered object (A, F) the underlying object of $\mathcal{A}$. The category $\mathrm{Fil}^f(\mathcal{A})$ is an additive category, but not abelian in general, see Homology, Example 12.3.12.

Because the functors gr^p, gr, (forget F) are additive they induce exact functors of triangulated categories

$$\mathrm{gr}^p, (\text{forget } F) : K(\mathrm{Fil}^f(\mathcal{A})) \to K(\mathcal{A}) \quad \text{and} \quad \mathrm{gr} : K(\mathrm{Fil}^f(\mathcal{A})) \to K(\mathrm{Gr}(\mathcal{A}))$$

by Lemma 13.9.6. By analogy with the case of the homotopy category of an abelian category we make the following definitions.

Definition 13.13.2. Let $\mathcal{A}$ be an abelian category.

(1) Let $\alpha : K^\bullet \to L^\bullet$ be a morphism of $K(\mathrm{Fil}^f(\mathcal{A}))$. We say that α is a *filtered quasi-isomorphism* if the morphism $\mathrm{gr}(\alpha)$ is a quasi-isomorphism.
(2) Let $K^\bullet$ be an object of $K(\mathrm{Fil}^f(\mathcal{A}))$. We say that $K^\bullet$ is *filtered acyclic* if the complex $\mathrm{gr}(K^\bullet)$ is acyclic.

Note that $\alpha : K^\bullet \to L^\bullet$ is a filtered quasi-isomorphism if and only if each $\mathrm{gr}^p(\alpha)$ is a quasi-isomorphism. Similarly a complex $K^\bullet$ is filtered acyclic if and only if each $\mathrm{gr}^p(K^\bullet)$ is acyclic.

Lemma 13.13.3. *Let $\mathcal{A}$ be an abelian category.*

(1) *The functor* $K(Fil^f(\mathcal{A})) \longrightarrow Gr(\mathcal{A})$, $K^\bullet \longmapsto H^0(gr(K^\bullet))$ *is homological.*
(2) *The functor* $K(Fil^f(\mathcal{A})) \to \mathcal{A}$, $K^\bullet \longmapsto H^0(gr^p(K^\bullet))$ *is homological.*
(3) *The functor* $K(Fil^f(\mathcal{A})) \longrightarrow \mathcal{A}$, $K^\bullet \longmapsto H^0((forget\ F)K^\bullet)$ *is homological.*

Proof. This follows from the fact that $H^0 : K(\mathcal{A}) \to \mathcal{A}$ is homological, see Lemma 13.10.1 and the fact that the functors $\mathrm{gr}, \mathrm{gr}^p, (\text{forget } F)$ are exact functors of triangulated categories. See Lemma 13.4.18. □

Lemma 13.13.4. *Let $\mathcal{A}$ be an abelian category. The full subcategory $FAc(\mathcal{A})$ of $K(Fil^f(\mathcal{A}))$ consisting of filtered acyclic complexes is a strictly full saturated triangulated subcategory of $K(Fil^f(\mathcal{A}))$. The corresponding saturated multiplicative system (see Lemma 13.6.10) of $K(Fil^f(\mathcal{A}))$ is the set $FQis(\mathcal{A})$ of filtered quasi-isomorphisms. In particular, the kernel of the localization functor*

$$Q : K(Fil^f(\mathcal{A})) \longrightarrow FQis(\mathcal{A})^{-1}K(Fil^f(\mathcal{A}))$$

is $FAc(\mathcal{A})$ and the functor $H^0 \circ gr$ factors through Q.

Proof. We know that $H^0 \circ \mathrm{gr}$ is a homological functor by Lemma 13.13.3. Thus this lemma is a special case of Lemma 13.6.11. □

Definition 13.13.5. Let $\mathcal{A}$ be an abelian category. Let $\mathrm{FAc}(\mathcal{A})$ and $\mathrm{FQis}(\mathcal{A})$ be as in Lemma 13.13.4. The *filtered derived category of $\mathcal{A}$* is the triangulated category

$$DF(\mathcal{A}) = K(\mathrm{Fil}^f(\mathcal{A}))/\mathrm{FAc}(\mathcal{A}) = \mathrm{FQis}(\mathcal{A})^{-1}K(\mathrm{Fil}^f(\mathcal{A})).$$

Lemma 13.13.6. *The functors $gr^p, gr, (forget\ F)$ induce canonical exact functors*

$$gr^p, gr, (forget\ F) : DF(\mathcal{A}) \longrightarrow D(\mathcal{A})$$

which commute with the localization functors.

Proof. This follows from the universal property of localization, see Lemma 13.5.6, provided we can show that a filtered quasi-isomorphism is turned into a quasi-isomorphism by each of the functors $\mathrm{gr}^p, \mathrm{gr}, (\text{forget } F)$. This is true by definition for the first two. For the last one the statement we have to do a little bit of work. Let $f : K^\bullet \to L^\bullet$ be a filtered quasi-isomorphism in $K(\mathrm{Fil}^f(\mathcal{A}))$. Choose a distinguished triangle $(K^\bullet, L^\bullet, M^\bullet, f, g, h)$ which contains f. Then $M^\bullet$ is filtered acyclic, see Lemma 13.13.4. Hence by the corresponding lemma for $K(\mathcal{A})$ it suffices to show that a filtered acyclic complex is an acyclic complex if we forget the filtration. This follows from Homology, Lemma 12.16.15. □

Definition 13.13.7. Let $\mathcal{A}$ be an abelian category. The *bounded filtered derived category* $DF^b(\mathcal{A})$ is the full subcategory of $DF(\mathcal{A})$ with objects those X such that $\mathrm{gr}(X) \in D^b(\mathcal{A})$. Similarly for the bounded below filtered derived category $DF^+(\mathcal{A})$ and the bounded above filtered derived category $DF^-(\mathcal{A})$.

Lemma 13.13.8. *Let $\mathcal{A}$ be an abelian category. Let $K^\bullet \in K(Fil^f(\mathcal{A}))$.*

(1) *If $H^n(gr(K^\bullet)) = 0$ for all $n < a$, then there exists a filtered quasi-isomorphism $K^\bullet \to L^\bullet$ with $L^n = 0$ for all $n < a$.*
(2) *If $H^n(gr(K^\bullet)) = 0$ for all $n > b$, then there exists a filtered quasi-isomorphism $M^\bullet \to K^\bullet$ with $M^n = 0$ for all $n > b$.*

(3) *If $H^n(gr(K^\bullet)) = 0$ for all $|n| \gg 0$, then there exists a commutative diagram of morphisms of complexes*

$$\begin{array}{ccc} K^\bullet & \longrightarrow & L^\bullet \\ \uparrow & & \uparrow \\ M^\bullet & \longrightarrow & N^\bullet \end{array}$$

where all the arrows are filtered quasi-isomorphisms, $L^\bullet$ bounded below, $M^\bullet$ bounded above, and $N^\bullet$ a bounded complex.

Proof. Suppose that $H^n(\mathrm{gr}(K^\bullet)) = 0$ for all $n < a$. By Homology, Lemma 12.16.15 the sequence

$$K^{a-1} \xrightarrow{d^{a-2}} K^{a-1} \xrightarrow{d^{a-1}} K^a$$

is an exact sequence of objects of $\mathcal{A}$ and the morphisms d^{a-2} and d^{a-1} are strict. Hence $\mathrm{Coim}(d^{a-1}) = \mathrm{Im}(d^{a-1})$ in $\mathrm{Fil}^f(\mathcal{A})$ and the map $\mathrm{gr}(\mathrm{Im}(d^{a-1})) \to \mathrm{gr}(K^a)$ is injective with image equal to the image of $\mathrm{gr}(K^{a-1}) \to \mathrm{gr}(K^a)$, see Homology, Lemma 12.16.13. This means that the map $K^\bullet \to \tau_{\geq a}K^\bullet$ into the truncation

$$\tau_{\geq a}K^\bullet = (\ldots \to 0 \to K^a/\mathrm{Im}(d^{a-1}) \to K^{a+1} \to \ldots)$$

is a filtered quasi-isomorphism. This proves (1). The proof of (2) is dual to the proof of (1). Part (3) follows formally from (1) and (2). □

To state the following lemma denote $\mathrm{FAc}^+(\mathcal{A})$, $\mathrm{FAc}^-(\mathcal{A})$, resp. $\mathrm{FAc}^b(\mathcal{A})$ the intersection of $K^+(\mathrm{Fil}^f\mathcal{A})$, $K^-(\mathrm{Fil}^f\mathcal{A})$, resp. $K^b(\mathrm{Fil}^f\mathcal{A})$ with $\mathrm{FAc}(\mathcal{A})$. Denote $\mathrm{FQis}^+(\mathcal{A})$, $\mathrm{FQis}^-(\mathcal{A})$, resp. $\mathrm{FQis}^b(\mathcal{A})$ the intersection of $K^+(\mathrm{Fil}^f\mathcal{A})$, $K^-(\mathrm{Fil}^f\mathcal{A})$, resp. $K^b(\mathrm{Fil}^f\mathcal{A})$ with $\mathrm{FQis}(\mathcal{A})$.

Lemma 13.13.9. *Let $\mathcal{A}$ be an abelian category. The subcategories $FAc^+(\mathcal{A})$, $FAc^-(\mathcal{A})$, resp. $FAc^b(\mathcal{A})$ are strictly full saturated triangulated subcategories of $K^+(Fil^f\mathcal{A})$, $K^-(Fil^f\mathcal{A})$, resp. $K^b(Fil^f\mathcal{A})$. The corresponding saturated multiplicative systems (see Lemma 13.6.10) are the sets $FQis^+(\mathcal{A})$, $FQis^-(\mathcal{A})$, resp. $FQis^b(\mathcal{A})$.*

(1) *The kernel of the functor $K^+(Fil^f\mathcal{A}) \to DF^+(\mathcal{A})$ is $FAc^+(\mathcal{A})$ and this induces an equivalence of triangulated categories*

$$K^+(Fil^f\mathcal{A})/FAc^+(\mathcal{A}) = FQis^+(\mathcal{A})^{-1}K^+(Fil^f\mathcal{A}) \longrightarrow DF^+(\mathcal{A})$$

(2) *The kernel of the functor $K^-(Fil^f\mathcal{A}) \to DF^-(\mathcal{A})$ is $FAc^-(\mathcal{A})$ and this induces an equivalence of triangulated categories*

$$K^-(Fil^f\mathcal{A})/FAc^-(\mathcal{A}) = FQis^-(\mathcal{A})^{-1}K^-(Fil^f\mathcal{A}) \longrightarrow DF^-(\mathcal{A})$$

(3) *The kernel of the functor $K^b(Fil^f\mathcal{A}) \to DF^b(\mathcal{A})$ is $FAc^b(\mathcal{A})$ and this induces an equivalence of triangulated categories*

$$K^b(Fil^f\mathcal{A})/FAc^b(\mathcal{A}) = FQis^b(\mathcal{A})^{-1}K^b(Fil^f\mathcal{A}) \longrightarrow DF^b(\mathcal{A})$$

Proof. This follows from the results above, in particular Lemma 13.13.8, by exactly the same arguments as used in the proof of Lemma 13.10.6. □

13.14. Derived functors in general

A reference for this section is Deligne's exposé XVII in [**AGV71**]. A very general notion of right and left derived functors exists where we have an exact functor between triangulated categories, a multiplicative system in the source category and we want to find the "correct" extension of the exact functor to the localized category.

Situation 13.14.1. Here $F : \mathcal{D} \to \mathcal{D}'$ is an exact functor of triangulated categories and S is a saturated multiplicative system in $\mathcal{D}$ compatible with the structure of triangulated category on $\mathcal{D}$.

Let $X \in \mathrm{Ob}(\mathcal{D})$. Recall from Categories, Remark 4.25.5 the filtered category X/S of arrows $s : X \to X'$ in S with source X. Dually, in Categories, Remark 4.25.12 we defined the cofiltered category S/X of arrows $s : X' \to X$ in S with target X.

Definition 13.14.2. Assumptions and notation as in Situation 13.14.1. Let $X \in \mathrm{Ob}(\mathcal{D})$.

(1) we say the *right derived functor RF is defined at X* if the ind-object
$$(X/S) \longrightarrow \mathcal{D}', \quad (s : X \to X') \longmapsto F(X')$$
is essentially constant[3]; in this case the value Y in $\mathcal{D}'$ is called the *value of RF at X*.

(2) we say the *left derived functor LF is defined at X* if the pro-object
$$(S/X) \longrightarrow \mathcal{D}', \quad (s : X' \to X) \longmapsto F(X')$$
is essentially constant; in this case the value Y in $\mathcal{D}'$ is called the *value of LF at X*.

By abuse of notation we often denote the values simply $RF(X)$ or $LF(X)$.

It will turn out that the full subcategory of $\mathcal{D}$ consisting of objects where RF is defined is a triangulated subcategory, and RF will define functor on this subcategory which transforms morphisms of s into isomorphisms.

Lemma 13.14.3. *Assumptions and notation as in Situation 13.14.1. Let $f : X \to Y$ be a morphism of $\mathcal{D}$.*

(1) *If RF is defined at X and Y then there exists a unique morphism $RF(f) : RF(X) \to RF(Y)$ between the values such that for any commutative diagram*
$$\begin{array}{ccc} X & \xrightarrow[s]{} & X' \\ {\scriptstyle f}\downarrow & & \downarrow{\scriptstyle f'} \\ Y & \xrightarrow{s'} & Y' \end{array}$$
with $s, s' \in S$ the diagram
$$\begin{array}{ccccc} F(X) & \longrightarrow & F(X') & \longrightarrow & RF(X) \\ \downarrow & & \downarrow & & \downarrow \\ F(Y) & \longrightarrow & F(Y') & \longrightarrow & RF(Y) \end{array}$$

[3]For a discussion of when an ind-object or pro-object of a category is essentially constant we refer to Categories, Section 4.22.

commutes.

(2) *If LF is defined at X and Y then there exists a unique morphism $LF(f) : LF(X) \to LF(Y)$ between the values such that for any commutative diagram*

$$\begin{array}{ccc} X' & \xrightarrow{s} & X \\ {\scriptstyle f'}\downarrow & & \downarrow{\scriptstyle f} \\ Y' & \xrightarrow{s'} & Y \end{array}$$

with s, s' in S the diagram

$$\begin{array}{ccccc} LF(X) & \longrightarrow & F(X') & \longrightarrow & F(X) \\ \downarrow & & \downarrow & & \downarrow \\ LF(Y) & \longrightarrow & F(Y') & \longrightarrow & F(Y) \end{array}$$

commutes.

Proof. Part (1) holds if we only assume that the colimits

$$RF(X) = \text{colim}_{s:X'\to X} F(X') \quad \text{and} \quad RF(Y) = \text{colim}_{s':Y'\to Y} F(Y')$$

exist. Namely, to give a morphism $RF(X) \to RF(Y)$ between the colimits is the same thing as giving for each $s : X \to X'$ in $\text{Ob}(X/S)$ a morphism $F(X') \to RF(Y)$ compatible with morphisms in the category X/S. To get the morphism we choose a commutative diagram

$$\begin{array}{ccc} X & \xrightarrow{s} & X' \\ {\scriptstyle f}\downarrow & & \downarrow{\scriptstyle f'} \\ Y & \xrightarrow{s'} & Y' \end{array}$$

with s, s' in S as is possible by MS2 and we set $F(X') \to RF(Y)$ equal to the composition $F(X') \to F(Y') \to RF(Y)$. To see that this is independent of the choice of the diagram above use MS3. Details omitted. The proof of (2) is dual. □

Lemma 13.14.4. *Assumptions and notation as in Situation 13.14.1. Let $s : X \to Y$ be an element of S.*

(1) *RF is defined at X if and only if it is defined at Y. In this case the map $RF(s) : RF(X) \to RF(Y)$ between values is an isomorphism.*
(2) *LF is defined at X if and only if it is defined at Y. In this case the map $LF(s) : LF(X) \to LF(Y)$ between values is an isomorphism.*

Proof. Omitted. □

Lemma 13.14.5. *Assumptions and notation as in Situation 13.14.1. Let X be an object of $\mathcal{D}$ and $n \in \mathbf{Z}$.*

(1) *RF is defined at X if and only if it is defined at $X[n]$. In this case there is a canonical isomorphism $RF(X)[n] = RF(X[n])$ between values.*
(2) *LF is defined at X if and only if it is defined at $X[n]$. In this case there is a canonical isomorphism $LF(X)[n] \to LF(X[n])$ between values.*

Proof. Omitted. □

Lemma 13.14.6. *Assumptions and notation as in Situation 13.14.1. Let (X, Y, Z, f, g, h) be a distinguished triangle of $\mathcal{D}$. If RF is defined at two out of three of X, Y, Z, then it is defined at the third. Moreover, in this case*

$$(RF(X), RF(Y), RF(Z), RF(f), RF(g), RF(h))$$

is a distinguished triangle in $\mathcal{D}'$. Similarly for LF.

Proof. Say RF is defined at X, Y with values A, B. Let $RF(f) : A \to B$ be the induced morphism, see Lemma 13.14.3. We may choose a distinguished triangle $(A, B, C, RF(f), b, c)$ in $\mathcal{D}'$. We claim that C is a value of RF at Z.

To see this pick $s : X \to X'$ in S such that there exists a morphism $\alpha : A \to F(X')$ as in Categories, Definition 4.22.1. We may choose a commutative diagram

$$\begin{array}{ccc} X & \xrightarrow{s} & X' \\ {\scriptstyle f}\downarrow & & \downarrow{\scriptstyle f'} \\ Y & \xrightarrow{s'} & Y' \end{array}$$

with $s' \in S$ by MS2. Using that Y/S is filtered we can (after replacing s' by some $s'' : Y \to Y''$ in S) assume that there exists a morphism $\beta : B \to F(Y')$ as in Categories, Definition 4.22.1. Picture

$$\begin{array}{ccccc} A & \xrightarrow{\alpha} & F(X') & \longrightarrow & A \\ {\scriptstyle RF(f)}\downarrow & & \downarrow{\scriptstyle F(f')} & & \downarrow{\scriptstyle RF(f)} \\ B & \xrightarrow{\beta} & F(Y') & \longrightarrow & B \end{array}$$

It may not be true that the left square commutes, but the outer and right squares commute. The assumption that the ind-object $\{F(Y')\}_{s' : Y' \to Y}$ is essentially constant means that there exists a $s'' : Y \to Y''$ in S and a morphism $h : Y' \to Y''$ with such that $s'' = h \circ s'$ and $F(h)$ equal to $F(Y') \to B \to F(Y') \to F(Y'')$. Hence after replacing Y' by Y'' and β by $F(h) \circ \beta$ the diagram will commute (by direct computation with arrows).

Using MS6 choose a morphism of triangles

$$(s, s', s'') : (X, Y, Z, f, g, h) \longrightarrow (X', Y', Z', f', g', h')$$

with $s'' \in S$. By TR3 choose a morphism of triangles

$$(\alpha, \beta, \gamma) : (A, B, C, RF(f), b, c) \longrightarrow (F(X'), F(Y'), F(Z'), F(f'), F(g'), F(h'))$$

By Lemma 13.14.4 it suffices to prove that $RF(Z')$ is defined and has value C. Consider the category $\mathcal{I}$ of Lemma 13.5.8 of triangles

$$\mathcal{I} = \{(t, t', t'') : (X', Y', Z', f', g', h') \to (X'', Y'', Z'', f'', g'', h'') \mid (t, t', t'') \in S\}$$

To show that the system $F(Z'')$ is essentially constant over the category Z'/S is equivalent to showing that the system of $F(Z'')$ is essentially constant over $\mathcal{I}$ by using the surjectivity of the functor $\mathcal{I} \to Z'/S$. For any object W in $\mathcal{D}'$ we can

consider the diagram

$$\begin{array}{ccc}
\lim_{\mathcal{I}} \mathrm{Mor}_{\mathcal{D}'}(F(X''), W) & \longrightarrow & \mathrm{Mor}_{\mathcal{D}'}(A, W) \\
\uparrow & & \uparrow \\
\lim_{\mathcal{I}} \mathrm{Mor}_{\mathcal{D}'}(F(Y''), W) & \longrightarrow & \mathrm{Mor}_{\mathcal{D}'}(B, W) \\
\uparrow & & \uparrow \\
\lim_{\mathcal{I}} \mathrm{Mor}_{\mathcal{D}'}(F(Z''), W) & \longrightarrow & \mathrm{Mor}_{\mathcal{D}'}(C, W) \\
\uparrow & & \uparrow \\
\lim_{\mathcal{I}} \mathrm{Mor}_{\mathcal{D}'}(F(X''[1]), W) & \longrightarrow & \mathrm{Mor}_{\mathcal{D}'}(A[1], W) \\
\uparrow & & \uparrow \\
\lim_{\mathcal{I}} \mathrm{Mor}_{\mathcal{D}'}(F(Y''[1]), W) & \longrightarrow & \mathrm{Mor}_{\mathcal{D}'}(B[1], W)
\end{array}$$

which shows that the middle arrow is an isomorphism by the 5 lemma. In this way we conclude that C is the colimit $\mathrm{colim}_{\mathcal{I}} F(Z'')$. To see that the ind-object is essentially constant it now suffices to show that for any object W in $\mathcal{D}'$ the map

$$\mathrm{colim}_{\mathcal{I}} \mathrm{Mor}_{\mathcal{D}'}(W, F(Z'')) \longrightarrow \mathrm{Mor}_{\mathcal{D}'}(W, C)$$

is bijective, see Categories, Lemma 4.22.6. To see this we can use again the 5 lemma and the commutative diagram

$$\begin{array}{ccc}
\mathrm{colim}_{\mathcal{I}} \mathrm{Mor}_{\mathcal{D}'}(W, F(X'')) & \longrightarrow & \mathrm{Mor}_{\mathcal{D}'}(W, A) \\
\uparrow & & \uparrow \\
\mathrm{colim}_{\mathcal{I}} \mathrm{Mor}_{\mathcal{D}'}(W, F(Y'')) & \longrightarrow & \mathrm{Mor}_{\mathcal{D}'}(W, B) \\
\uparrow & & \uparrow \\
\mathrm{colim}_{\mathcal{I}} \mathrm{Mor}_{\mathcal{D}'}(W, F(Z'')) & \longrightarrow & \mathrm{Mor}_{\mathcal{D}'}(W, C) \\
\uparrow & & \uparrow \\
\mathrm{colim}_{\mathcal{I}} \mathrm{Mor}_{\mathcal{D}'}(W, F(X''[1])) & \longrightarrow & \mathrm{Mor}_{\mathcal{D}'}(W, A[1]) \\
\uparrow & & \uparrow \\
\mathrm{colim}_{\mathcal{I}} \mathrm{Mor}_{\mathcal{D}'}(W, F(Y''[1])) & \longrightarrow & \mathrm{Mor}_{\mathcal{D}'}(W, B[1])
\end{array}$$

and the fact that Categories, Lemma 4.22.6 guarantees that the other horizontal arrows are isomorphisms. □

Lemma 13.14.7. *Assumptions and notation as in Situation 13.14.1. Let X, Y be objects of $\mathcal{D}$. If RF is defined at $X \oplus Y$, then it is defined at X and Y. Moreover, in this case $RF(X \oplus Y) = RF(X) \oplus RF(Y)$. Similarly for LF.*

Proof. Since S is a saturated system for any $s : X \to X'$ and $s' : Y \to Y'$ in S the morphism $s \oplus s' : X \oplus Y \to X' \oplus Y'$ is an element of S (as S is the set of arrows which become invertible under the additive localization functor $Q : \mathcal{D} \to S^{-1}\mathcal{D}$, see Categories, Lemma 4.25.18). To prove the lemma for RF it suffices to show that

these arrows $s \oplus s'$ are cofinal in the filtered category $(X \oplus Y)/S$. To do this pick any $t : X \oplus Y \to Z$ in S. Using MS2 we can find morphisms $Z \to X'$, $Z \to Y'$ and $s : X \to X'$, $s' : Y \to Y'$ in S such that

$$\begin{array}{ccccc} X & \longleftarrow & X \oplus Y & \longrightarrow & Y \\ \downarrow{\scriptstyle s} & & \downarrow & & \downarrow{\scriptstyle s'} \\ X' & \longleftarrow & Z & \longrightarrow & Y' \end{array}$$

commutes. Hence the desired result. The proof for LF is dual. □

Proposition 13.14.8. *Assumptions and notation as in Situation 13.14.1. The full subcategory $\mathcal{E}$ of $\mathcal{D}$ consisting of objects at which RF is defined is a strictly full saturated triangulated subcategory of $\mathcal{D}$. Choosing values using the axiom of choice gives rise to an exact functor*

$$RF : \mathcal{E} \longrightarrow \mathcal{D}'$$

of triangulated categories. Elements of S with either source or target in $\mathcal{E}$ are morphisms of $\mathcal{E}$. Any element of $S_\mathcal{E} = Arrows(\mathcal{E}) \cap S$ is transformed into an isomorphism by RF. Hence an exact functor

$$RF : S_\mathcal{E}^{-1}\mathcal{E} \longrightarrow \mathcal{D}'.$$

A similar result holds for LF.

Proof. This is just a summary of the results obtained in Lemmas 13.14.3, 13.14.4, 13.14.5, 13.14.6, and 13.14.7. □

Definition 13.14.9. In Situation 13.14.1. We say F is *right deriveable*, or that *RF everywhere defined* if RF is defined at every object of $\mathcal{D}$. We say F is *left deriveable*, or that *LF everywhere defined* if LF is defined at every object of $\mathcal{D}$.

In this case we obtain a right (resp. left) derived functor

$$RF : S^{-1}\mathcal{D} \longrightarrow \mathcal{D}', \quad (\text{resp. } LF : S^{-1}\mathcal{D} \longrightarrow \mathcal{D}'), \tag{13.14.9.1}$$

see Proposition 13.14.8. In most interesting situations it is not the case that $RF \circ Q$ is equal to F. In fact, it might happen that the canonical map $F(X) \to RF(X)$ is never an isomorphism. In practice this does not happen, because in practice we only know how to prove F is right deriveable by showing that RF can be computed by evaluating F at judiciously chosen objects of the triangulated category $\mathcal{D}$. This warrants a definition.

Definition 13.14.10. In Situation 13.14.1.

(1) An object X of $\mathcal{D}$ *computes RF* if RF is defined at X and the canonical map $F(X) \to RF(X)$ is an isomorphism.
(2) An object X of $\mathcal{D}$ *computes LF* if LF is defined at X and the canonical map $LF(X) \to F(X)$ is an isomorphism.

Lemma 13.14.11. *Assumptions and notation as in Situation 13.14.1. Let X be an object of $\mathcal{D}$ and $n \in \mathbf{Z}$.*

(1) *X computes RF if and only if $X[n]$ computes RF.*
(2) *X computes LF if and only if $X[n]$ computes LF.*

Proof. Omitted. □

Lemma 13.14.12. *Assumptions and notation as in Situation 13.14.1. Let (X, Y, Z, f, g, h) be a distinguished triangle of $\mathcal{D}$. If X, Y compute RF then so does Z. Similar for LF.*

Proof. By Lemma 13.14.6 we know that RF is defined at Z and that RF applied to the triangle produces a distinguished triangle. Consider the morphism of distinguished triangles

$$\begin{array}{c} (F(X), F(Y), F(Z), F(f), F(g), F(h)) \\ \downarrow \\ (RF(X), RF(Y), RF(Z), RF(f), RF(g), RF(h)) \end{array}$$

Two out of three maps are isomorphisms, hence so is the third. □

Lemma 13.14.13. *Assumptions and notation as in Situation 13.14.1. Let X, Y be objects of $\mathcal{D}$. If $X \oplus Y$ computes RF, then X and Y compute RF. Similarly for LF.*

Proof. By Lemma 13.14.7 we know that RF is defined at X and Y and that $RF(X \oplus Y) = RF(X) \oplus RF(Y)$. Since the map

$$F(X) \oplus F(Y) = F(X \oplus Y) \longrightarrow RF(X \oplus Y) = RF(X) \oplus RF(Y)$$

is compatible with direct sum decompositions we win. □

Lemma 13.14.14. *Assumptions and notation as in Situation 13.14.1.*

(1) *If for every object $X \in \mathrm{Ob}(\mathcal{D})$ there exists an arrow $s : X \to X'$ in S such that X' computes RF, then RF is everywhere defined.*
(2) *If for every object $X \in \mathrm{Ob}(\mathcal{D})$ there exists an arrow $s : X' \to X$ in S such that X' computes LF, then LF is everywhere defined.*

Proof. This is clear from the definitions. □

Lemma 13.14.15. *Assumptions and notation as in Situation 13.14.1. If there exists a subset $\mathcal{I} \subset \mathrm{Ob}(\mathcal{D})$ such that*

(1) *for all $X \in \mathrm{Ob}(\mathcal{D})$ there exists $s : X \to X'$ in S with $X' \in \mathcal{I}$, and*
(2) *for every arrow $s : X \to X'$ in S with $X, X' \in \mathcal{I}$ the map $F(s) : F(X) \to F(X')$ is an isomorphism,*

then RF is everywhere defined and every $X \in \mathcal{I}$ computes RF. Dually, if there exists a subset $\mathcal{P} \subset \mathrm{Ob}(\mathcal{D})$ such that

(1) *for all $X \in \mathrm{Ob}(\mathcal{D})$ there exists $s : X' \to X$ in S with $X' \in \mathcal{P}$, and*
(2) *for every arrow $s : X \to X'$ in S with $X, X' \in \mathcal{P}$ the map $F(s) : F(X) \to F(X')$ is an isomorphism,*

then LF is everywhere defined and every $X \in \mathcal{P}$ computes LF.

Proof. Let X be an object of $\mathcal{D}$. Assumption (1) implies that the arrows $s : X \to X'$ in S with $X' \in \mathcal{I}$ are cofinal in the category X/S. Assumption (2) implies that F is constant on this cofinal subcategory. Clearly this implies that $F : (X/S) \to \mathcal{D}'$ is essentially constant with value $F(X')$ for any $s : X \to X'$ in S with $X' \in \mathcal{I}$. □

Lemma 13.14.16. *Let $\mathcal{A}, \mathcal{B}, \mathcal{C}$ be triangulated categories. Let S, resp. S' be a saturated multiplicative system in $\mathcal{A}$, resp. $\mathcal{B}$ compatible with the triangulated structure. Let $F : \mathcal{A} \to \mathcal{B}$ and $G : \mathcal{B} \to \mathcal{C}$ be exact functors. Denote $F' : \mathcal{A} \to (S')^{-1}\mathcal{B}$ the composition of F with the localization functor.*

1. *If RF', RG, $R(G \circ F)$ are everywhere defined, then there is a canonical transformation of functors $t : R(G \circ F) \longrightarrow RG \circ RF'$.*
2. *If LF', LG, $L(G \circ F)$ are everywhere defined, then there is a canonical transformation of functors $t : LG \circ LF' \to L(G \circ F)$.*

Proof. In this proof we try to be careful. Hence let us think of the derived functors as the functors

$$RF' : S^{-1}\mathcal{A} \to (S')^{-1}\mathcal{B}, \quad R(G \circ F) : S^{-1}\mathcal{A} \to \mathcal{C}, \quad RG : (S')^{-1}\mathcal{B} \to \mathcal{C}.$$

Let us denote $Q_A : \mathcal{A} \to S^{-1}\mathcal{A}$ and $Q_B : \mathcal{B} \to (S')^{-1}\mathcal{B}$ the localization functors. Then $F' = Q_B \circ F$. Note that for every object Y of $\mathcal{B}$ there is a canonical map

$$G(Y) \longrightarrow RG(Q_B(Y))$$

in other words, there is a transformation of functors $t' : G \to RG \circ Q_B$. Let X be an object of $\mathcal{A}$. We have

$$\begin{aligned} R(G \circ F)(Q_A(X)) &= \operatorname{colim}_{s:X \to X' \in S} G(F(X')) \\ &\xrightarrow{t'} \operatorname{colim}_{s:X \to X' \in S} RG(Q_B(F(X'))) \\ &= \operatorname{colim}_{s:X \to X' \in S} RG(F'(X')) \\ &= RG(\operatorname{colim}_{s:X \to X' \in S} F'(X')) \\ &= RG(RF'(X)). \end{aligned}$$

The system $F'(X')$ is essentially constant in the category $(S')^{-1}\mathcal{B}$. Hence we may pull the colimit inside the functor RG in the third equality of the diagram above, see Categories, Lemma 4.22.5 and its proof. We omit the proof this this defines a transformation of functors. The case of left derived functors is similar. □

13.15. Derived functors on derived categories

In practice derived functors come about most often when given an additive functor between abelian categories.

Situation 13.15.1. Here $F : \mathcal{A} \to \mathcal{B}$ is an additive functor between abelian categories. This induces exact functors

$$F : K(\mathcal{A}) \to K(\mathcal{B}), \quad K^+(\mathcal{A}) \to K^+(\mathcal{B}), \quad K^-(\mathcal{A}) \to K^-(\mathcal{B}).$$

We also denote F the composition $K(\mathcal{A}) \to D(\mathcal{B})$, $K^+(\mathcal{A}) \to D^+(\mathcal{B})$, and $K^-(\mathcal{A}) \to D(\mathcal{B})$ of F with the localization functor $K(\mathcal{B}) \to D(\mathcal{B})$, etc. This situation leads to four derived functors we will consider in the following.

1. The right derived functor of $F : K(\mathcal{A}) \to D(\mathcal{B})$ relative to the multiplicative system $\mathrm{Qis}(\mathcal{A})$.
2. The right derived functor of $F : K^+(\mathcal{A}) \to D^+(\mathcal{B})$ relative to the multiplicative system $\mathrm{Qis}^+(\mathcal{A})$.
3. The left derived functor of $F : K(\mathcal{A}) \to D(\mathcal{B})$ relative to the multiplicative system $\mathrm{Qis}(\mathcal{A})$.

(4) The left derived functor of $F : K^-(\mathcal{A}) \to D^-(\mathcal{B})$ relative to the multiplicative system $\mathrm{Qis}(\mathcal{A})$.

Each of these cases is an example of Situation 13.14.1.

Some of the ambiguity that may arise is alleviated by the following.

Lemma 13.15.2. *In Situation 13.15.1.*

(1) *Let X be an object of $K^+(\mathcal{A})$. The right derived functor of $K(\mathcal{A}) \to D(\mathcal{B})$ is defined at X if and only if the right derived functor of $K^+(\mathcal{A}) \to D^+(\mathcal{B})$ is defined at X. Moreover, the values are canonically isomorphic.*
(2) *Let X be an object of $K^+(\mathcal{A})$. Then X computes the right derived functor of $K(\mathcal{A}) \to D(\mathcal{B})$ if and only if X computes the right derived functor of $K^+(\mathcal{A}) \to D^+(\mathcal{B})$.*
(3) *Let X be an object of $K^-(\mathcal{A})$. The left derived functor of $K(\mathcal{A}) \to D(\mathcal{B})$ is defined at X if and only if the left derived functor of $K^-(\mathcal{A}) \to D^-(\mathcal{B})$ is defined at X. Moreover, the values are canonically isomorphic.*
(4) *Let X be an object of $K^-(\mathcal{A})$. Then X computes the left derived functor of $K(\mathcal{A}) \to D(\mathcal{B})$ if and only if X computes the left derived functor of $K^-(\mathcal{A}) \to D^-(\mathcal{B})$.*

Proof. Let X be an object of $K^+(\mathcal{A})$. Consider a quasi-isomorphism $s : X \to X'$ in $K(\mathcal{A})$. By Lemma 13.10.5 there exists quasi-isomorphism $X' \to X''$ with X'' bounded below. Hence we see that $X/\mathrm{Qis}^+(\mathcal{A})$ is cofinal in $X/\mathrm{Qis}(\mathcal{A})$. Thus it is clear that (1) holds. Part (2) follows directly from part (1). Parts (3) and (4) are dual to parts (1) and (2). □

Given an object A of an abelian category $\mathcal{A}$ we get a complex

$$A[0] = (\ldots \to 0 \to A \to 0 \to \ldots)$$

where A is placed in degree zero. Hence a functor $\mathcal{A} \to K(\mathcal{A})$, $A \mapsto A[0]$. Let us temporarily say that a partial functor is one that is defined on a subcategory.

Definition 13.15.3. In Situation 13.15.1.

(1) The *right derived functors of F* are the partial functors RF associated to cases (1) and (2) of Situation 13.15.1.
(2) The *left derived functors of F* are the partial functors LF associated to cases (3) and (4) of Situation 13.15.1.
(3) An object A of $\mathcal{A}$ is said to be *right acyclic for F*, or *acyclic for RF* if $A[0]$ computes RF.
(4) An object A of $\mathcal{A}$ is said to be *left acyclic for F*, or *acyclic for LF* if $A[0]$ computes RF.

The following few lemmas give some criteria for the existence of enough acyclics.

Lemma 13.15.4. *Let $\mathcal{A}$ be an abelian category. Let $\mathcal{I} \subset \mathrm{Ob}(\mathcal{A})$ be a subset such that every object of $\mathcal{A}$ is a subobject of an element of $\mathcal{I}$. For every $K^\bullet$ with $K^n = 0$ for $n < a$ there exists a quasi-isomorphism $K^\bullet \to I^\bullet$ with $I^n = 0$ for $n < a$, each $I^n \in \mathcal{I}$, and each $K^n \to I^n$ injective.*

Proof. Consider the following induction hypothesis IH_n: There are $I^j \in \mathcal{I}$, $j \leq n$ almost all zero, maps $d^j : I^j \to I^{j+1}$ for $j < n$ and injective maps $\alpha^j : K^j \to I^j$ for

$j \leq n$ such that the diagram

$$\begin{array}{ccccccccc} \ldots & \longrightarrow & K^{n-1} & \longrightarrow & K^n & \longrightarrow & K^{n+1} & \longrightarrow & \ldots \\ & & \downarrow{\scriptstyle \alpha} & & \downarrow{\scriptstyle \alpha} & & & & \\ \ldots & \longrightarrow & I^{n-1} & \longrightarrow & I^n & & & & \end{array}$$

is commutative, such that $d^j \circ d^{j-1} = 0$ for $j < n$ and such that α induces isomorphisms $H^j(K^\bullet) \to \mathrm{Ker}(d^j)/\mathrm{Im}(d^{j-1})$ for $j < n$. Note that this implies

$$\alpha(\mathrm{Im}(d_K^{n-1})) \subset \alpha(\mathrm{Ker}(d_K^n)) \cap \mathrm{Im}(d^{n-1}) \subset \alpha(K^n) \cap \mathrm{Im}(d^{n-1}). \tag{13.15.4.1}$$

If these inclusions are not equalities, then choose an injection

$$I^n \oplus K^n/\mathrm{Im}(d_K^{n-1}) \longrightarrow I$$

with $I \in \mathcal{I}$. Denote $\alpha' : K^n \to I$ the map obtained by composing $\alpha \oplus 1 : K^n \to I^n \oplus K^n/\mathrm{Im}(d_K^{n-1})$ with the displayed injection. Denote $d' : I^{n-1} \to I$ the composition $I^{n-1} \to I^n \to I$ of d^{n-1} by the inclusion of the first summand. Then $\alpha'(K^n) \cap \mathrm{Im}(d') = \alpha'(\mathrm{Im}(d_K^{n-1}))$ simply because the intersection of $\alpha'(K^n)$ with the first summand of $I^n \oplus K^n/\mathrm{Im}(d_K^{n-1})$ is equal to $\alpha'(\mathrm{Im}(d_K^{n-1}))$. Hence, after replacing I^n by I, α by α' and d^{n-1} by d' we may assume that we have equality in Equation (13.15.4.1). Once this is the case consider the solid diagram

$$\begin{array}{ccc} K^n/\mathrm{Ker}(d_K^n) & \longrightarrow & K^{n+1} \\ \downarrow & & \vdots \\ I^n/(\mathrm{Im}(d^{n-1}) + \alpha(\mathrm{Ker}(d_K^n))) & \dashrightarrow & M \end{array}$$

The horizontal arrow is injective by fiat and the vertical arrow is injective as we have equality in (13.15.4.1). Hence the push-out M of this diagram contains both K^{n+1} and $I^n/(\mathrm{Im}(d^{n-1}) + \alpha(\mathrm{Ker}(d_K^n)))$ as subobjects. Choose an injection $M \to I^{n+1}$ with $I^{n+1} \in \mathcal{I}$. By construction we get $d^n : I^n \to I^{n+1}$ and an injective map $\alpha^{n+1} : K^{n+1} \to I^{n+1}$. The equality in Equation (13.15.4.1) and the construction of d^n guarantee that $\alpha : H^n(K^\bullet) \to \mathrm{Ker}(d^n)/\mathrm{Im}(d^{n-1})$ is an isomorphism. In other words IH_{n+1} holds.

We finish the proof of by the following observations. First we note that IH_n is true for $n = a$ since we can just take $I^j = 0$ for $j < a$ and $K^a \to I^a$ an injection of K^a into an element of $\mathcal{A}$. Next, we note that in the proof of $IH_n \Rightarrow IH_{n+1}$ we only modified the object I^n, the map d^{n-1} and the map α^n. Hence we see that proceeding by induction we produce a complex $I^\bullet$ with $I^n = 0$ for $n < a$ consisting of objects from $\mathcal{I}$, and a termwise injective quasi-isomorphism $\alpha : K^\bullet \to I^\bullet$ as desired. □

Lemma 13.15.5. *Let $\mathcal{A}$ be an abelian category. Let $\mathcal{P} \subset \mathrm{Ob}(\mathcal{A})$ be a subset such that every object of $\mathcal{A}$ is a quotient of an element of $\mathcal{P}$. Then for every bounded above complex $K^\bullet$ there exists a quasi-isomorphism $P^\bullet \to K^\bullet$ with $P^\bullet$ bounded above, each $P^n \in \mathcal{P}$, and each $P^n \to K^n$ surjective.*

Proof. This lemma is dual to Lemma 13.15.4. □

Lemma 13.15.6. *In Situation 13.15.1. Let $\mathcal{I} \subset \mathrm{Ob}(\mathcal{A})$ be a subset with the following properties:*

(1) *every object of $\mathcal{A}$ is a subobject of an element of $\mathcal{I}$,*
(2) *for any short exact sequence $0 \to P \to Q \to R \to 0$ of $\mathcal{A}$ with $P, Q \in \mathcal{I}$, then $R \in \mathcal{I}$, and $0 \to F(P) \to F(Q) \to F(R) \to 0$ is exact.*

Then every object of $\mathcal{I}$ is acyclic for RF.

Proof. Pick $A \in \mathcal{I}$. Let $A[0] \to K^\bullet$ be a quasi-isomorphism with $L^\bullet$ bounded below. Then we can find a quasi-isomorphism $K^\bullet \to I^\bullet$ with $I^\bullet$ bounded below and each $I^n \in \mathcal{I}$, see Lemma 13.15.4. Hence we see that these resolutions are cofinal in the category $A[0]/\mathrm{Qis}^+(\mathcal{A})$. To finish the proof it therefore suffices to show that for any quasi-isomorphism $A[0] \to I^\bullet$ with $I^\bullet$ bounded above and $I^n \in \mathcal{I}$ we have $F(A)[0] \to F(I^\bullet)$ is a quasi-isomorphism. To see this suppose that $I^n = 0$ for $n < n_0$. Of course we may assume that $n_0 < 0$. Starting with $n = n_0$ we prove inductively that $\mathrm{Im}(d^{n-1}) = \mathrm{Ker}(d^n)$ and $\mathrm{Im}(d^{-1})$ are elements of $\mathcal{I}$ using property (2) and the exact sequences

$$0 \to \mathrm{Ker}(d^n) \to I^n \to \mathrm{Im}(d^n) \to 0.$$

Moreover, property (2) also guarantees that the complex

$$0 \to F(I^{n_0}) \to F(I^{n_0+1}) \to \ldots \to F(I^{-1}) \to F(\mathrm{Im}(d^{-1})) \to 0$$

is exact. The exact sequence $0 \to \mathrm{Im}(d^{-1}) \to I^0 \to I^0/\mathrm{Im}(d^{-1}) \to 0$ implies that $I^0/\mathrm{Im}(d^{-1})$ is an element of $\mathcal{I}$. The exact sequence $0 \to A \to I^0/\mathrm{Im}(d^{-1}) \to \mathrm{Im}(d^0) \to 0$ then implies that $\mathrm{Im}(d^0) = \mathrm{Ker}(d^1)$ is an elements of $\mathcal{I}$ and from then on one continues as before to show that $\mathrm{Im}(d^{n-1}) = \mathrm{Ker}(d^n)$ is an element of $\mathcal{I}$ for all $n > 0$. Applying F to each of the short exact sequences mentioned above and using (2) we observe that $F(A)[0] \to F(I^\bullet)$ is an isomorphism as desired. □

Lemma 13.15.7. *In Situation 13.15.1. Let $\mathcal{P} \subset \mathrm{Ob}(\mathcal{A})$ be a subset with the following properties:*

(1) *every object of $\mathcal{A}$ is a quotient of an element of $\mathcal{P}$,*
(2) *for any short exact sequence $0 \to P \to Q \to R \to 0$ of $\mathcal{A}$ with $Q, R \in \mathcal{P}$, then $P \in \mathcal{P}$, and $0 \to F(P) \to F(Q) \to F(R) \to 0$ is exact.*

Then every object of $\mathcal{P}$ is acyclic for LF.

Proof. Dual to the proof of Lemma 13.15.6. □

Proposition 13.15.8. *In Situation 13.15.1.*

(1) *If every object of $\mathcal{A}$ injects into an object acyclic for RF, then RF is defined on all of $K^+(\mathcal{A})$ and we obtain an exact functor*

$$RF : D^+(\mathcal{B}) \longrightarrow D^+(\mathcal{A})$$

see (13.14.9.1). Moreover, any bounded below complex $K^\bullet$ whose terms are acyclic for RF computes RF.
(2) *If every object of $\mathcal{A}$ is quotient of an object acyclic for LF, then LF is defined on all of $K^-(\mathcal{A})$ and we obtain an exact functor*

$$LF : D^-(\mathcal{B}) \longrightarrow D^-(\mathcal{A})$$

see (13.14.9.1). Moreover, any bounded above complex $K^\bullet$ whose terms are acyclic for LF computes LF.

Proof. Suppose every object of $\mathcal{A}$ injects into an object acyclic for RF. Let $\mathcal{I}$ be the set of objects acyclic for RF. Let $K^\bullet$ be a bounded below complex with $K^n \in \mathcal{I}$. By Lemma 13.15.4 the quasi-isomorphisms $\alpha : K^\bullet \to I^\bullet$ with $I^\bullet$ bounded below and $I^n \in \mathcal{I}$ are cofinal in the category $K^\bullet/\mathrm{Qis}^+(\mathcal{A})$. Hence in order to show that $K^\bullet$ computes RF it suffices to show that $F(K^\bullet) \to F(I^\bullet)$ is an isomorphism. Note that $C(\alpha)^\bullet$ is an acyclic bounded below complex all of whose terms are in $\mathcal{I}$. Hence it suffices to show: given an acyclic bounded below complex $I^\bullet$ all of whose terms are in $\mathcal{I}$ the complex $F(I^\bullet)$ is acyclic.

Say $I^n = 0$ for $n < n_0$. Then we break $I^\bullet$ into short exact sequences $0 \to \mathrm{Im}(d^n) \to I^{n+1} \to \mathrm{Im}(d^{n+1}) \to 0$ for $n \geq n_0$. These sequences induce distinguished triangles $(\mathrm{Im}(d^n), I^{n+1}, \mathrm{Im}(d^{n+1}))$ by Lemma 13.11.1. This implies inductively that each $\mathrm{Im}(d^n)$ is acyclic for RF by Lemma 13.14.12. Moreover, the long exact cohomology sequences (13.10.1.1) associated to the distinguished triangles $(F(\mathrm{Im}(d^n)), F(I^{n+1}), F(\mathrm{Im}(d^{n+1})))$ of $D^+(\mathcal{B})$ imply that

$$0 \to F(\mathrm{Im}(d^n)) \to F(I^{n+1}) \to F(\mathrm{Im}(d^{n+1})) \to 0$$

is short exact, and this in turn proves that $F(I^\bullet)$ is exact.

Finally, since by Lemma 13.15.4 every object of $K^+(\mathcal{A})$ is quasi-isomorphic to such a bounded below complex with terms in $\mathcal{I}$ we see that RF is everywhere defined, see Lemma 13.14.14. The proof in the case of LF is dual. □

13.16. Higher derived functors

The following simple lemma shows that right derived functors "move to the right".

Lemma 13.16.1. *Let $F : \mathcal{A} \to \mathcal{B}$ be an additive functor between abelian categories. Let $K^\bullet \in K^+(\mathcal{A})$ and $a \in \mathbf{Z}$.*

1. *If $H^i(K^\bullet) = 0$ for all $i < a$ and RF is defined at $K^\bullet$, then $H^i(RF(K^\bullet)) = 0$ for all $i < a$.*
2. *If RF is defined at $K^\bullet$ and $\tau_{\leq a}K^\bullet$, then $H^i(RF(\tau_{\leq a}K^\bullet)) = H^i(RF(K^\bullet))$ for all $i \leq a$.*

Proof. Assume $K^\bullet$ satisfies the assumptions of (1). Let $K^\bullet \to L^\bullet$ be any quasi-isomorphism. Then it is also true that $K^\bullet \to \tau_{\geq a}L^\bullet$ is a quasi-isomorphism by our assumption on $K^\bullet$. Hence in the category $K^\bullet/\mathrm{Qis}^+(\mathcal{A})$ the quasi-isomorphisms $s : K^\bullet \to L^\bullet$ with $L^n = 0$ for $n < a$ are cofinal. Thus RF is the value of the essentially constant ind-object $F(L^\bullet)$ for these s it follows that $H^i(RF(K^\bullet)) = 0$ for $i < 0$.

To prove (2) we use the distinguished triangle

$$\tau_{\leq a}K^\bullet \to K^\bullet \to \tau_{\geq a+1}K^\bullet \to (\tau_{\leq a}K^\bullet)[1]$$

of Remark 13.11.4 to conclude via Lemma 13.14.6 that RF is defined at $\tau_{\geq a+1}K^\bullet$ as well and that we have a distinguished triangle

$$RF(\tau_{\leq a}K^\bullet) \to RF(K^\bullet) \to RF(\tau_{\geq a+1}K^\bullet) \to RF(\tau_{\leq a}K^\bullet)[1]$$

in $D(\mathcal{B})$. By part (1) we see that $RF(\tau_{\geq a+1}K^\bullet)$ has vanishing cohomology in degrees $< a+1$. The long exact cohomology sequence of this distinguished triangle then shows what we want. □

Definition 13.16.2. Let $F : \mathcal{A} \to \mathcal{B}$ be an additive functor between abelian categories. Assume $RF : D^+(\mathcal{A}) \to D^+(\mathcal{B})$ is everywhere defined. Let $i \in \mathbf{Z}$. The *ith right derived functor* R^iF *of* F is the functor

$$R^iF = H^i \circ RF : \mathcal{A} \longrightarrow \mathcal{B}$$

The following lemma shows that it really does not make a lot of sense to take the right derived functor unless the functor is left exact.

Lemma 13.16.3. *Let $F : \mathcal{A} \to \mathcal{B}$ be an additive functor between abelian categories and assume $RF : D^+(\mathcal{A}) \to D^+(\mathcal{B})$ is everywhere defined.*

1. *We have $R^iF = 0$ for $i < 0$,*
2. *R^0F is left exact,*
3. *the map $F \to R^0F$ is an isomorphism if and only if F is left exact.*

Proof. Let A be an object of $\mathcal{A}$. Let $A[0] \to K^\bullet$ be any quasi-isomorphism. Then it is also true that $A[0] \to \tau_{\geq 0}K^\bullet$ is a quasi-isomorphism. Hence in the category $A[0]/\mathrm{Qis}^+(\mathcal{A})$ the quasi-isomorphisms $s : A[0] \to K^\bullet$ with $K^n = 0$ for $n < 0$ are cofinal. Thus it is clear that $H^i(RF(A[0])) = 0$ for $i < 0$. Moreover, for such an s the sequence

$$0 \to A \to K^0 \to K^1$$

is exact. Hence if F is left exact, then $0 \to F(A) \to F(K^0) \to F(K^1)$ is exact as well, and we see that $F(A) \to H^0(F(K^\bullet))$ is an isomorphism for every $s : A[0] \to K^\bullet$ as above which implies that $H^0(RF(A[0])) = F(A)$.

Let $0 \to A \to B \to C \to 0$ be a short exact sequence of $\mathcal{A}$. By Lemma 13.11.1 we obtain a distinguished triangle $(A[0], B[0], C[0], a, b, c)$ in $K^+(\mathcal{A})$. From the long exact cohomology sequence (and the vanishing for $i < 0$ proved above) we deduce that $0 \to R^0F(A) \to R^0F(B) \to R^0F(C)$ is exact. Hence R^0F is left exact. Of course this also proves that if $F \to R^0F$ is an isomorphism, then F is left exact. □

Lemma 13.16.4. *Let $F : \mathcal{A} \to \mathcal{B}$ be an additive functor between abelian categories and assume $RF : D^+(\mathcal{A}) \to D^+(\mathcal{B})$ is everywhere defined. Let A be an object of $\mathcal{A}$.*

1. *A is right acyclic for F if and only if $F(A) \to R^0F(A)$ is an isomorphism and $R^iF(A) = 0$ for all $i > 0$,*
2. *if F is left exact, then A is right acyclic for F if and only if $R^iF(A) = 0$ for all $i > 0$.*

Proof. If A is right acyclic for F, then $RF(A[0]) = F(A)[0]$ and in particular $F(A) \to R^0F(A)$ is an isomorphism and $R^iF(A) = 0$ for $i \neq 0$. Conversely, if $F(A) \to R^0F(A)$ is an isomorphism and $R^iF(A) = 0$ for all $i > 0$ then $F(A[0]) \to RF(A[0])$ is a quasi-isomorphism by Lemma 13.16.3 part (1) and hence A is acyclic. If F is left exact then $F = R^0F$, see Lemma 13.16.3. □

Lemma 13.16.5. *Let $F : \mathcal{A} \to \mathcal{B}$ be a left exact functor between abelian categories and assume $RF : D^+(\mathcal{A}) \to D^+(\mathcal{B})$ is everywhere defined. Let $0 \to A \to B \to C \to 0$ be a short exact sequence of $\mathcal{A}$.*

1. *If A and C are right acyclic for F then so is B.*
2. *If A and B are right acyclic for F then so is C.*
3. *If B and C are right acyclic for F and $F(B) \to F(C)$ is surjective then A is right acyclic for F.*

In each of the three cases

$$0 \to F(A) \to F(B) \to F(C) \to 0$$

is a short exact sequence of $\mathcal{B}$.

Proof. By Lemma 13.11.1 we obtain a distinguished triangle $(A[0], B[0], C[0], a, b, c)$ in $K^+(\mathcal{A})$. As RF is an exact functor and since $R^iF = 0$ for $i < 0$ and $R^0F = F$ (Lemma 13.16.3) we obtain an exact cohomology sequence

$$0 \to F(A) \to F(B) \to F(C) \to R^1F(A) \to \ldots$$

in the abelian category $\mathcal{B}$. Thus the lemma follows from the characterization of acyclic objects in Lemma 13.16.4. □

Lemma 13.16.6. *Let $F : \mathcal{A} \to \mathcal{B}$ be an additive functor between abelian categories and assume $RF : D^+(\mathcal{A}) \to D^+(\mathcal{B})$ is everywhere defined.*

(1) *The functors R^iF, $i \geq 0$ come equipped with a canonical structure of a δ-functor from $\mathcal{A} \to \mathcal{B}$, see Homology, Definition 12.11.1.*
(2) *If every object of $\mathcal{A}$ is a subobject of a right acyclic object for F, then $\{R^iF, \delta\}_{i\geq 0}$ is a universal δ-functor, see Homology, Definition 12.11.3.*

Proof. The functor $\mathcal{A} \to \mathrm{Comp}^+(\mathcal{A})$, $A \mapsto A[0]$ is exact. The functor $\mathrm{Comp}^+(\mathcal{A}) \to D^+(\mathcal{A})$ is a δ-functor, see Lemma 13.11.1. The functor $RF : D^+(\mathcal{A}) \to D^+(\mathcal{B})$ is exact. Finally, the functor $H^0 : D^+(\mathcal{B}) \to \mathcal{B}$ is a homological functor, see Definition 13.10.3. Hence we get the structure of a δ-functor from Lemma 13.4.20 and Lemma 13.4.19. Part (2) follows from Homology, Lemma 12.11.4 and the description of acyclics in Lemma 13.16.4. □

Lemma 13.16.7 (Leray's acyclicity lemma)**.** *Let $F : \mathcal{A} \to \mathcal{B}$ be an additive functor between abelian categories and assume $RF : D^+(\mathcal{A}) \to D^+(\mathcal{B})$ is everywhere defined. Let $A^\bullet$ be a bounded below complex of F-acyclic objects. The canonical map*

$$F(A^\bullet) \longrightarrow RF(A^\bullet)$$

is an isomorphism in $D^+(\mathcal{B})$, i.e., $A^\bullet$ computes RF.

Proof. First we claim the lemma holds for a bounded complex of acyclic objects. Namely, it holds for complexes with at most one nonzero object by definition. Suppose that $A^\bullet$ is a complex with $A^n = 0$ for $n \notin [a, b]$. Using the "stupid" truncations we obtain a termwise split short exact sequence of complexes

$$0 \to \sigma_{\geq a+1}A^\bullet \to A^\bullet \to \sigma_{\leq a}A^\bullet \to 0$$

see Homology, Section 12.13. Thus a distinguished triangle $(\sigma_{\geq a+1}A^\bullet, A^\bullet, \sigma_{\leq a}A^\bullet)$. By induction hypothesis the two outer complexes compute RF. Then the middle one does too by Lemma 13.14.12.

Suppose that $A^\bullet$ is a bounded below complex of acyclic objects. To show that $F(A) \to RF(A)$ is an isomorphism in $D^+(\mathcal{B})$ it suffices to show that $H^i(F(A)) \to H^i(RF(A))$ is an isomorphism for all i. Pick i. Consider the termwise split short exact sequence of complexes

$$0 \to \sigma_{\geq i+2}A^\bullet \to A^\bullet \to \sigma_{\leq i+1}A^\bullet \to 0.$$

Note that this induces a termwise split short exact sequence

$$0 \to \sigma_{\geq i+2}F(A^\bullet) \to F(A^\bullet) \to \sigma_{\leq i+1}F(A^\bullet) \to 0.$$

Hence we get distinguished triangles

$$(\sigma_{\geq i+2}A^\bullet, A^\bullet, \sigma_{\leq i+1}A^\bullet)$$
$$(\sigma_{\geq a+1}F(A^\bullet), F(A^\bullet), \sigma_{\leq a}F(A^\bullet))$$
$$(RF(\sigma_{\geq a+1}A^\bullet), RF(A^\bullet), RF(\sigma_{\leq a}A^\bullet))$$

Using the last two we obtain a map of exact sequences

$$\begin{array}{ccccccc}
H^i(\sigma_{\geq i+2}F(A^\bullet)) & \longrightarrow & H^i(F(A^\bullet)) & \longrightarrow & H^i(\sigma_{\leq i+1}F(A^\bullet)) & \longrightarrow & H^{i+1}(\sigma_{\geq i+2}F(A^\bullet)) \\
\downarrow & & \downarrow{\scriptstyle \alpha} & & \downarrow{\scriptstyle \beta} & & \downarrow \\
R^iF(\sigma_{\geq i+2}A^\bullet) & \longrightarrow & R^iF(A^\bullet) & \longrightarrow & R^iF(\sigma_{\leq i+1}A^\bullet) & \longrightarrow & R^{i+1}F(\sigma_{\geq i+2}A^\bullet)
\end{array}$$

By the results of the first paragraph the map β is an isomorphism. By inspection the objects on the upper left and the upper right are zero. Hence to finish the proof we have to show that $R^iF(\sigma_{\geq i+2}A^\bullet) = 0$ and $R^{i+1}F(\sigma_{\geq i+2}A^\bullet) = 0$. This follows immediately from Lemma 13.16.1. □

Lemma 13.16.8. *Let $F : \mathcal{A} \to \mathcal{B}$ be an exact functor of abelian categories. Then*

(1) *every object of $\mathcal{A}$ is right acyclic for F,*
(2) *$RF : D^+(\mathcal{A}) \to D^+(\mathcal{A})$ is everywhere defined,*
(3) *$RF : D(\mathcal{A}) \to D(\mathcal{A})$ is everywhere defined,*
(4) *every complex computes RF, in other words, the canonical map $F(K^\bullet) \to RF(K^\bullet)$ is an isomorphism for all complexes, and*
(5) *$R^iF = 0$ for $i \neq 0$.*

Proof. This is true because F transforms acyclic complexes into acyclic complexes and quasi-isomorphisms into quasi-isomorphisms. Details omitted. □

13.17. Injective resolutions

In this section we prove some lemmas regarding the existence of injective resolutions in abelian categories having enough injectives.

Definition 13.17.1. Let $\mathcal{A}$ be an abelian category. Let $A \in \mathrm{Ob}(\mathcal{A})$. An *injective resolution of A* is a complex $I^\bullet$ together with a map $A \to I^0$ such that:

(1) We have $I^n = 0$ for $n < 0$.
(2) Each I^n is an injective object of $\mathcal{A}$.
(3) The map $A \to I^0$ is an isomorphism onto $\mathrm{Ker}(d^0)$.
(4) We have $H^i(I^\bullet) = 0$ for $i > 0$.

Hence $A[0] \to I^\bullet$ is a quasi-isomorphism. In other words the complex

$$\ldots \to 0 \to A \to I^0 \to I^1 \to \ldots$$

is acyclic. Let $K^\bullet$ be a complex in $\mathcal{A}$. An *injective resolution of $K^\bullet$* is a complex $I^\bullet$ together with a map $\alpha : K^\bullet \to I^\bullet$ of complexes such that

(1) We have $I^n = 0$ for $n \ll 0$, i.e., $I^\bullet$ is bounded below.
(2) Each I^n is an injective object of $\mathcal{A}$.
(3) The map $\alpha : K^\bullet \to I^\bullet$ is a quasi-isomorphism.

In other words an injective resolution $K^\bullet \to I^\bullet$ gives rise to a diagram

$$\begin{array}{ccccccccc} \ldots & \longrightarrow & K^{n-1} & \longrightarrow & K^n & \longrightarrow & K^{n+1} & \longrightarrow & \ldots \\ & & \downarrow & & \downarrow & & \downarrow & & \\ \ldots & \longrightarrow & I^{n-1} & \longrightarrow & I^n & \longrightarrow & I^{n+1} & \longrightarrow & \ldots \end{array}$$

which induces an isomorphism on cohomology objects in each degree. An injective resolution of an object A of $\mathcal{A}$ is almost the same thing as an injective resolution of the complex $A[0]$.

Lemma 13.17.2. *Let $\mathcal{A}$ be an abelian category. Let $K^\bullet$ be a complex of $\mathcal{A}$.*

1. *If $K^\bullet$ has an injective resolution then $H^n(K^\bullet) = 0$ for $n \ll 0$.*
2. *If $H^n(K^\bullet) = 0$ for all $n \ll 0$ then there exists a quasi-isomorphism $K^\bullet \to L^\bullet$ with $L^\bullet$ bounded below.*

Proof. Omitted. For the second statement use $L^\bullet = \tau_{\geq n} K^\bullet$ for some $n \ll 0$. See Homology, Section 12.13 for the definition of the truncation $\tau_{\geq n}$. □

Lemma 13.17.3. *Let $\mathcal{A}$ be an abelian category. Assume $\mathcal{A}$ has enough injectives.*

1. *Any object of $\mathcal{A}$ has an injective resolution.*
2. *If $H^n(K^\bullet) = 0$ for all $n \ll 0$ then $K^\bullet$ has an injective resolution.*
3. *If $K^\bullet$ is a complex with $K^n = 0$ for $n < a$, then there exists an injective resolution $\alpha : K^\bullet \to I^\bullet$ with $I^n = 0$ for $n < a$ such that each $\alpha^n : K^n \to I^n$ is injective.*

Proof. Proof of (1). First choose an injection $A \to I^0$ of A into an injective object of $\mathcal{A}$. Next, choose an injection $I_0/A \to I^1$ into an injective object of $\mathcal{A}$. Denote d^0 the induced map $I^0 \to I^1$. Next, choose an injection $I^1/\mathrm{Im}(d^0) \to I^2$ into an injective object of $\mathcal{A}$. Denote d^1 the induced map $I^1 \to I^2$. And so on. By Lemma 13.17.2 part (2) follows from part (3). Part (3) is a special case of Lemma 13.15.4. □

Lemma 13.17.4. *Let $\mathcal{A}$ be an abelian category. Let $K^\bullet$ be an acyclic complex. Let $I^\bullet$ be bounded below and consisting of injective objects. Any morphism $K^\bullet \to I^\bullet$ is homotopic to zero.*

Proof. Let $\alpha : K^\bullet \to I^\bullet$ be a morphism of complexes. Assume that $\alpha^j = 0$ for $j < n$. We will show that there exists a morphism $h : K^{n+1} \to I^n$ such that $\alpha^n = h \circ d$. Thus α will be homotopic to the morphism of complexes β defined by

$$\beta^j = \begin{cases} 0 & \text{if} \quad j \leq n \\ \alpha^{n+1} - d \circ h & \text{if} \quad j = n+1 \\ \alpha^j & \text{if} \quad j > n+1 \end{cases}$$

This will clearly prove the lemma (by induction). To prove the existence of h note that $\alpha^n|_{d^{n-1}(K^{n-1})} = 0$ since $\alpha^{n-1} = 0$. Since $K^\bullet$ is acyclic we have $d^{n-1}(K^{n-1}) = \mathrm{Ker}(K^n \to K^{n+1})$. Hence we can think of α^n as a map into I^n defined on the subobject $\mathrm{Im}(K^n \to K^{n+1})$ of K^{n+1}. By injectivity of the object I^n we can extend this to a map $h : K^{n+1} \to I^n$ as desired. □

Remark 13.17.5. Let $\mathcal{A}$ be an abelian category. Using the fact that $K(\mathcal{A})$ is a triangulated category we may use Lemma 13.17.4 to obtain proofs of some of the

lemmas below which are usually proved by chasing through diagrams. Namely, suppose that $\alpha : K^\bullet \to L^\bullet$ is a quasi-isomorphism of complexes. Then

$$(K^\bullet, L^\bullet, C(\alpha)^\bullet, \alpha, i, -p)$$

is a distinguished triangle in $K(\mathcal{A})$ (Lemma 13.8.14) and $C(f)^\bullet$ is an acyclic complex (Lemma 13.10.2). Next, let $I^\bullet$ be a bounded below complex of injective objects. Then

$$\begin{aligned} &\mathrm{Hom}_{K(\mathcal{A})}(C(\alpha)^\bullet, I^\bullet) \longrightarrow \mathrm{Hom}_{K(\mathcal{A})}(L^\bullet, I^\bullet) \longrightarrow \mathrm{Hom}_{K(\mathcal{A})}(K^\bullet, I^\bullet) \\ &\qquad \mathrm{Hom}_{K(\mathcal{A})}(C(\alpha)^\bullet[-1], I^\bullet) \end{aligned}$$

is an exact sequence of abelian groups, see Lemma 13.4.2. At this point Lemma 13.17.4 guarantees that the outer two groups are zero and hence $\mathrm{Hom}_{K(\mathcal{A})}(L^\bullet, I^\bullet) = \mathrm{Hom}_{K(\mathcal{A})}(K^\bullet, I^\bullet)$.

Lemma 13.17.6. *Let $\mathcal{A}$ be an abelian category. Consider a solid diagram*

$$\begin{array}{ccc} K^\bullet & \xrightarrow{\alpha} & L^\bullet \\ {\scriptstyle\gamma}\downarrow & \swarrow_{\beta} & \\ I^\bullet & & \end{array}$$

where $I^\bullet$ is bounded below and consists of injective objects, and α is a quasi-isomorphism.

(1) *There exists a map of complexes β making the diagram commute up to homotopy.*
(2) *If α is injective in every degree then we can find a β which makes the diagram commute.*

Proof. The "correct" proof of part (1) is explained in Remark 13.17.5. We also give a direct proof here.

We first show that (2) implies (1). Namely, let $\tilde{\alpha} : K \to \tilde{L}^\bullet$, π, s be as in Lemma 13.8.6. Since $\tilde{\alpha}$ is injective by (2) there exists a morphism $\tilde{\beta} : \tilde{L}^\bullet \to I^\bullet$ such that $\gamma = \tilde{\beta} \circ \tilde{\alpha}$. Set $\beta = \tilde{\beta} \circ s$. Then we have

$$\beta \circ \alpha = \tilde{\beta} \circ s \circ \pi \circ \tilde{\alpha} \sim \tilde{\beta} \circ \tilde{\alpha} = \gamma$$

as desired.

Assume that $\alpha : K^\bullet \to L^\bullet$ is injective. Suppose we have already defined β in all degrees $\leq n-1$ compatible with differentials and such that $\gamma^j = \beta^j \circ \alpha^j$ for all $j \leq n-1$. Consider the commutative solid diagram

$$\begin{array}{ccc} K^{n-1} & \longrightarrow & K^n \\ \downarrow{\scriptstyle\alpha} & & \downarrow{\scriptstyle\alpha} \\ L^{n-1} & \longrightarrow & L^n \\ \downarrow{\scriptstyle\beta} & & \vdots \\ I^{n-1} & \longrightarrow & I^n \end{array} \qquad \gamma : K^{n-1} \to I^{n-1}, \quad \gamma : K^n \to I^n$$

Thus we see that the dotted arrow is prescribed on the subobjects $\alpha(K^n)$ and $d^{n-1}(L^{n-1})$. Moreover, these two arrows agree on $\alpha(d^{n-1}(K^{n-1}))$. Hence if

$$\alpha(d^{n-1}(K^{n-1})) = \alpha(K^n) \cap d^{n-1}(L^{n-1}) \tag{13.17.6.1}$$

then these morphisms glue to a morphism $\alpha(K^n) + d^{n-1}(L^{n-1}) \to I^n$ and, using the injectivity of I^n, we can extend this to a morphism from all of L^n into I^n. After this by induction we get the morphism β for all n simultaneously (note that we can set $\beta^n = 0$ for all $n \ll 0$ since $I^\bullet$ is bounded below – in this way starting the induction).

It remains to prove the equality (13.17.6.1). The reader is encouraged to argue this for themselves with a suitable diagram chase. Nonetheless here is our argument. Note that the inclusion $\alpha(d^{n-1}(K^{n-1})) \subset \alpha(K^n) \cap d^{n-1}(L^{n-1})$ is obvious. Take an object T of $\mathcal{A}$ and a morphism $x : T \to L^n$ whose image is contained in the subobject $\alpha(K^n) \cap d^{n-1}(L^{n-1})$. Since α is injective we see that $x = \alpha \circ x'$ for some $x' : T \to K^n$. Moreover, since x lies in $d^{n-1}(L^{n-1})$ we see that $d^n \circ x = 0$. Hence using injectivity of α again we see that $d^n \circ x' = 0$. Thus x' gives a morphism $[x'] : T \to H^n(K^\bullet)$. On the other hand the corresponding map $[x] : T \to H^n(L^\bullet)$ induced by x is zero by assumption. Since α is a quasi-isomorphism we conclude that $[x'] = 0$. This of course means exactly that the image of x' is contained in $d^{n-1}(K^{n-1})$ and we win. □

Lemma 13.17.7. *Let $\mathcal{A}$ be an abelian category. Consider a solid diagram*

$$\begin{array}{ccc} K^\bullet & \xrightarrow{\alpha} & L^\bullet \\ {\scriptstyle\gamma}\downarrow & \swarrow_{\beta_i} & \\ I^\bullet & & \end{array}$$

where $I^\bullet$ is bounded below and consists of injective objects, and α is a quasi-isomorphism. Any two morphisms β_1, β_2 making the diagram commute up to homotopy are homotopic.

Proof. This follows from Remark 13.17.5. We also give a direct argument here.

Let $\tilde{\alpha} : K \to \tilde{L}^\bullet$, π, s be as in Lemma 13.8.6. If we can show that $\beta_1 \circ \pi$ is homotopic to $\beta_2 \circ \pi$, then we deduce that $\beta_1 \sim \beta_2$ because $\pi \circ s$ is the identity. Hence we may assume $\alpha^n : K^n \to L^n$ is the inclusion of a direct summand for all n. Thus we get a short exact sequence of complexes

$$0 \to K^\bullet \to L^\bullet \to M^\bullet \to 0$$

which is termwise split and such that $M^\bullet$ is acyclic. We choose splittings $L^n = K^n \oplus M^n$, so we have $\beta_i^n : K^n \oplus M^n \to I^n$ and $\gamma^n : K^n \to I^n$. In this case the condition on β_i is that there are morphisms $h_i^n : K^n \to I^{n-1}$ such that

$$\gamma^n - \beta_i^n|_{K^n} = d \circ h_i^n + h_i^{n+1} \circ d$$

Thus we see that

$$\beta_1^n|_{K^n} - \beta_2^n|_{K^n} = d \circ (h_1^n - h_2^n) + (h_1^{n+1} - h_2^{n+1}) \circ d$$

Consider the map $h^n : K^n \oplus M^n \to I^{n-1}$ which equals $h_1^n - h_2^n$ on the first summand and zero on the second. Then we see that

$$\beta_1^n - \beta_2^n - (d \circ h^n + h^{n+1}) \circ d)$$

is a morphism of complexes $L^\bullet \to I^\bullet$ which is identically zero on the subcomplex $K^\bullet$. Hence it factors as $L^\bullet \to M^\bullet \to I^\bullet$. Thus the result of the lemma follows from Lemma 13.17.4. □

Lemma 13.17.8. *Let $\mathcal{A}$ be an abelian category. Let $I^\bullet$ be bounded below complex consisting of injective objects. Let $L^\bullet \in K(\mathcal{A})$. Then*

$$\mathrm{Mor}_{K(\mathcal{A})}(L^\bullet, I^\bullet) = \mathrm{Mor}_{D(\mathcal{A})}(L^\bullet, I^\bullet).$$

Proof. Let a be an element of the right hand side. We may represent $a = \gamma\alpha^{-1}$ where $\alpha : K^\bullet \to L^\bullet$ is a quasi-isomorphism and $\gamma : K^\bullet \to I^\bullet$ is a map of complexes. By Lemma 13.17.6 we can find a morphism $\beta : L^\bullet \to I^\bullet$ such that $\beta \circ \alpha$ is homotopic to γ. This proves that the map is surjective. Let b be an element of the left hand side which maps to zero in the right hand side. Then b is the homotopy class of a morphism $\beta : L^\bullet \to I^\bullet$ such that there exists a quasi-isomorphism $\alpha : K^\bullet \to L^\bullet$ with $\beta \circ \alpha$ homotopic to zero. Then Lemma 13.17.7 shows that β is homotopic to zero also, i.e., $b = 0$. □

Lemma 13.17.9. *Let $\mathcal{A}$ be an abelian category. Assume $\mathcal{A}$ has enough injectives. For any short exact sequence $0 \to A^\bullet \to B^\bullet \to C^\bullet \to 0$ of $Comp^+(\mathcal{A})$ there exists a commutative diagram in $Comp^+(\mathcal{A})$*

$$\begin{array}{ccccccccc} 0 & \longrightarrow & A^\bullet & \longrightarrow & B^\bullet & \longrightarrow & C^\bullet & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & I_1^\bullet & \longrightarrow & I_2^\bullet & \longrightarrow & I_3^\bullet & \longrightarrow & 0 \end{array}$$

where the vertical arrows are injective resolutions and the rows are short exact sequences of complexes. In fact, given any injective resolution $A^\bullet \to I^\bullet$ we may assume $I_1^\bullet = I^\bullet$.

Proof. Step 1. Choose an injective resolution $A^\bullet \to I^\bullet$ (see Lemma 13.17.3) or use the given one. Recall that $\mathrm{Comp}^+(\mathcal{A})$ is an abelian category, see Homology, Lemma 12.12.9. Hence we may form the pushout along the injective map $A^\bullet \to I^\bullet$ to get

$$\begin{array}{ccccccccc} 0 & \longrightarrow & A^\bullet & \longrightarrow & B^\bullet & \longrightarrow & C^\bullet & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & I^\bullet & \longrightarrow & E^\bullet & \longrightarrow & C^\bullet & \longrightarrow & 0 \end{array}$$

Note that the lower short exact sequence is termwise split, see Homology, Lemma 12.23.2. Hence it suffices to prove the lemma when $0 \to A^\bullet \to B^\bullet \to C^\bullet \to 0$ is termwise split.

Step 2. Choose splittings. In other words, write $B^n = A^n \oplus C^n$. Denote $\delta : C^\bullet \to A^\bullet[1]$ the morphism as in Homology, Lemma 12.14.10. Choose injective resolutions $f_1 : A^\bullet \to I_1^\bullet$ and $f_3 : C^\bullet \to I_3^\bullet$. (If $A^\bullet$ is a complex of injectives, then use $I_1^\bullet = A^\bullet$.) We may assume f_3 is injective in every degree. By Lemma 13.17.6 we may find a morphism $\delta' : I_3^\bullet \to I_1^\bullet[1]$ such that $\delta' \circ f_3 = f_1[1] \circ \delta$ (equality of morphisms of complexes). Set $I_2^n = I_1^n \oplus I_3^n$. Define

$$d_{I_2}^n = \begin{pmatrix} d_{I_1}^n & (\delta')^n \\ 0 & d_{I_3}^n \end{pmatrix}$$

and define the maps $B^n \to I_2^n$ to be given as the sum of the maps $A^n \to I_1^n$ and $C^n \to I_3^n$. Everything is clear. □

13.18. Projective resolutions

This section is dual to Section 13.17. We give definitions and state results, but we do not reprove the lemmas.

Definition 13.18.1. Let $\mathcal{A}$ be an abelian category. Let $A \in \mathrm{Ob}(\mathcal{A})$. An *projective resolution of* A is a complex $P^\bullet$ together with a map $P^0 \to A$ such that:

(1) We have $P^n = 0$ for $n > 0$.
(2) Each P^n is an projective object of $\mathcal{A}$.
(3) The map $P^0 \to A$ induces an isomorphism $\mathrm{Coker}(d^{-1}) \to A$.
(4) We have $H^i(P^\bullet) = 0$ for $i < 0$.

Hence $P^\bullet \to A[0]$ is a quasi-isomorphism. In other words the complex

$$\ldots \to P^{-1} \to P^0 \to A \to 0 \to \ldots$$

is acyclic. Let $K^\bullet$ be a complex in $\mathcal{A}$. An *projective resolution of* $K^\bullet$ is a complex $P^\bullet$ together with a map $\alpha : P^\bullet \to K^\bullet$ of complexes such that

(1) We have $P^n = 0$ for $n \gg 0$, i.e., $P^\bullet$ is bounded above.
(2) Each P^n is an projective object of $\mathcal{A}$.
(3) The map $\alpha : P^\bullet \to K^\bullet$ is a quasi-isomorphism.

Lemma 13.18.2. *Let $\mathcal{A}$ be an abelian category. Let $K^\bullet$ be a complex of $\mathcal{A}$.*

(1) *If $K^\bullet$ has a projective resolution then $H^n(K^\bullet) = 0$ for $n \gg 0$.*
(2) *If $H^n(K^\bullet) = 0$ for $n \gg 0$ then there exists a quasi-isomorphism $L^\bullet \to K^\bullet$ with $L^\bullet$ bounded above.*

Proof. Dual to Lemma 13.17.2. □

Lemma 13.18.3. *Let $\mathcal{A}$ be an abelian category. Assume $\mathcal{A}$ has enough projectives.*

(1) *Any object of $\mathcal{A}$ has a projective resolution.*
(2) *If $H^n(K^\bullet) = 0$ for all $n \gg 0$ then $K^\bullet$ has a projective resolution.*
(3) *If $K^\bullet$ is a complex with $K^n = 0$ for $n > a$, then there exists a projective resolution $\alpha : P^\bullet \to K^\bullet$ with $P^n = 0$ for $n > a$ such that each $\alpha^n : P^n \to K^n$ is surjective.*

Proof. Dual to Lemma 13.17.3. □

Lemma 13.18.4. *Let $\mathcal{A}$ be an abelian category. Let $K^\bullet$ be an acyclic complex. Let $P^\bullet$ be bounded above and consisting of projective objects. Any morphism $P^\bullet \to K^\bullet$ is homotopic to zero.*

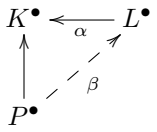
Proof. Dual to Lemma 13.17.4. □

Remark 13.18.5. Let $\mathcal{A}$ be an abelian category. Suppose that $\alpha : K^\bullet \to L^\bullet$ is a quasi-isomorphism of complexes. Let $P^\bullet$ be a bounded above complex of projectives. Then

$$\mathrm{Hom}_{K(\mathcal{A})}(P^\bullet, K^\bullet) \longrightarrow \mathrm{Hom}_{K(\mathcal{A})}(P^\bullet, L^\bullet)$$

is an isomorphism. This is dual to Remark 13.17.5.

Lemma 13.18.6. *Let $\mathcal{A}$ be an abelian category. Consider a solid diagram*

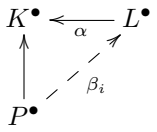


where $P^\bullet$ is bounded above and consists of projective objects, and α is a quasi-isomorphism.

(1) *There exists a map of complexes β making the diagram commute up to homotopy.*
(2) *If α is surjective in every degree then we can find a β which makes the diagram commute.*

Proof. Dual to Lemma 13.17.6. □

Lemma 13.18.7. *Let $\mathcal{A}$ be an abelian category. Consider a solid diagram*



where $P^\bullet$ is bounded above and consists of projective objects, and α is a quasi-isomorphism. Any two morphisms β_1, β_2 making the diagram commute up to homotopy are homotopic.

Proof. Dual to Lemma 13.17.7. □

Lemma 13.18.8. *Let $\mathcal{A}$ be an abelian category. Let $P^\bullet$ be bounded above complex consisting of projective objects. Let $L^\bullet \in K(\mathcal{A})$. Then*

$$\mathrm{Mor}_{K(\mathcal{A})}(P^\bullet, L^\bullet) = \mathrm{Mor}_{D(\mathcal{A})}(P^\bullet, L^\bullet).$$

Proof. Dual to Lemma 13.17.8. □

Lemma 13.18.9. *Let $\mathcal{A}$ be an abelian category. Assume $\mathcal{A}$ has enough projectives. For any short exact sequence $0 \to A^\bullet \to B^\bullet \to C^\bullet \to 0$ of $Comp^+(\mathcal{A})$ there exists a commutative diagram in $Comp^+(\mathcal{A})$*

$$\begin{array}{ccccccccc} 0 & \longrightarrow & P_1^\bullet & \longrightarrow & P_2^\bullet & \longrightarrow & P_3^\bullet & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & A^\bullet & \longrightarrow & B^\bullet & \longrightarrow & C^\bullet & \longrightarrow & 0 \end{array}$$

where the vertical arrows are projective resolutions and the rows are short exact sequences of complexes. In fact, given any projective resolution $P^\bullet \to C^\bullet$ we may assume $P_3^\bullet = P^\bullet$.

Proof. Dual to Lemma 13.17.9. □

Lemma 13.18.10. *Let $\mathcal{A}$ be an abelian category. Let $P^\bullet$, $K^\bullet$ be complexes. Let $n \in \mathbf{Z}$. Assume that*

(1) *$P^\bullet$ is a bounded complex consisting of projective objects,*
(2) *$P^i = 0$ for $i < n$, and*
(3) *$H^i(K^\bullet) = 0$ for $i \geq n$.*

Then $\mathrm{Hom}_{K(\mathcal{A})}(P^\bullet, K^\bullet) = \mathrm{Hom}_{D(\mathcal{A})}(P^\bullet, K^\bullet) = 0$.

Proof. The first equality follows from Lemma 13.18.8. Note that there is a distinguished triangle

$$(\tau_{\leq n-1}K^\bullet, K^\bullet, \tau_{\geq n}K^\bullet, f, g, h)$$

by Remark 13.11.4. Hence, by Lemma 13.4.2 it suffices to prove $\mathrm{Hom}_{K(\mathcal{A})}(P^\bullet, \tau_{\leq n-1}K^\bullet) = 0$ and $\mathrm{Hom}_{K(\mathcal{A})}(P^\bullet, \tau_{\geq n}K^\bullet) = 0$. The first vanishing is trivial and the second is Lemma 13.18.4. □

Lemma 13.18.11. *Let $\mathcal{A}$ be an abelian category. Let $\beta : P^\bullet \to L^\bullet$ and $\alpha : E^\bullet \to L^\bullet$ be maps of complexes. Let $n \in \mathbf{Z}$. Assume*

1. *$P^\bullet$ is a bounded complex of projectives and $P^i = 0$ for $i < n$,*
2. *$H^i(\alpha)$ is an isomorphism for $i > n$ and surjective for $i = n$.*

Then there exists a map of complexes $\gamma : P^\bullet \to E^\bullet$ such that $\alpha \circ \gamma$ and β are homotopic.

Proof. Consider the cone $C^\bullet = C(\alpha)^\bullet$ with map $i : L^\bullet \to C^\bullet$. Note that $i \circ \beta$ is zero by Lemma 13.18.10. Hence we can lift β to $E^\bullet$ by Lemma 13.4.2. □

13.19. Right derived functors and injective resolutions

At this point we can use the material above to define the right derived functors of an additive functor between an abelian category having enough injectives and a general abelian category.

Lemma 13.19.1. *Let $\mathcal{A}$ be an abelian category. Let $I \in \mathrm{Ob}(\mathcal{A})$ be an injective object. Let $I^\bullet$ be a bounded below complex of injectives in $\mathcal{A}$.*

1. *$I^\bullet$ computes RF relative to $Qis^+(\mathcal{A})$ for any exact functor $F : K^+(\mathcal{A}) \to \mathcal{D}$ into any triangulated category $\mathcal{D}$.*
2. *I is right acyclic for any additive functor $F : \mathcal{A} \to \mathcal{B}$ into any abelian category $\mathcal{B}$.*

Proof. Part (2) is a direct consequences of part (1) and Definition 13.15.3. To prove (1) let $\alpha : I^\bullet \to K^\bullet$ be a quasi-isomorphism into a complex. By Lemma 13.17.7 we see that α has a left inverse. Hence the category $I^\bullet/\mathrm{Qis}^+(\mathcal{A})$ is essentially constant with value $\mathrm{id} : I^\bullet \to I^\bullet$. Thus also the ind-object

$$I^\bullet/\mathrm{Qis}^+(\mathcal{A}) \longrightarrow \mathcal{D}, \quad (I^\bullet \to K^\bullet) \longmapsto F(K^\bullet)$$

is essentially constant with value $F(I^\bullet)$. This proves (1), see Definitions 13.14.2 and 13.14.10. □

Lemma 13.19.2. *Let $\mathcal{A}$ be an abelian category with enough injectives.*

1. *For any exact functor $F : K^+(\mathcal{A}) \to \mathcal{D}$ into a triangulated category $\mathcal{D}$ the right derived functor*
$$RF : D^+(\mathcal{A}) \longrightarrow \mathcal{D}$$
is everywhere defined.
2. *For any additive functor $F : \mathcal{A} \to \mathcal{B}$ into an abelian category $\mathcal{B}$ the right derived functor*
$$RF : D^+(\mathcal{A}) \longrightarrow D^+(\mathcal{B})$$
is everywhere defined.

Proof. Combine Lemma 13.19.1 and Proposition 13.15.8 for the second assertion. To see the first assertion combine Lemma 13.17.3, Lemma 13.19.1, Lemma 13.14.14, and Equation (13.14.9.1). □

Lemma 13.19.3. *Let $\mathcal{A}$ be an abelian category with enough injectives. Let $F : \mathcal{A} \to \mathcal{B}$ be an additive functor.*

(1) *The functor RF is an exact functor $D^+(\mathcal{A}) \to D^+(\mathcal{B})$.*
(2) *The functor RF induces an exact functor $K^+(\mathcal{A}) \to D^+(\mathcal{B})$.*
(3) *The functor RF induces a δ-functor $Comp^+(\mathcal{A}) \to D^+(\mathcal{B})$.*
(4) *The functor RF induces a δ-functor $\mathcal{A} \to D^+(\mathcal{B})$.*

Proof. This lemma simply reviews some of the results obtained so far. Note that by Lemma 13.19.2 RF is everywhere defined. Here are some references:

(1) The derived functor is exact: This boils down to Lemma 13.14.6.
(2) This is true because $K^+(\mathcal{A}) \to D^+(\mathcal{A})$ is exact and compositions of exact functors are exact.
(3) This is true because $\text{Comp}^+(\mathcal{A}) \to D^+(\mathcal{A})$ is a δ-functor, see Lemma 13.11.1.
(4) This is true because $\mathcal{A} \to \text{Comp}^+(\mathcal{A})$ is exact and precomposing a δ-functor by an exact functor gives a δ-functor.

□

Lemma 13.19.4. *Let $\mathcal{A}$ be an abelian category with enough injectives. Let $F : \mathcal{A} \to \mathcal{B}$ be a left exact functor.*

(1) *For any short exact sequence $0 \to A^\bullet \to B^\bullet \to C^\bullet \to 0$ of complexes in $Comp^+(\mathcal{A})$ there is an associated long exact sequence*

$$\ldots \to H^i(RF(A^\bullet)) \to H^i(RF(B^\bullet)) \to H^i(RF(C^\bullet)) \to H^{i+1}(RF(A^\bullet)) \to \ldots$$

(2) *The functors $R^iF : \mathcal{A} \to \mathcal{B}$ are zero for $i < 0$. Also $R^0F = F : \mathcal{A} \to \mathcal{B}$.*
(3) *We have $R^iF(I) = 0$ for $i > 0$ and I injective.*
(4) *The sequence (R^iF, δ) forms a universal δ-functor (see Homology, Definition 12.11.3) from $\mathcal{A}$ to $\mathcal{B}$.*

Proof. This lemma simply reviews some of the results obtained so far. Note that by Lemma 13.19.2 RF is everywhere defined. Here are some references:

(1) This follows from Lemma 13.19.3 part (3) combined with the long exact cohomology sequence (13.10.1.1) for $D^+(\mathcal{B})$.
(2) This is Lemma 13.16.3.
(3) This is the fact that injective objects are acyclic.
(4) This is Lemma 13.16.6.

□

13.20. Cartan-Eilenberg resolutions

This section can be expanded. The material can be generalized and applied in more cases. Resolutions need not use injectives and the method also works in the unbounded case in some situations.

Definition 13.20.1. Let $\mathcal{A}$ be an abelian category. Let $K^\bullet$ be a bounded below complex. A *Cartan-Eilenberg resolution* of $K^\bullet$ is given by a double complex $I^{\bullet,\bullet}$ and a morphism of complexes $\epsilon : K^\bullet \to I^{\bullet,0}$ with the following properties:

(1) There exists a $i \ll 0$ such that $I^{p,q} = 0$ for all $p < i$ and all q.
(2) We have $I^{p,q} = 0$ if $q < 0$.
(3) The complex $I^{p,\bullet}$ is an injective resolution of K^p.
(4) The complex $\mathrm{Ker}(d_1^{p,\bullet})$ is an injective resolution of $\mathrm{Ker}(d_K^p)$.
(5) The complex $\mathrm{Im}(d_1^{p,\bullet})$ is an injective resolution of $\mathrm{Im}(d_K^p)$.
(6) The complex $H_I^p(I^{\bullet,\bullet})$ is an injective resolution of $H^p(K^\bullet)$.

Lemma 13.20.2. *Let $\mathcal{A}$ be an abelian category with enough injectives. Let $K^\bullet$ be a bounded below complex. There exists a Cartan-Eilenberg resolution of $K^\bullet$.*

Proof. Suppose that $K^p = 0$ for $p < n$. Decompose $K^\bullet$ into short exact sequences as follows: Set $Z^p = \mathrm{Ker}(d^p)$, $B^p = \mathrm{Im}(d^{p-1})$, $H^p = Z^p/B^p$, and consider

$$\begin{aligned} 0 \to Z^n \to K^n \to B^{n+1} \to 0 \\ 0 \to B^{n+1} \to Z^{n+1} \to H^{n+1} \to 0 \\ 0 \to Z^{n+1} \to K^{n+1} \to B^{n+2} \to 0 \\ 0 \to B^{n+2} \to Z^{n+2} \to H^{n+2} \to 0 \\ \ldots \end{aligned}$$

Set $I^{p,q} = 0$ for $p < n$. Inductively we choose injective resolutions as follows:

(1) Choose an injective resolution $Z^n \to J_Z^{n,\bullet}$.
(2) Using Lemma 13.17.9 choose injective resolutions $K^n \to I^{n,\bullet}$, $B^{n+1} \to J_B^{n+1,\bullet}$, and an exact sequence of complexes $0 \to J_Z^{n,\bullet} \to I^{n,\bullet} \to J_B^{n+1,\bullet} \to 0$ compatible with the short exact sequence $0 \to Z^n \to K^n \to B^{n+1} \to 0$.
(3) Using Lemma 13.17.9 choose injective resolutions $Z^{n+1} \to J_Z^{n+1,\bullet}$, $H^{n+1} \to J_H^{n+1,\bullet}$, and an exact sequence of complexes $0 \to J_B^{n+1,\bullet} \to J_Z^{n+1,\bullet} \to J_H^{n+1,\bullet} \to 0$ compatible with the short exact sequence $0 \to B^{n+1} \to Z^{n+1} \to H^{n+1} \to 0$.
(4) Etc.

Taking as maps $d_1^\bullet : I^{p,\bullet} \to I^{p+1,\bullet}$ the compositions $I^{p,\bullet} \to J_B^{p+1,\bullet} \to J_Z^{p+1,\bullet} \to I^{p+1,\bullet}$ everything is clear. □

Lemma 13.20.3. *Let $F : \mathcal{A} \to \mathcal{B}$ be a left exact functor of abelian categories. Let $K^\bullet$ be a bounded below complex of $\mathcal{A}$. Let $I^{\bullet,\bullet}$ be a Cartan-Eilenberg resolution for $K^\bullet$. The spectral sequences $({}'E_r, {}'d_r)_{r\geq 0}$ and $({}''E_r, {}''d_r)_{r\geq 0}$ associated to the double complex $F(I^{\bullet,\bullet})$ satisfy the relations*

$$ {}'E_2^{p,q} = H^p(R^qF(K^\bullet))^4 \quad \text{and} \quad {}''E_2^{p,q} = R^pF(H^q(K^\bullet)) $$

Moreover, these spectral sequences converge to $H^{p+q}(RF(K^\bullet))$ and the associated induced filtrations on $H^{p+q}(RF(K^\bullet))$ are finite.

Proof. We will use the following remarks without further mention:

(1) As $I^{p,\bullet}$ is an injective resolution of K^p we see that RF is defined at $K^p[0]$ with value $F(I^{p,\bullet})$.
(2) As $H_I^p(I^{\bullet,\bullet})$ is an injective resolution of $H^p(K^\bullet)$ the derived functor RF is defined at $H^p(K^\bullet)[0]$ with value $F(H_I^p(I^{\bullet,\bullet}))$.
(3) By Homology, Lemma 12.22.6 the total complex $sI^\bullet$ is an injective resolution of $K^\bullet$. Hence RF is defined at $K^\bullet$ with value $F(sI^\bullet)$.

[4]This notation sucks! It really means the pth cohomology group of the complex with terms $R^qF(K^n)$. Not the pth cohomology of the qth derived functor of F applied to $K^\bullet$...

Consider the spectral sequences associated to the double complex $K^{\bullet,\bullet} = F(I^{\bullet,\bullet})$, see Homology, Lemma 12.22.3. These both converge, see Homology, Lemma 12.22.5, to the cohomology groups of the associated total complex $s(F(I^{\bullet,\bullet}) = F(sI^\bullet)$ which computes $H^n(RF(K^\bullet))$.

Computation of the first spectral sequence. We have ${}'E_1^{p,q} = H^q(K^{p,\bullet})$ in other words

$$ {}'E_1^{p,q} = H^q(F(I^{p,\bullet})) = R^qF(K^p) $$

and the maps ${}'E_1^{p,q} \to {}'E^{p+1,q}$ are the maps $R^qF(K^p) \to R^qF(K^{p+1})$ as desired.

Computation of the second spectral sequence. We have ${}''E_1^{p,q} = H^q(K^{\bullet,p}) = H^q(F(I^{\bullet,p}))$. Note that the complex $I^{\bullet,p}$ is bounded below, consists of injectives, and moreover each kernel, image, and cohomology group of the differentials is an injective object of $\mathcal{A}$. Hence we can split the differentials, i.e., each differential is a split surjection onto a direct summand. It follows that the same is true after applying F. Hence ${}''E_1^{p,q} = F(H^q(I^{\bullet,p})) = F(H_I^q(I^{\bullet,p}))$. The differentials on this are $(-1)^q$ times F applied to the differential of the complex $H_I^p(I^{\bullet,\bullet})$ which is an injective resolution of $H^p(K^\bullet)$. Hence the description of the E_2 terms. □

Remark 13.20.4. The spectral sequences of Lemma 13.20.3 are functorial in the complex $K^\bullet$. This follows from functoriality properties of Cartan-Eilenberg resolutions. On the other hand, they are both examples of a more general spectral sequence which may be associated to a filtered complex of $\mathcal{A}$. The functoriality will follow from its construction. We will return to this in the section on the filtered derived category, see Remark 13.25.15.

13.21. Composition of right derived functors

Sometimes we can compute the right derived functor of a composition. Suppose that $\mathcal{A}, \mathcal{B}, \mathcal{C}$ be abelian categories. Let $F : \mathcal{A} \to \mathcal{B}$ and $G : \mathcal{B} \to \mathcal{C}$ be left exact functors. Assume that the right derived functors $RF : D^+(\mathcal{A}) \to D^+(\mathcal{B})$, $RG : D^+(\mathcal{B}) \to D^+(\mathcal{C})$, and $R(G \circ F) : D^+(\mathcal{A}) \to D^+(\mathcal{C})$ are everywhere defined. Then there exists a canonical transformation

$$ t : R(G \circ F) \longrightarrow RG \circ RF $$

of functors from $D^+(\mathcal{A})$ to $D^+(\mathcal{C})$, see Lemma 13.14.16. This transformation need not always be an isomorphism.

Lemma 13.21.1. *Let $\mathcal{A}, \mathcal{B}, \mathcal{C}$ be abelian categories. Let $F : \mathcal{A} \to \mathcal{B}$ and $G : \mathcal{B} \to \mathcal{C}$ be left exact functors. Assume $\mathcal{A}$, $\mathcal{B}$ have enough injectives. If $F(I)$ is right acyclic for G for each injective object I of $\mathcal{A}$, then we have an isomorphism of functors*

$$ t : R(G \circ F) \longrightarrow RG \circ RF. $$

of functors from $D^+(\mathcal{A})$ to $D^+(\mathcal{C})$.

Proof. Let $A^\bullet$ be a bounded below complex of $\mathcal{A}$. Choose an injective resolution $A^\bullet \to I^\bullet$. The map t is given (see proof of Lemma 13.14.16) by the maps

$$ R(G \circ F)(A^\bullet) = (G \circ F)(I^\bullet) = G(F(I^\bullet))) \to RG(F(I^\bullet)) = RG(RF(A^\bullet)) $$

where the arrow is an isomorphism by Lemma 13.16.7. □

Lemma 13.21.2 (Grothendieck spectral sequence)**.** *With assumptions as in Lemma 13.21.1. Let A be an object of $\mathcal{A}$. There exists a spectral sequence $(E_r^{p,q}, d_r^{p,q})_{r\geq 0}$ associated to a filtered complex with*

$$E_2^{p,q} = R^pG(R^qF(A))$$

converging to $R^{p+q}(G \circ F)(A)$. Moreover, the induced filtration on each $R^n(G \circ F)(A)$ is finite.

Proof. Choose an injective resolution $A \to I^\bullet$. Choose a Cartan-Eilenberg resolution $F(I^\bullet) \to I^{\bullet,\bullet}$ using Lemma 13.20.2. Apply Lemma 13.20.3 (use the second spectral sequence). Details omitted. □

13.22. Resolution functors

Let $\mathcal{A}$ be an abelian category with enough injectives. Denote $\mathcal{I}$ the full additive subcategory of $\mathcal{A}$ whose objects are the injective objects of $\mathcal{A}$. It turns out that $K^+(\mathcal{I})$ and $D^+(\mathcal{A})$ are equivalent in this case (see Proposition 13.22.1). For many purposes it therefore makes sense to think of $D^+(\mathcal{A})$ as the (easier to grok) category $K^+(\mathcal{I})$ in this case.

Proposition 13.22.1. *Let $\mathcal{A}$ be an abelian category. Assume $\mathcal{A}$ has enough injectives. Denote $\mathcal{I} \subset \mathcal{A}$ the strictly full additive subcategory whose objects are the injective objects of $\mathcal{A}$. The functor*

$$K^+(\mathcal{I}) \longrightarrow D^+(\mathcal{A})$$

is exact, fully faithful and essentially surjective, i.e., an equivalence of triangulated categories.

Proof. It is clear that the functor is exact. It is essentially surjective by Lemma 13.17.3. Fully faithfulness is a consequence of Lemma 13.17.8. □

Proposition 13.22.1 implies that we can find resolution functors. It turns out that we can prove resolution functors exist even in some cases where the abelian category $\mathcal{A}$ is a "big" category, i.e., has a class of objects.

Definition 13.22.2. Let $\mathcal{A}$ be an abelian category with enough injectives. A *resolution functor*[5] for $\mathcal{A}$ is given by the following data:

(1) for all $K^\bullet \in \mathrm{Ob}(K^+(\mathcal{A}))$ a bounded below complex of injectives $j(K^\bullet)$, and
(2) for all $K^\bullet \in \mathrm{Ob}(K^+(\mathcal{A}))$ a quasi-isomorphism $i_{K^\bullet} : K^\bullet \to j(K^\bullet)$.

Lemma 13.22.3. *Let $\mathcal{A}$ be an abelian category with enough injectives. Given a resolution functor (j, i) there is a unique way to turn j into a functor and i into a 2-isomorphism producing a 2-commutative diagram*

$$\begin{array}{ccc} K^+(\mathcal{A}) & \xrightarrow[j]{} & K^+(\mathcal{I}) \\ & \searrow \quad \swarrow & \\ & D^+(\mathcal{A}) & \end{array}$$

where $\mathcal{I}$ is the full additive subcategory of $\mathcal{A}$ consisting of injective objects.

[5] This is likely nonstandard terminology.

Proof. For every morphism $\alpha : K^\bullet \to L^\bullet$ of $K^+(\mathcal{A})$ there is a unique morphism $j(\alpha) : j(K^\bullet) \to j(L^\bullet)$ in $K^+(\mathcal{I})$ such that

$$\begin{array}{ccc} K^\bullet & \xrightarrow{\alpha} & L^\bullet \\ {\scriptstyle i_{K^\bullet}}\downarrow & & \downarrow{\scriptstyle i_{L^\bullet}} \\ j(K^\bullet) & \xrightarrow{j(\alpha)} & j(L^\bullet) \end{array}$$

is commutative in $K^+(\mathcal{A})$. To see this either use Lemmas 13.17.6 and 13.17.7 or the equivalent Lemma 13.17.8. The uniqueness implies that j is a functor, and the commutativity of the diagram implies that i gives a 2-morphism which witnesses the 2-commutativity of the diagram of categories in the statement of the lemma. □

Lemma 13.22.4. *Let $\mathcal{A}$ be an abelian category. Assume $\mathcal{A}$ has enough injectives. Then a resolution functor j exists and is unique up to unique isomorphism of functors.*

Proof. Consider the set of all objects $K^\bullet$ of $K^+(\mathcal{A})$. (Recall that by our conventions any category has a set of objects unless mentioned otherwise.) By Lemma 13.17.3 every object has an injective resolution. By the axiom of choice we can choose for each $K^\bullet$ an injective resolution $i_{K^\bullet} : K^\bullet \to j(K^\bullet)$. □

Lemma 13.22.5. *Let $\mathcal{A}$ be an abelian category with enough injectives. Any resolution functor $j : K^+(\mathcal{A}) \to K^+(\mathcal{I})$ is exact.*

Proof. Denote $i_{K^\bullet} : K^\bullet \to j(K^\bullet)$ the canonical maps of Definition 13.22.2. First we discuss the existence of the functorial isomorphism $j(K^\bullet[1]) \to j(K^\bullet)[1]$. Consider the diagram

$$\begin{array}{ccc} K^\bullet[1] & = & K^\bullet[1] \\ {\scriptstyle i_{K^\bullet[1]}}\downarrow & & \downarrow{\scriptstyle i_{K^\bullet}[1]} \\ j(K^\bullet[1]) & \xrightarrow{\xi_{K^\bullet}} & j(K^\bullet)[1] \end{array}$$

By Lemmas 13.17.6 and 13.17.7 there exists a unique dotted arrow $\xi_{K^\bullet}$ in $K^+(\mathcal{I})$ making the diagram commute in $K^+(\mathcal{A})$. We omit the verification that this gives a functorial isomorphism. (Hint: use Lemma 13.17.7 again.)

Let $(K^\bullet, L^\bullet, M^\bullet, f, g, h)$ be a distinguished triangle of $K^+(\mathcal{A})$. We have to show that $(j(K^\bullet), j(L^\bullet), j(M^\bullet), j(f), j(g), \xi_{K^\bullet} \circ j(h))$ is a distinguished triangle of $K^+(\mathcal{I})$. Note that we have a commutative diagram

$$\begin{array}{ccccccc} K^\bullet & \xrightarrow{f} & L^\bullet & \xrightarrow{g} & M^\bullet & \xrightarrow{h} & K^\bullet[1] \\ \downarrow & & \downarrow & & \downarrow & & \downarrow \\ j(K^\bullet) & \xrightarrow{j(f)} & j(L^\bullet) & \xrightarrow{j(g)} & j(M^\bullet) & \xrightarrow{\xi_{K^\bullet} \circ j(h)} & j(K^\bullet)[1] \end{array}$$

in $K^+(\mathcal{A})$ whose vertical arrows are the quasi-isomorphisms i_K, i_L, i_M. Hence we see that the image of $(j(K^\bullet), j(L^\bullet), j(M^\bullet), j(f), j(g), \xi_{K^\bullet} \circ j(h))$ in $D^+(\mathcal{A})$ is isomorphic to a distinguished triangle and hence a distinguished triangle by TR1. Thus we see from Lemma 13.4.16 that $(j(K^\bullet), j(L^\bullet), j(M^\bullet), j(f), j(g), \xi_{K^\bullet} \circ j(h))$ is a distinguished triangle in $K^+(\mathcal{I})$. □

Lemma 13.22.6. *Let $\mathcal{A}$ be an abelian category which has enough injectives. Let j be a resolution functor. Write $Q : K^+(\mathcal{A}) \to D^+(\mathcal{A})$ for the natural functor. Then $j = j' \circ Q$ for a unique functor $j' : D^+(\mathcal{A}) \to K^+(\mathcal{I})$ which is quasi-inverse to the canonical functor $K^+(\mathcal{I}) \to D^+(\mathcal{A})$.*

Proof. By Lemma 13.10.6 Q is a localization functor. To prove the existence of j' it suffices to show that any element of $\mathrm{Qis}^+(\mathcal{A})$ is mapped to an isomorphism under the functor j, see Lemma 13.5.6. This is true by the remarks following Definition 13.22.2. □

Remark 13.22.7. Suppose that $\mathcal{A}$ is a "big" abelian category with enough injectives such as the category of abelian groups. In this case we have to be slightly more careful in constructing our resolution functor since we cannot use the axiom of choice with a quantifier ranging over a class. But note that the proof of the lemma does show that any two localization functors are canonically isomorphic. Namely, given quasi-isomorphisms $i : K^\bullet \to I^\bullet$ and $i' : K^\bullet \to J^\bullet$ of a bounded below complex $K^\bullet$ into bounded below complexes of injectives there exists a unique(!) morphism $a : I^\bullet \to J^\bullet$ in $K^+(\mathcal{I})$ such that $i' = i \circ a$ as morphisms in $K^+(\mathcal{I})$. Hence the only issue is existence, and we will see how to deal with this in the next section.

13.23. Functorial injective embeddings and resolution functors

In this section we redo the construction of a resolution functor $K^+(\mathcal{A}) \to K^+(\mathcal{I})$ in case the category $\mathcal{A}$ has functorial injective embeddings. There are two reasons for this: (1) the proof is easier and (2) the construction also works if $\mathcal{A}$ is a "big" abelian category. See Remark 13.23.3 below.

Let $\mathcal{A}$ be an abelian category. As before denote $\mathcal{I}$ the additive full subcategory of $\mathcal{A}$ consisting of injective objects. Consider the category $\mathrm{InjRes}(\mathcal{A})$ of arrows $\alpha : K^\bullet \to I^\bullet$ where $K^\bullet$ is a bounded below complex of $\mathcal{A}$, $I^\bullet$ is a bounded below complex of injectives of $\mathcal{A}$ and α is a quasi-isomorphism. In other words, α is an injective resolution and $K^\bullet$ is bounded below. There is an obvious functor

$$s : \mathrm{InjRes}(\mathcal{A}) \longrightarrow \mathrm{Comp}^+(\mathcal{A})$$

defined by $(\alpha : K^\bullet \to I^\bullet) \mapsto K^\bullet$. There is also a functor

$$t : \mathrm{InjRes}(\mathcal{A}) \longrightarrow K^+(\mathcal{I})$$

defined by $(\alpha : K^\bullet \to I^\bullet) \mapsto I^\bullet$.

Lemma 13.23.1. *Let $\mathcal{A}$ be an abelian category. Assume $\mathcal{A}$ has functorial injective embeddings, see Homology, Definition 12.23.5.*

1. *There exists a functor $inj : Comp^+(\mathcal{A}) \to InjRes(\mathcal{A})$ such that $s \circ inj = id$.*
2. *For any functor $inj : Comp^+(\mathcal{A}) \to InjRes(\mathcal{A})$ such that $s \circ inj = id$ we obtain a resolution functor, see Definition 13.22.2.*

Proof. Let $A \mapsto (A \to J(A))$ be a functorial injective embedding, see Homology, Definition 12.23.5. We first note that we may assume $J(0) = 0$. Namely, if not then for any object A we have $0 \to A \to 0$ which gives a direct sum decomposition $J(A) = J(0) \oplus \mathrm{Ker}(J(A) \to J(0))$. Note that the functorial morphism $A \to J(A)$ has to map into the second summand. Hence we can replace our functor by $J'(A) = \mathrm{Ker}(J(A) \to J(0))$ if needed.

Let $K^\bullet$ be a bounded below complex of $\mathcal{A}$. Say $K^p = 0$ if $p < B$. We are going to construct a double complex $I^{\bullet,\bullet}$ of injectives, together with a map $\alpha : K^\bullet \to I^{\bullet,0}$ such that α induces a quasi-isomorphism of $K^\bullet$ with the associated total complex of $I^{\bullet,\bullet}$. First we set $I^{p,q} = 0$ whenever $q < 0$. Next, we set $I^{p,0} = J(K^p)$ and $\alpha^p : K^p \to I^{p,0}$ the functorial embedding. Since J is a functor we see that $I^{\bullet,0}$ is a complex and that α is a morphism of complexes. Each α^p is injective. And $I^{p,0} = 0$ for $p < B$ because $J(0) = 0$. Next, we set $I^{p,1} = J(\mathrm{Coker}(K^p \to I^{p,0}))$. Again by functoriality we see that $I^{\bullet,1}$ is a complex. And again we get that $I^{p,1} = 0$ for $p < B$. It is also clear that K^p maps isomorphically onto $\mathrm{Ker}(I^{p,0} \to I^{p,1})$. As our third step we take $I^{p,2} = J(\mathrm{Coker}(I^{p,0} \to I^{p,1}))$. And so on and so forth.

At this point we can apply Homology, Lemma 12.22.6 to get that the map

$$\alpha : K^\bullet \to sI^\bullet$$

is a quasi-isomorphism. To prove we get a functor inj it rests to show that the construction above is functorial. This verification is omitted.

Suppose we have a functor inj such that $s \circ inj = \mathrm{id}$. For every object $K^\bullet$ of $\mathrm{Comp}^+(\mathcal{A})$ we can write

$$inj(K^\bullet) = (i_{K^\bullet} : K^\bullet \to j(K^\bullet))$$

This provides us with a resolution functor as in Definition 13.22.2. □

Remark 13.23.2. Suppose inj is a functor such that $s \circ inj = \mathrm{id}$ as in part (2) of Lemma 13.23.1. Write $inj(K^\bullet) = (i_{K^\bullet} : K^\bullet \to j(K^\bullet))$ as in the proof of that lemma. Suppose $\alpha : K^\bullet \to L^\bullet$ is a map of bounded below complexes. Consider the map $inj(\alpha)$ in the category $\mathrm{InjRes}(\mathcal{A})$. It induces a commutative diagram

$$\begin{array}{ccc} K^\bullet & \xrightarrow{\alpha} & L^\bullet \\ {\scriptstyle i_K}\downarrow & & \downarrow{\scriptstyle i_L} \\ j(K)^\bullet & \xrightarrow{inj(\alpha)} & j(L)^\bullet \end{array}$$

of morphisms of complexes. Hence, looking at the proof of Lemma 13.22.3 we see that the functor $j : K^+(\mathcal{A}) \to K^+(\mathcal{I})$ is given by the rule

$$j(\alpha \text{ up to homotopy}) = inj(\alpha) \text{ up to homotopy} \in \mathrm{Hom}_{K^+(\mathcal{I})}(j(K^\bullet), j(L^\bullet))$$

Hence we see that j matches $t \circ inj$ in this case, i.e., the diagram

$$\begin{array}{ccc} \mathrm{Comp}^+(\mathcal{A}) & \xrightarrow[t\circ inj]{} & K^+(\mathcal{I}) \\ & \searrow \quad \nearrow_{j} & \\ & K^+(\mathcal{A}) & \end{array}$$

is commutative.

Remark 13.23.3. Let $Mod(\mathcal{O}_X)$ be the category of $\mathcal{O}_X$-modules on a ringed space $(X, \mathcal{O}_X)$ (or more generally on a ringed site). We will see later that $Mod(\mathcal{O}_X)$ has enough injectives and in fact functorial injective embeddings, see Injectives, Theorem 19.8.4. Note that the proof of Lemma 13.22.4 does not apply to $Mod(\mathcal{O}_X)$. But the proof of Lemma 13.23.1 does apply to $Mod(\mathcal{O}_X)$. Thus we obtain

$$j : K^+(Mod(\mathcal{O}_X)) \longrightarrow K^+(\mathcal{I})$$

which is a resolution functor where $\mathcal{I}$ is the additive category of injective $\mathcal{O}_X$-modules. This argument also works in the following cases:

(1) The category Mod_R of R-modules over a ring R.
(2) The category $PMod(\mathcal{O})$ of presheaves of $\mathcal{O}$-modules on a site endowed with a presheaf of rings.
(3) The category $Mod(\mathcal{O})$ of sheaves of $\mathcal{O}$-modules on a ringed site.
(4) Add more here as needed.

13.24. Right derived functors via resolution functors

The content of the following lemma is that we can simply define $RF(K^\bullet) = F(j(K^\bullet)$ if we are given a resolution functor j.

Lemma 13.24.1. *Let $\mathcal{A}$ be an abelian category with enough injectives Let $F : \mathcal{A} \to \mathcal{B}$ be an additive functor into an abelian category. Let (i, j) be a resolution functor, see Definition 13.22.2. The right derived functor RF of F fits into the following 2-commutative diagram*

$$\begin{array}{ccc} D^+(\mathcal{A}) & \xrightarrow{j'} & K^+(\mathcal{I}) \\ & {\scriptstyle RF} \searrow \quad \swarrow {\scriptstyle F} & \\ & D^+(\mathcal{B}) & \end{array}$$

where j' is the functor from Lemma 13.22.6.

Proof. By Lemma 13.19.1 we have $RF(K^\bullet) = F(j(K^\bullet))$. □

Remark 13.24.2. In the situation of Lemma 13.24.1 we see that we have actually lifted the right derived functor to an exact functor $F \circ j' : D^+(\mathcal{A}) \to K^+(\mathcal{B})$. It is occasionally useful to use such a factorization.

13.25. Filtered derived category and injective resolutions

Let $\mathcal{A}$ be an abelian category. In this section we will show that if $\mathcal{A}$ has enough injectives, then so does the category $\mathrm{Fil}^f(\mathcal{A})$ in some sense. One can use this observation to compute in the filtered derived category of $\mathcal{A}$.

The category $\mathrm{Fil}^f(\mathcal{A})$ is an example of an exact category, see Injectives, Remark 19.9.6. A special role is played by the strict morphisms, see Homology, Definition 12.16.3, i.e., the morphisms f such that $\mathrm{Coim}(f) = \mathrm{Im}(f)$. We will say that a complex $A \to B \to C$ in $\mathrm{Fil}^f(\mathcal{A})$ is *exact* if the sequence $\mathrm{gr}(A) \to \mathrm{gr}(B) \to \mathrm{gr}(C)$ is exact in $\mathcal{A}$. This implies that $A \to B$ and $B \to C$ are strict morphisms, see Homology, Lemma 12.16.15.

Definition 13.25.1. Let $\mathcal{A}$ be an abelian category. We say an object I of $\mathrm{Fil}^f(\mathcal{A})$ is *filtered injective* if each $\mathrm{gr}^p(I)$ is an injective object of $\mathcal{A}$.

Lemma 13.25.2. *Let $\mathcal{A}$ be an abelian category. An object I of $Fil^f(\mathcal{A})$ is filtered injective if and only if there exist $a \leq b$, injective objects I_n, $a \leq n \leq b$ of $\mathcal{A}$ and an isomorphism $I \cong \bigoplus_{a \leq n \leq b} I_n$ such that $F^pI = \bigoplus_{n \geq p} I_n$.*

Proof. Follows from the fact that any injection $J \to M$ of $\mathcal{A}$ is split if J is an injective object. Details omitted. □

Lemma 13.25.3. *Let $\mathcal{A}$ be an abelian category. Any strict monomorphism $u : I \to A$ of $Fil^f(\mathcal{A})$ where I is a filtered injective object is a split injection.*

Proof. Let p be the largest integer such that $F^pI \neq 0$. In particular $\mathrm{gr}^p(I) = F^pI$. Let I' be the object of $\mathrm{Fil}^f(\mathcal{A})$ whose underlying object of $\mathcal{A}$ is F^pI and with filtration given by $F^nI' = 0$ for $n > p$ and $F^nI' = I' = F^pI$ for $n \leq p$. Note that $I' \to I$ is a strict monomorphism too. The fact that u is a strict monomorphism implies that $F^pI \to A/F^{p+1}(A)$ is injective, see Homology, Lemma 12.16.13. Choose a splitting $s : A/F^{p+1}A \to F^pI$ in $\mathcal{A}$. The induced morphism $s' : A \to I'$ is a strict morphism of filtered objects splitting the composition $I' \to I \to A$. Hence we can write $A = I' \oplus \mathrm{Ker}(s')$ and $I = I' \oplus \mathrm{Ker}(s'|_I)$. Note that $\mathrm{Ker}(s'|_I) \to \ker(s')$ is a strict monomorphism and that $\mathrm{Ker}(s'|_I)$ is a filtered injective object. By induction on the length of the filtration on I the map $\mathrm{Ker}(s'|_I) \to \ker(s')$ is a split injection. Thus we win. □

Lemma 13.25.4. *Let $\mathcal{A}$ be an abelian category. Let $u : A \to B$ be a strict monomorphism of $Fil^f(\mathcal{A})$ and $f : A \to I$ a morphism from A into a filtered injective object in $Fil^f(\mathcal{A})$. Then there exists a morphism $g : B \to I$ such that $f = g \circ u$.*

Proof. The pushout $f' : I \to I \amalg_A B$ of f by u is a strict monomorphism, see Homology, Lemma 12.16.10. Hence the result follows formally from Lemma 13.25.3. □

Lemma 13.25.5. *Let $\mathcal{A}$ be an abelian category with enough injectives. For any object A of $Fil^f(\mathcal{A})$ there exists a strict monomorphism $A \to I$ where I is a filtered injective object.*

Proof. Pick $a \leq b$ such that $\mathrm{gr}^p(A) = 0$ unless $p \in \{a, a+1, \ldots, b\}$. For each $n \in \{a, a+1, \ldots, b\}$ choose an injection $u_n : A/F^nA \to I_n$ with I_n and injective object. Set $I = \bigoplus_{a \leq n \leq b} I_p$ with filtration $F^pI = \bigoplus_{n \geq p} I_n$ and set $u : A \to I$ equal to the direct sum of the maps u_n. □

Lemma 13.25.6. *Let $\mathcal{A}$ be an abelian category with enough injectives. For any object A of $Fil^f(\mathcal{A})$ there exists a filtered quasi-isomorphism $A[0] \to I^\bullet$ where $I^\bullet$ is a complex of filtered injective objects with $I^n = 0$ for $n < 0$.*

Proof. First choose a strict monomorphism $u_0 : A \to I^0$ of A into a filtered injective object, see Lemma 13.25.5. Next, choose a strict monomorphism $u_1 : \mathrm{Coker}(u_0) \to I^1$ into a filtered injective object of $\mathcal{A}$. Denote d^0 the induced map $I^0 \to I^1$. Next, choose a strict monomorphism $u_2 : \mathrm{Coker}(u_1) \to I^2$ into a filtered injective object of $\mathcal{A}$. Denote d^1 the induced map $I^1 \to I^2$. And so on. This works because each of the sequences

$$0 \to \mathrm{Coker}(u_n) \to I^{n+1} \to \mathrm{Coker}(u_{n+1}) \to 0$$

is short exact, i.e., induces a short exact sequence on applying gr. To see this use Homology, Lemma 12.16.13. □

Lemma 13.25.7. *Let $\mathcal{A}$ be an abelian category with enough injectives. Let $f : A \to B$ be a morphism of $Fil^f(\mathcal{A})$. Given filtered quasi-isomorphisms $A[0] \to I^\bullet$ and*

$B[0] \to J^\bullet$ where $I^\bullet, J^\bullet$ are complexes of filtered injective objects with $I^n = J^n = 0$ for $n < 0$, then there exists a commutative diagram

$$\begin{array}{ccc} A[0] & \longrightarrow & B[0] \\ \downarrow & & \downarrow \\ I^\bullet & \longrightarrow & J^\bullet \end{array}$$

Proof. As $A[0] \to I^\bullet$ and $C[0] \to J^\bullet$ are filtered quasi-isomorphisms we conclude that $a : A \to I^0$, $b : B \to J^0$ and all the morphisms d_I^n, d_J^n are strict, see Homology, Lemma 13.13.4. We will inductively construct the maps f^n in the following commutative diagram

$$\begin{array}{ccccccccc} A & \xrightarrow[a]{} & I^0 & \longrightarrow & I^1 & \longrightarrow & I^2 & \longrightarrow & \ldots \\ {\scriptstyle f}\downarrow & & \downarrow{\scriptstyle f^0} & & \downarrow{\scriptstyle f^1} & & \downarrow{\scriptstyle f^2} & & \\ B & \xrightarrow{b} & J^0 & \longrightarrow & J^1 & \longrightarrow & J^2 & \longrightarrow & \ldots \end{array}$$

Because $A \to I^0$ is a strict monomorphism and because J^0 is filtered injective, we can find a morphism $f^0 : I^0 \to J^0$ such that $f^0 \circ a = b \circ f$, see Lemma 13.25.4. The composition $d_J^0 \circ b \circ f$ is zero, hence $d_J^0 \circ f^0 \circ a = 0$, hence $d_J^0 \circ f^0$ factors through a unique morphism

$$\mathrm{Coker}(a) = \mathrm{Coim}(d_I^0) = \mathrm{Im}(d_I^0) \longrightarrow J^1.$$

As $\mathrm{Im}(d_I^0) \to I^1$ is a strict monomorphism we can extend the displayed arrow to a morphism $f^1 : I^1 \to J^1$ by Lemma 13.25.4 again. And so on. □

Lemma 13.25.8. *Let $\mathcal{A}$ be an abelian category with enough injectives. Let $0 \to A \to B \to C \to 0$ be a short exact sequence in $Fil^f(\mathcal{A})$. Given filtered quasi-isomorphisms $A[0] \to I^\bullet$ and $C[0] \to J^\bullet$ where $I^\bullet, J^\bullet$ are complexes of filtered injective objects with $I^n = J^n = 0$ for $n < 0$, then there exists a commutative diagram*

$$\begin{array}{ccccccccc} 0 & \longrightarrow & A[0] & \longrightarrow & B[0] & \longrightarrow & C[0] & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & I^\bullet & \longrightarrow & M^\bullet & \longrightarrow & J^\bullet & \longrightarrow & 0 \end{array}$$

where the lower row is a termwise split sequence of complexes.

Proof. As $A[0] \to I^\bullet$ and $C[0] \to J^\bullet$ are filtered quasi-isomorphisms we conclude that $a : A \to I^0$, $c : C \to J^0$ and all the morphisms d_I^n, d_J^n are strict, see Homology, Lemma 13.13.4. We are going to step by step construct the south-east and the south arrows in the following commutative diagram

$$\begin{array}{ccccccccc} B & \longrightarrow & C & \xrightarrow[c]{} & J^0 & \longrightarrow & J^1 & \longrightarrow & \ldots \\ & {\scriptstyle\beta}\searrow & & \searrow{\scriptstyle\overline{b}} & \downarrow{\scriptstyle\delta^0} & & \downarrow{\scriptstyle\delta^1} & & \\ {\scriptstyle\alpha}\uparrow & & {\scriptstyle b}\searrow & & & & & & \\ A & \xrightarrow{a} & I^0 & \longrightarrow & I^1 & \longrightarrow & I^2 & \longrightarrow & \ldots \end{array}$$

As $A \to B$ is a strict monomorphism, we can find a morphism $b : B \to I^0$ such that $b \circ \alpha = a$, see Lemma 13.25.4. As A is the kernel of the strict morphism $I^0 \to I^1$ and $\beta = \mathrm{Coker}(\alpha)$ we obtain a unique morphism $\overline{b} : C \to I^1$ fitting into the diagram. As c is a strict monomorphism and I^1 is filtered injective we can find

$\delta^0 : J^0 \to I^1$, see Lemma 13.25.4. Because $B \to C$ is a strict epimorphism and because $B \to I^0 \to I^1 \to I^2$ is zero, we see that $C \to I^1 \to I^2$ is zero. Hence $d_I^1 \circ \delta^0$ is zero on $C \cong \mathrm{Im}(c)$. Hence $d_I^1 \circ \delta^0$ factors through a unique morphism

$$\mathrm{Coker}(c) = \mathrm{Coim}(d_J^0) = \mathrm{Im}(d_J^0) \longrightarrow I^2.$$

As I^2 is filtered injective and $\mathrm{Im}(d_J^0) \to J^1$ is a strict monomorphism we can extend the displayed morphism to a morphism $\delta^1 : J^1 \to I^2$, see Lemma 13.25.4. And so on. We set $M^\bullet = I^\bullet \oplus J^\bullet$ with differential

$$d_M^n = \begin{pmatrix} d_I^n & (-1)^{n+1}\delta^n \\ 0 & d_J^n \end{pmatrix}$$

Finally, the map $B[0] \to M^\bullet$ is given by $b \oplus c \circ \beta : M \to I^0 \oplus J^0$. □

Lemma 13.25.9. *Let $\mathcal{A}$ be an abelian category with enough injectives. For every $K^\bullet \in K^+(Fil^f(\mathcal{A}))$ there exists a filtered quasi-isomorphism $K^\bullet \to I^\bullet$ with $I^\bullet$ bounded below, each I^n a filtered injective object, and each $K^n \to I^n$ a strict monomorphism.*

Proof. After replacing $K^\bullet$ by a shift (which is harmless for the proof) we may assume that $K^n = 0$ for $n < 0$. Consider the short exact sequences

$$\begin{aligned} 0 \to \mathrm{Ker}(d_K^0) \to K^0 \to \mathrm{Coim}(d_K^0) \to 0 \\ 0 \to \mathrm{Ker}(d_K^1) \to K^1 \to \mathrm{Coim}(d_K^1) \to 0 \\ 0 \to \mathrm{Ker}(d_K^2) \to K^2 \to \mathrm{Coim}(d_K^2) \to 0 \\ \ldots \end{aligned}$$

of the exact category $\mathrm{Fil}^f(\mathcal{A})$ and the maps $u_i : \mathrm{Coim}(d_K^i) \to \mathrm{Ker}(d_K^{i+1})$. For each $i \geq 0$ we may choose filtered quasi-isomorphisms

$$\begin{aligned} \mathrm{Ker}(d_K^i)[0] \to I_{ker,i}^\bullet \\ \mathrm{Coim}(d_K^i)[0] \to I_{coim,i}^\bullet \end{aligned}$$

with $I_{ker,i}^n, I_{coim,i}^n$ filtered injective and zero for $n < 0$, see Lemma 13.25.6. By Lemma 13.25.7 we may lift u_i to a morphism of complexes $u_i^\bullet : I_{coim,i}^\bullet \to I_{ker,i+1}^\bullet$. Finally, for each $i \geq 0$ we may complete the diagrams

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \mathrm{Ker}(d_K^i)[0] & \longrightarrow & K^i[0] & \longrightarrow & \mathrm{Coim}(d_K^i)[0] & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & I_{ker,i}^\bullet & \xrightarrow{\alpha_i} & I_i^\bullet & \xrightarrow{\beta_i} & I_{coim,i}^\bullet & \longrightarrow & 0 \end{array}$$

with the lower sequence a termwise split exact sequence, see Lemma 13.25.8. For $i \geq 0$ set $d_i : I_i^\bullet \to I_{i+1}^\bullet$ equal to $d_i = \alpha_{i+1} \circ u_i^\bullet \circ \beta_i$. Note that $d_i \circ d_{i-1} = 0$ because $\beta_i \circ \alpha_i = 0$. Hence we have constructed a commutative diagram

$$\begin{array}{ccccccc} I_0^\bullet & \longrightarrow & I_1^\bullet & \longrightarrow & I_2^\bullet & \longrightarrow & \ldots \\ \uparrow & & \uparrow & & \uparrow & & \\ K^0[0] & \longrightarrow & K^1[0] & \longrightarrow & K^2[0] & \longrightarrow & \ldots \end{array}$$

Here the vertical arrows are filtered quasi-isomorphisms. The upper row is a complex of complexes and each complex consists of filtered injective objects with no

nonzero objects in degree < 0. Thus we obtain a double complex by setting $I^{a,b} = I^b_a$ and using

$$d_1^{a,b} : I^{a,b} = I^b_a \to I^b_{a+1} = I^{a+1,b}$$

the map d^b_a and using for

$$d_2^{a,b} : I^{a,b} = I^b_a \to I^{b+1}_a = I^{a,b+1}$$

the map $d^b_{I_a}$. Denote $\mathrm{Tot}(I^{\bullet,\bullet})$ the total complex associated to this double complex, see Homology, Definition 12.22.2. Observe that the maps $K^n[0] \to I^\bullet_n$ come from maps $K^n \to I^{n,0}$ which give rise to a map of complexes

$$K^\bullet \longrightarrow \mathrm{Tot}(I^{\bullet,\bullet})$$

We claim this is a filtered quasi-isomorphism. As $\mathrm{gr}(-)$ is an additive functor, we see that $\mathrm{gr}(\mathrm{Tot}(I^{\bullet,\bullet})) = \mathrm{Tot}(\mathrm{gr}(I^{\bullet,\bullet}))$. Thus we can use Homology, Lemma 12.22.6 to conclude that $\mathrm{gr}(K^\bullet) \to \mathrm{gr}(\mathrm{Tot}(I^{\bullet,\bullet}))$ is a quasi-isomorphism as desired. □

Lemma 13.25.10. *Let $\mathcal{A}$ be an abelian category. Let $K^\bullet, I^\bullet \in K(Fil^f(\mathcal{A}))$. Assume $K^\bullet$ is filtered acyclic and $I^\bullet$ bounded below and consisting of filtered injective objects. Any morphism $K^\bullet \to I^\bullet$ is homotopic to zero: $\mathrm{Hom}_{K(Fil^f(\mathcal{A}))}(K^\bullet, I^\bullet) = 0$.*

Proof. Let $\alpha : K^\bullet \to I^\bullet$ be a morphism of complexes. Assume that $\alpha^j = 0$ for $j < n$. We will show that there exists a morphism $h : K^{n+1} \to I^n$ such that $\alpha^n = h \circ d$. Thus α will be homotopic to the morphism of complexes β defined by

$$\beta^j = \begin{cases} 0 & \text{if} \quad j \leq n \\ \alpha^{n+1} - d \circ h & \text{if} \quad j = n+1 \\ \alpha^j & \text{if} \quad j > n+1 \end{cases}$$

This will clearly prove the lemma (by induction). To prove the existence of h note that $\alpha^n \circ d^{n-1}_K = 0$ since $\alpha^{n-1} = 0$. Since $K^\bullet$ is filtered acyclic we see that d^{n-1}_K and d^n_K are strict and that

$$0 \to \mathrm{Im}(d^{n-1}_K) \to K^n \to \mathrm{Im}(d^n_K) \to 0$$

is an exact sequence of the exact category $\mathrm{Fil}^f(\mathcal{A})$, see Homology, Lemma 12.16.15. Hence we can think of α^n as a map into I^n defined on $\mathrm{Im}(d^n_K)$. Using that $\mathrm{Im}(d^n_K) \to K^{n+1}$ is a strict monomorphism and that I^n is filtered injective we may lift this map to a map $h : K^{n+1} \to I^n$ as desired, see Lemma 13.25.4. □

Lemma 13.25.11. *Let $\mathcal{A}$ be an abelian category. Let $I^\bullet \in K(Fil^f(\mathcal{A}))$ be a bounded below complex consisting of filtered injective objects.*

(1) *Let $\alpha : K^\bullet \to L^\bullet$ in $K(Fil^f(\mathcal{A}))$ be a filtered quasi-isomorphism. Then the map*

$$\mathrm{Hom}_{K(Fil^f(\mathcal{A}))}(L^\bullet, I^\bullet) \to \mathrm{Hom}_{K(Fil^f(\mathcal{A}))}(K^\bullet, I^\bullet)$$

is bijective.

(2) *Let $L^\bullet \in K(\mathcal{A})$. Then*

$$\mathrm{Hom}_{K(Fil^f(\mathcal{A}))}(L^\bullet, I^\bullet) = \mathrm{Hom}_{DF(\mathcal{A})}(L^\bullet, I^\bullet).$$

Proof. Proof of (1). Note that

$$(K^\bullet, L^\bullet, C(\alpha)^\bullet, \alpha, i, -p)$$

is a distinguished triangle in $K(\mathrm{Fil}^f(\mathcal{A}))$ (Lemma 13.8.14) and $C(f)^\bullet$ is a filtered acyclic complex (Lemma 13.13.4). Then

$$\begin{array}{ccccc} \mathrm{Hom}_{K(\mathrm{Fil}^f(\mathcal{A}))}(C(\alpha)^\bullet, I^\bullet) & \longrightarrow & \mathrm{Hom}_{K(\mathrm{Fil}^f(\mathcal{A}))}(L^\bullet, I^\bullet) & \longrightarrow & \mathrm{Hom}_{K(\mathrm{Fil}^f(\mathcal{A}))}(K^\bullet, I^\bullet) \\ & & & \swarrow & \\ \mathrm{Hom}_{K(\mathrm{Fil}^f(\mathcal{A}))}(C(\alpha)^\bullet[-1], I^\bullet) & & & & \end{array}$$

is an exact sequence of abelian groups, see Lemma 13.4.2. At this point Lemma 13.25.10 guarantees that the outer two groups are zero and hence $\mathrm{Hom}_{K(\mathcal{A})}(L^\bullet, I^\bullet) = \mathrm{Hom}_{K(\mathcal{A})}(K^\bullet, I^\bullet)$.

Proof of (2). Let a be an element of the right hand side. We may represent $a = \gamma\alpha^{-1}$ where $\alpha : K^\bullet \to L^\bullet$ is a filtered quasi-isomorphism and $\gamma : K^\bullet \to I^\bullet$ is a map of complexes. By part (1) we can find a morphism $\beta : L^\bullet \to I^\bullet$ such that $\beta \circ \alpha$ is homotopic to γ. This proves that the map is surjective. Let b be an element of the left hand side which maps to zero in the right hand side. Then b is the homotopy class of a morphism $\beta : L^\bullet \to I^\bullet$ such that there exists a filtered quasi-isomorphism $\alpha : K^\bullet \to L^\bullet$ with $\beta \circ \alpha$ homotopic to zero. Then part (1) shows that β is homotopic to zero also, i.e., $b = 0$. □

Lemma 13.25.12. *Let $\mathcal{A}$ be an abelian category with enough injectives. Let $\mathcal{I}^f \subset Fil^f(\mathcal{A})$ denote the strictly full additive subcategory whose objects are the filtered injective objects. The canonical functor*

$$K^+(\mathcal{I}^f) \longrightarrow DF^+(\mathcal{A})$$

is exact, fully faithful and essentially surjective, i.e., an equivalence of triangulated categories. Furthermore the diagrams

$$\begin{array}{ccc} K^+(\mathcal{I}^f) & \longrightarrow & DF^+(\mathcal{A}) \\ {\scriptstyle gr^p}\downarrow & & \downarrow{\scriptstyle gr^p} \\ K^+(\mathcal{I}) & \longrightarrow & D^+(\mathcal{A}) \end{array} \qquad \begin{array}{ccc} K^+(\mathcal{I}^f) & \longrightarrow & DF^+(\mathcal{A}) \\ \downarrow{\scriptstyle forget\ F} & & \downarrow{\scriptstyle forget\ F} \\ K^+(\mathcal{I}) & \longrightarrow & D^+(\mathcal{A}) \end{array}$$

are commutative, where $\mathcal{I} \subset \mathcal{A}$ is the strictly full additive subcategory whose objects are the injective objects.

Proof. The functor $K^+(\mathcal{I}^f) \to DF^+(\mathcal{A})$ is essentially surjective by Lemma 13.25.9. It is fully faithful by Lemma 13.25.11. It is an exact functor by our definitions regarding distinguished triangles. The commutativity of the squares is immediate. □

Remark 13.25.13. We can invert the arrow of the lemma only if $\mathcal{A}$ is a category in our sense, namely if it has a set of objects. However, suppose given a big abelian category $\mathcal{A}$ with enough injectives, such as $Mod(\mathcal{O}_X)$ for example. Then for any given set of objects $\{A_i\}_{i\in I}$ there is an abelian subcategory $\mathcal{A}' \subset \mathcal{A}$ containing all of them and having enough injectives, see Sets, Lemma 3.12.1. Thus we may use the lemma above for $\mathcal{A}'$. This essentially means that if we use a set worth of diagrams, etc then we will never run into trouble using the lemma.

Let $\mathcal{A}, \mathcal{B}$ be abelian categories. Let $T : \mathcal{A} \to \mathcal{B}$ be a left exact functor. (We cannot use the letter F for the functor since this would conflict too much with our use of the letter F to indicate filtrations.) Note that T induces an additive functor

$$T : \mathrm{Fil}^f(\mathcal{A}) \to \mathrm{Fil}^f(\mathcal{B})$$

by the rule $T(A, F) = (T(A), F)$ where $F^pT(A) = T(F^pA)$ which makes sense as T is left exact. (Warning: It may not be the case that $\mathrm{gr}(T(A)) = T(\mathrm{gr}(A))$.) This induces functors of triangulated categories

$$T : K^+(\mathrm{Fil}^f(\mathcal{A})) \longrightarrow K^+(\mathrm{Fil}^f(\mathcal{B})) \tag{13.25.13.1}$$

The filtered right derived functor of T is the right derived functor of Definition 13.14.2 for this exact functor composed with the exact functor $K^+(\mathrm{Fil}^f(\mathcal{B})) \to DF^+(\mathcal{B})$ and the multiplicative set $\mathrm{FQis}^+(\mathcal{A})$. Assume $\mathcal{A}$ has enough injectives. At this point we can redo the discussion of Section 13.19 to define the *filtered right derived functors*

$$RT : DF^+(\mathcal{A}) \longrightarrow DF^+(\mathcal{B}) \tag{13.25.13.2}$$

of our functor T.

However, instead we will proceed as in Section 13.24, and it will turn out that we can define RT even if T is just additive. Namely, we first choose a quasi-inverse $j' : DF^+(\mathcal{A}) \to K^+(\mathcal{I}^f)$ of the equivalence of Lemma 13.25.12. By Lemma 13.4.16 we see that j' is an exact functor of triangulated categories. Next, we note that for a filtered injective object I we have a (noncanonical) decomposition

$$I \cong \bigoplus\nolimits_{p \in \mathbf{Z}} I_p, \quad \text{with} \quad F^pI = \bigoplus\nolimits_{q \geq p} I_q \tag{13.25.13.3}$$

by Lemma 13.25.2. Hence if T is any additive functor $T : \mathcal{A} \to \mathcal{B}$ then we get an additive functor

$$T_{ext} : \mathcal{I}^f \to \mathrm{Fil}^f(\mathcal{B}) \tag{13.25.13.4}$$

by setting $T_{ext}(I) = \bigoplus T(I_p)$ with $F^pT_{ext}(I) = \bigoplus_{q \geq p} T(I_q)$. Note that we have the property $\mathrm{gr}(T_{ext}(I)) = T(\mathrm{gr}(I))$ by construction. Hence we obtain a functor

$$T_{ext} : K^+(\mathcal{I}^f) \to K^+(\mathrm{Fil}^f(\mathcal{B})) \tag{13.25.13.5}$$

which commutes with gr. Then we define (13.25.13.2) by the composition

$$RT = T_{ext} \circ j'. \tag{13.25.13.6}$$

Since $RT : D^+(\mathcal{A}) \to D^+(\mathcal{B})$ is computed by injective resolutions as well, see Lemmas 13.19.1, the commutation of T with gr, and the commutative diagrams of Lemma 13.25.12 imply that

$$\mathrm{gr}^p \circ RT \cong RT \circ \mathrm{gr}^p \tag{13.25.13.7}$$

and

$$(\text{forget } F) \circ RT \cong RT \circ (\text{forget } F) \tag{13.25.13.8}$$

as functors $DF^+(\mathcal{A}) \to D^+(\mathcal{B})$.

The filtered derived functor RT (13.25.13.2) induces functors

$$\begin{gathered} RT : \mathrm{Fil}^f(\mathcal{A}) \to DF^+(\mathcal{B}), \\ RT : \mathrm{Comp}^+(\mathrm{Fil}^f(\mathcal{A})) \to DF^+(\mathcal{B}), \\ RT : KF^+(\mathcal{A}) \to DF^+(\mathcal{B}). \end{gathered}$$

Note that since $\mathrm{Fil}^f(\mathcal{A})$, and $\mathrm{Comp}^+(\mathrm{Fil}^f(\mathcal{A}))$ are no longer abelian it does not make sense to say that RT restricts to a δ-functor on them. (This can be repaired by thinking of these categories as exact categories and formulating the notion of a δ-functor from an exact category into a triangulated category.) But it does make sense, and it is true by construction, that RT is an exact functor on the triangulated category $KF^+(\mathcal{A})$.

Lemma 13.25.14. *Let $\mathcal{A}, \mathcal{B}$ be abelian categories. Let $T : \mathcal{A} \to \mathcal{B}$ be a left exact functor. Assume $\mathcal{A}$ has enough injectives. Let $(K^\bullet, F)$ be an object of $Comp^+(Fil^f(\mathcal{A}))$. There exists a spectral sequence $(E_r^{p,q}, d_r)_{r\geq 0}$ which is the spectral sequence associated to an object of $Comp^+(Fil^f(\mathcal{B}))$ with*

$$E_1^{p,q} = R^{p+q}T(gr^p(K^\bullet))$$

which converges to $R^{p+q}T(K^\bullet)$ inducing a finite filtration on each $R^nT(K^\bullet)$. Moreover the construction of this spectral sequence is functorial in the object $K^\bullet$ of $Comp^+(Fil^f(\mathcal{A}))$. In fact the terms (E_r, d_r) for $r \geq 2$ do not depend on any choices.

Proof. Choose a filtered quasi-isomorphism $K^\bullet \to I^\bullet$ with $I^\bullet$ a bounded below complex of filtered injective objects, see Lemma 13.25.9. Consider the complex $RT(K^\bullet) = T_{ext}(I^\bullet)$, see (13.25.13.6). Thus we can consider the spectral sequence $(E_r, d_r)_{r\geq 0}$ associated to this as a filtered complex in $\mathcal{B}$, see Homology, Section 12.21. By Homology, Lemma 12.21.2 we have $E_1^{p,q} = H^{p+q}(\mathrm{gr}^p(T(I^\bullet)))$. By Equation (13.25.13.3) we have $E_1^{p,q} = H^{p+q}(T(\mathrm{gr}^p(I^\bullet)))$, and by definition of a filtered injective resolution the map $\mathrm{gr}^p(K^\bullet) \to \mathrm{gr}^p(I^\bullet)$ is an injective resolution. Hence $E_1^{p,q} = R^{p+q}T(\mathrm{gr}^p(K^\bullet))$.

On the other hand, each I^n has a finite filtration and hence each $T(I^n)$ has a finite filtration. Thus we may apply Homology, Lemma 12.21.9 to conclude that the spectral sequence converges to $H^n(T(I^\bullet)) = R^nT(K^\bullet)$ moreover inducing finite filtrations on each of the terms.

Suppose that $K^\bullet \to L^\bullet$ is a morphism of $\mathrm{Comp}^+(\mathrm{Fil}^f(\mathcal{A}))$. Choose a filtered quasi-isomorphism $L^\bullet \to J^\bullet$ with $J^\bullet$ a bounded below complex of filtered injective objects, see Lemma 13.25.9. By our results above, for example Lemma 13.25.11, there exists a diagram

$$\begin{array}{ccc} K^\bullet & \longrightarrow & L^\bullet \\ \downarrow & & \downarrow \\ I^\bullet & \longrightarrow & J^\bullet \end{array}$$

which commutes up to homotopy. Hence we get a morphism of filtered complexes $T(I^\bullet) \to T(J^\bullet)$ which gives rise to the morphism of spectral sequences, see Homology, Lemma 12.21.4. The last statement follows from this. □

Remark 13.25.15. As promised in Remark 13.20.4 we discuss the connection of the lemma above with the constructions using Cartan-Eilenberg resolutions. Namely, assume the notations of Lemma 13.20.3. In particular $K^\bullet$ is a bounded below complex of $\mathcal{A}$ and $T : \mathcal{A} \to \mathcal{B}$ is a left exact functor. We give an alternative construction of the spectral sequences ${}'E$ and ${}''E$

First spectral sequence. Consider the "stupid" filtration on $K^\bullet$ obtained by setting $F^p(K^\bullet) = \sigma_{\geq p}(K^\bullet)$, see Homology, Section 12.13. Note that this stupid in the sense that $d(F^p(K^\bullet)) \subset F^{p+1}(K^\bullet)$, compare Homology, Lemma 12.21.3. Note that

$\mathrm{gr}^p(K^\bullet) = K^p[p]$ with this filtration. According to the above there is a spectral sequence with E_1 term $E_1^{p,q} = R^{p+q}T(K^p[p])$. Then the E_2 term is clearly $E_2^{p,q} = H^p(R^{p+q}T(K^\bullet))$ as in the spectral sequence ${}'E_r$.

Second spectral sequence. Consider the filtration on the complex $K^\bullet$ obtained by setting $F^p(K^\bullet) = \tau_{\leq -p}(K^\bullet)$, see Homology, Section 12.13. The minus sign is necessary to get a decreasing filtration. Note that $\mathrm{gr}^p(K^\bullet)$ is quasi-isomorphic to $H^{-p}(K^\bullet)[-p]$ with this filtration. According to the above there is a spectral sequence with E_1 term

$$E_1^{p,q} = R^{p+q}T(H^{-p}(K^\bullet)[-p]) = R^{2p+q}T(H^{-p}(K^\bullet)) = {}''E_2^{i,j}$$

with $i = 2p+q$ and $j = -p$. (This looks unnatural, but note that we could just have well developed the whole theory of filtered complexes using increasing filtrations, with the end result that this then looks natural, but the other one doesn't.) We leave it to the reader to see that the differentials match up.

Actually, given a Cartan-Eilenberg resolution $K^\bullet \to I^{\bullet,\bullet}$ the induced morphism $K^\bullet \to sI^\bullet$ into the associated simple complex will be a filtered injective resolution for either filtration using suitable filtrations on $sI^\bullet$. This can be used to match up the spectral sequences exactly.

13.26. Ext groups

In this section we start describing the ext groups of objects of an abelian category. First we have the following very general definition.

Definition 13.26.1. Let $\mathcal{A}$ be an abelian category. Let $i \in \mathbf{Z}$. Let X, Y be objects of $D(\mathcal{A})$. The *ith extension group* of X by Y is the group

$$\mathrm{Ext}^i_{\mathcal{A}}(X, Y) = \mathrm{Hom}_{D(\mathcal{A})}(X, Y[i]) = \mathrm{Hom}_{D(\mathcal{A})}(X[-i], Y).$$

If $A, B \in \mathrm{Ob}(\mathcal{A})$ we set $\mathrm{Ext}^i_{\mathcal{A}}(A, B) = \mathrm{Ext}^i_{\mathcal{A}}(A[0], B[0])$.

Since $\mathrm{Hom}_{D(\mathcal{A})}(X, -)$, resp. $\mathrm{Hom}_{D(\mathcal{A})}(-, Y)$ is a homological, resp. cohomological functor, see Lemma 13.4.2, we see that a distinguished triangle (Y, Y', Y''), resp. (X, X', X'') leads to a long exact sequence

$$\ldots \to \mathrm{Ext}^i_{\mathcal{A}}(X, Y) \to \mathrm{Ext}^i_{\mathcal{A}}(X, Y') \to \mathrm{Ext}^i_{\mathcal{A}}(X, Y'') \to \mathrm{Ext}^{i+1}_{\mathcal{A}}(X, Y) \to \ldots$$

respectively

$$\ldots \to \mathrm{Ext}^i_{\mathcal{A}}(X'', Y) \to \mathrm{Ext}^i_{\mathcal{A}}(X', Y) \to \mathrm{Ext}^i_{\mathcal{A}}(X, Y) \to \mathrm{Ext}^{i+1}_{\mathcal{A}}(X'', Y) \to \ldots$$

Note that since $D^+(\mathcal{A})$, $D^-(\mathcal{A})$, $D^b(\mathcal{A})$ are full subcategories we may compute the ext groups by Hom groups in these categories provided X, Y are contained in them. In case the category $\mathcal{A}$ has enough injectives or enough projectives we can compute the Ext groups using injective or projective resolutions. To avoid confusion, recall that having an injective (resp. projective) resolution implies vanishing of homology in all low (resp. high) degrees, see Lemmas 13.17.2 and 13.18.2.

Lemma 13.26.2. *Let $\mathcal{A}$ be an abelian category. Let $X^\bullet, Y^\bullet \in \mathrm{Ob}(K(\mathcal{A}))$.*

(1) *Let $Y^\bullet \to I^\bullet$ be an injective resolution (Definition 13.17.1). Then*
$$Ext^i_{\mathcal{A}}(X^\bullet, Y^\bullet) = \mathrm{Hom}_{K(\mathcal{A})}(X^\bullet, I^\bullet[i]).$$
(2) *Let $P^\bullet \to X^\bullet$ be a projective resolution (Definition 13.18.1). Then*
$$Ext^i_{\mathcal{A}}(X^\bullet, Y^\bullet) = \mathrm{Hom}_{K(\mathcal{A})}(P^\bullet[-i], Y^\bullet).$$

Proof. Follows immediately from Lemma 13.17.8 and Lemma 13.18.8. □

In the rest of this section we discuss extensions of objects of the abelian category itself. First we observe the following.

Lemma 13.26.3. *Let $\mathcal{A}$ be an abelian category.*

(1) *Let X, Y be objects of $D(\mathcal{A})$. Given $a, b \in \mathbf{Z}$ such that $H^i(X) = 0$ for $i > a$ and $H^j(Y) = 0$ for $j < b$, we have $Ext^n_{\mathcal{A}}(X, Y) = 0$ for $n < b - a$ and*
$$Ext^{b-a}_{\mathcal{A}}(X, Y) = \mathrm{Hom}_{\mathcal{A}}(H^a(X), H^b(Y))$$
(2) *Let $A, B \in \mathrm{Ob}(\mathcal{A})$. For $i < 0$ we have $Ext^i_{\mathcal{A}}(B, A) = 0$. We have $Ext^0_{\mathcal{A}}(B, A) = \mathrm{Hom}_{\mathcal{A}}(B, A)$.*

Proof. Choose complexes $X^\bullet$ and $Y^\bullet$ representing X and Y. Since $Y^\bullet \to \tau_{\geq b} Y^\bullet$ is a quasi-isomorphism, we may assume that $Y^j = 0$ for $j < b$. Let $L^\bullet \to X^\bullet$ be any quasi-isomorphism. Then $\tau_{\leq a} L^\bullet \to X^\bullet$ is a quasi-isomorphism. Hence a morphism $X \to Y[n]$ in $D(\mathcal{A})$ can be represented as fs^{-1} where $s : L^\bullet \to X^\bullet$ is a quasi-isomorphism, $f : L^\bullet \to Y^\bullet[n]$ a morphism, and $L^i = 0$ for $i < a$. Note that f maps L^i to Y^{i+n}. Thus $f = 0$ if $n < b - a$ because always either L^i or Y^{i+n} is zero. If $n = b - a$, then f corresponds exactly to a morphism $H^a(X) \to H^b(Y)$. Part (2) is a special case of (1). □

Let $\mathcal{A}$ be an abelian category. Suppose that $0 \to A \to A' \to A'' \to 0$ is a short exact sequence of objects of $\mathcal{A}$. Then $0 \to A[0] \to A'[0] \to A''[0] \to 0$ leads to a distinguished triangle in $D(\mathcal{A})$ (see Lemma 13.11.1) hence a long exact sequence of Ext groups
$$0 \to \mathrm{Ext}^0_{\mathcal{A}}(B, A) \to \mathrm{Ext}^0_{\mathcal{A}}(B, A') \to \mathrm{Ext}^0_{\mathcal{A}}(B, A'') \to \mathrm{Ext}^1_{\mathcal{A}}(B, A) \to \ldots$$
Similarly, given a short exact sequence $0 \to B \to B' \to B'' \to 0$ we obtain a long exact sequence of Ext groups
$$0 \to \mathrm{Ext}^0_{\mathcal{A}}(B'', A) \to \mathrm{Ext}^0_{\mathcal{A}}(B', A) \to \mathrm{Ext}^0_{\mathcal{A}}(B, A) \to \mathrm{Ext}^1_{\mathcal{A}}(B'', A) \to \ldots$$
We may view these Ext groups as an application of the construction of the derived category. It shows one can define Ext groups and construct the long exact sequence of Ext groups without needing the existence of enough injectives or projectives. There is an alternative construction of the Ext groups due to Yoneda which avoids the use of the derived category, see [**Yon60**].

Definition 13.26.4. Let $\mathcal{A}$ be an abelian category. Let $A, B \in \mathrm{Ob}(\mathcal{A})$. A degree i *Yoneda extension* of B by A is an exact sequence
$$E : 0 \to A \to Z_{i-1} \to Z_{i-2} \to \ldots \to Z_0 \to B \to 0$$
in $\mathcal{A}$. We say two Yoneda extensions E and E' of the same degree are *equivalent* if there exists a commutative diagram
$$\begin{array}{ccccccccccc}
0 & \longrightarrow & A & \longrightarrow & Z_{i-1} & \longrightarrow & \ldots & \longrightarrow & Z_0 & \longrightarrow & B & \longrightarrow & 0 \\
 & & \uparrow{\scriptstyle \mathrm{id}} & & \uparrow & & & & \uparrow & & \uparrow{\scriptstyle \mathrm{id}} & & \\
0 & \longrightarrow & A & \longrightarrow & Z''_{i-1} & \longrightarrow & \ldots & \longrightarrow & Z''_0 & \longrightarrow & B & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle \mathrm{id}} & & \downarrow & & & & \downarrow & & \downarrow{\scriptstyle \mathrm{id}} & & \\
0 & \longrightarrow & A & \longrightarrow & Z'_{i-1} & \longrightarrow & \ldots & \longrightarrow & Z'_0 & \longrightarrow & B & \longrightarrow & 0
\end{array}$$

where the middle row is a Yoneda extension as well.

It is not immediately clear that the equivalence of the definition is an equivalence relation. Although it is instructive to prove this directly this will also follow from Lemma 13.26.5 below.

Let $\mathcal{A}$ be an abelian category with objects A, B. Given a Yoneda extension $E : 0 \to A \to Z_{i-1} \to Z_{i-2} \to \ldots \to Z_0 \to B \to 0$ we define an associated element $\delta(E) \in \mathrm{Ext}^i(B, A)$ as the morphism $\delta(E) = fs^{-1} : B[0] \to A[i]$ where s is the quasi-isomorphism

$$(\ldots \to 0 \to A \to Z_{i-1} \to \ldots \to Z_0 \to 0 \to \ldots) \longrightarrow B[0]$$

and f is the morphism of complexes

$$(\ldots \to 0 \to A \to Z_{i-1} \to \ldots \to Z_0 \to 0 \to \ldots) \longrightarrow A[i]$$

We call $\delta(E) = fs^{-1}$ the *class* of the Yoneda extension. It turns out that this class characterizes the equivalence class of the Yoneda extension.

Lemma 13.26.5. *Let $\mathcal{A}$ be an abelian category with objects A, B. Any element in $Ext^i_{\mathcal{A}}(B, A)$ is $\delta(E)$ for some degree i Yoneda extension of B by A. Given two Yoneda extensions E, E' of the same degree then E is equivalent to E' if and only if $\delta(E) = \delta(E')$.*

Proof. Let $\xi : B[0] \to A[i]$ be an element of $\mathrm{Ext}^i_{\mathcal{A}}(B, A)$. We may write $\xi = fs^{-1}$ for some quasi-isomorphism $s : L^\bullet \to B[0]$ and map $f : L^\bullet \to A[i]$. After replacing $L^\bullet$ by $\tau_{\leq 0}L^\bullet$ we may assume that $L^i = 0$ for $i > 0$. Picture

$$\begin{array}{ccccccccc} L^{-i-1} & \longrightarrow & L^{-i} & \longrightarrow & \ldots & \longrightarrow & L^0 & \longrightarrow & B \longrightarrow 0 \\ & & \downarrow & & & & & & \\ & & A & & & & & & \end{array}$$

Then setting $Z_{i-1} = (L^{-i+1} \oplus A)/L^{-i}$ and $Z_j = L^{-j}$ for $j = i-2, \ldots, 0$ we see that we obtain a degree i extension E of B by A whose class $\delta(E)$ equals ξ.

It is immediate from the definitions that equivalent Yoneda extensions have the same class. Suppose that $E : 0 \to A \to Z_{i-1} \to Z_{i-2} \to \ldots \to Z_0 \to B \to 0$ and $E' : 0 \to A \to Z'_{i-1} \to Z'_{i-2} \to \ldots \to Z'_0 \to B \to 0$ are Yoneda extensions with the same class. By construction of $D(\mathcal{A})$ as the localization of $K(\mathcal{A})$ at the set of quasi-isomorphisms, this means there exists a complex $L^\bullet$ and quasi-isomorphisms

$$t : L^\bullet \to (\ldots \to 0 \to A \to Z_{i-1} \to \ldots \to Z_0 \to 0 \to \ldots)$$

and

$$t' : L^\bullet \to (\ldots \to 0 \to A \to Z'_{i-1} \to \ldots \to Z'_0 \to 0 \to \ldots)$$

such that $s \circ t = s' \circ t'$ and $f \circ t = f' \circ t'$, see Categories, Section 4.25. Let E'' be the degree i extension of B by A constructed from the pair $L^\bullet \to B[0]$ and $L^\bullet \to A[i]$ in the first paragraph of the proof. Then the reader sees readily that there exists "morphisms" of degree i Yoneda extensions $E'' \to E$ and $E'' \to E'$ as in the definition of equivalent Yoneda extensions (details omitted). This finishes the proof. □

Lemma 13.26.6. *Let $\mathcal{A}$ be an abelian category. Let A, B be objects of $\mathcal{A}$. Then $Ext^1_{\mathcal{A}}(B, A)$ is the group $Ext_{\mathcal{A}}(B, A)$ constructed in Homology, Definition 12.6.2.*

Proof. This is the case $i = 1$ of Lemma 13.26.5. □

13.27. Unbounded complexes

A reference for the material in this section is [**Spa88**]. The following lemma is useful to find "good" left resolutions of unbounded complexes.

Lemma 13.27.1. *Let $\mathcal{A}$ be an abelian category. Let $\mathcal{P} \subset \mathrm{Ob}(\mathcal{A})$ be a subset. Assume that every object of $\mathcal{A}$ is a quotient of an element of $\mathcal{P}$. Let $K^\bullet$ be a complex. There exists a commutative diagram*

$$\begin{array}{ccccc} P_1^\bullet & \longrightarrow & P_2^\bullet & \longrightarrow & \ldots \\ \downarrow & & \downarrow & & \\ \tau_{\leq 1}K^\bullet & \longrightarrow & \tau_{\leq 2}K^\bullet & \longrightarrow & \ldots \end{array}$$

in the category of complexes such that

(1) *the vertical arrows are quasi-isomorphisms,*
(2) *$P_1^\bullet$ is a bounded above complex with terms in $\mathcal{P}$,*
(3) *the arrows $P_n^\bullet \to P_{n+1}^\bullet$ are termwise split injections and each cokernel P_{n+1}^i/P_n^i is an element of $\mathcal{P}$.*

Proof. By Lemma 13.15.5 any bounded above complex has a resolution by a bounded above complex whose terms are in $\mathcal{P}$. Thus we obtain the first complex $P_1^\bullet$. By induction it suffices, given $P_1^\bullet, \ldots, P_n^\bullet$ to construct $P_{n+1}^\bullet$ and the maps $P_n^\bullet \to P_{n+1}^\bullet$ and $P_n^\bullet \to \tau_{\leq n+1}K^\bullet$. Consider the cone $C_1^\bullet$ of the composition $P_n^\bullet \to \tau_{\leq n}K^\bullet \to \tau_{\leq n+1}K^\bullet$. This fits into the distinguished triangle

$$P_n^\bullet \to \tau_{\leq n+1}K^\bullet \to C_1^\bullet \to P_n^\bullet[1]$$

Note that $C_1^\bullet$ is bounded above, hence we can choose a quasi-isomorphism $Q^\bullet \to C_1^\bullet$ where $Q^\bullet$ is a bounded above complex whose terms are elements of $\mathcal{P}$. Take the cone $C_2^\bullet$ of the map of complexes $Q^\bullet \to P_n^\bullet[1]$ to get the distinguished triangle

$$Q^\bullet \to P_n^\bullet[1] \to C_2^\bullet \to Q^\bullet[1]$$

By the axioms of triangulated categories we obtain a map of distinguished triangles

$$\begin{array}{ccccccc} P_n^\bullet & \longrightarrow & C_2^\bullet[-1] & \longrightarrow & Q^\bullet & \longrightarrow & P_n^\bullet[1] \\ \downarrow & & \downarrow & & \downarrow & & \downarrow \\ P_n^\bullet & \longrightarrow & \tau_{\leq n+1}K^\bullet & \longrightarrow & C_1^\bullet & \longrightarrow & P_n^\bullet[1] \end{array}$$

in the triangulated category $K(\mathcal{A})$. Set $P_{n+1}^\bullet = C_2^\bullet[-1]$. Note that (3) holds by construction. Choose an actual morphism of complexes $f : P_{n+1}^\bullet \to \tau_{\leq n+1}K^\bullet$. The left square of the diagram above commutes up to homotopy, but as $P_n^\bullet \to P_{n+1}^\bullet$ is a termwise split injection we can lift the homotopy and modify our choice of f to make it commute. Finally, f is a quasi-isomorphism, because both $P_n^\bullet \to P_n^\bullet$ and $Q^\bullet \to C_1^\bullet$ are. □

In some cases we can use the lemma above to show that a left derived functor is everywhere defined.

Proposition 13.27.2. *Let $F : \mathcal{A} \to \mathcal{B}$ be a right exact functor of abelian categories. Let $\mathcal{P} \subset \mathrm{Ob}(\mathcal{A})$ be a subset. Assume*

(1) *every object of $\mathcal{A}$ is a quotient of an element of $\mathcal{P}$,*
(2) *for any bounded above acyclic complex $P^\bullet$ of $\mathcal{A}$ with $P^n \in \mathcal{P}$ for all n the complex $F(P^\bullet)$ is exact,*
(3) *$\mathcal{A}$ and $\mathcal{B}$ have colimits of systems over $\mathbf{N}$,*
(4) *colimits over $\mathbf{N}$ are exact in both $\mathcal{A}$ and $\mathcal{B}$, and*
(5) *F commutes with colimits over $\mathbf{N}$.*

Then LF is defined on all of $D(\mathcal{A})$.

Proof. By (1) and Lemma 13.15.5 for any bounded above complex $K^\bullet$ there exists a quasi-isomorphism $P^\bullet \to K^\bullet$ with $P^\bullet$ bounded above and $P^n \in \mathcal{P}$ for all n. Suppose that $s : P^\bullet \to (P')^\bullet$ is a quasi-isomorphism of bounded above complexes consisting of objects of $\mathcal{P}$. Then $F(P^\bullet) \to F((P')^\bullet)$ is a quasi-isomorphism because $F(C(s)^\bullet)$ is acyclic by assumption (2). This already shows that LF is defined on $D^-(\mathcal{A})$ and that a bounded above complex consisting of objects of $\mathcal{P}$ computes LF, see Lemma 13.14.15.

Next, let $K^\bullet$ be an arbitrary complex of $\mathcal{A}$. Choose a diagram

$$\begin{array}{ccccc} P_1^\bullet & \longrightarrow & P_2^\bullet & \longrightarrow & \ldots \\ \downarrow & & \downarrow & & \\ \tau_{\leq 1}K^\bullet & \longrightarrow & \tau_{\leq 2}K^\bullet & \longrightarrow & \ldots \end{array}$$

as in Lemma 13.27.1. Note that the map $\operatorname{colim} P_n^\bullet \to K^\bullet$ is a quasi-isomorphism because colimits over $\mathbf{N}$ in $\mathcal{A}$ are exact and $H^i(P_n^\bullet) = H^i(K^\bullet)$ for $n > i$. We claim that

$$F(\operatorname{colim} P_n^\bullet) = \operatorname{colim} F(P_n^\bullet)$$

(termwise colimits) is $LF(K^\bullet)$, i.e., that $\operatorname{colim} P_n^\bullet$ computes LF. To see this, by Lemma 13.14.15, it suffices to prove the following claim. Suppose that

$$\operatorname{colim} Q_n^\bullet = Q^\bullet \xrightarrow{\alpha} P^\bullet = \operatorname{colim} P_n^\bullet$$

is a quasi-isomorphism of complexes, such that each $P_n^\bullet$, $Q_n^\bullet$ is a bounded above complex whose terms are in $\mathcal{P}$ and the maps $P_n^\bullet \to \tau_{\leq n}P^\bullet$ and $Q_n^\bullet \to \tau_{\leq n}Q^\bullet$ are quasi-isomorphisms. Claim: $F(\alpha)$ is a quasi-isomorphism.

The problem is that we do not assume that α is given as a colimit of maps between the complexes $P_n^\bullet$ and $Q_n^\bullet$. However, for each n we know that the solid arrows in the diagram

$$\begin{array}{ccccc} & & R^\bullet & & \\ & & \vdots & & \\ P_n^\bullet & \leftarrow & L^\bullet & \rightarrow & Q_n^\bullet \\ \downarrow & & & & \downarrow \\ \tau_{\leq n}P^\bullet & & \xrightarrow{\tau_{\leq n}\alpha} & & \tau_{\leq n}Q^\bullet \end{array}$$

are quasi-isomorphisms. Because quasi-isomorphisms form a multiplicative system in $K(\mathcal{A})$ (see Lemma 13.10.2) we can find a quasi-isomorphism $L^\bullet \to P_n^\bullet$ and map of complexes $L^\bullet \to Q_n^\bullet$ such that the diagram above commutes up to homotopy. Then $\tau_{\leq n}L^\bullet \to L^\bullet$ is a quasi-isomorphism. Hence (by the first part of the proof)

we can find a bounded above complex $R^\bullet$ whose terms are in $\mathcal{P}$ and a quasi-isomorphism $R^\bullet \to L^\bullet$ (as indicated in the diagram). Using the result of the first paragraph of the proof we see that $F(R^\bullet) \to F(P_n^\bullet)$ and $F(R^\bullet) \to F(Q_n^\bullet)$ are quasi-isomorphisms. Thus we obtain a isomorphisms $H^i(F(P_n^\bullet)) \to H^i(F(Q_n^\bullet))$ fitting into the commutative diagram

$$\begin{array}{ccc} H^i(F(P_n^\bullet)) & \longrightarrow & H^i(F(Q_n^\bullet)) \\ \downarrow & & \downarrow \\ H^i(F(P^\bullet)) & \longrightarrow & H^i(F(Q^\bullet)) \end{array}$$

The exact same argument shows that these maps are also compatible as n varies. Since by (4) and (5) we have

$$H^i(F(P^\bullet)) = H^i(F(\operatorname{colim} P_n^\bullet)) = H^i(\operatorname{colim} F(P_n^\bullet)) = \operatorname{colim} H^i(F(P_n^\bullet))$$

and similarly for $Q^\bullet$ we conclude that $H^i(\alpha) : H^i(F(P^\bullet) \to H^i(F(Q^\bullet)$ is an isomorphism and the claim follows. □

Lemma 13.27.3. *Let $\mathcal{A}$ be an abelian category. Let $\mathcal{I} \subset \mathrm{Ob}(\mathcal{A})$ be a subset. Assume that every object of $\mathcal{A}$ is a subobject of an element of $\mathcal{I}$. Let $K^\bullet$ be a complex. There exists a commutative diagram*

$$\begin{array}{ccccc} \ldots & \longrightarrow & \tau_{\geq -2}K^\bullet & \longrightarrow & \tau_{\geq -1}K^\bullet \\ & & \downarrow & & \downarrow \\ \ldots & \longrightarrow & I_2^\bullet & \longrightarrow & I_1^\bullet \end{array}$$

in the category of complexes such that

1. *the vertical arrows are quasi-isomorphisms,*
2. *$I_1^\bullet$ is a bounded above complex with terms in $\mathcal{I}$,*
3. *the arrows $I_{n+1}^\bullet \to I_n^\bullet$ are termwise split surjections and $Ker(I_{n+1}^i \to I_n^i)$ is an element of $\mathcal{I}$.*

Proof. This lemma is dual to Lemma 13.27.1. □

The following lemma is an example of why it is easier to work with unbounded derived categories. Namely, without having the unbounded derived functors, the lemma could not even be stated.

Lemma 13.27.4. *Let $F : \mathcal{A} \to \mathcal{B}$ and $G : \mathcal{B} \to \mathcal{A}$ be functors such that F is a right adjoint to G. If the derived functors $RF : D(\mathcal{A}) \to D(\mathcal{B})$ and $LG : D(\mathcal{B}) \to D(\mathcal{A})$ exist, then RF is a right adjoint to LG.*

Proof. Let $K^\bullet$ be a complex of $\mathcal{A}$ and let $M^\bullet$ be a complex of $\mathcal{B}$. Since RF is defined at $K^\bullet$, we see that the rule which assigns to a quasi-isomorphism $s : K^\bullet \to I^\bullet$ the object $F(I^\bullet)$ is essentially constant as an ind-object of $D(\mathcal{B})$ with value $RF(K^\bullet)$. Similarly, the rule which assigns to a quasi-isomorphism $t : P^\bullet \to M^\bullet$ the object $G(P^\bullet)$ is essentially constant as a pro-object of $D(\mathcal{A})$ with value $LG(M^\bullet)$.

Thus we have

$$\begin{aligned}
\mathrm{Hom}_{D(\mathcal{B})}(M^\bullet, RF(K^\bullet)) &= \mathrm{colim}_{s:K^\bullet \to I^\bullet} \mathrm{Hom}_{D(\mathcal{B})}(M^\bullet, F(I^\bullet)) \\
&= \mathrm{colim}_{s:K^\bullet \to I^\bullet} \mathrm{colim}_{t:P^\bullet \to M^\bullet} \mathrm{Hom}_{K(\mathcal{B})}(P^\bullet, F(I^\bullet)) \\
&= \mathrm{colim}_{t:P^\bullet \to M^\bullet} \mathrm{colim}_{s:K^\bullet \to I^\bullet} \mathrm{Hom}_{K(\mathcal{B})}(P^\bullet, F(I^\bullet)) \\
&= \mathrm{colim}_{t:P^\bullet \to M^\bullet} \mathrm{colim}_{s:K^\bullet \to I^\bullet} \mathrm{Hom}_{K(\mathcal{A})}(G(P^\bullet), I^\bullet) \\
&= \mathrm{colim}_{s:K^\bullet \to I^\bullet} \mathrm{Hom}_{D(\mathcal{A})}(G(P^\bullet), K^\bullet) \\
&= \mathrm{Hom}_{D(\mathcal{A})}(LG(M^\bullet), K^\bullet)
\end{aligned}$$

The first equality holds by Categories, Lemma 4.22.6. The second equality holds by the definition of morphisms in $D(\mathcal{B})$. The third equality holds by Categories, Lemma 4.14.9. The fourth equality holds because F and G are adjoint. The fifth equality holds by definition of morphism in $D(\mathcal{A})$. The sixth equality holds by Categories, Lemma 4.22.7. □

13.28. K-injective complexes

The following types of complexes can be used to compute right derived functors on the unbounded derived category.

Definition 13.28.1. Let $\mathcal{A}$ be an abelian category. A complex $I^\bullet$ is *K-injective* if for every acyclic complex $M^\bullet$ we have $\mathrm{Hom}_{K(\mathcal{A})}(M^\bullet, I^\bullet) = 0$.

In the situation of the definition we have in fact $\mathrm{Hom}_{K(\mathcal{A})}(M^\bullet[i], I^\bullet) = 0$ for all i as the translate of an acyclic complex is acyclic.

Lemma 13.28.2. *Let $\mathcal{A}$ be an abelian category. Let $I^\bullet$ be a complex. The following are equivalent*

(1) *$I^\bullet$ is K-injective,*
(2) *for every quasi-isomorphism $M^\bullet \to N^\bullet$ the map*
$$\mathrm{Hom}_{K(\mathcal{A})}(N^\bullet, I^\bullet) \to \mathrm{Hom}_{K(\mathcal{A})}(M^\bullet, I^\bullet)$$
is bijective, and
(3) *for every complex $N^\bullet$ the map*
$$\mathrm{Hom}_{K(\mathcal{A})}(N^\bullet, I^\bullet) \to \mathrm{Hom}_{D(\mathcal{A})}(N^\bullet, I^\bullet)$$
is an isomorphism.

Proof. Assume (1). Then (2) holds because the functor $\mathrm{Hom}_{K(\mathcal{A})}(-, I^\bullet)$ is cohomological and the cone on a quasi-isomorphism is acyclic.

Assume (2). A morphism $N^\bullet \to I^\bullet$ in $D(\mathcal{A})$ is of the form $fs^{-1} : N^\bullet \to I^\bullet$ where $s : M^\bullet \to N^\bullet$ is a quasi-isomorphism and $f : M^\bullet \to I^\bullet$ is a map. By (2) this corresponds to a unique morphism $N^\bullet \to I^\bullet$ in $K(\mathcal{A})$, i.e., (3) holds.

Assume (3). If $M^\bullet$ is acyclic then $M^\bullet$ is isomorphic to the zero complex in $D(\mathcal{A})$ hence $\mathrm{Hom}_{D(\mathcal{A})}(N^\bullet, I^\bullet) = 0$, whence $\mathrm{Hom}_{K(\mathcal{A})}(N^\bullet, I^\bullet) = 0$ by (3), i.e., (1) holds. □

Lemma 13.28.3. *Let $\mathcal{A}$ be an abelian category. Let (K, L, M, f, g, h) be a distinguished triangle of $K(\mathcal{A})$. If two out of K, L, M are K-injective complexes, then the third is too.*

Proof. Follows from the definition, Lemma 13.4.2, and the fact that $K(\mathcal{A})$ is a triangulated category (Proposition 13.9.3). □

Lemma 13.28.4. *Let $\mathcal{A}$ be an abelian category. A bounded below complex of injectives is K-injective.*

Proof. Follows from Lemmas 13.28.2 and 13.17.8. □

Lemma 13.28.5. *Let $\mathcal{A}$ be an abelian category. Let $F : K(\mathcal{A}) \to \mathcal{D}'$ be an exact functor of triangulated categories. Then RF is defined at every complex in $K(\mathcal{A})$ which is quasi-isomorphic to a K-injective complex. In fact, every K-injective complex computes RF.*

Proof. By Lemma 13.14.4 it suffices to show that RF is defined at a K-injective complex, i.e., it suffices to show a K-injective complex $I^\bullet$ computes RF. Any quasi-isomorphism $I^\bullet \to N^\bullet$ is a homotopy equivalence as it has an inverse by Lemma 13.28.2. Thus $I^\bullet \to I^\bullet$ is a final object of $I^\bullet/\text{Qis}(\mathcal{A})$ and we win. □

Lemma 13.28.6. *Let $\mathcal{A}$ be an abelian category. Assume every complex has a quasi-isomorphism towards a K-injective complex. Then any exact functor $F : K(\mathcal{A}) \to \mathcal{D}'$ of triangulated categories has a right derived functor*

$$RF : D(\mathcal{A}) \longrightarrow \mathcal{D}'$$

and $RF(I^\bullet) = F(I^\bullet)$ for K-injective complexes $I^\bullet$.

Proof. To see this we apply Lemma 13.14.15 with $\mathcal{I}$ the collection of K-injective complexes. Since (1) holds by assumption, it suffices to prove that if $I^\bullet \to J^\bullet$ is a quasi-isomorphism of K-injective complexes, then $F(I^\bullet) \to F(J^\bullet)$ is an isomorphism. This is clear because $I^\bullet \to J^\bullet$ is a homotopy equivalence, i.e., an isomorphism in $K(\mathcal{A})$, by Lemma 13.28.2. □

The following lemma can be generalized to limits over bigger ordinals.

Lemma 13.28.7. *Let $\mathcal{A}$ be an abelian category. Let*

$$\ldots \to I_3^\bullet \to I_2^\bullet \to I_1^\bullet$$

be an inverse system of K-injective complexes. Assume

(1) *each $I_n^\bullet$ is K-injective,*
(2) *each map $I_{n+1}^m \to I_n^m$ is a split surjection,*
(3) *the limits $I^m = \lim I_n^m$ exist.*

Then the complex $I^\bullet$ is K-injective.

Proof. Let $M^\bullet$ be an acyclic complex. Let us abbreviate $H_n(a, b) = \text{Hom}_{\mathcal{A}}(M^a, I_n^b)$. With this notation $\text{Hom}_{K(\mathcal{A})}(M^\bullet, I^\bullet)$ is the cohomology of the complex

$$\prod_m \lim_n H_n(m, m-2) \to \prod_m \lim_n H_n(m, m-1) \to \prod_m \lim_n H_n(m, m) \to \prod_m \lim_n H_n(m, m+1)$$

in the third spot from the left. We may exchange the order of $\prod$ and lim and each of the complexes

$$\prod_m H_n(m, m-2) \to \prod_m H_n(m, m-1) \to \prod_m H_n(m, m) \to \prod_m H_n(m, m+1)$$

is exact by assumption (1). By assumption (2) the maps in the systems

$$\ldots \to \prod_m H_3(m, m-2) \to \prod_m H_2(m, m-2) \to \prod_m H_1(m, m-2)$$

are surjective. Thus the lemma follows from Homology, Lemma 12.26.4. □

Remark 13.28.8. It appears that a combination of Lemmas 13.27.3, 13.28.4, and 13.28.7 produces "enough K-injectives" for any abelian category with enough injectives and countable limits. Actually, this may not work! Namely, suppose that $K^\bullet$ is a complex and $I_n^\bullet$ is the system of bounded above complexes of injectives produced by Lemma 13.27.3. Each complex $I_n^\bullet$ is K-injective by Lemma 13.28.4. Let $I^\bullet = \lim I_n^\bullet$ be the termwise limit which is K-injective by Lemma 13.28.7. The problem is that the map $K^\bullet \to I^\bullet$ may not be a quasi-isomorphism. The following lemma shows that it does work if $\mathcal{A}$ has countable exact products.

Lemma 13.28.9. *Let $\mathcal{A}$ be an abelian category having enough injectives and exact countable products. Then for every complex there is a quasi-isomorphism to a K-injective complex.*

Proof. Let $K^\bullet$ be a complex. Let $I_n^\bullet$ be the system of bounded above complexes of injectives produced by Lemma 13.27.3. Choose direct sum decompositions $I_{n+1}^p = C_{n+1}^p \oplus I_n^p$ for all $n \geq 1$. Set $C_1^p = I_1^p$. Each complex $I_n^\bullet$ is K-injective by Lemma 13.28.4. Then $I^\bullet = \lim I_n^\bullet$ exists because $I^p = \prod_{n \geq 1} C_n^p$. The $I^\bullet$ is K-injective by Lemma 13.28.7. Fix $p \in \mathbf{Z}$. We claim there is a short exact sequence

$$0 \to I^p \to \prod I_n^p \to \prod I_n^p \to 0$$

of objects of $\mathcal{A}$. Here the first map is given by the projection maps $I^p \to I_n^p$ and the second map by $(x_n) \mapsto (x_n - f_{n+1}^p(x_{n+1}))$ where $f_n^p : I_n^p \to I_{n-1}^p$ are the transition maps. In fact, the sequence is split exact as $I_n^p = \prod_{m \leq n} C_m^p$ and the f_n are the projection maps (details omitted). We obtain a short exact sequence of complexes

$$0 \to I^\bullet \to \prod I_n^\bullet \to \prod I_n^\bullet \to 0$$

Now, using that countable products are exact, we see that we get a long exact cohomology sequence

$$\ldots \to H^p(I^\bullet) \to \prod H^p(I_m^\bullet) \to \prod H^p(I_m^\bullet) \to \ldots$$

Since $I_m^\bullet$ is quasi-isomorphic to $\tau_{\geq -m} K^\bullet$ for each m we see that the system of objects $H^p(I_m^\bullet)$ (fixed p varying m) is eventually constant with value $H^p(K)$. Thus the map $\prod H^p(I_m^\bullet) \to \prod H^p(I_m^\bullet)$ is surjective with kernel $H^p(K^\bullet)$ (details omitted). This proves that $K^\bullet \to I^\bullet$ is a quasi-isomorphism as desired. □

Lemma 13.28.10. *Let $\mathcal{A}$ and $\mathcal{B}$ be abelian categories. Let $u : \mathcal{A} \to \mathcal{B}$ and $v : \mathcal{B} \to \mathcal{A}$ be additive functors. Assume*

1. *u is right adjoint to v, and*
2. *v is exact.*

Then u transforms K-injective complexes into K-injective complexes.

Proof. Let $I^\bullet$ be a K-injective complex of $\mathcal{A}$. Let $M^\bullet$ be a acyclic complex of $\mathcal{B}$. As v is exact we see that $v(M^\bullet)$ is an acyclic complex. By adjointness we get

$$0 = \mathrm{Hom}_{K(\mathcal{A})}(v(M^\bullet), I^\bullet) = \mathrm{Hom}_{K(\mathcal{B})}(M^\bullet, u(I^\bullet))$$

hence the lemma follows. □

13.29. Bounded cohomological dimension

There is another case where the unbounded derived functor exists. Namely, when the functor has bounded cohomological dimension.

Lemma 13.29.1. *Let $\mathcal{A}$ be an abelian category. Let $d : \mathrm{Ob}(\mathcal{A}) \to \{0, 1, 2, \ldots, \infty\}$ be a function. Assume that*

(1) *every object of $\mathcal{A}$ is a subobject of an object A with $d(A) = 0$,*
(2) *if $0 \to A \to B \to C \to 0$ is a short exact sequence then $d(C) \leq \max\{d(A) - 1, d(B)\}$.*

Let $K^\bullet$ be a complex such that $n + d(K^n)$ tends to $-\infty$ as $n \to -\infty$. Then there exists a quasi-isomorphism $K^\bullet \to L^\bullet$ with $d(L^n) = 0$ for all $n \in \mathbf{Z}$.

Proof. By Lemma 13.15.4 we can find a quasi-isomorphism $\sigma_{\geq 0}K^\bullet \to M^\bullet$ with $M^n = 0$ for $n < 0$ and $d(M^n) = 0$ for $n \geq 0$. Then $K^\bullet$ is quasi-isomorphic to the complex

$$\ldots \to K^{-2} \to K^{-1} \to M^0 \to M^1 \to \ldots$$

Hence we may assume that $d(K^n) = 0$ for $n \gg 0$. Note that the condition $n + d(K^n) \to -\infty$ as $n \to -\infty$ is not violated by this replacement.

We are going to improve $K^\bullet$ by an (infinite) sequence of elementary replacements. An *elementary replacement* is the following. Choose an index n such that $d(K^n) > 0$. Choose an injection $K^n \to M$ where $d(M) = 0$. Set $M' = \mathrm{Coker}(K^n \to M \oplus K^{n+1})$. Consider the map of complexes

$$\begin{array}{ccccccccc} K^\bullet : & & K^{n-1} & \longrightarrow & K^n & \longrightarrow & K^{n+1} & \longrightarrow & K^{n+2} \\ \downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow \\ (K')^\bullet : & & K^{n-1} & \longrightarrow & M & \longrightarrow & M' & \longrightarrow & K^{n+2} \end{array}$$

It is clear that $K^\bullet \to (K')^\bullet$ is a quasi-isomorphism. Moreover, it is clear that $d((K')^n) = 0$ and $d((K')^{n+1}) \leq \max\{d(K^{n+1}), d(K^n) - 1\}$ and the other values are unchanged.

To finish the proof we carefuly choose the order in which to do the elementary replacements so that for every integer m the complex $\sigma_{\geq m}K^\bullet$ is changed only a finite number of times. To do this set

$$\xi(K^\bullet) = \max\{n + d(K^n) \mid d(K^n) > 0\}$$

and

$$I = \{n \in \mathbf{Z} \mid \xi(K^\bullet) = n + d(K^n) \wedge d(K^n) > 0\}$$

Our assumption that $n + d(K^n)$ tends to $-\infty$ as $n \to -\infty$ and the fact that $d(K^n) = 0$ for $n >> 0$ implies $\xi(K^\bullet) < +\infty$ and that I is a finite set. It is clear that $\xi((K')^\bullet) \leq \xi(K^\bullet)$ for an elementary transformation as above. An elementary transformation changes the complex in degrees $\leq \xi(K^\bullet) + 1$. Hence if we can find finite sequence of elementary transformations which decrease $\xi(K^\bullet)$, then we win. However, note that if we do an elementary transformation starting with the smallest element $n \in I$, then we either decrease the size of I, or we increase $\min I$. Since every element of I is $\leq \xi(K^\bullet)$ we see that we win after a finite number of steps. $\square$

Lemma 13.29.2. *Let $F : \mathcal{A} \to \mathcal{B}$ be a left exact functor of abelian categories. If*

(1) *every object of $\mathcal{A}$ is a subobject of an object which is right acyclic for F,*

(2) *there exists an integer n such that $R^nF = 0$,*

then $RF : D(\mathcal{A}) \to D(\mathcal{B})$ exists. Any complex consisting of right acyclic objects for F computes RF and any complex is the source of a quasi-isomorphism into such a complex.

Proof. Note that the first condition implies that $RF : D^+(\mathcal{A}) \to D^+(\mathcal{B})$ exists, see Proposition 13.15.8. Let A be an object of $\mathcal{A}$. Choose an injection $A \to A'$ with A' acyclic. Then we see that $R^{n+1}F(A) = R^nF(A'/A) = 0$ by the long exact cohomology sequence. Hence we conclude that $R^{n+1}F = 0$. Continuing like this using induction we find that $R^mF = 0$ for all $m \geq n$.

We are going to use Lemma 13.29.1 with the function $d : \mathrm{Ob}(\mathcal{A}) \to \{0, 1, 2, \ldots\}$ given by $d(A) = \min\{0\} \cup \{i \mid R^iF(A) \neq 0\}$. The first assumption of Lemma 13.29.1 is our assumption (1) and the second assumption of Lemma 13.29.1 follows from the long exact cohomology sequence. Hence for every complex $K^\bullet$ there exists a quasi-isomorphism $K^\bullet \to L^\bullet$ with L^n right acyclic for F. We claim that if $L^\bullet \to M^\bullet$ is a quasi-isomorphism of complexes of right acyclic objects for F, then $F(L^\bullet) \to F(M^\bullet)$ is a quasi-isomorphism. If we prove this claim then we are done by Lemma 13.14.15. To prove the claim pick an integer $i \in \mathbf{Z}$. Consider the distinguished triangle

$$\sigma_{\geq i-n-1}L^\bullet \to \sigma_{\geq i-n-1}M^\bullet \to Q^\bullet,$$

i.e., let $Q^\bullet$ be the cone of the first map. Note that $Q^\bullet$ is bounded below and that $H^j(Q^\bullet)$ is zero except possibly for $j = i - n - 1$ or $j = i - n - 2$. We may apply RF to $Q^\bullet$. Using the spectral sequence of Lemma 13.20.3 and the assumed vanishing of cohomology (2) we conclude that $R^jF(Q^\bullet)$ is zero except possibly for $j \in \{i - n - 2, \ldots, i - 1\}$. Hence we see that $RF(\sigma_{\geq i-n-1}L^\bullet) \to RF(\sigma_{\geq i-n-1}M^\bullet)$ induces an isomorphism of cohomology objects in degrees $\geq i$. By Proposition 13.15.8 we know that $RF(\sigma_{\geq i-n-1}L^\bullet) = \sigma_{\geq i-n-1}F(L^\bullet)$ and $RF(\sigma_{\geq i-n-1}M^\bullet) = \sigma_{\geq i-n-1}F(M^\bullet)$. We conclude that $F(L^\bullet) \to F(M^\bullet)$ is an isomorphism in degree i as desired. □

Lemma 13.29.3. *Let $F : \mathcal{A} \to \mathcal{B}$ be a right exact functor of abelian categories. If*

(1) *every object of $\mathcal{A}$ is a quotient of an object which is left acyclic for F,*
(2) *there exists an integer n such that $L^nF = 0$,*

then $LF : D(\mathcal{A}) \to D(\mathcal{B})$ exists. Any complex consisting of left acyclic objects for F computes LF and any complex is the target of a quasi-isomorphism into such a complex.

Proof. This is dual to Lemma 13.29.2. □

13.30. Derived limits and colimits

In a triangulated category there is a notion of derived (co)limit which is sometimes useful.

Definition 13.30.1. Let $\mathcal{D}$ be a triangulated category. Let (K_n, f_n) be a system of objects of $\mathcal{D}$. We say an object K is a *derived colimit*, or a *homotopy colimit* of the system (K_n) if the direct sum $\bigoplus K_n$ exists and there is a distinguished triangle

$$\bigoplus K_n \to \bigoplus K_n \to K \to \bigoplus K_n[1]$$

where the map $\bigoplus K_n \to \bigoplus K_n$ is given by $1 - f_n$ in degree n. If this is the case, then we sometimes indicate this by the notation $K = \text{hocolim} K_n$.

By TR3 a derived colimit, if it exists, is unique up to (non-unique) isomorphism. Moreover, by TR1 a derived colimit of K_n exists as soon as $\bigoplus K_n$ exists. The derived category $D(Ab)$ of the category of abelian groups is an example (see lemma below).

Definition 13.30.2. Let $\mathcal{D}$ be a triangulated category. Let (K_n, f_n) be an inverse system of objects of $\mathcal{D}$. We say an object K is a *derived limit*, or a *homotopy limit* of the system (K_n) if the product $\prod K_n$ exists and there is a distinguished triangle

$$K \to \prod K_n \to \prod K_n \to K[1]$$

where the map $\prod K_n \to \prod K_n$ is given by $(k_n) \mapsto (k_n - f_{n+1}(k_{n+1}))$. If this is the case, then we sometimes indicate this by the notation $K = R\lim K_n$.

By TR3 a derived limit, if it exists, is unique up to (non-unique) isomorphism. Moreover, by TR1 a derived limit $R\lim K_n$ exists as soon as $\prod K_n$ exists. The derived category $D(Ab)$ of the category of abelian groups is an example. More generally, we have the following lemma.

Lemma 13.30.3. *Let $\mathcal{A}$ be an abelian category.*

(1) *If $\mathcal{A}$ has exact countable products, then $D(\mathcal{A})$ has countable products. In fact given a collection of complexes $K_i^\bullet$ indexed by a countable index set I the termwise product $\prod K_i^\bullet$ is the product of $K_i^\bullet$ in $D(\mathcal{A})$.*
(2) *If $\mathcal{A}$ has exact countable direct sums, then $D(\mathcal{A})$ has countable direct sums. In fact given a collection of complexes $K_i^\bullet$ indexed by a countable index set I the termwise direct sum $\bigoplus K_i^\bullet$ is the direct sum of $K_i^\bullet$ in $D(\mathcal{A})$.*

Proof. Proof of (1). Let $L^\bullet$ be a complex. Suppose given maps $\alpha_i : L^\bullet \to K_i^\bullet$ in $D(\mathcal{A})$. This means there exist quasi-isomorphisms $s_i : K_i^\bullet \to M_i^\bullet$ of complexes and maps of complexes $f_i : L^\bullet \to M_i^\bullet$ such that $\alpha_i = s_i^{-1} f_i$. By assumption the map of complexes

$$s : \prod K_i^\bullet \longrightarrow \prod M_i^\bullet$$

is a quasi-isomorphism. Hence setting $f = \prod f_i$ we see that $\alpha = s^{-1} f$ is a map in $D(\mathcal{A})$ whose composition with the projection $\prod K_i^\bullet \to K_i^\bullet$ is α_i. We omit the verification that α is unique.

Proof of (2). This is dual to (1). □

13.31. Generators of triangulated categories

In this section we briefly introduce a few of the different notions of a generator for a triangulated category. Our terminology is taken from [**BV03**] (except that we use "saturated" for what they call "épaisse", see Definition 13.6.1).

Let $\mathcal{D}$ be a triangulated category. Let E be an object of $\mathcal{D}$. Denote $\langle E \rangle_1$ the strictly full subcategory of $\mathcal{D}$ consisting of objects in $\mathcal{D}$ isomorphic to direct summands of finite direct sums

$$\bigoplus E[n_i]$$

of shifts of E. For $n > 1$ let $\langle E \rangle_n$ denote the full subcategory of $\mathcal{D}$ consisting of objects of $\mathcal{D}$ isomorphic to direct summands of objects X which fit into a distinguished triangle

$$A \to X \to B \to A[1]$$

where A and B are objects of $\langle E \rangle_{n-1}$. Each of the categories $\langle E \rangle_n$ is a strictly full additive subcategory of $\mathcal{D}$ preserved under shifts and under taking summands. But, $\langle E \rangle_n$ is not necessarily closed under "taking cones", hence not necessarily a triangulated subcategory. By the same token, the subcategory

$$\langle E \rangle = \bigcup\nolimits_n \langle E \rangle_n$$

is a strictly full, saturated, triangulated subcategory of $\mathcal{D}$ and it is the smallest such subcategory of $\mathcal{D}$ containing the object E.

Definition 13.31.1. Let $\mathcal{D}$ be a triangulated category. Let E be an object of $\mathcal{D}$.

(1) We say E is a *classical generator* of $\mathcal{D}$ if the smallest strictly full, saturated, triangulated subcategory of $\mathcal{D}$ containing E is equal to $\mathcal{D}$, in other words, if $\langle E \rangle = \mathcal{D}$.
(2) We say E is a *strong generator* of $\mathcal{D}$ if $\langle E \rangle_n = \mathcal{D}$ for some $n \geq 1$.
(3) We say E is a *weak generator* or a *generator* of $\mathcal{D}$ if for any nonzero object K of $\mathcal{D}$ there exists an integer n and a nonzero map $E \to K[n]$.

This definition can be generalized to the case of a family of objects.

Lemma 13.31.2. *Let $\mathcal{D}$ be a triangulated category. Let E, K be objects of $\mathcal{D}$. The following are equivalent*

(1) $\mathrm{Hom}(E, K[i]) = 0$ *for all* $i \in \mathbf{Z}$,
(2) $\mathrm{Hom}(E', K) = 0$ *for all* $E' \in \langle E \rangle$.

Proof. The implication (2) $\Rightarrow$ (1) is immediate. Conversely, assume (1). Then $\mathrm{Hom}(X, K) = 0$ for all X in $\langle E \rangle_1$. Arguing by induction on n and using Lemma 13.4.2 we see that $\mathrm{Hom}(X, K) = 0$ for all X in $\langle E \rangle_n$. □

Lemma 13.31.3. *Let $\mathcal{D}$ be a triangulated category. Let E be an object of $\mathcal{D}$. If E is a classical generator of $\mathcal{D}$, then E is a generator.*

Proof. Assume E is a classical generator. Let K be an object of $\mathcal{D}$ such that $\mathrm{Hom}(E, K[i]) = 0$ for all $i \in \mathbf{Z}$. By Lemma 13.31.2 $\mathrm{Hom}(E', K) = 0$ for all E' in $\langle E \rangle$. However, since $\mathcal{D} = \langle E \rangle$ we conclude that $\mathrm{id}_K = 0$, i.e., $K = 0$. □

13.32. Compact objects

Here is the definition.

Definition 13.32.1. Let $\mathcal{D}$ be an additive category with arbitrary direct sums. A *compact object* of $\mathcal{D}$ is an object K such that the map

$$\bigoplus\nolimits_{i \in I} \mathrm{Hom}_{\mathcal{D}}(K, E_i) \longrightarrow \mathrm{Hom}_{\mathcal{D}}(K, \bigoplus\nolimits_{i \in I} E_i)$$

is bijective for any set I and objects $E_i \in \mathrm{Ob}(\mathcal{D})$ parametrized by $i \in I$.

This notion turns out to be very useful in algebraic geometry. It is an intrinsic condition on objects that forces the objects to be, well, compact.

Lemma 13.32.2. *Let $\mathcal{D}$ be a (pre-)triangulated category with direct sums. Then the compact objects of $\mathcal{D}$ form the objects of a Karoubian, saturated, strictly full, (pre-)triangulated subcategory $\mathcal{D}_c$ of $\mathcal{D}$.*

Proof. Let (X, Y, Z, f, g, h) be a distinguished triangle of $\mathcal{D}$ with X and Y compact. Then it follows from Lemma 13.4.2 and the five lemma (Homology, Lemma 12.5.20) that Z is a compact object too. It is clear that if $X \oplus Y$ is compact, then X, Y are compact objects too. Hence $\mathcal{D}_c$ is a saturated triangulated subcategory. Since $\mathcal{D}$ is Karoubian by Lemma 13.4.12 we conclude that the same is true for $\mathcal{D}_c$. □

Lemma 13.32.3. *Let $\mathcal{D}$ be a triangulated category with direct sums. Let E_i, $i \in I$ be a family of compact objects of $\mathcal{D}$ such that $\bigoplus E_i$ generates $\mathcal{D}$. Then every object X of $\mathcal{D}$ can be written as*

$$X = \mathit{hocolim} X_n$$

where X_1 is a direct sum of shifts of the E_i and each transition morphism fits into a distinguished triangle $Y_n \to X_n \to X_{n+1} \to Y_n[1]$ where Y_n is a direct sum of shifts of the E_i.

Proof. Set $X_1 = \bigoplus_{(i,m,\varphi)} E_i[m]$ where the direct sum is over all triples (i, m, φ) such that $i \in I$, $m \in \mathbf{Z}$ and $\varphi : E_i[m] \to X$. Then X_1 comes equipped with a canonical morphism $X_1 \to X$. Given $X_n \to X$ we set $Y_n = \bigoplus_{(i,m,\varphi)} E_i[m]$ where the direct sum is over all triples (i, m, φ) such that $i \in I$, $m \in \mathbf{Z}$, and $\varphi : E_i[m] \to X_n$ is a morphism such that $E_i[m] \to X_n \to X$ is zero. Choose a distinguished triangle $Y_n \to X_n \to X_{n+1} \to Y_n[1]$ and let $X_{n+1} \to X$ be any morphism such that $X_n \to X_{n+1} \to X$ is the given one; such a morphism exists by our choice of Y_n. We obtain a morphism $\mathrm{hocolim} X_n \to X$ by the construction of our maps $X_n \to X$. Choose a distinguished triangle

$$C \to \mathrm{hocolim} X_n \to X \to C[1]$$

Let $E_i[m] \to C$ be a morphism. Since E_i is compact, the composition $E_i[m] \to X \to \mathrm{hocolim} X_n$ factors through X_n for some n, say by $E_i[m] \to X_n$. Then the construction of Y_n shows that the composition $E_i[m] \to X_n \to X_{n+1}$ is zero. In other words, the composition $E_i[m] \to C \to \mathrm{hocolim} X_n$ is zero. This means that our morphism $E_i[m] \to C$ comes from a morphism $E_i[m] \to X[-1]$. The construction of X_1 then shows that such morphism lifts to $\mathrm{hocolim} X_n$ and we conclude that our morphism $E_i[m] \to C$ is zero. The assumption that $\bigoplus E_i$ generates $\mathcal{D}$ implies that C is zero and the proof is done. □

Lemma 13.32.4. *With assumptions and notation as in Lemma 13.32.3. If C is a compact object and $C \to X_n$ is a morphism, then there is a factorization $C \to E \to X_n$ where E is an object of $\langle E_{i_1} \oplus \ldots \oplus E_{i_t} \rangle$ for some $i_1, \ldots, i_t \in I$.*

Proof. We prove this by induction on n. The base case $n = 1$ is clear. If $n > 1$ consider the composition $C \to X_n \to Y_{n-1}[1]$. This can be factored through some $E'[1] \to Y_{n-1}[1]$ where E' is a finite direct sum of shifts of the E_i. Let $I' \subset I$ be the finite set of indices that occur in this direct sum. Thus we obtain

$$\begin{array}{ccccccc} E' & \longrightarrow & C' & \longrightarrow & C & \longrightarrow & E'[1] \\ \downarrow & & \downarrow & & \downarrow & & \downarrow \\ Y_{n-1} & \longrightarrow & X_{n-1} & \longrightarrow & X_n & \longrightarrow & Y_{n-1}[1] \end{array}$$

By induction the morphism $C' \to X_{n-1}$ factors through $E'' \to X_{n-1}$ with E'' an object of $\langle \bigoplus_{i \in I''} E_i \rangle$ for some finite subset $I'' \subset I$. Choose a distinguished triangle

$$E' \to E'' \to E \to E'[1]$$

then E is an object of $\langle \bigoplus_{i \in I' \cup I''} E_i \rangle$. By construction and the axioms of a triangulated category we can choose morphisms $C \to E$ and a morphism $E \to X_n$ fitting into morphisms of triangles $(E', C', C) \to (E', E'', E)$ and $(E', E'', E) \to (Y_{n-1}, X_{n-1}, X_n)$. The composition $C \to E \to X_n$ may not equal the given morphism $C \to X_n$, but the compositions into Y_{n-1} are equal. Let $C \to X_{n-1}$ be a morphism that lifts the difference. By induction assumption we can factor this through a morphism $E''' \to X_{n-1}$ with E'' an object of $\langle \bigoplus_{i \in I'''} E_i \rangle$ for some finite subset $I' \subset I$. Thus we see that we get a solution on considering $E \oplus E''' \to X_n$ because $E \oplus E'''$ is an object of $\langle \bigoplus_{i \in I' \cup I'' \cup I'''} E_i \rangle$. □

Definition 13.32.5. Let $\mathcal{D}$ be a triangulated category with arbitrary direct sums. We say $\mathcal{D}$ is *compactly generated* if there exists a set E_i, $i \in I$ of compact objects such that $\bigoplus E_i$ generates $\mathcal{D}$.

The following proposition clarifies the relationship between classical generators and weak generators.

Proposition 13.32.6. *Let $\mathcal{D}$ be a triangulated category with direct sums. Let E be a compact object of $\mathcal{D}$. The following are equivalent*

1. *E is a classical generator for $\mathcal{D}_c$ and $\mathcal{D}$ is compactly generated, and*
2. *E is a generator for $\mathcal{D}$.*

Proof. If E is a classical generator for $\mathcal{D}_c$, then $\mathcal{D}_c = \langle E \rangle$. It follows formally from the assumption that $\mathcal{D}$ is compactly generated and Lemma 13.31.2 that E is a generator for $\mathcal{D}$.

The converse is more interesting. Assume that E is a generator for $\mathcal{D}$. Let X be a perfect object of $\mathcal{D}$. Apply Lemma 13.32.3 with $I = \{1\}$ and $E_1 = E$ to write

$$X = \text{hocolim} X_n$$

as in the lemma. Since X is compact we find that $X \to \text{hocolim} X_n$ factors through X_n for some n. Thus X is a direct summand of X_n. By Lemma 13.32.4 we see that X is a direct summand of an object of $\langle E \rangle$ and the lemma is proven. □

13.33. Other chapters

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CHAPTER 14

Simplicial Methods

14.1. Introduction

This is a minimal introduction to simplicial methods. We just add here whenever something is needed later on. A general reference to this material is perhaps [**GJ99**]. An example of the things you can do is the paper by Quillen on Homotopical Algebra, see [**Qui67**] or the paper on Étale Homotopy by Artin and Mazur, see [**AM69**].

14.2. The category of finite ordered sets

The category Δ is the category with

(1) objects $[0], [1], [2], \ldots$ with $[n] = \{0, 1, 2, \ldots, n\}$ and
(2) a morphism $[n] \to [m]$ is a nondecreasing map $\{0, 1, 2, \ldots, n\} \to \{0, 1, 2, \ldots, m\}$ between the corresponding sets.

Here *nondecreasing* for a map $\varphi : [n] \to [m]$ means by definition that $\varphi(i) \geq \varphi(j)$ if $i \geq j$. In other words, Δ is a category equivalent to the "big" category of finite totally ordered sets and nondecreasing maps. There are exactly $n + 1$ morphisms $[0] \to [n]$ and there is exactly 1 morphism $[n] \to [0]$. There are exactly $(n + 1)(n + 2)/2$ morphisms $[1] \to [n]$ and there are exactly $n + 2$ morphisms $[n] \to [1]$. And so on and so forth.

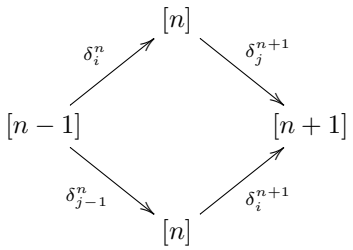
Definition 14.2.1. For any integer $n \geq 1$, and any $0 \leq j \leq n$ we let $\delta^n_j : [n-1] \to [n]$ denote the injective order preserving map skipping j. For any integer $n \geq 0$, and any $0 \leq j \leq n$ we denote $\sigma^n_j : [n+1] \to [n]$ the surjective order preserving map with $(\sigma^n_j)^{-1}(\{j\}) = \{j, j + 1\}$.

Lemma 14.2.2. *Any morphism in Δ can be written as a composition of the morphisms δ^n_j and σ^n_j.*

Proof. Let $\varphi : [n] \to [m]$ be a morphism of Δ. If $j \notin \mathrm{Im}(\varphi)$, then we can write φ as $\delta^m_j \circ \psi$ for some morphism $\psi : [n] \to [m - 1]$. If $\varphi(j) = \varphi(j + 1)$ then we can write φ as $\psi \circ \sigma^{n-1}_j$ for some morphism $\psi : [n - 1] \to [m]$. The result follows because each replacement as above lowers $n+m$ and hence at some point φ is both injective and surjective, hence an identity morphism. □

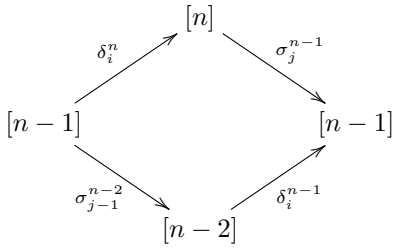
Lemma 14.2.3. *The morphisms δ^n_j and σ^n_j satisfy the following relations.*

(1) *If* $0 \leq i < j \leq n+1$, *then* $\delta_j^{n+1} \circ \delta_i^n = \delta_i^{n+1} \circ \delta_{j-1}^n$. *In other words the diagram*



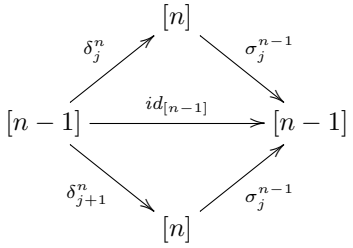
commutes.

(2) *If* $0 \leq i < j \leq n-1$, *then* $\sigma_j^{n-1} \circ \delta_i^n = \delta_i^{n-1} \circ \sigma_{j-1}^{n-2}$. *In other words the diagram*



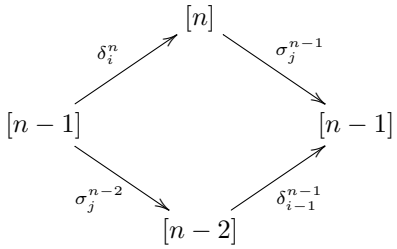
commutes.

(3) *If* $0 \leq j \leq n-1$, *then* $\sigma_j^{n-1} \circ \delta_j^n = id_{[n-1]}$ *and* $\sigma_j^{n-1} \circ \delta_{j+1}^n = id_{[n-1]}$. *In other words the diagram*



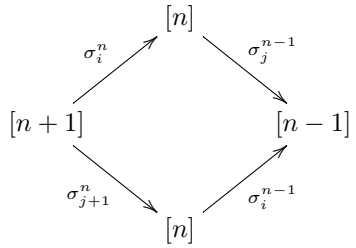
commutes.

(4) *If* $0 < j+1 < i \leq n$, *then* $\sigma_j^{n-1} \circ \delta_i^n = \delta_{i-1}^{n-1} \circ \sigma_j^{n-2}$. *In other words the diagram*



commutes.

(5) *If* $0 \leq i \leq j \leq n-1$, *then* $\sigma_j^{n-1} \circ \sigma_i^n = \sigma_i^{n-1} \circ \sigma_{j+1}^n$. *In other words the diagram*



commutes.

Proof. Omitted. □

Lemma 14.2.4. *The category* Δ *is the universal category with objects* $[n]$, $n \geq 0$ *and morphisms* δ_j^n *and* σ_j^n *such that (a) every morphism is a composition of these morphisms, (b) the relations listed in Lemma 14.2.3 are satisfied, and (c) any relation among the morphisms is a consequence of those relations.*

Proof. Omitted. □

14.3. Simplicial objects

Definition 14.3.1. Let $\mathcal{C}$ be a category.

(1) A *simplicial object* U *of* $\mathcal{C}$ is a contravariant functor U from Δ to $\mathcal{C}$, in a formula:
$$U : \Delta^{opp} \longrightarrow \mathcal{C}$$
(2) If $\mathcal{C}$ is the category of sets, then we call U a *simplicial set*.
(3) If $\mathcal{C}$ is the category of abelian groups, then we call U a *simplicial abelian group*.
(4) A *morphism of simplicial objects* $U \to U'$ is a transformation of functors.
(5) The *category of simplicial objects of* $\mathcal{C}$ is denoted $\mathrm{Simp}(\mathcal{C})$.

This means there are objects $U([0]), U([1]), U([2]), \ldots$ and for φ any nondecreasing map $\varphi : [m] \to [n]$ a morphism $U(\varphi) : U([n]) \to U([m])$, satisfying $U(\varphi \circ \psi) = U(\psi) \circ U(\varphi)$.

In particular there is a unique morphism $U([0]) \to U([n])$ and there are exactly $n+1$ morphisms $U([n]) \to U([0])$ corresponding to the $n+1$ maps $[0] \to [n]$. Obviously we need some more notation to be able to talk intelligently about these simplicial objects. We do this by considering the morphisms we singled out in Section 14.2 above.

Lemma 14.3.2. *Let* $\mathcal{C}$ *be a category.*

(1) *Given a simplicial object* U *in* $\mathcal{C}$ *we obtain a sequence of objects* $U_n = U([n])$ *endowed with the morphisms* $d_j^n = U(\delta_j^n) : U_n \to U_{n-1}$ *and* $s_j^n = U(\sigma_j^n) : U_n \to U_{n+1}$. *These morphisms satisfy the opposites of the relations displayed in Lemma 14.2.3.*
(2) *Conversely, given a sequence of objects* U_n *and morphisms* d_j^n, s_j^n *satisfying these relations there exists a unique simplicial object* U *in* $\mathcal{C}$ *such that* $U_n = U([n])$, $d_j^n = U(\delta_j^n)$, *and* $s_j^n = U(\sigma_j^n)$.

(3) *A morphism between simplicial objects U and U' is given by a family of morphisms $U_n \to U'_n$ commuting with the morphisms d_j^n and s_j^n.*

Proof. This follows from Lemma 14.2.4. □

Remark 14.3.3. By abuse of notation we sometimes write $d_i : U_n \to U_{n-1}$ instead of d_i^n, and similarly for $s_i : U_n \to U_{n+1}$. The relations among the morphisms d_i^n and s_i^n may be expressed as follows:

(1) If $i < j$, then $d_i \circ d_j = d_{j-1} \circ d_i$.
(2) If $i < j$, then $d_i \circ s_j = s_{j-1} \circ d_i$.
(3) We have $\mathrm{id} = d_j \circ s_j = d_{j+1} \circ s_j$.
(4) If $i > j + 1$, then $d_i \circ s_j = s_j \circ d_{i-1}$.
(5) If $i \leq j$, then $s_i \circ s_j = s_{j+1} \circ s_i$.

This means that whenever the compositions on both the left and the right are defined then the corresponding equality should hold.

We get a unique morphism $s_0^0 = U(\sigma_0^0) : U_0 \to U_1$ and two morphisms $d_0^1 = U(\delta_0^1)$, and $d_1^1 = U(\delta_1^1)$ which are morphisms $U_1 \to U_0$. There are two morphisms $s_0^1 = U(\sigma_0^1)$, $s_1^1 = U(\sigma_1^1)$ which are morphisms $U_1 \to U_2$. Three morphisms $d_0^2 = U(\delta_0^2)$, $d_1^2 = U(\delta_1^2)$, $d_2^2 = U(\delta_2^2)$ which are morphisms $U_3 \to U_2$. And so on.

Pictorially we think of U as follows:

$$U_2 \;\substack{\longrightarrow\\ \longleftarrow\\ \longrightarrow\\ \longleftarrow\\ \longrightarrow}\; U_1 \;\substack{\longrightarrow\\ \longleftarrow\\ \longleftarrow}\; U_0$$

Here the d-morphisms are the arrows pointing right and the s-morphisms are the arrows pointing left.

Example 14.3.4. The simplest example is the *constant* simplicial object with value $X \in \mathrm{Ob}(\mathcal{C})$. In other words, $U_n = X$ and all maps are id_X.

Example 14.3.5. Suppose that $Y \to X$ is a morphism of $\mathcal{C}$ such that all the fibred products $Y \times_X Y \times_X \ldots \times_X Y$ exist. Then we set U_n equal to the $(n+1)$-fold fibre product, and we let $\varphi : [n] \to [m]$ correspond to the map (on "coordinates") $(y_0, \ldots, y_m) \mapsto (y_{\varphi(0)}, \ldots, y_{\varphi(n)})$. In other words, the map $U_0 = Y \to U_1 = Y \times_X Y$ is the diagonal map. The two maps $U_1 = Y \times_X Y \to U_0 = Y$ are the projection maps.

Geometrically Example 14.3.5 above is an important example. It tells us that it is a good idea to think of the maps $d_j^n : U_n \to U_{n-1}$ as projection maps (forgetting the jth component), and to think of the maps $s_j^n : U_n \to U_{n+1}$ as diagonal maps (repeating the jth coordinate). We will return to this in the sections below.

Lemma 14.3.6. *Let $\mathcal{C}$ be a category. Let U be a simplicial object of $\mathcal{C}$. Each of the morphisms $s_i^n : U_n \to U_{n+1}$ has a left inverse. In particular s_i^n is a monomorphism.*

Proof. This is true because $d_i^{n+1} \circ s_i^n = \mathrm{id}_{U_n}$. □

14.4. Simplicial objects as presheaves

Another observation is that we may think of a simplicial object of $\mathcal{C}$ as a presheaf with values in $\mathcal{C}$ over Δ. See Sites, Definition 7.2.2. And in fact, if U, U' are simplicial objects of $\mathcal{C}$, then we have

$$\mathrm{Mor}(U, U') = \mathrm{Mor}_{PSh(\Delta)}(U, U'). \tag{14.4.0.1}$$

Some of the material below could be replace by the more general constructions in the chapter on sites. However, it seems a clearer picture arises from the arguments specific to simplicial objects.

14.5. Cosimplicial objects

A cosimplicial object of a category $\mathcal{C}$ could be defined simply as a simplicial object of the opposite category $\mathcal{C}^{opp}$. This is not really how the human brain works, so we introduce them separately here and point out some simple properties.

Definition 14.5.1. Let $\mathcal{C}$ be a category.

(1) A *cosimplicial object U of $\mathcal{C}$* is a covariant functor U from Δ to $\mathcal{C}$, in a formula:
$$U : \Delta \longrightarrow \mathcal{C}$$
(2) If $\mathcal{C}$ is the category of sets, then we call U a *cosimplicial set*.
(3) If $\mathcal{C}$ is the category of abelian groups, then we call U a *cosimplicial abelian group*.
(4) A *morphism of cosimplicial objects* $U \to U'$ is a transformation of functors.
(5) The *category of cosimplicial objects of $\mathcal{C}$* is denoted $\mathrm{CoSimp}(\mathcal{C})$.

This means there are objects $U([0]), U([1]), U([2]), \ldots$ and for φ any nondecreasing map $\varphi : [m] \to [n]$ a morphism $U(\varphi) : U([m]) \to U([n])$, satisfying $U(\varphi \circ \psi) = U(\varphi) \circ U(\psi)$.

In particular there is a unique morphism $U([n]) \to U([0])$ and there are exactly $n+1$ morphisms $U([0]) \to U([n])$ corresponding to the $n+1$ maps $[0] \to [n]$. Obviously we need some more notation to be able to talk intelligently about these simplicial objects. We do this by considering the morphisms we singled out in Section 14.2 above.

Lemma 14.5.2. *Let $\mathcal{C}$ be a category.*

(1) *Given a cosimplicial object U in $\mathcal{C}$ we obtain a sequence of objects $U_n = U([n])$ endowed with the morphisms $\delta^n_j = U(\delta^n_j) : U_{n-1} \to U_n$ and $\sigma^n_j = U(\sigma^n_j) : U_{n+1} \to U_n$. These morphisms satisfy the relations displayed in Lemma 14.2.3.*
(2) *Conversely, given a sequence of objects U_n and morphisms δ^n_j, σ^n_j satisfying these relations there exists a unique cosimplicial object U in $\mathcal{C}$ such that $U_n = U([n])$, $\delta^n_j = U(\delta^n_j)$, and $\sigma^n_j = U(\sigma^n_j)$.*
(3) *A morphism between cosimplicial objects U and U' is given by a family of morphisms $U_n \to U'_n$ commuting with the morphisms δ^n_j and σ^n_j.*

Proof. This follows from Lemma 14.2.4. □

Remark 14.5.3. By abuse of notation we sometimes write $\delta_i : U_{n-1} \to U_n$ instead of δ^n_i, and similarly for $\sigma_i : U_{n+1} \to U_n$. The relations among the morphisms δ^n_i and σ^n_i may be expressed as follows:

(1) If $i < j$, then $\delta_j \circ \delta_i = \delta_i \circ \delta_{j-1}$.
(2) If $i < j$, then $\sigma_j \circ \delta_i = \delta_i \circ \sigma_{j-1}$.
(3) We have $\mathrm{id} = \sigma_j \circ \delta_j = \sigma_j \circ \delta_{j+1}$.
(4) If $i > j+1$, then $\sigma_j \circ \delta_i = \delta_{i-1} \circ \sigma_j$.
(5) If $i \leq j$, then $\sigma_j \circ \sigma_i = \sigma_i \circ \sigma_{j+1}$.

This means that whenever the compositions on both the left and the right are defined then the corresponding equality should hold.

We get a unique morphism $\sigma_0^0 = U(\sigma_0^0) : U_1 \to U_0$ and two morphisms $\delta_0^1 = U(\delta_0^1)$, and $\delta_1^1 = U(\delta_1^1)$ which are morphisms $U_0 \to U_1$. There are two morphisms $\sigma_0^1 = U(\sigma_0^1)$, $\sigma_1^1 = U(\sigma_1^1)$ which are morphisms $U_2 \to U_1$. Three morphisms $\delta_0^2 = U(\delta_0^2)$, $\delta_1^2 = U(\delta_1^2)$, $\delta_2^2 = U(\delta_2^2)$ which are morphisms $U_2 \to U_3$. And so on.

Pictorially we think of U as follows:

$$U_0 \;\substack{\longrightarrow\\ \longleftarrow\\ \longrightarrow}\; U_1 \;\substack{\longrightarrow\\ \longleftarrow\\ \longleftarrow\\ \longrightarrow}\; U_2$$

Here the δ-morphisms are the arrows pointing right and the σ-morphisms are the arrows pointing left.

Example 14.5.4. The simplest example is the *constant* cosimplicial object with value $X \in \mathrm{Ob}(\mathcal{C})$. In other words, $U_n = X$ and all maps are id_X.

Example 14.5.5. Suppose that $Y \to X$ is a morphism of C such that all the pushouts $Y \coprod_X Y \coprod_X \ldots \coprod_X Y$ exist. Then we set U_n equal to the $(n+1)$-fold pushout, and we let $\varphi : [n] \to [m]$ correspond to the map

$$(y \text{ in } i\text{th component}) \mapsto (y \text{ in } \varphi(i)\text{th component})$$

on "coordinates". In other words, the map $U_1 = Y \coprod_X Y \to U_0 = Y$ is the identity on each component. The two maps $U_0 = Y \to U_1 = Y \coprod_X Y$ are the two natural maps.

Lemma 14.5.6. *Let $\mathcal{C}$ be a category. Let U be a cosimplicial object of $\mathcal{C}$. Each of the morphisms $\delta_i^n : U_{n-1} \to U_n$ has a left inverse. In particular δ_i^n is a monomorphism.*

Proof. This is true because $\sigma_i^{n-1} \circ \delta_i^n = \mathrm{id}_{U_n}$ for $j < n$. □

14.6. Products of simplicial objects

Of course we should define the product of simplicial objects as the product in the category of simplicial objects. This may lead to the potentially confusing situation where the product exists but is not described as below. To avoid this we define the product directly as follows.

Definition 14.6.1. Let $\mathcal{C}$ be a category. Let U and V be simplicial objects of $\mathcal{C}$. Assume the products $U_n \times V_n$ exist in $\mathcal{C}$. The *product of U and V* is the simplicial object $U \times V$ defined as follows:

(1) $(U \times V)_n = U_n \times V_n$,
(2) $d_i^n = (d_i^n, d_i^n)$, and
(3) $s_i^n = (s_i^n, s_i^n)$.

In other words, $U \times V$ is the product of the presheaves U and V on Δ.

Lemma 14.6.2. *If U and V are simplicial objects in the category $\mathcal{C}$, and if $U \times V$ exists, then we have*

$$\mathrm{Mor}(W, U \times V) = \mathrm{Mor}(W, U) \times \mathrm{Mor}(W, V)$$

for any third simplicial object W of $\mathcal{C}$.

Proof. Omitted. □

14.7. Fibre products of simplicial objects

Of course we should define the fibre product of simplicial objects as the fibre product in the category of simplicial objects. This may lead to the potentially confusing situation where the fibre product exists but is not described as below. To avoid this we define the fibre product directly as follows.

Definition 14.7.1. Let $\mathcal{C}$ be a category. Let U, V, W be simplicial objects of $\mathcal{C}$. Let $a : V \to U$, $b : W \to U$ be morphisms. Assume the fibre products $V_n \times_{U_n} W_n$ exist in $\mathcal{C}$. The *fibre product of V and W over U* is the simplicial object $V \times_U W$ defined as follows:

(1) $(V \times_U W)_n = V_n \times_{U_n} W_n$,
(2) $d_i^n = (d_i^n, d_i^n)$, and
(3) $s_i^n = (s_i^n, s_i^n)$.

In other words, $V \times_U W$ is the fibre product of the presheaves V and W over the presheaf U on Δ.

Lemma 14.7.2. *If U, V, W are simplicial objects in the category $\mathcal{C}$, and if $a : V \to U$, $b : W \to U$ are morphisms and if $V \times_U W$ exists, then we have*

$$\mathrm{Mor}(T, V \times_U V) = \mathrm{Mor}(T, V) \times_{\mathrm{Mor}(T,U)} \mathrm{Mor}(T, W)$$

for any fourth simplicial object T of $\mathcal{C}$.

Proof. Omitted. □

14.8. Pushouts of simplicial objects

Of course we should define the pushout of simplicial objects as the pushout in the category of simplicial objects. This may lead to the potentially confusing situation where the pushouts exist but are not as described below. To avoid this we define the pushout directly as follows.

Definition 14.8.1. Let $\mathcal{C}$ be a category. Let U, V, W be simplicial objects of $\mathcal{C}$. Let $a : U \to V$, $b : U \to W$ be morphisms. Assume the pushouts $V_n \amalg_{U_n} W_n$ exist in $\mathcal{C}$. The *pushout of V and W over U* is the simplicial object $V \amalg_U W$ defined as follows:

(1) $(V \amalg_U W)_n = V_n \amalg_{U_n} W_n$,
(2) $d_i^n = (d_i^n, d_i^n)$, and
(3) $s_i^n = (s_i^n, s_i^n)$.

In other words, $V \amalg_U W$ is the pushout of the presheaves V and W over the presheaf U on Δ.

Lemma 14.8.2. *If U, V, W are simplicial objects in the category $\mathcal{C}$, and if $a : U \to V$, $b : U \to W$ are morphisms and if $V \amalg_U W$ exists, then we have*

$$\mathrm{Mor}(V \amalg_U W, T) = \mathrm{Mor}(V, T) \times_{\mathrm{Mor}(U,T)} \mathrm{Mor}(W, T)$$

for any fourth simplicial object T of $\mathcal{C}$.

Proof. Omitted. □

14.9. Products of cosimplicial objects

Of course we should define the product of cosimplicial objects as the product in the category of cosimplicial objects. This may lead to the potentially confusing situation where the product exists but is not described as below. To avoid this we define the product directly as follows.

Definition 14.9.1. Let $\mathcal{C}$ be a category. Let U and V be cosimplicial objects of $\mathcal{C}$. Assume the products $U_n \times V_n$ exist in $\mathcal{C}$. The *product of U and V* is the cosimplicial object $U \times V$ defined as follows:

(1) $(U \times V)_n = U_n \times V_n$,
(2) for any $\varphi : [n] \to [m]$ the map $(U \times V)(\varphi) : U_n \times V_n \to U_m \times V_m$ is the product $U(\varphi) \times V(\varphi)$.

Lemma 14.9.2. *If U and V are cosimplicial objects in the category $\mathcal{C}$, and if $U \times V$ exists, then we have*

$$\mathrm{Mor}(W, U \times V) = \mathrm{Mor}(W, U) \times \mathrm{Mor}(W, V)$$

for any third cosimplicial object W of $\mathcal{C}$.

Proof. Omitted. □

14.10. Fibre products of cosimplicial objects

Of course we should define the fibre product of cosimplicial objects as the fibre product in the category of cosimplicial objects. This may lead to the potentially confusing situation where the product exists but is not described as below. To avoid this we define the fibre product directly as follows.

Definition 14.10.1. Let $\mathcal{C}$ be a category. Let U, V, W be cosimplicial objects of $\mathcal{C}$. Let $a : V \to U$ and $b : W \to U$ be morphisms. Assume the fibre products $V_n \times_{U_n} W_n$ exist in $\mathcal{C}$. The *fibre product of V and W over U* is the cosimplicial object $V \times_U W$ defined as follows:

(1) $(V \times_U W)_n = V_n \times_{U_n} W_n$,
(2) for any $\varphi : [n] \to [m]$ the map $(V \times_U W)(\varphi) : V_n \times_{U_n} W_n \to V_m \times_{U_m} W_m$ is the product $V(\varphi) \times_{U(\varphi)} W(\varphi)$.

Lemma 14.10.2. *If U, V, W are cosimplicial objects in the category $\mathcal{C}$, and if $a : V \to U$, $b : W \to U$ are morphisms and if $V \times_U W$ exists, then we have*

$$\mathrm{Mor}(T, V \times_U W) = \mathrm{Mor}(T, V) \times_{\mathrm{Mor}(T,U)} \mathrm{Mor}(T, W)$$

for any fourth cosimplicial object T of $\mathcal{C}$.

Proof. Omitted. □

14.11. Simplicial sets

Let U be a simplicial set. It is a good idea to think of U_0 as the 0-*simplices*, the set U_1 as the 1-*simplices*, the set U_2 as the 2-*simplices*, and so on.

We think of the maps $s_j^n : U_n \to U_{n+1}$ as the map that associates to an n-simplex A the degenerate $(n+1)$-simplex B whose $(j, j+1)$-edge is collapsed to the vertex j of A. We think of the map $d_j^n : U_n \to U_{n-1}$ as the map that associates to an n-simplex A one of the faces, namely the face that omits the vertex j. In this way it become possible to visualize the relations among the maps s_j^n and d_j^n geometrically.

Definition 14.11.1. Let U be a simplicial set. We say x is an *n-simplex of U* to signify that x is an element of U_n. We say that y is the $j$the *face of x* to signify that $d_j^n x = y$. We say that z is the jth *degeneracy of x* if $z = s_j^n x$. A simplex is called *degenerate* if it is the degeneracy of another simplex.

Here are a few fundamental examples.

Example 14.11.2. For every $n \geq 0$ we denote $\Delta[n]$ the simplicial set

$$\begin{aligned} \Delta^{opp} &\longrightarrow Sets \\ [k] &\longmapsto \mathrm{Mor}_\Delta([k], [n]) \end{aligned}$$

We leave it to the reader to verify the following statements. Every m-simplex of $\Delta[n]$ with $m > n$ is degenerate. There is a unique nondegenerate n-simplex of $\Delta[n]$, namely $\mathrm{id}_{[n]}$.

Lemma 14.11.3. *Let U be a simplicial set. Let $n \geq 0$ be an integer. There is a canonical bijection*

$$\mathrm{Mor}(\Delta[n], U) \longrightarrow U_n$$

which maps a morphism φ to the value of φ on the unique nondegenerate n-simplex of $\Delta[n]$.

Proof. Omitted. □

Example 14.11.4. Consider the category $\Delta/[n]$ of objects over $[n]$ in Δ, see Categories, Example 4.2.13. There is a functor $p : \Delta/[n] \to \Delta$. The fibre category of p over $[k]$, see Categories, Section 4.33, has as objects the set $\Delta[n]_k$ of k-simplices in $\Delta[n]$, and as morphisms only identities. For every morphism $\varphi : [k] \to [l]$ of Δ, and every object $\psi : [l] \to [n]$ in the fibre category over $[l]$ there is a unique object over $[k]$ with a morphism covering φ, namely $\psi \circ \varphi : [k] \to [n]$. Thus $\Delta/[n]$ is fibred in sets over Δ. In other words, we may think of $\Delta/[n]$ as a presheaf of sets over Δ. See also, Categories, Example 4.36.7. And this presheaf of sets agrees with the simplicial set $\Delta[n]$. In particular, from Equation (14.4.0.1) and Lemma 14.11.3 above we get the formula

$$\mathrm{Mor}_{PSh(\Delta)}(\Delta/[n], U) = U_n$$

for any simplicial set U.

Lemma 14.11.5. *Let U, V be simplicial sets. Let $a, b \geq 0$ be integers. Assume every n-simplex of U is degenerate if $n > a$. Assume every n-simplex of V is degenerate if $n > b$. Then every n-simplex of $U \times V$ is degenerate if $n > a + b$.*

Proof. Suppose $n > a+b$. Let $(u, v) \in (U \times V)_n = U_n \times V_n$. By assumption, there exists a $\alpha : [n] \to [a]$ and a $u' \in U_a$ and a $\beta : [n] \to [b]$ and a $v' \in V_b$ such that $u = U(\alpha)(u')$ and $v = V(\beta)(v')$. Because $n > a + b$, there exists an $0 \leq i \leq a + b$ such that $\alpha(i) = \alpha(i + 1)$ and $\beta(i) = \beta(i + 1)$. It follows immediately that (u, v) is in the image of s_i^{n-1}. □

14.12. Products with simplicial sets

Let $\mathcal{C}$ be a category. Let U be a simplicial set. Let V be a simplicial object of $\mathcal{C}$. We can consider the covariant functor which associates to a simplicial object W of

$\mathcal{C}$ the set
(14.12.0.1)

$$\left\{(f_{n,u} : V_n \to W_n)_{n\geq 0, u\in U_n} \text{ such that } \begin{array}{c}\forall \varphi : [m] \to [n] \\ f_{m,\varphi(u)} \circ V(\varphi) = W(\varphi) \circ f_{n,u}\end{array}\right\}$$

If this functor is of the form $\mathrm{Mor}_{\mathrm{Simp}(\mathcal{C})}(Q, -)$ then we can think of Q as the product of U with V. Instead of formalizing this in this way we just directly define the product as follows.

Definition 14.12.1. Let $\mathcal{C}$ be a category such that the coproduct of any two objects of $\mathcal{C}$ exists. Let U be a simplicial set. Let V be a simplicial object of $\mathcal{C}$. Assume that each U_n is finite nonempty. In this case we define the *product $U \times V$ of U and V* to be the simplicial object of $\mathcal{C}$ whose nth term is the object

$$(U \times V)_n = \coprod_{u\in U_n} V_n$$

with maps for $\varphi : [m] \to [n]$ given by the morphism

$$\coprod_{u\in U_n} V_n \longrightarrow \coprod_{u'\in U_m} V_m$$

which maps the component V_n corresponding to u to the component V_m corresponding to $u' = U(\varphi)(u)$ via the morphism $V(\varphi)$. More loosely, if all of the coproducts displayed above exist (without assuming anything about $\mathcal{C}$) we will say that the *product $U \times V$ exists*.

Lemma 14.12.2. *Let $\mathcal{C}$ be a category such that the coproduct of any two objects of $\mathcal{C}$ exists. Let U be a simplicial set. Let V be a simplicial object of $\mathcal{C}$. Assume that each U_n is finite nonempty. The functor $W \mapsto \mathrm{Mor}_{Simp(\mathcal{C})}(U \times V, W)$ is canonically isomorphic to the functor which maps W to the set in Equation (14.12.0.1).*

Proof. Omitted. □

Lemma 14.12.3. *Let $\mathcal{C}$ be a category such that the coproduct of any two objects of $\mathcal{C}$ exists. Let us temporarily denote FSSets the category of simplicial sets all of whose components are finite nonempty.*

(1) *The rule $(U, V) \mapsto U \times V$ defines a functor FSSets $\times$ Simp($\mathcal{C}$) $\to$ Simp($\mathcal{C}$).*
(2) *For every U, V as above there is a canonical map of simplicial objects*

$$U \times V \longrightarrow V$$

defined by taking the identity on each component of $(U \times V)_n = \coprod_u V_n$.

Proof. Omitted. □

We briefly study a special case of the construction above. Let $\mathcal{C}$ be a category. Let X be an object of $\mathcal{C}$. Let $k \geq 0$ be an integer. If all coproducts $X \coprod \ldots \coprod X$ exist then according to the definition above the product

$$X \times \Delta[k]$$

exists, where we think of X as the corresponding constant simplicial object.

Lemma 14.12.4. *With X and k as above. For any simplicial object V of $\mathcal{C}$ we have the following canonical bijection*

$$\mathrm{Mor}_{Simp(\mathcal{C})}(X \times \Delta[k], V) \longrightarrow \mathrm{Mor}_{\mathcal{C}}(X, V_k).$$

wich maps γ to the restriction of the morphism γ_k to the component corresponding to $id_{[k]}$. Similarly, for any $n \geq k$, if W is an n-truncated simplicial object of $\mathcal{C}$, then we have

$$\mathrm{Mor}_{Simp_n(\mathcal{C})}(sk_n(X \times \Delta[k]), W) = \mathrm{Mor}_{\mathcal{C}}(X, W_k).$$

Proof. A morphism $\gamma : X \times \Delta[k] \to V$ is given by a family of morphisms $\gamma_\alpha : X \to V_n$ where $\alpha : [n] \to [k]$. The morphisms have to satisfy the rules that for all $\varphi : [m] \to [n]$ the diagrams

$$\begin{array}{ccc} X & \xrightarrow{\gamma_\alpha} & V_n \\ \downarrow{\scriptstyle \mathrm{id}_X} & & \downarrow{\scriptstyle V(\varphi)} \\ X & \xrightarrow{\gamma_{\alpha\circ\varphi}} & V_m \end{array}$$

commute. Taking $\alpha = \mathrm{id}_{[k]}$, we see that for any $\varphi : [m] \to [k]$ we have $\gamma_\varphi = V(\varphi) \circ \gamma_{\mathrm{id}_{[k]}}$. Thus the morphism γ is determined by the value of γ on the component corresponding to $\mathrm{id}_{[k]}$. Conversely, given such a morphism $f : X \to V_k$ we easily construct a morphism γ by putting $\gamma_\alpha = V(\alpha) \circ f$.

The truncated case is similar, and left to the reader. □

A particular example of this is the case $k = 0$. In this case the formula of the lemma just says that

$$\mathrm{Mor}_{\mathcal{C}}(X, V_0) = \mathrm{Mor}_{\mathrm{Simp}(\mathcal{C})}(X, V)$$

where on the right hand side X indicates the constant simplicial object with value X. We will use this formula without further mention in the following.

14.13. Hom from simplicial sets into cosimplicial objects

Let $\mathcal{C}$ be a category. Let U be a simplicial object of $\mathcal{C}$, and let V be a cosimplicial object of $\mathcal{C}$. Then we get a cosimplicial set $\mathrm{Hom}_{\mathcal{C}}(U, V)$ as follows:

(1) we set $\mathrm{Hom}_{\mathcal{C}}(U, V)_n = \mathrm{Mor}_{\mathcal{C}}(U_n, V_n)$, and
(2) for $\varphi : [m] \to [n]$ we take the map $\mathrm{Hom}_{\mathcal{C}}(U, V)_m \to \mathrm{Hom}_{\mathcal{C}}(U, V)_n$ given by $f \mapsto V(\varphi) \circ f \circ U(\varphi)$.

This is our motivation for the following definition.

Definition 14.13.1. Let $\mathcal{C}$ be a category with finite products. Let V be a cosimplicial object of $\mathcal{C}$. Let U be a simplicial set such that each U_n is finite nonempty. We define $\mathrm{Hom}(U, V)$ to be the cosimplicial object of $\mathcal{C}$ defined as follows:

(1) we set $\mathrm{Hom}(U, V)_n = \prod_{u \in U_n} V_n$, in other words the unique object of $\mathcal{C}$ such that its X-valued points satisfy

$$\mathrm{Mor}_{\mathcal{C}}(X, \mathrm{Hom}(U, V)_n) = \mathrm{Map}(U_n, \mathrm{Mor}_{\mathcal{C}}(X, V_n))$$

and
(2) for $\varphi : [m] \to [n]$ we take the map $\mathrm{Hom}(U, V)_m \to \mathrm{Hom}(U, V)_n$ given by $f \mapsto V(\varphi) \circ f \circ U(\varphi)$ on X-valued points as above.

We leave it to the reader to spell out the definition in terms of maps between products. We also point out that the construction is functorial in both U (contravariantly) and V (covariantly), exactly as in Lemma 14.12.3 in the case of products of simplicial sets with simplicial objects.

14.14. Internal Hom

Let $\mathcal{C}$ be a category with finite nonempty products. Let U, V be simplicial objects $\mathcal{C}$. In some cases the functor

$$\begin{aligned} \mathrm{Simp}(\mathcal{C})^{opp} &\longrightarrow \mathit{Sets} \\ W &\longmapsto \mathrm{Mor}_{\mathrm{Simp}(\mathcal{C})}(W \times V, U) \end{aligned}$$

is representable. In this case we denote $\mathcal{H}om(V, U)$ the resulting simplicial object of $\mathcal{C}$, and we say that the *internal hom of V into U exists*. Moreover, in this case we would have

$$\begin{aligned} \mathrm{Mor}_{\mathcal{C}}(X, \mathcal{H}om(V,U)_n) &= \mathrm{Mor}_{\mathrm{Simp}(\mathcal{C})}(X \times \Delta[n], \mathcal{H}om(V,U)) \\ &= \mathrm{Mor}_{\mathrm{Simp}(\mathcal{C})}(X \times \Delta[n] \times V, U) \\ &= \mathrm{Mor}_{\mathrm{Simp}(\mathcal{C})}(X, \mathcal{H}om(\Delta[n] \times V, U)) \\ &= \mathrm{Mor}_{\mathcal{C}}(X, \mathcal{H}om(\Delta[n] \times V, U)_0) \end{aligned}$$

provided that $\mathcal{H}om(\Delta[n] \times V, U)$ exists also. Here we have used the material from Section 14.12.

The lesson we learn from this is that, given U and V, if we want to construct the internal hom then we should try to construct the objects

$$\mathcal{H}om(\Delta[n] \times V, U)_0$$

because these should be the nth term of $\mathcal{H}om(V, U)$. In the next section we study a construction of simplicial objects "$\mathrm{Hom}(\Delta[n], U)$".

14.15. Hom from simplicial sets into simplicial objects

Motivated by the discussion on internal hom we define what should be the simplicial object classifying morphisms from a simplicial set into a given simplicial object of the category $\mathcal{C}$.

Definition 14.15.1. Let $\mathcal{C}$ be a category such that the coproduct of any two objects exists. Let U be a simplicial set, with U_n finite nonempty for all $n \geq 0$. Let V be a simplicial object of $\mathcal{C}$. We denote $\mathrm{Hom}(U, V)$ any simplicial object of $\mathcal{C}$ such that

$$\mathrm{Mor}_{\mathrm{Simp}(\mathcal{C})}(W, \mathrm{Hom}(U, V)) = \mathrm{Mor}_{\mathrm{Simp}(\mathcal{C})}(W \times U, V)$$

functorially in the simplicial object W of $\mathcal{C}$.

Of course $\mathrm{Hom}(U, V)$ need not exist. Also, by the discussion in Section 14.14 we expect that if it does exist, then $\mathrm{Hom}(U, V)_n = \mathrm{Hom}(U \times \Delta[n], V)_0$. We do not use the italic notation for these Hom objects since $\mathrm{Hom}(U, V)$ is not an internal hom.

Lemma 14.15.2. *Assume the category $\mathcal{C}$ has coproducts of any two objects and countable limits. Let U be a simplicial set, with U_n finite nonempty for all $n \geq 0$. Let V be a simplicial object of $\mathcal{C}$. Then the functor*

$$\begin{aligned} \mathcal{C}^{opp} &\longrightarrow \mathit{Sets} \\ X &\longmapsto \mathrm{Mor}_{\mathit{Simp}(\mathcal{C})}(X \times U, V) \end{aligned}$$

is representable.

Proof. A morphism from $X \times U$ into V is given by a collection of morphisms $f_u : X \to V_n$ with $n \geq 0$ and $u \in U_n$. And such a collection actually defines a morphism if and only if for all $\varphi : [m] \to [n]$ all the diagrams

$$\begin{array}{ccc} X & \xrightarrow{f_u} & V_n \\ {\scriptstyle \mathrm{id}_X}\downarrow & & \downarrow{\scriptstyle V(\varphi)} \\ X & \xrightarrow{f_{U(\varphi)(u)}} & V_m \end{array}$$

commute. Thus it is natural to introduce a category $\mathcal{U}$ and a functor $\mathcal{V} : \mathcal{U}^{opp} \to \mathcal{C}$ as follows:

(1) The set of objects of $\mathcal{U}$ is $\coprod_{n \geq 0} U_n$,
(2) a morphism from $u' \in U_m$ to $u \in U_n$ is a $\varphi : [m] \to [n]$ such that $U(\varphi)(u) = u'$
(3) for $u \in U_n$ we set $\mathcal{V}(u) = V_n$, and
(4) for $\varphi : [m] \to [n]$ such that $U(\varphi)(u) = u'$ we set $\mathcal{V}(\varphi) = V(\varphi) : V_n \to V_m$.

At this point it is clear that our functor is nothing but the functor defining

$$\lim_{\mathcal{U}^{opp}} \mathcal{V}$$

Thus if $\mathcal{C}$ has countable limits then this limit and hence an object representing the functor of the lemma exist. □

Lemma 14.15.3. *Assume the category $\mathcal{C}$ has coproducts of any two objects and finite limits. Let U be a simplicial set, with U_n finite nonempty for all $n \geq 0$. Assume that all n-simplices of U are degenerate for all $n \gg 0$. Let V be a simplicial object of $\mathcal{C}$. Then the functor*

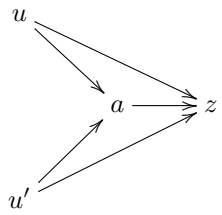
$$\begin{array}{ccc} \mathcal{C}^{opp} & \longrightarrow & \mathit{Sets} \\ X & \longmapsto & \mathrm{Mor}_{Simp(\mathcal{C})}(X \times U, V) \end{array}$$

is representable.

Proof. We have to show that the category $\mathcal{U}$ described in the proof of Lemma 14.15.2 has a finite subcategory $\mathcal{U}'$ such that the limit of $\mathcal{V}$ over $\mathcal{U}'$ is the same as the limit of $\mathcal{V}$ over $\mathcal{U}$. We will use Categories, Lemma 4.17.4. For $m > 0$ let $\mathcal{U}_{\leq m}$ denote the full subcategory with objects $\coprod_{0 \leq n \leq m} U_m$. Let m_0 be an integer such that every n-simplex of the simplicial set U is degenerate if $n > m_0$. For any $m \geq m_0$ large enough, the subcategory $\mathcal{U}_{\leq m}$ satisfies property (1) of Categories, Definition 4.17.3.

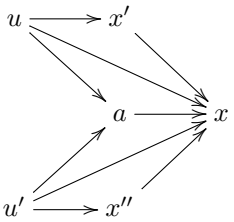
Suppose that $u \in U_n$ and $u' \in U_{n'}$ with $n, n' \leq m_0$ and suppose that $\varphi : [k] \to [n]$, $\varphi' : [k] \to [n']$ are morphisms such that $U(\varphi)(u) = U(\varphi')(u')$. A simple combinatorial argument shows that if $k > 2m_0$, then there exists an index $0 \leq i \leq 2m_0$ such that $\varphi(i) = \varphi(i+1)$ and $\varphi'(i) = \varphi'(i+1)$. (The pigeon hole principle would tell you this works if $k > m_0^2$ which is good enough for the argument below anyways.) Hence, if $k > 2m_0$, we may write $\varphi = \psi \circ \sigma_i^{k-1}$ and $\varphi' = \psi' \circ \sigma_i^{k-1}$ for some $\psi : [k-1] \to [n]$ and some $\psi' : [k-1] \to [n']$. Since $s_i^{k-1} : U_{k-1} \to U_k$ is injective, see Lemma 14.3.6, we conclude that $U(\psi)(u) = U(\psi')(u')$ also. Continuing in this fashion we conclude that given morphisms $u \to z$ and $u' \to z$ of $\mathcal{U}$ with $u, u' \in \mathcal{U}_{\leq m_0}$,

there exists a commutative diagram



with $a \in \mathcal{U}_{\leq 2m_0}$.

It is easy to deduce from this that the finite subcategory $\mathcal{U}_{\leq 2m_0}$ works. Namely, suppose given $x' \in U_n$ and $x'' \in U_{n'}$ with $n, n' \leq 2m_0$ as well as morphisms $x' \to x$ and $x'' \to x$ of $\mathcal{U}$ with the same target. By our choice of m_0 we can find objects u, u' of $\mathcal{U}_{\leq m_0}$ and morphisms $u \to x'$, $u' \to x''$. By the above we can find $a \in \mathcal{U}_{\leq 2m_0}$ and morphisms $u \to a$, $u' \to a$ such that



is commutative. Turning this diagram 90 degrees clockwise we get the desired diagram as in (2) of Categories, Definition 4.17.3. □

Lemma 14.15.4. *Assume the category $\mathcal{C}$ has coproducts of any two objects and finite limits. Let U be a simplicial set, with U_n finite nonempty for all $n \geq 0$. Assume that all n-simplices of U are degenerate for all $n \gg 0$. Let V be a simplicial object of $\mathcal{C}$. Then* $\mathrm{Hom}(U, V)$ *exists, moreover we have the expected equalities*

$$\mathrm{Hom}(U, V)_n = \mathrm{Hom}(U \times \Delta[n], V)_0.$$

Proof. We construct this simplicial object as follows. For $n \geq 0$ let $\mathrm{Hom}(U, V)_n$ denote the object of $\mathcal{C}$ representing the functor

$$X \longmapsto \mathrm{Mor}_{\mathrm{Simp}(\mathcal{C})}(X \times U \times \Delta[n], V)$$

This exists by Lemma 14.15.3 because $U \times \Delta[n]$ is a simplicial set with finite sets of simplices and no nondegenerate simplices in high enough degree, see Lemma 14.11.5. For $\varphi : [m] \to [n]$ we obtain an induced map of simplicial sets $\varphi : \Delta[m] \to \Delta[n]$. Hence we obtain a morphism $X \times U \times \Delta[m] \to X \times U \times \Delta[n]$ functorial in X, and hence a transformation of functors, which in turn gives

$$\mathrm{Hom}(U, V)(\varphi) : \mathrm{Hom}(U, V)_n \longrightarrow \mathrm{Hom}(U, V)_m.$$

Clearly this defines a contravariant functor $\mathrm{Hom}(U, V)$ from Δ into the category $\mathcal{C}$. In other words, we have a simplicial object of $\mathcal{C}$.

We have to show that $\mathrm{Hom}(U, V)$ satisfies the desired universal property

$$\mathrm{Mor}_{\mathrm{Simp}(\mathcal{C})}(W, \mathrm{Hom}(U, V)) = \mathrm{Mor}_{\mathrm{Simp}(\mathcal{C})}(W \times U, V)$$

To see this, let $f : W \to \mathrm{Hom}(U, V)$ be given. We want to construct the element $f' : W \times U \to V$ of the right hand side. By construction, each $f_n : W_n \to \mathrm{Hom}(U, V)_n$

corresponds to a morphism $f_n : W_n \times U \times \Delta[n] \to V$. Further, for every morphism $\varphi : [m] \to [n]$ the diagram

$$\begin{array}{ccc} W_n \times U \times \Delta[m] & \xrightarrow{W(\varphi)\times \mathrm{id}\times \mathrm{id}} & W_m \times U \times \Delta[m] \\ \downarrow{\scriptstyle \mathrm{id}\times \mathrm{id}\times \varphi} & & \downarrow{\scriptstyle f_m} \\ W_n \times U \times \Delta[n] & \xrightarrow{f_n} & V \end{array}$$

is commutative. For $\psi : [n] \to [k]$ in $(\Delta[n])_k$ we denote $(f_n)_{k,\psi} : W_n \times U_k \to V_k$ the component of $(f_n)_k$ corresponding to the element ψ. We define $f'_n : W_n \times U_n \to V_n$ as $f'_n = (f_n)_{n,\mathrm{id}}$, in other words, as the restriction of $(f_n)_n : W_n \times U_n \times (\Delta[n])_n \to V_n$ to $W_n \times U_n \times \mathrm{id}_{[n]}$. To see that the collection (f'_n) defines a morphism of simplicial objects, we have to show for any $\varphi : [m] \to [n]$ that $V(\varphi) \circ f'_n = f'_m \circ W(\varphi) \times U(\varphi)$. The commutative diagram above says that $(f_n)_{m,\varphi} : W_n \times U_m \to V_m$ is equal to $(f_m)_{m,\mathrm{id}} \circ W(\varphi) : W_n \times U_m \to V_m$. But then the fact that f_n is a morphism of simplicial objects implies that the diagram

$$\begin{array}{ccc} W_n \times U_n \times (\Delta[n])_n & \xrightarrow{(f_n)_n} & V_n \\ \downarrow{\scriptstyle \mathrm{id}\times U(\varphi)\times \varphi} & & \downarrow{\scriptstyle V(\varphi)} \\ W_n \times U_m \times (\Delta[n])_m & \xrightarrow{(f_n)_m} & V_m \end{array}$$

is commutative. And this implies that $(f_n)_{m,\varphi} \circ U(\varphi)$ is equal to $V(\varphi) \circ (f_n)_{n,\mathrm{id}}$. Altogether we obtain $V(\varphi) \circ (f_n)_{n,\mathrm{id}} = (f_n)_{m,\varphi} \circ U(\varphi) = (f_m)_{m,\mathrm{id}} \circ W(\varphi) \circ U(\varphi) = (f_m)_{m,\mathrm{id}} \circ W(\varphi) \times U(\varphi)$ as desired.

On the other hand, given a morphism $f' : W \times U \to V$ we define a morphism $f : W \to \mathrm{Hom}(U, V)$ as follows. By Lemma 14.12.4 the morphisms $\mathrm{id} : W_n \to W_n$ corresponds to a unique morphism $c_n : W_n \times \Delta[n] \to W$. Hence we can consider the composition

$$W_n \times \Delta[n] \times U \xrightarrow{c_n} W \times U \xrightarrow{f'} V.$$

By construction this corresponds to a unique morphism $f_n : W_n \to \mathrm{Hom}(U, V)_n$. We leave it to the reader to see that these define a morphism of simplicial sets as desired.

We also leave it to the reader to see that $f \mapsto f'$ and $f' \mapsto f$ are mutually inverse operations. □

We spell out the construction above in a special case. Let X be an object of a category $\mathcal{C}$. Assume that self products $X \times \ldots \times X$ exist. Let k be an integer. Consider the simplicial object U with terms

$$U_n = \prod\nolimits_{\alpha \in \mathrm{Mor}([k],[n])} X$$

and maps given $\varphi : [m] \to [n]$

$$\begin{array}{rcl} U(\varphi) : \prod_{\alpha \in \mathrm{Mor}([k],[n])} X & \longrightarrow & \prod_{\alpha' \in \mathrm{Mor}([k],[m])} X \\ (f_\alpha)_\alpha & \longmapsto & (f_{\varphi \circ \alpha'})_{\alpha'} \end{array}$$

In terms of "coordinates", the element $(x_\alpha)_\alpha$ is mapped to the element $(x_{\varphi\circ\alpha'})_{\alpha'}$. We claim this object is equal to

$$\mathrm{Hom}(\Delta[k], X)$$

where we think of X as the constant simplicial object X.

Lemma 14.15.5. *With X, k and U as above.*

(1) *For any simplicial object V of $\mathcal{C}$ we have the following canonical bijection*

$$\mathrm{Mor}_{Simp(\mathcal{C})}(V, U) \longrightarrow \mathrm{Mor}_{\mathcal{C}}(V_k, X).$$

wich maps γ to the morphism γ_k composed with the projection onto the factor corresponding to $id_{[k]}$.

(2) *Similarly, if W is an k-truncated simplicial object of $\mathcal{C}$, then we have*

$$\mathrm{Mor}_{Simp_k(\mathcal{C})}(W, sk_k U) = \mathrm{Mor}_{\mathcal{C}}(W_k, X).$$

(3) *The object U constructed above is an incarnation of* $\mathrm{Hom}(\Delta[k], X)$.

Proof. We first prove (1). Suppose that $\gamma : V \to U$ is a morphism. This is given by a family of morphisms $\gamma_\alpha : V_n \to X$ for $\gamma : [k] \to [n]$. The morphisms have to satisfy the rules that for all $\varphi : [m] \to [n]$ the diagrams

$$\begin{array}{ccc} X & \xleftarrow{\gamma_{\varphi\circ\alpha'}} & V_n \\ \downarrow{\scriptstyle \mathrm{id}_X} & & \downarrow{\scriptstyle V(\varphi)} \\ X & \xleftarrow{\gamma_{\alpha'}} & V_m \end{array}$$

commute for all $\alpha' : [k] \to [m]$. Taking $\alpha' = \mathrm{id}_{[k]}$, we see that for any $\varphi : [k] \to [n]$ we have $\gamma_\varphi = \gamma_{\mathrm{id}_{[k]}} \circ V(\varphi)$. Thus the morphism γ is determined by the component of γ_k corresponding to $\mathrm{id}_{[k]}$. Conversely, given such a morphism $f : V_k \to X$ we easily construct a morphism γ by putting $\gamma_\alpha = f \circ V(\alpha)$.

The truncated case is similar, and left to the reader.

To see (3) we argue as follows:

$$\begin{aligned} \mathrm{Mor}(V, \mathrm{Hom}(\Delta[k], X)) &= \mathrm{Mor}(V \times \Delta[k], X) \\ &= \{(f_n : V_n \times \Delta[k]_n \to X) \mid f_n \text{ compatible}\} \\ &= \{(f_n : V_n \to \textstyle\prod_{\Delta[k]_n} X) \mid f_n \text{ compatible}\} \\ &= \mathrm{Mor}(V, U) \end{aligned}$$

Thus U and $\mathrm{Hom}(\Delta[k], X)$ define the same functor on the category of simplicial objects and hence are canonically isomorphic. □

Lemma 14.15.6. *Assume the category $\mathcal{C}$ has coproducts of any two objects and finite limits. Let $a : U \to V$, $b : U \to W$ be morphisms of simplicial sets. Assume U_n, V_n, W_n finite nonempty for all $n \geq 0$. Assume that all n-simplices of U, V, W are degenerate for all $n \gg 0$. Let T be a simplicial object of $\mathcal{C}$. Then*

$$\mathrm{Hom}(V, T) \times_{\mathrm{Hom}(U,T)} \mathrm{Hom}(W, T) = \mathrm{Hom}(V \amalg_U W, T)$$

In other words, the fibre product on the left hand side is represented by the Hom object on the right hand side.

Proof. By Lemma 14.15.4 all the required Hom objects exist and satisfy the correct functorial properties. Now we can identify the nth term on the left hand side as the

object representing the functor that associates to X the first set of the following sequence of functorial equalities

$$\begin{aligned}
&\mathrm{Mor}(X \times \Delta[n], \mathrm{Hom}(V,T) \times_{\mathrm{Hom}(U,T)} \mathrm{Hom}(W,T)) \\
&= \mathrm{Mor}(X \times \Delta[n], \mathrm{Hom}(V,T)) \times_{\mathrm{Mor}(X \times \Delta[n], \mathrm{Hom}(U,T))} \mathrm{Mor}(X \times \Delta[n], \mathrm{Hom}(W,T)) \\
&= \mathrm{Mor}(X \times \Delta[n] \times V, T) \times_{\mathrm{Mor}(X \times \Delta[n] \times U, T)} \mathrm{Mor}(X \times \Delta[n] \times W, T) \\
&= \mathrm{Mor}(X \times \Delta[n] \times (V \amalg_U W), T))
\end{aligned}$$

Here we have used the fact that

$$(X \times \Delta[n] \times V) \times_{X \times \Delta[n] \times U} (X \times \Delta[n] \times W) = X \times \Delta[n] \times (V \amalg_U W)$$

which is easy to verify term by term. The result of the lemma follows as the last term in the displayed sequence of equalities corresponds to $\mathrm{Hom}(V \amalg_U W, T)_n$. □

14.16. Splitting simplicial objects

A subobject N of an object X of the category $\mathcal{C}$ is an object N of $\mathcal{C}$ together with a monomorphism $N \to X$. Of course we say (by abouse of notation) that the subobjects N, N' are equal if there exists an isomorphism $N \to N'$ compatible with the morphisms to X. The collection of subobjects forms a partially ordered set. (Because of our conventions on categories; not true for category of spaces up to homotopy for example.)

Definition 14.16.1. Let $\mathcal{C}$ be a category which admits finite nonempty coproducts. We say a simplicial object U of $\mathcal{C}$ is *split* if there exist subobjects $N(U_m)$ of U_m, $m \geq 0$ with the property that

$$\coprod_{\varphi:[n]\to[m] \text{ surjective}} N(U_m) \longrightarrow U_n \tag{14.16.1.1}$$

is an isomorphism for all $n \geq 0$.

If this is the case, then $N(U_0) = U_0$. Next, we have $U_1 = U_0 \coprod N(U_1)$. Second we have

$$U_2 = U_0 \coprod N(U_1) \coprod N(U_1) \coprod N(U_2).$$

It turns out that in many categories $\mathcal{C}$ every simplicial object is split.

Lemma 14.16.2. *Let U be a simplicial set. Then U has a splitting with $N(U_m)$ equal to the set of nondegenerate m-simplices.*

Proof. Let $x \in U_n$. Suppose that there are surjections $\varphi : [n] \to [k]$ and $\psi : [n] \to [l]$ and nondegenerate simplices $y \in U_k$, $z \in U_l$ such that $x = U(\varphi)(y)$ and $x = U(\psi)(z)$. Choose a right inverse $\xi : [l] \to [n]$ of ψ, i.e., $\psi \circ \xi = \mathrm{id}_{[l]}$. Then $z = U(\xi)(x)$. Hence $z = U(\xi)(x) = U(\varphi \circ \xi)(y)$. Since z is nondegenerate we conclude that $\varphi \circ \xi : [l] \to [k]$ is surjective, and hence $l \geq k$. Similarly $k \geq l$. Hence we see that $\varphi \circ \xi : [l] \to [k]$ has to be the identity map for any choice of right inverse ξ of ψ. This easily implies that $\psi = \varphi$. □

Of course it can happen that a map of simplicial sets maps a nondegenerate n-simplex to a degenerate n-simplex. Thus the splitting of Lemma 14.16.2 is not functorial. Here is a case where it is functorial.

Lemma 14.16.3. *Let $f : U \to V$ be a morphism of simplicial sets. Suppose that (a) the image of every nondegenerate simplex of U is a nondegenerate simplex of V and (b) no two nondegenerate simplices of U are mapped to the same simplex of V. Then f_n is injective for all n. Same holds with "injective" replaced by "surjective" or "bijective".*

Proof. Under hypothesis (a) we see that the map f preserves the disjoint union decompositions of the splitting of Lemma 14.16.2, in other words that we get commutative diagrams

$$\begin{array}{ccc} \coprod_{\varphi:[n]\to[m] \text{ surjective}} N(U_m) & \longrightarrow & U_n \\ \downarrow & & \downarrow \\ \coprod_{\varphi:[n]\to[m] \text{ surjective}} N(V_m) & \longrightarrow & V_n. \end{array}$$

And then (b) clearly shows that the left vertical arrow is injective (resp. surjective, resp. bijective). □

Lemma 14.16.4. *Let U be a simplicial set. Let $n \geq 0$ be an integer. The rule*

$$U'_m = \bigcup\nolimits_{\varphi:[m]\to[i],\ i\leq n} Im(U(\varphi))$$

defines a sub simplicial set $U' \subset U$ with $U'_i = U_i$ for $i \leq n$. Moreover, all m-simplices of U' are degenerate for all $m > n$.

Proof. If $x \in U_m$ and $x = U(\varphi)(y)$ for some $y \in U_i$, $i \leq n$ and some $\varphi : [m] \to [i]$ then any image $U(\psi)(x)$ for any $\psi : [m'] \to [m]$ is equal to $U(\varphi \circ \psi)(y)$ and $\varphi \circ \psi : [m'] \to [i]$. Hence U' is a simplicial set. By construction all simplices in dimension $n + 1$ and higher are degenerate. □

Lemma 14.16.5. *Let U be a simplicial abelian group. Then U has a splitting obtained by taking $N(U_0) = U_0$ and for $m \geq 1$ taking*

$$N(U_m) = \bigcap\nolimits_{i=0}^{m-1} Ker(d_i^m).$$

Moreover, this splitting is functorial on the category of simplicial abelian groups.

Proof. By induction on n we will show that the choice of $N(U_m)$ in the lemma guarantees that (14.16.1.1) is an isomorphism for $m \leq n$. This is clear for $n = 0$. In the rest of this proof we are going to drop the superscripts from the maps d_i and s_i in order to improve readability. We will also repeatedly use the relations from Remark 14.3.3.

First we make a general remark. For $0 \leq i \leq m$ and $z \in U_m$ we have $d_i(s_i(z)) = z$. Hence we can write any $x \in U_{m+1}$ uniquely as $x = x' + x''$ with $d_i(x') = 0$ and $x'' \in \mathrm{Im}(s_i)$ by taking $x' = (x - s_i(d_i(x)))$ and $x'' = s_i(d_i(x))$. Moreover, the element $z \in U_m$ such that $x'' = s_i(z)$ is unique because s_i is injective.

Here is a procedure for decomposing any $x \in U_{n+1}$. First, write $x = x_0 + s_0(z_0)$ with $d_0(x_0) = 0$. Next, write $x_0 = x_1 + s_1(z_1)$ with $d_n(x_1) = 0$. Continue like this

to get

$$\begin{aligned} x &= x_0 + s_0(z_0), \\ x_0 &= x_1 + s_1(z_1), \\ x_1 &= x_2 + s_2(z_2), \\ \ldots &\quad \ldots \quad \ldots \\ x_{n-1} &= x_n + s_n(z_n) \end{aligned}$$

where $d_i(x_i) = 0$ for all $i = n, \ldots, 0$. By our general remark above all of the x_i and z_i are determined uniquely by x. We claim that $x_i \in \mathrm{Ker}(d_0) \cap \mathrm{Ker}(d_1) \cap \ldots \cap \mathrm{Ker}(d_i)$ and $z_i \in \mathrm{Ker}(d_0) \cap \ldots \cap \mathrm{Ker}(d_{i-1})$ for $i = n, \ldots, 0$. Here and in the following an empty intersection of kernels indicates the whole space; i.e., the notation $z_0 \in \mathrm{Ker}(d_0) \cap \ldots \cap \mathrm{Ker}(d_{i-1})$ when $i = 0$ means $z_0 \in U_n$ with no restriction.

We prove this by ascending induction on i. It is clear for $i = 0$ by construction of x_0 and z_0. Let us prove it for $0 < i \leq n$ assuming the result for $i-1$. First of all we have $d_i(x_i) = 0$ by construction. So pick a j with $0 \leq j < i$. We have $d_j(x_{i-1}) = 0$ by induction. Hence

$$0 = d_j(x_{i-1}) = d_j(x_i) + d_j(s_i(z_i)) = d_j(x_i) + s_{i-1}(d_j(z_i)).$$

The last equality by the relations of Remark 14.3.3. These relations also imply that $d_{i-1}(d_j(x_i)) = d_j(d_i(x_i)) = 0$ because $d_i(x_i) = 0$ by construction. Then the uniqueness in the general remark above shows the equality $0 = x' + x'' = d_j(x_i) + s_{i-1}(d_j(z_i))$ can only hold if both terms are zero. We conclude that $d_j(x_i) = 0$ and by injectivity of s_{i-1} we also conclude that $d_j(z_i) = 0$. This proves the claim.

The claim implies we can uniquely write

$$x = s_0(z_0) + s_1(z_1) + \ldots + s_n(z_n) + x_0$$

with $x_0 \in N(U_{n+1})$ and $z_i \in \mathrm{Ker}(d_0) \cap \ldots \cap \mathrm{Ker}(d_{i-1})$. We can reformulate this as saying that we have found a direct sum decomposition

$$U_{n+1} = N(U_{n+1}) \oplus \bigoplus\nolimits_{i=0}^{i=n} s_i\Big(\mathrm{Ker}(d_0) \cap \ldots \cap \mathrm{Ker}(d_{i-1})\Big)$$

with the property that

$$\mathrm{Ker}(d_0) \cap \ldots \cap \mathrm{Ker}(d_j) = N(U_{n+1}) \oplus \bigoplus\nolimits_{i=j+1}^{i=n} s_i\Big(\mathrm{Ker}(d_n) \cap \ldots \cap \mathrm{Ker}(d_{i-1})\Big)$$

for $j = 0, \ldots, n$. The result follows from this statement as follows. Each of the z_i in the expression for x can be written uniquely as

$$z_i = s_i(z'_{i,i}) + \ldots + s_{n-1}(z'_{i,n-1}) + z_{i,0}$$

with $z_{i,0} \in N(U_n)$ and $z'_{i,j} \in \mathrm{Ker}(d_0) \cap \ldots \cap \mathrm{Ker}(d_{j-1})$. The first few steps in the decomposition of z_i are zero because z_i already is in the kernel of $d_0, \ldots, d_i$. This in turn uniquely gives

$$x = x_0 + s_0(z_{0,0}) + s_1(z_{1,0}) + \ldots + s_n(z_{n,0}) + \sum\nolimits_{0 \leq i \leq j \leq n-1} s_i(s_j(z'_{i,j})).$$

Continuing in this fashion we see that we in the end obtain a decomposition of x as a sum of terms of the form

$$s_{i_1} s_{i_2} \ldots s_{i_k}(z)$$

with $0 \leq i_1 \leq i_2 \leq \ldots \leq i_k \leq n-k+1$ and $z \in N(U_{n+1-k})$. This is exactly the required decomposition, because any surjective map $[n+1] \to [n+1-k]$ can be uniquely expressed in the form

$$\sigma_{i_k}^{n-k} \ldots \sigma_{i_2}^{n-1} \sigma_{i_1}^{n}$$

with $0 \leq i_1 \leq i_2 \leq \ldots \leq i_k \leq n-k+1$. □

Lemma 14.16.6. *Let $\mathcal{A}$ be an abelian category. Let U be a simplicial object in $\mathcal{A}$. Then U has a splitting obtained by taking $N(U_0) = U_0$ and for $m \geq 1$ taking*

$$N(U_m) = \bigcap\nolimits_{i=0}^{m-1} Ker(d_i^m).$$

Moreover, this splitting is functorial on the category of simplicial objects of $\mathcal{A}$.

Proof. For any object A of $\mathcal{A}$ we obtain a simplicial abelian group $\mathrm{Mor}_{\mathcal{A}}(A, U)$. Each of these are canonically split by Lemma 14.16.5. Moreover,

$$N(\mathrm{Mor}_{\mathcal{A}}(A, U_m)) = \bigcap\nolimits_{i=0}^{m-1} \mathrm{Ker}(d_i^m) = \mathrm{Mor}_{\mathcal{A}}(A, N(U_m)).$$

Hence we see that the morphism (14.16.1.1) becomes an isomorphism after applying the functor $\mathrm{Mor}_{\mathcal{A}}(A, -)$ for any object of $\mathcal{A}$. Hence it is an isomorphism by the Yoneda lemma. □

Lemma 14.16.7. *Let $\mathcal{A}$ be an abelian category. Let $f : U \to V$ be a morphism of simplicial objects of $\mathcal{A}$. If the induced morphisms $N(f)_i : N(U)_i \to N(V)_i$ are injective for all i, then f_i is injective for all i. Same holds with "injective" replaced with "surjective", or "isomorphism".*

Proof. This is clear from Lemma 14.16.6 and the definition of a splitting. □

Lemma 14.16.8. *Let $\mathcal{A}$ be an abelian category. Let U be a simplicial object in $\mathcal{A}$. Let $N(U_m)$ as in Lemma 14.16.6 above. Then $d_m^m(N(U_m)) \subset N(U_{m-1})$.*

Proof. For $j = 0, \ldots, m-2$ we have $d_j^{m-1} d_m^m = d_{m-1}^{m-1} d_j^m$ by the relations in Remark 14.3.3. The result follows. □

Lemma 14.16.9. *Let $\mathcal{A}$ be an abelian category. Let U be a simplicial object of $\mathcal{A}$. Let $n \geq 0$ be an integer. The rule*

$$U'_m = \sum\nolimits_{\varphi:[m]\to[i],\ i\leq n} Im(U(\varphi))$$

defines a sub simplicial object $U' \subset U$ with $U'_i = U_i$ for $i \leq n$. Moreover, $N(U'_m) = 0$ for all $m > n$.

Proof. Pick m, $i \leq n$ and some $\varphi : [m] \to [i]$. The image under $U(\psi)$ of $\mathrm{Im}(U(\varphi))$ for any $\psi : [m'] \to [m]$ is equal to the image of $U(\varphi \circ \psi)$ and $\varphi \circ \psi : [m'] \to [i]$. Hence U' is a simplicial object. Pick $m > n$. We have to show $N(U'_m) = 0$. By definition of $N(U_m)$ and $N(U'_m)$ we have $N(U'_m) = U'_m \cap N(U_m)$ (intersection of subobjects). Since U is split by Lemma 14.16.6, it suffices to show that U'_m is contained in the sum

$$\sum\nolimits_{\varphi:[m]\to[m'] \text{ surjective},\ m'<m} \mathrm{Im}(U(\varphi)|_{N(U_{m'})}).$$

By the splitting each $U_{m'}$ is the sum of images of $N(U_{m''})$ via $U(\psi)$ for surjective maps $\psi : [m'] \to [m'']$. Hence the displayed sum above is the same as

$$\sum\nolimits_{\varphi:[m]\to[m'] \text{ surjective},\ m'<m} \mathrm{Im}(U(\varphi)).$$

Clearly U'_m is contained in this by the simple fact that any $\varphi : [m] \to [i]$, $i \leq n$ occurring in the definition of U'_m may be factored as $[m] \to [m'] \to [i]$ with $[m] \to [m']$ surjective and $m' < m$ as in the last displayed sum above. $\square$

14.17. Skelet and coskelet functors

Let $\Delta_{\leq n}$ denote the full subcategory of Δ with objects $[0], [1], [2], \ldots, [n]$. Let $\mathcal{C}$ be a category.

Definition 14.17.1. An *n-truncated simplicial object of $\mathcal{C}$* is a contravariant functor from $\Delta_{\leq n}$ to $\mathcal{C}$. A *morphism of n-truncated simplicial objects* is a transformation of functors. We denote the category of n-truncated simplicial objects of $\mathcal{C}$ by the symbol $\mathrm{Simp}_n(\mathcal{C})$.

Given a simplicial object U of $\mathcal{C}$ the truncation $\mathrm{sk}_n U$ is the restriction of U to the subcategory $\Delta_{\leq n}$. This defines a *skelet functor*

$$\mathrm{sk}_n : \mathrm{Simp}(\mathcal{C}) \longrightarrow \mathrm{Simp}_n(\mathcal{C})$$

from the category of simplicial objects of $\mathcal{C}$ to the category of n-truncated simplicial objects of $\mathcal{C}$. See Remark 14.19.6 to avoid possible confusion with other functors in the literature.

The *coskelet functor* (if it exists) is a functor

$$\mathrm{cosk}_n : \mathrm{Simp}(\mathcal{C}) \longrightarrow \mathrm{Simp}_n(\mathcal{C})$$

which is right adjoint to the skelet functor. In a formula

$$\mathrm{Mor}_{\mathrm{Simp}(\mathcal{C})}(U, \mathrm{cosk}_n V) = \mathrm{Mor}_{\mathrm{Simp}_n(\mathcal{C})}(\mathrm{sk}_n U, V) \tag{14.17.1.1}$$

Given a n-truncated simplicial object V we say that *$\mathrm{cosk}_n V$ exists* if there exists a $\mathrm{cosk}_n V \in \mathrm{Ob}(\mathrm{Simp}(\mathcal{C}))$ and a morphism $\mathrm{sk}_n \mathrm{cosk}_n V \to V$ such that the displayed formula holds, in other words if the functor $U \mapsto \mathrm{Mor}_{\mathrm{Simp}_n(\mathcal{C})}(\mathrm{sk}_n U, V)$ is representable. If it exists it is unique up to unique isomorphism by the Yoneda lemma. See Categories, Section 4.3.

Example 14.17.2. Suppose the category $\mathcal{C}$ has finite nonempty self products. A 0-truncated simplicial object of $\mathcal{C}$ is the same as an object X of $\mathcal{C}$. In this case we claim that $\mathrm{cosk}_0(X)$ is the simplicial object U with $U_n = X^{n+1}$ the $(n+1)$-fold self product of X, and structure of simplicial object as in Example 14.3.5. Namely, a morphism $V \to U$ where V is a simplicial object is given by morphisms $V_n \to X^{n+1}$, such that all the diagrams

$$\begin{array}{ccc} V_n & \longrightarrow & X^{n+1} \\ {\scriptstyle V([0]\to[n],0\mapsto i)}\downarrow & & \downarrow{\scriptstyle \mathrm{pr}_i} \\ V_0 & \longrightarrow & X \end{array}$$

commute. Clearly this means that the map determines and is determined by a unique morphism $V_0 \to X$. This proves that formula (14.17.1.1) holds.

Recall the category $\Delta/[n]$, see Example 14.11.4. We let $(\Delta/[n])_{\leq m}$ denote the full subcategory of $\Delta/[n]$ consisting of objects $[k] \to [n]$ of $\Delta/[n]$ with $k \leq m$. In other

words we have the following commutative diagram of categories and functors

$$\begin{array}{ccc} (\Delta/[n])_{\leq m} & \longrightarrow & \Delta/[n] \\ \downarrow & & \downarrow \\ \Delta_{\leq m} & \longrightarrow & \Delta \end{array}$$

Given a m-truncated simplicial object U of $\mathcal{C}$ we define a functor

$$U(n) : (\Delta/[n])^{opp}_{\leq m} \longrightarrow \mathcal{C}$$

by the rules

$$\begin{array}{rcl} ([k] \to [n]) & \longmapsto & U_k \\ (\psi : ([k'] \to [n]) \to ([k] \to [n])) & \longmapsto & U(\psi) : U_k \to U_{k'} \end{array}$$

For a given morphism $\varphi : [n] \to [n']$ of Δ we have an associated functor

$$"\varphi" : (\Delta/[n])_{\leq m} \longrightarrow (\Delta/[n'])_{\leq m}$$

which maps $\alpha : [k] \to [n]$ to $\varphi \circ \alpha : [k] \to [n']$. The composition $U(n') \circ "\varphi"$ is equal to the functor $U(n)$.

Lemma 14.17.3. *If the category $\mathcal{C}$ has finite limits, then $cosk_m$ functors exist for all m. Moreover, for any m-truncated simplicial object U the simplicial object $cosk_m U$ is described by the formula*

$$(cosk_m U)_n = \lim_{(\Delta/[n])^{opp}_{\leq m}} U(n)$$

and for $\varphi : [n] \to [n']$ the map $cosk_m U(\varphi)$ comes from the identification $U(n') \circ "\varphi" = U(n)$ above via Categories, Lemma 4.14.8.

Proof. During the proof of this lemma we denote $\mathrm{cosk}_m U$ the simplicial object with $(\mathrm{cosk}_m U)_n$ equal to $\lim_{(\Delta/[n])^{opp}_{\leq m}} U(n)$. We will conclude at the end of the proof that it does satisfy the required mapping property.

Suppose that V is a simplicial object. A morphism $\gamma : V \to \mathrm{cosk}_m U$ is given by a sequence of morphisms $\gamma_n : V_n \to (\mathrm{cosk}_m U)_n$. By definition of a limit, this is given by a collection of morphisms $\gamma(\alpha) : V_n \to U_k$ where α ranges over all $\alpha : [k] \to [n]$ with $k \leq m$. These morphisms then also satisfy the rules that

$$\begin{array}{ccc} V_n & \xrightarrow[\gamma(\alpha)]{} & U_k \\ {\scriptstyle V(\varphi)}\uparrow & & \uparrow{\scriptstyle U(\psi)} \\ V_{n'} & \xrightarrow{\gamma(\alpha')} & U_{k'} \end{array}$$

are commutative, given any $0 \leq k, k' \leq m$, $0 \leq n, n'$ and any $\psi : [k] \to [k']$, $\varphi : [n] \to [n']$, $\alpha : [k] \to [n]$ and $\alpha' : [k'] \to [n']$ in Δ such that $\varphi \circ \alpha = \alpha' \circ \psi$. Taking $n = k$, $\varphi = \alpha'$, and $\alpha = \psi = \mathrm{id}_{[k]}$ we deduce that $\gamma(\alpha') = \gamma(\mathrm{id}_{[k]}) \circ V(\alpha')$. In other words, the morphisms $\gamma(\mathrm{id}_{[k]})$, $k \leq m$ determine the morphism γ. And it is easy to see that these morphisms form a morphism $\mathrm{sk}_m V \to U$.

Conversely, given a morphism $\gamma : \mathrm{sk}_m V \to U$, we obtain a family of morphisms $\gamma(\alpha)$ where α ranges over all $\alpha : [k] \to [n]$ with $k \leq m$ by setting $\gamma(\alpha) = \gamma(\mathrm{id}_{[k]}) \circ V(\alpha)$. These morphisms satisfy all the displayed commutativity restraints pictured above, and hence give rise to a morphism $V \to \mathrm{cosk}_m U$. $\square$

Lemma 14.17.4. *Let $\mathcal{C}$ be a category. Let U be an m-truncated simplicial object of $\mathcal{C}$. For $n \leq m$ the limit $\lim_{(\Delta/[n])^{opp}_{\leq m}} U(n)$ exists and is canonically isomorphic to U_n.*

Proof. This is true because the category $(\Delta/[n])_{\leq m}$ has an final object in this case, namely the identity map $[n] \to [n]$. □

Lemma 14.17.5. *Let $\mathcal{C}$ be a category with finite limits. Let U be an n-truncated simplicial object of $\mathcal{C}$. The morphism $sk_n cosk_n U \to U$ is an isomorphism.*

Proof. Combine Lemmas 14.17.3 and 14.17.4. □

Let us describe a particular instance of the coskelet functor in more detail. By abuse of notation we will denote sk_n also the restriction functor $\mathrm{Simp}_{n'}(\mathcal{C}) \to \mathrm{Simp}_n(\mathcal{C})$ for any $n' \geq n$. We are going to describe a right adjoint of the functor $\mathrm{sk}_n : \mathrm{Simp}_{n+1}(\mathcal{C}) \to \mathrm{Simp}_n(\mathcal{C})$. For $n \geq 1$, $0 \leq i < j \leq n+1$ define $\delta^{n+1}_{i,j} : [n-1] \to [n+1]$ to be the increasing map omitting i and j. Note that $\delta^{n+1}_{i,j} = \delta^{n+1}_j \circ \delta^n_i = \delta^{n+1}_i \circ \delta^n_{j-1}$, see Lemma 14.2.3. This motivates the following lemma.

Lemma 14.17.6. *Let n be an integer ≥ 1. Let U be a n-truncated simplicial object of $\mathcal{C}$. Consider the contravariant functor from $\mathcal{C}$ to Sets which associates to an object T the set*

$$\{(f_0, \ldots, f_{n+1}) \in \mathrm{Mor}_{\mathcal{C}}(T, U_n) \mid d^n_{j-1} \circ f_i = d^n_i \circ f_j \ \forall\ 0 \leq i < j \leq n+1\}$$

If this functor is representable by some object U_{n+1} of $\mathcal{C}$, then

$$U_{n+1} = \lim_{(\Delta/[n+1])^{opp}_{\leq n}} U(n)$$

Proof. The limit, if it exists, represents the functor that associates to an object T the set

$$\{(f_\alpha)_{\alpha:[k]\to[n+1],k\leq n} \mid f_{\alpha\circ\psi} = U(\psi) \circ f_\alpha \ \forall\ \psi : [k'] \to [k], \alpha : [k] \to [n+1]\}.$$

In fact we will show this functor is isomorphic to the one displayed in the lemma. The map in one direction is given by the rule

$$(f_\alpha)_\alpha \longmapsto (f_{\delta^{n+1}_0}, \ldots, f_{\delta^{n+1}_{n+1}}).$$

This satisfies the conditions of the lemma because

$$d^n_{j-1} \circ f_{\delta^{n+1}_i} = f_{\delta^{n+1}_i \circ \delta^n_{j-1}} = f_{\delta^{n+1}_j \circ \delta^n_i} = d^n_i \circ f_{\delta^{n+1}_j}$$

by the relations we recalled above the lemma. To construct a map in the other direction we have to associate to a system $(f_0, \ldots, f_{n+1})$ as in the displayed formula of the lemma a system of maps f_α. Let $\alpha : [k] \to [n+1]$ be given. Since $k \leq n$ the map α is not surjective. Hence we can write $\alpha = \delta^{n+1}_i \circ \psi$ for some $0 \leq i \leq n+1$ and some $\psi : [k] \to [n]$. We have no choice but to define

$$f_\alpha = U(\psi) \circ f_i.$$

Of course we have to check that this is independent of the choice of the pair (i, ψ). First, observe that given i there is a unique ψ which works. Second, suppose that (j, ϕ) is another pair. Then $i \neq j$ and we may assume $i < j$. Since both i, j are

not in the image of α we may actually write $\alpha = \delta^{n+1}_{i,j} \circ \xi$ and then we see that $\psi = \delta^n_{j-1} \circ \xi$ and $\phi = \delta^n_i \circ \xi$. Thus

$$\begin{aligned} U(\psi) \circ f_i &= U(\delta^n_{j-1} \circ \xi) \circ f_i \\ &= U(\xi) \circ d^n_{j-1} \circ f_i \\ &= U(\xi) \circ d^n_i \circ f_j \\ &= U(\delta^n_i \circ \xi) \circ f_j \\ &= U(\phi) \circ f_j \end{aligned}$$

as desired. We still have to verify that the maps f_α so defined satisfy the rules of a system of maps $(f_\alpha)_\alpha$. To see this suppose that $\psi : [k'] \to [k]$, $\alpha : [k] \to [n+1]$ with $k, k' \leq n$. Set $\alpha' = \alpha \circ \psi$. Choose i not in the image of α. Then clearly i is not in the image of α' also. Write $\alpha = \delta^n_i \circ \phi$ (we cannot use the letter ψ here because we've already used it). Then obviously $\alpha' = \delta^n_i \circ \phi \circ \psi$. By construction above we then have

$$U(\psi) \circ f_\alpha = U(\psi) \circ U(\phi) \circ f_i = U(\phi \circ \psi) \circ f_i = f_{\alpha \circ \psi} = f_{\alpha'}$$

as desired. We leave to the reader the pleasant task of verifying that our constructions are mutually inverse bijections, and are functorial in T. □

Lemma 14.17.7. *Let n be an integer ≥ 1. Let U be a n-truncated simplicial object of $\mathcal{C}$. Consider the contravariant functor from $\mathcal{C}$ to Sets which associates to an object T the set*

$$\{(f_0, \ldots, f_{n+1}) \in \mathrm{Mor}_{\mathcal{C}}(T, U_n) \mid d^n_{j-1} \circ f_i = d^n_i \circ f_j \ \forall\ 0 \leq i < j \leq n+1\}$$

If this functor is representable by some object U_{n+1} of $\mathcal{C}$, then there exists an $(n+1)$-truncated simplicial object $\tilde{U}$, with $sk_n\tilde{U} = U$ and $\tilde{U}_{n+1} = U_{n+1}$ such that the following adjointness holds

$$\mathrm{Mor}_{Simp_{n+1}(\mathcal{C})}(V, \tilde{U}) = \mathrm{Mor}_{Simp_n(\mathcal{C})}(sk_n V, U)$$

Proof. By Lemma 14.17.4 there are identifications

$$U_i = \lim_{(\Delta/[i])^{opp}_{\leq n}} U(i)$$

for $0 \leq i \leq n$. By Lemma 14.17.6 we have

$$U_{n+1} = \lim_{(\Delta/[n+1])^{opp}_{\leq n}} U(n).$$

Thus we may define for any $\varphi : [i] \to [j]$ with $i, j \leq n+1$ the corresponding map $\tilde{U}(\varphi) : \tilde{U}_j \to \tilde{U}_i$ exactly as in Lemma 14.17.3. This defines an $(n+1)$-truncated simplicial object $\tilde{U}$ with $\mathrm{sk}_n\tilde{U} = U$.

To see the adjointness we argue as follows. Given any element $\gamma : \mathrm{sk}_n V \to U$ of the right hand side of the formula consider the morphisms $f_i = \gamma_n \circ d^{n+1}_i : V_{n+1} \to V_n \to U_n$. These clearly satisfy the relations $d^n_{j-1} \circ f_i = d^n_i \circ f_j$ and hence define a unique morphism $V_{n+1} \to U_{n+1}$ by our choice of U_{n+1}. Conversely, given a morphism $\gamma' : V \to \tilde{U}$ of the left hand side we can simply restrict to $\Delta_{\leq n}$ to get an element of the right hand side. We leave it to the reader to show these are mutually inverse constructions. □

Remark 14.17.8. Let U, and U_{n+1} be as in Lemma 14.17.7. On T-valued points we can easily describe the face and degeneracy maps of $\tilde{U}$. Explicitly, the maps $d_i^{n+1} : U_{n+1} \to U_n$ are given by

$$(f_0, \ldots, f_{n+1}) \longmapsto f_i.$$

And the maps $s_j^n : U_n \to U_{n+1}$ are given by

$$\begin{aligned} f \quad \longmapsto \quad & (s_{j-1}^{n-1} \circ d_0^{n-1} \circ f, \\ & s_{j-1}^{n-1} \circ d_1^{n-1} \circ f, \\ & \ldots \\ & s_{j-1}^{n-1} \circ d_{j-1}^{n-1} \circ f, \\ & f, \\ & f, \\ & s_j^{n-1} \circ d_{j+1}^{n-1} \circ f, \\ & s_j^{n-1} \circ d_{j+2}^{n-1} \circ f, \\ & \ldots \\ & s_j^{n-1} \circ d_n^{n-1} \circ f) \end{aligned}$$

where we leave it to the reader to verify that the RHS is an element of the displayed set of Lemma 14.17.7. For $n = 0$ there is one map, namely $f \mapsto (f, f)$. For $n = 1$ there are two maps, namely $f \mapsto (f, f, s_0 d_1 f)$ and $f \mapsto (s_0 d_0 f, f, f)$. For $n = 2$ there are three maps, namely $f \mapsto (f, f, s_0 d_1 f, s_0 d_2 f)$, $f \mapsto (s_0 d_0 f, f, f, s_1 d_2 f)$, and $f \mapsto (s_1 d_0 f, s_1 d_1 f, f, f)$. And so on and so forth.

Remark 14.17.9. The construction of Lemma 14.17.7 above in the case of simplicial sets is the following. Given an n-truncated simplicial set U, we make a canonical $(n+1)$-truncated simplicial set $\tilde{U}$ as follows. We add a set of $(n+1)$-simplices U_{n+1} by the formula of the lemma. Namely, an element of U_{n+1} is a numbered collection of $(f_0, \ldots, f_{n+1})$ of n-simplices, with the property that they glue as they would in a $(n+1)$-simplex. In other words, the ith face of f_j is the $(j-1)$st face of f_i for $i < j$. Geometrically it is obvious how to define the face and degeneracy maps for $\tilde{U}$. If V is an $(n+1)$-truncated simplicial set, then its $(n+1)$-simplices give rise to compatible collections of n-simplices $(f_0, \ldots, f_{n+1})$ with $f_i \in V_n$. Hence there is a natural map $\mathrm{Mor}(\mathrm{sk}_n V, U) \to \mathrm{Mor}(V, \tilde{U})$ which is inverse to the canonical restriction mapping the other way.

Also, it is enough to do the combinatorics of the construction in the case of truncated simplicial sets. Namely, for any object T of the category $\mathcal{C}$, and any n-truncated simplicial object U of $\mathcal{C}$ we can consider the n-truncated simplicial set $\mathrm{Mor}(T, U)$. We may apply the construction to this, and take its set of $(n+1)$-simplices, and require this to be representable. This is a good way to think about the result of Lemma 14.17.7.

Remark 14.17.10. *Inductive construction of coskelets.* Suppose that $\mathcal{C}$ is a category with finite limits. Suppose that U is an m-truncated simplicial object in $\mathcal{C}$. Then we can inductively construct n-truncated objects U^n as follows:

(1) To start, set $U^m = U$.
(2) Given U^n for $n \geq m$ set $U^{n+1} = \tilde{U}^n$, where $\tilde{U}^n$ is constructed from U^n as in Lemma 14.17.7.

Since the construction of Lemma 14.17.7 has the property that it leaves the n-skeleton of U^n unchanged, we can then define $\text{cosk}_m U$ to be the simplicial object with $(\text{cosk}_m U)_n = U^n_n = U^{n+1}_n = \ldots$. And it follows formally from Lemma 14.17.7 that U^n satisfies the formula

$$\text{Mor}_{\text{Simp}_n(\mathcal{C})}(V, U^n) = \text{Mor}_{\text{Simp}_m(\mathcal{C})}(\text{sk}_m V, U)$$

for all $n \geq m$. It also then follows formally from this that

$$\text{Mor}_{\text{Simp}(\mathcal{C})}(V, \text{cosk}_m U) = \text{Mor}_{\text{Simp}_m(\mathcal{C})}(\text{sk}_m V, U)$$

with $\text{cosk}_m U$ chosen as above.

Lemma 14.17.11. *Let $\mathcal{C}$ be a category which has finite limits.*

1. *For every n the functor $sk_n : Simp(\mathcal{C}) \to Simp_n(\mathcal{C})$ has a right adjoint $cosk_n$.*
2. *For every $n' \geq n$ the functor $sk_n : Simp_{n'}(\mathcal{C}) \to Simp_n(\mathcal{C})$ has a right adjoint, namely $sk_{n'} cosk_n$.*
3. *For every $m \geq n \geq 0$ and every n-truncated simplicial object U of $\mathcal{C}$ we have $cosk_m sk_m cosk_n U = cosk_n U$.*
4. *If U is a simplicial object of $\mathcal{C}$ such that the canonical map $U \to cosk_n sk_n U$ is an isomorphism for some $n \geq 0$, then the canonical map $U \to cosk_m sk_m U$ is an isomorphism for all $m \geq n$.*

Proof. The existence in (1) follows from Lemma 14.17.3 above and the equality in (2), and (3) follows from the discussion in Remark 14.17.10. After this (4) is obvious. □

Remark 14.17.12. We do not need all finite limits in order to be able to define the coskelet functors. Here are some remarks

1. We have seen in Examples 14.17.2 that if $\mathcal{C}$ has products of pairs of objects then cosk_0 exists.
2. For $k > 0$ the functor cosk_k exists if $\mathcal{C}$ has finite connected limits.

This is clear from the inductive procedure of constructing coskelets (Remarks 14.17.9 and 14.17.10) but it also follows from the fact that the categories $(\Delta/[n])_{\leq k}$ for $k \geq 1$ and $n \geq k+1$ used in Lemma 14.17.3 are connected. Observe that we do not need the categories for $n \leq k$ by Lemma 14.17.4 or Lemma 14.17.5. (As k gets higher the categories $(\Delta/[n])_{\leq k}$ for $k \geq 1$ and $n \geq k+1$ are more and more connected in a topological sense.)

Lemma 14.17.13. *Let U, V be n-truncated simplicial objects of a category $\mathcal{C}$. Then*

$$cosk_n(U \times V) = cosk_n U \times cosk_n V$$

whenever the left and right hand sides exist.

Proof. Let W be a simplicial object. We have

$$\begin{aligned} \text{Mor}(W, \text{cosk}_n(U \times V)) &= \text{Mor}(\text{sk}_n W, U \times V) \\ &= \text{Mor}(\text{sk}_n W, U) \times \text{Mor}(\text{sk}_n W, V) \\ &= \text{Mor}(W, \text{cosk}_n U) \times \text{Mor}(W, \text{cosk}_n V) \\ &= \text{Mor}(W, \text{cosk}_n U \times \text{cosk}_n V) \end{aligned}$$

The lemma follows. □

Lemma 14.17.14. *Assume $\mathcal{C}$ has fibre products. Let U, V, W be n-truncated simplicial objects of the category $\mathcal{C}$. Then*

$$cosk_n(V \times_U W) = cosk_n U \times_{cosk_n U} cosk_n V$$

whenever the left and right hand side exist.

Proof. Omitted, but very similar to the proof of Lemma 14.17.13 above. □

Lemma 14.17.15. *Let $\mathcal{C}$ be a category with finite limits. Let $X \in \mathrm{Ob}(\mathcal{C})$. The functor $\mathcal{C}/X \to \mathcal{C}$ commutes with the coskelet functors $cosk_k$ for $k \geq 1$.*

Proof. The statement means that if U is a simplicial object of $\mathcal{C}/X$ which we can think of as a simplicial object of $\mathcal{C}$ with a morphism towards the constant simplicial object X, then $\mathrm{cosk}_k U$ computed in $\mathcal{C}/X$ is the same as computed in $\mathcal{C}$. This follows for example from Categories, Lemma 4.16.2 because the categories $(\Delta/[n])_{\leq k}$ for $k \geq 1$ and $n \geq k+1$ used in Lemma 14.17.3 are connected. Observe that we do not need the categories for $n \leq k$ by Lemma 14.17.4 or Lemma 14.17.5. □

Lemma 14.17.16. *The canonical map $\Delta[n] \to cosk_1 sk_1 \Delta[n]$ is an isomorphism.*

Proof. Consider a simplicial set U and a morphism $f : U \to \Delta[n]$. This is a rule that associates to each $u \in U_i$ a map $f_u : [i] \to [n]$ in Δ. Furthermore, these maps should have the property that $f_u \circ \varphi = f_{U(\varphi)(u)}$ for any $\varphi : [j] \to [i]$. Denote $\epsilon^i_j : [0] \to [i]$ the map which maps 0 to j. Denote $F : U_0 \to [n]$ the map $u \mapsto f_u(0)$. Then we see that

$$f_u(j) = F(\epsilon^i_j(u))$$

for all $0 \leq j \leq i$ and $u \in U_i$. In particular, if we know the function F then we know the maps f_u for all $u \in U_i$ all i. Conversely, given a map $F : U_0 \to [n]$, we can set for any i, and any $u \in U_i$ and any $0 \leq j \leq i$

$$f_u(j) = F(\epsilon^i_j(u))$$

This does not in general define a morphism f of simplicial sets as above. Namely, the condition is that all the maps f_u are nondecreasing. This clearly is equivalent to the condition that $F(\epsilon^i_j(u)) \leq F(\epsilon^i_{j'}(u))$ whenever $0 \leq j \leq j' \leq i$ and $u \in U_i$. But in this case the morphisms

$$\epsilon^i_j, \epsilon^i_{j'} : [0] \to [i]$$

both factor through the map $\epsilon^i_{j,j'} : [1] \to [i]$ defined by the rules $0 \mapsto j$, $1 \mapsto j'$. In other words, it is enough to check the inequalities for $i = 1$ and $u \in X_1$. In other words, we have

$$\mathrm{Mor}(U, \Delta[n]) = \mathrm{Mor}(\mathrm{sk}_1 U, \mathrm{sk}_1 \Delta[n])$$

as desired. □

14.18. Augmentations

Definition 14.18.1. Let $\mathcal{C}$ be a category. Let U be a simplicial object of $\mathcal{C}$. An *augmentation $\epsilon : U \to X$ of U towards an object X of $\mathcal{C}$* is a morphism from U into the constant simplicial object X.

Lemma 14.18.2. *Let $\mathcal{C}$ be a category. Let $x \in \mathrm{Ob}(\mathcal{C})$. Let U be a simplicial object of $\mathcal{C}$. To give an augmentation of U towards X is the same as giving a morphism $\epsilon_0 : U_0 \to X$ such that $\epsilon_0 \circ d^1_0 = \epsilon_0 \circ d^1_1$.*

Proof. Given a morphism $\epsilon : U \to X$ we certainly obtain an ϵ_0 as in the lemma. Conversely, given ϵ_0 as in the lemma, define $\epsilon_n : U_n \to X$ by choosing any morphism $\alpha : [0] \to [n]$ and taking $\epsilon_n = \epsilon_0 \circ U(\alpha)$. Namely, if $\beta : [0] \to [n]$ is another choice, then there exists a morphism $\gamma : [1] \to [n]$ such that α and β both factor as $[0] \to [1] \to [n]$. Hence the condition on ϵ_0 shows that ϵ_n is well defined. Then it is easy to show that $(\epsilon_n) : U \to X$ is a morphism of simplicial objects. □

Lemma 14.18.3. *Let $\mathcal{C}$ be a category with fibred products. Let $f : Y \to X$ be a morphism of $\mathcal{C}$. Let U be the simplicial object of $\mathcal{C}$ whose nth term is the $(n+1)$fold fibred product $Y \times_X Y \times_X \ldots \times_X Y$. See Example 14.3.5. For any simplicial object V of $\mathcal{C}$ we have*

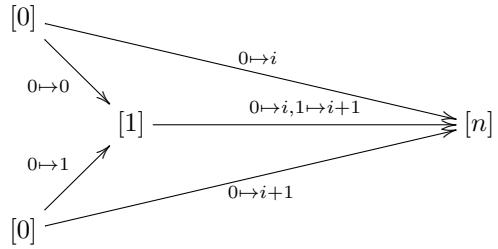
$$\begin{aligned} \mathrm{Mor}_{Simp(\mathcal{C})}(V, U) &= \mathrm{Mor}_{Simp_{\leq 1}(\mathcal{C})}(sk_1 V, sk_1 U) \\ &= \{g_0 : V_0 \to Y \mid f \circ g_0 \circ d_0^1 = f \circ g_0 \circ d_1^1\} \end{aligned}$$

In particular we have $U = cosk_1 sk_1 U$.

Proof. Suppose that $g : \mathrm{sk}_1 V \to \mathrm{sk}_1 U$ is a morphism of 1-truncated simplicial objects. Then the diagram

$$\begin{array}{ccccc} & & \xrightarrow{d_0^1} & & \\ V_1 & & \xrightarrow[d_1^1]{} & V_0 & \\ {\scriptstyle g_1}\downarrow & & & \downarrow{\scriptstyle g_0} & \\ & & \xrightarrow{pr_1} & & \\ Y \times_X Y & & \xrightarrow[pr_0]{} & Y & \longrightarrow X \end{array}$$

is commutative, which proves that the relation shown in the lemma holds. We have to show that, conversely, given a morphism g_0 satisfying the relation $f \circ g_0 \circ d_0^1 = f \circ g_0 \circ d_1^1$ we get a unique morphism of simplicial objects $g : V \to U$. This is done as follows. For any $n \geq 1$ let $g_{n,i} = g_0 \circ V([0] \to [n], 0 \mapsto i) : V_n \to Y$. The equality above implies that $f \circ g_{n,i} = f \circ g_{n,i+1}$ because of the commutative diagram



Hence we get $(g_{n,0}, \ldots, g_{n,n}) : V_n \to Y \times_X \ldots \times_X Y = U_n$. We leave it to the reader to see that this is a morphism of simplicial objects. The last assertion of the lemma is equivalent to the first equality in the displayed formula of the lemma. □

Remark 14.18.4. Let $\mathcal{C}$ be a category with fibre products. Let V be a simplicial object. Let $\epsilon : V \to X$ be an augmentation. Let U be the simplicial object whose nth term is the $(n+1)$st fibred product of V_0 over X. By a simple combination of Lemmas 14.18.2 and 14.18.3 we obtain a canonical morphism $V \to U$.

14.19. Left adjoints to the skeleton functors

In this section we construct a left adjoint $i_{m!}$ of the skeleton functor sk_m in certain cases. The adjointness formula is

$$\mathrm{Mor}_{\mathrm{Simp}_m(\mathcal{C})}(U, \mathrm{sk}_m V) = \mathrm{Mor}_{\mathrm{Simp}(\mathcal{C})}(i_{m!}U, V).$$

It turns out that this left adjoint exists when the category $\mathcal{C}$ has finite colimits.

We use a similar construction as in Section 14.17. Recall the category $[n]/\Delta$ of objects under $[n]$, see Categories, Example 4.2.14. Its objects are morphisms $\alpha : [n] \to [k]$ and its morphisms are commutative triangles. We let $([n]/\Delta)_{\leq m}$ denote the full subcategory of $[n]/\Delta$ consisting of objects $[n] \to [k]$ with $k \leq m$. Given a m-truncated simplicial object U of $\mathcal{C}$ we define a functor

$$U(n) : ([n]/\Delta)^{opp}_{\leq m} \longrightarrow \mathcal{C}$$

by the rules

$$\begin{aligned} ([n] \to [k]) &\longmapsto U_k \\ (\psi : ([n] \to [k']) \to ([n] \to [k])) &\longmapsto U(\psi) : U_k \to U_{k'} \end{aligned}$$

For a given morphism $\varphi : [n] \to [n']$ of Δ we have an associated functor

$$"\varphi" : ([n']/\Delta)_{\leq m} \longrightarrow ([n]/\Delta)_{\leq m}$$

which maps $\alpha : [n'] \to [k]$ to $\varphi \circ \alpha : [n] \to [k]$. The composition $U(n) \circ "\varphi"$ is equal to the functor $U(n')$.

Lemma 14.19.1. *Let $\mathcal{C}$ be a category which has finite colimits. The functors $i_{m!}$ exist for all m. Let U be an m-truncated simplicial object of $\mathcal{C}$. The simplicial object $i_{m!}U$ is described by the formula*

$$(i_{m!}U)_n = \mathrm{colim}_{([n]/\Delta)^{opp}_{\leq m}} U(n)$$

and for $\varphi : [n] \to [n']$ the map $i_{m!}U(\varphi)$ comes from the identification $U(n) \circ "\varphi" = U(n')$ above via Categories, Lemma 4.14.7.

Proof. In this proof we denote $i_{m!}U$ the simplicial object whose nth term is given by the displayed formula of the lemma. We will show it satisfies the adjointness property.

Let V be a simplicial object of $\mathcal{C}$. Let $\gamma : U \to \mathrm{sk}_m V$ be given. A morphism

$$\mathrm{colim}_{([n]/\Delta)^{opp}_{\leq m}} U(n) \to T$$

is given by a compatible system of morphisms $f_\alpha : U_k \to T$ where $\alpha : [n] \to [k]$ with $k \leq m$. Certainly, we have such a system of morphisms by taking the compositions

$$U_k \xrightarrow{\gamma_k} V_k \xrightarrow{V(\alpha)} V_n.$$

Hence we get an induced morphism $(i_{m!}U)_n \to V_n$. We leave it to the reader to see that these form a morphism of simplicial objects $\gamma' : i_{m!}U \to V$.

Conversely, given a morphism $\gamma' : i_{m!}U \to V$ we obtain a morphism $\gamma : U \to \mathrm{sk}_m V$ by setting $\gamma_i : U_i \to V_i$ equal to the composition

$$U_i \xrightarrow{\mathrm{id}_{[i]}} \mathrm{colim}_{([i]/\Delta)^{opp}_{\leq m}} U(i) \xrightarrow{\gamma'_i} V_i$$

for $0 \leq i \leq n$. We leave it to the reader to see that this is the inverse of the construction above. □

Lemma 14.19.2. *Let $\mathcal{C}$ be a category. Let U be an m-truncated simplicial object of $\mathcal{C}$. For any $n \leq m$ the colimit*

$$\operatorname{colim}_{([n]/\Delta)^{opp}_{\leq m}} U(n)$$

exists and is equal to U_n.

Proof. This is so because the category $([n]/\Delta)_{\leq m}$ has an initial object, namely $\mathrm{id} : [n] \to [n]$. □

Lemma 14.19.3. *Let $\mathcal{C}$ be a category which has finite colimits. Let U be an m-truncated simplicial object of $\mathcal{C}$. The map $U \to sk_m i_{m!} U$ is an isomorphism.*

Proof. Combine Lemmas 14.19.1 and 14.19.2. □

Lemma 14.19.4. *If U is an m-truncated simplicial set and $n > m$ then all n-simplices of $i_{m!}U$ are degenerate.*

Proof. This can be seen from the construction of $i_{m!}U$ in Lemma 14.19.1, but we can also argue directly as follows. Write $V = i_{m!}U$. Let $V' \subset V$ be the simplicial subset with $V'_i = V_i$ for $i \leq m$ and all i simplices degenerate for $i > m$, see Lemma 14.16.4. By the adjunction formula, since $\mathrm{sk}_m V' = U$, there is an inverse to the injection $V' \to V$. Hence $V' = V$. □

Lemma 14.19.5. *Let U be a simplicial set. Let $n \geq 0$ be an integer. The morphism $i_{n!}sk_n U \to U$ identifies $i_{n!}sk_n U$ with the simplicial set $U' \subset U$ defined in Lemma 14.16.4.*

Proof. By Lemma 14.19.4 the only nondegenerate simplices of $i_{n!}\mathrm{sk}_n U$ are in degrees $\leq n$. The map $i_{n!}\mathrm{sk}_n U \to U$ is an isomorphism in degrees $\leq n$. Combined we conclude that the map $i_{n!}\mathrm{sk}_n U \to U$ maps nondegenerate simplices to nondegenerate simplices and no two nondegenerate simplices have the same image. Hence Lemma 14.16.3 applies. Thus $i_{n!}\mathrm{sk}_n U \to U$ is injective. The result follows easily from this. □

Remark 14.19.6. In some texts the composite functor

$$\mathrm{Simp}(\mathcal{C}) \xrightarrow{\mathrm{sk}_m} \mathrm{Simp}_m(\mathcal{C}) \xrightarrow{i_{m!}} \mathrm{Simp}(\mathcal{C})$$

is denoted sk_m. This makes sense because Lemma 14.19.5 says that $i_{m!}\mathrm{sk}_m V$ is just the sub simplicial set of V consisting of all i-simplices of V, $i \leq m$ and their degeneracies. In those texts it is also customary to denote the composition

$$\mathrm{Simp}(\mathcal{C}) \xrightarrow{\mathrm{sk}_m} \mathrm{Simp}_m(\mathcal{C}) \xrightarrow{\mathrm{cosk}_m} \mathrm{Simp}(\mathcal{C})$$

by cosk_m.

Lemma 14.19.7. *Let $U \subset V$ be simplicial sets. Suppose $n \geq 0$ and $x \in V_n$, $x \notin U_n$ are such that*

1. *$V_i = U_i$ for $i < n$,*
2. *$V_n = U_n \cup \{x\}$,*
3. *any $z \in V_j$, $z \notin U_j$ for $j > n$ is degenerate.*

Let $\Delta[n] \to V$ be the unique morphism mapping the nondegenerate n-simplex of $\Delta[n]$ to x. In this case the diagram

$$\begin{array}{ccc} \Delta[n] & \longrightarrow & V \\ \uparrow & & \uparrow \\ i_{(n-1)!}sk_{n-1}\Delta[n] & \longrightarrow & U \end{array}$$

is a pushout diagram.

Proof. Let us denote $\partial\Delta[n] = i_{(n-1)!}\mathrm{sk}_{n-1}\Delta[n]$ for convenience. There is a natural map $U \amalg_{\partial\Delta[n]} \Delta[n] \to V$. We have to show that it is bijective in degree j for all j. This is clear for $j \leq n$. Let $j > n$. The third condition means that any $z \in V_j$, $z \notin U_j$ is a degenerate simplex, say $z = s_i^{j-1}(z')$. Of course $z' \notin U_{j-1}$. By induction it follows that z' is a degeneracy of x. Thus we conclude that all j-simplices of V are either in U or degeneracies of x. This implies that the map $U \amalg_{\partial\Delta[n]} \Delta[n] \to V$ is surjective. Note that a nondegenerate simplex of $U \amalg_{\partial\Delta[n]} \Delta[n]$ is either the image of a nondegenerate simplex of U, or the image of the (unique) nondegenerate n-simplex of $\Delta[n]$. Since clearly x is nondegenerate we deduce that $U \amalg_{\partial\Delta[n]} \Delta[n] \to V$ maps nondegenerate simplices to nondegenerate simplices and is injective on nondegenerate simplices. Hence it is injective, by Lemma 14.16.3. □

Lemma 14.19.8. *Let $U \subset V$ be simplicial sets, with U_n, V_n finite nonempty for all n. Assume that U and V have finitely many nondegenerate simplices. Then there exists a sequence of sub simplicial sets*

$$U = W^0 \subset W^1 \subset W^2 \subset \ldots W^r = V$$

such that Lemma 14.19.7 applies to each of the inclusions $W^i \subset W^{i+1}$.

Proof. Let n be the smallest integer such that V has a nondegenerate simplex that does not belong to U. Let $x \in V_n$, $x \notin U_n$ be such a nondegenerate simplex. Let $W \subset V$ be the set of elements which are either in U, or are a (repeated) degeneracy of x (in other words, are of the form $V(\varphi)(x)$ with $\varphi : [m] \to [n]$ surjective). It is easy to see that W is a simplicial set. The inclusion $U \subset W$ satisfies the conditions of Lemma 14.19.7. Moreover the number of nondegenerate simplices of V which are not contained in W is exactly one less than the number of nondegenerate simplices of V which are not contained in U. Hence we win by induction on this number. □

Lemma 14.19.9. *Let $\mathcal{A}$ be an abelian category Let U be an m-truncated simplicial object of $\mathcal{A}$. For $n > m$ we have $N(i_{m!}U)_n = 0$.*

Proof. Write $V = i_{m!}U$. Let $V' \subset V$ be the simplicial subobject of V with $V'_i = V_i$ for $i \leq m$ and $N(V'_i) = 0$ for $i > m$, see Lemma 14.16.9. By the adjunction formula, since $\mathrm{sk}_m V' = U$, there is an inverse to the injection $V' \to V$. Hence $V' = V$. □

Lemma 14.19.10. *Let $\mathcal{A}$ be an abelian category. Let U be a simplicial object of $\mathcal{A}$. Let $n \geq 0$ be an integer. The morphism $i_{n!}sk_nU \to U$ identifies $i_{n!}sk_nU$ with the simplicial subobject $U' \subset U$ defined in Lemma 14.16.9.*

Proof. By Lemma 14.19.9 we have $N(i_{n!}\mathrm{sk}_n U)_i = 0$ for $i > n$. The map $i_{n!}\mathrm{sk}_n U \to U$ is an isomorphism in degrees $\leq n$, see Lemma 14.19.3. Combined we conclude that the map $i_{n!}\mathrm{sk}_n U \to U$ induces injective maps $N(i_{n!}\mathrm{sk}_n U)_i \to N(U)_i$ for all i.

Hence Lemma 14.16.7 applies. Thus $i_{n!}\mathrm{sk}_n U \to U$ is injective. The result follows easily from this. □

Here is another way to think about the coskelet functor using the material above.

Lemma 14.19.11. *Let $\mathcal{C}$ be a category with finite coproducts and finite limits. Let V be a simplicial object of $\mathcal{C}$. In this case*

$$(cosk_n sk_n V)_{n+1} = \mathrm{Hom}(i_{n!} sk_n \Delta[n+1], V)_0.$$

Proof. By Lemma 14.12.4 the object on the left represents the functor which assigns to X the first set of the following equalities

$$\begin{aligned} \mathrm{Mor}(X \times \Delta[n+1], \mathrm{cosk}_n \mathrm{sk}_n V) &= \mathrm{Mor}(X \times \mathrm{sk}_n \Delta[n+1], \mathrm{sk}_n V) \\ &= \mathrm{Mor}(X \times i_{n!}\mathrm{sk}_n \Delta[n+1], V). \end{aligned}$$

The object on the right in the formula of the lemma is represented by the functor which assigns to X the last set in the sequence of equalities. This proves the result.

In the sequence of equalities we have used that $\mathrm{sk}_n(X \times \Delta[n+1]) = X \times \mathrm{sk}_n \Delta[n+1]$ and that $i_{n!}(X \times \mathrm{sk}_n \Delta[n+1]) = X \times i_{n!}\mathrm{sk}_n \Delta[n+1]$. The first equality is obvious. For any (possibly truncated) simplicial object W of $\mathcal{C}$ and any object X of $\mathcal{C}$ denote temporarily $\mathrm{Mor}_{\mathcal{C}}(X, W)$ the (possibly truncated) simplicial set $[n] \mapsto \mathrm{Mor}_{\mathcal{C}}(X, W_n)$. From the definitions it follows that $\mathrm{Mor}(U \times X, W) = \mathrm{Mor}(U, \mathrm{Mor}_{\mathcal{C}}(X, W))$ for any (possibly truncated) simplicial set U. Hence

$$\begin{aligned} \mathrm{Mor}(X \times i_{n!}\mathrm{sk}_n \Delta[n+1], W) &= \mathrm{Mor}(i_{n!}\mathrm{sk}_n \Delta[n+1], \mathrm{Mor}_{\mathcal{C}}(X, W)) \\ &= \mathrm{Mor}(\mathrm{sk}_n \Delta[n+1], \mathrm{sk}_n \mathrm{Mor}_{\mathcal{C}}(X, W)) \\ &= \mathrm{Mor}(X \times \mathrm{sk}_n \Delta[n+1], \mathrm{sk}_n W) \\ &= \mathrm{Mor}(i_{n!}(X \times \mathrm{sk}_n \Delta[n+1]), W). \end{aligned}$$

This proves the second equality used, and ends the proof of the lemma. □

Lemma 14.19.12. *Let $\mathcal{C}$ be a category with finite coproducts and finite limits. Let X be an object of $\mathcal{C}$. Let $k \geq 0$. The canonical map*

$$\mathrm{Hom}(\Delta[k], X) \longrightarrow cosk_1 sk_1 \mathrm{Hom}(\Delta[k], X)$$

is an isomorphism.

Proof. For any simplicial object V we have

$$\begin{aligned} \mathrm{Mor}(V, \mathrm{cosk}_1 \mathrm{sk}_1 \mathrm{Hom}(\Delta[k], X)) &= \mathrm{Mor}(\mathrm{sk}_1 V, \mathrm{sk}_1 \mathrm{Hom}(\Delta[k], X)) \\ &= \mathrm{Mor}(i_{1!}\mathrm{sk}_1 V, \mathrm{Hom}(\Delta[k], X)) \\ &= \mathrm{Mor}(i_{1!}\mathrm{sk}_1 V \times \Delta[k], X) \end{aligned}$$

The first equality by the adjointness of sk and cosk, the second equality by the adjointness of $i_{1!}$ and sk_1, and the first equality by Definition 14.15.1 where the last X denotes the constant simplicial object with value X. By Lemma 14.18.2 an element in this set depends only on the terms of degree 0 and 1 of $i_{1!}\mathrm{sk}_1 V \times \Delta[k]$. These agree with the degree 0 and 1 terms of $V \times \Delta[k]$, see Lemma 14.19.3. Thus the set above is equal to $\mathrm{Mor}(V \times \Delta[k], X) = \mathrm{Mor}(V, \mathrm{Hom}(\Delta[k], X))$. □

Lemma 14.19.13. *Let $\mathcal{C}$ be a category with finite coproducts and finite limits. Let X be an object of $\mathcal{C}$. Let $k \geq 0$. The canonical map*

$$\mathrm{Hom}(\Delta[k], X)_1 \longrightarrow (cosk_0 sk_0 \mathrm{Hom}(\Delta[k], X))_1$$

is identified with the map

$$\prod_{\alpha:[k]\to[1]} X \longrightarrow X \times X$$

which is the projection onto the factors where α is a constant map.

Proof. It is shown in Example 14.17.2 that $\mathrm{cosk}_0 Z$ equals $Z \times Z$ in degree 1. Moreover, it is true in general that the morphism $V_1 \to (\mathrm{cosk}_0\mathrm{sk}_0 V)_1$ is the morphism $(d_0^1, d_1^1) : V_1 \to V_0 \times V_0$ (left to the reader). Thus we simply have to compute the 0th and 1st term of $\mathrm{Hom}(\Delta[k], X)$. According to Lemma 14.15.5 we have $\mathrm{Hom}(\Delta[k], X)_0 = \prod_{\alpha:[k]\to[0]} X = X$, and $\mathrm{Hom}(\Delta[k], X)_0 = \prod_{\alpha:[k]\to[1]} X$. The lemma follows from the description of the morphisms of the simplicial object just above Lemma 14.15.5. □

14.20. Simplicial objects in abelian categories

Recall that an abelian category is defined in Homology, Section 12.5.

Lemma 14.20.1. *Let $\mathcal{A}$ be an abelian category.*

1. *The categories Simp($\mathcal{A}$) and CoSimp($\mathcal{A}$) are abelian.*
2. *A morphism of (co)simplicial objects $f : A \to B$ is injective if and only if each $f_n : A_n \to B_n$ is injective.*
3. *A morphism of (co)simplicial objects $f : A \to B$ is surjective if and only if each $f_n : A_n \to B_n$ is surjective.*
4. *A sequence of (co)simplicial objects*

$$A \xrightarrow{f} B \xrightarrow{g} C$$

is exact at B if and only if each sequence

$$A_i \xrightarrow{f_i} B_i \xrightarrow{g_i} C_i$$

is exact at B_i.

Proof. Pre-additivity is easy. A final object is given by $U_n = 0$ in all degrees. Existence of direct products we saw in Lemmas 14.6.2 and 14.9.2. Kernels and cokernels are obtained by taking termwise kernels and cokernels. □

For an object A of $\mathcal{A}$ and an integer k consider the k-truncated simplicial object U with

1. $U_i = 0$ for $i < k$,
2. $U_k = A$,
3. all morphisms $U(\varphi)$ equal to zero, except $U(\mathrm{id}_{[k]}) = \mathrm{id}_A$.

Since $\mathcal{A}$ has both finite limits and finite colimits we see that both $\mathrm{cosk}_k U$ and $i_{k!}U$ exist. We will describe both of these and the canonical map $i_{k!}U \to \mathrm{cosk}_k U$.

Lemma 14.20.2. *With A, k and U as above, so $U_i = 0$, $i < k$ and $U_k = A$.*

1. *Given a k-truncated simplicial object V we have*

$$\mathrm{Mor}(U, V) = \{f : A \to V_k \mid d_i^k \circ f = 0,\ i = 0, \ldots, k\}$$

and

$$\mathrm{Mor}(V, U) = \{f : V_k \to A \mid f \circ s_i^{k-1} = 0,\ i = 0, \ldots, k-1\}.$$

2. *The object $i_{k!}U$ has nth term equal to $\bigoplus_\alpha A$ where α runs over all surjective morphisms $\alpha : [n] \to [k]$.*

(3) *For any $\varphi : [m] \to [n]$ the map $i_{k!}U(\varphi)$ is described as the mapping $\bigoplus_\alpha A \to \bigoplus_{\alpha'} A$ which maps to component corresponding to $\alpha : [n] \to [k]$ to zero if $\alpha \circ \varphi$ is not surjective and by the identity to the component corresponding to $\alpha \circ \varphi$ if it is surjective.*
(4) *The object $cosk_k U$ has nth term equal to $\bigoplus_\beta A$, where β runs over all injective morphisms $\beta : [k] \to [n]$.*
(5) *For any $\varphi : [m] \to [n]$ the map $cosk_k U(\varphi)$ is described as the mapping $\bigoplus_\beta A \to \bigoplus_{\beta'} A$ which maps to component corresponding to $\beta : [k] \to [n]$ to zero if β does not factor through φ and by the identity to each of the components corresponding to β' such that $\beta = \varphi \circ \beta'$ if it does.*
(6) *The canonical map $c : i_{k!}U \to cosk_k U$ in degree n has (α, β) coefficient $A \to A$ equal to zero if $\alpha \circ \beta$ is not the identity and equal to id_A if it is.*
(7) *The canonical map $c : i_{k!}U \to cosk_k U$ is injective.*

Proof. The proof of (1) is left to the reader.

Let us take the rules of (2) and (3) as the definition of a simplicial object, call it $\tilde{U}$. We will show that it is an incarnation of $i_{k!}U$. This will prove (2), (3) at the same time. We have to show that given a morphism $f : U \to \mathrm{sk}_k V$ there exists a unique morphism $\tilde{f} : \tilde{U} \to V$ which recovers f upon taking the k-skeleton. From (1) we see that f corresponds with a morphism $f_k : A \to V_k$ which maps into the kernel of d_i^k for all i. For any surjective $\alpha : [n] \to [k]$ we set $\tilde{f}_\alpha : A \to V_n$ equal to the composition $\tilde{f}_\alpha = V(\alpha) \circ f_k : A \to V_n$. We define $\tilde{f}_n : \tilde{U}_n \to V_n$ as the sum of the $\tilde{f}_\alpha$ over $\alpha : [n] \to [k]$ surjective. Such a collection of $\tilde{f}_\alpha$ defines a morphism of simplicial objects if and only if for any $\varphi : [m] \to [n]$ the diagram

$$\begin{array}{ccc} \bigoplus_{\alpha:[n]\to[k] \text{ surjective}} A & \xrightarrow{\tilde{f}_n} & V_n \\ {\scriptstyle (3)}\downarrow & & \downarrow{\scriptstyle V(\varphi)} \\ \bigoplus_{\alpha':[m]\to[k] \text{ surjective}} A & \xrightarrow{\tilde{f}_m} & V_m \end{array}$$

is commutative. Choosing $\varphi = \alpha$ shows our choice of $\tilde{f}_\alpha$ is uniquely determined by f_k. The commutativity in general may be checked for each summand of the left upper corner separately. It is clear for the summands corresponding to α where $\alpha \circ \varphi$ is surjective, because those get mapped by id_A to the summand with $\alpha' = \alpha \circ \varphi$, and we have $\tilde{f}_{\alpha'} = V(\alpha') \circ f_k = V(\alpha \circ \varphi) \circ f_k = V(\varphi) \circ \tilde{f}_\alpha$. For those where $\alpha \circ \varphi$ is not surjective, we have to show that $V(\varphi) \circ \tilde{f}_\alpha = 0$. By definition this is equal to $V(\varphi) \circ V(\alpha) \circ f_k = V(\alpha \circ \varphi) \circ f_k$. Since $\alpha \circ \varphi$ is not surjective we can write it as $\delta_i^k \circ \psi$, and we deduce that $V(\varphi) \circ V(\alpha) \circ f_k = V(\psi) \circ d_i^k \circ f_k = 0$ see above.

Let us take the rules of (4) and (5) as the definition of a simplicial object, call it $\tilde{U}$. We will show that it is an incarnation of $\mathrm{cosk}_k U$. This will prove (4), (5) at the same time. The argument is completely dual to the proof of (2), (3) above, but we give it anyway. We have to show that given a morphism $f : \mathrm{sk}_k V \to U$ there exists a unique morphism $\tilde{f} : V \to \tilde{U}$ which recovers f upon taking the k-skeleton. From (1) we see that f corresponds with a morphism $f_k : V_k \to A$ which is zero on the image of s_i^{k-1} for all i. For any injective $\beta : [k] \to [n]$ we set $\tilde{f}_\beta : V_n \to A$ equal to the composition $\tilde{f}_\beta = f_k \circ V(\beta) : V_n \to A$. We define $\tilde{f}_n : V_n \to \tilde{U}_n$ as the sum of the $\tilde{f}_\beta$ over $\beta : [k] \to [n]$ injective. Such a collection of $\tilde{f}_\beta$ defines a morphism of

simplicial objects if and only if for any $\varphi : [m] \to [n]$ the diagram

$$\begin{array}{ccc} V_n & \xrightarrow[\tilde{f}_n]{} & \bigoplus_{\beta:[k]\to[n] \text{ injective}} A \\ {\scriptstyle V(\varphi)}\downarrow & & \downarrow{\scriptstyle (5)} \\ V_m & \xrightarrow{\tilde{f}_m} & \bigoplus_{\beta':[k]\to[m] \text{ injective}} A \end{array}$$

is commutative. Choosing $\varphi = \beta$ shows our choice of $\tilde{f}_\beta$ is uniquely determined by f_k. The commutativity in general may be checked for each summand of the right lower corner separately. It is clear for the summands corresponding to β' where $\varphi \circ \beta'$ is injective, because these summands get mapped into by exactly the summand with $\beta = \varphi \circ \beta'$ and we have in that case $\tilde{f}_{\beta'} \circ V(\varphi) = f_k \circ V(\beta') \circ V(\varphi) = f_k \circ V(\beta) = \tilde{f}_\beta$. For those where $\varphi \circ \beta'$ is not injective, we have to show that $\tilde{f}_{\beta'} \circ V(\varphi) = 0$. By definition this is equal to $f_k \circ V(\beta') \circ V(\varphi) = f_k \circ V(\varphi \circ \beta')$. Since $\varphi \circ \beta'$ is not injective we can write it as $\psi \circ \sigma_i^{k-1}$, and we deduce that $f_k \circ V(\beta') \circ V(\varphi) = f_k \circ s_i^{k-1} \circ V(\psi) = 0$ see above.

The composition $i_{k!}U \to \text{cosk}_k U$ is the unique map of simplicial objects which is the identity on $A = U_k = (i_{k!}U)_k = (\text{cosk}_k U)_k$. Hence it suffices to check that the proposed rule defines a morphism of simplicial objects. To see this we have to show that for any $\varphi : [m] \to [n]$ the diagram

$$\begin{array}{ccc} \bigoplus_{\alpha:[n]\to[k] \text{ surjective}} A & \xrightarrow[(6)]{} & \bigoplus_{\beta:[k]\to[n] \text{ injective}} A \\ {\scriptstyle (3)}\downarrow & & \downarrow{\scriptstyle (5)} \\ \bigoplus_{\alpha':[m]\to[k] \text{ surjective}} A & \xrightarrow{(6)} & \bigoplus_{\beta':[k]\to[m] \text{ injective}} A \end{array}$$

is commutative. Now we can think of this in terms of matrices filled with only 0's and 1's as follows: The matrix of (3) has a nonzero (α', α) entry if and only if $\alpha' = \alpha \circ \varphi$. Likewise the matrix of (5) has a nonzero (β', β) entry if and only if $\beta = \varphi \circ \beta'$. The upper matrix of (6) has a nonzero (α, β) entry if and only if $\alpha \circ \beta = \text{id}_{[k]}$. Similarly for the lower matrix of (6). The commutativity of the diagram then comes down to computing the (α, β') entry for both compositions and seeing they are equal. This comes down to the following equality

$$\#\left\{\beta \mid \beta = \varphi \circ \beta' \wedge \alpha \circ \beta = \text{id}_{[k]}\right\} = \#\left\{\alpha' \mid \alpha' = \alpha \circ \varphi \wedge \alpha' \circ \beta' = \text{id}_{[k]}\right\}$$

whose proof may safely be left to the reader.

Finally, we prove (7). This follows directly from Lemmas 14.16.7, 14.17.5, 14.19.3 and 14.19.9. □

Definition 14.20.3. Let $\mathcal{A}$ be an abelian category. Let A be an object of $\mathcal{A}$ and let k be an integer ≥ 0. The *Eilenberg-Maclane object* $K(A, k)$ is given by the object $K(A, k) = i_{k!}U$ which is described in Lemma 14.20.2 above.

Lemma 14.20.4. *Let $\mathcal{A}$ be an abelian category. Let A be an object of $\mathcal{A}$ and let k be an integer ≥ 0. Consider the simplicial object E defined by the following rules*

(1) *$E_n = \bigoplus_\alpha A$, where the sum is over $\alpha : [n] \to [k+1]$ whose image is either $[k]$ or $[k+1]$.*

(2) *Given $\varphi : [m] \to [n]$ the map $E_n \to E_m$ maps the summand corresponding to α via id_A to the summand corresponding to $\alpha \circ \varphi$, provided $Im(\alpha \circ \varphi)$ is equal to $[k]$ or $[k+1]$.*

Then there exists a short exact sequence

$$0 \to K(A,k) \to E \to K(A,k+1) \to 0$$

which is term by term split exact.

Proof. The maps $K(A,k)_n \to E_n$ resp. $E_n \to K(A,k+1)_n$ are given by the inclusion of direct sums, resp. projection of direct sums which is obvious from the inclusions of index sets. It is clear that these are maps of simplicial objects. □

Lemma 14.20.5. *Let $\mathcal{A}$ be an abelian category. For any simplicial object V of $\mathcal{A}$ we have*

$$V = \operatorname{colim}_n i_{n!} sk_n V$$

where all the transition maps are injections.

Proof. This is true simply because each V_m is equal to $(i_{n!}\mathrm{sk}_n V)_m$ as soon as $n \geq m$. See also Lemma 14.19.10 for the transition maps. □

14.21. Simplicial objects and chain complexes

Let $\mathcal{A}$ be an abelian category. See Homology, Section 12.12 for conventions and notation regarding chain complexes. Let U be a simplicial object of $\mathcal{A}$. The *associated chain complex $s(U)$ of U*, sometimes called the *Moore complex*, is the chain complex

$$\ldots \to U_2 \to U_1 \to U_0 \to 0 \to 0 \to \ldots$$

with boundary maps $d_n : U_n \to U_{n-1}$ given by the formula

$$d_n = \sum\nolimits_{i=0}^{n} (-1)^i d_i^n.$$

This is a complex because, by the relations listed in Remark 14.3.3, we have

$$\begin{aligned} d_n \circ d_{n+1} &= (\sum\nolimits_{i=0}^{n} (-1)^i d_i^n) \circ (\sum\nolimits_{j=0}^{n+1} (-1)^j d_j^{n+1}) \\ &= \sum\nolimits_{0 \leq i < j \leq n+1} (-1)^{i+j} d_{j-1}^n \circ d_i^{n+1} + \sum\nolimits_{n \geq i \geq j \geq 0} (-1)^{i+j} d_i^n \circ d_j^{n+1} \\ &= 0. \end{aligned}$$

The signs cancel! We denote the associated chain complex $s(U)$. Clearly, the construction is functorial and hence defines a functor

$$s : \mathrm{Simp}(\mathcal{A}) \longrightarrow \mathrm{Ch}_{\geq 0}(\mathcal{A}).$$

Thus we have the confusing but correct formula $s(U)_n = U_n$.

Lemma 14.21.1. *The functor s is exact.*

Proof. Clear from Lemma 14.20.1. □

Lemma 14.21.2. *Let $\mathcal{A}$ be an abelian category. Let A be an object of $\mathcal{A}$ and let k be an integer. Let E be the object described in Lemma 14.20.4. Then the complex $s(E)$ is acyclic.*

Proof. For a morphism $\alpha : [n] \to [k+1]$ we define $\alpha' : [n+1] \to [k+1]$ to be the map such that $\alpha'|_{[n]} = \alpha$ and $\alpha'(n+1) = k+1$. Note that if the image of α is $[k]$ or $[k+1]$, then the image of α' is $[k+1]$. Consider the family of maps $h_n : E_n \to E_{n+1}$ which maps the summand corresponding to α to the summand corresponding to α' via the identity on A. Let us compute $d_{n+1} \circ h_n - h_{n-1} \circ d_n$. We will first do this in case the category $\mathcal{A}$ is the category of abelian groups. Let us use the notation x_α to indicate the element $x \in A$ in the summand of E_n corresponding to the map α occurring in the index set. Let us also adopt the convention that x_α designates the zero element of E_n whenever $\mathrm{Im}(\alpha)$ is not $[k]$ or $[k+1]$. With these conventions we see that

$$d_{n+1}(h_n(x_\alpha)) = \sum\nolimits_{i=0}^{n+1} (-1)^i x_{\alpha' \circ \delta_i^{n+1}}$$

and

$$h_{n-1}(d_n(x_\alpha)) = \sum\nolimits_{i=0}^{n} (-1)^i x_{(\alpha \circ \delta_i^n)'}$$

It is easy to see that $\alpha' \circ \delta_i^{n+1} = (\alpha \circ \delta_i^n)'$ for $i = 0, \ldots, n$. It is also easy to see that $\alpha' \circ \delta_{n+1}^{n+1} = \alpha$. Thus we see that

$$(d_{n+1} \circ h_n - h_{n-1} \circ d_n)(x_\alpha) = (-1)^{n+1} x_\alpha$$

These identities continue to hold if $\mathcal{A}$ is any abelian category because they hold in the simplicial abelian group $[n] \mapsto \mathrm{Hom}(A, E_n)$; details left to the reader. We conclude that the identity map on E is homotopic to zero, with homotopy given by the system of maps $h'_n = (-1)^{n+1} h_n : E_n \to E_{n+1}$. Hence we see that E is acyclic, for example by Homology, Lemma 12.12.5. □

Lemma 14.21.3. *Let $\mathcal{A}$ be an abelian category. Let A be an object of $\mathcal{A}$ and let k be an integer. We have $H_i(s(K(A,k))) = A$ if $i = k$ and 0 else.*

Proof. First, let us prove this if $k = 0$. In this case we have $K(A,0)_n = A$ for all n. Furthermore, all the maps in this simplicial abelian group are id_A, in other words $K(A,0)$ is the constant simplicial object with value A. The boundary maps $d_n = \sum_{i=0}^n (-1)^i \mathrm{id}_A = 0$ if n odd and $= \mathrm{id}_A$ if n is even. Thus $s(K(A,0))$ looks like this

$$\ldots \to A \xrightarrow{0} A \xrightarrow{1} A \xrightarrow{0} A \to 0$$

and the result is clear.

Next, we prove the result for all k by induction. Given the result for k consider the short exact sequence

$$0 \to K(A,k) \to E \to K(A,k+1) \to 0$$

from Lemma 14.20.4. By Lemma 14.20.1 the associated sequence of chain complexes is exact. By Lemma 14.21.2 we see that $s(E)$ is acyclic. Hence the result for $k+1$ follows from the long exact sequence of homology, see Homology, Lemma 12.12.6. □

There is a second chain complex we can associate to a simplicial object of $\mathcal{A}$. Recall that by Lemma 14.16.6 any simplicial object U of $\mathcal{A}$ is canonically split with $N(U_m) = \bigcap_{i=0}^{m-1} \mathrm{Ker}(d_i^m)$. We define the *normalized chain complex* $N(U)$ to be the chain complex

$$\ldots \to N(U_2) \to N(U_1) \to N(U_0) \to 0 \to 0 \to \ldots$$

with boundary map $d_n : N(U_n) \to N(U_{n-1})$ given by the restriction of $(-1)^n d_n^n$ to the direct summand $N(U_n)$ of U_n. Note that Lemma 14.16.8 implies that $d_n^n(N(U_n)) \subset N(U_{n-1})$. It is a complex because $d_n^n \circ d_{n+1}^{n+1} = d_n^n \circ d_n^{n+1}$ and d_n^{n+1} is zero on $N(U_{n+1})$ by definition. Thus we obtain a second functor

$$N : \mathrm{Simp}(\mathcal{A}) \longrightarrow \mathrm{Ch}_{\geq 0}(\mathcal{A}).$$

Here is the reason for the sign in the differential.

Lemma 14.21.4. *Let $\mathcal{A}$ be an abelian category. Let U be a simplicial object of $\mathcal{A}$. The canonical map $N(U_n) \to U_n$ gives rise to a morphism of complexes $N(U) \to s(U)$.*

Proof. This is clear because the differential on $s(U)_n = U_n$ is $\sum(-1)^i d_i^n$ and the maps d_i^n, $i < n$ are zero on $N(U_n)$, whereas the restriction of $(-1)^n d_n^n$ is the boundary map of $N(U)$ by definition. □

Lemma 14.21.5. *Let $\mathcal{A}$ be an abelian category. Let A be an object of $\mathcal{A}$ and let k be an integer. We have $N(K(A,k))_i = A$ if $i = k$ and 0 else.*

Proof. It is clear that $N(K(A,k))_i = 0$ when $i < k$ because $K(A,k)_i = 0$ in that case. It is clear that $N(K(A,k))_k = A$ since $K(A,k)_{k-1} = 0$ and $K(A,k)_k = A$. For $i > k$ we have $N(K(A,k))_i = 0$ by Lemma 14.19.9 and the definition of $K(A,k)$, see Definition 14.20.3. □

Lemma 14.21.6. *Let $\mathcal{A}$ be an abelian category. Let U be a simplicial object of $\mathcal{A}$. The canonical morphism of chain complexes $N(U) \to s(U)$ is split. In fact,*

$$s(U) = N(U) \oplus A(U)$$

for some complex $A(U)$. The construction $U \mapsto A(U)$ is functorial.

Proof. Define $A(U)_n$ to be the image of

$$\bigoplus\nolimits_{\varphi:[n]\to[m] \text{ surjective},\ m<n} N(U_m) \xrightarrow{\bigoplus U(\varphi)} U_n$$

which is a subobject of U_n complementary to $N(U_n)$ according to Lemma 14.16.6 and Definition 14.16.1. We show that $A(U)$ is a subcomplex. Pick a surjective map $\varphi : [n] \to [m]$ with $m < n$ and consider the composition

$$N(U_m) \xrightarrow{U(\varphi)} U_n \xrightarrow{d_n} U_{n-1}.$$

This composition is the sum of the maps

$$N(U_m) \xrightarrow{U(\varphi \circ \delta_i^n)} U_{n-1}$$

with sign $(-1)^i$, $i = 0, \ldots, n$.

First we will prove by ascending induction on m, $0 \leq m < n-1$ that all the maps $U(\varphi \circ \delta_i^n)$ map $N(U_m)$ into $A(U)_{n-1}$. (The case $m = n-1$ is treated below.) Whenever the map $\varphi \circ \delta_i^n : [n-1] \to [m]$ is surjective then the image of $N(U_m)$ under $U(\varphi \circ \delta_i^n)$ is contained in $A(U)_{n-1}$ by definition. If $\varphi \circ \delta_i^n : [n-1] \to [m]$ is not surjective, set $j = \varphi(i)$ and observe that i is the unique index whose image under φ is j. We may write $\varphi \circ \delta_i^n = \delta_j^m \circ \psi \circ \delta_i^n$ for some $\psi : [n-1] \to [m-1]$. Hence $U(\varphi \circ \delta_i^n) = U(\psi \circ \delta_i^n) \circ d_j^m$ which is zero on $N(U_m)$ unless $j = m$. If $j = m$, then $d_m^m(N(U_m)) \subset N(U_{m-1})$ and hence $U(\varphi \circ \delta_i^n)(N(U_m)) \subset U(\psi \circ \delta_i^n)(N(U_{m-1}))$ and we win by induction hypothesis.

To finish proving that $A(U)$ is a subcomplex we still have to deal with the composition

$$N(U_m) \xrightarrow{U(\varphi)} U_n \xrightarrow{d_n} U_{n-1}.$$

in case $m = n-1$. In this case $\varphi = \sigma_j^{n-1}$ for some $0 \leq j \leq n-1$ and $U(\varphi) = s_j^{n-1}$. Thus the composition is given by the sum

$$\sum (-1)^i d_i^n \circ s_j^{n-1}$$

Recall from Remark 14.3.3 that $d_j^n \circ s_j^{n-1} = d_{j+1}^n \circ s_j^{n-1} = \mathrm{id}$ and these drop out because the corresponding terms have opposite signs. The map $d_n^n \circ s_j^{n-1}$, if $j < n-1$, is equal to $s_j^{n-2} \circ d_{n-1}^{n-1}$. Since d_{n-1}^{n-1} maps $N(U_{n-1})$ into $N(U_{n-2})$, we see that the image $d_n^n(s_j^{n-1}(N(U_{n-1}))$ is contained in $s_j^{n-2}(N(U_{n-2}))$ which is contained in $A(U_{n-1})$ by definition. For all other combinations of (i, j) we have either $d_i^n \circ s_j^{n-1} = s_{j-1}^{n-2} \circ d_i^{n-1}$ (if $i < j$), or $d_i^n \circ s_j^{n-1} = s_j^{n-2} \circ d_{i-1}^{n-1}$ (if $n > i > j+1$) and in these cases the map is zero because of the definition of $N(U_{n-1})$. □

Lemma 14.21.7. *The functor N is exact.*

Proof. By Lemma 14.21.1 and the functorial decomposition of Lemma 14.21.6. □

Lemma 14.21.8. *Let $\mathcal{A}$ be an abelian category. Let V be a simplicial object of $\mathcal{A}$. The canonical morphism of chain complexes $N(V) \to s(V)$ is a quasi-isomorphism. In other words, the complex $A(V)$ of Lemma 14.21.6 is acyclic.*

Proof. Note that the result holds for $K(A, k)$ for any object A and any $k \geq 0$, by Lemmas 14.21.3 and 14.21.5. Consider the hypothesis $IH_{n,m}$: for all V such that $V_j = 0$ for $j \leq m$ and all $i \leq n$ the map $N(V) \to s(V)$ induces an isomorphism $H_i(N(V)) \to H_i(s(V))$.

To start of the induction, note that $IH_{n,n}$ is trivially true, because in that case $N(V)_n = 0$ and $s(V)_n = 0$.

Assume $IH_{n,m}$, with $m \leq n$. Pick a simplicial object V such that $V_j = 0$ for $j < m$. By Lemma 14.20.2 and Definition 14.20.3 we have $K(V_m, m) = i_{m!}\mathrm{sk}_m V$. By Lemma 14.19.10 the natural morphism

$$K(V_m, m) = i_{m!}\mathrm{sk}_m V \to V$$

is injective. Thus we get a short exact sequence

$$0 \to K(V_m, m) \to V \to W \to 0$$

for some W with $W_i = 0$ for $i = 0, \ldots, m$. This short exact sequence induces a morphism of short exact sequence of associated complexes

$$\begin{array}{ccccccccc} 0 & \longrightarrow & N(K(V_m, m)) & \longrightarrow & N(V) & \longrightarrow & N(W) & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & s(K(V_m, m)) & \longrightarrow & s(V) & \longrightarrow & s(W) & \longrightarrow & 0 \end{array}$$

see Lemmas 14.21.1 and 14.21.7. Hence we deduce the result for V from the result on the ends. □

14.22. Dold-Kan

Lemma 14.22.1. *Let $\mathcal{A}$ be an abelian category. The functor N is faithful, and reflects isomorphisms, injections and surjections.*

Proof. The faithfulness is immediate from the canonical splitting of Lemma 14.16.6. The statement on reflecting injections, surjections, and isomorphisms follows from Lemma 14.16.7. □

Lemma 14.22.2. *Let $\mathcal{A}$ and $\mathcal{B}$ be abelian categories. Let $N : \mathcal{A} \to \mathcal{B}$, and $S : \mathcal{B} \to \mathcal{A}$ be functors. Suppose that*

1. *the functors S and N are exact,*
2. *there is an isomorphism $g : N \circ S \to id_{\mathcal{B}}$ to the identity functor of $\mathcal{B}$,*
3. *N is faithful, and*
4. *S is essentially surjective.*

Then S and N are quasi-inverse equivalences of categories.

Proof. It suffices to construct a functorial isomorphism $S(N(A)) \cong A$. To do this choose B and an isomorphism $f : A \to S(B)$. Consider the map

$$f^{-1} \circ g_{S(B)} \circ S(N(f)) : S(N(A)) \to S(N(S(B))) \to S(B) \to A.$$

It is easy to show this does not depend on the choice of f, B and gives the desired isomorphism $S \circ N \to \mathrm{id}_{\mathcal{A}}$. □

Theorem 14.22.3. *Let $\mathcal{A}$ be an abelian category. The functor N induces an equivalence of categories*

$$N : Simp(\mathcal{A}) \longrightarrow Ch_{\geq 0}(\mathcal{A})$$

Proof. We will describe a functor in the reverse direction inspired by the construction of Lemma 14.20.4 (except that we throw in a sign to get the boundaries right). Let $A_\bullet$ be a chain complex with boundary maps $d_{A,n} : A_n \to A_{n-1}$. For each $n \geq 0$ denote

$$I_n = \Big\{ \alpha : [n] \to \{0, 1, 2, \ldots\} \mid \mathrm{Im}(\alpha) = [k] \text{ for some } k \Big\}.$$

For $\alpha \in I_n$ we denote $k(\alpha)$ the unique integer such that $\mathrm{Im}(\alpha) = [k]$. We define a simplicial object $S(A_\bullet)$ as follows:

1. $S(A_\bullet)_n = \bigoplus_{\alpha \in I_n} A_{k(\alpha)}$, which we will write as $\bigoplus_{\alpha \in I_n} A_{k(\alpha)} \cdot \alpha$ to suggest thinking of "α" as a basis vector for the summand corresponding to it,
2. given $\varphi : [m] \to [n]$ we define $S(A_\bullet)(\varphi)$ by its restriction to the direct summand $A_{k(\alpha)} \cdot \alpha$ of $S(A_\bullet)_n$ as follows
   (a) $\alpha \circ \varphi \notin I_m$ then we set it equal to zero,
   (b) $\alpha \circ \varphi \in I_m$ but $k(\alpha \circ \varphi)$ not equal to either $k(\alpha)$ or $k(\alpha) - 1$ then we set it equal to zero as well,
   (c) if $\alpha \circ \varphi \in I_m$ and $k(\alpha \circ \varphi) = k(\alpha)$ then we use the identity map to the summand $A_{k(\alpha \circ \varphi)} \cdot (\alpha \circ \varphi)$ of $S(A_\bullet)_m$, and
   (d) if $\alpha \circ \varphi \in I_m$ and $k(\alpha \circ \varphi) = k(\alpha) - 1$ then we use $(-1)^{k(\alpha)} d_{A,k(\alpha)}$ to the summand $A_{k(\alpha \circ \varphi)} \cdot (\alpha \circ \varphi)$ of $S(A_\bullet)_m$.

It is an exercise (FIXME) to show that this is a simplicial complex; one has to use in particular that the compositions $d_{A,k} \circ d_{A,k-1}$ are all zero.

Having verified this, the correct way to proceed with the proof would be to prove directly that N and S are quasi-inverse functors (FIXME). Instead we prove this

by an indirect method using Eilenberg-Maclane objects and truncations. It is clear that $A_\bullet \mapsto S(A_\bullet)$ is an exact functor from chain complexes to simplicial objects. If $A_i = 0$ for $i = 0, \ldots, n$ then $S(A_\bullet)_i = 0$ for $i = 0, \ldots, n$. The objects $K(A, k)$, see Definition 14.20.3, are equal to $S(A[-k])$ where $A[-k]$ is the chain complex with A in degree k and zero elsewhere.

Moreover, for each integer k we get a sub simplicial object $S_{\leq k}(A_\bullet)$ by considering only those α with $k(\alpha) \leq k$. In fact this is nothing but $S(\sigma_{\leq k} A_\bullet)$, where $\sigma_{\leq k} A_\bullet$ is the "stupid" truncation of $A_\bullet$ at k (which simply replaces A_i by 0 for $i > k$). Also, by Lemma 14.19.10 we see that it is equal to $i_{k!}\mathrm{sk}_k S(A_\bullet)$. Clearly, the quotient $S_{\leq k}(A_\bullet)/S_{\leq k-1}(A_\bullet) = K(A_k, k)$ and the quotient $S(A_\bullet)/S_{\leq k}(A_\bullet) = S(A/\sigma_{\leq k} A_\bullet)$ is a simplicial object whose ith term is zero for $i = 0, \ldots, k$. Since $S_{\leq k-1}(A_\bullet)$ is filtered with subquotients $K(A_i, i)$, $i < k$ we see that $N(S_{\leq k-1}(A_\bullet))_k = 0$ by exactness of the functor N, see Lemma 14.21.7. All in all we conclude that the maps

$$N(S(A_\bullet))_k \leftarrow N(S_{\leq k}(A_\bullet))_k \to N(S(A_k[-k])) = N(K(A_k, k))_k = A_k$$

are functorial isomorphisms.

It is actually easy to identify the map $A_k \to N(S(A_\bullet))_k$. Note that there is a unique map $A_k \to S(A_\bullet)_k$ corresponding to the summand $\alpha = \mathrm{id}_{[k]}$. Note that $\mathrm{Im}(\mathrm{id}_{[k]} \circ \delta_i^k)$ has cardinality $k - 1$ but does not have image $[k - 1]$ unless $i = k$. Hence d_i^k kills the summand $A_k \cdot \mathrm{id}_{[k]}$ for $i = 0, \ldots, k - 1$. From the abstract computation of $N(S(A_\bullet))_k$ above we conclude that the summand $A_k \cdot \mathrm{id}_{[k]}$ is equal to $N(S(A_\bullet))_k$.

In order to show that $N \circ S$ is the identity functor on $\mathrm{Ch}_{\geq 0}(\mathcal{A})$, the last thing we have to verify is that we recover the map $d_{A,k+1} : A_{k+1} \to A_k$ as the differential on the complex $N(S(A_\bullet))$ as follows

$$A_{k+1} = N(S(A_\bullet))_{k+1} \to N(S(A_\bullet))_k = A_k$$

By definition the map $N(S(A_\bullet))_{k+1} \to N(S(A_\bullet))_k$ corresponds to the restriction of $(-1)^{k+1} d_{k+1}^{k+1}$ to $N(S(A_\bullet))$ which is the summand $A_{k+1} \cdot \mathrm{id}_{[k+1]}$. And by the definition of $S(A_\bullet)$ above the map d_{k+1}^{k+1} maps $A_{k+1} \cdot \mathrm{id}_{[k+1]}$ into $A_k \cdot \mathrm{id}_{[k]}$ by $(-1)^{k+1} d_{A,k+1}$. The signs cancel and hence the desired equality.

We know that N is faithful, see Lemma 14.22.1. If we can show that S is essentially surjective, then it will follow that N is an equivalence, see Homology, Lemma 14.22.2. Note that if $A_\bullet$ is a chain complex then $S(A_\bullet) = \mathrm{colim}_n S_{\leq n}(A_\bullet) = \mathrm{colim}_n S(\sigma_{\leq n} A_\bullet) = \mathrm{colim}_n i_{n!}\mathrm{sk}_n S(A_\bullet)$ by construction of S. By Lemma 14.20.5 it suffices to show that $i_{n!}V$ is in the essential image for any n-truncated simplicial object V. By induction on n it suffices to show that any extension

$$0 \to S(A_\bullet) \to V \to K(A, n) \to 0$$

where $A_i = 0$ for $i \geq n$ is in the essential image of S. By Homology, Lemma 12.7.2 we have abelian group homomorphisms

$$\mathrm{Ext}_{\mathrm{Simp}(\mathcal{A})}(K(A, n), S(A_\bullet)) \underset{S}{\overset{N}{\rightleftarrows}} \mathrm{Ext}_{\mathrm{Ch}_{\geq 0}(\mathcal{A})}(A[-n], A_\bullet)$$

between ext groups (see Homology, Definition 12.6.2). We want to show that S is surjective. We know that $N \circ S = \mathrm{id}$. Hence it suffices to show that $\mathrm{Ker}(N) = 0$. Clearly an extension

$$E: \quad \begin{array}{ccccccccccc} 0 & \longrightarrow & 0 & \longrightarrow & A_{n-1} & \longrightarrow & A_{n-2} & \longrightarrow & \ldots & \longrightarrow & A_0 & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & & & \downarrow & & \\ 0 & \longrightarrow & A & \longrightarrow & A_{n-1} & \longrightarrow & A_{n-2} & \longrightarrow & \ldots & \longrightarrow & A_0 & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & & & \downarrow & & \\ 0 & \longrightarrow & A & \longrightarrow & 0 & \longrightarrow & 0 & \longrightarrow & \ldots & \longrightarrow & 0 & \longrightarrow & 0 \end{array}$$

of $A_\bullet$ by $A[-n]$ in $\mathrm{Ch}(\mathcal{A})$ is zero if and only if the map $A \to A_{n-1}$ is zero. Thus we have to show that any extension

$$0 \to S(A_\bullet) \to V \to K(A, n) \to 0$$

such that $A = N(V)_n \to N(V)_{n-1}$ is zero is split. By Lemma 14.20.2 we have

$$\mathrm{Mor}(K(A,n), V) = \left\{ f : A \to \bigcap\nolimits_{i=0}^{n} \mathrm{Ker}(d_i^n : V_n \to V_{n-1}) \right\}$$

and if $A = N(V)_n \to N(V)_{n-1}$ is zero, then the intersection occurring in the formula above is equal to A. Let $i : K(A,n) \to V$ be the morphism corresponding to id_A on the right hand side of the displayed formula. Clearly this is a section to the map $V \to K(A,n)$ and the extension is split as desired. □

14.23. Dold-Kan for cosimplicial objects

Let $\mathcal{A}$ be an abelian category. According to Homology, Lemma 12.5.2 also $\mathcal{A}^{opp}$ is abelian. It follows formally from the definitions that

$$\mathrm{CoSimp}(\mathcal{A}) = \mathrm{Simp}(\mathcal{A}^{opp})^{opp}.$$

Thus Dold-Kan (Theorem 14.22.3) implies that $\mathrm{CoSimp}(\mathcal{A})$ is equivalent to the category $\mathrm{Ch}_{\geq 0}(\mathcal{A}^{opp})^{opp}$. And it follows formally from the definitions that

$$\mathrm{CoCh}_{\geq 0}(\mathcal{A}) = \mathrm{Ch}_{\geq 0}(\mathcal{A}^{opp})^{opp}.$$

Putting these arrows together we obtain an equivalence

$$Q : \mathrm{CoSimp}(\mathcal{A}) \longrightarrow \mathrm{CoCh}_{\geq 0}(\mathcal{A}).$$

In this section we describe Q.

First we define the *cochain complex $s(U)$ associated to a cosimplicial object U*. It is the cochain complex with terms zero in negative degrees, and $s(U)^n = U_n$ for $n \geq 0$. As differentials we use the maps $d^n : s(U)^n \to s(U)^{n+1}$ defined by $d^n = \sum_{i=0}^{n+1} (-1)^i \delta_i^{n+1}$. In other words the complex $s(U)$ looks like

$$0 \longrightarrow U_0 \xrightarrow{\delta_0^1 - \delta_1^1} U_1 \xrightarrow{\delta_0^2 - \delta_1^2 + \delta_2^2} U_2 \longrightarrow \ldots$$

This is sometimes also called the *Moore complex* associated to U.

On the other hand, given a cosimplicial object U of $\mathcal{A}$ set $Q(U)^0 = U_0$ and

$$Q(U)^n = \mathrm{Coker}\Big(\bigoplus\nolimits_{i=0}^{n-1} U_{n-1} \xrightarrow{\delta_i^n} U_n \Big).$$

The differential $d^n : Q(U)^n \to Q(U)^{n+1}$ is induced by $(-1)^{n+1}\delta^{n+1}_{n+1}$, i.e., by fitting the morphism $(-1)^{n+1}\delta^{n+1}_{n+1}$ into a commutative diagram

$$\begin{array}{ccc} U_n & \xrightarrow[(-1)^{n+1}\delta^{n+1}_{n+1}]{} & U_{n+1} \\ \downarrow & & \downarrow \\ Q(U)^n & \xrightarrow{d_n} & Q(U)^{n+1}. \end{array}$$

We leave it to the reader to show that this diagram makes sense, i.e., that the image of δ^n_i maps into the kernel of the right vertical arrow for $i = 0, \ldots, n-1$. (This is dual to Lemma 14.16.8.) Thus our cochain complex $Q(U)$ looks like this

$$0 \to Q(U)^0 \to Q(U)^1 \to Q(U)^2 \to \ldots$$

This is called the *normalized cochain complex associated to* U. The dual to the Dold-Kan Theorem 14.22.3 is the following.

Lemma 14.23.1. *Let $\mathcal{A}$ be an abelian category.*

1. *The functor $s : CoSimp(\mathcal{A}) \to CoCh_{\geq 0}(\mathcal{A})$ is exact.*
2. *The maps $s(U)^n \to Q(U)^n$ define a morphism of cochain complexes.*
3. *There exists a functorial direct sum decomposition $s(U) = A(U) \oplus Q(U)$ in $CoCh_{\geq 0}(\mathcal{A})$.*
4. *The functor Q is exact.*
5. *The morphism of complexes $s(U) \to Q(U)$ is a quasi-isomorphism.*
6. *The functor $U \mapsto Q(U)^\bullet$ defines an equivalence of categories $CoSimp(\mathcal{A}) \to CoCh_{\geq 0}(\mathcal{A})$.*

Proof. Omitted. But the results are the exact dual statements to Lemmas 14.21.1, 14.21.4, 14.21.6, 14.21.7, 14.21.8, and Theorem 14.22.3. □

14.24. Homotopies

Consider the simplicial sets $\Delta[0]$ and $\Delta[1]$. Recall that there are two morphisms

$$e_0, e_1 : \Delta[0] \longrightarrow \Delta[1],$$

coming from the morphisms $[0] \to [1]$ mapping 0 to an element of $[1] = \{0, 1\}$. Recall also that each set $\Delta[1]_k$ is finite. Hence, if the category $\mathcal{C}$ has finite coproducts, then we can form the product

$$U \times \Delta[1]$$

for any simplicial object U of $\mathcal{C}$, see Definition 14.12.1. Note that $\Delta[0]$ has the property that $\Delta[0]_k = \{*\}$ is a singleton for all $k \geq 0$. Hence $U \times \Delta[0] = U$. Thus e_0, e_1 above gives rise to morphisms

$$e_0, e_1 : U \to U \times \Delta[1].$$

Definition 14.24.1. Let $\mathcal{C}$ be a category having finite coproducts. Suppose that U and V are two simplicial objects of $\mathcal{C}$. Let $a, b : U \to V$ be two morphisms.

1. We say a morphism
$$h : U \times \Delta[1] \longrightarrow V$$
is a *homotopy connecting a to b* if $a = h \circ e_0$ and $b = h \circ e_1$.
2. We say morphisms a and b are *homotopic* if there exists a homotopy connecting a to b or a homotopy connecting b to a.

Warning: Being homotopic is **not** an equivalence relation on the set of all morphisms from U to V! The relation "there exists a homotopy from a to b" is not symmetric.

It turns out we can define homotopies between pairs of maps of simplicial objects in any category. To do this you just work out what it means to have the morphisms $h_n : (U \times \Delta[1])_n \to V_n$ in terms of the mapping property of coproducts.

Let $\mathcal{C}$ be a category with finite coproducts. Let U, V be simplicial objects of $\mathcal{C}$. Let $a, b : U \to V$ be morphisms. Further, suppose that $h : U \times \Delta[1] \to V$ is a homotopy connecting a to b. For every $n \geq 0$ let us write

$$\Delta[1]_n = \{\alpha_0^n, \ldots, \alpha_{n+1}^n\}$$

where $\alpha_i^n : [n] \to [1]$ is the map such that

$$\alpha_i^n(j) = \begin{cases} 0 & \text{if} \quad j < i \\ 1 & \text{if} \quad j \geq i \end{cases}$$

Thus

$$h_n : (U \times \Delta[1])_n = \coprod U_n \cdot \alpha_i^n \longrightarrow V_n$$

has a component $h_{n,i} : U_n \to V_n$ which is the restriction to the summand corresponding to α_i^n for all $i = 0, \ldots, n+1$.

Lemma 14.24.2. *In the situation above, we have the following relations:*

1. *We have $h_{n,0} = b_n$ and $h_{n,n+1} = a_n$.*
2. *We have $d_j^n \circ h_{n,i} = h_{n-1,i-1} \circ d_j^n$ for $i > j$.*
3. *We have $d_j^n \circ h_{n,i} = h_{n-1,i} \circ d_j^n$ for $i \leq j$.*
4. *We have $s_j^n \circ h_{n,i} = h_{n+1,i+1} \circ s_j^n$ for $i > j$.*
5. *We have $s_j^n \circ h_{n,i} = h_{n+1,i} \circ s_j^n$ for $i \leq j$.*

Conversely, given a system of maps $h_{n,i}$ satisfying the properties listed above, then these define a morphisms h which is a homotopy between a and b.

Proof. Omitted. You can prove the last statement using the fact, see Lemma 14.2.4 that to give a morphism of simplicial objects is the same as giving a sequence of morphisms h_n commuting with all d_j^n and s_j^n. □

Example 14.24.3. Suppose in the situation above $a = b$. Then there is a *trivial* homotopy between a and b, namely the one with $h_{n,i} = a_n = b_n$.

Remark 14.24.4. Let $\mathcal{C}$ be any category (no assumptions whatsoever). We say that a pair of morphisms $a, b : U \to V$ of simplicial objects are *homotopic* if there exist morphisms[1] $h_{n,i} : U_n \to V_n$, for $n \geq 0$, $i = 0, \ldots, n+1$ satisfying the relations of Lemma 14.24.2 (potentially with the roles of a and b switched). This is a "better" definition, because it applies to any category. Also it has the following property: if $F : \mathcal{C} \to \mathcal{C}'$ is any functor then a homotopic to b implies trivially that $F(a)$ is homotopic to $F(b)$. Since the lemma says that the newer notion is the same as the old one in case finite coproduct exist, we deduce in particular that functors preserve the old notion whenever both categories have finite coproducts.

[1] In the literature, often the maps $h_{n+1,i} \circ s_i : U_n \to V_{n+1}$ are used instead of the maps $h_{n,i}$. Of course the relations these maps satisfy are different from the ones in Lemma 14.24.2.

Remark 14.24.5. Let $\mathcal{C}$ be any category. Suppose two morphisms $a, a' : U \to V$ of simplicial objects are homotopic. Then for any morphism $b : V \to W$ the two maps $b \circ a, b \circ a' : U \to W$ are homotopic. Similarly, for any morphism $c : X \to U$ the two maps $a \circ c, a' \circ c : X \to V$ are homotopic. In fact the maps $b \circ a \circ c, b \circ a' \circ c : X \to W$ are homotopic. Namely, if the maps $h_{n,i} : U \to U$ define a homotopy between a and a' then the maps $b \circ h_{n,i} \circ c$ define a homotopy between $b \circ a \circ c$ and $b \circ a' \circ c$.

Definition 14.24.6. Let U and V be two simplicial objects of a category $\mathcal{C}$. We say a morphism $a : U \to V$ is a *homotopy equivalence* if there exists a morphism $b : V \to U$ such that $a \circ b$ is homotopic to id_V and $b \circ a$ is homotopic to id_U. If there exists such a morphism between U and V, then we say that U and V are *homotopy equivalent*[2].

Example 14.24.7. The simplicial set $\Delta[m]$ is homotopy equivalent to $\Delta[0]$. Namely, there is a unique morphism $f : \Delta[m] \to \Delta[0]$ and we take $g : \Delta[0] \to \Delta[m]$ to be given by the inclusion of the last 0-simplex of $\Delta[m]$. We have $f \circ g = \mathrm{id}$ and we will give a homotopy $h : \Delta[m] \times \Delta[1] \to \Delta[m]$ between $\mathrm{id}_{\Delta[m]}$ and $g \circ f$. Namely h given by the maps

$$\mathrm{Mor}_\Delta([n], [m]) \times \mathrm{Mor}_\Delta([n], [1]) \to \mathrm{Mor}_\Delta([n], [m])$$

which send (φ, α) to

$$k \mapsto \begin{cases} \varphi(k) & \text{if} \quad \alpha(k) = 0 \\ m & \text{if} \quad \alpha(k) = 1 \end{cases}$$

Note that this only works because we took g to be the inclusion of the last 0-simplex. If we took g to be the inclusion of the first 0-simplex we could find a homotopy from $g \circ f$ to $\mathrm{id}_{\Delta[m]}$. This is an illustration of the asymmetry inherent in homotopies in the category of simplicial sets.

The following lemma says that $U \times \Delta[1]$ is homotopy equivalent to U.

Lemma 14.24.8. *Let $\mathcal{C}$ be a category with finite coproducts. Let U be a simplicial object of $\mathcal{C}$. Consider the maps $e_1, e_0 : U \to U \times \Delta[1]$, and $\pi : U \times \Delta[1] \to U$, see Lemma 14.12.3.*

1. *We have $\pi \circ e_1 = \pi \circ e_0 = id_U$, and*
2. *The morphisms $id_{U \times \Delta[1]}$, and $e_0 \circ \pi$ are homotopic.*
3. *The morphisms $id_{U \times \Delta[1]}$, and $e_1 \circ \pi$ are homotopic.*

Proof. The first assertion is trivial. For the second, consider the map of simplicial sets $\Delta[1] \times \Delta[1] \longrightarrow \Delta[1]$ which in degree n assigns to a pair (β_1, β_2), $\beta_i : [n] \to [1]$ the morphism $\beta : [n] \to [1]$ defined by the rule

$$\beta(i) = \max\{\beta_1(i), \beta_2(i)\}.$$

It is a morphism of simplicial sets, because the action $\Delta[1](\varphi) : \Delta[1]_n \to \Delta[1]_m$ of $\varphi : [m] \to [n]$ is by precomposing. Clearly, using notation from Section 14.24, we have $\beta = \beta_1$ if $\beta_2 = \alpha_0^n$ and $\beta = \alpha_{n+1}^n$ if $\beta_2 = \alpha_{n+1}^n$. This implies easily that the induced morphism

$$U \times \Delta[1] \times \Delta[1] \longrightarrow U \times \Delta[1]$$

of Lemma 14.12.3 is a homotopy between $\mathrm{id}_{U \times \Delta[1]}$ and $e_0 \circ \pi$. Similarly for $e_1 \circ \pi$ (use minimum instead of maximum). □

[2]Warning: This notion is **not** an equivalence relation on objects in general.

Lemma 14.24.9. *Let $f : Y \to X$ be a morphism of a category $\mathcal{C}$ with fibre products. Assume f has a section s. Consider the simplicial object U constructed in Example 14.3.5 starting with f. The morphism $U \to U$ which in each degree is the self map $(s \circ f)^{n+1}$ of $Y \times_X \ldots \times_X Y$ given by $s \circ f$ on each factor is homotopic to the identity on U. In particular, U is homotopy equivalent to the constant simplicial object X.*

Proof. Set $g^0 = \mathrm{id}_Y$ and $g^1 = s \circ f$. We use the morphisms

$$\begin{aligned} Y \times_X \ldots \times_X Y \times \mathrm{Mor}([n], [1]) &\to Y \times_X \ldots \times_X Y \\ (y_0, \ldots, y_n) \times \alpha &\mapsto (g^{\alpha(0)}(y_0), \ldots, g^{\alpha(n)}(y_n)) \end{aligned}$$

where we use the functor of points point of view to define the maps. Another way to say this is to say that $h_{n,0} = \mathrm{id}$, $h_{n,n+1} = (s \circ f)^{n+1}$ and $h_{n,i} = \mathrm{id}_Y^{i+1} \times (s \circ f)^{n+1-i}$. We leave it to the reader to show that these satisfy the relations of Lemma 14.24.2. Hence they define the desired homotopy. See also Remark 14.24.4 which shows that we do not need to assume anything else on the category $\mathcal{C}$. □

Lemma 14.24.10. *Let $\mathcal{C}$ be a category.*

1. *If $a_t, b_t : X_t \to Y_t$, $t \in T$ are homotopic morphisms between simplicial objects of $\mathcal{C}$, then $\prod a_t, \prod b_t : \prod X_t \to \prod Y_t$ are homotopic morphisms between simplicial objects of $\mathcal{C}$, provided $\prod X_t$ and $\prod Y_t$ exist in Simp($\mathcal{C}$).*
2. *If (X_t, Y_t), $t \in T$ are homotopy equivalent pairs of simplicial objects of $\mathcal{C}$, then $\prod X_t$ and $\prod Y_t$ are homotopy equivalent pairs of simplicial objects of $\mathcal{C}$, provided $\prod X_t$ and $\prod Y_t$ exist in Simp($\mathcal{C}$).*

Proof. If $h_t = (h_{t,n,i})$ are homotopies connecting a_t and b_t (see Remark 14.24.4), then $h = (\prod_t h_{t,n,i})$ is a homotopy connecting $\prod a_t$ and $\prod b_t$. This proves (1). Part (2) follows from part (1) and the definitions. □

14.25. Homotopies in abelian categories

Let $\mathcal{A}$ be an abelian category. Let U, V be simplicial objects of $\mathcal{A}$. Let $a, b : U \to V$ be morphisms. Further, suppose that $h : U \times \Delta[1] \to V$ is a homotopy connecting a and b. Consider the two morphisms of chain complexes $s(a), s(b) : s(U) \longrightarrow s(V)$. Using the notation introduced above Lemma 14.24.2 we define

$$s(h)_n : U_n \longrightarrow V_{n+1}$$

by the formula

$$s(h)_n = \sum\nolimits_{i=0}^{n} (-1)^{i+1} h_{n+1,i+1} \circ s_i^n. \tag{14.25.0.1}$$

Let us compute $d_{n+1} \circ s(h)_n + s(h)_{n-1} \circ d_n$. We first compute

$$\begin{aligned} d_{n+1} \circ s(h)_n &= \sum\nolimits_{j=0}^{n+1} \sum\nolimits_{i=0}^{n} (-1)^{j+i+1} d_j^{n+1} \circ h_{n+1,i+1} \circ s_i^n \\ &= \sum\nolimits_{1 \leq i+1 \leq j \leq n+1} (-1)^{j+i+1} h_{n,i+1} \circ d_j^{n+1} \circ s_i^n \\ &\quad + \sum\nolimits_{n \geq i \geq j \geq 0} (-1)^{i+j+1} h_{n,i} \circ d_j^{n+1} \circ s_i^n \\ &= \sum\nolimits_{1 \leq i+1 < j \leq n+1} (-1)^{j+i+1} h_{n,i+1} \circ s_i^{n-1} \circ d_{j-1}^n \\ &\quad + \sum\nolimits_{1 \leq i+1 = j \leq n+1} (-1)^{j+i+1} h_{n,i+1} \\ &\quad + \sum\nolimits_{n \geq i = j \geq 0} (-1)^{i+j+1} h_{n,i} \\ &\quad + \sum\nolimits_{n \geq i > j \geq 0} (-1)^{i+j+1} h_{n,i} \circ s_{i-1}^{n-1} \circ d_j^n \end{aligned}$$

We leave it to the reader to see that the first and the last of the four sums cancel exactly against all the terms of

$$s(h)_{n-1} \circ d_n = \sum_{i=0}^{n-1} \sum_{j=0}^{n} (-1)^{i+1+j} h_{n,i+1} \circ s_i^{n-1} \circ d_j^n.$$

Hence we obtain

$$\begin{aligned} d_{n+1} \circ s(h)_n + s(h)_{n-1} \circ d_n &= \sum_{j=1}^{n+1} (-1)^{2j} h_{n,j} + \sum_{i=0}^{n} (-1)^{2i+1} h_{n,i} \\ &= h_{n,n+1} - h_{n,0} \\ &= a_n - b_n \end{aligned}$$

Thus we've proved part of the following lemma.

Lemma 14.25.1. *Let $\mathcal{A}$ be an abelian category. Let $a, b : U \to V$ be morphisms of simplicial objects of $\mathcal{A}$. If a, b are homotopic, then $s(a), s(b) : s(U) \to s(V)$, and $N(a), N(b) : N(U) \to N(V)$ are homotopic maps of chain complexes.*

Proof. The part about $s(a)$ and $s(b)$ is clear from the calculation above the lemma. On the other hand, if follows from Lemma 14.21.6 that $N(a)$, $N(b)$ are compositions

$$N(U) \to s(U) \to s(V) \to N(V)$$

where we use $s(a)$, $s(b)$ in the middle. Hence the assertion follows from Homology, Lemma 12.12.1. □

Lemma 14.25.2. *Let $\mathcal{A}$ be an abelian category. Let $a : U \to V$ be a morphism of simplicial objects of $\mathcal{A}$. If a is a homotopy equivalence, then $s(a) : s(U) \to s(V)$, and $N(a) : N(U) \to N(V)$ are homotopy equivalences of chain complexes.*

Proof. Omitted. See Lemma 14.25.1 above. □

14.26. Homotopies and cosimplicial objects

Let $\mathcal{C}$ be a category with finite products. Let V be a cosimplicial object and consider $\mathrm{Hom}(\Delta[1], V)$, see Section 14.13. The morphisms $e_0, e_1 : \Delta[0] \to \Delta[1]$ produce two morphisms $e_0, e_1 : \mathrm{Hom}(\Delta[1], V) \to V$.

Definition 14.26.1. Let $\mathcal{C}$ be a category having finite products. Suppose that U and V are two cosimplicial objects of $\mathcal{C}$. We say morphisms $a, b : U \to V$ are *homotopic* if there exists a morphism

$$h : U \longrightarrow \mathrm{Hom}(\Delta[1], V)$$

such that $a = e_0 \circ h$ and $b = e_1 \circ h$. In this case h is called a *homotopy connecting a and b*.

This is really exactly the same as the notion we introduced for simplicial objects earlier. In particular, recall that $\Delta[1]_n$ is a finite set, and that

$$h_n = (h_{n,\alpha}) : U \longrightarrow \prod\nolimits_{\alpha \in \Delta[1]_n} V_n$$

is given by a collection of maps $h_{n,\alpha} : U_n \to V_n$ parametrized by elements of $\Delta[1]_n = \mathrm{Mor}_\Delta([n], [1])$. As in Lemma 14.24.2 these morphisms satisfy some relations. Namely, for every $f : [n] \to [m]$ in Δ we should have

$$h_{m,\alpha} \circ U(f) = V(f) \circ h_{n,\alpha \circ f} \tag{14.26.1.1}$$

The condition that $a = e_0 \circ h$ means that $a_n = h_{n,0:[n]\to[1]}$ where $0 : [n] \to [1]$ is the constant map with value zero. Similarly, we should have $b_n = h_{n,1:[n]\to[1]}$. In particular we deduce once more that the notion of homotopy can be formulated between cosimplicial objects of any category, i.e., existence of products is not necessary. Here is a precise formulation of why this is dual to the notion of a homotopy between morphisms of simplicial objects.

Lemma 14.26.2. *Let $\mathcal{C}$ be a category having finite products. Suppose that U and V are two cosimplicial objects of $\mathcal{C}$. Let $a, b : U \to V$ be morphisms of cosimplicial objects. Recall that U, V correspond to simplicial objects U', V' of $\mathcal{C}^{opp}$. Moreover a, b correspond to morphisms $a', b' : V' \to U'$. The following are equivalent*

1. *The morphisms $a, b : U \to V$ of cosimplicial objects are homotopic.*
2. *The morphisms $a', b' : V' \to U'$ of simplicial objects of $\mathcal{C}^{opp}$ are homotopic.*

Proof. If $\mathcal{C}$ has finite products, then $\mathcal{C}^{opp}$ has finite coproducts. And the contravariant functor $(-)' : \mathcal{C} \to \mathcal{C}^{opp}$ transforms products into coproducts. Then it is immediate from the definitions that $(\mathrm{Hom}(\Delta[1], V))' = V' \times \Delta[1]$. And so on and so forth. □

Lemma 14.26.3. *Let $\mathcal{C}, \mathcal{C}', \mathcal{D}\mathcal{D}'$ be categories such that $\mathcal{C}, \mathcal{C}'$ have finite products, and $\mathcal{D}, \mathcal{D}'$ have finite coproducts.*

1. *Let $a, b : U \to V$ be morphisms of simplicial objects of $\mathcal{D}$. Let $F : \mathcal{D} \to \mathcal{D}'$ be a covariant functor. If a and b are homotopic, then $F(a)$, $F(b)$ are homotopic morphisms $F(U) \to F(V)$ of simplicial objects.*
2. *Let $a, b : U \to V$ be morphisms of cosimplicial objects of $\mathcal{C}$. Let $F : \mathcal{C} \to \mathcal{C}'$ be a covariant functor. If a and b are homotopic, then $F(a)$, $F(b)$ are homotopic morphisms $F(U) \to F(V)$ of cosimplicial objects.*
3. *Let $a, b : U \to V$ be morphisms of simplicial objects of $\mathcal{D}$. Let $F : \mathcal{D} \to \mathcal{C}$ be a contravariant functor. If a and b are homotopic, then $F(a)$, $F(b)$ are homotopic morphisms $F(V) \to F(U)$ of cosimplicial objects.*
4. *Let $a, b : U \to V$ be morphisms of cosimplicial objects of $\mathcal{C}$. Let $F : \mathcal{C} \to \mathcal{D}$ be a contravariant functor. If a and b are homotopic, then $F(a)$, $F(b)$ are homotopic morphisms $F(V) \to F(U)$ of simplicial objects.*

Proof. By Lemma 14.26.2 above, we can turn F into a covariant functor between a pair of categories which have finite coproducts, and we have to show that the functor preserves homotopic pairs of maps. It is explained in Remark 14.24.4 how this is the case. Even if the functor does not commute with coproducts! □

Lemma 14.26.4. *Let $f : Y \to X$ be a morphism of a category $\mathcal{C}$ with pushouts. Assume f has a section s. Consider the cosimplicial object U constructed in Example 14.5.5 starting with f. The morphism $U \to U$ which in each degree is the self map of $Y \amalg_X \ldots \amalg_X Y$ given by $s \circ f$ on each factor is homotopic to the identity on U. In particular, U is homotopy equivalent to the constant cosimplicial object X.*

Proof. The dual statement which is Lemma 14.24.9. Hence this lemma follows on applying Lemma 14.26.2. □

Lemma 14.26.5. *Let $\mathcal{A}$ be an abelian category. Let $a, b : U \to V$ be morphisms of cosimplicial objects of $\mathcal{A}$. If a, b are homotopic, then $s(a), s(b) : s(U) \to s(V)$, and $Q(a), Q(b) : Q(U) \to Q(V)$ are homotopic maps of cochain complexes.*

Proof. Let $(-)' : \mathcal{A} \to \mathcal{A}^{opp}$ be the contravariant functor $A \mapsto A$. By Lemma 14.26.4 the maps a' and b' are homotopic. By Lemma 14.25.1 we see that $s(a')$ and $s(b')$ are homotopic maps of chain complexes. Since $s(a') = (s(a))'$ and $s(b') = (s(b))'$ we conclude that also $s(a)$ and $s(b)$ are homotopic by applying the additive contravariant functor $(-)'' : \mathcal{A}^{opp} \to \mathcal{A}$. The result for the Q-complexes follows from the direct sum decomposition of Lemma 14.23.1 for example. □

14.27. More homotopies in abelian categories

Let $\mathcal{A}$ be an abelian category. In this section we show that a homotopy between morphisms in $\mathrm{Ch}_{\geq 0}(\mathcal{A})$ always comes from a morphism $U \times \Delta[1] \to V$ in the category of simplicial objects. In some sense this will provide a converse to Lemma 14.25.1. We first develop some material on homotopies between morphisms of chain complexes.

Lemma 14.27.1. *Let $\mathcal{A}$ be an abelian category. Let A be a chain complex. Consider the covariant functor*

$$B \longmapsto \{(a, b, h) \mid a, b : A \to B \text{ and } h \text{ a homotopy between } a, b\}$$

There exists a chain complex $\diamond A$ such that $\mathrm{Mor}_{Ch(\mathcal{A})}(\diamond A, -)$ is isomorphic to the displayed functor. The construction $A \mapsto \diamond A$ is functorial.

Proof. We set $\diamond A_n = A_n \oplus A_n \oplus A_{n-1}$, and we define $d_{\diamond A, n}$ by the matrix

$$d_{\diamond A,n} = \begin{pmatrix} d_{A,n} & 0 & \mathrm{id}_{A_{n-1}} \\ 0 & d_{A,n} & -\mathrm{id}_{A_{n-1}} \\ 0 & 0 & -d_{A,n-1} \end{pmatrix} : A_n \oplus A_n \oplus A_{n-1} \to A_{n-1} \oplus A_{n-1} \oplus A_{n-2}$$

If $\mathcal{A}$ is the category of abelian groups, and $(x, y, z) \in A_n \oplus A_n \oplus A_{n-1}$ then $d_{\diamond A,n}(x, y, z) = (d_n(x) + z, d_n(y) - z, -d_{n-1}(z))$. It is easy to verify that $d^2 = 0$. Clearly, there are two maps $\diamond a, \diamond b : A \to \diamond A$ (first summand and second summand), and a map $\diamond A \to A[-1]$ which give a short exact sequence

$$0 \to A \oplus A \to \diamond A \to A[-1] \to 0$$

which is termwise split. Moreover, there is a sequence of maps $\diamond h_n : A_n \to \diamond A_{n+1}$, namely the identity from A_n to the summand A_n of $\diamond A_{n+1}$, such that $\diamond h$ is a homotopy between $\diamond a$ and $\diamond b$.

We conclude that any morphism $f : \diamond A \to B$ gives rise to a triple (a, b, h) by setting $a = f \circ \diamond a$, $b = f \circ \diamond b$ and $h_n = f_{n+1} \circ \diamond h_n$. Conversely, given a triple (a, b, h) we get a morphism $f : \diamond A \to B$ by taking

$$f_n = (a_n, b_n, h_{n-1}).$$

To see that this is a morphism of chain complexes you have to do a calculation. We only do this in case $\mathcal{A}$ is the category of abelian groups: Say $(x, y, z) \in \diamond A_n = A_n \oplus A_n \oplus A_{n-1}$. Then

$$\begin{aligned} f_{n-1}(d_n(x, y, z)) &= f_{n-1}(d_n(x) + z, d_n(y) - z, -d_{n-1}(z)) \\ &= a_n(d_n(x)) + a_n(z) + b_n(d_n(y)) - b_n(z) - h_{n-2}(d_{n-1}(z)) \end{aligned}$$

and

$$\begin{aligned} d_n(f_n(x, y, z) &= d_n(a_n(x) + b_n(y) + h_{n-1}(z)) \\ &= d_n(a_n(x)) + d_n(b_n(y)) + d_n(h_{n-1}(z)) \end{aligned}$$

which are the same by definition of a homotopy. □

Note that the extension

$$0 \to A \oplus A \to \diamond A \to A[-1] \to 0$$

comes with sections of the morphisms $\diamond A_n \to A[-1]_n$ with the property that the associated morphism $\delta : A[-1] \to (A \oplus A)[-1]$, see Homology, Lemma 12.14.4 equals the morphism $(1, -1) : A[-1] \to A[-1] \oplus A[-1]$.

Lemma 14.27.2. *Let $\mathcal{A}$ be an abelian category. Let*

$$0 \to A \oplus A \to B \to C \to 0$$

be a short exact sequence of chain complexes of $\mathcal{A}$. Suppose given in addition morphisms $s_n : C_n \to B_n$ splitting the associated short exact sequence in degree n. Let $\delta(s) : C \to (A \oplus A)[-1] = A[-1] \oplus A[-1]$ be the associated morphism of complexes, see Homology, Lemma 12.14.4. If $\delta(s)$ factors through the morphism $(1, -1) : A[-1] \to A[-1] \oplus A[-1]$, then there is a unique morphism $B \to \diamond A$ fitting into a commutative diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & A \oplus A & \longrightarrow & B & \longrightarrow & C & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & A \oplus A & \longrightarrow & \diamond A & \longrightarrow & A[-1] & \longrightarrow & 0 \end{array}$$

where the vertical maps are compatible with the splittings s_n and the splittings of $\diamond A_n \to A[-1]_n$ as well.

Proof. Denote $(p_n, q_n) : B_n \to A_n \oplus A_n$ the morphism π_n of Homology, Lemma 12.14.4. Also write $(a, b) : A \oplus A \to B$, and $r : B \to C$ for the maps in the short exact sequence. Write the factorization of $\delta(s)$ as $\delta(s) = (1, -1) \circ f$. This means that $p_{n-1} \circ d_{B,n} \circ s_n = f_n$, and $q_{n-1} \circ d_{B,n} \circ s_n = -f_n$, and Set $B_n \to \diamond A_n = A_n \oplus A_n \oplus A_{n-1}$ equal to $(p_n, q_n, f_n \circ r_n)$.

Now we have to check that this actually defines a morphism of complexes. We will only do this in the case of abelian groups. Pick $x \in B_n$. Then $x = a_n(x_1) + b_n(x_2) + s_n(x_3)$ and it suffices to show that our definition commutes with differential for each term separately. For the term $a_n(x_1)$ we have $(p_n, q_n, f_n \circ r_n)(a_n(x_1)) = (x_1, 0, 0)$

and the result is obvious. Similarly for the term $b_n(x_2)$. For the term $s_n(x_3)$ we have

$$\begin{aligned}(p_n, q_n, f_n \circ r_n)(d_n(s_n(x_3))) &= (p_n, q_n, f_n \circ r_n)(\\ &\qquad a_n(f_n(x_3)) - b_n(f_n(x_3)) + s_n(d_n(x_3))) \\ &= (f_n(x_3), -f_n(x_3), f_n(d_n(x_3)))\end{aligned}$$

by definition of f_n. And

$$\begin{aligned}d_n(p_n, q_n, f_n \circ r_n)(s_n(x_3)) &= d_n(0, 0, f_n(x_3)) \\ &= (f_n(x_3), -f_n(x_3), d_{A[-1],n}(f_n(x_3)))\end{aligned}$$

The result follows as f is a morphism of complexes. □

Lemma 14.27.3. *Let $\mathcal{A}$ be an abelian category. Let U, V be simplicial objects of $\mathcal{A}$. Let $a, b : U \to V$ be a pair of morphisms. Assume the corresponding maps of chain complexes $N(a), N(b) : N(U) \to N(V)$ are homotopic by a homotopy $\{N_n : N(U)_n \to N(V)_{n+1}\}$. Then a, b are homotopic in the sense of Definition 14.24.1. Moreover, one can choose the homotopy $h : U \times \Delta[1] \to V$ such that $N_n = N(h)_n$ where $N(h)$ is the homotopy coming from h as in Section 14.25.*

Proof. Let $(\diamond N(U), \diamond a, \diamond b, \diamond h)$ be as in Lemma 14.27.1 and its proof. By that lemma there exists a morphism $\diamond N(U) \to N(V)$ representing the triple $(N(a), N(b), \{N_n\})$. We will show there exists a morphism $\psi : N(U \times \Delta[1]) \to \diamond N(U)$ such that $\diamond a = \psi \circ N(e_0)$, and $\diamond b = \psi \circ N(e_1)$. Moreover, we will show that the homotopy between $N(e_0), N(e_1) : N(U) \to N(U \times \Delta[1])$ coming from (14.25.0.1) and Lemma 14.25.1 with $h = \mathrm{id}_{U \times \Delta[1]}$ is mapped via ψ to the canonical homotopy $\diamond h$ between the two maps $\diamond a, \diamond b : N(U) \to \diamond N(U)$. Certainly this will imply the lemma.

Note that $N : \mathrm{Simp}(\mathcal{A}) \to \mathrm{Ch}_{\geq 0}(\mathcal{A})$ as a functor is a direct summand of the functor $N : \mathrm{Simp}(\mathcal{A}) \to \mathrm{Ch}_{\geq 0}(\mathcal{A})$. Also, the functor $\diamond$ is compatible with direct sums. Thus it suffices instead to construct a morphism $\Psi : s(U \times \Delta[1]) \to \diamond s(U)$ with the corresponding properties. This is what we do below.

By Definition 14.24.1 the morphisms $e_0 : U \to U \times \Delta[1]$ and $e_1 : U \to U \times \Delta[1]$ are homotopic with homotopy $\mathrm{id}_{U \times \Delta[1]}$. By Lemma 14.25.1 we get an explicit homotopy $\{h_n : s(U)_n \to s(U \times \Delta[1])_{n+1}\}$ between the morphisms of chain complexes $s(e_0) : s(U) \to s(U \times \Delta[1])$ and $s(e_1) : s(U) \to s(U \times \Delta[1])$. By Lemma 14.27.2 above we get a corresponding morphism

$$\Phi : \diamond s(U) \to s(U \times \Delta[1])$$

According to the construction, Φ_n restricted to the summand $s(U)[-1]_n = s(U)_{n-1}$ of $\diamond s(U)_n$ is equal to h_{n-1}. And

$$h_{n-1} = \sum\nolimits_{i=0}^{n-1} (-1)^{i+1} s_i^n \cdot \alpha_{i+1}^n : U_{n-1} \to \bigoplus\nolimits_j U_n \cdot \alpha_j^n.$$

with obvious notation.

On the other hand, the morphisms $e_i : U \to U \times \Delta[1]$ induce a morphism $(e_0, e_1) : U \oplus U \to U \times \Delta[1]$. Denote W the cokernel. Note that, if we write $(U \times \Delta[1])_n =$

$\bigoplus_{\alpha:[n]\to[1]} U_n \cdot \alpha$, then we may identify $W_n = \bigoplus_{i=1}^n U_n \cdot \alpha_i^n$ with α_i^n as in Section 14.24. We have a commutative diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & U \oplus U & \longrightarrow & U \times \Delta[1] & \longrightarrow & W & \longrightarrow & 0 \\ & & & \searrow^{(1,1)} & \downarrow \pi & & & & \\ & & & & U & & & & \end{array}$$

This implies we have a similar commutative diagram after applying the functor s. Next, we choose the splittings $\sigma_n : s(W)_n \to s(U \times \Delta[1])_n$ by mapping the summand $U_n \cdot \alpha_i^n \subset W_n$ via $(-1, 1)$ to the summands $U_n \cdot \alpha_0^n \oplus U_n \cdot \alpha_i^n \subset (U \times \Delta[1])_n$. Note that $s(\pi)_n \circ \sigma_n = 0$. It follows that $(1, 1) \circ \delta(\sigma)_n = 0$. Hence $\delta(\sigma)$ factors as in Lemma 14.27.2. By that lemma we obtain a canonical morphism $\Psi : s(U \times \Delta[1]) \to \diamond s(U)$.

To compute Ψ we first compute the morphism $\delta(\sigma) : s(W) \to s(U)[-1] \oplus s(U)[-1]$. According to Homology, Lemma 12.14.4 and its proof, to do this we have compute

$$d_{s(U\times\delta[1]),n} \circ \sigma_n - \sigma_{n-1} \circ d_{s(W),n}$$

and write it as a morphism into $U_{n-1} \cdot \alpha_0^{n-1} \oplus U_{n-1} \cdot \alpha_n^{n-1}$. We only do this in case $\mathcal{A}$ is the category of abelian groups. We use the short hand notation x_α for $x \in U_n$ to denote the element x in the summand $U_n \cdot \alpha$ of $(U \times \Delta[1])_n$. Recall that

$$d_{s(U\times\delta[1]),n} = \sum\nolimits_{i=0}^{n} (-1)^i d_i^n$$

where d_i^n maps the summand $U_n \cdot \alpha$ to the summand $U_{n-1} \cdot (\alpha \circ \delta_i^n)$ via the morphism d_i^n of the simplicial object U. In terms of the notation above this means

$$d_{s(U\times\delta[1]),n}(x_\alpha) = \sum\nolimits_{i=0}^{n} (-1)^i (d_i^n(x))_{\alpha\circ\delta_i^n}$$

Starting with $x_\alpha \in W_n$, in other words $\alpha = \alpha_j^n$ for some $j \in \{1, \ldots, n\}$, we see that $\sigma_n(x_\alpha) = x_\alpha - x_{\alpha_0^n}$ and hence

$$(d_{s(U\times\delta[1]),n} \circ \sigma_n)(x_\alpha) = \sum\nolimits_{i=0}^{n} (-1)^i (d_i^n(x))_{\alpha\circ\delta_i^n} - \sum\nolimits_{i=0}^{n} (-1)^i (d_i^n(x))_{\alpha_0^n\circ\delta_i^n}$$

To compute $d_{s(W),n}(x_\alpha)$, we have to omit all terms where $\alpha \circ \delta_i^n = \alpha_0^{n-1}, \alpha_n^{n-1}$. Hence we get

$$(\sigma_{n-1} \circ d_{s(W),n})(x_\alpha) =$$
$$\sum\nolimits_{i=0,\ldots,n \text{ and } \alpha\circ\delta_i^n \neq \alpha_0^{n-1} \text{ or } \alpha_n^{n-1}} \left((-1)^i (d_i^n(x))_{\alpha\circ\delta_i^n} - (-1)^i (d_i^n(x))_{\alpha_0^{n-1}} \right)$$

Clearly the difference of the two terms is the sum

$$\sum\nolimits_{i=0,\ldots,n \text{ and } \alpha\circ\delta_i^n = \alpha_0^{n-1} \text{ or } \alpha_n^{n-1}} \left((-1)^i (d_i^n(x))_{\alpha\circ\delta_i^n} - (-1)^i (d_i^n(x))_{\alpha_0^{n-1}} \right)$$

Of course, if $\alpha \circ \delta_i^n = \alpha_0^{n-1}$ then the term drops out. Recall that $\alpha = \alpha_j^n$ for some $j \in \{1, \ldots, n\}$. The only way $\alpha_j^n \circ \delta_i^n = \alpha_n^{n-1}$ is if $j = n$ and $i = n$. Thus we actually get 0 unless $j = n$ and in that case we get $(-1)^n (d_n^n(x))_{\alpha_n^{n-1}} - (-1)^n (d_n^n(x))_{\alpha_0^{n-1}}$. In other words, we conclude the morphism

$$\delta(\sigma)_n : W_n \to (s(U)[-1] \oplus s(U)[-1])_n = U_{n-1} \oplus U_{n-1}$$

is zero on all summands except $U_n \cdot \alpha_n^n$ and on that summand it is equal to $((-1)^n d_n^n, -(-1)^n d_n^n)$. (Namely, the first summand of the two corresponds to the factor with α_n^{n-1} because that is the map $[n-1] \to [1]$ which maps everybody to 0, and hence corresponds to e_0.)

We obtain a canonical diagram

$$\begin{array}{ccccccccc}
0 & \longrightarrow & s(U) \oplus s(U) & \longrightarrow & \diamond s(U) & \longrightarrow & s(U)[-1] & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow{\scriptstyle \Phi} & & \downarrow & & \\
0 & \longrightarrow & s(U) \oplus s(U) & \longrightarrow & s(U \times \Delta[1]) & \longrightarrow & s(W) & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow{\scriptstyle \Psi} & & \downarrow & & \\
0 & \longrightarrow & s(U) \oplus s(U) & \longrightarrow & \diamond s(U) & \longrightarrow & s(U)[-1] & \longrightarrow & 0
\end{array}$$

We claim that $\Phi \circ \Psi$ is the identity. To see this it is enough to prove that the composition of Φ and $\delta(\sigma)$ as a map $s(U)[-1] \to s(W) \to s(U)[-1] \oplus s(U)[-1]$ is the identity in the first factor and minus identity in the second. By the computations above it is $((-1)^n d_0^n, -(-1)^n d_0^n) \circ (-1)^n s_n^n = (1, -1)$ as desired. □

14.28. Trivial Kan fibrations

Recall that for $n \geq 0$ the simplicial set $\Delta[n]$ is given by the rule $[k] \mapsto \mathrm{Mor}_\Delta([k], [n])$, see Example 14.11.2. Recall that $\Delta[n]$ has a unique nondegenerate n-simplex and all nondegenerate simplices are faces of this n-simplex. In fact, the nondegenerate simplices of $\Delta[n]$ correspond exactly to injective morphisms $[k] \to [n]$, which we may identify with subsets of $[n]$. Moreover, recall that $\mathrm{Mor}(\Delta[n], X) = X_n$ for any simplicial set X (Lemma 14.11.3). We set

$$\partial\Delta[n] = i_{(n-1)!}\mathrm{sk}_{n-1}\Delta[n]$$

and we call it the *boundary* of $\Delta[n]$. From Lemma 14.19.5 we see that $\partial\Delta[n] \subset \Delta[n]$ is the simplicial subset having the same nondegenerate simplices in degrees $\leq n-1$ but not containing the nondegenerate n-simplex.

Definition 14.28.1. A map $X \to Y$ of simplicial sets is called a *trivial Kan fibration* if $X_0 \to Y_0$ is surjective and for all $n \geq 1$ and any commutative solid diagram

$$\begin{array}{ccc}
\partial\Delta[n] & \longrightarrow & X \\
\downarrow & \nearrow & \downarrow \\
\Delta[n] & \longrightarrow & Y
\end{array}$$

a dotted arrow exists making the diagram commute.

A trivial Kan fibration satisfies a very general lifting property.

Lemma 14.28.2. *Let $f : X \to Y$ be a trivial Kan fibration of simplicial sets. For any solid commutative diagram*

$$\begin{array}{ccc}
Z & \longrightarrow & X \\
\downarrow & \overset{b}{\nearrow} & \downarrow \\
W & \overset{a}{\longrightarrow} & Y
\end{array}$$

of simplicial sets with $Z \to W$ (termwise) injective a dotted arrow exists making the diagram commute.

Proof. Suppose that $Z \neq W$. Let n be the smallest integer such that $Z_n \neq W_n$. Let $x \in W_n$, $x \notin Z_n$. Denote $Z' \subset W$ the simplicial subset containing Z, x, and all degeneracies of x. Let $\varphi : \Delta[n] \to Z'$ be the morphism corresponding to x (Lemma 14.11.3). Then $\varphi|_{\partial\Delta[n]}$ maps into Z as all the nondegenerate simplices of $\partial\Delta[n]$ end up in Z. By assumption we can extend $b \circ \varphi|_{\partial\Delta[n]}$ to $\beta : \Delta[n] \to X$. By Lemma 14.19.7 the simplicial set Z' is the pushout of $\Delta[n]$ and Z along $\partial\Delta[n]$. Hence b and β define a morphism $b' : Z' \to X$. In other words, we have extended the morphism b to a bigger simplicial subset of Z.

The proof is finished by an application of Zorn's lemma (omitted). □

Lemma 14.28.3. *Let $f : X \to Y$ be a trivial Kan fibration of simplicial sets. Let $Y' \to Y$ be a morphism of simplicial sets. Then $X \times_Y Y' \to Y'$ is a trivial Kan fibration.*

Proof. This follows immediately from the functorial properties of the fibre product (Lemma 14.7.2) and the definitions. □

Lemma 14.28.4. *The composition of two trivial Kan fibrations is a trivial Kan fibration.*

Proof. Omitted. □

Lemma 14.28.5. *Let $\ldots \to U^2 \to U^1 \to U^0$ be a sequence of trivial Kan fibrations. Let $U = \lim U^t$ defined by taking $U_n = \lim U^t_n$. Then $U \to U^0$ is a trivial Kan fibration.*

Proof. Omitted. Hint: use that for a countable sequence of surjections of sets the inverse limit is nonempty. □

Lemma 14.28.6. *Let $X_i \to Y_i$ be a set of trivial Kan fibrations. Then $\prod X_i \to \prod Y_i$ is a trivial Kan fibration.*

Proof. Omitted. □

Lemma 14.28.7. *A filtered colimit of trivial Kan fibrations is a trivial Kan fibration.*

Proof. Omitted. Hint: See description of filtered colimits of sets in Categories, Section 4.19. □

Lemma 14.28.8. *Let $f : X \to Y$ be a trivial Kan fibration of simplicial sets. Then f is a homotopy equivalence.*

Proof. By Lemma 14.28.2 we can choose an right inverse $g : Y \to X$ to f. Consider the diagram

$$\begin{array}{ccc} \partial\Delta[1] \times X & \longrightarrow & X \\ \downarrow & \nearrow & \downarrow \\ \Delta[1] \times X & \longrightarrow & Y \end{array}$$

Here the top horizontal arrow is given by id_X and $g \circ f$ where we use that $(\partial\Delta[1] \times X)_n = X_n \amalg X_n$ for all $n \geq 0$. The bottom horizontal arrow is given by the map $\Delta[1] \to \Delta[0]$ and $f : X \to Y$. The diagram commutes as $f \circ g \circ f = f$. By Lemma 14.28.2 we can fill in the dotted arrow and we win. □

14.29. Kan fibrations

Let n, k be integers with $0 \leq k \leq n$ and $1 \leq n$. Let $\sigma_0, \ldots, \sigma_n$ be the $n+1$ faces of the unique nondegenerate n-simplex σ of $\Delta[n]$, i.e., $\sigma_i = d_i\sigma$. We let

$$\Lambda_k[n] \subset \Delta[n]$$

be the *kth horn* of the n-simplex $\Delta[n]$. It is the simplicial subset of $\Delta[n]$ generated by $\sigma_0, \ldots, \hat{\sigma}_k, \ldots, \sigma_n$. In other words, the image of the displayed inclusion contains all the nondegenerate simplices of $\Delta[n]$ except for σ and σ_k.

Definition 14.29.1. A map $X \to Y$ of simplicial sets is called a *Kan fibration* if for all k, n with $1 \leq n$, $0 \leq k \leq n$ and any commutative solid diagram

$$\begin{array}{ccc} \Lambda_k[n] & \longrightarrow & X \\ \downarrow & \nearrow & \downarrow \\ \Delta[n] & \longrightarrow & Y \end{array}$$

a dotted arrow exists making the diagram commute. A *Kan complex* is a simplicial set X such that $X \to *$ is a Kan fibration, where $*$ is the constant simplicial set on a singleton.

Note that $\Lambda_k[n]$ is always nonempty. This a morphism from the empty simplicial set to any simplicial set is always a Kan fibration. It follows from Lemma 14.28.2 that a trivial Kan fibration is a Kan fibration.

Lemma 14.29.2. *Let $f : X \to Y$ be a Kan fibration of simplicial sets. Let $Y' \to Y$ be a morphism of simplicial sets. Then $X \times_Y Y' \to Y'$ is a Kan fibration.*

Proof. This follows immediately from the functorial properties of the fibre product (Lemma 14.7.2) and the definitions. □

Lemma 14.29.3. *The composition of two Kan fibrations is a Kan fibration.*

Proof. Omitted. □

Lemma 14.29.4. *Let $\ldots \to U^2 \to U^1 \to U^0$ be a sequence of Kan fibrations. Let $U = \lim U^t$ defined by taking $U_n = \lim U^t_n$. Then $U \to U^0$ is a Kan fibration.*

Proof. Omitted. Hint: use that for a countable sequence of surjections of sets the inverse limit is nonempty. □

Lemma 14.29.5. *Let $X_i \to Y_i$ be a set of Kan fibrations. Then $\prod X_i \to \prod Y_i$ is a Kan fibration.*

Proof. Omitted. □

The following lemma is due to J.C. Moore, see [**Moo55**].

Lemma 14.29.6. *Let X be a simplicial group. Then X is a Kan complex.*

Proof. The following proof is basically just a translation into English of the proof in the reference mentioned above. Using the terminology as explained in the introduction to this section, suppose $f : \Lambda_k[n] \to X$ is a morphism from a horn. Set $x_i = f(\sigma_i) \in X_{n-1}$ for $i = 0, \ldots, \hat{k}, \ldots, n$. This means that for $i < j$ we have $d_i x_j = d_{j-1} x_i$ whenever $i, j \neq k$. We have to find an $x \in X_n$ such that $x_i = d_i x$ for $i = 0, \ldots, \hat{k}, \ldots, n$.

We first prove there exists a $u \in X_n$ such that $d_i u = x_i$ for $i < k$. This is trivial for $k = 0$. If $k > 0$, one defines by induction an element $u^r \in X_n$ such that $d_i u^r = x_i$ for $0 \leq i \leq r$. Start with $u^0 = s_0 x_0$. If $r < k - 1$, we set

$$y^r = s_{r+1}((d_{r+1}u^r)^{-1}x_{r+1}), \quad u^{r+1} = u^r y^r.$$

An easy calculation shows that $d_i y^r = 1$ (unit element of the group X_{n-1}) for $i \leq r$ and $d_{r+1}y^r = (d_{r+1}u^r)^{-1}x_{r+1}$. It follows that $d_i u^{r+1} = x_i$ for $i \leq r + 1$. Finally, take $u = u^{k-1}$ to get u as promised.

Next we prove, by induction on the integer r, $0 \leq r \leq n - k$, there exists a $x^r \in X_n$ such that

$$d_i x^r = x_i \quad \text{for } i < k \text{ and } i > n - r.$$

Start with $x^0 = u$ for $r = 0$. Having defined x^r for $r \leq n - k - 1$ we set

$$z^r = s_{n-r-1}((d_{n-r}x^r)^{-1}x_{n-r}), \quad x^{r+1} = x^r z^r$$

A simple calculation, using the given relations, shows that $d_i z^r = 1$ for $i < k$ and $i > n - r$ and that $d_{n-r}(z^r) = (d_{n-r}x^r)^{-1}x_{n-r}$. It follows that $d_i x^{r+1} = x_i$ for $i < k$ and $i > n - r - 1$. Finally, we take $x = x^{n-k}$ which finishes the proof. □

Lemma 14.29.7. *Let $f : X \to Y$ be a homomorphism of simplicial abelian groups which is termwise surjective. Then f is a Kan fibration of simplicial sets.*

Proof. Consider a commutative solid diagram

$$\begin{array}{ccc} \Lambda_k[n] & \xrightarrow{a} & X \\ \downarrow & \nearrow & \downarrow \\ \Delta[n] & \xrightarrow{b} & Y \end{array}$$

as in Definition 14.29.1. The map a corresponds to $x_0, \ldots, \hat{x}_k, \ldots, x_n \in X_{n-1}$ satisfying $d_i x_j = d_{j-1}x_i$ for $i < j$, $i, j \neq k$. The map b corresponds to an element $y \in Y_n$ such that $d_i y = f(x_i)$ for $i \neq k$. Our task is to produce an $x \in X_n$ such that $d_i x = x_i$ for $i \neq k$ and $f(x) = y$.

Since f is termwise surjective we can find $x \in X_n$ with $f(x) = y$. Replace y by $0 = y - f(x)$ and x_i by $x_i - d_i x$ for $i \neq k$. Then we see that we may assume $y = 0$. In particular $f(x_i) = 0$. In other words, we can replace X by $\mathrm{Ker}(f) \subset X$ and Y by 0. In this case the statement become Lemma 14.29.6. □

Lemma 14.29.8. *Let $f : X \to Y$ be a homomorphism of simplicial abelian groups which is termwise surjective and induces a quasi-isomorphism on associated chain complexes. Then f is a trivial Kan fibration of simplicial sets.*

Proof. Consider a commutative solid diagram

$$\begin{array}{ccc} \partial\Delta[n] & \xrightarrow{a} & X \\ \downarrow & \nearrow & \downarrow \\ \Delta[n] & \xrightarrow{b} & Y \end{array}$$

as in Definition 14.28.1. The map a corresponds to $x_0, \ldots, x_n \in X_{n-1}$ satisfying $d_i x_j = d_{j-1}x_i$ for $i < j$. The map b corresponds to an element $y \in Y_n$ such that $d_i y = f(x_i)$. Our task is to produce an $x \in X_n$ such that $d_i x = x_i$ and $f(x) = y$.

Since f is termwise surjective we can find $x \in X_n$ with $f(x) = y$. Replace y by $0 = y - f(x)$ and x_i by $x_i - d_i x$. Then we see that we may assume $y = 0$. In particular $f(x_i) = 0$. In other words, we can replace X by $\mathrm{Ker}(f) \subset X$ and Y by 0. This works, because by Homology, Lemma 12.12.6 the homology of the chain complex associated to $\mathrm{Ker}(f)$ is zero and hence $\mathrm{Ker}(f) \to 0$ induces a quasi-isomorphism on associated chain complexes.

Since X is a Kan complex (Lemma 14.29.6) we can find $x \in X_n$ with $d_i x = x_i$ for $i = 0, \ldots, n-1$. After replacing x_i by $x_i - d_i x$ for $i = 0, \ldots, n$ we may assume that $x_0 = x_1 = \ldots = x_{n-1} = 0$. In this case we see that $d_i x_n = 0$ for $i = 0, \ldots, n-1$. Thus $x_n \in N(X)_{n-1}$ and lies in the kernel of the differential $N(X)_{n-1} \to N(X)_{n-2}$. Here $N(X)$ is the normalized chain complex associated to X, see Section 14.21. Since $N(X)$ is quasi-isomorphic to $s(X)$ (Lemma 14.21.8) and thus acyclic we find $x \in N(X_n)$ whose differential is x_n. This x answers the question posed by the lemma and we are done. □

Lemma 14.29.9. *Let $f : X \to Y$ be a map of simplicial abelian groups. If f is termwise surjective*[3] *and a homotopy equivalence of simplicial sets, then f induces a quasi-isomorphism of associated chain complexes.*

Proof. By assumption there exists a map $g : Y \to X$ of simplicial sets, a homotopy $h : X \times \Delta[1] \to X$ between $g \circ f$ and id_X, and a homotopy $h' : Y \times \Delta[1] \to Y$ between $f \circ g$ and id_Y. During this proof we will write $H_n(X) = H_n(s(X)) = H_n(N(X))$, see Section 14.21.

Note that $H_0(X)$ is the cokernel of the difference map $d_1 - d_0 : X_1 \to X_0$. Observe that $x \in X_0$ corresponds to a morphism $\Delta[0] \to X$. Composing h with the induced map $\Delta[0] \times \Delta[1] \to X \times \Delta[1]$ we see that x and $g(f(x))$ are equal to $d_0 x'$ and $d_1 x'$ for some $x' \in X_1$. Similarly for $y \in Y_0$. We conclude that f defines a bijection $H_0(X) \to H_0(Y)$.

Let $n \geq 1$. Consider the simplicial set S which is the pushout of

$$\begin{array}{ccc} \partial\Delta[n] & \longrightarrow & * \\ \downarrow & & \downarrow \\ \Delta[n] & \longrightarrow & S \end{array}$$

Concretely, we take

$$S_k = \{\varphi : [k] \to [n] \mid \varphi \text{ is surjective}\} \amalg \{*\}.$$

Denote $E = \mathbf{Z}[S]$ the free abelian group on S. The inclusion $\Delta[0] \to S$ coming from $* \in S_0$ determines an injection $K(\mathbf{Z}, 0) \to E$ whose cokernel is the object $K(\mathbf{Z}, n)$, i.e., we have a short exact sequence

$$0 \to K(\mathbf{Z}, 0) \to E \to K(\mathbf{Z}, n) \to 0$$

See Definition 14.20.3 and the description of the Eilenberg-Maclane objects in Lemma 14.20.2. Note that the extension above is split, for example because the

[3] This assumption is not necessary. Also the proof as currently given is not the right one. A better proof is to define the homotopy groups of Kan complex and show that these are equal to the homology groups of the associated complex for a simplicial abelian group.

element $\xi = [\mathrm{id}_{[n]}] - [*] \in E_n$ satisfies $d_i\xi = 0$ and maps to the "generator" of $K(\mathbf{Z}, n)$. We have

$$\mathrm{Mor}_{\mathrm{Simp}(Sets)}(S, X) = \mathrm{Mor}_{\mathrm{Simp}(Ab)}(E, X) = X_0 \times \bigcap\nolimits_{i=0,\ldots,n} \mathrm{Ker}(d_i : X_n \to X_{n-1})$$

This uses the choice of our splitting above and the description of morphisms out of Eilenberg-Maclane objects given in Lemma 14.20.2. Note that we can think of $\bigcap_{i=0,\ldots,n} \mathrm{Ker}(d_i : X_n \to X_{n-1})$ as the cycles in degree n in the normalized chain complex associated to X, see Section 14.21. If two maps $a, b : S \to X$ are homotopic (as maps of simplicial sets), then the corresponding maps $a', b' : E \to X$ are homotopic as maps of simplicial abelian groups (because taking the free abelian group on is a functor). Thus if a, resp. b correspond to (a_0, a_n), resp. (b_0, b_n) in the formula above, then a_0 and b_0 define the same element of $H_0(X)$ and a_n and b_n define the same class in $H_n(X)$. See Lemma 14.25.1.

We come the final arguments of the proof. An element y of $H_n(Y)$ can be represented by an element y_n in $\bigcap_{i=0,\ldots,n} \mathrm{Ker}(d_i : Y_n \to Y_{n-1})$. Let $a : S \to Y$ be the map of simplicial sets corresponding to $(0, y_n)$. Then $b = g \circ a$ corresponds to some (b_0, b_n) as above for X. Using the homotopy h' we see $(f(b_0), f(b_n))$ and $(0, y_n)$ come from homotopic maps $S \to Y$ and hence y_n and $f(b_n)$ define the same element of $H_n(Y)$. Clearly this shows that $H_n(f)$ is surjective. Conversely, suppose x_n in $\bigcap_{i=0,\ldots,n} \mathrm{Ker}(d_i : X_n \to X_{n-1})$ and $f(x_n) = d(y')$ with $y' \in N(Y_{n+1})$. Since f is termwise surjective so is the induced map $f : N(X_{n+1}) \to N(Y_{n+1})$ (see Lemma 14.21.6). Thus we can pick $x' \in N(X_{n+1})$ mapping to y'. After replacing x_n by $x_n - d(x')$ we reach the point where $f(x_n) = 0$. This means that the morphism $a : S \to X$ corresponding to $(0, x_n)$ has the property that $f \circ a$ is the constant morphism with value 0 in Y. Hence $g \circ f \circ a$ is also a constant morphism, i.e., corresponds to a pair $(b_0, 0)$. Since as before x_n and 0 represent the same element of $H_n(X)$ we conclude. □

14.30. A homotopy equivalence

Suppose that A, B are sets, and that $f : A \to B$ is a map. Consider the associated map of simplicial sets

$$\begin{array}{ccc} \mathrm{cosk}_0(A) & = & \Big(\ldots A \times A \times A \rightrightarrows\rightrightarrows A \times A \rightrightarrows A \Big) \\ & & \downarrow \qquad\quad \downarrow \qquad \downarrow \\ \mathrm{cosk}_0(B) & = & \Big(\ldots B \times B \times B \rightrightarrows\rightrightarrows B \times B \rightrightarrows B \Big) \end{array}$$

See Example 14.17.2. The case $n = 0$ of the following lemma says that this map of simplicial sets is a trivial Kan fibration if f is surjective.

Lemma 14.30.1. *Let $f : V \to U$ be a morphism of simplicial sets. Let $n \geq 0$ be an integer. Assume*

1. *The map $f_i : V_i \to U_i$ is a bijection for $i < n$.*
2. *The map $f_n : V_n \to U_n$ is a surjection.*
3. *The canonical morphism $U \to cosk_n sk_n U$ is an isomorphism.*
4. *The canonical morphism $V \to cosk_n sk_n V$ is an isomorphism.*

Then f is a trivial Kan fibration.

Proof. Consider a solid diagram

$$\begin{array}{ccc} \partial\Delta[k] & \longrightarrow & V \\ \downarrow & \nearrow & \downarrow \\ \Delta[k] & \longrightarrow & U \end{array}$$

as in Definition 14.28.1. Let $x \in U_k$ be the k-simplex corresponding to the lower horizontal arrow. If $k \leq n$ then the dotted arrow is the one corresponding to a lift $y \in V_k$ of x; the diagram will commute as the other nondegenerate simplices of $\Delta[k]$ are in degrees $< k$ where f is an isomorphism. If $k > n$, then by conditions (3) and (4) we have (using adjointness of skeleton and coskeleton functors)

$$\mathrm{Mor}(\Delta[k], U) = \mathrm{Mor}(\mathrm{sk}_n\Delta[k], \mathrm{sk}_n U) = \mathrm{Mor}(\mathrm{sk}_n\partial\Delta[k], \mathrm{sk}_n U) = \mathrm{Mor}(\partial\Delta[k], U)$$

and similarly for V because $\mathrm{sk}_n\Delta[k] = \mathrm{sk}_n\partial\Delta[k]$ for $k > n$. Thus we obtain a unique dotted arrow fitting into the diagram in this case also. □

Let A, B be sets. Let $f^0, f^1 : A \to B$ be maps of sets. Consider the induced maps $f^0, f^1 : \mathrm{cosk}_0(A) \to \mathrm{cosk}_0(B)$ abusively denoted by the same symbols. The following lemma for $n = 0$ says that f^0 is homotopic to f^1. In fact, the homotopy is given by the map $h : \mathrm{cosk}_0(A) \times \Delta[1] \to \mathrm{cosk}_0(A)$ with components

$$\begin{aligned} h_m : A \times \ldots \times A \times \mathrm{Mor}_\Delta([m], [1]) &\longrightarrow A \times \ldots \times A, \\ (a_0, \ldots, a_m, \alpha) &\longmapsto (f^{\alpha(0)}(a_0), \ldots, f^{\alpha(m)}(a_m)) \end{aligned}$$

To check that this works, note that for a map $\varphi : [k] \to [m]$ the induced maps are $(a_0, \ldots, a_m) \mapsto (a_{\varphi(0)}, \ldots, a_{\varphi(k)})$ and $\alpha \mapsto \alpha \circ \varphi$. Thus $h = (h_m)_{m \geq 0}$ is clearly a map of simplicial sets as desired.

Lemma 14.30.2. *Let $f^0, f^1 : V \to U$ be maps of a simplicial sets. Let $n \geq 0$ be an integer. Assume*

1. *The maps $f_i^j : V_i \to U_i$, $j = 0, 1$ are equal for $i < n$.*
2. *The canonical morphism $U \to cosk_n sk_n U$ is an isomorphism.*
3. *The canonical morphism $V \to cosk_n sk_n V$ is an isomorphism.*

Then f^0 is homotopic to f^1.

First proof. Let W be the n-truncated simplicial set with $W_i = U_i$ for $i < n$ and $W_n = U_n/\sim$ where $\sim$ is the equivalence relation generated by $f^0(y) \sim f^1(y)$ for $y \in V_n$. This makes sense as the morphisms $U(\varphi) : U_n \to U_i$ corresponding to $\varphi : [i] \to [n]$ for $i < n$ factor through the quotient map $U_n \to W_n$ because f^0 and f^1 are morphisms of simplicial sets and equal in degrees $< n$. Next, we upgrade W to a simplicial set by taking $\mathrm{cosk}_n W$. By Lemma 14.30.1 the morphism $g : U \to W$ is a trivial Kan fibration. Observe that $g \circ f^0 = g \circ f^1$ by construction and denote this morphism $f : V \to W$. Consider the diagram

$$\begin{array}{ccc} \partial\Delta[1] \times V & \xrightarrow[f^0, f^1]{} & U \\ \downarrow & \nearrow & \downarrow \\ \Delta[1] \times V & \xrightarrow{f} & W \end{array}$$

By Lemma 14.28.2 the dotted arrow exists and the proof is done. □

Second proof. We have to construct a morphism of simplicial sets $h : V \times \Delta[1] \to U$ which recovers f^i on composing with e_i. The case $n = 0$ was dealt with above the lemma. Thus we may assume that $n \geq 1$. The map $\Delta[1] \to \text{cosk}_1\text{sk}_1\Delta[1]$ is an isomorphism, see Lemma 14.17.16. Thus we see that $\Delta[1] \to \text{cosk}_n\text{sk}_n\Delta[1]$ is an isomorphism as $n \geq 1$, see Lemma 14.17.11. And hence $V \times \Delta[1] \to \text{cosk}_n\text{sk}_n(V \times \Delta[1])$ is an isomorphism too, see Lemma 14.17.13. In other words, in order to construct the homotopy it suffices to construct a suitable morphism of n-truncated simplicial sets $h : \text{sk}_n V \times \text{sk}_n\Delta[1] \to \text{sk}_n U$.

For $k = 0, \ldots, n-1$ we define h_k by the formula $h_k(v, \alpha) = f^0(v) = f^1(v)$. The map $h_n : V_n \times \text{Mor}_\Delta([k], [1]) \to U_n$ is defined as follows. Pick $v \in V_n$ and $\alpha : [n] \to [1]$:

(1) If $\text{Im}(\alpha) = \{0\}$, then we set $h_n(v, \alpha) = f^0(v)$.
(2) If $\text{Im}(\alpha) = \{0, 1\}$, then we set $h_n(v, \alpha) = f^0(v)$.
(3) If $\text{Im}(\alpha) = \{1\}$, then we set $h_n(v, \alpha) = f^1(v)$.

Let $\varphi : [k] \to [l]$ be a morphism of $\Delta_{\leq n}$. We will show that the diagram

$$\begin{array}{ccc} V_l \times \text{Mor}([l], [1]) & \longrightarrow & U_l \\ \downarrow & & \downarrow \\ V_k \times \text{Mor}([k], [1]) & \longrightarrow & U_k \end{array}$$

commutes. Pick $v \in V_l$ and $\alpha : [l] \to [1]$. The commutativity means that

$$h_k(V(\varphi)(v), \alpha \circ \varphi) = U(\varphi)(h_l(v, \alpha)).$$

In almost every case this holds because $h_k(V(\varphi)(v), \alpha \circ \varphi) = f^0(V(\varphi)(v))$ and $U(\varphi)(h_l(v, \alpha)) = U(\varphi)(f^0(v))$, combined with the fact that f^0 is a morphism of simplicial sets. The only cases where this does not hold is when either (A) $\text{Im}(\alpha) = \{1\}$ and $l = n$ or (B) $\text{Im}(\alpha \circ \varphi) = \{1\}$ and $k = n$. Observe moreover that necessarily $f^0(v) = f^1(v)$ for any degenerate n-simplex of V. Thus we can narrow the cases above down even further to the cases (A) $\text{Im}(\alpha) = \{1\}$, $l = n$ and v nondegenerate, and (B) $\text{Im}(\alpha \circ \varphi) = \{1\}$, $k = n$ and $V(\varphi)(v)$ nondegenerate.

In case (A), we see that also $\text{Im}(\alpha \circ \varphi) = \{1\}$. Hence we see that not only $h_l(v, \alpha) = f^1(v)$ but also $h_k(V(\varphi)(v), \alpha \circ \varphi) = f^1(V(\varphi)(v))$. Thus we see that the relation holds because f^1 is a morphism of simplicial sets.

In case (B) we conclude that $l = k = n$ and φ is bijective, since otherwise $V(\varphi)(v)$ is degenerate. Thus $\varphi = \text{id}_{[n]}$, which is a trivial case. □

Lemma 14.30.3. *Let A, B be sets, and that $f : A \to B$ is a map. Consider the simplicial set U with n-simplices*

$$A \times_B A \times_B \ldots \times_B A \ (n + 1 \text{ factors}).$$

see Example 14.3.5. If f is surjective, the morphism $U \to B$ where B indicates the constant simplicial set with value B is a trivial Kan fibration.

Proof. Observe that U fits into a cartesian square

$$\begin{array}{ccc} U & \longrightarrow & \text{cosk}_0(B) \\ \downarrow & & \downarrow \\ B & \longrightarrow & \text{cosk}_0(A) \end{array}$$

Since the right vertical arrow is a trivial Kan fibration by Lemma 14.30.1, so is the left by Lemma 14.28.3. □

14.31. Standard resolutions

Some of the material in this section can be found in [**God73**, Appendix 1] and [**Ill72**, I 1.5].

Situation 14.31.1. Let $\mathcal{A}$, $\mathcal{S}$ be categories and let $i : \mathcal{A} \to \mathcal{S}$ be a functor with a left adjoint $F : \mathcal{S} \to \mathcal{A}$.

In this very general situation we will construct a simplicial object X in the category of functors from $\mathcal{A}$ to $\mathcal{A}$. Please keep the following example in mind while we do this.

Example 14.31.2. As an example of the above we can take $i : \mathit{Rings} \to \mathit{Sets}$ to be the forgetful functor and $F : \mathit{Sets} \to \mathit{Rings}$ to be the functor that associates to a set E the polynomial algebra $\mathbf{Z}[E]$ on E over $\mathbf{Z}$. The simplicial object X when evaluated on an ring A will give the simplicial ring

$$\mathbf{Z}[\mathbf{Z}[\mathbf{Z}[A]]] \;\substack{\longrightarrow\\ \longleftarrow\\ \longrightarrow\\ \longleftarrow\\ \longrightarrow}\; \mathbf{Z}[\mathbf{Z}[A]] \;\substack{\longrightarrow\\ \longleftarrow\\ \longrightarrow}\; \mathbf{Z}[A]$$

which comes with an augmentation towards A. We will also show this augmentation is a homotopy equivalence.

For the general construction we will use the horizontal composition as defined in Categories, Section 4.26. The definition of the adjunction morphisms $k : F \circ i \to \mathrm{id}_{\mathcal{A}}$ and $t : \mathrm{id}_{\mathcal{S}} \to i \circ F$ in Categories, Section 4.24 shows that the compositions

$$i \xrightarrow{t \star 1_i} i \circ F \circ i \xrightarrow{1_i \star k} i \quad \text{and} \quad F \xrightarrow{1_F \star t} F \circ i \circ F \xrightarrow{k \star 1_F} F \tag{14.31.2.1}$$

are the identity morphisms. Here to define the morphism $t \star 1$ we silently identify i with $\mathrm{id}_{\mathcal{S}} \circ i$ and 1 stands for $\mathrm{id}_i : i \to i$. We will use this notation and these relations repeatedly in what follows. For $n \geq 0$ we set

$$X_n = (F \circ i)^{\circ(n+1)} = F \circ i \circ F \circ \ldots \circ i \circ F$$

In other words, X_n is the $(n+1)$-fold composition of $F \circ i$ with itself. We also set $X_{-1} = \mathrm{id}_{\mathcal{A}}$. We have $X_{n+m+1} = X_n \circ X_m$ for all $n, m \geq -1$. We will endow this sequence of functors with the structure of a simplicial object of $\mathrm{Fun}(\mathcal{A}, \mathcal{A})$ by constructing the morphisms of functors

$$d_j^n : X_n \to X_{n-1}, \quad s_j^n : X_n \to X_{n+1}$$

satisfying the relations displayed in Lemma 14.2.3. Namely, we set

$$d_j^n = 1_{X_{j-1}} \star k \star 1_{X_{n-j-1}} \quad \text{and} \quad s_j^n = 1_{X_{j-1} \circ F} \star t \star 1_{i \circ X_{n-j-1}}$$

Finally, write $\epsilon_0 = k : X_0 \to X_{-1}$.

Example 14.31.3. In Example 14.31.2 we have $X_n(A) = \mathbf{Z}[\mathbf{Z}[\ldots[A]\ldots]]$ with $n+1$ brackets. We describe the maps constructed above using a typical element $\xi = \sum n_i[n_{ij}[a_{ij}]]$ of $X_1(A)$. The maps $d_0, d_1 : \mathbf{Z}[\mathbf{Z}[A]] \to \mathbf{Z}[A]$ are given by

$$d_0(\xi) = \sum n_i n_{ij}[a_{ij}] \quad \text{and} \quad d_1(\xi) = \sum n_i[n_{ij} a_{ij}].$$

The maps $s_0, s_1 : \mathbf{Z}[\mathbf{Z}[A]] \to \mathbf{Z}[\mathbf{Z}[\mathbf{Z}[A]]]$ are given by

$$s_0(\xi) = \sum n_i[[n_{ij}[a_{ij}]]] \quad \text{and} \quad s_1(\xi) = \sum n_i[n_{ij}[[a_{ij}]]].$$

Lemma 14.31.4. *In Situation 14.31.1 the system $X = (X_n, d_j^n, s_j^n)$ is a simplicial object of $Fun(\mathcal{A}, \mathcal{A})$ and ϵ_0 defines an augmentation ϵ from X to the constant simplicial object with value $X_{-1} = id_{\mathcal{A}}$.*

Proof. Suppose that we have shown that X is a simplicial object. Then to prove that $\epsilon_0 = k$ defines an augmentation we have to check that $\epsilon_0 \circ d_0^1 = \epsilon_0 \circ d_1^1$ as morphisms $X_1 \to X_{-1}$, see Lemma 14.18.2. In other words, we have to check that the diagram

$$\begin{array}{ccc} F \circ i \circ F \circ i & \xrightarrow[1_{F\circ i} \star k]{} & F \circ i \\ {\scriptstyle k \star 1_{F\circ i}}\downarrow & & \downarrow{\scriptstyle k} \\ F \circ i & \xrightarrow{k} & \mathrm{id}_{\mathcal{A}} \end{array}$$

is commutative. More precisely we should write this as the equality

$$(k \star 1_{\mathrm{id}_{\mathcal{A}}}) \circ (1_{F\circ i} \star k) = (1_{\mathrm{id}_{\mathcal{A}}} \star k) \circ (k \star 1_{F\circ i})$$

as morphisms $(F \circ i) \circ (F \circ i) \to \mathrm{id}_{\mathcal{A}} \circ \mathrm{id}_{\mathcal{A}}$. Applying the general property of Categories, Lemma 4.26.2 both sides expand to $k \star k$ when equality holds.

To prove that X is a simplicial object we have to check (see Remark 14.3.3):

(1) If $i < j$, then $d_i \circ d_j = d_{j-1} \circ d_i$.
(2) If $i < j$, then $d_i \circ s_j = s_{j-1} \circ d_i$.
(3) We have $\mathrm{id} = d_j \circ s_j = d_{j+1} \circ s_j$.
(4) If $i > j + 1$, then $d_i \circ s_j = s_j \circ d_{i-1}$.
(5) If $i \leq j$, then $s_i \circ s_j = s_{j+1} \circ s_i$.

Relation (1) is proved in exactly the same manner as the proof of the equality $\epsilon_0 \circ d_0^1 = \epsilon_0 \circ d_1^1$ above.

The simplest case of equality (5) is the commutativity of the diagram

$$\begin{array}{ccc} F \circ i & \xrightarrow[1_F \star t \star 1_i]{} & F \circ i \circ F \circ i \\ {\scriptstyle 1_F \star t \star 1_i}\downarrow & & \downarrow{\scriptstyle 1_F \star t \star 1_{i\circ F\circ i}} \\ F \circ i \circ F \circ i & \xrightarrow{1_{F\circ i\circ F} \star t \star 1_i} & F \circ i \circ F \circ i \circ F \circ i \end{array}$$

which holds because both compositions expand to the morphism $1_F \star t \star t \star 1_i$ from $F \circ \mathrm{id}_{\mathcal{A}} \circ \mathrm{id}_{\mathcal{A}} \circ i$ to $F \circ (i \circ F) \circ (i \circ F) \circ i$. All other cases of (5) are proved in the same manner.

The simplest case of equalities (2) and (4) is the commutativity of the diagram

$$\begin{array}{ccc} F \circ i \circ F \circ i & \xrightarrow[1_F \star k \star 1_i]{} & F \circ i \\ {\scriptstyle 1_{F\circ i\circ F\circ i} \star t}\downarrow & & \downarrow{\scriptstyle 1_{F\circ i} \star t} \\ F \circ i \circ F \circ i \circ F \circ i & \xrightarrow{1_F \star k \star 1_{i\circ F\circ i\circ F}} & F \circ i \circ F \circ i \end{array}$$

which again holds because both compositions expand to give $1_F \star k \star 1_i \star t$ as maps from $F \circ (i \circ F) \circ i \circ \mathrm{id}_{\mathcal{A}}$ to $F \circ \mathrm{id}_{\mathcal{A}} \circ i \circ (F \circ i)$. All other cases of (2) and (4) are proved in the same manner.

The relations (3) are the only nontrivial ones and these are consequences of the fact that the compositions in (14.31.2.1) are the identity. For example, the simplest case

of (3) states that the compositions

$$\begin{array}{ccc} F \circ i & \xrightarrow{1_{F\circ i} \star t} & F \circ i \circ F \circ i \\ {\scriptstyle t\star 1_{F\circ i}}\downarrow & & \downarrow{\scriptstyle 1_F \star k \star 1_i} \\ F \circ i \circ F \circ i & \xrightarrow{1_F \star k \star 1_i} & F \circ i \end{array}$$

go around the diagram either way evaluate out to the identity. Going around the top the composition evaluates to $1_F \star ((k \star 1_i) \circ (1_i \star t))$ which is the identity by what was said above. The other cases of (3) are proved in the same manner. □

Before reading the proof of the following lemma, we strongly urge the reader to look at the example discussed in Example 14.31.6 in order to understand the purpose of the lemma.

Lemma 14.31.5. *In Situation 14.31.1 the maps*

$$1_i \star \epsilon : i \circ X \to i, \quad \textit{and} \quad \epsilon \star 1_F : X \circ F \to F$$

are homotopy equivalences.

Proof. Denote $\epsilon_n : X_n \to X_{-1}$ the components of the augmentation morphism. We observe that $\epsilon_n = k^{\star(n+1)}$, the $(n+1)$-fold $\star$-composition of k. Recall that $t : \mathrm{id}_{\mathcal{S}} \to i \circ F$ is the adjunction map. We have the morphisms

$$t^{\star(n+1)} \star 1_i : i \longrightarrow i \circ (F \circ i)^{\circ(n+1)} = i \circ X_n$$

which are right inverse to $1_i \star \epsilon_n$ and the morphisms

$$1_F \star t^{\star(n+1)} : F \longrightarrow (F \circ i)^{\circ(n+1)} \circ F = X_n \circ F$$

which are right inverse to $\epsilon_n \star 1_F$. These morphisms determine morphisms of simplicial objects $b : i \to i \circ X$ and $c : F \to X \circ F$ (proof omitted). To finish it suffices to construct a homotopy between the morphisms $1, b \circ (1_i \star \epsilon) : i \circ X \to i \circ X$ and between the two morphisms $1, c \circ (\epsilon \star 1_F) : X \circ F \to X \circ F$.

To show the morphisms $b \circ (1_i \star \epsilon), 1 : i \circ X \to i \circ X$ are homotopic we have to construct morphisms

$$h_{n,j} : i \circ X_n \to i \circ X_n$$

for $n \geq 0$ and $0 \leq j \leq n+1$ satisfying the relations described in Lemma 14.24.2. See also Remark 14.24.4. We are forced to set $h_{n,0} = 1$ and

$$h_{n,n+1} = b_n \circ (1_i \star \epsilon_n) = (t^{\star(n+1)} \star 1_i) \circ (1_i \star k^{\star(n+1)})$$

Thus a logical choice is

$$h_{n,j} = (t^{\star(j)} \star 1) \circ (1_i \star k^{\star(j)} \star 1)$$

Here and in the rest of the proof we drop the subscript from 1 if it is clear by knowing the source and the target of the morphism what this subscript should be. Writing

$$i \circ X_n = i \circ F \circ i \circ \ldots \circ F \circ i$$

we can think of the morphism $h_{n,j}$ as collapsing the first j pairs $(F \circ i)$ to $\mathrm{id}_{\mathcal{S}}$ using $k^{\star(j)}$, then adding a $\mathrm{id}_{\mathcal{S}}$ in front and expanding this to j pairs $(i \circ F)$ using $t^{\star(j)}$. We have to prove

(1) We have $d^n_m \circ h_{n,j} = h_{n-1,j-1} \circ d^n_m$ for $j > m$.
(2) We have $d^n_m \circ h_{n,j} = h_{n-1,j} \circ d^n_m$ for $j \leq m$.

(3) We have $s^n_m \circ h_{n,j} = h_{n+1,j+1} \circ s^n_m$ for $j > m$.
(4) We have $s^n_m \circ h_{n,j} = h_{n+1,j} \circ s^n_m$ for $j \leq m$.

Recall that d^n_m is given by applying k to the $(m+1)$st pair $(F \circ i)$ in the functor $X_n = (F \circ i)^{\circ(n+1)}$. Thus it is clear that (2) holds (because k does $\star$-commute with k, but not with t). Similarly, s^n_m is given by applying $1_F \star t \star i_i$ to the $(m+1)$st pair $(F \circ i)$ in $X_n = (F \circ i)^{\circ(n+1)}$. Thus it is clear that (4) holds. In the two remaining cases one uses the fact that the compositions in (14.31.2.1) are the identity causes the drop in the index j. Some details omitted.

To show the morphisms $1, c \circ (\epsilon \star 1_F) : X \circ F \to X \circ F$ are homotopic we have to construct morphisms

$$h_{n,j} : X_n \circ F \longrightarrow X_n \circ F$$

for $n \geq 0$ and $0 \leq j \leq n+1$ satisfying the relations described in Lemma 14.24.2. See also Remark 14.24.4. We are forced to set $h_{n,0} = 1$ and

$$h_{n,n+1} = c_n \circ (\epsilon_n \star 1_F) = (1_F \star t^{\star(n+1)}) \circ (k^{\star(n+1)} \star 1_F)$$

Thus a logical choice is

$$h_{n,j} = (1_F \star t^{\star(j)} \star 1) \circ (k^{\star(j)} \star 1)$$

Here and in the rest of the proof we drop the subscript from 1 if it is clear by knowing the source and the target of the morphism what this subscript should be. Writing

$$X_n \circ F = F \circ i \circ F \circ \ldots \circ i \circ F$$

we can think of the morphism $h_{n,j}$ as collapsing the first j pairs $(F \circ i)$ to $\mathrm{id}_{\mathcal{S}}$ using $k^{\star(j)}$, then inserting a $\mathrm{id}_{\mathcal{S}}$ just after the first F and expanding this to j pairs $(i \circ F)$ using $t^{\star(j)}$. We have to prove

(1) We have $d^n_m \circ h_{n,j} = h_{n-1,j-1} \circ d^n_m$ for $j > m$.
(2) We have $d^n_m \circ h_{n,j} = h_{n-1,j} \circ d^n_m$ for $j \leq m$.
(3) We have $s^n_m \circ h_{n,j} = h_{n+1,j+1} \circ s^n_m$ for $j > m$.
(4) We have $s^n_m \circ h_{n,j} = h_{n+1,j} \circ s^n_m$ for $j \leq m$.

Recall that d^n_m is given by applying k to the $(m+1)$st pair $(F \circ i)$ in the functor $X_n = (F \circ i)^{\circ(n+1)}$. Thus it is clear that (2) holds (because k does $\star$-commute with k, but not with t). Similarly, s^n_m is given by applying $1_F \star t \star i_i$ to the $(m+1)$st pair $(F \circ i)$ in $X_n = (F \circ i)^{\circ(n+1)}$. Thus it is clear that (4) holds. In the two remaining cases one uses the fact that the compositions in (14.31.2.1) are the identity causes the drop in the index j. Some details omitted. □

Example 14.31.6. Going back to the example discussed in Example 14.31.2 our Lemma 14.31.5 signifies that for any ring A the map of simplicial rings

$$\begin{array}{ccccc} \mathbf{Z}[\mathbf{Z}[\mathbf{Z}[A]]] & \rightleftarrows & \mathbf{Z}[\mathbf{Z}[A]] & \rightleftarrows & \mathbf{Z}[A] \\ \downarrow & & \downarrow & & \downarrow \\ A & \rightleftarrows & A & \rightleftarrows & A \end{array}$$

is a homotopy equivalence on underlying simplicial sets. Moreover, the inverse map constructed in Lemma 14.31.5 is in degree n given by

$$a \longmapsto [\ldots [a] \ldots]$$

with obvious notation. In the other direction the lemma tells us that for every set E there is a homotopy equivalence

$$\begin{array}{ccccc} \mathbf{Z}[\mathbf{Z}[\mathbf{Z}[\mathbf{Z}[E]]]] & \rightleftarrows & \mathbf{Z}[\mathbf{Z}[\mathbf{Z}[E]]] & \rightleftarrows & \mathbf{Z}[\mathbf{Z}[E]] \\ \downarrow & & \downarrow & & \downarrow \\ \mathbf{Z}[E] & \rightleftarrows & \mathbf{Z}[E] & \rightleftarrows & \mathbf{Z}[E] \end{array}$$

of rings. The inverse map constructed in the lemma is in degree n given by the ring map

$$\sum m_{e_1,\ldots,e_p}[e_1][e_2]\ldots[e_p] \longmapsto \sum m_{e_1,\ldots,e_p}[\ldots[e_1]\ldots][\ldots[e_2]\ldots]\ldots[\ldots[e_p]\ldots]$$

(with obvious notation).

14.32. Other chapters

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CHAPTER 15

More on Algebra

15.1. Introduction

In this chapter we prove some results in commutative algebra which are less elementary than those in the first chapter on commutative algebra, see Algebra, Section 10.1. A reference is [**Mat70**].

15.2. Advice for the reader

More than in the chapter on commutative algebra, each of the sections in this chapter stands on its own. Starting with Section 15.41 we freely use the (unbounded) derived category of modules over rings and all the machinery that comes with it.

15.3. A comment on the Artin-Rees property

Some of this material is taken from [**CdJ02**]. A general discussion with additional references can be found in [**EH05**, Section 1].

Let A be a Noetherian ring and let $I \subset A$ be an ideal. Given a homomorphism $f : M \to N$ of finite A-modules there exists a $c \geq 0$ such that

$$f(M) \cap I^n N \subset f(I^{n-c} M)$$

for all $n \geq c$, see Algebra, Lemma 10.47.5. In this situation we will say *c works for f in the Artin-Rees lemma.*

Lemma 15.3.1. *Let A be a Noetherian ring. Let $I \subset A$ be an ideal contained in the Jacobson radical of A. Let*

$$S : L \xrightarrow{f} M \xrightarrow{g} N \quad \text{and} \quad S' : L \xrightarrow{f'} M \xrightarrow{g'} N$$

be two complexes of finite A-modules as shown. Assume that

1. *c works in the Artin-Rees lemma for f and g,*
2. *the complex S is exact, and*
3. *$f' = f \bmod I^{c+1} M$ and $g' = g \bmod I^{c+1} N$.*

Then c works in the Artin-Rees lemma for g' and the complex S' is exact.

Proof. We first show that $g'(L) \cap I^n M \subset g'(I^{n-c} L)$ for $n \geq c$. Let a be an element of M such that $g'(a) \in I^n N$. We want to adjust a by an element of $f'(L)$, i.e, without changing $g'(a)$, so that $a \in I^{n-c} M$. Assume that $a \in I^r M$, where $r < n - c$. Then

$$g(a) = g'(a) + (g - g')(a) \in I^n N + I^{r+c+1} N = I^{r+c+1} N.$$

By Artin-Rees for g we have $g(a) \in g(I^{r+1} M)$. Say $g(a) = g(a_1)$ with $a_1 \in I^{r+1} M$. Since the sequence S is exact, $a - a_1 \in f(L)$. Accordingly, we write $a = f(b) + a_1$ for some $b \in L$. Then $f(b) = a - a_1 \in I^r M$. Artin-Rees for f shows that if $r \geq c$,

we may replace b by an element of $I^{r-c}L$. Then in all cases, $a = f'(b) + a_2$, where $a_2 = (f - f')(b) + a_1 \in I^{r+1}M$. (Namely, either $c \geq r$ and $(f - f')(b) \in I^{r+1}M$ by assumption, or $c < r$ and $b \in I^{r-c}$, whence again $(f - f')(b) \in I^{c+1}I^{r-c}M = I^{r+1}M$.) So we can adjust a by the element $f'(b) \in f'(L)$ to increase r by 1.

In fact, the argument above shows that $(g')^{-1}(I^nM) \subset f'(L)+I^{n-c}M$ for all $n \geq c$. Hence S' is exact because

$$(g')^{-1}(0) = (g')^{-1}(\bigcap I^nN) \subset \bigcap f'(L) + I^{n-c}M = f'(L)$$

as $I \subset \mathrm{rad}(A)$, see Algebra, Lemma 10.47.7. □

Given an ideal $I \subset A$ of a ring A and an A-module M we set

$$\mathrm{Gr}_I(M) = \bigoplus I^nM/I^{n+1}M.$$

We think of this as a graded $\mathrm{Gr}_I(A)$-module.

Lemma 15.3.2. *Assumptions as in Lemma 15.3.1. Let $Q = Coker(g)$ and $Q' = Coker(g')$. Then $Gr_I(Q) \cong Gr_I(Q')$ as graded $Gr_I(A)$-modules.*

Proof. In degree n we have $\mathrm{Gr}_I(Q)_n = I^nN/(I^{n+1}N + g(M) \cap I^nN)$ and similarly for Q'. We claim that

$$g(M) \cap I^nN \subset I^{n+1}N + g'(M) \cap I^nN.$$

By symmetry (the proof of the claim will only use that c works for g which also holds for g' by the lemma) this will imply that

$$I^{n+1}N + g(M) \cap I^nN = I^{n+1}N + g'(M) \cap I^nN$$

whence $\mathrm{Gr}_I(Q)_n$ and $\mathrm{Gr}_I(Q')_n$ agree as subquotients of N, implying the lemma. Observe that the claim is clear for $n \leq c$ as $f = f' \bmod I^{c+1}N$. If $n > c$, then suppose $b \in g(M) \cap I^nN$. Write $b = g(a)$ for $a \in I^{n-c}M$. Set $b' = g'(a)$. We have $b - b' = (g - g')(a) \in I^{n+1}N$ as desired. □

Lemma 15.3.3. *Let $A \to B$ be a flat map of Noetherian rings. Let $I \subset A$ be an ideal. Let $f : M \to N$ be a homomorphism of finite A-modules. Assume that c works for f in the Artin-Rees lemma. Then c works for $f \otimes 1 : M \otimes_A B \to N \otimes_A B$ in the Artin-Rees lemma for the ideal IB.*

Proof. Note that

$$(f \otimes 1)(M) \cap I^nN \otimes_A B = (f \otimes 1)\left((f \otimes 1)^{-1}(I^nN \otimes_A B)\right)$$

On the other hand,

$$\begin{aligned}(f \otimes 1)^{-1}(I^nN \otimes_A B) &= \mathrm{Ker}(M \otimes_A B \to N \otimes_A B/(I^nN \otimes_A B)) \\ &= \mathrm{Ker}(M \otimes_A B \to (N/I^nN) \otimes_A B)\end{aligned}$$

As $A \to B$ is flat taking kernels and cokernels commutes with tensoring with B, whence this is equal to $f^{-1}(I^nN) \otimes_A B$. By assumption $f^{-1}(I^nN)$ is contained in $\mathrm{Ker}(f) + I^{n-c}M$. Thus the lemma holds. □

15.4. Fibre products of rings

Fibre products of rings have to do with pushouts of schemes. A special case of pushouts of schemes is discussed in More on Morphisms, Section 36.11.

Lemma 15.4.1. *Let R be a ring. Let $A \to B$ and $C \to B$ be R-algebra maps. Assume*

(1) *R is Noetherian,*
(2) *A, B, C are of finite type over R,*
(3) *$A \to B$ is surjective, and*
(4) *B is finite over C.*

Then $A \times_B C$ is of finite type over R.

Proof. Set $D = A \times_B C$. There is a commutative diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & I & \longrightarrow & A & \longrightarrow & B & \longrightarrow & 0 \\ & & \uparrow & & \uparrow & & \uparrow & & \\ 0 & \longrightarrow & I & \longrightarrow & D & \longrightarrow & C & \longrightarrow & 0 \end{array}$$

with exact rows. Choose $y_1, \ldots, y_n \in B$ which are generators for B as a C-module. Choose $x_i \in A$ mapping to y_i. Then $1, x_1, \ldots, x_n$ are generators for A as a D-module. The map $D \to A \times C$ is injective, and the ring $A \times C$ is finite as a D-module (because it is the direct sum of the finite D-modules A and C). Hence the lemma follows from the Artin-Tate lemma (Algebra, Lemma 10.47.9). □

Lemma 15.4.2. *Let R be a Noetherian ring. Let I be a finite set. Suppose given a cartesian diagram*

$$\begin{array}{ccc} P & \longrightarrow & \prod A_i \\ \downarrow & & \downarrow \prod \varphi_i \\ Q & \xrightarrow{\prod \psi_i} & \prod B_i \end{array}$$

with ψ_i and φ_i surjective, and Q, A_i, B_i of finite type over R. Then P is of finite type over R.

Proof. Follows from Lemma 15.4.1 and induction on the size of I. Namely, let $I = I' \amalg \{i_0\}$. Let P' be the ring defined by the diagram of the lemma using I'. Then P' is of finite type by the lemma. Finally, P sits in a fibre product diagram

$$\begin{array}{ccc} P & \longrightarrow & A_{i_0} \\ \downarrow & & \downarrow \\ P' & \longrightarrow & B_{i_0} \end{array}$$

to which the lemma applies. □

Lemma 15.4.3. *Suppose given a cartesian diagram of rings*

$$\begin{array}{ccc} B & \xrightarrow{s} & R \\ \uparrow & & \uparrow t \\ B' & \longrightarrow & R', \end{array}$$

i.e., $B' = B \times_R R'$. *If* $h \in B'$ *corresponds to* $g \in B$ *and* $f \in R'$ *such that* $s(g) = t(f)$, *then the diagram*

$$\begin{array}{ccc} B_g & \xrightarrow[s]{} & R_{s(g)} = R_{t(f)} \\ \uparrow & & \uparrow{\scriptstyle t} \\ (B')_h & \longrightarrow & (R')_f \end{array}$$

is cartesian too.

Proof. Note that $B' = \{(b, r') \in B \times R' \mid s(b) = t(r')\}$. So $h = (g, f) \in B'$. First we show that $(B')_h$ maps injectively into $B_g \times (R')_f$. Namely, suppose that $(x, y)/h^n$ maps to zero. This means that $g^N x = 0$ for some N and $f^M y$ is zero for some M. Thus $h^{\max(N,M)}(x, y) = 0$ in B' and hence $(x, y)/h^n = 0$ in B'_h. Next, suppose that x/g^n and y/f^m are elements which map to the same element of $R_{s(g)}$. This means that $s(g)^N(t(f)^m s(x) - s(g)^n t(y)) = 0$ in R' for some $N \gg 0$. We can rewrite this as $s(g^{m+N}x) = t(f^{n+N}y)$. Hence we see that the pair $(x/g^n, y/f^m)$ is the image of the element $(g^{m+N}x, f^{n+N}y)/h^{n+m+N}$ of $(B')_h$. □

Situation 15.4.4. In the following we will consider ring maps

$$B \longrightarrow A \longleftarrow A'$$

where we assume $A' \to A$ is surjective with kernel I. In this situation we set $B' = B \times_A A'$ to obtain a cartesian square

$$\begin{array}{ccc} A & \longleftarrow & A' \\ \uparrow & & \uparrow \\ B & \longleftarrow & B' \end{array}$$

We'd like to understand B'-modules in terms of modules over A', A, and B. In order to do this we consider the functor (where the fibre product of categories as constructed in Categories, Example 4.29.3)

$$\text{(15.4.4.1)} \quad \text{Mod}_{B'} \longrightarrow \text{Mod}_B \times_{\text{Mod}_A} \text{Mod}_{A'}, \quad L' \longmapsto (L' \otimes_{B'} B, L' \otimes_{B'} A', can)$$

where *can* is the canonical identification $L' \otimes_{B'} B \otimes_B A = L' \otimes_{B'} A' \otimes_{A'} A$. In the following we will write (N, M', φ) for an object of the right hand side, i.e., N is a B-module, M' is an A'-module and $\varphi : N \otimes_B A \to M' \otimes_{A'} A$ is an isomorphism. However, it is often more convenient think of φ as a B-linear map $\varphi : N \to M'/IM'$ which induces an isomorphism $N \otimes_B A \to M' \otimes_{A'} A = M'/IM'$.

Lemma 15.4.5. *In Situation 15.4.4 the functor (15.4.4.1) has a right adjoint, namely the functor*

$$F : (N, M', \varphi) \longmapsto N \times_{\varphi, M} M'$$

where $M = M'/IM'$. *Moreover, the composition of* F *with (15.4.4.1) is the identity functor on* $Mod_B \times_{Mod_A} Mod_{A'}$. *In other words, setting* $N' = N \times_{\varphi,M} M'$ *we have* $N' \otimes_{B'} B = N$ *and* $N' \otimes_{B'} A' = M'$.

Proof. The adjointness statement is that for a B'-module L' and a triple (N, M', φ) we have

$$\text{Hom}_{B'}(L', N \times_{\varphi,M} M') = \text{Hom}_B(L' \otimes_{B'} B, N) \times_{\text{Hom}_A(L' \otimes_{B'} A, M)} \text{Hom}_{A'}(L' \otimes_{B'} A', M')$$

This follows from Algebra, Lemma 10.13.3 and the fact that an element of the left hand side is given by a pair of B'-linear maps $L' \to N$ and $L' \to M'$ agreeing as maps to M. To prove the final assertion, recall that $B' = B \times_A A'$ and $N' = N \times_{\varphi,M} M'$ and extend these equalities to

$$\begin{array}{ccccc} A & \longleftarrow & A' & \longleftarrow & I \\ \uparrow & & \uparrow & & \uparrow \\ B & \longleftarrow & B' & \longleftarrow & J \end{array} \quad \text{and} \quad \begin{array}{ccccc} M & \longleftarrow & M' & \longleftarrow & K \\ \uparrow{\scriptstyle \varphi} & & \uparrow & & \uparrow \\ N & \longleftarrow & N' & \longleftarrow & L \end{array}$$

where I, J, K, L are the kernels of the horizontal maps of the original diagrams. We present the proof as a sequence of observations:

(1) $K = IM'$ (see statement lemma),
(2) $B' \to B$ is surjective with kernel J and $J \to I$ is bijective,
(3) $N' \to N$ is surjective with kernel L and $L \to K$ is bijective,
(4) $JN' \subset L$,
(5) $\mathrm{Im}(N \to M)$ generates M as an A-module (because $N \otimes_B A = M$),
(6) $\mathrm{Im}(N' \to M')$ generates M' as an A'-module (because it holds modulo K and L maps isomorphically to K),
(7) $JN' = L$ (because $L \cong K = IM'$ is generated by images of elements xn' with $x \in I$ and $n' \in N'$ by the previous statement),
(8) $N' \otimes_{B'} B = N$ (because $N = N'/L$, $B = B'/J$, and the previous statement),
(9) there is a map $\gamma : N' \otimes_{B'} A' \to M'$,
(10) γ is surjective (see above),
(11) the kernel of the composition $N' \otimes_{B'} A' \to M' \to M$ is generated by elements $l \otimes 1$ and $n' \otimes x$ with $l \in K$, $n' \in N'$, $x \in I$ (because $M = N \otimes_B A$ by assumption and because $N' \to N$ and $A' \to A$ are surjective with kernels L and I),
(12) any element of $N' \otimes_{B'} A'$ in the submodule generated by the elements $l \otimes 1$ and $n' \otimes x$ with $l \in L$, $n' \in N'$, $x \in I$ can be written as $l \otimes 1$ for some $l \in L$ (because J maps isomorphically to I we see that $n' \otimes x = n'x \otimes 1$ in $N' \otimes_{B'} A'$; similarly $xn' \otimes a' = n' \otimes xa' = n'(xa') \otimes 1$ in $N' \otimes_{B'} A'$ when $n' \in N'$, $x \in J$ and $a' \in A'$; since we have seen that $JN' = L$ this proves the assertion),
(13) the kernel of γ is zero (because by (10) and (11) any element of the kernel is of the form $l \otimes 1$ with $l \in L$ which is mapped to $l \in K \subset M'$ by γ).

This finishes the proof. □

Lemma 15.4.6. *In the situation of Lemma 15.4.5 for a B'-module L' the adjunction map*

$$L' \longrightarrow (L' \otimes_{B'} B) \times_{(L' \otimes_{B'} A)} (L' \otimes_{B'} A')$$

is surjective but in general not injective.

Proof. As in the proof of Lemma 15.4.5 let $J \subset B'$ be the kernel of the map $B' \to B$. Then $L' \otimes_{B'} B = L'/JL'$. Hence to prove surjectivity it suffices to show that elements of the form $(0, z)$ of the fibre product are in the image of the map of the lemma. The kernel of the map $L' \otimes_{B'} A' \to L' \otimes_{B'} A$ is the image of $L' \otimes_{B'} I \to L' \otimes_{B'} A'$. Since the map $J \to I$ induced by $B' \to A'$ is an isomorphism

the composition

$$L' \otimes_{B'} J \to L' \to (L' \otimes_{B'} B) \times_{(L' \otimes_{B'} A)} (L' \otimes_{B'} A')$$

induces a surjection of $L' \otimes_{B'} J$ onto the set of elements of the form $(0, z)$. To see the map is not injective in general we present a simple example. Namely, take a field k, set $B' = k[x, y]/(xy)$, $A = B'/(x)$, $B = B'/(y)$, $A = B'/(x, y)$ and $L' = B'/(x - y)$. In that case the class of x in L' is nonzero but is mapped to zero under the displayed arrow. □

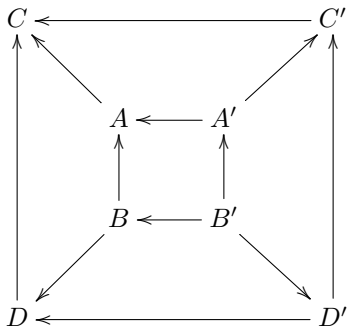
Lemma 15.4.7. *In Situation 15.4.4 let $(N_1, M_1', \varphi_1) \to (N_2, M_2', \varphi_2)$ be a morphism of $Mod_B \times_{Mod_A} Mod_{A'}$ with $N_1 \to N_2$ and $M_1' \to M_2'$ surjective. Then*

$$N_1 \times_{M_1} M_1' \to N_2 \times_{M_2} M_2'$$

is surjective.

Proof. Pick $(x_2, y_2) \in N_2 \times_{M_2} M_2'$. Choose $x_1 \in N_1$ mapping to x_2. Since $M_1' \to M_1$ is surjective we can find $y_1 \in M_1'$ mapping to $\varphi_1(x_1)$. Then (x_1, y_1) maps to (x_2, y_2') in $N_2 \times_{M_2} M_2'$. Thus it suffices to show that elements of the form $(0, y_2)$ are in the image of the map. Here we see that $y_2 \in IM_2'$. Write $y_2 = \sum t_i y_{2,i}$ with $t_i \in I$. Choose $y_{1,i} \in M_1'$ mapping to $y_{2,i}$. Then $y_1 = \sum t_i y_{1,i} \in IM_1'$ and the element $(0, y_1)$ does the job. □

Situation 15.4.8. Let A, A', B, B', I be as in Situation 15.4.4. Let $B' \to D'$ be a ring map. Set $D = D' \otimes_{B'} B$, $C' = D' \otimes_{B'} A'$, and $C = D' \otimes_{B'} A$. This leads to a big commutative diagram



of rings. Observe that we do **not** assume that the map $D' \to D \times_C C'$ is an isomorphism. In this situation we have the functor

$$\text{(15.4.8.1)} \quad \text{Mod}_{D'} \longrightarrow \text{Mod}_D \times_{\text{Mod}_C} \text{Mod}_{C'}, \quad L' \longmapsto (L' \otimes_{D'} D, L' \otimes_{D'} C', can)$$

analogous to (15.4.4.1). Note that $L' \otimes_{D'} D = L \otimes_{D'} (D' \otimes_{B'} B) = L \otimes_{B'} B$ and similarly $L' \otimes_{D'} C' = L \otimes_{D'} (D' \otimes_{B'} A') = L \otimes_{B'} A'$ hence the diagram

$$\begin{array}{ccc} \text{Mod}_{D'} & \longrightarrow & \text{Mod}_D \times_{\text{Mod}_C} \text{Mod}_{C'} \\ \downarrow & & \downarrow \\ \text{Mod}_{B'} & \longrightarrow & \text{Mod}_B \times_{\text{Mod}_A} \text{Mod}_{A'} \end{array}$$

is commutative. In the following we will write (N, M', φ) for an object of $\text{Mod}_D \times_{\text{Mod}_C} \text{Mod}_{C'}$, i.e., N is a D-module, M' is an C'-module and $\varphi : N \otimes_B A \to M' \otimes_{A'} A$ is an isomorphism of C-modules. However, it is often more convenient think of φ

as a D-linear map $\varphi : N \to M'/IM'$ which induces an isomorphism $N \otimes_B A \to M' \otimes_{A'} A = M'/IM'$.

Lemma 15.4.9. *In Situation 15.4.8 the functor (15.4.8.1) has a right adjoint, namely the functor*

$$F : (N, M', \varphi) \longmapsto N \times_{\varphi, M} M'$$

where $M = M'/IM'$. Moreover, the composition of F with (15.4.8.1) is the identity functor on $Mod_D \times_{Mod_C} Mod_{C'}$. In other words, setting $N' = N \times_{\varphi,M} M'$ we have $N' \otimes_{D'} D = N$ and $N' \otimes_{D'} C' = M'$.

Proof. The adjointness statement is that for a D'-module L' and a triple (N, M', φ) we have

$$\mathrm{Hom}_{D'}(L', N \times_{\varphi,M} M') = \mathrm{Hom}_D(L' \otimes_{D'} D, N) \times_{\mathrm{Hom}_C(L' \otimes_{D'} C, M)} \mathrm{Hom}_{C'}(L' \otimes_{D'} C', M')$$

This follows from Algebra, Lemma 10.13.3 and the fact that an element of the left hand side is given by a pair of D'-linear maps $L' \to N$ and $L' \to M'$ agreeing as maps to M. The final assertion follows from the corresponding assertion of Lemma 15.4.5. □

Lemma 15.4.10. *In Situation 15.4.8 the map $JD' \to IC'$ is surjective where $J = Ker(B' \to B)$.*

Proof. Since $C' = D' \otimes_{B'} A'$ we have that IC' is the image of $D' \otimes_{B'} I = C' \otimes_{A'} I \to C'$. As the ring map $B' \to A'$ induces an isomorphism $J \to I$ the lemma follows. □

Lemma 15.4.11. *Let $A, A', B, B', C, C', D, D', I, M', M, N, \varphi$ be as in Lemma 15.4.9. If N finite over D and M' finite over C', then $N' = N \times_M M'$ is finite over D'.*

Proof. We will use the results of Lemma 15.4.9 without further mention. Choose generators $x_1, \ldots, x_r$ of N over B and generators $y_1, \ldots, y_s$ of M' over A'. Using that $N = N' \otimes_{D'} D$ and $D' \to D$ is surjective we can find $u_1, \ldots, u_r \in N'$ mapping to $x_1, \ldots, x_r$ in N. Using that $M' = N' \otimes_{D'} C'$ we can find $v_1, \ldots, v_t \in N'$ such that $y_i = \sum v_j \otimes c'_{ij}$ for some $c'_{ij} \in C'$. In particular we see that the images $\overline{v}_j$ of the v_j generate M' over C'. We claim that $u_1, \ldots, u_r, v_1, \ldots, v_t$ generate N' as a D'-module. Namely, pick $\xi \in N'$. We first choose $d'_1, \ldots, d'_r \in D'$ such that ξ and $\sum d'_i u_i$ map to the same element of N. This is possible because $D' \to D$ is surjective and $x_1, \ldots, x_r$ generate N. The difference $\xi - \sum d'_i u_i$ is of the form $(0, \theta)$ for some θ in IM'. Say θ is $\sum t_j \overline{v}_j$ with $t_j \in IC'$. By Lemma 15.4.10 we can choose $s_j \in JD'$ mapping to t_j. Because $N' = N \times_M M'$ it follows that $\xi = \sum b'_i u_i + \sum s_j v_j$ as desired. □

Lemma 15.4.12. *With $A, A', B, B', C, C', D, D', I$ as in Situation 15.4.8.*

1. *Let (N, M', φ) be an object of $Mod_D \times_{Mod_C} Mod_{C'}$. If M' is flat over A' and N is flat over B, then $N' = N \times_M M'$ is flat over B'.*
2. *If L' is a D'-module flat over B', then $L' = (L \otimes_{D'} D) \times_{(L \otimes_{D'} C)} (L \otimes_{D'} C')$.*
3. *The category of D'-modules flat over B' is equivalent to the categories of objects (N, M', φ) of $Mod_D \times_{Mod_C} Mod_{C'}$ with N flat over B and M' flat over A'.*

Proof. Proof of (1). Let $J \subset B'$ be an ideal. We have to show that $J \otimes_{B'} N' \to N'$ is injective, see Algebra, Lemma 10.36.4. We know that

$$J/(J \cap I) \otimes_{B'} N' = J/(J \cap I) \otimes_B N \to N$$

is injective as N is flat over B. As $J \cap I \to J \to J/(J \cap I) \to 0$ is exact, we conclude that it suffices to show that $(J \cap I) \otimes_{B'} N' \to N'$ is injective. Thus we may assume that $J \subset I$; in particular we can think of J as an A'-module and an ideal of A' and

$$J \otimes_{B'} N' = J \otimes_{A'} A' \otimes_{B'} N' = J \otimes_{A'} M'$$

which maps injectively into M' by our assumption that M' is flat over A'. We conclude that $J \otimes_{B'} N' \to N' \to M'$ is injective and hence the first map is injective as desired.

Proof of (2). This follows by tensoring the short exact sequence $0 \to B' \to B \oplus A' \to A \to 0$ with L' over B' and using that $L' \otimes_{D'} D = L' \otimes_{B'} B$, $L' \otimes_{D'} C' = L' \otimes_{B'} A'$, and $L' \otimes_{D'} C = L' \otimes_{B'} A$, see discussion in Situation 15.4.8.

Proof of (3). Immediate consequence of (1) and (2). □

Lemma 15.4.13. *Let $A, A', B, B', C, C', D, D', I, M', M, N, \varphi$ be as in Lemma 15.4.9. If*

(1) *N is finitely presented over D and flat over B,*
(2) *M' finitely presented over C' and flat over A', and*
(3) *the ring map $B' \to D'$ factors as $B' \to D'' \to D''$ with $B' \to D''$ flat and $D'' \to D'$ of finite presentation,*

then $N' = N \times_M M'$ is finitely presented over D'.

Proof. Choose a surjection $D''' = D''[x_1, \ldots, x_n] \to D'$ with finitely generated kernel J. By Algebra, Lemma 10.7.4 it suffices to show that N' is finitely presented as a D'''-module. Moreover, $D''' \otimes_{B'} B \to D' \otimes_{B'} B = D$ and $D''' \otimes_{B'} A' \to D' \otimes_{B'} A' = C'$ are surjections whose kernels are generated by the image of J, hence N is a finitely presented $D''' \otimes_{B'} B$-module and M' is a finitely presented $D''' \otimes_{B'} A'$-module by Algebra, Lemma 10.7.4 again. Thus we may replace D' by D''' and D by $D''' \otimes_{B'} B$, etc. Since D''' is flat over B', it follows that we may assume that $B' \to D'$ is flat.

Assume $B' \to D'$ is flat. By Lemma 15.4.11 the module N' is finite over D'. Choose a surjection $(D')^{\oplus n} \to N'$ with kernel K'. By base change we obtain maps $D^{\oplus n} \to N$, $(C')^{\oplus n} \to M'$, and $C^{\oplus n} \to M$ with kernels K_D, $K_{C'}$, and K_C. There is a canonical map

$$K' \longrightarrow K_D \times_{K_C} K_{C'}$$

On the other hand, since $N' = N \times_M M'$ and $D' = D \times_C C'$ (by Lemma 15.4.12) there is also a canonical map $K_D \times_{K_C} K_{C'} \to K'$ inverse to the displayed arrow. Hence the displayed map is an isomorphism. By Algebra, Lemma 10.5.3 the modules K_D and $K_{C'}$ are finite. We conclude from Lemma 15.4.11 that K' is a finite D'-module provided that $K_D \to K_C$ and $K_{C'} \to K_C$ induce isomorphisms $K_D \otimes_B A = K_C = K_{C'} \otimes_{A'} A$. This is true because the flatness assumptions implies the sequences

$$0 \to K_D \to D^{\oplus n} \to N \to 0 \quad \text{and} \quad 0 \to K_{C'} \to (C')^{\oplus n} \to M' \to 0$$

stay exact upon tensoring, see Algebra, Lemma 10.36.11. □

Lemma 15.4.14. *Let A, A', B, B', I be as in Situation 15.4.4. Let (D, C', φ) be a system consisting of an B-algebra D, a A'-algebra C' and an isomorphism $D \otimes_B A \to C'/IC = C$. Set $D' = D \times_C C'$ (as in Lemma 15.4.5). Then*

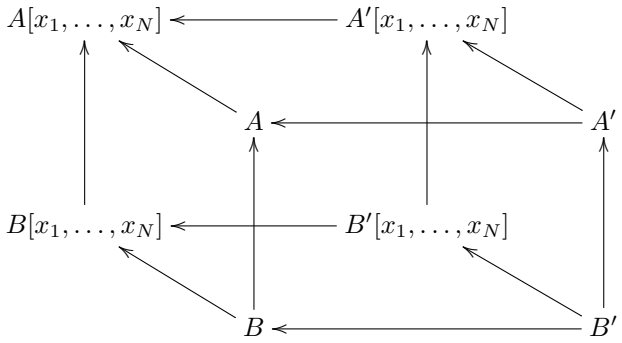
(1) *$B' \to D'$ is finite type if and only if $B \to D$ and $A' \to C'$ are finite type,*

(2) $B' \to D'$ *is flat if and only if* $B \to D$ *and* $A' \to C'$ *are flat,*
(3) $B' \to D'$ *is flat and of finite presentation if and only if* $B \to D$ *and* $A' \to C'$ *are flat and of finite presentation,*
(4) $B' \to D'$ *is smooth if and only if* $B \to D$ *and* $A' \to C'$ *are smooth,*
(5) $B' \to D'$ *is étale if and only if* $B \to D$ *and* $A' \to C'$ *are étale.*

Moreover, if D' *is a flat* B'*-algebra, then* $D' \to (D' \otimes_{B'} B) \times_{(D' \otimes_{B'} A)} (D' \otimes_{B'} A')$ *is an isomorphism. In this way the category of flat* B'*-algebras is equivalent to the categories of systems* (D, C', φ) *as above with* D *flat over* B *and* C' *flat over* A'*.*

Proof. The implication "⇒" follows from Algebra, Lemmas 10.13.2, 10.36.6, 10.130.4, and 10.136.3 because we have $D' \otimes_{B'} B = D$ and $D' \otimes_{B'} A' = C'$ by Lemma 15.4.5. Thus it suffices to prove the implications in the other direction.

Ad (1). Assume D of finite type over B and C' of finite type over A'. We will use the results of Lemma 15.4.5 without further mention. Choose generators $x_1, \ldots, x_r$ of D over B and generators $y_1, \ldots, y_s$ of C' over A'. Using that $N = N' \otimes_{B'} B$ and $B' \to B$ is surjective we can find $u_1, \ldots, u_r \in D'$ mapping to $x_1, \ldots, x_r$ in D. Using that $C' = D' \otimes_{B'} A'$ we can find $v_1, \ldots, v_t \in D'$ such that $y_i = \sum v_j \otimes a'_{ij}$ for some $a'_{ij} \in A'$. In particular, the images of v_j in C' generate C' as an A'-algebra. Set $N = r + t$ and consider the cube of rings



Observe that the back square is cartesian as well. Consider the ring map

$$B'[x_1, \ldots, x_N] \to D', \quad x_i \mapsto u_i \quad \text{and} \quad x_{r+j} \mapsto v_j.$$

Then we see that the induced maps $B[x_1, \ldots, x_N] \to D$ and $A'[x_1, \ldots, x_N] \to C'$ are surjective, in particular finite. We conclude from Lemma 15.4.11 that $B'[x_1, \ldots, x_N] \to D'$ is finite, which implies that D' is of finite type over B' for example by Algebra, Lemma 10.6.2.

Ad (2). The implication "⇐" follows from Lemma 15.4.12. Moreover, the final statement follows from the final statement of Lemma 15.4.12.

Ad (3). Assume $B \to D$ and $A' \to C'$ are flat and of finite presentation. The flatness of $B' \to D'$ we've seen in (2). We know $B' \to D'$ is of finite type by (1). Choose a surjection $B'[x_1, \ldots, x_N] \to D'$. By Algebra, Lemma 10.6.3 the ring D is of finite presentation as a $B[x_1, \ldots, x_N]$-module and the ring C' is of finite presentation as a $A'[x_1, \ldots, x_N]$-module. By Lemma 15.4.13 we see that D' is of finite presentation as a $B'[x_1, \ldots, x_N]$-module, i.e., $B' \to D'$ is of finite presentation.

Ad (4). Assume $B \to D$ and $A' \to C'$ smooth. By (3) we see that $B' \to D'$ is flat and of finite presentation. By Algebra, Lemma 10.130.16 it suffices to check that

$D' \otimes_{B'} k$ is smooth for any field k over B'. If the composition $J \to B' \to k$ is zero, then $B' \to k$ factors as $B' \to B \to k$ and we see that

$$D' \otimes_{B'} k = D' \otimes_{B'} B \otimes_B k = D \otimes_B k$$

is smooth as $B \to D$ is smooth. If the composition $J \to B' \to k$ is nonzero, then there exists an $h \in J$ which does not map to zero in k. Then $B' \to k$ factors as $B' \to B'_h \to k$. Observe that h maps to zero in B, hence $B_h = 0$. Thus by Lemma 15.4.3 we have $B'_h = A'_h$ and we get

$$D' \otimes_{B'} k = D' \otimes_{B'} B'_h \otimes_{B'_h} k = C'_h \otimes_{A'_h} k$$

is smooth as $A' \to C'$ is smooth.

Ad (5). Assume $B \to D$ and $A' \to C'$ are étale. By (4) we see that $B' \to D'$ is smooth. As we can read off whether or not a smooth map is étale from the dimension of fibres we see that (5) holds (argue as in the proof of (4) to identify fibres – some details omitted). □

Remark 15.4.15. In Situation 15.4.8. Assume $B' \to D'$ is of finite presentation and suppose we are given a D'-module L'. We claim there is a bijective correspondence between

(1) surjections of D'-modules $L' \to Q'$ with Q' of finite presentation over D' and flat over B', and
(2) pairs of surjections of modules $(L' \otimes_{D'} D \to Q_1, L' \otimes_{D'} C' \to Q_2)$ with
   (a) Q_1 of finite presentation over D and flat over B,
   (b) Q_2 of finite presentation over C' and flat over A',
   (c) $Q_1 \otimes_D C = Q_2 \otimes_{C'} C$ as quotients of $L' \otimes_{D'} C$.

The correspondence between these is given by $Q \mapsto (Q_1, Q_2)$ with $Q_1 = Q \otimes_{D'} D$ and $Q_2 = Q \otimes_{D'} C'$. And for the converse we use $Q = Q_1 \times_{Q_{12}} Q_2$ where Q_{12} the common quotient $Q_1 \otimes_D C = Q_2 \otimes_{C'} C$ of $L' \otimes_{D'} C$. As quotient map we use

$$L' \longrightarrow (L' \otimes_{D'} D) \times_{(L' \otimes_{D'} C)} (L' \otimes_{D'} C') \longrightarrow Q_1 \times_{Q_{12}} Q_2 = Q$$

where the first arrow is surjective by Lemma 15.4.6 and the second by Lemma 15.4.7. The claim follows by Lemmas 15.4.12 and 15.4.13.

15.5. Fitting ideals

The fitting ideals of a finite module are the ideals determined by the construction of Lemma 15.5.2.

Lemma 15.5.1. *Let R be a ring. Let A be an $n \times m$ matrix with coefficients in R. Let $I_r(A)$ be the ideal generated by the $r \times r$-minors of A with the convention that $I_0(A) = R$ and $I_r(A) = 0$ if $r > \min(n, m)$. Then*

(1) *$I_0(A) \supset I_1(A) \supset I_2(A) \ldots$,*
(2) *if B is an $(n + n') \times m$ matrix, and A is the first n rows of B, then $I_{r+n'}(B) \subset I_r(A)$,*
(3) *if C is an $n \times n$ matrix then $I_r(CA) \subset I_r(A)$.*
(4) *If A is a block matrix*
$$\begin{pmatrix} A_1 & 0 \\ 0 & A_2 \end{pmatrix}$$
then $I_r(A) = \sum_{r_1+r_2=r} I_{r_1}(A_1) I_{r_2}(A_2)$.
(5) *Add more here.*

Proof. Omitted. (Hint: Use that a determinant can be computed by expanding along a column or a row.) □

Lemma 15.5.2. *Let R be a ring. Let M be a finite R-module. Choose a presentation*

$$\bigoplus\nolimits_{j\in J} R \longrightarrow R^{\oplus n} \longrightarrow M \longrightarrow 0.$$

of M. Let $A = (a_{ij})_{i=1,\ldots,n,j\in J}$ be the matrix of the map $\bigoplus_{j\in J} R \to R^{\oplus n}$. The ideal $Fit_k(M)$ generated by the $(n-k)\times(n-k)$ minors of A is independent of the choice of the presentation.

Proof. Let $K \subset R^{\oplus n}$ be the kernel of the surjection $R^{\oplus n} \to M$. Pick $z_1, \ldots, z_{n-k} \in K$ and write $z_j = (z_{1j}, \ldots, z_{nj})$. Another description of the ideal $\mathrm{Fit}_k(M)$ is that it is the ideal generated by the $(n-k)\times(n-k)$ minors of all the matrices (z_{ij}) we obtain in this way.

Suppose we change the surjection into the surjection $R^{\oplus n+n'} \to M$ with kernel K' where we use the original map on the first n standard basis elements of $R^{\oplus n+n'}$ and 0 on the last n' basis vectors. Then the corresponding ideals are the same. Namely, if $z_1, \ldots, z_{n-k} \in K$ as above, let $z'_j = (z_{1j}, \ldots, z_{nj}, 0, \ldots, 0) \in K'$ for $j = 1, \ldots, n-k$ and $z'_{n+j'} = (0, \ldots, 0, 1, 0, \ldots, 0) \in K'$. Then we see that the ideal of $(n-k)\times(n-k)$ minors of (z_{ij}) agrees with the ideal of $(n+n'-k)\times(n+n'-k)$ minors of (z'_{ij}). This gives one of the inclusions. Conversely, given $z'_1, \ldots, z'_{n+n'-k}$ in K' we can project these to $R^{\oplus n}$ to get $z_1, \ldots, z_{n+n'-k}$ in K. By Lemma 15.5.1 we see that the ideal generated by the $(n+n'-k)\times(n+n'-k)$ minors of (z'_{ij}) is contained in the ideal generated by the $(n-k)\times(n-k)$ minors of (z_{ij}). This gives the other inclusion.

Let $R^{\oplus m} \to M$ be another surjection with kernel L. By the previous paragraph we may assume $m = n$. By Algebra, Lemma 10.5.2 we can choose a map $R^{\oplus n} \to R^{\oplus m}$ commuting with the surjections to M. Let $C = (c_{li})$ be the matrix of this map (it is a square matrix as $n = m$). Then given $z_1, \ldots, z_{n-k} \in K$ as above we get $Cz_1, \ldots, Cz_{n-k} \in L$. By Lemma 15.5.1 we get one of the inclusions. By symmetry we get the other. □

Definition 15.5.3. Let R be a ring. Let M be a finite R-module. Let $k \geq 0$. The *kth fitting ideal* of M is the ideal $\mathrm{Fit}_k(M)$ constructed in Lemma 15.5.1.

Since the fitting ideals are the ideals of minors of a big matrix (numbered in reverse ordering from the ordering in Lemma 15.5.1) we see that

$$\mathrm{Fit}_0(M) \subset \mathrm{Fit}_1(M) \subset \ldots \subset \mathrm{Fit}_t(M) = R$$

for some $t \gg 0$. Here are some basic properties of fitting ideals.

Lemma 15.5.4. *Let R be a ring. Let M be a finite R-module.*

(1) *If M can be generated by n elements, then $Fit_n(M) = R$.*
(2) *Given a second finite R-module M' we have*

$$Fit_k(M \oplus M') = \sum\nolimits_{k+k'=l} Fit_k(M) Fit_{k'}(M')$$

(3) *If $R \to R'$ is a ring map, then $Fit_k(M \otimes_R R')$ is the ideal of R' generated by the image of $Fit_k(M)$.*

(4) *If M is an R-module of finite presentation, then $Fit_k(M)$ is a finitely generated ideal.*
(5) *If $M \to M'$ is a surjection, then $Fit_k(M) \subset Fit_k(M')$.*
(6) *Add more here.*

Proof. Part (1) follows from the fact that $I_0(A) = R$ in Lemma 15.5.1. part (2) follows form the corresponding statement in Lemma 15.5.1. Part (3) follows from the fact that $\otimes_R R'$ is right exact, so the base change of a presentation of M is a presentation of $M \otimes_R R'$. Proof of (4). Let $R^{\oplus m} \xrightarrow{A} R^{\oplus n} \to M \to 0$ be a presentation. Then $\mathrm{Fit}_k(M)$ is the ideal generated by the $n - k \times n - k$ minors of the matrix A. Part (5) is immediate from the definition. □

Example 15.5.5. Let R be a ring. The fitting ideals of the finite free module $M = R^{\oplus n}$ are are $\mathrm{Fit}_k(M) = 0$ for $k < n$ and $\mathrm{Fit}_k(M) = R$ for $k \geq n$.

Lemma 15.5.6. *Let R be a ring. Let M be a finite R-module. Let $k \geq 0$. Let $\mathfrak{p}$ be a prime ideal with $Fit_k(M) \not\subset \mathfrak{p}$. Then there exists an $f \in R$, $f \notin \mathfrak{p}$ such that M_f can be generated by k elements over R_f.*

Proof. By Nakayama's lemma (Algebra, Lemma 10.18.1) we see that M_f can be generated by k elements over R_f for some $f \in R$, $f \notin \mathfrak{p}$ if $M \otimes_R \kappa(\mathfrak{p})$ can be generated by k elements. This reduces the problem to the case where R is a field and $\mathfrak{p} = (0)$. In this case the result follows from Example 15.5.5. □

Lemma 15.5.7. *Let R be a ring. Let M be a finite R-module. Let $r \geq 0$. The following are equivalent*

(1) *M is finite locally free of rank k (Algebra, Definition 10.73.1),*
(2) *$Fit_{r-1}(M) = 0$ and $Fit_r(M) = R$, and*
(3) *$Fit_k(M) = 0$ for $k < r$ and $Fit_k(M) = R$ for $k \geq r$.*

Proof. It is immediate that (2) is equivalent to (3) because the fitting ideals form an increasing sequence of ideals. Since the formation of $\mathrm{Fit}_k(M)$ commutes with base change (Lemma 15.5.4) we see that (1) implies (2) by Example 15.5.5 and glueing results (Algebra, Section 10.22). Conversely, assume (2). By Lemma 15.5.6 we may assume that M is generated by r elements. Thus a presentation $\bigoplus_{j \in J} R \to R^{\oplus r} \to M \to 0$. But now the assumption that $\mathrm{Fit}_{r-1}(M) = 0$ implies that all entries of the matrix of the map $\bigoplus_{j \in J} R \to R^{\oplus r}$ are zero. Thus M is free. □

15.6. Lifting

In this section we collection some lemmas concerning lifting statements of the following kind: If A is a ring and $I \subset A$ is an ideal, and $\overline{\xi}$ is some kind of structure over A/I, then we can lift $\overline{\xi}$ to a similar kind of structure ξ over A or over some étale extension of A. Here are some types of structure for which we have already proved some results:

(1) idempotents, see Algebra, Lemmas 10.49.5 and 10.49.6,
(2) projective modules, see Algebra, Lemma 10.72.4,
(3) basis elements, see Algebra, Lemmas 10.95.1 and 10.95.3,
(4) ring maps, i.e., proving certain algebras are formally smooth, see Algebra, Lemma 10.131.4, Proposition 10.131.13, and Lemma 10.131.16,
(5) syntomic ring maps, see Algebra, Lemma 10.129.18,
(6) smooth ring maps, see Algebra, Lemma 10.130.19,

(7) étale ring maps, see Algebra, Lemma 10.136.11,
(8) factoring polynomials, see Algebra, Lemma 10.136.20, and
(9) Algebra, Section 10.143 discusses henselian local rings.

The interested reader will find more results of this nature in Smoothing Ring Maps, Section 16.4 in particular Smoothing Ring Maps, Proposition 16.4.2.

Let A be a ring and let $I \subset A$ be an ideal. Let $\overline{\xi}$ be some kind of structure over A/I. In the following lemmas we look for étale ring maps $A \to A'$ which induce isomorphisms $A/I \to A'/IA'$ and objects ξ' over A' lifting $\overline{\xi}$. A general remark is that given étale ring maps $A \to A' \to A''$ such that $A/I \cong A'/IA'$ and $A'/IA' \cong A''/IA''$ the composition $A \to A''$ is also étale (Algebra, Lemma 10.136.3) and also satisfies $A/I \cong A''/IA''$. We will frequently use this in the following lemmas without further mention. Here is a trivial example of the type of result we are looking for.

Lemma 15.6.1. *Let A be a ring, let $I \subset A$ be an ideal, let $\overline{u} \in A/I$ be an invertible element. There exists an étale ring map $A \to A'$ which induces an isomorphism $A/I \to A'/IA'$ and an invertible element $u' \in A'$ lifting $\overline{u}$.*

Proof. Choose any lift $f \in A$ of $\overline{u}$ and set $A' = A_f$ and u the image of f in A'. □

Lemma 15.6.2. *Let A be a ring, let $I \subset A$ be an ideal, let $\overline{e} \in A/I$ be an idempotent. There exists an étale ring map $A \to A'$ which induces an isomorphism $A/I \to A'/IA'$ and an idempotent $e' \in A'$ lifting $\overline{e}$.*

Proof. Choose any lift $x \in A$ of $\overline{e}$. Set

$$A' = A[t]/(t^2 - t)\left[\frac{1}{t - 1 + x}\right].$$

The ring map $A \to A'$ is étale because $(2t-1)\mathrm{d}t = 0$ and $(2t-1)(2t-1) = 1$ which is invertible. We have $A'/IA' = A/I[t]/(t^2 - t)[\frac{1}{t-1+\overline{e}}] \cong A/I$ the last map sending t to $\overline{e}$ which works as $\overline{e}$ is a root of $t^2 - t$. This also shows that setting e' equal to the class of t in A' works. □

Lemma 15.6.3. *Let A be a ring, let $I \subset A$ be an ideal. Let $\mathrm{Spec}(A/I) = \coprod_{j \in J} \overline{U}_j$ be a finite disjoint open covering. Then there exists an étale ring map $A \to A'$ which induces an isomorphism $A/I \to A'/IA'$ and a finite disjoint open covering $\mathrm{Spec}(A') = \coprod_{j \in J} U'_j$ lifting the given covering.*

Proof. This follows from Lemma 15.6.2 and the fact that open and closed subsets of Spectra correspond to idempotents, see Algebra, Lemma 10.19.3. □

Lemma 15.6.4. *Let $A \to B$ be a ring map and $J \subset B$ an ideal. If $A \to B$ is étale at every prime of $V(J)$, then there exists a $g \in B$ mapping to an invertible element of B/J such that $A' = B_g$ is étale over A.*

Proof. The set of points of $\mathrm{Spec}(B)$ where $A \to B$ is not étale is a closed subset of $\mathrm{Spec}(B)$, see Algebra, Definition 10.136.1. Write this as $V(J')$ for some ideal $J' \subset B$. Then $V(J') \cap V(J) = \emptyset$ hence $J + J' = B$ by Algebra, Lemma 10.16.2. Write $1 = f + g$ with $f \in J$ and $g \in J'$. Then g works. □

The assumption on the leading coefficient in the following lemma will be removed in Lemma 15.6.6.

Lemma 15.6.5. *Let A be a ring, let $I \subset A$ be an ideal. Let $f \in A[x]$ be a monic polynomial. Let $\overline{f} = \overline{g}\overline{h}$ be a factorization of f in $A/I[x]$ and assume*

(1) *the leading coefficient of $\overline{g}$ is an invertible element of A/I, and*
(2) *$\overline{g}$, $\overline{h}$ generate the unit ideal in $A/I[x]$.*

Then there exists an étale ring map $A \to A'$ which induces an isomorphism $A/I \to A'/IA'$ and a factorization $f = g'h'$ in $A'[x]$ lifting the given factorization over A/I.

Proof. Applying Lemma 15.6.1 we may assume that the leading coefficient of $\overline{g}$ is the reduction of an invertible element $u \in A$. Then we may replace $\overline{g}$ by $\overline{u}^{-1}\overline{g}$ and $\overline{h}$ by $\overline{u}\overline{h}$. Thus we may assume that $\overline{g}$ is monic. Since f is monic we conclude that $\overline{h}$ is monic too. Say $\deg(\overline{g}) = n$ and $\deg(\overline{h}) = m$ so that $\deg(f) = n + m$. Write $f = x^{n+m} + \sum \alpha_i x^{n+m-i}$ for some $\alpha_1, \ldots, \alpha_{n+m} \in A$. Consider the ring map

$$R = \mathbf{Z}[a_1, \ldots, a_{n+m}] \longrightarrow S = \mathbf{Z}[b_1, \ldots, b_n, c_1, \ldots, c_m]$$

of Algebra, Example 10.136.13. Let $R \to A$ be the ring map which sends a_i to α_i. Set

$$B = A \otimes_R S$$

By construction the image of f in $B[x]$ factors. Write $\overline{g} = x^n + \sum \overline{\beta}_i x^{n-i}$ and $\overline{h} = x^m + \sum \overline{\gamma}_i x^{m-i}$. The A-algebra map

$$B \longrightarrow A/I, \quad 1 \otimes b_i \mapsto \overline{\beta}_i, \quad 1 \otimes c_i \mapsto \overline{\gamma}_i$$

maps the factorization of f over B to the given factorization over A/I. The displayed map is surjective; denote $J \subset B$ its kernel. From the discussion in Algebra, Example 10.136.13 it is clear that $A \to B$ is etale at all points of $V(J) \subset \mathrm{Spec}(B)$. Choose $g \in B$ as in Lemma 15.6.4 and set $A' = B_g$. □

Lemma 15.6.6. *Let A be a ring, let $I \subset A$ be an ideal. Let $f \in A[x]$ be a monic polynomial. Let $\overline{f} = \overline{g}\overline{h}$ be a factorization of f in $A/I[x]$ and assume that $\overline{g}$, $\overline{h}$ generate the unit ideal in $A/I[x]$. Then there exists an étale ring map $A \to A'$ which induces an isomorphism $A/I \to A'/IA'$ and a factorization $f = g'h'$ in $A'[x]$ lifting the given factorization over A/I.*

Proof. Say $f = x^d + a_1 x^{d-1} + \ldots + a_d$ has degree d. Write $\overline{g} = \sum \overline{b}_j x^j$ and $\overline{h} = \sum \overline{c}_j x^j$. Then we see that $1 = \sum \overline{b}_j \overline{c}_{d-j}$. It follows that $\mathrm{Spec}(A/I)$ is covered by the standard opens $D(\overline{b}_j \overline{c}_{d-j})$. However, each point $\mathfrak{p}$ of $\mathrm{Spec}(A/I)$ is contained in at most one of these as by looking at the induced factorization of f over the field $\kappa(\mathfrak{p})$ we see that $\deg(\overline{g} \bmod \mathfrak{p}) + \deg(\overline{h} \bmod \mathfrak{p}) = d$. Hence our open covering is a disjoint open covering. Applying Lemma 15.6.3 (and replacing A by A') we see that we may assume there is a corresponding disjoint open covering of $\mathrm{Spec}(A)$. This disjoint open covering corresponds to a product decomposition of A, see Algebra, Lemma 10.21.3. It follows that

$$A = A_0 \times \ldots \times A_d, \quad I = I_0 \times \ldots \times I_d,$$

where the image of $\overline{g}$, resp. $\overline{h}$ in A_j/I_j has degree j, resp. $d-j$ with invertible leading coefficient. Clearly, it suffices to prove the result for each factor A_j separatedly. Hence the lemma follows from Lemma 15.6.5. □

Lemma 15.6.7. *Let $R \to S$ be a ring map. Let $I \subset R$ be an ideal of R and let $J \subset S$ be an ideal of S. If the closure of the image of $V(J)$ in $\mathrm{Spec}(R)$ is disjoint*

from $V(I)$, then there exists an element $f \in R$ which maps to 1 in R/I and to an element of J in S.

Proof. Let $I' \subset R$ be an ideal such that $V(I')$ is the closure of the image of $V(J)$. Then $V(I) \cap V(I') = \emptyset$ by assumption and hence $I + I' = R$ by Algebra, Lemma 10.16.2. Write $1 = g + f$ with $g \in I$ and $f \in I'$. We have $V(f') \supset V(J)$ where f' is the image of f in S. Hence $(f')^n \in J$ for some n, see Algebra, Lemma 10.16.2. Replacing f by f^n we win. □

Lemma 15.6.8. *Let A be a ring, let $I \subset A$ be an ideal. Let $A \to B$ be an integral ring map. Let $\overline{e} \in B/IB$ be an idempotent. Then there exists an étale ring map $A \to A'$ which induces an isomorphism $A/I \to A'/IA'$ and an idempotent $e' \in B \otimes_A A'$ lifting $\overline{e}$.*

Proof. Choose an element $y \in B$ lifting $\overline{e}$. Then $z = y^2 - y$ is an element of IB. By Algebra, Lemma 10.35.4 there exist a monic polynomial $g(x) = x^d + \sum a_j x^j$ of degree d with $a_j \in I$ such that $g(z) = 0$ in B. Hence $f(x) = g(x^2 - x) \in A[x]$ is a monic polynomial such that $f(x) \equiv x^d(x-1)^d \bmod I$ and such that $f(y) = 0$ in B. By Lemma 15.6.5 we can find an étale ring map $A \to A'$ which induces an isomorphism $A/I \to A'/IA'$ and such that $f = gh$ in $A[x]$ with $g(x) = x^d \bmod IA'$ and $h(x) = (x-1)^d \bmod IA'$. After replacing A by A' we may assume that the factorization is defined over A. In that case we see that $b_1 = g(y) \in B$ is a lift of $\overline{e}^d = \overline{e}$ and $b_2 = h(y) \in B$ is a lift of $(\overline{e} - 1)^d = (-1)^d(1 - \overline{e})^d = (-1)^d(1 - \overline{e})$ and moreover $b_1 b_2 = 0$. Thus $(b_1, b_2)B/IB = B/IB$ and $V(b_1, b_2) \subset \mathrm{Spec}(B)$ is disjoint from $V(IB)$. Since $\mathrm{Spec}(B) \to \mathrm{Spec}(A)$ is closed (see Algebra, Lemmas 10.33.20 and 10.38.6) we can find an $a \in A$ which maps to an invertible element of A/I whose image in B lies in (b_1, b_2), see Lemma 15.6.7. After replacing A by the localization A_a we get that $(b_1, b_2) = B$. Then $\mathrm{Spec}(B) = D(b_1) \amalg D(b_2)$; disjoint union because $b_1 b_2 = 0$. Let $e \in B$ be the idempotent corresponding to the open and closed subset $D(b_1)$, see Algebra, Lemma 10.19.3. Since b_1 is a lift of $\overline{e}$ and b_2 is a lift of $\pm(1 - \overline{e})$ we conclude that e is a lift of $\overline{e}$ by the uniqueness statement in Algebra, Lemma 10.19.3. □

Lemma 15.6.9. *Let A be a ring, let $I \subset A$ be an ideal. Let $\overline{P}$ be finite projective A/I-module. Then there exists an étale ring map $A \to A'$ which induces an isomorphism $A/I \to A'/IA'$ and a finite projective A'-module P' lifting $\overline{P}$.*

Proof. We can choose an integer n and a direct sum decomposition $(A/I)^{\oplus n} = \overline{P} \oplus \overline{K}$ for some R/I-module $\overline{K}$. Choose a lift $\varphi : A^{\oplus n} \to A^{\oplus n}$ of the projector $\overline{p}$ associated to the direct summand $\overline{P}$. Let $f \in A[x]$ be the characteristic polynomial of φ. Set $B = A[x]/(f)$. By Cayley-Hamilton (Algebra, Lemma 10.15.1) there is a map $B \to \mathrm{End}_A(A^{\oplus n})$ mapping x to φ. For every prime $\mathfrak{p} \supset I$ the image of f in $\kappa(\mathfrak{p})$ is $(x-1)^r x^{n-r}$ where r is the dimension of $\overline{P} \otimes_{A/I} \kappa(\mathfrak{p})$. Hence $(x-1)^n x^n$ maps to zero in $B \otimes_A \kappa(\mathfrak{p})$ for all $\mathfrak{p} \supset I$. Hence the image of $(x-1)^n x^n$ in B is contained in

$$\bigcup\nolimits_{\mathfrak{p} \supset I} \mathfrak{p}B = (\bigcup\nolimits_{\mathfrak{p} \supset I} \mathfrak{p})B = \sqrt{I}B$$

the first equality because B is a free A-module and the second by Algebra, Lemma 10.16.2. Thus $(x-1)^N x^N$ is contained in IB for some N. It follows that $x^N + (1 -$

$x)^N$ is a unit in B/IB and that

$$\overline{e} = \text{image of } \frac{x^N}{x^N + (1-x)^N} \text{ in } B/IB$$

is an idempotent as both assertions hold in $\mathbf{Z}[x]/(x^n(x-1)^N)$. The image of $\overline{e}$ in $\mathrm{End}_{A/I}((A/I)^{\oplus n})$ is

$$\frac{\overline{p}^N}{\overline{p}^N + (1-\overline{p})^N} = \overline{p}$$

as $\overline{p}$ is an idempotent. After replacing A by an étale extension A' as in the lemma, we may assume there exists an idempotent $e \in B$ which maps to $\overline{e}$ in B/IB, see Lemma 15.6.8. Then the image of e under the map

$$B = A[x]/(f) \longrightarrow \mathrm{End}_A(A^{\oplus n}).$$

is an idempotent element p which lifts $\overline{p}$. Setting $P = \mathrm{Im}(p)$ we win. □

Lemma 15.6.10. *Let A be a ring. Let $0 \to K \to A^{\oplus m} \to M \to 0$ be a sequence of A-modules. Consider the A-algebra $C = Sym^*_A(M)$ with its presentation $\alpha : A[y_1, \ldots, y_m] \to C$ coming from the surjection $A^{\oplus m} \to M$. Then*

$$NL(\alpha) = (K \otimes_A C \to \bigoplus\nolimits_{j=1,\ldots,m} C\, dy_j)$$

(see Algebra, Section 10.127) in particular $\Omega_{C/A} = M \otimes_A C$.

Proof. Let $J = \mathrm{Ker}(\alpha)$. The lemma asserts that $J/J^2 \cong K \otimes_A C$. Note that α is a homomorphism of graded algebras. We will prove that in degree d we have $(J/J^2)_d = K \otimes_A C_{d-1}$. Note that

$$J_d = \mathrm{Ker}(\mathrm{Sym}^d_A(A^{\oplus m}) \to \mathrm{Sym}^d_A(M)) = \mathrm{Im}(K \otimes_A \mathrm{Sym}^{d-1}_A(A^{\oplus m}) \to \mathrm{Sym}^d_A(A^{\oplus m})),$$

see Algebra, Lemma 10.12.2. It follows that $(J^2)_d = \sum_{a+b=d} J_a \cdot J_b$ is the image of

$$K \otimes_A K \otimes_A \mathrm{Sym}^{d-2}_A(A^{\otimes m}) \to \mathrm{Sym}^d_A(A^{\oplus m}).$$

The cokernel of the map $K \otimes_A \mathrm{Sym}^{d-2}_A(A^{\otimes m}) \to \mathrm{Sym}^{d-1}_A(A^{\oplus m})$ is $\mathrm{Sym}^{d-1}_A(M)$ by the lemma referenced above. Hence it is clear that $(J/J^2)_d = J_d/(J^2)_d$ is equal to

$$\begin{aligned}\mathrm{Coker}(K \otimes_A K \otimes_A \mathrm{Sym}^{d-2}_A(A^{\otimes m}) \to K \otimes_A \mathrm{Sym}^{d-1}_A(A^{\otimes m})) &= K \otimes_A \mathrm{Sym}^{d-1}_A(M) \\ &= K \otimes_A C_{d-1}\end{aligned}$$

as desired. □

Lemma 15.6.11. *Let A be a ring. Let M be an A-module. Then $C = Sym^*_A(M)$ is smooth over A if and only if M is a finite projective A-module.*

Proof. Let $\sigma : C \to A$ be the projection onto the degree 0 part of C. Then $J = \mathrm{Ker}(\sigma)$ is the part of degree > 0 and we see that $J/J^2 = M$ as an A-module. Hence if $A \to C$ is smooth then M is a finite projective A-module by Algebra, Lemma 10.132.4.

Conversely, assume that M is finite projective and choose a surjection $A^{\oplus n} \to M$ with kernel K. Of course the sequence $0 \to K \to A^{\oplus n} \to M \to 0$ is split as M is projective. In particular we see that K is a finite A-module and hence C is of finite presentation over A as C is a quotient of $A[x_1, \ldots, x_n]$ by the ideal generated by $K \subset \bigoplus Ax_i$. The computation of Lemma 15.6.10 shows that $NL_{C/A}$ is homotopy equivalent to $(K \to M) \otimes_A C$. Hence $NL_{C/A}$ is quasi-isomorphic to $C \otimes_A M$

placed in degree 0 which means that C is smooth over A by Algebra, Definition 10.130.1. □

Lemma 15.6.12. *Let A be a ring, let $I \subset A$ be an ideal. Consider a commutative diagram*

$$\begin{array}{ccc} B & & \\ \uparrow & \searrow & \\ A & \longrightarrow & A/I \end{array}$$

where B is a smooth A-algebra. Then there exists an étale ring map $A \to A'$ which induces an isomorphism $A/I \to A'/IA'$ and an A-algebra map $B \to A'$ lifting the ring map $B \to A/I$.

Proof. Let $J \subset B$ be the kernel of $B \to A/I$ so that $B/J = A/I$. By Algebra, Lemma 10.132.3 the sequence

$$0 \to I/I^2 \to J/J^2 \to \Omega_{B/A} \otimes_B B/J \to 0$$

is split exact. Thus $\overline{P} = J/(J^2 + IB) = \Omega_{B/A} \otimes_B B/J$ is a finite projective A/I-module. Choose an integer n and a direct sum decomposition $A/I^{\oplus n} = \overline{P} \oplus \overline{K}$. By Lemma 15.6.9 we can find an étale ring map $A \to A'$ which induces an isomorphism $A/I \to A'/IA'$ and a finite projective A-module K which lifts $\overline{K}$. We may and do replace A by A'. Set $B' = B \otimes_A \mathrm{Sym}^*_A(K)$. Since $A \to \mathrm{Sym}^*_A(K)$ is smooth by Lemma 15.6.11 we see that $B \to B'$ is smooth which in turn implies that $A \to B'$ is smooth (see Algebra, Lemmas 10.130.4 and 10.130.13). Moreover the section $\mathrm{Sym}^*_A(K) \to A$ determines a section $B' \to B$ and we let $B' \to A/I$ be the composition $B' \to B \to A/I$. Let $J' \subset B'$ be the kernel of $B' \to A/I$. We have $JB' \subset J'$ and $B \otimes_A K \subset J'$. These maps combine to give an isomorphism

$$(A/I)^{\oplus n} \cong J/J^2 \oplus \overline{K} \longrightarrow J'/((J')^2 + IB')$$

Thus, after replacing B by B' we may assume that $J/(J^2 + IB) = \Omega_{B/A} \otimes_B B/J$ is a free A/I-module of rank n.

In this case, choose $f_1, \ldots, f_n \in J$ which map to a basis of $J/(J^2 + IB)$. Consider the finitely presented A-algebra $C = B/(f_1, \ldots, f_n)$. Note that we have an exact sequence

$$0 \to H_1(L_{C/A}) \to (f_1, \ldots, f_n)/(f_1, \ldots, f_n)^2 \to \Omega_{B/A} \otimes_B C \to \Omega_{C/A} \to 0$$

see Algebra, Lemma 10.127.4 (note that $H_1(L_{B/A}) = 0$ and that $\Omega_{B/A}$ is finite projective, in particular flat so the Tor group vanishes). For any prime $\mathfrak{q} \supset J$ of B the module $\Omega_{B/A,\mathfrak{q}}$ is free of rank n because $\Omega_{B/A}$ is finite projective and because $\Omega_{B/A} \otimes_B B/J$ is free of rank n. By our choice of $f_1, \ldots, f_n$ the map

$$\big((f_1, \ldots, f_n)/(f_1, \ldots, f_n)^2\big)_{\mathfrak{q}} \to \Omega_{B/A,\mathfrak{q}}$$

is surjective modulo I. Hence we see that this map of modules over the local ring $C_{\mathfrak{q}}$ has to be an isomorphism. Thus $H_1(L_{C/A})_{\mathfrak{q}} = 0$ and $\Omega_{C/A,\mathfrak{q}} = 0$. By Algebra, Lemma 10.130.12 we see that $A \to C$ is smooth at the prime $\overline{\mathfrak{q}}$ of C corresponding to $\mathfrak{q}$. Since $\Omega_{C/A,\mathfrak{q}} = 0$ it is actually étale at $\overline{\mathfrak{q}}$. Thus $A \to C$ is étale at all primes of C containing JC. By Lemma 15.6.4 we can find an $f \in C$ mapping to an invertible element of C/JC such that $A \to C_f$ is étale. By our choice of f it is still true that $C_f/JC_f = A/I$. The map $C_f/IC_f \to A/I$ is surjective and étale by Algebra,

Lemma 10.136.9. Hence A/I is isomorphic to the localization of C_f/IC_f at some element $g \in C$, see Algebra, Lemma 10.136.10. Set $A' = C_{fg}$ to conclude the proof. □

15.7. Henselian pairs

Some of the results of Section 15.6 may be viewed as results about henselian pairs. In this section a *pair* is a pair (A, I) where A is a ring and $I \subset A$ is an ideal. A *morphism of pairs* $(A, I) \to (B, J)$ is a ring map $\varphi : A \to B$ with $\varphi(I) \subset J$. As in Section 15.6 given an object ξ over A we denote $\overline{\xi}$ the "base change" of ξ to an object over A/I (provided this makes sense).

Definition 15.7.1. A *henselian pair* is a pair (A, I) satisfying

(1) I is contained in the radical ideal of A, and
(2) for any monic polynomial $f \in A[T]$ and factorization $\overline{f} = g_0 h_0$ with $g_0, h_0 \in A/I[T]$ monic generating the unit ideal in $A/I[T]$, there exists a factorization $f = gh$ in $A[T]$ with g, h monic and $g_0 = \overline{g}$ and $h_0 = \overline{h}$.

Observe that if A is a local ring and $I = \mathfrak{m}$ is the maximal ideal, then (A, I) is a henselian pair if and only if A is a henselian local ring, see Algebra, Lemma 10.143.3. In Lemma 15.7.5 we give a number of equivalent characterizations of henselian pairs (and we will add more as time goes on).

Lemma 15.7.2. *Let (A, I) be a pair. If I is contained in the radical ideal of A, then the map from idempotents of A to idempotents of A/I is injective.*

Proof. An idempotent of a local ring is either 0 or 1. Thus an idempotent is determined by the set of maximal ideals where it vanishes, by Algebra, Lemma 10.22.1. □

Lemma 15.7.3. *Let (A, I) be a pair. Let $A \to B$ be an integral ring map such that $B/IB = C_1 \times C_2$ as A/I-algebra with $A/I \to C_1$ injective. Any element $b \in B$ mapping to $(0, 1)$ in B/IB is the zero of a monic polynomial $f \in A[T]$ with $f \bmod I = gT^n$ and $g(0)$ a unit in A/I.*

Proof. Let $b \in B$ map to $(0, 1)$ in $C_1 \times C_2$. Let $J \subset A[T]$ be the kernel of the map $A[T] \to B$, $T \mapsto b$. Since B is integral over A, it is integral over $A[T]$. Hence the image of $\mathrm{Spec}(B)$ in $\mathrm{Spec}(A[T])$ is closed by Algebra, Lemmas 10.38.6 and 10.33.20. Hence this image is equal to $V(J) = \mathrm{Spec}(A[T]/J)$ by Algebra, Lemma 10.28.5. Intersecting with the inverse image of $V(I)$ our choice of b shows we have $V(J + IA[T]) \subset V(T^2 - T)$. Hence there exists an $n \geq 1$ and $g \in J$ with $g \bmod IA[T] = (T^2 - T)^n$. On the other hand, as $A \to B$ is integral there exists a monic polynomial $h \in J$. Note that $h(0) \bmod I$ maps to zero under the composition $A[T] \to B \to B/IB \to C_1$. Since $A/I \to C_1$ is injective we conclude $h \bmod IA[T] = h_0 T$ for some $h_0 \in A/I[T]$. Set

$$f = g + h^m$$

for $m > n$. If m is large enough, this is a monic polynomial and

$$f \bmod IA[T] = (T^2 - T)^n + h_0^m T^m = T^n((T - 1)^n + h_0^m T^{m-n})$$

and hence the desired conclusion. □

Lemma 15.7.4. *Let (A, I) be a pair. Let $A \to B$ be a finite type ring map such that $B/IB = C_1 \times C_2$ with $A/I \to C_1$ finite. Let B' be the integral closure of A in B. Then we can write $B'/IB' = C_1 \times C_2'$ such that the map $B'/IB' \to B/IB$ preserves product decompositions and there exists a $g \in B'$ mapping to $(1, 0)$ in $C_1 \times C_2'$ with $B'_g \to B_g$ an isomorphism.*

Proof. Observe that $A \to B$ is quasi-finite at every prime of the closed subset $T = \mathrm{Spec}(C_1) \subset \mathrm{Spec}(B)$ (this follows by looking at fibre rings, see Algebra, Definition 10.116.3). Consider the diagram of topological spaces

$$\begin{array}{ccc} \mathrm{Spec}(B) & \xrightarrow{\phi} & \mathrm{Spec}(B') \\ & {\scriptstyle\psi}\searrow \quad \swarrow{\scriptstyle\psi'} & \\ & \mathrm{Spec}(A) & \end{array}$$

By Algebra, Theorem 10.117.13 for every $\mathfrak{p} \in T$ there is a $h_\mathfrak{p} \in B'$, $h_\mathfrak{p} \notin \mathfrak{p}$ such that $B'_h \to B_h$ is an isomorphism. The union $U = \bigcup D(h_\mathfrak{p})$ gives an open $U \subset \mathrm{Spec}(B')$ such that $\phi^{-1}(U) \to U$ is a homeomorphism and $T \subset \phi^{-1}(U)$. Since T is open in $\psi^{-1}(V(I))$ we conclude that $\phi(T)$ is open in $U \cap (\psi')^{-1}(V(I))$. Thus $\phi(T)$ is open in $(\psi')^{-1}(V(I))$. On the other hand, since C_1 is finite over A/I it is finite over B'. Hence $\phi(T)$ is a closed subset of $\mathrm{Spec}(B')$ by Algebra, Lemmas 10.38.6 and 10.33.20. We conclude that $\mathrm{Spec}(B'/IB') \supset \phi(T)$ is open and closed. By Algebra, Lemma 10.21.3 we get a corresponding product decomposition $B'/IB' = C_1' \times C_2'$. The map $B'/IB' \to B/IB$ maps C_1' into C_1 and C_2' into C_2 as one sees by looking at what happens on spectra (hint: the inverse image of $\phi(T)$ is exactly T; some details omitted). Pick a $g \in B'$ mapping to $(1, 0)$ in $C_1' \times C_2'$ such that $D(g) \subset U$; this is possible because $\mathrm{Spec}(C_1')$ and $\mathrm{Spec}(C_2')$ are disjoint and closed in $\mathrm{Spec}(B')$ and $\mathrm{Spec}(C_1')$ is contained in U. Then $B'_g \to B_g$ defines a homeomorphism on spectra and an isomorphism on local rings (by our choice of U above). Hence it is an isomorphism, as follows for example from Algebra, Lemma 10.22.1. Finally, it follows that $C_1' = C_1$ and the proof is complete. □

Lemma 15.7.5. *Let (A, I) be a pair. The following are equivalent*

1. *(A, I) is a henselian pair,*
2. *given an étale ring map $A \to A'$ and an A-algebra map $\sigma : A' \to A/I$, there exists an A-algebra map $A' \to A$ lifting σ,*
3. *for any finite A-algebra B the map $B \to B/IB$ induces a bijection on idempotents, and*
4. *for any integral A-algebra B the map $B \to B/IB$ induces a bijection on idempotents.*

Proof. Assume (2) holds. Then I is contained in the radical of A, since otherwise there would be a nonunit $f \in A$ not contained in I and the map $A \to A_f$ would contradict (2). Hence $IB \subset B$ is contained in the radical of B for B integral over A because $\mathrm{Spec}(B) \to \mathrm{Spec}(A)$ is closed by Algebra, Lemmas 10.38.6 and 10.33.20. Thus the map from idempotents of B to idempotents of B/IB is injective by Lemma 15.7.2. On the other hand, since (2) holds, every idempotent of B lifts to an idempotent of B/IB by by Lemma 15.6.8. In this way we see that (2) implies (4).

The implication (4) ⇒ (3) is trivial.

Assume (3). Let $\mathfrak{m}$ be a maximal ideal and consider the finite map $A \to B = A/(I \cap \mathfrak{m})$. The condition that $B \to B/IB$ induces a bijection on idempotents implies that $I \subset \mathfrak{m}$ (if not, then $B = A/I \times A/\mathfrak{m}$ and $B/IB = A/I$). Thus we see that I is contained in the radical ideal of A. Let $f \in A[T]$ be monic and suppose given a factorization $\overline{f} = g_0 h_0$ with $g_0, h_0 \in A/I[T]$ monic. Set $B = A[T]/(f)$. Let $\overline{e}$ be the nontrivial idempotent of B/IB corresponding to the decomposition

$$B/IB = A/I[T]/(g_0) \times A[T]/(h_0)$$

of A-algebras. Let $e \in B$ be an idempotent lifting $\overline{e}$ which exists as we assumed (3). This gives a product decomposition

$$B = eB \times (1-e)B$$

Note that B is free of rank $\deg(f)$ as an A-module. Hence eB and $(1-e)B$ are finite locally free A-modules. However, since eB and $(1-e)B$ have constant rank $\deg(g_0)$ and $\deg(h_0)$ over A/I we find that the same is true over $\mathrm{Spec}(A)$. We conclude that

$$f = \det_A(T : B \to B) = \det_A(T : eB \to eB) \det_A(T : (1-e)B \to (1-e)B)$$

is a factorization into monic polynomials reducing to the given factorization modulo I. Thus (3) implies (1).

Assume (1). Let $A \to A'$ be an étale ring map and let $\sigma : A' \to A/I$ be an A-algebra map. This implies that $A'/IA' = A/I \times C$ for some ring C. Let $A'' \subset A'$ be the integral closure of A in A'. By Lemma 15.7.4 we can write $A''/IA'' = A/I \times C'$ such that $A''/IA'' \to A'/IA'$ maps A/I isomorphically to A'/IA' and C' to C and such that there exists a $a \in A''$ mapping to $(1,0)$ in $A/I \times C'$ such that $A''_a \cong A'_a$. By Lemma 15.7.3 we see that a satisfies a monic polynomial $f \in A[T]$ whose reduction modulo I factors as $\overline{f} = g_0 T^n$ where T, g_0 generate the unit ideal in $A/I[T]$. Thus by assumption we can factor f as $f = gh$ where g is a monic lift of g_0 and h is a monic lift of T^n. Because I is contained in the radical of A, we find that g and h generate the unit ideal in $A[T]$ (details omitted; hint: use that $A[T]/(g,h)$ is finite over A). Thus $A[T]/(f) = A[T]/(h) \times A[T]/(g)$ and we find a corresponding product decomposition $A'' = A''_1 \times A''_2$. By construction we have $A''_1/IA''_1 = A/I$ and $A''_2/IA''_2 = C'$. Since A''_1 is integral over A and I is contained in the radical of A we see that a maps to an invertible element of A''_1. Hence $A''_a = A''_1 \times (A''_2)_a$. It follows that $A \to A''_1$ is integral as well as étale, hence finite locally free. However, $A''_1/IA''_1 = A/I$ thus A''_1 has rank 1 as an A-module along $V(I)$. Since I is contained in the radical of A we conclude that A''_1 has rank 1 everywhere and it follows that $A \to A''_1$ is an isomorphism. Thus $A' \to A'_a \cong A''_a \to (A''_1)_a = A''_1 = A$ is the desired lift of σ. In this way we see that (1) implies (2). □

Lemma 15.7.6. *Let A be a ring. Let $I, J \subset A$ be ideals with $V(I) = V(J)$. Then (A, I) is henselian if and only if (A, J) is henselian.*

Proof. For any integral ring map $A \to B$ we see that $V(IB) = V(JB)$. Hence idempotents of B/IB and B/JB are in bijective correspondence (Algebra, Lemma 10.19.3). It follows that $B \to B/IB$ induces a bijection on sets of idempotents if and only if $B \to B/JB$ induces a bijection on sets of idempotents. Thus we conclude by Lemma 15.7.5. □

Lemma 15.7.7. *Let (A, I) be a henselian pair and let $A \to B$ be an integral ring map. Then (B, IB) is a henselian pair.*

Proof. Immediate from the fourth characterization of henselian pairs in Lemma 15.7.5 and the fact that the composition of integral ring maps is integral. □

Lemma 15.7.8. *Let (A, I) be a henselian pair. Let $\mathfrak{p} \subset A$ be a prime ideal. Then $V(\mathfrak{p} + I)$ is connected.*

Proof. By Lemma 15.7.7 we see that $(A/\mathfrak{p}, I + \mathfrak{p}/\mathfrak{p})$ is a henselian pair. Thus it suffices to prove: If (A, I) is a henselian pair and A is a domain, then $\mathrm{Spec}(A/I) = V(I)$ is connected. If not, then A/I has a nontrivial idempotent, whence by Lemma 15.7.5 A has a nontrivial idempotent. This is a contradiction. □

Lemma 15.7.9. *Let (A, I) be a henselian pair. The functor $B \to B/IB$ determines an equivalence between finite étale A-algebras and finite étale A/I-algebras.*

Proof. Let B, B' be two A-algebras finite étale over A. Then $B' \to B'' = B \otimes_A B'$ is finite étale as well (Algebra, Lemmas 10.136.3 and 10.33.11). Now we have 1-to-1 correspondences between

(1) A-algebra maps $B \to B'$,
(2) sections of $B' \to B''$, and
(3) idempotents e of B'' such that $B' \to B'' \to eB''$ is an isomorphism.

The bijection between (2) and (3) sends $\sigma : B'' \to B'$ to e such that $(1 - e)$ is the idempotent that generates the kernel of σ which exists by Algebra, Lemmas 10.136.9 and 10.136.10. There is a similar correspondence between A/I-algebra maps $B/IB \to B'/IB'$ and idempotents $\overline{e}$ of B''/IB'' such that $B'/IB' \to B''/IB'' \to \overline{e}(B''/IB'')$ is an isomorphism. However every idempotent $\overline{e}$ of B''/IB'' lifts uniquely to an idempotent e of B'' (Lemma 15.7.5). Moreover, if $B''/IB'' \to \overline{e}(B''/IB'')$ is an isomorphism, then $B' \to eB''$ is an isomorphism too by Nakayama's lemma (Algebra, Lemma 10.18.1). In this way we see that the functor is fully faithful.

Essential surjectivity. Let $A/I \to C$ be a finite étale map. By Algebra, Lemma 10.136.11 there exists an étale map $A \to B$ such that $B/IB \cong C$. Let B' be the integral closure of A in B. By Lemma 15.7.4 we have $B'/IB' = C \times C'$ for some ring C' and $B'_g \cong B_g$ for some $g \in B'$ mapping to $(1, 0) \in C \times C'$. Since idempotents lift (Lemma 15.7.5) we get $B' = B'_1 \times B'_2$ with $C = B'_1/IB'_1$ and $C' = B'_2/IB'_2$. The image of g in B'_1 is invertible and $(B'_2)_g = 0$ because IB' is contained in the radical of B' (for example because (B', IB') is a henselian pair by Lemma 15.7.7). We conclude that $B'_1 = B_g$ is finite étale over A and the proof is done. □

Lemma 15.7.10. *The inclusion functor*

$$\textit{category of henselian pairs} \longrightarrow \textit{category of pairs}$$

has a left adjoint $(A, I) \mapsto (A^h, I^h)$.

Proof. Let (A, I) be a pair. Consider the category $\mathcal{C}$ consisting of étale ring maps $A \to B$ such that $A/I \to B/IB$ is an isomorphism. We will show that the category $\mathcal{C}$ is directed and that $A^h = \mathrm{colim}_{B \in \mathcal{C}} B$ with ideal $I^h = IA^h$ gives the desired adjoint.

Suppose that B and B' are two objects of $\mathcal{C}$. Then $B \otimes_A B'$ is another. Suppose that $f, g : B \to B'$ are two maps between objects of $\mathcal{C}$. Then a coequalizer is $B' \otimes_{f,B,g} B'$ which is étale over A by Algebra, Lemmas 10.136.3 and 10.136.9. Thus the category $\mathcal{C}$ is directed.

Next, we show that $A^h = \operatorname{colim}_{B \in \mathcal{C}} B$ with $I^h = IA^h$ is a henselian pair. To do this we will verify one of the equivalent condtions of Lemma 15.7.5. Namely, suppose given an étale ring map $A^h \to A'$ and $\sigma A^h \to A'/I^h A'$. Then there exists a $B \in \mathcal{C}$ and an étale ring map $B \to B'$ such that $A' = B' \otimes_B A^h$. See Algebra, Lemma 10.136.3. Note that $A^h/I^h = A/I = B/IB$, hence $A'/I^h A' = B'/IB'$ and we see that σ induces a map $B \to B'/IB'$.

Let $(A, I) \to (A', I')$ be a morphism of pairs with (A', I') henselian. We will show there is a unique factorization $A \to A^h \to A'$ which will finish the proof. Namely, for each $A \to B$ in $\mathcal{C}$ the ring map $A' \to B' = A' \otimes_A B$ is étale and induces an isomorphism $A'/I' \to B'/I'B'$. Hence there is a section $\sigma_B : B' \to A'$ by Lemma 15.7.5. Given a morphism $B_1 \to B_2$ in $\mathcal{C}$ we claim the diagram

$$\begin{array}{ccc} B_1' & \longrightarrow & B_2' \\ & {\scriptstyle \sigma_{B_1}} \searrow \quad \swarrow {\scriptstyle \sigma_{B_2}} & \\ & A' & \end{array}$$

commutes. This follows once we prove that for every B in $\mathcal{C}$ the section σ_B is the unique A'-algebra map $B' \to A'$. We have $B' \otimes_{A'} B' = B' \times R$ for some ring R, see Algebra, Lemma 10.142.4. In our case $R/I'R = 0$ as $B'/I'B' = A'/I'$. Since I' is contained in the radical of A' we see that $\operatorname{Hom}_{A'\text{-modules}}(R, A') = 0$ for example by Algebra, Lemma 10.22.1. Thus there is at most one A'-algebra map $B' \to A'$. Using the commutativity we obtain

$$A^h = \operatorname{colim}_{B \in \mathcal{C}} B \to \operatorname{colim}_{B \in \mathcal{C}} A' \otimes_A B \xrightarrow{\operatorname{colim} \sigma_B} A'$$

as desired. The uniqueness of the maps σ_B also guarantees that this map is unique. Hence $(A, I) \mapsto (A^h, I^h)$ is the desired adjoint. □

Lemma 15.7.11. *The functor of Lemma 15.7.10 associates to a local ring $(A, \mathfrak{m})$ its henselization.*

Proof. This is clear as the constructions of A^h in Lemma 15.7.10 and Algebra, Lemma 10.143.16 are the same. □

Lemma 15.7.12. *Let $(A, I) = \operatorname{colim}(A_i, I_i)$ be a colimit of pairs. The functor of Lemma 15.7.10 gives $A^h = \operatorname{colim} A_i^h$ and $I^h = \operatorname{colim} I_i^h$.*

Proof. This is true for any left adjoint, see Categories, Lemma 4.24.3. □

15.8. Auto-associated rings

Some of this material is in [**Laz69**].

Definition 15.8.1. A ring R is said to be *auto-associated* if R is local and its maximal ideal $\mathfrak{m}$ is weakly associated to R.

Lemma 15.8.2. *An auto-associated ring R has the following property: (P) Every proper finitely generated ideal $I \subset R$ has a nonzero annihilator.*

Proof. By assumption there exists a nonzero element $x \in R$ such that for every $f \in \mathfrak{m}$ we have $f^n x = 0$. Say $I = (f_1, \ldots, f_r)$. Then x is in the kernel of $R \to \bigoplus R_{f_i}$. Hence we see that there exists a nonzero $y \in R$ such that $f_i y = 0$ for all i, see Algebra, Lemma 10.21.4. As $y \in \text{Ann}_R(I)$ we win. □

Lemma 15.8.3. *Let R be a ring having property (P) of Lemma 15.8.2. Let $u : N \to M$ be a homomorphism of projective R-modules. Then u is universally injective if and only if u is injective.*

Proof. Assume u is injective. Our goal is to show u is universally injective. First we choose a module Q such that $N \oplus Q$ is free. On considering the map $N \oplus Q \to M \oplus Q$ we see that it suffices to prove the lemma in case N is free. In this case N is a directed colimit of finite free R-modules. Thus we reduce to the case that N is a finite free R-module, say $N = R^{\oplus n}$. We prove the lemma by induction on n. The case $n = 0$ is trivial.

Let $u : R^{\oplus n} \to M$ be an injective module map with M projective. Choose an R-module Q such that $M \oplus Q$ is free. After replacing u by the composition $R^{\oplus n} \to M \to M \oplus Q$ we see that we may assume that M is free. Then we can find a direct summand $R^{\oplus m} \subset M$ such that $u(R^{\oplus n}) \subset R^{\oplus m}$. Hence we may assume that $M = R^{\oplus m}$. In this case u is given by a matrix $A = (a_{ij})$ so that $u(x_1, \ldots, x_n) = (\sum x_i a_{i1}, \ldots, \sum x_i a_{im})$. As u is injective, in particular $u(x, 0, \ldots, 0) = (xa_{11}, xa_{12}, \ldots, xa_{1m}) \neq 0$ if $x \neq 0$, and as R has property (P) we see that $a_{11}R + a_{12}R + \ldots + a_{1m}R = R$. Hence see that $R(a_{11}, \ldots, a_{1m}) \subset R^{\oplus m}$ is a direct summand of $R^{\oplus m}$, in particular $R^{\oplus m}/R(a_{11}, \ldots, a_{1m})$ is a projective R-module. We get a commutative diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & R & \longrightarrow & R^{\oplus n} & \longrightarrow & R^{\oplus n-1} & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle 1} & & \downarrow{\scriptstyle u} & & \downarrow & & \\ 0 & \longrightarrow & R & \xrightarrow{(a_{11},\ldots,a_{1m})} & R^{\oplus m} & \longrightarrow & R^{\oplus m}/R(a_{11}, \ldots, a_{1m}) & \longrightarrow & 0 \end{array}$$

with split exact rows. Thus the right vertical arrow is injective and we may apply the induction hypothesis to conclude that the right vertical arrow is universally injective. It follows that the middle vertical arrow is universally injective. □

Lemma 15.8.4. *Let R be a ring. The following are equivalent*

1. *R has property (P) of Lemma 15.8.2,*
2. *any injective map of projective R-modules is universally injective,*
3. *if $u : N \to M$ is injective and N, M are finite projective R-modules then Coker(u) is a finite projective R-module,*
4. *if $N \subset M$ and N, M are finite projective as R-modules, then N is a direct summand of M, and*
5. *any injective map $R \to R^{\oplus n}$ is a split injection.*

Proof. The implication (1) ⇒ (2) is Lemma 15.8.3. It is clear that (3) and (4) are equivalent. We have (2) ⇒ (3), (4) by Algebra, Lemma 10.77.4. Part (5) is a special case of (4). Assume (5). Let $I = (a_1, \ldots, a_n)$ be a proper finitely generated ideal of R. As $I \neq R$ we see that $R \to R^{\oplus n}$, $x \mapsto (xa_1, \ldots, xa_n)$ is not a split injection. Hence it has a nonzero kernel and we conclude that $\text{Ann}_R(I) \neq 0$. Thus (1) holds. □

Example 15.8.5. If the equivalent conditions of Lemma 15.8.4 hold, then it is not always the case that every injective map of free R-modules is a split injection. For example suppose that $R = k[x_1, x_2, x_3, \ldots]/(x_i^2)$. This is an auto-associated ring. Consider the map of free R-modules

$$u : \bigoplus\nolimits_{i\geq 1} Re_i \longrightarrow \bigoplus\nolimits_{i\geq 1} Rf_i, \quad e_i \longmapsto f_i - x_i f_{i+1}.$$

For any integer n the restriction of u to $\bigoplus_{i=1,\ldots,n} Re_i$ is injective as the images $u(e_1), \ldots, u(e_n)$ are R-linearly independent. Hence u is injective and hence universally injective by the lemma. Since $u \otimes \mathrm{id}_k$ is bijective we see that if u were a split injection then u would be surjective. But u is not surjective because the inverse image of f_1 would be the element

$$\sum\nolimits_{i\geq 0} x_1 \ldots x_i e_{i+1} = e_1 + x_1 e_2 + x_1 x_2 e_3 + \ldots$$

which is not an element of the direct sum. A side remark is that $\mathrm{Coker}(u)$ is a flat (because u is universally injective), countably generated R-module which is not projective (as u is not split), hence not Mittag-Leffler (see Algebra, Lemma 10.88.1).

15.9. Flattening stratification

Let $R \to S$ be a ring map and let N be an S-module. For any R-algebra R' we can consider the base changes $S' = S \otimes_R R'$ and $M' = M \otimes_R R'$. We say $R \to R'$ *flattens* M if the module M' is flat over R'. We would like to understand the structure of the collection of ring maps $R \to R'$ which flatten M. In particular we would like to know if there exists a *universal flattening* $R \to R_{univ}$ *of* M, i.e., a ring map $R \to R_{univ}$ which flattens M and has the property that any ring map $R \to R'$ which flattens M factors through $R \to R_{univ}$. It turns out that such a universal solution usually does not exist.

We will discuss *universal flattenings* and *flattening stratifications* in a scheme theoretic setting $\mathcal{F}/X/S$ in More on Flatness, Section 37.20. If the universal flattening $R \to R_{univ}$ exists then the morphism of schemes $\mathrm{Spec}(R_{univ}) \to \mathrm{Spec}(R)$ is the universal flattening of the quasi-coherent module $\widetilde{M}$ on $\mathrm{Spec}(S)$.

In this and the next few sections we prove some basic algebra facts related to this. The most basic result is perhaps the following.

Lemma 15.9.1. *Let R be a ring. Let M be an R-module. Let I_1, I_2 be ideals of R. If M/I_1M is flat over R/I_1 and M/I_2M is flat over R/I_2, then $M/(I_1 \cap I_2)M$ is flat over $R/(I_1 \cap I_2)$.*

Proof. By replacing R with $R/(I_1 \cap I_2)$ and M by $M/(I_1 \cap I_2)M$ we may assume that $I_1 \cap I_2 = 0$. Let $J \subset R$ be an ideal. To prove that M is flat over R we have to show that $J \otimes_R M \to M$ is injective, see Algebra, Lemma 10.36.4. By flatness of M/I_1M over R/I_1 the map

$$J/(J \cap I_1) \otimes_R M = (J + I_1)/I_1 \otimes_{R/I_1} M/I_1M \longrightarrow M/I_1M$$

is injective. As $0 \to (J \cap I_1) \to J \to J/(J \cap I_1) \to 0$ is exact we obtain a diagram

$$\begin{array}{ccccccc} (J \cap I_1) \otimes_R M & \longrightarrow & J \otimes_R M & \longrightarrow & J/(J \cap I_1) \otimes_R M & \longrightarrow & 0 \\ \downarrow & & \downarrow & & \downarrow & & \\ M & = & M & \longrightarrow & M/I_1M & & \end{array}$$

hence it suffices to show that $(J \cap I_1) \otimes_R M \to M$ is injective. Since $I_1 \cap I_2 = 0$ the ideal $J \cap I_1$ maps isomorphically to an ideal $J' \subset R/I_2$ and we see that $(J \cap I_1) \otimes_R M = J' \otimes_{R/I_2} M/I_2M$. By flatness of M/I_2M over R/I_2 the map $J' \otimes_{R/I_2} M/I_2M \to M/I_2M$ is injective, which clearly implies that $(J \cap I_1) \otimes_R M \to M$ is injective. □

15.10. Flattening over an Artinian ring

A universal flattening exists when the base ring is an Artinian local ring. It exists for an arbitrary module. Hence, as we will see later, a flatting stratification exists when the base scheme is the spectrum of an Artinian local ring.

Lemma 15.10.1. *Let R be an Artinian ring. Let M be an R-module. Then there exists a smallest ideal $I \subset R$ such that M/IM is flat over R/I.*

Proof. This follows directly from Lemma 15.9.1 and the Artinian property. □

This ideal has the following universal property.

Lemma 15.10.2. *Let R be an Artinian ring. Let M be an R-module. Let $I \subset R$ be the smallest ideal $I \subset R$ such that M/IM is flat over R/I. Then I has the following universal property: For every ring map $\varphi : R \to R'$ we have*

$$R' \otimes_R M \text{ is flat over } R' \Leftrightarrow \text{we have } \varphi(I) = 0.$$

Proof. Note that I exists by Lemma 15.10.1. The implication $\Rightarrow$ follows from Algebra, Lemma 10.36.6. Let $\varphi : R \to R'$ be such that $M \otimes_R R'$ is flat over R'. Let $J = \mathrm{Ker}(\varphi)$. By Algebra, Lemma 10.95.7 and as $R' \otimes_R M = R' \otimes_{R/J} M/JM$ is flat over R' we conclude that M/JM is flat over R/J. Hence $I \subset J$ as desired. □

15.11. Flattening over a closed subset of the base

Let $R \to S$ be a ring map. Let $I \subset R$ be an ideal. Let M be an S-module. In the following we will consider the following condition

$$\forall \mathfrak{q} \in V(IS) \subset \mathrm{Spec}(S) : M_\mathfrak{q} \text{ is flat over } R. \tag{15.11.0.1}$$

Geometrically, this means that M is flat over R along the inverse image of $V(I)$ in $\mathrm{Spec}(S)$. If R and S are Noetherian rings and M is a finite S-module, then (15.11.0.1) is equivalent to the condition that M/I^nM is flat over R/I^n for all $n \geq 1$, see Algebra, Lemma 10.93.10.

Lemma 15.11.1. *Let $R \to S$ be a ring map. Let $I \subset R$ be an ideal. Let M be an S-module. Let $R \to R'$ be a ring map and $IR' \subset I' \subset R'$ an ideal. If (15.11.0.1) holds for $(R \to S, I, M)$, then (15.11.0.1) holds for $(R' \to S \otimes_R R', I', M \otimes_R R')$.*

Proof. Assume (15.11.0.1) holds for $(R \to S, I \subset R, M)$. Let $I'(S \otimes_R R') \subset \mathfrak{q}'$ be a prime of $S \otimes_R R'$. Let $\mathfrak{q} \subset S$ be the corresponding prime of S. Then $IS \subset \mathfrak{q}$. Note that $(M \otimes_R R')_{\mathfrak{q}'}$ is a localization of the base change $M_{\mathfrak{q}} \otimes_R R'$. Hence $(M \otimes_R R')_{\mathfrak{q}'}$ is flat over R' as a localization of a flat module, see Algebra, Lemmas 10.36.6 and 10.36.19. □

Lemma 15.11.2. *Let $R \to S$ be a ring map. Let $I \subset R$ be an ideal. Let M be an S-module. Let $R \to R'$ be a ring map and $IR' \subset I' \subset R'$ an ideal such that*

1. *the map $V(I') \to V(I)$ induced by $\mathrm{Spec}(R') \to \mathrm{Spec}(R)$ is surjective, and*
2. *$R'_{\mathfrak{p}'}$ is flat over R for all primes $\mathfrak{p}' \in V(I')$.*

If (15.11.0.1) holds for $(R' \to S \otimes_R R', I', M \otimes_R R')$, then (15.11.0.1) holds for $(R \to S, I, M)$.

Proof. Assume (15.11.0.1) holds for $(R' \to S \otimes_R R', IR', M \otimes_R R')$. Pick a prime $IS \subset \mathfrak{q} \subset S$. Let $I \subset \mathfrak{p} \subset R$ be the corresponding prime of R. By assumption there exists a prime $\mathfrak{p}' \in V(I')$ of R' lying over $\mathfrak{p}$ and $R_{\mathfrak{p}} \to R'_{\mathfrak{p}'}$ is flat. Choose a prime $\overline{\mathfrak{q}}' \subset \kappa(\mathfrak{q}) \otimes_{\kappa(\mathfrak{p})} \kappa(\mathfrak{p}')$ which corresponds to a prime $\mathfrak{q}' \subset S \otimes_R R'$ which lies over $\mathfrak{q}$ and over $\mathfrak{p}'$. Note that $(S \otimes_R R')_{\mathfrak{q}'}$ is a localization of $S_{\mathfrak{q}} \otimes_{R_{\mathfrak{p}}} R'_{\mathfrak{p}'}$. By assumption the module $(M \otimes_R R')_{\mathfrak{q}'}$ is flat over $R'_{\mathfrak{p}'}$. Hence Algebra, Lemma 10.94.1 implies that $M_{\mathfrak{q}}$ is flat over $R_{\mathfrak{p}}$ which is what we wanted to prove. □

Lemma 15.11.3. *Let $R \to S$ be a ring map of finite presentation. Let M be an S-module of finite presentation. Let $R' = \mathrm{colim}_{\lambda \in \Lambda} R_\lambda$ be a directed colimit of R-algebras. Let $I_\lambda \subset R_\lambda$ be ideals such that $I_\lambda R_\mu \subset I_\mu$ for all $\mu \geq \lambda$ and set $I' = \mathrm{colim}_\lambda I_\lambda$. If (15.11.0.1) holds for $(R' \to S \otimes_R R', I', M \otimes_R R')$, then there exists a $\lambda \in \Lambda$ such that (15.11.0.1) holds for $(R_\lambda \to S \otimes_R R_\lambda, I_\lambda, M \otimes_R R_\lambda)$.*

Proof. We are going to write $S_\lambda = S \otimes_R R_\lambda$, $S' = S \otimes_R R'$, $M_\lambda = M \otimes_R R_\lambda$, and $M' = M \otimes_R R'$. The base change S' is of finite presentation over R' and M' is of finite presentation over S' and similarly for the versions with subscript λ, see Algebra, Lemma 10.13.2. By Algebra, Theorem 10.123.4 the set

$$U' = \{\mathfrak{q}' \in \mathrm{Spec}(S') \mid M'_{\mathfrak{q}'} \text{ is flat over } R'\}$$

is open in $\mathrm{Spec}(S')$. Note that $V(I'S')$ is a quasi-compact space which is contained in U' by assumption. Hence there exist finitely many $g'_j \in S'$, $j = 1, \ldots, m$ such that $D(g'_j) \subset U'$ and such that $V(I'S') \subset \bigcup D(g'_j)$. Note that in particular $(M')_{g'_j}$ is a flat module over R'.

We are going to pick increasingly large elements $\lambda \in \Lambda$. First we pick it large enough so that we can find $g_{j,\lambda} \in S_\lambda$ mapping to g'_j. The inclusion $V(I'S') \subset \bigcup D(g'_j)$ means that $I'S' + (g'_1, \ldots, g'_m) = S'$ which can be expressed as $1 = \sum z_s h_s + \sum f_j g'_j$ for some $z_s \in I'$, $h_s, f_j \in S'$. After increasing λ we may assume such an equation holds in S_λ. Hence we may assume that $V(I_\lambda S_\lambda) \subset \bigcup D(g_{j,\lambda})$. By Algebra, Lemma 10.154.1 we see that for some sufficiently large λ the modules $(M_\lambda)_{g_{j,\lambda}}$ are flat over R_λ. In particular the module M_λ is flat over R_λ at all the primes lying over the ideal I_λ. □

15.12. Flattening over a closed subsets of source and base

In this section we slightly generalize the discussion in Section 15.11. We strongly suggest the reader first read and understand that section.

Situation 15.12.1. Let $R \to S$ be a ring map. Let $J \subset S$ be an ideal. Let M be an S-module.

In this situation, given an R-algebra R' and an ideal $I' \subset R'$ we set $S' = S \otimes_R R'$ and $M' = M \otimes_R R'$. We will consider the condition

$$\forall \mathfrak{q}' \in V(I'S' + JS') \subset \operatorname{Spec}(S') : M'_{\mathfrak{q}'} \text{ is flat over } R'. \tag{15.12.1.1}$$

Geometrically, this means that M' is flat over R' along the intersection of the inverse image of $V(I')$ with the inverse image of $V(J)$. Since $(R \to S, J, M)$ are fixed, condition (15.12.1.1) only depends on the pair (R', I') where R' is viewed as an R-algebra.

Lemma 15.12.2. *In Situation 15.12.1 let $R' \to R''$ be an R-algebra map. Let $I' \subset R'$ and $I'R'' \subset I'' \subset R''$ be ideals. If (15.12.1.1) holds for (R', I'), then (15.12.1.1) holds for (R'', I'').*

Proof. Assume (15.12.1.1) holds for (R', I'). Let $I''S'' + JS'' \subset \mathfrak{q}''$ be a prime of S''. Let $\mathfrak{q}' \subset S'$ be the corresponding prime of S'. Then both $I'S' \subset \mathfrak{q}'$ and $JS' \subset \mathfrak{q}'$ because the corresponding conditions hold for $\mathfrak{q}''$. Note that $(M'')_{\mathfrak{q}''}$ is a localization of the base change $M'_{\mathfrak{q}'} \otimes_R R''$. Hence $(M'')_{\mathfrak{q}''}$ is flat over R'' as a localization of a flat module, see Algebra, Lemmas 10.36.6 and 10.36.19. □

Lemma 15.12.3. *In Situation 15.12.1 let $R' \to R''$ be an R-algebra map. Let $I' \subset R'$ and $I'R'' \subset I'' \subset R''$ be ideals. Assume*

1. *the map $V(I'') \to V(I')$ induced by $\operatorname{Spec}(R'') \to \operatorname{Spec}(R')$ is surjective, and*
2. *$R''_{\mathfrak{p}''}$ is flat over R' for all primes $\mathfrak{p}'' \in V(I'')$.*

If (15.12.1.1) holds for (R'', I''), then (15.12.1.1) holds for (R', I').

Proof. Assume (15.12.1.1) holds for (R'', I''). Pick a prime $I'S' + JS' \subset \mathfrak{q}' \subset S'$. Let $I' \subset \mathfrak{p}' \subset R'$ be the corresponding prime of R'. By assumption there exists a prime $\mathfrak{p}'' \in V(I'')$ of R'' lying over $\mathfrak{p}'$ and $R'_{\mathfrak{p}'} \to R''_{\mathfrak{p}''}$ is flat. Choose a prime $\overline{\mathfrak{q}}'' \subset \kappa(\mathfrak{q}') \otimes_{\kappa(\mathfrak{p}')} \kappa(\mathfrak{p}'')$. This corresponds to a prime $\mathfrak{q}'' \subset S'' = S' \otimes_{R'} R''$ which lies over $\mathfrak{q}'$ and over $\mathfrak{p}''$. In particular we see that $I''S'' \subset \mathfrak{q}''$ and that $JS'' \subset \mathfrak{q}''$. Note that $(S' \otimes_{R'} R'')_{\mathfrak{q}''}$ is a localization of $S'_{\mathfrak{q}'} \otimes_{R'_{\mathfrak{p}'}} R''_{\mathfrak{p}''}$. By assumption the module $(M' \otimes_{R'} R'')_{\mathfrak{q}''}$ is flat over $R''_{\mathfrak{p}''}$. Hence Algebra, Lemma 10.94.1 implies that $M'_{\mathfrak{q}'}$ is flat over $R'_{\mathfrak{p}'}$ which is what we wanted to prove. □

Lemma 15.12.4. *In Situation 15.12.1 assume $R \to S$ is essentially of finite presentation and M is an S-module of finite presentation. Let $R' = \operatorname{colim}_{\lambda \in \Lambda} R_\lambda$ be a directed colimit of R-algebras. Let $I_\lambda \subset R_\lambda$ be ideals such that $I_\lambda R_\mu \subset I_\mu$ for all $\mu \geq \lambda$ and set $I' = \operatorname{colim}_\lambda I_\lambda$. If (15.12.1.1) holds for (R', I'), then there exists a $\lambda \in \Lambda$ such that (15.12.1.1) holds for (R_λ, I_λ).*

Proof. We first prove the lemma in case $R \to S$ is of finite presentation and then we explain what needs to be changed in the general case. We are going to write $S_\lambda = S \otimes_R R_\lambda$, $S' = S \otimes_R R'$, $M_\lambda = M \otimes_R R_\lambda$, and $M' = M \otimes_R R'$. The base change S' is of finite presentation over R' and M' is of finite presentation over S' and similarly for the versions with subscript λ, see Algebra, Lemma 10.13.2. By Algebra, Theorem 10.123.4 the set

$$U' = \{\mathfrak{q}' \in \operatorname{Spec}(S') \mid M'_{\mathfrak{q}'} \text{ is flat over } R'\}$$

is open in $\mathrm{Spec}(S')$. Note that $V(I'S' + JS')$ is a quasi-compact space which is contained in U' by assumption. Hence there exist finitely many $g'_j \in S'$, $j = 1, \ldots, m$ such that $D(g'_j) \subset U'$ and such that $V(I'S' + JS') \subset \bigcup D(g'_j)$. Note that in particular $(M')_{g'_j}$ is a flat module over R'.

We are going to pick increasingly large elements $\lambda \in \Lambda$. First we pick it large enough so that we can find $g_{j,\lambda} \in S_\lambda$ mapping to g'_j. The inclusion $V(I'S'+JS') \subset \bigcup D(g'_j)$ means that $I'S' + JS' + (g'_1, \ldots, g'_m) = S'$ which can be expressed as

$$1 = \sum y_t k_t + \sum z_s h_s + \sum f_j g'_j$$

for some $z_s \in I'$, $y_t \in J$, $k_t, h_s, f_j \in S'$. After increasing λ we may assume such an equation holds in S_λ. Hence we may assume that $V(I_\lambda S_\lambda + JS_\lambda) \subset \bigcup D(g_{j,\lambda})$. By Algebra, Lemma 10.154.1 we see that for some sufficiently large λ the modules $(M_\lambda)_{g_{j,\lambda}}$ are flat over R_λ. In particular the module M_λ is flat over R_λ at all the primes corresponding to points of $V(I_\lambda S_\lambda + JS_\lambda)$.

In the case that S is essentially of finite presentation, we can write $S = \Sigma^{-1}C$ where $R \to C$ is of finite presentation and $\Sigma \subset C$ is a multiplicative subset. We can also write $M = \Sigma^{-1}N$ for some finitely presented C-module N, see Algebra, Lemma 10.120.3. At this point we introduce C_λ, C', N_λ, N'. Then in the discussion above we obtain an open $U' \subset \mathrm{Spec}(C')$ over which N' is flat over R'. The assumption that (15.12.1.1) is true means that $V(I'S'+JS')$ maps into U', because for a prime $\mathfrak{q}' \subset S'$, corresponding to a prime $\mathfrak{r}' \subset C'$ we have $M'_{\mathfrak{q}'} = N'_{\mathfrak{r}'}$. Thus we can find $g'_j \in C'$ such that $\bigcup D(g'_j)$ contains the image of $V(I'S' + JS')$. The rest of the proof is exactly the same as before. □

Lemma 15.12.5. *In Situation 15.12.1. Let $I \subset R$ be an ideal. Assume*

(1) *R is a Noetherian ring,*
(2) *S is a Noetherian ring,*
(3) *M is a finite S-module, and*
(4) *for each $n \geq 1$ and any prime $\mathfrak{q} \in V(J + IS)$ the module $(M/I^nM)_\mathfrak{q}$ is flat over R/I^n.*

Then (15.12.1.1) holds for (R, I), i.e., for every prime $\mathfrak{q} \in V(J+IS)$ the localization $M_\mathfrak{q}$ is flat over R.

Proof. Let $\mathfrak{q} \in V(J + IS)$. Then Algebra, Lemma 10.93.10 applied to $R \to S_\mathfrak{q}$ and $M_\mathfrak{q}$ implies that $M_\mathfrak{q}$ is flat over R. □

15.13. Flattening over a Noetherian complete local ring

The following three lemmas give a completely algebraic proof of the existence of the "local" flattening stratification when the base is a complete local Noetherian ring R and the given module is finite over a finite type R-algebra S.

Lemma 15.13.1. *Let $R \to S$ be a ring map. Let M be an S-module. Assume*

(1) *$(R, \mathfrak{m})$ is a complete local Noetherian ring,*
(2) *S is a Noetherian ring, and*
(3) *M is finite over S.*

Then there exists an ideal $I \subset \mathfrak{m}$ such that

(1) *$(M/IM)_\mathfrak{q}$ is flat over R/I for all primes $\mathfrak{q}$ of S/IS lying over $\mathfrak{m}$, and*

(2) *if $J \subset R$ is an ideal such that $(M/JM)_\mathfrak{q}$ is flat over R/J for all primes $\mathfrak{q}$ lying over $\mathfrak{m}$, then $I \subset J$.*

In other words, I is the smallest ideal of R such that (15.11.0.1) holds for $(\overline{R} \to \overline{S}, \overline{\mathfrak{m}}, \overline{M})$ where $\overline{R} = R/I$, $\overline{S} = S/IS$, $\overline{\mathfrak{m}} = \mathfrak{m}/I$ and $\overline{M} = M/IM$.

Proof. Let $J \subset R$ be an ideal. Apply Algebra, Lemma 10.93.10 to the module M/JM over the ring R/J. Then we see that $(M/JM)_\mathfrak{q}$ is flat over R/J for all primes $\mathfrak{q}$ of S/JS if and only if $M/(J+\mathfrak{m}^n)M$ is flat over $R/(J+\mathfrak{m}^n)$ for all $n \geq 1$. We will use this remark below.

For every $n \geq 1$ the local ring $R/\mathfrak{m}^n$ is Artinian. Hence, by Lemma 15.10.1 there exists a smallest ideal $I_n \supset \mathfrak{m}^n$ such that M/I_nM is flat over R/I_n. It is clear that $I_{n+1} + \mathfrak{m}^n$ is contains I_n and applying Lemma 15.9.1 we see that $I_n = I_{n+1} + \mathfrak{m}^n$. Since $R = \lim_n R/\mathfrak{m}^n$ we see that $I = \lim_n I_n/\mathfrak{m}^n$ is an ideal in R such that $I_n = I + \mathfrak{m}^n$ for all $n \geq 1$. By the initial remarks of the proof we see that I verifies (1) and (2). Some details omitted. □

Lemma 15.13.2. *With notation $R \to S$, M, and I and assumptions as in Lemma 15.13.1. Consider a local homomorphism of local rings $\varphi : (R, \mathfrak{m}) \to (R', \mathfrak{m}')$ such that R' is Noetherian. Then the following are equivalent*

(1) *condition (15.11.0.1) holds for $(R' \to S \otimes_R R', \mathfrak{m}', M \otimes_R R')$, and*
(2) *$\varphi(I) = 0$.*

Proof. The implication (2) $\Rightarrow$ (1) follows from Lemma 15.11.1. Let $\varphi : R \to R'$ be as in the lemma satisfying (1). We have to show that $\varphi(I) = 0$. This is equivalent to the condition that $\varphi(I)R' = 0$. By Artin-Rees in the Noetherian local ring R' (see Algebra, Lemma 10.47.6) this is equivalent to the condition that $\varphi(I)R' + (\mathfrak{m}')^n = (\mathfrak{m}')^n$ for all $n > 0$. Hence this is equivalent to the condition that the composition $\varphi_n : R \to R' \to R'/(\mathfrak{m}')^n$ annihilates I for each n. Now assumption (1) for φ implies assumption (1) for φ_n by Lemma 15.11.1. This reduces us to the case where R' is Artinian local.

Assume R' Artinian. Let $J = \mathrm{Ker}(\varphi)$. We have to show that $I \subset J$. By the construction of I in Lemma 15.13.1 it suffices to show that $(M/JM)_\mathfrak{q}$ is flat over R/J for every prime $\mathfrak{q}$ of S/JS lying over $\mathfrak{m}$. As R' is Artinian, condition (1) signifies that $M \otimes_R R'$ is flat over R'. As R' is Artinian and $R/J \to R'$ is a local injective ring map, it follows that R/J is Artinian too. Hence the flatness of $M \otimes_R R' = M/JM \otimes_{R/J} R'$ over R' implies that M/JM is flat over R/J by Algebra, Lemma 10.95.7. This concludes the proof. □

Lemma 15.13.3. *With notation $R \to S$, M, and I and assumptions as in Lemma 15.13.1. In addition assume that $R \to S$ is of finite type. Then for any local homomorphism of local rings $\varphi : (R, \mathfrak{m}) \to (R', \mathfrak{m}')$ the following are equivalent*

(1) *condition (15.11.0.1) holds for $(R' \to S \otimes_R R', \mathfrak{m}', M \otimes_R R')$, and*
(2) *$\varphi(I) = 0$.*

Proof. The implication (2) $\Rightarrow$ (1) follows from Lemma 15.11.1. Let $\varphi : R \to R'$ be as in the lemma satisfying (1). As R is Noetherian we see that $R \to S$ is of finite presentation and M is an S-module of finite presentation. Write $R' = \mathrm{colim}_\lambda R_\lambda$ as a directed colimit of local R-subalgebras $R_\lambda \subset R'$, with maximal ideals $\mathfrak{m}_\lambda = R_\lambda \cap \mathfrak{m}'$ such that each R_λ is essentially of finite type over R. By Lemma 15.11.3 we see that condition (15.11.0.1) holds for $(R_\lambda \to S \otimes_R R_\lambda, \mathfrak{m}_\lambda, M \otimes_R R_\lambda)$ for some λ.

Hence Lemma 15.13.2 applies to the ring map $R \to R_\lambda$ and we see that I maps to zero in R_λ, a fortiori it maps to zero in R'. □

15.14. Descent flatness along integral maps

First a few simple lemmas.

Lemma 15.14.1. *Let R be a ring. Let $P(T)$ be a monic polynomial with coefficients in R. If there exists an $\alpha \in R$ such that $P(\alpha) = 0$, then $P(T) = (T - \alpha)Q(T)$ for some monic polynomial $Q(T) \in R[T]$.*

Proof. By induction on the degree of P. If $\deg(P) = 1$, then $P(T) = T - \alpha$ and the result is true. If $\deg(P) > 1$, then we can write $P(T) = (T - \alpha)Q(T) + r$ for some polynomial $Q \in R[T]$ of degree $< \deg(P)$ and some $r \in R$ by long division. By assumption $0 = P(\alpha) = (\alpha - \alpha)Q(\alpha) + r = r$ and we conclude that $r = 0$ as desired. □

Lemma 15.14.2. *Let R be a ring. Let $P(T)$ be a monic polynomial with coefficients in R. There exists a finite free ring map $R \to R'$ such that $P(T) = (T - \alpha)Q(T)$ for some $\alpha \in R'$ and some monic polynomial $Q(T) \in R'[T]$.*

Proof. Write $P(T) = T^d + a_1T^{d-1} + \ldots + a_0$. Set $R' = R[x]/(x^d + a_1x^{d-1} + \ldots + a_0)$. Set α equal to the congruence class of x. Then it is clear that $P(\alpha) = 0$. Thus we win by Lemma 15.14.1. □

Lemma 15.14.3. *Let $R \to S$ be a finite ring map. There exists a finite free ring extension $R \subset R'$ such that $S \otimes_R R'$ is a quotient of a ring of the form*

$$R'[T_1, \ldots, T_n]/(P_1(T_1), \ldots, P_n(T_n))$$

with $P_i(T) = \prod_{j=1,\ldots,d_i}(T - \alpha_{ij})$ for some $\alpha_{ij} \in R'$.

Proof. Let $x_1, \ldots, x_n \in S$ be generators of S over R. For each i we can choose a monic polynomial $P_i(T) \in R[T]$ such that $P(x_i) = 0$ in S, see Algebra, Lemma 10.33.3. Say $\deg(P_i) = d_i$. By Lemma 15.14.2 (applied $\sum d_i$ times) there exists a finite free ring extension $R \subset R'$ such that each P_i splits completely:

$$P_i(T) = \prod\nolimits_{j=1,\ldots,d_i}(T - \alpha_{ij})$$

for certain $\alpha_{ik} \in R'$. Let $R'[T_1, \ldots, T_n] \to S \otimes_R R'$ be the R'-algebra map which maps T_i to $x_i \otimes 1$. As this maps $P_i(T_i)$ to zero, this induces the desired surjection. □

Lemma 15.14.4. *Let R be a ring. Let $S = R[T_1, \ldots, T_n]/J$. Assume J contains elements of the form $P_i(T_i)$ with $P_i(T) = \prod_{j=1,\ldots,d_i}(T - \alpha_{ij})$ for some $\alpha_{ij} \in R$. For $\underline{k} = (k_1, \ldots, k_n)$ with $1 \leq k_i \leq d_i$ consider the ring map*

$$\Phi_{\underline{k}} : R[T_1, \ldots, T_n] \to R, \quad T_i \longmapsto \alpha_{ik_i}$$

Set $J_{\underline{k}} = \Phi_{\underline{k}}(J)$. Then the image of $\operatorname{Spec}(S) \to \operatorname{Spec}(R)$ is equal to $V(\bigcap J_{\underline{k}})$.

Proof. This lemma proves itself. Hint: $V(\bigcap J_{\underline{k}}) = \bigcup V(J_{\underline{k}})$. □

The following result is due to Ferrand, see [**Fer69**].

Lemma 15.14.5. *Let $R \to S$ be a finite injective homomorphism of Noetherian rings. Let M be an R-module. If $M \otimes_R S$ is a flat S-module, then M is a flat R-module.*

Proof. Let M be an R-module such that $M \otimes_R S$ is flat over S. By Algebra, Lemma 10.36.7 in order to prove that M is flat we may replace R by any faithfully flat ring extension. By Lemma 15.14.3 we can find a finite locally free ring extension $R \subset R'$ such that $S' = S \otimes_R R' = R'[T_1, \ldots, T_n]/J$ for some ideal $J \subset R'[T_1, \ldots, T_n]$ which contains the elements of the form $P_i(T_i)$ with $P_i(T) = \prod_{j=1,\ldots,d_i}(T - \alpha_{ij})$ for some $\alpha_{ij} \in R'$. Note that R' is Noetherian and that $R' \subset S'$ is a finite extension of rings. Hence we may replace R by R' and assume that S has a presentation as in Lemma 15.14.4. Note that $\mathrm{Spec}(S) \to \mathrm{Spec}(R)$ is surjective, see Algebra, Lemma 10.33.15. Thus, using Lemma 15.14.4 we conclude that $I = \bigcap J_{\underline{k}}$ is an ideal such that $V(I) = \mathrm{Spec}(R)$. This means that $I \subset \sqrt{(0)}$, and since R is Noetherian that I is nilpotent. The maps $\Phi_{\underline{k}}$ induce commutative diagrams

$$\begin{array}{ccc} S & \longrightarrow & R/J_{\underline{k}} \\ & \nwarrow \quad \nearrow & \\ & R & \end{array}$$

from which we conclude that $M/J_{\underline{k}}M$ is flat over $R/J_{\underline{k}}$. By Lemma 15.9.1 we see that M/IM is flat over R/I. Finally, applying Algebra, Lemma 10.95.5 we conclude that M is flat over R. □

Lemma 15.14.6. *Let $R \to S$ be an injective integral ring map. Let M be a finitely presented module over $R[x_1, \ldots, x_n]$. If $M \otimes_R S$ is flat over S, then M is flat over R.*

Proof. Choose a presentation

$$R[x_1, \ldots, x_n]^{\oplus t} \to R[x_1, \ldots, x_n]^{\oplus r} \to M \to 0.$$

Let's say that the first map is given by the $r \times t$-matrix $T = (f_{ij})$ with $f_{ij} \in R[x_1, \ldots, x_n]$. Write $f_{ij} = \sum f_{ij,I} x^I$ with $f_{ij,I} \in R$ (multi-index notation). Consider diagrams

$$\begin{array}{ccc} R & \longrightarrow & S \\ \uparrow & & \uparrow \\ R_\lambda & \longrightarrow & S_\lambda \end{array}$$

where R_λ is a finitely generated $\mathbf{Z}$-subalgebra of R containing all $f_{ij,I}$ and S_λ is a finite R_λ-subalgebra of S. Let M_λ be the finite $R_\lambda[x_1, \ldots, x_n]$-module defined by a presentation as above, using the same matrix T but now viewed as a matrix over $R_\lambda[x_1, \ldots, x_n]$. Note that S is the directed colimit of the S_λ (details omitted). By Algebra, Lemma 10.154.1 we see that for some λ the module $M_\lambda \otimes_{R_\lambda} S_\lambda$ is flat over S_λ. By Lemma 15.14.5 we conclude that M_λ is flat over R_λ. Since $M = M_\lambda \otimes_{R_\lambda} R$ we win by Algebra, Lemma 10.36.6. □

15.15. Torsion and flatness

In this section we discuss the relationship between torsion and flatness.

Definition 15.15.1. Let R be a domain. Let M be an R-module.

(1) We say an element $x \in M$ is *torsion* if there exists a nonzero $f \in R$ such that $fx = 0$.
(2) We say M is *torsion free* if the only torsion element of M is 0.

Lemma 15.15.2. *Let R be a domain. Let M be an R-module. The set of torsion elements of M forms a submodule $M_{tors} \subset M$. The quotient module M/M_{tors} is torsion free.*

Proof. Omitted. □

Lemma 15.15.3. *Let R be a domain. Any flat R-module is torsion free.*

Proof. If $x \in R$ is nonzero, then $x : R \to R$ is injective, and hence if M is flat over R, then $x : M \to M$ is injective. Thus if M is flat over R, then M is torsion free. □

Lemma 15.15.4. *Let A be a valuation ring. An A-module M is flat over A if and only if M is torsion free.*

Proof. The implication "flat ⇒ torsion free" is Lemma 15.15.3. For the converse, assume M is torsion free. By the equational criterion of flatness (see Algebra, Lemma 10.36.10) we have to show that every relation in M is trivial. To do this assume that $\sum_{i=1,\ldots,n} a_i x_i = 0$ with $x_i \in M$ and $f_i \in A$. After renumbering we may assume that $v(a_1) \leq v(a_i)$ for all i. Hence we can write $a_i = a_i' a_1$ for some $a_i' \in A$. Note that $a_1' = 1$. As A is torsion free we see that $x_1 = -\sum_{i \geq 2} a_i' x_i$. Thus, if we choose $y_i = x_i$, $i = 2, \ldots, n$ then

$$x_1 = \sum\nolimits_{j \geq 2} -a_j' y_j, \quad x_i = y_i, (i \geq 2) \quad 0 = a_1 \cdot (-a_j') + a_j \cdot 1(j \geq 2)$$

shows that the relation was trivial (to be explicit the elements a_{ij} are defined by setting $a_{1j} = -a_j'$ and $a_{ij} = \delta_{ij}$ for $i, j \geq 2$). □

15.16. Flatness and finiteness conditions

In this section we discuss some implications of the type "flat + finite type ⇒ finite presentation". We will revisit this result in the chapter on flatness, see More on Flatness, Section 37.1. A first result of this type was proved in Algebra, Lemma 10.103.6.

Lemma 15.16.1. *Let R be a ring. Let $S = R[x_1, \ldots, x_n]$ be a polynomial ring over R. Let M be an S-module. Assume*

1. *there exist finitely many primes $\mathfrak{p}_1, \ldots, \mathfrak{p}_m$ of R such that the map $R \to \prod R_{\mathfrak{p}_j}$ is injective,*
2. *M is a finite S-module,*
3. *M flat over R, and*
4. *for every prime $\mathfrak{p}$ of R the module $M_{\mathfrak{p}}$ is of finite presentation over $S_{\mathfrak{p}}$.*

Then M is of finite presentation over S.

Proof. Choose a presentation

$$0 \to K \to S^{\oplus r} \to M \to 0$$

of M as an S-module. Let $\mathfrak{q}$ be a prime ideal of S lying over a prime $\mathfrak{p}$ of R. By assumption there exist finitely many elements $k_1, \ldots, k_t \in K$ such that if we set $K' = \sum Sk_j \subset K$ then $K'_{\mathfrak{p}} = K_{\mathfrak{p}}$ and $K'_{\mathfrak{p}_j} = K_{\mathfrak{p}_j}$ for $j = 1, \ldots, m$. Setting $M' = S^{\oplus r}/K'$ we deduce that in particular $M'_{\mathfrak{q}} = M_{\mathfrak{q}}$. By openness of flatness, see Algebra, Theorem 10.123.4 we conclude that there exists a $g \in S$, $g \notin \mathfrak{q}$ such that

M'_g is flat over R. Thus $M'_g \to M_g$ is a surjective map of flat R-modules. Consider the commutative diagram

$$\begin{array}{ccc} M'_g & \longrightarrow & M_g \\ \downarrow & & \downarrow \\ \prod (M'_g)_{\mathfrak{p}_j} & \longrightarrow & \prod (M_g)_{\mathfrak{p}_j} \end{array}$$

The bottom arrow is an isomorphism by choice of $k_1, \ldots, k_t$. The left vertical arrow is an injective map as $R \to \prod R_{\mathfrak{p}_j}$ is injective and M'_g is flat over R. Hence the top horizontal arrow is injective, hence an isomorphism. This proves that M_g is of finite presentation over S_g. We conclude by applying Algebra, Lemma 10.22.2. $\square$

Lemma 15.16.2. *Let $R \to S$ be a ring homomorphism. Assume*

1. *there exist finitely many primes $\mathfrak{p}_1, \ldots, \mathfrak{p}_m$ of R such that the map $R \to \prod R_{\mathfrak{p}_j}$ is injective,*
2. *$R \to S$ is of finite type,*
3. *S flat over R, and*
4. *for every prime $\mathfrak{p}$ of R the ring $S_{\mathfrak{p}}$ is of finite presentation over $R_{\mathfrak{p}}$.*

Then S is of finite presentation over R.

Proof. By assumption S is a quotient of a polynomial ring over R. Thus the result follows directly from Lemma 15.16.1. $\square$

Lemma 15.16.3. *Let R be a ring. Let $S = R[x_1, \ldots, x_n]$ be a graded polynomial algebra over R, i.e., $\deg(x_i) > 0$ but not necessarily equal to 1. Let M be a graded S-module. Assume*

1. *R is a local ring,*
2. *M is a finite S-module, and*
3. *M is flat over R.*

Then M is finitely presented as an S-module.

Proof. Let $M = \bigoplus M_d$ be the grading on M. Pick homogeneous generators $m_1, \ldots, m_r \in M$ of M. Say $\deg(m_i) = d_i \in \mathbf{Z}$. This gives us a presentation

$$0 \to K \to \bigoplus\nolimits_{i=1,\ldots,r} S(-d_i) \to M \to 0$$

which in each degree d leads to the short exact sequence

$$0 \to K_d \to \bigoplus\nolimits_{i=1,\ldots,r} S_{d-d_i} \to M_d \to 0.$$

By assumption each M_d is a finite flat R-module. By Algebra, Lemma 10.73.4 this implies each M_d is a finite free R-module. Hence we see each K_d is a finite R-module. Also each K_d is flat over R by Algebra, Lemma 10.36.12. Hence we conclude that each K_d is finite free by Algebra, Lemma 10.73.4 again.

Let $\mathfrak{m}$ be the maximal ideal of R. By the flatness of M over R the short exact sequences above remain short exact after tensoring with $\kappa = \kappa(\mathfrak{m})$. As the ring $S \otimes_R \kappa$ is Noetherian we see that there exist homogeneous elements $k_1, \ldots, k_t \in K$ such that the images $\overline{k}_j$ generate $K \otimes_R \kappa$ over $S \otimes_R \kappa$. Say $\deg(k_j) = e_j$. Thus for any d the map

$$\bigoplus\nolimits_{j=1,\ldots,t} S_{d-e_j} \longrightarrow K_d$$

becomes surjective after tensoring with κ. By Nakayama's lemma (Algebra, Lemma 10.18.1) this implies the map is surjective over R. Hence K is generated by $k_1, \ldots, k_t$ over S and we win. □

Lemma 15.16.4. *Let R be a ring. Let $S = \bigoplus_{n \geq 0} S_n$ be a graded R-algebra. Let $M = \bigoplus_{d \in \mathbf{Z}} M_d$ be a graded S-module. Assume S is finitely generated as an R-algebra, assume S_0 is a finite R-algebra, and assume there exist finitely many primes $\mathfrak{p}_j$, $i = 1, \ldots, m$ such that $R \to \prod R_{\mathfrak{p}_j}$ is injective.*

1. *If S is flat over R, then S is a finitely presented R-algebra.*
2. *If M is flat as an R-module and finite as an S-module, then M is finitely presented as an S-module.*

Proof. As S is finitely generated as an R-algebra, it is finitely generated as an S_0 algebra, say by homogeneous elements $t_1, \ldots, t_n \in S$ of degrees $d_1, \ldots, d_n > 0$. Set $P = R[x_1, \ldots, x_n]$ with $\deg(x_i) = d_i$. The ring map $P \to S$, $x_i \to t_i$ is finite as S_0 is a finite R-module. To prove (1) it suffices to prove that S is a finitely presented P-module. To prove (2) it suffices to prove that M is a finitely presented P-module. Thus it suffices to prove that if $S = P$ is a graded polynomial ring and M is a finite S-module flat over R, then M is finitely presented as an S-module. By Lemma 15.16.3 we see $M_{\mathfrak{p}}$ is a finitely presented $S_{\mathfrak{p}}$-module for every prime $\mathfrak{p}$ of R. Thus the result follows from Lemma 15.16.1. □

Remark 15.16.5. Let R be a ring. When does R satisfy the condition mentioned in Lemmas 15.16.1, 15.16.2, and 15.16.4? This holds if

1. R is local,
2. R is Noetherian,
3. R is a domain,
4. R is a reduced ring with finitely many minimal primes, or
5. R has finitely many weakly associated primes, see Algebra, Lemma 10.63.16.

Thus these lemmas hold in all cases listed above.

The following lemma will be improved below, see Proposition 15.16.8.

Lemma 15.16.6. *Let A be a valuation ring. Let $A \to B$ be a ring map of finite type. Let M be a finite B-module.*

1. *If B is flat over A, then B is a finitely presented A-algebra.*
2. *If M is flat as an A-module, then M is finitely presented as a B-module.*

Proof. We are going to use that an A-module is flat if and only if it is torsion free, see Lemma 15.15.4. By Algebra, Lemma 10.53.10 we can find a graded A-algebra S with $S_0 = A$ and generated by finitely many elements in degree 1, an element $f \in S_1$ and a finite graded S-module N such that $B \cong S_{(f)}$ and $M \cong N_{(f)}$. If M is torsion free, then we can take N torsion free by replacing it by N/N_{tors}, see Lemma 15.15.2. Similarly, if B is torsion free, then we can take S torsion free by replacing it by S/S_{tors}. Hence in case (1), we may apply Lemma 15.16.4 to see that S is a finitely presented A-algebra, which implies that $B = S_{(f)}$ is a finitely presented A-algebra. To see (2) we may first replace S by a graded polynomial ring, and then we may apply Lemma 15.16.3 to conclude. □

Lemma 15.16.7. *Let R be a domain with fraction field K. Let $S = R[x_1, \ldots, x_n]$ be a polynomial ring over R. Let M be a finite S-module. Assume that M is flat*

over R. If for every subring $R \subset R' \subset K$, $R \neq R'$ the module $M \otimes_R R'$ is finitely presented over $S \otimes_R R'$, then M is finitely presented over S.

Proof. Suppose that $f_1, \ldots, f_n \in R$ are elements which generate the unit ideal. If $R \neq R_{f_i}$ for each $i = 1, \ldots, n$, then we conclude that M_{f_i} is finitely presented over S_{f_i} for each i, and hence M is finitely presented over S by Algebra, Lemma 10.22.2. Thus we are done if such a sequence of elements exists. Assume this is not the case. In particular, for every $x \in R$ we have either $x \in R^*$, or $1 - x \in R^*$. This implies that R is local, see Algebra, Lemma 10.17.2.

Choose a presentation

$$0 \to K \to R[x_1, \ldots, x_n]^{\oplus r} \to M \to 0.$$

Throughout the rest of the proof we will use that this sequence stays exact after tensoring with any R-algebra, see Algebra, Lemma 10.36.11. Let R' be the integral closure of R in its fraction field. If $R \neq R'$, then we see that $M \otimes_R R'$ is finitely presented over $R'[x_1, \ldots, x_n]$. In particular, the module $K \otimes_R R'$ is finitely generated. Thus we may pick $k_1, \ldots, k_t \in K$ such that $k_1 \otimes 1, \ldots, k_t \otimes 1$ generate $K \otimes_R R'$. Set $K' = \sum R[x_1, \ldots, x_n]k_i \subset K$. Set $M' = R[x_1, \ldots, x_n]^{\oplus t}/K'$. Then M' is a finitely presented module over $R[x_1, \ldots, x_n]$ such that $M' \otimes_R R' \cong M \otimes_R R'$ is flat over R'. By Lemma 15.14.6 we conclude that M' is flat over R. Hence the surjective map $M' \to M$ is also injective as M' is torsion free, see Lemma 15.15.3. In other words, $M' \cong M$ and we conclude that M is finitely presented. Thus we are done if R is not a normal domain. Assume this is not the case. This reduces us to the case where R is a normal local domain.

Pick any pair of nonzero elements $x, y \in R$. Consider the inclusions $R \subset R[x/y]$ and $R[y/x]$. As R is a normal domain we get a short exact sequence

$$0 \to R \xrightarrow{(-1,1)} R[x/y] \oplus R[y/x] \xrightarrow{(1,1)} R[x/y, y/x] \to 0$$

see Algebra, Lemma 10.33.21. If $R \neq R[x/y]$ and $R \neq R[y/x]$ then we see that $K \otimes_R R[x/y]$ and $K \otimes_R R[y/x]$ are finitely generated as $R[x/y][x_1, \ldots, x_n]$ and $R[y/x][x_1, \ldots, x_n]$ modules. Thus we can find $k_1, \ldots, k_t \in K$ such that the elements $k_i \otimes 1$ generate $K \otimes_R R[x/y]$ and $K \otimes_R R[y/x]$ as $R[x/y][x_1, \ldots, x_n]$ and $R[y/x][x_1, \ldots, x_n]$ modules. Set $K' = \sum R[x_1, \ldots, x_n]k_i \subset K$. Tensoring the sequence above with $K' \subset K$ we get the diagram

$$\begin{array}{ccccccccc}
 & & K' & \longrightarrow & K' \otimes_R R[x/y] \oplus K' \otimes_R R[y/x] & \longrightarrow & K' \otimes_R R[x/y, y/x] & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & K & \longrightarrow & K \otimes_R R[x/y] \oplus K \otimes_R R[y/x] & \longrightarrow & K \otimes_R R[x/y, y/x] & \longrightarrow & 0
\end{array}$$

Now we know that the vertical arrows in the middle and on the right are isomorphisms. The lower row is exact as K is flat over R. Hence the left vertical arrow is surjective, i.e., an isomorphism. Thus we win if there exists a pair of nonzero elements such that neither x/y nor y/x is an element of R. Assume this is not the case. Then we know that $R \subset f.f.(R)$ is a normal local domain such that for every $x \in f.f.(R)$ either $x \in R$, or $x^{-1} \in R$. In other words, R is a valuation ring, see Algebra, Lemma 10.46.4. In this case M is finitely presented by Lemma 15.16.6 and we win. $\square$

The following result is a special case of results in [**GR71**] which we discuss in great detail in More on Flatness, Section 37.1.

Proposition 15.16.8. *Let R be a domain. Let $R \to S$ be a ring map of finite type. Let M be a finite S-module.*

(1) *If S is flat over R, then S is a finitely presented R-algebra.*
(2) *If M is flat as an R-module, then M is finitely presented as a S-module.*

Proof. It suffices to prove part (2) in case $S = R[x_1, \ldots, x_n]$. Choose a presentation

$$0 \to K \to R[x_1, \ldots, x_n]^{\oplus r} \to M \to 0.$$

Throughout the rest of the proof we will use that this sequence stays exact after tensoring with any R-algebra, see Algebra, Lemma 10.36.11. Let L be the fraction field of R. Consider the set

$$\mathcal{R} = \{R' \mid R \subset R' \subset L \text{ and } M \otimes_R R' \text{ not of finite presentation over } S \otimes_R R'\}$$

We order $\mathcal{R}$ by inclusion. Suppose that $\{R_i\}_{i \in I}$ is a totally ordered subset of $\mathcal{R}$. Set $R_\infty = \bigcup_{i \in I} R_i$. We claim that $R_\infty \in \mathcal{R}$. Namely, if $M \otimes_R R_\infty$ is finitely presented over $S \otimes_R R_\infty$, then $K \otimes_R R_\infty$ is finitely generated, say by $k_1, \ldots, k_t$. Then for some $i \in I$ we have $k_1, \ldots, k_t \in K \otimes_R R_i$. For any $i' \geq i$ set $M_{i'} = R_i[x_1, \ldots, x_n]^{\oplus r} / \sum R_i[x_1, \ldots, x_n]k_i$. By Algebra, Lemma 10.154.1 we see that $M_{i'}$ is flat over R_i for some sufficiently large $i' \in I$. For such an i' the surjective map $M_{i'} \to M \otimes_R R_i$ is also injective as $M_{i'}$ is torsion free. Hence we conclude that $M \otimes_R R_i$ is finitely presented which is a contradiction. In other words $R_\infty \in \mathcal{R}$. This shows that Zorn's lemma applies to $\mathcal{R}$ if $\mathcal{R}$ is not empty. But Lemma 15.16.7 shows that $\mathcal{R}$ does not have any maximal elements and the proposition is proved. □

15.17. Blowing up and flatness

In this section we begin our discussion of results of the form: "After a blow up the strict transform becomes flat". More results of this type may be found in More on Flatness, Section 37.26.

Definition 15.17.1. Let R be a domain. Let M be an R-module. Let $R \subset R'$ be an extension of domains. The *strict transform of M along $R \to R'$*[1] is the torsion free R'-module

$$M' = (M \otimes_R R')/(M \otimes_R R')_{tors}.$$

The following is a very weak version of flattening by blowing up, but it is already sometimes a useful result.

Lemma 15.17.2. *Let $(R, \mathfrak{m})$ be a local domain with fraction field K. Let S be a finite type R-algebra. Let M be a finite S-module. For every valuation ring $A \subset K$ dominating R there exists an ideal $I \subset \mathfrak{m}$ and a nonzero element $a \in I$ such that*

(1) *I is finitely generated,*
(2) *A has center on $R[\frac{I}{a}]$,*
(3) *the fibre ring of $R \to R[\frac{I}{a}]$ at $\mathfrak{m}$ is not zero, and*
(4) *the strict transform $S_{I,a}$ of S along $R \to R[\frac{I}{a}]$ is flat and of finite presentation over R, and the strict transform $M_{I,a}$ of M along $R \to R[\frac{I}{a}]$ is flat over R and finitely presented over $S_{I,a}$.*

[1] This is somewhat nonstandard notation.

Proof. Note that the assertion makes sense as $R[\frac{I}{a}]$ is a domain, and $R \to R[\frac{I}{a}]$ is injective, see Algebra, Lemmas 10.54.4 and 10.54.6. Before we start the proof of the Lemma, note that there is no loss in generality assuming that $S = R[x_1, \ldots, x_n]$ is a polynomial ring over R. We also fix a presentation

$$0 \to K \to S^{\oplus r} \to M \to 0.$$

Let M_A be the strict transform of M along $R \to A$. It is a finite module over $S_A = A[x_1, \ldots, x_n]$. By Lemma 15.15.4 we see that M_A is flat over A. By Lemma 15.16.6 we see that M_A is finitely presented. Hence there exist finitely many elements $k_1, \ldots, k_t \in S_A^{\oplus r}$ which generate the kernel of the presentation $S_A^{\oplus r} \to M_A$ as an S_A-module. For any choice of $a \in I \subset \mathfrak{m}$ satisfying (1), (2), and (3) we denote $M_{I,a}$ the strict transform of M along $R \to R[\frac{I}{a}]$. It is a finite module over $S_{I,a} = R[\frac{I}{a}][x_1, \ldots, x_n]$. By Algebra, Lemma 10.54.7 we have $A = \operatorname{colim}_{I,a} R[\frac{I}{a}]$. This implies that $S_A = \operatorname{colim} S_{I,a}$ and $M_A = \operatorname{colim}_{I,a} M_{I,a}$. Thus we may choose $a \in I \subset R$ such that $k_1, \ldots, k_t$ are elements of $S_{I,a}^{\oplus r}$ and map to zero in $M_{I,a}$. For any such pair (I, a) we set

$$M'_{I,a} = S_{I,a}^{\oplus r} / \sum S_{I,a} k_j.$$

Since $M_A = S_A^{\oplus r} / \sum S_A k_j$ we see that also $M_A = \operatorname{colim}_{I,a} M'_{I,a}$. At this point we may apply Algebra, Lemma 10.154.1 (3) to conclude that $M'_{I,a}$ is flat for some pair (I, a). (This lemma does not apply a priori to the system $M_{I,a}$ as the transition maps may not satisfy the assumptions of the lemma.) Since flatness implies torsion free (Lemma 15.15.3), we also conclude that $M'_{I,a} = M_{I,a}$ for such a pair and we win. □

15.18. Completion and flatness

In this section we discuss when the completion of a "big" flat module is flat.

Lemma 15.18.1. *Let R be a ring. Let $I \subset R$ be an ideal. Let A be a set. Assume R is Noetherian and complete with respect to I. There is a canonical map*

$$\left(\bigoplus\nolimits_{\alpha \in A} R\right)^\wedge \longrightarrow \prod\nolimits_{\alpha \in A} R$$

from the I-adic completion of the direct sum into the product which is universally injective.

Proof. By definition an element x of the left hand side is $x = (x_n)$ where $x_n = (x_{n,\alpha}) \in \bigoplus_{\alpha \in A} R/I^n$ such that $x_{n,\alpha} = x_{n+1,\alpha} \bmod I^n$. As $R = R^\wedge$ we see that for any α there exists a $y_\alpha \in R$ such that $x_{n,\alpha} = y_\alpha \bmod I^n$. Note that for each n there are only finitely many α such that the elements $x_{n,\alpha}$ are nonzero. Conversely, given $(y_\alpha) \in \prod_\alpha R$ such that for each n there are only finitely many α such that $y_\alpha \bmod I^n$ is nonzero, then this defines an element of the left hand side. Hence we can think of an element of the left hand side as infinite "convergent sums" $\sum_\alpha y_\alpha$ with $y_\alpha \in R$ such that for each n there are only finitely many y_α which are nonzero modulo I^n. The displayed map maps this element to the element to (y_α) in the product. In particular the map is injective.

Let Q be a finite R-module. We have to show that the map

$$Q \otimes_R \left(\bigoplus\nolimits_{\alpha \in A} R\right)^\wedge \longrightarrow Q \otimes_R \left(\prod\nolimits_{\alpha \in A} R\right)$$

is injective, see Algebra, Theorem 10.77.3. Choose a presentation $R^{\oplus k} \to R^{\oplus m} \to Q \to 0$ and denote $q_1, \ldots, q_m \in Q$ the corresponding generators for Q. By Artin-Rees (Algebra, Lemma 10.47.4) there exists a constant c such that $\mathrm{Im}(R^{\oplus k} \to R^{\oplus m}) \cap (I^N)^{\oplus m} \subset \mathrm{Im}((I^{N-c})^{\oplus k} \to R^{\oplus m})$. Let us contemplate the diagram

$$\begin{array}{ccccccc} \bigoplus_{l=1}^k \left(\bigoplus_{\alpha\in A} R\right)^\wedge & \longrightarrow & \bigoplus_{j=1}^m \left(\bigoplus_{\alpha\in A} R\right)^\wedge & \longrightarrow & Q \otimes_R \left(\bigoplus_{\alpha\in A} R\right)^\wedge & \longrightarrow & 0 \\ \downarrow & & \downarrow & & \downarrow & & \\ \bigoplus_{l=1}^k \left(\prod_{\alpha\in A} R\right) & \longrightarrow & \bigoplus_{j=1}^m \left(\prod_{\alpha\in A} R\right) & \longrightarrow & Q \otimes_R \left(\prod_{\alpha\in A} R\right) & \longrightarrow & 0 \end{array}$$

with exact rows. Pick an element $\sum_j \sum_\alpha y_{j,\alpha}$ of $\bigoplus_{j=1,\ldots,m} \left(\bigoplus_{\alpha\in A} R\right)^\wedge$. If this element maps to zero in the module $Q \otimes_R \left(\prod_{\alpha\in A} R\right)$, then we see in particular that $\sum_j q_j \otimes y_{j,\alpha} = 0$ in Q for each α. Thus we can find an element $(z_{1,\alpha}, \ldots, z_{k,\alpha}) \in \bigoplus_{l=1,\ldots,k} R$ which maps to $(y_{1,\alpha}, \ldots, y_{m,\alpha}) \in \bigoplus_{j=1,\ldots,m} R$. Moreover, if $y_{j,\alpha} \in I^{N_\alpha}$ for $j = 1, \ldots, m$, then we may assume that $z_{l,\alpha} \in I^{N_\alpha - c}$ for $l = 1, \ldots, k$. Hence the sum $\sum_l \sum_\alpha z_{l,\alpha}$ is "convergent" and defines an element of $\bigoplus_{l=1,\ldots,k} \left(\bigoplus_{\alpha\in A} R\right)^\wedge$ which maps to the element $\sum_j \sum_\alpha y_{j,\alpha}$ we started out with. Thus the right vertical arrow is injective and we win. □

The following lemma can also be deduced from Lemma 15.18.4 below.

Lemma 15.18.2. *Let R be a ring. Let $I \subset R$ be an ideal. Let A be a set. Assume R is Noetherian. The completion $(\bigoplus_{\alpha\in A} R)^\wedge$ is a flat R-module.*

Proof. Denote $R^\wedge$ the completion of R with respect to I. As $R \to R^\wedge$ is flat by Algebra, Lemma 10.91.3 it suffices to prove that $(\bigoplus_{\alpha\in A} R)^\wedge$ is a flat $R^\wedge$-module (use Algebra, Lemma 10.36.3). Since

$$(\bigoplus\nolimits_{\alpha\in A} R)^\wedge = (\bigoplus\nolimits_{\alpha\in A} R^\wedge)^\wedge$$

we may replace R by $R^\wedge$ and assume that R is complete with respect to I (see Algebra, Lemma 10.91.8). In this case Lemma 15.18.1 tells us the map $(\bigoplus_{\alpha\in A} R)^\wedge \to \prod_{\alpha\in A} R$ is universally injective. Thus, by Algebra, Lemma 10.77.7 it suffices to show that $\prod_{\alpha\in A} R$ is flat. By Algebra, Proposition 10.85.5 (and Algebra, Lemma 10.85.4) we see that $\prod_{\alpha\in A} R$ is flat. □

Lemma 15.18.3. *Let A be a Noetherian ring. Let I be an ideal of A. Let M be a finite A-module. For every $p > 0$ there exists a $c > 0$ such that $Tor_p^A(M, A/I^{n+c}) \to Tor_p^A(M, A/I^n)$ is zero.*

Proof. Proof for $p = 1$. Choose a short exact sequence $0 \to K \to R^{\oplus t} \to M \to 0$. Then $\mathrm{Tor}_1^A(M, A/I^n) = K \cap (I^n)^{\oplus t}/I^n K$. By Artin-Rees (Algebra, Lemma 10.47.4) there is a constant $c \geq 0$ such that $K \cap (I^{n+c})^{\oplus t} \subset I^n K$. Thus the result for $p = 1$. For $p > 1$ we have $\mathrm{Tor}_p^A(M, A/I^n) = \mathrm{Tor}_{p-1}^A(K, A/I^n)$. Thus the lemma follows by induction. □

Lemma 15.18.4. *Let A be a Noetherian ring. Let I be an ideal of A. Let (M_n) be an inverse system of A-modules such that*

1. *M_n is a flat A/I^n-module,*
2. *$M_{n+1} \to M_n$ is surjective.*

Then $M = \lim M_n$ is a flat A-module and $Q \otimes_A M = \lim Q \otimes_A M_n$ for every finite A-module Q.

Proof. We first show that $Q \otimes_A M = \lim Q \otimes_A M_n$ for every finite A-module Q. Choose a resolution $F_2 \to F_1 \to F_0 \to Q \to 0$ by finite free A-modules F_i. Then

$$F_2 \otimes_A M_n \to F_1 \otimes_A M_n \to F_0 \otimes_A M_n$$

is a chain complex whose homology in degree 0 is $Q \otimes_A M_n$ and whose homology in degree 1 is

$$\mathrm{Tor}_1^A(Q, M_n) = \mathrm{Tor}_1^A(Q, A/I^n) \otimes_{A/I^n} M_n$$

as M_n is flat over A/I^n. By Lemma 15.18.3 we see that this system is essentially constant (with value 0). It follows from Homology, Lemma 12.26.7 that $\lim Q \otimes_A A/I^n = \mathrm{Coker}(\lim F_1 \otimes_A M_n \to \lim F_0 \otimes_A M_n)$. Since F_i is finite free this equals $\mathrm{Coker}(F_1 \otimes_A M \to F_0 \otimes_A M) = Q \otimes_A M$.

Next, let $Q \to Q'$ be an injective map of finite A-modules. We have to show that $Q \otimes_A M \to Q' \otimes_A M$ is injective (Algebra, Lemma 10.36.4). By the above we see

$$\mathrm{Ker}(Q \otimes_A M \to Q' \otimes_A M) = \mathrm{Ker}(\lim Q \otimes_A M_n \to \lim Q' \otimes_A M_n).$$

For each n we have an exact sequence

$$\mathrm{Tor}_1^A(Q', M_n) \to \mathrm{Tor}_1^A(Q'', M_n) \to Q \otimes_A M_n \to Q' \otimes_A M_n$$

where $Q'' = \mathrm{Coker}(Q \to Q')$. Above we have seen that the inverse systems of Tor's are essentially constant with value 0. It follows from Homology, Lemma 12.26.7 that the inverse limit of the right most maps is injective. □

15.19. The Koszul complex

We define the Koszul complex as follows.

Definition 15.19.1. Let R be a ring. Let $\varphi : E \to R$ be an R-module map. The *Koszul complex* $K_\bullet(\varphi)$ associated to φ is the commutative differential graded algebra defined as follows:

(1) the underlying graded algebra is the exterior algebra $K_\bullet(\varphi) = \wedge(E)$,
(2) the differential $d : K_\bullet(\varphi) \to K_\bullet(\varphi)$ is the unique derivation such that $d(e) = \varphi(e)$ for all $e \in E = K_1(\varphi)$.

Explicitly, if $e_1 \wedge \ldots \wedge e_n$ is one of the generators of degree n in $K_\bullet(\varphi)$, then

$$d(e_1 \wedge \ldots \wedge e_n) = \sum\nolimits_{i=1,\ldots,n} (-1)^{i+1} \varphi(e_i) e_1 \wedge \ldots \wedge \widehat{e_i} \wedge \ldots \wedge e_n.$$

It is straightforward to see that this gives a well defined derivation on the tensor algebra, which annihilates $e \otimes e$ and hence factors through the exterior algebra.

We often assume that E is a finite free module, say $E = R^{\oplus n}$. In this case the map φ is given by a sequence of elements $f_1, \ldots, f_n \in R$.

Definition 15.19.2. Let R be a ring and let $f_1, \ldots, f_r \in R$. The *Koszul complex on $f_1, \ldots, f_r$* is the Koszul complex associated to the map $(f_1, \ldots, f_r) : R^{\oplus r} \to R$. Notation $K_\bullet(f_\bullet)$, $K_\bullet(f_1, \ldots, f_r)$, $K_\bullet(R, f_1, \ldots, f_r)$, or $K_\bullet(R, f_\bullet)$.

Of course, if E is finite locally free, then $K_\bullet(\varphi)$ is locally on $\mathrm{Spec}(R)$ isomorphic to a Koszul complex $K_\bullet(f_1, \ldots, f_r)$. This complex has many interesting formal properties.

Lemma 15.19.3. *Let $\varphi : E \to R$ and $\varphi : E' \to R$ be an R-module maps. Let $\psi : E \to E'$ be an R-module map such that $\varphi' \circ \psi = \varphi$. Then ψ induces a homomorphism of differential graded algebras $K_\bullet(\varphi) \to K_\bullet(\varphi')$.*

Proof. This is immediate from the definitions. □

Lemma 15.19.4. *Let $f_1, \ldots, f_r \in R$ be a sequence. Let (x_{ij}) be an invertible $r \times r$-matrix with coefficients in R. Then the complexes $K_\bullet(f_\bullet)$ and*

$$K_\bullet(\sum x_{1j} f_j, \sum x_{2j} f_j, \ldots, \sum x_{cj} f_j)$$

are isomorphic.

Proof. Set $g_i = \sum x_{ij} f_j$. The matrix (x_{ij}) gives an isomorphism $x : R^{\oplus r} \to R^{\oplus r}$ such that $(g_1, \ldots, g_r) \circ x = (f_1, \ldots, f_r)$. Hence this follows from the functoriality of the Koszul complex described in Lemma 15.19.3. □

Lemma 15.19.5. *Let R be a ring. Let $\varphi : E \to R$ be an R-module map. Let $e \in E$ with image $f = \varphi(e)$ in R. Then*

$$f = de + ed$$

as endomorphisms of $K_\bullet(\varphi)$.

Proof. This is true because $d(ea) = d(e)a - ed(a) = fa - ed(a)$. □

Lemma 15.19.6. *Let R be a ring. Let $f_1, \ldots, f_r \in R$ be a sequence. Multiplication by f_i on $K_\bullet(f_\bullet)$ is homotopic to zero, and in particular the cohomology modules $H_i(K_\bullet(f_\bullet))$ are annihilated by the ideal $(f_1, \ldots, f_r)$.*

Proof. Special case of Lemma 15.19.5. □

In Derived Categories, Section 13.8 we defined the cone of a morphism of cochain complexes. The cone $C(f)_\bullet$ of a morphism of chain complexes $f : A_\bullet \to B_\bullet$ is the complex $C(f)_\bullet$ given by $C(f)_n = B_n \oplus A_{n-1}$ and differential

$$d_{C(f),n} = \begin{pmatrix} d_{B,n} & f_{n-1} \\ 0 & -d_{A,n-1} \end{pmatrix} \tag{15.19.6.1}$$

It comes equipped with canonical morphisms of complexes $i : B_\bullet \to C(f)_\bullet$ and $p : C(f)_\bullet \to A_\bullet[-1]$ induced by the obvious maps $B_n \to C(f)_n \to A_{n-1}$.

Lemma 15.19.7. *Let R be a ring. Let $\varphi : E \to R$ be an R-module map. Let $f \in R$. Set $E' = E \oplus R$ and define $\varphi' : E' \to R$ by φ on E and multiplication by f on R. The complex $K_\bullet(\varphi')$ is isomorphic to the cone of the map of complexes*

$$f : K_\bullet(\varphi) \longrightarrow K_\bullet(\varphi).$$

Proof. Denote $e_0 \in E'$ the element $1 \in R \subset R \oplus E$. By our definition of the cone above we see that

$$C(f)_n = K_n(\varphi) \oplus K_{n-1}(\varphi) = \wedge^n(E) \oplus \wedge^{n-1}(E) = \wedge^n(E')$$

where in the last $=$ we map $(0, e_1 \wedge \ldots \wedge e_{n-1})$ to $e_0 \wedge e_1 \wedge \ldots \wedge e_{n-1}$ in $\wedge^n(E')$. A computation shows that this isomorphism is compatible with differentials. Namely, this is clear for elements of the first summand as $\varphi'|_E = \varphi$ and $d_{C(f)}$ restricted to the first summand is just $d_{K_\bullet(\varphi)}$. On the other hand, if $e_1 \wedge \ldots \wedge e_{n-1}$ is in the first summand, then

$$d_{C(f)}(0, e_1 \wedge \ldots \wedge e_{n-1}) = f e_1 \wedge \ldots \wedge e_{n-1} - d_{K_\bullet(\varphi)}(e_1 \wedge \ldots \wedge e_{n-1})$$

and on the other hand

$$\begin{aligned}
&d_{K_\bullet(\varphi')}(e_0 \wedge e_1 \wedge \ldots \wedge e_{n-1}) \\
&= \textstyle\sum_{i=0,\ldots,n-1}(-1)^i \varphi'(e_i) e_0 \wedge \ldots \wedge \widehat{e_i} \wedge \ldots \wedge e_{n-1} \\
&= f e_1 \wedge \ldots \wedge e_{n-1} + \textstyle\sum_{i=1,\ldots,n-1}(-1)^i \varphi(e_i) e_0 \wedge \ldots \wedge \widehat{e_i} \wedge \ldots \wedge e_{n-1} \\
&= f e_1 \wedge \ldots \wedge e_{n-1} - e_0 \left(\textstyle\sum_{i=1,\ldots,n-1}(-1)^{i+1} \varphi(e_i) e_1 \wedge \ldots \wedge \widehat{e_i} \wedge \ldots \wedge e_{n-1} \right)
\end{aligned}$$

which is the image of the result of the previous computation. □

Lemma 15.19.8. *Let R be a ring. Let $f_1, \ldots, f_r$ be a sequence of elements of R. The complex $K_\bullet(f_1, \ldots, f_r)$ is isomorphic to the cone of the map of complexes*

$$f_n : K_\bullet(f_1, \ldots, f_{r-1}) \longrightarrow K_\bullet(f_1, \ldots, f_{r-1}).$$

Proof. Special case of Lemma 15.19.7. □

Lemma 15.19.9. *Let R be a ring. Let $A_\bullet$ be a complex of R-modules. Let $f, g \in R$. Let $C(f)_\bullet$ be the cone of $f : A_\bullet \to A_\bullet$. Define similarly $C(g)_\bullet$ and $C(fg)_\bullet$. Then $C(fg)_\bullet$ is homotopy equivalent to the cone of a map*

$$C(f)_\bullet[1] \longrightarrow C(g)_\bullet$$

Proof. We first prove this if $A_\bullet$ is the complex consisting of R placed in degree 0. In this case the map we use is

$$\begin{array}{ccccccccc}
0 & \longrightarrow & 0 & \longrightarrow & R & \xrightarrow{f} & R & \longrightarrow & 0 \\
\downarrow & & \downarrow & & \downarrow{\scriptstyle 1} & & \downarrow & & \downarrow \\
0 & \longrightarrow & R & \xrightarrow{g} & R & \longrightarrow & 0 & \longrightarrow & 0
\end{array}$$

The cone of this is the chain complex consisting of $R \oplus R$ placed in degrees 1 and 0 and differential (15.19.6.1)

$$\begin{pmatrix} g & 1 \\ 0 & -f \end{pmatrix} : R^{\oplus 2} \longrightarrow R^{\oplus 2}$$

We leave it to the reader to show this this chain complex is homotopic to the complex $fg : R \to R$. In general we write $C(f)_\bullet$ and $C(g)_\bullet$ as the total complex of the double complexes

$$(R \xrightarrow{f} R) \otimes_R A_\bullet \quad \text{and} \quad (R \xrightarrow{g} R) \otimes_R A_\bullet$$

and in this way we deduce the result from the special case discussed above. Some details omitted. □

Lemma 15.19.10. *Let R be a ring. Let $\varphi : E \to R$ be an R-module map. Let $f, g \in R$. Set $E' = E \oplus R$ and define $\varphi'_f, \varphi'_g, \varphi'_{fg} : E' \to R$ by φ on E and multiplication by f, g, fg on R. The complex $K_\bullet(\varphi'_{fg})$ is isomorphic to the cone of a map of complexes*

$$K_\bullet(\varphi'_f)[1] \longrightarrow K_\bullet(\varphi'_g).$$

Proof. By Lemma 15.19.7 the complex $K_\bullet(\varphi'_f)$ is isomorphic to the cone of multiplication by f on $K_\bullet(\varphi)$ and similarly for the other two cases. Hence the lemma follows from Lemma 15.19.9. □

Lemma 15.19.11. *Let R be a ring. Let $f_1, \ldots, f_{r-1}$ be a sequence of elements of R. Let $f, g \in R$. The complex $K_\bullet(f_1, \ldots, f_{r-1}, fg)$ is homotopy equivalent to the cone of a map of complexes*

$$K_\bullet(f_1, \ldots, f_{r-1}, f)[1] \longrightarrow K_\bullet(f_1, \ldots, f_{r-1}, g)$$

Proof. Special case of Lemma 15.19.10. □

Lemma 15.19.12. *Let A be a ring. Let $f_1, \ldots, f_r$, $g_1, \ldots, g_s$ be elements of A. Then there is an isomorphism of Koszul complexes*

$$K_\bullet(A, f_1, \ldots, f_r, g_1, \ldots, g_s) = Tot(K_\bullet(A, f_1, \ldots, f_r) \otimes_A K_\bullet(A, g_1, \ldots, g_s)).$$

Proof. Omitted. Hint: If $K_\bullet(A, f_1, \ldots, f_r)$ is generated as a differential graded algebra by $x_1, \ldots, x_r$ with $\mathrm{d}(x_i) = f_i$ and $K_\bullet(A, g_1, \ldots, g_s)$ is generated as a differential graded algebra by $y_1, \ldots, y_s$ with $\mathrm{d}(y_j) = g_j$, then we can think of $K_\bullet(A, f_1, \ldots, f_r, g_1, \ldots, g_s)$ as the differential graded algebra generated by the sequence of elements $x_1, \ldots, x_r, y_1, \ldots, y_r$ with $\mathrm{d}(x_i) = f_i$ and $\mathrm{d}(y_j) = g_j$. □

Lemma 15.19.13. *Let R be a ring. Let $f_1, \ldots, f_r \in R$. The extended alternating Čech complex*

$$R \to \prod\nolimits_{i_0} R_{f_{i_0}} \to \prod\nolimits_{i_0 < i_1} R_{f_{i_0} f_{i_1}} \to \ldots \to R_{f_1 \ldots f_r}$$

is a colimit of the Koszul complexes $K(R, f_1^n, \ldots, f_r^n)$.

Proof. The transition maps $K(R, f_1^n, \ldots, f_r^n) \to K(R, f_1^{n+1}, \ldots, f_r^{n+1})$ are the maps sending $e_{i_1} \wedge \ldots \wedge e_{i_p}$ to $f_{i_{p+1}} \ldots f_{i_r} e_{i_1} \wedge \ldots \wedge e_{i_p}$ where the indices are such that $\{1, \ldots, r\} = \{i_1, \ldots, i_r\}$. In particular the transition maps are always 1 in degree r and equal to $f_1 \ldots f_r$ in degree 0. The terms of the colimit are equal to the terms of the extended alternating Čech complex by Algebra, Lemma 10.9.9. □

15.20. Koszul regular sequences

Please take a look at Algebra, Sections 10.65 and 10.66 before looking at this one.

Definition 15.20.1. Let R be a ring. Let $r \geq 0$ and let $f_1, \ldots, f_r \in R$ be a sequence of elements. Let M be an R-module. The sequence $f_1, \ldots, f_r$ is called

(1) *M-Koszul-regular* if $H_i(K_\bullet(f_1, \ldots, f_r) \otimes_R M) = 0$ for all $i \neq 0$,
(2) *M-H_1-regular* if $H_1(K_\bullet(f_1, \ldots, f_r) \otimes_R M) = 0$,
(3) *Koszul-regular* if $H_i(K_\bullet(f_1, \ldots, f_r)) = 0$ for all $i \neq 0$, and
(4) *H_1-Koszul-regular* if $H_1(K_\bullet(f_1, \ldots, f_r)) = 0$.

We will see in Lemmas 15.20.2 and 15.20.5 that for elements $f_1, \ldots, f_r$ of a ring R we have the following implications

$$\begin{aligned} f_1, \ldots, f_r \text{ is a regular sequence} &\Rightarrow f_1, \ldots, f_r \text{ is a Koszul-regular sequence} \\ &\Rightarrow f_1, \ldots, f_r \text{ is an } H_1\text{-regular sequence} \\ &\Rightarrow f_1, \ldots, f_r \text{ is a quasi-regular sequence.} \end{aligned}$$

In general none of these implications can be reversed, but if R is a Noetherian local ring and $f_1, \ldots, f_r \in \mathfrak{m}_R$, then the 4 conditions are all equivalent (Lemma 15.20.6). If $f = f_1 \in R$ is a length 1 sequence then it is clear that the following are all equivalent

(1) f is a regular sequence of length one,

(2) f is a Koszul-regular sequence of length one, and
(3) f is a H_1-regular sequence of length one.

It is also clear that these imply that f is a quasi-regular sequence of length one. But there do exist quasi-regular sequences of length 1 which are not regular sequences. Namely, let

$$R = k[x, y_0, y_1, \ldots]/(xy_0, xy_1 - y_0, xy_2 - y_1, \ldots)$$

and let f be the image of x in R. Then f is a zerodivisor, but $\bigoplus_{n\geq 0}(f^n)/(f^{n+1}) \cong k[x]$ is a polynomial ring.

Lemma 15.20.2. *An M-regular sequence is M-Koszul-regular. A regular sequence is Koszul-regular.*

Proof. Let R be a ring and let M be an R-module. It is immediate that an M-regular sequence of length 1 is M-Koszul-regular. Let $f_1, \ldots, f_r$ be an M-regular sequence. Then f_1 is a nonzero divisor on M. Hence

$$0 \to K_\bullet(f_2, \ldots, f_r) \otimes M \xrightarrow{f_1} K_\bullet(f_2, \ldots, f_r) \otimes M \to K_\bullet(\overline{f}_2, \ldots, \overline{f}_r) \otimes M/f_1M \to 0$$

is a short exact sequence of complexes where $\overline{f}_i$ is the image of f_i in $R/(f_1)$. By Lemma 15.19.8 the complex $K_\bullet(R, f_1, \ldots, f_r)$ is isomorphic to the cone of multiplication by f_1 on $K_\bullet(f_2, \ldots, f_r)$. Thus $K_\bullet(R, f_1, \ldots, f_r) \otimes M$ is isomorphic to the cone on the first map. Hence $K_\bullet(\overline{f}_2, \ldots, \overline{f}_r) \otimes M/f_1M$ is quasi-isomorphic to $K_\bullet(f_1, \ldots, f_r) \otimes M$. As $\overline{f}_2, \ldots, \overline{f}_r$ is an M/f_1M-regular sequence in $R/(f_1)$ the result follows from the case $r = 1$ and induction. □

Lemma 15.20.3. *Let $f_1, \ldots, f_{r-1} \in R$ be a sequence and $f, g \in R$. Let M be an R-module.*

(1) *If $f_1, \ldots, f_{r-1}, f$ and $f_1, \ldots, f_{r-1}, g$ are M-H_1-regular then $f_1, \ldots, f_{r-1}, fg$ is M-H_1-regular too.*
(2) *If $f_1, \ldots, f_{r-1}, f$ and $f_1, \ldots, f_{r-1}, f$ are M-Koszul-regular then $f_1, \ldots, f_{r-1}, fg$ is M-Koszul-regular too.*

Proof. By Lemma 15.19.11 we have exact sequences

$$H_i(K_\bullet(f_1, \ldots, f_{r-1}, f) \otimes M) \to H_i(K_\bullet(f_1, \ldots, f_{r-1}, fg) \otimes M) \to H_i(K_\bullet(f_1, \ldots, f_{r-1}, g) \otimes M)$$

for all i. □

Lemma 15.20.4. *Let $\varphi : R \to S$ be a flat ring map. Let $f_1, \ldots, f_r \in R$. Let M be an R-module and set $N = M \otimes_R S$.*

(1) *If $f_1, \ldots, f_r$ in R is an M-H_1-regular sequence, then $\varphi(f_1), \ldots, \varphi(f_r)$ is an N-H_1-regular sequence in S.*
(2) *If $f_1, \ldots, f_r$ is an M-Koszul-regular sequence in R, then $\varphi(f_1), \ldots, \varphi(f_r)$ is an N-Koszul-regular sequence in S.*

Proof. This is true because $K_\bullet(f_1, \ldots, f_r) \otimes_R S = K_\bullet(\varphi(f_1), \ldots, \varphi(f_r))$ and therefore $(K_\bullet(f_1, \ldots, f_r) \otimes_R M) \otimes_R S = K_\bullet(\varphi(f_1), \ldots, \varphi(f_r)) \otimes_S N$. □

Lemma 15.20.5. *An M-H_1-regular sequence is M-quasi-regular.*

Proof. Let R be a ring and let M be an R-module. Let $f_1, \ldots, f_r$ be an M-H_1-regular sequence. Denote $J = (f_1, \ldots, f_r)$. The assumption means that we have an exact sequence

$$\wedge^2(R^r) \otimes M \to R^{\oplus r} \otimes M \to JM \to 0$$

where the first arrow is given by $e_i \wedge e_j \otimes m \mapsto (f_i e_j - f_j e_i) \otimes m$. In particular this implies that

$$JM/J^2M = JM \otimes_R R/J = (M/JM)^{\oplus r}$$

is a finite free module. To finish the proof we have to prove for every $n \geq 2$ the following: if

$$\xi = \sum\nolimits_{|I|=n, I=(i_1,\ldots,i_r)} m_I f_1^{i_1} \ldots f_r^{i_r} \in J^{n+1}M$$

then $m_I \in JM$ for all I. Note that $f_1, \ldots, f_{r-1}, f_r^n$ is an M-H_1-regular sequence by Lemma 15.20.3. Hence we see that the required result holds for the multi-index $I = (0, \ldots, 0, n)$. It turns out that we can reduce the general case to this case as follows.

Let $S = R[x_1, x_2, \ldots, x_r, 1/x_r]$. The ring map $R \to S$ is faithfully flat, hence $f_1, \ldots, f_r$ is an M-H_1-regular sequence in S, see Lemma 15.20.4. By Lemma 15.19.4 we see that

$$g_1 = f_1 - x_1/x_r f_r, \ldots g_{r-1} = f_{r-1} - x_{r-1}/x_r f_r, g_r = (1/x_r) f_r$$

is an M-H_1-regular sequence in S. Finally, note that our element ξ can be rewritten

$$\xi = \sum\nolimits_{|I|=n, I=(i_1,\ldots,i_r)} m_I (g_1 + x_r g_r)^{i_1} \ldots (g_{r-1} + x_r g_r)^{i_{r-1}} (x_r g_r)^{i_r}$$

and the coefficient of g_r^n in this expression is

$$\sum m_I x_1^{i_1} \ldots x_r^{i_r} \in J(M \otimes_R S).$$

Since the monomials $x_1^{i_1} \ldots x_r^{i_r}$ form part of an R-basis of S over R we conclude that $m_I \in J$ for all I as desired. □

For nonzero finite modules over Noetherian local rings all of the types of regular sequences introduced so far are equivalent.

Lemma 15.20.6. *Let $(R, \mathfrak{m})$ be a Noetherian local ring. Let M be a nonzero finite R-module. Let $f_1, \ldots, f_r \in \mathfrak{m}$. The following are equivalent*

1. *$f_1, \ldots, f_r$ is an M-regular sequence,*
2. *$f_1, \ldots, f_r$ is a M-Koszul-regular sequence,*
3. *$f_1, \ldots, f_r$ is an M-H_1-regular sequence,*
4. *$f_1, \ldots, f_r$ is an M-quasi-regular sequence.*

In particular the sequence $f_1, \ldots, f_r$ is a regular sequence in R if and only if it is a Koszul regular sequence, if and only if it is a H_1-regular sequence, if and only if it is a quasi-regular sequence.

Proof. The implication (1) ⇒ (2) is Lemma 15.20.2. The implication (2) ⇒ (3) is immediate. The implication (3) ⇒ (4) is Lemma 15.20.5. The implication (4) ⇒ (1) is Algebra, Lemma 10.66.6. □

Lemma 15.20.7. *Let A be a ring. Let $I \subset A$ be an ideal. Let $g_1, \ldots, g_m$ be a sequence in A whose image in A/I is H_1-regular. Then $I \cap (g_1, \ldots, g_m) = I(g_1, \ldots, g_m)$.*

Proof. Consider the exact sequence of complexes

$$0 \to I \otimes_A K_\bullet(A, g_1, \ldots, g_m) \to K_\bullet(A, g_1, \ldots, g_m) \to K_\bullet(A/I, g_1, \ldots, g_m) \to 0$$

Since the complex on the right has $H_1 = 0$ by assumption we see that

$$\mathrm{Coker}(I^{\oplus m} \to I) \longrightarrow \mathrm{Coker}(A^{\oplus m} \to A)$$

is injective. This is equivalent to the assertion of the lemma. □

Lemma 15.20.8. *Let A be a ring. Let $I \subset J \subset A$ be ideals. Assume that $J/I \subset A/I$ is generated by an H_1-regular sequence. Then $I \cap J^2 = IJ$.*

Proof. To prove this choose $g_1, \ldots, g_m \in J$ whose images in A/I form a H_1-regular sequence which generates J/I. In particular $J = I + (g_1, \ldots, g_m)$. Suppose that $x \in I \cap J^2$. Because $x \in J^2$ can write

$$x = \sum a_{ij} g_i g_j + \sum a_j g_j + a$$

with $a_{ij} \in A$, $a_j \in I$ and $a \in I^2$. Then $\sum a_{ij} g_i g_j \in I \cap (g_1, \ldots, g_m)$ hence by Lemma 15.20.7 we see that $\sum a_{ij} g_i g_j \in I(g_1, \ldots, g_m)$. Thus $x \in IJ$ as desired. □

Lemma 15.20.9. *Let A be a ring. Let I be an ideal generated by a quasi-regular sequence $f_1, \ldots, f_n$ in A. Let $g_1, \ldots, g_m \in A$ be elements whose images $\overline{g}_1, \ldots, \overline{g}_m$ form an H_1-regular sequence in A/I. Then $f_1, \ldots, f_n, g_1, \ldots, g_m$ is a quasi-regular sequence in A.*

Proof. We claim that $g_1, \ldots, g_m$ forms an H_1-regular sequence in A/I^d for every d. By induction assume that this holds in A/I^{d-1}. We have a short exact sequence of complexes

$$0 \to K_\bullet(A, g_\bullet) \otimes_A I^{d-1}/I^d \to K_\bullet(A/I^d, g_\bullet) \to K_\bullet(A/I^{d-1}, g_\bullet) \to 0$$

Since $f_1, \ldots, f_n$ is quasi-regular we see that the first complex is a direct sum of copies of $K_\bullet(A/I, g_1, \ldots, g_m)$ hence acyclic in degree 1. By induction hypothesis the last complex is acyclic in degree 1. Hence also the middle complex is. In particular, the sequence $g_1, \ldots, g_m$ forms a quasi-regular sequence in A/I^d for every $d \geq 1$, see Lemma 15.20.5. Now we are ready to prove that $f_1, \ldots, f_n, g_1, \ldots, g_m$ is a quasi-regular sequence in A. Namely, set $J = (f_1, \ldots, f_n, g_1, \ldots, g_m)$ and suppose that (with multinomial notation)

$$\sum\nolimits_{|N|+|M|=d} a_{N,M} f^N g^M \in J^{d+1}$$

for some $a_{N,M} \in A$. We have to show that $a_{N,M} \in J$ for all N, M. Let $e \in \{0, 1, \ldots, d\}$. Then

$$\sum\nolimits_{|N|=d-e,\ |M|=e} a_{N,M} f^N g^M \in (g_1, \ldots, g_m)^{e+1} + I^{d-e+1}$$

Because $g_1, \ldots, g_m$ is a quasi-regular sequence in A/I^{d-e+1} we deduce

$$\sum\nolimits_{|N|=d-e} a_{N,M} f^N \in (g_1, \ldots, g_m) + I^{d-e+1}$$

for each M with $|M| = e$. By Lemma 15.20.7 applied to I^{d-e}/I^{d-e+1} in the ring A/I^{d-e+1} this implies $\sum_{|N|=d-e} a_{N,M} f^N \in I^{d-e}(g_1, \ldots, g_m)$. Since $f_1, \ldots, f_n$ is quasi-regular in A this implies that $a_{N,M} \in J$ for each N, M with $|N| = d - e$ and $|M| = e$. This proves the lemma. □

Lemma 15.20.10. *Let A be a ring. Let I be an ideal generated by an H_1-regular sequence $f_1, \ldots, f_n$ in A. Let $g_1, \ldots, g_m \in A$ be elements whose images $\overline{g}_1, \ldots, \overline{g}_m$ form an H_1-regular sequence in A/I. Then $f_1, \ldots, f_n, g_1, \ldots, g_m$ is an H_1-regular sequence in A.*

Proof. We have to show that $H_1(A, f_1, \ldots, f_n, g_1, \ldots, g_m) = 0$. To do this consider the commutative diagram

$$\begin{array}{ccccccc} \wedge^2(A^{\oplus n+m}) & \longrightarrow & A^{\oplus n+m} & \longrightarrow & A & \longrightarrow & 0 \\ \downarrow & & \downarrow & & \downarrow & & \\ \wedge^2(A/I^{\oplus m}) & \longrightarrow & A/I^{\oplus m} & \longrightarrow & A/I & \longrightarrow & 0 \end{array}$$

Consider an element $(a_1, \ldots, a_{n+m}) \in A^{\oplus n+m}$ which maps to zero in A. Because $\overline{g}_1, \ldots, \overline{g}_m$ form an H_1-regular sequence in A/I we see that $(\overline{a}_{n+1}, \ldots, \overline{a}_{n+m})$ is the image of some element $\overline{\alpha}$ of $\wedge^2(A/I^{\oplus m})$. We can lift $\overline{\alpha}$ to an element $\alpha \in \wedge^2(A^{\oplus n+m})$ and substract the image of it in $A^{\oplus n+m}$ from our element $(a_1, \ldots, a_{n+m})$. Thus we may assume that $a_{n+1}, \ldots, a_{n+m} \in I$. Since $I = (f_1, \ldots, f_n)$ we can modify our element $(a_1, \ldots, a_{n+m})$ by linear combinations of the elements

$$(0, \ldots, g_j, 0, \ldots, 0, f_i, 0, \ldots, 0)$$

in the image of the top left horizontal arrow to reduce to the case that $a_{n+1}, \ldots, a_{n+m}$ are zero. In this case $(a_1, \ldots, a_n, 0, \ldots, 0)$ defines an element of $H_1(A, f_1, \ldots, f_n)$ which we assumed to be zero. □

Lemma 15.20.11. *Let A be a ring. Let $f_1, \ldots, f_n, g_1, \ldots, g_m \in A$ be an H_1-regular sequence. Then the images $\overline{g}_1, \ldots, \overline{g}_m$ in $A/(f_1, \ldots, f_n)$ form an H_1-regular sequence.*

Proof. Set $I = (f_1, \ldots, f_n)$. We have to show that any relation $\sum_{j=1,\ldots,m} \overline{a}_j \overline{g}_j$ in A/I is a linear combination of trivial relations. Because $I = (f_1, \ldots, f_n)$ we can lift this relation to a relation

$$\sum\nolimits_{j=1,\ldots,m} a_j g_j + \sum\nolimits_{i=1,\ldots,n} b_i f_i = 0$$

in A. By assumption this relation in A is a linear combination of trivial relations. Taking the image in A/I we obtain what we want. □

Lemma 15.20.12. *Let A be a ring. Let I be an ideal generated by a Koszul-regular sequence $f_1, \ldots, f_n$ in A. Let $g_1, \ldots, g_m \in A$ be elements whose images $\overline{g}_1, \ldots, \overline{g}_m$ form a Koszul-regular sequence in A/I. Then $f_1, \ldots, f_n, g_1, \ldots, g_m$ is a Koszul-regular sequence in A.*

Proof. Our assumptions say that $K_\bullet(A, f_1, \ldots, f_n)$ is a finite free resolution of A/I and $K_\bullet(A/I, \overline{g}_1, \ldots, \overline{g}_m)$ is a finite free resolution of $A/(f_i, g_j)$ over A/I. Then

$$\begin{aligned} K_\bullet(A, f_1, \ldots, f_n, g_1, \ldots, g_m) &= \mathrm{Tot}(K_\bullet(A, f_1, \ldots, f_n) \otimes_A K_\bullet(A, g_1, \ldots, g_m)) \\ &\cong A/I \otimes_A K_\bullet(A, g_1, \ldots, g_m) \\ &= K_\bullet(A/I, \overline{g}_1, \ldots, \overline{g}_m) \\ &\cong A/(f_i, g_j) \end{aligned}$$

The first equality by Lemma 15.19.12. The first quasi-isomorphism $\cong$ by (the dual of) Homology, Lemma 12.22.6 as the qth row of the double complex $K_\bullet(A, f_1, \ldots, f_n) \otimes_A K_\bullet(A, g_1, \ldots, g_m)$ is a resolution of $A/I \otimes_A K_q(A, g_1, \ldots, g_m)$. The second equality is clear. The last quasi-isomorphism by assumption. Hence we win. □

To conclude in the following lemma it is necessary to assume that both $f_1, \ldots, f_n$ and $f_1, \ldots, f_n, g_1, \ldots, g_m$ are Koszul-regular. A counter example to dropping the assumption that $f_1, \ldots, f_n$ is Koszul-regular is Examples, Lemma 78.8.1.

Lemma 15.20.13. *Let A be a ring. Let $f_1, \ldots, f_n, g_1, \ldots, g_m \in A$. If both $f_1, \ldots, f_n$ and $f_1, \ldots, f_n, g_1, \ldots, g_m$ are Koszul-regular sequences in A, then $\overline{g}_1, \ldots, \overline{g}_m$ in $A/(f_1, \ldots, f_n)$ form a Koszul-regular sequence.*

Proof. Set $I = (f_1, \ldots, f_n)$. Our assumptions say that $K_\bullet(A, f_1, \ldots, f_n)$ is a finite free resolution of A/I and $K_\bullet(A, f_1, \ldots, f_n, g_1, \ldots, g_m)$ is a finite free resolution of $A/(f_i, g_j)$ over A. Then

$$\begin{aligned} A/(f_i, g_j) &\cong K_\bullet(A, f_1, \ldots, f_n, g_1, \ldots, g_m) \\ &= \mathrm{Tot}(K_\bullet(A, f_1, \ldots, f_n) \otimes_A K_\bullet(A, g_1, \ldots, g_m)) \\ &\cong A/I \otimes_A K_\bullet(A, g_1, \ldots, g_m) \\ &= K_\bullet(A/I, \overline{g}_1, \ldots, \overline{g}_m) \end{aligned}$$

The first quasi-isomorphism $\cong$ by assumption. The first equality by Lemma 15.19.12. The second quasi-isomorphism by (the dual of) Homology, Lemma 12.22.6 as the qth row of the double complex $K_\bullet(A, f_1, \ldots, f_n) \otimes_A K_\bullet(A, g_1, \ldots, g_m)$ is a resolution of $A/I \otimes_A K_q(A, g_1, \ldots, g_m)$. The second equality is clear. Hence we win. □

Lemma 15.20.14. *Let R be a ring. Let I be an ideal generated by $f_1, \ldots, f_r \in R$.*

1. *If I can be generated by a quasi-regular sequence of length r, then $f_1, \ldots, f_r$ is a quasi-regular sequence.*
2. *If I can be generated by an H_1-regular sequence of length r, then $f_1, \ldots, f_r$ is an H_1-regular sequence.*
3. *If I can be generated by a Koszul-regular sequence of length r, then $f_1, \ldots, f_r$ is a Koszul-regular sequence.*

Proof. If I can be generated by a quasi-regular sequence of length r, then I/I^2 is free of rank r over R/I. Since $f_1, \ldots, f_r$ generate by assumption we see that the images $\overline{f}_i$ form a basis of I/I^2 over R/I. It follows that $f_1, \ldots, f_r$ is a quasi-regular sequence as all this means, besides the freeness of I/I^2, is that the maps $\mathrm{Sym}^n_{R/I}(I/I^2) \to I^n/I^{n+1}$ are isomorphisms.

We continue to assume that I can be generated by a quasi-regular sequence, say $g_1, \ldots, g_r$. Write $g_j = \sum a_{ij} f_i$. As $f_1, \ldots, f_r$ is quasi-regular according to the previous paragraph, we see that $\det(a_{ij})$ is invertible mod I. The matrix a_{ij} gives a map $R^{\oplus r} \to R^{\oplus r}$ which induces a map of Koszul complexes $\alpha : K_\bullet(R, f_1, \ldots, f_r) \to K_\bullet(R, g_1, \ldots, g_r)$, see Lemma 15.19.3. This map becomes an isomorphism on inverting $\det(a_{ij})$. Since the cohomology modules of both $K_\bullet(R, f_1, \ldots, f_r)$ and $K_\bullet(R, g_1, \ldots, g_r)$ are annihilated by I, see Lemma 15.19.6, we see that α is a quasi-isomorphism. Hence if $g_1, \ldots, g_r$ is H_1-regular, then so is $f_1, \ldots, f_r$. Similarly for Koszul-regular. □

Lemma 15.20.15. *Let $A \to B$ be a ring map. Let $f_1, \ldots, f_r$ be a sequence in B such that $B/(f_1, \ldots, f_r)$ is A-flat. Let $A \to A'$ be a ring map. Then the canonical map*

$$H_1(K_\bullet(B, f_1, \ldots, f_r)) \otimes_A A' \longrightarrow H_1(K_\bullet(B', f'_1, \ldots, f'_r))$$

is surjective, where $B' = B \otimes_A A'$ and $f'_i \in B'$ is the image of f_i.

Proof. The sequence

$$\wedge^2(B^{\oplus r}) \to B^{\oplus r} \to B \to B/J \to 0$$

is a complex of A-modules with B/J flat over A and cohomology group $H_1 = H_1(K_\bullet(B, f_1, \ldots, f_r))$ in the spot $B^{\oplus r}$. If we tensor this with A' we obtain a complex

$$\wedge^2((B')^{\oplus r}) \to (B')^{\oplus r} \to B' \to B'/J' \to 0$$

which is exact at B' and B'/J'. In order to compute its cohomology group $H'_1 = H_1(K_\bullet(B', f'_1, \ldots, f'_r))$ at $(B')^{\oplus r}$ we split the first sequence above into short exact sequences $0 \to J \to B \to B/J \to 0$ and $0 \to K \to B^{\oplus r} \to J \to 0$ and $\wedge^2(B^{\oplus r}) \to K \to H_1 \to 0$. Tensoring with A' over A we obtain the exact sequences

$$0 \to J \otimes_A A' \to B \otimes_A A' \to (B/J) \otimes_A A' \to 0$$
$$K \otimes_A A' \to B^{\oplus r} \otimes_A A' \to J \otimes_A A' \to 0$$
$$\wedge^2(B^{\oplus r}) \otimes_A A' \to K \otimes_A A' \to H_1 \otimes_A A' \to 0$$

where the first one is exact as B/J is flat over A, see Algebra, Lemma 10.36.11. Hence we conclude what we want. □

Lemma 15.20.16. *Let R be a ring. Let $a_1, \ldots, a_n \in R$ be elements such that $R \to R^{\oplus n}$, $x \mapsto (xa_1, \ldots, xa_n)$ is injective. Then the element $\sum a_i t_i$ of the polynomial ring $R[t_1, \ldots, t_n]$ is a nonzerodivisor.*

Proof. If one of the a_i is a unit this is just the statement that any element of the form $t_1 + a_2 t_2 + \ldots + a_n t_n$ is a nonzero divisor in the polynomial ring over R.

Case I: R is Noetherian. Let $\mathfrak{q}_j$, $j = 1, \ldots, m$ be the associated primes of R. We have to show that each of the maps

$$\sum a_i t_i : \mathrm{Sym}^d(R^{\oplus n}) \longrightarrow \mathrm{Sym}^{d+1}(R^{\oplus n})$$

is injective. As $\mathrm{Sym}^d(R^{\oplus n})$ is a free R-module its associated primes are $\mathfrak{q}_j$, $j = 1, \ldots, m$. For each j there exists an $i = i(j)$ such that $a_i \notin \mathfrak{q}_j$ because there exists an $x \in R$ with $\mathfrak{q}_j x = 0$ but $a_i x \neq 0$ for some i by assumption. Hence a_i is a unit in $R_{\mathfrak{q}_j}$ and the map is injective after localizing at $\mathfrak{q}_j$. Thus the map is injective, see Algebra, Lemma 10.60.18.

Case II: R general. We can write R as the union of Noetherian rings R_λ with $a_1, \ldots, a_n \in R_\lambda$. For each R_λ the result holds, hence the result holds for R. □

Lemma 15.20.17. *Let R be a ring. Let $f_1, \ldots, f_n$ be a Koszul-regular sequence in R. Consider the faithfully flat, smooth ring map*

$$R \longrightarrow S = R[\{t_{ij}\}_{i \leq j}, t_{11}^{-1}, t_{22}^{-1}, \ldots, t_{nn}^{-1}]$$

For $1 \leq i \leq n$ set

$$g_i = \sum\nolimits_{i \leq j} t_{ij} f_j \in S.$$

Then $g_1, \ldots, g_n$ is a regular sequence in S and $(f_1, \ldots, f_n)S = (g_1, \ldots, g_n)$.

Proof. The equality of ideals is obvious as the matrix

$$\begin{pmatrix} t_{11} & t_{12} & t_{13} & \ldots \\ 0 & t_{22} & t_{23} & \ldots \\ 0 & 0 & t_{33} & \ldots \\ \ldots & \ldots & \ldots & \ldots \end{pmatrix}$$

is invertible in S. Because $f_1, \ldots, f_n$ is a Koszul-regular sequence we see that the kernel of $R \to R^{\oplus n}$, $x \mapsto (xf_1, \ldots, xf_n)$ is zero (as it computes the $n$the Koszul homology of R w.r.t. $f_1, \ldots, f_n$). Hence by Lemma 15.20.16 we see that $g_1 = f_1 t_{11} + \ldots + f_n t_{1n}$ is a nonzerodivisor in $S' = R[t_{11}, t_{12}, \ldots, t_{1n}, t_{11}^{-1}]$. We see that $g_1, f_2, \ldots, f_n$ is a Koszul-sequence in S' by Lemma 15.20.4 and 15.20.14. We conclude that $\overline{f}_2, \ldots, \overline{f}_n$ is a Koszul-regular sequence in $S'/(g_2)$ by Lemma 15.20.13. Hence by induction on n we see that the images $\overline{g}_2, \ldots, \overline{g}_n$ of $g_2, \ldots, g_n$ in $S'/(g_2)[\{t_{ij}\}_{2 \leq i \leq j}, t_{22}^{-1}, \ldots, t_{nn}^{-1}]$ form a regular sequence. This in turn means that $g_1, \ldots, g_n$ forms a regular sequence in S. □

15.21. Regular ideals

We will discuss the notion of a regular ideal sheaf in great generality in Divisors, Section 30.12. Here we define the corresponding notion in the affine case, i.e., in the case of an ideal in a ring.

Definition 15.21.1. Let R be a ring and let $I \subset R$ be an ideal.

(1) We say I is a *regular ideal* if for every $\mathfrak{p} \in V(I)$ there exists a $g \in R$, $g \notin \mathfrak{p}$ and a regular sequence $f_1, \ldots, f_r \in R_g$ such that I_g is generated by $f_1, \ldots, f_r$.
(2) We say I is a *Koszul-regular ideal* if for every $\mathfrak{p} \in V(I)$ there exists a $g \in R$, $g \notin \mathfrak{p}$ and a Koszul-regular sequence $f_1, \ldots, f_r \in R_g$ such that I_g is generated by $f_1, \ldots, f_r$.
(3) We say I is a *H_1-regular ideal* if for every $\mathfrak{p} \in V(I)$ there exists a $g \in R$, $g \notin \mathfrak{p}$ and an H_1-regular sequence $f_1, \ldots, f_r \in R_g$ such that I_g is generated by $f_1, \ldots, f_r$.
(4) We say I is a *quasi-regular ideal* if for every $\mathfrak{p} \in V(I)$ there exists a $g \in R$, $g \notin \mathfrak{p}$ and a quasi-regular sequence $f_1, \ldots, f_r \in R_g$ such that I_g is generated by $f_1, \ldots, f_r$.

It is clear that given $I \subset R$ we have the implications

$$\begin{aligned} I \text{ is a regular ideal} &\Rightarrow I \text{ is a Koszul-regular ideal} \\ &\Rightarrow I \text{ is a } H_1\text{-regular ideal} \\ &\Rightarrow I \text{ is a quasi-regular ideal} \end{aligned}$$

see Lemmas 15.20.2 and 15.20.5. Such an ideal is always finitely generated.

Lemma 15.21.2. *A quasi-regular ideal is finitely generated.*

Proof. Let $I \subset R$ be a quasi-regular ideal. Since $V(I)$ is quasi-compact, there exist $g_1, \ldots, g_m \in R$ such that $V(I) \subset D(g_1) \cup \ldots \cup D(g_m)$ and such that I_{g_j} is generated by a quasi-regular sequence $g_{j1}, \ldots, g_{jr_j} \in R_{g_j}$. Write $g_{ji} = g'_{ji}/g_j^{e_{ij}}$ for some $g'_{ij} \in I$. Write $1 + x = \sum g_j h_j$ for some $x \in I$ which is possible as $V(I) \subset D(g_1) \cup \ldots \cup D(g_m)$. Note that $\mathrm{Spec}(R) = D(g_1) \cup \ldots \cup D(g_m) \bigcup D(x)$ Then I is generated by the elements g'_{ij} and x as these generate on each of the pieces of the cover, see Algebra, Lemma 10.22.2. □

Lemma 15.21.3. *Let $I \subset R$ be a quasi-regular ideal of a ring. Then I/I^2 is a finite projective R/I-module.*

Proof. This follows from Algebra, Lemma 10.73.2 and the definitions. □

We prove flat descent for Koszul-regular, H_1-regular, quasi-regular ideals.

Lemma 15.21.4. *Let $A \to B$ be a faithfully flat ring map. Let $I \subset A$ be an ideal. If IB is a Koszul-regular (resp. H_1-regular, resp. quasi-regular) ideal in B, then I is a Koszul-regular (resp. H_1-regular, resp. quasi-regular) ideal in A.*

Proof. We fix the prime $\mathfrak{p} \supset I$ throughout the proof. Assume IB is quasi-regular. By Lemma 15.21.2 IB is a finite module, hence I is a finite A-module by Algebra, Lemma 10.78.2. As $A \to B$ is flat we see that

$$I/I^2 \otimes_{A/I} B/IB = I/I^2 \otimes_A B = IB/(IB)^2.$$

As IB is quasi-regular, the B/IB-module $IB/(IB)^2$ is finite locally free. Hence I/I^2 is finite projective, see Algebra, Proposition 10.78.3. In particular, after replacing A by A_f for some $f \in A$, $f \notin \mathfrak{p}$ we may assume that I/I^2 is free of rank r. Pick $f_1, \ldots, f_r \in I$ which give a basis of I/I^2. By Nakayama's lemma (see Algebra, Lemma 10.18.1) we see that, after another replacement $A \rightsquigarrow A_f$ as above, I is generated by $f_1, \ldots, f_r$.

Proof of the "quasi-regular" case. Above we have seen that I/I^2 is free on the r-generators $f_1, \ldots, f_r$. To finish the proof in this case we have to show that the maps $\mathrm{Sym}^d(I/I^2) \to I^d/I^{d+1}$ are isomorphisms for each $d \geq 2$. This is clear as the faithfully flat base changes $\mathrm{Sym}^d(IB/(IB)^2) \to (IB)^d/(IB)^{d+1}$ are isomorphisms locally on B by assumption. Details omitted.

Proof of the "H_1-regular" and "Koszul-regular" case. Consider the sequence of elements $f_1, \ldots, f_r$ generating I we constructed above. By Lemma 15.20.14 we see that $f_1, \ldots, f_r$ map to a H_1-regular or Koszul-regular sequence in B_g for any $g \in B$ such that IB is generated by an H_1-regular or Koszul-regular sequence. Hence $K_\bullet(A, f_1, \ldots, f_r) \otimes_A B_g$ has vanishing H_1 or H_i, $i > 0$. Since the homology of $K_\bullet(B, f_1, \ldots, f_r) = K_\bullet(A, f_1, \ldots, f_r) \otimes_A B$ is annihilated by IB (see Lemma 15.19.6) and since $V(IB) \subset \bigcup_{g \text{ as above}} D(g)$ we conclude that $K_\bullet(A, f_1, \ldots, f_r) \otimes_A B$ has vanishing homology in degree 1 or all positive degrees. Using that $A \to B$ is faithfully flat we conclude that the same is true for $K_\bullet(A, f_1, \ldots, f_r)$. □

Lemma 15.21.5. *Let A be a ring. Let $I \subset J \subset A$ be ideals. Assume that $J/I \subset A/I$ is a H_1-regular ideal. Then $I \cap J^2 = IJ$.*

Proof. Follows immediately from Lemma 15.20.8 by localizing. □

15.22. Local complete intersection maps

We can use the material above to define a local complete intersection map between rings using presentations by (finite) polynomial algebras.

Lemma 15.22.1. *Let $A \to B$ be a finite type ring map. If for some presentation $\alpha : A[x_1, \ldots, x_n] \to B$ the kernel I is a Koszul-regular ideal then for any presentation $\beta : A[y_1, \ldots, y_m] \to B$ the kernel J is a Koszul-regular ideal.*

Proof. Choose $f_j \in A[x_1, \ldots, x_n]$ with $\alpha(f_j) = \beta(y_j)$ and $g_i \in A[y_1, \ldots, y_m]$ with $\beta(g_i) = \alpha(x_i)$. Then we get a commutative diagram

$$\begin{array}{ccc} A[x_1, \ldots, x_n, y_1, \ldots, y_m] & \xrightarrow[y_j \mapsto f_j]{} & A[x_1, \ldots, x_n] \\ \downarrow{\scriptstyle x_i \mapsto g_i} & & \downarrow \\ A[y_1, \ldots, y_m] & \longrightarrow & B \end{array}$$

Note that the kernel K of $A[x_i, y_j] \to B$ is equal to $K = (I, y_j - f_j) = (J, x_i - f_i)$. In particular, as I is finitely generated by Lemma 15.21.2 we see that $J = K/(x_i - f_i)$ is finitely generated too.

Pick a prime $\mathfrak{q} \subset B$. Since $I/I^2 \oplus B^{\oplus m} = J/J^2 \oplus B^{\oplus n}$ (Algebra, Lemma 10.127.15) we see that

$$\dim J/J^2 \otimes_B \kappa(\mathfrak{q}) + n = \dim I/I^2 \otimes_B \kappa(\mathfrak{q}) + m.$$

Pick $p_1, \ldots, p_t \in I$ which map to a basis of $I/I^2 \otimes \kappa(\mathfrak{q}) = I \otimes_{A[x_i]} \kappa(\mathfrak{q})$. Pick $q_1, \ldots, q_s \in J$ which map to a basis of $J/J^2 \otimes \kappa(\mathfrak{q}) = J \otimes_{A[y_j]} \kappa(\mathfrak{q})$. So $s+n = t+m$. By Nakayama's lemma there exist $h \in A[x_i]$ and $h' \in A[y_j]$ both mapping to a nonzero element of $\kappa(\mathfrak{q})$ such that $I_h = (p_1, \ldots, p_t)$ in $A[x_i, 1/h]$ and $J_{h'} = (q_1, \ldots, q_s)$ in $A[y_j, 1/h']$. As I is Koszul-regular we may also assume that I_h is generated by a Koszul regular sequence. This sequence must necessarily have length $t = \dim I/I^2 \otimes_B \kappa(\mathfrak{q})$, hence we see that $p_1, \ldots, p_t$ is a Koszul-regular sequence by Lemma 15.20.14. As also $y_1 - f_1, \ldots, y_m - f_m$ is a regular sequence we conclude

$$y_1 - f_1, \ldots, y_m - f_m, p_1, \ldots, p_t$$

is a Koszul-regular sequence in $A[x_i, y_j, 1/h]$ (see Lemma 15.20.12). This sequence generates the ideal K_h. Hence the ideal $K_{hh'}$ is generated by a Koszul-regular sequence of length $m + t = n + s$. But it is also generated by the sequence

$$x_1 - g_1, \ldots, x_n - g_n, q_1, \ldots, q_s$$

of the same length which is thus a Koszul-regular sequence by Lemma 15.20.14. Finally, by Lemma 15.20.13 we conclude that the images of $q_1, \ldots, q_s$ in

$$A[x_i, y_j, 1/hh'](x_1 - g_1, \ldots, x_n - g_n) \cong A[y_j, 1/h'']$$

form a Koszul-regular sequence generating $J_{h''}$. Since h'' is the image of hh' it doesn't map to zero in $\kappa(\mathfrak{q})$ and we win. □

This lemma allows us to make the following definition.

Definition 15.22.2. A ring map $A \to B$ is called a *local complete intersection* if it is of finite type and for some (equivalently any) presentation $B = A[x_1, \ldots, x_n]/I$ the ideal I is Koszul-regular.

This notion is local.

Lemma 15.22.3. *Let $R \to S$ be a ring map. Let $g_1, \ldots, g_m \in S$ generate the unit ideal. If each $R \to S_{g_j}$ is a local complete intersection so is $R \to S$.*

Proof. Let $S = R[x_1, \ldots, x_n]/I$ be a presentation. Pick $h_j \in R[x_1, \ldots, x_n]$ mapping to g_j in S. Then $R[x_1, \ldots, x_n, x_{n+1}]/(I, x_{n+1}h_j - 1)$ is a presentation of S_{g_j}. Hence $I_j = (I, x_{n+1}h_j - 1)$ is a Koszul-regular ideal in $R[x_1, \ldots, x_n, x_{n+1}]$. Pick a prime $I \subset \mathfrak{q} \subset R[x_1, \ldots, x_n]$. Then $h_j \notin \mathfrak{q}$ for some j and $\mathfrak{q}_j = (\mathfrak{q}, x_{n+1}h_j - 1)$ is a prime ideal of $V(I_j)$ lying over $\mathfrak{q}$. Pick $f_1, \ldots, f_r \in I$ which map to a basis of $I/I^2 \otimes \kappa(\mathfrak{q})$. Then $x_{n+1}h_j - 1, f_1, \ldots, f_r$ is a sequence of elements of I_j which map to a basis of $I_j \otimes \kappa(\mathfrak{q}_j)$. By Nakayama's lemma there exists an $h \in R[x_1, \ldots, x_n, x_{n+1}]$ such that $(I_j)_h$ is generated by $x_{n+1}h_j - 1, f_1, \ldots, f_r$. We may also assume that $(I_j)_h$ is generated by a Koszul regular sequence of some length e. Looking at the dimension of $I_j \otimes \kappa(\mathfrak{q}_j)$ we see that $e = r + 1$. Hence by Lemma 15.20.14 we see that $x_{n+1}h_j - 1, f_1, \ldots, f_r$ is a Koszul-regular sequence generating $(I_j)_h$ for some $h \in R[x_1, \ldots, x_n, x_{n+1}]$, $h \notin \mathfrak{q}_j$. By Lemma 15.20.13 we see that $I_{h'}$ is generated by a Koszul-regular sequence for some $h' \in R[x_1, \ldots, x_n]$, $h' \notin \mathfrak{q}$ as desired. □

Lemma 15.22.4. *Let R be a ring. Let $R[x_1, \ldots, x_n]$. If $R[x_1, \ldots, x_n]/(f_1, \ldots, f_c)$ be a relative global complete intersection. Then $f_1, \ldots, f_c$ is a Koszul regular sequence.*

Proof. Recall that the homology groups $H_i(K_\bullet(f_\bullet))$ are annihilated by the ideal $(f_1, \ldots, f_c)$. Hence it suffices to show that $H_i(K_\bullet(f_\bullet))_{\mathfrak{q}}$ is zero for all primes $\mathfrak{q} \subset R[x_1, \ldots, x_n]$ containing $(f_1, \ldots, f_c)$. This follows from Algebra, Lemma 10.129.13 and the fact that a regular sequence is Koszul regular (Lemma 15.20.2). □

Lemma 15.22.5. *A syntomic ring map is a local complete intersection.*

Proof. Combine Lemmas 15.22.4 and 15.22.3 and Algebra, Lemma 10.129.15. □

For a local complete intersection $R \to S$ we have $H_n(L_{S/R}) = 0$ for $n \geq 2$. Since we haven't (yet) defined the full cotangent complex we can't state and prove this, but we can deduce one of the consequences.

Lemma 15.22.6. *Let $A \to B \to C$ be ring maps. Assume $B \to C$ is a local complete intersection homomorphism. Choose a presentation $\alpha : A[x_s, s \in S] \to B$ with kernel I. Choose a presentation $\beta : B[y_1, \ldots, y_m] \to C$ with kernel J. Let $\gamma : A[x_s, y_t] \to C$ be the induced presentation of C with kernel K. Then we get a canonical commutative diagram*

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \Omega_{A[x_s]/A} \otimes C & \longrightarrow & \Omega_{A[x_s,y_t]/A} \otimes C & \longrightarrow & \Omega_{B[y_t]/B} \otimes C & \longrightarrow & 0 \\ & & \uparrow & & \uparrow & & \uparrow & & \\ 0 & \longrightarrow & I/I^2 \otimes C & \longrightarrow & K/K^2 & \longrightarrow & J/J^2 & \longrightarrow & 0 \end{array}$$

with exact rows. In particular, the six term exact sequence of Algebra, Lemma 10.127.4 can be completed with a zero on the left, i.e., the sequence

$$0 \to H_1(NL_{B/A} \otimes_B C) \to H_1(L_{C/A}) \to H_1(L_{C/B}) \to \Omega_{B/A} \otimes_B C \to \Omega_{C/A} \to \Omega_{C/B} \to 0$$

is exact.

Proof. The only thing to prove is the injectivity of the map $I/I^2 \otimes C \to K/K^2$. By assumption the ideal J is Koszul-regular. Hence we have $IA[x_s, y_j] \cap K^2 = IK$ by Lemma 15.21.5. This means that the kernel of $K/K^2 \to J/J^2$ is isomorphic to $IA[x_s, y_j]/IK$. Since $I/I^2 \otimes_A C = IA[x_s, y_j]/IK$ this provides us with the desired injectivity of $I/I^2 \otimes_A C \to K/K^2$ so that the result follows from the snake lemma, see Homology, Lemma 12.5.17. □

Lemma 15.22.7. *Let $A \to B \to C$ be ring maps. If $B \to C$ is a filtered colimit of local complete intersection homomorphisms then the conclusion of Lemma 15.22.6 remains valid.*

Proof. Follows from Lemma 15.22.6 and Algebra, Lemma 10.127.9. □

15.23. Cartier's equality and geometric regularity

A reference for this section and the next is [**Mat70**, Section 39]. In order to comfortably read this section the reader should be familiar with the naive cotangent complex and its properties, see Algebra, Section 10.127.

Lemma 15.23.1 (Cartier equality)**.** *Let K/k be a finitely generated field extension. Then $\Omega_{K/k}$ and $H_1(L_{K/k})$ are finite dimensional and $trdeg_k(K) = \dim_K \Omega_{K/k} - \dim_K H_1(L_{K/k})$.*

Proof. We can find a global complete intersection $A = k[x_1, \ldots, x_n]/(f_1, \ldots, f_c)$ over k such that K is isomorphic to the fraction field of A, see Algebra, Lemma 10.145.11 and its proof. In this case we see that $NL_{K/k}$ is homotopy equivalent to the complex

$$\bigoplus\nolimits_{j=1,\ldots,c} K \longrightarrow \bigoplus\nolimits_{i=1,\ldots,n} K\mathrm{d}x_i$$

by Algebra, Lemmas 10.127.2 and 10.127.13. The transcendence degree of K over k is the dimension of A (by Algebra, Lemma 10.110.1) which is $n - c$ and we win. □

Lemma 15.23.2. *Let $K \subset L \subset M$ be field extensions. Then the Jacobi-Zariski sequence*

$$0 \to H_1(L_{L/K}) \otimes_L M \to H_1(L_{M/K}) \to H_1(L_{M/L}) \to \Omega_{L/K} \otimes_L M \to \Omega_{M/K} \to \Omega_{M/L} \to 0$$

is exact.

Proof. Combine Lemma 15.22.7 with Algebra, Lemma 10.145.11. □

Lemma 15.23.3. *Given a commutative diagram of fields*

$$\begin{array}{ccc} K & \longrightarrow & K' \\ \uparrow & & \uparrow \\ k & \longrightarrow & k' \end{array}$$

with $k \subset k'$ and $K \subset K'$ finitely generated field extensions the kernel and cokernel of the maps

$$\alpha : \Omega_{K/k} \otimes_K K' \to \Omega_{K'/k'} \quad \text{and} \quad \beta : H_1(L_{K/k}) \otimes_K K' \to H_1(L_{K'/k'})$$

are finite dimensional and

$$\dim \mathit{Ker}(\alpha) - \dim \mathit{Coker}(\alpha) - \dim \mathit{Ker}(\beta) + \dim \mathit{Coker}(\beta) = \mathit{trdeg}_k(k') - \mathit{trdeg}_K(K')$$

Proof. The Jacobi-Zariski sequences for $k \subset k' \subset K'$ and $k \subset K \subset K'$ are

$$0 \to H_1(L_{k'/k}) \otimes K' \to H_1(L_{K'/k}) \to H_1(L_{K'/k'}) \to \Omega_{k'/k} \otimes K' \to \Omega_{K'/k} \to \Omega_{K'/k} \to 0$$

and

$$0 \to H_1(L_{K/k}) \otimes K' \to H_1(L_{K'/k}) \to H_1(L_{K'/K}) \to \Omega_{K/k} \otimes K' \to \Omega_{K'/k} \to \Omega_{K'/K} \to 0$$

By Lemma 15.23.1 the vector spaces $\Omega_{k'/k}$, $\Omega_{K'/K}$, $H_1(L_{K'/K})$, and $H_1(L_{k'/k})$ are finite dimensional and the alternating sum of their dimensions is $\mathrm{trdeg}_k(k') - \mathrm{trdeg}_K(K')$. The lemma follows. □

15.24. Geometric regularity

Let k be a field. Let $(A, \mathfrak{m}, K)$ be a Noetherian local k-algebra. The Jacobi-Zariski sequence (Algebra, Lemma 10.127.4) is a canonical exact sequence

$$H_1(L_{K/k}) \to \mathfrak{m}/\mathfrak{m}^2 \to \Omega_{A/k} \otimes_A K \to \Omega_{K/k} \to 0$$

because $H_1(L_{K/A}) = \mathfrak{m}/\mathfrak{m}^2$ by Algebra, Lemma 10.127.6. We will show that exactness on the left of this sequence characterizes whether or not a regular local ring A is geometrically regular over k. We will link this to the notion of formal smoothness in Section 15.28.

Proposition 15.24.1. *Let k be a field of characteristic $p > 0$. Let $(A, \mathfrak{m}, K)$ be a Noetherian local k-algebra. The following are equivalent*

(1) *A is geometrically regular over k,*
(2) *for all $k \subset k' \subset k^{1/p}$ finite over k the ring $A \otimes_k k'$ is regular,*
(3) *A is regular and the canonical map $H_1(L_{K/k}) \to \mathfrak{m}/\mathfrak{m}^2$ is injective, and*
(4) *A is regular and the map $\Omega_{k/\mathbf{F}_p} \otimes_k K \to \Omega_{A/\mathbf{F}_p} \otimes_A K$ is injective.*

Proof. Proof of (3) $\Rightarrow$ (1). Assume (3). Let $k \subset k'$ be a finite purely inseparable extension. Set $A' = A \otimes_k k'$. This is a local ring with maximal ideal $\mathfrak{m}'$. Set $K' = A'/\mathfrak{m}'$. We get a commutative diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & H_1(L_{K/k}) \otimes K' & \longrightarrow & \mathfrak{m}/\mathfrak{m}^2 \otimes K' & \longrightarrow & \Omega_{A/k} \otimes_A K' & \longrightarrow & \Omega_{K/k} \otimes K' \longrightarrow 0 \\ & & \downarrow{\scriptstyle \beta} & & \downarrow & & \downarrow{\scriptstyle \cong} & & \downarrow{\scriptstyle \alpha} \\ & & H_1(L_{K'/k'}) & \longrightarrow & \mathfrak{m}'/(\mathfrak{m}')^2 & \longrightarrow & \Omega_{A'/k'} \otimes_{A'} K' & \longrightarrow & \Omega_{K'/k'} \longrightarrow 0 \end{array}$$

with exact rows. The third vertical arrow is an isomorphism by base change for modules of differentials (Algebra, Lemma 10.125.12). Thus α is surjective. By Lemma 15.23.3 we have

$$\dim \mathrm{Ker}(\alpha) - \dim \mathrm{Ker}(\beta) + \dim \mathrm{Coker}(\beta) = 0$$

(and these dimensions are all finite). A diagram chase shows that $\dim \mathfrak{m}'/(\mathfrak{m}')^2 \leq \dim \mathfrak{m}/\mathfrak{m}^2$. However, since $A \to A'$ is finite flat we see that $\dim(A) = \dim(A')$, see Algebra, Lemma 10.106.6. Hence A' is regular by definition.

Equivalence of (3) and (4). Consider the Jacobi-Zariski sequences for rows of the commutative diagram

$$\begin{array}{ccccc} \mathbf{F}_p & \longrightarrow & A & \longrightarrow & K \\ \uparrow & & \uparrow & & \uparrow \\ \mathbf{F}_p & \longrightarrow & k & \longrightarrow & K \end{array}$$

to get a commutative diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \mathfrak{m}/\mathfrak{m}^2 & \longrightarrow & \Omega_{A/\mathbf{F}_p} \otimes_A K & \longrightarrow & \Omega_{K/\mathbf{F}_p} & \longrightarrow & 0 \\ & & \uparrow & & \uparrow & & \uparrow & & \uparrow \\ 0 & \longrightarrow & H_1(L_{K/k}) & \longrightarrow & \Omega_{k/\mathbf{F}_p} \otimes_k K & \longrightarrow & \Omega_{K/\mathbf{F}_p} & \longrightarrow & \Omega_{K/k} \longrightarrow 0 \end{array}$$

with exact rows. We have used that $H_1(L_{K/A}) = \mathfrak{m}/\mathfrak{m}^2$ and that $H_1(L_{K/\mathbf{F}_p}) = 0$ as $K/\mathbf{F}_p$ is separable, see Algebra, Proposition 10.145.9. Thus it is clear that the kernels of $H_1(L_{K/k}) \to \mathfrak{m}/\mathfrak{m}^2$ and $\Omega_{k/\mathbf{F}_p} \otimes_k K \to \Omega_{A/\mathbf{F}_p} \otimes_A K$ have the same dimension.

Proof of (2) $\Rightarrow$ (4) following Faltings, see [**Fal78**]. Let $a_1, \ldots, a_n \in k$ be elements such that $\mathrm{d}a_1, \ldots, \mathrm{d}a_n$ are linearly independent in $\Omega_{k/\mathbf{F}_p}$. Consider the field extension $k' = k(a_1^{1/p}, \ldots, a_n^{1/p})$. By Algebra, Lemma 10.145.3 we see that $k' = k[x_1, \ldots, x_n]/(x_1^p - a_1, \ldots, x_n^p - a_n)$. In particular we see that the naive cotangent complex of k'/k is homotopic to the complex $\bigoplus_{j=1,\ldots,n} k' \to \bigoplus_{i=1,\ldots,n} k'$ with the zero differential as $\mathrm{d}(x_j^p - a_j) = 0$ in $\Omega_{k[x_1,\ldots,x_n]/k}$. Set $A' = A \otimes_k k'$ and

$K' = A'/\mathfrak{m}'$ as above. By Algebra, Lemma 10.127.8 we see that $NL_{A'/A}$ is homotopy equivalent to the complex $\bigoplus_{j=1,\ldots,n} A' \to \bigoplus_{i=1,\ldots,n} A'$ with the zero differential, i.e., $H_1(L_{A'/A})$ and $\Omega_{A'/A}$ are free of rank n. The Jacobi-Zariski sequence for $\mathbf{F}_p \to A \to A'$ is

$$H_1(L_{A'/A}) \to \Omega_{A/\mathbf{F}_p} \otimes_A A' \to \Omega_{A'/\mathbf{F}_p} \to \Omega_{A'/A} \to 0$$

Using the presentation $A[x_1, \ldots, x_n] \to A'$ with kernel $(x_j^p - a_j)$ we see, unwinding the maps in Algebra, Lemma 10.127.4, that the jth basis vector of $H_1(L_{A'/A})$ maps to $\mathrm{d}a_j \otimes 1$ in $\Omega_{A/\mathbf{F}_p} \otimes A'$. As $\Omega_{A'/A}$ is free (hence flat) we get on tensoring with K' an exact sequence

$$K'^{\oplus n} \to \Omega_{A/\mathbf{F}_p} \otimes_A K' \xrightarrow{\beta} \Omega_{A'/\mathbf{F}_p} \otimes_{A'} K' \to K'^{\oplus n} \to 0$$

We conclude that the elements $\mathrm{d}a_j \otimes 1$ generate $\mathrm{Ker}(\beta)$ and we have to show that are linearly independent, i.e., we have to show $\dim(\mathrm{Ker}(\beta)) = n$. Consider the following big diagram

$$\begin{array}{ccccccccc}
0 & \longrightarrow & \mathfrak{m}'/(\mathfrak{m}')^2 & \longrightarrow & \Omega_{A'/\mathbf{F}_p} \otimes K' & \longrightarrow & \Omega_{K'/\mathbf{F}_p} & \longrightarrow & 0 \\
 & & \uparrow{\scriptstyle \alpha} & & \uparrow{\scriptstyle \beta} & & \uparrow{\scriptstyle \gamma} & & \\
0 & \longrightarrow & \mathfrak{m}/\mathfrak{m}^2 \otimes K' & \longrightarrow & \Omega_{A/\mathbf{F}_p} \otimes K' & \longrightarrow & \Omega_{K/\mathbf{F}_p} \otimes K' & \longrightarrow & 0
\end{array}$$

By Lemma 15.23.1 and the Jacobi-Zariski sequence for $\mathbf{F}_p \to K \to K'$ we see that the kernel and cokernel of γ have the same finite dimension. By assumption A' is regular (and of the same dimension as A, see above) hence the kernel and cokernel of α have the same dimension. It follows that the kernel and cokernel of β have the same dimension which is what we wanted to show.

The implication (1) $\Rightarrow$ (2) is trivial. This finishes the proof of the proposition. $\square$

Lemma 15.24.2. *Let k be a field of characteristic $p > 0$. Let $(A, \mathfrak{m}, K)$ be a Noetherian local k-algebra. Assume A is geometrically regular over k. Let $k \subset F \subset K$ be a finitely generated subextension. Let $\varphi : k[y_1, \ldots, y_m] \to A$ be a k-algebra map such that y_i maps to an element of F in K and such that $dy_1, \ldots, dy_m$ map to a basis of $\Omega_{F/k}$. Set $\mathfrak{p} = \varphi^{-1}(\mathfrak{m})$. Then*

$$k[y_1, \ldots, y_m]_{\mathfrak{p}} \to A$$

is flat and $A/\mathfrak{p}A$ is regular.

Proof. Set $A_0 = k[y_1, \ldots, y_m]_{\mathfrak{p}}$ with maximal ideal $\mathfrak{m}_0$ and residue field K_0. Note that $\Omega_{A_0/k}$ is free of rank m and $\Omega_{A_0/k} \otimes K_0 \to \Omega_{K_0/k}$ is an isomorphism. It is clear that A_0 is geometrically regular over k. Hence $H_1(L_{K_0/k}) \to \mathfrak{m}_0/\mathfrak{m}_0^2$ is an isomorphism, see Proposition 15.24.1. Now consider

$$\begin{array}{ccc}
H_1(L_{K_0/k}) \otimes K & \longrightarrow & \mathfrak{m}_0/\mathfrak{m}_0^2 \otimes K \\
\downarrow & & \downarrow \\
H_1(L_{K/k}) & \longrightarrow & \mathfrak{m}/\mathfrak{m}^2
\end{array}$$

Since the left vertical arrow is injective by Lemma 15.23.2 and the lower horizontal by Proposition 15.24.1 we conclude that the right vertical one is too. Hence a

regular system of parameters in A_0 maps to part of a regular system of parameters in A. We win by Algebra, Lemmas 10.122.2 and 10.101.3. □

15.25. Topological rings and modules

Let's quickly discuss some properties of topological abelian groups. An abelian group M is a *topological abelian group* if M is endowed with a topology such that addition $M \times M \to M$ is continuous. A *homomorphism of topological abelian groups* is just a homomorphism of abelian groups which is continuous. The category of commutative topological groups is additive and has kernels and cokernels, but is not abelian (as the axiom Im = Coim doesn't hold). If $N \subset M$ is a subgroup, then we think of N and M/N as topological groups also, namely using the induced topology on N and the quotient topology on M/N (i.e., such that $M \to M/N$ is submersive). Note that if $N \subset M$ is an open subgroup, then the topology on M/N is discrete.

We say the topology on M is *linear* if there exists a fundamental system of neighbourhoods of 0 consisting of subgroups. If so then these subgroups are also open. An example is the following. Let I be a directed partially ordered set and let G_i be an inverse system of (discrete) abelian groups over I. Then

$$G = \lim_{i \in I} G_i$$

with the inverse limit topology is linearly topologized with a fundamental system of neighbourhoods of 0 given by $\mathrm{Ker}(G \to G_i)$. Conversely, let M be a linearly topologized abelian group. Choose any fundamental system of open subgroups $U_i \subset M$, $i \in I$ (i.e., the U_i form a fundamental system of open neighbourhoods and each U_i is a subgroup of M). Setting $i \geq i' \Leftrightarrow U_i \subset U_{i'}$ we see that I is a directed partially ordered set. We obtain a homomorphism of linearly topologized abelian groups

$$c : M \longrightarrow \lim_{i \in I} M/U_i.$$

It is clear that M is *separated* (as a topological space) if and only if c is injective. We say that M is *complete* if c is an isomorphism[2]. We leave it to the reader to check that this condition is independent of the choice of fundamental system of open subgroups $\{U_i\}_{i \in I}$ chosen above. In fact the topological abelian group $M^\wedge = \lim_{i \in I} M/U_i$ is independent of this choice and is sometimes called the *completion* of M. Any $G = \lim G_i$ as above is complete, in particular, the completion $M^\wedge$ is always complete.

Definition 15.25.1 (Topological rings)**.** Let R be a ring and let M be an R-module.

(1) We say R is a *topological ring* if R is endowed with a topology such that both addition and multiplication are continuous as maps $R \times R \to R$ where $R \times R$ has the product topology. In this case we say M is a *topological module* if M is endowed with a topology such that addition $M \times M \to M$ and scalar multiplication $R \times M \to M$ are continuous.

[2]We include being separated as part of being complete as we'd like to have a unique limits in complete groups. There is a definition of completeness for any topological group, agreeing, modulo the separation issue, with this one in our special case.

(2) A *homomorphism of topological modules* is just a continuous R-module map. A *homomorphism of topological rings* is a ring homomorphism which is continuous for the given topologies.
(3) We say M is *linearly topologized* if 0 has a fundamental system of neighbourhoods consisting of submodules. We say R is *linearly topologized* if 0 has a fundamental system of neighbourhoods consisting of ideals.
(4) If R is linearly topologized, we say that $I \subset R$ is an *ideal of definition* if I is open and if every neighbourhood of 0 contains I^n for some n.
(5) If R is linearly topologized, we say that R is *pre-admissible* if R has an ideal of definition.
(6) If R is linearly topologized, we say that R is *admissible* if it is pre-admissible and complete[3].
(7) If R is linearly topologized, we say that R is *pre-adic* if there exists an ideal of definition I such that $\{I^n\}_{n\geq 0}$ forms a fundamental system of neighbourhoods of 0.
(8) If R is linearly topologized, we say that R is *adic* if R is pre-adic and complete.

Note that a (pre)adic ring is the same thing as a (pre)admissible ring which has an ideal of definition I such that I^n is open for all $n \geq 1$.

Let R be a ring and let M be an R-module. Let $I \subset R$ be an ideal. Then we can consider the linear topology on R which has $\{I^n\}_{n\geq 0}$ as a fundamental system of neighbourhoods of 0. This topology is called the I-adic topology; R is a pre-adic topological ring in the I-adic topology[4]. Moreover, the linear topology on M which has $\{I^n M\}_{n\geq 0}$ as a fundamental system of open neighbourhoods of 0 turns M into a topological R-module. This is called the *I-adic topology* on M. We see that M is I-adically complete (as defined in Algebra, Definition 10.91.5) if and only M is complete in the I-adic topology[5]. In particular, we see that R is I-adically complete if and only if R is an adic topological ring in the I-adic topology.

As a special case, note that the discrete topology is the 0-adic topology and that any ring in the discrete topology is adic.

Lemma 15.25.2. *Let $\varphi : R \to S$ be a ring map. Let $I \subset R$ and $J \subset S$ be ideals and endow R with the I-adic topology and S with the J-adic topology. Then φ is a homomorphism of topological rings if and only if $\varphi(I^n) \subset J$ for some $n \geq 1$.*

Proof. Omitted. □

15.26. Formally smooth maps of topological rings

There is a version of formal smoothness which applies to homomorphisms of topological rings.

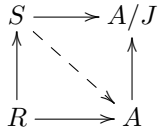
Definition 15.26.1. Let $R \to S$ be a homomorphism of topological rings with R and S linearly topologized. We say S is *formally smooth over* R if for every

[3] By our conventions this includes separated.

[4] Thus the I-adic topology is sometimes called the I-pre-adic topology.

[5] It may happen that the I-adic completion $M^\wedge$ is not I-adically complete, even though $M^\wedge$ is always complete with respect to the limit topology. If I is finitely generated then the I-adic topology and the limit topology on $M^\wedge$ agree, see Algebra, Lemma 10.91.7 and its proof.

commutative solid diagram



of homomorphisms of topological rings where A is a discrete ring and $J \subset A$ is an ideal of square zero, a dotted arrow exists which makes the diagram commute.

We will mostly use this notion when given ideals $\mathfrak{m} \subset R$ and $\mathfrak{n} \subset S$ and we endow R with the $\mathfrak{m}$-adic topology and S with the $\mathfrak{n}$-adic topology. Continuity of $\varphi : R \to S$ holds if and only if $\varphi(\mathfrak{m}^m) \subset \mathfrak{n}$ for some $m \geq 1$, see Lemma 15.25.2. It turns out that in this case only the topology on S is relevant.

Lemma 15.26.2. *Let $\varphi : R \to S$ be a ring map.*

(1) *If $R \to S$ is formally smooth in the sense of Algebra, Definition 10.131.1, then $R \to S$ is formally smooth for any linear topology on R and any pre-adic topology on S such that $R \to S$ is continuous.*
(2) *Let $\mathfrak{n} \subset S$ and $\mathfrak{m} \subset R$ ideals such that φ is continuous for the $\mathfrak{m}$-adic topology on R and the $\mathfrak{n}$-adic topology on S. Then the following are equivalent*
   (a) *φ is formally smooth for the $\mathfrak{m}$-adic topology on R and the $\mathfrak{n}$-adic topology on S, and*
   (b) *φ is formally smooth for the discrete topology on R and the $\mathfrak{n}$-adic topology on S.*

Proof. Assume $R \to S$ is formally smooth in the sense of Algebra, Definition 10.131.1. If S has a pre-adic topology, then there exists an ideal $\mathfrak{n} \subset S$ such that S has the $\mathfrak{n}$-adic topology. Suppose given a solid commutative diagram as in Definition 15.26.1. Continuity of $S \to A/J$ means that $\mathfrak{n}^k$ maps to zero in A/J for some $k \geq 1$, see Lemma 15.25.2. We obtain a ring map $\psi : S \to A$ from the assumed formal smoothness of S over R. Then $\psi(\mathfrak{n}^k) \subset J$ hence $\psi(\mathfrak{n}^{2k}) = 0$ as $J^2 = 0$. Hence ψ is continuous by Lemma 15.25.2. This proves (1).

The proof of (2)(b) $\Rightarrow$ (2)(a) is the same as the proof of (1). Assume (2)(a). Suppose given a solid commutative diagram as in Definition 15.26.1 where we use the discrete topology on R. Since φ is continuous we see that $\varphi(\mathfrak{m}^n) \subset \mathfrak{n}$ for some $m \geq 1$. As $S \to A/J$ is continuous we see that $\mathfrak{n}^k$ maps to zero in A/J for some $k \geq 1$. Hence $\mathfrak{m}^{nk}$ maps in J under the map $R \to A$. Thus $\mathfrak{m}^{2nk}$ maps to zero in A and we see that $R \to A$ is continuous in the $\mathfrak{m}$-adic topology. Thus (2)(a) gives a dotted arrow as desired. □

Definition 15.26.3. Let $R \to S$ be a ring map. Let $\mathfrak{n} \subset S$ be an ideal. If the equivalent conditions (2)(a) and (2)(b) of Lemma 15.26.2 hold, then we say $R \to S$ is *formally smooth for the $\mathfrak{n}$-adic topology.*

This property is inherited by the completions.

Lemma 15.26.4. *Let $(R, \mathfrak{m})$ and $(S, \mathfrak{n})$ be rings endowed with finitely generated ideals. Endow R and S with the $\mathfrak{m}$-adic and $\mathfrak{n}$-adic topologies. Let $R \to S$ be a homomorphism of topological rings. The following are equivalent*

(1) *$R \to S$ is formally smooth for the $\mathfrak{n}$-adic topology,*
(2) *$R \to S^\wedge$ is formally smooth for the $\mathfrak{n}^\wedge$-adic topology,*

(3) $R^\wedge \to S^\wedge$ *is formally smooth for the* $\mathfrak{n}^\wedge$*-adic topology.*

Here $R^\wedge$ *and* $S^\wedge$ *are the* $\mathfrak{m}$*-adic and* $\mathfrak{n}$*-adic completions of* R *and* S*.*

Proof. The assumption that $\mathfrak{m}$ is finitely generated implies that $R^\wedge$ is $\mathfrak{m}R^\wedge$-adically complete, that $\mathfrak{m}R^\wedge = \mathfrak{m}^\wedge$ and that $R^\wedge/\mathfrak{m}^n R^\wedge = R/\mathfrak{m}^n$, see Algebra, Lemma 10.91.7 and its proof. Similarly for $(S, \mathfrak{n})$. Thus it is clear that diagrams as in Definition 15.26.1 for the cases (1), (2), and (3) are in 1-to-1 correspondence. □

The advantage of working with adic rings is that one gets a stronger lifting property.

Lemma 15.26.5. *Let* $R \to S$ *be a ring map. Let* $\mathfrak{n}$ *be an ideal of* S*. Assume that* $R \to S$ *is formally smooth in the* $\mathfrak{n}$*-adic topology. Consider a solid commutative diagram*

$$\begin{array}{ccc} S & \longrightarrow & A/J \\ \uparrow & \overset{\psi}{\nearrow} & \uparrow \\ R & \longrightarrow & A \end{array}$$

of homomorphisms of topological rings where A *is adic and* A/J *is the quotient (as topological ring) of* A *by a closed ideal* $J \subset A$ *such that* J^t *is contained in an ideal of definition of* A *for some* $t \geq 1$*. Then there exists a dotted arrow in the category of topological rings which makes the diagram commute.*

Proof. Let $I \subset A$ be an ideal of definition so that $I \supset J^t$ for some n. Then $A = \lim A/I^n$ and $A/J = \lim A/J + I^n$ because J is assumed closed. Consider the following diagram of discrete R algebras $A_{n,m} = A/J^n + I^m$:

$$\begin{array}{ccccc} A/J^3 + I^3 & \longrightarrow & A/J^2 + I^3 & \longrightarrow & A/J + I^3 \\ \downarrow & & \downarrow & & \downarrow \\ A/J^3 + I^2 & \longrightarrow & A/J^2 + I^2 & \longrightarrow & A/J + I^2 \\ \downarrow & & \downarrow & & \downarrow \\ A/J^3 + I & \longrightarrow & A/J^2 + I & \longrightarrow & A/J + I \end{array}$$

Note that each of the commutative squares defines a surjection

$$A_{n+1,m+1} \longrightarrow A_{n+1,m} \times_{A_{n,m}} A_{n,m+1}$$

of R-algebras whose kernel has square zero. We will inductively construct R-algebra maps $\varphi_{n,m} : S \to A_{n,m}$. Namely, we have the maps $\varphi_{1,m} = \psi \bmod J + I^m$. Note that each of these maps is continuous as ψ is. We can inductively choose the maps $\varphi_{n,1}$ by starting with our choice of $\varphi_{1,1}$ and lifting up, using the formal smoothness of S over R, along the right column of the diagram above. We construct the remaining maps $\varphi_{n,m}$ by induction on $n + m$. Namely, we choose $\varphi_{n+1,m+1}$ by lifting the pair $(\varphi_{n+1,m}, \varphi_{n,m+1})$ along the displayed surjection above (again using the formal smoothness of S over R). In this way all of the maps $\varphi_{n,m}$ are compatible with the transition maps of the system. As $J^t \subset I$ we see that for example $\varphi_n = \varphi_{nt,n} \bmod I^n$ induces a map $S \to A/I^n$. Taking the limit $\varphi = \lim \varphi_n$ we obtain a map $S \to A = \lim A/I^n$. The composition into A/J agrees with ψ as we have seen that $A/J = \lim A/J + I^n$. Finally we show that φ is continuous. Namely, we know that $\psi(\mathfrak{n}^r) \subset J + I^r/J$ for some r by our assumption that ψ is

a morphism of topological rings, see Lemma 15.25.2. Hence $\varphi(\mathfrak{n}^r) \subset J + I$ hence $\varphi(\mathfrak{n}^{rt}) \subset I$ as desired. □

Lemma 15.26.6. *Let $R \to S$ be a ring map. Let $\mathfrak{n} \subset \mathfrak{n}' \subset S$ be ideals. If $R \to S$ is formally smooth for the $\mathfrak{n}$-adic topology, then $R \to S$ is formally smooth for the $\mathfrak{n}'$-adic topology.*

Proof. Omitted. □

Lemma 15.26.7. *A composition of formally smooth continuous homomorphisms of linearly topologized rings is formally smooth.*

Proof. Omitted. (Hint: This is completely formal, and follows from considering a suitable diagram.) □

Lemma 15.26.8. *Let R, S be rings. Let $\mathfrak{n} \subset S$ be an ideal. Let $R \to S$ be formally smooth for the $\mathfrak{n}$-adic topology. Let $R \to R'$ be any ring map. Then $R' \to S' = S \otimes_R R'$ is formally smooth in the $\mathfrak{n}' = \mathfrak{n}S'$-adic topology.*

Proof. Let a solid diagram

$$\begin{array}{ccccc} S & \longrightarrow & S' & \longrightarrow & A/J \\ \uparrow & \nearrow & \uparrow & \nearrow & \uparrow \\ R & \longrightarrow & R' & \longrightarrow & A \end{array}$$

as in Definition 15.26.1 be given. Then the composition $S \to S' \to A/J$ is continuous. By assumption the longer dotted arrow exists. By the universal property of tensor product we obtain the shorter dotted arrow. □

We have seen descent for formal smoothness along faithfully flat ring maps in Algebra, Lemma 10.131.15. Something similar holds in the current setting of topological rings. However, here we just prove the following very simple and easy to prove version which is already quite useful.

Lemma 15.26.9. *Let R, S be rings. Let $\mathfrak{n} \subset S$ be an ideal. Let $R \to R'$ be a ring map. Set $S' = S \otimes_R R'$ and $\mathfrak{n}' = \mathfrak{n}S$. If*

1. *the map $R \to R'$ embeds R as a direct summand of R' as an R-module, and*
2. *$R' \to S'$ is formally smooth for the $\mathfrak{n}'$-adic topology,*

then $R \to S$ is formally smooth in the $\mathfrak{n}$-adic topology.

Proof. Let a solid diagram

$$\begin{array}{ccc} S & \longrightarrow & A/J \\ \uparrow & & \uparrow \\ R & \longrightarrow & A \end{array}$$

as in Definition 15.26.1 be given. Set $A' = A \otimes_R R'$ and $J' = \mathrm{Im}(J \otimes_R R' \to A')$. The base change of the diagram above is the diagram

$$\begin{array}{ccc} S' & \longrightarrow & A'/J' \\ \uparrow & \overset{\psi'}{\nearrow} & \uparrow \\ R' & \longrightarrow & A' \end{array}$$

with continuous arrows. By condition (2) we obtain the dotted arrow $\psi' : S' \to A'$. Using condition (1) choose a direct summand decomposition $R' = R \oplus C$ as R-modules. (Warning: C isn't an ideal in R'.) Then $A' = A \oplus A \otimes_R C$. Set

$$J'' = \mathrm{Im}(J \otimes_R C \to A \otimes_R C) \subset J' \subset A'.$$

Then $J' = J \oplus J''$ as A-modules. The image of the composition $\psi : S \to A'$ of ψ' with $S \to S'$ is contained in $A + J' = A \oplus J''$. However, in the ring $A + J' = A \oplus J''$ the A-submodule J'' is an ideal! (Use that $J^2 = 0$.) Hence the composition $S \to A + J' \to (A + J')/J'' = A$ is the arrow we were looking for. □

The following lemma will be improved on in Section 15.28.

Lemma 15.26.10. *Let k be a field and let $(A, \mathfrak{m}, K)$ be a Noetherian local k-algebra. If $k \to A$ is formally smooth for the $\mathfrak{m}$-adic topology, then A is a regular local ring.*

Proof. Let $k_0 \subset k$ be the prime field. Then k_0 is perfect, hence k/k_0 is separable, hence formally smooth by Algebra, Lemma 10.145.7. By Lemmas 15.26.2 and 15.26.7 we see that $k_0 \to A$ is formally smooth for the $\mathfrak{m}$-adic topology on A. Hence we may assume $k = \mathbf{Q}$ or $k = \mathbf{F}_p$.

By Algebra, Lemmas 10.91.4 and 10.105.8 it suffices to prove the completion $A^\wedge$ is regular. By Lemma 15.26.4 we may replace A by $A^\wedge$. Thus we may assume that A is a Noetherian complete local ring. By the Cohen structure theorem (Algebra, Theorem 10.147.8) there exist a map $K \to A$. As k is the prime field we see that $K \to A$ is a k-algebra map.

Let $x_1, \ldots, x_n \in \mathfrak{m}$ be elements whose images form a basis of $\mathfrak{m}/\mathfrak{m}^2$. Set $T = K[[X_1, \ldots, X_n]]$. Note that

$$A/\mathfrak{m}^2 \cong K[x_1, \ldots, x_n]/(x_i x_j)$$

and

$$T/\mathfrak{m}_T^2 \cong K[X_1, \ldots, X_n]/(X_i X_j).$$

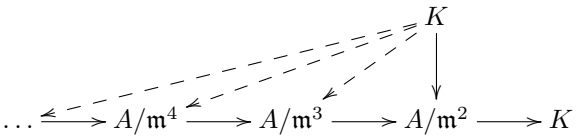
Let $A/\mathfrak{m}^2 \to T/m_T^2$ be the local K-algebra isomorphism given by mapping the class of x_i to the class of X_i. Denote $f_1 : A \to T/\mathfrak{m}_T^2$ the composition of this isomorphism with the quotient map $A \to A/\mathfrak{m}^2$. The assumption that $k \to A$ is formally smooth in the $\mathfrak{m}$-adic topology means we can lift f_1 to a map $f_2 : A \to T/\mathfrak{m}_T^3$, then to a map $f_3 : A \to T/\mathfrak{m}_T^4$, and so on, for all $n \geq 1$. Warning: the maps f_n are continuous k-algebra maps and may not be K-algebra maps. We get an induced map $f : A \to T = \lim T/\mathfrak{m}_T^n$ of local k-algebras. By our choice of f_1, the map f induces an isomorphism $\mathfrak{m}/\mathfrak{m}^2 \to \mathfrak{m}_T/\mathfrak{m}_T^2$ hence each f_n is surjective and we conclude f is surjective as A is complete. This implies $\dim(A) \geq \dim(T) = n$. Hence A is regular by definition. (It also follows that f is an isomorphism.) □

The following result will be improved on in Section 15.28

Lemma 15.26.11. *Let k be a field. Let $(A, \mathfrak{m}, K)$ be a regular local k-algebra such that K/k is separable. Then $k \to A$ is formally smooth in the $\mathfrak{m}$-adic topology.*

Proof. It suffices to prove that the completion of A is formally smooth over k, see Lemma 15.26.4. Hence we may assume that A is a complete local regular k-algebra with residue field K separable over k. Since K is formally smooth over k

by Algebra, Proposition 10.145.9 we can successively find maps



of k-algebras. Since A is complete this defines a k-algebra map $K \to A$. Pick $a_1, \ldots, a_n \in \mathfrak{m}$ which map to a K-basis of $\mathfrak{m}/\mathfrak{m}^2$. Consider the K-algebra map

$$c : K[[x_1, \ldots, x_n]] \longrightarrow A$$

which maps x_i to a_i (existence of c follows from the universal property of the powerseries ring). By construction the maps $K[[x_1, \ldots, x_n]] \to A/\mathfrak{m}^e$ are surjective for all $e \geq 1$. Since $K[[x_1, \ldots, x_n]]$ is complete we see that c is surjective. Since $\dim(A) = n$ as A is regular and since $K[[x_1, \ldots, x_n]]$ is a domain of dimension n we see that the kernel of c is zero. Hence c is an isomorphism.

We win because the power series ring $K[[x_1, \ldots, x_n]]$ is formally smooth over k. Namely, K is formally smooth over k and $K[x_1, \ldots, x_n]$ is formally smooth over K as a polynomial algebra. Hence $K[x_1, \ldots, x_n]$ is formally smooth over k by Algebra, Lemma 10.131.3. It follows that $k \to K[x_1, \ldots, x_n]$ is formally smooth for the $(x_1, \ldots, x_n)$-adic topology by Lemma 15.26.2. Finally, it follows that $k \to K[[x_1, \ldots, x_n]]$ is formally smooth for the $(x_1, \ldots, x_n)$-adic topology by Lemma 15.26.4. □

Lemma 15.26.12. *Let $A \to B$ be a finite type ring map with A Noetherian. Let $\mathfrak{q} \subset B$ be a prime ideal lying over $\mathfrak{p} \subset A$. The following are equivalent*

1. *$A \to B$ is smooth at $\mathfrak{q}$, and*
2. *$A_\mathfrak{p} \to B_\mathfrak{q}$ is formally smooth in the $\mathfrak{q}$-adic topology.*

Proof. The implication (2) $\Rightarrow$ (1) follows from Algebra, Lemma 10.134.2. Conversely, if $A \to B$ is smooth at $\mathfrak{q}$, then $A \to B_g$ is smooth for some $g \in B$, $g \notin \mathfrak{q}$. Then $A \to B_g$ is formally smooth by Algebra, Proposition 10.131.13. Hence $A_\mathfrak{p} \to B_\mathfrak{q}$ is formally smooth as localization preserves formal smoothness (for example by the criterion of Algebra, Proposition 10.131.8 and the fact that the cotangent complex behaves well with respect to localization, see Algebra, Lemmas 10.127.11 and 10.127.13). Finally, Lemma 15.26.2 implies that $A_\mathfrak{p} \to B_\mathfrak{q}$ is formally smooth in the $\mathfrak{q}$-adic topology. □

15.27. Some results on power series rings

Questions on formally smooth maps between Noetherian local rings can often be reduced to questions on maps between power series rings. In this section we prove some helper lemmas to facilitate this kind of argument.

Lemma 15.27.1. *Let K be a field of characteristic 0 and $A = K[[x_1, \ldots, x_n]]$. Let L be a field of characteristic $p > 0$ and $B = L[[x_1, \ldots, x_n]]$. Let Λ be a Cohen ring. Let $C = \Lambda[[x_1, \ldots, x_n]]$.*

1. *$\mathbf{Q} \to A$ is formally smooth in the $\mathfrak{m}$-adic topology.*
2. *$\mathbf{F}_p \to B$ is formally smooth in the $\mathfrak{m}$-adic topology.*
3. *$\mathbf{Z} \to C$ is formally smooth in the $\mathfrak{m}$-adic topology.*

Proof. By the universal property of power series rings it suffices to prove:

(1) $\mathbf{Q} \to K$ is formally smooth.
(2) $\mathbf{F}_p \to L$ is formally smooth.
(3) $\mathbf{Z} \to \Lambda$ is formally smooth in the $\mathfrak{m}$-adic topology.

The first two are Algebra, Proposition 10.145.9. The third follows from Algebra, Lemma 10.147.7 since for any test diagram as in Definition 15.26.1 some power of p will be zero in A/J and hence some power of p will be zero in A. □

Lemma 15.27.2. *Let K be a field and $A = K[[x_1, \ldots, x_n]]$. Let Λ be a Cohen ring and let $B = \Lambda[[x_1, \ldots, x_n]]$.*

(1) *If $y_1, \ldots, y_n \in A$ is a regular system of parameters then $K[[y_1, \ldots, y_n]] \to A$ is an isomorphism.*
(2) *If $z_1, \ldots, z_r \in A$ form part of a regular system of parameters for A, then $r \leq n$ and $A/(z_1, \ldots, z_r) \cong K[[y_1, \ldots, y_{n-r}]]$.*
(3) *If $p, y_1, \ldots, y_n \in B$ is a regular system of parameters then $\Lambda[[y_1, \ldots, y_n]] \to B$ is an isomorphism.*
(4) *If $p, z_1, \ldots, z_r \in B$ form part of a regular system of parameters for B, then $r \leq n$ and $B/(z_1, \ldots, z_r) \cong \Lambda[[y_1, \ldots, y_{n-r}]]$.*

Proof. Proof of (1). Set $A' = K[[y_1, \ldots, y_n]]$. It is clear that the map $A' \to A$ induces an isomorphism $A'/\mathfrak{m}_{A'}^n \to A/\mathfrak{m}_A^n$ for all $n \geq 1$. Since A and A' are both complete we deduce that $A' \to A$ is an isomorphism. Proof of (2). Extend $z_1, \ldots, z_r$ to a regular system of parameters $z_1, \ldots, z_r, y_1, \ldots, y_{n-r}$ of A. Consider the map $A' = K[[z_1, \ldots, z_r, y_1, \ldots, y_{n-r}]] \to A$. This is an isomorphism by (1). Hence (2) follows as it is clear that $A'/(z_1, \ldots, z_r) \cong K[[y_1, \ldots, y_{n-r}]]$. The proofs of (3) and (4) are exactly the same as the proofs of (1) and (2). □

Lemma 15.27.3. *Let $A \to B$ be a local homomorphism of Noetherian complete local rings. Then there exists a commutative diagram*

$$\begin{array}{ccc} S & \longrightarrow & B \\ \uparrow & & \uparrow \\ R & \longrightarrow & A \end{array}$$

with the following properties:

(1) *the horizontal arrows are surjective,*
(2) *if the characteristic of $A/\mathfrak{m}_A$ is zero, then S and R are power series rings over fields,*
(3) *if the characteristic of $A/\mathfrak{m}_A$ is $p > 0$, then S and R are power series rings over Cohen rings, and*
(4) *$R \to S$ maps a regular system of parameters of R to part of a regular system of parameters of S.*

In particular $R \to S$ is flat (see Algebra, Lemma 10.122.2) with regular fibre $S/\mathfrak{m}_R S$ (see Algebra, Lemma 10.101.3).

Proof. Use the Cohen structure theorem (Algebra, Theorem 10.147.8) to choose a surjection $S \to B$ as in the statement of the lemma where we choose S to be a power series over a Cohen ring if the residue characteristic is $p > 0$ and a power series over a field else. Let $J \subset S$ be the kernel of $S \to B$. Next, choose a surjection $R = \Lambda[[x_1, \ldots, x_n]] \to A$ where we choose Λ to be a Cohen ring if the residue characteristic of A is $p > 0$ and Λ equal to the residue field of A otherwise.

We lift the composition $\Lambda[[x_1, \ldots, x_n]] \to A \to B$ to a map $\varphi : R \to S$. This is possible because $\Lambda[[x_1, \ldots, x_n]]$ is formally smooth over $\mathbf{Z}$ in the $\mathfrak{m}$-adic topology (see Lemma 15.27.1) by an application of Lemma 15.26.5. Finally, we replace φ by the map $\varphi' : R = \Lambda[[x_1, \ldots, x_n]] \to S' = S[[y_1, \ldots, y_n]]$ with $\varphi'|_\Lambda = \varphi|_\Lambda$ and $\varphi'(x_i) = \varphi(x_i) + y_i$. We also replace $S \to B$ by the map $S' \to B$ which maps y_i to zero. After this replacement it is clear that a regular system of parameters of R maps to part of a regular sequence in S' and we win. □

There should be an elementary proof of the following lemma.

Lemma 15.27.4. *Let $S \to R$ and $S' \to R$ be surjective maps of complete Noetherian local rings. Then $S \times_R S'$ is a complete Noetherian local ring.*

Proof. Let k be the residue field of R. If the characteristic of k is $p > 0$, then we denote Λ a Cohen ring (Algebra, Definition 10.147.5) with residue field k (Algebra, Lemma 10.147.6). If the characteristic of k is 0 we set $\Lambda = k$. Choose a surjection $\Lambda[[x_1, \ldots, x_n]] \to R$ (as in the Cohen structure theorem, see Algebra, Theorem 10.147.8) and lift this to maps $\Lambda[[x_1, \ldots, x_n]] \to S$ and $\varphi : \Lambda[[x_1, \ldots, x_n]] \to S$ and $\varphi' : \Lambda[[x_1, \ldots, x_n]] \to S'$ using Lemmas 15.27.1 and 15.26.5. Next, choose $f_1, \ldots, f_m \in S$ generating the kernel of $S \to R$ and $f'_1, \ldots, f'_{m'} \in S'$ generating the kernel of $S' \to R$. Then the map

$$\Lambda[[x_1, \ldots, x_n, y_1, \ldots, y_m, z_1, \ldots, z_{m'}]] \longrightarrow S \times_R S,$$

which sends x_i to $(\varphi(x_i), \varphi'(x_i))$ and y_j to $(f_j, 0)$ and $z_{j'}$ to $(0, f'_j)$ is surjective. Thus $S \times_R S'$ is a quotient of a complete local ring, whence complete. □

15.28. Geometric regularity and formal smoothness

In this section we combine the results of the previous sections to prove the following characterization of geometrically regular local rings over fields. We then recycle some of our arguments to prove a characterization of formally smooth maps in the $\mathfrak{m}$-adic topology between Noetherian local rings.

Theorem 15.28.1. *Let k be a field. Let $(A, \mathfrak{m}, K)$ be a Noetherian local k-algebra. If the characteristic of k is zero then the following are equivalent*

1. *A is a regular local ring, and*
2. *$k \to A$ is formally smooth in the $\mathfrak{m}$-adic topology.*

If the characteristic of k is $p > 0$ then the following are equivalent

1. *A is geometrically regular over k,*
2. *$k \to A$ is formally smooth in the $\mathfrak{m}$-adic topology.*
3. *for all $k \subset k' \subset k^{1/p}$ finite over k the ring $A \otimes_k k'$ is regular,*
4. *A is regular and the canonical map $H_1(L_{K/k}) \to \mathfrak{m}/\mathfrak{m}^2$ is injective, and*
5. *A is regular and the map $\Omega_{k/\mathbf{F}_p} \otimes_k K \to \Omega_{A/\mathbf{F}_p} \otimes_A K$ is injective.*

Proof. If the characteristic of k is zero, then the equivalence of (1) and (2) follows from Lemmas 15.26.10 and 15.26.11.

If the characteristic of k is $p > 0$, then it follows from Proposition 15.24.1 that (1), (3), (4), and (5) are equivalent. Assume (2) holds. By Lemma 15.26.8 we see that $k' \to A' = A \otimes_k k'$ is formally smooth for the $\mathfrak{m}' = \mathfrak{m}A$-adic topology. Hence if $k \subset k'$ is finite purely inseparable, then A' is a regular local ring by Lemma 15.26.10. Thus we see that (1) holds.

Finally, we will prove that (5) implies (2). Choose a solid diagram

$$\begin{array}{ccc} A & \longrightarrow & B/J \\ {\scriptstyle i}\uparrow & \nearrow_{\bar\psi} & \uparrow{\scriptstyle \pi} \\ k & \xrightarrow[\varphi]{} & B \end{array}$$

as in Definition 15.26.1. As $J^2 = 0$ we see that J has a canonical B/J module structure and via $\bar\psi$ an A-module structure. As $\bar\psi$ is continuous for the $\mathfrak{m}$-adic topology we see that $\mathfrak{m}^n J = 0$ for some n. Hence we can filter J by B/J-submodules $0 \subset J_1 \subset J_2 \subset \ldots \subset J_n = J$ such that each quotient J_{t+1}/J_t is annihilated by $\mathfrak{m}$. Considering the sequence of ring maps $B \to B/J_1 \to B/J_2 \to \ldots \to B/J$ we see that it suffices to prove the existence of the dotted arrow when J is annihilated by $\mathfrak{m}$, i.e., when J is a K-vector space.

Assume given a diagram as above such that J is annihilated by $\mathfrak{m}$. By Lemma 15.26.11 we see that $\mathbf{F}_p \to A$ is formally smooth in the $\mathfrak{m}$-adic topology. Hence we can find a ring map $\psi : A \to B$ such that $\pi \circ \psi = \bar\psi$. Then $\psi \circ i, \varphi : k \to B$ are two maps whose compositions with π are equal. Hence $D = \psi \circ i - \varphi : k \to J$ is a derivation. By Algebra, Lemma 10.125.3 we can write $D = \xi \circ \mathrm{d}$ for some k-linear map $\xi : \Omega_{k/\mathbf{F}_p} \to J$. Using the K-vector space structure on J we extend ξ to a K-linear map $\xi' : \Omega_{k/\mathbf{F}_p} \otimes_k K \to J$. Using (5) we can find a K-linear map $\xi'' : \Omega_{A/\mathbf{F}_p} \otimes_A K$ whose restriction to $\Omega_{k/\mathbf{F}_p} \otimes_k K$ is ξ'. Write

$$D' : A \xrightarrow{\mathrm{d}} \Omega_{A/\mathbf{F}_p} \to \Omega_{A/\mathbf{F}_p} \otimes_A K \xrightarrow{\xi''} J.$$

Finally, set $\psi' = \psi - D' : A \to B$. The reader verifies that ψ' is a ring map such that $\pi \circ \psi' = \bar\psi$ and such that $\psi' \circ i = \varphi$ as desired. □

Example 15.28.2. Let k be a field of characteristic $p > 0$. Suppose that $a \in k$ is an element which is not a pth power. A standard example of a geometrically regular local k-algebra whose residue field is purely inseparable over k is the ring

$$A = k[x, y]_{(x, y^p - a)}/(y^p - a - x)$$

Namely, A is a localization of a smooth algebra over k hence $k \to A$ is formally smooth, hence $k \to A$ is formally smooth for the $\mathfrak{m}$-adic topology. A closely related example is the following. Let $k = \mathbf{F}_p(s)$ and $K = \mathbf{F}_p(t)^{perf}$. We claim the ring map

$$k \longrightarrow A = K[[x]], \quad s \longmapsto t + x$$

is formally smooth for the (x)-adic topology on A. Namely, $\Omega_{k/\mathbf{F}_p}$ is 1-dimensional with basis $\mathrm{d}s$. It maps to the element $\mathrm{d}x + \mathrm{d}t = \mathrm{d}x$ in $\Omega_{A/\mathbf{F}_p}$. We leave it to the reader to show that $\Omega_{A/\mathbf{F}_p}$ is free on $\mathrm{d}x$ as an A-module. Hence we see that condition (5) of Theorem 15.28.1 holds and we conclude that $k \to A$ is formally smooth in the (x)-adic topology.

Lemma 15.28.3. *Let $A \to B$ be a local homomorphism of Noetherian local rings. Assume $A \to B$ is formally smooth in the $\mathfrak{m}_B$-adic topology. Then $A \to B$ is flat.*

Proof. We may assume that A and B a Noetherian complete local rings by Lemma 15.26.4 and Algebra, Lemma 10.91.10 (this also uses Algebra, Lemma 10.36.8 and

10.91.4 to see that flatness of the map on completions implies flatness of $A \to B$). Choose a commutative diagram

$$\begin{array}{ccc} S & \longrightarrow & B \\ \uparrow & & \uparrow \\ R & \longrightarrow & A \end{array}$$

as in Lemma 15.27.3 with $R \to S$ flat. Let $I \subset R$ be the kernel of $R \to A$. Because B is formally smooth over A we see that the A-algebra map

$$S/IS \longrightarrow B$$

has a section, see Lemma 15.26.5. Hence B is a direct summand of the flat A-module S/IS (by base change of flatness, see Algebra, Lemma 10.36.6), whence flat. □

Proposition 15.28.4. *Let $A \to B$ be a local homomorphism of Noetherian local rings. Let k be the residue field of A and $\overline{B} = B \otimes_A k$ the special fibre. The following are equivalent*

1. *$A \to B$ is flat and $\overline{B}$ is geometrically regular over k,*
2. *$A \to B$ is flat and $k \to \overline{B}$ is formally smooth in the $\mathfrak{m}_{\overline{B}}$-adic topology, and*
3. *$A \to B$ is formally smooth in the $\mathfrak{m}_B$-adic topology.*

Proof. The equivalence of (1) and (2) follows from Theorem 15.28.1.

Assume (3). By Lemma 15.28.3 we see that $A \to B$ is flat. By Lemma 15.26.8 we see that $k \to \overline{B}$ is formally smooth in the $\mathfrak{m}_{\overline{B}}$-adic topology. Thus (2) holds.

Assume (2). Lemma 15.26.4 tells us formal smoothness is preserved under completion. The same is true for flatness by Algebra, Lemma 10.91.4. Hence we may replace A and B by their respective completions and assume that A and B are Noetherian complete local rings. In this case choose a diagram

$$\begin{array}{ccc} S & \longrightarrow & B \\ \uparrow & & \uparrow \\ R & \longrightarrow & A \end{array}$$

as in Lemma 15.27.3. We will use all of the properties of this diagram without further mention. Fix a regular system of parameters $t_1, \ldots, t_d$ of R with $t_1 = p$ in case the characteristic of k is $p > 0$. Set $\overline{S} = S \otimes_R k$. Consider the short exact sequence

$$0 \to J \to S \to B \to 0$$

Since B is flat over A we see that $J \otimes_R k$ is the kernel of $\overline{S} \to \overline{B}$. As $\overline{B}$ and $\overline{S}$ are regular we see that $J \otimes_R k$ is generated by elements $\overline{x}_1, \ldots, \overline{x}_r$ which form part of a regular system of parameters of $\overline{S}$, see Algebra, Lemma 10.101.4. Lift these elements to $x_1, \ldots, x_r \in J$. Then $t_1, \ldots, t_d, x_1, \ldots, x_r$ is part of a regular system of parameters for S. Hence $S/(x_1, \ldots, x_r)$ is a power series ring over a field (if the characteristic of k is zero) or a power series ring over a Cohen ring (if the characteristic of k is $p > 0$), see Lemma 15.27.2. Moreover, it is still the case that $R \to S/(x_1, \ldots, x_r)$ maps $t_1, \ldots, t_d$ to a part of a regular system of parameters of

$S/(x_1, \ldots, x_r)$. In other words, we may replace S by $S/(x_1, \ldots, x_r)$ and assume we have a diagram

$$\begin{array}{ccc} S & \longrightarrow & B \\ \uparrow & & \uparrow \\ R & \longrightarrow & A \end{array}$$

as in Lemma 15.27.3 with moreover $\overline{S} = \overline{B}$. In this case the map

$$S \otimes_R A \longrightarrow B$$

is an isomorphism as it is surjective and an isomorphism on special fibres, see Algebra, Lemma 10.93.1. Thus by Lemma 15.26.8 it suffices to show that $R \to S$ is formally smooth in the $\mathfrak{m}_S$-adic topology. Of course, since $\overline{S} = \overline{B}$, we have that $\overline{S}$ is formally smooth over $k = R/\mathfrak{m}_R$.

Choose elements $y_1, \ldots, y_m \in S$ such that $t_1, \ldots, t_d, y_1, \ldots, y_m$ is a regular system of parameters for S. If the characteristic of k is zero, choose a coefficient field $K \subset S$ and if the characteristic of k is $p > 0$ choose a Cohen ring $\Lambda \subset S$ with residue field K. At this point the map $K[[t_1, \ldots, t_d, y_1, \ldots, y_m]] \to S$ (characteristic zero case) or $\Lambda[[t_2, \ldots, t_d, y_1, \ldots, y_m]] \to S$ (characteristic $p > 0$ case) is an isomorphism, see Lemma 15.27.2. From now on we think of S as the above power series ring.

The rest of the proof is analogous to the argument in the proof of Theorem 15.28.1. Choose a solid diagram

$$\begin{array}{ccc} S & \longrightarrow & N/J \\ \uparrow{\scriptstyle i} & \overset{\bar{\psi}}{\searrow} & \uparrow{\scriptstyle \pi} \\ R & \xrightarrow[\varphi]{} & N \end{array}$$

as in Definition 15.26.1. As $J^2 = 0$ we see that J has a canonical N/J module structure and via $\bar{\psi}$ a S-module structure. As $\bar{\psi}$ is continuous for the $\mathfrak{m}_S$-adic topology we see that $\mathfrak{m}_S^n J = 0$ for some n. Hence we can filter J by N/J-submodules $0 \subset J_1 \subset J_2 \subset \ldots \subset J_n = J$ such that each quotient J_{t+1}/J_t is annihilated by $\mathfrak{m}_S$. Considering the sequence of ring maps $N \to N/J_1 \to N/J_2 \to \ldots \to N/J$ we see that it suffices to prove the existence of the dotted arrow when J is annihilated by $\mathfrak{m}_S$, i.e., when J is a K-vector space.

Assume given a diagram as above such that J is annihilated by $\mathfrak{m}_S$. As $\mathbf{Q} \to S$ (characteristic zero case) or $\mathbf{Z} \to S$ (characteristic $p > 0$ case) is formally smooth in the $\mathfrak{m}_S$-adic topology (see Lemma 15.27.1), we can find a ring map $\psi : S \to N$ such that $\pi \circ \psi = \bar{\psi}$. Since S is a power series ring in $t_1, \ldots, t_d$ (characteristic zero) or $t_2, \ldots, t_d$ (characteristic $p > 0$) over a subring, it follows from the universal property of power series rings that we can change our choice of ψ so that $\psi(t_i)$ equals $\varphi(t_i)$ (automatic for $t_1 = p$ in the characteristic p case). Then $\psi \circ i$ and $\varphi : R \to N$ are two maps whose compositions with π are equal and which agree on $t_1, \ldots, t_d$. Hence $D = \psi \circ i - \varphi : R \to J$ is a derivation which annihilates $t_1, \ldots, t_d$. By Algebra, Lemma 10.125.3 we can write $D = \xi \circ \mathrm{d}$ for some R-linear map $\xi : \Omega_{R/\mathbf{Z}} \to J$ which annihilates $\mathrm{d}t_1, \ldots, \mathrm{d}t_d$ (by construction) and $\mathfrak{m}_R \Omega_{R/\mathbf{Z}}$ (as J is annihilated by $\mathfrak{m}_R$). Hence ξ factors as a composition

$$\Omega_{R/\mathbf{Z}} \to \Omega_{k/\mathbf{Z}} \xrightarrow{\xi'} J$$

where ξ' is k-linear. Using the K-vector space structure on J we extend ξ' to a K-linear map

$$\xi'' : \Omega_{k/\mathbf{Z}} \otimes_k K \longrightarrow J.$$

Using that $\overline{S}/k$ is formally smooth we see that

$$\Omega_{k/\mathbf{Z}} \otimes_k K \to \Omega_{\overline{S}/\mathbf{Z}} \otimes_S K$$

is injective by Theorem 15.28.1 (this is true also in the characteristic zero case as it is even true that $\Omega_{k/\mathbf{Z}} \to \Omega_{K/\mathbf{Z}}$ is injective in characteristic zero, see Algebra, Proposition 10.145.9). Hence we can find a K-linear map $\xi''' : \Omega_{\overline{S}/\mathbf{Z}} \otimes_S K \to J$ whose restriction to $\Omega_{k/\mathbf{Z}} \otimes_k K$ is ξ''. Write

$$D' : S \xrightarrow{\mathrm{d}} \Omega_{S/\mathbf{Z}} \to \Omega_{\overline{S}/\mathbf{Z}} \to \Omega_{\overline{S}/\mathbf{Z}} \otimes_S K \xrightarrow{\xi'''} J.$$

Finally, set $\psi' = \psi - D' : S \to N$. The reader verifies that ψ' is a ring map such that $\pi \circ \psi' = \bar{\psi}$ and such that $\psi' \circ i = \varphi$ as desired. □

As an application of the result above we prove that deformations of formally smooth algebras are unobstructed.

Lemma 15.28.5. *Let A be a Noetherian complete local ring with residue field k. Let B be a Noetherian complete local k-algebra. Assume $k \to B$ is formally smooth in the $\mathfrak{m}_B$-adic topology. Then there exists a Noetherian complete local ring C and a local homomorphism $A \to C$ which is formally smooth in the $\mathfrak{m}_C$-adic topology such that $C \otimes_A k \cong B$.*

Proof. Choose a diagram

$$\begin{array}{ccc} S & \longrightarrow & B \\ \uparrow & & \uparrow \\ R & \longrightarrow & A \end{array}$$

as in Lemma 15.27.3. Let $t_1, \ldots, t_d$ be a regular system of parameters for R with $t_1 = p$ in case the characteristic of k is $p > 0$. As B and $\overline{S} = S \otimes_A k$ are regular we see that $\mathrm{Ker}(\overline{S} \to B)$ is generated by elements $\overline{x}_1, \ldots, \overline{x}_r$ which form part of a regular system of parameters of $\overline{S}$, see Algebra, Lemma 10.101.4. Lift these elements to $x_1, \ldots, x_r \in S$. Then $t_1, \ldots, t_d, x_1, \ldots, x_r$ is part of a regular system of parameters for S. Hence $S/(x_1, \ldots, x_r)$ is a power series ring over a field (if the characteristic of k is zero) or a power series ring over a Cohen ring (if the characteristic of k is $p > 0$), see Lemma 15.27.2. Moreover, it is still the case that $R \to S/(x_1, \ldots, x_r)$ maps $t_1, \ldots, t_d$ to a part of a regular system of parameters of $S/(x_1, \ldots, x_r)$. In other words, we may replace S by $S/(x_1, \ldots, x_r)$ and assume we have a diagram

$$\begin{array}{ccc} S & \longrightarrow & B \\ \uparrow & & \uparrow \\ R & \longrightarrow & A \end{array}$$

as in Lemma 15.27.3 with moreover $\overline{S} = B$. In this case $R \to S$ is formally smooth in the $\mathfrak{m}_S$-adic topology by Proposition 15.28.4. Hence the base change $C = S \otimes_R A$ is formally smooth over A in the $\mathfrak{m}_C$-adic topology by Lemma 15.26.8. □

Remark 15.28.6. The assertion of Lemma 15.28.5 is quite strong. Namely, suppose that we have a diagram

$$\begin{array}{ccc} & & B \\ & & \uparrow \\ A & \longrightarrow & A' \end{array}$$

of local homomorphisms of Noetherian complete local rings where $A \to A'$ induces an isomorphism of residue fields $k = A/\mathfrak{m}_A = A'/\mathfrak{m}_{A'}$ and with $B \otimes_{A'} k$ formally smooth over k. Then we can extend this to a commutative diagram

$$\begin{array}{ccc} C & \longrightarrow & B \\ \uparrow & & \uparrow \\ A & \longrightarrow & A' \end{array}$$

of local homomorphisms of Noetherian complete local rings where $A \to C$ is formally smooth in the $\mathfrak{m}_C$-adic topology and where $C \otimes_A k \cong B \otimes_{A'} k$. Namely, pick $A \to C$ as in Lemma 15.28.5 lifting $B \otimes_{A'} k$ over k. By formal smoothness we can find the arrow $C \to B$, see Lemma 15.26.5. Denote $C \otimes_A^\wedge A'$ the completion of $C \otimes_A A'$ with respect to the ideal $C \otimes_A \mathfrak{m}_{A'}$. Note that $C \otimes_A^\wedge A'$ is a Noetherian complete local ring (see Algebra, Lemma 10.91.9) which is flat over A' (see Algebra, Lemma 10.93.10). We have moreover

(1) $C \otimes_A^\wedge A' \to B$ is surjective,
(2) if $A \to A'$ is surjective, then $C \to B$ is surjective,
(3) if $A \to A'$ is finite, then $C \to B$ is finite, and
(4) if $A' \to B$ is flat, then $C \otimes_A^\wedge A' \cong B$.

Namely, by Nakayama's lemma for nilpotent ideals (see Algebra, Lemma 10.18.1) we see that $C \otimes_A k \cong B \otimes_{A'} k$ implies that $C \otimes_A A'/\mathfrak{m}_{A'}^n \to B/\mathfrak{m}_{A'}^n B$ is surjective for all n. This proves (1). Parts (2) and (3) follow from part (1). Part (4) follows from Algebra, Lemma 10.93.1.

15.29. Regular ring maps

Let k be a field. Recall that a Noetherian k-algebra A is said to be *geometrically regular* over k if and only if $A \otimes_k k'$ is regular for all finite purely inseparable extensions k' of k, see Algebra, Definition 10.152.2. Moreover, if this is the case then $A \otimes_k k'$ is regular for every finitely generated field extension $k \subset k'$, see Algebra, Lemma 10.152.1. We use this notion in the following definition.

Definition 15.29.1. A ring map $R \to \Lambda$ is *regular* if it is flat and for every prime $\mathfrak{p} \subset R$ the fibre ring

$$\Lambda \otimes_R \kappa(\mathfrak{p}) = \Lambda_\mathfrak{p}/\mathfrak{p}\Lambda_\mathfrak{p}$$

is Noetherian and geometrically regular over $\kappa(\mathfrak{p})$.

If $R \to \Lambda$ is a ring map with Λ Noetherian, then the fibre rings are always Noetherian.

Lemma 15.29.2 (Regular is a local property)**.** *Let $R \to \Lambda$ be a ring map with Λ Noetherian. Then $R \to \Lambda$ is regular if and only if the local ring maps $R_\mathfrak{p} \to \Lambda_\mathfrak{q}$ are regular for all $\mathfrak{q} \subset \Lambda$ lying over $\mathfrak{p} \subset R$.*

Proof. This is true because a Noetherian ring is regular if and only if all the local rings are regular local rings, see Algebra, Definition 15.29.1 and a ring map is flat if and only if all the induced maps of local rings are flat, see Algebra, Lemma 10.36.19. □

Lemma 15.29.3 (Regular maps and base change)**.** *Let $R \to \Lambda$ be a regular ring map. For any finite type ring map $R \to R'$ the base change $R' \to \Lambda \otimes_R R'$ is regular too.*

Proof. Flatness is preserved under any base change, see Algebra, Lemma 10.36.6. Consider a prime $\mathfrak{p}' \subset R'$ lying over $\mathfrak{p} \subset R$. The residue field extension $\kappa(\mathfrak{p}) \subset \kappa(\mathfrak{p}')$ is finitely generated as R' is of finite type over R. Hence the fibre ring

$$(\Lambda \otimes_R R') \otimes_{R'} \kappa(\mathfrak{p}') = \Lambda \otimes_R \kappa(\mathfrak{p}) \otimes_{\kappa(\mathfrak{p})} \kappa(\mathfrak{p}')$$

is Noetherian by Algebra, Lemma 10.29.7 and the assumption on the fibre rings of $R \to \Lambda$. Geometric regularity of the fibres is preserved by Algebra, Lemma 10.152.1. □

Lemma 15.29.4 (Composition of regular maps)**.** *Let $A \to B \to C$ be regular ring maps. If the fibre rings of $A \to C$ are Noetherian, then $A \to C$ is regular.*

Proof. Let $\mathfrak{p} \subset A$ be a prime. Let $\kappa(\mathfrak{p}) \subset k$ be a finite purely inseparable extension. We have to show that $C \otimes_A k$ is regular. By Lemma 15.29.3 we may assume that $A = k$ and we reduce to proving that C is regular. The assumption is that B is regular and that $B \to C$ is flat with regular fibres. Then C is regular by Algebra, Lemma 10.106.8. Some details omitted. □

Lemma 15.29.5. *Let R be a ring. Let $(A_i, \varphi_{ii'})$ be a directed system of smooth R-algebras. Set $\Lambda = \operatorname{colim} A_i$. If the fibre rings $\Lambda \otimes_R \kappa(\mathfrak{p})$ are Noetherian for all $\mathfrak{p} \subset R$, then $R \to \Lambda$ is regular.*

Proof. Note that Λ is flat over R by Algebra, Lemmas 10.36.2 and 10.130.10. Let $\kappa(\mathfrak{p}) \subset k$ be a finite purely inseparable extension. Note that

$$\Lambda \otimes_R \kappa(\mathfrak{p}) \otimes_{\kappa(\mathfrak{p})} k = \Lambda \otimes_R k = \operatorname{colim} A_i \otimes_R k$$

is a colimit of smooth k-algebras, see Algebra, Lemma 10.130.4. Since each local ring of a smooth k-algebra is regular by Algebra, Lemma 10.133.3 we conclude that all local rings of $\Lambda \otimes_R k$ are regular by Algebra, Lemma 10.101.8. This proves the lemma. □

Let's see when a field extension defines a regular ring map.

Lemma 15.29.6. *Let $k \subset K$ be a field extension. Then $k \to K$ is a regular ring map if and only if K is a separable field extension of k.*

Proof. If $k \to K$ is regular, then K is geometrically reduced over k, hence K is separable over k by Algebra, Proposition 10.145.9. Conversely, if K/k is separable, then K is a colimit of smooth k-algebras, see Algebra, Lemma 10.145.11 hence is regular by Lemma 15.29.5. □

Lemma 15.29.7. *Let $A \to B \to C$ be ring maps. If $A \to C$ is regular and $B \to C$ is flat and surjective on spectra, then $A \to B$ is regular.*

Proof. By Algebra, Lemma 10.36.9 we see that $A \to B$ is flat. Let $\mathfrak{p} \subset A$ be a prime. The ring map $B \otimes_A \kappa(\mathfrak{p}) \to C \otimes_A \kappa(\mathfrak{p})$ is flat and surjective on spectra. Hence $B \otimes_A \kappa(\mathfrak{p})$ is geometrically regular by Algebra, Lemma 10.152.3. □

15.30. Ascending properties along regular ring maps

This section is the analogue of Algebra, Section 10.149 but where the ring map $R \to S$ is regular.

Lemma 15.30.1. *Let $\varphi : R \to S$ be a ring map. Assume*

(1) *φ is regular,*
(2) *S is Noetherian, and*
(3) *R is Noetherian and reduced.*

Then S is reduced.

Proof. For Noetherian rings being reduced is the same as having properties (S_1) and (R_0), see Algebra, Lemma 10.144.3. Hence we may apply Algebra, Lemmas 10.149.4 and 10.149.5. □

15.31. Permanence of properties under completion

Given a Noetherian local ring A we denote $A^\wedge$ the completion of A with respect to its maximal ideal. We will use without further mention that $A^\wedge$ is a Noetherian complete local ring (Algebra, Lemmas 10.91.10 and 10.91.7) and that $A \to A^\wedge$ is flat (Algebra, Lemma 10.91.3).

Lemma 15.31.1. *Let A be a Noetherian local ring. Then $\dim(A) = \dim(A^\wedge)$.*

Proof. See for example Algebra, Lemma 10.106.7. □

Lemma 15.31.2. *Let A be a Noetherian local ring. Then $depth(A) = depth(A^\wedge)$.*

Proof. See Algebra, Lemma 10.149.1. □

Lemma 15.31.3. *Let A be a Noetherian local ring. Then A is Cohen-Macaulay if and only if $A^\wedge$ is so.*

Proof. A local ring A is Cohen-Macaulay if and only $\dim(A) = \text{depth}(A)$. As both of these invariants are preserved under completion (Lemmas 15.31.1 and 15.31.2) the claim follows. □

Lemma 15.31.4. *Let A be a Noetherian local ring. Then A is regular if and only if $A^\wedge$ is so.*

Proof. If $A^\wedge$ is regular, then A is regular by Algebra, Lemma 10.105.8. Assume A is regular. Let $\mathfrak{m}$ be the maximal ideal of A. Then $\dim_{\kappa(\mathfrak{m})} \mathfrak{m}/\mathfrak{m}^2 = \dim(A) = \dim(A^\wedge)$ (Lemma 15.31.1). On the other hand, $\mathfrak{m}A^\wedge$ is the maximal ideal of $A^\wedge$ and hence $\mathfrak{m}_{A^\wedge}$ is generated by at most $\dim(A^\wedge)$ elements. Thus $A^\wedge$ is regular. (You can also use Algebra, Lemma 10.106.8.) □

Lemma 15.31.5. *Let A be a Noetherian local ring.*

(1) *If $A^\wedge$ is reduced, then so is A.*
(2) *In general A reduced does not imply $A^\wedge$ is reduced.*
(3) *If A is Nagata, then A is reduced if and only if $A^\wedge$ is reduced.*

Proof. As $A \to A^\wedge$ is faithfully flat we have (1) by Algebra, Lemma 10.150.2. For (2) see Algebra, Example 10.113.4 (there are also examples in characteristic zero, see Algebra, Remark 10.113.5). For (3) see Algebra, Lemmas 10.148.27 and 10.148.24. □

15.32. Permanence of properties under henselization

Given a local ring R we denote R^h, resp. R^{sh} the henselization, resp. strict henselization of R, see Algebra, Definition 10.143.18. Many of the properties of R are reflected in R^h and R^{sh} as we will show in this section.

Lemma 15.32.1. *Let $(R, \mathfrak{m}, \kappa)$ be a local ring. Then we have the following*

(1) *$R \to R^h \to R^{sh}$ are faithfully flat ring maps,*
(2) *$\mathfrak{m}R^h = \mathfrak{m}^h$ and $\mathfrak{m}R^{sh} = \mathfrak{m}^h R^{sh} = \mathfrak{m}^{sh}$,*
(3) *$R/\mathfrak{m}^n = R^h/\mathfrak{m}^n R^h$ for all n,*
(4) *there exist elements $x_i \in R^{sh}$ such that $R^{sh}/\mathfrak{m}^n R^{sh}$ is a free $R/\mathfrak{m}^n$-module on $x_i \bmod \mathfrak{m}^n R^{sh}$.*

Proof. By construction R^h is a colimit of étale R-algebras, see Algebra, Lemma 10.143.16. Since étale ring maps are flat (Algebra, Lemma 10.136.3) we see that R^h is flat over R by Algebra, Lemma 10.36.2. As a flat local ring homomorphism is faithfully flat (Algebra, Lemma 10.36.16) we see that $R \to R^h$ is faithfully flat. The ring map $R^h \to R^{sh}$ is a colimit of finite étale ring maps, see proof of Algebra, Lemma 10.143.17. Hence the same arguments as above show that $R^h \to R^{sh}$ is faithfully flat.

Part (2) follows from Algebra, Lemmas 10.143.16 and 10.143.17. Part (3) follows from Algebra, Lemma 10.95.1 because $R/\mathfrak{m} \to R^h/\mathfrak{m}R^h$ is an isomorphism and $R/\mathfrak{m}^n \to R^h/\mathfrak{m}^n R^h$ is flat as a base change of the flat ring map $R \to R^h$ (Algebra, Lemma 10.36.6). Let κ^{sep} be the residue field of R^{sh} (it is a separable algebraic closure of κ). Choose $x_i \in R^{sh}$ mapping to a basis of κ^{sep} as a κ-vector space. Then (4) follows from Algebra, Lemma 10.95.1 in exactly the same way as above. □

Lemma 15.32.2. *Let $(R, \mathfrak{m}, \kappa)$ be a local ring. Then*

(1) *$R \to R^h$, $R^h \to R^{sh}$, and $R \to R^{sh}$ are formally étale,*
(2) *$R \to R^h$, $R^h \to R^{sh}$, resp. $R \to R^{sh}$ are formally smooth in the $\mathfrak{m}^h$, $\mathfrak{m}^{sh}$, resp. $\mathfrak{m}^{sh}$-topology.*

Proof. Part (1) follows from the fact that R^h and R^{sh} are directed colimits of étale algebras (by construction), that étale algebras are formally étale (Algebra, Lemma 10.141.2), and that colimits of formally étale algebras are formally etale (Algebra, Lemma 10.141.3). Part (2) follows from the fact that a formally étale ring map is formally smooth and Lemma 15.26.2. □

Lemma 15.32.3. *Let R be a local ring. The following are equivalent*

(1) *R is Noetherian,*
(2) *R^h is Noetherian, and*
(3) *R^{sh} is Noetherian.*

In this case we have

(a) *$(R^h)^\wedge$ and $(R^{sh})^\wedge$ are Noetherian complete local rings,*
(b) *$R^\wedge \to (R^h)^\wedge$ is an isomorphism,*
(c) *$R^h \to (R^h)^\wedge$ and $R^{sh} \to (R^{sh})^\wedge$ are flat,*

(d) $R^\wedge \to (R^{sh})^\wedge$ *is formally smooth in the* $\mathfrak{m}_{(R^{sh})^\wedge}$*-adic topology.*

Proof. Since $R \to R^h \to R^{sh}$ are faithfully flat (Lemma 15.32.1), we see that R^h or R^{sh} being Noetherian implies that R is Noetherian, see Algebra, Lemma 10.150.1. In the rest of the proof we assume R is Noetherian.

As $\mathfrak{m} \subset R$ is finitely generated it follows that $\mathfrak{m}^h = \mathfrak{m}R^h$ and $\mathfrak{m}^{sh} = \mathfrak{m}R^{sh}$ are finitely generated, see Lemma 15.32.1. Hence $(R^h)^\wedge$ and $(R^{sh})^\wedge$ are Noetherian by Algebra, Lemma 10.147.3. This proves (a).

Note that (b) is immediate from Lemma 15.32.1. In particular we see that $(R^h)^\wedge$ is flat over R, see Algebra, Lemma 10.91.4.

Next, we show that $R^h \to (R^h)^\wedge$ is flat. Write $R^h = \operatorname{colim}_i R_i$ as a directed colimit of localizations of étale R-algebras. By Algebra, Lemma 10.36.5 if $(R^h)^\wedge$ is flat over each R_i, then $R^h \to (R^h)^\wedge$ is flat. Note that $R^h = R_i^h$ (by construction). Hence $R_i^\wedge = (R^h)^\wedge$ by part (b) is flat over R_i as desired. To finish the proof of (c) we show that $R^{sh} \to (R^{sh})^\wedge$ is flat. To do this, by a limit argument as above, it suffices to show that $(R^{sh})^\wedge$ is flat over R. Note that it follows from Lemma 15.32.1 that $(R^{sh})^\wedge$ is the completion of a free R-module. By Lemma 15.18.2 we see this is flat over R as desired. This finishes the proof of (c).

At this point we know (c) is true and that $(R^h)^\wedge$ and $(R^{sh})^\wedge$ are Noetherian. It follows from Algebra, Lemma 10.150.1 that R^h and R^{sh} are Noetherian.

Part (d) follows from Lemma 15.32.2 and Lemma 15.26.4. □

Lemma 15.32.4. *Let R be a local ring. The following are equivalent: R is reduced, the henselization R^h of R is reduced, and the strict henselization R^{sh} of R is reduced.*

Proof. The ring maps $R \to R^h \to R^{sh}$ are faithfully flat. Hence one direction of the implications follows from Algebra, Lemma 10.150.2. Conversely, assume R is reduced. Since R^h and R^{sh} are filtered colimits of étale, hence smooth R-algebras, the result follows from Algebra, Lemma 10.149.6. □

Lemma 15.32.5. *Let R be a local ring. The following are equivalent: R is a normal domain, the henselization R^h of R is a normal domain, and the strict henselization R^{sh} of R is a normal domain.*

Proof. A preliminary remark is that a local ring is normal if and only if it is a normal domain (see Algebra, Definition 10.34.10). The ring maps $R \to R^h \to R^{sh}$ are faithfully flat. Hence one direction of the implications follows from Algebra, Lemma 10.150.3. Conversely, assume R is normal. Since R^h and R^{sh} are filtered colimits of étale, hence smooth R-algebras, the result follows from Algebra, Lemma 10.149.7. □

Lemma 15.32.6. *If $A \to B$ is an étale ring map and $\mathfrak{q}$ is a prime of B lying over $\mathfrak{p} \subset A$, then* $\dim(A_\mathfrak{p}) = \dim(B_\mathfrak{q})$.

Proof. Namely, because $A_\mathfrak{p} \to B_\mathfrak{q}$ is flat we have going down, and hence the inequality $\dim(A_\mathfrak{p}) \leq \dim(B_\mathfrak{q})$, see Algebra, Lemma 10.106.1. On the other hand, suppose that $\mathfrak{q}_0 \subset \mathfrak{q}_1 \subset \ldots \subset \mathfrak{q}_n$ is a chain of primes in $B_\mathfrak{q}$. Then the corresponding sequence of primes $\mathfrak{p}_0 \subset \mathfrak{p}_1 \subset \ldots \subset \mathfrak{p}_n$ (with $\mathfrak{p}_i = \mathfrak{q}_i \cap A_\mathfrak{p}$) is chain also (i.e., no equalities in the sequence) as an étale ring map is quasi-finite (see Algebra, Lemma 10.136.6) and a quasi-finite ring map induces a map of spectra with discrete fibres (by definition). This means that $\dim(A_\mathfrak{p}) \geq \dim(B_\mathfrak{q})$ as desired. □

Lemma 15.32.7. *Given any local ring R we have* $\dim(R) = \dim(R^h) = \dim(R^{sh})$.

Proof. Since $R \to R^{sh}$ is faithfully flat (Lemma 15.32.1) we see that $\dim(R^{sh}) \geq \dim(R)$ by going down, see Algebra, Lemma 10.106.1. For the converse, we write $R^{sh} = \operatorname{colim} R_i$ as a directed colimit of local rings R_i each of which is a localization of an étale R-algebra. Now if $\mathfrak{q}_0 \subset \mathfrak{q}_1 \subset \ldots \subset \mathfrak{q}_n$ is a chain of prime ideals in R^{sh}, then for some sufficiently large i the sequence

$$R_i \cap \mathfrak{q}_0 \subset R_i \cap \mathfrak{q}_1 \subset \ldots \subset R_i \cap \mathfrak{q}_n$$

is a chain of primes in R_i. Thus we see that $\dim(R^{sh}) \leq \sup_i \dim(R_i)$. But by the result of Lemma 15.32.6 we have $\dim(R_i) = \dim(R)$ for each i and we win. □

Lemma 15.32.8. *Given a Noetherian local ring R we have $depth(R) = depth(R^h) = depth(R^{sh})$.*

Proof. By Lemma 15.32.3 we know that R^h and R^{sh} are Noetherian. Hence the lemma follows from Algebra, Lemma 10.149.1. □

Lemma 15.32.9. *Let R be a Noetherian local ring. The following are equivalent: R is Cohen-Macaulay, the henselization R^h of R is Cohen-Macaulay, and the strict henselization R^{sh} of R is Cohen-Macaulay.*

Proof. By Lemma 15.32.3 we know that R^h and R^{sh} are Noetherian, hence the lemma makes sense. Since we have $\operatorname{depth}(R) = \operatorname{depth}(R^h) = \operatorname{depth}(R^{sh})$ and $\dim(R) = \dim(R^h) = \dim(R^{sh})$ by Lemmas 15.32.8 and 15.32.7 we conclude. □

Lemma 15.32.10. *Let R be a Noetherian local ring. The following are equivalent: R is a regular local ring, the henselization R^h of R is a regular local ring, and the strict henselization R^{sh} of R is a regular local ring.*

Proof. By Lemma 15.32.3 we know that R^h and R^{sh} are Noetherian, hence the lemma makes sense. Let $\mathfrak{m}$ be the maximal ideal of R. Let $x_1, \ldots, x_t \in \mathfrak{m}$ be a minimal system of generators of $\mathfrak{m}$, i.e., such that the images in $\mathfrak{m}/\mathfrak{m}^2$ form a basis over $\kappa = R/\mathfrak{m}$. Because $R \to R^h$ and $R \to R^{sh}$ are faithfully flat, it follows that the images $x_1^h, \ldots, x_t^h$ in R^h, resp. $x_1^{sh}, \ldots, x_t^{sh}$ in R^{sh} are a minimal system of generators for $\mathfrak{m}^h = \mathfrak{m}R^h$, resp. $\mathfrak{m}^{sh} = \mathfrak{m}R^{sh}$. Regularity of R by definition means $t = \dim(R)$ and similarly for R^h and R^{sh}. Hence the lemma follows from the equality of dimensions $\dim(R) = \dim(R^h) = \dim(R^{sh})$ of Lemma 15.32.7 □

Lemma 15.32.11. *Let R be a Noetherian local ring. Let $\mathfrak{p} \subset R$ be a prime. Then*

$$R^h \otimes_R \kappa(\mathfrak{p}) = \prod\nolimits_{i=1,\ldots,t} \kappa(\mathfrak{q}_i) \quad \textit{resp.} \quad R^{sh} \otimes_R \kappa(\mathfrak{p}) = \prod\nolimits_{i=1,\ldots,s} \kappa(\mathfrak{r}_i)$$

where $\mathfrak{q}_1, \ldots, \mathfrak{q}_t$, resp. $\mathfrak{r}_1, \ldots, \mathfrak{r}_s$ are the prime of R^h, resp. R^{sh} lying over $\mathfrak{p}$. Moreover, the field extensions $\kappa(\mathfrak{p}) \subset \kappa(\mathfrak{q}_i)$ resp. $\kappa(\mathfrak{p}) \subset \kappa(\mathfrak{q}_i)$ are separable algebraic.

Proof. We will use without further mention the results of Lemmas 15.32.1 and 15.32.3. Note that $R^h/\mathfrak{p}R^h$, resp. $R^{sh}/\mathfrak{p}R^{sh}$ is the henselization, resp. strict henselization of $R/\mathfrak{p}$, see Algebra, Lemma 10.143.24 resp. Algebra, Lemma 10.143.30. Hence we may replace R by $R/\mathfrak{p}$ and assume that R is a Noetherian local domain and that $\mathfrak{p} = (0)$. Since R^h, resp. R^{sh} is Noetherian, it has finitely many minimal primes $\mathfrak{q}_1, \ldots, \mathfrak{q}_t$, resp. $\mathfrak{r}_1, \ldots, \mathfrak{r}_s$. Since $R \to R^h$, resp. $R \to R^{sh}$ is flat these are exactly the primes lying over $\mathfrak{p} = (0)$ (by going down). Finally, as R is a domain, we see that R^h, resp. R^{sh} is reduced, see Lemma 15.32.4. Thus we see that

$R^h \otimes_R f.f.(R) = R^h \otimes_R \kappa(\mathfrak{p})$ resp. $R^{sh} \otimes_R f.f.(R) = R^{sh} \otimes_R \kappa(\mathfrak{p})$ is a reduced Noetherian ring with finitely many primes, all of which are minimal (and hence maximal). Thus these rings are Artinian and are products of their localizations at maximal ideals, each necessarily a field (see Algebra, Proposition 10.57.6 and Algebra, Lemma 10.24.4).

The final statement follows from the fact that $R \to R^h$, resp. $R \to R^{sh}$ is a colimit of étale ring maps and hence the induced residue field extensions are colimits of finite separable extensions, see Algebra, Lemma 10.136.5. □

15.33. Field extensions, revisited

In this section we study some peculiarities of field extensions in characteristic $p > 0$.

Definition 15.33.1. Let p be a prime number. Let $k \to K$ be an extension of fields of characteristic p. Denote kK^p the compositum of k and K^p in K.

(1) A subset $\{x_i\} \subset K$ is called *p-independent over k* if the elements $x^E = \prod x_i^{e_i}$ where $0 \leq e_i < p$ are linearly independent over kK^p.
(2) A subset $\{x_i\}$ of K is called a *p-basis of K over k* if the elements x^E form a basis of K over kK^p.

This is related to the notion of a p-basis of a $\mathbf{F}_p$-algebra which we will discuss later (insert future reference here).

Lemma 15.33.2. *Let $k \subset K$ be a field extension. Assume k has characteristic $p > 0$. Let $\{x_i\}$ be a subset of K. The following are equivalent*

(1) *the elements $\{x_i\}$ are p-independent over k, and*
(2) *the elements dx_i are K-linearly independent in $\Omega_{K/k}$.*

Any p-independent collection can be extended to a p-basis of K over k. In particular, the field K has a p-basis over k. Moreover, the following are equivalent:

(a) *$\{x_i\}$ is a p-basis of K over k, and*
(b) *dx_i is a basis of the K-vector space $\Omega_{K/k}$.*

Proof. Assume (2) and suppose that $\sum a_E x^E = 0$ is a linear relation with $a_E \in kK^p$. Let $\theta_i : K \to K$ be a k-derivation such that $\theta_i(x_j) = \delta_{ij}$ (Kronecker delta). Note that any k-derivation of K annihilates kK^p. Applying θ_i to the given relation we obtain new relations

$$\sum\nolimits_{E, e_i > 0} e_i a_E x_1^{e_1} \ldots x_i^{e_i - 1} \ldots x_n^{e_n} = 0$$

Hence if we pick $\sum a_E x^E$ as the relation with minimal total degree $|E| = \sum e_i$ for some $a_E \neq 0$, then we get a contradiction. Hence (2) holds.

If $\{x_i\}$ is a p-basis for K over k, then $K \cong kK^p[X_i]/(X_i^p - x_i^p)$. Hence we see that $\mathrm{d}x_i$ forms a basis for $\Omega_{K/k}$ over K. Thus (a) implies (b).

Let $\{x_i\}$ be a p-independent subset of K over k. An application of Zorn's lemma shows that we can enlarge this to a maximal p-independent subset of K over k. We claim that any maximal p-independent subset $\{x_i\}$ of K is a p-basis of K over k. The claim will imply that (1) implies (2) and establish the existence of p-bases. To prove the claim let L be the subfield of K generated by kK^p and the x_i. We have to show that $L = K$. If $x \in K$ but $x \notin L$, then $x^p \in L$ and $L(x) \cong L[z]/(z^p - x)$. Hence $\{x_i\} \cup \{x\}$ is p-independent over k, a contradiction.

Finally, we have to show that (b) implies (a). By the equivalence of (1) and (2) we see that $\{x_i\}$ is a maximal p-independent subset of K over k. Hence by the claim above it is a p-basis. □

Lemma 15.33.3. *Let $k \subset K$ be a field extension. Let $\{K_\alpha\}_{\alpha \in A}$ be a collection of subfields of K with the following properties*

(1) *$k \subset K_\alpha$ for all $\alpha \in A$,*
(2) *$k = \bigcap_{\alpha \in A} K_\alpha$,*
(3) *for $\alpha, \alpha' \in A$ there exists an $\alpha'' \in A$ such that $K_{\alpha''} \subset K_\alpha \cap K_{\alpha'}$.*

Then for $n \geq 1$ and $V \subset K^{\oplus n}$ a K-vector space we have $V \cap k^{\oplus n} \neq 0$ if and only if $V \cap K_\alpha^{\oplus n} \neq 0$ for all $\alpha \in A$.

Proof. By induction on n. The case $n = 1$ follows from the assumptions. Assume the result proven for subspaces of $K^{\oplus n-1}$. Assume that $V \subset K^{\oplus n}$ has nonzero intersection with $K_\alpha^{\oplus n}$ for all $\alpha \in A$. If $V \cap 0 \oplus k^{\oplus n-1}$ is nonzero then we win. Hence we may assume this is not the case. By induction hypothesis we can find an α such that $V \cap 0 \oplus K_\alpha^{\oplus n-1}$ is zero. Let $v = (x_1, \ldots, x_n) \in V \cap K_\alpha$ be a nonzero element. By our choice of α we see that x_1 is not zero. Replace v by $x_1^{-1}v$ so that $v = (1, x_2, \ldots, x_n)$. Note that if $v' = (x'_1, \ldots, x'_n) \in V \cap K_\alpha$, then $v' - x'_1 v = 0$ by our choice of α. Hence we see that $V \cap K_\alpha^{\oplus n} = K_\alpha v$. If we choose some α' such that $K_{\alpha'} \subset K_\alpha$, then we see that necessarily $v \in V \cap K_{\alpha'}^{\oplus n}$ (by the same arguments applied to α'). Hence

$$x_2, \ldots, x_n \in \bigcap\nolimits_{\alpha' \in A, K_{\alpha'} \subset K_\alpha} K_{\alpha'}$$

which equals k by (2) and (3). □

Lemma 15.33.4. *Let K be a field of characteristic p. Let $\{K_\alpha\}_{\alpha \in A}$ be a collection of subfields of K with the following properties*

(1) *$K^p \subset K_\alpha$ for all $\alpha \in A$,*
(2) *$K^p = \bigcap_{\alpha \in A} K_\alpha$,*
(3) *for $\alpha, \alpha' \in A$ there exists an $\alpha'' \in A$ such that $K_{\alpha''} \subset K_\alpha \cap K_{\alpha'}$.*

Then

(1) *the intersection of the kernels of the maps $\Omega_{K/\mathbf{F}_p} \to \Omega_{K/K_\alpha}$ is zero,*
(2) *for any finite extension $K \subset L$ we have $L^p = \bigcap_{\alpha \in A} L^p K_\alpha$.*

Proof. Proof of (1). Choose a p-basis $\{x_i\}$ for K over $\mathbf{F}_p$. Suppose that $\eta = \sum_{i \in I'} y_i \mathrm{d}x_i$ maps to zero in Ω_{K/K_α} for every $\alpha \in A$. Here the index set I' is finite. By Lemma 15.33.2 this means that for every α there exists a relation

$$\sum\nolimits_E a_{E,\alpha} x^E, \quad a_{E,\alpha} \in K_\alpha$$

where E runs over multi-indices $E = (e_i)_{i \in I'}$ with $0 \leq e_i < p$. On the other hand, Lemma 15.33.2 guarantees there is no such relation $\sum a_E x^E = 0$ with $a_E \in K^p$. This is a contradiction by Lemma 15.33.3.

Proof of (2). Suppose that we have a tower $K \subset M \subset L$ of finite extensions of fields. Set $M_\alpha = M^p K_\alpha$ and $L_\alpha = L^p K_\alpha = L^p M_\alpha$. Then we can first prove that $M^p = \bigcap_{\alpha \in A} M_\alpha$, and after that prove that $L^p = \bigcap_{\alpha \in A} L_\alpha$. Hence it suffices to prove (2) for primitive field extensions having no nontrivial subfields. First, assume

that $L = K(\theta)$ is separable over K. Then L is generated by θ^p over K, hence we may assume that $\theta \in L^p$. In this case we see that

$$L^p = K^p \oplus K^p\theta \oplus \ldots K^p\theta^{d-1} \quad \text{and} \quad L^pK_\alpha = K_\alpha \oplus K_\alpha\theta \oplus \ldots K_\alpha\theta^{d-1}$$

where $d = [L : K]$. Thus the conclusion is clear in this case. The other case is where $L = K(\theta)$ with $\theta^p = t \in K$, $t \notin K^p$. In this case we have

$$L^p = K^p \oplus K^pt \oplus \ldots K^pt^{p-1} \quad \text{and} \quad L^pK_\alpha = K_\alpha \oplus K_\alpha t \oplus \ldots K_\alpha t^{p-1}$$

Again the result is clear. □

Lemma 15.33.5. *Let k be a field of characteristic $p > 0$. Let $n, m \geq 0$. As k' ranges through all subfields $k^p \subset k' \subset k$ with $[k : k'] < \infty$ the subfields*

$$f.f.(k'[[x_1^p, \ldots, x_n^p]][y_1^p, \ldots, y_m^p]) \subset f.f.(k[[x_1, \ldots, x_d]][y_1, \ldots, y_m])$$

form a family of subfields as in Lemma 15.33.4. Moreover, each of the ring extensions $k'[[x_1^p, \ldots, x_n^p]][y_1^p, \ldots, y_m^p] \subset k[[x_1, \ldots, x_n]][y_1, \ldots, y_m]$ is finite.

Proof. Write $A = k[[x_1, \ldots, x_n]][y_1, \ldots, y_m]$ and $A' = k'[[x_1^p, \ldots, x_n^p]][y_1^p, \ldots, y_m^p]$. We also set $K = f.f.(A)$ and $K' = f.f.(A')$. The ring extension $k'[[x_1^p, \ldots, x_d^p]] \subset k[[x_1, \ldots, x_d]]$ is finite by Algebra, Lemma 10.91.18 which implies that $A \to A'$ is finite. For $f \in A$ we see that $f^p \in A'$. Hence $K^p \subset K'$. Any element of K' can be written as a/b^p with $a \in A'$ and $b \in A$ nonzero. Suppose that $f/g^p \in K$, $f, g \in A$, $g \neq 0$ is contained in K' for every choice of k'. Fix a choice of k' for the moment. By the above we see $f/g^p = a/b^p$ for some $a \in A'$ and some nonzero $b \in A$. Hence $b^pf \in A'$. For any A'-derivation $D : A \to A$ we see that $0 = D(b^pf) = b^pD(f)$ hence $D(f) = 0$ as A is a domain. Taking $D = \partial_{x_i}$ and $D = \partial_{y_j}$ we conclude that that $f \in k[[x_1^p, \ldots, x_n^p]][y_1^p, \ldots, y_d^p]$. Applying a k'-derivation $\theta : k \to k$ we similarly conclude that all coefficients of f are in k', i.e., $f \in A'$. Since it is clear that $A = \bigcap_{k'} A'$ where k' ranges over all subfields as in the lemma we win. □

15.34. The singular locus

Let R be a Noetherian ring. The *regular locus* $\mathrm{Reg}(X)$ of $X = \mathrm{Spec}(R)$ is the set of primes $\mathfrak{p}$ such that $R_\mathfrak{p}$ is a regular local ring. The *singular locus* $\mathrm{Sing}(X)$ of $X = \mathrm{Spec}(R)$ is the complement $X \setminus \mathrm{Reg}(X)$, i.e., the set of primes $\mathfrak{p}$ such that $R_\mathfrak{p}$ is not a regular local ring. By the discussion preceding Algebra, Definition 10.105.6 we see that $\mathrm{Reg}(X)$ is stable under generalization In the section we study conditions that guarantee that $\mathrm{Reg}(X)$ is open.

Definition 15.34.1. Let R be a Noetherian ring. Let $X = \mathrm{Spec}(R)$.

(1) We say R is *J-0* if $\mathrm{Reg}(X)$ contains a nonempty open.
(2) We say R is *J-1* if $\mathrm{Reg}(X)$ is open.
(3) We say R is *J-2* if any finite type R-algebra is J-1.

The ring $\mathbf{Q}[x]/(x^2)$ does not satisfy J-0. On the other hand J-1 implies J-0 for domains and even reduced rings as such a ring is regular at the minimal primes. Here is a characterization of the J-1 property.

Lemma 15.34.2. *Let R be a Noetherian ring. Let $X = \mathrm{Spec}(R)$. The ring R is J-1 if and only if $V(\mathfrak{p}) \cap Reg(X)$ contains a nonempty open subset of $V(\mathfrak{p})$ for all $\mathfrak{p} \in Reg(X)$.*

Proof. This follows immediately from Topology, Lemma 5.15.5. □

Lemma 15.34.3. *Let R be a Noetherian ring. Let $X = \mathrm{Spec}(R)$. Assume that for all $\mathfrak{p} \subset R$ the ring $R/\mathfrak{p}$ is J-0. Then R is J-1.*

Proof. We will show that the criterion of Lemma 15.34.2 applies. Let $\mathfrak{p} \in \mathrm{Reg}(X)$ be a prime of height r. Pick $f_1, \ldots, f_r \in \mathfrak{p}$ which map to generators of $\mathfrak{p}R_\mathfrak{p}$. Since $\mathfrak{p} \in \mathrm{Reg}(X)$ we see that $f_1, \ldots, f_r$ maps to a regular sequence in $R_\mathfrak{p}$, see Algebra, Lemma 10.101.3. Thus by Algebra, Lemma 10.65.8 we see that after replacing R by R_g for some $g \in R$, $g \notin \mathfrak{p}$ the sequence $f_1, \ldots, f_r$ is a regular sequence in R. Next, let $\mathfrak{p} \subset \mathfrak{q}$ be a prime ideal such that $(R/\mathfrak{p})_\mathfrak{q}$ is a regular local ring. By the assumption of the lemma there exists a non-empty open subset of $V(\mathfrak{p})$ consisting of such primes, hence it suffices to prove $R_\mathfrak{q}$ is regular. Note that $f_1, \ldots, f_r$ is a regular sequence in $R_\mathfrak{q}$ such that $R_\mathfrak{q}/(f_1, \ldots, f_r)R_\mathfrak{q}$ is regular. Hence $R_\mathfrak{q}$ is regular by Algebra, Lemma 10.101.7. □

Lemma 15.34.4. *Let $R \to S$ be a ring map. Assume that*

1. *R is a Noetherian domain,*
2. *$R \to S$ is injective and of finite type, and*
3. *S is a domain and J-0.*

Then R is J-0.

Proof. After replacing S by S_g for some nonzero $g \in S$ we may assume that S is a regular ring. By generic flatness we may assume that also $R \to S$ is faithfully flat, see Algebra, Lemma 10.112.1. Then R is regular by Algebra, Lemma 10.150.4. □

Lemma 15.34.5. *Let $R \to S$ be a ring map. Assume that*

1. *R is a Noetherian domain and J-0,*
2. *$R \to S$ is injective and of finite type, and*
3. *S is a domain and $f.f.(R) \to f.f.(S)$ is separable.*

Then S is J-0.

Proof. We may replace R by a principal localization and assume R is a regular ring. By Algebra, Lemma 10.133.9 the ring map $R \to S$ is smooth at (0). Hence after replacing S by a principal localization we may assume that S is smooth over R. Then S is regular too, see Algebra, Lemma 10.149.8. □

Lemma 15.34.6. *Let R be a Noetherian ring. The following are equivalent*

1. *R is J-2,*
2. *every finite type R-algebra which is a domain is J-0,*
3. *every finite R-algebra is J-1,*
4. *for every prime $\mathfrak{p}$ and every finite purely inseparable extension $\kappa(\mathfrak{p}) \subset L$ there exists a finite R-algebra R' which is a domain, which is J-0, and whose field of fractions is L.*

Proof. It is clear that we have the implications (1) ⇒ (2) and (2) ⇒ (4). Recall that a domain which is J-1 is J-0. Hence we also have the implications (1) ⇒ (3) and (3) ⇒ (4).

Let $R \to S$ be a finite type ring map and let's try to show S is J-1. By Lemma 15.34.3 it suffices to prove that $S/\mathfrak{q}$ is J-0 for every prime $\mathfrak{q}$ of S. In this way we see (2) ⇒ (1).

Assume (4). We will show that (2) holds which will finish the proof. Let $R \to S$ be a finite type ring map with S a domain. Let $\mathfrak{p} = \mathrm{Ker}(R \to S)$. Set $K = f.f.(S)$. There exists a diagram of fields

$$\begin{array}{ccc} K & \longrightarrow & K' \\ \uparrow & & \uparrow \\ \kappa(\mathfrak{p}) & \longrightarrow & L \end{array}$$

where the horizontal arrows are finite purely inseparable field extensions and where K'/L is separable, see Algebra, Lemma 10.39.4. Choose $R' \subset L$ as in (4) and let S' be the image of the map $S \otimes_R R' \to K'$. Then S' is a domain whose fraction field is K', hence S' is J-0 by Lemma 15.34.5 and our choice of R'. Then we apply Lemma 15.34.4 to see that S is J-0 as desired. □

15.35. Regularity and derivations

Let $R \to S$ be a ring map. Let $D : R \to R$ be a derivation. We say that D *extends to* S if there exists a derivation $D' : S \to S$ such that

$$\begin{array}{ccc} S & \xrightarrow{D'} & S \\ \uparrow & & \uparrow \\ R & \xrightarrow{D} & R \end{array}$$

is commutative.

Lemma 15.35.1. *Let R be a ring. Let $D : R \to R$ be a derivation.*

(1) *For any ideal $I \subset R$ the derivation D extends canonically to a derivation $D^\wedge : R^\wedge \to R^\wedge$ on the I-adic completion.*
(2) *For any multiplicative subset $S \subset R$ the derivation D extends uniquely to the localization $S^{-1}R$ of R.*

If $R \subset R'$ is an finite type extension of rings such that $R_g \cong R'_g$ for some nonzerodivisor $g \in R$, then $g^N D$ extends to R' for some $N \geq 0$.

Proof. Proof of (1). For $n \geq 2$ we have $D(I^n) \subset I^{n-1}$ by the Leibniz rule. Hence D induces maps $D_n : R/I^n \to R/I^{n-1}$. Taking the limit we obtain $D^\wedge$. We omit the verification that $D^\wedge$ is a derivation.

Proof of (2). To extend D to $S^{-1}R$ just set $D(r/s) = D(r)/s - rD(s)/s^2$ and check the axioms.

Proof of the final statement. Let $x_1, \ldots, x_n \in R'$ be generators of R' over R. Choose an N such that $g^N x_i \in R$. Consider $g^{N+1}D$. By (2) this extends to R_g. Moreover, by the Leibniz rule and our construction of the extension above we have

$$g^{N+1}D(x_i) = g^{N+1}D(g^{-N}g^N x_i) = -Ng^N x_i D(g) + gD(g^N x_i)$$

and both terms are in R. This implies that

$$g^{N+1}D(x_1^{e_1} \ldots x_n^{e_n}) = \sum e_i x_1^{e_1} \ldots x_i^{e_i - 1} \ldots x_n^{e_n} g^{N+1} D(x_i)$$

is an element of R'. Hence every element of R' (which can be written as a sum of monomials in the x_i with coefficients in R) is mapped to an element of R' by $g^{N+1}D$ and we win. □

Lemma 15.35.2. *Let R be a regular ring. Let $f \in R$. Assume there exists a derivation $D : R \to R$ such that $D(f)$ is a unit of $R/(f)$. Then $R/(f)$ is regular.*

Proof. It suffices to prove this when R is a local ring with maximal ideal $\mathfrak{m}$ and residue field κ. In this case it suffices to prove that $f \notin \mathfrak{m}^2$, see Algebra, Lemma 10.101.3. However, if $f \in \mathfrak{m}^2$ then $D(f) \in \mathfrak{m}$ by the Leibniz rule, a contradiction. □

Lemma 15.35.3. *Let R be a regular $\mathbf{F}_p$-algebra. Let $f \in R$. Assume there exists a derivation $D : R \to R$ such that $D(f)$ is a unit of R. Then $R[z]/(z^p - f)$ is regular.*

Proof. Apply Lemma 15.35.2 to the extension of D to $R[z]$ which maps z to zero. □

Lemma 15.35.4. *Let p be a prime number. Let B be a domain with $p = 0$ in B. Let $f \in B$ be an element which is not a pth power in the fraction field of B. If B is of finite type over a Noetherian complete local ring, then there exists a derivation $D : B \to B$ such that $D(f)$ is not zero.*

Proof. Let R be a Noetherian complete local ring such that there exists a finite type ring map $R \to B$. Of course we may replace R by its image in B, hence we may assume R is a domain of characteristic $p > 0$ (as well as Noetherian complete local). By Algebra, Lemma 10.147.10 we can write R as a finite extension of $k[[x_1, \ldots, x_n]]$ for some field k and integer n. Hence we may replace R by $k[[x_1, \ldots, x_n]]$. Next, we use Algebra, Lemma 10.109.7 to factor $R \to B$ as

$$R \subset R[y_1, \ldots, y_d] \subset B' \subset B$$

with B' finite over $R[y_1, \ldots, y_d]$ and $B'_g \cong B_g$ for some nonzero $g \in R$. Note that $f' = g^{pN} f \in B'$ for some large integer N. It is clear that f' is not a pth power in $f.f.(B') = f.f.(B)$. If we can find a derivation $D' : B' \to B'$ with $D'(f') \neq 0$, then Lemma 15.35.1 guarantees that $D = g^M D'$ extends to S for some $M > 0$. Then $D(f) = g^N D'(f) = g^M D'(g^{-pN} f') = g^{M-pN} D'(f')$ is nonzero. Thus it suffices to prove the lemma in case B is a finite extension of $A = k[[x_1, \ldots, x_n]][y_1, \ldots, y_m]$.

Note that $\mathrm{d}f$ is not zero in $\Omega_{f.f.(B)/\mathbf{F}_p}$, see Algebra, Lemma 10.145.2. We apply Lemma 15.33.5 to find a subfield $k' \subset k$ of finite index such that with $A' = k'[[x_1^p, \ldots, x_n^p]][y_1^p, \ldots, y_m^p]$ the element $\mathrm{d}f$ does not map to zero in $\Omega_{f.f.(B)/f.f.(A')}$. Thus we can choose a $f.f.(A')$-derivation $D' : f.f.(B) \to f.f.(B)$ with $D'(f) \neq 0$. Since $A' \subset A$ and $A \subset B$ are finite by construction we see that $A' \subset B$ is finite. Choose $b_1, \ldots, b_t \in B$ which generate B as an A'-module. Then $D'(b_i) = f_i/g_i$ for some $f_i, g_i \in B$ with $g_i \neq 0$. Setting $D = g_1 \ldots g_t D'$ we win. □

Lemma 15.35.5. *Let A be a Noetherian complete local domain. Then A is J-0.*

Proof. By Algebra, Lemma 10.147.10 we can find a regular subring $A_0 \subset A$ with A finite over A_0. If $f.f.(A_0) \subset f.f.(A)$ is separable, then we are done by Lemma 15.34.5. If not, then A_0 and A have characteristic $p > 0$. For any subextension $f.f.(A_0) \subset M \subset f.f.(A)$ there exists a finite subextension $A_0 \subset B \subset A$ such that $f.f.(B) = M$. Hence, arguing by induction on $[f.f.(A) : f.f.(A_0)]$ we may assume there exists $A_0 \subset B \subset A$ such that B is J-0 and $f.f.(B) \subset f.f.(A)$ has no nontrivial subextensions. In this case, if $f.f.(B) \subset f.f.(A)$ is separable, then we see that A is J-0 by Lemma 15.34.5. If not, then $f.f.(A) = f.f.(B)[z]/(z^p - b)$ for some $b \in B$ which is not a pth power in $f.f.(B)$. By Lemma 15.35.4 we can find a derivation

$D : B \to B$ with $D(f) \neq 0$. Applying Lemma 15.35.3 we see that $A_{\mathfrak{p}}$ is regular for any prime $\mathfrak{p}$ of A lying over a regular prime of B and not containing $D(f)$. As B is J-0 we conclude A is too. □

Proposition 15.35.6. *The following types of rings are J-2:*

(1) *fields,*
(2) *Noetherian complete local rings,*
(3) $\mathbf{Z}$*,*
(4) *Dedekind domains with fraction field of characteristic zero,*
(5) *finite type ring extensions of any of the above.*

Proof. For fields, $\mathbf{Z}$ and Dedekind domains of characteristic zero you just check condition (4) of Lemma 15.34.6. In the case of Noetherian complete local rings, note that if $R \to R'$ is finite and R is a Noetherian complete local ring, then R' is a product of Noetherian complete local rings, see Algebra, Lemma 10.147.2. Hence it suffices to prove that a Noetherian complete local ring which is a domain is J-0, which is Lemma 15.35.5. □

15.36. Formal smoothness and regularity

The title of this section refers to Proposition 15.36.2.

Lemma 15.36.1. *Let $A \to B$ be a local homomorphism of Noetherian local rings. Let $D : A \to A$ be a derivation. Assume that B is complete and $A \to B$ is formally smooth in the $\mathfrak{m}_B$-adic topology. Then there exists an extension $D' : B \to B$ of D.*

Proof. Denote $B[\epsilon] = B[x]/(x^2)$ the ring of dual numbers over B. Consider the ring map $\psi : A \to B[\epsilon]$, $a \mapsto a + \epsilon D(a)$. Consider the commutative diagram

$$\begin{array}{ccc} B & \xrightarrow{1} & B \\ \uparrow & & \uparrow \\ A & \xrightarrow{\psi} & B[\epsilon] \end{array}$$

By Lemma 15.26.5 and the assumption of formal smoothness of B/A we find a map $\varphi : B \to B[\epsilon]$ fitting into the diagram. Write $\varphi(b) = b + \epsilon D'(b)$. Then $D' : B \to B$ is the desired extension. □

Proposition 15.36.2. *Let $A \to B$ be a local homomorphism of Noetherian complete local rings. The following are equivalent*

(1) *$A \to B$ is regular,*
(2) *$A \to B$ is flat and $\overline{B}$ is geometrically regular over k,*
(3) *$A \to B$ is flat and $k \to \overline{B}$ is formally smooth in the $\mathfrak{m}_{\overline{B}}$-adic topology, and*
(4) *$A \to B$ is formally smooth in the $\mathfrak{m}_B$-adic topology.*

Proof. We have seen the equivalence of (2), (3), and (4) in Proposition 15.28.4. It is clear that (1) implies (2). Thus we assume the equivalent conditions (2), (3), and (4) hold and we prove (1).

Let $\mathfrak{p}$ be a prime of A. We will show that $B \otimes_A \kappa(\mathfrak{p})$ is geometrically regular over $\kappa(\mathfrak{p})$. By Lemma 15.26.8 we may replace A by $A/\mathfrak{p}$ and B by $B/\mathfrak{p}B$. Thus we may assume that A is a domain and that $\mathfrak{p} = (0)$.

Choose $A_0 \subset A$ as in Algebra, Lemma 10.147.10. We will use all the properties stated in that lemma without further mention. As $A_0 \to A$ induces an isomorphism on residue fields, and as $B/\mathfrak{m}_A B$ is geometrically regular over $A/\mathfrak{m}_A$ we can find a diagram

$$\begin{array}{ccc} C & \longrightarrow & B \\ \uparrow & & \uparrow \\ A_0 & \longrightarrow & A \end{array}$$

with $A_0 \to C$ formally smooth in the $\mathfrak{m}_C$-adic topology such that $B = C \otimes_{A_0} A$, see Remark 15.28.6. (Completion in the tensor product is not needed as $A_0 \to A$ is finite, see Algebra, Lemma 10.91.2.) Hence it suffices to show that $C \otimes_{A_0} f.f.(A_0)$ is a geometrically regular algebra over $f.f.(A_0)$.

The upshot of the preceding paragraph is that we may assume that $A = k[[x_1, \ldots, x_n]]$ where k is a field or $A = \Lambda[[x_1, \ldots, x_n]]$ where Λ is a Cohen ring. In this case B is a regular ring, see Algebra, Lemma 10.106.8. Hence $B \otimes_A f.f.(A)$ is a regular ring too and we win if the characteristic of $f.f.(A)$ is zero.

Thus we are left with the case where $A = k[[x_1, \ldots, x_n]]$ and k is a field of characteristic $p > 0$. Set $K = f.f.(A)$. Let $L \supset K$ be a finite purely inseparable field extension. We will show by induction on $[L : K]$ that $B \otimes_A L$ is regular. The base case is $L = K$ which we've seen above. Let $K \subset M \subset L$ be a subfield such that L is a degree p extension of M obtained by adjoining a pth root of an element $f \in M$. Let A' be a finite A-subalgebra of M with fraction field M. Clearing denominators, we may and do assume $f \in A'$. Set $A'' = A'[z]/(z^p - f)$ and note that $A' \subset A''$ is finite and that the fraction field of A'' is L. By induction we know that $B \otimes_A M$ ring is regular. We have

$$B \otimes_A L = B \otimes_A M[z]/(z^p - f)$$

By Lemma 15.35.4 we know there exists a derivation $D : A' \to A'$ such that $D(f) \neq 0$. As $A' \to B \otimes_A A'$ is formally smooth in the $\mathfrak{m}$-adic topology by Lemma 15.26.9 we can use Lemma 15.36.1 to extend D to a derivation $D' : B \otimes_A A' \to B \otimes_A A'$. Note that $D'(f) = D(f)$ is a unit in $B \otimes_A M$ as $D(f)$ is not zero in $A' \subset M$. Hence $B \otimes_A L$ is regular by Lemma 15.35.3 and we win. □

15.37. G-rings

Let A be a Noetherian local ring A. In Section 15.31 we have seen that some but not all properties of A are reflected in the completion $A^\wedge$ of A. To study this further we introduce some terminology. For a prime $\mathfrak{q}$ of A the fibre ring

$$(A^\wedge) \otimes_A \kappa(\mathfrak{q}) = (A^\wedge)_\mathfrak{q}/\mathfrak{q}(A^\wedge)_\mathfrak{q}$$

is called a *formal fibre* of A. We think of the formal fibre as an algebra over $\kappa(\mathfrak{q})$. Thus $A \to A^\wedge$ is a regular ring homomorphism if and only if all the formal fibres are geometrically regular algebras.

Definition 15.37.1. A ring R is called a *G-ring* if R is Noetherian and for every prime $\mathfrak{p}$ of R the ring map $R_\mathfrak{p} \to (R_\mathfrak{p})^\wedge$ is regular.

By the discussion above we see that R is a G-ring if and only if every local ring $R_\mathfrak{p}$ has geometrically regular formal fibres. Note that if $\mathbf{Q} \subset R$, then it suffices to

check the formal fibres are regular. Another way to express the G-ring condition is described in the following lemma.

Lemma 15.37.2. *Let R be a Noetherian ring. Then R is a G-ring if and only if for every pair of primes $\mathfrak{q} \subset \mathfrak{p} \subset R$ the algebra*

$$(R/\mathfrak{q})^\wedge_\mathfrak{p} \otimes_{R/\mathfrak{q}} \kappa(\mathfrak{q})$$

is geometrically regular over $\kappa(\mathfrak{q})$.

Proof. This follows from the fact that

$$R^\wedge_\mathfrak{p} \otimes_R \kappa(\mathfrak{q}) = (R/\mathfrak{q})^\wedge_\mathfrak{p} \otimes_{R/\mathfrak{q}} \kappa(\mathfrak{q})$$

as algebras over $\kappa(\mathfrak{q})$. □

Lemma 15.37.3. *Let $R \to R'$ be a finite type map of Noetherian rings and let*

$$\begin{array}{ccccc} \mathfrak{q}' & \longrightarrow & \mathfrak{p}' & \longrightarrow & R' \\ \downarrow & & \downarrow & & \uparrow \\ \mathfrak{q} & \longrightarrow & \mathfrak{p} & \longrightarrow & R \end{array}$$

be primes. Assume $R \to R'$ is quasi-finite at $\mathfrak{p}'$.

1. *If the formal fibre $R^\wedge_\mathfrak{p} \otimes_R \kappa(\mathfrak{q})$ is geometrically regular over $\kappa(\mathfrak{q})$, then the formal fibre $R'_{\mathfrak{p}'} \otimes_{R'} \kappa(\mathfrak{q}')$ is geometrically regular over $\kappa(\mathfrak{q}')$.*
2. *If the formal fibres of $R_\mathfrak{p}$ are geometrically regular, then the formal fibres of $R'_{\mathfrak{p}'}$ are geometrically regular.*
3. *If $R \to R'$ is quasi-finite and R is a G-ring, then R' is a G-ring.*

Proof. It is clear that (1) ⇒ (2) ⇒ (3). Assume $R^\wedge_\mathfrak{p} \otimes_R \kappa(\mathfrak{q})$ is geometrically regular over $\kappa(\mathfrak{q})$. By Algebra, Lemma 10.118.3 we see that

$$R^\wedge_\mathfrak{p} \otimes_R R' = (R'_{\mathfrak{p}'})^\wedge \times B$$

for some $R^\wedge_\mathfrak{p}$-algebra B. Hence $R'_{\mathfrak{p}'} \to (R'_{\mathfrak{p}'})^\wedge$ is a factor of a base change of the map $R_\mathfrak{p} \to R^\wedge_\mathfrak{p}$. It follows that $(R'_{\mathfrak{p}'})^\wedge \otimes_{R'} \kappa(\mathfrak{q}')$ is a factor of

$$R^\wedge_\mathfrak{p} \otimes_R R' \otimes_{R'} \kappa(\mathfrak{q}') = R^\wedge_\mathfrak{p} \otimes_R \kappa(\mathfrak{q}) \otimes_{\kappa(\mathfrak{q})} \kappa(\mathfrak{q}').$$

Thus the result follows as extension of base field preserves geometric regularity, see Algebra, Lemma 10.152.1. □

Lemma 15.37.4. *Let R be a Noetherian ring. Then R is a G-ring if and only if for every finite free ring map $R \to S$ the formal fibres of S are regular rings.*

Proof. Assume that for any finite free ring map $R \to S$ the ring S has regular formal fibres. Let $\mathfrak{q} \subset \mathfrak{p} \subset R$ be primes and let $\kappa(\mathfrak{q}) \subset L$ be a finite purely inseparable extension. To show that R is a G-ring it suffices to show that

$$R^\wedge_\mathfrak{p} \otimes_R \kappa(\mathfrak{q}) \otimes_{\kappa(\mathfrak{q})} L$$

is a regular ring. Choose a finite free extension $R \to R'$ such that $\mathfrak{q}' = \mathfrak{q}R'$ is a prime and such that $\kappa(\mathfrak{q}')$ is isomorphic to L over $\kappa(\mathfrak{q})$, see Algebra, Lemma 10.146.3. By Algebra, Lemma 10.91.19 we have

$$R^\wedge_\mathfrak{p} \otimes_R R' = \prod (R'_{\mathfrak{p}'_i})^\wedge$$

where $\mathfrak{p}'_i$ are the primes of R' lying over $\mathfrak{p}$. Thus we have

$$R^\wedge_\mathfrak{p} \otimes_R \kappa(\mathfrak{q}) \otimes_{\kappa(\mathfrak{q})} L = R^\wedge_\mathfrak{p} \otimes_R R' \otimes_{R'} \kappa(\mathfrak{q}') = \prod (R'_{\mathfrak{p}'_i})^\wedge \otimes_{R'_{\mathfrak{p}'_i}} \kappa(\mathfrak{q}')$$

Our assumption is that the rings on the right are regular, hence the ring on the left is regular too. Thus R is a G-ring. The converse follows from Lemma 15.37.3. □

Lemma 15.37.5. *Let k be a field of characteristic p. Let $A = k[[x_1, \ldots, x_n]][y_1, \ldots, y_n]$ and denote $K = f.f.(A)$. Let $\mathfrak{p} \subset A$ be a prime. Then $A^\wedge_\mathfrak{p} \otimes_A K$ is geometrically regular over K.*

Proof. Let $L \supset K$ be a finite purely inseparable field extension. We will show by induction on $[L : K]$ that $A^\wedge_\mathfrak{p} \otimes L$ is regular. The base case is $L = K$: as A is regular, $A^\wedge_\mathfrak{p}$ is regular (Lemma 15.31.4), hence the localization $A^\wedge_\mathfrak{p} \otimes K$ is regular. Let $K \subset M \subset L$ be a subfield such that L is a degree p extension of M obtained by adjoining a pth root of an element $f \in M$. Let B be a finite A-subalgebra of M with fraction field M. Clearing denominators, we may and do assume $f \in B$. Set $C = B[z]/(z^p - f)$ and note that $B \subset C$ is finite and that the fraction field of C is L. Since $A \subset B \subset C$ are finite and $L/M/K$ are purely inseparable we see that for every element of B or C some power of it lies in A. Hence there is a unique prime $\mathfrak{r} \subset B$, resp. $\mathfrak{q} \subset C$ lying over $\mathfrak{p}$. Note that

$$A^\wedge_\mathfrak{p} \otimes_A M = B^\wedge_\mathfrak{r} \otimes_B M$$

see Algebra, Lemma 10.91.19. By induction we know that this ring is regular. In the same manner we have

$$A^\wedge_\mathfrak{p} \otimes_A L = C^\wedge_\mathfrak{r} \otimes_C L = B^\wedge_\mathfrak{r} \otimes_B M[z]/(z^p - f)$$

the last equality because the completion of $C = B[z]/(z^p - f)$ equals $B^\wedge_\mathfrak{r}[z]/(z^p - f)$. By Lemma 15.35.4 we know there exists a derivation $D : B \to B$ such that $D(f) \neq 0$. In other words, $g = D(f)$ is a unit in M! By Lemma 15.35.1 D extends to a derivation of $B_\mathfrak{r}$, $B^\wedge_\mathfrak{r}$ and $B^\wedge_\mathfrak{r} \otimes_B M$ (successively extending through a localization, a completion, and a localization). Since it is an extension we end up with a derivation of $B^\wedge_\mathfrak{r} \otimes_B M$ which maps f to g and g is a unit of the ring $B^\wedge_\mathfrak{r} \otimes_B M$. Hence $A^\wedge_\mathfrak{p} \otimes_A L$ is regular by Lemma 15.35.3 and we win. □

Proposition 15.37.6. *A Noetherian complete local ring is a G-ring.*

Proof. Let A be a Noetherian complete local ring. By Lemma 15.37.2 it suffices to check that $B = A/\mathfrak{q}$ has geometrically regular formal fibres over the minimal prime (0) of B. Thus we may assume that A is a domain and it suffices to check the condition for the formal fibres over the minimal prime (0) of A. Set $K = f.f(A)$.

We can choose a subring $A_0 \subset A$ which is a regular complete local ring such that A is finite over A_0, see Algebra, Lemma 10.147.10. Moreover, we may assume that A_0 is a power series ring over a field or a Cohen ring. By Lemma 15.37.3 we see that it suffices to prove the result for A_0.

Assume that A is a power series ring over a field or a Cohen ring. Since A is regular the localizations $A_\mathfrak{p}$ are regular (see Algebra, Definition 10.105.6 and the discussion preceding it). Hence the completions $A^\wedge_\mathfrak{p}$ are regular, see Lemma 15.31.4. Hence the fibre $A^\wedge_\mathfrak{p} \otimes_A K$ is, as a localization of $A^\wedge_\mathfrak{p}$, also regular. Thus we are done if the characteristic of K is 0. The positive characteristic case is the case $A = k[[x_1, \ldots, x_d]]$ which is a special case of Lemma 15.37.5. □

Lemma 15.37.7. *Let R be a Noetherian ring. Then R is a G-ring if and only if $R_\mathfrak{m}$ has geometrically regular formal fibres for every maximal ideal $\mathfrak{m}$ of R.*

Proof. Assume $R_\mathfrak{m} \to R^\wedge_\mathfrak{m}$ is regular for every maximal ideal $\mathfrak{m}$ of R. Let $\mathfrak{p}$ be a prime of R and choose a maximal ideal $\mathfrak{p} \subset \mathfrak{m}$. Since $R_\mathfrak{m} \to R^\wedge_\mathfrak{m}$ is faithfully flat we can choose a prime $\mathfrak{p}'$ if $R^\wedge_\mathfrak{m}$ lying over $\mathfrak{p}R_\mathfrak{m}$. Consider the commutative diagram

$$\begin{array}{ccccc} R^\wedge_\mathfrak{m} & \longrightarrow & (R^\wedge_\mathfrak{m})_{\mathfrak{p}'} & \longrightarrow & (R^\wedge_\mathfrak{m})^\wedge_{\mathfrak{p}'} \\ \uparrow & & \uparrow & & \uparrow \\ R_\mathfrak{m} & \longrightarrow & R_\mathfrak{p} & \longrightarrow & R^\wedge_\mathfrak{p} \end{array}$$

By assumption the ring map $R_\mathfrak{m} \to R^\wedge_\mathfrak{m}$ is regular. By Proposition 15.37.6 $(R^\wedge_\mathfrak{m})_{\mathfrak{p}'} \to (R^\wedge_\mathfrak{m})^\wedge_{\mathfrak{p}'}$ is regular. Hence $R_\mathfrak{m} \to (R^\wedge_\mathfrak{m})^\wedge_{\mathfrak{p}'}$ is regular and since it factors through the localization $R_\mathfrak{p}$, also the ring map $R_\mathfrak{p} \to (R^\wedge_\mathfrak{m})^\wedge_{\mathfrak{p}'}$ is regular. Thus we may apply Lemma 15.29.7 to see that $R_\mathfrak{p} \to R^\wedge_\mathfrak{p}$ is regular. □

Lemma 15.37.8. *Let R be a Noetherian local ring ring which is a G-ring. Then the henselization R^h and the strict henselization R^{sh} are G-rings.*

Proof. We will use the criterion of Lemma 15.37.7. Let $\mathfrak{q} \subset R^h$ be a prime and set $\mathfrak{p} = R \cap \mathfrak{q}$. Set $\mathfrak{q}_1 = \mathfrak{q}$ and let $\mathfrak{q}_2, \ldots, \mathfrak{q}_t$ be the other primes of R^h lying over $\mathfrak{p}$, so that $R^h \otimes_R \kappa(\mathfrak{p}) = \prod_{i=1,\ldots,t} \kappa(\mathfrak{q}_i)$, see Lemma 15.32.11. Using that $(R^h)^\wedge = R^\wedge$ (Lemma 15.32.3) we see

$$\prod\nolimits_{i=1,\ldots,t} (R^h)^\wedge \otimes_{R^h} \kappa(\mathfrak{q}_i) = (R^h)^\wedge \otimes_{R^h} (R^h \otimes_R \kappa(\mathfrak{p})) = R^\wedge \otimes_R \kappa(\mathfrak{p})$$

Hence $(R^h)^\wedge \otimes_{R^h} \kappa(\mathfrak{q}_i)$ is geometrically regular over $\kappa(\mathfrak{p})$ by assumption. Since $\kappa(\mathfrak{q}_i)$ is separable algebraic over $\kappa(\mathfrak{p})$ it follows from Algebra, Lemma 10.152.6 that $(R^h)^\wedge \otimes_{R^h} \kappa(\mathfrak{q}_i)$ is geometrically regular over $\kappa(\mathfrak{q}_i)$.

Let $\mathfrak{r} \subset R^{sh}$ be a prime and set $\mathfrak{p} = R \cap \mathfrak{r}$. Set $\mathfrak{r}_1 = \mathfrak{r}$ and let $\mathfrak{r}_2, \ldots, \mathfrak{r}_s$ be the other primes of R^{sh} lying over $\mathfrak{p}$, so that $R^{sh} \otimes_R \kappa(\mathfrak{p}) = \prod_{i=1,\ldots,t} \kappa(\mathfrak{q}_i)$, see Lemma 15.32.11. Then we see that

$$\prod\nolimits_{i=1,\ldots,t} (R^{sh})^\wedge \otimes_{R^{sh}} \kappa(\mathfrak{r}_i) = (R^{sh})^\wedge \otimes_{R^{sh}} (R^{sh} \otimes_R \kappa(\mathfrak{p})) = (R^{sh})^\wedge \otimes_R \kappa(\mathfrak{p})$$

Note that $R^\wedge \to (R^{sh})^\wedge$ is formally smooth in the $\mathfrak{m}_{(R^{sh})^\wedge}$-adic topology, see Lemma 15.32.3. Hence $R^\wedge \to (R^{sh})^\wedge$ is regular by Proposition 15.36.2. We conclude that $(R^{sh})^\wedge \otimes_{R^h} \kappa(\mathfrak{q}_i)$ is regular over $\kappa(\mathfrak{p})$ by Lemma 15.29.4 as $R^\wedge \otimes_R \kappa(\mathfrak{p})$ is regular over $\kappa(\mathfrak{p})$ by assumption. Since $\kappa(\mathfrak{r}_i)$ is separable algebraic over $\kappa(\mathfrak{p})$ it follows from Algebra, Lemma 10.152.6 that $(R^{sh})^\wedge \otimes_{R^{sh}} \kappa(\mathfrak{r}_i)$ is geometrically regular over $\kappa(\mathfrak{r}_i)$. □

Lemma 15.37.9. *Let p be a prime number. Let A be a Noetherian complete local domain with fraction field K of characteristic p. Let $\mathfrak{q} \subset A[x]$ be a maximal ideal lying over the maximal ideal of A and let $\mathfrak{r} \subset \mathfrak{q}$ be a prime lying over $(0) \subset A$. Then $A[x]^\wedge_\mathfrak{q} \otimes_{A[x]} \kappa(\mathfrak{r})$ is geometrically regular over $\kappa(\mathfrak{r})$.*

Proof. Note that $K \subset \kappa(r)$ is finite. Hence, given a finite purely inseparable extension $\kappa(r) \subset L$ there exists a finite extension of Noetherian complete local domains $A \subset B$ such that $\kappa(r) \otimes_A B$ surjects onto L. Namely, you take $B \subset L$ a

finite A-subalgebra whose field of fractions is L. Denote $\mathfrak{r}' \subset B[x]$ the kernel of the map $B[x] = A[x] \otimes_A B \to \kappa(\mathfrak{r}) \otimes_A B \to L$ so that $\kappa(\mathfrak{r}') = L$. Then

$$A[x]^\wedge_\mathfrak{q} \otimes_{A[x]} L = A[x]^\wedge_\mathfrak{q} \otimes_{A[x]} B[x] \otimes_{B[x]} \kappa(\mathfrak{r}') = \prod B[x]^\wedge_{\mathfrak{q}_i} \otimes_{B[x]} \kappa(\mathfrak{r}')$$

where $\mathfrak{q}_1, \ldots, \mathfrak{q}_t$ are the primes of $B[x]$ lying over $\mathfrak{q}$, see Algebra, Lemma 10.91.19. Thus we see that it suffices to prove the rings $B[x]^\wedge_{\mathfrak{q}_i} \otimes_{B[x]} \kappa(\mathfrak{r}')$ are regular. This reduces us to showing that $A[x]^\wedge_\mathfrak{q} \otimes_{A[x]} \kappa(\mathfrak{r})$ is regular in the special case that $K = \kappa(\mathfrak{r})$.

Assume $K = \kappa(\mathfrak{r})$. In this case we see that $\mathfrak{r}K[x]$ is generated by $x - f$ for some $f \in K$ and

$$A[x]^\wedge_\mathfrak{q} \otimes_{A[x]} \kappa(\mathfrak{r}) = (A[x]^\wedge_\mathfrak{q} \otimes_A K)/(x - f)$$

The derivation $D = \mathrm{d}/\mathrm{d}x$ of $A[x]$ extends to $K[x]$ and maps $x - f$ to a unit of $K[x]$. Moreover D extends to $A[x]^\wedge_\mathfrak{q} \otimes_A K$ by Lemma 15.35.1. As $A \to A[x]^\wedge_\mathfrak{q}$ is formally smooth (see Lemmas 15.26.2 and 15.26.4) the ring $A[x]^\wedge_\mathfrak{q} \otimes_A K$ is regular by Proposition 15.36.2 (the arguments of the proof of that proposition simplify significantly in this particular case). We conclude by Lemma 15.35.2. □

Proposition 15.37.10. *Let R be a G-ring. If $R \to S$ is essentially of finite type then S is a G-ring.*

Proof. Since being a G-ring is a property of the local rings it is clear that a localization of a G-ring is a G-ring. Conversely, if every localization at a prime is a G-ring, then the ring is a G-ring. Thus it suffices to show that $S_\mathfrak{q}$ is a G-ring for every finite type R-algebra S and every prime $\mathfrak{q}$ of S. Writing S as a quotient of $R[x_1, \ldots, x_n]$ we see from Lemma 15.37.3 that it suffices to prove that $R[x_1, \ldots, x_n]$ is a G-ring. By induction on n it suffices to prove that $R[x]$ is a G-ring. Let $\mathfrak{q} \subset R[x]$ be a maximal ideal. By Lemma 15.37.7 it suffices to show that

$$R[x]_\mathfrak{q} \longrightarrow R[x]^\wedge_\mathfrak{q}$$

is regular. If $\mathfrak{q}$ lies over $\mathfrak{p} \subset R$, then we may replace R by $R_\mathfrak{p}$. Hence we may assume that R is a Noetherian local G-ring with maximal ideal $\mathfrak{m}$ and that $\mathfrak{q} \subset R[x]$ lies over $\mathfrak{m}$. Note that there is a unique prime $\mathfrak{q}' \subset R^\wedge[x]$ lying over $\mathfrak{q}$. Consider the diagram

$$\begin{array}{ccc} R[x]^\wedge_\mathfrak{q} & \longrightarrow & (R^\wedge[x]_{\mathfrak{q}'})^\wedge \\ \uparrow & & \uparrow \\ R[x]_\mathfrak{q} & \longrightarrow & R^\wedge[x]_{\mathfrak{q}'} \end{array}$$

Since R is a G-ring the lower horizontal arrow is regular (as a localization of a base change of the regular ring map $R \to R^\wedge$). Suppose we can prove the right vertical arrow is regular. Then it follows that the composition $R[x]_\mathfrak{q} \to (R^\wedge[x]_{\mathfrak{q}'})^\wedge$ is regular, and hence the left vertical arrow is regular by Lemma 15.29.7. Hence we see that we may assume R is a Noetherian complete local ring and $\mathfrak{q}$ a prime lying over the maximal ideal of R.

Let R be a Noetherian complete local ring and let $\mathfrak{q} \subset R[x]$ be a maximal ideal lying over the maximal ideal of R. Let $\mathfrak{r} \subset \mathfrak{q}$ be a prime ideal. We want to show that $R[x]^\wedge_\mathfrak{q} \otimes_{R[x]} \kappa(\mathfrak{r})$ is a geometrically regular algebra over $\kappa(\mathfrak{r})$. Set $\mathfrak{p} = R \cap \mathfrak{r}$. Then we can replace R by $R/\mathfrak{p}$ and $\mathfrak{q}$ and $\mathfrak{r}$ by their images in $R/\mathfrak{p}[x]$, see Lemma 15.37.2. Hence we may assume that R is a domain and that $\mathfrak{r} \cap R = (0)$.

By Algebra, Lemma 10.147.10 we can find $R_0 \subset R$ which is regular and such that R is finite over R_0. Applying Lemma 15.37.3 we see that it suffices to prove $R[x]^\wedge_{\mathfrak{q}} \otimes_{R[x]} \kappa(\mathfrak{r})$ is geometrically regular over $\kappa(r)$ when, in addition to the above, R is a regular complete local ring.

Now R is a regular complete local ring, we have $\mathfrak{q} \subset \mathfrak{r} \subset R[x]$, we have $(0) = R \cap \mathfrak{r}$ and $\mathfrak{q}$ is a maximal ideal lying over the maximal ideal of R. Since R is regular the ring $R[x]$ is regular (Algebra, Lemma 10.149.8). Hence the localization $R[x]_{\mathfrak{q}}$ is regular. Hence the completions $R[x]^\wedge_{\mathfrak{q}}$ are regular, see Lemma 15.31.4. Hence the fibre $R[x]^\wedge_{\mathfrak{q}} \otimes_{R[x]} \kappa(\mathfrak{r})$ is, as a localization of $R[x]^\wedge_{\mathfrak{q}}$, also regular. Thus we are done if the characteristic of $f.f.(R)$ is 0.

If the characteristic of R is positive, then $R = k[[x_1, \ldots, x_n]]$. In this case we split the argument in two subcases:

(1) The case $\mathfrak{r} = (0)$. The result is a direct consequence of Lemma 15.37.5.
(2) The case $\mathfrak{r} \neq (0)$. This is Lemma 15.37.9.

□

Remark 15.37.11. Let R be a G-ring and let $I \subset R$ be an ideal. In general it is not the case that the I-adic completion $R^\wedge$ is a G-ring. An example was given by Nishimura in [**Nis81**]. A generalization and, in some sense, clarification of this example can be found in the last section of [**Dum00**].

Proposition 15.37.12. *The following types of rings are G-rings:*

(1) *fields,*
(2) *Noetherian complete local rings,*
(3) $\mathbf{Z}$*,*
(4) *Dedekind domains with fraction field of characteristic zero,*
(5) *finite type ring extensions of any of the above.*

Proof. For fields, $\mathbf{Z}$ and Dedekind domains of characteristic zero this follows immediately from the definition and the fact that the completion of a discrete valuation ring is a discrete valuation ring. A Noetherian complete local ring is a G-ring by Proposition 15.37.6. The statement on finite type overrings is Proposition 15.37.10. □

15.38. Excellent rings

In this section we discuss Grothendieck's notion of excellent rings. For the definitions of G-rings, J-2 rings, and universally catenary rings we refer to Definition 15.37.1, Definition 15.34.1, and Algebra, Definition 10.100.5.

Definition 15.38.1. Let R be a ring.

(1) We say R is *quasi-excellent* if R is Noetherian, a G-ring, and J-2.
(2) We say R is *excellent* if R is quasi-excellent and universally catenary.

Thus a Noetherian ring is quasi-excellent if it has geometrically regular formal fibres and if any finite type algebra over it has closed singular set. For such a ring to be excellent we require in addition that there exists (locally) a good dimension function.

Lemma 15.38.2. *Any localization of a finite type ring over a (quasi-)excellent ring is (quasi-)excellent.*

Proof. For finite type algebras this follows from the definitions for the properties J-2 and universally catenary. For G-rings, see Proposition 15.37.10. We omit the proof that localization preserves (quasi-)excellency. □

Lemma 15.38.3. *A quasi-excellent ring is Nagata.*

Proof. Let R be quasi-excellent. Using that a finite type algebra over R is quasi-excellent (Lemma 15.38.2) we see that it suffices to show that any quasi-excellent domain is N-1, see Algebra, Lemma 10.148.17. Applying Algebra, Lemma 10.148.29 (and using that a quasi-excellent ring is J-2) we reduce to showing that a quasi-excellent local domain R is N-1. As $R \to R^\wedge$ is regular we see that $R^\wedge$ is reduced by Lemma 15.30.1. In other words, R is analytically unramified. Hence R is N-1 by Algebra, Lemma 10.148.24. □

Proposition 15.38.4. *The following types of rings are excellent:*

(1) *fields,*
(2) *Noetherian complete local rings,*
(3) $\mathbf{Z}$*,*
(4) *Dedekind domains with fraction field of characteristic zero,*
(5) *finite type ring extensions of any of the above.*

Proof. See Propositions 15.37.12 and 15.35.6 to see that these rings are G-rings and have J-2. Any Cohen-Macaulay ring is universally catenary (in particular fields, Dedekind rings, and more generally regular rings are universally catenary). Via the Cohen structure theorem we see that complete local rings are universally catenary, see Algebra, Remark 10.147.9. □

15.39. Injective abelian groups

In this section we show the category of abelian groups has enough injectives. Recall that an abelian group M is *divisible* if and only if for every $x \in M$ and every $n \in \mathbf{N}$ there exists a $y \in M$ such that $ny = x$.

Lemma 15.39.1. *An abelian group J is an injective object in the category of abelian groups if and only if J is divisible.*

Proof. Suppose that J is not divisible. Then there exists an $x \in J$ and $n \in \mathbf{N}$ such that there is no $y \in J$ with $ny = x$. Then the morphism $\mathbf{Z} \to J$, $m \mapsto mx$ does not extend to $\frac{1}{n}\mathbf{Z} \supset \mathbf{Z}$. Hence J is not injective.

Let $A \subset B$ be abelian groups. Assume that J is a divisible abelian group. Let $\varphi : A \to J$ be a morphism. Consider the set of homomorphisms $\varphi' : A' \to J$ with $A \subset A' \subset B$ and $\varphi'|_A = \varphi$. Define $(A', \varphi') \geq (A'', \varphi'')$ if and only if $A' \supset A''$ and $\varphi'|_{A''} = \varphi''$. If $(A_i, \varphi_i)_{i \in I}$ is a totally ordered collection of such pairs, then we obtain a map $\bigcup_{i \in I} A_i \to J$ defined by $a \in A_i$ maps to $\varphi_i(a)$. Thus Zorn's lemma applies. To conclude we have to show that if the pair (A', φ') is maximal then $A' = B$. In other words, it suffices to show, given any subgroup $A \subset B$, $A \neq B$ and any $\varphi : A \to J$, then we can find $\varphi' : A' \to J$ with $A \subset A' \subset B$ such that (a) the inclusion $A \subset A'$ is strict, and (b) the morphism φ' extends φ.

To prove this, pick $x \in B$, $x \notin A$. If there exists no $n \in \mathbf{N}$ such that $nx \in A$, then $A \oplus \mathbf{Z} \cong A + \mathbf{Z}x$. Hence we can extend φ to $A' = A + \mathbf{Z}x$ by using φ on A and mapping x to zero for example. If there does exist an $n \in \mathbf{N}$ such that $nx \in A$, then

let n be the minimal such integer. Let $z \in J$ be an element such that $nz = \varphi(nx)$. Define a morphism $\tilde{\varphi} : A \oplus \mathbf{Z} \to J$ by $(a, m) \mapsto \varphi(a) + mz$. By our choice of z the kernel of $\tilde{\varphi}$ contains the kernel of the map $A \oplus \mathbf{Z} \to B$, $(a, m) \mapsto a + mx$. Hence $\tilde{\varphi}$ factors through the image $A' = A + \mathbf{Z}x$, and this extends the morphism φ. □

We can use this lemma to show that every abelian group can be embedded in a injective abelian group. But this is a special case of the result of the following section.

15.40. Injectives in the category of modules

In this section we prove that there are enough injective modules over a ring R. We start with the fact that $\mathbf{Q}/\mathbf{Z}$ is an injective abelian group. This follows from Lemma 15.39.1.

Definition 15.40.1. Let R be a ring.

(1) For any R-module M over R we denote $M^\vee = \mathrm{Hom}(M, \mathbf{Q}/\mathbf{Z})$ with its natural R-module structure. We think of $M \mapsto M^\vee$ as a contravariant functor from the category of R-modules to itself.
(2) For any R-module M we denote

$$F(M) = \bigoplus\nolimits_{m \in M} R[m]$$

the *free module* with basis given by the elements $[m]$ with $m \in M$. We let $F(M) \to M$, $\sum f_i[m_i] \mapsto \sum f_i m_i$ be the natural surjection of R-modules. We think of $M \mapsto (F(M) \to M)$ as a functor from the category of R-modules to the category of arrows in R-modules.

Lemma 15.40.2. *Let R be a ring. The functor $M \mapsto M^\vee$ is exact.*

Proof. This because $\mathbf{Q}/\mathbf{Z}$ is an injective abelian group by Lemma 15.39.1. □

There is a canonical map $ev : M \to (M^\vee)^\vee$ given by evaluation: given $x \in M$ we let $ev(x) \in (M^\vee)^\vee = \mathrm{Hom}(M^\vee, \mathbf{Q}/\mathbf{Z})$ be the map $\varphi \mapsto \varphi(x)$.

Lemma 15.40.3. *For any R-module M the evaluation map $ev : M \to (M^\vee)^\vee$ is injective.*

Proof. You can check this using that $\mathbf{Q}/\mathbf{Z}$ is an injective abelian group. Namely, if $x \in M$ is not zero, then let $M' \subset M$ be the cyclic group it generates. There exists a nonzero map $M' \to \mathbf{Q}/\mathbf{Z}$ which necessarily does not annihilate x. This extends to a map $\varphi : M \to \mathbf{Q}/\mathbf{Z}$ And then $ev(x)(\varphi) = \varphi(x) \neq 0$. □

The canonical surjection $F(M) \to M$ of R-modules turns into a a canonical injection, see above, of R-modules

$$(M^\vee)^\vee \longrightarrow (F(M^\vee))^\vee.$$

Set $J(M) = (F(M^\vee))^\vee$. The composition of ev with this the displayed map gives $M \to J(M)$ functorially in M.

Lemma 15.40.4. *Let R be a ring. For every R-module M the R-module $J(M)$ is injective.*

Proof. Note that $J(M) \cong \prod_{\varphi \in M^\vee} R^\vee$ as an R-module. As the product of injective modules is injective, it suffices to show that $R^\vee$ is injective. For this we use that

$$\mathrm{Hom}_R(N, R^\vee) = \mathrm{Hom}_R(N, \mathrm{Hom}_{\mathbf{Z}}(R, \mathbf{Q}/\mathbf{Z})) = N^\vee$$

and the fact that $(-)^\vee$ is an exact functor by Lemma 15.40.2. □

Lemma 15.40.5. *Let R be a ring. The construction above defines a covariant functor $M \mapsto (M \to J(M))$ from the category of R-modules to the category of arrows of R-modules such that for every module M the output $M \to J(M)$ is an injective map of M into an injective R-module $J(M)$.*

Proof. Follows from the above. □

In particular, for any map of R-modules $M \to N$ there is an associated morphism $J(M) \to J(N)$ making the following diagram commute:

$$\begin{array}{ccc} M & \longrightarrow & N \\ \downarrow & & \downarrow \\ J(M) & \longrightarrow & J(N) \end{array}$$

This is the kind of construction we would like to have in general. In Homology, Section 12.23 we introduced terminology to express this. Namely, we say this means that the category of R-modules has functorial injective embeddings.

15.41. Derived categories of modules

In this section we put some generalities concerning the derived category of modules over a ring.

Let A be a ring. The category of A-modules has products and products are exact. The category of A-modules has enough injectives by Lemma 15.40.5. Hence every complex of A-modules is quasi-isomorphic to a K-injective complex and $D(A)$ has products (Derived Categories, Lemmas 13.30.3 and 13.28.9). This implies that every inverse system of objects of $D(A)$ has a derived limit (well defined up to isomorphism), see Derived Categories, Section 13.30.

Lemma 15.41.1. *Let $R \to S$ be a flat ring map. If $I^\bullet$ is a K-injective complex of S-modules, then $I^\bullet$ is K-injective as a complex of R-modules.*

Proof. This is true because $\mathrm{Hom}_{K(R)}(M^\bullet, I^\bullet) = \mathrm{Hom}_{K(S)}(M^\bullet \otimes_R S, I^\bullet)$ by Algebra, Lemma 10.13.3 and the fact that tensoring with S is exact. □

Lemma 15.41.2. *Let $R \to S$ be an epimorphism of rings. Let $I^\bullet$ be a complex of S-modules. If $I^\bullet$ is K-injective as a complex of R-modules, then $I^\bullet$ is a K-injective complex of S-modules.*

Proof. This is true because $\mathrm{Hom}_{K(R)}(N^\bullet, I^\bullet) = \mathrm{Hom}_{K(S)}(N^\bullet, I^\bullet)$ for any complex of S-modules $N^\bullet$, see Algebra, Lemma 10.102.14. □

Lemma 15.41.3. *Let $A \to B$ be a ring map. If $I^\bullet$ is a K-injective complex of A-modules, then $\mathrm{Hom}_A(B, I^\bullet)$ is a K-injective complex of B-modules.*

Proof. This is true because $\mathrm{Hom}_{K(B)}(N^\bullet, \mathrm{Hom}_A(B, I^\bullet)) = \mathrm{Hom}_{K(A)}(N^\bullet, I^\bullet)$ by Algebra, Lemma 10.13.4. □

15.42. Computing Tor

Let R be a ring. We denote $D(R)$ the derived category of the abelian category Mod_R of R-modules. Note that Mod_R has enough projectives as every free R-module is projective. Thus we can define the left derived functors of any additive functor from Mod_R to any abelian category.

This implies in particular to the functor $- \otimes_R M : \mathrm{Mod}_R \to \mathrm{Mod}_R$ whose right derived functors are the Tor functors $\mathrm{Tor}_i^R(-, M)$, see Algebra, Section 10.70. There is also a total right derived functor

$$- \otimes_R^{\mathbf{L}} M : D^-(R) \longrightarrow D^-(R) \tag{15.42.0.1}$$

which is denoted $- \otimes_R^{\mathbf{L}} M$. Its satellites are the Tor modules, i.e., we have

$$H^{-p}(N \otimes_R^{\mathbf{L}} M) = \mathrm{Tor}_p^R(N, M).$$

A special situation occurs when we consider the tensor product with an R-algebra A. In this case we think of $- \otimes_R A$ as a functor from Mod_R to Mod_A. Hence the total right derived functor

$$- \otimes_R^{\mathbf{L}} A : D^-(R) \longrightarrow D^-(A) \tag{15.42.0.2}$$

which is denoted $- \otimes_R^{\mathbf{L}} A$. Its satellites are the tor groups, i.e., we have

$$H^{-p}(N \otimes_R^{\mathbf{L}} A) = \mathrm{Tor}_p^R(N, A).$$

In particular these Tor groups naturally have the structure of A-modules.

15.43. Derived tensor product

We can construct the derived tensor product in greater generality. In fact, it turns out that the boundedness assumptions are not necessary, provided we choose K-flat resolutions.

Lemma 15.43.1. *Let R be a ring. Let $P^\bullet$ be a complex of R-modules. Let $\alpha, \beta : L^\bullet \to M^\bullet$ be homotopy equivalent maps of complexes. Then α and β induce homotopy equivalent maps*

$$Tot(\alpha \otimes id_P), Tot(\beta \otimes id_P) : Tot(L^\bullet \otimes_R P^\bullet) \longrightarrow Tot(M^\bullet \otimes_R P^\bullet).$$

In particular the construction $L^\bullet \mapsto Tot(L^\bullet \otimes_R P^\bullet)$ defines an endo-functor of the homotopy category of complexes.

Proof. Say $\alpha = \beta + dh + hd$ for some homotopy h defined by $h^n : L^n \to M^{n-1}$. Set

$$H^n = \bigoplus\nolimits_{a+b=n} h^a \otimes \mathrm{id}_{P^b} : \bigoplus\nolimits_{a+b=n} L^a \otimes_R P^b \longrightarrow \bigoplus\nolimits_{a+b=n} M^{a-1} \otimes_R P^b$$

Then a straightforward computation shows that

$$\mathrm{Tot}(\alpha \otimes \mathrm{id}_P) = \mathrm{Tot}(\beta \otimes \mathrm{id}_P) + dH + Hd$$

as maps $\mathrm{Tot}(L^\bullet \otimes_R P^\bullet) \to \mathrm{Tot}(M^\bullet \otimes_R P^\bullet)$. □

Lemma 15.43.2. *Let R be a ring. Let $P^\bullet$ be a complex of R-modules. The functor*

$$K(Mod_R) \longrightarrow K(Mod_R), \quad L^\bullet \longmapsto Tot(L^\bullet \otimes_R P^\bullet)$$

is an exact functor of triangulated categories.

Proof. By our definition of the triangulated structure on $K(\text{Mod}_R)$ we have to check that our functor maps a termwise split short exact sequence of complexes to a termwise split short exact sequence of complexes. As the terms of $\text{Tot}(L^\bullet \otimes_R P^\bullet)$ are direct sums of the tensor products $L^a \otimes_R P^b$ this is clear. □

The following definition will allow us to think intelligently about derived tensor products of unbounded complexes.

Definition 15.43.3. Let R be a ring. A complex $K^\bullet$ is called *K-flat* if for every acyclic complex $M^\bullet$ the total complex $\text{Tot}(M^\bullet \otimes_R K^\bullet)$ is acyclic.

Lemma 15.43.4. *Let R be a ring. Let $K^\bullet$ be a K-flat complex. Then the functor*

$$K(Mod_R) \longrightarrow K(Mod_R), \quad L^\bullet \longmapsto Tot(L^\bullet \otimes_R K^\bullet)$$

transforms quasi-isomorphisms into quasi-isomorphisms.

Proof. Follows from Lemma 15.43.2 and the fact that quasi-isomorphisms in $K(\text{Mod}_R)$ and $K(\text{Mod}_A)$ are characterized by having acyclic cones. □

Lemma 15.43.5. *Let $R \to R'$ be a ring map. If $K^\bullet$ is a K-flat complex of R-modules, then $K^\bullet \otimes_R R'$ is a K-flat complex of R'-modules.*

Proof. Follows from the definitions and the fact that $(K^\bullet \otimes_R R') \otimes_{R'} L^\bullet = K^\bullet \otimes_R L^\bullet$ for any complex $L^\bullet$ of R'-modules. □

Lemma 15.43.6. *Let R be a ring. If $K^\bullet$, $L^\bullet$ are K-flat complexes of R-modules, then $Tot(K^\bullet \otimes_R L^\bullet)$ is a K-flat complex of R-modules.*

Proof. Follows from the isomorphism

$$\text{Tot}(M^\bullet \otimes_R \text{Tot}(K^\bullet \otimes_R L^\bullet)) = \text{Tot}(\text{Tot}(M^\bullet \otimes_R K^\bullet) \otimes_R L^\bullet)$$

and the definition. □

Lemma 15.43.7. *Let R be a ring. Let $(K_1^\bullet, K_2^\bullet, K_3^\bullet)$ be a distinguished triangle in $K(Mod_R)$. If two out of three of $K_i^\bullet$ are K-flat, so is the third.*

Proof. Follows from Lemma 15.43.2 and the fact that in a distinguished triangle in $K(\text{Mod}_A)$ if two out of three are acyclic, so is the third. □

Lemma 15.43.8. *Let R be a ring. Let $P^\bullet$ be a bounded above complex of flat R-modules. Then $P^\bullet$ is K-flat.*

Proof. Let $L^\bullet$ be an acyclic complex of R-modules. Let $\xi \in H^n(\text{Tot}(L^\bullet \otimes_R P^\bullet))$. We have to show that $\xi = 0$. Since $\text{Tot}^n(L^\bullet \otimes_R P^\bullet)$ is a direct sum with terms $L^a \otimes_R P^b$ we see that ξ comes from an element in $H^n(\text{Tot}(\tau_{\leq m} L^\bullet \otimes_R P^\bullet))$ for some $m \in \mathbf{Z}$. Since $\tau_{\leq m} L^\bullet$ is also acyclic we may replace $L^\bullet$ by $\tau_{\leq m} L^\bullet$. Hence we may assume that $L^\bullet$ is bounded above. In this case the spectral sequence of Homology, Lemma 12.22.5 has

$${}'E_1^{p,q} = H^p(L^\bullet \otimes_R P^q)$$

which is zero as P^q is flat and $L^\bullet$ acyclic. Hence $H^*(\text{Tot}(L^\bullet \otimes_R P^\bullet)) = 0$. □

In the following lemma by a colimit of a system of complexes we mean the termwise colimit.

Lemma 15.43.9. *Let R be a ring. Let $K_1^\bullet \to K_2^\bullet \to \ldots$ be a system of K-flat complexes. Then $\text{colim}_i K_i^\bullet$ is K-flat.*

Proof. Because we are taking termwise colimits it is clear that

$$\operatorname{colim}_i \operatorname{Tot}(M^\bullet \otimes_R K_i^\bullet) = \operatorname{Tot}(M^\bullet \otimes_R \operatorname{colim}_i K_i^\bullet)$$

Hence the lemma follows from the fact that filtered colimits are exact. □

Lemma 15.43.10. *Let R be a ring. For any complex $M^\bullet$ there exists a K-flat complex $K^\bullet$ and a quasi-isomorphism $K^\bullet \to M^\bullet$. Moreover each K^n is a flat R-module.*

Proof. Let $\mathcal{P} \subset \operatorname{Ob}(\operatorname{Mod}_R)$ be the class of flat R-modules. By Derived Categories, Lemma 13.27.1 there exists a system $K_1^\bullet \to K_2^\bullet \to \ldots$ and a diagram

$$\begin{array}{ccccc} K_1^\bullet & \longrightarrow & K_2^\bullet & \longrightarrow & \ldots \\ \downarrow & & \downarrow & & \\ \tau_{\leq 1} M^\bullet & \longrightarrow & \tau_{\leq 2} M^\bullet & \longrightarrow & \ldots \end{array}$$

with the properties (1), (2), (3) listed in that lemma. These properties imply each complex $K_i^\bullet$ is a bounded above complex of flat modules. Hence $K_i^\bullet$ is K-flat by Lemma 15.43.8. The induced map $\operatorname{colim}_i K_i^\bullet \to M^\bullet$ is a quasi-isomorphism by construction. The complex $\operatorname{colim}_i K_i^\bullet$ is K-flat by Lemma 15.43.9. The final assertion of the lemma is true because the colimit of a system of flat modules is flat, see Algebra, Lemma 10.36.2. □

Remark 15.43.11. In fact, we can do better than Lemma 15.43.10. Namely, we can find a quasi-isomorphism $P^\bullet \to M^\bullet$ where $P^\bullet$ is a complex of A-modules endowed with a filtration

$$0 = F_{-1}P^\bullet \subset F_0 P^\bullet \subset F_1 P^\bullet \subset \ldots \subset P^\bullet$$

by differential graded submodules such that

(1) $P^\bullet = \bigcup F_p P^\bullet$,
(2) the inclusions $F_i P^\bullet \to F_{i+1} P^\bullet$ are termwise split injections,
(3) the quotients $F_{i+1}P^\bullet / F_i P^\bullet$ are isomorphic to direct sums of shifts $A[k]$ (as complexes, so differentials are zero).

This was shown in Differential Graded Algebra, Lemma 22.13.4. Moreover, given such a complex we obtain a distinguished triangle

$$\bigoplus F_i P^\bullet \to \bigoplus F_i P^\bullet \to M^\bullet \to \bigoplus F_i P^\bullet[1]$$

in $D(A)$. Using this we can sometimes reduce statements about general complexes to statements about $A[k]$ (this of course only works if the statement is preserved under taking direct sums). More precisely, let T be a property of objects of $D(A)$. Suppose that

(1) if $K_i \in D(A)$, $i \in I$ is a family of objects with $T(K_i)$ for all $i \in I$, then $T(\bigoplus K_i)$,
(2) if $K \to L \to M \to K[1]$ is a distinguished triangle and T holds for two, then T holds for the third object,
(3) $T(A[k])$ holds for all k.

Then T holds for all objects of $D(A)$.

Lemma 15.43.12. *Let R be a ring. Let $\alpha : P^\bullet \to Q^\bullet$ be a quasi-isomorphism of K-flat complexes of R-modules. For every complex $L^\bullet$ of R-modules the induced map*

$$Tot(id_L \otimes \alpha) : Tot(L^\bullet \otimes_R P^\bullet) \longrightarrow Tot(L^\bullet \otimes_R Q^\bullet)$$

is a quasi-isomorphism.

Proof. Choose a quasi-isomorphism $K^\bullet \to L^\bullet$ with $K^\bullet$ a K-flat complex, see Lemma 15.43.10. Consider the commutative diagram

$$\begin{array}{ccc} \mathrm{Tot}(K^\bullet \otimes_R P^\bullet) & \longrightarrow & \mathrm{Tot}(K^\bullet \otimes_R Q^\bullet) \\ \downarrow & & \downarrow \\ \mathrm{Tot}(L^\bullet \otimes_R P^\bullet) & \longrightarrow & \mathrm{Tot}(L^\bullet \otimes_R Q^\bullet) \end{array}$$

The result follows as by Lemma 15.43.4 the vertical arrows and the top horizontal arrow are quasi-isomorphisms. □

Let R be a ring. Let $M^\bullet$ be an object of $D(R)$. Choose a K-flat resolution $K^\bullet \to M^\bullet$, see Lemma 15.43.10. By Lemmas 15.43.1 and 15.43.2 we obtain an exact functor of triangulated categories

$$K(\mathrm{Mod}_R) \longrightarrow K(\mathrm{Mod}_R), \quad L^\bullet \longmapsto \mathrm{Tot}(L^\bullet \otimes_R K^\bullet)$$

By Lemma 15.43.4 this functor induces a functor $D(R) \to D(R)$ simply because $D(R)$ is the localization of $K(\mathrm{Mod}_R)$ at quasi-isomorphism. By Lemma 15.43.12 the resulting functor (up to isomorphism) does not depend on the choice of the K-flat resolution.

Definition 15.43.13. Let R be a ring. Let $M^\bullet$ be an object of $D(R)$. The *derived tensor product*

$$- \otimes^{\mathbf{L}}_R M^\bullet : D(R) \longrightarrow D(R)$$

is the exact functor of triangulated categories described above.

This functor extends the functor (15.42.0.1). It is clear from our explicit constructions that there is a canonical isomorphism

$$M^\bullet \otimes^{\mathbf{L}}_R L^\bullet \cong L^\bullet \otimes^{\mathbf{L}}_R M^\bullet$$

whenever both $L^\bullet$ and $M^\bullet$ are in $D(R)$. Hence when we write $M^\bullet \otimes^{\mathbf{L}}_R L^\bullet$ we will usually be agnostic about which variable we are using to define the derived tensor product with.

15.44. Derived change of rings

Let $R \to A$ be a ring map. Let N be an A-module. We can also use K-flat resolutions to define a functor

$$- \otimes^{\mathbf{L}}_R N : D(R) \to D(A)$$

which is the left derived functor of the functor $\mathrm{Mod}_R \to \mathrm{Mod}_A$, $M \mapsto M \otimes_R N$. In particular, taking $N = A$ we obtain a derived base change functor

$$- \otimes^{\mathbf{L}}_R A : D(R) \to D(A)$$

extending the functor (15.42.0.2). Namely, for every complex of R-modules $M^\bullet$ we can choose a K-flat resolution $K^\bullet \to M^\bullet$ and set $M^\bullet \otimes^{\mathbf{L}}_R N = K^\bullet \otimes_R N$. You can use Lemmas 15.43.10 and 15.43.12 to see that this is well defined. However, to

cross all the t's and dot all the i's it is perhaps more convenient to use some general theory.

Lemma 15.44.1. *The construction above is independent of choices and defines an exact functor of triangulated categories $- \otimes^{\mathbf{L}}_R N : D(R) \to D(A)$. There is a functorial isomorphism*

$$E \otimes^{\mathbf{L}}_R N = (E \otimes^{\mathbf{L}}_R A) \otimes^{\mathbf{L}}_A N$$

for E in $D(R)$.

Proof. To prove the existence of the derived functor $- \otimes^{\mathbf{L}}_R N$ we use the general theory developed in Derived Categories, Section 13.14. Set $\mathcal{D} = K(\mathrm{Mod}_R)$ and $\mathcal{D}' = D(A)$. Let us write $F : \mathcal{D} \to \mathcal{D}'$ the exact functor of triangulated categories defined by the rule $F(M^\bullet) = M^\bullet \otimes_R N$. We let S be the set of quasi-isomorphisms in $\mathcal{D} = K(\mathrm{Mod}_R)$. This gives a situation as in Derived Categories, Situation 13.14.1 so that Derived Categories, Definition 13.14.2 applies. We claim that LF is everywhere defined. This follows from Derived Categories, Lemma 13.14.15 with $\mathcal{P} \subset \mathrm{Ob}(\mathcal{D})$ the collection of K-flat complexes: (1) follows from Lemma 15.43.10 and (2) follows from Lemma 15.43.12. Thus we obtain a derived functor

$$LF : D(R) = S^{-1}\mathcal{D} \longrightarrow \mathcal{D}' = D(A)$$

see Derived Categories, Equation (13.14.9.1). Finally, Derived Categories, Lemma 13.14.15 guarantees that $LF(K^\bullet) = F(K^\bullet) = K^\bullet \otimes_R N$ when $K^\bullet$ is K-flat, i.e., LF is indeed computed in the way described above. Moreover, by Lemma 15.43.5 the complex $K^\bullet \otimes_R A$ is a K-flat complex of A-modules. Hence

$$(K^\bullet \otimes^{\mathbf{L}}_R A) \otimes^{\mathbf{L}}_A N = (K^\bullet \otimes_R A) \otimes_A N = K^\bullet \otimes_A N = K^\bullet \otimes^{\mathbf{L}}_A N$$

which proves the final statement of the lemma. □

Remark 15.44.2 (Warning)**.** Let $R \to A$ be a ring map, and let N and N' be A-modules. Denote N_R and N'_R the restriction of N and N' to R-modules, see Algebra, Section 10.13. In this situation, the objects $N_R \otimes^{\mathbf{L}}_R N'$ and $N \otimes^{\mathbf{L}}_R N'_R$ of $D(A)$ are in general not isomorphic! In other words, one has to pay careful attention as to which of the two sides is being used to provide the A-module structure.

For a specific example, set $R = k[x, y]$, $A = R/(xy)$, $N = R/(x)$ and $N' = A = R/(xy)$. The resolution $0 \to R \xrightarrow{xy} R \to N'_R \to 0$ shows that $N \otimes^{\mathbf{L}}_R N'_R = N[1] \oplus N$ in $D(A)$. The resolution $0 \to R \xrightarrow{x} R \to N_R \to 0$ shows that $N_R \otimes^{\mathbf{L}}_R N'$ is represented by the complex $A \xrightarrow{x} A$. To see these two complexes are not isomorphic, one can show that the second complex is not isomorphic in $D(A)$ to the direct sum of its cohomology groups, or one can show that the first complex is not a perfect object of $D(A)$ whereas the second one is. Some details omitted.

Lemma 15.44.3. *Let $A \to B \to C$ be ring maps. Let M be an A-module, N a B-module, and K a C-module. Then*

$$(M \otimes^{\mathbf{L}}_A N) \otimes^{\mathbf{L}}_B K = (M \otimes^{\mathbf{L}}_A K) \otimes^{\mathbf{L}}_C (N \otimes^{\mathbf{L}}_B C) = (M \otimes^{\mathbf{L}}_A C) \otimes^{\mathbf{L}}_C (N \otimes^{\mathbf{L}}_B K)$$

in $D(C)$.

Proof. Let $M^\bullet \to M$ be a free resolution of M as an A-module and let $N^\bullet$ be a free resolution of N as a B-module. We have

$$\begin{aligned} M \otimes^{\mathbf{L}}_A N &= M^\bullet \otimes_A N \\ &= M^\bullet \otimes_A B \otimes_B N \\ &\leftarrow \mathrm{Tot}((M^\bullet \otimes_A B) \otimes_B N^\bullet) \\ &= \mathrm{Tot}(M^\bullet \otimes_A N^\bullet) \end{aligned}$$

Here the arrow is a quasi-isomorphism in $D(B)$ as $M^\bullet \otimes_A B$ is a bounded above complex of free B-modules, hence K-flat (Lemma 15.43.8) and hence Lemma 15.43.4 applies. Now the complex $\mathrm{Tot}(M^\bullet \otimes_A N^\bullet)$ is a complex of free B-modules hence we see that

$$(M \otimes^{\mathbf{L}}_A N) \otimes^{\mathbf{L}}_B K = \mathrm{Tot}(M^\bullet \otimes_A N^\bullet) \otimes_B K = \mathrm{Tot}(M^\bullet \otimes_A N^\bullet \otimes_B K)$$

On the other hand,

$$M \otimes^{\mathbf{L}}_A K = M^\bullet \otimes_A K \quad \text{and} \quad N \otimes^{\mathbf{L}}_B C = N^\bullet \otimes_B C$$

and the second is a bounded above complex of free C-modules hence we see that

$$(M \otimes^{\mathbf{L}}_A K) \otimes^{\mathbf{L}}_C (N \otimes^{\mathbf{L}}_B C) = \mathrm{Tot}((M^\bullet \otimes_A K) \otimes_C (N^\bullet \otimes_B C)) = \mathrm{Tot}(M^\bullet \otimes_A N^\bullet \otimes_B K)$$

which proves the first equality of the statement of the lemma. To prove the second we use that

$$M \otimes^{\mathbf{L}}_A C = M^\bullet \otimes_A C \quad \text{and} \quad N \otimes^{\mathbf{L}}_B K = N^\bullet \otimes_B K$$

and the first is a bounded above complex of free C-modules so that

$$(M \otimes^{\mathbf{L}}_A C) \otimes^{\mathbf{L}}_C (N \otimes^{\mathbf{L}}_B K) = \mathrm{Tot}((M^\bullet \otimes_A C) \otimes_C (N^\bullet \otimes_B K)) = \mathrm{Tot}(M^\bullet \otimes_A N^\bullet \otimes_B K)$$

as before. □

15.45. Tor independence

Consider a commutative diagram

$$\begin{array}{ccc} A & \longrightarrow & A' \\ \uparrow & & \uparrow \\ R & \longrightarrow & R' \end{array}$$

of rings. Given an object K of $D(A)$ we can consider its restriction to an object of $D(R)$. We can then consider take the derived change of rings of K to an object of $D(A')$ and $D(R')$. We claim there is a functorial comparison map

$$K \otimes^{\mathbf{L}}_R R' \longrightarrow K^\bullet \otimes^{\mathbf{L}}_A A' \tag{15.45.0.1}$$

in $D(R')$. To construct this comparison map choose a K-flat complex $K^\bullet$ of A-modules representing K. Next, choose a quasi-isomorphism $E^\bullet \to K^\bullet$ where $E^\bullet$ is a K-flat complex of R-modules. The map above is the map

$$K \otimes^{\mathbf{L}}_R R' = E^\bullet \otimes_R R' \longrightarrow K^\bullet \otimes_A A' = K \otimes^{\mathbf{L}}_A A'$$

In general there is no chance that this map is an isomorphism.

However, we often encounter the situation where the diagram above is a "base change" diagram of rings, i.e., $A' = A \otimes_R R'$. In this situation, for any A-module M we have $M \otimes_A A' = M \otimes_R R'$. Thus $- \otimes_R R'$ is equal to $- \otimes_A A'$ as a functor

$\mathrm{Mod}_A \to \mathrm{Mod}_{A'}$. In general this equality **does not extend** to derived tensor products. In other words, the comparison map is not an isomorphism. A simple example is to take $R = k[x]$, $A = R' = A' = k[x]/(x) = k$ and $K^\bullet = A[0]$. Clearly, a necessary condition is that $\mathrm{Tor}_p^R(A, R') = 0$ for all $p > 0$.

Definition 15.45.1. Let R be a ring. Let A, B be R-algebras. We say A and B are *Tor independent over R* if $\mathrm{Tor}_p^R(A, B) = 0$ for all $p > 0$.

Lemma 15.45.2. *The comparison map (15.45.0.1) is an isomorphism if $A' = A \otimes_R R'$ and A and R' are Tor independent over R.*

Proof. To prove this we choose a free resolution $F^\bullet \to R'$ of R' as an R-module. Because A and R' are Tor independent over R we see that $F^\bullet \otimes_R A$ is a free A-module resolution of A' over A. By our general construction of the derived tensor product above we see that

$$K^\bullet \otimes_A A' \cong \mathrm{Tot}(K^\bullet \otimes_A (F^\bullet \otimes_R A)) = \mathrm{Tot}(K^\bullet \otimes_R F^\bullet) \cong \mathrm{Tot}(E^\bullet \otimes_R F^\bullet) \cong E^\bullet \otimes_R R'$$

as desired. □

Lemma 15.45.3. *Consider a commutative diagram of rings*

$$\begin{array}{ccccc} A' & \longleftarrow & R' & \longrightarrow & B' \\ \uparrow & & \uparrow & & \uparrow \\ A & \longleftarrow & R & \longrightarrow & B \end{array}$$

Assume that R' is flat over R and A' is flat over $A \otimes_R R'$ and B' is flat over $R' \otimes_R B$. Then

$$Tor_i^R(A, B) \otimes_{(A \otimes_R B)} (A' \otimes_{R'} B') = Tor_i^{R'}(A', B')$$

Proof. By Algebra, Section 10.71 there are canonical maps

$$\mathrm{Tor}_i^R(A, B) \longrightarrow \mathrm{Tor}_i^{R'}(A \otimes_R R', B \otimes_R R') \longrightarrow \mathrm{Tor}_i^{R'}(A', B')$$

These induce a map from left to right in the formula of the lemma.

Take a free resolution $F_\bullet \to A$ of A as an R-module. Then we see that $F_\bullet \otimes_R R'$ is a resolution of $A \otimes_R R'$. Hence $\mathrm{Tor}_i^{R'}(A \otimes_R R', B \otimes_R R')$ is computed by $F_\bullet \otimes_R B \otimes_R R'$. By our assumption that R' is flat over R, this computes $\mathrm{Tor}_i^R(A, B) \otimes_R R'$. Thus $\mathrm{Tor}_i^{R'}(A \otimes_R R', B \otimes_R R') = \mathrm{Tor}_i^R(A, B) \otimes_R R'$ (uses only flatness of R' over R).

By Lazard's theorem (Algebra, Theorem 10.76.4) we can write A', resp. B' as a filtered colimit of finite free $A \otimes_R R'$, resp. $B \otimes_R R'$-modules. Say $A' = \mathrm{colim}\, M_i$ and $B' = \mathrm{colim}\, N_j$. The result above gives

$$\mathrm{Tor}_i^{R'}(M_i, N_j) = \mathrm{Tor}_i^R(A, B) \otimes_{A \otimes_R B} (M_i \otimes_{R'} N_j)$$

as one can see by writing everything out in terms of bases. Taking the colimit we get the result of the lemma. □

Lemma 15.45.4. *Let R be a ring. Let A, B be R-algebras. The following are equivalent*

1. *A and B are Tor independent over R,*
2. *for every pair of primes $\mathfrak{p} \subset A$ and $\mathfrak{q} \subset B$ lying over the same prime $\mathfrak{r} \subset R$ the rings $A_\mathfrak{p}$ and $B_\mathfrak{q}$ are Tor independent over $R_\mathfrak{r}$, and*

(3) *For every prime* $\mathfrak{s}$ *of* $A \otimes_R B$ *the module*
$$Tor_i^R(A, B)_\mathfrak{s} = Tor_i^{R_\mathfrak{r}}(A_\mathfrak{p}, B_\mathfrak{q})_\mathfrak{s}$$
(where $\mathfrak{p} = A \cap \mathfrak{s}$, $\mathfrak{q} = B \cap \mathfrak{s}$ *and* $\mathfrak{r} = R \cap \mathfrak{s}$*) is zero.*

Proof. Let $\mathfrak{s}$ be a prime of $A \otimes_R B$ as in (3). The equality
$$\mathrm{Tor}_i^R(A, B)_\mathfrak{s} = \mathrm{Tor}_i^{R_\mathfrak{r}}(A_\mathfrak{p}, B_\mathfrak{q})_\mathfrak{s}$$
where $\mathfrak{p} = A \cap \mathfrak{s}$, $\mathfrak{q} = B \cap \mathfrak{s}$ and $\mathfrak{r} = R \cap \mathfrak{s}$ follows from Lemma 15.45.3. Hence (2) implies (3). Since we can test the vanishing of modules by localizing at primes (Algebra, Lemma 10.22.1) we conclude that (3) implies (1). For (1) $\Rightarrow$ (2) we use that
$$\mathrm{Tor}_i^{R_\mathfrak{r}}(A_\mathfrak{p}, B_\mathfrak{q}) = \mathrm{Tor}_i^R(A, B) \otimes_{(A \otimes_R B)} (A_\mathfrak{p} \otimes_{R_\mathfrak{r}} B_\mathfrak{q})$$
again by Lemma 15.45.3. □

15.46. Spectral sequences for Tor

In this section we collect various spectral sequences that come up when considering the Tor functors.

Example 15.46.1. Let R be a ring. Let $K_\bullet$ be a bounded above chain complex of R-modules. Let M be an R-module. Then there is a spectral sequence with E_2-page
$$\mathrm{Tor}_i^R(H_j(K_\bullet), M) \Rightarrow H_{i+j}(K_\bullet \otimes_R^{\mathbf{L}} M)$$
and another spectral sequence with E_1-page
$$\mathrm{Tor}_i^R(K_j, M) \Rightarrow H_{i+j}(K_\bullet \otimes_R^{\mathbf{L}} M)$$
This follows from the dual to Derived Categories, Lemma 13.20.3.

Example 15.46.2. Let $R \to S$ be a ring map. Let M be an R-module and let N be an S-module. Then there is a spectral sequence
$$\mathrm{Tor}_n^S(\mathrm{Tor}_m^R(M, S), N) \Rightarrow \mathrm{Tor}_{n+m}^R(M, N).$$
To construct it choose a R-free resolution $P_\bullet$ of M. Then we have
$$M \otimes_R^{\mathbf{L}} N = P^\bullet \otimes_R N = (P^\bullet \otimes_R S) \otimes_S N$$
and then apply the first spectral sequence of Example 15.46.1.

Example 15.46.3. Consider a commutative diagram
$$\begin{array}{ccc} B & \longrightarrow & B' = B \otimes_A A' \\ \uparrow & & \uparrow \\ A & \longrightarrow & A' \end{array}$$
and B-modules M, N. Set $M' = M \otimes_A A' = M \otimes_B B'$ and $N' = N \otimes_A A' = N \otimes_B B'$. *Assume that* $A \to B$ *is flat and that* M *and* N *are* A*-flat.* Then there is a spectral sequence
$$\mathrm{Tor}_i^A(\mathrm{Tor}_j^B(M, N), A') \Rightarrow \mathrm{Tor}_{i+j}^{B'}(M', N')$$
The reason is as follows. Choose free resolution $F_\bullet \to M$ as a B-module. As B and M are A-flat we see that $F_\bullet \otimes_A A'$ is a free B'-resolution of M'. Hence we see that the groups $\mathrm{Tor}_n^{B'}(M', N')$ are computed by the complex
$$(F_\bullet \otimes_A A') \otimes_{B'} N' = (F_\bullet \otimes_B N) \otimes_A A' = (F_\bullet \otimes_B N) \otimes_A^{\mathbf{L}} A'$$

the last equality because $F_\bullet \otimes_B N$ is a complex of flat A-modules as N is flat over A. Hence we obtain the spectral sequence by applying the spectral sequence of Example 15.46.1.

Example 15.46.4. Let $K^\bullet, L^\bullet$ be objects of $D^-(R)$. Then there are spectral sequences

$$E_2^{p,q} = H^p(K^\bullet \otimes_R^{\mathbf{L}} H^q(L^\bullet)) \Rightarrow H^{p+q}(K^\bullet \otimes_R^{\mathbf{L}} L^\bullet)$$

with $d_2^{p,q} : E_2^{p,q} \to E_2^{p+2,q-1}$ and

$$H^q(H^p(K^\bullet) \otimes_R^{\mathbf{L}} L^\bullet) \Rightarrow H^{p+q}(K^\bullet \otimes_R^{\mathbf{L}} L^\bullet)$$

After replacing $K^\bullet$ and $L^\bullet$ by bounded above complexes of projectives, these spectral sequences are simply the two spectral sequences for computing the cohomology of $\mathrm{Tot}(K^\bullet \otimes L^\bullet)$ discussed in Homology, Section 12.22.

15.47. Products and Tor

The simplest example of the product maps comes from the following situation. Suppose that $K^\bullet, L^\bullet \in D(R)$ with one of them contained in $D^-(R)$. Then there are maps

$$H^i(K^\bullet) \otimes_R H^j(L^\bullet) \longrightarrow H^{i+j}(K^\bullet \otimes_R^{\mathbf{L}} L^\bullet) \tag{15.47.0.1}$$

Namely, to define these maps we may assume that one of $K^\bullet, L^\bullet$ is a bounded above complex of projective R-modules. In that case $K^\bullet \otimes_R^{\mathbf{L}} L^\bullet$ is represented by the complex $\mathrm{Tot}(K^\bullet \otimes_R L^\bullet)$, see Section 15.42. Next, suppose that $\xi \in H^i(K^\bullet)$ and $\zeta \in H^j(L^\bullet)$. Choose $k \in \mathrm{Ker}(K^i \to K^{i+1})$ and $l \in \mathrm{Ker}(L^j \to L^{j+1})$ representing ξ and ζ. Then we set

$$\xi \cup \zeta = \text{class of } k \otimes l \text{ in } H^{i+j}(\mathrm{Tot}(K^\bullet \otimes_R L^\bullet)).$$

This make sense because the formula (see Homology, Definition 12.22.2) for the differential d on the total complex shows that $k \otimes l$ is a cocycle. Moreover, if $k' = d_K(k'')$ for some $k'' \in K^{i-1}$, then $k' \otimes l = \mathrm{d}(k'' \otimes l)$ because l is a cocycle. Similarly, altering the choice of l representing ζ does not change the class of $k \otimes l$. It is equally clear that $\cup$ is bilinear, and hence to a general element of $H^i(K^\bullet) \otimes_R H^j(L^\bullet)$ we assign

$$\sum \xi_i \otimes \zeta_i \longmapsto \sum \xi_i \cup \zeta_i$$

in $H^{i+j}(\mathrm{Tot}(K^\bullet \otimes_R L^\bullet))$.

Let $R \to A$ be a ring map. Let $K^\bullet, L^\bullet \in D^-(R)$. Then we have a canonical identification

$$(K^\bullet \otimes_R^{\mathbf{L}} A) \otimes_A^{\mathbf{L}} (L^\bullet \otimes_R^{\mathbf{L}} A) = (K^\bullet \otimes_R^{\mathbf{L}} L^\bullet) \otimes_R^{\mathbf{L}} A \tag{15.47.0.2}$$

in $D(A)$. It is constructed as follows. First, choose projective resolutions $P^\bullet \to K^\bullet$ and $Q^\bullet \to L^\bullet$ over R. Then the left hand side is represented by the complex $\mathrm{Tot}((P^\bullet \otimes_R A) \otimes_A (Q^\bullet \otimes_R A))$ and the right hand side by the complex $\mathrm{Tot}(P^\bullet \otimes_R Q^\bullet) \otimes_R A$. These complexes are canonically isomorphic. Thus the construction above induces products

$$\mathrm{Tor}_n^R(K^\bullet, A) \otimes_A \mathrm{Tor}_m^R(L^\bullet, A) \longrightarrow \mathrm{Tor}_{n+m}^R(K^\bullet \otimes_R L^\bullet, A)$$

which are occasionally useful.

Let M, N be R-modules. Using the general construction above and functoriality of Tor we obtain canonical maps

$$\mathrm{Tor}_n^R(M, A) \otimes_A \mathrm{Tor}_m^R(N, A) \longrightarrow \mathrm{Tor}_{n+m}^R(M \otimes_R N, A) \tag{15.47.0.3}$$

Here is a direct construction using projective resolutions. First, choose projective resolutions

$$P_\bullet \to M, \quad Q_\bullet \to N, \quad T_\bullet \to M \otimes_R N$$

over R. We have $H_0(\mathrm{Tot}(P_\bullet \otimes_R Q_\bullet)) = M \otimes_R N$ by right exactness of $\otimes_R$. Hence Derived Categories, Lemmas 13.18.6 and 13.18.7 guarantee the existence and uniqueness of a map of complexes $\mu : \mathrm{Tot}(P_\bullet \otimes_R Q_\bullet) \to T_\bullet$ such that $H_0(\mu) = \mathrm{id}_{M \otimes_R N}$. This induces a canonical map

$$\begin{aligned}
(M \otimes_R^{\mathbf{L}} A) \otimes_A^{\mathbf{L}} (N \otimes_R^{\mathbf{L}} A) &= \mathrm{Tot}((P_\bullet \otimes_R A) \otimes_A (Q_\bullet \otimes_R A)) \\
&= \mathrm{Tot}(P_\bullet \otimes_R Q_\bullet) \otimes_R A \\
&\to T_\bullet \otimes_R A \\
&= (M \otimes_R N) \otimes_R^{\mathbf{L}} A
\end{aligned}$$

in $D(A)$. Hence the products (15.47.0.3) above are constructed using (15.47.0.1) over A to construct

$$\mathrm{Tor}_n^R(M, A) \otimes_A \mathrm{Tor}_m^R(N, A) \to H^{-n-m}((M \otimes_R^{\mathbf{L}} A) \otimes_A^{\mathbf{L}} (N \otimes_R^{\mathbf{L}} A))$$

and then composing by the displayed map above to end up in $\mathrm{Tor}_{n+m}^R(M \otimes_R N, A)$.

An interesting special case of the above occurs when $M = N = B$ where B is an R-algebra. In this case we obtain maps

$$\mathrm{Tor}_n^R(B, A) \otimes_A \mathrm{Tor}_m^R(B, A) \longrightarrow \mathrm{Tor}_n^R(B \otimes_R B, A) \longrightarrow \mathrm{Tor}_n^R(B, A)$$

the second arrow being induced by the multiplication map $B \otimes_R B \to B$ via functoriality for Tor. In other words we obtain an A-algebra structure on $\mathrm{Tor}_\star^R(B, A)$. This algebra structure has many intriguing properties (associativity, graded commutative, B-algebra structure, divided powers in some case, etc) which we will discuss elsewhere (insert future reference here).

Lemma 15.47.1. *Let R be a ring. Let A, B, C be R-algebras and let $B \to C$ be an R-algebra map. Then the induced map*

$$Tor_\star^R(B, A) \longrightarrow Tor_\star^R(C, A)$$

is an A-algebra homomorphism.

Proof. Omitted. Hint: You can prove this by working through the definitions, writing all the complexes explicitly. □

15.48. Pseudo-coherent modules

Suppose that R is a ring. Recall that an R-module M is of finite type if there exists a surjection $R^{\oplus a} \to M$ and of finite presentation if there exists a presentation $R^{\oplus a_1} \to R^{\oplus a_0} \to M \to 0$. Similarly, we can consider those R-modules for which there exists a length n resolution

$$R^{\oplus a_n} \to R^{\oplus a_{n-1}} \to \ldots \to R^{\oplus a_0} \to M \to 0 \tag{15.48.0.1}$$

by finite free R-modules. A module is called *pseudo-coherent* of we can find such a resolution for every n. Here is the formal definition.

Definition 15.48.1. Let R be a ring. Denote $D(R)$ its derived category. Let $m \in \mathbf{Z}$.

(1) An object $K^\bullet$ of $D(R)$ is *m-pseudo-coherent* if there exists a bounded complex $E^\bullet$ of finite free R-modules and a morphism $\alpha : E^\bullet \to K^\bullet$ such that $H^i(\alpha)$ is an isomorphism for $i > m$ and $H^m(\alpha)$ is surjective.
(2) An object $K^\bullet$ of $D(R)$ is *pseudo-coherent* if it is quasi-isomorphic to a bounded above complex of finite free R-modules.
(3) An R-module M is called *m-pseudo-coherent* if if $M[0]$ is an m-pseudo-coherent object of $D(R)$.
(4) An R-module M is called *pseudo-coherent*[6] if $M[0]$ is a pseudo-coherent object of $D(R)$.

As usual we apply this terminology also to complexes of R-modules. Since any morphism $E^\bullet \to K^\bullet$ in $D(R)$ is represented by an actual map of complexes, see Derived Categories, Lemma 13.18.8, there is no ambiguity. It turns out that $K^\bullet$ is pseudo-coherent if and only if $K^\bullet$ is m-pseudo-coherent for all $m \in \mathbf{Z}$, see Lemma 15.48.5. Also, if the ring is Noetherian the condition can be understood as a finite generation condition on the cohomology, see Lemma 15.48.16. Let us first relate this to the informal discussion above.

Lemma 15.48.2. *Let R be a ring and $m \in \mathbf{Z}$. Let $(K^\bullet, L^\bullet, M^\bullet, f, g, h)$ be a distinguished triangle in $D(R)$.*

(1) *If $K^\bullet$ is $(m+1)$-pseudo-coherent and $L^\bullet$ is m-pseudo-coherent then $M^\bullet$ is m-pseudo-coherent.*
(2) *If $K^\bullet, M^\bullet$ are m-pseudo-coherent, then $L^\bullet$ is m-pseudo-coherent.*
(3) *If $L^\bullet$ is $(m+1)$-pseudo-coherent and $M^\bullet$ is m-pseudo-coherent, then $K^\bullet$ is $(m+1)$-pseudo-coherent.*

Proof. Proof of (1). Choose $\alpha : P^\bullet \to K^\bullet$ with $P^\bullet$ a bounded complex of finite free modules such that $H^i(\alpha)$ is an isomorphism for $i > m+1$ and surjective for $i = m+1$. We may replace $P^\bullet$ by $\sigma_{\geq m+1}P^\bullet$ and hence we may assume that $P^i = 0$ for $i < m+1$. Choose $\beta : E^\bullet \to L^\bullet$ with $E^\bullet$ a bounded complex of finite free modules such that $H^i(\beta)$ is an isomorphism for $i > m$ and surjective for $i = m$. By Derived Categories, Lemma 13.18.11 we can find a map $\alpha : P^\bullet \to E^\bullet$ such that the diagram

$$\begin{array}{ccc} K^\bullet & \longrightarrow & L^\bullet \\ \uparrow & & \uparrow \\ P^\bullet & \xrightarrow{\alpha} & E^\bullet \end{array}$$

is commutative in $D(R)$. The cone $C(\alpha)^\bullet$ is a bounded complex of finite free R-modules, and the commutativity of the diagram implies that there exists a morphism of distinguished triangles

$$(P^\bullet, E^\bullet, C(\alpha)^\bullet) \longrightarrow (K^\bullet, L^\bullet, M^\bullet).$$

It follows from the induced map on long exact cohomology sequences and Homology, Lemmas 12.5.19 and 12.5.20 that $C(\alpha)^\bullet \to M^\bullet$ induces an isomorphism on cohomology in degrees $> m$ and a surjection in degree m. Hence $M^\bullet$ is m-pseudo-coherent.

[6] This clashes with what is meant by a pseudo-coherent module in [**Bou61**].

Assertions (2) and (3) follow from (1) by rotating the distinguished triangle. □

Lemma 15.48.3. *Let R be a ring. Let $K^\bullet$ be a complex of R-modules. Let $m \in \mathbf{Z}$.*

(1) *If $K^\bullet$ is m-pseudo-coherent and $H^i(K^\bullet) = 0$ for $i > m$, then $H^m(K^\bullet)$ is a finite type R-module.*
(2) *If $K^\bullet$ is m-pseudo-coherent and $H^i(K^\bullet) = 0$ for $i > m+1$, then $H^{m+1}(K^\bullet)$ is a finitely presented R-module.*

Proof. Proof of (1). Choose a bounded complex $E^\bullet$ of finite projective R-modules and a map $\alpha : E^\bullet \to K^\bullet$ which induces an isomorphism on cohomology in degrees $> m$ and a surjection in degree m. It is clear that it suffices to prove the result for $E^\bullet$. Let n be the largest integer such that $E^n \neq 0$. If $n = m$, then the result is clear. If $n > m$, then $E^{n-1} \to E^n$ is surjective as $H^n(E^\bullet) = 0$. As E^n is finite projective we see that $E^{n-1} = E' \oplus E^n$. Hence it suffices to prove the result for the complex $(E')^\bullet$ which is the same as $E^\bullet$ except has E' in degree $n-1$ and 0 in degree n. We win by induction on n.

Proof of (2). Choose a bounded complex $E^\bullet$ of finite projective R-modules and a map $\alpha : E^\bullet \to K^\bullet$ which induces an isomorphism on cohomology in degrees $> m$ and a surjection in degree m. As in the proof of (1) we can reduce to the case that $E^i = 0$ for $i > m+1$. Then we see that $H^{m+1}(K^\bullet) \cong H^{m+1}(E^\bullet) = \mathrm{Coker}(E^m \to E^{m+1})$ which is of finite presentation. □

Lemma 15.48.4. *Let R be a ring. Let M be an R-module. Then*

(1) *M is 0-pseudo-coherent if and only if M is a finite type R-module,*
(2) *M is (-1)-pseudo-coherent if and only if M is a finitely presented R-module,*
(3) *M is $(-d)$-pseudo-coherent if and only if there exists a resolution*
$$R^{\oplus a_d} \to R^{\oplus a_{d-1}} \to \ldots \to R^{\oplus a_0} \to M \to 0$$
of length d, and
(4) *M is pseudo-coherent if and only if there exists an infinite resolution*
$$\ldots \to R^{\oplus a_1} \to R^{\oplus a_0} \to M \to 0$$
by finite free R-modules.

Proof. If M is of finite type (resp. of finite presentation), then M is 0-pseudo-coherent (resp. (-1)-pseudo-coherent) as follows from the discussion preceding Definition 15.48.1. Conversely, if M is 0-pseudo-coherent, then $M = H^0(M[0])$ is of finite type by Lemma 15.48.3. If M is (-1)-pseudo-coherent, then it is 0-pseudo-coherent hence of finite type. Choose a surjection $R^{\oplus a} \to M$ and denote $K = \mathrm{Ker}(R^{\oplus a} \to M)$. By Lemma 15.48.2 we see that K is 0-pseudo-coherent, hence of finite type, whence M is of finite presentation.

To prove the third and fourth statement use induction and an argument similar to the above (details omitted). □

Lemma 15.48.5. *Let R be a ring. Let $K^\bullet$ be a complex of R-modules. The following are equivalent*

(1) *$K^\bullet$ is pseudo-coherent,*
(2) *$K^\bullet$ is m-pseudo-coherent for every $m \in \mathbf{Z}$, and*

(3) *$K^\bullet$ is quasi-isomorphic to a bounded above complex of finite projective R-modules.*

Proof. We see that (1) $\Rightarrow$ (3) as a finite free module is a finite projective R-module. Conversely, suppose $P^\bullet$ is a bounded above complex of finite projective R-modules. Say $P^i = 0$ for $i > n_0$. We choose a direct sum decompositions $F^{n_0} = P^{n_0} \oplus C^{n_0}$ with F^{n_0} a finite free R-module, and inductively

$$F^{n-1} = P^{n-1} \oplus C^n \oplus C^{n-1}$$

for $n \leq n_0$ with F^{n_0} a finite free R-module. As a complex $F^\bullet$ has maps $F^{n-1} \to F^n$ which agree with $P^{n-1} \to P^n$, induce the identity $C^n \to C^n$, and are zero on C^{n-1}. The map $F^\bullet \to P^\bullet$ is a quasi-isomorphism (even a homotopy equivalence) and hence (3) implies (1).

Assume (1). Let $E^\bullet$ be a bounded above complex of finite free R-modules and let $E^\bullet \to K^\bullet$ be a quasi-isomorphism. Then the induced maps $\sigma_{\geq m}E^\bullet \to K^\bullet$ from the stupid truncation of $E^\bullet$ to $K^\bullet$ show that $K^\bullet$ is m-pseudo-coherent. Hence (1) implies (2).

Assume (2). We first apply (2) for $n = 0$ to obtain a map of complexes $\alpha : F^\bullet \to K^\bullet$ where $F^\bullet$ is bounded above, consists of finite free R-modules and such that $H^i(\alpha)$ is an isomorphism for $i > 0$ and surjective for $i = 0$. Note that these conditions remain satisfied after replacing $F^\bullet$ by $\sigma_{\geq 0}F^\bullet$. Picture

$$\begin{array}{ccccccc} & & F^0 & \longrightarrow & F^1 & \longrightarrow & \ldots \\ & & \downarrow{\scriptstyle \alpha} & & \downarrow{\scriptstyle \alpha} & & \\ K^{-1} & \longrightarrow & K^0 & \longrightarrow & K^1 & \longrightarrow & \ldots \end{array}$$

By induction on $n < 0$ we are going to extend $F^\bullet$ to a complex $F^n \to F^{n+1} \to \ldots \to F^{-1} \to F^0 \to \ldots$ of finite free R-modules and extend α such that $H^i(\alpha)$ is an isomorphism for $i > n$ and surjective for $i = n$. By shifting it suffices to prove the induction step for $n = -1$, i.e., it suffices to extend the diagram above by adding F^{-1}. Let $C^\bullet$ be the cone on α (Derived Categories, Definition 13.8.1). The long exact sequence of cohomology shows that $H^i(C^\bullet) = 0$ for $i \geq 0$. By Lemma 15.48.2 we see that $C^\bullet$ is (-1)-pseudo-coherent. By Lemma 15.48.3 we see that $H^{-1}(C^\bullet)$ is a finite R-module. Choose a finite free R-module F^{-1} and a map $\beta : F^{-1} \to C^{-1}$ such that the composition $F^{-1} \to C^{-1} \to C^0$ is zero and such that F^{-1} surjects onto $H^{-1}(C^\bullet)$. Since $C^{-1} = K^{-1} \oplus F^0$ we can write $\beta = (\alpha^{-1}, -d^{-1})$. The vanishing of the composition $F^{-1} \to C^{-1} \to C^0$ implies these maps fit into a morphism of complexes

$$\begin{array}{ccccccccc} & & F^{-1} & \xrightarrow[d^{-1}]{} & F^0 & \longrightarrow & F^1 & \longrightarrow & \ldots \\ & & \downarrow{\scriptstyle \alpha^{-1}} & & \downarrow{\scriptstyle \alpha} & & \downarrow{\scriptstyle \alpha} & & \\ \ldots & \longrightarrow & K^{-1} & \longrightarrow & K^0 & \longrightarrow & K^1 & \longrightarrow & \ldots \end{array}$$

Moreover, these maps define a morphism of distinguished triangles

$$\begin{array}{ccccccc} (F^0 \to \ldots) & \longrightarrow & (F^{-1} \to \ldots) & \longrightarrow & F^{-1} & \longrightarrow & (F^0 \to \ldots)[1] \\ \downarrow & & \downarrow & & \downarrow{\scriptstyle \beta} & & \downarrow \\ (F^0 \to \ldots) & \longrightarrow & K^\bullet & \longrightarrow & C^\bullet & \longrightarrow & (F^0 \to \ldots)[1] \end{array}$$

Hence our choice of β implies that the map of complexes $(F^{-1} \to \ldots) \to K^\bullet$ induces an isomorphism on cohomology in degrees ≥ 0 and a surjection in degree -1. This finishes the proof of the lemma. □

Lemma 15.48.6. *Let R be a ring. Let $(K^\bullet, L^\bullet, M^\bullet, f, g, h)$ be a distinguished triangle in $D(R)$. If two out of three of $K^\bullet, L^\bullet, M^\bullet$ are pseudo-coherent then the third is also pseudo-coherent.*

Proof. Combine Lemmas 15.48.2 and 15.48.5. □

Lemma 15.48.7. *Let R be a ring. Let $K^\bullet$ be a complex of R-modules. Let $m \in \mathbf{Z}$.*

1. *If $H^i(K^\bullet) = 0$ for all $i \geq m$, then $K^\bullet$ is m-pseudo-coherent.*
2. *If $H^i(K^\bullet) = 0$ for $i > m$ and $H^m(K^\bullet)$ is a finite R-module, then $K^\bullet$ is m-pseudo-coherent.*
3. *If $H^i(K^\bullet) = 0$ for $i > m+1$, the module $H^{m+1}(K^\bullet)$ is of finite presentation, and $H^m(K^\bullet)$ is of finite type, then $K^\bullet$ is m-pseudo-coherent.*

Proof. It suffices to prove (3). Set $M = H^{m+1}(K^\bullet)$. Note that $\tau_{\geq m+1}K^\bullet$ is quasi-isomorphic to $M[-m-1]$. By Lemma 15.48.4 we see that $M[-m-1]$ is m-pseudo-coherent. Since we have the distinguished triangle

$$(\tau_{\leq m}K^\bullet, K^\bullet, \tau_{\geq m+1}K^\bullet)$$

(Derived Categories, Remark 13.11.4) by Lemma 15.48.2 it suffices to prove that $\tau_{\leq m}K^\bullet$ is pseudo-coherent. By assumption $H^m(\tau_{\leq m}K^\bullet)$ is a finite type R-module. Hence we can find a finite free R-module E and a map $E \to \mathrm{Ker}(d_K^m)$ such that the composition $E \to \mathrm{Ker}(d_K^m) \to H^m(\tau_{\leq m}K^\bullet)$ is surjective. Then $E[-m] \to \tau_{\leq m}K^\bullet$ witnesses the fact that $\tau_{\leq m}K^\bullet$ is m-pseudo-coherent. □

Lemma 15.48.8. *Let R be a ring. Let $m \in \mathbf{Z}$. If $K^\bullet \oplus L^\bullet$ is m-pseudo-coherent (resp. pseudo-coherent) so are $K^\bullet$ and $L^\bullet$.*

Proof. In this proof we drop the superscript $^\bullet$. Assume that $K \oplus L$ is m-pseudo-coherent. It is clear that $K, L \in D^-(R)$. Note that there is a distinguished triangle

$$(K \oplus L, K \oplus L, L \oplus L[1]) = (K, K, 0) \oplus (L, L, L \oplus L[1])$$

see Derived Categories, Lemma 13.4.9. By Lemma 15.48.2 we see that $L \oplus L[1]$ is m-pseudo-coherent. Hence also $L[1] \oplus L[2]$ is m-pseudo-coherent. By induction $L[n] \oplus L[n+1]$ is m-pseudo-coherent. By Lemma 15.48.7 we see that $L[n]$ is m-pseudo-coherent for large n. Hence working backwards, using the distinguished triangles

$$(L[n], L[n] \oplus L[n-1], L[n-1])$$

we conclude that $L[n], L[n-1], \ldots, L$ are m-pseudo-coherent as desired. The pseudo-coherent case follows from this and Lemma 15.48.5. □

Lemma 15.48.9. *Let R be a ring. Let $m \in \mathbf{Z}$. Let $K^\bullet$ be a bounded above complex of R-modules such that K^i is $(m-i)$-pseudo-coherent for all i. Then $K^\bullet$ is m-pseudo-coherent. In particular, if $K^\bullet$ is a bounded above complex of pseudo-coherent R-modules, then $K^\bullet$ is pseudo-coherent.*

Proof. We may replace $K^\bullet$ by $\sigma_{\geq m-1}K^\bullet$ (for example) and hence assume that $K^\bullet$ is bounded. Then the complex $K^\bullet$ is m-pseudo-coherent as each $K^i[-i]$ is m-pseudo-coherent by induction on the length of the complex: use Lemma 15.48.2 and the stupid truncations. For the final statement, it suffices to prove that $K^\bullet$ is m-pseudo-coherent for all $m \in \mathbf{Z}$, see Lemma 15.48.5. This follows from the first part. □

Lemma 15.48.10. *Let R be a ring. Let $m \in \mathbf{Z}$. Let $K^\bullet \in D^-(R)$ such that $H^i(K^\bullet)$ is $(m-i)$-pseudo-coherent (resp. pseudo-coherent) for all i. Then $K^\bullet$ is m-pseudo-coherent (resp. pseudo-coherent).*

Proof. Assume $K^\bullet$ is an object of $D^-(R)$ such that each $H^i(K^\bullet)$ is $(m-i)$-pseudo-coherent. Let n be the largest integer such that $H^n(K^\bullet)$ is nonzero. We will prove the lemma by induction on n. If $n < m$, then $K^\bullet$ is m-pseudo-coherent by Lemma 15.48.7. If $n \geq m$, then we have the distinguished triangle

$$(\tau_{\leq n-1}K^\bullet, K^\bullet, H^n(K^\bullet)[-n])$$

(Derived Categories, Remark 13.11.4) Since $H^n(K^\bullet)[-n]$ is m-pseudo-coherent by assumption, we can use Lemma 15.48.2 to see that it suffices to prove that $\tau_{\leq n-1}K^\bullet$ is m-pseudo-coherent. By induction on n we win. (The pseudo-coherent case follows from this and Lemma 15.48.5.) □

Lemma 15.48.11. *Let $A \to B$ be a ring map. Assume that B is pseudo-coherent as an A-module. Let $K^\bullet$ be a complex of B-modules. The following are equivalent*

1. *$K^\bullet$ is m-pseudo-coherent as a complex of B-modules, and*
2. *$K^\bullet$ is m-pseudo-coherent as a complex of A-modules.*

The same equivalence holds for pseudo-coherence.

Proof. Assume (1). Choose a bounded complex of finite free B-modules $E^\bullet$ and a map $\alpha : E^\bullet \to K^\bullet$ which is an isomorphism on cohomology in degrees $> m$ and a surjection in degree m. Consider the distinguished triangle $(E^\bullet, K^\bullet, C(\alpha)^\bullet)$. By Lemma 15.48.7 $C(\alpha)^\bullet$ is m-pseudo-coherent as a complex of A-modules. Hence it suffices to prove that $E^\bullet$ is pseudo-coherent as a complex of A-modules, which follows from Lemma 15.48.9. The pseudo-coherent case of (1) $\Rightarrow$ (2) follows from this and Lemma 15.48.5.

Assume (2). Let n be the largest integer such that $H^n(K^\bullet) \neq 0$. We will prove that $K^\bullet$ is m-pseudo-coherent as a complex of B-modules by induction on $n - m$. The case $n < m$ follows from Lemma 15.48.7. Choose a bounded complex of finite free A-modules $E^\bullet$ and a map $\alpha : E^\bullet \to K^\bullet$ which is an isomorphism on cohomology in degrees $> m$ and a surjection in degree m. Consider the induced map of complexes

$$\alpha \otimes 1 : E^\bullet \otimes_A B \to K^\bullet.$$

Note that $C(\alpha \otimes 1)^\bullet$ is acyclic in degrees $\geq n$ as $H^n(E) \to H^n(E^\bullet \otimes_A B) \to H^n(K^\bullet)$ is surjective by construction and since $H^i(E^\bullet \otimes_A B) = 0$ for $i > n$ by the spectral sequence of Example 15.46.4. On the other hand, $C(\alpha \otimes 1)^\bullet$ is m-pseudo-coherent as a complex of A-modules because both $K^\bullet$ and $E^\bullet \otimes_A B$ (see Lemma 15.48.9) are so,

see Lemma 15.48.2. Hence by induction we see that $C(\alpha \otimes 1)^\bullet$ is m-pseudo-coherent as a complex of B-modules. Finally another application of Lemma 15.48.2 shows that $K^\bullet$ is m-pseudo-coherent as a complex of B-modules (as clearly $E^\bullet \otimes_A B$ is pseudo-coherent as a complex of B-modules). The pseudo-coherent case of (2) $\Rightarrow$ (1) follows from this and Lemma 15.48.5. $\square$

Lemma 15.48.12. *Let $A \to B$ be a ring map. Let $K^\bullet$ be an m-pseudo-coherent (resp. pseudo-coherent) complex of A-modules. Then $K^\bullet \otimes^{\mathbf{L}}_A B$ is an m-pseudo-coherent (resp. pseudo-coherent) complex of B-modules.*

Proof. First we note that the statement of the lemma makes sense as $K^\bullet$ is bounded above and hence $K^\bullet \otimes^{\mathbf{L}}_A B$ is defined by Equation (15.42.0.2). Having said this, choose a bounded complex $E^\bullet$ of finite free A-modules and $\alpha : E^\bullet \to K^\bullet$ with $H^i(\alpha)$ an isomorphism for $i > m$ and surjective for $i = m$. Then the cone $C(\alpha)^\bullet$ is acyclic in degrees $\geq m$. Since $- \otimes^{\mathbf{L}}_A B$ is an exact functor we get a distinguished triangle

$$(E^\bullet \otimes^{\mathbf{L}}_A B, K^\bullet \otimes^{\mathbf{L}}_A B, C(\alpha)^\bullet \otimes^{\mathbf{L}}_A B)$$

of complexes of B-modules. By the dual to Derived Categories, Lemma 13.16.1 we see that $H^i(C(\alpha)^\bullet \otimes^{\mathbf{L}}_A B) = 0$ for $i \geq m$. Since $E^\bullet$ is a complex of projective R-modules we see that $E^\bullet \otimes^{\mathbf{L}}_A B = E^\bullet \otimes_A B$ and hence

$$E^\bullet \otimes_A B \longrightarrow K^\bullet \otimes^{\mathbf{L}}_A B$$

is a morphism of complexes of B-modules that witnesses the fact that $K^\bullet \otimes^{\mathbf{L}}_A B$ is m-pseudo-coherent. The case of pseudo-coherent complexes follows from the case of m-pseudo-coherent complexes via Lemma 15.48.5. $\square$

Lemma 15.48.13. *Let $A \to B$ be a flat ring map. Let M be an m-pseudo-coherent (resp. pseudo-coherent) A-module. Then $M \otimes_A B$ is an m-pseudo-coherent (resp. pseudo-coherent) B-module.*

Proof. Immediate consequence of Lemma 15.48.12 and the fact that $M \otimes^{\mathbf{L}}_A B = M \otimes_A B$ because B is flat over A. $\square$

The following lemma also follows from the stronger Lemma 15.48.14.

Lemma 15.48.14. *Let R be a ring. Let $f_1, \ldots, f_r \in R$ be elements which generate the unit ideal. Let $m \in \mathbf{Z}$. Let $K^\bullet$ be a complex of R-modules. If for each i the complex $K^\bullet \otimes_R R_{f_i}$ is m-pseudo-coherent (resp. pseudo-coherent), then $K^\bullet$ is m-pseudo-coherent (resp. pseudo-coherent).*

Proof. We will use without further mention that $- \otimes_R R_{f_i}$ is an exact functor and that therefore

$$H^i(K^\bullet)_{f_i} = H^i(K^\bullet) \otimes_R R_{f_i} = H^i(K^\bullet \otimes_R R_{f_i}).$$

Assume $K^\bullet \otimes_R R_{f_i}$ is m-pseudo-coherent for $i = 1, \ldots, r$. Let $n \in \mathbf{Z}$ be the largest integer such that $H^n(K^\bullet \otimes_R R_{f_i})$ is nonzero for some i. This implies in particular that $H^i(K^\bullet) = 0$ for $i > n$ (and that $H^n(K^\bullet) \neq 0$) see Algebra, Lemma 10.22.2. We will prove the lemma by induction on $n - m$. If $n < m$, then the lemma is true by Lemma 15.48.7. If $n \geq m$, then $H^n(K^\bullet)_{f_i}$ is a finite R_{f_i}-module for each i, see Lemma 15.48.3. Hence $H^n(K^\bullet)$ is a finite R-module, see Algebra, Lemma 10.22.2. Choose a finite free R-module E and a surjection $E \to H^n(K^\bullet)$. As E is projective we can lift this to a map of complexes $\alpha : E[-n] \to K^\bullet$. Then the

cone $C(\alpha)^\bullet$ has vanishing cohomology in degrees $\geq n$. On the other hand, the complexes $C(\alpha)^\bullet \otimes_R R_{f_i}$ are m-pseudo-coherent for each i, see Lemma 15.48.2. Hence by induction we see that $C(\alpha)^\bullet$ is m-pseudo-coherent as a complex of R-modules. Applying Lemma 15.48.2 once more we conclude. □

Lemma 15.48.15. *Let R be a ring. Let $m \in \mathbf{Z}$. Let $K^\bullet$ be a complex of R-modules. Let $R \to R'$ be a faithfully flat ring map. If the complex $K^\bullet \otimes_R R'$ is m-pseudo-coherent (resp. pseudo-coherent), then $K^\bullet$ is m-pseudo-coherent (resp. pseudo-coherent).*

Proof. We will use without further mention that $- \otimes_R R'$ is an exact functor and that therefore

$$H^i(K^\bullet) \otimes_R R' = H^i(K^\bullet \otimes_R R').$$

Assume $K^\bullet \otimes_R R'$ is m-pseudo-coherent. Let $n \in \mathbf{Z}$ be the largest integer such that $H^n(K^\bullet)$ is nonzero; then n is also the largest integer such that $H^n(K^\bullet \otimes_R R')$ is nonzero. We will prove the lemma by induction on $n - m$. If $n < m$, then the lemma is true by Lemma 15.48.7. If $n \geq m$, then $H^n(K^\bullet) \otimes_R R'$ is a finite R'-module, see Lemma 15.48.3. Hence $H^n(K^\bullet)$ is a finite R-module, see Algebra, Lemma 10.78.2. Choose a finite free R-module E and a surjection $E \to H^n(K^\bullet)$. As E is projective we can lift this to a map of complexes $\alpha : E[-n] \to K^\bullet$. Then the cone $C(\alpha)^\bullet$ has vanishing cohomology in degrees $\geq n$. On the other hand, the complex $C(\alpha)^\bullet \otimes_R R'$ is m-pseudo-coherent, see Lemma 15.48.2. Hence by induction we see that $C(\alpha)^\bullet$ is m-pseudo-coherent as a complex of R-modules. Applying Lemma 15.48.2 once more we conclude. □

Lemma 15.48.16. *Let R be a Noetherian ring. Then*

1. *A complex of R-modules $K^\bullet$ is m-pseudo-coherent if and only if $K^\bullet \in D^-(R)$ and $H^i(K^\bullet)$ is a finite R-module for $i \geq m$.*
2. *A complex of R-modules $K^\bullet$ is pseudo-coherent if and only if $K^\bullet \in D^-(R)$ and $H^i(K^\bullet)$ is a finite R-module for all i.*
3. *An R-module is pseudo-coherent if and only if it is finite.*

Proof. In Algebra, Lemma 10.67.1 we have seen that any finite R-module is pseudo-coherent. On the other hand, a pseudo-coherent module is finite, see Lemma 15.48.4. Hence (3) holds. Suppose that $K^\bullet$ is an m-pseudo-coherent complex. Then there exists a bounded complex of finite free R-modules $E^\bullet$ such that $H^i(K^\bullet)$ is isomorphic to $H^i(E^\bullet)$ for $i > m$ and such that $H^m(K^\bullet)$ is a quotient of $H^m(E^\bullet)$. Thus it is clear that each $H^i(K^\bullet)$, $i \geq m$ is a finite module. The converse implication in (1) follows from Lemma 15.48.10 and part (3). Part (2) follows from (1) and Lemma 15.48.5. □

Remark 15.48.17. Let R be ring map. Let L, M, N be R-modules. Consider the canonical map

$$\mathrm{Hom}_R(M, N) \otimes_R L \to \mathrm{Hom}_R(M, N \otimes_R L)$$

Choose a two term free resolution $F_1 \to F_0 \to M \to 0$. Assuming L flat over R we obtain a commutative diagram

$$\begin{array}{ccccccc}
0 & \longrightarrow & \mathrm{Hom}_R(M, N) \otimes_R L & \longrightarrow & \mathrm{Hom}_R(F_0, N) \otimes_R L & \longrightarrow & \mathrm{Hom}_R(F_1, N) \otimes_R L \\
 & & \downarrow & & \downarrow & & \downarrow \\
0 & \longrightarrow & \mathrm{Hom}_R(M, N \otimes_R L) & \longrightarrow & \mathrm{Hom}_R(F_0, N \otimes_R L) & \longrightarrow & \mathrm{Hom}_R(F_1, N \otimes_R L)
\end{array}$$

with exact rows. We conclude that if F_0 and F_1 are finite free, i.e., if M is finitely presented, then the first displayed map is an isomorphism. Similarly, if M is $(-m)$-pseudo-coherent and still assuming L is flat over R, then the map

$$\mathrm{Ext}^i_R(M, N) \otimes_R L \to \mathrm{Ext}^i_R(M, N \otimes_R L)$$

is an isomorphism for $i < m$.

Remark 15.48.18. Let R be ring map. Let M, N be R-modules. Let $R \to R'$ be a flat ring map. By Algebra, Lemma 10.68.1 we have $\mathrm{Ext}^i_{R'}(M \otimes_R R', N \otimes_R R') = \mathrm{Ext}^i_R(M, N \otimes_R R')$. Combined with Remark 15.48.17 we conclude that

$$\mathrm{Hom}_R(M, N) \otimes_R R' = \mathrm{Hom}_{R'}(M \otimes_R R', N \otimes_R R')$$

if M is a finitely presented R-module and that

$$\mathrm{Ext}^i_R(M, N) \otimes_R R' = \mathrm{Ext}^i_{R'}(M \otimes_R R', N \otimes_R R')$$

is an isomorphism for $i < m$ if M is $(-m)$-pseudo-coherent. In particular if R is Noetherian and M is a finite module this holds for all i.

15.49. Tor dimension

Instead of resolving by projective modules we can look at resolutions by flat modules. This leads to the following concept.

Definition 15.49.1. Let R be a ring. Denote $D(R)$ its derived category. Let $a, b \in \mathbf{Z}$.

(1) An object $K^\bullet$ of $D(R)$ has *tor-amplitude in* $[a, b]$ if $H^i(K^\bullet \otimes^{\mathbf{L}}_R M) = 0$ for all R-modules M and all $i \notin [a, b]$.
(2) An object $K^\bullet$ of $D(R)$ has *finite tor dimension* if it has tor-amplitude in $[a, b]$ for some a, b.
(3) An R-module M has *tor dimension* $\leq d$ if if $M[0]$ as an object of $D(R)$ has tor-amplitude in $[-d, 0]$.
(4) An R-module M has *finite tor dimension* if $M[0]$ as an object of $D(R)$ has finite tor dimension.

We observe that if $K^\bullet$ has finite tor dimension, then $K^\bullet \in D^b(R)$.

Lemma 15.49.2. *Let R be a ring. Let $K^\bullet$ be a bounded above complex of flat R-modules with tor-amplitude in $[a, b]$. Then $\mathrm{Coker}(d^{a-1}_K)$ is a flat R-module.*

Proof. As $K^\bullet$ is a bounded above complex of flat modules we see that $K^\bullet \otimes_R M = K^\bullet \otimes^{\mathbf{L}}_R M$. Hence for every R-module M the sequence

$$K^{a-2} \otimes_R M \to K^{a-1} \otimes_R M \to K^a \otimes_R M$$

is exact in the middle. Since $K^{a-2} \to K^{a-1} \to K^a \to \mathrm{Coker}(d^{a-1}_K) \to 0$ is a flat resolution this implies that $\mathrm{Tor}^R_1(\mathrm{Coker}(d^{a-1}_K), M) = 0$ for all R-modules M. This means that $\mathrm{Coker}(d^{a-1}_K)$ is flat, see Algebra, Lemma 10.70.7. □

Lemma 15.49.3. *Let R be a ring. Let $K^\bullet$ be an object of $D(R)$. Let $a, b \in \mathbf{Z}$. The following are equivalent*

(1) *$K^\bullet$ has tor-amplitude in $[a, b]$.*
(2) *$K^\bullet$ is quasi-isomorphic to a complex $E^\bullet$ of flat R-modules with $E^i = 0$ for $i \notin [a, b]$.*

Proof. If (2) holds, then we may compute $K^\bullet \otimes^{\mathbf{L}}_R M = E^\bullet \otimes_R M$ and it is clear that (1) holds. Assume that (1) holds. We may replace $K^\bullet$ by a projective resolution. Let n be the largest integer such that $K^n \neq 0$. If $n > b$, then $K^{n-1} \to K^n$ is surjective as $H^n(K^\bullet) = 0$. As K^n is projective we see that $K^{n-1} = K' \oplus K^n$. Hence it suffices to prove the result for the complex $(K')^\bullet$ which is the same as $K^\bullet$ except has K' in degree $n-1$ and 0 in degree n. Thus, by induction on n, we reduce to the case that $K^\bullet$ is a complex of projective R-modules with $K^i = 0$ for $i > b$.

Set $E^\bullet = \tau_{\geq a} K^\bullet$. Everything is clear except that E^a is flat which follows immediately from Lemma 15.49.2 and the definitions. □

Lemma 15.49.4. *Let R be a ring. Let $(K^\bullet, L^\bullet, M^\bullet, f, g, h)$ be a distinguished triangle in $D(R)$. Let $a, b \in \mathbf{Z}$.*

1. *If $K^\bullet$ has tor-amplitude in $[a+1, b+1]$ and $L^\bullet$ has tor-amplitude in $[a, b]$ then $M^\bullet$ has tor-amplitude in $[a, b]$.*
2. *If $K^\bullet, M^\bullet$ have tor-amplitude in $[a, b]$, then $L^\bullet$ has tor-amplitude in $[a, b]$.*
3. *If $L^\bullet$ has tor-amplitude in $[a+1, b+1]$ and $M^\bullet$ has tor-amplitude in $[a, b]$, then $K^\bullet$ has tor-amplitude in $[a+1, b+1]$.*

Proof. Omitted. Hint: This just follows from the long exact cohomology sequence associated to a distinguished triangle and the fact that $- \otimes^{\mathbf{L}}_R M$ preserves distinguished triangles. The easiest one to prove is (2) and the others follow from it by translation. □

Lemma 15.49.5. *Let R be a ring. Let M be an R-module. Let $d \geq 0$. The following are equivalent*

1. *M has tor dimension $\leq d$, and*
2. *there exists a resolution*

$$0 \to F_d \to \ldots \to F_1 \to F_0 \to M \to 0$$

with F_i a flat R-module.

In particular an R-module has tor dimension 0 if and only if it is a flat R-module.

Proof. Assume (2). Then the complex $E^\bullet$ with $E^{-i} = F_i$ is quasi-isomorphic to M. Hence the Tor dimension of M is at most d by Lemma 15.49.3. Conversely, assume (1). Let $P^\bullet \to M$ be a projective resolution of M. By Lemma 15.49.2 we see that $\tau_{\geq -d} P^\bullet$ is a flat resolution of M of length d, i.e., (2) holds. □

Lemma 15.49.6. *Let R be a ring. Let $a, b \in \mathbf{Z}$. If $K^\bullet \oplus L^\bullet$ has tor amplitude in $[a, b]$ so do $K^\bullet$ and $L^\bullet$.*

Proof. Clear from the fact that the Tor functors are additive. □

Lemma 15.49.7. *Let R be a ring. Let $K^\bullet$ be a bounded complex of R-modules such that K^i has tor amplitude in $[a-i, b-i]$ for all i. Then $K^\bullet$ has tor amplitude in $[a, b]$. In particular if $K^\bullet$ is a finite complex of R-modules of finite tor dimension, then $K^\bullet$ has finite tor dimension.*

Proof. Follows by induction on the length of the finite complex: use Lemma 15.49.4 and the stupid truncations. □

Lemma 15.49.8. *Let R be a ring. Let $a, b \in \mathbf{Z}$. Let $K^\bullet \in D^b(R)$ such that $H^i(K^\bullet)$ has tor amplitude in $[a-i, b-i]$ for all i. Then $K^\bullet$ has tor amplitude in $[a, b]$. In particular if $K^\bullet \in D^-(R)$ and all its cohomology groups have finite tor dimension then $K^\bullet$ has finite tor dimension.*

Proof. Follows by induction on the length of the finite complex: use Lemma 15.49.4 and the canonical truncations. □

Lemma 15.49.9. *Let $A \to B$ be a ring map. Assume that B is flat as an A-module. Let $K^\bullet$ be a complex of B-modules. Let $a, b \in \mathbf{Z}$. If $K^\bullet$ as a complex of B-modules has tor amplitude in $[a, b]$, then $K^\bullet$ as a complex of A-modules has tor amplitude in $[a, b]$.*

Proof. This is true because $K^\bullet \otimes^{\mathbf{L}}_A M = K^\bullet \otimes^{\mathbf{L}}_B (M \otimes_A B)$ since any projective resolution of $K^\bullet$ as a complex of B-modules is a flat resolution of $K^\bullet$ as a complex of A-modules and can be used to compute $K^\bullet \otimes^{\mathbf{L}}_A M$. □

Lemma 15.49.10. *Let $A \to B$ be a ring map. Assume that B has tor dimension $\leq d$ as an A-module. Let $K^\bullet$ be a complex of B-modules. Let $a, b \in \mathbf{Z}$. If $K^\bullet$ as a complex of B-modules has tor amplitude in $[a, b]$, then $K^\bullet$ as a complex of A-modules has tor amplitude in $[a-d, b]$.*

Proof. Let M be an A-module. Choose a free resolution $F^\bullet \to M$. Then

$$K^\bullet \otimes^{\mathbf{L}}_A M = \mathrm{Tot}(K^\bullet \otimes_A F^\bullet) = \mathrm{Tot}(K^\bullet \otimes_B (F^\bullet \otimes_A B)) = K^\bullet \otimes^{\mathbf{L}}_B (M \otimes^{\mathbf{L}}_A B).$$

By our assumption on B as an A-module we see that $M \otimes^{\mathbf{L}}_A B$ has cohomology only in degrees $-d, -d+1, \ldots, 0$. Because $K^\bullet$ has tor amplitude in $[a, b]$ we see from the spectral sequence in Example 15.46.4 that $K^\bullet \otimes^{\mathbf{L}}_B (M \otimes^{\mathbf{L}}_A B)$ has cohomology only in degrees $[-d+a, b]$ as desired. □

Lemma 15.49.11. *Let $A \to B$ be a ring map. Let $a, b \in \mathbf{Z}$. Let $K^\bullet$ be a complex of A-modules with tor amplitude in $[a, b]$. Then $K^\bullet \otimes^{\mathbf{L}}_A B$ as a complex of B-modules has tor amplitude in $[a, b]$.*

Proof. By Lemma 15.49.3 we can find a quasi-isomorphism $E^\bullet \to K^\bullet$ where $E^\bullet$ is a complex of flat A-modules with $E^i = 0$ for $i \notin [a, b]$. Then $E^\bullet \otimes_A B$ computes $K^\bullet \otimes^{\mathbf{L}}_A B$ by construction and each $E^i \otimes_A B$ is a flat B-module by Algebra, Lemma 10.36.6. Hence we conclude by Lemma 15.49.3. □

Lemma 15.49.12. *Let $A \to B$ be a flat ring map. Let $d \geq 0$. Let M be an A-module of tor dimension $\leq d$. Then $M \otimes_A B$ is a B-module of tor dimension $\leq d$.*

Proof. Immediate consequence of Lemma 15.49.11 and the fact that $M \otimes^{\mathbf{L}}_A B = M \otimes_A B$ because B is flat over A. □

Lemma 15.49.13. *Let R be a ring. Let $f_1, \ldots, f_r \in R$ be elements which generate the unit ideal. Let $a, b \in \mathbf{Z}$. Let $K^\bullet$ be a complex of R-modules. If for each i the complex $K^\bullet \otimes_R R_{f_i}$ has tor amplitude in $[a, b]$, then $K^\bullet$ has tor amplitude in $[a, b]$.*

Proof. Note that $- \otimes_R R_{f_i}$ is an exact functor and that therefore

$$H^i(K^\bullet)_{f_i} = H^i(K^\bullet) \otimes_R R_{f_i} = H^i(K^\bullet \otimes_R R_{f_i}).$$

and similarly for every R-module M we have

$$H^i(K^\bullet \otimes^{\mathbf{L}}_R M)_{f_i} = H^i(K^\bullet \otimes^{\mathbf{L}}_R M) \otimes_R R_{f_i} = H^i(K^\bullet \otimes_R R_{f_i} \otimes^{\mathbf{L}}_{R_{f_i}} M_{f_i}).$$

Hence the result follows from the fact that an R-module N is zero if and only if N_{f_i} is zero for each i, see Algebra, Lemma 10.22.2. □

Lemma 15.49.14. *Let R be a ring. Let $a, b \in \mathbf{Z}$. Let $K^\bullet$ be a complex of R-modules. Let $R \to R'$ be a faithfully flat ring map. If the complex $K^\bullet \otimes_R R'$ has tor amplitude in $[a, b]$, then $K^\bullet$ has tor amplitude in $[a, b]$.*

Proof. Let M be an R-module. Since $R \to R'$ is flat we see that

$$(M \otimes^{\mathbf{L}}_R K^\bullet) \otimes_R R' = ((M \otimes_R R') \otimes^{\mathbf{L}}_{R'} (K^\bullet \otimes_R R')$$

and taking cohomology commutes with tensoring with R'. Hence $\mathrm{Tor}_i^R(M, K^\bullet) = \mathrm{Tor}_i^{R'}(M \otimes_R R', K^\bullet \otimes_R R')$. Since $R \to R'$ is faithfully flat, the vanishing of $\mathrm{Tor}_i^{R'}(M \otimes_R R', K^\bullet \otimes_R R')$ for $i \notin [a, b]$ implies the same thing for $\mathrm{Tor}_i^R(M, K^\bullet)$. □

Lemma 15.49.15. *Let R be a ring of finite global dimension d. Then*

1. *every module has finite tor dimension $\leq d$,*
2. *a complex of R-modules $K^\bullet$ with $H^i(K^\bullet) \neq 0$ only if $i \in [a, b]$ has tor amplitude in $[a - d, b]$, and*
3. *a complex of R-modules $K^\bullet$ has finite tor dimension if and only if $K^\bullet \in D^b(R)$.*

Proof. The assumption on R means that every module has a finite projective resolution of length at most d, in particular every module has finite tor dimension. The second statement follows from Lemma 15.49.8 and the definitions. The third statement is a rephrasing of the second. □

15.50. Perfect complexes

A perfect complex is a pseudo-coherent complex of finite tor dimension. But we can also define the directly as follows.

Definition 15.50.1. Let R be a ring. Denote $D(R)$ the derived category of the abelian category of R-modules.

1. An object K of $D(R)$ is *perfect* if it is quasi-isomorphic to a bounded complex of finite projective R-modules.
2. An R-module M is *perfect* if $M[0]$ is a perfect object in $D(R)$.

For example, over a Noetherian ring a finite module is perfect if and only if it has finite projective dimension, see Lemma 15.50.3 and Algebra, Definition 10.104.2.

Lemma 15.50.2. *Let $K^\bullet$ be an object of $D(R)$. The following are equivalent*

1. *$K^\bullet$ is perfect, and*
2. *$K^\bullet$ is pseudo-coherent and has finite tor dimension.*

Proof. It is clear that (1) implies (2), see Lemmas 15.48.5 and 15.49.3. Assume (2). Choose a bounded above complex $F^\bullet$ of finite free R-modules and a quasi-isomorphism $F^\bullet \to K^\bullet$. Assume that $K^\bullet$ has tor-amplitude in $[a, b]$. Set $E^\bullet = \tau_{\geq a} F^\bullet$. Note that E^i is finite free except E^a which is a finitely presented R-module. By Lemma 15.49.2 E^a is flat. Hence by Algebra, Lemma 10.73.2 we see that E^a is finite projective. □

Lemma 15.50.3. *Let M be a module over a ring R. The following are equivalent*

1. *M is a perfect module, and*

(2) *there exists a resolution*

$$0 \to F_d \to \ldots \to F_1 \to F_0 \to M \to 0$$

with each F_i a finite projective R-module.

Proof. Assume (2). Then the complex $E^\bullet$ with $E^{-i} = F_i$ is quasi-isomorphic to $M[0]$. Hence M is perfect. Conversely, assume (1). By Lemmas 15.50.2 and 15.48.4 we can find resolution $E^\bullet \to M$ with E^{-i} a finite free R-module. By Lemma 15.49.2 we see that $F_d = \mathrm{Coker}(E^{d-1} \to E^d)$ is flat for some d sufficiently large. By Algebra, Lemma 10.73.2 we see that F_d is finite projective. Hence

$$0 \to F_d \to E^{-d+1} \to \ldots \to E^0 \to M \to 0$$

is the desired resolution. □

Lemma 15.50.4. *Let R be a ring. Let $(K^\bullet, L^\bullet, M^\bullet, f, g, h)$ be a distinguished triangle in $D(R)$. If two out of three of $K^\bullet, L^\bullet, M^\bullet$ are perfect then the third is also perfect.*

Proof. Combine Lemmas 15.50.2, 15.48.6, and 15.49.4. □

Lemma 15.50.5. *Let R be a ring. If $K^\bullet \oplus L^\bullet$ is perfect, then so are $K^\bullet$ and $L^\bullet$.*

Proof. Follows from Lemmas 15.50.2, 15.48.8, and 15.49.6. □

Lemma 15.50.6. *Let R be a ring. Let $K^\bullet$ be a bounded complex of perfect R-modules. Then $K^\bullet$ is a perfect complex.*

Proof. Follows by induction on the length of the finite complex: use Lemma 15.50.4 and the stupid truncations. □

Lemma 15.50.7. *Let R be a ring. If $K^\bullet \in D^b(R)$ and all its cohomology modules are perfect, then $K^\bullet$ is perfect.*

Proof. Follows by induction on the length of the finite complex: use Lemma 15.50.4 and the canonical truncations. □

Lemma 15.50.8. *Let $A \to B$ be a ring map. Assume that B is perfect as an A-module. Let $K^\bullet$ be a perfect complex of B-modules. Then $K^\bullet$ is perfect as a complex of A-modules.*

Proof. Using Lemma 15.50.2 this translates into the corresponding results for pseudo-coherent modules and modules of finite tor dimension. See Lemma 15.49.10 and Lemma 15.48.11 for those results. □

Lemma 15.50.9. *Let $A \to B$ be a ring map. Let $K^\bullet$ be a perfect complex of A-modules. Then $K^\bullet \otimes^{\mathbf{L}}_A B$ is a perfect complex of B-modules.*

Proof. Using Lemma 15.50.2 this translates into the corresponding results for pseudo-coherent modules and modules of finite tor dimension. See Lemma 15.49.11 and Lemma 15.48.12 for those results. □

Lemma 15.50.10. *Let $A \to B$ be a flat ring map. Let M be a perfect A-module. Then $M \otimes_A B$ is a perfect B-module.*

Proof. By Lemma 15.50.3 the assumption implies that M has a finite resolution $F_\bullet$ by finite projective R-modules. As $A \to B$ is flat the complex $F_\bullet \otimes_A B$ is a finite length resolution of $M \otimes_A B$ by finite projective modules over B. Hence $M \otimes_A B$ is perfect. □

Lemma 15.50.11. *Let R be a ring. Let $f_1, \ldots, f_r \in R$ be elements which generate the unit ideal. Let $K^\bullet$ be a complex of R-modules. If for each i the complex $K^\bullet \otimes_R R_{f_i}$ is perfect, then $K^\bullet$ is perfect.*

Proof. Using Lemma 15.50.2 this translates into the corresponding results for pseudo-coherent modules and modules of finite tor dimension. See Lemma 15.49.13 and Lemma 15.48.14 for those results. □

Lemma 15.50.12. *Let R be a ring. Let $a, b \in \mathbf{Z}$. Let $K^\bullet$ be a complex of R-modules. Let $R \to R'$ be a faithfully flat ring map. If the complex $K^\bullet \otimes_R R'$ has tor amplitude in $[a, b]$, then $K^\bullet$ has tor amplitude in $[a, b]$.*

Proof. Using Lemma 15.50.2 this translates into the corresponding results for pseudo-coherent modules and modules of finite tor dimension. See Lemma 15.49.14 and Lemma 15.48.15 for those results. □

Lemma 15.50.13. *Let R be a regular ring of finite dimension. Then*

1. *an R-module is perfect if and only if it is a finite R-module, and*
2. *a complex of R-modules $K^\bullet$ is perfect if and only if $K^\bullet \in D^b(R)$ and each $H^i(K^\bullet)$ is a finite R-module.*

Proof. By Algebra, Lemma 10.105.7 the assumption on R means that R has finite global dimension. Hence every module has finite tor dimension, see Lemma 15.49.15. On the other hand, as R is Noetherian, a module is pseudo-coherent if and only if it is finite, see Lemma 15.48.16. This proves part (1).

Let $K^\bullet$ be a complex of R-modules. If $K^\bullet$ is perfect, then it is in $D^b(R)$ and it is quasi-isomorphic to a finite complex of finite projective R-modules so certainly each $H^i(K^\bullet)$ is a finite R-module (as R is Noetherian). Conversely, suppose that $K^\bullet$ is in $D^b(R)$ and each $H^i(K^\bullet)$ is a finite R-module. Then by (1) each $H^i(K^\bullet)$ is a perfect R-module, whence $K^\bullet$ is perfect by Lemma 15.50.7 □

Lemma 15.50.14. *Let R be a ring. Let $\mathfrak{p} \subset R$ be a prime ideal. Let $i \in \mathbf{Z}$. Let $K^\bullet$ be a pseudo-coherent complex of R-modules such that $H^i(K^\bullet \otimes^{\mathbf{L}}_R \kappa(\mathfrak{p})) = 0$. Then there exists an $f \in R$, $f \notin \mathfrak{p}$ such that*

$$K^\bullet \otimes_R R_f = \tau_{\geq i+1} K^\bullet \otimes_R R_f \oplus \tau_{\leq i-1} K^\bullet \otimes_R R_f$$

in $D(R_f)$ with $\tau_{\geq i+1} K^\bullet \otimes_R R_f$ a perfect complex with tor amplitude in $[i+1, j]$ for some $j \in \mathbf{Z}$.

Proof. We may assume that $K^\bullet$ is a bounded above complex of finite free R-modules. Let us inspect what is happening in degree i:

$$\ldots \to K^{i-2} \to R^{\oplus l} \to R^{\oplus m} \to R^{\oplus n} \to K^{i+2} \to \ldots$$

Let A be the $m \times l$ matrix corresponding to $K^{i-1} \to K^i$ and let B be the $n \times m$ matrix corresponding to $K^i \to K^{i+1}$. The assumption is that $A \bmod \mathfrak{p}$ has rank r and that $B \bmod \mathfrak{p}$ has rank $m - r$. In other words, there is some $r \times r$ minor a of A which is not in $\mathfrak{p}$ and there is some $(m-r) \times (m-r)$-minor b of B which is not in $\mathfrak{p}$. Set $f = ab$. Then after inverting f we can find direct sum decompositions

$K^{i-1} = R^{\oplus l-r} \oplus R^{\oplus r}$, $K^i = R^{\oplus r} \oplus R^{\oplus m-r}$, $K^{i+1} = R^{\oplus m-r} \oplus R^{\oplus n-m+r}$ such that the module map $K^{i-1} \to K^i$ kills of $R^{\oplus l-r}$ and induces an isomorphism of $R^{\oplus r}$ onto the corresponding summand of K^i and such that the module map $K^i \to K^{i+1}$ kills of $R^{\oplus r}$ and induces an isomorphism of $R^{\oplus m-r}$ onto the corresponding summand of K^{i+1}. Thus $K^\bullet$ becomes quasi-isomorphic to

$$\ldots \to K^{i-2} \to R^{\oplus l-r} \to 0 \to R^{\oplus n-m+r} \to K^{i+2} \to \ldots$$

and everything is clear. □

Lemma 15.50.15. *Let R be a ring. Let $a, b \in \mathbf{Z}$. Let $K^\bullet$ be a pseudo-coherent complex of R-modules. The following are equivalent*

(1) *$K^\bullet$ is perfect with tor amplitude in $[a, b]$,*
(2) *for every prime $\mathfrak{p}$ we have $H^i(K^\bullet \otimes^{\mathbf{L}}_R \kappa(\mathfrak{p})) = 0$ for all $i \notin [a, b]$, and*
(3) *for every maximal ideal $\mathfrak{m}$ we have $H^i(K^\bullet \otimes^{\mathbf{L}}_R \kappa(\mathfrak{m})) = 0$ for all $i \notin [a, b]$.*

Proof. We omit the proof of the implications (1) $\Rightarrow$ (2) $\Rightarrow$ (3). Assume (3). Let $i \in \mathbf{Z}$ with $i \notin [a, b]$. By Lemma 15.50.14 we see that the assumption implies that $H^i(K^\bullet)_\mathfrak{m} = 0$ for all maximal ideals of R. Hence $H^i(K^\bullet) = 0$, see Algebra, Lemma 10.22.1. Moreover, Lemma 15.50.14 now also implies that for every maximal ideal $\mathfrak{m}$ there exists an element $f \in R$, $f \notin \mathfrak{m}$ such that $K^\bullet \otimes_R R_f$ is perfect with tor amplitude in $[a, b]$. Hence we conclude by appealing to Lemmas 15.50.11 and 15.49.13. □

Lemma 15.50.16. *Let R be a ring. Let $K^\bullet$ be a pseudo-coherent complex of R-modules. The following are equivalent*

(1) *$K^\bullet$ is perfect,*
(2) *for every prime ideal $\mathfrak{p}$ the complex $K^\bullet \otimes_R R_\mathfrak{p}$ is perfect,*
(3) *for every prime $\mathfrak{p}$ we have $H^i(K^\bullet \otimes^{\mathbf{L}}_R \kappa(\mathfrak{p})) = 0$ for all $i \ll 0$,*
(4) *for every maximal ideal $\mathfrak{m}$ the complex $K^\bullet \otimes_R R_\mathfrak{m}$ is perfect,*
(5) *for every maximal ideal $\mathfrak{m}$ we have $H^i(K^\bullet \otimes^{\mathbf{L}}_R \kappa(\mathfrak{m})) = 0$ for all $i \ll 0$.*

Proof. Assume (5). Pick a maximal ideal $\mathfrak{m}$ of R. By Lemma 15.50.14 we see that the assumption implies that $K^\bullet \otimes_R R_f$ is a perfect complex for some $f \in R$, $f \notin \mathfrak{m}$. Since $\mathrm{Spec}(R)$ is quasi-compact we conclude that $K^\bullet$ is perfect by Lemmas 15.50.11. The proof of the other implications is omitted. □

The following lemma useful in order to find perfect complexes over a polynomial ring $B = A[x_1, \ldots, x_d]$.

Lemma 15.50.17. *Let $A \to B$ be a ring map. Let $a, b \in \mathbf{Z}$. Let $d \geq 0$. Let $K^\bullet$ be a complex of B-modules. Assume*

(1) *the ring map $A \to B$ is flat,*
(2) *for every prime $\mathfrak{p} \subset A$ the ring $B \otimes_A \kappa(\mathfrak{p})$ has finite global dimension $\leq d$,*
(3) *$K^\bullet$ is pseudo-coherent as a complex of B-modules, and*
(4) *$K^\bullet$ has tor amplitude in $[a, b]$ as a complex of A-modules.*

Then $K^\bullet$ is perfect as a complex of B-modules with tor amplitude in $[a - d, b]$.

Proof. We may assume that $K^\bullet$ is a bounded above complex of finite free B-modules. In particular, $K^\bullet$ is flat as a complex of A-modules and $K^\bullet \otimes_A M = K^\bullet \otimes^{\mathbf{L}}_A M$ for any A-module M. For every prime $\mathfrak{p}$ of A the complex

$$K^\bullet \otimes_A \kappa(\mathfrak{p})$$

is a bounded above complex of finite free modules over $B \otimes_A \kappa(\mathfrak{p})$ with vanishing H^i except for $i \in [a, b]$. As $B \otimes_A \kappa(\mathfrak{p})$ has global dimension d we see from Lemma 15.49.15 that $K^\bullet \otimes_A \kappa(\mathfrak{p})$ has tor amplitude in $[a-d, b]$. Let $\mathfrak{q}$ be a prime of B lying over $\mathfrak{p}$. Since $K^\bullet \otimes_A \kappa(\mathfrak{p})$ is a bounded above complex of free $B \otimes_A \kappa(\mathfrak{q})$-modules we see that

$$\begin{aligned} K^\bullet \otimes_B^{\mathbf{L}} \kappa(\mathfrak{q}) &= K^\bullet \otimes_B \kappa(\mathfrak{q}) \\ &= (K^\bullet \otimes_A \kappa(\mathfrak{p})) \otimes_{B \otimes_A \kappa(\mathfrak{q})} \kappa(\mathfrak{q}) \\ &= (K^\bullet \otimes_A \kappa(\mathfrak{p})) \otimes^{\mathbf{L}}_{B \otimes_A \kappa(\mathfrak{q})} \kappa(\mathfrak{q}) \end{aligned}$$

Hence the arguments above imply that $H^i(K^\bullet \otimes_B^{\mathbf{L}} \kappa(\mathfrak{q})) = 0$ for $i \notin [a-d, b]$. We conclude by Lemma 15.50.15. □

The following lemma is a local version of Lemma 15.50.17. It can be used to find perfect complexes over reglar local rings.

Lemma 15.50.18. *Let $A \to B$ be a local ring homomorphism. Let $a, b \in \mathbf{Z}$. Let $d \geq 0$. Let $K^\bullet$ be a complex of B-modules. Assume*

1. *the ring map $A \to B$ is flat,*
2. *the ring $B/\mathfrak{m}_A B$ is regular of dimension d,*
3. *$K^\bullet$ is pseudo-coherent as a complex of B-modules, and*
4. *$K^\bullet$ has tor amplitude in $[a, b]$ as a complex of A-modules, in fact it suffices if $H^i(K^\bullet \otimes^{\mathbf{L}}_A \kappa(\mathfrak{m}_A))$ is nonzero only for $i \in [a, b]$.*

Then $K^\bullet$ is perfect as a complex of B-modules with tor amplitude in $[a-d, b]$.

Proof. By (3) we may assume that $K^\bullet$ is a bounded above complex of finite free B-modules. We compute

$$\begin{aligned} K^\bullet \otimes_B^{\mathbf{L}} \kappa(\mathfrak{m}_B) &= K^\bullet \otimes_B \kappa(\mathfrak{m}_B) \\ &= (K^\bullet \otimes_A \kappa(\mathfrak{m}_A)) \otimes_{B/\mathfrak{m}_A B} \kappa(\mathfrak{m}_B) \\ &= (K^\bullet \otimes_A \kappa(\mathfrak{m}_A)) \otimes^{\mathbf{L}}_{B/\mathfrak{m}_A B} \kappa(\mathfrak{m}_B) \end{aligned}$$

The first equality because $K^\bullet$ is a bounded above complex of flat B-modules. The second equality follows from basic properties of the tensor product. The third equality holds because $K^\bullet \otimes_A \kappa(\mathfrak{m}_A) = K^\bullet/\mathfrak{m}_A K^\bullet$ is a bounded above complex of flat $B/\mathfrak{m}_A B$-modules. Since $K^\bullet$ is a bounded above complex of flat A-modules by (1), the cohomology modules H^i of the complex $K^\bullet \otimes_A \kappa(\mathfrak{m}_A)$ are nonzero only for $i \in [a, b]$ by assumption (4). Thus the spectral sequence of Example 15.46.1 and the fact that $B/\mathfrak{m}_A B$ has finite global dimension d (by (2) and Algebra, Proposition 10.105.1) shows that $H^j(K^\bullet \otimes_B^{\mathbf{L}} \kappa(\mathfrak{m}_B))$ is zero for $j \notin [a-d, b]$. This finishes the proof by Lemma 15.50.15. □

Lemma 15.50.19. *Let $K^\bullet$ be a perfect complex over a ring A. There exists a perfect complex $E^\bullet$ such that we have functorial isomorphisms*

$$H^0(K^\bullet \otimes^{\mathbf{L}}_A L^\bullet) = Ext^0_A(E^\bullet, L^\bullet)$$

for $L^\bullet \in D(A)$.

Proof. We may assume that $K^\bullet$ is a finite complex of finite projective A-modules. The cohomology group on the left is simply $H^0(\mathrm{Tot}(K^\bullet \otimes_A L^\bullet))$. Set $E^\bullet = \mathrm{Hom}_A(K^\bullet, A)$, i.e., $E^n = \mathrm{Hom}_A(K^{-n}, A)$ with differentials the transpose of the

differentials of $K^\bullet$. Observe that $E^\bullet$ is a finite complex of finite projective A-modules. The group on the right is $\mathrm{Mor}_{K(\mathrm{Mod}_A)}(E^\bullet, L^\bullet)$ by Derived Categories, Lemma 13.18.8 and the definition of Ext groups, see Derived Categories, Section 13.26. By definition this is the cohomology of

$$\prod\nolimits_n \mathrm{Hom}_A(E^n, L^{n-1}) \to \prod\nolimits_n \mathrm{Hom}_A(E^n, L^n) \to \prod\nolimits_n \mathrm{Hom}_A(E^n, L^{n+1})$$

Using $\mathrm{Hom}_A(E^n, L) = K^{-n} \otimes_A L$ as K^{-n} is finite projective, we see that the cohomology groups are the same. □

15.51. Characterizing perfect complexes

Let R be a ring. Recall that $D(R)$ has direct sums which are given simply by taking direct sums of complexes, see Derived Categories, Lemma 13.30.3. We will use this in the lemmas of this section without further mention.

Lemma 15.51.1. *Let R be a ring. Let $K \in D(R)$ be an object such that for every countable set of objects $E_n \in D(R)$ the canonical map*

$$\bigoplus \mathrm{Hom}_{D(R)}(K, E_n) \longrightarrow \mathrm{Hom}_{D(R)}(K, \bigoplus E_n)$$

is a bijection. Then, given any system $L_n^\bullet$ of complexes over $\mathbf{N}$ we have that

$$\mathrm{colim}\, \mathrm{Hom}_{D(R)}(K, L_n^\bullet) \longrightarrow \mathrm{Hom}_{D(R)}(K, L^\bullet)$$

is a bijection, where $L^\bullet$ is the termwise colimit, i.e., $L^m = \mathrm{colim}\, L_n^m$ for all $m \in \mathbf{Z}$.

Proof. Consider the short exact sequence of complexes

$$0 \to \bigoplus L_n^\bullet \to \bigoplus L_n^\bullet \to L^\bullet \to 0$$

where the first map is given by $1 - t_n$ in degree n where $t_n : L_n^\bullet \to L_{n+1}^\bullet$ is the transition map. By Derived Categories, Lemma 13.11.1 this is a distinguished triangle in $D(R)$. Apply the homological functor $\mathrm{Hom}_{D(R)}(K, -)$, see Derived Categories, Lemma 13.4.2. Thus a long exact cohomology sequence

$$\begin{array}{ccccc} & & \ldots & \longrightarrow & \mathrm{Hom}_{D(R)}(K, \mathrm{colim}\, L_n^\bullet[-1]) \\ & \swarrow & & & \\ \mathrm{Hom}_{D(R)}(K, \bigoplus L_n^\bullet) & \longrightarrow & \mathrm{Hom}_{D(R)}(K, \bigoplus L_n^\bullet) & \longrightarrow & \mathrm{Hom}_{D(R)}(K, \mathrm{colim}\, L_n^\bullet) \\ & \swarrow & & & \\ \mathrm{Hom}_{D(R)}(K, \bigoplus L_n^\bullet[1]) & \longrightarrow & \ldots & & \end{array}$$

Since we have assumed that $\mathrm{Hom}_{D(R)}(K, \bigoplus L_n^\bullet)$ is equal to $\bigoplus \mathrm{Hom}_{D(R)}(K, L_n^\bullet)$ we see that the first map on every row of the diagram is injective (by the explicit description of this map as the sum of the maps induced by $1 - t_n$). Hence we conclude that $\mathrm{Hom}_{D(R)}(K, \mathrm{colim}\, L_n^\bullet)$ is the cokernel of the first map of the middle row in the diagram above which is what we had to show. □

The following proposition, characterizing perfect complexes as the compact objects (Derived Categories, Definition 13.32.1) of the derived category, shows up in various places. See for example [**Ric89b**, proof of Proposition 6.3] (this treats the bounded case), [**TT90**, Theorem 2.4.3] (the statement doesn't match exactly), and [**BN93**, Proposition 6.4] (watch out for horrendous notational conventions).

Proposition 15.51.2. *Let R be a ring. For an object K of $D(R)$ the following are equivalent*

(1) *K is perfect, and*
(2) *K is a compact object of $D(R)$.*

Proof. Assume K is perfect, i.e., K is quasi-isomorphic to a bounded complex $P^\bullet$ of finite projective modules, see Definition 15.50.1. If E_i is represented by the complex $E_i^\bullet$, then $\bigoplus E_i$ is represented by the complex whose degree n term is $\bigoplus E_i^n$. On the other hand, as P^n is projective for all n we have $\mathrm{Hom}_{D(R)}(P^\bullet, K^\bullet) = \mathrm{Hom}_{K(R)}(P^\bullet, K^\bullet)$ for every complex of R-modules $K^\bullet$, see Derived Categories, Lemma 13.18.8. Thus $\mathrm{Hom}_{D(R)}(P^\bullet, E^\bullet)$ is the cohomology of the complex

$$\prod \mathrm{Hom}_R(P^n, E^{n-1}) \to \prod \mathrm{Hom}_R(P^n, E^n) \to \prod \mathrm{Hom}_R(P^n, E^{n+1}).$$

Since $P^\bullet$ is bounded we see that we may replace the $\prod$ signs by $\bigoplus$ signs in the complex above. Since each P^n is a finite R-module we see that $\mathrm{Hom}_R(P^n, \bigoplus_i E_i^m) = \bigoplus_i \mathrm{Hom}_R(P^n, E_i^m)$ for all n, m. Combining these remarks we see that the map of Derived Categories, Definition 13.32.1 is a bijection.

Conversely, assume K is compact. Represent K by a complex $K^\bullet$ and consider the map

$$K^\bullet \longrightarrow \bigoplus\nolimits_{n \geq 0} \tau_{\geq n} K^\bullet$$

where we have used the canonical truncations, see Homology, Section 12.13. This makes sense as in each degree the direct sum on the right is finite. By assumption this map factors through a finite direct sum. We conclude that $K \to \tau_{\geq n} K$ is zero for at least one n, i.e., K is in $D^-(R)$.

Since $K \in D^-(R)$ and since every R-module is a quotient of a free module, we may represent K by a bounded above complex $K^\bullet$ of free R-modules, see Derived Categories, Lemma 13.15.5. Note that we have

$$K^\bullet = \bigcup\nolimits_{n \leq 0} \sigma_{\geq n} K^\bullet$$

where we have used the stupid truncations, see Homology, Section 12.13. Hence by Lemma 15.51.1 we see that $1 : K^\bullet \to K^\bullet$ factors through $\sigma_{\geq n} K^\bullet \to K^\bullet$ in $D(R)$. Thus we see that $1 : K^\bullet \to K^\bullet$ factors as

$$K^\bullet \xrightarrow{\varphi} L^\bullet \xrightarrow{\psi} K^\bullet$$

in $D(R)$ for some complex $L^\bullet$ which is bounded and whose terms are free R-modules. Say $L^i = 0$ for $i \notin [a, b]$. Fix a, b from now on. Let c be the largest integer $\leq b + 1$ such that we can find a factorization of $1_{K^\bullet}$ as above with L^i is finite free for $i < c$. We will show by induction that $c = b + 1$. Namely, write $L^c = \bigoplus_{\lambda \in \Lambda} R$. Since L^{c-1} is finite free we can find a finite subset $\Lambda' \subset \Lambda$ such that $L^{c-1} \to L^c$ factors through $\bigoplus_{\lambda \in \Lambda'} R \subset L^c$. Consider the map of complexes

$$\pi : L^\bullet \longrightarrow (\bigoplus\nolimits_{\lambda \in \Lambda \setminus \Lambda'} R)[-i]$$

given by the projection onto the factors corresponding to $\Lambda \setminus \Lambda'$ in degree i. By our assumption on K we see that, after possibly replacing Λ' by a larger finite subset, we may assume that $\pi \circ \varphi = 0$ in $D(R)$. Let $(L')^\bullet \subset L^\bullet$ be the kernel of π. Since π is surjective we get a short exact sequence of complexes, which gives a distinguished triangle in $D(R)$ (see Derived Categories, Lemma 13.11.1).

Since $\mathrm{Hom}_{D(R)}(K, -)$ is homological (see Derived Categories, Lemma 13.4.2) and $\pi \circ \varphi = 0$, we can find a morphism $\varphi' : K^\bullet \to (L')^\bullet$ in $D(R)$ whose composition with $(L')^\bullet \to L^\bullet$ gives φ. Setting ψ' equal to the composition of ψ with $(L')^\bullet \to L^\bullet$ we obtain a new factorization. Since $(L')^\bullet$ agrees with $L^\bullet$ except in degree c and since $(L')^c = \bigoplus_{\lambda \in \Lambda'} R$ the induction step is proved.

The conclusion of the discussion of the preceding paragraph is that $1_K : K \to K$ factors as

$$K \xrightarrow{\varphi} L \xrightarrow{\psi} K$$

in $D(R)$ where L can be represented by a finite complex of free R-modules. In particular we see that L is perfect. Note that $e = \varphi \circ \psi \in \mathrm{End}_{D(R)}(L)$ is an idempotent. By Derived Categories, Lemma 13.4.12 we see that $L = \mathrm{Ker}(e) \oplus \mathrm{Ker}(1 - e)$. The map $\varphi : K \to L$ induces an isomorphism with $\mathrm{Ker}(1 - e)$ in $D(R)$. Hence we finally conclude that K is perfect by Lemma 15.50.5. □

Lemma 15.51.3. *Let R be a ring. Let $I \subset R$ be an ideal. Let $\overline{P}^\bullet$ be a complex of R/I-modules. Let K be an object of $D(R)$. Assume that*

(1) *$\overline{P}^\bullet$ is a bounded above complex of projective R/I-modules,*
(2) *$K \otimes^{\mathbf{L}}_R R/I$ is represented by $\overline{P}^\bullet$ in $D(R/I)$, and*
(3) *I is a nilpotent ideal.*

Then there exists a bounded above complex $P^\bullet$ of projective R-modules representing K in $D(R)$ such that $P^\bullet \otimes_R R/I$ is isomorphic to $\overline{P}^\bullet$.

Proof. Assumption (3) means that $I^n = 0$ for some n. The result holds if $n = 1$. Below we will prove the result holds for $n = 2$. This implies for $n > 2$ that $\overline{P}^\bullet$ lifts to a complex of projective R/I^2-modules representing $K \otimes^{\mathbf{L}}_R R/I^2$. Then the result will follow by induction on n as $(I^2)^{\lceil n/2 \rceil} = 0$. Thus we may and do assume $I^2 = 0$.

Let us represent K by a K-flat complex $K^\bullet$ with all K^n flat, see Lemma 15.43.10. Then we have a short exact sequence of complexes

$$0 \to IK^\bullet \to K^\bullet \to K^\bullet/IK^\bullet \to 0$$

and $K^\bullet/IK^\bullet$ represents $K \otimes^{\mathbf{L}}_R R/I$ by construction of the derived tensor product. By flatness we see that

$$IK^\bullet = K^\bullet \otimes_R I = K^\bullet \otimes_R R/I \otimes_{R/I} I$$

represents $K \otimes^{\mathbf{L}}_R R/I \otimes^{\mathbf{L}}_{R/I} I$ because $K^\bullet \otimes_R R/I$ is a K-flat complex over R/I, see Lemma 15.43.5. By assumption (2) and Derived Categories, Lemma 13.18.8 there is a quasi-isomorphism $\overline{\alpha} : \overline{P}^\bullet \to K^\bullet/IK^\bullet$. Since $\overline{P}^\bullet$ is K-flat we see that the induced map $\overline{P}^\bullet \otimes_{R/I} I \to IK^\bullet$ is a quasi-isomorphism too.

Suppose that $\overline{P}^i = 0$ for $i > b$. We will show by induction on a that we can find a complex $P^a \to P^{a+1} \to \ldots \to P^b$ lifting $\overline{P}^a \to \overline{P}^{a+1} \to \ldots \to \overline{P}^b$ and a map of complexes

$$\alpha : (P^a \to P^{a+1} \to \ldots \to P^b) \to K$$

lifting $\overline{\alpha}$ such that $H^i(\alpha)$ is an isomorphism for $i > a$ and surjective for $i = a$. If $a > b$ then we just take $P^i = 0$ for all i. Suppose we have a solution for some a. Consider the cone $C^\bullet$ of α, see Derived Categories, Definition 13.8.1. Note that $C^\bullet/IC^\bullet$ and $IC^\bullet$ are the cones of the induced maps $\overline{\alpha}$ and $\alpha \otimes \mathrm{id}_I$. Then $H^i(C^\bullet)$, $H^i(C^\bullet/IC^\bullet)$, and $H^i(IC^\bullet)$ are 0 for $i \geq a$ (use long exact cohomology sequence

associated to the cones; details omitted), in particular $H^{a-1}(C^\bullet) \to H^{a-1}(C^\bullet/IC^\bullet)$ is surjective. The given maps $\overline{P}^{a-1} \to K^{a-1}/IK^{a-1}$ and $\overline{P}^{a-1} \to \overline{P}^a$ induce a map

$$\overline{P}^{a-1}[a-1] \to C^\bullet/IC^\bullet$$

in $D(R/I)$. Choose a lift P^{a-1} of $\overline{P}^{a-1}$, see Algebra, Lemma 10.72.4. Because $H^{a-1}(C^\bullet) \to H^{a-1}(C^\bullet/IC^\bullet)$ is surjective, see above, we can lift the displayed map to a map

$$P^{a-1}[a-1] \to C^\bullet$$

in $D(R)$. The composition with the map $C^\bullet \to (P^a \to \ldots \to P^b)[1]$ determines a map $P^{a-1} \to P^a$ whose composition with $P^a \to P^{a+1}$ is zero. On the other hand, we have $C^{a-1} = K^{a-1} \oplus P^a$ and a calculation shows that the resulting map $P^{a-1} \to K^{a-1}$ gives the desired extension of α. □

Lemma 15.51.4. *Let R be a ring. Let $I \subset R$ be an ideal. Let K be an object of $D(R)$. Assume that*

(1) *$K \otimes^{\mathbf{L}}_R R/I$ is perfect in $D(R/I)$, and*
(2) *I is a nilpotent ideal.*

Then K is perfect in $D(R)$.

Proof. Choose a finite complex $\overline{P}^\bullet$ of finite projective R/I-modules representing $K \otimes^{\mathbf{L}}_R R/I$, see Definition 15.50.1. By Lemma 15.51.3 there exists a complex $P^\bullet$ of projective R-modules representing K such that $\overline{P}^\bullet = P^\bullet/IP^\bullet$. It follows from Nakayama's lemma (Algebra, Lemma 10.18.1) that $P^\bullet$ is a finite complex of finite projective R-modules. □

Lemma 15.51.5. *Let R be a ring. Let $I, J \subset R$ be ideals. Let K be an object of $D(R)$. Assume that*

(1) *$K \otimes^{\mathbf{L}}_R R/I$ is perfect in $D(R/I)$, and*
(2) *$K \otimes^{\mathbf{L}}_R R/J$ is perfect in $D(R/J)$.*

Then $K \otimes^{\mathbf{L}}_R R/IJ$ is perfect in $D(R/IJ)$.

Proof. It is clear that we may assume replace R by R/IJ and K by $K \otimes^{\mathbf{L}}_R R/IJ$. Then $R \to R/(I \cap J)$ is a surjection whose kernel has square zero. Hence by Lemma 15.51.4 it suffices to prove that $K \otimes^{\mathbf{L}}_R R/(I \cap J)$ is perfect. Thus we may assume that $I \cap J = 0$.

We prove the lemma in case $I \cap J = 0$. First, we may represent K by a K-flat complex $K^\bullet$ with all K^n flat, see Lemma 15.43.10. Then we see that we have a short exact sequence of complexes

$$0 \to K^\bullet \to K^\bullet/IK^\bullet \oplus K^\bullet/JK^\bullet \to K^\bullet/(I+J)K^\bullet \to 0$$

Note that $K^\bullet/IK^\bullet$ represents $K \otimes^{\mathbf{L}}_R R/I$ by construction of the derived tensor product. Similarly for $K^\bullet/JK^\bullet$ and $K^\bullet/(I+J)K^\bullet$. Note that $K^\bullet/(I+J)K^\bullet$ is a perfect complex of $R/(I+J)$-modules, see Lemma 15.50.9. Hence the complexes $K^\bullet/IK^\bullet$, and $K^\bullet/JK^\bullet$ and $K^\bullet/(I+J)K^\bullet$. have finitely many nonzero cohomology groups (since a perfect complex has finite Tor-amplitude, see Lemma 15.50.2). We conclude that $K \in D^b(R)$ by the long exact cohomology sequence associated to short exact sequence of complexes displayed above. In particular we assume $K^\bullet$ is a bounded above complex of free R-modules (see Derived Categories, Lemma 13.15.5).

We will now show that K is perfect using the criterion of Proposition 15.51.2. Thus we let $E_j \in D(R)$ be a family of objects parametrized by a set J. We choose complexes $E_j^\bullet$ with flat terms representing E_j, see for example Lemma 15.43.10. It is clear that

$$0 \to E_j^\bullet \to E_j^\bullet/IE_j^\bullet \oplus E_j^\bullet/JE_j^\bullet \to E_j^\bullet/(I+J)E_j^\bullet \to 0$$

is a short exact sequence of complexes. Taking direct sums we obtain a similar short exact sequence

$$0 \to \bigoplus E_j^\bullet \to \bigoplus E_j^\bullet/IE_j^\bullet \oplus E_j^\bullet/JE_j^\bullet \to \bigoplus E_j^\bullet/(I+J)E_j^\bullet \to 0$$

(Note that $- \otimes_R R/I$ commutes with direct sums.) This short exact sequence determines a distinguished triangle in $D(R)$, see Derived Categories, Lemma 13.11.1. Apply the homological functor $\mathrm{Hom}_{D(R)}(K, -)$ (see Derived Categories, Lemma 13.4.2) to get a commutative diagram

$$\begin{array}{ccc}
\bigoplus \mathrm{Hom}_{D(R)}(K^\bullet, E_j^\bullet/(I+J))[-1] & \longrightarrow & \mathrm{Hom}_{D(R)}(K^\bullet, \bigoplus E_j^\bullet/(I+J))[-1] \\
\downarrow & & \downarrow \\
\bigoplus \mathrm{Hom}_{D(R)}(K^\bullet, E_j^\bullet/I \oplus E_j^\bullet/J)[-1] & \longrightarrow & \mathrm{Hom}_{D(R)}(K^\bullet, \bigoplus E_j^\bullet/I \oplus E_j^\bullet/J)[-1] \\
\downarrow & & \downarrow \\
\bigoplus \mathrm{Hom}_{D(R)}(K^\bullet, E_j^\bullet) & \longrightarrow & \mathrm{Hom}_{D(R)}(K^\bullet, \bigoplus E_j^\bullet) \\
\downarrow & & \downarrow \\
\bigoplus \mathrm{Hom}_{D(R)}(K^\bullet, E_j^\bullet/I \oplus E_j^\bullet/J) & \longrightarrow & \mathrm{Hom}_{D(R)}(K^\bullet, \bigoplus E_j^\bullet/I \oplus E_j^\bullet/J) \\
\downarrow & & \downarrow \\
\bigoplus \mathrm{Hom}_{D(R)}(K^\bullet, E_j^\bullet/(I+J)) & \longrightarrow & \mathrm{Hom}_{D(R)}(K^\bullet, \bigoplus E_j^\bullet/(I+J))
\end{array}$$

with exact columns. It is clear that, for any complex $E^\bullet$ of R-modules we have

$$\begin{aligned}
\mathrm{Hom}_{D(R)}(K^\bullet, E^\bullet/I) &= \mathrm{Hom}_{K(R)}(K^\bullet, E^\bullet/I) \\
&= \mathrm{Hom}_{K(R/I)}(K^\bullet/IK^\bullet, E^\bullet/I) \\
&= \mathrm{Hom}_{D(R/I)}(K^\bullet/IK^\bullet, E^\bullet/I)
\end{aligned}$$

and similarly for when dividing by J or $I+J$, see Derived Categories, Lemma 13.18.8. Derived Categories. Thus all the horizontal arrows, except for possibly the middle one, are isomorphisms as the complexes $K^\bullet/IK^\bullet$, $K^\bullet/JK^\bullet$, $K^\bullet/(I+J)K^\bullet$ are perfect complexes of R/I, R/J, $R/(I+J)$-modules, see Proposition 15.51.2. It follows from the 5-lemma (Homology, Lemma 12.5.20) that the middle map is an isomorphism and the lemma follows by Proposition 15.51.2. □

15.52. Relatively finitely presented modules

Let R be a ring. Let $A \to B$ be a finite map of finite type R-algebras. Let M be a finite B-module. In this case it is **not true** that

$$M \text{ of finite presentation over } B \Leftrightarrow M \text{ of finite presentation over } A$$

A counter example is $R = k[x_1, x_2, x_3, \ldots]$, $A = R$, $B = R/(x_i)$, and $M = B$. To "fix" this we introduce a relative notion of finite presentation.

Lemma 15.52.1. *Let $R \to A$ be a ring map of finite type. Let M be an A-module. The following are equivalent*

(1) *for some presentation $\alpha : R[x_1, \ldots, x_n] \to A$ the module M is a finitely presented $R[x_1, \ldots, x_n]$-module,*
(2) *for all presentations $\alpha : R[x_1, \ldots, x_n] \to A$ the module M is a finitely presented $R[x_1, \ldots, x_n]$-module, and*
(3) *for any surjection $A' \to A$ where A' is a finitely presented R-algebra, the module M is finitely presented as A'-module.*

In this case M is a finitely presented A-module.

Proof. If $\alpha : R[x_1, \ldots, x_n] \to A$ and $\beta : R[y_1, \ldots, y_m] \to A$ are presentations. Choose $f_j \in R[x_1, \ldots, x_n]$ with $\alpha(f_j) = \beta(y_j)$ and $g_i \in R[y_1, \ldots, y_m]$ with $\beta(g_i) = \alpha(x_i)$. Then we get a commutative diagram

$$\begin{array}{ccc} R[x_1, \ldots, x_n, y_1, \ldots, y_m] & \xrightarrow[y_j \mapsto f_j]{} & R[x_1, \ldots, x_n] \\ \downarrow{\scriptstyle x_i \mapsto g_i} & & \downarrow \\ R[y_1, \ldots, y_m] & \longrightarrow & A \end{array}$$

Hence the equivalence of (1) and (2) follows by applying Algebra, Lemmas 10.6.4 and 10.7.4. The equivalence of (2) and (3) follows by choosing a presentation $A' = R[x_1, \ldots, x_n]/(f_1, \ldots, f_m)$ and using Algebra, Lemma 10.7.4 to show that M is finitely presented as A'-module if and only if M is finitely presented as a $R[x_1, \ldots, x_n]$-module. □

Definition 15.52.2. Let $R \to A$ be a finite type ring map. Let M be an A-module. We say M is an A-module *finitely presented relative to R* if the equivalent conditions of Lemma 15.52.1 hold.

Note that if $R \to A$ is of finite presentation, then M is an A-module finitely presented relative to R if and only if M is a finitely presented A-module. It is equally clear that A as an A-module is finitely presented relative to R if and only if A is of finite presentation over R. If R is Noetherian the notion is uninteresting. Now we can formulate the result we were looking for.

Lemma 15.52.3. *Let R be a ring. Let $A \to B$ be a finite map of finite type R-algebras. Let M be a B-module. Then M is an A-module finitely presented relative to R if and only if M is a B-module finitely presented relative to R.*

Proof. Choose a surjection $R[x_1, \ldots, x_n] \to A$. Choose $y_1, \ldots, y_m \in B$ which generate B over A. As $A \to B$ is finite each y_i satisfies a monic equation with coefficients in A. Hence we can find monic polynomials $P_j(T) \in R[x_1, \ldots, x_n][T]$ such that $P_j(y_j) = 0$ in B. Then we get a commutative diagram

$$\begin{array}{ccc} R[x_1, \ldots, x_n] & \longrightarrow & R[x_1, \ldots, x_n, y_1, \ldots, y_m]/(P_j(y_j)) \\ \downarrow & & \downarrow \\ A & \longrightarrow & B \end{array}$$

Since the top arrow is a finite and finitely presented ring map we conclude by Algebra, Lemma 10.7.4 and the definition. □

With this result in hand we see that the relative notion makes sense and behaves well with regards to finite maps of rings of finite type over R. It is also stable under localization, stable under base change, and "glues" well.

Lemma 15.52.4. *Let R be a ring, $f \in R$ an element, $R_f \to A$ is a finite type ring map, $g \in A$, and M an A-module. If M of finite presentation relative to R_f, then M_g is an A_g-module of finite presentation relative to R.*

Proof. Choose a presentation $R_f[x_1, \ldots, x_n] \to A$. We write $R_f = R[x_0]/(fx_0 - 1)$. Consider the presentation $R[x_0, x_1, \ldots, x_n, x_{n+1}] \to A_g$ which extends the given map, maps x_0 to the image of $1/f$, and maps x_{n+1} to $1/g$. Choose $g' \in R[x_0, x_1, \ldots, x_n]$ which maps to g (this is possible). Suppose that

$$R_f[x_1, \ldots, x_n]^{\oplus s} \to R_f[x_1, \ldots, x_n]^{\oplus t} \to M \to 0$$

is a presentation of M given by a matrix (h_{ij}). Pick $h'_{ij} \in R[x_0, x_1, \ldots, x_n]$ which map to h_{ij}. Then

$$R[x_0, x_1, \ldots, x_n, x_{n+1}]^{\oplus s+2t} \to R[x_0, x_1, \ldots, x_n, x_{n+1}]^{\oplus t} \to M_g \to 0$$

is a presentation of M_f. Here the $t \times (s + 2t)$ matrix defining the map has a first $t \times s$ block consisting of the matrix h'_{ij}, a second $t \times t$ block which is $(x_0 f -)I_t$, and a third block which is $(x_{n+1}g' - 1)I_t$. □

Lemma 15.52.5. *Let $R \to A$ be a finite type ring map. Let M be an A-module finitely presented relative to R. For any ring map $R \to R'$ the $A \otimes_R R'$-module*

$$M \otimes_A A' = M \otimes_R R'$$

is finitely presented relative to R'.

Proof. Choose a surjection $R[x_1, \ldots, x_n] \to A$. Choose a presentation

$$R[x_1, \ldots, x_n]^{\oplus s} \to R[x_1, \ldots, x_n]^{\oplus t} \to M \to 0$$

Then

$$R'[x_1, \ldots, x_n]^{\oplus s} \to R'[x_1, \ldots, x_n]^{\oplus t} \to M \otimes_R R' \to 0$$

is a presentation of the base change and we win. □

Lemma 15.52.6. *Let $R \to A$ be a finite type ring map. Let M be an A-module finitely presented relative to R. Let $A \to A'$ be a ring map of finite presentation. The A'-module $M \otimes_A A'$ is finitely presented relative to R.*

Proof. Choose a surjection $R[x_1, \ldots, x_n] \to A$. Choose a presentation $A' = A[y_1, \ldots, y_m]/(g_1, \ldots, g_l)$. Pick $g'_i \in R[x_1, \ldots, x_n, y_1, \ldots, y_m]$ mapping to g_i. Say

$$R[x_1, \ldots, x_n]^{\oplus s} \to R[x_1, \ldots, x_n]^{\oplus t} \to M \to 0$$

is a presentation of M given by a matrix (h_{ij}). Then

$$R[x_1, \ldots, x_n, y_1, \ldots, y_m]^{\oplus s+tl} \to R[x_0, x_1, \ldots, x_n, y_1, \ldots, y_m]^{\oplus t} \to M \otimes_A A' \to 0$$

is a presentation of $M \otimes_A A'$. Here the $t \times (s + lt)$ matrix defining the map has a first $t \times s$ block consisting of the matrix h_{ij}, followed by l blocks of size $t \times t$ which are $g'_i I_t$. □

Lemma 15.52.7. *Let $R \to A \to B$ be finite type ring maps. Let M be a B-module. If M is finitely presented relative to A and A is of finite presentation over R, then M is finitely presented relative to R.*

Proof. Choose a surjection $A[x_1, \ldots, x_n] \to B$. Choose a presentation

$$A[x_1, \ldots, x_n]^{\oplus s} \to A[x_1, \ldots, x_n]^{\oplus t} \to M \to 0$$

given by a matrix (h_{ij}). Choose a presentation

$$A = R[y_1, \ldots, y_m]/(g_1, \ldots, g_u).$$

Choose $h'_{ij} \in R[y_1, \ldots, y_m, x_1, \ldots, x_n]$ mapping to h_{ij}. Then we obtain the presentation

$$R[y_1, \ldots, y_m, x_1, \ldots, x_n]^{\oplus s+tu} \to R[y_1, \ldots, y_m, x_1, \ldots, x_n]^{\oplus t} \to M \to 0$$

where the $t \times (s + tu)$-matrix is given by a first $t \times s$ block consisting of h'_{ij} followed by u blocks of size $t \times t$ given by $g_i I_t$, $i = 1, \ldots, u$. □

Lemma 15.52.8. *Let $R \to A$ be a finite type ring map. Let M be an A-module. Let $f_1, \ldots, f_r \in A$ generate the unit ideal. The following are equivalent*

(1) *each M_{f_i} is finitely presented relative to R, and*
(2) *M is finitely presented relative to R.*

Proof. The implication (2) $\Rightarrow$ (1) is in Lemma 15.52.4. Assume (1). Write $1 = \sum f_i g_i$ in A. Choose a surjection $R[x_1, \ldots, x_n, y_1, \ldots, y_r, z_1, \ldots, z_r] \to A$. such that y_i maps to f_i and z_i maps to g_i. Then we see that there exists a surjection

$$P = R[x_1, \ldots, x_n, y_1, \ldots, y_r, z_1, \ldots, z_r]/(\sum y_i z_i - 1) \longrightarrow A.$$

By Lemma 15.52.1 we see that M_{f_i} is a finitely presented A_{f_i}-module, hence by Algebra, Lemma 10.22.2 we see that M is a finitely presented A-module. Hence M is a finite P-module (with P as above). Choose a surjection $P^{\oplus t} \to M$. We have to show that the kernel K of this map is a finite P-module. Since P_{y_i} surjects onto A_{f_i} we see by Lemma 15.52.1 and Algebra, Lemma 10.5.3 that the localization K_{y_i} is a finitely generated P_{y_i}-module. Choose elements $k_{i,j} \in K$, $i = 1, \ldots, r$, $j = 1, \ldots, s_i$ such that the images of $k_{i,j}$ in K_{y_i} generate. Set $K' \subset K$ equal to the P-module generated by the elements $k_{i,j}$. Then K/K' is a module whose localization at y_i is zero for all i. Since $(y_1, \ldots, y_r) = P$ we see that $K/K' = 0$ as desired. □

Lemma 15.52.9. *Let $R \to A$ be a finite type ring map. Let $0 \to M' \to M \to M'' \to 0$ be a short exact sequence of A-modules.*

(1) *If M', M'' are finitely presented relative to R, then so is M.*
(2) *If M' is a finite type A-module and M is finitely presented relative to R, then M'' is finitely presented relative to R.*

Proof. Follows immediately from Algebra, Lemma 10.5.3. □

Lemma 15.52.10. *Let $R \to A$ be a finite type ring map. Let M, M' be A-modules. If $M \oplus M'$ is finitely presented relative to R, then so are M and M'.*

Proof. Omitted. □

15.53. Relatively pseudo-coherent modules

This section is the analogue of Section 15.52 for pseudo-coherence.

Lemma 15.53.1. *Let R be a ring. Let $K^\bullet$ be an object of $D^-(R)$. Consider the R-algebra map $R[x] \to R$ which maps x to zero. Then*

$$K^\bullet \otimes^{\mathbf{L}}_{R[x]} R \cong K^\bullet \oplus K^\bullet[1]$$

in $D(R)$.

Proof. Choose a projective resolution $P^\bullet \to K^\bullet$ over R. Then

$$P^\bullet \otimes_R R[x] \xrightarrow{x} P^\bullet \otimes_R R[x]$$

is a double complex of projective $R[x]$-modules whose associated total complex is quasi-isomorphic to $P^\bullet$. Hence

$$\begin{aligned} K^\bullet \otimes^{\mathbf{L}}_{R[x]} R &\cong \mathrm{Tot}(P^\bullet \otimes_R R[x] \xrightarrow{x} P^\bullet \otimes_R R[x]) \otimes_{R[x]} R = \mathrm{Tot}(P^\bullet \xrightarrow{0} P^\bullet) \\ &= P^\bullet \oplus P^\bullet[1] \cong K^\bullet \oplus K^\bullet[1] \end{aligned}$$

as desired. □

Lemma 15.53.2. *Let R be a ring and $K^\bullet$ a complex of R-modules. Let $m \in \mathbf{Z}$. Consider the R-algebra map $R[x] \to R$ which maps x to zero. Then $K^\bullet$ is m-pseudo-coherent as a complex of R-modules if and only if $K^\bullet$ is m-pseudo-coherent as a complex of $R[x]$-modules.*

Proof. This is a special case of Lemma 15.48.11. We also prove it in another way as follows.

Note that $0 \to R[x] \to R[x] \to R \to 0$ is exact. Hence R is pseudo-coherent as an $R[x]$-module. Thus one implication of the lemma follows from Lemma 15.48.11. To prove the other implication, assume that $K^\bullet$ is m-pseudo-coherent as a complex of $R[x]$-modules. By Lemma 15.48.12 we see that $K^\bullet \otimes^{\mathbf{L}}_{R[x]} R$ is m-pseudo-coherent as a complex of R-modules. By Lemma 15.53.1 we see that $K^\bullet \oplus K^\bullet[1]$ is m-pseudo-coherent as a complex of R-modules. Finally, we conclude that $K^\bullet$ is m-pseudo-coherent as a complex of R-modules from Lemma 15.48.8. □

Lemma 15.53.3. *Let $R \to A$ be a ring map of finite type. Let $K^\bullet$ be a complex of A-modules. Let $m \in \mathbf{Z}$. The following are equivalent*

1. *for some presentation $\alpha : R[x_1, \ldots, x_n] \to A$ the complex $K^\bullet$ is an m-pseudo-coherent complex of $R[x_1, \ldots, x_n]$-modules,*
2. *for all presentations $\alpha : R[x_1, \ldots, x_n] \to A$ the complex $K^\bullet$ is an m-pseudo-coherent complex of $R[x_1, \ldots, x_n]$-modules.*

In particular the same equivalence holds for pseudo-coherence.

Proof. If $\alpha : R[x_1, \ldots, x_n] \to A$ and $\beta : R[y_1, \ldots, y_m] \to A$ are presentations. Choose $f_j \in R[x_1, \ldots, x_n]$ with $\alpha(f_j) = \beta(y_j)$ and $g_i \in R[y_1, \ldots, y_m]$ with $\beta(g_i) = \alpha(x_i)$. Then we get a commutative diagram

$$\begin{array}{ccc} R[x_1, \ldots, x_n, y_1, \ldots, y_m] & \xrightarrow[y_j \mapsto f_j]{} & R[x_1, \ldots, x_n] \\ \downarrow{\scriptstyle x_i \mapsto g_i} & & \downarrow \\ R[y_1, \ldots, y_m] & \longrightarrow & A \end{array}$$

After a change of coordinates the ring homomorphism $R[x_1, \ldots, x_n, y_1, \ldots, y_m] \to R[x_1, \ldots, x_n]$ is isomorphic to the ring homomorphism which maps each y_i to zero. Similarly for the left vertical map in the diagram. Hence, by induction on the

number of variables this lemma follows from Lemma 15.53.2. The pseudo-coherent case follows from this and Lemma 15.48.5. □

Definition 15.53.4. Let $R \to A$ be a finite type ring map. Let $K^\bullet$ be a complex of A-modules. Let M be an A-module. Let $m \in \mathbf{Z}$.

(1) We say $K^\bullet$ is *m-pseudo-coherent relative to R* if the equivalent conditions of Lemma 15.53.3 hold.
(2) We say $K^\bullet$ is *pseudo-coherent relative to R* if $K^\bullet$ is m-pseudo-coherent relative to R for all $m \in \mathbf{Z}$.
(3) We say M is *m-pseudo-coherent relative to R* if $M[0]$ is m-pseudo-coherent.
(4) We say M is *pseudo-coherent relative to R* if $M[0]$ is pseudo-coherent relative to R.

Part (2) means that $K^\bullet$ is pseudo-coherent as a complex of $R[x_1, \ldots, x_n]$-modules for any surjection $R[y_1, \ldots, y_m] \to A$, see Lemma 15.48.5. This definition has the following pleasing property.

Lemma 15.53.5. *Let R be a ring. Let $A \to B$ be a finite map of finite type R-algebras. Let $m \in \mathbf{Z}$. Let $K^\bullet$ be a complex of B-modules. Then $K^\bullet$ is m-pseudo-coherent (resp. pseudo-coherent) relative to R if and only if $K^\bullet$ seen as a complex of A-modules is m-pseudo-coherent (pseudo-coherent) relative to R.*

Proof. Choose a surjection $R[x_1, \ldots, x_n] \to A$. Choose $y_1, \ldots, y_m \in B$ which generate B over A. As $A \to B$ is finite each y_i satisfies a monic equation with coefficients in A. Hence we can find monic polynomials $P_j(T) \in R[x_1, \ldots, x_n][T]$ such that $P_j(y_j) = 0$ in B. Then we get a commutative diagram

$$\begin{array}{ccc} & & R[x_1, \ldots, x_n, y_1, \ldots, y_m] \\ & & \downarrow \\ R[x_1, \ldots, x_n] & \longrightarrow & R[x_1, \ldots, x_n, y_1, \ldots, y_m]/(P_j(y_j)) \\ \downarrow & & \downarrow \\ A & \longrightarrow & B \end{array}$$

The top horizontal arrow and the top right vertical arrow satisfy the assumptions of Lemma 15.48.11. Hence $K^\bullet$ is m-pseudo-coherent (resp. pseudo-coherent) as a complex of $R[x_1, \ldots, x_n]$-modules if and only if $K^\bullet$ is m-pseudo-coherent (resp. pseudo-coherent) as a complex of $R[x_1, \ldots, x_n, y_1, \ldots, y_m]$-modules. □

Lemma 15.53.6. *Let R be a ring. Let $R \to A$ be a finite type ring map. Let $m \in \mathbf{Z}$. Let $(K^\bullet, L^\bullet, M^\bullet, f, g, h)$ be a distinguished triangle in $D(A)$.*

(1) *If $K^\bullet$ is $(m+1)$-pseudo-coherent relative to R and $L^\bullet$ is m-pseudo-coherent relative to R then $M^\bullet$ is m-pseudo-coherent relative to R.*
(2) *If $K^\bullet, M^\bullet$ are m-pseudo-coherent relative to R, then $L^\bullet$ is m-pseudo-coherent relative to R.*
(3) *If $L^\bullet$ is $(m+1)$-pseudo-coherent relative to R and $M^\bullet$ is m-pseudo-coherent relative to R, then $K^\bullet$ is $(m+1)$-pseudo-coherent relative to R.*

Moreover, if two out of three of $K^\bullet, L^\bullet, M^\bullet$ are pseudo-coherent relative to R, the so is the third.

Proof. Follows immediately from Lemma 15.48.2 and the definitions. □

Lemma 15.53.7. *Let $R \to A$ be a finite type ring map. Let M be an A-module. Then*

(1) *M is 0-pseudo-coherent relative to R if and only if M is a finite type A-module,*
(2) *M is (-1)-pseudo-coherent relative to R if and only if M is a finitely presented relative to R,*
(3) *M is $(-d)$-pseudo-coherent relative to R if and only if for every surjection $R[x_1, \ldots, x_n] \to A$ there exists a resolution*

$$R[x_1, \ldots, x_n]^{\oplus a_d} \to R[x_1, \ldots, x_n]^{\oplus a_{d-1}} \to \ldots \to R[x_1, \ldots, x_n]^{\oplus a_0} \to M \to 0$$

of length d, and
(4) *M is pseudo-coherent relative to R if and only if for every presentation $R[x_1, \ldots, x_n] \to A$ there exists an infinite resolution*

$$\ldots \to R[x_1, \ldots, x_n]^{\oplus a_1} \to R[x_1, \ldots, x_n]^{\oplus a_0} \to M \to 0$$

by finite free $R[x_1, \ldots, x_n]$-modules.

Proof. Follows immediately from Lemma 15.48.4 and the definitions. □

Lemma 15.53.8. *Let $R \to A$ be a finite type ring map. Let $m \in \mathbf{Z}$. Let $K^\bullet, L^\bullet \in D(A)$. If $K^\bullet \oplus L^\bullet$ is m-pseudo-coherent (resp. pseudo-coherent) relative to R so are $K^\bullet$ and $L^\bullet$.*

Proof. Immediate from Lemma 15.48.8 and the definitions. □

Lemma 15.53.9. *Let $R \to A$ be a finite type ring map. Let $m \in \mathbf{Z}$. Let $K^\bullet$ be a bounded above complex of A-modules such that K^i is $(m - i)$-pseudo-coherent relative to R for all i. Then $K^\bullet$ is m-pseudo-coherent relative to R. In particular, if $K^\bullet$ is a bounded above complex of A-modules pseudo-coherent relative to R, then $K^\bullet$ is pseudo-coherent relative to R.*

Proof. Immediate from Lemma 15.48.9 and the definitions. □

Lemma 15.53.10. *Let $R \to A$ be a finite type ring map. Let $m \in \mathbf{Z}$. Let $K^\bullet \in D^-(A)$ such that $H^i(K^\bullet)$ is $(m-i)$-pseudo-coherent (resp. pseudo-coherent) relative to R for all i. Then $K^\bullet$ is m-pseudo-coherent (resp. pseudo-coherent) relative to R.*

Proof. Immediate from Lemma 15.48.10 and the definitions. □

Lemma 15.53.11. *Let R be a ring, $f \in R$ an element, $R_f \to A$ is a finite type ring map, $g \in A$, and $K^\bullet$ a complex of A-modules. If $K^\bullet$ is m-pseudo-coherent (resp. pseudo-coherent) relative to R_f, then $K^\bullet \otimes_A A_g$ is m-pseudo-coherent (resp. pseudo-coherent) relative to R.*

Proof. First we show that $K^\bullet$ is m-pseudo-coherent relative to R. Namely, suppose $R_f[x_1, \ldots, x_n] \to A$ is surjective. Write $R_f = R[x_0]/(fx_0 - 1)$. Then $R[x_0, x_1, \ldots, x_n] \to A$ is surjective, and $R_f[x_1, \ldots, x_n]$ is pseudo-coherent as an $R[x_0, \ldots, x_n]$-module. Hence by Lemma 15.48.11 we see that $K^\bullet$ is m-pseudo-coherent as a complex of $R[x_0, x_1, \ldots, x_n]$-modules.

Choose an element $g' \in R[x_0, x_1, \ldots, x_n]$ which maps to $g \in A$. By Lemma 15.48.12 we see that

$$\begin{aligned} K^\bullet \otimes^{\mathbf{L}}_{R[x_0,x_1,\ldots,x_n]} R[x_0, x_1, \ldots, x_n, \frac{1}{g'}] &= K^\bullet \otimes_{R[x_0,x_1,\ldots,x_n]} R[x_0, x_1, \ldots, x_n, \frac{1}{g'}] \\ &= K^\bullet \otimes_A A_f \end{aligned}$$

is m-pseudo-coherent as a complex of $R[x_0, x_1, \ldots, x_n, \frac{1}{g'}]$-modules. write

$$R[x_0, x_1, \ldots, x_n, \frac{1}{g'}] = R[x_0, \ldots, x_n, x_{n+1}]/(x_{n+1}g' - 1).$$

As $R[x_0, x_1, \ldots, x_n, \frac{1}{g'}]$ is pseudo-coherent as a $R[x_0, \ldots, x_n, x_{n+1}]$-module we conclude (see Lemma 15.48.11) that $K^\bullet \otimes_A A_g$ is m-pseudo-coherent as a complex of $R[x_0, \ldots, x_n, x_{n+1}]$-modules as desired. □

Lemma 15.53.12. *Let $R \to A$ be a finite type ring map. Let $m \in \mathbf{Z}$. Let $K^\bullet$ be a complex of A-modules which is m-pseudo-coherent (resp. pseudo-coherent) relative to R. Let $R \to R'$ be a ring map such that A and R' are Tor independent over R. Set $A' = A \otimes_R R'$. Then $K^\bullet \otimes^{\mathbf{L}}_A A'$ is is m-pseudo-coherent (resp. pseudo-coherent) relative to R'.*

Proof. Choose a surjection $R[x_1, \ldots, x_n] \to A$. Note that

$$K^\bullet \otimes^{\mathbf{L}}_A A' = K^\bullet \otimes^{\mathbf{L}}_R R' = K^\bullet \otimes^{\mathbf{L}}_{R[x_1,\ldots,x_n]} R'[x_1, \ldots, x_n]$$

by Lemma 15.45.2 applied twice. Hence we win by Lemma 15.48.12. □

Lemma 15.53.13. *Let $R \to A \to B$ be finite type ring maps. Let $m \in \mathbf{Z}$. Let $K^\bullet$ be a complex of A-modules. Assume B as a B-module is pseudo-coherent relative to A. If $K^\bullet$ is m-pseudo-coherent (resp. pseudo-coherent) relative to R, then $K^\bullet \otimes^{\mathbf{L}}_A B$ is m-pseudo-coherent (resp. pseudo-coherent) relative to R.*

Proof. Choose a surjection $A[y_1, \ldots, y_m] \to B$. Choose a surjection $R[x_1, \ldots, x_n] \to A$. Combined we get a surjection $R[x_1, \ldots, x_n, y_1, \ldots y_m] \to B$. Choose a resolution $E^\bullet \to B$ of B by a complex of finite free $A[y_1, \ldots, y_n]$-modules (which is possible by our assumption on the ring map $A \to B$). We may assume that $K^\bullet$ is a bounded above complex of flat A-modules. Then

$$\begin{aligned} K^\bullet \otimes^{\mathbf{L}}_A B &= \mathrm{Tot}(K^\bullet \otimes_A B[0]) \\ &= \mathrm{Tot}(K^\bullet \otimes_A A[y_1, \ldots, y_m] \otimes_{A[y_1,\ldots,y_m]} B[0]) \\ &\cong \mathrm{Tot}\left((K^\bullet \otimes_A A[y_1, \ldots, y_m]) \otimes_{A[y_1,\ldots,y_m]} E^\bullet\right) \\ &= \mathrm{Tot}(K^\bullet \otimes_A E^\bullet) \end{aligned}$$

in $D(A[y_1, \ldots, y_m])$. The quasi-isomorphism $\cong$ comes from an application of Lemma 15.43.8. Thus we have to show that $\mathrm{Tot}(K^\bullet \otimes_A E^\bullet)$ is m-pseudo-coherent as a complex of $R[x_1, \ldots, x_n, y_1, \ldots y_m]$-modules. Note that $\mathrm{Tot}(K^\bullet \otimes_A E^\bullet)$ has a filtration by subcomplexes with successive quotients the complexes $K^\bullet \otimes_A E^i[-i]$. Note that for $i \ll 0$ the complexes $K^\bullet \otimes_A E^i[-i]$ have zero cohomology in degrees $\leq m$ and hence are m-pseudo-coherent (over any ring). Hence, applying Lemma 15.53.6 and induction, it suffices to show that $K^\bullet \otimes_A E^i[-i]$ is pseudo-coherent relative to R for all i. Note that $E^i = 0$ for $i > 0$. Since also E^i is finite free this reduces to proving that $K^\bullet \otimes_A A[y_1, \ldots, y_m]$ is m-pseudo-coherent relative to R which follows from Lemma 15.53.12 for instance. □

Lemma 15.53.14. *Let $R \to A \to B$ be finite type ring maps. Let $m \in \mathbf{Z}$. Let M be an A-module. Assume B is flat over A and B as a B-module is pseudo-coherent relative to A. If M is m-pseudo-coherent (resp. pseudo-coherent) relative to R, then $M \otimes_A B$ is m-pseudo-coherent (resp. pseudo-coherent) relative to R.*

Proof. Immediate from Lemma 15.53.13. □

Lemma 15.53.15. *Let R be a ring. Let $A \to B$ be a map of finite type R-algebras. Let $m \in \mathbf{Z}$. Let $K^\bullet$ be a complex of B-modules. Assume A is pseudo-coherent relative to R. Then the following are equivalent*

1. *$K^\bullet$ is m-pseudo-coherent (resp. pseudo-coherent) relative to A, and*
2. *$K^\bullet$ is m-pseudo-coherent (resp. pseudo-coherent) relative to R.*

Proof. Choose a surjection $R[x_1, \ldots, x_n] \to A$. Choose a surjection $A[y_1, \ldots, y_m] \to B$. Then we get a surjection

$$R[x_1, \ldots, x_n, y_1, \ldots, y_m] \to A[y_1, \ldots, y_m]$$

which is a flat base change of $R[x_1, \ldots, x_n] \to A$. By assumption A is a pseudo-coherent module over $R[x_1, \ldots, x_n]$ hence by Lemma 15.48.13 we see that $A[y_1, \ldots, y_m]$ is pseudo-coherent over $R[x_1, \ldots, x_n, y_1, \ldots, y_m]$. Thus the lemma follows from Lemma 15.48.11 and the definitions. □

Lemma 15.53.16. *Let $R \to A$ be a finite type ring map. Let $K^\bullet$ be a complex of A-modules. Let $m \in \mathbf{Z}$. Let $f_1, \ldots, f_r \in A$ generate the unit ideal. The following are equivalent*

1. *each $K^\bullet \otimes_A A_{f_i}$ is m-pseudo-coherent relative to R, and*
2. *$K^\bullet$ is m-pseudo-coherent relative to R.*

The same equivalence holds for pseudo-coherence.

Proof. The implication (2) $\Rightarrow$ (1) is in Lemma 15.53.11. Assume (1). Write $1 = \sum f_i g_i$ in A. Choose a surjection $R[x_1, \ldots, x_n, y_1, \ldots, y_r, z_1, \ldots, z_r] \to A$. such that y_i maps to f_i and z_i maps to g_i. Then we see that there exists a surjection

$$P = R[x_1, \ldots, x_n, y_1, \ldots, y_r, z_1, \ldots, z_r]/(\sum y_i z_i - 1) \longrightarrow A.$$

Note that P is pseudo-coherent as an $R[x_1, \ldots, x_n, y_1, \ldots, y_r, z_1, \ldots, z_r]$-module and that $P[1/y_i]$ is pseudo-coherent as an $R[x_1, \ldots, x_n, y_1, \ldots, y_r, z_1, \ldots, z_r, 1/y_i]$-module. Hence by Lemma 15.48.11 we see that $K^\bullet \otimes_A A_{f_i}$ is an m-pseudo-coherent complex of $P[1/y_i]$-modules for each i. Thus by Lemma 15.48.14 we see that $K^\bullet$ is pseudo-coherent as a complex of P-modules, and Lemma 15.48.11 shows that $K^\bullet$ is pseudo-coherent as a complex of $R[x_1, \ldots, x_n, y_1, \ldots, y_r, z_1, \ldots, z_r]$-modules. □

Lemma 15.53.17. *Let R be a Noetherian ring. Let $R \to A$ be a finite type ring map. Then*

1. *A complex of A-modules $K^\bullet$ is m-pseudo-coherent relative to R if and only if $K^\bullet \in D^-(A)$ and $H^i(K^\bullet)$ is a finite A-module for $i \geq m$.*
2. *A complex of A-modules $K^\bullet$ is pseudo-coherent relative to R if and only if $K^\bullet \in D^-(A)$ and $H^i(K^\bullet)$ is a finite A-module for all i.*
3. *An A-module is pseudo-coherent relative to R if and only if it is finite.*

Proof. Immediate consequence of Lemma 15.48.16 and the definitions. □

15.54. Pseudo-coherent and perfect ring maps

We can define these types of ring maps as follows.

Definition 15.54.1. Let $A \to B$ be a ring map.

(1) We say $A \to B$ is a *pseudo-coherent ring map* if it is of finite type and B, as a B-module, is pseudo-coherent relative to A.
(2) We say $A \to B$ is a *perfect ring map* if it is a pseudo-coherent ring map such that B as an A-module has finite tor dimension.

This terminology may be nonstandard. Using Lemma 15.53.7 we see that $A \to B$ is pseudo-coherent if and only if $B = A[x_1, \ldots, x_n]/I$ and B as an $A[x_1, \ldots, x_n]$-module has a resolution by finite free $A[x_1, \ldots, x_n]$-modules. The motivation for the definition of a perfect ring map is Lemma 15.50.2. The following lemmas gives a more useful and intuitive characterization of a perfect ring map.

Lemma 15.54.2. *A ring map $A \to B$ is perfect if and only if $B = A[x_1, \ldots, x_n]/I$ and B as an $A[x_1, \ldots, x_n]$-module has a finite resolution by finite projective $A[x_1, \ldots, x_n]$-modules.*

Proof. If $A \to B$ is perfect, then $B = A[x_1, \ldots, x_n]/I$ and B is pseudo-coherent as an $A[x_1, \ldots, x_n]$-module and has finite tor dimension as an A-module. Hence Lemma 15.50.17 implies that B is perfect as a $A[x_1, \ldots, x_n]$-module, i.e., it has a finite resolution by finite projective $A[x_1, \ldots, x_n]$-modules (Lemma 15.50.3). Conversely, if $B = A[x_1, \ldots, x_n]/I$ and B as an $A[x_1, \ldots, x_n]$-module has a finite resolution by finite projective $A[x_1, \ldots, x_n]$-modules then B is pseudo-coherent as an $A[x_1, \ldots, x_n]$-module, hence $A \to B$ is pseudo-coherent. Moreover, the given resolution over $A[x_1, \ldots, x_n]$ is a finite resolution by flat A-modules and hence B has finite tor dimension as an A-module. □

Lots of the results of the preceding sections can be reformulated in terms of this terminology. We also refer to More on Morphisms, Sections 36.40 and 36.41 for the corresponding discussion concerning morphisms of schemes.

Lemma 15.54.3. *A finite type ring map of Noetherian rings is pseudo-coherent.*

Proof. See Lemma 15.53.17. □

Lemma 15.54.4. *A ring map which is flat and of finite presentation is perfect.*

Proof. Let $A \to B$ be a ring map which is flat and of finite presentation. It is clear that B has finite tor dimension. By Algebra, Lemma 10.154.1 there exists a finite type $\mathbf{Z}$-algebra $A_0 \subset A$ and a flat finite type ring map $A_0 \to B_0$ such that $B = B_0 \otimes_{A_0} A$. By Lemma 15.53.17 we see that $A_0 \to B_0$ is pseudo-coherent. As $A_0 \to B_0$ is flat we see that B_0 and A are tor independent over A_0, hence we may use Lemma 15.53.12 to conclude that $A \to B$ is pseudo-coherent. □

Lemma 15.54.5. *Let $A \to B$ be a finite type ring map with A a regular ring of finite dimension. Then $A \to B$ is perfect.*

Proof. By Algebra, Lemma 10.105.7 the assumption on A means that A has finite global dimension. Hence every module has finite tor dimension, see Lemma 15.49.15, in particular B does. By Lemma 15.54.3 the map is pseudo-coherent. □

Lemma 15.54.6. *A local complete intersection homomorphism is perfect.*

Proof. Let $A \to B$ he a local complete intersection homomorphism. By Definition 15.22.2 this means that $B = A[x_1, \ldots, x_n]/I$ where I is a Koszul ideal in $A[x_1, \ldots, x_n]$. By Lemmas 15.54.2 and 15.50.3 it suffices to show that I is a perfect module over $A[x_1, \ldots, x_n]$. By Lemma 15.50.11 this is a local question. Hence we may assume that I is generated by a Koszul-regular sequence (by Definition 15.21.1). Of course this means that I has a finite free resolution and we win. □

15.55. Rlim of abelian groups and modules

We briefly discuss $R\lim$ on abelian groups and modules. In this section we will denote $Ab(\mathbf{N})$ the abelian category of inverse systems of abelian groups. This makes sense as an inverse system of abelian groups is the same thing as a sheaf of groups on the category $\mathbf{N}$ (with a unique morphism $i \to j$ if $i \leq j$), see Remark 15.55.4. Many of the arguments in this section duplicate the arguments used to construct the cohomological machinery for modules on ringed sites.

Lemma 15.55.1. *The functor* $\lim : Ab(\mathbf{N}) \to Ab$ *has a right derived functor*

$$R\lim : D(Ab(\mathbf{N})) \longrightarrow D(Ab) \tag{15.55.1.1}$$

As usual we set $R^p \lim(K) = H^p(R\lim(K))$. *Moreover, we have*

(1) *for any* (A_n) *in* $Ab(\mathbf{N})$ *we have* $R^p \lim A_n = 0$ *for* $p > 1$,
(2) *the object* $R\lim A_n$ *of* $D(Ab)$ *is represented by the complex*
$$\prod A_n \to \prod A_n, \quad (x_n) \mapsto (x_n - f_{n+1}(x_{n+1}))$$
sitting in degrees 0 *and* 1,
(3) *if* (A_n) *is ML, then* $R^1 \lim A_n = 0$, *i.e.,* (A_n) *is right acyclic for* lim,
(4) *every* $K^\bullet \in D(Ab(\mathbf{N}))$ *is quasi-isomorphic to a complex whose terms are right acyclic for* lim, *and*
(5) *if each* $K^p = (K^p_n)$ *is right acyclic for* lim, *i.e., of* $R^1 \lim_n K^p_n = 0$, *then* $R\lim K$ *is represented by the complex whose term in degree* p *is* $\lim_n K^p_n$.

Proof. Let (A_n) be an arbitrary inverse system. Let (B_n) be the inverse system with

$$B_n = A_n \oplus A_{n-1} \oplus \ldots \oplus A_1$$

and transition maps given by projections. Let $A_n \to B_n$ be given by $(1, f_n, f_{n-1} \circ f_n, \ldots, f_2 \circ \ldots \circ f_n$ where $f_i : A_i \to A_{i-1}$ are the transition maps. In this way we see that every inverse system is a subobject of a ML system (Homology, Section 12.26). It follows from Derived Categories, Lemma 13.15.6 using Homology, Lemma 12.26.3 that every ML system is right acyclic for lim, i.e., (3) holds. This already implies that RF is defined on $D^+(Ab(\mathbf{N}))$, see Derived Categories, Proposition 13.15.8. Set $C_n = A_{n-1} \oplus \ldots \oplus A_1$ for $n > 1$ and $C_1 = 0$ with transition maps given by projections as well. Then there is a short exact sequence of inverse systems $0 \to (A_n) \to (B_n) \to (C_n) \to 0$ where $B_n \to C_n$ is given by $(x_i) \mapsto (x_i - f_{i+1}(x_{i+1}))$. Since (C_n) is ML as well, we conclude that (2) holds (by proposition reference above) which also implies (1). Finally, this implies by Derived Categories, Lemma 13.29.2 that $R\lim$ is in fact defined on all of $D(Ab(\mathbf{N}))$. In fact, the proof of Derived Categories, Lemma 13.29.2 proceeds by proving assertions (4) and (5). □

We give two simple applications. The first is the "correct" formulation of Homology, Lemma 12.26.7.

Lemma 15.55.2. *Let*

$$(A_n^{-2} \to A_n^{-1} \to A_n^0 \to A_n^1)$$

be an inverse system of complexes of abelian groups and denote $A^{-2} \to A^{-1} \to A^0 \to A^1$ *its limit. Denote* (H_n^{-1}), (H_n^0) *the inverse systems of cohomologies, and denote* H^{-1}, H^0 *the cohomologies of* $A^{-2} \to A^{-1} \to A^0 \to A^1$. *If*

(1) (A_n^{-2}) *and* (A_n^{-1}) *have vanishing* $R^1\lim$,
(2) (H_n^{-1}) *has vanishing* $R^1\lim$,

then $H^0 = \lim H_i^0$.

Proof. Let $K \in D(Ab(\mathbf{N}))$ be the object represented by the system of complexes whose nth constituent is the complex $A_n^{-2} \to A_n^{-1} \to A_n^0 \to A_n^1$. We will compute $H^0(R\lim K)$ using both spectral sequences[7] of Derived Categories, Lemma 13.20.3. The first has E_1-page

$$\begin{array}{cccc} 0 & 0 & R^1\lim A_n^0 & R^1\lim A_n^1 \\ A^{-2} & A^{-1} & A^0 & A^1 \end{array}$$

with horizontal differentials and all higher differentials are zero. The second has E_2 page

$$\begin{array}{cccc} R^1\lim H_n^{-2} & 0 & R^1\lim H_n^0 & R^1\lim H_n^1 \\ \lim H_n^{-2} & \lim H_n^{-1} & \lim H_n^0 & \lim H_n^1 \end{array}$$

and degenerates at this point. The result follows. □

Lemma 15.55.3. *Let* $\mathcal{D}$ *be a triangulated category. Let* (K_n) *be an inverse system of objects of* $\mathcal{D}$. *Let* K *be a derived limit of the system* (K_n). *Then for every* L *in* $\mathcal{D}$ *we have short exact sequences*

$$0 \to R^1\lim \mathrm{Hom}_{\mathcal{D}}(L, K_n[1]) \to \mathrm{Hom}_{\mathcal{D}}(L, K) \to \lim \mathrm{Hom}_{\mathcal{D}}(L, K_n) \to 0$$

Proof. This follows from Derived Categories, Definition 13.30.2 and Lemma 13.4.2, and the description of lim and $R^1\lim$ in Lemma 15.55.1 above. □

Remark 15.55.4 (Rlim as cohomology)**.** Consider the category $\mathbf{N}$ whose objects are natural numbers and whose morphisms are unique arrows $i \to j$ if $j \geq i$. Endow $\mathbf{N}$ with the chaotic topology (Sites, Example 7.6.6) so that a sheaf $\mathcal{F}$ is the same thing as an inverse system

$$\mathcal{F}_1 \leftarrow \mathcal{F}_2 \leftarrow \mathcal{F}_3 \leftarrow \ldots$$

of sets over $\mathbf{N}$. Note that $\Gamma(\mathbf{N}, \mathcal{F}) = \lim \mathcal{F}_n$. For an inverse system of abelian groups $\mathcal{F}_n$ we have

$$R^p\lim \mathcal{F}_n = H^p(\mathbf{N}, \mathcal{F})$$

because both sides are the higher right derived functors of $\mathcal{F} \mapsto \lim \mathcal{F}_n = H^0(\mathbf{N}, \mathcal{F})$. Thus the existence of $R\lim$ also follows from the general material in Cohomology on Sites, Sections 21.3 and 21.19.

Warning. An object of $D(Ab(\mathbf{N}))$ is a complex of inverse systems of abelian groups. You can also think of this as an inverse system $(K_n^\bullet)$ of complexes. However, this is **not** the same thing as an inverse system of objects of $D(Ab)$; we will come back and explain the difference later.

[7] To use these spectral sequences we have to show that $Ab(\mathbf{N})$ has enough injectives. A inverse system (I_n) of abelian groups is injective if and only if each I_n is an injective abelian group and the transition maps are split surjections. Every system embeds in one of these. Details omitted.

The products in the following lemma can be seen as termwise products of complexes or as products in the derived category $D(Ab)$, see Derived Categories, Lemma 13.30.3. This lemma in particular shows the notation in this section is compatible with the notation introduced in Derived Categories, Section 13.30. See Remark 15.55.16 for more explanation.

Lemma 15.55.5. *Let $K = (K_n^\bullet)$ be an object of $D(Ab(\mathbf{N}))$. There exists a canonical distinguished triangle*

$$R\lim K \to \prod\nolimits_n K_n^\bullet \to \prod\nolimits_n K_n^\bullet \to R\lim K[1]$$

in $D(Ab)$ where the middle map fits into the commutative diagrams

$$\begin{array}{ccc} \prod_n K_n^\bullet & \longrightarrow & \prod_n K_n^\bullet \\ \downarrow & & \downarrow \\ K_n^\bullet \oplus K_{n+1}^\bullet & \xrightarrow{1-\pi} & K_n^\bullet \end{array}$$

whose vertical maps are projections and where $\pi : K_{n+1}^\bullet \to K_n^\bullet$ is the transition map of the system.

Proof. Suppose that for each p the inverse system (K_n^p) is right acyclic for lim. By Lemma 15.55.1 this gives a short exact sequence

$$0 \to \lim\nolimits_n K_n^p \to \prod\nolimits_n K_n^p \to \prod\nolimits_n K_n^p \to 0$$

for each p. Since the complex consisting of $\lim_n K_n^p$ computes $R\lim K$ by Lemma 15.55.1 we see that the lemma holds in this case.

Next, assume $K = (K_n^\bullet)$ is general. By Lemma 15.55.1 there is a quasi-isomorphism $K \to L$ in $D(Ab(\mathbf{N}))$ such that (L_n^p) is acyclic for each p. Then $\prod K_n^\bullet$ is quasi-isomorphic to $\prod L_n^\bullet$ as products are exact in Ab, whence the result for L (proved above) implies the result for K. □

Lemma 15.55.6. *With notation as in Lemma 15.55.5 the long exact cohomology sequence associated to the distinguished triangle breaks up into short exact sequences*

$$0 \to R^1 \lim\nolimits_n H^{p-1}(K_n^\bullet) \to H^p(R\lim K) \to \lim\nolimits_n H^p(K_n^\bullet) \to 0$$

Proof. The long exact sequence of the distinguished triangle is

$$\ldots \to H^p(R\lim K) \to \prod\nolimits_n H^p(K_n^\bullet) \to \prod\nolimits_n H^p(K_n^\bullet) \to H^{p+1}(R\lim K) \to \ldots$$

The map in the middle has kernel $\lim_n H^p(K_n^\bullet)$ by its explicit description given in the lemma. The cokernel of this map is $R^1 \lim_n H^p(K_n^\bullet)$ by Lemma 15.55.1. □

Lemma 15.55.7. *Let $E \to D$ be a morphism of $D(Ab(\mathbf{N}))$. Let (E_n), resp. (D_n) be the system of objects of $D(Ab)$ associated to E, resp. D. If $(E_n) \to (D_n)$ is an isomorphism of pro-objects, then $R\lim E \to R\lim D$ is an isomorphism in $D(Ab)$.*

Proof. The assumption in particular implies that the pro-objects $H^p(E_n)$ and $H^p(D_n)$ are isomorphic. By the short exact sequences of Lemma 15.55.6 it suffices to show that given a map $(A_n) \to (B_n)$ of inverse systems of abelian groupsc which induces an isomorphism of pro-objects, then $\lim A_n \cong \lim B_n$ and $R^1 \lim A_n \cong R^1 \lim B_n$.

The assumption implies there are $1 \leq m_1 < m_2 < m_3 < \ldots$ and maps $\varphi_n : B_{m_n} \to A_n$ such that $(\varphi_n) : (B_{m_n}) \to (A_n)$ is a map of systems which is inverse to the given map $\psi = (\psi_n) : (A_n) \to (B_n)$ as a morphism of pro-objects. What this means is that (after possibly replacing m_n by larger integers) we may assume that the compositions $A_{m_n} \to B_{m_n} \to A_n$ and $B_{m_n} \to A_n \to B_n$ are equal to the transition maps of the inverse systems. Now, if $(b_n) \in \lim B_n$ we can set $a_n = \varphi_{m_n}(b_{m_n})$. This defines an inverse $\lim B_n \to \lim A_n$ (computation omitted). Let us use the cokernel of the map

$$\prod B_n \longrightarrow \prod B_n$$

as an avatar of $R^1 \lim B_n$ (Lemma 15.55.1). Any element in this cokernel can be represented by an element (b_i) with $b_i = 0$ if $i \neq m_n$ for some n (computation omitted). We can define a map $R^1 \lim B_n \to R^1 \lim A_n$ by mapping the class of such a special element (b_n) to the class of $(\varphi_n(b_{m_n}))$. We omit the verification this map is inverse to the map $R^1 \lim A_n \to R^1 \lim B_n$. □

Lemma 15.55.8. *Let (A_n) be an inverse system of abelian groups. The following are equivalent*

(1) *(A_n) is zero as a pro-object,*
(2) *$\lim A_n = 0$ and $R^1 \lim A_n = 0$ and the same holds for $\bigoplus_{i \in \mathbf{N}} (A_n)$.*

Proof. It follows from Lemma 15.55.7 that (1) implies (2). For $m \geq n$ let $A_{n,m} = \mathrm{Im}(A_m \to A_n)$ so that $A_n = A_{n,n} \supset A_{n,n+1} \supset \ldots$. Note that (A_n) is zero as a pro-object if and only if for every n there is an $m \geq n$ such that $A_{n,m} = 0$. Note that (A_n) is ML if and only if for every n there is an $m_n \geq n$ such that $A_{n,m} = A_{n,m+1} = \ldots$. In the ML case it is clear that $\lim A_n = 0$ implies that $A_{n,m_n} = 0$ because the maps $A_{n+1,m_{n+1}} \to A_{n,m}$ are surjective.

Assume (A_n) is not zero as a pro-object and not ML. Then we can pick an n and a sequence of integers $n < m_1 < m_2 < \ldots$ and elements $x_i \in A_{m_i}$ whose image $y_i \in A_n$ is not in A_{n,m_i+1}. Set $B_n = \bigoplus_{i \in \mathbf{N}} A_n$. Let $\xi = (\xi_n) \in \prod B_n$ be the element with $\xi_n = 0$ unless $n = m_i$ and $\xi_{m_i} = (0, \ldots, 0, x_i, 0, \ldots)$ with x_i placed in the ith summand. We claim that ξ is not in the image of the map $\prod B_n \to \prod B_n$ of Lemma 15.55.1. This shows that $R^1 \lim B_n$ is nonzero and finishes the proof. Namely, suppose that ξ is the image of $\eta = (z_1, z_2, \ldots)$ with $z_n = \sum z_{n,i} \in \bigoplus_i A_n$. Observe that $x_i = z_{m_i,i} \bmod A_{m_i,m_i+1}$. Then $z_{m_i-1,i}$ is the image of $z_{m_i,i}$ under $A_{m_i} \to A_{m_i-1}$, and so on, and we conclude that $z_{n,i}$ is the image of $z_{m_i,i}$ under $A_{m_i} \to A_n$. We conclude that $z_{n,i}$ is congruent to y_i modulo A_{n,m_i+1}. In particular $z_{n,i} \neq 0$. This is impossible as $\sum z_{n,i} \in \bigoplus_i A_n$ hence only a finite number of $z_{n,i}$ can be nonzero. □

Let (A_n) be an inverse system of rings. We will denote $Mod(\mathbf{N}, (A_n))$ the category of inverse systems (M_n) of abelian groups such that each M_n is given the structure of a A_n-module and the transition maps $M_{n+1} \to M_n$ are A_{n+1}-module maps. This is an abelian category. Set $A = \lim A_n$. Given an object (M_n) of $Mod(\mathbf{N}, (A_n))$ the limit $\lim M_n$ is an A-module.

Lemma 15.55.9. *In the situation above. The functor $\lim : Mod(\mathbf{N}, (A_n)) \to Mod_A$ has a right derived functor*

$$R\lim : D(Mod(\mathbf{N}, (A_n))) \longrightarrow D(A)$$

As usual we set $R^p \lim(K) = H^p(R\lim(K))$. Moreover, we have

(1) *for any* (A_n) *in* $Mod(\mathbf{N}, (A_n))$ *we have* $R^p \lim A_n = 0$ *for* $p > 1$,
(2) *the object* $R\lim A_n$ *of* $D(Mod_A)$ *is represented by the complex*

$$\prod A_n \to \prod A_n, \quad (x_n) \mapsto (x_n - f_{n+1}(x_{n+1}))$$

sitting in degrees 0 *and* 1,
(3) *if* (A_n) *is ML, then* $R^1 \lim A_n = 0$, *i.e.,* (A_n) *is right acyclic for* $\lim$,
(4) *every* $K^\bullet \in D(Mod(\mathbf{N}, (A_n)))$ *is quasi-isomorphic to a complex whose terms are right acyclic for* $\lim$, *and*
(5) *if each* $K^p = (K^p_n)$ *is right acyclic for* $\lim$, *i.e., of* $R^1 \lim_n K^p_n = 0$, *then* $R\lim K$ *is represented by the complex whose term in degree* p *is* $\lim_n K^p_n$.

Proof. The proof of this is word for word the same as the proof of Lemma 15.55.1. □

Remark 15.55.10. This remark is a continuation of Remark 15.55.4. A sheaf of rings on $\mathbf{N}$ is just an inverse system of rings (A_n). A sheaf of modules over (A_n) is exactly the same thing as an object of the category $Mod(\mathbf{N}, (A_n))$ defined above. The derived functor $R\lim$ of Lemma 15.55.9 is simply $R\Gamma(\mathbf{N}, -)$ from the derived category of modules to the derived category of modules over the global sections of the structure sheaf. is true in general that cohomology of groups and modules agree, see Cohomology on Sites, Lemma 21.12.4.

Lemma 15.55.11. *Let* (A_n) *be an inverse system of rings. Every* $K \in D(Mod(\mathbf{N}, (A_n)))$ *can be represented by a system of complexes* $(M^\bullet_n)$ *such that all the transition maps* $M^\bullet_{n+1} \to M^\bullet_n$ *are surjective.*

Proof. Let K be represented by the system $(K^\bullet_n)$. Set $M^\bullet_1 = K^\bullet_1$. Suppose we have constructed surjective maps of complexes $M^\bullet_n \to M^\bullet_{n-1} \to \ldots \to M^\bullet_1$ and homotopy equivalences $\psi_e : K^\bullet_e \to M^\bullet_e$ such that the diagrams

$$\begin{array}{ccc} K^\bullet_{e+1} & \longrightarrow & K^\bullet_e \\ \downarrow & & \downarrow \\ M^\bullet_{e+1} & \longrightarrow & M^\bullet_e \end{array}$$

commute for all $e < n$. Then we consider the diagram

$$\begin{array}{ccc} K^\bullet_{n+1} & \longrightarrow & K^\bullet_n \\ & & \downarrow \\ & & M^\bullet_n \end{array}$$

By Derived Categories, Lemma 13.8.8 we can factor the composition $K^\bullet_{n+1} \to M^\bullet_n$ as $K^\bullet_{n+1} \to M^\bullet_{n+1} \to M^\bullet_n$ such that the first arrow is a homotopy equivalence and the second a termwise split surjection. The lemma follows from this and induction. □

Lemma 15.55.12. *Let* (A_n) *be an inverse system of rings. Every* $K \in D(Mod(\mathbf{N}, (A_n)))$ *can be represented by a system of complexes* $(K^\bullet_n)$ *such that each* $K^\bullet_n$ *is K-flat.*

Proof. First use Lemma 15.55.11 to represent K by a system of complexes $(M^\bullet_n)$ such that all the transition maps $M^\bullet_{n+1} \to M^\bullet_n$ are surjective. Next, let $K^\bullet_1 \to M^\bullet_1$ be a quasi-isomorphism with $K^\bullet_1$ a K-flat complex of A_1-modules (Lemma 15.43.10).

Suppose we have constructed $K_n^\bullet \to K_{n-1}^\bullet \to \ldots \to K_1^\bullet$ and maps of complexes $\psi_e : K_e^\bullet \to M_e^\bullet$ such that

$$\begin{array}{ccc} K_{e+1}^\bullet & \longrightarrow & K_e^\bullet \\ \downarrow & & \downarrow \\ M_{e+1}^\bullet & \longrightarrow & M_e^\bullet \end{array}$$

commutes for all $e < n$. Then we consider the diagram

$$\begin{array}{ccc} C^\bullet & \dashrightarrow & K_n^\bullet \\ \vdots & & \downarrow \psi_n \\ M_{n+1}^\bullet & \xrightarrow{\varphi_n} & M_n^\bullet \end{array}$$

in $D(A_{n+1})$. As $M_{n+1}^\bullet \to M_n^\bullet$ is termwise surjective, the complex $C^\bullet$ fitting into the left upper corner with terms

$$C^p = M_{n+1}^p \times_{M_n^p} K_n^p$$

is quasi-isomorphic to $M_{n+1}^\bullet$ (details omitted). Choose a quasi-isomorphism $K_{n+1}^\bullet \to C^\bullet$ with $K_{n+1}^\bullet$ K-flat. Thus the lemma holds by induction. □

Lemma 15.55.13. *Let (A_n) be an inverse system of rings. Given $K, L \in D(Mod(\mathbf{N}, (A_n)))$ there is a canonical derived tensor product $K \otimes^{\mathbf{L}} L$ in $D(\mathbf{N}, (A_n))$ compatible with the maps to $D(A_n)$. The construction is symmetric in K and L and an exact functor of triangulated categories in each variable.*

Proof. Choose a representive $(K_n^\bullet)$ for K such that each $K_n^\bullet$ is a K-flat complex (Lemma 15.55.12). Then you can define $K \otimes^{\mathbf{L}} L$ as the object represented by the system of complexes

$$(\mathrm{Tot}(K_n^\bullet \otimes_{A_n} L_n^\bullet))$$

for any choice of representative $(L_n^\bullet)$ for L. This is well defined in both variables by Lemmas 15.43.4 and 15.43.12. Compatibility with the map to $D(A_n)$ is clear. Exactness follows exactly as in Lemma 15.43.2. □

As in the case of abelian groups an object $M = (M_n^\bullet)$ of $D(Mod(\mathbf{N}, (A_n)))$ is an inverse system of complexes of modules, which is **not** the same thing as an inverse system of objects in the derived categories. In the following lemma we show how an inverse system of objects in derived categories always lifts to an object of $D(Mod(\mathbf{N}, (A_n)))$.

Lemma 15.55.14. *Let (A_n) be an inverse system of rings. Suppose that we are given*

1. *for every n an object $K_n^\bullet$ of $D(A_n)$, and*
2. *for every n a map $\varphi_n : K_{n+1}^\bullet \to K_n^\bullet$ of $D(A_{n+1})$ where we think of $K_n^\bullet$ as an object of $D(A_{n+1})$ by restriction via the restriction map $A_{n+1} \to A_n$.*

There exists an object $M = (M_n^\bullet) \in D(Mod(\mathbf{N}, (A_n)))$ *and isomorphisms* $\psi_n : M_n^\bullet \to K_n^\bullet$ *in* $D(A_n)$ *such that the diagrams*

$$\begin{array}{ccc} M_{n+1}^\bullet & \longrightarrow & M_n^\bullet \\ {\scriptstyle \psi_{n+1}}\downarrow & & \downarrow{\scriptstyle \psi_n} \\ K_n^\bullet & \xrightarrow{\varphi_n} & K_n^\bullet \end{array}$$

commute in $D(A_{n+1})$.

Proof. Namely, set $M_1^\bullet = K_1^\bullet$. Suppose we have constructed $M_n^\bullet \to M_{n-1}^\bullet \to \ldots \to M_1^\bullet$ and maps of complexes $\psi_e : M_e^\bullet \to K_e^\bullet$ such that the diagrams above commute for all $e < n$. Then we consider the diagram

$$\begin{array}{ccc} & & M_n^\bullet \\ & & \downarrow{\scriptstyle \psi_n} \\ K_{n+1}^\bullet & \xrightarrow{\varphi_n} & K_n^\bullet \end{array}$$

in $D(A_{n+1})$. By the definition of morphisms in $D(A_{n+1})$ we can find a quasi-isomorphism $\psi_{n+1} : M_{n+1}^\bullet \to K_{n+1}^\bullet$ of complexes of A_{n+1}-modules such that there exists a morphism of complexes $M_{n+1}^\bullet \to M_n^\bullet$ of A_{n+1}-modules representing the composition $\psi_n^{-1} \circ \varphi_n \circ \psi_{n+1}$ in $D(A_{n+1})$. Thus the lemma holds by induction. □

Remark 15.55.15. With assumptions as in Lemma 15.55.14. A priori there are many isomorphism classes of objects M of $D(Mod(\mathbf{N}, (A_n)))$ which give rise to the system $(K_n^\bullet, \varphi_n)$ as above. For each such M we can consider the complex $R\lim M \in D(A)$ where $A = \lim A_n$. By Lemma 15.55.5 there exists a canonical distinguished triangle

$$R\lim M \to \prod\nolimits_n K_n^\bullet \to \prod\nolimits_n K_n^\bullet \to R\lim M[1]$$

in $D(A)$. Hence we see that the isomorphism class of $R\lim M$ in $D(A)$ is independent of the choices made in constructing M, by axiom TR3 of triangulated categories and Derived Categories, Lemma 13.4.3.

Remark 15.55.16. Let (K_n) be an inverse system of objects of $D(Ab)$. Let $K = R\lim K_n$ be a derived limit of this system (see Derived Categories, Section 13.30). Such a derived limit exists because $D(Ab)$ has countable products (Derived Categories, Lemma 13.30.3). By Lemma 15.55.14 we can also lift (K_n) to an object M of $D(\mathbf{N})$. Then $K \cong R\lim M$ where $R\lim$ is the functor (15.55.1.1) because $R\lim M$ is also a derived limit of the system (K_n) (by Lemma 15.55.5) and derived limits are unique up to isomorphism. In particular for every $p \in \mathbf{Z}$ there is a canonical short exact sequence

$$0 \to R^1 \lim H^{p-1}(K_n) \to H^p(K) \to \lim H^p(K_n) \to 0$$

as follows from Lemma 15.55.5 for M. This can also been seen directly, without invoking the existence of M, by applying the argument of the proof of Lemma 15.55.5 to the (defining) distinguished triangle $K \to \prod K_n \to \prod K_n \to K[1]$.

Remark 15.55.17. Let A be a ring. Let (E_n) be an inverse system of objects of $D(A)$. We've seen above that a derived limit $R\lim E_n$ exists. Thus for every object

K of $D(A)$ also the derived limit $R\lim(K \otimes^{\mathbf{L}}_A E_n)$ exists. It turns out that we can construct these derived limits functorially in K and obtain an exact functor

$$R\lim(- \otimes^{\mathbf{L}}_A E_n) : D(A) \longrightarrow D(A)$$

of triangulated categories. Namely, we first lift (E_n) to an object E of $D(\mathbf{N}, A)$, see Lemma 15.55.14. (The functor will depend on the choice of this lift.) Next, observe that there is a "diagonal" or "constant" functor

$$\Delta : D(A) \longrightarrow D(\mathbf{N}, A)$$

mapping the complex $K^\bullet$ to the constant inverse system of complexes with value $K^\bullet$. Then we simply define

$$R\lim(K \otimes^{\mathbf{L}}_A E_n) = R\lim(\Delta(K) \otimes^{\mathbf{L}} E)$$

where on the right hand side we use the functor $R\lim$ of Lemma 15.55.9 and the functor $- \otimes^{\mathbf{L}} -$ of Lemma 15.55.13.

Lemma 15.55.18. *Let A be a ring. Let $E \to D \to F \to E[1]$ be a distinguished triangle of $D(\mathbf{N}, A)$. Let (E_n), resp. (D_n), resp. (F_n) be the system of objects of $D(A)$ associated to E, resp. D, resp. F. Then for every $K \in D(A)$ there is a canonical distinguished triangle*

$$R\lim(K \otimes^{\mathbf{L}}_A E_n) \to R\lim(K \otimes^{\mathbf{L}}_A D_n) \to R\lim(K \otimes^{\mathbf{L}}_A F_n) \to R\lim(K \otimes^{\mathbf{L}}_A E_n)[1]$$

in $D(A)$ with notation as in Remark 15.55.17.

Proof. This is clear from the construction in Remark 15.55.17 and the fact that $\Delta : D(A) \to D(\mathbf{N}, A)$, $- \otimes^{\mathbf{L}} -$, and $R\lim$ are exact functors of triangulated categories. □

Lemma 15.55.19. *Let A be a ring. Let $E \to D$ be a morphism of $D(\mathbf{N}, A)$. Let (E_n), resp. (D_n) be the system of objects of $D(A)$ associated to E, resp. D. If $(E_n) \to (D_n)$ is an isomorphism of pro-objects, then for every $K \in D(A)$ the corresponding map*

$$R\lim(K \otimes^{\mathbf{L}}_A E_n) \longrightarrow R\lim(K \otimes^{\mathbf{L}}_A D_n)$$

in $D(A)$ is an isomorphism (notation as in Remark 15.55.17).

Proof. Follows from the definitions and Lemma 15.55.7. □

15.56. Formal glueing of module categories

Fix a noetherian scheme X, and a closed subscheme Z with complement U. Our goal is to explain a result of Artin that describes how coherent sheaves on X can be constructed (uniquely) from coherent sheaves on the formal completion of X along Z, and those on U with a suitable compatibility on the overlap.

Here are some references treating some of the material in this section: [**Art70**, Section 2], [**FR70**, Appendix], [**MB96**], and [**dJ95**, Section 4.6].

Definition 15.56.1. Let R be a ring. Let M be an R-module.

(1) Let $I \subset R$ be an ideal. We say M is an *I-power torsion module* if for every $m \in M$ there exists an $n > 0$ such that $I^n m = 0$.
(2) Let $f \in R$. We say M is *an f-power torsion module* if for each $m \in M$, there exists an $n > 0$ such that $f^n m = 0$.

Thus an f-power torsion module is the same thing as a I-power torsion module for $I = (f)$. We sometimes use the notation $M[I^n] = \{m \in M \mid I^n m = 0\}$ and $M[I^\infty] = \bigcup M[I^n]$ for an R-module M. Thus M is I-power torsion if and only if $M = M[I^\infty]$ if and only if $M = \bigcup M[I^n]$.

Lemma 15.56.2. *Let $\varphi : R \to S$ be a ring map. Let $I \subset R$ be an ideal. The following are equivalent*

1. *φ is flat and $R/I \to S/IS$ is faithfully flat,*
2. *φ is flat, and the map $\mathrm{Spec}(S/IS) \to \mathrm{Spec}(R/I)$ is surjective.*
3. *φ is flat, and the base change functor $M \mapsto M \otimes_R S$ is faithful on modules annihilated by I, and*
4. *φ is flat, and the base change functor $M \mapsto M \otimes_R S$ is faithful on I-power torsion modules.*

Proof. If $R \to S$ is flat, then $R/I^n \to S/I^nS$ is flat for every n, see Algebra, Lemma 10.36.6. Hence (1) and (2) are equivalent by Algebra, Lemma 10.36.15. The equivalence of (1) with (3) follows by identifying I-torsion R-modules with R/I-modules, using that

$$M \otimes_R S = M \otimes_{R/I} S/IS$$

for R-modules M annihilated by I, and Algebra, Lemma 10.36.13. The implication (4) $\Rightarrow$ (3) is immediate. Assume (3). We have seen above that $R/I^n \to S/I^nS$ is flat, and by assumption it induces a surjection on spectra, as $\mathrm{Spec}(R/I^n) = \mathrm{Spec}(R/I)$ and similarly for S. Hence the base change functor is faithful on modules annihilated by I^n. Since any I-power torsion module M is the union $M = \bigcup M_n$ where M_n is annihilated by I^n we see that the base change functor is faithful on the category of all I-power torsion modules (as tensor product commutes with colimits). □

Lemma 15.56.3. *Let R be a ring. Let I be an ideal of R. Let M be an I-power torsion module. Then M admits a resolution*

$$\ldots \to K_2 \to K_1 \to K_0 \to M \to 0$$

with each K_i a direct sum of copies of R/I^n for n variable.

Proof. There is a canonical surjection

$$\oplus_{m \in M} R/I^{n_m} \to M \to 0$$

where n_m is the smallest positive integer such that $I^{n_m} \cdot m = 0$. The kernel of the preceding surjection is also an I-power torsion module. Proceeding inductively, we construct the desired resolution of M. □

Lemma 15.56.4. *Assume $(\varphi : R \to S, I)$ satisfies the equivalent conditions of Lemma 15.56.2. The following are equivalent*

1. *for any I-power torsion module M, the natural map $M \to M \otimes_R S$ is an isomorphism, and*
2. *$R/I \to S/IS$ is an isomorphism.*

Proof. The implication (1) $\Rightarrow$ (2) is immediate. Assume (2). First assume that M is annihilated by I. In this case, M is an R/I-module. Hence, we have an isomorphism

$$M \otimes_R S = M \otimes_{R/I} S/IS = M \otimes_{R/I} R/I = M$$

proving the claim. Next we prove by induction that $M \to M \otimes_R S$ is an isomorphism for any module M is annihilated by I^n. Assume the induction hypothesis holds for n and assume M is annihilated by I^{n+1}. Then we have a short exact sequence

$$0 \to I^n M \to M \to M/I^n M \to 0$$

and as $R \to S$ is flat this gives rise to a short exact sequence

$$0 \to I^n M \otimes_R S \to M \otimes_R S \to M/I^n M \otimes_R S \to 0$$

Using that the canonical map is an isomorphism for $M' = I^n M$ and $M'' = M/I^n M$ (by induction hypothesis) we conclude the same thing is true for M. Finally, suppose that M is a general I-power torsion module. Then $M = \bigcup M_n$ where M_n is annihilated by I^n and we conclude using that tensor products commute with colimits. □

Lemma 15.56.5. *Let R be a ring. Let I be an ideal of R. For any R-module M set $M[I^n] = \{m \in M \mid I^n m = 0\}$. If I is finitely generated then the following are equivalent*

1. *$M[I] = 0$,*
2. *$M[I^n] = 0$ for all $n \geq 1$, and*
3. *if $I = (f_1, \ldots, f_t)$, then the map $M \to \bigoplus M_{f_i}$ is injective.*

Proof. This follows from Algebra, Lemma 10.21.4. □

Lemma 15.56.6. *Let R be a ring. Let I be an ideal of R. For any R-module M set $M[I^\infty] = \bigcup_{n \geq 1} M[I^n]$. If I is finitely generated, then $(M/M[I^\infty])[I] = 0$.*

Proof. Let $m \in M$. If m maps to an element of $(M/M[I^\infty])[I]$ then $Im \subset M[I^\infty]$. Write $I = (f_1, \ldots, f_t)$. Then we see that $f_i m \in M[I^\infty]$, i.e., $I^{n_i} f_i m = 0$ for some $n_i > 0$. Thus we see that $I^N m = 0$ with $N = \sum n_i + 2$. Hence m maps to zero in $(M/M[I^\infty])$ which proves the lemma. □

Lemma 15.56.7. *Assume $\varphi : R \to S$ is a flat ring map and $I \subset R$ is a finitely generated ideal such that $R/I \to S/IS$ is an isomorphism. Then*

1. *for any R-module M the map $M \to M \otimes_R S$ induces an isomorphism $M[I^\infty] \to (M \otimes_R S)[(IS)^\infty]$ of I-power torsion submodules,*
2. *the natural map*
$$\mathrm{Hom}_R(M, N) \longrightarrow \mathrm{Hom}_S(M \otimes_R S, N \otimes_R S)$$
is an isomorphism if either M or N is I-power torsion, and
3. *the base change functor $M \mapsto M \otimes_R S$ defines an equivalence of categories between I-power torsion modules and IS-power torsion modules.*

Proof. Note that the equivalent conditions of both Lemma 15.56.2 and Lemma 15.56.4 are satisfied. We will use these without further mention. We first prove (1). Let M be any R-module. Set $M' = M/M[I^\infty]$ and consider the exact sequence

$$0 \to M[I^\infty] \to M \to M' \to 0$$

As $M[I^\infty] = M[I^\infty] \otimes_R S$ we see that it suffices to show that $(M' \otimes_R S)[(IS)^\infty] = 0$. Write $I = (f_1, \ldots, f_t)$. By Lemma 15.56.6 we see that $M'[I^\infty] = 0$. Hence for every $n > 0$ the map

$$M' \longrightarrow \bigoplus\nolimits_{i=1,\ldots t} M', \quad x \longmapsto (f_1^n x, \ldots, f_t^n x)$$

is injective. As S is flat over R also the corresponding map $M' \otimes_R S \to \bigoplus_{i=1,\ldots t} M' \otimes_R S$ is injective. This means that $(M' \otimes_R S)[I^n] = 0$ as desired.

Next we prove (2). If N is I-power torsion, then $N \otimes_R S = N$ and the displayed map of (2) is an isomorphism by Algebra, Lemma 10.13.3. If M is I-power torsion, then the image of any map $M \to N$ factors through $M[I^\infty]$ and the image of any map $M \otimes_R S \to N \otimes_R S$ factors through $(N \otimes_R S)[(IS)^\infty]$. Hence in this case part (1) guarantees that we may replace N by $N[I^\infty]$ and the result follows from the case where N is I-power torsion we just discussed.

Next we prove (3). The functor is fully faithful by (2). For essential surjectivity, we simply note that for any IS-power torsion S-module N, the natural map $N \otimes_R S \to N$ is an isomorphism. □

Lemma 15.56.8. *Assume $\varphi : R \to S$ is a flat ring map and $I \subset R$ is a finitely generated ideal such that $R/I \to S/IS$ is an isomorphism. For any $f_1, \ldots, f_r \in R$ such that $V(f_1, \ldots, f_r) = V(I)$*

1. *the map of Koszul complexes $K(R, f_1, \ldots, f_r) \to K(S, f_1, \ldots, f_r)$ is a quasi-isomorphism, and*
2. *The map of extended alternating Čech complexes*

$$\begin{array}{ccccccc} R & \to & \prod_{i_0} R_{f_{i_0}} & \to & \prod_{i_0<i_1} R_{f_{i_0}f_{i_1}} & \to \ldots \to & R_{f_1\ldots f_r} \\ & & \downarrow & & \downarrow & & \\ S & \to & \prod_{i_0} S_{f_{i_0}} & \to & \prod_{i_0<i_1} S_{f_{i_0}f_{i_1}} & \to \ldots \to & S_{f_1\ldots f_r} \end{array}$$

is a quasi-isomorphism.

Proof. In both cases we have a complex $K_\bullet$ of R modules and we want to show that $K_\bullet \to K_\bullet \otimes_R S$ is a quasi-isomorphism. By Lemma 15.56.4 and the flatness of $R \to S$ this will hold as soon as all homology groups of K are I-power torsion. This is true for the Koszul complex by Lemma 15.19.6. Since the alternating Čech complex is a colimit of Koszul complexes (Lemma 15.19.13) the case of the Koszul complex implies the second statement too. □

Lemma 15.56.9. *Let R be a ring. Let $I = (f_1, \ldots, f_n)$ be a finitely generated ideal of R. Let M be the R-module generated by elements $e_1, \ldots, e_n$ subject to the relations $f_i e_j - f_j e_i = 0$. There exists a short exact sequence*

$$0 \to K \to M \to I \to 0$$

such that K is annihilated by I.

Proof. This is just a truncation of the Koszul complex. The map $M \to I$ is is determined by the rule $e_i \mapsto f_i$. If $m = \sum a_i e_i$ is in the kernel of $M \to I$, i.e., $\sum a_i f_i = 0$, then $f_j m = \sum f_j a_i e_i = (\sum f_i a_i) e_j = 0$. □

Lemma 15.56.10. *Let R be a ring. Let $I = (f_1, \ldots, f_n)$ be a finitely generated ideal of R. For any R-module N set*

$$H_1(N, f_\bullet) = \frac{\{(x_1, \ldots, x_n) \in N^{\oplus n} \mid f_i x_j = f_j x_i\}}{\{f_1 x, \ldots, f_n x) \mid x \in N\}}$$

For any R-module N there exists a canonical short exact sequence

$$0 \to Ext_R(R/I, N) \to H_1(N, f_\bullet) \to \mathrm{Hom}_R(K, N)$$

where K is as in Lemma 15.56.9.

Proof. The notation above indicates the Ext-groups in Mod_R as defined in Homology, Section 12.6. These are denoted $\text{Ext}_R(M, N)$. Using the long exact sequence of Homology, Lemma 12.6.4 associated to the short exact sequence $0 \to I \to R \to R/I \to 0$ and the fact that $\text{Ext}_R(R, N) = 0$ we see that

$$\text{Ext}_R(R/I, N) = \text{Coker}(N \longrightarrow \text{Hom}(I, N))$$

Using the short exact sequence of Lemma 15.56.9 we see that we get a complex

$$N \to \text{Hom}(M, N) \to \text{Hom}_R(K, N)$$

whose homology in the middle is canonically isomorphic to $\text{Ext}_R(R/I, N)$. The proof of the lemma is now complete as the cokernel of the first map is canonically isomorphic to $H_1(N, f_\bullet)$. □

Lemma 15.56.11. *Let R be a ring. Let $I = (f_1, \ldots, f_n)$ be a finitely generated ideal of R. For any R-module N the Koszul homology group $H_1(N, f_\bullet)$ defined in Lemma 15.56.10 is annihilated by I.*

Proof. Let $(x_1, \ldots, x_n) \in N^{\oplus n}$ with $f_i x_j = f_j x_i$. Then we have $f_i(x_1, \ldots, x_n) = (f_i x_i, \ldots, f_i x_n)$. In other words f_i annihilates $H_1(N, f_\bullet)$. □

We can improve on the full faithfulness of Lemma 15.56.7 by showing that Ext-groups whose source is I-power torsion are insensitive to passing to S as well. See Remark 15.56.13 below for a more highbrow version of the following lemma.

Lemma 15.56.12. *Assume $\varphi : R \to S$ is a flat ring map and $I \subset R$ is a finitely generated ideal such that $R/I \to S/IS$ is an isomorphism. Let M, N be R-modules. Assume M is I-power torsion. Given an short exact sequence*

$$0 \to N \otimes_R S \to \tilde{E} \to M \otimes_R S \to 0$$

there exists a commutative diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & N & \longrightarrow & E & \longrightarrow & M & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & N \otimes_R S & \longrightarrow & \tilde{E} & \longrightarrow & M \otimes_R S & \longrightarrow & 0 \end{array}$$

with exact rows.

Proof. As M is I-power torsion we see that $M \otimes_R S = M$, see Lemma 15.56.4. We will use this identification without further mention. As $R \to S$ is flat, the base change functor is exact and we obtain a functorial map of Ext-groups

$$\text{Ext}_R(M, N) \longrightarrow \text{Ext}_S(M \otimes_R S, N \otimes_R S),$$

see Homology, Lemma 12.7.2. The claim of the lemma is that this map is surjective when M is I-power torsion. In fact we will show that it is an isomorphism. By Lemma 15.56.3 we can find a surjection $M' \to M$ with M' a direct sum of modules of the form R/I^n. Using the long exact sequence of Homology, Lemma 12.6.4 we see that it suffices to prove the lemma for M'. Using compatibility of Ext with direct sums (details omitted) we reduce to the case where $M = R/I^n$ for some n.

Let $f_1, \ldots, f_t$ be generators for I^n. By Lemma 15.56.10 we have a commutative diagram

$$\begin{array}{ccccccc} 0 & \longrightarrow & \mathrm{Ext}_R(R/I^n, N) & \longrightarrow & H_1(N, f_\bullet) & \longrightarrow & \mathrm{Hom}_R(K, N) \\ & & \downarrow & & \downarrow & & \downarrow \\ 0 & \longrightarrow & \mathrm{Ext}_S(S/I^nS, N \otimes S) & \longrightarrow & H_1(N \otimes S, f_\bullet) & \longrightarrow & \mathrm{Hom}_S(K \otimes S, N \otimes S) \end{array}$$

with exact rows where K is as in Lemma 15.56.9. Hence it suffices to prove that the two right vertical arrows are isomorphisms. Since K is annihilated by I^n we see that $\mathrm{Hom}_R(K, N) = \mathrm{Hom}_S(K \otimes_R S, N \otimes_R S)$ by Lemma 15.56.7. As $R \to S$ is flat we have $H_1(N, f_\bullet) \otimes_R S = H_1(N \otimes_R S, f_\bullet)$. As $H_1(N, f_\bullet)$ is annihilated by I^n, see Lemma 15.56.11 we have $H_1(N, f_\bullet) \otimes_R S = H_1(N, f_\bullet)$ by Lemma 15.56.4. □

Remark 15.56.13. Assume $\varphi : R \to S$ is a flat ring map and $I \subset R$ is a finitely generated ideal such that $R/I \to S/IS$ is an isomorphism. Let M, N be R-modules and assume M is I-power torsion. Then the canonical map

$$\mathrm{Ext}^i_R(M, N) \longrightarrow \mathrm{Ext}^i_S(M \otimes_R S, N \otimes_R S)$$

is an isomorphism for all i. Namely, we can construct an inverse to this map as follows. Think of an element ξ of the right hand side as a morphism $\xi : M \otimes_R S \to N \otimes_R S[i]$ in $D(S)$. Let $\check{\mathcal{C}}_R \to \check{\mathcal{C}}_S$ be the quasi-isomorphism of extended alternating Čech complexes of Lemma 15.56.8. Tensoring over S with $\check{\mathcal{C}}_S$ is equal to derived tensor, hence we obtain

$$\xi \otimes \mathrm{id} : M \otimes_R \check{\mathcal{C}}_S \to N \otimes_R \check{\mathcal{C}}_S[i]$$

in $D(S)$. On the one hand the complex $M \otimes_R \check{\mathcal{C}}_S$ is equal to $M = M \otimes_R S$ because M is I-power torsion. On the other hand, in $D(R)$ we have

$$N \otimes_R \check{\mathcal{C}}_S = N \otimes_R \check{\mathcal{C}}_R \longrightarrow N$$

just because there is a map of complexes $\check{\mathcal{C}}_R \to R$. Combining all of the above we obtain a map $M \to N[i]$ in $D(R)$ as desired. We omit the verification that this construction indeed produces an inverse to the first map. If we ever need to use this result we will put in a more detailed proof.

Let $R \to S$ be a ring map. Let $f_1, \ldots, f_t \in R$ and $I = (f_1, \ldots, f_t)$. Then for any R-module M we can define a complex

$$0 \to M \xrightarrow{\alpha} M \otimes_R S \times \prod M_{f_i} \xrightarrow{\beta} \prod (M \otimes_R S)_{f_i} \times \prod M_{f_i f_j} \tag{15.56.13.1}$$

where $\alpha(m) = (m \otimes 1, m/1, \ldots, m/1)$ and

$$\beta(m', m_1, \ldots, m_t) = ((m'/1 - m_1 \otimes 1, \ldots, m'/1 - m_t \otimes 1), (m_1 - m_2, \ldots, m_{t-1} - m_t).$$

We would like to know when this complex is exact.

Lemma 15.56.14. *Assume $\varphi : R \to S$ is a flat ring map and $I = (f_1, \ldots, f_t) \subset R$ is an ideal such that $R/I \to S/IS$ is an isomorphism. Let M be an R-module. Then the complex (15.56.13.1) is exact.*

Proof. First proof. Denote $\check{\mathcal{C}}_R \to \check{\mathcal{C}}_S$ the quasi-isomorphism of extended alternating Čech complexes of Lemma 15.56.8. Since these complexes are bounded with flat terms, we see that $M \otimes_R \check{\mathcal{C}}_R \to M \otimes_R \check{\mathcal{C}}_S$ is a quasi-isomorphism too (Lemmas

15.43.8 and 15.43.12). Now the complex (15.56.13.1) is a truncation of the cone of the map $M \otimes_R \check{\mathcal{C}}_R \to M \otimes_R \check{\mathcal{C}}_S$ and we win.

Second computational proof. Let $m \in M$. If $\alpha(m) = 0$, then $m \in M[I^\infty]$, see Lemma 15.56.5. Pick n such that $I^n m = 0$ and consider the map $\varphi : R/I^n \to M$. If $m \otimes 1 = 0$, then $\varphi \otimes 1_S = 0$, hence $\varphi = 0$ (see Lemma 15.56.7) hence $m = 0$. In this way we see that α is injective.

Let $(m', m'_1, \ldots, m'_t) \in \mathrm{Ker}(\beta)$. Write $m'_i = m_i/f_i^n$ for some $n > 0$ and $m_i \in M$. We may, after possibly enlarging n assume that $f_i^n m' = m_i \otimes 1$ in $M \otimes_R S$ and $f_j^n m_i - f_i^n m_j = 0$ in M. In particular we see that $(m_1, \ldots, m_t)$ defines an element ξ of $H_1(M, (f_1^n, \ldots, f_t^n))$. Since $H_1(M, (f_1^n, \ldots, f_t^n))$ is annihilated by I^{tn+1} (see Lemma 15.56.11) and since $R \to S$ is flat we see that

$$H_1(M, (f_1^n, \ldots, f_t^n)) = H_1(M, (f_1^n, \ldots, f_t^n)) \otimes_R S = H_1(M \otimes_R S, (f_1^n, \ldots, f_t^n))$$

by Lemma 15.56.4 The existence of m' implies that ξ maps to zero in the last group, i.e., the element ξ is zero. Thus there exists an $m \in M$ such that $m_i = f_i^n m$. Then $(m', m'_1, \ldots, m'_t) - \alpha(m) = (m'', 0, \ldots, 0)$ for some $m'' \in (M \otimes_R S)[(IS)^\infty]$. By Lemma 15.56.7 we conclude that $m'' \in M[I^\infty]$ and we win. □

Remark 15.56.15. In this remark we define a category of glueing data. Let $R \to S$ be a ring map. Let $f_1, \ldots, f_t \in R$ and $I = (f_1, \ldots, f_t)$. Consider the category $\mathrm{Glue}(R \to S, f_1, \ldots, f_t)$ as the category whose

(1) objects are systems $(M', M_i, \alpha_i, \alpha_{ij})$, where M' is an S-module, M_i is an R_{f_i}-module, $\alpha_i : (M')_{f_i} \to M_i \otimes_R S$ is an isomorphism, and $\alpha_{ij} : (M_i)_{f_j} \to (M_j)_{f_i}$ are isomorphisms such that
   (a) $\alpha_{ij} \circ \alpha_i = \alpha_j$ as maps $(M')_{f_i f_j} \to (M_j)_{f_i}$, and
   (b) $\alpha_{jk} \circ \alpha_{ij} = \alpha_{ik}$ as maps $(M_i)_{f_j f_k} \to (M_k)_{f_i f_j}$ (cocycle condition).

(2) morphisms $(M', M_i, \alpha_i, \alpha_{ij}) \to (N', N_i, \beta_i, \beta_{ij})$ are given by maps $\varphi' : M' \to N'$ and $\varphi_i : M_i \to N_i$ compatible with the given maps $\alpha_i, \beta_i, \alpha_{ij}, \beta_{ij}$.

There is a canonical functor

$$\mathrm{Can} : \mathrm{Mod}_R \longrightarrow \mathrm{Glue}(R \to S, f_1, \ldots, f_t), \quad M \longmapsto (M \otimes_R S, M_{f_i}, \mathrm{can}_i, \mathrm{can}_{ij})$$

where $\mathrm{can}_i : (M \otimes_R S)_{f_i} \to M_{f_i} \otimes_R S$ and $\mathrm{can}_{ij} : (M_{f_i})_{f_j} \to (M_{f_j})_{f_i}$ are the canonical isomorphisms. For any object $\mathbf{M} = (M', M_i, \alpha_i, \alpha_{ij})$ of the category $\mathrm{Glue}(R \to S, f_1, \ldots, f_t)$ we define

$$H^0(\mathbf{M}) = \{(m', m_i) \mid \alpha_i(m') = m_i \otimes 1, \alpha_{ij}(m_i) = m_j\}$$

in other words defined by the exact sequence

$$0 \to H^0(\mathbf{M}) \to M' \times \prod M_i \to \prod M'_{f_i} \times \prod (M_i)_{f_j}$$

similar to (15.56.13.1). We think of $H^0(\mathbf{M})$ as an R-module. Thus we also get a functor

$$H^0 : \mathrm{Glue}(R \to S, f_1, \ldots, f_t) \longrightarrow \mathrm{Mod}_R$$

Our next goal is to show that the functors Can and H^0 are sometimes quasi-inverse to each other.

Lemma 15.56.16. *Assume $\varphi : R \to S$ is a flat ring map and $I = (f_1, \ldots, f_t) \subset R$ is an ideal such that $R/I \to S/IS$ is an isomorphism. Then the functor H^0 is a left quasi-inverse to the functor Can of Remark 15.56.15.*

Proof. This is a reformulation of Lemma 15.56.14. □

Lemma 15.56.17. *Assume $\varphi : R \to S$ is a flat ring map and let $I = (f_1, \ldots, f_t) \subset R$ be an ideal. Then Glue$(R \to S, f_1, \ldots, f_t)$ is an abelian category, and the functor Can is exact and commutes with arbitrary colimits.*

Proof. Given a morphism $(\varphi', \varphi_i) : (M', M_i, \alpha_i, \alpha_{ij}) \to (N', N_i, \beta_i, \beta_{ij})$ of the category $\mathrm{Glue}(R \to S, f_1, \ldots, f_t)$ we see that its kernel exists and is equal to the object $(\mathrm{Ker}(\varphi'), \mathrm{Ker}(\varphi_i), \alpha_i, \alpha_{ij})$ and its cokernel exists and is equal to the object $(\mathrm{Coker}(\varphi'), \mathrm{Coker}(\varphi_i), \beta_i, \beta_{ij})$. This works because $R \to S$ is flat, hence taking kernels/cokernels commutes with $- \otimes_R S$. Details omitted. The exactness follows from the R-flatness of R_{f_i} and S, while commuting with colimits follows as tensor products commute with colimits. □

Lemma 15.56.18. *Let $\varphi : R \to S$ be a flat ring map and $(f_1, \ldots, f_t) = R$. Then Can and H^0 are quasi-inverse equivalences of categories*

$$Mod_R = Glue(R \to S, f_1, \ldots, f_t)$$

Proof. Consider an object $\mathbf{M} = (M', M_i, \alpha_i, \alpha_{ij})$ of $\mathrm{Glue}(R \to S, f_1, \ldots, f_t)$. By Algebra, Lemma 10.22.4 there exists a unique module M and isomorphisms $M_{f_i} \to M_i$ which recover the glueing data α_{ij}. Then both M' and $M \otimes_R S$ are S-modules which recover the modules $M_i \otimes_R S$ upon localizing at f_i. Whence there is a canonical isomorphism $M \otimes_R S \to M'$. This shows that $\mathbf{M}$ is in the essential image of Can. Combined with Lemma 15.56.16 the lemma follows. □

Lemma 15.56.19. *Let $\varphi : R \to S$ be a flat ring map and $I = (f_1, \ldots, f_t)$ and ideal. Let $R \to R'$ be a flat ring map, and set $S' = S \otimes_R R'$. Then we obtain a commutative diagram of categories and functors*

$$\begin{array}{ccccc} Mod_R & \xrightarrow[Can]{} & Glue(R \to S, f_1, \ldots, f_t) & \xrightarrow[H^0]{} & Mod_R \\ \downarrow{\scriptstyle -\otimes_R R'} & & \downarrow{\scriptstyle -\otimes_R R'} & & \downarrow{\scriptstyle -\otimes_R R'} \\ Mod_{R'} & \xrightarrow{Can} & Glue(R' \to S', f_1, \ldots, f_t) & \xrightarrow{H^0} & Mod_{R'} \end{array}$$

Proof. Omitted. □

Proposition 15.56.20. *Assume $\varphi : R \to S$ is a flat ring map and $I = (f_1, \ldots, f_t) \subset R$ is an ideal such that $R/I \to S/IS$ is an isomorphism. Then Can and H^0 are quasi-inverse equivalences of categories*

$$Mod_R = Glue(R \to S, f_1, \ldots, f_t)$$

Proof. We have already seen that $H^0 \circ \mathrm{Can}$ is isomorphic to the identity functor, see Lemma 15.56.16. Consider an object $\mathbf{M} = (M', M_i, \alpha_i, \alpha_{ij})$ of $\mathrm{Glue}(R \to S, f_1, \ldots, f_t)$. We get a natural morphism

$$\Psi : (H^0(\mathbf{M}) \otimes_R S, H^0(\mathbf{M})_{f_i}, \mathrm{can}_i, \mathrm{can}_{ij}) \longrightarrow (M', M_i, \alpha_i, \alpha_{ij}).$$

Namely, by definition $H^0(\mathbf{M})$ comes equipped with compatible R-module maps $H^0(\mathbf{M}) \to M'$ and $H^0(\mathbf{M}) \to M_i$. We have to show that this map is an isomorphism.

Pick an index i and set $R' = R_{f_i}$. Combining Lemmas 15.56.19 and 15.56.18 we see that $\Psi \otimes_R R'$ is an isomorphism. Hence the kernel, resp. cokernel of Ψ is a system of the form $(K, 0, 0, 0)$, resp. $(Q, 0, 0, 0)$. Note that $H^0((K, 0, 0, 0)) = K$, that H^0

is left exact, and that by construction $H^0(\Psi)$ is bijective. Hence we see $K = 0$, i.e., the kernel of Ψ is zero.

The conclusion of the above is that we obtain a short exact sequence

$$0 \to H^0(\mathbf{M}) \otimes_R S \to M' \to Q \to 0$$

and that $M_i = H^0(\mathbf{M})_{f_i}$. Note that we may think of Q as an R-module which is I-power torsion so that $Q = Q \otimes_R S$. By Lemma 15.56.12 we see that there exists a commutative diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & H^0(\mathbf{M}) & \longrightarrow & E & \longrightarrow & Q & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & H^0(\mathbf{M}) \otimes_R S & \longrightarrow & M' & \longrightarrow & Q & \longrightarrow & 0 \end{array}$$

with exact rows. This clearly determines an isomorphism $\mathrm{Can}(E) \to (M', M_i, \alpha_i, \alpha_{ij})$ in the category $\mathrm{Glue}(R \to S, f_1, \ldots, f_t)$ and we win. (Of course, a posteriori we have $Q = 0$.) □

Next, we specialize this very general proposition to get something more useable. Namely, if $I = (f)$ is a principal ideal then the objects of $\mathrm{Glue}(R \to S, f)$ are simply triples (M', M_1, α_1) and there is *no* cocycle condition to check!

Theorem 15.56.21. *Let R be a ring, and let $f \in R$. Let $\varphi : R \to S$ be a flat ring map inducing an isomorphism $R/fR \to S/fS$. Then the functor*

$$Mod_R \longrightarrow Mod_S \times_{Mod_{S_f}} Mod_{R_f}, \quad M \longmapsto (M \otimes_R S, M_f, can)$$

is an equivalence.

Proof. The category appearing on the right side of the arrow is the category of triples (M', M_1, α_1) where M' is an S-module, M_1 is a R_f-module, and $\alpha_1 : M'_f \to M_1 \otimes_R S$ is a S_f-isomorphism, see Categories, Example 4.29.3. Hence this theorem is a special case of Proposition 15.56.20. □

A useful special case of Theorem 15.56.21 is when R is noetherian, and S is a completion of R at an element f. The completion $R \to S$ is flat, and the functor $M \mapsto M \otimes_R S$ can be identified with the f-adic completion functor when M is finitely generated. To state this more precisely, let Mod_R^{fg} denote the category of finitely generated R-modules.

Proposition 15.56.22. *Let R be a noetherian ring. Let $f \in R$ be an element. Let $R^\wedge$ be the f-adic completion of R. Then the functor $M \mapsto (M^\wedge, M_f, can)$ defines an equivalence*

$$Mod_R^{fg} \longrightarrow Mod_{R^\wedge}^{fg} \times_{Mod_{R^\wedge_f}^{fg}} Mod_{R_f}^{fg}$$

Proof. The ring map $R \to R^\wedge$ is flat by Algebra, Lemma 10.91.3. It is clear that $R/fR = R^\wedge/fR^\wedge$. By Algebra, Lemma 10.91.2 the completion of a finite R-module M is equal to $M \otimes_R R^\wedge$. Hence the displayed functor of the proposition is equal to the functor occurring in Theorem 15.56.21. In particular it is fully faithful. Let (M_1, M_2, ψ) be an object of the right hand side. By Theorem 15.56.21 there exists an R-module M such that $M_1 = M \otimes_R R^\wedge$ and $M_2 = M_f$. As $R \to R^\wedge \times R_f$ is faithfully flat we conclude from Algebra, Lemma 10.22.2 that M is finitely generated, i.e., $M \in \mathrm{Mod}_R^{fg}$. This proves the proposition. □

Remark 15.56.23. The equivalences of Proposition 15.56.20, Theorem 15.56.21, and Proposition 15.56.22 preserve the $\otimes$-structures on either side. Thus, it defines equivalences of various categories built out of the pair $(\mathrm{Mod}_R, \otimes)$, such as the category of R-algebras.

Remark 15.56.24. Given a differential manifold X with a compact closed submanifold Z having complement U, specifying a sheaf on X is the same as specifying a sheaf on U, a sheaf on an unspecified tubular neighbourhood T of Z in X, and an isomorphism between the two resulting sheaves along $T \cap U$. Tubular neighbourhoods do not exist in algebraic geometry as such, but results such as Proposition 15.56.20, Theorem 15.56.21, and Proposition 15.56.22 allow us to work with formal neighbourhoods instead.

15.57. Derived Completion

Some references for the material in this section are [**DG02**], [**GM92**], [**Lur11**]. Our exposition follows [**BS13**].

Let $K \in D(A)$. Let $f \in A$. We denote $T(K, f)$ a derived limit of the system

$$\ldots \to K \xrightarrow{f} K \xrightarrow{f} K$$

in $D(A)$.

Lemma 15.57.1. *Let A be a ring. Let $f \in A$. Let $K \in D(A)$. The following are equivalent*

1. *$Ext^n_A(A_f, K) = 0$ for all n,*
2. *$\mathrm{Hom}_{D(A)}(E, K) = 0$ for all E in $D(A_f)$,*
3. *$T(K, f) = 0$,*
4. *for every $p \in \mathbf{Z}$ we have $T(H^p(K), f) = 0$,*
5. *for every $p \in \mathbf{Z}$ we have $\mathrm{Hom}_A(A_f, H^p(K)) = 0$ and $Ext^1_A(A_f, H^p(K)) = 0$,*
6. *add more here.*

Proof. It is clear that (2) implies (1). Assume (1). Let $I^\bullet$ be a K-injective complex of A-modules representing K. Condition (1) signifies that $\mathrm{Hom}_A(A_f, I^\bullet)$ is acyclic. Let $M^\bullet$ be a complex of A_f-modules representing E. Then

$$\mathrm{Hom}_{D(A)}(E, K) = \mathrm{Hom}_{K(A)}(M^\bullet, I^\bullet) = \mathrm{Hom}_{K(A_f)}(M^\bullet, \mathrm{Hom}_A(A_f, I^\bullet))$$

by Algebra, Lemma 10.13.4. As $\mathrm{Hom}_A(A_f, I^\bullet)$ is a K-injective complex of A_f-modules by Lemma 15.41.3 the fact that it is acyclic implies that it is homotopy equivalent to zero (Derived Categories, Lemma 13.28.2). Thus we get (2).

A free resolution of the A-module A_f is given by

$$0 \to \bigoplus\nolimits_{n \in \mathbf{N}} A \to \bigoplus\nolimits_{n \in \mathbf{N}} A \to A_f \to 0$$

where the first map sends the $(x_0, x_1, \ldots)$ to $(fx_0 - x_1, fx_1 - x_2, \ldots)$ and the second map sends $(x_0, x_1, \ldots)$ to $x_0 + x_1/f + x_2/f^2 + \ldots$. Applying $\mathrm{Hom}_A(-, I^\bullet)$ we get

$$0 \to \mathrm{Hom}_A(A_f, I^\bullet) \to \prod I^\bullet \to \prod I^\bullet \to 0$$

This means that the object $T(K, f)$ is a representative of $R\mathrm{Hom}_A(A_f, K)$ in $D(A)$. Thus the equivalence of (1) and (3).

There is a spectral sequence

$$E_2^{p,q} = \mathrm{Ext}_A^q(A_f, H^p(K)) \Rightarrow \mathrm{Ext}_A^{p+q}(A_f, K)$$

(details omitted). This spectral sequence degenerates at E_2 because A_f has a length 1 resolution by projective A-modules (see above) hence the E_2-page has only 2 nonzero rows. Thus we obtain short exact sequences

$$0 \to \mathrm{Ext}_A^1(A_f, H^{p-1}(K)) \to \mathrm{Ext}_A^p(A_f, K) \to \mathrm{Hom}_A(A_f, H^p(K)) \to 0$$

This proves (4) and (5) are equivalent to (1). □

Lemma 15.57.2. *Let A be a ring. Let $K \in D(A)$. The set I of $f \in A$ such that $T(K, f) = 0$ is an ideal of A.*

Proof. We will use the results of Lemma 15.57.1 without further mention. If $f \in I$, and $g \in A$, then A_{gf} is an A_f-module hence $\mathrm{Ext}_A^n(A_{gf}, K) = 0$ for all n, hence $gf \in I$. Suppose $f, g \in I$. Then there is a short exact sequence

$$0 \to A_{f+g} \to A_{f(f+g)} \oplus A_{g(f+g)} \to A_{gf(f+g)} \to 0$$

because f, g generate the unit ideal in A_{f+g}. This follows from Algebra, Lemma 10.21.1 and the easy fact that the last arrow is surjective. By the long exact sequence of Ext and the vanishing of $\mathrm{Ext}_A^n(A_{f(f+g)}, K)$, $\mathrm{Ext}_A^n(A_{g(f+g)}, K)$, and $\mathrm{Ext}_A^n(A_{gf(f+g)}, K)$ we get the vanishing of $\mathrm{Ext}_A^n(A_{f+g}, K)$. □

Lemma 15.57.3. *Let A be a ring. Let $I \subset A$ be an ideal. Let M be an A-module.*

1. *If M is I-adically complete, then $T(M, f) = 0$ for all $f \in I$.*
2. *Conversely, if $T(M, f) = 0$ for all $f \in I$ and I is finitely generated, then $M \to \lim M/I^n M$ is surjective.*

Proof. Proof of (1). Assume M is I-adically complete. By Lemma 15.57.1 it suffices to prove $\mathrm{Ext}_A^1(A_f, M) = 0$ and $\mathrm{Hom}_A(A_f, M) = 0$. Since $M = \lim M/I^n M$ and since $\mathrm{Hom}_A(A_f, M/I^n M) = 0$ it follows that $\mathrm{Hom}_A(A_f, M) = 0$. Suppose we have an extension

$$0 \to M \to E \to A_f \to 0$$

For $n \geq 0$ pick $e_n \in E$ mapping to $1/f^n$. Set $\delta_n = fe_{n+1} - e_n \in M$ for $n \geq 0$. Replace e_n by

$$e_n' = e_n + \delta_n + f\delta_{n+1} + f^2\delta_{n+2} + \ldots$$

The infinite sum exists as M is complete with respect to I and $f \in I$. A simple calculation shows that $fe_{n+1}' = e_n'$. Thus we get a splitting of the extension by mapping $1/f^n$ to e_n'.

Proof of (2). Assume that $I = (f_1, \ldots, f_r)$ and that $T(M, f_i) = 0$ for $i = 1, \ldots, r$. By Algebra, Lemma 10.91.12 we may assume $I = (f)$ and $T(M, f) = 0$. Let $x_n \in M$ for $n \geq 0$. Consider the extension

$$0 \to M \to E \to A_f \to 0$$

given by

$$E = M \oplus \bigoplus Ae_n \Big/ \langle x_n - fe_{n+1} + e_n \rangle$$

mapping e_n to $1/f^n$ in A_f (see above). By assumption and Lemma 15.57.1 this extension is split, hence we obtain an element $x + e_0$ which generates a copy of A_f in E. Then

$$x + e_0 = x - x_0 + fe_1 = x - x_0 - x_1 + f^2 e_2 = \ldots$$

Since $M/f^nM = E/f^nE$ by the snake lemma, we see that $x = x_0 + fx_1 + \ldots + f^{n-1}x_{n-1}$ modulo f^nM. In other words, the map $M \to M/(f^n)M$ is surjective as desired. □

Motivated by the results above we make the following definition.

Definition 15.57.4. Let A be a ring. Let $K \in D(A)$. Let $I \subset A$ be an ideal. We say K is *derived complete with respect to* I if for every $f \in I$ we have $T(K, f) = 0$. If M is an A-module, then we say M is *derived complete with respect to* I if $M[0] \in D(A)$ is derived complete with respect to I.

The full subcategory $D_{comp}(A) = D_{comp}(A, I) \subset D(A)$ consisting of derived complete objects is a strictly full, saturated triangulated subcategory, see Derived Categories, Definitions 13.3.4 and 13.6.1. This subcategory is preserved under products and homotopy limits in $D(A)$. But it is not preserved under countable direct sums in general. We will often simply say M is a derived complete module if the choice of the ideal I is clear from the context.

Proposition 15.57.5. *Let $I \subset A$ be a finitely generated ideal of a ring A. Let M be an A-module. The following are equivalent*

1. *M is I-adically complete, and*
2. *M is derived complete with respect to I and $\bigcap I^nM = 0$.*

Proof. This is clear from the results of Lemma 15.57.3. □

Lemma 15.57.6. *Let I be an ideal of a ring A.*

1. *The derived complete A-modules form a weak Serre subcategory $\mathcal{C}$ of Mod_A.*
2. *$D_{\mathcal{C}}(A) \subset D(A)$ is the full subcategory of derived complete objects.*

Proof. Part (2) is immediate from Lemma 15.57.1 and the definitions. For part (1), suppose that $M \to N$ is a map of derived complete modules. Denote $K = (M \to N)$ the corresponding object of $D(A)$. Pick $f \in I$. Then $\mathrm{Ext}^n_A(A_f, K)$ is zero for all n because $\mathrm{Ext}^n_A(A_f, M)$ and $\mathrm{Ext}^n_A(A_f, N)$ are zero for all n. Hence K is derived complete. By (2) we see that $\mathrm{Ker}(M \to N)$ and $\mathrm{Coker}(M \to N)$ are objects of $\mathcal{C}$. Finally, suppose that $0 \to M_1 \to M_2 \to M_3 \to 0$ is a short exact sequence of A-modules and M_1, M_3 are derived complete. Then it follows from the long exact sequence of Ext's that M_2 is derived complete. Thus $\mathcal{C}$ is a weak Serre subcategory by Homology, Lemma 12.9.3. □

Lemma 15.57.7. *Let I be a finitely generated ideal of a ring A. Let $D_{comp}(A)$ denote the full subcategory of derived complete objects. The inclusion functor $D_{comp}(A) \to D(A)$ has a left adjoint, i.e., given any object K of $D(A)$ there exists a map $K \to K^\wedge$ of K into a derived complete object of $D(A)$ such that the map*

$$\mathrm{Hom}_{D(A)}(K^\wedge, E) \longrightarrow \mathrm{Hom}_{D(A)}(K, E)$$

is bijective whenever E is a derived complete object of $D(A)$.

Proof. Let $I = (f_1, \ldots, f_r)$. For each $i \in \{1, \ldots, r\}$ let $D_i(A)$ denote the full subcategory of objects which are derived complete with respect to (f_i). Then $D_{comp}(A) = D_1(A) \cap \ldots \cap D_r(A)$. It formally follows from this that it suffices to construct a left adjoint of the inclusion functor $D_i(A) \to D(A)$. In other words: we may assume I is generated by a single element.

Let $I = (f)$. Any object of $D(A)$ can be represented by a K-injective complex of A-modules $K^\bullet$. Let $L^\bullet = \mathrm{Hom}_A(A_f, K^\bullet)$. Note that $L^\bullet$ is a K-injective complex of A_f-modules and K-injective as a complex of A-modules, by Lemmas 15.41.3 and 15.41.1. There is a canonical map of complexes $L^\bullet \to K^\bullet$ given by evaluation at 1. We claim that

$$M^\bullet = \mathrm{Cone}(L^\bullet \to K^\bullet)$$

endowed with the canonical map $K^\bullet \to M^\bullet$ has the required universal property. First, observe we have a distinguished triangle

$$L^\bullet \to K^\bullet \to M^\bullet \to L^\bullet[1]$$

Next, for any complex of A_f-modules $N^\bullet$ we have

$$\begin{aligned}
\mathrm{Hom}_{D(A)}(N^\bullet, L^\bullet) &= \mathrm{Hom}_{K(A)}(N^\bullet, L^\bullet) \\
&= \mathrm{Hom}_{K(A_f)}(N^\bullet, L^\bullet) \\
&= \mathrm{Hom}_{K(A_f)}(N^\bullet, \mathrm{Hom}_A(A_f, K^\bullet)) \\
&= \mathrm{Hom}_{K(A)}(N^\bullet, K^\bullet) \\
&= \mathrm{Hom}_{D(A)}(N^\bullet, K^\bullet)
\end{aligned}$$

The first equality holds as $L^\bullet$ is K-injective as a complex of A-modules, the second holds by Algebra, Lemma 10.102.14, the third by definition of $L^\bullet$, the fourth by Algebra, Lemma 10.13.4, the fifth as $K^\bullet$ is a K-injective complex of A-modules. From the distinguished triangle and Derived Categories, Lemma 13.4.2 it follows that $\mathrm{Hom}_{D(A)}(N^\bullet, M^\bullet) = 0$, i.e., $M^\bullet$ is derived complete by Lemma 15.57.1. Next, assume that E is a derived complete object. Then $\mathrm{Hom}_{D(A)}(L^\bullet, E) = 0$ since $L^\bullet$ is a complex of A_f-modules. Hence from the distinguished triangle we obtain $\mathrm{Hom}_{D(A)}(K^\bullet, E) = \mathrm{Hom}_{D(A)}(M^\bullet, E)$. This finishes the proof. □

Let A be a ring and let $I \subset A$ be an ideal. For any $K \in D(A)$ we can consider the derived limit

$$K' = R\lim(K \otimes^{\mathbf{L}}_A A/I^n)$$

This is a functor in K, see Remark 15.55.17. The system of maps $A \to A/I^n$ induces a map $K \to K'$. It turns out that K' is derived complete with respect to I. This "naive" derived completion construction does not agree with the adjoint of Lemma 15.57.7 in general. For example, if $A = \mathbf{Z}_p \oplus \mathbf{Q}_p/\mathbf{Z}_p$ with the second summand an ideal of square zero, $K = A[0]$, and $I = (p)$, then the naive derived completion gives $\mathbf{Z}_p[0]$, but the construction of Lemma 15.57.7 gives $K^\wedge \cong \mathbf{Z}_p[1] \oplus \mathbf{Z}_p[0]$ (computation omitted).

Lemma 15.57.8. *Let A be a ring and let $I \subset A$ be an ideal. For $K \in D(A)$ the naive derived completion $K' = R\lim(K \otimes^{\mathbf{L}}_A A/I^n)$ is derived complete with respect to I.*

Proof. Let $f \in I$. The groups $\mathrm{Ext}^p_A(A_f, K')$ sit in short exact sequences

$$0 \to R^1\lim \mathrm{Ext}^{p-1}_A(A_f, K \otimes^{\mathbf{L}}_A A/I^n) \to \mathrm{Ext}^p_A(A_f, K') \to \lim \mathrm{Ext}^p_A(A_f, K \otimes^{\mathbf{L}}_A A/I^n) \to 0$$

by Lemma 15.55.3. We conclude since f acts both as an isomorphism and nilpotently on the outer terms. □

Lemma 15.57.9. *Let A be a ring. Let $f \in A$. If there exists an integer $c \geq 1$ such that $A[f^c] = A[f^{c+1}] = A[f^{c+2}] = \ldots$ (for example if A is Noetherian), then for all $n \geq 1$ there exist maps*

$$(A \xrightarrow{f^n} A) \longrightarrow A/(f^n), \quad \text{and} \quad A/(f^{n+c}) \longrightarrow (A \xrightarrow{f^n} A)$$

in $D(A)$ inducing an isomorphism of the pro-objects $\{A/(f^n)\}$ and $\{(f^n : A \to A)\}$ in $D(A)$.

Proof. The first displayed arrow is obvious. We can define the second arrow of the lemma by the diagram

$$\begin{array}{ccc} A/A[f^c] & \xrightarrow[f^{n+c}]{} & A \\ {\scriptstyle f^c}\downarrow & & \downarrow{\scriptstyle 1} \\ A & \xrightarrow{f^n} & A \end{array}$$

Since the top horizontal arrow is injective the complex in the top row is quasi-isomorphic to $A/f^{n+c}A$. We omit the calculation of compositions needed to show the statement on pro objects. □

Let A be a ring. Let $f_1, \ldots, f_r \in A$. We are going to consider the sequence of Koszul complexes $K_n^\bullet = K_\bullet(A, f_1^n, \ldots, f_r^n)$ placed in cohomological degrees $-r, -r+1, \ldots, 0$. Using the functoriality of Lemma 15.19.3 we get maps

$$\ldots \to K_3^\bullet \to K_2^\bullet \to K_1^\bullet$$

compatible with $H^0(K_n^\bullet) = A/(f_1^n, \ldots, f_r^n)$ and the natural maps between these quotients. A key feature of the discussion below will use that for $m > n$ the map

$$K_m^{-p} = \wedge^p(A^{\oplus r}) \to \wedge^p(A^{\oplus r}) = K_n^{-p}$$

is given by multiplication by $f_{i_1}^{m-n} \ldots f_{i_p}^{m-n}$ on the basis element $e_{i_1} \wedge \ldots \wedge e_{i_p}$. Finally, note that there is a compatible system of maps $A \to K_n^\bullet$.

Lemma 15.57.10. *With notation as above, let $I = (f_1, \ldots, f_r) \subset A$. For $K \in D(A)$ the object $K' = R\lim(K \otimes_A^{\mathbf{L}} K_n^\bullet)$ is derived complete with respect to I.*

Proof. Let $f \in I$. We have short exact sequences

$$0 \to R^1 \lim \mathrm{Ext}_A^{p-1}(A_f, K \otimes_A^{\mathbf{L}} K_n^\bullet) \to \mathrm{Ext}_A^p(A_f, K') \to \lim \mathrm{Ext}_A^p(A_f, K \otimes_A^{\mathbf{L}} K_n^\bullet) \to 0$$

by Lemma 15.55.3. We conclude the middle term is zero as f acts both as an isomorphism and nilpotently on the outer terms (recall that f_i^n acts by an endomorphism of $K_n^\bullet$ which is homotopic to zero). Thus K is derived complete with respect to I. □

Lemma 15.57.11. *With notation as above. Let $I = (f_1, \ldots, f_r)$. Let $K \in D(A)$. The following are equivalent*

1. *K is derived complete with respect to I, and*
2. *the canonical map $K \to R\lim(K \otimes_A^{\mathbf{L}} K_n^\bullet)$ is an isomorphism of $D(A)$.*

Proof. If (2) holds, then K is derived complete with respect to I by Lemma 15.57.10. Conversely, assume that K is derived complete with respect to I. Consider the filtrations

$$K_n^\bullet \supset \sigma_{\geq -r+1} K_n^\bullet \supset \sigma_{\geq -r+2} K_n^\bullet \supset \ldots \supset \sigma_{\geq -1} K_n^\bullet \supset \sigma_{\geq 0} K_n^\bullet = A$$

by stupid truncations (Homology, Section 12.13). Because the construction $R\lim(K \otimes E)$ is exact in the second variable (Lemma 15.55.18) we see that it suffices to show

$$R\lim\left(K \otimes^{\mathbf{L}}_A (\sigma_{\geq p}K^\bullet_n/\sigma_{\geq p+1}K^\bullet_n)\right) = 0$$

for $p < 0$. The explicit description of the Koszul complexes above shows that

$$R\lim\left(K \otimes^{\mathbf{L}}_A (\sigma_{\geq p}K^\bullet_n/\sigma_{\geq p+1}K^\bullet_n)\right) = \bigoplus\nolimits_{i_1,\ldots,i_{-p}} T(K, f_{i_1} \ldots f_{i_{-p}})$$

which is zero for $p < 0$ by assumption on K. □

Lemma 15.57.12. *With notation as above. Let $I = (f_1, \ldots, f_r) \subset A$. The functor which sends $K \in D(A)$ to the derived limit $K' = R\lim(K \otimes^{\mathbf{L}}_A K^\bullet_n)$ is the left adjoint to the inclusion functor $D_{comp}(A) \to D(A)$ constructed in Lemma 15.57.7.*

Proof. The assingment $K \rightsquigarrow K'$ is a functor and K' is derived complete with respect to I by Lemma 15.57.10. By a formal argument (omitted) we see that it suffices to show $K \to K'$ is an isomorphism if K is derived complete with respect to I. This is Lemma 15.57.11. □

Lemma 15.57.13. *With notation as above. If A is Noetherian, then for every n there exists an $m \geq n$ such that $K^\bullet_m \to K^\bullet_n$ factors through the map $K^\bullet_m \to A/(f_1^m, \ldots, f_r^m)$. In other words, the pro-objects $\{K^\bullet_n\}$ and $\{A/(f_1^n, \ldots, f_r^n)\}$ of $D(A)$ are isomorphic.*

Proof. Note that the Koszul complexes have length r. Thus the dual of Derived Categories, Lemma 13.11.5 implies it suffices to show that for every $p < 0$ and $n \in \mathbf{N}$ there exists an $m \geq n$ such that $H^p(K^\bullet_m) \to H^p(K^\bullet_n)$ is zero. Since A is Noetherian, we see that

$$H^p(K^\bullet_n) = \frac{\mathrm{Ker}(K^p_n \to K^{p+1}_n)}{\mathrm{Im}(K^{p-1}_n \to K^p_n)}$$

is a finite A-module. Moreover, the map $K^p_m \to K^p_n$ is given by a diagonal matrix whose entries are in the ideal $(f_1^{m-n}, \ldots, f_r^{m-n})$ if $p < 0$ (in fact they are in the $|p|$th power of that ideal). Note that $H^p(K^\bullet_n)$ is annihilated by $I = (f_1^n, \ldots, f_r^n)$, see Lemma 15.19.6. Now $I^t \subset (f_1^{m-n}, \ldots, f_r^{m-n})$ for $m = n + tr$. Thus by Artin-Rees (Algebra, Lemma 10.47.4) for some m large enough we see that the image of $K^p_m \to K^p_n$ intersected with $\mathrm{Ker}(K^p_n \to K^{p+1}_n)$ is contained in $I\mathrm{Ker}(K^p_n \to K^{p+1}_n)$. For this m we get the zero map. □

Proposition 15.57.14. *Let A be a Noetherian ring. Let $I \subset A$ be an ideal. The functor which sends $K \in D(A)$ to the derived limit $K' = R\lim(K \otimes^{\mathbf{L}}_A A/I^n)$ is the left adjoint to the inclusion functor $D_{comp}(A) \to D(A)$ constructed in Lemma 15.57.7.*

Proof. Say $(f_1, \ldots, f_r) = I$ and let $K^\bullet_n$ be the Koszul complex with respect to $f_1^n, \ldots, f_r^n$. By Lemma 15.57.12 it suffices to prove that

$$R\lim(K \otimes^{\mathbf{L}}_A K^\bullet_n) = R\lim(K \otimes^{\mathbf{L}}_A A/(f_1^n, \ldots, f_r^n)) = R\lim(K \otimes^{\mathbf{L}}_A A/I^n).$$

By Lemma 15.57.13 the pro-objects $\{K^\bullet_n\}$ and $\{A/(f_1^n, \ldots, f_r^n)\}$ of $D(A)$ are isomorphic. It is clear that the pro-objects $\{A/(f_1^n, \ldots, f_r^n)\}$ and $\{A/I^n\}$ are isomorphic. Thus the map from left to right is an isomorphism by Lemma 15.55.19. □

Lemma 15.57.15. *Let I be a finitely generated ideal of a ring A. Let M be a derived complete A-module. If $M/IM = 0$, then $M = 0$.*

Proof. Assume that M/IM is zero. Let $I = (f_1, \ldots, f_r)$. Let $i < r$ be the largest integer such that $N = M/(f_1, \ldots, f_i)M$ is nonzero. If i does not exist, then $M = 0$ which is what we want to show. Then N is derived complete as a cokernel of a map between derived complete modules, see Lemma 15.57.6. By our choice of i we have that $f_{i+1} : N \to N$ is surjective. Hence

$$\lim(\ldots \to N \xrightarrow{f_{i+1}} N \xrightarrow{f_{i+1}} N)$$

is nonzero, contradicting the derived completeness of N. □

Lemma 15.57.16. *Let I be an ideal of a Noetherian ring A. Let M be a derived complete A-module. If M/IM is a finite A/I-module, then $M = \lim M/I^nM$ and M is a finite $A^\wedge$-module.*

Proof. Assume M/IM is finite. Pick $x_1, \ldots, x_t \in M$ which map to generators of M/IM. We obtain a map $A^{\oplus t} \to M$ mapping the ith basis vector to x_i. By Proposition 15.57.14 the derived completion of A is $A^\wedge = \lim A/I^n$. As M is derived complete, we see that our map factors through a map $q : (A^\wedge)^{\oplus t} \to M$. The module $\mathrm{Coker}(q)$ is zero by Lemma 15.57.15. Thus M is a finite $A^\wedge$-module. Since $A^\wedge$ is Noetherian and complete with respect to $IA^\wedge$, it follows that M is I-adically complete (use Algebra, Lemmas 10.91.9, 10.91.16, and 10.47.4). □

Remark 15.57.17. Let A be a ring and $f \in A$. Set $I = (f)$. In this situation we have the naive derived I-adic completion functor $K \mapsto K' = R\lim(K \otimes^{\mathbf{L}}_A A/f^nA)$ and the functor

$$K \mapsto K^\wedge = R\lim(K \otimes^{\mathbf{L}}_A (A \xrightarrow{f^n} A))$$

of Lemmas 15.57.12 and 15.57.7. There is a natural transformation of functors $K^\wedge \to K'$. Thus we ask: When is this transformation an isomorphism of functors? We have seen in Proposition 15.57.14 that this is true if A is Noetherian. More generally, the same argument shows this is true if the pro-objects $\{(f^n : A \to A)\}$ and $\{A/f^nA\}$ are equal, for example if f-torsion is bounded (Lemma 15.57.9). Conversely, we see from Lemma 15.55.18 that the condition is exactly that

$$R\lim(K \otimes^{\mathbf{L}}_A A[f^n])$$

is zero for all $K \in D(A)$. Here the maps of the system $(A[f^n])$ are given by multiplication by f. Taking $K = A$ and $K = \bigoplus_{i \in \mathbf{N}} A$ we see from Lemma 15.55.8 this implies $(A[f^n])$ is zero as a pro-object, i.e., $f^{n-1}A[f^n] = 0$ for some n, i.e., $A[f^{n-1}] = A[f^n]$, i.e., the f-torsion is bounded.

Example 15.57.18. Let A be a ring. Let $f \in A$ be a nonzero divisor. An example to keep in mind is $A = \mathbf{Z}_p$ and $f = p$. Let M be an A-module. Claim: M is derived complete with respect to f if and only if there exists a short exact sequence

$$0 \to K \to L \to M \to 0$$

where K, L are f-adically complete modules whose f-torsion is zero. Namely, if there is a such a short exact sequence, then

$$M \otimes^{\mathbf{L}}_A (A \xrightarrow{f^n} A) = (K/f^nK \to L/f^nL)$$

because f is a nonzero divisor on K and L and we conclude that $R\lim(M \otimes^{\mathbf{L}}_A (A \xrightarrow{f^n} A))$ is quasi-isomorphic to $K \to L$, i.e., M. This shows that M is derived complete by Lemma 15.57.11. Conversely, suppose that M is derived complete. Choose a

surjection $F \to M$ where F is a free A-module. Since f is a nonzero divisor on F the derived completion of F is $L = \lim F/f^nF$. Note that L is f-torsion free: if (x_n) with $x_n \in F$ represents an element ξ of L and $f\xi = 0$, then $x_n = x_{n+1} + f^n z_n$ and $fx_n = f^n y_n$ for some $z_n, y_n \in F$. Then $f^n y_n = fx_n = fx_{n+1} + f^{n+1}z_n = f^{n+1}y_{n+1} + f^{n+1}z_n$ and since f is a nonzero divisor on F we see that $y_n \in fF$ which implies that $x_n \in f^nF$, i.e., $\xi = 0$. Since L is the derived completion, the universal property gives a map $L \to M$ factoring $F \to M$. Let $K = \mathrm{Ker}(L \to M)$ be the kernel. Again K is f-torsion free, hence the derived completion of K is $\lim K/f^nK$. On the other hand, both K and L are derived complete, hence K is too by Lemma 15.57.6. It follows that $K = \lim K/f^nK$ and the claim is proved.

Lemma 15.57.19. *Let $A \to B$ be a ring map. Let $I \subset A$ be an ideal. The inverse image of $D_{comp}(B, IB)$ under the restriction functor $D(B) \to D(A)$ is $D_{comp}(A, I)$.*

Proof. Using Lemma 15.57.2 we see that $L \in D(B)$ is in $D_{comp}(B, IB)$ if and only if $T(L, f)$ is zero for every local section $f \in I$. Observe that the cohomology of $T(L, f)$ is computed in the category of abelian groups, so it doesn't matter whether we think of f as an element of A or take the image of f in B. The lemma follows immediately from this and the definition of derived complete objects. □

Lemma 15.57.20. *Let $A \to B$ be a ring map. Let $I \subset A$ be a finitely generated ideal. If $A \to B$ is flat and $A/I \cong B/IB$, then the restriction functor $D(B) \to D(A)$ induces an equivalence $D_{comp}(B, IB) \to D_{comp}(A, I)$.*

Proof. Choose generators $f_1, \ldots, f_r$ of I. Denote $\check{\mathcal{C}}^\bullet_A \to \check{\mathcal{C}}^\bullet_B$ the quasi-isomorphism of extended alternating Čech complexes of Lemma 15.56.8. Let $K \in D_{comp}(A, I)$. Let $I^\bullet$ be a K-injective complex of A-modules representing K. Since $\mathrm{Ext}^n_A(A_f, K)$ and $\mathrm{Ext}^n_A(B_f, K)$ are zero for all $f \in I$ and $n \in \mathbf{Z}$ (Lemma 15.57.1) we conclude that $\check{\mathcal{C}}^\bullet_A \to A$ and $\check{\mathcal{C}}^\bullet_B \to B$ induce quasi-isomorphisms

$$I^\bullet = \mathrm{Hom}_A(A, I^\bullet) \longrightarrow \mathrm{Tot}(\mathrm{Hom}_A(\check{\mathcal{C}}^\bullet_A, I^\bullet))$$

and

$$\mathrm{Hom}_A(B, I^\bullet) \longrightarrow \mathrm{Tot}(\mathrm{Hom}_A(\check{\mathcal{C}}^\bullet_B, I^\bullet))$$

Some details omitted. Since $\check{\mathcal{C}}^\bullet_A \to \check{\mathcal{C}}^\bullet_B$ is a quasi-isomorphism and $I^\bullet$ is K-injective we conclude that $\mathrm{Hom}_A(B, I^\bullet) \to I^\bullet$ is a quasi-isomorphism. As the complex $\mathrm{Hom}_A(B, I^\bullet)$ is a complex of B-modules we conclude that K is in the image of the restriction map, i.e., the functor is essentially surjective

In fact, the argument shows that $F : D_{comp}(A, I) \to D_{comp}(B, IB)$, $K \mapsto \mathrm{Hom}_A(B, I^\bullet)$ is a left inverse to restriction. Finally, suppose that $L \in D_{comp}(B, IB)$. Represent L by a K-injective complex $J^\bullet$ of B-modules. Then $J^\bullet$ is also K-injective as a complex of A-modules (Lemma 15.41.1) hence $F(\text{restriction of } L) = \mathrm{Hom}_A(B, J^\bullet)$. There is a map $J^\bullet \to \mathrm{Hom}_A(B, J^\bullet)$ of complexes of B-modules, whose composition with $\mathrm{Hom}_A(B, J^\bullet) \to J^\bullet$ is the identity. We conclude that F is also a right inverse to restriction and the proof is finished. □

15.58. Taking limits of complexes

In this section we discuss what happens when we have a "formal deformation" of a complex and we take its limit. More precisely, we have a ring A an ideal I and objects $K_n \in D(A/I^n)$ which fit together in the sense that

$$K_n = K_{n+1} \otimes^{\mathbf{L}}_{A/I^{n+1}} A/I^n.$$

Under some additional hypotheses we can show that $K = R\lim K_n$ reproduces the system in the sense that $K_n = K \otimes^{\mathbf{L}}_A A/I^n$. We do not know if the following lemma holds for unbounded complexes.

Lemma 15.58.1. *Let A be a ring and $I \subset A$ an ideal. Suppose given $K_n \in D(A/I^n)$ and maps $K_{n+1} \to K_n$ in $D(A/I^{n+1})$. If*

(1) *A is Noetherian,*
(2) *K_1 is bounded above, and*
(3) *the maps induce isomorphisms $K_{n+1} \otimes^{\mathbf{L}}_{A/I^{n+1}} A/I^n \to K_n$,*

then $K = R\lim K_n$ is a derived complete object of $D^-(A)$ and $K \otimes^{\mathbf{L}}_A A/I^n \to K_n$ is an isomorphism for all n.

Proof. Suppose that $H^i(K_1) = 0$ for $i > b$. Then we can find a complex of free A/I-modules $P_1^\bullet$ representing K_1 with $P_1^i = 0$ for $i > b$. By Lemma 15.51.3 we can, by induction on $n > 1$, find complexes $P_n^\bullet$ of free A/I^n-modules representing K_n and maps $P_n^\bullet \to P_{n-1}^\bullet$ representing the maps $K_n \to K_{n-1}$ inducing isomorphisms (!) of complexes $P_n^\bullet/I^{n-1}P_n^\bullet \to P_{n-1}^\bullet$.

Thus we have arrived at the situation where $R\lim K_n$ is represented by $P^\bullet = \lim P_n^\bullet$, see Lemma 15.55.9 and Remark 15.55.15. The complexes $P_n^\bullet$ are uniformly bounded above complexes of flat A/I^n-modules and the transition maps are termwise surjective. Then $P^\bullet$ is a bounded above complex of flat A-modules by Lemma 15.18.4. It follows that $K \otimes^{\mathbf{L}}_A A/I^t$ is represented by $P^\bullet \otimes_A A/I^t$. We have $P^\bullet \otimes_A A/I^t = \lim P_n^\bullet \otimes_A A/I^t$ termwise by Lemma 15.18.4. The transition maps $P_{n+1}^\bullet \otimes_A A/I^t \to P_n^\bullet \otimes_A A/I^t$ are isomorphisms for $n \geq t$. Hence we have $\lim P_n^\bullet \otimes_A A/I^t = R\lim P_n^\bullet \otimes_A A/I^t$. By assumption and our choice of $P_n^\bullet$ the complex $P_n^\bullet \otimes_A A/I^t = P_n^\bullet \otimes_{A/I^n} A/I^t$ represents $K_n \otimes^{\mathbf{L}}_{A/I^n} A/I^t = K_t$ for all $n \geq t$. We conclude

$$P^\bullet \otimes_A A/I^t = R\lim P_n^\bullet \otimes_A A/I^t = R\lim K_t = K_t$$

In other words, we have $K \otimes^{\mathbf{L}}_A A/I^t = K_t$. This proves the lemma as it follows that K is derived complete by Proposition 15.57.14. □

Lemma 15.58.2. *Let A be a ring and $I \subset A$ an ideal. Suppose given $K_n \in D(A/I^n)$ and maps $K_{n+1} \to K_n$ in $D(A/I^{n+1})$. Assume*

(1) *A is I-adically complete,*
(2) *K_1 is pseudo-coherent, and*
(3) *the maps induce isomorphisms $K_{n+1} \otimes^{\mathbf{L}}_{A/I^{n+1}} A/I^n \to K_n$.*

Then $K = R\lim K_n$ is a pseudo-coherent, derived complete object of $D(A)$ and $K \otimes^{\mathbf{L}}_A A/I^n \to K_n$ is an isomorphism for all n.

Proof. By assumption we can find a bounded above complex of finite free A/I-modules $P_1^\bullet$ representing K_1, see Definition 15.48.1. By Lemma 15.51.3 we can, by induction on $n > 1$, find complexes $P_n^\bullet$ of finite free A/I^n-modules representing K_n and maps $P_n^\bullet \to P_{n-1}^\bullet$ representing the maps $K_n \to K_{n-1}$ inducing isomorphisms (!) of complexes $P_n^\bullet/I^{n-1}P_n^\bullet \to P_{n-1}^\bullet$.

Thus $R\lim K_n$ is represented by $P^\bullet = \lim P_n^\bullet$, see Lemma 15.55.9 and Remark 15.55.15. Since A is I-adically complete the modules P^i are finite free A-modules. Thus K is pseudo-coherent. Moreover, $P^\bullet$ is a bounded above complex of flat A-modules. It follows that $K \otimes^{\mathbf{L}}_A A/I^t$ is represented by $P^\bullet \otimes_A A/I^t$. We have $P^\bullet \otimes_A$

$A/I^t = \lim P_n^\bullet \otimes_A A/I^t$ termwise. The transition maps $P_{n+1}^\bullet \otimes_A A/I^t \to P_n^\bullet \otimes_A A/I^t$ are isomorphisms for $n \geq t$. Hence we have $\lim P_n^\bullet \otimes_A A/I^t = R\lim P_n^\bullet \otimes_A A/I^t$. By assumption and our choice of $P_n^\bullet$ the complex $P_n^\bullet \otimes_A A/I^t = P_n^\bullet \otimes_{A/I^n} A/I^t$ represents $K_n \otimes^{\mathbf{L}}_{A/I^n} A/I^t = K_t$ for all $n \geq t$. We conclude

$$P^\bullet \otimes_A A/I^t = R\lim P_n^\bullet \otimes_A A/I^t = R\lim K_t = K_t$$

In other words, we have $K \otimes^{\mathbf{L}}_A A/I^t = K_t$. Finally, K_n is a derived complete object of $D(A)$ as it is annihilated by I^n. Since the category of derived objects is preserved under homotopy limits we see that K is derived complete. This proves the lemma. □

Lemma 15.58.3. *Let A be a ring and $I \subset A$ an ideal. Suppose given $K_n \in D(A/I^n)$ and maps $K_{n+1} \to K_n$ in $D(A/I^{n+1})$. Assume*

1. *A is I-adically complete,*
2. *K_1 is a perfect object, and*
3. *the maps induce isomorphisms $K_{n+1} \otimes^{\mathbf{L}}_{A/I^{n+1}} A/I^n \to K_n$.*

Then $K = R\lim K_n$ is a perfect, derived complete object of $D(A)$ and $K \otimes^{\mathbf{L}}_A A/I^n \to K_n$ is an isomorphism for all n.

Proof. By Lemma 15.58.2 we see that K is bounded above, pseudo-coherent, and that $K \otimes^{\mathbf{L}}_A A/I^n \to K_n$ is an isomorphism for all n. Thus it suffices to show that $H^i(K \otimes^{\mathbf{L}}_A \kappa) = 0$ for $i \ll 0$ and every surjective map $A \to \kappa$ whose kernel is a maximal ideal $\mathfrak{m}$, see Lemma 15.50.16. Since A is I-adically complete we have $I \subset \mathfrak{m}$, see Algebra, Lemma 10.91.11. Hence

$$K \otimes^{\mathbf{L}}_A \kappa = K \otimes^{\mathbf{L}}_A A/I \otimes^{\mathbf{L}}_{A/I} \kappa = K_1 \otimes^{\mathbf{L}}_{A/I} \kappa$$

and we get what we want as K_1 has finite tor dimension by Lemma 15.50.2. □

15.59. Miscellany

Some results which do not fit anywhere else.

Lemma 15.59.1. *Let A be a Noetherian ring. Let $I \subset A$ be an ideal. Let M, N be finite A-modules. Set $M_n = M/I^nM$ and $N_n = N/I^nN$. Then the systems $(\mathrm{Hom}_A(M_n, N_n))$ and $(Isom_A(M_n, N_n))$ are Mittag-Leffler.*

Proof. Note that $\mathrm{Hom}_A(M_n, N_n) = \mathrm{Hom}_A(M, N_n)$. Choose a presentation

$$A^{\oplus t} \xrightarrow{T} A^{\oplus s} \to M \to 0$$

The transpose of T induces a map $\varphi : N^{\oplus s} \to N^{\oplus t}$ such that

$$\mathrm{Hom}_A(M, N_n) = \varphi^{-1}(I^nN^{\oplus t})/I^nN^{\oplus s}.$$

By Artin-Rees there exists an integer c such that

$$\varphi^{-1}(I^nN^{\oplus t}) = \mathrm{Ker}(\varphi) + I^{n-c}\varphi^{-1}(I^cN^{\oplus t})$$

see Algebra, Lemma 10.47.5. Thus it is clear that the images of $\mathrm{Hom}_A(M, N_n) \to \mathrm{Hom}_A(M, N_m)$ stabilize for $n \geq m + c$. The result for isomorphisms follows from this and a consideration of $\mathrm{Hom}_A(N_n, M_n)$. □

Lemma 15.59.2. *Let A be a Noetherian ring. Let $I \subset A$ be an ideal. Let M, N be finite A-modules. Set $M_n = M/I^nM$ and $N_n = N/I^nN$. If $M_n \cong N_n$ for all n, then $M^\wedge \cong N^\wedge$ as $A^\wedge$-modules.*

Proof. By Lemma 15.59.1 the system $(\mathrm{Isom}_A(M_n, N_n))$ is Mittag-Leffler. By assumption each of the sets $\mathrm{Isom}_A(M_n, N_n)$ is nonempty. Hence $\lim \mathrm{Isom}_A(M_n, N_n)$ is nonempty. Since $\lim \mathrm{Isom}_A(M_n, N_n) = \mathrm{Isom}(M^\wedge, N^\wedge)$ (use Algebra, Lemma 10.92.1) we obtain an isomorphism. □

Lemma 15.59.3. *Let A be a Noetherian ring. Let $I \subset A$ be an ideal. Let M, N be finite A-modules with N annihilated by I. For each $p > 0$ there exists an n such that the map $Ext^p_A(M, N) \to Ext^p_A(I^n M, N)$ is zero.*

Proof. The result is clear for $p = 0$ (with $n = 1$). Choose a short exact sequence $0 \to K \to A^{\oplus t} \to M \to 0$. For n pick a short exact sequence $0 \to L \to A^{\oplus s} \to I^n M \to 0$. It is clear that we can construct a map of short exact sequences

$$\begin{array}{ccccccccc} 0 & \longrightarrow & L & \longrightarrow & A^{\oplus s} & \longrightarrow & I^n M & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & K & \longrightarrow & A^{\oplus s} & \longrightarrow & M & \longrightarrow & 0 \end{array}$$

such that $A^{\oplus s} \to A^{\oplus t}$ has image in $(I^n)^{\oplus t}$. By Artin-Rees (Algebra, Lemma 10.47.4) we see that $L \to K$ has image contained in $I^{n-c}K$ if $n \geq c$. At this point the exact sequence

$$\mathrm{Hom}_A(A^{\oplus t}, N) \to \mathrm{Hom}_A(K, N) \to \mathrm{Ext}^1_A(M, N) \to 0$$

and the corresponding sequence for $\mathrm{Ext}^1_A(I^n M, N)$ show that the lemma holds for $p = 1$ with $n = c + 1$. Moreover, we see that the result for $p - 1$ and the module K implies the result for p and the module M by the commutativity of the diagram

$$\begin{array}{ccccc} & & \mathrm{Ext}^{p-1}_A(L, N) & \xrightarrow[\cong]{} & \mathrm{Ext}^p_A(I^n M, N) \\ & \swarrow & & & \downarrow \\ \mathrm{Ext}^{p-1}_A(I^{n-c}K, N) & \longrightarrow & \mathrm{Ext}^{p-1}_A(K, N) & \longrightarrow & \mathrm{Ext}^p_A(M, N) \end{array}$$

for $p > 1$. Some details omitted. □

Lemma 15.59.4. *Let A be a Noetherian ring. Let $I \subset A$ be an ideal. Let M be a finite A-module. There exists an integer $n > 0$ such that $I^n M \to M$ factors through the map $I \otimes^{\mathbf{L}}_A M \to M$ in $D(A)$.*

Proof. Consider the distinguished triangle

$$I \otimes^{\mathbf{L}}_A M \to M \to A/I \otimes^{\mathbf{L}}_A M \to I \otimes^{\mathbf{L}}_A M[1]$$

By the axioms of a triangulated category it suffices to prove that $I^n M \to A/I \otimes^{\mathbf{L}}_A M$ is zero in $D(A)$ for some n. Choose generators $f_1, \ldots, f_r$ of I and let $K = K_\bullet(A, f_1, \ldots, f_r)$ be the Koszul complex. and consider the factorization $A \to K \to A/I$ of the quotient map. Then we see that it suffices to show that $I^n M \to K \otimes_A M$ is zero in $D(A)$ for some $n > 0$. Suppose that we have found an $n > 0$ such that $I^n M \to K \otimes_A M$ factors through $\tau_{\geq t}(K \otimes_A M)$ in $D(A)$. Then the obstruction to factoring through $\tau_{\geq t+1}(K \otimes_A M)$ is an element in $\mathrm{Ext}^t(I^n M, H_t(K \otimes_A M))$. The finite A-module $H_t(K \otimes_A M)$ is annihilated by I. Then by Lemma 15.59.3 we can after increasing n assume this obstruction element is zero. Repeating this a finite number of times we find n such that $I^n M \to K \otimes_A M$ factors through $0 = \tau_{\geq r+1}(K \otimes_A M)$ in $D(A)$ and we win. □

Lemma 15.59.5. *Let R be a Noetherian local ring. Let $I \subset R$ be an ideal and let E be a module over R/I. If R/I has finite projective dimension and E has finite projective dimension over R/I, then E has finite projective dimension over R and*

$$pd_R(E) = pd_R(R/I) + pd_{R/I}(E)$$

Proof. We will use that, for a finite module, having finite projective dimension over R, resp. R/I is the same as being a perfect module, see discussion following Definition 15.50.1. We see that E has finite projective dimension over R by Lemma 15.50.7. Thus we can apply Auslander-Buchsbaum (Algebra, Proposition 10.97.1) to see that

$$\mathrm{pd}_R(E) + \mathrm{depth}_R(E) = \mathrm{depth}(R), \quad \mathrm{pd}_{R/I}(E) + \mathrm{depth}_{R/I}(E) = \mathrm{depth}(R/I),$$

and

$$\mathrm{pd}_R(R/I) + \mathrm{depth}_R(R/I) = \mathrm{depth}(R)$$

Combined with the trivial observation that $\mathrm{depth}_R(E) = \mathrm{depth}_{R/I}(E)$ this concludes the proof. □

15.60. Weakly étale ring maps

Most of the results in this section are from the paper [**Oli83**] by Olivier. See also the related paper [**Fer67**].

Definition 15.60.1. A ring A is called *absolutely flat* if every A-module is flat over A. A ring map $A \to B$ is *weakly étale* or *absolutely flat* if both $A \to B$ and $B \to B \otimes_A B$ are flat.

For example a localization is weakly étale. An étale ring map is weakly étale. Here is a simple, yet key property.

Lemma 15.60.2. *Let $A \to B$ be a ring map such that $B \otimes_A B \to B$ is flat. Let N be a B-module. If N is flat as an A-module, then N is flat as a B-module.*

Proof. Assume N is a flat as an A-module. Then the functor

$$\mathrm{Mod}_B \longrightarrow \mathrm{Mod}_{B \otimes_A B}, \quad N' \mapsto N \otimes_A N'$$

is exact. As $B \otimes_A B \to B$ is flat we conclude that the functor

$$\mathrm{Mod}_B \longrightarrow \mathrm{Mod}_B, \quad N' \mapsto (N \otimes_A N') \otimes_{B \otimes_A B} B = N \otimes_B N'$$

is exact, hence N is flat over B. □

Definition 15.60.3. Let A be a ring. Let $d \geq 0$ be an integer. We say that A has *weak dimension* $\leq d$ if every A-module has tor dimension $\leq d$.

Lemma 15.60.4. *Let $A \to B$ be a weakly étale ring map. If A has weak dimension at most d, then so does B.*

Proof. Let N be a B-module. If $d = 0$, then N is flat as an A-module, hence flat as a B-module by Lemma 15.60.2. Assume $d > 0$. Choose a resolution $F_\bullet \to N$ by free B-modules. Our assumption implies that $K = \mathrm{Im}(F_d \to F_{d-1})$ is A-flat, see Lemma 15.49.2. Hence it is B-flat by Lemma 15.60.2. Thus $0 \to K \to F_{d-1} \to \ldots \to F_0 \to N \to 0$ is a flat resolution of length d and we see that N has tor dimension at most d. □

Lemma 15.60.5. *Let A be a ring. The following are equivalent*

(1) *A has weak dimension ≤ 0,*
(2) *A is absolutely flat, and*
(3) *A is reduced and every prime is maximal.*

In this case every local ring of A is a field.

Proof. The equivalence of (1) and (2) is immediate. Assume A is absolutely flat. This implies every ideal of A is pure, see Algebra, Definition 10.103.1. Hence every finitely generated ideal is generated by an idempotent by Algebra, Lemma 10.103.5. If $f \in A$, then $(f) = (e)$ for some idempotent $e \in A$ and $D(f) = D(e)$ is open and closed (Algebra, Lemma 10.19.1). This already implies every ideal of A is maximal for example by Algebra, Lemma 10.24.6. Moreover, if f is nilpotent, then $e = 0$ hence $f = 0$. Thus A is reduced.

Assume A is reduced and every prime of A is maximal. Let M be an A-module. Our goal is to show that M is flat. We may write M as a filtered colimit of finite A-modules, hence we may assume M is finite (Algebra, Lemma 10.36.2). There is a finite filtration of M by modules of the form A/I (Algebra, Lemma 10.5.4), hence we may assume that $M = A/I$ (Algebra, Lemma 10.36.12). Thus it suffices to show every ideal of A is pure. Since A every local ring of A is a field (by Algebra, Lemma 10.24.4 and the fact that every prime of A is minimal), we see that every ideal $I \subset A$ is radical. Note that every closed subset of $\mathrm{Spec}(A)$ is closed under specialization. Thus every (radical) ideal of A is pure by Algebra, Lemma 10.103.4. □

Lemma 15.60.6. *A product of fields is an absolutely flat ring.*

Proof. Let K_i be a family of fields. If $f = (f_i) \in \prod K_i$, then the ideal generated by f is the same as the ideal generated by the idempotent $e = (e_i)$ with $e_i = 0, 1$ according to whether f_i is $0, 1$. Thus $D(f) = D(e)$ is open and closed and we conclude by Lemma 15.60.5 and Algebra, Lemma 10.24.6. □

Lemma 15.60.7. *Let $A \to B$ and $A \to A'$ be ring maps. Let $B' = B \otimes_A A'$ be the base change of B.*

(1) *If $B \otimes_A B \to B$ is flat, then $B' \otimes_{A'} B' \to B'$ is flat.*
(2) *If $A \to B$ is weakly étale, then $A' \to B'$ is weakly étale.*

Proof. Assume $B \otimes_A B \to B$ is flat. The ring map $B' \otimes_{A'} B' \to B'$ is the base change of $B \otimes_A B \to B$ by $A \to A'$. Hence it is flat by Algebra, Lemma 10.36.6. This proves (1). Part (2) follows from (1) and the fact (just used) that the base change of a flat ring map is flat. □

Lemma 15.60.8. *Let $A \to B$ be a ring map such that $B \otimes_A B \to B$ is flat.*

(1) *If A is an absolutely flat ring, then so is B.*
(2) *If A is reduced and $A \to B$ is weakly étale, then B is reduced.*

Proof. Part (1) follows immediately from Lemma 15.60.2 and the definitions. If A is reduced, then there exists an injection $A \to A' = \prod_{\mathfrak{p} \subset A \text{ minimal}} A_{\mathfrak{p}}$ of A into an absolutely flat ring (Algebra, Lemma 10.24.7 and Lemma 15.60.6). If $A \to B$ is flat, then the induced map $B \to B' = B \otimes_A A'$ is injective too. By Lemma 15.60.7 the ring map $A' \to B'$ is weakly étale. By part (1) we see that B' is absolutely flat. By Lemma 15.60.5 the ring B' is reduced. Hence B is reduced. □

Lemma 15.60.9. *Let $A \to B$ and $B \to C$ be ring maps.*

(1) *If $B \otimes_A B \to B$ and $C \otimes_B C \to C$ are flat, then $C \otimes_A C \to C$ is flat.*
(2) *If $A \to B$ and $B \to C$ are weakly étale, then $A \to C$ is weakly étale.*

Proof. Part (1) follows from the factorization

$$C \otimes_A C \longrightarrow C \otimes_B C \longrightarrow C$$

of the multiplication map, the fact that

$$C \otimes_B C = (C \otimes_A C) \otimes_{B \otimes_A B} B,$$

the fact that a base change of a flat map is flat, and the fact that the composition of flat ring maps is flat. See Algebra, Lemmas 10.36.6 and 10.36.3. Part (2) follows from (1) and the fact (just used) that the composition of flat ring maps is flat. □

Lemma 15.60.10. *Let $A \to B \to C$ be ring maps.*

(1) *If $B \to C$ is faithfully flat and $C \otimes_A C \to C$ is flat, then $B \otimes_A B \to B$ is flat.*
(2) *If $B \to C$ is faithfully flat and $A \to C$ is weakly étale, then $A \to B$ is weakly étale.*

Proof. Assume $B \to C$ is faithfully flat and $C \otimes_A C \to C$ is flat. Consider the commutative diagram

$$\begin{array}{ccc} C \otimes_A C & \longrightarrow & C \\ \uparrow & & \uparrow \\ B \otimes_A B & \longrightarrow & B \end{array}$$

The vertical arrows are flat, the top horizontal arrow is flat. Hence C is flat as a $B \otimes_A B$-module. The map $B \to C$ is faithfully flat and $C = B \otimes_B C$. Hence B is flat as a $B \otimes_A B$-module by Algebra, Lemma 10.36.8. This proves (1). Part (2) follows from (1) and the fact that $A \to B$ is flat if $A \to C$ is flat and $B \to C$ is faithfully flat (Algebra, Lemma 10.36.8). □

Lemma 15.60.11. *Let A be a ring. Let $B \to C$ be an A-algebra map of weakly étale A-algebras. Then $B \to C$ is weakly étale.*

Proof. Write $B \to C$ as the composition $B \to B \otimes_A C \to C$. The first map is flat as the base change of the flat ring map $A \to C$. The second is the base change of the flat ring map $B \otimes_A B \to B$ by the ring map $B \otimes_A B \to B \otimes_A C$, hence flat. Thus $B \to C$ is flat. The ring map $C \otimes_A C \to C \otimes_B C$ is surjective, hence an epimorphism. Thus Lemma 15.60.2 implies, that since C is flat over $C \otimes_A C$ it follows that C is flat over $C \otimes_B C$. □

Lemma 15.60.12. *Let $A \to B$ be a ring map such that $B \otimes_A B \to B$ is flat. Then $\Omega_{B/A} = 0$, i.e., B is formally unramified over A.*

Proof. Let $I \subset B \otimes_A B$ be the kernel of the flat surjective map $B \otimes_A B \to B$. Then I is a pure ideal (Algebra, Definition 10.103.1), so $I^2 = I$ (Algebra, Lemma 10.103.2). Since $\Omega_{B/A} = I/I^2$ (Algebra, Lemma 10.125.13) we obtain the vanishing. This means B is formally unramified over A by Algebra, Lemma 10.139.2. □

Lemma 15.60.13. *Let $A \to B$ be a ring map. Then $A \to B$ is weakly étale in each of the following cases*

(1) *$B = S^{-1}A$ is a localization of A,*

(2) $A \to B$ *is étale,*
(3) B *is a filtered colimit of weakly étale* A*-algebras.*

Proof. An étale ring map is flat and the map $B \otimes_A B \to B$ is also étale as a map between étale A-algebras (Algebra, Lemma 10.136.9). This proves (2).

Let B_i be a directed system of weakly étale A-algebras. Then $B = \operatorname{colim} B_i$ is flat over A by Algebra, Lemma 10.36.2. Note that the transition maps $B_i \to B_{i'}$ are flat by Lemma 15.60.11. Hence B is flat over B_i for each i, and we see that B is flat over $B_i \otimes_A B_i$ by Algebra, Lemma 10.36.3. Thus B is flat over $B \otimes_A B = \operatorname{colim} B_i \otimes_A B_i$ by Algebra, Lemma 10.36.5.

Part (1) can be proved directly, but also follows by combining (2) and (3). □

Lemma 15.60.14. *Let $K \subset L$ be an extension of fields. If $L \otimes_K L \to L$ is flat, then L is an algebraic separable extension of K.*

Proof. By Lemma 15.60.10 we see that any subfield $K \subset L' \subset L$ the map $L' \otimes_K L' \to L'$ is flat. Thus we may assume L is a finitely generated field extension of K. In this case the fact that L/K is formally unramified (Lemma 15.60.12) implies that L/K is finite separable, see Algebra, Lemma 10.145.1. □

Lemma 15.60.15. *Let K be a field. Let $K \to B$ be a ring map such that $B \otimes_K B \to B$ is flat. Then B is a filtered colimit of étale K-algebras.*

Proof. A field is absolutely flat ring, hence B is a absolutely flat ring by Lemma 15.60.8. Hence B is reduced and every local ring is a field, see Lemma 15.60.5.

Let $\mathfrak{q} \subset B$ be a prime. The ring map $B \to B_{\mathfrak{q}}$ is weakly étale, hence $B_{\mathfrak{q}}$ is weakly étale over K (Lemma 15.60.9). Thus $B_{\mathfrak{p}}$ is a separable algebraic extension of K by Lemma 15.60.14.

Let $K \subset A \subset B$ be a finitely generated K-sub algebra. Then every minimal prime $\mathfrak{p} \subset A$ is the image of a prime $\mathfrak{q}$ of B, see Algebra, Lemma 10.28.5. Thus $\kappa(\mathfrak{p})$ as a subfield of $B_{\mathfrak{q}} = \kappa(\mathfrak{q})$ is separable algebraic over K. Hence every generic point of $\operatorname{Spec}(A)$ is closed (Algebra, Lemma 10.32.9). Thus $\dim(A) = 0$. Then A is the product of its local rings, e.g., by Algebra, Proposition 10.57.6. Moreover, since A is reduced, all local rings are equal to their residue fields wich are finite separable over K. This means that A is étale over K by Algebra, Lemma 10.136.4 and finishes the proof. □

Lemma 15.60.16. *Let $A \to B$ be a ring map. If $A \to B$ is weakly étale, then $A \to B$ induces separable algebraic residue field extensions.*

Proof. Let $\mathfrak{p}$ be a prime of A. Then $\kappa(\mathfrak{p}) \to B \otimes_A \kappa(\mathfrak{p})$ is weakly étale by Lemma 15.60.7. Hence $B \otimes_A \kappa(\mathfrak{p})$ is a filtered colimit of étale $\kappa(\mathfrak{p})$-algebras by Lemma 15.60.15. Hence for $\mathfrak{q} \subset B$ lying over $\mathfrak{p}$ the extension $\kappa(\mathfrak{p}) \subset \kappa(\mathfrak{q})$ is a filtered colimit of finite separable extensions by Algebra, Lemma 10.136.4. □

Lemma 15.60.17. *Let A be a ring. The following are equivalent*

(1) A *has weak dimension* ≤ 1,
(2) *every ideal of A is flat,*
(3) *every finitely generated ideal of A is flat,*
(4) *every submodule of a flat A-module is flat.*

Proof. If A has weak dimension ≤ 1, then the resolution $0 \to I \to A \to A/I \to 0$ shows that every ideal I is is flat by Lemma 15.49.2. Hence (1) $\Rightarrow$ (2).

Assume (4). Let M be an A-module. Choose a surjection $F \to M$ where F is a free A-module. Then $\mathrm{Ker}(F \to M)$ is flat by assumption, and we see that M has tor dimension ≤ 1 by Lemma 15.49.5. Hence (4) $\Rightarrow$ (1).

Every ideal is the union of the finitely generated ideals contained in it. Hence (3) implies (2) by Algebra, Lemma 10.36.2. Thus (3) and (2) are equivalent.

Assume (2). Suppose that $N \subset M$ with M a flat A-module. We will prove that N is flat. We can write $M = \mathrm{colim}\, M_i$ with each M_i finite free, see Algebra, Theorem 10.76.4. Setting $N_i \subset M_i$ the inverse image of N we see that $N = \mathrm{colim}\, N_i$. By Algebra, Lemma 10.36.2. it suffices to prove N_i is flat and we reduce to the case $M = R^{\oplus n}$. In this case the module N has a finite filtration by the submodules $R^{\oplus j} \cap N$ whose subquotients are ideals. By (2) these ideals are flat and hence N is flat by Algebra, Lemma 10.36.12. □

Lemma 15.60.18. *Let J be a set. For each $j \in J$ let A_j be a valuation ring with fraction field K_j. Set $A = \prod A_j$ and $K = \prod K_j$. Then A is has weak dimension at most 1 and $A \to K$ is a localization.*

Proof. Let $I \subset A$ be a finitely generated ideal. By Lemma 15.60.17 it suffices to show that I is a flat A-module. Let $I_j \subset A_j$ be the image of I. Observe that $I_j = I \otimes_A A_j$, hence $I \to \prod I_j$ is surjective by Algebra, Proposition 10.84.2. Thus $I = \prod I_j$. Since A_j is a valuation ring, the ideal I_j is generated by a single element (Algebra, Lemma 10.46.13). Say $I_j = (f_j)$. Then I is generated by the element $f = (f_j)$. Let $e \in A$ be the idempotent which has a 0 or 1 in A_j depending on whether f_j is 0 or 1. Then $f = ge$ for some nonzero divisor $g \in A$: take $g = (g_j)$ with $g_j = 1$ if $f_j = 0$ and $g_j = f_j$ else. Thus $I \cong (e)$ as a module. We conclude I is flat as (e) is a direct summand of A. The final statement is true because $K = S^{-1}A$ where $S = \prod(A_j \setminus \{0\})$. □

Lemma 15.60.19. *Let A be a normal domain with fraction field K. There exists a cartesian diagram*

$$\begin{array}{ccc} A & \longrightarrow & K \\ \downarrow & & \downarrow \\ V & \longrightarrow & L \end{array}$$

of rings where V has weak dimension at most 1 and $V \to L$ is a flat, injective, epimorphism of rings.

Proof. For every $x \in K$, $x \notin A$ pick $V_x \subset K$ as in Algebra, Lemma 10.46.9. Set $V = \prod_{x \in K \setminus A} V_x$ and $L = \prod_{x \in K \setminus A} K$. The ring V has weak dimension at most 1 by Lemma 15.60.18 which also shows that $V \to K$ is a localization. A localization is flat and an epimorphism, see Algebra, Lemmas 10.36.19 and 10.102.5. □

Lemma 15.60.20. *Let A be a ring of weak dimension at most 1. If $A \to B$ is a flat, injective, epimorphism of rings, then A is integrally closed in B.*

Proof. Let $x \in B$ be integral over A. Let $A' = A[x] \subset B$. Then A' is a finite ring extension of A by Algebra, Lemma 10.33.5. To show $A = A'$ it suffices to show $A \to A'$ is an epimorphism by Algebra, Lemma 10.102.6. Note that A' is flat over

A by assumption on A and the fact that B is flat over A (Lemma 15.60.17). Hence the composition

$$A' \otimes_A A' \to B \otimes_A A' \to B \otimes_A B \to B$$

is injective, i.e., $A' \otimes_A A' \cong A'$ and the lemma is proved. □

Lemma 15.60.21. *Let A be a normal domain with fraction field K. Let $A \to B$ be weakly étale. Then B is integrally closed in $B \otimes_A K$.*

Proof. Choose a diagram as in Lemma 15.60.19. As $A \to B$ is flat, the base change gives a cartesian diagram

$$\begin{array}{ccc} B & \longrightarrow & B \otimes_A K \\ \downarrow & & \downarrow \\ B \otimes_A V & \longrightarrow & B \otimes_A L \end{array}$$

of rings. Note that $V \to B \otimes_A V$ is weakly étale (Lemma 15.60.7), hence $B \otimes_A V$ has weak dimension at most 1 by Lemma 15.60.4. Note that $B \otimes_A V \to B \otimes_A L$ is a flat, injective, epimorphism of rings as a flat base change of such (Algebra, Lemmas 10.36.6 and 10.102.3). By Lemma 15.60.20 we see that $B \otimes_A V$ is integrally closed in $B \otimes_A L$. It follows from the cartesian property of the diagram that B is integrally closed in $B \otimes_A K$. □

Lemma 15.60.22. *Let $A \to B$ be a ring homomorphism. Assume*

1. *A is a henselian local ring,*
2. *$A \to B$ is integral,*
3. *B is a domain.*

Then B is a henselian local ring and $A \to B$ is a local homomorphism. If A is strictly henselian, then B is a strictly henselian local ring and the extension $\kappa(\mathfrak{m}_A) \subset \kappa(\mathfrak{m}_B)$ of residue fields is purely inseparable.

Proof. Write B as a filtered colimit $B = \operatorname{colim} B_i$ of finite A-sub algebras. If we prove the results for each B_i, then the result follows for B. See Algebra, Lemma 10.143.5. If $A \to B$ is finite, then B is a product of local henselian rings by Algebra, Lemma 10.143.4. Since B is a domain we see that B is a local ring. The maximal ideal of B lies over the maximal ideal of A by going up for $A \to B$ (Algebra, Lemma 10.33.20). If A is strictly henselian, then the field extension $\kappa(\mathfrak{m}_A) \subset \kappa(\mathfrak{m}_B)$ being algebraic, has to be purely inseparable. Of course, then $\kappa(\mathfrak{m}_B)$ is separably algebraically closed and B is strictly henselian. □

Lemma 15.60.23. *Let $A \to B$ and $A \to C$ be local homomorphisms of local rings. If $A \to C$ is integral and $\kappa(\mathfrak{m}_A) \subset \kappa(\mathfrak{m}_C)$ is purely inseparable, then $D = B \otimes_A C$ is a local ring and $B \to D$ and $C \to D$ are local.*

Proof. Any maximal ideal of D lies over the maximal ideal of B by going up for the integral ring map $B \to D$ (Algebra, Lemma 10.33.20). Now $D/\mathfrak{m}_B D = \kappa(\mathfrak{m}_B) \otimes_A C = \kappa(\mathfrak{m}_B) \otimes_{\kappa(\mathfrak{m}_A)} C/\mathfrak{m}_A C$. The spectrum of $C/\mathfrak{m}_A C$ consists of a single point, namely $\mathfrak{m}_C$. Thus the spectrum of $D/\mathfrak{m}_B D$ is the same as the spectrum of $\kappa(\mathfrak{m}_B) \otimes_{\kappa(\mathfrak{m}_A)} \kappa(\mathfrak{m}_C)$ which is a single point by our assumption that $\kappa(\mathfrak{m}_A) \subset \kappa(\mathfrak{m}_C)$ is purely inseparable. This proves that D is local and that the ring maps $B \to D$ and $C \to D$ are local. □

Theorem 15.60.24 (Olivier)**.** *Let $A \to B$ be a local homomorphism of local rings. If A is strictly henselian and $A \to B$ is weakly étale, then $A = B$.*

Proof. We will show that for all $\mathfrak{p} \subset A$ there is a unique prime $\mathfrak{q} \subset B$ lying over $\mathfrak{p}$ and $\kappa(\mathfrak{p}) = \kappa(\mathfrak{q})$. This implies that $B \otimes_A B \to B$ is bijective on spectra as well as surjective and flat. Hence it is an isomorphism for example by the description of pure ideals in Algebra, Lemma 10.103.4. Hence $A \to B$ is a faithfully flat epimorphism of rings. We get $A = B$ by Algebra, Lemma 10.102.7.

Note that the fibre ring $B \otimes_A \kappa(\mathfrak{p})$ is a colimit of étale extensions of $\kappa(\mathfrak{p})$ by Lemmas 15.60.7 and 15.60.15. Hence, if there exists more than one prime lying over $\mathfrak{p}$ or if $\kappa(\mathfrak{p}) \neq \kappa(\mathfrak{q})$ for some $\mathfrak{q}$, then $B \otimes_A L$ has a nontrivial idempotent for some (separable) algebraic field extension $L \supset \kappa(\mathfrak{p})$.

Let $\kappa(\mathfrak{p}) \subset L$ be an algebraic field extension. Let $A' \subset L$ be the integral closure of $A/\mathfrak{p}$ in L. By Lemma 15.60.22 we see that A' is a strictly henselian local ring whose residue field is a purely inseparable extension of the residue field of A. Thus $B \otimes_A A'$ is a local ring by Lemma 15.60.23. On the other hand, $B \otimes_A A'$ is integrally closed in $B \otimes_A L$ by Lemma 15.60.21. Since $B \otimes_A A'$ is local, it follows that the ring $B \otimes_A L$ does not have nontrivial idempotents which is what we wanted to prove. □

15.61. Ramification theory

In this section and the next we use the following definitions.

Definition 15.61.1. We say that $A \to B$ or $A \subset B$ is an *extension of discrete valuation rings* if A and B are discrete valuation rings and $A \to B$ is injective and local. In particular, if π_A and π_B are uniformizers of A and B, then $\pi_A = u\pi_B^e$ for some $e \geq 1$ and unit u of B. The integer e does not depend on the choice of the uniformizers as it is also the unique integer ≥ 1 such that

$$\mathfrak{m}_A B = \mathfrak{m}_B^e$$

The integer e is called the *ramification index* of B over A. We say that B is *weakly unramified* over A if $e = 1$. If the extension of residue fields $\kappa_A = A/\mathfrak{m}_A \subset \kappa_B = B/\mathfrak{m}_B$ is finite, then we set $f = [\kappa_B : \kappa_A]$ and we call it the *residual degree* or *residue degree* of the extension $A \subset B$.

Note that we do not require the extension of fraction fields to be finite.

Lemma 15.61.2. *Let $A \subset B$ be an extension of discrete valuation rings with fraction fields $K \subset L$. If the extension $K \subset L$ is finite, then the residue field extension is finite and we have $ef \leq [L : K]$.*

Proof. Finiteness of the residue field extension is Algebra, Lemma 10.113.9. The inequality follows from Algebra, Lemmas 10.113.8 and 10.48.12. □

Lemma 15.61.3. *Let $A \subset B$ be an extension of discrete valuation rings inducing the field extension $K \subset L$. If the characteristic of K is $p > 0$ and L is purely inseparable over K, then the ramification index e is a power of p.*

Proof. Write $\pi_A = u\pi_B^e$ for some $u \in B^*$. On the other hand, we have $\pi_B^q \in K$ for some p-power q. Write $\pi_B^q = v\pi_A^k$ for some $v \in A^*$ and $k \in \mathbf{Z}$. Then $\pi_A^q = u^q \pi_B^{qe} = u^q v^e \pi_A^{ke}$. Taking valuations in B we conclude that $ke = q$. □

Lemma 15.61.4. *Let $A \subset B$ be an extension of discrete valuation rings. The following are equivalent*

(1) *$A \to B$ is formally smooth in the $\mathfrak{m}_B$-adic topology, and*
(2) *$A \to B$ is weakly unramified and $\kappa_A \subset \kappa_B$ is a separable field extension.*

Proof. This follows from Proposition 15.28.4 and Algebra, Proposition 10.145.9. □

Remark 15.61.5. Let A be a discrete valution ring with fraction field K. Let $K \subset L$ be a finite separable field extension. Let $B \subset L$ be the integral closure of A in L. Picture:

$$\begin{array}{ccc} B & \longrightarrow & L \\ \uparrow & & \uparrow \\ A & \longrightarrow & K \end{array}$$

By Algebra, Lemma 10.148.8 the ring extension $A \subset B$ is finite, hence B is Noetherian. By Algebra, Lemma 10.106.4 the dimension of B is 1, hence B is a Dedekind domain, see Algebra, Lemma 10.114.10. Let $\mathfrak{m}_1, \ldots, \mathfrak{m}_n$ be the maximal ideals of B (i.e., the primes lying over $\mathfrak{m}_A$). We obtain extensions of discrete valuation rings

$$A \subset B_{\mathfrak{m}_i}$$

and hence ramification indices e_i and residue degrees f_i. We have

$$[L : K] = \sum e_i f_i$$

by Algebra, Lemma 10.118.1. We observe that $n = 1$ if A is henselian (by Algebra, Lemma 10.143.4), e.g. if A is complete.

Definition 15.61.6. Let A be a discrete valuation ring with fraction field K. Let $L \supset K$ be a finite separable extension. With B and $\mathfrak{m}_i$, $i = 1, \ldots, n$ as in Remark 15.61.5 we say the extension L/K is

(1) *unramified* if $e_i = 1$ and the extension $\kappa_A \subset \kappa(\mathfrak{m}_i)$ is separable for all i,
(2) *totally ramified* if $n = 1$ and the residue field extension $\kappa_A \subset \kappa(\mathfrak{m}_1)$ is trivial,
(3) *tamely ramified* if either the characteristic of κ_A is 0 or the characteristic of κ_A is $p > 0$ and the field extensions $\kappa_A \subset \kappa(\mathfrak{m}_i)$ are separable and the ramification indices e_i are prime to p.

Lemma 15.61.7. *Let A be a discrete valuation ring with fraction field K. Let $K \subset L$ be a Galois extension with Galois group G. Then G acts on the ring B of Remark 15.61.5 and acts transitively on the set of maximal ideals of B.*

Proof. If there are two or more orbits of the action, then we can find an element $b \in B$ which vanishes at all the maximal ideals of one orbit and has residue 1 at all the maximal ideals in another orbit. Then $b' = \prod_{\sigma \in G} \sigma(b)$ is a G-invariant element of $B \subset L$ which is in some maximal ideals of B but not in all maximal ideals of B. Since $K = L^G$ we conclude that $b' \in K$. Since b' maps to an element of B we see that $b' \in A$. Then on the one hand it must be true that $b' \in \mathfrak{m}_A$ as b' is in some maximal ideal of B and on the other hand it must be true that $b' \notin \mathfrak{m}_A$ as b' is not in all maximal ideals of B. This contradiction finishes the proof of the lemma. □

Lemma 15.61.8. *Let A be a discrete valuation ring with fraction field K. Let $K \subset L$ be a Galois extension. Then there are $e \geq 1$ and $f \geq 1$ such that $e_i = e$ and $f_i = f$ for all i (notation as in Remark 15.61.5). In particular $[L : K] = nef$.*

Proof. Immediate consequence of Lemma 15.61.7 and the definitions. □

Definition 15.61.9. Let A be a discrete valuation ring with fraction field K. Let $K \subset L$ be a Galois extension with Galois group G. Let B be the integral closure of A in L.

(1) For a maximal ideal $\mathfrak{m} \subset B$ the *decomposition group* associated to $\mathfrak{m}$ is the subgroup $D = \{\sigma \in G \mid \sigma(\mathfrak{m}) = \mathfrak{m}\}$ of G.
(2) The kernel I of the map $D \to \mathrm{Aut}(\kappa(\mathfrak{m})/\kappa_A)$ is called the *inertia group*.

Note that the field $\kappa(\mathfrak{m})$ may be inseparable over κ_A. In particular the field extension $\kappa_A \subset \kappa(\mathfrak{m})$ need not be Galois. If κ_A is perfect, then it is.

Lemma 15.61.10. *Let A be a discrete valuation ring with fraction field K and residue field κ. Let $K \subset L$ be a Galois extension with Galois group G. Let B be the integral closure of A in L.*

(1) *the field extension $\kappa \subset \kappa(\mathfrak{m})$ is normal, and*
(2) *$D \to Aut(\kappa(\mathfrak{m})/\kappa)$ is surjective.*

If for some (equivalently all) maximal ideal(s) $\mathfrak{m} \subset B$ the field extension $\kappa \subset \kappa(\mathfrak{m})$ is separable, then

(3) *$\kappa \subset \kappa(\mathfrak{m})$ is Galois, and*
(4) *$D \to Gal(\kappa(\mathfrak{m})/\kappa)$ is surjective.*

Here $D \subset G$ is the decomposition group of $\mathfrak{m}$.

Proof. Pick an element $\theta \in \kappa(\mathfrak{m})$. Pick $b \in B$ mapping to θ and with zero residue modulo all other maximal ideals of B. Let $P(t) \in K[t]$ be the minimal polynomial of b over K. By Algebra, Lemma 10.35.6 the polynomial P has coefficients in A. Thus θ is a root of the image $\overline{P} \in \kappa[t]$. Thus the minimal polynomial of θ divides $\overline{P}$. Since P splits completely as $P(t) = \prod(t - b_i)$ over B where the b_i are the conjugates of b in L, we conclude that the minimal polynomial of θ over κ splits completely in $\kappa(\mathfrak{m})$. This shows $\kappa(\mathfrak{m})$ is a normal extension of κ.

Since $\kappa(\mathfrak{m})/\kappa_A$ is normal we may assume $\kappa(\mathfrak{m}) = \kappa_1 \otimes_\kappa \kappa_2$ with $\kappa \subset \kappa_1$ purely inseparable and $\kappa \subset \kappa_2$ Galois, see Fields, Lemma 9.24.3. Pick $\theta \in \kappa_2$ which generates κ_2 over κ. If $\theta' \in \kappa_2$ is a conjugate of θ, then the above shows there exists a $\sigma \in G$ such that $\sigma(b)$ maps to θ'. By our choice of b (vanishing at other maximal ideals) this implies $\sigma \in D$ and that the image of σ in $\mathrm{Aut}(\kappa(\mathfrak{m})/\kappa_A)$ maps θ to θ'. Hence the surjectivity.

The "equivalently all" part of the lemma follows from Lemma 15.61.7. Assume $\kappa_A \subset \kappa(\mathfrak{m})$ is separable. Parts (3) and (4) follow immediately from (1) and (2). □

Lemma 15.61.11. *Let A be a discrete valuation ring with fraction field K. Let $K \subset L$ be a Galois extension with Galois group G. Let B be the integral closure of A in L. Let $\mathfrak{m} \subset B$ be a maximal ideal. The inertia group I of $\mathfrak{m}$ has the following structure*

(1) *if the characteristic of κ_A is 0, then I is finite cyclic of order e,*

(2) *if the characteristic of κ_A is $p > 0$, then there is a short exact sequence of groups $1 \to P \to I \to I_t \to 0$ where P is a p-group and I_t is cyclic of order prime to p. In fact, the order of I_t is the prime to p part of the integer e.*

Here e is the integer of Lemma 15.61.8.

Proof. Recall that $|G| = [L : K] = nef$, see Lemma 15.61.8. Since G acts transitively on the set $\{\mathfrak{m}_1, \ldots, \mathfrak{m}_n\}$ of maximal ideals of B (Lemma 15.61.7) and since D is the stabilizer of an element we see that $|D| = ef$. By Lemma 15.61.10 we have

$$ef = |D| = |I| \cdot |\mathrm{Aut}(\kappa(\mathfrak{m})/\kappa_A)|$$

As $\kappa(\mathfrak{m})$ is normal over κ_A the order of $\mathrm{Aut}(\kappa(\mathfrak{m})/\kappa_A)$ differs from f by a power of p (small detail omitted). Hence the prime to p part of $|I|$ is equal to the prime to p part of e.

Set $C = B_{\mathfrak{m}}$. Then I acts on C over A and trivially on the residue field of C. Let $\pi_A \in A$ and $\pi_C \in C$ be uniformizers. Write $\pi_A = u\pi_C^e$ for some unit u in B. For $\sigma \in I$ write $\sigma(\pi_C) = u_\sigma \pi_C$. Then we have

$$\pi_A = \sigma(\pi_A) = \sigma(u)(u_\sigma \pi_C)^e = \sigma(u) u_\sigma^e \pi_C^e = \frac{\sigma(u)}{u} u_\sigma^e \pi_A$$

Since $\sigma(u) \equiv u \bmod \mathfrak{m}_C$ we see that u_σ maps to an eth root of unity in κ_C. We obtain a homomorphism

$$\chi : I \longrightarrow \mu_e(\kappa_C)$$

Since κ_C has characteristic p, the group $\mu_e(\kappa_C)$ is cyclic of order at most the prime to p part of e (some facts about roots of unity in fields omitted). Thus it suffices to prove that the kernel of χ is a p-group. Let σ be a nontrivial element of the kernel. Then $\sigma(\mathfrak{m}_C^i) \subset \mathfrak{m}_C^{i+1}$ for all i. Let m be the order of σ. Pick $c \in C$ such that $\sigma(c) \neq c$. Then $\sigma(c) - c \in \mathfrak{m}_C^i$, $\sigma(c) - c \notin \mathfrak{m}_C^{i+1}$ for some i and we have

$$\begin{aligned} 0 &= \sigma^m(c) - c \\ &= \sigma^m(c) - \sigma^{m-1}(c) + \ldots + \sigma(c) - c \\ &= \sum\nolimits_{j=0,\ldots,m-1} \sigma^j(\sigma(c) - c) \\ &\equiv m(\sigma(c) - c) \bmod \mathfrak{m}_C^{i+1} \end{aligned}$$

It follows that $p|m$ (or $m = 0$ if $p = 1$). Thus every element of the kernel of χ has order divisible by p, i.e., $\mathrm{Ker}(\chi)$ is a p-group. □

Lemma 15.61.12. *Let $\varphi : R \to S$ be a ring map such that*

(1) *the kernel of φ is locally nilpotent, and*
(2) *S is generated as an R-algebra by elements $s \in S$ for which there exists a polynomial $P(T) \in R[T]$ whose image in $S[T]$ is $(T - s)^n$ for some $n = n(s) > 0$.*

Then $\mathrm{Spec}(S) \to \mathrm{Spec}(R)$ is a homeomorphism and $R \to S$ induces purely inseparable extensions of residue fields. Moreover, conditions (1) and (2) remain true on arbitrary base change.

Proof. We may replace R by $R/\mathrm{Ker}(\varphi)$, see Algebra, Lemma 10.43.2. Assumption (2) implies S is generated over R by elements which are integral over R. Hence $R \subset S$ is integral (Algebra, Lemma 10.33.7). In particular $\mathrm{Spec}(S) \to \mathrm{Spec}(R)$

is surjective and closed (Algebra, Lemmas 10.33.15, 10.38.6, and 10.33.20). Let $\mathfrak{p} \subset R$ be a prime and let $k \supset \kappa(\mathfrak{p})$ be the perfection. Set $A = S \otimes_R k$. By the surjectivity the ring $A' = S \otimes_R \kappa(\mathfrak{p})$ is not zero (Algebra, Lemma 10.16.9) whence $A = A' \otimes_{\kappa(\mathfrak{p})} k$ is nonzero. Assumption (2) implies that A is generated over k by elements $a \in A$ for which there exist a $P(T) \in k[T]$ with $P(T) \mapsto (T-a)^{n(a)}$ in $A[T]$. Since $A \neq 0$ and k is perfect, this implies that $P = (T - \lambda_a)^{n(a)}$ for some $\lambda_a \in k$ (some details omitted). Hence $k = A_{red}$ because a and λ_a map to the same element of A_{red} for all of these generaters. It follows that $\mathrm{Spec}(A)$ is a singleton. Hence $\mathrm{Spec}(S) \to \mathrm{Spec}(R)$ is bijective and all residue field extensions are purely inseparable. Since a bijective, continuous, closed map of topological spaces is a homeomorphism the proof is finished. □

Lemma 15.61.13. *Let R be a ring. Let G be a finite group acting on R. Let $I \subset R$ be an ideal such that $\sigma(I) \subset I$ for all $\sigma \in G$. The ring maps*

$$R^G/I^G \longrightarrow R^G/(R^G \cap I) \longrightarrow (R/I)^G$$

induce homeomorphisms on spectra and purely inseparable extensions of residue fields.

Proof. Let n be the order of G. Let $x \in R^G \cap I$. Then $x^n = \prod_{\sigma \in G} \sigma(x)$ is in I^G. Hence the kernel of the first map is locally nilpotent. Since the first map is also surjective, we see that the result of the lemma is true for the first map.

The second map is injective. Let $x \in R$ be an element which maps to an element of $(R/I)^G$. Consider the polynomial $P(T) = \prod(T - \sigma(x)) \in R^G[T]$. The image of P in $(R/I)^G[T]$ is $(T-x)^n$. Thus Lemma 15.61.12 applies to both the second map and the composition. □

Lemma 15.61.14. *Let A be a discrete valuation ring with fraction field K. Let L be a Galois extension of K. Let $\mathfrak{m} \subset B$ be a maximal ideal of the integral closure of A in L. Let $I \subset G$ be the corresponding inertia subgroup. Then B^I is the integral closure of A in L^I and $A \to (B^I)_{B^I \cap \mathfrak{m}}$ is étale.*

Proof. It follows from the definitions that B^I is the integral closure of A in L^I.

We first prove the final statement in case B is a discrete valuation ring, i.e., when G is the decomposition group of $\mathfrak{m}$. As I acts trivially on κ_B it follows from Lemma 15.61.13 that the extension $\kappa_{B^I} = B^I/(B^I \cap \mathfrak{m}) \subset \kappa_B$ is purely inseparable. Since G/I acts faithfully on κ_B, we conclude that G/I acts faithfully on κ_{B^I} over κ_A. By Galois theory we see that $[\kappa_{B^I} : \kappa_A] \geq |G/I|$. On the other hand, we have $[L^I : K] = |G/I|$ by Galois theory. By Lemma 15.61.2 we see that $A \subset B^I$ is weakly unramified and that $[\kappa_{B^I} : \kappa_A] = |G/I|$. Thus κ_{B^I} is Galois over κ_A (with group G/I) and we conclude that L^I is unramified over K as desired.

In general we reduce to the case discussed in the previous paragraph by splitting B using Algebra, Lemma 10.136.23. (An alternative is to use completion to do this.) We omit the details. □

Lemma 15.61.15 (Krasner's lemma)**.** *Let A be a complete local domain of dimension 1. Let $P(t) \in A[t]$ be a polynomial with coefficients in A. Let $\alpha \in A$ be a root of P but not a root of the derivative $P' = dP/dt$. For every $c \geq 0$ there exists an integer n such that for any $Q \in A[t]$ whose coefficients are in $\mathfrak{m}_A^n$ the polynomial $P + Q$ has a root $\beta \in A$ with $\beta - \alpha \in \mathfrak{m}_A^c$.*

Proof. Choose a nonzero $\pi \in \mathfrak{m}$. Since the dimension of A is 1 we have $\mathfrak{m} = \sqrt{(\pi)}$. By assumption we may write $P'(\alpha)^{-1} = \pi^{-m}a$ for some $m \geq 0$ and $a \in A$. We may and do assume that $c \geq m + 1$. Pick n such that $\mathfrak{m}_A^n \subset (\pi^{c+m})$. Pick any Q as in the statement. For later use we observe that we can write

$$P(x + y) = P(x) + P'(x)y + R(x, y)y^2$$

for some $R(x, y) \in A[x, y]$. We will show by induction that we can find a sequence $\alpha_m, \alpha_{m+1}, \alpha_{m+2}, \ldots$ such that

(1) $\alpha_k \equiv \alpha \bmod \pi^c$,
(2) $\alpha_{k+1} - \alpha_k \in (\pi^k)$, and
(3) $(P + Q)(\alpha_k) \in (\pi^{m+k})$.

Setting $\beta = \lim \alpha_k$ will finish the proof.

Base case. Since the coefficients of Q are in (π^{c+m}) we have $(P + Q)(\alpha) \in (\pi^{c+m})$. Hence $\alpha_m = \alpha$ works. This choice guarantees that $\alpha_k \equiv \alpha \bmod \pi^c$ for all $k \geq m$.

Induction step. Given α_k we write $\alpha_{k+1} = \alpha_k + \delta$ for some $\delta \in (\pi^k)$. Then we have

$$(P + Q)(\alpha_{k+1}) = P(\alpha_k + \delta) + Q(\alpha_k + \delta)$$

Because the coefficients of Q are in (π^{c+m}) we see that $Q(\alpha_k + \delta) \equiv Q(\alpha_k) \bmod \pi^{c+m+k}$. On the other hand we have

$$P(\alpha_k + \delta) = P(\alpha_k) + P'(\alpha_k)\delta + R(\alpha_k, \delta)\delta^2$$

Note that $P'(\alpha_k) \equiv P'(\alpha) \bmod (\pi^{m+1})$ as $\alpha_k \equiv \alpha \bmod \pi^{m+1}$. Hence we obtain

$$P(\alpha_k + \delta) \equiv P(\alpha_k) + P'(\alpha)\delta \bmod \pi^{k+m+1}$$

Recombining the two terms we see that

$$(P + Q)(\alpha_{k+1}) \equiv (P + Q)(\alpha_k) + P'(\alpha)\delta \bmod \pi^{k+m+1}$$

Thus a solution is to take $\delta = -P'(\alpha)^{-1}(P + Q)(\alpha_k) = -\pi^{-m}a(P + Q)(\alpha_k)$ which is contained in (π^k) by induction assumption. □

Lemma 15.61.16. *Let A be a discrete valuation ring with field of fractions K. Let $A^\wedge$ be the completion of A with fraction field $K^\wedge$. If $K^\wedge \subset M$ is a finite separable extension, then there exists a finite separable extension $K \subset L$ such that $M = K^\wedge \otimes_K L$.*

Proof. Note that $A^\wedge$ is a discrete valuation ring too (by Lemmas 15.31.4 and 15.31.1). In particular $A^\wedge$ is a domain. The proof will work more generally for Noetherian local rings A such that $A^\wedge$ is a local domain of dimension 1.

Let $\theta \in M$ be an element that generates M over $K^\wedge$. (Theorem of the primitive element.) Let $P(t) \in K^\wedge[t]$ be the minimal polynomial of θ over $K^\wedge$. Let $\pi \in \mathfrak{m}_A$ be a nonzero element. After replacing θ by $\pi^n\theta$ we may assume that the coefficients of $P(t)$ are in $A^\wedge$. Let $B = A^\wedge[\theta] = A^\wedge[t]/(P(t))$. Note that B is a complete local domain of dimension 1 because it is finite over A and contained in M. Since M is separable over K the element θ is not a root of the derivative of P. For any integer n we can find a monic polynomial $P_1 \in A[t]$ such that $P - P_1$ has coefficients in $\pi^n A^\wedge[t]$. By Krasner's lemma (Lemma 15.61.15) we see that P_1 has a root β in B for n sufficiently large. Moreover, we may assume (if n is chosen large enough) that $\theta - \beta \in \pi B$. Consider the map $\Phi : A^\wedge[t]/(P_1) \to B$ of $A^\wedge$-algebras which maps t to β. Since $B = \pi B + \sum_{i<\deg(P)} A^\wedge\theta^i$, the map Φ is surjective by Nakayama's lemma.

As $\deg(P_1) = \deg(P)$ it follows that Φ is an isomorphism. We conclude that the ring extension $L = K[t]/(P_1(t))$ satisfies $K^\wedge \otimes_K L \cong M$. This implies that L is a field and the proof is complete. $\square$

Definition 15.61.17. Let A be a discrete valution ring. We say A has *mixed characteristic* if the characteristic of the residue field of A is $p > 0$ and the characteristic of the fraction field of A is 0. In this case we obtain an extension of discrete valution rings $\mathbf{Z}_{(p)} \subset A$ and the *absolute ramification index* of A is the ramification index of this extension.

15.62. Eliminating ramification

In this section we discuss a result of Helmut Epp, see [**Epp73**]. We strongly encourage the reader to read the original. Our approach is slightly different as we try to handle the mixed and equicharacteristic cases by the same method. For related results, see also [**Pon98**], [**Pon99**], [**Kuh03**], and [**ZK99**].

Remark 15.62.1. Let $A \to B$ be an extension of discrete valuation rings with fraction fields $K \subset L$. Let $K \subset K_1$ be a finite extension of fields. Let $A_1 \subset K_1$ be the integral closure of A in K_1. On the other hand, let $L_1 = (L \otimes_K K_1)_{red}$. Then L_1 is a nonempty finite product of finite field extensions of L. Let B_1 be the integral closure of B in L_1. We obtain compatible commutative diagrams

$$\begin{array}{ccc} L & \longrightarrow & L_1 \\ \uparrow & & \uparrow \\ K & \longrightarrow & K_1 \end{array} \quad \text{and} \quad \begin{array}{ccc} B & \longrightarrow & B_1 \\ \uparrow & & \uparrow \\ A & \longrightarrow & A_1 \end{array}$$

In this situation we have the following

(1) By Algebra, Lemma 10.114.11 the ring A_1 is a Dedekind domain and B_1 is a finite product of Dedekind domains.
(2) Note that $L \otimes_K K_1 = (B \otimes_A A_1)_\pi$ where $\pi \in A$ is a uniformizer and that π is a nonzero divisor on $B \otimes_A A_1$. Thus the ring map $B \otimes_A A_1 \to B_1$ is integral with kernel consisting of nilpotent elements. Hence $\mathrm{Spec}(B_1) \to \mathrm{Spec}(B \otimes_A A_1)$ is surjective on spectra (Algebra, Lemma 10.33.15). The map $\mathrm{Spec}(B \otimes_A A_1) \to \mathrm{Spec}(A_1)$ is surjective as $A_1/\mathfrak{m}_A A_1 \to B/\mathfrak{m}_A B \otimes_{\kappa_A} A_1/\mathfrak{m}_A A_1$ is an injective ring map with $A_1/\mathfrak{m}_A A_1$ Artinian. We conclude that $\mathrm{Spec}(B_1) \to \mathrm{Spec}(A_1)$ is surjective.
(3) Let $\mathfrak{m}_i$, $i = 1, \ldots n$ with $n \geq 1$ be the maximal ideals of A_1. For each $i = 1, \ldots, n$ let $\mathfrak{m}_{ij}$, $j = 1, \ldots, m_i$ with $m_i \geq 1$ be the maximal ideals of B_1 lying over m_i. We obtain diagrams
$$\begin{array}{ccc} B & \longrightarrow & (B_1)_{\mathfrak{m}_{ij}} \\ \uparrow & & \uparrow \\ A & \longrightarrow & (A_1)_{\mathfrak{m}_i} \end{array}$$
of extensions of discrete valuation rings.
(4) If A is henselian (for example complete), then A_1 is a discrete valuation ring, i.e., $n = 1$. Namely, A_1 is a union of finite extensions of A which are domains, hence local by Algebra, Lemma 10.143.4.

(5) If B is henselian (for example complete), then B_1 is a product of discrete valuation rings, i.e., $m_i = 1$ for $i = 1, \ldots, n$.
(6) If $K \subset K_1$ is purely inseparable, then A_1 and B_1 are both discrete valuation rings, i.e., $n = 1$ and $m_1 = 1$. This is true because for every $b \in B_1$ a p-power power of b is in B, hence B_1 can only have one maximal ideal.
(7) If $K \subset K_1$ is finite separable, then $L_1 = L \otimes_K K_1$ and is a finite product of finite separable extensions too. Hence $A \subset A_1$ and $B \subset B_1$ are finite by Algebra, Lemma 10.148.8.
(8) If A is Nagata, then $A \subset A_1$ is finite.
(9) If B is Nagata, then $B \subset B_1$ is finite.

The goal in this section is to find extensions $K \subset K_1$ as in Remark 15.62.1 such that the extensions $(A_1)_{\mathfrak{m}_i} \subset (B_1)_{\mathfrak{m}_{ij}}$ are all weakly unramified or even formally smooth.

Definition 15.62.2. Let $A \to B$ be an extension of discrete valuation rings with fraction fields $K \subset L$.

(1) We say a finite field extension $K \subset K_1$ is a *weak solution for* $A \subset B$ if all the extensions $(A_1)_{\mathfrak{m}_i} \subset (B_1)_{\mathfrak{m}_{ij}}$ of Remark 15.62.1 are weakly unramified.
(2) We say a finite field extension $K \subset K_1$ is a *solution for* $A \subset B$ if each extension $(A_1)_{\mathfrak{m}_i} \subset (B_1)_{\mathfrak{m}_{ij}}$ of Remark 15.62.1 is formally smooth in the $\mathfrak{m}_{ij}$-adic topology.

We say a solution $K \subset K_1$ is a *separable solution* if $K \subset K_1$ is separable.

In general (weak) solutions do not exist. The following example shows that in general one needs inseparable extensions to get a solution.

Example 15.62.3. Let k be a perfect field of characteristic $p > 0$. Let $A = k[[x]]$ and $K = k((x))$. Let $B = A[x^{1/p}]$. Any weak solution $K \subset K_1$ for $A \to B$ is inseparable (and any finite inseparable extension of K is a solution). We omit the proof.

Solutions are stable under further extensions, but this may not be true for weak solutions. Weak solutions are stable under totally ramified solutions.

Lemma 15.62.4. *Let $A \to B$ be an extension of discrete valuation rings with fraction fields $K \subset L$. Assume that $A \to B$ is formally smooth in the $\mathfrak{m}_B$-adic topology. Then for any finite extension $K \subset K_1$ we have $L_1 = L \otimes_K K_1$, $B_1 = B \otimes_A A_1$, and each extension $(A_1)_{\mathfrak{m}_i} \subset (B_1)_{\mathfrak{m}_{ij}}$ (see Remark 15.62.1) is formally smooth in the $\mathfrak{m}_{ij}$-adic topology.*

Proof. We will use the equivalence of Lemma 15.61.4 without further mention. Let $\pi \in A$ and $\pi_i \in (A_1)_{\mathfrak{m}_i}$ be uniformizers. As $\kappa_A \subset \kappa_B$ is separable, the ring

$$(B \otimes_A (A_1)_{\mathfrak{m}_i})/\pi_i(B \otimes_A (A_1)_{\mathfrak{m}_i}) = B/\pi B \otimes_{A/\pi A} (A_1)_{\mathfrak{m}_i}/\pi_i(A_1)_{\mathfrak{m}_i}$$

is a product of fields each separable over $\kappa_{\mathfrak{m}_i}$. Hence the element π_i in $B \otimes_A (A_1)_{\mathfrak{m}_i}$ is a nonzerodivisor and the quotient by this element is a product of fields. It follows that $B \otimes_A A_1$ is a Dedekind domain in particular reduced. Thus $B \otimes_A A_1 \subset B_1$ is an equality. □

Lemma 15.62.5. *Let $A \to B$ be an extension of discrete valuation rings with fraction fields $K \subset L$. Assume that $A \to B$ is weakly unramified. Then for any totally ramified separable extension $K \subset K_1$ we have that $L_1 = L \otimes_K K_1$ is a field,*

A_1 and $B_1 = B \otimes_A A_1$ are discrete valuation rings, and the extension $A_1 \subset B_1$ (see Remark 15.62.1) is weakly unramified.

Proof. Let $\pi \in A$ and $\pi_1 \in A_1$ be uniformizers. As $K \subset K_1$ is totally ramified we have $\pi_1^e = u\pi$ for some unit u_1 in A_1. Hence A_1 is generated by π_1 over A and the minimal polynomial $P(t)$ of π_1 over K has the form

$$P(t) = t^e + a_{e-1}t^{e-1} + \ldots + a_0$$

with $a_i \in (\pi)$ and $a_0 = u\pi$ for some unit u of A. Note that $e = [K_1 : K]$ as well. Since $A \to B$ is weakly unramified we see that π is a uniformizer of B and hence $B_1 = B[t]/(P(t))$ is a discrete valuation ring with uniformizer the class of t. Thus the lemma is clear. □

Lemma 15.62.6. *Let $A \to B \to C$ be extensions of discrete valuation rings with fraction fields $K \subset L \subset M$. Let $K \subset K_1$ be a finite extension.*

1. *If K_1 is a (weak) solution for $A \to C$, then K_1 is a (weak) solution for $A \to B$.*
2. *If K_1 is a (weak) solution for $A \to B$ and $L_1 = (L \otimes_K K_1)_{red}$ is a product of fields which are (weak) solutions for $B \to C$, then K_1 is a weak solution for $A \to C$.*

Proof. Let $L_1 = (L \otimes_K K_1)_{red}$ and $M_1 = (M \otimes_K K_1)_{red}$ and let $B_1 \subset L_1$ and $C_1 \subset M_1$ be the integral closure of B and C. Note that $M_1 = (M \otimes_L L_1)_{red}$ and that L_1 is a (nonempty) finite product of finite extensions of L. Hence the ring map $B_1 \to C_1$ is a finite product of ring maps of the form discussed in Remark 15.62.1. In particular, the map $\mathrm{Spec}(C_1) \to \mathrm{Spec}(B_1)$ is surjective. Choose a maximal ideal $\mathfrak{m} \subset C_1$ and consider the extensions of discrete valuation rings

$$(A_1)_{A_1 \cap \mathfrak{m}} \to (B_1)_{B_1 \cap \mathfrak{m}} \to (C_1)_{\mathfrak{m}}$$

If the composition is weakly unramified, so is the map $(A_1)_{A_1 \cap \mathfrak{m}} \to (B_1)_{B_1 \cap \mathfrak{m}}$. If the residue field extension $\kappa_{A_1 \cap \mathfrak{m}} \to \kappa_{\mathfrak{m}}$ is separable, so is the subextension $\kappa_{A_1 \cap \mathfrak{m}} \to \kappa_{B_1 \cap \mathfrak{m}}$. Taking into account Lemma 15.61.4 this proves (1). A similar argument works for (2). □

Lemma 15.62.7. *Let $A \to B$ be an extension of discrete valuation rings. There exists a commutative diagram*

$$\begin{array}{ccc} B & \longrightarrow & B' \\ \uparrow & & \uparrow \\ A & \longrightarrow & A' \end{array}$$

of extensions of discrete valuation rings such that

1. *the extensions $K \subset K'$ and $L \subset L'$ of fraction fields are separable algebraic,*
2. *the residue fields of A' and B' are separable algebraic closures of the residue fields of A and B, and*
3. *if a solution, weak solution, or separable solution exists for $A' \to B'$, then a solution, weak solution, or separable solution exists for $A \to B$.*

Proof. By Algebra, Lemma 10.146.2 there exists an extension $A \subset A'$ which is a filtered colimit of finite étale extensions such that the residue field of A' is a separable algebraic closure of the residue field of A. Then $A \subset A'$ is an extension of

discrete valuation rings such that the induced extension $K \subset K'$ of fraction fields is separable algebraic.

Let $B \subset B'$ be a strict henselization of B. Then $B \subset B'$ is an extension of discrete valuation rings whose fraction field extension is separable algebraic. By Algebra, Lemma 10.143.25 there exists a commutative diagram as in the statement of the lemma. Parts (1) and (2) of the lemma are clear.

Let $K' \subset K'_1$ be a (weak) solution for $A' \to B'$. Since A' is a colimit, we can find a finite étale extension $A \subset A'_1$ and a finite extension $f.f.(A'_1) \subset K_1$ such that $K'_1 = K' \otimes_{f.f.(A'_1)} K_1$. As $A \subset A'_1$ is finite étale and B' strictly henselian, it follows that $B' \otimes_A A'_1$ is a finite product of rings isomorphic to B'. Hence

$$L' \otimes_K K_1 = L' \otimes_K f.f.(A'_1) \otimes_{f.f.(A'_1)} K_1$$

is a finite product of rings isomorphic to $L' \otimes_{K'} K'_1$. Thus we see that $K \subset K_1$ is a (weak) solution for $A \to B'$. Hence it is also a (weak) solution for $A \to B$ by Lemma 15.62.6. □

Lemma 15.62.8. *Let $A \to B$ be an extension of discrete valuation rings with fraction fields $K \subset L$. Let $K \subset K_1$ be a normal extension. Say $G = Aut(K_1/K)$. Then G acts on the rings K_1, L_1, A_1 and B_1 of Remark 15.62.1 and acts transitively on the set of maximal ideals of B_1.*

Proof. Everything is clear apart from the last assertion. If there are two or more orbits of the action, then we can find an element $b \in B_1$ which vanishes at all the maximal ideals of one orbit and has residue 1 at all the maximal ideals in another orbit. Then $b' = \prod_{\sigma \in G} \sigma(b)$ is a G-invariant element of $B_1 \subset L_1 = (L \otimes_K K_1)_{red}$ which is in some maximal ideals of B_1 but not in all maximal ideals of B_1. Lifting it to an element of $L \otimes_K K_1$ and raising to a high power we obtain a G-invariant element b'' of $L \otimes_K K_1$ mapping to $(b')^N$ for some $N > 0$; in fact, we only need to do this in case the characteristic is $p > 0$ and in this case raising to a suitably large p-power q defines a canonical map $(L \otimes_K K_1)_{red} \to L \otimes_K K_1$. Since $K = (K_1)^G$ we conclude that $b'' \in L$. Since b'' maps to an element of B_1 we see that $b'' \in B$ (as B is normal). Then on the one hand it must be true that $b'' \in \mathfrak{m}_B$ as b' is in some maximal ideal of B_1 and on the other hand it must be true that $b'' \notin \mathfrak{m}_B$ as b' is not in all maximal ideals of B_1. This contradiction finishes the proof of the lemma. □

Lemma 15.62.9. *Let A be a discrete valuation ring with uniformizer π. Let $n \geq 2$. Then $K_1 = K[\pi^{1/n}]$ is a degree n extension of K and the integral closure A_1 of A in K_1 is the ring $A[\pi^{1/n}]$ which is a discrete valuation ring with ramification index n over A.*

Proof. This lemma proves itself. □

Lemma 15.62.10. *Let A be a discrete valuation ring with uniformizer π. If the residue characteristic of A is $p > 0$, then for every $n > 1$ and p-power q there exists a degree q separable totally ramified extension $K \subset L$ such that the integral closure B of A in L has ramification index q and a uniformizer π_B such that $\pi_B^q = \pi + \pi^n b$ and $\pi_B^q = \pi + (\pi_B)^{nq} b'$ for some $b, b' \in B$.*

Proof. If the characteristic of K is zero, then we can take the extension given by $\pi_B^q = \pi$, see Lemma 15.62.9. If the characteristic of K is $p > 0$, then we can take the extension of K given by $z^q - \pi^n z = \pi^{1-q}$. Namely, then we see that $y^q - \pi^{n+q-1}y = \pi$ where $y = \pi z$. Taking $\pi_B = y$ we obtain the desired result. □

Lemma 15.62.11. *Let A be a discrete valuation ring. Assume the reside field κ_A has characteristic $p > 0$ and that $a \in A$ is an element whose residue class in κ_A is not a pth power. Then a is not a pth power in K and the integral closure of A in $K[a^{1/p}]$ is the ring $A[a^{1/p}]$ which is a discrete valuation ring weakly unramified over A.*

Proof. This lemma proves itself. □

Lemma 15.62.12. *Let $A \subset B \subset C$ be extensions of discrete valuation rings with fractions fields $K \subset L \subset M$. Let $\pi \in A$ be a uniformizer. Assume*

1. *B is a Nagata ring,*
2. *$A \subset B$ is weakly unramified,*
3. *M is a degree p purely inseparable extension of L.*

Then either

1. *$A \to C$ is weakly unramified, or*
2. *$C = B[\pi^{1/p}]$, or*
3. *there exists a degree p separable, totally ramified extension $K \subset K_1$ such that $L_1 = L \otimes_K K_1$ and $M_1 = M \otimes_K K_1$ are fields and the maps of integral closures $A_1 \to B_1 \to C_1$ are weakly unramified extensions of discrete valuation rings.*

Proof. Let e be the ramification index of C over B. If $e = 1$, then we are done. If not, then $e = p$ by Lemmas 15.61.2 and 15.61.3. This in turn implies that the residue fields of B and C agree. Choose a uniformizer π_C of C. Write $\pi_C^p = u\pi$ for some unit u of C. Since $\pi_C^p \in L$, we see that $u \in B^*$. Also $M = L[\pi_C]$.

Suppose there exists an integer $m \geq 0$ such that

$$u = \sum\nolimits_{0 \leq i < m} b_i^p \pi^i + b\pi^m$$

with $b_i \in B$ and with $b \in B$ an element whose image in κ_B is not a pth power. Choose an extension $K \subset K_1$ as in Lemma 15.62.10 with $n = m + 2$ and denote π' the uniformizer of the integral closure A_1 of A in K_1 such that $\pi = (\pi')^p + (\pi')^{np}a$ for some $a \in A_1$. Let B_1 be the integral closure of B in $L \otimes_K K_1$. Observe that $A_1 \to B_1$ is weakly unramified by Lemma 15.62.5. In B_1 we have

$$u\pi = \left(\sum\nolimits_{0 \leq i < m} b_i (\pi')^{i+1}\right)^p + b(\pi')^{(m+1)p} + (\pi')^{np} b_1$$

for some $b_1 \in B_1$ (computation omitted). We conclude that M_1 is obtained from L_1 by adjoining a pth root of

$$b + (\pi')^{n-m-1} b_1$$

Since the residue field of B_1 equals the residue field of B we see from Lemma 15.62.11 that M_1/L_1 has degree p and the integral closure C_1 of B_1 is weakly unramified over B_1. Thus we conclude in this case.

If there does not exist an integer m as in the preceding paragraph, then u is a pth power in the π-adic completion of B_1. Since B is Nagata, this means that u is a

pth power in B_1 by Algebra, Lemma 10.148.33. Whence the second case of the statement of the lemma holds. □

Lemma 15.62.13. *Let A be a local ring annihilated by a prime p whose maximal ideal is nilpotent. There exists a ring map $\sigma : \kappa_A \to A$ which is a section to the residue map $A \to \kappa_A$. If $A \to A'$ is a local homomorphism of local rings, then we can choose a similar ring map $\sigma' : \kappa_{A'} \to A'$ compatible with σ provided that the extension $\kappa_A \subset \kappa_{A'}$ is separable.*

Proof. Separable extensions are formally smooth by Algebra, Proposition 10.145.9. Thus the existence of σ follows from the fact that $\mathbf{F}_p \to \kappa_A$ is separable. Similarly for the existence of σ' compatible with σ. □

Lemma 15.62.14. *Let A be a discrete valuation ring with fraction field K of characteristic $p > 0$. Let $\xi \in K$. Let L be an extension of K obtained by adjoining a root of $z^p - z = \xi$. Then L/K is Galois and one of the following happens*

1. *$L = K$,*
2. *L/K is unramified of degree p,*
3. *L/K is totally ramified with ramification index p, and*
4. *L/K is weakly unramified, the integral closure B of A in L is a discrete valuation ring and $A \to B$ induces a purely inseparable residue field extension of degree p.*

Let π be a uniformizer of A. We have the following implications:

(A) *If $\xi \in A$, then we are in case (1) or (2).*
(B) *If $\xi = \pi^{-n}a$ where $n > 0$ is not divisible by p and a is a unit in A, then we are in case (3)*
(C) *If $\xi = \pi^{-n}a$ where $n > 0$ is divisible by p and the image of a in κ_A is not a pth power, then we are in case (4).*

Proof. The extension is Galois of order dividing p by the discussion in Fields, Section 9.22. It immediately follows from the discussion in Section 15.61 that we are in one of the cases (1) – (4) listed in the lemma.

Case (A). Here we see that $A \to A[x]/(x^p - x - \xi)$ is a finite étale ring extension. Hence we are in cases (1) or (2).

Case (B). Write $\xi = \pi^{-n}a$ where p does not divide n. Let $B \subset L$ be the integral closure of A in L. If $C = B_{\mathfrak{m}}$ for some maximal ideal $\mathfrak{m}$, then it is clear that $p\text{ord}_C(z) = -n\text{ord}_C(\pi)$. In particular $A \subset C$ has ramification index divisible by p. It follows that it is p and that $B = C$.

Case (C). Set $k = n/p$. Then we can rewrite the equation as

$$(\pi^k z)^p - \pi^{n-k}(\pi^k z) = a$$

Since $A[y]/(y^p - \pi^{n-k}y - a)$ is a discrete valuation ring weakly unramified over A, the lemma follows. □

Lemma 15.62.15. *Let $A \subset B \subset C$ be extensions of discrete valuation rings with fractions fields $K \subset L \subset M$. Assume*

1. *$A \subset B$ weakly unramified,*
2. *the characteristic of K is p,*
3. *M is a degree p Galois extension of L, and*

(4) $\kappa_A = \bigcap_{n\geq 1} \kappa_B^{p^n}$.

Then there exists a totally ramified Galois extension K_1 of K which is a weak solution for $A \to C$.

Proof. Since the characteristic of L is p we know that M is an Artin-Schreier extension of L (Fields, Lemma 9.22.1). Thus we may pick $z \in M$, $z \notin L$ such that $\xi = z^p - z \in L$. Choose $n \geq 0$ such that $\pi^n \xi \in B$. We pick z such that n is minimal. If $n = 0$, then M is unramified over L (Lemma 15.62.14) and we are done. Thus we have $n > 0$.

Assumption (4) implies that κ_A is perfect. Thus we may choose compatible ring maps $\overline{\sigma} : \kappa_A \to A/\pi^n A$ and $\overline{\sigma} : \kappa_B \to B/\pi^n B$ as in Lemma 15.62.13. We lift the second of these to a map of sets $\sigma : \kappa_B \to B$[8]. Then we can write

$$\xi = \sum\nolimits_{i=n,\ldots,1} \sigma(\lambda_i)\pi^{-i} + b$$

for some $\lambda_i \in \kappa_B$ and $b \in B$. Let

$$I = \{i \in \{n, \ldots, 1\} \mid \lambda_i \in \kappa_A\}$$

and

$$J = \{j \in \{n, \ldots, 1\} \mid \lambda_i \notin \kappa_A\}$$

We will argue by induction on the size of the finite set J.

The case $J = \emptyset$. Here for all $i \in \{n, \ldots, 1\}$ we have $\sigma(\lambda_i) = a_i + \pi^n b_i$ for some $a_i \in A$ and $b_i \in B$ by our choice of σ. Thus $\xi = \pi^{-n}a + b$ for some $a \in A$ and $b \in B$. If $p|n$, then we write $a = a_0^p + \pi a_1$ for some $a_0, a_1 \in A$ (as the residue field of A is perfect). We compute

$$(z - \pi^{-n/p}a_0)^p - (z - \pi^{-n/p}a_0) = \pi^{-(n-1)}(a_1 + \pi^{n-1-n/p}a_0) + b'$$

for some $b' \in B$. This would contradict the minimality of n. Thus p does not divide n. Consider the degree p extension K_1 of K given by $w^p - w = \pi^{-n}a$. By Lemma 15.62.14 this extension is a totally ramified Galois extension. Thus $L_1 = L \otimes_K K_1$ is a field and $A_1 \subset B_1$ is weakly unramified (Lemma 15.62.5). By Lemma 15.62.14 the ring $M_1 = M \otimes_K K_1$ is either a product of p copies of L_1 (in which case we are done) or a field extension of L_1 of degree p. Moreover, in the second case, M_1 is either weakly unramified over L_1 (in which case we are done) or a degree p totally ramified Galois extension. In this last case the extension M_1/L_1 is generated by the element $z - w$ and

$$(z - w)^p - (z - w) = z^p - z - (w^p - w) = b$$

with $b \in B$ (see above). Thus by Lemma 15.62.14 once more the extension M_1/L_1 is unramified and we conclude that K_1 is a weak solution for $A \to C$. From now on we assume $J \neq \emptyset$.

Suppose that $j', j \in J$ such that $j' = p^r j$ for some $r > 0$. Then we change our choice of z into

$$z' = z - (\sigma(\lambda_j)\pi^{-j} + \sigma(\lambda_j^p)\pi^{-pj} + \ldots + \sigma(\lambda_j^{p^{r-1}})\pi^{-p^{r-1}j})$$

Then ξ changes into $\xi' = (z')^p - (z')$ as follows

$$\xi' = \xi - \sigma(\lambda_j)\pi^{-j} + \sigma(\lambda_j^{p^r})\pi^{-j'} + \text{something in } B$$

[8] If B is complete, then we can choose σ to be a ring map. If A is also complete and σ is a ring map, then σ maps κ_A into A.

Writing $\xi' = \sum_{i=n,\ldots,1} \sigma(\lambda_i')\pi^{-i} + b'$ as before we find that $\lambda_i' = \lambda_i$ for $i \neq j, j'$ and $\lambda_j' = 0$. Thus the set J has gotten smaller. By induction on the size of J we may assume no such pair j, j' exists. (Please observe that in this procedure we may get thrown back into the case that $J = \emptyset$ we treated above.)

For $j \in J$ write $\lambda_j = \mu_j^{p^{r_j}}$ for some $r_j \geq 0$ and $\mu_j \in \kappa_B$ which is not a pth power. This is possible by our assumption (4). Let $j \in J$ be the unique index such that jp^{-r_j} is maximal. (The index is unique by the result of the preceding paragraph.) Choose $r > \max(r_j + 1)$ and such that $jp^{r-r_j} > n$ for $j \in J$. Choose a separable totally ramified extension $K \subset K_1$ of degree p^r such that the corresponding discrete valuation ring $A_1 \subset K_1$ has uniformizer π' with $(\pi')^{p^r} = \pi + \pi^{n+1}a$ for some $a \in A_1$ (Lemma 15.62.10). Observe that $L_1 = L \otimes_K K_1$ is a field and $B_1 \subset L_1$ a discrete valuation ring totally ramified over B (Lemma 15.62.5). Computing in B_1 we get

$$\xi = \sum\nolimits_{i \in I} \sigma(\lambda_i)(\pi')^{-ip^r} + \sum\nolimits_{j \in J} \sigma(\mu_j)^{p^{r_j}} (\pi')^{-jp^r} + b_1$$

for some $b_1 \in B_1$. Note that $\sigma(\lambda_i)$ for $i \in I$ is a qth power modulo π^n, i.e., modulo $(\pi')^{np^r}$. Hence we can rewrite the above as

$$\xi = \sum\nolimits_{i \in I} x_i^{p^r} (\pi')^{-ip^r} + \sum\nolimits_{j \in J} \sigma(\mu_j)^{p^{r_j}} (\pi')^{-jp^r} + b_1$$

As in the previous paragraph we change our choice of z into

$$\begin{aligned} z' = z & \\ & - \sum\nolimits_{i \in I} \left(x_i(\pi')^{-i} + \ldots + x_i^{p^{r-1}} (\pi')^{-ip^{r-1}} \right) \\ & - \sum\nolimits_{j \in J} \left(\sigma(\mu_j)(\pi')^{-jp^{r-r_j}} + \ldots + \sigma(\mu_j)^{p^{r_j-1}} (\pi')^{-jp^{r-1}} \right) \end{aligned}$$

to obtain

$$(z')^p - z' = \sum\nolimits_{i \in I} x_i(\pi')^{-i} + \sum\nolimits_{j \in J} \sigma(\mu_j)(\pi')^{-jp^{r-r_j}} + b_1'$$

for some $b_1' \in B_1$. Since there is a unique j such that jp^{r-r_j} is maximal and since jp^{r-r_j} is bigger than $i \in I$ and divisible by p, we see that M_1/L_1 falls into case (C) of Lemma 15.62.14. This finishes the proof. □

Lemma 15.62.16. *Let A be a ring which contains a primitive pth root of unity ζ. Set $w = 1 - \zeta$. Then*

$$P(z) = \frac{(1 + wz)^p - 1}{w^p} = z^p - z + \sum\nolimits_{0 < i < p} a_i z^i$$

is an element of $A[z]$ and in fact $a_i \in (w)$. Moreover, we have

$$P(z_1 + z_2 + wz_1z_2) = P(z_1) + P(z_2) + w^p P(z_1)P(z_2)$$

in the polynomial ring $A[z_1, z_2]$.

Proof. It suffices to prove this when

$$A = \mathbf{Z}[\zeta] = \mathbf{Z}[x]/(x^{p-1} + \ldots + x + 1)$$

is the ring of integers of the cyclotomic field. The polynomial identity $t^p - 1 = (t-1)(t-\zeta)\ldots(t-\zeta^{p-1})$ (which is proved by looking at the roots on both sides) shows that $t^{p-1} + \ldots + t + 1 = (t-\zeta)\ldots(t-\zeta^{p-1})$. Substituting $t = 1$ we obtain $p = (1-\zeta)(1-\zeta^2)\ldots(1-\zeta^{p-1})$. The maximal ideal $(p, w) = (w)$ is the unique

prime ideal of A lying over p (as fields of characteristic p do not have nontrivial pth roots of 1). It follows that $p = uw^{p-1}$ for some unit u. This implies that

$$a_i = \frac{1}{p}\binom{p}{i} uw^{i-1}$$

for $p > i > 1$ and $-1 + a_1 = pw/w^p = u$. Since $P(-1) = 0$ we see that $0 = (-1)^p - u$ modulo (w). Hence $a_1 \in (w)$ and the proof if the first part is done. The second part follows from a direct computation we omit. □

Lemma 15.62.17. *Let A be a discrete valuation ring of mixed characteristic $(0, p)$ which contains a primitive pth root of 1. Let $P(t) \in A[t]$ be the polynomial of Lemma 15.62.16. Let $\xi \in K$. Let L be an extension of K obtained by adjoining a root of $P(z) = \xi$. Then L/K is Galois and one of the following happens*

1. *$L = K$,*
2. *L/K is unramified of degree p,*
3. *L/K is totally ramified with ramification index p, and*
4. *L/K is weakly unramified, the integral closure B of A in L is a discrete valuation ring and $A \to B$ induces a purely inseparable residue field extension of degree p.*

Let π be a uniformizer of A. We have the following implications:

(A) *If $\xi \in A$, then we are in case (1) or (2).*
(B) *If $\xi = \pi^{-n}a$ where $n > 0$ is not divisible by p and a is a unit in A, then we are in case (3)*
(C) *If $\xi = \pi^{-n}a$ where $n > 0$ is divisible by p and the image of a in κ_A is not a pth power, then we are in case (4).*

Proof. Adjoining a root of $P(z) = \xi$ is the same thing as adjoining a root of $y^p = w^p(1 + \xi)$. Since K contains a primitive pth root of 1 the extension is Galois of order dividing p by the discussion in Fields, Section 9.21. It immediately follows from the discussion in Section 15.61 that we are in one of the cases (1) – (4) listed in the lemma.

Case (A). Here we see that $A \to A[x]/(P(x) - \xi)$ is a finite étale ring extension. Hence we are in cases (1) or (2).

Case (B). Write $\xi = \pi^{-n}a$ where p does not divide n. Let $B \subset L$ be the integral closure of A in L. If $C = B_{\mathfrak{m}}$ for some maximal ideal $\mathfrak{m}$, then it is clear that $p\text{ord}_C(z) = -n\text{ord}_C(\pi)$. In particular $A \subset C$ has ramification index divisible by p. It follows that it is p and that $B = C$.

Case (C). Set $k = n/p$. Then we can rewrite the equation as

$$(\pi^k z)^p - \pi^{n-k}(\pi^k z) + \sum a_i \pi^{n-ik}(\pi^k z)^i = a$$

Since $A[y]/(y^p - \pi^{n-k}y - \sum a_i \pi^{n-ik} y^i - a)$ is a discrete valuation ring weakly unramified over A, the lemma follows. □

Let A be a discrete valuation ring of mixed characteristic $(0, p)$ containing a primitive pth root of 1. Let $w \in A$ and $P(t) \in A[t]$ be as in Lemma 15.62.16. Let L be a finite extension of K. We say L/K is a *degree p extension of finite level* if L is a degree p extension of K obtained by adjoining a root of the equation $P(z) = \xi$ where $\xi \in K$ is an element with $w^p\xi \in \mathfrak{m}_A$.

This definition is relevant to the discussion in this section due to the following straightforward lemma.

Lemma 15.62.18. *Let $A \subset B \subset C$ be extensions of discrete valuation rings with fractions fields $K \subset L \subset M$. Assume that*

(1) *A has mixed characteristic $(0, p)$,*
(2) *$A \subset B$ is weakly unramified,*
(3) *B contains a primitive pth root of 1, and*
(4) *M/L is Galois of degree p.*

Then there exists a totally ramified Galois extension $K \subset K_1$ which is either a weak solution for $A \to C$ or is such that M_1/L_1 is a degree p extension of finite level.

Proof. Let $\pi \in A$ be a uniformizer. By Kummer theory (Fields, Lemma 9.21.1) M is obtained from L by adjoining the root of $y^p = b$ for some $b \in L$.

If $\text{ord}_B(b)$ is prime to p, then we choose a degree p separable, totally ramified extension $K \subset K_1$ (for example using Lemma 15.62.10). Let A_1 be the integral closure of A in K_1. By Lemma 15.62.5 the integral closure B_1 of B in $L_1 = L \otimes_K K_1$ is a discrete valuation ring weakly unramified over A_1. If $K \subset K_1$ is not a weak solution for $A \to C$, then the integral closure C_1 of C in $M_1 = M \otimes_K K_1$ is a discrete valuation ring and $B_1 \to C_1$ has ramification index p. In this case, the field M_1 is obtained from L_1 by adjoining the pth root of b with $\text{ord}_{B_1}(b)$ divisible by p. Replacing A by A_1, etc we may assume that $b = \pi^n u$ where $u \in B$ is a unit and n is divisible by p. Of course, in this case the extension M is obtained from L by adjoining the pth root of a unit.

Suppose M is obtained from L by adjoining the root of $y^p = u$ for some unit u of B. If the residue class of u in κ_B is not a pth power, then $B \subset C$ is weakly unramified (Lemma 15.62.11) and we are done. Otherwise, we can replace our choice of y by y/v where v^p and u have the same image in κ_B. After such a replacement we have

$$y^p = 1 + \pi b$$

for some $b \in B$. Then we see that $P(z) = \pi b/w^p$ where $z = (y-1)/w$. Thus we see that the extension is a degree p extension of finite level with $\xi = \pi b/w^p$. □

Let A be a discrete valuation ring of mixed characteristic $(0, p)$ containing a primitive pth root of 1. Let $w \in A$ and $P(t) \in A[t]$ be as in Lemma 15.62.16. Let L be a degree p extension of K of finite level. Choose $z \in L$ generating L over K with $\xi = P(z) \in K$. Choose a uniformizer π for A and write $w = u\pi^{e_1}$ for some integer $e_1 = \text{ord}_A(w)$ and unit $u \in A$. Finally, pick $n \geq 0$ such that

$$\pi^n \xi \in A$$

The *level* of L/K is the smallest value of the quantity n/e_1 taking over all z generating L/K with $\xi = P(z) \in K$.

We make a couple of remarks. Since the extension is of finite level we know that we can choose z such that $n < pe_1$. Thus the level is a rational number contained in $[0, p)$. If the level is zero then L/K is unramified by Lemma 15.62.17. Our next goal is to lower the level.

Lemma 15.62.19. *Let $A \subset B \subset C$ be extensions of discrete valuation rings with fractions fields $K \subset L \subset M$. Assume*

(1) *A has mixed characteristic $(0, p)$,*

(2) $A \subset B$ *weakly unramified,*
(3) B *contains a primitive* p*th root of* 1,
(4) M/L *is a degree* p *extension of finite level* $l > 0$,
(5) $\kappa_A = \bigcap_{n\geq 1} \kappa_B^{p^n}$.

Then there exists a separable, totally ramified extension K_1 *of* K *such that either* K_1 *is a weak solution for* $A \to C$*, or the extension* M_1/L_1 *is a degree* p *extension of finite level* $\leq \max(0, l-1, 2l-p)$.

Proof. Let $\pi \in A$ be a uniformizer. Let $w \in B$ and $P \in B[t]$ be as in Lemma 15.62.16 (for B). Set $e_1 = \mathrm{ord}_B(w)$, so that w and π^{e_1} are associates in B. Pick $z \in M$ generating M over L with $\xi = P(z) \in K$ and n such that $\pi^n \xi \in B$ as in the definition of the level of M over L, i.e., $l = n/e_1$.

The proof of this lemma is completely similar to the proof of Lemma 15.62.15. To explain what is going on, observe that

$$P(z) \equiv z^p - z \bmod \pi^{-n+e_1} B \tag{15.62.19.1}$$

for any $z \in L$ such that $\pi^{-n} P(z) \in B$ (use that z has valuation at worst $-n/p$ and the shape of the polynomial P). Moreover, we have

$$\xi_1 + \xi_2 + w^p \xi_1 \xi_2 \equiv \xi_1 + \xi_2 \bmod \pi^{-2n+pe_1} B \tag{15.62.19.2}$$

for $\xi_1, \xi_2 \in \pi^{-n} B$. Finally, observe that $n - e_1 = (l-1)/e_1$ and $-2n + pe_1 = -(2l-p)e_1$. Write $m = n - e_1 \max(0, l-1, 2l-p)$. The above shows that doing calculations in $\pi^{-n}B/\pi^{-n+m}B$ the polynomial P behaves exactly as the polynomial $z^p - z$. This explains why the lemma is true but we also give the details below.

Assumption (4) implies that κ_A is perfect. Observe that $m \leq e_1$ and hence A/π^m is annihilated by w and hence p. Thus we may choose compatible ring maps $\overline{\sigma} : \kappa_A \to A/\pi^m A$ and $\overline{\sigma} : \kappa_B \to B/\pi^m B$ as in Lemma 15.62.13. We lift the second of these to a map of sets $\sigma : \kappa_B \to B$. Then we can write

$$\xi = \sum\nolimits_{i=n,\ldots,n-m+1} \sigma(\lambda_i)\pi^{-i} + \pi^{-n+m)} b$$

for some $\lambda_i \in \kappa_B$ and $b \in B$. Let

$$I = \{i \in \{n, \ldots, n-m+1\} \mid \lambda_i \in \kappa_A\}$$

and

$$J = \{j \in \{n, \ldots, n-m+1\} \mid \lambda_i \notin \kappa_A\}$$

We will argue by induction on the size of the finite set J.

The case $J = \emptyset$. Here for all $i \in \{n, \ldots, n-m+1\}$ we have $\sigma(\lambda_i) = a_i + \pi^{n-m} b_i$ for some $a_i \in A$ and $b_i \in B$ by our choice of $\overline{\sigma}$. Thus $\xi = \pi^{-n} a + \pi^{-n+m} b$ for some $a \in A$ and $b \in B$. If $p | n$, then we write $a = a_0^p + \pi a_1$ for some $a_0, a_1 \in A$ (as the residue field of A is perfect). Set $z_1 = -\pi^{-n/p} a_0$. Note that $P(z_1) \in \pi^{-n} B$ and that $z + z_1 + wzz_1$ is an element generating M over L (note that $wz_1 \neq -1$ as $n < pe_1$). Moveover, by Lemma 15.62.16 we have

$$P(z + z_1 + wzz_1) = P(z) + P(z_1) + w^p P(z) P(z_1) \in K$$

and by equations (15.62.19.1) and (15.62.19.2) we have

$$P(z) + P(z_1) + w^p P(z) P(z_1) \equiv \xi + z_1^p - z_1 \bmod \pi^{-n+m} B$$

for some $b' \in B$. This contradict the minimality of n! Thus p does not divide n. Consider the degree p extension K_1 of K given by $P(y) = -\pi^{-n} a$. By Lemma

15.62.17 this extension is separable, totally ramified. Thus $L_1 = L \otimes_K K_1$ is a field and $A_1 \subset B_1$ is weakly unramified (Lemma 15.62.5). By Lemma 15.62.17 the ring $M_1 = M \otimes_K K_1$ is either a product of p copies of L_1 (in which case we are done) or a field extension of L_1 of degree p. Moreover, in the second case, M_1 is either weakly unramified over L_1 (in which case we are done) or a degree p totally ramified Galois extension. In this last case the extension M_1/L_1 is generated by the element $z + y + wzy$ and we see that $P(z + y + wzy) \in L_1$ and

$$\begin{aligned} P(z + y + wzy) &= P(z) + P(y) + w^p P(z)P(y) \\ &\equiv \xi - \pi^{-n} a \bmod \pi^{-n+m} B_1 \\ &\equiv 0 \bmod \pi^{-n+m} B_1 \end{aligned}$$

in exactly the same manner as above. By our choice of m this means exactly that M_1/L_1 has level at most $\max(0, l-1, 2l-p)$. From now on we assume that $J \neq \emptyset$.

Suppose that $j', j \in J$ such that $j' = p^r j$ for some $r > 0$. Then we set

$$z_1 = -\sigma(\lambda_j)\pi^{-j} - \sigma(\lambda_j^p)\pi^{-pj} - \ldots - \sigma(\lambda_j^{p^{r-1}})\pi^{-p^{r-1}j}$$

and we change z into $z' = z + z_1 + wzz_1$. Observe that $z' \in M$ generates M over L and that we have $\xi' = P(z') = P(z) + P(z_1) + wP(z)P(z_1) \in L$ with

$$\xi' \equiv \xi - \sigma(\lambda_j)\pi^{-j} + \sigma(\lambda_j^{p^r})\pi^{-j'} \bmod \pi^{-n+m} B$$

by using equations (15.62.19.1) and (15.62.19.2) as above. Writing

$$\xi' = \sum\nolimits_{i=n,\ldots,n-m+1} \sigma(\lambda_i')\pi^{-i} + \pi^{-n+m} b'$$

as before we find that $\lambda_i' = \lambda_i$ for $i \neq j, j'$ and $\lambda_j' = 0$. Thus the set J has gotten smaller. By induction on the size of J we may assume there is no pair j, j' of J such that j'/j is a power of p. (Please observe that in this procedure we may get thrown back into the case that $J = \emptyset$ we treated above.)

For $j \in J$ write $\lambda_j = \mu_j^{p^{r_j}}$ for some $r_j \geq 0$ and $\mu_j \in \kappa_B$ which is not a pth power. This is possible by our assumption (4). Let $j \in J$ be the unique index such that jp^{-r_j} is maximal. (The index is unique by the result of the preceding paragraph.) Choose $r > \max(r_j + 1)$ and such that $jp^{r-r_j} > n$ for $j \in J$. Let $K \subset K_1$ be the totally ramified extension of degree p^r defined by $(\pi')^{p^r} = \pi$. Observe that π' is the uniformizer of the corresponding discrete valuation ring $A_1 \subset K_1$. Observe that $L_1 = L \otimes_K K_1$ is a field and $B_1 \subset L_1$ a discrete valuation ring totally ramified over B (Lemma 15.62.5). Computing in B_1 we get

$$\xi = \sum\nolimits_{i \in I} \sigma(\lambda_i)(\pi')^{-ip^r} + \sum\nolimits_{j \in J} \sigma(\mu_j)^{p^{r_j}} (\pi')^{-jp^r} + \pi^{-n+m} b_1$$

for some $b_1 \in B_1$. Note that $\sigma(\lambda_i)$ for $i \in I$ is a qth power modulo π^m, i.e., modulo $(\pi')^{mp^r}$. Hence we can rewrite the above as

$$\xi = \sum\nolimits_{i \in I} x_i^{p^r} (\pi')^{-ip^r} + \sum\nolimits_{j \in J} \sigma(\mu_j)^{p^{r_j}} (\pi')^{-jp^r} + \pi^{-n+m} b_1$$

Similar to our choice in the previous paragraph we set

$$\begin{aligned} z_1 &- \sum\nolimits_{i \in I} \left(x_i(\pi')^{-i} + \ldots + x_i^{p^{r-1}} (\pi')^{-ip^{r-1}} \right) \\ &- \sum\nolimits_{j \in J} \left(\sigma(\mu_j)(\pi')^{-jp^{r-r_j}} + \ldots + \sigma(\mu_j)^{p^{r_j-1}} (\pi')^{-jp^{r-1}} \right) \end{aligned}$$

and we change our choice of z into $z' = z + z_1 + wzz_1$. Then z' generates M_1 over L_1 and $\xi' = P(z') = P(z) + P(z_1) + w^p P(z)P(z_1) \in L_1$ and a calculation shows that

$$\xi' \equiv \sum\nolimits_{i \in I} x_i (\pi')^{-i} + \sum\nolimits_{j \in J} \sigma(\mu_j)(\pi')^{-jp^{r-r_j}} + (\pi')^{(-n+m)p^r} b_1'$$

for some $b_1' \in B_1$. There is a unique j such that jp^{r-r_j} is maximal and jp^{r-r_j} is bigger than $i \in I$. If $jp^{r-r_j} \leq (n-m)p^r$ then the level of the extension M_1/L_1 is less than $\max(0, l-1, 2l-p)$. If not, then, as p divides jp^{r-r_j}, we see that M_1/L_1 falls into case (C) of Lemma 15.62.17. This finishes the proof. □

Lemma 15.62.20. *Let $A \subset B \subset C$ be extensions of discrete valuation rings with fraction fields $K \subset L \subset M$. Assume*

1. *the residue field k of A is algebraically closed of characteristic $p > 0$,*
2. *A and B are complete,*
3. *$A \to B$ is weakly unramified,*
4. *M is a finite extension of L,*
5. *$k = \bigcap_{n \geq 1} \kappa_B^{p^n}$*

Then there exists a finite extension $K \subset K_1$ which is a weak solution for $A \to C$.

Proof. Let M' be any finite extension of L and consider the integral closure C' of B in M'. Then C' is finite over B as B is Nagata by Algebra, Lemma 10.148.22. Moreover, C' is a discrete valuation ring, see discussion in Remark 15.62.1. Moreover C' is complete as a B-module, hence complete as a discrete valuation ring, see Algebra, Section 10.91. It follows in particular that C is the integral closure of B in M (by definition of valuation rings as maximal for the relation of domination).

Let $M \subset M'$ be a finite extension and let $C' \subset M'$ be the integral closure of B as above. By Lemma 15.62.6 tt suffices to prove the result for $A \to B \to C'$. Hence we may assume that M/L is normal, see Fields, Lemma 9.15.3.

If M/L is normal, we can find a chain of finite extensions

$$L = L^0 \subset L^1 \subset L^2 \subset \ldots \subset L^r = M$$

such that each extension L^{j+1}/L^j is either: (a) purely inseparable of degree p, (b) totally ramified Galois of degree p, (c) totally ramified Galois cyclic of order prime to p, or (d) Galois and unramified. Namely, since M/L is normal we can write it as a compositum of a Galois extension and a purely inseparable extension (Fields, Lemma 9.24.3). For the purely inseparable extension the existence of the filtration is clear. In the Galois case, note that G is "the" decomposition group and let $I \subset G$ be the inertia group. Then on the one hand I is solvable by Lemma 15.61.11 and on the other hand the extension M^I/L is unramfied by Lemma 15.61.14. This proves we have a filtration as stated.

We are going to argue by induction on the integer r. Suppose that we can find a finite extension $K \subset K_1$ which is a weak solution for $A \to B^1$ where B^1 is the integral closure of B in L^1. Let K_1' be the normal closure of K_1/K (Fields, Lemma 9.15.3). Since A is complete and the residue field of A is algebraically closed we see that $K_1 \subset K_1'$ is separable and totally ramified (some details omitted). Hence $K \subset K_1'$ is a weak solution for $A \to B^1$ as well by Lemma 15.62.5. In other words,

we may and do assume that K_1 is a normal extension of K. Having done so we consider the sequence

$$L_1^0 = (L^0 \otimes_K K_1)_{red} \subset L_1^1 = (L^1 \otimes_K K_1)_{red} \subset \ldots \subset L_1^r = (L^r \otimes_K K_1)_{red}$$

and the corresponding integral closures B_1^i. Note that $C_1 = B_1^r$ is a product of discrete valuation rings which are transitively permuted by $G = \mathrm{Aut}(K_1/K)$ by Lemma 15.62.8. In particular all the extensions of discrete valution rings $A_1 \to (C_1)_{\mathfrak{m}}$ are isomorphic and a solution for one will be a solution for all of them. We can apply the induction hypothesis to the sequence

$$A_1 \to (B_1^1)_{B_1^1 \cap \mathfrak{m}} \to (B_1^2)_{B_1^2 \cap \mathfrak{m}} \to \ldots \to (B_1^r)_{B_1^r \cap \mathfrak{m}} = (C_1)_{\mathfrak{m}}$$

to get a solution $K_1 \subset K_2$ for $A_1 \to (C_1)_{\mathfrak{m}}$. The extension $K \subset K_2$ will then be a solution for $A \to C$ by what we said before. Note that the induction hypothesis applies: the ring map $A_1 \to (B_1^1)_{B_1^1 \cap \mathfrak{m}}$ is weakly unramified by our choice of K_1 and the sequence of fraction field extensions each still have one of the properties (a), (b), (c), or (d) listed above. Moreover, observe that for any finite extension $\kappa_B \subset \kappa$ we still have $k = \bigcap \kappa^{p^n}$.

Thus everything boils down to finding a weak solution when the field extension $L \subset M$ satisfies one of the properties (a), (b), (c), or (d).

Case (d). This case is trivial as here $B \to C$ is unramified already.

Case (c). This is the tamely ramified case. Say M/L is cyclic of order n prime to p. Choose uniformizers $\pi \in A$ and $\pi_C \in C$. Then $\pi_C^n = u\pi$ for some unit u of C. Let $K_1 = K[\pi^{1/n}]$. Then the ring $M_1 = M \otimes_K K_1$ is a product of fields each of which is obtained from M by adjoining a root of $z^n - u = 0$. Since $C[x]/(x^n - u)$ is finite étale over C we conclude that each of these extensions is unramified over M. Considering the commutative diagram

$$\begin{array}{ccc} M & \longrightarrow & M_1 \\ \uparrow & & \uparrow \\ L & \longrightarrow & L_1 \end{array}$$

and using multiplicativity of ramification indices, it follows that each factor of M_1 is unramified over L_1 as well.

Case (b). We divide this case into the mixed characteristic case and the equicharacteristic case. In the equicharacteristic case this is Lemma 15.62.15. In the mixed characteristic case, we first replace K by a finite extension to get to the situation where M/L is a degree p extension of finite level using Lemma 15.62.18. Then the level is a rational number $l \in [0, p)$, see discussion preceding Lemma 15.62.19. If the level is 0, then $B \to C$ is weakly unramified and we're done. If not, then we can replacing the field K by a finite extension to obtain a new situation with level $l' \leq \max(0, l-1, 2l-p)$ by Lemma 15.62.19. If $l = p - \epsilon$ for $\epsilon < 1$ then we see that $l' \leq p - 2\epsilon$. Hence after a finite number of replacements we obtain a case with level $\leq p - 1$. Then after at most $p - 1$ more such replacements we reach the situation where the level is zero.

Case (a) is Lemma 15.62.12. This is the only case where we possibly need a purely inseparable extension of K, namely, in case (2) of the statement of the lemma we

win by adjoining a pth power of the element π. This finishes the proof of the lemma. $\square$

At this point we have collected all the lemmas we need to prove the main result of this section.

Theorem 15.62.21 (Epp)**.** *Let $A \subset B$ be an extension of discrete valuation rings with fraction fields $K \subset L$. If the characteristic of κ_A is $p > 0$, assume that every element of*

$$\bigcap\nolimits_{n \geq 1} \kappa_B^{p^n}$$

is separable algebraic over κ_A. Then there exists a finite extension $K \subset K_1$ which is a weak solution for $A \to B$ as defined in Definition 15.62.2.

Proof. We first prove the result in case the characteristic of κ_A is zero as the result is easy in that case. Namely, suppose the ramification index is e. Choose a uniformizer $\pi_B \in B$ and a uniformizer $\pi \in A$. Write $\pi_B^e = u\pi$ for some unit $u \in B$. Let $K \subset K_1$ be be the totally ramified extension obtained by adjoining an eth root of π (see Lemma 15.62.9). Then $L_1 = L \otimes_K K_1$ is a product of fields, each obtained by adjoining a root of $z^e = u$ to L. However, since $B[x]/(x^e - u)$ is finite étale over B (since the characteristic is zero) we conclude that each of these extensions is unramified over L. Considering the commutative diagram

$$\begin{array}{ccc} L & \longrightarrow & L_1 \\ \uparrow & & \uparrow \\ K & \longrightarrow & K_1 \end{array}$$

and using multiplicativity of ramification indices, it follows that each factor of L_1 is unramified over K_1 as well. This finishes the proof in residue characteristic 0.

From now on we let p be a prime number and we assume that κ_A has characteristic p. We first apply Lemma 15.62.7 to reduce to the case that A and B have separably closed residue fields. Since κ_A and κ_B are replaced by their separable algebraic closures by this procedure we see that we obtain

$$\kappa_A \supset \bigcap\nolimits_{n \geq 1} \kappa_B^{p^n}$$

from the condition of the theorem.

Let $\pi \in A$ be a uniformizer. Let $A^\wedge$ and $B^\wedge$ be the completions of A and B. We have a commutative diagram

$$\begin{array}{ccc} B & \longrightarrow & B^\wedge \\ \uparrow & & \uparrow \\ A & \longrightarrow & A^\wedge \end{array}$$

of extensions of discrete valuation rings. Let $K^\wedge$ be the fraction field of $A^\wedge$. Suppose that we can find a finite extension $K^\wedge \subset M$ which is (a) a weak solution for $A^\wedge \to B^\wedge$ and (b) a compositum of a separable extension and an extension obtained by adjoining a p-power root of π. Then by Lemma 15.61.16 we can find a finite extension $K \subset K_1$ such that $K^\wedge \otimes_K K_1 = M$. Let A_1, resp. $A_1^\wedge$ be the integral closure of A, resp. $A^\wedge$ in K_1, resp. M. Since $A \to A^\wedge$ is formally smooth (Lemma 15.61.4) we see that $A_1 \to A_1^\wedge$ is formally smooth (Lemma 15.62.4 and A_1 and $A_1^\wedge$

are discrete valuation rings by discussion in Remark 15.62.1). We conclude from Lemma 15.62.6 part (2) that $K \subset K_1$ is a weak solution for $A \to B^\wedge$. Applying Lemma 15.62.6 part (1) we see that $K \subset K_1$ is a weak solution for $A \to B$.

Thus we may assume A and B are complete discrete valution rings with separably closed residue fields of characteristic p and with $\kappa_A \supset \bigcap_{n\geq 1} \kappa_B^{p^n}$. We are also given a uniformizer $\pi \in A$ and we have to find a weak solution for $A \to B$ which is a compositum of a separable extension and a field obtained by taking p-power roots of π. Note that the second condition is automatic if A has mixed characteristic.

Set $k = \bigcap_{n\geq 1} \kappa_B^{p^n}$. Observe that k is an algebraically closed field of characteristic p. If A has mixed characteristic let Λ be a Cohen ring for k and in the equicharacteristic case set $\Lambda = k[[t]]$. We can choose a ring map $\Lambda \to A$ which maps t to π in the equicharacteristic case. In the equicharacteristic case this follows from the Cohen structure theorem (Algebra, Theorem 10.147.8) and in the mixed characteristic case this follows as $\mathbf{Z}_p \to \Lambda$ is formally smooth in the adic topology (Lemmas 15.61.4 and 15.26.5). Applying Lemma 15.62.6 we see that it suffices to prove the existence of a weak solution for $\Lambda \to B$ which in the equicharacteristic p case is a compositum of a separable extension and a field obtained by taking p-power roots of t. However, since $\Lambda = k[[t]]$ in the equicharacteristic case and any extension of $k((t))$ is such a compositum, we can now drop this requirement!

Thus we arrive at the situation where A and B are complete, the residue field k of A is algebraically closed of characteristic $p > 0$, we have $k = \bigcap \kappa_B^{p^n}$, and in the mixed characteristic case p is a uniformizer of A (i.e., A is a Cohen ring for k). If A has mixed characteristic choose a Cohen ring Λ for κ_B and in the equicharacteristic case set $\Lambda = \kappa_B[[t]]$. Arguing as above we may choose a ring map $A \to \Lambda$ lifting $k \to \kappa_B$ and mapping a uniformizer to a uniformizer. Since $k \subset \kappa_B$ is separable the ring map $A \to \Lambda$ is formally smooth in the adic topology (Lemma 15.61.4). Hence we can find a ring map $\Lambda \to B$ such that the composition $A \to \Lambda \to B$ is the given ring map $A \to B$ (see Lemma 15.26.5). Since Λ and B are complete discrete valution rings with the same residue field, B is finite over Λ (Algebra, Lemma 10.91.17). This reduces us to the special case discussed in Lemma 15.62.20. □

Lemma 15.62.22. *Let $A \to B$ be an extension of discrete valuation rings with fraction fields $K \subset L$. Assume B is essentially of finite type over A. Let $K \subset K'$ be an algebraic extension of fields such that the integral closure A' of A in K' is Noetherian. Then the integral closure B' of B in $L' = (L \otimes_K K')_{red}$ is Noetherian as well. Moreover, the map $\mathrm{Spec}(B') \to \mathrm{Spec}(A')$ is surjective and the corresponding residue field extensions are finitely generated field extensions.*

Proof. Let $A \to C$ be a finite type ring map such that B is a localization of C at a prime $\mathfrak{p}$. Then $C' = C \otimes_A A'$ is a finite type A'-algebra, in particular Noetherian. Since $A \to A'$ is integral, so is $C \to C'$. Thus $B = C_\mathfrak{p} \subset C'_\mathfrak{p}$ is integral too. It follows that the dimension of $C'_\mathfrak{p}$ is 1 (Algebra, Lemma 10.106.4). Of course $C'_\mathfrak{p}$ is Noetherian. Let $\mathfrak{q}_1, \ldots, \mathfrak{q}_n$ be the minimal primes of $C'_\mathfrak{p}$. Let B'_i be the integral closure of $B = C_\mathfrak{p}$, or equivalently by the above of $C'_\mathfrak{p}$ in the field of fractions of $C'_{\mathfrak{p}'}/\mathfrak{q}_i$. It follows from Krull-Akizuki (Algebra, Lemma 10.113.11 applied to the finitely many localizations of $C'_\mathfrak{p}$ at its maximal ideals) that each B'_i is Noetherian. Moreover the residue field extensions in $C'_\mathfrak{p} \to B'_i$ are finite

by Algebra, Lemma 10.113.9. Finally, we observe that $B' = \prod B'_i$ is the integral closure of B in $L' = (L \otimes_K K')_{red}$. □

Proposition 15.62.23. *Let $A \to B$ be an extension of discrete valuation rings with fraction fields $K \subset L$. If B is essentially of finite type over A, then there exists a finite extension $K \subset K_1$ which is a solution for $A \to B$ as defined in Definition 15.62.2.*

Proof. Observe that a weak solution is a solution if the residue field of A is perfect, see Lemma 15.61.4. Thus the proposition follows immediately from Theorem 15.62.21 if the residue characteristic of A is 0 (and in fact we do not need the assumption that $A \to B$ is essentially of finite type). If the residue characteristic of A is $p > 0$ we will also deduce it from Epp's theorem.

Let $x_i \in A$, $i \in I$ be a set of elements mapping to a p-base of the residue field κ of A. Set

$$A' = \bigcup\nolimits_{n \geq 1} A[t_{i,n}]/(t_{i,n}^{p^n} - x_i)$$

where the transition maps send $t_{i,n+1}$ to $t_{i,n}^p$. Observe that A' is a filtered colimit of weakly unramified finite extensions of discrete valuation rings over A. Thus A' is a discrete valuation ring and $A \to A'$ is weakly unramfied. By construction the residue field $\kappa' = A'/\mathfrak{m}_A A'$ is the perfection of κ.

Let $K' = f.f.(A')$. We may apply Lemma 15.62.22 to the extension $K \subset K'$. Thus B' is a finite product of Dedekind domains. Let $\mathfrak{m}_1, \ldots, \mathfrak{m}_n$ be the maximal ideals of B'. Using Epp's theorem (Theorem 15.62.21) we find a weak solution $K' = f.f.(A') \subset K'_i$ for each of the extensions $A' \subset B'_{\mathfrak{m}_i}$. Since the residue field of A' is perfect, these are actually solutions. Let $K' \subset K'_1$ be a finite extension which contains each K'_i. Then $K' \subset K'_1$ is still a solution for each $A' \subset B'_{\mathfrak{m}_i}$ by Lemma 15.62.4.

Let A'_1 be the integral closure of A in K'_1. Note that A'_1 is a Dedekind domain by the discussion in Remark 15.62.1 applied to $K' \subset K'_1$. Thus Lemma 15.62.22 applies to $K \subset K'_1$. Therefore the integral closure B'_1 of B in $L'_1 = (L \otimes_K K'_1)_{red}$ is a Dedekind domain and because $K' \subset K'_1$ is a solution for each $A' \subset B'_{\mathfrak{m}_i}$ we see that $(A'_1)_{A'_1 \cap \mathfrak{m}} \to (B'_1)_{\mathfrak{m}}$ is formally smooth for each maximal ideal $\mathfrak{m} \subset B'_1$.

By construction, the field K'_1 is a filtered colimit of finite extensions of K. Say $K'_1 = \operatorname{colim}_{i \in I} K_i$. For each i let A_i, resp. B_i be the integral closure of A, resp. B in K_i, resp. $L_i = (L \otimes_K K_i)_{red}$. Then it is clear that

$$A'_1 = \operatorname{colim} A_i \quad \text{and} \quad B'_1 = \operatorname{colim} B_i$$

Since the ring maps $A_i \to A'_1$ and $B_i \to B'_1$ are injective integral ring maps and since A'_1 and B'_1 have finite spectra, we see that for all i large enough the ring maps $A_i \to A'_1$ and $B_i \to B'_1$ are bijective on spectra. Once this is true, for all i large enough the maps $A_i \to A'_1$ and $B_i \to B'_1$ will be weakly unramified (once the uniformizer is in the image). It follows from multiplicativity of ramification indices that $A_i \to B_i$ induces weakly unramified maps on all localizations at maximal ideals of B_i for such i. Increasing i a bit more we see that

$$B_i \otimes_{A_i} A'_1 \longrightarrow B'_1$$

induces surjective maps on residue fields (because the residue fields of B_1' are finitely generated over those of A_1' by Lemma 15.62.22). Picture of residue fields at maximal ideals lying under a chosen maximal ideal of B_1':

$$\begin{array}{ccccccc} \kappa_{B_i} & \longrightarrow & \kappa_{B_{i'}} & \longrightarrow & \ldots & & \kappa_{B_1'} \\ \uparrow & & \uparrow & & & & \uparrow \\ \kappa_{A_i} & \longrightarrow & \kappa_{A_{i'}} & \longrightarrow & \ldots & & \kappa_{A_1'} \end{array}$$

Thus κ_{B_i} is a finitely generated extension of κ_{A_i} such that the compositum of κ_{B_i} and $\kappa_{A_1'}$ in $\kappa_{B_1'}$ is separable over $\kappa_{A_1'}$. Then that happens already at a finite stage: for example, say $\kappa_{B_1'}$ is finite separable over $\kappa_{A_1'}(x_1, \ldots, x_n)$, then just increase i such that $x_1, \ldots, x_n$ are in κ_{B_i} and such that all generators satisfy separable polynomial equations over $\kappa_{A_i}(x_1, \ldots, x_n)$. This means that $A_i \to B_i$ is formally smooth at all maximal ideals of B_i and the proof is complete. □

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CHAPTER 16

Smoothing Ring Maps

16.1. Introduction

The main result of this chapter is the following:

A regular map of Noetherian rings is a filtered colimit of smooth ones.

This theorem is due to Popescu, see [**Pop90**]. A readable exposition of Popescu's proof was given by Richard Swan, see [**Swa98**] who used notes by André and a paper of Ogoma, see [**Ogo94**].

Our exposition follows Swan's, but we first prove an intermediate result which lets us work in a slightly simpler situation. Here is an overview. We first solve the following "lifting problem": A flat infinitesimal deformation of a filtered colimit of smooth algebras is a filtered colimit of smooth algebras. This result essentially says that it suffices to prove the main theorem for maps between reduced Noetherian rings. Next we prove two very clever lemmas called the "lifting lemma" and the "desingularization lemma". We show that these lemmas combined reduce the main theorem to proving a Noetherian, geometrically regular k-algebra Λ is a filtered limit of smooth k-algebras. Next, we discuss the necessary local tricks that go into the Popescu-Ogoma-Swan-André proof. Finally, in the last three sections we give the proof.

We end this introduction with some pointers to references. Let A be a henselian Noetherian local ring. We say A has the *approximation property* if for any $f_1, \ldots, f_m \in A[x_1, \ldots, x_n]$ the system of equations $f_1 = 0, \ldots, f_m = 0$ has a solution in the completion of A if and only if it has a solution in A. This definition is due to Artin. Artin first proved the approximation property for analytic systems of equations, see [**Art68**]. In [**Art69a**] Artin proved the approximation property for local rings essentially of finite type over an excellent discrete valuation ring. Artin conjectured (page 26 of [**Art69a**]) that every excellent henselian local ring should have the approximation property.

At some point in time it became a conjecture that that every regular homomorphism of Noetherian rings is a filtered colimit of smooth algebras (see for example [**Ray72**], [**Pop81**], [**Art82**], [**AD83**]). We're not sure who this conjecture[1] is due to. The relationship with the approximation property is that if $A \to A^\wedge$ is a colimit of smooth algebras, then the approximation property holds (insert future reference here). Moreover, the main theorem applies to the map $A \to A^\wedge$ if A is an excellent local ring, as one of the conditions of an excellent local ring is that the formal

[1]The question/conjecture as formulated in [**Art82**], [**AD83**], and [**Pop81**] is stronger and was shown to be equivalent to the original version in [**CP84**].

fibres are geometrically regular. Note that excellent local rings were defined by Grothendieck and their definition appeared in print in 1965.

In [**Art82**] it was shown that $R \to R^\wedge$ is a filtered colimit of smooth algebras for any local ring R essentially of finite type over a field. In [**AR88**] it was shown that $R \to R^\wedge$ is a filtered colimit of smooth algebras for any local ring R essentially of finite type over an excellent discrete valuation ring. Finally, the main theorem was shown in [**Pop85**], [**Pop86**], [**Pop90**], [**Ogo94**], and [**Swa98**] as discussed above.

Conversely, using some of the results above, in [**Rot90**] it was shown that any local ring with the approximation property is excellent.

The paper [**Spi99**] provides an alternative approach to the main theorem, but it seems hard to read (for example [**Spi99**, Lemma 5.2] appears to be an incorrectly reformulated version of [**Elk73**, Lemma 3]). There is also a Bourbaki lecture about this material, see [**Tei95**].

16.2. Colimits

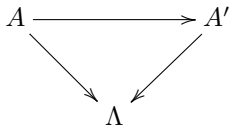
In Categories, Section 4.19 we discuss filtered colimits. In particular, note that Categories, Lemma 4.21.3 tells us that colimits over filtered index categories are the same thing as colimits over directed partially ordered sets.

Lemma 16.2.1. *Let $R \to \Lambda$ be a ring map. Let $\mathcal{E}$ be a set of R-algebras such that each $A \in \mathcal{E}$ is of finite presentation over R. Then the following two statements are equivalent*

1. *Λ is a filtered colimit of elements of $\mathcal{E}$, and*
2. *for any R algebra map $A \to \Lambda$ with A of finite presentation over R we can find a factorization $A \to B \to \Lambda$ with $B \in \mathcal{E}$.*

Proof. Suppose that $\mathcal{I} \to \mathcal{E}$, $i \mapsto A_i$ is a diagram such that $\Lambda = \operatorname{colim}_i A_i$. Let $A \to \Lambda$ with A of finite presentation over R. Pick a presentation $A = R[x_1, \ldots, x_n]/(f_1, \ldots, f_m)$. Say $A \to \Lambda$ maps x_s to $\lambda_s \in \Lambda$. We can find an $i \in \mathrm{Ob}(\mathcal{I})$ and elements $a_s \in A_i$ whose image in Λ is λ_s. Increasing i if necessary we may also assume that $f_t(a_1, \ldots, a_n) = 0$ in A_i. Hence we can factor $A \to \Lambda$ through A_i by mapping x_s to a_s.

Conversely, suppose that (2) holds. Consider the category $\mathcal{I}$ whose objects are R-algebra maps $A \to \Lambda$ with $A \in \mathcal{E}$ and whose morphisms are commutative diagrams



of R-algebras. We claim that $\mathcal{I}$ is a filtered index category and that $\Lambda = \operatorname{colim}_{\mathcal{I}} A$. To see that $\mathcal{I}$ is filtered, let $A \to \Lambda$ and $A' \to \Lambda$ be two objects. Then we can factor $A \otimes_R A' \to \Lambda$ through an object of $\mathcal{I}$ by assumption (2) and the fact that the elements of $\mathcal{E}$ are of finite presentation over R. Suppose that $\varphi, \psi : A \to A'$ are two morphisms of $\mathcal{I}$. Let $x_1, \ldots, x_n$ be generators of A as an R-algebra. By assumption (2) we can factor the R-algebra map $A'/(\varphi(x_i) - \psi(x_i)) \to \Lambda$ through an object of $\mathcal{I}$. This proves that $\mathcal{I}$ is filtered. We omit the proof that $\Lambda = \operatorname{colim}_{\mathcal{I}} A$. □

16.3. Singular ideals

Let $R \to A$ be a ring map. The singular ideal of A over R is the radical ideal in A cutting out the singular locus of the morphism $\mathrm{Spec}(A) \to \mathrm{Spec}(R)$. Here is a formal definition.

Definition 16.3.1. Let $R \to A$ be a ring map. The *singular ideal of A over R*, denoted $H_{A/R}$ is the unique radical ideal $H_{A/R} \subset A$ with

$$V(H_{A/R}) = \{\mathfrak{q} \in \mathrm{Spec}(A) \mid R \to A \text{ not smooth at } \mathfrak{q}\}$$

This makes sense because the set of primes where $R \to A$ is smooth is open, see Algebra, Definition 10.130.11. In order to find an explicit set of generators for the singular ideal we first prove the following lemma.

Lemma 16.3.2. *Let R be a ring. Let $A = R[x_1, \ldots, x_n]/(f_1, \ldots, f_m)$. Let $\mathfrak{q} \subset A$. Assume $R \to A$ is smooth at $\mathfrak{q}$. Then there exists an $a \in A$, $a \notin \mathfrak{q}$, an integer c, $0 \leq c \leq \min(n, m)$, subsets $U \subset \{1, \ldots, n\}$, $V \subset \{1, \ldots, m\}$ of cardinality c such that*

$$a = a' \det(\partial f_j/\partial x_i)_{j \in V, i \in U}$$

for some $a' \in A$ and

$$af_\ell \in (f_j, j \in V) + (f_1, \ldots, f_m)^2$$

for all $\ell \in \{1, \ldots, m\}$.

Proof. Set $I = (f_1, \ldots, f_m)$ so that the naive cotangent complex of A over R is homotopy equivalent to $I/I^2 \to \bigoplus A\mathrm{d}x_i$, see Algebra, Lemma 10.127.2. We will use the formation of the naive cotangent complex commutes with localization, see Algebra, Section 10.127, especially Algebra, Lemma 10.127.13. By Algebra, Definitions 10.130.1 and 10.130.11 we see that $(I/I^2)_a \to \bigoplus A_a\mathrm{d}x_i$ is a split injection for some $a \in A$, $a \notin \mathfrak{p}$. After renumbering $x_1, \ldots, x_n$ and $f_1, \ldots, f_m$ we may assume that $f_1, \ldots, f_c$ form a basis for the vector space $I/I^2 \otimes_A \kappa(\mathfrak{q})$ and that $\mathrm{d}x_{c+1}, \ldots, \mathrm{d}x_n$ map to a basis of $\Omega_{A/R} \otimes_A \kappa(\mathfrak{q})$. Hence after replacing a by aa' for some $a' \in A$, $a' \notin \mathfrak{q}$ we may assume $f_1, \ldots, f_c$ form a basis for $(I/I^2)_a$ and that $\mathrm{d}x_{c+1}, \ldots, \mathrm{d}x_n$ map to a basis of $(\Omega_{A/R})_a$. In this situation a^N for some large integer N satisfies the conditions of the lemma (with $U = V = \{1, \ldots, c\}$). □

We will use the notion of a *strictly standard* element in a A over R. Our notion is slightly weaker than the one in Swan's paper [**Swa98**]. We also define an *elementary standard* element to be one of the type we found in the lemma above. We compare the different types of elements in Lemma 16.4.7.

Definition 16.3.3. Let $R \to A$ be a ring map of finite presentation. We say an element $a \in A$ is *elementary standard in A over R* if there exists a presentation $A = R[x_1, \ldots, x_n]/(f_1, \ldots, f_m)$ and $0 \leq c \leq \min(n, m)$ such that

$$a = a' \det(\partial f_j/\partial x_i)_{i,j=1,\ldots,c} \tag{16.3.3.1}$$

for some $a' \in A$ and

$$af_{c+j} \in (f_1, \ldots, f_c) + (f_1, \ldots, f_m)^2 \tag{16.3.3.2}$$

for $j = 1, \ldots, m - c$. We say $a \in A$ is *strictly standard in A over R* if there exists a presentation $A = R[x_1, \ldots, x_n]/(f_1, \ldots, f_m)$ and $0 \leq c \leq \min(n, m)$ such that

$$a = \sum\nolimits_{I \subset \{1,\ldots,n\},\ |I|=c} a_I \det(\partial f_j/\partial x_i)_{j=1,\ldots,c,\ i \in I} \tag{16.3.3.3}$$

for some $a_I \in A$ and

$$af_{c+j} \in (f_1, \ldots, f_c) + (f_1, \ldots, f_m)^2 \tag{16.3.3.4}$$

for $j = 1, \ldots, m - c$.

The following lemma is useful to find implications of (16.3.3.3).

Lemma 16.3.4. *Let R be a ring. Let $A = R[x_1, \ldots, x_n]/(f_1, \ldots, f_m)$ and write $I = (f_1, \ldots, f_n)$. Let $a \in A$. Then (16.3.3.3) implies there exists an A-linear map $\psi : \bigoplus_{i=1,\ldots,n} A dx_i \to A^{\oplus c}$ such that the composition*

$$A^{\oplus c} \xrightarrow{(f_1,\ldots,f_c)} I/I^2 \xrightarrow{f \mapsto df} \bigoplus\nolimits_{i=1,\ldots,n} A dx_i \xrightarrow{\psi} A^{\oplus c}$$

is multiplication by a. Conversely, if such a ψ exists, then a^c satisfies (16.3.3.3).

Proof. This is a special case of Algebra, Lemma 10.14.5. □

Lemma 16.3.5 (Elkik)**.** *Let $R \to A$ be a ring map of finite presentation. The singular ideal $H_{A/R}$ is the radical of the ideal generated by strictly standard elements in A over R and also the radical of the ideal generated by elementary standard elements in A over R.*

Proof. Assume a is strictly standard in A over R. We claim that A_a is smooth over R, which proves that $a \in H_{A/R}$. Namely, let $A = R[x_1, \ldots, x_n]/(f_1, \ldots, f_m)$, c, and $a' \in A$ be as in Definition 16.3.3. Write $I = (f_1, \ldots, f_m)$ so that the naive cotangent complex of A over R is given by $I/I^2 \to \bigoplus A \mathrm{d}x_i$. Assumption (16.3.3.4) implies that $(I/I^2)_a$ is generated by the classes of $f_1, \ldots, f_c$. Assumption (16.3.3.3) implies that the differential $(I/I^2)_a \to \bigoplus A_a \mathrm{d}x_i$ has a left inverse, see Lemma 16.3.4. Hence $R \to A_a$ is smooth by definition and Algebra, Lemma 10.127.13.

Let $H_e, H_s \subset A$ be the radical of the ideal generated by elementary, resp. strictly standard elements of A over R. By definition and what we just proved we have $H_e \subset H_s \subset H_{A/R}$. The inclusion $H_{A/R} \subset H_e$ follows from Lemma 16.3.2. □

Example 16.3.6. The set of points where a finitely presented ring map is smooth needn't be a quasi-compact open. For example, let $R = k[x, y_1, y_2, y_3, \ldots]/(xy_i)$ and $A = R/(x)$. Then the smooth locus of $R \to A$ is $\bigcup D(y_i)$ which is not quasi-compact.

Lemma 16.3.7. *Let $R \to A$ be a ring map of finite presentation. Let $R \to R'$ be a ring map. If $a \in A$ is elementary, resp. strictly standard in A over R, then $a \otimes 1$ is elementary, resp. strictly standard in $A \otimes_R R'$ over R'.*

Proof. If $A = R[x_1, \ldots, x_n]/(f_1, \ldots, f_m)$ is a presentation of A over R, then $A \otimes_R R' = R'[x_1, \ldots, x_n]/(f'_1, \ldots, f'_m)$ is a presentation of $A \otimes_R R'$ over R'. Here f'_j is the image of f_j in $R'[x_1, \ldots, x_n]$. Hence the result follows from the definitions. □

Lemma 16.3.8. *Let $R \to A \to \Lambda$ be ring maps with A of finite presentation over R. Assume that $H_{A/R}\Lambda = \Lambda$. Then there exists a factorization $A \to B \to \Lambda$ with B smooth over R.*

Proof. Choose $f_1, \ldots, f_r \in H_{A/R}$ and $\lambda_1, \ldots, \lambda_r \in \Lambda$ such that $\sum f_i \lambda_i = 1$ in Λ. Set $B = A[x_1, \ldots, x_r]/(f_1 x_1 + \ldots + f_r x_r - 1)$ and define $B \to \Lambda$ by mapping x_i to λ_i. Details omitted. □

16.4. Presentations of algebras

Some of the results in this section are due to Elkik. Note that the algebra C in the following lemma is a symmetric algebra over A. Moreover, if R is Noetherian, then C is of finite presentation over R.

Lemma 16.4.1. *Let R be a ring and let A be a finitely presented R-algebra. There exists finite type R-algebra map $A \to C$ which has a retraction with the following two properties*

1. *for each $a \in A$ such that $R \to A_a$ is a local complete intersection (More on Algebra, Definition 15.22.2) the ring C_a is smooth over A_a and has a presentation $C_a = R[y_1, \ldots, y_m]/J$ such that J/J^2 is free over C_a, and*
2. *for each $a \in A$ such that A_a is smooth over R the module $\Omega_{C_a/R}$ is free over C_a.*

Proof. Choose a presentation $A = R[x_1, \ldots, x_n]/I$ and write $I = (f_1, \ldots, f_m)$. Define the A-module K by the short exact sequence

$$0 \to K \to A^{\oplus m} \to I/I^2 \to 0$$

where the jth basis vector e_j in the middle is mapped to the class of f_j on the right. Set

$$C = \mathrm{Sym}^*_A(I/I^2).$$

The retraction is just the projection onto the degree 0 part of C. We have a surjection $R[x_1, \ldots, x_n, y_1, \ldots, y_m] \to C$ which maps y_j to the class of f_j in I/I^2. The kernel J of this map is generated by the elements $f_1, \ldots, f_m$ and by elements $\sum h_j y_j$ with $h_j \in R[x_1, \ldots, x_n]$ such that $\sum h_j e_j$ defines an element of K. By Algebra, Lemma 10.127.4 applied to $R \to A \to C$ and the presentations above and More on Algebra, Lemma 15.6.10 there is a short exact sequence

$$I/I^2 \otimes_A C \to J/J^2 \to K \otimes_A C \to 0 \tag{16.4.1.1}$$

of C-modules. Let $h \in R[x_1, \ldots, x_n]$ be an element with image $a \in A$. We will use as presentations for the localized rings

$$A_a = R[x_0, x_1, \ldots, x_n]/I' \quad \text{and} \quad C_a = R[x_0, x_1, \ldots, x_n, y_1, \ldots, y_m]/J'$$

where $I' = (hx_0 - 1, I)$ and $J' = (hx_0 - 1, J)$. Hence $I'/(I')^2 = C_a \oplus I/I^2 \otimes_A C_a$ and $J'/(J')^2 = C_a \oplus (J/J^2)_a$ as C_a-modules. Thus we obtain

$$C_a \oplus I/I^2 \otimes_A C_a \to C_a \oplus (J/J^2)_a \to K \otimes_A C_a \to 0 \tag{16.4.1.2}$$

as the sequence of Algebra, Lemma 10.127.4 corresponding to $R \to A_a \to C_a$ and the presentations above.

Next, assume that $a \in A$ is such that A_a is a local complete intersection over R. Then $(I/I^2)_a$ is finite projective over A_a, see More on Algebra, Lemma 15.21.3. Hence we see $K_a \oplus (I/I^2)_a \cong A_a^{\oplus m}$ is free. In particular K_a is finite projective too. By More on Algebra, Lemma 15.22.6 the sequence (16.4.1.2) is exact on the left. Hence

$$J'/(J')^2 \cong C_a \oplus I/I^2 \otimes_A C_a \oplus K \otimes_A C_a \cong C_a^{\oplus m+1}$$

This proves (1). Finally, suppose that in addition A_a is smooth over R. Then the same presentation shows that $\Omega_{C_a/R}$ is the cokernel of the map

$$J'/(J')^2 \longrightarrow \bigoplus\nolimits_i C_a \mathrm{d}x_i \oplus \bigoplus\nolimits_j C_a \mathrm{d}y_j$$

The summand C_a of $J'/(J')^2$ in the decomposition above corresponds to $hx_0 - 1$ and hence maps isomorphically to the summand $C_a \mathrm{d}x_0$. The summand $I/I^2 \otimes_A C_a$ of $J'/(J')^2$ maps injectively to $\bigoplus_{i=1,\ldots,n} C_a \mathrm{d}x_i$ with quotient $\Omega_{A_a/R} \otimes_{A_a} C_a$. The summand $K \otimes_A C_a$ maps injectively to $\bigoplus_{j \geq 1} C_a \mathrm{d}y_j$ with quotient isomorphic to $I/I^2 \otimes_A C_a$. Thus the cokernel of the last displayed map is the module $I/I^2 \otimes_A C_a \oplus \Omega_{A_a/R} \otimes_{A_a} C_a$. Since $(I/I^2)_a \oplus \Omega_{A_a/R}$ is free (from the definition of smooth ring maps) we see that (2) holds. □

The following proposition was proved for smooth ring maps over henselian pairs by Elkik in [**Elk73**]. For smooth ring maps it can be found in [**Ara01**], where it is also proven that ring maps between smooth algebras can be lifted.

Proposition 16.4.2. *Let $R \to R_0$ be a surjective ring map with kernel I.*

(1) *If $R_0 \to A_0$ is a syntomic ring map, then there exists a syntomic ring map $R \to A$ such that $A/IA \cong A_0$.*
(2) *If $R_0 \to A_0$ is a smooth ring map, then there exists a smooth ring map $R \to A$ such that $A/IA \cong A_0$.*

Proof. Assume $R_0 \to A_0$ syntomic, in particular a local complete intersection (More on Algebra, Lemma 15.22.5). Choose a presentation $A_0 = R_0[x_1, \ldots, x_n]/J_0$. Set $C_0 = \mathrm{Sym}^*_{A_0}(J_0/J_0^2)$. Note that J_0/J_0^2 is a finite projective A_0-module (Algebra, Lemma 10.129.16). By Lemma 16.4.1 the ring map $A_0 \to C_0$ is smooth and we can find a presentation $C_0 = R_0[y_1, \ldots, y_m]/K_0$ with K_0/K_0^2 free over C_0. By Algebra, Lemma 10.129.6 we can assume $C_0 = R_0[y_1, \ldots, y_m]/(\overline{f}_1, \ldots, \overline{f}_c)$ where $\overline{f}_1, \ldots, \overline{f}_c$ maps to a basis of K_0/K_0^2 over C_0. Choose $f_1, \ldots, f_c \in R[y_1, \ldots, y_c]$ lifting $\overline{f}_1, \ldots, \overline{f}_c$ and set

$$C = R[y_1, \ldots, y_m]/(f_1, \ldots, f_c)$$

By construction $C_0 = C/IC$. By Algebra, Lemma 10.129.11 we can after replacing C by C_g assume that C is a relative global complete intersection over R. We conclude that there exists a finite projective A_0-module P_0 such that $C_0 = \mathrm{Sym}^*_{A_0}(P_0)$ is isomorphic to C/IC for some syntomic R-algebra C.

Choose an integer n and a direct sum decomposition $A_0^{\oplus n} = P_0 \oplus Q_0$. By More on Algebra, Lemma 15.6.9 we can find an étale ring map $C \to C'$ which induces an isomorphism $C/IC \to C'/IC'$ and a finite projective C'-module Q such that Q/IQ is isomorphic to $Q_0 \otimes_{A_0} C/IC$. Then $D = \mathrm{Sym}^*_{C'}(Q)$ is a smooth C'-algebra (see More on Algebra, Lemma 15.6.11). Picture

$$\begin{array}{ccccccc} R & \longrightarrow & C & \longrightarrow & C' & \longrightarrow & D \\ \downarrow & & \downarrow & & \downarrow & & \downarrow \\ R/I & \longrightarrow & A_0 & \longrightarrow & C/IC & \xrightarrow{\cong} & C'/IC' \longrightarrow D/ID \end{array}$$

Observe that our choice of Q gives

$$\begin{aligned} D/ID &= \mathrm{Sym}^*_{C/IC}(Q_0 \otimes_{A_0} C/IC) \\ &= \mathrm{Sym}^*_{A_0}(Q_0) \otimes_{A_0} C/IC \\ &= \mathrm{Sym}^*_{A_0}(Q_0) \otimes_{A_0} \mathrm{Sym}^*_{A_0}(P_0) \\ &= \mathrm{Sym}^*_{A_0}(Q_0 \oplus P_0) \\ &= \mathrm{Sym}^*_{A_0}(A_0^{\oplus n}) \\ &= A_0[x_1, \ldots, x_n] \end{aligned}$$

Choose $f_1, \ldots, f_n \in D$ which map to $x_1, \ldots, x_n$ in $D/ID = A_0[x_1, \ldots, x_n]$. Set $A = D/(f_1, \ldots, f_n)$. Note that $A_0 = A/IA$. We claim that $R \to A$ is syntomic in a neighbourhood of $V(IA)$. If the claim is true, then we can find a $f \in A$ mapping to $1 \in A_0$ such that A_f is syntomic over R and the proof of (1) is finished.

Proof of the claim. Observe that $R \to D$ is syntomic as a composition of the syntomic ring map $R \to C$, the étale ring map $C \to C'$ and the smooth ring map $C' \to D$ (Algebra, Lemmas 10.129.17 and 10.130.10). The question is local on $\mathrm{Spec}(D)$, hence we may assume that D is a relative global complete intersection (Algebra, Lemma 10.129.15). Say $D = R[y_1, \ldots, y_m]/(g_1, \ldots, g_s)$. Let $f'_1, \ldots, f'_n \in R[y_1, \ldots, y_m]$ be lifts of $f_1, \ldots, f_n$. Then we can apply Algebra, Lemma 10.129.11 to get the claim.

Proof of (2). Since a smooth ring map is syntomic, we can find a syntomic ring map $R \to A$ such that $A_0 = A/IA$. By assumption the fibres of $R \to A$ are smooth over primes in $V(I)$ hence $R \to A$ is smooth in an open neighbourhood of $V(IA)$ (Algebra, Lemma 10.130.16). Thus we can replace A by a localization to obtain the result we want. □

We know that any syntomic ring map $R \to A$ is locally a relative global complete intersection, see Algebra, Lemma 10.129.15. The next lemma says that a vector bundle over $\mathrm{Spec}(A)$ is a relative global complete intersection.

Lemma 16.4.3. *Let $R \to A$ be a syntomic ring map. Then there exists a smooth R-algebra map $A \to C$ with a retraction such that C is a global relative complete intersection over R, i.e.,*

$$C \cong R[x_1, \ldots, x_n]/(f_1, \ldots, f_c)$$

flat over R and all fibres of dimension $n - c$.

Proof. Apply Lemma 16.4.1 to get $A \to C$. By Algebra, Lemma 10.129.6 we can write $C = R[x_1, \ldots, x_n]/(f_1, \ldots, f_c)$ with f_i mapping to a basis of J/J^2. The ring map $R \to C$ is syntomic (hence flat) as it is a composition of a syntomic and a smooth ring map. The dimension of the fibres is $n - c$ by Algebra, Lemma 10.128.4 (the fibres are local complete intersections, so the lemma applies). □

Lemma 16.4.4. *Let $R \to A$ be a smooth ring map. Then there exists a smooth R-algebra map $A \to B$ with a retraction such that B is standard smooth over R, i.e.,*

$$B \cong R[x_1, \ldots, x_n]/(f_1, \ldots, f_c)$$

and $\det(\partial f_j/\partial x_i)_{i,j=1,\ldots,c}$ is invertible in B.

Proof. Apply Lemma 16.4.3 to get a smooth R-algebra map $A \to C$ with a retraction such that $C = R[x_1, \ldots, x_n]/(f_1, \ldots, f_c)$ is a relative global complete intersection over R. As C is smooth over R we have a short exact sequence

$$0 \to \bigoplus\nolimits_{j=1,\ldots,c} Cf_j \to \bigoplus\nolimits_{i=1,\ldots,n} C\mathrm{d}x_i \to \Omega_{C/R} \to 0$$

Since $\Omega_{C/R}$ is a projective C-module this sequence is split. Choose a left inverse t to the first map. Say $t(\mathrm{d}x_i) = \sum c_{ij} f_j$ so that $\sum_i \frac{\partial f_j}{\partial x_i} c_{i\ell} = \delta_{j\ell}$ (Kronecker delta). Let

$$B' = C[y_1, \ldots, y_c] = R[x_1, \ldots, x_n, y_1, \ldots, y_c]/(f_1, \ldots, f_c)$$

The R-algebra map $C \to B'$ has a retraction given by mapping y_j to zero. We claim that the map

$$R[z_1, \ldots, z_n] \longrightarrow B', \quad z_i \longmapsto x_i - \sum\nolimits_j c_{ij} y_j$$

is étale at every point in the image of $\mathrm{Spec}(C) \to \mathrm{Spec}(B')$. In $\Omega_{B'/R[z_1,\ldots,z_n]}$ we have

$$0 = \mathrm{d}f_j - \sum\nolimits_i \frac{\partial f_j}{\partial x_i} \mathrm{d}z_i \equiv \sum\nolimits_{i,\ell} \frac{\partial f_j}{\partial x_i} c_{i\ell} \mathrm{d}y_\ell \equiv \mathrm{d}y_j \bmod (y_1, \ldots, y_c)\Omega_{B'/R[z_1,\ldots,z_n]}$$

Since $0 = \mathrm{d}z_i = \mathrm{d}x_i$ modulo $\sum B' \mathrm{d}y_j + (y_1, \ldots, y_c)\Omega_{B'/R[z_1,\ldots,z_n]}$ we conclude that

$$\Omega_{B'/R[z_1,\ldots,z_n]}/(y_1, \ldots, y_c)\Omega_{B'/R[z_1,\ldots,z_n]} = 0.$$

As $\Omega_{B'/R[z_1,\ldots,z_n]}$ is a finite B'-module by Nakayama's lemma there exists a $g \in 1 + (y_1, \ldots, y_c)$ that $(\Omega_{B'/R[z_1,\ldots,z_n]})_g = 0$. This proves that $R[z_1, \ldots, z_n] \to B'_g$ is unramified, see Algebra, Definition 10.142.1. For any ring map $R \to k$ where k is a field we obtain an unramified ring map $k[z_1, \ldots, z_n] \to (B'_g) \otimes_R k$ between smooth k-algebras of dimension n. It follows that $k[z_1, \ldots, z_n] \to (B'_g) \otimes_R k$ is flat by Algebra, Lemmas 10.122.1 and 10.133.2. By the critère de platitude par fibre (Algebra, Lemma 10.122.8) we conclude that $R[z_1, \ldots, z_n] \to B'_g$ is flat. Finally, Algebra, Lemma 10.136.7 implies that $R[z_1, \ldots, z_n] \to B'_g$ is étale. Set $B = B'_g$. Note that $C \to B$ is smooth and has a retraction, so also $A \to B$ is smooth and has a retraction. Moreover, $R[z_1, \ldots, z_n] \to B$ is étale. By Algebra, Lemma 10.136.2 we can write

$$B = R[z_1, \ldots, z_n, w_1, \ldots, w_c]/(g_1, \ldots, g_c)$$

with $\det(\partial g_j/\partial w_i)$ invertible in B. This proves the lemma. □

Lemma 16.4.5. *Let $R \to \Lambda$ be a ring map. If Λ is a filtered colimit of smooth R-algebras, then Λ is a filtered colimit of standard smooth R-algebras.*

Proof. Let $A \to \Lambda$ be an R-algebra map with A of finite presentation over R. According to Lemma 16.2.1 we have to factor this map through a standard smooth algebra, and we know we can factor it as $A \to B \to \Lambda$ with B smooth over R. Choose an R-algebra map $B \to C$ with a retraction $C \to B$ such that C is standard smooth over R, see Lemma 16.4.4. Then the desired factorization is $A \to B \to C \to B \to \Lambda$. □

Lemma 16.4.6. *Let $R \to A$ be a standard smooth ring map. Let $E \subset A$ be a finite subset of order $|E| = n$. Then there exists a presentation $A = R[x_1, \ldots, x_{n+m}]/(f_1, \ldots, f_c)$ with $c \geq n$, with $\det(\partial f_j/\partial x_i)_{i,j=1,\ldots,c}$ invertible in A, and such that E is the set of congruence classes of $x_1, \ldots, x_n$.*

Proof. Choose a presentation $A = R[y_1, \ldots, y_m]/(g_1, \ldots, g_d)$ such that the image of $\det(\partial g_j/\partial y_i)_{i,j=1,\ldots,d}$ is invertible in A. Choose an enumerations $E = \{a_1, \ldots, a_n\}$ and choose $h_i \in R[y_1, \ldots, y_m]$ whose image in A is a_i. Consider the presentation

$$A = R[x_1, \ldots, x_n, y_1, \ldots, y_m]/(x_1 - h_1, \ldots, x_n - h_n, g_1, \ldots, g_d)$$

and set $c = n + d$. □

Lemma 16.4.7. *Let $R \to A$ be a ring map of finite presentation. Let $a \in A$. Consider the following conditions on a:*

(1) *A_a is smooth over R,*
(2) *A_a is smooth over R and $\Omega_{A_a/R}$ is stably free,*
(3) *A_a is smooth over R and $\Omega_{A_a/R}$ is free,*
(4) *A_a is standard smooth over R,*
(5) *a is strictly standard in A over R,*
(6) *a is elementary standard in A over R.*

Then we have

(a) *(4) $\Rightarrow$ (3) $\Rightarrow$ (2) $\Rightarrow$ (1),*
(b) *(6) $\Rightarrow$ (5),*
(c) *(6) $\Rightarrow$ (4),*
(d) *(5) $\Rightarrow$ (2),*
(e) *(2) $\Rightarrow$ the elements a^e, $e \geq e_0$ are strictly standard in A over R,*
(f) *(4) $\Rightarrow$ the elements a^e, $e \geq e_0$ are elementary standard in A over R.*

Proof. Part (a) is clear from the definitions and Algebra, Lemma 10.130.7. Part (b) is clear from Definition 16.3.3.

Proof of (c). Choose a presentation $A = R[x_1, \ldots, x_n]/(f_1, \ldots, f_m)$ such that (16.3.3.1) and (16.3.3.2) hold. Choose $h \in R[x_1, \ldots, x_n]$ mapping to a. Then

$$A_a = R[x_0, x_1, \ldots, x_n]/(x_0 h - 1, f_1, \ldots, f_n).$$

Write $J = (x_0 h - 1, f_1, \ldots, f_n)$. By (16.3.3.2) we see that the A_a-module J/J^2 is generated by $x_0 h - 1, f_1, \ldots, f_c$ over A_a. Hence, as in the proof of Algebra, Lemma 10.129.6, we can choose a $g \in 1 + J$ such that

$$A_a = R[x_0, \ldots, x_n, x_{n+1}]/(x_0 h - 1, f_1, \ldots, f_n, g x_{n+1} - 1).$$

At this point (16.3.3.1) implies that $R \to A_a$ is standard smooth (use the coordinates $x_0, x_1, \ldots, x_c, x_{n+1}$ to take derivatives).

Proof of (d). Choose a presentation $A = R[x_1, \ldots, x_n]/(f_1, \ldots, f_m)$ such that (16.3.3.3) and (16.3.3.4) hold. We already know that A_a is smooth over R, see Lemma 16.3.5. As above we get a presentation $A_a = R[x_0, x_1, \ldots, x_n]/J$ with J/J^2 free. Then $\Omega_{A_a/R} \oplus J/J^2 \cong A_a^{\oplus n+1}$ by the definition of smooth ring maps, hence we see that $\Omega_{A_a/R}$ is stably free.

Proof of (e). Choose a presentation $A = R[x_1, \ldots, x_n]/I$ with I finitely generated. By assumption we have a short exact sequence

$$0 \to (I/I^2)_a \to \bigoplus\nolimits_{i=1,\ldots,n} A_a \mathrm{d}x_i \to \Omega_{A_a/R} \to 0$$

which is split exact. Hence we see that $(I/I^2)_a \oplus \Omega_{A_a/R}$ is a free A_a-module. Since $\Omega_{A_a/R}$ is stably free we see that $(I/I^2)_a$ is stably free as well. Thus replacing the presentation chosen above by $A = R[x_1, \ldots, x_n, x_{n+1}, \ldots, x_{n+r}]/J$ with

$J = (I, x_{n+1}, \ldots, x_{n+r})$ for some r we get that $(J/J^2)_a$ is (finite) free. Choose $f_1, \ldots, f_c \in J$ which map to a basis of $(J/J^2)_a$. Extend this to a list of generators $f_1, \ldots, f_m \in J$. Consider the presentation $A = R[x_1, \ldots, x_{n+r}]/(f_1, \ldots, f_m)$. Then (16.3.3.4) holds for a^e for all sufficiently large e by construction. Moreover, since $(J/J^2)_a \to \bigoplus_{i=1,\ldots,n} A_a \mathrm{d}x_i$ is a split injection we can find an A_a-linear left inverse. Writing this left inverse in terms of the basis $f_1, \ldots, f_c$ and clearing denominators we find a linear map $\psi_0 : A^{\oplus n} \to A^{\oplus c}$ such that

$$A^{\oplus c} \xrightarrow{(f_1,\ldots,f_c)} J/J^2 \xrightarrow{f \mapsto \mathrm{d}f} \bigoplus\nolimits_{i=1,\ldots,n} A\mathrm{d}x_i \xrightarrow{\psi_0} A^{\oplus c}$$

is multiplication by a^{e_0} for some $e_0 \geq 1$. By Lemma 16.3.4 we see (16.3.3.3) holds for all a^{ce_0} and hence for a^e for all e with $e \geq ce_0$.

Proof of (f). Choose a presentation $A_a = R[x_1, \ldots, x_n]/(f_1, \ldots, f_c)$ such that $\det(\partial f_j/\partial x_i)_{i,j=1,\ldots,c}$ is invertible in A_a. We may assume that for some $m < n$ the classes of the elements $x_1, \ldots, x_m$ correspond $a_i/1$ where $a_1, \ldots, a_m \in A$ are generators of A over R, see Lemma 16.4.6. After replacing x_i by $a^N x_i$ for $m < i \leq n$ we may assume the class of x_i is $a_i/1 \in A_a$ for some $a_i \in A$. Consider the ring map

$$\Psi : R[x_1, \ldots, x_n] \longrightarrow A, \quad x_i \longmapsto a_i.$$

This is a surjective ring map. By replacing f_j by $a^N f_j$ we may assume that $f_j \in R[x_1, \ldots, x_n]$ and that $\Psi(f_j) = 0$ (since after all $f_j(a_1/1, \ldots, a_n/1) = 0$ in A_a). Let $J = \mathrm{Ker}(\Psi)$. Then $A = R[x_1, \ldots, x_n]/J$ is a presentation and $f_1, \ldots, f_c \in J$ are elements such that $(J/J^2)_a$ is freely generated by $f_1, \ldots, f_c$ and such that $\det(\partial f_j/\partial x_i)_{i,j=1,\ldots,c}$ maps to an invertible element of A_a. It follows that (16.3.3.1) and (16.3.3.2) hold for a^e and all large enough e as desired. □

16.5. The lifting problem

The goal in this section is to prove (Proposition 16.5.3) that the collection of algebras which are filtered colimits of smooth algebras is closed under infinitesimal flat deformations. The proof is elementary and only uses the results on presentations of smooth algebras from Section 16.4.

Lemma 16.5.1. *Let $R \to \Lambda$ be a ring map. Let $I \subset R$ be an ideal. Assume that*

1. *$I^2 = 0$, and*
2. *$\Lambda/I\Lambda$ is a filtered colimit of smooth R/I-algebras.*

Let $\varphi : A \to \Lambda$ be an R-algebra map with A of finite presentation over R. Then there exists a factorization

$$A \to B/J \to \Lambda$$

where B is a smooth R-algebra and $J \subset IB$ is a finitely generated ideal.

Proof. Choose a factorization

$$A/IA \to \bar{B} \to \Lambda/I\Lambda$$

with $\bar{B}$ standard smooth over R/I; this is possible by assumption and Lemma 16.4.5. Write

$$\bar{B} = A/IA[t_1, \ldots, t_r]/(\bar{g}_1, \ldots, \bar{g}_s)$$

and say $\bar{B} \to \Lambda/I\Lambda$ maps t_i to the class of λ_i modulo $I\Lambda$. Choose $g_1, \ldots, g_s \in A[t_1, \ldots, t_r]$ lifting $\bar{g}_1, \ldots, \bar{g}_s$. Write $\varphi(g_i)(\lambda_1, \ldots, \lambda_r) = \sum \epsilon_{ij}\mu_{ij}$ for some $\epsilon_{ij} \in I$ and $\mu_{ij} \in \Lambda$. Define

$$A' = A[t_1, \ldots, t_r, \delta_{i,j}]/(g_i - \sum \epsilon_{ij}\delta_{ij})$$

and consider the map

$$A' \longrightarrow \Lambda, \quad a \longmapsto \varphi(a), \quad t_i \longmapsto \lambda_i, \quad \delta_{ij} \longmapsto \mu_{ij}$$

We have

$$A'/IA' = A/IA[t_1, \ldots, t_r]/(\bar{g}_1, \ldots, \bar{g}_s)[\delta_{ij}] \cong \bar{B}[\delta_{ij}]$$

This is a standard smooth algebra over R/I as $\bar{B}$ is standard smooth. Choose a presentation $A'/IA' = R/I[x_1, \ldots, x_n]/(\bar{f}_1, \ldots, \bar{f}_c)$ with $\det(\partial \bar{f}_j/\partial x_i)_{i,j=1,\ldots,c}$ invertible in A'/IA'. Choose lifts $f_1, \ldots, f_c \in R[x_1, \ldots, x_n]$ of $\bar{f}_1, \ldots, \bar{f}_c$. Then

$$B = R[x_1, \ldots, x_n, x_{n+1}]/(f_1, \ldots, f_c, x_{n+1} \det(\partial f_j/\partial x_i)_{i,j=1,\ldots,c} - 1)$$

is smooth over R. Since smooth ring maps are formally smooth (Algebra, Proposition 10.131.13) there exists an R-algebra map $B \to A'$ which is an isomorphism modulo I. Then $B \to A'$ is surjective by Nakayama's lemma (Algebra, Lemma 10.18.1). Thus $A' = B/J$ with $J \subset IB$ finitely generated (see Algebra, Lemma 10.6.3). □

Lemma 16.5.2. *Let $R \to \Lambda$ be a ring map. Let $I \subset R$ be an ideal. Assume that*

(1) $I^2 = 0$,
(2) $\Lambda/I\Lambda$ *is a filtered colimit of smooth R/I-algebras, and*
(3) $R \to \Lambda$ *is flat.*

Let $\varphi : B \to \Lambda$ be an R-algebra map with B smooth over R. Let $J \subset IB$ be a finitely generated ideal. Then there exists R-algebra maps

$$B \xrightarrow{\alpha} B' \xrightarrow{\beta} \Lambda$$

such that B' is smooth over R, such that $\alpha(J) = 0$ and such that $\beta \circ \alpha = \varphi \bmod I\Lambda$.

Proof. If we can prove the lemma in case $J = (h)$, then we can prove the lemma by induction on the number of generators of J. Namely, suppose that J can be generated by n elements $h_1, \ldots, h_n$ and the lemma holds for all cases where J is generated by $n-1$ elements. Then we apply the case $n = 1$ to produce $B \to B' \to \Lambda$ where the first map kills of h_n. Then we let J' be the ideal of B' generated by the images of $h_1, \ldots, h_{n-1}$ and we apply the case for $n - 1$ to produce $B' \to B'' \to \Lambda$. It is easy to verify that $B \to B'' \to \Lambda$ does the job.

Assume $J = (h)$ and write $h = \sum \epsilon_i b_i$ for some $\epsilon_i \in I$ and $b_i \in B$. Note that $0 = \varphi(h) = \sum \epsilon_i \varphi(b_i)$. As Λ is flat over R, the equational criterion for flatness (Algebra, Lemma 10.36.10) implies that we can find $\lambda_j \in \Lambda$, $j = 1, \ldots, m$ and $a_{ij} \in R$ such that $\varphi(b_i) = \sum_j a_{ij}\lambda_j$ and $\sum_i \epsilon_i a_{ij} = 0$. Set

$$C = B[x_1, \ldots, x_m]/(b_i - \sum a_{ij}x_j)$$

with $C \to \Lambda$ given by φ and $x_j \mapsto \lambda_j$. Choose a factorization

$$C \to B'/J' \to \Lambda$$

as in Lemma 16.5.1. Since B is smooth over R we can lift the map $B \to C \to B'/J'$ to a map $\psi : B \to B'$. We claim that $\psi(h) = 0$. Namely, the fact that ψ agrees with $B \to C \to B'/J'$ mod I implies that

$$\psi(b_i) = \sum a_{ij}\xi_j + \theta_i$$

for some $\xi_i \in B'$ and $\theta_i \in IB'$. Hence we see that

$$\psi(h) = \psi(\sum \epsilon_i b_i) = \sum \epsilon_i a_{ij}\xi_j + \sum \epsilon_i \theta_i = 0$$

because of the relations above and the fact that $I^2 = 0$. □

Proposition 16.5.3. *Let $R \to \Lambda$ be a ring map. Let $I \subset R$ be an ideal. Assume that*

(1) *I is nilpotent,*
(2) *$\Lambda/I\Lambda$ is a filtered colimit of smooth R/I-algebras, and*
(3) *$R \to \Lambda$ is flat.*

Then Λ is a colimit of smooth R-algebras.

Proof. Since $I^n = 0$ for some n, it follows by induction on n that it suffices to consider the case where $I^2 = 0$. Let $\varphi : A \to \Lambda$ be an R-algebra map with A of finite presentation over R. We have to find a factorization $A \to B \to \Lambda$ with B smooth over R, see Lemma 16.2.1. By Lemma 16.5.1 we may assume that $A = B/J$ with B smooth over R and $J \subset IB$ a finitely generated ideal. By Lemma 16.5.2 we can find a (possibly noncommutative) diagram

$$\begin{array}{ccc} B & \xrightarrow{\alpha} & B' \\ & {\scriptstyle\varphi}\searrow \quad \swarrow{\scriptstyle\beta} & \\ & \Lambda & \end{array}$$

of R-algebras which commutes modulo I and such that $\alpha(J) = 0$. The map

$$D : B \longrightarrow I\Lambda, \quad b \longmapsto \varphi(b) - \beta(\alpha(b))$$

is a derivation over R hence we can write it as $D = \xi \circ \mathrm{d}_{B/R}$ for some B-linear map $\xi : \Omega_{B/R} \to I\Lambda$. Since $\Omega_{B/R}$ is a finite projective B-module we can write $\xi = \sum_{i=1,\ldots,n} \epsilon_i \Xi_i$ for some $\epsilon_i \in I$ and B-linear maps $\Xi_i : \Omega_{B/R} \to \Lambda$. (Details omitted. Hint: write $\Omega_{B/R}$ as a direct sum of a finite free module to reduce to the finite free case.) We define

$$B'' = \mathrm{Sym}^*_{B'}\left(\bigoplus\nolimits_{i=1,\ldots,n} \Omega_{B/R} \otimes_{B,\alpha} B'\right)$$

and we define $\beta' : B'' \to \Lambda$ by β on B' and by

$$\beta'|_{i\text{th summand } \Omega_{B/R} \otimes_{B,\alpha} B'} = \Xi_i \otimes \beta$$

and $\alpha' : B \to B''$ by

$$\alpha'(b) = \alpha(b) \oplus \sum \epsilon_i \mathrm{d}_{B/R}(b) \otimes 1 \oplus 0 \oplus \ldots$$

At this point the diagram

$$\begin{array}{ccc} B & \xrightarrow{\alpha'} & B'' \\ & {\scriptstyle\varphi}\searrow \quad \swarrow{\scriptstyle\beta'} & \\ & \Lambda & \end{array}$$

does commute. Moreover, it is direct from the definitions that $\alpha'(J) = 0$ as $I^2 = 0$. Hence the desired factorization. □

16.6. The lifting lemma

Here is a fiendishly clever lemma.

Lemma 16.6.1. *Let R be a Noetherian ring. Let Λ be an R-algebra. Let $\pi \in R$ and assume that $Ann_R(\pi) = Ann_R(\pi^2)$ and $Ann_\Lambda(\pi) = Ann_\Lambda(\pi^2)$. Suppose we have R-algebra maps $R/\pi^2R \to \bar{C} \to \Lambda/\pi^2\Lambda$ with $\bar{C}$ of finite presentation. Then there exists an R-algebra homomorphism $D \to \Lambda$ and a commutative diagram*

$$\begin{array}{ccccc} R/\pi^2R & \longrightarrow & \bar{C} & \longrightarrow & \Lambda/\pi^2\Lambda \\ \downarrow & & \downarrow & & \downarrow \\ R/\pi R & \longrightarrow & D/\pi D & \longrightarrow & \Lambda/\pi\Lambda \end{array}$$

with the following properties

(a) *D is of finite presentation,*
(b) *$R \to D$ is smooth at any prime $\mathfrak{q}$ with $\pi \notin \mathfrak{q}$,*
(c) *$R \to D$ is smooth at any prime $\mathfrak{q}$ with $\pi \in \mathfrak{q}$ lying over a prime of $\bar{C}$ where $R/\pi^2R \to \bar{C}$ is smooth, and*
(d) *$\bar{C}/\pi\bar{C} \to D/\pi D$ is smooth at any prime lying over a prime of $\bar{C}$ where $R/\pi^2R \to \bar{C}$ is smooth.*

Proof. We choose a presentation

$$\bar{C} = R[x_1, \ldots, x_n]/(f_1, \ldots, f_m)$$

We also denote $I = (f_1, \ldots, f_m)$ and $\bar{I}$ the image of I in $R/\pi^2R[x_1, \ldots, x_n]$. Since R is Noetherian, so is $\bar{C}$. Hence the smooth locus of $R/\pi^2R \to \bar{C}$ is quasi-compact, see Topology, Lemma 5.8.2. Applying Lemma 16.3.2 we may choose a finite list of elements $a_1, \ldots, a_r \in R[x_1, \ldots, x_n]$ such that

(1) the union of the open subspaces $\mathrm{Spec}(\bar{C}_{a_k}) \subset \mathrm{Spec}(\bar{C})$ cover the smooth locus of $R/\pi^2R \to \bar{C}$, and
(2) for each $k = 1, \ldots, r$ there exists a finite subset $E_k \subset \{1, \ldots, m\}$ such that $(\bar{I}/\bar{I}^2)_{a_k}$ is freely generated by the classes of f_j, $j \in E_k$.

Set $I_k = (f_j, j \in E_k) \subset I$ and denote $\bar{I}_k$ the image of I_k in $R/\pi^2R[x_1, \ldots, x_n]$. By (2) and Nakayama's lemma we see that $(\bar{I}/\bar{I}_k)_{a_k}$ is annihilated by $1 + b'_k$ for some $b'_k \in \bar{I}_{a_k}$. Suppose b'_k is the image of $b_k/(a_k)^N$ for some $b_k \in I$ and some integer N. After replacing a_k by a_kb_k we get

(3) $(\bar{I}_k)_{a_k} = (\bar{I})_{a_k}$.

Thus, after possibly replacing a_k by a high power, we may write

(4) $a_kf_\ell = \sum_{j \in E_k} h^j_{k,\ell} f_j + \pi^2 g_{k,\ell}$

for any $\ell \in \{1, \ldots, m\}$ and some $h^j_{i,\ell}, g_{i,\ell} \in R[x_1, \ldots, x_n]$. If $\ell \in E_k$ we choose $h^j_{k,\ell} = a_k\delta_{\ell,j}$ (Kronecker delta) and $g_{k,\ell} = 0$. Set

$$D = R[x_1, \ldots, x_n, z_1, \ldots, z_m]/(f_j - \pi z_j, p_{k,\ell}).$$

Here $j \in \{1, \ldots, m\}$, $k \in \{1, \ldots, r\}$, $\ell \in \{1, \ldots, m\}$, and

$$p_{k,\ell} = a_kz_\ell - \sum\nolimits_{j \in E_k} h^j_{k,\ell} z_j - \pi g_{k,\ell}.$$

Note that for $\ell \in E_k$ we have $p_{k,\ell} = 0$ by our choices above.

The map $R \to D$ is the given one. Say $\bar{C} \to \Lambda/\pi^2\Lambda$ maps x_i to the class of λ_i modulo π^2. For an element $f \in R[x_1, \ldots, x_n]$ we denote $f(\lambda) \in \Lambda$ the result of substituting λ_i for x_i. Then we know that $f_j(\lambda) = \pi^2\mu_j$ for some $\mu_j \in \Lambda$. Define $D \to \Lambda$ by the rules $x_i \mapsto \lambda_i$ and $z_j \mapsto \pi\mu_j$. This is well defined because

$$\begin{aligned} p_{k,\ell} &\mapsto a_k(\lambda)\pi\mu_\ell - \sum\nolimits_{j\in E_k} h^j_{k,\ell}(\lambda)\pi\mu_j - \pi g_{k,\ell}(\lambda) \\ &= \pi\left(a_k(\lambda)\mu_\ell - \sum\nolimits_{j\in E_k} h^j_{k,\ell}(\lambda)\mu_j - g_{k,\ell}(\lambda)\right) \end{aligned}$$

Substituting $x_i = \lambda_i$ in (4) above we see that the expression inside the brackets is annihilated by π^2, hence it is annihilated by π as we have assumed $\mathrm{Ann}_\Lambda(\pi) = \mathrm{Ann}_\Lambda(\pi^2)$. The map $\bar{C} \to D/\pi D$ is determined by $x_i \mapsto x_i$ (clearly well defined). Thus we are done if we can prove (b), (c), and (d).

Using (4) we obtain the following key equality

$$\begin{aligned} \pi p_{k,\ell} &= \pi a_k z_\ell - \sum\nolimits_{j\in E_k} \pi h^j_{k,\ell} z_j - \pi^2 g_{k,\ell} \\ &= -a_k(f_\ell - \pi z_\ell) + a_k f_\ell + \sum\nolimits_{j\in E_k} h^j_{k,\ell}(f_j - \pi z_j) - \sum\nolimits_{j\in E_k} h^j_{k,\ell} f_j - \pi^2 g_{k,\ell} \\ &= -a_k(f_\ell - \pi z_\ell) + \sum\nolimits_{j\in E_k} h^j_{k,\ell}(f_j - \pi z_j) \end{aligned}$$

The end result is an element of the ideal generated by $f_j - \pi z_j$. In particular, we see that $D[1/\pi]$ is isomorphic to $R[1/\pi][x_1, \ldots, x_n, z_1, \ldots, z_m]/(f_j - \pi z_j)$ which is isomorphic to $R[1/\pi][x_1, \ldots, x_n]$ hence smooth over R. This proves (b).

For fixed $k \in \{1, \ldots, r\}$ consider the ring

$$D_k = R[x_1, \ldots, x_n, z_1, \ldots, z_m]/(f_j - \pi z_j, j \in E_k, p_{k,\ell})$$

The number of equations is $m = |E_k| + (m - |E_k|)$ as $p_{k,\ell}$ is zero if $\ell \in E_k$. Also, note that

$$\begin{aligned} (D_k/\pi D_k)_{a_k} &= R/\pi R[x_1, \ldots, x_n, 1/a_k, z_1, \ldots, z_m]/(f_j, j \in E_k, p_{k,\ell}) \\ &= (\bar{C}/\pi\bar{C})_{a_k}[z_1, \ldots, z_m]/(a_k z_\ell - \sum\nolimits_{j\in E_k} h^j_{k,\ell} z_j) \\ &\cong (\bar{C}/\pi\bar{C})_{a_k}[z_j, j \in E_k] \end{aligned}$$

In particular $(D_k/\pi D_k)_{a_k}$ is smooth over $(\bar{C}/\pi\bar{C})_{a_k}$. By our choice of a_k we have that $(\bar{C}/\pi\bar{C})_{a_k}$ is smooth over $R/\pi R$ of relative dimension $n - |E_k|$, see (2). Hence for a prime $\mathfrak{q}_k \subset D_k$ containing π and lying over $\mathrm{Spec}(\bar{C}_{a_k})$ the fibre ring of $R \to D_k$ is smooth at $\mathfrak{q}_k$ of dimension n. Thus $R \to D_k$ is syntomic at $\mathfrak{q}_k$ by our count of the number of equations above, see Algebra, Lemma 10.129.11. Hence $R \to D_k$ is smooth at $\mathfrak{q}_k$, see Algebra, Lemma 10.130.16.

To finish the proof, let $\mathfrak{q} \subset D$ be a prime containing π lying over a prime where $R/\pi^2 R \to \bar{C}$ is smooth. Then $a_k \notin \mathfrak{q}$ for some k by (1). We will show that the surjection $D_k \to D$ induces an isomorphism on local rings at $\mathfrak{q}$. Since we know that the ring maps $\bar{C}/\pi\bar{C} \to D_k/\pi D_k$ and $R \to D_k$ are smooth at the corresponding prime $\mathfrak{q}_k$ by the preceding paragraph this will prove (c) and (d) and thus finish the proof.

First, note that for any ℓ the equation $\pi p_{k,\ell} = -a_k(f_\ell - \pi z_\ell) + \sum_{j\in E_k} h^j_{k,\ell}(f_j - \pi z_j)$ proved above shows that $f_\ell - \pi z_\ell$ maps to zero in $(D_k)_{a_k}$ and in particular in $(D_k)_{\mathfrak{q}_k}$.

The relations (4) imply that $a_k f_\ell = \sum_{j\in E_k} h^j_{k,\ell} f_j$ in I/I^2. Since $(\bar{I}_k/\bar{I}_k^2)_{a_k}$ is free on f_j, $j \in E_k$ we see that

$$a_{k'} h^j_{k,\ell} - \sum\nolimits_{j'\in E_{k'}} h^{j'}_{k',\ell} h^j_{k,j'}$$

is zero in $\bar{C}_{a_k}$ for every k, k', ℓ and $j \in E_k$. Hence we can find a large integer N such that

$$a_k^N \left(a_{k'} h^j_{k,\ell} - \sum\nolimits_{j'\in E_{k'}} h^{j'}_{k',\ell} h^j_{k,j'} \right)$$

is in $I_k + \pi^2 R[x_1, \ldots, x_n]$. Computing modulo π we have

$$\begin{aligned} & a_k p_{k',\ell} - a_{k'} p_{k,\ell} + \sum h^{j'}_{k',\ell} p_{k,j'} \\ &= -a_k \sum h^{j'}_{k',\ell} z_{j'} + a_{k'} \sum h^j_{k,\ell} z_j + \sum h^{j'}_{k',\ell} a_k z_{j'} - \sum\sum h^{j'}_{k',\ell} h^j_{k,j'} z_j \\ &= \sum \left(a_{k'} h^j_{k,\ell} - \sum h^{j'}_{k',\ell} h^j_{k,j'} \right) z_j \end{aligned}$$

with Einstein summation convention. Combining with the above we see $a_k^{N+1} p_{k',\ell}$ is contained in the ideal generated by I_k and π in $R[x_1, \ldots, x_n, z_1, \ldots, z_m]$. Thus $p_{k',\ell}$ maps into $\pi(D_k)_{a_k}$. On the other hand, the equation

$$\pi p_{k',\ell} = -a_{k'}(f_\ell - \pi z_\ell) + \sum\nolimits_{j'\in E_{k'}} h^{j'}_{k',\ell}(f_{j'} - \pi z_{j'})$$

shows that $\pi p_{k',\ell}$ is zero in $(D_k)_{a_k}$. Since we have assumed that $\mathrm{Ann}_R(\pi) = \mathrm{Ann}_R(\pi^2)$ and since $(D_k)_{\mathfrak{q}_k}$ is smooth hence flat over R we see that $\mathrm{Ann}_{(D_k)_{\mathfrak{q}_k}}(\pi) = \mathrm{Ann}_{(D_k)_{\mathfrak{q}_k}}(\pi^2)$. We conclude that $p_{k',\ell}$ maps to zero as well, hence $D_{\mathfrak{q}} = (D_k)_{\mathfrak{q}_k}$ and we win. □

16.7. The desingularization lemma

Here is another fiendishly clever lemma.

Lemma 16.7.1. *Let R be a Noetherian ring. Let Λ be an R-algebra. Let $\pi \in R$ and assume that $Ann_\Lambda(\pi) = Ann_\Lambda(\pi^2)$. Let $A \to \Lambda$ be an R-algebra map with A of finite presentation. Assume*

1. *the image of π is strictly standard in A over R, and*
2. *there exists a section $\rho : A/\pi^4 A \to R/\pi^4 R$ which is compatible with the map to $\Lambda/\pi^4\Lambda$.*

Then we can find R-algebra maps $A \to B \to \Lambda$ with B of finite presentation such that $\mathfrak{a}B \subset H_{B/R}$ where $\mathfrak{a} = Ann_R(Ann_R(\pi^2)/Ann_R(\pi))$.

Proof. Choose a presentation

$$A = R[x_1, \ldots, x_n]/(f_1, \ldots, f_m)$$

and $0 \leq c \leq \min(n, m)$ such that (16.3.3.3) holds for π and such that

$$\pi f_{c+j} \in (f_1, \ldots, f_c) + (f_1, \ldots, f_m)^2 \tag{16.7.1.1}$$

for $j = 1, \ldots, m-c$. Say ρ maps x_i to the class of $r_i \in R$. Then we can replace x_i by $x_i - r_i$. Hence we may assume $\rho(x_i) = 0$ in $R/\pi^4 R$. This implies that $f_j(0) \in \pi^4 R$ and that $A \to \Lambda$ maps x_i to $\pi^4 \lambda_i$ for some $\lambda_i \in \Lambda$. Write

$$f_j = f_j(0) + \sum\nolimits_{i=1,\ldots,n} r_{ji} x_i + \text{h.o.t.}$$

This implies that the constant term of $\partial f_j/\partial x_i$ is r_{ji}. Apply ρ to (16.3.3.3) for π and we see that

$$\pi = \sum\nolimits_{I\subset\{1,\ldots,n\},\ |I|=c} r_I \det(r_{ji})_{j=1,\ldots,c,\ i\in I} \bmod \pi^4 R$$

for some $r_I \in R$. Thus we have

$$u\pi = \sum\nolimits_{I\subset\{1,\ldots,n\},\ |I|=c} r_I \det(r_{ji})_{j=1,\ldots,c,\ i\in I}$$

for some $u \in 1+\pi^3 R$. By Algebra, Lemma 10.14.5 this implies there exists a $n \times c$ matrix (s_{ik}) such that

$$u\pi\delta_{jk} = \sum\nolimits_{i=1,\ldots,n} r_{ji}c_{ik} \quad \text{for all } j,k = 1,\ldots,c$$

(Kronecker delta). We introduce auxiliary variables $v_1,\ldots,v_c,w_1,\ldots,w_n$ and we set

$$h_i = x_i - \pi^2 \sum\nolimits_{j=1,\ldots c} s_{ij}v_j - \pi^3 w_i$$

In the following we will use that

$$R[x_1,\ldots,x_n,v_1,\ldots,v_c,w_1,\ldots,w_n]/(h_1,\ldots,h_n) = R[v_1,\ldots,v_c,w_1,\ldots,w_n]$$

without further mention. In $R[x_1,\ldots,x_n,v_1,\ldots,v_c,w_1,\ldots,w_n]/(h_1,\ldots,h_n)$ we have

$$\begin{aligned} f_j &= f_j(x_1-h_1,\ldots,x_n-h_n) \\ &= \sum\nolimits_i \pi^2 r_{ji}s_{ik}v_k + \sum\nolimits_i \pi^3 r_{ji}w_i \bmod \pi^4 \\ &= \pi^3 v_j + \sum \pi^3 r_{ji}w_i \bmod \pi^4 \end{aligned}$$

for $1 \le j \le c$. Hence we can choose elements $g_j \in R[v_1,\ldots,v_c,w_1,\ldots,w_n]$ such that $g_j = v_j + \sum r_{ji}w_i \bmod \pi$ and such that $f_j = \pi^3 g_j$ in the R-algebra $R[x_1,\ldots,x_n,v_1,\ldots,v_c,w_1,\ldots,w_n]/(h_1,\ldots,h_n)$. We set

$$B = R[x_1,\ldots,x_n,v_1,\ldots,v_c,w_1,\ldots,w_n]/(f_1,\ldots,f_n,h_1,\ldots,h_n,g_1,\ldots,g_c).$$

The map $A \to B$ is clear. We define $B \to \Lambda$ by mapping $x_i \to \pi^4\lambda_i$, $v_i \mapsto 0$, and $w_i \mapsto \pi\lambda_i$. Then it is clear that the elements f_j and h_i are mapped to zero in Λ. Moreover, it is clear that g_i is mapped to an element t of $\pi\Lambda$ such that $\pi^3 t = 0$ (as $f_i = \pi^3 g_i$ modulo the ideal generated by the h's). Hence our assumption that $\mathrm{Ann}_\Lambda(\pi) = \mathrm{Ann}_\Lambda(\pi^2)$ implies that $t = 0$. Thus we are done if we can prove the statement about smoothness.

Note that $B_\pi \cong A_\pi[v_1,\ldots,v_c]$ because the equations $g_i = 0$ are implied by $f_i = 0$. Hence B_π is smooth over R as A_π is smooth over R by the assumption that π is strictly standard in A over R, see Lemma 16.3.5.

Set $B' = R[v_1,\ldots,v_c,w_1,\ldots,w_n]/(g_1,\ldots,g_c)$. As $g_i = v_i + \sum r_{ji}w_i \bmod \pi$ we see that $B'/\pi B' = R/\pi R[w_1,\ldots,w_n]$. Hence $R \to B'$ is smooth of relative dimension n at every point of $V(\pi)$ by Algebra, Lemmas 10.129.11 and 10.130.16 (the first lemma shows it is syntomic at those primes, in particular flat, whereupon the second lemma shows it is smooth).

Let $\mathfrak{q} \subset B$ be a prime with $\pi \in \mathfrak{q}$ and for some $r \in \mathfrak{a}$, $r \notin \mathfrak{q}$. Denote $\mathfrak{q}' = B' \cap \mathfrak{q}$. We claim the surjection $B' \to B$ induces an isomorphism of local rings $(B')_{\mathfrak{q}'} \to B_\mathfrak{q}$. This will conclude the proof of the lemma. Note that $B_\mathfrak{q}$ is the quotient of $(B')_{\mathfrak{q}'}$ by the ideal generated by f_{c+j}, $j = 1,\ldots,m-c$. We observe two things: first the

image of f_{c+j} in $(B')_{\mathfrak{q}'}$ is divisible by π^2 and second the image of πf_{c+j} in $(B')_{\mathfrak{q}'}$ can be written as $\sum b_{j_1 j_2} f_{c+j_1} f_{c+j_2}$ by (16.7.1.1). Thus we see that the image of each πf_{c+j} is contained in the ideal generated by the elements $\pi^2 f_{c+j'}$. Hence $\pi f_{c+j} = 0$ in $(B')_{\mathfrak{q}'}$ as this is a Noetherian local ring, see Algebra, Lemma 10.47.6. As $R \to (B')_{\mathfrak{q}'}$ is flat we see that

$$\big(\mathrm{Ann}_R(\pi^2)/\mathrm{Ann}_R(\pi)\big) \otimes_R (B')_{\mathfrak{q}'} = \mathrm{Ann}_{(B')_{\mathfrak{q}'}}(\pi^2)/\mathrm{Ann}_{(B')_{\mathfrak{q}'}}(\pi)$$

Because $r \in \mathfrak{a}$ is invertible in $(B')_{\mathfrak{q}'}$ we see that this module is zero. Hence we see that the image of f_{c+j} is zero in $(B')_{\mathfrak{q}'}$ as desired. □

Lemma 16.7.2. *Let R be a Noetherian ring. Let Λ be an R-algebra. Let $\pi \in R$ and assume that $Ann_R(\pi) = Ann_R(\pi^2)$ and $Ann_\Lambda(\pi) = Ann_\Lambda(\pi^2)$. Let $A \to \Lambda$ and $D \to \Lambda$ be R-algebra maps with A and D of finite presentation. Assume*

1. *π is strictly standard in A over R, and*
2. *there exists an R-algebra map $A/\pi^4 A \to D/\pi^4 D$ compatible with the maps to $\Lambda/\pi^4\Lambda$.*

Then we can find an R-algebra map $B \to \Lambda$ with B of finite presentation and R-algebra maps $A \to B$ and $D \to B$ compatible with the maps to Λ such that $H_{D/R}B \subset H_{B/D}$ and $H_{D/R}B \subset H_{B/R}$.

Proof. We apply Lemma 16.7.1 to

$$D \longrightarrow A \otimes_R D \longrightarrow \Lambda$$

and the image of π in D. By Lemma 16.3.7 we see that π is strictly standard in $A \otimes_R D$ over D. As our section $\rho : (A \otimes_R D)/\pi^4(A \otimes_R D) \to D/\pi^4 D$ we take the map induced by the map in (2). Thus Lemma 16.7.1 applies and we obtain a factorization $A \otimes_R D \to B \to \Lambda$ with B of finite presentation and $\mathfrak{a}B \subset H_{B/D}$ where

$$\mathfrak{a} = \mathrm{Ann}_D(\mathrm{Ann}_D(\pi^2)/\mathrm{Ann}_D(\pi)).$$

For any prime $\mathfrak{q}$ of D such that $D_\mathfrak{q}$ is flat over R we have $\mathrm{Ann}_{D_\mathfrak{q}}(\pi^2)/\mathrm{Ann}_{D_\mathfrak{q}}(\pi) = 0$ because annihilators of elements commutes with flat base change and we assumed $\mathrm{Ann}_R(\pi) = \mathrm{Ann}_R(\pi^2)$. Because D is Noetherian we see that $\mathrm{Ann}_D(\pi^2)/\mathrm{Ann}_D(\pi)$ is a finite D-module, hence formation of its annihilator commutes with localization. Thus we see that $\mathfrak{a} \not\subset \mathfrak{q}$. Hence we see that $D \to B$ is smooth at any prime of B lying over $\mathfrak{q}$. Since any prime of D where $R \to D$ is smooth is one where $D_\mathfrak{q}$ is flat over R we conclude that $H_{D/R}B \subset H_{B/D}$. The final inclusion $H_{D/R}B \subset H_{B/R}$ follows because compositions of smooth ring maps are smooth (Algebra, Lemma 10.130.14). □

Lemma 16.7.3. *Let R be a Noetherian ring. Let Λ be an R-algebra. Let $\pi \in R$ and assume that $Ann_R(\pi) = Ann_R(\pi^2)$ and $Ann_\Lambda(\pi) = Ann_\Lambda(\pi^2)$. Let $A \to \Lambda$ be an R-algebra map with A of finite presentation and assume π is strictly standard in A over R. Let*

$$A/\pi^8 A \to \bar{C} \to \Lambda/\pi^8\Lambda$$

be a factorization with $\bar{C}$ of finite presentation. Then we can find a factorization $A \to B \to \Lambda$ with B of finite presentation such that $R_\pi \to B_\pi$ is smooth and such that

$$H_{\bar{C}/(R/\pi^8 R)} \cdot \Lambda/\pi^8\Lambda \subset \sqrt{H_{B/R}\Lambda} \bmod \pi^8\Lambda.$$

Proof. Apply Lemma 16.6.1 to get $R \to D \to \Lambda$ with a factorization $\bar{C}/\pi^4\bar{C} \to D/\pi^4 D \to \Lambda/\pi^4\Lambda$ such that $R \to D$ is smooth at any prime not containing π and at any prime lying over a prime of $\bar{C}/\pi^4\bar{C}$ where $R/\pi^8 R \to \bar{C}$ is smooth. By Lemma 16.7.2 we can find a finitely presented R-algebra B and factorizations $A \to B \to \Lambda$ and $D \to B \to \Lambda$ such that $H_{D/R}B \subset H_{B/R}$. We omit the verification that this is a solution to the problem posed by the lemma. □

16.8. Warmup: reduction to a base field

In this section we apply the lemmas in the previous sections to prove that it suffices to prove the main result when the base ring is a field, see Lemma 16.8.4.

Situation 16.8.1. Here $R \to \Lambda$ is a regular ring map of Noetherian rings.

Let $R \to \Lambda$ be as in Situation 16.8.1. We say *PT holds for* $R \to \Lambda$ if Λ is a filtered colimit of smooth R-algebras.

Lemma 16.8.2. *Let $R_i \to \Lambda_i$, $i = 1, 2$ be as in Situation 16.8.1. If PT holds for $R_i \to \Lambda_i$, $i = 1, 2$, then PT holds for $R_1 \times R_2 \to \Lambda_1 \times \Lambda_2$.*

Proof. Omitted. Hint: A product of colimits is a colimit. □

Lemma 16.8.3. *Let $R \to A \to \Lambda$ be ring maps with A of finite presentation over R. Let $S \subset R$ be a multiplicative set. Let $S^{-1}A \to B' \to S^{-1}\Lambda$ be a factorization with B' smooth over $S^{-1}R$. Then we can find a factorization $A \to B \to \Lambda$ such that some $s \in S$ maps to an elementary standard element in B over R.*

Proof. We first apply Lemma 16.4.4 to $S^{-1}R \to B'$. Thus we may assume B' is standard smooth over $S^{-1}R$. Write $A = R[x_1, \ldots, x_n]/(g_1, \ldots, g_t)$ and say $x_i \mapsto \lambda_i$ in Λ. We may write $B' = S^{-1}R[x_1, \ldots, x_{n+m}]/(f_1, \ldots, f_c)$ for some $c \geq n$ where $\det(\partial f_j/\partial x_i)_{i,j=1,\ldots,c}$ is invertible in B' and such that $A \to B'$ is given by $x_i \mapsto x_i$, see Lemma 16.4.6. After multiplying x_i, $i > n$ by an element of S and correspondingly modifying the equations f_j we may assume $B' \to S^{-1}\Lambda$ maps x_i to $\lambda_i/1$ for some $\lambda_i \in \Lambda$ for $i > n$. Choose a relation

$$1 = a_0 \det(\partial f_j/\partial x_i)_{i,j=1,\ldots,c} + \sum\nolimits_{j=1,\ldots,c} a_j f_j$$

for some $a_j \in S^{-1}R[x_1, \ldots, x_{n+m}]$. Since each element of S is invertible in B' we may (by clearing denominators) assume that $f_j, a_j \in R[x_1, \ldots, x_{n+m}]$ and that

$$s_0 = a_0 \det(\partial f_j/\partial x_i)_{i,j=1,\ldots,c} + \sum\nolimits_{j=1,\ldots,c} a_j f_j$$

for some $s_0 \in S$. Since g_j maps to zero in $S^{-1}R[x_1, \ldots, x_{n+m}]/(f_1, \ldots, x_c)$ we can find elements $s_j \in S$ such that $s_j g_j = 0$ in $R[x_1, \ldots, x_{n+m}]/(f_1, \ldots, f_c)$. Since f_j maps to zero in $S^{-1}\Lambda$ we can find $s'_j \in S$ such that $s'_j f_j(\lambda_1, \ldots, \lambda_{n+m}) = 0$ in Λ. Consider the ring

$$B = R[x_1, \ldots, x_{n+m}]/(s'_1 f_1, \ldots, s'_c f_c, g_1, \ldots, g_t)$$

and the factorization $A \to B \to \Lambda$ with $B \to \Lambda$ given by $x_i \mapsto \lambda_i$. We claim that $s = s_0 s_1 \ldots s_t s'_1 \ldots s'_c$ is elementary standard in B over R which finishes the proof. Namely, $s_j g_j \in (f_1, \ldots, f_c)$ and hence $s g_j \in (s'_1 f_1, \ldots, s'_c f_c)$. Finally, we have

$$a_0 \det(\partial s'_j f_j/\partial x_i)_{i,j=1,\ldots,c} + \sum\nolimits_{j=1,\ldots,c} (s'_1 \ldots \hat{s'_j} \ldots s'_c) a_j s'_j f_j = s_0 s'_1 \ldots s'_c$$

which divides s as desired. □

Lemma 16.8.4. *If for every Situation 16.8.1 where R is a field PT holds, then PT holds in general.*

Proof. Assume PT holds for any Situation 16.8.1 where R is a field. Let $R \to \Lambda$ be as in Situation 16.8.1 arbitrary. Note that $R/I \to \Lambda/I\Lambda$ is another regular ring map of Noetherian rings, see More on Algebra, Lemma 15.29.3. Consider the set of ideals

$$\mathcal{I} = \{I \subset R \mid R/I \to \Lambda/I\Lambda \text{ does not have PT}\}$$

We have to show that $\mathcal{I}$ is empty. If this set is nonempty, then it contains a maximal element because R is Noetherian. Replacing R by R/I and Λ by Λ/I we obtain a situation where PT holds for $R/I \to \Lambda/I\Lambda$ for any nonzero ideal of R. In particular, we see by applying Proposition 16.5.3 that R is a reduced ring.

Let $A \to \Lambda$ be an R-algebra homomorphism with A of finite presentation. We have to find a factorization $A \to B \to \Lambda$ with B smooth over R, see Lemma 16.2.1.

Let $S \subset R$ be the set of nonzerodivisors and consider the total ring of fractions $Q = S^{-1}R$ of R. We know that $Q = K_1 \times \ldots \times K_n$ is a product of fields, see Algebra, Lemmas 10.23.2 and 10.29.6. By Lemma 16.8.2 and our assumption PT holds for the ring map $S^{-1}R \to S^{-1}\Lambda$. Hence we can find a factorization $S^{-1}A \to B' \to \Lambda$ with B' smooth over $S^{-1}R$.

We apply Lemma 16.8.3 and find a factorization $A \to B \to \Lambda$ such that some $\pi \in S$ is elementary standard in B over R. After replacing A by B we may assume that π is elementary standard, hence strictly standard in A. We know that $R/\pi^8R \to \Lambda/\pi^8\Lambda$ satisfies PT. Hence we can find a factorization $R/\pi^8R \to A/\pi^8A \to \bar{C} \to \Lambda/\pi^8\Lambda$ with $R/\pi^8R \to \bar{C}$ smooth. By Lemma 16.6.1 we can find an R-algebra map $D \to \Lambda$ with D smooth over R and a factorization $R/\pi^4R \to A/\pi^4A \to D/\pi^4D \to \Lambda/\pi^4\Lambda$. By Lemma 16.7.2 we can find $A \to B \to \Lambda$ with B smooth over R which finishes the proof. □

16.9. Local tricks

Situation 16.9.1. We are given a Noetherian ring R and an R-algebra map $A \to \Lambda$ and a prime $\mathfrak{q} \subset \Lambda$. We assume A is of finite presentation over R. In this situation we denote $\mathfrak{h}_A = \sqrt{H_{A/R}\Lambda}$.

Let $R \to A \to \Lambda \supset \mathfrak{q}$ be as in Situation 16.9.1. We say $R \to A \to \Lambda \supset \mathfrak{q}$ *can be resolved* if there exists a factorization $A \to B \to \Lambda$ with B of finite presentation and $\mathfrak{h}_A \subset \mathfrak{h}_B \not\subset \mathfrak{q}$. In this case we will call the factorization $A \to B \to \Lambda$ a *resolution of* $R \to A \to \Lambda \supset \mathfrak{q}$.

Lemma 16.9.2. *Let $R \to A \to \Lambda \supset \mathfrak{q}$ be as in Situation 16.9.1. Let $r \geq 1$ and $\pi_1, \ldots, \pi_r \in R$ map to elements of $\mathfrak{q}$. Assume*

(1) *for $i = 1, \ldots, r$ we have*

$$Ann_{R/(\pi_1^8,\ldots,\pi_{i-1}^8)R}(\pi_i) = Ann_{R/(\pi_1^8,\ldots,\pi_{i-1}^8)R}(\pi_i^2)$$

and

$$Ann_{\Lambda/(\pi_1^8,\ldots,\pi_{i-1}^8)\Lambda}(\pi_i) = Ann_{\Lambda/(\pi_1^8,\ldots,\pi_{i-1}^8)\Lambda}(\pi_i^2)$$

(2) *for $i = 1, \ldots, r$ the element π_i maps to a strictly standard element in A over R.*

Then, if

$$R/(\pi_1^8,\ldots,\pi_r^8)R \to A/(\pi_1^8,\ldots,\pi_r^8)A \to \Lambda/(\pi_1^8,\ldots,\pi_r^8)\Lambda \supset \mathfrak{q}/(\pi_1^8,\ldots,\pi_r^8)\Lambda$$

can be resolved, so can $R \to A \to \Lambda \supset \mathfrak{q}$.

Proof. We are going to prove this by induction on r.

The case $r = 1$. Here the assumption is that there exists a factorization $A/\pi_1^8 \to \bar{C} \to \Lambda/\pi_1^8$ which resolves the situation modulo π_1^8. Conditions (1) and (2) are the assumptions needed to apply Lemma 16.7.3. Thus we can "lift" the resolution $\bar{C}$ to a resolution of $R \to A \to \Lambda \supset \mathfrak{q}$.

The case $r > 1$. In this case we apply the induction hypothesis for $r-1$ to the situation $R/\pi_1^8 \to A/\pi_1^8 \to \Lambda/\pi_1^8 \supset \mathfrak{q}/\pi_1^8\Lambda$. Note that property (2) is preserved by Lemma 16.3.7. □

Lemma 16.9.3. *Let* $R \to A \to \Lambda \supset \mathfrak{q}$ *be as in Situation 16.9.1. Let* $\mathfrak{p} = R \cap \mathfrak{q}$. *Assume that* $\mathfrak{q}$ *is minimal over* $\mathfrak{h}_A$ *and that* $R_\mathfrak{p} \to A_\mathfrak{p} \to \Lambda_\mathfrak{q} \supset \mathfrak{q}\Lambda_\mathfrak{q}$ *can be resolved. Then there exists a factorization* $A \to C \to \Lambda$ *with* C *of finite presentation such that* $H_{C/R}\Lambda \not\subset \mathfrak{q}$.

Proof. Let $A_\mathfrak{p} \to C \to \Lambda_\mathfrak{q}$ be a resolution of $R_\mathfrak{p} \to A_\mathfrak{p} \to \Lambda_\mathfrak{q} \supset \mathfrak{q}\Lambda_\mathfrak{q}$. By our assumption that $\mathfrak{q}$ is minimal over $\mathfrak{h}_A$ this means that $H_{C/R_\mathfrak{p}}\Lambda_\mathfrak{q} = \Lambda_\mathfrak{q}$. By Lemma 16.3.8 we may assume that C is smooth over $\Lambda_\mathfrak{p}$. By Lemma 16.4.4 we may assume that C is standard smooth over $R_\mathfrak{p}$. Write $A = R[x_1,\ldots,x_n]/(g_1,\ldots,g_t)$ and say $A \to \Lambda$ is given by $x_i \mapsto \lambda_i$. Write $C = R_\mathfrak{p}[x_1,\ldots,x_{n+m}]/(f_1,\ldots,f_c)$ for some $c \geq n$ such that $A \to C$ maps x_i to x_i and such that $\det(\partial f_j/\partial x_i)_{i,j=1,\ldots,c}$ is invertible in C, see Lemma 16.4.6. After clearing denominators we may assume $f_1,\ldots,f_c$ are elements of $R[x_1,\ldots,x_{n+m}]$. Of course $\det(\partial f_j/\partial x_i)_{i,j=1,\ldots,c}$ is not invertible in $R[x_1,\ldots,x_{n+m}]/(f_1,\ldots,f_c)$ but it becomes invertible after inverting some element $s_0 \in R$, $s_0 \notin \mathfrak{p}$. As g_j maps to zero under $R[x_1,\ldots,x_n] \to A \to C$ we can find $s_j \in R$, $s_j \notin \mathfrak{p}$ such that $s_j g_j$ is zero in $R[x_1,\ldots,x_{n+m}]/(f_1,\ldots,f_c)$. Write $f_j = F_j(x_1,\ldots,x_{n+m},1)$ for some polynomial $F_j \in R[x_1,\ldots,x_n,X_{n+1},\ldots,X_{n+m+1}]$ homogeneous in $X_{n+1},\ldots,X_{n+m+1}$. Pick $\lambda_{n+i} \in \Lambda$, $i = 1,\ldots,m+1$ with $\lambda_{n+m+1} \notin \mathfrak{q}$ such that x_{n+i} maps to $\lambda_{n+i}/\lambda_{n+m+1}$ in $\Lambda_\mathfrak{q}$. Then

$$\begin{aligned} F_j(\lambda_1,\ldots,\lambda_{n+m+1}) &= (\lambda_{n+m+1})^{\deg(F_j)} F_j(\lambda_1,\ldots,\lambda_n,\frac{\lambda_{n+1}}{\lambda_{n+m+1}},\ldots,\frac{\lambda_{n+m}}{\lambda_{n+m+1}},1) \\ &= (\lambda_{n+m+1})^{\deg(F_j)} f_j(\lambda_1,\ldots,\lambda_n,\frac{\lambda_{n+1}}{\lambda_{n+m+1}},\ldots,\frac{\lambda_{n+m}}{\lambda_{n+m+1}}) \\ &= 0 \end{aligned}$$

in $\Lambda_\mathfrak{q}$. Thus we can find $\lambda_0 \in \Lambda$, $\lambda_0 \notin \mathfrak{q}$ such that $\lambda_0 F_j(\lambda_1,\ldots,\lambda_{n+m+1}) = 0$ in Λ. Now we set B equal to

$$R[x_0,\ldots,x_{n+m+1}]/(g_1,\ldots,g_t,x_0F_1(x_1,\ldots,x_{n+m+1}),\ldots,x_0F_c(x_1,\ldots,x_{n+m+1}))$$

which we map to Λ by mapping x_i to λ_i. Let b be the image of $x_0x_1s_0s_1\ldots s_t$ in B. Then B_b is isomorphic to

$$R_{s_0s_1}[x_0,x_1,\ldots,x_{n+m+1},1/x_0x_{n+m+1}]/(f_1,\ldots,f_c)$$

which is smooth over R by construction. Since b does not map to an element of $\mathfrak{q}$, we win. □

Lemma 16.9.4. *Let $R \to A \to \Lambda \supset \mathfrak{q}$ be as in Situation 16.9.1. Let $\mathfrak{p} = R \cap \mathfrak{q}$. Assume*

(1) *$\mathfrak{q}$ is minimal over $\mathfrak{h}_A$,*
(2) *$R_\mathfrak{p} \to A_\mathfrak{p} \to \Lambda_\mathfrak{q} \supset \mathfrak{q}\Lambda_\mathfrak{q}$ can be resolved, and*
(3) *$\dim(\Lambda_\mathfrak{q}) = 0$.*

Then $R \to A \to \Lambda \supset \mathfrak{q}$ can be resolved.

Proof. By (3) the ring $\Lambda_\mathfrak{q}$ is Artinian local hence $\mathfrak{q}\Lambda_\mathfrak{q}$ is nilpotent. Thus $(\mathfrak{h}_A)^N\Lambda_\mathfrak{q} = 0$ for some $N > 0$. Thus there exists a $\lambda \in \Lambda$, $\lambda \notin \mathfrak{q}$ such that $\lambda(\mathfrak{h}_A)^N = 0$ in Λ. Say $H_{A/R} = (a_1, \ldots, a_r)$ so that $\lambda a_i^N = 0$ in Λ. By Lemma 16.9.3 we can find a factorization $A \to C \to \Lambda$ with C of finite presentation such that $\mathfrak{h}_C \not\subset \mathfrak{q}$. Write $C = A[x_1, \ldots, x_n]/(f_1, \ldots, f_m)$. Set

$$B = A[x_1, \ldots, x_n, y_1, \ldots, y_r, z, t_{ij}]/(f_j - \sum y_i t_{ij}, zy_i)$$

where t_{ij} is a set of rm variables. Note that there is a map $B \to C[y_i, z]/(y_i z)$ given by setting t_{ij} equal to zero. The map $B \to \Lambda$ is the composition $B \to C[y_i, z]/(y_i z) \to \Lambda$ where $C[y_i, z]/(y_i z) \to \Lambda$ is the given map $C \to \Lambda$, maps z to λ, and maps y_i to the image of a_i^N in Λ.

We claim that B is a solution for $R \to A \to \Lambda \supset \mathfrak{q}$. First note that B_z is isomorphic to $C[y_1, \ldots, y_r, z, z^{-1}]$ and hence is smooth. On the other hand, $B_{y_\ell} \cong A[x_i, y_i, y_\ell^{-1}, t_{ij}, i \neq \ell]$ which is smooth over A. Thus we see that z and $a_\ell y_\ell$ (compositions of smooth maps are smooth) are all elements of $H_{B/R}$. This proves the lemma. □

16.10. Separable residue fields

In this section we explain how to solve a local problem in the case of a separable residue field extension.

Lemma 16.10.1 (Ogoma)**.** *Let A be a Noetherian ring and let M be a finite A-module. Let $S \subset A$ be a multiplicative set. If $\pi \in A$ and $Ker(\pi : S^{-1}M \to S^{-1}M) = Ker(\pi^2 : S^{-1}M \to S^{-1}M)$ then there exists an $s \in S$ such that for any $n > 0$ we have $Ker(s^n\pi : M \to M) = Ker((s^n\pi)^2 : M \to M)$.*

Proof. Let $K = \mathrm{Ker}(\pi : M \to M)$ and $K' = \{m \in M \mid \pi^2 m = 0 \text{ in } S^{-1}M\}$ and $Q = K'/K$. Note that $S^{-1}Q = 0$ by assumption. Since A is Noetherian we see that Q is a finite A-module. Hence we can find an $s \in S$ such that s annihilates Q. Then s works. □

Lemma 16.10.2. *Let Λ be a Noetherian ring. Let $I \subset \Lambda$ be an ideal. Let $I \subset \mathfrak{q}$ be a prime. Let n, e be positive integers Assume that $\mathfrak{q}^n\Lambda_\mathfrak{q} \subset I\Lambda_\mathfrak{q}$ and that $\Lambda_\mathfrak{q}$ is a regular local ring of dimension d. Then there exists an $n > 0$ and $\pi_1, \ldots, \pi_d \in \Lambda$ such that*

(1) *$(\pi_1, \ldots, \pi_d)\Lambda_\mathfrak{q} = \mathfrak{q}\Lambda_\mathfrak{q}$,*
(2) *$\pi_1^n, \ldots, \pi_d^n \in I$, and*
(3) *for $i = 1, \ldots, d$ we have*

$$Ann_{\Lambda/(\pi_1^e, \ldots, \pi_{i-1}^e)\Lambda}(\pi_i) = Ann_{\Lambda/(\pi_1^e, \ldots, \pi_{i-1}^e)\Lambda}(\pi_i^2).$$

Proof. Set $S = \Lambda \setminus \mathfrak{q}$ so that $\Lambda_\mathfrak{q} = S^{-1}\Lambda$. First pick $\pi_1, \ldots, \pi_d$ with (1) which is possible as $\Lambda_\mathfrak{q}$ is regular. By assumption $\pi_i^n \in I\Lambda_\mathfrak{q}$. Thus we can find $s_1, \ldots, s_d \in S$ such that $s_i\pi_i^n \in I$. Replacing π_i by $s_i\pi_i$ we get (2). Note that (1) and (2) are preserved by further multiplying by elements of S. Suppose that (3) holds for $i = 1, \ldots, t$ for some $t \in \{0, \ldots, d\}$. Note that $\pi_1, \ldots, \pi_d$ is a regular sequence in $S^{-1}\Lambda$, see Algebra, Lemma 10.101.3. In particular $\pi_1^e, \ldots, \pi_t^e, \pi_{t+1}$ is a regular sequence in $S^{-1}\Lambda = \Lambda_\mathfrak{q}$ by Algebra, Lemma 10.65.10. Hence we see that

$$\mathrm{Ann}_{S^{-1}\Lambda/(\pi_1^e,\ldots,\pi_{i-1}^e)}(\pi_i) = \mathrm{Ann}_{S^{-1}\Lambda/(\pi_1^e,\ldots,\pi_{i-1}^e)}(\pi_i^2).$$

Thus we get (3) for $i = t+1$ after replacing π_{t+1} by $s\pi_{t+1}$ for some $s \in S$ by Lemma 16.10.1. By induction on t this produces a sequence satisfying (1), (2), and (3). □

Lemma 16.10.3. *Let $k \to A \to \Lambda \supset \mathfrak{q}$ be as in Situation 16.9.1 where*

(1) *k is a field,*
(2) *Λ is Noetherian,*
(3) *$\mathfrak{q}$ is minimal over $\mathfrak{h}_A$,*
(4) *$\Lambda_\mathfrak{q}$ is a regular local ring, and*
(5) *the field extension $k \subset \kappa(\mathfrak{q})$ is separable.*

Then $k \to A \to \Lambda \supset \mathfrak{q}$ can be resolved.

Proof. Set $d = \dim \Lambda_\mathfrak{q}$. Set $R = k[x_1, \ldots, x_d]$. Choose $n > 0$ such that $\mathfrak{q}^n\Lambda_\mathfrak{q} \subset \mathfrak{h}_A\Lambda_\mathfrak{q}$ which is possible as $\mathfrak{q}$ is minimal over $\mathfrak{h}_A$. Choose generators $a_1, \ldots, a_r$ of $H_{A/R}$. Set

$$B = A[x_1, \ldots, x_d, z_{ij}]/(x_i^n - \sum z_{ij}a_j)$$

Each B_{a_j} is smooth over R it is a polynomial algebra over $A_{a_j}[x_1, \ldots, x_d]$ and A_{a_j} is smooth over k. Hence B_{x_i} is smooth over R. Let $B \to C$ be the R-algebra map constructed in Lemma 16.4.1 which comes with a R-algebra retraction $C \to B$. In particular a map $C \to \Lambda$ fitting into the diagram above. By construction C_{x_i} is a smooth R-algebra with $\Omega_{C_{x_i}/R}$ free. Hence we can find $c > 0$ such that x_i^c is strictly standard in C/R, see Lemma 16.4.7. Now choose $\pi_1, \ldots, \pi_d \in \Lambda$ as in Lemma 16.10.2 where $n = n$, $e = 8c$, $\mathfrak{q} = \mathfrak{q}$ and $I = \mathfrak{h}_A$. Write $\pi_i^n = \sum \lambda_{ij}a_j$ for some $\pi_{ij} \in \Lambda$. There is a map $B \to \Lambda$ given by $x_i \mapsto \pi_i$ and $z_{ij} \mapsto \lambda_{ij}$. Set $R = k[x_1, \ldots, x_d]$. Diagram

$$\begin{array}{ccccc} R & \longrightarrow & B & & \\ \uparrow & & \uparrow & \searrow & \\ k & \longrightarrow & A & \longrightarrow & \Lambda \end{array}$$

Now we apply Lemma 16.9.2 to $R \to C \to \Lambda \supset \mathfrak{q}$ and the sequence of elements $x_1^c, \ldots, x_d^c$ of R. Assumption (2) is clear. Assumption (1) holds for R by inspection and for Λ by our choice of $\pi_1, \ldots, \pi_d$. (Note that if $\mathrm{Ann}_\Lambda(\pi) = \mathrm{Ann}_\Lambda(\pi^2)$, then we have $\mathrm{Ann}_\Lambda(\pi) = \mathrm{Ann}_\Lambda(\pi^c)$ for all $c > 0$.) Thus it suffices to resolve

$$R/(x_1^e, \ldots, x_d^e) \to C/(x_1^e, \ldots, x_d^e) \to \Lambda/(\pi_1^e, \ldots, \pi_d^e) \supset \mathfrak{q}/(\pi_1^e, \ldots, \pi_d^e)$$

for $e = 8c$. By Lemma 16.9.4 it suffices to resolve this after localizing at $\mathfrak{q}$. But since $x_1, \ldots, x_d$ map to a regular sequence in $\Lambda_\mathfrak{q}$ we see that $R \to \Lambda$ is flat, see Algebra, Lemma 10.122.2. Hence

$$R/(x_1^e, \ldots, x_d^e) \to \Lambda_\mathfrak{q}/(\pi_1^e, \ldots, \pi_d^e)$$

is a flat ring map of Artinian local rings. Moreover, this map induces a separable field extension on residue fields by assumption. Thus this map is a filtered colimit of smooth algebras by Algebra, Lemma 10.145.11 and Proposition 16.5.3. Existence of the desired solution follows from Lemma 16.2.1. □

16.11. Inseparable residue fields

In this section we explain how to solve a local problem in the case of an inseparable residue field extension.

Lemma 16.11.1. *Let k be a field of characteristic $p > 0$. Let $(\Lambda, \mathfrak{m}, K)$ be an Artinian local k-algebra. Assume that $\dim H_1(L_{K/k}) < \infty$. Then Λ is a filtered colimit of Artinian local k-algebras A with each map $A \to \Lambda$ flat, with $\mathfrak{m}_A\Lambda = \mathfrak{m}$, and with A essentially of finite type over k.*

Proof. Note that the flatness of $A \to \Lambda$ implies that $A \to \Lambda$ is injective, so the lemma really tells us that Λ is a directed union of these types of subrings $A \subset \Lambda$. Let n be the minimal integer such that $\mathfrak{m}^n = 0$. We will prove this lemma by induction on n. The case $n = 1$ is clear as a field extension is a union of finitely generated field extensions.

Pick $\lambda_1, \ldots, \lambda_d \in \mathfrak{m}$ which generate $\mathfrak{m}$. As K is formally smooth over $\mathbf{F}_p$ (see Algebra, Lemma 10.145.7) we can find a ring map $\sigma : K \to \Lambda$ which is a section of the quotient map $\Lambda \to K$. In general σ is **not** a k-algebra map. Given σ we define

$$\Psi_\sigma : K[x_1, \ldots, x_d] \longrightarrow \Lambda$$

using σ on elements of K and mapping x_i to λ_i. Claim: there exists a $\sigma : K \to \Lambda$ and a subfield $k \subset F \subset K$ finitely generated over k such that the image of k in Λ is contained in $\Psi_\sigma(F[x_1, \ldots, x_d])$.

We will prove the claim by induction on the least integer n such that $\mathfrak{m}^n = 0$. It is clear for $n = 1$. If $n > 1$ set $I = \mathfrak{m}^{n-1}$ and $\Lambda' = \Lambda/I$. By induction we may assume given $\sigma' : K \to \Lambda'$ and $k \subset F' \subset K$ finitely generated such that the image of $k \to \Lambda \to \Lambda'$ is contained in $A' = \Psi_{\sigma'}(F'[x_1, \ldots, x_d])$. Denote $\tau' : k \to A'$ the induced map. Choose a lift $\sigma : K \to \Lambda$ of σ' (this is possible by the formal smoothness of $K/\mathbf{F}_p$ we mentioned above). For later reference we note that we can change σ to $\sigma + D$ for some derivation $D : K \to I$. Set $A = F[x_1, \ldots, x_d]/(x_1, \ldots, x_d)^n$. Then Ψ_σ induces a ring map $\Psi_\sigma : A \to \Lambda$. The composition with the quotient map $\Lambda \to \Lambda'$ induces a surjective map $A \to A'$ with nilpotent kernel. Choose a lift $\tau : k \to A$ of τ' (possible as $k/\mathbf{F}_p$ is formally smooth). Thus we obtain two maps $k \to \Lambda$, namely $\Psi_\sigma \circ \tau : k \to \Lambda$ and the given map $i : k \to \Lambda$. These maps agree modulo I, whence the difference is a derivation $\theta = i - \Psi_\sigma \circ \tau : k \to I$. Note that if we change σ into $\sigma + D$ then we change θ into $\theta - D|_k$.

Choose a set of elements $\{y_j\}_{j \in J}$ of k whose differentials $\mathrm{d}y_j$ form a basis of $\Omega_{k/\mathbf{F}_p}$. The Jacobi-Zariski sequence for $\mathbf{F}_p \subset k \subset K$ is

$$0 \to H_1(L_{K/k}) \to \Omega_{k/\mathbf{F}_p} \otimes K \to \Omega_{K/\mathbf{F}_p} \to \Omega_{K/k} \to 0$$

As $\dim H_1(L_{K/k}) < \infty$ we can find a finite subset $J_0 \subset J$ such that the image of the first map is contained in $\bigoplus_{j \in J_0} K\mathrm{d}y_j$. Hence the elements $\mathrm{d}y_j$, $j \in J \setminus J_0$ map to

K-linearly independent elements of $\Omega_{K/\mathbf{F}_p}$. Therefore we can choose a $D : K \to I$ such that $\theta - D|_k = \xi \circ \mathrm{d}$ where ξ is a composition

$$\Omega_{k/\mathbf{F}_p} = \bigoplus\nolimits_{j \in J} k\mathrm{d}y_j \longrightarrow \bigoplus\nolimits_{j \in J_0} k\mathrm{d}y_j \longrightarrow I$$

Let $f_j = \xi(\mathrm{d}y_j) \in I$ for $j \in J_0$. Change σ into $\sigma + D$ as above. Then we see that $\theta(a) = \sum_{j \in J_0} a_j f_j$ for $a \in k$ where $\mathrm{d}a = \sum a_j \mathrm{d}y_j$ in $\Omega_{k/\mathbf{F}_p}$. Note that I is generated by the monomials $\lambda^E = \lambda_1^{e_1} \ldots \lambda_d^{e_d}$ of total degree $|E| = \sum e_i = n - 1$ in $\lambda_1, \ldots, \lambda_d$. Write $f_j = \sum_E c_{j,E} \lambda^E$ with $c_{j,E} \in K$. Replace F' by $F = F'(c_{j,E})$. Then the claim holds.

Choose σ and F as in the claim. The kernel of Ψ_σ is generated by finitely many polynomials $g_1, \ldots, g_t \in K[x_1, \ldots, x_d]$ and we may assume their coefficients are in F after enlarging F by adjoining finitely many elements. In this case it is clear that the map $A = F[x_1, \ldots, x_d]/(g_1, \ldots, g_t) \to K[x_1, \ldots, x_d]/(g_1, \ldots, g_t) = \Lambda$ is flat. By the claim A is a k-subalgebra of Λ. It is clear that Λ is the filtered colimit of these algebras, as K is the filtered union of the subfields F. Finally, these algebras are essentially of finite type over k by Algebra, Lemma 10.50.3. □

Lemma 16.11.2. *Let k be a field of characteristic $p > 0$. Let Λ be a Noetherian geometrically regular k-algebra. Let $\mathfrak{q} \subset \Lambda$ be a prime ideal. Let $n \geq 1$ be an integer and let $E \subset \Lambda_\mathfrak{q}/\mathfrak{q}^n\Lambda_\mathfrak{q}$ be a finite subset. Then we can find $m \geq 0$ and $\varphi : k[y_1, \ldots, y_m] \to \Lambda$ with the following properties*

1. *setting $\mathfrak{p} = \varphi^{-1}(\mathfrak{q})$ we have $\mathfrak{q}\Lambda_\mathfrak{q} = \mathfrak{p}\Lambda_\mathfrak{q}$ and $k[y_1, \ldots, y_m]_\mathfrak{p} \to \Lambda_\mathfrak{q}$ is flat,*
2. *there is a factorization by homomorphisms of local Artinian rings*
$$k[y_1, \ldots, y_m]_\mathfrak{p}/\mathfrak{p}^n k[y_1, \ldots, y_m]_\mathfrak{p} \to D \to \Lambda_\mathfrak{q}/\mathfrak{q}^n\Lambda_\mathfrak{q}$$
where the first arrow is essentially smooth and the second is flat,
3. *E is contained in D modulo $\mathfrak{q}^n\Lambda_\mathfrak{q}$.*

Proof. Set $\bar{\Lambda} = \Lambda_\mathfrak{q}/\mathfrak{q}^n\Lambda_\mathfrak{q}$. Note that $\dim H_1(L_{\kappa(\mathfrak{q})/k}) < \infty$ by More on Algebra, Proposition 15.24.1. Pick $A \subset \bar{\Lambda}$ containing E such that A is local Artinian, essentially of finite type over k, the map $A \to \bar{\Lambda}$ is flat, and $\mathfrak{m}_A$ generates the maximal ideal of $\bar{\Lambda}$, see Lemma 16.11.1. Denote $F = A/\mathfrak{m}_A$ the residue field so that $k \subset F \subset K$. Pick $\lambda_1, \ldots, \lambda_t \in \Lambda$ which map to elements of A in $\bar{\Lambda}$ such that moreover the images of $\mathrm{d}\lambda_1, \ldots, \mathrm{d}\lambda_t$ form a basis of $\Omega_{F/k}$. Consider the map $\varphi' : k[y_1, \ldots, y_t] \to \Lambda$ sending y_j to λ_j. Set $\mathfrak{p}' = (\varphi')^{-1}(\mathfrak{q})$. By More on Algebra, Lemma 15.24.2 the ring map $k[y_1, \ldots, y_t]_{\mathfrak{p}'} \to \Lambda_\mathfrak{q}$ is flat and $\Lambda_\mathfrak{q}/\mathfrak{p}'\Lambda_\mathfrak{q}$ is regular. Thus we can choose further elements $\lambda_{t+1}, \ldots, \lambda_m \in \Lambda$ which map into $A \subset \bar{\Lambda}$ and which map to a regular system of parameters of $\Lambda_\mathfrak{q}/\mathfrak{p}'\Lambda_\mathfrak{q}$. We obtain $\varphi : k[y_1, \ldots, y_m] \to \Lambda$ having property (1) such that $k[y_1, \ldots, y_m]_\mathfrak{p}/\mathfrak{p}^n k[y_1, \ldots, y_m]_\mathfrak{p} \to \bar{\Lambda}$ factors through A. Thus $k[y_1, \ldots, y_m]_\mathfrak{p}/\mathfrak{p}^n k[y_1, \ldots, y_m]_\mathfrak{p} \to A$ is flat by Algebra, Lemma 10.36.8. By construction the residue field extension $\kappa(\mathfrak{p}) \subset F$ is finitely generated and $\Omega_{F/\kappa(\mathfrak{p})} = 0$. Hence it is finite separable by More on Algebra, Lemma 15.23.1. Thus $k[y_1, \ldots, y_m]_\mathfrak{p}/\mathfrak{p}^n k[y_1, \ldots, y_m]_\mathfrak{p} \to A$ is finite by Algebra, Lemma 10.50.3. Finally, we conclude that it is étale by Algebra, Lemma 10.136.7. Since an étale ring map is certainly essentially smooth we win. □

Lemma 16.11.3. *Let $\varphi : k[y_1, \ldots, y_m] \to \Lambda$, n, $\mathfrak{q}$, $\mathfrak{p}$ and*

$$k[y_1, \ldots, y_m]_\mathfrak{p}/\mathfrak{p}^n \to D \to \Lambda_\mathfrak{q}/\mathfrak{q}^n\Lambda_\mathfrak{q}$$

be as in Lemma 16.11.2. Then for any $\lambda \in \Lambda \setminus \mathfrak{q}$ there exists an integer $q > 0$ and a factorization

$$k[y_1, \ldots, y_m]_{\mathfrak{p}}/\mathfrak{p}^n \to D \to D' \to \Lambda_{\mathfrak{q}}/\mathfrak{q}^n\Lambda_{\mathfrak{q}}$$

such that $D \to D'$ is an essentially smooth map of local Artinian rings, the last arrow is flat, and λ^q is in D'.

Proof. Set $\bar{\Lambda} = \Lambda_{\mathfrak{q}}/\mathfrak{q}^n\Lambda_{\mathfrak{q}}$. Let $\bar{\lambda}$ be the image of λ in $\bar{\Lambda}$. Let $\alpha \in \kappa(\mathfrak{q})$ be the image of λ in the residue field. Let $k \subset F \subset \kappa(\mathfrak{q})$ be the residue field of D. If α is in F then we can find an $x \in D$ such that $x\bar{\lambda} = 1 \bmod \mathfrak{q}$. Hence $(x\bar{\lambda})^q = 1 \bmod (\mathfrak{q})^q$ if q is divisible by p. Hence $\bar{\lambda}^q$ is in D. If α is transcendental over F, then we can take $D' = (D[\bar{\lambda}])_{\mathfrak{m}}$ equal to the subring generated by D and $\bar{\lambda}$ localized at $\mathfrak{m} = D[\bar{\lambda}] \cap \mathfrak{q}\bar{\Lambda}$. This works because $D[\bar{\lambda}]$ is in fact a polynomial algebra over D in this case. Finally, if $\lambda \bmod \mathfrak{q}$ is algebraic over F, then we can find a p-power q such that α^q is separable algebraic over F, see Fields, Section 9.25. Note that D and $\bar{\Lambda}$ are henselian local rings, see Algebra, Lemma 10.143.11. Let $D \to D'$ be a finite étale extension whose residue field extension is $F \subset F(\alpha^q)$, see Algebra, Lemma 10.143.8. Since $\bar{\Lambda}$ is henselian and $F(\alpha^q)$ is contained in its residue field we can find a factorization $D' \to \bar{\Lambda}$. By the first part of the argument we see that $\bar{\lambda}^{qq'} \in D'$ for some $q' > 0$. □

Lemma 16.11.4. *Let $k \to A \to \Lambda \supset \mathfrak{q}$ be as in Situation 16.9.1 where*

1. *k is a field of characteristic $p > 0$,*
2. *Λ is Noetherian and geometrically regular over k,*
3. *$\mathfrak{q}$ is minimal over $\mathfrak{h}_A$.*

Then $k \to A \to \Lambda \supset \mathfrak{q}$ can be resolved.

Proof. The lemma is proven by the following steps in the given order. We will justify each of these steps below.

1. Pick an integer $N > 0$ such that $\mathfrak{q}^N\Lambda_{\mathfrak{q}} \subset H_{A/k}\Lambda_{\mathfrak{q}}$.
2. Pick generators $a_1, \ldots, a_t \in A$ of the ideal $H_{A/R}$.
3. Set $d = \dim(\Lambda_{\mathfrak{q}})$.
4. Set $B = A[x_1, \ldots, x_d, z_{ij}]/(x_i^{2N} - \sum z_{ij}a_j)$.
5. Consider B as a $k[x_1, \ldots, x_d]$-algebra and let $B \to C$ be as in Lemma 16.4.1. We also obtain a section $C \to B$.
6. Choose $c > 0$ such that each x_i^c is strictly standard in C over $k[x_1, \ldots, x_d]$.
7. Set $n = N + dc$ and $e = 8c$.
8. Let $E \subset \Lambda_{\mathfrak{q}}/\mathfrak{q}^n\Lambda_{\mathfrak{q}}$ be the images of generators of A as a k-algebra.
9. Choose an integer m and a k-algebra map $\varphi : k[y_1, \ldots, y_m] \to \Lambda$ and a factorization by local Artinian rings
$$k[y_1, \ldots, y_m]_{\mathfrak{p}}/\mathfrak{p}^n k[y_1, \ldots, y_m]_{\mathfrak{p}} \to D \to \Lambda_{\mathfrak{q}}/\mathfrak{q}^n\Lambda_{\mathfrak{q}}$$
such that the first arrow is essentially smooth, the second is flat, E is contained in D, with $\mathfrak{p} = \varphi^{-1}(\mathfrak{q})$ the map $k[y_1, \ldots, y_m]_{\mathfrak{p}} \to \Lambda_{\mathfrak{q}}$ is flat, and $\mathfrak{p}\Lambda_{\mathfrak{q}} = \mathfrak{q}\Lambda_{\mathfrak{q}}$.
10. Choose $\pi_1, \ldots, \pi_d \in \mathfrak{p}$ which map to a regular system of parameters of $k[y_1, \ldots, y_m]_{\mathfrak{p}}$.
11. Let $R = k[y_1, \ldots, y_m, t_1, \ldots, t_m]$ and $\gamma_i = \pi_i t_i$.
12. If necessary modify the choice of π_i such that for $i = 1, \ldots, d$ we have
$$\mathrm{Ann}_{R/(\gamma_1^e, \ldots, \gamma_{i-1}^e)R}(\gamma_i) = \mathrm{Ann}_{R/(\gamma_1^e, \ldots, \gamma_{i-1}^e)R}(\gamma_i^2)$$

(13) There exist $\delta_1, \ldots, \delta_d \in \Lambda$, $\delta_i \notin \mathfrak{q}$ and a factorization $D \to D' \to \Lambda_\mathfrak{q}/\mathfrak{q}^n\Lambda_\mathfrak{q}$ with D' local Artinian, $D \to D'$ essentially smooth, the map $D' \to \Lambda_\mathfrak{q}/\mathfrak{q}^n\Lambda_\mathfrak{q}$ flat such that, with $\pi_i' = \delta_i\pi_i$, we have for $i = 1, \ldots, d$
   (a) $(\pi_i')^{2N} = \sum a_j\lambda_{ij}$ in Λ where $\lambda_{ij} \bmod \mathfrak{q}^n\Lambda_\mathfrak{q}$ is an element of D',
   (b) $\mathrm{Ann}_{\Lambda/(\pi'^e_1,\ldots,\pi'^e_{i-1})}(\pi'_i) = \mathrm{Ann}_{\Lambda/(\pi'^e_1,\ldots,\pi'^e_{i-1})}(\pi'^2_i)$,
   (c) $\delta_i \bmod \mathfrak{q}^n\Lambda_\mathfrak{q}$ is an element of D'.

(14) Define $B \to \Lambda$ by sending x_i to π_i' and z_{ij} to λ_{ij} found above. Define $C \to \Lambda$ by composing the map $B \to \Lambda$ with the retraction $C \to B$.

(15) Map $R \to \Lambda$ by φ on $k[y_1, \ldots, y_m]$ and by sending t_i to δ_i. Further introduce a map
$$k[x_1, \ldots, x_d] \longrightarrow R = k[y_1, \ldots, y_m, t_1, \ldots, t_d]$$
by sending x_i to $\gamma_i = \pi_i t_i$.

(16) It suffices to resolve
$$R \to C \otimes_{k[x_1,\ldots,x_d]} R \to \Lambda \supset \mathfrak{q}$$

(17) Set $I = (\gamma_1^e, \ldots, \gamma_d^e) \subset R$.

(18) It suffices to resolve
$$R/I \to C \otimes_{k[x_1,\ldots,x_d]} R/I \to \Lambda/I\Lambda \supset \mathfrak{q}/I\Lambda$$

(19) We denote $\mathfrak{r} \subset R = k[y_1, \ldots, y_m, t_1, \ldots, t_d]$ the inverse image of $\mathfrak{q}$.

(20) It suffices to resolve
$$(R/I)_\mathfrak{r} \to C \otimes_{k[x_1,\ldots,x_d]} (R/I)_\mathfrak{r} \to \Lambda_\mathfrak{q}/I\Lambda_\mathfrak{q} \supset \mathfrak{q}\Lambda_\mathfrak{q}/I\Lambda_\mathfrak{q}$$

(21) Set $J = (\pi_1^e, \ldots, \pi_d^e)$ in $k[y_1, \ldots, y_m]$.

(22) It suffices to resolve
$$(R/JR)_\mathfrak{p} \to C \otimes_{k[x_1,\ldots,x_d]} (R/JR)_\mathfrak{p} \to \Lambda_\mathfrak{q}/J\Lambda_\mathfrak{q} \supset \mathfrak{q}\Lambda_\mathfrak{q}/J\Lambda_\mathfrak{q}$$

(23) It suffices to resolve
$$(R/\mathfrak{p}^nR)_\mathfrak{p} \to C \otimes_{k[x_1,\ldots,x_d]} (R/\mathfrak{p}^nR)_\mathfrak{p} \to \Lambda_\mathfrak{q}/\mathfrak{q}^n\Lambda_\mathfrak{q} \supset \mathfrak{q}\Lambda_\mathfrak{q}/\mathfrak{q}^n\Lambda_\mathfrak{q}$$

(24) It suffices to resolve
$$(R/\mathfrak{p}^nR)_\mathfrak{p} \to B \otimes_{k[x_1,\ldots,x_d]} (R/\mathfrak{p}^nR)_\mathfrak{p} \to \Lambda_\mathfrak{q}/\mathfrak{q}^n\Lambda_\mathfrak{q} \supset \mathfrak{q}\Lambda_\mathfrak{q}/\mathfrak{q}^n\Lambda_\mathfrak{q}$$

(25) The ring $D'[t_1, \ldots, t_d]$ is given the structure of an $R_\mathfrak{p}/\mathfrak{p}^nR_\mathfrak{p}$-algebra by the given map $k[y_1, \ldots, y_m]_\mathfrak{p}/\mathfrak{p}^nk[y_1, \ldots, y_m]_\mathfrak{p} \to D'$ and by sending t_i to t_i. It suffices to find a factorization
$$B \otimes_{k[x_1,\ldots,x_d]} (R/\mathfrak{p}^nR)_\mathfrak{p} \to D'[t_1, \ldots, t_d] \to \Lambda_\mathfrak{q}/\mathfrak{q}^n\Lambda_\mathfrak{q}$$
where the second arrow sends t_i to δ_i and induces the given homomorphism $D' \to \Lambda_\mathfrak{q}/\mathfrak{q}^n\Lambda_\mathfrak{q}$.

(26) Such a factorization exists by our choice of D' above.

We now give the justification for each of the steps, except that we skip justifying the steps which just introduce notation.

Ad (1). This is possible as $\mathfrak{q}$ is minimal over $\mathfrak{h}_A = \sqrt{H_{A/k}\Lambda}$.

Ad (6). Note that A_{a_i} is smooth over k. Hence B_{a_j}, which is isomorphic to a polynomial algebra over $A_{a_j}[x_1, \ldots, x_d]$, is smooth over $k[x_1, \ldots, x_d]$. Thus B_{x_i} is smooth over $k[x_1, \ldots, x_d]$. By Lemma 16.4.1 we see that C_{x_i} is smooth over $k[x_1, \ldots, x_d]$ with finite free module of differentials. Hence some power of x_i is strictly standard in C over $k[x_1, \ldots, x_n]$ by Lemma 16.4.7.

Ad (9). This follows by applying Lemma 16.11.2.

Ad (10). Since $k[y_1, \ldots, y_m]_{\mathfrak{p}} \to \Lambda_{\mathfrak{q}}$ is flat and $\mathfrak{p}\Lambda_{\mathfrak{q}} = \mathfrak{q}\Lambda_{\mathfrak{q}}$ by construction we see that $\dim(k[y_1, \ldots, y_m]_{\mathfrak{p}}) = d$ by Algebra, Lemma 10.106.7. Thus we can find $\pi_1, \ldots, \pi_d \in \Lambda$ which map to a regular system of parameters in $\Lambda_{\mathfrak{q}}$.

Ad (12). By Algebra, Lemma 10.101.3 any permutation of the sequence $\pi_1, \ldots, \pi_d$ is a regular sequence in $k[y_1, \ldots, y_m]_{\mathfrak{p}}$. Hence $\gamma_1 = \pi_1 t_1, \ldots, \gamma_d = \pi_d t_d$ is a regular sequence in $R_{\mathfrak{p}} = k[y_1, \ldots, y_m]_{\mathfrak{p}}[t_1, \ldots, t_d]$, see Algebra, Lemma 10.65.11. Let $S = k[y_1, \ldots, y_m] \setminus \mathfrak{p}$ so that $R_{\mathfrak{p}} = S^{-1}R$. Note that $\pi_1, \ldots, \pi_d$ and $\gamma_1, \ldots, \gamma_d$ remain regular sequences if we multiply our π_i by elements of S. Suppose that

$$\mathrm{Ann}_{R/(\gamma_1^e, \ldots, \gamma_{i-1}^e)R}(\gamma_i) = \mathrm{Ann}_{R/(\gamma_1^e, \ldots, \gamma_{i-1}^e)R}(\gamma_i^2)$$

holds for $i = 1, \ldots, t$ for some $t \in \{0, \ldots, d\}$. Note that $\gamma_1^e, \ldots, \gamma_t^e, \gamma_{t+1}$ is a regular sequence in $S^{-1}R$ by Algebra, Lemma 10.65.10. Hence we see that

$$\mathrm{Ann}_{S^{-1}R/(\gamma_1^e, \ldots, \gamma_{i-1}^e)}(\gamma_i) = \mathrm{Ann}_{S^{-1}R/(\gamma_1^e, \ldots, \gamma_{i-1}^e)}(\gamma_i^2).$$

Thus we get

$$\mathrm{Ann}_{R/(\gamma_1^e, \ldots, \gamma_t^e)R}(\gamma_{t+1}) = \mathrm{Ann}_{R/(\gamma_1^e, \ldots, \gamma_t^e)R}(\gamma_{t+1}^2)$$

after replacing π_{t+1} by $s\pi_{t+1}$ for some $s \in S$ by Lemma 16.10.1. By induction on t this produces the desired sequence.

Ad (13). Let $S = \Lambda \setminus \mathfrak{q}$ so that $\Lambda_{\mathfrak{q}} = S^{-1}\Lambda$. Set $\bar{\Lambda} = \Lambda_{\mathfrak{q}}/\mathfrak{q}^n\Lambda_{\mathfrak{q}}$. Suppose that we have a $t \in \{0, \ldots, d\}$ and $\delta_1, \ldots, \delta_t \in S$ and a factorization $D \to D' \to \bar{\Lambda}$ as in (13) such that (a), (b), (c) hold for $i = 1, \ldots, t$. We have $\pi_{t+1}^N \in H_{A/k}\Lambda_{\mathfrak{q}}$ as $\mathfrak{q}^N\Lambda_{\mathfrak{q}} \subset H_{A/k}\Lambda_{\mathfrak{q}}$ by (1). Hence $\pi_{t+1}^N \in H_{A/k}\bar{\Lambda}$. Hence $\pi_{t+1}^N \in H_{A/k}D'$ as $D' \to \bar{\Lambda}$ is faithfully flat, see Algebra, Lemma 10.77.11. Recall that $H_{A/k} = (a_1, \ldots, a_t)$. Say $\pi_{t+1}^N = \sum a_j d_j$ in D' and choose $c_j \in \Lambda_{\mathfrak{q}}$ lifting $d_j \in D'$. Then $\pi_{t+1}^N = \sum c_j a_j + \epsilon$ with $\epsilon \in \mathfrak{q}^n\Lambda_{\mathfrak{q}} \subset \mathfrak{q}^{n-N}H_{A/k}\Lambda_{\mathfrak{q}}$. Write $\epsilon = \sum a_j c_j'$ for some $c_j' \in \mathfrak{q}^{n-N}\Lambda_{\mathfrak{q}}$. Hence $\pi_{t+1}^{2N} = \sum(\pi_{t+1}^N c_j + \pi_{t+1}^N c_j')a_j$. Note that $\pi_{t+1}^N c_j'$ maps to zero in $\bar{\Lambda}$; this trivial but key observation will ensure later that (a) holds. Now we choose $s \in S$ such that there exist $\mu_{t+1j} \in \Lambda$ such that on the one hand $\pi_{t+1}^N c_j + \pi_{t+1}^N c_j' = \mu_{t+1j}/s^{2N}$ in $S^{-1}\Lambda$ and on the other $(s\pi_{t+1})^{2N} = \sum \mu_{t+1j} a_j$ in Λ (minor detail omitted). We may further replace s by a power and enlarge D' such that s maps to an element of D'. With these choices μ_{t+1j} maps to $s^{2N}d_j$ which is an element of D'. Note that $\pi_1, \ldots, \pi_d$ are a regular sequence of parameters in $S^{-1}\Lambda$ by our choice of φ. Hence $\pi_1, \ldots, \pi_d$ forms a regular sequence in $\Lambda_{\mathfrak{q}}$ by Algebra, Lemma 10.101.3. It follows that ${\pi'}_1^e, \ldots, {\pi'}_t^e, s\pi_{t+1}$ is a regular sequence in $S^{-1}\Lambda$ by Algebra, Lemma 10.65.10. Thus we get

$$\mathrm{Ann}_{S^{-1}\Lambda/({\pi'}_1^e, \ldots, {\pi'}_t^e)}(s\pi_{t+1}) = \mathrm{Ann}_{S^{-1}\Lambda/({\pi'}_1^e, \ldots, {\pi'}_t^e)}((s\pi_{t+1})^2).$$

Hence we may apply Lemma 16.10.1 to find an $s' \in S$ such that

$$\mathrm{Ann}_{\Lambda/({\pi'}_1^e, \ldots, {\pi'}_t^e)}((s')^q s\pi_{t+1}) = \mathrm{Ann}_{\Lambda/({\pi'}_1^e, \ldots, {\pi'}_t^e)}(((s')^q s\pi_{t+1})^2).$$

for any $q > 0$. By Lemma 16.11.3 we can choose q and enlarge D' such that $(s')^q$ maps to an element of D'. Setting $\delta_{t+1} = (s')^q s$ and we conclude that (a), (b), (c) hold for $i = 1, \ldots, t+1$. For (a) note that $\lambda_{t+1j} = (s')^{2Nq}\mu_{t+1j}$ works. By induction on t we win.

Ad (16). By construction the radical of $H_{(C \otimes_{k[x_1, \ldots, x_d]} R)/R}\Lambda$ contains $\mathfrak{h}_A$. Namely, the elements $a_j \in H_{A/k}$ map to elements of $H_{B/k[x_1, \ldots, x_n]}$, hence map to elements

of $H_{C/k[x_1,\ldots,x_n]}$, hence $a_j \otimes 1$ map to elements of $H_{C\otimes_{k[x_1,\ldots,x_d]}R/R}$. Moreover, if we have a solution $C \otimes_{k[x_1,\ldots,x_n]} R \to T \to \Lambda$ of

$$R \to C \otimes_{k[x_1,\ldots,x_d]} R \to \Lambda \supset \mathfrak{q}$$

then $H_{T/R} \subset H_{T/k}$ as R is smooth over k. Hence T will also be a solution for the original situation $k \to A \to \Lambda \supset \mathfrak{q}$.

Ad (18). Follows on applying Lemma 16.9.2 to $R \to C \otimes_{k[x_1,\ldots,x_d]} R \to \Lambda \supset \mathfrak{q}$ and the sequence of elements $\gamma_1^c, \ldots, \gamma_d^c$. We note that since x_i^c are strictly standard in C over $k[x_1, \ldots, x_d]$ the elements γ_i^c are strictly standard in $C \otimes_{k[x_1,\ldots,x_d]} R$ over R by Lemma 16.3.7. The other assumption of Lemma 16.9.2 holds by steps (12) and (13).

Ad (20). Apply Lemma 16.9.4 to the situation in (18). In the rest of the arguments the target ring is local Artinian, hence we are looking for a factorization by a smooth algebra T over the source ring.

Ad (22). Suppose that $C \otimes_{k[x_1,\ldots,x_d]} (R/JR)_\mathfrak{p} \to T \to \Lambda_\mathfrak{q}/J\Lambda_\mathfrak{q}$ is a solution to

$$(R/JR)_\mathfrak{p} \to C \otimes_{k[x_1,\ldots,x_d]} (R/JR)_\mathfrak{p} \to \Lambda_\mathfrak{q}/J\Lambda_\mathfrak{q} \supset \mathfrak{q}\Lambda_\mathfrak{q}/J\Lambda_\mathfrak{q}$$

Then $C \otimes_{k[x_1,\ldots,x_d]} (R/I)_\mathfrak{r} \to T_\mathfrak{r} \to \Lambda_\mathfrak{q}/I\Lambda_\mathfrak{q}$ is a solution to the situation in (20).

Ad (23). Our $n = N + dc$ is large enough so that $\mathfrak{p}^n k[y_1, \ldots, y_m]_\mathfrak{p} \subset J_\mathfrak{p}$ and $\mathfrak{q}^n\Lambda_\mathfrak{q} \subset J\Lambda_\mathfrak{q}$. Hence if we have a solution $C \otimes_{k[x_1,\ldots,x_d]} (R/\mathfrak{p}^n R)_\mathfrak{p} \to T \to \Lambda_\mathfrak{q}/\mathfrak{q}^n\Lambda_\mathfrak{q}$ of (22 then we can take T/JT as the solution for (23).

Ad (24). This is true because we have a section $C \to B$ in the category of R-algebras.

Ad (25). This is true because D' is essentially smooth over the local Artinian ring $k[y_1, \ldots, y_m]_\mathfrak{p}/\mathfrak{p}^n k[y_1, \ldots, y_m]_\mathfrak{p}$ and

$$R_\mathfrak{p}/\mathfrak{p}^n R_\mathfrak{p} = k[y_1, \ldots, y_m]_\mathfrak{p}/\mathfrak{p}^n k[y_1, \ldots, y_m]_\mathfrak{p}[t_1, \ldots, t_d].$$

Hence $D'[t_1, \ldots, t_d]$ is a filtered colimit of smooth $R_\mathfrak{p}/\mathfrak{p}^n R_\mathfrak{p}$-algebras and $B \otimes_{k[x_1,\ldots,x_d]} (R_\mathfrak{p}/\mathfrak{p}^n R_\mathfrak{p})$ factors through one of these.

Ad (26). The final twist of the proof is that we cannot just use the map $B \to D'$ which maps x_i to the image of π_i' in D' and z_{ij} to the image of λ_{ij} in D' because we need the diagram

$$\begin{array}{ccc} B & \longrightarrow & D'[t_1, \ldots, t_d] \\ \uparrow & & \uparrow \\ k[x_1, \ldots, x_d] & \longrightarrow & R_\mathfrak{p}/\mathfrak{p}^n R_\mathfrak{p} \end{array}$$

to commute and we need the composition $B \to D'[t_1, \ldots, t_d] \to \Lambda_\mathfrak{q}/\mathfrak{q}^n\Lambda_\mathfrak{q}$ to be the map of (14). This requires us to map x_i to the image of $\pi_i t_i$ in $D'[t_1, \ldots, t_d]$. Hence we map z_{ij} to the image of $\lambda_{ij} t_i^{2N}/\delta_i^{2N}$ in $D'[t_1, \ldots, t_d]$ and everything is clear. $\square$

16.12. The main theorem

In this section we wrap up the discussion.

Theorem 16.12.1 (Popescu)**.** *Any regular homomorphism of Noetherian rings is a filtered colimit of smooth ring maps.*

Proof. By Lemma 16.8.4 it suffices to prove this for $k \to \Lambda$ where Λ is Noetherian and geometrically regular over k. Let $k \to A \to \Lambda$ be a factorization with A a finite type k-algebra. It suffices to construct a factorization $A \to B \to \Lambda$ with B of finite type such that $\mathfrak{h}_B = \Lambda$, see Lemma 16.3.8. Hence we may perform Noetherian induction on the ideal $\mathfrak{h}_A$. Pick a prime $\mathfrak{q} \supset \mathfrak{h}_A$ such that $\mathfrak{q}$ is minimal over $\mathfrak{h}_A$. It now suffices to resolve $k \to A \to \Lambda \supset \mathfrak{q}$ (as defined in the text following Situation 16.9.1). If the characteristic of k is zero, this follows from Lemma 16.10.3. If the characteristic of k is $p > 0$, this follows from Lemma 16.11.4. □

16.13. The approximation property for G-rings

Let R be a Noetherian local ring. In this case R is a G-ring if and only if the ring map $R \to R^\wedge$ is regular, see More on Algebra, Lemma 15.37.7. In this case it is true that the henselization R^h and the strict henselization R^{sh} of R are G-rings, see More on Algebra, Lemma 15.37.8. Moreover, any algebra essentially of finite type over a field, over a complete local ring, over $\mathbf{Z}$, or over a characteristic zero Dedekind ring is a G-ring, see More on Algebra, Proposition 15.37.12. This gives an ample supply of rings to which the result below applies.

Let R be a ring. Let $f_1, \ldots, f_m \in R[x_1, \ldots, x_n]$. Let S be an R-algebra. In this situation we say a vector $(a_1, \ldots, a_n) \in S^n$ is a *solution in* S if and only if

$$f_j(a_1, \ldots, a_n) = 0 \text{ in } S, \text{ for } j = 1, \ldots, m$$

Of course an important question in algebraic geometry is to see when systems of polynomial equations have solutions. The following theorem tells us that having solutions in the completion of a local Noetherian ring is often enough to show there exist solutions in the henselization of the ring.

Theorem 16.13.1. *Let R be a Noetherian local ring. Let $f_1, \ldots, f_m \in R[x_1, \ldots, x_n]$. Suppose that $(a_1, \ldots, a_n) \in (R^\wedge)^n$ is a solution in $R^\wedge$. If R is a henselian G-ring, then for every integer N there exists a solution $(b_1, \ldots, b_n) \in R^n$ in R such that $a_i - b_i \in \mathfrak{m}^N R^\wedge$.*

Proof. Let $c_i \in R$ be an element such that $a_i - c_i \in \mathfrak{m}^N$. Choose generators $\mathfrak{m}^N = (d_1, \ldots, d_M)$. Write $a_i = c_i + \sum a_{i,l} d_l$. Consider the polynomial ring $R[x_{i,l}]$ and the elements

$$g_j = f_j(c_1 + \sum x_{1,l} d_l, \ldots, c_n + \sum x_{n,l} d_{n,l}) \in R[x_{i,l}]$$

The system of equations $g_j = 0$ has the solution $(a_{i,l})$. Suppose that we can show that g_j as a solution $(b_{i,l})$ in R. Then it follows that $b_i = c_i + \sum b_{i,l} d_l$ is a solution of $f_j = 0$ which is congruent to a_i modulo $\mathfrak{m}^N$. Thus it suffices to show that solvability over $R^\wedge$ implies solvability over R.

Let $A \subset R^\wedge$ be the R-subalgebra generated by $a_1, \ldots, a_n$. Since we've assumed R is a G-ring, i.e., that $R \to R^\wedge$ is regular, we see that there exists a factorization

$$A \to B \to R^\wedge$$

with B smooth over R, see Theorem 16.12.1. Denote $\kappa = R/\mathfrak{m}$ the residue field. It is also the residue field of $R^\wedge$, so we get a commutative diagram

$$\begin{array}{ccc} B & \dashrightarrow & R' \\ \uparrow & \searrow & \downarrow \\ R & \longrightarrow & \kappa \end{array}$$

Since the vertical arrow is smooth, More on Algebra, Lemma 15.6.12 implies that there exists an étale ring map $R \to R'$ which induces an isomorphism $R/\mathfrak{m} \to R'/\mathfrak{m}R'$ and an R-algebra map $B \to R'$ making the diagram above commute. Since R is henselian we see that $R \to R'$ has a section, see Algebra, Lemma 10.143.3. Let $b_i \in R$ be the image of a_i under the ring maps $A \to B \to R' \to R$. Since all of these maps are R-algebra maps, we see that $(b_1, \ldots, b_n)$ is a solution in R. □

Given a Noetherian local ring $(R, \mathfrak{m})$, an étale ring map $R \to R'$, and a maximal ideal $\mathfrak{m}' \subset R'$ lying over $\mathfrak{m}$ with $\kappa(\mathfrak{m}) = \kappa(\mathfrak{m}')$, then we have inclusions

$$R \subset R_{\mathfrak{m}'} \subset R^h \subset R^\wedge,$$

by Algebra, Lemma 10.143.19 and More on Algebra, Lemma 15.32.3.

Theorem 16.13.2. *Let R be a Noetherian local ring. Let $f_1, \ldots, f_m \in R[x_1, \ldots, x_n]$. Suppose that $(a_1, \ldots, a_n) \in (R^\wedge)^n$ is a solution. If R is a G-ring, then for every integer N there exist*

1. *an étale ring map $R \to R'$,*
2. *a maximal ideal $\mathfrak{m}' \subset R'$ lying over $\mathfrak{m}$*
3. *a solution $(b_1, \ldots, b_n) \in (R')^n$ in R'*

such that $\kappa(\mathfrak{m}) = \kappa(\mathfrak{m}')$ and $a_i - b_i \in (\mathfrak{m}')^N R^\wedge$.

Proof. We could deduce this theorem from Theorem 16.13.1 using that the henselization R^h is a G-ring by More on Algebra, Lemma 15.37.8 and writing R^h as a directed colimit of étale extension R'. Instead we prove this by redoing the proof of the previous theorem in this case.

Let $c_i \in R$ be an element such that $a_i - c_i \in \mathfrak{m}^N$. Choose generators $\mathfrak{m}^N = (d_1, \ldots, d_M)$. Write $a_i = c_i + \sum a_{i,l} d_l$. Consider the polynomial ring $R[x_{i,l}]$ and the elements

$$g_j = f_j(c_1 + \sum x_{1,l} d_l, \ldots, c_n + \sum x_{n,l} d_{n,l}) \in R[x_{i,l}]$$

The system of equations $g_j = 0$ has the solution $(a_{i,l})$. Suppose that we can show that g_j as a solution $(b_{i,l})$ in R' for some étale ring map $R \to R'$ endowed with a maximal ideal $\mathfrak{m}'$ such that $\kappa(\mathfrak{m}) = \kappa(\mathfrak{m}')$. Then it follows that $b_i = c_i + \sum b_{i,l} d_l$ is a solution of $f_j = 0$ which is congruent to a_i modulo $(\mathfrak{m}')^N$. Thus it suffices to show that solvability over $R^\wedge$ implies solvability over some étale ring extension which induces a trivial residue field extension at some prime over $\mathfrak{m}$.

Let $A \subset R^\wedge$ be the R-subalgebra generated by $a_1, \ldots, a_n$. Since we've assumed R is a G-ring, i.e., that $R \to R^\wedge$ is regular, we see that there exists a factorization

$$A \to B \to R^\wedge$$

with B smooth over R, see Theorem 16.12.1. Denote $\kappa = R/\mathfrak{m}$ the residue field. It is also the residue field of $R^\wedge$, so we get a commutative diagram

$$\begin{array}{ccc} B & \dashrightarrow & R' \\ \uparrow & \searrow & \downarrow \\ R & \longrightarrow & \kappa \end{array}$$

Since the vertical arrow is smooth, More on Algebra, Lemma 15.6.12 implies that there exists an étale ring map $R \to R'$ which induces an isomorphism $R/\mathfrak{m} \to R'/\mathfrak{m}R'$ and an R-algebra map $B \to R'$ making the diagram above commute. Let $b_i \in R'$ be the image of a_i under the ring maps $A \to B \to R'$. Since all of these maps are R-algebra maps, we see that $(b_1, \ldots, b_n)$ is a solution in R'. □

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CHAPTER 17

Sheaves of Modules

17.1. Introduction

In this chapter we work out basic notions of sheaves of modules. This in particular includes the case of abelian sheaves, since these may be viewed as sheaves of $\underline{\mathbf{Z}}$-modules. Basic references are [**Ser55b**], [**DG67**] and [**AGV71**].

We work out what happens for sheaves of modules on ringed topoi in another chapter (see Modules on Sites, Section 18.1), although there we will mostly just duplicate the discussion from this chapter.

17.2. Pathology

A ringed space is a pair consisting of a topological space X and a sheaf of rings $\mathcal{O}$. We allow $\mathcal{O} = 0$ in the definition. In this case the category of modules has a single object (namely 0). It is still an abelian category etc, but it is a little degenerate. Similarly the sheaf $\mathcal{O}$ may be zero over open subsets of X, etc.

This doesn't happen when considering locally ringed spaces (as we will do later).

17.3. The abelian category of sheaves of modules

Let $(X, \mathcal{O}_X)$ be a ringed space, see Sheaves, Definition 6.25.1. Let $\mathcal{F}$, $\mathcal{G}$ be sheaves of $\mathcal{O}_X$-modules, see Sheaves, Definition 6.10.1. Let $\varphi, \psi : \mathcal{F} \to \mathcal{G}$ be morphisms of sheaves of $\mathcal{O}_X$-modules. We define $\varphi + \psi : \mathcal{F} \to \mathcal{G}$ to be the map which on each open $U \subset X$ is the sum of the maps induced by φ, ψ. This is clearly again a map of sheaves of $\mathcal{O}_X$-modules. It is also clear that composition of maps of $\mathcal{O}_X$-modules is bilinear with respect to this addition. Thus $\mathit{Mod}(\mathcal{O}_X)$ is a pre-additive category, see Homology, Definition 12.3.1.

We will denote 0 the sheaf of $\mathcal{O}_X$-modules which has constant value $\{0\}$ for all open $U \subset X$. Clearly this is both a final and an initial object of $\mathit{Mod}(\mathcal{O}_X)$. Given a morphism of $\mathcal{O}_X$-modules $\varphi : \mathcal{F} \to \mathcal{G}$ the following are equivalent: (a) φ is zero, (b) φ factors through 0, (c) φ is zero on sections over each open U, and (d) $\varphi_x = 0$ for all $x \in X$. See Sheaves, Lemma 6.16.1.

Moreover, given a pair $\mathcal{F}$, $\mathcal{G}$ of sheaves of $\mathcal{O}_X$-modules we may define the direct sum as

$$\mathcal{F} \oplus \mathcal{G} = \mathcal{F} \times \mathcal{G}$$

with obvious maps (i, j, p, q) as in Homology, Definition 12.3.5. Thus $\mathit{Mod}(\mathcal{O}_X)$ is an additive category, see Homology, Definition 12.3.8.

Let $\varphi : \mathcal{F} \to \mathcal{G}$ be a morphism of $\mathcal{O}_X$-modules. We may define $\mathrm{Ker}(\varphi)$ to be the subsheaf of $\mathcal{F}$ with sections

$$\mathrm{Ker}(\varphi)(U) = \{s \in \mathcal{F}(U) \mid \varphi(s) = 0 \text{ in } \mathcal{G}(U)\}$$

for all open $U \subset X$. It is easy to see that this is indeed a kernel in the category of $\mathcal{O}_X$-modules. In other words, a morphism $\alpha : \mathcal{H} \to \mathcal{F}$ factors through $\text{Ker}(\varphi)$ if and only if $\varphi \circ \alpha = 0$. Moreover, on the level of stalks we have $\text{Ker}(\varphi)_x = \text{Ker}(\varphi_x)$.

On the other hand, we define $\text{Coker}(\varphi)$ as the sheaf of $\mathcal{O}_X$-modules associated to the presheaf of $\mathcal{O}_X$-modules defined by the rule

$$U \longmapsto \text{Coker}(\mathcal{G}(U) \to \mathcal{F}(U)) = \mathcal{F}(U)/\varphi(\mathcal{G}(U)).$$

Since taking stalks commutes with taking sheafification, see Sheaves, Lemma 6.17.2 we see that $\text{Coker}(\varphi)_x = \text{Coker}(\varphi_x)$. Thus the map $\mathcal{G} \to \text{Coker}(\varphi)$ is surjective (as a map of sheaves of sets), see Sheaves, Section 6.16. To show that this is a cokernel, note that if $\beta : \mathcal{G} \to \mathcal{H}$ is a morphism of $\mathcal{O}_X$-modules such that $\beta \circ \varphi$ is zero, then you get for every open $U \subset X$ a map induced by β from $\mathcal{G}(U)/\varphi(\mathcal{F}(U))$ into $\mathcal{H}(U)$. By the universal property of sheafification (see Sheaves, Lemma 6.20.1) we obtain a canonical map $\text{Coker}(\varphi) \to \mathcal{H}$ such that the original β is equal to the composition $\mathcal{G} \to \text{Coker}(\varphi) \to \mathcal{H}$. The morphism $\text{Coker}(\varphi) \to \mathcal{H}$ is unique because of the surjectivity mentioned above.

Lemma 17.3.1. *Let $(X, \mathcal{O}_X)$ be a ringed space. The category $Mod(\mathcal{O}_X)$ is an abelian category. Moreover a complex*

$$\mathcal{F} \to \mathcal{G} \to \mathcal{H}$$

is exact at $\mathcal{G}$ if and only if for all $x \in X$ the complex

$$\mathcal{F}_x \to \mathcal{G}_x \to \mathcal{H}_x$$

is exact at $\mathcal{G}_x$.

Proof. By Homology, Definition 12.5.1 we have to show that image and coimage agree. By Sheaves, Lemma 6.16.1 it is enough to show that image and coimage have the same stalk at every $x \in X$. By the constructions of kernels and cokernels above these stalks are the coimage and image in the categories of $\mathcal{O}_{X,x}$-modules. Thus we get the result from the fact that the category of modules over a ring is abelian. □

Actually the category $Mod(\mathcal{O}_X)$ has many more properties. Here are two constructions we can do.

(1) Given any set I and for each $i \in I$ a $\mathcal{O}_X$-module we can form the product

$$\prod_{i \in I} \mathcal{F}_i$$

which is the sheaf that associates to each open U the product of the modules $\mathcal{F}_i(U)$. This is also the categorical product, as in Categories, Definition 4.14.5.

(2) Given any set I and for each $i \in I$ a $\mathcal{O}_X$-module we can form the direct sum

$$\bigoplus_{i \in I} \mathcal{F}_i$$

which is the *sheafification* of the presheaf that associates to each open U the direct sum of the modules $\mathcal{F}_i(U)$. This is also the categorical coproduct, as in Categories, Definition 4.14.6. To see this you use the universal property of sheafification.

Using these we conclude that all limits and colimits exist in $Mod(\mathcal{O}_X)$.

Lemma 17.3.2. *Let $(X, \mathcal{O}_X)$ be a ringed space.*

(1) *All limits exist in $Mod(\mathcal{O}_X)$. Limits are the same as the corresponding limits of presheaves of $\mathcal{O}_X$-modules (i.e., commute with taking sections over opens).*
(2) *All colimits exist in $Mod(\mathcal{O}_X)$. Colimits are the sheafification of the corresponding colimit in the category of presheaves. Taking colimits commutes with taking stalks.*
(3) *Filtered colimits are exact.*
(4) *Finite direct sums are the same as the corresponding finite direct sums of presheaves of $\mathcal{O}_X$-modules.*

Proof. As $Mod(\mathcal{O}_X)$ is abelian (Lemma 17.3.1) it has all finite limits and colimits (Homology, Lemma 12.5.5). Thus the existence of limits and colimits and their description follows from the existence of products and coproducts and their description (see discussion above) and Categories, Lemmas 4.14.10 and 4.14.11. Since sheafification commutes with taking stalks we see that colimits commute with taking stalks. Part (3) signifies that given a system $0 \to \mathcal{F}_i \to \mathcal{G}_i \to \mathcal{H}_i \to 0$ of exact sequences of $\mathcal{O}_X$-modules over a directed partially ordered set I the sequence $0 \to \operatorname{colim} \mathcal{F}_i \to \operatorname{colim} \mathcal{G}_i \to \operatorname{colim} \mathcal{H}_i \to 0$ is exact as well. Since we can check exactness on stalks (Lemma 17.3.1) this follows from the case of modules which is Algebra, Lemma 10.8.9. We omit the proof of (4). □

The existence of limits and colimits allows us to consider exactness properties of functors defined on the category of $\mathcal{O}$-modules in terms of limits and colimits, as in Categories, Section 4.23. See Homology, Lemma 12.7.1 for a description of exactness properties in terms of short exact sequences.

Lemma 17.3.3. *Let $f : (X, \mathcal{O}_X) \to (Y, \mathcal{O}_Y)$ be a morphism of ringed spaces.*

(1) *The functor $f_* : Mod(\mathcal{O}_X) \to Mod(\mathcal{O}_Y)$ is left exact. In fact it commutes with all limits.*
(2) *The functor $f^* : Mod(\mathcal{O}_Y) \to Mod(\mathcal{O}_X)$ is right exact. In fact it commutes with all colimits.*
(3) *Pullback $f^{-1} : Ab(Y) \to Ab(X)$ on abelian sheaves is exact.*

Proof. Parts (1) and (2) hold because (f^*, f_*) is an adjoint pair of functors, see Sheaves, Lemma 6.26.2 and Categories, Section 4.24. Part (3) holds because exactness can be checked on stalks (Lemma 17.3.1) and the description of stalks of the pullback, see Sheaves, Lemma 6.22.1. □

Lemma 17.3.4. *Let $j : U \to X$ be an open immersion of topological spaces. The functor $j_! : Ab(U) \to Ab(X)$ is exact.*

Proof. Follows from the description of stalks given in Sheaves, Lemma 6.31.6. □

Lemma 17.3.5. *Let $(X, \mathcal{O}_X)$ be a ringed space. Let I be a set. For $i \in I$, let $\mathcal{F}_i$ be a sheaf of $\mathcal{O}_X$-modules. For $U \subset X$ quasi-compact open the map*

$$\bigoplus\nolimits_{i \in I} \mathcal{F}_i(U) \longrightarrow \left(\bigoplus\nolimits_{i \in I} \mathcal{F}_i\right)(U)$$

is bijective.

Proof. If s is an element of the right hand side, then there exists an open covering $U = \bigcup_{j \in J} U_j$ such that $s|_{U_j}$ is a finite sum $\sum_{i \in I_j} s_{ji}$ with $s_{ji} \in \mathcal{F}_i(U_j)$. Because

U is quasi-compact we may assume that the covering is finite, i.e., that J is finite. Then $I' = \bigcup_{j \in J} I_j$ is a finite subset of I. Clearly, s is a section of the subsheaf $\bigoplus_{i \in I'} \mathcal{F}_i$. The result follows from the fact that for a finite direct sum sheafification is not needed, see Lemma 17.3.2 above. □

17.4. Sections of sheaves of modules

Let $(X, \mathcal{O}_X)$ be a ringed space. Let $\mathcal{F}$ be a sheaf of $\mathcal{O}_X$-modules. Let $s \in \Gamma(X, \mathcal{F}) = \mathcal{F}(X)$ be a global section. There is a unique *map of $\mathcal{O}_X$-modules*

$$\mathcal{O}_X \longrightarrow \mathcal{F}, \ f \longmapsto fs$$

associated to s. The notation above signifies that a local section f of $\mathcal{O}_X$, i.e., a section f over some open U, is mapped to the multiplication of f with the restriction of s to U. Conversely, any map $\varphi : \mathcal{O}_X \to \mathcal{F}$ gives rise to a section $s = \varphi(1)$ such that φ is the morphism associated to s.

Definition 17.4.1. Let $(X, \mathcal{O}_X)$ be a ringed space. Let $\mathcal{F}$ be a sheaf of $\mathcal{O}_X$-modules. We say that $\mathcal{F}$ is *generated by global sections* if there exist a set I, and global sections $s_i \in \Gamma(X, \mathcal{F})$, $i \in I$ such that the map

$$\bigoplus\nolimits_{i \in I} \mathcal{O}_X \longrightarrow \mathcal{F}$$

which is the map associated to s_i on the summand corresponding to i, is surjective. In this case we say that the sections s_i *generate* $\mathcal{F}$.

We often use the abuse of notation introduced in Sheaves, Section 6.11 where, given a local section s of $\mathcal{F}$ defined in an open neighbourhood of a point $x \in X$, we denote s_x, or even s the image of s in the stalk $\mathcal{F}_x$.

Lemma 17.4.2. *Let $(X, \mathcal{O}_X)$ be a ringed space. Let $\mathcal{F}$ be a sheaf of $\mathcal{O}_X$-modules. Let I be a set. Let $s_i \in \Gamma(X, \mathcal{F})$, $i \in I$ be global sections. The sections s_i generate $\mathcal{F}$ if and only if for all $x \in X$ the elements $s_{i,x} \in \mathcal{F}_x$ generate the $\mathcal{O}_{X,x}$-module $\mathcal{F}_x$.*

Proof. Omitted. □

Lemma 17.4.3. *Let $(X, \mathcal{O}_X)$ be a ringed space. Let $\mathcal{F}$, $\mathcal{G}$ be sheaves of $\mathcal{O}_X$-modules. If $\mathcal{F}$ and $\mathcal{G}$ are generated by global sections then so is $\mathcal{F} \otimes_{\mathcal{O}_X} \mathcal{G}$.*

Proof. Omitted. □

Lemma 17.4.4. *Let $(X, \mathcal{O}_X)$ be a ringed space. Let $\mathcal{F}$ be a sheaf of $\mathcal{O}_X$-modules. Let I be a set. Let s_i, $i \in I$ be a collection of local sections of $\mathcal{F}$, i.e., $s_i \in \mathcal{F}(U_i)$ for some opens $U_i \subset X$. There exists a unique smallest subsheaf of $\mathcal{O}_X$-modules $\mathcal{G}$ such that each s_i corresponds to a local section of $\mathcal{G}$.*

Proof. Consider the subpresheaf of $\mathcal{O}_X$-modules defined by the rule

$$U \longmapsto \{\text{sums } \sum\nolimits_{i \in J} f_i(s_i|_U) \text{ where } J \text{ is finite, } U \subset U_i \text{ for } i \in J, \text{ and } f_i \in \mathcal{O}_X(U)\}$$

Let $\mathcal{G}$ be the sheafification of this subpresheaf. This is a subsheaf of $\mathcal{F}$ by Sheaves, Lemma 6.16.3. Since all the finite sums clearly have to be in $\mathcal{G}$ this is the smallest subsheaf as desired. □

Definition 17.4.5. Let $(X, \mathcal{O}_X)$ be a ringed space. Let $\mathcal{F}$ be a sheaf of $\mathcal{O}_X$-modules. Given a set I, and local sections s_i, $i \in I$ of $\mathcal{F}$ we say that the subsheaf $\mathcal{G}$ of Lemma 17.4.4 above is the *subsheaf generated by the s_i*.

Lemma 17.4.6. *Let $(X, \mathcal{O}_X)$ be a ringed space. Let $\mathcal{F}$ be a sheaf of $\mathcal{O}_X$-modules. Given a set I, and local sections s_i, $i \in I$ of $\mathcal{F}$. Let $\mathcal{G}$ be the subsheaf generated by the s_i and let $x \in X$. Then $\mathcal{G}_x$ is the $\mathcal{O}_{X,x}$-submodule of $\mathcal{F}_x$ generated by the elements $s_{i,x}$ for those i such that s_i is defined at x.*

Proof. This is clear from the construction of $\mathcal{G}$ in the proof of Lemma 17.4.4. □

17.5. Supports of modules and sections

Definition 17.5.1. Let $(X, \mathcal{O}_X)$ be a ringed space. Let $\mathcal{F}$ be a sheaf of $\mathcal{O}_X$-modules.

(1) The *support of* $\mathcal{F}$ is the set of points $x \in X$ such that $\mathcal{F}_x \neq 0$.
(2) We denote $\text{Supp}(\mathcal{F})$ the support of $\mathcal{F}$.
(3) Let $s \in \Gamma(X, \mathcal{F})$ be a global section. The *support of* s is the set of points $x \in X$ such that the image $s_x \in \mathcal{F}_x$ of s is not zero.

Of course the support of a local section is then defined also since a local section is a global section of the restriction of $\mathcal{F}$.

Lemma 17.5.2. *Let $(X, \mathcal{O}_X)$ be a ringed space. Let $\mathcal{F}$ be a sheaf of $\mathcal{O}_X$-modules. Let $U \subset X$ open.*

(1) *The support of $s \in \mathcal{F}(U)$ is closed in U.*
(2) *The support of fs is contained in the intersections of the supports of $f \in \mathcal{O}_X(U)$ and $s \in \mathcal{F}(U)$*
(3) *The support of $s + s'$ is contained in the union of the supports of $s, s' \in \mathcal{F}(U)$.*
(4) *The support of $\mathcal{F}$ is the union of the supports of all local sections of $\mathcal{F}$.*
(5) *If $\varphi : \mathcal{F} \to \mathcal{G}$ is a morphism of $\mathcal{O}_X$-modules, then the support of $\varphi(s)$ is contained in the support of $s \in \mathcal{F}(U)$.*

Proof. This is true because if $s_x = 0$, then s is zero in an open neighbourhood of x by definition of stalks. Similarly for f. Details omitted. □

In general the support of a sheaf of modules is not closed. Namely, the sheaf could be an abelian sheaf on $\mathbf{R}$ (with the usual archimedean topology) which is the direct sum of infinitely many nonzero skyscraper sheaves each supported at a single point p_i of $\mathbf{R}$. Then the support would be the set of points p_i which may not be closed.

Another example is to consider the open immersion $j : U = (0, \infty) \to \mathbf{R} = X$, and the abelian sheaf $j_!\underline{\mathbf{Z}}_U$. By Sheaves, Section 6.31 the support of this sheaf is exactly U.

Lemma 17.5.3. *Let X be a topological space. The support of a sheaf of rings is closed.*

Proof. This is true because (according to our conventions) a ring is 0 if and only if $1 = 0$, and hence the support of a sheaf of rings is the support of the unit section. □

17.6. Closed immersions and abelian sheaves

Recall that we think of an abelian sheaf on a topological space X as a sheaf of $\underline{\mathbf{Z}}_X$-modules. Thus we may apply any results, definitions for sheaves of modules to abelian sheaves.

Lemma 17.6.1. *Let X be a topological space. Let $Z \subset X$ be a closed subset. Denote $i : Z \to X$ the inclusion map. The functor*

$$i_* : Ab(Z) \longrightarrow Ab(X)$$

is exact, fully faithful, with essential image exactly those abelian sheaves whose support is contained in Z.

Proof. Exactness follows from the description of stalks in Sheaves, Lemma 6.32.1 and Lemma 17.3.1. The rest was shown in Sheaves, Lemma 6.31.10. □

Let $\mathcal{F}$ be a sheaf on X. There is a canonical subsheaf of $\mathcal{F}$ which consists of exactly those sections whose support is contained in Z. Here is the exact statement.

Lemma 17.6.2. *Let X be a topological space. Let $Z \subset X$ be a closed subset. Let $\mathcal{F}$ be a sheaf on X. For $U \subset X$ open set*

$$\Gamma(U, \mathcal{H}_Z(\mathcal{F})) = \{s \in \mathcal{F}(U) \mid \text{ the support of } s \text{ is contained in } Z \cap U\}$$

Then $\mathcal{H}_Z(\mathcal{F})$ is an abelian subsheaf of $\mathcal{F}$. It is the largest abelian subsheaf of $\mathcal{F}$ whose support is contained in Z. The construction $\mathcal{F} \mapsto \mathcal{H}_Z(\mathcal{F})$ is functorial in the abelian sheaf $\mathcal{F}$.

Proof. This follows from Lemma 17.5.2. □

This seems like a good opportunity to show that the functor i_* has a right adjoint on abelian sheaves.

Lemma 17.6.3. *Let $i : Z \to X$ be the inclusion of a closed subset into the topological space X. Denote[1] $i^! : Ab(X) \to Ab(Z)$ the functor $\mathcal{F} \mapsto i^{-1}\mathcal{H}_Z(\mathcal{F})$. Then $i^!$ is a right adjoint to i_*, in a formula*

$$\mathrm{Mor}_{Ab(X)}(i_*\mathcal{G}, \mathcal{F}) = \mathrm{Mor}_{Ab(Z)}(\mathcal{G}, i^!\mathcal{F}).$$

In particular i_* commutes with arbitrary colimits.

Proof. Note that $i_*i^!\mathcal{F} = \mathcal{H}_Z(\mathcal{F})$. Since i_* is fully faithful we are reduced to showing that

$$\mathrm{Mor}_{Ab(X)}(i_*\mathcal{G}, \mathcal{F}) = \mathrm{Mor}_{Ab(X)}(i_*\mathcal{G}, \mathcal{H}_Z(\mathcal{F})).$$

This follows since the support of the image via any homomorphism of a section of $i_*\mathcal{G}$ is supported on Z, see Lemma 17.5.2. □

Remark 17.6.4. In Sheaves, Remark 6.32.5 we showed that i_* as a functor on the categories of sheaves of sets does not have a right adjoint simply because it is not exact. However, it is very close to being true, in fact, the functor i_* is exact on sheaves of pointed sets, sections with support in Z can be defined for sheaves of pointed sets, and $i^!$ makes sense and is a right adjoint to i_*.

[1] This is likely nonstandard notation.

17.7. A canonical exact sequence

We give this exact sequence its own section.

Lemma 17.7.1. *Let X be a topological space. Let $U \subset X$ be an open subset with complement $Z \subset X$. Denote $j : U \to X$ the open immersion and $i : Z \to X$ the closed immersion. For any sheaf of abelian groups $\mathcal{F}$ on X the adjunction mappings $j_!j^*\mathcal{F} \to \mathcal{F}$ and $\mathcal{F} \to i_*i^*\mathcal{F}$ give a short exact sequence*

$$0 \to j_!j^*\mathcal{F} \to \mathcal{F} \to i_*i^*\mathcal{F} \to 0$$

of sheaves of abelian groups. For any morphism $\varphi : \mathcal{F} \to \mathcal{G}$ of abelian sheaves on X we obtain a morphism of short exact sequences

$$\begin{array}{ccccccccc} 0 & \longrightarrow & j_!j^*\mathcal{F} & \longrightarrow & \mathcal{F} & \longrightarrow & i_*i^*\mathcal{F} & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & j_!j^*\mathcal{G} & \longrightarrow & \mathcal{G} & \longrightarrow & i_*i^*\mathcal{G} & \longrightarrow & 0 \end{array}$$

Proof. We may check exactness on stalks (Lemma 17.3.1). For a description of the stalks in question see Sheaves, Lemmas 6.31.6 and 6.32.1. We omit the proof of the functorial behaviour of the exact sequence. □

17.8. Modules locally generated by sections

Let $(X, \mathcal{O}_X)$ be a ringed space. In this and the following section we will often restrict sheaves to open subspaces $U \subset X$, see Sheaves, Section 6.31. In particular, we will often denote the open subspace by $(U, \mathcal{O}_U)$ instead of the more correct notation $(U, \mathcal{O}_X|_U)$, see Sheaves, Definition 6.31.2.

Consider the open immersion $j : U = (0, \infty) \to \mathbf{R} = X$, and the abelian sheaf $j_!\underline{\mathbf{Z}}_U$. By Sheaves, Section 6.31 the stalk of $j_!\underline{\mathbf{Z}}_U$ at $x = 0$ is 0. In fact the sections of this sheaf over any open interval containing 0 are 0. Thus there is no open neighbourhood of the point 0 over which the sheaf can be generated by sections.

Definition 17.8.1. Let $(X, \mathcal{O}_X)$ be a ringed space. Let $\mathcal{F}$ be a sheaf of $\mathcal{O}_X$-modules. We say that $\mathcal{F}$ is *locally generated by sections* if for every $x \in X$ there exists an open neighbourhood U such that $\mathcal{F}|_U$ is globally generated as a sheaf of $\mathcal{O}_U$-modules.

In other words there exists a set I and for each i a section $s_i \in \mathcal{F}(U)$ such that the associated map

$$\bigoplus\nolimits_{i \in I} \mathcal{O}_U \longrightarrow \mathcal{F}|_U$$

is surjective.

Lemma 17.8.2. *Let $f : (X, \mathcal{O}_X) \to (Y, \mathcal{O}_Y)$ be a morphism of ringed spaces. The pullback $f^*\mathcal{G}$ is locally generated by sections if $\mathcal{G}$ is locally generated by sections.*

Proof. Given an open subspace V of Y we may consider the commutative diagram of ringed spaces

$$\begin{array}{ccc} (f^{-1}V, \mathcal{O}_{f^{-1}V}) & \xrightarrow[j']{} & (X, \mathcal{O}_X) \\ {\scriptstyle f'}\downarrow & & \downarrow{\scriptstyle f} \\ (V, \mathcal{O}_V) & \xrightarrow{j} & (Y, \mathcal{O}_Y) \end{array}$$

We know that $f^*\mathcal{G}|_{f^{-1}V} \cong (f')^*(\mathcal{G}|_V)$, see Sheaves, Lemma 6.26.3. Thus we may assume that $\mathcal{G}$ is globally generated.

We have seen that f^* commutes with all colimits, and is right exact, see Lemma 17.3.3. Thus if we have a surjection

$$\bigoplus\nolimits_{i\in I} \mathcal{O}_Y \to \mathcal{G} \to 0$$

then upon applying f^* we obtain the surjection

$$\bigoplus\nolimits_{i\in I} \mathcal{O}_X \to f^*\mathcal{G} \to 0.$$

This implies the lemma. □

17.9. Modules of finite type

Definition 17.9.1. Let $(X, \mathcal{O}_X)$ be a ringed space. Let $\mathcal{F}$ be a sheaf of $\mathcal{O}_X$-modules. We say that $\mathcal{F}$ is of *finite type* if for every $x \in X$ there exists an open neighbourhood U such that $\mathcal{F}|_U$ is generated by finitely many sections.

Lemma 17.9.2. *Let $f : (X, \mathcal{O}_X) \to (Y, \mathcal{O}_Y)$ be a morphism of ringed spaces. The pullback $f^*\mathcal{G}$ of a finite type $\mathcal{O}_Y$-module is a finite type $\mathcal{O}_X$-module.*

Proof. Arguing as in the proof of Lemma 17.8.2 we may assume $\mathcal{G}$ is globally generated by finitely many sections. We have seen that f^* commutes with all colimits, and is right exact, see Lemma 17.3.3. Thus if we have a surjection

$$\bigoplus\nolimits_{i=1,\ldots,n} \mathcal{O}_Y \to \mathcal{G} \to 0$$

then upon applying f^* we obtain the surjection

$$\bigoplus\nolimits_{i=1,\ldots,n} \mathcal{O}_X \to f^*\mathcal{G} \to 0.$$

This implies the lemma. □

Lemma 17.9.3. *Let X be a ringed space. The image of a morphism of $\mathcal{O}_X$-modules of finite type is of finite type. Let $0 \to \mathcal{F}_1 \to \mathcal{F}_2 \to \mathcal{F}_3 \to 0$ be a short exact sequence of $\mathcal{O}_X$-modules. If $\mathcal{F}_1$ and $\mathcal{F}_3$ are of finite type, so is $\mathcal{F}_2$.*

Proof. The statement on images is trivial. The statement on short exact sequences comes from the fact that sections of $\mathcal{F}_3$ locally lift to sections of $\mathcal{F}_2$ and the corresponding result in the category of modules over a ring (applied to the stalks for example). □

Lemma 17.9.4. *Let X be a ringed space. Let $\varphi : \mathcal{G} \to \mathcal{F}$ be a homomorphism of $\mathcal{O}_X$-modules. Let $x \in X$. Assume $\mathcal{F}$ of finite type and the map on stalks $\varphi_x : \mathcal{G}_x \to \mathcal{F}_x$ surjective. Then there exists an open neighbourhood $x \in U \subset X$ such that $\varphi|_U$ is surjective.*

Proof. Choose an open neighbourhood $U \subset X$ such that $\mathcal{F}$ is generated by $s_1, \ldots, s_n \in \mathcal{F}(U)$ over U. By assumption of surjectivity of φ_x, after shrinking V we may assume that $s_i = \varphi(t_i)$ for some $t_i \in \mathcal{G}(U)$. Then U works. □

Lemma 17.9.5. *Let X be a ringed space. Let $\mathcal{F}$ be an $\mathcal{O}_X$-module. Let $x \in X$. Assume $\mathcal{F}$ of finite type and $\mathcal{F}_x = 0$. Then there exists an open neighbourhood $x \in U \subset X$ such that $\mathcal{F}|_U$ is zero.*

Proof. This is a special case of Lemma 17.9.4 applied to the morphism $0 \to \mathcal{F}$. □

Lemma 17.9.6. *Let $(X, \mathcal{O}_X)$ be a ringed space. Let $\mathcal{F}$ be a sheaf of $\mathcal{O}_X$-modules. If $\mathcal{F}$ is of finite type then support of $\mathcal{F}$ is closed.*

Proof. This is a reformulation of Lemma 17.9.5. □

Lemma 17.9.7. *Let X be a ringed space. Let I be a partially ordered set and let $(\mathcal{F}_i, f_{ii'})$ be a system over I consisting of sheaves of $\mathcal{O}_X$-modules (see Categories, Section 4.21). Let $\mathcal{F} = \operatorname{colim} \mathcal{F}_i$ be the colimit. Assume (a) I is directed, (b) $\mathcal{F}$ is a finite type $\mathcal{O}_X$-module, and (c) X is quasi-compact. Then there exists an i such that $\mathcal{F}_i \to \mathcal{F}$ is surjective. If the transition maps $f_{ii'}$ are injective then we conclude that $\mathcal{F} = \mathcal{F}_i$ for some $i \in I$.*

Proof. Let $x \in X$. There exists an open neighbourhood $U \subset X$ of x and finitely many sections $s_j \in \mathcal{F}(U)$, $j = 1, \ldots, m$ such that $s_1, \ldots, s_m$ generate $\mathcal{F}$ as $\mathcal{O}_U$-module. After possibly shrinking U to a smaller open neighbourhood of x we may assume that each s_j comes from a section of $\mathcal{F}_i$ for some $i \in I$. Hence, since X is quasi-compact we can find a finite open covering $X = \bigcup_{j=1,\ldots,m} U_j$, and for each j an index i_j and finitely many sections $s_{jl} \in \mathcal{F}_{i_j}(U_j)$ whose images generate the restriction of $\mathcal{F}$ to U_j. Clearly, the lemma holds for any index $i \in I$ which is $\geq$ all i_j. □

Lemma 17.9.8. *Let X be a ringed space. There exists a set of $\mathcal{O}_X$-modules $\{\mathcal{F}_i\}_{i \in I}$ of finite type such that each finite type $\mathcal{O}_X$-module on X is isomorphic to exactly one of the $\mathcal{F}_i$.*

Proof. For each open covering $\mathcal{U} : X = \bigcup U_j$ consider the sheaves of $\mathcal{O}_X$-modules $\mathcal{F}$ such that each restriction $\mathcal{F}|_{U_j}$ is a quotient of $\mathcal{O}_{U_j}^{\oplus r}$ for some $r_j \geq 0$. These are parametrized by subsheaves $\mathcal{K}_i \subset \mathcal{O}_{U_j}^{\oplus r_j}$ and glueing data

$$\varphi_{jj'} : \mathcal{O}_{U_j \cap U_{j'}}^{\oplus r_j} / (\mathcal{K}_j|_{U_j \cap U_{j'}}) \longrightarrow \mathcal{O}_{U_j \cap U_{j'}}^{\oplus r_{j'}} / (\mathcal{K}_{j'}|_{U_j \cap U_{j'}})$$

see Sheaves, Section 6.33. Note that the collection of all glueing data forms a set. The collection of all coverings $\mathcal{U} : X = \bigcup_{j \in J} U_i$ where $J \to \mathcal{P}(X)$, $j \mapsto U_j$ is injective forms a set as well. Hence the collection of all sheaves of $\mathcal{O}_X$-modules gotten from glueing quotients as above forms a set $\mathcal{I}$. By definition every finite type $\mathcal{O}_X$-module is isomorphic to an element of $\mathcal{I}$. Choosing an element out of each isomorphism class inside $\mathcal{I}$ gives the desired set of sheaves (uses axiom of choice). □

17.10. Quasi-coherent modules

In this section we introduce an abstract notion of quasi-coherent $\mathcal{O}_X$-module. This notion is very useful in algebraic geometry, since quasi-coherent modules on a scheme have a good description on any affine open. However, we warn the reader that in the general setting of (locally) ringed spaces this notion is not well behaved at all. The category of quasi-coherent sheaves is not abelian in general, infinite direct sums of quasi-coherent sheaves aren't quasi-coherent, etc, etc.

Definition 17.10.1. Let $(X, \mathcal{O}_X)$ be a ringed space. Let $\mathcal{F}$ be a sheaf of $\mathcal{O}_X$-modules. We say that $\mathcal{F}$ is a *quasi-coherent sheaf of $\mathcal{O}_X$-modules* if for every point

$x \in X$ there exists an open neighbourhood $x \in U \subset X$ such that $\mathcal{F}|_U$ is isomorphic to the cokernel of a map

$$\bigoplus\nolimits_{j\in J} \mathcal{O}_U \longrightarrow \bigoplus\nolimits_{i\in I} \mathcal{O}_U$$

The category of quasi-coherent $\mathcal{O}_X$-modules is denoted $QCoh(\mathcal{O}_X)$.

The definition means that X is covered by open sets U such that $\mathcal{F}|_U$ has a *presentation* of the form

$$\bigoplus\nolimits_{j\in J} \mathcal{O}_U \longrightarrow \bigoplus\nolimits_{i\in I} \mathcal{O}_U \to \mathcal{F}|_U \to 0.$$

Here presentation signifies that the displayed sequence is exact. In other words

(1) for every point x of X there exists an open neighbourhood such that $\mathcal{F}|_U$ is generated by global sections, and
(2) for a suitable choice of these sections the kernel of the associated surjection is also generated by global sections.

Lemma 17.10.2. *Let $(X, \mathcal{O}_X)$ be a ringed space. The direct sum of two quasi-coherent $\mathcal{O}_X$-modules is a quasi-coherent $\mathcal{O}_X$-module.*

Proof. Omitted. □

Remark 17.10.3. Warning: It is not true in general that an infinite direct sum of quasi-coherent $\mathcal{O}_X$-modules is quasi-coherent. For more esoteric behaviour of quasi-coherent modules see Example 17.10.9.

Lemma 17.10.4. *Let $f : (X, \mathcal{O}_X) \to (Y, \mathcal{O}_Y)$ be a morphism of ringed spaces. The pullback $f^*\mathcal{G}$ of a quasi-coherent $\mathcal{O}_Y$-module is quasi-coherent.*

Proof. Arguing as in the proof of Lemma 17.8.2 we may assume $\mathcal{G}$ has a global presentation by direct sums of copies of $\mathcal{O}_Y$. We have seen that f^* commutes with all colimits, and is right exact, see Lemma 17.3.3. Thus if we have an exact sequence

$$\bigoplus\nolimits_{j\in J} \mathcal{O}_Y \longrightarrow \bigoplus\nolimits_{i\in I} \mathcal{O}_Y \to \mathcal{G} \to 0$$

then upon applying f^* we obtain the exact sequence

$$\bigoplus\nolimits_{j\in J} \mathcal{O}_X \longrightarrow \bigoplus\nolimits_{i\in I} \mathcal{O}_X \to f^*\mathcal{G} \to 0.$$

This implies the lemma. □

This gives plenty of examples of quasi-coherent sheaves.

Lemma 17.10.5. *Let $(X, \mathcal{O}_X)$ be ringed space. Let $\alpha : R \to \Gamma(X, \mathcal{O}_X)$ be a ring homomorphism from a ring R into the ring of global sections on X. Let M be an R-module. The following three constructions give canonically isomorphic sheaves of $\mathcal{O}_X$-modules:*

(1) *Let $\pi : (X, \mathcal{O}_X) \longrightarrow (\{*\}, R)$ be the morphism of ringed spaces with $\pi : X \to \{*\}$ the unique map and with π-map $\pi^\sharp$ the given map $\alpha : R \to \Gamma(X, \mathcal{O}_X)$. Set $\mathcal{F}_1 = \pi^*M$.*
(2) *Choose a presentation $\bigoplus_{j\in J} R \to \bigoplus_{i\in I} R \to M \to 0$. Set*

$$\mathcal{F}_2 = Coker\left(\bigoplus\nolimits_{j\in J} \mathcal{O}_X \to \bigoplus\nolimits_{i\in I} \mathcal{O}_X\right).$$

Here the map on the component $\mathcal{O}_X$ corresponding to $j \in J$ given by the section $\sum_i \alpha(r_{ij})$ where the r_{ij} are the matrix coefficients of the map in the presentation of M.

(3) *Set $\mathcal{F}_3$ equal to the sheaf associated to the presheaf $U \mapsto \mathcal{O}_X(U) \otimes_R M$, where the map $R \to \mathcal{O}_X(U)$ is the composition of α and the restriction map $\mathcal{O}_X(X) \to \mathcal{O}_X(U)$.*

This construction has the following properties:

(1) *The resulting sheaf of $\mathcal{O}_X$-modules $\mathcal{F}_M = \mathcal{F}_1 = \mathcal{F}_2 = \mathcal{F}_3$ is quasi-coherent.*
(2) *The construction gives a functor from the category of R-modules to the category of quasi-coherent sheaves on X which commutes with arbitrary colimits.*
(3) *For any $x \in X$ we have $\mathcal{F}_{M,x} = \mathcal{O}_{X,x} \otimes_R M$ functorial in M.*
(4) *Given any $\mathcal{O}_X$-module $\mathcal{G}$ we have*

$$\mathrm{Mor}_{\mathcal{O}_X}(\mathcal{F}_M, \mathcal{G}) = \mathrm{Hom}_R(M, \Gamma(X, \mathcal{G}))$$

where the R-module structure on $\Gamma(X, \mathcal{G})$ comes from the $\Gamma(X, \mathcal{O}_X)$-module structure via α.

Proof. The isomorphism between $\mathcal{F}_1$ and $\mathcal{F}_3$ comes from the fact that π^* is defined as the sheafification of the presheaf in (3), see Sheaves, Section 6.26. The isomorphism between the constructions in (2) and (1) comes from the fact that the functor π^* is right exact, so $\pi^*(\bigoplus_{j\in J} R) \to \pi^*(\bigoplus_{i\in I} R) \to \pi^*M \to 0$ is exact, π^* commutes with arbitrary direct sums, see Lemma 17.3.3, and finally the fact that $\pi^*(R) = \mathcal{O}_X$.

Assertion (1) is clear from construction (2). Assertion (2) is clear since π^* has these properties. Assertion (3) follows from the description of stalks of pullback sheaves, see Sheaves, Lemma 6.26.4. Assertion (4) follows from adjointness of π_* and π^*. □

Definition 17.10.6. In the situation of Lemma 17.10.5 we say $\mathcal{F}_M$ is the *sheaf associated to the module M and the ring map α*. If $R = \Gamma(X, \mathcal{O}_X)$ and $\alpha = \mathrm{id}_R$ we simply say $\mathcal{F}_M$ is the *sheaf associated to the module M*.

Lemma 17.10.7. *Let $(X, \mathcal{O}_X)$ be ringed space. Set $R = \Gamma(X, \mathcal{O}_X)$. Let M be an R-module. Let $\mathcal{F}_M$ be the quasi-coherent sheaf of $\mathcal{O}_X$-modules associated to M. If $g : (Y, \mathcal{O}_Y) \to (X, \mathcal{O}_X)$ is a morphism of ringed spaces, then $g^*\mathcal{F}_M$ is the sheaf associated to the $\Gamma(Y, \mathcal{O}_Y)$-module $\Gamma(Y, \mathcal{O}_Y) \otimes_R M$.*

Proof. The assertion follows from the first description of $\mathcal{F}_M$ in Lemma 17.10.5 as π^*M, and the following commutative diagram of ringed spaces

$$\begin{array}{ccc} (X, \mathcal{O}_X) & \xrightarrow{\pi} & (\{*\}, \Gamma(X, \mathcal{O}_X)) \\ \downarrow{\scriptstyle g} & & \downarrow{\scriptstyle g^\sharp} \\ (Y, \mathcal{O}_Y) & \xrightarrow{\pi} & (\{*\}, \Gamma(Y, \mathcal{O}_Y)) \end{array}$$

(Also use Sheaves, Lemma 6.26.3.) □

Lemma 17.10.8. *Let $(X, \mathcal{O}_X)$ be a ringed space. Let $x \in X$ be a point. Assume that x has a fundamental system of quasi-compact neighbourhoods. Consider any quasi-coherent $\mathcal{O}_X$-module $\mathcal{F}$. Then there exists an open neighbourhood U of x such*

that $\mathcal{F}|_U$ is isomorphic to the sheaf of modules $\mathcal{F}_M$ on $(U, \mathcal{O}_U)$ associated to some $\Gamma(U, \mathcal{O}_U)$-module M.

Proof. First we may replace X by an open neighbourhood of x and assume that $\mathcal{F}$ is isomorphic to the cokernel of a map

$$\Psi : \bigoplus\nolimits_{j \in J} \mathcal{O}_X \longrightarrow \bigoplus\nolimits_{i \in I} \mathcal{O}_X.$$

The problem is that this map may not be given by a "matrix", because the module of global sections of a direct sum is in general different from the direct sum of the modules of global sections.

Let $x \in E \subset X$ be a quasi-compact neighbourhood of x (note: E may not be open). Let $x \in U \subset E$ be an open neighbourhood of x contained in E. Next, we proceed as in the proof of Lemma 17.3.5. For each $j \in J$ denote $s_j \in \Gamma(X, \bigoplus_{i \in I} \mathcal{O}_X)$ the image of the section 1 in the summand $\mathcal{O}_X$ corresponding to j. There exists a finite collection of opens U_{jk}, $k \in K_j$ such that $E \subset \bigcup_{k \in K_j} U_{jk}$ and such that each restriction $s_j|_{U_{jk}}$ is a finite sum $\sum_{i \in I_{jk}} f_{jki}$ with $I_{jk} \subset I$, and f_{jki} in the summand $\mathcal{O}_X$ corresponding to $i \in I$. Set $I_j = \bigcup_{k \in k_j} I_{jk}$. This is a finite set. Since $U \subset E \subset \bigcup_{k \in K_j} U_{jk}$ the section $s_j|_U$ is a section of the finite direct sum $\bigoplus_{i \in I_j} \mathcal{O}_X$. By Lemma 17.3.2 we see that actually $s_j|_U$ is a sum $\sum_{i \in I_j} f_{ij}$ and $f_{ij} \in \mathcal{O}_X(U) = \Gamma(U, \mathcal{O}_U)$.

At this point we can define a module M as the cokernel of the map

$$\bigoplus\nolimits_{j \in J} \Gamma(U, \mathcal{O}_U) \longrightarrow \bigoplus\nolimits_{i \in I} \Gamma(U, \mathcal{O}_U)$$

with matrix given by the (f_{ij}). By construction (2) of Lemma 17.10.5 we see that $\mathcal{F}_M$ has the same presentation as $\mathcal{F}|_U$ and therefore $\mathcal{F}_M \cong \mathcal{F}|_U$. □

Example 17.10.9. Let X be countably many copies $L_1, L_2, L_3, \ldots$ of the real line all glued together at 0; a fundamental system of neighbourhoods of 0 being the collection $\{U_n\}_{n \in \mathbf{N}}$, with $U_n \cap L_i = (-1/n, 1/n)$. Let $\mathcal{O}_X$ be the sheaf of continuous real valued functions. Let $f : \mathbf{R} \to \mathbf{R}$ be a continuous function which is identically zero on $(-1, 1)$ and identically 1 on $(-\infty, -2) \cup (2, \infty)$. Denote f_n the continuous function on X which is equal to $x \mapsto f(nx)$ on each $L_j = \mathbf{R}$. Let 1_{L_j} be the characteristic function of L_j. We consider the map

$$\bigoplus\nolimits_{j \in \mathbf{N}} \mathcal{O}_X \longrightarrow \bigoplus\nolimits_{j,i \in \mathbf{N}} \mathcal{O}_X, \quad e_j \longmapsto \sum\nolimits_{j \in \mathbf{N}} f_i 1_{L_j} e_{ij}$$

with obvious notation. This makes sense because this sum is locally finite. Clearly, there is no neighbourhood of $0 \in X$ such that this map is given by a "matrix" as in the proof of Lemma 17.10.8 above.

Note that $\bigoplus_{j \in \mathbf{N}} \mathcal{O}_X$ is the sheaf associated to the free module with basis e_j and similarly for the other direct sum. Thus we see that a morphism of sheaves associated to modules in general even locally on X does not come from a morphism of modules. Similarly there should be an example of a ringed space X and a quasi-coherent $\mathcal{O}_X$-module $\mathcal{F}$ such that $\mathcal{F}$ is not locally of the form $\mathcal{F}_M$. (Please email if you find one.) Moreover, there should be examples of locally compact spaces X and maps $\mathcal{F}_M \to \mathcal{F}_N$ which also do not locally come from maps of modules (the proof of Lemma 17.10.8 shows this cannot happen if N is free).

17.11. Modules of finite presentation

Definition 17.11.1. Let $(X, \mathcal{O}_X)$ be a ringed space. Let $\mathcal{F}$ be a sheaf of $\mathcal{O}_X$-modules. We say that $\mathcal{F}$ is of *finite presentation* if for every point $x \in X$ there exists an open neighbourhood $x \in U \subset X$, and $n, m \in \mathbf{N}$ such that $\mathcal{F}|_U$ is isomorphic to the cokernel of a map

$$\bigoplus\nolimits_{j=1,\ldots,m} \mathcal{O}_U \longrightarrow \bigoplus\nolimits_{i=1,\ldots,n} \mathcal{O}_U$$

This means that X is covered by open sets U such that $\mathcal{F}|_U$ has a *presentation* of the form

$$\bigoplus\nolimits_{j=1,\ldots,m} \mathcal{O}_U \longrightarrow \bigoplus\nolimits_{i=1,\ldots,n} \mathcal{O}_U \to \mathcal{F}|_U \to 0.$$

Here presentation signifies that the displayed sequence is exact. In other words

(1) for every point x of X there exists an open neighbourhood such that $\mathcal{F}|_U$ is generated by finitely many global sections, and
(2) for a suitable choice of these sections the kernel of the associated surjection is also generated by finitely many global sections.

Lemma 17.11.2. *Let $(X, \mathcal{O}_X)$ be a ringed space. Any $\mathcal{O}_X$-module of finite presentation is quasi-coherent.*

Proof. Immediate from definitions. □

Lemma 17.11.3. *Let $(X, \mathcal{O}_X)$ be a ringed space. Let $\mathcal{F}$ be a $\mathcal{O}_X$-module of finite presentation.*

(1) *If $\psi : \mathcal{O}_X^{\oplus r} \to \mathcal{F}$ is a surjection, then $Ker(\psi)$ is of finite type.*
(2) *If $\theta : \mathcal{G} \to \mathcal{F}$ is surjective with $\mathcal{G}$ of finite type, then $Ker(\theta)$ is of finite type.*

Proof. Proof of (1). Let $x \in X$. Choose an open neighbourhood $U \subset X$ of x such that there exists a presentation

$$\mathcal{O}_U^{\oplus m} \xrightarrow{\chi} \mathcal{O}_U^{\oplus n} \xrightarrow{\varphi} \mathcal{F}|_U \to 0.$$

Let e_k be the section generating the kth factor of $\mathcal{O}_X^{\oplus r}$. For every $k = 1, \ldots, r$ we can, after shrinking U to a small neighbourhood of x, lift $\psi(e_k)$ to a section $\tilde{e}_k$ of $\mathcal{O}_U^{\oplus n}$ over U. This gives a morphism of sheaves $\alpha : \mathcal{O}_U^{\oplus r} \to \mathcal{O}_U^{\oplus n}$ such that $\varphi \circ \alpha = \psi$. Similarly, after shrinking U, we can find a morphism $\beta : \mathcal{O}_U^{\oplus n} \to \mathcal{O}_U^{\oplus r}$ such that $\psi \circ \beta = \varphi$. Then the map

$$\mathcal{O}_U^{\oplus m} \oplus \mathcal{O}_U^{\oplus r} \xrightarrow{\beta \circ \chi, 1 - \beta \circ \alpha} \mathcal{O}_U^{\oplus r}$$

is a surjection onto the kernel of ψ.

To prove (2) we may locally choose a surjection $\eta : \mathcal{O}_X^{\oplus r} \to \mathcal{G}$. By part (1) we see $\mathrm{Ker}(\theta \circ \eta)$ is of finite type. Since $\mathrm{Ker}(\theta) = \eta(\mathrm{Ker}(\theta \circ \eta))$ we win. □

Lemma 17.11.4. *Let $f : (X, \mathcal{O}_X) \to (Y, \mathcal{O}_Y)$ be a morphism of ringed spaces. The pullback $f^*\mathcal{G}$ of a module of finite presentation is of finite presentation.*

Proof. Exactly the same as the proof of Lemma 17.10.4 but with finite index sets. □

Lemma 17.11.5. *Let $(X, \mathcal{O}_X)$ be a ringed space. Set $R = \Gamma(X, \mathcal{O}_X)$. Let M be an R-module. The $\mathcal{O}_X$-module $\mathcal{F}_M$ associated to M is a directed colimit of finitely presented $\mathcal{O}_X$-modules.*

Proof. This follows immediately from Lemma 17.10.5 and the fact that any module is a directed colimit of finitely presented modules, see Algebra, Lemma 10.8.13. □

Lemma 17.11.6. *Let X be a ringed space. Let I be a partially ordered set and let $(\mathcal{F}_i, \varphi_{ii'})$ be a system over I consisting of sheaves of $\mathcal{O}_X$-modules (see Categories, Section 4.21). Assume*

(1) *I is directed,*
(2) *$\mathcal{G}$ is an $\mathcal{O}_X$-module of finite presentation, and*
(3) *X has a cofinal system of open coverings $\mathcal{U} : X = \bigcup_{j \in J} U_j$ with J finite and $U_j \cap U_{j'}$ quasi-compact for all $j, j' \in J$*

Then we have

$$\operatorname{colim}_i \operatorname{Hom}_X(\mathcal{G}, \mathcal{F}_i) = \operatorname{Hom}_X(\mathcal{G}, \operatorname{colim}_i \mathcal{F}_i).$$

Proof. Let α be an element of the right hand side. For every point $x \in X$ we may choose an open neighbourhood $U \subset X$ and finitely many sections $s_j \in \mathcal{G}(U)$ which generate $\mathcal{G}$ over U and finitely many relations $\sum f_{kj} s_j = 0$, $k = 1, \ldots, n$ with $f_{kj} \in \mathcal{O}_X(U)$ which generate the kernel of $\bigoplus_{j=1,\ldots,m} \mathcal{O}_U \to \mathcal{G}$. After possibly shrinking U to a smaller open neighbourhood of x we may assume there exists an index $i \in I$ such that the sections $\alpha(s_j)$ all come from sections $s'_j \in \mathcal{F}_i(U)$. After possibly shrinking U to a smaller open neighbourhood of x and increasing i we may assume the relations $\sum f_{kj} s'_j = 0$ hold in $\mathcal{F}_i(U)$. Hence we see that $\alpha|_U$ lifts to a morphism $\mathcal{G}|_U \to \mathcal{F}_i|_U$ for some index $i \in I$.

By condition (3) and the preceding arguments, we may choose a finite open covering $X = \bigcup_{j=1,\ldots,m} U_j$ such that (a) $\mathcal{G}|_{U_j}$ is generated by finitely many sections $s_{jk} \in \mathcal{G}(U_j)$, (b) the restriction $\alpha|_{U_j}$ comes from a morphism $\alpha_j : \mathcal{G} \to \mathcal{F}_{i_j}$ for some $i_j \in I$, and (c) the intersections $U_j \cap U_{j'}$ are all quasi-compact. For every pair $(j, j') \in \{1, \ldots, m\}^2$ and any k we can find we can find an index $i \geq \max(i_j, i_{j'})$ such that

$$\varphi_{i_j i}(\alpha_j(s_{jk}|_{U_j \cap U_{j'}})) = \varphi_{i_{j'} i}(\alpha_{j'}(s_{jk}|_{U_j \cap U_{j'}}))$$

see Sheaves, Lemma 6.29.1 (2). Since there are finitely many of these pairs (j, j') and finitely many s_{jk} we see that we can find a single i which works for all of them. For this index i all of the maps $\varphi_{i_j i} \circ \alpha_j$ agree on the overlaps $U_j \cap U_{j'}$ as the sections s_{jk} generate $\mathcal{G}$ over this overlap. Hence we get a morphism $\mathcal{G} \to \mathcal{F}_i$ as desired. □

Remark 17.11.7. In the lemma above some condition beyond the condition that X is quasi-compact is necessary. See Sheaves, Example 6.29.2.

17.12. Coherent modules

The category of coherent sheaves on a ringed space X is a more reasonable object than the category of quasi-coherent sheaves, in the sense that it is at least an abelian subcategory of $Mod(\mathcal{O}_X)$ no matter what X is. On the other hand, the pullback of a coherent module is "almost never" coherent in the general setting of ringed spaces.

Definition 17.12.1. Let $(X, \mathcal{O}_X)$ be a ringed space. Let $\mathcal{F}$ be a sheaf of $\mathcal{O}_X$-modules. We say that $\mathcal{F}$ is a *coherent $\mathcal{O}_X$-module* if the following two conditions hold:

(1) $\mathcal{F}$ is of finite type, and

(2) for every open $U \subset X$ and every finite collection $s_i \in \mathcal{F}(U)$, $i = 1, \ldots, n$ the kernel of the associated map $\bigoplus_{i=1,\ldots,n} \mathcal{O}_U \to \mathcal{F}|_U$ is of finite type.

The category of coherent $\mathcal{O}_X$-modules is denoted $\mathrm{Coh}(\mathcal{O}_X)$.

Lemma 17.12.2. *Let $(X, \mathcal{O}_X)$ be a ringed space. Any coherent $\mathcal{O}_X$-module is of finite presentation and hence quasi-coherent.*

Proof. Let $\mathcal{F}$ be a coherent sheaf on X. Pick a point $x \in X$. By (1) of the definition of coherent, we may find an open neighbourhood U and sections s_i, $i = 1, \ldots, n$ of $\mathcal{F}$ over U such that $\Psi : \bigoplus_{i=1,\ldots,n} \mathcal{O}_U \to \mathcal{F}$ is surjective. By (2) of the definition of coherent, we may find an open neighbourhood V, $x \in V \subset U$ and sections $t_1, \ldots, t_m$ of $\bigoplus_{i=1,\ldots,n} \mathcal{O}_V$ which generate the kernel of $\Psi|_V$. Then over V we get the presentation

$$\bigoplus\nolimits_{j=1,\ldots,m} \mathcal{O}_V \longrightarrow \bigoplus\nolimits_{i=1,\ldots,n} \mathcal{O}_V \to \mathcal{F}|_V \to 0$$

as desired. □

Example 17.12.3. Suppose that X is a point. In this case the definition above gives a notion for modules over rings. What does the definition of coherent mean? It is closely related to the notion of Noetherian, but it is not the same: Namely, the ring $R = \mathbf{C}[x_1, x_2, x_3, \ldots]$ is coherent as a module over itself but not Noetherian as a module over itself. See Algebra, Section 10.85 for more discussion.

Lemma 17.12.4. *Let $(X, \mathcal{O}_X)$ be a ringed space.*

(1) *Any finite type subsheaf of a coherent sheaf is coherent.*
(2) *Let $\varphi : \mathcal{F} \to \mathcal{G}$ be a morphism from a finite type sheaf $\mathcal{F}$ to a coherent sheaf $\mathcal{G}$. Then $Ker(\varphi)$ is finite type.*
(3) *Let $\varphi : \mathcal{F} \to \mathcal{G}$ be a morphism of coherent $\mathcal{O}_X$-modules. Then $Ker(\varphi)$ and $Coker(\varphi)$ are coherent.*
(4) *The category of coherent sheaves on X is abelian.*
(5) *Given a short exact sequence of $\mathcal{O}_X$-modules $0 \to \mathcal{F}_1 \to \mathcal{F}_2 \to \mathcal{F}_3 \to 0$ if two out of three are coherent so is the third.*

Proof. Condition (2) of Definition 17.12.1 holds for any subsheaf of a coherent sheaf. Thus we get the first statement.

Assume the hypotheses of (2). Let us show that $\mathrm{Ker}(\varphi)$ is of finite type. Pick $x \in X$. Choose an open neighbourhood U of x in X such that $\mathcal{F}|_U$ is generated by $s_1, \ldots, s_n$. By Definition 17.12.1 the kernel $\mathcal{K}$ of the induced map $\bigoplus_{i=1}^n \mathcal{O}_U \to \mathcal{G}$, $e_i \mapsto \varphi(s_i)$ is of finite type. Hence $\mathrm{Ker}(\varphi)$ which is the image of the composition $\mathcal{K} \to \bigoplus_{i=1}^n \mathcal{O}_U \to \mathcal{F}$ is of finite type.

Assume the hypotheses of (3). By (2) the kernel of φ is of finite type and hence by (1) it is coherent.

With the same hypotheses let us show that $\mathrm{Coker}(\varphi)$ is coherent. Since $\mathcal{G}$ is of finite type so is $\mathrm{Coker}(\varphi)$. Let $U \subset X$ be open and let $\overline{s}_i \in \mathrm{Coker}(\varphi)(U)$, $i = 1, \ldots, n$ be sections. We have to show that the kernel of the associated morphism $\overline{\Psi} : \bigoplus_{i=1}^n \mathcal{O}_U \to \mathrm{Coker}(\varphi)$ has finite type. There exists an open covering of U such that on each open all the sections $\overline{s}_i$ lift to sections s_i of $\mathcal{G}$. Hence we may assume

this is the case over U. Thus $\overline{\Psi}$ lifts to $\Psi : \bigoplus_{i=1}^n \mathcal{O}_U \to \mathcal{G}$ Consider the following diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \mathrm{Ker}(\Psi) & \longrightarrow & \bigoplus_{i=1}^n \mathcal{O}_U & \longrightarrow & \mathcal{G} & \longrightarrow & 0 \\ & & \downarrow & & \| & & \downarrow & & \\ 0 & \longrightarrow & \mathrm{Ker}(\overline{\Psi}) & \longrightarrow & \bigoplus_{i=1}^n \mathcal{O}_U & \longrightarrow & \mathrm{Coker}(\varphi) & \longrightarrow & 0 \end{array}$$

By the snake lemma we get a short exact sequence $0 \to \mathrm{Ker}(\Psi) \to \mathrm{Ker}(\overline{\Psi}) \to \mathrm{Im}(\varphi) \to 0$. Hence by Lemma 17.9.3 we see that $\mathrm{Ker}(\overline{\Psi})$ has finite type.

Statement (4) follows from (3).

Let $0 \to \mathcal{F}_1 \to \mathcal{F}_2 \to \mathcal{F}_3 \to 0$ be a short exact sequence of $\mathcal{O}_X$-modules. It suffices to prove that if $\mathcal{F}_1$ and $\mathcal{F}_3$ are coherent so is $\mathcal{F}_2$. By Lemma 17.9.3 we see that $\mathcal{F}_2$ has finite type. Let $s_1, \ldots, s_n$ be finitely many local sections of $\mathcal{F}_2$ defined over a common open U of X. We have to show that the module of relations $\mathcal{K}$ between them is of finite type. Consider the following commutative diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & 0 & \longrightarrow & \bigoplus_{i=1}^n \mathcal{O}_U & \longrightarrow & \bigoplus_{i=1}^n \mathcal{O}_U & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & \mathcal{F}_1 & \longrightarrow & \mathcal{F}_2 & \longrightarrow & \mathcal{F}_3 & \longrightarrow & 0 \end{array}$$

with obvious notation. By the snake lemma we get a short exact sequence $0 \to \mathcal{K} \to \mathcal{K}_3 \to \mathcal{F}_1$ where $\mathcal{K}_3$ is the module of relations among the images of the sections s_i in $\mathcal{F}_3$. Since $\mathcal{F}_3$ is coherent we see that $\mathcal{K}_3$ is finite type. Since $\mathcal{F}_1$ is coherent we see that the image $\mathcal{I}$ of $\mathcal{K}_3 \to \mathcal{F}_1$ is coherent. Hence $\mathcal{K}$ is the kernel of the map $\mathcal{K}_3 \to \mathcal{I}$ between a finite type sheaf and a coherent sheaves and hence finite type by (2). □

Lemma 17.12.5. *Let $(X, \mathcal{O}_X)$ be a ringed space. Let $\mathcal{F}$ be an $\mathcal{O}_X$-module. Assume $\mathcal{O}_X$ is a coherent $\mathcal{O}_X$-module. Then $\mathcal{F}$ is coherent if and only if it is of finite presentation.*

Proof. Omitted. □

Lemma 17.12.6. *Let X be a ringed space. Let $\varphi : \mathcal{G} \to \mathcal{F}$ be a homomorphism of $\mathcal{O}_X$-modules. Let $x \in X$. Assume $\mathcal{G}$ of finite type, $\mathcal{F}$ coherent and the map on stalks $\varphi_x : \mathcal{G}_x \to \mathcal{F}_x$ injective. Then there exists an open neighbourhood $x \in U \subset X$ such that $\varphi|_U$ is injective.*

Proof. Denote $\mathcal{K} \subset \mathcal{G}$ the kernel of φ. By Lemma 17.12.4 we see that $\mathcal{K}$ is a finite type $\mathcal{O}_X$-module. Our assumption is that $\mathcal{K}_x = 0$. By Lemma 17.9.5 there exists an open neighbourhood U of x such that $\mathcal{K}|_U = 0$. Then U works. □

17.13. Closed immersions of ringed spaces

When do we declare a morphism of ringed spaces $i : (Z, \mathcal{O}_Z) \to (X, \mathcal{O}_X)$ to be a closed immersion?

Motivated by the example of a closed immersion of normal topological spaces (ringed with the sheaf of continuous functors), or differential manifolds (ringed with the sheaf of differentiable functions), it seems natural to assume at least:

(1) The map i is a closed immersion of topological spaces.

(2) The associated map $\mathcal{O}_X \to i_*\mathcal{O}_Z$ is surjective. Denote the kernel by $\mathcal{I}$.

Already these conditions imply a number of pleasing results: For example we prove that the category of $\mathcal{O}_Z$-modules is equivalent to the category of $\mathcal{O}_X$-modules annihilated by $\mathcal{I}$ generalizing the result on abelian sheaves of Section 17.6

However, in the Stacks project we choose the definition that guarantees that if i is a closed immersion and $(X, \mathcal{O}_X)$ is a scheme, then also $(Z, \mathcal{O}_Z)$ is a scheme. Moreover, in this situation we want i_* and i^* to provide an equivalence between the category of quasi-coherent $\mathcal{O}_Z$-modules and the category of quasi-coherent $\mathcal{O}_X$-modules annihilated by $\mathcal{I}$. A minimal condition is that $i_*\mathcal{O}_Z$ is a quasi-coherent sheaf of $\mathcal{O}_X$-modules. A good way to guarantee that $i_*\mathcal{O}_Z$ is a quasi-coherent $\mathcal{O}_X$-module is to assume that $\mathcal{I}$ is locally generated by sections. We can interpret this condition as saying "$(Z, \mathcal{O}_Z)$ is locally on $(X, \mathcal{O}_X)$ defined by setting some regular functions f_i, i.e., local sections of $\mathcal{O}_X$, equal to zero". This leads to the following definition.

Definition 17.13.1. A *closed immersion of ringed spaces*[2] is a morphism $i : (Z, \mathcal{O}_Z) \to (X, \mathcal{O}_X)$ with the following properties:

(1) The map i is a closed immersion of topological spaces.
(2) The associated map $\mathcal{O}_X \to i_*\mathcal{O}_Z$ is surjective. Denote the kernel by $\mathcal{I}$.
(3) The $\mathcal{O}_X$-module $\mathcal{I}$ is locally generated by sections.

Actually, this definition still does not guarantee that i_* of a quasi-coherent $\mathcal{O}_Z$-module is a quasi-coherent $\mathcal{O}_X$-module. The problem is that it is not clear how to convert a local presentation of a quasi-coherent $\mathcal{O}_Z$-module into a local presentation for the pushforward. However, the following is trivial.

Lemma 17.13.2. *Let $i : (Z, \mathcal{O}_Z) \to (X, \mathcal{O}_X)$ be a closed immersion of locally ringed spaces. Let $\mathcal{F}$ be a quasi-coherent $\mathcal{O}_Z$-module. Then $i_*\mathcal{F}$ is locally on X the cokernel of a map of quasi-coherent $\mathcal{O}_X$-modules.*

Proof. This is true because $i_*\mathcal{O}_Z$ is quasi-coherent by definition. And locally on Z the sheaf $\mathcal{F}$ is a cokernel of a map between direct sums of copies of $\mathcal{O}_Z$. Moreover, any direct sum of copies of the *the same* quasi-coherent sheaf is quasi-coherent. And finally, i_* commutes with arbitrary colimits, see Lemma 17.6.3. Some details omitted. □

Lemma 17.13.3. *Let $i : (Z, \mathcal{O}_Z) \to (X, \mathcal{O}_X)$ be a morphism of ringed spaces. Assume i is a homeomorphism onto a closed subset of X and that $\mathcal{O}_X \to i_*\mathcal{O}_Z$ is surjective. Let $\mathcal{F}$ be an $\mathcal{O}_Z$-module. Then $i_*\mathcal{F}$ is of finite type if and only if $\mathcal{F}$ is of finite type.*

Proof. Suppose that $\mathcal{F}$ is of finite type. Pick $x \in X$. If $x \notin Z$, then $i_*\mathcal{F}$ is zero in a neighbourhood of x and hence finitely generated in a neighbourhood of x. If $x = i(z)$, then choose an open neighbourhood $z \in V \subset Z$ and sections $s_1, \ldots, s_n \in \mathcal{F}(V)$ which generate $\mathcal{F}$ over V. Write $V = Z \cap U$ for some open $U \subset X$. Note that U is a neighbourhood of x. Clearly the sections s_i give sections s_i of $i_*\mathcal{F}$ over U. The resulting map

$$\bigoplus\nolimits_{i=1,\ldots,n} \mathcal{O}_U \longrightarrow i_*\mathcal{F}|_U$$

[2]This is nonstandard notation; see discussion above.

is surjective by inspection of what it does on stalks (here we use that $\mathcal{O}_X \to i_*\mathcal{O}_Z$ is surjective). Hence $i_*\mathcal{F}$ is of finite type.

Conversely, suppose that $i_*\mathcal{F}$ is of finite type. Choose $z \in Z$. Set $x = i(z)$. By assumption there exists an open neighbourhood $U \subset X$ of x, and sections $s_1, \ldots, s_n \in (i_*\mathcal{F})(U)$ which generate $i_*\mathcal{F}$ over U. Set $V = Z \cap U$. By definition of i_* the sections s_i correspond to sections s_i of $\mathcal{F}$ over V. The resulting map

$$\bigoplus\nolimits_{i=1,\ldots,n} \mathcal{O}_V \longrightarrow \mathcal{F}|_V$$

is surjective by inspection of what it does on stalks. Hence $\mathcal{F}$ is of finite type. □

Lemma 17.13.4. *Let $i : (Z, \mathcal{O}_X) \to (X, \mathcal{O}_X)$ be a morphism of ringed spaces. Assume i is a homeomorphism onto a closed subset of X and $i^\sharp : \mathcal{O}_X \to i_*\mathcal{O}_Z$ is surjective. Denote $\mathcal{I} \subset \mathcal{O}_X$ the kernel of $i^\sharp$. The functor*

$$i_* : \mathit{Mod}(\mathcal{O}_Z) \longrightarrow \mathit{Mod}(\mathcal{O}_X)$$

is exact, fully faithful, with essential image those $\mathcal{O}_X$-modules $\mathcal{G}$ such that $\mathcal{I}\mathcal{G} = 0$.

Proof. We claim that for a $\mathcal{O}_Z$-module $\mathcal{F}$ the canonical map

$$i^*i_*\mathcal{F} \longrightarrow \mathcal{F}$$

is an isomorphism. We check this on stacks. Say $z \in Z$ and $x = i(z)$. We have

$$(i^*i_*\mathcal{F})_z = (i_*\mathcal{F})_x \otimes_{\mathcal{O}_{X,x}} \mathcal{O}_{Z,z} = \mathcal{F}_z \otimes_{\mathcal{O}_{X,x}} \mathcal{O}_{Z,z} = \mathcal{F}_z$$

by Sheaves, Lemma 6.26.4, the fact that $\mathcal{O}_{Z,z}$ is a quotient of $\mathcal{O}_{X,x}$, and Sheaves, Lemma 6.32.1. It follows that i_* is fully faithful.

Let $\mathcal{G}$ be a $\mathcal{O}_X$-module with $\mathcal{I}\mathcal{G} = 0$. If $x \in X$, $x \notin i(Z)$, then $\mathcal{G}_x = 0$ because $\mathcal{I}_x = \mathcal{O}_{X,x}$ in this case. Thus we see that $\mathcal{G}$ us supported on Z. By Lemma 17.6.1 we can write $\mathcal{G} = i_*\mathcal{F}$ for a unique abelian sheaf $\mathcal{F}$ on Z. Let $W \subset Z$ be open, $f \in \mathcal{O}_Z(W)$ and $s \in \mathcal{F}(W)$. We define $fs \in \mathcal{F}(W)$. Since $i^\sharp$ is surjective we can find opens $U_j \subset X$ such that $W = \bigcup i^{-1}(U_j)$ and $f|i^{-1}(U_j)$ is the image of $f_j \in \mathcal{O}_X(U_j)$. Note that $s|_{i^{-1}(U_j)}$ is an element of $\mathcal{F}(i^{-1}(U_j)) = \mathcal{G}(U_i)$. Thus we can form $s_j = f_j s \in \mathcal{F}(i^{-1}(U_j)) = \mathcal{G}(U_i)$. By our assumption that $\mathcal{I}\mathcal{G} = 0$ the sections s_j are independent of the choice of f_j lifting $f|i^{-1}(U_j)$ and glue to a section fs of $\mathcal{F}$ over W. In this way $\mathcal{F}$ becomes an $\mathcal{O}_Z$-module such that $\mathcal{G} \cong i_*\mathcal{F}$. □

17.14. Locally free sheaves

Let $(X, \mathcal{O}_X)$ be a ringed space. Our conventions allow (some of) the stalks $\mathcal{O}_{X,x}$ to be the zero ring. This means we have to be a little careful when defining the rank of a locally free sheaf.

Definition 17.14.1. Let $(X, \mathcal{O}_X)$ be a ringed space. Let $\mathcal{F}$ be a sheaf of $\mathcal{O}_X$-modules.

(1) We say $\mathcal{F}$ is *locally free* if for every point $x \in X$ there exists a set I and an open neighbourhood $x \in U \subset X$ such that $\mathcal{F}|_U$ is isomorphic to $\bigoplus_{i \in I} \mathcal{O}_X|_U$ as an $\mathcal{O}_X|_U$-module.
(2) We say $\mathcal{F}$ is *finite locally free* if we may choose the index sets I to be finite.
(3) We say $\mathcal{F}$ is *finite locally free of rank r* if we may choose the index sets I to have cardinality r.

A finite direct sum of (finite) locally free sheaves is (finite) locally free. However, it may not be the case that an infinite direct sum of locally free sheaves is locally free.

Lemma 17.14.2. *Let $(X, \mathcal{O}_X)$ be a ringed space. Let $\mathcal{F}$ be a sheaf of $\mathcal{O}_X$-modules. If $\mathcal{F}$ is locally free then it is quasi-coherent.*

Proof. Omitted. □

Lemma 17.14.3. *Let $f : (X, \mathcal{O}_X) \to (Y, \mathcal{O}_Y)$ be a morphism of ringed spaces. If $\mathcal{G}$ is a locally free $\mathcal{O}_Y$-module, then $f^*\mathcal{G}$ is a locally free $\mathcal{O}_X$-module.*

Proof. Omitted. □

Lemma 17.14.4. *Let $(X, \mathcal{O}_X)$ be a ringed space. Suppose that the support of $\mathcal{O}_X$ is X, i.e., all stalk of $\mathcal{O}_X$ are nonzero rings. Let $\mathcal{F}$ be a locally free sheaf of $\mathcal{O}_X$-modules. There exists a locally constant function*

$$rank_{\mathcal{F}} : X \longrightarrow \{0, 1, 2, \ldots\} \cup \{\infty\}$$

such that for any point $x \in X$ the cardinality of any set I such that $\mathcal{F}$ is isomorphic to $\bigoplus_{i \in I} \mathcal{O}_X$ in a neighbourhood of x is $rank_{\mathcal{F}}(x)$.

Proof. Under the assumption of the lemma the cardinality of I can be read off from the rank of the free module $\mathcal{F}_x$ over the nonzero ring $\mathcal{O}_{X,x}$, and it is constant in a neighbourhood of x. □

Lemma 17.14.5. *Let $(X, \mathcal{O}_X)$ be a ringed space. Let $r \geq 0$. Let $\varphi : \mathcal{F} \to \mathcal{G}$ be a map of finite locally free $\mathcal{O}_X$-modules of rank r. Then φ is an isomorphism if and only if φ is surjective.*

Proof. Assume φ is surjective. Pick $x \in X$. There exists an open neighbourhood U of x such that both $\mathcal{F}|_U$ and $\mathcal{G}|_U$ are isomorphic to $\mathcal{O}_U^{\oplus r}$. Pick lifts of the free generators of $\mathcal{G}|_U$ to obtain a map $\psi : \mathcal{G}|_U \to \mathcal{F}|_U$ such that $\varphi|_U \circ \psi = \mathrm{id}$. Hence we conclude that the map $\Gamma(U, \mathcal{F}) \to \Gamma(U, \mathcal{G})$ induced by φ is surjective. Since both $\Gamma(U, \mathcal{F})$ and $\Gamma(U, \mathcal{G})$ are isomorphic to $\Gamma(U, \mathcal{O}_U)^{\oplus n}$ as an $\Gamma(U, \mathcal{O}_U)$-module we may apply Algebra, Lemma 10.15.4 to see that $\Gamma(U, \mathcal{F}) \to \Gamma(U, \mathcal{G})$ is injective. This finishes the proof. □

17.15. Tensor product

Let $(X, \mathcal{O}_X)$ be a ringed space. Let $\mathcal{F}$, $\mathcal{G}$ be $\mathcal{O}_X$-modules. We have briefly discussed the tensor product in the setting of change of rings in Sheaves, Sections 6.6 and 6.20. In exactly the same way we define first the *tensor product presheaf*

$$\mathcal{F} \otimes_{p, \mathcal{O}_X} \mathcal{G}$$

as the rule which assigns to $U \subset X$ open the $\mathcal{O}_X(U)$-module $\mathcal{F}(U) \otimes_{\mathcal{O}_X(U)} \mathcal{G}(U)$. Having defined this we define the *tensor product sheaf* as the sheafification of the above:

$$\mathcal{F} \otimes_{\mathcal{O}_X} \mathcal{G} = (\mathcal{F} \otimes_{p, \mathcal{O}_X} \mathcal{G})^{\#}$$

This can be characterized as the sheaf of $\mathcal{O}_X$-modules such that for any third sheaf of $\mathcal{O}_X$-modules $\mathcal{H}$ we have

$$\mathrm{Hom}_{\mathcal{O}_X}(\mathcal{F} \otimes_{\mathcal{O}_X} \mathcal{G}, \mathcal{H}) = \mathrm{Bilin}_{\mathcal{O}_X}(\mathcal{F} \times \mathcal{G}, \mathcal{H}).$$

Here the right hand side indicates the set of bilinear maps of sheaves of $\mathcal{O}_X$-modules (definition omitted).

The tensor product of modules M, N over a ring R satisfies symmetry, namely $M \otimes_R N = N \otimes_R M$, hence the same holds for tensor products of sheaves of modules, i.e., we have

$$\mathcal{F} \otimes_{\mathcal{O}_X} \mathcal{G} = \mathcal{G} \otimes_{\mathcal{O}_X} \mathcal{F}$$

functorial in $\mathcal{F}$, $\mathcal{G}$. And since tensor product of modules satisfies associativity we also get canonical functorial isomorphisms

$$(\mathcal{F} \otimes_{\mathcal{O}_X} \mathcal{G}) \otimes_{\mathcal{O}_X} \mathcal{H} = \mathcal{F} \otimes_{\mathcal{O}_X} (\mathcal{G} \otimes_{\mathcal{O}_X} \mathcal{H})$$

functorial in $\mathcal{F}$, $\mathcal{G}$, and $\mathcal{H}$.

Lemma 17.15.1. *Let $(X, \mathcal{O}_X)$ be a ringed space. Let $\mathcal{F}$, $\mathcal{G}$ be $\mathcal{O}_X$-modules. Let $x \in X$. There is a canonical isomorphism of $\mathcal{O}_{X,x}$-modules*

$$(\mathcal{F} \otimes_{\mathcal{O}_X} \mathcal{G})_x = \mathcal{F}_x \otimes_{\mathcal{O}_{X,x}} \mathcal{G}_x$$

functorial in $\mathcal{F}$ and $\mathcal{G}$.

Proof. Omitted. □

Lemma 17.15.2. *Let $(X, \mathcal{O}_X)$ be a ringed space. Let $\mathcal{F}'$, $\mathcal{G}'$ be presheaves of $\mathcal{O}_X$-modules with sheafifications $\mathcal{F}$, $\mathcal{G}$. Then $\mathcal{F} \otimes_{\mathcal{O}_X} \mathcal{G} = (\mathcal{F}' \otimes_{p,\mathcal{O}_X} \mathcal{G}')^\#$.*

Proof. Omitted. □

Lemma 17.15.3. *Let $(X, \mathcal{O}_X)$ be a ringed space. Let $\mathcal{G}$ be an $\mathcal{O}_X$-module. If $\mathcal{F}_1 \to \mathcal{F}_2 \to \mathcal{F}_3 \to 0$ is an exact sequence of $\mathcal{O}_X$-modules then the induced sequence*

$$\mathcal{F}_1 \otimes_{\mathcal{O}_X} \mathcal{G} \to \mathcal{F}_2 \otimes_{\mathcal{O}_X} \mathcal{G} \to \mathcal{F}_3 \otimes_{\mathcal{O}_X} \mathcal{G} \to 0$$

is exact.

Proof. This follows from the fact that exactness may be checked at stalks (Lemma 17.3.1), the description of stalks (Lemma 17.15.1) and the corresponding result for tensor products of modules (Algebra, Lemma 10.11.10). □

Lemma 17.15.4. *Let $f : (X, \mathcal{O}_X) \to (Y, \mathcal{O}_Y)$ be a morphism of ringed spaces. Let $\mathcal{F}$, $\mathcal{G}$ be $\mathcal{O}_Y$-modules. Then $f^*(\mathcal{F} \otimes_{\mathcal{O}_Y} \mathcal{G}) = f^*\mathcal{F} \otimes_{\mathcal{O}_X} f^*\mathcal{G}$ functorially in $\mathcal{F}$, $\mathcal{G}$.*

Proof. Omitted. □

Lemma 17.15.5. *Let $(X, \mathcal{O}_X)$ be a ringed space. Let $\mathcal{F}$, $\mathcal{G}$ be $\mathcal{O}_X$-modules.*

(1) *If $\mathcal{F}$, $\mathcal{G}$ are locally generated by sections, so is $\mathcal{F} \otimes_{\mathcal{O}_X} \mathcal{G}$.*
(2) *If $\mathcal{F}$, $\mathcal{G}$ are of finite type, so is $\mathcal{F} \otimes_{\mathcal{O}_X} \mathcal{G}$.*
(3) *If $\mathcal{F}$, $\mathcal{G}$ are quasi-coherent, so is $\mathcal{F} \otimes_{\mathcal{O}_X} \mathcal{G}$.*
(4) *If $\mathcal{F}$, $\mathcal{G}$ are of finite presentation, so is $\mathcal{F} \otimes_{\mathcal{O}_X} \mathcal{G}$.*
(5) *If $\mathcal{F}$ is of finite presentation and $\mathcal{G}$ is coherent, then $\mathcal{F} \otimes_{\mathcal{O}_X} \mathcal{G}$ is coherent.*
(6) *If $\mathcal{F}$, $\mathcal{G}$ are coherent, so is $\mathcal{F} \otimes_{\mathcal{O}_X} \mathcal{G}$.*
(7) *If $\mathcal{F}$, $\mathcal{G}$ are locally free, so is $\mathcal{F} \otimes_{\mathcal{O}_X} \mathcal{G}$.*

Proof. We first prove that the tensor product of locally free $\mathcal{O}_X$-modules is locally free. This follows if we show that $(\bigoplus_{i \in I} \mathcal{O}_X) \otimes_{\mathcal{O}_X} (\bigoplus_{j \in J} \mathcal{O}_X) \cong \bigoplus_{(i,j) \in I \times J} \mathcal{O}_X$.

The sheaf $\bigoplus_{i\in I}\mathcal{O}_X$ is the sheaf associated to the presheaf $U \mapsto \bigoplus_{i\in I}\mathcal{O}_X(U)$. Hence the tensor product is the sheaf associated to the presheaf

$$U \longmapsto (\bigoplus\nolimits_{i\in I}\mathcal{O}_X(U)) \otimes_{\mathcal{O}_X(U)} (\bigoplus\nolimits_{j\in J}\mathcal{O}_X(U)).$$

We deduce what we want since for any ring R we have $(\bigoplus_{i\in I} R) \otimes_R (\bigoplus_{j\in J} R) = \bigoplus_{(i,j)\in I\times J} R$.

If $\mathcal{F}_2 \to \mathcal{F}_1 \to \mathcal{F} \to 0$ is exact, then by Lemma 17.15.3 the complex $\mathcal{F}_2 \otimes \mathcal{G} \to \mathcal{F}_1 \otimes \mathcal{G} \to \mathcal{F} \otimes \mathcal{G} \to 0$ is exact. Using this we can prove (5). Namely, in this case there exists locally such an exact sequence with $\mathcal{F}_i$, $i = 1, 2$ finite free. Hence the two terms $\mathcal{F}_2 \otimes \mathcal{G}$ are isomorphic to finite direct sums of $\mathcal{G}$. Since finite direct sums are coherent sheaves, these are coherent and so is the cokernel of the map, see Lemma 17.12.4.

And if also $\mathcal{G}_2 \to \mathcal{G}_1 \to \mathcal{G} \to 0$ is exact, then we see that

$$\mathcal{F}_2 \otimes_{\mathcal{O}_X} \mathcal{G}_1 \oplus \mathcal{F}_1 \otimes_{\mathcal{O}_X} \mathcal{G}_2 \to \mathcal{F}_1 \otimes_{\mathcal{O}_X} \mathcal{G}_1 \to \mathcal{F} \otimes_{\mathcal{O}_X} \mathcal{G} \to 0$$

is exact. Using this we can for example prove (3). Namely, the assumption means that we can locally find presentations as above with $\mathcal{F}_i$ and $\mathcal{G}_i$ free $\mathcal{O}_X$-modules. Hence the displayed presentation is a presentation of the tensor product by free sheaves as well.

The proof of the other statements is omitted. □

Lemma 17.15.6. *Let $(X, \mathcal{O}_X)$ be a ringed space. For any $\mathcal{O}_X$-module $\mathcal{F}$ the functor*

$$Mod(\mathcal{O}_X) \longrightarrow Mod(\mathcal{O}_X), \quad \mathcal{G} \longmapsto \mathcal{F} \otimes_{\mathcal{O}} \mathcal{G}$$

commutes with arbitrary colimits.

Proof. Let I be a partially ordered set and let $\{\mathcal{G}_i\}$ be a system over I. Set $\mathcal{G} = \operatorname{colim}_i \mathcal{G}_i$. Recall that $\mathcal{G}$ is the sheaf associated to the presheaf $\mathcal{G}' : U \mapsto \operatorname{colim}_i \mathcal{G}_i(U)$, see Sheaves, Section 6.29. By Lemma 17.15.2 the tensor product $\mathcal{F} \otimes_{\mathcal{O}_X} \mathcal{G}$ is the sheafification of the presheaf

$$U \longmapsto \mathcal{F}(U) \otimes_{\mathcal{O}_X(U)} \operatorname{colim}_i \mathcal{G}_i(U) = \operatorname{colim}_i \mathcal{F}(U) \otimes_{\mathcal{O}_X(U)} \mathcal{G}_i(U)$$

where the equality sign is Algebra, Lemma 10.11.9. Hence the lemma follows from the description of colimits in $Mod(\mathcal{O}_X)$. □

17.16. Flat modules

We can define flat modules exactly as in the case of modules over rings.

Definition 17.16.1. Let $(X, \mathcal{O}_X)$ be a ringed space. An $\mathcal{O}_X$-module $\mathcal{F}$ is *flat* if the functor

$$Mod(\mathcal{O}_X) \longrightarrow Mod(\mathcal{O}_X), \quad \mathcal{G} \mapsto \mathcal{G} \otimes_{\mathcal{O}} \mathcal{F}$$

is exact.

We can characterize flatness by looking at the stalks.

Lemma 17.16.2. *Let $(X, \mathcal{O}_X)$ be a ringed space. An $\mathcal{O}_X$-module $\mathcal{F}$ is flat if and only if the stalk $\mathcal{F}_x$ is a flat $\mathcal{O}_{X,x}$-module for all $x \in X$.*

Proof. Assume $\mathcal{F}_x$ is a flat $\mathcal{O}_{X,x}$-module for all $x \in X$. In this case, if $\mathcal{G} \to \mathcal{H} \to \mathcal{K}$ is exact, then also $\mathcal{G} \otimes_{\mathcal{O}_X} \mathcal{F} \to \mathcal{G} \otimes_{\mathcal{O}_X} \mathcal{F} \to \mathcal{G} \otimes_{\mathcal{O}_X} \mathcal{F}$ is exact because we can check exactness at stalks and because tensor product commutes with taking stalks, see Lemma 17.15.1. Conversely, suppose that $\mathcal{F}$ is flat, and let $x \in X$. Consider the skyscraper sheaves $i_{x,*}M$ where M is a $\mathcal{O}_{X,x}$-module. Note that

$$M \otimes_{\mathcal{O}_{X,x}} \mathcal{F}_x = (i_{x,*}M \otimes_{\mathcal{O}_X} \mathcal{F}))_x$$

again by Lemma 17.15.1. Since $i_{x,*}$ is exact, we see that the fact that $\mathcal{F}$ is flat implies that $M \mapsto M \otimes_{\mathcal{O}_{X,x}} \mathcal{F}_x$ is exact. Hence $\mathcal{F}_x$ is a flat $\mathcal{O}_{X,x}$-module. □

Thus the following definition makes sense.

Definition 17.16.3. Let $(X, \mathcal{O}_X)$ be a ringed space. Let $x \in X$. An $\mathcal{O}_X$-module $\mathcal{F}$ is *flat at* x if $\mathcal{F}_x$ is a flat $\mathcal{O}_{X,x}$-module.

Hence we see that $\mathcal{F}$ is a flat $\mathcal{O}_X$-module if and only if it is flat at every point.

Lemma 17.16.4. *Let $(X, \mathcal{O}_X)$ be a ringed space. A filtered colimit of flat $\mathcal{O}_X$-modules is flat. A direct sum of flat $\mathcal{O}_X$-modules is flat.*

Proof. This follows from Lemma 17.15.6, Lemma 17.15.1, Algebra, Lemma 10.8.9, and the fact that we can check exactness at stalks. □

Lemma 17.16.5. *Let $(X, \mathcal{O}_X)$ be a ringed space. Let $U \subset X$ be open. The sheaf $j_{U!}\mathcal{O}_U$ is a flat sheaf of $\mathcal{O}_X$-modules.*

Proof. The stalks of $j_!\mathcal{O}_U$ are either zero of equal to $\mathcal{O}_{X,x}$. Apply Lemma 17.16.2. □

Lemma 17.16.6. *Let $(X, \mathcal{O}_X)$ be a ringed space.*

(1) *Any sheaf of $\mathcal{O}_X$-modules is a quotient of a direct sum $\bigoplus j_{U_i!}\mathcal{O}_{U_i}$.*
(2) *Any $\mathcal{O}_X$-module is a quotient of a flat $\mathcal{O}_X$-module.*

Proof. Let $\mathcal{F}$ be an $\mathcal{O}_X$-module. For every open $U \subset X$ and every $s \in \mathcal{F}(U)$ we get a morphism $j_{U!}\mathcal{O}_U \to \mathcal{F}$, namely the adjoint to the morphism $\mathcal{O}_U \to \mathcal{F}|_U$, $1 \mapsto s$. Clearly the map

$$\bigoplus\nolimits_{(U,s)} j_{U!}\mathcal{O}_U \longrightarrow \mathcal{F}$$

is surjective, and the source is flat by combining Lemmas 17.16.4 and 17.16.5. □

Lemma 17.16.7. *Let $(X, \mathcal{O}_X)$ be a ringed space. Let*

$$0 \to \mathcal{F}'' \to \mathcal{F}' \to \mathcal{F} \to 0$$

be a short exact sequence of $\mathcal{O}_X$-modules. Assume $\mathcal{F}$ is flat. Then for any $\mathcal{O}_X$-module $\mathcal{G}$ the sequence

$$0 \to \mathcal{F}'' \otimes_{\mathcal{O}} \mathcal{G} \to \mathcal{F}' \otimes_{\mathcal{O}} \mathcal{G} \to \mathcal{F} \otimes_{\mathcal{O}} \mathcal{G} \to 0$$

is exact.

Proof. Using that $\mathcal{F}_x$ is a flat $\mathcal{O}_{X,x}$-module for every $x \in X$ and that exactness can be checked on stalks, this follows from Algebra, Lemma 10.36.11. □

Lemma 17.16.8. *Let $(X, \mathcal{O}_X)$ be a ringed space. Let*

$$0 \to \mathcal{F}_2 \to \mathcal{F}_1 \to \mathcal{F}_0 \to 0$$

be a short exact sequence of $\mathcal{O}_X$-modules.

(1) *If $\mathcal{F}_2$ and $\mathcal{F}_0$ are flat so is $\mathcal{F}_1$.*
(2) *If $\mathcal{F}_1$ and $\mathcal{F}_0$ are flat so is $\mathcal{F}_2$.*

Proof. Since exactness and flatness may be checked at the level of stalks this follows from Algebra, Lemma 10.36.12. □

Lemma 17.16.9. *Let $(X, \mathcal{O}_X)$ be a ringed space. Let*

$$\ldots \to \mathcal{F}_2 \to \mathcal{F}_1 \to \mathcal{F}_0 \to \mathcal{Q} \to 0$$

be an exact complex of $\mathcal{O}_X$-modules. If $\mathcal{Q}$ and all $\mathcal{F}_i$ are flat $\mathcal{O}_X$-modules, then for any $\mathcal{O}_X$-module $\mathcal{G}$ the complex

$$\ldots \to \mathcal{F}_2 \otimes_{\mathcal{O}_X} \mathcal{G} \to \mathcal{F}_1 \otimes_{\mathcal{O}_X} \mathcal{G} \to \mathcal{F}_0 \otimes_{\mathcal{O}_X} \mathcal{G} \to \mathcal{Q} \otimes_{\mathcal{O}_X} \mathcal{G} \to 0$$

is exact also.

Proof. Follows from Lemma 17.16.7 by splitting the complex into short exact sequences and using Lemma 17.16.8 to prove inductively that $\mathrm{Im}(\mathcal{F}_{i+1} \to \mathcal{F}_i)$ is flat. □

The following lemma gives one direction of the equational criterion of flatness (Algebra, Lemma 10.36.10).

Lemma 17.16.10. *Let $(X, \mathcal{O}_X)$ be a ringed space. Let $\mathcal{F}$ be a flat $\mathcal{O}_X$-module. Let $U \subset X$ be open and let*

$$\mathcal{O}_U \xrightarrow{(f_1, \ldots, f_n)} \mathcal{O}_U^{\oplus n} \xrightarrow{(s_1, \ldots, s_n)} \mathcal{F}|_U$$

be a complex of $\mathcal{O}_U$-modules. For every $x \in U$ there exists an open neighbourhood $V \subset U$ of x and a factorization

$$\mathcal{O}_V^{\oplus n} \xrightarrow{A} \mathcal{O}_V^{\oplus m} \xrightarrow{(t_1, \ldots, t_m)} \mathcal{F}|_V$$

of $(s_1, \ldots, s_n)|_V$ such that $A \circ (f_1, \ldots, f_n)|_V = 0$.

Proof. Let $\mathcal{I} \subset \mathcal{O}_U$ be the sheaf of ideals generated by $f_1, \ldots, f_n$. Then $\sum f_i \otimes s_i$ is a section of $\mathcal{I} \otimes_{\mathcal{O}_U} \mathcal{F}|_U$ which maps to zero in $\mathcal{F}|_U$. As $\mathcal{F}|_U$ is flat the map $\mathcal{I} \otimes_{\mathcal{O}_U} \mathcal{F}|_U \to \mathcal{F}|_U$ is injective. Since $\mathcal{I} \otimes_{\mathcal{O}_U} \mathcal{F}|_U$ is the sheaf associated to the presheaf tensor product, we see there exists an open neighbourhood $V \subset U$ of x such that $\sum f_i|_V \otimes s_i|_V$ is zero in $\mathcal{I}(V) \otimes_{\mathcal{O}(V)} \mathcal{F}(V)$. Unwinding the definitions using Algebra, Lemma 10.102.10 we find $t_1, \ldots, t_m \in \mathcal{F}(V)$ and $a_{ij} \in \mathcal{O}(V)$ such that $\sum a_{ij} f_i|_V = 0$ and $s_i|_V = \sum a_{ij} t_j$. □

Lemma 17.16.11. *Let $(X, \mathcal{O}_X)$ be a ringed space. Let $\mathcal{F}$ be locally of finite presentation and flat. Then $\mathcal{F}$ is locally a direct summand of a finite free $\mathcal{O}_X$-module.*

Proof. After replacing X by the members of an open covering, we may assume there exists a presentation

$$\mathcal{O}_X^{\oplus r} \to \mathcal{O}_X^{\oplus n} \to \mathcal{F} \to 0$$

Let $x \in X$. By Lemma 17.16.10 we can, after shrinking X to an open neighbourhood of x, assume there exists a factorization

$$\mathcal{O}_X^{\oplus n} \to \mathcal{O}_X^{\oplus n_1} \to \mathcal{F}$$

such that the composition $\mathcal{O}_X^{\oplus r} \to \mathcal{O}_X^{\oplus n} \to \mathcal{O}_X^{\oplus n_1}$ annihilates the first summand of $\mathcal{O}_X^{\oplus r}$. Repeating this argument $r - 1$ more times we obtain a factorization

$$\mathcal{O}_X^{\oplus n} \to \mathcal{O}_X^{\oplus n_r} \to \mathcal{F}$$

such that the composition $\mathcal{O}_X^{\oplus r} \to \mathcal{O}_X^{\oplus n} \to \mathcal{O}_X^{\oplus n_r}$ is zero. This means that the surjection $\mathcal{O}_X^{\oplus n_r} \to \mathcal{F}$ has a section and we win. □

17.17. Flat morphisms of ringed spaces

The pointwise definition is motivated by Lemma 17.16.2 and Definition 17.16.3 above.

Definition 17.17.1. Let $f : X \to Y$ be a morphism of ringed spaces. Let $x \in X$. We say f is said to be *flat at* x if the map of rings $\mathcal{O}_{Y,f(x)} \to \mathcal{O}_{X,x}$ is flat. We say f is *flat* if f is flat at every $x \in X$.

Consider the map of sheaves of rings $f^\sharp : f^{-1}\mathcal{O}_Y \to \mathcal{O}_X$. We see that the stalk at x is the ring map $f_x^\sharp : \mathcal{O}_{Y,f(x)} \to \mathcal{O}_{X,x}$. Hence f is flat at x if and only if $\mathcal{O}_X$ is flat at x as an $f^{-1}\mathcal{O}_Y$-module. And f is flat if and only if $\mathcal{O}_X$ is flat as an $f^{-1}\mathcal{O}_Y$-module. A very special case of a flat morphism is an open immersion.

Lemma 17.17.2. *Let $f : X \to Y$ be a flat morphism of ringed spaces. Then the pullback functor $f^* : Mod(\mathcal{O}_Y) \to Mod(\mathcal{O}_X)$ is exact.*

Proof. The functor f^* is the composition of the exact functor $f^{-1} : Mod(\mathcal{O}_Y) \to Mod(f^{-1}\mathcal{O}_Y)$ and the change of rings functor

$$Mod(f^{-1}\mathcal{O}_Y) \to Mod(\mathcal{O}_X), \quad \mathcal{F} \longmapsto \mathcal{F} \otimes_{f^{-1}\mathcal{O}_Y} \mathcal{O}_X.$$

Thus the result follows from the discussion following Definition 17.17.1. □

Definition 17.17.3. Let $f : (X, \mathcal{O}_X) \to (Y, \mathcal{O}_Y)$ be a morphism of ringed spaces. Let $\mathcal{F}$ be a sheaf of $\mathcal{O}_X$-modules.

(1) We say that $\mathcal{F}$ is *flat over Y at a point $x \in X$* if the stalk $\mathcal{F}_x$ is a flat $\mathcal{O}_{Y,f(x)}$-module.
(2) We say that $\mathcal{F}$ is *flat over Y* if $\mathcal{F}$ is flat over Y at every point x of X.

With this definition we see that $\mathcal{F}$ is flat over Y at x if and only if $\mathcal{F}$ is flat at x as an $f^{-1}\mathcal{O}_Y$-module because $(f^{-1}\mathcal{O}_Y)_x = \mathcal{O}_{Y,f(x)}$ by Sheaves, Lemma 6.21.5.

17.18. Symmetric and exterior powers

Let $(X, \mathcal{O}_X)$ be a ringed space. Let $\mathcal{F}$ be an $\mathcal{O}_X$-algebra. We define the *tensor algebra* of $\mathcal{F}$ to be the sheaf of noncommutative $\mathcal{O}_X$-algebras

$$\mathrm{T}(\mathcal{F}) = \mathrm{T}_{\mathcal{O}_X}(\mathcal{F}) = \bigoplus\nolimits_{n \geq 0} \mathrm{T}^n(\mathcal{F}).$$

Here $\mathrm{T}^0(\mathcal{F}) = \mathcal{O}_X$, $\mathrm{T}^1(\mathcal{F}) = \mathcal{F}$ and for $n \geq 2$ we have

$$\mathrm{T}^n(\mathcal{F}) = \mathcal{F} \otimes_{\mathcal{O}_X} \ldots \otimes_{\mathcal{O}_X} \mathcal{F} \quad (n \text{ factors})$$

We define $\wedge(\mathcal{F})$ to be the quotient of $\mathrm{T}(\mathcal{F})$ by the two sided ideal generated by local sections $s \otimes s$ of $\mathrm{T}^2(\mathcal{F})$ where s is a local section of $\mathcal{F}$. This is called the *exterior algebra* of $\mathcal{F}$. Similarly, we define $\mathrm{Sym}(\mathcal{F})$ to be the quotient of $\mathrm{T}(\mathcal{F})$ by the two sided ideal generated by local sections of the form $s \otimes t - t \otimes s$ of $\mathrm{T}^2(\mathcal{F})$.

Both $\wedge(\mathcal{F})$ and $\mathrm{Sym}(\mathcal{F})$ are graded $\mathcal{O}_X$-algebras, with grading inherited from $\mathrm{T}(\mathcal{F})$. Moreover $\mathrm{Sym}(\mathcal{F})$ is commutative, and $\wedge(\mathcal{F})$ is graded commutative.

Lemma 17.18.1. *In the situation described above. The sheaf $\wedge^n\mathcal{F}$ is the sheafification of the presheaf*

$$U \longmapsto \wedge^n_{\mathcal{O}_X(U)}(\mathcal{F}(U)).$$

See Algebra, Section 10.12. Similarly, the sheaf $Sym^n\mathcal{F}$ is the sheafification of the presheaf

$$U \longmapsto Sym^n_{\mathcal{O}_X(U)}(\mathcal{F}(U)).$$

Proof. Omitted. It may be more efficient to define $\mathrm{Sym}(\mathcal{F})$ and $\wedge(\mathcal{F})$ in this way instead of the method given above. □

Lemma 17.18.2. *In the situation described above. Let $x \in X$. There are canonical isomorphisms of $\mathcal{O}_{X,x}$-modules $T(\mathcal{F})_x = T(\mathcal{F}_x)$, $Sym(\mathcal{F})_x = Sym(\mathcal{F}_x)$, and $\wedge(\mathcal{F})_x = \wedge(\mathcal{F}_x)$.*

Proof. Clear from Lemma 17.18.1 above, and Algebra, Lemma 10.12.4. □

Lemma 17.18.3. *Let $f : (X, \mathcal{O}_X) \to (Y, \mathcal{O}_Y)$ be a morphism of ringed spaces. Let $\mathcal{F}$ be a sheaf of $\mathcal{O}_Y$-modules. Then $f^*T(\mathcal{F}) = T(f^*\mathcal{F})$, and similarly for the exterior and symmetric algebras associated to $\mathcal{F}$.*

Proof. Omitted. □

Lemma 17.18.4. *Let $(X, \mathcal{O}_X)$ be a ringed space. Let $\mathcal{F}_2 \to \mathcal{F}_1 \to \mathcal{F} \to 0$ be an exact sequence of sheaves of $\mathcal{O}_X$-modules. For each $n \geq 1$ there is an exact sequence*

$$\mathcal{F}_2 \otimes_{\mathcal{O}_X} Sym^{n-1}(\mathcal{F}_1) \to Sym^n(\mathcal{F}_1) \to Sym^n(\mathcal{F}) \to 0$$

and similarly an exact sequence

$$\mathcal{F}_2 \otimes_{\mathcal{O}_X} \wedge^{n-1}(\mathcal{F}_1) \to \wedge^n(\mathcal{F}_1) \to \wedge^n(\mathcal{F}) \to 0$$

Proof. See Algebra, Lemma 10.12.2. □

Lemma 17.18.5. *Let $(X, \mathcal{O}_X)$ be a ringed space. Let $\mathcal{F}$ be a sheaf of $\mathcal{O}_X$-modules.*

1. *If $\mathcal{F}$ is locally generated by sections, then so is each $T^n(\mathcal{F})$, $\wedge^n(\mathcal{F})$, and $Sym^n(\mathcal{F})$.*
2. *If $\mathcal{F}$ is of finite type, then so is each $T^n(\mathcal{F})$, $\wedge^n(\mathcal{F})$, and $Sym^n(\mathcal{F})$.*
3. *If $\mathcal{F}$ is of finite presentation, then so is each $T^n(\mathcal{F})$, $\wedge^n(\mathcal{F})$, and $Sym^n(\mathcal{F})$.*
4. *If $\mathcal{F}$ is coherent, then for $n > 0$ each $T^n(\mathcal{F})$, $\wedge^n(\mathcal{F})$, and $Sym^n(\mathcal{F})$ is coherent.*
5. *If $\mathcal{F}$ is quasi-coherent, then so is each $T^n(\mathcal{F})$, $\wedge^n(\mathcal{F})$, and $Sym^n(\mathcal{F})$.*
6. *If $\mathcal{F}$ is locally free, then so is each $T^n(\mathcal{F})$, $\wedge^n(\mathcal{F})$, and $Sym^n(\mathcal{F})$.*

Proof. These statements for $\mathrm{T}^n(\mathcal{F})$ follow from Lemma 17.15.5.

Statements (1) and (2) follow from the fact that $\wedge^n(\mathcal{F})$ and $\mathrm{Sym}^n(\mathcal{F})$ are quotients of $\mathrm{T}^n(\mathcal{F})$.

Statement (6) follows from Algebra, Lemma 10.12.1.

For (3) and (5) we will use Lemma 17.18.4 above. By locally choosing a presentation $\mathcal{F}_2 \to \mathcal{F}_1 \to \mathcal{F} \to 0$ with $\mathcal{F}_i$ free, or finite free and applying the lemma we see that $\mathrm{Sym}^n(\mathcal{F})$, $\wedge^n(\mathcal{F})$ has a similar presentation; here we use (6) and Lemma 17.15.5.

To prove (4) we will use Algebra, Lemma 10.12.3. We may localize on X and assume that $\mathcal{F}$ is generated by a finite set $(s_i)_{i\in I}$ of global sections. The lemma

mentioned above combined with Lemma 17.18.1 above implies that for $n \geq 2$ there exists an exact sequence

$$\bigoplus_{j \in J} \mathrm{T}^{n-2}(\mathcal{F}) \to \mathrm{T}^n(\mathcal{F}) \to \mathrm{Sym}^n(\mathcal{F}) \to 0$$

where the index set J is finite. Now we know that $\mathrm{T}^{n-2}(\mathcal{F})$ is finitely generated and hence the image of the first arrow is a coherent subsheaf of $\mathrm{T}^n(\mathcal{F})$, see Lemma 17.12.4. By that same lemma we conclude that $\mathrm{Sym}^n(\mathcal{F})$ is coherent. □

Lemma 17.18.6. *Let $(X, \mathcal{O}_X)$ be a ringed space. Let $\mathcal{F}$ be a sheaf of $\mathcal{O}_X$-modules.*

(1) *If $\mathcal{F}$ is quasi-coherent, then so is each $T(\mathcal{F})$, $\wedge(\mathcal{F})$, and $Sym(\mathcal{F})$.*
(2) *If $\mathcal{F}$ is locally free, then so is each $T(\mathcal{F})$, $\wedge(\mathcal{F})$, and $Sym(\mathcal{F})$.*

Proof. It is not true that an infinite direct sum $\bigoplus \mathcal{G}_i$ of locally free modules is locally free, or that an infinite direct sum of quasi-coherent modules is quasi-coherent. The problem is that given a point $x \in X$ the open neighbourhoods U_i of x on which $\mathcal{G}_i$ becomes free (resp. has a suitable presentation) may have an intersection which is not an open neighbourhood of x. However, in the proof of Lemma 17.18.5 we saw that once a suitable open neighbourhood for $\mathcal{F}$ has been chosen, then this open neighbourhood works for each of the sheaves $\mathrm{T}^n(\mathcal{F})$, $\wedge^n(\mathcal{F})$ and $\mathrm{Sym}^n(\mathcal{F})$. The lemma follows. □

17.19. Internal Hom

Let $(X, \mathcal{O}_X)$ be a ringed space. Let $\mathcal{F}$, $\mathcal{G}$ be $\mathcal{O}_X$-modules. Consider the rule

$$U \longmapsto \mathrm{Hom}_{\mathcal{O}_X|_U}(\mathcal{F}|_U, \mathcal{G}|_U).$$

It follows from the discussion in Sheaves, Section 6.33 that this is a sheaf of abelian groups. In addition, given an element $\varphi \in \mathrm{Hom}_{\mathcal{O}_X|_U}(\mathcal{F}|_U, \mathcal{G}|_U)$ and a section $f \in \mathcal{O}_X(U)$ then we can define $f\varphi \in \mathrm{Hom}_{\mathcal{O}_X|_U}(\mathcal{F}|_U, \mathcal{G}|_U)$ by either precomposing with multiplication by f on $\mathcal{F}|_U$ or postcomposing with multiplication by f on $\mathcal{G}|_U$ (it gives the same result). Hence we in fact get a sheaf of $\mathcal{O}_X$-modules. We will denote this sheaf $\mathcal{H}om_{\mathcal{O}_X}(\mathcal{F}, \mathcal{G})$. There is a canonical "evaluation" morphism

$$\mathcal{F} \otimes_{\mathcal{O}_X} \mathcal{H}om_{\mathcal{O}_X}(\mathcal{F}, \mathcal{G}) \longrightarrow \mathcal{G}.$$

For every $x \in X$ there is also a canonical morphism

$$\mathcal{H}om_{\mathcal{O}_X}(\mathcal{F}, \mathcal{G})_x \to \mathrm{Hom}_{\mathcal{O}_{X,x}}(\mathcal{F}_x, \mathcal{G}_x)$$

which is rarely an isomorphism.

Lemma 17.19.1. *Let $(X, \mathcal{O}_X)$ be a ringed space. Let $\mathcal{F}$, $\mathcal{G}$, $\mathcal{H}$ be $\mathcal{O}_X$-modules. There is a canonical isomorphism*

$$\mathcal{H}om_{\mathcal{O}_X}(\mathcal{F} \otimes_{\mathcal{O}_X} \mathcal{G}, \mathcal{H}) \longrightarrow \mathcal{H}om_{\mathcal{O}_X}(\mathcal{F}, \mathcal{H}om_{\mathcal{O}_X}(\mathcal{G}, \mathcal{H}))$$

which is functorial in all three entries (sheaf Hom in all three spots). In particular, to give a morphism $\mathcal{F} \otimes_{\mathcal{O}_X} \mathcal{G} \to \mathcal{H}$ is the same as giving a morphism $\mathcal{F} \to \mathcal{H}om_{\mathcal{O}_X}(\mathcal{G}, \mathcal{H})$.

Proof. This is the analogue of Algebra, Lemma 10.11.8. The proof is the same, and is omitted. □

Lemma 17.19.2. *Let $(X, \mathcal{O}_X)$ be a ringed space. Let $\mathcal{F}$, $\mathcal{G}$ be $\mathcal{O}_X$-modules.*

(1) *If* $\mathcal{F}_2 \to \mathcal{F}_1 \to \mathcal{F} \to 0$ *is an exact sequence of* $\mathcal{O}_X$*-modules, then*
$$0 \to \mathcal{H}om_{\mathcal{O}_X}(\mathcal{F}, \mathcal{G}) \to \mathcal{H}om_{\mathcal{O}_X}(\mathcal{F}_1, \mathcal{G}) \to \mathcal{H}om_{\mathcal{O}_X}(\mathcal{F}_2, \mathcal{G})$$
is exact.

(2) *If* $0 \to \mathcal{G} \to \mathcal{G}_1 \to \mathcal{G}_2$ *is an exact sequence of* $\mathcal{O}_X$*-modules, then*
$$0 \to \mathcal{H}om_{\mathcal{O}_X}(\mathcal{F}, \mathcal{G}) \to \mathcal{H}om_{\mathcal{O}_X}(\mathcal{F}, \mathcal{G}_1) \to \mathcal{H}om_{\mathcal{O}_X}(\mathcal{F}, \mathcal{G}_2)$$
is exact.

Proof. Omitted. □

Lemma 17.19.3. *Let* $(X, \mathcal{O}_X)$ *be a ringed space. Let* $\mathcal{F}$*,* $\mathcal{G}$ *be* $\mathcal{O}_X$*-modules. If* $\mathcal{F}$ *is finitely presented then the canonical map*
$$\mathcal{H}om_{\mathcal{O}_X}(\mathcal{F}, \mathcal{G})_x \to \mathrm{Hom}_{\mathcal{O}_{X,x}}(\mathcal{F}_x, \mathcal{G}_x)$$
is an isomorphism.

Proof. By localizing on X we may assume that $\mathcal{F}$ has a presentation
$$\bigoplus\nolimits_{j=1,\ldots,m} \mathcal{O}_X \longrightarrow \bigoplus\nolimits_{i=1,\ldots,n} \mathcal{O}_X \to \mathcal{F} \to 0.$$
By Lemma 17.19.2 this gives an exact sequence $0 \to \mathcal{H}om_{\mathcal{O}_X}(\mathcal{F}, \mathcal{G}) \to \bigoplus_{i=1,\ldots,n} \mathcal{G} \longrightarrow \bigoplus_{j=1,\ldots,m} \mathcal{G}$. Taking stalks we get an exact sequence $0 \to \mathcal{H}om_{\mathcal{O}_X}(\mathcal{F}, \mathcal{G})_x \to \bigoplus_{i=1,\ldots,n} \mathcal{G}_x \longrightarrow \bigoplus_{j=1,\ldots,m} \mathcal{G}_x$ and the result follows since $\mathcal{F}_x$ sits in an exact sequence $\bigoplus_{j=1,\ldots,m} \mathcal{O}_{X,x} \longrightarrow \bigoplus_{i=1,\ldots,n} \mathcal{O}_{X,x} \to \mathcal{F}_x \to 0$ which induces the exact sequence $0 \to \mathrm{Hom}_{\mathcal{O}_{X,x}}(\mathcal{F}_x, \mathcal{G}_x) \to \bigoplus_{i=1,\ldots,n} \mathcal{G}_x \longrightarrow \bigoplus_{j=1,\ldots,m} \mathcal{G}_x$ which is the same as the one above. □

Lemma 17.19.4. *Let* $(X, \mathcal{O}_X)$ *be a ringed space. Let* $\mathcal{F}$*,* $\mathcal{G}$ *be* $\mathcal{O}_X$*-modules. If* $\mathcal{F}$ *is finitely presented then the sheaf* $\mathcal{H}om_{\mathcal{O}_X}(\mathcal{F}, \mathcal{G})$ *is locally a kernel of a map between finite direct sums of copies of* $\mathcal{G}$*. In particular, if* $\mathcal{G}$ *is coherent then* $\mathcal{H}om_{\mathcal{O}_X}(\mathcal{F}, \mathcal{G})$ *is coherent too.*

Proof. The first assertion we saw in the proof of Lemma 17.19.3. And the result for coherent sheaves then follows from Lemma 17.12.4. □

17.20. Koszul complexes

We suggest first reading the section on Koszul complexes in More on Algebra, Section 15.19. We define the Koszul complex in the category of $\mathcal{O}_X$-modules as follows.

Definition 17.20.1. Let X be a ringed space. Let $\varphi : \mathcal{E} \to \mathcal{O}_X$ be an $\mathcal{O}_X$-module map. The *Koszul complex* $K_\bullet(\varphi)$ associated to φ is the sheaf of commutative differential graded algebras defined as follows:

(1) the underlying graded algebra is the exterior algebra $K_\bullet(\varphi) = \wedge(\mathcal{E})$,
(2) the differential $d : K_\bullet(\varphi) \to K_\bullet(\varphi)$ is the unique derivation such that $d(e) = \varphi(e)$ for all local sections e of $\mathcal{E} = K_1(\varphi)$.

Explicitly, if $e_1 \wedge \ldots \wedge e_n$ is a wedge product of local sections of $\mathcal{E}$, then
$$d(e_1 \wedge \ldots \wedge e_n) = \sum\nolimits_{i=1,\ldots,n} (-1)^{i+1} \varphi(e_i) e_1 \wedge \ldots \wedge \widehat{e_i} \wedge \ldots \wedge e_n.$$
It is straightforward to see that this gives a well defined derivation on the tensor algebra, which annihilates $e \wedge e$ and hence factors through the exterior algebra.

Definition 17.20.2. Let X be a ringed space and let $f_1, \ldots, f_n \in \Gamma(X, \mathcal{O}_X)$. The *Koszul complex on* $f_1, \ldots, f_r$ is the Koszul complex associated to the map $(f_1, \ldots, f_n) : \mathcal{O}_X^{\oplus n} \to \mathcal{O}_X$. Notation $K_\bullet(\mathcal{O}_X, f_1, \ldots, f_n)$, or $K_\bullet(\mathcal{O}_X, f_\bullet)$.

Of course, given an $\mathcal{O}_X$-module map $\varphi : \mathcal{E} \to \mathcal{O}_X$, if $\mathcal{E}$ is finite locally free, then $K_\bullet(\varphi)$ is locally on X isomorphic to a Koszul complex $K_\bullet(\mathcal{O}_X, f_1, \ldots, f_n)$.

17.21. Invertible sheaves

Definition 17.21.1. Let $(X, \mathcal{O}_X)$ be a ringed space. Assume that all stalks $\mathcal{O}_{X,x}$ are local rings[3]. An *invertible $\mathcal{O}_X$-module* is a sheaf of $\mathcal{O}_X$-modules $\mathcal{L}$ such that for each point $x \in X$ there exists an open neighbourhood $U \subset X$ and an isomorphism $\mathcal{L}|_U \cong \mathcal{O}_X|_U$. We say that $\mathcal{L}$ is *trivial* if it is isomorphic as an $\mathcal{O}_X$-module to $\mathcal{O}_X$.

Lemma 17.21.2. *Let $(X, \mathcal{O}_X)$ be a ringed space. Assume that all stalks $\mathcal{O}_{X,x}$ are local rings.*

1. *If $\mathcal{L}$, $\mathcal{N}$ are invertible $\mathcal{O}_X$-modules, then so is $\mathcal{L} \otimes_{\mathcal{O}_X} \mathcal{N}$.*
2. *If $\mathcal{L}$ is an invertible $\mathcal{O}_X$-module, then so is $\mathcal{L}^{\otimes -1} = \mathcal{H}om_{\mathcal{O}_X}(\mathcal{L}, \mathcal{O}_X)$.*
3. *If $\mathcal{L}$ is an invertible $\mathcal{O}_X$-module, then the evaluation map $\mathcal{L} \otimes_{\mathcal{O}_X} \mathcal{L}^{\otimes -1} \to \mathcal{O}_X$ is an isomorphism.*

Proof. Omitted. □

Definition 17.21.3. Let $(X, \mathcal{O}_X)$ be a ringed space. Assume that all stalks $\mathcal{O}_{X,x}$ are local rings. Given an invertible sheaf $\mathcal{L}$ on X we define the nth *tensor power of* $\mathcal{L}$ by the rule

$$\mathcal{L}^{\otimes n} = \begin{cases} \mathcal{O}_X & \text{if} \quad n = 0 \\ \mathcal{H}om_{\mathcal{O}_X}(\mathcal{L}, \mathcal{O}_X) & \text{if} \quad n = -1 \\ \mathcal{L} \otimes_{\mathcal{O}_X} \ldots \otimes_{\mathcal{O}_X} \mathcal{L} & \text{if} \quad n > 0 \\ \mathcal{L}^{\otimes -1} \otimes_{\mathcal{O}_X} \ldots \otimes_{\mathcal{O}_X} \mathcal{L}^{\otimes -1} & \text{if} \quad n < -1 \end{cases}$$

With this definition we have canonical isomorphisms $\mathcal{L}^{\otimes n} \otimes_{\mathcal{O}_X} \mathcal{L}^{\otimes m} \to \mathcal{L}^{\otimes n+m}$, and these isomorphisms satisfy a commutativity and an associativity constraint (formulation omitted). Thus we can define a $\mathbf{Z}$-graded ring structure on $\bigoplus \Gamma(X, \mathcal{L}^{\otimes n})$ by mapping $s \in \Gamma(X, \mathcal{L}^{\otimes n})$ and $t \in \Gamma(X, \mathcal{L}^{\otimes m})$ to the section corresponding to $s \otimes t$ in $\Gamma(X, \mathcal{L}^{\otimes n+m})$. We omit the verification that this defines a commutative and associative ring with 1. However, by our conventions in Algebra, Section 10.52 a graded ring has no nonzero elements in negative degrees. This leads to the following definition.

Definition 17.21.4. Let $(X, \mathcal{O}_X)$ be a ringed space. Assume that all stalks $\mathcal{O}_{X,x}$ are local rings. Given an invertible sheaf $\mathcal{L}$ on X we define the *associated graded ring* to be

$$\Gamma_*(X, \mathcal{L}) = \bigoplus\nolimits_{n \geq 0} \Gamma(X, \mathcal{L}^{\otimes n})$$

Given a sheaf of $\mathcal{O}_X$-modules $\mathcal{F}$ we set

$$\Gamma_*(X, \mathcal{L}, \mathcal{F}) = \bigoplus\nolimits_{n \in \mathbf{Z}} \Gamma(X, \mathcal{F} \otimes_{\mathcal{O}_X} \mathcal{L}^{\otimes n})$$

which we think of as a graded $\Gamma_*(X, \mathcal{L})$-module.

[3] We should at least assume that they are nonzero. However, in this generality the stalks $\mathcal{O}_{X,x}$ can have nontrivial Picard groups, and then there are two possible definitions. One were we require $\mathcal{L}$ to be locally free of rank 1, and the other where we require $\mathcal{L}$ to be a flat, finite presentation $\mathcal{O}_X$-module such that there exists a second such sheaf $\mathcal{L}^{\otimes -1}$ with $\mathcal{L} \otimes_{\mathcal{O}_X} \mathcal{L}^{\otimes -1} \cong \mathcal{O}_X$.

We often write simply $\Gamma_*(\mathcal{L})$ and $\Gamma_*(\mathcal{F})$ (although this is ambiguous if $\mathcal{F}$ is invertible). The multiplication of $\Gamma_*(\mathcal{L})$ on $\Gamma_*(\mathcal{F})$ is defined using the isomorphisms above. If $\gamma : \mathcal{F} \to \mathcal{G}$ is a $\mathcal{O}_X$-module map, then we get an $\Gamma_*(\mathcal{L})$-module homomorphism $\gamma : \Gamma_*(\mathcal{F}) \to \Gamma_*(\mathcal{G})$. If $\alpha : \mathcal{L} \to \mathcal{N}$ is an $\mathcal{O}_X$-module map between invertible $\mathcal{O}_X$-modules, then we obtain a graded ring homomorphism $\Gamma_*(\mathcal{L}) \to \Gamma_*(\mathcal{N})$. If $f : (Y, \mathcal{O}_Y) \to (X, \mathcal{O}_X)$ is a morphism of locally ringed spaces (see Schemes, Definition 25.2.1), and if $\mathcal{L}$ is invertible on X, then we get an invertible sheaf $f^*\mathcal{L}$ on Y and an induced homomorphism of graded rings

$$f^* : \Gamma_*(X, \mathcal{L}) \longrightarrow \Gamma_*(Y, f^*\mathcal{L})$$

Furthermore, there are some compatibilities between the constructions above whose statements we omit.

Lemma 17.21.5. *Let $(X, \mathcal{O}_X)$ be a ringed space. Assume that all stalks $\mathcal{O}_{X,x}$ are local rings. There exists a set of invertible modules $\{\mathcal{L}_i\}_{i \in I}$ such that each invertible module on X is isomorphic to exactly one of the $\mathcal{L}_i$.*

Proof. For each open covering $\mathcal{U} : X = \bigcup U_j$ consider the sheaves of $\mathcal{O}_X$-modules gotten from glueing the sheaves $\mathcal{O}_X|_{U_j}$, see Sheaves, Section 6.33. Note that the collection of all glueing data forms a set. The collection of all coverings $\mathcal{U} : X = \bigcup_{j \in J} U_i$ where $J \to \mathcal{P}(X)$, $j \mapsto U_j$ is injective forms a set as well. Hence the collection of all sheaves of of $\mathcal{O}_X$-modules gotten from glueing trivial invertible $\mathcal{O}_X$-modules forms a set $\mathcal{I}$. By definition every invertible $\mathcal{O}_X$-module is isomorphic to an element of $\mathcal{I}$. Choosing an element out of each isomorphism class inside $\mathcal{I}$ gives the desired set of invertible sheaves (uses axiom of choice). □

This lemma says roughly speaking that the collection of isomorphism classes of invertible sheaves forms a set. Lemma 17.21.2 says that tensor product defines the structure of an abelian group on this set.

Definition 17.21.6. Let $(X, \mathcal{O}_X)$ be a ringed space. Assume all stalks $\mathcal{O}_{X,x}$ are local rings. The *Picard group* $\mathrm{Pic}(X)$ of X is the abelian group whose elements are isomorphism classes of invertible $\mathcal{O}_X$-modules, with addition corresponding to tensor product.

Lemma 17.21.7. *Let X be a ringed space. Assume that each stalk $\mathcal{O}_{X,x}$ is a local ring with maximal ideal $\mathfrak{m}_x$. Let $\mathcal{L}$ be an invertible $\mathcal{O}_X$-module. For any section $s \in \Gamma(X, \mathcal{L})$ the set*

$$X_s = \{x \in X \mid \text{image } s \notin \mathfrak{m}_x \mathcal{L}_x\}$$

is open in X. The map $s : \mathcal{O}_{X_s} \to \mathcal{L}|_{X_s}$ is an isomorphism, and there exists a section s' of $\mathcal{L}^{\otimes -1}$ over X_s such that $s'(s|_{X_s}) = 1$.

Proof. Suppose $x \in X_s$. We have an isomorphism

$$\mathcal{L}_x \otimes_{\mathcal{O}_{X,x}} (\mathcal{L}^{\otimes -1})_x \longrightarrow \mathcal{O}_{X,x}$$

by Lemma 17.21.2. Both $\mathcal{L}_x$ and $(\mathcal{L}^{\otimes -1})_x$ are free $\mathcal{O}_{X,x}$-modules of rank 1. We conclude from Algebra, Nakayama's Lemma 10.18.1 that s_x is a basis for $\mathcal{L}_x$. Hence there exists a basis element $t_x \in (\mathcal{L}^{\otimes -1})_x$ such that $s_x \otimes t_x$ maps to 1. Choose an open neighbourhood U of x such that t_x comes from a section t of $(\mathcal{L}^{\otimes -1})_x$ over U and such that $s \otimes t$ maps to $1 \in \mathcal{O}_X(U)$. Clearly, for every $x' \in U$ we see that s generates the module $\mathcal{L}_{x'}$. Hence $U \subset X_s$. This proves that X_s is open. Moreover,

the section t constructed over U above is unique, and hence these glue to give te section s' of the lemma. □

It is also true that, given a morphism of locally ringed spaces $f : Y \to X$ (see Schemes, Definition 25.2.1) that the inverse image $f^{-1}(X_s)$ is equal to Y_{f^*s}, where $f^*s \in \Gamma(Y, f^*\mathcal{L})$ is the pullback of s.

17.22. Localizing sheaves of rings

Let X be a topological space and let $\mathcal{O}_X$ be a presheaf of rings. Let $\mathcal{S} \subset \mathcal{O}_X$ be a presheaf of sets contained in $\mathcal{O}_X$. Suppose that for every open $U \subset X$ the set $\mathcal{S}(U) \subset \mathcal{O}_X(U)$ is a multiplicative subset, see Algebra, Definition 10.9.1. In this case we can consider the presheaf of rings

$$\mathcal{S}^{-1}\mathcal{O}_X : U \longmapsto \mathcal{S}(U)^{-1}\mathcal{O}_X(U).$$

The restriction mapping sends the section f/s, $f \in \mathcal{O}_X(U)$, $s \in \mathcal{S}(U)$ to $(f|_V)/(s|_V)$ if $V \subset U$ are opens of X.

Lemma 17.22.1. *Let X be a topological space and let $\mathcal{O}_X$ be a presheaf of rings. Let $\mathcal{S} \subset \mathcal{O}_X$ be a pre-sheaf of sets contained in $\mathcal{O}_X$. Suppose that for every open $U \subset X$ the set $\mathcal{S}(U) \subset \mathcal{O}_X(U)$ is a multiplicative subset.*

1. *There is a map of presheaves of rings $\mathcal{O}_X \to \mathcal{S}^{-1}\mathcal{O}_X$ such that every local section of $\mathcal{S}$ maps to an invertible section of $\mathcal{O}_X$.*
2. *For any homomorphism of presheaves of rings $\mathcal{O}_X \to \mathcal{A}$ such that each local section of $\mathcal{S}$ maps to an invertible section of $\mathcal{A}$ there exists a unique factorization $\mathcal{S}^{-1}\mathcal{O}_X \to \mathcal{A}$.*
3. *For any $x \in X$ we have*
$$(\mathcal{S}^{-1}\mathcal{O}_X)_x = \mathcal{S}_x^{-1}\mathcal{O}_{X,x}.$$
4. *The sheafification $(\mathcal{S}^{-1}\mathcal{O}_X)^\#$ is a sheaf of rings with a map of sheaves of rings $(\mathcal{O}_X)^\# \to (\mathcal{S}^{-1}\mathcal{O}_X)^\#$ which is universal for maps of $(\mathcal{O}_X)^\#$ into sheaves of rings such that each local section of $\mathcal{S}$ maps to an invertible section.*
5. *For any $x \in X$ we have*
$$(\mathcal{S}^{-1}\mathcal{O}_X)_x^\# = \mathcal{S}_x^{-1}\mathcal{O}_{X,x}.$$

Proof. Omitted. □

Let X be a topological space and let $\mathcal{O}_X$ be a presheaf of rings. Let $\mathcal{S} \subset \mathcal{O}_X$ be a presheaf of sets contained in $\mathcal{O}_X$. Suppose that for every open $U \subset X$ the set $\mathcal{S}(U) \subset \mathcal{O}_X(U)$ is a multiplicative subset. Let $\mathcal{F}$ be a presheaf of $\mathcal{O}_X$-modules In this case we can consider the presheaf of $\mathcal{S}^{-1}\mathcal{O}_X$-modules

$$\mathcal{S}^{-1}\mathcal{F} : U \longmapsto \mathcal{S}(U)^{-1}\mathcal{F}(U).$$

The restriction mapping sends the section t/s, $t \in \mathcal{F}(U)$, $s \in \mathcal{S}(U)$ to $(t|_V)/(s|_V)$ if $V \subset U$ are opens of X.

Lemma 17.22.2. *Let X be a topological space. Let $\mathcal{O}_X$ be a presheaf of rings. Let $\mathcal{S} \subset \mathcal{O}_X$ be a pre-sheaf of sets contained in $\mathcal{O}_X$. Suppose that for every open $U \subset X$ the set $\mathcal{S}(U) \subset \mathcal{O}_X(U)$ is a multiplicative subset. For any presheaf of $\mathcal{O}_X$-modules $\mathcal{F}$ we have*

$$\mathcal{S}^{-1}\mathcal{F} = \mathcal{S}^{-1}\mathcal{O}_X \otimes_{p,\mathcal{O}_X} \mathcal{F}$$

(see Sheaves, Section 6.6 for notation) and if $\mathcal{F}$ and $\mathcal{O}_X$ are sheaves then

$$(\mathcal{S}^{-1}\mathcal{F})^\# = (\mathcal{S}^{-1}\mathcal{O}_X)^\# \otimes_{\mathcal{O}_X} \mathcal{F}$$

(see Sheaves, Section 6.20 for notation).

Proof. Omitted. □

17.23. Modules of differentials

In this section we briefly explain how to define the module of relative differentials for a morphism of ringed spaces. We suggest the reader take a look at the corresponding section in the chapter on commutative algebra (Algebra, Section 10.125).

Definition 17.23.1. Let X be a topological space. Let $\varphi : \mathcal{O}_1 \to \mathcal{O}_2$ be a homomorphism of sheaves of rings. Let $\mathcal{F}$ be an $\mathcal{O}_2$-module. A $\mathcal{O}_1$-*derivation* or more precisely a φ-*derivation* into $\mathcal{F}$ is a map $D : \mathcal{O}_2 \to \mathcal{F}$ which is additive, annihilates the image of $\mathcal{O}_1 \to \mathcal{O}_2$, and satisfies the *Leibniz rule*

$$D(ab) = aD(b) + D(a)b$$

for all a, b local sections of $\mathcal{O}_2$ (wherever they are both defined). We denote $\mathrm{Der}_{\mathcal{O}_1}(\mathcal{O}_2, \mathcal{F})$ the set of φ-derivations into $\mathcal{F}$.

This is the sheaf theoretic analogue of Algebra, Definition 17.23.1. Given a derivation $D : \mathcal{O}_2 \to \mathcal{F}$ as in the definition the map on global sections

$$D : \Gamma(X, \mathcal{O}_2) \longrightarrow \Gamma(X, \mathcal{F})$$

is a $\Gamma(X, \mathcal{O}_1)$-derivation as in the algebra definition. Note that if $\alpha : \mathcal{F} \to \mathcal{G}$ is a map of $\mathcal{O}_2$-modules, then there is an induced map

$$\mathrm{Der}_{\mathcal{O}_1}(\mathcal{O}_2, \mathcal{F}) \longrightarrow \mathrm{Der}_{\mathcal{O}_1}(\mathcal{O}_2, \mathcal{G})$$

given by the rule $D \mapsto \alpha \circ D$. In other words we obtain a functor.

Lemma 17.23.2. *Let X be a topological space. Let $\varphi : \mathcal{O}_1 \to \mathcal{O}_2$ be a homomorphism of sheaves of rings. The functor*

$$Mod(\mathcal{O}_2) \longrightarrow Ab, \quad \mathcal{F} \longmapsto Der_{\mathcal{O}_1}(\mathcal{O}_2, \mathcal{F})$$

is representable.

Proof. This is proved in exactly the same way as the analogous statement in algebra. During this proof, for any sheaf of sets $\mathcal{F}$ on X, let us denote $\mathcal{O}_2[\mathcal{F}]$ the sheafification of the presheaf $U \mapsto \mathcal{O}_2(U)[\mathcal{F}(U)]$ where this denotes the free $\mathcal{O}_1(U)$-module on the set $\mathcal{F}(U)$. For $s \in \mathcal{F}(U)$ we denote $[s]$ the corresponding section of $\mathcal{O}_2[\mathcal{F}]$ over U. If $\mathcal{F}$ is a sheaf of $\mathcal{O}_2$-modules, then there is a canonical map

$$c : \mathcal{O}_2[\mathcal{F}] \longrightarrow \mathcal{F}$$

which on the presheaf level is given by the rule $\sum f_s[s] \mapsto \sum f_s s$. We will employ the short hand $[s] \mapsto s$ to describe this map and similarly for other maps below. Consider the map of $\mathcal{O}_2$-modules

$$\begin{array}{ccc} \mathcal{O}_2[\mathcal{O}_2 \times \mathcal{O}_2] \oplus \mathcal{O}_2[\mathcal{O}_2 \times \mathcal{O}_2] \oplus \mathcal{O}_2[\mathcal{O}_1] & \longrightarrow & \mathcal{O}_2[\mathcal{O}_2] \\ [(a,b)] \oplus [(f,g)] \oplus [h] & \longmapsto & [a+b] - [a] - [b] + \\ & & [fg] - g[f] - f[g] + \\ & & [\varphi(h)] \end{array} \tag{17.23.2.1}$$

with short hand notation as above. Set $\Omega_{\mathcal{O}_2/\mathcal{O}_1}$ equal to the cokernel of this map. Then it is clear that there exists a map of sheaves of sets

$$\mathrm{d} : \mathcal{O}_2 \longrightarrow \Omega_{\mathcal{O}_2/\mathcal{O}_1}$$

mapping a local section f to the image of $[f]$ in $\Omega_{\mathcal{O}_2/\mathcal{O}_1}$. By construction d is a $\mathcal{O}_1$-derivation. Next, let $\mathcal{F}$ be a sheaf of $\mathcal{O}_2$-modules and let $D : \mathcal{O}_2 \to \mathcal{F}$ be a $\mathcal{O}_1$-derivation. Then we can consider the $\mathcal{O}_2$-linear map $\mathcal{O}_2[\mathcal{O}_2] \to \mathcal{F}$ which sends $[g]$ to $D(g)$. It follows from the definition of a derivation that this map annihilates sections in the image of the map (17.23.2.1) and hence defines a map

$$\alpha_D : \Omega_{\mathcal{O}_2/\mathcal{O}_1} \longrightarrow \mathcal{F}$$

Since it is clear that $D = \alpha_D \circ \mathrm{d}$ the lemma is proved. □

Definition 17.23.3. Let X be a topological space. Let $\varphi : \mathcal{O}_1 \to \mathcal{O}_2$ be a homomorphism of sheaves of rings on X. The *module of differentials* of φ is the object representing the functor $\mathcal{F} \mapsto \mathrm{Der}_{\mathcal{O}_1}(\mathcal{O}_2, \mathcal{F})$ which exists by Lemma 17.23.2. It is denoted $\Omega_{\mathcal{O}_2/\mathcal{O}_1}$, and the *universal φ-derivation* is denoted $\mathrm{d} : \mathcal{O}_2 \to \Omega_{\mathcal{O}_2/\mathcal{O}_1}$.

Note that $\Omega_{\mathcal{O}_2/\mathcal{O}_1}$ is the cokernel of the map (17.23.2.1) of $\mathcal{O}_2$-modules. Moreover the map d is described by the rule that $\mathrm{d}f$ is the image of the local section $[f]$.

Lemma 17.23.4. *Let X be a topological space. Let $\varphi : \mathcal{O}_1 \to \mathcal{O}_2$ be a homomorphism of sheaves of rings on X. Then $\Omega_{\mathcal{O}_2/\mathcal{O}_1}$ is the sheaf associated to the presheaf $U \mapsto \Omega_{\mathcal{O}_2(U)/\mathcal{O}_1(U)}$.*

Proof. Consider the map (17.23.2.1). There is a similar map of presheaves whose value on the open U is

$$\mathcal{O}_2(U)[\mathcal{O}_2(U)\times\mathcal{O}_2(U)]\oplus\mathcal{O}_2(U)[\mathcal{O}_2(U)\times\mathcal{O}_2(U)]\oplus\mathcal{O}_2(U)[\mathcal{O}_1(U)] \longrightarrow \mathcal{O}_2(U)[\mathcal{O}_2(U)]$$

The cokernel of this map has value $\Omega_{\mathcal{O}_2(U)/\mathcal{O}_1(U)}$ over U by the construction of the module of differentials in Algebra, Definition 10.125.2. On the other hand, the sheaves in (17.23.2.1) are the sheafifications of the presheaves above. Thus the result follows as sheafification is exact. □

Lemma 17.23.5. *Let X be a topological space. Let $\varphi : \mathcal{O}_1 \to \mathcal{O}_2$ be a homomorphism of sheaves of rings. For $U \subset X$ open there is a canonical isomorphism*

$$\Omega_{\mathcal{O}_2/\mathcal{O}_1}|_U = \Omega_{(\mathcal{O}_2|_U)/(\mathcal{O}_1|_U)}$$

compatible with universal derivations.

Proof. Holds because $\Omega_{\mathcal{O}_2/\mathcal{O}_1}$ is the cokernel of the map (17.23.2.1). □

Lemma 17.23.6. *Let $f : Y \to X$ be a continuous map of topological spaces. Let $\varphi : \mathcal{O}_1 \to \mathcal{O}_2$ be a homomorphism of sheaves of rings on X. Then there is a canonical identification $f^{-1}\Omega_{\mathcal{O}_2/\mathcal{O}_1} = \Omega_{f^{-1}\mathcal{O}_2/f^{-1}\mathcal{O}_1}$ compatible with universal derivations.*

Proof. This holds because the sheaf $\Omega_{\mathcal{O}_2/\mathcal{O}_1}$ is the cokernel of the map (17.23.2.1) and a similar statement holds for $\Omega_{f^{-1}\mathcal{O}_2/f^{-1}\mathcal{O}_1}$, because the functor f^{-1} is exact, and because $f^{-1}(\mathcal{O}_2[\mathcal{O}_2]) = f^{-1}\mathcal{O}_2[f^{-1}\mathcal{O}_2]$, $f^{-1}(\mathcal{O}_2[\mathcal{O}_2 \times \mathcal{O}_2]) = f^{-1}\mathcal{O}_2[f^{-1}\mathcal{O}_2 \times f^{-1}\mathcal{O}_2]$, and $f^{-1}(\mathcal{O}_2[\mathcal{O}_1]) = f^{-1}\mathcal{O}_2[f^{-1}\mathcal{O}_1]$. □

Lemma 17.23.7. *Let X be a topological space. Let $\mathcal{O}_1 \to \mathcal{O}_2$ be a homomorphism of sheaves of rings on X. Let $x \in X$. Then we have $\Omega_{\mathcal{O}_2/\mathcal{O}_1,x} = \Omega_{\mathcal{O}_{2,x}/\mathcal{O}_{1,x}}$.*

Proof. This is a special case of Lemma 17.23.6 for the inclusion map $\{x\} \to X$. An alternative proof is the use Lemma 17.23.4, Sheaves, Lemma 6.17.2, and Algebra, Lemma 10.125.4 □

Lemma 17.23.8. *Let X be a topological space. Let*

$$\begin{array}{ccc} \mathcal{O}_2 & \xrightarrow{\varphi} & \mathcal{O}_2' \\ \uparrow & & \uparrow \\ \mathcal{O}_1 & \longrightarrow & \mathcal{O}_1' \end{array}$$

be a commutative diagram of sheaves of rings on X. The map $\mathcal{O}_2 \to \mathcal{O}_2'$ composed with the map $d : \mathcal{O}_2' \to \Omega_{\mathcal{O}_2'/\mathcal{O}_1'}$ is a $\mathcal{O}_1$-derivation. Hence we obtain a canonical map of $\mathcal{O}_2$-modules $\Omega_{\mathcal{O}_2/\mathcal{O}_1} \to \Omega_{\mathcal{O}_2'/\mathcal{O}_1'}$. It is uniquely characterized by the property that $d(f)$ mapsto $d(\varphi(f))$ for any local section f of $\mathcal{O}_2$. In this way $\Omega_{-/-}$ becomes a functor on the category of arrows of sheaves of rings.

Proof. This lemma proves itself. □

Lemma 17.23.9. *In Lemma 17.23.8 suppose that $\mathcal{O}_2 \to \mathcal{O}_2'$ is surjective with kernel $\mathcal{I} \subset \mathcal{O}_2$ and assume that $\mathcal{O}_1 = \mathcal{O}_1'$. Then there is a canonical exact sequence of $\mathcal{O}_2'$-modules*

$$\mathcal{I}/\mathcal{I}^2 \longrightarrow \Omega_{\mathcal{O}_2/\mathcal{O}_1} \otimes_{\mathcal{O}_2} \mathcal{O}_2' \longrightarrow \Omega_{\mathcal{O}_2'/\mathcal{O}_1} \longrightarrow 0$$

The leftmost map is characterized by the rule that a local section f of $\mathcal{I}$ maps to $df \otimes 1$.

Proof. For a local section f of $\mathcal{I}$ denote $\overline{f}$ the image of f in $\mathcal{I}/\mathcal{I}^2$. To show that the map $\overline{f} \mapsto \mathrm{d}f \otimes 1$ is well defined we just have to check that $\mathrm{d}f_1 f_2 \otimes 1 = 0$ if f_1, f_2 are local sections of $\mathcal{I}$. And this is clear from the Leibniz rule $\mathrm{d}f_1 f_2 \otimes 1 = (f_1 \mathrm{d}f_2 + f_2 \mathrm{d}f_1) \otimes 1 = \mathrm{d}f_2 \otimes f_1 + \mathrm{d}f_2 \otimes f_1 = 0$. A similar computation show this map is $\mathcal{O}_2' = \mathcal{O}_2/\mathcal{I}$-linear. The map on the right is the one from Lemma 17.23.8. To see that the sequence is exact, we can check on stalks (Lemma 17.3.1). By Lemma 17.23.7 this follows from Algebra, Lemma 10.125.9. □

Definition 17.23.10. Let $(f, f^\sharp) : (X, \mathcal{O}_X) \to (S, \mathcal{O}_S)$ be a morphism of ringed spaces.

(1) Let $\mathcal{F}$ be an $\mathcal{O}_X$-module. An *S-derivation* into $\mathcal{F}$ is a $f^{-1}\mathcal{O}_S$-derivation, or more precisely a $f^\sharp$-derivation in the sense of Definition 17.23.1. We denote $\mathrm{Der}_S(\mathcal{O}_X, \mathcal{F})$ the set of S-derivations into $\mathcal{F}$.
(2) The *sheaf of differentials $\Omega_{X/S}$ of X over S* is the module of differentials $\Omega_{\mathcal{O}_X/f^{-1}\mathcal{O}_S}$ endowed with its universal S-derivation $\mathrm{d}_{X/S} : \mathcal{O}_X \to \Omega_{X/S}$.

Here is a particular situation where derivations come up naturally.

Lemma 17.23.11. *Let $(f, f^\sharp) : (X, \mathcal{O}_X) \to (S, \mathcal{O}_S)$ be a morphism of ringed spaces. Consider a short exact sequence*

$$0 \to \mathcal{I} \to \mathcal{A} \to \mathcal{O}_X \to 0$$

Here $\mathcal{A}$ is a sheaf of $f^{-1}\mathcal{O}_S$-algebras, $\pi : \mathcal{A} \to \mathcal{O}_X$ is a surjection of sheaves of $f^{-1}\mathcal{O}_S$-algebras, and $\mathcal{I} = Ker(\pi)$ is its kernel. Assume $\mathcal{I}$ an ideal sheaf with square zero in $\mathcal{A}$. So $\mathcal{I}$ has a natural structure of an $\mathcal{O}_X$-module. A section $s : \mathcal{O}_X \to \mathcal{A}$

of π is a $f^{-1}\mathcal{O}_S$-algebra map such that $\pi \circ s = id$. Given any section $s : \mathcal{O}_X \to \mathcal{I}$ of π and any S-derivation $D : \mathcal{O}_X \to \mathcal{I}$ the map

$$s + D : \mathcal{O}_X \to \mathcal{A}$$

is a section of π and every section s' is of the form $s + D$ for a unique S-derivation D.

Proof. Recall that the $\mathcal{O}_X$-module structure on $\mathcal{I}$ is given by $h\tau = \tilde{h}\tau$ (multiplication in $\mathcal{A}$) where h is a local section of $\mathcal{O}_X$, and $\tilde{h}$ is a local lift of h to a local section of $\mathcal{A}$, and τ is a local section of $\mathcal{I}$. In particular, given s, we may use $\tilde{h} = s(h)$. To verify that $s + D$ is a homomorphism of sheaves of rings we compute

$$\begin{aligned}(s + D)(ab) &= s(ab) + D(ab) \\ &= s(a)s(b) + aD(b) + D(a)b \\ &= s(a)s(b) + s(a)D(b) + D(a)s(b) \\ &= (s(a) + D(a))(s(b) + D(b))\end{aligned}$$

by the Leibniz rule. In the same manner one shows $s + D$ is a $f^{-1}\mathcal{O}_S$-algebra map because D is an S-derivation. Conversely, given s' we set $D = s' - s$. Details omitted. □

Lemma 17.23.12. *Let*

$$\begin{array}{ccc} X' & \xrightarrow{f} & X \\ {\scriptstyle h'}\downarrow & & \downarrow{\scriptstyle h} \\ S' & \xrightarrow{g} & S \end{array}$$

be a commutative diagram of ringed spaces.

1. *The canonical map $\mathcal{O}_X \to f_*\mathcal{O}_{X'}$ composed with $f_*d_{X'/S'} : f_*\mathcal{O}_{X'} \to f_*\Omega_{X'/S'}$ is a S-derivation and we obtain a canonical map of $\mathcal{O}_X$-modules $\Omega_{X/S} \to f_*\Omega_{X'/S'}$.*
2. *The commutative diagram*

$$\begin{array}{ccc} f^{-1}\mathcal{O}_X & \longrightarrow & \mathcal{O}_{X'} \\ \uparrow & & \uparrow \\ f^{-1}h^{-1}\mathcal{O}_S & \longrightarrow & (h')^{-1}\mathcal{O}_{S'} \end{array}$$

   induces by Lemmas 17.23.6 and 17.23.8 a canonical map $f^{-1}\Omega_{X/S} \to \Omega_{X'/S'}$.

These two maps correspond (via adjointness of f_* and f^* and via $f^*\Omega_{X/S} = f^{-1}\Omega_{X/S} \otimes_{f^{-1}\mathcal{O}_X} \mathcal{O}_{X'}$ and Sheaves, Lemma 6.20.2) to the same $\mathcal{O}_{X'}$-module homomorphism

$$c_f : f^*\Omega_{X/S} \longrightarrow \Omega_{X'/S'}$$

which is uniquely characterized by the property that $f^*d_{X/S}(a)$ mapsto $d_{X'/S'}(f^*a)$ for any local section a of $\mathcal{O}_X$.

Proof. Omitted. □

Lemma 17.23.13. *Let*

$$\begin{array}{ccccc} X'' & \xrightarrow{g} & X' & \xrightarrow{f} & X \\ \downarrow & & \downarrow & & \downarrow \\ S'' & \longrightarrow & S' & \longrightarrow & S \end{array}$$

be a commutative diagram of ringed spaces. With notation as in Lemma 17.23.12 we have

$$c_{f \circ g} = c_g \circ g^* c_f$$

as maps $(f \circ g)^* \Omega_{X/S} \to \Omega_{X''/S''}$.

Proof. Omitted. □

17.24. The naive cotangent complex

This section is the analogue of Algebra, Section 10.127 for morphisms of ringed spaces. We urge the reader to read that section first.

Let X be a topological space. Let $\mathcal{A} \to \mathcal{B}$ be a homomorphism of sheaves of rings. In this section, for any sheaf of sets $\mathcal{E}$ on X we denote $\mathcal{A}[\mathcal{E}]$ the sheafification of the presheaf $U \mapsto \mathcal{A}(U)[\mathcal{E}(U)]$. Here $\mathcal{A}(U)[\mathcal{E}(U)]$ denotes the polynomial algebra over $\mathcal{A}(U)$ whose variables correspond to the elements of $\mathcal{E}(U)$. We denote $[e] \in \mathcal{A}(U)[\mathcal{E}(U)]$ the variable corresponding to $e \in \mathcal{E}(U)$. There is a canonical surjection of $\mathcal{A}$-algebras

$$\mathcal{A}[\mathcal{B}] \longrightarrow \mathcal{B}, \quad [b] \longmapsto b \tag{17.24.0.1}$$

whose kernel we denote $\mathcal{I} \subset \mathcal{A}[\mathcal{B}]$. It is a simple observation that $\mathcal{I}$ is generated by the local sections $[b][b'] - [bb']$ and $[a] - a$. According to Lemma 17.23.9 there is a canonical map

$$\mathcal{I}/\mathcal{I}^2 \longrightarrow \Omega_{\mathcal{A}[\mathcal{B}]/\mathcal{A}} \otimes_{\mathcal{A}[\mathcal{B}]} \mathcal{B} \tag{17.24.0.2}$$

whose cokernel is canonically isomorphic to $\Omega_{\mathcal{B}/\mathcal{A}}$.

Definition 17.24.1. Let X be a topological space. Let $\mathcal{A} \to \mathcal{B}$ be a homomorphism of sheaves of rings. The *naive cotangent complex* $NL_{\mathcal{B}/\mathcal{A}}$ is the chain complex (17.24.0.2)

$$NL_{\mathcal{B}/\mathcal{A}} = \left(\mathcal{I}/\mathcal{I}^2 \longrightarrow \Omega_{\mathcal{A}[\mathcal{B}]/\mathcal{A}} \otimes_{\mathcal{A}[\mathcal{B}]} \mathcal{B}\right)$$

with $\mathcal{I}/\mathcal{I}^2$ placed in (homological) degree 1 and $\Omega_{\mathcal{A}[\mathcal{B}]/\mathcal{A}} \otimes_{\mathcal{A}[\mathcal{B}]} \mathcal{B}$ placed in degree 0.

This construction satisfies a functoriality similar to that discussed in Lemma 17.23.8 for modules of differentials. Namely, given a commutative diagram

$$\begin{array}{ccc} \mathcal{B} & \longrightarrow & \mathcal{B}' \\ \uparrow & & \uparrow \\ \mathcal{A} & \longrightarrow & \mathcal{A}' \end{array} \tag{17.24.1.1}$$

of sheaves of rings on X there is a canonical $\mathcal{B}$-linear map of complexes

$$NL_{\mathcal{B}/\mathcal{A}} \longrightarrow NL_{\mathcal{B}'/\mathcal{A}'}$$

Namely, the maps in the commutative diagram give rise to a canonical map $\mathcal{A}[\mathcal{B}] \to \mathcal{A}'[\mathcal{B}']$ which maps $\mathcal{I}$ into $\mathcal{I}' = \mathrm{Ker}(\mathcal{A}'[\mathcal{B}'] \to \mathcal{B}')$. Thus a map $\mathcal{I}/\mathcal{I}^2 \to \mathcal{I}'/(\mathcal{I}')^2$

and a map between modules of differentials, which together give the desired map between the naive cotangent complexes.

We can choose a different presentation of $\mathcal{B}$ as a quotient of a polynomial algebra over $\mathcal{A}$ and still obtain the same object of $D(\mathcal{B})$. To explain this, suppose that $\mathcal{E}$ is a sheaves of sets on X and $\alpha : \mathcal{E} \to \mathcal{B}$ a map of sheaves of sets. Then we obtain an $\mathcal{A}$-algebra homomorphism $\mathcal{A}[\mathcal{E}] \to \mathcal{B}$. Assume this map is surjective, and let $\mathcal{J} \subset \mathcal{A}[\mathcal{E}]$ be the kernel. Set

$$NL(\alpha) = \left(\mathcal{J}/\mathcal{J}^2 \longrightarrow \Omega_{\mathcal{A}[\mathcal{E}]/\mathcal{A}} \otimes_{\mathcal{A}[\mathcal{E}]} \mathcal{B}\right)$$

Here is the result.

Lemma 17.24.2. *In the situation above there is a canonical isomorphism $NL(\alpha) = NL_{\mathcal{B}/\mathcal{A}}$ in $D(\mathcal{B})$.*

Proof. Observe that $NL_{\mathcal{B}/\mathcal{A}} = NL(\mathrm{id}_{\mathcal{B}})$. Thus it suffices to show that given two maps $\alpha_i : \mathcal{E}_i \to \mathcal{B}$ as above, there is a canonical quasi-isomorphism $NL(\alpha_1) = NL(\alpha_2)$ in $D(\mathcal{B})$. To see this set $\mathcal{E} = \mathcal{E}_1 \amalg \mathcal{E}_2$ and $\alpha = \alpha_1 \amalg \alpha_2 : \mathcal{E} \to \mathcal{B}$. Set $\mathcal{J}_i = \mathrm{Ker}(\mathcal{A}[\mathcal{E}_i] \to \mathcal{B})$ and $\mathcal{J} = \mathrm{Ker}(\mathcal{A}[\mathcal{E}] \to \mathcal{B})$. We obtain maps $\mathcal{A}[\mathcal{E}_i] \to \mathcal{A}[\mathcal{E}]$ which send $\mathcal{J}_i$ into $\mathcal{J}$. Thus we obtain canonical maps of complexes

$$NL(\alpha_i) \longrightarrow NL(\alpha)$$

and it suffices to show these maps are quasi-isomorphism. To see this it suffices to check on stalks (Lemma 17.3.1). Here by Lemma 17.23.7 we see the result holds by Algebra, Lemma 10.127.2. □

Lemma 17.24.3. *Let $f : X \to Y$ be a continuous map of topological spaces. Let $\mathcal{A} \to \mathcal{B}$ be a homomorphism of sheaves of rings on Y. Then $f^{-1} NL_{\mathcal{B}/\mathcal{A}} = NL_{f^{-1}\mathcal{B}/f^{-1}\mathcal{A}}$.*

Proof. Omitted. Hint: Use Lemma 17.23.6. □

The cotangent complex of a morphism of ringed spaces is defined in terms of the cotangent complex we defined above.

Definition 17.24.4. The *naive cotangent complex* $NL_f = NL_{X/Y}$ of a morphism of ringed spaces $f : (X, \mathcal{O}_X) \to (Y, \mathcal{O}_Y)$ is $NL_{\mathcal{O}_X/f^{-1}\mathcal{O}_Y}$.

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CHAPTER 18

Modules on Sites

18.1. Introduction

In this document we work out basic notions of sheaves of modules on ringed topoi or ringed sites. We first work out some basic facts on abelian sheaves. After this we introduce ringed sites and ringed topoi. We work through some of the very basic notions on (pre)sheaves of $\mathcal{O}$-modules, analogous to the material on (pre)sheaves of $\mathcal{O}$-modules in the chapter on sheaves on spaces. Having done this, we duplicate much of the discussion in the chapter on sheaves of modules (see Modules, Section 17.1). Basic references are [**Ser55b**], [**DG67**] and [**AGV71**].

18.2. Abelian presheaves

Let $\mathcal{C}$ be a category. Abelian presheaves were introduced in Sites, Sections 7.2 and 7.7 and discussed a bit more in Sites, Section 7.43. We will follow the convention of this last reference, in that we think of an abelian presheaf as a presheaf of sets endowed with addition rules on all sets of sections compatible with the restriction mappings. Recall that the category of abelian presheaves on $\mathcal{C}$ is denoted $PAb(\mathcal{C})$.

The category $PAb(\mathcal{C})$ is abelian as defined in Homology, Definition 12.5.1. Given a map of presheaves $\varphi : \mathcal{G}_1 \to \mathcal{G}_2$ the kernel of φ is the abelian presheaf $U \mapsto \mathrm{Ker}(\mathcal{G}_1(U) \to \mathcal{G}_2(U))$ and the cokernel of φ is the presheaf $U \mapsto \mathrm{Coker}(\mathcal{G}_1(U) \to \mathcal{G}_2(U))$. Since the category of abelian groups is abelian it follows that Coim = Im because this holds over each U. A sequence of abelian presheaves

$$\mathcal{G}_1 \longrightarrow \mathcal{G}_2 \longrightarrow \mathcal{G}_3$$

is exact if and only if $\mathcal{G}_1(U) \to \mathcal{G}_2(U) \to \mathcal{G}_3(U)$ is an exact sequence of abelian groups for all $U \in \mathrm{Ob}(\mathcal{C})$. We leave the verifications to the reader.

Lemma 18.2.1. *Let $\mathcal{C}$ be a category.*

(1) *All limits and colimits exist in $PAb(\mathcal{C})$.*
(2) *All limits and colimits commute with taking sections over objects of $\mathcal{C}$.*

Proof. Let $\mathcal{I} \to PAb(\mathcal{C})$, $i \mapsto \mathcal{F}_i$ be a diagram. We can simply define abelian presheaves L and C by the rules

$$L : U \longmapsto \lim_i \mathcal{F}_i(U)$$

and

$$C : U \longmapsto \mathrm{colim}_i \mathcal{F}_i(U).$$

It is clear that there are maps of abelian presheaves $L \to \mathcal{F}_i$ and $\mathcal{F}_i \to C$, by using the corresponding maps on groups of sections over each U. It is straightforward to check that L and C endowed with these maps are the limit and colimit of the diagram in $PAb(\mathcal{C})$. This proves (1) and (2). Details omitted. □

18.3. Abelian sheaves

Let $\mathcal{C}$ be a site. The category of abelian sheaves on $\mathcal{C}$ is denoted $Ab(\mathcal{C})$. It is the full subcategory of $PAb(\mathcal{C})$ consisting of those abelian presheaves whose underlying presheaves of sets are sheaves. Properties (α) – (ζ) of Sites, Section 7.43 hold, see Sites, Proposition 7.43.3. In particular the inclusion functor $Ab(\mathcal{C}) \to PAb(\mathcal{C})$ has a left adjoint, namely the sheafification functor $\mathcal{G} \mapsto \mathcal{G}^\#$.

We suggest the reader prove the lemma on a piece of scratch paper rather than reading the proof.

Lemma 18.3.1. *Let $\mathcal{C}$ be a site. Let $\varphi : \mathcal{F} \to \mathcal{G}$ be a morphism of abelian sheaves on $\mathcal{C}$.*

1. *The category $Ab(\mathcal{C})$ is an abelian category.*
2. *The kernel $Ker(\varphi)$ of φ is the same as the kernel of φ as a morphism of presheaves.*
3. *The morphism φ is injective (Homology, Definition 12.5.3) if and only if φ is injective as a map of presheaves (Sites, Definition 7.3.1), if and only if φ is injective as a map of sheaves (Sites, Definition 7.12.1).*
4. *The cokernel $Coker(\varphi)$ of φ is the sheafification of the cokernel of φ as a morphism of presheaves.*
5. *The morphism φ is surjective (Homology, Definition 12.5.3) if and only if φ is surjective as a map of sheaves (Sites, Definition 7.12.1).*
6. *A complex of abelian sheaves*
$$\mathcal{F} \to \mathcal{G} \to \mathcal{H}$$
is exact at $\mathcal{G}$ if and only if for all $U \in \mathrm{Ob}(\mathcal{C})$ and all $s \in \mathcal{G}(U)$ mapping to zero in $\mathcal{H}(U)$ there exists a covering $\{U_i \to U\}_{i \in I}$ in $\mathcal{C}$ such that each $s|_{U_i}$ is in the image of $\mathcal{F}(U_i) \to \mathcal{G}(U_i)$.

Proof. We claim that Homology, Lemma 12.7.3 applies to the categories $\mathcal{A} = Ab(\mathcal{C})$ and $\mathcal{B} = PAb(\mathcal{C})$, and the functors $a : \mathcal{A} \to \mathcal{B}$ (inclusion), and $b : \mathcal{B} \to \mathcal{A}$ (sheafification). Let us check the assumptions of Homology, Lemma 12.7.3. Assumption (1) is that $\mathcal{A}$, $\mathcal{B}$ are additive categories, a, b are additive functors, and a is right adjoint to b. The first two statements are clear and adjointness is Sites, Section 7.43 (ϵ). Assumption (2) says that $PAb(\mathcal{C})$ is abelian which we saw in Section 18.2 and that sheafification is left exact, which is Sites, Section 7.43 (ζ). The final assumption is that $ba \cong \mathrm{id}_{\mathcal{A}}$ which is Sites, Section 7.43 (δ). Hence Homology, Lemma 12.7.3 applies and we conclude that $Ab(\mathcal{C})$ is abelian.

In the proof of Homology, Lemma 12.7.3 it is shown that $\mathrm{Ker}(\varphi)$ and $\mathrm{Coker}(\varphi)$ are equal to the sheafification of the kernel and cokernel of φ as a morphism of abelian presheaves. This proves (4). Since the kernel is a equalizer (i.e., a limit) and since sheafification commutes with finite limits, we conclude that (2) holds.

Statement (2) implies (3). Statement (4) implies (5) by our description of sheafification. The characterization of exactness in (6) follows from (2) and (5), and the fact that the sequence is exact if and only if $\mathrm{Im}(\mathcal{F} \to \mathcal{G}) = \mathrm{Ker}(\mathcal{G} \to \mathcal{H})$. □

Another way to say part (6) of the lemma is that a sequence of abelian sheaves
$$\mathcal{F}_1 \longrightarrow \mathcal{F}_2 \longrightarrow \mathcal{F}_3$$

is exact if and only if the sheafification of $U \mapsto \mathcal{F}_2(U)/\mathcal{F}_1(U)$ is equal to the kernel of $\mathcal{F}_2 \to \mathcal{F}_3$.

Lemma 18.3.2. *Let $\mathcal{C}$ be a site.*

(1) *All limits and colimits exist in $Ab(\mathcal{C})$.*
(2) *Limits are the same as the corresponding limits of abelian presheaves over $\mathcal{C}$ (i.e., commute with taking sections over objects of $\mathcal{C}$).*
(3) *Finite direct sums are the same as the corresponding finite direct sums in the category of abelian pre-sheaves over $\mathcal{C}$.*
(4) *A colimit is the sheafification of the corresponding colimit in the category of abelian presheaves.*
(5) *Filtered colimits are exact.*

Proof. By Lemma 18.2.1 limits and colimits of abelian presheaves exist, and are described by taking limits and colimits on the level of sections over objects.

Let $\mathcal{I} \to Ab(\mathcal{C})$, $i \mapsto \mathcal{F}_i$ be a diagram. Let $\lim_i \mathcal{F}_i$ be the limit of the diagram as an abelian presheaf. By Sites, Lemma 7.10.1 this is an abelian sheaf. Then it is quite easy to see that $\lim_i \mathcal{F}_i$ is the limit of the diagram in $Ab(\mathcal{C})$. This proves limits exist and (2) holds.

By Categories, Lemma 4.24.3, and because sheafification is left adjoint to the inclusion functor we see that $\operatorname{colim}_i \mathcal{F}$ exists and is the sheafification of the colimit in $PAb(\mathcal{C})$. This proves colimits exist and (4) holds.

Finite direct sums are the same thing as finite products in any abelian category. Hence (3) follows from (2).

Proof of (5). The statement means that given a system $0 \to \mathcal{F}_i \to \mathcal{G}_i \to \mathcal{H}_i \to 0$ of exact sequences of abelian sheaves over a directed partially ordered set I the sequence $0 \to \operatorname{colim} \mathcal{F}_i \to \operatorname{colim} \mathcal{G}_i \to \operatorname{colim} \mathcal{H}_i \to 0$ is exact as well. A formal argument using Homology, Lemma 12.5.8 and the definition of colimits shows that the sequence $\operatorname{colim} \mathcal{F}_i \to \operatorname{colim} \mathcal{G}_i \to \operatorname{colim} \mathcal{H}_i \to 0$ is exact. Note that $\operatorname{colim} \mathcal{F}_i \to \operatorname{colim} \mathcal{G}_i$ is the sheafification of the map of presheaf colimits which is injective as each of the maps $\mathcal{F}_i \to \mathcal{G}_i$ is injective. Since sheafification is exact we conclude. □

18.4. Free abelian presheaves

In order to prepare notation for the following definition, let us agree to denote the free abelian group on a set S as[1] $\mathbf{Z}[S] = \bigoplus_{s \in S} \mathbf{Z}$. It is characterized by the property

$$\operatorname{Mor}_{Ab}(\mathbf{Z}[S], A) = \operatorname{Mor}_{Sets}(S, A)$$

In other words the construction $S \mapsto \mathbf{Z}[S]$ is a left adjoint to the forgetful functor $Ab \to Sets$.

Definition 18.4.1. Let $\mathcal{C}$ be a category. Let $\mathcal{G}$ be a presheaf of sets. The *free abelian presheaf* $\mathbf{Z}_\mathcal{G}$ on $\mathcal{G}$ is the abelian presheaf defined by the rule

$$U \longmapsto \mathbf{Z}[\mathcal{G}(U)].$$

In the special case $\mathcal{G} = h_X$ of a representable presheaf associated to an object X of $\mathcal{C}$ we use the notation $\mathbf{Z}_X = \mathbf{Z}_{h_X}$. In other words

$$\mathbf{Z}_X(U) = \mathbf{Z}[\operatorname{Mor}_\mathcal{C}(U, X)].$$

[1] In other chapters the notation $\mathbf{Z}[S]$ sometimes indicates the polynomial ring over $\mathbf{Z}$ on S.

This construction is clearly functorial in the presheaf $\mathcal{G}$. In fact it is adjoint to the forgetful functor $PAb(\mathcal{C}) \to PSh(\mathcal{C})$. Here is the precise statement.

Lemma 18.4.2. *Let $\mathcal{C}$ be a category. Let $\mathcal{G}$, $\mathcal{F}$ be a presheaves of sets. Let $\mathcal{A}$ be an abelian presheaf. Let U be an object of $\mathcal{C}$. Then we have*

$$\begin{aligned} \mathrm{Mor}_{PSh(\mathcal{C})}(h_U, \mathcal{F}) &= \mathcal{F}(U), \\ \mathrm{Mor}_{PAb(\mathcal{C})}(\mathbf{Z}_{\mathcal{G}}, \mathcal{A}) &= \mathrm{Mor}_{PSh(\mathcal{C})}(\mathcal{G}, \mathcal{A}), \\ \mathrm{Mor}_{PAb(\mathcal{C})}(\mathbf{Z}_U, \mathcal{A}) &= \mathcal{A}(U). \end{aligned}$$

All of these equalities are functorial.

Proof. Omitted. □

Lemma 18.4.3. *Let $\mathcal{C}$ be a category. Let I be a set. For each $i \in I$ let $\mathcal{G}_i$ be a presheaf of sets. Then*

$$\mathbf{Z}_{\coprod_i \mathcal{G}_i} = \bigoplus\nolimits_{i \in I} \mathbf{Z}_{\mathcal{G}_i}$$

in $PAb(\mathcal{C})$.

Proof. Omitted. □

18.5. Free abelian sheaves

Here is the notion of a free abelian sheaf on a sheaf of sets.

Definition 18.5.1. Let $\mathcal{C}$ be a site. Let $\mathcal{G}$ be a presheaf of sets. The *free abelian sheaf* $\mathbf{Z}^\#_{\mathcal{G}}$ on $\mathcal{G}$ is the abelian sheaf $\mathbf{Z}^\#_{\mathcal{G}}$ which is the sheafification of the abelian presheaf on $\mathcal{G}$. In the special case $\mathcal{G} = h_X$ of a representable presheaf associated to an object X of $\mathcal{C}$ we use the notation $\mathbf{Z}^\#_X$.

This construction is clearly functorial in the presheaf $\mathcal{G}$. In fact it provides an adjoint to the forgetful functor $Ab(\mathcal{C}) \to Sh(\mathcal{C})$. Here is the precise statement.

Lemma 18.5.2. *Let $\mathcal{C}$ be a site. Let $\mathcal{G}$, $\mathcal{F}$ be a sheaves of sets. Let $\mathcal{A}$ be an abelian sheaf. Let U be an object of $\mathcal{C}$. Then we have*

$$\begin{aligned} \mathrm{Mor}_{Sh(\mathcal{C})}(h^\#_U, \mathcal{F}) &= \mathcal{F}(U), \\ \mathrm{Mor}_{Ab(\mathcal{C})}(\mathbf{Z}^\#_{\mathcal{G}}, \mathcal{A}) &= \mathrm{Mor}_{Sh(\mathcal{C})}(\mathcal{G}, \mathcal{A}), \\ \mathrm{Mor}_{Ab(\mathcal{C})}(\mathbf{Z}^\#_U, \mathcal{A}) &= \mathcal{A}(U). \end{aligned}$$

All of these equalities are functorial.

Proof. Omitted. □

Lemma 18.5.3. *Let $\mathcal{C}$ be a site. Let $\mathcal{G}$ be a presheaf of sets. Then $\mathbf{Z}^\#_{\mathcal{G}} = (\mathbf{Z}_{\mathcal{G}^\#})^\#$.*

Proof. Omitted. □

18.6. Ringed sites

In this chapter we mainly work with sheaves of modules on a ringed site. Hence we need to define this notion.

Definition 18.6.1. Ringed sites.

(1) A *ringed site* is a pair $(\mathcal{C}, \mathcal{O})$ where $\mathcal{C}$ is a site and $\mathcal{O}$ is a sheaf of rings on $\mathcal{C}$. The sheaf $\mathcal{O}$ is called the *structure sheaf* of the ringed site.

(2) Let $(\mathcal{C}, \mathcal{O})$, $(\mathcal{C}', \mathcal{O}')$ be ringed sites. A *morphism of ringed sites*
$$(f, f^\sharp) : (\mathcal{C}, \mathcal{O}) \longrightarrow (\mathcal{C}', \mathcal{O}')$$
is given by a morphism of sites $f : \mathcal{C} \to \mathcal{C}'$ (see Sites, Definition 7.15.1) together with a map of sheaves of rings $f^\sharp : f^{-1}\mathcal{O}' \to \mathcal{O}$, which by adjunction is the same thing as a map of sheaves of rings $f^\sharp : \mathcal{O}' \to f_*\mathcal{O}$.

(3) Let $(f, f^\sharp) : (\mathcal{C}_1, \mathcal{O}_1) \to (\mathcal{C}_2, \mathcal{O}_2)$ and $(g, g^\sharp) : (\mathcal{C}_2, \mathcal{O}_2) \to (\mathcal{C}_3, \mathcal{O}_3)$ be morphisms of ringed sites. Then we define the *composition of morphisms of ringed sites* by the rule
$$(g, g^\sharp) \circ (f, f^\sharp) = (g \circ f, f^\sharp \circ g^\sharp).$$
Here we use composition of morphisms of sites defined in Sites, Definition 7.15.4 and $f^\sharp \circ g^\sharp$ indicates the morphism of sheaves of rings
$$\mathcal{O}_3 \xrightarrow{g^\sharp} g_*\mathcal{O}_2 \xrightarrow{g_* f^\sharp} g_* f_* \mathcal{O}_1 = (g \circ f)_* \mathcal{O}_1$$

18.7. Ringed topoi

A ringed topos is just a ringed site, except that the notion of a morphism of ringed topoi is different from the notion of a morphism of ringed sites.

Definition 18.7.1. Ringed topoi.

(1) A *ringed topos* is a pair $(Sh(\mathcal{C}), \mathcal{O})$ where $\mathcal{C}$ is a site and $\mathcal{O}$ is a sheaf of rings on $\mathcal{C}$. The sheaf $\mathcal{O}$ is called the *structure sheaf* of the ringed site.

(2) Let $(Sh(\mathcal{C}), \mathcal{O})$, $(Sh(\mathcal{C}'), \mathcal{O}')$ be ringed topoi. A *morphism of ringed topoi*
$$(f, f^\sharp) : (Sh(\mathcal{C}), \mathcal{O}) \longrightarrow (Sh(\mathcal{C}'), \mathcal{O}')$$
is given by a morphism of topoi $f : \mathcal{C} \to \mathcal{C}'$ (see Sites, Definition 7.16.1) together with a map of sheaves of rings $f^\sharp : f^{-1}\mathcal{O}' \to \mathcal{O}$, which by adjunction is the same thing as a map of sheaves of rings $f^\sharp : \mathcal{O}' \to f_*\mathcal{O}$.

(3) Let $(f, f^\sharp) : (Sh(\mathcal{C}_1), \mathcal{O}_1) \to (Sh(\mathcal{C}_2), \mathcal{O}_2)$ and $(g, g^\sharp) : (Sh(\mathcal{C}_2), \mathcal{O}_2) \to (Sh(\mathcal{C}_3), \mathcal{O}_3)$ be morphisms of ringed topoi. Then we define the *composition of morphisms of ringed topoi* by the rule
$$(g, g^\sharp) \circ (f, f^\sharp) = (g \circ f, f^\sharp \circ g^\sharp).$$
Here we use composition of morphisms of topoi defined in Sites, Definition 7.16.1 and $f^\sharp \circ g^\sharp$ indicates the morphism of sheaves of rings
$$\mathcal{O}_3 \xrightarrow{g^\sharp} g_*\mathcal{O}_2 \xrightarrow{g_* f^\sharp} g_* f_* \mathcal{O}_1 = (g \circ f)_* \mathcal{O}_1$$

Every morphism of ringed topoi is the composition of an equivalence of ringed topoi with a morphism of ringed topoi associated to a morphism of ringed sites. Here is the precise statement.

Lemma 18.7.2. *Let $(f, f^\sharp) : (Sh(\mathcal{C}), \mathcal{O}_\mathcal{C}) \to (Sh(\mathcal{D}), \mathcal{O}_\mathcal{D})$ be a morphism of ringed topoi. There exists a factorization*
$$\begin{array}{ccc} (Sh(\mathcal{C}), \mathcal{O}_\mathcal{C}) & \xrightarrow[(f,f^\sharp)]{} & (Sh(\mathcal{D}), \mathcal{O}_\mathcal{D}) \\ \downarrow{\scriptstyle (g,g^\sharp)} & & \downarrow{\scriptstyle (e,e^\sharp)} \\ (Sh(\mathcal{C}'), \mathcal{O}_{\mathcal{C}'}) & \xrightarrow[(h,h^\sharp)]{} & (Sh(\mathcal{D}'), \mathcal{O}_{\mathcal{D}'}) \end{array}$$
where

(1) $g : Sh(\mathcal{C}) \to Sh(\mathcal{C}')$ *is an equivalence of topoi induced by a special cocontinuous functor* $\mathcal{C} \to \mathcal{C}'$ *(see Sites, Definition 7.28.2),*
(2) $e : Sh(\mathcal{D}) \to Sh(\mathcal{D}')$ *is an equivalence of topoi induced by a special cocontinuous functor* $\mathcal{D} \to \mathcal{D}'$ *(see Sites, Definition 7.28.2),*
(3) $\mathcal{O}_{\mathcal{C}'} = g_*\mathcal{O}_{\mathcal{C}}$ *and* $g^\sharp$ *is the obvious map,*
(4) $\mathcal{O}_{\mathcal{D}'} = e_*\mathcal{O}_{\mathcal{D}}$ *and* $e^\sharp$ *is the obvious map,*
(5) *the sites* $\mathcal{C}'$ *and* $\mathcal{D}'$ *have final objects and fibre products (i.e., all finite limits),*
(6) h *is a morphism of sites induced by a continuous functor* $u : \mathcal{D}' \to \mathcal{C}'$ *which commutes with all finite limits (i.e., it satisfies the assumptions of Sites, Proposition 7.15.6), and*
(7) *given any set of sheaves* $\mathcal{F}_i$ *(resp.* $\mathcal{G}_j$*) on* $\mathcal{C}$ *(resp.* $\mathcal{D}$*) we may assume each of these is a representable sheaf on* $\mathcal{C}'$ *(resp.* $\mathcal{D}'$*).*

Moreover, if $(f, f^\sharp)$ *is an equivalence of ringed topoi, then we can choose the diagram such that* $\mathcal{C}' = \mathcal{D}'$, $\mathcal{O}_{\mathcal{C}'} = \mathcal{O}_{\mathcal{D}'}$ *and* $(h, h^\sharp)$ *is the identity.*

Proof. This follows from Sites, Lemma 7.28.6, and Sites, Remarks 7.28.7 and 7.28.8. You just have to carry along the sheaves of rings. Some details omitted. □

18.8. 2-morphisms of ringed topoi

This is a brief section concerning the notion of a 2-morphism of ringed topoi.

Definition 18.8.1. Let $f, g : (Sh(\mathcal{C}), \mathcal{O}_{\mathcal{C}}) \to (Sh(\mathcal{D}), \mathcal{O}_{\mathcal{D}})$ be two morphisms of ringed topoi. A *2-morphism from* f *to* g is given by a transformation of functors $t : f_* \to g_*$ such that

$$\begin{array}{ccc} & \mathcal{O}_{\mathcal{D}} & \\ {\scriptstyle f^\sharp}\swarrow & & \searrow{\scriptstyle g^\sharp} \\ f_*\mathcal{O}_{\mathcal{C}} & \xrightarrow{\quad t \quad} & g_*\mathcal{O}_{\mathcal{C}} \end{array}$$

is commutative.

Pictorially we sometimes represent t as follows:

$$(Sh(\mathcal{C}), \mathcal{O}_{\mathcal{C}}) \;\overset{f}{\underset{g}{\Downarrow t}}\; (Sh(\mathcal{D}), \mathcal{O}_{\mathcal{D}})$$

As in Sites, Section 7.35 giving a 2-morphism $t : f_* \to g_*$ is equivalent to giving $t : g^{-1} \to f^{-1}$ (usually denoted by the same symbol) such that the diagram

$$\begin{array}{ccc} f^{-1}\mathcal{O}_{\mathcal{D}} & \xleftarrow{\quad t \quad} & g^{-1}\mathcal{O}_{\mathcal{D}} \\ & {\scriptstyle f^\sharp}\searrow \quad \swarrow{\scriptstyle g^\sharp} & \\ & \mathcal{O}_{\mathcal{C}} & \end{array}$$

is commutative. As in Sites, Section 7.35 the axioms of a strict 2-category hold with horizontal and vertical compositions defined as explained in loc. cit.

18.9. Presheaves of modules

Let $\mathcal{C}$ be a category. Let $\mathcal{O}$ be a presheaf of rings on $\mathcal{C}$. At this point we have not yet defined a presheaf of $\mathcal{O}$-modules. Thus we do so right now.

Definition 18.9.1. Let $\mathcal{C}$ be a category, and let $\mathcal{O}$ be a presheaf of rings on $\mathcal{C}$.

(1) A *presheaf of $\mathcal{O}$-modules* is given by an abelian presheaf $\mathcal{F}$ together with a map of presheaves of sets
$$\mathcal{O} \times \mathcal{F} \longrightarrow \mathcal{F}$$
such that for every object U of $\mathcal{C}$ the map $\mathcal{O}(U) \times \mathcal{F}(U) \to \mathcal{F}(U)$ defines the structure of an $\mathcal{O}(U)$-module structure on the abelian group $\mathcal{F}(U)$.

(2) A *morphism $\varphi : \mathcal{F} \to \mathcal{G}$ of presheaves of $\mathcal{O}$-modules* is a morphism of abelian presheaves $\varphi : \mathcal{F} \to \mathcal{G}$ such that the diagram
$$\begin{array}{ccc} \mathcal{O} \times \mathcal{F} & \longrightarrow & \mathcal{F} \\ {\scriptstyle \mathrm{id} \times \varphi}\downarrow & & \downarrow{\scriptstyle \varphi} \\ \mathcal{O} \times \mathcal{G} & \longrightarrow & \mathcal{G} \end{array}$$
commutes.

(3) The set of $\mathcal{O}$-module morphisms as above is denoted $\mathrm{Hom}_{\mathcal{O}}(\mathcal{F}, \mathcal{G})$.

(4) The category of presheaves of $\mathcal{O}$-modules is denoted $\mathit{PMod}(\mathcal{O})$.

Suppose that $\mathcal{O}_1 \to \mathcal{O}_2$ is a morphism of presheaves of rings on the category $\mathcal{C}$. In this case, if $\mathcal{F}$ is a presheaf of $\mathcal{O}_2$-modules then we can think of $\mathcal{F}$ as a presheaf of $\mathcal{O}_1$-modules by using the composition
$$\mathcal{O}_1 \times \mathcal{F} \to \mathcal{O}_2 \times \mathcal{F} \to \mathcal{F}.$$
We sometimes denote this by $\mathcal{F}_{\mathcal{O}_1}$ to indicate the restriction of rings. We call this the *restriction of $\mathcal{F}$*. We obtain the restriction functor
$$\mathit{PMod}(\mathcal{O}_2) \longrightarrow \mathit{PMod}(\mathcal{O}_1)$$

On the other hand, given a presheaf of $\mathcal{O}_1$-modules $\mathcal{G}$ we can construct a presheaf of $\mathcal{O}_2$-modules $\mathcal{O}_2 \otimes_{p,\mathcal{O}_1} \mathcal{G}$ by the rule
$$U \longmapsto (\mathcal{O}_2 \otimes_{p,\mathcal{O}_1} \mathcal{G})(U) = \mathcal{O}_2(U) \otimes_{\mathcal{O}_1(U)} \mathcal{G}(U)$$
where $U \in \mathrm{Ob}(\mathcal{C})$, with obvious restriction mappings. The index p stands for "presheaf" and not "point". This presheaf is called the tensor product presheaf. We obtain the *change of rings* functor
$$\mathit{PMod}(\mathcal{O}_1) \longrightarrow \mathit{PMod}(\mathcal{O}_2)$$

Lemma 18.9.2. *With $\mathcal{C}$, $\mathcal{O}_1 \to \mathcal{O}_2$, $\mathcal{F}$ and $\mathcal{G}$ as above there exists a canonical bijection*
$$\mathrm{Hom}_{\mathcal{O}_1}(\mathcal{G}, \mathcal{F}_{\mathcal{O}_1}) = \mathrm{Hom}_{\mathcal{O}_2}(\mathcal{O}_2 \otimes_{p,\mathcal{O}_1} \mathcal{G}, \mathcal{F})$$
In other words, the restriction and change of rings functors defined above are adjoint to each other.

Proof. This follows from the fact that for a ring map $A \to B$ the restriction functor and the change of ring functor are adjoint to each other. □

18.10. Sheaves of modules

Definition 18.10.1. Let $\mathcal{C}$ be a site. Let $\mathcal{O}$ be a sheaf of rings on $\mathcal{C}$.

(1) A *sheaf of $\mathcal{O}$-modules* is a presheaf of $\mathcal{O}$-modules $\mathcal{F}$, see Definition 18.9.1, such that the underlying presheaf of abelian groups $\mathcal{F}$ is a sheaf.
(2) A *morphism of sheaves of $\mathcal{O}$-modules* is a morphism of presheaves of $\mathcal{O}$-modules.
(3) Given sheaves of $\mathcal{O}$-modules $\mathcal{F}$ and $\mathcal{G}$ we denote $\mathrm{Hom}_{\mathcal{O}}(\mathcal{F}, \mathcal{G})$ the set of morphism of sheaves of $\mathcal{O}$-modules.
(4) The category of sheaves of $\mathcal{O}$-modules is denoted $\mathit{Mod}(\mathcal{O})$.

This definition kind of makes sense even if $\mathcal{O}$ is just a presheaf of rings, although we do not know any examples where this is useful, and we will avoid using the terminology "sheaves of $\mathcal{O}$-modules" in case $\mathcal{O}$ is not a sheaf of rings.

18.11. Sheafification of presheaves of modules

Lemma 18.11.1. *Let $\mathcal{C}$ be a site. Let $\mathcal{O}$ be a presheaf of rings on $\mathcal{C}$ Let $\mathcal{F}$ be a presheaf $\mathcal{O}$-modules. Let $\mathcal{O}^\#$ be the sheafification of $\mathcal{O}$ as a presheaf of rings, see Sites, Section 7.43. Let $\mathcal{F}^\#$ be the sheafification of $\mathcal{F}$ as a presheaf of abelian groups. There exists a map of sheaves of sets*

$$\mathcal{O}^\# \times \mathcal{F}^\# \longrightarrow \mathcal{F}^\#$$

which makes the diagram

$$\begin{array}{ccc} \mathcal{O} \times \mathcal{F} & \longrightarrow & \mathcal{F} \\ \downarrow & & \downarrow \\ \mathcal{O}^\# \times \mathcal{F}^\# & \longrightarrow & \mathcal{F}^\# \end{array}$$

commute and which makes $\mathcal{F}^\#$ into a sheaf of $\mathcal{O}^\#$-modules. In addition, if $\mathcal{G}$ is a sheaf of $\mathcal{O}^\#$-modules, then any morphism of presheaves of $\mathcal{O}$-modules $\mathcal{F} \to \mathcal{G}$ (into the restriction of $\mathcal{G}$ to a $\mathcal{O}$-module) factors uniquely as $\mathcal{F} \to \mathcal{F}^\# \to \mathcal{G}$ where $\mathcal{F}^\# \to \mathcal{G}$ is a morphism of $\mathcal{O}^\#$-modules.

Proof. Omitted. □

This actually means that the functor $i : \mathit{Mod}(\mathcal{O}^\#) \to \mathit{PMod}(\mathcal{O})$ (combining restriction and including sheaves into presheaves) and the sheafification functor of the lemma $^\# : \mathit{PMod}(\mathcal{O}) \to \mathit{Mod}(\mathcal{O}^\#)$ are adjoint. In a formula

$$\mathrm{Mor}_{\mathit{PMod}(\mathcal{O})}(\mathcal{F}, i\mathcal{G}) = \mathrm{Mor}_{\mathit{Mod}(\mathcal{O}^\#)}(\mathcal{F}^\#, \mathcal{G})$$

An important case happens when $\mathcal{O}$ is already a sheaf of rings. In this case the formula reads

$$\mathrm{Mor}_{\mathit{PMod}(\mathcal{O})}(\mathcal{F}, i\mathcal{G}) = \mathrm{Mor}_{\mathit{Mod}(\mathcal{O})}(\mathcal{F}^\#, \mathcal{G})$$

because $\mathcal{O} = \mathcal{O}^\#$ in this case.

Lemma 18.11.2. *Let $\mathcal{C}$ be a site. Let $\mathcal{O}$ be a presheaf of rings on $\mathcal{C}$ The sheafification functor*

$$\mathit{PMod}(\mathcal{O}) \longrightarrow \mathit{Mod}(\mathcal{O}^\#), \quad \mathcal{F} \longmapsto \mathcal{F}^\#$$

is exact.

Proof. This is true because it holds for sheafification $\mathit{PAb}(\mathcal{C}) \to \mathit{Ab}(\mathcal{C})$. See the discussion in Section 18.3. □

Let $\mathcal{C}$ be a site. Let $\mathcal{O}_1 \to \mathcal{O}_2$ be a morphism of sheaves of rings on $\mathcal{C}$. In Section 18.9 we defined a restriction functor and a change of rings functor on presheaves of modules associated to this situation.

If $\mathcal{F}$ is a sheaf of $\mathcal{O}_2$-modules then the restriction $\mathcal{F}_{\mathcal{O}_1}$ of $\mathcal{F}$ is clearly a sheaf of $\mathcal{O}_1$-modules. We obtain the restriction functor

$$Mod(\mathcal{O}_2) \longrightarrow Mod(\mathcal{O}_1)$$

On the other hand, given a sheaf of $\mathcal{O}_1$-modules $\mathcal{G}$ the presheaf of $\mathcal{O}_2$-modules $\mathcal{O}_2 \otimes_{p,\mathcal{O}_1} \mathcal{G}$ is in general not a sheaf. Hence we define the *tensor product sheaf* $\mathcal{O}_2 \otimes_{\mathcal{O}_1} \mathcal{G}$ by the formula

$$\mathcal{O}_2 \otimes_{\mathcal{O}_1} \mathcal{G} = (\mathcal{O}_2 \otimes_{p,\mathcal{O}_1} \mathcal{G})^\#$$

as the sheafification of our construction for presheaves. We obtain the *change of rings* functor

$$Mod(\mathcal{O}_1) \longrightarrow Mod(\mathcal{O}_2)$$

Lemma 18.11.3. *With X, $\mathcal{O}_1$, $\mathcal{O}_2$, $\mathcal{F}$ and $\mathcal{G}$ as above there exists a canonical bijection*

$$\mathrm{Hom}_{\mathcal{O}_1}(\mathcal{G}, \mathcal{F}_{\mathcal{O}_1}) = \mathrm{Hom}_{\mathcal{O}_2}(\mathcal{O}_2 \otimes_{\mathcal{O}_1} \mathcal{G}, \mathcal{F})$$

In other words, the restriction and change of rings functors are adjoint to each other.

Proof. This follows from Lemma 18.9.2 and the fact that $\mathrm{Hom}_{\mathcal{O}_2}(\mathcal{O}_2 \otimes_{\mathcal{O}_1} \mathcal{G}, \mathcal{F}) = \mathrm{Hom}_{\mathcal{O}_2}(\mathcal{O}_2 \otimes_{p,\mathcal{O}_1} \mathcal{G}, \mathcal{F})$ because $\mathcal{F}$ is a sheaf. □

Lemma 18.11.4. *Let $\mathcal{C}$ be a site. Let $\mathcal{O} \to \mathcal{O}'$ be an epimorphism of sheaves of rings. Let $\mathcal{G}_1, \mathcal{G}_2$ be $\mathcal{O}'$-modules. Then*

$$\mathrm{Hom}_{\mathcal{O}'}(\mathcal{G}_1, \mathcal{G}_2) = \mathrm{Hom}_{\mathcal{O}}(\mathcal{G}_1, \mathcal{G}_2).$$

In other words, the restriction functor $Mod(\mathcal{O}') \to Mod(\mathcal{O})$ is fully faithful.

Proof. This is the sheaf version of Algebra, Lemma 10.102.14 and is proved in exactly the same way. □

18.12. Morphisms of topoi and sheaves of modules

All of this material is completely straightforward. We formulate everything in the case of morphisms of topoi, but of course the results also hold in the case of morphisms of sites.

Lemma 18.12.1. *Let $\mathcal{C}$, $\mathcal{D}$ be sites. Let $f : Sh(\mathcal{C}) \to Sh(\mathcal{D})$ be a morphism of topoi. Let $\mathcal{O}$ be a sheaf of rings on $\mathcal{C}$. Let $\mathcal{F}$ be a sheaf of $\mathcal{O}$-modules. There is a natural map of sheaves of sets*

$$f_*\mathcal{O} \times f_*\mathcal{F} \longrightarrow f_*\mathcal{F}$$

which turns $f_*\mathcal{F}$ into a sheaf of $f_*\mathcal{O}$-modules. This construction is functorial in $\mathcal{F}$.

Proof. Denote $\mu : \mathcal{O} \times \mathcal{F} \to \mathcal{F}$ the multiplication map. Recall that f_* (on sheaves of sets) is left exact and hence commutes with products. Hence $f_*\mu$ is a map as indicated. This proves the lemma. □

Lemma 18.12.2. *Let $\mathcal{C}$, $\mathcal{D}$ be sites. Let $f : Sh(\mathcal{C}) \to Sh(\mathcal{D})$ be a morphism of topoi. Let $\mathcal{O}$ be a sheaf of rings on $\mathcal{D}$. Let $\mathcal{G}$ be a sheaf of $\mathcal{O}$-modules. There is a natural map of sheaves of sets*

$$f^{-1}\mathcal{O} \times f^{-1}\mathcal{G} \longrightarrow f^{-1}\mathcal{G}$$

which turns $f^{-1}\mathcal{G}$ into a sheaf of $f^{-1}\mathcal{O}$-modules. This construction is functorial in $\mathcal{G}$.

Proof. Denote $\mu : \mathcal{O} \times \mathcal{G} \to \mathcal{G}$ the multiplication map. Recall that f^{-1} (on sheaves of sets) is exact and hence commutes with products. Hence $f^{-1}\mu$ is a map as indicated. This proves the lemma. □

Lemma 18.12.3. *Let $\mathcal{C}$, $\mathcal{D}$ be sites. Let $f : Sh(\mathcal{C}) \to Sh(\mathcal{D})$ be a morphism of topoi. Let $\mathcal{O}$ be a sheaf of rings on $\mathcal{D}$. Let $\mathcal{G}$ be a sheaf of $\mathcal{O}$-modules. Let $\mathcal{F}$ be a sheaf of $f^{-1}\mathcal{O}$-modules. Then*

$$\mathrm{Mor}_{Mod(f^{-1}\mathcal{O})}(f^{-1}\mathcal{G}, \mathcal{F}) = \mathrm{Mor}_{Mod(\mathcal{O})}(\mathcal{G}, f_*\mathcal{F}).$$

Here we use Lemmas 18.12.2 and 18.12.1, and we think of $f_*\mathcal{F}$ as an $\mathcal{O}$-module by restriction via $\mathcal{O} \to f_*f^{-1}\mathcal{O}$.

Proof. First we note that we have

$$\mathrm{Mor}_{Ab(\mathcal{C})}(f^{-1}\mathcal{G}, \mathcal{F}) = \mathrm{Mor}_{Ab(\mathcal{D})}(\mathcal{G}, f_*\mathcal{F}).$$

by Sites, Proposition 7.43.3. Suppose that $\alpha : f^{-1}\mathcal{G} \to \mathcal{F}$ and $\beta : \mathcal{G} \to f_*\mathcal{F}$ are morphisms of abelian sheaves which correspond via the formula above. We have to show that α is $f^{-1}\mathcal{O}$-linear if and only if β is $\mathcal{O}$-linear. For example, suppose α is $f^{-1}\mathcal{O}$-linear, then clearly $f_*\alpha$ is $f_*f^{-1}\mathcal{O}$-linear, and hence (as restriction is a functor) is $\mathcal{O}$-linear. Hence it suffices to prove that the adjunction map $\mathcal{G} \to f_*f^{-1}\mathcal{G}$ is $\mathcal{O}$-linear. Using that both f_* and f^{-1} commute with products (on sheaves of sets) this comes down to showing that

$$\begin{array}{ccc} \mathcal{O} \times \mathcal{G} & \longrightarrow & f_*f^{-1}(\mathcal{O} \times \mathcal{G}) \\ \downarrow & & \downarrow \\ \mathcal{G} & \longrightarrow & f_*f^{-1}\mathcal{G} \end{array}$$

is commutative. This holds because the adjunction mapping $\mathrm{id}_{Sh(\mathcal{D})} \to f_*f^{-1}$ is a transformation of functors. We omit the proof of the implication β linear $\Rightarrow$ α linear. □

Lemma 18.12.4. *Let $\mathcal{C}$, $\mathcal{D}$ be sites. Let $f : Sh(\mathcal{C}) \to Sh(\mathcal{D})$ be a morphism of topoi. Let $\mathcal{O}$ be a sheaf of rings on $\mathcal{C}$. Let $\mathcal{F}$ be a sheaf of $\mathcal{O}$-modules. Let $\mathcal{G}$ be a sheaf of $f_*\mathcal{O}$-modules. Then*

$$\mathrm{Mor}_{Mod(\mathcal{O})}(\mathcal{O} \otimes_{f^{-1}f_*\mathcal{O}} f^{-1}\mathcal{G}, \mathcal{F}) = \mathrm{Mor}_{Mod(f_*\mathcal{O})}(\mathcal{G}, f_*\mathcal{F}).$$

Here we use Lemmas 18.12.2 and 18.12.1, and we use the canonical map $f^{-1}f_*\mathcal{O} \to \mathcal{O}$ in the definition of the tensor product.

Proof. Note that we have

$$\mathrm{Mor}_{Mod(\mathcal{O})}(\mathcal{O} \otimes_{f^{-1}f_*\mathcal{O}} f^{-1}\mathcal{G}, \mathcal{F}) = \mathrm{Mor}_{Mod(f^{-1}f_*\mathcal{O})}(f^{-1}\mathcal{G}, \mathcal{F}_{f^{-1}f_*\mathcal{O}})$$

by Lemma 18.11.3. Hence the result follows from Lemma 18.12.3. □

18.13. Morphisms of ringed topoi and modules

We have now introduced enough notation so that we are able to define the pullback and pushforward of modules along a morphism of ringed topoi.

Definition 18.13.1. Let $(f, f^\sharp) : (Sh(\mathcal{C}), \mathcal{O}_\mathcal{C}) \to (Sh(\mathcal{D}), \mathcal{O}_\mathcal{D})$ be a morphism of ringed topoi or ringed sites.

(1) Let $\mathcal{F}$ be a sheaf of $\mathcal{O}_\mathcal{C}$-modules. We define the *pushforward* of $\mathcal{F}$ as the sheaf of $\mathcal{O}_\mathcal{D}$-modules which as a sheaf of abelian groups equals $f_*\mathcal{F}$ and with module structure given by the restriction via $f^\sharp : \mathcal{O}_\mathcal{D} \to f_*\mathcal{O}_\mathcal{C}$ of the module structure

$$f_*\mathcal{O}_\mathcal{C} \times f_*\mathcal{F} \longrightarrow f_*\mathcal{F}$$

from Lemma 18.12.1.

(2) Let $\mathcal{G}$ be a sheaf of $\mathcal{O}_\mathcal{D}$-modules. We define the *pullback* $f^*\mathcal{G}$ to be the sheaf of $\mathcal{O}_\mathcal{C}$-modules defined by the formula

$$f^*\mathcal{F} = \mathcal{O}_\mathcal{C} \otimes_{f^{-1}\mathcal{O}_\mathcal{D}} f^{-1}\mathcal{F}$$

where the ring map $f^{-1}\mathcal{O}_\mathcal{D} \to \mathcal{O}_\mathcal{C}$ is $f^\sharp$, and where the module structure is given by Lemma 18.12.2.

Thus we have defined functors

$$\begin{array}{rcl} f_* : Mod(\mathcal{O}_\mathcal{C}) & \longrightarrow & Mod(\mathcal{O}_\mathcal{D}) \\ f^* : Mod(\mathcal{O}_\mathcal{D}) & \longrightarrow & Mod(\mathcal{O}_\mathcal{C}) \end{array}$$

The final result on these functors is that they are indeed adjoint as expected.

Lemma 18.13.2. *Let $(f, f^\sharp) : (Sh(\mathcal{C}), \mathcal{O}_\mathcal{C}) \to (Sh(\mathcal{D}), \mathcal{O}_\mathcal{D})$ be a morphism of ringed topoi or ringed sites. Let $\mathcal{F}$ be a sheaf of $\mathcal{O}_\mathcal{C}$-modules. Let $\mathcal{G}$ be a sheaf of $\mathcal{O}_\mathcal{D}$-modules. There is a canonical bijection*

$$\mathrm{Hom}_{\mathcal{O}_\mathcal{C}}(f^*\mathcal{G}, \mathcal{F}) = \mathrm{Hom}_{\mathcal{O}_\mathcal{D}}(\mathcal{G}, f_*\mathcal{F}).$$

In other words: the functor f^* is the left adjoint to f_*.

Proof. This follows from the work we did before:

$$\begin{array}{rcl} \mathrm{Hom}_{\mathcal{O}_\mathcal{C}}(f^*\mathcal{G}, \mathcal{F}) & = & \mathrm{Mor}_{Mod(\mathcal{O}_\mathcal{C})}(\mathcal{O}_\mathcal{C} \otimes_{f^{-1}\mathcal{O}_\mathcal{D}} f^{-1}\mathcal{G}, \mathcal{F}) \\ & = & \mathrm{Mor}_{Mod(f^{-1}\mathcal{O}_\mathcal{D})}(f^{-1}\mathcal{G}, \mathcal{F}_{f^{-1}\mathcal{O}_\mathcal{D}}) \\ & = & \mathrm{Hom}_{\mathcal{O}_\mathcal{D}}(\mathcal{G}, f_*\mathcal{F}). \end{array}$$

Here we use Lemmas 18.11.3 and 18.12.3. □

Lemma 18.13.3. *$(f, f^\sharp) : (Sh(\mathcal{C}_1), \mathcal{O}_1) \to (Sh(\mathcal{C}_2), \mathcal{O}_2)$ and $(g, g^\sharp) : (Sh(\mathcal{C}_2), \mathcal{O}_2) \to (Sh(\mathcal{C}_3), \mathcal{O}_3)$ be morphisms of ringed topoi. There are canonical isomorphisms of functors $(g \circ f)_* \cong g_* \circ f_*$ and $(g \circ f)^* \cong f^* \circ g^*$.*

Proof. This is clear from the definitions. □

18.14. The abelian category of sheaves of modules

Let $(Sh(\mathcal{C}), \mathcal{O})$ be a ringed topos. Let $\mathcal{F}$, $\mathcal{G}$ be sheaves of $\mathcal{O}$-modules, see Sheaves, Definition 6.10.1. Let $\varphi, \psi : \mathcal{F} \to \mathcal{G}$ be morphisms of sheaves of $\mathcal{O}$-modules. We define $\varphi + \psi : \mathcal{F} \to \mathcal{G}$ to be the sum of φ and ψ as morphisms of abelian sheaves. This is clearly again a map of $\mathcal{O}$-modules. It is also clear that composition of maps of $\mathcal{O}$-modules is bilinear with respect to this addition. Thus $Mod(\mathcal{O})$ is a pre-additive category, see Homology, Definition 12.3.1.

We will denote 0 the sheaf of $\mathcal{O}$-modules which has constant value $\{0\}$ for all objects U of $\mathcal{C}$. Clearly this is both a final and an initial object of $Mod(\mathcal{O})$. Given a morphism of $\mathcal{O}$-modules $\varphi : \mathcal{F} \to \mathcal{G}$ the following are equivalent: (a) φ is zero, (b) φ factors through 0, (c) φ is zero on sections over each object U.

Moreover, given a pair $\mathcal{F}$, $\mathcal{G}$ of sheaves of $\mathcal{O}$-modules we may define the direct sum as

$$\mathcal{F} \oplus \mathcal{G} = \mathcal{F} \times \mathcal{G}$$

with obvious maps (i, j, p, q) as in Homology, Definition 12.3.5. Thus $Mod(\mathcal{O})$ is an additive category, see Homology, Definition 12.3.8.

Let $\varphi : \mathcal{F} \to \mathcal{G}$ be a morphism of $\mathcal{O}$-modules. We may define $\mathrm{Ker}(\varphi)$ to be the kernel of φ as a map of abelian sheaves. By Section 18.3 this is the subsheaf of $\mathcal{F}$ with sections

$$\mathrm{Ker}(\varphi)(U) = \{s \in \mathcal{F}(U) \mid \varphi(s) = 0 \text{ in } \mathcal{G}(U)\}$$

for all objects U of $\mathcal{C}$. It is easy to see that this is indeed a kernel in the category of $\mathcal{O}$-modules. In other words, a morphism $\alpha : \mathcal{H} \to \mathcal{F}$ factors through $\mathrm{Ker}(\varphi)$ if and only if $\varphi \circ \alpha = 0$.

Similarly, we define $\mathrm{Coker}(\varphi)$ as the cokernel of φ as a map of abelian sheaves. There is a unique multiplication map

$$\mathcal{O} \times \mathrm{Coker}(\varphi) \longrightarrow \mathrm{Coker}(\varphi)$$

such that the map $\mathcal{G} \to \mathrm{Coker}(\varphi)$ becomes a morphism of $\mathcal{O}$-modules (verification omitted). The map $\mathcal{G} \to \mathrm{Coker}(\varphi)$ is surjective (as a map of sheaves of sets, see Section 18.3). To show that $\mathrm{Coker}(\varphi)$ is a cokernel in $Mod(\mathcal{O})$, note that if $\beta : \mathcal{G} \to \mathcal{H}$ is a morphism of $\mathcal{O}$-modules such that $\beta \circ \varphi$ is zero, then you get for every object U of $\mathcal{C}$ a map induced by β from $\mathcal{G}(U)/\varphi(\mathcal{F}(U))$ into $\mathcal{H}(U)$. By the universal property of sheafification (see Sheaves, Lemma 6.20.1) we obtain a canonical map $\mathrm{Coker}(\varphi) \to \mathcal{H}$ such that the original β is equal to the composition $\mathcal{G} \to \mathrm{Coker}(\varphi) \to \mathcal{H}$. The morphism $\mathrm{Coker}(\varphi) \to \mathcal{H}$ is unique because of the surjectivity mentioned above.

Lemma 18.14.1. *Let $(Sh(\mathcal{C}), \mathcal{O})$ be a ringed topos. The category $Mod(\mathcal{O})$ is an abelian category. The forgetful functor $Mod(\mathcal{O}) \to Ab(\mathcal{C})$ is exact, hence kernels, cokernels and exactness of $\mathcal{O}$-modules, correspond to the corresponding notions for abelian sheaves.*

Proof. Above we have seen that $Mod(\mathcal{O})$ is an additive category, with kernels and cokernels and that $Mod(\mathcal{O}) \to Ab(\mathcal{C})$ preserves kernels and cokernels. By Homology, Definition 12.5.1 we have to show that image and coimage agree. This is clear because it is true in $Ab(\mathcal{C})$. The lemma follows. □

Lemma 18.14.2. *Let $(Sh(\mathcal{C}), \mathcal{O})$ be a ringed topos. All limits and colimits exist in $Mod(\mathcal{O})$ and the forgetful functor $Mod(\mathcal{O}) \to Ab(\mathcal{C})$ commutes with them. Moreover, filtered colimits are exact.*

Proof. The final statement follows from the first as filtered colimits are exact in $Ab(\mathcal{C})$ by Lemma 18.3.2. Let $\mathcal{I} \to Mod(\mathcal{C})$, $i \mapsto \mathcal{F}_i$ be a diagram. Let $\lim_i \mathcal{F}_i$ be the limit of the diagram in $Ab(\mathcal{C})$. By the description of this limit in Lemma 18.3.2 we see immediately that there exists a multiplication

$$\mathcal{O} \times \lim_i \mathcal{F}_i \longrightarrow \lim_i \mathcal{F}_i$$

which turns $\lim_i \mathcal{F}_i$ into a sheaf of $\mathcal{O}$-modules. It is easy to see that this is the limit of the diagram in $Mod(\mathcal{C})$. Let $\operatorname{colim}_i \mathcal{F}_i$ be the colimit of the diagram in $PAb(\mathcal{C})$. By the description of this colimit in the proof of Lemma 18.2.1 we see immediately that there exists a multiplication

$$\mathcal{O} \times \operatorname{colim}_i \mathcal{F}_i \longrightarrow \operatorname{colim}_i \mathcal{F}_i$$

which turns $\operatorname{colim}_i \mathcal{F}_i$ into a presheaf of $\mathcal{O}$-modules. Applying sheafification we get a sheaf of $\mathcal{O}$-modules $(\operatorname{colim}_i \mathcal{F}_i)^\#$, see Lemma 18.11.1. It is easy to see that $(\operatorname{colim}_i \mathcal{F}_i)^\#$ is the colimit of the diagram in $Mod(\mathcal{O})$, and by Lemma 18.3.2 forgetting the $\mathcal{O}$-module structure is the colimit in $Ab(\mathcal{C})$. □

The existence of limits and colimits allows us to consider exactness properties of functors defined on the category of $\mathcal{O}$-modules in terms of limits and colimits, as in Categories, Section 4.23. See Homology, Lemma 12.7.1 for a description of exactness properties in terms of short exact sequences.

Lemma 18.14.3. *Let $f : (Sh(\mathcal{C}), \mathcal{O}_\mathcal{C}) \to (Sh(\mathcal{D}), \mathcal{O}_\mathcal{D})$ be a morphism of ringed topoi.*

(1) *The functor f_* is left exact. In fact it commutes with all limits.*
(2) *The functor f^* is right exact. In fact it commutes with all colimits.*

Proof. This is true because (f^*, f_*) is an adjoint pair of functors, see Lemma 18.13.2. See Categories, Section 4.24. □

Lemma 18.14.4. *Let $\mathcal{C}$ be a site. If $\{p_i\}_{i \in I}$ is a conservative family of points, then we may check exactness of a sequence of abelian sheaves on the stalks at the points p_i, $i \in I$. If $\mathcal{C}$ has enough points, then exactness of a sequence of abelian sheaves may be checked on stalks.*

Proof. This is immediate from Sites, Lemma 7.37.2. □

18.15. Exactness of pushforward

Some technical lemmas concerning exactness properties of pushforward.

Lemma 18.15.1. *Let $f : Sh(\mathcal{C}) \to Sh(\mathcal{D})$ be a morphism of topoi. The following are equivalent:*

(1) *$f^{-1} f_* \mathcal{F} \to \mathcal{F}$ is surjective for all $\mathcal{F}$ in $Ab(\mathcal{C})$, and*
(2) *$f_* : Ab(\mathcal{C}) \to Ab(\mathcal{D})$ reflects surjections.*

In this case the functor $f_* : Ab(\mathcal{C}) \to Ab(\mathcal{D})$ is faithful.

Proof. Assume (1). Suppose that $a : \mathcal{F} \to \mathcal{F}'$ is a map of abelian sheaves on $\mathcal{C}$ such that f_*a is surjective. As f^{-1} is exact this implies that $f^{-1}f_*a : f^{-1}f_*\mathcal{F} \to f^{-1}f_*\mathcal{F}'$ is surjective. Combined with (1) this implies that a is surjective. This means that (2) holds.

Assume (2). Let $\mathcal{F}$ be an abelian sheaf on $\mathcal{C}$. We have to show that the map $f^{-1}f_*\mathcal{F} \to \mathcal{F}$ is surjective. By (2) it suffices to show that $f_*f^{-1}f_*\mathcal{F} \to f_*\mathcal{F}$ is surjective. And this is true because there is a canonical map $f_*\mathcal{F} \to f_*f^{-1}f_*\mathcal{F}$ which is a one-sided inverse.

We omit the proof of the final assertion. □

Lemma 18.15.2. *Let $f : Sh(\mathcal{C}) \to Sh(\mathcal{D})$ be a morphism of topoi. Assume at least one of the following properties holds*

(1) *f_* transforms surjections of sheaves of sets into surjections,*
(2) *f_* transforms surjections of abelian sheaves into surjections,*
(3) *f_* commutes with coequalizers on sheaves of sets,*
(4) *f_* commutes with pushouts on sheaves of sets,*

Then $f_* : Ab(\mathcal{C}) \to Ab(\mathcal{D})$ is exact.

Proof. Since $f_* : Ab(\mathcal{C}) \to Ab(\mathcal{D})$ is a right adjoint we already know that it transforms a short exact sequence $0 \to \mathcal{F}_1 \to \mathcal{F}_2 \to \mathcal{F}_3 \to 0$ of abelian sheaves on $\mathcal{C}$ into an exact sequence

$$0 \to f_*\mathcal{F}_1 \to f_*\mathcal{F}_2 \to f_*\mathcal{F}_3$$

see Categories, Sections 4.23 and 4.24 and Homology, Section 12.7. Hence it suffices to prove that the map $f_*\mathcal{F}_2 \to f_*\mathcal{F}_3$ is surjective. If (1), (2) holds, then this is clear from the definitions. By Sites, Lemma 7.40.1 we see that either (3) or (4) formally implies (1), hence in these cases we are done also. □

Lemma 18.15.3. *Let $f : \mathcal{D} \to \mathcal{C}$ be a morphism of sites associated to the continuous functor $u : \mathcal{C} \to \mathcal{D}$. Assume u is almost cocontinuous. Then*

(1) *$f_* : Ab(\mathcal{D}) \to Ab(\mathcal{C})$ is exact.*
(2) *if $f^\sharp : f^{-1}\mathcal{O}_\mathcal{C} \to \mathcal{O}_\mathcal{D}$ is given so that f becomes a morphism of ringed sites, then $f_* : Mod(\mathcal{O}_\mathcal{D}) \to Mod(\mathcal{O}_\mathcal{C})$ is exact.*

Proof. Part (2) follows from part (1) by Lemma 18.14.2. Part (1) follows from Sites, Lemmas 7.41.6 and 7.40.1. □

18.16. Exactness of lower shriek

Let $u : \mathcal{C} \to \mathcal{D}$ be a functor between sites. Assume that

(a) u is cocontinuous, and
(b) u is continuous.

Let $g : Sh(\mathcal{C}) \to Sh(\mathcal{D})$ be the morphism of topoi associated with u, see Sites, Lemma 7.20.1. Recall that $g^{-1} = u^p$, i.e., g^{-1} is given by the simple formula $(g^{-1}\mathcal{G})(U) = \mathcal{G}(u(U))$, see Sites, Lemma 7.20.5. We would like to show that $g^{-1} : Ab(\mathcal{D}) \to Ab(\mathcal{C})$ has a left adjoint $g_!$. By Sites, Lemma 7.20.5 the functor $g_!^{Sh} = (u_p\ )^\#$ is a left adjoint on sheaves of sets. Moreover, we know that $g_!^{Sh}\mathcal{F}$ is the sheaf associated to the presheaf

$$V \longmapsto \operatorname{colim}_{V \to u(U)} \mathcal{F}(U)$$

where the colimit is over $(\mathcal{I}_V^u)^{opp}$ and is taken in the category of sets. Hence the following definition is natural.

Definition 18.16.1. With $u : \mathcal{C} \to \mathcal{D}$ satisfying (a), (b) above. For $\mathcal{F} \in PAb(\mathcal{C})$ we define $g_{p!}\mathcal{F}$ as the presheaf

$$V \longmapsto \operatorname{colim}_{V \to u(U)} \mathcal{F}(U)$$

with colimits over $(\mathcal{I}_V^u)^{opp}$ taken in Ab. For $\mathcal{F} \in PAb(\mathcal{C})$ we set $g_!\mathcal{F} = (g_{p!}\mathcal{F})^\#$.

The reason for being so explicit with this is that the functors $g_!^{Sh}$ and $g_!$ are different. Whenever we use both we have to be careful to make the distinction clear.

Lemma 18.16.2. *The functor $g_{p!}$ is a left adjoint to the functor u^p. The functor $g_!$ is a left adjoint to the functor g^{-1}. In other words the formulas*

$$\operatorname{Mor}_{PAb(\mathcal{C})}(\mathcal{F}, u^p\mathcal{G}) = \operatorname{Mor}_{PAb(\mathcal{D})}(g_{p!}\mathcal{F}, \mathcal{G}),$$
$$\operatorname{Mor}_{Ab(\mathcal{C})}(\mathcal{F}, g^{-1}\mathcal{G}) = \operatorname{Mor}_{Ab(\mathcal{D})}(g_!\mathcal{F}, \mathcal{G})$$

hold bifunctorially in $\mathcal{F}$ and $\mathcal{G}$.

Proof. The second formula follows formally from the first, since if $\mathcal{F}$ and $\mathcal{G}$ are abelian sheaves then

$$\begin{aligned}\operatorname{Mor}_{Ab(\mathcal{C})}(\mathcal{F}, g^{-1}\mathcal{G}) &= \operatorname{Mor}_{PAb(\mathcal{D})}(g_{p!}\mathcal{F}, \mathcal{G}) \\ &= \operatorname{Mor}_{Ab(\mathcal{D})}(g_!\mathcal{F}, \mathcal{G})\end{aligned}$$

by the universal property of sheafification.

To prove the first formula, let $\mathcal{F}$, $\mathcal{G}$ be abelian presheaves. To prove the lemma we will construct maps from the group on the left to the group on the right and omit the verification that these are mutually inverse.

Note that there is a canonical map of abelian presheaves $\mathcal{F} \to u^p g_{p!}\mathcal{F}$ which on sections over U is the natural map $\mathcal{F}(U) \to \operatorname{colim}_{u(U) \to u(U')} \mathcal{F}(U')$, see Sites, Lemma 7.5.3. Given a map $\alpha : g_{p!}\mathcal{F} \to \mathcal{G}$ we get $u^p\alpha : u^p g_{p!}\mathcal{F} \to u^p\mathcal{G}$. which we can precompose by the map $\mathcal{F} \to u^p g_{p!}\mathcal{F}$.

Note that there is a canonical map of abelian presheaves $g_{p!}u^p\mathcal{G} \to \mathcal{G}$ which on sections over V is the natural map $\operatorname{colim}_{V \to u(U)} \mathcal{G}(u(U)) \to \mathcal{G}(V)$. It maps a section $s \in u(U)$ in the summand corresponding to $t : V \to u(U)$ to $t^*s \in \mathcal{G}(V)$. Hence, given a map $\beta : \mathcal{F} \to u^p\mathcal{G}$ we get a map $g_{p!}\beta : g_{p!}\mathcal{F} \to g_{p!}u^p\mathcal{G}$ which we can postcompose with the map $g_{p!}u^p\mathcal{G} \to \mathcal{G}$ above. □

Lemma 18.16.3. *Let $\mathcal{C}$ and $\mathcal{D}$ be sites. Let $u : \mathcal{C} \to \mathcal{D}$ be a functor. Assume that*

(a) *u is cocontinuous,*
(b) *u is continuous, and*
(c) *fibre products and equalizers exist in $\mathcal{C}$ and u commutes with them.*

In this case the functor $g_! : Ab(\mathcal{C}) \to Ab(\mathcal{D})$ is exact.

Proof. Compare with Sites, Lemma 7.20.6. Assume (a), (b), and (c). We already know that $g_!$ is right exact as it is a left adjoint, see Categories, Lemma 4.24.4 and Homology, Section 12.7. We have $g_! = (g_{p!}\ )^\#$. We have to show that $g_!$ transforms injective maps of abelian sheaves into injective maps of abelian presheaves. Recall that sheafification of abelian presheaves is exact, see Lemma 18.3.2. Thus it suffices to show that $g_{p!}$ transforms injective maps of abelian presheaves into injective maps

of abelian presheaves. To do this it suffices that colimits over the categories $(\mathcal{I}_V^u)^{opp}$ of Sites, Section 7.5 transform injective maps between diagrams into injections. This follows from Sites, Lemma 7.5.1 and Algebra, Lemma 10.8.11. □

Lemma 18.16.4. *Let $\mathcal{C}$ and $\mathcal{D}$ be sites. Let $u : \mathcal{C} \to \mathcal{D}$ be a functor. Assume that*

(a) *u is cocontinuous,*
(b) *u is continuous, and*
(c) *u is fully faithful.*

For g and $g_!$ as above the canonical map $\mathcal{F} \to g^{-1}g_!\mathcal{F}$ is an isomorphism for all abelian sheaves $\mathcal{F}$ on $\mathcal{C}$.

Proof. Pick $U \in \mathrm{Ob}(\mathcal{C})$. We will show that $g^{-1}g_!\mathcal{F}(U) = \mathcal{F}(U)$. First, note that $g^{-1}g_!\mathcal{F}(U) = g_!\mathcal{F}(u(U))$. Hence it suffices to show that $g_!\mathcal{F}(u(U)) = \mathcal{F}(U)$. We know that $g_!\mathcal{F}$ is the (abelian) sheaf associated to the presheaf $g_{p!}\mathcal{F}$ which is defined by the rule

$$V \longmapsto \mathrm{colim}_{V \to u(U')} \mathcal{F}(U')$$

with colimit taken in Ab. If $V = u(U)$, then, as u is fully faithful this colimit is over $U \to U'$. Hence we conclude that $g_{p!}\mathcal{F}(u(U) = \mathcal{F}(U)$. Since u is cocontinuous and continuous any covering of $u(U)$ in $\mathcal{D}$ can be refined by a covering (!) $\{u(U_i) \to u(U)\}$ of $\mathcal{D}$ where $\{U_i \to U\}$ is a covering in $\mathcal{C}$. This implies that $(g_{p!}\mathcal{F})^+(u(U)) = \mathcal{F}(U)$ also, since in the colimit defining the value of $(g_{p!}\mathcal{F})^+$ on $u(U)$ we may restrict to the cofinal system of coverings $\{u(U_i) \to u(U)\}$ as above. Hence we see that $(g_{p!}\mathcal{F})^+(u(U)) = \mathcal{F}(U)$ for all objects U of $\mathcal{C}$ as well. Repeating this argument one more time gives the equality $(g_{p!}\mathcal{F})^\#(u(U)) = \mathcal{F}(U)$ for all objects U of $\mathcal{C}$. This produces the desired equality $g^{-1}g_!\mathcal{F} = \mathcal{F}$. □

Remark 18.16.5. In general the functor $g_!$ cannot be extended to categories of modules in case g is (part of) a morphism of ringed topoi. Namely, given any ring map $A \to B$ the functor $M \mapsto B \otimes_A M$ has a right adjoint (restriction) but not in general a left adjoint (because its existence would imply that $A \to B$ is flat). We will see in Section 18.19 below that it is possible to define $j_!$ on sheaves of modules in the case of a localization of sites. We will discuss this in greater generality in Section 18.40 below.

Lemma 18.16.6. *Let $\mathcal{C}$ and $\mathcal{D}$ be sites. Let $g : Sh(\mathcal{C}) \to Sh(\mathcal{D})$ be the morphism of topoi associated to a continuous and cocontinuous functor $u : \mathcal{C} \to \mathcal{D}$.*

(1) *If u has a left adjoint w, then $g_!$ agrees with $g_!^{Sh}$ on underlying sheaves of sets and $g_!$ is exact.*
(2) *If in addition w is cocontinuous, then $g_! = h^{-1}$ and $g^{-1} = h_*$ where $h : Sh(\mathcal{D}) \to Sh(\mathcal{C})$ is the morphism of topoi associated to w.*

Proof. This Lemma is the analogue of Sites, Lemma 7.22.1. From Sites, Lemma 7.18.3 we see that the categories $\mathcal{I}_V^u$ have an initial object. Thus the underlying set of a colimit of a system of abelian groups over $(\mathcal{I}_V^u)^{opp}$ is the colimit of the underlying sets. Whence the agreement of $g_!^{Sh}$ and $g_!$ by our construction of $g_!$ in Definition 18.16.1. The exactness and (2) follow immediately from the corresponding statements of Sites, Lemma 7.22.1. □

18.17. Global types of modules

Definition 18.17.1. Let $(Sh(\mathcal{C}), \mathcal{O})$ be a ringed topos. Let $\mathcal{F}$ be a sheaf of $\mathcal{O}$-modules.

(1) We say $\mathcal{F}$ is a *free $\mathcal{O}$-module* if $\mathcal{F}$ is isomorphic as an $\mathcal{O}$-module to a sheaf of the form $\bigoplus_{i\in I} \mathcal{O}$.
(2) We say $\mathcal{F}$ is *finite free* if $\mathcal{F}$ is isomorphic as an $\mathcal{O}$-module to a sheaf of the form $\bigoplus_{i\in I} \mathcal{O}$ with a finite index set I.
(3) We say $\mathcal{F}$ is *generated by global sections* if there exists a surjection
$$\bigoplus\nolimits_{i\in I} \mathcal{O} \longrightarrow \mathcal{F}$$
from a free $\mathcal{O}$-module onto $\mathcal{F}$.
(4) Given $r \geq 0$ we say $\mathcal{F}$ is *generated by r global sections* if there exists a surjection $\mathcal{O}^{\oplus r} \to \mathcal{F}$.
(5) We say $\mathcal{F}$ is *generated by finitely many global sections* if it is generated by r global sections for some $r \geq 0$.
(6) We say $\mathcal{F}$ has a *global presentation* if there exists an exact sequence
$$\bigoplus\nolimits_{j\in J} \mathcal{O} \longrightarrow \bigoplus\nolimits_{i\in I} \mathcal{O} \longrightarrow \mathcal{F}$$
of $\mathcal{O}$-modules.
(7) We say $\mathcal{F}$ has a *global finite presentation* if there exists an exact sequence
$$\bigoplus\nolimits_{j\in J} \mathcal{O} \longrightarrow \bigoplus\nolimits_{i\in I} \mathcal{O} \longrightarrow \mathcal{F}$$
of $\mathcal{O}$-modules with I and J finite sets.

Note that for any set I the direct sum $\bigoplus_{i\in I} \mathcal{O}$ exists (Lemma 18.14.2) and is the sheafification of the presheaf $U \mapsto \bigoplus_{i\in I} \mathcal{O}(U)$. This module is called the *free $\mathcal{O}$-module on the set I*.

Lemma 18.17.2. *Let $(f, f^\sharp) : (Sh(\mathcal{C}), \mathcal{O}_\mathcal{C}) \to (Sh(\mathcal{D}), \mathcal{O}_\mathcal{D})$ be a morphism of ringed topoi. Let $\mathcal{F}$ be an $\mathcal{O}_\mathcal{D}$-module.*

(1) *If $\mathcal{F}$ is free then $f^*\mathcal{F}$ is free.*
(2) *If $\mathcal{F}$ is finite free then $f^*\mathcal{F}$ is finite free.*
(3) *If $\mathcal{F}$ is generated by global sections then $f^*\mathcal{F}$ is generated by global sections.*
(4) *Given $r \geq 0$ if $\mathcal{F}$ is generated by r global sections, then $f^*\mathcal{F}$ is generated by r global sections.*
(5) *If $\mathcal{F}$ is generated by finitely many global sections then $f^*\mathcal{F}$ is generated by finitely many global sections.*
(6) *If $\mathcal{F}$ has a global presentation then $f^*\mathcal{F}$ has a global presentation.*
(7) *If $\mathcal{F}$ has a finite global presentation then $f^*\mathcal{F}$ has a finite global presentation.*

Proof. This is true because f^* commutes with arbitrary colimits (Lemma 18.14.3) and $f^*\mathcal{O}_\mathcal{D} = \mathcal{O}_\mathcal{C}$. □

18.18. Intrinsic properties of modules

Let $\mathcal{P}$ be a property of sheaves of modules on ringed topoi. We say $\mathcal{P}$ is an *intrinsic property* if we have $\mathcal{P}(\mathcal{F}) \Leftrightarrow \mathcal{P}(f^*\mathcal{F})$ whenever $(f, f^\sharp) : (Sh(\mathcal{C}'), \mathcal{O}') \to (Sh(\mathcal{C}), \mathcal{O})$ is

an equivalence of ringed topoi. For example, the property of being free is intrinsic. Indeed, the free $\mathcal{O}$-module on the set I is characterized by the property that

$$\mathrm{Mor}_{Mod(\mathcal{O})}(\bigoplus\nolimits_{i\in I}\mathcal{O}, \mathcal{F}) = \prod\nolimits_{i\in I}\mathrm{Mor}_{Sh(\mathcal{C})}(\{*\}, \mathcal{F})$$

for a variable $\mathcal{F}$ in $Mod(\mathcal{O})$. Alternatively, we can also use Lemma 18.17.2 to see that being free is intrinsic. In fact, each of the properties defined in Definition 18.17.1 is intrinsic for the same reason. How will we go about defining other intrinsic properties of $\mathcal{O}$-modules?

The upshot of Lemma 18.7.2 is the following: Suppose you want to define an intrinsic property $\mathcal{P}$ of an $\mathcal{O}$-module on a topos. Then you can proceed as follows:

(1) Given any site $\mathcal{C}$, any sheaf of rings $\mathcal{O}$ on $\mathcal{C}$ and any $\mathcal{O}$-module $\mathcal{F}$ define the corresponding property $\mathcal{P}(\mathcal{C}, \mathcal{O}, \mathcal{F})$.
(2) For any pair of sites $\mathcal{C}$, $\mathcal{C}'$, any special cocontinuous functor $u : \mathcal{C} \to \mathcal{C}'$, any sheaf of rings $\mathcal{O}$ on $\mathcal{C}$ any $\mathcal{O}$-module $\mathcal{F}$, show that
$$\mathcal{P}(\mathcal{C}, \mathcal{O}, \mathcal{F}) \Leftrightarrow \mathcal{P}(\mathcal{C}', g_*\mathcal{O}, g_*\mathcal{F})$$
where $g : Sh(\mathcal{C}) \to Sh(\mathcal{C}')$ is the equivalence of topoi associated to u.

In this case, given any ringed topos $(Sh(\mathcal{C}), \mathcal{O})$ and any sheaf of $\mathcal{O}$-modules $\mathcal{F}$ we simply say that $\mathcal{F}$ has property $\mathcal{P}$ if $\mathcal{P}(\mathcal{C}, \mathcal{O}, \mathcal{F})$ is true. And Lemma 18.7.2 combined with (2) above guarantees that this is well defined.

Moreover, the same Lemma 18.7.2 also guarantees that if in addition

(3) For any morphism of ringed sites $(f, f^\sharp) : (\mathcal{C}, \mathcal{O}_\mathcal{C}) \to (\mathcal{D}, \mathcal{O}_\mathcal{D})$ such that f is given by a functor $u : \mathcal{D} \to \mathcal{C}$ satisfying the assumptions of Sites, Proposition 7.15.6, and any $\mathcal{O}_\mathcal{D}$-module $\mathcal{G}$ we have
$$\mathcal{P}(\mathcal{D}, \mathcal{O}_\mathcal{D}, \mathcal{F}) \Rightarrow \mathcal{P}(\mathcal{C}, \mathcal{O}_\mathcal{C}, f^*\mathcal{F})$$

then it is true that $\mathcal{P}$ is preserved under pullback of modules w.r.t. arbitrary morphisms of ringed topoi.

We will use this method in the following sections to see that: locally free, locally generated by sections, locally generated by r sections, finite type, finite presentation, quasi-coherent, and coherent are intrinsic properties of modules.

Perhaps a more satisfying method would be to find an intrinsic definition of these notions, rather than the laborious process sketched here. On the other hand, in many geometric situations where we want to apply these definitions we are given a definite ringed site, and a definite sheaf of modules, and it is nice to have a definition already adapted to this language.

18.19. Localization of ringed sites

Let $(\mathcal{C}, \mathcal{O})$ be a ringed site. Let $U \in \mathrm{Ob}(\mathcal{C})$. We explain the counterparts of the results in Sites, Section 7.24 in this setting.

Denote $\mathcal{O}_U = j_U^{-1}\mathcal{O}$ the restriction of $\mathcal{O}$ to the site $\mathcal{C}/U$. It is described by the simple rule $\mathcal{O}_U(V/U) = \mathcal{O}(V)$. With this notation the localization morphism j_U becomes a morphism of ringed topoi

$$(j_U, j_U^\sharp) : (Sh(\mathcal{C}/U), \mathcal{O}_U) \longrightarrow (Sh(\mathcal{C}), \mathcal{O})$$

namely, we take $j_U^\sharp : j_U^{-1}\mathcal{O} \to \mathcal{O}_U$ the identity map. Moreover, we obtain the following descriptions for pushforward and pullback of modules.

Definition 18.19.1. Let $(\mathcal{C}, \mathcal{O})$ be a ringed site. Let $U \in \mathrm{Ob}(\mathcal{C})$.

(1) The ringed site $(\mathcal{C}/U, \mathcal{O}_U)$ is called the *localization of the ringed site* $(\mathcal{C}, \mathcal{O})$ *at the object* U.
(2) The morphism of ringed topoi $(j_U, j_U^\sharp) : (Sh(\mathcal{C}/U), \mathcal{O}_U) \to (Sh(\mathcal{C}), \mathcal{O})$ is called the *localization morphism*.
(3) The functor $j_{U*} : Mod(\mathcal{O}_U) \to Mod(\mathcal{O})$ is called the *direct image functor*.
(4) For a sheaf of $\mathcal{O}$-modules $\mathcal{F}$ on $\mathcal{C}$ the sheaf $j_U^*\mathcal{F}$ is called the *restriction of* $\mathcal{F}$ *to* $\mathcal{C}/U$. We will sometimes denote it by $\mathcal{F}|_{\mathcal{C}/U}$ or even $\mathcal{F}|_U$. It is described by the simple rule $j_U^*(\mathcal{F})(X/U) = \mathcal{F}(X)$.
(5) The left adjoint $j_{U!} : Mod(\mathcal{O}_U) \to Mod(\mathcal{O})$ of restriction is called *extension by zero*. It exists and is exact by Lemmas 18.19.2 and 18.19.3.

As in the topological case, see Sheaves, Section 6.31, the extension by zero $j_{U!}$ functor is different from extension by the empty set $j_{U!}$ defined on sheaves of sets. Here is the lemma defining extension by zero.

Lemma 18.19.2. *Let $(\mathcal{C}, \mathcal{O})$ be a ringed site. Let $U \in \mathrm{Ob}(\mathcal{C})$. The restriction functor $j_U^* : Mod(\mathcal{O}) \to Mod(\mathcal{O}_U)$ has a left adjoint $j_{U!} : Mod(\mathcal{O}_U) \to Mod(\mathcal{O})$. So*

$$\mathrm{Mor}_{Mod(\mathcal{O}_U)}(\mathcal{G}, j_U^*\mathcal{F}) = \mathrm{Mor}_{Mod(\mathcal{O})}(j_{U!}\mathcal{G}, \mathcal{F})$$

for $\mathcal{F} \in \mathrm{Ob}(Mod(\mathcal{O}))$ and $\mathcal{G} \in \mathrm{Ob}(Mod(\mathcal{O}_U))$. Moreover, the extension by zero $j_{U!}\mathcal{G}$ of $\mathcal{G}$ is the sheaf associated to the presheaf

$$V \longmapsto \bigoplus\nolimits_{\varphi \in \mathrm{Mor}_{\mathcal{C}}(V,U)} \mathcal{G}(V \xrightarrow{\varphi} U)$$

with obvious restriction mappings and an obvious $\mathcal{O}$-module structure.

Proof. The $\mathcal{O}$-module structure on the presheaf is defined as follows. If $f \in \mathcal{O}(V)$ and $s \in \mathcal{G}(V \xrightarrow{\varphi} U)$, then we define $f \cdot s = fs$ where $f \in \mathcal{O}_U(\varphi : V \to U) = \mathcal{O}(V)$ (because $\mathcal{O}_U$ is the restriction of $\mathcal{O}$ to $\mathcal{C}/U$).

Similarly, let $\alpha : \mathcal{G} \to \mathcal{F}|_U$ be a morphism of $\mathcal{O}_U$-modules. In this case we can define a map from the presheaf of the lemma into $\mathcal{F}$ by mapping

$$\bigoplus\nolimits_{\varphi \in \mathrm{Mor}_{\mathcal{C}}(V,U)} \mathcal{G}(V \xrightarrow{\varphi} U) \longrightarrow \mathcal{F}(V)$$

by the rule that $s \in \mathcal{G}(V \xrightarrow{\varphi} U)$ maps to $\alpha(s) \in \mathcal{F}(V)$. It is clear that this is $\mathcal{O}$-linear, and hence induces a morphism of $\mathcal{O}$-modules $\alpha' : j_{U!}\mathcal{G} \to \mathcal{F}$ by the properties of sheafification of modules (Lemma 18.11.1).

Conversely, let $\beta : j_{U!}\mathcal{G} \to \mathcal{F}$ by a map of $\mathcal{O}$-modules. Recall from Sites, Section 7.24 that there exists an extension by the empty set $j_{U!}^{Sh} : Sh(\mathcal{C}/U) \to Sh(\mathcal{C})$ on sheaves of sets which is left adjoint to j_U^{-1}. Moreover, $j_{U!}^{Sh}\mathcal{G}$ is the sheaf associated to the presheaf

$$V \longmapsto \coprod\nolimits_{\varphi \in \mathrm{Mor}_{\mathcal{C}}(V,U)} \mathcal{G}(V \xrightarrow{\varphi} U)$$

Hence there is a natural map $j_{U!}^{Sh}\mathcal{G} \to j_{U!}\mathcal{G}$ of sheaves of sets. Hence precomposing β by this map we get a map of sheaves of sets $j_{U!}^{Sh}\mathcal{G} \to \mathcal{F}$ which by adjunction corresponds to a map of sheaves of sets $\beta' : \mathcal{G} \to \mathcal{F}|_U$. We claim that β' is $\mathcal{O}_U$-linear. Namely, suppose that $\varphi : V \to U$ is an object of $\mathcal{C}/U$ and that $s, s' \in \mathcal{G}(\varphi : V \to U)$, and $f \in \mathcal{O}(V) = \mathcal{O}_U(\varphi : V \to U)$. Then by the discussion above we see that $\beta'(s + s')$, resp. $\beta'(fs)$ in $\mathcal{F}|_U(\varphi : V \to U)$ correspond to $\beta(s + s')$, resp. $\beta(fs)$ in $\mathcal{F}(V)$. Since β is a homomorphism we conclude.

To conclude the proof of the lemma we have to show that the constructions $\alpha \mapsto \alpha'$ and $\beta \mapsto \beta'$ are mutually inverse. We omit the verifications. □

Lemma 18.19.3. *Let $(\mathcal{C}, \mathcal{O})$ be a ringed site. Let $U \in \mathrm{Ob}(\mathcal{C})$. The functor $j_{U!} : Mod(\mathcal{O}_U) \to Mod(\mathcal{O})$ is exact.*

Proof. Since $j_{U!}$ is a left adjoint to j_U^* we see that it is right exact (see Categories, Lemma 4.24.4 and Homology, Section 12.7). Hence it suffices to show that if $\mathcal{G}_1 \to \mathcal{G}_2$ is an injective map of $\mathcal{O}_U$-modules, then $j_{U!}\mathcal{G}_1 \to j_{U!}\mathcal{G}_2$ is injective. The map on sections of presheaves over an object V (as in Lemma 18.19.2) is the map

$$\bigoplus\nolimits_{\varphi \in \mathrm{Mor}_{\mathcal{C}}(V,U)} \mathcal{G}_1(V \xrightarrow{\varphi} U) \longrightarrow \bigoplus\nolimits_{\varphi \in \mathrm{Mor}_{\mathcal{C}}(V,U)} \mathcal{G}_2(V \xrightarrow{\varphi} U)$$

which is injective by assumption. Since sheafification is exact by Lemma 18.11.2 we conclude $j_{U!}\mathcal{G}_1 \to j_{U!}\mathcal{G}_2$ is injective and we win. □

Lemma 18.19.4. *Let $(\mathcal{C}, \mathcal{O})$ be a ringed site. Let $f : V \to U$ be a morphism of $\mathcal{C}$. Then there exists a commutative diagram*

$$\begin{array}{ccc} (Sh(\mathcal{C}/V), \mathcal{O}_V) & \xrightarrow{\quad (j, j^\sharp) \quad} & (Sh(\mathcal{C}/U), \mathcal{O}_U) \\ & {\scriptstyle (j_V, j_V^\sharp)} \searrow \quad \swarrow {\scriptstyle (j_U, j_U^\sharp)} & \\ & (Sh(\mathcal{C}), \mathcal{O}) & \end{array}$$

of ringed topoi. Here $(j, j^\sharp)$ is the localization morphism associated to the object U/V of the ringed site $(\mathcal{C}/V, \mathcal{O}_V)$.

Proof. The only thing to check is that $j_V^\sharp = j^\sharp \circ j^{-1}(j_U^\sharp)$, since everything else follows directly from Sites, Lemma 7.24.7 and Sites, Equation (7.24.7.1). We omit the verification of the equality. □

Remark 18.19.5. In the situation of Lemma 18.19.2 the diagram

$$\begin{array}{ccc} Mod(\mathcal{O}_U) & \xrightarrow{j_{U!}} & Mod(\mathcal{O}_\mathcal{C}) \\ {\scriptstyle forget}\downarrow & & \downarrow{\scriptstyle forget} \\ Ab(\mathcal{C}/U) & \xrightarrow{j_{U!}^{Ab}} & Ab(\mathcal{C}) \end{array}$$

commutes. This is clear from the explicit description of the functor $j_{U!}$ in the lemma.

Remark 18.19.6. Localization and presheaves of modules; see Sites, Remark 7.24.9. Let $\mathcal{C}$ be a category. Let $\mathcal{O}$ be a presheaf of rings. Let U be an object of $\mathcal{C}$. Strictly speaking the functors j_U^*, j_{U*} and $j_{U!}$ have not been defined for presheaves of $\mathcal{O}$-modules. But of course, we can think of a presheaf as a sheaf for the chaotic topology on $\mathcal{C}$ (see Sites, Examples 7.6.6). Hence we also obtain a functor

$$j_U^* : PMod(\mathcal{O}) \longrightarrow PMod(\mathcal{O}_U)$$

and functors

$$j_{U*}, j_{U!} : PMod(\mathcal{O}_U) \longrightarrow PMod(\mathcal{O})$$

which are right, left adjoint to j_U^*. Inspecting the proof of Lemma 18.19.2 we see that $j_{U!}\mathcal{G}$ is the presheaf

$$V \longmapsto \bigoplus\nolimits_{\varphi \in \mathrm{Mor}_{\mathcal{C}}(V,U)} \mathcal{G}(V \xrightarrow{\varphi} U)$$

In addition the functor $j_{U!}$ is exact (by Lemma 18.19.3 in the case of the discrete topologies). Moreover, if $\mathcal{C}$ is actually a site, and $\mathcal{O}$ is actually a sheaf of rings, then the diagram

$$\begin{array}{ccc} Mod(\mathcal{O}_U) & \xrightarrow[j_{U!}]{} & Mod(\mathcal{O}) \\ {\scriptstyle forget}\downarrow & & \uparrow{\scriptstyle (\ )^\#} \\ PMod(\mathcal{O}_U) & \xrightarrow{j_{U!}} & PMod(\mathcal{O}) \end{array}$$

commutes.

Remark 18.19.7 (Map from lower shriek to pushforward)**.** Let U be an object of $\mathcal{C}$. For any abelian sheaf $\mathcal{G}$ on $\mathcal{C}/U$ there is a canonical map

$$c : j_{U!}\mathcal{G} \longrightarrow j_{U*}\mathcal{G}$$

Namely, this is the same thing as a map $j_U^{-1}j_{U!}\mathcal{G} \to \mathcal{G}$. Note that restriction commutes with sheafification. Thus we can use the presheaf of Lemma 18.19.2. Hence it suffices to define for V/U a map

$$\bigoplus\nolimits_{\varphi \in \mathrm{Mor}_{\mathcal{C}}(V,U)} \mathcal{G}(V) \longrightarrow \mathcal{G}(V)$$

compatible with restrictions. We simply take the map which is zero on all summands except for the one where φ is the structure morphism $V \to U$ where we take 1. Moreover, if $\mathcal{O}$ is a sheaf of rings on $\mathcal{C}$ and $\mathcal{G}$ is an $\mathcal{O}_U$-module, then the displayed map above is a map of $\mathcal{O}$-modules.

18.20. Localization of morphisms of ringed sites

This section is the analogue of Sites, Section 7.27.

Lemma 18.20.1. *Let $(f, f^\sharp) : (\mathcal{C}, \mathcal{O}) \longrightarrow (\mathcal{D}, \mathcal{O}')$ be a morphism of ringed sites where f is given by the continuous functor $u : \mathcal{D} \to \mathcal{C}$. Let V be an object of $\mathcal{D}$ and set $U = u(V)$. Then there is a canonical map of sheaves of rings $(f')^\sharp$ such that the diagram of Sites, Lemma 7.27.1 is turned into a commutative diagram of ringed topoi*

$$\begin{array}{ccc} (Sh(\mathcal{C}/U), \mathcal{O}_U) & \xrightarrow[(j_U, j_U^\sharp)]{} & (Sh(\mathcal{C}), \mathcal{O}) \\ {\scriptstyle (f', (f')^\sharp)}\downarrow & & \downarrow{\scriptstyle (f, f^\sharp)} \\ (Sh(\mathcal{D}/V), \mathcal{O}'_V) & \xrightarrow{(j_V, j_V^\sharp)} & (Sh(\mathcal{D}), \mathcal{O}'). \end{array}$$

Moreover, in this situation we have $f'_* j_U^{-1} = j_V^{-1} f_*$ and $f'_* j_U^* = j_V^* f_*$.

Proof. Just take $(f')^\sharp$ to be

$$(f')^{-1}\mathcal{O}'_V = (f')^{-1} j_V^{-1}\mathcal{O}' = j_U^{-1} f^{-1}\mathcal{O}' \xrightarrow{j_U^{-1} f^\sharp} j_U^{-1}\mathcal{O} = \mathcal{O}_U$$

and everything else follows from Sites, Lemma 7.27.1. (Note that $j^{-1} = j^*$ on sheaves of modules if j is a localization morphism, hence the first equality of functors implies the second.) □

Lemma 18.20.2. *Let $(f, f^\sharp) : (\mathcal{C}, \mathcal{O}) \longrightarrow (\mathcal{D}, \mathcal{O}')$ be a morphism of ringed sites where f is given by the continuous functor $u : \mathcal{D} \to \mathcal{C}$. Let $V \in \mathrm{Ob}(\mathcal{D})$, $U \in \mathrm{Ob}(\mathcal{C})$*

and $c : U \to u(V)$ a morphism of $\mathcal{C}$. There exists a commutative diagram of ringed topoi

$$\begin{array}{ccc} (Sh(\mathcal{C}/U), \mathcal{O}_U) & \xrightarrow[(j_U, j_U^\sharp)]{} & (Sh(\mathcal{C}), \mathcal{O}) \\ \downarrow{\scriptstyle (f_c, f_c^\sharp)} & & \downarrow{\scriptstyle (f, f^\sharp)} \\ (Sh(\mathcal{D}/V), \mathcal{O}'_V) & \xrightarrow{(j_V, j_V^\sharp)} & (Sh(\mathcal{D}), \mathcal{O}'). \end{array}$$

The morphism $(f_c, f_c^\sharp)$ is equal to the composition of the morphism

$$(f', (f')^\sharp) : (Sh(\mathcal{C}/u(V)), \mathcal{O}_{u(V)}) \longrightarrow (Sh(\mathcal{D}/V), \mathcal{O}'_V)$$

of Lemma 18.20.1 and the morphism

$$(j, j^\sharp) : (Sh(\mathcal{C}/U), \mathcal{O}_U) \to (Sh(\mathcal{C}/u(V)), \mathcal{O}_{u(V)})$$

of Lemma 18.19.4. Given any morphisms $b : V' \to V$, $a : U' \to U$ and $c' : U' \to u(V')$ such that

$$\begin{array}{ccc} U' & \xrightarrow[c']{} & u(V') \\ {\scriptstyle a}\downarrow & & \downarrow{\scriptstyle u(b)} \\ U & \xrightarrow{c} & u(V) \end{array}$$

commutes, then the following diagram of ringed topoi

$$\begin{array}{ccc} (Sh(\mathcal{C}/U'), \mathcal{O}_{U'}) & \xrightarrow[(j_{U'/U}, j_{U'/U}^\sharp)]{} & (Sh(\mathcal{C}/U), \mathcal{O}_U) \\ \downarrow{\scriptstyle (f_{c'}, f_{c'}^\sharp)} & & \downarrow{\scriptstyle (f_c, f_c^\sharp)} \\ (Sh(\mathcal{D}/V'), \mathcal{O}'_{V'}) & \xrightarrow{(j_{V'/V}, j_{V'/V}^\sharp)} & (Sh(\mathcal{D}/V), \mathcal{O}'_{V'}) \end{array}$$

commutes.

Proof. On the level of morphisms of topoi this is Sites, Lemma 7.27.3. To check that the diagrams commute as morphisms of ringed topoi use Lemmas 18.19.4 and 18.20.1 exactly as in the proof of Sites, Lemma 7.27.3. □

18.21. Localization of ringed topoi

This section is the analogue of Sites, Section 7.29 in the setting of ringed topoi.

Lemma 18.21.1. *Let $(Sh(\mathcal{C}), \mathcal{O})$ be a ringed topos. Let $\mathcal{F} \in Sh(\mathcal{C})$ be a sheaf. For a sheaf $\mathcal{H}$ on $\mathcal{C}$ denote $\mathcal{H}_\mathcal{F}$ the sheaf $\mathcal{H} \times \mathcal{F}$ seen as an object of the category $Sh(\mathcal{C})/\mathcal{F}$. The pair $(Sh(\mathcal{C})/\mathcal{F}, \mathcal{O}_\mathcal{F})$ is a ringed topos and there is a canonical morphism of ringed topoi*

$$(j_\mathcal{F}, j_\mathcal{F}^\sharp) : (Sh(\mathcal{C})/\mathcal{F}, \mathcal{O}_\mathcal{F}) \longrightarrow (Sh(\mathcal{C}), \mathcal{O})$$

which is a localization as in Section 18.19 such that

1. *the functor $j_\mathcal{F}^{-1}$ is the functor $\mathcal{H} \mapsto \mathcal{H}_\mathcal{F}$,*
2. *the functor $j_\mathcal{F}^*$ is the functor $\mathcal{H} \mapsto \mathcal{H}_\mathcal{F}$,*
3. *the functor $j_{\mathcal{F}!}$ on sheaves of sets is the forgetful functor $\mathcal{G}/\mathcal{F} \mapsto \mathcal{G}$,*
4. *the functor $j_{\mathcal{F}!}$ on sheaves of modules associates to the $\mathcal{O}_\mathcal{F}$-module $\varphi : \mathcal{G} \to \mathcal{F}$ the $\mathcal{O}$-module which is the sheafification of the presheaf*

$$V \longmapsto \bigoplus\nolimits_{s \in \mathcal{F}(V)} \{\sigma \in \mathcal{G}(V) \mid \varphi(\sigma) = s\}$$

for $V \in \mathrm{Ob}(\mathcal{C})$.

Proof. By Sites, Lemma 7.29.1 we see that $Sh(\mathcal{C})/\mathcal{F}$ is a topos and that (1) and (3) are true. In particular this shows that $j_{\mathcal{F}}^{-1}\mathcal{O} = \mathcal{O}_{\mathcal{F}}$ and shows that $\mathcal{O}_{\mathcal{F}}$ is a sheaf of rings. Thus we may choose the map $j_{\mathcal{F}}^\sharp$ to be the identity, in particular we see that (2) is true. Moreover, the proof of Sites, Lemma 7.29.1 shows that we may assume $\mathcal{C}$ is a site with all finite limits and a subcanonical topology and that $\mathcal{F} = h_U$ for some object U of $\mathcal{C}$. Then (4) follows from the description of $j_{\mathcal{F}!}$ in Lemma 18.19.2. Alternatively one could show directly that the functor described in (4) is a left adjoint to $j_{\mathcal{F}}^*$. □

Definition 18.21.2. Let $(Sh(\mathcal{C}), \mathcal{O})$ be a ringed topos. Let $\mathcal{F} \in Sh(\mathcal{C})$.

(1) The ringed topos $(Sh(\mathcal{C})/\mathcal{F}, \mathcal{O}_{\mathcal{F}})$ is called the *localization of the ringed topos* $(Sh(\mathcal{C}), \mathcal{O})$ *at* $\mathcal{F}$.
(2) The morphism of ringed topoi $(j_{\mathcal{F}}, j_{\mathcal{F}}^\sharp) : (Sh(\mathcal{C})/\mathcal{F}, \mathcal{O}_{\mathcal{F}}) \to (Sh(\mathcal{C}), \mathcal{O})$ of Lemma 18.21.1 is called the *localization morphism*.

We continue the tradition, established in the chapter on sites, that we check the localization constructions on topoi are compatible with the constructions of localization on sites, whenever this makes sense.

Lemma 18.21.3. *With $(Sh(\mathcal{C}), \mathcal{O})$ and $\mathcal{F} \in Sh(\mathcal{C})$ as in Lemma 18.21.1. If $\mathcal{F} = h_U^\#$ for some object U of $\mathcal{C}$ then via the identification $Sh(\mathcal{C}/U) = Sh(\mathcal{C})/h_U^\#$ of Sites, Lemma 7.24.4 we have*

(1) *canonically $\mathcal{O}_U = \mathcal{O}_{\mathcal{F}}$, and*
(2) *with these identifications we have $(j_{\mathcal{F}}, j_{\mathcal{F}}^\sharp) = (j_U, j_U^\sharp)$.*

Proof. The assertion for underlying topoi is Sites, Lemma 7.29.5. Note that $\mathcal{O}_U$ is the restriction of $\mathcal{O}$ which by Sites, Lemma 7.24.6 corresponds to $\mathcal{O} \times h_U^\#$ under the equivalence of Sites, Lemma 7.24.4. By definition of $\mathcal{O}_{\mathcal{F}}$ we get (1). What's left is to prove that $j_{\mathcal{F}}^\sharp = j_U^\sharp$ under this identification. We omit the verification. □

Localization is functorial in the following two ways: We can "relocalize" a localization (see Lemma 18.21.4) or we can given a morphism of ringed topoi, localize upstairs at the inverse image of a sheaf downstairs and get a commutative diagram of locally ringed spaces (see Lemma 18.22.1).

Lemma 18.21.4. *Let $(Sh(\mathcal{C}), \mathcal{O})$ be a ringed topos. If $s : \mathcal{G} \to \mathcal{F}$ is a morphism of sheaves on $\mathcal{C}$ then there exists a natural commutative diagram of morphisms of ringed topoi*

$$\begin{array}{ccc} (Sh(\mathcal{C})/\mathcal{G}, \mathcal{O}_{\mathcal{G}}) & \xrightarrow{(j,j^\sharp)} & (Sh(\mathcal{C})/\mathcal{F}, \mathcal{O}_{\mathcal{F}}) \\ & {\scriptstyle (j_{\mathcal{G}}, j_{\mathcal{G}}^\sharp)} \searrow \quad \swarrow {\scriptstyle (j_{\mathcal{F}}, j_{\mathcal{F}}^\sharp)} & \\ & (Sh(\mathcal{C}), \mathcal{O}) & \end{array}$$

where $(j, j^\sharp)$ is the localization morphism of the ringed topos $(Sh(\mathcal{C})/\mathcal{F}, \mathcal{O}_{\mathcal{F}})$ at the object $\mathcal{G}/\mathcal{F}$.

Proof. All assertions follow from Sites, Lemma 7.29.6 except the assertion that $j_{\mathcal{G}}^\sharp = j^\sharp \circ j^{-1}(j_{\mathcal{F}}^\sharp)$. We omit the verification. □

Lemma 18.21.5. *With $(Sh(\mathcal{C}), \mathcal{O})$, $s : \mathcal{G} \to \mathcal{F}$ as in Lemma 18.21.4. If there exist a morphism $f : V \to U$ of $\mathcal{C}$ such that $\mathcal{G} = h_V^\#$ and $\mathcal{F} = h_U^\#$ and s is*

induced by f, then the diagrams of Lemma 18.19.4 and Lemma 18.21.4 agree via the identifications $(j_{\mathcal{F}}, j_{\mathcal{F}}^\sharp) = (j_U, j_U^\sharp)$ and $(j_{\mathcal{G}}, j_{\mathcal{G}}^\sharp) = (j_V, j_V^\sharp)$ of Lemma 18.21.3.

Proof. All assertions follow from Sites, Lemma 7.29.7 except for the assertion that the two maps $j^\sharp$ agree. This holds since in both cases the map $j^\sharp$ is simply the identity. Some details omitted. □

18.22. Localization of morphisms of ringed topoi

This section is the analogue of Sites, Section 7.30.

Lemma 18.22.1. *Let*

$$f : (Sh(\mathcal{C}), \mathcal{O}) \longrightarrow (Sh(\mathcal{D}), \mathcal{O}')$$

be a morphism of ringed topoi. Let $\mathcal{G}$ be a sheaf on $\mathcal{D}$. Set $\mathcal{F} = f^{-1}\mathcal{G}$. Then there exists a commutative diagram of ringed topoi

$$\begin{array}{ccc} (Sh(\mathcal{C})/\mathcal{F}, \mathcal{O}_{\mathcal{F}}) & \xrightarrow{(j_{\mathcal{F}}, j_{\mathcal{F}}^\sharp)} & (Sh(\mathcal{C}), \mathcal{O}) \\ \downarrow{\scriptstyle (f', (f')^\sharp)} & & \downarrow{\scriptstyle (f, f^\sharp)} \\ (Sh(\mathcal{D})/\mathcal{G}, \mathcal{O}'_{\mathcal{G}}) & \xrightarrow{(j_{\mathcal{G}}, j_{\mathcal{G}}^\sharp)} & (Sh(\mathcal{D}), \mathcal{O}') \end{array}$$

We have $f'_* j_{\mathcal{F}}^{-1} = j_{\mathcal{G}}^{-1} f_*$ and $f'_* j_{\mathcal{F}}^* = j_{\mathcal{G}}^* f_*$. Moreover, the morphism f' is characterized by the rule

$$(f')^{-1}(\mathcal{H} \xrightarrow{\varphi} \mathcal{G}) = (f^{-1}\mathcal{H} \xrightarrow{f^{-1}\varphi} \mathcal{F}).$$

Proof. By Sites, Lemma 7.30.1 we have the diagram of underlying topoi, the equality $f'_* j_{\mathcal{F}}^{-1} = j_{\mathcal{G}}^{-1} f_*$, and the description of $(f')^{-1}$. To define $(f')^\sharp$ we use the map

$$(f')^\sharp : \mathcal{O}'_{\mathcal{G}} = j_{\mathcal{G}}^{-1}\mathcal{O}' \xrightarrow{j_{\mathcal{G}}^{-1} f^\sharp} j_{\mathcal{G}}^{-1} f_* \mathcal{O} = f'_* j_{\mathcal{F}}^{-1} \mathcal{O} = f'_* \mathcal{O}_{\mathcal{F}}$$

or equivalently the map

$$(f')^\sharp : (f')^{-1}\mathcal{O}'_{\mathcal{G}} = (f')^{-1} j_{\mathcal{G}}^{-1}\mathcal{O}' = j_{\mathcal{F}}^{-1} f^{-1}\mathcal{O}' \xrightarrow{j_{\mathcal{F}}^{-1} f^\sharp} j_{\mathcal{F}}^{-1}\mathcal{O} = \mathcal{O}_{\mathcal{F}}.$$

We omit the verification that these two maps are indeed adjoint to each other. The second construction of $(f')^\sharp$ shows that the diagram commutes in the 2-category of ringed topoi (as the maps $j_{\mathcal{F}}^\sharp$ and $j_{\mathcal{G}}^\sharp$ are identities). Finally, the equality $f'_* j_{\mathcal{F}}^* = j_{\mathcal{G}}^* f_*$ follows from the equality $f'_* j_{\mathcal{F}}^{-1} = j_{\mathcal{G}}^{-1} f_*$ and the fact that pullbacks of sheaves of modules and sheaves of sets agree, see Lemma 18.21.1. □

Lemma 18.22.2. *Let*

$$f : (Sh(\mathcal{C}), \mathcal{O}) \longrightarrow (Sh(\mathcal{D}), \mathcal{O}')$$

be a morphism of ringed topoi. Let $\mathcal{G}$ be a sheaf on $\mathcal{D}$. Set $\mathcal{F} = f^{-1}\mathcal{G}$. If f is given by a continuous functor $u : \mathcal{D} \to \mathcal{C}$ and $\mathcal{G} = h_V^\#$, then the commutative diagrams of Lemma 18.20.1 and Lemma 18.22.1 agree via the identifications of Lemma 18.21.3.

Proof. At the level of morphisms of topoi this is Sites, Lemma 7.30.2. This works also on the level of morphisms of ringed topoi since the formulas defining $(f')^\sharp$ in the proofs of Lemma 18.20.1 and Lemma 18.22.1 agree. □

Lemma 18.22.3. *Let $(f, f^\sharp) : (Sh(\mathcal{C}), \mathcal{O}) \to (Sh(\mathcal{D}), \mathcal{O}')$ be a morphism of ringed topoi. Let $\mathcal{G}$ be a sheaf on $\mathcal{D}$, let $\mathcal{F}$ be a sheaf on $\mathcal{C}$, and let $s : \mathcal{F} \to f^{-1}\mathcal{G}$ a morphism of sheaves. There exists a commutative diagram of ringed topoi*

$$\begin{array}{ccc} (Sh(\mathcal{C})/\mathcal{F}, \mathcal{O}_\mathcal{F}) & \xrightarrow{(j_\mathcal{F}, j_\mathcal{F}^\sharp)} & (Sh(\mathcal{C}), \mathcal{O}) \\ \downarrow{\scriptstyle (f_c, f_c^\sharp)} & & \downarrow{\scriptstyle (f, f^\sharp)} \\ (Sh(\mathcal{D})/\mathcal{G}, \mathcal{O}'_\mathcal{G}) & \xrightarrow{(j_\mathcal{G}, j_\mathcal{G}^\sharp)} & (Sh(\mathcal{D}), \mathcal{O}'). \end{array}$$

The morphism $(f_s, f_s^\sharp)$ is equal to the composition of the morphism

$$(f', (f')^\sharp) : (Sh(\mathcal{C})/f^{-1}\mathcal{G}, \mathcal{O}_{f^{-1}\mathcal{G}}) \longrightarrow (Sh(\mathcal{D})/\mathcal{G}, \mathcal{O}'_\mathcal{G})$$

of Lemma 18.22.1 and the morphism

$$(j, j^\sharp) : (Sh(\mathcal{C})/\mathcal{F}, \mathcal{O}_\mathcal{F}) \to (Sh(\mathcal{C})/f^{-1}\mathcal{G}, \mathcal{O}_{f^{-1}\mathcal{G}})$$

of Lemma 18.21.4. Given any morphisms $b : \mathcal{G}' \to \mathcal{G}$, $a : \mathcal{F}' \to \mathcal{F}$, and $s' : \mathcal{F}' \to f^{-1}\mathcal{G}'$ such that

$$\begin{array}{ccc} \mathcal{F}' & \xrightarrow{s'} & f^{-1}\mathcal{G}' \\ \downarrow{\scriptstyle a} & & \downarrow{\scriptstyle f^{-1}b} \\ \mathcal{F} & \xrightarrow{s} & f^{-1}\mathcal{G} \end{array}$$

commutes, then the following diagram of ringed topoi

$$\begin{array}{ccc} (Sh(\mathcal{C})/\mathcal{F}', \mathcal{O}_{\mathcal{F}'}) & \xrightarrow{(j_{\mathcal{F}'/\mathcal{F}}, j_{\mathcal{F}'/\mathcal{F}}^\sharp)} & (Sh(\mathcal{C})/\mathcal{F}, \mathcal{O}_\mathcal{F}) \\ \downarrow{\scriptstyle (f_{s'}, f_{s'}^\sharp)} & & \downarrow{\scriptstyle (f_s, f_s^\sharp)} \\ (Sh(\mathcal{D})/\mathcal{G}', \mathcal{O}'_{\mathcal{G}'}) & \xrightarrow{(j_{\mathcal{G}'/\mathcal{G}}, j_{\mathcal{G}'/\mathcal{G}}^\sharp)} & (Sh(\mathcal{D})/\mathcal{G}, \mathcal{O}'_{\mathcal{G}'}) \end{array}$$

commutes.

Proof. On the level of morphisms of topoi this is Sites, Lemma 7.30.3. To check that the diagrams commute as morphisms of ringed topoi use the commutative diagrams of Lemmas 18.21.4 and 18.22.1. □

Lemma 18.22.4. *Let $(f, f^\sharp) : (Sh(\mathcal{C}), \mathcal{O}) \to (Sh(\mathcal{D}), \mathcal{O}')$, $s : \mathcal{F} \to f^{-1}\mathcal{G}$ be as in Lemma 18.22.3. If f is given by a continuous functor $u : \mathcal{D} \to \mathcal{C}$ and $\mathcal{G} = h_V^\#$, $\mathcal{F} = h_U^\#$ and s comes from a morphism $c : U \to u(V)$, then the commutative diagrams of Lemma 18.20.2 and Lemma 18.22.3 agree via the identifications of Lemma 18.21.3.*

Proof. This is formal using Lemmas 18.21.5 and 18.22.2. □

18.23. Local types of modules

According to our general strategy explained in Section 18.18 we first define the local types for sheaves of modules on a ringed site, and then we immediately show that these types are intrinsic, hence make sense for sheaves of modules on ringed topoi.

Definition 18.23.1. Let $(\mathcal{C}, \mathcal{O})$ be a ringed site. Let $\mathcal{F}$ be a sheaf of $\mathcal{O}$-modules. We will freely use the notions defined in Definition 18.17.1.

(1) We say $\mathcal{F}$ is *locally free* if for every object U of $\mathcal{C}$ there exists a covering $\{U_i \to U\}_{i\in I}$ of $\mathcal{C}$ such that each restriction $\mathcal{F}|_{\mathcal{C}/U_i}$ is a free $\mathcal{O}_{U_i}$-module.
(2) We say $\mathcal{F}$ is *finite locally free* if for every object U of $\mathcal{C}$ there exists a covering $\{U_i \to U\}_{i\in I}$ of $\mathcal{C}$ such that each restriction $\mathcal{F}|_{\mathcal{C}/U_i}$ is a finite free $\mathcal{O}_{U_i}$-module.
(3) We say $\mathcal{F}$ is *locally generated by sections* if for every object U of $\mathcal{C}$ there exists a covering $\{U_i \to U\}_{i\in I}$ of $\mathcal{C}$ such that each restriction $\mathcal{F}|_{\mathcal{C}/U_i}$ is an $\mathcal{O}_{U_i}$-module generated by global sections.
(4) Given $r \geq 0$ we sat $\mathcal{F}$ is *locally generated by r sections* if for every object U of $\mathcal{C}$ there exists a covering $\{U_i \to U\}_{i\in I}$ of $\mathcal{C}$ such that each restriction $\mathcal{F}|_{\mathcal{C}/U_i}$ is an $\mathcal{O}_{U_i}$-module generated by r global sections.
(5) We say $\mathcal{F}$ is *of finite type* if for every object U of $\mathcal{C}$ there exists a covering $\{U_i \to U\}_{i\in I}$ of $\mathcal{C}$ such that each restriction $\mathcal{F}|_{\mathcal{C}/U_i}$ is an $\mathcal{O}_{U_i}$-module generated by finitely many global sections.
(6) We say $\mathcal{F}$ is *quasi-coherent* if for every object U of $\mathcal{C}$ there exists a covering $\{U_i \to U\}_{i\in I}$ of $\mathcal{C}$ such that each restriction $\mathcal{F}|_{\mathcal{C}/U_i}$ is an $\mathcal{O}_{U_i}$-module which has a global presentation.
(7) We say $\mathcal{F}$ is *of finite presentation* if for every object U of $\mathcal{C}$ there exists a covering $\{U_i \to U\}_{i\in I}$ of $\mathcal{C}$ such that each restriction $\mathcal{F}|_{\mathcal{C}/U_i}$ is an $\mathcal{O}_{U_i}$-module which has a finite global presentation.
(8) We say $\mathcal{F}$ is *coherent* if and only if $\mathcal{F}$ is of finite type, and for every object U of $\mathcal{C}$ and any $s_1, \ldots, s_n \in \mathcal{F}(U)$ the kernel of the map $\bigoplus_{i=1,\ldots,n} \mathcal{O}_U \to \mathcal{F}|_U$ is of finite type on $(\mathcal{C}/U, \mathcal{O}_U)$.

Lemma 18.23.2. *Any of the properties (1) – (8) of Definition 18.23.1 is intrinsic (see discussion in Section 18.18).*

Proof. Let $\mathcal{C}$, $\mathcal{D}$ be sites. Let $u : \mathcal{C} \to \mathcal{D}$ be a special cocontinuous functor. Let $\mathcal{O}$ be a sheaf of rings on $\mathcal{C}$. Let $\mathcal{F}$ be a sheaf of $\mathcal{O}$-modules on $\mathcal{C}$. Let $g : Sh(\mathcal{C}) \to Sh(\mathcal{D})$ be the equivalence of topoi associated to u. Set $\mathcal{O}' = g_*\mathcal{O}$, and let $g^\sharp : \mathcal{O}' \to g_*\mathcal{O}$ be the identity. Finally, set $\mathcal{F}' = g_*\mathcal{F}$. Let $\mathcal{P}_l$ be one of the properties (1) – (7) listed in Definition 18.23.1. (We will discuss the coherent case at the end of the proof.) Let $\mathcal{P}_g$ denote the corresponding property listed in Definition 18.17.1. We have already seen that $\mathcal{P}_g$ is intrinsic. We have to show that $\mathcal{P}_l(\mathcal{C}, \mathcal{O}, \mathcal{F})$ holds if and only if $\mathcal{P}_l(\mathcal{D}, \mathcal{O}', \mathcal{F}')$ holds.

Assume that $\mathcal{F}$ has $\mathcal{P}_l$. Let V be an object of $\mathcal{D}$. One of the properties of a special cocontinuous functor is that there exists a covering $\{u(U_i) \to V\}_{i\in I}$ in the site $\mathcal{D}$. By assumption, for each i there exists a covering $\{U_{ij} \to U_i\}_{j\in J_i}$ in $\mathcal{C}$ such that each restriction $\mathcal{F}|_{U_{ij}}$ is $\mathcal{P}_g$. By Sites, Lemma 7.28.3 we have commutative diagrams of ringed topoi

$$\begin{array}{ccc} (Sh(\mathcal{C}/U_{ij}), \mathcal{O}_{U_{ij}}) & \longrightarrow & (Sh(\mathcal{C}), \mathcal{O}) \\ \downarrow & & \downarrow \\ (Sh(\mathcal{D}/u(U_{ij})), \mathcal{O}'_{u(U_{ij})}) & \longrightarrow & (Sh(\mathcal{D}), \mathcal{O}') \end{array}$$

where the vertical arrows are equivalences. Hence we conclude that $\mathcal{F}'|_{u(U_{ij})}$ has property $\mathcal{P}_g$ also. And moreover, $\{u(U_{ij}) \to V\}_{i\in I, j\in J_i}$ is a covering of the site $\mathcal{D}$. Hence $\mathcal{F}'$ has property $\mathcal{P}_l$.

Assume that $\mathcal{F}'$ has $\mathcal{P}_l$. Let U be an object of $\mathcal{C}$. By assumption, there exists a covering $\{V_i \to u(U)\}_{i\in I}$ such that $\mathcal{F}'|_{V_i}$ has property $\mathcal{P}_g$. Because u is cocontinuous we can refine this covering by a family $\{u(U_j) \to u(U)\}_{j\in J}$ where $\{U_j \to U\}_{j\in J}$ is a covering in $\mathcal{C}$. Say the refinement is given by $\alpha : J \to I$ and $u(U_j) \to V_{\alpha(j)}$. Restricting is transitive, i.e., $(\mathcal{F}'|_{V_{\alpha(j)}})|_{u(U_j)} = \mathcal{F}'|_{u(U_j)}$. Hence by Lemma 18.17.2 we see that $\mathcal{F}'|_{u(U_j)}$ has property $\mathcal{P}_g$. Hence the diagram

$$\begin{array}{ccc} (Sh(\mathcal{C}/U_j), \mathcal{O}_{U_j}) & \longrightarrow & (Sh(\mathcal{C}), \mathcal{O}) \\ \downarrow & & \downarrow \\ (Sh(\mathcal{D}/u(U_j)), \mathcal{O}'_{u(U_j)}) & \longrightarrow & (Sh(\mathcal{D}), \mathcal{O}') \end{array}$$

where the vertical arrows are equivalences shows that $\mathcal{F}|_{U_j}$ has property $\mathcal{P}_g$ also. Thus $\mathcal{F}$ has property $\mathcal{P}_l$ as desired.

Finally, we prove the lemma in case $\mathcal{P}_l = coherent$[2]. Assume $\mathcal{F}$ is coherent. This implies that $\mathcal{F}$ is of finite type and hence $\mathcal{F}'$ is of finite type also by the first part of the proof. Let V be an object of $\mathcal{D}$ and let $s_1, \ldots, s_n \in \mathcal{F}'(V)$. We have to show that the kernel $\mathcal{K}'$ of $\bigoplus_{j=1,\ldots,n} \mathcal{O}_V \to \mathcal{F}'|_V$ is of finite type on $\mathcal{D}/V$. This means we have to show that for any V'/V there exists a covering $\{V'_i \to V'\}$ such that $\mathcal{F}'|_{V'_i}$ is generated by finitely many sections. Replacing V by V' (and restricting the sections s_j to V') we reduce to the case where $V' = V$. Since u is a special cocontinuous functor, there exists a covering $\{u(U_i) \to V\}_{i\in I}$ in the site $\mathcal{D}$. Using the isomorphism of topoi $Sh(\mathcal{C}/U_i) = Sh(\mathcal{D}/u(U_i))$ we see that $\mathcal{K}'|_{u(U_i)}$ corresponds to the kernel $\mathcal{K}_i$ of a map $\bigoplus_{j=1,\ldots,n} \mathcal{O}_{U_i} \to \mathcal{F}|_{U_i}$. Since $\mathcal{F}$ is coherent we see that $\mathcal{K}_i$ is of finite type. Hence we conclude (by the first part of the proof again) that $\mathcal{K}|_{u(U_i)}$ is of finite type. Thus there exist coverings $\{V_{il} \to u(U_i)\}$ such that $\mathcal{K}|_{V_{il}}$ is generated by finitely many global sections. Since $\{V_{il} \to V\}$ is a covering of $\mathcal{D}$ we conclude that $\mathcal{K}$ is of finite type as desired.

Assume $\mathcal{F}'$ is coherent. This implies that $\mathcal{F}'$ is of finite type and hence $\mathcal{F}$ is of finite type also by the first part of the proof. Let U be an object of $\mathcal{C}$, and let $s_1, \ldots, s_n \in \mathcal{F}(U)$. We have to show that the kernel $\mathcal{K}$ of $\bigoplus_{j=1,\ldots,n} \mathcal{O}_U \to \mathcal{F}|_U$ is of finite type on $\mathcal{C}/U$. Using the isomorphism of topoi $Sh(\mathcal{C}/U) = Sh(\mathcal{D}/u(U))$ we see that $\mathcal{K}|_U$ corresponds to the kernel $\mathcal{K}'$ of a map $\bigoplus_{j=1,\ldots,n} \mathcal{O}_{u(U)} \to \mathcal{F}'|_{u(U)}$. As $\mathcal{F}'$ is coherent, we see that $\mathcal{K}'$ is of finite type. Hence, by the first part of the proof again, we conclude that $\mathcal{K}$ is of finite type. □

Hence from now on we may refer to the properties of $\mathcal{O}$-modules defined in Definition 18.23.1 without specifying a site.

Lemma 18.23.3. *Let $(Sh(\mathcal{C}), \mathcal{O})$ be a ringed topos. Let $\mathcal{F}$ be an $\mathcal{O}$-module. Assume that the site $\mathcal{C}$ has a final object X. Then*

(1) *The following are equivalent*
   (a) *$\mathcal{F}$ is locally free,*
   (b) *there exists a covering $\{X_i \to X\}$ in $\mathcal{C}$ such that each restriction $\mathcal{F}|_{\mathcal{C}/X_i}$ is a locally free $\mathcal{O}_{X_i}$-module, and*

[2]The mechanics of this are a bit awkward, and we suggest the reader skip this part of the proof.

(c) *there exists a covering* $\{X_i \to X\}$ *in* $\mathcal{C}$ *such that each restriction* $\mathcal{F}|_{\mathcal{C}/X_i}$ *is a free* $\mathcal{O}_{X_i}$*-module.*

(2) *The following are equivalent*
   (a) $\mathcal{F}$ *is finite locally free,*
   (b) *there exists a covering* $\{X_i \to X\}$ *in* $\mathcal{C}$ *such that each restriction* $\mathcal{F}|_{\mathcal{C}/X_i}$ *is a finite locally free* $\mathcal{O}_{X_i}$*-module, and*
   (c) *there exists a covering* $\{X_i \to X\}$ *in* $\mathcal{C}$ *such that each restriction* $\mathcal{F}|_{\mathcal{C}/X_i}$ *is a finite free* $\mathcal{O}_{X_i}$*-module.*

(3) *The following are equivalent*
   (a) $\mathcal{F}$ *is locally generated by sections,*
   (b) *there exists a covering* $\{X_i \to X\}$ *in* $\mathcal{C}$ *such that each restriction* $\mathcal{F}|_{\mathcal{C}/X_i}$ *is an* $\mathcal{O}_{X_i}$*-module locally generated by sections, and*
   (c) *there exists a covering* $\{X_i \to X\}$ *in* $\mathcal{C}$ *such that each restriction* $\mathcal{F}|_{\mathcal{C}/X_i}$ *is an* $\mathcal{O}_{X_i}$*-module globally generated by sections.*

(4) *Given* $r \geq 0$*, the following are equivalent*
   (a) $\mathcal{F}$ *is locally generated by* r *sections,*
   (b) *there exists a covering* $\{X_i \to X\}$ *in* $\mathcal{C}$ *such that each restriction* $\mathcal{F}|_{\mathcal{C}/X_i}$ *is an* $\mathcal{O}_{X_i}$*-module locally generated by* r *sections, and*
   (c) *there exists a covering* $\{X_i \to X\}$ *in* $\mathcal{C}$ *such that each restriction* $\mathcal{F}|_{\mathcal{C}/X_i}$ *is an* $\mathcal{O}_{X_i}$*-module globally generated by* r *sections.*

(5) *The following are equivalent*
   (a) $\mathcal{F}$ *is of finite type,*
   (b) *there exists a covering* $\{X_i \to X\}$ *in* $\mathcal{C}$ *such that each restriction* $\mathcal{F}|_{\mathcal{C}/X_i}$ *is an* $\mathcal{O}_{X_i}$*-module of finite type, and*
   (c) *there exists a covering* $\{X_i \to X\}$ *in* $\mathcal{C}$ *such that each restriction* $\mathcal{F}|_{\mathcal{C}/X_i}$ *is an* $\mathcal{O}_{X_i}$*-module globally generated by finitely many sections.*

(6) *The following are equivalent*
   (a) $\mathcal{F}$ *is quasi-coherent,*
   (b) *there exists a covering* $\{X_i \to X\}$ *in* $\mathcal{C}$ *such that each restriction* $\mathcal{F}|_{\mathcal{C}/X_i}$ *is a quasi-coherent* $\mathcal{O}_{X_i}$*-module, and*
   (c) *there exists a covering* $\{X_i \to X\}$ *in* $\mathcal{C}$ *such that each restriction* $\mathcal{F}|_{\mathcal{C}/X_i}$ *is an* $\mathcal{O}_{X_i}$*-module which has a global presentation.*

(7) *The following are equivalent*
   (a) $\mathcal{F}$ *is of finite presentation,*
   (b) *there exists a covering* $\{X_i \to X\}$ *in* $\mathcal{C}$ *such that each restriction* $\mathcal{F}|_{\mathcal{C}/X_i}$ *is an* $\mathcal{O}_{X_i}$*-module of finite presentation, and*
   (c) *there exists a covering* $\{X_i \to X\}$ *in* $\mathcal{C}$ *such that each restriction* $\mathcal{F}|_{\mathcal{C}/X_i}$ *is an* $\mathcal{O}_{X_i}$*-module has a finite global presentation.*

(8) *The following are equivalent*
   (a) $\mathcal{F}$ *is coherent, and*
   (b) *there exists a covering* $\{X_i \to X\}$ *in* $\mathcal{C}$ *such that each restriction* $\mathcal{F}|_{\mathcal{C}/X_i}$ *is a coherent* $\mathcal{O}_{X_i}$*-module.*

Proof. In each case we have (a) $\Rightarrow$ (*b*). In each of the cases (1) - (6) condition (b) implies condition (c) by axiom (2) of a site (see Sites, Definition 7.6.2) and the definition of the local types of modules. Suppose $\{X_i \to X\}$ is a covering. Then for every object U of $\mathcal{C}$ we get an induced covering $\{X_i \times_X U \to U\}$. Moreover, the global property for $\mathcal{F}|_{\mathcal{C}/X_i}$ in part (c) implies the corresponding global property for

$\mathcal{F}|_{\mathcal{C}/X_i \times_X U}$ by Lemma 18.17.2, hence the sheaf has property (a) by definition. We omit the proof of (b) $\Rightarrow$ (a) in case (7). $\square$

Lemma 18.23.4. *Let $(f, f^\sharp) : (Sh(\mathcal{C}), \mathcal{O}_\mathcal{C}) \to (Sh(\mathcal{D}), \mathcal{O}_\mathcal{D})$ be a morphism of ringed topoi. Let $\mathcal{F}$ be an $\mathcal{O}_\mathcal{D}$-module.*

1. *If $\mathcal{F}$ is locally free then $f^*\mathcal{F}$ is locally free.*
2. *If $\mathcal{F}$ is finite locally free then $f^*\mathcal{F}$ is finite locally free.*
3. *If $\mathcal{F}$ is locally generated by sections then $f^*\mathcal{F}$ is locally generated by sections.*
4. *If $\mathcal{F}$ is locally generated by r sections then $f^*\mathcal{F}$ is locally generated by r sections.*
5. *If $\mathcal{F}$ is of finite type then $f^*\mathcal{F}$ is of finite type.*
6. *If $\mathcal{F}$ is quasi-coherent then $f^*\mathcal{F}$ is quasi-coherent.*
7. *If $\mathcal{F}$ is of finite presentation then $f^*\mathcal{F}$ is of finite presentation.*

Proof. According to the discussion in Section 18.18 we need only check preservation under pullback for a morphism of ringed sites $(f, f^\sharp) : (\mathcal{C}, \mathcal{O}_\mathcal{C}) \to (\mathcal{D}, \mathcal{O}_\mathcal{D})$ such that f is given by a left exact, continuous functor $u : \mathcal{D} \to \mathcal{C}$ between sites which have all finite limits. Let $\mathcal{G}$ be a sheaf of $\mathcal{O}_\mathcal{D}$-modules which has one of the properties (1) – (6) of Definition 18.23.1. We know $\mathcal{D}$ has a final object Y and $X = u(Y)$ is a final object for $\mathcal{C}$. By assumption we have a covering $\{Y_i \to Y\}$ such that $\mathcal{G}|_{\mathcal{D}/Y_i}$ has the corresponding global property. Set $X_i = u(Y_i)$ so that $\{X_i \to X\}$ is a covering in $\mathcal{C}$. We get a commutative diagram of morphisms ringed sites

$$\begin{array}{ccc} (\mathcal{C}/X_i, \mathcal{O}_\mathcal{C}|_{X_i}) & \longrightarrow & (\mathcal{C}, \mathcal{O}_\mathcal{C}) \\ \downarrow & & \downarrow \\ (\mathcal{D}/Y_i, \mathcal{O}_\mathcal{D}|_{Y_i}) & \longrightarrow & (\mathcal{D}, \mathcal{O}_\mathcal{D}) \end{array}$$

by Sites, Lemma 7.27.2. Hence by Lemma 18.17.2 that $f^*\mathcal{G}|_{X_i}$ has the corresponding global property. Hence we conclude that $\mathcal{G}$ has the local property we started out with by Lemma 18.23.3. $\square$

18.24. Basic results on local types of modules

Basic lemmas related to the definitions made above.

Lemma 18.24.1. *Let $(\mathcal{C}, \mathcal{O})$ be a ringed site. Let $\theta : \mathcal{G} \to \mathcal{F}$ be a surjective $\mathcal{O}$-module map with $\mathcal{F}$ of finite presentation and $\mathcal{G}$ of finite type. Then $Ker(\theta)$ is of finite type.*

Proof. Omitted. Hint: See Modules, Lemma 17.11.3. $\square$

18.25. Closed immersions of ringed topoi

When do we declare a morphism of ringed topoi $i : (Sh(\mathcal{C}), \mathcal{O}) \to (Sh(\mathcal{D}), \mathcal{O}')$ to be a closed immersion? By analogy with the discussion in Modules, Section 17.13 it seems natural to assume at least:

1. The functor i is a closed immersion of topoi (Sites, Definition 7.42.7).
2. The associated map $\mathcal{O}' \to i_*\mathcal{O}$ is surjective.

These conditions already imply a number of pleasing results which we discuss in this section. However, it seems prudent to not actually define the notion of a closed immersion of ringed topoi as there are many different definitions we could use.

Lemma 18.25.1. *Let $i : (Sh(\mathcal{C}), \mathcal{O}) \to (Sh(\mathcal{D}), \mathcal{O}')$ be a morphism of ringed topoi. Assume i is a closed immersion of topi and $i^\sharp : \mathcal{O}' \to i_*\mathcal{O}$ is surjective. Denote $\mathcal{I} \subset \mathcal{O}'$ the kernel of $i^\sharp$. The functor*

$$i_* : Mod(\mathcal{O}) \longrightarrow Mod(\mathcal{O}')$$

is exact, fully faithful, with essential image those $\mathcal{O}'$-modules $\mathcal{G}$ such that $\mathcal{I}\mathcal{G} = 0$.

Proof. By Lemma 18.15.2 and Sites, Lemma 7.42.8 we see that i_* is exact. From the fact that i_* is fully faithful on sheaves of sets, and the fact that $i^\sharp$ is surjective it follows that i_* is fully faithful as a functor $Mod(\mathcal{O}) \to Mod(\mathcal{O}')$. Namely, suppose that $\alpha : i_*\mathcal{F}_1 \to i_*\mathcal{F}_2$ is an $\mathcal{O}'$-module map. By the fully faithfulness of i_* we obtain a map $\beta : \mathcal{F}_1 \to \mathcal{F}_2$ of sheaves of sets. To prove β is a map of modules we have to show that

$$\begin{array}{ccc} \mathcal{O} \times \mathcal{F}_1 & \longrightarrow & \mathcal{F}_1 \\ \downarrow & & \downarrow \\ \mathcal{O} \times \mathcal{F}_2 & \longrightarrow & \mathcal{F}_2 \end{array}$$

commutes. It suffices to prove commutativity after applying i_*. Consider

$$\begin{array}{ccccc} \mathcal{O}' \times i_*\mathcal{F}_1 & \longrightarrow & i_*\mathcal{O} \times i_*\mathcal{F}_1 & \longrightarrow & i_*\mathcal{F}_1 \\ \downarrow & & \downarrow & & \downarrow \\ \mathcal{O}' \times i_*\mathcal{F}_2 & \longrightarrow & i_*\mathcal{O} \times i_*\mathcal{F}_2 & \longrightarrow & i_*\mathcal{F}_2 \end{array}$$

We know the outer rectangle commutes. Since $i^\sharp$ is surjective we conclude.

To finish the proof we have to prove the statement on the essential image of i_*. It is clear that $i_*\mathcal{F}$ is annihilated by $\mathcal{I}$ for any $\mathcal{O}$-module $\mathcal{F}$. Conversely, let $\mathcal{G}$ be a $\mathcal{O}'$-module with $\mathcal{I}\mathcal{G} = 0$. By definition of a closed subtopos there exists a subsheaf $\mathcal{U}$ of the final object of $\mathcal{D}$ such that the essential image of i_* on sheaves of sets is the class of sheaves of sets $\mathcal{H}$ such that $\mathcal{H} \times \mathcal{U} \to \mathcal{U}$ is an isomorphism. In particular, $i_*\mathcal{O} \times \mathcal{U} = \mathcal{U}$. This implies that $\mathcal{I} \times \mathcal{U} = \mathcal{O} \times \mathcal{U}$. Hence our module $\mathcal{G}$ satisfies $\mathcal{G} \times \mathcal{U} = \{0\} \times \mathcal{U} = \mathcal{U}$ (because the zero module is isomorphic to the final object of sheaves of sets). Thus there exists a sheaf of sets $\mathcal{F}$ on $\mathcal{C}$ with $i_*\mathcal{F} = \mathcal{G}$. Since i_* is fully faithful on sheaves of sets, we see that in order to define the addition $\mathcal{F} \times \mathcal{F} \to \mathcal{F}$ and the multiplication $\mathcal{O} \times \mathcal{F} \to \mathcal{F}$ it suffices to use the addition

$$\mathcal{G} \times \mathcal{G} \longrightarrow \mathcal{G}$$

(given to us as $\mathcal{G}$ is a $\mathcal{O}'$-module) and the multiplication

$$i_*\mathcal{O} \times \mathcal{G} \to \mathcal{G}$$

which is given to us as we have the multiplication by $\mathcal{O}'$ which annihilates $\mathcal{I}$ by assumption and $i_*\mathcal{O} = \mathcal{O}'/\mathcal{I}$. By construction $\mathcal{G}$ is isomorphic to the pushforward of the $\mathcal{O}$-module $\mathcal{F}$ so constructed. □

18.26. Tensor product

In Sections 18.9 and 18.11 we defined the change of rings functor by a tensor product construction. To be sure this construction makes sense also to define the tensor product of presheaves of $\mathcal{O}$-modules. To be precise, suppose $\mathcal{C}$ is a category, $\mathcal{O}$ is a presheaf of rings, and $\mathcal{F}$, $\mathcal{G}$ are presheaves of $\mathcal{O}$-modules. In this case we define $\mathcal{F} \otimes_{p,\mathcal{O}} \mathcal{G}$ to be the presheaf

$$U \longmapsto (\mathcal{F} \otimes_{p,\mathcal{O}} \mathcal{G})(U) = \mathcal{F}(U) \otimes_{\mathcal{O}(U)} \mathcal{G}(U)$$

If $\mathcal{C}$ is a site, $\mathcal{O}$ is a sheaf of rings and $\mathcal{F}$, $\mathcal{G}$ are sheaves of $\mathcal{O}$-modules then we define

$$\mathcal{F} \otimes_{\mathcal{O}} \mathcal{G} = (\mathcal{F} \otimes_{p,\mathcal{O}} \mathcal{G})^{\#}$$

to be the sheaf of $\mathcal{O}$-modules associated to the presheaf $\mathcal{F} \otimes_{p,\mathcal{O}} \mathcal{G}$.

Here are some formulas which we will use below without further mention:

$$(\mathcal{F} \otimes_{p,\mathcal{O}} \mathcal{G}) \otimes_{p,\mathcal{O}} \mathcal{H} = \mathcal{F} \otimes_{p,\mathcal{O}} (\mathcal{G} \otimes_{p,\mathcal{O}} \mathcal{H}),$$

and similarly for sheaves. If $\mathcal{O}_1 \to \mathcal{O}_2$ is a map of presheaves of rings, then

$$(\mathcal{F} \otimes_{p,\mathcal{O}_1} \mathcal{G}) \otimes_{p,\mathcal{O}_1} \mathcal{O}_2 = (\mathcal{F} \otimes_{p,\mathcal{O}_1} \mathcal{O}_2) \otimes_{p,\mathcal{O}_2} (\mathcal{G} \otimes_{p,\mathcal{O}_1} \mathcal{O}_2),$$

and similarly for sheaves. These follow from their algebraic counterparts and sheafification.

Let $\mathcal{C}$ be a site, let $\mathcal{O}$ be a sheaf of rings and let $\mathcal{F}$, $\mathcal{G}$, $\mathcal{H}$ be sheaves of $\mathcal{O}$-modules. In this case we define

$$\text{Bilin}_{\mathcal{O}}(\mathcal{F} \times \mathcal{G}, \mathcal{H}) = \{\varphi \in \text{Mor}_{Sh(\mathcal{C})}(\mathcal{F} \times \mathcal{G}, \mathcal{H}) \mid \varphi \text{ is } \mathcal{O}\text{-bilinear}\}.$$

With this definition we have

$$\text{Hom}_{\mathcal{O}}(\mathcal{F} \otimes_{\mathcal{O}} \mathcal{G}, \mathcal{H}) = \text{Bilin}_{\mathcal{O}}(\mathcal{F} \times \mathcal{G}, \mathcal{H}).$$

In other words $\mathcal{F} \otimes_{\mathcal{O}} \mathcal{G}$ represents the functor which associates to $\mathcal{H}$ the set of bilinear maps $\mathcal{F} \times \mathcal{G} \to \mathcal{H}$. In particular, since the notion of a bilinear map makes sense for a pair of modules on a ringed topos, we see that the tensor product of sheaves of modules is intrinsic to the topos (compare the discussion in Section 18.18). In fact we have the following.

Lemma 18.26.1. *Let $f : (Sh(\mathcal{C}), \mathcal{O}_{\mathcal{C}}) \to (Sh(\mathcal{D}), \mathcal{O}_{\mathcal{D}})$ be a morphism of ringed topoi. Let $\mathcal{F}$, $\mathcal{G}$ be $\mathcal{O}_{\mathcal{D}}$-modules. Then $f^*(\mathcal{F} \otimes_{\mathcal{O}_{\mathcal{D}}} \mathcal{G}) = f^*\mathcal{F} \otimes_{\mathcal{O}_{\mathcal{C}}} f^*\mathcal{G}$ functorially in $\mathcal{F}$, $\mathcal{G}$.*

Proof. For a sheaf $\mathcal{H}$ of $\mathcal{O}_{\mathcal{C}}$ modules we have

$$\begin{aligned}
\text{Hom}_{\mathcal{O}_{\mathcal{C}}}(f^*(\mathcal{F} \otimes_{\mathcal{O}} \mathcal{G}), \mathcal{H}) &= \text{Hom}_{\mathcal{O}_{\mathcal{D}}}(\mathcal{F} \otimes_{\mathcal{O}} \mathcal{G}, f_*\mathcal{H}) \\
&= \text{Bilin}_{\mathcal{O}_{\mathcal{D}}}(\mathcal{F} \times \mathcal{G}, f_*\mathcal{H}) \\
&= \text{Bilin}_{f^{-1}\mathcal{O}_{\mathcal{D}}}(f^{-1}\mathcal{F} \times f^{-1}\mathcal{G}, \mathcal{H}) \\
&= \text{Hom}_{f^{-1}\mathcal{O}_{\mathcal{D}}}(f^{-1}\mathcal{F} \otimes_{f^{-1}\mathcal{O}_{\mathcal{D}}} f^{-1}\mathcal{G}, \mathcal{H}) \\
&= \text{Hom}_{\mathcal{O}_{\mathcal{C}}}(f^*\mathcal{F} \otimes_{f^*\mathcal{O}_{\mathcal{D}}} f^*\mathcal{G}, \mathcal{H})
\end{aligned}$$

The interesting "=" in this sequence of equalities is the third equality. It follows from the definition and adjointness of f_* and f^{-1} (as discussed in previous sections) in a straightforward manner. □

Lemma 18.26.2. *Let $(\mathcal{C}, \mathcal{O})$ be a ringed site. Let $\mathcal{F}$, $\mathcal{G}$ be sheaves of $\mathcal{O}$-modules.*

(1) *If $\mathcal{F}$, $\mathcal{G}$ are locally free, so is $\mathcal{F} \otimes_{\mathcal{O}} \mathcal{G}$.*
(2) *If $\mathcal{F}$, $\mathcal{G}$ are finite locally free, so is $\mathcal{F} \otimes_{\mathcal{O}} \mathcal{G}$.*
(3) *If $\mathcal{F}$, $\mathcal{G}$ are locally generated by sections, so is $\mathcal{F} \otimes_{\mathcal{O}} \mathcal{G}$.*
(4) *If $\mathcal{F}$, $\mathcal{G}$ are of finite type, so is $\mathcal{F} \otimes_{\mathcal{O}} \mathcal{G}$.*
(5) *If $\mathcal{F}$, $\mathcal{G}$ are quasi-coherent, so is $\mathcal{F} \otimes_{\mathcal{O}} \mathcal{G}$.*
(6) *If $\mathcal{F}$, $\mathcal{G}$ are of finite presentation, so is $\mathcal{F} \otimes_{\mathcal{O}} \mathcal{G}$.*
(7) *If $\mathcal{F}$ is of finite presentation and $\mathcal{G}$ is coherent, then $\mathcal{F} \otimes_{\mathcal{O}} \mathcal{G}$ is coherent.*
(8) *If $\mathcal{F}$, $\mathcal{G}$ are coherent, so is $\mathcal{F} \otimes_{\mathcal{O}} \mathcal{G}$.*

Proof. Omitted. Hint: Compare with Sheaves of Modules, Lemma 17.15.5. □

18.27. Internal Hom

Let $\mathcal{C}$ be a category and let $\mathcal{O}$ be a presheaf of rings. Let $\mathcal{F}$, $\mathcal{G}$ be presheaves of $\mathcal{O}$-modules. Consider the rule

$$U \longmapsto \mathrm{Hom}_{\mathcal{O}_U}(\mathcal{F}|_U, \mathcal{G}|_U).$$

For $\varphi : V \to U$ in $\mathcal{C}$ we define a restriction mapping

$$\mathrm{Hom}_{\mathcal{O}_U}(\mathcal{F}|_U, \mathcal{G}|_U) \longrightarrow \mathrm{Hom}_{\mathcal{O}_V}(\mathcal{F}|_V, \mathcal{G}|_V)$$

by restricting via the relocalization morphism $j : \mathcal{C}/V \to \mathcal{C}/U$, see Sites, Lemma 7.24.7. Hence this defines a presheaf $\mathcal{H}om_{\mathcal{O}}(\mathcal{F}, \mathcal{G})$. In addition, given an element $\varphi \in \mathrm{Hom}_{\mathcal{O}|_U}(\mathcal{F}|_U, \mathcal{G}|_U)$ and a section $f \in \mathcal{O}(U)$ then we can define $f\varphi \in \mathrm{Hom}_{\mathcal{O}|_U}(\mathcal{F}|_U, \mathcal{G}|_U)$ by either precomposing with multiplication by f on $\mathcal{F}|_U$ or postcomposing with multiplication by f on $\mathcal{G}|_U$ (it gives the same result). Hence we in fact get a presheaf of $\mathcal{O}$-modules. There is a canonical "evaluation" morphism

$$\mathcal{F} \otimes_{p,\mathcal{O}} \mathcal{H}om_{\mathcal{O}}(\mathcal{F}, \mathcal{G}) \longrightarrow \mathcal{G}.$$

Lemma 18.27.1. *If $\mathcal{C}$ is a site, $\mathcal{O}$ is a sheaf of rings, $\mathcal{F}$ is a presheaf of $\mathcal{O}$-modules, and $\mathcal{G}$ is a sheaf of $\mathcal{O}$-modules, then $\mathcal{H}om_{\mathcal{O}}(\mathcal{F}, \mathcal{G})$ is a sheaf of $\mathcal{O}$-modules.*

Proof. Omitted. Hints: Note first that $\mathcal{H}om_{\mathcal{O}}(\mathcal{F}, \mathcal{G}) = \mathcal{H}om_{\mathcal{O}}(\mathcal{F}^{\#}, \mathcal{G})$, which reduces the question to the case where both $\mathcal{F}$ and $\mathcal{G}$ are sheaves. The result for sheaves of sets is Sites, Lemma 7.25.1. □

In the situation of the lemma the "evaluation" morphism factors through the tensor product of sheaves of modules

$$\mathcal{F} \otimes_{\mathcal{O}} \mathcal{H}om_{\mathcal{O}}(\mathcal{F}, \mathcal{G}) \longrightarrow \mathcal{G}.$$

Lemma 18.27.2. *Internal hom and (co)limits. Let $\mathcal{C}$ be a category and let $\mathcal{O}$ be a presheaf of rings.*

(1) *For any presheaf of $\mathcal{O}$-modules $\mathcal{F}$ the functor*

$$PMod(\mathcal{O}) \longrightarrow PMod(\mathcal{O}), \quad \mathcal{G} \longmapsto \mathcal{H}om_{\mathcal{O}}(\mathcal{F}, \mathcal{G})$$

commutes with arbitrary limits.

(2) *For any presheaf of $\mathcal{O}$-modules $\mathcal{G}$ the functor*

$$PMod(\mathcal{O}) \longrightarrow PMod(\mathcal{O})^{opp}, \quad \mathcal{F} \longmapsto \mathcal{H}om_{\mathcal{O}}(\mathcal{F}, \mathcal{G})$$

commutes with arbitrary colimits, in a formula

$$\mathcal{H}om_{\mathcal{O}}(\mathrm{colim}_i \mathcal{F}_i, \mathcal{G}) = \lim_i \mathcal{H}om_{\mathcal{O}}(\mathcal{F}_i, \mathcal{G}).$$

Suppose that $\mathcal{C}$ is a site, and $\mathcal{O}$ is a sheaf of rings.

(3) *For any sheaf of $\mathcal{O}$-modules $\mathcal{F}$ the functor*
$$Mod(\mathcal{O}) \longrightarrow Mod(\mathcal{O}), \quad \mathcal{G} \longmapsto \mathcal{H}om_{\mathcal{O}}(\mathcal{F}, \mathcal{G})$$
commutes with arbitrary limits.

(4) *For any sheaf of $\mathcal{O}$-modules $\mathcal{G}$ the functor*
$$Mod(\mathcal{O}) \longrightarrow Mod(\mathcal{O})^{opp}, \quad \mathcal{F} \longmapsto \mathcal{H}om_{\mathcal{O}}(\mathcal{F}, \mathcal{G})$$
commutes with arbitrary colimits, in a formula
$$\mathcal{H}om_{\mathcal{O}}(\operatorname{colim}_i \mathcal{F}_i, \mathcal{G}) = \lim_i \mathcal{H}om_{\mathcal{O}}(\mathcal{F}_i, \mathcal{G}).$$

Proof. Let $\mathcal{I} \to PMod(\mathcal{O})$, $i \mapsto \mathcal{G}_i$ be a diagram. Let U be an object of the category $\mathcal{C}$. As j_U^* is both a left and a right adjoint we see that $\lim_i j_U^* \mathcal{G}_i = j_U^* \lim_i \mathcal{G}_i$. Hence we have
$$\begin{aligned} \mathcal{H}om_{\mathcal{O}}(\mathcal{F}, \lim_i \mathcal{G}_i)(U) &= \operatorname{Hom}_{\mathcal{O}_U}(\mathcal{F}|_U, \lim_i \mathcal{G}_i|_U) \\ &= \lim_i \operatorname{Hom}_{\mathcal{O}_U}(\mathcal{F}|_U, \mathcal{G}_i|_U) \\ &= \lim_i \mathcal{H}om_{\mathcal{O}}(\mathcal{F}, \mathcal{G}_i)(U) \end{aligned}$$
by definition of a limit. This proves (1). Part (2) is proved in exactly the same way. Part (3) follows from (1) because the limit of a diagram of sheaves is the same as the limit in the category of presheaves. Finally, (4) follow because, in the formula we have
$$\operatorname{Mor}_{Mod(\mathcal{O})}(\operatorname{colim}_i \mathcal{F}_i, \mathcal{G}) = \operatorname{Mor}_{PMod(\mathcal{O})}(\operatorname{colim}_i^{PSh} \mathcal{F}_i, \mathcal{G})$$
as the colimit $\operatorname{colim}_i \mathcal{F}_i$ is the sheafification of the colimit $\operatorname{colim}_i^{PSh} \mathcal{F}_i$ in $PMod(\mathcal{O})$. Hence (4) follows from (2) (by the remark on limits above again). □

Lemma 18.27.3. *Let $\mathcal{C}$ be a category. Let $\mathcal{O}$ be a presheaf of rings.*

(1) *Let $\mathcal{F}$, $\mathcal{G}$, $\mathcal{H}$ be presheaves of $\mathcal{O}$-modules. There is a canonical isomorphism*
$$\mathcal{H}om_{\mathcal{O}}(\mathcal{F} \otimes_{p,\mathcal{O}} \mathcal{G}, \mathcal{H}) \longrightarrow \mathcal{H}om_{\mathcal{O}}(\mathcal{F}, \mathcal{H}om_{\mathcal{O}}(\mathcal{G}, \mathcal{H}))$$
which is functorial in all three entries (sheaf Hom in all three spots). In particular,
$$\operatorname{Mor}_{PMod(\mathcal{O})}(\mathcal{F} \otimes_{p,\mathcal{O}} \mathcal{G}, \mathcal{H}) = \operatorname{Mor}_{PMod(\mathcal{O})}(\mathcal{F}, \mathcal{H}om_{\mathcal{O}}(\mathcal{G}, \mathcal{H}))$$

(2) *Suppose that $\mathcal{C}$ is a site, $\mathcal{O}$ is a sheaf of rings, and $\mathcal{F}$, $\mathcal{G}$, $\mathcal{H}$ are sheaves of $\mathcal{O}$-modules. There is a canonical isomorphism*
$$\mathcal{H}om_{\mathcal{O}}(\mathcal{F} \otimes_{\mathcal{O}} \mathcal{G}, \mathcal{H}) \longrightarrow \mathcal{H}om_{\mathcal{O}}(\mathcal{F}, \mathcal{H}om_{\mathcal{O}}(\mathcal{G}, \mathcal{H}))$$
which is functorial in all three entries (sheaf Hom in all three spots). In particular,
$$\operatorname{Mor}_{Mod(\mathcal{O})}(\mathcal{F} \otimes_{\mathcal{O}} \mathcal{G}, \mathcal{H}) = \operatorname{Mor}_{Mod(\mathcal{O})}(\mathcal{F}, \mathcal{H}om_{\mathcal{O}}(\mathcal{G}, \mathcal{H}))$$

Proof. This is the analogue of Algebra, Lemma 10.11.8. The proof is the same, and is omitted. □

Lemma 18.27.4. *Tensor product and (co)limits. Let $\mathcal{C}$ be a category and let $\mathcal{O}$ be a presheaf of rings.*

(1) *For any presheaf of $\mathcal{O}$-modules $\mathcal{F}$ the functor*
$$PMod(\mathcal{O}) \longrightarrow PMod(\mathcal{O}), \quad \mathcal{G} \longmapsto \mathcal{F} \otimes_{p,\mathcal{O}} \mathcal{G}$$
commutes with arbitrary colimits.

(2) *Suppose that $\mathcal{C}$ is a site, and $\mathcal{O}$ is a sheaf of rings. For any sheaf of $\mathcal{O}$-modules $\mathcal{F}$ the functor*

$$Mod(\mathcal{O}) \longrightarrow Mod(\mathcal{O}), \quad \mathcal{G} \longmapsto \mathcal{F} \otimes_{\mathcal{O}} \mathcal{G}$$

commutes with arbitrary colimits.

Proof. This is because tensor product is adjoint to internal hom according to Lemma 18.27.3. See Categories, Lemma 4.24.3. □

Lemma 18.27.5. *Let $\mathcal{C}$ be a category, resp. a site Let $\mathcal{O} \to \mathcal{O}'$ be a map of presheaves, resp. sheaves of rings. Then*

$$\mathrm{Hom}_{\mathcal{O}}(\mathcal{G}, \mathcal{F}) = \mathrm{Hom}_{\mathcal{O}'}(\mathcal{G}, \mathcal{H}om_{\mathcal{O}}(\mathcal{O}', \mathcal{F}))$$

for any $\mathcal{O}'$-module $\mathcal{G}$ and $\mathcal{O}$-module $\mathcal{F}$.

Proof. This is the analogue of Algebra, Lemma 10.13.4. The proof is the same, and is omitted. □

18.28. Flat modules

We can define flat modules exactly as in the case of modules over rings.

Definition 18.28.1. Let $\mathcal{C}$ be a category. Let $\mathcal{O}$ be a presheaf of rings.

(1) A presheaf $\mathcal{F}$ of $\mathcal{O}$-modules is called *flat* if the functor

$$PMod(\mathcal{O}) \longrightarrow PMod(\mathcal{O}), \quad \mathcal{G} \mapsto \mathcal{G} \otimes_{p,\mathcal{O}} \mathcal{F}$$

is exact.

(2) A map $\mathcal{O} \to \mathcal{O}'$ of presheaves of rings is called *flat* if $\mathcal{O}'$ is flat as a presheaf of $\mathcal{O}$-modules.

(3) If $\mathcal{C}$ is a site, $\mathcal{O}$ is a sheaf of rings and $\mathcal{F}$ is a sheaf of $\mathcal{O}$-modules, then we say $\mathcal{F}$ is *flat* if the functor

$$Mod(\mathcal{O}) \longrightarrow Mod(\mathcal{O}), \quad \mathcal{G} \mapsto \mathcal{G} \otimes_{\mathcal{O}} \mathcal{F}$$

is exact.

(4) A map $\mathcal{O} \to \mathcal{O}'$ of sheaves of rings on a site is called *flat* if $\mathcal{O}'$ is flat as a sheaf of $\mathcal{O}$-modules.

The notion of a flat module or flat ring map is intrinsic (Section 18.18).

Lemma 18.28.2. *Let $\mathcal{C}$ be a category. Let $\mathcal{O}$ be a presheaf of rings. Let $\mathcal{F}$ be a presheaf of $\mathcal{O}$-modules. If each $\mathcal{F}(U)$ is a flat $\mathcal{O}(U)$-module, then $\mathcal{F}$ is flat.*

Proof. This is immediate from the definitions. □

Lemma 18.28.3. *Let $\mathcal{C}$ be a category. Let $\mathcal{O}$ be a presheaf of rings. Let $\mathcal{F}$ be a presheaf of $\mathcal{O}$-modules. If $\mathcal{F}$ is a flat $\mathcal{O}$-module, then $\mathcal{F}^\#$ is a flat $\mathcal{O}^\#$-module.*

Proof. Omitted. (Hint: Sheafification is exact.) □

Lemma 18.28.4. *Colimits and tensor product.*

(1) *A filtered colimit of flat presheaves of modules is flat. A direct sum of flat presheaves of modules is flat.*

(2) *A filtered colimit of flat sheaves of modules is flat. A direct sum of flat sheaves of modules is flat.*

Proof. Part (1) follows from Lemma 18.27.4 and Algebra, Lemma 10.8.9 by looking at sections over objects. To see part (2), use Lemma 18.27.4 and the fact that a filtered colimit of exact complexes is an exact complex (this uses that sheafification is exact and commutes with colimits). Some details omitted. □

Lemma 18.28.5. *Let $\mathcal{C}$ be a category. Let $\mathcal{O}$ be a presheaf of rings. Let U be an object of $\mathcal{C}$. Consider the functor $j_U : \mathcal{C}/U \to \mathcal{C}$.*

(1) *The presheaf of $\mathcal{O}$-modules $j_{U!}\mathcal{O}_U$ (see Remark 18.19.6) is flat.*
(2) *If $\mathcal{C}$ is a site, $\mathcal{O}$ is a sheaf of rings, $j_{U!}\mathcal{O}_U$ is a flat sheaf of $\mathcal{O}$-modules.*

Proof. Proof of (1). By the discussion in Remark 18.19.6 we see that

$$j_{U!}\mathcal{O}_U(V) = \bigoplus\nolimits_{\varphi \in \mathrm{Mor}_{\mathcal{C}}(V,U)} \mathcal{O}(V)$$

which is a flat $\mathcal{O}(V)$-module. Hence (1) follows from Lemma 18.28.2. Then (2) follows as $j_{U!}\mathcal{O}_U = (j_{U!}\mathcal{O}_U)^\#$ (the first $j_{U!}$ on sheaves, the second on presheaves) and Lemma 18.28.3. □

Lemma 18.28.6. *Let $\mathcal{C}$ be a category. Let $\mathcal{O}$ be a presheaf of rings.*

(1) *Any presheaf of $\mathcal{O}$-modules is a quotient of a direct sum $\bigoplus j_{U_i!}\mathcal{O}_{U_i}$.*
(2) *Any presheaf of $\mathcal{O}$-modules is a quotient of a flat presheaf of $\mathcal{O}$-modules.*
(3) *If $\mathcal{C}$ is a site, $\mathcal{O}$ is a sheaf of rings, then any sheaf of $\mathcal{O}$-modules is a quotient of a direct sum $\bigoplus j_{U_i!}\mathcal{O}_{U_i}$.*
(4) *If $\mathcal{C}$ is a site, $\mathcal{O}$ is a sheaf of rings, then any sheaf of $\mathcal{O}$-modules is a quotient of a flat sheaf of $\mathcal{O}$-modules.*

Proof. Proof of (1). For every object U of $\mathcal{C}$ and every $s \in \mathcal{F}(U)$ we get a morphism $j_{U!}\mathcal{O}_U \to \mathcal{F}$, namely the adjoint to the morphism $\mathcal{O}_U \to \mathcal{F}|_U$, $1 \mapsto s$. Clearly the map

$$\bigoplus\nolimits_{(U,s)} j_{U!}\mathcal{O}_U \longrightarrow \mathcal{F}$$

is surjective. The source is flat by combining Lemmas 18.28.4 and 18.28.5 which proves (2). The sheaf case follows from this either by sheafifying or repeating the same argument. □

Lemma 18.28.7. *Let $\mathcal{C}$ be a category. Let $\mathcal{O}$ be a presheaf of rings. Let*

$$0 \to \mathcal{F}'' \to \mathcal{F}' \to \mathcal{F} \to 0$$

be a short exact sequence of presheaves of $\mathcal{O}$-modules. Assume $\mathcal{F}$ is flat. Then

(1) *For any presheaf $\mathcal{G}$ of $\mathcal{O}$-modules, the sequence*

$$0 \to \mathcal{F}'' \otimes_{p,\mathcal{O}} \mathcal{G} \to \mathcal{F}' \otimes_{p,\mathcal{O}} \mathcal{G} \to \mathcal{F} \otimes_{p,\mathcal{O}} \mathcal{G} \to 0$$

is exact.
(2) *If $\mathcal{C}$ is a site, and $\mathcal{O}$, $\mathcal{F}$, $\mathcal{F}'$, $\mathcal{F}''$, and $\mathcal{G}$ are all sheaves, the sequence*

$$0 \to \mathcal{F}'' \otimes_{\mathcal{O}} \mathcal{G} \to \mathcal{F}' \otimes_{\mathcal{O}} \mathcal{G} \to \mathcal{F} \otimes_{\mathcal{O}} \mathcal{G} \to 0$$

is exact.

Proof. Choose a flat presheaf of $\mathcal{O}$-modules $\mathcal{G}'$ which surjects onto $\mathcal{G}$. This is possible by Lemma 18.28.6. Let $\mathcal{G}'' = \mathrm{Ker}(\mathcal{G}' \to \mathcal{G})$. The lemma follows by applying the snake lemma to the following diagram

$$\begin{array}{ccccccccc} & & 0 & & 0 & & 0 & & \\ & & \uparrow & & \uparrow & & \uparrow & & \\ & & \mathcal{F}'' \otimes_{p,\mathcal{O}} \mathcal{G} & \to & \mathcal{F}' \otimes_{p,\mathcal{O}} \mathcal{G} & \to & \mathcal{F} \otimes_{p,\mathcal{O}} \mathcal{G} & \to & 0 \\ & & \uparrow & & \uparrow & & \uparrow & & \\ 0 & \to & \mathcal{F}'' \otimes_{p,\mathcal{O}} \mathcal{G}' & \to & \mathcal{F}' \otimes_{p,\mathcal{O}} \mathcal{G}' & \to & \mathcal{F} \otimes_{p,\mathcal{O}} \mathcal{G}' & \to & 0 \\ & & \uparrow & & \uparrow & & \uparrow & & \\ & & \mathcal{F}'' \otimes_{p,\mathcal{O}} \mathcal{G}'' & \to & \mathcal{F}' \otimes_{p,\mathcal{O}} \mathcal{G}'' & \to & \mathcal{F} \otimes_{p,\mathcal{O}} \mathcal{G}'' & \to & 0 \\ & & & & & & \uparrow & & \\ & & & & & & 0 & & \end{array}$$

with exact rows and columns. The middle row is exact because tensoring with the flat module $\mathcal{G}'$ is exact. The sheaf case follows from the presheaf case as sheafification is exact. □

Lemma 18.28.8. *Let $\mathcal{C}$ be a category. Let $\mathcal{O}$ be a presheaf of rings. Let*

$$0 \to \mathcal{F}_2 \to \mathcal{F}_1 \to \mathcal{F}_0 \to 0$$

be a short exact sequence of presheaves of $\mathcal{O}$-modules.

(1) *If $\mathcal{F}_2$ and $\mathcal{F}_0$ are flat so is $\mathcal{F}_1$.*
(2) *If $\mathcal{F}_1$ and $\mathcal{F}_0$ are flat so is $\mathcal{F}_2$.*

If $\mathcal{C}$ is a site and $\mathcal{O}$ is a sheaf of rings then the same result holds Mod($\mathcal{O}$).

Proof. Let $\mathcal{G}^\bullet$ be an arbitrary exact complex of presheaves of $\mathcal{O}$-modules. Assume that $\mathcal{F}_0$ is flat. By Lemma 18.28.7 we see that

$$0 \to \mathcal{G}^\bullet \otimes_{p,\mathcal{O}} \mathcal{F}_2 \to \mathcal{G}^\bullet \otimes_{p,\mathcal{O}} \mathcal{F}_1 \to \mathcal{G}^\bullet \otimes_{p,\mathcal{O}} \mathcal{F}_0 \to 0$$

is a short exact sequence of complexes of presheaves of $\mathcal{O}$-modules. Hence (1) and (2) follow from the snake lemma. The case of sheaves of modules is proved in the same way. □

Lemma 18.28.9. *Let $\mathcal{C}$ be a category. Let $\mathcal{O}$ be a presheaf of rings. Let*

$$\ldots \to \mathcal{F}_2 \to \mathcal{F}_1 \to \mathcal{F}_0 \to \mathcal{Q} \to 0$$

be an exact complex of presheaves of $\mathcal{O}$-modules. If $\mathcal{Q}$ and all $\mathcal{F}_i$ are flat $\mathcal{O}$-modules, then for any presheaf $\mathcal{G}$ of $\mathcal{O}$-modules the complex

$$\ldots \to \mathcal{F}_2 \otimes_{p,\mathcal{O}} \mathcal{G} \to \mathcal{F}_1 \otimes_{p,\mathcal{O}} \mathcal{G} \to \mathcal{F}_0 \otimes_{p,\mathcal{O}} \mathcal{G} \to \mathcal{Q} \otimes_{p,\mathcal{O}} \mathcal{G} \to 0$$

is exact also. If $\mathcal{C}$ is a site and $\mathcal{O}$ is a sheaf of rings then the same result holds Mod($\mathcal{O}$).

Proof. Follows from Lemma 18.28.7 by splitting the complex into short exact sequences and using Lemma 18.28.8 to prove inductively that $\mathrm{Im}(\mathcal{F}_{i+1} \to \mathcal{F}_i)$ is flat. □

Lemma 18.28.10. *Let $\mathcal{O}_1 \to \mathcal{O}_2$ be a map of sheaves of rings on a site $\mathcal{C}$. If $\mathcal{G}$ is a flat $\mathcal{O}_1$-module, then $\mathcal{G} \otimes_{\mathcal{O}_1} \mathcal{O}_2$ is a flat $\mathcal{O}_2$-module.*

Proof. This is true because

$$(\mathcal{G} \otimes_{\mathcal{O}_1} \mathcal{O}_2) \otimes_{\mathcal{O}_2} \mathcal{H} = \mathcal{G} \otimes_{\mathcal{O}_1} \mathcal{F}$$

(as sheaves of abelian groups for example). □

The following lemma gives one direction of the equational criterion of flatness (Algebra, Lemma 10.36.10).

Lemma 18.28.11. *Let $(\mathcal{C}, \mathcal{O})$ be a ringed site. Let $\mathcal{F}$ be a flat $\mathcal{O}$-module. Let U be an object of $\mathcal{C}$ and let*

$$\mathcal{O}_U \xrightarrow{(f_1,\ldots,f_n)} \mathcal{O}_U^{\oplus n} \xrightarrow{(s_1,\ldots,s_n)} \mathcal{F}|_U$$

be a complex of $\mathcal{O}_U$-modules. There exists a covering $\{U_i \to U\}$ and for each i a factorization

$$\mathcal{O}_{U_i}^{\oplus n} \xrightarrow{A} \mathcal{O}_{U_i}^{\oplus m} \xrightarrow{(t_1,\ldots,t_m)} \mathcal{F}|_{U_i}$$

of $(s_1, \ldots, s_n)|_{U_i}$ such that $A \circ (f_1, \ldots, f_n)|_{U_i} = 0$.

Proof. Let $\mathcal{I} \subset \mathcal{O}_U$ be the sheaf of ideals generated by $f_1, \ldots, f_n$. Then $\sum f_j \otimes s_j$ is a section of $\mathcal{I} \otimes_{\mathcal{O}_U} \mathcal{F}|_U$ which maps to zero in $\mathcal{F}|_U$. As $\mathcal{F}|_U$ is flat the map $\mathcal{I} \otimes_{\mathcal{O}_U} \mathcal{F}|_U \to \mathcal{F}|_U$ is injective. Since $\mathcal{I} \otimes_{\mathcal{O}_U} \mathcal{F}|_U$ is the sheaf associated to the presheaf tensor product, we see there exists a covering $\{U_i \to U\}$ such that $\sum f_j|_{U_i} \otimes s_j|_{U_i}$ is zero in $\mathcal{I}(U_i) \otimes_{\mathcal{O}(U_i)} \mathcal{F}(U_i)$. Unwinding the definitions using Algebra, Lemma 10.102.10 we find $t_1, \ldots, t_m \in \mathcal{F}(U_i)$ and $a_{jk} \in \mathcal{O}(U_i)$ such that $\sum a_{jk} f_j|_{U_i} = 0$ and $s_j|_{U_i} = \sum a_{jk} t_k$. □

Lemma 18.28.12. *Let $(\mathcal{C}, \mathcal{O})$ be a ringed site. Let $\mathcal{F}$ be locally of finite presentation and flat. Then given an object U of $\mathcal{C}$ there exists a covering $\{U_i \to U\}$ such that $\mathcal{F}|_{U_i}$ is a direct summand of a finite free $\mathcal{O}_{U_i}$-module.*

Proof. Choose an object U of $\mathcal{C}$. After replacing U by the members of a covering, we may assume there exists a presentation

$$\mathcal{O}_U^{\oplus r} \to \mathcal{O}_U^{\oplus n} \to \mathcal{F} \to 0$$

By Lemma 18.28.11 we may assume, after replacing U by the members of a covering, assume there exists a factorization

$$\mathcal{O}_U^{\oplus n} \to \mathcal{O}_U^{\oplus n_1} \to \mathcal{F}$$

such that the composition $\mathcal{O}_U^{\oplus r} \to \mathcal{O}_U^{\oplus n} \to \mathcal{O}_U^{\oplus n_1}$ annihilates the first summand of $\mathcal{O}_U^{\oplus r}$. Repeating this argument $r - 1$ more times we obtain a factorization

$$\mathcal{O}_U^{\oplus n} \to \mathcal{O}_U^{\oplus n_r} \to \mathcal{F}$$

such that the composition $\mathcal{O}_U^{\oplus r} \to \mathcal{O}_U^{\oplus n} \to \mathcal{O}_U^{\oplus n_r}$ is zero. This means that the surjection $\mathcal{O}_U^{\oplus n_r} \to \mathcal{F}$ has a section and we win. □

Lemma 18.28.13. *Let $\mathcal{C}$ be a site. Let $\mathcal{O}' \to \mathcal{O}$ be a surjection of sheaves of rings whose kernel $\mathcal{I}$ is an ideal of square zero. Let $\mathcal{F}'$ be an $\mathcal{O}'$-module and set $\mathcal{F} = \mathcal{F}'/\mathcal{I}\mathcal{F}'$. The following are equivalent*

1. *$\mathcal{F}'$ is a flat $\mathcal{O}'$-module, and*
2. *$\mathcal{F}$ is a flat $\mathcal{O}$-module and $\mathcal{I} \otimes_{\mathcal{O}} \mathcal{F} \to \mathcal{F}'$ is injective.*

Proof. If (1) holds, then $\mathcal{F} = \mathcal{F}' \otimes_{\mathcal{O}'} \mathcal{O}$ is flat over $\mathcal{O}$ by Lemma 18.28.10 and we see the map $\mathcal{I} \otimes_{\mathcal{O}} \mathcal{F} \to \mathcal{F}'$ is injective by applying $- \otimes_{\mathcal{O}'} \mathcal{F}'$ to the exact sequence $0 \to \mathcal{I} \to \mathcal{O}' \to \mathcal{O} \to 0$, see Lemma 18.28.7. Assume (2). In the rest of the proof we will use without further mention that $\mathcal{K} \otimes_{\mathcal{O}'} \mathcal{F}' = \mathcal{K} \otimes_{\mathcal{O}} \mathcal{F}$ for any $\mathcal{O}'$-module $\mathcal{K}$ annihilated by $\mathcal{I}$/ Let $\alpha : \mathcal{G}' \to \mathcal{H}'$ be an injective map of $\mathcal{O}'$-modules. Let $\mathcal{G} \subset \mathcal{G}'$, resp. $\mathcal{H} \subset \mathcal{H}'$ be the subsheaf of sections annihilated by $\mathcal{I}$. Consider the diagram

$$\begin{array}{ccccccc} \mathcal{G} \otimes_{\mathcal{O}'} \mathcal{F}' & \longrightarrow & \mathcal{G}' \otimes_{\mathcal{O}'} \mathcal{F}' & \longrightarrow & \mathcal{G}'/\mathcal{G} \otimes_{\mathcal{O}'} \mathcal{F}' & \longrightarrow & 0 \\ \downarrow & & \downarrow & & \downarrow & & \\ \mathcal{H} \otimes_{\mathcal{O}'} \mathcal{F}' & \longrightarrow & \mathcal{H}' \otimes_{\mathcal{O}'} \mathcal{F}' & \longrightarrow & \mathcal{H}'/\mathcal{H} \otimes_{\mathcal{O}'} \mathcal{F}' & \longrightarrow & 0 \end{array}$$

Note that $\mathcal{G}'/\mathcal{G}$ and $\mathcal{H}'/\mathcal{H}$ are annihilated by $\mathcal{I}$ and that $\mathcal{G}'/\mathcal{G} \to \mathcal{H}'/\mathcal{H}$ is injective. Thus the right vertical arrow is injective as $\mathcal{F}$ is flat over $\mathcal{O}$. The same is true for the left vertical arrow. Hence the middle vertical arrow is injective and $\mathcal{F}'$ is flat. □

18.29. Towards constructible modules

Recall that a quasi-compact object of a site is one such that every covering of it can be refined by a finite covering. It turns out that if every object of a site has a covering by quasi-compact objects, then the modules $j_!\mathcal{O}_U$ with U quasi-compact form a particularly nice set of generators for the category of all modules.

Lemma 18.29.1. *Let $(\mathcal{C}, \mathcal{O})$ be a ringed site. Let $\{U_i \to U\}$ be a covering of $\mathcal{C}$. Then the sequence*

$$\bigoplus j_{U_i \times_U U_j !}\mathcal{O}_{U_i \times_U U_j} \to \bigoplus j_{U_i !}\mathcal{O}_{U_i} \to j_!\mathcal{O}_U \to 0$$

is exact.

Proof. This holds because for any $\mathcal{O}$-module $\mathcal{F}$ the functor $\mathrm{Hom}_{\mathcal{O}}(-, \mathcal{F})$ turns our sequence into the exact sequence $0 \to \mathcal{F}(U) \to \prod \mathcal{F}(U_i) \to \prod \mathcal{F}(U_i \times_U U_j)$. Then the lemma follows from Homology, Lemma 12.5.8. □

Lemma 18.29.2. *Let $\mathcal{C}$ be a site. Let W be a quasi-compact object of $\mathcal{C}$.*

1. *The functor $Sh(\mathcal{C}) \to Sets$, $\mathcal{F} \mapsto \mathcal{F}(W)$ commutes with coproducts.*
2. *Let $\mathcal{O}$ be a sheaf of rings on $\mathcal{C}$. The functor $Mod(\mathcal{O}) \to Ab$, $\mathcal{F} \mapsto \mathcal{F}(W)$ commutes with direct sums.*

Proof. Proof of (1). Taking sections over W commutes with filtered colimits with injective transition maps by Sites, Lemma 7.11.2. If $\mathcal{F}_i$ is a family of sheaves of sets indexed by a set I. Then $\coprod \mathcal{F}_i$ is the filtered colimit over the partially ordered set of finite subsets $E \subset I$ of the coproducts $\mathcal{F}_E = \coprod_{i \in E} \mathcal{F}_i$. Since the transition maps are injective we conclude.

Proof of (2). Let $\mathcal{F}_i$ be a family of sheaves of $\mathcal{O}$-modules indexed by a set I. Then $\bigoplus \mathcal{F}_i$ is the filtered colimit over the partially ordered set of finite subsets $E \subset I$ of the direct sums $\mathcal{F}_E = \bigoplus_{i \in E} \mathcal{F}_i$. A filtered colimit of abelian sheaves can be computed in the category of sheaves of sets. Moreover, for $E \subset E'$ the transition map $\mathcal{F}_E \to \mathcal{F}_{E'}$ is injective (as sheafification is exact and the injectivity is clear on underlying presheaves). Hence it suffices to show the result for a finite index set by Sites, Lemma 7.11.2. The finite case is dealt with in Lemma 18.3.2 (it holds over any object of $\mathcal{C}$). □

Lemma 18.29.3. *Let $(\mathcal{C}, \mathcal{O})$ be a ringed site. Let U be a quasi-compact object of $\mathcal{C}$. Then the functor $\mathrm{Hom}_{\mathcal{O}}(j_!\mathcal{O}_U, -)$ commutes with direct sums.*

Proof. This is true because $\mathrm{Hom}_{\mathcal{O}}(j_!\mathcal{O}_U, \mathcal{F}) = \mathcal{F}(U)$ and because the functor $\mathcal{F} \mapsto \mathcal{F}(U)$ commutes with direct sums by Lemma 18.29.2. □

In order to state the sharpest possible results in the following we introduce some notation.

Situation 18.29.4. Let $\mathcal{C}$ be a site. Let $\mathcal{B} \subset \mathrm{Ob}(\mathcal{C})$ be a set of objects. We consider the following conditions

(18.29.4.1) Every object of $\mathcal{C}$ has a covering by elements of $\mathcal{B}$.
(18.29.4.2) Every $U \in \mathcal{B}$ is quasi-compact.
(18.29.4.3) For a finite covering $\{U_i \to U\}$ with $U_i, U \in \mathcal{B}$ the fibre products $U_i \times_U U_j$ are quasi-compact.

Lemma 18.29.5. *In Situation 18.29.4 assume (18.29.4.1) holds.*

(1) *Every sheaf of sets is the target of a surjective map whose source is a coproduct $\coprod h^\#_{U_i}$ with U_i in $\mathcal{B}$.*
(2) *If $\mathcal{O}$ is a sheaf of rings, then every $\mathcal{O}$-module is a quotient of a direct sum $\bigoplus j_{U_i!}\mathcal{O}_{U_i}$ with U_i in $\mathcal{B}$.*

Proof. Follows immediately from Lemmas 18.28.6 and 18.29.1. □

Lemma 18.29.6. *In Situation 18.29.4 assume (18.29.4.1) and (18.29.4.2) hold.*

(1) *Every sheaf of sets is a filtered colimit of sheaves of the form*

$$\text{Coequalizer}\left(\coprod\nolimits_{j=1,\ldots,m} h^\#_{V_j} \longrightarrow\!\!\!\!\!\rightarrow \coprod\nolimits_{i=1,\ldots,n} h^\#_{U_i} \right) \tag{18.29.6.1}$$

with U_i and V_j in $\mathcal{B}$.

(2) *If $\mathcal{O}$ is a sheaf of rings, then every $\mathcal{O}$-module is a filtered colimit of sheaves of the form*

$$\text{Coker}\left(\bigoplus\nolimits_{j=1,\ldots,m} j_{V_j!}\mathcal{O}_{V_j} \longrightarrow \bigoplus\nolimits_{i=1,\ldots,n} j_{U_i!}\mathcal{O}_{U_i} \right) \tag{18.29.6.2}$$

with U_i and V_j in $\mathcal{B}$.

Proof. Proof of (1). By Lemma 18.29.5 every sheaf of sets $\mathcal{F}$ is the target of a surjection whose source is a coprod $\mathcal{F}_0$ of sheaves the form $h^\#_U$ with $U \in \mathcal{B}$. Applying this to $\mathcal{F}_0 \times_{\mathcal{F}} \mathcal{F}_0$ we find that $\mathcal{F}$ is a coequalizer of a pair of maps

$$\coprod\nolimits_{j\in J} h^\#_{V_j} \longrightarrow\!\!\!\!\!\rightarrow \coprod\nolimits_{i\in I} h^\#_{U_i}$$

for some index sets I, J and V_j and U_i in $\mathcal{B}$. For every finite subset $J' \subset J$ there is a finite subset $I' \subset I$ such that the coproduct over $j \in J'$ maps into the coprod over $i \in I'$ via both maps, see Lemma 18.29.3. Thus our sheaf is the colimit of the cokernels of these maps between finite coproducts.

Proof of (2). By Lemma 18.29.5 every module is a quotient of a direct sum of modules of the form $j_{U!}\mathcal{O}_U$ with $U \in \mathcal{B}$. Thus every module is a cokernel

$$\mathrm{Coker}\left(\bigoplus\nolimits_{j\in J} j_{V_j!}\mathcal{O}_{V_j} \longrightarrow \bigoplus\nolimits_{i\in I} j_{U_i!}\mathcal{O}_{U_i} \right)$$

for some index sets I, J and V_j and U_i in $\mathcal{B}$. For every finite subset $J' \subset J$ there is a finite subset $I' \subset I$ such that the direct sum over $j \in J'$ maps into the direct sum

over $i \in I'$, see Lemma 18.29.3. Thus our module is the colimit of the cokernels of these maps between finite direct sums. $\square$

Lemma 18.29.7. *In Situation 18.29.4 assume (18.29.4.1) and (18.29.4.2) hold. Let $\mathcal{O}$ be a sheaf of rings. Then a cokernel of a map between modules as in (18.29.6.2) is another module as in (18.29.6.2).*

Proof. Let $\mathcal{F} = \mathrm{Coker}(\bigoplus j_{V_j!}\mathcal{O}_{V_j} \to \bigoplus j_{U_i!}\mathcal{O}_{U_i})$ as in (18.29.6.2). It suffices to show that the cokernel of a map $\varphi : j_{W!}\mathcal{O}_W \to \mathcal{F}$ with $W \in \mathcal{B}$ is another module of the same type. The map φ corresponds to $s \in \mathcal{F}(W)$. Since W is quasi-compact we can find a finite covering $\{W_k \to W\}$ with $W_k \in \mathcal{B}$ such that $s|_{W_k}$ comes from a section $\sum s_{ki}$ of $\bigoplus j_{U_i!}\mathcal{O}_{U_i})$. This determines maps $j_{W_k!}\mathcal{O}_{W_k} \to \bigoplus j_{U_i!}\mathcal{O}_{U_i}$. Since $\bigoplus j_{W_k!}\mathcal{O}_{W_k} \to j_{W!}\mathcal{O}_W$ is surjective (Lemma 18.29.1) we see that $\mathrm{Coker}(\varphi)$ is equal to

$$\mathrm{Coker}\left(\bigoplus j_{W_k!}\mathcal{O}_{W_k} \oplus \bigoplus j_{V_j!}\mathcal{O}_{V_j} \longrightarrow \bigoplus j_{U_i!}\mathcal{O}_{U_i}\right)$$

as desired. $\square$

Lemma 18.29.8. *In Situation 18.29.4 assume (18.29.4.1), (18.29.4.2), and (18.29.4.3) hold. Let $\mathcal{O}$ be a sheaf of rings. Then given a map*

$$\bigoplus\nolimits_{j=1,\ldots,m} j_{V_j!}\mathcal{O}_{V_j} \longrightarrow \bigoplus\nolimits_{i=1,\ldots,n} j_{U_i!}\mathcal{O}_{U_i}$$

with U_i and V_j in $\mathcal{B}$, and finite coverings $\{U_{ik} \to U_i\}$ by $U_{ik} \in \mathcal{B}$, there exist a finite set of $W_l \in \mathcal{B}$ and a commutative diagram

$$\begin{array}{ccc} \bigoplus j_{W_l!}\mathcal{O}_{W_l} & \longrightarrow & \bigoplus j_{U_i!}\mathcal{O}_{U_{ik}} \\ \downarrow & & \downarrow \\ \bigoplus j_{V_j!}\mathcal{O}_{V_j} & \longrightarrow & \bigoplus j_{U_i!}\mathcal{O}_{U_i} \end{array}$$

inducing an isomorphism on cokernels of the horizontal maps.

Proof. Since $\bigoplus j_{U_{ik}!}\mathcal{O}_{U_{ik}} \to \bigoplus j_{U_i!}\mathcal{O}_{U_i}$ is surjective (Lemma 18.29.1), we can find finite coverings $\{V_{jm} \to V_j\}$ with $V_{jm} \in \mathcal{B}$ such that we can find a commutative diagram

$$\begin{array}{ccc} \bigoplus j_{V_{jm}!}\mathcal{O}_{V_{jm}} & \longrightarrow & \bigoplus j_{U_i!}\mathcal{O}_{U_{ik}} \\ \downarrow & & \downarrow \\ \bigoplus j_{V_j!}\mathcal{O}_{V_j} & \longrightarrow & \bigoplus j_{U_i!}\mathcal{O}_{U_i} \end{array}$$

Adding

$$\bigoplus j_{U_{ik} \times_{U_i} U_{ik'}!}\mathcal{O}_{U_{ik} \times_{U_i} U_{ik'}}$$

to the upper left corner finishes the proof by Lemma 18.29.1. $\square$

Lemma 18.29.9. *In Situation 18.29.4 assume (18.29.4.1), (18.29.4.2), and (18.29.4.3) hold. Let $\mathcal{O}$ be a sheaf of rings. Then an extension of modules as in (18.29.6.2) is another module as in (18.29.6.2).*

Proof. Let $0 \to \mathcal{F}_1 \to \mathcal{F}_2 \to \mathcal{F}_3 \to 0$ be a short exact sequence of $\mathcal{O}$-modules with $\mathcal{F}_1$ and $\mathcal{F}_3$ as in (18.29.6.2). Choose presentations

$$\bigoplus A_{V_j} \to \bigoplus A_{U_i} \to \mathcal{F}_1 \to 0 \quad \text{and} \quad \bigoplus A_{T_j} \to \bigoplus A_{W_i} \to \mathcal{F}_3 \to 0$$

In this proof the direct sums are always finite, and we write $A_U = j_{U!}\mathcal{O}_U$ for $U \in \mathcal{B}$. By Lemma 18.29.8 we may replace W_i by finite coverings $\{W_{ik} \to W_i\}$ with $W_{ik} \in \mathcal{B}$. Thus we may assume the map $\bigoplus A_{W_i} \to \mathcal{F}_3$ lifts to a map into $\mathcal{F}_2$. Consider the kernel

$$\mathcal{K}_2 = \mathrm{Ker}(\bigoplus A_{U_i} \oplus \bigoplus A_{W_i} \longrightarrow \mathcal{F}_2)$$

By the snake lemma this kernel surjections onto $\mathcal{K}_3 = \mathrm{Ker}(\bigoplus A_{W_i} \to \mathcal{F}_3)$. Thus after replacing each T_j by a finite covering with elements of $\mathcal{B}$ (permissible by Lemma 18.29.1) we may assume there is a map $\bigoplus A_{T_j} \to \mathcal{K}_2$ lifting the given map $\bigoplus A_{T_j} \to \mathcal{K}_3$. Then $\bigoplus A_{V_j} \oplus \bigoplus A_{T_j} \to \mathcal{K}_2$ is surjective which finishes the proof. □

Lemma 18.29.10. *In Situation 18.29.4 assume (18.29.4.1), (18.29.4.2), and (18.29.4.3) hold. Let $\mathcal{O}$ be a sheaf of rings. Let $\mathcal{A} \subset Mod(\mathcal{O})$ be the full subcategory of modules isomorphic to a cokernel as in (18.29.6.2). If the kernel of every map of $\mathcal{O}$-modules of the form*

$$\bigoplus\nolimits_{j=1,\ldots,m} j_{V_j!}\mathcal{O}_{V_j} \longrightarrow \bigoplus\nolimits_{i=1,\ldots,n} j_{U_i!}\mathcal{O}_{U_i}$$

with U_i and V_j in $\mathcal{B}$, is in $\mathcal{A}$, then $\mathcal{A}$ is weak Serre subcategory of $Mod(\mathcal{O})$.

Proof. We will use the criterion of Homology, Lemma 12.9.3. By the results of Lemmas 18.29.7 and 18.29.9 it suffices to see that the kernel of a map $\mathcal{F} \to \mathcal{G}$ between objects of $\mathcal{A}$ is in $\mathcal{A}$. To prove this choose presentations

$$\bigoplus A_{V_j} \to \bigoplus A_{U_i} \to \mathcal{F} \to 0 \quad \text{and} \quad \bigoplus A_{T_j} \to \bigoplus A_{W_i} \to \mathcal{G} \to 0$$

In this proof the direct sums are always finite, and we write $A_U = j_{U!}\mathcal{O}_U$ for $U \in \mathcal{B}$. Using Lemmas 18.29.1 and 18.29.8 and arguing as in the proof of Lemma 18.29.9 we may assume that the map $\mathcal{F} \to \mathcal{G}$ lifts to a map of presentations

$$\begin{array}{ccccccc}
\bigoplus A_{V_j} & \longrightarrow & \bigoplus A_{U_i} & \longrightarrow & \mathcal{F} & \longrightarrow & 0 \\
\downarrow & & \downarrow & & \downarrow & & \\
\bigoplus A_{T_j} & \longrightarrow & \bigoplus A_{W_i} & \longrightarrow & \mathcal{G} & \longrightarrow & 0
\end{array}$$

Then we see that

$$\mathrm{Ker}(\mathcal{F} \to \mathcal{G}) = \mathrm{Coker}\left(\bigoplus A_{V_j} \to \mathrm{Ker}\left(\bigoplus A_{T_j} \oplus \bigoplus A_{U_i} \to \bigoplus A_{W_i}\right)\right)$$

and the lemma follows from the assumption and Lemma 18.29.7. □

18.30. Flat morphisms

Definition 18.30.1. Let $(f, f^\sharp) : (Sh(\mathcal{C}), \mathcal{O}) \longrightarrow (Sh(\mathcal{C}'), \mathcal{O}')$ be a morphism of ringed topoi. We say $(f, f^\sharp)$ is *flat* if the ring map $f^\sharp : f^{-1}\mathcal{O}' \to \mathcal{O}$ is flat. We say a morphism of ringed sites is *flat* if the associated morphism of ringed topoi is flat.

Lemma 18.30.2. *Let $f : Sh(\mathcal{C}) \to Sh(\mathcal{C}')$ be a morphism of ringed topoi. Then*

$$f^{-1} : Ab(\mathcal{C}') \longrightarrow Ab(\mathcal{C}), \quad \mathcal{F} \longmapsto f^{-1}\mathcal{F}$$

is exact. If $(f, f^\sharp) : (Sh(\mathcal{C}), \mathcal{O}) \to (Sh(\mathcal{C}'), \mathcal{O}')$ is a flat morphism of ringed topoi then

$$f^* : Mod(\mathcal{O}') \longrightarrow Mod(\mathcal{O}), \quad \mathcal{F} \longmapsto f^*\mathcal{F}$$

is exact.

Proof. Given an abelian sheaf $\mathcal{G}$ on $\mathcal{C}'$ the underlying sheaf of sets of $f^{-1}\mathcal{G}$ is the same as f^{-1} of the underlying sheaf of sets of $\mathcal{G}$, see Sites, Section 7.43. Hence the exactness of f^{-1} for sheaves of sets (required in the definition of a morphism of topoi, see Sites, Definition 7.16.1) implies the exactness of f^{-1} as a functor on abelian sheaves.

To see the statement on modules recall that $f^*\mathcal{F}$ is defined as the tensor product $f^{-1}\mathcal{F} \otimes_{f^{-1}\mathcal{O}', f^\sharp} \mathcal{O}$. Hence f^* is a composition of functors both of which are exact. □

Definition 18.30.3. Let $f : (Sh(\mathcal{C}), \mathcal{O}) \to (Sh(\mathcal{D}), \mathcal{O}')$ be a morphism of ringed topoi. Let $\mathcal{F}$ be a sheaf of $\mathcal{O}$-modules. We say that $\mathcal{F}$ is *flat over* $(Sh(\mathcal{D}), \mathcal{O}')$ if $\mathcal{F}$ is flat as an $f^{-1}\mathcal{O}'$-module.

This is compatible with the notion as defined for morphisms of ringed spaces, see Modules, Definition 17.17.3 and the discussion following.

18.31. Invertible modules

Here is the definition.

Definition 18.31.1. Let $(\mathcal{C}, \mathcal{O})$ be a ringed site.

(1) A finite locally free $\mathcal{O}$-module $\mathcal{F}$ is said to have *rank* r if for every object U of $\mathcal{C}$ there exists a covering $\{U_i \to U\}$ of U such that $\mathcal{F}|_{U_i}$ is isomorphic to $\mathcal{O}_{U_i}^{\oplus r}$ as an $\mathcal{O}_{U_i}$-module.
(2) An *invertible $\mathcal{O}$-module* is a finite locally free $\mathcal{O}$-module of rank 1.
(3) The sheaf $\mathcal{O}^*$ is the subsheaf of $\mathcal{O}$ defined by the rule

$$U \longmapsto \mathcal{O}^*(U) = \{f \in \mathcal{O}(U) \mid \exists g \in \mathcal{O}(U) \text{ such that } fg = 1\}$$

It is a sheaf of abelian groups with multiplication as the group law.

Lemma 18.31.2. *Let $(\mathcal{C}, \mathcal{O})$ be a ringed space.*

(1) *If $\mathcal{L}$, $\mathcal{N}$ are invertible $\mathcal{O}$-modules, then so is $\mathcal{L} \otimes_{\mathcal{O}} \mathcal{N}$.*
(2) *If $\mathcal{L}$ is an invertible $\mathcal{O}$-modules, then so is $\mathcal{L}^{\otimes -1} = \mathcal{H}om_{\mathcal{O}}(\mathcal{L}, \mathcal{O})$.*
(3) *If $\mathcal{L}$ is an invertible $\mathcal{O}$-module, then the evaluation map $\mathcal{L} \otimes_{\mathcal{O}} \mathcal{L}^{\otimes -1} \to \mathcal{O}$ is an isomorphism.*

Proof. Omitted. □

Lemma 18.31.3. *Let $(\mathcal{C}, \mathcal{O})$ be a ringed space. There exists a set of invertible modules $\{\mathcal{L}_i\}_{i \in I}$ such that each invertible module on $(\mathcal{C}, \mathcal{O})$ is isomorphic to exactly one of the $\mathcal{L}_i$.*

Proof. Omitted, but see Sheaves of Modules, Lemma 17.21.5. □

This lemma says roughly speaking that the collection of isomorphism classes of invertible sheaves forms a set. Lemma 18.31.2 says that tensor product defines the structure of an abelian group on this set.

Definition 18.31.4. Let $(\mathcal{C}, \mathcal{O})$ be a ringed site. The *Picard group* $\mathrm{Pic}(\mathcal{O})$ the ringed site is the abelian group whose elements are isomorphism classes of invertible $\mathcal{O}$-modules, with addition corresponding to tensor product.

18.32. Modules of differentials

In this section we briefly explain how to define the module of relative differentials for a morphism of ringed topoi. We suggest the reader take a look at the corresponding section in the chapter on commutative algebra (Algebra, Section 10.125).

Definition 18.32.1. Let $\mathcal{C}$ be a site. Let $\varphi : \mathcal{O}_1 \to \mathcal{O}_2$ be a homomorphism of sheaves of rings. Let $\mathcal{F}$ be an $\mathcal{O}_2$-module. A $\mathcal{O}_1$-*derivation* or more precisely a φ-*derivation* into $\mathcal{F}$ is a map $D : \mathcal{O}_2 \to \mathcal{F}$ which is additive, annihilates the image of $\mathcal{O}_1 \to \mathcal{O}_2$, and satisfies the *Leibniz rule*

$$D(ab) = aD(b) + D(a)b$$

for all a, b local sections of $\mathcal{O}_2$ (wherever they are both defined). We denote $\mathrm{Der}_{\mathcal{O}_1}(\mathcal{O}_2, \mathcal{F})$ the set of φ-derivations into $\mathcal{F}$.

This is the sheaf theoretic analogue of Algebra, Definition 18.32.1. Given a derivation $D : \mathcal{O}_2 \to \mathcal{F}$ as in the definition the map on global sections

$$D : \Gamma(\mathcal{O}_2) \longrightarrow \Gamma(\mathcal{F})$$

clearly is a $\Gamma(\mathcal{O}_1)$-derivation as in the algebra definition. Note that if $\alpha : \mathcal{F} \to \mathcal{G}$ is a map of $\mathcal{O}_2$-modules, then there is an induced map

$$\mathrm{Der}_{\mathcal{O}_1}(\mathcal{O}_2, \mathcal{F}) \longrightarrow \mathrm{Der}_{\mathcal{O}_1}(\mathcal{O}_2, \mathcal{G})$$

given by the rule $D \mapsto \alpha \circ D$. In other words we obtain a functor.

Lemma 18.32.2. *Let $\mathcal{C}$ be a site. Let $\varphi : \mathcal{O}_1 \to \mathcal{O}_2$ be a homomorphism of sheaves of rings. The functor*

$$Mod(\mathcal{O}_2) \longrightarrow Ab, \quad \mathcal{F} \longmapsto Der_{\mathcal{O}_1}(\mathcal{O}_2, \mathcal{F})$$

is representable.

Proof. This is proved in exactly the same way as the analogous statement in algebra. During this proof, for any sheaf of sets $\mathcal{F}$ on $\mathcal{C}$, let us denote $\mathcal{O}_2[\mathcal{F}]$ the sheafification of the presheaf $U \mapsto \mathcal{O}_2(U)[\mathcal{F}(U)]$ where this denotes the free $\mathcal{O}_1(U)$-module on the set $\mathcal{F}(U)$. For $s \in \mathcal{F}(U)$ we denote $[s]$ the corresponding section of $\mathcal{O}_2[\mathcal{F}]$ over U. If $\mathcal{F}$ is a sheaf of $\mathcal{O}_2$-modules, then there is a canonical map

$$c : \mathcal{O}_2[\mathcal{F}] \longrightarrow \mathcal{F}$$

which on the presheaf level is given by the rule $\sum f_s[s] \mapsto \sum f_s s$. We will employ the short hand $[s] \mapsto s$ to describe this map and similarly for other maps below. Consider the map of $\mathcal{O}_2$-modules

$$\begin{array}{ccc} \mathcal{O}_2[\mathcal{O}_2 \times \mathcal{O}_2] \oplus \mathcal{O}_2[\mathcal{O}_2 \times \mathcal{O}_2] \oplus \mathcal{O}_2[\mathcal{O}_1] & \longrightarrow & \mathcal{O}_2[\mathcal{O}_2] \\ [(a,b)] \oplus [(f,g)] \oplus [h] & \longmapsto & \begin{array}{c}[a+b] - [a] - [b] + \\ [fg] - g[f] - f[g] + \\ [\varphi(h)]\end{array} \end{array} \tag{18.32.2.1}$$

with short hand notation as above. Set $\Omega_{\mathcal{O}_2/\mathcal{O}_1}$ equal to the cokernel of this map. Then it is clear that there exists a map of sheaves of sets

$$\mathrm{d} : \mathcal{O}_2 \longrightarrow \Omega_{\mathcal{O}_2/\mathcal{O}_1}$$

mapping a local section f to the image of $[f]$ in $\Omega_{\mathcal{O}_2/\mathcal{O}_1}$. By construction d is a $\mathcal{O}_1$-derivation. Next, let $\mathcal{F}$ be a sheaf of $\mathcal{O}_2$-modules and let $D : \mathcal{O}_2 \to \mathcal{F}$ be a $\mathcal{O}_1$-derivation. Then we can consider the $\mathcal{O}_2$-linear map $\mathcal{O}_2[\mathcal{O}_2] \to \mathcal{F}$ which sends

$[g]$ to $D(g)$. It follows from the definition of a derivation that this map annihilates sections in the image of the map (18.32.2.1) and hence defines a map

$$\alpha_D : \Omega_{\mathcal{O}_2/\mathcal{O}_1} \longrightarrow \mathcal{F}$$

Since it is clear that $D = \alpha_D \circ \mathrm{d}$ the lemma is proved. □

Definition 18.32.3. Let $\mathcal{C}$ be a site. Let $\varphi : \mathcal{O}_1 \to \mathcal{O}_2$ be a homomorphism of sheaves of rings. The *module of differentials* of the ring map φ is the object representing the functor $\mathcal{F} \mapsto \mathrm{Der}_{\mathcal{O}_1}(\mathcal{O}_2, \mathcal{F})$ which exists by Lemma 18.32.2. It is denoted $\Omega_{\mathcal{O}_2/\mathcal{O}_1}$, and the *universal φ-derivation* is denoted $\mathrm{d} : \mathcal{O}_2 \to \Omega_{\mathcal{O}_2/\mathcal{O}_1}$.

Since this module and the derivation form the universal object representing a functor, this notion is clearly intrinsic (i.e., does not depend on the choice of the site underlying the ringed topos, see Section 18.18). Note that $\Omega_{\mathcal{O}_2/\mathcal{O}_1}$ is the cokernel of the map (18.32.2.1) of $\mathcal{O}_2$-modules. Moreover the map d is described by the rule that $\mathrm{d}f$ is the image of the local section $[f]$.

Lemma 18.32.4. *Let $\mathcal{C}$ be a site. Let $\varphi : \mathcal{O}_1 \to \mathcal{O}_2$ be a homomorphism of presheaves of rings. Then $\Omega_{\mathcal{O}_2^\#/\mathcal{O}_1^\#}$ is the sheaf associated to the presheaf $U \mapsto \Omega_{\mathcal{O}_2(U)/\mathcal{O}_1(U)}$.*

Proof. Consider the map (18.32.2.1). There is a similar map of presheaves whose value on $U \in \mathrm{Ob}(\mathcal{C})$ is

$$\mathcal{O}_2(U)[\mathcal{O}_2(U)\times\mathcal{O}_2(U)]\oplus\mathcal{O}_2(U)[\mathcal{O}_2(U)\times\mathcal{O}_2(U)]\oplus\mathcal{O}_2(U)[\mathcal{O}_1(U)] \longrightarrow \mathcal{O}_2(U)[\mathcal{O}_2(U)]$$

The cokernel of this map has value $\Omega_{\mathcal{O}_2(U)/\mathcal{O}_1(U)}$ over U by the construction of the module of differentials in Algebra, Definition 10.125.2. On the other hand, the sheaves in (18.32.2.1) are the sheafifications of the presheaves above. Thus the result follows as sheafification is exact. □

Lemma 18.32.5. *Let $f : Sh(\mathcal{D}) \to Sh(\mathcal{C})$ be a morphism of topoi. Let $\varphi : \mathcal{O}_1 \to \mathcal{O}_2$ be a homomorphism of sheaves of rings on $\mathcal{C}$. Then there is a canonical identification $f^{-1}\Omega_{\mathcal{O}_2/\mathcal{O}_1} = \Omega_{f^{-1}\mathcal{O}_2/f^{-1}\mathcal{O}_1}$ compatible with universal derivations.*

Proof. This holds because the sheaf $\Omega_{\mathcal{O}_2/\mathcal{O}_1}$ is the cokernel of the map (18.32.2.1) and a similar statement holds for $\Omega_{f^{-1}\mathcal{O}_2/f^{-1}\mathcal{O}_1}$, because the functor f^{-1} is exact, and because $f^{-1}(\mathcal{O}_2[\mathcal{O}_2]) = f^{-1}\mathcal{O}_2[f^{-1}\mathcal{O}_2]$, $f^{-1}(\mathcal{O}_2[\mathcal{O}_2 \times \mathcal{O}_2]) = f^{-1}\mathcal{O}_2[f^{-1}\mathcal{O}_2 \times f^{-1}\mathcal{O}_2]$, and $f^{-1}(\mathcal{O}_2[\mathcal{O}_1]) = f^{-1}\mathcal{O}_2[f^{-1}\mathcal{O}_1]$. □

Lemma 18.32.6. *Let $\mathcal{C}$ be a site. Let $\varphi : \mathcal{O}_1 \to \mathcal{O}_2$ be a homomorphism of sheaves of rings. For any object U of $\mathcal{C}$ there is a canonical isomorphism*

$$\Omega_{\mathcal{O}_2/\mathcal{O}_1}|_U = \Omega_{(\mathcal{O}_2|_U)/(\mathcal{O}_1|_U)}$$

compatible with universal derivations.

Proof. This is a special case of Lemma 18.32.5. □

Lemma 18.32.7. *Let $\mathcal{C}$ be a site. Let*

$$\begin{array}{ccc} \mathcal{O}_2 & \xrightarrow{\varphi} & \mathcal{O}_2' \\ \uparrow & & \uparrow \\ \mathcal{O}_1 & \longrightarrow & \mathcal{O}_1' \end{array}$$

be a commutative diagram of sheaves of rings on $\mathcal{C}$. The map $\mathcal{O}_2 \to \mathcal{O}'_2$ composed with the map $d : \mathcal{O}'_2 \to \Omega_{\mathcal{O}'_2/\mathcal{O}'_1}$ is a $\mathcal{O}_1$-derivation. Hence we obtain a canonical map of $\mathcal{O}_2$-modules $\Omega_{\mathcal{O}_2/\mathcal{O}_1} \to \Omega_{\mathcal{O}'_2/\mathcal{O}'_1}$. It is uniquely characterized by the property that $d(f)$ mapsto $d(\varphi(f))$ for any local section f of $\mathcal{O}_2$. In this way $\Omega_{-/-}$ becomes a functor on the category of arrows of sheaves of rings.

Proof. This lemma proves itself. □

Lemma 18.32.8. *In Lemma 18.32.7 suppose that $\mathcal{O}_2 \to \mathcal{O}'_2$ is surjective with kernel $\mathcal{I} \subset \mathcal{O}_2$ and assume that $\mathcal{O}_1 = \mathcal{O}'_1$. Then there is a canonical exact sequence of $\mathcal{O}'_2$-modules*

$$\mathcal{I}/\mathcal{I}^2 \longrightarrow \Omega_{\mathcal{O}_2/\mathcal{O}_1} \otimes_{\mathcal{O}_2} \mathcal{O}'_2 \longrightarrow \Omega_{\mathcal{O}'_2/\mathcal{O}_1} \longrightarrow 0$$

The leftmost map is characterized by the rule that a local section f of $\mathcal{I}$ maps to $df \otimes 1$.

Proof. For a local section f of $\mathcal{I}$ denote $\overline{f}$ the image of f in $\mathcal{I}/\mathcal{I}^2$. To show that the map $\overline{f} \mapsto \mathrm{d}f \otimes 1$ is well defined we just have to check that $\mathrm{d}f_1 f_2 \otimes 1 = 0$ if f_1, f_2 are local sections of $\mathcal{I}$. And this is clear from the Leibniz rule $\mathrm{d}f_1 f_2 \otimes 1 = (f_1 \mathrm{d}f_2 + f_2 \mathrm{d}f_1) \otimes 1 = \mathrm{d}f_2 \otimes f_1 + \mathrm{d}f_2 \otimes f_1 = 0$. A similar computation show this map is $\mathcal{O}'_2 = \mathcal{O}_2/\mathcal{I}$-linear. The map on the right is the one from Lemma 18.32.7.

To see that the sequence is exact, we argue as follows. Let $\mathcal{O}''_2 \subset \mathcal{O}'_2$ be the presheaf of $\mathcal{O}_1$-algebras whose value on U is the image of $\mathcal{O}_2(U) \to \mathcal{O}'_2(U)$. By Algebra, Lemma 10.125.9 the sequences

$$\mathcal{I}(U)/\mathcal{I}(U)^2 \longrightarrow \Omega_{\mathcal{O}_2(U)/\mathcal{O}_1(U)} \otimes_{\mathcal{O}_2(U)} \mathcal{O}''_2(U) \longrightarrow \Omega_{\mathcal{O}''_2(U)/\mathcal{O}_1(U)} \longrightarrow 0$$

are exact for all objects U of $\mathcal{C}$. Since sheafification is exact this gives an exact sequence of sheaves of $(\mathcal{O}'_2)^\#$-modules. By Lemma 18.32.4 and the fact that $(\mathcal{O}''_2)^\# = \mathcal{O}'_2$ we conclude. □

Here is a particular situation where derivations come up naturally.

Lemma 18.32.9. *Let $\mathcal{C}$ be a site. Let $\varphi : \mathcal{O}_1 \to \mathcal{O}_2$ be a homomorphism of sheaves of rings. Consider a short exact sequence*

$$0 \to \mathcal{F} \to \mathcal{A} \to \mathcal{O}_2 \to 0$$

Here $\mathcal{A}$ is a sheaf of $\mathcal{O}_1$-algebras, $\pi : \mathcal{A} \to \mathcal{O}_2$ is a surjection of sheaves of $\mathcal{O}_1$-algebras, and $\mathcal{F} = Ker(\pi)$ is its kernel. Assume $\mathcal{F}$ an ideal sheaf with square zero in $\mathcal{A}$. So $\mathcal{F}$ has a natural structure of an $\mathcal{O}_2$-module. A section $s : \mathcal{O}_2 \to \mathcal{A}$ of π is a $\mathcal{O}_1$-algebra map such that $\pi \circ s = id$. Given any section $s : \mathcal{O}_2 \to \mathcal{F}$ of π and any φ-derivation $D : \mathcal{O}_1 \to \mathcal{F}$ the map

$$s + D : \mathcal{O}_1 \to \mathcal{A}$$

is a section of π and every section s' is of the form $s + D$ for a unique φ-derivation D.

Proof. Recall that the $\mathcal{O}_2$-module structure on $\mathcal{F}$ is given by $h\tau = \tilde{h}\tau$ (multiplication in $\mathcal{A}$) where h is a local section of $\mathcal{O}_2$, and $\tilde{h}$ is a local lift of h to a local section of $\mathcal{A}$, and τ is a local section of $\mathcal{F}$. In particular, given s, we may use $\tilde{h} = s(h)$. To

verify that $s + D$ is a homomorphism of sheaves of rings we compute

$$\begin{aligned}(s + D)(ab) &= s(ab) + D(ab) \\ &= s(a)s(b) + aD(b) + D(a)b \\ &= s(a)s(b) + s(a)D(b) + D(a)s(b) \\ &= (s(a) + D(a))(s(b) + D(b))\end{aligned}$$

by the Leibniz rule. In the same manner one shows $s + D$ is a $\mathcal{O}_1$-algebra map because D is an $\mathcal{O}_1$-derivation. Conversely, given s' we set $D = s' - s$. Details omitted. □

Definition 18.32.10. Let $X = (Sh(\mathcal{C}), \mathcal{O})$ and $Y = (Sh(\mathcal{C}'), \mathcal{O}')$ be ringed topoi. Let $(f, f^\sharp) : X \to Y$ be a morphism of ringed topoi. In this situation

(1) for a sheaf $\mathcal{F}$ of $\mathcal{O}$-modules a *Y-derivation* $D : \mathcal{O} \to \mathcal{F}$ is just a $f^\sharp$-derivation, and
(2) the *sheaf of differentials* $\Omega_{X/Y}$ *of X over Y* is the module of differentials of $f^\sharp : f^{-1}\mathcal{O}' \to \mathcal{O}$, see Definition 18.32.3.

Thus $\Omega_{X/Y}$ comes equipped with a *universal Y-derivation* $\mathrm{d}_{X/Y} : \mathcal{O} \longrightarrow \Omega_{X/Y}$. We sometimes write $\Omega_{X/Y} = \Omega_f$.

Recall that $f^\sharp : f^{-1}\mathcal{O}' \to \mathcal{O}$ so that this definition makes sense.

Lemma 18.32.11. *Let $X = (Sh(\mathcal{C}_X), \mathcal{O}_X)$, $Y = (Sh(\mathcal{C}_Y), \mathcal{O}_Y)$, $X' = (Sh(\mathcal{C}_{X'}), \mathcal{O}_{X'})$, and $Y' = (Sh(\mathcal{C}_{Y'}), \mathcal{O}_{Y'})$ be ringed topoi. Let*

$$\begin{array}{ccc} X' & \xrightarrow{f} & X \\ \downarrow & & \downarrow \\ Y' & \longrightarrow & Y \end{array}$$

be a commutative diagram of morphisms of ringed topoi. The map $f^\sharp : \mathcal{O}_X \to f_*\mathcal{O}_{X'}$ composed with the map $f_*\mathrm{d}_{X'/Y'} : f_*\mathcal{O}_{X'} \to f_*\Omega_{X'/Y'}$ is a Y-derivation. Hence we obtain a canonical map of $\mathcal{O}_X$-modules $\Omega_{X/Y} \to f_*\Omega_{X'/Y'}$, and by adjointness of f_* and f^* a canonical $\mathcal{O}_{X'}$-module homomorphism

$$c_f : f^*\Omega_{X/Y} \longrightarrow \Omega_{X'/Y'}.$$

It is uniquely characterized by the property that $f^*\mathrm{d}_{X/Y}(t)$ mapsto $\mathrm{d}_{X'/Y'}(f^*t)$ for any local section t of $\mathcal{O}_X$.

Proof. This is clear except for the last assertion. Let us explain the meaning of this. Let $U \in \mathrm{Ob}(\mathcal{C}_X)$ and let $t \in \mathcal{O}_X(U)$. This is what it means for t to be a local section of $\mathcal{O}_X$. Now, we may think of t as a map of sheaves of sets $t : h_U^\# \to \mathcal{O}_X$. Then $f^{-1}t : f^{-1}h_U^\# \to f^{-1}\mathcal{O}_X$. By f^*t we mean the composition

$$f^{-1}h_U^\# \xrightarrow{f^{-1}t} f^{-1}\mathcal{O}_X \xrightarrow{f^\sharp} \mathcal{O}_{X'}, \qquad f^{-1}h_U^\# \xrightarrow{f^*t} \mathcal{O}_{X'}$$

Note that $\mathrm{d}_{X/Y}(t) \in \Omega_{X/Y}(U)$. Hence we may think of $\mathrm{d}_{X/Y}(t)$ as a map $\mathrm{d}_{X/Y}(t) : h_U^\# \to \Omega_{X/Y}$. Then $f^{-1}\mathrm{d}_{X/Y}(t) : f^{-1}h_U^\# \to f^{-1}\Omega_{X/Y}$. By $f^*\mathrm{d}_{X/Y}(t)$ we mean the

composition

$$f^{-1}h_U^\# \xrightarrow{f^{-1}\mathrm{d}_{X/Y}(t)} f^{-1}\Omega_{X/Y} \xrightarrow{1\otimes\mathrm{id}} f^*\Omega_{X/Y}, \qquad f^{-1}h_U^\# \xrightarrow{f^*\mathrm{d}_{X/Y}(t)} f^*\Omega_{X/Y}$$

OK, and now the statement of the lemma means that we have

$$c_f \circ f^*t = f^*\mathrm{d}_{X/Y}(t)$$

as maps from $f^{-1}h_U^\#$ to $\Omega_{X'/Y'}$. We omit the verification that this property holds for c_f as defined in the lemma. (Hint: The first map $c'_f : \Omega_{X/Y} \to f_*\Omega_{X'/Y'}$ satisfies $c'_f(\mathrm{d}_{X/Y}(t)) = f_*\mathrm{d}_{X'/Y'}(f^\sharp(t))$ as sections of $f_*\Omega_{X'/Y'}$ over U, and you have to turn this into the equality above by using adjunction.) The reason that this uniquely characterizes c_f is that the images of $f^*\mathrm{d}_{X/Y}(t)$ generate the $\mathcal{O}_{X'}$-module $f^*\Omega_{X/Y}$ simply because the local sections $\mathrm{d}_{X/Y}(t)$ generate the $\mathcal{O}_X$-module $\Omega_{X/Y}$. □

18.33. Finite order differential operators

In this section we introduce differential operators of finite order. We suggest the reader take a look at the corresponding section in the chapter on commutative algebra (Algebra, Section 10.126).

Definition 18.33.1. Let $\mathcal{C}$ be a site. Let $\varphi : \mathcal{O}_1 \to \mathcal{O}_2$ be a homomorphism of sheaves of rings. Let $k \geq 0$ be an integer. Let $\mathcal{F}$, $\mathcal{G}$ be sheaves of $\mathcal{O}_2$-modules. A *differential operator* $D : \mathcal{F} \to \mathcal{G}$ *of order* k is an is an $\mathcal{O}_1$-linear map such that for all local sections g of $\mathcal{O}_2$ the map $s \mapsto D(gs) - gD(s)$ is a differential operator of order $k-1$. For the base case $k = 0$ we define a differential operator of oder 0 to be an $\mathcal{O}_2$-linear map.

If $D : \mathcal{F} \to \mathcal{G}$ is a differential operator of order k, then for all local sections g of $\mathcal{O}_2$ the map gD is a differential operator of order k. The sum of two differential operators of order k is another. Hence the set of all these

$$\mathrm{Diff}^k(\mathcal{F}, \mathcal{G}) = \mathrm{Diff}^k_{\mathcal{O}_2/\mathcal{O}_1}(\mathcal{F}, \mathcal{G})$$

is a $\Gamma(\mathcal{C}, \mathcal{O}_2)$-module. We have

$$\mathrm{Diff}^0(\mathcal{F}, \mathcal{G}) \subset \mathrm{Diff}^1(\mathcal{F}, \mathcal{G}) \subset \mathrm{Diff}^2(\mathcal{F}, \mathcal{G}) \subset \ldots$$

The rule which maps $U \in \mathrm{Ob}(\mathcal{C})$ to the module of differential operators $D : \mathcal{F}|_U \to \mathcal{G}|_U$ of order k is a sheaf of $\mathcal{O}_2$-modules on the site $\mathcal{C}$. Thus we obtain a sheaf of differential operators (if we ever need this we will add a definition here).

Lemma 18.33.2. *Let $\mathcal{C}$ be a site. Let $\mathcal{O}_1 \to \mathcal{O}_2$ be a map of sheaves of rings. Let $\mathcal{E}, \mathcal{F}, \mathcal{G}$ be sheaves of $\mathcal{O}_2$-modules. If $D : \mathcal{E} \to \mathcal{F}$ and $D' : \mathcal{F} \to \mathcal{G}$ are differential operators of order k and k', then $D' \circ D$ is a differential operator of order $k + k'$.*

Proof. Let g be a local section of $\mathcal{O}_2$. Then the map which sends a local section x of $\mathcal{E}$ to

$$D'(D(gx)) - gD'(D(x)) = D'(D(gx)) - D'(gD(x)) + D'(gD(x)) - gD'(D(x))$$

is a sum of two compositions of differential operators of lower order. Hence the lemma follows by induction on $k + k'$. □

Lemma 18.33.3. *Let $\mathcal{C}$ be a site. Let $\mathcal{O}_1 \to \mathcal{O}_2$ be a map of sheaves of rings. Let $\mathcal{F}$ be a sheaf of $\mathcal{O}_2$-modules. Let $k \geq 0$. There exists a sheaf of $\mathcal{O}_2$-modules $\mathcal{P}^k_{\mathcal{O}_2/\mathcal{O}_1}(\mathcal{F})$ and a canonical isomorphism*

$$Diff^k_{\mathcal{O}_2/\mathcal{O}_1}(\mathcal{F}, \mathcal{G}) = \mathrm{Hom}_{\mathcal{O}_2}(\mathcal{P}^k_{\mathcal{O}_2/\mathcal{O}_1}(\mathcal{F}), \mathcal{G})$$

functorial in the $\mathcal{O}_2$-module $\mathcal{G}$.

Proof. The existence follows from general category theoretic arguments (insert future reference here), but we will also give a direct construction as this construction will be useful in the future proofs. We will freely use the notation introduced in the proof of Lemma 18.32.2. Given any differential operator $D : \mathcal{F} \to \mathcal{G}$ we obtain an $\mathcal{O}_2$-linear map $L_D : \mathcal{O}_2[\mathcal{F}] \to \mathcal{G}$ sending $[m]$ to $D(m)$. If D has order 0 then L_D annihilates the local sections

$$[m + m'] - [m] - [m'], \quad g_0[m] - [g_0 m]$$

where g_0 is a local section of $\mathcal{O}_2$ and m, m' are local sections of $\mathcal{F}$. If D has order 1, then L_D annihilates the local sections

$$[m + m' - [m] - [m'], \quad f[m] - [fm], \quad g_0 g_1[m] - g_0[g_1 m] - g_1[g_0 m] + [g_1 g_0 m]$$

where f is a local section of $\mathcal{O}_1$, g_0, g_1 are local sections of $\mathcal{O}_2$, and m, m' are local sections of $\mathcal{F}$. If D has order k, then L_D annihilates the local sections $[m + m'] - [m] - [m']$, $f[m] - [fm]$, and the local sections

$$g_0 g_1 \ldots g_k[m] - \sum g_0 \ldots \hat{g}_i \ldots g_k[g_i m] + \ldots + (-1)^{k+1}[g_0 \ldots g_k m]$$

Conversely, if $L : \mathcal{O}_2[\mathcal{F}] \to \mathcal{G}$ is an $\mathcal{O}_2$-linear map annihilating all the local sections listed in the previous sentence, then $m \mapsto L([m])$ is a differential operator of order k. Thus we see that $\mathcal{P}^k_{\mathcal{O}_2/\mathcal{O}_1}(\mathcal{F})$ is the quotient of $\mathcal{O}_2[\mathcal{F}]$ by the $\mathcal{O}_2$-submodule generated by these local sections. □

Definition 18.33.4. Let $\mathcal{C}$ be a site. Let $\mathcal{O}_1 \to \mathcal{O}_2$ be a map of sheaves of rings. Let $\mathcal{F}$ be a sheaf of $\mathcal{O}_2$-modules. The module $\mathcal{P}^k_{\mathcal{O}_2/\mathcal{O}_1}(\mathcal{F})$ constructed in Lemma 18.33.3 is called the *module of principal parts of order k* of $\mathcal{F}$.

Note that the inclusions

$$\mathrm{Diff}^0(\mathcal{F}, \mathcal{G}) \subset \mathrm{Diff}^1(\mathcal{F}, \mathcal{G}) \subset \mathrm{Diff}^2(\mathcal{F}, \mathcal{G}) \subset \ldots$$

correspond via Yoneda's lemma (Categories, Lemma 4.3.5) to surjections

$$\ldots \to \mathcal{P}^2_{\mathcal{O}_2/\mathcal{O}_1}(\mathcal{F}) \to \mathcal{P}^1_{\mathcal{O}_2/\mathcal{O}_1}(\mathcal{F}) \to \mathcal{P}^0_{\mathcal{O}_2/\mathcal{O}_1}(\mathcal{F}) = \mathcal{F}$$

Lemma 18.33.5. *Let $\mathcal{C}$ be a site. Let $\mathcal{O}_1 \to \mathcal{O}_2$ be a homomorphism of presheaves of rings. Let $\mathcal{F}$ be a presheaf of $\mathcal{O}_2$-modules. Then $\mathcal{P}^k_{\mathcal{O}_2^\#/\mathcal{O}_1^\#}(\mathcal{F}^\#)$ is the sheaf associated to the presheaf $U \mapsto P^k_{\mathcal{O}_2(U)/\mathcal{O}_1(U)}(\mathcal{F}(U))$.*

Proof. This can be proved in exactly the same way as is done for the sheaf of differentials in Lemma 18.32.4. Perhaps a more pleasing approach is to use the universal property of Lemma 18.33.3 directly to see the equality. We omit the details. □

Lemma 18.33.6. *Let $\mathcal{C}$ be a site. Let $\mathcal{O}_1 \to \mathcal{O}_2$ be a homomorphism of presheaves of rings. Let $\mathcal{F}$ be a presheaf of $\mathcal{O}_2$-modules. There is a canonical short exact sequence*

$$0 \to \Omega_{\mathcal{O}_2/\mathcal{O}_1} \otimes_{\mathcal{O}_2} \mathcal{F} \to \mathcal{P}^1_{\mathcal{O}_2/\mathcal{O}_1}(\mathcal{F}) \to \mathcal{F} \to 0$$

functorial in $\mathcal{F}$ called the sequence of principal parts.

Proof. Follows from the commutative algebra version (Algebra, Lemma 10.126.6) and Lemmas 18.32.4 and 18.33.5. □

Remark 18.33.7. Let $\mathcal{C}$ be a site. Suppose given a commutative diagram of sheaves of rings

$$\begin{array}{ccc} \mathcal{B} & \longrightarrow & \mathcal{B}' \\ \uparrow & & \uparrow \\ \mathcal{A} & \longrightarrow & \mathcal{A}' \end{array}$$

a $\mathcal{B}$-module $\mathcal{F}$, a $\mathcal{B}'$-module $\mathcal{F}'$, and a $\mathcal{B}$-linear map $\mathcal{F} \to \mathcal{F}'$. Then we get a compatible system of module maps

$$\begin{array}{ccccccc} \ldots & \longrightarrow & \mathcal{P}^2_{\mathcal{B}'/\mathcal{A}'}(\mathcal{F}') & \longrightarrow & \mathcal{P}^1_{\mathcal{B}'/\mathcal{A}'}(\mathcal{F}') & \longrightarrow & \mathcal{P}^0_{\mathcal{B}'/\mathcal{A}'}(\mathcal{F}') \\ & & \uparrow & & \uparrow & & \uparrow \\ \ldots & \longrightarrow & \mathcal{P}^2_{\mathcal{B}/\mathcal{A}}(\mathcal{F}) & \longrightarrow & \mathcal{P}^1_{\mathcal{B}/\mathcal{A}}(\mathcal{F}) & \longrightarrow & \mathcal{P}^0_{\mathcal{B}/\mathcal{A}}(\mathcal{F}) \end{array}$$

These maps are compatible with further composition of maps of this type. The easiest way to see this is to use the description of the modules $\mathcal{P}^k_{\mathcal{B}/\mathcal{A}}(\mathcal{M})$ in terms of (local) generators and relations in the proof of Lemma 18.33.3 but it can also be seen directly from the universal property of these modules. Moreover, these maps are compatible with the short exact sequences of Lemma 18.33.6.

18.34. The naive cotangent complex

This section is the analogue of Algebra, Section 10.127 and Modules, Section 17.24. We advise the reader to read those sections first.

Let $\mathcal{C}$ be a site. Let $\mathcal{A} \to \mathcal{B}$ be a homomorphism of sheaves of rings on $\mathcal{C}$. In this section, for any sheaf of sets $\mathcal{E}$ on $\mathcal{C}$ we denote $\mathcal{A}[\mathcal{E}]$ the sheafification of the presheaf $U \mapsto \mathcal{A}(U)[\mathcal{E}(U)]$. Here $\mathcal{A}(U)[\mathcal{E}(U)]$ denotes the polynomial algebra over $\mathcal{A}(U)$ whose variables correspond to the elements of $\mathcal{E}(U)$. We denote $[e] \in \mathcal{A}(U)[\mathcal{E}(U)]$ the variable corresponding to $e \in \mathcal{E}(U)$. There is a canonical surjection of $\mathcal{A}$-algebras

$$\mathcal{A}[\mathcal{B}] \longrightarrow \mathcal{B}, \quad [b] \longmapsto b \tag{18.34.0.1}$$

whose kernel we denote $\mathcal{I} \subset \mathcal{A}[\mathcal{B}]$. It is a simple observation that $\mathcal{I}$ is generated by the local sections $[b][b'] - [bb']$ and $[a] - a$. According to Lemma 18.32.8 there is a canonical map

$$\mathcal{I}/\mathcal{I}^2 \longrightarrow \Omega_{\mathcal{A}[\mathcal{B}]/\mathcal{A}} \otimes_{\mathcal{A}[\mathcal{B}]} \mathcal{B} \tag{18.34.0.2}$$

whose cokernel is canonically isomorphic to $\Omega_{\mathcal{B}/\mathcal{A}}$.

Definition 18.34.1. Let $\mathcal{C}$ be a site. Let $\mathcal{A} \to \mathcal{B}$ be a homomorphism of sheaves of rings on $\mathcal{C}$. The *naive cotangent complex* $NL_{\mathcal{B}/\mathcal{A}}$ is the chain complex (18.34.0.2)

$$NL_{\mathcal{B}/\mathcal{A}} = \left(\mathcal{I}/\mathcal{I}^2 \longrightarrow \Omega_{\mathcal{A}[\mathcal{B}]/\mathcal{A}} \otimes_{\mathcal{A}[\mathcal{B}]} \mathcal{B}\right)$$

with $\mathcal{I}/\mathcal{I}^2$ placed in (homological) degree 1 and $\Omega_{\mathcal{A}[\mathcal{B}]/\mathcal{A}} \otimes_{\mathcal{A}[\mathcal{B}]} \mathcal{B}$ placed in degree 0.

This construction satisfies a functoriality similar to that discussed in Lemma 18.32.7 for modules of differentials. Namely, given a commutative diagram

$$\begin{array}{ccc} \mathcal{B} & \longrightarrow & \mathcal{B}' \\ \uparrow & & \uparrow \\ \mathcal{A} & \longrightarrow & \mathcal{A}' \end{array} \tag{18.34.1.1}$$

of sheaves of rings on $\mathcal{C}$ there is a canonical $\mathcal{B}$-linear map of complexes

$$NL_{\mathcal{B}/\mathcal{A}} \longrightarrow NL_{\mathcal{B}'/\mathcal{A}'}$$

Namely, the maps in the commutative diagram give rise to a canonical map $\mathcal{A}[\mathcal{B}] \to \mathcal{A}'[\mathcal{B}']$ which maps $\mathcal{I}$ into $\mathcal{I}' = \mathrm{Ker}(\mathcal{A}'[\mathcal{B}'] \to \mathcal{B}')$. Thus a map $\mathcal{I}/\mathcal{I}^2 \to \mathcal{I}'/(\mathcal{I}')^2$ and a map between modules of differentials, which together give the desired map between the naive cotangent complexes.

We can choose a different presentation of $\mathcal{B}$ as a quotient of a polynomial algebra over $\mathcal{A}$ and still obtain the same object of $D(\mathcal{B})$. To explain this, suppose that $\mathcal{E}$ is a sheaves of sets on $\mathcal{C}$ and $\alpha : \mathcal{E} \to \mathcal{B}$ a map of sheaves of sets. Then we obtain an $\mathcal{A}$-algebra homomorphism $\mathcal{A}[\mathcal{E}] \to \mathcal{B}$. Assume this map is surjective, and let $\mathcal{J} \subset \mathcal{A}[\mathcal{E}]$ be the kernel. Set

$$NL(\alpha) = \left(\mathcal{J}/\mathcal{J}^2 \longrightarrow \Omega_{\mathcal{A}[\mathcal{E}]/\mathcal{A}} \otimes_{\mathcal{A}[\mathcal{E}]} \mathcal{B}\right)$$

Here is the result.

Lemma 18.34.2. *In the situation above there is a canonical isomorphism $NL(\alpha) = NL_{\mathcal{B}/\mathcal{A}}$ in $D(\mathcal{B})$.*

Proof. Observe that $NL_{\mathcal{B}/\mathcal{A}} = NL(\mathrm{id}_{\mathcal{B}})$. Thus it suffices to show that given two maps $\alpha_i : \mathcal{E}_i \to \mathcal{B}$ as above, there is a canonical quasi-isomorphism $NL(\alpha_1) = NL(\alpha_2)$ in $D(\mathcal{B})$. To see this set $\mathcal{E} = \mathcal{E}_1 \amalg \mathcal{E}_2$ and $\alpha = \alpha_1 \amalg \alpha_2 : \mathcal{E} \to \mathcal{B}$. Set $\mathcal{J}_i = \mathrm{Ker}(\mathcal{A}[\mathcal{E}_i] \to \mathcal{B})$ and $\mathcal{J} = \mathrm{Ker}(\mathcal{A}[\mathcal{E}] \to \mathcal{B})$. We obtain maps $\mathcal{A}[\mathcal{E}_i] \to \mathcal{A}[\mathcal{E}]$ which send $\mathcal{J}_i$ into $\mathcal{J}$. Thus we obtain canonical maps of complexes

$$NL(\alpha_i) \longrightarrow NL(\alpha)$$

and it suffices to show these maps are quasi-isomorphism. To see this we argue as follows. First, observe that $H_0(NL(\alpha_i)) = \Omega_{\mathcal{B}/\mathcal{A}}$ and $H_0(NL(\alpha)) = \Omega_{\mathcal{B}/\mathcal{A}}$ by Lemma 18.32.8 hence the map is an isomorphism on 0th homology sheaves. Similarly, we claim that $H_1(NL(\alpha_i))$ and $H_1(NL(\alpha))$ are the sheaves associated to the presheaf $U \mapsto H_1(L_{\mathcal{B}(U)/\mathcal{A}(U)})$. If the claim holds, then the proof is finished.

Proof of the claim. Let $\alpha : \mathcal{E} \to \mathcal{B}$ be as above. Let $\mathcal{B}' \subset \mathcal{B}$ be the subpresheaf of $\mathcal{A}$-algebras whose value on U is the image of $\mathcal{A}(U)[\mathcal{E}(U)] \to \mathcal{B}(U)$. Let $\mathcal{I}'$ be the presheaf whose value on U is the kernel of $\mathcal{A}(U)[\mathcal{E}(U)] \to \mathcal{B}(U)$. Then $\mathcal{I}$ is the

sheafification of $\mathcal{I}'$ and $\mathcal{B}$ is the sheafification of $\mathcal{B}'$. Similarly, $H_1(NL(\alpha))$ is the sheafification of the presheaf

$$U \longmapsto \mathrm{Ker}(\mathcal{I}'(U)/\mathcal{I}'(U)^2 \to \Omega_{\mathcal{A}(U)[\mathcal{E}(U)]/\mathcal{A}(U)} \otimes_{\mathcal{A}(U)[\mathcal{E}(U)]} \mathcal{B}'(U))$$

by Lemma 18.32.4. By Algebra, Lemma 10.127.2 we conclude $H_1(NL(\alpha))$ is the sheaf associated to the presheaf $U \mapsto H_1(L_{\mathcal{B}'(U)/\mathcal{A}(U)})$. Thus we have to show that the maps $H_1(L_{\mathcal{B}'(U)/\mathcal{A}(U)}) \to H_1(L_{\mathcal{B}(U)/\mathcal{A}(U)})$ induce an isomorphism $\mathcal{H}'_1 \to \mathcal{H}_1$ of sheafifications.

Injectivity of $\mathcal{H}'_1 \to \mathcal{H}_1$. Let $f \in H_1(L_{\mathcal{B}'(U)/\mathcal{A}(U)})$ map to zero in $\mathcal{H}_1(U)$. To show: f maps to zero in $\mathcal{H}'_1(U)$. The assumption means there is a covering $\{U_i \to U\}$ such that f maps to zero in $H_1(L_{\mathcal{B}(U_i)/\mathcal{A}(U_i)})$ for all i. Replace U by U_i to get to the point where f maps to zero in $H_1(L_{\mathcal{B}(U)/\mathcal{A}(U)})$. By Algebra, Lemma 10.127.9 we can find a finitely generated subalgebra $\mathcal{B}'(U) \subset B \subset \mathcal{B}(U)$ such that f maps to zero in $H_1(L_{B/\mathcal{A}(U)})$. Since $\mathcal{B} = (\mathcal{B}')^\#$ we can find a covering $\{U_i \to U\}$ such that $B \to \mathcal{B}(U_i)$ factors through $\mathcal{B}'(U_i)$. Hence f maps to zero in $H_1(L_{\mathcal{B}'(U_i)/\mathcal{A}(U_i)})$ as desired.

The surjectivity of $\mathcal{H}'_1 \to \mathcal{H}_1$ is proved in exactly the same way. □

Lemma 18.34.3. *Let $f : Sh(\mathcal{C}) \to Sh(\mathcal{D})$ be morphism of topoi. Let $\mathcal{A} \to \mathcal{B}$ be a homomorphism of sheaves of rings on $\mathcal{D}$. Then $f^{-1} NL_{\mathcal{B}/\mathcal{A}} = NL_{f^{-1}\mathcal{B}/f^{-1}\mathcal{A}}$.*

Proof. Omitted. Hint: Use Lemma 18.32.5. □

The cotangent complex of a morphism of ringed topoi is defined in terms of the cotangent complex we defined above.

Definition 18.34.4. Let $X = (Sh(\mathcal{C}), \mathcal{O})$ and $Y = (Sh(\mathcal{C}'), \mathcal{O}')$ be ringed topoi. Let $(f, f^\sharp) : X \to Y$ be a morphism of ringed topoi. The *naive cotangent complex* $NL_f = NL_{X/Y}$ of the given morphism of ringed topoi is $NL_{\mathcal{O}_X/f^{-1}\mathcal{O}_Y}$. We sometimes write $NL_{X/Y} = NL_{\mathcal{O}_X/\mathcal{O}_Y}$.

18.35. Stalks of modules

We have to be a bit careful when taking stalks at points, since the colimit defining a stalk (see Sites, Equation 7.31.1.1) may not be filtered[3]. On the other hand, by definition of a point of a site the stalk functor is exact and commutes with arbitrary colimits. In other words, it behaves exactly as if the colimit were filtered.

Lemma 18.35.1. *Let $\mathcal{C}$ be a site. Let p be a point of $\mathcal{C}$.*

1. *We have $(\mathcal{F}^\#)_p = \mathcal{F}_p$ for any presheaf of sets on $\mathcal{C}$.*
2. *The stalk functor $Sh(\mathcal{C}) \to Sets$, $\mathcal{F} \mapsto \mathcal{F}_p$ is exact (see Categories, Definition 4.23.1) and commutes with arbitrary colimits.*
3. *The stalk functor $PSh(\mathcal{C}) \to Sets$, $\mathcal{F} \mapsto \mathcal{F}_p$ is exact (see Categories, Definition 4.23.1) and commutes with arbitrary colimits.*

Proof. By Sites, Lemma 7.31.5 we have (1). By Sites, Lemmas 7.31.4 we see that $PSh(\mathcal{C}) \to Sets$, $\mathcal{F} \mapsto \mathcal{F}_p$ is a left adjoint, and by Sites, Lemma 7.31.5 we see the same thing for $PSh(\mathcal{C}) \to Sets$, $\mathcal{F} \mapsto \mathcal{F}_p$. Hence the stalk functor commutes with arbitrary colimits (see Categories, Lemma 4.24.3). It follows from the definition of

[3]Of course in almost any naturally occurring case the colimit is filtered and some of the discussion in this section may be simplified.

a point of a site, see Sites, Definition 7.31.2 that $Sh(\mathcal{C}) \to Sets$, $\mathcal{F} \mapsto \mathcal{F}_p$ is exact. Since sheafification is exact (Sites, Lemma 7.10.14) it follows that $PSh(\mathcal{C}) \to Sets$, $\mathcal{F} \mapsto \mathcal{F}_p$ is exact. □

In particular, since the stalk functor $\mathcal{F} \mapsto \mathcal{F}_p$ on presheaves commutes with all finite limits and colimits we may apply the reasoning of the proof of Sites, Proposition 7.43.3. The result of such an argument is that if $\mathcal{F}$ is a (pre)sheaf of algebraic structures listed in Sites, Proposition 7.43.3 then the stalk $\mathcal{F}_p$ is naturally an algebraic structure of the same kind. Let us explain this in detail when $\mathcal{F}$ is an abelian presheaf. In this case the addition map $+ : \mathcal{F} \times \mathcal{F} \to \mathcal{F}$ induces a map

$$+ : \mathcal{F}_p \times \mathcal{F}_p = (\mathcal{F} \times \mathcal{F})_p \longrightarrow \mathcal{F}_p$$

where the equal sign uses that stalk functor on presheaves of sets commutes with finite limits. This defines a group structure on the stalk $\mathcal{F}_p$. In this way we obtain our stalk functor

$$PAb(\mathcal{C}) \longrightarrow Ab, \quad \mathcal{F} \longmapsto \mathcal{F}_p$$

By construction the underlying set of $\mathcal{F}_p$ is the stalk of the underlying presheaf of sets. This also defines our stalk functor for sheaves of abelian groups by precomposing with the inclusion $Ab(\mathcal{C}) \subset PAb(\mathcal{C})$.

Lemma 18.35.2. *Let $\mathcal{C}$ be a site. Let p be a point of $\mathcal{C}$.*

1. *The functor $Ab(\mathcal{C}) \to Ab$, $\mathcal{F} \mapsto \mathcal{F}_p$ is exact.*
2. *The stalk functor $PAb(\mathcal{C}) \to Ab$, $\mathcal{F} \mapsto \mathcal{F}_p$ is exact.*
3. *For $\mathcal{F} \in \mathrm{Ob}(PAb(\mathcal{C}))$ we have $\mathcal{F}_p = \mathcal{F}_p^\#$.*

Proof. This is formal from the results of Lemma 18.35.1 and the construction of the stalk functor above. □

Next, we turn to the case of sheaves of modules. Let $(\mathcal{C}, \mathcal{O})$ be a ringed site. (It suffices for the discussion that $\mathcal{O}$ be a presheaf of rings.) Let $\mathcal{F}$ be a presheaf of $\mathcal{O}$-modules. Let p be a point of $\mathcal{C}$. In this case we get a map

$$\cdot : \mathcal{O}_p \times \mathcal{O}_p = (\mathcal{O} \times \mathcal{O})_p \longrightarrow \mathcal{O}_p$$

which is the stalk of the multiplication map and

$$\cdot : \mathcal{O}_p \times \mathcal{F}_p = (\mathcal{O} \times \mathcal{F})_p \longrightarrow \mathcal{F}_p$$

which is the stalk of the multiplication map. We omit the verification that this defines a ring structure on $\mathcal{O}_p$ and an $\mathcal{O}_p$-module structure on $\mathcal{F}_p$. In this way we obtain a functor

$$PMod(\mathcal{O}) \longrightarrow Mod(\mathcal{O}_p), \quad \mathcal{F} \longmapsto \mathcal{F}_p$$

By construction the underlying set of $\mathcal{F}_p$ is the stalk of the underlying presheaf of sets. This also defines our stalk functor for sheaves of $\mathcal{O}$-modules by precomposing with the inclusion $Mod(\mathcal{O}) \subset PMod(\mathcal{O})$.

Lemma 18.35.3. *Let $(\mathcal{C}, \mathcal{O})$ be a ringed site. Let p be a point of $\mathcal{C}$.*

1. *The functor $Mod(\mathcal{O}) \to Mod(\mathcal{O}_p)$, $\mathcal{F} \mapsto \mathcal{F}_p$ is exact.*
2. *The stalk functor $PMod(\mathcal{O}) \to Mod(\mathcal{O}_p)$, $\mathcal{F} \mapsto \mathcal{F}_p$ is exact.*
3. *For $\mathcal{F} \in \mathrm{Ob}(PMod(\mathcal{O}))$ we have $\mathcal{F}_p = \mathcal{F}_p^\#$.*

Proof. This is formal from the results of Lemma 18.35.2, the construction of the stalk functor above, and Lemma 18.14.1. □

Lemma 18.35.4. *Let $(f, f^\sharp) : (Sh(\mathcal{C}), \mathcal{O}_\mathcal{C}) \to (Sh(\mathcal{D}), \mathcal{O}_\mathcal{D})$ be a morphism of ringed topoi or ringed sites. Let p be a point of $\mathcal{C}$ or $Sh(\mathcal{C})$ and set $q = f \circ p$. Then*

$$(f^*\mathcal{F})_p = \mathcal{F}_q \otimes_{\mathcal{O}_{\mathcal{D},q}} \mathcal{O}_{\mathcal{C},p}$$

for any $\mathcal{O}_\mathcal{D}$-module $\mathcal{F}$.

Proof. We have

$$f^*\mathcal{F} = f^{-1}\mathcal{F} \otimes_{f^{-1}\mathcal{O}_\mathcal{D}} \mathcal{O}_\mathcal{C}$$

by definition. Since taking stalks at p (i.e., applying p^{-1}) commutes with $\otimes$ by Lemma 18.26.1 we win by the relation between the stalk of pullbacks at p and stalks at q explained in Sites, Lemma 7.33.1 or Sites, Lemma 7.33.2. □

18.36. Skyscraper sheaves

Let p be a point of a site $\mathcal{C}$ or a topos $Sh(\mathcal{C})$. In this section we study the exactness properties of the functor which associates to an abelian group A the skyscraper sheaf p_*A. First, recall that $p_* : Sets \to Sh(\mathcal{C})$ has a lot of exactness properties, see Sites, Lemmas 7.31.9 and 7.31.10.

Lemma 18.36.1. *Let $\mathcal{C}$ be a site. Let p be a point of $\mathcal{C}$ or of its associated topos.*

(1) *The functor $p_* : Ab \to Ab(\mathcal{C})$, $A \mapsto p_*A$ is exact.*
(2) *There is a functorial direct sum decomposition*

$$p^{-1}p_*A = A \oplus I(A)$$

for $A \in \mathrm{Ob}(Ab)$.

Proof. By Sites, Lemma 7.31.9 there are functorial maps $A \to p^{-1}p_*A \to A$ whose composition equals id_A. Hence a functorial direct sum decomposition as in (2) with $I(A)$ the kernel of the adjunction map $p^{-1}p_*A \to A$. The functor p_* is left exact by Lemma 18.14.3. The functor p_* transforms surjections into surjections by Sites, Lemma 7.31.10. Hence (1) holds. □

To do the same thing for sheaves of modules, suppose given a point p of a ringed topos $(Sh(\mathcal{C}), \mathcal{O})$. Recall that p^{-1} is just the stalk functor. Hence we can think of p as a morphism of ringed topoi

$$(p, \mathrm{id}_{\mathcal{O}_p}) : (Sh(pt), \mathcal{O}_p) \longrightarrow (Sh(\mathcal{C}), \mathcal{O}).$$

Thus we get a pullback functor $p^* : Mod(\mathcal{O}) \to Mod(\mathcal{O}_p)$ which equals the stalk functor, and which we discussed in Lemma 18.35.3. In this section we consider the functor $p_* : Mod(\mathcal{O}_p) \to Mod(\mathcal{O})$.

Lemma 18.36.2. *Let $(Sh(\mathcal{C}), \mathcal{O})$ be a ringed topos. Let p be a point of the topos $Sh(\mathcal{C})$.*

(1) *The functor $p_* : Mod(\mathcal{O}_p) \to Mod(\mathcal{O})$, $M \mapsto p_*M$ is exact.*
(2) *There is a functorial direct sum decomposition of $\mathcal{O}_p$-modules*

$$p^{-1}p_*M = M \oplus I(M)$$

for M a $\mathcal{O}_p$-module.

Proof. This follows immediately from the corresponding result for abelian sheaves in Lemma 18.36.1. □

Example 18.36.3. Let G be a group. Consider the site $\mathcal{T}_G$ and its point p, see Sites, Example 7.32.6. Let R be a ring with a G-action which corresponds to a sheaf of rings $\mathcal{O}$ on $\mathcal{T}_G$. Then $\mathcal{O}_p = R$ where we forget the G-action. In this case $p^{-1}p_*M = \mathrm{Map}(G, M)$ and $I(M) = \{f : G \to M \mid f(1_G) = 0\}$ and $M \to \mathrm{Map}(G, M)$ assigns to $m \in M$ the constant function with value m.

18.37. Localization and points

Lemma 18.37.1. *Let $(\mathcal{C}, \mathcal{O})$ be a ringed site. Let p be a point of $\mathcal{C}$. Let U be an object of $\mathcal{C}$. For $\mathcal{G}$ in $Mod(\mathcal{O}_U)$ we have*

$$(j_{U!}\mathcal{G})_p = \bigoplus\nolimits_q \mathcal{G}_q$$

where the coproduct is over the points q of $\mathcal{C}/U$ lying over p, see Sites, Lemma 7.34.2.

Proof. We use the description of $j_{U!}\mathcal{G}$ as the sheaf associated to the presheaf $V \mapsto \bigoplus_{\varphi \in \mathrm{Mor}_{\mathcal{C}}(V,U)} \mathcal{G}(V/_\varphi U)$ of Lemma 18.19.2. The stalk of $j_{U!}\mathcal{G}$ at p is equal to the stalk of this presheaf, see Lemma 18.35.3. Let $u : \mathcal{C} \to Sets$ be the functor corresponding to p (see Sites, Section 7.31). Hence we see that

$$(j_{U!}\mathcal{G})_p = \mathrm{colim}_{(V,y)} \bigoplus\nolimits_{\varphi : V \to U} \mathcal{G}(V/_\varphi U)$$

where the colimit is taken in the category of abelian groups. To a quadruple (V, y, φ, s) occurring in this colimit, we can assign $x = u(\varphi)(y) \in u(U)$. Hence we obtain

$$(j_{U!}\mathcal{G})_p = \bigoplus\nolimits_{x \in u(U)} \mathrm{colim}_{(\varphi : V \to U, y),\ u(\varphi)(y) = x} \mathcal{G}(V/_\varphi U).$$

This is equal to the expression of the lemma by the description of the points q lying over x in Sites, Lemma 7.34.2. □

Remark 18.37.2. Warning: The result of Lemma 18.37.1 has no analogue for $j_{U,*}$.

18.38. Pullbacks of flat modules

The pullback of a flat module along a morphism of ringed topoi is flat. This is quite tricky to prove, except when there are enough points. Here we prove it only in this case and we will add the general case if we ever need it.

Lemma 18.38.1. *Let $(\mathcal{C}, \mathcal{O})$ be a ringed site. Let p be a point of $\mathcal{C}$. If $\mathcal{F}$ is a flat $\mathcal{O}$-module, then $\mathcal{F}_p$ is a flat $\mathcal{O}_p$-module.*

Proof. Let M be an $\mathcal{O}_p$-module. Then

$$\begin{aligned}(p_*M \otimes_{\mathcal{O}} \mathcal{F})_p &= p^{-1}(p_*M \otimes_{\mathcal{O}} \mathcal{F}) \\ &= p^{-1}p_*M \otimes_{\mathcal{O}_p} \mathcal{F}_p \\ &= M \otimes_{\mathcal{O}_p} \mathcal{F}_p \oplus I(M) \otimes_{\mathcal{O}_p} \mathcal{F}_p\end{aligned}$$

where we have used the description of the stalk functor as a pullback, Lemma 18.26.1, and Lemma 18.36.2. Since p_* is exact by Lemma 18.36.2, it is clear that if $\mathcal{F}$ is flat, then also the functor $M \mapsto M \otimes_{\mathcal{O}_p} \mathcal{F}_p$ is exact, i.e., $\mathcal{F}_p$ is flat. □

Lemma 18.38.2. *Let $(\mathcal{C}, \mathcal{O})$ be a ringed site. Let $\mathcal{F}$ be a sheaf of $\mathcal{O}$-modules. Let $\{p_i\}_{i \in I}$ be a conservative family of points of $\mathcal{C}$. Then $\mathcal{F}$ is flat if and only if $\mathcal{F}_{p_i}$ is a flat $\mathcal{O}_{p_i}$-module for all $i \in I$.*

Proof. By Lemma 18.38.1 we see one of the implications. For the converse, use that $(\mathcal{F} \otimes_{\mathcal{O}} \mathcal{G})_p = \mathcal{F}_p \otimes_{\mathcal{O}_p} \mathcal{G}_p$ by Lemma 18.26.1 (as taking stalks at p is given by p^{-1}) and Lemma 18.14.4. □

Lemma 18.38.3. *Let $(f, f^\sharp) : (Sh(\mathcal{C}), \mathcal{O}_\mathcal{C}) \to (Sh(\mathcal{D}), \mathcal{O}_\mathcal{D})$ be a morphism of ringed topoi or ringed sites. Assume $\mathcal{C}$ has enough points[4]. Then $f^*\mathcal{F}$ is a flat $\mathcal{O}_\mathcal{C}$-module whenever $\mathcal{F}$ is a flat $\mathcal{O}_\mathcal{D}$-module.*

Proof. Let p be a point of $\mathcal{C}$ and set $q = f \circ p$. Then

$$(f^*\mathcal{F})_p = \mathcal{F}_q \otimes_{\mathcal{O}_{\mathcal{D},q}} \mathcal{O}_{\mathcal{C},p}$$

by Lemma 18.35.4. Hence if $\mathcal{F}$ is flat, then $\mathcal{F}_q$ is a flat $\mathcal{O}_{\mathcal{D},q}$-module by Lemma 18.38.1 and hence by Algebra, Lemma 10.36.6 we see that $(f^*\mathcal{F})_p$ is a flat $\mathcal{O}_{\mathcal{C},p}$-module. This implies that $f^*\mathcal{F}$ is a flat $\mathcal{O}_\mathcal{C}$-module by Lemma 18.38.2. □

18.39. Locally ringed topoi

A reference for this section is [**AGV71**, Exposé IV, Exercice 13.9].

Lemma 18.39.1. *Let $(\mathcal{C}, \mathcal{O})$ be a ringed site. The following are equivalent*

1. *For every object U of $\mathcal{C}$ and $f \in \mathcal{O}(U)$ there exists a covering $\{U_j \to U\}$ such that for each j either $f|_{U_j}$ is invertible or $(1-f)|_{U_j}$ is invertible.*
2. *For every object U of $\mathcal{C}$ and $f_1, \ldots, f_n \in \mathcal{O}(U)$ which generate the unit ideal in $\mathcal{O}(U)$ there exists a covering $\{U_j \to U\}$ such that for each j there exists an i such that $f_i|_{U_j}$ is invertible.*
3. *The map of sheaves of sets*

$$(\mathcal{O} \times \mathcal{O}) \amalg (\mathcal{O} \times \mathcal{O}) \longrightarrow \mathcal{O} \times \mathcal{O}$$

which maps (f, a) in the first component to (f, af) and (f, b) in the second component to $(f, b(1-f))$ is surjective.

Proof. It is clear that (2) implies (1). To show that (1) implies (2) we argue by induction on n. The first case is $n = 2$ (since $n = 1$ is trivial). In this case we have $a_1 f_1 + a_2 f_2 = 1$ for some $a_1, a_2 \in \mathcal{O}(U)$. By assumption we can find a covering $\{U_j \to U\}$ such that for each j either $a_1 f_1|_{U_j}$ is invertible or $a_2 f_2|_{U_j}$ is invertible. Hence either $f_1|_{U_j}$ is invertible or $f_2|_{U_j}$ is invertible as desired. For $n > 2$ we have $a_1 f_1 + \ldots + a_n f_n = 1$ for some $a_1, \ldots, a_n \in \mathcal{O}(U)$. By the case $n = 2$ we see that we have some covering $\{U_j \to U\}_{j \in J}$ such that for each j either $f_n|_{U_j}$ is invertible or $a_1 f_1 + \ldots + a_{n-1} f_{n-1}|_{U_j}$ is invertible. Say the first case happens for $j \in J_n$. Set $J' = J \setminus J_n$. By induction hypothesis, for each $j \in J'$ we can find a covering $\{U_{jk} \to U_j\}_{k \in K_j}$ such that for each $k \in K_j$ there exists an $i \in \{1, \ldots, n-1\}$ such that $f_i|_{U_{jk}}$ is invertible. By the axioms of a site the family of morphisms $\{U_j \to U\}_{j \in J_n} \cup \{U_{jk} \to U\}_{j \in J', k \in K_j}$ is a covering which has the desired property.

Assume (1). To see that the map in (3) is surjective, let (f, c) be a section of $\mathcal{O} \times \mathcal{O}$ over U. By assumption there exists a covering $\{U_j \to U\}$ such that for each j either f or $1 - f$ restricts to an invertible section. In the first case we can take $a = c|_{U_j}(f|_{U_j})^{-1}$, and in the second case we can take $b = c|_{U_j}(1 - f|_{U_j})^{-1}$. Hence (f, c) is in the image of the map on each of the members. Conversely, assume (3) holds. For any U and $f \in \mathcal{O}(U)$ there exists a covering $\{U_j \to U\}$ of U such that the section $(f, 1)|_{U_j}$ is in the image of the map in (3) on sections over U_j. This

[4] This assumption is not necessary, see introduction to this section.

means precisely that either f or $1-f$ restricts to an invertible section over U_j, and we see that (1) holds. □

Lemma 18.39.2. *Let $(\mathcal{C}, \mathcal{O})$ be a ringed site. Consider the following conditions*

(1) *For every object U of $\mathcal{C}$ and $f \in \mathcal{O}(U)$ there exists a covering $\{U_j \to U\}$ such that for each j either $f|_{U_j}$ is invertible or $(1-f)|_{U_j}$ is invertible.*
(2) *For every point p of $\mathcal{C}$ the stalk $\mathcal{O}_p$ is either the zero ring or a local ring.*

We always have (1) $\Rightarrow$ (2). If $\mathcal{C}$ has enough points then (1) and (2) are equivalent.

Proof. Assume (1). Let p be a point of $\mathcal{C}$ given by a functor $u : \mathcal{C} \to Sets$. Let $f_p \in \mathcal{O}_p$. Since $\mathcal{O}_p$ is computed by Sites, Equation (7.31.1.1) we may represent f_p by a triple (U, x, f) where $x \in U(U)$ and $f \in \mathcal{O}(U)$. By assumption there exists a covering $\{U_i \to U\}$ such that for each i either f or $1-f$ is invertible on U_i. Because u defines a point of the site we see that for some i there exists an $x_i \in u(U_i)$ which maps to $x \in u(U)$. By the discussion surrounding Sites, Equation (7.31.1.1) we see that (U, x, f) and $(U_i, x_i, f|_{U_i})$ define the same element of $\mathcal{O}_p$. Hence we conclude that either f_p or $1 - f_p$ is invertible. Thus $\mathcal{O}_p$ is a ring such that for every element a either a or $1-a$ is invertible. This means that $\mathcal{O}_p$ is either zero or a local ring, see Algebra, Lemma 10.17.2.

Assume (2) and assume that $\mathcal{C}$ has enough points. Consider the map of sheaves of sets

$$\mathcal{O} \times \mathcal{O} \amalg \mathcal{O} \times \mathcal{O} \longrightarrow \mathcal{O} \times \mathcal{O}$$

of Lemma 18.39.1 part (3). For any local ring R the corresponding map $(R \times R) \amalg (R \times R) \to R \times R$ is surjective, see for example Algebra, Lemma 10.17.2. Since each $\mathcal{O}_p$ is a local ring or zero the map is surjective on stalks. Hence, by our assumption that $\mathcal{C}$ has enough points it is surjective and we win. □

In Modules, Section 17.2 we pointed out how in a ringed space $(X, \mathcal{O}_X)$ there can be an open subspace over which the structure sheaf is zero. To prevent this we can require the sections 1 and 0 to have different values in every stalk of the space X. In the setting of ringed topoi and ringed sites the condition is that

$$\emptyset^\# \longrightarrow \text{Equalizer}(0, 1 : * \longrightarrow \mathcal{O}) \tag{18.39.2.1}$$

is an isomorphism of sheaves. Here $*$ is the singleton sheaf, resp. $\emptyset^\#$ is the "empty sheaf", i.e., the final, resp. initial object in the category of sheaves, see Sites, Example 7.10.2, resp. Section 7.41. In other words, the condition is that whenever $U \in \text{Ob}(\mathcal{C})$ is not sheaf theoretically empty, then $1, 0 \in \mathcal{O}(U)$ are not equal. Let us state the obligatory lemma.

Lemma 18.39.3. *Let $(\mathcal{C}, \mathcal{O})$ be a ringed site. Consider the statements*

(1) *(18.39.2.1) is an isomorphism, and*
(2) *for every point p of $\mathcal{C}$ the stalk $\mathcal{O}_p$ is not the zero ring.*

We always have (1) $\Rightarrow$ (2) and if $\mathcal{C}$ has enough points then (1) $\Leftrightarrow$ (2).

Proof. Omitted. □

Lemmas 18.39.1, 18.39.2, and 18.39.3 motivate the following definition.

Definition 18.39.4. A ringed site $(\mathcal{C}, \mathcal{O})$ is said to be *locally ringed site* if (18.39.2.1) is an isomorphism, and the equivalent properties of Lemma 18.39.1 are satisfied.

In [**AGV71**, Exposé IV, Exercice 13.9] the condition that (18.39.2.1) be an isomorphism is missing leading to a slightly different notion of a locally ringed site and locally ringed topos. As we are motivated by the notion of a locally ringed space we decided to add this condition (see explanation above).

Lemma 18.39.5. *Being a locally ringed site is an intrinsic property. More precisely,*

(1) *if $f : Sh(\mathcal{C}') \to Sh(\mathcal{C})$ is a morphism of topoi and $(\mathcal{C}, \mathcal{O})$ is a locally ringed site, then $(\mathcal{C}', f^{-1}\mathcal{O})$ is a locally ringed site, and*
(2) *if $(f, f^\sharp) : (Sh(\mathcal{C}'), \mathcal{O}') \to (Sh(\mathcal{C}), \mathcal{O})$ is an equivalence of ringed topoi, then $(\mathcal{C}, \mathcal{O})$ is locally ringed if and only if $(\mathcal{C}', \mathcal{O}')$ is locally ringed.*

Proof. It is clear that (2) follows from (1). To prove (1) note that as f^{-1} is exact we have $f^{-1}* = *$, $f^{-1}\emptyset^\# = \emptyset^\#$, and f^{-1} commutes with products, equalizers and transforms isomorphisms and surjections into isomorphisms and surjections. Thus f^{-1} transforms the isomorphism (18.39.2.1) into its analogue for $f^{-1}\mathcal{O}$ and transforms the surjection of Lemma 18.39.1 part (3) into the corresponding surjection for $f^{-1}\mathcal{O}$. □

In fact Lemma 18.39.5 part (2) is the analogue of Schemes, Lemma 25.2.2. It assures us that the following definition makes sense.

Definition 18.39.6. A ringed topos $(Sh(\mathcal{C}), \mathcal{O})$ is said to be *locally ringed* if the underlying ringed site $(\mathcal{C}, \mathcal{O})$ is locally ringed.

Next, we want to work out what it means to have a morphism of locally ringed spaces. In order to do this we have the following lemma.

Lemma 18.39.7. *Let $(f, f^\sharp) : (Sh(\mathcal{C}), \mathcal{O}_\mathcal{C}) \to (Sh(\mathcal{D}), \mathcal{O}_\mathcal{D})$ be a morphism of ringed topoi. Consider the following conditions*

(1) *The diagram of sheaves*
$$\begin{array}{ccc} f^{-1}(\mathcal{O}_\mathcal{D}^*) & \xrightarrow{f^\sharp} & \mathcal{O}_\mathcal{C}^* \\ \downarrow & & \downarrow \\ f^{-1}(\mathcal{O}_\mathcal{D}) & \xrightarrow{f^\sharp} & \mathcal{O}_\mathcal{C} \end{array}$$
is cartesian.
(2) *For any point p of $\mathcal{C}$, setting $q = f \circ p$, the diagram*
$$\begin{array}{ccc} \mathcal{O}_{\mathcal{D},q}^* & \longrightarrow & \mathcal{O}_{\mathcal{C},p}^* \\ \downarrow & & \downarrow \\ \mathcal{O}_{\mathcal{D},q} & \longrightarrow & \mathcal{O}_{\mathcal{C},p} \end{array}$$
of sets is cartesian.

We always have (1) ⇒ (2). If $\mathcal{C}$ has enough points then (1) and (2) are equivalent. If $(Sh(\mathcal{C}), \mathcal{O}_\mathcal{C})$ and $(Sh(\mathcal{D}), \mathcal{O}_\mathcal{D})$ are locally ringed topoi then (2) is equivalent to

(3) *For any point p of $\mathcal{C}$, setting $q = f \circ p$, the ring map $\mathcal{O}_{\mathcal{D},q} \to \mathcal{O}_{\mathcal{C},p}$ is a local ring map.*

In fact, properties (2), or (3) for a conservative family of points implies (1).

Proof. This lemma proves itself, in other words, it follows by unwinding the definitions. □

Definition 18.39.8. Let $(f, f^\sharp) : (Sh(\mathcal{C}), \mathcal{O}_\mathcal{C}) \to (Sh(\mathcal{D}), \mathcal{O}_\mathcal{D})$ be a morphism of ringed topoi. Assume $(Sh(\mathcal{C}), \mathcal{O}_\mathcal{C})$ and $(Sh(\mathcal{D}), \mathcal{O}_\mathcal{D})$ are locally ringed topoi. We say that $(f, f^\sharp)$ is a *morphism of locally ringed topoi* if and only if the diagram of sheaves

$$\begin{array}{ccc} f^{-1}(\mathcal{O}_\mathcal{D}^*) & \xrightarrow[f^\sharp]{} & \mathcal{O}_\mathcal{C}^* \\ \downarrow & & \downarrow \\ f^{-1}(\mathcal{O}_\mathcal{D}) & \xrightarrow{f^\sharp} & \mathcal{O}_\mathcal{C} \end{array}$$

(see Lemma 18.39.7) is cartesian. If $(f, f^\sharp)$ is a morphism of ringed sites, then we say that it is a *morphism of locally ringed sites* if the associated morphism of ringed topoi is a morphism of locally ringed topoi.

It is clear that an isomorphism of ringed topoi between locally ringed topoi is automatically an isomorphism of locally ringed topoi.

Lemma 18.39.9. *Let $(f, f^\sharp) : (Sh(\mathcal{C}_1), \mathcal{O}_1) \to (Sh(\mathcal{C}_2), \mathcal{O}_2)$ and $(g, g^\sharp) : (Sh(\mathcal{C}_2), \mathcal{O}_2) \to (Sh(\mathcal{C}_3), \mathcal{O}_3)$ be morphisms of locally ringed topoi. Then the composition $(g, g^\sharp) \circ (f, f^\sharp)$ (see Definition 18.7.1) is also a morphism of locally ringed topoi.*

Proof. Omitted. □

Lemma 18.39.10. *If $f : Sh(\mathcal{C}') \to Sh(\mathcal{C})$ is a morphism of topoi. If $\mathcal{O}$ is a sheaf of rings on $\mathcal{C}$, then*

$$f^{-1}(\mathcal{O}^*) = (f^{-1}\mathcal{O})^*.$$

In particular, if $\mathcal{O}$ turns $\mathcal{C}$ into a locally ringed site, then setting $f^\sharp = id$ the morphism of ringed topoi

$$(f, f^\sharp) : (Sh(\mathcal{C}'), f^{-1}\mathcal{O}) \to (Sh(\mathcal{C}, \mathcal{O})$$

is a morphism of locally ringed topoi.

Proof. Note that the diagram

$$\begin{array}{ccc} \mathcal{O}^* & \longrightarrow & * \\ {\scriptstyle u\mapsto(u,u^{-1})}\downarrow & & \downarrow{\scriptstyle 1} \\ \mathcal{O} \times \mathcal{O} & \xrightarrow{(a,b)\mapsto ab} & \mathcal{O} \end{array}$$

is cartesian. Since f^{-1} is exact we conclude that

$$\begin{array}{ccc} f^{-1}(\mathcal{O}^*) & \longrightarrow & * \\ {\scriptstyle u\mapsto(u,u^{-1})}\downarrow & & \downarrow{\scriptstyle 1} \\ f^{-1}\mathcal{O} \times f^{-1}\mathcal{O} & \xrightarrow{(a,b)\mapsto ab} & f^{-1}\mathcal{O} \end{array}$$

is cartesian which implies the first assertion. For the second, note that $(\mathcal{C}', f^{-1}\mathcal{O})$ is a locally ringed site by Lemma 18.39.5 so that the assertion makes sense. Now the first part implies that the morphism is a morphism of locally ringed topoi. □

Lemma 18.39.11. *Localization of locally ringed sites and topoi.*

(1) *Let $(\mathcal{C}, \mathcal{O})$ be a locally ringed site. Let U be an object of $\mathcal{C}$. Then the localization $(\mathcal{C}/U, \mathcal{O}_U)$ is a locally ringed site, and the localization morphism*
$$(j_U, j_U^\sharp) : (Sh(\mathcal{C}/U), \mathcal{O}_U) \to (Sh(\mathcal{C}), \mathcal{O})$$
is a morphism of locally ringed topoi.

(2) *Let $(\mathcal{C}, \mathcal{O})$ be a locally ringed site. Let $f : V \to U$ be a morphism of $\mathcal{C}$. Then the morphism*
$$(j, j^\sharp) : (Sh(\mathcal{C}/V), \mathcal{O}_V) \to (Sh(\mathcal{C}/U), \mathcal{O}_U)$$
of Lemma 18.19.4 is a morphism of locally ringed topoi.

(3) *Let $(f, f^\sharp) : (\mathcal{C}, \mathcal{O}) \longrightarrow (\mathcal{D}, \mathcal{O}')$ be a morphism of locally ringed sites where f is given by the continuous functor $u : \mathcal{D} \to \mathcal{C}$. Let V be an object of $\mathcal{D}$ and let $U = u(V)$. Then the morphism*
$$(f', (f')^\sharp) : (Sh(\mathcal{C}/U), \mathcal{O}_U) \to (Sh(\mathcal{D}/V), \mathcal{O}'_V)$$
of Lemma 18.20.1 is a morphism of locally ringed sites.

(4) *Let $(f, f^\sharp) : (\mathcal{C}, \mathcal{O}) \longrightarrow (\mathcal{D}, \mathcal{O}')$ be a morphism of locally ringed sites where f is given by the continuous functor $u : \mathcal{D} \to \mathcal{C}$. Let $V \in \mathrm{Ob}(\mathcal{D})$, $U \in \mathrm{Ob}(\mathcal{C})$, and $c : U \to u(V)$. Then the morphism*
$$(f_c, (f_c)^\sharp) : (Sh(\mathcal{C}/U), \mathcal{O}_U) \to (Sh(\mathcal{D}/V), \mathcal{O}'_V)$$
of Lemma 18.20.2 is a morphism of locally ringed topoi.

(5) *Let $(Sh(\mathcal{C}), \mathcal{O})$ be a locally ringed topos. Let $\mathcal{F}$ be a sheaf on $\mathcal{C}$. Then the localization $(Sh(\mathcal{C})/\mathcal{F}, \mathcal{O}_\mathcal{F})$ is a locally ringed topos and the localization morphism*
$$(j_\mathcal{F}, j_\mathcal{F}^\sharp) : (Sh(\mathcal{C})/\mathcal{F}, \mathcal{O}_\mathcal{F}) \to (Sh(\mathcal{C}), \mathcal{O})$$
is a morphism of locally ringed topoi.

(6) *Let $(Sh(\mathcal{C}), \mathcal{O})$ be a locally ringed topos. Let $s : \mathcal{G} \to \mathcal{F}$ be a map of sheaves on $\mathcal{C}$. Then the morphism*
$$(j, j^\sharp) : (Sh(\mathcal{C})/\mathcal{G}, \mathcal{O}_\mathcal{G}) \longrightarrow (Sh(\mathcal{C})/\mathcal{F}, \mathcal{O}_\mathcal{F})$$
of Lemma 18.21.4 is a morphism of locally ringed topoi.

(7) *Let $f : (Sh(\mathcal{C}), \mathcal{O}) \longrightarrow (Sh(\mathcal{D}), \mathcal{O}')$ be a morphism of locally ringed topoi. Let $\mathcal{G}$ be a sheaf on $\mathcal{D}$. Set $\mathcal{F} = f^{-1}\mathcal{G}$. Then the morphism*
$$(f', (f')^\sharp) : (Sh(\mathcal{C})/\mathcal{F}, \mathcal{O}_\mathcal{F}) \longrightarrow (Sh(\mathcal{D})/\mathcal{G}, \mathcal{O}'_\mathcal{G})$$
of Lemma 18.22.1 is a morphism of locally ringed topoi.

(8) *Let $f : (Sh(\mathcal{C}), \mathcal{O}) \longrightarrow (Sh(\mathcal{D}), \mathcal{O}')$ be a morphism of locally ringed topoi. Let $\mathcal{G}$ be a sheaf on $\mathcal{D}$, let $\mathcal{F}$ be a sheaf on $\mathcal{C}$, and let $s : \mathcal{F} \to f^{-1}\mathcal{G}$ be a morphism of sheaves. Then the morphism*
$$(f_s, (f_s)^\sharp) : (Sh(\mathcal{C})/\mathcal{F}, \mathcal{O}_\mathcal{F}) \longrightarrow (Sh(\mathcal{D})/\mathcal{G}, \mathcal{O}'_\mathcal{G})$$
of Lemma 18.22.3 is a morphism of locally ringed topoi.

Proof. Part (1) is clear since $\mathcal{O}_U$ is just the restriction of $\mathcal{O}$, so Lemmas 18.39.5 and 18.39.10 apply. Part (2) is clear as the morphism $(j, j^\sharp)$ is actually a localization of a locally ringed site so (1) applies. Part (3) is clear also since $(f')^\sharp$ is just the restriction of $f^\sharp$ to the topos $Sh(\mathcal{C})/\mathcal{F}$, see proof of Lemma 18.22.1 (hence the diagram of Definition 18.39.8 for the morphism f' is just the restriction of the corresponding diagram for f, and restriction is an exact functor). Part (4)

follows formally on combining (2) and (3). Parts (5), (6), (7), and (8) follow from their counterparts (1), (2), (3), and (4) by enlarging the sites as in Lemma 18.7.2 and translating everything in terms of sites and morphisms of sites using the comparisons of Lemmas 18.21.3, 18.21.5, 18.22.2, and 18.22.4. (Alternatively one could use the same arguments as in the proofs of (1), (2), (3), and (4) to prove (5), (6), (7), and (8) directly.) □

18.40. Lower shriek for modules

In this section we extend the construction of $g_!$ discussed in Section 18.16 to the case of sheaves of modules.

Lemma 18.40.1. *Let $u : \mathcal{C} \to \mathcal{D}$ be a continuous and cocontinuous functor between sites. Denote $g : Sh(\mathcal{C}) \to Sh(\mathcal{O}_\mathcal{D})$ the associated morphism of topoi. Let $\mathcal{O}_\mathcal{D}$ be a sheaf of rings on $\mathcal{D}$. Set $\mathcal{O}_\mathcal{C} = g^{-1}\mathcal{O}_\mathcal{D}$. Hence g becomes a morphism of ringed topoi with $g^* = g^{-1}$. In this case there exists a functor*

$$g_! : Mod(\mathcal{O}_\mathcal{C}) \longrightarrow Mod(\mathcal{O}_\mathcal{D})$$

which is left adjoint to g^*.

Proof. Let U be an object of $\mathcal{C}$. For any $\mathcal{O}_\mathcal{D}$-module $\mathcal{G}$ we have

$$\begin{aligned} \mathrm{Hom}_{\mathcal{O}_\mathcal{C}}(j_{U!}\mathcal{O}_U, g^{-1}\mathcal{G}) &= g^{-1}\mathcal{G}(U) \\ &= \mathcal{G}(u(U)) \\ &= \mathrm{Hom}_{\mathcal{O}_\mathcal{D}}(j_{u(U)!}\mathcal{O}_{u(U)}, \mathcal{G}) \end{aligned}$$

because g^{-1} is described by restriction, see Sites, Lemma 7.20.5. Of course a similar formula holds a direct sum of modules of the form $j_{U!}\mathcal{O}_U$. By Homology, Lemma 12.25.6 and Lemma 18.28.6 we see that $g_!$ exists. □

Remark 18.40.2. Warning! Let $u : \mathcal{C} \to \mathcal{D}$, g, $\mathcal{O}_\mathcal{D}$, and $\mathcal{O}_\mathcal{C}$ be as in Lemma 18.40.1. In general it is **not** the case that the diagram

$$\begin{array}{ccc} Mod(\mathcal{O}_\mathcal{C}) & \xrightarrow{g_!} & Mod(\mathcal{O}_\mathcal{D}) \\ {\scriptstyle forget}\downarrow & & \downarrow{\scriptstyle forget} \\ Ab(\mathcal{C}) & \xrightarrow{g_!^{Ab}} & Ab(\mathcal{D}) \end{array}$$

commutes (here $g_!^{Ab}$ is the one from Lemma 18.16.2). There is a transformation of functors

$$g_!^{Ab} \circ forget \longrightarrow forget \circ g_!$$

From the proof of Lemma 18.40.1 we see that this is an isomorphism if and only if $g_! j_{U!}\mathcal{O}_U = g_!^{Ab} j_{U!}\mathcal{O}_U$ for all objects U of $\mathcal{C}$, in other words, if and only if

$$g_!^{Ab} j_{U!}\mathcal{O}_U = j_{u(U)!}\mathcal{O}_{u(U)}$$

for all objects U of $\mathcal{C}$. Note that for such a U we obtain a commutative diagram

$$\begin{array}{ccc} \mathcal{C}/U & \xrightarrow{u'} & \mathcal{D}/u(U) \\ {\scriptstyle j_U}\downarrow & & \downarrow{\scriptstyle j_{u(U)}} \\ \mathcal{C} & \xrightarrow{u} & \mathcal{D} \end{array}$$

of cocontinuous functors of sites, see Sites, Lemma 7.27.4. Hence we see that $g_! = g_!^{Ab}$ if the canonical map

$$(g')_!^{Ab}\mathcal{O}_U \longrightarrow \mathcal{O}_{u(U)} \tag{18.40.2.1}$$

is an isomorphism for all objects U of $\mathcal{C}$. Here $g' : Sh(\mathcal{C}/U) \to Sh(\mathcal{D}/u(U))$ is the morphism of topoi induced by the cocontinuous functor u'.

18.41. Constant sheaves

Let E be a set and let $\mathcal{C}$ be a site. We will denote $\underline{E}$ the constant sheaf with value E on $\mathcal{C}$. If E is an abelian group, ring, module, etc, then $\underline{E}$ is a sheaf of abelian groups, rings, modules, etc.

Lemma 18.41.1. *Let $\mathcal{C}$ be a site. If $0 \to A \to B \to C \to 0$ is a short exact sequence of abelian groups, then $0 \to \underline{A} \to \underline{B} \to \underline{C} \to 0$ is an exact sequence of abelian sheaves and in fact it is even exact as a sequence of abelian presheaves.*

Proof. Since sheafification is exact it is clear that $0 \to \underline{A} \to \underline{B} \to \underline{C} \to 0$ is an exact sequence of abelian sheaves. Thus $0 \to \underline{A} \to \underline{B} \to \underline{C}$ is an exact sequence of abelian presheaves. To see that $\underline{B} \to \underline{C}$ is surjective, pick a set theoretical section $s : C \to B$. This induces a section $\underline{s} : \underline{C} \to \underline{B}$ of sheaves of sets left inverse to the surjection $\underline{B} \to \underline{C}$. □

Lemma 18.41.2. *Let $\mathcal{C}$ be a site. Let Λ be a ring and let M and Q be Λ-modules. If Q is a finitely presented Λ-module, then we have $\underline{M \otimes_\Lambda Q}(U) = \underline{M}(U) \otimes_\Lambda Q$ for all $U \in \mathrm{Ob}(\mathcal{C})$.*

Proof. Choose a presentation $\Lambda^{\oplus m} \to \Lambda^{\oplus n} \to Q \to 0$. This gives an exact sequence $M^{\oplus m} \to M^{\oplus n} \to M \otimes Q \to 0$. By Lemma 18.41.1 we obtain an exact sequence

$$\underline{M}(U)^{\oplus m} \to \underline{M}(U)^{\oplus n} \to \underline{M \otimes Q}(U) \to 0$$

which proves the lemma. (Note that taking sections over U always commutes with finite direct sums, but not arbitrary direct sums.) □

Lemma 18.41.3. *Let $\mathcal{C}$ be a site. Let Λ be a coherent ring. Let M be a flat Λ-module. For $U \in \mathrm{Ob}(\mathcal{C})$ the module $\underline{M}(U)$ is a flat Λ-module.*

Proof. Let $I \subset \Lambda$ be a finitely generated ideal. By Algebra, Lemma 10.36.4 it suffices to show that $\underline{M}(U) \otimes_\Lambda I \to \underline{M}(U)$ is injective. As Λ is coherent I is finitely presented as a Λ-module. By Lemma 18.41.2 we see that $\underline{M}(U) \otimes I = \underline{M \otimes I}$. Since M is flat the map $M \otimes I \to M$ is injective, whence $\underline{M \otimes I} \to \underline{M}$ is injective. □

Lemma 18.41.4. *Let $\mathcal{C}$ be a site. Let Λ be a Noetherian ring. Let $I \subset \Lambda$ be an ideal. The sheaf $\underline{\Lambda}^\wedge = \lim \underline{\Lambda/I^n}$ is a flat $\underline{\Lambda}$-algebra. Moreover we have canonical identifications*

$$\underline{\Lambda}/I\underline{\Lambda} = \underline{\Lambda}/\underline{I} = \underline{\Lambda}^\wedge/I\underline{\Lambda}^\wedge = \underline{\Lambda}^\wedge/\underline{I} \cdot \underline{\Lambda}^\wedge = \underline{\Lambda}^\wedge/\underline{I}^\wedge = \underline{\Lambda/I}$$

where $\underline{I}^\wedge = \lim \underline{I/I^n}$.

Proof. To prove $\underline{\Lambda}^\wedge$ is flat, it suffices to show that $\underline{\Lambda}^\wedge(U)$ is flat as a Λ-module for each $U \in \mathrm{Ob}(\mathcal{C})$, see Lemmas 18.28.2 and 18.28.3. By Lemma 18.41.3 we see that

$$\underline{\Lambda}^\wedge(U) = \lim \underline{\Lambda/I^n}(U)$$

is a limit of a system of flat Λ/I^n-modules. By Lemma 18.41.1 we see that the transition maps are surjective. We conclude by More on Algebra, Lemma 15.18.4.

To see the equalities, note that $\underline{\Lambda}(U)/I\underline{\Lambda}(U) = \underline{\Lambda/I}(U)$ by Lemma 18.41.2. It follows that $\underline{\Lambda}/I\underline{\Lambda} = \underline{\Lambda}/\underline{I} = \underline{\Lambda/I}$. The system of short exact sequences

$$0 \to \underline{I/I^n}(U) \to \underline{\Lambda/I^n}(U) \to \underline{\Lambda/I}(U) \to 0$$

has surjective transition maps, hence gives a short exact sequence

$$0 \to \lim \underline{I/I^n}(U) \to \lim \underline{\Lambda/I^n}(U) \to \lim \underline{\Lambda/I}(U) \to 0$$

see Homology, Lemma 12.26.3. Thus we see that $\underline{\Lambda}^\wedge/\underline{I}^\wedge = \underline{\Lambda/I}$. Since

$$I\underline{\Lambda}^\wedge \subset \underline{I} \cdot \underline{\Lambda}^\wedge \subset \underline{I}^\wedge$$

it suffices to show that $I\underline{\Lambda}^\wedge(U) = \underline{I}^\wedge(U)$ for all U. Choose generators $I = (f_1, \ldots, f_r)$. This gives a short exact sequence $0 \to K \to \Lambda^{\oplus r} \to I \to 0$. We obtain short exact sequences

$$0 \to \underline{(K \cap I^n)/I^nK}(U) \to \underline{(\Lambda/I^n)^{\oplus r}}(U) \to \underline{I/I^n}(U) \to 0$$

By Artin-Rees (Algebra, Lemma 10.47.4) the system of modules on the left hand side has ML. (It is zero as a pro-object.) Thus we see that $(\underline{\Lambda}^\wedge)^{\oplus r}(U) \to \underline{I}^\wedge(U)$ is surjective by Homology, Lemma 12.26.3 which is what we wanted to show. □

Lemma 18.41.5. *Let $\mathcal{C}$ be a site. Let Λ be a ring and let M be a Λ-module. Assume $Sh(\mathcal{C})$ is not the empty topos. Then*

1. *$\underline{M}$ is a finite type sheaf of $\underline{\Lambda}$-modules if and only if M is a finite Λ-module, and*
2. *$\underline{M}$ is a finitely presented sheaf of $\underline{\Lambda}$-modules if and only if M is a finitely presented Λ-module.*

Proof. Proof of (1). If M is generated by $x_1, \ldots, x_r$ then $x_1, \ldots, x_r$ define global sections of $\underline{M}$ which generate it, hence $\underline{M}$ is of finite type. Conversely, assume $\underline{M}$ is of finite type. Let $U \in \mathcal{C}$ be an object which is not sheaf theoretically empty (Sites, Definition 7.41.1). Such an object exists as we assumed $Sh(\mathcal{C})$ is not the empty topos. Then there exists a covering $\{U_i \to U\}$ and finitely many sections $s_{ij} \in \underline{M}(U_i)$ generating $\underline{M}|_{U_i}$. After refining the covering we may assume that s_{ij} come from elements x_{ij} of M. Then x_{ij} define global sections of $\underline{M}$ whose restriction to U generate $\underline{M}$.

Assume there exist elements $x_1, \ldots, x_r$ of M which define global sections of $\underline{M}$ generating $\underline{M}$ as a sheaf of $\underline{\Lambda}$-modules. We will show that $x_1, \ldots, x_r$ generate M as a Λ-module. Let $x \in M$. We can find a covering $\{U_i \to U\}_{i \in I}$ and $f_{i,j} \in \underline{\Lambda}(U_i)$ such that $x|_{U_i} = \sum f_{i,j} x_j|_{U_i}$. After refining the covering we may assume $f_{i,j} \in \Lambda$. Since U is not sheaf theoretically empty we see that $I \neq \emptyset$. Thus we can pick $i \in I$ and we see that $x = \sum f_{i,j} x_j$ in M as desired.

Proof of (2). Assume $\underline{M}$ is a $\underline{\Lambda}$-module of finite presentation. By (1) we see that M is of finite type. Choose generators $x_1, \ldots, x_r$ of M as a Λ-module. This determines a short exact sequence $0 \to K \to \Lambda^{\oplus r} \to M \to 0$ which turns into a short exact sequence

$$0 \to \underline{K} \to \underline{\Lambda}^{\oplus r} \to \underline{M} \to 0$$

by Lemma 18.41.1. By Lemma 18.24.1 we see that $\underline{K}$ is of finite type. Hence K is a finite Λ-module by (1). Thus M is a Λ-module of finite presentation. □

18.42. Locally constant sheaves

Here is the general definition.

Definition 18.42.1. Let $\mathcal{C}$ be a site. Let $\mathcal{F}$ be a sheaf of sets, groups, abelian groups, rings, modules over a fixed ring Λ, etc.

(1) We say $\mathcal{F}$ is a *constant sheaf* of sets, groups, abelian groups, rings, modules over a fixed ring Λ, etc if it is isomorphic as a sheaf of sets, groups, abelian groups, rings, modules over a fixed ring Λ, etc to a constant sheaf $\underline{E}$ as in Section 18.41.
(2) We say $\mathcal{F}$ is *locally constant* if for every object U of $\mathcal{C}$ there exists a covering $\{U_i \to U\}$ such that $\mathcal{F}|_{U_i}$ is a constant sheaf.
(3) If $\mathcal{F}$ is a sheaf of sets or groups, then we say $\mathcal{F}$ is *finite locally constant* if the constant values are finite sets or finite groups.

Lemma 18.42.2. *Let $f : Sh(\mathcal{C}) \to Sh(\mathcal{D})$ be a morphism of topoi. If $\mathcal{G}$ is a locally constant sheaf of sets, groups, abelian groups, rings, modules over a fixed ring Λ, etc on $\mathcal{D}$, the same is true for $f^{-1}\mathcal{G}$ on $\mathcal{C}$.*

Proof. Omitted. □

Lemma 18.42.3. *Let $\mathcal{C}$ be a site with a final object X.*

(1) *Let $\varphi : \mathcal{F} \to \mathcal{G}$ be a map of locally constant sheaves of sets on $\mathcal{C}$. If $\mathcal{F}$ is finite locally constant, there exists a covering $\{U_i \to X\}$ such that $\varphi|_{U_i}$ is the map of constant sheaves associated to a map of sets.*
(2) *Let $\varphi : \mathcal{F} \to \mathcal{G}$ be a map of locally constant sheaves of abelian groups on $\mathcal{C}$. If $\mathcal{F}$ is finite locally constant, there exists a covering $\{U_i \to X\}$ such that $\varphi|_{U_i}$ is the map of constant abelian sheaves associated to a map of abelian groups.*
(3) *Let Λ be a ring. Let $\varphi : \mathcal{F} \to \mathcal{G}$ be a map of locally constant sheaves of Λ-modules on $\mathcal{C}$. If $\mathcal{F}$ is of finite type, then there exists a covering $\{U_i \to X\}$ such that $\varphi|_{U_i}$ is the map of constant sheaves of Λ-modules associated to a map of Λ-modules.*

Proof. Proof omitted. □

Lemma 18.42.4. *Let $\mathcal{C}$ be a site. Let Λ be a ring. Let M, N be Λ-modules. Let $\mathcal{F}, \mathcal{G}$ be a locally constant sheaves of Λ-modules.*

(1) *If M is of finite presentation, then*
$$\underline{\mathrm{Hom}_\Lambda(M, N)} = \mathcal{H}om_{\underline{\Lambda}}(\underline{M}, \underline{N})$$
(2) *If M and N are both of finite presentation, then*
$$\underline{Isom_\Lambda(M, N)} = Isom_{\underline{\Lambda}}(\underline{M}, \underline{N})$$
(3) *If $\mathcal{F}$ is of finite presentation, then $\mathcal{H}om_{\underline{\Lambda}}(\mathcal{F}, \mathcal{G})$ is a locally constant sheaf of Λ-modules.*
(4) *If $\mathcal{F}$ and $\mathcal{G}$ are both of finite presentation, then $Isom_{\underline{\Lambda}}(\mathcal{F}, \mathcal{G})$ is a locally constant sheaf of sets.*

Proof. Proof of (1). Set $E = \mathrm{Hom}_\Lambda(M, N)$. We want to show the canonical map
$$\underline{E} \longrightarrow \mathcal{H}om_{\underline{\Lambda}}(\underline{M}, \underline{N})$$

is an isomorphism. The module M has a presentation $\Lambda^{\oplus s} \to \Lambda^{\oplus t} \to M \to 0$. Then E sits in an exact sequence

$$0 \to E \to \mathrm{Hom}_\Lambda(\Lambda^{\oplus t}, N) \to \mathrm{Hom}_\Lambda(\Lambda^{\oplus s}, N)$$

and we have similarly

$$0 \to \mathcal{H}om_{\underline{\Lambda}}(\underline{M}, \underline{N}) \to \mathcal{H}om_{\underline{\Lambda}}(\underline{\Lambda^{\oplus t}}, \underline{N}) \to \mathcal{H}om_{\underline{\Lambda}}(\underline{\Lambda^{\oplus s}}, \underline{N})$$

This reduces the question to the case where M is a finite free module where the result is clear.

Proof of (3). The question is local on $\mathcal{C}$, hence we may assume $\mathcal{F} = \underline{M}$ and $\mathcal{G} = \underline{N}$ for some Λ-modules M and N. By Lemma 18.41.5 the module M is of finite presentation. Thus the result follows from (1).

Parts (2) and (4) follow from parts (1) and (3) and the fact that *Isom* can be viewed as the subsheaf of sections of $\mathcal{H}om_{\underline{\Lambda}}(\mathcal{F}, \mathcal{G})$ which have an inverse in $\mathcal{H}om_{\underline{\Lambda}}(\mathcal{G}, \mathcal{F})$. □

Lemma 18.42.5. *Let $\mathcal{C}$ be a site.*

(1) *The category of finite locally constant sheaves of sets is closed under finite limits and colimits inside $Sh(\mathcal{C})$.*
(2) *The category of finite locally constant abelian sheaves is a weak Serre subcategory of $Ab(\mathcal{C})$.*
(3) *Let Λ be a Noetherian ring. The category of finite type, locally constant sheaves of Λ-modules on $\mathcal{C}$ is a weak Serre subcategory of $Mod(\mathcal{C}, \Lambda)$.*

Proof. Proof of (1). We may work locally on $\mathcal{C}$. Hence by Lemma 18.42.3 we may assume we are given a finite diagram of finite sets such that our diagram of sheaves is the associated diagram of constant sheaves. Then we just take the limit or colimit in the category of sets and take the associated constant sheaf. Some details omitted.

To prove (2) and (3) we use the criterion of Homology, Lemma 12.9.3. Existence of kernels and cokernels is argued in the same way as above. Of course, the reason for using a Noetherian ring in (3) is to assure us that the kernel of a map of finite Λ-modules is a finite Λ-module. To see that the category is closed under extensions (in the case of sheaves Λ-modules), assume given an extension of sheaves of Λ-modules

$$0 \to \mathcal{F} \to \mathcal{E} \to \mathcal{G} \to 0$$

on $\mathcal{C}$ with $\mathcal{F}$, $\mathcal{G}$ finite type and locally constant. Localizing on $\mathcal{C}$ we may assume $\mathcal{F}$ and $\mathcal{G}$ are constant, i.e., we get

$$0 \to \underline{M} \to \mathcal{E} \to \underline{N} \to 0$$

for some Λ-modules M, N. Choose generators $y_1, \ldots, y_m$ of N, so that we get a short exact sequence $0 \to K \to \Lambda^{\oplus m} \to N \to 0$ of Λ-modules. Localizing further we may assume y_j lifts to a section s_j of $\mathcal{E}$. Thus we see that $\mathcal{E}$ is a pushout as in the following diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \underline{K} & \longrightarrow & \underline{\Lambda^{\oplus m}} & \longrightarrow & \underline{N} & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & \underline{M} & \longrightarrow & \mathcal{E} & \longrightarrow & \underline{N} & \longrightarrow & 0 \end{array}$$

By Lemma 18.42.3 again (and the fact that K is a finite Λ-module as Λ is Noetherian) we see that the map $\underline{K} \to \underline{M}$ is locally constant, hence we conclude. □

Lemma 18.42.6. *Let $\mathcal{C}$ be a site. Let Λ be a ring. The tensor product of two locally constant sheaves of Λ-modules on $\mathcal{C}$ is a locally constant sheaf of Λ-modules.*

Proof. Omitted. □

18.43. Other chapters

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(7) Sites and Sheaves
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Schemes

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CHAPTER 19

Injectives

19.1. Introduction

In future chapters we will use the existence of injectives and K-injective complexes to do cohomology of sheaves of modules on ringed sites. In this chapter we explain how to produce injectives and K-injective complexes first for modules on sites and later more generally for Grothendieck abelian categories.

We observe that we already know that the category of abelian groups and the category of modules over a ring have enough injectives, see More on Algebra, Sections 15.39 and 15.40

19.2. Baer's argument for modules

There is another, more set-theoretic approach to showing that any R-module M can be imbedded in an injective module. This approach constructs the injective module by a transfinite colimit of push-outs. While this method is somewhat abstract and more complicated than the one of More on Algebra, Section 15.40, it is also more general. Apparently this method originates with Baer, and was revisited by Cartan and Eilenberg in [**CE56**] and by Grothendieck in [**Gro57**]. There Grothendieck uses it to show that many other abelian categories have enough injectives. We will get back to the general case later (insert future reference here).

We begin with a few set theoretic remarks. Let $\{B_\beta\}_{\beta\in\alpha}$ be an inductive system of objects in some category $\mathcal{C}$, indexed by an ordinal α. Assume that $\operatorname{colim}_{\beta\in\alpha} B_\beta$ exists in $\mathcal{C}$. If A is an object of $\mathcal{C}$, then there is a natural map

$$\operatorname{colim}_{\beta\in\alpha} \operatorname{Mor}_{\mathcal{C}}(A, B_\beta) \longrightarrow \operatorname{Mor}_{\mathcal{C}}(A, \operatorname{colim}_{\beta\in\alpha} B_\beta). \tag{19.2.0.1}$$

because if one is given a map $A \to B_\beta$ for some β, one naturally gets a map from A into the colimit by composing with $B_\beta \to \operatorname{colim}_{\beta\in\alpha} B_\alpha$. Note that the left colimit is one of sets! In general, (19.2.0.1) is neither injective or surjective.

Example 19.2.1. Consider the category of sets. Let $A = \mathbf{N}$ and $B_n = \{1, \ldots, n\}$ be the inductive system indexed by the natural numbers where $B_n \to B_m$ for $n \leq m$ is the obvious map. Then $\operatorname{colim} B_n = \mathbf{N}$, so there is a map $A \to \varinjlim B_n$, which does not factor as $A \to B_m$ for any m. Consequently, $\operatorname{colim} \operatorname{Mor}(A, \overrightarrow{B_n}) \to \operatorname{Mor}(A, \operatorname{colim} B_n)$ is not surjective.

Example 19.2.2. Next we give an example where the map fails to be injective. Let $B_n = \mathbf{N}/\{1, 2, \ldots, n\}$, that is, the quotient set of $\mathbf{N}$ with the first n elements collapsed to one element. There are natural maps $B_n \to B_m$ for $n \leq m$, so the $\{B_n\}$ form a system of sets over $\mathbf{N}$. It is easy to see that $\operatorname{colim} B_n = \{*\}$: it is the one-point set. So it follows that $\operatorname{Mor}(A, \operatorname{colim} B_n)$ is a one-element set for every set A. However, $\operatorname{colim} \operatorname{Mor}(A, B_n)$ is **not** a one-element set. Consider the family

of maps $A \to B_n$ which are just the natural projections $\mathbf{N} \to \mathbf{N}/\{1, 2, \ldots, n\}$ and the family of maps $A \to B_n$ which map the whole of A to the class of 1. These two families of maps are distinct at each step and thus are distinct in $\operatorname{colim} \operatorname{Mor}(A, B_n)$, but they induce the same map $A \to \operatorname{colim} B_n$.

Nonetheless, if we map out of a finite set then (19.2.0.1) is an isomorphism always.

Lemma 19.2.3. *Suppose that, in (19.2.0.1), $\mathcal{C}$ is the category of sets and A is a finite set, then the map is a bijection.*

Proof. Let $f : A \to \operatorname{colim} B_\beta$. The range of f is finite, containing say elements $c_1, \ldots, c_r \in \operatorname{colim} B_\beta$. These all come from some elements in B_β for $\beta \in \alpha$ large by definition of the colimit. Thus we can define $\widetilde{f} : A \to B_\beta$ lifting f at a finite stage. This proves that (19.2.0.1) is surjective. Next, suppose two maps $f : A \to B_\gamma, f' : A \to B_{\gamma'}$ define the same map $A \to \operatorname{colim} B_\beta$. Then each of the finitely many elements of A gets sent to the same point in the colimit. By definition of the colimit for sets, there is $\beta \geq \gamma, \gamma'$ such that the finitely many elements of A get sent to the same points in B_β under f and f'. This proves that (19.2.0.1) is injective. □

The most interesting case of the lemma is when $\alpha = \omega$, i.e., when the system $\{B_\beta\}$ is a system $\{B_n\}_{n \in \mathbf{N}}$ over the natural numbers as in Examples 19.2.1 and 19.2.2. The essential idea is that A is "small" relative to the long chain of compositions $B_1 \to B_2 \to \ldots$, so that it has to factor through a finite step. A more general version of this lemma can be found in Sets, Lemma 3.7.1. Next, we generalize this to the category of modules.

Definition 19.2.4. Let $\mathcal{C}$ be a category, let $I \subset \operatorname{Arrow}(\mathcal{C})$, and let α be an ordinal. An object A of $\mathcal{C}$ is said to be *α-small with respect to I* if whenever $\{B_\beta\}$ is a system over α with transition maps in I, then the map (19.2.0.1) is an isomorphism.

In the rest of this section we shall restrict ourselves to the category of R-modules for a fixed commutative ring R. We shall also take I to be the collection of injective maps, i.e., the *monomorphisms* in the category of modules over R. In this case, for any system $\{B_\beta\}$ as in the definition each of the maps

$$B_\beta \to \operatorname{colim}_{\beta \in \alpha} B_\beta$$

is an injection. It follows that the map (19.2.0.1) is an *injection*. We can in fact interpret the B_β's as submodules of the module $B = \operatorname{colim}_{\beta \in \alpha} B_\beta$, and then we have $B = \bigcup_{\beta \in \alpha} B_\beta$. This is not an abuse of notation if we identify B_α with the image in the colimit. We now want to show that modules are always small for "large" ordinals α.

Proposition 19.2.5. *Let R be a ring. Let M be an R-module. Let κ the cardinality of the set of submodules of M. If α is an ordinal whose cofinality is bigger than κ, then M is α-small with respect to injections.*

Proof. The proof is straightforward, but let us first think about a special case. If M is finite, then the claim is that for any inductive system $\{B_\beta\}$ with injections between them, parametrized by a limit ordinal, any map $M \to \operatorname{colim} B_\beta$ factors through one of the B_β. And this we proved in Lemma 19.2.3.

Now we start the proof in the general case. We need only show that the map (19.2.0.1) is a surjection. Let $f : M \to \operatorname{colim} B_\beta$ be a map. Consider the subobjects

$\{f^{-1}(B_\beta)\}$ of M, where B_β is considered as a subobject of the colimit $B = \bigcup_\beta B_\beta$. If one of these, say $f^{-1}(B_\beta)$, fills M, then the map factors through B_β.

So suppose to the contrary that all of the $f^{-1}(B_\beta)$ were proper subobjects of M. However, we know that

$$\bigcup f^{-1}(B_\beta) = f^{-1}\left(\bigcup B_\beta\right) = M.$$

Now there are at most κ different subobjects of M that occur among the $f^{-1}(B_\alpha)$, by hypothesis. Thus we can find a subset $S \subset \alpha$ of cardinality at most κ such that as β' ranges over S, the $f^{-1}(B_{\beta'})$ range over *all* the $f^{-1}(B_\alpha)$.

However, S has an upper bound $\widetilde{\alpha} < \alpha$ as α has cofinality bigger than κ. In particular, all the $f^{-1}(B_{\beta'})$, $\beta' \in S$ are contained in $f^{-1}(B_{\widetilde{\alpha}})$. It follows that $f^{-1}(B_{\widetilde{\alpha}}) = M$. In particular, the map f factors through $B_{\widetilde{\alpha}}$. □

From this lemma we will be able to deduce the existence of lots of injectives. Let us recall the criterion of Baer.

Lemma 19.2.6. *Let R be a ring. An R-module Q is injective if and only if in every commutative diagram*

$$\begin{array}{ccc} \mathfrak{a} & \longrightarrow & Q \\ \downarrow & \nearrow & \\ R & & \end{array}$$

for $\mathfrak{a} \subset R$ an ideal, the dotted arrow exists.

Proof. Assume Q satisfies the assumption of the lemma. Let $M \subset N$ be R-modules, and let $\varphi : M \to Q$ be an R-module map. Arguing as in the proof of More on Algebra, Lemma 15.39.1 we see that it suffices to prove that if $M \neq N$, then we can find an R-module M', $M \subset M' \subset N$ such that (a) the inclusion $M \subset M'$ is strict, and (b) φ can be extended to M'. To find M', let $x \in N$, $x \notin M$. Let $\psi : R \to N$, $r \mapsto rx$. Set $\mathfrak{a} = \psi^{-1}(M)$. By assumption the morphism

$$\mathfrak{a} \xrightarrow{\psi} M \xrightarrow{\varphi} Q$$

can be extended to a morphism $\varphi' : R \to Q$. Note that φ' annihilates the kernel of ψ (as this is true for φ). Thus φ' gives rise to a morphism $\varphi'' : \mathrm{Im}(\psi) \to Q$ which agrees with φ on the intersection $M \cap \mathrm{Im}(\psi)$ by construction. Thus φ and φ'' glue to give an extension of φ to the strictly bigger module $M' = \mathcal{F} + \mathrm{Im}(\psi)$. □

If M is an R-module, then in general we may have a semi-complete diagram as in Lemma 19.2.6. In it, we can form the *push-out*

$$\begin{array}{ccc} \mathfrak{a} & \longrightarrow & Q \\ \downarrow & & \downarrow \\ R & \longrightarrow & R \oplus_{\mathfrak{a}} Q. \end{array}$$

Here the vertical map is injective, and the diagram commutes. The point is that we can extend $\mathfrak{a} \to Q$ to R *if* we extend Q to the larger module $R \oplus_{\mathfrak{a}} Q$.

The key point of Baer's argument is to repeat this procedure transfinitely many times. To do this we first define, given an R-module M the following (huge) pushout

$$\begin{array}{ccc} \bigoplus_{\mathfrak{a}} \bigoplus_{\varphi \in \mathrm{Hom}_R(\mathfrak{a}, M)} \mathfrak{a} & \longrightarrow & M \\ \downarrow & & \downarrow \\ \bigoplus_{\mathfrak{a}} \bigoplus_{\varphi \in \mathrm{Hom}_R(\mathfrak{a}, M)} R & \longrightarrow & \mathbf{M}(M). \end{array} \tag{19.2.6.1}$$

Here the top horizontal arrow maps the element $a \in \mathfrak{a}$ in the summand corresponding to φ to the element $\varphi(a) \in M$. The left vertical arrow maps $a \in \mathfrak{a}$ in the summand corresponding to φ simply to the element $a \in R$ in the summand corresponding to φ. The fundamental properties of this construction are formulated in the following lemma.

Lemma 19.2.7. *Let R be a ring.*

1. *The construction $M \mapsto (M \to \mathbf{M}(M))$ is functorial in M.*
2. *The map $M \to \mathbf{M}(M)$ is injective.*
3. *For any ideal $\mathfrak{a}$ and any R-module map $\varphi : \mathfrak{a} \to M$ there is an R-module map $\varphi' : R \to \mathbf{M}(M)$ such that*

$$\begin{array}{ccc} \mathfrak{a} & \xrightarrow{\varphi} & M \\ \downarrow & & \downarrow \\ R & \xrightarrow{\varphi'} & \mathbf{M}(M) \end{array}$$

commutes.

Proof. Parts (2) and (3) are immediate from the construction. To see (1), let $\chi : M \to N$ be an R-module map. We claim there exists a canonical commutative diagram

$$\begin{array}{ccccc} \bigoplus_{\mathfrak{a}} \bigoplus_{\varphi \in \mathrm{Hom}_R(\mathfrak{a}, M)} \mathfrak{a} & \longrightarrow & M & & \\ & \searrow & & \overset{\chi}{\searrow} & \\ \downarrow & & \bigoplus_{\mathfrak{a}} \bigoplus_{\psi \in \mathrm{Hom}_R(\mathfrak{a}, N)} \mathfrak{a} & \longrightarrow & N \\ \bigoplus_{\mathfrak{a}} \bigoplus_{\varphi \in \mathrm{Hom}_R(\mathfrak{a}, M)} R & & \downarrow & & \\ & \searrow & & & \\ & & \bigoplus_{\mathfrak{a}} \bigoplus_{\psi \in \mathrm{Hom}_R(\mathfrak{a}, N)} R & & \end{array}$$

which induces the desired map $\mathbf{M}(M) \to \mathbf{M}(N)$. The middle east-south-east arrow maps the summand $\mathfrak{a}$ corresponding to φ via $\mathrm{id}_{\mathfrak{a}}$ to the summand $\mathfrak{a}$ corresponding to $\psi = \chi \circ \varphi$. Similarly for the lower east-south-east arrow. Details omitted. □

The idea will now be to apply the functor $\mathbf{M}$ a transfinite number of times. We define for each ordinal α a functor $\mathbf{M}_\alpha$ on the category of R-modules, together with a natural injection $N \to \mathbf{M}_\alpha(N)$. We do this by transfinite induction. First, $\mathbf{M}_1 = \mathbf{M}$ is the functor defined above. Now, suppose given an ordinal α, and suppose $\mathbf{M}_{\alpha'}$ is defined for $\alpha' < \alpha$. If α has an immediate predecessor $\widetilde{\alpha}$, we let

$$\mathbf{M}_\alpha = \mathbf{M} \circ \mathbf{M}_{\widetilde{\alpha}}.$$

If not, i.e., if α is a limit ordinal, we let

$$\mathbf{M}_\alpha(N) = \operatorname{colim}_{\alpha'<\alpha} \mathbf{M}_{\alpha'}(N).$$

It is clear (e.g., inductively) that the $\mathbf{M}_\alpha(N)$ form an inductive system over ordinals, so this is reasonable.

Theorem 19.2.8. *Let κ be the cardinality of the set of ideals in R, and let α be an ordinal whose cofinality is greater than κ. Then $\mathbf{M}_\alpha(N)$ is an injective R-module, and $N \to \mathbf{M}_\alpha(N)$ is a functorial injective embedding.*

Proof. By Baer's criterion Lemma 19.2.6, it suffices to show that if $\mathfrak{a} \subset R$ is an ideal, then any map $f : \mathfrak{a} \to \mathbf{M}_\alpha(N)$ extends to $R \to \mathbf{M}_\alpha(N)$. However, we know since α is a limit ordinal that

$$\mathbf{M}_\alpha(N) = \operatorname{colim}_{\beta<\alpha} \mathbf{M}_\beta(N),$$

so by Proposition 19.2.5, we find that

$$\operatorname{Hom}_R(\mathfrak{a}, \mathbf{M}_\alpha(N)) = \operatorname{colim}_{\beta<\alpha} \operatorname{Hom}_R(\mathfrak{a}, \mathbf{M}_\beta(N)).$$

This means in particular that there is some $\beta' < \alpha$ such that f factors through the submodule $\mathbf{M}_{\beta'}(N)$, as

$$f : \mathfrak{a} \to \mathbf{M}_{\beta'}(N) \to \mathbf{M}_\alpha(N).$$

However, by the fundamental property of the functor $\mathbf{M}$, see Lemma 19.2.7 part (3), we know that the map $\mathfrak{a} \to \mathbf{M}_{\beta'}(N)$ can be extended to

$$R \to \mathbf{M}(\mathbf{M}_{\beta'}(N)) = \mathbf{M}_{\beta'+1}(N),$$

and the last object imbeds in $\mathbf{M}_\alpha(N)$ (as $\beta' + 1 < \alpha$ since α is a limit ordinal). In particular, f can be extended to $\mathbf{M}_\alpha(N)$. □

19.3. G-modules

We will see later (Differential Graded Algebra, Section 22.12) that the category of modules over an algebra has functorial injective embeddings. The construction is exactly the same as the construction in More on Algebra, Section 15.40.

Lemma 19.3.1. *Let G be a topological group. The category Mod_G of discrete G-modules, see Étale Cohomology, Definition 44.58.1 has functorial injective hulls.*

Proof. By the remark above the lemma the category $\operatorname{Mod}_{\mathbf{Z}[G]}$ has functorial injective embeddings. Consider the forgetful functor $v : Mod_G \to \operatorname{Mod}_{\mathbf{Z}[G]}$. This functor is fully faithful, transforms injective maps into injective maps and has a right adjoint, namely

$$u : M \mapsto u(M) = \{x \in M \mid \text{stabilizer of } x \text{ is open}\}$$

Since it is true that $v(M) = 0 \Rightarrow M = 0$ we conclude by Homology, Lemma 12.25.5. □

19.4. Abelian sheaves on a space

Lemma 19.4.1. *Let X be a topological space. The category of abelian sheaves on X has enough injectives. In fact it has functorial injective embeddings.*

Proof. For an abelian group A we denote $j : A \to J(A)$ the functorial injective embedding constructed in More on Algebra, Section 15.40. Let $\mathcal{F}$ be an abelian sheaf on X. By Sheaves, Example 6.7.5 the assignment

$$\mathcal{I} : U \mapsto \mathcal{I}(U) = \prod\nolimits_{x \in U} J(\mathcal{F}_x)$$

is an abelian sheaf. There is a canonical map $\mathcal{F} \to \mathcal{I}$ given by mapping $s \in \mathcal{F}(U)$ to $\prod_{x \in U} j(s_x)$ where $s_x \in \mathcal{F}_x$ denotes the germ of s at x. This map is injective, see Sheaves, Lemma 6.11.1 for example.

It remains to prove the following: Given a rule $x \mapsto I_x$ which assigns to each point $x \in X$ an injective abelian group the sheaf $\mathcal{I} : U \mapsto \prod_{x \in U} I_x$ is injective. Note that

$$\mathcal{I} = \prod\nolimits_{x \in X} i_{x,*} I_x$$

is the product of the skyscraper sheaves $i_{x,*}I_x$ (see Sheaves, Section 6.27 for notation.) We have

$$\mathrm{Mor}_{Ab}(\mathcal{F}_x, I_x) = \mathrm{Mor}_{Ab(X)}(\mathcal{F}, i_{x,*} I_x).$$

see Sheaves, Lemma 6.27.3. Hence it is clear that each $i_{x,*}I_x$ is injective. Hence the injectivity of $\mathcal{I}$ follows from Homology, Lemma 12.23.3. □

19.5. Sheaves of modules on a ringed space

Lemma 19.5.1. *Let $(X, \mathcal{O}_X)$ be a ringed space, see Sheaves, Section 6.25. The category of sheaves of $\mathcal{O}_X$-modules on X has enough injectives. In fact it has functorial injective embeddings.*

Proof. For any ring R and any R-module M we denote $j : M \to J_R(M)$ the functorial injective embedding constructed in More on Algebra, Section 15.40. Let $\mathcal{F}$ be a sheaf of $\mathcal{O}_X$-modules on X. By Sheaves, Examples 6.7.5 and 6.15.6 the assignment

$$\mathcal{I} : U \mapsto \mathcal{I}(U) = \prod\nolimits_{x \in U} J_{\mathcal{O}_{X,x}}(\mathcal{F}_x)$$

is an abelian sheaf. There is a canonical map $\mathcal{F} \to \mathcal{I}$ given by mapping $s \in \mathcal{F}(U)$ to $\prod_{x \in U} j(s_x)$ where $s_x \in \mathcal{F}_x$ denotes the germ of s at x. This map is injective, see Sheaves, Lemma 6.11.1 for example.

It remains to prove the following: Given a rule $x \mapsto I_x$ which assigns to each point $x \in X$ an injective $\mathcal{O}_{X,x}$-module the sheaf $\mathcal{I} : U \mapsto \prod_{x \in U} I_x$ is injective. Note that

$$\mathcal{I} = \prod\nolimits_{x \in X} i_{x,*} I_x$$

is the product of the skyscraper sheaves $i_{x,*}I_x$ (see Sheaves, Section 6.27 for notation.) We have

$$\mathrm{Hom}_{\mathcal{O}_{X,x}}(\mathcal{F}_x, I_x) = \mathrm{Hom}_{\mathcal{O}_X}(\mathcal{F}, i_{x,*} I_x).$$

see Sheaves, Lemma 6.27.3. Hence it is clear that each $i_{x,*}I_x$ is an injective $\mathcal{O}_X$-module (see Homology, Lemma 12.25.1 or argue directly). Hence the injectivity of $\mathcal{I}$ follows from Homology, Lemma 12.23.3. □

19.6. Abelian presheaves on a category

Let $\mathcal{C}$ be a category. Recall that this means that $\mathrm{Ob}(\mathcal{C})$ is a set. On the one hand, consider abelian presheaves on $\mathcal{C}$, see Sites, Section 7.2. On the other hand, consider families of abelian groups indexed by elements of $\mathrm{Ob}(\mathcal{C})$; in other words presheaves on the discrete category with underlying set of objects $\mathrm{Ob}(\mathcal{C})$. Let us denote this discrete category simply $\mathrm{Ob}(\mathcal{C})$. There is a natural functor

$$i : \mathrm{Ob}(\mathcal{C}) \longrightarrow \mathcal{C}$$

and hence there is a natural restriction or forgetful functor

$$v = i^p : PAb(\mathcal{C}) \longrightarrow PAb(\mathrm{Ob}(\mathcal{C}))$$

compare Sites, Section 7.5. We will denote presheaves on $\mathcal{C}$ by B and presheaves on $\mathrm{Ob}(\mathcal{C})$ by A.

There are also two functors, namely i_p and ${}_pi$ which assign an abelian presheaf on $\mathcal{C}$ to an abelian presheaf on $\mathrm{Ob}(\mathcal{C})$, see Sites, Sections 7.5 and 7.18. Here we will use $u = {}_pi$ which is defined (in the case at hand) as follows:

$$uA(U) = \prod\nolimits_{U' \to U} A(U').$$

So an element is a family $(a_\phi)_\phi$ with ϕ ranging through all morphisms in $\mathcal{C}$ with target U. The restriction map on uA corresponding to $g : V \to U$ maps our element $(a_\phi)_\phi$ to the element $(a_{g \circ \psi})_\psi$.

There is a canonical surjective map $vuA \to A$ and a canonical injective map $B \to uvB$. We leave it to the reader to show that

$$\mathrm{Mor}_{PAb(\mathrm{Ob}(\mathcal{C}))}(B, uA) = \mathrm{Mor}_{PAb(\mathcal{C})}(vB, A).$$

in this simple case; the general case is in Sites, Section 7.5. Thus the pair (u, v) is an example of a pair of adjoint functors, see Categories, Section 4.24.

At this point we can list the following facts about the situation above.

(1) The functors u and v are exact. This follows from the explicit description of these functors given above.
(2) In particular the functor v transforms injective maps into injective maps.
(3) The category $PAb(\mathrm{Ob}(\mathcal{C}))$ has enough injectives.
(4) In fact there is a functorial injective embedding $A \mapsto \big(A \to J(A)\big)$ as in Homology, Definition 12.23.5. Namely, we can take $J(A)$ to be the presheaf $U \mapsto J(A(U))$, where $J(-)$ is the functor constructed in More on Algebra, Section 15.40 for the ring $\mathbf{Z}$.

Putting all of this together gives us the following procedure for embedding objects B of $PAb(\mathcal{C}))$ into an injective object: $B \to uJ(vB)$. See Homology, Lemma 12.25.5.

Proposition 19.6.1. *For abelian presheaves on a category there is a functorial injective embedding.*

Proof. See discussion above. □

19.7. Abelian Sheaves on a site

Let $\mathcal{C}$ be a site. In this section we prove that there are enough injectives for abelian sheaves on $\mathcal{C}$.

Denote $i : Ab(\mathcal{C}) \longrightarrow PAb(\mathcal{C})$ the forgetful functor from abelian sheaves to abelian presheaves. Let $^{\#} : PAb(\mathcal{C}) \longrightarrow Ab(\mathcal{C})$ denote the sheafification functor. Recall that $^{\#}$ is a left adjoint to i, that $^{\#}$ is exact, and that $i\mathcal{F}^{\#} = \mathcal{F}$ for any abelian sheaf $\mathcal{F}$. Finally, let $\mathcal{G} \to J(\mathcal{G})$ denote the canonical embedding into an injective presheaf we found in Section 19.6.

For any sheaf $\mathcal{F}$ in $Ab(\mathcal{C})$ and any ordinal β we define a sheaf $J_\beta(\mathcal{F})$ by transfinite induction. We set $J_0(\mathcal{F}) = \mathcal{F}$. We define $J_1(\mathcal{F}) = J(i\mathcal{F})^{\#}$. Sheafification of the canonical map $i\mathcal{F} \to J(i\mathcal{F})$ gives a functorial map

$$\mathcal{F} \longrightarrow J_1(\mathcal{F})$$

which is injective as $\#$ is exact. We set $J_{\alpha+1}(\mathcal{F}) = J_1(J_\alpha(\mathcal{F}))$. So that there are canonical injective maps $J_\alpha(\mathcal{F}) \to J_{\alpha+1}(\mathcal{F})$. For a limit ordinal β, we define

$$J_\beta(\mathcal{F}) = \text{colim}_{\alpha<\beta} J_\alpha(\mathcal{F}).$$

Note that this is a directed colimit. Hence for any ordinals $\alpha < \beta$ we have an injective map $J_\alpha(\mathcal{F}) \to J_\beta(\mathcal{F})$.

Lemma 19.7.1. *With notation as above. Suppose that $\mathcal{G}_1 \to \mathcal{G}_2$ is an injective map of abelian sheaves on $\mathcal{C}$. Let α be an ordinal and let $\mathcal{G}_1 \to J_\alpha(\mathcal{F})$ be a morphism of sheaves. There exists a morphism $\mathcal{G}_2 \to J_{\alpha+1}(\mathcal{F})$ such that the following diagram commutes*

$$\begin{array}{ccc} \mathcal{G}_1 & \longrightarrow & \mathcal{G}_2 \\ \downarrow & & \downarrow \\ J_\alpha(\mathcal{F}) & \longrightarrow & J_{\alpha+1}(\mathcal{F}) \end{array}$$

Proof. This is because the map $i\mathcal{G}_1 \to i\mathcal{G}_2$ is injective and hence $i\mathcal{G}_1 \to iJ_\alpha(\mathcal{F})$ extends to $i\mathcal{G}_2 \to J(iJ_\alpha(\mathcal{F}))$ which gives the desired map after applying the sheafification functor. □

This lemma says that somehow the system $\{J_\alpha(\mathcal{F})\}$ is an injective embedding of $\mathcal{F}$. Of course we cannot take the limit over all α because they form a class and not a set. However, the idea is now that you don't have to check injectivity on all injections $\mathcal{G}_1 \to \mathcal{G}_2$, plus the following lemma.

Lemma 19.7.2. *Suppose that $\mathcal{G}_i$, $i \in I$ is set of abelian sheaves on $\mathcal{C}$. There exists an ordinal β such that for any sheaf $\mathcal{F}$, any $i \in I$, and any map $\varphi : \mathcal{G}_i \to J_\beta(\mathcal{F})$ there exists an $\alpha < \beta$ such that φ factors through $J_\alpha(\mathcal{F})$.*

Proof. This reduces to the case of a single sheaf $\mathcal{G}$ by taking the direct sum of all the $\mathcal{G}_i$.

Consider the sets

$$S = \coprod\nolimits_{U \in \text{Ob}(\mathcal{C})} \mathcal{G}(U).$$

and

$$T_\beta = \coprod\nolimits_{U \in \text{Ob}(\mathcal{C})} J_\beta(\mathcal{F})(U)$$

Then $T_\beta = \operatorname{colim}_{\alpha<\beta} T_\alpha$ with injective transition maps. A morphism $\mathcal{G} \to J_\beta(\mathcal{F})$ factors through $J_\alpha(\mathcal{F})$ if and only if the associated map $S \to T_\beta$ factors through T_α. By Sets, Lemma 3.7.1 the cofinality of β is bigger than the cardinality of S, then the result of the lemma is true. Hence the lemma follows from the fact that there are ordinals with arbitrarily large cofinality, see Sets, Proposition 3.7.2. □

Recall that for an object X of $\mathcal{C}$ we denote $\mathbf{Z}_X$ the presheaf of abelian groups $\Gamma(U, \mathbf{Z}_X) = \oplus_{U \to X} \mathbf{Z}$, see Modules on Sites, Section 18.4. The sheaf associated to this presheaf is denoted $\mathbf{Z}_X^\#$, see Modules on Sites, Section 18.5. It can be characterized by the property

$$\operatorname{Mor}_{Ab(\mathcal{C})}(\mathbf{Z}_X^\#, \mathcal{G}) = \mathcal{G}(X) \tag{19.7.2.1}$$

where the element φ of the left hand side is mapped to $\varphi(1 \cdot \mathrm{id}_X)$ in the right hand side. We can use these sheaves to characterize injective abelian sheaves.

Lemma 19.7.3. *Suppose $\mathcal{J}$ is a sheaf of abelian groups with the following property: For all $X \in \mathrm{Ob}(\mathcal{C})$, for any abelian subsheaf $\mathcal{S} \subset \mathbf{Z}_X^\#$ and any morphism $\varphi : \mathcal{S} \to \mathcal{J}$, there exists a morphism $\mathbf{Z}_X^\# \to \mathcal{J}$ extending φ. Then $\mathcal{J}$ is an injective sheaf of abelian groups.*

Proof. Let $\mathcal{F} \to \mathcal{G}$ be an injective map of abelian sheaves. Suppose $\varphi : \mathcal{F} \to \mathcal{J}$ is a morphism. Arguing as in the proof of More on Algebra, Lemma 15.39.1 we see that it suffices to prove that if $\mathcal{F} \neq \mathcal{G}$, then we can find an abelian sheaf $\mathcal{F}'$, $\mathcal{F} \subset \mathcal{F}' \subset \mathcal{G}$ such that (a) the inclusion $\mathcal{F} \subset \mathcal{F}'$ is strict, and (b) φ can be extended to $\mathcal{F}'$. To find $\mathcal{F}'$, let X be an object of $\mathcal{C}$ such that the inclusion $\mathcal{F}(X) \subset \mathcal{G}(X)$ is strict. Pick $s \in \mathcal{G}(X)$, $s \notin \mathcal{F}(X)$. Let $\psi : \mathbf{Z}_X^\# \to \mathcal{G}$ be the morphism corresponding to the section s via (19.7.2.1). Set $\mathcal{S} = \psi^{-1}(\mathcal{F})$. By assumption the morphism

$$\mathcal{S} \xrightarrow{\psi} \mathcal{F} \xrightarrow{\varphi} \mathcal{J}$$

can be extended to a morphism $\varphi' : \mathbf{Z}_X^\# \to \mathcal{J}$. Note that φ' annihilates the kernel of ψ (as this is true for φ). Thus φ' gives rise to a morphism $\varphi'' : \mathrm{Im}(\psi) \to \mathcal{J}$ which agrees with φ on the intersection $\mathcal{F} \cap \mathrm{Im}(\psi)$ by construction. Thus φ and φ'' glue to give an extension of φ to the strictly bigger subsheaf $\mathcal{F}' = \mathcal{F} + \mathrm{Im}(\psi)$. □

Theorem 19.7.4. *The category of sheaves of abelian groups on a site has enough injectives. In fact there exists a functorial injective embedding, see Homology, Definition 12.23.5.*

Proof. Let $\mathcal{G}_i$, $i \in I$ be a set of abelian sheaves such that every subsheaf of every $\mathbf{Z}_X^\#$ occurs as one of the $\mathcal{G}_i$. Apply Lemma 19.7.2 to this collection to get an ordinal β. We claim that for any sheaf of abelian groups $\mathcal{F}$ the map $\mathcal{F} \to J_\beta(\mathcal{F})$ is an injection of $\mathcal{F}$ into an injective. Note that by construction the assingment $\mathcal{F} \mapsto \big(\mathcal{F} \to J_\beta(\mathcal{F})\big)$ is indeed functorial.

The proof of the claim comes from the fact that by Lemma 19.7.3 it suffices to extend any morphism $\gamma : \mathcal{G} \to J_\beta(\mathcal{F})$ from a subsheaf $\mathcal{G}$ of some $\mathbf{Z}_X^\#$ to all of $\mathbf{Z}_X^\#$. Then by Lemma 19.7.2 the map γ lifts into $J_\alpha(\mathcal{F})$ for some $\alpha < \beta$. Finally, we apply Lemma 19.7.1 to get the desired extension of γ to a morphism into $J_{\alpha+1}(\mathcal{F}) \to J_\beta(\mathcal{F})$. □

19.8. Modules on a ringed site

Let $\mathcal{C}$ be a site. Let $\mathcal{O}$ be a sheaf of rings on $\mathcal{C}$. By analogy with More on Algebra, Section 15.40 let us try to prove that there are enough injective $\mathcal{O}$-modules. First of all, we pick an injective embedding

$$\bigoplus\nolimits_{U,\mathcal{I}} j_{U!}\mathcal{O}_U/\mathcal{I} \longrightarrow \mathcal{J}$$

where $\mathcal{J}$ is an injective abelian sheaf (which exists by the previous section). Here the direct sum is over all objects U of $\mathcal{C}$ and over all $\mathcal{O}$-submodules $\mathcal{I} \subset j_{U!}\mathcal{O}_U$. Please see Modules on Sites, Section 18.19 to read about the functors restriction and extension by 0 for the localization functor $j_U : \mathcal{C}/U \to \mathcal{C}$.

For any sheaf of $\mathcal{O}$-modules $\mathcal{F}$ denote

$$\mathcal{F}^\vee = \mathcal{H}om(\mathcal{F}, \mathcal{J})$$

with its natural $\mathcal{O}$-module structure. Insert here future reference to internal hom. We will also need a canonical flat resolution of a sheaf of $\mathcal{O}$-modules. This we can do as follows: For any $\mathcal{O}$-module $\mathcal{F}$ we denote

$$F(\mathcal{F}) = \bigoplus\nolimits_{U \in \mathrm{Ob}(\mathcal{C}), s \in \mathcal{F}(U)} j_{U!}\mathcal{O}_U.$$

This is a flat sheaf of $\mathcal{O}$-modules which comes equipped with a canonical surjection $F(\mathcal{F}) \to \mathcal{F}$, see Modules on Sites, Lemma 18.28.6. Moreover the construction $\mathcal{F} \mapsto F(\mathcal{F})$ is functorial in $\mathcal{F}$.

Lemma 19.8.1. *The functor $\mathcal{F} \mapsto \mathcal{F}^\vee$ is exact.*

Proof. This because $\mathcal{J}$ is an injective abelian sheaf. □

There is a canonical map $ev : \mathcal{F} \to (\mathcal{F}^\vee)^\vee$ given by evaluation: given $x \in \mathcal{F}(U)$ we let $ev(x) \in (\mathcal{F}^\vee)^\vee = \mathcal{H}om(\mathcal{F}^\vee, \mathcal{J})$ be the map $\varphi \mapsto \varphi(x)$.

Lemma 19.8.2. *For any $\mathcal{O}$-module $\mathcal{F}$ the evaluation map $ev : \mathcal{F} \to (\mathcal{F}^\vee)^\vee$ is injective.*

Proof. You can check this using the definition of $\mathcal{J}$. Namely, if $s \in \mathcal{F}(U)$ is not zero, then let $j_{U!}\mathcal{O}_U \to \mathcal{F}$ be the map of $\mathcal{O}$-modules it corresponds to via adjunction. Let $\mathcal{I}$ be the kernel of this map. There exists a nonzero map $\mathcal{F} \supset j_{U!}\mathcal{O}_U/\mathcal{I} \to \mathcal{J}$ which does not annihilate s. As $\mathcal{J}$ is an injective $\mathcal{O}$-module, this extends to a map $\varphi : \mathcal{F} \to \mathcal{J}$. Then $ev(s)(\varphi) = \varphi(s) \neq 0$ which is what we had to prove. □

The canonical surjection $F(\mathcal{F}) \to \mathcal{F}$ of $\mathcal{O}$-modules turns into a a canonical injection, see above, of $\mathcal{O}$-modules

$$(\mathcal{F}^\vee)^\vee \longrightarrow (F(\mathcal{F}^\vee))^\vee.$$

Set $J(\mathcal{F}) = (F(\mathcal{F}^\vee))^\vee$. The composition of ev with this the displayed map gives $\mathcal{F} \to J(\mathcal{F})$ functorially in $\mathcal{F}$.

Lemma 19.8.3. *Let $\mathcal{O}$ be a sheaf of rings. For every $\mathcal{O}$-module $\mathcal{F}$ the $\mathcal{O}$-module $J(\mathcal{F})$ is injective.*

Proof. We have to show that the functor $\mathrm{Hom}_{\mathcal{O}}(\mathcal{G}, J(\mathcal{F}))$ is exact. Note that

$$\begin{aligned} \mathrm{Hom}_{\mathcal{O}}(\mathcal{G}, J(\mathcal{F})) &= \mathrm{Hom}_{\mathcal{O}}(\mathcal{G}, (F(\mathcal{F}^\vee))^\vee) \\ &= \mathrm{Hom}_{\mathcal{O}}(\mathcal{G}, \mathcal{H}om(F(\mathcal{F}^\vee), \mathcal{J})) \\ &= \mathrm{Hom}(\mathcal{G} \otimes_{\mathcal{O}} F(\mathcal{F}^\vee), \mathcal{J}) \end{aligned}$$

Thus what we want follows from the fact that $F(\mathcal{F}^\vee)$ is flat and $\mathcal{J}$ is injective. □

Theorem 19.8.4. *Let $\mathcal{C}$ be a site. Let $\mathcal{O}$ be a sheaf of rings on $\mathcal{C}$. The category of sheaves of $\mathcal{O}$-modules on a site has enough injectives. In fact there exists a functorial injective embedding, see Homology, Definition 12.23.5.*

Proof. From the discussion in this section. □

Proposition 19.8.5. *Let $\mathcal{C}$ be a category. Let $\mathcal{O}$ be a presheaf of rings on $\mathcal{C}$. The category $PMod(\mathcal{O})$ of presheaves of $\mathcal{O}$-modules has functorial injective embeddings.*

Proof. We could prove this along the lines of the discussion in Section 19.6. But instead we argue using the theorem above. Endow $\mathcal{C}$ with the structure of a site by letting the set of coverings of an object U consist of all singletons $\{f : V \to U\}$ where f is an isomorphism. We omit the verification that this defines a site. A sheaf for this topology is the same as a presheaf (proof omitted). Hence the theorem applies. □

19.9. Embedding abelian categories

In this section we show that an abelian category embeds in the category of abelian sheaves on a site having enough points. The site will be the one described in the following lemma.

Lemma 19.9.1. *Let $\mathcal{A}$ be an abelian category. Let*

$$Cov = \{\{f : V \to U\} \mid f \text{ is surjective}\}.$$

Then $(\mathcal{A}, Cov)$ is a site, see Sites, Definition 7.6.2.

Proof. Note that $\mathrm{Ob}(\mathcal{A})$ is a set by our conventions about categories. An isomorphism is a surjective morphism. The composition of surjective morphisms is surjective. And the base change of a surjective morphism in $\mathcal{A}$ is surjective, see Homology, Lemma 12.5.14. □

Let $\mathcal{A}$ be a pre-additive category. In this case the Yoneda embedding $\mathcal{A} \to PSh(\mathcal{A})$, $X \mapsto h_X$ factors through a functor $\mathcal{A} \to PAb(\mathcal{A})$.

Lemma 19.9.2. *Let $\mathcal{A}$ be an abelian category. Let $\mathcal{C} = (\mathcal{A}, Cov)$ be the site defined in Lemma 19.9.1. Then $X \mapsto h_X$ defines a fully faithful, exact functor*

$$\mathcal{A} \longrightarrow Ab(\mathcal{C}).$$

Moreover, the site $\mathcal{C}$ has enough points.

Proof. Suppose that $f : V \to U$ is a surjective morphism of $\mathcal{A}$. Let $K = \mathrm{Ker}(f)$. Recall that $V \times_U V = \mathrm{Ker}((f, -f) : V \oplus V \to U)$, see Homology, Example 12.5.6. In particular there exists an injection $K \oplus K \to V \times_U V$. Let $p, q : V \times_U V \to V$ be the two projection morphisms. Note that $p - q : V \times_U V \to V$ is a morphism such that $f \circ (p - q) = 0$. Hence $p - q$ factors through $K \to V$. Let us denote this morphism by $c : V \times_K V \to K$. And since the composition $K \oplus K \to V \times_U V \to K$ is surjective, we conclude that c is surjective. It follows that

$$V \times_K V \xrightarrow{p-q} V \to U \to 0$$

is an exact sequence of $\mathcal{A}$. Hence for an object X of $\mathcal{A}$ the sequence

$$0 \to \mathrm{Hom}_{\mathcal{A}}(U, X) \to \mathrm{Hom}_{\mathcal{A}}(V, X) \to \mathrm{Hom}_{\mathcal{A}}(V \times_U V, X)$$

is an exact sequence of abelian groups, see Homology, Lemma 12.5.8. This means that h_X satisfies the sheaf condition on $\mathcal{C}$.

The functor is fully faithful by Categories, Lemma 4.3.5. The functor is a left exact functor between abelian categories by Homology, Lemma 12.5.8. To show that it is right exact, let $X \to Y$ be a surjective morphism of $\mathcal{A}$. Let U be an object of $\mathcal{A}$, and let $s \in h_Y(U) = \mathrm{Mor}_{\mathcal{A}}(U, Y)$ be a section of h_Y over U. By Homology, Lemma 12.5.14 the projection $U \times_Y X \to U$ is surjective. Hence $\{V = U \times_Y X \to U\}$ is a covering of U such that $s|_V$ lifts to a section of h_X. This proves that $h_X \to h_Y$ is a surjection of abelian sheaves, see Sites, Lemma 7.12.2.

The site $\mathcal{C}$ has enough points by Sites, Proposition 7.38.3. □

Remark 19.9.3. The Freyd-Mitchell embedding theorem says there exists a fully faithful exact functor from any abelian category $\mathcal{A}$ to the category of modules over a ring. Lemma 19.9.2 is not quite as strong. But the result is suitable for the stacks project as we have to understand sheaves of abelian groups on sites in detail anyway. Moreover, "diagram chasing" works in the category of abelian sheaves on $\mathcal{C}$, for example by working with sections over objects, or by working on the level of stalks using that $\mathcal{C}$ has enough points. To see how to deduce the Freyd-Mitchell embedding theorem from Lemma 19.9.2 see Remark 19.9.5.

Remark 19.9.4. If $\mathcal{A}$ is a "big" abelian category, i.e., if $\mathcal{A}$ has a class of objects, then Lemma 19.9.2 does not work. In this case, given any set of objects $E \subset \mathrm{Ob}(\mathcal{A})$ there exists an abelian full subcategory $\mathcal{A}' \subset \mathcal{A}$ such that $\mathrm{Ob}(\mathcal{A}')$ is a set and $E \subset \mathrm{Ob}(\mathcal{A}')$. Then one can apply Lemma 19.9.2 to $\mathcal{A}'$. One can use this to prove that results depending on a diagram chase hold in $\mathcal{A}$.

Remark 19.9.5. Let $\mathcal{C}$ be a site. Note that $Ab(\mathcal{C})$ has enough injectives, see Theorem 19.7.4. (In the case that $\mathcal{C}$ has enough points this is straightforward because p_*I is an injective sheaf if I is an injective $\mathbf{Z}$-module and p is a point.) Also, $Ab(\mathcal{C})$ has a cogenerator (details omitted). Hence Lemma 19.9.2 proves that we have a fully faithful, exact embedding $\mathcal{A} \to \mathcal{B}$ where $\mathcal{B}$ has a cogenerator and enough injectives. We can apply this to $\mathcal{A}^{opp}$ and we get a fully faithful exact functor $i : \mathcal{A} \to \mathcal{D} = \mathcal{B}^{opp}$ where $\mathcal{D}$ has enough projectives and a generator. Hence $\mathcal{D}$ has a projective generator P. Set $R = \mathrm{Mor}_{\mathcal{D}}(P, P)$. Then

$$\mathcal{A} \longrightarrow \mathrm{Mod}_R, \quad X \longmapsto \mathrm{Hom}_{\mathcal{D}}(P, X).$$

One can check this is a fully faithful, exact functor. In other words, one retrieves the Freyd-Mitchell theorem mentioned in Remark 19.9.3 above.

Remark 19.9.6. The arguments proving Lemmas 19.9.1 and 19.9.2 work also for *exact categories*, see [**Büh10**, Appendix A] and [**BBD82**, 1.1.4]. We quickly review this here and we add more details if we ever need it in the stacks project.

Let $\mathcal{A}$ be an additive category. A *kernel-cokernel* pair is a pair (i, p) of morphisms of $\mathcal{A}$ with $i : A \to B$, $p : B \to C$ such that i is the kernel of p and p is the cokernel of i. Given a set $\mathcal{E}$ of kernel-cokernel pairs we say $i : A \to B$ is an *admissible monomorphism* if $(i, p) \in \mathcal{E}$ for some morphism p. Similarly we say a morphism $p : B \to C$ is an *admissible epimorphism* if $(i, p) \in \mathcal{E}$ for some morphism i. The pair $(\mathcal{A}, \mathcal{E})$ is said to be an *exact category* if the following axioms hold

(1) $\mathcal{E}$ is closed under isomorphisms of kernel-cokernel pairs,

(2) for any object A the morphism 1_A is both an admissible epimorphism and an admissible monomorphism,
(3) admissible monomorphisms are stable under composition,
(4) admissible epimorphisms are stable under composition,
(5) the push-out of an admissible monomorphism $i : A \to B$ via any morphism $A \to A'$ exist and the induced morphism $i' : A' \to B'$ is an admissible monomorphism, and
(6) the base change of an admissible epimorphism $p : B \to C$ via any morphism $C' \to C$ exist and the induced morphism $p' : B' \to C'$ is an admissible epimorphism.

Given such a structure let $\mathcal{C} = (\mathcal{A}, \text{Cov})$ where coverings (i.e., elements of Cov) are given by admissible epimorphisms. The axioms listed above immediately imply that this is a site. Consider the functor

$$F : \mathcal{A} \longrightarrow Ab(\mathcal{C}), \quad X \longmapsto h_X$$

exactly as in Lemma 19.9.2. It turns out that this functor is fully faithful, exact, and reflects exactness. Moreover, any extension of objects in the essential image of F is in the essential image of F.

19.10. Grothendieck's AB conditions

This and the next few sections are mostly interesting for "big" abelian categories, i.e., those categories listed in Categories, Remark 4.2.2. A good case to keep in mind is the category of sheaves of modules on a ringed site.

Grothendieck proved the existence of injectives in great generality in the paper [**Gro57**]. He used the following conditions to single out abelian categories with special properties.

Definition 19.10.1. Let $\mathcal{A}$ be an abelian category. We name some conditions

AB3 $\mathcal{A}$ has direct sums,
AB4 $\mathcal{A}$ has AB3 and direct sums are exact,
AB5 $\mathcal{A}$ has AB3 and filtered colimits are exact.

Here are the dual notions

AB3* $\mathcal{A}$ has products,
AB4* $\mathcal{A}$ has AB3* and products are exact,
AB5* $\mathcal{A}$ has AB3* and filtered limits are exact.

We say an object U of $\mathcal{A}$ is a *generator* if for every $N \subset M$, $N \neq M$ in $\mathcal{A}$ there exists a morphism $U \to M$ which does not factor through N. We say $\mathcal{A}$ is a *Grothendieck abelian category* if it has AB5 and a generator.

Discussion: A direct sum in an abelian category is a coproduct. If an abelian category has direct sums (i.e., AB3), then it has colimits, see Categories, Lemma 4.14.11. Similarly if $\mathcal{A}$ has AB3* then it has limits, see Categories, Lemma 4.14.10. Exactness of direct sums means the following: given an index set I and short exact sequences

$$0 \to A_i \to B_i \to C_i \to 0, \quad i \in I$$

in $\mathcal{A}$ then the sequence

$$0 \to \bigoplus\nolimits_{i \in I} A_i \to \bigoplus\nolimits_{i \in I} B_i \to \bigoplus\nolimits_{i \in I} C_i \to 0$$

is exact as well. Without assuming AB4 it is only true in general that the sequence is exact on the right (i.e., taking direct sums is a right exact functor if direct sums exist). Similarly, exactness of filtered colimits means the following: given a directed partially ordered set I and a system of short exact sequences

$$0 \to A_i \to B_i \to C_i \to 0$$

over I in $\mathcal{A}$ then the sequence

$$0 \to \operatorname{colim}_{i\in I} A_i \to \operatorname{colim}_{i\in I} B_i \to \operatorname{colim}_{i\in I} C_i \to 0$$

is exact as well. Without assuming AB5 it is only true in general that the sequence is exact on the right (i.e., taking colimits is a right exact functor if colimits exist). A similar explanation holds for AB4* and AB5*.

Lemma 19.10.2. *Let $\mathcal{A}$ be an abelian category with AB5. Let $A_1 \to A_2 \to A_3 \to \ldots$ be a system of objects of $\mathcal{A}$. Then there is a short exact sequence*

$$0 \to \bigoplus A_n \to \bigoplus A_n \to \operatorname{colim} A_n \to 0$$

where the first map in degree n is given by $1 - f_n$.

Proof. For each n we have the short exact sequence

$$0 \to A_1 \oplus \ldots \oplus A_{n-1} \to A_1 \oplus \ldots \oplus A_n \to A_n \to 0$$

where the second map is the sum of the transition maps. Take the colimit to get the sequence of the lemma. □

19.11. Injectives in Grothendieck categories

The existence of a generator implies that given an object M of a Grothendieck abelian category $\mathcal{A}$ there is a set of subobjects. (This may not be true for a general "big" abelian category.)

Definition 19.11.1. Let $\mathcal{A}$ be a Grothendieck abelian category. Let M be an object of $\mathcal{A}$. The *size* $|M|$ of M is the cardinality of the set of subobjects of M.

Lemma 19.11.2. *Let $\mathcal{A}$ be a Grothendieck abelian category. If $0 \to M' \to M \to M'' \to 0$ is a short exact sequence of $\mathcal{A}$, then $|M'|, |M''| \leq |M|$.*

Proof. Immediate from the definitions. □

Lemma 19.11.3. *Let $\mathcal{A}$ be a Grothendieck abelian category with generator U.*

1. *If $|M| \leq \kappa$, then M is the quotient of a direct sum of at most κ copies of U.*
2. *For every cardinal κ there exists a set of isomorphism classes of objects M with $|M| \leq \kappa$.*

Proof. For (1) choose for every proper subobject $M' \subset M$ a morphism $\varphi_{M'} : U \to M$ whose image is not contained in M'. Then $\bigoplus_{M'\subset M} \varphi_{M'} : \bigoplus_{M'\subset N} U \to M$ is surjective. It is clear that (1) implies (2). □

Proposition 19.11.4. *Let $\mathcal{A}$ be a Grothendieck abelian category. Let M be an object of $\mathcal{A}$. Let $\kappa = |M|$. If α is an ordinal whose cofinality is bigger than κ, then M is α-small with respect to injections.*

Proof. Please compare with Proposition 19.2.5. We need only show that the map (19.2.0.1) is a surjection. Let $f : M \to \operatorname{colim} B_\beta$ be a map. Consider the subobjects $\{f^{-1}(B_\beta)\}$ of M, where B_β is considered as a subobject of the colimit $B = \bigcup_\beta B_\beta$. If one of these, say $f^{-1}(B_\beta)$, fills M, then the map factors through B_β.

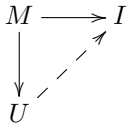
So suppose to the contrary that all of the $f^{-1}(B_\beta)$ were proper subobjects of M. However, because $\mathcal{A}$ has AB5 we have

$$\operatorname{colim} f^{-1}(B_\beta) = f^{-1}\left(\operatorname{colim} B_\beta\right) = M.$$

Now there are at most κ different subobjects of M that occur among the $f^{-1}(B_\alpha)$, by hypothesis. Thus we can find a subset $S \subset \alpha$ of cardinality at most κ such that as β' ranges over S, the $f^{-1}(B_{\beta'})$ range over *all* the $f^{-1}(B_\alpha)$.

However, S has an upper bound $\widetilde{\alpha} < \alpha$ as α has cofinality bigger than κ. In particular, all the $f^{-1}(B_{\beta'})$, $\beta' \in S$ are contained in $f^{-1}(B_{\widetilde{\alpha}})$. It follows that $f^{-1}(B_{\widetilde{\alpha}}) = M$. In particular, the map f factors through $B_{\widetilde{\alpha}}$. □

Lemma 19.11.5. *Let $\mathcal{A}$ be a Grothendieck abelian category with generator U. An object I of $\mathcal{A}$ is injective if and only if in every commutative diagram*



for $M \subset U$ a subobject, the dotted arrow exists.

Proof. Please see Lemma 19.2.6 for the case of modules. Choose an injection $A \subset B$ and a morphism $\varphi : A \to I$. Consider the set S of pairs (A', φ') consisting of subobjects $A \subset A' \subset B$ and a morphism $\varphi' : A' \to I$ extending φ. Define a partial ordering on this set in the obvious manner. Choose a totally ordered subset $T \subset S$. Then

$$A' = \operatorname{colim}_{t \in T} A_t \xrightarrow{\operatorname{colim}_{t \in T} \varphi_t} I$$

is an upper bound. Hence by Zorn's lemma the set S has a maximal element (A', φ'). We claim that $A' = B$. If not, then choose a morphism $\psi : U \to B$ which does not factor through A'. Set $N = A' \cap \psi(U)$. Set $M = \psi^{-1}(N)$. Then the map

$$M \to N \to A' \xrightarrow{\varphi'} I$$

can be extended to a morphism $\chi : U \to I$. Since $\chi|_{\operatorname{Ker}(\psi)} = 0$ we see that χ factors as

$$U \to \operatorname{Im}(\psi) \xrightarrow{\varphi''} I$$

Since φ' and φ'' agree on $N = A' \cap \operatorname{Im}(\psi)$ we see that combined the define a morphism $A' + \operatorname{Im}(\psi) \to I$ contradicting the assumed maximality of A'. □

Theorem 19.11.6. *Let $\mathcal{A}$ be a Grothendieck abelian category. Then $\mathcal{A}$ has functorial injective embeddings.*

Proof. Please compare with the proof of Theorem 19.2.8. Choose a generator U of $\mathcal{A}$. For an object M we define $\mathbf{M}(M)$ by the following pushout diagram

$$\begin{array}{ccc} \bigoplus_{N\subset U}\bigoplus_{\varphi\in\mathrm{Hom}(N,M)} N & \longrightarrow & M \\ \downarrow & & \downarrow \\ \bigoplus_{N\subset U}\bigoplus_{\varphi\in\mathrm{Hom}(U,M)} U & \longrightarrow & \mathbf{M}(M). \end{array}$$

Note that $M \to \mathbf{M}(N)$ is a functor and that there exist functorial injective maps $M \to \mathbf{M}(M)$. By transfinite induction we define functors $\mathbf{M}_\alpha(M)$ for every ordinal α. Namely, set $\mathbf{M}_0(M) = M$. Given $\mathbf{M}_\alpha(M)$ set $\mathbf{M}_{\alpha+1}(M) = \mathbf{M}(\mathbf{M}_\alpha(M))$. For a limit ordinal β set

$$\mathbf{M}_\beta(M) = \mathrm{colim}_{\alpha<\beta}\, \mathbf{M}_\alpha(M).$$

Finally, choose an ordinal α whose cofinality is greater than $|U|$, see Sets, Proposition 3.7.2. We claim that $M \to \mathbf{M}_\alpha(M)$ is the desired functorial injective embedding. Namely, if $N \subset U$ is a subobject and $\varphi : N \to \mathbf{M}_\alpha(M)$ is a morphism, then we see that φ factors through $\mathbf{M}_{\alpha'}(M)$ for some $\alpha' < \alpha$ by Proposition 19.11.4. By construction of $\mathbf{M}(-)$ we see that φ extends to a morphism from U into $\mathbf{M}_{\alpha'+1}(M)$ and hence into $\mathbf{M}_\alpha(M)$. By Lemma 19.11.5 we conclude that $\mathbf{M}_\alpha(M)$ is injective. □

19.12. K-injectives in Grothendieck categories

Most of the material in this section is taken from the paper [**Ser03**] which generalizes some of the results in the paper [**Spa88**] by Spaltenstein to general Grothendieck abelian categories. The exposition is also simplified somewhat: Lemma 19.12.3 gives a new characterization of K-injective complexes, and we have consistently tried to mimic Grothendieck's approach (see proof of Theorem 19.11.6).

Lemma 19.12.1. *Let $\mathcal{A}$ be a Grothendieck abelian category with generator U. Let c be the function on cardinals defined by $c(\kappa) = |\bigoplus_{\alpha\in\kappa} U|$. If $\pi : M \to N$ is a surjection then there exists a subobject $M' \subset M$ which surjects onto N with $|N'| \leq c(|N|)$.*

Proof. For every proper subobject $N' \subset N$ choose a morphism $\varphi_{N'} : U \to M$ such that $U \to M \to N$ does not factor through N'. Set

$$N' = \mathrm{Im}\left(\bigoplus\nolimits_{N'\subset N} \varphi_{N'} : \bigoplus\nolimits_{N'\subset N} U \longrightarrow M\right)$$

Then N' works. □

Lemma 19.12.2. *Let $\mathcal{A}$ be a Grothendieck abelian category. The exists a cardinal κ such that given any acyclic complex $M^\bullet$ there exists a surjection of complexes*

$$\bigoplus\nolimits_{i\in I} M_i^\bullet \longrightarrow M^\bullet$$

where $M_i^\bullet$ is bounded above, acyclic and $|M_i^n| \leq \kappa$.

Proof. Choose a generator U of $\mathcal{A}$. Denote c the function of Lemma 19.12.1. Set $\kappa = \sup\{c^n(|U|), n = 1, 2, 3, \ldots\}$. It suffices to prove that for every $\varphi : U \to M^n$ there exists a morphism of complexes $\varphi^\bullet : N^\bullet \to M^\bullet$ with $N^n = U$, $\varphi^n = \varphi$, $N^\bullet$ bounded above, acyclic and $|N^m| \leq \kappa$ for all m. To do this set $N^n = U$, $\varphi^n = \varphi$, $N^{n+1} = \mathrm{Im}(U \to M^n \to M^{n+1})$ and $\varphi^{n+1} : N^{n+1} \to M^{n+1}$ the inclusion

map. Moreover, we set $N^m = 0$ for $m \geq n+2$. Suppose we have constructed $\varphi^m : N^m \to M^m$ and $N^m \to M^m$ for all $m \geq k$ such that

$$\begin{array}{ccccccc} & & N^k & \longrightarrow & N^{k+1} & \longrightarrow & \ldots \\ & & \downarrow & & \downarrow & & \\ M^{k-1} & \longrightarrow & M^k & \longrightarrow & M^{k+1} & \longrightarrow & \ldots \end{array}$$

commutes, such that $|N^m| \leq c^{\max\{n-m,1\}}(|U|)$, and such that $\mathrm{Im}(N^{m-1} \to N^m) = \mathrm{Ker}(N^m \to N^{m+1})$ for all $m \geq k+1$. Then we can choose

$$N^{k-1} \longrightarrow \mathrm{Ker}(N^k \to N^{k+1}) \times_{M^k} M^{k-1}$$

surjecting onto $\mathrm{Ker}(N^k \to N^{k+1})$ as in Lemma 19.12.1. This is possible as the complex $M^\bullet$ is exact by assumption. Hence we win by induction. □

Lemma 19.12.3. *Let $\mathcal{A}$ be a Grothendieck abelian category. Let κ be a cardinal as in Lemma 19.12.2. Suppose that $I^\bullet$ is a complex such that*

1. *each I^j is injective, and*
2. *for every bounded above acyclic complex $M^\bullet$ such that $|M^n| \leq \kappa$ we have $\mathrm{Hom}_{K(\mathcal{A})}(M^\bullet, I^\bullet) = 0$.*

Then $I^\bullet$ is an K-injective complex.

Proof. Let $M^\bullet$ be an acyclic complex. By Lemma 19.12.2 and Derived Categories, Lemma 13.15.5 we can find a resolution

$$\ldots \to M_2^\bullet \to M_1^\bullet \to M_0^\bullet \to M^\bullet \to 0$$

(in the abelian category of complexes) where each $M_k^\bullet$ is a direct sum of complexes which satisfy the hypotheses of (2). Then we obtain a short exact sequence of complexes

$$0 \to \bigoplus\nolimits_{i \geq 1} M_i^\bullet \to \bigoplus\nolimits_{i \geq 0} M_i^\bullet \to M^\bullet \to 0$$

In other words we have constructed a short exact sequence of complexes $0 \to K^\bullet \to L^\bullet \to M^\bullet \to 0$ where $K^\bullet$ and $L^\bullet$ is a direct sum of complexes as in (2). Since each component of $I^\bullet$ is injective we see that we obtain an exact sequence

$$\mathrm{Hom}_{K(\mathcal{A})}(K^\bullet[1], I^\bullet) \to \mathrm{Hom}_{K(\mathcal{A})}(M^\bullet, I^\bullet) \to \mathrm{Hom}_{K(\mathcal{A})}(L^\bullet, I^\bullet)$$

Hence we see that it suffices to prove $\mathrm{Hom}_{K(\mathcal{A})}(M^\bullet, I^\bullet)$ is zero when $M^\bullet$ is a direct sum of complexes as in (2). This follows from assumption (2). □

Lemma 19.12.4. *Let $\mathcal{A}$ be a Grothendieck abelian category. Let $(K_i^\bullet)_{i \in I}$ be a set of acyclic complexes. There exists a functor $M^\bullet \mapsto \mathbf{M}^\bullet(M^\bullet)$ and a natural transformation $j_{M^\bullet} : M^\bullet \to \mathbf{M}^\bullet(M^\bullet)$ such*

1. *$j_{M^\bullet}$ is a (termwise) injective quasi-isomorphism, and*
2. *for every $i \in I$ and $w : K_i^\bullet \to M^\bullet$ the morphism $j_{M^\bullet} \circ w$ is homotopic to zero.*

Proof. For every $i \in I$ choose a (termwise) injective map of complexes $K_i^\bullet \to L_i^\bullet$ which is homotopic to zero with $L_i^\bullet$ quasi-isomorphic to zero. For example, take $L_i^\bullet$

to be the cone on the identity of $K_i^\bullet$. We define $\mathbf{M}^\bullet(M^\bullet)$ by the following pushout diagram

$$\begin{array}{ccc} \bigoplus_{i\in I}\bigoplus_{w:K_i^\bullet\to M^\bullet} K_i^\bullet & \longrightarrow & M^\bullet \\ \downarrow & & \downarrow \\ \bigoplus_{i\in I}\bigoplus_{w:K_i^\bullet\to M^\bullet} L_i^\bullet & \longrightarrow & \mathbf{M}^\bullet(M^\bullet). \end{array}$$

Then $M^\bullet \to \mathbf{M}^\bullet(M^\bullet)$ is a functor. The right vertical arrow defines the functorial injective map $j_{M^\bullet}$. The cokernel of $j_{M^\bullet}$ is isomorphic to the direct sum of the cokernels of the maps $K_i^\bullet \to L_i^\bullet$ hence acyclic. Thus $j_{M^\bullet}$ is a quasi-isomorphism. Part (2) holds by construction. □

Lemma 19.12.5. *Let $\mathcal{A}$ be a Grothendieck abelian category. There exists a functor $M^\bullet \mapsto \mathbf{N}^\bullet(M^\bullet)$ and a natural transformation $j_{M^\bullet} : M^\bullet \to \mathbf{N}^\bullet(M^\bullet)$ such*

1. *$j_{M^\bullet}$ is a (termwise) injective quasi-isomorphism, and*
2. *for every $n \in \mathbf{Z}$ the map $M^n \to \mathbf{N}^n(M^\bullet)$ factors through a subobject $I^n \subset \mathbf{N}^n(M^\bullet)$ where I^n is an injective object of $\mathcal{A}$.*

Proof. Choose a functorial injective embeddings $i_M : M \to I(M)$, see Theorem 19.11.6. For every complex $M^\bullet$ denote $J^\bullet(M^\bullet)$ the complex with terms $J^n(M^\bullet) = I(M^n) \oplus I(M^{n+1})$ and differential

$$d_{J^\bullet(M^\bullet)} = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$$

There exists a canonical injective map of complexes $u_{M^\bullet} : M^\bullet \to J^\bullet(M^\bullet)$ by mapping M^n to $I(M^n) \oplus I(M^{n+1})$ via the maps $i_{M^n} : M^n \to I(M^n)$ and $i_{M^{n+1}} \circ d : M^n \to M^{n+1} \to I(M^{n+1})$. Hence a short exact sequence of complexes

$$0 \to M^\bullet \xrightarrow{u_{M^\bullet}} J^\bullet(M^\bullet) \xrightarrow{v_{M^\bullet}} Q^\bullet(M^\bullet) \to 0$$

functorial in $M^\bullet$. Set

$$\mathbf{N}^\bullet(M^\bullet) = C(v_{M^\bullet})^\bullet[-1].$$

Note that

$$\mathbf{N}^n(M^\bullet) = Q^{n-1}(M^\bullet) \oplus J^n(M^\bullet)$$

with differential

$$\begin{pmatrix} -d^{n-1}_{Q^\bullet(M^\bullet)} & -v^n_{M^\bullet} \\ 0 & d^n_{J^\bullet(M)} \end{pmatrix}$$

Hence we see that there is a map of complexes $j_{M^\bullet} : M^\bullet \to \mathbf{N}^\bullet(M^\bullet)$ induced by u. It is injective and factors through an injective subobject by construction. The map $j_{M^\bullet}$ is a quasi-isomorphism as one can prove by looking at the long exact sequence of cohomology associated to the short exact sequences of complexes above. □

Theorem 19.12.6. *Let $\mathcal{A}$ be a Grothendieck abelian category. For every complex $M^\bullet$ there exists a quasi-isomorphism $M^\bullet \to I^\bullet$ where $I^\bullet$ is a K-injective complex. In fact, we may also assume that I^n is an injective object of $\mathcal{A}$ for all n. Moreover, there exists a functorial injective quasi-isomorphism into such a K-injective complex.*

Proof. Please compare with the proof of Theorem 19.2.8 and Theorem 19.11.6. Choose a cardinal κ as in Lemmas 19.12.2 and 19.12.3. Choose a set $(K_i^\bullet)_{i\in I}$ of bounded above, acyclic complexes such that every bounded above acyclic complex $K^\bullet$ such that $|K^n| \leq \kappa$ is isomorphic to $K_i^\bullet$ for some $i \in I$. This is possible by Lemma 19.11.3. Denote $\mathbf{M}^\bullet(-)$ the functor constructed in Lemma 19.12.4. Denote $\mathbf{N}^\bullet(-)$ the functor constructed in Lemma 19.12.5. Both of these functors come with injective transformations $\mathrm{id} \to \mathbf{M}$ and $\mathrm{id} \to \mathbf{N}$.

By transfinite induction we define a sequence of functors $\mathbf{T}_\alpha(-)$ and corresponding transformations $\mathrm{id} \to \mathbf{T}_\alpha$. Namely we set $\mathbf{T}_0(M^\bullet) = M^\bullet$. If $\mathbf{T}_\alpha$ is given then we set

$$\mathbf{T}_{\alpha+1}(M^\bullet) = \mathbf{N}^\bullet(\mathbf{M}^\bullet(\mathbf{T}_\alpha(M^\bullet)))$$

If β is a limit ordinal we set

$$\mathbf{T}_\beta(M^\bullet) = \mathrm{colim}_{\alpha<\beta} \mathbf{T}_\alpha(M^\bullet)$$

The transition maps of the system are injective quasi-isomorphisms. By AB5 we see that the colimit is still quasi-isomorphic to $M^\bullet$. We claim that $M^\bullet \to \mathbf{T}_\alpha(M^\bullet)$ does the job if the cofinality of α is larger than $\max(\kappa, |U|)$ where U is a generator of $\mathcal{A}$. Namely, it suffices to check conditions (1) and (2) of Lemma 19.12.3.

For (1) we use the criterion of Lemma 19.11.5. Suppose that $M \subset U$ and $\varphi : M \to \mathbf{T}_\alpha^n(M^\bullet)$ is a morphism for some $n \in \mathbf{Z}$. By Proposition 19.11.4 we see that φ factor through $\mathbf{T}_{\alpha'}^n(M^\bullet)$ for some $\alpha' < \alpha$. In particular, by the construction of the functor $\mathbf{N}^\bullet(-)$ we see that φ factors through an injective object of $\mathcal{A}$ which shows that φ lifts to a morphism on U.

For (2) let $w : K^\bullet \to \mathbf{T}_\alpha(M^\bullet)$ be a morphism of complexes where $K^\bullet$ is a bounded above acyclic complex such that $|K^n| \leq \kappa$. Then $K^\bullet \cong K_i^\bullet$ for some $i \in I$. Moreover, by Proposition 19.11.4 once again we see that w factor through $\mathbf{T}_{\alpha'}^n(M^\bullet)$ for some $\alpha' < \alpha$. In particular, by the construction of the functor $\mathbf{M}^\bullet(-)$ we see that w is homotopic to zero. This finishes the proof. □

19.13. Additional remarks on Grothendieck abelian categories

In this section we put some results on Grothendieck abelian categories which are folklore.

Lemma 19.13.1. *Let $\mathcal{A}$ be a Grothendieck abelian category. Let $F : \mathcal{A}^{opp} \to Sets$ be a functor. Then F is representable if and only if F commutes with colimits, i.e.,*

$$F(\mathrm{colim}_i N_i) = \lim F(N_i)$$

for any diagram $\mathcal{I} \to \mathcal{A}$, $i \in \mathcal{I}$.

Proof. If F is representable, then it commutes with colimits by definition of colimits.

Assume that F commutes with colimits. Then $F(M \oplus N) = F(M) \prod F(N)$ and we can use this to define a group structure on $F(M)$. Hence we get $F : \mathcal{A} \to Ab$ which is additive and right exact, i.e., transforms a short exact sequence $0 \to K \to L \to M \to 0$ into an exact sequence $F(K) \leftarrow F(L) \leftarrow F(M) \leftarrow 0$ (compare with Homology, Section 12.7).

Let U be a generator for $\mathcal{A}$. Set $A = \bigoplus_{s\in F(U)} U$. Let $s_{univ} = (s)_{s\in F(U)} \in F(A) = \prod_{s\in F(U)} F(U)$. Let $A' \subset A$ be the largest subobject such that s_{univ} restricts to zero

on A'. This exists because $\mathcal{A}$ is a grothendieck category and because F commutes with colimits. Because F commutes with colimits there exists a unique element $\overline{s}_{univ} \in F(A/A')$ which maps to s_{univ} in $F(A)$. We claim that A/A' represents F, in other words, the Yoneda map

$$\overline{s}_{univ} : h_{A/A'} \longrightarrow F$$

is an isomorphism. Let $M \in \mathrm{Ob}(\mathcal{A})$ and $s \in F(M)$. Consider the surjection

$$c_M : A_M = \bigoplus\nolimits_{\varphi \in \mathrm{Hom}_{\mathcal{A}}(U,M)} U \longrightarrow M.$$

This gives $F(c_M)(s) = (s_\varphi) \in \prod_\varphi F(U)$. Consider the map

$$\psi : A_M = \bigoplus\nolimits_{\varphi \in \mathrm{Hom}_{\mathcal{A}}(U,M)} U \longrightarrow \bigoplus\nolimits_{s \in F(U)} U = A$$

which maps the summand corresponding to φ to the summand corresponding to s_φ by the identity map on U. Then s_{univ} maps to $(s_\varphi)_\varphi$ by construction. in other words the right square in the diagram

$$\begin{array}{ccccc} A' & \longrightarrow & A & \overset{s_{univ}}{\dashrightarrow} & F \\ \uparrow{\scriptstyle ?} & & \uparrow{\scriptstyle \psi} & & \uparrow{\scriptstyle s} \\ K & \longrightarrow & A_M & \longrightarrow & M \end{array}$$

commutes. Let $K = \mathrm{Ker}(A_M \to M)$. Since s restricts to zero on K we see that $\psi(K) \subset A'$ by definition of A'. Hence there is an induced morphism $M \to A/A'$. This construction gives an inverse to the map $h_{A/A'}(M) \to F(M)$ (details omitted). □

Lemma 19.13.2. *A Grothendieck abelian category has Ab3*.*

Proof. Let M_i, $i \in I$ be a family of objects of $\mathcal{A}$ indexed by a set I. The functor $F = \prod_{i \in I} h_{M_i}$ commutes with colimits. Hence Lemma 19.13.1 applies. □

Remark 19.13.3. In the chapter on derived categories we consistently work with "small" abelian categories (as is the convention in the Stacks project). For a "big" abelian category $\mathcal{A}$ it isn't clear that the derived category $D(\mathcal{A})$ exists because it isn't clear that morphisms in the derived category are sets. In general this isn't true, see Examples, Lemma 78.46.1. However, if $\mathcal{A}$ is a Grothendieck abelian category, and given $K^\bullet, L^\bullet$ in $K(\mathcal{A})$, then by Theorem 19.12.6 there exists a quasi-isomorphism $L^\bullet \to I^\bullet$ to a K-injective complex $I^\bullet$ and Derived Categories, Lemma 13.28.2 shows that

$$\mathrm{Hom}_{D(\mathcal{A})}(K^\bullet, L^\bullet) = \mathrm{Hom}_{K(\mathcal{A})}(K^\bullet, I^\bullet)$$

which is a set. Some examples of Grothendieck abelian categories are the category of modules over a ring, or more generally the category of sheaves of modules on a ringed site.

Lemma 19.13.4. *Let $\mathcal{A}$ be a Grothendieck abelian category. Then*

1. *$D(\mathcal{A})$ has both direct sums and products,*
2. *direct sums are obtained by taking termwise direct sums of any complexes,*
3. *products are obtained by taking termwise products of K-injective complexes.*

Proof. Let $K_i^\bullet$, $i \in I$ be a family of objects of $D(\mathcal{A})$ indexed by a set I. We claim that the termwise direct sum $\bigoplus_{i\in I} K_i^\bullet$ is a direct sum in $D(\mathcal{A})$. Namely, let $I^\bullet$ be a K-injective complex. Then we have

$$\begin{aligned}\mathrm{Hom}_{D(\mathcal{A})}(\bigoplus\nolimits_{i\in I} K_i^\bullet, I^\bullet) &= \mathrm{Hom}_{K(\mathcal{A})}(\bigoplus\nolimits_{i\in I} K_i^\bullet, I^\bullet) \\ &= \prod\nolimits_{i\in I} \mathrm{Hom}_{K(\mathcal{A})}(K_i^\bullet, I^\bullet) \\ &= \prod\nolimits_{i\in I} \mathrm{Hom}_{D(\mathcal{A})}(K_i^\bullet, I^\bullet)\end{aligned}$$

as desired. This is sufficient since any complex can be represented by a K-injective complex by Theorem 19.12.6. To construct the product, choose a K-injective resolution $K_i^\bullet \to I_i^\bullet$ for each i. Then we claim that $\prod_{i\in I} I_i^\bullet$ is a product in $D(\mathcal{A})$. Namely, let $K^\bullet$ be an complex. Note that a product of K-injective complexes is K-injective (follows immediately from the definition). Thus we have

$$\begin{aligned}\mathrm{Hom}_{D(\mathcal{A})}(K^\bullet, \prod\nolimits_{i\in I} I_i^\bullet) &= \mathrm{Hom}_{K(\mathcal{A})}(K^\bullet, \prod\nolimits_{i\in I} I_i^\bullet) \\ &= \prod\nolimits_{i\in I} \mathrm{Hom}_{K(\mathcal{A})}(K^\bullet, I_i^\bullet) \\ &= \prod\nolimits_{i\in I} \mathrm{Hom}_{D(\mathcal{A})}(K^\bullet, I_i^\bullet)\end{aligned}$$

which proves the result. □

Remark 19.13.5. Let R be a ring. Suppose that M_n, $n \in \mathbf{Z}$ are R-modules. Denote $E_n = M_n[-n] \in D(R)$. We claim that $E = \bigoplus M_n[-n]$ is *both* the direct sum and the product of the objects E_n in $D(R)$. To see that it is the direct sum, take a look at the proof of Lemma 19.13.4. To see that it is the direct product, take injective resolutions $M_n \to I_n^\bullet$. By the proof of Lemma 19.13.4 we have

$$\prod E_n = \prod I_n^\bullet[-n]$$

in $D(R)$. Since products in Mod_R are exact, we see that $\prod I_n^\bullet$ is quasi-isomorphic to E. This works more generally in $D(\mathcal{A})$ where $\mathcal{A}$ is a Grothendieck abelian category with Ab4*.

Lemma 19.13.6. *Let $F : \mathcal{A} \to \mathcal{B}$ be an additive functor of abelian categories. Assume*

(1) *$\mathcal{A}$ is a Grothendieck abelian category,*
(2) *$\mathcal{B}$ has exact countable products, and*
(3) *F commutes with countable products.*

Then $RF : D(\mathcal{A}) \to D(\mathcal{B})$ commutes with derived limits.

Proof. Observe that RF exists as $\mathcal{A}$ has enough K-injectives (Theorem 19.12.6 and Derived Categories, Lemma 13.28.5). The statement means that if $K = R\lim K_n$, then $RF(K) = R\lim RF(K_n)$. See Derived Categories, Definition 13.30.2 for notation. Since RF is an exact functor of triangulated categories it suffices to see that RF commutes with countable products of objects of $D(\mathcal{A})$. In the proof of Lemma 19.13.4 we have seen that products in $D(\mathcal{A})$ are computed by taking products of K-injective complexes and moreover that a product of K-injective complexes is K-injective. Moreover, in Derived Categories, Lemma 13.30.3 we have seen that products in $D(\mathcal{B})$ are computed by taking termwise products. Since RF is computed by applying F to a K-injective representative and since we've assumed F commutes with countable products, the lemma follows. □

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CHAPTER 20

Cohomology of Sheaves

20.1. Introduction

In this document we work out some topics on cohomology of sheaves on topological spaces. We mostly work in the generality of modules over a sheaf of rings and we work with morphisms of ringed spaces. To see what happens for sheaves on sites take a look at the chapter Cohomology on Sites, Section 21.1. Basic references are [**God73**] and [**Ive86**].

20.2. Topics

Here are some topics that should be discussed in this chapter, and have not yet been written.

(1) Ext-groups.
(2) Ext sheaves.
(3) Tor functors.
(4) Derived pullback for morphisms between ringed spaces.
(5) Cup-product.
(6) Etc, etc, etc.

20.3. Cohomology of sheaves

Let X be a topological space. Let $\mathcal{F}$ be a abelian sheaf. We know that the category of abelian sheaves on X has enough injectives, see Injectives, Lemma 19.4.1. Hence we can choose an injective resolution $\mathcal{F}[0] \to \mathcal{I}^\bullet$. As is customary we define

$$H^i(X, \mathcal{F}) = H^i(\Gamma(X, \mathcal{I}^\bullet)) \tag{20.3.0.1}$$

to be the *ith cohomology group of the abelian sheaf* $\mathcal{F}$. The family of functors $H^i((X, -)$ forms a universal δ-functor from $Ab(X) \to Ab$.

Let $f : X \to Y$ be a continuous map of topological spaces. With $\mathcal{F}[0] \to \mathcal{I}^\bullet$ as above we define

$$R^i f_* \mathcal{F} = H^i(f_* \mathcal{I}^\bullet) \tag{20.3.0.2}$$

to be the *ith higher direct image of* $\mathcal{F}$. The family of functors $R^i f_*$ forms a universal δ-functor from $Ab(X) \to Ab(Y)$.

Let $(X, \mathcal{O}_X)$ be a ringed space. Let $\mathcal{F}$ be an $\mathcal{O}_X$-module. We know that the category of $\mathcal{O}_X$-modules on X has enough injectives, see Injectives, Lemma 19.5.1. Hence we can choose an injective resolution $\mathcal{F}[0] \to \mathcal{I}^\bullet$. As is customary we define

$$H^i(X, \mathcal{F}) = H^i(\Gamma(X, \mathcal{I}^\bullet)) \tag{20.3.0.3}$$

to be the *ith cohomology group of* $\mathcal{F}$. The family of functors $H^i((X, -)$ forms a universal δ-functor from $Mod(\mathcal{O}_X) \to \mathrm{Mod}_{\mathcal{O}_X(X)}$.

Let $f : (X, \mathcal{O}_X) \to (Y, \mathcal{O}_Y)$ be a morphism of ringed spaces. With $\mathcal{F}[0] \to \mathcal{I}^\bullet$ as above we define

$$R^i f_* \mathcal{F} = H^i(f_* \mathcal{I}^\bullet) \tag{20.3.0.4}$$

to be the *ith higher direct image of* $\mathcal{F}$. The family of functors $R^i f_*$ forms a universal δ-functor from $Mod(\mathcal{O}_X) \to Mod(\mathcal{O}_Y)$.

20.4. Derived functors

We briefly explain an approach to right derived functors using resolution functors. Let $(X, \mathcal{O}_X)$ be a ringed space. The category $Mod(\mathcal{O}_X)$ is abelian, see Modules, Lemma 17.3.1. In this chapter we will write

$$K(X) = K(\mathcal{O}_X) = K(Mod(\mathcal{O}_X)) \quad \text{and} \quad D(X) = D(\mathcal{O}_X) = D(Mod(\mathcal{O}_X)).$$

and similarly for the bounded versions for the triangulated categories introduced in Derived Categories, Definition 13.7.1 and Definition 13.10.3. By Derived Categories, Remark 13.23.3 there exists a resolution functor

$$j = j_X : K^+(Mod(\mathcal{O}_X)) \longrightarrow K^+(\mathcal{I})$$

where $\mathcal{I}$ is the strictly full additive subcategory of $Mod(\mathcal{O}_X)$ consisting of injective sheaves. For any left exact functor $F : Mod(\mathcal{O}_X) \to \mathcal{B}$ into any abelian category $\mathcal{B}$ we will denote RF the right derived functor described in Derived Categories, Section 13.19 and constructed using the resolution functor j_X just described:

$$RF = F \circ j'_X : D^+(X) \longrightarrow D^+(\mathcal{B}) \tag{20.4.0.5}$$

see Derived Categories, Lemma 13.24.1 for notation. Note that we may think of RF as defined on $Mod(\mathcal{O}_X)$, $\mathrm{Comp}^+(Mod(\mathcal{O}_X))$, $K^+(X)$, or $D^+(X)$ depending on the situation. According to Derived Categories, Definition 13.16.2 we obtain the ith right derived functor

$$R^i F = H^i \circ RF : Mod(\mathcal{O}_X) \longrightarrow \mathcal{B} \tag{20.4.0.6}$$

so that $R^0 F = F$ and $\{R^i F, \delta\}_{i \geq 0}$ is universal δ-functor, see Derived Categories, Lemma 13.19.4.

Here are two special cases of this construction. Given a ring R we write $K(R) = K(\mathrm{Mod}_R)$ and $D(R) = D(\mathrm{Mod}_R)$ and similarly for bounded versions. For any open $U \subset X$ we have a left exact functor $\Gamma(U, -) : Mod(\mathcal{O}_X) \longrightarrow \mathrm{Mod}_{\mathcal{O}_X(U)}$ which gives rise to

$$R\Gamma(U, -) : D^+(X) \longrightarrow D^+(\mathcal{O}_X(U)) \tag{20.4.0.7}$$

by the discussion above. We set $H^i(U, -) = R^i\Gamma(U, -)$. If $U = X$ we recover (20.3.0.3). If $f : X \to Y$ is a morphism of ringed spaces, then we have the left exact functor $f_* : Mod(\mathcal{O}_X) \longrightarrow Mod(\mathcal{O}_Y)$ which gives rise to the *derived pushforward*

$$Rf_* : D^+(X) \longrightarrow D^+(Y) \tag{20.4.0.8}$$

The ith cohomology sheaf of $Rf_*\mathcal{F}^\bullet$ is denoted $R^i f_* \mathcal{F}^\bullet$ and called the ith *higher direct image* in accordance with (20.3.0.4). The two displayed functors above are exact functor of derived categories.

Abuse of notation: When the functor Rf_*, or any other derived functor, is applied to a sheaf $\mathcal{F}$ on X or a complex of sheaves it is understood that $\mathcal{F}$ has been replaced by a suitable resolution of $\mathcal{F}$. To facilitate this kind of operation we will say, given an object $\mathcal{F}^\bullet \in D(X)$, that a bounded below complex $\mathcal{I}^\bullet$ of injectives of

$Mod(\mathcal{O}_X)$ represents $\mathcal{F}^\bullet$ in the derived category if there exists a quasi-isomorphism $\mathcal{F}^\bullet \to \mathcal{I}^\bullet$. In the same vein the phrase "let $\alpha : \mathcal{F}^\bullet \to \mathcal{G}^\bullet$ be a morphism of $D(X)$" does not mean that α is represented by a morphism of complexes. If we have an actual morphism of complexes we will say so.

20.5. First cohomology and torsors

Definition 20.5.1. Let X be a topological space. Let $\mathcal{G}$ be a sheaf of (possibly non-commutative) groups on X. A *torsor*, or more precisely a *$\mathcal{G}$-torsor*, is a sheaf of sets $\mathcal{F}$ on X endowed with an action $\mathcal{G} \times \mathcal{F} \to \mathcal{F}$ such that

(1) whenever $\mathcal{F}(U)$ is nonempty the action $\mathcal{G}(U) \times \mathcal{F}(U) \to \mathcal{F}(U)$ is simply transitive, and
(2) for every $x \in X$ the stalk $\mathcal{F}_x$ is nonempty.

A *morphism of $\mathcal{G}$-torsors $\mathcal{F} \to \mathcal{F}'$* is simply a morphism of sheaves of sets compatible with the $\mathcal{G}$-actions. The *trivial $\mathcal{G}$-torsor* is the sheaf $\mathcal{G}$ endowed with the obvious left $\mathcal{G}$-action.

It is clear that a morphism of torsors is automatically an isomorphism.

Lemma 20.5.2. *Let X be a topological space. Let $\mathcal{G}$ be a sheaf of (possibly non-commutative) groups on X. A $\mathcal{G}$-torsor $\mathcal{F}$ is trivial if and only if $\mathcal{F}(X) \neq \emptyset$.*

Proof. Omitted. □

Lemma 20.5.3. *Let X be a topological space. Let $\mathcal{H}$ be an abelian sheaf on X. There is a canonical bijection between the set of isomorphism classes of $\mathcal{H}$-torsors and $H^1(X, \mathcal{H})$.*

Proof. Let $\mathcal{F}$ be a $\mathcal{H}$-torsor. Consider the free abelian sheaf $\mathbf{Z}[\mathcal{F}]$ on $\mathcal{F}$. It is the sheafification of the rule which associates to $U \subset X$ open the collection of finite formal sums $\sum n_i[s_i]$ with $n_i \in \mathbf{Z}$ and $s_i \in \mathcal{F}(U)$. There is a natural map

$$\sigma : \mathbf{Z}[\mathcal{F}] \longrightarrow \underline{\mathbf{Z}}$$

which to a local section $\sum n_i[s_i]$ associates $\sum n_i$. The kernel of σ is generated by the local section of the form $[s] - [s']$. There is a canonical map $a : \mathrm{Ker}(\sigma) \to \mathcal{H}$ which maps $[s] - [s'] \mapsto h$ where h is the local section of $\mathcal{H}$ such that $h \cdot s = s'$. Consider the pushout diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \mathrm{Ker}(\sigma) & \longrightarrow & \mathbf{Z}[\mathcal{F}] & \longrightarrow & \underline{\mathbf{Z}} & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle a} & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & \mathcal{H} & \longrightarrow & \mathcal{E} & \longrightarrow & \underline{\mathbf{Z}} & \longrightarrow & 0 \end{array}$$

Here $\mathcal{E}$ is the extension obtained by pushout. From the long exact cohomology sequence associated to the lower short exact sequence we obtain an element $\xi = \xi_{\mathcal{F}} \in H^1(X, \mathcal{H})$ by applying the boundary operator to $1 \in H^0(X, \underline{\mathbf{Z}})$.

Conversely, given $\xi \in H^1(X, \mathcal{H})$ we can associate to ξ a torsor as follows. Choose an embedding $\mathcal{H} \to \mathcal{I}$ of $\mathcal{H}$ into an injective abelian sheaf $\mathcal{I}$. We set $\mathcal{Q} = \mathcal{I}/\mathcal{H}$ so that we have a short exact sequence

$$0 \longrightarrow \mathcal{H} \longrightarrow \mathcal{I} \longrightarrow \mathcal{Q} \longrightarrow 0$$

The element ξ is the image of a global section $q \in H^0(X, \mathcal{Q})$ because $H^1(X, \mathcal{I}) = 0$ (see Derived Categories, Lemma 13.19.4). Let $\mathcal{F} \subset \mathcal{I}$ be the subsheaf (of sets) of sections that map to q in the sheaf $\mathcal{Q}$. It is easy to verify that $\mathcal{F}$ is a torsor.

We omit the verification that the two constructions given above are mutually inverse. □

20.6. First cohomology and invertible sheaves

The Picard group of a ringed space is defined in Modules, Section 17.21.

Lemma 20.6.1. *Let $(X, \mathcal{O}_X)$ be a ringed space. There is a canonical isomorphism*

$$H^1(X, \mathcal{O}_X^*) = Pic(X).$$

of abelian groups.

Proof. Let $\mathcal{L}$ be an invertible $\mathcal{O}_X$-module. Consider the presheaf $\mathcal{L}^*$ defined by the rule

$$U \longmapsto \{s \in \mathcal{L}(U) \text{ such that } \mathcal{O}_U \xrightarrow{s \cdot -} \mathcal{L}_U \text{ is an isomorphism}\}$$

This presheaf satisfies the sheaf condition. Moreover, if $f \in \mathcal{O}_X^*(U)$ and $s \in \mathcal{L}^*(U)$, then clearly $fs \in \mathcal{L}^*(U)$. By the same token, if $s, s' \in \mathcal{L}^*(U)$ then there exists a unique $f \in \mathcal{O}_X^*(U)$ such that $fs = s'$. Moreover, the sheaf $\mathcal{L}^*$ has sections locally by the very definition of an invertible sheaf. In other words we see that $\mathcal{L}^*$ is a $\mathcal{O}_X^*$-torsor. Thus we get a map

$$\begin{array}{c}\text{invertible sheaves on } (X, \mathcal{O}_X) \\ \text{up to isomorphism}\end{array} \longrightarrow \begin{array}{c}\mathcal{O}_X^*\text{-torsors} \\ \text{up to isomorphism}\end{array}$$

We omit the verification that this is a homomorphism of abelian groups. By Lemma 20.5.3 the right hand side is canonically bijective to $H^1(X, \mathcal{O}_X^*)$. Thus we have to show this map is injective and surjective.

Injective. If the torsor $\mathcal{L}^*$ is trivial, this means by Lemma 20.5.2 that $\mathcal{L}^*$ has a global section. Hence this means exactly that $\mathcal{L} \cong \mathcal{O}_X$ is the neutral element in $\mathrm{Pic}(X)$.

Surjective. Let $\mathcal{F}$ be an $\mathcal{O}_X^*$-torsor. Consider the presheaf of sets

$$\mathcal{L}_1 : U \longmapsto (\mathcal{F}(U) \times \mathcal{O}_X(U))/\mathcal{O}_X^*(U)$$

where the action of $f \in \mathcal{O}_X^*(U)$ on (s, g) is $(fs, f^{-1}g)$. Then $\mathcal{L}_1$ is a presheaf of $\mathcal{O}_X$-modules by setting $(s, g) + (s', g') = (s, g + (s'/s)g')$ where s'/s is the local section f of $\mathcal{O}_X^*$ such that $fs = s'$, and $h(s, g) = (s, hg)$ for h a local section of $\mathcal{O}_X$. We omit the verification that the sheafification $\mathcal{L} = \mathcal{L}_1^\#$ is an invertible $\mathcal{O}_X$-module whose associated $\mathcal{O}_X^*$-torsor $\mathcal{L}^*$ is isomorphic to $\mathcal{F}$. □

20.7. Locality of cohomology

The following lemma says there is no ambiguity in defining the cohomology of a sheaf $\mathcal{F}$ over an open.

Lemma 20.7.1. *Let X be a ringed space. Let $U \subset X$ be an open subspace.*

1. *If $\mathcal{I}$ is an injective $\mathcal{O}_X$-module then $\mathcal{I}|_U$ is an injective $\mathcal{O}_U$-module.*
2. *For any sheaf of $\mathcal{O}_X$-modules $\mathcal{F}$ we have $H^p(U, \mathcal{F}) = H^p(U, \mathcal{F}|_U)$.*

Proof. Denote $j : U \to X$ the open immersion. Recall that the functor j^{-1} of restriction to U is a right adjoint to the functor $j_!$ of extension by 0, see Sheaves, Lemma 6.31.8. Moreover, $j_!$ is exact. Hence (1) follows from Homology, Lemma 12.25.1.

By definition $H^p(U, \mathcal{F}) = H^p(\Gamma(U, \mathcal{I}^\bullet))$ where $\mathcal{F} \to \mathcal{I}^\bullet$ is an injective resolution in $Mod(\mathcal{O}_X)$. By the above we see that $\mathcal{F}|_U \to \mathcal{I}^\bullet|_U$ is an injective resolution in $Mod(\mathcal{O}_U)$. Hence $H^p(U, \mathcal{F}|_U)$ is equal to $H^p(\Gamma(U, \mathcal{I}^\bullet|_U))$. Of course $\Gamma(U, \mathcal{F}) = \Gamma(U, \mathcal{F}|_U)$ for any sheaf $\mathcal{F}$ on X. Hence the equality in (2). □

Let X be a ringed space. Let $\mathcal{F}$ be a sheaf of $\mathcal{O}_X$-modules. Let $U \subset V \subset X$ be open subsets. Then there is a canonical *restriction mapping*

$$H^n(V, \mathcal{F}) \longrightarrow H^n(U, \mathcal{F}), \quad \xi \longmapsto \xi|_U \tag{20.7.1.1}$$

functorial in $\mathcal{F}$. Namely, choose any injective resolution $\mathcal{F} \to \mathcal{I}^\bullet$. The restriction mappings of the sheaves $\mathcal{I}^p$ give a morphism of complexes

$$\Gamma(V, \mathcal{I}^\bullet) \longrightarrow \Gamma(U, \mathcal{I}^\bullet)$$

The LHS is a complex representing $R\Gamma(V, \mathcal{F})$ and the RHS is a complex representing $R\Gamma(U, \mathcal{F})$. We get the map on cohomology groups by applying the functor H^n. As indicated we will use the notation $\xi \mapsto \xi|_U$ to denote this map. Thus the rule $U \mapsto H^n(U, \mathcal{F})$ is a presheaf of $\mathcal{O}_X$-modules. This presheaf is customarily denoted $\underline{H}^n(\mathcal{F})$. We will give another interpretation of this presheaf in Lemma 20.12.3.

Lemma 20.7.2. *Let X be a ringed space. Let $\mathcal{F}$ be a sheaf of $\mathcal{O}_X$-modules. Let $U \subset X$ be an open subspace. Let $n > 0$ and let $\xi \in H^n(U, \mathcal{F})$. Then there exists an open covering $U = \bigcup_{i \in I} U_i$ such that $\xi|_{U_i} = 0$ for all $i \in I$.*

Proof. Let $\mathcal{F} \to \mathcal{I}^\bullet$ be an injective resolution. Then

$$H^n(U, \mathcal{F}) = \frac{\mathrm{Ker}(\mathcal{I}^n(U) \to \mathcal{I}^{n+1}(U))}{\mathrm{Im}(\mathcal{I}^{n-1}(U) \to \mathcal{I}^n(U))}.$$

Pick an element $\tilde{\xi} \in \mathcal{I}^n(U)$ representing the cohomology class in the presentation above. Since $\mathcal{I}^\bullet$ is an injective resolution of $\mathcal{F}$ and $n > 0$ we see that the complex $\mathcal{I}^\bullet$ is exact in degree n. Hence $\mathrm{Im}(\mathcal{I}^{n-1} \to \mathcal{I}^n) = \mathrm{Ker}(\mathcal{I}^n \to \mathcal{I}^{n+1})$ as sheaves. Since $\tilde{\xi}$ is a section of the kernel sheaf over U we conclude there exists an open covering $U = \bigcup_{i \in I} U_i$ such that $\tilde{\xi}|_{U_i}$ is the image under d of a section $\xi_i \in \mathcal{I}^{n-1}(U_i)$. By our definition of the restriction $\xi|_{U_i}$ as corresponding to the class of $\tilde{\xi}|_{U_i}$ we conclude. □

Lemma 20.7.3. *Let $f : X \to Y$ be a morphism of ringed spaces. Let $\mathcal{F}$ be a $\mathcal{O}_X$-module. The sheaves $R^i f_* \mathcal{F}$ are the sheaves associated to the presheaves*

$$V \longmapsto H^i(f^{-1}(V), \mathcal{F})$$

with restriction mappings as in Equation (20.7.1.1). There is a similar statement for $R^i f_*$ applied to a bounded below complex $\mathcal{F}^\bullet$.

Proof. Let $\mathcal{F} \to \mathcal{I}^\bullet$ be an injective resolution. Then $R^i f_* \mathcal{F}$ is by definition the ith cohomology sheaf of the complex

$$f_* \mathcal{I}^0 \to f_* \mathcal{I}^1 \to f_* \mathcal{I}^2 \to \ldots$$

By definition of the abelian category structure on $\mathcal{O}_Y$-modules this cohomology sheaf is the sheaf associated to the presheaf

$$V \longmapsto \frac{\mathrm{Ker}(f_*\mathcal{I}^i(V) \to f_*\mathcal{I}^{i+1}(V))}{\mathrm{Im}(f_*\mathcal{I}^{i-1}(V) \to f_*\mathcal{I}^i(V))}$$

and this is obviously equal to

$$\frac{\mathrm{Ker}(\mathcal{I}^i(f^{-1}(V)) \to \mathcal{I}^{i+1}(f^{-1}(V)))}{\mathrm{Im}(\mathcal{I}^{i-1}(f^{-1}(V)) \to \mathcal{I}^i(f^{-1}(V)))}$$

which is equal to $H^i(f^{-1}(V), \mathcal{F})$ and we win. □

Lemma 20.7.4. *Let $f : X \to Y$ be a morphism of ringed spaces. Let $\mathcal{F}$ be an $\mathcal{O}_X$-module. Let $V \subset Y$ be an open subspace. Denote $g : f^{-1}(V) \to V$ the restriction of f. Then we have*

$$R^p g_*(\mathcal{F}|_{f^{-1}(V)}) = (R^p f_*\mathcal{F})|_V$$

There is a similar statement for the derived image $Rf_*\mathcal{F}^\bullet$ where $\mathcal{F}^\bullet$ is a bounded below complex of $\mathcal{O}_X$-modules.

Proof. First proof. Apply Lemmas 20.7.3 and 20.7.1 to see the displayed equality. Second proof. Choose an injective resolution $\mathcal{F} \to \mathcal{I}^\bullet$ and use that $\mathcal{F}|_{f^{-1}(V)} \to \mathcal{I}^\bullet|_{f^{-1}(V)}$ is an injective resolution also. □

Remark 20.7.5. Here is a different approach to the proofs of Lemmas 20.7.2 and 20.7.3 above. Let $(X, \mathcal{O}_X)$ be a ringed space. Let $i_X : \mathit{Mod}(\mathcal{O}_X) \to \mathit{Mod}(\mathcal{O}_X)$ be the inclusion functor and let $\#$ be the sheafification functor. Recall that i_X is left exact and $\#$ is exact.

(1) First prove Lemma 20.12.3 below which says that the right derived functors of i_X are given by $R^p i_X\mathcal{F} = \underline{H}^p(\mathcal{F})$. Here is another proof: The equality is clear for $p = 0$. Both $(R^p i_X)_{p\geq 0}$ and $(\underline{H}^p)_{p\geq 0}$ are delta functors vanishing on injectives, hence both are universal, hence they are isomorphic. See Homology, Section 12.11.

(2) A restatement of Lemma 20.7.2 is that $(\underline{H}^p(\mathcal{F}))^\# = 0$, $p > 0$ for any sheaf of $\mathcal{O}_X$-modules $\mathcal{F}$. To see this is true, use that $^\#$ is exact so

$$(\underline{H}^p(\mathcal{F}))^\# = (R^p i_X\mathcal{F})^\# = R^p(\# \circ i_X)(\mathcal{F}) = 0$$

because $\# \circ i_X$ is the identity functor.

(3) Let $f : X \to Y$ be a morphism of ringed spaces. Let $\mathcal{F}$ be an $\mathcal{O}_X$-module. The presheaf $V \mapsto H^p(f^{-1}V, \mathcal{F})$ is equal to $R^p(i_Y \circ f_*)\mathcal{F}$. You can prove this by noticing that both give universal delta functors as in the argument of (1) above. Hence Lemma 20.7.3 says that $R^p f_*\mathcal{F} = (R^p(i_Y \circ f_*)\mathcal{F})^\#$. Again using that $\#$ is exact a that $\# \circ i_Y$ is the identity functor we see that

$$R^p f_*\mathcal{F} = R^p(\# \circ i_Y \circ f_*)\mathcal{F} = (R^p(i_Y \circ f_*)\mathcal{F})^\#$$

as desired.

20.8. Projection formula

In this section we collect variants of the projection formula. The most basic version is Lemma 20.8.2.

Lemma 20.8.1. *Let X be a ringed space. Let $\mathcal{I}$ be an injective $\mathcal{O}_X$-module. Let $\mathcal{E}$ be an $\mathcal{O}_X$-module. Assume $\mathcal{E}$ is finite locally free on X, see Modules, Definition 17.14.1. Then $\mathcal{E} \otimes_{\mathcal{O}_X} \mathcal{I}$ is an injective $\mathcal{O}_X$-module.*

Proof. This is true because under the assumptions of the lemma we have

$$\mathrm{Hom}_{\mathcal{O}_X}(\mathcal{F}, \mathcal{E} \otimes_{\mathcal{O}_X} \mathcal{I}) = \mathrm{Hom}_{\mathcal{O}_X}(\mathcal{F} \otimes_{\mathcal{O}_X} \mathcal{E}^\wedge, \mathcal{I})$$

where $\mathcal{E}^\wedge = \mathcal{H}om_{\mathcal{O}_X}(\mathcal{E}, \mathcal{O}_X)$ is the dual of $\mathcal{E}$ which is finite locally free also. Since tensoring with a finite locally free sheaf is an exact functor we win by Homology, Lemma 12.23.2. □

Lemma 20.8.2. *Let $f : X \to Y$ be a morphism of ringed spaces. Let $\mathcal{F}$ be an $\mathcal{O}_X$-module. Let $\mathcal{E}$ be an $\mathcal{O}_Y$-module. Assume $\mathcal{E}$ is finite locally free on Y, see Modules, Definition 17.14.1. Then there exist isomorphisms*

$$\mathcal{E} \otimes_{\mathcal{O}_Y} R^q f_* \mathcal{F} \longrightarrow R^q f_*(f^*\mathcal{E} \otimes_{\mathcal{O}_X} \mathcal{F})$$

for all $q \geq 0$. In fact there exists an isomorphism

$$\mathcal{E} \otimes_{\mathcal{O}_Y} Rf_* \mathcal{F} \longrightarrow Rf_*(f^*\mathcal{E} \otimes_{\mathcal{O}_X} \mathcal{F})$$

in $D^+(Y)$ functorial in $\mathcal{F}$.

Proof. Choose an injective resolution $\mathcal{F} \to \mathcal{I}^\bullet$ on X. Note that $f^*\mathcal{E}$ is finite locally free also, hence we get a resolution

$$f^*\mathcal{E} \otimes_{\mathcal{O}_X} \mathcal{F} \longrightarrow f^*\mathcal{E} \otimes_{\mathcal{O}_X} \mathcal{I}^\bullet$$

which is an injective resolution by Lemma 20.8.1. Apply f_* to see that

$$Rf_*(f^*\mathcal{E} \otimes_{\mathcal{O}_X} \mathcal{F}) = f_*(f^*\mathcal{E} \otimes_{\mathcal{O}_X} \mathcal{I}^\bullet).$$

Hence the lemma follows if we can show that $f_*(f^*\mathcal{E} \otimes_{\mathcal{O}_X} \mathcal{F}) = \mathcal{E} \otimes_{\mathcal{O}_Y} f_*(\mathcal{F})$ functorially in the $\mathcal{O}_X$-module $\mathcal{F}$. This is clear when $\mathcal{E} = \mathcal{O}_Y^{\oplus n}$, and follows in general by working locally on Y. Details omitted. □

20.9. Mayer-Vietoris

Below will construct the Čech-to-cohomology spectral sequence, see Lemma 20.12.4. A special case of that spectral sequence is the Mayer-Vietoris long exact sequence. Since it is such a basic, useful and easy to understand variant of the spectral sequence we treat it here separately.

Lemma 20.9.1. *Let X be a ringed space. Let $U' \subset U \subset X$ be open subspaces. For any injective $\mathcal{O}_X$-module $\mathcal{I}$ the restriction mapping $\mathcal{I}(U) \to \mathcal{I}(U')$ is surjective.*

Proof. Let $j : U \to X$ and $j' : U' \to X$ be the open immersions. Recall that $j_!\mathcal{O}_U$ is the extension by zero of $\mathcal{O}_U = \mathcal{O}_X|_U$, see Sheaves, Section 6.31. Since $j_!$ is a left adjoint to restriction we see that for any sheaf $\mathcal{F}$ of $\mathcal{O}_X$-modules

$$\mathrm{Hom}_{\mathcal{O}_X}(j_!\mathcal{O}_U, \mathcal{F}) = \mathrm{Hom}_{\mathcal{O}_U}(\mathcal{O}_U, \mathcal{F}|_U) = \mathcal{F}(U)$$

see Sheaves, Lemma 6.31.8. Similarly, the sheaf $j'_!\mathcal{O}_{U'}$ represents the functor $\mathcal{F} \mapsto \mathcal{F}(U')$. Moreover there is an obvious canonical map of $\mathcal{O}_X$-modules

$$j'_!\mathcal{O}_{U'} \longrightarrow j_!\mathcal{O}_U$$

which corresponds to the restriction mapping $\mathcal{F}(U) \to \mathcal{F}(U')$ via Yoneda's lemma (Categories, Lemma 4.3.5). By the description of the stalks of the sheaves $j'_!\mathcal{O}_{U'}$, $j_!\mathcal{O}_U$ we see that the displayed map above is injective (see lemma cited above). Hence if $\mathcal{I}$ is an injective $\mathcal{O}_X$-module, then the map

$$\mathrm{Hom}_{\mathcal{O}_X}(j_!\mathcal{O}_U, \mathcal{F}) \longrightarrow \mathrm{Hom}_{\mathcal{O}_X}(j'_!\mathcal{O}_{U'}, \mathcal{F})$$

is surjective, see Homology, Lemma 12.23.2. Putting everything together we obtain the lemma. □

Lemma 20.9.2 (Mayer-Vietoris)**.** *Let X be a ringed space. Suppose that $X = U \cup V$ is a union of two open subsets. For every $\mathcal{O}_X$-module $\mathcal{F}$ there exists a long exact cohomology sequence*

$$0 \to H^0(X, \mathcal{F}) \to H^0(U, \mathcal{F}) \oplus H^0(V, \mathcal{F}) \to H^0(U \cap V, \mathcal{F}) \to H^1(X, \mathcal{F}) \to \ldots$$

This long exact sequence is functorial in $\mathcal{F}$.

Proof. The sheaf condition says that the kernel of $(1, -1) : \mathcal{F}(U) \oplus \mathcal{F}(V) \to \mathcal{F}(U \cap V)$ is equal to the image of $\mathcal{F}(X)$ by the first map for any abelian sheaf $\mathcal{F}$. Lemma 20.9.1 above implies that the map $(1, -1) : \mathcal{I}(U) \oplus \mathcal{I}(V) \to \mathcal{I}(U \cap V)$ is surjective whenever $\mathcal{I}$ is an injective $\mathcal{O}_X$-module. Hence if $\mathcal{F} \to \mathcal{I}^\bullet$ is an injective resolution of $\mathcal{F}$, then we get a short exact sequence of complexes

$$0 \to \mathcal{I}^\bullet(X) \to \mathcal{I}^\bullet(U) \oplus \mathcal{I}^\bullet(V) \to \mathcal{I}^\bullet(U \cap V) \to 0.$$

Taking cohomology gives the result (use Homology, Lemma 12.12.12). We omit the proof of the functoriality of the sequence. □

Lemma 20.9.3 (Relative Mayer-Vietoris)**.** *Let $f : X \to Y$ be a morphism of ringed spaces. Suppose that $X = U \cup V$ is a union of two open subsets. Denote $a = f|_U : U \to Y$, $b = f|_V : V \to Y$, and $c = f|_{U \cap V} : U \cap V \to Y$. For every $\mathcal{O}_X$-module $\mathcal{F}$ there exists a long exact sequence*

$$0 \to f_*\mathcal{F} \to a_*(\mathcal{F}|_U) \oplus b_*(\mathcal{F}|_V) \to c_*(\mathcal{F}|_{U \cap V}) \to R^1 f_*\mathcal{F} \to \ldots$$

This long exact sequence is functorial in $\mathcal{F}$.

Proof. Let $\mathcal{F} \to \mathcal{I}^\bullet$ be an injective resolution of $\mathcal{F}$. We claim that we get a short exact sequence of complexes

$$0 \to f_*\mathcal{I}^\bullet \to a_*\mathcal{I}^\bullet|_V \oplus b_*\mathcal{I}^\bullet|_V \to c_*\mathcal{I}^\bullet|_{U \cap V} \to 0.$$

Namely, for any open $W \subset Y$, and for any $n \geq 0$ the corresponding sequence of groups of sections over W

$$0 \to \mathcal{I}^n(f^{-1}(W)) \to \mathcal{I}^n(V \cap f^{-1}(W)) \oplus \mathcal{I}^n(V \cap f^{-1}(W)) \to \mathcal{I}^n(U \cap V \cap f^{-1}(W)) \to 0$$

was shown to be short exact in the proof of Lemma 20.9.2. The lemma follows by taking cohomology sheaves and using the fact that $\mathcal{I}^\bullet|_U$ is an injective resolution of $\mathcal{F}|_U$ and similarly for $\mathcal{I}^\bullet|_V$, $\mathcal{I}^\bullet|_{U \cap V}$ see Lemma 20.7.1. □

20.10. The Čech complex and Čech cohomology

Let X be a topological space. Let $\mathcal{U} : U = \bigcup_{i\in I} U_i$ be an open covering, see Topology, Basic notion (10). As is customary we denote $U_{i_0\ldots i_p} = U_{i_0} \cap \ldots \cap U_{i_p}$ for the $(p+1)$-fold intersection of members of $\mathcal{U}$. Let $\mathcal{F}$ be an abelian presheaf on X. Set

$$\check{\mathcal{C}}^p(\mathcal{U}, \mathcal{F}) = \prod\nolimits_{(i_0,\ldots,i_p)\in I^{p+1}} \mathcal{F}(U_{i_0\ldots i_p}).$$

This is an abelian group. For $s \in \check{\mathcal{C}}^p(\mathcal{U}, \mathcal{F})$ we denote $s_{i_0\ldots i_p}$ its value in $\mathcal{F}(U_{i_0\ldots i_p})$. Note that if $s \in \check{\mathcal{C}}^2(\mathcal{U}, \mathcal{F})$ and $i, j \in I$ then s_{ij} and s_{ji} are both elements of $\mathcal{F}(U_i \cap U_j)$ but there is no imposed relation between s_{ij} and s_{ji}. In other words, we are *not* working with alternating cochains (these will be defined in Section 20.22). We define

$$d : \check{\mathcal{C}}^p(\mathcal{U}, \mathcal{F}) \longrightarrow \check{\mathcal{C}}^{p+1}(\mathcal{U}, \mathcal{F})$$

by the formula

$$d(s)_{i_0\ldots i_{p+1}} = \sum\nolimits_{j=0}^{p+1} (-1)^j s_{i_0\ldots\hat{i}_j\ldots i_{p+1}}|_{U_{i_0\ldots i_{p+1}}} \tag{20.10.0.1}$$

It is straightforward to see that $d \circ d = 0$. In other words $\check{\mathcal{C}}^\bullet(\mathcal{U}, \mathcal{F})$ is a complex.

Definition 20.10.1. Let X be a topological space. Let $\mathcal{U} : U = \bigcup_{i\in I} U_i$ be an open covering. Let $\mathcal{F}$ be an abelian presheaf on X. The complex $\check{\mathcal{C}}^\bullet(\mathcal{U}, \mathcal{F})$ is the *Čech complex* associated to $\mathcal{F}$ and the open covering $\mathcal{U}$. Its cohomology groups $H^i(\check{\mathcal{C}}^\bullet(\mathcal{U}, \mathcal{F}))$ are called the *Čech cohomology groups* associated to $\mathcal{F}$ and the covering $\mathcal{U}$. They are denoted $\check{H}^i(\mathcal{U}, \mathcal{F})$.

Lemma 20.10.2. *Let X be a topological space. Let $\mathcal{F}$ be an abelian presheaf on X. The following are equivalent*

1. *$\mathcal{F}$ is an abelian sheaf and*
2. *for every open covering $\mathcal{U} : U = \bigcup_{i\in I} U_i$ the natural map*
$$\mathcal{F}(U) \to \check{H}^0(\mathcal{U}, \mathcal{F})$$
is bijective.

Proof. This is true since the sheaf condition is exactly that $\mathcal{F}(U) \to \check{H}^0(\mathcal{U}, \mathcal{F})$ is bijective for every open covering. □

20.11. Čech cohomology as a functor on presheaves

Warning: In this section we work almost exclusively with presheaves and categories of presheaves and the results are completely wrong in the setting of sheaves and categories of sheaves!

Let X be a ringed space. Let $\mathcal{U} : U = \bigcup_{i\in I} U_i$ be an open covering. Let $\mathcal{F}$ be a presheaf of $\mathcal{O}_X$-modules. We have the Čech complex $\check{\mathcal{C}}^\bullet(\mathcal{U}, \mathcal{F})$ of $\mathcal{F}$ just by thinking of $\mathcal{F}$ as a presheaf of abelian groups. However, each term $\check{\mathcal{C}}^p(\mathcal{U}, \mathcal{F})$ has a natural structure of a $\mathcal{O}_X(U)$-module and the differential is given by $\mathcal{O}_X(U)$-module maps. Moreover, it is clear that the construction

$$\mathcal{F} \longmapsto \check{\mathcal{C}}^\bullet(\mathcal{U}, \mathcal{F})$$

is functorial in $\mathcal{F}$. In fact, it is a functor

$$\check{\mathcal{C}}^\bullet(\mathcal{U}, -) : PMod(\mathcal{O}_X) \longrightarrow \mathrm{Comp}^+(\mathrm{Mod}_{\mathcal{O}_X(U)}) \tag{20.11.0.1}$$

see Derived Categories, Definition 13.7.1 for notation. Recall that the category of bounded below complexes in an abelian category is an abelian category, see Homology, Lemma 12.12.9.

Lemma 20.11.1. *The functor given by Equation (20.11.0.1) is an exact functor (see Homology, Lemma 12.7.1).*

Proof. For any open $W \subset U$ the functor $\mathcal{F} \mapsto \mathcal{F}(W)$ is an additive exact functor from $PMod(\mathcal{O}_X)$ to $\mathrm{Mod}_{\mathcal{O}_X(U)}$. The terms $\check{\mathcal{C}}^p(\mathcal{U}, \mathcal{F})$ of the complex are products of these exact functors and hence exact. Moreover a sequence of complexes is exact if and only if the sequence of terms in a given degree is exact. Hence the lemma follows. □

Lemma 20.11.2. *Let X be a ringed space. Let $\mathcal{U} : U = \bigcup_{i \in I} U_i$ be an open covering. The functors $\mathcal{F} \mapsto \check{H}^n(\mathcal{U}, \mathcal{F})$ form a δ-functor from the abelian category of presheaves of $\mathcal{O}_X$-modules to the category of $\mathcal{O}_X(U)$-modules (see Homology, Definition 12.11.1).*

Proof. By Lemma 20.11.1 a short exact sequence of presheaves of $\mathcal{O}_X$-modules $0 \to \mathcal{F}_1 \to \mathcal{F}_2 \to \mathcal{F}_3 \to 0$ is turned into a short exact sequence of complexes of $\mathcal{O}_X(U)$-modules. Hence we can use Homology, Lemma 12.12.12 to get the boundary maps $\delta_{\mathcal{F}_1 \to \mathcal{F}_2 \to \mathcal{F}_3} : \check{H}^n(\mathcal{U}, \mathcal{F}_3) \to \check{H}^{n+1}(\mathcal{U}, \mathcal{F}_1)$ and a corresponding long exact sequence. We omit the verification that these maps are compatible with maps between short exact sequences of presheaves. □

In the formulation of the following lemma we use the functor $j_{p!}$ of extension by 0 for presheaves of modules relative to an open immersion $j : U \to X$. See Sheaves, Section 6.31. For any open $W \subset X$ and any presheaf $\mathcal{G}$ of $\mathcal{O}_X|_U$-modules we have

$$(j_{p!}\mathcal{G})(W) = \begin{cases} \mathcal{G}(W) & \text{if } W \subset U \\ 0 & \text{else.} \end{cases}$$

Moreover, the functor $j_{p!}$ is a left adjoint to the restriction functor see Sheaves, Lemma 6.31.8. In particular we have the following formula

$$\mathrm{Hom}_{\mathcal{O}_X}(j_{p!}\mathcal{O}_U, \mathcal{F}) = \mathrm{Hom}_{\mathcal{O}_U}(\mathcal{O}_U, \mathcal{F}|_U) = \mathcal{F}(U).$$

Since the functor $\mathcal{F} \mapsto \mathcal{F}(U)$ is an exact functor on the category of presheaves we conclude that the presheaf $j_{p!}\mathcal{O}_U$ is a projective object in the category $PMod(\mathcal{O}_X)$, see Homology, Lemma 12.24.2.

Note that if we are given open subsets $U \subset V \subset X$ with associated open immersions j_U, j_V, then we have a canonical map $(j_U)_{p!}\mathcal{O}_U \to (j_V)_{p!}\mathcal{O}_V$. It is the identity on sections over any open $W \subset U$ and 0 else. In terms of the identification $\mathrm{Hom}_{\mathcal{O}_X}((j_U)_{p!}\mathcal{O}_U, (j_V)_{p!}\mathcal{O}_V) = (j_V)_{p!}\mathcal{O}_V(U) = \mathcal{O}_V(U)$ it corresponds to the element $1 \in \mathcal{O}_V(U)$.

Lemma 20.11.3. *Let X be a ringed space. Let $\mathcal{U} : U = \bigcup_{i \in I} U_i$ be a covering. Denote $j_{i_0 \ldots i_p} : U_{i_0 \ldots i_p} \to X$ the open immersion. Consider the chain complex $K(\mathcal{U})_\bullet$ of presheaves of $\mathcal{O}_X$-modules*

$$\ldots \to \bigoplus_{i_0 i_1 i_2} (j_{i_0 i_1 i_2})_{p!}\mathcal{O}_{U_{i_0 i_1 i_2}} \to \bigoplus_{i_0 i_1} (j_{i_0 i_1})_{p!}\mathcal{O}_{U_{i_0 i_1}} \to \bigoplus_{i_0} (j_{i_0})_{p!}\mathcal{O}_{U_{i_0}} \to 0 \to \ldots$$

where the last nonzero term is placed in degree 0 and where the map

$$(j_{i_0 \ldots i_{p+1}})_{p!}\mathcal{O}_{U_{i_0 \ldots i_{p+1}}} \longrightarrow (j_{i_0 \ldots \hat{i}_j \ldots i_{p+1}})_{p!}\mathcal{O}_{U_{i_0 \ldots \hat{i}_j \ldots i_{p+1}}}$$

is given by $(-1)^j$ times the canonical map. Then there is an isomorphism

$$\mathrm{Hom}_{\mathcal{O}_X}(K(\mathcal{U})_\bullet, \mathcal{F}) = \check{\mathcal{C}}^\bullet(\mathcal{U}, \mathcal{F})$$

functorial in $\mathcal{F} \in \mathrm{Ob}(PMod(\mathcal{O}_X))$.

Proof. We saw in the discussion just above the lemma that

$$\mathrm{Hom}_{\mathcal{O}_X}((j_{i_0\ldots i_p})_{p!}\mathcal{O}_{U_{i_0\ldots i_p}}, \mathcal{F}) = \mathcal{F}(U_{i_0\ldots i_p}).$$

Hence we see that it is indeed the case that the direct sum

$$\bigoplus\nolimits_{i_0\ldots i_p} (j_{i_0\ldots i_p})_{p!}\mathcal{O}_{U_{i_0\ldots i_p}}$$

represents the functor

$$\mathcal{F} \longmapsto \prod\nolimits_{i_0\ldots i_p} \mathcal{F}(U_{i_0\ldots i_p}).$$

Hence by Categories, Yoneda Lemma 4.3.5 we see that there is a complex $K(\mathcal{U})_\bullet$ with terms as given. It is a simple matter to see that the maps are as given in the lemma. □

Lemma 20.11.4. *Let X be a ringed space. Let $\mathcal{U} : U = \bigcup_{i\in I} U_i$ be a covering. Let $\mathcal{O}_\mathcal{U} \subset \mathcal{O}_X$ be the image presheaf of the map $\bigoplus j_{p!}\mathcal{O}_{U_i} \to \mathcal{O}_X$. The chain complex $K(\mathcal{U})_\bullet$ of presheaves of Lemma 20.11.3 above has homology presheaves*

$$H_i(K(\mathcal{U})_\bullet) = \begin{cases} 0 & \text{if} \quad i \neq 0 \\ \mathcal{O}_\mathcal{U} & \text{if} \quad i = 0 \end{cases}$$

Proof. Consider the extended complex $K^{ext}_\bullet$ one gets by putting $\mathcal{O}_\mathcal{U}$ in degree -1 with the obvious map $K(\mathcal{U})_0 = \bigoplus_{i_0}(j_{i_0})_{p!}\mathcal{O}_{U_{i_0}} \to \mathcal{O}_\mathcal{U}$. It suffices to show that taking sections of this extended complex over any open $W \subset X$ leads to an acyclic complex. In fact, we claim that for every $W \subset X$ the complex $K^{ext}_\bullet(W)$ is homotopy equivalent to the zero complex. Write $I = I_1 \sqcup I_2$ where $W \subset U_i$ if and only if $i \in I_1$.

If $I_1 = \emptyset$, then the complex $K^{ext}_\bullet(W) = 0$ so there is nothing to prove.

If $I_1 \neq \emptyset$, then $\mathcal{O}_\mathcal{U}(W) = \mathcal{O}_X(W)$ and

$$K^{ext}_p(W) = \bigoplus\nolimits_{i_0\ldots i_p \in I_1} \mathcal{O}_X(W).$$

This is true because of the simple description of the presheaves $(j_{i_0\ldots i_p})_{p!}\mathcal{O}_{U_{i_0\ldots i_p}}$. Moreover, the differential of the complex $K^{ext}_\bullet(W)$ is given by

$$d(s)_{i_0\ldots i_p} = \sum\nolimits_{j=0,\ldots,p+1} \sum\nolimits_{i\in I_1} (-1)^j s_{i_0\ldots i_{j-1} i i_j \ldots i_p}.$$

The sum is finite as the element s has finite support. Fix an element $i_{\mathrm{fix}} \in I_1$. Define a map

$$h : K^{ext}_p(W) \longrightarrow K^{ext}_{p+1}(W)$$

by the rule

$$h(s)_{i_0\ldots i_{p+1}} = \begin{cases} 0 & \text{if} \quad i_0 \neq i \\ s_{i_1\ldots i_{p+1}} & \text{if} \quad i_0 = i_{\mathrm{fix}} \end{cases}$$

We will use the shorthand $h(s)_{i_0 \ldots i_{p+1}} = (i_0 = i_{\text{fix}}) s_{i_1 \ldots i_p}$ for this. Then we compute

$$\begin{aligned} & (dh + hd)(s)_{i_0 \ldots i_p} \\ = & \sum_j \sum_{i \in I_1} (-1)^j h(s)_{i_0 \ldots i_{j-1} i i_j \ldots i_p} + (i = i_0) d(s)_{i_1 \ldots i_p} \\ = & s_{i_0 \ldots i_p} + \sum_{j \geq 1} \sum_{i \in I_1} (-1)^j (i_0 = i_{\text{fix}}) s_{i_1 \ldots i_{j-1} i i_j \ldots i_p} + (i_0 = i_{\text{fix}}) d(s)_{i_1 \ldots i_p} \end{aligned}$$

which is equal to $s_{i_0 \ldots i_p}$ as desired. □

Lemma 20.11.5. *Let X be a ringed space. Let $\mathcal{U} : U = \bigcup_{i \in I} U_i$ be an open covering of $U \subset X$. The Čech cohomology functors $\check{H}^p(\mathcal{U}, -)$ are canonically isomorphic as a δ-functor to the right derived functors of the functor*

$$\check{H}^0(\mathcal{U}, -) : PMod(\mathcal{O}_X) \longrightarrow Mod_{\mathcal{O}_X(U)}.$$

Moreover, there is a functorial quasi-isomorphism

$$\check{\mathcal{C}}^\bullet(\mathcal{U}, \mathcal{F}) \longrightarrow R\check{H}^0(\mathcal{U}, \mathcal{F})$$

where the right hand side indicates the right derived functor

$$R\check{H}^0(\mathcal{U}, -) : D^+(PMod(\mathcal{O}_X)) \longrightarrow D^+(\mathcal{O}_X(U))$$

of the left exact functor $\check{H}^0(\mathcal{U}, -)$.

Proof. Note that the category of presheaves of $\mathcal{O}_X$-modules has enough injectives, see Injectives, Proposition 19.8.5. Note that $\check{H}^0(\mathcal{U}, -)$ is a left exact functor from the category of presheaves of $\mathcal{O}_X$-modules to the category of $\mathcal{O}_X(U)$-modules. Hence the derived functor and the right derived functor exist, see Derived Categories, Section 13.19.

Let $\mathcal{I}$ be a injective presheaf of $\mathcal{O}_X$-modules. In this case the functor $\text{Hom}_{\mathcal{O}_X}(-, \mathcal{I})$ is exact on $PMod(\mathcal{O}_X)$. By Lemma 20.11.3 we have

$$\text{Hom}_{\mathcal{O}_X}(K(\mathcal{U})_\bullet, \mathcal{I}) = \check{\mathcal{C}}^\bullet(\mathcal{U}, \mathcal{I}).$$

By Lemma 20.11.4 we have that $K(\mathcal{U})_\bullet$ is quasi-isomorphic to $\mathcal{O}_\mathcal{U}[0]$. Hence by the exactness of Hom into $\mathcal{I}$ mentioned above we see that $\check{H}^i(\mathcal{U}, \mathcal{I}) = 0$ for all $i > 0$. Thus the δ-functor $(\check{H}^n, \delta)$ (see Lemma 20.11.2) satisfies the assumptions of Homology, Lemma 12.11.4, and hence is a universal δ-functor.

By Derived Categories, Lemma 13.19.4 also the sequence $R^i\check{H}^0(\mathcal{U}, -)$ forms a universal δ-functor. By the uniqueness of universal δ-functors, see Homology, Lemma 12.11.5 we conclude that $R^i\check{H}^0(\mathcal{U}, -) = \check{H}^i(\mathcal{U}, -)$. This is enough for most applications and the reader is suggested to skip the rest of the proof.

Let $\mathcal{F}$ be any presheaf of $\mathcal{O}_X$-modules. Choose an injective resolution $\mathcal{F} \to \mathcal{I}^\bullet$ in the category $PMod(\mathcal{O}_X)$. Consider the double complex $A^{\bullet,\bullet}$ with terms

$$A^{p,q} = \check{\mathcal{C}}^p(\mathcal{U}, \mathcal{I}^q).$$

Consider the simple complex $sA^\bullet$ associated to this double complex. There is a map of complexes

$$\check{\mathcal{C}}^\bullet(\mathcal{U}, \mathcal{F}) \longrightarrow sA^\bullet$$

coming from the maps $\check{\mathcal{C}}^p(\mathcal{U}, \mathcal{F}) \to A^{p,0} = \check{\mathcal{C}}^\bullet(\mathcal{U}, \mathcal{I}^0)$ and there is a map of complexes

$$\check{H}^0(\mathcal{U}, \mathcal{I}^\bullet) \longrightarrow sA^\bullet$$

coming from the maps $\check{H}^0(\mathcal{U}, \mathcal{I}^q) \to A^{0,q} = \check{\mathcal{C}}^0(\mathcal{U}, \mathcal{I}^q)$. Both of these maps are quasi-isomorphisms by an application of Homology, Lemma 12.22.6. Namely, the columns of the double complex are exact in positive degrees because the Čech complex as a functor is exact (Lemma 20.11.1) and the rows of the double complex are exact in positive degrees since as we just saw the higher Čech cohomology groups of the injective presheaves $\mathcal{I}^q$ are zero. Since quasi-isomorphisms become invertible in $D^+(\mathcal{O}_X(U))$ this gives the last displayed morphism of the lemma. We omit the verification that this morphism is functorial. □

20.12. Čech cohomology and cohomology

Lemma 20.12.1. *Let X be a ringed space. Let $\mathcal{U} : U = \bigcup_{i \in I} U_i$ be a covering. Let $\mathcal{I}$ be an injective $\mathcal{O}_X$-module. Then*

$$\check{H}^p(\mathcal{U}, \mathcal{I}) = \begin{cases} \mathcal{I}(U) & \text{if} \quad p = 0 \\ 0 & \text{if} \quad p > 0 \end{cases}$$

Proof. An injective $\mathcal{O}_X$-module is also injective as an object in the category $PMod(\mathcal{O}_X)$ (for example since sheafification is an exact left adjoint to the inclusion functor, using Homology, Lemma 12.25.1). Hence we can apply Lemma 20.11.5 (or its proof) to see the result. □

Lemma 20.12.2. *Let X be a ringed space. Let $\mathcal{U} : U = \bigcup_{i \in I} U_i$ be a covering. There is a transformation*

$$\check{\mathcal{C}}^\bullet(\mathcal{U}, -) \longrightarrow R\Gamma(U, -)$$

of functors $Mod(\mathcal{O}_X) \to D^+(\mathcal{O}_X(U))$. In particular this provides canonical maps $\check{H}^p(\mathcal{U}, \mathcal{F}) \to H^p(U, \mathcal{F})$ for $\mathcal{F}$ ranging over $Mod(\mathcal{O}_X)$.

Proof. Let $\mathcal{F}$ be an $\mathcal{O}_X$-module. Choose an injective resolution $\mathcal{F} \to \mathcal{I}^\bullet$. Consider the double complex $\check{\mathcal{C}}^\bullet(\mathcal{U}, \mathcal{I}^\bullet)$ with terms $\check{\mathcal{C}}^p(\mathcal{U}, \mathcal{I}^q)$. There is a map of complexes

$$\alpha : \Gamma(U, \mathcal{I}^\bullet) \longrightarrow \mathrm{Tot}(\check{\mathcal{C}}^\bullet(\mathcal{U}, \mathcal{I}^\bullet))$$

coming from the maps $\mathcal{I}^q(U) \to \check{H}^0(\mathcal{U}, \mathcal{I}^q)$ and a map of complexes

$$\beta : \check{\mathcal{C}}^\bullet(\mathcal{U}, \mathcal{F}) \longrightarrow \mathrm{Tot}(\check{\mathcal{C}}^\bullet(\mathcal{U}, \mathcal{I}^\bullet))$$

coming from the map $\mathcal{F} \to \mathcal{I}^0$. We can apply Homology, Lemma 12.22.6 to see that α is a quasi-isomorphism. Namely, Lemma 20.12.1 implies that the qth row of the double complex $\check{\mathcal{C}}^\bullet(\mathcal{U}, \mathcal{I}^\bullet)$ is a resolution of $\Gamma(U, \mathcal{I}^q)$. Hence α becomes invertible in $D^+(\mathcal{O}_X(U))$ and the transformation of the lemma is the composition of β followed by the inverse of α. We omit the verification that this is functorial. □

Lemma 20.12.3. *Let X be a ringed space. Consider the functor $i : Mod(\mathcal{O}_X) \to PMod(\mathcal{O}_X)$. It is a left exact functor with right derived functors given by*

$$R^p i(\mathcal{F}) = \underline{H}^p(\mathcal{F}) : U \longmapsto H^p(U, \mathcal{F})$$

see discussion in Section 20.7.

Proof. It is clear that i is left exact. Choose an injective resolution $\mathcal{F} \to \mathcal{I}^\bullet$. By definition $R^p i$ is the pth cohomology *presheaf* of the complex $\mathcal{I}^\bullet$. In other words, the sections of $R^p i(\mathcal{F})$ over an open U are given by

$$\frac{\mathrm{Ker}(\mathcal{I}^n(U) \to \mathcal{I}^{n+1}(U))}{\mathrm{Im}(\mathcal{I}^{n-1}(U) \to \mathcal{I}^n(U))}.$$

which is the definition of $H^p(U, \mathcal{F})$. □

Lemma 20.12.4. *Let X be a ringed space. Let $\mathcal{U} : U = \bigcup_{i \in I} U_i$ be a covering. For any sheaf of $\mathcal{O}_X$-modules $\mathcal{F}$ there is a spectral sequence $(E_r, d_r)_{r \geq 0}$ with*

$$E_2^{p,q} = \check{H}^p(\mathcal{U}, \underline{H}^q(\mathcal{F}))$$

converging to $H^{p+q}(U, \mathcal{F})$. This spectral sequence is functorial in $\mathcal{F}$.

Proof. This is a Grothendieck spectral sequence (see Derived Categories, Lemma 13.21.2) for the functors

$$i : Mod(\mathcal{O}_X) \to PMod(\mathcal{O}_X) \quad \text{and} \quad \check{H}^0(\mathcal{U}, -) : PMod(\mathcal{O}_X) \to \mathrm{Mod}_{\mathcal{O}_X(U)}.$$

Namely, we have $\check{H}^0(\mathcal{U}, i(\mathcal{F})) = \mathcal{F}(U)$ by Lemma 20.10.2. We have that $i(\mathcal{I})$ is Čech acyclic by Lemma 20.12.1. And we have that $\check{H}^p(\mathcal{U}, -) = R^p\check{H}^0(\mathcal{U}, -)$ as functors on $PMod(\mathcal{O}_X)$ by Lemma 20.11.5. Putting everything together gives the lemma. □

Lemma 20.12.5. *Let X be a ringed space. Let $\mathcal{U} : U = \bigcup_{i \in I} U_i$ be a covering. Let $\mathcal{F}$ be an $\mathcal{O}_X$-module. Assume that $H^i(U_{i_0 \ldots i_p}, \mathcal{F}) = 0$ for all $i > 0$, all $p \geq 0$ and all $i_0, \ldots, i_p \in I$. Then $\check{H}^p(\mathcal{U}, \mathcal{F}) = H^p(U, \mathcal{F})$ as $\mathcal{O}_X(U)$-modules.*

Proof. We will use the spectral sequence of Lemma 20.12.4. The assumptions mean that $E_2^{p,q} = 0$ for all (p, q) with $q \neq 0$. Hence the spectral sequence degenerates at E_2 and the result follows. □

Lemma 20.12.6. *Let X be a ringed space. Let*

$$0 \to \mathcal{F} \to \mathcal{G} \to \mathcal{H} \to 0$$

be a short exact sequence of $\mathcal{O}_X$-modules. Let $U \subset X$ be an open subset. If there exists a cofinal system of open coverings $\mathcal{U}$ of U such that $\check{H}^1(\mathcal{U}, \mathcal{F}) = 0$, then the map $\mathcal{G}(U) \to \mathcal{H}(U)$ is surjective.

Proof. Take an element $s \in \mathcal{H}(U)$. Choose an open covering $\mathcal{U} : U = \bigcup_{i \in I} U_i$ such that (a) $\check{H}^1(\mathcal{U}, \mathcal{F}) = 0$ and (b) $s|_{U_i}$ is the image of a section $s_i \in \mathcal{G}(U_i)$. Since we can certainly find a covering such that (b) holds it follows from the assumptions of the lemma that we can find a covering such that (a) and (b) both hold. Consider the sections

$$s_{i_0 i_1} = s_{i_1}|_{U_{i_0 i_1}} - s_{i_0}|_{U_{i_0 i_1}}.$$

Since s_i lifts s we see that $s_{i_0 i_1} \in \mathcal{F}(U_{i_0 i_1})$. By the vanishing of $\check{H}^1(\mathcal{U}, \mathcal{F})$ we can find sections $t_i \in \mathcal{F}(U_i)$ such that

$$s_{i_0 i_1} = t_{i_1}|_{U_{i_0 i_1}} - t_{i_0}|_{U_{i_0 i_1}}.$$

Then clearly the sections $s_i - t_i$ satisfy the sheaf condition and glue to a section of $\mathcal{G}$ over U which maps to s. Hence we win. □

Lemma 20.12.7. *Let X be a ringed space. Let $\mathcal{F}$ be an $\mathcal{O}_X$-module such that*

$$\check{H}^p(\mathcal{U}, \mathcal{F}) = 0$$

for all $p > 0$ and any open covering $\mathcal{U} : U = \bigcup_{i \in I} U_i$ of an open of X. Then $H^p(U, \mathcal{F}) = 0$ for all $p > 0$ and any open $U \subset X$.

Proof. Let $\mathcal{F}$ be a sheaf satisfying the assumption of the lemma. We will indicate this by saying "$\mathcal{F}$ has vanishing higher Čech cohomology for any open covering". Choose an embedding $\mathcal{F} \to \mathcal{I}$ into an injective $\mathcal{O}_X$-module. By Lemma 20.12.1 $\mathcal{I}$ has vanishing higher Čech cohomology for any open covering. Let $\mathcal{Q} = \mathcal{I}/\mathcal{F}$ so that we have a short exact sequence

$$0 \to \mathcal{F} \to \mathcal{I} \to \mathcal{Q} \to 0.$$

By Lemma 20.12.6 and our assumptions this sequence is actually exact as a sequence of presheaves! In particular we have a long exact sequence of Čech cohomology groups for any open covering $\mathcal{U}$, see Lemma 20.11.2 for example. This implies that $\mathcal{Q}$ is also an $\mathcal{O}_X$-module with vanishing higher Čech cohomology for all open coverings.

Next, we look at the long exact cohomology sequence

$$\begin{array}{ccccccc} 0 & \longrightarrow & H^0(U,\mathcal{F}) & \longrightarrow & H^0(U,\mathcal{I}) & \longrightarrow & H^0(U,\mathcal{Q}) \\ & & & \searrow & & & \\ & & H^1(U,\mathcal{F}) & \longrightarrow & H^1(U,\mathcal{I}) & \longrightarrow & H^1(U,\mathcal{Q}) \\ & & & \searrow & & & \\ & & \ldots & & \ldots & & \ldots \end{array}$$

for any open $U \subset X$. Since $\mathcal{I}$ is injective we have $H^n(U,\mathcal{I}) = 0$ for $n > 0$ (see Derived Categories, Lemma 13.19.4). By the above we see that $H^0(U,\mathcal{I}) \to H^0(U,\mathcal{Q})$ is surjective and hence $H^1(U,\mathcal{F}) = 0$. Since $\mathcal{F}$ was an arbitrary $\mathcal{O}_X$-module with vanishing higher Čech cohomology we conclude that also $H^1(U,\mathcal{Q}) = 0$ since $\mathcal{Q}$ is another of these sheaves (see above). By the long exact sequence this in turn implies that $H^2(U,\mathcal{F}) = 0$. And so on and so forth. □

Lemma 20.12.8. *(Variant of Lemma 20.12.7.) Let X be a ringed space. Let $\mathcal{B}$ be a basis for the topology on X. Let $\mathcal{F}$ be an $\mathcal{O}_X$-module. Assume there exists a set of open coverings Cov with the following properties:*

1. *For every $\mathcal{U} \in Cov$ with $\mathcal{U} : U = \bigcup_{i\in I} U_i$ we have $U, U_i \in \mathcal{B}$ and every $U_{i_0 \ldots i_p} \in \mathcal{B}$.*
2. *For every $U \in \mathcal{B}$ the open coverings of U occurring in Cov is a cofinal system of open coverings of U.*
3. *For every $\mathcal{U} \in Cov$ we have $\check{H}^p(\mathcal{U},\mathcal{F}) = 0$ for all $p > 0$.*

Then $H^p(U,\mathcal{F}) = 0$ for all $p > 0$ and any $U \in \mathcal{B}$.

Proof. Let $\mathcal{F}$ and Cov be as in the lemma. We will indicate this by saying "$\mathcal{F}$ has vanishing higher Čech cohomology for any $\mathcal{U} \in$ Cov". Choose an embedding $\mathcal{F} \to \mathcal{I}$ into an injective $\mathcal{O}_X$-module. By Lemma 20.12.1 $\mathcal{I}$ has vanishing higher Čech cohomology for any $\mathcal{U} \in$ Cov. Let $\mathcal{Q} = \mathcal{I}/\mathcal{F}$ so that we have a short exact sequence

$$0 \to \mathcal{F} \to \mathcal{I} \to \mathcal{Q} \to 0.$$

By Lemma 20.12.6 and our assumption (2) this sequence gives rise to an exact sequence

$$0 \to \mathcal{F}(U) \to \mathcal{I}(U) \to \mathcal{Q}(U) \to 0.$$

for every $U \in \mathcal{B}$. Hence for any $\mathcal{U} \in \text{Cov}$ we get a short exact sequence of Čech complexes

$$0 \to \check{\mathcal{C}}^\bullet(\mathcal{U}, \mathcal{F}) \to \check{\mathcal{C}}^\bullet(\mathcal{U}, \mathcal{I}) \to \check{\mathcal{C}}^\bullet(\mathcal{U}, \mathcal{Q}) \to 0$$

since each term in the Čech complex is made up out of a product of values over elements of $\mathcal{B}$ by assumption (1). In particular we have a long exact sequence of Čech cohomology groups for any open covering $\mathcal{U} \in \text{Cov}$. This implies that $\mathcal{Q}$ is also an $\mathcal{O}_X$-module with vanishing higher Čech cohomology for all $\mathcal{U} \in \text{Cov}$.

Next, we look at the long exact cohomology sequence

$$\begin{array}{ccccccc} 0 & \longrightarrow & H^0(U, \mathcal{F}) & \longrightarrow & H^0(U, \mathcal{I}) & \longrightarrow & H^0(U, \mathcal{Q}) \\ & & & \searrow & & & \\ & & H^1(U, \mathcal{F}) & \longrightarrow & H^1(U, \mathcal{I}) & \longrightarrow & H^1(U, \mathcal{Q}) \\ & & & \searrow & & & \\ & & \ldots & & \ldots & & \ldots \end{array}$$

for any $U \in \mathcal{B}$. Since $\mathcal{I}$ is injective we have $H^n(U, \mathcal{I}) = 0$ for $n > 0$ (see Derived Categories, Lemma 13.19.4). By the above we see that $H^0(U, \mathcal{I}) \to H^0(U, \mathcal{Q})$ is surjective and hence $H^1(U, \mathcal{F}) = 0$. Since $\mathcal{F}$ was an arbitrary $\mathcal{O}_X$-module with vanishing higher Čech cohomology for all $\mathcal{U} \in \text{Cov}$ we conclude that also $H^1(U, \mathcal{Q}) = 0$ since $\mathcal{Q}$ is another of these sheaves (see above). By the long exact sequence this in turn implies that $H^2(U, \mathcal{F}) = 0$. And so on and so forth. □

Lemma 20.12.9. *Let $f : X \to Y$ be a morphism of ringed spaces. Let $\mathcal{I}$ be an injective $\mathcal{O}_X$-module. Then*

1. *$\check{H}^p(\mathcal{V}, f_*\mathcal{I}) = 0$ for all $p > 0$ and any open covering $\mathcal{V} : V = \bigcup_{j \in J} V_j$ of Y.*
2. *$H^p(V, f_*\mathcal{I}) = 0$ for all $p > 0$ and every open $V \subset Y$.*

In other words, $f_*\mathcal{I}$ is right acyclic for $\Gamma(U, -)$ (see Derived Categories, Definition 13.15.3) for any $U \subset X$ open.

Proof. Set $\mathcal{U} : f^{-1}(V) = \bigcup_{j \in J} f^{-1}(V_j)$. It is an open covering of X and

$$\check{\mathcal{C}}^\bullet(\mathcal{V}, f_*\mathcal{I}) = \check{\mathcal{C}}^\bullet(\mathcal{U}, \mathcal{I}).$$

This is true because

$$f_*\mathcal{I}(V_{j_0 \ldots j_p}) = \mathcal{I}(f^{-1}(V_{j_0 \ldots j_p})) = \mathcal{I}(f^{-1}(V_{j_0}) \cap \ldots \cap f^{-1}(V_{j_p})) = \mathcal{I}(U_{j_0 \ldots j_p}).$$

Thus the first statement of the lemma follows from Lemma 20.12.1. The second statement follows from the first and Lemma 20.12.7. □

The following lemma implies in particular that $f_* : Ab(X) \to Ab(Y)$ transforms injective abelian sheaves into injective abelian sheaves.

Lemma 20.12.10. *Let $f : X \to Y$ be a morphism of ringed spaces. Assume f is flat. Then $f_*\mathcal{I}$ is an injective $\mathcal{O}_Y$-module for any injective $\mathcal{O}_X$-module $\mathcal{I}$.*

Proof. In this case the functor f^* transforms injections into injections (Modules, Lemma 17.17.2). Hence the result follows from Homology, Lemma 12.25.1. □

20.13. Flasque sheaves

Here is the definition.

Definition 20.13.1. Let X be a topological space. We say a presheaf of sets $\mathcal{F}$ is *flasque* or *flabby* if for every $U \subset V$ open in X the restriction map $\mathcal{F}(V) \to \mathcal{F}(U)$ is surjective.

We will use this terminology also for abelian sheaves and sheaves of modules if X is a ringed space. Clearly it suffices to assume the restriction maps $\mathcal{F}(X) \to \mathcal{F}(U)$ is surjective for every open $U \subset X$.

Lemma 20.13.2. *Let $(X, \mathcal{O}_X)$ be a ringed space. Then any injective $\mathcal{O}_X$-module is flasque.*

Proof. This is a reformulation of Lemma 20.9.1. □

Lemma 20.13.3. *Let $(X, \mathcal{O}_X)$ be a ringed space. Any flasque $\mathcal{O}_X$-module is acyclic for $R\Gamma(X, -)$ as well as $R\Gamma(U, -)$ for any open U of X.*

Proof. We will prove this using Derived Categories, Lemma 13.15.6. Since every injective module is flasque we see that we can embed every $\mathcal{O}_X$-module into a flasque module, see Injectives, Lemma 19.4.1. Thus it suffices to show that given a short exact sequence

$$0 \to \mathcal{F} \to \mathcal{G} \to \mathcal{H} \to 0$$

with $\mathcal{F}$, $\mathcal{G}$ flasque, then $\mathcal{H}$ is flasque and the sequence remains short exact after taking sections on any open of X. In fact, the second statement implies the first. Thus, let $U \subset X$ be an open subspace. Let $s \in \mathcal{H}(U)$. We will show that we can lift s to a sequence of $\mathcal{G}$ over U. To do this consider the set T of pairs (V, t) where $V \subset U$ is open and $t \in \mathcal{G}(V)$ is a section mapping to $s|_V$ in $\mathcal{H}$. We put a partial ordering on T by setting $(V, t) \leq (V', t')$ if and only if $V \subset V'$ and $t'|_V = t$. If (V_α, t_α), $\alpha \in A$ is a totally ordered subset of T, then $V = \bigcup V_\alpha$ is open and there is a unique section $t \in \mathcal{G}(V)$ restricting to t_α over V_α by the sheaf condition on $\mathcal{G}$. Thus by Zorn's lemma there exists a maximal element (V, t) in T. We will show that $V = U$ thereby finishing the proof. Namely, pick any $x \in U$. We can find a small open neighbourhood $W \subset U$ of x and $t' \in \mathcal{H}(W)$ mapping to $s|_W$ in $\mathcal{H}$. Then $t'|_{W \cap V} - t|_{W \cap V}$ maps to zero in $\mathcal{H}$, hence comes from some section $r' \in \mathcal{F}(W \cap V)$. Using that $\mathcal{F}$ is flasque we find a section $r \in \mathcal{F}(W)$ restricting to r' over $W' \cap V$. Modifying t' by the image of r we may assume that t and t' restrict to the same section over $W \cap V$. By the sheaf condition of $\mathcal{G}$ we can find a section $\tilde{t}$ of $\mathcal{G}$ over $W \cup V$ restricting to t and t'. By maximality of (V, t) we see that $V \cap W = V$. Thus $x \in V$ and we are done. □

The following lemma does not hold for flasque presheaves.

Lemma 20.13.4. *Let $(X, \mathcal{O}_X)$ be a ringed space. Let $\mathcal{F}$ be a sheaf of $\mathcal{O}_X$-modules. Let $\mathcal{U} : U = \bigcup U_i$ be an open covering. If $\mathcal{F}$ is flasque, then $\check{H}^p(\mathcal{U}, \mathcal{F}) = 0$ for $p > 0$.*

Proof. The presheaves $\underline{H}^q(\mathcal{F})$ used in the statement of Lemma 20.12.4 are zero by Lemma 20.13.3. Hence $\check{H}^p(U, \mathcal{F}) = H^p(U, \mathcal{F}) = 0$ by Lemma 20.13.3 again. □

Lemma 20.13.5. *Let $(X, \mathcal{O}_X) \to (Y, \mathcal{O}_Y)$ be a morphism of ringed spaces. Let $\mathcal{F}$ be a sheaf of $\mathcal{O}_X$-modules. If $\mathcal{F}$ is flasque, then $R^p f_* \mathcal{F} = 0$ for $p > 0$.*

Proof. Immediate from Lemma 20.7.3 and Lemma 20.13.3. □

20.14. The Leray spectral sequence

Lemma 20.14.1. *Let $f : X \to Y$ be a morphism of ringed spaces. There is a commutative diagram*

$$\begin{array}{ccc} D^+(X) & \xrightarrow[R\Gamma(X,-)]{} & D^+(\mathcal{O}_X(X)) \\ {\scriptstyle Rf_*}\downarrow & & \downarrow{\scriptstyle restriction} \\ D^+(Y) & \xrightarrow[R\Gamma(Y,-)]{} & D^+(\mathcal{O}_Y(Y)) \end{array}$$

More generally for any $V \subset Y$ open and $U = f^{-1}(V)$ there is a commutative diagram

$$\begin{array}{ccc} D^+(X) & \xrightarrow[R\Gamma(U,-)]{} & D^+(\mathcal{O}_X(U)) \\ {\scriptstyle Rf_*}\downarrow & & \downarrow{\scriptstyle restriction} \\ D^+(Y) & \xrightarrow[R\Gamma(V,-)]{} & D^+(\mathcal{O}_Y(V)) \end{array}$$

See also Remark 20.14.2 for more explanation.

Proof. Let $\Gamma_{res} : Mod(\mathcal{O}_X) \to \mathrm{Mod}_{\mathcal{O}_Y(Y)}$ be the functor which associates to an $\mathcal{O}_X$-module $\mathcal{F}$ the global sections of $\mathcal{F}$ viewed as a $\mathcal{O}_Y(Y)$-module via the map $f^\sharp : \mathcal{O}_Y(Y) \to \mathcal{O}_X(X)$. Let $restriction : \mathrm{Mod}_{\mathcal{O}_X(X)} \to \mathrm{Mod}_{\mathcal{O}_Y(Y)}$ be the restriction functor induced by $f^\sharp : \mathcal{O}_Y(Y) \to \mathcal{O}_X(X)$. Note that $restriction$ is exact so that its right derived functor is computed by simply applying the restriction functor, see Derived Categories, Lemma 13.16.8. It is clear that

$$\Gamma_{res} = restriction \circ \Gamma(X, -) = \Gamma(Y, -) \circ f_*$$

We claim that Derived Categories, Lemma 13.21.1 applies to both compositions. For the first this is clear by our remarks above. For the second, it follows from Lemma 20.12.9 which implies that injective $\mathcal{O}_X$-modules are mapped to $\Gamma(Y, -)$-acyclic sheaves on Y. □

Remark 20.14.2. Here is a down-to-earth explanation of the meaning of Lemma 20.14.1. It says that given $f : X \to Y$ and $\mathcal{F} \in Mod(\mathcal{O}_X)$ and given an injective resolution $\mathcal{F} \to \mathcal{I}^\bullet$ we have

$$\begin{array}{ccc} R\Gamma(X, \mathcal{F}) & \text{is represented by} & \Gamma(X, \mathcal{I}^\bullet) \\ Rf_*\mathcal{F} & \text{is represented by} & f_*\mathcal{I}^\bullet \\ R\Gamma(Y, Rf_*\mathcal{F}) & \text{is represented by} & \Gamma(Y, f_*\mathcal{I}^\bullet) \end{array}$$

the last fact coming from Leray's acyclicity lemma (Derived Categories, Lemma 13.16.7) and Lemma 20.12.9. Finally, it combines this with the trivial observation that

$$\Gamma(X, \mathcal{I}^\bullet) = \Gamma(Y, f_*\mathcal{I}^\bullet).$$

to arrive at the commutativity of the diagram of the lemma.

Lemma 20.14.3. *Let X be a ringed space. Let $\mathcal{F}$ be an $\mathcal{O}_X$-module.*

1. *The cohomology groups $H^i(U, \mathcal{F})$ for $U \subset X$ open of $\mathcal{F}$ computed as an $\mathcal{O}_X$-module, or computed as an abelian sheaf are identical.*
2. *Let $f : X \to Y$ be a morphism of ringed spaces. The higher direct images $R^i f_*\mathcal{F}$ of $\mathcal{F}$ computed as an $\mathcal{O}_X$-module, or computed as an abelian sheaf are identical.*

There are similar statements in the case of bounded below complexes of $\mathcal{O}_X$-modules.

Proof. Consider the morphism of ringed spaces $(X, \mathcal{O}_X) \to (X, \underline{\mathbf{Z}}_X)$ given by the identity on the underlying topological space and by the unique map of sheaves of rings $\underline{\mathbf{Z}}_X \to \mathcal{O}_X$. Let $\mathcal{F}$ be an $\mathcal{O}_X$-module. Denote $\mathcal{F}_{ab}$ the same sheaf seen as an $\underline{\mathbf{Z}}_X$-module, i.e., seen as a sheaf of abelian groups. Let $\mathcal{F} \to \mathcal{I}^\bullet$ be an injective resolution. By Remark 20.14.2 we see that $\Gamma(X, \mathcal{I}^\bullet)$ computes both $R\Gamma(X, \mathcal{F})$ and $R\Gamma(X, \mathcal{F}_{ab})$. This proves (1).

To prove (2) we use (1) and Lemma 20.7.3. The result follows immediately. □

Lemma 20.14.4 (Leray spectral sequence)**.** *Let $f : X \to Y$ be a morphism of ringed spaces. Let $\mathcal{F}^\bullet$ be a bounded below complex of $\mathcal{O}_X$-modules. There is a spectral sequence*

$$E_2^{p,q} = H^p(Y, R^q f_*(\mathcal{F}^\bullet))$$

converging to $H^{p+q}(X, \mathcal{F}^\bullet)$.

Proof. This is just the Grothendieck spectral sequence Derived Categories, Lemma 13.21.2 coming from the composition of functors $\Gamma_{res} = \Gamma(Y, -) \circ f_*$ where Γ_{res} is as in the proof of Lemma 20.14.1. To see that the assumptions of Derived Categories, Lemma 13.21.2 are satisfied, see the proof of Lemma 20.14.1 or Remark 20.14.2. □

Remark 20.14.5. The Leray spectral sequence, the way we proved it in Lemma 20.14.4 is a spectral sequence of $\Gamma(Y, \mathcal{O}_Y)$-modules. However, it is quite easy to see that it is in fact a spectral sequence of $\Gamma(X, \mathcal{O}_X)$-modules. For example f gives rise to a morphism of ringed spaces $f' : (X, \mathcal{O}_X) \to (Y, f_*\mathcal{O}_X)$. By Lemma 20.14.3 the terms $E_r^{p,q}$ of the Leray spectral sequence for an $\mathcal{O}_X$-module $\mathcal{F}$ and f are identical with those for $\mathcal{F}$ and f' at least for $r \geq 2$. Namely, they both agree with the terms of the Leray spectral sequence for $\mathcal{F}$ as an abelian sheaf. And since $(f_*\mathcal{O}_X)(Y) = \mathcal{O}_X(X)$ we see the result. It is often the case that the Leray spectral sequence carries additional structure.

Lemma 20.14.6. *Let $f : X \to Y$ be a morphism of ringed spaces. Let $\mathcal{F}$ be an $\mathcal{O}_X$-module.*

(1) *If $R^q f_*\mathcal{F} = 0$ for $q > 0$, then $H^p(X, \mathcal{F}) = H^p(Y, f_*\mathcal{F})$ for all p.*
(2) *If $H^p(Y, R^q f_*\mathcal{F}) = 0$ for all q and $p > 0$, then $H^q(X, \mathcal{F}) = H^0(Y, R^q f_*\mathcal{F})$ for all q.*

Proof. These are two simple conditions that force the Leray spectral sequence to converge. You can also prove these facts directly (without using the spectral sequence) which is a good exercise in cohomology of sheaves. □

Lemma 20.14.7. *Let $f : X \to Y$ and $g : Y \to Z$ be morphisms of ringed spaces. In this case $Rg_* \circ Rf_* = R(g \circ f)_*$ as functors from $D^+(X) \to D^+(Z)$.*

Proof. We are going to apply Derived Categories, Lemma 13.21.1. It is clear that $g_* \circ f_* = (g \circ f)_*$, see Sheaves, Lemma 6.21.2. It remains to show that $f_*\mathcal{I}$ is g_*-acyclic. This follows from Lemma 20.12.9 and the description of the higher direct images $R^i g_*$ in Lemma 20.7.3. □

Lemma 20.14.8 (Relative Leray spectral sequence)**.** *Let $f : X \to Y$ and $g : Y \to Z$ be morphisms of ringed spaces. Let $\mathcal{F}$ be an $\mathcal{O}_X$-module. There is a spectral sequence with*

$$E_2^{p,q} = R^p g_*(R^q f_*\mathcal{F})$$

converging to $R^{p+q}(g \circ f)_*\mathcal{F}$. This spectral sequence is functorial in $\mathcal{F}$, and there is a version for bounded below complexes of $\mathcal{O}_X$-modules.

Proof. This is a Grothendieck spectral sequence for composition of functors and follows from Lemma 20.14.7 and Derived Categories, Lemma 13.21.2. □

20.15. Functoriality of cohomology

Lemma 20.15.1. *Let $f : X \to Y$ be a morphism of ringed spaces. Let $\mathcal{G}^\bullet$, resp. $\mathcal{F}^\bullet$ be a bounded below complex of $\mathcal{O}_Y$-modules, resp. $\mathcal{O}_X$-modules. Let $\varphi : \mathcal{G}^\bullet \to f_*\mathcal{F}^\bullet$ be a morphism of complexes. There is a canonical morphism*

$$\mathcal{G}^\bullet \longrightarrow Rf_*(\mathcal{F}^\bullet)$$

in $D^+(Y)$. Moreover this construction is functorial in the triple $(\mathcal{G}^\bullet, \mathcal{F}^\bullet, \varphi)$.

Proof. Choose an injective resolution $\mathcal{F}^\bullet \to \mathcal{I}^\bullet$. By definition $Rf_*(\mathcal{F}^\bullet)$ is represented by $f_*\mathcal{I}^\bullet$ in $K^+(\mathcal{O}_Y)$. The composition

$$\mathcal{G}^\bullet \to f_*\mathcal{F}^\bullet \to f_*\mathcal{I}^\bullet$$

is a morphism in $K^+(Y)$ which turns into the morphism of the lemma upon applying the localization functor $j_Y : K^+(Y) \to D^+(Y)$. □

Let $f : X \to Y$ be a morphism of ringed spaces. Let $\mathcal{G}$ be an $\mathcal{O}_Y$-module and let $\mathcal{F}$ be an $\mathcal{O}_X$-module. Recall that an f-map φ from $\mathcal{G}$ to $\mathcal{F}$ is a map $\varphi : \mathcal{G} \to f_*\mathcal{F}$, or what is the same thing, a map $\varphi : f^*\mathcal{G} \to \mathcal{F}$. See Sheaves, Definition 6.21.7. Such an f-map gives rise to a morphism of complexes

$$\varphi : R\Gamma(Y, \mathcal{G}) \longrightarrow R\Gamma(X, \mathcal{F}) \tag{20.15.1.1}$$

in $D^+(\mathcal{O}_Y(Y))$. Namely, we use the morphism $\mathcal{G} \to Rf_*\mathcal{F}$ in $D^+(Y)$ of Lemma 20.15.1, and we apply $R\Gamma(Y, -)$. By Lemma 20.14.1 we see that $R\Gamma(X, \mathcal{F}) = R\Gamma(Y, Rf_*\mathcal{F})$ and we get the displayed arrow. We spell this out completely in Remark 20.15.2 below. In particular it gives rise to maps on cohomology

$$\varphi : H^i(Y, \mathcal{G}) \longrightarrow H^i(X, \mathcal{F}). \tag{20.15.1.2}$$

Remark 20.15.2. Let $f : X \to Y$ be a morphism of ringed spaces. Let $\mathcal{G}$ be an $\mathcal{O}_Y$-module. Let $\mathcal{F}$ be an $\mathcal{O}_X$-module. Let φ be an f-map from $\mathcal{G}$ to $\mathcal{F}$. Choose a resolution $\mathcal{F} \to \mathcal{I}^\bullet$ by a complex of injective $\mathcal{O}_X$-modules. Choose resolutions $\mathcal{G} \to \mathcal{J}^\bullet$ and $f_*\mathcal{I} \to (\mathcal{J}')^\bullet$ by complexes of injective $\mathcal{O}_Y$-modules. By Derived Categories, Lemma 13.17.6 there exists a map of complexes β such that the diagram

$$\begin{array}{ccccc} \mathcal{G} & \longrightarrow & f_*\mathcal{F} & \longrightarrow & f_*\mathcal{I}^\bullet \\ \downarrow & & & & \downarrow \\ \mathcal{J}^\bullet & & \xrightarrow{\beta} & & (\mathcal{J}')^\bullet \end{array} \tag{20.15.2.1}$$

commutes. Applying global section functors we see that we get a diagram

$$\begin{array}{ccc} & & \Gamma(Y, f_*\mathcal{I}^\bullet) = \Gamma(X, \mathcal{I}^\bullet) \\ & & \downarrow{\scriptstyle qis} \\ \Gamma(Y, \mathcal{J}^\bullet) & \xrightarrow{\beta} & \Gamma(Y, (\mathcal{J}')^\bullet) \end{array}$$

The complex on the bottom left represents $R\Gamma(Y, \mathcal{G})$ and the complex on the top right represents $R\Gamma(X, \mathcal{F})$. The vertical arrow is a quasi-isomorphism by Lemma 20.14.1 which becomes invertible after applying the localization functor $K^+(\mathcal{O}_Y(Y)) \to D^+(\mathcal{O}_Y(Y))$. The arrow (20.15.1.1) is given by the composition of the horizontal map by the inverse of the vertical map.

20.16. Refinements and Čech cohomology

Let $(X, \mathcal{O}_X)$ be a ringed space. Let $\mathcal{U} : X = \bigcup_{i \in I} U_i$ and $\mathcal{V} : X = \bigcup_{j \in J} V_j$ be open coverings. Assume that $\mathcal{U}$ is a refinement of $\mathcal{V}$. Choose a map $c : I \to J$ such that $U_i \subset V_{c(i)}$ for all $i \in I$. This induces a map of Čech complexes

$$\gamma : \check{\mathcal{C}}^\bullet(\mathcal{V}, \mathcal{F}) \longrightarrow \check{\mathcal{C}}^\bullet(\mathcal{U}, \mathcal{F}), \quad (\xi_{j_0 \ldots j_p}) \longmapsto (\xi_{c(i_0) \ldots c(i_p)}|_{U_{i_0 \ldots i_p}})$$

functorial in the sheaf of $\mathcal{O}_X$-modules $\mathcal{F}$. Suppose that $c' : I \to J$ is a second map such that $U_i \subset V_{c'(i)}$ for all $i \in I$. Then the corresponding maps γ and γ' are homotopic. Namely, $\gamma - \gamma' = \mathrm{d} \circ h + h \circ \mathrm{d}$ with $h : \check{\mathcal{C}}^{p+1}(\mathcal{V}, \mathcal{F}) \to \check{\mathcal{C}}^p(\mathcal{U}, \mathcal{F})$ given by the rule

$$h(\xi)_{i_0 \ldots i_p} = \sum\nolimits_{a=0}^{p} (-1)^a \alpha_{c(i_0) \ldots c(i_a) c'(i_a) \ldots c'(i_p)}$$

We omit the computation showing this works; please see the discussion following (20.24.0.2) for the proof in a more general case. In particular, the map on Čech cohomology groups is independent of the choice of c. Moreover, it is clear that if $\mathcal{W} : X = \bigcup_{k \in K} W_k$ is a third open covering and $\mathcal{V}$ is a refinement of $\mathcal{W}$, then the composition of the maps

$$\check{\mathcal{C}}^\bullet(\mathcal{W}, \mathcal{F}) \longrightarrow \check{\mathcal{C}}^\bullet(\mathcal{V}, \mathcal{F}) \longrightarrow \check{\mathcal{C}}^\bullet(\mathcal{U}, \mathcal{F})$$

associated to maps $I \to J$ and $J \to K$ is the map associated to the composition $I \to K$. In particular, we can define the Čech cohomology groups

$$\check{H}^p(X, \mathcal{F}) = \mathrm{colim}_{\mathcal{U}} \check{H}^p(\mathcal{U}, \mathcal{F})$$

where the colimit is over all open coverings of X partially ordered by refinement.

It turns out that the maps γ defined above are compatible with the map to cohomology, in other words, the composition

$$\check{H}^p(\mathcal{V}, \mathcal{F}) \to \check{H}^p(\mathcal{U}, \mathcal{F}) \xrightarrow{\text{Lemma 20.12.2}} H^p(X, \mathcal{F})$$

is the canonical map from the first group to cohomology of Lemma 20.12.2. In the lemma below we will prove this in a slightly more general setting. A consequence is that we obtain a well defined map

$$\check{H}^p(X, \mathcal{F}) = \mathrm{colim}_{\mathcal{U}} \check{H}^p(\mathcal{U}, \mathcal{F}) \longrightarrow H^p(X, \mathcal{F}) \tag{20.16.0.2}$$

from Čech cohomology to cohomology.

Lemma 20.16.1. *Let $f : X \to Y$ be a morphism of ringed spaces. Let $\varphi : f^*\mathcal{G} \to \mathcal{F}$ be an f-map from an $\mathcal{O}_Y$-module $\mathcal{G}$ to an $\mathcal{O}_X$-module $\mathcal{F}$. Let $\mathcal{U} : X = \bigcup_{i \in I} U_i$ and $\mathcal{V} : Y = \bigcup_{j \in J} V_j$ be open coverings. Assume that $\mathcal{U}$ is a refinement of $f^{-1}\mathcal{V} : X =$*

$\bigcup_{j\in J} f^{-1}(V_j)$. *In this case there exists a commutative diagram*

$$\begin{array}{ccc} \check{\mathcal{C}}^\bullet(\mathcal{U},\mathcal{F}) & \longrightarrow & R\Gamma(X,\mathcal{F}) \\ \uparrow{\scriptstyle \gamma} & & \uparrow \\ \check{\mathcal{C}}^\bullet(\mathcal{V},\mathcal{G}) & \longrightarrow & R\Gamma(Y,\mathcal{G}) \end{array}$$

in $D^+(\mathcal{O}_X(X))$ *with horizontal arrows given by Lemma 20.12.2 and right vertical arrow by (20.15.1.1). In particular we get commutative diagrams of cohomology groups*

$$\begin{array}{ccc} \check{H}^p(\mathcal{U},\mathcal{F}) & \longrightarrow & H^p(X,\mathcal{F}) \\ \uparrow{\scriptstyle \gamma} & & \uparrow \\ \check{H}^p(\mathcal{V},\mathcal{G}) & \longrightarrow & H^p(Y,\mathcal{G}) \end{array}$$

where the right vertical arrow is (20.15.1.2)

Proof. We first define the left vertical arrow. Namely, choose a map $c : I \to J$ such that $U_i \subset f^{-1}(V_{c(i)})$ for all $i \in I$. In degree p we define the map by the rule

$$\gamma(s)_{i_0\ldots i_p} = \varphi(s)_{c(i_0)\ldots c(i_p)}$$

This makes sense because φ does indeed induce maps $\mathcal{G}(V_{c(i_0)\ldots c(i_p)}) \to \mathcal{F}(U_{i_0\ldots i_p})$ by assumption. It is also clear that this defines a morphism of complexes. Choose injective resolutions $\mathcal{F} \to \mathcal{I}^\bullet$ on X and $\mathcal{G} \to J^\bullet$ on Y. According to the proof of Lemma 20.12.2 we introduce the double complexes $A^{\bullet,\bullet}$ and $B^{\bullet,\bullet}$ with terms

$$B^{p,q} = \check{\mathcal{C}}^p(\mathcal{V},\mathcal{J}^q) \quad \text{and} \quad A^{p,q} = \check{\mathcal{C}}^p(\mathcal{U},\mathcal{I}^q).$$

As in Remark 20.15.2 above we also choose an injective resolution $f_*\mathcal{I} \to (\mathcal{J}')^\bullet$ on Y and a morphism of complexes $\beta : \mathcal{J} \to (\mathcal{J}')^\bullet$ making (20.15.2.1) commutes. We introduce some more double complexes, namely $(B')^{\bullet,\bullet}$ and $(B'')\bullet,\bullet$ with

$$(B')^{p,q} = \check{\mathcal{C}}^p(\mathcal{V},(\mathcal{J}')^q) \quad \text{and} \quad (B'')^{p,q} = \check{\mathcal{C}}^p(\mathcal{V},f_*\mathcal{I}^q).$$

Note that there is an f-map of complexes from $f_*\mathcal{I}^\bullet$ to $\mathcal{I}^\bullet$. Hence it is clear that the same rule as above defines a morphism of double complexes

$$\gamma : (B'')^{\bullet,\bullet} \longrightarrow A^{\bullet,\bullet}.$$

Consider the diagram of complexes

$$\begin{array}{ccccccc} \check{\mathcal{C}}^\bullet(\mathcal{U},\mathcal{F}) & \longrightarrow & sA^\bullet & & \xleftarrow{qis} & & \Gamma(X,\mathcal{I}^\bullet) \\ & & & \nwarrow^{s\gamma} & & & \\ \uparrow{\scriptstyle \gamma} & & & & & & \\ \check{\mathcal{C}}^\bullet(\mathcal{V},\mathcal{G}) & \longrightarrow & sB^\bullet & \xrightarrow{\beta} & s(B')^\bullet & \xleftarrow{} & s(B'')^\bullet \\ & & \uparrow{\scriptstyle qis} & & \uparrow & & \uparrow \\ & & \Gamma(Y,\mathcal{J}^\bullet) & \xrightarrow{\beta} & \Gamma(Y,(\mathcal{J}')^\bullet) & \xleftarrow{qis} & \Gamma(Y,f_*\mathcal{I}^\bullet) \end{array}$$

The two horizontal arrows with targets $sA^\bullet$ and $sB^\bullet$ are the ones explained in Lemma 20.12.2. The left upper shape (a pentagon) is commutative simply because (20.15.2.1) is commutative. The two lower squares are trivially commutative. It is also immediate from the definitions that the right upper shape (a square) is

commutative. The result of the lemma now follows from the definitions and the fact that going around the diagram on the outer sides from $\check{\mathcal{C}}^\bullet(\mathcal{V}, \mathcal{G})$ to $\Gamma(X, \mathcal{I}^\bullet)$ either on top or on bottom is the same (where you have to invert any quasi-isomorphisms along the way). □

20.17. Cohomology on Hausdorff quasi-compact spaces

For such a space Čech cohomology agrees with cohomology.

Lemma 20.17.1. *Let X be a topological space. Let $\mathcal{F}$ be an abelian sheaf. Then the map $\check{H}^1(X, \mathcal{F}) \to H^1(X, \mathcal{F})$ defined in (20.16.0.2) is an isomorphism.*

Proof. Let $\mathcal{U}$ be an open covering of X. By Lemma 20.12.4 there is an exact sequence

$$0 \to \check{H}^1(\mathcal{U}, \mathcal{F}) \to H^1(X, \mathcal{F}) \to \check{H}^0(\mathcal{U}, \underline{H}^1(\mathcal{F}))$$

Thus the map is injective. To show surjectivity it suffices to show that any element of $\check{H}^0(\mathcal{U}, \underline{H}^1(\mathcal{F}))$ maps to zero after replacing $\mathcal{U}$ by a refinement. This is immediate from the definitions and the fact that $\underline{H}^1(\mathcal{F})$ is a presheaf of abelian groups whose sheafification is zero by locality of cohomology, see Lemma 20.7.2. □

Lemma 20.17.2. *Let X be a Hausdorff and quasi-compact topological space. Let $\mathcal{F}$ be an abelian sheaf on X. Then the map $\check{H}^p(X, \mathcal{F}) \to H^p(X, \mathcal{F})$ defined in (20.16.0.2) is an isomorphism for all p.*

Proof. We argue by induction on p that the map $c^p_{\mathcal{F}} : \check{H}^p(X, \mathcal{F}) \to H^p(X, \mathcal{F})$ is an isomorphism. For $p = 0$ the result is clear and for $p = 1$ the result holds by Lemma 20.17.1. Thus we may assume $p > 1$.

Choose an injective map $a : \mathcal{F} \to \mathcal{I}$, where $\mathcal{I}$ is an injective abelian sheaf. Let $b : \mathcal{I} \to \mathcal{G}$ be the quotient by $\mathcal{F}$. Let $\xi = (\xi_{i_0 \ldots i_p})$ be a cocycle of the Čech complex, giving rise to an element $\overline{\xi}$ of $\check{H}^p(\mathcal{U}, \mathcal{F})$. Then $a(\xi) = \mathrm{d}(\eta)$ for some cochain η for $\mathcal{I}$ by Lemma 20.12.1. The image $\theta = b(\eta)$ of η in the Čech complex for $\mathcal{G}$ is a cocyle, hence gives rise to an element $\overline{\theta}$ in $\check{H}^{p-1}(\mathcal{U}, \mathcal{G})$. A straightforward argument (using $p \geq 2$ and hence the Čech complex of $\mathcal{I}$ is acyclic in degree $p-1$) shows that the rule which assigns the element $\overline{\theta} \in \check{H}^{p-1}(\mathcal{U}, \mathcal{G})$ of θ to the class is well defined. It follows from the construction that $c^p_{\mathcal{F}}(\overline{\xi}) = \partial(c^{p-1}_{\mathcal{G}}(\overline{\theta}))$ where $\partial : H^{p-1}(X, \mathcal{G}) \to H^p(X, \mathcal{F})$ is the boundary coming from the short exact sequence $0 \to \mathcal{F} \to \mathcal{I} \to \mathcal{G} \to 0$ (details omitted).

Conversely, let $\theta = (\theta_{i_0 \ldots i_{p-1}})$ a cocycle of the Čech complex of $\mathcal{G}$ for some open covering $\mathcal{U}$. We would like to lift θ to a cochain for $\mathcal{I}$. The problem is that the sequence of complexes

$$0 \to \check{\mathcal{C}}^\bullet(\mathcal{U}, \mathcal{F}) \to \check{\mathcal{C}}^\bullet(\mathcal{U}, \mathcal{I}) \to \check{\mathcal{C}}^\bullet(\mathcal{U}, \mathcal{G}) \to 0$$

may not be exact on the right. Howeover, we know that for all p-tuples $i_0 \ldots i_{p-1}$ of I there exists an open covering

$$U_{i_0} \cap \ldots \cap U_{i_{p-1}} = \bigcup W_{i_0 \ldots i_{p-1}, k}$$

such that $\theta_{i_0 \ldots i_{p-1}}|_{W_{i_0 \ldots i_{p-1}, k}}$ does lift to a section of $\mathcal{I}$ over $W_{i_0 \ldots i_{p-1}, k}$. Thus, by Topology, Lemma 5.12.4 after refining $\mathcal{U}$, we can lift θ to a $(p-1)$-cochain η in the Čech complex of $\mathcal{I}$. Then $\mathrm{d}(\eta) = a(\xi)$ for some p-cocycle ξ for $\mathcal{F}$. In other words,

every element of colim $\check{H}^{p-1}(\mathcal{U}, \mathcal{G})$ comes about by the construction of the previous paragraph from an element of colim $\check{H}^p(\mathcal{U}, \mathcal{F})$.

By the compatibility of the construction with the boundary map $\partial : H^{p-1}(X, \mathcal{G}) \to H^p(X, \mathcal{F})$, the surjectivity of the map, the induction hypothesis saying $\gamma_{\mathcal{G}}^{p-1}$ is an isomorphism, and the fact that $H^{p-1}(X, \mathcal{I}) = H^p(X, \mathcal{I}) = 0$, it follows formally that $c_{\mathcal{F}}^p$ is surjective. To show injectivity one has to show that, given ξ, η, θ linked as above, if θ is a boundary, then ξ becomes a boundary after replacing $\mathcal{U}$ by a refinement. To do this argue as above, once more appealing to Topology, Lemma 5.12.4. Some details omitted. □

Lemma 20.17.3. *Let X be a Hausdorff and locally quasi-compact space. Let $Z \subset X$ be a quasi-compact (hence closed) subset. For every abelian sheaf $\mathcal{F}$ on X we have*

$$\operatorname{colim} H^p(U, \mathcal{F}) \longrightarrow H^p(Z, \mathcal{F}|_Z)$$

where the colimit is over open neighbourhoods U of Z in X.

Proof. We first prove this for $p = 0$. Injectivity follows from the definition of $\mathcal{F}|_Z$ and holds in general (for any subset of any topological space X). Next, suppose that $s \in H^0(Z, \mathcal{F}|_Z)$. Then we can find opens $U_i \subset X$ such that $Z \subset \bigcup U_i$ and such that $s|_{Z \cap U_i}$ comes from $s_i \in \mathcal{F}(U_i)$. It follows that there exist opens $W_{ij} \subset U_i \cap U_j$ with $W_{ij} \cap Z = U_i \cap U_j \cap Z$ such that $s_i|_{W_{ij}} = s_j|_{W_{ij}}$. Applying Topology, Lemma 5.12.5 we find opens V_i of X such that $V_i \subset U_i$ and such that $V_i \cap V_j \subset W_{ij}$. Hence we see that $s_i|_{V_i}$ glue to a section of $\mathcal{F}$ over the open neighbourhood $\bigcup V_i$ of Z.

To finish the proof, it suffices to show that if $\mathcal{I}$ is an injective abelian sheaf on X, then $H^p(Z, \mathcal{I}|_Z) = 0$ for $p > 0$. This follows using short exact sequences and dimension shifting; details omitted. Thus, suppose $\overline{\xi}$ is an element of $H^p(Z, \mathcal{I}|_Z)$ for some $p > 0$. By Lemma 20.17.2 the element $\overline{\xi}$ comes from $\check{H}^p(\mathcal{V}, \mathcal{I}|_Z)$ for some open covering $\mathcal{V} : Z = \bigcup V_i$ of Z. Say $\overline{\xi}$ is the image of the class of a cocycle $\xi = (\xi_{i_0 \ldots i_p})$ in $\check{\mathcal{C}}^p(\mathcal{V}, \mathcal{I}|_Z)$.

Let $\mathcal{I}' \subset \mathcal{I}|_Z$ be the subpresheaf defined by the rule

$$\mathcal{I}'(V) = \{s \in \mathcal{I}|_Z(V) \mid \exists (U, t),\ U \subset X \text{ open},\ t \in \mathcal{I}(U),\ V = Z \cap U,\ s = t|_{Z \cap U}\}$$

Then $\mathcal{I}|_Z$ is the sheafification of $\mathcal{I}'$. Thus for every $(p+1)$-tuple $i_0 \ldots i_p$ we can find an open covering $V_{i_0 \ldots i_p} = \bigcup W_{i_0 \ldots i_p, k}$ such that $\xi_{i_0 \ldots i_p}|_{W_{i_0 \ldots i_p, k}}$ is a section of $\mathcal{I}'$. Applying Topology, Lemma 5.12.4 we may after refining $\mathcal{V}$ assume that each $\xi_{i_0 \ldots i_p}$ is a section of the presheaf $\mathcal{I}'$.

Write $V_i = Z \cap U_i$ for some opens $U_i \subset X$. Since $\mathcal{I}$ is flasque (Lemma 20.13.2) and since $\xi_{i_0 \ldots i_p}$ is a section of $\mathcal{I}'$ for every $(p+1)$-tuple $i_0 \ldots i_p$ we can choose a section $s_{i_0 \ldots i_p} \in \mathcal{I}(U_{i_0 \ldots i_p})$ which restricts to $\xi_{i_0 \ldots i_p}$ on $V_{i_0 \ldots i_p} = Z \cap U_{i_0 \ldots i_p}$. (This appeal to injectives being flasque can be avoided by an additional application of Topology, Lemma 5.12.5.) Let $s = (s_{i_0 \ldots i_p})$ be the corresponding cochain for the open covering $U = \bigcup U_i$. Since $\mathrm{d}(\xi) = 0$ we see that the sections $\mathrm{d}(s)_{i_0 \ldots i_{p+1}}$ restrict to zero on $Z \cap U_{i_0 \ldots i_{p+1}}$. Hence, by the initial remarks of the proof, there exists open subsets $W_{i_0 \ldots i_{p+1}} \subset U_{i_0 \ldots i_{p+1}}$ with $Z \cap W_{i_0 \ldots i_{p+1}} = Z \cap U_{i_0 \ldots i_{p+1}}$ such that $\mathrm{d}(s)_{i_0 \ldots i_{p+1}}|_{W_{i_0 \ldots i_{p+1}}} = 0$. By Topology, Lemma 5.12.5 we can find $U_i' \subset U_i$ such that $Z \subset \bigcup U_i'$ and such that $U'_{i_0 \ldots i_{p+1}} \subset W_{i_0 \ldots i_{p+1}}$. Then $s' = (s'_{i_0 \ldots i_p})$ with $s'_{i_0 \ldots i_p} = s_{i_0 \ldots i_p}|_{U'_{i_0 \ldots i_p}}$ is a cocycle for $\mathcal{I}$ for the open covering $U' = \bigcup U_i'$ of an open

neighbourhood of Z. Since $\mathcal{I}$ has trivial higher Čech cohomology groups (Lemma 20.12.1) we conclude that s' is a coboundary. It follows that the image of ξ in the Čech complex for the open covering $Z = \bigcup Z \cap U_i'$ is a coboundary and we are done. □

20.18. The base change map

We will need to know how to construct the base change map in some cases. Since we have not yet discussed derived pullback we only discuss this in the case of a base change by a flat morphism of ringed spaces. Before we state the result, let us discuss flat pullback on the derived category. Namely, suppose that $g : X \to Y$ is a flat morphism of ringed spaces. By Modules, Lemma 17.17.2 the functor $g^* : \mathit{Mod}(\mathcal{O}_Y) \to \mathit{Mod}(\mathcal{O}_X)$ is exact. Hence it has a derived functor

$$g^* : D^+(Y) \to D^+(X)$$

which is computed by simply pulling back an representative of a given object in $D^+(Y)$, see Derived Categories, Lemma 13.16.8. Hence as indicated we indicate this functor by g^* rather than Lg^*.

Lemma 20.18.1. *Let*

$$\begin{array}{ccc} X' & \xrightarrow{g'} & X \\ {\scriptstyle f'}\downarrow & & \downarrow{\scriptstyle f} \\ S' & \xrightarrow{g} & S \end{array}$$

be a commutative diagram of ringed spaces. Let $\mathcal{F}^\bullet$ be a bounded below complex of $\mathcal{O}_X$-modules. Assume both g and g' are flat. Then there exists a canonical base change map

$$g^* Rf_* \mathcal{F}^\bullet \longrightarrow R(f')_*(g')^* \mathcal{F}^\bullet$$

in $D^+(S')$.

Proof. Choose injective resolutions $\mathcal{F}^\bullet \to \mathcal{I}^\bullet$ and $(g')^*\mathcal{F}^\bullet \to \mathcal{J}^\bullet$. By Lemma 20.12.10 we see that $(g')_*\mathcal{J}^\bullet$ is a complex of injectives representing $R(g')_*(g')^*\mathcal{F}^\bullet$. Hence by Derived Categories, Lemmas 13.17.6 and 13.17.7 the arrow β in the diagram

$$\begin{array}{ccc} (g')_*(g')^*\mathcal{F}^\bullet & \longrightarrow & (g')_*\mathcal{J}^\bullet \\ {\scriptstyle adjunction}\uparrow & & \uparrow{\scriptstyle \beta} \\ \mathcal{F}^\bullet & \longrightarrow & \mathcal{I}^\bullet \end{array}$$

exists and is unique up to homotopy. Pushing down to S we get

$$f_*\beta : f_*\mathcal{I}^\bullet \longrightarrow f_*(g')_*\mathcal{J}^\bullet = g_*(f')_*\mathcal{J}^\bullet$$

By adjunction of g^* and g_* we get a map of complexes $g^* f_* \mathcal{I}^\bullet \to (f')_*\mathcal{J}^\bullet$. Note that this map is unique up to homotopy since the only choice in the whole process was the choice of the map β and everything was done on the level of complexes. □

Remark 20.18.2. The "correct" version of the base change map is map

$$Lg^* Rf_* \mathcal{F}^\bullet \longrightarrow R(f')_* L(g')^* \mathcal{F}^\bullet.$$

The construction of this map involves unbounded complexes, see Remark 20.27.2.

20.19. Proper base change in topology

In this section we prove a very general version of the proper base change theorem in topology. It tells us that the stalks of the higher direct images $R^p f_*$ can be computed on the fibre.

Lemma 20.19.1. *Let $f : (X, \mathcal{O}_X) \to (Y, \mathcal{O}_Y)$ be a morphism of ringed spaces. Let $y \in Y$. Assume that*

1. *X is Hausdorff and locally quasi-compact,*
2. *$f^{-1}(y)$ is quasi-compact, and*
3. *f is closed.*

Then for E in $D^+(\mathcal{O}_X)$ we have $(Rf_*E)_y = R\Gamma(f^{-1}(y), E|_{f^{-1}(y)})$ in $D^+(\mathcal{O}_{Y,y})$.

Proof. The base change map of Lemma 20.18.1 gives a canonical map $(Rf_*E)_y \to R\Gamma(f^{-1}(y), E|_{f^{-1}(y)})$. To prove this map is an isomorphism, we represent E by a bounded below complex of injectives $\mathcal{I}^\bullet$. By Lemma 20.17.3 the restrictions $\mathcal{I}^n|_{f^{-1}(y)}$ are acyclic for $\Gamma(f^{-1}(y), -)$. Thus $R\Gamma(f^{-1}(y), E|_{f^{-1}(y)})$ is represented by the complex $\Gamma(f^{-1}(y), \mathcal{I}^\bullet|_{f^{-1}(y)})$, see Derived Categories, Lemma 13.16.7. In other words, we have to show the map

$$\operatorname{colim}_V \mathcal{I}^\bullet(f^{-1}(V)) \longrightarrow \Gamma(f^{-1}(y), \mathcal{I}^\bullet|_{f^{-1}(y)})$$

is an isomorphism. Using Lemma 20.17.3 we see that it suffices to show that the collection of open neighbourhoods $f^{-1}(V)$ of $f^{-1}(y)$ is cofinal in the system of all open neighbourhoods. If $f^{-1}(y) \subset U$ is an open neighbourhood, then as f is closed the set $V = Y \setminus f(X \setminus U)$ is an open neighbourhood of y with $f^{-1}(V) \subset U$. This proves the lemma. □

Theorem 20.19.2 (Proper base change)**.** *Consider a cartesian square of Hausdorff, locally quasi-compact topological spaces*

$$\begin{array}{ccc} X' = Y' \times_Y X & \xrightarrow{g'} & X \\ \downarrow{\scriptstyle f'} & & \downarrow{\scriptstyle f} \\ Y' & \xrightarrow{g} & Y \end{array}$$

and assume that f is proper. Let E be an object of $D^+(X)$. Then the base change map

$$g^{-1}Rf_*E \longrightarrow Rf'_*(g')^{-1}E$$

of Lemma 20.18.1 is an isomorphism in $D^+(Y')$.

Proof. Let $y' \in Y'$ be a point with image $y \in Y$. It suffices to show that the base change map induces an isomorphism on stalks at y'. As f is proper it follows that f' is proper, the fibres of f and f' are quasi-compact and f and f' are closed, see Topology, Theorem 5.16.5. Thus we can apply Lemma 20.19.1 twice to see that

$$(Rf'_*(g')^{-1}E)_{y'} = R\Gamma((f')^{-1}(y'), (g')^{-1}E|_{(f')^{-1}(y')})$$

and

$$(Rf_*E)_y = R\Gamma(f^{-1}(y), E|_{f^{-1}(y)})$$

The induced map of fibres $(f')^{-1}(y') \to f^{-1}(y)$ is a homeomorphism of topological spaces and the pull back of $E|_{f^{-1}(y)}$ is $(g')^{-1}E|_{(f')^{-1}(y')}$. The desired result follows. □

20.20. Cohomology and colimits

Let X be a ringed space. Let $(\mathcal{F}_i, \varphi_{ii'})$ be a directed system of sheaves of $\mathcal{O}_X$-modules over the partially ordered set I, see Categories, Section 4.21. Since for each i there is a canonical map $\mathcal{F}_i \to \operatorname{colim}_i \mathcal{F}_i$ we get a canonical map

$$\operatorname{colim}_i H^p(X, \mathcal{F}_i) \longrightarrow H^p(X, \operatorname{colim}_i \mathcal{F}_i)$$

for every $p \geq 0$. Of course there is a similar map for every open $U \subset X$. These maps are in general not isomorphisms, even for $p = 0$. In this section we generalize the results of Sheaves, Lemma 6.29.1. See also Modules, Lemma 17.11.6 (in the special case $\mathcal{G} = \mathcal{O}_X$).

Lemma 20.20.1. *Let X be a ringed space. Assume that the underlying topological space of X has the following properties:*

(1) *there exists a basis of quasi-compact open subsets, and*
(2) *the intersection of any two quasi-compact opens is quasi-compact.*

Then for any directed system $(\mathcal{F}_i, \varphi_{ii'})$ of sheaves of $\mathcal{O}_X$-modules and for any quasi-compact open $U \subset X$ the canonical map

$$\operatorname{colim}_i H^q(U, \mathcal{F}_i) \longrightarrow H^q(U, \operatorname{colim}_i \mathcal{F}_i)$$

is an isomorphism for every $q \geq 0$.

Proof. It is important in this proof to argue for all quasi-compact opens $U \subset X$ at the same time. The result is true for $i = 0$ and any quasi-compact open $U \subset X$ by Sheaves, Lemma 6.29.1 (combined with Topology, Lemma 5.25.1). Assume that we have proved the result for all $q \leq q_0$ and let us prove the result for $q = q_0 + 1$.

By our conventions on directed systems the index set I is directed, and any system of $\mathcal{O}_X$-modules $(\mathcal{F}_i, \varphi_{ii'})$ over I is directed. By Injectives, Lemma 19.5.1 the category of $\mathcal{O}_X$-modules has functorial injective embeddings. Thus for any system $(\mathcal{F}_i, \varphi_{ii'})$ there exists a system $(\mathcal{I}_i, \varphi_{ii'})$ with each $\mathcal{I}_i$ an injective $\mathcal{O}_X$-module and a morphism of systems given by injective $\mathcal{O}_X$-module maps $\mathcal{F}_i \to \mathcal{I}_i$. Denote $\mathcal{Q}_i$ the cokernel so that we have short exact sequences

$$0 \to \mathcal{F}_i \to \mathcal{I}_i \to \mathcal{Q}_i \to 0.$$

We claim that the sequence

$$0 \to \operatorname{colim}_i \mathcal{F}_i \to \operatorname{colim}_i \mathcal{I}_i \to \operatorname{colim}_i \mathcal{Q}_i \to 0.$$

is also a short exact sequence of $\mathcal{O}_X$-modules. We may check this on stalks. By Sheaves, Sections 6.28 and 6.29 taking stalks commutes with colimits. Since a directed colimit of short exact sequences of abelian groups is short exact (see Algebra, Lemma 10.8.9) we deduce the result. We claim that $H^q(U, \operatorname{colim}_i \mathcal{I}_i) = 0$ for all quasi-compact open $U \subset X$ and all $q \geq 1$. Accepting this claim for the moment consider the diagram

$$\begin{array}{ccccccc}
\operatorname{colim}_i H^{q_0}(U, \mathcal{I}_i) & \longrightarrow & \operatorname{colim}_i H^{q_0}(U, \mathcal{Q}_i) & \longrightarrow & \operatorname{colim}_i H^{q_0+1}(U, \mathcal{F}_i) & \longrightarrow & 0 \\
\downarrow & & \downarrow & & \downarrow & & \downarrow \\
H^{q_0}(U, \operatorname{colim}_i \mathcal{I}_i) & \longrightarrow & H^{q_0}(U, \operatorname{colim}_i \mathcal{Q}_i) & \longrightarrow & H^{q_0+1}(U, \operatorname{colim}_i \mathcal{F}_i) & \longrightarrow & 0
\end{array}$$

The zero at the lower right corner comes from the claim and the zero at the upper right corner comes from the fact that the sheaves $\mathcal{I}_i$ are injective. The top row is

exact by an application of Algebra, Lemma 10.8.9. Hence by the snake lemma we deduce the result for $q = q_0 + 1$.

It remains to show that the claim is true. We will use Lemma 20.12.8. Let $\mathcal{B}$ be the collection of all quasi-compact open subsets of X. This is a basis for the topology on X by assumption. Let Cov be the collection of finite open coverings $\mathcal{U} : U = \bigcup_{j=1,\ldots,m} U_j$ with each of U, U_j quasi-compact open in X. By the result for $q = 0$ we see that for $\mathcal{U} \in \text{Cov}$ we have

$$\check{\mathcal{C}}^\bullet(\mathcal{U}, \text{colim}_i \mathcal{I}_i) = \text{colim}_i \check{\mathcal{C}}^\bullet(\mathcal{U}, \mathcal{I}_i)$$

because all the multiple intersections $U_{j_0 \ldots j_p}$ are quasi-compact. By Lemma 20.12.1 each of the complexes in the colimit of Čech complexes is acyclic in degree ≥ 1. Hence by Algebra, Lemma 10.8.9 we see that also the Čech complex $\check{\mathcal{C}}^\bullet(\mathcal{U}, \text{colim}_i \mathcal{I}_i)$ is acyclic in degrees ≥ 1. In other words we see that $\check{H}^p(\mathcal{U}, \text{colim}_i \mathcal{I}_i) = 0$ for all $p \geq 1$. Thus the assumptions of Lemma 20.12.8 are satisfied and the claim follows. □

20.21. Vanishing on Noetherian topological spaces

The aim is to prove a theorem of Grothendieck namely Lemma 20.21.5. See [**Gro57**].

Lemma 20.21.1. *Let $i : Z \to X$ be a closed immersion of topological spaces. For any abelian sheaf $\mathcal{F}$ on Z we have $H^p(Z, \mathcal{F}) = H^p(X, i_*\mathcal{F})$.*

Proof. This is true because i_* is exact (see Modules, Lemma 17.6.1), and hence $R^p i_* = 0$ as a functor (Derived Categories, Lemma 13.16.8). Thus we may apply Lemma 20.14.6. □

Lemma 20.21.2. *Let X be an irreducible topological space. Then $H^p(X, \underline{A}) = 0$ for all $p > 0$ and any abelian group A.*

Proof. Recall that $\underline{A}$ is the constant sheaf as defined in Sheaves, Definition 6.7.4. It is clear that for any nonempty open $U \subset X$ we have $\underline{A}(U) = A$ as X is irreducible (and hence U is connected). We will show that the higher Čech cohomology groups $\check{H}^p(\mathcal{U}, \underline{A})$ are zero for any open covering $\mathcal{U} : U = \bigcup_{i \in I} U_i$ of an open $U \subset X$. Then the lemma will follow from Lemma 20.12.7.

Recall that the value of an abelian sheaf on the empty open set is 0. Hence we may clearly assume $U_i \neq \emptyset$ for all $i \in I$. In this case we see that $U_i \cap U_{i'} \neq \emptyset$ for all $i, i' \in I$. Hence we see that the Čech complex is simply the complex

$$\prod_{i_0 \in I} A \to \prod_{(i_0, i_1) \in I^2} A \to \prod_{(i_0, i_1, i_2) \in I^3} A \to \ldots$$

We have to see this has trivial higher cohomology groups. We can see this for example because this is the cech complex for the covering of a 1-point space and Čech cohomology agrees with cohomology on such a space. (You can also directly verify it by writing an explicit homotopy.) □

Lemma 20.21.3. *Let X be a topological space. Let $n \geq 0$ be an integer. Assume*

1. *there exists a basis of quasi-compact open subsets, and*
2. *the intersection of any two quasi-compact opens is quasi-compact.*

(3) $H^p(X, \mathcal{F}) = 0$ *for* $p \geq n$ *for any abelian sheaf* $\mathcal{F}$ *which is a quotient of* $j_!\underline{\mathbf{Z}}_U$ *for some open* $j : U \to X$.

Then $H^p(X, \mathcal{F}) = 0$ *for all* $p \geq n$ *and any abelian sheaf* $\mathcal{F}$ *on* X.

Proof. Let $S = \coprod_{U \subset X} \mathcal{F}(U)$. For any finite subset $A = \{s_1, \ldots, s_n\} \subset S$, let $\mathcal{F}_A$ be the subsheaf of $\mathcal{F}$ generated by all s_i (see Modules, Definition 17.4.5). Note that if $A \subset A'$, then $\mathcal{F}_A \subset \mathcal{F}_{A'}$. Hence $\{\mathcal{F}_A\}$ forms a system over the partially ordered set of finite subsets of S. By Modules, Lemma 17.4.6 it is clear that

$$\operatorname{colim}_A \mathcal{F}_A = \mathcal{F}$$

by looking at stalks. By Lemma 20.20.1 we have

$$H^p(X, \mathcal{F}) = \operatorname{colim}_A H^p(X, \mathcal{F}_A)$$

Hence it suffices to prove the vanishing for the abelian sheaves $\mathcal{F}_A$. In other words, it suffices to prove the result when $\mathcal{F}$ is generated by finitely many local sections.

Suppose that $\mathcal{F}$ is generated by the local sections $s_1, \ldots, s_n$. Let $\mathcal{F}' \subset \mathcal{F}$ be the subsheaf generated by $s_1, \ldots, s_{n-1}$. Then we have a short exact sequence

$$0 \to \mathcal{F}' \to \mathcal{F} \to \mathcal{F}/\mathcal{F}' \to 0$$

From the long exact sequence of cohomology we see that it suffices to prove the vanishing for the abelian sheaves $\mathcal{F}'$ and $\mathcal{F}/\mathcal{F}'$ which are generated by fewer than n local sections. Hence it suffices to prove the vanishing for sheaves generated by at most one local section. These sheaves are exactly the quotients of the sheaves $j_!\underline{\mathbf{Z}}_U$ mentioned in the lemma. □

Lemma 20.21.4. *Let* X *be an irreducible topological space. Let* $\mathcal{H} \subset \underline{\mathbf{Z}}$ *be an abelian subsheaf of the constant sheaf. Then there exists a nonempty open* $U \subset X$ *such that* $\mathcal{H}|_U = \underline{d\mathbf{Z}}_U$ *for some* $d \in \mathbf{Z}$.

Proof. Recall that $\underline{\mathbf{Z}}(V) = \mathbf{Z}$ for any nonempty open V of X (see proof of Lemma 20.21.2). If $\mathcal{H} = 0$, then the lemma holds with $d = 0$. If $\mathcal{H} \neq 0$, then there exists a nonempty open $U \subset X$ such that $\mathcal{H}(U) \neq 0$. Say $\mathcal{H}(U) = n\mathbf{Z}$ for some $n \geq 1$. Hence we see that $\underline{n\mathbf{Z}}_U \subset \mathcal{H}|_U \subset \underline{\mathbf{Z}}_U$. If the first inclusion is strict we can find a nonempty $U' \subset U$ and an integer $1 \leq n' < n$ such that $\underline{n'\mathbf{Z}}_{U'} \subset \mathcal{H}|_{U'} \subset \underline{\mathbf{Z}}_{U'}$. This process has to stop after a finite number of steps, and hence we get the lemma. □

Lemma 20.21.5. *Let* X *be a Noetherian topological space. If* $\dim(X) \leq n$, *then* $H^p(X, \mathcal{F}) = 0$ *for all* $p > n$ *and any abelian sheaf* $\mathcal{F}$ *on* X.

Proof. We prove this lemma by induction on n. So fix n and assume the lemma holds for all Noetherian topological spaces of dimension $< n$.

Let $\mathcal{F}$ be an abelian sheaf on X. Suppose $U \subset X$ is an open. Let $Z \subset X$ denote the closed complement. Denote $j : U \to X$ and $i : Z \to X$ the inclusion maps. Then there is a short exact sequence

$$0 \to j_!j^*\mathcal{F} \to \mathcal{F} \to i_*i^*\mathcal{F} \to 0$$

see Modules, Lemma 17.7.1. Note that $j_!j^*\mathcal{F}$ is supported on the topological closure Z' of U, i.e., it is of the form $i'_*\mathcal{F}'$ for some abelian sheaf $\mathcal{F}'$ on Z', where $i' : Z' \to X$ is the inclusion.

We can use this to reduce to the case where X is irreducible. Namely, according to Topology, Lemma 5.8.2 X has finitely many irreducible components. If X has

more than one irreducible component, then let $Z \subset X$ be an irreducible component of X and set $U = X \setminus Z$. By the above, and the long exact sequence of cohomology, it suffices to prove the vanishing of $H^p(X, i_*i^*\mathcal{F})$ and $H^p(X, i'_*\mathcal{F}')$ for $p > n$. By Lemma 20.21.1 it suffices to prove $H^p(Z, i^*\mathcal{F})$ and $H^p(Z', \mathcal{F}')$ for $p > n$. Since Z' and Z have fewer irreducible components we indeed reduce to the case of an irreducible X.

If $n = 0$ and $X = \{*\}$, then every sheaf is constant and higher cohomology groups vanish (for example by Lemma 20.21.2).

Suppose X is irreducible of dimension n. By Lemma 20.21.3 we reduce to the case where $\mathcal{F}$ is generated by a single local section, i.e., to the case where there is an exact sequence

$$0 \to \mathcal{H} \to j'_!\mathbf{Z}_V \to \mathcal{F} \to 0$$

for some open $j' : V \to X$. By Lemma 20.21.4 (applied to the restriction of $\mathcal{H}$ to V) there exists a nonempty open $U \subset V$, and $d \in \mathbf{Z}$ such that $\mathcal{H}|_U = \underline{d\mathbf{Z}}_U$. Hence we see that $\mathcal{F}|_U \cong \underline{\mathbf{Z}/d\mathbf{Z}}_U$. Let Z be the complement of U in X. Denote $j : U \to X$ and $i : Z \to X$ the inclusion maps. As in the first paragraph of the proof we obtain a short exact sequence

$$0 \to j_!\underline{\mathbf{Z}/d\mathbf{Z}} \to \mathcal{F} \to i_*i^*\mathcal{F} \to 0$$

OK, and now $\dim(Z) < n$ so by induction we have $H^p(X, i_*i^*\mathcal{F}) = H^p(Z, i^*\mathcal{F}) = 0$ for all $p \geq n$. Hence it suffices to prove the vanishing for sheaves of the form $j_!(\underline{A}_U)$ where $j : U \to X$ is an open immersion and A is an abelian group.

In this case we again look at the short exact sequence

$$0 \to j_!(\underline{A}_U) \to \underline{A} \to i_*\underline{A}_Z \to 0$$

By Lemma 20.21.2 we have the vanishing of $H^p(X, \underline{A})$ for all $p \geq 1$. By induction we have $H^p(X, i_*\underline{A}_Z) = H^p(Z, \underline{A}_Z) = 0$ for $p \leq n$. Hence we win by the long exact cohomology sequence. □

20.22. The alternating Čech complex

This section compares the Čech complex with the alternating Čech complex and some related complexes.

Let X be a topological space. Let $\mathcal{U} : U = \bigcup_{i \in I} U_i$ be an open covering. For $p \geq 0$ set

$$\check{\mathcal{C}}^p_{alt}(\mathcal{U}, \mathcal{F}) = \left\{ \begin{array}{c} s \in \check{\mathcal{C}}^p(\mathcal{U}, \mathcal{F}) \text{ such that } s_{i_0 \ldots i_p} = 0 \text{ if } i_n = i_m \text{ for some } n \neq m \\ \text{and } s_{i_0 \ldots i_n \ldots i_m \ldots i_p} = -s_{i_0 \ldots i_m \ldots i_n \ldots i_p} \text{ in any case.} \end{array} \right\}$$

We omit the verification that the differential d of Equation (20.10.0.1) maps $\check{\mathcal{C}}^p_{alt}(\mathcal{U}, \mathcal{F})$ into $\check{\mathcal{C}}^{p+1}_{alt}(\mathcal{U}, \mathcal{F})$.

Definition 20.22.1. Let X be a topological space. Let $\mathcal{U} : U = \bigcup_{i \in I} U_i$ be an open covering. Let $\mathcal{F}$ be an abelian presheaf on X. The complex $\check{\mathcal{C}}^\bullet_{alt}(\mathcal{U}, \mathcal{F})$ is the *alternating Čech complex* associated to $\mathcal{F}$ and the open covering $\mathcal{U}$.

Hence there is a canonical morphism of complexes

$$\check{\mathcal{C}}^\bullet_{alt}(\mathcal{U}, \mathcal{F}) \longrightarrow \check{\mathcal{C}}^\bullet(\mathcal{U}, \mathcal{F})$$

namely the inclusion of the alternating Čech complex into the usual Čech complex.

Suppose our covering $\mathcal{U} : U = \bigcup_{i \in I} U_i$ comes equipped with a total ordering $<$ on I. In this case, set

$$\check{\mathcal{C}}^p_{ord}(\mathcal{U}, \mathcal{F}) = \prod\nolimits_{(i_0,\ldots,i_p) \in I^{p+1}, i_0 < \ldots < i_p} \mathcal{F}(U_{i_0 \ldots i_p}).$$

This is an abelian group. For $s \in \check{\mathcal{C}}^p_{ord}(\mathcal{U}, \mathcal{F})$ we denote $s_{i_0 \ldots i_p}$ its value in $\mathcal{F}(U_{i_0 \ldots i_p})$. We define

$$d : \check{\mathcal{C}}^p_{ord}(\mathcal{U}, \mathcal{F}) \longrightarrow \check{\mathcal{C}}^{p+1}_{ord}(\mathcal{U}, \mathcal{F})$$

by the formula

$$d(s)_{i_0 \ldots i_{p+1}} = \sum\nolimits_{j=0}^{p+1} (-1)^j s_{i_0 \ldots \hat{i}_j \ldots i_p}|_{U_{i_0 \ldots i_{p+1}}}$$

for any $i_0 < \ldots < i_{p+1}$. Note that this formula is identical to Equation (20.10.0.1). It is straightforward to see that $d \circ d = 0$. In other words $\check{\mathcal{C}}^\bullet_{ord}(\mathcal{U}, \mathcal{F})$ is a complex.

Definition 20.22.2. Let X be a topological space. Let $\mathcal{U} : U = \bigcup_{i \in I} U_i$ be an open covering. Assume given a total ordering on I. Let $\mathcal{F}$ be an abelian presheaf on X. The complex $\check{\mathcal{C}}^\bullet_{ord}(\mathcal{U}, \mathcal{F})$ is the *ordered Čech complex* associated to $\mathcal{F}$, the open covering $\mathcal{U}$ and the given total ordering on I.

This complex is sometimes called the alternating Čech complex. The reason is that there is an obvious comparison map between the ordered Čech complex and the alternating Čech complex. Namely, consider the map

$$c : \check{\mathcal{C}}^\bullet_{ord}(\mathcal{U}, \mathcal{F}) \longrightarrow \check{\mathcal{C}}^\bullet(\mathcal{U}, \mathcal{F})$$

given by the rule

$$c(s)_{i_0 \ldots i_p} = \begin{cases} 0 & \text{if} \quad i_n = i_m \text{ for some } n \neq m \\ \operatorname{sgn}(\sigma) s_{i_{\sigma(0)} \ldots i_{\sigma(p)}} & \text{if} \quad i_{\sigma(0)} < i_{\sigma(1)} < \ldots < i_{\sigma(p)} \end{cases}$$

Here σ denotes a permutation of $\{0, \ldots, p\}$ and $\operatorname{sgn}(\sigma)$ denotes its sign. The alternating and ordered Čech complexes are often identified in the literature via the map c. Namely we have the following easy lemma.

Lemma 20.22.3. *Let X be a topological space. Let $\mathcal{U} : U = \bigcup_{i \in I} U_i$ be an open covering. Assume I comes equipped with a total ordering. The map c is a morphism of complexes. In fact it induces an isomorphism*

$$c : \check{\mathcal{C}}^\bullet_{ord}(\mathcal{U}, \mathcal{F}) \to \check{\mathcal{C}}^\bullet_{alt}(\mathcal{U}, \mathcal{F})$$

of complexes.

Proof. Omitted. □

There is also a map

$$\pi : \check{\mathcal{C}}^\bullet(\mathcal{U}, \mathcal{F}) \longrightarrow \check{\mathcal{C}}^\bullet_{ord}(\mathcal{U}, \mathcal{F})$$

which is described by the rule

$$\pi(s)_{i_0 \ldots i_p} = s_{i_0 \ldots i_p}$$

whenever $i_0 < i_1 < \ldots < i_p$.

Lemma 20.22.4. *Let X be a topological space. Let $\mathcal{U} : U = \bigcup_{i\in I} U_i$ be an open covering. Assume I comes equipped with a total ordering. The map $\pi : \check{\mathcal{C}}^\bullet(\mathcal{U}, \mathcal{F}) \to \check{\mathcal{C}}^\bullet_{ord}(\mathcal{U}, \mathcal{F})$ is a morphism of complexes. It induces an isomorphism*

$$\pi : \check{\mathcal{C}}^\bullet_{alt}(\mathcal{U}, \mathcal{F}) \to \check{\mathcal{C}}^\bullet_{ord}(\mathcal{U}, \mathcal{F})$$

of complexes which is a left inverse to the morphism c.

Proof. Omitted. □

Remark 20.22.5. This means that if we have two total orderings $<_1$ and $<_2$ on the index set I, then we get an isomorphism of complexes $\tau = \pi_2 \circ c_1 : \check{\mathcal{C}}_{ord\text{-}1}(\mathcal{U}, \mathcal{F}) \to \check{\mathcal{C}}_{ord\text{-}2}(\mathcal{U}, \mathcal{F})$. It is clear that

$$\tau(s)_{i_0 \ldots i_p} = \text{sign}(\sigma) s_{i_{\sigma(0)} \ldots i_{\sigma(p)}}$$

where $i_0 <_1 i_1 <_1 \ldots <_1 i_p$ and $i_{\sigma(0)} <_2 i_{\sigma(1)} <_2 \ldots <_2 i_{\sigma(p)}$. This is the sense in which the ordered Čech complex is independent of the chosen total ordering.

Lemma 20.22.6. *Let X be a topological space. Let $\mathcal{U} : U = \bigcup_{i\in I} U_i$ be an open covering. Assume I comes equipped with a total ordering. The map $c \circ \pi$ is homotopic to the identity on $\check{\mathcal{C}}^\bullet(\mathcal{U}, \mathcal{F})$. In particular the inclusion map $\check{\mathcal{C}}^\bullet_{alt}(\mathcal{U}, \mathcal{F}) \to \check{\mathcal{C}}^\bullet(\mathcal{U}, \mathcal{F})$ is a homotopy equivalence.*

Proof. For any multi-index $(i_0, \ldots, i_p) \in I^{p+1}$ there exists a unique permutation $\sigma : \{0, \ldots, p\} \to \{0, \ldots, p\}$ such that

$$i_{\sigma(0)} \leq i_{\sigma(1)} \leq \ldots \leq i_{\sigma(p)} \quad \text{and} \quad \sigma(j) < \sigma(j+1) \quad \text{if} \quad i_{\sigma(j)} = i_{\sigma(j+1)}.$$

We denote this permutation $\sigma = \sigma^{i_0 \ldots i_p}$.

For any permutation $\sigma : \{0, \ldots, p\} \to \{0, \ldots, p\}$ and any a, $0 \leq a \leq p$ we denote σ_a the permutation of $\{0, \ldots, p\}$ such that

$$\sigma_a(j) = \begin{cases} \sigma(j) & \text{if} \quad 0 \leq j < a, \\ \min\{j' \mid j' > \sigma_a(j-1), j' \neq \sigma(k), \forall k < a\} & \text{if} \quad a \leq j \end{cases}$$

So if $p = 3$ and σ, τ are given by

$$\begin{array}{cccccc} \text{id} & 0 & 1 & 2 & 3 \\ \sigma & 3 & 2 & 1 & 0 \end{array} \quad \text{and} \quad \begin{array}{cccccc} \text{id} & 0 & 1 & 2 & 3 \\ \tau & 3 & 0 & 2 & 1 \end{array}$$

then we have

$$\begin{array}{cccccc} \text{id} & 0 & 1 & 2 & 3 \\ \sigma_0 & 0 & 1 & 2 & 3 \\ \sigma_1 & 3 & 0 & 1 & 2 \\ \sigma_2 & 3 & 2 & 0 & 1 \\ \sigma_3 & 3 & 2 & 1 & 0 \end{array} \quad \text{and} \quad \begin{array}{cccccc} \text{id} & 0 & 1 & 2 & 3 \\ \tau_0 & 0 & 1 & 2 & 3 \\ \tau_1 & 3 & 0 & 1 & 2 \\ \tau_2 & 3 & 0 & 1 & 2 \\ \tau_3 & 3 & 0 & 2 & 1 \end{array}$$

It is clear that always $\sigma_0 = \text{id}$ and $\sigma_p = \sigma$.

Having introduced this notation we define for $s \in \check{\mathcal{C}}^{p+1}(\mathcal{U}, \mathcal{F})$ the element $h(s) \in \check{\mathcal{C}}^p(\mathcal{U}, \mathcal{F})$ to be the element with components

$$h(s)_{i_0 \ldots i_p} = \sum\nolimits_{0 \leq a \leq p} (-1)^a \text{sign}(\sigma_a) s_{i_{\sigma(0)} \ldots i_{\sigma(a)} i_{\sigma_a(a)} \ldots i_{\sigma_a(p)}} \tag{20.22.6.1}$$

where $\sigma = \sigma^{i_0 \ldots i_p}$. The index $i_{\sigma(a)}$ occurs twice in $i_{\sigma(0)} \ldots i_{\sigma(a)} i_{\sigma_a(a)} \ldots i_{\sigma_a(p)}$ once in the first group of $a+1$ indices and once in the second group of $p-a+1$ indices since $\sigma_a(j) = \sigma(a)$ for some $j \geq a$ by definition of σ_a. Hence the sum makes sense

since each of the elements $s_{i_{\sigma(0)}\ldots i_{\sigma(a)}i_{\sigma a(a)}\ldots i_{\sigma a(p)}}$ is defined over the open $U_{i_0\ldots i_p}$. Note also that for $a = 0$ we get $s_{i_0\ldots i_p}$ and for $a = p$ we get $(-1)^p\text{sign}(\sigma)s_{i_{\sigma(0)}\ldots i_{\sigma(p)}}$.

We claim that

$$(dh + hd)(s)_{i_0\ldots i_p} = s_{i_0\ldots i_p} - \text{sign}(\sigma)s_{i_{\sigma(0)}\ldots i_{\sigma(p)}}$$

where $\sigma = \sigma^{i_0\ldots i_p}$. We omit the verification of this claim. (There is a PARI/gp script called first-homotopy.gp in the stacks-project subdirectory scripts which can be used to check finitely many instances of this claim. We wrote this script to make sure the signs are correct.) Write

$$\kappa : \check{\mathcal{C}}^\bullet(\mathcal{U}, \mathcal{F}) \longrightarrow \check{\mathcal{C}}^\bullet(\mathcal{U}, \mathcal{F})$$

for the operator given by the rule

$$\kappa(s)_{i_0\ldots i_p} = \text{sign}(\sigma^{i_0\ldots i_p})s_{i_{\sigma(0)}\ldots i_{\sigma(p)}}.$$

The claim above implies that κ is a morphism of complexes and that κ is homotopic to the identity map of the Čech complex. This does not immediately imply the lemma since the image of the operator κ is not the alternating subcomplex. Namely, the image of κ is the "semi-alternating" complex $\check{\mathcal{C}}^p_{semi\text{-}alt}(\mathcal{U}, \mathcal{F})$ where s is a p-cochain of this complex if and only if

$$s_{i_0\ldots i_p} = \text{sign}(\sigma)s_{i_{\sigma(0)}\ldots i_{\sigma(p)}}$$

for any $(i_0, \ldots, i_p) \in I^{p+1}$ with $\sigma = \sigma^{i_0\ldots i_p}$. We introduce yet another variant Čech complex, namely the semi-ordered Čech complex defined by

$$\check{\mathcal{C}}^p_{semi\text{-}ord}(\mathcal{U}, \mathcal{F}) = \prod\nolimits_{i_0 \leq i_1 \leq \ldots \leq i_p} \mathcal{F}(U_{i_0\ldots i_p})$$

It is easy to see that Equation (20.10.0.1) also defines a differential and hence that we get a complex. It is also clear (analogous to Lemma 20.22.4) that the projection map

$$\check{\mathcal{C}}^\bullet_{semi\text{-}alt}(\mathcal{U}, \mathcal{F}) \longrightarrow \check{\mathcal{C}}^\bullet_{semi\text{-}ord}(\mathcal{U}, \mathcal{F})$$

is an isomorphism of complexes.

Hence the Lemma follows if we can show that the obvious inclusion map

$$\check{\mathcal{C}}^p_{ord}(\mathcal{U}, \mathcal{F}) \longrightarrow \check{\mathcal{C}}^p_{semi\text{-}ord}(\mathcal{U}, \mathcal{F})$$

is a homotopy equivalence. To see this we use the homotopy

(20.22.6.2)

$$h(s)_{i_0\ldots i_p} = \begin{cases} 0 & \text{if} \quad i_0 < i_1 < \ldots < i_p \\ (-1)^a s_{i_0\ldots i_{a-1}i_a i_a i_{a+1}\ldots i_p} & \text{if} \quad i_0 < i_1 < \ldots < i_{a-1} < i_a = i_{a+1} \end{cases}$$

We claim that

$$(dh + hd)(s)_{i_0\ldots i_p} = \begin{cases} 0 & \text{if} \quad i_0 < i_1 < \ldots < i_p \\ s_{i_0\ldots i_p} & \text{else} \end{cases}$$

We omit the verification. (There is a PARI/gp script called second-homotopy.gp in the stacks-project subdirectory scripts which can be used to check finitely many instances of this claim. We wrote this script to make sure the signs are correct.) The claim clearly shows that the composition

$$\check{\mathcal{C}}^\bullet_{semi\text{-}ord}(\mathcal{U}, \mathcal{F}) \longrightarrow \check{\mathcal{C}}^\bullet_{ord}(\mathcal{U}, \mathcal{F}) \longrightarrow \check{\mathcal{C}}^\bullet_{semi\text{-}ord}(\mathcal{U}, \mathcal{F})$$

of the projection with the natural inclusion is homotopic to the identity map as desired. □

20.23. Locally finite coverings and the Čech complex

In this section we discuss an alternative way to establish the relationship between the Čech complex and cohomology in case the covering is locally finite.

Definition 20.23.1. Let X be a topological space. An open covering $X = \bigcup_{i \in I} U_i$ is said to be *locally finite* if for every $x \in X$ there exists an open neighbourhood W of x such that $\{i \in I \mid W \cap U_i \neq \emptyset\}$ is finite.

Remark 20.23.2. Let $X = \bigcup_{i \in I} U_i$ be a locally finite open covering. Denote $j_i : U_i \to X$ the inclusion map. Suppose that for each i we are given an abelian sheaf $\mathcal{F}_i$ on U_i. Consider the abelian sheaf $\mathcal{G} = \bigoplus_{i \in I} (j_i)_* \mathcal{F}_i$. Then actually

$$\Gamma(X, \mathcal{G}) = \prod\nolimits_{i \in I} \mathcal{F}_i(U_i).$$

This seems strange until you realize that the direct sum of a collection of sheaves is the sheafification of what you think it should be. See discussion in Modules, Section 17.3.

Lemma 20.23.3. *Let X be a ringed space. Let $\mathcal{U} : X = \bigcup_{i \in I} U_i$ be a locally finite open covering of X. Let $\mathcal{F}$ be an $\mathcal{O}_X$-module. Denote $\mathcal{F}_{i_0 \ldots i_p}$ the restriction of $\mathcal{F}$ to $U_{i_0 \ldots i_p}$. There exists a complex $\mathfrak{C}^\bullet(\mathcal{U}, \mathcal{F})$ of $\mathcal{O}_X$-modules with*

$$\mathfrak{C}^p(\mathcal{U}, \mathcal{F}) = \bigoplus\nolimits_{i_0 \ldots i_p} (j_{i_0 \ldots i_p})_* \mathcal{F}_{i_0 \ldots i_p}$$

and differential $d : \mathfrak{C}^p(\mathcal{U}, \mathcal{F}) \to \mathfrak{C}^{p+1}(\mathcal{U}, \mathcal{F})$ as in Equation (20.10.0.1). Moreover, there exists a canonical map

$$\mathcal{F} \to \mathfrak{C}^\bullet(\mathcal{U}, \mathcal{F})$$

which is a quasi-isomorphism, i.e., $\mathfrak{C}^\bullet(\mathcal{U}, \mathcal{F})$ is a resolution of $\mathcal{F}$.

Proof. Omitted. □

With this lemma it is easy to reprove the Čech to cohomology spectral sequence of Lemma 20.12.4 in the special case of locally finite coverings. Namely, let X, $\mathcal{U}$, $\mathcal{F}$ as in Lemma 20.23.3 and let $\mathcal{F} \to \mathcal{I}^\bullet$ be an injective resolution. Then we may consider the double complex

$$A^{\bullet,\bullet} = \Gamma(X, \mathfrak{C}^\bullet(\mathcal{U}, \mathcal{I}^\bullet)).$$

Note that

$$A^{p,q} = \prod\nolimits_{i_0 \ldots i_p} \mathcal{I}^q(U_{i_0 \ldots i_p})$$

see Remark 20.23.2. Consider the two spectral sequences of Homology, Section 12.22 associated to this double complex. See especially Homology, Lemma 12.22.3. For the spectral sequence $({}'E_r, {}'d_r)_{r \geq 0}$ we get ${}'E_2^{p,q} = \check{H}^p(\mathcal{U}, \underline{H}^q(\mathcal{F}))$ and for the spectral sequence $({}''E_r, {}''d_r)_{r \geq 0}$ we get ${}''E_2^{p,q} = 0$ if $p > 0$ and ${}''E_2^{0,q} = H^q(X, \mathcal{F})$. Whence the result since both spectral sequences converge to the cohomology of the simple complex $sA^\bullet$.

20.24. Čech cohomology of complexes

In general for sheaves of abelian groups $\mathcal{F}$ and $\mathcal{G}$ on X there is a cupproduct map

$$H^i(X, \mathcal{F}) \times H^j(X, \mathcal{G}) \longrightarrow H^{i+j}(X, \mathcal{F} \otimes_{\mathbf{Z}} \mathcal{G}).$$

In this section we define it using Čech cocycles by an explicit formula for the cup product. If you are worried about the fact that cohomology may not equal Čech cohomology, then you can use hypercoverings and still use the cocycle notation. This also has the advantage that it works to define the cup product for hypercohomology on any topos (insert future reference here).

Let $\mathcal{F}^\bullet$ be a bounded below complex of presheaves of abelian groups on X. We can often compute $H^n(X, \mathcal{F}^\bullet)$ using Čech cocycles. Namely, let $\mathcal{U} : X = \bigcup_{i \in I} U_i$ be an open covering of X. Since the Čech complex $\check{\mathcal{C}}^\bullet(\mathcal{U}, \mathcal{F})$ (Definition 20.10.1) is functorial in the presheaf $\mathcal{F}$ we obtain a double complex $\check{\mathcal{C}}^\bullet(\mathcal{U}, \mathcal{F}^\bullet)$. The associated total complex to $\check{\mathcal{C}}^\bullet(\mathcal{U}, \mathcal{F}^\bullet)$ is the complex with degree n term

$$\mathrm{Tot}^n(\check{\mathcal{C}}^\bullet(\mathcal{U}, \mathcal{F}^\bullet)) = \bigoplus\nolimits_{p+q=n} \prod\nolimits_{i_0 \ldots i_p} \mathcal{F}^q(U_{i_0 \ldots i_p})$$

see Homology, Definition 12.22.2. A typical element in Tot^n will be denoted $\alpha = \{\alpha_{i_0 \ldots i_p}\}$ where $\alpha_{i_0 \ldots i_p} \in \mathcal{F}^q(U_{i_0 \ldots i_p})$. In other words the $\mathcal{F}$-degree of $\alpha_{i_0 \ldots i_p}$ is $q = n - p$. This notation requires us to be aware of the degree α lives in at all times. We indicate this situation by the formula $\deg_{\mathcal{F}}(\alpha_{i_0 \ldots i_p}) = q$. According to our conventions in Homology, Definition 12.22.2 the differential of an element α of degree n is given by

$$d(\alpha)_{i_0 \ldots i_{p+1}} = \sum\nolimits_{j=0}^{p+1} (-1)^j \alpha_{i_0 \ldots \hat{i}_j \ldots i_{p+1}} + (-1)^{p+1} d_{\mathcal{F}}(\alpha_{i_0 \ldots i_{p+1}})$$

where $d_{\mathcal{F}}$ denotes the differential on the complex $\mathcal{F}^\bullet$. The expression $\alpha_{i_0 \ldots \hat{i}_j \ldots i_{p+1}}$ means the restriction of $\alpha_{i_0 \ldots \hat{i}_j \ldots i_{p+1}} \in \mathcal{F}(U_{i_0 \ldots \hat{i}_j \ldots i_{p+1}})$ to $U_{i_0 \ldots i_{p+1}}$.

The construction of $\mathrm{Tot}(\check{\mathcal{C}}^\bullet(\mathcal{U}, \mathcal{F}^\bullet))$ is functorial in $\mathcal{F}^\bullet$. As well there is a functorial transformation

$$\Gamma(X, \mathcal{F}^\bullet) \longrightarrow \mathrm{Tot}(\check{\mathcal{C}}^\bullet(\mathcal{U}, \mathcal{F}^\bullet)) \tag{20.24.0.1}$$

of complexes defined by the following rule: The section $s \in \Gamma(X, \mathcal{F}^n)$ is mapped to the element $\alpha = \{\alpha_{i_0 \ldots i_p}\}$ with $\alpha_{i_0} = s|_{U_{i_0}}$ and $\alpha_{i_0 \ldots i_p} = 0$ for $p > 0$.

Refinements. Let $\mathcal{V} = \{V_j\}_{j \in J}$ be a refinement of $\mathcal{U}$. This means there is a map $t : J \to I$ such that $V_j \subset U_{t(j)}$ for all $j \in J$. This gives rise to a functorial transformation

$$T_t : \mathrm{Tot}(\check{\mathcal{C}}^\bullet(\mathcal{U}, \mathcal{F}^\bullet)) \longrightarrow \mathrm{Tot}(\check{\mathcal{C}}^\bullet(\mathcal{V}, \mathcal{F}^\bullet)). \tag{20.24.0.2}$$

defined by the rule

$$T_t(\alpha)_{j_0 \ldots j_p} = \alpha_{t(j_0) \ldots t(j_p)}|_{V_{j_0 \ldots j_p}}.$$

Given two maps $t, t' : J \to I$ as above the maps T_t and $T_{t'}$ constructed above are homotopic. The homotopy is given by

$$h(\alpha)_{j_0 \ldots j_p} = \sum\nolimits_{a=0}^{p} (-1)^a \alpha_{t(j_0) \ldots t(j_a) t'(j_a) \ldots t'(j_p)}$$

for an element α of degree n. This works because of the following computation, again with α an element of degree n (so $d(\alpha)$ has degree $n+1$ and $h(\alpha)$ has degree $n-1$):

$$\begin{aligned}
(d(h(\alpha)) + h(d(\alpha)))_{j_0 \ldots j_p} = & \sum_{k=0}^{p} (-1)^k h(\alpha)_{j_0 \ldots \hat{j}_k \ldots j_p} + \\
& (-1)^p d_{\mathcal{F}}(h(\alpha)_{j_0 \ldots j_p}) + \\
& \sum_{a=0}^{p} (-1)^a d(\alpha)_{t(j_0)\ldots t(j_a)t'(j_a)\ldots t'(j_p)} \\
= & \sum_{k=0}^{p} \sum_{a=0}^{k-1} (-1)^{k+a} \alpha_{t(j_0)\ldots t(j_a)t'(j_a)\ldots t'(\hat{j}_k)\ldots t'(j_p)} + \\
& \sum_{k=0}^{p} \sum_{a=k+1}^{p} (-1)^{k+a-1} \alpha_{t(j_0)\ldots t(\hat{j}_k)\ldots t(j_a)t'(j_a)\ldots t'(j_p)} + \\
& \sum_{a=0}^{p} (-1)^{p+a} d_{\mathcal{F}}(\alpha_{t(j_0)\ldots t(j_a)t'(j_a)\ldots t'(j_p)}) + \\
& \sum_{a=0}^{p} \sum_{k=0}^{a} (-1)^{a+k} \alpha_{t(j_0)\ldots t(\hat{j}_k)\ldots t(j_a)t'(j_a)\ldots t'(j_p)} + \\
& \sum_{a=0}^{p} \sum_{k=a}^{p} (-1)^{a+k+1} \alpha_{t(j_0)\ldots t(j_a)t'(j_a)\ldots t'(\hat{j}_k)\ldots t'(j_p)} + \\
& \sum_{a=0}^{p} (-1)^{a+p+1} d_{\mathcal{F}}(\alpha_{t(j_0)\ldots t(j_a)t'(j_a)\ldots t'(j_p)}) \\
& = \alpha_{t'(j_0)\ldots t'(j_p)} + (-1)^{2p+1} \alpha_{t(j_0)\ldots t(j_p)} \\
& = T_{t'}(\alpha)_{j_0 \ldots j_p} - T_t(\alpha)_{j_0 \ldots j_p}
\end{aligned}$$

We leave it to the reader to verify the cancellations. (Note that the terms having both k and a in the 1st, 2nd and 4th, 5th summands cancel, except the ones where $a = k$ which only occur in the 4th and 5th and these cancel against each other except for the two desired terms.) It follows that the induced map

$$H^n(T_t) : H^n(\mathrm{Tot}(\check{\mathcal{C}}^\bullet(\mathcal{U}, \mathcal{F}^\bullet))) \to H^n(\mathrm{Tot}(\check{\mathcal{C}}^\bullet(\mathcal{V}, \mathcal{F}^\bullet)))$$

is independent of the choice of t. We define *Čech hypercohomology* as the limit of the Čech cohomology groups over all refinements via the maps $H^\bullet(T_t)$.

In the limit (over all open coverings of X) the following lemma provides a map of Čech hypercohomology into cohomology, which is often an isomorphism and is always an isomorphism if we use hypercoverings.

Lemma 20.24.1. *Let $(X, \mathcal{O}_X)$ be a ringed space. Let $\mathcal{U} : X = \bigcup_{i \in I} U_i$ be an open covering. For a bounded below complex $\mathcal{F}^\bullet$ of $\mathcal{O}_X$-modules there is a canonical map*

$$Tot(\check{\mathcal{C}}^\bullet(\mathcal{U}, \mathcal{F}^\bullet)) \longrightarrow R\Gamma(X, \mathcal{F}^\bullet)$$

functorial in $\mathcal{F}^\bullet$ and compatible with (20.24.0.1) and (20.24.0.2). There is a spectral sequence $(E_r, d_r)_{r \geq 0}$ with

$$E_2^{p,q} = H^p(\, Tot(\check{\mathcal{C}}^\bullet(\mathcal{U}, \underline{H}^q(\mathcal{F}^\bullet))$$

converging to $H^{p+q}(X, \mathcal{F}^\bullet)$.

Proof. Let $\mathcal{I}^\bullet$ be a bounded below complex of injectives. The map (20.24.0.1) for $\mathcal{I}^\bullet$ is a map $\Gamma(X, \mathcal{I}^\bullet) \to \mathrm{Tot}(\check{\mathcal{C}}^\bullet(\mathcal{U}, \mathcal{I}^\bullet))$. This is a quasi-isomorphism of complexes of abelian groups as follows from Homology, Lemma 12.22.6 applied to the double complex $\check{\mathcal{C}}^\bullet(\mathcal{U}, \mathcal{I}^\bullet)$ using Lemma 20.12.1. Suppose $\mathcal{F}^\bullet \to \mathcal{I}^\bullet$ is a quasi-isomorphism

of $\mathcal{F}^\bullet$ into a bounded below complex of injectives. Since $R\Gamma(X, \mathcal{F}^\bullet)$ is represented by the complex $\Gamma(X, \mathcal{I}^\bullet)$ we obtain the map of the lemma using

$$\mathrm{Tot}(\check{\mathcal{C}}^\bullet(\mathcal{U}, \mathcal{F}^\bullet)) \longrightarrow \mathrm{Tot}(\check{\mathcal{C}}^\bullet(\mathcal{U}, \mathcal{I}^\bullet)).$$

We omit the verification of functoriality and compatibilities. To construct the spectral sequence of the lemma, choose a Cartan-Eilenberg resolution $\mathcal{F}^\bullet \to \mathcal{I}^{\bullet,\bullet}$, see Derived Categories, Lemma 13.20.2. In this case $\mathcal{F}^\bullet \to \mathrm{Tot}(\mathcal{I}^{\bullet,\bullet})$ is an injective resolution and hence

$$\mathrm{Tot}(\check{\mathcal{C}}^\bullet(\mathcal{U}, \mathrm{Tot}(\mathcal{I}^{\bullet,\bullet})))$$

computes $R\Gamma(X, \mathcal{F}^\bullet)$ as we've seen above. By Homology, Remark 12.22.8 we can view this as the total complex associated to the triple complex $\check{\mathcal{C}}^\bullet(\mathcal{U}, \mathcal{I}^{\bullet,\bullet})$ hence, using the same remark we can view it as the total complex associate to the double complex $A^{\bullet,\bullet}$ with terms

$$A^{n,m} = \bigoplus\nolimits_{p+q=n} \check{\mathcal{C}}^p(\mathcal{U}, \mathcal{I}^{q,m})$$

Since $\mathcal{I}^{q,\bullet}$ is an injective resolution of $\mathcal{F}^q$ we can apply the first spectral sequence associated to $A^{\bullet,\bullet}$ (Homology, Lemma 12.22.3) to get a spectral sequence with

$$E_1^{n,m} = \bigoplus\nolimits_{p+q=n} \check{\mathcal{C}}^p(\mathcal{U}, \underline{H}^m(\mathcal{F}^q))$$

which is the nth term of the complex $\mathrm{Tot}(\check{\mathcal{C}}^\bullet(\mathcal{U}, \underline{H}^m(\mathcal{F}^\bullet))$. Hence we obtain E_2 terms as described in the lemma. Convergence by Homology, Lemma 12.22.5. □

Let X be a topological space, let $\mathcal{U} : X = \bigcup_{i\in I} U_i$ be an open covering, and let $\mathcal{F}^\bullet$ be a bounded below complex of presheaves of abelian groups. Consider the map $\tau : \mathrm{Tot}(\check{\mathcal{C}}^\bullet(\mathcal{U}, \mathcal{F}^\bullet)) \to \mathrm{Tot}(\check{\mathcal{C}}^\bullet(\mathcal{U}, \mathcal{F}^\bullet))$ defined by

$$\tau(\alpha)_{i_0 \ldots i_p} = (-1)^{p(p+1)/2} \alpha_{i_p \ldots i_0}.$$

Then we have for an element α of degree n that

$$\begin{aligned}
& d(\tau(\alpha))_{i_0 \ldots i_{p+1}} \\
& = \sum\nolimits_{j=0}^{p+1} (-1)^j \tau(\alpha)_{i_0 \ldots \hat{i}_j \ldots i_{p+1}} + (-1)^{p+1} d_{\mathcal{F}}(\tau(\alpha)_{i_0 \ldots i_{p+1}}) \\
& = \sum\nolimits_{j=0}^{p+1} (-1)^{j + \frac{p(p+1)}{2}} \alpha_{i_{p+1} \ldots \hat{i}_j \ldots i_0} + (-1)^{p+1+\frac{(p+1)(p+2)}{2}} d_{\mathcal{F}}(\alpha_{i_{p+1} \ldots i_0})
\end{aligned}$$

On the other hand we have

$$\begin{aligned}
& \tau(d(\alpha))_{i_0 \ldots i_{p+1}} \\
& = (-1)^{\frac{(p+1)(p+2)}{2}} d(\alpha)_{i_{p+1} \ldots i_0} \\
& = (-1)^{\frac{(p+1)(p+2)}{2}} \left(\sum\nolimits_{j=0}^{p+1} (-1)^j \alpha_{i_{p+1} \ldots \hat{i}_{p+1-j} \ldots i_0} + (-1)^{p+1} d_{\mathcal{F}}(\alpha_{i_{p+1} \ldots i_0}) \right)
\end{aligned}$$

Thus we conclude that $d(\tau(\alpha)) = \tau(d(\alpha))$ because $p(p+1)/2 \equiv (p+1)(p+2)/2 + p + 1 \bmod 2$. In other words τ is an endomorphism of the complex $\mathrm{Tot}(\check{\mathcal{C}}^\bullet(\mathcal{U}, \mathcal{F}^\bullet))$. Note that the diagram

$$\begin{array}{ccc}
\Gamma(X, \mathcal{F}^\bullet) & \longrightarrow & \mathrm{Tot}(\check{\mathcal{C}}^\bullet(\mathcal{U}, \mathcal{F}^\bullet)) \\
\downarrow \mathrm{id} & & \downarrow \tau \\
\Gamma(X, \mathcal{F}^\bullet) & \longrightarrow & \mathrm{Tot}(\check{\mathcal{C}}^\bullet(\mathcal{U}, \mathcal{F}^\bullet))
\end{array}$$

commutes. In addition τ is clearly compatible with refinements. This suggests that τ acts as the identity on Čech cohomology (i.e., in the limit – provided Čech

hypercohomology agrees with hypercohomology, which is always the case if we use hypercoverings). We claim that τ actually is homotopic to the identity on the total Čech complex $\mathrm{Tot}(\check{\mathcal{C}}^\bullet(\mathcal{U}, \mathcal{F}^\bullet))$. To prove this, we use as homotopy

$$h(\alpha)_{i_0 \ldots i_p} = \sum\nolimits_{a=0}^{p} \epsilon_p(a) \alpha_{i_0 \ldots i_a i_p \ldots i_a} \quad \text{with} \quad \epsilon_p(a) = (-1)^{\frac{(p-a)(p-a-1)}{2} + p}$$

for α of degree n. As usual we omit writing $|_{U_{i_0 \ldots i_p}}$. This works because of the following computation, again with α an element of degree n:

$$\begin{aligned}
(d(h(\alpha)) + h(d(\alpha)))_{i_0 \ldots i_p} = & \sum\nolimits_{k=0}^{p} (-1)^k h(\alpha)_{i_0 \ldots \hat{i_k} \ldots i_p} + \\
& (-1)^p d_{\mathcal{F}}(h(\alpha)_{i_0 \ldots i_p}) + \\
& \sum\nolimits_{a=0}^{p} \epsilon_p(a) d(\alpha)_{i_0 \ldots i_a i_p \ldots i_a} \\
= & \sum\nolimits_{k=0}^{p} \sum\nolimits_{a=0}^{k-1} (-1)^k \epsilon_{p-1}(a) \alpha_{i_0 \ldots i_a i_p \ldots \hat{i_k} \ldots i_a} + \\
& \sum\nolimits_{k=0}^{p} \sum\nolimits_{a=k+1}^{p} (-1)^k \epsilon_{p-1}(a-1) \alpha_{i_0 \ldots \hat{i_k} \ldots i_a i_p \ldots i_a} + \\
& \sum\nolimits_{a=0}^{p} (-1)^p \epsilon_p(a) d_{\mathcal{F}}(\alpha_{i_0 \ldots i_a i_p \ldots i_a}) + \\
& \sum\nolimits_{a=0}^{p} \sum\nolimits_{k=0}^{a} \epsilon_p(a) (-1)^k \alpha_{i_0 \ldots \hat{i_k} \ldots i_a i_p \ldots i_a} + \\
& \sum\nolimits_{a=0}^{p} \sum\nolimits_{k=a}^{p} \epsilon_p(a) (-1)^{p+a+1-k} \alpha_{i_0 \ldots i_a i_p \ldots \hat{i_k} \ldots i_a} + \\
& \sum\nolimits_{a=0}^{p} \epsilon_p(a) (-1)^{p+1} d_{\mathcal{F}}(\alpha_{i_0 \ldots i_a i_p \ldots i_a}) \\
= & \epsilon_p(0) \alpha_{i_p \ldots i_0} + \epsilon_p(p) (-1)^{p+1} \alpha_{i_0 \ldots i_p} \\
= & (-1)^{\frac{p(p+1)}{2}} \alpha_{i_p \ldots i_0} - \alpha_{i_0 \ldots i_p}
\end{aligned}$$

The cancellations follow because

$$(-1)^k \epsilon_{p-1}(a) + \epsilon_p(a)(-1)^{p+a+1-k} = 0 \quad \text{and} \quad (-1)^k \epsilon_{p-1}(a-1) + \epsilon_p(a)(-1)^k = 0$$

We leave it to the reader to verify the cancellations.

Suppose we have two bounded below complexes complexes of abelian sheaves $\mathcal{F}^\bullet$ and $\mathcal{G}^\bullet$. We define the complex $\mathrm{Tot}(\mathcal{F}^\bullet \otimes_{\mathbf{Z}} \mathcal{G}^\bullet)$ to be to complex with terms $\bigoplus_{p+q=n} \mathcal{F}^p \otimes \mathcal{G}^q$ and differential according to the rule

$$d(\alpha \otimes \beta) = d(\alpha) \otimes \beta + (-1)^{\deg(\alpha)} \alpha \otimes d(\beta) \tag{20.24.1.1}$$

when α and β are homogeneous, see Homology, Definition 12.22.2.

Suppose that $M^\bullet$ and $N^\bullet$ are two bounded below complexes of abelian groups. Then if m, resp. n is a cocycle for $M^\bullet$, resp. $N^\bullet$, it is immediate that $m \otimes n$ is a cocycle for $\mathrm{Tot}(M^\bullet \otimes N^\bullet)$. Hence a cupproduct

$$H^i(M^\bullet) \times H^j(N^\bullet) \longrightarrow H^{i+j}(Tot(M^\bullet \otimes N^\bullet)).$$

This is discussed also in More on Algebra, Section 15.47.

So the construction of the cup product in hypercohomology of complexes rests on a construction of a map of complexes

$$\mathrm{Tot}\left(\mathrm{Tot}(\check{\mathcal{C}}^\bullet(\mathcal{U}, \mathcal{F}^\bullet)) \otimes_{\mathbf{Z}} \mathrm{Tot}(\check{\mathcal{C}}^\bullet(\mathcal{U}, \mathcal{G}^\bullet))\right) \longrightarrow \mathrm{Tot}(\check{\mathcal{C}}^\bullet(\mathcal{U}, \mathrm{Tot}(\mathcal{F}^\bullet \otimes \mathcal{G}^\bullet))) \tag{20.24.1.2}$$

This map is denoted $\cup$ and is given by the rule

$$(\alpha \cup \beta)_{i_0 \ldots i_p} = \sum\nolimits_{r=0}^{p} \epsilon(n, m, p, r) \alpha_{i_0 \ldots i_r} \otimes \beta_{i_r \ldots i_p}.$$

where α has degree n and β has degree m and with

$$\epsilon(n, m, p, r) = (-1)^{(p+r)n + rp + r}.$$

Note that $\epsilon(n, m, p, n) = 1$. Hence if $\mathcal{F}^\bullet = \mathcal{F}[0]$ is the complex consisting in a single abelian sheaf $\mathcal{F}$ placed in degree 0, then there no signs in the formula for $\cup$ (as in that case $\alpha_{i_0 \ldots i_r} = 0$ unless $r = n$). For an explanation of why there has to be a sign and how to compute it see [**AGV71**, Exposee XVII] by Deligne. To check (20.24.1.2) is a map of complexes we have to show that

$$d(\alpha \cup \beta) = d(\alpha) \cup \beta + (-1)^{\deg(\alpha)} \alpha \cup d(\beta)$$

by the definition of the differential on $\mathrm{Tot}(\mathrm{Tot}(\check{\mathcal{C}}^\bullet(\mathcal{U}, \mathcal{F}^\bullet)) \otimes_{\mathbf{Z}} \mathrm{Tot}(\check{\mathcal{C}}^\bullet(\mathcal{U}, \mathcal{G}^\bullet)))$ as given in Homology, Definition 12.22.2. We compute first

$$\begin{aligned}
d(\alpha \cup \beta)_{i_0 \ldots i_{p+1}} &= \sum\nolimits_{j=0}^{p+1} (-1)^j (\alpha \cup \beta)_{i_0 \ldots \hat{i}_j \ldots i_{p+1}} + (-1)^{p+1} d_{\mathcal{F} \otimes \mathcal{G}}((\alpha \cup \beta)_{i_0 \ldots i_{p+1}}) \\
&= \sum\nolimits_{j=0}^{p+1} \sum\nolimits_{r=0}^{j-1} (-1)^j \epsilon(n, m, p, r) \alpha_{i_0 \ldots i_r} \otimes \beta_{i_r \ldots \hat{i}_j \ldots i_{p+1}} + \\
&\quad \sum\nolimits_{j=0}^{p+1} \sum\nolimits_{r=j+1}^{p+1} (-1)^j \epsilon(n, m, p, r-1) \alpha_{i_0 \ldots \hat{i}_j \ldots i_r} \otimes \beta_{i_r \ldots i_{p+1}} + \\
&\quad \sum\nolimits_{r=0}^{p+1} (-1)^{p+1} \epsilon(n, m, p+1, r) d_{\mathcal{F} \otimes \mathcal{G}}(\alpha_{i_0 \ldots i_r} \otimes \beta_{i_r \ldots i_{p+1}})
\end{aligned}$$

and note that the summands in the last term equal

$$(-1)^{p+1} \epsilon(n, m, p+1, r) \left(d_{\mathcal{F}}(\alpha_{i_0 \ldots i_r}) \otimes \beta_{i_r \ldots i_{p+1}} + (-1)^{n-r} \alpha_{i_0 \ldots i_r} \otimes d_{\mathcal{G}}(\beta_{i_r \ldots i_{p+1}}) \right).$$

because $\deg_{\mathcal{F}}(\alpha_{i_0 \ldots i_r}) = n - r$. On the other hand

$$\begin{aligned}
(d(\alpha) \cup \beta)_{i_0 \ldots i_{p+1}} &= \sum\nolimits_{r=0}^{p+1} \epsilon(n+1, m, p+1, r) d(\alpha)_{i_0 \ldots i_r} \otimes \beta_{i_r \ldots i_{p+1}} \\
&= \sum\nolimits_{r=0}^{p+1} \sum\nolimits_{j=0}^{r} \epsilon(n+1, m, p+1, r) (-1)^j \alpha_{i_0 \ldots \hat{i}_j \ldots i_r} \otimes \beta_{i_r \ldots i_{p+1}} + \\
&\quad \sum\nolimits_{r=0}^{p+1} \epsilon(n+1, m, p+1, r) (-1)^r d_{\mathcal{F}}(\alpha_{i_0 \ldots i_r}) \otimes \beta_{i_r \ldots i_{p+1}}
\end{aligned}$$

and

$$\begin{aligned}
(\alpha \cup d(\beta))_{i_0 \ldots i_{p+1}} &= \sum\nolimits_{r=0}^{p+1} \epsilon(n, m+1, p+1, r) \alpha_{i_0 \ldots i_r} \otimes d(\beta)_{i_r \ldots i_{p+1}} \\
&= \sum\nolimits_{r=0}^{p+1} \sum\nolimits_{j=r}^{p+1} \epsilon(n, m+1, p+1, r) (-1)^{j-r} \alpha_{i_0 \ldots i_r} \otimes \beta_{i_r \ldots \hat{i}_j \ldots i_{p+1}} + \\
&\quad \sum\nolimits_{r=0}^{p+1} \epsilon(n, m+1, p+1, r) (-1)^{p+1-r} \alpha_{i_0 \ldots i_r} \otimes d_{\mathcal{G}}(\beta_{i_r \ldots i_{p+1}})
\end{aligned}$$

The desired equality holds if we have

$$\begin{aligned}
(-1)^{p+1} \epsilon(n, m, p+1, r) &= \epsilon(n+1, m, p+1, r) (-1)^r \\
(-1)^{p+1} \epsilon(n, m, p+1, r) (-1)^{n-r} &= (-1)^n \epsilon(n, m+1, p+1, r) (-1)^{p+1-r} \\
\epsilon(n+1, m, p+1, r) (-1)^r &= (-1)^{1+n} \epsilon(n, m+1, p+1, r-1) \\
(-1)^j \epsilon(n, m, p, r) &= (-1)^n \epsilon(n, m+1, p+1, r) (-1)^{j-r} \\
(-1)^j \epsilon(n, m, p, r-1) &= \epsilon(n+1, m, p+1, r) (-1)^j
\end{aligned}$$

(The third equality is necessary to get the terms with $r = j$ from $d(\alpha) \cup \beta$ and $(-1)^n \alpha \cup d(\beta)$ to cancel each other.) We leave the verifications to the reader. (Alternatively, check the script signs.gp in the scripts subdirectory of the stacks project.)

Associativity of the cupproduct. Suppose that $\mathcal{F}^\bullet$, $\mathcal{G}^\bullet$ and $\mathcal{H}^\bullet$ are bounded below complexes of abelian groups on X. The obvious map (without the intervention of signs) is an isomorphism of complexes

$$\mathrm{Tot}(\mathrm{Tot}(\mathcal{F}^\bullet \otimes_{\mathbf{Z}} \mathcal{G}^\bullet) \otimes_{\mathbf{Z}} \mathcal{H}^\bullet) \longrightarrow \mathrm{Tot}(\mathcal{F}^\bullet \otimes_{\mathbf{Z}} \mathrm{Tot}(\mathcal{G}^\bullet \otimes_{\mathbf{Z}} \mathcal{H}^\bullet)).$$

Another way to say this is that the triple complex $\mathcal{F}^\bullet \otimes_{\mathbf{Z}} \mathcal{G}^\bullet \otimes_{\mathbf{Z}} \mathcal{H}^\bullet$ gives rise to a well defined total complex with differential satisfying

$$d(\alpha \otimes \beta \otimes \gamma) = d(\alpha) \otimes \beta \otimes \gamma + (-1)^{\deg(\alpha)} \alpha \otimes d(\beta) \otimes \gamma + (-1)^{\deg(\alpha)+\deg(\beta)} \alpha \otimes \beta \otimes d(\gamma)$$

for homogeneous elements. Using this map it is easy to verify that

$$(\alpha \cup \beta) \cup \gamma = \alpha \cup (\beta \cup \gamma)$$

namely, if α has degree a, β has degree b and γ has degree c, then

$$\begin{aligned}((\alpha \cup \beta) \cup \gamma)_{i_0 \ldots i_p} &= \sum\nolimits_{r=0}^{p} \epsilon(a+b, c, p, r)(\alpha \cup \beta)_{i_0 \ldots i_r} \otimes \gamma_{i_r \ldots i_p} \\ &= \sum\nolimits_{r=0}^{p} \sum\nolimits_{s=0}^{r} \epsilon(a+b, c, p, r)\epsilon(a, b, r, s)\alpha_{i_0 \ldots i_s} \otimes \beta_{i_s \ldots i_r} \otimes \gamma_{i_r \ldots i_p}\end{aligned}$$

and

$$\begin{aligned}(\alpha \cup (\beta \cup \gamma)_{i_0 \ldots i_p} &= \sum\nolimits_{s=0}^{p} \epsilon(a, b+c, p, s)\alpha_{i_0 \ldots i_s} \otimes (\beta \cup \gamma)_{i_s \ldots i_p} \\ &= \sum\nolimits_{s=0}^{p} \sum\nolimits_{r=s}^{p} \epsilon(a, b+c, p, s)\epsilon(b, c, p-s, r-s)\alpha_{i_0 \ldots i_s} \otimes \beta_{i_s \ldots i_r} \otimes \gamma_{i_r \ldots i_p}\end{aligned}$$

and a trivial mod 2 calculation shows the signs match up. (Alternatively, check the script signs.gp in the scripts subdirectory of the stacks project.)

Finally, we indicate why the cup product preserves a graded commutative structure, at least on a cohomological level. For this we use the operator τ introduced above. Let $\mathcal{F}^\bullet$ be a bounded below complexes of abelian groups, and assume we are given a graded commutative multiplication

$$\wedge^\bullet : \mathrm{Tot}(\mathcal{F}^\bullet \otimes \mathcal{F}^\bullet) \longrightarrow \mathcal{F}^\bullet.$$

This means the following: For s a local section of $\mathcal{F}^a$, and t a local section of $\mathcal{F}^b$ we have $s \wedge t$ a local section of $\mathcal{F}^{a+b}$. Graded commutative means we have $s \wedge t = (-1)^{ab} t \wedge s$. Since $\wedge$ is a map of complexes we have $d(s \wedge t) = d(s) \wedge t + (-1)^a s \wedge t$. The composition

$$\mathrm{Tot}(\mathrm{Tot}(\check{\mathcal{C}}^\bullet(\mathcal{U}, \mathcal{F}^\bullet)) \otimes \mathrm{Tot}(\check{\mathcal{C}}^\bullet(\mathcal{U}, \mathcal{F}^\bullet))) \to \mathrm{Tot}(\check{\mathcal{C}}^\bullet(\mathcal{U}, \mathrm{Tot}(\mathcal{F}^\bullet \otimes_{\mathbf{Z}} \mathcal{F}^\bullet))) \to \mathrm{Tot}(\check{\mathcal{C}}^\bullet(\mathcal{U}, \mathcal{F}^\bullet))$$

induces a cup product on cohomology

$$H^n(\mathrm{Tot}(\check{\mathcal{C}}^\bullet(\mathcal{U}, \mathcal{F}^\bullet))) \times H^m(\mathrm{Tot}(\check{\mathcal{C}}^\bullet(\mathcal{U}, \mathcal{F}^\bullet))) \longrightarrow H^{n+m}(\mathrm{Tot}(\check{\mathcal{C}}^\bullet(\mathcal{U}, \mathcal{F}^\bullet)))$$

and so in the limit also a product on Čech cohomology and therefore (using hypercoverings if needed) a product in cohomology of $\mathcal{F}^\bullet$. We claim this product (on cohomology) is graded commutative as well. To prove this we first consider

an element α of degree n in $\mathrm{Tot}(\check{\mathcal{C}}^\bullet(\mathcal{U}, \mathcal{F}^\bullet))$ and an element β of degree m in $\mathrm{Tot}(\check{\mathcal{C}}^\bullet(\mathcal{U}, \mathcal{F}^\bullet))$ and we compute

$$\begin{aligned}\wedge^\bullet(\alpha \cup \beta)_{i_0 \ldots i_p} &= \sum\nolimits_{r=0}^{p} \epsilon(n, m, p, r) \alpha_{i_0 \ldots i_r} \wedge \beta_{i_r \ldots i_p} \\ &= \sum\nolimits_{r=0}^{p} \epsilon(n, m, p, r) (-1)^{\deg(\alpha_{i_0 \ldots i_r}) \deg(\beta_{i_r \ldots i_p})} \beta_{i_r \ldots i_p} \wedge \alpha_{i_0 \ldots i_r}\end{aligned}$$

because $\wedge$ is graded commutative. On the other hand we have

$$\begin{aligned}\tau(\wedge^\bullet(\tau(\beta) \cup \tau(\alpha)))_{i_0 \ldots i_p} =& \chi(p) \sum\nolimits_{r=0}^{p} \epsilon(m, n, p, r) \tau(\beta)_{i_p \ldots i_{p-r}} \wedge \tau(\alpha)_{i_{p-r} \ldots i_0} \\ =& \chi(p) \sum\nolimits_{r=0}^{p} \epsilon(m, n, p, r) \chi(r) \chi(p-r) \beta_{i_{p-r} \ldots i_p} \wedge \alpha_{i_0 \ldots i_{p-r}} \\ =& \chi(p) \sum\nolimits_{r=0}^{p} \epsilon(m, n, p, p-r) \chi(r) \chi(p-r) \beta_{i_r \ldots i_p} \wedge \alpha_{i_0 \ldots i_r}\end{aligned}$$

where $\chi(t) = (-1)^{\frac{t(t+1)}{2}}$. Since we proved earlier that τ acts as the identity on cohomology we have to verify that

$$\epsilon(n, m, p, r)(-1)^{(n-r)(m-(p-r))} = (-1)^{nm} \chi(p) \epsilon(m, n, p, p-r) \chi(r) \chi(p-r)$$

A trivial mod 2 calculation shows these signs match up. (Alternatively, check the script signs.gp in the scripts subdirectory of the stacks project.)

Finally, we study the compatibility of cup product with boundary maps. Suppose that

$$0 \to \mathcal{F}_1^\bullet \to \mathcal{F}_2^\bullet \to \mathcal{F}_3^\bullet \to 0 \quad \text{and} \quad 0 \leftarrow \mathcal{G}_1^\bullet \leftarrow \mathcal{G}_2^\bullet \leftarrow \mathcal{G}_3^\bullet \leftarrow 0$$

are short exact sequences of bounded below complexes of abelian sheaves on X. Let $\mathcal{H}^\bullet$ be another bounded below complex of abelian sheaves, and suppose we have maps of complexes

$$\gamma_i : \mathrm{Tot}(\mathcal{F}_i^\bullet \otimes_{\mathbf{Z}} \mathcal{G}_i^\bullet) \longrightarrow \mathcal{H}^\bullet$$

which are compatible with the maps between the complexes, namely such that the diagrams

$$\begin{array}{ccc} \mathrm{Tot}(\mathcal{F}_1^\bullet \otimes_{\mathbf{Z}} \mathcal{G}_1^\bullet) & \longleftarrow & \mathrm{Tot}(\mathcal{F}_1^\bullet \otimes_{\mathbf{Z}} \mathcal{G}_2^\bullet) \\ \downarrow{\scriptstyle \gamma_1} & & \downarrow \\ \mathcal{H}^\bullet & \xleftarrow{\gamma_2} & \mathrm{Tot}(\mathcal{F}_2^\bullet \otimes_{\mathbf{Z}} \mathcal{G}_2^\bullet) \end{array}$$

and

$$\begin{array}{ccc} \mathrm{Tot}(\mathcal{F}_2^\bullet \otimes_{\mathbf{Z}} \mathcal{G}_2^\bullet) & \longleftarrow & \mathrm{Tot}(\mathcal{F}_2^\bullet \otimes_{\mathbf{Z}} \mathcal{G}_3^\bullet) \\ \downarrow{\scriptstyle \gamma_2} & & \downarrow \\ \mathcal{H}^\bullet & \xleftarrow{\gamma_3} & \mathrm{Tot}(\mathcal{F}_3^\bullet \otimes_{\mathbf{Z}} \mathcal{G}_3^\bullet) \end{array}$$

are commutative.

Lemma 20.24.2. *In the situation above, assume Čech cohomology agrees with cohomology for the sheaves $\mathcal{F}_i^p$ and $\mathcal{G}_j^q$. Let $a_3 \in H^n(X, \mathcal{F}_3^\bullet)$ and $b_1 \in H^m(X, \mathcal{G}_1^\bullet)$. Then we have*

$$\gamma_1(\partial a_3 \cup b_1) = (-1)^{n+1} \gamma_3(a_3 \cup \partial b_1)$$

in $H^{n+m}(X, \mathcal{H}^\bullet)$ where ∂ indicates the boundary map on cohomology associated to the short exact sequences of complexes above.

Proof. We will use the following conventions and notation. We think of $\mathcal{F}_1^p$ as a subsheaf of $\mathcal{F}_2^p$ and we think of $\mathcal{G}_3^q$ as a subsheaf of $\mathcal{G}_2^q$. Hence if s is a local section of $\mathcal{F}_1^p$ we use s to denote the corresponding section of $\mathcal{F}_2^p$ as well. Similarly for local sections of $\mathcal{G}_3^q$. Furthermore, if s is a local section of $\mathcal{F}_2^p$ then we denote $\bar{s}$ its image in $\mathcal{F}_3^p$. Similarly for the map $\mathcal{G}_2^q \to \mathcal{G}_1^q$. In particular if s is a local section of $\mathcal{F}_2^p$ and $\bar{s} = 0$ then s is a local section of $\mathcal{F}_1^p$. The commutativity of the diagrams above implies, for local sections s of $\mathcal{F}_2^p$ and t of $\mathcal{G}_3^q$ that $\gamma_2(s \otimes t) = \gamma_3(\bar{s} \otimes t)$ as sections of $\mathcal{H}^{p+q}$.

Let $\mathcal{U} : X = \bigcup_{i \in I} U_i$ be an open covering of X. Suppose that α_3, resp. β_1 is a degree n, resp. m cocycle of $\mathrm{Tot}(\check{\mathcal{C}}^\bullet(\mathcal{U}, \mathcal{F}_3^\bullet))$, resp. $\mathrm{Tot}(\check{\mathcal{C}}^\bullet(\mathcal{U}, \mathcal{G}_1^\bullet))$ representing a_3, resp. b_1. After refining $\mathcal{U}$ if necessary, we can find cochains α_2, resp. β_2 of degree n, resp. m in $\mathrm{Tot}(\check{\mathcal{C}}^\bullet(\mathcal{U}, \mathcal{F}_2^\bullet))$, resp. $\mathrm{Tot}(\check{\mathcal{C}}^\bullet(\mathcal{U}, \mathcal{G}_2^\bullet))$ mapping to α_3, resp. β_1. Then we see that

$$\overline{d(\alpha_2)} = d(\bar{\alpha}_2) = 0 \quad \text{and} \quad \overline{d(\beta_2)} = d(\bar{\beta}_2) = 0.$$

This means that $\alpha_1 = d(\alpha_2)$ is a degree $n+1$ cocycle in $\mathrm{Tot}(\check{\mathcal{C}}^\bullet(\mathcal{U}, \mathcal{F}_1^\bullet))$ representing ∂a_3. Similarly, $\beta_3 = d(\beta_2)$ is a degree $m+1$ cocycle in $\mathrm{Tot}(\check{\mathcal{C}}^\bullet(\mathcal{U}, \mathcal{G}_3^\bullet))$ representing ∂b_1. Thus we may compute

$$\begin{aligned} d(\gamma_2(\alpha_2 \cup \beta_2)) &= \gamma_2(d(\alpha_2 \cup \beta_2)) \\ &= \gamma_2(d(\alpha_2) \cup \beta_2 + (-1)^n \alpha_2 \cup d(\beta_2)) \\ &= \gamma_2(\alpha_1 \cup \beta_2) + (-1)^n \gamma_2(\alpha_2 \cup \beta_3) \\ &= \gamma_1(\alpha_1 \cup \beta_1) + (-1)^n \gamma_3(\alpha_3 \cup \beta_3) \end{aligned}$$

So this even tells us that the sign is $(-1)^{n+1}$ as indicated in the lemma[1]. □

20.25. Flat resolutions

A reference for the material in this section is [**Spa88**]. Let $(X, \mathcal{O}_X)$ be a ringed space. By Modules, Lemma 17.16.6 any $\mathcal{O}_X$-module is a quotient of a flat $\mathcal{O}_X$-module. By Derived Categories, Lemma 13.15.5 any bounded above complex of $\mathcal{O}_X$-modules has a left resolution by a bounded above complex of flat $\mathcal{O}_X$-modules. However, for unbounded complexes, it turns out that flat resolutions aren't good enough.

Lemma 20.25.1. *Let $(X, \mathcal{O}_X)$ be a ringed space. Let $\mathcal{G}^\bullet$ be a complex of $\mathcal{O}_X$-modules. The functor*

$$K(\mathit{Mod}(\mathcal{O}_X)) \longrightarrow K(\mathit{Mod}(\mathcal{O}_X)), \quad \mathcal{F}^\bullet \longmapsto \mathit{Tot}(\mathcal{F}^\bullet \otimes_{\mathcal{O}_X} \mathcal{G}^\bullet)$$

is an exact functor of triangulated categories.

Proof. Omitted. Hint: See More on Algebra, Lemmas 15.43.1 and 15.43.2. □

Definition 20.25.2. Let $(X, \mathcal{O}_X)$ be a ringed space. A complex $\mathcal{K}^\bullet$ of $\mathcal{O}_X$-modules is called *K-flat* if for every acyclic complex $\mathcal{F}^\bullet$ of $\mathcal{O}_X$-modules the complex

$$\mathrm{Tot}(\mathcal{F}^\bullet \otimes_{\mathcal{O}_X} \mathcal{K}^\bullet)$$

[1] The sign depends on the convention for the signs in the long exact sequence in cohomology associated to a triangle in $D(X)$. The conventions in the stacks project are (a) distinguished triangles correspond to termwise split exact sequences and (b) the boundary maps in the long exact sequence are given by the maps in the snake lemma without the intervention of signs. See Derived Categories, Section 13.9.

is acyclic.

Lemma 20.25.3. *Let $(X, \mathcal{O}_X)$ be a ringed space. Let $\mathcal{K}^\bullet$ be a K-flat complex. Then the functor*

$$K(Mod(\mathcal{O}_X)) \longrightarrow K(Mod(\mathcal{O}_X)), \quad \mathcal{F}^\bullet \longmapsto Tot(\mathcal{F}^\bullet \otimes_{\mathcal{O}_X} \mathcal{K}^\bullet)$$

transforms quasi-isomorphisms into quasi-isomorphisms.

Proof. Follows from Lemma 20.25.1 and the fact that quasi-isomorphisms are characterized by having acyclic cones. □

Lemma 20.25.4. *Let $(X, \mathcal{O}_X)$ be a ringed space. Let $\mathcal{K}^\bullet$ be a complex of $\mathcal{O}_X$-modules. Then $\mathcal{K}^\bullet$ is K-flat if and only if for all $x \in X$ the complex $\mathcal{K}^\bullet_x$ of $\mathcal{O}_{X,x}$ is K-flat (More on Algebra, Definition 15.43.3).*

Proof. If $\mathcal{K}^\bullet_x$ is K-flat for all $x \in X$ then we see that $\mathcal{K}^\bullet$ is K-flat because $\otimes$ and direct sums commute with taking stalks and because we can check exactness at stalks, see Modules, Lemma 17.3.1. Conversely, assume $\mathcal{K}^\bullet$ is K-flat. Pick $x \in X$ $M^\bullet$ be an acyclic complex of $\mathcal{O}_{X,x}$-modules. Then $i_{x,*}M^\bullet$ is an acyclic complex of $\mathcal{O}_X$-modules. Thus $\mathrm{Tot}(i_{x,*}M^\bullet \otimes_{\mathcal{O}_X} \mathcal{K}^\bullet)$ is acyclic. Taking stalks at x shows that $\mathrm{Tot}(M^\bullet \otimes_{\mathcal{O}_{X,x}} \mathcal{K}^\bullet_x)$ is acyclic. □

Lemma 20.25.5. *Let $(X, \mathcal{O}_X)$ be a ringed space. If $\mathcal{K}^\bullet$, $\mathcal{L}^\bullet$ are K-flat complexes of $\mathcal{O}_X$-modules, then $Tot(\mathcal{K}^\bullet \otimes_{\mathcal{O}_X} \mathcal{L}^\bullet)$ is a K-flat complex of $\mathcal{O}_X$-modules.*

Proof. Follows from the isomorphism

$$\mathrm{Tot}(\mathcal{M}^\bullet \otimes_{\mathcal{O}_X} \mathrm{Tot}(\mathcal{K}^\bullet \otimes_{\mathcal{O}_X} \mathcal{L}^\bullet)) = \mathrm{Tot}(\mathrm{Tot}(\mathcal{M}^\bullet \otimes_{\mathcal{O}_X} \mathcal{K}^\bullet) \otimes_{\mathcal{O}_X} \mathcal{L}^\bullet)$$

and the definition. □

Lemma 20.25.6. *Let $(X, \mathcal{O}_X)$ be a ringed space. Let $(\mathcal{K}^\bullet_1, \mathcal{K}^\bullet_2, \mathcal{K}^\bullet_3)$ be a distinguished triangle in $K(Mod(\mathcal{O}_X))$. If two out of three of $\mathcal{K}^\bullet_i$ are K-flat, so is the third.*

Proof. Follows from Lemma 20.25.1 and the fact that in a distinguished triangle in $K(Mod(\mathcal{O}_X))$ if two out of three are acyclic, so is the third. □

Lemma 20.25.7. *Let $f : (X, \mathcal{O}_X) \to (Y, \mathcal{O}_Y)$ be a morphism of ringed spaces. The pullback of a K-flat complex of $\mathcal{O}_Y$-modules is a K-flat complex of $\mathcal{O}_X$-modules.*

Proof. We can check this on stalks, see Lemma 20.25.4. Hence this follows from Sheaves, Lemma 6.26.4 and More on Algebra, Lemma 15.43.5. □

Lemma 20.25.8. *Let $(X, \mathcal{O}_X)$ be a ringed space. A bounded above complex of flat $\mathcal{O}_X$-modules is K-flat.*

Proof. We can check this on stalks, see Lemma 20.25.4. Thus this lemma follows from Modules, Lemma 17.16.2 and More on Algebra, Lemma 15.43.8. □

In the following lemma by a colimit of a system of complexes we mean the termwise colimit.

Lemma 20.25.9. *Let $(X, \mathcal{O}_X)$ be a ringed space. Let $\mathcal{K}^\bullet_1 \to \mathcal{K}^\bullet_2 \to \ldots$ be a system of K-flat complexes. Then $\mathrm{colim}_i \mathcal{K}^\bullet_i$ is K-flat.*

Proof. Because we are taking termwise colimits it is clear that

$$\operatorname{colim}_i \operatorname{Tot}(\mathcal{F}^\bullet \otimes_{\mathcal{O}_X} \mathcal{K}_i^\bullet) = \operatorname{Tot}(\mathcal{F}^\bullet \otimes_{\mathcal{O}_X} \operatorname{colim}_i \mathcal{K}_i^\bullet)$$

Hence the lemma follows from the fact that filtered colimits are exact. □

Lemma 20.25.10. *Let $(X, \mathcal{O}_X)$ be a ringed space. For any complex $\mathcal{G}^\bullet$ of $\mathcal{O}_X$-modules there exists a commutative diagram of complexes of $\mathcal{O}_X$-modules*

$$\begin{array}{ccccc} \mathcal{K}_1^\bullet & \longrightarrow & \mathcal{K}_2^\bullet & \longrightarrow & \ldots \\ \downarrow & & \downarrow & & \\ \tau_{\leq 1}\mathcal{G}^\bullet & \longrightarrow & \tau_{\leq 2}\mathcal{G}^\bullet & \longrightarrow & \ldots \end{array}$$

with the following properties: (1) the vertical arrows are quasi-isomorphisms, (2) each $\mathcal{K}_n^\bullet$ is a bounded above complex whose terms are direct sums of $\mathcal{O}_X$-modules of the form $j_{U!}\mathcal{O}_U$, and (3) the maps $\mathcal{K}_n^\bullet \to \mathcal{K}_{n+1}^\bullet$ are termwise split injections whose cokernels are direct sums of $\mathcal{O}_X$-modules of the form $j_{U!}\mathcal{O}_U$. Moreover, the map $\operatorname{colim} \mathcal{K}_n^\bullet \to \mathcal{G}^\bullet$ is a quasi-isomorphism.

Proof. The existence of the diagram and properties (1), (2), (3) follows immediately from Modules, Lemma 17.16.6 and Derived Categories, Lemma 13.27.1. The induced map $\operatorname{colim} \mathcal{K}_n^\bullet \to \mathcal{G}^\bullet$ is a quasi-isomorphism because filtered colimits are exact. □

Lemma 20.25.11. *Let $(X, \mathcal{O}_X)$ be a ringed space. For any complex $\mathcal{G}^\bullet$ there exists a K-flat complex $\mathcal{K}^\bullet$ and a quasi-isomorphism $\mathcal{K}^\bullet \to \mathcal{G}^\bullet$.*

Proof. Choose a diagram as in Lemma 20.25.10. Each complex $\mathcal{K}_n^\bullet$ is a bounded above complex of flat modules, see Modules on Sites, Lemma 18.28.5. Hence $\mathcal{K}_n^\bullet$ is K-flat by Lemma 20.25.8. The induced map $\operatorname{colim} \mathcal{K}_n^\bullet \to \mathcal{G}^\bullet$ is a quasi-isomorphism by construction. Since $\operatorname{colim} \mathcal{K}_n^\bullet$ is K-flat by Lemma 20.25.9 we win. □

Lemma 20.25.12. *Let $(X, \mathcal{O}_X)$ be a ringed space. Let $\alpha : \mathcal{P}^\bullet \to \mathcal{Q}^\bullet$ be a quasi-isomorphism of K-flat complexes of $\mathcal{O}_X$-modules. For every complex $\mathcal{F}^\bullet$ of $\mathcal{O}_X$-modules the induced map*

$$\mathit{Tot}(id_{\mathcal{F}^\bullet} \otimes \alpha) : \mathit{Tot}(\mathcal{F}^\bullet \otimes_{\mathcal{O}_X} \mathcal{P}^\bullet) \longrightarrow \mathit{Tot}(\mathcal{F}^\bullet \otimes_{\mathcal{O}_X} \mathcal{Q}^\bullet)$$

is a quasi-isomorphism.

Proof. Choose a quasi-isomorphism $\mathcal{K}^\bullet \to \mathcal{F}^\bullet$ with $\mathcal{K}^\bullet$ a K-flat complex, see Lemma 20.25.11. Consider the commutative diagram

$$\begin{array}{ccc} \operatorname{Tot}(\mathcal{K}^\bullet \otimes_{\mathcal{O}_X} \mathcal{P}^\bullet) & \longrightarrow & \operatorname{Tot}(\mathcal{K}^\bullet \otimes_{\mathcal{O}_X} \mathcal{Q}^\bullet) \\ \downarrow & & \downarrow \\ \operatorname{Tot}(\mathcal{F}^\bullet \otimes_{\mathcal{O}_X} \mathcal{P}^\bullet) & \longrightarrow & \operatorname{Tot}(\mathcal{F}^\bullet \otimes_{\mathcal{O}_X} \mathcal{Q}^\bullet) \end{array}$$

The result follows as by Lemma 20.25.3 the vertical arrows and the top horizontal arrow are quasi-isomorphisms. □

Let $(X, \mathcal{O}_X)$ be a ringed space. Let $\mathcal{F}^\bullet$ be an object of $D(\mathcal{O}_X)$. Choose a K-flat resolution $\mathcal{K}^\bullet \to \mathcal{F}^\bullet$, see Lemma 20.25.11. By Lemma 20.25.1 we obtain an exact functor of triangulated categories

$$K(\mathcal{O}_X) \longrightarrow K(\mathcal{O}_X), \quad \mathcal{G}^\bullet \longmapsto \operatorname{Tot}(\mathcal{G}^\bullet \otimes_{\mathcal{O}_X} \mathcal{K}^\bullet)$$

By Lemma 20.25.3 this functor induces a functor $D(\mathcal{O}_X) \to D(\mathcal{O}_X)$ simply because $D(\mathcal{O}_X)$ is the localization of $K(\mathcal{O}_X)$ at quasi-isomorphisms. By Lemma 20.25.12 the resulting functor (up to isomorphism) does not depend on the choice of the K-flat resolution.

Definition 20.25.13. Let $(X, \mathcal{O}_X)$ be a ringed space. Let $\mathcal{F}^\bullet$ be an object of $D(\mathcal{O}_X)$. The *derived tensor product*

$$- \otimes^{\mathbf{L}}_{\mathcal{O}_X} \mathcal{F}^\bullet : D(\mathcal{O}_X) \longrightarrow D(\mathcal{O}_X)$$

is the exact functor of triangulated categories described above.

It is clear from our explicit constructions that there is a canonical isomorphism

$$\mathcal{F}^\bullet \otimes^{\mathbf{L}}_{\mathcal{O}_X} \mathcal{G}^\bullet \cong \mathcal{G}^\bullet \otimes^{\mathbf{L}}_{\mathcal{O}_X} \mathcal{F}^\bullet$$

for $\mathcal{G}^\bullet$ and $\mathcal{F}^\bullet$ in $D(\mathcal{O}_X)$. Hence when we write $\mathcal{F}^\bullet \otimes^{\mathbf{L}}_{\mathcal{O}_X} \mathcal{G}^\bullet$ we will usually be agnostic about which variable we are using to define the derived tensor product with.

Definition 20.25.14. Let $(X, \mathcal{O}_X)$ be a ringed space. Let $\mathcal{F}$, $\mathcal{G}$ be $\mathcal{O}_X$-modules. The *Tor*'s of $\mathcal{F}$ and $\mathcal{G}$ are define by the formula

$$\mathrm{Tor}_p^{\mathcal{O}_X}(\mathcal{F}, \mathcal{G}) = H^{-p}(\mathcal{F} \otimes^{\mathbf{L}}_{\mathcal{O}_X} \mathcal{G})$$

with derived tensor product as defined above.

This definition implies that for every short exact sequence of $\mathcal{O}_X$-modules $0 \to \mathcal{F}_1 \to \mathcal{F}_2 \to \mathcal{F}_3 \to 0$ we have a long exact cohomology sequence

$$\begin{array}{ccccccc} \mathcal{F}_1 \otimes_{\mathcal{O}_X} \mathcal{G} & \longrightarrow & \mathcal{F}_2 \otimes_{\mathcal{O}_X} \mathcal{G} & \longrightarrow & \mathcal{F}_3 \otimes_{\mathcal{O}_X} \mathcal{G} & \longrightarrow & 0 \\ & \searrow & & & & & \\ \mathrm{Tor}_1^{\mathcal{O}_X}(\mathcal{F}_1, \mathcal{G}) & \longrightarrow & \mathrm{Tor}_1^{\mathcal{O}_X}(\mathcal{F}_2, \mathcal{G}) & \longrightarrow & \mathrm{Tor}_1^{\mathcal{O}_X}(\mathcal{F}_3, \mathcal{G}) & & \end{array}$$

for every $\mathcal{O}_X$-module $\mathcal{G}$. This will be called the long exact sequence of Tor associated to the situation.

Lemma 20.25.15. *Let $(X, \mathcal{O}_X)$ be a ringed space. Let $\mathcal{F}$ be an $\mathcal{O}_X$-module. The following are equivalent*

1. *$\mathcal{F}$ is a flat $\mathcal{O}_X$-module, and*
2. *$\mathit{Tor}_1^{\mathcal{O}_X}(\mathcal{F}, \mathcal{G}) = 0$ for every $\mathcal{O}_X$-module $\mathcal{G}$.*

Proof. If $\mathcal{F}$ is flat, then $\mathcal{F} \otimes_{\mathcal{O}_X} -$ is an exact functor and the satellites vanish. Conversely assume (2) holds. Then if $\mathcal{G} \to \mathcal{H}$ is injective with cokernel $\mathcal{Q}$, the long exact sequence of Tor shows that the kernel of $\mathcal{F} \otimes_{\mathcal{O}_X} \mathcal{G} \to \mathcal{F} \otimes_{\mathcal{O}_X} \mathcal{H}$ is a quotient of $\mathrm{Tor}_1^{\mathcal{O}_X}(\mathcal{F}, \mathcal{Q})$ which is zero by assumption. Hence $\mathcal{F}$ is flat. □

20.26. Derived pullback

Let $f : (X, \mathcal{O}_X) \to (Y, \mathcal{O}_Y)$ be a morphism of ringed spaces. We can use K-flat resolutions to define a derived pullback functor

$$Lf^* : D(\mathcal{O}_Y) \to D(\mathcal{O}_X)$$

Namely, for every complex of $\mathcal{O}_Y$-modules $\mathcal{G}^\bullet$ we can choose a K-flat resolution $\mathcal{K}^\bullet \to \mathcal{G}^\bullet$ and set $Lf^*\mathcal{G}^\bullet = f^*\mathcal{K}^\bullet$. You can use Lemmas 20.25.7, 20.25.11, and 20.25.12 to see that this is well defined. However, to cross all the t's and dot all the i's it is perhaps more convenient to use some general theory.

Lemma 20.26.1. *The construction above is independent of choices and defines an exact functor of triangulated categories $Lf^* : D(\mathcal{O}_Y) \to D(\mathcal{O}_X)$.*

Proof. To see this we use the general theory developed in Derived Categories, Section 13.14. Set $\mathcal{D} = K(\mathcal{O}_Y)$ and $\mathcal{D}' = D(\mathcal{O}_X)$. Let us write $F : \mathcal{D} \to \mathcal{D}'$ the exact functor of triangulated categories defined by the rule $F(\mathcal{G}^\bullet) = f^*\mathcal{G}^\bullet$. We let S be the set of quasi-isomorphisms in $\mathcal{D} = K(\mathcal{O}_Y)$. This gives a situation as in Derived Categories, Situation 13.14.1 so that Derived Categories, Definition 13.14.2 applies. We claim that LF is everywhere defined. This follows from Derived Categories, Lemma 13.14.15 with $\mathcal{P} \subset \mathrm{Ob}(\mathcal{D})$ the collection of K-flat complexes: (1) follows from Lemma 20.25.11 and to see (2) we have to show that for a quasi-isomorphism $\mathcal{K}_1^\bullet \to \mathcal{K}_2^\bullet$ between K-flat complexes of $\mathcal{O}_Y$-modules the map $f^*\mathcal{K}_1^\bullet \to f^*\mathcal{K}_2^\bullet$ is a quasi-isomorphism. To see this write this as

$$f^{-1}\mathcal{K}_1^\bullet \otimes_{f^{-1}\mathcal{O}_Y} \mathcal{O}_X \longrightarrow f^{-1}\mathcal{K}_2^\bullet \otimes_{f^{-1}\mathcal{O}_Y} \mathcal{O}_X$$

The functor f^{-1} is exact, hence the map $f^{-1}\mathcal{K}_1^\bullet \to f^{-1}\mathcal{K}_2^\bullet$ is a quasi-isomorphism. By Lemma 20.25.7 applied to the morphism $(X, f^{-1}\mathcal{O}_Y) \to (Y, \mathcal{O}_Y)$ the complexes $f^{-1}\mathcal{K}_1^\bullet$ and $f^{-1}\mathcal{K}_2^\bullet$ are K-flat complexes of $f^{-1}\mathcal{O}_Y$-modules. Hence Lemma 20.25.12 guarantees that the displayed map is a quasi-isomorphism. Thus we obtain a derived functor

$$LF : D(\mathcal{O}_Y) = S^{-1}\mathcal{D} \longrightarrow \mathcal{D}' = D(\mathcal{O}_X)$$

see Derived Categories, Equation (13.14.9.1). Finally, Derived Categories, Lemma 13.14.15 also guarantees that $LF(\mathcal{K}^\bullet) = F(\mathcal{K}^\bullet) = f^*\mathcal{K}^\bullet$ when $\mathcal{K}^\bullet$ is K-flat, i.e., $Lf^* = LF$ is indeed computed in the way described above. □

Lemma 20.26.2. *Let $f : (X, \mathcal{O}_X) \to (Y, \mathcal{O}_Y)$ be a morphism of ringed spaces. There is a canonical bifunctorial isomorphism*

$$Lf^*(\mathcal{F}^\bullet \otimes^{\mathbf{L}}_{\mathcal{O}_Y} \mathcal{G}^\bullet) = Lf^*\mathcal{F}^\bullet \otimes^{\mathbf{L}}_{\mathcal{O}_X} Lf^*\mathcal{G}^\bullet$$

for $\mathcal{F}^\bullet, \mathcal{G}^\bullet \in \mathrm{Ob}(D(X))$.

Proof. We may assume that $\mathcal{F}^\bullet$ and $\mathcal{G}^\bullet$ are K-flat complexes. In this case $\mathcal{F}^\bullet \otimes^{\mathbf{L}}_{\mathcal{O}_Y} \mathcal{G}^\bullet$ is just the total complex associated to the double complex $\mathcal{F}^\bullet \otimes_{\mathcal{O}_Y} \mathcal{G}^\bullet$. By Lemma 20.25.5 $\mathrm{Tot}(\mathcal{F}^\bullet \otimes_{\mathcal{O}_Y} \mathcal{G}^\bullet)$ is K-flat also. Hence the isomorphism of the lemma comes from the isomorphism

$$\mathrm{Tot}(f^*\mathcal{F}^\bullet \otimes_{\mathcal{O}_X} f^*\mathcal{G}^\bullet) \longrightarrow f^*\mathrm{Tot}(\mathcal{F}^\bullet \otimes_{\mathcal{O}_Y} \mathcal{G}^\bullet)$$

whose constituents are the isomorphisms $f^*\mathcal{F}^p \otimes_{\mathcal{O}_X} f^*\mathcal{G}^q \to f^*(\mathcal{F}^p \otimes_{\mathcal{O}_Y} \mathcal{G}^q)$ of Modules, Lemma 17.15.4. □

Lemma 20.26.3. *Let $f : (X, \mathcal{O}_X) \to (Y, \mathcal{O}_Y)$ be a morphism of ringed spaces. There is a canonical bifunctorial isomorphism*

$$\mathcal{F}^\bullet \otimes^{\mathbf{L}}_{\mathcal{O}_X} Lf^*\mathcal{G}^\bullet = \mathcal{F}^\bullet \otimes^{\mathbf{L}}_{f^{-1}\mathcal{O}_Y} f^{-1}\mathcal{G}^\bullet$$

for $\mathcal{F}^\bullet$ in $D(X)$ and $\mathcal{G}^\bullet$ in $D(Y)$.

Proof. Let $\mathcal{F}$ be an $\mathcal{O}_X$-module and let $\mathcal{G}$ be an $\mathcal{O}_Y$-module. Then $\mathcal{F} \otimes_{\mathcal{O}_X} f^*\mathcal{G} = \mathcal{F} \otimes_{f^{-1}\mathcal{O}_Y} f^{-1}\mathcal{G}$ because $f^*\mathcal{G} = \mathcal{O}_X \otimes_{f^{-1}\mathcal{O}_Y} f^{-1}\mathcal{G}$. The lemma follows from this and the definitions. □

20.27. Cohomology of unbounded complexes

Let $(X, \mathcal{O}_X)$ be a ringed space. The category $Mod(\mathcal{O}_X)$ is a Grothendieck abelian category: it has all colimits, filtered colimits are exact, and it has a generator, namely

$$\bigoplus\nolimits_{U \subset X \text{ open}} j_{U!}\mathcal{O}_U,$$

see Modules, Section 17.3 and Lemmas 17.16.5 and 17.16.6. By Injectives, Theorem 19.12.6 for every complex $\mathcal{F}^\bullet$ of $\mathcal{O}_X$-modules there exists an injective quasi-isomorphism $\mathcal{F}^\bullet \to \mathcal{I}^\bullet$ to a K-injective complex of $\mathcal{O}_X$-modules. Hence we can define

$$R\Gamma(X, \mathcal{F}^\bullet) = \Gamma(X, \mathcal{I}^\bullet)$$

and similarly for any left exact functor, see Derived Categories, Lemma 13.28.6. For any morphism of ringed spaces $f : (X, \mathcal{O}_X) \to (Y, \mathcal{O}_Y)$ we obtain

$$Rf_* : D(X) \longrightarrow D(Y)$$

on the unbounded derived categories.

Lemma 20.27.1. *Let $f : (X, \mathcal{O}_X) \to (Y, \mathcal{O}_Y)$ be a morphism of ringed spaces. The functor Rf_* defined above and the functor Lf^* defined in Lemma 20.26.1 are adjoint:*

$$\mathrm{Hom}_{D(X)}(Lf^*\mathcal{G}^\bullet, \mathcal{F}^\bullet) = \mathrm{Hom}_{D(Y)}(\mathcal{G}^\bullet, Rf_*\mathcal{F}^\bullet)$$

bifunctorially in $\mathcal{F}^\bullet \in \mathrm{Ob}(D(X))$ and $\mathcal{G}^\bullet \in \mathrm{Ob}(D(Y))$.

Proof. This follows formally from the fact that Rf_* and Lf^* exist, see Derived Categories, Lemma 13.27.4. □

Remark 20.27.2. The construction of unbounded derived functor Lf^* and Rf_* allows one to construct the base change map in full generality. Namely, suppose that

$$\begin{array}{ccc} X' & \xrightarrow{g'} & X \\ {\scriptstyle f'}\downarrow & & \downarrow{\scriptstyle f} \\ S' & \xrightarrow{g} & S \end{array}$$

is a commutative diagram of ringed spaces. Let $\mathcal{F}^\bullet$ be a complex of $\mathcal{O}_X$-modules. Then there exists a canonical base change map

$$Lg^*Rf_*\mathcal{F}^\bullet \longrightarrow R(f')_*L(g')^*\mathcal{F}^\bullet$$

in $D(\mathcal{O}_{S'})$. Namely, this map is adjoint to a map $L(f')^*Lg^*Rf_*\mathcal{F}^\bullet \to L(g')^*\mathcal{F}^\bullet$ Since $L(f')^*Lg^* = L(g')^*Lf^*$ we see this is the same as a map $L(g')^*Lf^*Rf_*\mathcal{F}^\bullet \to L(g')^*\mathcal{F}^\bullet$ which we can take to be $L(g')^*$ of the adjunction map $Lf^*Rf_*\mathcal{F}^\bullet \to \mathcal{F}^\bullet$.

20.28. Unbounded Mayer-Vietoris

Let $(X, \mathcal{O}_X)$ be a ringed space. Let $U \subset X$ be an open subset. Denote $j : (U, \mathcal{O}_U) \to (X, \mathcal{O}_X)$ the corresponding open immersion. The pullback functor j^* is exact as it is just the restriction functor. Thus derived pullback Lj^* is computed on any complex by simply restricting the complex. We often simply denote the corresponding functor

$$D(\mathcal{O}_X) \to D(\mathcal{O}_U), \quad E \mapsto j^*E = E|_U$$

Similarly, extension by zero $j_! : Mod(\mathcal{O}_U) \to Mod(\mathcal{O}_X)$ (see Sheaves, Section 6.31) is an exact functor (Modules, Lemma 17.3.4). Thus it induces a functor

$$j_! : D(\mathcal{O}_U) \to D(\mathcal{O}_X), \quad F \mapsto j_!F$$

by simply applying $j_!$ to any complex representing the object F.

Lemma 20.28.1. *Let X be a ringed space. Let $U \subset X$ be an open subspace. The restriction of a K-injective complex of $\mathcal{O}_X$-modules to U is a K-injective complex of $\mathcal{O}_U$-modules.*

Proof. Follows immediately from Derived Categories, Lemma 13.28.10 and the fact that the restriction functor has the exact adjoint $j_!$. See discussion above. □

Lemma 20.28.2. *Let $f : (X, \mathcal{O}_X) \to (Y, \mathcal{O}_Y)$ be a morphism of ringed spaces. Given an open subspace $V \subset Y$, set $U = f^{-1}(V)$ and denote $g : U \to V$ the induced morphism. Then $(Rf_*E)|_V = Rg_*(E|_U)$ for E in $D(\mathcal{O}_X)$.*

Proof. Represent E by a K-injective complex $\mathcal{I}^\bullet$ of $\mathcal{O}_X$-modules. Then $Rf_*(E) = f_*\mathcal{I}^\bullet$ and $Rg_*(E|_U) = g_*(\mathcal{I}^\bullet|_U)$ by Lemma 20.28.1. Hence the result follows from Lemma 20.7.4 (with $p = 0$). □

Lemma 20.28.3. *Let $(X, \mathcal{O}_X)$ be a ringed space. Let $U \subset X$ be an open subset. Denote $j : (U, \mathcal{O}_U) \to (X, \mathcal{O}_X)$ the corresponding open immersion. The restriction functor $D(\mathcal{O}_X) \to D(\mathcal{O}_U)$ is a right adjoint to extension by zero $j_! : D(\mathcal{O}_U) \to D(\mathcal{O}_X)$.*

Proof. We have to show that

$$\mathrm{Hom}_{D(\mathcal{O}_X)}(j_!E, F) = \mathrm{Hom}_{D(\mathcal{O}_U)}(E, F|_U)$$

Choose a complex $\mathcal{E}^\bullet$ of $\mathcal{O}_U$-modules representing E and choose a K-injective complex $\mathcal{I}^\bullet$ representing F. By Lemma 20.28.1 the complex $\mathcal{I}^\bullet|_U$ is K-injective as well. Hence we see that the formula above becomes

$$\mathrm{Hom}_{D(\mathcal{O}_X)}(j_!\mathcal{E}^\bullet, \mathcal{I}^\bullet) = \mathrm{Hom}_{D(\mathcal{O}_U)}(\mathcal{E}^\bullet, \mathcal{I}^\bullet|_U)$$

which holds as $|_U$ and $j_!$ are adjoint functors (Sheaves, Lemma 6.31.8) and Derived Categories, Lemma 13.28.2. □

Lemma 20.28.4. *Let $(X, \mathcal{O}_X)$ be a ringed space. Let $X = U \cup V$ be the union of two open subspaces. For any object E of $D(\mathcal{O}_X)$ we have a distinguished triangle*

$$j_{U\cap V!}E|_{U\cap V} \to j_{U!}E|_U \oplus j_{V!}E|_V \to E \to j_{U\cap V!}E|_{U\cap V}[1]$$

in $D(\mathcal{O}_X)$.

Proof. We have seen above that the restriction functors and the extension by zero functors are computed by just applying the functors to any complex. Let $\mathcal{E}^\bullet$ be a complex of $\mathcal{O}_X$-modules representing E. The distinguished triangle of the lemma is the distinguished triangle associated (by Derived Categories, Section 13.11 and especially Lemma 13.11.1) to the short exact sequence of complexes of $\mathcal{O}_X$-modules

$$0 \to j_{U\cap V!}\mathcal{E}^\bullet|_{U\cap V} \to j_{U!}\mathcal{E}^\bullet|_U \oplus j_{V!}\mathcal{E}^\bullet|_V \to \mathcal{E}^\bullet \to 0$$

To see this sequence is exact one checks on stalks using Sheaves, Lemma 6.31.8 (computation omitted). □

Lemma 20.28.5. *Let $(X, \mathcal{O}_X)$ be a ringed space. Let $X = U \cup V$ be the union of two open subspaces. For any object E of $D(\mathcal{O}_X)$ we have a distinguished triangle*

$$E \to Rj_{U,*}E|_U \oplus Rj_{V,*}E|_V \to Rj_{U\cap V,*}E|_{U\cap V} \to E[1]$$

in $D(\mathcal{O}_X)$.

Proof. Choose a K-injective complex $\mathcal{I}^\bullet$ representing E whose terms $\mathcal{I}^n$ are injective objects of $Mod(\mathcal{O}_X)$, see Injectives, Theorem 19.12.6. We have seen that $\mathcal{I}^\bullet|U$ is a K-injective complex as well (Lemma 20.28.1). Hence $Rj_{U,*}E|_U$ is represented by $j_{U,*}\mathcal{I}^\bullet|_U$. Similarly for V and $U \cap V$. Hence the distinguished triangle of the lemma is the distinguished triangle associated (by Derived Categories, Section 13.11 and especially Lemma 13.11.1) to the short exact sequence of complexes

$$0 \to \mathcal{I}^\bullet \to j_{U,*}\mathcal{I}^\bullet|_U \oplus j_{V,*}\mathcal{I}^\bullet|_V \to j_{U\cap V,*}\mathcal{I}^\bullet|_{U\cap V} \to 0.$$

This sequence is exact because for any $W \subset X$ open and any n the sequence

$$0 \to \mathcal{I}^n(W) \to \mathcal{I}^n(W \cap U) \oplus \mathcal{I}^n(W \cap V) \to \mathcal{I}^n(W \cap U \cap V) \to 0$$

is exact (see proof of Lemma 20.9.2). □

Lemma 20.28.6. *Let $(X, \mathcal{O}_X)$ be a ringed space. Let $X = U \cup V$ be the union of two open subspaces of X. For objects E, F of $D(\mathcal{O}_X)$ we have a Mayer-Vietoris sequence*

$$\begin{array}{ccccc} & & \ldots & \longrightarrow & Ext^{-1}(E_{U\cap V}, F_{U\cap V}) \\ & \swarrow & & & \\ \mathrm{Hom}(E,F) & \longrightarrow & \mathrm{Hom}(E_U,F_U) \oplus \mathrm{Hom}(E_V,F_V) & \longrightarrow & \mathrm{Hom}(E_{U\cap V},F_{U\cap V}) \end{array}$$

where the subscripts denote restrictions to the relevant opens and the Hom*'s are taken in the relevant derived categories.*

Proof. Use the distinguished triangle of Lemma 20.28.4 to obtain a long exact sequence of Hom's (from Derived Categories, Lemma 13.4.2) and use that $\mathrm{Hom}(j_{U!}E|_U, F) = \mathrm{Hom}(E|_U, F|_U)$ by Lemma 20.28.3. □

Lemma 20.28.7. *Let $(X, \mathcal{O}_X)$ be a ringed space. Suppose that $X = U \cup V$ is a union of two open subsets. For an object E of $D(\mathcal{O}_X)$ we have a distinguished triangle*

$$R\Gamma(X, E) \to R\Gamma(U, E) \oplus R\Gamma(V, E) \to R\Gamma(U \cap V, E) \to R\Gamma(X, E)[1]$$

and in particular a long exact cohomology sequence

$$\ldots \to H^n(X, E) \to H^n(U, E) \oplus H^0(V, E) \to H^n(U \cap V, E) \to H^{n+1}(X, E) \to \ldots$$

The construction of the distinguished triangle and the long exact sequence is functorial in E.

Proof. Choose a K-injective complex $\mathcal{I}^\bullet$ representing E. We may assume $\mathcal{I}^n$ is an injective object of $Mod(\mathcal{O}_X)$ for all n, see Injectives, Theorem 19.12.6. Then $R\Gamma(X, E)$ is computed by $\Gamma(X, \mathcal{I}^\bullet)$. Similarly for U, V, and $U \cap V$ by Lemma 20.28.1. Hence the distinguished triangle of the lemma is the distinguished triangle associated (by Derived Categories, Section 13.11 and especially Lemma 13.11.1) to the short exact sequence of complexes

$$0 \to \mathcal{I}^\bullet(X) \to \mathcal{I}^\bullet(U) \oplus \mathcal{I}^\bullet(V) \to \mathcal{I}^\bullet(U \cap V) \to 0.$$

We have seen this is a short exact sequence in the proof of Lemma 20.9.2. The final statement follows from the functoriality of the construction in Injectives, Theorem 19.12.6. □

Lemma 20.28.8. *Let $f : X \to Y$ be a morphism of ringed spaces. Suppose that $X = U \cup V$ is a union of two open subsets. Denote $a = f|_U : U \to Y$, $b = f|_V : V \to Y$, and $c = f|_{U\cap V} : U \cap V \to Y$. For every object E of $D(\mathcal{O}_X)$ there exists a distinguished triangle*

$$Rf_*E \to Ra_*(E|_U) \oplus Rb_*(E|_V) \to Rc_*(E|_{U\cap V}) \to Rf_*E[1]$$

This triangle is functorial in E.

Proof. Choose a K-injective complex $\mathcal{I}^\bullet$ representing E. We may assume $\mathcal{I}^n$ is an injective object of $Mod(\mathcal{O}_X)$ for all n, see Injectives, Theorem 19.12.6. Then Rf_*E is computed by $f_*\mathcal{I}^\bullet$. Similarly for U, V, and $U \cap V$ by Lemma 20.28.1. Hence the distinguished triangle of the lemma is the distinguished triangle associated (by Derived Categories, Section 13.11 and especially Lemma 13.11.1) to the short exact sequence of complexes

$$0 \to f_*\mathcal{I}^\bullet \to a_*\mathcal{I}^\bullet|_U \oplus b_*\mathcal{I}^\bullet|_V \to c_*\mathcal{I}^\bullet|_{U\cap V} \to 0.$$

This is a short exact sequence of complexes by Lemma 20.9.3 and the fact that $R^1 f_*\mathcal{I} = 0$ for an injective object $\mathcal{I}$ of $Mod(\mathcal{O}_X)$. The final statement follows from the functoriality of the construction in Injectives, Theorem 19.12.6. □

Lemma 20.28.9. *Let $(X, \mathcal{O}_X)$ be a ringed space. Let $j : U \to X$ be an open subspace. Let $T \subset X$ be a closed subset contained in U.*

1. *If E is an object of $D(\mathcal{O}_X)$ whose cohomology sheaves are supported on T, then $E \to Rj_*(E|_U)$ is an isomorphism.*
2. *If F is an object of $D(\mathcal{O}_U)$ whose cohomology sheaves are supported on T, then $j_!F \to Rj_*F$ is an isomorphism.*

Proof. Let $V = X \setminus T$ and $W = U \cap V$. Note that $X = U \cup V$ is an open covering of X. Denote $j_W : W \to V$ the open immersion. Let E be an object of $D(\mathcal{O}_X)$ whose cohomology sheaves are supported on T. By Lemma 20.28.2 we have $(Rj_*E|_U)|_V = Rj_{W,*}(E|_W) = 0$ because $E|_W = 0$ by our assumption. On the other hand, $Rj_*(E|_U)|_U = E|_U$. Thus (1) is clear. Let F be an object of $D(\mathcal{O}_U)$ whose cohomology sheaves are supported on T. By Lemma 20.28.2 we have $(Rj_*F)|_V = Rj_{W,*}(F|_W) = 0$ because $F|_W = 0$ by our assumption. We also have $(j_!F)|_V = j_{W!}(F|_W) = 0$ (the first equality is immediate from the definition of extension by zero). Since both $(Rj_*F)|_U = F$ and $(j_!F)|_U = F$ we see that (2) holds. □

We can glue complexes!

Lemma 20.28.10. *Let $(X, \mathcal{O}_X)$ be a ringed space. Let $X = U \cup V$ be the union of two open subspaces of X. Suppose given*

1. *an object E of $D(\mathcal{O}_X)$,*
2. *a morphism $a : A \to E|_U$ of $D(\mathcal{O}_U)$,*
3. *a morphism $b : B \to E|_V$ of $D(\mathcal{O}_V)$,*
4. *an isomorphism $c : A|_{U\cap V} \to B|_{U\cap V}$*

such that

$$a|_{U\cap V} = b|_{U\cap V} \circ c.$$

Then there exists a morphism $F \to E$ in $D(\mathcal{O}_X)$ whose restriction to U is isomorphic to a and whose restriction to V is isomorphic to b.

Proof. Denote j_U, j_V, $j_{U\cap V}$ the corresponding open immersions. Choose a distinguished triangle

$$F \to Rj_{U,*}A \oplus Rj_{V,*}B \to Rj_{U\cap V,*}(B|_{U\cap V}) \to F[1]$$

where the map $Rj_{V,*}B \to Rj_{U\cap V,*}(B|_{U\cap V})$ is the obvious one and where $Rj_{U,*}A \to Rj_{U\cap V,*}(B|_{U\cap V})$ is the composition of $Rj_{U,*}A \to Rj_{U\cap V,*}(A|_{U\cap V})$ with $Rj_{U\cap V,*}c$. Restricting to U we obtain

$$F|_U \to A \oplus (Rj_{V,*}B)|_U \to (Rj_{U\cap V,*}(B|_{U\cap V}))|_U \to F|_U[1]$$

Denote $j : U \cap V \to U$. Compatibility of restriction to opens and cohomology shows that both $(Rj_{V,*}B)|_U$ and $(Rj_{U\cap V,*}(B|_{U\cap V}))|_U$ are canonically isomorphic to $Rj_*(B|_{U\cap V})$. Hence the second arrow of the last displayed diagram has a section, and we conclude that the morphism $F|_U \to A$ is an isomorphism. Similarly, the morphism $F|_V \to B$ is an isomorphism. The existence of the morphism $F \to E$ follows from the Mayer-Vietoris sequence for Hom, see Lemma 20.28.6. □

20.29. Producing K-injective resolutions

First a technical lemma.

Lemma 20.29.1. *Let $(X, \mathcal{O}_X)$ be a ringed space. Let $(\mathcal{F}_n^\bullet)$ be an inverse system of complexes of $\mathcal{O}_X$-modules. Let $m \in \mathbf{Z}$. Assume there exist a set $\mathcal{B}$ of open subsets of X and an integer n_0 such that*

1. *every open in X has a covering whose members are elements of $\mathcal{B}$,*
2. *for every $U \in \mathcal{B}$*
   (a) *the systems $\mathcal{F}_n^{m-2}(U)$ and $\mathcal{F}_n^{m-1}(U)$ are ML,*
   (b) *the system $H^{m-1}(\mathcal{F}_n^\bullet(U))$ is essentially constant, and*
   (c) *we have $H^m(\mathcal{F}_n^\bullet(U)) = H^m(\mathcal{F}_{n_0}^\bullet(U))$ for all $n \geq n_0$.*

Then $H^m(\lim \mathcal{F}_n^\bullet) \to \lim H^m(\mathcal{F}_n^\bullet) \to H^m(\mathcal{F}_{n_0}^\bullet)$ are isomorphisms of sheaves.

Proof. Set $\mathcal{F}^\bullet = \lim \mathcal{F}_n^\bullet$. Let $U \in \mathcal{B}$. Note that $H^m(\mathcal{F}^\bullet(U))$ is the cohomology of

$$\lim_n \mathcal{F}_n^{m-2}(U) \to \lim_n \mathcal{F}_n^{m-1}(U) \to \lim_n \mathcal{F}_n^m(U) \to \lim_n \mathcal{F}_n^{m+1}(U)$$

in the third spot from the left. By assumptions (2)(a) and (2)(b) we may apply Homology, Lemma 12.26.7 to conclude that

$$H^m(\mathcal{F}^\bullet(U)) = \lim H^m(\mathcal{F}_n^\bullet(U))$$

By assumption (2)(c) we conclude

$$H^m(\mathcal{F}^\bullet(U)) = H^m(\mathcal{F}_n^\bullet(U))$$

for all $n \geq n_0$. By assumption (1) we conclude that the sheafification of $U \mapsto H^m(\mathcal{F}^\bullet(U))$ is equal to the sheafification of $U \mapsto H^m(\mathcal{F}_n^\bullet(U))$ for all $n \geq n_0$. Thus the inverse system of sheaves $H^m(\mathcal{F}_n^\bullet)$ is constant for $n \geq n_0$ with value $H^m(\mathcal{F}^\bullet)$ which proves the lemma. □

Let $(X, \mathcal{O}_X)$ be a ringed space. Let $\mathcal{F}^\bullet$ be a complex of $\mathcal{O}_X$-modules. The category $Mod(\mathcal{O}_X)$ has enough injectives, hence we can use Derived Categories, Lemma 13.27.3 produce a diagram

$$\begin{array}{ccccc} \ldots & \longrightarrow & \tau_{\geq -2}\mathcal{F}^\bullet & \longrightarrow & \tau_{\geq -1}\mathcal{F}^\bullet \\ & & \downarrow & & \downarrow \\ \ldots & \longrightarrow & \mathcal{I}_2^\bullet & \longrightarrow & \mathcal{I}_1^\bullet \end{array}$$

in the category of complexes of $\mathcal{O}_X$-modules such that

(1) the vertical arrows are quasi-isomorphisms,
(2) $\mathcal{I}_n^\bullet$ is a bounded above complex of injectives,
(3) the arrows $\mathcal{I}_{n+1}^\bullet \to \mathcal{I}_n^\bullet$ are termwise split surjections.

The category of $\mathcal{O}_X$-modules has limits (they are computed on the level of presheaves), hence we can form the termwise limit $\mathcal{I}^\bullet = \lim_n \mathcal{I}_n^\bullet$. By Derived Categories, Lemmas 13.28.4 and 13.28.7 this is a K-injective complex. In general the canonical map

$$\mathcal{F}^\bullet \to \mathcal{I}^\bullet \tag{20.29.1.1}$$

may not be a quasi-isomorphism. In the following lemma we describe some conditions under which it is.

Lemma 20.29.2. *In the situation described above. Denote $\mathcal{H}^m = H^m(\mathcal{F}^\bullet)$ the mth cohomology sheaf. Let $\mathcal{B}$ be a set of open subsets of X. Let $d \in \mathbf{N}$. Assume*

(1) *every open in X has a covering whose members are elements of $\mathcal{B}$,*
(2) *for every $U \in \mathcal{B}$ we have $H^p(U, \mathcal{H}^q) = 0$ for $p > d$[2].*

Then (20.29.1.1) is a quasi-isomorphism.

Proof. Let $m \in \mathbf{Z}$. We have to show that the map $\mathcal{F}^\bullet \to \mathcal{I}^\bullet$ induces an isomorphism $\mathcal{H}^m \to H^m(\mathcal{I}^\bullet)$. Since $\mathcal{I}_n^\bullet$ is quasi-isomorphic to $\tau_{\geq -n}\mathcal{F}^\bullet$ it suffices to show that $H^m(\mathcal{I}^\bullet) \to H^m(\mathcal{I}_n^\bullet)$ is an isomorphism for n large enough. To do this we will verify the hypotheses (1), (2)(a), (2)(b), (2)(c) of Lemma 20.29.1.

Hypothesis (1) is assumption (1) above. Hypothesis (2)(a) follows from the fact that the maps $\mathcal{I}_{n+1}^k \to \mathcal{I}_n^k$ are split surjections. We will prove hypothesis (2)(b) and (2)(c) simultaneously by proving that for $U \in \mathcal{B}$ the system $H^m(\mathcal{I}_n^\bullet(U))$ becomes constant for $n \geq -m + d$. Namely, recalling that $\mathcal{I}_n^\bullet$ is quasi-isomorphic to $\tau_{\geq -n}\mathcal{F}^\bullet$ we obtain for all n a distinguished triangle

$$\mathcal{H}^{-n}[n] \to \mathcal{I}_n^\bullet \to \mathcal{I}_{n-1}^\bullet \to \mathcal{H}^{-n}[n+1]$$

(Derived Categories, Remark 13.11.4) in $D(\mathcal{O}_X)$. By assumption (2) we see that if $m > d - n$ then

$$H^m(U, \mathcal{H}^{-n}[n]) = 0 \quad \text{and} \quad H^m(U, \mathcal{H}^{-n}[n+1]) = 0.$$

Observe that $H^m(\mathcal{I}_n^\bullet(U)) = H^m(U, \mathcal{I}_n^\bullet)$ as $\mathcal{I}_n^\bullet$ is a bounded below complex of injectives. Unwinding the long exact sequence of cohomology associated to the distinguished triangle above this implies that

$$H^m(\mathcal{I}_n^\bullet(U)) \to H^m(\mathcal{I}_{n-1}^\bullet(U))$$

[2] In fact, analyzing the proof we see that it suffices if there exists a function $d : \mathbf{Z} \to \mathbf{Z} \cup \{+\infty\}$ such that $H^p(U, \mathcal{H}^q) = 0$ for $p > d(q)$ where $q + d(q) \to -\infty$ as $q \to -\infty$

is an isomorphism for $m > d - n$, i.e., $n > d - m$ and we win. □

Lemma 20.29.3. *With assumptions and notation as in Lemma 20.29.2. Let K denote the object of $D(\mathcal{O}_X)$ represented by the complex $\mathcal{F}^\bullet$. Then there exists a distinguished triangle*

$$K \to \prod\nolimits_{n\geq 0} \tau_{\geq -n}K \to \prod\nolimits_{n\geq 0} \tau_{\geq -n}K \to K[1]$$

in $D(\mathcal{O}_X)$. In other words, K is the derived limit of its canonical truncations.

Proof. The proof of Injectives, Lemma 19.13.4 shows that $\prod \tau_{\geq -n}K$ is represented by the complex $\prod \mathcal{I}_n^\bullet$. Because the transition maps $\mathcal{I}_{n+1}^\bullet \to \mathcal{I}_n^\bullet$ are termwise split surjections, we have a short exact sequence of complexes

$$0 \to \mathcal{I}^\bullet \to \prod \mathcal{I}_n^\bullet \to \prod \mathcal{I}_n^\bullet \to 0$$

Since $\mathcal{I}^\bullet$ represents K by Lemma 20.29.2 the distinguished triangle of the lemma is the distinguished triangle associated to the short exact sequence above (Derived Categories, Lemma 13.11.1). □

20.30. Čech cohomology of unbounded complexes

The construction of Section 20.24 isn't the "correct" one for unbounded complexes. The problem is that in the Stacks project we use direct sums in the totalization of a double complex and we would have to replace this by a product. Instead of doing so in this section we assume the covering is finite and we use the alternating Čech complex.

Let $(X, \mathcal{O}_X)$ be a ringed space. Let $\mathcal{F}^\bullet$ be a complex of presheaves of $\mathcal{O}_X$-modules. Let $\mathcal{U} : X = \bigcup_{i\in I} U_i$ be a **finite** open covering of X. Since the alternating Čech complex $\check{\mathcal{C}}^\bullet_{alt}(\mathcal{U}, \mathcal{F})$ (Section 20.22) is functorial in the presheaf $\mathcal{F}$ we obtain a double complex $\check{\mathcal{C}}^\bullet_{alt}(\mathcal{U}, \mathcal{F}^\bullet)$. In this section we work with the associated total complex. The construction of $\mathrm{Tot}(\check{\mathcal{C}}^\bullet_{alt}(\mathcal{U}, \mathcal{F}^\bullet))$ is functorial in $\mathcal{F}^\bullet$. As well there is a functorial transformation

$$\Gamma(X, \mathcal{F}^\bullet) \longrightarrow \mathrm{Tot}(\check{\mathcal{C}}^\bullet_{alt}(\mathcal{U}, \mathcal{F}^\bullet)) \tag{20.30.0.1}$$

of complexes defined by the following rule: The section $s \in \Gamma(X, \mathcal{F}^n)$ is mapped to the element $\alpha = \{\alpha_{i_0\ldots i_p}\}$ with $\alpha_{i_0} = s|_{U_{i_0}}$ and $\alpha_{i_0\ldots i_p} = 0$ for $p > 0$.

Lemma 20.30.1. *Let $(X, \mathcal{O}_X)$ be a ringed space. Let $\mathcal{U} : X = \bigcup_{i\in I} U_i$ be a finite open covering. For a complex $\mathcal{F}^\bullet$ of $\mathcal{O}_X$-modules there is a canonical map*

$$Tot(\check{\mathcal{C}}^\bullet_{alt}(\mathcal{U}, \mathcal{F}^\bullet)) \longrightarrow R\Gamma(X, \mathcal{F}^\bullet)$$

functorial in $\mathcal{F}^\bullet$ and compatible with (20.30.0.1).

Proof. Let $\mathcal{I}^\bullet$ be a K-injective complex whose terms are injective $\mathcal{O}_X$-modules. The map (20.30.0.1) for $\mathcal{I}^\bullet$ is a map $\Gamma(X, \mathcal{I}^\bullet) \to \mathrm{Tot}(\check{\mathcal{C}}^\bullet_{alt}(\mathcal{U}, \mathcal{I}^\bullet))$. This is a quasi-isomorphism of complexes of abelian groups as follows from Homology, Lemma 12.22.6 applied to the double complex $\check{\mathcal{C}}^\bullet_{alt}(\mathcal{U}, \mathcal{I}^\bullet)$ using Lemmas 20.12.1 and 20.22.6. Suppose $\mathcal{F}^\bullet \to \mathcal{I}^\bullet$ is a quasi-isomorphism of $\mathcal{F}^\bullet$ into a K-injective complex whose terms are injectives (Injectives, Theorem 19.12.6). Since $R\Gamma(X, \mathcal{F}^\bullet)$ is represented by the complex $\Gamma(X, \mathcal{I}^\bullet)$ we obtain the map of the lemma using

$$\mathrm{Tot}(\check{\mathcal{C}}^\bullet_{alt}(\mathcal{U}, \mathcal{F}^\bullet)) \longrightarrow \mathrm{Tot}(\check{\mathcal{C}}^\bullet_{alt}(\mathcal{U}, \mathcal{I}^\bullet)).$$

We omit the verification of functoriality and compatibilities. □

Lemma 20.30.2. *Let $(X, \mathcal{O}_X)$ be a ringed space. Let $\mathcal{U} : X = \bigcup_{i \in I} U_i$ be a finite open covering. Let $\mathcal{F}^\bullet$ be a complex of $\mathcal{O}_X$-modules. Let $\mathcal{B}$ be a set of open subsets of X. Assume*

(1) *every open in X has a covering whose members are elements of $\mathcal{B}$,*
(2) *we have $U_{i_0 \ldots i_p} \in \mathcal{B}$ for all $i_0, \ldots, i_p \in I$,*
(3) *for every $U \in \mathcal{B}$ and $p > 0$ we have*
   (a) *$H^p(U, \mathcal{F}^q) = 0$,*
   (b) *$H^p(U, Coker(\mathcal{F}^{q-1} \to \mathcal{F}^q)) = 0$, and*
   (c) *$H^p(U, H^q(\mathcal{F})) = 0$.*

Then the map

$$Tot(\check{\mathcal{C}}^\bullet_{alt}(\mathcal{U}, \mathcal{F}^\bullet)) \longrightarrow R\Gamma(X, \mathcal{F}^\bullet)$$

of Lemma 20.30.1 is an isomorphism in $D(Ab)$.

Proof. If $\mathcal{F}^\bullet$ is bounded below, this follows from assumption (3)(a) and the spectral sequence of Lemma 20.24.1 and the fact that

$$\mathrm{Tot}(\check{\mathcal{C}}^\bullet_{alt}(\mathcal{U}, \mathcal{F}^\bullet)) \longrightarrow \mathrm{Tot}(\check{\mathcal{C}}^\bullet(\mathcal{U}, \mathcal{F}^\bullet))$$

is a quasi-isomorphism by Lemma 20.22.6 (some details omitted). In general, by assumption (3)(c) we may choose a resolution $\mathcal{F}^\bullet \to \mathcal{I}^\bullet = \lim \mathcal{I}^\bullet_n$ as in Lemma 20.29.2. Then the map of the lemma becomes

$$\lim_n \mathrm{Tot}(\check{\mathcal{C}}^\bullet_{alt}(\mathcal{U}, \tau_{\geq -n}\mathcal{F}^\bullet)) \longrightarrow \lim_n \Gamma(X, \mathcal{I}^\bullet_n)$$

Note that (3)(b) shows that $\tau_{\geq -n}\mathcal{F}^\bullet$ is a bounded below complex satisfying the hypothesis of the lemma. Thus the case of bounded below complexes shows each of the maps

$$\mathrm{Tot}(\check{\mathcal{C}}^\bullet_{alt}(\mathcal{U}, \tau_{\geq -n}\mathcal{F}^\bullet)) \longrightarrow \Gamma(X, \mathcal{I}^\bullet_n)$$

is a quasi-isomorphism. The cohomologies of the complexes on the left hand side in given degree are eventually constant (as the alternating Čech complex is finite). Hence the same is true on the right hand side. Thus the cohomology of the limit on the right hand side is this constant value by Homology, Lemma 12.26.7 and we win. □

20.31. Internal hom in the derived category

Let $(X, \mathcal{O}_X)$ be a ringed space. Let L, M be objects of $D(\mathcal{O}_X)$. We would like to construct an object $R\mathcal{H}om(L, M)$ of $D(\mathcal{O}_X)$ such that for every third object K of $D(\mathcal{O}_X)$ there exists a canonical bijection

$$\mathrm{Hom}_{D(\mathcal{O}_X)}(K, R\mathcal{H}om(L, M)) = \mathrm{Hom}_{D(\mathcal{O}_X)}(K \otimes^{\mathbf{L}}_{\mathcal{O}_X} L, M) \tag{20.31.0.1}$$

Observe that this formula defines $R\mathcal{H}om(L, M)$ up to unique isomorphism by the Yoneda lemma (Categories, Lemma 4.3.5).

To construct such an object, choose a K-injective complex $\mathcal{I}^\bullet$ representing M and any complex of $\mathcal{O}_X$-modules $\mathcal{L}^\bullet$ representing L. Consider the complex $\mathcal{H}^\bullet$ of $\mathcal{O}_X$-modules with term

$$\mathcal{H}^n = \prod\nolimits_{n=-p+q} \mathcal{H}om_{\mathcal{O}_X}(\mathcal{L}^p, \mathcal{I}^q)$$

in degree n and differential given on a local section $\alpha = \prod \alpha_{p,q}$ by the rule

$$\alpha_{p,q} \longmapsto \alpha_{p,q} \circ d^{p-1}_{\mathcal{L}} + (-1)^p d^q_{\mathcal{I}} \circ \alpha_{p,q}$$

With this definition we have

$$H^0(\mathcal{H}^\bullet(U)) = \mathrm{Hom}_{K(\mathcal{O}_U)}(\mathcal{L}^\bullet|_U, \mathcal{I}^\bullet|_U) = \mathrm{Hom}_{D(\mathcal{O}_U)}(\mathcal{L}^\bullet|_U, \mathcal{I}^\bullet|_U)$$

for every open $U \subset X$. The second equality comes from the fact that $\mathcal{I}^\bullet|_U$ is K-injective (Lemma 20.28.1). Hence the sheafification $H^0(\mathcal{H}^\bullet)$ of the presheaf $U \mapsto H^0(\mathcal{H}^\bullet(U))$ is independent of the choices made in the complexes representing L and M. A similar argument can be made for the other cohomology sheaves $H^i(\mathcal{H}^\bullet)$. Thus we obtain a well defined object $R\mathcal{H}om(L, M)$ of $D(\mathcal{O}_X)$. Note that the construction is functorial in L and M in an obvious manner.

Lemma 20.31.1. *Let $(X, \mathcal{O}_X)$ be a ringed space. Let K, L, M be objects of $D(\mathcal{O}_X)$. With the construction as described above there is a canonical isomorphism (20.31.0.1) functorial in K, L, M.*

Proof. Choose a K-injective complex $\mathcal{I}^\bullet$ representing M and a K-flat complex of $\mathcal{O}_X$-modules $\mathcal{L}^\bullet$ representing L. Let $\mathcal{H}^\bullet$ be the complex described above. For any complex of $\mathcal{O}_X$-modules $\mathcal{K}^\bullet$ we see that

$$\mathrm{Hom}_{K(\mathcal{O}_X)}(\mathcal{K}^\bullet, \mathcal{H}^\bullet) = \mathrm{Hom}_{K(\mathcal{O}_X)}(\mathrm{Tot}(\mathcal{K}^\bullet \otimes_{\mathcal{O}_X} \mathcal{L}^\bullet), \mathcal{I}^\bullet)$$

simply because

$$\mathrm{Hom}_{\mathcal{O}_X}(\mathcal{K}^a, \mathcal{H}om_{\mathcal{O}_X}(\mathcal{L}^b, \mathcal{I}^c)) = \mathrm{Hom}_{\mathcal{O}_X}(\mathcal{K}^a \otimes_{\mathcal{O}_X} \mathcal{L}^b, \mathcal{I}^c)$$

by Modules, Lemma 17.19.1. Since $\mathcal{L}^\bullet$ is K-flat, the complex $\mathrm{Tot}(\mathcal{K}^\bullet \otimes_{\mathcal{O}_X} \mathcal{L}^\bullet)$ (a) is acyclic if $\mathcal{K}^\bullet$ is acyclic, and (b) represents the derived tensor product $K \otimes^{\mathbf{L}}_{\mathcal{O}_X} L$. Thus $\mathcal{H}^\bullet$ is a K-injective complex, as $\mathcal{I}^\bullet$ is K-injective. Hence the first displayed formula of this proof implies (20.31.0.1) in $D(\mathcal{O}_X)$ which is what we wanted to prove. □

Lemma 20.31.2. *Let $(X, \mathcal{O}_X)$ be a ringed space. Let K, L be objects of $D(\mathcal{O}_X)$. For every open U we have*

$$\mathrm{Hom}_{D(\mathcal{O}_U)}(K|_U, L|_U) = H^0(U, R\mathcal{H}om(K, L))$$

Proof. This is clear from the discussion above, but it can also be seen by the following argument using only (20.31.0.1). Namely, we have

$$\begin{aligned} H^0(U, R\mathcal{H}om(K, L)) &= \mathrm{Hom}_{D(\mathcal{O}_X)}(j_!\mathcal{O}_U, R\mathcal{H}om(K, L)) \\ &= \mathrm{Hom}_{D(\mathcal{O}_X)}(j_!\mathcal{O}_U \otimes^{\mathbf{L}}_{\mathcal{O}_X} K, L) \\ &= \mathrm{Hom}_{D(\mathcal{O}_X)}(j_!(K|_U), L) \\ &= \mathrm{Hom}_{D(\mathcal{O}_U)}(K|_U, L|_U) \end{aligned}$$

The first and last equalities by Lemma 20.28.3. The second equality by Lemma 20.31.1. The remaining equality holds because $j_!\mathcal{O}_U \otimes^{\mathbf{L}}_{\mathcal{O}_X} K = j_!(K|_U)$. To see this use that $j_!\mathcal{O}_U$ is flat, see Modules, Lemma 17.16.5. Some details omitted. □

Lemma 20.31.3. *Let $(X, \mathcal{O}_X)$ be a ringed space. Let K, L be objects of $D(\mathcal{O}_X)$. The construction of $R\mathcal{H}om(K, L)$ commutes with restrictions to opens, i.e., for every open U we have $R\mathcal{H}om(K|_U, L|_U) = R\mathcal{H}om(K, L)|_U$.*

Proof. Omitted. Hint: Use Lemma 20.28.1. □

Lemma 20.31.4. *Let $(X, \mathcal{O}_X)$ be a ringed space. The bifunctor $R\mathcal{H}om(-, -)$ transforms distinguished triangles into distinguished triangles in both variables.*

Proof. Omitted. Hint: Prove that the construction above transforms a termwise split exact sequence of complexes in the first variable into a termwise split exact sequence of complexes and that it transforms a cone of a morphism between K-injective complexes in the second variable into the corresponding cone. □

Lemma 20.31.5. *Let $(X, \mathcal{O}_X)$ be a ringed space. Let L be an object of $D(\mathcal{O}_X)$. Set $L^\wedge = R\mathcal{H}om(L, \mathcal{O}_X)$. For M in $D(\mathcal{O}_X)$ there is a canonical map*

$$L^\wedge \otimes^{\mathbf{L}}_{\mathcal{O}_X} M \longrightarrow R\mathcal{H}om(L, M) \tag{20.31.5.1}$$

which induces a canonical map

$$H^0(X, L^\wedge \otimes^{\mathbf{L}}_{\mathcal{O}_X} M) \longrightarrow \mathrm{Hom}_{D(\mathcal{O}_X)}(L, M)$$

functorial in M in $D(\mathcal{O}_X)$.

Proof. There is a canonical map

$$ev : L^\wedge \otimes^{\mathbf{L}}_{\mathcal{O}_X} L \longrightarrow \mathcal{O}_X$$

coming from the identification $L^\wedge \to R\mathcal{H}om(L, \mathcal{O}_X)$ by the equality (20.31.0.1) proved in Lemma 20.31.1. This gives a map

$$L^\wedge \otimes^{\mathbf{L}} M \otimes^{\mathbf{L}} L = M \otimes^{\mathbf{L}} L^\wedge \otimes^{\mathbf{L}} L \xrightarrow{1_M \otimes ev} M$$

which by (20.31.0.1) with $K = L^\wedge \otimes^{\mathbf{L}} M$ corresponds to a map as in (20.31.5.1). □

Remark 20.31.6. Let $h : X \to Y$ be a morphism of ringed spaces. Let K, L be objects of $D(\mathcal{O}_Y)$. We claim there is a canonical map

$$Lh^* R\mathcal{H}om(K, L) \longrightarrow R\mathcal{H}om(Lh^*K, Lh^*L)$$

in $D(\mathcal{O}_X)$. Namely, by (20.31.0.1) proved in Lemma 20.31.1 such a map is the same thing as a map

$$Lh^* R\mathcal{H}om(K, L) \otimes^{\mathbf{L}} Lh^*K \longrightarrow Lh^*L$$

The source of this arrow is $Lh^*(\mathcal{H}om(K, L) \otimes^{\mathbf{L}} K)$ by Lemma 20.26.2 hence it suffices to construct a canonical map

$$R\mathcal{H}om(K, L) \otimes^{\mathbf{L}} K \longrightarrow L.$$

For this we take the arrow corresponding to

$$\mathrm{id} : R\mathcal{H}om(K, L) \longrightarrow R\mathcal{H}om(K, L)$$

via (20.31.0.1).

Remark 20.31.7. Suppose that

$$\begin{array}{ccc} X' & \xrightarrow[h]{} & X \\ {\scriptstyle f'}\downarrow & & \downarrow{\scriptstyle f} \\ S' & \xrightarrow{g} & S \end{array}$$

is a commutative diagram of ringed spaces. Let K, L be objects of $D(\mathcal{O}_X)$. We claim there exists a canonical base change map

$$Lg^* Rf_* R\mathcal{H}om(K, L) \longrightarrow R(f')_* R\mathcal{H}om(Lh^*K, Lh^*L)$$

in $D(\mathcal{O}_{S'})$. Namely, we take the map adjoint to the composition

$$\begin{aligned} L(f')^*Lg^*Rf_*R\mathcal{H}om(K,L) &= Lh^*Lf^*Rf_*R\mathcal{H}om(K,L) \\ &\to Lh^*R\mathcal{H}om(K,L) \\ &\to R\mathcal{H}om(Lh^*K,Lh^*L) \end{aligned}$$

where the first arrow uses the adjunction mapping $Lf^*Rf_* \to \mathrm{id}$ and the second arrow is the canonical map constructed in Remark 20.31.6.

20.32. Strictly perfect complexes

Strictly perfect complexes of modules are used to define the notions of pseudo-coherent and perfect complexes later on. They are defined as follows.

Definition 20.32.1. Let $(X, \mathcal{O}_X)$ be a ringed space. Let $\mathcal{E}^\bullet$ be a complex of $\mathcal{O}_X$-modules. We say $\mathcal{E}^\bullet$ is *strictly perfect* if $\mathcal{E}^i$ is zero for all but finitely many i and $\mathcal{E}^i$ is a direct summand of a finite free $\mathcal{O}_X$-module for all i.

Warning: Since we do not assume that X is a locally ringed space, it may not be true that a direct summand of a finite free $\mathcal{O}_X$-module is finite locally free.

Lemma 20.32.2. *The cone on a morphism of strictly perfect complexes is strictly perfect.*

Proof. This is immediate from the definitions. □

Lemma 20.32.3. *The total complex associated to the tensor product of two strictly perfect complexes is strictly perfect.*

Proof. Omitted. □

Lemma 20.32.4. *Let $f : (X, \mathcal{O}_X) \to (Y, \mathcal{O}_Y)$ be a morphism of ringed spaces. If $\mathcal{F}^\bullet$ is a strictly perfect complex of $\mathcal{O}_Y$-modules, then $f^*\mathcal{F}^\bullet$ is a strictly perfect complex of $\mathcal{O}_X$-modules.*

Proof. The pullback of a finite free module is finite free. The functor f^* is additive functor hence preserves direct summands. The lemma follows. □

Lemma 20.32.5. *Let $(X, \mathcal{O}_X)$ be a ringed space. Given a solid diagram of $\mathcal{O}_X$-modules*

$$\begin{array}{ccc} \mathcal{E} & \longrightarrow & \mathcal{F} \\ & \searrow & \uparrow{\scriptstyle p} \\ & & \mathcal{G} \end{array}$$

with $\mathcal{E}$ a direct summand of a finite free $\mathcal{O}_X$-module and p surjective, then a dotted arrow making the diagram commute exists locally on X.

Proof. We may assume $\mathcal{E} = \mathcal{O}_X^{\oplus n}$ for some n. In this case finding the dotted arrow is equivalent to lifting the images of the basis elements in $\Gamma(X, \mathcal{F})$. This is locally possible by the characterization of surjective maps of sheaves (Sheaves, Section 6.16). □

Lemma 20.32.6. *Let $(X, \mathcal{O}_X)$ be a ringed space.*

(1) *Let $\alpha : \mathcal{E}^\bullet \to \mathcal{F}^\bullet$ be a morphism of complexes of $\mathcal{O}_X$-modules with $\mathcal{E}^\bullet$ strictly perfect and $\mathcal{F}^\bullet$ acyclic. Then α is locally on X homotopic to zero.*

(2) *Let $\alpha : \mathcal{E}^\bullet \to \mathcal{F}^\bullet$ be a morphism of complexes of $\mathcal{O}_X$-modules with $\mathcal{E}^\bullet$ strictly perfect, $\mathcal{E}^i = 0$ for $i < a$, and $H^i(\mathcal{F}^\bullet) = 0$ for $i \geq a$. Then α is locally on X homotopic to zero.*

Proof. The first statement follows from the second, hence we only prove (2). We will prove this by induction on the length of the complex $\mathcal{E}^\bullet$. If $\mathcal{E}^\bullet \cong \mathcal{E}[-n]$ for some direct summand $\mathcal{E}$ of a finite free $\mathcal{O}_X$-module and integer $n \geq a$, then the result follows from Lemma 20.32.5 and the fact that $\mathcal{F}^{n-1} \to \mathrm{Ker}(\mathcal{F}^n \to \mathcal{F}^{n+1})$ is surjective by the assumed vanishing of $H^n(\mathcal{F}^\bullet)$. If $\mathcal{E}^i$ is zero except for $i \in [a, b]$, then we have a split exact sequence of complexes

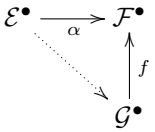
$$0 \to \mathcal{E}^b[-b] \to \mathcal{E}^\bullet \to \sigma_{\leq b-1}\mathcal{E}^\bullet \to 0$$

which determines a distinguished triangle in $K(\mathcal{O}_X)$. Hence an exact sequence

$$\mathrm{Hom}_{K(\mathcal{O}_X)}(\sigma_{\leq b-1}\mathcal{E}^\bullet, \mathcal{F}^\bullet) \to \mathrm{Hom}_{K(\mathcal{O}_X)}(\mathcal{E}^\bullet, \mathcal{F}^\bullet) \to \mathrm{Hom}_{K(\mathcal{O}_X)}(\mathcal{E}^b[-b], \mathcal{F}^\bullet)$$

by the axioms of triangulated categories. The composition $\mathcal{E}^b[-b] \to \mathcal{F}^\bullet$ is locally homotopic to zero, whence we may assume our map comes from an element in the left hand side of the displayed exact sequence above. This element is locally zero by induction hypothesis. □

Lemma 20.32.7. *Let $(X, \mathcal{O}_X)$ be a ringed space. Given a solid diagram of complexes of $\mathcal{O}_X$-modules*



with $\mathcal{E}^\bullet$ strictly perfect, $\mathcal{E}^j = 0$ for $j < a$ and $H^j(f)$ an isomorphism for $j > a$ and surjective for $j = a$, then a dotted arrow making the diagram commute up to homotopy exists locally on X.

Proof. Our assumptions on f imply the cone $C(f)^\bullet$ has vanishing cohomology sheaves in degrees $\geq a$. Hence Lemma 20.32.6 guarantees there is an open covering $X = \bigcup U_i$ such that the composition $\mathcal{E}^\bullet \to \mathcal{F}^\bullet \to C(f)^\bullet$ is homotopic to zero over U_i. Since

$$\mathcal{G}^\bullet \to \mathcal{F}^\bullet \to C(f)^\bullet \to \mathcal{G}^\bullet[1]$$

restricts to a distinguished triangle in $K(\mathcal{O}_{U_i})$ we see that we can lift $\alpha|_{U_i}$ up to homotopy to a map $\alpha_i : \mathcal{E}^\bullet|_{U_i} \to \mathcal{G}^\bullet|_{U_i}$ as desired. □

Lemma 20.32.8. *Let $(X, \mathcal{O}_X)$ be a ringed space. Let $\mathcal{E}^\bullet$, $\mathcal{F}^\bullet$ be complexes of $\mathcal{O}_X$-modules with $\mathcal{E}^\bullet$ strictly perfect.*

(1) *For any element $\alpha \in \mathrm{Hom}_{D(\mathcal{O}_X)}(\mathcal{E}^\bullet, \mathcal{F}^\bullet)$ there exists an open covering $X = \bigcup U_i$ such that $\alpha|_{U_i}$ is given by a morphism of complexes $\alpha_i : \mathcal{E}^\bullet|_{U_i} \to \mathcal{F}^\bullet|_{U_i}$.*

(2) *Given a morphism of complexes $\alpha : \mathcal{E}^\bullet \to \mathcal{F}^\bullet$ whose image in the group $\mathrm{Hom}_{D(\mathcal{O}_X)}(\mathcal{E}^\bullet, \mathcal{F}^\bullet)$ is zero, there exists an open covering $X = \bigcup U_i$ such that $\alpha|_{U_i}$ is homotopic to zero.*

Proof. Proof of (1). By the construction of the derived category we can find a quasi-isomorphism $f : \mathcal{F}^\bullet \to \mathcal{G}^\bullet$ and a map of complexes $\beta : \mathcal{E}^\bullet \to \mathcal{G}^\bullet$ such that $\alpha = f^{-1}\beta$. Thus the result follows from Lemma 20.32.7. We omit the proof of (2). □

Lemma 20.32.9. *Let $(X, \mathcal{O}_X)$ be a ringed space. Let $\mathcal{E}^\bullet$, $\mathcal{F}^\bullet$ be complexes of $\mathcal{O}_X$-modules with $\mathcal{E}^\bullet$ strictly perfect. Then the internal hom $R\mathcal{H}om(\mathcal{E}^\bullet, \mathcal{F}^\bullet)$ is represented by the complex $\mathcal{H}^\bullet$ with terms*

$$\mathcal{H}^n = \bigoplus\nolimits_{n=-p+q} \mathcal{H}om_{\mathcal{O}_X}(\mathcal{E}^p, \mathcal{F}^q)$$

and differential as described in Section 20.31.

Proof. Choose a quasi-isomorphism $\mathcal{F}^\bullet \to \mathcal{I}^\bullet$ into a K-injective complex. Let $(\mathcal{H}')^\bullet$ be the complex with terms

$$(\mathcal{H}')^n = \prod\nolimits_{n=-p+q} \mathcal{H}om_{\mathcal{O}_X}(\mathcal{L}^p, \mathcal{I}^q)$$

which represents $R\mathcal{H}om(\mathcal{E}^\bullet, \mathcal{F}^\bullet)$ by the construction in Section 20.31. It suffices to show that the map

$$\mathcal{H}^\bullet \longrightarrow (\mathcal{H}')^\bullet$$

is a quasi-isomorphism. Given an open $U \subset X$ we have by inspection

$$H^0(\mathcal{H}^\bullet(U)) = \mathrm{Hom}_{K(\mathcal{O}_U)}(\mathcal{E}^\bullet|_U, \mathcal{K}^\bullet|_U) \to H^0((\mathcal{H}')^\bullet(U)) = \mathrm{Hom}_{D(\mathcal{O}_U)}(\mathcal{E}^\bullet|_U, \mathcal{K}^\bullet|_U)$$

By Lemma 20.32.8 the sheafification of $U \mapsto H^0(\mathcal{H}^\bullet(U))$ is equal to the sheafification of $U \mapsto H^0((\mathcal{H}')^\bullet(U))$. A similar argument can be given for the other cohomology sheaves. Thus $\mathcal{H}^\bullet$ is quasi-isomorphic to $(\mathcal{H}')^\bullet$ which proves the lemma. □

Lemma 20.32.10. *Let $(X, \mathcal{O}_X)$ be a ringed space. Let $\mathcal{E}^\bullet$, $\mathcal{F}^\bullet$ be complexes of $\mathcal{O}_X$-modules with*

1. *$\mathcal{F}^n = 0$ for $n \ll 0$,*
2. *$\mathcal{E}^n = 0$ for $n \gg 0$, and*
3. *$\mathcal{E}^n$ isomorphic to a direct summand of a finite free $\mathcal{O}_X$-module.*

Then the internal hom $R\mathcal{H}om(\mathcal{E}^\bullet, \mathcal{F}^\bullet)$ is represented by the complex $\mathcal{H}^\bullet$ with terms

$$\mathcal{H}^n = \bigoplus\nolimits_{n=-p+q} \mathcal{H}om_{\mathcal{O}_X}(\mathcal{E}^p, \mathcal{F}^q)$$

and differential as described in Section 20.31.

Proof. Choose a quasi-isomorphism $\mathcal{F}^\bullet \to \mathcal{I}^\bullet$ where $\mathcal{I}^\bullet$ is a bounded below complex of injectives. Note that $\mathcal{I}^\bullet$ is K-injective (Derived Categories, Lemma 13.28.4). Hence the construction in Section 20.31 shows that $R\mathcal{H}om(\mathcal{E}^\bullet, \mathcal{F}^\bullet)$ is represented by the complex $(\mathcal{H}')^\bullet$ with terms

$$(\mathcal{H}')^n = \prod\nolimits_{n=-p+q} \mathcal{H}om_{\mathcal{O}_X}(\mathcal{E}^p, \mathcal{I}^q) = \bigoplus\nolimits_{n=-p+q} \mathcal{H}om_{\mathcal{O}_X}(\mathcal{E}^p, \mathcal{I}^q)$$

(equality because there are only finitely many nonzero terms). Note that $\mathcal{H}^\bullet$ is the total complex associated to the double complex with terms $\mathcal{H}om_{\mathcal{O}_X}(\mathcal{E}^p, \mathcal{F}^q)$ and similarly for $(\mathcal{H}')^\bullet$. The natural map $(\mathcal{H}')^\bullet \to \mathcal{H}^\bullet$ comes from a map of double complexes. Thus to show this map is a quasi-isomorphism, we may use the spectral sequence of a double complex (Homology, Lemma 12.22.5)

$${}'E_1^{p,q} = H^q(\mathcal{H}om_{\mathcal{O}_X}(\mathcal{E}^p, \mathcal{F}^\bullet))$$

converging to $H^{p+q}(\mathcal{H}^\bullet)$ and similarly for $(\mathcal{H}')^\bullet$. To finish the proof of the lemma it suffices to show that $\mathcal{F}^\bullet \to \mathcal{I}^\bullet$ induces an isomorphism

$$H^q(\mathcal{H}om_{\mathcal{O}_X}(\mathcal{E}, \mathcal{F}^\bullet)) \longrightarrow H^q(\mathcal{H}om_{\mathcal{O}_X}(\mathcal{E}, \mathcal{I}^\bullet))$$

on cohomology sheaves whenever $\mathcal{E}$ is a direct summand of a finite free $\mathcal{O}_X$-module. Since this is clear when $\mathcal{E}$ is finite free the result follows. □

20.33. Pseudo-coherent modules

In this section we discuss pseudo-coherent complexes.

Definition 20.33.1. Let $(X, \mathcal{O}_X)$ be a ringed space. Let $\mathcal{E}^\bullet$ be a complex of $\mathcal{O}_X$-modules. Let $m \in \mathbf{Z}$.

(1) We say $\mathcal{E}^\bullet$ is *m-pseudo-coherent* if there exists an open covering $X = \bigcup U_i$ and for each i a morphism of complexes $\alpha_i : \mathcal{E}_i^\bullet \to \mathcal{E}^\bullet|_{U_i}$ where $\mathcal{E}_i$ is strictly perfect on U_i and $H^j(\alpha_i)$ is an isomorphism for $j > m$ and $H^m(\alpha_i)$ is surjective.
(2) We say $\mathcal{E}^\bullet$ is *pseudo-coherent* if it is m-pseudo-coherent for all m.
(3) We say an object E of $D(\mathcal{O}_X)$ is *m-pseudo-coherent* (resp. *pseudo-coherent*) if and only if it can be represented by a m-pseudo-coherent (resp. pseudo-coherent) complex of $\mathcal{O}_X$-modules.

If X is quasi-compact, then an m-pseudo-coherent object of $D(\mathcal{O}_X)$ is in $D^-(\mathcal{O}_X)$. But this need not be the case if X is not quasi-compact.

Lemma 20.33.2. *Let $(X, \mathcal{O}_X)$ be a ringed space. Let E be an object of $D(\mathcal{O}_X)$.*

(1) *If there exists an open covering $X = \bigcup U_i$, strictly perfect complexes $\mathcal{E}_i^\bullet$ on U_i, and maps $\alpha_i : \mathcal{E}_i^\bullet \to E|_{U_i}$ in $D(\mathcal{O}_{U_i})$ with $H^j(\alpha_i)$ an isomorphism for $j > m$ and $H^m(\alpha_i)$ surjective, then E is m-pseudo-coherent.*
(2) *If E is m-pseudo-coherent, then any complex representing E is m-pseudo-coherent.*

Proof. Let $\mathcal{F}^\bullet$ be any complex representing E and let $X = \bigcup U_i$ and $\alpha_i : \mathcal{E}_i \to E|_{U_i}$ be as in (1). We will show that $\mathcal{F}^\bullet$ is m-pseudo-coherent as a complex, which will prove (1) and (2) simultaneously. By Lemma 20.32.8 we can after refining the open covering $X = \bigcup U_i$ represent the maps α_i by maps of complexes $\alpha_i : \mathcal{E}_i^\bullet \to \mathcal{F}^\bullet|_{U_i}$. By assumption $H^j(\alpha_i)$ are isomorphisms for $j > m$, and $H^m(\alpha_i)$ is surjective whence $\mathcal{F}^\bullet$ is m-pseudo-coherent. □

Lemma 20.33.3. *Let $f : (X, \mathcal{O}_X) \to (Y, \mathcal{O}_Y)$ be a morphism of ringed spaces. Let E be an object of $D(\mathcal{O}_Y)$. If E is m-pseudo-coherent, then Lf^*E is m-pseudo-coherent.*

Proof. Represent E by a complex $\mathcal{E}^\bullet$ of $\mathcal{O}_Y$-modules and choose an open covering $Y = \bigcup V_i$ and $\alpha_i : \mathcal{E}_i^\bullet \to \mathcal{E}^\bullet|_{V_i}$ as in Definition 20.33.1. Set $U_i = f^{-1}(V_i)$. By Lemma 20.33.2 it suffices to show that $Lf^*\mathcal{E}^\bullet|_{U_i}$ is m-pseudo-coherent. Choose a distinguished triangle

$$\mathcal{E}_i^\bullet \to \mathcal{E}^\bullet|_{V_i} \to C \to \mathcal{E}_i^\bullet[1]$$

The assumption on α_i means exactly that the cohomology sheaves $H^j(C)$ are zero for all $j \geq m$. Denote $f_i : U_i \to V_i$ the restriction of f. Note that $Lf^*\mathcal{E}^\bullet|_{U_i} = Lf_i^*(\mathcal{E}|_{V_i})$. Applying Lf_i^* we obtain the distinguished triangle

$$Lf_i^*\mathcal{E}_i^\bullet \to Lf_i^*\mathcal{E}|_{V_i} \to Lf_i^*C \to Lf_i^*\mathcal{E}_i^\bullet[1]$$

By the construction of Lf_i^* as a left derived functor we see that $H^j(Lf_i^*C) = 0$ for $j \geq m$ (by the dual of Derived Categories, Lemma 13.16.1). Hence $H^j(Lf_i^*\alpha_i)$ is an isomorphism for $j > m$ and $H^m(Lf^*\alpha_i)$ is surjective. On the other hand, $Lf_i^*\mathcal{E}_i^\bullet = f_i^*\mathcal{E}_i^\bullet$. is strictly perfect by Lemma 20.32.4. Thus we conclude. □

Lemma 20.33.4. *Let $(X, \mathcal{O}_X)$ be a ringed space and $m \in \mathbf{Z}$. Let (K, L, M, f, g, h) be a distinguished triangle in $D(\mathcal{O}_X)$.*

(1) *If K is $(m+1)$-pseudo-coherent and L is m-pseudo-coherent then M is m-pseudo-coherent.*
(2) *If K anf M are m-pseudo-coherent, then L is m-pseudo-coherent.*
(3) *If L is $(m+1)$-pseudo-coherent and M is m-pseudo-coherent, then K is $(m+1)$-pseudo-coherent.*

Proof. Proof of (1). Choose an open covering $X = \bigcup U_i$ and maps $\alpha_i : \mathcal{K}_i^\bullet \to K|_{U_i}$ in $D(\mathcal{O}_{U_i})$ with $\mathcal{K}_i^\bullet$ strictly perfect and $H^j(\alpha_i)$ isomorphisms for $j > m+1$ and surjective for $j = m+1$. We may replace $\mathcal{K}_i^\bullet$ by $\sigma_{\geq m+1}\mathcal{K}_i^\bullet$ and hence we may assume that $\mathcal{K}_i^j = 0$ for $j < m+1$. After refining the open covering we may choose maps $\beta_i : \mathcal{L}_i^\bullet \to L|_{U_i}$ in $D(\mathcal{O}_{U_i})$ with $\mathcal{L}_i^\bullet$ strictly perfect such that $H^j(\beta)$ is an isomorphism for $j > m$ and surjective for $j = m$. By Lemma 20.32.7 we can, after refining the covering, find maps of complexes $\gamma_i : \mathcal{K}^\bullet \to \mathcal{L}^\bullet$ such that the diagrams

$$\begin{array}{ccc} K|_{U_i} & \longrightarrow & L|_{U_i} \\ \uparrow{\scriptstyle \alpha_i} & & \uparrow{\scriptstyle \beta_i} \\ \mathcal{K}_i^\bullet & \xrightarrow{\gamma_i} & \mathcal{L}_i^\bullet \end{array}$$

are commutative in $D(\mathcal{O}_{U_i})$ (this requires representing the maps α_i, β_i and $K|_{U_i} \to L|_{U_i}$ by actual maps of complexes; some details omitted). The cone $C(\gamma_i)^\bullet$ is strictly perfect (Lemma 20.32.2). The commutativity of the diagram implies that there exists a morphism of distinguished triangles

$$(\mathcal{K}_i^\bullet, \mathcal{L}_i^\bullet, C(\gamma_i)^\bullet) \longrightarrow (K|_{U_i}, L|_{U_i}, M|_{U_i}).$$

It follows from the induced map on long exact cohomology sequences and Homology, Lemmas 12.5.19 and 12.5.20 that $C(\gamma_i)^\bullet \to M|_{U_i}$ induces an isomorphism on cohomology in degrees $> m$ and a surjection in degree m. Hence M is m-pseudo-coherent by Lemma 20.33.2.

Assertions (2) and (3) follow from (1) by rotating the distinguished triangle. □

Lemma 20.33.5. *Let $(X, \mathcal{O}_X)$ be a ringed space. Let K, L be objects of $D(\mathcal{O}_X)$.*
(1) *If K is n-pseudo-coherent and $H^i(K) = 0$ for $i > a$ and L is m-pseudo-coherent and $H^j(L) = 0$ for $j > b$, then $K \otimes^{\mathbf{L}}_{\mathcal{O}_X} L$ is t-pseudo-coherent with $t = \max(m + a, n + b)$.*
(2) *If K and L are pseudo-coherent, then $K \otimes^{\mathbf{L}}_{\mathcal{O}_X} L$ is pseudo-coherent.*

Proof. Proof of (1). By replacing X by the members of an open covering we may assume there exist strictly perfect complexes $\mathcal{K}^\bullet$ and $\mathcal{L}^\bullet$ and maps $\alpha : \mathcal{K}^\bullet \to K$ and $\beta : \mathcal{L}^\bullet \to L$ with $H^i(\alpha)$ and isomorphism for $i > n$ and surjective for $i = n$ and with $H^i(\beta)$ and isomorphism for $i > m$ and surjective for $i = m$. Then the map

$$\alpha \otimes^{\mathbf{L}} \beta : \mathrm{Tot}(\mathcal{K}^\bullet \otimes_{\mathcal{O}_X} \mathcal{L}^\bullet) \to K \otimes^{\mathbf{L}}_{\mathcal{O}_X} L$$

induces isomorphisms on cohomology sheaves in degree i for $i > t$ and a surjection for $i = t$. This follows from the spectral sequence of tors (details omitted).

Proof of (2). We may first replace X by the members of an open covering to reduce to the case that K and L are bounded above. Then the statement follows immediately from case (1). □

Lemma 20.33.6. *Let $(X, \mathcal{O}_X)$ be a ringed space. Let $m \in \mathbf{Z}$. If $K \oplus L$ is m-pseudo-coherent (resp. pseudo-coherent) in $D(\mathcal{O}_X)$ so are K and L.*

Proof. Assume that $K \oplus L$ is m-pseudo-coherent. After replacing X by the members of an open covering we may assume $K \oplus L \in D^-(\mathcal{O}_X)$, hence $L \in D^-(\mathcal{O}_X)$. Note that there is a distinguished triangle

$$(K \oplus L, K \oplus L, L \oplus L[1]) = (K, K, 0) \oplus (L, L, L \oplus L[1])$$

see Derived Categories, Lemma 13.4.9. By Lemma 20.33.4 we see that $L \oplus L[1]$ is m-pseudo-coherent. Hence also $L[1] \oplus L[2]$ is m-pseudo-coherent. By induction $L[n] \oplus L[n+1]$ is m-pseudo-coherent. Since L is bounded above we see that $L[n]$ is m-pseudo-coherent for large n. Hence working backwards, using the distinguished triangles

$$(L[n], L[n] \oplus L[n-1], L[n-1])$$

we conclude that $L[n-1], L[n-2], \ldots, L$ are m-pseudo-coherent as desired. □

Lemma 20.33.7. *Let $(X, \mathcal{O}_X)$ be a ringed space. Let $m \in \mathbf{Z}$. Let $\mathcal{F}^\bullet$ be a (locally) bounded above complex of $\mathcal{O}_X$-modules such that $\mathcal{F}^i$ is $(m-i)$-pseudo-coherent for all i. Then $\mathcal{F}^\bullet$ is m-pseudo-coherent.*

Proof. Omitted. Hint: use Lemma 20.33.4 and truncations as in the proof of More on Algebra, Lemma 15.48.9. □

Lemma 20.33.8. *Let $(X, \mathcal{O}_X)$ be a ringed space. Let $m \in \mathbf{Z}$. Let E be an object of $D(\mathcal{O}_X)$. If E is (locally) bounded above and $H^i(E)$ is $(m-i)$-pseudo-coherent for all i, then E is m-pseudo-coherent.*

Proof. Omitted. Hint: use Lemma 20.33.4 and truncations as in the proof of More on Algebra, Lemma 15.48.10. □

Lemma 20.33.9. *Let $(X, \mathcal{O}_X)$ be a ringed space. Let K be an object of $D(\mathcal{O}_X)$. Let $m \in \mathbf{Z}$.*

1. *If K is m-pseudo-coherent and $H^i(K) = 0$ for $i > m$, then $H^m(K)$ is a finite type $\mathcal{O}_X$-module.*
2. *If K is m-pseudo-coherent and $H^i(K) = 0$ for $i > m+1$, then $H^{m+1}(K)$ is a finitely presented $\mathcal{O}_X$-module.*

Proof. Proof of (1). We may work locally on X. Hence we may assume there exists a strictly perfect complex $\mathcal{E}^\bullet$ and a map $\alpha : \mathcal{E}^\bullet \to K$ which induces an isomorphism on cohomology in degrees $> m$ and a surjection in degree m. It suffices to prove the result for $\mathcal{E}^\bullet$. Let n be the largest integer such that $\mathcal{E}^n \neq 0$. If $n = m$, then $H^m(\mathcal{E}^\bullet)$ is a quotient of $\mathcal{E}^n$ and the result is clear. If $n > m$, then $\mathcal{E}^{n-1} \to \mathcal{E}^n$ is surjective as $H^n(E^\bullet) = 0$. By Lemma 20.32.5 we can locally find a section of this surjection and write $\mathcal{E}^{n-1} = \mathcal{E}' \oplus \mathcal{E}^n$. Hence it suffices to prove the result for the complex $(\mathcal{E}')^\bullet$ which is the same as $\mathcal{E}^\bullet$ except has $\mathcal{E}'$ in degree $n-1$ and 0 in degree n. We win by induction on n.

Proof of (2). We may work locally on X. Hence we may assume there exists a strictly perfect complex $\mathcal{E}^\bullet$ and a map $\alpha : \mathcal{E}^\bullet \to K$ which induces an isomorphism on cohomology in degrees $> m$ and a surjection in degree m. As in the proof of (1) we can reduce to the case that $\mathcal{E}^i = 0$ for $i > m+1$. Then we see that $H^{m+1}(K) \cong H^{m+1}(\mathcal{E}^\bullet) = \mathrm{Coker}(\mathcal{E}^m \to \mathcal{E}^{m+1})$ which is of finite presentation. □

Lemma 20.33.10. *Let $(X, \mathcal{O}_X)$ be a ringed space. Let $\mathcal{F}$ be a sheaf of $\mathcal{O}_X$-modules.*

1. *$\mathcal{F}$ viewed as an object of $D(\mathcal{O}_X)$ is 0-pseudo-coherent if and only if $\mathcal{F}$ is a finite type $\mathcal{O}_X$-module, and*

(2) *$\mathcal{F}$ viewed as an object of $D(\mathcal{O}_X)$ is (-1)-pseudo-coherent if and only if $\mathcal{F}$ is an $\mathcal{O}_X$-module of finite presentation.*

Proof. Use Lemma 20.33.9 to prove the implications in one direction and Lemma 20.33.8 for the other. □

20.34. Tor dimension

In this section we take a closer look at resolutions by flat modules.

Definition 20.34.1. Let $(X, \mathcal{O}_X)$ be a ringed space. Let E be an object of $D(\mathcal{O}_X)$. Let $a, b \in \mathbf{Z}$ with $a \leq b$.

(1) We say E has *tor-amplitude in* $[a, b]$ if $H^i(E \otimes^{\mathbf{L}}_{\mathcal{O}_X} \mathcal{F}) = 0$ for all $\mathcal{O}_X$-modules $\mathcal{F}$ and all $i \notin [a, b]$.
(2) We say E has *finite tor dimension* if it has tor-amplitude in $[a, b]$ for some a, b.
(3) We say E *locally has finite tor dimension* if there exists an open covering $X = \bigcup U_i$ such that $E|_{U_i}$ has finite tor dimension for all i.

Note that if E has finite tor dimension, then E is an object of $D^b(\mathcal{O}_X)$ as can be seen by taking $\mathcal{F} = \mathcal{O}_X$ in the definition above.

Lemma 20.34.2. *Let $(X, \mathcal{O}_X)$ be a ringed space. Let $\mathcal{E}^\bullet$ be a bounded above complex of flat $\mathcal{O}_X$-modules with tor-amplitude in $[a, b]$. Then $Coker(d^{a-1}_{\mathcal{E}^\bullet})$ is a flat $\mathcal{O}_X$-module.*

Proof. As $\mathcal{E}^\bullet$ is a bounded above complex of flat modules we see that $\mathcal{E}^\bullet \otimes_{\mathcal{O}_X} \mathcal{F} = \mathcal{E}^\bullet \otimes^{\mathbf{L}}_{\mathcal{O}_X} \mathcal{F}$ for any $\mathcal{O}_X$-module $\mathcal{F}$. Hence for every $\mathcal{O}_X$-module $\mathcal{F}$ the sequence

$$\mathcal{E}^{a-2} \otimes_{\mathcal{O}_X} \mathcal{F} \to \mathcal{E}^{a-1} \otimes_{\mathcal{O}_X} \mathcal{F} \to \mathcal{E}^a \otimes_{\mathcal{O}_X} \mathcal{F}$$

is exact in the middle. Since $\mathcal{E}^{a-2} \to \mathcal{E}^{a-1} \to \mathcal{E}^a \to \mathrm{Coker}(d^{a-1}) \to 0$ is a flat resolution this implies that $\mathrm{Tor}_1^{\mathcal{O}_X}(\mathrm{Coker}(d^{a-1}), \mathcal{F}) = 0$ for all $\mathcal{O}_X$-modules $\mathcal{F}$. This means that $\mathrm{Coker}(d^{a-1})$ is flat, see Lemma 20.25.15. □

Lemma 20.34.3. *Let $(X, \mathcal{O}_X)$ be a ringed space. Let E be an object of $D(\mathcal{O}_X)$. Let $a, b \in \mathbf{Z}$ with $a \leq b$. The following are equivalent*

(1) *E has tor-amplitude in $[a, b]$.*
(2) *E is represented by a complex $\mathcal{E}^\bullet$ of flat $\mathcal{O}_X$-modules with $\mathcal{E}^i = 0$ for $i \notin [a, b]$.*

Proof. If (2) holds, then we may compute $E \otimes^{\mathbf{L}}_{\mathcal{O}_X} \mathcal{F} = \mathcal{E}^\bullet \otimes_{\mathcal{O}_X} \mathcal{F}$ and it is clear that (1) holds.

Assume that (1) holds. We may represent E by a bounded above complex of flat $\mathcal{O}_X$-modules $\mathcal{K}^\bullet$, see Section 20.25. Let n be the largest integer such that $\mathcal{K}^n \neq 0$. If $n > b$, then $\mathcal{K}^{n-1} \to \mathcal{K}^n$ is surjective as $H^n(\mathcal{K}^\bullet) = 0$. As $\mathcal{K}^n$ is flat we see that $\mathrm{Ker}(\mathcal{K}^{n-1} \to \mathcal{K}^n)$ is flat (Modules, Lemma 17.16.8). Hence we may replace $\mathcal{K}^\bullet$ by $\tau_{\leq n-1}\mathcal{K}^\bullet$. Thus, by induction on n, we reduce to the case that $K^\bullet$ is a complex of flat $\mathcal{O}_X$-modules with $\mathcal{K}^i = 0$ for $i > b$.

Set $\mathcal{E}^\bullet = \tau_{\geq a}\mathcal{K}^\bullet$. Everything is clear except that $\mathcal{E}^a$ is flat which follows immediately from Lemma 20.34.2 and the definitions. □

Lemma 20.34.4. *Let $f : (X, \mathcal{O}_X) \to (Y, \mathcal{O}_Y)$ be a morphism of ringed spaces. Let E be an object of $D(\mathcal{O}_Y)$. If E has tor amplitude in $[a, b]$, then Lf^*E has tor amplitude in $[a, b]$.*

Proof. Assume E has tor amplitude in $[a, b]$. By Lemma 20.34.3 we can represent E by a complex of $\mathcal{E}^\bullet$ of flat $\mathcal{O}$-modules with $\mathcal{E}^i = 0$ for $i \notin [a, b]$. Then Lf^*E is represented by $f^*\mathcal{E}^\bullet$. By Modules, Lemma 17.17.2 the modules $f^*\mathcal{E}^i$ are flat. Thus by Lemma 20.34.3 we conclude that Lf^*E has tor amplitude in $[a, b]$. □

Lemma 20.34.5. *Let $(X, \mathcal{O}_X)$ be a ringed space. Let E be an object of $D(\mathcal{O}_X)$. Let $a, b \in \mathbf{Z}$ with $a \leq b$. The following are equivalent*

1. *E has tor-amplitude in $[a, b]$.*
2. *for every $x \in X$ the object E_x of $D(\mathcal{O}_{X,x})$ has tor-amplitude in $[a, b]$.*

Proof. Taking stalks at x is the same thing as pulling back by the morphism of ringed spaces $(x, \mathcal{O}_{X,x}) \to (X, \mathcal{O}_X)$. Hence the implication (1) $\Rightarrow$ (2) follows from Lemma 20.34.4. For the converse, note that taking stalks commutes with tensor products (Modules, Lemma 17.15.1). Hence

$$(E \otimes^{\mathbf{L}}_{\mathcal{O}_X} \mathcal{F})_x = E_x \otimes^{\mathbf{L}}_{\mathcal{O}_{X,x}} \mathcal{F}_x$$

On the other hand, taking stalks is exact, so

$$H^i(E \otimes^{\mathbf{L}}_{\mathcal{O}_X} \mathcal{F})_x = H^i((E \otimes^{\mathbf{L}}_{\mathcal{O}_X} \mathcal{F})_x) = H^i(E_x \otimes^{\mathbf{L}}_{\mathcal{O}_{X,x}} \mathcal{F}_x)$$

and we can check whether $H^i(E \otimes^{\mathbf{L}}_{\mathcal{O}_X} \mathcal{F})$ is zero by checking whether all of its stalks are zero (Modules, Lemma 17.3.1). Thus (2) implies (1). □

Lemma 20.34.6. *Let $(X, \mathcal{O}_X)$ be a ringed space. Let (K, L, M, f, g, h) be a distinguished triangle in $D(\mathcal{O}_X)$. Let $a, b \in \mathbf{Z}$.*

1. *If K has tor-amplitude in $[a + 1, b + 1]$ and L has tor-amplitude in $[a, b]$ then M has tor-amplitude in $[a, b]$.*
2. *If K and M have tor-amplitude in $[a, b]$, then L has tor-amplitude in $[a, b]$.*
3. *If L has tor-amplitude in $[a + 1, b + 1]$ and M has tor-amplitude in $[a, b]$, then K has tor-amplitude in $[a + 1, b + 1]$.*

Proof. Omitted. Hint: This just follows from the long exact cohomology sequence associated to a distinguished triangle and the fact that $- \otimes^{\mathbf{L}}_{\mathcal{O}_X} \mathcal{F}$ preserves distinguished triangles. The easiest one to prove is (2) and the others follow from it by translation. □

Lemma 20.34.7. *Let $(X, \mathcal{O}_X)$ be a ringed space. Let K, L be objects of $D(\mathcal{O}_X)$. If K has tor-amplitude in $[a, b]$ and L has tor-amplitude in $[c, d]$ then $K \otimes^{\mathbf{L}}_{\mathcal{O}_X} L$ has tor amplitude in $[a + c, b + d]$.*

Proof. Omitted. Hint: use the spectral sequence for tors. □

Lemma 20.34.8. *Let $(X, \mathcal{O}_X)$ be a ringed space. Let $a, b \in \mathbf{Z}$. For K, L objects of $D(\mathcal{O}_X)$ if $K \oplus L$ has tor amplitude in $[a, b]$ so do K and L.*

Proof. Clear from the fact that the Tor functors are additive. □

20.35. Perfect complexes

In this section we discuss properties of perfect complexes on ringed spaces.

Definition 20.35.1. Let $(X, \mathcal{O}_X)$ be a ringed space. Let $\mathcal{E}^\bullet$ be a complex of $\mathcal{O}_X$-modules. We say $\mathcal{E}^\bullet$ is *perfect* if there exists an open covering $X = \bigcup U_i$ such that for each i there exists a morphism of complexes $\mathcal{E}_i^\bullet \to \mathcal{E}^\bullet|_{U_i}$ which is a quasi-isomorphism with $\mathcal{E}_i^\bullet$ strictly perfect. An object E of of $D(\mathcal{O}_X)$ is *perfect* if it can be represented by a perfect complex of $\mathcal{O}_X$-modules.

Lemma 20.35.2. *Let $(X, \mathcal{O}_X)$ be a ringed space. Let E be an object of $D(\mathcal{O}_X)$.*

1. *If there exists an open covering $X = \bigcup U_i$, strictly perfect complexes $\mathcal{E}_i^\bullet$ on U_i, and isomorphisms $\alpha_i : \mathcal{E}_i^\bullet \to E|_{U_i}$ in $D(\mathcal{O}_{U_i})$, then E is perfect.*
2. *If E is perfect, then any complex representing E is perfect.*

Proof. Identical to the proof of Lemma 20.33.2. □

Lemma 20.35.3. *Let $(X, \mathcal{O}_X)$ be a ringed space. Let E be an object of $D(\mathcal{O}_X)$. Let $a \leq b$ be integers. If E has tor amplitude in $[a, b]$ and is $(a-1)$-pseudo-coherent, then E is perfect.*

Proof. After replacing X by the members of an open covering we may assume there exists a strictly perfect complex $\mathcal{E}^\bullet$ and a map $\alpha : \mathcal{E}^\bullet \to E$ such that $H^i(\alpha)$ is an isomorphism for $i \geq a$. We may and do replace $\mathcal{E}^\bullet$ by $\sigma_{\geq a-1}\mathcal{E}^\bullet$. Choose a distinguished triangle

$$\mathcal{E}^\bullet \to E \to C \to \mathcal{E}^\bullet[1]$$

From the vanishing of cohomology sheaves of E and $\mathcal{E}^\bullet$ and the assumption on α we obtain $C \cong \mathcal{K}[a-2]$ with $\mathcal{K} = \mathrm{Ker}(\mathcal{E}^{a-1} \to \mathcal{E}^a)$. Let $\mathcal{F}$ be an $\mathcal{O}_X$-module. Applying $- \otimes^{\mathbf{L}}_{\mathcal{O}_X} \mathcal{F}$ the assumption that E has tor amplitude in $[a, b]$ implies $\mathcal{K} \otimes_{\mathcal{O}_X} \mathcal{F} \to \mathcal{E}^{a-1} \otimes_{\mathcal{O}_X} \mathcal{F}$ has image $\mathrm{Ker}(\mathcal{E}^{a-1} \otimes_{\mathcal{O}_X} \mathcal{F} \to \mathcal{E}^a \otimes_{\mathcal{O}_X} \mathcal{F})$. It follows that $\mathrm{Tor}_1^{\mathcal{O}_X}(\mathcal{E}', \mathcal{F}) = 0$ where $\mathcal{E}' = \mathrm{Coker}(\mathcal{E}^{a-1} \to \mathcal{E}^a)$. Hence $\mathcal{E}'$ is flat (Lemma 20.25.15). Thus $\mathcal{E}'$ is locally a direct summand of a finite free module by Modules, Lemma 17.16.11. Thus locally the complex

$$\mathcal{E}' \to \mathcal{E}^{a-1} \to \ldots \to \mathcal{E}^b$$

is quasi-isomorphic to E and E is perfect. □

Lemma 20.35.4. *Let $(X, \mathcal{O}_X)$ be a ringed space. Let E be an object of $D(\mathcal{O}_X)$. The following are equivalent*

1. *E is perfect, and*
2. *E is pseudo-coherent and locally has finite tor dimension.*

Proof. Assume (1). By definition this means there exists an open covering $X = \bigcup U_i$ such that $E|_{U_i}$ is represented by a strictly perfect complex. Thus E is pseudo-coherent (i.e., m-pseudo-coherent for all m) by Lemma 20.33.2. Moreover, a direct summand of a finite free module is flat, hence $E|_{U_i}$ has finite Tor dimension by Lemma 20.34.3. Thus (2) holds.

Assume (2). After replacing X by the members of an open covering we may assume there exist integers $a \leq b$ such that E has tor amplitude in $[a, b]$. Since E is m-pseudo-coherent for all m we conclude using Lemma 20.35.3. □

Lemma 20.35.5. *Let $f : (X, \mathcal{O}_X) \to (Y, \mathcal{O}_Y)$ be a morphism of ringed spaces. Let E be an object of $D(\mathcal{O}_Y)$. If E is perfect in $D(\mathcal{O}_Y)$, then Lf^*E is perfect in $D(\mathcal{O}_X)$.*

Proof. This follows from Lemma 20.35.4, 20.34.4, and 20.33.3. (An alternative proof is to copy the proof of Lemma 20.33.3.) □

Lemma 20.35.6. *Let $(X, \mathcal{O}_X)$ be a ringed space. Let (K, L, M, f, g, h) be a distinguished triangle in $D(\mathcal{O}_X)$. If two out of three of K, L, M are perfect then the third is also perfect.*

Proof. First proof: Combine Lemmas 20.35.4, 20.33.4, and 20.34.6. Second proof (sketch): Say K and L are perfect. After replacing X by the members of an open covering we may assume that K and L are represented by strictly perfect complexes $\mathcal{K}^\bullet$ and $\mathcal{L}^\bullet$. After replacing X by the members of an open covering we may assume the map $K \to L$ is given by a map of complexes $\alpha : \mathcal{K}^\bullet \to \mathcal{L}^\bullet$, see Lemma 20.32.8. Then M is isomorphic to the cone of α which is strictly perfect by Lemma 20.32.2. □

Lemma 20.35.7. *Let $(X, \mathcal{O}_X)$ be a ringed space. If K, L are perfect objects of $D(\mathcal{O}_X)$, then so is $K \otimes^{\mathbf{L}}_{\mathcal{O}_X} L$.*

Proof. Follows from Lemmas 20.35.4, 20.33.5, and 20.34.7. □

Lemma 20.35.8. *Let $(X, \mathcal{O}_X)$ be a ringed space. If $K \oplus L$ is a perfect object of $D(\mathcal{O}_X)$, then so are K and L.*

Proof. Follows from Lemmas 20.35.4, 20.33.6, and 20.34.8. □

Lemma 20.35.9. *Let $(X, \mathcal{O}_X)$ be a ringed space. Let $j : U \to X$ be an open subspace. Let E be a perfect object of $D(\mathcal{O}_U)$ whose cohomology sheaves are supported on a closed subset $T \subset U$ with $j(T)$ closed in X. Then Rj_*E is a perfect object of $D(\mathcal{O}_X)$.*

Proof. Being a perfect complex is local on X. Thus it suffices to check that Rj_*E is perfect when restricted to U and $V = X \setminus j(T)$. We have $Rj_*E|_U = E$ which is perfect. We have $Rj_*E|_V = 0$ because $E|_{U \setminus T} = 0$. □

Lemma 20.35.10. *Let $(X, \mathcal{O}_X)$ be a ringed space. Let K be a perfect object of $D(\mathcal{O}_X)$. Then $K^\wedge = R\mathcal{H}om(K, \mathcal{O}_X)$ is a perfect object too and $(K^\wedge)^\wedge = K$. There are functorial isomorphisms*

$$H^0(X, K^\wedge \otimes^{\mathbf{L}}_{\mathcal{O}_X} M) = \mathrm{Hom}_{D(\mathcal{O}_X)}(K, M)$$

for M in $D(\mathcal{O}_X)$.

Proof. We will us without further mention that formation of internal hom commutes with restriction to opens (Lemma 20.31.3). In particular we may check the first two statements locally on X. By Lemma 20.31.5 to see the final statement it suffices to check that the map (20.31.5.1)

$$K^\wedge \otimes^{\mathbf{L}}_{\mathcal{O}_X} M \longrightarrow R\mathcal{H}om(K, M)$$

is an isomorphism. This is local on X as well. Hence it suffices to prove the lemma when K is represented by a strictly perfect complex.

Assume K is represented by the strictly perfect complex $\mathcal{E}^\bullet$. Then it follows from Lemma 20.32.9 that $K^\wedge$ is represented by the complex whose terms are $(\mathcal{E}^n)^\wedge = \mathcal{H}om_{\mathcal{O}_X}(\mathcal{E}^n, \mathcal{O}_X)$ in degree $-n$. Since $\mathcal{E}^n$ is a direct summand of a finite free $\mathcal{O}_X$-module, so is $(\mathcal{E}^n)^\wedge$. Hence $K^\wedge$ is represented by a strictly perfect complex too. It is also clear that $(K^\wedge)^\wedge = K$ as we have $((\mathcal{E}^n)^\wedge)^\wedge = \mathcal{E}^n$. To see that (20.31.5.1) is an isomorphism, represent M by a K-flat complex $\mathcal{F}^\bullet$. By Lemma 20.32.9 the complex $R\mathcal{H}om(K, M)$ is represented by the complex with terms

$$\bigoplus\nolimits_{n=-p+q} \mathcal{H}om_{\mathcal{O}_X}(\mathcal{E}^p, \mathcal{F}^q)$$

On the other hand, then object $K^\wedge \otimes^{\mathbf{L}} M$ is represented by the complex with terms

$$\bigoplus\nolimits_{n=-p+q} (\mathcal{E}^p)^\wedge \otimes_{\mathcal{O}_X} \mathcal{F}^q$$

Thus the assertion that (20.31.5.1) is an isomorphism reduces to the assertion that the canonical map

$$\mathcal{H}om_{\mathcal{O}_X}(\mathcal{E}, \mathcal{O}_X) \otimes_{\mathcal{O}_X} \mathcal{F} \longrightarrow \mathcal{H}om_{\mathcal{O}_X}(\mathcal{E}, \mathcal{F})$$

is an isomorphism when $\mathcal{E}$ is a direct summand of a finite free $\mathcal{O}_X$-module and $\mathcal{F}$ is any $\mathcal{O}_X$-module. This follows immediately from the corresponding statement when $\mathcal{E}$ is finite free. □

20.36. Compact objects

n this section we study compact objects in the derived category of modules on a ringed space. We recall that compact objects are defined in Derived Categories, Definition 13.32.1. On suitable ringed spaces the perfect objects are compact.

Lemma 20.36.1. *Let X be a ringed space. Assume that the underlying topological space of X has the following properties:*

1. *X is quasi-compact,*
2. *there exists a basis of quasi-compact open subsets, and*
3. *the intersection of any two quasi-compact opens is quasi-compact.*

Then any perfect object of $D(\mathcal{O}_X)$ is compact.

Proof. Let K be a perfect object and let $K^\wedge$ be its dual, see Lemma 20.35.10. Then we have

$$\mathrm{Hom}_{D(\mathcal{O}_X)}(K, M) = H^0(X, K^\wedge \otimes^{\mathbf{L}}_{\mathcal{O}_X} M)$$

functorially in M in $D(\mathcal{O}_X)$. Since $K^\wedge \otimes^{\mathbf{L}}_{\mathcal{O}_X} -$ commutes with direct sums (by construction) and H^0 does by Lemma 20.20.1 and the construction of direct sums in Injectives, Lemma 19.13.4 we obtain the result of the lemma. □

20.37. Other chapters

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CHAPTER 21

Cohomology on Sites

21.1. Introduction

In this document we work out some topics on cohomology of sheaves. We work out what happens for sheaves on sites, although often we will simply duplicate the discussion, the constructions, and the proofs from the topological case in the case. Basic references are [**AGV71**], [**God73**] and [**Ive86**].

21.2. Topics

Here are some topics that should be discussed in this chapter, and have not yet been written.

(1) Cohomology of a sheaf of modules on a site is the same as the cohomology of the underlying abelian sheaf.
(2) Hypercohomology on a site.
(3) Ext-groups.
(4) Ext sheaves.
(5) Tor functors.
(6) Higher direct images for a morphism of sites.
(7) Derived pullback for morphisms between ringed sites.
(8) Cup-product.
(9) Group cohomology.
(10) Comparison of group cohomology and cohomology on $\mathcal{T}_G$.
(11) Cech cohomology on sites.
(12) Cech to cohomology spectral sequence on sites.
(13) Leray Spectral sequence for a morphism between ringed sites.
(14) Etc, etc, etc.

21.3. Cohomology of sheaves

Let $\mathcal{C}$ be a site, see Sites, Definition 7.6.2. Let $\mathcal{F}$ be a abelian sheaf on $\mathcal{C}$. We know that the category of abelian sheaves on $\mathcal{C}$ has enough injectives, see Injectives, Theorem 19.7.4. Hence we can choose an injective resolution $\mathcal{F}[0] \to \mathcal{I}^\bullet$. For any object U of the site $\mathcal{C}$ we define

$$H^i(U, \mathcal{F}) = H^i(\Gamma(U, \mathcal{I}^\bullet)) \tag{21.3.0.1}$$

to be the *ith cohomology group of the abelian sheaf $\mathcal{F}$ over the object U*. In other words, these are the right derived functors of the functor $\mathcal{F} \mapsto \mathcal{F}(U)$. The family of functors $H^i(U, -)$ forms a universal δ-functor $Ab(\mathcal{C}) \to Ab$.

It sometimes happens that the site $\mathcal{C}$ does not have a final object. In this case we define the *global sections* of a presheaf of sets $\mathcal{F}$ over $\mathcal{C}$ to be the set

$$\Gamma(\mathcal{C}, \mathcal{F}) = \mathrm{Mor}_{PSh(\mathcal{C})}(e, \mathcal{F}) \tag{21.3.0.2}$$

where e is a final object in the category of presheaves on $\mathcal{C}$. In this case, given an abelian sheaf $\mathcal{F}$ on $\mathcal{C}$, we define the *ith cohomology group of $\mathcal{F}$ on $\mathcal{C}$* as follows

$$H^i(\mathcal{C}, \mathcal{F}) = H^i(\Gamma(\mathcal{C}, \mathcal{I}^\bullet)) \tag{21.3.0.3}$$

in other words, it is the ith right derived functor of the global sections functor. The family of functors $H^i(\mathcal{C}, -)$ forms a universal δ-functor $Ab(\mathcal{C}) \to Ab$.

Let $f : Sh(\mathcal{C}) \to Sh(\mathcal{D})$ be a morphism of topoi, see Sites, Definition 7.16.1. With $\mathcal{F}[0] \to \mathcal{I}^\bullet$ as above we define

$$R^i f_* \mathcal{F} = H^i(f_* \mathcal{I}^\bullet) \tag{21.3.0.4}$$

to be the *ith higher direct image of $\mathcal{F}$*. These are the right derived functors of f_*. The family of functors $R^i f_*$ forms a universal δ-functor from $Ab(\mathcal{C}) \to Ab(\mathcal{D})$.

Let $(\mathcal{C}, \mathcal{O})$ be a ringed site, see Modules on Sites, Definition 18.6.1. Let $\mathcal{F}$ be an $\mathcal{O}$-module. We know that the category of $\mathcal{O}$-modules has enough injectives, see Injectives, Theorem 19.8.4. Hence we can choose an injective resolution $\mathcal{F}[0] \to \mathcal{I}^\bullet$. For any object U of the site $\mathcal{C}$ we define

$$H^i(U, \mathcal{F}) = H^i(\Gamma(U, \mathcal{I}^\bullet)) \tag{21.3.0.5}$$

to be the *the ith cohomology group of $\mathcal{F}$ over U*. The family of functors $H^i(U, -)$ forms a universal δ-functor $Mod(\mathcal{O}) \to \mathrm{Mod}_{\mathcal{O}(U)}$. Similarly

$$H^i(\mathcal{C}, \mathcal{F}) = H^i(\Gamma(\mathcal{C}, \mathcal{I}^\bullet)) \tag{21.3.0.6}$$

it the *ith cohomology group of $\mathcal{F}$ on $\mathcal{C}$*. The family of functors $H^i(\mathcal{C}, -)$ forms a universal δ-functor $Mod(\mathcal{C}) \to \mathrm{Mod}_{\Gamma(\mathcal{C}, \mathcal{O})}$.

Let $f : (Sh(\mathcal{C}), \mathcal{O}) \to (Sh(\mathcal{D}), \mathcal{O}')$ be a morphism of ringed topoi, see Modules on Sites, Definition 18.7.1. With $\mathcal{F}[0] \to \mathcal{I}^\bullet$ as above we define

$$R^i f_* \mathcal{F} = H^i(f_* \mathcal{I}^\bullet) \tag{21.3.0.7}$$

to be the *ith higher direct image of $\mathcal{F}$*. These are the right derived functors of f_*. The family of functors $R^i f_*$ forms a universal δ-functor from $Mod(\mathcal{O}) \to Mod(\mathcal{O}')$.

21.4. Derived functors

We briefly explain an approach to right derived functors using resolution functors. Namely, suppose that $(\mathcal{C}, \mathcal{O})$ is a ringed site. In this chapter we will write

$$K(\mathcal{O}) = K(Mod(\mathcal{O})) \quad \text{and} \quad D(\mathcal{O}) = D(Mod(\mathcal{O}))$$

and similarly for the bounded versions for the triangulated categories introduced in Derived Categories, Definition 13.7.1 and Definition 13.10.3. By Derived Categories, Remark 13.23.3 there exists a resolution functor

$$j = j_{(\mathcal{C}, \mathcal{O})} : K^+(Mod(\mathcal{O})) \longrightarrow K^+(\mathcal{I})$$

where $\mathcal{I}$ is the strictly full additive subcategory of $Mod(\mathcal{O})$ which consists of injective $\mathcal{O}$-modules. For any left exact functor $F : Mod(\mathcal{O}) \to \mathcal{B}$ into any abelian category

$\mathcal{B}$ we will denote RF the right derived functor of Derived Categories, Section 13.19 constructed using the resolution functor j just described:

$$RF = F \circ j' : D^+(\mathcal{O}) \longrightarrow D^+(\mathcal{B}) \tag{21.4.0.8}$$

see Derived Categories, Lemma 13.24.1 for notation. Note that we may think of RF as defined on $Mod(\mathcal{O})$, $\mathrm{Comp}^+(Mod(\mathcal{O}))$, or $K^+(\mathcal{O})$ depending on the situation. According to Derived Categories, Definition 13.16.2 we obtain the $i$the right derived functor

$$R^iF = H^i \circ RF : Mod(\mathcal{O}) \longrightarrow \mathcal{B} \tag{21.4.0.9}$$

so that $R^0F = F$ and $\{R^iF, \delta\}_{i\geq 0}$ is universal δ-functor, see Derived Categories, Lemma 13.19.4.

Here are two special cases of this construction. Given a ring R we write $K(R) = K(\mathrm{Mod}_R)$ and $D(R) = D(\mathrm{Mod}_R)$ and similarly for the bounded versions. For any object U of $\mathcal{C}$ have a left exact functor $\Gamma(U, -) : Mod(\mathcal{O}) \longrightarrow \mathrm{Mod}_{\mathcal{O}(U)}$ which gives rise to

$$R\Gamma(U, -) : D^+(\mathcal{O}) \longrightarrow D^+(\mathcal{O}(U))$$

by the discussion above. Note that $H^i(U, -) = R^i\Gamma(U, -)$ is compatible with (21.3.0.5) above. We similarly have

$$R\Gamma(\mathcal{C}, -) : D^+(\mathcal{O}) \longrightarrow D^+(\Gamma(\mathcal{C}, \mathcal{O}))$$

compatible with (21.3.0.6). If $f : (Sh(\mathcal{C}), \mathcal{O}) \to (Sh(\mathcal{D}), \mathcal{O}')$ is a morphism of ringed topoi then we get a left exact functor $f_* : Mod(\mathcal{O}) \to Mod(\mathcal{O}')$ which gives rise to *derived pushforward*

$$Rf_* : D^+(\mathcal{O}) \to D^+(\mathcal{O}')$$

The ith cohomology sheaf of $Rf_*\mathcal{F}^\bullet$ is denoted $R^if_*\mathcal{F}^\bullet$ and called the ith *higher direct image* in accordance with (21.3.0.7). The displayed functors above are exact functor of derived categories.

21.5. First cohomology and torsors

Definition 21.5.1. Let $\mathcal{C}$ be a site. Let $\mathcal{G}$ be a sheaf of (possibly non-commutative) groups on $\mathcal{C}$. A *pseudo torsor*, or more precisely a *pseudo $\mathcal{G}$-torsor*, is a sheaf of sets $\mathcal{F}$ on $\mathcal{C}$ endowed with an action $\mathcal{G} \times \mathcal{F} \to \mathcal{F}$ such that

(1) whenever $\mathcal{F}(U)$ is nonempty the action $\mathcal{G}(U) \times \mathcal{F}(U) \to \mathcal{F}(U)$ is simply transitive.

A *morphism of pseudo $\mathcal{G}$-torsors* $\mathcal{F} \to \mathcal{F}'$ is simply a morphism of sheaves of sets compatible with the $\mathcal{G}$-actions. A *torsor*, or more precisely a *$\mathcal{G}$-torsor*, is a pseudo G-torsor such that in addition

(2) for every $U \in \mathrm{Ob}(\mathcal{C})$ there exists a covering $\{U_i \to U\}_{i\in I}$ of U such that $\mathcal{F}(U_i)$ is nonempty for all $i \in I$.

A *morphism of G-torsors* is simply a morphism of pseudo G-torsors. The *trivial $\mathcal{G}$-torsor* is the sheaf $\mathcal{G}$ endowed with the obvious left $\mathcal{G}$-action.

It is clear that a morphism of torsors is automatically an isomorphism.

Lemma 21.5.2. *Let $\mathcal{C}$ be a site. Let $\mathcal{G}$ be a sheaf of (possibly non-commutative) groups on $\mathcal{C}$. A $\mathcal{G}$-torsor $\mathcal{F}$ is trivial if and only if $\Gamma(\mathcal{C}, \mathcal{F}) \neq \emptyset$.*

Proof. Omitted. □

Lemma 21.5.3. *Let $\mathcal{C}$ be a site. Let $\mathcal{H}$ be an abelian sheaf on $\mathcal{C}$. There is a canonical bijection between the set of isomorphism classes of $\mathcal{H}$-torsors and $H^1(\mathcal{C}, \mathcal{H})$.*

Proof. Let $\mathcal{F}$ be a $\mathcal{H}$-torsor. Consider the free abelian sheaf $\mathbf{Z}[\mathcal{F}]$ on $\mathcal{F}$. It is the sheafification of the rule which associates to $U \in \mathrm{Ob}(\mathcal{C})$ the collection of finite formal sums $\sum n_i[s_i]$ with $n_i \in \mathbf{Z}$ and $s_i \in \mathcal{F}(U)$. There is a natural map

$$\sigma : \mathbf{Z}[\mathcal{F}] \longrightarrow \underline{\mathbf{Z}}$$

which to a local section $\sum n_i[s_i]$ associates $\sum n_i$. The kernel of σ is generated by sections of the form $[s] - [s']$. There is a canonical map $a : \mathrm{Ker}(\sigma) \to \mathcal{H}$ which maps $[s] - [s'] \mapsto h$ where h is the local section of $\mathcal{H}$ such that $h \cdot s = s'$. Consider the pushout diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \mathrm{Ker}(\sigma) & \longrightarrow & \mathbf{Z}[\mathcal{F}] & \longrightarrow & \underline{\mathbf{Z}} & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle a} & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & \mathcal{H} & \longrightarrow & \mathcal{E} & \longrightarrow & \underline{\mathbf{Z}} & \longrightarrow & 0 \end{array}$$

Here $\mathcal{E}$ is the extension obtained by pushout. From the long exact cohomology sequence associated to the lower short exact sequence we obtain an element $\xi = \xi_{\mathcal{F}} \in H^1(\mathcal{C}, \mathcal{H})$ by applying the boundary operator to $1 \in H^0(\mathcal{C}, \underline{\mathbf{Z}})$.

Conversely, given $\xi \in H^1(\mathcal{C}, \mathcal{H})$ we can associate to ξ a torsor as follows. Choose an embedding $\mathcal{H} \to \mathcal{I}$ of $\mathcal{H}$ into an injective abelian sheaf $\mathcal{I}$. We set $\mathcal{Q} = \mathcal{I}/\mathcal{H}$ so that we have a short exact sequence

$$0 \longrightarrow \mathcal{H} \longrightarrow \mathcal{I} \longrightarrow \mathcal{Q} \longrightarrow 0$$

The element ξ is the image of a global section $q \in H^0(\mathcal{C}, \mathcal{Q})$ because $H^1(\mathcal{C}, \mathcal{I}) = 0$ (see Derived Categories, Lemma 13.19.4). Let $\mathcal{F} \subset \mathcal{I}$ be the subsheaf (of sets) of sections that map to q in the sheaf $\mathcal{Q}$. It is easy to verify that $\mathcal{F}$ is a $\mathcal{H}$-torsor.

We omit the verification that the two constructions given above are mutually inverse. □

21.6. First cohomology and extensions

Lemma 21.6.1. *Let $(\mathcal{C}, \mathcal{O})$ be a ringed site. Let $\mathcal{F}$ be a sheaf of $\mathcal{O}$-modules on $\mathcal{C}$. There is a canonical bijection*

$$Ext^1_{Mod(\mathcal{O})}(\mathcal{O}, \mathcal{F}) \longrightarrow H^1(\mathcal{C}, \mathcal{F})$$

which associates to the extension

$$0 \to \mathcal{F} \to \mathcal{E} \to \mathcal{O} \to 0$$

the image of $1 \in \Gamma(\mathcal{C}, \mathcal{O})$ in $H^1(\mathcal{C}, \mathcal{F})$.

Proof. Let us construct the inverse of the map given in the lemma. Let $\xi \in H^1(\mathcal{C}, \mathcal{F})$. Choose an injection $\mathcal{F} \subset \mathcal{I}$ with $\mathcal{I}$ injective in $Mod(\mathcal{O})$. Set $\mathcal{Q} = \mathcal{I}/\mathcal{F}$. By the long exact sequence of cohomology, we see that ξ is the image of of a section $\tilde{\xi} \in \Gamma(\mathcal{C}, \mathcal{Q}) = \mathrm{Hom}_{\mathcal{O}}(\mathcal{O}, \mathcal{Q})$. Now, we just form the pullback

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \mathcal{F} & \longrightarrow & \mathcal{E} & \longrightarrow & \mathcal{O} & \longrightarrow & 0 \\ & & \| & & \downarrow & & \downarrow{\scriptstyle \tilde{\xi}} & & \\ 0 & \longrightarrow & \mathcal{F} & \longrightarrow & \mathcal{I} & \longrightarrow & \mathcal{Q} & \longrightarrow & 0 \end{array}$$

see Homology, Section 12.6. □

The following lemma will be superseded by the more general Lemma 21.12.4.

Lemma 21.6.2. *Let $(\mathcal{C}, \mathcal{O})$ be a ringed site. Let $\mathcal{F}$ be a sheaf of $\mathcal{O}$-modules on $\mathcal{C}$. Let $\mathcal{F}_{ab}$ denote the underlying sheaf of abelian groups. Then there is a functorial isomorphism*

$$H^1(\mathcal{C}, \mathcal{F}_{ab}) = H^1(\mathcal{C}, \mathcal{F})$$

where the left hand side is cohomology computed in $Ab(\mathcal{C})$ and the right hand side is cohomology computed in $Mod(\mathcal{O})$.

Proof. Let $\underline{\mathbf{Z}}$ denote the constant sheaf $\mathbf{Z}$. As $Ab(\mathcal{C}) = Mod(\underline{\mathbf{Z}})$ we may apply Lemma 21.6.1 twice, and it follows that we have to show

$$\mathrm{Ext}^1_{Mod(\mathcal{O})}(\mathcal{O}, \mathcal{F}) = \mathrm{Ext}^1_{Mod(\underline{\mathbf{Z}})}(\underline{\mathbf{Z}}, \mathcal{F}_{ab}).$$

Suppose that $0 \to \mathcal{F} \to \mathcal{E} \to \mathcal{O} \to 0$ is an extension in $Mod(\mathcal{O})$. Then we can use the obvious map of abelian sheaves $1 : \underline{\mathbf{Z}} \to \mathcal{O}$ and pullback to obtain an extension $\mathcal{E}_{ab}$, like so:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \mathcal{F}_{ab} & \longrightarrow & \mathcal{E}_{ab} & \longrightarrow & \underline{\mathbf{Z}} & \longrightarrow & 0 \\ & & \| & & \downarrow & & \downarrow{\scriptstyle 1} & & \\ 0 & \longrightarrow & \mathcal{F} & \longrightarrow & \mathcal{E} & \longrightarrow & \mathcal{O} & \longrightarrow & 0 \end{array}$$

The converse is a little more fun. Suppose that $0 \to \mathcal{F}_{ab} \to \mathcal{E}_{ab} \to \underline{\mathbf{Z}} \to 0$ is an extension in $Mod(\underline{\mathbf{Z}})$. Since $\underline{\mathbf{Z}}$ is a flat $\underline{\mathbf{Z}}$-module we see that the sequence

$$0 \to \mathcal{F}_{ab} \otimes_{\underline{\mathbf{Z}}} \mathcal{O} \to \mathcal{E}_{ab} \otimes_{\underline{\mathbf{Z}}} \mathcal{O} \to \underline{\mathbf{Z}} \otimes_{\underline{\mathbf{Z}}} \mathcal{O} \to 0$$

is exact, see Modules on Sites, Lemma 18.28.7. Of course $\underline{\mathbf{Z}} \otimes_{\underline{\mathbf{Z}}} \mathcal{O} = \mathcal{O}$. Hence we can form the pushout via the ($\mathcal{O}$-linear) multiplication map $\mu : \mathcal{F} \otimes_{\underline{\mathbf{Z}}} \mathcal{O} \to \mathcal{F}$ to get an extension of $\mathcal{O}$ by $\mathcal{F}$, like this

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \mathcal{F}_{ab} \otimes_{\underline{\mathbf{Z}}} \mathcal{O} & \longrightarrow & \mathcal{E}_{ab} \otimes_{\underline{\mathbf{Z}}} \mathcal{O} & \longrightarrow & \mathcal{O} & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle \mu} & & \downarrow & & \| & & \\ 0 & \longrightarrow & \mathcal{F} & \longrightarrow & \mathcal{E} & \longrightarrow & \mathcal{O} & \longrightarrow & 0 \end{array}$$

which is the desired extension. We omit the verification that these constructions are mutually inverse. □

21.7. First cohomology and invertible sheaves

The Picard group of a ringed site is defined in Modules on Sites, Section 18.31.

Lemma 21.7.1. *Let $(\mathcal{C}, \mathcal{O})$ be a ringed site. There is a canonical isomorphism*

$$H^1(\mathcal{C}, \mathcal{O}^*) = Pic(\mathcal{O}).$$

of abelian groups.

Proof. Let $\mathcal{L}$ be an invertible $\mathcal{O}$-module. Consider the presheaf $\mathcal{L}^*$ defined by the rule

$$U \longmapsto \{s \in \mathcal{L}(U) \text{ such that } \mathcal{O}_U \xrightarrow{s \cdot -} \mathcal{L}_U \text{ is an isomorphism}\}$$

This presheaf satisfies the sheaf condition. Moreover, if $f \in \mathcal{O}^*(U)$ and $s \in \mathcal{L}^*(U)$, then clearly $fs \in \mathcal{L}^*(U)$. By the same token, if $s, s' \in \mathcal{L}^*(U)$ then there exists a unique $f \in \mathcal{O}^*(U)$ such that $fs = s'$. Moreover, the sheaf $\mathcal{L}^*$ has sections locally

by the very definition of an invertible sheaf. In other words we see that $\mathcal{L}^*$ is a $\mathcal{O}^*$-torsor. Thus we get a map

$$\begin{array}{c}\text{set of invertible sheaves on } (\mathcal{C}, \mathcal{O}) \\ \text{up to isomorphism}\end{array} \longrightarrow \begin{array}{c}\text{set of } \mathcal{O}^*\text{-torsors} \\ \text{up to isomorphism}\end{array}$$

We omit the verification that this is a homomorphism of abelian groups. By Lemma 21.5.3 the right hand side is canonically bijective to $H^1(\mathcal{C}, \mathcal{O}^*)$. Thus we have to show this map is injective and surjective.

Injective. If the torsor $\mathcal{L}^*$ is trivial, this means by Lemma 21.5.2 that $\mathcal{L}^*$ has a global section. Hence this means exactly that $\mathcal{L} \cong \mathcal{O}$ is the neutral element in $\mathrm{Pic}(\mathcal{O})$.

Surjective. Let $\mathcal{F}$ be an $\mathcal{O}^*$-torsor. Consider the presheaf of sets

$$\mathcal{L}_1 : U \longmapsto (\mathcal{F}(U) \times \mathcal{O}(U))/\mathcal{O}^*(U)$$

where the action of $f \in \mathcal{O}^*(U)$ on (s, g) is $(fs, f^{-1}g)$. Then $\mathcal{L}_1$ is a presheaf of $\mathcal{O}$-modules by setting $(s, g)+(s', g') = (s, g+(s'/s)g')$ where s'/s is the local section f of $\mathcal{O}^*$ such that $fs = s'$, and $h(s, g) = (s, hg)$ for h a local section of $\mathcal{O}$. We omit the verification that the sheafification $\mathcal{L} = \mathcal{L}_1^\#$ is an invertible $\mathcal{O}$-module whose associated $\mathcal{O}^*$-torsor $\mathcal{L}^*$ is isomorphic to $\mathcal{F}$. □

21.8. Locality of cohomology

The following lemma says there is no ambiguity in defining the cohomology of a sheaf $\mathcal{F}$ over an object of the site.

Lemma 21.8.1. *Let $(\mathcal{C}, \mathcal{O})$ be a ringed site. Let U be an object of $\mathcal{C}$.*

1. *If $\mathcal{I}$ is an injective $\mathcal{O}$-module then $\mathcal{I}|_U$ is an injective $\mathcal{O}_U$-module.*
2. *For any sheaf of $\mathcal{O}$-modules $\mathcal{F}$ we have $H^p(U, \mathcal{F}) = H^p(\mathcal{C}/U, \mathcal{F}|_U)$.*

Proof. Recall that the functor j_U^{-1} of restriction to U is a right adjoint to the functor $j_{U!}$ of extension by 0, see Modules on Sites, Section 18.19. Moreover, $j_{U!}$ is exact. Hence (1) follows from Homology, Lemma 12.25.1.

By definition $H^p(U, \mathcal{F}) = H^p(\mathcal{I}^\bullet(U))$ where $\mathcal{F} \to \mathcal{I}^\bullet$ is an injective resolution in $Mod(\mathcal{O})$. By the above we see that $\mathcal{F}|_U \to \mathcal{I}^\bullet|_U$ is an injective resolution in $Mod(\mathcal{O}_U)$. Hence $H^p(U, \mathcal{F}|_U)$ is equal to $H^p(\mathcal{I}^\bullet|_U(U))$. Of course $\mathcal{F}(U) = \mathcal{F}|_U(U)$ for any sheaf $\mathcal{F}$ on $\mathcal{C}$. Hence the equality in (2). □

The following lemma will be use to see what happens if we change a partial universe, or to compare cohomology of the small and big étale sites.

Lemma 21.8.2. *Let $\mathcal{C}$ and $\mathcal{D}$ be sites. Let $u : \mathcal{C} \to \mathcal{D}$ be a functor. Assume u satisfies the hypotheses of Sites, Lemma 7.20.8. Let $g : Sh(\mathcal{C}) \to Sh(\mathcal{D})$ be the associated morphism of topoi. For any abelian sheaf $\mathcal{F}$ on $\mathcal{D}$ we have isomorphisms*

$$R\Gamma(\mathcal{C}, g^{-1}\mathcal{F}) = R\Gamma(\mathcal{D}, \mathcal{F}),$$

in particular $H^p(\mathcal{C}, g^{-1}\mathcal{F}) = H^p(\mathcal{D}, \mathcal{F})$ and for any $U \in \mathrm{Ob}(\mathcal{C})$ we have isomorphisms

$$R\Gamma(U, g^{-1}\mathcal{F}) = R\Gamma(u(U), \mathcal{F}),$$

in particular $H^p(U, g^{-1}\mathcal{F}) = H^p(u(U), \mathcal{F})$. All of these isomorphisms are functorial in $\mathcal{F}$.

Proof. Since it is clear that $\Gamma(\mathcal{C}, g^{-1}\mathcal{F}) = \Gamma(\mathcal{D}, \mathcal{F})$ by hypothesis (e), it suffices to show that g^{-1} transforms injective abelian sheaves into injective abelian sheaves. As usual we use Homology, Lemma 12.25.1 to see this. The left adjoint to g^{-1} is $g_! = f^{-1}$ with the notation of Sites, Lemma 7.20.8 which is an exact functor. Hence the lemma does indeed apply. □

Let $(\mathcal{C}, \mathcal{O})$ be a ringed site. Let $\mathcal{F}$ be a sheaf of $\mathcal{O}$-modules. Let $\varphi : U \to V$ be a morphism of $\mathcal{O}$. Then there is a canonical *restriction mapping*

$$H^n(V, \mathcal{F}) \longrightarrow H^n(U, \mathcal{F}), \quad \xi \longmapsto \xi|_U \tag{21.8.2.1}$$

functorial in $\mathcal{F}$. Namely, choose any injective resolution $\mathcal{F} \to \mathcal{I}^\bullet$. The restriction mappings of the sheaves $\mathcal{I}^p$ give a morphism of complexes

$$\Gamma(V, \mathcal{I}^\bullet) \longrightarrow \Gamma(U, \mathcal{I}^\bullet)$$

The LHS is a complex representing $R\Gamma(V, \mathcal{F})$ and the RHS is a complex representing $R\Gamma(U, \mathcal{F})$. We get the map on cohomology groups by applying the functor H^n. As indicated we will use the notation $\xi \mapsto \xi|_U$ to denote this map. Thus the rule $U \mapsto H^n(U, \mathcal{F})$ is a presheaf of $\mathcal{O}$-modules. This presheaf is customarily denoted $\underline{H}^n(\mathcal{F})$. We will give another interpretation of this presheaf in Lemma 21.11.4.

The following lemma says that it is possible to kill higher cohomology classes by going to a covering.

Lemma 21.8.3. *Let $(\mathcal{C}, \mathcal{O})$ be a ringed site. Let $\mathcal{F}$ be a sheaf of $\mathcal{O}$-modules. Let U be an object of $\mathcal{C}$. Let $n > 0$ and let $\xi \in H^n(U, \mathcal{F})$. Then there exists a covering $\{U_i \to U\}$ of $\mathcal{C}$ such that $\xi|_{U_i} = 0$ for all $i \in I$.*

Proof. Let $\mathcal{F} \to \mathcal{I}^\bullet$ be an injective resolution. Then

$$H^n(U, \mathcal{F}) = \frac{\mathrm{Ker}(\mathcal{I}^n(U) \to \mathcal{I}^{n+1}(U))}{\mathrm{Im}(\mathcal{I}^{n-1}(U) \to \mathcal{I}^n(U))}.$$

Pick an element $\tilde{\xi} \in \mathcal{I}^n(U)$ representing the cohomology class in the presentation above. Since $\mathcal{I}^\bullet$ is an injective resolution of $\mathcal{F}$ and $n > 0$ we see that the complex $\mathcal{I}^\bullet$ is exact in degree n. Hence $\mathrm{Im}(\mathcal{I}^{n-1} \to \mathcal{I}^n) = \mathrm{Ker}(\mathcal{I}^n \to \mathcal{I}^{n+1})$ as sheaves. Since $\tilde{\xi}$ is a section of the kernel sheaf over U we conclude there exists a covering $\{U_i \to U\}$ of the site such that $\tilde{\xi}|_{U_i}$ is the image under d of a section $\xi_i \in \mathcal{I}^{n-1}(U_i)$. By our definition of the restriction $\xi|_{U_i}$ as corresponding to the class of $\tilde{\xi}|_{U_i}$ we conclude. □

Lemma 21.8.4. *Let $f : (\mathcal{C}, \mathcal{O}_\mathcal{C}) \to (\mathcal{D}, \mathcal{O}_\mathcal{D})$ be a morphism of ringed sites corresponding to the continuous functor $u : \mathcal{D} \to \mathcal{C}$. For any $\mathcal{F} \in \mathrm{Ob}(Mod(\mathcal{O}_\mathcal{C}))$ the sheaf $R^i f_* \mathcal{F}$ is the sheaf associated to the presheaf*

$$V \longmapsto H^i(u(V), \mathcal{F})$$

Proof. Let $\mathcal{F} \to \mathcal{I}^\bullet$ be an injective resolution. Then $R^i f_* \mathcal{F}$ is by definition the ith cohomology sheaf of the complex

$$f_*\mathcal{I}^0 \to f_*\mathcal{I}^1 \to f_*\mathcal{I}^2 \to \ldots$$

By definition of the abelian category structure on $\mathcal{O}_\mathcal{D}$-modules this cohomology sheaf is the sheaf associated to the presheaf

$$V \longmapsto \frac{\mathrm{Ker}(f_*\mathcal{I}^i(V) \to f_*\mathcal{I}^{i+1}(V))}{\mathrm{Im}(f_*\mathcal{I}^{i-1}(V) \to f_*\mathcal{I}^i(V))}$$

and this is obviously equal to

$$\frac{\mathrm{Ker}(\mathcal{I}^i(u(V)) \to \mathcal{I}^{i+1}(u(V)))}{\mathrm{Im}(\mathcal{I}^{i-1}(u(V)) \to \mathcal{I}^i(u(V)))}$$

which is equal to $H^i(u(V), \mathcal{F})$ and we win. □

21.9. The Cech complex and Cech cohomology

Let $\mathcal{C}$ be a category. Let $\mathcal{U} = \{U_i \to U\}_{i \in I}$ be a family of morphisms with fixed target, see Sites, Definition 7.6.1. Assume that all fibre products $U_{i_0} \times_U \ldots \times_U U_{i_p}$ exist in $\mathcal{C}$. Let $\mathcal{F}$ be an abelian presheaf on $\mathcal{C}$. Set

$$\check{\mathcal{C}}^p(\mathcal{U}, \mathcal{F}) = \prod\nolimits_{(i_0, \ldots, i_p) \in I^{p+1}} \mathcal{F}(U_{i_0} \times_U \ldots \times_U U_{i_p}).$$

This is an abelian group. For $s \in \check{\mathcal{C}}^p(\mathcal{U}, \mathcal{F})$ we denote $s_{i_0 \ldots i_p}$ its value in the factor $\mathcal{F}(U_{i_0} \times_U \ldots \times_U U_{i_p})$. We define

$$d : \check{\mathcal{C}}^p(\mathcal{U}, \mathcal{F}) \longrightarrow \check{\mathcal{C}}^{p+1}(\mathcal{U}, \mathcal{F})$$

by the formula

$$d(s)_{i_0 \ldots i_{p+1}} = \tag{21.9.0.1}$$

$$\sum\nolimits_{j=0}^{p+1} (-1)^j (U_{i_0} \times_U \ldots \times_U U_{i_{p+1}} \longrightarrow U_{i_0} \times_U \ldots \widehat{U_{i_j}} \ldots \times_U U_{i_{p+1}})^* s_{i_0 \ldots \hat{i}_j \ldots i_p}$$

It is straightforward to see that $d \circ d = 0$. In other words $\check{\mathcal{C}}^\bullet(\mathcal{U}, \mathcal{F})$ is a complex.

Definition 21.9.1. Let $\mathcal{C}$ be a category. Let $\mathcal{U} = \{U_i \to U\}_{i \in I}$ be a family of morphisms with fixed target such that all fibre products $U_{i_0} \times_U \ldots \times_U U_{i_p}$ exist in $\mathcal{C}$. Let $\mathcal{F}$ be an abelian presheaf on $\mathcal{C}$. The complex $\check{\mathcal{C}}^\bullet(\mathcal{U}, \mathcal{F})$ is the *Cech complex* associated to $\mathcal{F}$ and the family $\mathcal{U}$. Its cohomology groups $H^i(\check{\mathcal{C}}^\bullet(\mathcal{U}, \mathcal{F}))$ are called the *Cech cohomology groups* of $\mathcal{F}$ with respect to $\mathcal{U}$. They are denoted $\check{H}^i(\mathcal{U}, \mathcal{F})$.

We observe that any covering $\{U_i \to U\}$ of a site $\mathcal{C}$ is a family of morphisms with fixed target to which the definition applies.

Lemma 21.9.2. *Let $\mathcal{C}$ be a site. Let $\mathcal{F}$ be an abelian presheaf on $\mathcal{C}$. The following are equivalent*

1. *$\mathcal{F}$ is an abelian sheaf on $\mathcal{C}$ and*
2. *for every covering $\mathcal{U} = \{U_i \to U\}_{i \in I}$ of the site $\mathcal{C}$ the natural map*

$$\mathcal{F}(U) \to \check{H}^0(\mathcal{U}, \mathcal{F})$$

(see Sites, Section 7.10) is bijective.

Proof. This is true since the sheaf condition is exactly that $\mathcal{F}(U) \to \check{H}^0(\mathcal{U}, \mathcal{F})$ is bijective for every open covering. □

Let $\mathcal{C}$ be a category. Let $\mathcal{U} = \{U_i \to U\}_{i \in I}$ be a family of morphisms of $\mathcal{C}$ with fixed target such that all fibre products $U_{i_0} \times_U \ldots \times_U U_{i_p}$ exist in $\mathcal{C}$. Let $\mathcal{V} = \{V_j \to V\}_{j \in J}$ be another. Let $f : U \to V$, $\alpha : I \to J$ and $f_i : U_i \to V_{\alpha(i)}$ be a morphism of families of morphisms with fixed target, see Sites, Section 7.8. In this case we get a map of Cech complexes

$$\varphi : \check{\mathcal{C}}^\bullet(\mathcal{V}, \mathcal{F}) \longrightarrow \check{\mathcal{C}}^\bullet(\mathcal{U}, \mathcal{F}) \tag{21.9.2.1}$$

which in degree p is given by

$$\varphi(s)_{i_0 \ldots i_p} = (f_{i_0} \times \ldots \times f_{i_p})^* s_{\alpha(i_0)\ldots\alpha(i_p)}$$

21.10. Cech cohomology as a functor on presheaves

Warning: In this section we work exclusively with abelian presheaves on a category. The results are completely wrong in the setting of sheaves and categories of sheaves!

Let $\mathcal{C}$ be a category. Let $\mathcal{U} = \{U_i \to U\}_{i \in I}$ be a family of morphisms with fixed target such that all fibre products $U_{i_0} \times_U \ldots \times_U U_{i_p}$ exist in $\mathcal{C}$. Let $\mathcal{F}$ be an abelian presheaf on $\mathcal{C}$. The construction

$$\mathcal{F} \longmapsto \check{\mathcal{C}}^\bullet(\mathcal{U}, \mathcal{F})$$

is functorial in $\mathcal{F}$. In fact, it is a functor

$$\check{\mathcal{C}}^\bullet(\mathcal{U}, -) : PAb(\mathcal{C}) \longrightarrow \mathrm{Comp}^+(Ab) \tag{21.10.0.2}$$

see Derived Categories, Definition 13.7.1 for notation. Recall that the category of bounded below complexes in an abelian category is an abelian category, see Homology, Lemma 12.12.9.

Lemma 21.10.1. *The functor given by Equation (21.10.0.2) is an exact functor (see Homology, Lemma 12.7.1).*

Proof. For any object W of $\mathcal{C}$ the functor $\mathcal{F} \mapsto \mathcal{F}(W)$ is an additive exact functor from $PAb(\mathcal{C})$ to Ab. The terms $\check{\mathcal{C}}^p(\mathcal{U}, \mathcal{F})$ of the complex are products of these exact functors and hence exact. Moreover a sequence of complexes is exact if and only if the sequence of terms in a given degree is exact. Hence the lemma follows. □

Lemma 21.10.2. *Let $\mathcal{C}$ be a category. Let $\mathcal{U} = \{U_i \to U\}_{i \in I}$ be a family of morphisms with fixed target such that all fibre products $U_{i_0} \times_U \ldots \times_U U_{i_p}$ exist in $\mathcal{C}$. The functors $\mathcal{F} \mapsto \check{H}^n(\mathcal{U}, \mathcal{F})$ form a δ-functor from the abelian category $PAb(\mathcal{C})$ to the category of $\mathbf{Z}$-modules (see Homology, Definition 12.11.1).*

Proof. By Lemma 21.10.1 a short exact sequence of abelian presheaves $0 \to \mathcal{F}_1 \to \mathcal{F}_2 \to \mathcal{F}_3 \to 0$ is turned into a short exact sequence of complexes of $\mathbf{Z}$-modules. Hence we can use Homology, Lemma 12.12.12 to get the boundary maps $\delta_{\mathcal{F}_1 \to \mathcal{F}_2 \to \mathcal{F}_3} : \check{H}^n(\mathcal{U}, \mathcal{F}_3) \to \check{H}^{n+1}(\mathcal{U}, \mathcal{F}_1)$ and a corresponding long exact sequence. We omit the verification that these maps are compatible with maps between short exact sequences of presheaves. □

Lemma 21.10.3. *Let $\mathcal{C}$ be a category. Let $\mathcal{U} = \{U_i \to U\}_{i \in I}$ be a family of morphisms with fixed target such that all fibre products $U_{i_0} \times_U \ldots \times_U U_{i_p}$ exist in $\mathcal{C}$. Consider the chain complex $\mathbf{Z}_{\mathcal{U},\bullet}$ of abelian presheaves*

$$\ldots \to \bigoplus_{i_0 i_1 i_2} \mathbf{Z}_{U_{i_0} \times_U U_{i_1} \times_U U_{i_2}} \to \bigoplus_{i_0 i_1} \mathbf{Z}_{U_{i_0} \times_U U_{i_1}} \to \bigoplus_{i_0} \mathbf{Z}_{U_{i_0}} \to 0 \to \ldots$$

where the last nonzero term is placed in degree 0 and where the map

$$\mathbf{Z}_{U_{i_0} \times_U \ldots \times_u U_{i_{p+1}}} \longrightarrow \mathbf{Z}_{U_{i_0} \times_U \ldots \widehat{U_{i_j}} \ldots \times_U U_{i_{p+1}}}$$

is given by $(-1)^j$ times the canonical map. Then there is an isomorphism

$$\mathrm{Hom}_{PAb(\mathcal{C})}(\mathbf{Z}_{U,\bullet}, \mathcal{F}) = \check{\mathcal{C}}^\bullet(\mathcal{U}, \mathcal{F})$$

functorial in $\mathcal{F} \in \mathrm{Ob}(PAb(\mathcal{C}))$.

Proof. This is a tautology based on the fact that

$$\begin{aligned}\mathrm{Hom}_{PAb(\mathcal{C})}(\bigoplus_{i_0\ldots i_p} \mathbf{Z}_{U_{i_0}\times_U\ldots\times_U U_{i_p}}, \mathcal{F}) &= \prod_{i_0\ldots i_p} \mathrm{Hom}_{PAb(\mathcal{C})}(\mathbf{Z}_{U_{i_0}\times_U\ldots\times_U U_{i_p}}, \mathcal{F}) \\ &= \prod_{i_0\ldots i_p} \mathcal{F}(U_{i_0}\times_U\ldots\times_U U_{i_p})\end{aligned}$$

see Modules on Sites, Lemma 18.4.2. □

Lemma 21.10.4. *Let $\mathcal{C}$ be a category. Let $\mathcal{U} = \{f_i : U_i \to U\}_{i\in I}$ be a family of morphisms with fixed target such that all fibre products $U_{i_0}\times_U\ldots\times_U U_{i_p}$ exist in $\mathcal{C}$. The chain complex $\mathbf{Z}_{\mathcal{U},\bullet}$ of presheaves of Lemma 21.10.3 above is exact in positive degrees, i.e., the homology presheaves $H_i(\mathbf{Z}_{\mathcal{U},\bullet})$ are zero for $i > 0$.*

Proof. Let V be an object of $\mathcal{C}$. We have to show that the chain complex of abelian groups $\mathbf{Z}_{\mathcal{U},\bullet}(V)$ is exact in degrees > 0. This is the complex

$$\begin{array}{c}\vdots\\ \downarrow\\ \bigoplus_{i_0i_1i_2}\mathbf{Z}[\mathrm{Mor}_{\mathcal{C}}(V, U_{i_0}\times_U U_{i_1}\times_U U_{i_2})]\\ \downarrow\\ \bigoplus_{i_0i_1}\mathbf{Z}[\mathrm{Mor}_{\mathcal{C}}(V, U_{i_0}\times_U U_{i_1})]\\ \downarrow\\ \bigoplus_{i_0}\mathbf{Z}[\mathrm{Mor}_{\mathcal{C}}(V, U_{i_0})]\\ \downarrow\\ 0\end{array}$$

For any morphism $\varphi : V \to U$ denote $\mathrm{Mor}_\varphi(V, U_i) = \{\varphi_i : V \to U_i \mid f_i\circ\varphi_i = \varphi\}$. We will use a similar notation for $\mathrm{Mor}_\varphi(V, U_{i_0}\times_U\ldots\times_U U_{i_p})$. Note that composing with the various projection maps between the fibred products $U_{i_0}\times_U\ldots\times_U U_{i_p}$ preserves these morphism sets. Hence we see that the complex above is the same as the complex

$$\begin{array}{c}\vdots\\ \downarrow\\ \bigoplus_\varphi\bigoplus_{i_0i_1i_2}\mathbf{Z}[\mathrm{Mor}_\varphi(V, U_{i_0}\times_U U_{i_1}\times_U U_{i_2})]\\ \downarrow\\ \bigoplus_\varphi\bigoplus_{i_0i_1}\mathbf{Z}[\mathrm{Mor}_\varphi(V, U_{i_0}\times_U U_{i_1})]\\ \downarrow\\ \bigoplus_\varphi\bigoplus_{i_0}\mathbf{Z}[\mathrm{Mor}_\varphi(V, U_{i_0})]\\ \downarrow\\ 0\end{array}$$

Next, we make the remark that we have

$$\mathrm{Mor}_\varphi(V, U_{i_0} \times_U \ldots \times_U U_{i_p}) = \mathrm{Mor}_\varphi(V, U_{i_0}) \times \ldots \times \mathrm{Mor}_\varphi(V, U_{i_p})$$

Using this and the fact that $\mathbf{Z}[A] \oplus \mathbf{Z}[B] = \mathbf{Z}[A \coprod B]$ we see that the complex becomes

$$\begin{array}{c} \ldots \\ \downarrow \\ \bigoplus_\varphi \mathbf{Z}\left[\coprod_{i_0 i_1 i_2} \mathrm{Mor}_\varphi(V, U_{i_0}) \times \mathrm{Mor}_\varphi(V, U_{i_2})\right] \\ \downarrow \\ \bigoplus_\varphi \mathbf{Z}\left[\coprod_{i_0 i_1} \mathrm{Mor}_\varphi(V, U_{i_0}) \times \mathrm{Mor}_\varphi(V, U_{i_1})\right] \\ \downarrow \\ \bigoplus_\varphi \mathbf{Z}\left[\coprod_{i_0} \mathrm{Mor}_\varphi(V, U_{i_0})\right] \\ \downarrow \\ 0 \end{array}$$

Finally, on setting $S_\varphi = \coprod_{i \in I} \mathrm{Mor}_\varphi(V, U_i)$ we see that we get

$$\bigoplus\nolimits_\varphi (\ldots \to \mathbf{Z}[S_\varphi \times S_\varphi \times S_\varphi] \to \mathbf{Z}[S_\varphi \times S_\varphi] \to \mathbf{Z}[S_\varphi] \to 0 \to \ldots)$$

Thus we have simplified our task. Namely, it suffices to show that for any nonempty set S the (extended) complex of free abelian groups

$$\ldots \to \mathbf{Z}[S \times S \times S] \to \mathbf{Z}[S \times S] \to \mathbf{Z}[S] \xrightarrow{\Sigma} \mathbf{Z} \to 0 \to \ldots$$

is exact in all degrees. To see this fix an element $s \in S$, and use the homotopy

$$n_{(s_0, \ldots, s_p)} \longmapsto n_{(s, s_0, \ldots, s_p)}$$

with obvious notations. □

Lemma 21.10.5. *Let $\mathcal{C}$ be a category. Let $\mathcal{U} = \{f_i : U_i \to U\}_{i \in I}$ be a family of morphisms with fixed target such that all fibre products $U_{i_0} \times_U \ldots \times_U U_{i_p}$ exist in $\mathcal{C}$. Let $\mathcal{O}$ be a presheaf of rings on $\mathcal{C}$. The chain complex*

$$\mathbf{Z}_{\mathcal{U},\bullet} \otimes_{p,\mathbf{Z}} \mathcal{O}$$

is exact in positive degrees. Here $\mathbf{Z}_{\mathcal{U},\bullet}$ is the cochain complex of Lemma 21.10.3, and the tensor product is over the constant presheaf of rings with value $\mathbf{Z}$.

Proof. Let V be an object of $\mathcal{C}$. In the proof of Lemma 21.10.4 we saw that $\mathbf{Z}_{\mathcal{U},\bullet}(V)$ is isomorphic as a complex to a direct sum of complexes which are homotopic to $\mathbf{Z}$ placed in degree zero. Hence also $\mathbf{Z}_{\mathcal{U},\bullet}(V) \otimes_{\mathbf{Z}} \mathcal{O}(V)$ is isomorphic as a complex to a direct sum of complexes which are homotopic to $\mathcal{O}(V)$ placed in degree zero. Or you can use Modules on Sites, Lemma 18.28.9, which applies since the presheaves $\mathbf{Z}_{\mathcal{U},i}$ are flat, and the proof of Lemma 21.10.4 shows that $H_0(\mathbf{Z}_{\mathcal{U},\bullet})$ is a flat presheaf also. □

Lemma 21.10.6. *Let $\mathcal{C}$ be a category. Let $\mathcal{U} = \{f_i : U_i \to U\}_{i \in I}$ be a family of morphisms with fixed target such that all fibre products $U_{i_0} \times_U \ldots \times_U U_{i_p}$ exist in $\mathcal{C}$. The Cech cohomology functors $\check{H}^p(\mathcal{U}, -)$ are canonically isomorphic as a δ-functor to the right derived functors of the functor*

$$\check{H}^0(\mathcal{U}, -) : PAb(\mathcal{C}) \longrightarrow Ab.$$

Moreover, there is a functorial quasi-isomorphism

$$\check{\mathcal{C}}^\bullet(\mathcal{U}, \mathcal{F}) \longrightarrow R\check{H}^0(\mathcal{U}, \mathcal{F})$$

where the right hand side indicates the derived functor

$$R\check{H}^0(\mathcal{U}, -) : D^+(PAb(\mathcal{C})) \longrightarrow D^+(\mathbf{Z})$$

of the left exact functor $\check{H}^0(\mathcal{U}, -)$.

Proof. Note that the category of abelian presheaves has enough injectives, see Injectives, Proposition 19.6.1. Note that $\check{H}^0(\mathcal{U}, -)$ is a left exact functor from the category of abelian presheaves to the category of $\mathbf{Z}$-modules. Hence the derived functor and the right derived functor exist, see Derived Categories, Section 13.19.

Let $\mathcal{I}$ be a injective abelian presheaf. In this case the functor $\mathrm{Hom}_{PAb(\mathcal{C})}(-, \mathcal{I})$ is exact on $PAb(\mathcal{C})$. By Lemma 21.10.3 we have

$$\mathrm{Hom}_{PAb(\mathcal{C})}(\mathbf{Z}_{\mathcal{U},\bullet}, \mathcal{I}) = \check{\mathcal{C}}^\bullet(\mathcal{U}, \mathcal{I}).$$

By Lemma 21.10.4 we have that $\mathbf{Z}_{\mathcal{U},\bullet}$ is exact in positive degrees. Hence by the exactness of Hom into $\mathcal{I}$ mentioned above we see that $\check{H}^i(\mathcal{U}, \mathcal{I}) = 0$ for all $i > 0$. Thus the δ-functor $(\check{H}^n, \delta)$ (see Lemma 21.10.2) satisfies the assumptions of Homology, Lemma 12.11.4, and hence is a universal δ-functor.

By Derived Categories, Lemma 13.19.4 also the sequence $R^i\check{H}^0(\mathcal{U}, -)$ forms a universal δ-functor. By the uniqueness of universal δ-functors, see Homology, Lemma 12.11.5 we conclude that $R^i\check{H}^0(\mathcal{U}, -) = \check{H}^i(\mathcal{U}, -)$. This is enough for most applications and the reader is suggested to skip the rest of the proof.

Let $\mathcal{F}$ be any abelian presheaf on $\mathcal{C}$. Choose an injective resolution $\mathcal{F} \to \mathcal{I}^\bullet$ in the category $PAb(\mathcal{C})$. Consider the double complex $A^{\bullet,\bullet}$ with terms

$$A^{p,q} = \check{\mathcal{C}}^p(\mathcal{U}, \mathcal{I}^q).$$

Consider the simple complex $sA^\bullet$ associated to this double complex. There is a map of complexes

$$\check{\mathcal{C}}^\bullet(\mathcal{U}, \mathcal{F}) \longrightarrow sA^\bullet$$

coming from the maps $\check{\mathcal{C}}^p(\mathcal{U}, \mathcal{F}) \to A^{p,0} = \check{\mathcal{C}}^\bullet(\mathcal{U}, \mathcal{I}^0)$ and there is a map of complexes

$$\check{H}^0(\mathcal{U}, \mathcal{I}^\bullet) \longrightarrow sA^\bullet$$

coming from the maps $\check{H}^0(\mathcal{U}, \mathcal{I}^q) \to A^{0,q} = \check{\mathcal{C}}^0(\mathcal{U}, \mathcal{I}^q)$. Both of these maps are quasi-isomorphisms by an application of Homology, Lemma 12.22.6. Namely, the columns of the double complex are exact in positive degrees because the Cech complex as a functor is exact (Lemma 21.10.1) and the rows of the double complex are exact in positive degrees since as we just saw the higher Cech cohomology groups of the injective presheaves $\mathcal{I}^q$ are zero. Since quasi-isomorphisms become invertible in $D^+(\mathbf{Z})$ this gives the last displayed morphism of the lemma. We omit the verification that this morphism is functorial. □

21.11. Cech cohomology and cohomology

The relationship between cohomology and Cech cohomology comes from the fact that the Cech cohomology of an injective abelian sheaf is zero. To see this we note that an injective abelian sheaf is an injective abelian presheaf and then we apply results in Cech cohomology in the preceding section.

Lemma 21.11.1. *Let $\mathcal{C}$ be a site. An injective abelian sheaf is also injective as an object in the category $PAb(\mathcal{C})$.*

Proof. Apply Homology, Lemma 12.25.1 to the categories $\mathcal{A} = Ab(\mathcal{C})$, $\mathcal{B} = PAb(\mathcal{C})$, the inclusion functor and sheafification. (See Modules on Sites, Section 18.3 to see that all assumptions of the lemma are satisfied.) □

Lemma 21.11.2. *Let $\mathcal{C}$ be a site. Let $\mathcal{U} = \{U_i \to U\}_{i \in I}$ be a covering of $\mathcal{C}$. Let $\mathcal{I}$ be an injective abelian sheaf, i.e., an injective object of $Ab(\mathcal{C})$. Then*

$$\check{H}^p(\mathcal{U}, \mathcal{I}) = \begin{cases} \mathcal{I}(U) & \text{if} \quad p = 0 \\ 0 & \text{if} \quad p > 0 \end{cases}$$

Proof. By Lemma 21.11.1 we see that $\mathcal{I}$ is an injective object in $PAb(\mathcal{C})$. Hence we can apply Lemma 21.10.6 (or its proof) to see the vanishing of higher Cech cohomology group. For the zeroth see Lemma 21.9.2. □

Lemma 21.11.3. *Let $\mathcal{C}$ be a site. Let $\mathcal{U} = \{U_i \to U\}_{i \in I}$ be a covering of $\mathcal{C}$. There is a transformation*

$$\check{\mathcal{C}}^\bullet(\mathcal{U}, -) \longrightarrow R\Gamma(U, -)$$

of functors $Ab(\mathcal{C}) \to D^+(\mathbf{Z})$. In particular this gives a transformation of functors $\check{H}^p(U, \mathcal{F}) \to H^p(U, \mathcal{F})$ for $\mathcal{F}$ ranging over $Ab(\mathcal{C})$.

Proof. Let $\mathcal{F}$ be an abelian sheaf. Choose an injective resolution $\mathcal{F} \to \mathcal{I}^\bullet$. Consider the double complex $A^{\bullet,\bullet}$ with terms $A^{p,q} = \check{\mathcal{C}}^p(\mathcal{U}, \mathcal{I}^q)$. Moreover, consider the associated simple complex $sA^\bullet$, see Homology, Definition 12.22.2. There is a map of complexes

$$\alpha : \Gamma(U, \mathcal{I}^\bullet) \longrightarrow sA^\bullet$$

coming from the maps $\mathcal{I}^q(U) \to \check{H}^0(\mathcal{U}, \mathcal{I}^q)$ and a map of complexes

$$\beta : \check{\mathcal{C}}^\bullet(\mathcal{U}, \mathcal{F}) \longrightarrow sA^\bullet$$

coming from the map $\mathcal{F} \to \mathcal{I}^0$. We can apply Homology, Lemma 12.22.6 to see that α is a quasi-isomorphism. Namely, Lemma 21.11.2 implies that the qth row of the double complex $A^{\bullet,\bullet}$ is a resolution of $\Gamma(U, \mathcal{I}^q)$. Hence α becomes invertible in $D^+(\mathbf{Z})$ and the transformation of the lemma is the composition of β followed by the inverse of α. We omit the verification that this is functorial. □

Lemma 21.11.4. *Let $\mathcal{C}$ be a site. Consider the functor $i : Ab(\mathcal{C}) \to PAb(\mathcal{C})$. It is a left exact functor with right derived functors given by*

$$R^p i(\mathcal{F}) = \underline{H}^p(\mathcal{F}) : U \longmapsto H^p(U, \mathcal{F})$$

see discussion in Section 21.8.

Proof. It is clear that i is left exact. Choose an injective resolution $\mathcal{F} \to \mathcal{I}^\bullet$. By definition $R^p i$ is the pth cohomology *presheaf* of the complex $\mathcal{I}^\bullet$. In other words, the sections of $R^p i(\mathcal{F})$ over an open U are given by

$$\frac{\mathrm{Ker}(\mathcal{I}^n(U) \to \mathcal{I}^{n+1}(U))}{\mathrm{Im}(\mathcal{I}^{n-1}(U) \to \mathcal{I}^n(U))}.$$

which is the definition of $H^p(U, \mathcal{F})$. □

Lemma 21.11.5. *Let $\mathcal{C}$ be a site. Let $\mathcal{U} = \{U_i \to U\}_{i \in I}$ be a covering of $\mathcal{C}$. For any abelian sheaf $\mathcal{F}$ there is a spectral sequence $(E_r, d_r)_{r \geq 0}$ with*

$$E_2^{p,q} = \check{H}^p(\mathcal{U}, \underline{H}^q(\mathcal{F}))$$

converging to $H^{p+q}(U, \mathcal{F})$. This spectral sequence is functorial in $\mathcal{F}$.

Proof. This is a Grothendieck spectral sequence (see Derived Categories, Lemma 13.21.2) for the functors

$$i : Ab(\mathcal{C}) \to PAb(\mathcal{C}) \quad \text{and} \quad \check{H}^0(\mathcal{U}, -) : PAb(\mathcal{C}) \to Ab.$$

Namely, we have $\check{H}^0(\mathcal{U}, i(\mathcal{F})) = \mathcal{F}(U)$ by Lemma 21.9.2. We have that $i(\mathcal{I})$ is Cech acyclic by Lemma 21.11.2. And we have that $\check{H}^p(\mathcal{U}, -) = R^p\check{H}^0(\mathcal{U}, -)$ as functors on $PAb(\mathcal{C})$ by Lemma 21.10.6. Putting everything together gives the lemma. □

Lemma 21.11.6. *Let $\mathcal{C}$ be a site. Let $\mathcal{U} = \{U_i \to U\}_{i \in I}$ be a covering. Let $\mathcal{F} \in \mathrm{Ob}(Ab(\mathcal{C}))$. Assume that $H^i(U_{i_0} \times_U \ldots \times_U U_{i_p}, \mathcal{F}) = 0$ for all $i > 0$, all $p \geq 0$ and all $i_0, \ldots, i_p \in I$. Then $\check{H}^p(\mathcal{U}, \mathcal{F}) = H^p(U, \mathcal{F})$.*

Proof. We will use the spectral sequence of Lemma 21.11.5. The assumptions mean that $E_2^{p,q} = 0$ for all (p, q) with $q \neq 0$. Hence the spectral sequence degenerates at E_2 and the result follows. □

Lemma 21.11.7. *Let $\mathcal{C}$ be a site. Let*

$$0 \to \mathcal{F} \to \mathcal{G} \to \mathcal{H} \to 0$$

be a short exact sequence of abelian sheaves on $\mathcal{C}$. Let U be an object of $\mathcal{C}$. If there exists a cofinal system of coverings $\mathcal{U}$ of U such that $\check{H}^1(\mathcal{U}, \mathcal{F}) = 0$, then the map $\mathcal{G}(U) \to \mathcal{H}(U)$ is surjective.

Proof. Take an element $s \in \mathcal{H}(U)$. Choose a covering $\mathcal{U} = \{U_i \to U\}_{i \in I}$ such that (a) $\check{H}^1(\mathcal{U}, \mathcal{F}) = 0$ and (b) $s|_{U_i}$ is the image of a section $s_i \in \mathcal{G}(U_i)$. Since we can certainly find a covering such that (b) holds it follows from the assumptions of the lemma that we can find a covering such that (a) and (b) both hold. Consider the sections

$$s_{i_0 i_1} = s_{i_1}|_{U_{i_0} \times_U U_{i_1}} - s_{i_0}|_{U_{i_0} \times_U U_{i_1}}.$$

Since s_i lifts s we see that $s_{i_0 i_1} \in \mathcal{F}(U_{i_0} \times_U U_{i_1})$. By the vanishing of $\check{H}^1(\mathcal{U}, \mathcal{F})$ we can find sections $t_i \in \mathcal{F}(U_i)$ such that

$$s_{i_0 i_1} = t_{i_1}|_{U_{i_0} \times_U U_{i_1}} - t_{i_0}|_{U_{i_0} \times_U U_{i_1}}.$$

Then clearly the sections $s_i - t_i$ satisfy the sheaf condition and glue to a section of $\mathcal{G}$ over U which maps to s. Hence we win. □

Lemma 21.11.8. *(Variant of Cohomology, Lemma 20.12.7.) Let $\mathcal{C}$ be a site. Let $Cov_{\mathcal{C}}$ be the set of coverings of $\mathcal{C}$ (see Sites, Definition 7.6.2). Let $\mathcal{B} \subset \mathrm{Ob}(\mathcal{C})$, and $Cov \subset Cov_{\mathcal{C}}$ be subsets. Let $\mathcal{F}$ be an abelian sheaf on $\mathcal{C}$. Assume that*

(1) *For every* $\mathcal{U} \in Cov$, $\mathcal{U} = \{U_i \to U\}_{i \in I}$ *we have* $U, U_i \in \mathcal{B}$ *and every* $U_{i_0} \times_U \ldots \times_U U_{i_p} \in \mathcal{B}$.
(2) *For every* $U \in \mathcal{B}$ *the coverings of* U *occurring in* Cov *is a cofinal system of coverings of* U.
(3) *For every* $\mathcal{U} \in Cov$ *we have* $\check{H}^p(\mathcal{U}, \mathcal{F}) = 0$ *for all* $p > 0$.

Then $H^p(U, \mathcal{F}) = 0$ *for all* $p > 0$ *and any* $U \in \mathcal{B}$.

Proof. Let $\mathcal{F}$ and Cov be as in the lemma. We will indicate this by saying "$\mathcal{F}$ has vanishing higher Cech cohomology for any $\mathcal{U} \in$ Cov". Choose an embedding $\mathcal{F} \to \mathcal{I}$ into an injective abelian sheaf. By Lemma 21.11.2 $\mathcal{I}$ has vanishing higher Cech cohomology for any $\mathcal{U} \in$ Cov. Let $\mathcal{Q} = \mathcal{I}/\mathcal{F}$ so that we have a short exact sequence

$$0 \to \mathcal{F} \to \mathcal{I} \to \mathcal{Q} \to 0.$$

By Lemma 21.11.7 and our assumption (2) this sequence gives rise to an exact sequence

$$0 \to \mathcal{F}(U) \to \mathcal{I}(U) \to \mathcal{Q}(U) \to 0.$$

for every $U \in \mathcal{B}$. Hence for any $\mathcal{U} \in$ Cov we get a short exact sequence of Cech complexes

$$0 \to \check{\mathcal{C}}^\bullet(\mathcal{U}, \mathcal{F}) \to \check{\mathcal{C}}^\bullet(\mathcal{U}, \mathcal{I}) \to \check{\mathcal{C}}^\bullet(\mathcal{U}, \mathcal{Q}) \to 0$$

since each term in the Cech complex is made up out of a product of values over elements of $\mathcal{B}$ by assumption (1). In particular we have a long exact sequence of Cech cohomology groups for any covering $\mathcal{U} \in$ Cov. This implies that $\mathcal{Q}$ is also an abelian sheaf with vanishing higher Cech cohomology for all $\mathcal{U} \in$ Cov.

Next, we look at the long exact cohomology sequence

$$\begin{array}{ccccccc} 0 & \longrightarrow & H^0(U, \mathcal{F}) & \longrightarrow & H^0(U, \mathcal{I}) & \longrightarrow & H^0(U, \mathcal{Q}) \\ & & & \searrow & & & \\ & & H^1(U, \mathcal{F}) & \longrightarrow & H^1(U, \mathcal{I}) & \longrightarrow & H^1(U, \mathcal{Q}) \\ & & & \searrow & & & \\ & & \ldots & & \ldots & & \ldots \end{array}$$

for any $U \in \mathcal{B}$. Since $\mathcal{I}$ is injective we have $H^n(U, \mathcal{I}) = 0$ for $n > 0$ (see Derived Categories, Lemma 13.19.4). By the above we see that $H^0(U, \mathcal{I}) \to H^0(U, \mathcal{Q})$ is surjective and hence $H^1(U, \mathcal{F}) = 0$. Since $\mathcal{F}$ was an arbitrary abelian sheaf with vanishing higher Cech cohomology for all $\mathcal{U} \in$ Cov we conclude that also $H^1(U, \mathcal{Q}) = 0$ since $\mathcal{Q}$ is another of these sheaves (see above). By the long exact sequence this in turn implies that $H^2(U, \mathcal{F}) = 0$. And so on and so forth. □

21.12. Cohomology of modules

Everything that was said for cohomology of abelian sheaves goes for cohomology of modules, since the two agree.

Lemma 21.12.1. *Let* $(\mathcal{C}, \mathcal{O})$ *be a ringed site. An injective sheaf of modules is also injective as an object in the category* $PMod(\mathcal{O})$.

Proof. Apply Homology, Lemma 12.25.1 to the categories $\mathcal{A} = Mod(\mathcal{O})$, $\mathcal{B} = PMod(\mathcal{O})$, the inclusion functor and sheafification. (See Modules on Sites, Section 18.11 to see that all assumptions of the lemma are satisfied.) □

Lemma 21.12.2. *Let $(\mathcal{C}, \mathcal{O})$ be a ringed site. Consider the functor $i : Mod(\mathcal{C}) \to PMod(\mathcal{C})$. It is a left exact functor with right derived functors given by*

$$R^p i(\mathcal{F}) = \underline{H}^p(\mathcal{F}) : U \longmapsto H^p(U, \mathcal{F})$$

see discussion in Section 21.8.

Proof. It is clear that i is left exact. Choose an injective resolution $\mathcal{F} \to \mathcal{I}^\bullet$ in $Mod(\mathcal{O})$. By definition $R^p i$ is the pth cohomology *presheaf* of the complex $\mathcal{I}^\bullet$. In other words, the sections of $R^p i(\mathcal{F})$ over an open U are given by

$$\frac{\mathrm{Ker}(\mathcal{I}^n(U) \to \mathcal{I}^{n+1}(U))}{\mathrm{Im}(\mathcal{I}^{n-1}(U) \to \mathcal{I}^n(U))}.$$

which is the definition of $H^p(U, \mathcal{F})$. □

Lemma 21.12.3. *Let $(\mathcal{C}, \mathcal{O})$ be a ringed site. Let $\mathcal{U} = \{U_i \to U\}_{i \in I}$ be a covering of $\mathcal{C}$. Let $\mathcal{I}$ be an injective $\mathcal{O}$-module, i.e., an injective object of $Mod(\mathcal{O})$. Then*

$$\check{H}^p(\mathcal{U}, \mathcal{I}) = \begin{cases} \mathcal{I}(U) & \text{if } p = 0 \\ 0 & \text{if } p > 0 \end{cases}$$

Proof. Lemma 21.10.3 gives the first equality in the following sequence of equalities

$$\begin{aligned} \check{\mathcal{C}}^\bullet(\mathcal{U}, \mathcal{I}) &= \mathrm{Mor}_{PAb(\mathcal{C})}(\mathbf{Z}_{\mathcal{U},\bullet}, \mathcal{I}) \\ &= \mathrm{Mor}_{PMod(\mathbf{Z})}(\mathbf{Z}_{\mathcal{U},\bullet}, \mathcal{I}) \\ &= \mathrm{Mor}_{PMod(\mathcal{O})}(\mathbf{Z}_{\mathcal{U},\bullet} \otimes_{p,\mathbf{Z}} \mathcal{O}, \mathcal{I}) \end{aligned}$$

The third equality by Modules on Sites, Lemma 18.9.2. By Lemma 21.12.1 we see that $\mathcal{I}$ is an injective object in $PMod(\mathcal{O})$. Hence $\mathrm{Hom}_{PMod(\mathcal{O})}(-, \mathcal{I})$ is an exact functor. By Lemma 21.10.5 we see the vanishing of higher Cech cohomology groups. For the zeroth see Lemma 21.9.2. □

Lemma 21.12.4. *Let $\mathcal{C}$ be a site. Let $\mathcal{O}$ be a sheaf of rings on $\mathcal{C}$. Let $\mathcal{F}$ be an $\mathcal{O}$-module, and denote $\mathcal{F}_{ab}$ the underlying sheaf of abelian groups. Then we have*

$$H^i(\mathcal{C}, \mathcal{F}_{ab}) = H^i(\mathcal{C}, \mathcal{F})$$

and for any object U of $\mathcal{C}$ we also have

$$H^i(U, \mathcal{F}_{ab}) = H^i(U, \mathcal{F}).$$

Here the left hand side is cohomology computed in $Ab(\mathcal{C})$ and the right hand side is cohomology computed in $Mod(\mathcal{O})$.

Proof. By Derived Categories, Lemma 13.19.4 the δ-functor $(\mathcal{F} \mapsto H^p(U, \mathcal{F}))_{p \geq 0}$ is universal. The functor $Mod(\mathcal{O}) \to Ab(\mathcal{C})$, $\mathcal{F} \mapsto \mathcal{F}_{ab}$ is exact. Hence $(\mathcal{F} \mapsto H^p(U, \mathcal{F}_{ab}))_{p \geq 0}$ is a δ-functor also. Suppose we show that $(\mathcal{F} \mapsto H^p(U, \mathcal{F}_{ab}))_{p \geq 0}$ is also universal. This will imply the second statement of the lemma by uniqueness of universal δ-functors, see Homology, Lemma 12.11.5. Since $Mod(\mathcal{O})$ has enough injectives, it suffices to show that $H^i(U, \mathcal{I}_{ab}) = 0$ for any injective object $\mathcal{I}$ in $Mod(\mathcal{O})$, see Homology, Lemma 12.11.4.

Let $\mathcal{I}$ be an injective object of $Mod(\mathcal{O})$. Apply Lemma 21.11.8 with $\mathcal{F} = \mathcal{I}$, $\mathcal{B} = \mathcal{C}$ and $\mathrm{Cov} = \mathrm{Cov}_{\mathcal{C}}$. Assumption (3) of that lemma holds by Lemma 21.12.3. Hence we see that $H^i(U, \mathcal{I}_{ab}) = 0$ for every object U of $\mathcal{C}$.

If $\mathcal{C}$ has a final object then this also implies the first equality. If not, then according to Sites, Lemma 7.28.5 we see that the ringed topos $(Sh(\mathcal{C}), \mathcal{O})$ is equivalent to a

ringed topos where the underlying site does have a final object. Hence the lemma follows. □

Lemma 21.12.5. *Let $\mathcal{F}_i$ be a family of abelian sheaves on a site $\mathcal{C}$. Then there are canonical maps*

$$H^p(U, \prod\nolimits_{i\in I} \mathcal{F}_i) \longrightarrow \prod\nolimits_{i\in I} H^p(U, \mathcal{F}_i)$$

for any object U of $\mathcal{C}$. For $p = 0$ this map is an isomorphism and for $p = 1$ this map is injective.

Proof. Choose injective resolutions $\mathcal{F}_i \to \mathcal{I}_i^\bullet$. Then $\mathcal{F} = \prod \mathcal{F}_i$ maps to the complex $(\prod \mathcal{I}_i)^\bullet$ which consists of injectives, see Homology, Lemma 12.23.3. Choose an injective resolution $\mathcal{F} \to \mathcal{I}^\bullet$. There exists a map of complexes $\beta : \mathcal{I}^\bullet \to (\prod \mathcal{I}_i)^\bullet$ which induces the identity on $\prod \mathcal{F}_i$, see Derived Categories, Lemma 13.17.6. Since $\Gamma(U, \prod \mathcal{I}_i^p) = \prod \Gamma(U, \mathcal{I}_i^p)$ and since H^p commutes with products (see Homology, Lemma 12.27.1) we obtain a canonical map

$$H^p(U, \prod \mathcal{F}_i) = H^p(\Gamma(U, \mathcal{I}^\bullet)) \longrightarrow H^p(\Gamma(U, (\prod \mathcal{I}_i)^\bullet)) = \prod H^p(U, \mathcal{F}_i).$$

To prove the assertion for H^1, pick an element $\xi \in H^1(U, \mathcal{F})$ which maps to zero in $\prod H^1(U, \mathcal{F}_i)$. By locality of cohomology, see Lemma 21.8.3, there exists a covering $\mathcal{U} = \{U_j \to U\}$ such that $\xi|_{U_j} = 0$ for all j. Hence ξ comes from an element of $\check{\xi} \in \check{H}^1(\mathcal{U}, \mathcal{F})$ in the spectral sequence of Lemma 21.11.5. Since the edge maps $\check{H}^1(\mathcal{U}, \mathcal{F}_i) \to H^1(U, \mathcal{F}_i)$ are injective for all i, and since the image of ξ is zero in $\prod H^1(U, \mathcal{F}_i)$ we see that the image $\check{\xi}_i = 0$ in $\check{H}^1(\mathcal{U}, \mathcal{F}_i)$. However, since $\mathcal{F} = \prod \mathcal{F}_i$ we see that $\check{\mathcal{C}}(\mathcal{U}, \mathcal{F})$ is the product of the complexes $\check{\mathcal{C}}(\mathcal{U}, \mathcal{F}_i)$, hence by Homology, Lemma 12.27.1 we conclude that $\check{\xi} = 0$ as desired. □

Lemma 21.12.6. *Let $(\mathcal{C}, \mathcal{O})$ be a ringed site. Let $a : U' \to U$ be a monomorphism in $\mathcal{C}$. Then for any injective $\mathcal{O}$-module $\mathcal{I}$ the restriction mapping $\mathcal{I}(U) \to \mathcal{I}(U')$ is surjective.*

Proof. Let $j : \mathcal{C}/U \to \mathcal{C}$ and $j' : \mathcal{C}/U' \to \mathcal{C}$ be the localization morphisms (Modules on Sites, Section 18.19). Since $j_!$ is a left adjoint to restriction we see that for any sheaf $\mathcal{F}$ of $\mathcal{O}$-modules

$$\mathrm{Hom}_{\mathcal{O}}(j_!\mathcal{O}_U, \mathcal{F}) = \mathrm{Hom}_{\mathcal{O}_U}(\mathcal{O}_U, \mathcal{F}|_U) = \mathcal{F}(U)$$

Similarly, the sheaf $j'_!\mathcal{O}_{U'}$ represents the functor $\mathcal{F} \mapsto \mathcal{F}(U')$. Moreover below we describe a canonical map of $\mathcal{O}$-modules

$$j'_!\mathcal{O}_{U'} \longrightarrow j_!\mathcal{O}_U$$

which corresponds to the restriction mapping $\mathcal{F}(U) \to \mathcal{F}(U')$ via Yoneda's lemma (Categories, Lemma 4.3.5). It suffices to prove the displayed map of modules is injective, see Homology, Lemma 12.23.2.

To construct our map it suffices to construct a map between the presheaves which assign to an object V of $\mathcal{C}$ the $\mathcal{O}(V)$-module

$$\bigoplus\nolimits_{\varphi' \in \mathrm{Mor}_{\mathcal{C}}(V,U')} \mathcal{O}(V) \quad \text{and} \quad \bigoplus\nolimits_{\varphi \in \mathrm{Mor}_{\mathcal{C}}(V,U)} \mathcal{O}(V)$$

see Modules on Sites, Lemma 18.19.2. We take the map which maps the summand corresponding to φ' to the summand corresponding to $\varphi = a \circ \varphi'$ by the identity

map on $\mathcal{O}(V)$. As a is a monomorphism, this map is injective. As sheafification is exact, the result follows. □

21.13. Limp sheaves

Let $(\mathcal{C}, \mathcal{O})$ be a ringed site. Let K be a sheaf of sets on $\mathcal{C}$ (we intentionally use a roman capital here to distinguish from abelian sheaves). Given an abelian sheaf $\mathcal{F}$ we denote $\mathcal{F}(K) = \mathrm{Mor}_{Sh(\mathcal{C})}(K, \mathcal{F})$. The functor $\mathcal{F} \mapsto \mathcal{F}(K)$ is a left exact functor $Mod(\mathcal{O}) \to Ab$ hence we have its right derived functors. We will denote these $H^p(K, \mathcal{F})$ so that $H^0(K, \mathcal{F}) = \mathcal{F}(K)$.

We mention two special cases. The first is the case where $K = h_U^\#$ for some object U of $\mathcal{C}$. In this case $H^p(K, \mathcal{F}) = H^p(U, \mathcal{F})$, because $\mathrm{Mor}_{Sh(\mathcal{C})}(h_U^\#, \mathcal{F}) = \mathcal{F}(U)$, see Sites, Section 7.13. The second is the case $\mathcal{O} = \mathbf{Z}$ (the constant sheaf). In this case the cohomology groups are functors $H^p(K, -) : Ab(\mathcal{C}) \to Ab$. Here is the analogue of Lemma 21.12.4.

Lemma 21.13.1. *Let $(\mathcal{C}, \mathcal{O})$ be a ringed site. Let K be a sheaf of sets on $\mathcal{C}$. Let $\mathcal{F}$ be an $\mathcal{O}$-module and denote $\mathcal{F}_{ab}$ the underlying sheaf of abelian groups. Then $H^p(K, \mathcal{F}) = H^p(K, \mathcal{F}_{ab})$.*

Proof. Note that both $H^p(K, \mathcal{F})$ and $H^p(K, \mathcal{F}_{ab})$ depend only on the topos, not on the underlying site. Hence by Sites, Lemma 7.28.5 we may replace $\mathcal{C}$ by a "larger" site such that $K = h_U$ for some object U of $\mathcal{C}$. In this case the result follows from Lemma 21.12.4. □

Lemma 21.13.2. *Let $\mathcal{C}$ be a site. Let $K' \to K$ be a surjective map of sheaves of sets on $\mathcal{C}$. Set $K'_p = K' \times_K \ldots \times_K K'$ ($p + 1$-factors). For every abelian sheaf $\mathcal{F}$ there is a spectral sequence with $E_1^{p,q} = H^q(K'_p, \mathcal{F})$ converging to $H^{p+q}(K, \mathcal{F})$.*

Proof. After replacing $\mathcal{C}$ by a "larger" site as in Sites, Lemma 7.28.5 we may assume that K, K' are objects of $\mathcal{C}$ and that $\mathcal{U} = \{K' \to K\}$ is a covering. Then we have the Čech to cohomology spectral sequence of Lemma 21.11.5 whose E_1 page is as indicated in the statement of the lemma. □

Lemma 21.13.3. *Let $\mathcal{C}$ be a site. Let K be a sheaf of sets on $\mathcal{C}$. Consider the morphism of topoi $j : Sh(\mathcal{C}/K) \to Sh(\mathcal{C})$, see Sites, Lemma 7.29.3. Then j^{-1} preserves injectives and $H^p(K, \mathcal{F}) = H^p(\mathcal{C}/K, j^{-1}\mathcal{F})$ for any abelian sheaf $\mathcal{F}$ on $\mathcal{C}$.*

Proof. By Sites, Lemmas 7.29.1 and 7.29.3 the morphism of topoi j is equivalent to a localization. Hence this follows from Lemma 21.8.1. □

Keeping in mind Lemma 21.13.1 we see that the following definition is the "correct one" also for sheaves of modules on ringed sites.

Definition 21.13.4. Let $\mathcal{C}$ be a site. We say an abelian sheaf $\mathcal{F}$ is *limp*[1] if for every sheaf of sets K we have $H^p(K, \mathcal{F}) = 0$ for all $p \geq 1$.

It is clear that being limp is an intrinsic property, i.e., preserved under equivalences of topoi. A limp sheaf has vanishing higher cohomology on all objects of the site, but in general the condition of being limp is strictly stronger. Here is a characterization of limp sheaves which is sometimes useful.

[1]This is probably nonstandard notation. Please email stacks.project@gmail.com if you know the correct terminology.

Lemma 21.13.5. *Let $\mathcal{C}$ be a site. Let $\mathcal{F}$ be an abelian sheaf. If*

(1) *$H^p(U, \mathcal{F}) = 0$ for $p > 0$ and $U \in \mathrm{Ob}(\mathcal{C})$, and*
(2) *for every surjection $K' \to K$ of sheaves of sets the extended Čech complex*

$$0 \to H^0(K, \mathcal{F}) \to H^0(K', \mathcal{F}) \to H^0(K' \times_K K', \mathcal{F}) \to \ldots$$

is exact,

then $\mathcal{F}$ is limp (and the converse holds too).

Proof. By assumption (1) we have $H^p(h_U^\#, g^{-1}\mathcal{I}) = 0$ for all $p > 0$ and all objects U of $\mathcal{C}$. Note that if $K = \coprod K_i$ is a coproduct of sheaves of sets on $\mathcal{C}$ then $H^p(K, g^{-1}\mathcal{I}) = \prod H^p(K_i, g^{-1}\mathcal{I})$. For any sheaf of sets K there exists a surjection

$$K' = \coprod h_{U_i}^\# \longrightarrow K$$

see Sites, Lemma 7.13.5. Thus we conclude that: (*) for every sheaf of sets K there exists a surjection $K' \to K$ of sheaves of sets such that $H^p(K', \mathcal{F}) = 0$ for $p > 0$. We claim that (*) and condition (2) imply that $\mathcal{F}$ is limp. Note that conditions (*) and (2) only depend on $\mathcal{F}$ as an object of the topos $Sh(\mathcal{C})$ and not on the underlying site. (We will not use property (1) in the rest of the proof.)

We are going to prove by induction on $n \geq 0$ that (*) and (2) imply the following induction hypothesis IH_n: $H^p(K, \mathcal{F}) = 0$ for all $0 < p \leq n$ and all sheaves of sets K. Note that IH_0 holds. Assume IH_n. Pick a sheaf of sets K. Pick a surjection $K' \to K$ such that $H^p(K', \mathcal{F}) = 0$ for all $p > 0$. We have a spectral sequence with

$$E_1^{p,q} = H^q(K'_p, \mathcal{F})$$

covering to $H^{p+q}(K, \mathcal{F})$, see Lemma 21.13.2. By IH_n we see that $E_1^{p,q} = 0$ for $0 < q \leq n$ and by assumption (2) we see that $E_2^{p,0} = 0$ for $p > 0$. Finally, we have $E_1^{0,q} = 0$ for $q > 0$ because $H^q(K', \mathcal{F}) = 0$ by choice of K'. Hence we conclude that $H^{n+1}(K, \mathcal{F}) = 0$ because all the terms $E_2^{p,q}$ with $p + q = n + 1$ are zero. □

21.14. The Leray spectral sequence

The key to proving the existence of the Leray spectral sequence is the following lemma.

Lemma 21.14.1. *Let $f : (Sh(\mathcal{C}), \mathcal{O}_\mathcal{C}) \to (Sh(\mathcal{D}), \mathcal{O}_\mathcal{D})$ be a morphism of ringed topoi. Then for any injective object $\mathcal{I}$ in $Mod(\mathcal{O}_\mathcal{C})$ the pushforward $f_*\mathcal{I}$ is limp.*

Proof. Let K be a sheaf of sets on $\mathcal{D}$. By Modules on Sites, Lemma 18.7.2 we may replace $\mathcal{C}$, $\mathcal{D}$ by "larger" sites such that f comes from a morphism of ringed sites induced by a continuous functor $u : \mathcal{D} \to \mathcal{C}$ such that $K = h_V$ for some object V of $\mathcal{D}$.

Thus we have to show that $H^q(V, f_*\mathcal{I})$ is zero for $q > 0$ and all objects V of $\mathcal{D}$ when f is given by a morphism of ringed sites. Let $\mathcal{V} = \{V_j \to V\}$ be any covering of $\mathcal{D}$. Since u is continuous we see that $\mathcal{U} = \{u(V_j) \to u(v)\}$ is a covering of $\mathcal{C}$. Then we have an equality of Čech complexes

$$\check{\mathcal{C}}^\bullet(\mathcal{V}, f_*\mathcal{I}) = \check{\mathcal{C}}^\bullet(\mathcal{U}, \mathcal{I})$$

by the definition of f_*. By Lemma 21.12.3 we see that the cohomology of this complex is zero in positive degrees. We win by Lemma 21.11.8. □

For flat morphisms the functor f_* preserves injective modules. In particular the functor $f_* : Ab(\mathcal{C}) \to Ab(\mathcal{D})$ always transforms injective abelian sheaves into injective abelian sheaves.

Lemma 21.14.2. *Let $f : (Sh(\mathcal{C}), \mathcal{O}_\mathcal{C}) \to (Sh(\mathcal{D}), \mathcal{O}_\mathcal{D})$ be a morphism of ringed topoi. If f is flat, then $f_*\mathcal{I}$ is an injective $\mathcal{O}_\mathcal{D}$-module for any injective $\mathcal{O}_\mathcal{C}$-module $\mathcal{I}$.*

Proof. In this case the functor f^* is exact, see Modules on Sites, Lemma 18.30.2. Hence the result follows from Homology, Lemma 12.25.1. □

Lemma 21.14.3. *Let $(Sh(\mathcal{C}), \mathcal{O}_\mathcal{C})$ be a ringed topos. A limp sheaf is right acyclic for the following functors:*

1. *the functor $H^0(U, -)$ for any object U of $\mathcal{C}$,*
2. *the functor $\mathcal{F} \mapsto \mathcal{F}(K)$ for any presheaf of sets K,*
3. *the functor $\Gamma(\mathcal{C}, -)$ of global sections,*
4. *the functor f_* for any morphism $f : (Sh(\mathcal{C}), \mathcal{O}_\mathcal{C}) \to (Sh(\mathcal{D}), \mathcal{O}_\mathcal{D})$ of ringed topoi.*

Proof. Part (2) is the definition of a limp sheaf. Part (1) is a consequence of (2) as pointed out in the discussion following the definition of limp sheaves. Part (3) is a special case of (2) where $K = e$ is the final object of $Sh(\mathcal{C})$.

To prove (4) we may assume, by Modules on Sites, Lemma 18.7.2 that f is given by a morphism of sites. In this case we see that $R^i f_*$, $i > 0$ of a limp sheaf are zero by the description of higher direct images in Lemma 21.8.4. □

Remark 21.14.4. As a consequence of the results above we find that Derived Categories, Lemma 13.21.1 applies to a number of situations. For example, given a morphism $f : (Sh(\mathcal{C}), \mathcal{O}_\mathcal{C}) \to (Sh(\mathcal{D}), \mathcal{O}_\mathcal{D})$ of ringed topoi we have

$$R\Gamma(\mathcal{D}, Rf_*\mathcal{F}) = R\Gamma(\mathcal{C}, \mathcal{F})$$

for any sheaf of $\mathcal{O}_\mathcal{C}$-modules $\mathcal{F}$. Namely, for an injective $\mathcal{O}_\mathcal{X}$-module $\mathcal{I}$ the $\mathcal{O}_\mathcal{D}$-module $f_*\mathcal{I}$ is limp by Lemma 21.14.1 and a limp sheaf is acyclic for $\Gamma(\mathcal{D}, -)$ by Lemma 21.14.3.

Lemma 21.14.5 (Leray spectral sequence)**.** *Let $f : (Sh(\mathcal{C}), \mathcal{O}_\mathcal{C}) \to (Sh(\mathcal{D}), \mathcal{O}_\mathcal{D})$ be a morphism of ringed topoi. Let $\mathcal{F}^\bullet$ be a bounded below complex of $\mathcal{O}_\mathcal{C}$-modules. There is a spectral sequence*

$$E_2^{p,q} = H^p(\mathcal{D}, R^q f_*(\mathcal{F}^\bullet))$$

converging to $H^{p+q}(\mathcal{C}, \mathcal{F}^\bullet)$.

Proof. This is just the Grothendieck spectral sequence Derived Categories, Lemma 13.21.2 coming from the composition of functors $\Gamma(\mathcal{C}, -) = \Gamma(\mathcal{D}, -) \circ f_*$. To see that the assumptions of Derived Categories, Lemma 13.21.2 are satisfied, see Lemmas 21.14.1 and 21.14.3. □

Lemma 21.14.6. *Let $f : (Sh(\mathcal{C}), \mathcal{O}_\mathcal{C}) \to (Sh(\mathcal{D}), \mathcal{O}_\mathcal{D})$ be a morphism of ringed topoi. Let $\mathcal{F}$ be an $\mathcal{O}_\mathcal{C}$-module.*

1. *If $R^q f_*\mathcal{F} = 0$ for $q > 0$, then $H^p(\mathcal{C}, \mathcal{F}) = H^p(\mathcal{D}, f_*\mathcal{F})$ for all p.*
2. *If $H^p(\mathcal{D}, R^q f_*\mathcal{F}) = 0$ for all q and $p > 0$, then $H^q(\mathcal{C}, \mathcal{F}) = H^0(\mathcal{D}, R^q f_*\mathcal{F})$ for all q.*

Proof. These are two simple conditions that force the Leray spectral sequence to converge. You can also prove these facts directly (without using the spectral sequence) which is a good exercise in cohomology of sheaves. □

Lemma 21.14.7 (Relative Leray spectral sequence)**.** *Let $f : (Sh(\mathcal{C}), \mathcal{O}_\mathcal{C}) \to (Sh(\mathcal{D}), \mathcal{O}_\mathcal{D})$ and $g : (Sh(\mathcal{D}), \mathcal{O}_\mathcal{D}) \to (Sh(\mathcal{E}), \mathcal{O}_\mathcal{E})$ be morphisms of ringed topoi. Let $\mathcal{F}$ be an $\mathcal{O}_\mathcal{C}$-module. There is a spectral sequence with*

$$E_2^{p,q} = R^p g_*(R^q f_* \mathcal{F})$$

converging to $R^{p+q}(g \circ f)_* \mathcal{F}$. This spectral sequence is functorial in $\mathcal{F}$, and there is a version for bounded below complexes of $\mathcal{O}_\mathcal{C}$-modules.

Proof. This is a Grothendieck spectral sequence for composition of functors, see Derived Categories, Lemma 13.21.2 and Lemmas 21.14.1 and 21.14.3. □

21.15. The base change map

In this section we construct the base change map in some cases; the general case is treated in Remark 21.19.2. The discussion in this section avoids using derived pullback by restricting to the case of a base change by a flat morphism of ringed sites. Before we state the result, let us discuss flat pullback on the derived category. Suppose $g : (Sh(\mathcal{C}), \mathcal{O}_\mathcal{C}) \to (Sh(\mathcal{D}), \mathcal{O}_\mathcal{D})$ is a flat morphism of ringed topoi. By Modules on Sites, Lemma 18.30.2 the functor $g^* : Mod(\mathcal{O}_\mathcal{D}) \to Mod(\mathcal{O}_\mathcal{C})$ is exact. Hence it has a derived functor

$$g^* : D(\mathcal{O}_\mathcal{C}) \to D(\mathcal{O}_\mathcal{D})$$

which is computed by simply pulling back an representative of a given object in $D(\mathcal{O}_\mathcal{D})$, see Derived Categories, Lemma 13.16.8. It preserved the bounded (above, below) subcategories. Hence as indicated we indicate this functor by g^* rather than Lg^*.

Lemma 21.15.1. *Let*

$$\begin{array}{ccc} (Sh(\mathcal{C}'), \mathcal{O}_{\mathcal{C}'}) & \xrightarrow{g'} & (Sh(\mathcal{C}), \mathcal{O}_\mathcal{C}) \\ \downarrow{\scriptstyle f'} & & \downarrow{\scriptstyle f} \\ (Sh(\mathcal{D}'), \mathcal{O}_{\mathcal{D}'}) & \xrightarrow{g} & (Sh(\mathcal{D}), \mathcal{O}_\mathcal{D}) \end{array}$$

be a commutative diagram of ringed topoi. Let $\mathcal{F}^\bullet$ be a bounded below complex of $\mathcal{O}_\mathcal{C}$-modules. Assume both g and g' are flat. Then there exists a canonical base change map

$$g^* R f_* \mathcal{F}^\bullet \longrightarrow R(f')_* (g')^* \mathcal{F}^\bullet$$

in $D^+(\mathcal{O}_{\mathcal{D}'})$.

Proof. Choose injective resolutions $\mathcal{F}^\bullet \to \mathcal{I}^\bullet$ and $(g')^* \mathcal{F}^\bullet \to \mathcal{J}^\bullet$. By Lemma 21.14.2 we see that $(g')_* \mathcal{J}^\bullet$ is a complex of injectives representing $R(g')_*(g')^* \mathcal{F}^\bullet$. Hence by Derived Categories, Lemmas 13.17.6 and 13.17.7 the arrow β in the diagram

$$\begin{array}{ccc} (g')_*(g')^* \mathcal{F}^\bullet & \longrightarrow & (g')_* \mathcal{J}^\bullet \\ {\scriptstyle adjunction}\uparrow & & \uparrow{\scriptstyle \beta} \\ \mathcal{F}^\bullet & \longrightarrow & \mathcal{I}^\bullet \end{array}$$

exists and is unique up to homotopy. Pushing down to $\mathcal{D}$ we get

$$f_*\beta : f_*\mathcal{I}^\bullet \longrightarrow f_*(g')_*\mathcal{J}^\bullet = g_*(f')_*\mathcal{J}^\bullet$$

By adjunction of g^* and g_* we get a map of complexes $g^*f_*\mathcal{I}^\bullet \to (f')_*\mathcal{J}^\bullet$. Note that this map is unique up to homotopy since the only choice in the whole process was the choice of the map β and everything was done on the level of complexes. $\square$

21.16. Cohomology and colimits

Let $(\mathcal{C}, \mathcal{O})$ be a ringed site. Let $\mathcal{I} \to \mathit{Mod}(\mathcal{O})$, $i \mapsto \mathcal{F}_i$ be a diagram over the index category $\mathcal{I}$, see Categories, Section 4.14. For each i there is a canonical map $\mathcal{F}_i \to \operatorname{colim}_i \mathcal{F}_i$ which induces a map on cohomology. Hence we get a canonical map

$$\operatorname{colim}_i H^p(U, \mathcal{F}_i) \longrightarrow H^p(U, \operatorname{colim}_i \mathcal{F}_i)$$

for every $p \geq 0$ and every object U of $\mathcal{C}$. These maps are in general not isomorphisms, even for $p = 0$.

The following lemma is the analogue of Sites, Lemma 7.11.2 for cohomology.

Lemma 21.16.1. *Let $\mathcal{C}$ be a site. Let $\mathit{Cov}_\mathcal{C}$ be the set of coverings of $\mathcal{C}$ (see Sites, Definition 7.6.2). Let $\mathcal{B} \subset \mathrm{Ob}(\mathcal{C})$, and $\mathit{Cov} \subset \mathit{Cov}_\mathcal{C}$ be subsets. Assume that*

1. *For every $\mathcal{U} \in \mathit{Cov}$ we have $\mathcal{U} = \{U_i \to U\}_{i \in I}$ with I finite, $U, U_i \in \mathcal{B}$ and every $U_{i_0} \times_U \ldots \times_U U_{i_p} \in \mathcal{B}$.*
2. *For every $U \in \mathcal{B}$ the coverings of U occurring in Cov is a cofinal system of coverings of U.*

Then the map

$$\operatorname{colim}_i H^p(U, \mathcal{F}_i) \longrightarrow H^p(U, \operatorname{colim}_i \mathcal{F}_i)$$

is an isomorphism for every $p \geq 0$, every $U \in \mathcal{B}$, and every filtered diagram $\mathcal{I} \to \mathit{Ab}(\mathcal{C})$.

Proof. To prove the lemma we will argue by induction on p. Note that we require in (1) the coverings $\mathcal{U} \in \mathrm{Cov}$ to be finite, so that all the elements of $\mathcal{B}$ are quasi-compact. Hence (2) and (1) imply that any $U \in \mathcal{B}$ satisfies the hypothesis of Sites, Lemma 7.11.2 (4). Thus we see that the result holds for $p = 0$. Now we assume the lemma holds for p and prove it for $p + 1$.

Choose a filtered diagram $\mathcal{F} : \mathcal{I} \to \mathit{Ab}(\mathcal{C})$, $i \mapsto \mathcal{F}_i$. Since $\mathit{Ab}(\mathcal{C})$ has functorial injective embeddings, see Injectives, Theorem 19.7.4, we can find a morphism of filtered diagrams $\mathcal{F} \to \mathcal{I}$ such that each $\mathcal{F}_i \to \mathcal{I}_i$ is an injective map of abelian sheaves into an injective abelian sheaf. Denote $\mathcal{Q}_i$ the cokernel so that we have short exact sequences

$$0 \to \mathcal{F}_i \to \mathcal{I}_i \to \mathcal{Q}_i \to 0.$$

Since colimits of sheaves are the sheafification of colimits on the level of presheaves, since sheafification is exact, and since filtered colimits of abelian groups are exact (see Algebra, Lemma 10.8.9), we see the sequence

$$0 \to \operatorname{colim}_i \mathcal{F}_i \to \operatorname{colim}_i \mathcal{I}_i \to \operatorname{colim}_i \mathcal{Q}_i \to 0.$$

is also a short exact sequence. We claim that $H^q(U, \mathrm{colim}_i \mathcal{I}_i) = 0$ for all $U \in \mathcal{B}$ and all $q \geq 1$. Accepting this claim for the moment consider the diagram

$$\begin{array}{ccccccc} \mathrm{colim}_i H^p(U, \mathcal{I}_i) & \longrightarrow & \mathrm{colim}_i H^p(U, \mathcal{Q}_i) & \longrightarrow & \mathrm{colim}_i H^{p+1}(U, \mathcal{F}_i) & \longrightarrow & 0 \\ \downarrow & & \downarrow & & \downarrow & & \downarrow \\ H^p(U, \mathrm{colim}_i \mathcal{I}_i) & \longrightarrow & H^p(U, \mathrm{colim}_i \mathcal{Q}_i) & \longrightarrow & H^{p+1}(U, \mathrm{colim}_i \mathcal{F}_i) & \longrightarrow & 0 \end{array}$$

The zero at the lower right corner comes from the claim and the zero at the upper right corner comes from the fact that the sheaves $\mathcal{I}_i$ are injective. The top row is exact by an application of Algebra, Lemma 10.8.9. Hence by the snake lemma we deduce the result for $p + 1$.

It remains to show that the claim is true. We will use Lemma 21.11.8. By the result for $p = 0$ we see that for $\mathcal{U} \in \mathrm{Cov}$ we have

$$\check{\mathcal{C}}^\bullet(\mathcal{U}, \mathrm{colim}_i \mathcal{I}_i) = \mathrm{colim}_i \check{\mathcal{C}}^\bullet(\mathcal{U}, \mathcal{I}_i)$$

because all the $U_{j_0} \times_U \ldots \times_U U_{j_p}$ are in $\mathcal{B}$. By Lemma 21.11.2 each of the complexes in the colimit of Cech complexes is acyclic in degree ≥ 1. Hence by Algebra, Lemma 10.8.9 we see that also the Cech complex $\check{\mathcal{C}}^\bullet(\mathcal{U}, \mathrm{colim}_i \mathcal{I}_i)$ is acyclic in degrees ≥ 1. In other words we see that $\check{H}^p(\mathcal{U}, \mathrm{colim}_i \mathcal{I}_i) = 0$ for all $p \geq 1$. Thus the assumptions of Lemma 21.11.8. are satisfied and the claim follows. □

Let $\mathcal{C}$ be a limit of sites $\mathcal{C}_i$ as in Sites, Lemma 7.11.4. In particular, all coverings in $\mathcal{C}$ and $\mathcal{C}_i$ have finite index sets. Moreover, assume given

(1) an abelian sheaf $\mathcal{F}_i$ on $\mathcal{C}_i$ for all $i \in I$,
(2) for $i' \geq i$ a map $\varphi_{i'i} : f_{i'i}^{-1}\mathcal{F}_i \to \mathcal{F}_{i'}$ of abelian sheaves on $\mathcal{C}_{i'}$

such that $\varphi_{i''i} = \varphi_{i''i'} \circ f_{i''i'}^{-1}\varphi_{i'i}$ whenever $i'' \geq i' \geq i$.

Lemma 21.16.2. *In the situation discussed above set $\mathcal{F} = \mathrm{colim}\, f_i^{-1}\mathcal{F}_i$. Let $0 \in I$, $X_0 \in Ob(\mathcal{C}_0)$, set $X = u_0(X_0)$ and $X_i = u_{i0}(X_0)$ for $i \geq 0$. Then*

$$\mathrm{colim}_{i \geq 0} H^p(X_i, \mathcal{F}_i) = H^p(X, \mathcal{F})$$

for all $p \geq 0$.

Proof. The case $p = 0$ is Sites, Lemma 7.11.4. We prove the result for $p > 0$ by induction on p.

For each i we pick an injection $f_{i0}^{-1}\mathcal{F}_0 \to \mathcal{I}_i$ where $\mathcal{I}_i$ is an injective abelian sheaf on $\mathcal{C}_i$. Then we can consider the family of maps

$$\gamma_i : \mathcal{F}_i \longrightarrow \prod\nolimits_{j \geq i} f_{ji,*}\mathcal{I}_j = \mathcal{G}_i$$

where the component maps are the maps adjoint to the maps $f_{ji}^{-1}\mathcal{F}_i \to \mathcal{F}_j \to \mathcal{I}_j$. There are maps

$$\psi_{i'i} : f_{i'i}^{-1}\mathcal{G}_i \to \mathcal{G}_{i'}$$

whose compenents are the canonical maps $f_{i'i}^{-1} f_{ji,*}\mathcal{I}_j \to f_{ji',*}\mathcal{I}_j$ for $j \geq i'$. Thus we find an injection $\{\gamma_i\} : \{\mathcal{F}_i, \varphi_{i'i}) \to (\mathcal{G}_i, \psi_{i'i})$ of systems of abelian sheaves. Note that $\mathcal{G}_i$ is an injective sheaf of abelian groups on $\mathcal{C}_i$, see Lemma 21.14.2 and Homology, Lemma 12.23.3. Arguing exactly as in the proof of Lemma 21.16.1 we see that it suffices to prove that $H^p(X, \mathrm{colim}\, f_i^{-1}\mathcal{G}_i) = 0$.

Set $\mathcal{G} = \operatorname{colim} f_i^{-1}\mathcal{G}_i$. To show vanishing of cohomology of $\mathcal{G}$ on every object of $\mathcal{C}$ we show that the Čech cohomology of $\mathcal{G}$ for any covering $\mathcal{U}$ of $\mathcal{C}$ is zero (Lemma 21.11.8). The covering $\mathcal{U}$ comes from a covering $\mathcal{U}_i$ of $\mathcal{C}_i$. We have

$$\check{\mathcal{C}}^\bullet(\mathcal{U}, \mathcal{G}) = \operatorname{colim}_{i' \geq i} \check{\mathcal{C}}^\bullet(\mathcal{U}_{i'}, \mathcal{G}_{i'})$$

(with obvious notation) by the case $p = 0$. The right hand side is acyclic in positive degrees as a filtered colimit of acyclic complexes by Lemma 21.11.2. See Algebra, Lemma 10.8.9. □

21.17. Flat resolutions

In this section we redo the arguments of Cohomology, Section 20.25 in the setting of ringed sites and ringed topoi.

Lemma 21.17.1. *Let $(\mathcal{C}, \mathcal{O})$ be a ringed site. Let $\mathcal{G}^\bullet$ be a complex of $\mathcal{O}$-modules. The functor*

$$K(\mathit{Mod}(\mathcal{O})) \longrightarrow K(\mathit{Mod}(\mathcal{O})), \quad \mathcal{F}^\bullet \longmapsto \mathit{Tot}(\mathcal{F}^\bullet \otimes_{\mathcal{O}} \mathcal{G}^\bullet)$$

is an exact functor of triangulated categories.

Proof. Omitted. Hint: See More on Algebra, Lemmas 15.43.1 and 15.43.2. □

Definition 21.17.2. Let $(\mathcal{C}, \mathcal{O})$ be a ringed site. A complex $\mathcal{K}^\bullet$ of $\mathcal{O}$-modules is called *K-flat* if for every acyclic complex $\mathcal{F}^\bullet$ of $\mathcal{O}$-modules the complex

$$\operatorname{Tot}(\mathcal{F}^\bullet \otimes_{\mathcal{O}} \mathcal{K}^\bullet)$$

is acyclic.

Lemma 21.17.3. *Let $(\mathcal{C}, \mathcal{O})$ be a ringed site. Let $\mathcal{K}^\bullet$ be a K-flat complex. Then the functor*

$$K(\mathit{Mod}(\mathcal{O})) \longrightarrow K(\mathit{Mod}(\mathcal{O})), \quad \mathcal{F}^\bullet \longmapsto \mathit{Tot}(\mathcal{F}^\bullet \otimes_{\mathcal{O}} \mathcal{K}^\bullet)$$

transforms quasi-isomorphisms into quasi-isomorphisms.

Proof. Follows from Lemma 21.17.1 and the fact that quasi-isomorphisms are characterized by having acyclic cones. □

Lemma 21.17.4. *Let $(\mathcal{C}, \mathcal{O})$ be a ringed site. If $\mathcal{K}^\bullet$, $\mathcal{L}^\bullet$ are K-flat complexes of $\mathcal{O}$-modules, then $\mathit{Tot}(\mathcal{K}^\bullet \otimes_{\mathcal{O}} \mathcal{L}^\bullet)$ is a K-flat complex of $\mathcal{O}$-modules.*

Proof. Follows from the isomorphism

$$\operatorname{Tot}(\mathcal{M}^\bullet \otimes_{\mathcal{O}} \operatorname{Tot}(\mathcal{K}^\bullet \otimes_{\mathcal{O}} \mathcal{L}^\bullet)) = \operatorname{Tot}(\operatorname{Tot}(\mathcal{M}^\bullet \otimes_{\mathcal{O}} \mathcal{K}^\bullet) \otimes_{\mathcal{O}} \mathcal{L}^\bullet)$$

and the definition. □

Lemma 21.17.5. *Let $(\mathcal{C}, \mathcal{O})$ be a ringed site. Let $(\mathcal{K}_1^\bullet, \mathcal{K}_2^\bullet, \mathcal{K}_3^\bullet)$ be a distinguished triangle in $K(\mathit{Mod}(\mathcal{O}))$. If two out of three of $\mathcal{K}_i^\bullet$ are K-flat, so is the third.*

Proof. Follows from Lemma 21.17.1 and the fact that in a distinguished triangle in $K(\mathit{Mod}(\mathcal{O}))$ if two out of three are acyclic, so is the third. □

Lemma 21.17.6. *Let $(\mathcal{C}, \mathcal{O})$ be a ringed site. A bounded above complex of flat $\mathcal{O}$-modules is K-flat.*

Proof. Let $\mathcal{K}^\bullet$ be a bounded above complex of flat $\mathcal{O}$-modules. Let $\mathcal{L}^\bullet$ be an acyclic complex of $\mathcal{O}$-modules. Note that $\mathcal{L}^\bullet = \operatorname{colim}_m \tau_{\leq m}\mathcal{L}^\bullet$ where we take termwise colimits. Hence also

$$\operatorname{Tot}(\mathcal{K}^\bullet \otimes_{\mathcal{O}} \mathcal{L}^\bullet) = \operatorname{colim}_m \operatorname{Tot}(\mathcal{K}^\bullet \otimes_{\mathcal{O}} \tau_{\leq m}\mathcal{L}^\bullet)$$

termwise. Hence to prove the complex on the left is acyclic it suffices to show each of the complexes on the right is acyclic. Since $\tau_{\leq m}\mathcal{L}^\bullet$ is acyclic this reduces us to the case where $\mathcal{L}^\bullet$ is bounded above. In this case the spectral sequence of Homology, Lemma 12.22.5 has

$${}'E_1^{p,q} = H^p(\mathcal{L}^\bullet \otimes_R \mathcal{K}^q)$$

which is zero as $\mathcal{K}^q$ is flat and $\mathcal{L}^\bullet$ acyclic. Hence we win. □

Lemma 21.17.7. *Let $(\mathcal{C}, \mathcal{O})$ be a ringed site. Let $\mathcal{K}_1^\bullet \to \mathcal{K}_2^\bullet \to \ldots$ be a system of K-flat complexes. Then $\operatorname{colim}_i \mathcal{K}_i^\bullet$ is K-flat.*

Proof. Because we are taking termwise colimits it is clear that

$$\operatorname{colim}_i \operatorname{Tot}(\mathcal{F}^\bullet \otimes_{\mathcal{O}} \mathcal{K}_i^\bullet) = \operatorname{Tot}(\mathcal{F}^\bullet \otimes_{\mathcal{O}} \operatorname{colim}_i \mathcal{K}_i^\bullet)$$

Hence the lemma follows from the fact that filtered colimits are exact. □

Lemma 21.17.8. *Let $(\mathcal{C}, \mathcal{O})$ be a ringed site. For any complex $\mathcal{G}^\bullet$ of $\mathcal{O}$-modules there exists a commutative diagram of complexes of $\mathcal{O}$-modules*

$$\begin{array}{ccccc} \mathcal{K}_1^\bullet & \longrightarrow & \mathcal{K}_2^\bullet & \longrightarrow & \ldots \\ \downarrow & & \downarrow & & \\ \tau_{\leq 1}\mathcal{G}^\bullet & \longrightarrow & \tau_{\leq 2}\mathcal{G}^\bullet & \longrightarrow & \ldots \end{array}$$

with the following properties: (1) the vertical arrows are quasi-isomorphisms, (2) each $\mathcal{K}_n^\bullet$ is a bounded above complex whose terms are direct sums of $\mathcal{O}$-modules of the form $j_{U!}\mathcal{O}_U$, and (3) the maps $\mathcal{K}_n^\bullet \to \mathcal{K}_{n+1}^\bullet$ are termwise split injections whose cokernels are direct sums of $\mathcal{O}$-modules of the form $j_{U!}\mathcal{O}_U$. Moreover, the map $\operatorname{colim} \mathcal{K}_n^\bullet \to \mathcal{G}^\bullet$ is a quasi-isomorphism.

Proof. The existence of the diagram and properties (1), (2), (3) follows immediately from Modules on Sites, Lemma 18.28.6 and Derived Categories, Lemma 13.27.1. The induced map $\operatorname{colim} \mathcal{K}_n^\bullet \to \mathcal{G}^\bullet$ is a quasi-isomorphism because filtered colimits are exact. □

Lemma 21.17.9. *Let $(\mathcal{C}, \mathcal{O})$ be a ringed site. For any complex $\mathcal{G}^\bullet$ of $\mathcal{O}$-modules there exists a K-flat complex $\mathcal{K}^\bullet$ and a quasi-isomorphism $\mathcal{K}^\bullet \to \mathcal{G}^\bullet$.*

Proof. Choose a diagram as in Lemma 21.17.8. Each complex $\mathcal{K}_n^\bullet$ is a bounded above complex of flat modules, see Modules on Sites, Lemma 18.28.5. Hence $\mathcal{K}_n^\bullet$ is K-flat by Lemma 21.17.6. The induced map $\operatorname{colim} \mathcal{K}_n^\bullet \to \mathcal{G}^\bullet$ is a quasi-isomorphism by construction. Since $\operatorname{colim} \mathcal{K}_n^\bullet$ is K-flat by Lemma 21.17.7 we win. □

Lemma 21.17.10. *Let $(\mathcal{C}, \mathcal{O})$ be a ringed site. Let $\alpha : \mathcal{P}^\bullet \to \mathcal{Q}^\bullet$ be a quasi-isomorphism of K-flat complexes of $\mathcal{O}$-modules. For every complex $\mathcal{F}^\bullet$ of $\mathcal{O}$-modules the induced map*

$$Tot(id_{\mathcal{F}^\bullet} \otimes \alpha) : Tot(\mathcal{F}^\bullet \otimes_{\mathcal{O}} \mathcal{P}^\bullet) \longrightarrow Tot(\mathcal{F}^\bullet \otimes_{\mathcal{O}} \mathcal{Q}^\bullet)$$

is a quasi-isomorphism.

Proof. Choose a quasi-isomorphism $\mathcal{K}^\bullet \to \mathcal{F}^\bullet$ with $\mathcal{K}^\bullet$ a K-flat complex, see Lemma 21.17.9. Consider the commutative diagram

$$\begin{array}{ccc} \mathrm{Tot}(\mathcal{K}^\bullet \otimes_{\mathcal{O}} \mathcal{P}^\bullet) & \longrightarrow & \mathrm{Tot}(\mathcal{K}^\bullet \otimes_{\mathcal{O}} \mathcal{Q}^\bullet) \\ \downarrow & & \downarrow \\ \mathrm{Tot}(\mathcal{F}^\bullet \otimes_{\mathcal{O}} \mathcal{P}^\bullet) & \longrightarrow & \mathrm{Tot}(\mathcal{F}^\bullet \otimes_{\mathcal{O}} \mathcal{Q}^\bullet) \end{array}$$

The result follows as by Lemma 21.17.3 the vertical arrows and the top horizontal arrow are quasi-isomorphisms. □

Let $(\mathcal{C}, \mathcal{O})$ be a ringed site. Let $\mathcal{F}^\bullet$ be an object of $D(\mathcal{O})$. Choose a K-flat resolution $\mathcal{K}^\bullet \to \mathcal{F}^\bullet$, see Lemma 21.17.9. By Lemma 21.17.1 we obtain an exact functor of triangulated categories

$$K(\mathcal{O}) \longrightarrow K(\mathcal{O}), \quad \mathcal{G}^\bullet \longmapsto \mathrm{Tot}(\mathcal{G}^\bullet \otimes_{\mathcal{O}} \mathcal{K}^\bullet)$$

By Lemma 21.17.3 this functor induces a functor $D(\mathcal{O}) \to D(\mathcal{O})$ simply because $D(\mathcal{O})$ is the localization of $K(\mathcal{O})$ at quasi-isomorphisms. By Lemma 21.17.10 the resulting functor (up to isomorphism) does not depend on the choice of the K-flat resolution.

Definition 21.17.11. Let $(\mathcal{C}, \mathcal{O})$ be a ringed site. Let $\mathcal{F}^\bullet$ be an object of $D(\mathcal{O})$. The *derived tensor product*

$$- \otimes^{\mathbf{L}}_{\mathcal{O}} \mathcal{F}^\bullet : D(\mathcal{O}) \longrightarrow D(\mathcal{O})$$

is the exact functor of triangulated categories described above.

It is clear from our explicit constructions that there is a canonical isomorphism

$$\mathcal{F}^\bullet \otimes^{\mathbf{L}}_{\mathcal{O}} \mathcal{G}^\bullet \cong \mathcal{G}^\bullet \otimes^{\mathbf{L}}_{\mathcal{O}} \mathcal{F}^\bullet$$

for $\mathcal{G}^\bullet$ and $\mathcal{F}^\bullet$ in $D(\mathcal{O})$. Hence when we write $\mathcal{F}^\bullet \otimes^{\mathbf{L}}_{\mathcal{O}} \mathcal{G}^\bullet$ we will usually be agnostic about which variable we are using to define the derived tensor product with.

Definition 21.17.12. Let $(\mathcal{C}, \mathcal{O})$ be a ringed site. Let $\mathcal{F}$, $\mathcal{G}$ be $\mathcal{O}$-modules. The *Tor*'s of $\mathcal{F}$ and $\mathcal{G}$ are define by the formula

$$\mathrm{Tor}^{\mathcal{O}}_p(\mathcal{F}, \mathcal{G}) = H^{-p}(\mathcal{F} \otimes^{\mathbf{L}}_{\mathcal{O}} \mathcal{G})$$

with derived tensor product as defined above.

This definition implies that for every short exact sequence of $\mathcal{O}$-modules $0 \to \mathcal{F}_1 \to \mathcal{F}_2 \to \mathcal{F}_3 \to 0$ we have a long exact cohomology sequence

$$\begin{array}{ccccccc} \mathcal{F}_1 \otimes_{\mathcal{O}} \mathcal{G} & \longrightarrow & \mathcal{F}_2 \otimes_{\mathcal{O}} \mathcal{G} & \longrightarrow & \mathcal{F}_3 \otimes_{\mathcal{O}} \mathcal{G} & \longrightarrow & 0 \\ & \searrow & & & & & \\ \mathrm{Tor}^{\mathcal{O}}_1(\mathcal{F}_1, \mathcal{G}) & \longrightarrow & \mathrm{Tor}^{\mathcal{O}}_1(\mathcal{F}_2, \mathcal{G}) & \longrightarrow & \mathrm{Tor}^{\mathcal{O}}_1(\mathcal{F}_3, \mathcal{G}) & & \end{array}$$

for every $\mathcal{O}$-module $\mathcal{G}$. This will be called the long exact sequence of Tor associated to the situation.

Lemma 21.17.13. *Let $(\mathcal{C}, \mathcal{O})$ be a ringed site. Let $\mathcal{F}$ be an $\mathcal{O}$-module. The following are equivalent*

(1) *$\mathcal{F}$ is a flat $\mathcal{O}$-module, and*
(2) *$\mathit{Tor}^{\mathcal{O}}_1(\mathcal{F}, \mathcal{G}) = 0$ for every $\mathcal{O}$-module $\mathcal{G}$.*

Proof. If $\mathcal{F}$ is flat, then $\mathcal{F} \otimes_{\mathcal{O}} -$ is an exact functor and the satellites vanish. Conversely assume (2) holds. Then if $\mathcal{G} \to \mathcal{H}$ is injective with cokernel $\mathcal{Q}$, the long exact sequence of Tor shows that the kernel of $\mathcal{F} \otimes_{\mathcal{O}} \mathcal{G} \to \mathcal{F} \otimes_{\mathcal{O}} \mathcal{H}$ is a quotient of $\mathrm{Tor}_1^{\mathcal{O}}(\mathcal{F}, \mathcal{Q})$ which is zero by assumption. Hence $\mathcal{F}$ is flat. □

21.18. Derived pullback

Let $f : (Sh(\mathcal{C}), \mathcal{O}) \to (Sh(\mathcal{C}'), \mathcal{O}')$ be a morphism of ringed topoi. We can use K-flat resolutions to define a derived pullback functor

$$Lf^* : D(\mathcal{O}') \to D(\mathcal{O})$$

However, we have to be a little careful since we haven't yet proved the pullback of a flat module is flat in complete generality, see Modules on Sites, Section 18.38. In this section, we will use the hypothesis that our sites have enough points, but once we improve the result of the aforementioned section, all of the results in this section will hold without the assumption on the existence of points.

Lemma 21.18.1. *Let $f : Sh(\mathcal{C}) \to Sh(\mathcal{C}')$ be a morphism of topoi. Let $\mathcal{O}'$ be a sheaf of rings on $\mathcal{C}'$. Assume $\mathcal{C}$ has enough points. For any complex of $\mathcal{O}'$-modules $\mathcal{G}^\bullet$, there exists a quasi-isomorphism $\mathcal{K}^\bullet \to \mathcal{G}^\bullet$ such that $\mathcal{K}^\bullet$ is a K-flat complex of $\mathcal{O}'$-modules and $f^{-1}\mathcal{K}^\bullet$ is a K-flat complex of $f^{-1}\mathcal{O}'$-modules.*

Proof. In the proof of Lemma 21.17.9 we find a quasi-isomorphism $\mathcal{K}^\bullet = \mathrm{colim}_i \mathcal{K}_i^\bullet \to \mathcal{G}^\bullet$ where each $\mathcal{K}_i^\bullet$ is a bounded above complex of flat $\mathcal{O}'$-modules. By Modules on Sites, Lemma 18.38.3 applied to the morphism of ringed topoi $(Sh(\mathcal{C}), f^{-1}\mathcal{O}') \to (Sh(\mathcal{C}'), \mathcal{O}')$ we see that $f^{-1}\mathcal{F}_i^\bullet$ is a bounded above complex of flat $f^{-1}\mathcal{O}'$-modules. Hence $f^{-1}\mathcal{K}^\bullet = \mathrm{colim}_i f^{-1}\mathcal{K}_i^\bullet$ is K-flat by Lemmas 21.17.6 and 21.17.7. □

Remark 21.18.2. It is straightforward to show that the pullback of a K-flat complex is K-flat for a morphism of ringed topoi with enough points; this slightly improves the result of Lemma 21.18.1. However, in applications it seems rather that the explicit form of the K-flat complexes constructed in Lemma 21.17.9 is what is useful (as in the proof above) and not the plain fact that they are K-flat. Note for example that the terms of the complex constructed are each direct sums of modules of the form $j_{U!}\mathcal{O}_U$, see Lemma 21.17.8.

Lemma 21.18.3. *Let $f : (Sh(\mathcal{C}), \mathcal{O}) \to (Sh(\mathcal{C}'), \mathcal{O}')$ be a morphism of ringed topoi. Assume $\mathcal{C}$ has enough points. There exists an exact functor*

$$Lf^* : D(\mathcal{O}') \longrightarrow D(\mathcal{O})$$

of triangulated categories so that $Lf^*\mathcal{K}^\bullet = f^*\mathcal{K}^\bullet$ for any complex as in Lemma 21.18.1 in particular for any bounded above complex of flat $\mathcal{O}'$-modules.

Proof. To see this we use the general theory developed in Derived Categories, Section 13.14. Set $\mathcal{D} = K(\mathcal{O}')$ and $\mathcal{D}' = D(\mathcal{O})$. Let us write $F : \mathcal{D} \to \mathcal{D}'$ the exact functor of triangulated categories defined by the rule $F(\mathcal{G}^\bullet) = f^*\mathcal{G}^\bullet$. We let S be the set of quasi-isomorphisms in $\mathcal{D} = K(\mathcal{O}')$. This gives a situation as in Derived Categories, Situation 13.14.1 so that Derived Categories, Definition 13.14.2 applies. We claim that LF is everywhere defined. This follows from Derived Categories, Lemma 13.14.15 with $\mathcal{P} \subset \mathrm{Ob}(\mathcal{D})$ the collection of complexes $\mathcal{K}^\bullet$ such that $f^{-1}\mathcal{K}^\bullet$ is a K-flat complex of $f^{-1}\mathcal{O}'$-modules: (1) follows from Lemma 21.18.1 and to see

(2) we have to show that for a quasi-isomorphism $\mathcal{K}_1^\bullet \to \mathcal{K}_2^\bullet$ between elements of $\mathcal{P}$ the map $f^*\mathcal{K}_1^\bullet \to f^*\mathcal{K}_2^\bullet$ is a quasi-isomorphism. To see this write this as

$$f^{-1}\mathcal{K}_1^\bullet \otimes_{f^{-1}\mathcal{O}'} \mathcal{O} \longrightarrow f^{-1}\mathcal{K}_2^\bullet \otimes_{f^{-1}\mathcal{O}'} \mathcal{O}$$

The functor f^{-1} is exact, hence the map $f^{-1}\mathcal{K}_1^\bullet \to f^{-1}\mathcal{K}_2^\bullet$ is a quasi-isomorphism. The complexes $f^{-1}\mathcal{K}_1^\bullet$ and $f^{-1}\mathcal{K}_2^\bullet$ are K-flat complexes of $f^{-1}\mathcal{O}'$-modules by our choice of $\mathcal{P}$. Hence Lemma 21.17.10 guarantees that the displayed map is a quasi-isomorphism. Thus we obtain a derived functor

$$LF : D(\mathcal{O}') = S^{-1}\mathcal{D} \longrightarrow \mathcal{D}' = D(\mathcal{O})$$

see Derived Categories, Equation (13.14.9.1). Finally, Derived Categories, Lemma 13.14.15 also guarantees that $LF(\mathcal{K}^\bullet) = F(\mathcal{K}^\bullet) = f^*\mathcal{K}^\bullet$ when $\mathcal{K}^\bullet$ is in $\mathcal{P}$. Since the proof of Lemma 21.18.1 shows that bounded above complexes of flat modules are in $\mathcal{P}$ we win. □

Lemma 21.18.4. *Let $f : (Sh(\mathcal{C}), \mathcal{O}) \to (Sh(\mathcal{D}), \mathcal{O}')$ be a morphism of ringed topoi. Assume $\mathcal{C}$ has enough points. There is a canonical bifunctorial isomorphism*

$$Lf^*(\mathcal{F}^\bullet \otimes^{\mathbf{L}}_{\mathcal{O}'} \mathcal{G}^\bullet) = Lf^*\mathcal{F}^\bullet \otimes^{\mathbf{L}}_{\mathcal{O}} Lf^*\mathcal{G}^\bullet$$

for $\mathcal{F}^\bullet, \mathcal{G}^\bullet \in \mathrm{Ob}(D(\mathcal{O}'))$.

Proof. By Lemma 21.18.1 we may assume that $\mathcal{F}^\bullet$ and $\mathcal{G}^\bullet$ are K-flat complexes of $\mathcal{O}'$-modules such that $f^*\mathcal{F}^\bullet$ and $f^*\mathcal{G}^\bullet$ are K-flat complexes of $\mathcal{O}$-modules. In this case $\mathcal{F}^\bullet \otimes^{\mathbf{L}}_{\mathcal{O}'} \mathcal{G}^\bullet$ is just the total complex associated to the double complex $\mathcal{F}^\bullet \otimes_{\mathcal{O}'} \mathcal{G}^\bullet$. By Lemma 21.17.4 $\mathrm{Tot}(\mathcal{F}^\bullet \otimes_{\mathcal{O}'} \mathcal{G}^\bullet)$ is K-flat also. Hence the isomorphism of the lemma comes from the isomorphism

$$\mathrm{Tot}(f^*\mathcal{F}^\bullet \otimes_{\mathcal{O}} f^*\mathcal{G}^\bullet) \longrightarrow f^*\mathrm{Tot}(\mathcal{F}^\bullet \otimes_{\mathcal{O}'} \mathcal{G}^\bullet)$$

whose constituents are the isomorphisms $f^*\mathcal{F}^p \otimes_{\mathcal{O}} f^*\mathcal{G}^q \to f^*(\mathcal{F}^p \otimes_{\mathcal{O}'} \mathcal{G}^q)$ of Modules on Sites, Lemma 18.26.1. □

Lemma 21.18.5. *Let $f : (Sh(\mathcal{C}), \mathcal{O}) \to (Sh(\mathcal{C}'), \mathcal{O}')$ be a morphism of ringed topoi. There is a canonical bifunctorial isomorphism*

$$\mathcal{F}^\bullet \otimes^{\mathbf{L}}_{\mathcal{O}} Lf^*\mathcal{G}^\bullet = \mathcal{F}^\bullet \otimes^{\mathbf{L}}_{f^{-1}\mathcal{O}_Y} f^{-1}\mathcal{G}^\bullet$$

for $\mathcal{F}^\bullet$ in $D(\mathcal{O})$ and $\mathcal{G}^\bullet$ in $D(\mathcal{O}')$.

Proof. Let $\mathcal{F}$ be an $\mathcal{O}$-module and let $\mathcal{G}$ be an $\mathcal{O}'$-module. Then $\mathcal{F} \otimes_{\mathcal{O}} f^*\mathcal{G} = \mathcal{F} \otimes_{f^{-1}\mathcal{O}'} f^{-1}\mathcal{G}$ because $f^*\mathcal{G} = \mathcal{O} \otimes_{f^{-1}\mathcal{O}'} f^{-1}\mathcal{G}$. The lemma follows from this and the definitions. □

21.19. Cohomology of unbounded complexes

Let $(\mathcal{C}, \mathcal{O})$ be a ringed site. The category $Mod(\mathcal{O})$ is a Grothendieck abelian category: it has all colimits, filtered colimits are exact, and it has a generator, namely

$$\bigoplus\nolimits_{U \in \mathrm{Ob}(\mathcal{C})} j_{U!}\mathcal{O}_U,$$

see Modules on Sites, Section 18.14 and Lemmas 18.28.5 and 18.28.6. By Injectives, Theorem 19.12.6 for every complex $\mathcal{F}^\bullet$ of $\mathcal{O}$-modules there exists an injective quasi-isomorphism $\mathcal{F}^\bullet \to \mathcal{I}^\bullet$ to a K-injective complex of $\mathcal{O}$-modules. Hence we can define

$$R\Gamma(\mathcal{C}, \mathcal{F}^\bullet) = \Gamma(\mathcal{C}, \mathcal{I}^\bullet)$$

and similarly for any left exact functor, see Derived Categories, Lemma 13.28.6. For any morphism of ringed topoi $f : (Sh(\mathcal{C}), \mathcal{O}) \to (Sh(\mathcal{D}), \mathcal{O}')$ we obtain

$$Rf_* : D(\mathcal{O}) \longrightarrow D(\mathcal{O}')$$

on the unbounded derived categories.

Lemma 21.19.1. *Let $f : (Sh(\mathcal{C}), \mathcal{O}) \to (Sh(\mathcal{D}), \mathcal{O}')$ be a morphism of ringed topoi. Assume $\mathcal{C}$ has enough points. The functor Rf_* defined above and the functor Lf^* defined in Lemma 21.18.3 are adjoint:*

$$\mathrm{Hom}_{D(\mathcal{O})}(Lf^*\mathcal{G}^\bullet, \mathcal{F}^\bullet) = \mathrm{Hom}_{D(\mathcal{O}')}(\mathcal{G}^\bullet, Rf_*\mathcal{F}^\bullet)$$

bifunctorially in $\mathcal{F}^\bullet \in \mathrm{Ob}(D(\mathcal{O}))$ and $\mathcal{G}^\bullet \in \mathrm{Ob}(D(\mathcal{O}'))$.

Proof. This follows formally from the fact that Rf_* and Lf^* exist, see Derived Categories, Lemma 13.27.4. □

Remark 21.19.2. The construction of unbounded derived functor Lf^* and Rf_* allows one to construct the base change map in full generality. Namely, suppose that

$$\begin{array}{ccc} (Sh(\mathcal{C}'), \mathcal{O}_{\mathcal{C}'}) & \xrightarrow{g'} & (Sh(\mathcal{C}), \mathcal{O}_{\mathcal{C}}) \\ {\scriptstyle f'}\downarrow & & \downarrow{\scriptstyle f} \\ (Sh(\mathcal{D}'), \mathcal{O}_{\mathcal{D}'}) & \xrightarrow{g} & (Sh(\mathcal{D}), \mathcal{O}_{\mathcal{D}}) \end{array}$$

is a commutative diagram of ringed topoi. Let $\mathcal{F}^\bullet$ be a complex of $\mathcal{O}_{\mathcal{C}}$-modules. Then there exists a canonical base change map

$$Lg^*Rf_*\mathcal{F}^\bullet \longrightarrow R(f')_*L(g')^*\mathcal{F}^\bullet$$

in $D(\mathcal{O}_{\mathcal{D}'})$. Namely, this map is adjoint to a map $L(f')^*Lg^*Rf_*\mathcal{F}^\bullet \to L(g')^*\mathcal{F}^\bullet$ Since $L(f')^*Lg^* = L(g')^*Lf^*$ we see this is the same as a map $L(g')^*Lf^*Rf_*\mathcal{F}^\bullet \to L(g')^*\mathcal{F}^\bullet$ which we can take to be $L(g')^*$ of the adjunction map $Lf^*Rf_*\mathcal{F}^\bullet \to \mathcal{F}^\bullet$.

21.20. Some properties of K-injective complexes

Let $(\mathcal{C}, \mathcal{O})$ be a ringed site. Let U be an object of $\mathcal{C}$. Denote $j : (Sh(\mathcal{C}/U), \mathcal{O}_U) \to (Sh(\mathcal{C}), \mathcal{O})$ the corresponding localization morphism. The pullback functor j^* is exact as it is just the restriction functor. Thus derived pullback Lj^* is computed on any complex by simply restricting the complex. We often simply denote the corresponding functor

$$D(\mathcal{O}) \to D(\mathcal{O}_U), \quad E \mapsto j^*E = E|_U$$

Similarly, extension by zero $j_! : Mod(\mathcal{O}_U) \to Mod(\mathcal{O})$ (see Modules on Sites, Definition 18.19.1) is an exact functor (Modules on Sites, Lemma 18.19.3). Thus it induces a functor

$$j_! : D(\mathcal{O}_U) \to D(\mathcal{O}), \quad F \mapsto j_!F$$

by simply applying $j_!$ to any complex representing the object F.

Lemma 21.20.1. *Let $(\mathcal{C}, \mathcal{O})$ be a ringed site. Let U be an object of $\mathcal{C}$. The restriction of a K-injective complex of $\mathcal{O}$-modules to $\mathcal{C}/U$ is a K-injective complex of $\mathcal{O}_U$-modules.*

Proof. Follows immediately from Derived Categories, Lemma 13.28.10 and the fact that the restriction functor has the exact left adjoint $j_!$. See discussion above. □

Lemma 21.20.2. *Let $(\mathcal{C}, \mathcal{O})$ be a ringed site. Let U be an object of $\mathcal{C}$. Denote $j : (Sh(\mathcal{C}/U), \mathcal{O}_U) \to (Sh(\mathcal{C}), \mathcal{O})$ the corresponding localization morphism. The restriction functor $D(\mathcal{O}) \to D(\mathcal{O}_U)$ is a right adjoint to extension by zero $j_! : D(\mathcal{O}_U) \to D(\mathcal{O})$.*

Proof. We have to show that

$$\mathrm{Hom}_{D(\mathcal{O})}(j_!E, F) = \mathrm{Hom}_{D(\mathcal{O}_U)}(E, F|_U)$$

Choose a complex $\mathcal{E}^\bullet$ of $\mathcal{O}_U$-modules representing E and choose a K-injective complex $\mathcal{I}^\bullet$ representing F. By Lemma 21.20.1 the complex $\mathcal{I}^\bullet|_U$ is K-injective as well. Hence we see that the formula above becomes

$$\mathrm{Hom}_{D(\mathcal{O})}(j_!\mathcal{E}^\bullet, \mathcal{I}^\bullet) = \mathrm{Hom}_{D(\mathcal{O}_U)}(\mathcal{E}^\bullet, \mathcal{I}^\bullet|_U)$$

which holds as $|_U$ and $j_!$ are adjoint functors (Modules on Sites, Lemma 18.19.2) and Derived Categories, Lemma 13.28.2. □

Lemma 21.20.3. *Let $\mathcal{C}$ be a site. Let $\mathcal{O} \to \mathcal{O}'$ be a flat map of sheaves of rings. If $\mathcal{I}^\bullet$ is a K-injective complex of $\mathcal{O}'$-modules, then $\mathcal{I}^\bullet$ is K-injective as a complex of $\mathcal{O}$-modules.*

Proof. This is true because $\mathrm{Hom}_{K(\mathcal{O})}(\mathcal{F}^\bullet, \mathcal{I}^\bullet) = \mathrm{Hom}_{K(\mathcal{O}')}(\mathcal{F}^\bullet \otimes_{\mathcal{O}} \mathcal{O}', \mathcal{I}^\bullet)$ by Modules on Sites, Lemma 18.11.3 and the fact that tensoring with $\mathcal{O}'$ is exact. □

Lemma 21.20.4. *Let $\mathcal{C}$ be a site. Let $\mathcal{O} \to \mathcal{O}'$ be a map of sheaves of rings. If $\mathcal{I}^\bullet$ is a K-injective complex of $\mathcal{O}$-modules, then $\mathcal{H}om_{\mathcal{O}}(\mathcal{O}', I^\bullet)$ is a K-injective complex of $\mathcal{O}'$-modules.*

Proof. This is true because $\mathrm{Hom}_{K(\mathcal{O}')}(\mathcal{G}^\bullet, \mathrm{Hom}_{\mathcal{O}}(\mathcal{O}', \mathcal{I}^\bullet)) = \mathrm{Hom}_{K(\mathcal{O})}(\mathcal{G}^\bullet, \mathcal{I}^\bullet)$ by Modules on Sites, Lemma 18.27.5. □

21.21. Derived and homotopy limits

Let $\mathcal{C}$ be a site. Consider the category $\mathcal{C} \times \mathbf{N}$ with $\mathrm{Mor}((U,n),(V,m)) = \emptyset$ if $n > m$ and $\mathrm{Mor}((U,n),(V,m)) = \mathrm{Mor}(U,V)$ else. We endow this with the structure of a site by letting coverings be families $\{(U_i, n) \to (U, n)\}$ such that $\{U_i \to U\}$ is a covering of $\mathcal{C}$. Then the reader verifies immediately that sheaves on $\mathcal{C} \times \mathbf{N}$ are the same thing as inverse systems of sheaves on $\mathcal{C}$. In particular $Ab(\mathcal{C} \times \mathbf{N})$ is inverse systems of abelian sheaves on $\mathcal{C}$. Consider now the functor

$$\lim : Ab(\mathcal{C} \times \mathbf{N}) \to Ab(\mathcal{C})$$

which takes an inverse system to its limit. This is nothing but g_* where $g : Sh(\mathcal{C} \times \mathbf{N}) \to Sh(\mathcal{C})$ is the morphism of topoi associated to the continuous and cocontinuous functor $\mathcal{C} \times \mathbf{N} \to \mathcal{C}$. (Observe that g^{-1} assigns to a sheaf on $\mathcal{C}$ the corresponding constant inverse system.)

By the general machinery explained above we obtain a derived functor

$$R\lim = Rg_* : D(\mathcal{C} \times \mathbf{N}) \to D(\mathcal{C}).$$

As indicated this functor is often denoted $R\lim$.

On the other hand, the continuous and cocontinuous functors $\mathcal{C} \to \mathcal{C} \times \mathbf{N}$, $U \mapsto (U, n)$ define morphisms of topoi $i_n : Sh(\mathcal{C}) \to Sh(\mathcal{C} \times \mathbf{N})$. Of course i_n^{-1} is the functor which picks the nth term of the inverse system. Thus there are transformations of functors $i_{n+1}^{-1} \to i_n^{-1}$. Hence given $K \in D(\mathcal{C} \times \mathbf{N})$ we get $K_n = i_n^{-1}K \in D(\mathcal{C})$

and maps $K_{n+1} \to K_n$. In Derived Categories, Definition 13.30.2 we have defined the notion of a homotopy limit

$$R\lim K_n \in D(\mathcal{C})$$

We claim the two notions agree (as far as it makes sense).

Lemma 21.21.1. *Let $\mathcal{C}$ be a site. Let K be an object of $D(\mathcal{C} \times \mathbf{N})$. Set $K_n = i_n^{-1}K$ as above. Then*

$$R\lim K \cong R\lim K_n$$

in $D(\mathcal{C})$.

Proof. To calculate $R\lim$ on an object K of $D(\mathcal{C} \times \mathbf{N})$ we choose a K-injective representative $\mathcal{I}^\bullet$ whose terms are injective objects of $Ab(\mathcal{C} \times \mathbf{N})$, see Injectives, Theorem 19.12.6. We may and do think of $\mathcal{I}^\bullet$ as an inverse system of complexes $(\mathcal{I}_n^\bullet)$ and then we see that

$$R\lim K = \lim \mathcal{I}_n^\bullet$$

where the right hand side is the termwise inverse limit.

Let $\mathcal{J} = (\mathcal{J}_n)$ be an injective object of $Ab(\mathcal{C} \times \mathbf{N})$. The morphisms $(U, n) \to (U, n+1)$ are monomorphisms of $\mathcal{C} \times \mathbf{N}$, hence $\mathcal{J}(U, n+1) \to \mathcal{J}(U, n)$ is surjective (Lemma 21.12.6). It follows that $\mathcal{J}_{n+1} \to \mathcal{J}_n$ is surjective as a map of presheaves.

Note that the functor i_n^{-1} has an exact left adjoint $i_{n,!}$. Namely, $i_{n,!}\mathcal{F}$ is the inverse system $\ldots 0 \to 0 \to \mathcal{F} \to \ldots \to \mathcal{F}$. Thus the complexes $i_n^{-1}\mathcal{I}^\bullet = \mathcal{I}_n^\bullet$ are K-injective by Derived Categories, Lemma 13.28.10.

Because we chose our K-injective complex to have injective terms we conclude that

$$0 \to \lim \mathcal{I}_n^\bullet \to \prod \mathcal{I}_n^\bullet \to \prod \mathcal{I}_n^\bullet \to 0$$

is a short exact sequence of complexes of abelian sheaves as it is a short exact sequence of complexes of abelian presheaves. Moreover, the products in the middle and the right represent the products in $D(\mathcal{C})$, see Injectives, Lemma 19.13.4 and its proof (this is where we use that $\mathcal{I}_n^\bullet$ is K-injective). Thus $R\lim K$ is a homotopy limit of the inverse system (K_n) by definition of homotopy limits in triangulated categories. □

21.22. Producing K-injective resolutions

First a technical lemma.

Lemma 21.22.1. *Let $(\mathcal{C}, \mathcal{O})$ be a ringed site. Let $(\mathcal{F}_n^\bullet)$ be an inverse system of complexes of $\mathcal{O}$-modules. Let $m \in \mathbf{Z}$. Suppose given $\mathcal{B} \subset \mathrm{Ob}(\mathcal{C})$ and an integer n_0 such that*

1. *every object of $\mathcal{C}$ has a covering whose members are elements of $\mathcal{B}$,*
2. *for every $U \in \mathcal{B}$*
   (a) *the systems $\mathcal{F}_n^{m-2}(U)$ and $\mathcal{F}_n^{m-1}(U)$ are ML,*
   (b) *the system $H^{m-1}(\mathcal{F}_n^\bullet(U))$ is essentially constant, and*
   (c) *we have $H^m(\mathcal{F}_n^\bullet(U)) = H^m(\mathcal{F}_{n_0}^\bullet(U))$ for all $n \geq n_0$.*

Then $H^m(\lim \mathcal{F}_n^\bullet) \to \lim H^m(\mathcal{F}_n^\bullet) \to H^m(\mathcal{F}_{n_0}^\bullet)$ are isomorphisms of sheaves.

Proof. Set $\mathcal{F}^\bullet = \lim \mathcal{F}_n^\bullet$. Let $U \in \mathcal{B}$. Note that $H^m(\mathcal{F}^\bullet(U))$ is the cohomology of

$$\lim_n \mathcal{F}_n^{m-2}(U) \to \lim_n \mathcal{F}_n^{m-1}(U) \to \lim_n \mathcal{F}_n^m(U) \to \lim_n \mathcal{F}_n^{m+1}(U)$$

in the third spot from the left. By assumptions (2)(a) and (2)(b) we may apply Homology, Lemma 12.26.7 to conclude that

$$H^m(\mathcal{F}^\bullet(U)) = \lim H^m(\mathcal{F}_n^\bullet(U))$$

By assumption (2)(c) we conclude

$$H^m(\mathcal{F}^\bullet(U)) = H^m(\mathcal{F}_n^\bullet(U))$$

for all $n \geq n_0$. By assumption (1) we conclude that the sheafification of $U \mapsto H^m(\mathcal{F}^\bullet(U))$ is equal to the sheafification of $U \mapsto H^m(\mathcal{F}_n^\bullet(U))$ for all $n \geq n_0$. Thus the inverse system of sheaves $H^m(\mathcal{F}_n^\bullet)$ is constant for $n \geq n_0$ with value $H^m(\mathcal{F}^\bullet)$ which proves the lemma. □

Let $(\mathcal{C}, \mathcal{O})$ be a ringed site. Let $\mathcal{F}^\bullet$ be a complex of $\mathcal{O}$-modules. The category $Mod(\mathcal{O})$ has enough injectives, hence we can use Derived Categories, Lemma 13.27.3 produce a diagram

$$\begin{array}{ccccc} \ldots & \longrightarrow & \tau_{\geq -2}\mathcal{F}^\bullet & \longrightarrow & \tau_{\geq -1}\mathcal{F}^\bullet \\ & & \downarrow & & \downarrow \\ \ldots & \longrightarrow & \mathcal{I}_2^\bullet & \longrightarrow & \mathcal{I}_1^\bullet \end{array}$$

in the category of complexes of $\mathcal{O}$-modules such that

(1) the vertical arrows are quasi-isomorphisms,
(2) $\mathcal{I}_n^\bullet$ is a bounded below complex of injectives,
(3) the arrows $\mathcal{I}_{n+1}^\bullet \to \mathcal{I}_n^\bullet$ are termwise split surjections.

The category of $\mathcal{O}$-modules has limits (they are computed on the level of presheaves), hence we can form the termwise limit $\mathcal{I}^\bullet = \lim_n \mathcal{I}_n^\bullet$. By Derived Categories, Lemmas 13.28.4 and 13.28.7 this is a K-injective complex. In general the canonical map

$$\mathcal{F}^\bullet \to \mathcal{I}^\bullet \tag{21.22.1.1}$$

may not be a quasi-isomorphism. In the following lemma we describe some conditions under which it is.

Lemma 21.22.2. *In the situation described above. Denote $\mathcal{H}^i = H^i(\mathcal{F}^\bullet)$ the ith cohomology sheaf. Let $\mathcal{B} \subset \mathrm{Ob}(\mathcal{C})$ be a subset. Let $d \in \mathbf{N}$. Assume*

(1) *every object of $\mathcal{C}$ has a covering whose members are elements of $\mathcal{B}$,*
(2) *for every $U \in \mathcal{B}$ we have $H^p(U, \mathcal{H}^q) = 0$ for $p > d$[2].*

Then (21.22.1.1) is a quasi-isomorphism.

Proof. Let $m \in \mathbf{Z}$. We have to show that the map $\mathcal{F}^\bullet \to \mathcal{I}^\bullet$ induces an isomorphism $\mathcal{H}^m \to H^m(\mathcal{I}^\bullet)$. Since $\mathcal{I}_n^\bullet$ is quasi-isomorphic to $\tau_{\geq -n}\mathcal{F}^\bullet$ it suffices to show that $H^m(\mathcal{I}^\bullet) \to H^m(\mathcal{I}_n^\bullet)$ is an isomorphism for n large enough. To do this we will verify the hypotheses (1), (2)(a), (2)(b), (2)(c) of Lemma 21.22.1.

Hypothesis (1) is assumption (1) above. Hypothesis (2)(a) follows from the fact that the maps $\mathcal{I}_{n+1}^k \to \mathcal{I}_n^k$ are split surjections. We will prove hypothesis (2)(b) and

[2] In fact, analyzing the proof we see that it suffices if there exists a function $d : \mathbf{Z} \to \mathbf{Z} \cup \{+\infty\}$ such that $H^p(U, \mathcal{H}^q) = 0$ for $p > d(q)$ where $q + d(q) \to -\infty$ as $q \to -\infty$

(2)(c) simultaneously by proving that for $U \in \mathcal{B}$ the system $H^m(\mathcal{I}_n^\bullet(U))$ becomes constant for $n \geq -m+d$. Namely, recalling that $\mathcal{I}_n^\bullet$ is quasi-isomorphic to $\tau_{\geq -n}\mathcal{F}^\bullet$ we obtain for all n a distinguished triangle

$$\mathcal{H}^{-n}[n] \to \mathcal{I}_n^\bullet \to \mathcal{I}_{n-1}^\bullet \to \mathcal{H}^{-n}[n+1]$$

(Derived Categories, Remark 13.11.4) in $D(\mathcal{O})$. By assumption (2) we see that if $m > d - n$ then

$$H^m(U, \mathcal{H}^{-n}[n]) = 0 \quad \text{and} \quad H^m(U, \mathcal{H}^{-n}[n+1]) = 0.$$

Observe that $H^m(\mathcal{I}_n^\bullet(U)) = H^m(U, \mathcal{I}_n^\bullet)$ as $\mathcal{I}_n^\bullet$ is a bounded below complex of injectives. Unwinding the long exact sequence of cohomology associated to the distinguished triangle above this implies that

$$H^m(\mathcal{I}_n^\bullet(U)) \to H^m(\mathcal{I}_{n-1}^\bullet(U))$$

is an isomorphism for $m > d - n$, i.e., $n > d - m$ and we win. □

Lemma 21.22.3. *With assumptions and notation as in Lemma 21.22.2. Let K denote the object of $D(\mathcal{O})$ represented by the complex $\mathcal{F}^\bullet$. Then there exists a distinguished triangle*

$$K \to \prod\nolimits_{n\geq 0} \tau_{\geq -n}K \to \prod\nolimits_{n\geq 0} \tau_{\geq -n}K \to K[1]$$

in $D(\mathcal{O})$. In other words, K is a derived limit of its canonical truncations (Derived Categories, Definition 13.30.2).

Proof. The proof of Injectives, Lemma 19.13.4 shows that $\prod \tau_{\geq -n}K$ is represented by the complex $\prod \mathcal{I}_n^\bullet$. Because the transition maps $\mathcal{I}_{n+1}^\bullet \to \mathcal{I}_n^\bullet$ are termwise split surjections, we have a short exact sequence of complexes

$$0 \to \mathcal{I}^\bullet \to \prod \mathcal{I}_n^\bullet \to \prod \mathcal{I}_n^\bullet \to 0$$

Since $\mathcal{I}^\bullet$ represents K by Lemma 21.22.2 the distinguished triangle of the lemma is the distinguished triangle associated to the short exact sequence above (Derived Categories, Lemma 13.11.1). □

The construction above can be used in the following setting. Let $\mathcal{C}$ be a category. Let $\mathrm{Cov}(\mathcal{C}) \supset \mathrm{Cov}'(\mathcal{C})$ be two ways to endow $\mathcal{C}$ with the structure of a site. Denote τ the topology corresponding to $\mathrm{Cov}(\mathcal{C})$ and τ' the topology corresponding to $\mathrm{Cov}'(\mathcal{C})$. Then the identity functor on $\mathcal{C}$ defines a morphism of sites

$$\epsilon : \mathcal{C}_\tau \longrightarrow \mathcal{C}_{\tau'}$$

where ϵ_* is the identity functor on underlying presheaves and where ϵ^{-1} is the τ-sheafification of a τ'-sheaf (hence clearly exact). Let $\mathcal{O}$ be a sheaf of rings for the τ-topology. Then $\mathcal{O}$ is also a sheaf for the τ'-topology and ϵ becomes a morphism of ringed sites

$$\epsilon : (\mathcal{C}_\tau, \mathcal{O}_\tau) \longrightarrow (\mathcal{C}_{\tau'}, \mathcal{O}_{\tau'})$$

In this situation we can sometimes point out subcategories of $D(\mathcal{O}_\tau)$ and $D(\mathcal{O}_{\tau'})$ which are identified by the functors ϵ^* and $R\epsilon_*$.

Lemma 21.22.4. *With $\epsilon : (\mathcal{C}_\tau, \mathcal{O}_\tau) \longrightarrow (\mathcal{C}_{\tau'}, \mathcal{O}_{\tau'})$ as above. Let $\mathcal{B} \subset \mathrm{Ob}(\mathcal{C})$ be a subset. Let $\mathcal{A} \subset PMod(\mathcal{O})$ be a full subcategory. Assume*

1. *every object of $\mathcal{A}$ is a sheaf for the τ-topology,*
2. *$\mathcal{A}$ is a weak Serre subcategory of $Mod(\mathcal{O}_\tau)$,*

(3) *every object of $\mathcal{C}$ has a τ'-covering whose members are elements of $\mathcal{B}$, and*
(4) *for every $U \in \mathcal{B}$ we have $H^p_\tau(U, \mathcal{F}) = 0$, $p > 0$ for all $\mathcal{F} \in \mathcal{A}$.*

Then $\mathcal{A}$ is a weak Serre subcategory of $Mod(\mathcal{O}_\tau)$ and there is an equivalence of triangulated categories $D_{\mathcal{A}}(\mathcal{O}_\tau) = D_{\mathcal{A}}(\mathcal{O}_{\tau'})$ given by ϵ^* and $R\epsilon_*$.

Proof. Note that for $A \in \mathcal{A}$ we can think of A as a sheaf in either topology and (abusing notation) that $\epsilon_* A = A$ and $\epsilon^* A = A$. Consider an exact sequence

$$A_0 \to A_1 \to A_2 \to A_3 \to A_4$$

in $Mod(\mathcal{O}_{\tau'})$ with A_0, A_1, A_3, A_4 in $\mathcal{A}$. We have to show that A_2 is an element of $\mathcal{A}$, see Homology, Definition 12.9.1. Apply the exact functor $\epsilon^* = \epsilon^{-1}$ to conclude that $\epsilon^* A_2$ is an object of $\mathcal{A}$. Consider the map of sequences

$$\begin{array}{ccccccccc} A_0 & \longrightarrow & A_1 & \longrightarrow & A_2 & \longrightarrow & A_3 & \longrightarrow & A_4 \\ \downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow \\ A_0 & \longrightarrow & A_1 & \longrightarrow & \epsilon_*\epsilon^* A_2 & \longrightarrow & A_3 & \longrightarrow & A_4 \end{array}$$

to conclude that $A_2 = \epsilon_*\epsilon^* A_2$ is an object of $\mathcal{A}$. At this point it makes sense to talk about the derived categories $D_{\mathcal{A}}(\mathcal{O}_\tau)$ and $D_{\mathcal{A}}(\mathcal{O}_{\tau'})$, see Derived Categories, Section 13.12.

Since ϵ^* is exact and preserves $\mathcal{A}$, it is clear that we obtain a functor $\epsilon^* : D_{\mathcal{A}}(\mathcal{O}_{\tau'}) \to D_{\mathcal{A}}(\mathcal{O}_\tau)$. We claim that $R\epsilon_*$ is a quasi-inverse. Namely, let $\mathcal{F}^\bullet$ be an object of $D_{\mathcal{A}}(\mathcal{O}_\tau)$. Construct a map $\mathcal{F}^\bullet \to \mathcal{I}^\bullet = \lim \mathcal{I}^\bullet_n$ as in (21.22.1.1). By Lemma 21.22.2 and assumption (4) we see that $\mathcal{F}^\bullet \to \mathcal{I}^\bullet$ is a quasi-isomorphism. Then

$$R\epsilon_* \mathcal{F}^\bullet = \epsilon_* \mathcal{I}^\bullet = \lim_n \epsilon_* \mathcal{I}^\bullet_n$$

For every $U \in \mathcal{B}$ we have

$$H^m(\epsilon_* \mathcal{I}^\bullet_n(U)) = H^m(\mathcal{I}^\bullet_n(U)) = \begin{cases} H^m(\mathcal{F}^\bullet)(U) & \text{if} \quad m \geq -n \\ 0 & \text{if} \quad m < n \end{cases}$$

by the assumed vanishing of (4), the spectral sequence Derived Categories, Lemma 13.20.3, and the fact that $\tau_{\geq -n}\mathcal{F}^\bullet \to \mathcal{I}^\bullet_n$ is a quasi-isomorphism. The maps $\epsilon_* \mathcal{I}^\bullet_{n+1} \to \epsilon_* \mathcal{I}^\bullet_n$ are termwise split surjections as ϵ_* is a functor. Hence we can apply Homology, Lemma 12.26.7 to the sequence of complexes

$$\lim_n \epsilon_* \mathcal{I}^{m-2}_n(U) \to \lim_n \epsilon_* \mathcal{I}^{m-1}_n(U) \to \lim_n \epsilon_* \mathcal{I}^m_n(U) \to \lim_n \epsilon_* \mathcal{I}^{m+1}_n(U)$$

to conclude that $H^m(\epsilon_* \mathcal{I}^\bullet(U)) = H^m(\mathcal{F}^\bullet)(U)$ for $U \in \mathcal{B}$. Sheafifying and using property (3) this proves that $H^m(\epsilon_* \mathcal{I}^\bullet)$ is isomorphic to $\epsilon_* H^m(\mathcal{F}^\bullet)$, i.e., is an object of $\mathcal{A}$. Thus $R\epsilon_*$ indeed gives rise to a functor

$$R\epsilon_* : D_{\mathcal{A}}(\mathcal{O}_\tau) \longrightarrow D_{\mathcal{A}}(\mathcal{O}_{\tau'})$$

For $\mathcal{F}^\bullet \in D_{\mathcal{A}}(\mathcal{O}_\tau)$ the adjunction map $\epsilon^* R\epsilon_* \mathcal{F}^\bullet \to \mathcal{F}^\bullet$ is a quasi-isomorphism as we've seen above that the cohomology sheaves of $R\epsilon_* \mathcal{F}^\bullet$ are $\epsilon_* H^m(\mathcal{F}^\bullet)$. For $\mathcal{G}^\bullet \in D_{\mathcal{A}}(\mathcal{O}_{\tau'})$ the adjunction map $\mathcal{G}^\bullet \to R\epsilon_* \epsilon^* \mathcal{G}^\bullet$ is a quasi-isomorphism for the same reason, i.e., because the cohomology sheaves of $R\epsilon_* \epsilon^* \mathcal{G}^\bullet$ are isomorphic to $\epsilon_* H^m(\epsilon^* \mathcal{G}) = H^m(\mathcal{G}^\bullet)$. $\square$

21.23. Cohomology on Hausdorff and locally quasi-compact spaces

We continue our convention to say "Hausdorff and locally quasi-compact" instead of saying "locally compact" as is often done in the literature. Let LC denote the category whose objects are Hausdorff and locally quasi-compact topological spaces and whose morphisms are continuous maps.

Lemma 21.23.1. *The category LC has fibre products and a final object and hence has arbitrary finite limits. Given morphisms $X \to Z$ and $Y \to Z$ in LC with X and Y quasi-compact, then $X \times_Z Y$ is quasi-compact.*

Proof. The final object is the singleton space. Given morphisms $X \to Z$ and $Y \to Z$ of LC the fibre product $X \times_Z Y$ is a subspace of $X \times Y$. Hence $X \times_Z Y$ is Hausdorff as $X \times Y$ is Hausdorff by Topology, Section 5.3.

If X and Y are quasi-compact, then $X \times Y$ is quasi-compact by Topology, Theorem 5.13.2. Since $X \times_Z Y$ is a closed subset of $X \times Y$ (Topology, Lemma 5.3.4) we find that $X \times_Z Y$ is quasi-compact by Topology, Lemma 5.11.3.

Finally, returning to the general case, if $x \in X$ and $y \in Y$ we can pick quasi-compact neighbourhoods $x \in E \subset X$ and $y \in F \subset Y$ and we find that $E \times_Z F$ is a quasi-compact neighbourhood of (x, y) by the result above. Thus $X \times_Z Y$ is an object of LC by Topology, Lemma 5.12.2. □

We can endow LC with a stronger topology than the usual one.

Definition 21.23.2. Let $\{f_i : X_i \to X\}$ be a family of morphisms with fixed target in the category LC. We say this family is a *qc covering*[3] if for every $x \in X$ there exist $i_1, \ldots, i_n \in I$ and quasi-compact subsets $E_j \subset X_{i_j}$ such that $\bigcup f_{i_j}(E_j)$ is a neighbourhood of x.

Observe that an open covering $X = \bigcup U_i$ of an object of LC gives a qc covering $\{U_i \to X\}$ because X is locally quasi-compact. We start with the obligatory lemma.

Lemma 21.23.3. *Let X be a Hausdorff and locally quasi-compact space, in other words, an object of LC.*

1. *If $X' \to X$ is an isomorphism in LC then $\{X' \to X\}$ is a qc covering.*
2. *If $\{f_i : X_i \to X\}_{i \in I}$ is a qc covering and for each i we have a qc covering $\{g_{ij} : X_{ij} \to X_i\}_{j \in J_i}$, then $\{X_{ij} \to X\}_{i \in I, j \in J_i}$ is a qc covering.*
3. *If $\{X_i \to X\}_{i \in I}$ is a qc covering and $X' \to X$ is a morphism of LC then $\{X' \times_X X_i \to X'\}_{i \in I}$ is a qc covering.*

Proof. Part (1) holds by the remark above that open coverings are qc coverings.

Proof of (2). Let $x \in X$. Choose $i_1, \ldots, i_n \in I$ and $E_a \subset X_{i_a}$ quasi-compact such that $\bigcup f_{i_a}(E_a)$ is a neighbourhood of x. For every $e \in E_a$ we can find a finite subset $J_e \subset J_{i_a}$ and quasi-compact $F_{e,j} \subset X_{ij}$, $j \in J_e$ such that $\bigcup g_{ij}(F_{e,j})$ is a neighbourhood of e. Since E_a is quasi-compact we find a finite collection $e_1, \ldots, e_{m_a}$ such that

$$E_a \subset \bigcup\nolimits_{k=1,\ldots,m_a} \bigcup\nolimits_{j \in J_{e_k}} g_{ij}(F_{e_k,j})$$

Then we find that

$$\bigcup\nolimits_{a=1,\ldots,n} \bigcup\nolimits_{k=1,\ldots,m_a} \bigcup\nolimits_{j \in J_{e_k}} f_i(g_{ij}(F_{e_k,j}))$$

[3] This is nonstandard notation. We chose it to remind the reader of fpqc coverings of schemes.

is a neighbourhood of x.

Proof of (3). Let $x' \in X'$ be a point. Let $x \in X$ be its image. Choose $i_1, \ldots, i_n \in I$ and quasi-compact subsets $E_j \subset X_{i_j}$ such that $\bigcup f_{i_j}(E_j)$ is a neighbourhood of x. Choose a quasi-compact neighbourhood $F \subset X'$ of x' which maps into the quasi-compact neighbourhood $\bigcup f_{i_j}(E_j)$ of x. Then $F \times_X E_j \subset X' \times_X X_{i_j}$ is a quasi-compact subset and F is the image of the map $\coprod F \times_X E_j \to F$. Hence the base change is a qc covering and the proof is finished. □

Besides some set theoretic issues the lemma above shows that LC with the collection of qc coverings forms a site. We will denote this site (suitably modified to overcome the set theoretical issues) LC_{qc}.

Remark 21.23.4 (Set theoretic issues)**.** The category LC is a "big" category as its objects form a proper class. Similarly, the coverings form a proper class. Let us define the *size* of a topological space X to be the cardinality of the set of points of X. Choose a function $Bound$ on cardinals, for example as in Sets, Equation (3.9.1.1). Finally, let S_0 be an initial set of objects objects of LC, for example $S_0 = \{(\mathbf{R}, \text{euclidean topology})\}$. Exactly as in Sets, Lemma 3.9.2 we can choose a limit ordinal α such that $LC_\alpha = LC \cap V_\alpha$ contains S_0 and is preserved under all countable limits and colimits which exist in LC. Moreover, if $X \in LC_\alpha$ and if $Y \in LC$ and $\text{size}(Y) \leq Bound(\text{size}(X))$, then Y is isomorphic to an object of LC_α. Next, we apply Sets, Lemma 3.11.1 to choose set Cov of qc covering on LC_α such that every qc covering in LC_α is combinatorially equivalent to a covering this set. In this way we obtain a site (LC_α, Cov) which we will denote LC_{qc}.

There is a second topology on the site LC_{qc} of Remark 21.23.4. Namely, given an object X we can consider all coverings $\{X_i \to X\}$ of LC_{qc} such that $X_i \to X$ is an open immersion. We denote this site LC_{Zar}. The identity functor $LC_{Zar} \to LC_{qc}$ is continuous and defines a morphism of sites

$$\epsilon : LC_{qc} \to LC_{Zar}$$

by an application of Sites, Proposition 7.15.6.

Consider an object X of the site LC_{qc} constructed in Remark 21.23.4. (Translation for those not worried about set theoretic issues: Let X be a Hausdorff and locally quasi-compact space.) Let X_{Zar} be the site whose objects are opens of X, see Sites, Example 7.6.4. There is a morphism of sites

$$\pi : LC_{Zar}/X \to X_{Zar}$$

given by the continuous functor

$$X_{Zar} \longrightarrow LC_{Zar}/X, \quad U \longmapsto U$$

Namely, X_{Zar} has fibre products and a final object and the functor above commutes with these and Sites, Proposition 7.15.6 applies.

Lemma 21.23.5. *Let X be an object of LC_{qc}. Let $\mathcal{F}$ be a sheaf on X_{Zar}. Then the sheaf $\pi^{-1}\mathcal{F}$ on LC_{Zar}/X is given by the rule*

$$\pi^{-1}\mathcal{F}(Y) = \Gamma(Y_{Zar}, f^{-1}\mathcal{F})$$

for $f : Y \to X$ in LC_{qc}. Moreover $\pi^{-1}\mathcal{F}$ is a sheaf for the qc topology, i.e., the sheaf $\epsilon^{-1}\pi^{-1}\mathcal{F}$ on LC_{qc} is given by the same formula.

Proof. Of course the pullback f^{-1} on the right hand side indicates usual pullback of sheaves on topological spaces (Sites, Example 7.15.2). The equality of the lemma follows directly from the defintions.

Let $\mathcal{V} = \{g_i : Y_i \to Y\}_{i \in I}$ be a covering of LC_{qc}/X. It suffices to show that $\pi^{-1}\mathcal{F}(Y) \to H^0(\mathcal{V}, \pi^{-1}\mathcal{F})$ is an isomorphism, see Sites, Section 7.10. We first point out that the map is injective as a qc covering is surjective and we can detect equality of sections at stalks (use Sheaves, Lemmas 6.11.1 and 6.21.4). Thus we see that $\pi^{-1}\mathcal{F}$ is a separated presheaf on LC_{qc} hence it suffices to show that any element $(s_i) \in H^0(\mathcal{V}, \pi^{-1}\mathcal{F})$ maps to an element in the image of $\pi^{-1}\mathcal{F}(Y)$ after replacing $\mathcal{V}$ by a refinement (Sites, Theorem 7.10.10).

Observe that $\pi^{-1}\mathcal{F}|_{Y_{i,Zar}}$ is the pullback of $f^{-1}\mathcal{F} = \pi^{-1}\mathcal{F}|_{Y_{Zar}}$ under the continuous map $g_i : Y_i \to Y$. Thus we can choose an open covering $Y_i = \bigcup V_{ij}$ such that for each j there is an open $W_{ij} \subset Y$ and a section $t_{ij} \in \pi^{-1}\mathcal{F}(W_{ij})$ such that $s|_{U_{ij}}$ is the pullback of t_{ij}. In other words, after refining the covering $\{Y_i \to Y\}$ we may assume there are opens $W_i \subset Y$ such that $Y_i \to Y$ factors through W_i and sections t_i of $\pi^{-1}\mathcal{F}$ over W_i which restrict to the given sections s_i. Moreover, if $y \in Y$ is in the image of both $Y_i \to Y$ and $Y_j \to Y$, then the images $t_{i,y}$ and $t_{j,y}$ in the stalk $f^{-1}\mathcal{F}_y$ agree (because s_i and s_j agree over $Y_i \times_Y Y_j$). Thus for $y \in Y$ there is a well defined element t_y of $f^{-1}\mathcal{F}_y$ agreeing with $t_{i,y}$ whenever $y \in Y_i$. We will show that the element (t_y) comes from a global section of $f^{-1}\mathcal{F}$ over Y which will finish the proof of the lemma.

It suffices to show that this is true locally on Y, see Sheaves, Section 6.17. Let $y_0 \in Y$. Pick $i_1, \ldots, i_n \in I$ and quasi-compact subsets $E_j \subset Y_{i_j}$ such that $\bigcup g_{i_j}(E_j)$ is a neighbourhood of y_0. Then we can find an open neighbourhood $V \subset Y$ of y_0 contained in $W_{i_1} \cap \ldots \cap W_{i_n}$ such that the sections $t_{i_j}|_V$, $j = 1, \ldots, n$ agree. Hence we see that $(t_y)_{y \in V}$ comes from this section and the proof is finished. □

Lemma 21.23.6. *Let X be an object of LC_{qc}. Let $\mathcal{F}$ be an abelian sheaf on X_{Zar}. Then we have*

$$H^q(X_{Zar}, \mathcal{F}) = H^q(LC_{qc}/X, \epsilon^{-1}\pi^{-1}\mathcal{F})$$

In particular, if A is an abelian group, then we have $H^q(X, \underline{A}) = H^q(LC_{qc}/X, \underline{A})$.

Proof. The statement is more precisely that the canonical map

$$H^q(X_{Zar}, \mathcal{F}) \longrightarrow H^q(LC_{qc}/X, \epsilon^{-1}\pi^{-1}\mathcal{F})$$

is an isomorphism for all q. The result holds for $q = 0$ by Lemma 21.23.5. We argue by induction on q. Pick $q_0 > 0$. We will assume the result holds for $q < q_0$ and prove it for q_0.

Injective. Let $\xi \in H^{q_0}(X, \mathcal{F})$. We may choose an open covering $\mathcal{U} : X = \bigcup U_i$ such that $\xi|_{U_i}$ is zero for all i (Cohomology, Lemma 20.7.2). Then $\mathcal{U}$ is also a covering for the qc topology. Hence we obtain a map

$$E_2^{p,q} = \check{H}^p(\mathcal{U}, \underline{H}^q(\mathcal{F})) \longrightarrow E_2^{p,q} = \check{H}^p(\mathcal{U}, \underline{H}^q(\epsilon^{-1}\pi^{-1}\mathcal{F}))$$

between the spectral sequences of Cohomology, Lemma 20.12.4 and Lemma 21.11.5. Since the maps $\underline{H}^q(\mathcal{F})(U_{i_0 \ldots i_p}) \to \underline{H}^q(\epsilon^{-1}\pi^{-1}\mathcal{F}))(U_{i_0 \ldots i_p})$ are isomorphisms for $q < q_0$ we see that

$$\mathrm{Ker}(H^{q_0}(X, \mathcal{F}) \to \prod H^{q_0}(U_i, \mathcal{F}))$$

maps isomorphically to the corresponding subgroup of $H^{q_0}(LC_{qc}/X, \epsilon^{-1}\pi^{-1}\mathcal{F})$. In this way we conclude that our map is injective for q_0.

Surjective. Let $\xi \in H^{q_0}(LC_{qc}/X, \epsilon^{-1}\pi^{-1}\mathcal{F})$. If for every $x \in X$ we can find a neighbourhood $x \in U \subset X$ such that $\xi|_U = 0$, then we can use the Čech complex argument of the previous paragraph to conclude that ξ is in the image of our map. Fix $x \in X$. We can find a qc covering $\{f_i : X_i \to X\}_{i \in I}$ such that $\xi|_{X_i}$ is zero (Lemma 21.8.3). Pick $i_1, \ldots, i_n \in I$ and $E_j \subset X_{i_j}$ such that $\bigcup f_{i_j}(E_j)$ is a neighbourhood of x. We may replace X by $\bigcup f_{i_j}(E_j)$ and set $Y = \coprod E_{i_j}$. Then $Y \to X$ is a surjective continuous map of Hausdorff and quasi-compact topological spaces, $\xi \in H^{q_0}(LC_{qc}/X, \epsilon^{-1}\pi^{-1}\mathcal{F})$, and $\xi|_Y = 0$. Set $Y_p = Y \times_X \ldots \times_X Y$ ($p+1$-factors) and denote $\mathcal{F}_p$ the pullback of $\mathcal{F}$ to Y_p. Then the spectral sequence

$$E_1^{p,q} = \check{C}^p(\{Y \to X\}, \underline{H}^q(\epsilon^{-1}\pi^{-1}\mathcal{F}))$$

of Lemma 21.11.5 has rows for $q < q_0$ which are (by induction) the complexes

$$H^q(Y_0, \mathcal{F}_0) \to H^q(Y_1, \mathcal{F}_1) \to H^q(Y_2, \mathcal{F}_2) \to \ldots$$

If these complexes were exact in degree $p = q_0 - q$, then the spectral sequence would collapse and ξ would be zero. This is not true in general, but we don't need to show ξ is zero, we just need to show ξ becomes zero after restricting X to a neighbourhood of x. Thus it suffices to show that the complexes

$$\mathrm{colim}_{x \in U \subset X} (H^q(Y_0 \times_X U, \mathcal{F}_0) \to H^q(Y_1 \times_X U, \mathcal{F}_1) \to H^q(Y_2 \times_X U, \mathcal{F}_2) \to \ldots)$$

are exact (some details omitted). By the proper base change theorem in topology (for example Cohomology, Lemma 20.19.1) the colimit is equal to

$$H^q(Y_x, \underline{\mathcal{F}_x}) \to H^q(Y_x^2, \underline{\mathcal{F}_x}) \to H^q(Y_x^3, \underline{\mathcal{F}_x}) \to \ldots$$

where $Y_x \subset Y$ is the fibre of $Y \to X$ over x and where $\underline{\mathcal{F}_x}$ denotes the constant sheaf with value $\mathcal{F}_x$. But the simplicial topological space (Y_x^n) is homotopy equivalent to the constant simplicial space on the singleton $\{x\}$, see Simplicial, Lemma 14.24.9. Since $H^q(-, \underline{\mathcal{F}_x})$ is a functor on the category of topological spaces, we conclude that the cosimplicial abelian group with values $H^q(Y_x^n, \underline{\mathcal{F}_x})$ is homotopy equivalent to the constant cosimplicial abelian group with value

$$H^q(\{x\}, \mathcal{F}_x) = \begin{cases} \mathcal{F}_x & \text{if } q = 0 \\ 0 & \text{else} \end{cases}$$

As the complex associated to a constant cosimplicial group has the required exactness properties this finishes the proof of the lemma. □

Lemma 21.23.7. *Let $f : X \to Y$ be a morphism of LC. If f is proper and surjective, then $\{f : X \to Y\}$ is a qc covering.*

Proof. Let $y \in Y$ be a point. For each $x \in X_y$ choose a quasi-compact neighbourhood $E_x \subset X$. Choose $x \in U_x \subset E_x$ open. Since f is proper the fibre X_y is quasi-compact and we find $x_1, \ldots, x_n \in X_y$ such that $X_y \subset U_{x_1} \cup \ldots \cup U_{x_n}$. We claim that $f(E_{x_1}) \cup \ldots \cup f(E_{x_n})$ is a neighbourhood of y. Namely, as f is closed (Topology, Theorem 5.16.5) we see that $Z = f(X \setminus U_{x_1} \cup \ldots \cup U_{x_n})$ is a closed subset of Y not containing y. As f is surjective we see that $Y \setminus Z$ is contained in $f(E_{x_1}) \cup \ldots \cup f(E_{x_n})$ as desired. □

21.24. Spectral sequences for Ext

In this section we collect various spectral sequences that come up when considering the Ext functors. For any pair of complexes $\mathcal{G}^\bullet, \mathcal{F}^\bullet$ of complexes of modules on a ringed site $(\mathcal{C}, \mathcal{O})$ we denote

$$\mathrm{Ext}^n_{\mathcal{O}}(\mathcal{G}^\bullet, \mathcal{F}^\bullet) = \mathrm{Hom}_{D(\mathcal{O})}(\mathcal{G}^\bullet, \mathcal{F}^\bullet[n])$$

according to our general conventions in Derived Categories, Section 13.26.

Example 21.24.1. Let $(\mathcal{C}, \mathcal{O})$ be a ringed site. Let $\mathcal{K}^\bullet$ be a bounded above complex of $\mathcal{O}$-modules. Let $\mathcal{F}$ be an $\mathcal{O}$-module. Then there is a spectral sequence with E_2-page

$$E_2^{i,j} = \mathrm{Ext}^i_{\mathcal{O}}(H^{-j}(\mathcal{K}^\bullet), \mathcal{F}) \Rightarrow \mathrm{Ext}^{i+j}_{\mathcal{O}}(\mathcal{K}^\bullet, \mathcal{F})$$

and another spectral sequence with E_1-page

$$E_1^{i,j} = \mathrm{Ext}^j_{\mathcal{O}}(\mathcal{K}^{-i}, \mathcal{F}) \Rightarrow \mathrm{Ext}^{i+j}_{\mathcal{O}}(\mathcal{K}^\bullet, \mathcal{F}).$$

To construct these spectral sequences choose an injective resolution $\mathcal{F} \to \mathcal{I}^\bullet$ and consider the two spectral sequences coming from the double complex $\mathrm{Hom}_{\mathcal{O}}(\mathcal{K}^\bullet, \mathcal{I}^\bullet)$, see Homology, Section 12.22.

21.25. Internal hom in the derived category

Let $(\mathcal{C}, \mathcal{O})$ be a ringed site. Let L, M be objects of $D(\mathcal{O})$. We would like to construct an object $R\mathcal{H}om(L, M)$ of $D(\mathcal{O})$ such that for every third object K of $D(\mathcal{O})$ there exists a canonical bijection

$$\mathrm{Hom}_{D(\mathcal{O})}(K, R\mathcal{H}om(L, M)) = \mathrm{Hom}_{D(\mathcal{O})}(K \otimes^{\mathbf{L}}_{\mathcal{O}} L, M) \tag{21.25.0.1}$$

Observe that this formula defines $R\mathcal{H}om(L, M)$ up to unique isomorphism by the Yoneda lemma (Categories, Lemma 4.3.5).

To construct such an object, choose a K-injective complex of $\mathcal{O}$-modules $\mathcal{I}^\bullet$ representing M and any complex of $\mathcal{O}$-modules $\mathcal{L}^\bullet$ representing L. Consider the complex $\mathcal{H}^\bullet$ of $\mathcal{O}$-modules with term

$$\mathcal{H}^n = \prod\nolimits_{n=-p+q} \mathcal{H}om_{\mathcal{O}}(\mathcal{L}^p, \mathcal{I}^q)$$

in degree n and differential given on a local section $\alpha = \prod \alpha_{p,q}$ by the rule

$$\alpha_{p,q} \longmapsto \alpha_{p,q} \circ d^{p-1}_{\mathcal{L}} + (-1)^p d^q_{\mathcal{I}} \circ \alpha_{p,q}$$

With this definition we have

$$H^0(\mathcal{H}^\bullet(U)) = \mathrm{Hom}_{K(\mathcal{O}_U)}(\mathcal{L}^\bullet|_U, \mathcal{I}^\bullet|_U) = \mathrm{Hom}_{D(\mathcal{O}_U)}(\mathcal{L}^\bullet|_U, \mathcal{I}^\bullet|_U)$$

for every object U of $\mathcal{C}$. The second equality comes from the fact that $\mathcal{I}^\bullet|_U$ is K-injective (Lemma 21.20.1). Hence the sheafification $H^0(\mathcal{H}^\bullet)$ of the presheaf $U \mapsto H^0(\mathcal{H}^\bullet(U))$ is independent of the choices made in the complexes representing L and M. A similar argument can be made for the other cohomology sheaves $H^i(\mathcal{H}^\bullet)$. Thus we obtain a well defined object $R\mathcal{H}om(L, M)$ of $D(\mathcal{O})$. Note that the construction is functorial in L and M in an obvious manner.

Lemma 21.25.1. *Let $(\mathcal{C}, \mathcal{O})$ be a ringed space. Let K, L, M be objects of $D(\mathcal{O})$. With the construction as described above there is a canonical isomorphism (21.25.0.1) functorial in K, L, M.*

Proof. Choose a K-injective complex $\mathcal{I}^\bullet$ representing M and a K-flat complex of $\mathcal{O}$-modules $\mathcal{L}^\bullet$ representing L. Let $\mathcal{H}^\bullet$ be the complex described above. For any complex of $\mathcal{O}$-modules $\mathcal{K}^\bullet$ we see that

$$\mathrm{Hom}_{K(\mathcal{O})}(\mathcal{K}^\bullet, \mathcal{H}^\bullet) = \mathrm{Hom}_{K(\mathcal{O})}(\mathrm{Tot}(\mathcal{K}^\bullet \otimes_{\mathcal{O}} \mathcal{L}^\bullet), \mathcal{I}^\bullet)$$

simply because

$$\mathrm{Hom}_{\mathcal{O}}(\mathcal{K}^a, \mathcal{H}om_{\mathcal{O}}(\mathcal{L}^b, \mathcal{I}^c)) = \mathrm{Hom}_{\mathcal{O}}(\mathcal{K}^a \otimes_{\mathcal{O}} \mathcal{L}^b, \mathcal{I}^c)$$

by Modules, Lemma 17.19.1. Since $\mathcal{L}^\bullet$ is K-flat, the complex $\mathrm{Tot}(\mathcal{K}^\bullet \otimes_{\mathcal{O}} \mathcal{L}^\bullet)$ (a) is acyclic if $\mathcal{K}^\bullet$ is acyclic, and (b) represents the derived tensor product $K \otimes^{\mathbf{L}}_{\mathcal{O}} L$. Thus $\mathcal{H}^\bullet$ is a K-injective complex, as $\mathcal{I}^\bullet$ is K-injective. Hence the first displayed formula of this proof implies (21.25.0.1) in $D(\mathcal{O})$ which is what we wanted to prove. □

Lemma 21.25.2. *Let $(\mathcal{C}, \mathcal{O})$ be a ringed site. Let K, L be objects of $D(\mathcal{O})$. For every object U of $\mathcal{C}$ we have*

$$\mathrm{Hom}_{D(\mathcal{O}_U)}(K|_U, L|_U) = H^0(U, R\mathcal{H}om(K, L))$$

Proof. This is clear from the discussion above, but it can also be seen by the following argument using only (21.25.0.1). Namely, denoting $j : Sh(\mathcal{C}/U) \to Sh(\mathcal{C})$ the localization morphism, we have

$$\begin{aligned} H^0(U, R\mathcal{H}om(K, L)) &= \mathrm{Hom}_{D(\mathcal{O})}(j_!\mathcal{O}_U, R\mathcal{H}om(K, L)) \\ &= \mathrm{Hom}_{D(\mathcal{O})}(j_!\mathcal{O}_U \otimes^{\mathbf{L}}_{\mathcal{O}} K, L) \\ &= \mathrm{Hom}_{D(\mathcal{O})}(j_!(K|_U), L) \\ &= \mathrm{Hom}_{D(\mathcal{O}_U)}(K|_U, L|_U) \end{aligned}$$

The first and last equalities by Lemma 21.20.2. The second equality by Lemma 21.25.1. The remaining equality holds because $j_!\mathcal{O}_U \otimes^{\mathbf{L}}_{\mathcal{O}} K = j_!(K|_U)$. To see this use that $j_!\mathcal{O}_U$ is flat, see Modules on Sites, Lemma 18.28.5. Some details omitted. □

Lemma 21.25.3. *Let $(\mathcal{C}, \mathcal{O})$ be a ringed site. Let K, L be objects of $D(\mathcal{O})$. The construction of $R\mathcal{H}om(K, L)$ commutes with restrictions, i.e., for every object U of $\mathcal{C}$ we have $R\mathcal{H}om(K|_U, L|_U) = R\mathcal{H}om(K, L)|_U$.*

Proof. Omitted. Hint: Use Lemma 21.20.1 and the construction of the internal hom. □

Lemma 21.25.4. *Let $(\mathcal{C}, \mathcal{O})$ be a ringed site. The bifunctor $R\mathcal{H}om(-, -)$ transforms distinguished triangles into distinguished triangles in both variables.*

Proof. Omitted. Hint: Prove that the construction above transforms a termwise split exact sequence of complexes in the first variable into a termwise split exact sequence of complexes and that it transforms a cone of a morphism between K-injective complexes in the second variable into the corresponding cone. □

Lemma 21.25.5. *Let $(\mathcal{C}, \mathcal{O})$ be a ringed site. Let L be an object of $D(\mathcal{O})$. Set $L^\wedge = R\mathcal{H}om(L, \mathcal{O})$. For M in $D(\mathcal{O})$ there is a canonical map*

$$L^\wedge \otimes^{\mathbf{L}}_{\mathcal{O}} M \longrightarrow R\mathcal{H}om(L, M) \tag{21.25.5.1}$$

which induces a canonical map

$$H^0(\mathcal{C}, L^\wedge \otimes^{\mathbf{L}}_{\mathcal{O}} M) \longrightarrow \mathrm{Hom}_{D(\mathcal{O})}(L, M)$$

functorial in M in $D(\mathcal{O})$.

Proof. There is a canonical map

$$ev : L^\wedge \otimes^{\mathbf{L}}_{\mathcal{O}} L \longrightarrow \mathcal{O}$$

coming from the identification $L^\wedge \to R\mathcal{H}om(L, \mathcal{O})$ by the equality (21.25.0.1) proved in Lemma 21.25.1. This gives a map

$$L^\wedge \otimes^{\mathbf{L}} M \otimes^{\mathbf{L}} L = M \otimes^{\mathbf{L}} L^\wedge \otimes^{\mathbf{L}} L \xrightarrow{1_M \otimes ev} M$$

which by (21.25.0.1) with $K = L^\wedge \otimes^{\mathbf{L}} M$ corresponds to a map as in (21.25.5.1). □

Remark 21.25.6. Let $h : (Sh(\mathcal{C}), \mathcal{O}) \to (Sh(\mathcal{C}'), \mathcal{O}')$ be a morphism of ringed topoi. Let K, L be objects of $D(\mathcal{O}')$. We claim there is a canonical map

$$Lh^* R\mathcal{H}om(K, L) \longrightarrow R\mathcal{H}om(Lh^* K, Lh^* L)$$

in $D(\mathcal{O})$. Namely, by (21.25.0.1) proved in Lemma 21.25.1 such a map is the same thing as a map

$$Lh^* R\mathcal{H}om(K, L) \otimes^{\mathbf{L}} Lh^* K \longrightarrow Lh^* L$$

The source of this arrow is $Lh^*(\mathcal{H}om(K, L) \otimes^{\mathbf{L}} K)$ by Lemma 21.18.4 hence it suffices to construct a canonical map

$$R\mathcal{H}om(K, L) \otimes^{\mathbf{L}} K \longrightarrow L.$$

For this we take the arrow corresponding to

$$\mathrm{id} : R\mathcal{H}om(K, L) \longrightarrow R\mathcal{H}om(K, L)$$

via (21.25.0.1).

Remark 21.25.7. Suppose that

$$\begin{array}{ccc} (Sh(\mathcal{C}'), \mathcal{O}_{\mathcal{C}'}) & \xrightarrow[h]{} & (Sh(\mathcal{C}), \mathcal{O}_{\mathcal{C}}) \\ {\scriptstyle f'}\downarrow & & \downarrow{\scriptstyle f} \\ (Sh(\mathcal{D}'), \mathcal{O}_{\mathcal{D}'}) & \xrightarrow{g} & (Sh(\mathcal{D}), \mathcal{O}_{\mathcal{D}}) \end{array}$$

is a commutative diagram of ringed topoi. Let K, L be objects of $D(\mathcal{O}_{\mathcal{C}})$. We claim there exists a canonical base change map

$$Lg^* Rf_* R\mathcal{H}om(K, L) \longrightarrow R(f')_* R\mathcal{H}om(Lh^* K, Lh^* L)$$

in $D(\mathcal{O}_{\mathcal{D}'})$. Namely, we take the map adjoint to the composition

$$\begin{aligned} L(f')^* Lg^* Rf_* R\mathcal{H}om(K, L) &= Lh^* Lf^* Rf_* R\mathcal{H}om(K, L) \\ &\to Lh^* R\mathcal{H}om(K, L) \\ &\to R\mathcal{H}om(Lh^* K, Lh^* L) \end{aligned}$$

where the first arrow uses the adjunction mapping $Lf^* Rf_* \to \mathrm{id}$ and the second arrow is the canonical map constructed in Remark 21.25.6.

21.26. Derived lower shriek

In this section we study some situations where besides Lf^* and Rf_* there also a derived functor $Lf_!$.

Lemma 21.26.1. *Let $u : \mathcal{C} \to \mathcal{D}$ be a continuous and cocontinuous functor of sites which induces a morphism of topoi $g : Sh(\mathcal{C}) \to Sh(\mathcal{D})$. Let $\mathcal{O}_\mathcal{D}$ be a sheaf of rings and set $\mathcal{O}_\mathcal{C} = g^{-1}\mathcal{O}_\mathcal{D}$. The functor $g_! : Mod(\mathcal{O}_\mathcal{C}) \to Mod(\mathcal{O}_\mathcal{D})$ (see Modules on Sites, Lemma 18.40.1) has a left derived functor*

$$Lg_! : D(\mathcal{O}_\mathcal{C}) \longrightarrow D(\mathcal{O}_\mathcal{D})$$

which is left adjoint to g^*. Moreover, for $U \in \mathrm{Ob}(\mathcal{C})$ we have

$$Lg_!(j_{U!}\mathcal{O}_U) = g_!j_{U!}\mathcal{O}_U = j_{u(U)!}\mathcal{O}_{u(U)}.$$

where $j_{U!}$ and $j_{u(U)!}$ are extension by zero associated to the localization morphism $j_U : \mathcal{C}/U \to \mathcal{C}$ and $j_{u(U)} : \mathcal{D}/u(U) \to \mathcal{D}$.

Proof. We are going to use Derived Categories, Proposition 13.27.2 to construct $Lg_!$. To do this we have to verify assumptions (1), (2), (3), (4), and (5) of that proposition. First, since $g_!$ is a left adjoint we see that it is right exact and commutes with all colimits, so (5) holds. Conditions (3) and (4) hold because the category of modules on a ringed site is a Grothendieck abelian category. Let $\mathcal{P} \subset \mathrm{Ob}(Mod(\mathcal{O}_\mathcal{C}))$ be the collection of $\mathcal{O}_\mathcal{C}$-modules which are direct sums of modules of the form $j_{U!}\mathcal{O}_U$. Note that $g_!j_{U!}\mathcal{O}_U = j_{u(U)!}\mathcal{O}_{u(U)}$, see proof of Modules on Sites, Lemma 18.40.1. Every $\mathcal{O}_\mathcal{C}$-module is a quotient of an object of $\mathcal{P}$, see Modules on Sites, Lemma 18.28.6. Thus (1) holds. Finally, we have to prove (2). Let $\mathcal{K}^\bullet$ be a bounded above acyclic complex of $\mathcal{O}_\mathcal{C}$-modules with $\mathcal{K}^n \in \mathcal{P}$ for all n. We have to show that $g_!\mathcal{K}^\bullet$ is exact. To do this it suffices to show, for every injective $\mathcal{O}_\mathcal{D}$-module $\mathcal{I}$ that

$$\mathrm{Hom}_{D(\mathcal{O}_\mathcal{D})}(g_!\mathcal{K}^\bullet, \mathcal{I}[n]) = 0$$

for all $n \in \mathbf{Z}$. Since $\mathcal{I}$ is injective we have

$$\begin{aligned}\mathrm{Hom}_{D(\mathcal{O}_\mathcal{D})}(g_!\mathcal{K}^\bullet, \mathcal{I}[n]) &= \mathrm{Hom}_{K(\mathcal{O}_\mathcal{D})}(g_!\mathcal{K}^\bullet, \mathcal{I}[n]) \\ &= H^n(\mathrm{Hom}_{\mathcal{O}_\mathcal{D}}(g_!\mathcal{K}^\bullet, \mathcal{I})) \\ &= H^n(\mathrm{Hom}_{\mathcal{O}_\mathcal{C}}(\mathcal{K}^\bullet, g^{-1}\mathcal{I}))\end{aligned}$$

the last equality by the adjointness of $g_!$ and g^{-1}.

The vanishing of this group would be clear if $g^{-1}\mathcal{I}$ were an injective $\mathcal{O}_\mathcal{C}$-module. But $g^{-1}\mathcal{I}$ isn't necessarily an injective $\mathcal{O}_\mathcal{C}$-module as $g_!$ isn't exact in general. We do know that

$$\mathrm{Ext}^p_{\mathcal{O}_\mathcal{C}}(j_{U!}\mathcal{O}_U, g^{-1}\mathcal{I}) = H^p(U, g^{-1}\mathcal{I}) = 0 \text{ for } p \geq 1$$

Namely, the first equality follows from $\mathrm{Hom}_{\mathcal{O}_\mathcal{C}}(j_{U!}\mathcal{O}_U, \mathcal{H}) = \mathcal{H}(U)$ and taking derived functors. The vanishing of $H^p(U, g^{-1}\mathcal{I})$ for all $U \in \mathrm{Ob}(\mathcal{C})$ comes from the vanishing of all higher Čech cohomology groups $\check{H}^p(\mathcal{U}, g^{-1}\mathcal{I})$ via Lemma 21.11.8. Namely, for a covering $\mathcal{U} = \{U_i \to U\}_{i\in I}$ in $\mathcal{C}$ we have $\check{H}^p(\mathcal{U}, g^{-1}\mathcal{I}) = \check{H}^p(u(\mathcal{U}), \mathcal{I})$. Since $\mathcal{I}$ is an injective $\mathcal{O}$-module these Čech cohomology groups vanish, see Lemma 21.12.3. Since each $\mathcal{K}^{-q}$ is a direct sum of modules of the form $j_{U!}\mathcal{O}_U$ we see that

$$\mathrm{Ext}^p_{\mathcal{O}_\mathcal{C}}(\mathcal{K}^{-q}, g^{-1}\mathcal{I}) = 0 \text{ for } p \geq 1 \text{ and all } q$$

Let us use the spectral sequence (see Example 21.24.1)

$$E_1^{p,q} = \mathrm{Ext}^p_{\mathcal{O}_\mathcal{C}}(\mathcal{K}^{-q}, g^{-1}\mathcal{I}) \Rightarrow \mathrm{Ext}^{p+q}_{\mathcal{O}_\mathcal{C}}(\mathcal{K}^\bullet, g^{-1}\mathcal{I}) = 0.$$

Note that the spectral sequence abuts to zero as $\mathcal{K}^\bullet$ is acyclic (hence vanishes in the derived category, hence produces vanishing ext groups). By the vanishing of higher exts proved above the only nonzero terms on the E_1 page are the terms $E_1^{0,q} = \mathrm{Hom}_{\mathcal{O}_\mathcal{C}}(\mathcal{K}^{-q}, g^{-1}\mathcal{I})$. We conclude that the complex $\mathrm{Hom}_{\mathcal{O}_\mathcal{C}}(\mathcal{K}^\bullet, g^{-1}\mathcal{I})$ is acyclic as desired.

Thus the left derived functor $Lg_!$ exists. We still have to show that it is left adjoint to $g^{-1} = g^* = Rg^* = Lg^*$, i.e., that we have

$$\mathrm{Hom}_{D(\mathcal{O}_\mathcal{C})}(\mathcal{H}^\bullet, g^{-1}\mathcal{E}^\bullet) = \mathrm{Hom}_{D(\mathcal{O}_\mathcal{D})}(Lg_!\mathcal{H}^\bullet, \mathcal{E}^\bullet) \tag{21.26.1.1}$$

This is actually a formal consequence of the discussion above. Choose a quasi-isomorphism $\mathcal{K}^\bullet \to \mathcal{H}^\bullet$ such that $\mathcal{K}^\bullet$ computes $Lg_!$. Moreover, choose a quasi-isomorphism $\mathcal{E}^\bullet \to \mathcal{I}^\bullet$ into a K-injective complex of $\mathcal{O}_\mathcal{D}$-modules $\mathcal{I}^\bullet$. Then the RHS of (21.26.1.1) is

$$\mathrm{Hom}_{K(\mathcal{O}_\mathcal{D})}(g_!\mathcal{K}^\bullet, \mathcal{I}^\bullet)$$

On the other hand, by the definition of morphisms in the derived category the LHS of (21.26.1.1) is

$$\begin{aligned}\mathrm{Hom}_{D(\mathcal{O}_\mathcal{C})}(\mathcal{K}^\bullet, g^{-1}\mathcal{I}^\bullet) &= \mathrm{colim}_{s:\mathcal{L}^\bullet\to\mathcal{K}^\bullet} \mathrm{Hom}_{K(\mathcal{O}_\mathcal{C})}(\mathcal{L}^\bullet, g^{-1}\mathcal{I}^\bullet) \\ &= \mathrm{colim}_{s:\mathcal{L}^\bullet\to\mathcal{K}^\bullet} \mathrm{Hom}_{K(\mathcal{O}_\mathcal{D})}(g_!\mathcal{L}^\bullet, \mathcal{I}^\bullet)\end{aligned}$$

by the adjointness of $g_!$ and g^* on the level of sheaves of modules. The colimit is over all quasi-isomorphisms with target $\mathcal{K}^\bullet$. Since for every complex $\mathcal{L}^\bullet$ there exists a quasi-isomorphism $(\mathcal{K}')^\bullet \to \mathcal{L}^\bullet$ such that $(\mathcal{K}')^\bullet$ computes $Lg_!$ we see that we may as well take the colimit over quasi-isomorphisms of the form $s : (\mathcal{K}')^\bullet \to \mathcal{K}^\bullet$ where $(\mathcal{K}')^\bullet$ computes $Lg_!$. In this case

$$\mathrm{Hom}_{K(\mathcal{O}_\mathcal{D})}(g_!\mathcal{K}^\bullet, \mathcal{I}^\bullet) \longrightarrow \mathrm{Hom}_{K(\mathcal{O}_\mathcal{D})}(g_!(\mathcal{K}')^\bullet, \mathcal{I}^\bullet)$$

is an isomorphism as $g_!(\mathcal{K}')^\bullet \to g_!\mathcal{K}^\bullet$ is a quasi-isomorphism and $\mathcal{I}^\bullet$ is K-injective. This finishes the proof. □

Remark 21.26.2. Warning! Let $u : \mathcal{C} \to \mathcal{D}$, g, $\mathcal{O}_\mathcal{D}$, and $\mathcal{O}_\mathcal{C}$ be as in Lemma 21.26.1. In general it is **not** the case that the diagram

$$\begin{array}{ccc} D(\mathcal{O}_\mathcal{C}) & \xrightarrow[Lg_!]{} & D(\mathcal{O}_\mathcal{D}) \\ {\scriptstyle forget}\downarrow & & \downarrow{\scriptstyle forget} \\ D(\mathcal{C}) & \xrightarrow{Lg_!^{Ab}} & D(\mathcal{D}) \end{array}$$

commutes where the functor $Lg_!^{Ab}$ is the one constructed in Lemma 21.26.1 but using the constant sheaf $\mathbf{Z}$ as the structure sheaf on both $\mathcal{C}$ and $\mathcal{D}$. In general it isn't even the case that $g_! = g_!^{Ab}$ (see Modules on Sites, Remark 18.40.2), but this phenomenon **can occur even if** $g_! = g_!^{Ab}$! Namely, the construction of $Lg_!$ in the proof of Lemma 21.26.1 shows that $Lg_!$ agrees with $Lg_!^{Ab}$ if and only if the canonical maps

$$Lg_!^{Ab} j_{U!}\mathcal{O}_U \longrightarrow j_{u(U)!}\mathcal{O}_{u(U)}$$

are isomorphisms in $D(\mathcal{D})$ for all objects U in $\mathcal{C}$. In general all we can say is that there exists a natural transformation

$$Lg_!^{Ab} \circ forget \longrightarrow forget \circ Lg_!$$

21.27. Derived lower shriek for fibred categories

In this section we work out some special cases of the situation discussed in Section 21.26. We make sure that we have equality between lower shriek on modules and sheaves of abelian groups. We encourage the reader to skip this section on a first reading.

Situation 21.27.1. Here $(\mathcal{D}, \mathcal{O}_\mathcal{D})$ be a ringed site and $p : \mathcal{C} \to \mathcal{D}$ is a fibred category. We endow $\mathcal{C}$ with the topology inherited from $\mathcal{D}$ (Stacks, Section 8.10). We denote $\pi : Sh(\mathcal{C}) \to Sh(\mathcal{D})$ the morphism of topoi associated to p (Stacks, Lemma 8.10.3). We set $\mathcal{O}_\mathcal{C} = \pi^{-1}\mathcal{O}_\mathcal{D}$ so that we obtain a morphism of ringed topoi

$$\pi : (Sh(\mathcal{C}), \mathcal{O}_\mathcal{C}) \longrightarrow (Sh(\mathcal{D}), \mathcal{O}_\mathcal{D})$$

Lemma 21.27.2. *Assumptions and notation as in Situation 21.27.1. For $U \in \mathrm{Ob}(\mathcal{C})$ consider the induced morphism of topoi*

$$\pi_U : Sh(\mathcal{C}/U) \longrightarrow Sh(\mathcal{D}/p(U))$$

Then there exists a morphism of topoi

$$\sigma : Sh(\mathcal{D}/p(U)) \to Sh(\mathcal{C}/U)$$

such that $\pi_U \circ \sigma = id$ and $\sigma^{-1} = \pi_{U,*}$.

Proof. Observe that π_U is the restriction of π to the localizations, see Sites, Lemma 7.27.4. For an object $V \to p(U)$ of $\mathcal{D}/p(U)$ denote $V \times_{p(U)} U \to U$ the strongly cartesian morphism of $\mathcal{C}$ over $\mathcal{D}$ which exists as p is a fibred category. The functor

$$v : \mathcal{D}/p(U) \to \mathcal{C}/U, \quad V/p(U) \mapsto V \times_{p(U)} U/U$$

is continuous by the definition of the topology on $\mathcal{C}$. Moreover, it is a right adjoint to p by the definition of strongly cartesian morphisms. Hence we are in the situation discussed in Sites, Section 7.21 and we see that the sheaf $\pi_{U,*}\mathcal{F}$ is equal to $V \mapsto \mathcal{F}(V \times_{p(U)} U)$ (see especially Sites, Lemma 7.21.2).

But here we have more. Namely, the functor v is also cocontinuous (as all morphisms in coverings of $\mathcal{C}$ are strongly cartesian). Hence v defines a morphism σ as indicated in the lemma. The equality $\sigma^{-1} = \pi_{U,*}$ is immediate from the definition. Since $\pi_U^{-1}\mathcal{G}$ is given by the rule $U'/U \mapsto \mathcal{G}(p(U')/p(U))$ it follows that $\sigma^{-1} \circ \pi_U^{-1} = \mathrm{id}$ which proves the equality $\pi_U \circ \sigma = \mathrm{id}$. □

Situation 21.27.3. Let $(\mathcal{D}, \mathcal{O}_\mathcal{D})$ be a ringed site. Let $u : \mathcal{C}' \to \mathcal{C}$ be a 1-morphism of fibred categories over $\mathcal{D}$ (Categories, Definition 4.31.9). Endow $\mathcal{C}$ and $\mathcal{C}'$ with their inherited topologies (Stacks, Definition 8.10.2) and let $\pi : Sh(\mathcal{C}) \to Sh(\mathcal{D})$, $\pi' : Sh(\mathcal{C}') \to Sh(\mathcal{D})$, and $g : Sh(\mathcal{C}') \to Sh(\mathcal{C})$ be the corresponding morphisms of topoi (Stacks, Lemma 8.10.3). Set $\mathcal{O}_\mathcal{C} = \pi^{-1}\mathcal{O}_\mathcal{D}$ and $\mathcal{O}_{\mathcal{C}'} = (\pi')^{-1}\mathcal{O}_\mathcal{D}$. Observe that $g^{-1}\mathcal{O}_\mathcal{C} = \mathcal{O}_{\mathcal{C}'}$ so that

$$\begin{array}{ccc} (Sh(\mathcal{C}'), \mathcal{O}_{\mathcal{C}'}) & \xrightarrow{\quad g \quad} & (Sh(\mathcal{C}), \mathcal{O}_\mathcal{C}) \\ & {\scriptstyle \pi'} \searrow \quad \swarrow {\scriptstyle \pi} & \\ & (Sh(\mathcal{D}), \mathcal{O}_\mathcal{D}) & \end{array}$$

is a commutative diagram of morphisms of ringed topoi.

Lemma 21.27.4. *Assumptions and notation as in Situation 21.27.3. For $U' \in \mathrm{Ob}(\mathcal{C}')$ set $U = u(U')$ and $V = p'(U')$ and consider the induced morphisms of ringed topoi*

$$\begin{array}{ccc} (Sh(\mathcal{C}'/U'), \mathcal{O}_{U'}) & \xrightarrow{g'} & (Sh(\mathcal{C}), \mathcal{O}_U) \\ & {\scriptstyle \pi'_{U'}} \searrow \quad \swarrow {\scriptstyle \pi_U} & \\ & (Sh(\mathcal{D}/V), \mathcal{O}_V) & \end{array}$$

Then there exists a morphism of topoi

$$\sigma' : Sh(\mathcal{D}/V) \to Sh(\mathcal{C}'/U'),$$

such that setting $\sigma = g' \circ \sigma'$ we have $\pi'_{U'} \circ \sigma' = id$, $\pi_U \circ \sigma = id$, $(\sigma')^{-1} = \pi'_{U',*}$, and $\sigma^{-1} = \pi_{U,*}$.

Proof. Let $v' : \mathcal{D}/V \to \mathcal{C}'/U'$ be the functor constructed in the proof of Lemma 21.27.2 starting with $p' : \mathcal{C}' \to \mathcal{D}'$ and the object U'. Since u is a 1-morphism of fibred categories over $\mathcal{D}$ it transforms strongly cartesian morphisms into strongly cartesian morphisms, hence the functor $v = u \circ v'$ is the functor of the proof of Lemma 21.27.2 relative to $p : \mathcal{C} \to \mathcal{D}$ and U. Thus our lemma follows from that lemma. □

Lemma 21.27.5. *Assumption and notation as in Situation 21.27.3.*

(1) *There are left adjoints $g_! : Mod(\mathcal{O}_{\mathcal{C}'}) \to Mod(\mathcal{O}_{\mathcal{C}})$ and $g_!^{Ab} : Ab(\mathcal{C}') \to Ab(\mathcal{C})$ to $g^* = g^{-1}$ on modules and on abelian sheaves.*

(2) *The diagram*

$$\begin{array}{ccc} Mod(\mathcal{O}_{\mathcal{C}'}) & \xrightarrow{g_!} & Mod(\mathcal{O}_{\mathcal{C}}) \\ \downarrow & & \downarrow \\ Ab(\mathcal{C}') & \xrightarrow{g_!^{Ab}} & Ab(\mathcal{C}) \end{array}$$

commutes.

(3) *There are left adjoints $Lg_! : D(\mathcal{O}_{\mathcal{C}'}) \to D(\mathcal{O}_{\mathcal{C}})$ and $Lg_!^{Ab} : D(\mathcal{C}') \to D(\mathcal{C})$ to $g^* = g^{-1}$ on derived categories of modules and abelian sheaves.*

(4) *The diagram*

$$\begin{array}{ccc} D(\mathcal{O}_{\mathcal{C}'}) & \xrightarrow{Lg_!} & D(\mathcal{O}_{\mathcal{C}}) \\ \downarrow & & \downarrow \\ D(\mathcal{C}') & \xrightarrow{Lg_!^{Ab}} & D(\mathcal{C}) \end{array}$$

commutes.

Proof. The functor u is continuous and cocontinuous Stacks, Lemma 8.10.3. Hence the existence of the functors $g_!$, $g_!^{Ab}$, $Lg_!$, and $Lg_!^{Ab}$ can be found in Modules on Sites, Sections 18.16 and 18.40 and Section 21.26.

To prove (2) it suffices to show that the canonical map

$$g_!^{Ab} j_{U'!} \mathcal{O}_{U'} \to j_{u(U')!} \mathcal{O}_{u(U')}$$

is an isomorphism for all objects U' of $\mathcal{C}'$, see Modules on Sites, Remark 18.40.2. Similarly, to prove (4) it suffices to show that the canonical map

$$Lg_!^{Ab} j_{U'!}\mathcal{O}_{U'} \to j_{u(U')!}\mathcal{O}_{u(U')}$$

is an isomorphism in $D(\mathcal{C})$ for all objects U' of $\mathcal{C}'$, see Remark 21.26.2. This will also imply the previous formula hence this is what we will show.

We will use that for a localization morphism j the functors $j_!$ and $j_!^{Ab}$ agree (see Modules on Sites, Remark 18.19.5) and that $j_!$ is exact (Modules on Sites, Lemma 18.19.3). Let us adopt the notation of Lemma 21.27.4. Since $Lg_!^{Ab} \circ j_{U'!} = j_{U!} \circ L(g')_!^{Ab}$ (by commutativity of Sites, Lemma 7.27.4 and uniqueness of adjoint functors) it suffices to prove that $L(g')_!^{Ab}\mathcal{O}_{U'} = \mathcal{O}_U$. Using the results of Lemma 21.27.4 we have for any object E of $D(\mathcal{C}/u(U'))$ the following sequence of equalities

$$\begin{aligned}
\mathrm{Hom}_{D(\mathcal{C}/U)}(L(g')_!^{Ab}\mathcal{O}_{U'}, E) &= \mathrm{Hom}_{D(\mathcal{C}'/U')}(\mathcal{O}_{U'}, (g')^{-1}E) \\
&= \mathrm{Hom}_{D(\mathcal{C}'/U')}((\pi'_{U'})^{-1}\mathcal{O}_V, (g')^{-1}E) \\
&= \mathrm{Hom}_{D(\mathcal{D}/V)}(\mathcal{O}_V, R\pi'_{U',*}(g')^{-1}E) \\
&= \mathrm{Hom}_{D(\mathcal{D}/V)}(\mathcal{O}_V, (\sigma')^{-1}(g')^{-1}E) \\
&= \mathrm{Hom}_{D(\mathcal{D}/V)}(\mathcal{O}_V, \sigma^{-1}E) \\
&= \mathrm{Hom}_{D(\mathcal{D}/V)}(\mathcal{O}_V, \pi_{U,*}E) \\
&= \mathrm{Hom}_{D(\mathcal{C}/U)}(\pi_U^{-1}\mathcal{O}_V, E) \\
&= \mathrm{Hom}_{D(\mathcal{C}/U)}(\mathcal{O}_U, E)
\end{aligned}$$

By Yoneda's lemma we conclude. □

Remark 21.27.6. Assumptions and notation as in Situation 21.27.1. Note that setting $\mathcal{C}' = \mathcal{D}$ and u equal to the structure functor of $\mathcal{C}$ gives a situation as in Situation 21.27.3. Hence Lemma 21.27.5 tells us we have functors $\pi_!$, $\pi_!^{Ab}$, $L\pi_!$, and $L\pi_!^{Ab}$ such that $forget \circ \pi_! = \pi_!^{Ab} \circ forget$ and $forget \circ L\pi_! = L\pi_!^{Ab} \circ forget$.

Remark 21.27.7. Assumptions and notation as in Situation 21.27.3. Let $\mathcal{F}$ be an abelian sheaf on $\mathcal{C}$, let $\mathcal{F}'$ be an abelian sheaf on $\mathcal{C}'$, and let $t : \mathcal{F}' \to g^{-1}\mathcal{F}$ be a map. Then we obtain a canonical map

$$L\pi'_!(\mathcal{F}') \longrightarrow L\pi_!(\mathcal{F})$$

by using the adjoint $g_!\mathcal{F}' \to \mathcal{F}$ of t, the map $Lg_!(\mathcal{F}') \to g_!\mathcal{F}'$, and the equality $L\pi'_! = L\pi_! \circ Lg_!$.

Lemma 21.27.8. *Assumptions and notation as in Situation 21.27.1. For $\mathcal{F}$ in $Ab(\mathcal{C})$ the sheaf $\pi_!\mathcal{F}$ is the sheaf associated the presheaf*

$$V \longmapsto \mathrm{colim}_{\mathcal{C}_V^{opp}} \mathcal{F}|_{\mathcal{C}_V}$$

with restriction maps as indicated in the proof.

Proof. Denote $\mathcal{H}$ be the rule of the lemma. For a morphism $h : V' \to V$ of $\mathcal{D}$ there is a pullback functor $h^* : \mathcal{C}_V \to \mathcal{C}_{V'}$ of fibre categories (Categories, Definition 4.31.6). Moreover for $U \in \mathrm{Ob}(\mathcal{C}_V)$ there is a strongly cartesian morphism $h^*U \to U$ covering h. Restriction along these strongly cartesian morphisms defines a transformation of functors

$$\mathcal{F}|_{\mathcal{C}_V} \longrightarrow \mathcal{F}|_{\mathcal{C}_{V'}} \circ h^*.$$

Hence a map $\mathcal{H}(V) \to \mathcal{H}(V')$ between colimits, see Categories, Lemma 4.14.7.

To prove the lemma we show that

$$\mathrm{Mor}_{PSh(\mathcal{D})}(\mathcal{H}, \mathcal{G}) = \mathrm{Mor}_{Sh(\mathcal{C})}(\mathcal{F}, \pi^{-1}\mathcal{G})$$

for every sheaf $\mathcal{G}$ on $\mathcal{C}$. An element of the left hand side is a compatible system of maps $\mathcal{F}(U) \to \mathcal{G}(p(U))$ for all U in $\mathcal{C}$. Since $\pi^{-1}\mathcal{G}(U) = \mathcal{G}(p(U))$ by our choice of topology on $\mathcal{C}$ we see the same thing is true for the right hand side and we win. $\square$

21.28. Homology on a category

In the case of a category over a point we will baptize the left derived lower shriek functors the homology functors.

Example 21.28.1 (Category over point)**.** Let $\mathcal{C}$ be a category. Endow $\mathcal{C}$ with the chaotic topology (Sites, Example 7.6.6). Thus presheaves and sheaves agree on $\mathcal{C}$. The functor $p : \mathcal{C} \to *$ where $*$ is the category with a single object and a single morphism is cocontinuous and continuous. Let $\pi : Sh(\mathcal{C}) \to Sh(*)$ be the corresponding morphism of topoi. Let B be a ring. We endow $*$ with the sheaf of rings B and $\mathcal{C}$ with $\mathcal{O}_\mathcal{C} = \pi^{-1}B$ which we will denote $\underline{B}$. In this way

$$\pi : (Sh(\mathcal{C}), \underline{B}) \to (*, B)$$

is an example of Situation 21.27.1. By Remark 21.27.6 we do not need to distinguish between $\pi_!$ on modules or abelian sheaves. By Lemma 21.27.8 we see that $\pi_!\mathcal{F} = \mathrm{colim}_{\mathcal{C}^{opp}} \mathcal{F}$. Thus $L_n\pi_!$ is the nth left derived functor of taking colimits. In the following, we write

$$H_n(\mathcal{C}, \mathcal{F}) = L_n\pi_!(\mathcal{F})$$

and we will name this the *nth homology group of $\mathcal{F}$* on $\mathcal{C}$.

Example 21.28.2 (Computing homology)**.** In Example 21.28.1 we can compute the functors $H_n(\mathcal{C}, -)$ as follows. Let $\mathcal{F} \in \mathrm{Ob}(Ab(\mathcal{C}))$. Consider the chain complex

$$K_\bullet(\mathcal{F}) : \ \ldots \to \bigoplus\nolimits_{U_2 \to U_1 \to U_0} \mathcal{F}(U_0) \to \bigoplus\nolimits_{U_1 \to U_0} \mathcal{F}(U_0) \to \bigoplus\nolimits_{U_0} \mathcal{F}(U_0)$$

where the transition maps are given by

$$(U_2 \to U_1 \to U_0, s) \longmapsto (U_1 \to U_0, s) - (U_2 \to U_0, s) + (U_2 \to U_1, s|_{U_1})$$

and similarly in other degrees. By construction

$$H_0(\mathcal{C}, \mathcal{F}) = \mathrm{colim}_{\mathcal{C}^{opp}} \mathcal{F} = H_0(K_\bullet(\mathcal{F})),$$

see Categories, Lemma 4.14.11. The construction of $K_\bullet(\mathcal{F})$ is functorial in $\mathcal{F}$ and transforms short exact sequences of $Ab(\mathcal{C})$ into short exact sequences of complexes. Thus the sequence of functors $\mathcal{F} \mapsto H_n(K_\bullet(\mathcal{F}))$ forms a δ-functor, see Homology, Definition 12.11.1 and Lemma 12.12.12. For $\mathcal{F} = j_{U!}\mathbf{Z}_U$ the complex $K_\bullet(\mathcal{F})$ is the complex associated to the free $\mathbf{Z}$-module on the simplicial set $X_\bullet$ with terms

$$X_n = \coprod\nolimits_{U_n \to \ldots \to U_1 \to U_0} \mathrm{Mor}_\mathcal{C}(U_0, U)$$

This simplicial set is homotopy equivalent to the constant simplicial set on a singleton $\{*\}$. Namely, the map $X_\bullet \to \{*\}$ is obvious, the map $\{*\} \to X_n$ is given by mapping $*$ to $(U \to \ldots \to U, \mathrm{id}_U)$, and the maps

$$h_{n,i} : X_n \longrightarrow X_n$$

(Simplicial, Lemma 14.24.2) defining the homotopy between the two maps $X_\bullet \to X_\bullet$ are given by the rule

$$h_{n,i} : (U_n \to \ldots \to U_0, f) \longmapsto (U_n \to \ldots \to U_i \to U \to \ldots \to U, \mathrm{id})$$

for $i > 0$ and $h_{n,0} = \mathrm{id}$. Verifications omitted. This implies that $K_\bullet(j_{U!}\mathbf{Z}_U)$ has trivial cohomology in negative degrees (by the functoriality of Simplicial, Remark 14.24.4 and the result of Simplicial, Lemma 14.25.1). Thus $K_\bullet(\mathcal{F})$ computes the left derived functors $H_n(\mathcal{C}, -)$ of $H_0(\mathcal{C}, -)$ for example by (the duals of) Homology, Lemma 12.11.4 and Derived Categories, Lemma 13.16.6.

Example 21.28.3. Let $u : \mathcal{C}' \to \mathcal{C}$ be a functor. Endow $\mathcal{C}'$ and $\mathcal{C}$ with the chaotic topology as in Example 21.28.1. The functors u, $\mathcal{C}' \to *$, and $\mathcal{C} \to *$ where $*$ is the category with a single object and a single morphism are cocontinuous and continuous. Let $g : Sh(\mathcal{C}') \to Sh(\mathcal{C})$, $\pi' : Sh(\mathcal{C}') \to Sh(*)$, and $\pi : Sh(\mathcal{C}) \to Sh(*)$, be the corresponding morphisms of topoi. Let B be a ring. We endow $*$ with the sheaf of rings B and $\mathcal{C}'$, $\mathcal{C}$ with the constant sheaf $\underline{B}$. In this way

$$\begin{array}{ccc} (Sh(\mathcal{C}'), \underline{B}) & \xrightarrow{\quad g \quad} & (Sh(\mathcal{C}), \underline{B}) \\ & {\scriptstyle \pi'} \searrow \quad \swarrow {\scriptstyle \pi} & \\ & (Sh(*), B) & \end{array}$$

is an example of Situation 21.27.3. Thus Lemma 21.27.5 applies to g so we do not need to distinguish between $g_!$ on modules or abelian sheaves. In particular Remark 21.27.7 produces canonical maps

$$H_n(\mathcal{C}', \mathcal{F}') \longrightarrow H_n(\mathcal{C}, \mathcal{F})$$

whenever we have $\mathcal{F}$ in $Ab(\mathcal{C})$, $\mathcal{F}'$ in $Ab(\mathcal{C}')$, and a map $t : \mathcal{F}' \to g^{-1}\mathcal{F}$. In terms of the computation of homology given in Example 21.28.2 we see that these maps come from a map of complexes

$$K_\bullet(\mathcal{F}') \longrightarrow K_\bullet(\mathcal{F})$$

given by the rule

$$(U'_n \to \ldots \to U'_0, s') \longmapsto (u(U'_n) \to \ldots \to u(U'_0), t(s'))$$

with obvious notation.

Remark 21.28.4. Notation and assumptions as in Example 21.28.1. Let $\mathcal{F}^\bullet$ be a bounded complex of abelian sheaves on $\mathcal{C}$. For any object U of $\mathcal{C}$ there is a canonical map

$$\mathcal{F}^\bullet(U) \longrightarrow L\pi_!(\mathcal{F}^\bullet)$$

in $D(Ab)$. If $\mathcal{F}^\bullet$ is a complex of $\underline{B}$-modules then this map is in $D(B)$. To prove this, note that we compute $L\pi_!(\mathcal{F}^\bullet)$ by taking a quasi-isomorphism $\mathcal{P}^\bullet \to \mathcal{F}^\bullet$ where $\mathcal{P}^\bullet$ is a complex of projectives. However, since the topology is chaotic this means that $\mathcal{P}^\bullet(U) \to \mathcal{F}^\bullet(U)$ is a quasi-isomorphism hence can be inverted in $D(Ab)$, resp. $D(B)$. Composing with the canonical map $\mathcal{P}^\bullet(U) \to \pi_!(\mathcal{P}^\bullet)$ coming from the computation of $\pi_!$ as a colimit we obtain the desired arrow.

Lemma 21.28.5. *Notation and assumptions as in Example 21.28.1. If $\mathcal{C}$ has either an initial or a final object, then $L\pi_! \circ \pi^{-1} = id$ on $D(Ab)$, resp. $D(B)$.*

Proof. If $\mathcal{C}$ has an initial object, then $\pi_!$ is computed by evaluating on this object and the statement is clear. If $\mathcal{C}$ has a final object, then $R\pi_*$ is computed by evaluating on this object, hence $R\pi_* \circ \pi^{-1} \cong \mathrm{id}$ on $D(Ab)$, resp. $D(B)$. This implies that $\pi^{-1} : D(Ab) \to D(\mathcal{C})$, resp. $\pi^{-1} : D(B) \to D(\underline{B})$ is fully faithful, see Categories, Lemma 4.24.2. Then the same lemma implies that $L\pi_! \circ \pi^{-1} = \mathrm{id}$ as desired. □

Lemma 21.28.6. *Notation and assumptions as in Example 21.28.1. Let $B \to B'$ be a ring map. Consider the commutative diagram of ringed topoi*

$$\begin{array}{ccc} (Sh(\mathcal{C}), \underline{B}) & \xleftarrow{h} & (Sh(\mathcal{C}), \underline{B'}) \\ \downarrow{\scriptstyle \pi} & & \downarrow{\scriptstyle \pi'} \\ (*, B) & \xleftarrow{f} & (*, B') \end{array}$$

Then $L\pi_! \circ Lh^* = Lf^* \circ L\pi'_!$.

Proof. Both functors are right adjoint to the obvious functor $D(B') \to D(\underline{B})$. □

Lemma 21.28.7. *Notation and assumptions as in Example 21.28.1. Let $U_\bullet$ be a cosimplicial object in $\mathcal{C}$ such that for every $U \in \mathrm{Ob}(\mathcal{C})$ the simplicial set $\mathrm{Mor}_{\mathcal{C}}(U_\bullet, U)$ is homotopy equivalent to the constant simplicial set on a singleton. Then*

$$L\pi_!(\mathcal{F}) = \mathcal{F}(U_\bullet)$$

in $D(Ab)$, resp. $D(B)$ functorially in $\mathcal{F}$ in $Ab(\mathcal{C})$, resp. $Mod(\underline{B})$.

Proof. As $L\pi_!$ agrees for modules and abelian sheaves by Lemma 21.27.5 it suffices to prove this when $\mathcal{F}$ is an abelian sheaf. For $U \in \mathrm{Ob}(\mathcal{C})$ the abelian sheaf $j_{U!}\mathbf{Z}_U$ is a projective object of $Ab(\mathcal{C})$ since $\mathrm{Hom}(j_{U!}\mathbf{Z}_U, \mathcal{F}) = \mathcal{F}(U)$ and taking sections is an exact functor as the topology is chaotic. Every abelian sheaf is a quotient of a direct sum of $j_{U!}\mathbf{Z}_U$ by Modules on Sites, Lemma 18.28.6. Thus we can compute $L\pi_!(\mathcal{F})$ by choosing a resolution

$$\ldots \to \mathcal{G}^{-1} \to \mathcal{G}^0 \to \mathcal{F} \to 0$$

whose terms are direct sums of sheaves of the form above and taking $L\pi_!(\mathcal{F}) = \pi_!(\mathcal{G}^\bullet)$. Consider the double complex $A^{\bullet,\bullet} = \mathcal{G}^\bullet(U_\bullet)$. The map $\mathcal{G}^0 \to \mathcal{F}$ gives a map of complexes $A^{0,\bullet} \to \mathcal{F}(U_\bullet)$. Since $\pi_!$ is computed by taking the colimit over $\mathcal{C}^{opp}$ (Lemma 21.27.8) we see that the two compositions $\mathcal{G}^m(U_1) \to \mathcal{G}^m(U_0) \to \pi_!\mathcal{G}^m$ are equal. Thus we obtain a canonical map of complexes

$$\mathrm{Tot}(A^{\bullet,\bullet}) \longrightarrow \pi_!(\mathcal{G}^\bullet) = L\pi_!(\mathcal{F})$$

To prove the lemma it suffices to show that the complexes

$$\ldots \to \mathcal{G}^m(U_1) \to \mathcal{G}^m(U_0) \to \pi_!\mathcal{G}^m \to 0$$

are exact, see Homology, Lemma 12.22.6. Since the sheaves $\mathcal{G}^m$ are direct sums of the sheaves $j_{U!}\mathbf{Z}_U$ we reduce to $\mathcal{G} = j_{U!}\mathbf{Z}_U$. The complex $j_{U!}\mathbf{Z}_U(U_\bullet)$ is the complex of abelian groups associated to the free $\mathbf{Z}$-module on the simplicial set $\mathrm{Mor}_{\mathcal{C}}(U_\bullet, U)$ which we assumed to be homotopy equivalent to a singleton. We conclude that

$$j_{U!}\mathbf{Z}_U(U_\bullet) \to \mathbf{Z}$$

is a homotopy equivalence of abelian groups hence a quasi-isomorphism (Simplicial, Remark 14.24.4 and Lemma 14.25.1). This finishes the proof since $\pi_! j_{U!}\mathbf{Z}_U = \mathbf{Z}$ as was shown in the proof of Lemma 21.27.5. □

Lemma 21.28.8. *Notation and assumptions as in Example 21.28.3. If there exists a cosimplicial object $U'_\bullet$ of $\mathcal{C}'$ such that Lemma 21.28.7 applies to both $U'_\bullet$ in $\mathcal{C}'$ and $u(U'_\bullet)$ in $\mathcal{C}$, then we have $L\pi'_! \circ g^{-1} = L\pi_!$ as functors $D(\mathcal{C}) \to D(Ab)$, resp. $D(\mathcal{C}, \underline{B}) \to D(B)$.*

Proof. Follows immediately from Lemma 21.28.7 and the fact that g^{-1} is given by precomposing with u. □

Lemma 21.28.9. *Let $\mathcal{C}_i$, $i = 1, 2$ be categories. Let $u_i : \mathcal{C}_1 \times \mathcal{C}_2 \to \mathcal{C}_i$ be the projection functors. Let B be a ring. Let $g_i : (Sh(\mathcal{C}_1 \times \mathcal{C}_2), \underline{B}) \to (Sh(\mathcal{C}_i), \underline{B})$ be the corresponding morphisms of ringed topoi, see Example 21.28.3. For $K_i \in D(\mathcal{C}_i, B)$ we have*

$$L(\pi_1 \times \pi_2)_!(g_1^{-1}K_1 \otimes^{\mathbf{L}}_{\underline{B}} g_2^{-1}K_2) = L\pi_{1,!}(K_1) \otimes^{\mathbf{L}}_B L\pi_{2,!}(K_2)$$

in $D(B)$ with obvious notation.

Proof. As both sides commute with colimits, it suffices to prove this for $K_1 = j_{U!}\underline{B}_U$ and $K_2 = j_{V!}\underline{B}_V$ for $U \in \mathrm{Ob}(\mathcal{C}_1)$ and $V \in \mathrm{Ob}(\mathcal{C}_2)$. See construction of $L\pi_!$ in Lemma 21.26.1. In this case

$$g_1^{-1}K_1 \otimes^{\mathbf{L}}_{\underline{B}} g_2^{-1}K_2 = g_1^{-1}K_1 \otimes_{\underline{B}} g_2^{-1}K_2 = j_{(U,V)!}\underline{B}_{(U,V)}$$

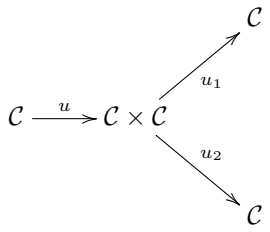
Verification omitted. Hence the result follows as both the left and the right hand side of the formula of the lemma evaluate to B, see construction of $L\pi_!$ in Lemma 21.26.1. □

Lemma 21.28.10. *Notation and assumptions as in Example 21.28.1. If there exists a cosimplicial object $U_\bullet$ of $\mathcal{C}$ such that Lemma 21.28.7 applies, then*

$$L\pi_!(K_1 \otimes^{\mathbf{L}}_{\underline{B}} K_2) = L\pi_!(K_1) \otimes^{\mathbf{L}}_B L\pi_!(K_2)$$

for all $K_i \in D(\underline{B})$.

Proof. Consider the diagram of categories and functors



where u is the diagonal functor and u_i are the projection functors. This gives morphisms of ringed topoi g, g_1, g_2. For any object (U_1, U_2) of $\mathcal{C}$ we have

$$\mathrm{Mor}_{\mathcal{C}\times\mathcal{C}}(u(U_\bullet), (U_1, U_2)) = \mathrm{Mor}_{\mathcal{C}}(U_\bullet, U_1) \times \mathrm{Mor}_{\mathcal{C}}(U_\bullet, U_2)$$

which is homotopy equivalent to a point by Simplicial, Lemma 14.24.10. Thus Lemma 21.28.8 gives $L\pi_!(g^{-1}K) = L(\pi \times \pi)_!(K)$ for any K in $D(\mathcal{C} \times \mathcal{C}, B)$. Take $K = g_1^{-1}K_1 \otimes^{\mathbf{L}}_B g_2^{-1}K_2$. Then $g^{-1}K = K_1 \otimes^{\mathbf{L}}_{\underline{B}} K_2$ because $g^{-1} = g^* = Lg^*$ commutes with derived tensor product (Lemma 21.18.4 – a site with chaotic topology has enough points). To finish we apply Lemma 21.28.9. □

Remark 21.28.11 (Simplicial modules)**.** Let $\mathcal{C} = \Delta$ and let B be any ring. This is a special case of Example 21.28.1 where the assumptions of Lemma 21.28.7 hold. Namely, let $U_\bullet$ be the cosimplicial object of Δ given by the identity functor. To verify the condition we have to show that for $[m] \in \mathrm{Ob}(\Delta)$ the simplicial set $\Delta[m] : n \mapsto \mathrm{Mor}_\Delta([n], [m])$ is homotopy equivalent to a point. This is explained in Simplicial, Example 14.24.7.

In this situation the category $Mod(\underline{B})$ is just the category of simplicial B-modules and the functor $L\pi_!$ sends a simplicial B-module $M_\bullet$ to its associated complex $s(M_\bullet)$ of B-modules. Thus the results above can be reinterpreted in terms of results on simplicial modules. For example a special case of Lemma 21.28.10 is: if $M_\bullet$, $M'_\bullet$ are flat simplicial B-modules, then the complex $s(M_\bullet \otimes_B M'_\bullet)$ is quasi-isomorphic to the total complex associated to the double complex $s(M_\bullet) \otimes_B s(M'_\bullet)$. (Hint: use flatness to convert from derived tensor products to usual tensor products.) This is a special case of the Eilenberg-Zilber theorem which can be found in [**EZ53**].

Lemma 21.28.12. *Let $\mathcal{C}$ be a category (endowed with chaotic topology). Let $\mathcal{O} \to \mathcal{O}'$ be a map of sheaves of rings on $\mathcal{C}$. Assume*

(1) *there exists a cosimplicial object $U_\bullet$ in $\mathcal{C}$ as in Lemma 21.28.7, and*
(2) *$L\pi_!\mathcal{O} \to L\pi_!\mathcal{O}'$ is an isomorphism.*

For K in $D(\mathcal{O})$ we have

$$L\pi_!(K) = L\pi_!(K \otimes^{\mathbf{L}}_{\mathcal{O}} \mathcal{O}')$$

in $D(Ab)$.

Proof. Note: in this proof $L\pi_!$ denotes the left derived functor of $\pi_!$ on abelian sheaves. Since $L\pi_!$ commutes with colimits, it suffices to prove this for bounded above complexes of $\mathcal{O}$-modules (compare with argument of Derived Categories, Proposition 13.27.2 or just stick to bounded above complexes). Every such complex is quasi-isomorphic to a bounded above complex whose terms are direct sums of $j_{U!}\mathcal{O}_U$ with $U \in \mathrm{Ob}(\mathcal{C})$, see Modules on Sites, Lemma 18.28.6. Thus it suffices to prove the lemma for $j_{U!}\mathcal{O}_U$. By assumption

$$S_\bullet = \mathrm{Mor}_{\mathcal{C}}(U_\bullet, U)$$

is a simplicial set homotopy equivalent to the constant simplicial set on a singleton. Set $P_n = \mathcal{O}(U_n)$ and $P'_n = \mathcal{O}'(U_n)$. Observe that the complex associated to the simplicial abelian group

$$X_\bullet : n \longmapsto \bigoplus\nolimits_{s \in S_n} P_n$$

computes $L\pi_!(j_{U!}\mathcal{O}_U)$ by Lemma 21.28.7. Since $j_{U!}\mathcal{O}_U$ is a flat $\mathcal{O}$-module we have $j_{U!}\mathcal{O}_U \otimes^{\mathbf{L}}_{\mathcal{O}} \mathcal{O}' = j_{U!}\mathcal{O}'_U$ and $L\pi_!$ of this is computed by the complex associated to the simplicial abelian group

$$X'_\bullet : n \longmapsto \bigoplus\nolimits_{s \in S_n} P'_n$$

As the rule which to a simplicial set $T_\bullet$ associated the simplicial abelian group with terms $\bigoplus_{t \in T_n} P_n$ is a functor, we see that $X_\bullet \to P_\bullet$ is a homotopy equivalence of simplicial abelian groups. Similarly, the rule which to a simplicial set $T_\bullet$ associates the simplicial abelian group with terms $\bigoplus_{t \in T_n} P'_n$ is a functor. Hence $X'_\bullet \to P'_\bullet$ is a homotopy equivalence of simplicial abelian groups. By assumption $P_\bullet \to P'_\bullet$ is a quasi-isomorphism (since $P_\bullet$, resp. $P'_\bullet$ computes $L\pi_!\mathcal{O}$, resp. $L\pi_!\mathcal{O}'$ by Lemma 21.28.7). We conclude that $X_\bullet$ and $X'_\bullet$ are quasi-isomorphic as desired. $\square$

Remark 21.28.13. Let $\mathcal{C}$ and B be as in Example 21.28.1. Assume there exists a cosimplicial object as in Lemma 21.28.7. Let $\mathcal{O} \to \underline{B}$ be a map sheaf of rings on $\mathcal{C}$ which induces an isomorphism $L\pi_!\mathcal{O} \to L\pi_!\underline{B}$. In this case we obtain an exact functor of triangulated categories

$$L\pi_! : D(\mathcal{O}) \longrightarrow D(B)$$

Namely, for any object K of $D(\mathcal{O})$ we have $L\pi_!^{Ab}(K) = L\pi_!^{Ab}(K \otimes^{\mathbf{L}}_{\mathcal{O}} \underline{B})$ by Lemma 21.28.12. Thus we can define the displayed functor as the composition of $- \otimes^{\mathbf{L}}_{\mathcal{O}} \underline{B}$ with the functor $L\pi_! : D(\underline{B}) \to D(B)$. In other words, we obtain a B-module structure on $L\pi_!(K)$ coming from the (canonical, functorial) identification of $L\pi_!(K)$ with $L\pi_!(K \otimes^{\mathbf{L}}_{\mathcal{O}} \underline{B})$ of the lemma.

21.29. Calculating derived lower shriek

In this section we apply the results from Section 21.28 to compute $L\pi_!$ in Situation 21.27.1 and $Lg_!$ in Situation 21.27.3.

Lemma 21.29.1. *Assumptions and notation as in Situation 21.27.1. For $\mathcal{F}$ in $PAb(\mathcal{C})$ and $n \geq 0$ consider the abelian sheaf $L_n(\mathcal{F})$ on $\mathcal{D}$ which is the sheaf associated to the presheaf*

$$V \longmapsto H_n(\mathcal{C}_V, \mathcal{F}|_{\mathcal{C}_V})$$

with restriction maps as indicated in the proof. Then $L_n(\mathcal{F}) = L_n(\mathcal{F}^\#)$.

Proof. For a morphism $h : V' \to V$ of $\mathcal{D}$ there is a pullback functor $h^* : \mathcal{C}_V \to \mathcal{C}_{V'}$ of fibre categories (Categories, Definition 4.31.6). Moreover for $U \in \mathrm{Ob}(\mathcal{C}_V)$ there is a strongly cartesian morphism $h^*U \to U$ covering h. Restriction along these strongly cartesian morphisms defines a transformation of functors

$$\mathcal{F}|_{\mathcal{C}_V} \longrightarrow \mathcal{F}|_{\mathcal{C}_{V'}} \circ h^*.$$

By Example 21.28.3 we obtain the desired restriction map

$$H_n(\mathcal{C}_V, \mathcal{F}|_{\mathcal{C}_V}) \longrightarrow H_n(\mathcal{C}_{V'}, \mathcal{F}|_{\mathcal{C}_{V'}})$$

Let us denote $L_{n,p}(\mathcal{F})$ this presheaf, so that $L_n(\mathcal{F}) = L_{n,p}(\mathcal{F})^\#$. The canonical map $\gamma : \mathcal{F} \to \mathcal{F}^+$ (Sites, Theorem 7.10.10) defines a canonical map $L_{n,p}(\mathcal{F}) \to L_{n,p}(\mathcal{F}^+)$. We have to prove this map becomes an isomorphism after sheafification.

Let us use the computation of homology given in Example 21.28.2. Denote $K_\bullet(\mathcal{F}|_{\mathcal{C}_V})$ the complex associated to the restriction of $\mathcal{F}$ to the fibre category $\mathcal{C}_V$. By the remarks above we obtain a presheaf $K_\bullet(\mathcal{F})$ of complexes

$$V \longmapsto K_\bullet(\mathcal{F}|_{\mathcal{C}_V})$$

whose cohomology presheaves are the presheaves $L_{n,p}(\mathcal{F})$. Thus it suffices to show that

$$K_\bullet(\mathcal{F}) \longrightarrow K_\bullet(\mathcal{F}^+)$$

becomes an isomorphism on sheafification.

Injectivity. Let V be an object of $\mathcal{D}$ and let $\xi \in K_n(\mathcal{F})(V)$ be an element which maps to zero in $K_n(\mathcal{F}^+)(V)$. We have to show there exists a covering $\{V_j \to V\}$ such that $\xi|_{V_j}$ is zero in $K_n(\mathcal{F})(V_j)$. We write

$$\xi = \sum (U_{i,n+1} \to \ldots \to U_{i,0}, \sigma_i)$$

with $\sigma_i \in \mathcal{F}(U_{i,0})$. We arrange it so that each sequence of morphisms $U_n \to \ldots \to U_0$ of $\mathcal{C}_V$ occurs are most once. Since the sums in the definition of the complex $K_\bullet$ are direct sums, the only way this can map to zero in $K_\bullet(\mathcal{F}^+)(V)$ is if all σ_i map to zero in $\mathcal{F}^+(U_{i,0})$. By construction of $\mathcal{F}^+$ there exist coverings $\{U_{i,0,j} \to U_{i,0}\}$ such that $\sigma_i|_{U_{i,0,j}}$ is zero. By our construction of the topology on $\mathcal{C}$ we can write $U_{i,0,j} \to U_{i,0}$ as the pullback (Categories, Definition 4.31.6) of some morphisms $V_{i,j} \to V$ and moreover each $\{V_{i,j} \to V\}$ is a covering. Choose a covering $\{V_j \to V\}$ dominating each of the coverings $\{V_{i,j} \to V\}$. Then it is clear that $\xi|_{V_j} = 0$.

Surjectivity. Proof omitted. Hint: Argue as in the proof of injectivity. □

Lemma 21.29.2. *Assumptions and notation as in Situation 21.27.1. For $\mathcal{F}$ in $Ab(\mathcal{C})$ and $n \geq 0$ the sheaf $L_n\pi_!(\mathcal{F})$ is equal to the sheaf $L_n(\mathcal{F})$ constructed in Lemma 21.29.1.*

Proof. Consider the sequence of functors $\mathcal{F} \mapsto L_n(\mathcal{F})$ from $PAb(\mathcal{C}) \to Ab(\mathcal{C})$. Since for each $V \in \mathrm{Ob}(\mathcal{D})$ the sequence of functors $H_n(\mathcal{C}_V, -)$ forms a δ-functor so do the functors $\mathcal{F} \mapsto L_n(\mathcal{F})$. Our goal is to show these form a universal δ-functor. In order to do this we construct some abelian presheaves on which these functors vanish.

For $U' \in \mathrm{Ob}(\mathcal{C})$ consider the abelian presheaf $\mathcal{F}_{U'} = j_{U'!}^{PAb}\mathbf{Z}_{U'}$ (Modules on Sites, Remark 18.19.6). Recall that

$$\mathcal{F}_{U'}(U) = \bigoplus\nolimits_{\mathrm{Mor}_\mathcal{C}(U,U')} \mathbf{Z}$$

If U lies over $V = p(U)$ in $\mathcal{D}$) and U' lies over $V' = p(U')$ then any morphism $a : U \to U'$ factors uniquely as $U \to h^*U' \to U'$ where $h = p(a) : V \to V'$ (see Categories, Definition 4.31.6). Hence we see that

$$\mathcal{F}_{U'}|_{\mathcal{C}_V} = \bigoplus\nolimits_{h \in \mathrm{Mor}_\mathcal{D}(V,V')} j_{h^*U'!}\mathbf{Z}_{h^*U'}$$

where $j_{h^*U'} : Sh(\mathcal{C}_V/h^*U') \to Sh(\mathcal{C}_V)$ is the localization morphism. The sheaves $j_{h^*U'!}\mathbf{Z}_{h^*U'}$ have vanishing higher homology groups (see Example 21.28.2). We conclude that $L_n(\mathcal{F}_{U'}) = 0$ for all $n > 0$ and all U'. It follows that any abelian presheaf $\mathcal{F}$ is a quotient of an abelian presheaf $\mathcal{G}$ with $L_n(\mathcal{G}) = 0$ for all $n > 0$ (Modules on Sites, Lemma 18.28.6). Since $L_n(\mathcal{F}) = L_n(\mathcal{F}^\#)$ we see that the same thing is true for abelian sheaves. Thus the sequence of functors $L_n(-)$ is a universal delta functor on $Ab(\mathcal{C})$ (Homology, Lemma 12.11.4). Since we have agreement with $H^{-n}(L\pi_!(-))$ for $n = 0$ by Lemma 21.27.8 we conclude by uniqueness of universal δ-functors (Homology, Lemma 12.11.5) and Derived Categories, Lemma 13.16.6. □

Lemma 21.29.3. *Assumptions and notation as in Situation 21.27.3. For an abelian sheaf $\mathcal{F}'$ on $\mathcal{C}'$ the sheaf $L_ng_!(\mathcal{F}')$ is the sheaf associated to the presheaf*

$$U \longmapsto H_n(\mathcal{I}_U, \mathcal{F}'_U)$$

For notation and restriction maps see proof.

Proof. Say $p(U) = V$. The category $\mathcal{I}_U$ is the category of pairs (U', φ) where $\varphi : U \to u(U')$ is a morphism of $\mathcal{C}$ with $p(\varphi) = \mathrm{id}_V$, i.e., φ is a morphism of the fibre category $\mathcal{C}_V$. Morphisms $(U'_1, \varphi_1) \to (U'_2, \varphi_2)$ are given by morphisms $a : U'_1 \to U'_2$ of the fibre category $\mathcal{C}'_V$ such that $\varphi_2 = u(a) \circ \varphi_1$. The presheaf $\mathcal{F}'_U$ sends (U', φ) to $\mathcal{F}'(U')$. We will construct the restriction mappings below.

Choose a factorization

$$\mathcal{C}' \underset{w}{\overset{u'}{\rightleftarrows}} \mathcal{C}'' \xrightarrow{u''} \mathcal{C}$$

of u as in Categories, Lemma 4.31.14. Then $g_! = g''_! \circ g'_!$ and similarly for derived functors. On the other hand, the functor $g'_!$ is exact, see Modules on Sites, Lemma 18.16.6. Thus we get $Lg_!(\mathcal{F}') = Lg''_!(\mathcal{F}'')$ where $\mathcal{F}'' = g'_!\mathcal{F}'$. Note that $\mathcal{F}'' = h^{-1}\mathcal{F}'$ where $h : Sh(\mathcal{C}'') \to Sh(\mathcal{C}')$ is the morphism of topoi associated to w, see Sites, Lemma 7.22.1. The functor u'' turns $\mathcal{C}''$ into a fibred category over $\mathcal{C}$, hence Lemma 21.29.2 applies to the computation of $L_n g''_!$. The result follows as the construction of $\mathcal{C}''$ in the proof of Categories, Lemma 4.31.14 shows that the fibre category $\mathcal{C}''_U$ is equal to $\mathcal{I}_U$. Moreover, $h^{-1}\mathcal{F}'|_{\mathcal{C}''_U}$ is given by the rule described above (as w is continuous and cocontinuous by Stacks, Lemma 8.10.3 so we may apply Sites, Lemma 7.20.5). □

21.30. Simplicial modules

Let $A_\bullet$ be a simplicial ring. Recall that we may think of $A_\bullet$ as a sheaf on Δ (endowed with the chaotic topology), see Simplicial, Section 14.4. Then a simplicial module $M_\bullet$ over $A_\bullet$ is just a sheaf of $A_\bullet$-modules on Δ. In other words, for every $n \geq 0$ we have an A_n-module M_n and for every map $\varphi : [n] \to [m]$ we have a corresponding map

$$M_\bullet(\varphi) : M_m \longrightarrow M_n$$

which is $A_\bullet(\varphi)$-linear such that these maps compose in the usual manner.

Let $\mathcal{C}$ be a site. A *simplicial sheaf of rings* $\mathcal{A}_\bullet$ on $\mathcal{C}$ is a simplicial object in the category of sheaves of rings on $\mathcal{C}$. In this case the assignment $U \mapsto \mathcal{A}_\bullet(U)$ is a sheaf of simplicial rings and in fact the two notions are equivalent. A similar discussion holds for simplicial abelian sheaves, simplicial sheaves of Lie algebras, and so on.

However, as in the case of simplicial rings above, there is another way to think about simplicial sheaves. Namely, consider the projection

$$p : \Delta \times \mathcal{C} \longrightarrow \mathcal{C}$$

This defines a fibred category with strongly cartesian morphisms exactly the morphisms of the form $([n], U) \to ([n], V)$. We endow the category $\Delta \times \mathcal{C}$ with the topology inherited from $\mathcal{C}$ (see Stacks, Section 8.10). The simple description of the coverings in $\Delta \times \mathcal{C}$ (Stacks, Lemma 8.10.1) immediately implies that a simplicial sheaf of rings on $\mathcal{C}$ is the same thing as a sheaf of rings on $\Delta \times \mathcal{C}$.

By analogy with the case of simplicial modules over a simplicial ring, we define simplicial modules over simplicial sheaves of rings as follows.

Definition 21.30.1. Let $\mathcal{C}$ be a site. Let $\mathcal{A}_\bullet$ be a simplicial sheaf of rings on $\mathcal{C}$. A *simplicial $\mathcal{A}_\bullet$-module* $\mathcal{F}_\bullet$ (sometimes called a *simplicial sheaf of $\mathcal{A}_\bullet$-modules*) is a sheaf of modules over the sheaf of rings on $\Delta \times \mathcal{C}$ associated to $\mathcal{A}_\bullet$.

We obtain a category $Mod(\mathcal{A}_\bullet)$ of simplicial modules and a corresponding derived category $D(\mathcal{A}_\bullet)$. Given a map $\mathcal{A}_\bullet \to \mathcal{B}_\bullet$ of simplicial sheaves of rings we obtain a functor

$$- \otimes^{\mathbf{L}}_{\mathcal{A}_\bullet} \mathcal{B}_\bullet : D(\mathcal{A}_\bullet) \longrightarrow D(\mathcal{B}_\bullet)$$

Moreover, the material of the preceding sections determines a functor

$$L\pi_! : D(\mathcal{A}_\bullet) \longrightarrow D(\mathcal{C})$$

Given a simplicial module $\mathcal{F}_\bullet$ the object $L\pi_!(\mathcal{F}_\bullet)$ is represented by the associated chain complex $s(\mathcal{F}_\bullet)$ (Simplicial, Section 14.21). This follows from Lemmas 21.29.2 and 21.28.7.

Lemma 21.30.2. *Let $\mathcal{C}$ be a site. Let $\mathcal{A}_\bullet \to \mathcal{B}_\bullet$ be a homomorphism of simplicial sheaves of rings on $\mathcal{C}$. If $L\pi_!\mathcal{A}_\bullet \to L\pi_!\mathcal{B}_\bullet$ is an isomorphism in $D(\mathcal{C})$, then we have*

$$L\pi_!(K) = L\pi_!(K \otimes^{\mathbf{L}}_{\mathcal{A}_\bullet} \mathcal{B}_\bullet)$$

for all K in $D(\mathcal{A}_\bullet)$.

Proof. Let $([n], U)$ be an object of $\Delta \times \mathcal{C}$. Since $L\pi_!$ commutes with colimits, it suffices to prove this for bounded above complexes of $\mathcal{O}$-modules (compare with argument of Derived Categories, Proposition 13.27.2 or just stick to bounded above complexes). Every such complex is quasi-isomorphic to a bounded above complex whose terms are flat modules, see Modules on Sites, Lemma 18.28.6. Thus it suffices to prove the lemma for a flat $\mathcal{A}_\bullet$-module $\mathcal{F}$. In this case the derived tensor product is the usual tensor product and is a sheaf also. Hence by Lemma 21.29.2 we can compute the cohomology sheaves of both sides of the equation by the procedure of Lemma 21.29.1. Thus it suffices to prove the result for the restriction of $\mathcal{F}$ to the fibre categories (i.e., to $\Delta \times U$). In this case the result follows from Lemma 21.28.12. □

Remark 21.30.3. Let $\mathcal{C}$ be a site. Let $\epsilon : \mathcal{A}_\bullet \to \mathcal{O}$ be an augmentation (Simplicial, Definition 14.18.1) in the category of sheaves of rings. Assume ϵ induces a quasi-isomorphism $s(\mathcal{A}_\bullet) \to \mathcal{O}$. In this case we obtain an exact functor of triangulated categories

$$L\pi_! : D(\mathcal{A}_\bullet) \longrightarrow D(\mathcal{O})$$

Namely, for any object K of $D(\mathcal{A}_\bullet)$ we have $L\pi_!(K) = L\pi_!(K \otimes^{\mathbf{L}}_{\mathcal{A}_\bullet} \mathcal{O})$ by Lemma 21.30.2. Thus we can define the displayed functor as the composition of $- \otimes^{\mathbf{L}}_{\mathcal{A}_\bullet} \mathcal{O}$ with the functor $L\pi_! : D(\Delta \times \mathcal{C}, \pi^{-1}\mathcal{O}) \to D(\mathcal{O})$ of Remark 21.27.6. In other words, we obtain a $\mathcal{O}$-module structure on $L\pi_!(K)$ coming from the (canonical, functorial) identification of $L\pi_!(K)$ with $L\pi_!(K \otimes^{\mathbf{L}}_{\mathcal{A}_\bullet} \mathcal{O})$ of the lemma.

21.31. Cohomology on a category

In the situation of Example 21.28.1 in addition to the derived functor $L\pi_!$, we also have the functor $R\pi_*$. For an abelian sheaf $\mathcal{F}$ on $\mathcal{C}$ we have $H_n(\mathcal{C}, \mathcal{F}) = H^{-n}(L\pi_!\mathcal{F})$ and $H^n(\mathcal{C}, \mathcal{F}) = H^n(R\pi_*\mathcal{F})$.

Example 21.31.1 (Computing cohomology)**.** In Example 21.28.1 we can compute the functors $H^n(\mathcal{C}, -)$ as follows. Let $\mathcal{F} \in \mathrm{Ob}(Ab(\mathcal{C}))$. Consider the cochain complex

$$K^\bullet(\mathcal{F}) : \prod\nolimits_{U_0} \mathcal{F}(U_0) \to \prod\nolimits_{U_0 \to U_1} \mathcal{F}(U_0) \to \prod\nolimits_{U_0 \to U_1 \to U_2} \mathcal{F}(U_0) \to \ldots$$

where the transition maps are given by

$$(s_{U_0 \to U_1}) \longmapsto ((U_0 \to U_1 \to U_2) \mapsto s_{U_0 \to U_1} - s_{U_0 \to U_2} + s_{U_1 \to U_2}|_{U_0})$$

and similarly in other degrees. By construction

$$H^0(\mathcal{C}, \mathcal{F}) = \lim_{\mathcal{C}^{opp}} \mathcal{F} = H^0(K^\bullet(\mathcal{F})),$$

see Categories, Lemma 4.14.10. The construction of $K^\bullet(\mathcal{F})$ is functorial in $\mathcal{F}$ and transforms short exact sequences of $Ab(\mathcal{C})$ into short exact sequences of complexes. Thus the sequence of functors $\mathcal{F} \mapsto H^n(K^\bullet(\mathcal{F}))$ forms a δ-functor, see Homology, Definition 12.11.1 and Lemma 12.12.12. For an object U of $\mathcal{C}$ denote $p_U : Sh(*) \to Sh(\mathcal{C})$ the corresponding point with p_U^{-1} equal to evaluation at U, see Sites, Example 7.32.7. Let A be an abelian group and set $\mathcal{F} = p_{U,*}A$. In this case the complex $K^\bullet(\mathcal{F})$ is the complex with terms $\mathrm{Map}(X_n, A)$ where

$$X_n = \coprod_{U_0 \to \ldots \to U_{n-1} \to U_n} \mathrm{Mor}_{\mathcal{C}}(U, U_0)$$

This simplicial set is homotopy equivalent to the constant simplicial set on a singleton $\{*\}$. Namely, the map $X_\bullet \to \{*\}$ is obvious, the map $\{*\} \to X_n$ is given by mapping $*$ to $(U \to \ldots \to U, \mathrm{id}_U)$, and the maps

$$h_{n,i} : X_n \longrightarrow X_n$$

(Simplicial, Lemma 14.24.2) defining the homotopy between the two maps $X_\bullet \to X_\bullet$ are given by the rule

$$h_{n,i} : (U_0 \to \ldots \to U_n, f) \longmapsto (U \to \ldots \to U \to U_i \to \ldots \to U_n, \mathrm{id})$$

for $i > 0$ and $h_{n,0} = \mathrm{id}$. Verifications omitted. Since $\mathrm{Map}(-, A)$ is a contravariant functor, implies that $K^\bullet(p_{U,*}A)$ has trivial cohomology in positive degrees (by the functoriality of Simplicial, Remark 14.24.4 and the result of Simplicial, Lemma 14.26.5). This implies that $K^\bullet(\mathcal{F})$ is acyclic in positive degrees also if $\mathcal{F}$ is a product of sheaves of the form $p_{U,*}A$. As every abelian sheaf on $\mathcal{C}$ embeds into such a product we conclude that $K^\bullet(\mathcal{F})$ computes the left derived functors $H^n(\mathcal{C}, -)$ of $H^0(\mathcal{C}, -)$ for example by Homology, Lemma 12.11.4 and Derived Categories, Lemma 13.16.6.

Example 21.31.2 (Computing Exts)**.** In Example 21.28.1 assume we are moreover given a sheaf of rings $\mathcal{O}$ on $\mathcal{C}$. Let $\mathcal{F}$, $\mathcal{G}$ be $\mathcal{O}$-modules. Consider the complex $K^\bullet(\mathcal{G}, \mathcal{F})$ with degree n term

$$\prod_{U_0 \to U_1 \to \ldots \to U_n} \mathrm{Hom}_{\mathcal{O}(U_n)}(\mathcal{G}(U_n), \mathcal{F}(U_0))$$

and transition map given by

$$(\varphi_{U_0 \to U_1}) \longmapsto ((U_0 \to U_1 \to U_2) \mapsto \varphi_{U_0 \to U_1} \circ \rho_{U_1}^{U_2} - \varphi_{U_0 \to U_2} + \rho_{U_0}^{U_1} \circ \varphi_{U_1 \to U_2})$$

and similarly in other degrees. Here the ρ's indicate restriction maps. By construction

$$\mathrm{Hom}_{\mathcal{O}}(\mathcal{G}, \mathcal{F}) = H^0(K^\bullet(\mathcal{G}, \mathcal{F}))$$

for all pairs of $\mathcal{O}$-modules $\mathcal{F}, \mathcal{G}$. The assignment $(\mathcal{G}, \mathcal{F}) \mapsto K^\bullet(\mathcal{G}, \mathcal{F})$ is a bifunctor which transforms direct sums in the first variable into products and commutes with products in the second variable. We claim that

$$\mathrm{Ext}^i_{\mathcal{O}}(\mathcal{G}, \mathcal{F}) = H^i(K^\bullet(\mathcal{G}, \mathcal{F}))$$

for $i \geq 0$ provided either

(1) $\mathcal{G}(U)$ is a projective $\mathcal{O}(U)$-module for all $U \in \mathrm{Ob}(\mathcal{C})$, or
(2) $\mathcal{F}(U)$ is an injective $\mathcal{O}(U)$-module for all $U \in \mathrm{Ob}(\mathcal{C})$.

Namely, case (1) the functor $K^\bullet(\mathcal{G}, -)$ is an exact functor from the category of $\mathcal{O}$-modules to the category of cochain complexes of abelian groups. Thus, arguing as

in Example 21.31.1, it suffices to show that $K^\bullet(\mathcal{G}, \mathcal{F})$ is acyclic in positive degrees when $\mathcal{F}$ is $p_{U,*}A$ for an $\mathcal{O}(U)$-module A. Choose a short exact sequence

$$0 \to \mathcal{G}' \to \bigoplus j_{U_i!}\mathcal{O}_{U_i} \to \mathcal{G} \to 0 \tag{21.31.2.1}$$

see Modules on Sites, Lemma 18.28.6. Since (1) holds for the middle and right sheaves, it also holds for $\mathcal{G}'$ and evaluating (21.31.2.1) on an object of $\mathcal{C}$ gives a split exact sequence of modules. We obtain a short exact sequence of complexes

$$0 \to K^\bullet(\mathcal{G}, \mathcal{F}) \to \prod K^\bullet(j_{U_i!}\mathcal{O}_{U_i}, \mathcal{F}) \to K^\bullet(\mathcal{G}', \mathcal{F}) \to 0$$

for any $\mathcal{F}$, in particular $\mathcal{F} = p_{U,*}A$. On H^0 we obtain

$$0 \to \mathrm{Hom}(\mathcal{G}, p_{U,*}A) \to \mathrm{Hom}(\prod j_{U_i!}\mathcal{O}_{U_i}, p_{U,*}A) \to \mathrm{Hom}(\mathcal{G}', p_{U,*}A) \to 0$$

which is exact as $\mathrm{Hom}(\mathcal{H}, p_{U,*}A) = \mathrm{Hom}_{\mathcal{O}(U)}(\mathcal{H}(U), A)$ and the sequence of sections of (21.31.2.1) over U is split exact. Thus we can use dimension shifting to see that it suffices to prove $K^\bullet(j_{U'!}\mathcal{O}_{U'}, p_{U,*}A)$ is acyclic in positive degrees for all $U, U' \in \mathrm{Ob}(\mathcal{C})$. In this case $K^n(j_{U'!}\mathcal{O}_{U'}, p_{U,*}A)$ is equal to

$$\prod\nolimits_{U \to U_0 \to U_1 \to \ldots \to U_n \to U'} A$$

In other words, $K^\bullet(j_{U'!}\mathcal{O}_{U'}, p_{U,*}A)$ is the complex with terms $\mathrm{Map}(X_\bullet, A)$ where

$$X_n = \coprod\nolimits_{U_0 \to \ldots \to U_{n-1} \to U_n} \mathrm{Mor}_{\mathcal{C}}(U, U_0) \times \mathrm{Mor}_{\mathcal{C}}(U_n, U')$$

This simplicial set is homotopy equivalent to the constant simplicial set on a singleton $\{*\}$ as can be proved in exactly the same way as the corresponding statement in Example 21.31.1. This finishes the proof of the claim.

The argument in case (2) is similar (but dual).

21.32. Strictly perfect complexes

This section is the analogue of Cohomology, Section 20.32.

Definition 21.32.1. Let $(\mathcal{C}, \mathcal{O})$ be a ringed site. Let $\mathcal{E}^\bullet$ be a complex of $\mathcal{O}$-modules. We say $\mathcal{E}^\bullet$ is *strictly perfect* if $\mathcal{E}^i$ is zero for all but finitely many i and $\mathcal{E}^i$ is a direct summand of a finite free $\mathcal{O}$-module for all i.

Let U be an object of $\mathcal{C}$. We will often say "Let $\mathcal{E}^\bullet$ be a strictly perfect complex of $\mathcal{O}_U$-modules" to mean $\mathcal{E}^\bullet$ is a strictly perfect complex of modules on the ringed site $(\mathcal{C}/U, \mathcal{O}_U)$, see Modules on Sites, Definition 18.19.1.

Lemma 21.32.2. *The cone on a morphism of strictly perfect complexes is strictly perfect.*

Proof. This is immediate from the definitions. □

Lemma 21.32.3. *The total complex associated to the tensor product of two strictly perfect complexes is strictly perfect.*

Proof. Omitted. □

Lemma 21.32.4. *Let $(f, f^\sharp) : (\mathcal{C}, \mathcal{O}_\mathcal{C}) \to (\mathcal{D}, \mathcal{O}_\mathcal{D})$ be a morphism of ringed sites. If $\mathcal{F}^\bullet$ is a strictly perfect complex of $\mathcal{O}_\mathcal{D}$-modules, then $f^*\mathcal{F}^\bullet$ is a strictly perfect complex of $\mathcal{O}_\mathcal{C}$-modules.*

Proof. We have seen in Modules on Sites, Lemma 18.17.2 that the pullback of a finite free module is finite free. The functor f^* is additive functor hence preserves direct summands. The lemma follows. □

Lemma 21.32.5. *Let $(\mathcal{C}, \mathcal{O})$ be a ringed site. Let U be an object of $\mathcal{C}$. Given a solid diagram of $\mathcal{O}_U$-modules*

$$\begin{array}{ccc} \mathcal{E} & \longrightarrow & \mathcal{F} \\ & \searrow & \uparrow{\scriptstyle p} \\ & & \mathcal{G} \end{array}$$

with $\mathcal{E}$ a direct summand of a finite free $\mathcal{O}_U$-module and p surjective, then there exists a covering $\{U_i \to U\}$ such that a dotted arrow making the diagram commute exists over each U_i.

Proof. We may assume $\mathcal{E} = \mathcal{O}_U^{\oplus n}$ for some n. In this case finding the dotted arrow is equivalent to lifting the images of the basis elements in $\Gamma(U, \mathcal{F})$. This is locally possible by the characterization of surjective maps of sheaves (Sites, Section 7.12). □

Lemma 21.32.6. *Let $(\mathcal{C}, \mathcal{O})$ be a ringed site. Let U be an object of $\mathcal{C}$.*

1. *Let $\alpha : \mathcal{E}^\bullet \to \mathcal{F}^\bullet$ be a morphism of complexes of $\mathcal{O}_U$-modules with $\mathcal{E}^\bullet$ strictly perfect and $\mathcal{F}^\bullet$ acyclic. Then there exists a covering $\{U_i \to U\}$ such that each $\alpha|_{U_i}$ is homotopic to zero.*
2. *Let $\alpha : \mathcal{E}^\bullet \to \mathcal{F}^\bullet$ be a morphism of complexes of $\mathcal{O}_U$-modules with $\mathcal{E}^\bullet$ strictly perfect, $\mathcal{E}^i = 0$ for $i < a$, and $H^i(\mathcal{F}^\bullet) = 0$ for $i \geq a$. Then there exists a covering $\{U_i \to U\}$ such that each $\alpha|_{U_i}$ is homotopic to zero.*

Proof. The first statement follows from the second, hence we only prove (2). We will prove this by induction on the length of the complex $\mathcal{E}^\bullet$. If $\mathcal{E}^\bullet \cong \mathcal{E}[-n]$ for some direct summand $\mathcal{E}$ of a finite free $\mathcal{O}$-module and integer $n \geq a$, then the result follows from Lemma 21.32.5 and the fact that $\mathcal{F}^{n-1} \to \mathrm{Ker}(\mathcal{F}^n \to \mathcal{F}^{n+1})$ is surjective by the assumed vanishing of $H^n(\mathcal{F}^\bullet)$. If $\mathcal{E}^i$ is zero except for $i \in [a, b]$, then we have a split exact sequence of complexes

$$0 \to \mathcal{E}^b[-b] \to \mathcal{E}^\bullet \to \sigma_{\leq b-1}\mathcal{E}^\bullet \to 0$$

which determines a distinguished triangle in $K(\mathcal{O}_U)$. Hence an exact sequence

$$\mathrm{Hom}_{K(\mathcal{O}_U)}(\sigma_{\leq b-1}\mathcal{E}^\bullet, \mathcal{F}^\bullet) \to \mathrm{Hom}_{K(\mathcal{O}_U)}(\mathcal{E}^\bullet, \mathcal{F}^\bullet) \to \mathrm{Hom}_{K(\mathcal{O}_U)}(\mathcal{E}^b[-b], \mathcal{F}^\bullet)$$

by the axioms of triangulated categories. The composition $\mathcal{E}^b[-b] \to \mathcal{F}^\bullet$ is homotopic to zero on the members of a covering of U by the above, whence we may assume our map comes from an element in the left hand side of the displayed exact sequence above. This element is zero on the members of a covering of U by induction hypothesis. □

Lemma 21.32.7. *Let $(\mathcal{C}, \mathcal{O})$ be a ringed site. Let U be an object of $\mathcal{C}$. Given a solid diagram of complexes of $\mathcal{O}_U$-modules*

$$\begin{array}{ccc} \mathcal{E}^\bullet & \xrightarrow{\alpha} & \mathcal{F}^\bullet \\ & \searrow & \uparrow{\scriptstyle f} \\ & & \mathcal{G}^\bullet \end{array}$$

with $\mathcal{E}^\bullet$ strictly perfect, $\mathcal{E}^j = 0$ for $j < a$ and $H^j(f)$ an isomorphism for $j > a$ and surjective for $j = a$, then there exists a covering $\{U_i \to U\}$ and for each i a dotted arrow over U_i making the diagram commute up to homotopy.

Proof. Our assumptions on f imply the cone $C(f)^\bullet$ has vanishing cohomology sheaves in degrees $\geq a$. Hence Lemma 21.32.6 guarantees there is a covering $\{U_i \to U\}$ such that the composition $\mathcal{E}^\bullet \to \mathcal{F}^\bullet \to C(f)^\bullet$ is homotopic to zero over U_i. Since

$$\mathcal{G}^\bullet \to \mathcal{F}^\bullet \to C(f)^\bullet \to \mathcal{G}^\bullet[1]$$

restricts to a distinguished triangle in $K(\mathcal{O}_{U_i})$ we see that we can lift $\alpha|_{U_i}$ up to homotopy to a map $\alpha_i : \mathcal{E}^\bullet|_{U_i} \to \mathcal{G}^\bullet|_{U_i}$ as desired. □

Lemma 21.32.8. *Let $(\mathcal{C}, \mathcal{O})$ be a ringed site. Let U be an object of $\mathcal{C}$. Let $\mathcal{E}^\bullet$, $\mathcal{F}^\bullet$ be complexes of $\mathcal{O}_U$-modules with $\mathcal{E}^\bullet$ strictly perfect.*

1. *For any element $\alpha \in \mathrm{Hom}_{D(\mathcal{O}_U)}(\mathcal{E}^\bullet, \mathcal{F}^\bullet)$ there exists a covering $\{U_i \to U\}$ such that $\alpha|_{U_i}$ is given by a morphism of complexes $\alpha_i : \mathcal{E}^\bullet|_{U_i} \to \mathcal{F}^\bullet|_{U_i}$.*
2. *Given a morphism of complexes $\alpha : \mathcal{E}^\bullet \to \mathcal{F}^\bullet$ whose image in the group $\mathrm{Hom}_{D(\mathcal{O}_U)}(\mathcal{E}^\bullet, \mathcal{F}^\bullet)$ is zero, there exists a covering $\{U_i \to U\}$ such that $\alpha|_{U_i}$ is homotopic to zero.*

Proof. Proof of (1). By the construction of the derived category we can find a quasi-isomorphism $f : \mathcal{F}^\bullet \to \mathcal{G}^\bullet$ and a map of complexes $\beta : \mathcal{E}^\bullet \to \mathcal{G}^\bullet$ such that $\alpha = f^{-1}\beta$. Thus the result follows from Lemma 21.32.7. We omit the proof of (2). □

Lemma 21.32.9. *Let $(\mathcal{C}, \mathcal{O})$ be a ringed site. Let $\mathcal{E}^\bullet$, $\mathcal{F}^\bullet$ be complexes of $\mathcal{O}$-modules with $\mathcal{E}^\bullet$ strictly perfect. Then the internal hom $R\mathcal{H}om(\mathcal{E}^\bullet, \mathcal{F}^\bullet)$ is represented by the complex $\mathcal{H}^\bullet$ with terms*

$$\mathcal{H}^n = \bigoplus\nolimits_{n=-p+q} \mathcal{H}om_{\mathcal{O}}(\mathcal{E}^p, \mathcal{F}^q)$$

and differential as described in Section 21.25.

Proof. Choose a quasi-isomorphism $\mathcal{F}^\bullet \to \mathcal{I}^\bullet$ into a K-injective complex. Let $(\mathcal{H}')^\bullet$ be the complex with terms

$$(\mathcal{H}')^n = \prod\nolimits_{n=-p+q} \mathcal{H}om_{\mathcal{O}}(\mathcal{L}^p, \mathcal{I}^q)$$

which represents $R\mathcal{H}om(\mathcal{E}^\bullet, \mathcal{F}^\bullet)$ by the construction in Section 21.25. It suffices to show that the map

$$\mathcal{H}^\bullet \longrightarrow (\mathcal{H}')^\bullet$$

is a quasi-isomorphism. Given an object U of $\mathcal{C}$ we have by inspection

$$H^0(\mathcal{H}^\bullet(U)) = \mathrm{Hom}_{K(\mathcal{O}_U)}(\mathcal{E}^\bullet|_U, \mathcal{K}^\bullet|_U) \to H^0((\mathcal{H}')^\bullet(U)) = \mathrm{Hom}_{D(\mathcal{O}_U)}(\mathcal{E}^\bullet|_U, \mathcal{K}^\bullet|_U)$$

By Lemma 21.32.8 the sheafification of $U \mapsto H^0(\mathcal{H}^\bullet(U))$ is equal to the sheafification of $U \mapsto H^0((\mathcal{H}')^\bullet(U))$. A similar argument can be given for the other cohomology sheaves. Thus $\mathcal{H}^\bullet$ is quasi-isomorphic to $(\mathcal{H}')^\bullet$ which proves the lemma. □

Lemma 21.32.10. *Let $(\mathcal{C}, \mathcal{O})$ be a ringed site. Let $\mathcal{E}^\bullet$, $\mathcal{F}^\bullet$ be complexes of $\mathcal{O}$-modules with*

1. *$\mathcal{F}^n = 0$ for $n \ll 0$,*
2. *$\mathcal{E}^n = 0$ for $n \gg 0$, and*
3. *$\mathcal{E}^n$ isomorphic to a direct summand of a finite free $\mathcal{O}$-module.*

Then the internal hom $R\mathcal{H}om(\mathcal{E}^\bullet, \mathcal{F}^\bullet)$ is represented by the complex $\mathcal{H}^\bullet$ with terms

$$\mathcal{H}^n = \bigoplus\nolimits_{n=-p+q} \mathcal{H}om_{\mathcal{O}}(\mathcal{E}^p, \mathcal{F}^q)$$

and differential as described in Section 21.25.

Proof. Choose a quasi-isomorphism $\mathcal{F}^\bullet \to \mathcal{I}^\bullet$ where $\mathcal{I}^\bullet$ is a bounded below complex of injectives. Note that $\mathcal{I}^\bullet$ is K-injective (Derived Categories, Lemma 13.28.4). Hence the construction in Section 21.25 shows that $R\mathcal{H}om(\mathcal{E}^\bullet, \mathcal{F}^\bullet)$ is represented by the complex $(\mathcal{H}')^\bullet$ with terms

$$(\mathcal{H}')^n = \prod\nolimits_{n=-p+q} \mathcal{H}om_{\mathcal{O}}(\mathcal{E}^p, \mathcal{I}^q) = \bigoplus\nolimits_{n=-p+q} \mathcal{H}om_{\mathcal{O}}(\mathcal{E}^p, \mathcal{I}^q)$$

(equality because there are only finitely many nonzero terms). Note that $\mathcal{H}^\bullet$ is the total complex associated to the double complex with terms $\mathcal{H}om_{\mathcal{O}}(\mathcal{E}^p, \mathcal{F}^q)$ and similarly for $(\mathcal{H}')^\bullet$. The natural map $(\mathcal{H}')^\bullet \to \mathcal{H}^\bullet$ comes from a map of double complexes. Thus to show this map is a quasi-isomorphism, we may use the spectral sequence of a double complex (Homology, Lemma 12.22.5)

$${}'E_1^{p,q} = H^q(\mathcal{H}om_{\mathcal{O}}(\mathcal{E}^p, \mathcal{F}^\bullet))$$

converging to $H^{p+q}(\mathcal{H}^\bullet)$ and similarly for $(\mathcal{H}')^\bullet$. To finish the proof of the lemma it suffices to show that $\mathcal{F}^\bullet \to \mathcal{I}^\bullet$ induces an isomorphism

$$H^q(\mathcal{H}om_{\mathcal{O}}(\mathcal{E}, \mathcal{F}^\bullet)) \longrightarrow H^q(\mathcal{H}om_{\mathcal{O}}(\mathcal{E}, \mathcal{I}^\bullet))$$

on cohomology sheaves whenever $\mathcal{E}$ is a direct summand of a finite free $\mathcal{O}$-module. Since this is clear when $\mathcal{E}$ is finite free the result follows. □

21.33. Pseudo-coherent modules

In this section we discuss pseudo-coherent complexes.

Definition 21.33.1. Let $(\mathcal{C}, \mathcal{O})$ be a ringed site. Let $\mathcal{E}^\bullet$ be a complex of $\mathcal{O}$-modules. Let $m \in \mathbf{Z}$.

1. We say $\mathcal{E}^\bullet$ is *m-pseudo-coherent* if for every object U of $\mathcal{C}$ there exists a covering $\{U_i \to U\}$ and for each i a morphism of complexes $\alpha_i : \mathcal{E}_i^\bullet \to \mathcal{E}^\bullet|_{U_i}$ where $\mathcal{E}_i$ is a strictly perfect complex of $\mathcal{O}_{U_i}$-modules and $H^j(\alpha_i)$ is an isomorphism for $j > m$ and $H^m(\alpha_i)$ is surjective.
2. We say $\mathcal{E}^\bullet$ is *pseudo-coherent* if it is m-pseudo-coherent for all m.
3. We say an object E of $D(\mathcal{O})$ is *m-pseudo-coherent* (resp. *pseudo-coherent*) if and only if it can be represented by a m-pseudo-coherent (resp. pseudo-coherent) complex of $\mathcal{O}$-modules.

If $\mathcal{C}$ has a final object X which is quasi-compact (i.e., every covering of X can be refined by a finite covering), then an m-pseudo-coherent object of $D(\mathcal{O})$ is in $D^-(\mathcal{O})$. But this need not be the case in general.

Lemma 21.33.2. *Let $(\mathcal{C}, \mathcal{O})$ be a ringed site. Let E be an object of $D(\mathcal{O})$.*

1. *If $\mathcal{C}$ has a final object X and if there exist a covering $\{U_i \to X\}$, strictly perfect complexes $\mathcal{E}_i^\bullet$ of $\mathcal{O}_{U_i}$-modules, and maps $\alpha_i : \mathcal{E}_i^\bullet \to E|_{U_i}$ in $D(\mathcal{O}_{U_i})$ with $H^j(\alpha_i)$ an isomorphism for $j > m$ and $H^m(\alpha_i)$ surjective, then E is m-pseudo-coherent.*
2. *If E is m-pseudo-coherent, then any complex of $\mathcal{O}$-modules representing E is m-pseudo-coherent.*

(3) *If for every object U of $\mathcal{C}$ there exists a covering $\{U_i \to U\}$ such that $E|_{U_i}$ is m-pseudo-coherent, then E is m-pseudo-coherent.*

Proof. Let $\mathcal{F}^\bullet$ be any complex representing E and let X, $\{U_i \to X\}$, and $\alpha_i : \mathcal{E}_i \to E|_{U_i}$ be as in (1). We will show that $\mathcal{F}^\bullet$ is m-pseudo-coherent as a complex, which will prove (1) and (2) in case $\mathcal{C}$ has a final object. By Lemma 21.32.8 we can after refining the covering $\{U_i \to X\}$ represent the maps α_i by maps of complexes $\alpha_i : \mathcal{E}_i^\bullet \to \mathcal{F}^\bullet|_{U_i}$. By assumption $H^j(\alpha_i)$ are isomorphisms for $j > m$, and $H^m(\alpha_i)$ is surjective whence $\mathcal{F}^\bullet$ is m-pseudo-coherent.

Proof of (2). By the above we see that $\mathcal{F}^\bullet|_U$ is m-pseudo-coherent as a complex of $\mathcal{O}_U$-modules for all objects U of $\mathcal{C}$. It is a formal consequence of the definitions that $\mathcal{F}^\bullet$ is m-pseudo-coherent.

Proof of (3). Follows from the definitions and Sites, Definition 7.6.2 part (2). □

Lemma 21.33.3. *Let $(f, f^\sharp) : (\mathcal{C}, \mathcal{O}_\mathcal{C}) \to (\mathcal{D}, \mathcal{O}_\mathcal{D})$ be a morphism of ringed sites. Let E be an object of $D(\mathcal{O}_\mathcal{C})$. If E is m-pseudo-coherent, then Lf^*E is m-pseudo-coherent.*

Proof. Say f is given by the functor $u : \mathcal{D} \to \mathcal{C}$. Let U be an object of $\mathcal{C}$. By Sites, Lemma 7.15.9 we can find a covering $\{U_i \to U\}$ and for each i a morphism $U_i \to u(V_i)$ for some object V_i of $\mathcal{D}$. By Lemma 21.33.2 it suffices to show that $Lf^*E|_{U_i}$ is m-pseudo-coherent. To do this it is enough to show that $Lf^*E|_{u(V_i)}$ is m-pseudo-coherent, since $Lf^*E|_{U_i}$ is the restriction of $Lf^*E|_{u(V_i)}$ to $\mathcal{C}/U_i$ (via Modules on Sites, Lemma 18.19.4). By the commutative diagram of Modules on Sites, Lemma 18.20.1 it suffices to prove the lemma for the morphism of ringed sites $(\mathcal{C}/u(V_i), \mathcal{O}_{u(V_i)}) \to (\mathcal{D}/V_i, \mathcal{O}_{V_i})$. Thus we may assume $\mathcal{D}$ has a final object Y such that $X = u(Y)$ is a final object of $\mathcal{C}$.

Let $\{V_i \to Y\}$ be a covering such that for each i there exists a strictly perfect complex $\mathcal{F}_i^\bullet$ of $\mathcal{O}_{V_i}$-modules and a morphism $\alpha_i : \mathcal{F}_i^\bullet \to E|_{V_i}$ of $D(\mathcal{O}_{V_i})$ such that $H^j(\alpha_i)$ is an isomorphism for $j > m$ and $H^m(\alpha_i)$ is surjective. Arguing as above it suffices to prove the result for $(\mathcal{C}/u(V_i), \mathcal{O}_{u(V_i)}) \to (\mathcal{D}/V_i, \mathcal{O}_{V_i})$. Hence we may assume that there exists a strictly perfect complex $\mathcal{F}^\bullet$ of $\mathcal{O}_\mathcal{D}$-modules and a morphism $\alpha : \mathcal{F}^\bullet \to E$ of $D(\mathcal{O}_\mathcal{D})$ such that $H^j(\alpha)$ is an isomorphism for $j > m$ and $H^m(\alpha)$ is surjective. In this case, choose a distinguished triangle

$$\mathcal{F}^\bullet \to E \to C \to \mathcal{F}^\bullet[1]$$

The assumption on α means exactly that the cohomology sheaves $H^j(C)$ are zero for all $j \geq m$. Applying Lf^* we obtain the distinguished triangle

$$Lf^*\mathcal{F}^\bullet \to Lf^*E \to Lf^*C \to Lf^*\mathcal{F}^\bullet[1]$$

By the construction of Lf^* as a left derived functor we see that $H^j(Lf^*C) = 0$ for $j \geq m$ (by the dual of Derived Categories, Lemma 13.16.1). Hence $H^j(Lf^*\alpha)$ is an isomorphism for $j > m$ and $H^m(Lf^*\alpha)$ is surjective. On the other hand, since $\mathcal{F}^\bullet$ is a bounded above complex of flat $\mathcal{O}_\mathcal{D}$-modules we see that $Lf^*\mathcal{F}^\bullet = f^*\mathcal{F}^\bullet$. Applying Lemma 21.32.4 we conclude. □

Lemma 21.33.4. *Let $(\mathcal{C}, \mathcal{O})$ be a ringed site and $m \in \mathbf{Z}$. Let (K, L, M, f, g, h) be a distinguished triangle in $D(\mathcal{O})$.*

(1) *If K is $(m+1)$-pseudo-coherent and L is m-pseudo-coherent then M is m-pseudo-coherent.*

(2) *If K anf M are m-pseudo-coherent, then L is m-pseudo-coherent.*
(3) *If L is $(m+1)$-pseudo-coherent and M is m-pseudo-coherent, then K is $(m+1)$-pseudo-coherent.*

Proof. Proof of (1). Let U be an object of $\mathcal{C}$. Choose a covering $\{U_i \to U\}$ and maps $\alpha_i : \mathcal{K}_i^\bullet \to K|_{U_i}$ in $D(\mathcal{O}_{U_i})$ with $\mathcal{K}_i^\bullet$ strictly perfect and $H^j(\alpha_i)$ isomorphisms for $j > m+1$ and surjective for $j = m+1$. We may replace $\mathcal{K}_i^\bullet$ by $\sigma_{\geq m+1}\mathcal{K}_i^\bullet$ and hence we may assume that $\mathcal{K}_i^j = 0$ for $j < m+1$. After refining the covering we may choose maps $\beta_i : \mathcal{L}_i^\bullet \to L|_{U_i}$ in $D(\mathcal{O}_{U_i})$ with $\mathcal{L}_i^\bullet$ strictly perfect such that $H^j(\beta)$ is an isomorphism for $j > m$ and surjective for $j = m$. By Lemma 21.32.7 we can, after refining the covering, find maps of complexes $\gamma_i : \mathcal{K}^\bullet \to \mathcal{L}^\bullet$ such that the diagrams

$$\begin{array}{ccc} K|_{U_i} & \longrightarrow & L|_{U_i} \\ \uparrow{\scriptstyle \alpha_i} & & \uparrow{\scriptstyle \beta_i} \\ \mathcal{K}_i^\bullet & \xrightarrow{\gamma_i} & \mathcal{L}_i^\bullet \end{array}$$

are commutative in $D(\mathcal{O}_{U_i})$ (this requires representing the maps α_i, β_i and $K|_{U_i} \to L|_{U_i}$ by actual maps of complexes; some details omitted). The cone $C(\gamma_i)^\bullet$ is strictly perfect (Lemma 21.32.2). The commutativity of the diagram implies that there exists a morphism of distinguished triangles

$$(\mathcal{K}_i^\bullet, \mathcal{L}_i^\bullet, C(\gamma_i)^\bullet) \longrightarrow (K|_{U_i}, L|_{U_i}, M|_{U_i}).$$

It follows from the induced map on long exact cohomology sequences and Homology, Lemmas 12.5.19 and 12.5.20 that $C(\gamma_i)^\bullet \to M|_{U_i}$ induces an isomorphism on cohomology in degrees $> m$ and a surjection in degree m. Hence M is m-pseudo-coherent by Lemma 21.33.2.

Assertions (2) and (3) follow from (1) by rotating the distinguished triangle. □

Lemma 21.33.5. *Let $(\mathcal{C}, \mathcal{O})$ be a ringed site. Let K, L be objects of $D(\mathcal{O})$.*

(1) *If K is n-pseudo-coherent and $H^i(K) = 0$ for $i > a$ and L is m-pseudo-coherent and $H^j(L) = 0$ for $j > b$, then $K \otimes^{\mathbf{L}}_{\mathcal{O}} L$ is t-pseudo-coherent with $t = \max(m + a, n + b)$.*
(2) *If K and L are pseudo-coherent, then $K \otimes^{\mathbf{L}}_{\mathcal{O}} L$ is pseudo-coherent.*

Proof. Proof of (1). Let U be an object of $\mathcal{C}$. By replacing U by the members of a covering and replacing $\mathcal{C}$ by the localization $\mathcal{C}/U$ we may assume there exist strictly perfect complexes $\mathcal{K}^\bullet$ and $\mathcal{L}^\bullet$ and maps $\alpha : \mathcal{K}^\bullet \to K$ and $\beta : \mathcal{L}^\bullet \to L$ with $H^i(\alpha)$ and isomorphism for $i > n$ and surjective for $i = n$ and with $H^i(\beta)$ and isomorphism for $i > m$ and surjective for $i = m$. Then the map

$$\alpha \otimes^{\mathbf{L}} \beta : \mathrm{Tot}(\mathcal{K}^\bullet \otimes_{\mathcal{O}} \mathcal{L}^\bullet) \to K \otimes^{\mathbf{L}}_{\mathcal{O}} L$$

induces isomorphisms on cohomology sheaves in degree i for $i > t$ and a surjection for $i = t$. This follows from the spectral sequence of tors (details omitted).

Proof of (2). Let U be an object of $\mathcal{C}$. We may first replace U by the members of a covering and $\mathcal{C}$ by the localization $\mathcal{C}/U$ to reduce to the case that K and L are bounded above. Then the statement follows immediately from case (1). □

Lemma 21.33.6. *Let $(\mathcal{C}, \mathcal{O})$ be a ringed site. Let $m \in \mathbf{Z}$. If $K \oplus L$ is m-pseudo-coherent (resp. pseudo-coherent) in $D(\mathcal{O})$ so are K and L.*

Proof. Assume that $K \oplus L$ is m-pseudo-coherent. Let U be an object of $\mathcal{C}$. After replacing U by the members of a covering we may assume $K \oplus L \in D^-(\mathcal{O}_U)$, hence $L \in D^-(\mathcal{O}_U)$. Note that there is a distinguished triangle

$$(K \oplus L, K \oplus L, L \oplus L[1]) = (K, K, 0) \oplus (L, L, L \oplus L[1])$$

see Derived Categories, Lemma 13.4.9. By Lemma 21.33.4 we see that $L \oplus L[1]$ is m-pseudo-coherent. Hence also $L[1] \oplus L[2]$ is m-pseudo-coherent. By induction $L[n] \oplus L[n+1]$ is m-pseudo-coherent. Since L is bounded above we see that $L[n]$ is m-pseudo-coherent for large n. Hence working backwards, using the distinguished triangles

$$(L[n], L[n] \oplus L[n-1], L[n-1])$$

we conclude that $L[n-1], L[n-2], \ldots, L$ are m-pseudo-coherent as desired. □

Lemma 21.33.7. *Let $(\mathcal{C}, \mathcal{O})$ be a ringed site. Let K be an object of $D(\mathcal{O})$. Let $m \in \mathbf{Z}$.*

1. *If K is m-pseudo-coherent and $H^i(K) = 0$ for $i > m$, then $H^m(K)$ is a finite type $\mathcal{O}$-module.*
2. *If K is m-pseudo-coherent and $H^i(K) = 0$ for $i > m+1$, then $H^{m+1}(K)$ is a finitely presented $\mathcal{O}$-module.*

Proof. Proof of (1). Let U be an object of $\mathcal{C}$. We have to show that $H^m(K)$ is can be generated by finitely many sections over the members of a covering of U (see Modules on Sites, Definition 18.23.1). Thus during the proof we may (finitely often) choose a covering $\{U_i \to U\}$ and replace $\mathcal{C}$ by $\mathcal{C}/U_i$ and U by U_i. In particular, by our definitions we may assume there exists a strictly perfect complex $\mathcal{E}^\bullet$ and a map $\alpha : \mathcal{E}^\bullet \to K$ which induces an isomorphism on cohomology in degrees $> m$ and a surjection in degree m. It suffices to prove the result for $\mathcal{E}^\bullet$. Let n be the largest integer such that $\mathcal{E}^n \neq 0$. If $n = m$, then $H^m(\mathcal{E}^\bullet)$ is a quotient of $\mathcal{E}^n$ and the result is clear. If $n > m$, then $\mathcal{E}^{n-1} \to \mathcal{E}^n$ is surjective as $H^n(E^\bullet) = 0$. By Lemma 21.32.5 we can (after replacing U by the members of a covering) find a section of this surjection and write $\mathcal{E}^{n-1} = \mathcal{E}' \oplus \mathcal{E}^n$. Hence it suffices to prove the result for the complex $(\mathcal{E}')^\bullet$ which is the same as $\mathcal{E}^\bullet$ except has $\mathcal{E}'$ in degree $n-1$ and 0 in degree n. We win by induction on n.

Proof of (2). Pick an object U of $\mathcal{C}$. As in the proof of (1) we may work locally on U. Hence we may assume there exists a strictly perfect complex $\mathcal{E}^\bullet$ and a map $\alpha : \mathcal{E}^\bullet \to K$ which induces an isomorphism on cohomology in degrees $> m$ and a surjection in degree m. As in the proof of (1) we can reduce to the case that $\mathcal{E}^i = 0$ for $i > m+1$. Then we see that $H^{m+1}(K) \cong H^{m+1}(\mathcal{E}^\bullet) = \mathrm{Coker}(\mathcal{E}^m \to \mathcal{E}^{m+1})$ which is of finite presentation. □

21.34. Tor dimension

In this section we take a closer look at resolutions by flat modules.

Definition 21.34.1. Let $(\mathcal{C}, \mathcal{O})$ be a ringed site. Let E be an object of $D(\mathcal{O})$. Let $a, b \in \mathbf{Z}$ with $a \leq b$.

1. We say E has *tor-amplitude in* $[a, b]$ if $H^i(E \otimes^{\mathbf{L}}_{\mathcal{O}} \mathcal{F}) = 0$ for all $\mathcal{O}$-modules $\mathcal{F}$ and all $i \notin [a, b]$.
2. We say E has *finite tor dimension* if it has tor-amplitude in $[a, b]$ for some a, b.

(3) We say E *locally has finite tor dimension* if for any object U of $\mathcal{C}$ there exists a covering $\{U_i \to U\}$ such that $E|_{U_i}$ has finite tor dimension for all i.

Note that if E has finite tor dimension, then E is an object of $D^b(\mathcal{O})$ as can be seen by taking $\mathcal{F} = \mathcal{O}$ in the definition above.

Lemma 21.34.2. *Let $(\mathcal{C}, \mathcal{O})$ be a ringed site. Let $\mathcal{E}^\bullet$ be a bounded above complex of flat $\mathcal{O}$-modules with tor-amplitude in $[a, b]$. Then Coker$(d^{a-1}_{\mathcal{E}^\bullet})$ is a flat $\mathcal{O}$-module.*

Proof. As $\mathcal{E}^\bullet$ is a bounded above complex of flat modules we see that $\mathcal{E}^\bullet \otimes_{\mathcal{O}} \mathcal{F} = \mathcal{E}^\bullet \otimes^{\mathbf{L}}_{\mathcal{O}} \mathcal{F}$ for any $\mathcal{O}$-module $\mathcal{F}$. Hence for every $\mathcal{O}$-module $\mathcal{F}$ the sequence

$$\mathcal{E}^{a-2} \otimes_{\mathcal{O}} \mathcal{F} \to \mathcal{E}^{a-1} \otimes_{\mathcal{O}} \mathcal{F} \to \mathcal{E}^a \otimes_{\mathcal{O}} \mathcal{F}$$

is exact in the middle. Since $\mathcal{E}^{a-2} \to \mathcal{E}^{a-1} \to \mathcal{E}^a \to \text{Coker}(d^{a-1}) \to 0$ is a flat resolution this implies that $\text{Tor}^{\mathcal{O}}_1(\text{Coker}(d^{a-1}), \mathcal{F}) = 0$ for all $\mathcal{O}$-modules $\mathcal{F}$. This means that $\text{Coker}(d^{a-1})$ is flat, see Lemma 21.17.13. □

Lemma 21.34.3. *Let $(\mathcal{C}, \mathcal{O})$ be a ringed site. Let E be an object of $D(\mathcal{O})$. Let $a, b \in \mathbf{Z}$ with $a \leq b$. The following are equivalent*

(1) *E has tor-amplitude in $[a, b]$.*
(2) *E is represented by a complex $\mathcal{E}^\bullet$ of flat $\mathcal{O}$-modules with $\mathcal{E}^i = 0$ for $i \notin [a, b]$.*

Proof. If (2) holds, then we may compute $E \otimes^{\mathbf{L}}_{\mathcal{O}} \mathcal{F} = \mathcal{E}^\bullet \otimes_{\mathcal{O}} \mathcal{F}$ and it is clear that (1) holds.

Assume that (1) holds. We may represent E by a bounded above complex of flat $\mathcal{O}$-modules $\mathcal{K}^\bullet$, see Section 21.17. Let n be the largest integer such that $\mathcal{K}^n \neq 0$. If $n > b$, then $\mathcal{K}^{n-1} \to \mathcal{K}^n$ is surjective as $H^n(\mathcal{K}^\bullet) = 0$. As $\mathcal{K}^n$ is flat we see that $\text{Ker}(\mathcal{K}^{n-1} \to \mathcal{K}^n)$ is flat (Modules on Sites, Lemma 18.28.8). Hence we may replace $\mathcal{K}^\bullet$ by $\tau_{\leq n-1}\mathcal{K}^\bullet$. Thus, by induction on n, we reduce to the case that $K^\bullet$ is a complex of flat $\mathcal{O}$-modules with $\mathcal{K}^i = 0$ for $i > b$.

Set $\mathcal{E}^\bullet = \tau_{\geq a}\mathcal{K}^\bullet$. Everything is clear except that $\mathcal{E}^a$ is flat which follows immediately from Lemma 21.34.2 and the definitions. □

Lemma 21.34.4. *Let $(f, f^\sharp) : (\mathcal{C}, \mathcal{O}_\mathcal{C}) \to (\mathcal{D}, \mathcal{O}_\mathcal{D})$ be a morphism of ringed sites. Assume $\mathcal{C}$ has enough points. Let E be an object of $D(\mathcal{O}_\mathcal{D})$. If E has tor amplitude in $[a, b]$, then Lf^*E has tor amplitude in $[a, b]$.*

Proof. Assume E has tor amplitude in $[a, b]$. By Lemma 21.34.3 we can represent E by a complex of $\mathcal{E}^\bullet$ of flat $\mathcal{O}$-modules with $\mathcal{E}^i = 0$ for $i \notin [a, b]$. Then Lf^*E is represented by $f^*\mathcal{E}^\bullet$. By Modules on Sites, Lemma 18.38.3 the module $f^*\mathcal{E}^i$ are flat (this is where we need the assumption on the existence of points). Thus by Lemma 21.34.3 we conclude that Lf^*E has tor amplitude in $[a, b]$. □

Lemma 21.34.5. *Let $(\mathcal{C}, \mathcal{O})$ be a ringed site. Let (K, L, M, f, g, h) be a distinguished triangle in $D(\mathcal{O})$. Let $a, b \in \mathbf{Z}$.*

(1) *If K has tor-amplitude in $[a + 1, b + 1]$ and L has tor-amplitude in $[a, b]$ then M has tor-amplitude in $[a, b]$.*
(2) *If K and M have tor-amplitude in $[a, b]$, then L has tor-amplitude in $[a, b]$.*
(3) *If L has tor-amplitude in $[a + 1, b + 1]$ and M has tor-amplitude in $[a, b]$, then K has tor-amplitude in $[a + 1, b + 1]$.*

Proof. Omitted. Hint: This just follows from the long exact cohomology sequence associated to a distinguished triangle and the fact that $- \otimes^{\mathbf{L}}_{\mathcal{O}} \mathcal{F}$ preserves distinguished triangles. The easiest one to prove is (2) and the others follow from it by translation. □

Lemma 21.34.6. *Let $(\mathcal{C}, \mathcal{O})$ be a ringed site. Let K, L be objects of $D(\mathcal{O})$. If K has tor-amplitude in $[a, b]$ and L has tor-amplitude in $[c, d]$ then $K \otimes^{\mathbf{L}}_{\mathcal{O}} L$ has tor amplitude in $[a + c, b + d]$.*

Proof. Omitted. Hint: use the spectral sequence for tors. □

Lemma 21.34.7. *Let $(\mathcal{C}, \mathcal{O})$ be a ringed site. Let $a, b \in \mathbf{Z}$. For K, L objects of $D(\mathcal{O})$ if $K \oplus L$ has tor amplitude in $[a, b]$ so do K and L.*

Proof. Clear from the fact that the Tor functors are additive. □

Lemma 21.34.8. *Let $(\mathcal{C}, \mathcal{O})$ be a ringed site. Let $\mathcal{I} \subset \mathcal{O}$ be a sheaf of ideals. Let K be an object of $D(\mathcal{O})$.*

1. *If $K \otimes^{\mathbf{L}}_{\mathcal{O}} \mathcal{O}/\mathcal{I}$ is bounded above, then $K \otimes^{\mathbf{L}}_{\mathcal{O}} \mathcal{O}/\mathcal{I}^n$ is uniformly bounded above for all n.*
2. *If $K \otimes^{\mathbf{L}}_{\mathcal{O}} \mathcal{O}/\mathcal{I}$ as an object of $D(\mathcal{O}/\mathcal{I})$ has tor amplitude in $[a, b]$, then $K \otimes^{\mathbf{L}}_{\mathcal{O}} \mathcal{O}/\mathcal{I}^n$ as an object of $D(\mathcal{O}/\mathcal{I}^n)$ has tor amplitude in $[a, b]$ for all n.*

Proof. Proof of (1). Assume that $K \otimes^{\mathbf{L}}_{\mathcal{O}} \mathcal{O}/\mathcal{I}$ is bounded above, say $H^i(K \otimes^{\mathbf{L}}_{\mathcal{O}} \mathcal{O}/\mathcal{I}) = 0$ for $i > b$. Note that we have distinguished triangles

$$K \otimes^{\mathbf{L}}_{\mathcal{O}} \mathcal{I}^n/\mathcal{I}^{n+1} \to K \otimes^{\mathbf{L}}_{\mathcal{O}} \mathcal{O}/\mathcal{I}^{n+1} \to K \otimes^{\mathbf{L}}_{\mathcal{O}} \mathcal{O}/\mathcal{I}^n \to K \otimes^{\mathbf{L}}_{\mathcal{O}} \mathcal{I}^n/\mathcal{I}^{n+1}[1]$$

and that

$$K \otimes^{\mathbf{L}}_{\mathcal{O}} \mathcal{I}^n/\mathcal{I}^{n+1} = \left(K \otimes^{\mathbf{L}}_{\mathcal{O}} \mathcal{O}/\mathcal{I}\right) \otimes^{\mathbf{L}}_{\mathcal{O}/\mathcal{I}} \mathcal{I}^n/\mathcal{I}^{n+1}$$

By induction we conclude that $H^i(K \otimes^{\mathbf{L}}_{\mathcal{O}} \mathcal{O}/\mathcal{I}^n) = 0$ for $i > b$ for all n.

Proof of (2). Assume $K \otimes^{\mathbf{L}}_{\mathcal{O}} \mathcal{O}/\mathcal{I}$ as an object of $D(\mathcal{O}/\mathcal{I})$ has tor amplitude in $[a, b]$. Let $\mathcal{F}$ be a sheaf of $\mathcal{O}/\mathcal{I}^n$-modules. Then we have a finite filtration

$$0 \subset \mathcal{I}^{n-1}\mathcal{F} \subset \ldots \subset \mathcal{I}\mathcal{F} \subset \mathcal{F}$$

whose successive quotients are sheaves of $\mathcal{O}/\mathcal{I}$-modules. Thus to prove that $K \otimes^{\mathbf{L}}_{\mathcal{O}} \mathcal{O}/\mathcal{I}^n$ has tor amplitude in $[a, b]$ it suffices to show $H^i(K \otimes^{\mathbf{L}}_{\mathcal{O}} \mathcal{O}/\mathcal{I}^n \otimes^{\mathbf{L}}_{\mathcal{O}/\mathcal{I}^n} \mathcal{G})$ is zero for $i \notin [a, b]$ for all $\mathcal{O}/\mathcal{I}$-modules $\mathcal{G}$. Since

$$\left(K \otimes^{\mathbf{L}}_{\mathcal{O}} \mathcal{O}/\mathcal{I}^n\right) \otimes^{\mathbf{L}}_{\mathcal{O}/\mathcal{I}^n} \mathcal{G} = \left(K \otimes^{\mathbf{L}}_{\mathcal{O}} \mathcal{O}/\mathcal{I}\right) \otimes^{\mathbf{L}}_{\mathcal{O}/\mathcal{I}} \mathcal{G}$$

for every sheaf of $\mathcal{O}/\mathcal{I}$-modules $\mathcal{G}$ the result follows. □

21.35. Perfect complexes

In this section we discuss properties of perfect complexes on ringed sites.

Definition 21.35.1. Let $(\mathcal{C}, \mathcal{O})$ be a ringed site. Let $\mathcal{E}^\bullet$ be a complex of $\mathcal{O}$-modules. We say $\mathcal{E}^\bullet$ is *perfect* if for every object U of $\mathcal{C}$ there exists a covering $\{U_i \to U\}$ such that for each i there exists a morphism of complexes $\mathcal{E}_i^\bullet \to \mathcal{E}^\bullet|_{U_i}$ which is a quasi-isomorphism with $\mathcal{E}_i^\bullet$ strictly perfect. An object E of of $D(\mathcal{O})$ is *perfect* if it can be represented by a perfect complex of $\mathcal{O}$-modules.

Lemma 21.35.2. *Let $(\mathcal{C}, \mathcal{O})$ be a ringed site. Let E be an object of $D(\mathcal{O})$.*

(1) *If $\mathcal{C}$ has a final object X and there exist a covering $\{U_i \to X\}$, strictly perfect complexes $\mathcal{E}_i^\bullet$ of $\mathcal{O}_{U_i}$-modules, and isomorphisms $\alpha_i : \mathcal{E}_i^\bullet \to E|_{U_i}$ in $D(\mathcal{O}_{U_i})$, then E is perfect.*
(2) *If E is perfect, then any complex representing E is perfect.*

Proof. Identical to the proof of Lemma 21.33.2. □

Lemma 21.35.3. *Let $(\mathcal{C}, \mathcal{O})$ be a ringed site. Let E be an object of $D(\mathcal{O})$. Let $a \leq b$ be integers. If E has tor amplitude in $[a, b]$ and is $(a-1)$-pseudo-coherent, then E is perfect.*

Proof. Let U be an object of $\mathcal{C}$. After replacing U by the members of a covering and $\mathcal{C}$ by the localization $\mathcal{C}/U$ we may assume there exists a strictly perfect complex $\mathcal{E}^\bullet$ and a map $\alpha : \mathcal{E}^\bullet \to E$ such that $H^i(\alpha)$ is an isomorphism for $i \geq a$. We may and do replace $\mathcal{E}^\bullet$ by $\sigma_{\geq a-1}\mathcal{E}^\bullet$. Choose a distinguished triangle

$$\mathcal{E}^\bullet \to E \to C \to \mathcal{E}^\bullet[1]$$

From the vanishing of cohomology sheaves of E and $\mathcal{E}^\bullet$ and the assumption on α we obtain $C \cong \mathcal{K}[a-2]$ with $\mathcal{K} = \mathrm{Ker}(\mathcal{E}^{a-1} \to \mathcal{E}^a)$. Let $\mathcal{F}$ be an $\mathcal{O}$-module. Applying $- \otimes^{\mathbf{L}}_{\mathcal{O}} \mathcal{F}$ the assumption that E has tor amplitude in $[a, b]$ implies $\mathcal{K} \otimes_{\mathcal{O}} \mathcal{F} \to \mathcal{E}^{a-1} \otimes_{\mathcal{O}} \mathcal{F}$ has image $\mathrm{Ker}(\mathcal{E}^{a-1} \otimes_{\mathcal{O}} \mathcal{F} \to \mathcal{E}^a \otimes_{\mathcal{O}} \mathcal{F})$. It follows that $\mathrm{Tor}_1^{\mathcal{O}}(\mathcal{E}', \mathcal{F}) = 0$ where $\mathcal{E}' = \mathrm{Coker}(\mathcal{E}^{a-1} \to \mathcal{E}^a)$. Hence $\mathcal{E}'$ is flat (Lemma 21.17.13). Thus there exists a covering $\{U_i \to U\}$ such that $\mathcal{E}'|_{U_i}$ is a direct summand of a finite free module by Modules on Sites, Lemma 18.28.12. Thus the complex

$$\mathcal{E}'|_{U_i} \to \mathcal{E}^{a-1}|_{U_i} \to \ldots \to \mathcal{E}^b|_{U_i}$$

is quasi-isomorphic to $E|_{U_i}$ and E is perfect. □

Lemma 21.35.4. *Let $(\mathcal{C}, \mathcal{O})$ be a ringed site. Let E be an object of $D(\mathcal{O})$. The following are equivalent*

(1) *E is perfect, and*
(2) *E is pseudo-coherent and locally has finite tor dimension.*

Proof. Assume (1). Let U be an object of $\mathcal{C}$. By definition there exists a covering $\{U_i \to U\}$ such that $E|_{U_i}$ is represented by a strictly perfect complex. Thus E is pseudo-coherent (i.e., m-pseudo-coherent for all m) by Lemma 21.33.2. Moreover, a direct summand of a finite free module is flat, hence $E|_{U_i}$ has finite Tor dimension by Lemma 21.34.3. Thus (2) holds.

Assume (2). Let U be an object of $\mathcal{C}$. After replacing U by the members of a covering we may assume there exist integers $a \leq b$ such that $E|_U$ has tor amplitude in $[a, b]$. Since $E|_U$ is m-pseudo-coherent for all m we conclude using Lemma 21.35.3. □

Lemma 21.35.5. *Let $(f, f^\sharp) : (\mathcal{C}, \mathcal{O}_\mathcal{C}) \to (\mathcal{D}, \mathcal{O}_\mathcal{D})$ be a morphism of ringed sites. Assume $\mathcal{C}$ has enough points. Let E be an object of $D(\mathcal{O}_\mathcal{D})$. If E is perfect in $D(\mathcal{O}_\mathcal{D})$, then Lf^*E is perfect in $D(\mathcal{O}_\mathcal{C})$.*

Proof. This follows from Lemma 21.35.4, 21.34.4, and 21.33.3. (An alternative proof is to copy the proof of Lemma 21.33.3. This gives a proof of the result without assuming the site $\mathcal{C}$ has enough points.) □

Lemma 21.35.6. *Let $(\mathcal{C}, \mathcal{O})$ be a ringed site. Let (K, L, M, f, g, h) be a distinguished triangle in $D(\mathcal{O})$. If two out of three of K, L, M are perfect then the third is also perfect.*

Proof. First proof: Combine Lemmas 21.35.4, 21.33.4, and 21.34.5. Second proof (sketch): Say K and L are perfect. Let U be an object of $\mathcal{C}$. After replacing U by the members of a covering we may assume that $K|_U$ and $L|_U$ are represented by strictly perfect complexes $\mathcal{K}^\bullet$ and $\mathcal{L}^\bullet$. After replacing U by the members of a covering we may assume the map $K|_U \to L|_U$ is given by a map of complexes $\alpha : \mathcal{K}^\bullet \to \mathcal{L}^\bullet$, see Lemma 21.32.8. Then $M|_U$ is isomorphic to the cone of α which is strictly perfect by Lemma 21.32.2. □

Lemma 21.35.7. *Let $(\mathcal{C}, \mathcal{O})$ be a ringed site. If K, L are perfect objects of $D(\mathcal{O})$, then so is $K \otimes^{\mathbf{L}}_{\mathcal{O}} L$.*

Proof. Follows from Lemmas 21.35.4, 21.33.5, and 21.34.6. □

Lemma 21.35.8. *Let $(\mathcal{C}, \mathcal{O})$ be a ringed site. If $K \oplus L$ is a perfect object of $D(\mathcal{O})$, then so are K and L.*

Proof. Follows from Lemmas 21.35.4, 21.33.6, and 21.34.7. □

Lemma 21.35.9. *Let $(\mathcal{C}, \mathcal{O})$ be a ringed site. Let K be a perfect object of $D(\mathcal{O})$. Then $K^\wedge = R\mathcal{H}om(K, \mathcal{O})$ is a perfect object too and $(K^\wedge)^\wedge = K$. There are functorial isomorphisms*

$$K^\wedge \otimes^{\mathbf{L}}_{\mathcal{O}} M = R\mathcal{H}om_{\mathcal{O}}(K, M)$$

and

$$H^0(\mathcal{C}, K^\wedge \otimes^{\mathbf{L}}_{\mathcal{O}} M) = \mathrm{Hom}_{D(\mathcal{O})}(K, M)$$

for M in $D(\mathcal{O})$.

Proof. We will us without further mention that formation of internal hom commutes with restriction (Lemma 21.25.3). In particular we may check the first two statements locally, i.e., given any object U of $\mathcal{C}$ it suffices to prove there is a covering $\{U_i \to U\}$ such that the statement is true after restricting to $\mathcal{C}/U_i$ for each i. By Lemma 21.25.5 to see the final statement it suffices to check that the map (21.25.5.1)

$$K^\wedge \otimes^{\mathbf{L}}_{\mathcal{O}} M \longrightarrow R\mathcal{H}om(K, M)$$

is an isomorphism. This is a local question as well. Hence it suffices to prove the lemma when K is represented by a strictly perfect complex.

Assume K is represented by the strictly perfect complex $\mathcal{E}^\bullet$. Then it follows from Lemma 21.32.9 that $K^\wedge$ is represented by the complex whose terms are $(\mathcal{E}^n)^\wedge = \mathcal{H}om_{\mathcal{O}}(\mathcal{E}^n, \mathcal{O})$ in degree $-n$. Since $\mathcal{E}^n$ is a direct summand of a finite free $\mathcal{O}$-module, so is $(\mathcal{E}^n)^\wedge$. Hence $K^\wedge$ is represented by a strictly perfect complex too. It is also clear that $(K^\wedge)^\wedge = K$ as we have $((\mathcal{E}^n)^\wedge)^\wedge = \mathcal{E}^n$. To see that (21.25.5.1) is an isomorphism, represent M by a K-flat complex $\mathcal{F}^\bullet$. By Lemma 21.32.9 the complex $R\mathcal{H}om(K, M)$ is represented by the complex with terms

$$\bigoplus\nolimits_{n=-p+q} \mathcal{H}om_{\mathcal{O}}(\mathcal{E}^p, \mathcal{F}^q)$$

On the other hand, the object $K^\wedge \otimes^{\mathbf{L}} M$ is represented by the complex with terms

$$\bigoplus\nolimits_{n=-p+q} (\mathcal{E}^p)^\wedge \otimes_{\mathcal{O}} \mathcal{F}^q$$

Thus the assertion that (21.25.5.1) is an isomorphism reduces to the assertion that the canonical map

$$\mathcal{H}om_{\mathcal{O}}(\mathcal{E}, \mathcal{O}) \otimes_{\mathcal{O}} \mathcal{F} \longrightarrow \mathcal{H}om_{\mathcal{O}}(\mathcal{E}, \mathcal{F})$$

is an isomorphism when $\mathcal{E}$ is a direct summand of a finite free $\mathcal{O}$-module and $\mathcal{F}$ is any $\mathcal{O}$-module. This follows immediately from the corresponding statement when $\mathcal{E}$ is finite free. □

21.36. Projection formula

A general version of the projection formula is the following.

Lemma 21.36.1. *Let $f : (Sh(\mathcal{C}), \mathcal{O}_{\mathcal{C}}) \to (Sh(\mathcal{D}), \mathcal{O}_{\mathcal{D}})$ be a morphism of ringed topoi. Let $E \in D(\mathcal{O}_{\mathcal{C}})$ and $K \in D(\mathcal{O}_{\mathcal{D}})$. If K is perfect, then*

$$Rf_*E \otimes^{\mathbf{L}}_{\mathcal{O}_{\mathcal{D}}} K = Rf_*(E \otimes^{\mathbf{L}}_{\mathcal{O}_{\mathcal{C}}} Lf^*K)$$

in $D(\mathcal{O}_{\mathcal{D}})$.

Proof. Without any assumptions there is a map $Rf_*(E) \otimes^{\mathbf{L}} K \to Rf_*(E \otimes^{\mathbf{L}} Lf^*K)$. Namely, it is the adjoint to the canonical map

$$Lf^*(Rf_*(E) \otimes^{\mathbf{L}} K) = Lf^*(Rf_*(E)) \otimes^{\mathbf{L}} Lf^*K \longrightarrow E \otimes^{\mathbf{L}} Lf^*K$$

coming from the map $Lf^*Rf_*E \to E$. See Lemmas 21.18.4 and 21.19.1. To check it is an isomorphism we may work locally on $\mathcal{D}$, i.e., for any object V of $\mathcal{D}$ we have to find a covering $\{V_j \to V\}$ such that the map restricts to an isomorphism on V_j. By definition of perfect objects, this means we may assume K is represented by a strictly perfect complex of $\mathcal{O}_{\mathcal{D}}$-modules. Note that, completely generally, the statement is true for $K = K_1 \oplus K_2$, if and only if the statement is true for K_1 and K_2. Hence we may assume K is a finite complex of finite free $\mathcal{O}_{\mathcal{D}}$-modules. In this case a simple argument involving stupid truncations reduces the statement to the case where K is represented by a finite free $\mathcal{O}_{\mathcal{D}}$-module. Since the statement is invariant under finite direct summands in the K variable, we conclude it suffices to prove it for $K = \mathcal{O}_{\mathcal{D}}[n]$ in which case it is trivial. □

21.37. Weakly contractible objects

An object U of a site is *weakly contractible* if every surjection $\mathcal{F} \to \mathcal{G}$ of sheaves of sets gives rise to a surjection $\mathcal{F}(U) \to \mathcal{G}(U)$, see Sites, Definition 7.39.2.

Lemma 21.37.1. *Let $\mathcal{C}$ be a site. Let U be a weakly contractible object of $\mathcal{C}$. Then*

1. *the functor $\mathcal{F} \mapsto \mathcal{F}(U)$ is an exact functor $Ab(\mathcal{C}) \to Ab$,*
2. *$H^p(U, \mathcal{F}) = 0$ for every abelian sheaf $\mathcal{F}$ and all $p \geq 1$, and*
3. *for any sheaf of groups $\mathcal{G}$ any $\mathcal{G}$-torsor has a section over U.*

Proof. The first statement follows immediately from the definition (see also Homology, Section 12.7). The higher derived functors vanish by Derived Categories, Lemma 13.16.8. Let $\mathcal{F}$ be a $\mathcal{G}$-torsor. Then $\mathcal{F} \to *$ is a surjective map of sheaves. Hence (3) follows from the definition as well. □

It is convenient to list some consequences of having enough weakly contractible objects here.

Proposition 21.37.2. *Let $\mathcal{C}$ be a site. Let $\mathcal{B} \subset \mathrm{Ob}(\mathcal{C})$ such that every $U \in \mathcal{B}$ is weakly contractible and every object of $\mathcal{C}$ has a covering by elements of $\mathcal{B}$. Let $\mathcal{O}$ be a sheaf of rings on $\mathcal{C}$. Then*

(1) *A complex $\mathcal{F}_1 \to \mathcal{F}_2 \to \mathcal{F}_3$ of $\mathcal{O}$-modules is exact, if and only if $\mathcal{F}_1(U) \to \mathcal{F}_2(U) \to \mathcal{F}_3(U)$ is exact for all $U \in \mathcal{B}$.*
(2) *Every object K of $D(\mathcal{O})$ is a derived limit of its canonical truncations: $K = R\lim \tau_{\geq -n} K$.*
(3) *Given an inverse system $\ldots \to \mathcal{F}_3 \to \mathcal{F}_2 \to \mathcal{F}_1$ with surjective transition maps, the projection $\lim \mathcal{F}_n \to \mathcal{F}_1$ is surjective.*
(4) *Products are exact on $Mod(\mathcal{O})$.*
(5) *Products on $D(\mathcal{O})$ can be computed by taking products of any representative complexes.*
(6) *If $(\mathcal{F}_n)$ is an inverse system of $\mathcal{O}$-modules, then $R^p \lim \mathcal{F}_n = 0$ for all $p > 1$ and*
$$R^1 \lim \mathcal{F}_n = Coker(\prod \mathcal{F}_n \to \prod \mathcal{F}_n)$$
where the map is $(x_n) \mapsto (x_n - f(x_{n+1}))$.
(7) *If (K_n) is an inverse system of objects of $D(\mathcal{O})$, then there are short exact sequences*
$$0 \to R^1 \lim H^{p-1}(K_n) \to H^p(R \lim K_n) \to \lim H^p(K_n) \to 0$$

Proof. Proof of (1). If the sequence is exact, then evaluating at any weakly contractible element of $\mathcal{C}$ gives an exact sequence by Lemma 21.37.1. Conversely, assume that $\mathcal{F}_1(U) \to \mathcal{F}_2(U) \to \mathcal{F}_3(U)$ is exact for all $U \in \mathcal{B}$. Let V be an object of $\mathcal{C}$ and let $s \in \mathcal{F}_2(V)$ be an element of the kernel of $\mathcal{F}_2 \to \mathcal{F}_3$. By assumption there exists a covering $\{U_i \to V\}$ with $U_i \in \mathcal{B}$. Then $s|_{U_i}$ lifts to a section $s_i \in \mathcal{F}_1(U_i)$. Thus s is a section of the image sheaf $\mathrm{Im}(\mathcal{F}_1 \to \mathcal{F}_2)$. In other words, the sequence $\mathcal{F}_1 \to \mathcal{F}_2 \to \mathcal{F}_3$ is exact.

Proof of (2). Lemma 21.22.2 applies to every complex of sheaves on $\mathcal{C}$. Thus (1) holds by Lemma 21.22.3.

Proof of (3). Let $(\mathcal{F}_n)$ be a system as in (2) and set $\mathcal{F} = \lim \mathcal{F}_n$. If $U \in \mathcal{B}$, then $\mathcal{F}(U) = \lim \mathcal{F}_n(U)$ surjects onto $\mathcal{F}_1(U)$ as all the transition maps $\mathcal{F}_{n+1}(U) \to \mathcal{F}_n(U)$ are surjective. Thus $\mathcal{F} \to \mathcal{F}_1$ is surjective by Sites, Definition 7.12.1 and the assumption that every object has a covering by elements of $\mathcal{B}$.

Proof of (4). Let $\mathcal{F}_{i,1} \to \mathcal{F}_{i,2} \to \mathcal{F}_{i,3}$ be a family of exact sequences of $\mathcal{O}$-modules. We want to show that $\prod \mathcal{F}_{i,1} \to \prod \mathcal{F}_{i,2} \to \prod \mathcal{F}_{i,3}$ is exact. We use the criterion of (1). Let $U \in \mathcal{B}$. Then
$$(\prod \mathcal{F}_{i,1})(U) \to (\prod \mathcal{F}_{i,2})(U) \to (\prod \mathcal{F}_{i,3})(U)$$
is the same as
$$\prod \mathcal{F}_{i,1}(U) \to \prod \mathcal{F}_{i,2}(U) \to \prod \mathcal{F}_{i,3}(U)$$
Each of the sequences $\mathcal{F}_{i,1}(U) \to \mathcal{F}_{i,2}(U) \to \mathcal{F}_{i,3}(U)$ are exact by (1). Thus the displayed sequences are exact by Homology, Lemma 12.27.1. We conclude by (1) again.

Proof of (5). Follows from (4) and (slightly generalized) Derived Categories, Lemma 13.30.3.

Proof of (6) and (7). We refer to Section 21.21 for a discussion of derived and homotopy limits and their relationship. By Derived Categories, Definition 13.30.2

we have a distinguished triangle

$$R\lim K_n \to \prod K_n \to \prod K_n \to R\lim K_n[1]$$

Taking the long exact sequence of cohomology sheaves we obtain

$$H^{p-1}(\prod K_n) \to H^{p-1}(\prod K_n) \to H^p(R\lim K_n) \to H^p(\prod K_n) \to H^p(\prod K_n)$$

Since products are exact by (4) this becomes

$$\prod H^{p-1}(K_n) \to \prod H^{p-1}(K_n) \to H^p(R\lim K_n) \to \prod H^p(K_n) \to \prod H^p(K_n)$$

Now we first apply this to the case $K_n = \mathcal{F}_n[0]$ where $(\mathcal{F}_n)$ is as in (6). We conclude that (6) holds. Next we apply it to (K_n) as in (7) and we conclude (7) holds. □

21.38. Compact objects

In this section we study compact objects in the derived category of modules on a ringed site. We recall that compact objects are defined in Derived Categories, Definition 13.32.1.

Lemma 21.38.1. *Let $(\mathcal{C}, \mathcal{O})$ be a ringed site. Assume $\mathcal{C}$ has the following properties*

1. *$\mathcal{C}$ has a quasi-compact final object X,*
2. *every object of $\mathcal{C}$ can be covered by quasi-compact objects,*
3. *for a finite covering $\{U_i \to U\}_{i\in I}$ with U, U_i quasi-compact the fibre products $U_i \times_U U_j$ are quasi-compact.*

Then any perfect object of $D(\mathcal{O})$ is compact.

Proof. Let K be a perfect object and let $K^\wedge$ be its dual, see Lemma 21.35.9. Then we have

$$\mathrm{Hom}_{D(\mathcal{O}_X)}(K, M) = H^0(X, K^\wedge \otimes^{\mathbf{L}}_{\mathcal{O}_X} M)$$

functorially in M in $D(\mathcal{O}_X)$. Since $K^\wedge \otimes^{\mathbf{L}}_{\mathcal{O}_X} -$ commutes with direct sums (by construction) and H^0 does by Lemma 21.16.1 and the construction of direct sums in Injectives, Lemma 19.13.4 we obtain the result of the lemma. □

Lemma 21.38.2. *Let $\mathcal{A}$ be a Grothendieck abelian category. Let $L^\bullet_n$ be a system of complexes of $\mathcal{A}$. Then the termwise colimit $L^\bullet = \mathrm{colim}\, L^\bullet_n$ is a homotopy colimit of the system in $D(\mathcal{A})$.*

Proof. We have an exact sequence of complexes

$$0 \to \bigoplus L^\bullet_n \to \bigoplus L^\bullet_n \to L^\bullet \to 0$$

by Injectives, Lemma 19.10.2. The direct sums are direct sums in $D(\mathcal{A})$ by Derived Categories, Lemma 13.30.3. Thus the result follows from the definition of derived colimits in Derived Categories, Definition 13.30.1 and the fact that a short exact sequence of complexes gives a distinguished triangle (Derived Categories, Lemma 13.11.1). □

Lemma 21.38.3. *Let $\mathcal{D}$ be a triangulated category with countable direct sums. Let $K \in \mathcal{D}$ be an object such that for every countable set of objects $E_n \in \mathcal{D}$ the canonical map*

$$\bigoplus \mathrm{Hom}_{\mathcal{D}}(K, E_n) \longrightarrow \mathrm{Hom}_{\mathcal{D}}(K, \bigoplus E_n)$$

is a bijection. Then, given any system L_n of $\mathcal{D}$ over $\mathbf{N}$ whose derived colimit $L = hocolim L_n$ exists we have that

$$\operatorname{colim} \mathrm{Hom}_{\mathcal{D}}(K, L_n) \longrightarrow \mathrm{Hom}_{\mathcal{D}}(K, L)$$

is a bijection.

Proof. This follows from the defining distinguished triangle

$$\bigoplus L_n \to \bigoplus L_n \to L \to \bigoplus L_n[1]$$

applying $\mathrm{Hom}_{\mathcal{D}}(K, -)$, Derived Categories, Lemma 13.4.2, and elementary considerations concerning colimits of abelian groups. □

Lemma 21.38.4. *Let $\mathcal{A}$ be a Grothendieck abelian category. Let $S \subset \mathrm{Ob}(\mathcal{A})$ be a set of objects such that*

(1) *any object of $\mathcal{A}$ is a quotient of a direct sum of elements of S, and*
(2) *for any $E \in S$ the functor $\mathrm{Hom}_{\mathcal{A}}(E, -)$ commutes with direct sums.*

Then every compact object of $D(\mathcal{A})$ is a direct summand in $D(\mathcal{A})$ of a finite complex of finite direct sums of elements of S.

Proof. Assume $K \in D(\mathcal{A})$ is a compact object. Represent K by a complex $K^\bullet$ and consider the map

$$K^\bullet \longrightarrow \bigoplus\nolimits_{n \geq 0} \tau_{\geq n} K^\bullet$$

where we have used the canonical truncations, see Homology, Section 12.13. This makes sense as in each degree the direct sum on the right is finite. By assumption this map factors through a finite direct sum. We conclude that $K \to \tau_{\geq n} K$ is zero for at least one n, i.e., K is in $D^-(R)$.

We may represent K by a bounded above complex $K^\bullet$ each of whose terms is a direct sum of objects from S, see Derived Categories, Lemma 13.15.5. Note that we have

$$K^\bullet = \bigcup\nolimits_{n \leq 0} \sigma_{\geq n} K^\bullet$$

where we have used the stupid truncations, see Homology, Section 12.13. Hence by Lemmas 21.38.2 and 21.38.3 we see that $1 : K^\bullet \to K^\bullet$ factors through $\sigma_{\geq n} K^\bullet \to K^\bullet$ in $D(R)$. Thus we see that $1 : K^\bullet \to K^\bullet$ factors as

$$K^\bullet \xrightarrow{\varphi} L^\bullet \xrightarrow{\psi} K^\bullet$$

in $D(\mathcal{A})$ for some complex $L^\bullet$ which is bounded and whose terms are direct sums of elements of S. Say L^i is zero for $i \notin [a, b]$. Let c be the largest integer $\leq b + 1$ such that L^i a finite direct sum of elements of S for $i < c$. Claim: if $c < b + 1$, then we can modify $L^\bullet$ to increase c. By induction this claim will show we have a factorization of 1_K as

$$K \xrightarrow{\varphi} L \xrightarrow{\psi} K$$

in $D(\mathcal{A})$ where L can be represented by a finite complex of finite direct sums of elements of S. Note that $e = \varphi \circ \psi \in \mathrm{End}_{D(\mathcal{A})}(L)$ is an idempotent. By Derived Categories, Lemma 13.4.12 we see that $L = \mathrm{Ker}(e) \oplus \mathrm{Ker}(1 - e)$. The map $\varphi : K \to L$ induces an isomorphism with $\mathrm{Ker}(1 - e)$ in $D(R)$ and we conclude.

Proof of the claim. Write $L^c = \bigoplus_{\lambda\in\Lambda} E_\lambda$. Since L^{c-1} is a finite direct sum of elements of S we can by assumption (2) find a finite subset $\Lambda' \subset \Lambda$ such that $L^{c-1} \to L^c$ factors through $\bigoplus_{\lambda\in\Lambda'} E_\lambda \subset L^c$. Consider the map of complexes

$$\pi : L^\bullet \longrightarrow (\bigoplus\nolimits_{\lambda\in\Lambda\setminus\Lambda'} E_\lambda)[-i]$$

given by the projection onto the factors corresponding to $\Lambda \setminus \Lambda'$ in degree i. By our assumption on K we see that, after possibly replacing Λ' by a larger finite subset, we may assume that $\pi \circ \varphi = 0$ in $D(\mathcal{A})$. Let $(L')^\bullet \subset L^\bullet$ be the kernel of π. Since π is surjective we get a short exact sequence of complexes, which gives a distinguished triangle in $D(\mathcal{A})$ (see Derived Categories, Lemma 13.11.1). Since $\mathrm{Hom}_{D(\mathcal{A})}(K, -)$ is homological (see Derived Categories, Lemma 13.4.2) and $\pi \circ \varphi = 0$, we can find a morphism $\varphi' : K^\bullet \to (L')^\bullet$ in $D(\mathcal{A})$ whose composition with $(L')^\bullet \to L^\bullet$ gives φ. Setting ψ' equal to the composition of ψ with $(L')^\bullet \to L^\bullet$ we obtain a new factorization. Since $(L')^\bullet$ agrees with $L^\bullet$ except in degree c and since $(L')^c = \bigoplus_{\lambda\in\Lambda'} E_\lambda$ the claim is proved. □

Lemma 21.38.5. *Let $(\mathcal{C}, \mathcal{O})$ be a ringed site. Assume every object of $\mathcal{C}$ has a covering by quasi-compact objects. Then every compact object of $D(\mathcal{O})$ is a direct summand in $D(\mathcal{O})$ of a finite complex whose terms are finite direct sums of $\mathcal{O}$-modules of the form $j_!\mathcal{O}_U$ where U is a quasi-compact object of $\mathcal{C}$.*

Proof. Apply Lemma 21.38.4 where $S \subset \mathrm{Ob}(\mathit{Mod}(\mathcal{O}))$ is the set of modules of the form $j_!\mathcal{O}_U$ with $U \in \mathrm{Ob}(\mathcal{C})$ quasi-compact. Assumption (1) holds by Modules on Sites, Lemma 18.28.6 and the assumption that every U can be covered by quasi-compact objects. Assumption (2) follows as

$$\mathrm{Hom}_{\mathcal{O}}(j_!\mathcal{O}_U, \mathcal{F}) = \mathcal{F}(U)$$

which commutes with direct sums by Sites, Lemma 7.11.2. □

In the situation of the lemma above it is not always true that the modules $j_!\mathcal{O}_U$ are compact objects of $D(\mathcal{O})$ (even if U is a quasi-compact object of $\mathcal{C}$). Here is a criterion.

Lemma 21.38.6. *Let $(\mathcal{C}, \mathcal{O})$ be a ringed site. Let U be an object of $\mathcal{C}$. The $\mathcal{O}$-module $j_!\mathcal{O}_U$ is a compact object of $D(\mathcal{O})$ if there exists an integer d such that*

(1) *$H^p(U, \mathcal{F}) = 0$ for all $p > d$, and*
(2) *the functors $\mathcal{F} \mapsto H^p(U, \mathcal{F})$ commute with direct sums.*

Proof. Assume (1) and (2). The first means that the functor $F = H^0(U, -)$ has finite cohomological dimension. Moreover, any direct sum of injective modules is acyclic for F by (2). Since we may compute RF by applying F to any complex of acyclics (Derived Categories, Lemma 13.29.2). Thus, if K_i be a family of objects of $D(\mathcal{O})$, then we can choose K-injective representatives $I_i^\bullet$ and we see that $\bigoplus K_i$ is represented by $\bigoplus I_i^\bullet$. Thus $H^0(U, -)$ commutes with direct sums. □

Lemma 21.38.7. *Let $(\mathcal{C}, \mathcal{O})$ be a ringed site. Let U be an object of $\mathcal{C}$ which is quasi-compact and weakly contractible. Then $j_!\mathcal{O}_U$ is a compact object of $D(\mathcal{O})$.*

Proof. Combine Lemmas 21.38.6 and 21.37.1 with Modules on Sites, Lemma 18.29.2. □

21.39. Complexes with locally constant cohomology sheaves

Locally constant sheaves are introduced in Modules on Sites, Section 18.42. Let $\mathcal{C}$ be a site. Let Λ be a ring. We denote $D(\mathcal{C}, \Lambda)$ the derived category of the abelian category of $\underline{\Lambda}$-modules on $\mathcal{C}$.

Lemma 21.39.1. *Let $\mathcal{C}$ be a site with final object X. Let Λ be a Noetherian ring. Let $K \in D^b(\mathcal{C}, \Lambda)$ with $H^i(K)$ locally constant sheaves of Λ-modules of finite type. Then there exists a covering $\{U_i \to X\}$ such that each $K|_{U_i}$ is represented by a complex of locally constant sheaves of Λ-modules of finite type.*

Proof. Let $a \leq b$ be such that $H^i(K) = 0$ for $i \notin [a, b]$. By induction on $b - a$ we will prove there exists a covering $\{U_i \to X\}$ such that $K|_{U_i}$ can be represented by a complex $\underline{M^\bullet}_{U_i}$ with M^p a finite type Λ-module and $M^p = 0$ for $p \notin [a, b]$. If $b = a$, then this is clear. In general, we may replace X by the members of a covering and assume that $H^b(K)$ is constant, say $H^b(K) = \underline{M}$. By Modules on Sites, Lemma 18.41.5 the module M is a finite Λ-module. Choose a surjection $\Lambda^{\oplus r} \to M$ given by generators $x_1, \ldots, x_r$ of M.

By a slight generalization of Lemma 21.8.3 (details omitted) there exists a covering $\{U_i \to X\}$ such that $x_i \in H^0(X, H^b(K))$ lifts to an element of $H^b(U_i, K)$. Thus, after replacing X by the U_i we reach the situation where there is a map $\underline{\Lambda^{\oplus r}}[-b] \to K$ inducing a surjection on cohomology sheaves in degree b. Choose a distinguished triangle

$$\underline{\Lambda^{\oplus r}}[-b] \to K \to L \to \underline{\Lambda^{\oplus r}}[-b+1]$$

Now the cohomology sheaves of L are nonzero only in the interval $[a, b-1]$, agree with the cohomology sheaves of K in the interval $[a, b-2]$ and there is a short exact sequence

$$0 \to H^{b-1}(K) \to H^{b-1}(L) \to \underline{\mathrm{Ker}(\Lambda^{\oplus r} \to M)} \to 0$$

in degree $b-1$. By Modules on Sites, Lemma 18.42.5 we see that $H^{b-1}(L)$ is locally constant of finite type. By induction hypothesis we obtain an isomorphism $\underline{M^\bullet} \to L$ in $D(\mathcal{C}, \underline{\Lambda})$ with M^p a finite Λ-module and $M^p = 0$ for $p \notin [a, b-1]$. The map $L \to \Lambda^{\oplus r}[-b+1]$ gives a map $\underline{M^{b-1}} \to \underline{\Lambda^{\oplus r}}$ which locally is constant (Modules on Sites, Lemma 18.42.3). Thus we may assume it is given by a map $M^{b-1} \to \Lambda^{\oplus r}$. The distinguished triangle shows that the composition $M^{b-2} \to M^{b-1} \to \Lambda^{\oplus r}$ is zero and the axioms of triangulated categories produce an isomorphism

$$\underline{M^a \to \ldots \to M^{b-1} \to \Lambda^{\oplus r}} \longrightarrow K$$

in $D(\mathcal{C}, \Lambda)$. □

Let $\mathcal{C}$ be a site. Let Λ be a ring. Using the morphism $Sh(\mathcal{C}) \to Sh(pt)$ we see that there is a functor $D(\Lambda) \to D(\mathcal{C}, \Lambda)$, $K \mapsto \underline{K}$.

Lemma 21.39.2. *Let $\mathcal{C}$ be a site with final object X. Let Λ be a ring. Let*

1. *K a perfect object of $D(\Lambda)$,*
2. *a finite complex $K^\bullet$ of finite projective Λ-modules representing K,*
3. *$\mathcal{L}^\bullet$ a complex of sheaves of Λ-modules, and*
4. *$\varphi : \underline{K} \to \mathcal{L}^\bullet$ a map in $D(\mathcal{C}, \Lambda)$.*

Then there exists a covering $\{U_i \to X\}$ and maps of complexes $\alpha_i : \underline{K^\bullet}|_{U_i} \to \mathcal{L}^\bullet|_{U_i}$ representing $\varphi|_{U_i}$.

Proof. Follows immediately from Lemma 21.32.8. □

Lemma 21.39.3. *Let $\mathcal{C}$ be a site with final object X. Let Λ be a ring. Let K, L be objects of $D(\Lambda)$ with K perfect. Let $\varphi : \underline{K} \to \underline{L}$ be map in $D(\mathcal{C}, \Lambda)$. There exists a covering $\{U_i \to X\}$ such that $\varphi|_{U_i}$ is equal to $\underline{\alpha_i}$ for some map $\alpha_i : K \to L$ in $D(\Lambda)$.*

Proof. Follows from Lemma 21.39.2 and Modules on Sites, Lemma 18.42.3. □

Lemma 21.39.4. *Let $\mathcal{C}$ be a site. Let Λ be a Noetherian ring. Let $K, L \in D^-(\mathcal{C}, \Lambda)$. If the cohomology sheaves of K and L are locally constant sheaves of Λ-modules of finite type, then the cohomology sheaves of $K \otimes^{\mathbf{L}}_{\Lambda} L$ are locally constant sheaves of Λ-modules of finite type.*

Proof. We'll prove this as an application of Lemma 21.39.1. Note that $H^i(K \otimes^{\mathbf{L}}_{\Lambda} L)$ is the same as $H^i(\tau_{\geq i-1} K \otimes^{\mathbf{L}}_{\Lambda} \tau_{\geq i-1} L)$. Thus we may assume K and L are bounded. By Lemma 21.39.1 we may assume that K and L are represented by complexes of locally constant sheaves of Λ-modules of finite type. Then we can replace these complexes by bounded above complexes of finite free Λ-modules. In this case the result is clear. □

Lemma 21.39.5. *Let $\mathcal{C}$ be a site. Let Λ be a Noetherian ring. Let $I \subset \Lambda$ be an ideal. Let $K \in D^-(\mathcal{C}, \Lambda)$. If the cohomology sheaves of $K \otimes^{\mathbf{L}}_{\Lambda} \underline{\Lambda/I}$ are locally constant sheaves of Λ/I-modules of finite type, then the cohomology sheaves of $K \otimes^{\mathbf{L}}_{\Lambda} \underline{\Lambda/I^n}$ are locally constant sheaves of Λ/I^n-modules of finite type for all $n \geq 1$.*

Proof. Recall that the locally constant sheaves of Λ-modules of finite type form a weak Serre subcategory of all $\underline{\Lambda}$-modules, see Modules on Sites, Lemma 18.42.5. Thus the subcategory of $D(\mathcal{C}, \Lambda)$ consisting of complexes whose cohomology sheaves are locally constant sheaves of Λ-modules of finite type forms a strictly full, saturated triangulated subcategory of $D(\mathcal{C}, \Lambda)$, see Derived Categories, Lemma 13.12.1. Next, consider the distinguished triangles

$$K \otimes^{\mathbf{L}}_{\Lambda} \underline{I^n/I^{n+1}} \to K \otimes^{\mathbf{L}}_{\Lambda} \underline{\Lambda/I^{n+1}} \to K \otimes^{\mathbf{L}}_{\Lambda} \underline{\Lambda/I^n} \to K \otimes^{\mathbf{L}}_{\Lambda} \underline{I^n/I^{n+1}}[1]$$

and the isomorphisms

$$K \otimes^{\mathbf{L}}_{\Lambda} \underline{I^n/I^{n+1}} = \left(K \otimes^{\mathbf{L}}_{\Lambda} \underline{\Lambda/I}\right) \otimes^{\mathbf{L}}_{\Lambda/I} \underline{I^n/I^{n+1}}$$

Combined with Lemma 21.39.4 we obtain the result. □

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CHAPTER 22

Differential Graded Algebra

22.1. Introduction

In this chapter we talk about differential graded algebras, modules, categories, etc. A basic reference is [**Kel94**]. A survey paper is [**Kel06**].

Since we do not worry about length of exposition in the Stacks project we first develop the material in the setting of categories of differential graded modules. After that we redo the constructions in the setting of differential graded modules over differential graded categories.

22.2. Conventions

In this chapter we hold on to the convention that *ring* means commutative ring with 1. If R is a ring, then an *R-algebra* A will be an R-module A endowed with an R-bilinear map $A \times A \to A$ (multiplication) such that multiplication is associative and has a unit. In other words, these are unital associative R-algebras such that the structure map $R \to A$ maps into the center of A.

22.3. Differential graded algebras

Just the definitions.

Definition 22.3.1. Let R be a commutative ring. A *differential graded algebra over R* is either

(1) a chain complex $A_\bullet$ of R-modules endowed with R-bilinear maps $A_n \times A_m \to A_{n+m}$, $(a, b) \mapsto ab$ such that
$$\mathrm{d}_{n+m}(ab) = \mathrm{d}_n(a)b + (-1)^n a\mathrm{d}_m(b)$$
and such that $\bigoplus A_n$ becomes an associative and unital R-algebra, or

(2) a cochain complex $A^\bullet$ of R-modules endowed with R-bilinear maps $A^n \times A^m \to A^{n+m}$, $(a, b) \mapsto ab$ such that
$$\mathrm{d}^{n+m}(ab) = \mathrm{d}^n(a)b + (-1)^n a\mathrm{d}^m(b)$$
and such that $\bigoplus A^n$ becomes an associative and unital R-algebra.

We often just write $A = \bigoplus A_n$ or $A = \bigoplus A^n$ and think of this as an associative unital R-algebra endowed with a $\mathbf{Z}$-grading and an R-linear operator d whose square is zero and which satisfies the Leibniz rule as explained above. In this case we often say "Let (A, d) be a differential graded algebra".

Definition 22.3.2. A *homomorphism of differential graded algebras* $f : (A, \mathrm{d}) \to (B, \mathrm{d})$ is an algebra map $f : A \to B$ compatible with the gradings and d.

Definition 22.3.3. Let R be a ring. Let (A, d) be a differential graded algebra over R. The *opposite differential graded algebra* is the differential graded algebra (A^{opp}, d) over R where $A^{opp} = A$ as an R-module, $\mathrm{d} = \mathrm{d}$, and multiplication is given by

$$a \cdot_{opp} b = (-1)^{\deg(a)\deg(b)} ba$$

for homogeneous elements $a, b \in A$.

This makes sense because

$$\begin{aligned} \mathrm{d}(a \cdot_{opp} b) &= (-1)^{\deg(a)\deg(b)} \mathrm{d}(ba) \\ &= (-1)^{\deg(a)\deg(b)} \mathrm{d}(b)a + (-1)^{\deg(a)\deg(b)+\deg(b)} b\mathrm{d}(a) \\ &= (-1)^{\deg(a)} a \cdot_{opp} \mathrm{d}(b) + \mathrm{d}(a) \cdot_{opp} b \end{aligned}$$

as desired.

Definition 22.3.4. A differential graded algebra (A, d) is *commutative* if $ab = (-1)^{nm} ba$ for a in degree n and b in degree m. We say A is *strictly commutative* if in addition $a^2 = 0$ for $\deg(a)$ odd.

The following definition makes sense in general but is perhaps "correct" only when tensoring commutative differential graded algebras.

Definition 22.3.5. Let R be a ring. Let (A, d), (B, d) be differential graded algebras over R. The *tensor product differential graded algebra* of A and B is the algebra $A \otimes_R B$ with multiplication defined by

$$(a \otimes b)(a' \otimes b') = (-1)^{\deg(a')\deg(b)} aa' \otimes bb'$$

endowed with differential d defined by the rule $\mathrm{d}(a \otimes b) = \mathrm{d}(a) \otimes b + (-1)^m a \otimes \mathrm{d}(b)$ where $m = \deg(b)$.

Lemma 22.3.6. *Let R be a ring. Let (A, d), (B, d) be differential graded algebras over R. Denote $A^\bullet$, $B^\bullet$ the underlying cochain complexes. As cochain complexes of R-modules we have*

$$(A \otimes_R B)^\bullet = Tot(A^\bullet \otimes_R B^\bullet).$$

Proof. Recall that the differential of the total complex is given by $\mathrm{d}_1^{p,q} + (-1)^p \mathrm{d}_2^{p,q}$ on $A^p \otimes_R B^q$. And this is exactly the same as the rule for the differential on $A \otimes_R B$ in Definition 22.3.5. □

22.4. Differential graded modules

Just the definitions.

Definition 22.4.1. Let R be a ring. Let (A, d) be a differential graded algebra over R. A (right) *differential graded module* M over A is a right A-module M which has a grading $M = \bigoplus M^n$ and a differential d such that $M^n A^m \subset M^{n+m}$, such that $\mathrm{d}(M^n) \subset M^{n+1}$, and such that

$$\mathrm{d}(ma) = \mathrm{d}(m)a + (-1)^n m\mathrm{d}(a)$$

for $a \in A$ and $m \in M^n$. A *homomorphism of differential graded modules* $f : M \to N$ is an A-module map compatible with gradings and differentials. The category of (right) differential graded A-modules is denoted $\mathrm{Mod}_{(A,\mathrm{d})}$.

Note that we can think of M as a cochain complex $M^\bullet$ of (right) R-modules. Namely, for $r \in R$ we have $\mathrm{d}(r) = 0$ and r maps to a degree 0 element of A, hence $\mathrm{d}(mr) = \mathrm{d}(m)r$.

We can define *left differential graded A-modules* in exactly the same manner. If M is a left A-module, then we can think of M as a right A^{opp}-module with mulitplication $\cdot_{opp}$ defined by the rule

$$m \cdot_{opp} a = (-1)^{\deg(a)\deg(m)} am$$

for a and m homogeneous. The category of left differential graded A-modules is equivalent to the category of right differential graded A^{opp}-modules. We prefer to work with right modules (essentially because of what happens in Example 22.19.8), but the reader is free to switch to left modules if (s)he so desires.

Lemma 22.4.2. *Let (A, d) be a differential graded algebra. The category $Mod_{(A,d)}$ is abelian and has arbitrary limits and colimits.*

Proof. Kernels and cokernels commute with taking underlying A-modules. Similarly for direct sums and colimits. In other words, these operations in $\mathrm{Mod}_{(A,\mathrm{d})}$ commute with the forgetful functor to the category of A-modules. This is not the case for products and limits. Namely, if N_i, $i \in I$ is a family of differential graded A-modules, then the product $\prod N_i$ in $\mathrm{Mod}_{(A,\mathrm{d})}$ is given by setting $(\prod N_i)^n = \prod N_i^n$ and $\prod N_i = \bigoplus_n (\prod N_i)^n$. Thus we see that the product does commute with the forgetful functor to the category of graded A-modules. A category with products and equalizers has limits, see Categories, Lemma 4.14.10. □

Thus, if (A, d) is a differential graded algebra over R, then there is an exact functor

$$\mathrm{Mod}_{(A,\mathrm{d})} \longrightarrow \mathrm{Comp}(R)$$

of abelian categories. For a differential graded module M the cohomology groups $H^n(M)$ are defined as the cohomology of the corresponding complex of R-modules. Therefore, a short exact sequence $0 \to K \to L \to M \to 0$ of differential graded modules gives rise to a long exact sequence

$$H^n(K) \to H^n(L) \to H^n(M) \to H^{n+1}(K) \tag{22.4.2.1}$$

of cohomology modules, see Homology, Lemma 12.12.12.

Moreover, from now on we borrow all the terminology used for complexes of modules. For example, we say that a differential graded A-module M is *acyclic* if $H^k(M) = 0$ for all $k \in \mathbf{Z}$. We say that a homomorphism $M \to N$ of differential graded A-modules is a *quasi-isomorphism* if it induces isomorphisms $H^k(M) \to H^k(N)$ for all $k \in \mathbf{Z}$. And so on and so forth.

Definition 22.4.3. Let (A, d) be a differential graded algebra. Let M be a differential graded module. For any $k \in \mathbf{Z}$ we define the *k-shifted module* $M[k]$ as follows

(1) as A-module $M[k] = M$,
(2) $M[k]^n = M^{n+k}$,
(3) $\mathrm{d}_{M[k]} = (-1)^k \mathrm{d}_M$.

For a morphism $f : M \to N$ of differential graded A-modules we let $f[k] : M[k] \to N[k]$ be the map equal to f on underlying A-modules. This defines a functor $[k] : \mathrm{Mod}_{(A,\mathrm{d})} \to \mathrm{Mod}_{(A,\mathrm{d})}$.

The remarks in Homology, Section 12.14 apply. In particular, we will identify the cohomology groups of all shifts $M[k]$ without the intervention of signs.

At this point we have enough structure to talk about *triangles*, see Derived Categories, Definition 13.3.1. In fact, our next goal is to develop enough theory to be able to state and prove that the homotopy category of differential graded modules is a triangulated category. First we define the homotopy category.

22.5. The homotopy category

Our homotopies take into account the A-module structure and the grading, but not the differential (of course).

Definition 22.5.1. Let (A, d) be a differential graded algebra. Let $f, g : M \to N$ be homomorphisms of differential graded A-modules. A *homotopy between f and g* is an A-module map $h : M \to N$ such that

(1) $h(M^n) \subset N^{n-1}$ for all n, and
(2) $f(x) - g(x) = \mathrm{d}_N(h(x)) + h(\mathrm{d}_M(x))$ for all $x \in M$.

If a homotopy exists, then we say f and g are *homotopic*.

Thus h is compatible with the A-module structure and the grading but not with the differential. If $f = g$ and h is a homotopy as in the definition, then h defines a morphism $h : M \to N[1]$ in $\mathrm{Mod}_{(A,\mathrm{d})}$.

Lemma 22.5.2. *Let (A, d) be a differential graded algebra. Let $f, g : L \to M$ be homomorphisms of differential graded A-modules. Suppose given further homomorphisms $a : K \to L$, and $c : M \to N$. If $h : M \to N$ defines a homotopy between f and g, then $c \circ h \circ a$ defines a homotopy between $c \circ f \circ a$ and $c \circ g \circ a$.*

Proof. Immediate from Homology, Lemma 12.12.7. □

This lemma allows us to define the homotopy category as follows.

Definition 22.5.3. Let (A, d) be a differential graded algebra. The *homotopy category*, denoted $K(\mathrm{Mod}_{(A,\mathrm{d})})$, is the category whose objects are the objects of $\mathrm{Mod}_{(A,\mathrm{d})}$ and whose morphisms are homotopy classes of homomorphisms of differential graded A-modules.

The notation $K(\mathrm{Mod}_{(A,\mathrm{d})})$ is not standard but at least is consistent with the use of $K(-)$ in other places of the Stacks project.

Lemma 22.5.4. *Let (A, d) be a differential graded algebra. The homotopy category $K(\mathit{Mod}_{(A,d)})$ has direct sums and products.*

Proof. Omitted. Hint: Just use the direct sums and products as in Lemma 22.4.2. This works because we saw that these functors commute with the forgetful functor to the category of graded A-modules and because $\prod$ is an exact functor on the category of families of abelian groups. □

22.6. Cones

We introduce cones for the category of differential graded modules.

Definition 22.6.1. Let (A, d) be a differential graded algebra. Let $f : K \to L$ be a homomorphism of differential graded A-modules. The *cone* of f is the differential graded A-module $C(f)$ given by $C(f) = L \oplus K$ with grading $C(f)^n = L^n \oplus K^{n+1}$ and differential

$$d_{C(f)} = \begin{pmatrix} \mathrm{d}_L & f \\ 0 & -\mathrm{d}_K \end{pmatrix}$$

It comes equipped with canonical morphisms of complexes $i : L \to C(f)$ and $p : C(f) \to K[1]$ induced by the obvious maps $L \to C(f)$ and $C(f) \to K$.

The formation of the cone triangle is functorial in the following sense.

Lemma 22.6.2. *Let (A, d) be a differential graded algebra. Suppose that*

$$\begin{array}{ccc} K_1 & \xrightarrow[f_1]{} & L_1 \\ {\scriptstyle a}\downarrow & & \downarrow{\scriptstyle b} \\ K_2 & \xrightarrow{f_2} & L_2 \end{array}$$

is a diagram of homomorphisms of diferential graded A-modules which is commutative up to homotopy. Then there exists a morphism $c : C(f_1) \to C(f_2)$ which gives rise to a morphism of triangles

$$(a, b, c) : (K_1, L_1, C(f_1), f_1, i_1, p_1) \to (K_1, L_1, C(f_1), f_2, i_2, p_2)$$

in $K(Mod_{(A,d)})$.

Proof. Let $h : K_1 \to L_2$ be a homotopy between $f_2 \circ a$ and $b \circ f_1$. Define c by the matrix

$$c = \begin{pmatrix} b & h \\ 0 & a \end{pmatrix} : L_1 \oplus K_1 \to L_2 \oplus K_2$$

A matrix computation show that c is a morphism of differential graded modules. It is trivial that $c \circ i_1 = i_2 \circ b$, and it is trivial also to check that $p_2 \circ c = a \circ p_1$. □

22.7. Admissible short exact sequences

An admissible short exact sequence is the analogue of termwise split exact sequences in the setting of differential graded modules.

Definition 22.7.1. Let (A, d) be a differential graded algebra.

(1) A homomorphism $K \to L$ of differential graded A-modules is an *admissible monomorphism* if there exists a graded A-module map $L \to K$ which is left inverse to $K \to L$.
(2) A homomorphism $L \to M$ of differential graded A-modules is an *admissible epimorphism* if there exists a graded A-module map $M \to L$ which is right inverse to $L \to M$.
(3) A short exact sequence $0 \to K \to L \to M \to 0$ of differential graded A-modules is an *admissible short exact sequence* if it is split as a sequence of graded A-modules.

Thus the splittings are compatible with all the data except for the differentials. Given an admissible short exact sequence we obtain a triangle; this is the reason that we require our splittings to be compatible with the A-module structure.

Lemma 22.7.2. *Let (A, d) be a differential graded algebra. Let $0 \to K \to L \to M \to 0$ be an admissible short exact sequence of differential graded A-modules. Let $s : M \to L$ and $\pi : L \to K$ be splittings such that $Ker(\pi) = Im(s)$. Then we obtain a morphism*

$$\delta = \pi \circ d_L \circ s : M \to K[1]$$

of $Mod_{(A,d)}$ which induces the boundary maps in the long exact sequence of cohomology (22.4.2.1).

Proof. The map $\pi \circ \mathrm{d}_L \circ s$ is compatible with the A-module structure and the gradings by construction. It is compatible with differentials by Homology, Lemmas 12.14.10. Let R be the ring that A is a differential graded algebra over. The equality of maps is a statement about R-modules. Hence this follows from Homology, Lemmas 12.14.10 and 12.14.11. □

Lemma 22.7.3. *Let (A, d) be a differential graded algebra. Let*

$$\begin{array}{ccc} K & \xrightarrow{f} & L \\ {\scriptstyle a}\downarrow & & \downarrow{\scriptstyle b} \\ M & \xrightarrow{g} & N \end{array}$$

be a diagram of homomorphisms of differential graded A-modules commuting up to homotopy.

1. *If f is an admissible monomorphism, then b is homotopic to a homomorphism which makes the diagram commute.*
2. *If g is an admissible epimorphism, then a is homotopic to a morphism which makes the diagram commute.*

Proof. Let $h : K \to N$ be a homotopy between bf and ga, i.e., $bf - ga = \mathrm{d}h + h\mathrm{d}$. Suppose that $\pi : L \to K$ is a graded A-module map left inverse to f. Take $b' = b - \mathrm{d}h\pi - h\pi\mathrm{d}$. Suppose $s : N \to M$ is a graded A-module map right inverse to g. Take $a' = a + \mathrm{d}sh + sh\mathrm{d}$. Computations omitted. □

Lemma 22.7.4. *Let (A, d) be a differential graded algebra. Let $\alpha : K \to L$ be a homomorphism of differential graded A-modules. There exists a factorization*

$$K \xrightarrow{\tilde{\alpha}} \tilde{L} \xrightarrow{\pi} L, \quad K \xrightarrow{\alpha} L$$

in $Mod_{(A,d)}$ such that

1. *$\tilde{\alpha}$ is an admissible monomorphism (see Definition 22.7.1),*
2. *there is a morphism $s : L \to \tilde{L}$ such that $\pi \circ s = id_L$ and such that $s \circ \pi$ is homotopic to $id_{\tilde{L}}$.*

Proof. The proof is identical to the proof of Derived Categories, Lemma 13.8.6. Namely, we set $\tilde{L} = L \oplus C(1_K)$ and we use elementary properties of the cone construction. □

Lemma 22.7.5. *Let (A, d) be a differential graded algebra. Let $L_1 \to L_2 \to \ldots \to L_n$ be a sequence of composable homomorphisms of differential graded A-modules.*

There exists a commutative diagram

$$\begin{array}{ccccccc} L_1 & \longrightarrow & L_2 & \longrightarrow & \ldots & \longrightarrow & L_n \\ \uparrow & & \uparrow & & & & \uparrow \\ M_1 & \longrightarrow & M_2 & \longrightarrow & \ldots & \longrightarrow & M_n \end{array}$$

in $Mod_{(A,d)}$ such that each $M_i \to M_{i+1}$ is an admissible monomorphism and each $M_i \to L_i$ is a homotopy equivalence.

Proof. The case $n = 1$ is without content. Lemma 22.7.4 is the case $n = 2$. Suppose we have constructed the diagram except for M_n. Apply Lemma 22.7.4 to the composition $M_{n-1} \to L_{n-1} \to L_n$. The result is a factorization $M_{n-1} \to M_n \to L_n$ as desired. □

Lemma 22.7.6. *Let (A, d) be a differential graded algebra. Let $0 \to K_i \to L_i \to M_i \to 0$, $i = 1, 2, 3$ be admissible short exact sequence of differential graded A-modules. Let $b : L_1 \to L_2$ and $b' : L_2 \to L_3$ be homomorphisms of differential graded modules such that*

$$\begin{array}{ccccc} K_1 & \longrightarrow & L_1 & \longrightarrow & M_1 \\ {\scriptstyle 0}\downarrow & & {\scriptstyle b}\downarrow & & {\scriptstyle 0}\downarrow \\ K_2 & \longrightarrow & L_2 & \longrightarrow & M_2 \end{array} \quad \text{and} \quad \begin{array}{ccccc} K_2 & \longrightarrow & L_2 & \longrightarrow & M_2 \\ \downarrow{\scriptstyle 0} & & \downarrow{\scriptstyle b'} & & \downarrow{\scriptstyle 0} \\ K_3 & \longrightarrow & L_3 & \longrightarrow & M_3 \end{array}$$

commute up to homotopy. Then $b' \circ b$ is homotopic to 0.

Proof. By Lemma 22.7.3 we can replace b and b' by homotopic maps such that the right square of the left diagram commutes and the left square of the right diagram commutes. In other words, we have $\mathrm{Im}(b) \subset \mathrm{Im}(K_2 \to L_2)$ and $\mathrm{Ker}((b')^n) \supset \mathrm{Im}(K_2 \to L_2)$. Then $b \circ b' = 0$ as a map of modules. □

22.8. Distinguished triangles

The following lemma produces our distinguished triangles.

Lemma 22.8.1. *Let (A, d) be a differential graded algebra. Let $0 \to K \to L \to M \to 0$ be an admissible short exact sequence of differential graded A-modules. The triangle*

$$K \to L \to M \xrightarrow{\delta} K[1] \tag{22.8.1.1}$$

with δ as in Lemma 22.7.2 is, up to canonical isomorphism in $K(Mod_{(A,d)})$, independent of the choices made in Lemma 22.7.2.

Proof. Namely, let (s', π') be a second choice of splittings as in Lemma 22.7.2. Then we claim that δ and δ' are homotopic. Namely, write $s' = s + \alpha \circ h$ and $\pi' = \pi + g \circ \beta$ for some unique homomorphisms of A-modules $h : M \to K$ and $g : M \to K$ of degree -1. Then $g = -h$ and g is a homotopy between δ and δ'. The computations are done in the proof of Homology, Lemma 12.14.12. □

Definition 22.8.2. Let (A, d) be a differential graded algebra.

(1) If $0 \to K \to L \to M \to 0$ is an admissible short exact sequence of differential graded A-modules, then the *triangle associated to* $0 \to K \to L \to M \to 0$ is the triangle (22.8.1.1) of $K(\mathrm{Mod}_{(A,\mathrm{d})})$.

(2) A triangle of $K(\mathrm{Mod}_{(A,\mathrm{d})})$ is called a *distinguished triangle* if it is isomorphic to a triangle associated to an admissible short exact sequence of differential graded A-modules.

22.9. Cones and distinguished triangles

Let (A, d) be a differential graded algebra. Let $f : K \to L$ be a homomorphism of differential graded A-modules. Then $(K, L, C(f), f, i, p)$ forms a triangle:

$$K \to L \to C(f) \to K[1]$$

in $\mathrm{Mod}_{(A,\mathrm{d})}$ and hence in $K(\mathrm{Mod}_{(A,\mathrm{d})})$. Cones are **not** distinguished triangles in general, but the difference is a sign or a rotation (your choice). Here are two precise statements.

Lemma 22.9.1. *Let (A, d) be a differential graded algebra. Let $f : K \to L$ be a homomorphism of differential graded modules. The triangle $(L, C(f), K[1], i, p, f[1])$ is the triangle associated to the admissible short exact sequence*

$$0 \to L \to C(f) \to K[1] \to 0$$

coming from the definition of the cone of f.

Proof. Immediate from the definitions. □

Lemma 22.9.2. *Let (A, d) be a differential graded algebra. Let $\alpha : K \to L$ and $\beta : L \to M$ define an admissible short exact sequence*

$$0 \to K \to L \to M \to 0$$

of differential graded A-modules. Let $(K, L, M, \alpha, \beta, \delta)$ be the associated triangle. Then the triangles

$$(M[-1], K, L, \delta[-1], \alpha, \beta) \quad \text{and} \quad (M[-1], K, C(\delta[-1]), \delta[-1], i, p)$$

are isomorphic.

Proof. Using a choice of splittings we write $L = K \oplus M$ and we identify α and β with the natural inclusion and projection maps. By construction of δ we have

$$d_B = \begin{pmatrix} d_K & \delta \\ 0 & d_M \end{pmatrix}$$

On the other hand the cone of $\delta[-1] : M[-1] \to K$ is given as $C(\delta[-1]) = K \oplus M$ with differential identical with the matrix above! Whence the lemma. □

Lemma 22.9.3. *Let (A, d) be a differential graded algebra. Let $f_1 : K_1 \to L_1$ and $f_2 : K_2 \to L_2$ be homomorphisms of differential graded A-modules. Let*

$$(a, b, c) : (K_1, L_1, C(f_1), f_1, i_1, p_1) \longrightarrow (K_1, L_1, C(f_1), f_2, i_2, p_2)$$

be any morphism of triangles of $K(Mod_{(A,d)})$. If a and b are homotopy equivalences then so is c.

Proof. Let $a^{-1} : K_2 \to K_1$ be a homomorphism of differential graded A-modules which is inverse to a in $K(\mathrm{Mod}_{(A,\mathrm{d})})$. Let $b^{-1} : L_2 \to L_1$ be a homomorphism of differential graded A-modules which is inverse to b in $K(\mathrm{Mod}_{(A,\mathrm{d})})$. Let $c' : C(f_2) \to C(f_1)$ be the morphism from Lemma 22.6.2 applied to $f_1 \circ a^{-1} = b^{-1} \circ f_2$. If we can show that $c \circ c'$ and $c' \circ c$ are isomorphisms in $K(\mathrm{Mod}_{(A,\mathrm{d})})$ then we win. Hence it suffices to prove the following: Given a morphism of triangles

$(1, 1, c) : (K, L, C(f), f, i, p)$ in $K(\mathrm{Mod}_{(A,\mathrm{d})})$ the morphism c is an isomorphism in $K(\mathrm{Mod}_{(A,\mathrm{d})})$. By assumption the two squares in the diagram

$$\begin{array}{ccccc} L & \longrightarrow & C(f) & \longrightarrow & K[1] \\ {\scriptstyle 1}\downarrow & & {\scriptstyle c}\downarrow & & {\scriptstyle 1}\downarrow \\ L & \longrightarrow & C(f) & \longrightarrow & K[1] \end{array}$$

commute up to homotopy. By construction of $C(f)$ the rows form admissible short exact sequences. Thus we see that $(c-1)^2 = 0$ in $K(\mathrm{Mod}_{(A,\mathrm{d})})$ by Lemma 22.7.6. Hence c is an isomorphism in $K(\mathrm{Mod}_{(A,\mathrm{d})})$ with inverse $2 - c$. □

The following lemma shows that the collection of triangles of the homotopy category given by cones and the distinguished triangles are the same up to isomorphisms, at least up to sign!

Lemma 22.9.4. *Let (A, d) be a differential graded algebra.*

(1) *Given an admissible short exact sequence $0 \to K \xrightarrow{\alpha} L \to M \to 0$ of differential graded A-modules there exists a homotopy equivalence $C(\alpha) \to M$ such that the diagram*

$$\begin{array}{ccccccc} K & \longrightarrow & L & \longrightarrow & C(\alpha) & \xrightarrow[-p]{} & K[1] \\ \downarrow & & \downarrow & & \downarrow & & \downarrow \\ K & \xrightarrow{\alpha} & L & \xrightarrow{\beta} & M & \xrightarrow{\delta} & K[1] \end{array}$$

defines an isomorphism of triangles in $K(Mod_{(A,d)})$.

(2) *Given a morphism of complexes $f : K \to L$ there exists an isomorphism of triangles*

$$\begin{array}{ccccccc} K & \longrightarrow & \tilde{L} & \longrightarrow & M & \xrightarrow[\delta]{} & K[1] \\ \downarrow & & \downarrow & & \downarrow & & \downarrow \\ K & \longrightarrow & L & \longrightarrow & C(f) & \xrightarrow{-p} & K[1] \end{array}$$

where the upper triangle is the triangle associated to a admissible short exact sequence $K \to \tilde{L} \to M$.

Proof. Proof of (1). We have $C(\alpha) = L \oplus K$ and we simply define $C(\alpha) \to M$ via the projection onto L followed by β. This defines a morphism of differential graded modules because the compositions $K^{n+1} \to L^{n+1} \to M^{n+1}$ are zero. Choose splittings $s : M \to L$ and $\pi : L \to K$ with $\mathrm{Ker}(\pi) = \mathrm{Im}(s)$ and set $\delta = \pi \circ \mathrm{d}_L \circ s$ as usual. To get a homotopy inverse we take $M \to C(\alpha)$ given by $(s, -\delta)$. This is compatible with differentials because δ^n can be characterized as the unique map $M^n \to K^{n+1}$ such that $\mathrm{d} \circ s^n - s^{n+1} \circ \mathrm{d} = \alpha \circ \delta^n$, see proof of Homology, Lemma 12.14.10. The composition $M \to C(f) \to M$ is the identity. The composition $C(f) \to M \to C(f)$ is equal to the morphism

$$\begin{pmatrix} s \circ \beta & 0 \\ -\delta \circ \beta & 0 \end{pmatrix}$$

To see that this is homotopic to the identity map use the homotopy $h : C(\alpha) \to C(\alpha)$ given by the matrix

$$\begin{pmatrix} 0 & 0 \\ \pi & 0 \end{pmatrix} : C(\alpha) = L \oplus K \to L \oplus K = C(\alpha)$$

It is trivial to verify that

$$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} - \begin{pmatrix} s \\ -\delta \end{pmatrix} \begin{pmatrix} \beta & 0 \end{pmatrix} = \begin{pmatrix} \mathrm{d} & \alpha \\ 0 & -\mathrm{d} \end{pmatrix} \begin{pmatrix} 0 & 0 \\ \pi & 0 \end{pmatrix} + \begin{pmatrix} 0 & 0 \\ \pi & 0 \end{pmatrix} \begin{pmatrix} \mathrm{d} & \alpha \\ 0 & -\mathrm{d} \end{pmatrix}$$

To finish the proof of (1) we have to show that the morphisms $-p : C(\alpha) \to K[1]$ (see Definition 22.6.1) and $C(\alpha) \to M \to K[1]$ agree up to homotopy. This is clear from the above. Namely, we can use the homotopy inverse $(s, -\delta) : M \to C(\alpha)$ and check instead that the two maps $M \to K[1]$ agree. And note that $p \circ (s, -\delta) = -\delta$ as desired.

Proof of (2). We let $\tilde{f} : K \to \tilde{L}$, $s : L \to \tilde{L}$ and $\pi : L \to L$ be as in Lemma 22.7.4. By Lemmas 22.6.2 and 22.9.3 the triangles $(K, L, C(f), i, p)$ and $(K, \tilde{L}, C(\tilde{f}), \tilde{i}, \tilde{p})$ are isomorphic. Note that we can compose isomorphisms of triangles. Thus we may replace L by $\tilde{L}$ and f by $\tilde{f}$. In other words we may assume that f is an admissible monomorphism. In this case the result follows from part (1). □

22.10. The homotopy category is triangulated

We first prove that it is pre-triangulated.

Lemma 22.10.1. *Let (A, d) be a differential graded algebra. The homotopy category $K(Mod_{(A,d)})$ with its natural translation functors and distinguished triangles is a pre-triangulated category.*

Proof. Proof of TR1. By definition every triangle isomorphic to a distinguished one is distinguished. Also, any triangle $(K, K, 0, 1, 0, 0)$ is distinguished since $0 \to K \to K \to 0 \to 0$ is an admissible short exact sequence. Finally, given any homomorphism $f : K \to L$ of differential graded A-modules the triangle $(K, L, C(f), f, i, -p)$ is distinguished by Lemma 22.9.4.

Proof of TR2. Let (X, Y, Z, f, g, h) be a triangle. Assume $(Y, Z, X[1], g, h, -f[1])$ is distinguished. Then there exists an admissible short exact sequence $0 \to K \to L \to M \to 0$ such that the associated triangle $(K, L, M, \alpha, \beta, \delta)$ is isomorphic to $(Y, Z, X[1], g, h, -f[1])$. Rotating back we see that (X, Y, Z, f, g, h) is isomorphic to $(M[-1], K, L, -\delta[-1], \alpha, \beta)$. It follows from Lemma 22.9.2 that the triangle $(M[-1], K, L, \delta[-1], \alpha, \beta)$ is isomorphic to $(M[-1], K, C(\delta[-1]), \delta[-1], i, p)$. Precomposing the previous isomorphism of triangles with -1 on Y it follows that (X, Y, Z, f, g, h) is isomorphic to $(M[-1], K, C(\delta[-1]), \delta[-1], i, -p)$. Hence it is distinguished by Lemma 22.9.4. On the other hand, suppose that (X, Y, Z, f, g, h) is distinguished. By Lemma 22.9.4 this means that it is isomorphic to a triangle of the form $(K, L, C(f), f, i, -p)$ for some morphism f of $\mathrm{Mod}_{(A,\mathrm{d})}$. Then the rotated triangle $(Y, Z, X[1], g, h, -f[1])$ is isomorphic to $(L, C(f), K[1], i, -p, -f[1])$ which is isomorphic to the triangle $(L, C(f), K[1], i, p, f[1])$. By Lemma 22.9.1 this triangle is distinguished. Hence $(Y, Z, X[1], g, h, -f[1])$ is distinguished as desired.

Proof of TR3. Let (X, Y, Z, f, g, h) and (X', Y', Z', f', g', h') be distinguished triangles of $K(\mathcal{A})$ and let $a : X \to X'$ and $b : Y \to Y'$ be morphisms such that $f' \circ a = b \circ f$. By Lemma 22.9.4 we may assume that $(X, Y, Z, f, g, h) = (X, Y, C(f), f, i, -p)$

and $(X', Y', Z', f', g', h') = (X', Y', C(f'), f', i', -p')$. At this point we simply apply Lemma 22.6.2 to the commutative diagram given by f, f', a, b. □

Before we prove TR4 in general we prove it in a special case.

Lemma 22.10.2. *Let (A, d) be a differential graded algebra. Suppose that $\alpha : K \to L$ and $\beta : L \to M$ are admissible monomorphisms of differential graded A-modules. Then there exist distinguished triangles $(K, L, Q_1, \alpha, p_1, d_1)$, $(K, M, Q_2, \beta \circ \alpha, p_2, d_2)$ and $(L, M, Q_3, \beta, p_3, d_3)$ for which TR4 holds.*

Proof. Say $\pi_1 : L \to K$ and $\pi_3 : M \to L$ are homomorphisms of graded A-modules which are left inverse to α and β. Then also $K \to M$ is an admissible monomorphism with left inverse $\pi_2 = \pi_1 \circ \pi_3$. Let us write Q_1, Q_2 and Q_3 for the cokernels of $K \to L$, $K \to M$, and $L \to M$. Then we obtain identifications (as graded A-modules) $Q_1 = \mathrm{Ker}(\pi_1)$, $Q_3 = \mathrm{Ker}(\pi_3)$ and $Q_2 = \mathrm{Ker}(\pi_2)$. Then $L = K \oplus Q_1$ and $M = L \oplus Q_3$ as graded A-modules. This implies $M = K \oplus Q_1 \oplus Q_3$. Note that $\pi_2 = \pi_1 \circ \pi_3$ is zero on both Q_1 and Q_3. Hence $Q_2 = Q_1 \oplus Q_3$. Consider the commutative diagram

$$\begin{array}{ccccccccc}
0 & \to & K & \to & L & \to & Q_1 & \to & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \to & K & \to & M & \to & Q_2 & \to & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \to & L & \to & M & \to & Q_3 & \to & 0
\end{array}$$

The rows of this diagram are admissible short exact sequences, and hence determine distinguished triangles by definition. Moreover downward arrows in the diagram above are compatible with the chosen splittings and hence define morphisms of triangles

$$(K \to L \to Q_1 \to K[1]) \longrightarrow (K \to M \to Q_2 \to K[1])$$

and

$$(K \to M \to Q_2 \to K[1]) \longrightarrow (L \to M \to Q_3 \to L[1]).$$

Note that the splittings $Q_3 \to M$ of the bottom sequence in the diagram provides a splitting for the split sequence $0 \to Q_1 \to Q_2 \to Q_3 \to 0$ upon composing with $M \to Q_2$. It follows easily from this that the morphism $\delta : Q_3 \to Q_1[1]$ in the corresponding distinguished triangle

$$(Q_1 \to Q_2 \to Q_3 \to Q_1[1])$$

is equal to the composition $Q_3 \to L[1] \to Q_1[1]$. Hence we get a structure as in the conclusion of axiom TR4. □

Here is the final result.

Proposition 22.10.3. *Let (A, d) be a differential graded algebra. The homotopy category $K(Mod_{(A,d)})$ of differential graded A-modules with its natural translation functors and distinguished triangles is a triangulated category.*

Proof. We know that $K(\mathrm{Mod}_{(A,\mathrm{d})})$ is a pre-triangulated category. Hence it suffices to prove TR4 and to prove it we can use Derived Categories, Lemma 13.4.13. Let $K \to L$ and $L \to M$ be composable morphisms of $K(\mathrm{Mod}_{(A,\mathrm{d})})$. By Lemma 22.7.5 we may assume that $K \to L$ and $L \to M$ are admissible monomorphisms. In this case the result follows from Lemma 22.10.2. □

22.11. Projective modules over algebras

In this section we discuss projective modules over algebras and over graded algebras. Thus it is the analogue of Algebra, Section 10.72 in the setting of this chapter.

Algebras and modules. Let R be a ring and let A be an R-algebra, see Section 22.2 for our conventions. It is clear that A is a projective right A-module since $\mathrm{Hom}_A(A, M) = M$ for any right A-module M (and thus $\mathrm{Hom}_A(A, -)$ is exact). Conversely, let P be a projective right A-module. Then we can choose a surjection $\bigoplus_{i\in I} A \to M$ by choosing a set $\{m_i\}_{i\in I}$ of generators of P over A. Since P is projective there is a left inverse to the surjection, and we find that P is isomorphic to a direct summand of a free module, exactly as in the commutative case (Algebra, Lemma 10.72.2).

Graded algebras and modules. Let R be a ring. Let A be a graded algebra over R. Let Mod_A denote the category of graded right A-modules. For an integer k let $A[k]$ denote the shift of A. For an graded right A-module we have

$$\mathrm{Hom}_{\mathrm{Mod}_A}(A[k], M) = M^{-k}$$

As the functor $M \mapsto M^{-k}$ is exact on Mod_A we conclude that $A[k]$ is a projective object of Mod_A. Conversely, suppose that P is a projective object of Mod_A. By choosing a set of homogeneous generators of P as an A-module, we can find a surjection

$$\bigoplus\nolimits_{i\in I} A[k_i] \longrightarrow P$$

Thus we conclude that a projective object of Mod_A is a direct summand of a direct sum of the shifts $A[k]$.

If (A, d) is a differential graded algebra and P is an object of $\mathrm{Mod}_{(A,\mathrm{d})}$ then we say *P is projective as a graded A-module* or sometimes *P is graded projective* to mean that P is a projective object of the abelian category Mod_A of graded A-modules.

Lemma 22.11.1. *Let (A, d) be a differential graded algebra. Let $M \to P$ be a surjective homomorphism of differential graded A-modules. If P is projective as a graded A-module, then $M \to P$ is an admissible epimorphism.*

Proof. This is immediate from the definitions. □

Lemma 22.11.2. *Let (A, d) be a differential graded algebra. Then we have*

$$\mathrm{Hom}_{Mod_{(A,d)}}(A[k], M) = Ker(d : M^{-k} \to M^{-k+1})$$

and

$$\mathrm{Hom}_{K(Mod_{(A,d)})}(A[k], M) = H^{-k}(M)$$

for any differential graded A-module M.

Proof. This is clear from the discussion above. □

22.12. Injective modules over algebras

In this section we discuss injective modules over algebras and over graded algebras. Thus it is the analogue of More on Algebra, Section 15.40 in the setting of this chapter.

Algebras and modules. Let R be a ring and let A be an R-algebra, see Section 22.2 for our conventions. For a right A-module M we set

$$M^\vee = \mathrm{Hom}_{\mathbf{Z}}(M, \mathbf{Q}/\mathbf{Z})$$

which we think of as a left A-module by the multiplication $(af)(x) = f(xa)$. Namely, $((ab)f)(x) = f(xab) = (bf)(xa) = (a(bf))(x)$. Conversely, if M is a left A-module, then $M^\vee$ is a right A-module. Since $\mathbf{Q}/\mathbf{Z}$ is an injective abelian group (More on Algebra, Lemma 15.39.1), the functor $M \mapsto M^\vee$ is exact (More on Algebra, Lemma 15.40.2). Moreover, the evaluation map $M \to (M^\vee)^\vee$ is injective for all modules M (More on Algebra, Lemma 15.40.3).

We claim that $A^\vee$ is an injective right A-module. Namely, given a right A-module N we have

$$\mathrm{Hom}_A(N, A^\vee) = \mathrm{Hom}_A(N, \mathrm{Hom}_{\mathbf{Z}}(A, \mathbf{Q}/\mathbf{Z})) = N^\vee$$

and we conclude because the functor $N \mapsto N^\vee$ is exact. The second equality holds because

$$\mathrm{Hom}_{\mathbf{Z}}(N, \mathrm{Hom}_{\mathbf{Z}}(A, \mathbf{Q}/\mathbf{Z})) = \mathrm{Hom}_{\mathbf{Z}}(N \otimes_{\mathbf{Z}} A, \mathbf{Q}/\mathbf{Z})$$

by Algebra, Lemma 10.11.8. Inside this module A-linearity exactly picks out the bilinear maps $\varphi : N \times A \to \mathbf{Q}/\mathbf{Z}$ which have the same value on $x \otimes a$ and $xa \otimes 1$, i.e., come from elements of $N^\vee$.

Finally, for every right A-module M we can choose a surjection $\bigoplus_{i \in I} A \to M^\vee$ to get an injection $M \to (M^\vee)^\vee \to \prod_{i \in I} A^\vee$.

We conclude

(1) the category of A-modules has enough injectives,
(2) $A^\vee$ is an injective A-module, and
(3) every A-module injects into a product of copies of $A^\vee$.

Graded algebras and modules. Let R be a ring. Let A be a graded algebra over R. If M is a graded A-module we set

$$M^\vee = \bigoplus\nolimits_{n \in \mathbf{Z}} \mathrm{Hom}_{\mathbf{Z}}(M^{-n}, \mathbf{Q}/\mathbf{Z}) = \bigoplus\nolimits_{n \in \mathbf{Z}} (M^{-n})^\vee$$

as a graded R-module with the A-module structure defined as above (for homogeneous elements). This again switches left and right modules. On the category of graded A-modules the functor $M \mapsto M^\vee$ is exact (check on graded pieces). Moreover, the evaluation map $M \to (M^\vee)^\vee$ is injective as before (because we can check this on the graded pieces).

We claim that $A^\vee$ is an injective object of the category Mod_A of graded right A-modules. Namely, given a graded right A-module N we have

$$\mathrm{Hom}_{\mathrm{Mod}_A}(N, A^\vee) = \mathrm{Hom}_{\mathrm{Mod}_A}(N, \bigoplus \mathrm{Hom}_{\mathbf{Z}}(A^{-n}, \mathbf{Q}/\mathbf{Z})) = (N^0)^\vee$$

and we conclude because the functor $N \mapsto (N^0)^\vee = (N^\vee)^0$ is exact. To see that the second equality holds we use the equalities

$$\mathrm{Hom}_{\mathbf{Z}}(N^n, \mathrm{Hom}_{\mathbf{Z}}(A^{-n}, \mathbf{Q}/\mathbf{Z})) = \mathrm{Hom}_{\mathbf{Z}}(N^n \otimes_{\mathbf{Z}} A^{-n}, \mathbf{Q}/\mathbf{Z})$$

of Algebra, Lemma 10.11.8. Thus an element of $\mathrm{Hom}_{\mathrm{Mod}_A}(N, A^\vee)$ corresponds to a family of $\mathbf{Z}$-bilinear maps $\psi_n : N^n \times A^{-n} \to \mathbf{Q}/\mathbf{Z}$ such that $\psi_n(x, a) = \psi_0(xa, 1)$ for all $x \in N^n$ and $a \in A^{-n}$. Moreover, $\psi_0(x, a) = \psi_0(xa, 1)$ for all $x \in N^0$, $a \in A^0$.

It follows that the maps ψ_n are determined by ψ_0 and that $\psi_0(x, a) = \varphi(xa)$ for a unique element $\varphi \in (N^0)^\vee$.

Finally, for every graded right A-module M we can choose a surjection (of graded left A-modules)

$$\bigoplus\nolimits_{i \in I} A[k_i] \to M^\vee$$

where $A[k_i]$ denotes the shift of A by $k_i \in \mathbf{Z}$. (We do this by choosing homogeneous generators for $M^\vee$.) In this way we get an injection

$$M \to (M^\vee)^\vee \to \prod A[k_i]^\vee = \prod A^\vee[-k_i]$$

Observe that the products in the formula above are products in the category of graded modules (in other words, take products in each degree and then take the direct sum of the pieces).

We conclude that

(1) the category of graded A-modules has enough injectives,
(2) for every $k \in \mathbf{Z}$ the module $A^\vee[k]$ is injective, and
(3) every A-module injects into a product in the category of graded modules of copies of shifts $A^\vee[k]$.

If (A, d) is a differential graded algebra and I is an object of $\mathrm{Mod}_{(A,\mathrm{d})}$ then we say *I is injective as a graded A-module* to mean that I is a injective object of the abelian category Mod_A of graded A-modules.

Lemma 22.12.1. *Let (A, d) be a differential graded algebra. Let $I \to M$ be an injective homomorphism of differential graded A-modules. If I is an injective object of the category of graded A-modules, then $I \to M$ is an admissible monomorphism.*

Proof. This is immediate from the definitions. □

Let (A, d) be a differential graded algebra. If M is a (right) differential graded A-module, then $M^\vee$ (as defined above for graded modules) is a differential graded module as well with differential given by the contragredient of d_M up to sign. The sign rule (see Example 22.19.8) is to set

$$\mathrm{d}_{M^\vee}(f) = -(-1)^n f \circ \mathrm{d}_M^{-n-1} \quad \text{in } (M^\vee)^{n+1}$$

where $f \in (M^\vee)^n = \mathrm{Hom}_{\mathbf{Z}}(M^{-n}, \mathbf{Q}/\mathbf{Z})$ and $\mathrm{d}_M^{-n-1} : M^{-n-1} \to M^{-n}$ is the differential of M. This does indeed work, namely, if $a \in A^m$, $x \in M^{-n-m-1}$ and f as above, then

$$\begin{aligned}
\mathrm{d}_{M^\vee}(af)(x) &= -(-1)^{n+m}(af)(\mathrm{d}(x)) \\
&= -(-1)^{n+m} f(\mathrm{d}(x)a) \\
&= -(-1)^{n+m} f(\mathrm{d}(xa) - (-1)^{n+m} x\mathrm{d}(a)) \\
&= (\mathrm{d}(a)f)(x) + (-1)^m (a\mathrm{d}_{M^\vee}(f))(x)
\end{aligned}$$

in other words we have $\mathrm{d}(af) = \mathrm{d}(a)f + (-1)^m a\mathrm{d}(f)$ as desired.

Similarly, if M is a left differential graded module, then $M^\vee$ becomes a right differential graded module with the same sign rule for $\mathrm{d}_{M^\vee}$. Computation omitted.

Lemma 22.12.2. *Let (A, d) be a differential graded algebra. If M is a left differential graded A-module and N is a right differential graded A-module, then*

$$\mathrm{Hom}_{Mod_{(A,d)}}(N, M^\vee)$$

is isomorphic to the set of sequences (ψ_n) of $\mathbf{Z}$-bilinear pairings

$$\psi_n : N^n \times M^{-n} \longrightarrow \mathbf{Q}/\mathbf{Z}$$

such that $\psi_{n+m}(y, ax) = \psi_{n+m}(ya, x)$ for all $y \in N^n$, $x \in M^{-m}$, and $a \in A^{m-n}$ and such that $\psi_{n+1}(d(y), x) + (-1)^n \psi_n(y, d(x)) = 0$ for all $y \in N^n$ and $x \in M^{-n-1}$.

Proof. If $f \in \mathrm{Hom}_{\mathrm{Mod}_{(A,\mathrm{d})}}(N, M^\vee)$, then we map this to the sequence of pairings defined by $\psi_n(y, x) = f(y)(x)$. It is a computation (omitted) to see that these pairings satisfy the conditions as in the lemma. For the converse, use Algebra, Lemma 10.11.8 to turn a sequence of pairings into a map $f : N \to M^\vee$. □

Lemma 22.12.3. *Let (A, d) be a differential graded algebra. Then we have*

$$\mathrm{Hom}_{Mod_{(A,d)}}(M, A^\vee[k]) = Ker(d : (M^\vee)^k \to (M^\vee)^{k+1})$$

and

$$\mathrm{Hom}_{K(Mod_{(A,d)})}(A[k], M) = H^k(M^\vee)$$

for any differential graded A-module M.

Proof. This is clear from the discussion above. □

22.13. P-resolutions

This section is the analogue of Derived Categories, Section 13.27.

Let (A, d) be a differential graded algebra. Let P be a differential graded A-module. We say P *has property (P)* if it there exists a filtration

$$0 = F_{-1}P \subset F_0P \subset F_1P \subset \ldots \subset P$$

by differential graded submodules such that

(1) $P = \bigcup F_pP$,
(2) the inclusions $F_iP \to F_{i+1}P$ are admissible monomorphisms,
(3) the quotients $F_{i+1}P/F_iP$ are isomorphic to direct sums of shifts $A[k]$.

In fact, condition (2) is a consequence of condition (3), see Lemma 22.11.1.

Lemma 22.13.1. *Let (A, d) be a differential graded algebra. Let P be a differential graded A-module. If $F_\bullet$ is a filtration as in property (P), then we obtain an admissible short exact sequence*

$$0 \to \bigoplus F_iP \to \bigoplus F_iP \to P \to 0$$

of differential graded A-modules.

Proof. The second map is the direct sum of the inclusion maps. The first map on the summand F_iP of the source is the sum of the identity $F_iP \to F_iP$ and the negative of the inclusion map $F_iP \to P_{i+1}P$. Choose homomorphisms $s_i : F_{i+1}P \to F_iP$ of graded A-modules which are left inverse to the inclusion maps. Composing gives maps $s_{j,i} : F_jP \to F_iP$ for all $j > i$. Then a left inverse of the first arrow maps $x \in F_jP$ to $(s_{j,0}(x), s_{j,1}(x), \ldots, s_{j,j-1}(x), 0, \ldots)$ in $\bigoplus F_iP$. □

The following lemma shows that differential graded modules with property (P) are the dual notion to K-injective modules (i.e., they are K-projective in some sense). See Derived Categories, Definition 13.28.1.

Lemma 22.13.2. *Let (A, d) be a differential graded algebra. Let P be a differential graded A-module with property (P). Then*

$$\mathrm{Hom}_{K(Mod_{(A,d)})}(P, N) = 0$$

for all acyclic differential graded A-modules N.

Proof. We will use that $K(\mathrm{Mod}_{(A,\mathrm{d})})$ is a triangulated category (Proposition 22.10.3). Let $F_\bullet$ be a filtration on P as in property (P). The short exact sequence of Lemma 22.13.1 produces a distinguished triangle. Hence by Derived Categories, Lemma 13.4.2 it suffices to show that

$$\mathrm{Hom}_{K(\mathrm{Mod}_{(A,\mathrm{d})})}(F_iP, N) = 0$$

for all acyclic differential graded A-modules N and all i. Each of the differential graded modules F_iP has a finite filtration by admissible monomorphisms, whose graded pieces are direct sums of shifts $A[k]$. Thus it suffices to prove that

$$\mathrm{Hom}_{K(\mathrm{Mod}_{(A,\mathrm{d})})}(A[k], N) = 0$$

for all acyclic differential graded A-modules N and all k. This follows from Lemma 22.11.2. □

Lemma 22.13.3. *Let (A, d) be a differential graded algebra. Let M be a differential graded A-module. There exists a homomorphism $P \to M$ of differential graded A-modules with the following properties*

1. *$P \to M$ is surjective,*
2. *$Ker(d_P) \to Ker(d_M)$ is surjective, and*
3. *P sits in an admissible short exact sequence $0 \to P' \to P \to P'' \to 0$ where P', P'' are direct sums of shifts of A.*

Proof. Let P_k be the free A-module with generators x, y in degrees k and $k+1$. Define the structure of a differential graded A-module on P_k by setting $\mathrm{d}(x) = y$ and $\mathrm{d}(y) = 0$. For every element $m \in M^k$ there is a homomorphism $P_k \to M$ sending x to m and y to $\mathrm{d}(m)$. Thus we see that there is a surjection from a direct sum of copies of P_k to M. This clearly produces $P \to M$ having properties (1) and (3). To obtain property (2) note that if $m \in \mathrm{Ker}(\mathrm{d}_M)$ has degree k, then there is a map $A[k] \to M$ mapping 1 to m. Hence we can achieve (2) by adding a direct sum of copies of shifts of A. □

Lemma 22.13.4. *Let (A, d) be a differential graded algebra. Let M be a differential graded A-module. There exists a homomorphism $P \to M$ of differential graded A-modules such that*

1. *$P \to M$ is a quasi-isomorphism, and*
2. *P has property (P).*

Proof. Set $M = M_0$. We inductively choose short exact sequences

$$0 \to M_{i+1} \to P_i \to M_i \to 0$$

where the maps $P_i \to M_i$ are chosen as in Lemma 22.13.3. This gives a "resolution"

$$\ldots \to P_2 \xrightarrow{f_2} P_1 \xrightarrow{f_1} P_0 \to M \to 0$$

Then we set

$$P = \bigoplus\nolimits_{i \geq 0} P_i$$

as an A-module with grading given by $P^n = \bigoplus_{a+b=n} P^b_{-a}$ and differential (as in the construction of the total complex associated to a double complex) by

$$\mathrm{d}_P(x) = f_{-a}(x) + (-1)^a \mathrm{d}_{P_{-a}}(x)$$

for $x \in P^b_{-a}$. With these conventions P is indeed a differential graded A-module. Recalling that each P_i has a two step filtration $0 \to P'_i \to P_i \to P''_i \to 0$ we set

$$F_{2i}P = \bigoplus\nolimits_{i \geq j \geq 0} P_j \subset \bigoplus\nolimits_{i \geq 0} P_i = P$$

and we add P'_{i+1} to $F_{2i}P$ to get F_{2i+1}. These are differential graded submodules and the successive quotients are direct sums of shifts of A. By Lemma 22.11.1 we see that the inclusions $F_iP \to F_{i+1}P$ are admissible monomorphisms. Finally, we have to show that the map $P \to M$ (given by the augmentation $P_0 \to M$) is a quasi-isomorphism. This follows from Homology, Lemma 12.22.9. □

22.14. I-resolutions

This section is the dual of the section on P-resolutions.

Let (A, d) be a differential graded algebra. Let I be a differential graded A-module. We say I *has property (I)* if it there exists a filtration

$$I = F_0I \supset F_1I \supset F_2I \supset \ldots \supset 0$$

by differential graded submodules such that

(1) $I = \lim I/F_pI$,
(2) the inclusions $F_iI \to F_{i-1}I$ are admissible monomorphisms,
(3) the quotients $F_{i-1}I/F_iI$ are direct sums of the modules $A^\vee[k]$.

In fact, condition (2) is a consequence of condition (3), see Lemma 22.12.1.

Lemma 22.14.1. *Let (A, d) be a differential graded algebra. Let I be a differential graded A-module. If $F_\bullet$ is a filtration as in property (I), then we obtain an admissible short exact sequence*

$$0 \to I \to \prod I/F_iI \to \prod I/F_iI \to 0$$

of differential graded A-modules.

Proof. Omitted. Hint: This is dual to Lemma 22.13.1. □

The following lemma shows that differential graded modules with property (I) are the analogue of K-injective modules. See Derived Categories, Definition 13.28.1.

Lemma 22.14.2. *Let (A, d) be a differential graded algebra. Let I be a differential graded A-module with property (I). Then*

$$\mathrm{Hom}_{K(Mod_{(A,d)})}(N, I) = 0$$

for all acyclic differential graded A-modules N.

Proof. We will use that $K(\mathrm{Mod}_{(A,\mathrm{d})})$ is a triangulated category (Proposition 22.10.3). Let $F_\bullet$ be a filtration on I as in property (I). The short exact sequence of Lemma 22.14.1 produces a distinguished triangle. Hence by Derived Categories, Lemma 13.4.2 it suffices to show that

$$\mathrm{Hom}_{K(\mathrm{Mod}_{(A,\mathrm{d})})}(N, I/F_iI) = 0$$

for all acyclic differential graded A-modules N and all i. Each of the differential graded modules I/F_iI has a finite filtration by admissible monomorphisms, whose graded pieces are direct sums of $A^\vee[k]$. Thus it suffices to prove that

$$\mathrm{Hom}_{K(\mathrm{Mod}_{(A,\mathrm{d})})}(N, A^\vee[k]) = 0$$

for all acyclic differential graded A-modules N and all k. This follows from Lemma 22.12.3 and the fact that $(-)^\vee$ is an exact functor. □

Lemma 22.14.3. *Let (A, d) be a differential graded algebra. Let M be a differential graded A-module. There exists a homomorphism $M \to I$ of differential graded A-modules with the following properties*

1. *$M \to I$ is injective,*
2. *$Coker(d_M) \to Coker(d_I)$ is injective, and*
3. *I sits in an admissible short exact sequence $0 \to I' \to I \to I'' \to 0$ where I', I'' are products of shifts of $A^\vee$.*

Proof. For every $k \in \mathbf{Z}$ let Q_k be the free left A-module with generators x, y in degrees k and $k+1$. Define the structure of a left differential graded A-module on Q_k by setting $\mathrm{d}(x) = y$ and $\mathrm{d}(y) = 0$. Let $I_k = Q_{-k}^\vee$ be the "dual" right differential graded A-module, see Section 22.12. The next paragraph shows that we can embed M into a product of copies of I_k (for varying k). The dual statement (that any differential graded module is a quotient of a direct sum of of P_k's) is easy to prove (see proof of Lemma 22.13.3) and using double duals there should be a noncomputational way to deduce what we want. Thus we suggest skipping the next paragraph.

Given a $\mathbf{Z}$-linear map $\lambda : M^k \to \mathbf{Q}/\mathbf{Z}$ we construct pairings

$$\psi_n : M^n \times Q_k^{-n} \longrightarrow \mathbf{Q}/\mathbf{Z}$$

by setting

$$\psi_n(m, ax + by) = \lambda(ma + (-1)^{k+1}\mathrm{d}(mb))$$

for $m \in M^n$, $a \in A^{-n-k}$, and $b \in A^{-n-k-1}$. We compute

$$\begin{aligned}\psi_{n+1}(\mathrm{d}(m), ax + by) &= \lambda\left(\mathrm{d}(m)a + (-1)^{k+1}\mathrm{d}(\mathrm{d}(m)b)\right) \\ &= \lambda\left(\mathrm{d}(m)a + (-1)^{k+n}\mathrm{d}(m)\mathrm{d}(b)\right)\end{aligned}$$

and because $\mathrm{d}(ax + by) = \mathrm{d}(a)x + (-1)^{-n-k}ay + \mathrm{d}(b)y$ we have

$$\begin{aligned}\psi_n(m, \mathrm{d}(ax + by)) &= \lambda\left(m\mathrm{d}(a) + (-1)^{k+1}\mathrm{d}(m((-1)^{-n-k}a + \mathrm{d}(b)))\right) \\ &= \lambda\left(m\mathrm{d}(a) + (-1)^{-n+1}\mathrm{d}(ma) + (-1)^{k+1}\mathrm{d}(m)\mathrm{d}(b))\right)\end{aligned}$$

and we see that

$$\psi_{n+1}(\mathrm{d}(m), ax + by) + (-1)^n\psi_n(m, \mathrm{d}(ax + by)) = 0$$

Thus these pairings define a homomorphism $f_\lambda : M \to I_k$ by Lemma 22.12.2 such that the composition

$$M^k \xrightarrow{f_\lambda^k} I_k^k = (Q_k^k)^\vee \xrightarrow{\text{evaluation at } x} \mathbf{Q}/\mathbf{Z}$$

is the given map λ. It is clear that we can find an embedding into a product of copies of I_k's by using a map of the form $\prod f_\lambda$ for a suitable choice of the maps λ.

The result of the previous paragraph produces $M \to I$ having properties (1) and (3). To obtain property (2), suppose $\overline{m} \in \mathrm{Coker}(\mathrm{d}_M)$ is a nonzero element of degree k. Pick a map $\lambda : M^k \to \mathbf{Q}/\mathbf{Z}$ which vanishes on $\mathrm{Im}(M^{k-1} \to M^k)$ but not on m. By Lemma 22.12.3 this corresponds to a homomorphism $M \to A^\vee[k]$ of differential graded A-modules which does not vanish on m. Hence we can achieve (2) by adding a product of copies of shifts of $A^\vee$. □

Lemma 22.14.4. *Let (A, d) be a differential graded algebra. Let M be a differential graded A-module. There exists a homomorphism $M \to I$ of differential graded A-modules such that*

(1) *$M \to I$ is a quasi-isomorphism, and*
(2) *I has property (I).*

Proof. Set $M = M_0$. We inductively choose short exact sequences

$$0 \to M_i \to I_i \to M_{i+1} \to 0$$

where the maps $M_i \to I_i$ are chosen as in Lemma 22.14.3. This gives a "resolution"

$$0 \to M \to I_0 \xrightarrow{f_0} I_1 \xrightarrow{f_1} I_1 \to \ldots$$

Then we set

$$I = \prod\nolimits_{i \geq 0} I_i$$

where we take the product in the category of graded A-modules and differential defined by

$$\mathrm{d}_I(x) = f_a(x) + (-1)^a \mathrm{d}_{I_a}(x)$$

for $x \in I_a^b$. With these conventions I is indeed a differential graded A-module. Recalling that each I_i has a two step filtration $0 \to I_i' \to I_i \to I_i'' \to 0$ we set

$$F_{2i}P = \prod\nolimits_{j \geq i} I_j \subset \prod\nolimits_{i \geq 0} I_i = I$$

and we add a factor I_{i+1}' to $F_{2i}I$ to get $F_{2i+1}I$. These are differential graded submodules and the successive quotients are products of shifts of $A^\vee$. By Lemma 22.12.1 we see that the inclusions $F_{i+1}I \to F_iI$ are admissible monomorphisms. Finally, we have to show that the map $M \to I$ (given by the augmentation $M \to I_0$) is a quasi-isomorphism. This follows from Homology, Lemma 12.22.10. □

22.15. The derived category

Recall that the notions of acyclic differential graded modules and quasi-isomorphism of differential graded modules make sense (see Section 22.4).

Lemma 22.15.1. *Let (A, d) be a differential graded algebra. The full subcategory Ac of $K(Mod_{(A,d)})$ consisting of acyclic modules is a strictly full saturated triangulated subcategory of $K(Mod_{(A,d)})$. The corresponding saturated multiplicative*

system (see Derived Categories, Lemma 13.6.10) of $K(Mod_{(A,d)})$ is the class Qis of quasi-isomorphisms. In particular, the kernel of the localization functor

$$Q : K(Mod_{(A,d)}) \to Qis^{-1}K(Mod_{(A,d)})$$

is Ac. Moreover, the functor H^0 factors through Q.

Proof. We know that H^0 is a homological functor by the long exact sequence of homology (22.4.2.1). The kernel of H^0 is the subcategory of acyclic objects and the arrows with induce isomorphisms on all H^i are the quasi-isomorphisms. Thus this lemma is a special case of Derived Categories, Lemma 13.6.11.

Set theoretical remark. The construction of the localization in Derived Categories, Proposition 13.5.5 assumes the given triangulated category is "small", i.e., that the underlying collection of objects forms a set. Let V_α be a partial universe (as in Sets, Section 3.5) containing (A, d) and where the cofinality of α is bigger than $\aleph_0$ (see Sets, Proposition 3.7.2). Then we can consider the category $\mathrm{Mod}_{(A,\mathrm{d}),\alpha}$ of differential graded A-modules contained in V_α. A straightforward check shows that all the constructions used in the proof of Proposition 22.10.3 work inside of $\mathrm{Mod}_{(A,\mathrm{d}),\alpha}$ (because at worst we take finite direct sums of differential graded modules). Thus we obtain a triangulated category $\mathrm{Qis}_\alpha^{-1}K(\mathrm{Mod}_{(A,\mathrm{d}),\alpha})$. We will see below that if $\beta > \alpha$, then the transition functors

$$\mathrm{Qis}_\alpha^{-1}K(\mathrm{Mod}_{(A,\mathrm{d}),\alpha}) \longrightarrow \mathrm{Qis}_\beta^{-1}K(\mathrm{Mod}_{(A,\mathrm{d}),\beta})$$

are fully faithful as the morphism sets in the quotient categories are computed by maps in the homotopy categories from P-resolutions (the construction of a P-resolution in the proof of Lemma 22.13.4 takes countable direct sums as well as direct sums indexed over subsets of the given module). The reader should therefore think of the category of the lemma as the union of these subcategories. □

Taking into account the set theoretical remark at the end of the proof of the preceding lemma we define the derived category as follows.

Definition 22.15.2. Let (A, d) be a differential graded algebra. Let Ac and Qis be as in Lemma 22.15.1. The *derived category of* (A, d) is the triangulated category

$$D(A, \mathrm{d}) = K(\mathrm{Mod}_{(A,\mathrm{d})})/\mathrm{Ac} = \mathrm{Qis}^{-1}K(\mathrm{Mod}_{(A,\mathrm{d})}).$$

We denote $H^0 : D(A, \mathrm{d}) \to \mathrm{Mod}_R$ the unique functor whose composition with the quotient functor gives back the functor H^0 defined above.

Here is the promised lemma computing morphism sets in the derived category.

Lemma 22.15.3. *Let (A, d) be a differential graded algebra. Let M and N be differential graded A-modules.*

(1) *Let $P \to M$ be a P-resolution as in Lemma 22.13.4. Then*

$$\mathrm{Hom}_{D(A,d)}(M, N) = \mathrm{Hom}_{K(Mod_{(A,d)})}(P, N)$$

(2) *Let $N \to I$ be an I-resolution as in Lemma 22.14.4. Then*

$$\mathrm{Hom}_{D(A,d)}(M, N) = \mathrm{Hom}_{K(Mod_{(A,d)})}(M, I)$$

Proof. Let $P \to M$ be as in (1). Since $P \to M$ is a quasi-isomorphism we see that

$$\mathrm{Hom}_{D(A,\mathrm{d})}(P, N) = \mathrm{Hom}_{D(A,\mathrm{d})}(M, N)$$

by definition of the derived category. A morphism $f : P \to N$ in $D(A, \mathrm{d})$ is equal to $s^{-1}f'$ where $f' : P \to N'$ is a morphism and $s : N \to N'$ is a quasi-isomorphism. Choose a distringuished triangle

$$N \to N' \to Q \to N[1]$$

As s is a quasi-isomorphism, we see that Q is acyclic. Thus $\mathrm{Hom}_{K(\mathrm{Mod}_{(A,\mathrm{d})})}(P, Q[k]) = 0$ for all k by Lemma 22.13.2. Since $\mathrm{Hom}_{K(\mathrm{Mod}_{(A,\mathrm{d})})}(P, -)$ is cohomological, we conclude that we can lift $f' : P \to N'$ uniquely to a morphism $f : P \to N$. This finishes the proof.

The proof of (2) is dual to that of (1) using Lemma 22.14.2 in stead of Lemma 22.13.2. □

Lemma 22.15.4. *Let (A, d) be a differential graded algebra. Then*

(1) *$D(A, d)$ has both direct sums and products,*
(2) *direct sums are obtained by taking direct sums of differential graded modules,*
(3) *products are obtained by taking products of differential graded modules.*

Proof. We will use that $\mathrm{Mod}_{(A,\mathrm{d})}$ is an abelian category with arbitrary direct sums and products, and that these give rise to direct sums and products in $K(\mathrm{Mod}_{(A,\mathrm{d})})$. See Lemmas 22.4.2 and 22.5.4.

Let M_j be a family of differential graded A-modules. Consider the graded direct sum $M = \bigoplus M_j$ which is a differential graded A-module with the obvious. For a differential graded A-module N choose a quasi-isomorphism $N \to I$ where I is a differential graded A-module with property (I). See Lemma 22.14.4. Using Lemma 22.15.3 we have

$$\begin{aligned}
\mathrm{Hom}_{D(A,\mathrm{d})}(M, N) &= \mathrm{Hom}_{K(A,\mathrm{d})}(M, I) \\
&= \prod \mathrm{Hom}_{K(A,\mathrm{d})}(M_j, I) \\
&= \prod \mathrm{Hom}_{D(A,\mathrm{d})}(M_j, N)
\end{aligned}$$

whence the existence of direct sums in $D(A, \mathrm{d})$ as given in part (2) of the lemma.

Let M_j be a family of differential graded A-modules. Consider the product $M = \prod M_j$ of differential graded A-modules. For a differential graded A-module N choose a quasi-isomorphism $P \to N$ where P is a differential graded A-module with property (P). See Lemma 22.13.4. Using Lemma 22.15.3 we have

$$\begin{aligned}
\mathrm{Hom}_{D(A,\mathrm{d})}(N, M) &= \mathrm{Hom}_{K(A,\mathrm{d})}(P, M) \\
&= \prod \mathrm{Hom}_{K(A,\mathrm{d})}(P, M_j) \\
&= \prod \mathrm{Hom}_{D(A,\mathrm{d})}(N, M_j)
\end{aligned}$$

whence the existence of direct sums in $D(A, \mathrm{d})$ as given in part (3) of the lemma. □

22.16. The canonical delta-functor

Let (A, d) be a differential graded algebra. Consider the functor $\mathrm{Mod}(\mathcal{A}) \to K(\mathrm{Mod}_{(A,\mathrm{d})})$. This functor is **not** a δ-functor in general. However, it turns out that the functor $\mathrm{Mod}_{(A,\mathrm{d})} \to D(A, \mathrm{d})$ is a δ-functor. In order to see this we have to define the morphisms δ associated to a short exact sequence

$$0 \to K \xrightarrow{a} L \xrightarrow{b} M \to 0$$

in the abelian category $\mathrm{Mod}_{(A,\mathrm{d})}$. Consider the cone $C(a)$ of the morphism a. We have $C(a) = L \oplus K$ and we define $q : C(a) \to M$ via the projection to L followed by b. Hence a homomorphism of differential graded A-modules

$$q : C(a) \longrightarrow M.$$

It is clear that $q \circ i = b$ where i is as in Definition 22.6.1. Note that, as a is injective, the kernel of q is identified with the cone of id_K which is acyclic. Hence we see that q is a quasi-isomorphism. According to Lemma 22.9.4 the triangle

$$(K, L, C(a), a, i, -p)$$

is a distinguished triangle in $K(\mathrm{Mod}_{(A,\mathrm{d})})$. As the localization functor $K(\mathrm{Mod}_{(A,\mathrm{d})}) \to D(A, \mathrm{d})$ is exact we see that $(K, L, C(a), a, i, -p)$ is a distinguished triangle in $D(A, \mathrm{d})$. Since q is a quasi-isomorphism we see that q is an isomorphism in $D(A, \mathrm{d})$. Hence we deduce that

$$(K, L, M, a, b, -p \circ q^{-1})$$

is a distinguished triangle of $D(A, \mathrm{d})$. This suggests the following lemma.

Lemma 22.16.1. *Let (A, d) be a differential graded algebra. The functor $Mod_{(A,d)} \to D(A, d)$ defined has the natural structure of a δ-functor, with*

$$\delta_{K \to L \to M} = -p \circ q^{-1}$$

with p and q as explained above.

Proof. We have already seen that this choice leads to a distinguished triangle whenever given a short exact sequence of complexes. We have to show functoriality of this construction, see Derived Categories, Definition 13.3.6. This follows from Lemma 22.6.2 with a bit of work. Compare with Derived Categories, Lemma 13.11.1. □

22.17. Linear categories

Just the definitions.

Definition 22.17.1. Let R be a ring. An *R-linear category* $\mathcal{A}$ is a category where every morphism set is given the structure of an R-module and where for $x, y, z \in \mathrm{Ob}(\mathcal{A})$ composition law

$$\mathrm{Hom}_{\mathcal{A}}(y, z) \times \mathrm{Hom}_{\mathcal{A}}(x, y) \longrightarrow \mathrm{Hom}_{\mathcal{A}}(x, z)$$

is R-bilinear.

Thus composition determines an R-linear map

$$\mathrm{Hom}_{\mathcal{A}}(y, z) \otimes_R \mathrm{Hom}_{\mathcal{A}}(x, y) \longrightarrow \mathrm{Hom}_{\mathcal{A}}(x, z)$$

of R-modules. Note that we do not assume R-linear categories to be additive.

Definition 22.17.2. Let R be a ring. A *functor of R-linear categories*, or an *R-linear* is a functor $F : \mathcal{A} \to \mathcal{B}$ where for all objects x, y of $\mathcal{A}$ the map $F : \mathrm{Hom}_{\mathcal{A}}(x, y) \to \mathrm{Hom}_{\mathcal{A}}(F(x), F(y))$ is a homomorphism of R-modules.

22.18. Graded categories

Just some definitions.

Definition 22.18.1. Let R be a ring. A *graded category $\mathcal{A}$ over R* is a category where every morphism set is given the structure of a graded R-module and where for $x, y, z \in \mathrm{Ob}(\mathcal{A})$ composition is R-bilinear and induces a homomorphism

$$\mathrm{Hom}_{\mathcal{A}}(y, z) \otimes_R \mathrm{Hom}_{\mathcal{A}}(x, y) \longrightarrow \mathrm{Hom}_{\mathcal{A}}(x, z)$$

of graded R-modules (i.e., preserving degrees).

In this situation we denote $\mathrm{Hom}^i_{\mathcal{A}}(x, y)$ the degree i part of the graded object $\mathrm{Hom}_{\mathcal{A}}(x, y)$, so that

$$\mathrm{Hom}_{\mathcal{A}}(x, y) = \bigoplus\nolimits_{i \in \mathbf{Z}} \mathrm{Hom}^i_{\mathcal{A}}(x, y)$$

is the direct sum decomposition into graded parts.

Definition 22.18.2. Let R be a ring. A *functor of graded categories over R*, or a *graded functor* is a functor $F : \mathcal{A} \to \mathcal{B}$ where for all objects x, y of $\mathcal{A}$ the map $F : \mathrm{Hom}_{\mathcal{A}}(x, y) \to \mathrm{Hom}_{\mathcal{A}}(F(x), F(y))$ is a homomorphism of graded R-modules.

Given a graded category we are often interested in the corresponding "usual" category of maps of degree 0. Here is a formal definition.

Definition 22.18.3. Let R be a ring. Let $\mathcal{A}$ be a differential graded category over R. We let $\mathcal{A}^0$ be the category with the same objects as $\mathcal{A}$ and with

$$\mathrm{Hom}_{\mathcal{A}^0}(x, y) = \mathrm{Hom}^0_{\mathcal{A}}(x, y)$$

the degree 0 graded piece of the graded module of morphisms of $\mathcal{A}$.

Definition 22.18.4. Let R be a ring. Let $\mathcal{A}$ be a graded category over R. A direct sum (x, y, z, i, j, p, q) in $\mathcal{A}$ (notation as in Homology, Remark 12.3.6) is a *graded direct sum* if i, j, p, q are homogeneous of degree 0.

Example 22.18.5 (Graded category of graded objects)**.** Let $\mathcal{B}$ be an additive category. Recall that we have defined the category $\mathrm{Gr}(\mathcal{B})$ of graded objects of $\mathcal{B}$ in Homology, Definition 12.15.1. In this example, we will construct a graded category $\mathrm{Gr}^{gr}(\mathcal{B})$ over $R = \mathbf{Z}$ whose associated category $\mathrm{Gr}^{gr}(\mathcal{B})^0$ recovers $\mathrm{Gr}(\mathcal{B})$. As objects of $\mathrm{Comp}^{gr}(\mathcal{B})$ we take graded objects of $\mathcal{B}$. Then, given graded objects $A = (A^i)$ and $B = (B^i)$ of $\mathcal{B}$ we set

$$\mathrm{Hom}_{\mathrm{Gr}^{gr}(\mathcal{B})}(A, B) = \bigoplus\nolimits_{n \in \mathbf{Z}} \mathrm{Hom}^n(A, B)$$

where the graded piece of degree n is the abelian group of homogeneous maps of degree n from A to B defined by the rule

$$\mathrm{Hom}^n(A, B) = \mathrm{Hom}_{\mathrm{Gr}(\mathcal{A})}(A, B[n]) = \mathrm{Hom}_{\mathrm{Gr}(\mathcal{A})}(A[-n], B)$$

see Homology, Equation (12.15.3.1). Explicitly we have

$$\mathrm{Hom}^n(A, B) = \prod\nolimits_{p+q=n} \mathrm{Hom}_{\mathcal{B}}(A^{-q}, B^p)$$

(observe reversal of indices and observe that we have a product here and not a direct sum). In other words, a degree n morphism f from A to B can be seen as a system $f = (f_{p,q})$ where $p, q \in \mathbf{Z}$, $p + q = n$ with $f_{p,q} : A^{-q} \to B^p$ a morphism of $\mathcal{B}$. Given graded objects A, B, C of $\mathcal{B}$ composition of morphisms in $\mathrm{Gr}^{gr}(\mathcal{B})$ is defined via the maps

$$\mathrm{Hom}^m(B, C) \times \mathrm{Hom}^n(A, B) \longrightarrow \mathrm{Hom}^{n+m}(A, C)$$

by simple composition $(g, f) \mapsto g \circ f$ of homogeneous maps of graded objects. In terms of components we have

$$(g \circ f)_{p,r} = g_{p,q} \circ f_{-q,r}$$

where q is such that $p + q = m$ and $-q + r = n$.

Example 22.18.6 (Graded category of graded modules)**.** Let A be a $\mathbf{Z}$-graded algebra over a ring R. We will construct a graded category Mod_A^{gr} over R whose associated category $(\mathrm{Mod}_A^{gr})^0$ is the category of graded A-modules. As objects of Mod_A^{gr} we take right graded A-modules (see Section 22.11). Given graded A-modules L and M we set

$$\mathrm{Hom}_{\mathrm{Mod}_A^{gr}}(L, M) = \bigoplus\nolimits_{n \in \mathbf{Z}} \mathrm{Hom}^n(L, M)$$

where $\mathrm{Hom}^n(L, M)$ is the set of right A-module maps $L \to M$ which are homogeneous of degree n, i.e., $f(L^i) \subset M^{i+n}$ for all $i \in \mathbf{Z}$. In terms of components, we have that

$$\mathrm{Hom}^n(L, M) \subset \prod\nolimits_{p+q=n} \mathrm{Hom}_R(L^{-q}, M^p)$$

(observe reversal of indices) is the subset consisting of those $f = (f_{p,q})$ such that

$$f_{p,q}(ma) = f_{p-i,q+i}(m)a$$

for $a \in A^i$ and $m \in L^{-q-i}$. For graded A-modules K, L, M we define composition in Mod_A^{gr} via the maps

$$\mathrm{Hom}^m(L, M) \times \mathrm{Hom}^n(K, L) \longrightarrow \mathrm{Hom}^{n+m}(K, M)$$

by simple composition of right A-module maps: $(g, f) \mapsto g \circ f$.

Remark 22.18.7. Let R be a ring. Let $\mathcal{D}$ be an R-linear category endowed with a collection of R-linear functors $[n] : \mathcal{D} \to \mathcal{D}$, $x \mapsto x[n]$ indexed by $n \in \mathbf{Z}$ such that $[n] \circ [m] = [n + m]$ and $[0] = \mathrm{id}_{\mathcal{D}}$ (equality as functors). This allows us to construct a graded category $\mathcal{D}^{gr}$ over R with the same objects of $\mathcal{D}$ setting

$$\mathrm{Hom}_{\mathcal{D}^{gr}}(x, y) = \bigoplus\nolimits_{n \in \mathbf{Z}} \mathrm{Hom}_{\mathcal{D}}(x, y[n])$$

for x, y in $\mathcal{D}$. Observe that $(\mathcal{D}^{gr})^0 = \mathcal{D}$ (see Definition 22.18.3). Moreover, the graded category $\mathcal{D}^{gr}$ inherits R-linear graded functors $[n]$ satisfying $[n] \circ [m] = [n + m]$ and $[0] = \mathrm{id}_{\mathcal{D}^{gr}}$ with the property that

$$\mathrm{Hom}_{\mathcal{D}^{gr}}(x, y[n]) = \mathrm{Hom}_{\mathcal{D}^{gr}}(x, y)[n]$$

as graded R-modules compatible with composition of morphisms.

Conversely, suppose given a graded category $\mathcal{A}$ over R endowed with a collection of R-linear graded functors $[n]$ satisfying $[n] \circ [m] = [n + m]$ and $[0] = \mathrm{id}_{\mathcal{A}}$ which are moreover equipped with isomorphisms

$$\mathrm{Hom}_{\mathcal{A}}(x, y[n]) = \mathrm{Hom}_{\mathcal{A}}(x, y)[n]$$

as graded R-modules compatible with composition of morphisms. Then the reader easily shows that $\mathcal{A} = (\mathcal{A}^0)^{gr}$.

Here are two examples of the relationship $\mathcal{D} \leftrightarrow \mathcal{A}$ we established above:

(1) Let $\mathcal{B}$ be an additive category. If $\mathcal{D} = \mathrm{Gr}(\mathcal{B})$, then $\mathcal{A} = \mathrm{Gr}^{gr}(\mathcal{B})$ as in Example 22.18.5.
(2) If A is a graded ring and $\mathcal{D} = \mathrm{Mod}_A$ is the category of graded right A-modules, then $\mathcal{A} = \mathrm{Mod}_A^{gr}$, see Example 22.18.6.

22.19. Differential graded categories

Note that if R is a ring, then R is a differential graded algebra over itself (with $R = R^0$ of course). In this case a differential graded R-module is the same thing as a complex of R-modules. In particular, given two differential graded R-modules M and N we denote $M \otimes_R N$ the differential graded R-module corresponding to the total complex associated to the double complex obtained by the tensor product of the complexes of R-modules associated to M and N.

Definition 22.19.1. Let R be a ring. A *differential graded category $\mathcal{A}$ over R* is a category where every morphism set is given the structure of a differential graded R-module and where for $x, y, z \in \mathrm{Ob}(\mathcal{A})$ composition is R-bilinear and induces a homomorphism

$$\mathrm{Hom}_{\mathcal{A}}(y, z) \otimes_R \mathrm{Hom}_{\mathcal{A}}(x, y) \longrightarrow \mathrm{Hom}_{\mathcal{A}}(x, z)$$

of differential graded R-modules.

The final condition of the definition signifies the following: if $f \in \mathrm{Hom}_{\mathcal{A}}^n(x, y)$ and $g \in \mathrm{Hom}_{\mathcal{A}}^m(y, z)$ are homogeneous of degrees n and m, then

$$\mathrm{d}(g \circ f) = \mathrm{d}(g) \circ f + (-1)^m g \circ \mathrm{d}(f)$$

in $\mathrm{Hom}_{\mathcal{A}}^{n+m+1}(x, z)$. This follows from the sign rule for the differential on the total complex of a double complex, see Homology, Definition 12.22.2.

Definition 22.19.2. Let R be a ring. A *functor of differential graded categories over R* is a functor $F : \mathcal{A} \to \mathcal{B}$ where for all objects x, y of $\mathcal{A}$ the map $F : \mathrm{Hom}_{\mathcal{A}}(x, y) \to \mathrm{Hom}_{\mathcal{A}}(F(x), F(y))$ is a homomorphism of differential graded R-modules.

Given a diffferential graded category we are often interested in the corresponding categories of complexes and homotopy category. Here is a formal definition.

Definition 22.19.3. Let R be a ring. Let $\mathcal{A}$ be a differential graded category over R. Then we let

(1) the *category of complexes of* $\mathcal{A}$[1] be the category $\mathrm{Comp}(\mathcal{A})$ whose objects are the same as the objects of $\mathcal{A}$ and with
$$\mathrm{Hom}_{\mathrm{Comp}(\mathcal{A})}(x, y) = \mathrm{Ker}(d : \mathrm{Hom}_{\mathcal{A}}^0(x, y) \to \mathrm{Hom}_{\mathcal{A}}^0(x, y))$$
(2) the *homotopy category of* $\mathcal{A}$ be the category $K(\mathcal{A})$ whose objects are the same as the objects of $\mathcal{A}$ and with
$$\mathrm{Hom}_{\mathrm{Comp}(\mathcal{A})}(x, y) = H^0(\mathrm{Hom}_{\mathcal{A}}(x, y))$$

[1]This may be nonstandard terminology.

Our use of the symbol $K(\mathcal{A})$ is nonstandard, but at least is compatible with the use of $K(-)$ in other chapters of the Stacks project.

Definition 22.19.4. Let R be a ring. Let $\mathcal{A}$ be a differential graded category over R. A direct sum (x, y, z, i, j, p, q) in $\mathcal{A}$ (notation as in Homology, Remark 12.3.6) is a *differential graded direct sum* if i, j, p, q are homogeneous of degree 0 and closed, i.e., $\mathrm{d}(i) = 0$, etc.

Lemma 22.19.5. *Let R be a ring. A functor $F : \mathcal{A} \to \mathcal{B}$ of differential graded categories over R induces functors $Comp(\mathcal{A}) \to Comp(\mathcal{B})$ and $K(\mathcal{A}) \to K(\mathcal{B})$.*

Proof. Omitted. □

Example 22.19.6 (Differential graded category of complexes)**.** Let $\mathcal{B}$ be an additive category. We will construct a differential graded category $\mathrm{Comp}^{dg}(\mathcal{B})$ over $R = \mathbf{Z}$ whose associated category of complexes is $\mathrm{Comp}(\mathcal{B})$ and whose associated homotopy category is $K(\mathcal{B})$. As objects of $\mathrm{Comp}^{dg}(\mathcal{B})$ we take complexes of $\mathcal{B}$. Given complexes $A^\bullet$ and $B^\bullet$ of $\mathcal{B}$, we sometimes also denote $A^\bullet$ and $B^\bullet$ the corresponding graded objects of $\mathcal{B}$ (i.e., forget about the differential). Using this abuse of notation, we set

$$\mathrm{Hom}_{\mathrm{Comp}^{dg}(\mathcal{B})}(A^\bullet, B^\bullet) = \mathrm{Hom}_{\mathrm{Gr}^{gr}(\mathcal{B})}(A^\bullet, B^\bullet)$$

as a graded $\mathbf{Z}$-module where the right hand side is defined in Example 22.18.5. In other words, the nth graded piece is the abelian group of homogeneous morphism of degree n of graded objects

$$\mathrm{Hom}^n(A^\bullet, B^\bullet) = \mathrm{Hom}_{\mathrm{Gr}(\mathcal{B})}(A^\bullet, B^\bullet[n]) = \prod\nolimits_{p+q=n} \mathrm{Hom}_{\mathcal{B}}(A^{-q}, B^p)$$

(observe reversal of indices and observe we have a direct product and not a direct sum). For an element $f \in \mathrm{Hom}^n(A^\bullet, B^\bullet)$ of degree n we set

$$\mathrm{d}(f) = \mathrm{d}_B \circ f - (-1)^n f \circ \mathrm{d}_A$$

To make sense of this we think of d_B and d_A as maps of graded objects of $\mathcal{B}$ homogeneous of degree 1 and we use composition in the category $\mathrm{Gr}^{gr}(\mathcal{B})$ on the right hand side. In terms of components, if $f = (f_{p,q})$ with $f_{p,q} : A^{-q} \to B^p$ we have

$$\mathrm{d}(f_{p,q}) = \mathrm{d}_B \circ f_{p,q} + (-1)^{p+q+1} f_{p,q} \circ \mathrm{d}_A \tag{22.19.6.1}$$

Note that the first term of this expression is in $\mathrm{Hom}_{\mathcal{B}}(A^{-q}, B^{p+1})$ and the second term is in $\mathrm{Hom}_{\mathcal{B}}(A^{-q-1}, B^p)$. In other words, given $p + q = n + 1$ we have

$$\mathrm{d}(f)_{p,q} = \mathrm{d}_B \circ f_{p-1,q} - (-1)^n f_{p,q-1} \circ \mathrm{d}_A$$

with obvious notation. The reader checks[2] that

(1) d has square zero,
(2) an element f in $\mathrm{Hom}^n(A^\bullet, B^\bullet)$ has $\mathrm{d}(f) = 0$ if and only if the morphism $f : A^\bullet \to B^\bullet[n]$ of graded objects of $\mathcal{B}$ is actually a map of complexes,

[2]What may be useful here is to think of the double complex $H^{\bullet,\bullet}$ with terms $H^{p,q} = \mathrm{Hom}_{\mathcal{B}}(A^{-q}, B^p)$ and differentials d_1 of degree $(1, 0)$ given by d_B and d_2 of degree $(0, 1)$ given by the contragredient of d_A. Up to sign and up to replacing the direct sum by a direct product, the differential graded $\mathbf{Z}$-module $\mathrm{Hom}_{\mathrm{Comp}^{dg}(\mathcal{B})}(A^\bullet, B^\bullet)$ is the total complex associated to $H^{\bullet,\bullet}$, see Homology, Definition 12.22.2. To get the sign correct, change $d_2^{p,q} : H^{p,q} \to H^{p,q+1}$ by $(-1)^{q+1}$ (after this change we still have a double complex).

(3) in particular, the category of complexes of $\mathrm{Comp}^{dg}(\mathcal{B})$ is equal to $\mathrm{Comp}(\mathcal{B})$,
(4) the morphism of complexes defined by f as in (2) is homotopy equivalent to zero if and only if $f = \mathrm{d}(g)$ for some $g \in \mathrm{Hom}^{n-1}(A^\bullet, B^\bullet)$.
(5) in particular, we obtain a canonical isomorphism

$$\mathrm{Hom}_{K(\mathcal{B})}(A^\bullet, B^\bullet) \longrightarrow H^0(\mathrm{Hom}_{\mathrm{Comp}^{dg}(\mathcal{B})}(A^\bullet, B^\bullet))$$

and the homotopy category of $\mathrm{Comp}^{dg}(\mathcal{B})$ is equal to $K(\mathcal{B})$.

Given complexes $A^\bullet$, $B^\bullet$, $C^\bullet$ we define composition

$$\mathrm{Hom}^m(B^\bullet, C^\bullet) \times \mathrm{Hom}^n(A^\bullet, B^\bullet) \longrightarrow \mathrm{Hom}^{n+m}(A^\bullet, C^\bullet)$$

by composition $(g, f) \mapsto g \circ f$ in the graded category $\mathrm{Gr}^{gr}(\mathcal{B})$, see Example 22.18.5. This defines a map of differential graded modules as in Definition 22.19.1 because

$$\begin{aligned} \mathrm{d}(g \circ f) &= \mathrm{d}_C \circ g \circ f - (-1)^{n+m} g \circ f \circ \mathrm{d}_A \\ &= (\mathrm{d}_C \circ g - (-1)^m g \circ \mathrm{d}_B) \circ f + (-1)^m g \circ (\mathrm{d}_B \circ f - (-1)^n f \circ \mathrm{d}_A) \\ &= \mathrm{d}(g) \circ f + (-1)^m g \circ \mathrm{d}(f) \end{aligned}$$

as desired.

Lemma 22.19.7. *Let $F : \mathcal{B} \to \mathcal{B}'$ be an additive functor between additive categories. Then F induces a functor of differential graded categories*

$$F : \mathit{Comp}^{dg}(\mathcal{B}) \to \mathit{Comp}^{dg}(\mathcal{B}')$$

of Example 22.19.6 inducing the usual functors on the category of complexes and the homotopy categories.

Proof. Omitted. □

Example 22.19.8 (Differential graded category of differential graded modules)**.** Let (A, d) be a differential graded algebra over a ring R. We will construct a differential graded category $\mathrm{Mod}^{dg}_{(A,\mathrm{d})}$ over R whose category of complexes is $\mathrm{Mod}_{(A,\mathrm{d})}$ and whose homotopy category is $K(\mathrm{Mod}_{(A,\mathrm{d})})$. As objects of $\mathrm{Mod}^{dg}_{(A,\mathrm{d})}$ we take the differential graded A-modules. Given differential graded A-modules L and M we set

$$\mathrm{Hom}_{\mathrm{Mod}^{dg}_{(A,\mathrm{d})}}(L, M) = \mathrm{Hom}_{\mathrm{Mod}^{gr}_{A}}(L, M) = \bigoplus \mathrm{Hom}^n(L, M)$$

as a graded R-module where the right hand side is defined as in Example 22.18.6. In other words, the nth graded piece $\mathrm{Hom}^n(L, M)$ is the R-module of right A-module maps homogeneous of degree n. For an element $f \in \mathrm{Hom}^n(L, M)$ we set

$$\mathrm{d}(f) = \mathrm{d}_M \circ f - (-1)^n f \circ \mathrm{d}_L$$

To make sense of this we think of d_M and d_L as graded R-module maps and we use composition of graded R-module maps. It is clear that $\mathrm{d}(f)$ is homogeneous of degree $n + 1$ as a graded R-module map, and it is linear because

$$\begin{aligned} \mathrm{d}(f)(xa) &= \mathrm{d}_M(f(x)a) - (-1)^n f(\mathrm{d}_L(xa)) \\ &= \mathrm{d}_M(f(x))a + (-1)^{\deg(x)+n} f(x)\mathrm{d}(a) - (-1)^n f(\mathrm{d}_L(x))a - (-1)^{n+\deg(x)} f(x)\mathrm{d}(a) \\ &= \mathrm{d}(f)(x)a \end{aligned}$$

as desired (observe that this calculation would not work without the sign in the definition of our differential on Hom). Similar formulae to those of Example 22.19.6

hold for the differential of f in terms of components. The reader checks (in the same way as in Example 22.19.6) that

(1) d has square zero,
(2) an element f in $\mathrm{Hom}^n(L, M)$ has $\mathrm{d}(f) = 0$ if and only if $f : L \to M[n]$ is a homomorphism of differential graded A-modules,
(3) in particular, the category of complexes of $\mathrm{Mod}^{dg}_{(A,\mathrm{d})}$ is $\mathrm{Mod}_{(A,\mathrm{d})}$,
(4) the homomorphism defined by f as in (2) is homotopy equivalent to zero if and only if $f = \mathrm{d}(g)$ for some $g \in \mathrm{Hom}^{n-1}(L, M)$.
(5) in particular, we obtain a canonical isomorphism

$$\mathrm{Hom}_{K(\mathrm{Mod}_{(A,\mathrm{d})})}(L, M) \longrightarrow H^0(\mathrm{Hom}_{\mathrm{Mod}^{dg}_{(A,\mathrm{d})}}(L, M))$$

and the homotopy category of $\mathrm{Mod}^{dg}_{(A,\mathrm{d})}$ is $K(\mathrm{Mod}_{(A,\mathrm{d})})$.

Given differential graded A-modules K, L, M we define composition

$$\mathrm{Hom}^m(L, M) \times \mathrm{Hom}^n(K, L) \longrightarrow \mathrm{Hom}^{n+m}(K, M)$$

by composition of homogeneous right A-module maps $(g, f) \mapsto g \circ f$. This defines a map of differential graded modules as in Definition 22.19.1 because

$$\begin{aligned} \mathrm{d}(g \circ f) &= \mathrm{d}_M \circ g \circ f - (-1)^{n+m} g \circ f \circ \mathrm{d}_K \\ &= (\mathrm{d}_M \circ g - (-1)^m g \circ \mathrm{d}_L) \circ f + (-1)^m g \circ (\mathrm{d}_L \circ f - (-1)^n f \circ \mathrm{d}_K) \\ &= \mathrm{d}(g) \circ f + (-1)^m g \circ \mathrm{d}(f) \end{aligned}$$

as desired.

Lemma 22.19.9. *Let $\varphi : (A, d) \to (E, d)$ be a homomorphism of differential graded algebras. Then φ induces a functor of differential graded categories*

$$F : Mod^{dg}_{(E,d)} \longrightarrow Mod^{dg}_{(A,d)}$$

of Example 22.19.8 inducing obvious restriction functors on the categories of differential graded modules and homotopy categories.

Proof. Omitted. □

Lemma 22.19.10. *Let R be a ring. Let $\mathcal{A}$ be a differential graded category over R. Let x be an object of $\mathcal{A}$. Let*

$$(E, d) = \mathrm{Hom}_{\mathcal{A}}(x, x)$$

be the differential graded R-algebra of endomorphisms of x. We obtain a functor

$$\mathcal{A} \longrightarrow Mod^{dg}_{(E,d)}, \quad y \longmapsto \mathrm{Hom}_{\mathcal{A}}(x, y)$$

of differential graded categories by letting E act on $\mathrm{Hom}_{\mathcal{A}}(x, y)$ via composition in $\mathcal{A}$. This functor induces functors

$$Comp(\mathcal{A}) \to Mod_{(A,d)} \quad \text{and} \quad K(\mathcal{A}) \to K(Mod_{(A,d)})$$

by an application of Lemma 22.19.5.

Proof. This lemma proves itself. □

22.20. Obtaining triangulated categories

In this section we discuss the most general setup to which the arguments proving Derived Categories, Proposition 22.10.3 and Proposition 22.10.3 apply.

Let R be a ring. Let $\mathcal{A}$ be a differential graded category over R. To make our argument work, we impose some axioms on $\mathcal{A}$:

(A) $\mathcal{A}$ has a zero object and differential graded direct sums of two objects (as in Definition 22.19.4).

(B) there are functors $[n] : \mathcal{A} \longrightarrow \mathcal{A}$ of differential graded categories such that $[0] = \mathrm{id}_{\mathcal{A}}$ and $[n + m] = [n] \circ [m]$ and given isomorphisms
$$\mathrm{Hom}_{\mathcal{A}}(x, y[n]) = \mathrm{Hom}_{\mathcal{A}}(x, y)[n]$$
of differential graded R-modules compatible with composition.

Given our differential graded category $\mathcal{A}$ we say

(1) a sequence $x \to y \to z$ of morphisms of $\mathrm{Comp}(\mathcal{A})$ is an *admissible short exact sequence* if there exists an isomorphism $y \cong x \oplus z$ in the underlying graded category such that $x \to z$ and $y \to z$ are (co)projections.

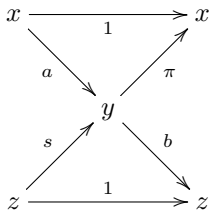
(2) a morphism $x \to y$ of $\mathrm{Comp}(\mathcal{A})$ is an *admissible monomorphism* if it extends to an admissible short exact sequence $x \to y \to z$.

(3) a morphism $y \to z$ of $\mathrm{Comp}(\mathcal{A})$ is an *admissible epimorphism* if it extends to an admissible short exact sequence $x \to y \to z$.

The next lemma tells us an admissible short exact sequence gives a triangle, provided we have axioms (A) and (B).

Lemma 22.20.1. *Let $\mathcal{A}$ be a differential graded category satisfying axioms (A) and (B). Given an admissible short exact sequence $x \to y \to z$ we obtain (see proof) a triangle*
$$x \to y \to z \to x[1]$$
in Comp($\mathcal{A}$) with the property that any two compositions in $z[-1] \to x \to y \to z \to x[1]$ are zero in $K(\mathcal{A})$.

Proof. Choose a diagram



giving the isomorphism of graded objects $y \cong x \oplus z$ as in the defintion of an admissible short exact sequence. Here are some equations that hold in this situation

(1) $1 = \pi a$ and hence $\mathrm{d}(\pi)a = 0$,
(2) $1 = bs$ and hence $b\mathrm{d}(s) = 0$,
(3) $1 = a\pi + sb$ and hence $a\mathrm{d}(\pi) + \mathrm{d}(s)b = 0$,
(4) $\pi s = 0$ and hence $\mathrm{d}(\pi)s + \pi\mathrm{d}(s) = 0$,
(5) $\mathrm{d}(s) = a\pi\mathrm{d}(s)$ because $\mathrm{d}(s) = (a\pi + sb)\mathrm{d}(s)$ and $b\mathrm{d}(s) = 0$,
(6) $\mathrm{d}(\pi) = \mathrm{d}(\pi)sb$ because $\mathrm{d}(\pi) = \mathrm{d}(\pi)(a\pi + sb)$ and $\mathrm{d}(\pi)a = 0$,
(7) $\mathrm{d}(\pi\mathrm{d}(s)) = 0$ because if we postcompose it with the monomorphism a we get $\mathrm{d}(a\pi\mathrm{d}(s)) = \mathrm{d}(\mathrm{d}(s)) = 0$, and

(8) $\mathrm{d}(\mathrm{d}(\pi)s) = 0$ as by (4) it is the negative of $\mathrm{d}(\pi\mathrm{d}(s))$ which is 0 by (7).

We've used repeatedly that $\mathrm{d}(a) = 0$, $\mathrm{d}(b) = 0$, and that $\mathrm{d}(1) = 0$. By (7) we see that

$$\delta = \pi\mathrm{d}(s) = -\mathrm{d}(\pi)s : z \to x[1]$$

is a morphism in $\mathrm{Comp}(\mathcal{A})$. By (5) we see that the composition $a\delta = a\pi\mathrm{d}(s) = \mathrm{d}(s)$ is homotopic to zero. By (6) we see that the composition $\delta b = -\mathrm{d}(\pi)sb = \mathrm{d}(-\pi)$ is homotopic to zero. □

Besides axioms (A) and (B) we need an axiom concerning the existence of cones. We formalize everything as follows.

Situation 22.20.2. Here R is a ring and $\mathcal{A}$ is a differential graded category over R having axioms (A), (B), and

(C) given an arrow $f : x \to y$ of degree 0 with $\mathrm{d}(f) = 0$ there exists an admissible short exact sequence $y \to c(f) \to x[1]$ in $\mathrm{Comp}(\mathcal{A})$ such that the map $x[1] \to y[1]$ of Lemma 22.20.1 is equal to $f[1]$.

We will call $c(f)$ a *cone* of the morphism f. If (A), (B), and (C) hold, then cones are functorial in a weak sense.

Lemma 22.20.3. *In Situation 22.20.2 suppose that*

$$\begin{array}{ccc} x_1 & \xrightarrow{f_1} & y_1 \\ {\scriptstyle a}\downarrow & & \downarrow{\scriptstyle b} \\ x_2 & \xrightarrow{f_2} & y_2 \end{array}$$

is a diagram of $\mathrm{Comp}(\mathcal{A})$ commutative up to homotopy. Then there exists a morphism $c : c(f_1) \to c(f_2)$ which gives rise to a morphism of triangles

$$(a, b, c) : (x_1, y_1, c(f_1)) \to (x_1, y_1, c(f_1))$$

in $K(\mathcal{A})$.

Proof. The assumption means there exists a morphism $h : x_1 \to y_2$ of degree -1 such that $\mathrm{d}(h) = bf_1 - f_2a$. Choose isomorphisms $c(f_i) = y_i \oplus x_i[1]$ of graded objects compatible with the morphisms $y_i \to c(f_i) \to x_i[1]$. Let's denote $a_i : y_i \to c(f_i)$, $b_i : c(f_i) \to x_i[1]$, $s_i : x_i[1] \to c(f_i)$, and $\pi_i : c(f_i) \to y_i$ the given morphisms. Recall that $x_i[1] \to y_i[1]$ is given by $\pi_i\mathrm{d}(s_i)$. By axiom (C) this means that

$$f_i = \pi_i\mathrm{d}(s_i) = -\mathrm{d}(\pi_i)s_i$$

(we identify $\mathrm{Hom}(x_i, y_i)$ with $\mathrm{Hom}(x_i[1], y_i[1])$ using the shift functor [1]). Set $c = a_2b\pi_1 + s_2ab_1 + a_2hb$. Then, using the equalities found in the proof of Lemma 22.20.1 we obtain

$$\begin{aligned} \mathrm{d}(c) &= a_2b\mathrm{d}(\pi_1) + \mathrm{d}(s_2)ab_1 + a_2\mathrm{d}(h)b_1 \\ &= -a_2bf_1b_1 + a_2f_2ab_1 + a_2(bf_1 - f_2a)b_1 \\ &= 0 \end{aligned}$$

(where we have used in particular that $\mathrm{d}(\pi_1) = \mathrm{d}(\pi_1)s_1b_1 = f_1b_1$ and $\mathrm{d}(s_2) = a_2\pi_2\mathrm{d}(s_2) = a_2f_2$). Thus c is a degree 0 morphism $c : c(f_1) \to c(f_2)$ of $\mathcal{A}$ compatible with the given morphisms $y_i \to c(f_i) \to x_i[1]$. □

In Situation 22.20.2 we say that a triangle (x, y, z, f, g, h) in $K(\mathcal{A})$ is a *distinguished triangle* if there exists an admissible short exact sequence $x' \to y' \to z'$ such that (x, y, z, f, g, h) is isomorphic as a triangle in $K(\mathcal{A})$ to the triangle $(x', y', z', x' \to y', y' \to z', \delta)$ constructed in Lemma 22.20.1. We will show below that

$$\boxed{K(\mathcal{A}) \text{ is a triangulated category}}$$

This result, although not as general as one might think, applies to a number of natural generalizations of the cases covered so far in the Stacks project. Here are some examples:

(1) Let $(X, \mathcal{O}_X)$ be a ringed space. Let (A, d) be a sheaf of differential graded $\mathcal{O}_X$-algebras. Let $\mathcal{A}$ be the differential graded category of differential graded A-modules. Then $K(\mathcal{A})$ is a triangulated category.
(2) Let $(\mathcal{C}, \mathcal{O})$ be a ringed site. Let (A, d) be a sheaf of differential graded $\mathcal{O}$-algebras. Let $\mathcal{A}$ be the differential graded category of differential graded A-modules. Then $K(\mathcal{A})$ is a triangulated category.
(3) Two examples with a different flavor may be found in Examples, Section 78.53.

The following simple lemma is a key to the construction.

Lemma 22.20.4. *In Situation 22.20.2 given any object x of $\mathcal{A}$, and the cone $C(1_x)$ of the identity morphism $1_x : x \to x$, the identity morphism on $C(1_x)$ is homotopic to zero.*

Proof. Consider the admissible short exact sequence given by axiom (C).

$$x \underset{\pi}{\overset{a}{\rightleftarrows}} C(1_x) \underset{s}{\overset{b}{\rightleftarrows}} x[1]$$

Then by Lemma 22.20.1, identifying hom-sets under shifting, we have $1_x = \pi d(s) = -d(\pi)s$ where s is regarded as a morphism in $\mathrm{Hom}^{-1}_{\mathcal{A}}(x, C(1_x))$. Therefore $a = a\pi d(s) = d(s)$ using formula (5) of Lemma 22.20.1, and $b = -d(\pi)sb = -d(\pi)$ by formula (6) of Lemma 22.20.1. Hence

$$1_{C(1_x)} = a\pi + sb = d(s)\pi - sd(\pi) = d(s\pi)$$

since s is of degree -1. □

A more general version of the above lemma will appear in Lemma 22.20.13. The following lemma is the analogue of Lemma 22.7.3.

Lemma 22.20.5. *In Situation 22.20.2 given a diagram*

$$\begin{array}{ccc} x & \xrightarrow{f} & y \\ {\scriptstyle a}\downarrow & & \downarrow{\scriptstyle b} \\ z & \xrightarrow{g} & w \end{array}$$

in Comp($\mathcal{A}$) commuting up to homotopy. Then

(1) *If f is an admissible monomorphism, then b is homotopic to a morphism b' which makes the diagram commute.*
(2) *If g is an admissible epimorphism, then a is homotopic to a morphism a' which makes the diagram commute.*

Proof. To prove (1), observe that the hypothesis implies that there is some $h \in \mathrm{Hom}_{\mathcal{A}}(x, w)$ of degree -1 such that $bf - ga = d(h)$. Since f is an admissible monomorphism, there is a morphism $\pi : y \to x$ in the category $\mathcal{A}$ of degree 0. Let $b' = b - d(h\pi)$. Then

$$\begin{aligned} b'f = bf - d(h\pi)f =& bf - d(h\pi f) \text{ (since } d(f) = 0) \\ =& bf - d(h) \\ =& ga \end{aligned}$$

as desired. The proof for (2) is omitted. □

The following lemma is the analogue of Lemma 22.7.4.

Lemma 22.20.6. *In Situation 22.20.2 let $\alpha : x \to y$ be a morphism in Comp($\mathcal{A}$). Then there exists a factorization in Comp($\mathcal{A}$):*

$$x \xrightarrow{\tilde{\alpha}} \tilde{y} \underset{s}{\overset{\pi}{\rightleftarrows}} y$$

such that

(1) *$\tilde{\alpha}$ is an admissible monomorphism, and $\pi\tilde{\alpha} = \alpha$.*
(2) *There exists a morphism $s : y \to \tilde{y}$ in Comp($\mathcal{A}$) such that $\pi s = 1_y$ and $s\pi$ is homotopic to $1_{\tilde{y}}$.*

Proof. By axiom (B), we may let $\tilde{y}$ be the differential graded direct sum of y and $C(1_x)$, i.e., there exists a diagram

$$y \underset{\pi}{\overset{s}{\rightleftarrows}} y \oplus C(1_x) \underset{t}{\overset{p}{\rightleftarrows}} C(1_x)$$

where all morphisms are of degree zero, and in Comp($\mathcal{A}$). Let $\tilde{y} = y \oplus C(1_x)$. Then $1_{\tilde{y}} = s\pi + tp$. Consider now the diagram

$$x \xrightarrow{\tilde{\alpha}} \tilde{y} \underset{s}{\overset{\pi}{\rightleftarrows}} y$$

where $\tilde{\alpha}$ is induced by the morphism $x \xrightarrow{\alpha} y$ and the natural morphism $x \to C(1_x)$ fitting in the admissible short exact sequence

$$x \rightleftarrows C(1_x) \rightleftarrows x[1]$$

So the morphism $C(1_x) \to x$ of degree 0 in this diagram, together with the zero morphism $y \to x$, induces a degree-0 morphism $\beta : \tilde{y} \to x$. $\tilde{\alpha}$ is an admissible monomorphism since it fits into the admissible short exact sequence

$$x \xrightarrow{\tilde{\alpha}} \tilde{y} \longrightarrow x[1]$$

Furthermore, $\pi\tilde{\alpha} = \alpha$ by the construction of $\tilde{\alpha}$, and $\pi s = 1_y$ by the first diagram. It remains to show that $s\pi$ is homotopic to $1_{\tilde{y}}$. Write 1_x as $d(h)$ for some degree -1 map. Then, our last statement follows from

$$\begin{aligned} 1_{\tilde{y}} - s\pi =& tp \\ =& t(dh)p \text{ (by Lemma 22.20.4)} \\ =& d(thp) \end{aligned}$$

since $dt = dp = 0$, and t is of degree zero. □

The following lemma is the analogue of Lemma 22.7.5.

Lemma 22.20.7. *In Situation 22.20.2 let $x_1 \to x_2 \to \cdots \to x_n$ be a sequence of composable morphisms in Comp($\mathcal{A}$). Then there exists a commutative diagram in Comp($\mathcal{A}$):*

$$\begin{array}{ccccccc} x_1 & \longrightarrow & x_2 & \longrightarrow & \cdots & \longrightarrow & x_n \\ \uparrow & & \uparrow & & & & \uparrow \\ y_1 & \longrightarrow & y_2 & \longrightarrow & \cdots & \longrightarrow & y_n \end{array}$$

such that each $y_i \to y_{i+1}$ is an admissible monomorphism and each $y_i \to x_i$ is a homotopy equivalence.

Proof. The case for $n = 1$ is trivial: one simply takes $y_1 = x_1$ and the identity morphism on x_1 is in particular a homotopy equivalence. The case $n = 2$ is given by Lemma 22.20.6. Suppose we have constructed the diagram up to x_{n-1}. We apply Lemma 22.20.6 to the composition $y_{n-1} \to x_{n-1} \to x_n$ to obtain y_n. Then $y_{n-1} \to y_n$ will be an admissible monomorphism, and $y_n \to x_n$ a homotopy equivalence. □

The following lemma is the analogue of Lemma 22.7.6.

Lemma 22.20.8. *In Situation 22.20.2 let $x_i \to y_i \to z_i$ be morphisms in $\mathcal{A}$ ($i = 1, 2, 3$) such that $x_2 \to y_2 \to z_2$ is an admissible short exact sequence. Let $b : y_1 \to y_2$ and $b' : y_2 \to y_3$ be morphisms in Comp($\mathcal{A}$) such that*

$$\begin{array}{ccccc} x_1 & \longrightarrow & y_1 & \longrightarrow & z_1 \\ {\scriptstyle 0}\downarrow & & \downarrow{\scriptstyle b} & & \downarrow{\scriptstyle 0} \\ x_2 & \longrightarrow & y_2 & \longrightarrow & z_2 \end{array} \quad \text{and} \quad \begin{array}{ccccc} x_2 & \longrightarrow & y_2 & \longrightarrow & z_2 \\ {\scriptstyle 0}\downarrow & & \downarrow{\scriptstyle b'} & & \downarrow{\scriptstyle 0} \\ x_3 & \longrightarrow & y_3 & \longrightarrow & z_3 \end{array}$$

commute up to homotopy. Then $b' \circ b$ is homotopic to 0.

Proof. By Lemma 22.20.5, we can replace b and b' by homotopic maps $\tilde{b}$ and $\tilde{b}'$, such that the right square of the left diagram commutes and the left square of the right diagram commutes. Say $b = \tilde{b} + d(h)$ and $b' = \tilde{b}' + d(h')$ for degree -1 morphisms h and h' in $\mathcal{A}$. Hence

$$b'b = \tilde{b}'\tilde{b} + d(\tilde{b}'h + h'\tilde{b} + h'd(h))$$

since $d(\tilde{b}) = d(\tilde{b}') = 0$, i.e. $b'b$ is homotopic to $\tilde{b}'\tilde{b}$. We now want to show that $\tilde{b}'\tilde{b} = 0$. Because $x_2 \xrightarrow{f} y_2 \xrightarrow{g} z_2$ is an admissible short exact sequence, there exist degree 0 morphisms $\pi : y_2 \to x_2$ and $s : z_2 \to y_2$ such that $\mathrm{id}_{y_2} = f\pi + sg$. Therefore

$$\tilde{b}'\tilde{b} = \tilde{b}'(f\pi + sg)\tilde{b} = 0$$

since $g\tilde{b} = 0$ and $\tilde{b}'f = 0$ as consequences of the two commuting squares. □

The following lemma is the analogue of Lemma 22.8.1.

Lemma 22.20.9. *In Situation 22.20.2 let $0 \to x \to y \to z \to 0$ be an admissible short exact sequence in Comp($\mathcal{A}$). The triangle*

$$x \longrightarrow y \longrightarrow z \xrightarrow{\delta} x[1]$$

with $\delta : z \to x[1]$ as defined in Lemma 22.20.1 is up to canonical isomorphism in $K(\mathcal{A})$, independent of the choices made in Lemma 22.20.1.

Proof. Suppose δ is defined by the splitting

$$x \underset{\pi}{\overset{a}{\rightleftarrows}} y \underset{s}{\overset{b}{\rightleftarrows}} z$$

and δ' is defined by the splitting with π', s' in place of π, s. Then

$$s' - s = (a\pi + sb)(s' - s) = a\pi s'$$

since $bs' = bs = 1_z$ and $\pi s = 0$. Similarly,

$$\pi' - \pi = (\pi' - \pi)(a\pi + sb) = \pi' sb$$

Since $\delta = \pi d(s)$ and $\delta' = \pi' d(s')$ as constructed in Lemma 22.20.1, we may compute

$$\delta' = \pi' d(s') = (\pi + \pi' sb) d(s + a\pi s') = \delta + d(\pi s')$$

using $\pi a = 1_x$, $ba = 0$, and $\pi' sbd(s') = \pi' sba\pi d(s') = 0$ by formula (5) in Lemma 22.20.1. □

The following lemma is the analogue of Lemma 22.9.1.

Lemma 22.20.10. *In Situation 22.20.2 let $f : x \to y$ be a morphism in $\mathit{Comp}(\mathcal{A})$. The triangle $(y, c(f), x[1], i, p, f[1])$ is the triangle associated to the admissible short exact sequence*

$$y \longrightarrow c(f) \longrightarrow x[1]$$

where the cone $c(f)$ is defined as in Lemma 22.20.1.

Proof. This follows from axiom (C). □

The following lemma is the analogue of Lemma 22.9.2.

Lemma 22.20.11. *In Situation 22.20.2 let $\alpha : x \to y$ and $\beta : y \to z$ define an admissible short exact sequence*

$$x \longrightarrow y \longrightarrow z$$

in $\mathit{Comp}(\mathcal{A})$. Let $(x, y, z, \alpha, \beta, \delta)$ be the associated triangle in $K(\mathcal{A})$. Then, the triangles

$$(z[-1], x, y, \delta[-1], \alpha, \beta) \quad \textit{and} \quad (z[-1], x, c(\delta[-1]), \delta[-1], i, p)$$

are isomorphic.

Proof. We have a diagram of the form

$$\begin{array}{ccccccc}
z[-1] & \xrightarrow{\delta[-1]} & x & \underset{\tilde{\alpha}}{\overset{\alpha}{\rightleftarrows}} & y & \underset{\tilde{\beta}}{\overset{\beta}{\rightleftarrows}} & z \\
\downarrow 1 & & \downarrow 1 & & \downarrow & & \downarrow 1 \\
z[-1] & \xrightarrow{\delta[-1]} & x & \underset{\tilde{i}}{\overset{i}{\rightleftarrows}} & c(\delta[-1]) & \underset{\tilde{p}}{\overset{p}{\rightleftarrows}} & z
\end{array}$$

with splittings to α, β, i, and p given by $\tilde{\alpha}, \tilde{\beta}, \tilde{i}$, and $\tilde{p}$ respectively. Define a morphism $y \to c(\delta[-1])$ by $i\tilde{\alpha} + \tilde{p}\beta$ and a morphism $c(\delta[-1]) \to y$ by $\alpha\tilde{i} + \tilde{\beta}p$. Let us

first check that these define morphisms in $\mathrm{Comp}(\mathcal{A})$. We remark that by identities from Lemma 22.20.1, we have the relation $\delta[-1] = \tilde{\alpha}d(\tilde{\beta}) = -d(\tilde{\alpha})\tilde{\beta}$ and the relation $\delta[-1] = \tilde{i}d(\tilde{p})$. Then

$$\begin{aligned} d(\tilde{\alpha}) &= d(\tilde{\alpha})\tilde{\beta}\beta \\ &= -\delta[-1]\beta \end{aligned}$$

where we have used equation (6) of Lemma 22.20.1 for the first equality and the preceeding remark for the second. Similarly, we obtain $d(\tilde{p}) = i\delta[-1]$. Hence

$$\begin{aligned} d(i\tilde{\alpha} + \tilde{p}\beta) &= d(i)\tilde{\alpha} + id(\tilde{\alpha}) + d(\tilde{p})\beta + \tilde{p}d(\beta) \\ &= id(\tilde{\alpha}) + d(\tilde{p})\beta \\ &= -i\delta[-1]\beta + i\delta[-1]\beta \\ &= 0 \end{aligned}$$

so $i\tilde{\alpha} + \tilde{p}\beta$ is indeed a morphism of $\mathrm{Comp}(\mathcal{A})$. By a similar calculation, $\alpha\tilde{i} + \tilde{\beta}p$ is also a morphism of $\mathrm{Comp}(\mathcal{A})$. It is immediate that these morphisms fit in the commutative diagram. We compute:

$$\begin{aligned} (i\tilde{\alpha} + \tilde{p}\beta)(\alpha\tilde{i} + \tilde{\beta}p) &= i\tilde{\alpha}\alpha\tilde{i} + i\tilde{\alpha}\tilde{\beta}p + \tilde{p}\beta\alpha\tilde{i} + \tilde{p}\beta\tilde{\beta}p \\ &= i\tilde{i} + \tilde{p}p \\ &= 1_{c(\delta[-1])} \end{aligned}$$

where we have freely used the identities of Lemma 22.20.1. Similarly, we compute $(\alpha\tilde{i} + \tilde{\beta}p)(i\tilde{\alpha} + \tilde{p}\beta) = 1_y$, so we conclude $y \cong c(\delta[-1])$. Hence, the two triangles in question are isomorphic. □

The following lemma is the analogue of Lemma 22.9.3.

Lemma 22.20.12. *In Situation 22.20.2 let $f_1 : x_1 \to y_1$ and $f_2 : x_2 \to y_2$ be morphisms in Comp($\mathcal{A}$). Let*

$$(a, b, c) : (x_1, y_1, c(f_1), f_1, i_1, p_1) \to (x_2, y_2, c(f_2), f_2, i_1, p_1)$$

be any morphism of triangles in $K(\mathcal{A})$. If a and b are homotopy equivalences, then so is c.

Proof. Since a and b are homotopy equivalences, they are invertible in $K(\mathcal{A})$ so let a^{-1} and b^{-1} denote their inverses in $K(\mathcal{A})$, giving us a commutative diagram

$$\begin{array}{ccccc} x_2 & \xrightarrow{f_2} & y_2 & \xrightarrow{i_2} & c(f_2) \\ \downarrow{\scriptstyle a^{-1}} & & \downarrow{\scriptstyle b^{-1}} & & \downarrow{\scriptstyle c'} \\ x_1 & \xrightarrow{f_1} & y_1 & \xrightarrow{i_1} & c(f_1) \end{array}$$

where the map c' is defined via Lemma 22.20.3 applied to the left commutative box of the above diagram. Since the diagram commutes in $K(\mathcal{A})$, it suffices by Lemma 22.20.8 to prove the following: given a morphism of triangle $(1, 1, c) : (x, y, c(f), f, i, p) \to (x, y, c(f), f, i, p)$ in $K(\mathcal{A})$, the map c is an isomorphism in

$K(\mathcal{A})$. We have the commutative diagrams in $K(\mathcal{A})$:

$$\begin{array}{ccccc} y & \longrightarrow & c(f) & \longrightarrow & x[1] \\ \downarrow{\scriptstyle 1} & & \downarrow{\scriptstyle c} & & \downarrow{\scriptstyle 1} \\ y & \longrightarrow & c(f) & \longrightarrow & x[1] \end{array} \quad \Longrightarrow \quad \begin{array}{ccccc} y & \longrightarrow & c(f) & \longrightarrow & x[1] \\ \downarrow{\scriptstyle 0} & & \downarrow{\scriptstyle c-1} & & \downarrow{\scriptstyle 0} \\ y & \longrightarrow & c(f) & \longrightarrow & x[1] \end{array}$$

Since the rows are admissible short exact sequences, we obtain the identity $(c-1)^2 = 0$ by Lemma 22.20.8, from which we conclude that $2 - c$ is inverse to c in $K(\mathcal{A})$ so that c is an isomorphism. □

The following lemma is the analogue of Lemma 22.9.4.

Lemma 22.20.13. *In Situation 22.20.2.*

(1) *Given an admissible short exact sequence $x \xrightarrow{\alpha} y \xrightarrow{\beta} z$. Then there exists a homotopy equivalence $e : C(\alpha) \to z$ such that the diagram*

$$\begin{array}{ccccccc} x & \xrightarrow{\alpha} & y & \xrightarrow{b} & C(\alpha) & \xrightarrow{-c} & x[1] \\ \downarrow & & \downarrow & & \vdots{\scriptstyle e} & & \downarrow \\ x & \xrightarrow{\alpha} & y & \xrightarrow{\beta} & z & \xrightarrow{\delta} & x[1] \end{array} \tag{22.20.13.1}$$

defines an isomorphism of triangles in $K(\mathcal{A})$. Here $y \xrightarrow{b} C(\alpha) \xrightarrow{c} x[1]$ is the admissible short exact sequence given as in axiom (C).

(2) *Given a morphism $\alpha : x \to y$ in $\mathit{Comp}(\mathcal{A})$, let $x \xrightarrow{\tilde{\alpha}} \tilde{y} \to y$ be the factorization given as in Lemma 22.20.6, where the admissible monomorphism $x \xrightarrow{\tilde{\alpha}} y$ extends to the admissible short exact sequence*

$$x \xrightarrow{\tilde{\alpha}} \tilde{y} \longrightarrow z$$

Then there exists an isomorphism of triangles

$$\begin{array}{ccccccc} x & \xrightarrow{\tilde{\alpha}} & \tilde{y} & \longrightarrow & z & \xrightarrow{\delta} & x[1] \\ \downarrow & & \downarrow & & \vdots{\scriptstyle e} & & \downarrow \\ x & \xrightarrow{\alpha} & y & \longrightarrow & C(\alpha) & \xrightarrow{-c} & x[1] \end{array}$$

where the upper triangle is the triangle associated to the sequence $x \xrightarrow{\tilde{\alpha}} \tilde{y} \to z$.

Proof. For (1), we consider the more complete diagram, *without* the sign change on c:

$$\begin{array}{ccccccccc} x & \underset{\pi}{\overset{\alpha}{\rightleftarrows}} & y & \underset{p}{\overset{b}{\rightleftarrows}} & C(\alpha) & \underset{\sigma}{\overset{c}{\rightleftarrows}} & x[1] & \underset{\pi}{\overset{\alpha}{\rightleftarrows}} & y[1] \\ \downarrow & & \downarrow & & {\scriptstyle f}\uparrow\vdots{\scriptstyle e} & & \downarrow & & \\ x & \underset{\pi}{\overset{\alpha}{\rightleftarrows}} & y & \underset{s}{\overset{\beta}{\rightleftarrows}} & z & \xrightarrow{\delta} & x[1] & & \end{array}$$

where the admissible short exact sequence $x \xrightarrow{\alpha} y \xrightarrow{\beta} z$ is given the splitting π, s, and the admissible short exact sequence $y \xrightarrow{b} C(\alpha) \xrightarrow{c} x[1]$ is given the splitting p,

σ. Note that (identifying hom-sets under shifting)

$$\alpha = pd(\sigma) = -d(p)\sigma, \quad \delta = \pi d(s) = -d(\pi)s$$

by the construction in Lemma 22.20.1.

We define $e = \beta p$ and $f = bs - \sigma\delta$. We first check that they are morphisms in $\mathrm{Comp}(\mathcal{A})$. To show that $d(e) = \beta d(p)$ vanishes, it suffices to show that $\beta d(p)b$ and $\beta d(p)\sigma$ both vanish, whereas

$$\beta d(p)b = \beta d(pb) = \beta d(1_y) = 0, \quad \beta d(p)\sigma = -\beta\alpha = 0$$

Similarly, to check that $d(f) = bd(s) - d(\sigma)\delta$ vanishes, it suffices to check the post-compositions by p and c both vanish, whereas

$$\begin{aligned} pbd(s) - pd(\sigma)\delta =& d(s) - \alpha\delta = d(s) - \alpha\pi d(s) = 0 \\ cbd(s) - cd(\sigma)\delta =& -cd(\sigma)\delta = -d(c\sigma)\delta = 0 \end{aligned}$$

The commutativity of left two squares of the diagram 22.20.13.1 follows directly from definition. Before we prove the commutativity of the right square (up to homotopy), we first check that e is a homotopy equivalence. Clearly,

$$ef = \beta p(bs - \sigma\delta) = \beta s = 1_z$$

To check that fe is homotopic to $1_{C(\alpha)}$, we first observe

$$b\alpha = bpd(\alpha) = d(\sigma), \quad \alpha c = -d(p)\sigma c = -d(p), \quad d(\pi)p = d(\pi)s\beta p = -\delta\beta p$$

Using these identities, we compute

$$\begin{aligned} 1_{C(\alpha)} =& bp + \sigma c \ \text{ (from } y \xrightarrow{b} C(\alpha) \xrightarrow{c} x[1]) \\ =& b(\alpha\pi + s\beta)p + \sigma(\pi\alpha)c \ \text{ (from } x \xrightarrow{\alpha} y \xrightarrow{\beta} z) \\ =& d(\sigma)\pi p + bs\beta p - \sigma\pi d(p) \ \text{ (by the first two identities above)} \\ =& d(\sigma)\pi p + bs\beta p - \sigma\delta\beta p + \sigma\delta\beta p - \sigma\pi d(p) \\ =& (bs - \sigma\delta)\beta p + d(\sigma)\pi p - \sigma d(\pi)p - \sigma\pi d(p) \ \text{ (by the third identity above)} \\ =& fe + d(\sigma\pi p) \end{aligned}$$

since $\sigma \in \mathrm{Hom}^{-1}(x, C(\alpha))$ (cf. proof of Lemma 22.20.4). Hence e and f are homotopy inverses. Finally, to check that the right square of diagram 22.20.13.1 commutes up to homotopy, it suffices to check that $-cf = \delta$. This follows from

$$-cf = -c(bs - \sigma\delta) = c\sigma\delta = \delta$$

since $cb = 0$.

For (2), consider the factorization $x \xrightarrow{\tilde{\alpha}} \tilde{y} \to y$ given as in Lemma 22.20.6, so the second morphism is a homotopy equivalence. By Lemmas 22.20.3 and 22.20.12, there exists an isomorphism of triangles between

$$x \xrightarrow{\alpha} y \to C(\alpha) \to x[1] \quad \text{and} \quad x \xrightarrow{\tilde{\alpha}} \tilde{y} \to C(\tilde{\alpha}) \to x[1]$$

Since we can compose isomorphisms of triangles, by replacing α by $\tilde{\alpha}$, y by $\tilde{y}$, and $C(\alpha)$ by $C(\tilde{\alpha})$, we may assume α is an admissible monomorphism. In this case, the result follows from (1). □

The following lemma is the analogue of Lemma 22.10.1.

Lemma 22.20.14. *In Situation 22.20.2 the homotopy category $K(\mathcal{A})$ with its natural translation functors and distinguished triangles is a pre-triangulated category.*

Proof. We will verify each of TR1, TR2, and TR3.

Proof of TR1. By definition every triangle isomorphic to a distinguished one is distinguished. Since

$$x \xrightarrow{1_x} x \longrightarrow 0$$

is an admissible short exact sequence, $(x, x, 0, 1_x, 0, 0)$ is a distinguished triangle. Moreover, given a morphism $\alpha : x \to y$ in $\text{Comp}(\mathcal{A})$, the triangle given by $(x, y, c(\alpha), \alpha, i, -p)$ is distinguished by Lemma 22.20.13.

Proof of TR2. Let $(x, y, z, \alpha, \beta, \gamma)$ be a triangle and suppose $(y, z, x[1], \beta, \gamma, -\alpha[1])$ is distinguished. Then there exists an admissible short exact sequence $0 \to x' \to y' \to z' \to 0$ such that the associated triangle $(x', y', z', \alpha', \beta', \gamma')$ is isomorphic to $(y, z, x[1], \beta, \gamma, -\alpha[1])$. After rotating, we conclude that $(x, y, z, \alpha, \beta, \gamma)$ is isomorphic to $(z'[-1], x', y', \gamma'[-1], \alpha', \beta')$. By Lemma 22.20.11, we deduce that $(z'[-1], x', y', \gamma'[-1], \alpha', \beta')$ is isomorphic to $(z'[-1], x', c(\gamma'[-1]), \gamma'[-1], i, p)$. Composing the two isomorphisms with sign changes as indicated in the following diagram:

$$\begin{array}{ccccccc}
x & \xrightarrow{\alpha} & y & \xrightarrow{\beta} & z & \xrightarrow{\gamma} & x[1] \\
\downarrow & & \downarrow & & \downarrow & & \downarrow \\
z'[-1] & \xrightarrow{-\gamma'[-1]} & x & \xrightarrow{\alpha'} & y' & \xrightarrow{\beta'} & z' \\
{\scriptstyle -1_{z'[-1]}}\downarrow & & \| & & \downarrow & & \downarrow{\scriptstyle -1_{z'}} \\
z'[-1] & \xrightarrow{\gamma'[-1]} & x & \xrightarrow{\alpha'} & c(\gamma'[-1]) & \xrightarrow{-p} & z'
\end{array}$$

We conclude that $(x, y, z, \alpha, \beta, \gamma)$ is distinguished by Lemma 22.20.13 (2). Conversely, suppose that $(x, y, z, \alpha, \beta, \gamma)$ is distinguished, so that by Lemma 22.20.13 (1), it is isomorphic to a triangle of the form $(x', y', c(\alpha'), \alpha', i, -p)$ for some morphism $\alpha' : x' \to y'$ in $\text{Comp}(\mathcal{A})$. The rotated triangle $(y, z, x[1], \beta, \gamma, -\alpha[1])$ is isomorphic to the triangle $(y', c(\alpha'), x'[1], i, -p, -\alpha[1])$ which is isomorphic to $(y', c(\alpha'), x'[1], i, p, \alpha[1])$. By Lemma 22.20.10, this triangle is distinguished, from which it follows that $(y, z, x[1], \beta, \gamma, -\alpha[1])$ is distinguished.

Proof of TR3: Suppose $(x, y, z, \alpha, \beta, \gamma)$ and $(x', y', z', \alpha', \beta', \gamma')$ are distinguished triangles of $\text{Comp}(\mathcal{A})$ and let $f : x \to x'$ and $g : y \to y'$ be morphisms such that $\alpha' \circ f = g \circ \alpha$. By Lemma 22.20.13, we may assume that $(x, y, z, \alpha, \beta, \gamma) = (x, y, c(\alpha), \alpha, i, -p)$ and $(x', y', z', \alpha', \beta', \gamma') = (x', y', c(\alpha'), \alpha', i', -p')$. Now apply Lemma 22.20.3 and we are done. □

The following lemma is the analogue of Lemma 22.10.2.

Lemma 22.20.15. *In Situation 22.20.2 given admissible monomorphisms $x \xrightarrow{\alpha} y$, $y \xrightarrow{\beta} z$ in $\mathcal{A}$, there exist distinguished triangles $(x, y, q_1, \alpha, p_1, \delta_1)$, $(x, z, q_2, \beta\alpha, p_2, \delta_2)$ and $(y, z, q_3, \beta, p_3, \delta_3)$ for which TR4 holds.*

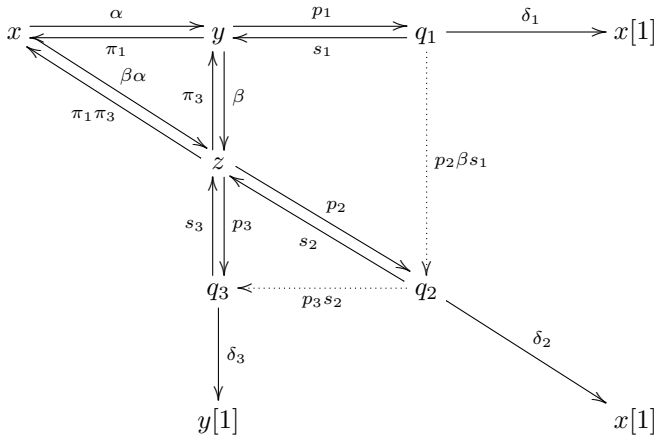
Proof. Given admissible monomorphisms $x \xrightarrow{\alpha} y$ and $y \xrightarrow{\beta} z$, we can find distinguished triangles, via their extensions to admissible short exact sequences,

$$x \underset{\pi_1}{\overset{\alpha}{\rightleftarrows}} y \underset{s_1}{\overset{p_1}{\rightleftarrows}} q_1 \xrightarrow{\delta_1} x[1]$$

$$x \underset{\pi_1\pi_3}{\overset{\beta\alpha}{\rightleftarrows}} z \underset{s_2}{\overset{p_2}{\rightleftarrows}} q_2 \xrightarrow{\delta_2} x[1]$$

$$y \underset{\pi_3}{\overset{\beta}{\rightleftarrows}} z \underset{s_3}{\overset{p_3}{\rightleftarrows}} q_3 \xrightarrow{\delta_3} x[1]$$

In these diagrams, the maps δ_i are defined as $\delta_i = \pi_i d(s_i)$ analagous to the maps defined in Lemma 22.20.1. They fit in the following solid commutative diagram



where we have defined the dashed arrows as indicated. Clearly, their composition $p_3 s_2 p_2 \beta s_1 = 0$ since $s_2 p_2 = 0$. We claim that they both are morphisms of $\mathrm{Comp}(\mathcal{A})$. We can check this using equations in Lemma 22.20.1:

$$d(p_2\beta s_1) = p_2\beta d(s_1) = p_2\beta\alpha\pi_1 d(s_1) = 0$$

since $p_2\beta\alpha = 0$, and

$$d(p_3 s_2) = p_3 d(s_2) = p_3\beta\alpha\pi_1\pi_3 d(s_2) = 0$$

since $p_3\beta = 0$. To check that $q_1 \to q_2 \to q_3$ is an admissible short exact sequence, it remains to show that in the underlying graded category, $q_2 = q_1 \oplus q_3$ with the above two morphisms as coprojection and projection. To do this, observe that in the underlying graded category $\mathcal{C}$, there hold

$$y = x \oplus q_1, \quad z = y \oplus q_3 = x \oplus q_1 \oplus q_3$$

where $\pi_1\pi_3$ gives the projection morphism onto the first factor: $x \oplus q_1 \oplus q_3 \to z$. By axiom (A) on $\mathcal{A}$, $\mathcal{C}$ is an additive category, hence we may apply Homology, Lemma 12.3.10 and conclude that

$$\ker(\pi_1\pi_3) = q_1 \oplus q_3$$

in $\mathcal{C}$. Another application of Homology, Lemma 12.3.10 to $z = x \oplus q_2$ gives $\ker(\pi_1\pi_3) = q_2$. Hence $q_2 \cong q_1 \oplus q_3$ in $\mathcal{C}$. It is clear that the dashed morphisms defined above give coprojection and projection.

Finally, we have to check that the morphism $\delta : q_3 \to q_1[1]$ induced by the admissible short exact sequence $q_1 \to q_2 \to q_3$ agrees with $p_1\delta_3$. By the construction in Lemma 22.20.1, the morphism δ is given by

$$\begin{aligned} p_1\pi_3 s_2 d(p_2 s_3) =& p_1\pi_3 s_2 p_2 d(s_3) \\ =& p_1\pi_3(1 - \beta\alpha\pi_1\pi_3)d(s_3) \\ =& p_1\pi_3 d(s_3) \text{ (since } \pi_3\beta = 0) \\ =& p_1\delta_3 \end{aligned}$$

as desired. The proof is complete. □

Putting everything together we finally obtain the analogue of Proposition 22.10.3.

Proposition 22.20.16. *In Situation 22.20.2 the homotopy category $K(\mathcal{A})$ with its natural translation functors and distinguished triangles is a triangulated category.*

Proof. By Lemma 22.20.14 we know that $K(\mathcal{A})$ is pre-triangulated. Combining Lemmas 22.20.7 and 22.20.15 with Derived Categories, Lemma 13.4.13, we conclude that $K(\mathcal{A})$ is a triangulated category. □

22.21. Derived Hom

Let R be a ring. Let (B, d) be a differential graded algebra over R. Denote $\mathcal{B} = \mathrm{Mod}^{dg}_{(B,\mathrm{d})}$ the differential graded category of differential graded B-modules, see Example 22.19.8. Let N be a differential graded B-module. Then the endomorphisms of N in $\mathcal{B}$

$$\mathrm{Hom}_{\mathcal{B}}(N, N)$$

is differential graded algebra over R. Now let N' be a second differential graded B-module. Then

$$\mathrm{Hom}_{\mathcal{B}}(N, N')$$

becomes a right differential graded $\mathrm{Hom}_{\mathcal{B}}(N, N)$-module by the composition

$$\mathrm{Hom}_{\mathcal{B}}(N, N') \times \mathrm{Hom}_{\mathcal{B}}(N, N) \longrightarrow \mathrm{Hom}_{\mathcal{B}}(N, N')$$

We need one more piece of data, in order to be able to formulate the results in the correct generality. Namely, let (A, d) be a differential graded R-algebra and let $A \to \mathrm{Hom}_{\mathcal{B}}(N, N)$ be a homomorphism of differential graded R-algebras[3]. Using this homomorphism we obtain a functor

$$\mathrm{Mod}_{(B,\mathrm{d})} \longrightarrow \mathrm{Mod}_{(A,\mathrm{d})}, \quad N' \longmapsto \mathrm{Hom}_{\mathcal{B}}(N, N') \tag{22.21.0.1}$$

where A acts on $\mathrm{Hom}_{\mathcal{B}}(N, N')$ via the given homomorphism and the action of $\mathrm{Hom}_{\mathcal{B}}(N, N)$ given above.

Lemma 22.21.1. *The functor (22.21.0.1) defines an exact functor of triangulated categories $K(Mod_{(B,d)}) \to K(Mod_{(A,d)})$.*

Proof. Combining Lemmas 22.19.9, 22.19.10, and 22.19.5 we obtain the functor of the statement. We have to show that (22.21.0.1) transforms distinguished triangles into distinguished triangles. To see this suppose that $0 \to N_1 \to N_2 \to N_3 \to 0$ is an admissible short exact sequence of differential graded B-modules. Let $s : N_3 \to N_2$ be a graded B-module homomorphism which is left inverse to $N_2 \to N_3$. Then s

[3] A very interesting case is when $A = \mathrm{Hom}_{\mathcal{B}}(N, N)$.

defines a graded A-module homomorphism $\mathrm{Hom}_{\mathcal{B}}(N, N_3) \to \mathrm{Hom}_{\mathcal{B}}(N, N_2)$ which is left inverse to $\mathrm{Hom}_{\mathcal{B}}(N, N_2) \to \mathrm{Hom}_{\mathcal{B}}(N, N_3)$. This finishes the proof. □

At this point we can consider the diagram

$$\begin{array}{ccc} K(\mathrm{Mod}_{(B,\mathrm{d})}) & \xrightarrow{\mathrm{Hom}_{\mathcal{B}}(N,-)} & K(\mathrm{Mod}_{(A,\mathrm{d})}) \\ \downarrow & \searrow^{F} & \downarrow \\ D(B,\mathrm{d}) & \dashrightarrow & D(A,\mathrm{d}) \end{array}$$

We would like to construct a dotted arrow as the *right derived functor* of the composition F. (*Warning*: the diagram will not commute.) Namely, in the general setting of Derived Categories, Section 13.14 we want to compute the right derived functor of F with respect to the multplicative system of quasi-isomorphisms in $K(\mathrm{Mod}_{(A,\mathrm{d})})$.

Lemma 22.21.2. *In the situation above, the right derived functor of F exists. We denote it $R\,\mathrm{Hom}(N, -) : D(B, d) \to D(A, d)$.*

Proof. We will use Derived Categories, Lemma 13.14.15 to prove this. As our collection $\mathcal{I}$ of objects we will use the objects with property (I). Property (1) was shown in Lemma 22.14.4. Property (2) holds because if $s : I \to I'$ is a quasi-isomorphism of modules with property (I), then s is a homotopy equivalence by Lemma 22.15.3. □

22.22. Variant of derived Hom

Let $\mathcal{A}$ be an abelian category. Consider the differential graded category $\mathrm{Comp}^{dg}(\mathcal{A})$ of complexes of $\mathcal{A}$, see Example 22.19.6. Let $K^\bullet$ be a complex of $\mathcal{A}$. Set

$$(E, \mathrm{d}) = \mathrm{Hom}_{\mathrm{Comp}^{dg}(\mathcal{A})}(K^\bullet, K^\bullet)$$

and consider the functor of differential graded categories

$$\mathrm{Comp}^{dg}(\mathcal{A}) \longrightarrow \mathrm{Mod}^{dg}_{(E,\mathrm{d})}, \quad X^\bullet \longmapsto \mathrm{Hom}_{\mathrm{Comp}^{dg}(\mathcal{A})}(K^\bullet, X^\bullet)$$

of Lemma 22.19.10.

Lemma 22.22.1. *In the situation above. If the right derived functor $R\,\mathrm{Hom}(K^\bullet, -)$ of $\mathrm{Hom}(K^\bullet, -) : K(\mathcal{A}) \to D(Ab)$ is everywhere defined on $D(\mathcal{A})$, then we obtain a canonical exact functor*

$$R\,\mathrm{Hom}(K^\bullet, -) : D(\mathcal{A}) \longrightarrow D(E, d)$$

of triangulated categories which reduces to the usual one on taking associated complexes of abelian groups.

Proof. Note that we have an associated functor $K(\mathcal{A}) \to K(\mathrm{Mod}_{(E,\mathrm{d})})$ by Lemma 22.19.10. We claim this functor is an exact functor of triangulated categories. Namely, let $f : A^\bullet \to B^\bullet$ be a map of complexes of $\mathcal{A}$. Then a computation shows that

$$\mathrm{Hom}_{\mathrm{Comp}^{dg}(\mathcal{A})}(K^\bullet, C(f)^\bullet) = C\left(\mathrm{Hom}_{\mathrm{Comp}^{dg}(\mathcal{A})}(K^\bullet, A^\bullet) \to \mathrm{Hom}_{\mathrm{Comp}^{dg}(\mathcal{A})}(K^\bullet, B^\bullet)\right)$$

where the right hand side is the cone in $\mathrm{Mod}_{(E,\mathrm{d})}$ defined earlier in this chapter. This shows that our functor is compatible with cones, hence with distinguished triangles. Let $X^\bullet$ be an object of $K(\mathcal{A})$. Consider the category of quasi-isomorphisms $s : X^\bullet \to Y^\bullet$. We are given that the functor $(s : X^\bullet \to Y^\bullet) \mapsto \mathrm{Hom}_{\mathcal{A}}(K^\bullet, Y^\bullet)$

is essentially constant when viewed in $D(Ab)$. But since the forgetful functor $D(E, \mathrm{d}) \to D(Ab)$ is compatible with taking cohomology, the same thing is true in $D(E, \mathrm{d})$. This proves the lemma. □

Warning: Although the lemma holds as stated and may be useful as stated, the differential algebra E isn't the "correct" one unless $H^n(E) = \mathrm{Ext}^n_{D(\mathcal{A})}(K^\bullet, K^\bullet)$ for all $n \in \mathbf{Z}$.

22.23. Tensor product

This section should be moved somewhere else. Let R be a ring. Let A be an R-algebra (see Section 22.2). Given a right A-module M and a left A-module N there is a *tensor product*

$$M \otimes_A N$$

This tensor product is a module over R. In fact, it is the receptacle of the universal A-bilinear map $M \times N \to M \otimes_A N$, $(m, n) \mapsto m \otimes n$.

We list some properties of the tensor product

(1) In each variable the tensor product is right exact, in fact commutes with direct sums and arbitrary colimits.
(2) If A, M, N are graded and the module structures are compatible with gradings then $M \otimes_A N$ is graded as well. Then nth graded piece $(M \otimes_A N)^n$ of $M \otimes_A N$ is the quotient of $\bigoplus_{p+q=n} M^p \otimes_{A^0} N^q$ by the submodule generated by $m \otimes an - ma \otimes n$ where $m \in M^p$, $n \in N^q$, and $a \in A^{n-p-q}$.
(3) If (A, d) is a differential graded algebra, and M and N are (left and right) differential graded A-modules, then $M \otimes_A N$ is a differential graded R-module with differential
$$\mathrm{d}(m \otimes n) = \mathrm{d}(m) \otimes n + (-1)^i m \otimes \mathrm{d}(n)$$
for $m \in M^i$ and $n \in N$.
(4) If N is a (A, B)-bimodule then $M \otimes_A N$ is a right B-module.
(5) If A and B are graded algebras, M is a graded A-module, and N is an (A, B)-bimodule which comes with a grading such that it is both a left graded A-module and a right graded B-module, then $M \otimes_A N$ is a graded B-module.
(6) If (A, d) and (B, d) are differential graded algebras, M is a differential graded A-module, and N is an (A, B)-bimodule which comes with a grading and a differential such that it is both a left differential graded A-module and a right differential graded B-module, then $M \otimes_A N$ is a differential graded B-module.

In the last item, the condition may be more succintly stated by saying that N is a differential graded module over $A^{opp} \otimes_R B$. We state the following as a lemma.

Lemma 22.23.1. *Let (A, d) and (B, d) be differential graded algebras, and let N be an (A, B)-bimodule which comes with a grading and a differential such that it is both a left differential graded A-module and a right differential graded B-module. Then $M \mapsto M \otimes_A N$ defines a functor*

$$- \otimes_A N : Mod^{dg}_{(A,d)} \longrightarrow Mod^{dg}_{(B,d)}$$

of differential graded categories. This functor induces functors

$$Mod_{(A,d)} \to Mod_{(B,d)} \quad and \quad K(Mod_{(A,d)}) \to K(Mod_{(B,d)})$$

by an application of Lemma 22.19.5.

Proof. This follows from the discussion above. □

If A is an algebra and M, M' are right A-modules, then we define

$$\mathrm{Hom}_A(M, M') = \{f : M \to M' \mid f \text{ is } A\text{-linear}\}$$

as usual. If A is graded and M and M' are graded A-modules, then we recall (Example 22.18.6) that

$$\mathrm{Hom}_{\mathrm{Mod}_A^{gr}}(M, M') = \bigoplus\nolimits_{n \in \mathbf{Z}} \mathrm{Hom}^n(M, M')$$

where $\mathrm{Hom}^n(M, M')$ is the collection of all A-module maps $M \to M'$ which are homogeneous of degree n.

Lemma 22.23.2. *Let A and B be algebras. Let M be a right A-module, N an (A, B)-bimodule, and N' a right B-module. Then we have*

$$\mathrm{Hom}_B(M \otimes_A N, N') = \mathrm{Hom}_A(M, \mathrm{Hom}_B(N, N'))$$

If A, B, M, N, N' are compatibly graded, then we have

$$\mathrm{Hom}_{Mod_B^{gr}}(M \otimes_A N, N') = \mathrm{Hom}_{Mod_A^{gr}}(M, \mathrm{Hom}_{Mod_B^{gr}}(N, N'))$$

for the graded versions.

Proof. This follows by interpreting both sides as A-bilinear maps $\psi : M \times N \to N'$ which are B-linear on the right. □

22.24. Derived tensor product

This section is analogous to More on Algebra, Section 15.44.

Let R be a ring. Let (A, d) and (B, d) be differential graded algebras over R. Let N be a (A, B)-bimodule equipped with a grading and differential such that N is a left differential graded A-module and a right differential graded B-module. In other words, N is a differential graded $A^{opp} \otimes_R B$-module. Consider the functor

$$\mathrm{Mod}_{(A,\mathrm{d})} \longrightarrow \mathrm{Mod}_{(B,\mathrm{d})}, \quad M \longmapsto M \otimes_A N \tag{22.24.0.1}$$

defined in Section 22.23.

Lemma 22.24.1. *The functor (22.24.0.1) defines an exact functor of triangulated categories $K(Mod_{(A,d)}) \to K(Mod_{(B,d)})$.*

Proof. The functor was constructed in Lemma 22.23.1. We have to show that $- \otimes_A N$ transforms distinguished triangles into distinguished triangles. Suppose that $0 \to K \to L \to M \to 0$ is an admissible short exact sequence of differential graded A-modules. Let $s : M \to L$ be a graded A-module homomorphism which is left inverse to $L \to M$. Then s defines a graded B-module homomorphism $M \otimes_A N \to L \otimes_A N$ which is left inverse to $L \otimes_A N \to M \otimes_A N$. □

At this point we can consider the diagram

$$\begin{array}{ccc} K(\mathrm{Mod}_{(A,\mathrm{d})}) & \xrightarrow{-\otimes_A N} & K(\mathrm{Mod}_{(B,\mathrm{d})}) \\ \downarrow & \searrow^{F} & \downarrow \\ D(A,\mathrm{d}) & \dashrightarrow & D(B,\mathrm{d}) \end{array}$$

The dotted arrow that we will construct below will be the *left derived functor* of the composition F. (*Warning*: the diagram will not commute.) Namely, in the general setting of Derived Categories, Section 13.14 we want to compute the left derived functor of F with respect to the multplicative system of quasi-isomorphisms in $K(\mathrm{Mod}_{(A,\mathrm{d})})$.

Lemma 22.24.2. *In the situation above, the left derived functor of F exists. We denote it $- \otimes^{\mathbf{L}}_A N : D(A, d) \to D(B, d)$.*

Proof. We will use Derived Categories, Lemma 13.14.15 to prove this. As our collection $\mathcal{P}$ of objects we will use the objects with property (P). Property (1) was shown in Lemma 22.13.4. Property (2) holds because if $s : P \to P'$ is a quasi-isomorphism of modules with property (P), then s is a homotopy equivalence by Lemma 22.15.3. □

Remark 22.24.3. Let (A, d) and (B, d) be differential graded algebras. Let $f : N \to N'$ be a homomorphism of differential graded $A^{opp} \otimes_R B$-modules. Then f induces a morphism of functors

$$1 \otimes f : - \otimes^{\mathbf{L}}_A N \longrightarrow - \otimes^{\mathbf{L}}_A N'$$

If f is a quasi-isomorphism, then $1 \otimes f$ is an isomorphism of functors.

Lemma 22.24.4. *Let (A, d) and (B, d) be differential graded algebras. Let N be an (A, B)-bimodule which comes with a grading and a differential such that it is a differential graded module for both A and B. Then the functors*

$$- \otimes^{\mathbf{L}}_A N : D(A, d) \longrightarrow D(B, d)$$

of Lemma 22.24.2 and

$$R\,\mathrm{Hom}(N, -) : D(B, d) \longrightarrow D(A, d)$$

of Lemma 22.21.2 are adjoint.

Proof. The statement means that we have

$$\mathrm{Hom}_{D(A,\mathrm{d})}(M, R\,\mathrm{Hom}(N, N')) = \mathrm{Hom}_{D(B,\mathrm{d})}(M \otimes^{\mathbf{L}}_A N, N')$$

bifunctorially in M and N'. To see this we may assume that M is a differential graded A-module with property (P) and that N' is a differential graded B-module with property (I). The computation of the derived functors given in the lemmas referenced in the statement combined with Lemma 22.15.3 translates the above into

$$\mathrm{Hom}_{K(\mathrm{Mod}_{(A,\mathrm{d})})}(M, \mathrm{Hom}_{\mathcal{B}}(N, N')) = \mathrm{Hom}_{K(\mathrm{Mod}_{(B,\mathrm{d})})}(M \otimes_A N, N')$$

where $\mathcal{B} = \mathrm{Mod}^{dg}_{(B,\mathrm{d})}$. Thus it is certainly sufficient to show that

$$\mathrm{Hom}_{\mathcal{A}}(M, \mathrm{Hom}_{\mathcal{B}}(N, N')) = \mathrm{Hom}_{\mathcal{B}}(M \otimes_A N, N')$$

as differential graded $\mathbf{Z}$-modules where $\mathcal{A} = \mathrm{Mod}^{dg}_{(A,\mathrm{d})}$. This follows from the fact that the isomorphism (Lemma 22.23.2)

$$\mathrm{Hom}_A(M, \mathrm{Hom}_B(N, N')) = \mathrm{Hom}_B(M \otimes_A N, N')$$

of internal homs of graded modules respects the differentials. □

Lemma 22.24.5. *Let R be a ring. Let (A, d), (B, d), and (C, d) be differential graded algebras over R. Let N be a differential graded $A^{opp} \otimes_R B$-module. Let N' be a differential graded $B^{opp} \otimes_R C$-module. If C is K-flat as a complex of R-modules, then the composition*

$$D(A, d) \xrightarrow{-\otimes^{\mathbf{L}}_A N} D(B, d) \xrightarrow{-\otimes^{\mathbf{L}}_B N'} D(C, d)$$

is isomorphic to $- \otimes^{\mathbf{L}}_A N''$ for some differential graded $A^{opp} \otimes_R C$-module N''.

Proof. We will use the construction of the functor $- \otimes^{\mathbf{L}} -$ of the proof of Lemma 22.24.2 without further mention. By Remark 22.24.3 we may replace N' by a quasi-isomorphic bimodule. Thus we assume that N' has property (P) as a differential graded $B^{opp} \otimes_R C$-module, see Lemma 22.13.4. Let $F_\bullet$ be the corresponding filtration on N'. We claim that $N'' = N \otimes_B N'$ works.

Let M be an object of $D(A, \mathrm{d})$. Using the lemma we may and do assume that M has property (P) as a differential graded A-module. Then $M \otimes^{\mathbf{L}}_A N = M \otimes_A N$. Next, we choose a quasi-isomorphism $P \to M \otimes_A N$ where P is a differential graded B-module with property (P). Then

$$(M \otimes^{\mathbf{L}}_A N) \otimes^{\mathbf{L}}_B N' = P \otimes_B N'$$

The map $P \to M \otimes_A N$ induces a map

$$P \otimes_B N' \to (M \otimes_A N) \otimes_B N' = M \otimes_A N''$$

This construction is functorial in M (details omitted) and hence it suffices to prove this map is a quasi-isomorphism.

Since $N' = \mathrm{colim}\, F_i N'$ it suffices to prove

$$P \otimes_B F_i N' \to M \otimes_A N \otimes_B F_i N'$$

is a quasi-isomorphism for all i. Using the short exact sequences $0 \to F_{i-1}N' \to F_i N' \to F_i N'/F_{i-1}N' \to 0$ which are graded split, we see that it suffices to prove that the maps

$$P \otimes_B F_i N'/F_{i-1}N' \to M \otimes_A N \otimes_B F_i N'/F_{i-1}N'$$

are quasi-isomorphisms for all i. Since $F_i N'/F_{i-1}N'$ is a direct sum of shifts of $B^{opp} \otimes_R C$ we finally reduce to showing that the map

$$P \otimes_B (B^{opp} \otimes_R C) \to M \otimes_A N \otimes_B (B^{opp} \otimes_R C)$$

is a quasi-isomorphism. In other words, we have to show that

$$P \otimes_R C \to M \otimes_A N \otimes_R C$$

is a quasi-isomorphism. Since $P \to M \otimes_A N$ is a quasi-isomorphism we conclude using More on Algebra, Lemma 15.43.4. □

Lemma 22.24.6. *With notation and assumptions as in Lemma 22.24.4. Assume*

(1) *N defines a compact object of $D(B, d)$, and*

(2) *the map* $H^k(A) \to \mathrm{Hom}_{D(B,d)}(N, N[k])$ *is an isomorphism for all* $k \in \mathbf{Z}$.

Then the functor $- \otimes^{\mathbf{L}}_A N$ *is fully faithful.*

Proof. Because our functor has a left adjoint given by $R\mathrm{Hom}(N, -)$ by Lemma 22.24.4 it suffices to show that for a differential graded A-module M the map

$$H^0(M) \longrightarrow \mathrm{Hom}_{D(B,\mathrm{d})}(N, M \otimes^{\mathbf{L}}_A N)$$

is an isomorphism. We may assume that $M = P$ is a differential graded A-module which has property (P). Since N defines a compact object, we reduce using Lemma 22.13.1 to the case where P has a finite filtration whose graded pieces are direct sums of $A[k]$. Again using compactness we reduce to the case $P = A[k]$. Assumption (2) on N is that the result holds for these. □

22.25. Variant of derived tensor product

Let $(\mathcal{C}, \mathcal{O})$ be a ringed site. Then we have the functors

$$\mathrm{Comp}(\mathcal{O}) \to K(\mathcal{O}) \to D(\mathcal{O})$$

and as we've seen above we have differential graded enhancement $\mathrm{Comp}^{dg}(\mathcal{O})$. Namely, this is the differential graded category of Example 22.19.6 associated to the abelian category $Mod(\mathcal{O})$. Let $K^\bullet$ be a complex of $\mathcal{O}$-modules in other words, an object of $\mathrm{Comp}^{dg}(\mathcal{O})$. Set

$$(E, \mathrm{d}) = \mathrm{Hom}_{\mathrm{Comp}^{dg}(\mathcal{O})}(K^\bullet, K^\bullet)$$

This is a differential graded $\mathbf{Z}$-algebra. We claim there is an analogue of the derived base change in this situation.

Lemma 22.25.1. *In the situation above there is a functor*

$$- \otimes_E K^\bullet : Mod^{dg}_{(E,d)} \longrightarrow Comp^{dg}(\mathcal{O})$$

of differential graded categories. This functor sends E *to* $K^\bullet$ *and commutes with direct sums.*

Proof. Let M be a differential graded E-module. For every object U of $\mathcal{C}$ the complex $K^\bullet(U)$ is a left differential graded E-module as well as a right $\mathcal{O}(U)$-module. The actions commute, so we have a bimodule. Thus, by the constructions in Section 22.23 we can form the tensor product

$$M \otimes_E K^\bullet(U)$$

which is a differential graded $\mathcal{O}(U)$-module, i.e., a complex of $\mathcal{O}(U)$-modules. This construction is functorial with respect to U, hence we can sheafify to get a complex of $\mathcal{O}$-modules which we denote

$$M \otimes_E K^\bullet$$

Moreover, for each U the construction determines a functor $\mathrm{Mod}^{dg}_{(E,\mathrm{d})} \to \mathrm{Comp}^{dg}(\mathcal{O}(U))$ of differential graded categories by Lemma 22.23.1. It is therefore clear that we obtain a functor as stated in the lemma. □

Lemma 22.25.2. *The functor of Lemma 22.25.1 defines an exact functor of triangulated categories* $K(Mod_{(E,d)}) \to K(\mathcal{O})$.

Proof. The functor induces a functor between homotopy categories by Lemma 22.19.5. We have to show that $- \otimes_E K^\bullet$ transforms distinguished triangles into distinguished triangles. Suppose that $0 \to K \to L \to M \to 0$ is an admissible short exact sequence of differential graded E-modules. Let $s : M \to L$ be a graded E-module homomorphism which is left inverse to $L \to M$. Then s defines a map $M \otimes_E K^\bullet \to L \otimes_E K^\bullet$ of graded $\mathcal{O}$-modules (i.e., respecting $\mathcal{O}$-module structure and grading, but not differentials) which is left inverse to $L \otimes_E K^\bullet \to M \otimes_E K^\bullet$. Thus we see that

$$0 \to K \otimes_E K^\bullet \to L \otimes_E K^\bullet \to M \otimes_E K^\bullet \to 0$$

is a termwise split short exact sequences of complexes, i.e., a defines a distinguished triangle in $K(\mathcal{O})$. □

Lemma 22.25.3. *The functor $K(Mod_{(E,d)}) \to K(\mathcal{O})$ of Lemma 22.25.2 has a left derived version defined on all of $D(E, d)$. We denote it $-\otimes^{\mathbf{L}}_E K^\bullet : D(E, d) \to D(\mathcal{O})$.*

Proof. We will use Derived Categories, Lemma 13.14.15 to prove this. As our collection $\mathcal{P}$ of objects we will use the objects with property (P). Property (1) was shown in Lemma 22.13.4. Property (2) holds because if $s : P \to P'$ is a quasi-isomorphism of modules with property (P), then s is a homotopy equivalence by Lemma 22.15.3. □

Lemma 22.25.4. *Let $(\mathcal{C}, \mathcal{O})$ be a ringed site. Let $K^\bullet$ be a complex of $\mathcal{O}$-modules. Then the functors*

$$- \otimes^{\mathbf{L}}_E K^\bullet : D(E, d) \longrightarrow D(\mathcal{O})$$

of Lemma 22.25.3 and

$$R\operatorname{Hom}(K^\bullet, -) : D(\mathcal{O}) \longrightarrow D(E, d)$$

of Lemma 22.22.1 are adjoint.

Proof. The statement means that we have

$$\operatorname{Hom}_{D(E,\mathrm{d})}(M, R\operatorname{Hom}(K^\bullet, L^\bullet)) = \operatorname{Hom}_{D(\mathcal{O})}(M \otimes^{\mathbf{L}}_E K^\bullet, L^\bullet)$$

bifunctorially in M and $L^\bullet$. To see this we may replace M by a differential graded E-module P with property (P). We also may replace $L^\bullet$ by a K-injective complex of $\mathcal{O}$-modules $I^\bullet$. The computation of the derived functors given in the lemmas referenced in the statement combined with Lemma 22.15.3 translates the above into

$$\operatorname{Hom}_{K(\mathrm{Mod}_{(E,\mathrm{d})})}(P, \operatorname{Hom}_{\mathcal{B}}(K^\bullet, I^\bullet)) = \operatorname{Hom}_{K(\mathcal{O})}(P \otimes_E K^\bullet, I^\bullet)$$

where $\mathcal{B} = \mathrm{Comp}^{dg}(\mathcal{O})$. There is an evalution map from right to left functorial in P and $I^\bullet$ (details omitted). Choose a filtration $F_\bullet$ on P as in the definition of property (P). By Lemma 22.13.1 and the fact that both sides of the equation are homological functors in P on $K(\mathrm{Mod}_{(E,\mathrm{d})})$ we reduce to the case where P is replaced by the differential graded E-module $\bigoplus F_i P$. Since both sides turn direct sums in the variable P into direct products we reduce to the case where P is one of the differential graded E-modules $F_i P$. Since each $F_i P$ has a finite filtration (given by admissible monomorpisms) whose graded pieces are graded projective E-modules we reduce to the case where P is a graded projective E-module. In this case we clearly have

$$\operatorname{Hom}_{\mathrm{Mod}^{dg}_{(E,\mathrm{d})}}(P, \operatorname{Hom}_{\mathcal{B}}(K^\bullet, I^\bullet)) = \operatorname{Hom}_{\mathrm{Comp}^{dg}(\mathcal{O})}(P \otimes_E K^\bullet, I^\bullet)$$

as graded $\mathbf{Z}$-modules (because this statement reduces to the case $P = E[k]$ where it is obvious). As the isomorphism is compatible with differentials we conclude. □

Lemma 22.25.5. *Let $(\mathcal{C}, \mathcal{O})$ be a ringed site. Let $K^\bullet$ be a complex of $\mathcal{O}$-modules. Assume*

(1) *$K^\bullet$ represents a compact object of $D(\mathcal{O})$, and*
(2) *$E = \mathrm{Hom}_{Comp^{dg}(\mathcal{O})}(K^\bullet, K^\bullet)$ computes the ext groups of $K^\bullet$ in $D(\mathcal{O})$.*

Then the functor

$$- \otimes^{\mathbf{L}}_E K^\bullet : D(E, d) \longrightarrow D(\mathcal{O})$$

of Lemma 22.25.3 is fully faithful.

Proof. Because our functor has a left adjoint given by $R\mathrm{Hom}(K^\bullet, -)$ by Lemma 22.25.4 it suffices to show for a differential graded E-module M that the map

$$H^0(M) \longrightarrow \mathrm{Hom}_{D(\mathcal{O})}(K^\bullet, M \otimes^{\mathbf{L}}_E K^\bullet)$$

is an isomorphism. We may assume that $M = P$ is a differential graded E-module which has property (P). Since $K^\bullet$ defines a compact object, we reduce using Lemma 22.13.1 to the case where P has a finite filtration whose graded pieces are direct sums of $E[k]$. Again using compactness we reduce to the case $P = E[k]$. The assumption on $K^\bullet$ is that the result holds for these. □

22.26. Characterizing compact objects

Compact objects of additive categories are defined in Derived Categories, Definition 13.32.1. In this section we characterize compact objects of the derived category of a differential graded algebra.

Lemma 22.26.1. *Let (A, d) be a differential graded algebra. Let P, M be differential graded A-modules. Assume P is projective as a graded A-module.*

(1) *If M is acyclic, then $\mathrm{Hom}_{K(A,d)}(P, M) = 0$.*
(2) *In general $\mathrm{Hom}_{K(A,d)}(P, M) = \mathrm{Hom}_{D(A,d)}(P, M)$.*

Proof. Consider the functor

$$N \longmapsto \mathrm{Hom}_{\mathrm{Mod}^{dg}_{(A,\mathrm{d})}}(P, N)$$

that to a differential graded A-module N associates the differential graded R-module whose degree n summand is the degree n homogeneous A-module maps from P to N, see Example 22.19.8. Since P is graded projective, this functor transforms any short exact sequence of differential graded A-modules into a short exact sequence of complexes of R-modules. In particular, if $Z \subset M$ is the kernel of d_M, then we get the short exact sequence $0 \to Z \to M \to Z[1] \to 0$ because M is acyclic. This in turn produces the short exact sequence

$$0 \to \mathrm{Hom}_{\mathrm{Mod}^{dg}_{(A,\mathrm{d})}}(P, Z) \to \mathrm{Hom}_{\mathrm{Mod}^{dg}_{(A,\mathrm{d})}}(P, M) \to \mathrm{Hom}_{\mathrm{Mod}^{dg}_{(A,\mathrm{d})}}(P, Z[1]) \to 0$$

The associated long exact sequence of cohomology modules is

$$\begin{aligned}\mathrm{Hom}_{K(A,\mathrm{d})}(P, Z) \to \mathrm{Hom}_{K(A,\mathrm{d})}(P, Z) \to \\ \mathrm{Hom}_{K(A,\mathrm{d})}(P, M) \to \mathrm{Hom}_{K(A,\mathrm{d})}(P, Z[1]) \to \mathrm{Hom}_{K(A,\mathrm{d})}(P, Z[1])\end{aligned}$$

A computation, which we omit, shows that the first and the last arrows are identity maps, thereby proving part (1).

Choose a quasi-isomorphism $M \to I$ where I has property (I) which is possible by Lemma 22.14.4. Choose a distinguished triangle

$$M \to I \to Q \to M[1]$$

in $K(A, \mathrm{d})$. Since $M \to I$ is a quasi-isomorphism, we see that Q is acyclic. By part (1) we have $\mathrm{Hom}_{K(A,\mathrm{d})}(P, Q[n]) = 0$ for all n. As $\mathrm{Hom}_{K(A,\mathrm{d})}(P, -)$ is homological (Derived Categories, Lemma 13.4.2) we see that $\mathrm{Hom}_{K(A,\mathrm{d})}(P, M) = \mathrm{Hom}_{K(A,\mathrm{d})}(P, I)$. We finish by applying Lemma 22.15.3. □

Let us say a differential graded A-module M is *finite* if M is generated, as a right A-module, by finitely many elements.

Lemma 22.26.2. *Let (A, d) be a differential graded algebra. If P is a differential graded A-module which is finite graded projective, then P is a compact object of $D(A, d)$.*

Proof. Suppose that P is finite and projective as a graded module. Choose a quasi-isomorphism $P' \to P$ where P' has property (P). Let $K = \mathrm{Ker}(P' \to P)$. Then K is an acyclic differential graded A-module, whence the boundary $\delta : P \to K[1]$ (see Lemmas 22.11.1 and 22.7.2) is zero by Lemma 22.26.1. Thus we see that $P' \to P$ is a homotopy equivalence (by general properties of triangulated categories). It follows that $\mathrm{Hom}_{K(A,\mathrm{d})}(P', -) = \mathrm{Hom}_{K(A,\mathrm{d})}(P, -)$ as functors, whence we see that

$$\mathrm{Hom}_{D(A,\mathrm{d})}(P, M) = \mathrm{Hom}_{K(A,\mathrm{d})}(P, M)$$

for any differential graded A-module M as this is true for P' by Lemma 22.15.3. Now, if $M = \bigoplus M_i$, then because P is a finite A-module, we see that

$$\mathrm{Hom}_{K(A,\mathrm{d})}(P, M) = \bigoplus \mathrm{Hom}_{K(A,\mathrm{d})}(P, M_i)$$

(look at the image of the generators of P for any map or homotopy). Moreover, the direct sum is the direct sum in the derived category, see Lemma 22.15.4. Thus P represents a compact object of $D(A, \mathrm{d})$. □

Lemma 22.26.3. *Let (A, d) be a differential graded algebra. Let E be a compact object of $D(A, d)$. Let P be a differential graded A-module which has a finite filtration*

$$0 = F_{-1}P \subset F_0P \subset F_1P \subset \ldots \subset F_nP = P$$

by differential graded submodules such that

$$F_{i+1}P/F_iP \cong \bigoplus\nolimits_{j \in J_i} A[k_{i,j}]$$

as differential graded A-modules for some sets J_i and integers $k_{i,j}$. Let $E \to P$ be a morphism of $D(A, d)$. Then there exists a differential graded submodule $P' \subset P$ such that $F_{i+1}P \cap P'/(F_iP \cap P')$ is equal to $\bigoplus_{j \in J'_i} A[k_{i,j}]$ for some finite subsets $J'_i \subset J_i$ and such that $E \to P$ factors through P'.

Proof. We will prove by induction on $-1 \leq m \leq n$ that there exists a differential graded submodule $P' \subset P$ such that

(1) $F_mP \subset P'$,
(2) for $i \geq m$ the quotient $F_{i+1}P \cap P'/(F_iP \cap P')$ is isomorphic to $\bigoplus_{j \in J'_i} A[k_{i,j}]$ for some finite subsets $J'_i \subset J_i$, and
(3) $E \to P$ factors through P'.

The base case is $m = n$ where we can take $P' = P$.

Induction step. Assume P' works for m. For $i \geq m$ and $j \in J'_i$ let $x_{i,j} \in F_{i+1}P \cap P'$ be a homogeneous element of degree $k_{i,j}$ whose image in $F_{i+1}P \cap P'/(F_iP \cap P')$ is the generator in the summand corresponding to $j \in J_i$. The $x_{i,j}$ generate P'/F_mP as an A-module. Write

$$\mathrm{d}(x_{i,j}) = \sum x_{i',j'} a_{i,j}^{i',j'} + y_{i,j}$$

with $y_{i,j} \in F_mP$ and $a_{i,j}^{i',j'} \in A$. There exists a finite subset $J'_{m-1} \subset J_{m-1}$ such that each $y_{i,j}$ maps to an element of the submodule $\bigoplus_{j \in J'_{m-1}} A[k_{m-1,j}]$ of $F_mP/F_{m-1}P$. Let $P'' \subset F_mP$ be the inverse image of $\bigoplus_{j \in J'_{m-1}} A[k_{m-1,j}]$ under the map $F_mP \to F_mP/F_{m-1}P$. Then we see that the A-submodule

$$P'' + \sum x_{i,j} A$$

is a differential graded submodule of the type we are looking for. Moreover

$$P'/(P'' + \sum x_{i,j}A) = \bigoplus\nolimits_{j \in J_{m-1} \setminus J'_{m-1}} A[k_{m-1,j}]$$

Since E is compact, the composition of the given map $E \to P'$ with the quotient map, factors through a finite direct subsum of the module displayed above. Hence after enlarging J'_{m-1} we may assume $E \to P'$ factors through $P'' + \sum x_{i,j}A$ as desired. □

It is not true that every compact object of $D(A, \mathrm{d})$ comes from a finite graded projective differential graded A-module, see Examples, Section 78.52.

Proposition 22.26.4. *Let (A, d) be a differential graded algebra. Let E be an object of $D(A, d)$. Then the followoing are equivalent*

1. *E is a compact object,*
2. *E is a direct summand of an object of $D(A, d)$ which is represented by a finite graded projective differential graded A-module, and*
3. *E is a direct summand of an object of $D(A, d)$ which is represented by a differential graded module P which has a finite filtration $F_\bullet$ by differential graded submodules such that $F_iP/F_{i-1}P$ are finite direct sums of shifts of A.*

Proof. Assume E is compact. By Lemma 22.13.4 we may assume that E is represented by a differential graded A-module P with propery (P). Consider the distinguished triangle

$$\bigoplus F_iP \to \bigoplus F_iP \to P \xrightarrow{\delta} \bigoplus F_iP[1]$$

coming from the admissible short exact sequence of Lemma 22.13.1. Since E is compact we have $\delta = \sum_{i=1,\ldots,n} \delta_i$ for some $\delta_i : P \to F_iP[1]$. Since the compostion of δ with the map $\bigoplus F_iP[1] \to \bigoplus F_iP[1]$ is zero (Derived Categories, Lemma 13.4.1) it follows that $\delta = 0$ (follows as $\bigoplus F_iP \to \bigoplus F_iP$ maps the summand F_iP via the difference of id and the inclusion map into $F_{i-1}P$). Thus we see that the identity on E factors through $\bigoplus F_iP$ in $D(A, \mathrm{d})$ (by Derived Categories, Lemma 13.4.10). Next, we use that P is compact again to see that the map $E \to \bigoplus F_iP$ factors through $\bigoplus_{i=1,\ldots,n} F_iP$ for some n. In other words, the identity on E factors through $\bigoplus_{i=1,\ldots,n} F_iP$. By Lemma 22.26.3 we see that the identity of E factors as

$E \to P \to E$ where P is as in part (3) of the statement of the lemma. In other words, we have proven that (1) implies (3).

It is clear that (3) implies (2). Finally, (2) implies (1) by Lemma 22.26.2 and Derived Categories, Lemma 13.32.2. □

Lemma 22.26.5. *Let (A, d) be a differential graded algebra. For every compact object E of $D(A, d)$ there exist integers $a \leq b$ such that $\mathrm{Hom}_{D(A,d)}(E, M) = 0$ if $H^i(M) = 0$ for $i \in [a, b]$.*

Proof. Observe that the collection of objects of $D(A, \mathrm{d})$ for which such a pair of integers exists is a saturated, strictly full triangulated subcategory of $D(A, \mathrm{d})$. Thus by Proposition 22.26.4 it suffices to prove this when E is represented by a differential graded module P which has a finite filtration $F_\bullet$ by differential graded submodules such that $F_iP/F_{i-1}P$ are finite direct sums of shifts of A. Using the compatibility with triangles, we see that it suffices to prove it for $P = A$. In this case $\mathrm{Hom}_{D(A,\mathrm{d})}(A, M) = H^0(M)$ and the result holds with $a = b = 0$. □

If (A, d) is just a graded algebra or more generally lives in only a finite number of degrees, then we do obtain the more precise description of compact objects.

Lemma 22.26.6. *Let (A, d) be a differential graded algebra. Assume that $A^n = 0$ for $|n| \gg 0$. Then an object of $D(A, d)$ is compact if and only if it can be represented by a finite graded projective differential graded A-module.*

Proof. One implication is Lemma 22.26.2. For the converse, let E be a compact object of $D(A, \mathrm{d})$. Fix $a \leq b$ as in Lemma 22.26.5. After decreasing a and increasing b if necessary, we may also assume that $H^i(E) = 0$ for $i \notin [a, b]$ (this follows from Proposition 22.26.4 and our assumption on A). Moreover, fix an integer $c > 0$ such that $A^n = 0$ if $|n| \geq c$.

By Proposition 22.26.4 we see that E is a direct summand, in $D(A, \mathrm{d})$, of a differential graded A-module P which has a finite filtration $F_\bullet$ by differential graded submodules such that $F_iP/F_{i-1}P$ are finite direct sums of shifts of A. Note that P is a finite graded free A-module. We will use without further mention that for such a differential graded A-module we have $\mathrm{Hom}_{D(A,\mathrm{d})}(P, P) = \mathrm{Hom}_{K(A,\mathrm{d})}(P, P)$ by Lemma 22.26.1.

Choose $n > 0$ such that $b + 4c - n < a$. Represent the projector onto E by an endomorphism $\varphi : P \to P$ of differential graded A-modules. Consider the distinguished triangle

$$P \xrightarrow{1-\varphi} P \to C \to P[1]$$

in $K(A, \mathrm{d})$ where C is the cone of the first arrow. Then $C \cong E \oplus E[1]$ in $D(A, \mathrm{d})$ and C is a finite graded free A-module. Next, consider a distinguished triangle

$$C[1] \to C \to C' \to C[2]$$

in $K(A, \mathrm{d})$ where C' is the cone on a morphism $C[1] \to C$ representing the composition

$$C[1] \cong E[1] \oplus E[2] \to E[1] \to E \oplus E[1] \cong C$$

in $D(A, \mathrm{d})$. Then we see that C' represents $E \oplus E[2]$. Continuing in this manner we see that we can find a differential graded A-module P which is a finite graded free A-module representing $E \oplus E[n]$.

Choose a basis $x_i, i \in I$ of homogeneous element for P as an A-module. Let $d_i = \deg(x_i)$. Let P_1 be the A-submodule of P generated by x_i and $\mathrm{d}(x_i)$ for $d_i \leq a - c - 1$. Let P_2 be the A-submodule of P generated by x_i and $\mathrm{d}(x_i)$ for $d_i \geq b - n + c$. We observe

(1) P_1 and P_2 are differential graded submodules of P,
(2) $P_1^t = 0$ for $t \geq a$,
(3) $P_1^t = P^t$ for $t \leq a - 2c$,
(4) $P_2^t = 0$ for $t \leq b - n$,
(5) $P_2^t = P^t$ for $t \geq b - n + 2c$.

As $b - n + 2c \geq a - 2c$ by our choice of n we obtain a short exact sequence of differential graded A-modules

$$0 \to P_1 \cap P_2 \to P_1 \oplus P_2 \xrightarrow{\pi} P \to 0$$

Since P is projective as a graded A-module this is an admissible short exact sequence (Lemma 22.11.1). Hence we obtain a boundary map $\delta : P \to (P_1 \cap P_2)[1]$ in $K(A, \mathrm{d})$, see Lemma 22.7.2. Since $P = E \oplus E[n]$ and since $P_1 \cap P_2$ lives in degrees $(b - n, a)$ we find that $\mathrm{Hom}_{D(A,\mathrm{d})}(E \oplus E[n], (P_1 \cap P_2)[1])$ is zero. Therefore $\delta = 0$ as a morphism in $K(A, \mathrm{d})$. By Derived Categories, Lemma 13.4.10 we can find a map $s : P \to P_1 \oplus P_2$ such that $\pi \circ s = \mathrm{id}_P + \mathrm{d}h + h\mathrm{d}$ for some $h : P \to P$ of degree -1. Since $P_1 \oplus P_2 \to P$ is surjective and since P is projective as a graded A-module we can choose a homogeneous lift $\tilde{h} : P \to P_1 \oplus P_2$ of h. Then we change s into $s + \mathrm{d}\tilde{h} + \tilde{h}\mathrm{d}$ to get $\pi \circ s = \mathrm{id}_P$. This means we obtain a direct sum decomposition $P = s^{-1}(P_1) \oplus s^{-1}(P_2)$. Since $s^{-1}(P_2)$ is equal to P in degrees $\geq b - n + 2c$ we see that $s^{-1}(P_2) \to P \to E$ is a quasi-isomorphism, i.e., an isomorphism in $D(A, \mathrm{d})$. This finishes the proof. □

22.27. Equivalences of derived categories

Let R be a ring. Let (A, d) and (B, d) be differential graded R-algebras. A natural question that arises in nature is what it means that $D(A, \mathrm{d})$ is equivalent to $D(B, \mathrm{d})$ as an R-linear triangulated category. This is a rather subtle question and it will turn out it isn't always the correct question to ask. Nonetheless, in this section we collection some conditions that guarantee this is the case.

We strongly urge the reader to take a look at the groundbreaking paper [**Ric89b**] on this topic.

Lemma 22.27.1. *Let R be a ring. Let $(A, d) \to (B, d)$ be a homomorphism of differential graded algebras over R, which induces an isomorphism on cohomology algebras. Then*

$$- \otimes^{\mathbf{L}}_A B : D(A, d) \to D(B, d)$$

gives an R-linear equivalence of triangulated categories with quasi-inverse the restriction functor $N \mapsto N_A$.

Proof. By Lemma 22.24.6 the functor $M \longmapsto M \otimes^{\mathbf{L}}_A B$ is fully faithful. By Lemma 22.24.4 the functor $N \longmapsto R\mathrm{Hom}(B, N) = N_A$ is a right adjoint. It is clear that the kernel of $R\mathrm{Hom}(B, -)$ is zero. Hence the result follows from Derived Categories, Lemma 13.6.12. □

When we analyze the proof above we see that we obtain the following generalization for free.

Lemma 22.27.2. *Let R be a ring. Let (A, d) and (B, d) be differential graded algebras over R. Let N be an (A, B)-bimodule which comes with a grading and a differential such that it is a differential graded module for both A and B. Assume that*

(1) *N defines a compact object of $D(B, d)$,*
(2) *if $N' \in D(B, d)$ and $\mathrm{Hom}_{D(B,d)}(N, N'[n]) = 0$ for $n \in \mathbf{Z}$, then $N' = 0$, and*
(3) *the map $H^k(A) \to \mathrm{Hom}_{D(B,d)}(N, N[k])$ is an isomorphism for all $k \in \mathbf{Z}$.*

Then

$$- \otimes^{\mathbf{L}}_A N : D(A, d) \to D(B, d)$$

gives an R-linear equivalence of triangulated categories.

Proof. By Lemma 22.24.6 the functor $M \longmapsto M \otimes^{\mathbf{L}}_A N$ is fully faithful. By Lemma 22.24.4 the functor $N' \longmapsto R\mathrm{Hom}(N, N')$ is a right adjoint. By assumption (3) the kernel of $R\mathrm{Hom}(N, -)$ is zero. Hence the result follows from Derived Categories, Lemma 13.6.12. □

Remark 22.27.3. In Lemma 22.27.2 we can replace condition (2) by the condition that N is a classical generator for $D_{compact}(B, d)$, see Derived Categories, Proposition 13.32.6. Moreover, if we knew that $R\mathrm{Hom}(N, B)$ is a compact object of $D(A, \mathrm{d})$, then it suffices to check that N is a weak generator for $D_{compact}(B, \mathrm{d})$. We omit the proof; we will add it here if we ever need it in the Stacks project.

Sometimes the B-module P in the lemma below is called an "(A, B)-tilting complex".

Lemma 22.27.4. *Let R be a ring. Let (A, d) and (B, d) be differential graded R-algebras. Assume that $A = H^0(A)$. The following are equivalent*

(1) *$D(A, d)$ and $D(B, d)$ are equivalent as R-linear triangulated categories, and*
(2) *there exists an object P of $D(B, d)$ such that*
   (a) *P is a compact object of $D(B, d)$,*
   (b) *if $N \in D(B, d)$ with $\mathrm{Hom}_{D(B,d)}(P, N[i]) = 0$ for $i \in \mathbf{Z}$, then $N = 0$,*
   (c) *$\mathrm{Hom}_{D(B,d)}(P, P[i]) = 0$ for $i \neq 0$ and equal to A for $i = 0$.*

Proof. Let $F : D(A, \mathrm{d}) \to D(B, \mathrm{d})$ be an equivalence. Then F maps compact objects to compact objects. Hence $P = F(A)$ is compact, i.e., (2)(a) holds. Conditions (2)(b) and (2)(c) are immediate from the fact that F is an equivalence.

Let P be an object as in (2). Represent P by a differential graded module with property (P). Set

$$(E, \mathrm{d}) = \mathrm{Hom}_{\mathrm{Mod}^{dg}_B}(P, P)$$

Then $H^0(E) = A$ and $H^k(E) = 0$ for $k \neq 0$ by Lemma 22.15.3 and assumption (2)(c). Viewing P as a (E, B)-bimodule and using Lemma 22.27.2 and assumption (2)(b) we obtain an equivalence

$$D(E, \mathrm{d}) \to D(B)$$

Let $E' \subset E$ be the differential graded R-subalgebra with

$$(E')^i = \begin{cases} E^i & \text{if } i < 0 \\ \mathrm{Ker}(E^0 \to E^1) & \text{if } i = 0 \\ 0 & \text{if } i > 0 \end{cases}$$

Then there are quasi-isomorphisms of differential graded algebras $(A, \mathrm{d}) \leftarrow (E', \mathrm{d}) \rightarrow (E, \mathrm{d})$ Thus we obtain equivalences

$$D(A, \mathrm{d}) \leftarrow D(E', \mathrm{d}) \rightarrow D(E, \mathrm{d}) \rightarrow D(B, \mathrm{d})$$

by Lemma 22.27.1. □

Remark 22.27.5. Let R be a ring. Let (A, d) and (B, d) be differential graded R-algebras. Suppose given an R-linear equivalence

$$F : D(A, \mathrm{d}) \longrightarrow D(B, \mathrm{d})$$

of triangulated categories. Set $N = F(A)$. Then N is a differential graded B-module. Since F is an equivalence and A is a compact object of $D(A, \mathrm{d})$, we conclude that N is a compact object of $D(B, \mathrm{d})$. Moreover, since $H^k(A) = \mathrm{Hom}_{D(A,\mathrm{d})}(A, A[k])$ and F an equivalence we see that F induces an isomorphism $H^k(A) = \mathrm{Hom}_{D(B,\mathrm{d})}(N, N[k])$ for all k. In order to conclude that there is an equivalence $D(A, \mathrm{d}) \longrightarrow D(B, \mathrm{d})$ which arises from the construction in Lemma 22.27.2 all we need is a right A-module structure on N or on any differential graded B-module quasi-isomorphic to B. This module structure can be constructed in certain cases. For example, if we assume that F can be lifted to a differential graded functor

$$F^{dg} : \mathrm{Mod}^{dg}_{(A,\mathrm{d})} \longrightarrow \mathrm{Mod}^{dg}_{(B,\mathrm{d})}$$

(for notation see Example 22.19.8) between the associated differential graded categories, then this holds. Another case is discussed in the proposition below.

Proposition 22.27.6. *Let R be a ring. Let (A, d) and (B, d) be differential graded R-algebras. Let $F : D(A, d) \to D(B, d)$ be an R-linear equivalence of triangulated cateories. Assume that*

(1) *$A = H^0(A)$, and*
(2) *B is K-flat as a complex of R-modules.*

Then there exists an (A, B)-bimodule N as in Lemma 22.27.2.

Proof. As in Remark 22.27.5 above, we set $N = F(A)$ in $D(B, \mathrm{d})$. We may assume that N is a differential graded B-module with property (P). Set

$$(E, \mathrm{d}) = \mathrm{Hom}_{\mathrm{Mod}^{dg}_{(B,\mathrm{d})}}(N, N)$$

Then $H^0(E) = A$ and $H^k(E) = 0$ for $k \neq 0$ by Lemma 22.15.3. Moreover, by the discussion preceding the proposition and Lemma 22.27.2 we see that N as a (E, B)-bimodule induces an equivalence $- \otimes^{\mathbf{L}}_E N : D(E, \mathrm{d}) \to D(B, \mathrm{d})$. Let $E' \subset E$ be the differential graded R-subalgebra with

$$(E')^i = \begin{cases} E^i & \text{if } i < 0 \\ \mathrm{Ker}(E^0 \to E^1) & \text{if } i = 0 \\ 0 & \text{if } i > 0 \end{cases}$$

Then there are quasi-isomorphisms of differential graded algebras $(A, \mathrm{d}) \leftarrow (E', \mathrm{d}) \rightarrow (E, \mathrm{d})$ Thus we obtain equivalences

$$D(A, \mathrm{d}) \leftarrow D(E', \mathrm{d}) \rightarrow D(E, \mathrm{d}) \rightarrow D(B, \mathrm{d})$$

by Lemma 22.27.1. Note that the quasi-inverse $D(A, \mathrm{d}) \to D(E', \mathrm{d})$ of the left vertical arrow is given by $M \mapsto M \otimes^{\mathbf{L}}_A A$ where A is viewed as a $A^{opp} \otimes_R E'$-module. On the other hand the functor $D(E', \mathrm{d}) \to D(B, \mathrm{d})$ is given by $M \mapsto M \otimes^{\mathbf{L}}_{E'} N$ where N is as above. We conclude by Lemma 22.24.5. □

Remark 22.27.7. Let A, B, F, N be as in Proposition 22.27.6. It is not clear that F and the functor $G(-) = - \otimes^{\mathbf{L}}_A N$ are isomorphic. By construction there is an isomorphism $N = G(A) \to F(A)$ in $D(B, \mathrm{d})$. It is straigthforward to extend this to a functorial isomorphism $G(M) \to F(M)$ for M is a differential graded A-module which is graded projective (e.g., a sum of shifts of A). Then one can conclude that $G(M) \cong F(M)$ when M is a cone of a map between such modules. We don't know whether more is true in general.

Lemma 22.27.8. *Let R be a ring. Let A and B be R-algebras. The following are equivalent*

(1) *there is an R-linear equivalence $D(A) \to D(B)$ of triangulated categories,*
(2) *there exists an object P of $D(B)$ such that*
   (a) *P can be represented by a finite complex of finite projective B-modules,*
   (b) *if $K \in D(B)$ with $Ext^i_B(P, K) = 0$ for $i \in \mathbf{Z}$, then $K = 0$, and*
   (c) *$Ext^i_B(P, P) = 0$ for $i \neq 0$ and equal to A for $i = 0$.*

Moreover, if B is flat as an R-module, then this is also equivalent to

(3) *there exists an (A, B)-bimodule N such that $- \otimes^{\mathbf{L}}_A N : D(A) \to D(B)$ is an equivalence.*

Proof. The equivalence of (1) and (2) is a special case of Lemma 22.27.4 combined with the result of Lemma 22.26.6 characterizing compact objects of $D(B)$. The equivalence with (3) if B is R-flat follows from Proposition 22.27.6. □

Remark 22.27.9. Let R be a ring. Let A and B be R-algebras. If $D(A)$ and $D(B)$ are equivalent as R-linear triangulated categories, then the centers of A and B are isomorphic as R-algebras. In particular, if A and B are commutative, then $A \cong B$. The rather tricky proof can be found in [**Ric89b**, Proposition 9.2] or [**KZ98**, Proposition 6.3.2]. Another approach might be to use Hochschild cohomology (see remark below).

Remark 22.27.10. Let R be a ring. Let (A, d) and (B, d) be differential graded R-algebras which are derived equivalent, i.e., such that there exists an R-linear equivalence $D(A, \mathrm{d}) \to D(B, \mathrm{d})$ of triangulated categories. We would like to show that certain invariants of (A, d) and (B, d) coincide. In many situations one has more control of the situation. For example, it may happen that there is an equivalence of the form

$$- \otimes_A \Omega : D(A, \mathrm{d}) \longrightarrow D(B, \mathrm{d})$$

for some differential graded $A^{opp} \otimes_R B$-module Ω (this happens in the situation of Proposition 22.27.6 and is often true if the equivalence comes from a geometric construction). If also the quasi-inverse of our functor is given as

$$- \otimes^{\mathbf{L}}_A \Omega' : D(B, \mathrm{d}) \longrightarrow D(A, \mathrm{d})$$

for a differential graded $B^{opp} \otimes_R A$-module Ω' (and as before such a module Ω' often exists in practice) then we can consider the functor

$$D(A^{opp} \otimes_R A, \mathrm{d}) \longrightarrow D(B^{opp} \otimes_R B, \mathrm{d}), \quad M \longmapsto \Omega' \otimes^{\mathbf{L}}_A M \otimes^{\mathbf{L}}_A \Omega$$

Observe that this functor sends the (A, A)-bimodule A to the (B, B)-bimodule B. Under suitable conditions (e.g., flatness of A, B, Ω, etc) this functor will be an

equivalence as well. If this is the case, then it follows that we have isomorphisms of Hochschild cohomology groups

$$HH^i(A, \mathrm{d}) = \mathrm{Hom}_{D(A^{opp} \otimes_R A, \mathrm{d})}(A, A[i]) \longrightarrow \mathrm{Hom}_{D(B^{opp} \otimes_R B, \mathrm{d})}(B, B[i]) = HH^i(B, \mathrm{d}).$$

For example, if $A = H^0(A)$, then $HH^0(A, \mathrm{d})$ is equal to the center of A, and this gives a conceptual proof of the result mentioned in Remark 22.27.9. If we ever need this remark we will provide a precise statement with a detailed proof here.

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CHAPTER 23

Divided Power Algebra

23.1. Introduction

In this chapter we talk about divided power algebras and what you can do with them. A reference is the book [**Ber74**].

23.2. Divided powers

In this section we collect some results on divided power rings. We will use the convention $0! = 1$ (as empty products should give 1).

Definition 23.2.1. Let A be a ring. Let I be an ideal of A. A collection of maps $\gamma_n : I \to I$, $n > 0$ is called a *divided power structure* on I if for all $n \geq 0$, $m > 0$, $x, y \in I$, and $a \in A$ we have

(1) $\gamma_1(x) = x$, we also set $\gamma_0(x) = 1$,
(2) $\gamma_n(x)\gamma_m(x) = \frac{(n+m)!}{n!m!}\gamma_{n+m}(x)$,
(3) $\gamma_n(ax) = a^n\gamma_n(x)$,
(4) $\gamma_n(x+y) = \sum_{i=0,\ldots,n} \gamma_i(x)\gamma_{n-i}(y)$,
(5) $\gamma_n(\gamma_m(x)) = \frac{(nm)!}{n!(m!)^n}\gamma_{nm}(x)$.

Note that the rational numbers $\frac{(n+m)!}{n!m!}$ and $\frac{(nm)!}{n!(m!)^n}$ occurring in the definition are in fact integers; the first is the number of ways to choose n out of $n+m$ and the second counts the number of ways to divide a group of nm objects into n groups of m. We make some remarks about the definition which show that $\gamma_n(x)$ is a replacement for $x^n/n!$ in I.

Lemma 23.2.2. *Let A be a ring. Let I be an ideal of A.*

(1) *If γ is a divided power structure on I, then $n!\gamma_n(x) = x^n$ for $n \geq 1$, $x \in I$.*

Assume A is torsion free as a $\mathbf{Z}$-module.

(2) *A divided power structure on I, if it exists, is unique.*
(3) *If $\gamma_n : I \to I$ are maps then*

$$\gamma \text{ is a divided power structure} \Leftrightarrow n!\gamma_n(x) = x^n \ \forall x \in I, n \geq 1.$$

(4) *The ideal I has a divided power structure if and only if there exists a set of generators x_i of I as an ideal such that for all $n \geq 1$ we have $x_i^n \in (n!)I$.*

Proof. Proof of (1). If γ is a divided power structure, then condition (2) implies that $n\gamma_n(x) = \gamma_1(x)\gamma_{n-1}(x)$. Hence by induction and condition (1) we get $n!\gamma_n(x) = x^n$.

Assume A is torsion free as a $\mathbf{Z}$-module. Proof of (2). This is clear from (1).

Proof of (3). Assume that $n!\gamma_n(x) = x^n$ for all $x \in I$ and $n \geq 1$. Since $A \subset A \otimes_{\mathbf{Z}} \mathbf{Q}$ it suffices to prove (1) – (5) in case A is a $\mathbf{Q}$-algebra. In this case $\gamma_n(x) = x^n/n!$

and it is straightforward to verify (1) – (5), for example (4) corresponds to the binomial formula

$$(x+y)^n = \sum \frac{n!}{i!(n-i)!} x^i y^{n-i}$$

We encourage the reader to do the verifications to make sure that we have the coefficients correct.

Proof of (4). Assume we have generators x_i of I as an ideal such that $x_i^n \in (n!)I$ for all $n \geq 1$. We claim that for all $x \in I$ we have $x^n \in (n!)I$. If the claim holds then we can set $\gamma_n(x) = x^n/n!$ which is a divided power structure by (3). To prove the claim we note that it holds for $x = ax_i$. Hence we see that the claim holds for a set of generators of I as an abelian group. By induction on the length of an expression in terms of these, it suffices to prove the claim for $x + y$ if it holds for x and y. This follows immediately from the binomial theorem. □

Example 23.2.3. Let p be a prime number. Let A be a ring such that every integer n not divisible by p is invertible, i.e., A is a $\mathbf{Z}_{(p)}$-algebra. Then $I = pA$ has a canonical divided power structure. Namely, given $x = pa \in A$ we set

$$\gamma_n(x) = \frac{p^n}{n!} a^n$$

The reader verifies immediately that $p^n/n! \in \mathbf{Z}_{(p)}$ so that the definition makes sense. It is a straightforward exercise to verify that conditions (1) – (5) of Definition 23.2.1 are satisfied. Alternatively, it is clear that the definition works for $A_0 = \mathbf{Z}_{(p)}$ and then the result follows from Lemma 23.4.2.

Lemma 23.2.4. *Let A be a ring. Let I be an ideal of A. Let $\gamma_n : I \to I$, $n \geq 1$ be a sequence of maps. Assume*

(a) *(1), (3), and (4) of Definition 23.2.1 hold for all $x, y \in I$, and*
(b) *properties (2) and (5) hold for x in set of generators of I as an ideal.*

Then γ is a divided power structure on I.

Proof. The numbers (1), (2), (3), (4), (5) in this proof refer to the conditions listed in Definition 23.2.1. Applying (3) we see that if (2) and (5) hold for x then (2) and (5) hold for ax for all $a \in A$. Hence we see (b) implies (2) and (5) hold for a set of generators of I as an abelian group. Hence, by induction of the length of an expression in terms of these it suffices to prove that, given $x, y \in I$ such that (2) and (5) hold for x and y, then (2) and (5) hold for $x + y$.

Proof of (2) for $x + y$. By (4) we have

$$\gamma_n(x+y)\gamma_m(x+y) = \sum\nolimits_{i+j=n,\ k+l=m} \gamma_i(x)\gamma_k(x)\gamma_j(y)\gamma_l(y)$$

Using (2) for x and y this equals

$$\sum \frac{(i+k)!}{i!k!} \frac{(j+l)!}{j!l!} \gamma_{i+k}(x)\gamma_{j+l}(y)$$

Comparing this with the expansion

$$\gamma_{n+m}(x+y) = \sum \gamma_a(x)\gamma_b(y)$$

we see that we have to prove that given $a + b = n + m$ we have

$$\sum\nolimits_{i+k=a,\ j+l=b,\ i+j=n,\ k+l=m} \frac{(i+k)!}{i!k!} \frac{(j+l)!}{j!l!} = \frac{(n+m)!}{n!m!}.$$

Instead of arguing this directly, we note that the result is true for the ideal $I = (x, y)$ in the polynomial ring $\mathbf{Q}[x, y]$ because $\gamma_n(f) = f^n/n!$, $f \in I$ defines a divided power structure on I. Hence the equality of rational numbers above is true.

Proof of (5) for $x + y$ given that (1) – (4) hold and that (5) holds for x and y. We will again reduce the proof to an equality of rational numbers. Namely, using (4) we can write $\gamma_n(\gamma_m(x+y)) = \gamma_n(\sum \gamma_i(x)\gamma_j(y))$. Using (4) we can write $\gamma_n(\gamma_m(x+y))$ as a sum of terms which are products of factors of the form $\gamma_k(\gamma_i(x)\gamma_j(y))$. If $i > 0$ then

$$\begin{aligned}\gamma_k(\gamma_i(x)\gamma_j(y)) &= \gamma_j(y)^k\gamma_k(\gamma_i(x)) \\ &= \frac{(ki)!}{k!(i!)^k}\gamma_j(y)^k\gamma_{ki}(x) \\ &= \frac{(ki)!}{k!(i!)^k}\frac{(kj)!}{k!(j!)^k}\gamma_{ik}(x)\gamma_{kj}(y)\end{aligned}$$

using (3) in the first equality, (5) for x in the second, and (2) exactly k times in the third. Using (5) for y we see the same equality holds when $i = 0$. Continuing like this using all axioms but (5) we see that we can write

$$\gamma_n(\gamma_m(x + y)) = \sum\nolimits_{i+j=nm} c_{ij}\gamma_i(x)\gamma_j(y)$$

for certain universal constants $c_{ij} \in \mathbf{Z}$. Again the fact that the equality is valid in the polynomial ring $\mathbf{Q}[x, y]$ implies that the coefficients c_{ij} are all equal to $(nm)!/n!(m!)^n$ as desired. □

Lemma 23.2.5. *Let A be a ring with two ideals $I, J \subset A$. Let γ be a divided power structure on I and let δ be a divided power structure on J. Then*

1. *γ and δ agree on IJ,*
2. *if γ and δ agree on $I \cap J$ then they are the restriction of a unique divided power structure ϵ on $I + J$.*

Proof. Let $x \in I$ and $y \in J$. Then

$$\gamma_n(xy) = y^n\gamma_n(x) = n!\delta_n(y)\gamma_n(x) = \delta_n(y)x^n = \delta_n(xy).$$

Hence γ and δ agree on a set of (additive) generators of IJ. By property (4) of Definition 23.2.1 it follows that they agree on all of IJ.

Let $z \in I + J$. Write $z = x + y$ with $x \in I$ and $y \in J$. Then we set

$$\epsilon_n(z) = \sum \gamma_i(x)\delta_{n-i}(y)$$

To see that this is well defined, suppose that $z = x' + y'$ is another representation with $x' \in I$ and $y' \in J$. Then $w = x - x' = y' - y \in I \cap J$. Hence

$$\begin{aligned}\sum\nolimits_{i+j=n} \gamma_i(x)\delta_j(y) &= \sum\nolimits_{i+j=n} \gamma_i(x' + w)\delta_j(y) \\ &= \sum\nolimits_{i'+l+j=n} \gamma_{i'}(x')\gamma_l(w)\delta_j(y) \\ &= \sum\nolimits_{i'+l+j=n} \gamma_{i'}(x')\delta_l(w)\delta_j(y) \\ &= \sum\nolimits_{i'+j'=n} \gamma_{i'}(x')\delta_{j'}(y + w) \\ &= \sum\nolimits_{i'+j'=n} \gamma_{i'}(x')\delta_{j'}(y')\end{aligned}$$

as desired. Next, we prove conditions (1) – (5) of Definition 23.2.1. Properties (1) and (3) are clear. To see (4), suppose that $z = x + y$ and $z' = x' + y'$ with $x, x' \in I$ and $y, y' \in J$ and compute

$$\begin{aligned}
\epsilon_n(z + z') &= \sum\nolimits_{a+b=n} \gamma_i(x + x')\delta_j(y + y') \\
&= \sum\nolimits_{i+i'+j+j'=n} \gamma_i(x)\gamma_{i'}(x')\delta_j(y)\delta_{j'}(y') \\
&= \sum\nolimits_{k=0,\ldots,n} \sum\nolimits_{i+j=k} \gamma_i(x)\delta_j(y) \sum\nolimits_{i'+j'=n-k} \gamma_{i'}(x')\delta_{j'}(y') \\
&= \sum\nolimits_{k=0,\ldots,n} \epsilon_k(z)\epsilon_{n-k}(z')
\end{aligned}$$

as desired. Now we see that it suffices to prove (2) and (5) for elements of I or J, see Lemma 23.2.4. This is clear because γ and δ are divided power structures. □

Lemma 23.2.6. *Let p be a prime number. Let A be a ring, let $I \subset A$ be an ideal, and let γ be a divided power structure on I. Assume p is nilpotent in A/I. Then I is locally nilpotent if and only if p is nilpotent in A.*

Proof. If $p^N = 0$ in A, then for $x \in I$ we have $x^{pN} = (pN)!\gamma_N(x) = 0$ because $(pN)!$ is divisible by p^N. Conversely, assume I is locally nilpotent. We've also assumed that p is nilpotent in A/I, hence $p^r \in I$ for some r, hence p^r nilpotent, hence p nilpotent. □

23.3. Divided power rings

There is a category of divided power rings. Here is the definition.

Definition 23.3.1. A *divided power ring* is a triple (A, I, γ) where A is a ring, $I \subset A$ is an ideal, and $\gamma = (\gamma_n)_{n\geq 1}$ is a divided power structure on I. A *homomorphism of divided power rings* $\varphi : (A, I, \gamma) \to (B, J, \delta)$ is a ring homomorphism $\varphi : A \to B$ such that $\varphi(I) \subset J$ and such that $\delta_n(\varphi(x)) = \varphi(\gamma_n(x))$ for all $x \in I$.

We sometimes say "let (B, J, δ) be a divided power algebra over (A, I, γ)" to indicate that (B, J, δ) is a divided power ring which comes equipped with a homomorphism of divided power rings $(A, I, \gamma) \to (B, J, \delta)$.

Lemma 23.3.2. *The category of divided power rings has all limits and they agree with limits in the category of rings.*

Proof. The empty limit is the zero ring (that's weird but we need it). The product of a collection of divided power rings (A_t, I_t, γ_t), $t \in T$ is given by $(\prod A_t, \prod I_t, \gamma)$ where $\gamma_n((x_t)) = (\gamma_{t,n}(x_t))$. The equalizer of $\alpha, \beta : (A, I, \gamma) \to (B, J, \delta)$ is just $C = \{a \in A \mid \alpha(a) = \beta(a)\}$ with ideal $C \cap I$ and induced divided powers. It follows that all limits exist, see Categories, Lemma 4.14.10. □

The following lemma illustrates a very general category theoretic phenomenon in the case of divided power algebras.

Lemma 23.3.3. *Let $\mathcal{C}$ be the category of divided power rings. Let $F : \mathcal{C} \to Sets$ be a functor. Assume that*

1. *there exists a cardinal κ such that for every $f \in F(A, I, \gamma)$ there exists a morphism $(A', I', \gamma') \to (A, I, \gamma)$ of $\mathcal{C}$ such that f is the image of $f' \in F(A', I', \gamma')$ and $|A'| \leq \kappa$, and*
2. *F commutes with limits.*

Then F is representable, i.e., there exists an object (B, J, δ) of $\mathcal{C}$ such that

$$F(A, I, \gamma) = \mathrm{Hom}_{\mathcal{C}}((B, J, \delta), (A, I, \gamma))$$

functorially in (A, I, γ).

Proof. Consider a set of objects $\mathcal{U}$ of $\mathcal{C}$ containing an object isomorphic to every (A, I, γ) with $|A| \leq \kappa$. Let $\mathcal{I}$ be the category of pairs (U, f) where $U \in \mathcal{U}$ and $f \in F(U)$. A morphism $(U, f) \to (U', f')$ of $\mathcal{I}$ is a map $u : U \to U'$ of $\mathcal{C}$ such that $F(u)(f) = f'$. Set

$$(B, J, \delta) = \lim_{(U,f)\in\mathcal{I}} U.$$

The limit exists by Lemma 23.3.2. As F commutes with limits we have

$$F(B, J, \delta) = \lim_{(U,f)\in\mathcal{I}} F(U).$$

Hence there is a universal element $\xi \in F(B, J, \delta)$ which for $U \in \mathcal{U}$ maps to $f \in F(U)$ under F applied to the projection map $(B, J, \delta) \to U$ of the limit corresponding to f. Using ξ we obtain a transformation of functors

$$\xi : \mathrm{Hom}_{\mathcal{C}}((B, J, \delta), -) \longrightarrow F(-)$$

see Categories, Section 4.3. Let (A, I, γ) be an arbitrary object of $\mathcal{C}$ and let $f \in F(A, I, \gamma)$. Choose $U \to (A, I, \gamma)$ with $U \in \mathcal{U}$ and $f' \in F(U)$ mapping to f which is possible by assumption (1). Then F applied to the maps

$$(B, J, \delta) \longrightarrow U \longrightarrow (A, I, \gamma)$$

(the first being the projection map of the limit defining B) sends ξ to f. Hence the transformation ξ is surjective. Finally, suppose that $a, b : (B, J, \delta) \to (A, I, \gamma)$ are two maps such that $F(a)(\xi) = F(b)(\xi)$. Since F commutes with limits, it commutes with equalizers. This means that ξ comes from an element $\xi' \in F(B', J', \delta')$ where $B' \subset B$ is the equalizer of a and b.

At this point there are two ways to finish the proof. The first is to show that $B' = B$ using compatibility of F with equalizers and the construction of B as a limit over $\mathcal{I}$ above; we omit the details. The second is to replace B by the smallest divided power subring $(B', J', \delta') \subset (B, J, \delta)$ such that ξ comes from an element $\xi' \in F(B', J', \delta')$. Since F commutes with limits F commutes with intersections hence a smallest divided power subring exists. It is clear that the transformation defined by ξ' is still surjective, and the argument above shows that it is also injective. □

Lemma 23.3.4. *The category of divided power rings has all colimits.*

Proof. The empty colimit is $\mathbf{Z}$ with divided power ideal (0). Let's discuss general colimits. Let $\mathcal{C}$ be a category and let $c \mapsto (A_c, I_c, \gamma_c)$ be a diagram. Consider the functor

$$F(B, J, \delta) = \lim_{c\in\mathcal{C}} Hom((A_c, I_c, \gamma_c), (B, J, \delta))$$

Note that any $f = (f_c)_{c\in C} \in F(B, J, \delta)$ has the property that all the images $f_c(A_c)$ generate a subring B' of B of bounded cardinality κ and that all the images $f_c(I_c)$ generate a divided power sub ideal J' of B'. And we get a factorization of f as a f' in $F(B')$ followed by the inclusion $B' \to B$. Also, F commutes with limits. Hence we may apply Lemma 23.3.3 to see that F is representable and we win. □

Remark 23.3.5. The forgetful functor $(A, I, \gamma) \mapsto A$ does not commute with colimits. For example, let

$$\begin{array}{ccc} (B, J, \delta) & \longrightarrow & (B'', J'', \delta'') \\ \uparrow & & \uparrow \\ (A, I, \gamma) & \longrightarrow & (B', J', \delta') \end{array}$$

be a pushout in the category of divided power rings. Then in general the map $B \otimes_A B' \to B''$ isn't an isomorphism. (It is always surjective.) An explicit example is given by $(A, I, \gamma) = (\mathbf{Z}, (0), \emptyset)$, $(B, J, \delta) = (\mathbf{Z}/4\mathbf{Z}, 2\mathbf{Z}/4\mathbf{Z}, \delta)$, and $(B', J', \delta') = (\mathbf{Z}/4\mathbf{Z}, 2\mathbf{Z}/4\mathbf{Z}, \delta')$ where $\delta_2(2) = 2$ and $\delta'_2(2) = 0$ and all higher divided powers equal to zero. Then $(B'', J'', \delta'') = (\mathbf{F}_2, (0), \emptyset)$ which doesn't agree with the tensor product. However, note that it is always true that

$$B''/J'' = B/J \otimes_{A/I} B'/J'$$

as can be seen from the universal property of the pushout by considering maps into divided power algebras of the form $(C, (0), \emptyset)$.

23.4. Extending divided powers

Here is the definition.

Definition 23.4.1. Given a divided power ring (A, I, γ) and a ring map $A \to B$ we say γ *extends* to B if there exists a divided power structure $\bar{\gamma}$ on IB such that $(A, I, \gamma) \to (B, IB, \bar{\gamma})$ is a homomorphism of divided power rings.

Lemma 23.4.2. *Let (A, I, γ) be a divided power ring. Let $A \to B$ be a ring map. If γ extends to B then it extends uniquely. Assume (at least) one of the following conditions holds*

1. *$IB = 0$,*
2. *I is principal, or*
3. *$A \to B$ is flat.*

Then γ extends to B.

Proof. Any element of IB can be written as a finite sum $\sum b_i x_i$ with $b_i \in B$ and $x_i \in I$. If γ extends to $\bar{\gamma}$ on IB then $\bar{\gamma}_n(x_i) = \gamma_n(x_i)$. Thus conditions (3) and (4) imply that

$$\bar{\gamma}_n(\sum b_i x_i) = \sum\nolimits_{n_1+\ldots+n_t=n} \prod\nolimits_{i=1}^{t} b_i^{n_i} \gamma_{n_i}(x_i)$$

Thus we see that $\bar{\gamma}$ is unique if it exists.

If $IB = 0$ then setting $\bar{\gamma}_n(0) = 0$ works. If $I = (x)$ then we define $\bar{\gamma}_n(bx) = b^n \gamma_n(x)$. This is well defined: if $b'x = bx$, i.e., $(b - b')x = 0$ then

$$\begin{aligned} b^n\gamma_n(x) - (b')^n\gamma_n(x) &= (b^n - (b')^n)\gamma_n(x) \\ &= (b^{n-1} + \ldots + (b')^{n-1})(b - b')\gamma_n(x) = 0 \end{aligned}$$

because $\gamma_n(x)$ is divisible by x and hence annihilated by $b - b'$. Next, we prove conditions (1) – (5) of Definition 23.2.1. Parts (1), (2), (3), (5) are obvious from

the construction. For (4) suppose that $y, z \in IB$, say $y = bx$ and $z = cx$. Then $y + z = (b + c)x$ hence

$$\begin{aligned}\bar{\gamma}_n(y + z) &= (b + c)^n \gamma_n(x) \\ &= \sum \frac{n!}{i!(n-i)!} b^i c^{n-i} \gamma_n(x) \\ &= \sum b^i c^{n-i} \gamma_i(x)\gamma_{n-i}(x) \\ &= \sum \bar{\gamma}_i(y)\bar{\gamma}_{n-i}(z)\end{aligned}$$

as desired.

Assume $A \to B$ is flat. Suppose that $b_1, \ldots, b_r \in B$ and $x_1, \ldots, x_r \in I$. Then

$$\bar{\gamma}_n(\sum b_i x_i) = \sum b_1^{e_1} \ldots b_r^{e_r} \gamma_{e_1}(x_1) \ldots \gamma_{e_r}(x_r)$$

where the sum is over $e_1 + \ldots + e_r = n$ if $\bar{\gamma}_n$ exists. Next suppose that we have $c_1, \ldots, c_s \in B$ and $a_{ij} \in A$ such that $b_i = \sum a_{ij} c_j$. Setting $y_j = \sum a_{ij} x_i$ we claim that

$$\sum b_1^{e_1} \ldots b_r^{e_r} \gamma_{e_1}(x_1) \ldots \gamma_{e_r}(x_r) = \sum c_1^{d_1} \ldots c_s^{d_s} \gamma_{d_1}(y_1) \ldots \gamma_{d_s}(y_s)$$

in B where on the right hand side we are summing over $d_1 + \ldots + d_s = n$. Namely, using the axioms of a divided power structure we can expand both sides into a sum with coefficients in $\mathbf{Z}[a_{ij}]$ of terms of the form $c_1^{d_1} \ldots c_s^{d_s} \gamma_{e_1}(x_1) \ldots \gamma_{e_r}(x_r)$. To see that the coefficients agree we note that the result is true in $\mathbf{Q}[x_1, \ldots, x_r, c_1, \ldots, c_s, a_{ij}]$ with γ the unique divided power structure on $(x_1, \ldots, x_r)$. By Lazard's theorem (Algebra, Theorem 10.76.4) we can write B as a directed colimit of finite free A-modules. In particular, if $z \in IB$ is written as $z = \sum x_i b_i$ and $z = \sum x'_{i'} b'_{i'}$, then we can find $c_1, \ldots, c_s \in B$ and $a_{ij}, a'_{i'j} \in A$ such that $b_i = \sum a_{ij} c_j$ and $b'_{i'} = \sum a'_{i'j} c_j$ such that $y_j = \sum x_i a_{ij} = \sum x'_{i'} a'_{i'j}$. Hence the procedure above gives a well defined map $\bar{\gamma}_n$ on IB. By construction $\bar{\gamma}$ satisfies conditions (1), (3), and (4). Moreover, for $x \in I$ we have $\bar{\gamma}_n(x) = \gamma_n(x)$. Hence it follows from Lemma 23.2.4 that $\bar{\gamma}$ is a divided power structure on IB. □

Lemma 23.4.3. *Let (A, I, γ) be a divided power ring.*

1. *If $\varphi : (A, I, \gamma) \to (B, J, \delta)$ is a homomorphism of divided power rings, then $Ker(\varphi) \cap I$ is preserved by γ_n for all $n \geq 1$.*
2. *Let $\mathfrak{a} \subset A$ be an ideal and set $I' = I \cap \mathfrak{a}$. The following are equivalent*
   (a) *I' is preserved by γ_n for all $n > 0$,*
   (b) *γ extends to $A/\mathfrak{a}$, and*
   (c) *there exist a set of generators x_i of I' as an ideal such that $\gamma_n(x_i) \in I'$ for all $n > 0$.*

Proof. Proof of (1). This is clear. Assume (2)(a). Define $\bar{\gamma}_n(x \bmod I') = \gamma_n(x) \bmod I'$ for $x \in I$. This is well defined since $\gamma_n(x + y) = \gamma_n(x) \bmod I'$ for $y \in I'$ by Definition 23.2.1 (4) and the fact that $\gamma_j(y) \in I'$ by assumption. It is clear that $\bar{\gamma}$ is a divided power structure as γ is one. Hence (2)(b) holds. Also, (2)(b) implies (2)(a) by part (1). It is clear that (2)(a) implies (2)(c). Assume (2)(c). Note that $\gamma_n(x) = a^n \gamma_n(x_i) \in I'$ for $x = ax_i$. Hence we see that $\gamma_n(x) \in I'$ for a set of generators of I' as an abelian group. By induction on the length of an expression in terms of these, it suffices to prove $\forall n : \gamma_n(x + y) \in I'$ if $\forall n : \gamma_n(x), \gamma_n(y) \in I'$. This follows immediately from the fourth axiom of a divided power structure. □

Lemma 23.4.4. *Let (A, I, γ) be a divided power ring. Let $E \subset I$ be a subset. Then the smallest ideal $J \subset I$ preserved by γ and containing all $f \in E$ is the ideal J generated by $\gamma_n(f)$, $n \geq 1$, $f \in E$.*

Proof. Follows immediately from Lemma 23.4.3. □

Lemma 23.4.5. *Let (A, I, γ) be a divided power ring. Let p be a prime. If p is nilpotent in A/I, then*

1. *the p-adic completion $A^\wedge = \lim_e A/p^eA$ surjects onto A/I,*
2. *the kernel of this map is the p-adic completion $I^\wedge$ of I, and*
3. *each γ_n is continuous for the p-adic topology and extends to $\gamma_n^\wedge : I^\wedge \to I^\wedge$ defining a divided power structure on $I^\wedge$.*

If moreover A is a $\mathbf{Z}_{(p)}$-algebra, then

4. *for e large enough the ideal $p^eA \subset I$ is preserved by the divided power structure γ and*
$$(A^\wedge, I^\wedge, \gamma^\wedge) = \lim_e (A/p^eA, I/p^eA, \bar{\gamma})$$
in the category of divided power rings.

Proof. Let $t \geq 1$ be an integer such that $p^tA/I = 0$, i.e., $p^tA \subset I$. The map $A^\wedge \to A/I$ is the composition $A^\wedge \to A/p^tA \to A/I$ which is surjective (for example by Algebra, Lemma 10.91.1). As $p^eI \subset p^eA \cap I \subset p^{e-t}I$ for $e \geq t$ we see that the kernel of the composition $A^\wedge \to A/I$ is the p-adic completion of I. The map γ_n is continuous because
$$\gamma_n(x + p^e y) = \sum\nolimits_{i+j=n} p^{je}\gamma_i(x)\gamma_j(y) = \gamma_n(x) \bmod p^eI$$
by the axioms of a divided power structure. It is clear that the axioms for divided power structures are inherited by the maps $\gamma_n^\wedge$ from the maps γ_n. Finally, to see the last statement say $e > t$. Then $p^eA \subset I$ and $\gamma_1(p^eA) \subset p^eA$ and for $n > 1$ we have
$$\gamma_n(p^e a) = p^n\gamma_n(p^{e-1}a) = \frac{p^n}{n!}p^{n(e-1)}a^n \in p^eA$$
as $p^n/n! \in \mathbf{Z}_{(p)}$ and as $n \geq 2$ and $e \geq 2$ so $n(e-1) \geq e$. This proves that γ extends to A/p^eA, see Lemma 23.4.3. The statement on limits is clear from the construction of limits in the proof of Lemma 23.3.2. □

23.5. Divided power polynomial algebras

A very useful example is the *divided power polynomial algebra*. Let A be a ring. Let $t \geq 1$. We will denote $A\langle x_1, \ldots, x_t\rangle$ the following A-algebra: As an A-module we set
$$A\langle x_1, \ldots, x_t\rangle = \bigoplus\nolimits_{n_1,\ldots,n_t \geq 0} A x_1^{[n_1]} \ldots x_t^{[n_t]}$$
with multiplication given by
$$x_i^{[n]}x_i^{[m]} = \frac{(n+m)!}{n!m!}x_i^{[n+m]}.$$
We also set $x_i = x_i^{[1]}$. Note that $1 = x_1^{[0]} \ldots x_t^{[0]}$. There is a similar construction which gives the divided power polynomial algebra in infinitely many variables. There is an canonical A-algebra map $A\langle x_1, \ldots, x_t\rangle \to A$ sending $x_i^{[n]}$ to zero for $n > 0$. The kernel of this map is denoted $A\langle x_1, \ldots, x_t\rangle_+$.

Lemma 23.5.1. *Let (A, I, γ) be a divided power ring. There exists a unique divided power structure δ on*

$$J = IA\langle x_1, \ldots, x_t\rangle + A\langle x_1, \ldots, x_t\rangle_+$$

such that

(1) $\delta_n(x_i) = x_i^{[n]}$, *and*
(2) $(A, I, \gamma) \to (A\langle x_1, \ldots, x_t\rangle, J, \delta)$ *is a homomorphism of divided power rings.*

Moreover, $(A\langle x_1, \ldots, x_t\rangle, J, \delta)$ has the following universal property: A homomorphism of divided power rings $\varphi : (A\langle x\rangle, J, \delta) \to (C, K, \epsilon)$ is the same thing as a homomorphism of divided power rings $A \to C$ and elements $k_1, \ldots, k_t \in K$.

Proof. We will prove the lemma in case of a divided power polynomial algebra in one variable. The result for the general case can be argued in exactly the same way, or by noting that $A\langle x_1, \ldots, x_t\rangle$ is isomorphic to the ring obtained by adjoining the divided power variables $x_1, \ldots, x_t$ one by one.

Let $A\langle x\rangle_+$ be the ideal generated by $x, x^{[2]}, x^{[3]}, \ldots$. Note that $J = IA\langle x\rangle + A\langle x\rangle_+$ and that

$$IA\langle x\rangle \cap A\langle x\rangle_+ = IA\langle x\rangle \cdot A\langle x\rangle_+$$

Hence by Lemma 23.2.5 it suffices to show that there exist divided power structures on the ideals $IA\langle x\rangle$ and $A\langle x\rangle_+$. The existence of the first follows from Lemma 23.4.2 as $A \to A\langle x\rangle$ is flat. For the second, note that if A is torsion free, then we can apply Lemma 23.2.2 (4) to see that δ exists. Namely, choosing as generators the elements $x^{[m]}$ we see that $(x^{[m]})^n = \frac{(nm)!}{(m!)^n} x^{[nm]}$ and $n!$ divides the integer $\frac{(nm)!}{(m!)^n}$. In general write $A = R/\mathfrak{a}$ for some torsion free ring R (e.g., a polynomial ring over $\mathbf{Z}$). The kernel of $R\langle x\rangle \to A\langle x\rangle$ is $\bigoplus \mathfrak{a} x^{[m]}$. Applying criterion (2)(c) of Lemma 23.4.3 we see that the divided power structure on $R\langle x\rangle_+$ extends to $A\langle x\rangle$ as desired.

Proof of the universal property. Given a homomorphism $\varphi : A \to C$ of divided power rings and $k_1, \ldots, k_t \in K$ we consider

$$A\langle x_1, \ldots, x_t\rangle \to C, \quad x_1^{[n_1]} \ldots x_t^{[n_t]} \longmapsto \epsilon_{n_1}(k_1) \ldots \epsilon_{n_t}(k_t)$$

using φ on coefficients. The only thing to check is that this is an A-algebra homomorphism (details omitted). The inverse construction is clear. □

Remark 23.5.2. Let (A, I, γ) be a divided power ring. There is a variant of Lemma 23.5.1 for infinitely many variables. First note that if $s < t$ then there is a canonical map

$$A\langle x_1, \ldots, x_s\rangle \to A\langle x_1, \ldots, x_t\rangle$$

Hence if W is any set, then we set

$$A\langle x_w, w \in W\rangle = \operatorname{colim}_{E \subset W} A\langle x_e, e \in E\rangle$$

(colimit over E finite subset of W) with transition maps as above. By the definition of a colimit we see that the universal mapping property of $A\langle x_w, w \in W\rangle$ is completely analogous to the mapping property stated in Lemma 23.5.1.

The following lemma can be found in [**BO83**].

Lemma 23.5.3. *Let p be a prime number. Let A be a ring such that every integer n not divisible by p is invertible, i.e., A is a $\mathbf{Z}_{(p)}$-algebra. Let $I \subset A$ be an ideal. Two divided power structures γ, γ' on I are equal if and only if $\gamma_p = \gamma'_p$. Moreover, given a map $\delta : I \to I$ such that*

(1) $p!\delta(x) = x^p$ *for all* $x \in I$,
(2) $\delta(ax) = a^p\delta(x)$ *for all* $a \in A$, $x \in I$, *and*
(3) $\delta(x+y) = \delta(x) + \sum_{i+j=p,i,j\geq 1} \frac{1}{i!j!}x^i y^j + \delta(y)$ *for all* $x, y \in I$,

then there exists a unique divided power structure γ *on* I *such that* $\gamma_p = \delta$.

Proof. If n is not divisible by p, then $\gamma_n(x) = cx\gamma_{n-1}(x)$ where c is a unit in $\mathbf{Z}_{(p)}$. Moreover,

$$\gamma_{pm}(x) = c\gamma_m(\gamma_p(x))$$

where c is a unit in $\mathbf{Z}_{(p)}$. Thus the first assertion is clear. For the second assertion, we can, working backwards, use these equalities to define all γ_n. More precisely, if $n = a_0 + a_1 p + \ldots + a_e p^e$ with $a_i \in \{0, \ldots, p-1\}$ then we set

$$\gamma_n(x) = c_n x^{a_0}\delta(x)^{a_1} \ldots \delta^e(x)^{a_e}$$

for $c_n \in \mathbf{Z}_{(p)}$ defined by

$$c_n = (p!)^{a_1 + a_2(1+p) + \ldots + a_e(1+\ldots+p^{e-1})}/n!.$$

Now we have to show the axioms (1) – (5) of a divided power structure, see Definition 23.2.1. We observe that (1) and (3) are immediate. Verification of (2) and (5) is by a direct calculation which we omit. Let $x, y \in I$. We claim there is a ring map

$$\varphi : \mathbf{Z}_{(p)}\langle u, v\rangle \longrightarrow A$$

which maps $u^{[n]}$ to $\gamma_n(x)$ and $v^{[n]}$ to $\gamma_n(y)$. By construction of $\mathbf{Z}_{(p)}\langle u, v\rangle$ this means we have to check that

$$\gamma_n(x)\gamma_m(x) = \frac{(n+m)!}{n!m!}\gamma_{n+m}(x)$$

in A and similarly for y. This is true because (2) holds for γ. Let ϵ denote the divided power structure on the ideal $\mathbf{Z}_{(p)}\langle u, v\rangle_+$ of $\mathbf{Z}_{(p)}\langle u, v\rangle$. Next, we claim that $\varphi(\epsilon_n(f)) = \gamma_n(\varphi(f))$ for $f \in \mathbf{Z}_{(p)}\langle u, v\rangle_+$ and all n. This is clear for $n = 0, 1, \ldots, p-1$. For $n = p$ it suffices to prove it for a set of generators of the ideal $\mathbf{Z}_{(p)}\langle u, v\rangle_+$ because both ϵ_p and $\gamma_p = \delta$ satisfy properties (1) and (3) of the lemma. Hence it suffices to prove that $\gamma_p(\gamma_n(x)) = \frac{(pn)!}{p!(n!)^p}\gamma_{pn}(x)$ and similarly for y, which follows as (5) holds for γ. Now, if $n = a_0 + a_1 p + \ldots + a_e p^e$ is an arbitrary integer written in p-adic expansion as above, then

$$\epsilon_n(f) = c_n f^{a_0}\gamma_p(f)^{a_1} \ldots \gamma_p^e(f)^{a_e}$$

because ϵ is a divided power structure. Hence we see that $\varphi(\epsilon_n(f)) = \gamma_n(\varphi(f))$ holds for all n. Applying this for $f = u + v$ we see that axiom (4) for γ follows from the fact that ϵ is a divided power structure. □

23.6. Tate resolutions

In this section we briefly discuss the resolutions constructed in [**Tat57**] which combine divided power structures with differential graded algebras. In this section we will use *homological notation* for differential graded algebras. Our differential graded algebras will sit in nonnegative homological degrees. Thus our differential graded algebras (A, d) will be given as chain complexes

$$\ldots \to A_2 \to A_1 \to A_0 \to 0 \to \ldots$$

endowed with a multiplication.

Let R be a ring. In this section we will often consider graded R-algebras $A = \bigoplus_{d\geq 0} A_d$ whose components are zero in negative degrees. We will set $A_+ = \bigoplus_{d>0} A_d$. We will write $A_{even} = \bigoplus_{d\geq 0} A_{2d}$ and $A_{odd} = \bigoplus_{d\geq 0} A_{2d+1}$. Recall that A is graded commutative if $xy = (-1)^{\deg(x)\deg(y)} yx$ for homogeneous elements x, y. Recall that A is strictly graded commutative if in addition $x^2 = 0$ for homogeneous elements x of odd degree. Finally, to understand the following definition, keep in mind that $\gamma_n(x) = x^n/n!$ if A is a $\mathbf{Q}$-algebra.

Definition 23.6.1. Let R be a ring. Let $A = \bigoplus_{d\geq 0} A_d$ be a graded R-algebra which is strictly graded commutative. A collection of maps $\gamma_n : A_{even,+} \to A_{even,+}$ defined for all $n > 0$ is called a *divided power structure* on A if we have

(1) $\gamma_n(x) \in A_{2nd}$ if $x \in A_{2d}$,
(2) $\gamma_1(x) = x$ for any x, we also set $\gamma_0(x) = 1$,
(3) $\gamma_n(x)\gamma_m(x) = \frac{(n+m)!}{n!m!}\gamma_{n+m}(x)$,
(4) $\gamma_n(xy) = x^n\gamma_n(y)$ for all $x \in A_{even}$ and $y \in A_{even,+}$,
(5) $\gamma_n(xy) = 0$ if $x, y \in A_{odd}$ homogeneous and $n > 1$
(6) if $x, y \in A_{even,+}$ then $\gamma_n(x + y) = \sum_{i=0,\ldots,n} \gamma_i(x)\gamma_{n-i}(y)$,
(7) $\gamma_n(\gamma_m(x)) = \frac{(nm)!}{n!(m!)^n}\gamma_{nm}(x)$ for $x \in A_{even,+}$.

Observe that conditions (2), (3), (4), (6), and (7) imply that γ is a "usual" divided power structure on the ideal $A_{even,+}$ of the (commutative) ring A_{even}, see Sections 23.2, 23.3, 23.4, and 23.5. In particular, we have $\gamma_n(x) = n!x^n$ for all $x \in A_{even,+}$. Condition (1) states that γ is compatible with grading and condition (5) tells us γ_n for $n > 1$ vanishes on products of homogeneous elements of odd degree. But note that it may happen that

$$\gamma_2(z_1z_2 + z_3z_4) = z_1z_2z_3z_4$$

is nonzero if z_1, z_2, z_3, z_4 are homogeneous elements of odd degree.

Example 23.6.2 (Adjoining odd variable)**.** Let R be a ring. Let (A, γ) be a strictly graded commutative graded R-algebra endowed with a divided power structure as in the definition above. Let $d > 0$ be an odd integer. In this setting we can adjoin a variable T of degree d to A. Namely, set

$$A\langle T\rangle = A \oplus AT$$

with grading given by $A\langle T\rangle_m = A_m \oplus A_{m-d}T$. We claim there is a unique divided power structure on $A\langle T\rangle$ compatible with the given divided power structure on A. Namely, we set

$$\gamma_n(x + yT) = \gamma_n(x) + \gamma_{n-1}(x)yT$$

for $x \in A_{even,+}$ and $y \in A_{odd}$.

Example 23.6.3 (Adjoining even variable)**.** Let R be a ring. Let (A, γ) be a strictly graded commutative graded R-algebra endowed with a divided power structure as in the definition above. Let $d > 0$ be an even integer. In this setting we can adjoin a variable T of degree d to A. Namely, set

$$A\langle T\rangle = A \oplus AT \oplus AT^{(2)} \oplus AT^{(3)} \oplus \ldots$$

with multiplication given by

$$T^{(n)}T^{(m)} = \frac{(n+m)!}{n!m!}T^{(n+m)}$$

and with grading given by

$$A\langle T\rangle_m = A_m \oplus A_{m-d}T \oplus A_{m-2d}T^{(2)} \oplus \ldots$$

We claim there is a unique divided power structure on $A\langle T\rangle$ compatible with the given divided power structure on A such that $\gamma_n(T^{(i)}) = T^{(ni)}$. To define the divided power structure we first set

$$\gamma_n\left(\sum\nolimits_{i>0} x_i T^{(i)}\right) = \sum \prod\nolimits_{n=\sum e_i} x_i^{e_i} T^{(ie_i)}$$

if x_i is in A_{even}. If $x_0 \in A_{even,+}$ then we take

$$\gamma_n\left(\sum\nolimits_{i\geq 0} x_i T^{(i)}\right) = \sum\nolimits_{a+b=n} \gamma_a(x_0)\gamma_b\left(\sum\nolimits_{i>0} x_i T^{(i)}\right)$$

where γ_b is as defined above.

At this point we tie in the definition of divided power structures with differentials. To understand the definition note that $\mathrm{d}(x^n/n!) = \mathrm{d}(x)x^{n-1}/(n-1)!$ if A is a $\mathbf{Q}$-algebra and $x \in A_{even,+}$.

Definition 23.6.4. Let R be a ring. Let $A = \bigoplus_{d\geq 0} A_d$ be a differential graded R-algebra which is strictly graded commutative. A divided power structure γ on A is *compatible with the differential graded structure* if $\mathrm{d}(\gamma_n(x)) = \mathrm{d}(x)\gamma_{n-1}(x)$ for all $x \in A_{even,+}$.

Warning: Let (A, d, γ) be as in Definition 23.6.4. It may not be true that $\gamma_n(x)$ is a boundary, if x is a boundary. Thus γ in general does not induce a divided power structure on the homology algebra $H(A)$. In some papers the authors put an additional compatibility condition in order to insure this is the case, but we elect not to do so.

Lemma 23.6.5. *Let (A, d, γ) and (B, d, γ) be as in Definition 23.6.4. Let $f : A \to B$ be a map of differential graded algebras compatible with divided power structures. Assume*

(1) *$H_k(A) = 0$ for $k > 0$, and*
(2) *f is surjective.*

Then γ induces a divided power structure on the graded R-algebra $H(B)$.

Proof. Suppose that x and x' are homogeneous of the same degree $2d$ and define the same cohomology class in $H(B)$. Say $x' - x = \mathrm{d}(w)$. Choose a lift $y \in A_{2d}$ of x and a lift $z \in A_{2d+1}$ of w. Then $y' = y + \mathrm{d}(z)$ is a lift of x'. Hence

$$\gamma_n(y') = \sum \gamma_i(y)\gamma_{n-i}(\mathrm{d}(z)) = \gamma_n(y) + \sum\nolimits_{i<n} \gamma_i(y)\gamma_{n-i}(\mathrm{d}(z))$$

Since A is acyclic in positive degrees and since $\mathrm{d}(\gamma_j(\mathrm{d}(z))) = 0$ for all j we can write this as

$$\gamma_n(y') = \gamma_n(y) + \sum\nolimits_{i<n} \gamma_i(y)\mathrm{d}(z_i)$$

for some z_i in A. Moreover, for $0 < i < n$ we have

$$\mathrm{d}(\gamma_i(y)z_i) = \mathrm{d}(\gamma_i(y))z_i + \gamma_i(y)\mathrm{d}(z_i) = \mathrm{d}(y)\gamma_{i-1}(y)z_i + \gamma_i(y)\mathrm{d}(z_i)$$

and the first term maps to zero in B as $\mathrm{d}(y)$ maps to zero in B. Hence $\gamma_n(x')$ and $\gamma_n(x)$ map to the same element of $H(B)$. Thus we obtain a well defined map $\gamma_n : H_{2d}(B) \to H_{2nd}(B)$ for all $d > 0$ and $n > 0$. We omit the verification that this defines a divided power structure on $H(B)$. □

Lemma 23.6.6. *Let (A, d, γ) is as in Definition 23.6.4. Let $R \to R'$ be a ring map. Then d and γ induce similar structures on $A' = A \otimes_R R'$ such that (A, d, γ) is as in Definition 23.6.4.*

Proof. Observe that $A'_{even} = A_{even} \otimes_R R'$ and $A'_{even,+} = A_{even,+} \otimes_R R'$. Hence we are trying to show that the divided powers γ extend to A'_{even} (terminology as in Definition 23.4.1). Once we have shown γ extends it follows easily that this extension has all the desired properties.

Choose a polynomial R-algebra P and a surjection of R-algebras $P \to R'$. The ring map $A_{even} \to A_{even} \otimes_R P$ is flat, hence the divided powers γ extend to $A_{even} \otimes_R P$ uniquely by Lemma 23.4.2. Let $J = \mathrm{Ker}(P \to R')$. To show that γ extends to $A \otimes_R R'$ it suffices to show that $I' = \mathrm{Ker}(A_{even,+} \otimes_R P \to A_{even,+} \otimes_R R')$ is generated by elements z such that $\gamma_n(z) \in I'$ for all $n > 0$. This is clear as I' is generated by elements of the form $x \otimes f$ with $x \in A_{even,+}$ and $f \in \mathrm{Ker}(P \to R')$. □

Lemma 23.6.7. *Let (A, d, γ) be as in Definition 23.6.4. Let $d \geq 1$ be an integer. Let $A\langle T \rangle$ be the graded divided power polynomial algebra on T with $\deg(T) = d$ constructed in Example 23.6.2 or 23.6.3. Let $f \in A_{d-1}$ be an element with $d(f) = 0$. There exists a unique differential d on $A\langle T \rangle$ such that $d(T) = f$ and such that d is compatible with the divided power structure on $A\langle T \rangle$.*

Proof. This is proved by a direct computation which is omitted. □

Here is the construction of Tate.

Lemma 23.6.8. *Assume that R is a Noetherian ring and $R \to S$ a ring map of finite type. There exists a factorization*

$$R \to A \to S$$

with the following properties

1. *(A, d, γ) is as in Definition 23.6.4,*
2. *$A \to S$ is a quasi-isomorphism (if we endow S with the zero differential),*
3. *A is a graded divided power polynomial algebra over R with finitely many variables in each degree.*

The last condition means that A is constructed out of R by successively adjoining variables T as in Examples 23.6.2 and 23.6.3.

Proof. Start of the construction. Let $A(0) = R[x_1, \ldots, x_n]$ be a (usual) polynomial ring and let $A(0) \to S$ be a surjection. As grading we take $A(0)_0 = A(0)$ and $A(0)_d = 0$ for $d \neq 0$. Thus $\mathrm{d} = 0$ and γ_n, $n > 0$ is zero as well.

Choose generators $f_1, \ldots, f_m \in R[x_1, \ldots, x_m]$ for the kernel of the given map $A(0) = R[x_1, \ldots, x_m] \to S$. We apply Examples 23.6.2 m times to get

$$A(1) = A(0)\langle T_1, \ldots, T_m \rangle$$

with $\deg(T_i) = 1$ as a graded divided power polynomial algebra. We set $\mathrm{d}(T_i) = f_i$. Since $A(1)$ is a divided power polynomial algebra over $A(0)$ and since $\mathrm{d}(f_i) = 0$ this extends uniquely to a differential on $A(1)$ by Lemma 23.6.7.

Induction hypothesis: Assume we are given factorizations

$$R \to A(0) \to A(1) \to \ldots \to A(m) \to S$$

where $A(0)$ and $A(1)$ are as above and each $R \to A(m') \to S$ for $2 \leq m' \leq m$ satisfies properties (1) and (3) of the statement of the lemma and (2) replaced by the condition that $H_i(A(m')) \to H_i(S)$ is an isomorphism for $m' > i \geq 0$. The base case is $m = 1$.

Induction step. Assume we have $R \to A(m) \to S$ as in the induction hypothesis. Consider the group $H_m(A(m))$. This is a module over $H_0(A(m)) = S$. In fact, it is a subquotient of $A(m)_m$ which is a finite type module over $A(m)_0 = R[x_1, \ldots, x_n]$. Thus we can pick finitely many elements

$$e_1, \ldots, e_t \in \mathrm{Ker}(\mathrm{d} : A(m)_m \to A(m)_{m-1})$$

which map to generators of this module. Applying Example 23.6.3 or 23.6.2 t times we get

$$A(m+1) = A(m)\langle T_1, \ldots, T_t \rangle$$

with $\deg(T_i) = m + 1$ as a graded divided power algebra. We set $\mathrm{d}(T_i) = e_i$. Since $A(1)$ is a divided power polynomial algebra over $A(0)$ and since $\mathrm{d}(e_i) = 0$ this extends uniquely to a differential on $A(m + 1)$ compatible with the divided power structure. Since we've added only material in degree $m + 1$ and higher we see that $H_i(A(m+1)) = H_i(A(m))$ for $i < m$. Moreover, it is clear that $H_m(A(m+1)) = 0$ by construction.

To finish the proof we observe that we have shown there exists a sequence of maps

$$R \to A(0) \to A(1) \to \ldots \to A(m) \to A(m + 1) \to \ldots \to S$$

and to finish the proof we set $A = \mathrm{colim}\, A(m)$. □

Lemma 23.6.9. *Let R be a ring. Suppose that (A, d, γ) and (B, d, γ) are as in Definition 23.6.4. Let $\overline{\varphi} : H_0(A) \to H_0(B)$ be an R-algebra map. Assume*

1. *A is a graded divided power polynomial algebra over R with finitely many variables in each degree,*
2. *$H_k(B) = 0$ for $k > 0$.*

Then there exists a map $\varphi : A \to B$ of differential graded R-algebras compatible with divided powers lifting φ.

Proof. Since A is obtained from R by adjoining divided power variables, we obtain filtrations $R \subset A(0) \subset A(1) \subset \ldots$ such that $A(m + 1)$ is obtained from $A(m)$ by adjoining finitely many divided power variables of degree $m + 1$. Then $A(0) \to S$ is a surjection from a (usual) polynomial algebra over R onto S. Thus we can lift $\overline{\varphi}$ to an R-algebra map $\varphi(0) : A(0) \to B(0)$.

Write $A(1) = A(0)\langle T_1, \ldots, T_m \rangle$ for some divided power variables T_j of degree 1. Let $f_j \in B_0$ be $f_j = \varphi(0)(\mathrm{d}(T_j))$. Observe that f_j maps to zero in $H_0(B)$ as $\mathrm{d}T_j$ maps to zero in $H_0(A)$. Thus we can find $b_j \in B_1$ with $\mathrm{d}(b_j) = f_j$. By the universal property of divided power polynomial algebras we find a lift $\varphi(1) : A(1) \to B$ of $\varphi(0)$ mapping T_j to f_j.

Having constructed $\varphi(m)$ for some $m \geq 1$ we can construct $\varphi(m+1) : A(m+1) \to B$ in exactly the same manner. We omit the details. □

Lemma 23.6.10. *Let R be a Noetherian ring. Let $R \to S$ and $R \to T$ be finite type ring maps. There exists a canonical structure of a divided power graded R-algebra on*

$$Tor_*^R(S, T)$$

Proof. Choose a factorization $R \to A \to S$ as above. Since $A \to S$ is a quasi-isomorphism and since A_d is a free R-module, we see that the differential graded algebra $B = A \otimes_R T$ computes the tor groups displayed in the lemma. Choose a surjection $R[y_1, \ldots, y_k] \to T$. Then we see that B is a quotient of the differential graded algebra $A[y_1, \ldots, y_k]$ whose homology sits in degree 0 (it is equal to $S[y_1, \ldots, y_k]$). By Lemma 23.6.6 the differential graded algebras B and $A[y_1, \ldots, y_k]$ have divided power structures compatible with the differentials. Hence we obtain our divided power structure on $H(B)$ by Lemma 23.6.5.

The divided power algebra structure constructed in this way is independent of the choice of A. Namely, if A' is a second choice, then Lemma 23.6.9 implies there is a map $A \to A'$ preserving all structure and the augmentations towards S. Then the induced map $B = A \otimes_R T \to A' \otimes_R T' = B'$ is likewise and is a quasi-isomorphism. The induced isomorphism of Tor algebras is therefore compatible with all multiplication and divided powers. □

23.7. Application to complete intersections

Let R be a ring. Let (A, d, γ) be as in Definition 23.6.4. A *derivation* of degree 2 is an R-linear map $\theta : A \to A$ with the following properties

(1) $\theta(A_d) \subset A_{d-2}$,
(2) $\theta(xy) = \theta(x)y + x\theta(y)$,
(3) θ commutes with d,
(4) $\theta(\gamma_n(x)) = \theta(x)\gamma_{n-1}(x)$ for all $x \in A_{2d}$ all d.

In the following lemma we construct a derivation.

Lemma 23.7.1. *Let R be a ring. Let (A, d, γ) be as in Definition 23.6.4. Let $R' \to R$ be a surjection of rings whose kernel has square zero and is generated by one element f. If A is a graded divided power polynomial algebra over R with finitely many variables in each degree, then we obtain a derivation $\theta : A/IA \to A/IA$ where I is the annihilator of f in R.*

Proof. Since A is a divided power polynomial algebra, we can find a divided power polynomial algebra A' over R' such that $A = A' \otimes_R R'$. Moreover, we can lift d to an R-linear operator d on A' such that

(1) $\mathrm{d}(xy) = \mathrm{d}(x)y + (-1)^{\deg(x)}x\mathrm{d}(y)$ for $x, y \in A'$ homogeneous, and
(2) $\mathrm{d}(\gamma_n(x)) = \mathrm{d}(x)\gamma_{n-1}(x)$ for $x \in A'_{even,+}$.

We omit the details (hint: proceed one variable at the time). However, it may not be the case that d^2 is zero on A'. It is clear that d^2 maps A' into $fA' \cong A/IA$. Hence d^2 annihilates fA' and factors as a map $A \to A/IA$. Since d^2 is R-linear we obtain our map $\theta : A/IA \to A/IA$. The verification of the properties of a derivation is immediate. □

Lemma 23.7.2. *Assumption and notation as in Lemma 23.7.1. Suppose $S = H_0(A)$ is isomorphic to $R[x_1, \ldots, x_n]/(f_1, \ldots, f_m)$ for some n, m, and $f_j \in R[x_1, \ldots, x_n]$. Moreover, suppose given a relation*

$$\sum r_j f_j = 0$$

with $r_j \in R[x_1, \ldots, x_n]$. Choose $r'_j, f'_j \in R'[x_1, \ldots, x_n]$ lifting r_j, f_j. Write $\sum r'_j f'_j = gf$ for some $g \in R/I[x_1, \ldots, x_n]$. If $H_1(A) = 0$ and all the coefficients of each r_j are in I, then there exists an element $\xi \in H_2(A/IA)$ such that $\theta(\xi) = g$ in S/IS.

Proof. Let $A(0) \subset A(1) \subset A(2) \subset \ldots$ be the filtration of A such that $A(m)$ is gotten from $A(m-1)$ by adjoining divided power variables of degree m. Then $A(0)$ is a polynomial algebra over R equipped with an R-algebra surjection $A(0) \to S$. Thus we can choose a map

$$\varphi : R[x_1, \ldots, x_n] \to A(0)$$

lifting the augmentations to S. Next, $A(1) = A(0)\langle T_1, \ldots, T_t\rangle$ for some divided power variables T_i of degree 1. Since $H_0(A) = S$ we can pick $\xi_j \in \sum A(0)T_i$ with $\mathrm{d}(\xi_j) = \varphi(f_j)$. Then

$$\mathrm{d}\left(\sum \varphi(r_j)\xi_j\right) = \sum \varphi(r_j)\varphi(f_j) = \sum \varphi(r_j f_j) = 0$$

Since $H_1(A) = 0$ we can pick $\xi \in A_2$ with $\mathrm{d}(\xi) = \sum \varphi(r_j)\xi_j$. If the coefficients of r_j are in I, then the same is true for $\varphi(r_j)$. In this case $\mathrm{d}(\xi)$ dies in A_1/IA_1 and hence ξ defines a class in $H_2(A/IA)$.

The construction of θ in the proof of Lemma 23.7.1 proceeds by successively lifting $A(i)$ to $A'(i)$ and lifting the differential d. We lift φ to $\varphi' : R'[x_1, \ldots, x_n] \to A'(0)$. Next, we have $A'(1) = A'(0)\langle T_1, \ldots, T_t\rangle$. Moreover, we can lift ξ_j to $\xi'_j \in \sum A'(0)T_i$. Then $\mathrm{d}(\xi'_j) = \varphi'(f'_j) + fa_j$ for some $a_j \in A'(0)$. Consider a lift $\xi' \in A'_2$ of ξ. Then we know that

$$\mathrm{d}(\xi') = \sum \varphi'(r'_j)\xi'_j + \sum fb_iT_i$$

for some $b_i \in A(0)$. Applying d again we find

$$\theta(\xi) = \sum \varphi'(r'_j)\varphi'(f'_j) + \sum f\varphi'(r'_j)a_j + \sum fb_i\mathrm{d}(T_i)$$

The first term gives us what we want. The second term is zero because the coefficients of r_j are in I and hence are annihilated by f. The third term maps to zero in H_0 because $\mathrm{d}(T_i)$ maps to zero. □

The method of proof of the following lemma is apparantly due to Gulliksen.

Lemma 23.7.3. *Let $R' \to R$ be a surjection of Noetherian rings whose kernel has square zero and is generated by one element f. Let $S = R[x_1, \ldots, x_n]/(f_1, \ldots, f_m)$. Let $\sum r_j f_j = 0$ be a relation in $R[x_1, \ldots, x_m]$. Assume that*

1. *each r_j has coeffients in the annihilator I of f in R,*
2. *for some lifts $r'_j, f'_j \in R'[x_1, \ldots, x_n]$ we have $\sum r'_j f'_j = gf$ where g is not nilpotent in S.*

Then S does not have finite tor dimension over R (i.e., S is not a perfect R-algebra).

Proof. Choose a Tate resolution $R \to A \to S$ as in Lemma 23.6.8. Let $\xi \in H_2(A/IA)$ and $\theta : A/IA \to A/IA$ be the element and derivation found in Lemmas 23.7.1 and 23.7.2. Observe that

$$\theta^n(\gamma_n(\xi)) = g^n$$

Hence if g is not nilpotent, then ξ^n is nonzero in $H_{2n}(A/IA)$ for all $n > 0$. Since $H_{2n}(A/IA) = \mathrm{Tor}^R_{2n}(S, R/I)$ we conclude. □

The following result can be found in [**Rod88**].

Lemma 23.7.4. *Let $(A, \mathfrak{m})$ be a Noetherian local ring. Let $I \subset J \subset A$ be proper ideals. If A/J has finite tor dimension over A/I, then $I/\mathfrak{m}I \to J/\mathfrak{m}J$ is injective.*

Proof. Let $f \in I$ be an element mapping to a nonzero element of $I/\mathfrak{m}I$ which is mapped to zero in $J/\mathfrak{m}J$. We can choose an ideal I' with $\mathfrak{m}I \subset I' \subset I$ such that I/I' is generated by the image of f. Set $R = A/I$ and $R' = A/I'$. Let $J = (a_1, \ldots, a_m)$ for some $a_j \in A$. Then $f = \sum b_j a_j$ for some $b_j \in \mathfrak{m}$. Let $r_j, f_j \in R$ resp. $r'_j, f'_j \in R'$ be the image of b_j, a_j. Then we see we are in the situation of Lemma 23.7.3 (with the ideal I of that lemma equal to $\mathfrak{m}_R$) and the lemma is proved. □

Lemma 23.7.5. *Let $(A, \mathfrak{m})$ be a Noetherian local ring. Let $I \subset J \subset A$ be proper ideals. Assume*

(1) *A/J has finite tor dimension over A/I, and*
(2) *J is generated by a regular sequence.*

Then I is generated by a regular sequence and J/I is generated by a regular sequence.

Proof. By Lemma 23.7.4 we see that $I/\mathfrak{m}I \to J/\mathfrak{m}J$ is injective. Thus we can find $s \leq r$ and a minimal system of generators $f_1, \ldots, f_r$ of J such that $f_1, \ldots, f_s$ are in I and form a minimal system of generators of I. The lemma follows as any minimal system of generators of J is a regular sequence by More on Algebra, Lemmas 15.20.14 and 15.20.6. □

Lemma 23.7.6. *Let $R \to S$ be a local ring map of Noetherian local rings. Let $I \subset R$ and $J \subset S$ be ideals with $IS \subset J$. If $R \to S$ is flat and $S/\mathfrak{m}_R S$ is regular, then the following are equivalent*

(1) *J is generated by a regular sequence and S/J has finite tor dimension as a module over R/I,*
(2) *J is generated by a regular sequence and $Tor_p^{R/I}(S/J, R/\mathfrak{m}_R)$ is nonzero for only finitely many p,*
(3) *I is generated by a regular sequence and J/IS is generated by a regular sequence in S/IS.*

Proof. If (3) holds, then J is generated by a regular sequence, see for example More on Algebra, Lemmas 15.20.12 and 15.20.6. Moreover, if (3) holds, then $S/J = (S/I)/(J/I)$ has finite projective dimension over S/IS because the Koszul complex will be a finite free resolution of S/J over S/IS. Since $R/I \to S/IS$ is flat, it then follows that S/J has finite tor dimension over R/I by More on Algebra, Lemma 15.49.9. Thus (3) implies (1).

The implication (1) $\Rightarrow$ (2) is trivial. Assume (2). By More on Algebra, Lemma 15.50.18 we find that S/J has finite tor dimension over S/IS. Thus we can apply Lemma 23.7.5 to conclude that IS and J/IS are generated by regular sequences. Let $f_1, \ldots, f_r \in I$ be a minimal system of generators of I. Since $R \to S$ is flat, we see that $f_1, \ldots, f_r$ form a minimal system of generators for IS in S. Thus $f_1, \ldots, f_r \in R$ is a sequence of elements whose images in S form a regular sequence by More on Algebra, Lemmas 15.20.14 and 15.20.6. Thus $f_1, \ldots, f_r$ is a regular sequence in R by Algebra, Lemma 10.65.7. □

23.8. Local complete intersection rings

Let $(A, \mathfrak{m})$ be a Noetherian complete local ring. By the Cohen structure theorem (see Algebra, Theorem 10.147.8) we can write A as the quotient of a regular Noetherian complete local ring R. Let us say that A is a *complete intersection* if there exists some surjection $R \to A$ with R a regular local ring such that the

kernel is generated by a regular sequence. The following lemma shows this notion is independent of the choice of the surjection.

Lemma 23.8.1. *Let $(A, \mathfrak{m})$ be a Noetherian complete local ring. The following are equivalent*

(1) *for every surjection of local rings $R \to A$ with R a regular local ring, the kernel of $R \to A$ is generated by a regular sequence, and*
(2) *for some surjection of local rings $R \to A$ with R a regular local ring, the kernel of $R \to A$ is generated by a regular sequence.*

Proof. Let k be the residue field of A. If the characteristic of k is $p > 0$, then we denote Λ a Cohen ring (Algebra, Definition 10.147.5) with residue field k (Algebra, Lemma 10.147.6). If the characteristic of k is 0 we set $\Lambda = k$. Recall that $\Lambda[[x_1, \ldots, x_n]]$ for any n is formally smooth over $\mathbf{Z}$, resp. $\mathbf{Q}$ in the $\mathfrak{m}$-adic topology, see More on Algebra, Lemma 15.27.1. Fix a surjection $\Lambda[[x_1, \ldots, x_n]] \to A$ as in the Cohen structure theorem (Algebra, Theorem 10.147.8).

Let $R \to A$ be a surjection from a regular local ring R. Let $f_1, \ldots, f_r$ be a minimal sequence of generators of $\mathrm{Ker}(R \to A)$. We will use without further mention that an ideal in a Noetherian local ring is generated by a regular sequence if and only if any minimal set of generators is a regular sequence. Observe that $f_1, \ldots, f_r$ is a regular sequence in R if and only if $f_1, \ldots, f_r$ is a regular sequence in the completion $R^\wedge$ by Algebra, Lemmas 10.65.7 and 10.91.3. Moreover, we have

$$R^\wedge/(f_1, \ldots, f_r)R^\wedge = (R/(f_1, \ldots, f_n))^\wedge = A^\wedge = A$$

because A is $\mathfrak{m}_A$-adically complete (first equality by Algebra, Lemma 10.91.2). Finally, the ring $R^\wedge$ is regular since R is regular (More on Algebra, Lemma 15.31.4). Hence we may assume R is complete.

If R is complete we can choose a map $\Lambda[[x_1, \ldots, x_n]] \to R$ lifting the given map $\Lambda[[x_1, \ldots, x_n]] \to A$, see More on Algebra, Lemma 15.26.5. By adding some more variables $y_1, \ldots, y_m$ mapping to generators of the kernel of $R \to A$ we may assume that $\Lambda[[x_1, \ldots, x_n, y_1, \ldots, y_m]] \to R$ is surjective (some details omitted). Then we can consider the commutative diagram

$$\begin{array}{ccc} \Lambda[[x_1, \ldots, x_n, y_1, \ldots, y_m]] & \longrightarrow & R \\ \downarrow & & \downarrow \\ \Lambda[[x_1, \ldots, x_n]] & \longrightarrow & A \end{array}$$

By Algebra, Lemma 10.128.6 we see that the condition for $R \to A$ is equivalent to the condition for the fixed chosen map $\Lambda[[x_1, \ldots, x_n]] \to A$. This finishes the proof of the lemma. □

The following two lemmas are sanity checks on the definition given above.

Lemma 23.8.2. *Let R be a regular ring. Let $\mathfrak{p} \subset R$ be a prime. Let $f_1, \ldots, f_r \in \mathfrak{p}$ be a regular sequence. Then the completion of*

$$A = (R/(f_1, \ldots, f_r))_\mathfrak{p} = R_\mathfrak{p}/(f_1, \ldots, f_r)R_\mathfrak{p}$$

is a complete intersection in the sense defined above.

Proof. The completion of A is equal to $A^\wedge = R^\wedge_\mathfrak{p}/(f_1, \ldots, f_r)R^\wedge_\mathfrak{p}$ because completion for finite modules over the Noetherian ring $R_\mathfrak{p}$ is exact (Algebra, Lemma 10.91.2). The image of the sequence $f_1, \ldots, f_r$ in $R_\mathfrak{p}$ is a regular sequence by Algebra, Lemmas 10.91.3 and 10.65.7. Moreover, $R^\wedge_\mathfrak{p}$ is a regular local ring by More on Algebra, Lemma 15.31.4. Hence the result holds by our definition of complete intersection for complete local rings. □

The following lemma is the analogue of Algebra, Lemma 10.128.4.

Lemma 23.8.3. *Let R be a regular ring. Let $\mathfrak{p} \subset R$ be a prime. Let $I \subset \mathfrak{p}$ be an ideal. Set $A = (R/I)_\mathfrak{p} = R_\mathfrak{p}/I_\mathfrak{p}$. The following are equivalent*

(1) *the completion of A is a complete intersection in the sense above,*
(2) *$I_\mathfrak{p} \subset R_\mathfrak{p}$ is generated by a regular sequence,*
(3) *the module $(I/I^2)_\mathfrak{p}$ can be generated by $\dim(R_\mathfrak{p}) - \dim(A)$ elements,*
(4) *add more here.*

Proof. We may and do replace R by its localization at $\mathfrak{p}$. Then $\mathfrak{p} = \mathfrak{m}$ is the maximal ideal of R and $A = R/I$. Let $f_1, \ldots, f_r \in I$ be a minimal sequence of generators. The completion of A is equal to $A^\wedge = R^\wedge/(f_1, \ldots, f_r)R^\wedge$ because completion for finite modules over the Noetherian ring $R_\mathfrak{p}$ is exact (Algebra, Lemma 10.91.2).

If (1) holds, then the image of the sequence $f_1, \ldots, f_r$ in $R^\wedge$ is a regular sequence by assumption. Hence it is a regular sequence in R by Algebra, Lemmas 10.91.3 and 10.65.7. Thus (1) implies (2).

Assume (3) holds. Set $c = \dim(R) - \dim(A)$ and let $f_1, \ldots, f_c \in I$ map to generators of I/I^2. by Nakayama's lemme (Algebra, Lemma 10.18.1) we see that $I = (f_1, \ldots, f_c)$. Since R is regular and hence Cohen-Macaulay (Algebra, Proposition 10.98.4) we see that $f_1, \ldots, f_c$ is a regular sequence by Algebra, Proposition 10.98.4. Thus (3) implies (2). Finally, (2) implies (1) by Lemma 23.8.2. □

The following result is due to Avramov, see [**Avr75**].

Proposition 23.8.4. *Let $A \to B$ be a flat local homomorphism of Noetherian local rings. Then the following are equivalent*

(1) *$B^\wedge$ is a complete intersection,*
(2) *$A^\wedge$ and $(B/\mathfrak{m}_A B)^\wedge$ are complete intersections.*

Proof. Consider the diagram

$$\begin{array}{ccc} B & \longrightarrow & B^\wedge \\ \uparrow & & \uparrow \\ A & \longrightarrow & A^\wedge \end{array}$$

Since the horizontal maps are faithfully flat (Algebra, Lemma 10.91.4) we conclude that the right vertical arrow is flat (for example by Algebra, Lemma 10.93.14). Moreover, we have $(B/\mathfrak{m}_A B)^\wedge = B^\wedge/\mathfrak{m}_{A^\wedge} B^\wedge$ by Algebra, Lemma 10.91.2. Thus we may assume A and B are complete local Noetherian rings.

Assume A and B are complete local Noetherian rings. Choose a diagram

$$\begin{array}{ccc} S & \longrightarrow & B \\ \uparrow & & \uparrow \\ R & \longrightarrow & A \end{array}$$

as in More on Algebra, Lemma 15.27.3. Let $I = \mathrm{Ker}(R \to A)$ and $J = \mathrm{Ker}(S \to B)$. Note that since $R/I = A \to B = S/J$ is flat the map $J/I \otimes_R R/\mathfrak{m}_R \to J/J \cap \mathfrak{m}_R S$ is an isomorphism. Hence a minimal system of generators of J/I maps to a minimal system of generators of $\mathrm{Ker}(S/\mathfrak{m}_R S \to B/\mathfrak{m}_A B)$. Finally, $S/\mathfrak{m}_R S$ is a regular local ring.

Assume (1) holds, i.e., J is generated by a regular sequence. Since $A = R/I \to B = S/J$ is flat we see Lemma 23.7.6 applies and we deduce that I and J/I are generated by regular sequences. We have $\dim(B) = \dim(A) + \dim(B/\mathfrak{m}_A B)$ and $\dim(S/IS) = \dim(A) + \dim(S/\mathfrak{m}_R S)$ (Algebra, Lemma 10.106.7). Thus J/I is generated by

$$\dim(S/J) - \dim(S/IS) = \dim(S/\mathfrak{m}_R S) - \dim(B/\mathfrak{m}_A B)$$

elements (Algebra, Lemma 10.57.11). It follows that $\mathrm{Ker}(S/\mathfrak{m}_R S \to B/\mathfrak{m}_A B)$ is generated by the same number of elements (see above). Hence $\mathrm{Ker}(S/\mathfrak{m}_R S \to B/\mathfrak{m}_A B)$ is generated by a regular sequence, see for example Lemma 23.8.3. In this way we see that (2) holds.

If (2) holds, then I and $J/J \cap \mathfrak{m}_R S$ are generated by regular sequences. Lifting these generators (see above), using flatness of $R/I \to S/IS$, and using Grothendieck's lemma (Algebra, Lemma 10.93.3) we find that J/I is generated by a regular sequence in S/IS. Thus Lemma 23.7.6 tells us that J is generated by a regular sequence, whence (1) holds. □

Definition 23.8.5. Let A be a Noetherian ring.

(1) If A is local, then we say A is a *complete intersection* if its completion is a complete intersection in the sense above.
(2) In general we say A is a *local complete intersection* if all of its local rings are complete intersections.

We will check below that this does not conflict with the terminology introduced in Algebra, Definitions 10.128.1 and 10.128.5. But first, we show this "makes sense" by showing that if A is a Noetherian local complete intersection, then A is a local complete intersection, i.e., all of its local rings are complete intersections.

Lemma 23.8.6. *Let $(A, \mathfrak{m})$ be a Noetherian local ring. Let $\mathfrak{p} \subset A$ be a prime ideal. If A is a complete intersection, then $A_\mathfrak{p}$ is a complete intersection too.*

Proof. Choose a prime $\mathfrak{q}$ of $A^\wedge$ lying over $\mathfrak{p}$ (this is possible as $A \to A^\wedge$ is faithfully flat by Algebra, Lemma 10.91.4). Then $A_\mathfrak{p} \to (A^\wedge)_\mathfrak{q}$ is a flat local ring homomorphism. Thus by Proposition 23.8.4 we see that $A_\mathfrak{p}$ is a complete intersection if and only if $(A^\wedge)_\mathfrak{q}$ is a complete intersection. Thus it suffices to prove the lemma in case A is complete (this is the key step of the proof).

Assume A is complete. By definition we may write $A = R/(f_1, \ldots, f_r)$ for some regular sequence $f_1, \ldots, f_r$ in a regular local ring R. Let $\mathfrak{q} \subset R$ be the prime

corresponding to $\mathfrak{p}$. Observe that $f_1, \ldots, f_r \in \mathfrak{q}$ and that $A_\mathfrak{p} = R_\mathfrak{q}/(f_1, \ldots, f_r)R_\mathfrak{q}$. Hence $A_\mathfrak{p}$ is a complete intersection by Lemma 23.8.2. □

Lemma 23.8.7. *Let A be a Noetherian ring. Then A is a local complete intersection if and only if $A_\mathfrak{m}$ is a complete intersection for every maximal ideal $\mathfrak{m}$ of A.*

Proof. This follows immediately from Lemma 23.8.6 and the definitions. □

Lemma 23.8.8. *Let S be a finite type algebra over a field k.*

1. *for a prime $\mathfrak{q} \subset S$ the local ring $S_\mathfrak{q}$ is a complete intersection in the sense of Algebra, Definition 10.128.5 if and only if $S_\mathfrak{q}$ is acomplete intersection in the sense of Definition 23.8.5, and*
2. *S is a local complete intersection in the sense of Algebra, Definition 10.128.1 if and only if S is a local complete intersection in the sense of Definition 23.8.5.*

Proof. Proof of (1). Let $k[x_1, \ldots, x_n] \to S$ be a surjection. Let $\mathfrak{p} \subset k[x_1, \ldots, x_n]$ be the prime ideal corresponding to $\mathfrak{q}$. Let $I \subset k[x_1, \ldots, x_n]$ be the kernel of our surjection. Note that $k[x_1, \ldots, x_n]_\mathfrak{p} \to S_\mathfrak{q}$ is surjective with kernel $I_\mathfrak{p}$. Observe that $k[x_1, \ldots, x_n]$ is a regular ring by Algebra, Proposition 10.108.2. Hence the equivalence of the two notions in (1) follows by combining Lemma 23.8.3 with Algebra, Lemma 10.128.7.

Having proved (1) the equivalence in (2) follows from the definition and Algebra, Lemma 10.128.9. □

Lemma 23.8.9. *Let $A \to B$ be a flat local homomorphism of Noetherian local rings. Then the following are equivalent*

1. *B is a complete intersection,*
2. *A and $B/\mathfrak{m}_A B$ are complete intersections.*

Proof. Now that the definition makes sense this is a trivial reformulation of the (nontrivial) Proposition 23.8.4. □

23.9. Local complete intersection maps

Let $A \to B$ be a local homomorphism of Noetherian complete local rings. A consequence of the Cohen structure theorem is that we can find a commutative diagram

$$\begin{array}{ccc} S & \longrightarrow & B \\ & \nwarrow & \uparrow \\ & & A \end{array}$$

of Noetherian complete local rings with $S \to B$ surjective, $A \to S$ flat, and $S/\mathfrak{m}_A S$ a regular local ring. This follows from More on Algebra, Lemma 15.27.3. Let us (temporarily) say $A \to S \to B$ is a *good factorization* of $A \to B$ if S is a Noetherian local ring, $A \to S \to B$ are local ring maps, $S \to B$ surjective, $A \to S$ flat, and $S/\mathfrak{m}_A S$ regular. Let us say that $A \to B$ is a *complete intersection homomorphism* if there exists some good factorization $A \to S \to B$ such that the kernel of $S \to B$ is generated by a regular sequence. The following lemma shows this notion is independent of the choice of the diagram.

Lemma 23.9.1. *Let $A \to B$ be a local homomorphism of Noetherian complete local rings. The following are equivalent*

(1) *for some good factorization $A \to S \to B$ the kernel of $S \to B$ is generated by a regular sequence, and*
(2) *for every good factorization $A \to S \to B$ the kernel of $S \to B$ is generated by a regular sequence.*

Proof. Let $A \to S \to B$ be a good factorization. As B is complete we obtain a factorization $A \to S^\wedge \to B$ where $S^\wedge$ is the completion of S. Note that this is also a good factorization: The ring map $S \to S^\wedge$ is flat (Algebra, Lemma 10.91.3), hence $A \to S^\wedge$ is flat. The ring $S^\wedge/\mathfrak{m}_A S^\wedge = (S/\mathfrak{m}_A S)^\wedge$ is regular since $S/\mathfrak{m}_A S$ is regular (More on Algebra, Lemma 15.31.4). Let $f_1, \ldots, f_r$ be a minimal sequence of generators of $\mathrm{Ker}(S \to B)$. We will use without further mention that an ideal in a Noetherian local ring is generated by a regular sequence if and only if any minimal set of generators is a regular sequence. Observe that $f_1, \ldots, f_r$ is a regular sequence in S if and only if $f_1, \ldots, f_r$ is a regular sequence in the completion $S^\wedge$ by Algebra, Lemma 10.65.7. Moreover, we have

$$S^\wedge/(f_1, \ldots, f_r)R^\wedge = (S/(f_1, \ldots, f_n))^\wedge = B^\wedge = B$$

because B is $\mathfrak{m}_B$-adically complete (first equality by Algebra, Lemma 10.91.2). Thus the kernel of $S \to B$ is generated by a regular sequence if and only if the kernel of $S^\wedge \to B$ is generated by a regular sequence. Hence it suffices to consider good factorizations where S is complete.

Assume we have two factorizations $A \to S \to B$ and $A \to S' \to B$ with S and S' complete. By More on Algebra, Lemma 15.27.4 the ring $S \times_B S'$ is a Noetherian complete local ring. Hence, using More on Algebra, Lemma 15.27.3 we can choose a good factorization $A \to S'' \to S \times_B S'$ with S'' complete. Thus it suffices to show: If $A \to S' \to S \to B$ are comparable good factorizations, then $\mathrm{Ker}(S \to B)$ is generated by a regular sequence if and only if $\mathrm{Ker}(S' \to B)$ is generated by a regular sequence.

Let $A \to S' \to S \to B$ be comparable good factorizations. First, since $S'/\mathfrak{m}_R S' \to S/\mathfrak{m}_R S$ is a surjection of regular local rings, the kernel is generated by a regular sequence $\overline{x}_1, \ldots, \overline{x}_c \in \mathfrak{m}_{S'}/\mathfrak{m}_R S'$ which can be extended to a regular system of parameters for the regular local ring $S'/\mathfrak{m}_R S'$, see (Algebra, Lemma 10.101.4). Set $I = \mathrm{Ker}(S' \to S)$. By flatness of S over R we have

$$I/\mathfrak{m}_R I = \mathrm{Ker}(S'/\mathfrak{m}_R S' \to S/\mathfrak{m}_R S) = (\overline{x}_1, \ldots, \overline{x}_c).$$

Choose lifts $x_1, \ldots, x_c \in I$. These lifts form a regular sequence generating I as S' is flat over R, see Algebra, Lemma 10.93.3.

We conclude that if also $\mathrm{Ker}(S \to B)$ is generated by a regular sequence, then so is $\mathrm{Ker}(S' \to B)$, see More on Algebra, Lemmas 15.20.12 and 15.20.6.

Conversely, assume that $J = \mathrm{Ker}(S' \to B)$ is generated by a regular sequence. Because the generators $x_1, \ldots, x_c$ of I map to linearly independent elements of $\mathfrak{m}_{S'}/\mathfrak{m}_{S'}^2$ we see that $I/\mathfrak{m}_{S'} I \to J/\mathfrak{m}_{S'} J$ is injective. Hence there exists a minimal system of generators $x_1, \ldots, x_c, y_1, \ldots, y_d$ for J. Then $x_1, \ldots, x_c, y_1, \ldots, y_d$ is a regular sequence and it follows that the images of $y_1, \ldots, y_d$ in S form a regular sequence generating $\mathrm{Ker}(S \to B)$. This finishes the proof of the lemma. $\square$

In the following proposition observe that the condition on vanishing of Tor's applies in particular if B has finite tor dimension over A and thus in particular if B is flat over A.

Proposition 23.9.2. *Let $A \to B$ be a local homomorphism of Noetherian local rings. Then the following are equivalent*

(1) *B is a complete intersection and $Tor_p^A(B, A/\mathfrak{m}_A)$ is nonzero for only finitely many p,*
(2) *A is a complete intersection and $A^\wedge \to B^\wedge$ is a complete intersection homomorphism in the sense defined above.*

Proof. Let $F_\bullet \to A/\mathfrak{m}_A$ be a resolution by finite free A-modules. Observe that $\mathrm{Tor}_p^A(B, A/\mathfrak{m}_A)$ is the pth homology of the complex $F_\bullet \otimes_A B$. Let $F_\bullet^\wedge = F_\bullet \otimes_A A^\wedge$ be the completion. Then $F_\bullet^\wedge$ is a resolution of $A^\wedge/\mathfrak{m}_{A^\wedge}$ by finite free $A^\wedge$-modules (as $A \to A^\wedge$ is flat and completion on finite modules is exact, see Algebra, Lemmas 10.91.2 and 10.91.3). It follows that

$$F_\bullet^\wedge \otimes_{A^\wedge} B^\wedge = F_\bullet \otimes_A B \otimes_B B^\wedge$$

By flatness of $B \to B^\wedge$ we conclude that

$$\mathrm{Tor}_p^{A^\wedge}(B^\wedge, A^\wedge/\mathfrak{m}_{A^\wedge}) = \mathrm{Tor}_p^A(B, A/\mathfrak{m}_A) \otimes_B B^\wedge$$

In this way we see that the condition in (1) on the local ring map $A \to B$ is equivalent to the same condition for the local ring map $A^\wedge \to B^\wedge$. Thus we may assume A and B are complete local Noetherian rings (since the other conditions are formulated in terms of the completions in any case).

Assume A and B are complete local Noetherian rings. Choose a diagram

$$\begin{array}{ccc} S & \longrightarrow & B \\ \uparrow & & \uparrow \\ R & \longrightarrow & A \end{array}$$

as in More on Algebra, Lemma 15.27.3. Let $I = \mathrm{Ker}(R \to A)$ and $J = \mathrm{Ker}(S \to B)$. The proposition now follows from Lemma 23.7.6. □

Remark 23.9.3. It appears difficult to define an good notion of "local complete intersection homomorphisms" for maps between general Noetherian rings. The reason is that, for a local Noetherian ring A, the fibres of $A \to A^\wedge$ are not local complete intersection rings. Thus, if $A \to B$ is a local homomorphism of local Noetherian rings, and the map of completions $A^\wedge \to B^\wedge$ is a complete intersection homomorphism in the sense defined above, then $(A_\mathfrak{p})^\wedge \to (B_\mathfrak{q})^\wedge$ is in general **not** a complete intersection homomorphism in the sense defined above. A solution can be had by working exclusively with excellent Noetherian rings. More generally, one could work with those Noetherian rings whose formal fibres are complete intersections, see [**Rod87**]. We will develop this theory here if we ever need it.

To finish of this section we compare the notion defined above with the notion introduced in More on Algebra, Section 23.8.

Lemma 23.9.4. *Consider a commutative diagram*

$$\begin{array}{ccc} S & \longrightarrow & B \\ & \nwarrow & \uparrow \\ & & A \end{array}$$

of Noetherian local rings with $S \to B$ surjective, $A \to S$ flat, and $S/\mathfrak{m}_A S$ a regular local ring. The following are equivalent

(1) *$Ker(S \to B)$ is generated by a regular sequence, and*
(2) *$A^\wedge \to B^\wedge$ is a complete intersection homomorphism as defined above.*

Proof. Omitted. Hint: the proof is indentical to the argument given in the first paragraph of the proof of Lemma 23.9.1. □

Lemma 23.9.5. *Let A be a Noetherian ring. Let $A \to B$ be a finite type ring map. The following are equivalent*

(1) *$A \to B$ is a local complete intersection in the sense of More on Algebra, Definition 15.22.2,*
(2) *for every prime $\mathfrak{q} \subset B$ and with $\mathfrak{p} = A \cap \mathfrak{q}$ the ring map $(A_\mathfrak{p})^\wedge \to (B_\mathfrak{q})^\wedge$ is a complete intersection homomorphism in the sense defined above.*

Proof. Choose a surjection $R = A[x_1, \ldots, x_n] \to B$. Observe that $A \to R$ is flat with regular fibres. Let I be the kernel of $R \to B$. Assume (2). Then we see that I is locally generated by a regular sequence by Lemma 23.9.4 and Algebra, Lemma 10.65.8. In other words, (1) holds. Conversely, assume (1). Then after localizing on R and B we can assume that I is generated by a Koszul regular sequence. By More on Algebra, Lemma 15.20.6 we find that I is locally generated by a regular sequence. Hence (2) hold by Lemma 23.9.4. Some details omitted. □

Lemma 23.9.6. *Let A be a Noetherian ring. Let $A \to B$ be a finite type ring map such that the image of* $\mathrm{Spec}(B) \to \mathrm{Spec}(A)$ *contains all closed points of* $\mathrm{Spec}(A)$*. Then the following are equivalent*

(1) *B is a complete intersection and $A \to B$ has finite tor dimension,*
(2) *A is a complete intersection and $A \to B$ is a complete intersection in the sense of More on Algebra, Definition 15.22.2.*

Proof. This is a reformulation of Proposition 23.9.2 via Lemma 23.9.5. We omit the details. □

23.10. Other chapters

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CHAPTER 24

Hypercoverings

24.1. Introduction

Let $\mathcal{C}$ be a site, see Sites, Definition 7.6.2. Let X be an object of $\mathcal{C}$. Given an abelian sheaf $\mathcal{F}$ on $\mathcal{C}$ we would like to compute its cohomology groups

$$H^i(X, \mathcal{F}).$$

According to our general definitions (Cohomology on Sites, Section 21.3) this cohomology group is computed by choosing an injective resolution $0 \to \mathcal{F} \to \mathcal{I}^0 \to \mathcal{I}^1 \to \ldots$ and setting

$$H^i(X, \mathcal{F}) = H^i(\Gamma(X, \mathcal{I}^0) \to \Gamma(X, \mathcal{I}^1) \to \Gamma(X, \mathcal{I}^2) \to \ldots)$$

The goal of this chapter is to show that we may also compute these cohomology groups without choosing an injective resolution (in the case that $\mathcal{C}$ has fibre products). To do this we will use hypercoverings.

A hypercovering in a site is a generalization of a covering, see [**AGV71**, Exposé V, Sec. 7]. Given a hypercovering K of an object X, there is a Čech to cohomology spectral sequence expressing the cohomogy of an abelian sheaf $\mathcal{F}$ over X in terms of the cohomology of the sheaf over the components K_n of K. It turns out that there are always enough hypercoverings, so that taking the colimit over all hypercoverings, the spectral sequence degenerates and the cohomology of $\mathcal{F}$ over X is computed by the colimit of the Čech cohomology groups.

A more general gadget one can consider is a simplicial augmentation where one has cohomological descent, see [**AGV71**, Exposé Vbis]. A nice manuscript on cohomological descent is the text by Brian Conrad, see `http://math.stanford.edu/~conrad/papers/hypercover.pdf`. We will come back to these issue in the chapter on simplicial spaces where we will show, for example, that proper hypercoverings of "locally compact" topological spaces are of cohomological descent (Simplicial Spaces, Section 63.6). Our method of attack will be to reduce this statement to the Čech to cohomology spectral sequence constructed in this chapter.

24.2. Hypercoverings

In order to start we make the following definition. The letters "SR" stand for Semi-Representable.

Definition 24.2.1. Let $\mathcal{C}$ be a site. We denote $\mathrm{SR}(\mathcal{C})$ the category of *semi-representable objects* defined as follows

(1) objects are families of objects $\{U_i\}_{i \in I}$, and
(2) morphisms $\{U_i\}_{i \in I} \to \{V_j\}_{j \in J}$ are given by a map $\alpha : I \to J$ and for each $i \in I$ a morphism $f_i : U_i \to V_{\alpha(i)}$ of $\mathcal{C}$.

Let $X \in \mathrm{Ob}(\mathcal{C})$ be an object of $\mathcal{C}$. The category of *semi-representable objects over* X is the category $\mathrm{SR}(\mathcal{C}, X) = \mathrm{SR}(\mathcal{C}/X)$.

This definition differs from the one in [**AGV71**, Exposé V, Sec. 7], but it seems flexible enough to do all the required arguments. Note that this is a "big" category. We will later "bound" the size of the index sets I that we need for hypercoverings of X. We can then redefine $\mathrm{SR}(\mathcal{C}, X)$ to become a category. Let's spell out the objects and morphisms $\mathrm{SR}(\mathcal{C}, X)$:

(1) objects are families of morphisms $\{U_i \to X\}_{i \in I}$, and
(2) morphisms $\{U_i \to X\}_{i \in I} \to \{V_j \to X\}_{j \in J}$ are given by a map $\alpha : I \to J$ and for each $i \in I$ a morphism $f_i : U_i \to V_{\alpha(i)}$ over X.

There is a forgetful functor $\mathrm{SR}(\mathcal{C}, X) \to \mathrm{SR}(\mathcal{C})$.

Definition 24.2.2. Let $\mathcal{C}$ be a site with fibre products. We denote F the functor *which associates a sheaf to a semi-representable object*. In a formula

$$\begin{array}{rcl} F : \mathrm{SR}(\mathcal{C}) & \longrightarrow & PSh(\mathcal{C}) \\ \{U_i\}_{i \in I} & \longmapsto & \amalg_{i \in I} h_{U_i} \end{array}$$

where h_U denotes the representable presheaf associated to the object U.

Given a morphism $U \to X$ we obtain a morphism $h_U \to h_X$ of representable presheaves. Thus we often think of F on $\mathrm{SR}(\mathcal{C}, X)$ as a functor into the category of presheaves of sets over h_X, namely $PSh(\mathcal{C})/h_X$. Here is a picture:

$$\begin{array}{ccc} \mathrm{SR}(\mathcal{C}, X) & \xrightarrow[F]{} & PSh(\mathcal{C})/h_X \\ \downarrow & & \downarrow \\ \mathrm{SR}(\mathcal{C}) & \xrightarrow{F} & PSh(\mathcal{C}) \end{array}$$

Next we discuss the existence of limits in the category of semi-representable objects.

Lemma 24.2.3. *Let $\mathcal{C}$ be a site.*

(1) *the category $SR(\mathcal{C})$ has coproducts and F commutes with them,*
(2) *the functor $F : SR(\mathcal{C}) \to PSh(\mathcal{C})$ commutes with limits,*
(3) *if $\mathcal{C}$ has fibre products, then $SR(\mathcal{C})$ has fibre products,*
(4) *if $\mathcal{C}$ has products of pairs, then $SR(\mathcal{C})$ has products of pairs,*
(5) *if $\mathcal{C}$ has equalizers, so does $SR(\mathcal{C})$, and*
(6) *if $\mathcal{C}$ has a final object, so does $SR(\mathcal{C})$.*

Let $X \in \mathrm{Ob}(\mathcal{C})$.

(1) *the category $SR(\mathcal{C}, X)$ has coproducts and F commutes with them,*
(2) *if $\mathcal{C}$ has fibre products, then $SR(\mathcal{C}, X)$ has finite limits and $F : SR(\mathcal{C}, X) \to PSh(\mathcal{C})/h_X$ commutes with them.*

Proof. Proof of the results on $\mathrm{SR}(\mathcal{C})$. Proof of (1). The coproduct of $\{U_i\}_{i \in I}$ and $\{V_j\}_{j \in J}$ is $\{U_i\}_{i \in I} \amalg \{V_j\}_{j \in J}$. Similarly for coproducts of families of objects. It is clear that F commutes with these.

Proof of (2). For U in $\mathrm{Ob}(\mathcal{C})$ consider the object $\{U\}$ of $\mathrm{SR}(\mathcal{C})$. It is clear that $\mathrm{Mor}_{\mathrm{SR}(\mathcal{C})}(\{U\}, K)) = F(K)(U)$ for $K \in \mathrm{Ob}(\mathrm{SR}(\mathcal{C}))$. Since limits of presheaves are computed at the level of sections (Sites, Section 7.4) we conclude that F commutes with limits.

Proof of (3). Suppose given a morphism $(\alpha, f_i) : \{U_i\}_{i\in I} \to \{V_j\}_{j\in J}$ and a morphism $(\beta, g_k) : \{W_k\}_{k\in K} \to \{V_j\}_{j\in J}$. The fibred product of these morphisms is given by

$$\{U_i \times_{f_i, V_j, g_k} W_k\}_{(i,j,k)\in I\times J\times K \text{ such that } k=\alpha(i)=\beta(j)}$$

The fibre products exist if $\mathcal{C}$ has fibre products.

Proof of (4). The product of $\{U_i\}_{i\in I}$ and $\{V_j\}_{j\in J}$ is $\{U_i \times V_j\}_{i\in I, j\in J}$. The products exist if $\mathcal{C}$ has products.

Proof of (5). The equalizer of two maps $(\alpha, f_i), (\alpha', f_i') : \{U_i\}_{i\in I} \to \{V_j\}_{j\in J}$ is

$$\{\mathrm{Eq}(f_i, f_i' : U_i \to V_{\alpha(i)})\}_{i\in I,\ \alpha(i)=\alpha'(i)}$$

The equalizers exist if $\mathcal{C}$ has equalizers.

Proof of (6). If X is a final object of $\mathcal{C}$, then $\{X\}$ is a final object of $\mathrm{SR}(\mathcal{C})$.

Proof of the statements about $\mathrm{SR}(\mathcal{C}, X)$. These follow from the results above applied to the category $\mathcal{C}/X$ using that $\mathrm{SR}(\mathcal{C}/X) = \mathrm{SR}(\mathcal{C}, X)$ and that $\mathrm{PSh}(\mathcal{C}/X) = \mathrm{PSh}(\mathcal{C})/h_X$ (Sites, Lemma 7.24.4 applied to $\mathcal{C}$ endowed with the chaotic topology). However we also argue directly as follows. It is clear that the coproduct of $\{U_i \to X\}_{i\in I}$ and $\{V_j \to X\}_{j\in J}$ is $\{U_i \to X\}_{i\in I} \amalg \{V_j \to X\}_{j\in J}$ and similarly for coproducts of families of families of morphisms with target X. The object $\{X \to X\}$ is a final object of $\mathrm{SR}(\mathcal{C}, X)$. Suppose given a morphism $(\alpha, f_i) : \{U_i \to X\}_{i\in I} \to \{V_j \to X\}_{j\in J}$ and a morphism $(\beta, g_k) : \{W_k \to X\}_{k\in K} \to \{V_j \to X\}_{j\in J}$. The fibred product of these morphisms is given by

$$\{U_i \times_{f_i, V_j, g_k} W_k \to X\}_{(i,j,k)\in I\times J\times K \text{ such that } k=\alpha(i)=\beta(j)}$$

The fibre products exist by the assumption that $\mathcal{C}$ has fibre products. Thus $\mathrm{SR}(\mathcal{C}, X)$ has finite limits, see Categories, Lemma 4.18.4. We omit verifying the statements on the functor F in this case. □

Definition 24.2.4. Let $\mathcal{C}$ be a site. Let $f = (\alpha, f_i) : \{U_i\}_{i\in I} \to \{V_j\}_{j\in J}$ be a morphism in the category $\mathrm{SR}(\mathcal{C})$. We say that f is a *covering* if for every $j \in J$ the family of morphisms $\{U_i \to V_j\}_{i\in I, \alpha(i)=j}$ is a covering for the site $\mathcal{C}$. Let X be an object of $\mathcal{C}$. A morphism $K \to L$ in $\mathrm{SR}(\mathcal{C}, X)$ is a *covering* if its image in $\mathrm{SR}(\mathcal{C}$ is a covering.

Lemma 24.2.5. *Let $\mathcal{C}$ be a site.*

1. *A composition of coverings in $SR(\mathcal{C})$ is a covering.*
2. *If $K \to L$ is a covering in $SR(\mathcal{C})$ and $L' \to L$ is a morphism, then $L' \times_L K$ exists and $L' \times_L K \to L'$ is a covering.*
3. *If $\mathcal{C}$ has products of pairs, and $A \to B$ and $K \to L$ are coverings in $SR(\mathcal{C})$, then $A \times K \to B \times L$ is a covering.*

Let $X \in \mathrm{Ob}(\mathcal{C})$. Then (1) and (2) holds for $SR(\mathcal{C}, X)$ and (3) holds if $\mathcal{C}$ has fibre products.

Proof. Part (1) is immediate from the axioms of a site. Part (2) follows by the construction of fibre products in $\mathrm{SR}(\mathcal{C})$ in the proof of Lemma 24.2.3 and the requirement that the morphisms in a covering of $\mathcal{C}$ are representable. Part (3) follows by thinking of $A \times K \to B \times L$ as the composition $A \times K \to B \times K \to B \times L$ and hence a composition of basechanges of coverings. The final statement follows because $\mathrm{SR}(\mathcal{C}, X) = \mathrm{SR}(\mathcal{C}/X)$. □

According to the results in the chapter on simplicial methods the coskelet of a truncated simplicial object of $\mathrm{SR}(\mathcal{C}, X)$ exists if $\mathcal{C}$ has fibre products. Hence the following definition makes sense.

Definition 24.2.6. Let $\mathcal{C}$ be a site. Assume $\mathcal{C}$ has fibre products. Let $X \in \mathrm{Ob}(\mathcal{C})$ be an object of $\mathcal{C}$. A *hypercovering of* X is a simplicial object K of $\mathrm{SR}(\mathcal{C}, X)$ such that

(1) The object K_0 is a covering of X for the site $\mathcal{C}$.
(2) For every $n \geq 0$ the canonical morphism
$$K_{n+1} \longrightarrow (\mathrm{cosk}_n \mathrm{sk}_n K)_{n+1}$$
is a covering in the sense defined above.

Condition (2) makes sense since each object of $\mathrm{SR}(\mathcal{C}, X)$ is after all a family of morphisms with target X. It could also be formulated as saying that the morphism of K_0 to the final object of $\mathrm{SR}(\mathcal{C}, X)$ is a covering.

Example 24.2.7. Let $\{U_i \to X\}_{i \in I}$ be a covering of the site $\mathcal{C}$. Set $K_0 = \{U_i \to X\}_{i \in I}$. Then K_0 is a 0-truncated simplicial object of $\mathrm{SR}(\mathcal{C}, X)$. Hence we may form
$$K = \mathrm{cosk}_0 K_0.$$
Clearly K passes condition (1) of Definition 24.2.6. Since all the morphisms $K_{n+1} \to (\mathrm{cosk}_n \mathrm{sk}_n K)_{n+1}$ are isomorphisms it also passes condition (2). Note that the terms K_n are the usual
$$K_n = \{U_{i_0} \times_X U_{i_1} \times_X \ldots \times_X U_{i_n} \to X\}_{(i_0, i_1, \ldots, i_n) \in I^{n+1}}$$

Lemma 24.2.8. *Let $\mathcal{C}$ be a site with fibre products. Let $X \in \mathrm{Ob}(\mathcal{C})$ be an object of $\mathcal{C}$. The collection of all hypercoverings of X forms a set.*

Proof. Since $\mathcal{C}$ is a site, the set of all coverings of X forms a set. Thus we see that the collection of possible K_0 forms a set. Suppose we have shown that the collection of all possible $K_0, \ldots, K_n$ form a set. Then it is enough to show that given $K_0, \ldots, K_n$ the collection of all possible K_{n+1} forms a set. And this is clearly true since we have to choose K_{n+1} among all possible coverings of $(\mathrm{cosk}_n \mathrm{sk}_n K)_{n+1}$. □

Remark 24.2.9. The lemma does not just say that there is a cofinal system of choices of hypercoverings that is a set, but that really the hypercoverings form a set.

The category of presheaves on $\mathcal{C}$ has finite (co)limits. Hence the functors cosk_n exists for presheaves of sets.

Lemma 24.2.10. *Let $\mathcal{C}$ be a site with fibre products. Let $X \in \mathrm{Ob}(\mathcal{C})$ be an object of $\mathcal{C}$. Let K be a hypercovering of X. Consider the simplicial object $F(K)$ of $PSh(\mathcal{C})$, endowed with its augmentation to the constant simplicial presheaf h_X.*

(1) *The morphism of presheaves $F(K)_0 \to h_X$ becomes a surjection after sheafification.*
(2) *The morphism*
$$(d_0^1, d_1^1) : F(K)_1 \longrightarrow F(K)_0 \times_{h_X} F(K)_0$$
becomes a surjection after sheafification.

(3) *For every $n \geq 1$ the morphism*

$$F(K)_{n+1} \longrightarrow (cosk_n sk_n F(K))_{n+1}$$

turns into a surjection after sheafification.

Proof. We will use the fact that if $\{U_i \to U\}_{i \in I}$ is a covering of the site $\mathcal{C}$, then the morphism

$$\amalg_{i \in I} h_{U_i} \to h_U$$

becomes surjective after sheafification, see Sites, Lemma 7.13.4. Thus the first assertion follows immediately.

For the second assertion, note that according to Simplicial, Example 14.17.2 the simplicial object $\text{cosk}_0\text{sk}_0 K$ has terms $K_0 \times \ldots \times K_0$. Thus according to the definition of a hypercovering we see that $(d_0^1, d_1^1) : K_1 \to K_0 \times K_0$ is a covering. Hence (2) follows from the claim above and the fact that F transforms products into fibred products over h_X.

For the third, we claim that $\text{cosk}_n\text{sk}_n F(K) = F(\text{cosk}_n\text{sk}_n K)$ for $n \geq 1$. To prove this, denote temporarily F' the functor $\text{SR}(\mathcal{C}, X) \to PSh(\mathcal{C})/h_X$. By Lemma 24.2.3 the functor F' commutes with finite limits. By our description of the cosk_n functor in Simplicial, Section 14.17 we see that $\text{cosk}_n\text{sk}_n F'(K) = F'(\text{cosk}_n\text{sk}_n K)$. Recall that the category used in the description of $(\text{cosk}_n U)_m$ in Simplicial, Lemma 14.17.3 is the category $(\Delta/[m])^{opp}_{\leq n}$. It is an amusing exercise to show that $(\Delta/[m])_{\leq n}$ is a connected category (see Categories, Definition 4.16.1) as soon as $n \geq 1$. Hence, Categories, Lemma 4.16.2 shows that $\text{cosk}_n\text{sk}_n F'(K) = \text{cosk}_n\text{sk}_n F(K)$. Whence the claim. Property (2) follows from this, because now we see that the morphism in (2) is the result of applying the functor F to a covering as in Definition 24.2.4, and the result follows from the first fact mentioned in this proof. □

24.3. Acyclicity

Let $\mathcal{C}$ be a site. For a presheaf of sets $\mathcal{F}$ we denote $\mathbf{Z}_\mathcal{F}$ the presheaf of abelian groups defined by the rule

$$\mathbf{Z}_\mathcal{F}(U) = \text{free abelian group on } \mathcal{F}(U).$$

We will sometimes call this the *free abelian presheaf on* $\mathcal{F}$. Of course the construction $\mathcal{F} \mapsto \mathbf{Z}_\mathcal{F}$ is a functor and it is left adjoint to the forgetful functor $PAb(\mathcal{C}) \to PSh(\mathcal{C})$. Of course the sheafification $\mathbf{Z}^\#_\mathcal{F}$ is a sheaf of abelian groups, and the functor $\mathcal{F} \mapsto \mathbf{Z}^\#_\mathcal{F}$ is a left adjoint as well. We sometimes call $\mathbf{Z}^\#_\mathcal{F}$ the *free abelian sheaf on* $\mathcal{F}$.

For an object X of the site $\mathcal{C}$ we denote $\mathbf{Z}_X$ the free abelian presheaf on h_X, and we denote $\mathbf{Z}^\#_X$ its sheafification.

Definition 24.3.1. Let $\mathcal{C}$ be a site. Let K be a simplicial object of $PSh(\mathcal{C})$. By the above we get a simplicial object $\mathbf{Z}^\#_K$ of $Ab(\mathcal{C})$. We can take its associated complex of abelian presheaves $s(\mathbf{Z}^\#_K)$, see Simplicial, Section 14.21. The *homology of* K is the homology of the complex of abelian sheaves $s(\mathbf{Z}^\#_K)$.

In other words, the *ith homology* $H_i(K)$ *of* K is the sheaf of abelian groups $H_i(K) = H_i(s(\mathbf{Z}^\#_K))$. In this section we worry about the homology in case K is a hypercovering of an object X of $\mathcal{C}$.

Lemma 24.3.2. *Let $\mathcal{C}$ be a site. Let $\mathcal{F} \to \mathcal{G}$ be a morphism of presheaves of sets. Denote K the simplicial object of $PSh(\mathcal{C})$ whose nth term is the $(n+1)$st fibre product of $\mathcal{F}$ over $\mathcal{G}$, see Simplicial, Example 14.3.5. Then, if $\mathcal{F} \to \mathcal{G}$ is surjective after sheafification, we have*

$$H_i(K) = \begin{cases} 0 & \text{if} \quad i > 0 \\ \mathbf{Z}_{\mathcal{G}}^{\#} & \text{if} \quad i = 0 \end{cases}$$

The isomorphism in degree 0 is given by the morphism $H_0(K) \to \mathbf{Z}_{\mathcal{G}}^{\#}$ coming from the map $(\mathbf{Z}_K^{\#})_0 = \mathbf{Z}_{\mathcal{F}}^{\#} \to \mathbf{Z}_{\mathcal{G}}^{\#}$.

Proof. Let $\mathcal{G}' \subset \mathcal{G}$ be the image of the morphism $\mathcal{F} \to \mathcal{G}$. Let $U \in \mathrm{Ob}(\mathcal{C})$. Set $A = \mathcal{F}(U)$ and $B = \mathcal{G}'(U)$. Then the simplicial set $K(U)$ is equal to the simplicial set with n-simplices given by

$$A \times_B A \times_B \ldots \times_B A \ (n+1 \text{ factors}).$$

By Simplicial, Lemma 14.30.3 the morphism $K(U) \to B$ is a trivial Kan fibration. Thus it is a homotopy equivalence (Simplicial, Lemma 14.30.3). Hence applying the functor "free abelian group on" to this we deduce that

$$\mathbf{Z}_K(U) \longrightarrow \mathbf{Z}_B$$

is a homotopy equivalence. Note that $s(\mathbf{Z}_B)$ is the complex

$$\ldots \to \bigoplus\nolimits_{b \in B} \mathbf{Z} \xrightarrow{0} \bigoplus\nolimits_{b \in B} \mathbf{Z} \xrightarrow{1} \bigoplus\nolimits_{b \in B} \mathbf{Z} \xrightarrow{0} \bigoplus\nolimits_{b \in B} \mathbf{Z} \to 0$$

see Simplicial, Lemma 14.21.3. Thus we see that $H_i(s(\mathbf{Z}_K(U))) = 0$ for $i > 0$, and $H_0(s(\mathbf{Z}_K(U))) = \bigoplus_{b \in B} \mathbf{Z} = \bigoplus_{s \in \mathcal{G}'(U)} \mathbf{Z}$. These identifications are compatible with restriction maps.

We conclude that $H_i(s(\mathbf{Z}_K)) = 0$ for $i > 0$ and $H_0(s(\mathbf{Z}_K)) = \mathbf{Z}_{\mathcal{G}'}$, where here we compute homology groups in $PAb(\mathcal{C})$. Since sheafification is an exact functor we deduce the result of the lemma. Namely, the exactness implies that $H_0(s(\mathbf{Z}_K))^{\#} = H_0(s(\mathbf{Z}_K^{\#}))$, and similarly for other indices. □

Lemma 24.3.3. *Let $\mathcal{C}$ be a site. Let $f : L \to K$ be a morphism of simplicial objects of $PSh(\mathcal{C})$. Let $n \geq 0$ be an integer. Assume that*

1. *For $i < n$ the morphism $L_i \to K_i$ is an isomorphism.*
2. *The morphism $L_n \to K_n$ is surjective after sheafification.*
3. *The canonical map $L \to cosk_n sk_n L$ is an isomorphism.*
4. *The canonical map $K \to cosk_n sk_n K$ is an isomorphism.*

Then $H_i(f) : H_i(L) \to H_i(K)$ is an isomorphism.

Proof. This proof is exactly the same as the proof of Lemma 24.3.2 above. Namely, we first let $K'_n \subset K_n$ be the sub presheaf which is the image of the map $L_n \to K_n$. Assumption (2) means that the sheafification of K'_n is equal to the sheafification of K_n. Moreover, since $L_i = K_i$ for all $i < n$ we see that get an n-truncated simplicial presheaf U by taking $U_0 = L_0 = K_0, \ldots, U_{n-1} = L_{n-1} = K_{n-1}, U_n = K'_n$. Denote $K' = \mathrm{cosk}_n U$, a simplicial presheaf. Because we can construct K'_m as a finite limit, and since sheafification is exact, we see that $(K'_m)^{\#} = K_m$. In other words, $(K')^{\#} = K^{\#}$. We conclude, by exactness of sheafification once more, that $H_i(K) = H_i(K')$. Thus it suffices to prove the lemma for the morphism $L \to K'$, in other words, we may assume that $L_n \to K_n$ is a surjective morphism of *presheaves*!

In this case, for any object U of $\mathcal{C}$ we see that the morphism of simplicial sets

$$L(U) \longrightarrow K(U)$$

satisfies all the assumptions of Simplicial, Lemma 14.30.1. Hence it is a trivial Kan fibration. In particular it is a homotopy equivalence (Simplicial, Lemma 14.28.8). Thus

$$\mathbf{Z}_L(U) \longrightarrow \mathbf{Z}_K(U)$$

is a homotopy equivalence too. This for all U. The result follows. □

Lemma 24.3.4. *Let $\mathcal{C}$ be a site. Let K be a simplicial presheaf. Let $\mathcal{G}$ be a presheaf. Let $K \to \mathcal{G}$ be an augmentation of K towards $\mathcal{G}$. Assume that*

(1) *The morphism of presheaves $K_0 \to \mathcal{G}$ becomes a surjection after sheafification.*
(2) *The morphism*

$$(d_0^1, d_1^1) : K_1 \longrightarrow K_0 \times_{\mathcal{G}} K_0$$

becomes a surjection after sheafification.
(3) *For every $n \geq 1$ the morphism*

$$K_{n+1} \longrightarrow (cosk_n sk_n K)_{n+1}$$

turns into a surjection after sheafification.

Then $H_i(K) = 0$ for $i > 0$ and $H_0(K) = \mathbf{Z}_{\mathcal{G}}^{\#}$.

Proof. Denote $K^n = \mathrm{cosk}_n \mathrm{sk}_n K$ for $n \geq 1$. Define K^0 as the simplicial object with terms $(K^0)_n$ equal to the $(n+1)$-fold fibred product $K_0 \times_{\mathcal{G}} \ldots \times_{\mathcal{G}} K_0$, see Simplicial, Example 14.3.5. We have morphisms

$$K \longrightarrow \ldots \to K^n \to K^{n-1} \to \ldots \to K^1 \to K^0.$$

The morphisms $K \to K^i$, $K^j \to K^i$ for $j \geq i \geq 1$ come from the universal properties of the cosk_n functors. The morphism $K^1 \to K^0$ is the canonical morphism from Simplicial, Remark 14.18.4. We also recall that $K^0 \to \mathrm{cosk}_1 \mathrm{sk}_1 K^0$ is an isomorphism, see Simplicial, Lemma 14.18.3.

By Lemma 24.3.2 we see that $H_i(K^0) = 0$ for $i > 0$ and $H_0(K^0) = \mathbf{Z}_{\mathcal{G}}^{\#}$.

Pick $n \geq 1$. Consider the morphism $K^n \to K^{n-1}$. It is an isomorphism on terms of degree $< n$. Note that $K^n \to \mathrm{cosk}_n \mathrm{sk}_n K^n$ and $K^{n-1} \to \mathrm{cosk}_n \mathrm{sk}_n K^{n-1}$ are isomorphisms. Note that $(K^n)_n = K_n$ and that $(K^{n-1})_n = (\mathrm{cosk}_{n-1} \mathrm{sk}_{n-1} K)_n$. Hence by assumption, we have that $(K^n)_n \to (K^{n-1})_n$ is a morphism of presheaves which becomes surjective after sheafification. By Lemma 24.3.3 we conclude that $H_i(K^n) = H_i(K^{n-1})$. Combined with the above this proves the lemma. □

Lemma 24.3.5. *Let $\mathcal{C}$ be a site with fibre products. Let X be an object of of $\mathcal{C}$. Let K be a hypercovering of X. The homology of the simplicial presheaf $F(K)$ is 0 in degrees > 0 and equal to $\mathbf{Z}_X^{\#}$ in degree 0.*

Proof. Combine Lemmas 24.3.4 and 24.2.10. □

24.4. Cech cohomology and hypercoverings

Let $\mathcal{C}$ be a site. Consider a presheaf of abelian groups $\mathcal{F}$ on the site $\mathcal{C}$. It defines a functor

$$\begin{array}{rcl} \mathcal{F} : \mathrm{SR}(\mathcal{C})^{opp} & \longrightarrow & Ab \\ \{U_i\}_{i\in I} & \longmapsto & \prod_{i\in I} \mathcal{F}(U_i) \end{array}$$

Thus a simplicial object K of $\mathrm{SR}(\mathcal{C})$ is turned into a cosimplicial object $\mathcal{F}(K)$ of Ab. The cochain complex $s(\mathcal{F})(K))$ associated to $\mathcal{F}(K)$ (Simplicial, Section 14.23) is called the Čech complex of $\mathcal{F}$ with respect to the simplicial object K. We set

$$\check{H}^i(K, \mathcal{F}) = H^i(s(\mathcal{F}(K))).$$

and we call it the ith Čech cohomology group of $\mathcal{F}$ with respect to K. In this section we prove analogues of some of the results for Cech cohomology of open coverings proved in Cohomology, Sections 20.10, 20.11 and 20.12.

Lemma 24.4.1. *Let $\mathcal{C}$ be a site with fibre products. Let X be an object of $\mathcal{C}$. Let K be a hypercovering of X. Let $\mathcal{F}$ be a sheaf of abelian groups on $\mathcal{C}$. Then $\check{H}^0(K, \mathcal{F}) = \mathcal{F}(X)$.*

Proof. We have

$$\check{H}^0(K, \mathcal{F}) = \mathrm{Ker}(\mathcal{F}(K_0) \longrightarrow \mathcal{F}(K_1))$$

Write $K_0 = \{U_i \to X\}$. It is a covering in the site $\mathcal{C}$. As well, we have that $K_1 \to K_0 \times K_0$ is a covering in $\mathrm{SR}(\mathcal{C}, X)$. Hence we may write $K_1 = \amalg_{i_0,i_1\in I}\{V_{i_0i_1j} \to X\}$ so that the morphism $K_1 \to K_0 \times K_0$ is given by coverings $\{V_{i_0i_1j} \to U_{i_0} \times_X U_{i_1}\}$ of the site $\mathcal{C}$. Thus we can further identify

$$\check{H}^0(K, \mathcal{F}) = \mathrm{Ker}(\prod\nolimits_i \mathcal{F}(U_i) \longrightarrow \prod\nolimits_{i_0i_1j} \mathcal{F}(V_{i_0i_1j}))$$

with obvious map. The sheaf property of $\mathcal{F}$ implies that $\check{H}^0(K, \mathcal{F}) = H^0(X, \mathcal{F})$. $\square$

In fact this property characterizes the abelian sheaves among all abelian presheaves on $\mathcal{C}$ of course. The analogue of Cohomology, Lemma 24.4.2 in this case is the following.

Lemma 24.4.2. *Let $\mathcal{C}$ be a site with fibre products. Let X be an object of $\mathcal{C}$. Let K be a hypercovering of X. Let $\mathcal{I}$ be an injective sheaf of abelian groups on $\mathcal{C}$. Then*

$$\check{H}^p(K, \mathcal{I}) = \begin{cases} \mathcal{I}(X) & \text{if } p = 0 \\ 0 & \text{if } p > 0 \end{cases}$$

Proof. Observe that for any object $Z = \{U_i \to X\}$ of $\mathrm{SR}(\mathcal{C}, X)$ and any abelian sheaf $\mathcal{F}$ on $\mathcal{C}$ we have

$$\begin{array}{rcl} \mathcal{F}(Z) & = & \prod \mathcal{F}(U_i) \\ & = & \prod \mathrm{Mor}_{PSh(\mathcal{C})}(h_{U_i}, \mathcal{F}) \\ & = & \mathrm{Mor}_{PSh(\mathcal{C})}(F(Z), \mathcal{F}) \\ & = & \mathrm{Mor}_{PAb(\mathcal{C})}(\mathbf{Z}_{F(Z)}, \mathcal{F}) \\ & = & \mathrm{Mor}_{Ab(\mathcal{C})}(\mathbf{Z}^{\#}_{F(Z)}, \mathcal{F}) \end{array}$$

Thus we see, for any simplicial object K of $\mathrm{SR}(\mathcal{C}, X)$ that we have

$$s(\mathcal{F}(K)) = \mathrm{Hom}_{Ab(\mathcal{C})}(s(\mathbf{Z}^{\#}_{F(K)}), \mathcal{F}) \tag{24.4.2.1}$$

see Definition 24.3.1 for notation. The complex of sheaves $s(\mathbf{Z}^\#_{F(K)})$ is quasi-isomorphic to $\mathbf{Z}^\#_X$ if K is a hypercovering, see Lemma 24.3.5. We conclude that if $\mathcal{I}$ is an injective abelian sheaf, and K a hypercovering, then the complex $s(\mathcal{I}(K))$ is acyclic except possibly in degree 0. In other words, we have

$$\check{H}^i(K, \mathcal{I}) = 0$$

for $i > 0$. Combined with Lemma 24.4.1 the lemma is proved. □

Next we come to the analogue of Cohomology on Sites, Lemma 21.11.5. Let $\mathcal{C}$ be a site. Let $\mathcal{F}$ be a sheaf of abelian groups on $\mathcal{C}$. Recall that $\underline{H}^i(\mathcal{F})$ indicates the presheaf of abelian groups on $\mathcal{C}$ which is defined by the rule $\underline{H}^i(\mathcal{F}) : U \longmapsto H^i(U, \mathcal{F})$. We extend this to $\mathrm{SR}(\mathcal{C})$ as in the introduction to this section.

Lemma 24.4.3. *Let $\mathcal{C}$ be a site with fibre products. Let X be an object of $\mathcal{C}$. Let K be a hypercovering of X. Let $\mathcal{F}$ be a sheaf of abelian groups on $\mathcal{C}$. There is a map*

$$s(\mathcal{F}(K)) \longrightarrow R\Gamma(X, \mathcal{F})$$

in $D^+(Ab)$ functorial in $\mathcal{F}$, which induces natural transformations

$$\check{H}^i(K, -) \longrightarrow H^i(X, -)$$

as functors $Ab(\mathcal{C}) \to Ab$. Moreover, there is a spectral sequence $(E_r, d_r)_{r \geq 0}$ with

$$E_2^{p,q} = \check{H}^p(K, \underline{H}^q(\mathcal{F}))$$

converging to $H^{p+q}(X, \mathcal{F})$. This spectral sequence is functorial in $\mathcal{F}$ and in the hypercovering K.

Proof. We could prove this by the same method as employed in the corresponding lemma in the chapter on cohomology. Instead let us prove this by a double complex argument.

Choose an injective resolution $\mathcal{F} \to \mathcal{I}^\bullet$ in the category of abelian sheaves on $\mathcal{C}$. Consider the double complex $A^{\bullet,\bullet}$ with terms

$$A^{p,q} = \mathcal{I}^q(K_p)$$

where the differential $d_1^{p,q} : A^{p,q} \to A^{p+1,q}$ is the one coming from the differential $\mathcal{I}^p \to \mathcal{I}^{p+1}$ and the differential $d_2^{p,q} : A^{p,q} \to A^{p,q+1}$ is the one coming from the differential on the complex $s(\mathcal{I}^p(K))$ associated to the cosimplicial abelian group $\mathcal{I}^p(K)$ as explained above. As usual we denote $sA^\bullet$ the simple complex associated to the double complex $A^{\bullet,\bullet}$. We will use the two spectral sequences $({}'E_r, {}'d_r)$ and $({}''E_r, {}''d_r)$ associated to this double complex, see Homology, Section 12.22.

By Lemma 24.4.2 the complexes $s(\mathcal{I}^p(K))$ are acyclic in positive degrees and have H^0 equal to $\mathcal{I}^p(X)$. Hence by Homology, Lemma 12.22.6 and its proof the spectral sequence $({}'E_r, {}'d_r)$ degenerates, and the natural map

$$\mathcal{I}^\bullet(X) \longrightarrow sA^\bullet$$

is a quasi-isomorphism of complexes of abelian groups. In particular we conclude that $H^n(sA^\bullet) = H^n(X, \mathcal{F})$.

The map $s(\mathcal{F}(K)) \longrightarrow R\Gamma(X, \mathcal{F})$ of the lemma is the composition of the natural map $s(\mathcal{F}(K)) \to sA^\bullet$ followed by the inverse of the displayed quasi-isomorphism above. This works because $\mathcal{I}^\bullet(X)$ is a representative of $R\Gamma(X, \mathcal{F})$.

Consider the spectral sequence $({}''E_r, {}''d_r)_{r\geq 0}$. By Homology, Lemma 12.22.3 we see that

$$ {}''E_2^{p,q} = H^p_{II}(H^q_I(A^{\bullet,\bullet}))$$

In other words, we first take cohomology with respect to d_1 which gives the groups ${}''E_1^{p,q} = \underline{H}^p(\mathcal{F})(K_q)$. Hence it is indeed the case (by the description of the differential ${}''d_1$) that ${}''E_2^{p,q} = \check{H}^p(K, \underline{H}^q(\mathcal{F}))$. And by the other spectral sequence above we see that this one converges to $H^n(X, \mathcal{F})$ as desired.

We omit the proof of the statements regarding the functoriality of the above constructions in the abelian sheaf $\mathcal{F}$ and the hypercovering K. □

24.5. Hypercoverings a la Verdier

The astute reader will have noticed that all we need in order to get the Čech to cohomology spectral sequence for a hypercovering of an object X, is the conclusion of Lemma 24.2.10. Therefore the following definition makes sense.

Definition 24.5.1. Let $\mathcal{C}$ be a site. Assume $\mathcal{C}$ has equalizers and fibre products. Let $\mathcal{G}$ be a presheaf of sets. A *hypercovering of* $\mathcal{G}$ is a simplicial object K of $\mathrm{SR}(\mathcal{C})$ endowed with an augmentation $F(K) \to \mathcal{G}$ such that

(1) $F(K_0) \to \mathcal{G}$ becomes surjective after sheafification,
(2) $F(K_1) \to F(K_0) \times_{\mathcal{G}} F(K_0)$ becomes surjective after sheafification, and
(3) $F(K_{n+1}) \longrightarrow F((\mathrm{cosk}_n \mathrm{sk}_n K)_{n+1})$ for $n \geq 1$ becomes surjective after sheafification.

We say that a simplicial object K of $\mathrm{SR}(\mathcal{C})$ is a *hypercovering* if K is a hypercovering of the final object $*$ of $PSh(\mathcal{C})$.

The assumption that $\mathcal{C}$ has fibre products and equalizers guarantees that $\mathrm{SR}(\mathcal{C})$ has fibre products and equalizers and F commutes with these (Lemma 24.2.3) which suffices to define the coskelet functors used (see Simplicial, Remark 14.17.12 and Categories, Lemma 4.18.2). If $\mathcal{C}$ is general, we can replace the condition (3) by the condition that $F(K_{n+1}) \longrightarrow ((\mathrm{cosk}_n \mathrm{sk}_n F(K))_{n+1})$ for $n \geq 1$ becomes surjective after sheafification and the results of this section remain valid.

Let $\mathcal{F}$ be an abelian sheaf on $\mathcal{C}$. In the previous section, we defined the Čech complex of $\mathcal{F}$ with respect to a simplicial object K of $\mathrm{SR}(\mathcal{C})$. Next, given a presheaf $\mathcal{G}$ we set

$$H^0(\mathcal{G}, \mathcal{F}) = \mathrm{Mor}_{PSh(\mathcal{C})}(\mathcal{G}, \mathcal{F}) = \mathrm{Mor}_{Sh(\mathcal{C})}(\mathcal{G}^\#, \mathcal{F}) = H^0(\mathcal{G}^\#, \mathcal{F})$$

with notation as in Cohomology on Sites, Section 21.13). This is a left exact functor and its higher derived functors (briefly studied in Cohomology on Sites, Section 21.13) will be denoted $H^i(\mathcal{G}, \mathcal{F})$. We will show that given a hypercovering K of $\mathcal{G}$, there is a Čech to cohomology spectral sequence converging to the cohomology $H^i(\mathcal{G}, \mathcal{F})$. Note that if $\mathcal{G} = *$, then $H^i(*, \mathcal{F}) = H^i(\mathcal{F})$ recovers the global cohomology of $\mathcal{F}$.

Lemma 24.5.2. *Let $\mathcal{C}$ be a site with equalizers and fibre products. Let $\mathcal{G}$ be a presheaf on $\mathcal{C}$. Let K be a hypercovering of $\mathcal{G}$. Let $\mathcal{F}$ be a sheaf of abelian groups on $\mathcal{C}$. Then $\check{H}^0(K, \mathcal{F}) = H^0(\mathcal{G}, \mathcal{F})$.*

Proof. This follows from the definition of $H^0(\mathcal{G}, \mathcal{F})$ and the fact that

$$F(K_1) \rightrightarrows F(K_0) \longrightarrow \mathcal{G}$$

becomes an coequalizer diagram after sheafification. □

Lemma 24.5.3. *Let $\mathcal{C}$ be a site with equalizers and fibre products. Let $\mathcal{G}$ be a presheaf on $\mathcal{C}$. Let K be a hypercovering of $\mathcal{G}$. Let $\mathcal{I}$ be an injective sheaf of abelian groups on $\mathcal{C}$. Then*

$$\check{H}^p(K, \mathcal{I}) = \begin{cases} H^0(\mathcal{G}, \mathcal{I}) & \text{if} \quad p = 0 \\ 0 & \text{if} \quad p > 0 \end{cases}$$

Proof. By (24.4.2.1) we have

$$s(\mathcal{F}(K)) = \mathrm{Hom}_{Ab(\mathcal{C})}(s(\mathbf{Z}^\#_{F(K)}), \mathcal{F})$$

The complex $s(\mathbf{Z}^\#_{F(K)})$ is quasi-isomorphic to $\mathbf{Z}^\#_{\mathcal{G}}$, see Lemma 24.3.4. We conclude that if $\mathcal{I}$ is an injective abelian sheaf, then the complex $s(\mathcal{I}(K))$ is acyclic except possibly in degree 0. In other words, we have $\check{H}^i(K, \mathcal{I}) = 0$ for $i > 0$. Combined with Lemma 24.5.2 the lemma is proved. □

Lemma 24.5.4. *Let $\mathcal{C}$ be a site with equalizers and fibre products. Let $\mathcal{G}$ be a presheaf on $\mathcal{C}$. Let K be a hypercovering of $\mathcal{G}$. Let $\mathcal{F}$ be a sheaf of abelian groups on $\mathcal{C}$. There is a map*

$$s(\mathcal{F}(K)) \longrightarrow R\Gamma(\mathcal{G}, \mathcal{F})$$

in $D^+(Ab)$ functorial in $\mathcal{F}$, which induces a natural transformation

$$\check{H}^i(K, -) \longrightarrow H^i(\mathcal{G}, -)$$

of functors $Ab(\mathcal{C}) \to Ab$. Moreover, there is a spectral sequence $(E_r, d_r)_{r \geq 0}$ with

$$E_2^{p,q} = \check{H}^p(K, \underline{H}^q(\mathcal{F}))$$

converging to $H^{p+q}(\mathcal{G}, \mathcal{F})$. This spectral sequence is functorial in $\mathcal{F}$ and in the hypercovering K.

Proof. Choose an injective resolution $\mathcal{F} \to \mathcal{I}^\bullet$ in the category of abelian sheaves on $\mathcal{C}$. Consider the double complex $A^{\bullet,\bullet}$ with terms

$$A^{p,q} = \mathcal{I}^q(K_p)$$

where the differential $d_1^{p,q} : A^{p,q} \to A^{p+1,q}$ is the one coming from the differential $\mathcal{I}^p \to \mathcal{I}^{p+1}$ and the differential $d_2^{p,q} : A^{p,q} \to A^{p,q+1}$ is the one coming from the differential on the complex $s(\mathcal{I}^p(K))$ associated to the cosimplicial abelian group $\mathcal{I}^p(K)$ as explained above. We will use the two spectral sequences $({}'E_r, {}'d_r)$ and $({}''E_r, {}''d_r)$ associated to this double complex, see Homology, Section 12.22.

By Lemma 24.5.3 the complexes $s(\mathcal{I}^p(K))$ are acyclic in positive degrees and have H^0 equal to $H^0(\mathcal{G}, \mathcal{I}^p)$. Hence by Homology, Lemma 12.22.6 and its proof the spectral sequence $({}'E_r, {}'d_r)$ degenerates, and the natural map

$$H^0(\mathcal{G}, \mathcal{I}^\bullet) \longrightarrow \mathrm{Tot}(A^{\bullet,\bullet})$$

is a quasi-isomorphism of complexes of abelian groups. The map $s(\mathcal{F}(K)) \longrightarrow R\Gamma(\mathcal{G}, \mathcal{F})$ of the lemma is the composition of the natural map $s(\mathcal{F}(K)) \to \mathrm{Tot}(A^{\bullet,\bullet})$ followed by the inverse of the displayed quasi-isomorphism above. This works because $H^0(\mathcal{G}, \mathcal{I}^\bullet)$ is a representative of $R\Gamma(\mathcal{G}, \mathcal{F})$.

Consider the spectral sequence $({}''E_r, {}''d_r)_{r \geq 0}$. By Homology, Lemma 12.22.3 we see that

$${}''E_2^{p,q} = H^p_{II}(H^q_I(A^{\bullet,\bullet}))$$

In other words, we first take cohomology with respect to d_1 which gives the groups ${}''E_1^{p,q} = \underline{H}^p(\mathcal{F})(K_q)$. Hence it is indeed the case (by the description of the differential ${}''d_1$) that ${}''E_2^{p,q} = \check{H}^p(K, \underline{H}^q(\mathcal{F}))$. Since this spectral sequence converges to the cohomology of $\mathrm{Tot}(A^{\bullet,\bullet})$ the proof is finished. □

Lemma 24.5.5. *Let $\mathcal{C}$ be a site with equalizers and fibre products. Let K be a hypercovering. Let $\mathcal{F}$ be an abelian sheaf. There is a spectral sequence $(E_r, d_r)_{r\geq 0}$ with*

$$E_2^{p,q} = \check{H}^p(K, \underline{H}^q(\mathcal{F}))$$

converging to the global cohomology groups $H^{p+q}(\mathcal{F})$.

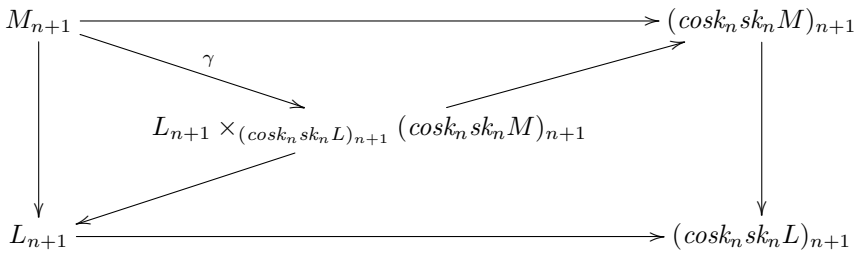
Proof. This is a special case of Lemma 24.5.4. □

24.6. Covering hypercoverings

Here are some ways to construct hypercoverings. We note that since the category $\mathrm{SR}(\mathcal{C}, X)$ has fibre products the category of simplicial objects of $\mathrm{SR}(\mathcal{C}, X)$ has fibre products as well, see Simplicial, Lemma 14.7.2.

Lemma 24.6.1. *Let $\mathcal{C}$ be a site with fibre products. Let X be an object of $\mathcal{C}$. Let K, L, M be simplicial objects of $SR(\mathcal{C}, X)$. Let $a : K \to L$, $b : M \to L$ be morphisms. Assume*

(1) *K is a hypercovering of X,*
(2) *the morphism $M_0 \to L_0$ is a covering, and*
(3) *for all $n \geq 0$ in the diagram*



the arrow γ is a covering.

Then the fibre product $K \times_L M$ is a hypercovering of X.

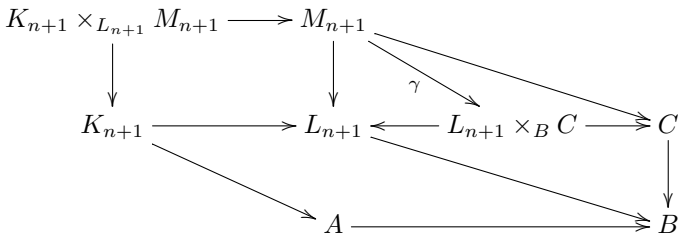
Proof. The morphism $(K \times_L M)_0 = K_0 \times_{L_0} M_0 \to K_0$ is a base change of a covering by (2), hence a covering, see Lemma 24.2.5. And $K_0 \to \{X \to X\}$ is a covering by (1). Thus $(K \times_L M)_0 \to \{X \to X\}$ is a covering by Lemma 24.2.5. Hence $K \times_L M$ satisfies the first condition of Definition 24.2.6.

We still have to check that

$$K_{n+1} \times_{L_{n+1}} M_{n+1} = (K \times_L M)_{n+1} \longrightarrow (\mathrm{cosk}_n \mathrm{sk}_n (K \times_L M))_{n+1}$$

is a covering for all $n \geq 0$. We abbreviate as follows: $A = (\mathrm{cosk}_n \mathrm{sk}_n K)_{n+1}$, $B = (\mathrm{cosk}_n \mathrm{sk}_n L)_{n+1}$, and $C = (\mathrm{cosk}_n \mathrm{sk}_n M)_{n+1}$. The functor $\mathrm{cosk}_n \mathrm{sk}_n$ commutes with fibre products, see Simplicial, Lemma 14.17.14. Thus the right hand side

above is equal to $A \times_B C$. Consider the following commutative diagram



This diagram shows that

$$K_{n+1} \times_{L_{n+1}} M_{n+1} = (K_{n+1} \times_B C) \times_{(L_{n+1} \times_B C), \gamma} M_{n+1}$$

Now, $K_{n+1} \times_B C \to A \times_B C$ is a base change of the covering $K_{n+1} \to A$ via the morphism $A \times_B C \to A$, hence is a covering. By assumption (3) the morphism γ is a covering. Hence the morphism

$$(K_{n+1} \times_B C) \times_{(L_{n+1} \times_B C), \gamma} M_{n+1} \longrightarrow K_{n+1} \times_B C$$

is a covering as a base change of a covering. The lemma follows as a composition of coverings is a covering. □

Lemma 24.6.2. *Let $\mathcal{C}$ be a site with fibre products. Let X be an object of $\mathcal{C}$. If K, L are hypercoverings of X, then $K \times L$ is a hypercovering of X.*

Proof. You can either verify this directly, or use Lemma 24.6.1 above and check that $L \to \{X \to X\}$ has property (3). □

Let $\mathcal{C}$ be a site with fibre products. Let X be an object of $\mathcal{C}$. Since the category $\mathrm{SR}(\mathcal{C}, X)$ has coproducts and finite limits, it is permissible to speak about the objects $U \times K$ and $\mathrm{Hom}(U, K)$ for certain simplicial sets U (for example those with finitely many nondegenerate simplices) and any simplicial object K of $\mathrm{SR}(\mathcal{C}, X)$. See Simplicial, Sections 14.12 and 14.15.

Lemma 24.6.3. *Let $\mathcal{C}$ be a site with fibre products. Let X be an object of $\mathcal{C}$. Let K be a hypercovering of X. Let $k \geq 0$ be an integer. Let $u : Z \to K_k$ be a covering in in $SR(\mathcal{C}, X)$. Then there exists a morphism of hypercoverings $f : L \to K$ such that $L_k \to K_k$ factors through u.*

Proof. Denote $Y = K_k$. There is a canonical morphism $K \to \mathrm{Hom}(\Delta[k], Y)$ corresponding to id_Y via Simplicial, Lemma 14.15.5. We will use the description of $\mathrm{Hom}(\Delta[k], Y)$ and $\mathrm{Hom}(\Delta[k], Z)$ given in that lemma. In particular there is a morphism $\mathrm{Hom}(\Delta[k], Y) \to \mathrm{Hom}(\Delta[k], Z)$ which on degree n terms is the morphism

$$\prod\nolimits_{\alpha:[k]\to[n]} Y \longrightarrow \prod\nolimits_{\alpha:[k]\to[n]} Z.$$

Set

$$L = K \times_{\mathrm{Hom}(\Delta[n], Y)} \mathrm{Hom}(\Delta[n], Z).$$

The morphism $L_k \to K_k$ sits in to a commutative diagram

$$\begin{array}{ccccc} L_k & \longrightarrow & \prod_{\alpha:[k]\to[n]} Y & \xrightarrow{\mathrm{pr}_{\mathrm{id}_{[k]}}} & Y \\ \downarrow & & \downarrow & & \downarrow \\ K_k & \longrightarrow & \prod_{\alpha:[k]\to[n]} Z & \xrightarrow{\mathrm{pr}_{\mathrm{id}_{[k]}}} & Z \end{array}$$

Since the composition of the two bottom arrows is the identity we conclude that we have the desired factorization.

We still have to show that L is a hypercovering of X. To see this we will use Lemma 24.6.1. Condition (1) is satisfied by assumption. For (2), the morphism

$$\mathrm{Hom}(\Delta[k], Y)_0 \to \mathrm{Hom}(\Delta[k], Z)_0$$

is a covering because it is a product of coverings, see Lemma 24.2.5. For (3) suppose first that $n \geq 1$. In this case by Simplicial, Lemma 14.19.12 we have $\mathrm{Hom}(\Delta[k], Y) = \mathrm{cosk}_n \mathrm{sk}_n \mathrm{Hom}(\Delta[k], Y)$ and similarly for Z. Thus condition (3) for $n > 0$ is clear. For $n = 0$, the diagram of condition (3) of Lemma 24.6.1 is, according to Simplicial, Lemma 14.19.13, the diagram

$$\begin{array}{ccc} \prod_{\alpha:[k]\to[1]} Z & \longrightarrow & Z \times Z \\ \downarrow & & \downarrow \\ \prod_{\alpha:[k]\to[1]} Y & \longrightarrow & Y \times Y \end{array}$$

with obvious horizontal arrows. Thus the morphism γ is the morphism

$$\prod\nolimits_{\alpha:[k]\to[1]} Z \longrightarrow \prod\nolimits_{\alpha:[k]\to[1] \text{ not onto}} Z \times \prod\nolimits_{\alpha:[k]\to[1] \text{ onto}} Y$$

which is a product of coverings and hence a covering according to Lemma 24.6.1 once again. □

Lemma 24.6.4. *Let $\mathcal{C}$ be a site with fibre products. Let X be an object of $\mathcal{C}$. Let K be a hypercovering of X. Let $n \geq 0$ be an integer. Let $u : \mathcal{F} \to F(K_n)$ be a morphism of presheaves which becomes surjective on sheafification. Then there exists a morphism of hypercoverings $f : L \to K$ such that $F(f_n) : F(L_n) \to F(K_n)$ factors through u.*

Proof. Write $K_n = \{U_i \to X\}_{i \in I}$. Thus the map u is a morphism of presheaves of sets $u : \mathcal{F} \to \amalg h_{u_i}$. The assumption on u means that for every $i \in I$ there exists a covering $\{U_{ij} \to U_i\}_{j \in I_i}$ of the site $\mathcal{C}$ and a morphism of presheaves $t_{ij} : h_{U_{ij}} \to \mathcal{F}$ such that $u \circ t_{ij}$ is the map $h_{U_{ij}} \to h_{U_i}$ coming from the morphism $U_{ij} \to U_i$. Set $J = \amalg_{i \in I} I_i$, and let $\alpha : J \to I$ be the obvious map. For $j \in J$ denote $V_j = U_{\alpha(j)j}$. Set $Z = \{V_j \to X\}_{j \in J}$. Finally, consider the morphism $u' : Z \to K_n$ given by $\alpha : J \to I$ and the morphisms $V_j = U_{\alpha(j)j} \to U_{\alpha(j)}$ above. Clearly, this is a covering in the category $\mathrm{SR}(\mathcal{C}, X)$, and by construction $F(u') : F(Z) \to F(K_n)$ factors through u. Thus the result follows from Lemma 24.6.3 above. □

24.7. Adding simplices

In this section we prove some technical lemmas which we will need later. Let $\mathcal{C}$ be a site with fibre products. Let X be an object of $\mathcal{C}$. As we pointed out in Section 24.6 above, the objects $U \times K$ and $\mathrm{Hom}(U, K)$ for certain simplicial sets U and any simplicial object K of $\mathrm{SR}(\mathcal{C}, X)$ are defined. See Simplicial, Sections 14.12 and 14.15.

Lemma 24.7.1. *Let $\mathcal{C}$ be a site with fibre products. Let X be an object of $\mathcal{C}$. Let K be a hypercovering of X. Let $U \subset V$ be simplicial sets, with U_n, V_n finite nonempty for all n. Assume that U has finitely many nondegenerate simplices. Suppose $n \geq 0$ and $x \in V_n$, $x \notin U_n$ are such that*

(1) $V_i = U_i$ *for* $i < n$,
(2) $V_n = U_n \cup \{x\}$,
(3) *any* $z \in V_j$, $z \notin U_j$ *for* $j > n$ *is degenerate.*

Then the morphism

$$\mathrm{Hom}(V, K)_0 \longrightarrow \mathrm{Hom}(U, K)_0$$

of $SR(\mathcal{C}, X)$ *is a covering.*

Proof. If $n = 0$, then it follows easily that $V = U \amalg \Delta[0]$ (see below). In this case $\mathrm{Hom}(V, K)_0 = \mathrm{Hom}(U, K)_0 \times K_0$. The result, in this case, then follows from Lemma 24.2.5.

Let $a : \Delta[n] \to V$ be the morphism associated to x as in Simplicial, Lemma 14.11.3. Let us write $\partial\Delta[n] = i_{(n-1)!}\mathrm{sk}_{n-1}\Delta[n]$ for the $(n-1)$-skeleton of $\Delta[n]$. Let $b : \partial\Delta[n] \to U$ be the restriction of a to the $(n-1)$ skeleton of $\Delta[n]$. By Simplicial, Lemma 14.19.7 we have $V = U \amalg_{\partial\Delta[n]} \Delta[n]$. By Simplicial, Lemma 14.15.6 we get that

$$\begin{array}{ccc} \mathrm{Hom}(V, K)_0 & \longrightarrow & \mathrm{Hom}(U, K)_0 \\ \downarrow & & \downarrow \\ \mathrm{Hom}(\Delta[n], K)_0 & \longrightarrow & \mathrm{Hom}(\partial\Delta[n], K)_0 \end{array}$$

is a fibre product square. Thus it suffices to show that the bottom horizontal arrow is a covering. By Simplicial, Lemma 14.19.11 this arrow is identified with

$$K_n \to (\mathrm{cosk}_{n-1}\mathrm{sk}_{n-1}K)_n$$

and hence is a covering by definition of a hypercovering. □

Lemma 24.7.2. *Let* $\mathcal{C}$ *be a site with fibre products. Let* X *be an object of* $\mathcal{C}$*. Let* K *be a hypercovering of* X*. Let* $U \subset V$ *be simplicial sets, with* U_n, V_n *finite nonempty for all* n*. Assume that* U *and* V *have finitely many nondegenerate simplices. Then the morphism*

$$\mathrm{Hom}(V, K)_0 \longrightarrow \mathrm{Hom}(U, K)_0$$

of $SR(\mathcal{C}, X)$ *is a covering.*

Proof. By Lemma 24.7.1 above, it suffices to prove a simple lemma about inclusions of simplicial sets $U \subset V$ as in the lemma. And this is exactly the result of Simplicial, Lemma 14.19.8. □

24.8. Homotopies

Let $\mathcal{C}$ be a site with fibre products. Let X be an object of $\mathcal{C}$. Let L be a simplicial object of $\mathrm{SR}(\mathcal{C}, X)$. According to Simplicial, Lemma 14.15.4 there exists an object $\mathrm{Hom}(\Delta[1], L)$ in the category $\mathrm{Simp}(\mathrm{SR}(\mathcal{C}, X))$ which represents the functor

$$T \longmapsto \mathrm{Mor}_{\mathrm{Simp}(\mathrm{SR}(\mathcal{C},X))}(\Delta[1] \times T, L)$$

There is a canonical morphism

$$\mathrm{Hom}(\Delta[1], L) \to L \times L$$

coming from $e_i : \Delta[0] \to \Delta[1]$ and the identification $\mathrm{Hom}(\Delta[0], L) = L$.

Lemma 24.8.1. *Let $\mathcal{C}$ be a site with fibre products. Let X be an object of $\mathcal{C}$. Let L be a simplicial object of $SR(\mathcal{C}, X)$. Let $n \geq 0$. Consider the commutative diagram*

$$\begin{array}{ccc} \mathrm{Hom}(\Delta[1], L)_{n+1} & \longrightarrow & (cosk_n sk_n \mathrm{Hom}(\Delta[1], L))_{n+1} \\ \downarrow & & \downarrow \\ (L \times L)_{n+1} & \longrightarrow & (cosk_n sk_n (L \times L))_{n+1} \end{array} \tag{24.8.1.1}$$

coming from the morphism defined above. We can identify the terms in this diagram as follows, where $\partial\Delta[n+1] = i_{n!} sk_n \Delta[n+1]$ is the n-skeleton of the $(n+1)$-simplex:

$$\begin{aligned} \mathrm{Hom}(\Delta[1], L)_{n+1} &= \mathrm{Hom}(\Delta[1] \times \Delta[n+1], L)_0 \\ (cosk_n sk_n \mathrm{Hom}(\Delta[1], L))_{n+1} &= \mathrm{Hom}(\Delta[1] \times \partial\Delta[n+1], L)_0 \\ (L \times L)_{n+1} &= \mathrm{Hom}((\Delta[n+1] \amalg \Delta[n+1], L)_0 \\ (cosk_n sk_n (L \times L))_{n+1} &= \mathrm{Hom}(\partial\Delta[n+1] \amalg \partial\Delta[n+1], L)_0 \end{aligned}$$

and the morphism between these objects of $SR(\mathcal{C}, X)$ come from the commutative diagram of simplicial sets

$$\begin{array}{ccc} \Delta[1] \times \Delta[n+1] & \longleftarrow & \Delta[1] \times \partial\Delta[n+1] \\ \uparrow & & \uparrow \\ \Delta[n+1] \amalg \Delta[n+1] & \longleftarrow & \partial\Delta[n+1] \amalg \partial\Delta[n+1] \end{array} \tag{24.8.1.2}$$

Moreover the fibre product of the bottom arrow and the right arrow in (24.8.1.1) is equal to

$$\mathrm{Hom}(U, L)_0$$

where $U \subset \Delta[1] \times \Delta[n+1]$ is the smallest simplicial subset such that both $\Delta[n+1] \amalg \Delta[n+1]$ and $\Delta[1] \times \partial\Delta[n+1]$ map into it.

Proof. The first and third equalities are Simplicial, Lemma 14.15.4. The second and fourth follow from the cited lemma combined with Simplicial, Lemma 14.19.11. The last assertion follows from the fact that U is the push-out of the bottom and right arrow of the diagram (24.8.1.2), via Simplicial, Lemma 14.15.6. To see that U is equal to this push-out it suffices to see that the intersection of $\Delta[n+1] \amalg \Delta[n+1]$ and $\Delta[1] \times \partial\Delta[n+1]$ in $\Delta[1] \times \Delta[n+1]$ is equal to $\partial\Delta[n+1] \amalg \partial\Delta[n+1]$. This we leave to the reader. □

Lemma 24.8.2. *Let $\mathcal{C}$ be a site with fibre products. Let X be an object of $\mathcal{C}$. Let K, L be hypercoverings of X. Let $a, b : K \to L$ be morphisms of hypercoverings. There exists a morphism of hypercoverings $c : K' \to K$ such that $a \circ c$ is homotopic to $b \circ c$.*

Proof. Consider the following commutative diagram

$$\begin{array}{ccccc} K' & \overset{def}{=\!=} & K \times_{(L \times L)} \mathrm{Hom}(\Delta[1], L) & \longrightarrow & \mathrm{Hom}(\Delta[1], L) \\ & \searrow^{c} & \downarrow & & \downarrow \\ & & K & \xrightarrow{(a,b)} & L \times L \end{array}$$

By the functorial property of $\mathrm{Hom}(\Delta[1], L)$ the composition of the horizontal morphisms corresponds to a morphism $K'\Delta[1] \to L$ which defines a homotopy between

$c \circ a$ and $c \circ b$. Thus if we can show that K' is a hypercovering of X, then we obtain the lemma. To see this we will apply Lemma 24.6.1 to the pair of morphisms $K \to L \times L$ and $\mathrm{Hom}(\Delta[1], L) \to L \times L$. Condition (1) of Lemma 24.6.1 is satisfied. Condition (2) of Lemma 24.6.1 is true because $\mathrm{Hom}(\Delta[1], L)_0 = L_1$, and the morphism $(d_0^1, d_1^1) : L_1 \to L_0 \times L_0$ is a covering of $\mathrm{SR}(\mathcal{C}, X)$ by our assumption that L is a hypercovering. To prove condition (3) of Lemma 24.6.1 we use Lemma 24.8.1 above. According to this lemma the morphism γ of condition (3) of Lemma 24.6.1 is the morphism

$$\mathrm{Hom}(\Delta[1] \times \Delta[n+1], L)_0 \longrightarrow \mathrm{Hom}(U, L)_0$$

where $U \subset \Delta[1] \times \Delta[n+1]$. According to Lemma 24.7.2 this is a covering and hence the claim has been proven. □

Remark 24.8.3. Note that the crux of the proof is to use Lemma 24.7.2. This lemma is completely general and does not care about the exact shape of the simplicial sets (as long as they have only finitely many nondegenerate simplices). It seems altogether reasonable to expect a result of the following kind: Given any morphism $a : K \times \partial\Delta[k] \to L$, with K and L hypercoverings, there exists a morphism of hypercoverings $c : K' \to K$ and a morphism $g : K' \times \Delta[k] \to L$ such that $g|_{K' \times \partial\Delta[k]} = a \circ (c \times \mathrm{id}_{\partial\Delta[k]})$. In other words, the category of hypercoverings is in a suitable sense contractible.

24.9. Cohomology and hypercoverings

Let $\mathcal{C}$ be a site with fibre products. Let X be an object of $\mathcal{C}$. Let $\mathcal{F}$ be a sheaf of abelian groups on $\mathcal{C}$. Let K, L be hypercoverings of X. If $a, b : K \to L$ are homotopic maps, then $\mathcal{F}(a), \mathcal{F}(b) : \mathcal{F}(K) \to \mathcal{F}(L)$ are homotopic maps, see Simplicial, Lemma 14.26.3. Hence have the same effect on cohomology groups of the associated cochain complexes, see Simplicial, Lemma 14.26.5. We are going to use this to define the colimit over all hypercoverings.

Let us temporarily denote $\mathrm{HC}(\mathcal{C}, X)$ the category of hypercoverings of X. We have seen that this is a category and not a "big" category, see Lemma 24.2.8. This will be the index category for our diagram, see Categories, Section 4.14 for notation. Consider the diagram

$$\check{H}^i(-, \mathcal{F}) : \mathrm{HC}(\mathcal{C}, X) \longrightarrow Ab.$$

By Lemma 24.6.2 and Lemma 24.8.2, and the remark on homotopies above, this diagram is directed, see Categories, Definition 4.19.1. Thus the colimit

$$\check{H}^i_{\mathrm{HC}}(X, \mathcal{F}) = \mathrm{colim}_{K \in \mathrm{HC}(\mathcal{C}, X)} \check{H}^i(K, \mathcal{F})$$

has a particularly simple description (see location cited).

Theorem 24.9.1. *Let $\mathcal{C}$ be a site with fibre products. Let X be an object of $\mathcal{C}$. Let $i \geq 0$. The functors*

$$\begin{aligned} Ab(\mathcal{C}) &\longrightarrow Ab \\ \mathcal{F} &\longmapsto H^i(X, \mathcal{F}) \\ \mathcal{F} &\longmapsto \check{H}^i_{HC}(X, \mathcal{F}) \end{aligned}$$

are canonically isomorphic.

Proof using spectral sequences. Suppose that $\xi \in H^p(X, \mathcal{F})$ for some $p \geq 0$. Let us show that ξ is in the image of the map $\check{H}^p(X, \mathcal{F}) \to H^p(X, \mathcal{F})$ of Lemma 24.4.3 for some hypercovering K of X.

This is true if $p = 0$ by Lemma 24.4.1. If $p = 1$, choose a Cech hypercovering K of X as in Example 24.2.7 starting with a covering $K_0 = \{U_i \to X\}$ in the site $\mathcal{C}$ such that $\xi|_{U_i} = 0$, see Cohomology on Sites, Lemma 21.8.3. It follows immediately from the spectral sequence in Lemma 24.4.3 that ξ comes from an element of $\check{H}^1(K, \mathcal{F})$ in this case. In general, choose any hypercovering K of X such that ξ maps to zero in $\underline{H}^p(\mathcal{F})(K_0)$ (using Example 24.2.7 and Cohomology on Sites, Lemma 21.8.3 again). By the spectral sequence of Lemma 24.4.3 the obstruction for ξ to come from an element of $\check{H}^p(K, \mathcal{F})$ is a sequence of elements $\xi_1, \ldots, \xi_{p-1}$ with $\xi_q \in \check{H}^{p-q}(K, \underline{H}^q(\mathcal{F}))$ (more precisely the images of the ξ_q in certain subquotients of these groups).

We can inductively replace the hypercovering K by refinements such that the obstructions $\xi_1, \ldots, \xi_{p-1}$ restrict to zero (and not just the images in the subquotients – so no subtlety here). Indeed, suppose we have already managed to reach the situation where $\xi_{q+1}, \ldots, \xi_{p-1}$ are zero. Note that $\xi_q \in \check{H}^{p-q}(K, \underline{H}^q(\mathcal{F}))$ is the class of some element

$$\tilde{\xi}_q \in \underline{H}^q(\mathcal{F})(K_{p-q}) = \prod H^q(U_i, \mathcal{F})$$

if $K_{p-q} = \{U_i \to X\}_{i \in I}$. Let $\xi_{q,i}$ be the component of $\tilde{\xi}_q$ in $H^q(U_i, \mathcal{F})$. As $q \geq 1$ we can use Cohomology on Sites, Lemma 21.8.3 yet again to choose coverings $\{U_{i,j} \to U_i\}$ of the site such that each restriction $\xi_{q,i}|_{U_{i,j}} = 0$. Consider the object $Z = \{U_{i,j} \to X\}$ of the category $\mathrm{SR}(\mathcal{C}, X)$ and its obvious morphism $u : Z \to K_{p-q}$. It is clear that u is a covering, see Definition 24.2.4. By Lemma 24.6.3 there exists a morphism $L \to K$ of hypercoverings of X such that $L_{p-q} \to K_{p-q}$ factors through u. Then clearly the image of ξ_q in $\underline{H}^q(\mathcal{F})(L_{p-q})$. is zero. Since the spectral sequence of Lemma 24.4.3 is functorial this means that after replacing K by L we reach the situation where $\xi_q, \ldots, \xi_{p-1}$ are all zero. Continuing like this we end up with a hypercovering where they are all zero and hence ξ is in the image of the map $\check{H}^p(X, \mathcal{F}) \to H^p(X, \mathcal{F})$.

Suppose that K is a hypercovering of X, that $\xi \in \check{H}^p(K, \mathcal{F})$ and that the image of ξ under the map $\check{H}^p(X, \mathcal{F}) \to H^p(X, \mathcal{F})$ of Lemma 24.4.3 is zero. To finish the proof of the theorem we have to show that there exists a morphism of hypercoverings $L \to K$ such that ξ restricts to zero in $\check{H}^p(L, \mathcal{F})$. By the spectral sequence of Lemma 24.4.3 the vanishing of the image of ξ in $H^p(X, \mathcal{F})$ means that there exist elements $\xi_1, \ldots, \xi_{p-2}$ with $\xi_q \in \check{H}^{p-1-q}(K, \underline{H}^q(\mathcal{F}))$ (more precisely the images of these in certain subquotients) such that the images $d_{q+1}^{p-1-q,q}\xi_q$ (in the spectral sequence) add up to ξ. Hence by exactly the same mechanism as above we can find a morphism of hypercoverings $L \to K$ such that the restrictions of the elements ξ_q, $q = 1, \ldots, p-2$ in $\check{H}^{p-1-q}(L, \underline{H}^q(\mathcal{F}))$ are zero. Then it follows that ξ is zero since the morphism $L \to K$ induces a morphism of spectral sequences according to Lemma 24.4.3. $\square$

Proof without using spectral sequences. We have seen the result for $i = 0$, see Lemma 24.4.1. We know that the functors $H^i(X, -)$ form a universal δ-functor, see Derived Categories, Lemma 13.19.4. In order to prove the theorem it suffices to show that the sequence of functors $\check{H}^i_{HC}(X, -)$ forms a δ-functor. Namely we

know that Cech cohomology is zero on injective sheaves (Lemma 24.4.2) and then we can apply Homology, Lemma 12.11.4.

Let

$$0 \to \mathcal{F} \to \mathcal{G} \to \mathcal{H} \to 0$$

be a short exact sequence of abelian sheaves on $\mathcal{C}$. Let $\xi \in \check{H}^p_{HC}(X, \mathcal{H})$. Choose a hypercovering K of X and an element $\sigma \in \mathcal{H}(K_p)$ representing ξ in cohomology. There is a corresponding exact sequence of complexes

$$0 \to s(\mathcal{F}(K)) \to s(\mathcal{G}(K)) \to s(\mathcal{H}(K))$$

but we are not assured that there is a zero on the right also and this is the only thing that prevents us from defining $\delta(\xi)$ by a simple application of the snake lemma. Recall that

$$\mathcal{H}(K_p) = \prod \mathcal{H}(U_i)$$

if $K_p = \{U_i \to X\}$. Let $\sigma = \prod \sigma_i$ with $\sigma_i \in \mathcal{H}(U_i)$. Since $\mathcal{G} \to \mathcal{H}$ is a surjection of sheaves we see that there exist coverings $\{U_{i,j} \to U_i\}$ such that $\sigma_i|_{U_{i,j}}$ is the image of some element $\tau_{i,j} \in \mathcal{G}(U_{i,j})$. Consider the object $Z = \{U_{i,j} \to X\}$ of the category $\mathrm{SR}(\mathcal{C}, X)$ and its obvious morphism $u : Z \to K_p$. It is clear that u is a covering, see Definition 24.2.4. By Lemma 24.6.3 there exists a morphism $L \to K$ of hypercoverings of X such that $L_p \to K_p$ factors through u. After replacing K by L we may therefore assume that σ is the image of an element $\tau \in \mathcal{G}(K_p)$. Note that $d(\sigma) = 0$, but not necessarily $d(\tau) = 0$. Thus $d(\tau) \in \mathcal{F}(K_{p+1})$ is a cocycle. In this situation we define $\delta(\xi)$ as the class of the cocycle $d(\tau)$ in $\check{H}^{p+1}_{HC}(X, \mathcal{F})$.

At this point there are several things to verify: (a) $\delta(\xi)$ does not depend on the choice of τ, (b) $\delta(\xi)$ does not depend on the choice of the hypercovering $L \to K$ such that σ lifts, and (c) $\delta(\xi)$ does not depend on the initial hypercovering and σ chosen to represent ξ. We omit the verification of (a), (b), and (c); the independence of the choices of the hypercoverings really comes down to Lemmas 24.6.2 and 24.8.2. We also omit the verification that δ is functorial with respect to morphisms of short exact sequences of abelian sheaves on $\mathcal{C}$.

Finally, we have to verify that with this definition of δ our short exact sequence of abelian sheaves above leads to a long exact sequence of Cech cohomology groups. First we show that if $\delta(\xi) = 0$ (with ξ as above) then ξ is the image of some element $\xi' \in \check{H}^p_{HC}(X, \mathcal{G})$. Namely, if $\delta(\xi) = 0$, then, with notation as above, we see that the class of $d(\tau)$ is zero in $\check{H}^{p+1}_{HC}(X, \mathcal{F})$. Hence there exists a morphism of hypercoverings $L \to K$ such that the restriction of $d(\tau)$ to an element of $\mathcal{F}(L_{p+1})$ is equal to $d(\upsilon)$ for some $\upsilon \in \mathcal{F}(L_p)$. This implies that $\tau|_{L_p} + \upsilon$ form a cocycle, and determine a class $\xi' \in \check{H}^p(L, \mathcal{G})$ which maps to ξ as desired.

We omit the proof that if $\xi' \in \check{H}^{p+1}_{HC}(X, \mathcal{F})$ maps to zero in $\check{H}^{p+1}_{HC}(X, \mathcal{G})$, then it is equal to $\delta(\xi)$ for some $\xi \in \check{H}^p_{HC}(X, \mathcal{H})$. □

Next, we deduce Verdier's case of Theorem 24.9.1 by a sleight of hand.

Proposition 24.9.2. *Let $\mathcal{C}$ be a site with fibre products and products of pairs. Let $\mathcal{F}$ be an abelian sheaf on $\mathcal{C}$. Let $i \geq 0$. Then*

(1) *for every $\xi \in H^i(\mathcal{F})$ there exists a hypercovering K such that ξ is in the image of the canonical map $\check{H}^i(K, \mathcal{F}) \to H^i(\mathcal{F})$, and*

(2) *if K, L are hypercoverings and $\xi_K \in \check{H}^i(K, \mathcal{F})$, $\xi_L \in \check{H}^i(L, \mathcal{F})$ are elements mapping to the same element of $H^i(\mathcal{F})$, then there exists a hypercovering M and morphisms $M \to K$ and $M \to L$ such that ξ_K and ξ_L map to the same element of $\check{H}^i(M, \mathcal{F})$.*

In other words, modulo set theoretical issues, the cohomology groups of $\mathcal{F}$ on $\mathcal{C}$ are the colimit of the Čech cohomology groups of $\mathcal{F}$ over all hypercoverings.

Proof. This result is a trivial consequence of Theorem 24.9.1. Namely, we can articially replace $\mathcal{C}$ with a slightly bigger site $\mathcal{C}'$ such that (I) $\mathcal{C}'$ has a final object X and (II) hypercoverings in $\mathcal{C}$ are more or less the same thing as hypercoverings of X in $\mathcal{C}'$. But due to the nature of things, there is quite a bit of bookkeeping to do.

Let us call a family of morphisms $\{U_i \to U\}$ in $\mathcal{C}$ with fixed target a *weak covering* if the sheafification of the map $\coprod_{i \in I} h_{U_i} \to h_U$ becomes surjective. We construct a new site $\mathcal{C}'$ as follows

(1) as a category set $\mathrm{Ob}(\mathcal{C}') = \mathrm{Ob}(\mathcal{C}) \amalg \{X\}$ and add a unique morphism to X from every object of $\mathcal{C}'$,
(2) $\mathcal{C}'$ has fibre products as fibre products and products of pairs exist in $\mathcal{C}$,
(3) coverings of $\mathcal{C}'$ are weak coverings of $\mathcal{C}$ together with those $\{U_i \to X\}_{i \in I}$ such that either $U_i = X$ for some i, or $U_i \neq X$ for all i and the map $\coprod h_{U_i} \to *$ of presheaves on $\mathcal{C}$ becomes surjective after sheafification on $\mathcal{C}$,
(4) we apply Sets, Lemma 3.11.1 to restrict the coverings to obtain our site $\mathcal{C}'$.

Then $Sh(\mathcal{C}') = Sh(\mathcal{C})$ because the inclusion functor $\mathcal{C} \to \mathcal{C}'$ is a special cocontinuous functor (see Sites, Definition 7.28.2). We omit the straightforward verifications.

Choose a covering $\{U_i \to X\}$ of $\mathcal{C}'$ such that U_i is an object of $\mathcal{C}$ for all i (possible because $\mathcal{C} \to \mathcal{C}'$ is special cocontinuous). Then $K_0 = \{U_i \to X\}$ is a covering in the site $\mathcal{C}'$ constructed above. We view K_0 as an object of $\mathrm{SR}(\mathcal{C}', X)$ and we set $K_{init} = \mathrm{cosk}_0(K_0)$. Then K_{init} is a hypercovering of X, see Example 24.2.7. Note that every $K_{init,n}$ has the shape $\{W_j \to X\}$ with $W_j \in \mathrm{Ob}(\mathcal{C})$.

Proof of (1). Choose $\xi \in H^i(\mathcal{F}) = H^i(X, \mathcal{F}')$ where $\mathcal{F}'$ is the abelian sheaf on $\mathcal{C}'$ corresponding to $\mathcal{F}$ on $\mathcal{C}$. By Theorem 24.9.1 there exists a morphism of hypercoverings $K' \to K_{init}$ of X in $\mathcal{C}'$ such that ξ comes from an element of $\check{H}^i(K', \mathcal{F})$. Write $K'_n = \{U_{n,j} \to X\}$. Now since K'_n maps to $K_{init,n}$ we see that $U_{n,j}$ is an object of $\mathcal{C}$. Hence we can define a simplicial object K of $\mathrm{SR}(\mathcal{C})$ by setting $K_n = \{U_{n,j}\}$. Since coverings in $\mathcal{C}'$ consisting of families of morphisms of $\mathcal{C}$ are weak coverings, we see that K is a hypercovering in the sense of Definition 24.5.1. Finally, since $\mathcal{F}'$ is the unique sheaf on $\mathcal{C}'$ whose restriction to $\mathcal{C}$ is equal to $\mathcal{F}$ we see that the Čech complexes $s(\mathcal{F}(K))$ and $s(\mathcal{F}'(K'))$ are identical and (1) follows. (Compatibility with map into cohomology groups omitted.)

Proof of (2). Let K and L be hypercoverings in $\mathcal{C}$. Let K' and L' be the simplicial objects of $\mathrm{SR}(\mathcal{C}', X)$ gotten from K and L by the functor $\mathrm{SR}(\mathcal{C}) \to \mathrm{SR}(\mathcal{C}', X)$, $\{U_i\} \mapsto \{U_i \to X\}$. As before we have equality of Čech complexes and hence we obtain $\xi_{K'}$ and $\xi_{L'}$ mapping to the same cohomology class of $\mathcal{F}'$ over $\mathcal{C}'$. After possibly enlarging our choice of coverings in $\mathcal{C}'$ (due to a set theoretical issue) we may assume that K' and L' are hypercoverings of X in $\mathcal{C}'$; this is true by our

definition of hypercoverings in Definition 24.5.1 and the fact that weak coverings in $\mathcal{C}$ give coverings in $\mathcal{C}'$. By Theorem 24.9.1 there exists a hypercovering M' of X in $\mathcal{C}'$ and morphisms $M' \to K'$, $M' \to L'$, and $M' \to K_{init}$ such that $\xi_{K'}$ and $\xi_{L'}$ restrict to the same element of $\check{H}^i(M', \mathcal{F})$. Unwinding this statement as above we find that (2) is true. □

24.10. Hypercoverings of spaces

The theory above is mildly interesting even in the case of topological spaces. In this case we can work out what a hypercovering is and see what the result actually says.

Let X be a topological space. Consider the site X_{Zar} of Sites, Example 7.6.4. Recall that an object of X_{Zar} is simply an open of X and that morphisms of X_{Zar} correspond simply to inclusions. So what is a hypercovering of X for the site X_{Zar}?

Let us first unwind Definition 24.2.1. An object of $\mathrm{SR}(X_{Zar}, X)$ is simply given by a set I and for each $i \in I$ an open $U_i \subset X$. Let us denote this by $\{U_i\}_{i \in I}$ since there can be no confusion about the morphism $U_i \to X$. A morphism $\{U_i\}_{i \in I} \to \{V_j\}_{j \in J}$ between two such objects is given by a map of sets $\alpha : I \to J$ such that $U_i \subset V_{\alpha(i)}$ for all $i \in I$. When is such a morphism a covering? This is the case if and only if for every $j \in J$ we have $V_j = \bigcup_{i \in I,\ \alpha(i) = j} U_i$ (and is a covering in the site X_{Zar}).

Using the above we get the following description of a hypercovering in the site X_{Zar}. A hypercovering of X in X_{Zar} is given by the following data

(1) a simplicial set I (see Simplicial, Section 14.11), and
(2) for each $n \geq 0$ and every $i \in I_n$ an open set $U_i \subset X$.

We will denote such a collection of data by the notation $(I, \{U_i\})$. In order for this to be a hypercovering of X we require the following properties

- for $i \in I_n$ and $0 \leq a \leq n$ we have $U_i \subset U_{d^n_a(i)}$,
- for $i \in I_n$ and $0 \leq a \leq n$ we have $U_i = U_{s^n_a(i)}$,
- we have

$$X = \bigcup\nolimits_{i \in I_0} U_i, \tag{24.10.0.1}$$

- for every $i_0, i_1 \in I_0$, we have

$$U_{i_0} \cap U_{i_1} = \bigcup\nolimits_{i \in I_1,\ d^1_0(i) = i_0,\ d^1_1(i) = i_1} U_i, \tag{24.10.0.2}$$

- for every $n \geq 1$ and every $(i_0, \ldots, i_{n+1}) \in (I_n)^{n+2}$ such that $d^n_{b-1}(i_a) = d^n_a(i_b)$ for all $0 \leq a < b \leq n+1$ we have

$$U_{i_0} \cap \ldots \cap U_{i_{n+1}} = \bigcup\nolimits_{i \in I_{n+1},\ d^{n+1}_a(i) = i_a,\ a = 0, \ldots, n+1} U_i, \tag{24.10.0.3}$$

- each of the open coverings (24.10.0.1), (24.10.0.2), and (24.10.0.3) is an element of $\mathrm{Cov}(X_{Zar})$ (this is a set theoretic condition, bounding the size of the index sets of the coverings).

Conditions (24.10.0.1) and (24.10.0.2) should be familiar from the chapter on sheaves on spaces for example, and condition (24.10.0.3) is the natural generalization.

Remark 24.10.1. One feature of this description is that if one of the multiple intersections $U_{i_0} \cap \ldots \cap U_{i_{n+1}}$ is empty then the covering on the right hand side may be the empty covering. Thus it is not automatically the case that the maps

$I_{n+1} \to (\mathrm{cosk}_n \mathrm{sk}_n I)_{n+1}$ are surjective. This means that the geometric realization of I may be an interesting (non-contractible) space.

In fact, let $I'_n \subset I_n$ be the subset consisting of those simplices $i \in I_n$ such that $U_i \neq \emptyset$. It is easy to see that $I' \subset I$ is a subsimplicial set, and that $(I', \{U_i\})$ is a hypercovering. Hence we can always refine a hypercovering to a hypercovering where none of the opens U_i is empty.

Remark 24.10.2. Let us repackage this information in yet another way. Namely, suppose that $(I, \{U_i\})$ is a hypercovering of the topological space X. Given this data we can construct a simplicial topological space $U_\bullet$ by setting

$$U_n = \coprod\nolimits_{i \in I_n} U_i,$$

and where for given $\varphi : [n] \to [m]$ we let morphisms $U(\varphi) : U_n \to U_m$ be the morphism coming from the inclusions $U_i \subset U_{\varphi(i)}$ for $i \in I_n$. This simplicial topological space comes with an augmentation $\epsilon : U_\bullet \to X$. With this morphism the simplicial space $U_\bullet$ becomes a hypercovering of X along which one has cohomological descent in the sense of [**AGV71**, Exposé Vbis]. In other words, $H^n(U_\bullet, \epsilon^*\mathcal{F}) = H^n(X, \mathcal{F})$. (Insert future reference here to cohomology over simplicial spaces and cohomological descent formulated in those terms.) Suppose that $\mathcal{F}$ is an abelian sheaf on X. In this case the spectral sequence of Lemma 24.4.3 becomes the spectral sequence with E_1-term

$$E_1^{p,q} = H^q(U_p, \epsilon_q^*\mathcal{F}) \Rightarrow H^{p+q}(U_\bullet, \epsilon^*\mathcal{F}) = H^{p+q}(X, \mathcal{F})$$

comparing the total cohomology of $\epsilon^*\mathcal{F}$ to the cohomology groups of $\mathcal{F}$ over the pieces of $U_\bullet$. (Insert future reference to this spectral sequence here.)

In topology we often want to find hypercoverings of X which have the property that all the U_i come from a given basis for the topology of X and that all the coverings (24.10.0.2) and (24.10.0.3) are from a given cofinal collection of coverings. Here are two example lemmas.

Lemma 24.10.3. *Let X be a topological space. Let $\mathcal{B}$ be a basis for the topology of X. There exists a hypercovering $(I, \{U_i\})$ of X such that each U_i is an element of $\mathcal{B}$.*

Proof. Let $n \geq 0$. Let us say that an *n-truncated hypercovering* of X is given by an n-truncated simplicial set I and for each $i \in I_a$, $0 \leq a \leq n$ an open U_i of X such that the conditions defining a hypercovering hold whenever they make sense. In other words we require the inclusion relations and covering conditions only when all simplices that occur in them are a-simplices with $a \leq n$. The lemma follows if we can prove that given a n-truncated hypercovering $(I, \{U_i\})$ with all $U_i \in \mathcal{B}$ we can extend it to an $(n+1)$-truncated hypercovering without adding any a-simplices for $a \leq n$. This we do as follows. First we consider the $(n+1)$-truncated simplicial set I' defined by $I' = \mathrm{sk}_{n+1}(\mathrm{cosk}_n I)$. Recall that

$$I'_{n+1} = \left\{ \begin{array}{c} (i_0, \ldots, i_{n+1}) \in (I_n)^{n+2} \text{ such that} \\ d^n_{b-1}(i_a) = d^n_a(i_b) \text{ for all } 0 \leq a < b \leq n+1 \end{array} \right\}$$

If $i' \in I'_{n+1}$ is degenerate, say $i' = s^n_a(i)$ then we set $U_{i'} = U_i$ (this is forced on us anyway by the second condition). We also set $J_{i'} = \{i'\}$ in this case. If $i' \in I'_{n+1}$

is nondegenerate, say $i' = (i_0, \ldots, i_{n+1})$, then we choose a set $J_{i'}$ and an open covering

$$U_{i_0} \cap \ldots \cap U_{i_{n+1}} = \bigcup\nolimits_{i \in J_{i'}} U_i, \tag{24.10.3.1}$$

with $U_i \in \mathcal{B}$ for $i \in J_{i'}$. Set

$$I_{n+1} = \coprod\nolimits_{i' \in I'_{n+1}} J_{i'}$$

There is a canonical map $\pi : I_{n+1} \to I'_{n+1}$ which is a bijection over the set of degenerate simplices in I'_{n+1} by construction. For $i \in I_{n+1}$ we define $d_a^{n+1}(i) = d_a^{n+1}(\pi(i))$. For $i \in I_n$ we define $s_a^n(i) \in I_{n+1}$ as the unique simplex lying over the degenerate simplex $s_a^n(i) \in I'_{n+1}$. We omit the verification that this defines an $(n+1)$-truncated hypercovering of X. □

Lemma 24.10.4. *Let X be a topological space. Let $\mathcal{B}$ be a basis for the topology of X. Assume that*

1. *X is quasi-compact,*
2. *each $U \in \mathcal{B}$ is quasi-compact open, and*
3. *the intersection of any two quasi-compact opens in X is quasi-compact.*

Then there exists a hypercovering $(I, \{U_i\})$ of X with the following properties

1. *each U_i is an element of the basis $\mathcal{B}$,*
2. *each of the I_n is a finite set, and in particular*
3. *each of the coverings (24.10.0.1), (24.10.0.2), and (24.10.0.3) is finite.*

Proof. This follows directly from the construction in the proof of Lemma 24.10.3 if we choose finite coverings by elements of $\mathcal{B}$ in (24.10.3.1). Details omitted. □

24.11. Hypercoverings and weakly contractible objects

In this section we construct hypercoverings in the presence of enough weakly contractible objects (Sites, Definition 7.39.2). With our conventions this is particularly straightforward if every object has a covering by a single weakly contractible object.

Lemma 24.11.1. *Let $\mathcal{C}$ be a site. Let $\mathcal{B} \subset \mathrm{Ob}(\mathcal{C})$ be a subset. Assume*

1. *$\mathcal{C}$ has fibre products,*
2. *for all $X \in \mathrm{Ob}(\mathcal{C})$ there exists a covering $\{U \to X\}$ with $U \in \mathcal{B}$,*
3. *every element of $\mathcal{B}$ is weakly contractible,*
4. *the topology is subcanonical.*

Then for every X there exists a hypercovering K of X such that each $K_n = \{U_n \to X\}$ with $U_n \in \mathcal{B}$.

Proof. We will construct K by induction. As a first approximation choose a covering $\{U_0 \to X\}$ with $U_0 \in \mathcal{B}$ and set $K_0 = \{U_0 \to X\}$ and $K = \mathrm{cosk}_0 K_0$, see Example 24.2.7. (This object will be denoted K^0 in the final paragraph of the proof.)

Suppose for some $n \geq 0$ we have constructed a hypercovering K such that K_k consists of a single object of $\mathcal{B}$ mapping to X for $0 \leq k \leq n$ and such that $K = \mathrm{cosk}_n \mathrm{sk}_n K$. In particular $K_{n+1} = (\mathrm{cosk}_n \mathrm{sk}_n K)_{n+1}$ is a finite limit of K_k for $k \leq n$, see Simplicial, Section 14.17 (especially Lemmas 14.17.3 and 14.17.6). By the description of finite limits in $\mathrm{SR}(\mathcal{C}, X)$ (see proof of Lemma 24.2.3) we see that $K_{n+1} = \{X_{n+1} \to X\}$ for some object X_{n+1} of $\mathcal{C}$. Choose a covering

$\{U_{n+1} \to X_{n+1}\}$ with $U_{n+1} \in \mathcal{B}$. Since U_n is weakly contractible, the topology is subcanonical, and $\{U_{n+1} \to X_{n+1}\}$ is a covering, the morphisms $s_i : U_n \to X_{n+1}$ lift to morphisms $s'_i : U_n \to U_{n+1}$. Set $d'_j : U_{n+1} \to U_n$ equal to the composition of $U_{n+1} \to X_{n+1}$ and $d_j : X_{n+1} \to U_n$. We obtain a truncated simplicial object K' of $\mathrm{SR}(\mathcal{C}, X)$ by setting $K'_k = K_k$ for $k \leq n$ and $K'_{n+1} = U_{n+1}$ and morphisms $d'_i = d_i$ and $s'_i = s_i$ in degrees $k \leq n-1$ and using the morphisms d'_j and s'_i in degree n. Extend this to a full simplicial object K' of $\mathrm{SR}(\mathcal{C}, X)$ using cosk_{n+1}. By functoriality of the coskelet functors there is a morphism $K' \to K$ of simplicial objects extending the given morphism of $(n+1)$-truncated simplicial objects.

It is immediately clear from the construction that the simplicial object K' so constructed is a hypercovering of X. Moreover, note that $K' \to K$ is the identity morphism in degrees $\leq n$.

To finish the proof we take the inverse limit $K = \lim K^n$ of the sequence of simplicial objects

$$\ldots \to K^2 \to K^1 \to K^0$$

constructed above. Since we have stabilization in each degree it is clear that K agrees with K^n in degrees $\leq n$ and therefore is a hypercovering of X. □

24.12. Other chapters

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CHAPTER 25

Schemes

25.1. Introduction

In this document we define schemes. A basic reference is [**DG67**].

25.2. Locally ringed spaces

Recall that we defined ringed spaces in Sheaves, Section 6.25. Briefly, a ringed space is a pair $(X, \mathcal{O}_X)$ consisting of a topological space X and a sheaf of rings $\mathcal{O}_X$. A morphism of ringed spaces $f : (X, \mathcal{O}_X) \to (Y, \mathcal{O}_Y)$ is given by a continuous map $f : X \to Y$ and an f-map of sheaves of rings $f^\sharp : \mathcal{O}_Y \to \mathcal{O}_X$. You can think of $f^\sharp$ as a map $\mathcal{O}_Y \to f_*\mathcal{O}_X$, see Sheaves, Definition 6.21.7 and Lemma 6.21.8.

A good geometric example of this to keep in mind is $\mathcal{C}^\infty$-manifolds and morphisms of $\mathcal{C}^\infty$-manifolds. Namely, if M is a $\mathcal{C}^\infty$-manifold, then the sheaf $\mathcal{C}^\infty_M$ of smooth functions is a sheaf of rings on M. And any map $f : M \to N$ of manifolds is smooth if and only if for every local section h of $\mathcal{C}^\infty_N$ the composition $h \circ f$ is a local section of $\mathcal{C}^\infty_M$. Thus a smooth map f gives rise in a natural way to a morphism of ringed spaces

$$f : (M, \mathcal{C}^\infty_M) \longrightarrow (N, \mathcal{C}^\infty_N)$$

see Sheaves, Example 6.25.2. It is instructive to consider what happens to stalks. Namely, let $m \in M$ with image $f(m) = n \in N$. Recall that the stalk $\mathcal{C}^\infty_{M,m}$ is the ring of germs of smooth functions at m, see Sheaves, Example 6.11.4. The algebra of germs of functions on (M, m) is a local ring with maximal ideal the functions which vanish at m. Similarly for $\mathcal{C}^\infty_{N,n}$. The map on stalks $f^\sharp : \mathcal{C}^\infty_{N,n} \to \mathcal{C}^\infty_{M,m}$ maps the maximal ideal into the maximal ideal, simply because $f(m) = n$.

In algebraic geometry we study schemes. On a scheme the sheaf of rings is not determined by an intrinsic property of the space. The spectrum of a ring R (see Algebra, Section 10.16) endowed with a sheaf of rings constructed out of R (see below), will be our basic building block. It will turn out that the stalks of $\mathcal{O}$ on $\mathrm{Spec}(R)$ are the local rings of R at its primes. There are two reasons to introduce locally ringed spaces in this setting: (1) There is in general no mechanism that assigns to a continuous map of spectra a map of the corresponding rings. This is why we add as an extra datum the map $f^\sharp$. (2) If we consider morphisms of these spectra in the category of ringed spaces, then the maps on stalks may not be local homomorphisms. Since our geometric intuition says it should we introduce locally ringed spaces as follows.

Definition 25.2.1. Locally ringed spaces.

(1) A *locally ringed space* $(X, \mathcal{O}_X)$ is a pair consisting of a topological space X and a sheaf of rings $\mathcal{O}_X$ all of whose stalks are local rings.

(2) Given a locally ringed space $(X, \mathcal{O}_X)$ we say that $\mathcal{O}_{X,x}$ is the *local ring of X at x*. We denote $\mathfrak{m}_{X,x}$ or simply $\mathfrak{m}_x$ the maximal ideal of $\mathcal{O}_{X,x}$. Moreover, the *residue field of X at x* is the residue field $\kappa(x) = \mathcal{O}_{X,x}/\mathfrak{m}_x$.
(3) A *morphism of locally ringed spaces* $(f, f^\sharp) : (X, \mathcal{O}_X) \to (Y, \mathcal{O}_Y)$ is a morphism of ringed spaces such that for all $x \in X$ the induced ring map $\mathcal{O}_{Y,f(x)} \to \mathcal{O}_{X,x}$ is a local ring map.

We will usually suppress the sheaf of rings $\mathcal{O}_X$ in the notation when discussing locally ringed spaces. We will simply refer to "the locally ringed space X". We will by abuse of notation think of X also as the underlying topological space. Finally we will denote the corresponding sheaf of rings $\mathcal{O}_X$ as the *structure sheaf of X*. In addition, it is customary to denote the maximal ideal of the local ring $\mathcal{O}_{X,x}$ by $\mathfrak{m}_{X,x}$ or simply $\mathfrak{m}_x$. We will say "let $f : X \to Y$ be a morphism of locally ringed spaces" thereby suppressing the structure sheaves even further. In this case, we will by abuse of notation think of $f : X \to Y$ also as the underlying continuous map of topological spaces. The f-map corresponding to f will customarily be denoted $f^\sharp$. The condition that f is a morphism of locally ringed spaces can then be expressed by saying that for every $x \in X$ the map on stalks

$$f_x^\sharp : \mathcal{O}_{Y,f(x)} \longrightarrow \mathcal{O}_{X,x}$$

maps the maximal ideal $\mathfrak{m}_{Y,f(x)}$ into $\mathfrak{m}_{X,x}$.

Let us use these notational conventions to show that the collection of locally ringed spaces and morphisms of locally ringed spaces forms a category. In order to see this we have to show that the composition of morphisms of locally ringed spaces is a morphism of locally ringed spaces. OK, so let $f : X \to Y$ and $g : Y \to Z$ be morphism of locally ringed spaces. The composition of f and g is defined in Sheaves, Definition 6.25.3. Let $x \in X$. By Sheaves, Lemma 6.21.10 the composition

$$\mathcal{O}_{Z,g(f(x))} \xrightarrow{g^\sharp} \mathcal{O}_{Y,f(x)} \xrightarrow{f^\sharp} \mathcal{O}_{X,x}$$

is the associated map on stalks for the morphism $g \circ f$. The result follows since a composition of local ring homomorphisms is a local ring homomorphism.

A pleasing feature of the definition is the fact that the functor

$$\textit{Locally ringed spaces} \longrightarrow \textit{Ringed spaces}$$

reflects isomorphisms (plus more). Here is a less abstract statement.

Lemma 25.2.2. *Let X, Y be locally ringed spaces. If $f : X \to Y$ is an isomorphism of ringed spaces, then f is an isomorphism of locally ringed spaces.*

Proof. This follows trivially from the corresponding fact in algebra: Suppose A, B are local rings. Any isomorphism of rings $A \to B$ is a local ring homomorphism. □

25.3. Open immersions of locally ringed spaces

Definition 25.3.1. Let $f : X \to Y$ be a morphism of locally ringed spaces. We say that f is an *open immersion* if f is a homeomorphism of X onto an open subset of Y, and the map $f^{-1}\mathcal{O}_Y \to \mathcal{O}_X$ is an isomorphism.

The following construction is parallel to Sheaves, Definition 6.31.2 (3).

Example 25.3.2. Let X be a locally ringed space. Let $U \subset X$ be an open subset. Let $\mathcal{O}_U = \mathcal{O}_X|_U$ be the restriction of $\mathcal{O}_X$ to U. For $u \in U$ the stalk $\mathcal{O}_{U,u}$ is equal to the stalk $\mathcal{O}_{X,u}$, and hence is a local ring. Thus $(U, \mathcal{O}_U)$ is a locally ringed space and the morphism $j : (U, \mathcal{O}_U) \to (X, \mathcal{O}_X)$ is an open immersion.

Definition 25.3.3. Let X be a locally ringed space. Let $U \subset X$ be an open subset. The locally ringed space $(U, \mathcal{O}_U)$ of Example 25.3.2 above is the *open subspace of X associated to U*.

Lemma 25.3.4. *Let $f : X \to Y$ be an open immersion of locally ringed spaces. Let $j : V = f(X) \to Y$ be the open subspace of Y associated to the image of f. There is a unique isomorphism $f' : X \cong V$ of locally ringed spaces such that $f = j \circ f'$.*

Proof. Let f' be the homeomorphism between X and V induced by f. Then $f = j \circ f'$ as maps of topological spaces. Since there is an isomorphism of sheaves $f^\sharp : f^{-1}(\mathcal{O}_Y) \to \mathcal{O}_X$, there is an isomorphism of rings $f^\sharp : \Gamma(U, f^{-1}(\mathcal{O}_Y)) \to \Gamma(U, \mathcal{O}_X)$ for each open subset $U \subset X$. Since $\mathcal{O}_V = j^{-1}\mathcal{O}_Y$ and $f^{-1} = f'^{-1}j^{-1}$ (Sheaves, Lemma 6.21.6) we see that $f^{-1}\mathcal{O}_Y = f'^{-1}\mathcal{O}_V$, hence $\Gamma(U, f'^{-1}(\mathcal{O}_V)) \to \Gamma(U, f^{-1}(\mathcal{O}_Y))$ for every $U \subset X$ open. By composing these we get an isomorphism of rings

$$\Gamma(U, f'^{-1}(\mathcal{O}_V)) \to \Gamma(U, \mathcal{O}_X)$$

for each open subset $U \subset X$, and therefore an isomorphism of sheaves $f^{-1}(\mathcal{O}_V) \to \mathcal{O}_X$. In other words, we have an isomorphism $f'^\sharp : f'^{-1}(\mathcal{O}_V) \to \mathcal{O}_X$ and therefore an isomorphism of locally ringed spaces $(f', f'^\sharp) : (X, \mathcal{O}_X) \to (V, \mathcal{O}_V)$ (use Lemma 25.2.2). Note that $f = j \circ f'$ as morphisms of locally ringed spaces by construction.

Suppose we have another morphism $f'' : (X, \mathcal{O}_X) \to (V, \mathcal{O}_Y)$ such that $f = j \circ f''$. At any point $x \in X$, we have $j(f'(x)) = j(f''(x))$ from which it follows that $f'(x) = f''(x)$ since j is the inclusion map; therefore f' and f'' are the same as morphisms of topological spaces. On structure sheaves, for each open subset $U \subset X$ we have a commutative diagram

$$\begin{array}{ccc} \Gamma(U, f^{-1}(\mathcal{O}_Y)) & \xrightarrow{\cong} & \Gamma(U, \mathcal{O}_X) \\ \Big\downarrow \cong & \nearrow_{f'^\sharp} & \\ \Gamma(U, f'^{-1}(\mathcal{O}_V)) & \nearrow_{f''^\sharp} & \end{array}$$

from which we see that $f'^\sharp$ and $f''^\sharp$ define the same morphism of sheaves. □

From now on we do not distinguish between open subsets and their associated subspaces.

Lemma 25.3.5. *Let $f : X \to Y$ be a morphism of locally ringed spaces. Let $U \subset X$, and $V \subset Y$ be open subsets. Suppose that $f(U) \subset V$. There exists a unique morphism of locally ringed spaces $f|_U : U \to V$ such that the following diagram is a commutative square of locally ringed spaces*

$$\begin{array}{ccc} U & \longrightarrow & X \\ {\scriptstyle f|_U}\downarrow & & \downarrow{\scriptstyle f} \\ V & \longrightarrow & Y \end{array}$$

Proof. Omitted. □

In the following we will use without further mention the following fact which follows from the lemma above. Given any morphism $f : Y \to X$ of locally ringed spaces, and any open subset $U \subset X$ such that $f(Y) \subset U$, then there exists a unique morphism of locally ringed spaces $Y \to U$ such that the composition $Y \to U \to X$ is equal to f. In fact, we will even by abuse of notation write $f : Y \to U$ since this rarely gives rise to confusion.

25.4. Closed immersions of locally ringed spaces

We follow our conventions introduced in Modules, Definition 17.13.1.

Definition 25.4.1. Let $i : Z \to X$ be a morphism of locally ringed spaces. We say that i is an *closed immersion* if:

(1) The map i is a homeomorphism of Z onto a closed subset of X.
(2) The map $\mathcal{O}_X \to i_*\mathcal{O}_Z$ is surjective; let $\mathcal{I}$ denote the kernel.
(3) The $\mathcal{O}_X$-module $\mathcal{I}$ is locally generated by sections.

Lemma 25.4.2. *Let $f : Z \to X$ be a morphism of locally ringed spaces. In order for f to be a closed immersion it suffices if there exists an open covering $X = \bigcup U_i$ such that each $f : f^{-1}U_i \to U_i$ is a closed immersion.*

Proof. Omitted. □

Example 25.4.3. Let X be a locally ringed space. Let $\mathcal{I} \subset \mathcal{O}_X$ be a sheaf of ideals which is locally generated by sections as a sheaf of $\mathcal{O}_X$-modules. Let Z be the support of the sheaf of rings $\mathcal{O}_X/\mathcal{I}$. This is a closed subset of X, by Modules, Lemma 17.5.3. Denote $i : Z \to X$ the inclusion map. By Modules, Lemma 17.6.1 there is a unique sheaf of rings $\mathcal{O}_Z$ on Z with $i_*\mathcal{O}_Z = \mathcal{O}_X/\mathcal{I}$. For any $z \in Z$ the local ring $\mathcal{O}_{Z,z}$ is equal to the quotient ring $\mathcal{O}_{X,i(z)}/\mathcal{I}_{i(z)}$ and nonzero, hence a local ring. Thus $i : (Z, \mathcal{O}_Z) \to (X, \mathcal{O}_X)$ is a closed immersion of locally ringed spaces.

Definition 25.4.4. Let X be a locally ringed space. Let $\mathcal{I}$ be a sheaf of ideals on X which is locally generated by sections. The locally ringed space $(Z, \mathcal{O}_Z)$ of Example 25.4.3 above is the *closed subspace of X associated to the sheaf of ideals $\mathcal{I}$.*

Lemma 25.4.5. *Let $f : X \to Y$ be a closed immersion of locally ringed spaces. Let $\mathcal{I}$ be the kernel of the map $\mathcal{O}_Y \to f_*\mathcal{O}_X$. Let $i : Z \to Y$ be the closed subspace of Y associated to $\mathcal{I}$. There is a unique isomorphism $f' : X \cong Z$ of locally ringed spaces such that $f = i \circ f'$.*

Proof. Omitted. □

Lemma 25.4.6. *Let X, Y be a locally ringed spaces. Let $\mathcal{I} \subset \mathcal{O}_X$ be a sheaf of ideals locally generated by sections. Let $i : Z \to X$ be the associated closed subspace. A morphism $f : Y \to X$ factors through Z if and only if the map $f^*\mathcal{I} \to f^*\mathcal{O}_X = \mathcal{O}_Y$ is zero. If this is the case the morphism $g : Y \to Z$ such that $f = i \circ g$ is unique.*

Proof. Clearly if f factors as $Y \to Z \to X$ then the map $f^*\mathcal{I} \to \mathcal{O}_Y$ is zero. Conversely suppose that $f^*\mathcal{I} \to \mathcal{O}_Y$ is zero. Pick any $y \in Y$, and consider the ring map $f_y^\sharp : \mathcal{O}_{X,f(y)} \to \mathcal{O}_{Y,y}$. Since the composition $\mathcal{I}_{f(y)} \to \mathcal{O}_{X,f(y)} \to \mathcal{O}_{Y,y}$ is zero

by assumption and since $f_y^\sharp(1) = 1$ we see that $1 \notin \mathcal{I}_{f(y)}$, i.e., $\mathcal{I}_{f(y)} \neq \mathcal{O}_{X,f(y)}$. We conclude that $f(Y) \subset Z = \mathrm{Supp}(\mathcal{O}_X/\mathcal{I})$. Hence $f = i \circ g$ where $g : Y \to Z$ is continuous. Consider the map $f^\sharp : \mathcal{O}_X \to f_*\mathcal{O}_Y$. The assumption $f^*\mathcal{I} \to \mathcal{O}_Y$ is zero implies that the composition $\mathcal{I} \to \mathcal{O}_X \to f_*\mathcal{O}_Y$ is zero by adjointness of f_* and f^*. In other words, we obtain a morphism of sheaves of rings $\overline{f^\sharp} : \mathcal{O}_X/\mathcal{I} \to f_*\mathcal{O}_Y$. Note that $f_*\mathcal{O}_Y = i_*g_*\mathcal{O}_Y$ and that $\mathcal{O}_X/\mathcal{I} = i_*\mathcal{O}_Z$. By Sheaves, Lemma 6.32.4 we obtain a unique morphism of sheaves of rings $g^\sharp : \mathcal{O}_Z \to g_*\mathcal{O}_Y$ whose pushforward under i is $\overline{f^\sharp}$. We omit the verification that $(g, g^\sharp)$ defines a morphism of locally ringed spaces and that $f = i \circ g$ as a morphism of locally ringed spaces. The uniqueness of $(g, g^\sharp)$ was pointed out above. □

Lemma 25.4.7. *Let $f : X \to Y$ be a morphism of locally ringed spaces. Let $\mathcal{I} \subset \mathcal{O}_Y$ be a sheaf of ideals which is locally generated by functions. Let $i : Z \to Y$ be the closed subspace associated to the sheaf of ideals $\mathcal{I}$. Let $\mathcal{J}$ be the image of the map $f^*\mathcal{I} \to f^*\mathcal{O}_Y = \mathcal{O}_X$. Then this ideal is locally generated by sections. Moreover, let $i' : Z' \to X$ be the associated closed subspace of X. There exists a unique morphism of locally ringed spaces $f' : Z' \to Z$ such that the following diagram is a commutative square of locally ringed spaces*

$$\begin{array}{ccc} Z' & \xrightarrow{i'} & X \\ {\scriptstyle f'}\downarrow & & \downarrow{\scriptstyle f} \\ Z & \xrightarrow{i} & Y \end{array}$$

Moreover, this diagram is a fibre square in the category of locally ringed spaces.

Proof. The ideal $\mathcal{J}$ is locally generated by sections by Modules, Lemma 17.8.2. The rest of the lemma follows from the characterization, in Lemma 25.4.6 above, of what it means for a morphism to factor through a closed subspace. □

25.5. Affine schemes

Let R be a ring. Consider the topological space $\mathrm{Spec}(R)$ associated to R, see Algebra, Section 10.16. We will endow this space with a sheaf of rings $\mathcal{O}_{\mathrm{Spec}(R)}$ and the resulting pair $(\mathrm{Spec}(R), \mathcal{O}_{\mathrm{Spec}(R)})$ will be an affine scheme.

Recall that $\mathrm{Spec}(R)$ has a basis of open sets $D(f)$, $f \in R$ which we call standard opens, see Algebra, Definition 10.16.3. In addition, the intersection of two standard opens is another: $D(f) \cap D(g) = D(fg)$, $f, g \in R$.

Lemma 25.5.1. *Let R be a ring. Let $f \in R$.*

1. *If $g \in R$ and $D(g) \subset D(f)$, then*
   (a) *f is invertible in R_g,*
   (b) *$g^e = af$ for some $e \geq 1$ and $a \in R$,*
   (c) *there is a canonical ring map $R_f \to R_g$, and*
   (d) *there is a canonical R_f-module map $M_f \to M_g$ for any R-module M.*
2. *Any open covering of $D(f)$ can be refined to a finite open covering of the form $D(f) = \bigcup_{i=1}^n D(g_i)$.*
3. *If $g_1, \ldots, g_n \in R$, then $D(f) \subset \bigcup D(g_i)$ if and only if $g_1, \ldots, g_n$ generate the unit ideal in R_f.*

Proof. Recall that $D(g) = \mathrm{Spec}(R_g)$ (see Algebra, Lemma 10.16.6). Thus (a) holds because f maps to an element of R_g which is not contained in any prime ideal, and hence invertible, see Algebra, Lemma 10.16.2. Write the inverse of f in R_g as a/g^d. This means $g^d - af$ is annihilated by a power of g, whence (b). For (c), the map $R_f \to R_g$ exists by (a) from the universal property of localization, or we can define it by mapping b/f^n to $a^n b/g^{ne}$. The equality $M_f = M \otimes_R R_f$ can be used to obtain the map on modules, or we can define $M_f \to M_g$ by mapping x/f^n to $a^n x/g^{ne}$.

Recall that $D(f)$ is quasi-compact, see Algebra, Lemma 10.27.1. Hence the second statement follows directly from the fact that the standard opens form a basis for the topology.

The third statement follows directly from Algebra, Lemma 10.16.2. □

In Sheaves, Section 6.30 we defined the notion of a sheaf on a basis, and we showed that it is essentially equivalent to the notion of a sheaf on the space, see Sheaves, Lemmas 6.30.6 and 6.30.9. Moreover, we showed in Sheaves, Lemma 6.30.4 that it is sufficient to check the sheaf condition on a cofinal system of open coverings for each standard open. By the lemma above it suffices to check on the finite coverings by standard opens.

Definition 25.5.2. Let R be a ring.

(1) A *standard open covering* of $\mathrm{Spec}(R)$ is a covering $\mathrm{Spec}(R) = \bigcup_{i=1}^n D(f_i)$, where $f_1, \ldots, f_n \in R$.
(2) Suppose that $D(f) \subset \mathrm{Spec}(R)$ is a standard open. A *standard open covering* of $D(f)$ is a covering $D(f) = \bigcup_{i=1}^n D(g_i)$, where $g_1, \ldots, g_n \in R$.

Let R be a ring. Let M be an R-module. We will define a presheaf $\widetilde{M}$ on the basis of standard opens. Suppose that $U \subset \mathrm{Spec}(R)$ is a standard open. If $f, g \in R$ are such that $D(f) = D(g)$, then by Lemma 25.5.1 above there are canonical maps $M_f \to M_g$ and $M_g \to M_f$ which are mutually inverse. Hence we may choose any f such that $U = D(f)$ and define

$$\widetilde{M}(U) = M_f.$$

Note that if $D(g) \subset D(f)$, then by Lemma 25.5.1 above we have a canonical map

$$\widetilde{M}(D(f)) = M_f \longrightarrow M_g = \widetilde{M}(D(g)).$$

Clearly, this defines a presheaf of abelian groups on the basis of standard opens. If $M = R$, then $\widetilde{R}$ is a presheaf of rings on the basis of standard opens.

Let us compute the stalk of $\widetilde{M}$ at a point $x \in \mathrm{Spec}(R)$. Suppose that x corresponds to the prime $\mathfrak{p} \subset R$. By definition of the stalk we see that

$$\widetilde{M}_x = \mathrm{colim}_{f \in R, f \notin \mathfrak{p}} M_f$$

Here the set $\{f \in R, f \notin \mathfrak{p}\}$ is partially ordered by the rule $f \geq f' \Leftrightarrow D(f) \subset D(f')$. If $f_1, f_2 \in R \setminus \mathfrak{p}$, then we have $f_1 f_2 \geq f_1$ in this ordering. Hence by Algebra, Lemma 10.9.9 we conclude that

$$\widetilde{M}_x = M_{\mathfrak{p}}.$$

Next, we check the sheaf condition for the standard open coverings. If $D(f) = \bigcup_{i=1}^n D(g_i)$, then the sheaf condition for this covering is equivalent with the exactness of the sequence

$$0 \to M_f \to \bigoplus M_{g_i} \to \bigoplus M_{g_i g_j}.$$

Note that $D(g_i) = D(fg_i)$, and hence we can rewrite this sequence as the sequence

$$0 \to M_f \to \bigoplus M_{fg_i} \to \bigoplus M_{fg_i g_j}.$$

In addition, by Lemma 25.5.1 above we see that $g_1, \ldots, g_n$ generate the unit ideal in R_f. Thus we may apply Algebra, Lemma 10.21.2 to the module M_f over R_f and the elements $g_1, \ldots, g_n$. We conclude that the sequence is exact. By the remarks made above, we see that $\widetilde{M}$ is a sheaf on the basis of standard opens.

Thus we conclude from the material in Sheaves, Section 6.30 that there exists a unique sheaf of rings $\mathcal{O}_{\mathrm{Spec}(R)}$ which agrees with $\widetilde{R}$ on the standard opens. Note that by our computation of stalks above, the stalks of this sheaf of rings are all local rings.

Similarly, for any R-module M there exists a unique sheaf of $\mathcal{O}_{\mathrm{Spec}(R)}$-modules $\mathcal{F}$ which agrees with $\widetilde{M}$ on the standard opens, see Sheaves, Lemma 6.30.12.

Definition 25.5.3. Let R be a ring.

(1) The *structure sheaf* $\mathcal{O}_{\mathrm{Spec}(R)}$ *of the spectrum of* R is the unique sheaf of rings $\mathcal{O}_{\mathrm{Spec}(R)}$ which agrees with $\widetilde{R}$ on the basis of standard opens.
(2) The locally ringed space $(\mathrm{Spec}(R), \mathcal{O}_{\mathrm{Spec}(R)})$ is called the *spectrum* of R and denoted $\mathrm{Spec}(R)$.
(3) The sheaf of $\mathcal{O}_{\mathrm{Spec}(R)}$-modules extending $\widetilde{M}$ to all opens of $\mathrm{Spec}(R)$ is called the sheaf of $\mathcal{O}_{\mathrm{Spec}(R)}$-modules associated to M. This sheaf is denoted $\widetilde{M}$ as well.

We summarize the results obtained so far.

Lemma 25.5.4. *Let R be a ring. Let M be an R-module. Let $\widetilde{M}$ be the sheaf of $\mathcal{O}_{\mathrm{Spec}(R)}$-modules associated to M.*

(1) *We have $\Gamma(\mathrm{Spec}(R), \mathcal{O}_{\mathrm{Spec}(R)}) = R$.*
(2) *We have $\Gamma(\mathrm{Spec}(R), \widetilde{M}) = M$ as an R-module.*
(3) *For every $f \in R$ we have $\Gamma(D(f), \mathcal{O}_{\mathrm{Spec}(R)}) = R_f$.*
(4) *For every $f \in R$ we have $\Gamma(D(f), \widetilde{M}) = M_f$ as an R_f-module.*
(5) *Whenever $D(g) \subset D(f)$ the restriction mappings on $\mathcal{O}_{\mathrm{Spec}(R)}$ and $\widetilde{M}$ are the maps $R_f \to R_g$ and $M_f \to M_g$ from Lemma 25.5.1.*
(6) *Let $\mathfrak{p}$ be a prime of R, and let $x \in \mathrm{Spec}(R)$ be the corresponding point. We have $\mathcal{O}_{\mathrm{Spec}(R),x} = R_{\mathfrak{p}}$.*
(7) *Let $\mathfrak{p}$ be a prime of R, and let $x \in \mathrm{Spec}(R)$ be the corresponding point. We have $\mathcal{F}_x = M_{\mathfrak{p}}$ as an $R_{\mathfrak{p}}$-module.*

Moreover, all these identifications are functorial in the R module M. In particular, the functor $M \mapsto \widetilde{M}$ is an exact functor from the category of R-modules to the category of $\mathcal{O}_{\mathrm{Spec}(R)}$-modules.

Proof. Assertions (1) - (7) are clear from the discussion above. The exactness of the functor $M \mapsto \widetilde{M}$ follows from the fact that the functor $M \mapsto M_{\mathfrak{p}}$ is exact and the fact that exactness of short exact sequences may be checked on stalks, see Modules, Lemma 17.3.1. □

Definition 25.5.5. An *affine scheme* is a locally ringed space isomorphic as a locally ringed space to $\mathrm{Spec}(R)$ for some ring R. A *morphism of affine schemes* is a morphism in the category of locally ringed spaces.

It turns out that affine schemes play a special role among all locally ringed spaces, which is what the next section is about.

25.6. The category of affine schemes

Note that if Y is an affine scheme, then its points are in canonical $1-1$ bijection with prime ideals in $\Gamma(Y, \mathcal{O}_Y)$.

Lemma 25.6.1. *Let X be a locally ringed space. Let Y be an affine scheme. Let $f \in \mathrm{Mor}(X, Y)$ be a morphism of locally ringed spaces. Given a point $x \in X$ consider the ring maps*

$$\Gamma(Y, \mathcal{O}_Y) \xrightarrow{f^\sharp} \Gamma(X, \mathcal{O}_X) \to \mathcal{O}_{X,x}$$

Let $\mathfrak{p} \subset \Gamma(Y, \mathcal{O}_Y)$ denote the inverse image of $\mathfrak{m}_x$. Let $y \in Y$ be the corresponding point. Then $f(x) = y$.

Proof. Consider the commutative diagram

$$\begin{array}{ccc} \Gamma(X, \mathcal{O}_X) & \longrightarrow & \mathcal{O}_{X,x} \\ \uparrow & & \uparrow \\ \Gamma(Y, \mathcal{O}_Y) & \longrightarrow & \mathcal{O}_{Y,f(x)} \end{array}$$

(see the discussion of f-maps below Sheaves, Definition 6.21.7). Since the right vertical arrow is local we see that $\mathfrak{m}_{f(x)}$ is the inverse image of $\mathfrak{m}_x$. The result follows. □

Lemma 25.6.2. *Let X be a locally ringed space. Let $f \in \Gamma(X, \mathcal{O}_X)$. The set*

$$D(f) = \{x \in X \mid \text{image } f \notin \mathfrak{m}_x\}$$

is open. Moreover $f|_{D(f)}$ has an inverse.

Proof. This is a special case of Modules, Lemma 17.21.7, but we also give a direct proof. Suppose that $U \subset X$ and $V \subset X$ are two open subsets such that $f|_U$ has an inverse g and $f|_V$ has an inverse h. Then clearly $g|_{U \cap V} = h|_{U \cap V}$. Thus it suffices to show that f is invertible in an open neighbourhood of any $x \in D(f)$. This is clear because $f \notin \mathfrak{m}_x$ implies that $f \in \mathcal{O}_{X,x}$ has an inverse $g \in \mathcal{O}_{X,x}$ which means there is some open neighbourhood $x \in U \subset X$ so that $g \in \mathcal{O}_X(U)$ and $g \cdot f|_U = 1$. □

Lemma 25.6.3. *In Lemma 25.6.2 above, if X is an affine scheme, then the open $D(f)$ agrees with the standard open $D(f)$ defined previously (in Algebra, Definition 10.16.1).*

Proof. Omitted. □

Lemma 25.6.4. *Let X be a locally ringed space. Let Y be an affine scheme. The map*

$$\mathrm{Mor}(X, Y) \longrightarrow \mathrm{Hom}(\Gamma(Y, \mathcal{O}_Y), \Gamma(X, \mathcal{O}_X))$$

which maps f to $f^\sharp$ (on global sections) is bijective.

Proof. Since Y is affine we have $(Y, \mathcal{O}_Y) \cong (\mathrm{Spec}(R), \mathcal{O}_{\mathrm{Spec}(R)})$ for some ring R. During the proof we will use facts about Y and its structure sheaf which are direct consequences of things we know about the spectrum of a ring, see e.g. Lemma 25.5.4.

Motivated by the lemmas above we construct the inverse map. Let $\psi_Y : \Gamma(Y, \mathcal{O}_Y) \to \Gamma(X, \mathcal{O}_X)$ be a ring map. First, we define the corresponding map of spaces

$$\Psi : X \longrightarrow Y$$

by the rule of Lemma 25.6.1. In other words, given $x \in X$ we define $\Psi(x)$ to be the point of Y corresponding to the prime in $\Gamma(Y, \mathcal{O}_Y)$ which is the inverse image of $\mathfrak{m}_x$ under the composition $\Gamma(Y, \mathcal{O}_Y) \xrightarrow{\psi_Y} \Gamma(X, \mathcal{O}_X) \to \mathcal{O}_{X,x}$.

We claim that the map $\Psi : X \to Y$ is continuous. The standard opens $D(g)$, for $g \in \Gamma(Y, \mathcal{O}_Y)$ are a basis for the topology of Y. Thus it suffices to prove that $\Psi^{-1}(D(g))$ is open. By construction of Ψ the inverse image $\Psi^{-1}(D(g))$ is exactly the set $D(\psi_Y(g)) \subset X$ which is open by Lemma 25.6.2. Hence Ψ is continuous.

Next we construct a Ψ-map of sheaves from $\mathcal{O}_Y$ to $\mathcal{O}_X$. By Sheaves, Lemma 6.30.14 it suffices to define ring maps $\psi_{D(g)} : \Gamma(D(g), \mathcal{O}_Y) \to \Gamma(\Psi^{-1}(D(g)), \mathcal{O}_X)$ compatible with restriction maps. We have a canonical isomorphism $\Gamma(D(g), \mathcal{O}_Y) = \Gamma(Y, \mathcal{O}_Y)_g$, because Y is an affine scheme. Because $\psi_Y(g)$ is invertible on $D(\psi_Y(g))$ we see that there is a canonical map

$$\Gamma(Y, \mathcal{O}_Y)_g \longrightarrow \Gamma(\Psi^{-1}(D(g)), \mathcal{O}_X) = \Gamma(D(\psi_Y(g)), \mathcal{O}_X)$$

extending the map ψ_Y by the universal property of localization. Note that there is no choice but to take the canonical map here! And we take this, combined with the canonical identification $\Gamma(D(g), \mathcal{O}_Y) = \Gamma(Y, \mathcal{O}_Y)_g$, to be $\psi_{D(g)}$. This is compatible with localization since the restriction mapping on the affine schemes are defined in terms of the universal properties of localization also, see Lemmas 25.5.4 and 25.5.1.

Thus we have defined a morphism of ringed spaces $(\Psi, \psi) : (X, \mathcal{O}_X) \to (Y, \mathcal{O}_Y)$ recovering ψ_Y on global sections. To see that it is a morphism of locally ringed spaces we have to show that the induced maps on local rings

$$\psi_x : \mathcal{O}_{Y,\Psi(x)} \longrightarrow \mathcal{O}_{X,x}$$

are local. This follows immediately from the commutative diagram of the proof of Lemma 25.6.1 and the definition of Ψ.

Finally, we have to show that the constructions $(\Psi, \psi) \mapsto \psi_Y$ and the construction $\psi_Y \mapsto (\Psi, \psi)$ are inverse to each other. Clearly, $\psi_Y \mapsto (\Psi, \psi) \mapsto \psi_Y$. Hence the only thing to prove is that given ψ_Y there is at most one pair (Ψ, ψ) giving rise to it. The uniqueness of Ψ was shown in Lemma 25.6.1 and given the uniqueness of Ψ the uniqueness of the map ψ was pointed out during the course of the proof above. □

Lemma 25.6.5. *The category of affine schemes is equivalent to the opposite of the category of rings. The equivalence is given by the functor that associates to an affine scheme the global sections of its structure sheaf.*

Proof. This is now clear from Definition 25.5.5 and Lemma 25.6.4. □

Lemma 25.6.6. *Let Y be an affine scheme. Let $f \in \Gamma(Y, \mathcal{O}_Y)$. The open subspace $D(f)$ is an affine scheme.*

Proof. We may assume that $Y = \mathrm{Spec}(R)$ and $f \in R$. Consider the morphism of affine schemes $\phi : U = \mathrm{Spec}(R_f) \to \mathrm{Spec}(R) = Y$ induced by the ring map $R \to R_f$. By Algebra, Lemma 10.16.6 we know that it is a homeomorphism onto $D(f)$. On the other hand, the map $f^{-1}\mathcal{O}_Y \to \mathcal{O}_U$ is an isomorphism on stalks, hence an isomorphism. Thus we see that ϕ is an open immersion. We conclude that $D(f)$ is isomorphic to U by Lemma 25.3.4. □

Lemma 25.6.7. *The category of affine schemes has finite products, and fibre products. In other words, it has finite limits. Moreover, the products and fibre products in the category of affine schemes are the same as in the category of locally ringed spaces. In a formula, we have (in the category of locally ringed spaces)*

$$\mathrm{Spec}(R) \times \mathrm{Spec}(S) = \mathrm{Spec}(R \otimes_{\mathbf{Z}} S)$$

and given ring maps $R \to A$, $R \to B$ we have

$$\mathrm{Spec}(A) \times_{\mathrm{Spec}(R)} \mathrm{Spec}(B) = \mathrm{Spec}(A \otimes_R B).$$

Proof. This is just an application of Lemma 25.6.4. First of all, by that lemma, the affine scheme $\mathrm{Spec}(\mathbf{Z})$ is the final object in the category of locally ringed spaces. Thus the first displayed formula follows from the second. To prove the second note that for any locally ringed space X we have

$$\begin{aligned} \mathrm{Mor}(X, \mathrm{Spec}(A \otimes_R B)) &= \mathrm{Hom}(A \otimes_R B, \mathcal{O}_X(X)) \\ &= \mathrm{Hom}(A, \mathcal{O}_X(X)) \times_{\mathrm{Hom}(R, \mathcal{O}_X(X))} \mathrm{Hom}(B, \mathcal{O}_X(X)) \\ &= \mathrm{Mor}(X, \mathrm{Spec}(A)) \times_{\mathrm{Mor}(X, \mathrm{Spec}(R))} \mathrm{Mor}(X, \mathrm{Spec}(B)) \end{aligned}$$

which proves the formula. See Categories, Section 4.6 for the relevant definitions. □

Lemma 25.6.8. *Let X be a locally ringed space. Assume $X = U \amalg V$ with U and V open and such that U, V are affine schemes. Then X is an affine scheme.*

Proof. Set $R = \Gamma(X, \mathcal{O}_X)$. Note that $R = \mathcal{O}_X(U) \times \mathcal{O}_X(V)$ by the sheaf property. By Lemma 25.6.4 there is a canonical morphism of locally ringed spaces $X \to \mathrm{Spec}(R)$. By Algebra, Lemma 10.19.2 we see that as a topological space $\mathrm{Spec}(\mathcal{O}_X(U)) \amalg \mathrm{Spec}(\mathcal{O}_X(V)) = \mathrm{Spec}(R)$ with the maps coming from the ring homomorphisms $R \to \mathcal{O}_X(U)$ and $R \to \mathcal{O}_X(V)$. This of course means that $\mathrm{Spec}(R)$ is the coproduct in the category of locally ringed spaces as well. By assumption the morphism $X \to \mathrm{Spec}(R)$ induces an isomorphism of $\mathrm{Spec}(\mathcal{O}_X(U))$ with U and similarly for V. Hence $X \to \mathrm{Spec}(R)$ is an isomorphism. □

25.7. Quasi-coherent sheaves on affines

Recall that we have defined the abstract notion of a quasi-coherent sheaf in Modules, Definition 17.10.1. In this section we show that any quasi-coherent sheaf on an affine scheme $\mathrm{Spec}(R)$ corresponds to the sheaf $\widetilde{M}$ associated to an R-module M.

Lemma 25.7.1. *Let $(X, \mathcal{O}_X) = (\mathrm{Spec}(R), \mathcal{O}_{\mathrm{Spec}(R)})$ be an affine scheme. Let M be an R-module. There exists a canonical isomorphism between the sheaf $\widetilde{M}$ associated to the R-module M (Definition 25.5.3) and the sheaf $\mathcal{F}_M$ associated to the R-module M (Modules, Definition 17.10.6). This isomorphism is functorial in M. In particular, the sheaves $\widetilde{M}$ are quasi-coherent. Moreover, they are characterized by the following mapping property*

$$\mathrm{Hom}_{\mathcal{O}_X}(\widetilde{M}, \mathcal{F}) = \mathrm{Hom}_R(M, \Gamma(X, \mathcal{F}))$$

for any sheaf of $\mathcal{O}_X$-modules $\mathcal{F}$. Here a map $\alpha : \widetilde{M} \to \mathcal{F}$ corresponds to its effect on global sections.

Proof. By Modules, Lemma 17.10.5 we have a morphism $\mathcal{F}_M \to \widetilde{M}$ corresponding to the map $M \to \Gamma(X, \widetilde{M}) = M$. Let $x \in X$ correspond to the prime $\mathfrak{p} \subset R$. The induced map on stalks are the maps $\mathcal{O}_{X,x} \otimes_R M \to M_{\mathfrak{p}}$ which are isomorphisms because $R_{\mathfrak{p}} \otimes_R M = M_{\mathfrak{p}}$. Hence the map $\mathcal{F}_M \to \widetilde{M}$ is an isomorphism. The mapping property follows from the mapping property of the sheaves $\mathcal{F}_M$. $\square$

Lemma 25.7.2. *Let $(X, \mathcal{O}_X) = (\mathrm{Spec}(R), \mathcal{O}_{\mathrm{Spec}(R)})$ be an affine scheme. There are canonical isomorphisms*

1. *$\widetilde{M \otimes_R N} \cong \widetilde{M} \otimes_{\mathcal{O}_X} \widetilde{N}$, see Modules, Section 17.15.*
2. *$\widetilde{T^n(M)} \cong T^n(\widetilde{M})$, $\widetilde{Sym^n(M)} \cong Sym^n(\widetilde{M})$, and $\widetilde{\wedge^n(M)} \cong \wedge^n(\widetilde{M})$, see Modules, Section 17.18.*
3. *if M is a finitely presented R-module, then $\mathcal{H}om_{\mathcal{O}_X}(\widetilde{M}, \widetilde{N}) \cong \widetilde{\mathrm{Hom}_R(M, N)}$, see Modules, Section 17.19.*

Proof. To give a map $\widetilde{M \otimes_R N}$ into $\widetilde{M} \otimes_{\mathcal{O}_X} \widetilde{N}$ we have to give a map on global sections $M \otimes_R N \to \Gamma(X, \widetilde{M} \otimes_{\mathcal{O}_X} \widetilde{N})$ which exists by definition of the tensor product of sheaves of modules. To see that this map is an isomorphism it suffices to check that it is an isomorphism on stalks. And this follows from the description of the stalks of $\widetilde{M}$ (as a functor) and Modules, Lemma 17.15.1.

The proof of (2) is similar, using Modules, Lemma 17.18.2.

For (3) note that if M is finitely presented as an R-module then $\widetilde{M}$ has a global finite presentation as an $\mathcal{O}_X$-module. Hence Modules, Lemma 17.19.3 applies. $\square$

Lemma 25.7.3. *Let $(X, \mathcal{O}_X) = (\mathrm{Spec}(S), \mathcal{O}_{\mathrm{Spec}(S)})$, $(Y, \mathcal{O}_Y) = (\mathrm{Spec}(R), \mathcal{O}_{\mathrm{Spec}(R)})$ be affine schemes. Let $\psi : (X, \mathcal{O}_X) \to (Y, \mathcal{O}_Y)$ be a morphism of affine schemes, corresponding to the ring map $\psi^\sharp : R \to S$ (see Lemma 25.6.5).*

1. *We have $\psi^*\widetilde{M} = \widetilde{S \otimes_R M}$ functorially in the R-module M.*
2. *We have $\psi_*\widetilde{N} = \widetilde{N_R}$ functorially in the S-module N.*

Proof. The first assertion follows from the identification in Lemma 25.7.1 and the result of Modules, Lemma 17.10.7. The second assertion follows from the fact that $\psi^{-1}(D(f)) = D(\psi^\sharp(f))$ and hence

$$\psi_*\widetilde{N}(D(f)) = \widetilde{N}(D(\psi^\sharp(f))) = N_{\psi^\sharp(f)} = (N_R)_f = \widetilde{N_R}(D(f))$$

as desired. □

Lemma 25.7.3 above says in particular that if you restrict the sheaf $\widetilde{M}$ to a standard affine open subspace $D(f)$, then you get $\widetilde{M_f}$. We will use this from now on without further mention.

Lemma 25.7.4. *Let $(X, \mathcal{O}_X) = (\mathrm{Spec}(R), \mathcal{O}_{\mathrm{Spec}(R)})$ be an affine scheme. Let $\mathcal{F}$ be a quasi-coherent $\mathcal{O}_X$-module. Then $\mathcal{F}$ is isomorphic to the sheaf associated to the R-module $\Gamma(X, \mathcal{F})$.*

Proof. Let $\mathcal{F}$ be a quasi-coherent $\mathcal{O}_X$-module. Since every standard open $D(f)$ is quasi-compact we see that X is a locally quasi-compact, i.e., every point has a fundamental system of quasi-compact neighbourhoods, see Topology, Definition 5.12.1. Hence by Modules, Lemma 17.10.8 for every prime $\mathfrak{p} \subset R$ corresponding to $x \in X$ there exists an open neighbourhood $x \in U \subset X$ such that $\mathcal{F}|_U$ is isomorphic to the quasi-coherent sheaf associated to some $\mathcal{O}_X(U)$-module M. In other words, we get an open covering by U's with this property. By Lemma 25.5.1 for example we can refine this covering to a standard open covering. Thus we get a covering $\mathrm{Spec}(R) = \bigcup D(f_i)$ and R_{f_i}-modules M_i and isomorphisms $\varphi_i : \mathcal{F}|_{D(f_i)} \to \mathcal{F}_{M_i}$ for some R_{f_i}-module M_i. On the overlaps we get isomorphisms

$$\mathcal{F}_{M_i}|_{D(f_if_j)} \xrightarrow{\varphi_i^{-1}|_{D(f_if_j)}} \mathcal{F}|_{D(f_if_j)} \xrightarrow{\varphi_j|_{D(f_if_j)}} \mathcal{F}_{M_j}|_{D(f_if_j)}.$$

Let us denote these ψ_{ij}. It is clear that we have the cocycle condition

$$\psi_{jk}|_{D(f_if_jf_k)} \circ \psi_{ij}|_{D(f_if_jf_k)} = \psi_{ik}|_{D(f_if_jf_k)}$$

on triple overlaps.

Recall that each of the open subspaces $D(f_i)$, $D(f_if_j)$, $D(f_if_jf_k)$ is an affine scheme. Hence the sheaves $\mathcal{F}_{M_i}$ are isomorphic to the sheaves $\widetilde{M_i}$ by Lemma 25.7.1 above. In particular we see that $\mathcal{F}_{M_i}(D(f_if_j)) = (M_i)_{f_j}$, etc. Also by Lemma 25.7.1 above we see that ψ_{ij} corresponds to a unique $R_{f_if_j}$-module isomorphism

$$\psi_{ij} : (M_i)_{f_j} \longrightarrow (M_j)_{f_i}$$

namely, the effect of ψ_{ij} on sections over $D(f_if_j)$. Moreover these then satisfy the cocycle condition that

$$\begin{array}{ccc} (M_i)_{f_jf_k} & \xrightarrow{\psi_{ik}} & (M_k)_{f_if_j} \\ & {\scriptstyle\psi_{ij}} \searrow \quad \nearrow {\scriptstyle\psi_{jk}} & \\ & (M_j)_{f_if_k} & \end{array}$$

commutes (for any triple i, j, k).

Now Algebra, Lemma 10.22.4 shows that there exist an R-module M such that $M_i = M_{f_i}$ compatible with the morphisms ψ_{ij}. Consider $\mathcal{F}_M = \widetilde{M}$. At this point it is a formality to show that $\widetilde{M}$ is isomorphic to the quasi-coherent sheaf $\mathcal{F}$ we

started out with. Namely, the sheaves $\mathcal{F}$ and $\widetilde{M}$ give rise to isomorphic sets of glueing data of sheaves of $\mathcal{O}_X$-modules with respect to the covering $X = \bigcup D(f_i)$, see Sheaves, Section 6.33 and in particular Lemma 6.33.4. Explicitly, in the current situation, this boils down to the following argument: Let us construct an R-module map

$$M \longrightarrow \Gamma(X, \mathcal{F}).$$

Namely, given $m \in M$ we get $m_i = m/1 \in M_{f_i} = M_i$ by construction of M. By construction of M_i this corresponds to a section $s_i \in \mathcal{F}(U_i)$. (Namely, $\varphi_i^{-1}(m_i)$.) We claim that $s_i|_{D(f_if_j)} = s_j|_{D(f_if_j)}$. This is true because, by construction of M, we have $\psi_{ij}(m_i) = m_j$, and by the construction of the ψ_{ij}. By the sheaf condition of $\mathcal{F}$ this collection of sections gives rise to a unique section s of $\mathcal{F}$ over X. We leave it to the reader to show that $m \mapsto s$ is a R-module map. By Lemma 25.7.1 we obtain an associated $\mathcal{O}_X$-module map

$$\widetilde{M} \longrightarrow \mathcal{F}.$$

By construction this map reduces to the isomorphisms φ_i^{-1} on each $D(f_i)$ and hence is an isomorphism. □

Lemma 25.7.5. *Let $(X, \mathcal{O}_X) = (\mathrm{Spec}(R), \mathcal{O}_{\mathrm{Spec}(R)})$ be an affine scheme. The functors $M \mapsto \widetilde{M}$ and $\mathcal{F} \mapsto \Gamma(X, \mathcal{F})$ define quasi-inverse equivalences of categories*

$$QCoh(\mathcal{O}_X) \rightleftarrows Mod\text{-}R$$

between the category of quasi-coherent $\mathcal{O}_X$-modules and the category of R-modules.

Proof. See Lemmas 25.7.1 and 25.7.4 above. □

From now on we will not distinguish between quasi-coherent sheaves on affine schemes and sheaves of the form $\widetilde{M}$.

Lemma 25.7.6. *Let $X = \mathrm{Spec}(R)$ be an affine scheme. Kernels and cokernels of maps of quasi-coherent $\mathcal{O}_X$-modules are quasi-coherent.*

Proof. This follows from the exactness of the functor $\widetilde{\ }$ since by Lemma 25.7.1 we know that any map $\psi : \widetilde{M} \to \widetilde{N}$ comes from an R-module map $\varphi : M \to N$. (So we have $\mathrm{Ker}(\psi) = \widetilde{\mathrm{Ker}(\varphi)}$ and $\mathrm{Coker}(\psi) = \widetilde{\mathrm{Coker}(\varphi)}$.) □

Lemma 25.7.7. *Let $X = \mathrm{Spec}(R)$ be an affine scheme. The direct sum of an arbitrary collection of quasi-coherent sheaves on X is quasi-coherent. The same holds for colimits.*

Proof. Suppose $\mathcal{F}_i$, $i \in I$ is a collection of quasi-coherent sheaves on X. By Lemma 25.7.5 above we can write $\mathcal{F}_i = \widetilde{M}_i$ for some R-module M_i. Set $M = \bigoplus M_i$. Consider the sheaf $\widetilde{M}$. For each standard open $D(f)$ we have

$$\widetilde{M}(D(f)) = M_f = \left(\bigoplus M_i\right)_f = \bigoplus M_{i,f}.$$

Hence we see that the quasi-coherent $\mathcal{O}_X$-module $\widetilde{M}$ is the direct sum of the sheaves $\mathcal{F}_i$. A similar argument works for general colimits. □

Lemma 25.7.8. *Let $(X, \mathcal{O}_X) = (\mathrm{Spec}(R), \mathcal{O}_{\mathrm{Spec}(R)})$ be an affine scheme. Suppose that*

$$0 \to \mathcal{F}_1 \to \mathcal{F}_2 \to \mathcal{F}_3 \to 0$$

is a short exact sequence of sheaves $\mathcal{O}_X$-modules. If two out of three are quasi-coherent then so is the third.

Proof. This is clear in case both $\mathcal{F}_1$ and $\mathcal{F}_2$ are quasi-coherent because the functor $M \mapsto \widetilde{M}$ is exact, see Lemma 25.5.4. Similarly in case both $\mathcal{F}_2$ and $\mathcal{F}_3$ are quasi-coherent. Now, suppose that $\mathcal{F}_1 = \widetilde{M}_1$ and $\mathcal{F}_3 = \widetilde{M}_3$ are quasi-coherent. Set $M_2 = \Gamma(X, \mathcal{F}_2)$. We claim it suffices to show that the sequence

$$0 \to M_1 \to M_2 \to M_3 \to 0$$

is exact. Namely, if this is the case, then (by using the mapping property of Lemma 25.7.1) we get a commutative diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \widetilde{M}_1 & \longrightarrow & \widetilde{M}_2 & \longrightarrow & \widetilde{M}_3 & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & \mathcal{F}_1 & \longrightarrow & \mathcal{F}_2 & \longrightarrow & \mathcal{F}_3 & \longrightarrow & 0 \end{array}$$

and we win by the snake lemma.

The "correct" argument here would be to show first that $H^1(X, \mathcal{F}) = 0$ for any quasi-coherent sheaf $\mathcal{F}$. This is actually not all that hard, but it is perhaps better to postpone this till later. Instead we use a small trick.

Pick $m \in M_3 = \Gamma(X, \mathcal{F}_3)$. Consider the following set

$$I = \{f \in R \mid \text{the element } fm \text{ comes from } M_2\}.$$

Clearly this is an ideal. It suffices to show $1 \in I$. Hence it suffices to show that for any prime $\mathfrak{p}$ there exists an $f \in I$, $f \notin \mathfrak{p}$. Let $x \in X$ be the point corresponding to $\mathfrak{p}$. Because surjectivity can be checked on stalks there exists an open neighbourhood U of x such that $m|_U$ comes from a local section $s \in \mathcal{F}_2(U)$. In fact we may assume that $U = D(f)$ is a standard open, i.e., $f \in R$, $f \notin \mathfrak{p}$. We will show that for some $N \gg 0$ we have $f^N \in I$, which will finish the proof.

Take any point $z \in V(f)$, say corresponding to the prime $\mathfrak{q} \subset R$. We can also find a $g \in R$, $g \notin \mathfrak{q}$ such that $m|_{D(g)}$ lifts to some $s' \in \mathcal{F}_2(D(g))$. Consider the difference $s|_{D(fg)} - s'|_{D(fg)}$. This is an element m' of $\mathcal{F}_1(D(fg)) = (M_1)_{fg}$. For some integer $n = n(z)$ the element $f^n m'$ comes from some $m'_1 \in (M_1)_g$. We see that $f^n s$ extends to a section σ of $\mathcal{F}_2$ on $D(f) \cup D(g)$ because it agrees with the restriction of $f^n s' + m'_1$ on $D(f) \cap D(g) = D(fg)$. Moreover, σ maps to the restriction of $f^n m$ to $D(f) \cup D(g)$.

Since $V(f)$ is quasi-compact, there exists a finite list of elements $g_1, \ldots, g_m \in R$ such that $V(f) \subset \bigcup D(g_j)$, an integer $n > 0$ and sections $\sigma_j \in \mathcal{F}_2(D(f) \cup D(g_j))$ such that $\sigma_j|_{D(f)} = f^n s$ and σ_j maps to the section $f^n m|_{D(f) \cup D(g_j)}$ of $\mathcal{F}_3$. Consider the differences

$$\sigma_j|_{D(f) \cup D(g_j g_k)} - \sigma_k|_{D(f) \cup D(g_j g_k)}.$$

These correspond to sections of $\mathcal{F}_1$ over $D(f) \cup D(g_j g_k)$ which are zero on $D(f)$. In particular their images in $\mathcal{F}_1(D(g_j g_k)) = (M_1)_{g_j g_k}$ are zero in $(M_1)_{g_j g_k f}$. Thus some high power of f kills each and every one of these. In other words, the elements

$f^N\sigma_j$, for some $N \gg 0$ satisfy the glueing condition of the sheaf property and give rise to a section σ of $\mathcal{F}_2$ over $\bigcup(D(f) \cup D(g_j)) = X$ as desired. □

25.8. Closed subspaces of affine schemes

Example 25.8.1. Let R be a ring. Let $I \subset R$ be an ideal. Consider the morphism of affine schemes $i : Z = \mathrm{Spec}(R/I) \to \mathrm{Spec}(R) = X$. By Algebra, Lemma 10.16.7 this is a homeomorphism of Z onto a closed subset of X. Moreover, if $I \subset \mathfrak{p} \subset R$ is a prime corresponding to a point $x = i(z)$, $x \in X$, $z \in Z$, then on stalks we get the map

$$\mathcal{O}_{X,x} = R_{\mathfrak{p}} \longrightarrow R_{\mathfrak{p}}/IR_{\mathfrak{p}} = \mathcal{O}_{Z,z}$$

Thus we see that i is a closed immersion of locally ringed spaces, see Definition 25.4.1. Clearly, this is (isomorphic) to the closed subspace associated to the quasi-coherent sheaf of ideals $\widetilde{I}$, as in Example 25.4.3.

Lemma 25.8.2. *Let $(X, \mathcal{O}_X) = (\mathrm{Spec}(R), \mathcal{O}_{\mathrm{Spec}(R)})$ be an affine scheme. Let $i : Z \to X$ be any closed immersion of locally ringed spaces. Then there exists a unique ideal $I \subset R$ such that the morphism $i : Z \to X$ can be identified with the closed immersion $\mathrm{Spec}(R/I) \to \mathrm{Spec}(R)$ constructed in Example 25.8.1 above.*

Proof. This is kind of silly! Namely, by Lemma 25.4.5 we can identify $Z \to X$ with the closed subspace associated to a sheaf of ideals $\mathcal{I} \subset \mathcal{O}_X$ as in Definition 25.4.4 and Example 25.4.3. By our conventions this sheaf of ideals is locally generated by sections as a sheaf of $\mathcal{O}_X$-modules. Hence the quotient sheaf $\mathcal{O}_X/\mathcal{I}$ is locally on X the cokernel of a map $\bigoplus_{j\in J} \mathcal{O}_U \to \mathcal{O}_U$. Thus by definition, $\mathcal{O}_X/\mathcal{I}$ is quasi-coherent. By our results in Section 25.7 it is of the form $\widetilde{S}$ for some R-module S. Moreover, since $\mathcal{O}_X = \widetilde{R} \to \widetilde{S}$ is surjective we see by Lemma 25.7.8 that also $\mathcal{I}$ is quasi-coherent, say $\mathcal{I} = \widetilde{I}$. Of course $I \subset R$ and $S = R/I$ and everything is clear. □

25.9. Schemes

Definition 25.9.1. A *scheme* is a locally ringed space with the property that every point has an open neighbourhood which is an affine scheme. A *morphism of schemes* is a morphism of locally ringed spaces. The category of schemes will be denoted *Sch*.

Let X be a scheme. We will use the following (very slight) abuse of language. We will say $U \subset X$ is an *affine open*, or an *open affine* if the open subspace U is an affine scheme. We will often write $U = \mathrm{Spec}(R)$ to indicate that U is isomorphic to $\mathrm{Spec}(R)$ and moreover that we will identify (temporarily) U and $\mathrm{Spec}(R)$.

Lemma 25.9.2. *Let X be a scheme. Let $j : U \to X$ be an open immersion of locally ringed spaces. Then U is a scheme. In particular, any open subspace of X is a scheme.*

Proof. Let $U \subset X$. Let $u \in U$. Pick an affine open neighbourhood $u \in V \subset X$. Because standard opens of V form a basis of the topology on V we see that there exists a $f \in \mathcal{O}_V(V)$ such that $u \in D(f) \subset U$. And $D(f)$ is an affine scheme by Lemma 25.6.6. This proves that every point of U has an open neighbourhood which is affine. □

Clearly the lemma (or its proof) shows that any scheme X has a basis (see Topology, Section 5.4) for the topology consisting of affine opens.

Example 25.9.3. Let k be a field. An example of a scheme which is not affine is given by the open subspace $U = \mathrm{Spec}(k[x, y]) \setminus \{(x, y)\}$ of the affine scheme $X = \mathrm{Spec}(k[x, y])$. It is covered by two affines, namely $D(x) = \mathrm{Spec}(k[x, y, 1/x])$ and $D(y) = \mathrm{Spec}(k[x, y, 1/y])$ whose intersection is $D(xy) = \mathrm{Spec}(k[x, y, 1/xy])$. By the sheaf property for $\mathcal{O}_U$ there is an exact sequence

$$0 \to \Gamma(U, \mathcal{O}_U) \to k[x, y, 1/x] \times k[x, y, 1/y] \to k[x, y, 1/xy]$$

We conclude that the map $k[x, y] \to \Gamma(U, \mathcal{O}_U)$ (coming from the morphism $U \to X$) is an isomorphism. Therefore U cannot be affine since if it was then by Lemma 25.6.5 we would have $U \cong X$.

25.10. Immersions of schemes

In Lemma 25.9.2 we saw that any open subspace of a scheme is a scheme. Below we will prove that the same holds for a closed subspace of a scheme.

Note that the notion of a quasi-coherent sheaf of $\mathcal{O}_X$-modules is defined for any ringed space X in particular when X is a scheme. By our efforts in Section 25.7 we know that such a sheaf is on any affine open $U \subset X$ of the form $\widetilde{M}$ for some $\mathcal{O}_X(U)$-module M.

Lemma 25.10.1. *Let X be a scheme. Let $i : Z \to X$ be a closed immersion of locally ringed spaces.*

1. *The locally ringed space Z is a scheme,*
2. *the kernel $\mathcal{I}$ of the map $\mathcal{O}_X \to i_*\mathcal{O}_Z$ is a quasi-coherent sheaf of ideals,*
3. *for any affine open $U = \mathrm{Spec}(R)$ of X the morphism $i^{-1}(U) \to U$ can be identified with $\mathrm{Spec}(R/I) \to \mathrm{Spec}(R)$ for some ideal $I \subset R$, and*
4. *we have $\mathcal{I}|_U = \widetilde{I}$.*

In particular, any sheaf of ideals locally generated by sections is a quasi-coherent sheaf of ideals (and vice versa), and any closed subspace of X is a scheme.

Proof. Let $i : Z \to X$ be a closed immersion. Let $z \in Z$ be a point. Choose any affine open neighbourhood $i(z) \in U \subset X$. Say $U = \mathrm{Spec}(R)$. By Lemma 25.8.2 we know that $i^{-1}(U) \to U$ can be identified with the morphism of affine schemes $\mathrm{Spec}(R/I) \to \mathrm{Spec}(R)$. First of all this implies that $z \in i^{-1}(U) \subset Z$ is an affine neighbourhood of z. Thus Z is a scheme. Second this implies that $\mathcal{I}|_U$ is $\widetilde{I}$. In other words for every point $x \in i(Z)$ there exists an open neighbourhood such that $\mathcal{I}$ is quasi-coherent in that neighbourhood. Note that $\mathcal{I}|_{X \setminus i(Z)} \cong \mathcal{O}_{X \setminus i(Z)}$. Thus the restriction of the sheaf of ideals is quasi-coherent on $X \setminus i(Z)$ also. We conclude that $\mathcal{I}$ is quasi-coherent. □

Definition 25.10.2. Let X be a scheme.

1. A morphism of schemes is called an *open immersion* if it is an open immersion of locally ringed spaces (see Definition 25.3.1).
2. An *open subscheme* of X is an open subspace of X which is a scheme by Lemma 25.9.2 above.
3. A morphism of schemes is called a *closed immersion* if it is a closed immersion of locally ringed spaces (see Definition 25.4.1).

(4) A *closed subscheme* of X is a closed subspace of X which is a scheme by Lemma 25.10.1 above.
(5) A morphism of schemes $f : X \to Y$ is called an *immersion*, or a *locally closed immersion* if it can be factored as $j \circ i$ where i is a closed immersion and j is an open immersion.

It follows from the lemmas in Sections 25.3 and 25.4 that any open (resp. closed) immersion of schemes is isomorphic to the inclusion of an open (resp. closed) subscheme of the target. We will define locally closed subschemes below.

Remark 25.10.3. If $f : X \to Y$ is an immersion of schemes, then it is in general not possible to factor f as an open immersion followed by a closed immersion. See Morphisms, Example 28.3.4.

Lemma 25.10.4. *Let $f : Y \to X$ be an immersion of schemes. Then f is a closed immersion if and only if $f(Y) \subset X$ is a closed subset.*

Proof. If f is a closed immersion then $f(Y)$ is closed by definition. Conversely, suppose that $f(Y)$ is closed. By definition there exists an open subscheme $U \subset X$ such that f is the composition of a closed immersion $i : Y \to U$ and the open immersion $j : U \to X$. Let $\mathcal{I} \subset \mathcal{O}_U$ be the quasi-coherent sheaf of ideals associated to the closed immersion i. Note that $\mathcal{I}|_{U \setminus i(Y)} = \mathcal{O}_{U \setminus i(Y)} = \mathcal{O}_{X \setminus i(Y)}|_{U \setminus i(Y)}$. Thus we may glue (see Sheaves, Section 6.33) $\mathcal{I}$ and $\mathcal{O}_{X \setminus i(Y)}$ to a sheaf of ideals $\mathcal{J} \subset \mathcal{O}_X$. Since every point of X has a neighbourhood where $\mathcal{J}$ is quasi-coherent, we see that $\mathcal{J}$ is quasi-coherent (in particular locally generated by sections). By construction $\mathcal{O}_X/\mathcal{J}$ is supported on U and equal to $\mathcal{O}_U/\mathcal{I}$. Thus we see that the closed subspaces associated to $\mathcal{I}$ and $\mathcal{J}$ are canonically isomorphic, see Example 25.4.3. In particular the closed subspace of U associated to $\mathcal{I}$ is isomorphic to a closed subspace of X. Since $Y \to U$ is identified with the closed subspace associated to $\mathcal{I}$, see Lemma 25.4.5, we conclude that $Y \to U \to X$ is a closed immersion. □

Let $f : Y \to X$ be an immersion. Let $Z = \overline{f(Y)} \setminus f(Y)$ which is a closed subset of X. Let $U = X \setminus Z$. The lemma implies that U is the biggest open subspace of X such that $f : Y \to X$ factors through a closed immersion into U. If we define a *locally closed subscheme of X* as a pair (Z, U) consisting of a closed subscheme Z of an open subscheme U of X such that in addition $\overline{Z} \cup U = X$. We usually just say "let Z be a locally closed subscheme of X" since we may recover U from the morphism $Z \to X$. The above then shows that any immersion $f : Y \to X$ factors uniquely as $Y \to Z \to X$ where Z is a locally closed subspace of X and $Y \to Z$ is an isomorphism.

The interest of this is that the collection of locally closed subschemes of X forms a set. We may define a partial ordering on this set, which we call inclusion for obvious reasons. To be explicit, if $Z \to X$ and $Z' \to X$ are two locally closed subschemes of X, then we say that Z *is contained in* Z' simply if the morphism $Z \to X$ factors through Z'. If it does, then of course Z is identified with a unique locally closed subscheme of Z', and so on.

25.11. Zariski topology of schemes

See Topology, Section 5.1 for some basic material in topology adapted to the Zariski topology of schemes.

Lemma 25.11.1. *Let X be a scheme. Any irreducible closed subset of X has a unique generic point. In other words, X is a sober topological space, see Topology, Definition 5.7.4.*

Proof. Let $Z \subset X$ be an irreducible closed subset. For every affine open $U \subset X$, $U = \mathrm{Spec}(R)$ we know that $Z \cap U = V(I)$ for a unique radical ideal $I \subset R$. Note that $Z \cap I$ is either empty or irreducible. In the second case (which occurs for at least one U) we see that $I = \mathfrak{p}$ is a prime ideal, which is a generic point ξ of $Z \cap U$. It follows that $Z = \overline{\{\xi\}}$, in other words ξ is a generic point of Z. If ξ' was a second generic point, then $\xi' \in Z \cap U$ and it follows immediately that $\xi' = \xi$. □

Lemma 25.11.2. *Let X be a scheme. The collection of affine opens of X forms a basis for the topology on X.*

Proof. This follows from the discussion on open subschemes in Section 25.9. □

Remark 25.11.3. In general the intersection of two affine opens in X is not affine open. See Example 25.14.3.

Lemma 25.11.4. *The underlying topological space of any scheme is locally quasi-compact, see Topology, Definition 5.12.1.*

Proof. This follows from Lemma 25.11.2 above and the fact that the spectrum of ring is quasi-compact, see Algebra, Lemma 10.16.10. □

Lemma 25.11.5. *Let X be a scheme. Let U, V be affine opens of X, and let $x \in U \cap V$. There exists an affine open neighbourhood W of x such that W is a standard open of both U and V.*

Proof. Write $U = \mathrm{Spec}(A)$ and $V = \mathrm{Spec}(B)$. Say x corresponds to the prime $\mathfrak{p} \subset A$ and the prime $\mathfrak{q} \subset B$. We may choose a $f \in A$, $f \notin \mathfrak{p}$ such that $D(f) \subset U \cap V$. Note that any standard open of $D(f)$ is a standard open of $\mathrm{Spec}(A) = U$. Hence we may assume that $U \subset V$. In other words, now we may think of U as an affine open of V. Next we choose a $g \in B$, $g \notin \mathfrak{q}$ such that $D(g) \subset U$. In this case we see that $D(g) = D(g_A)$ where $g_A \in A$ denotes the image of $g \in A$. Thus the lemma is proved. □

Lemma 25.11.6. *Let X be a scheme. Let $X = \bigcup_i U_i$ be an affine open covering. Let $V \subset X$ be an affine open. There exists a standard open covering $V = \bigcup_{j=1,\ldots,m} V_j$ (see Definition 25.5.2) such that each V_j is a standard open in one of the U_i.*

Proof. Pick $v \in V$. Then $v \in U_i$ for some i. By Lemma 25.11.5 above there exists an open $v \in W_v \subset V \cap U_i$ such that W_v is a standard open in both V and U_i. Since V is quasi-compact the lemma follows. □

Lemma 25.11.7. *Let X be a scheme whose underlying topological space is a finite discrete set. Then X is affine.*

Proof. Say $X = \{x_1, \ldots, x_n\}$. Then $U_i = \{x_i\}$ is an open neighbourhood of x_i. By Lemma 25.11.2 it is affine. Hence X is a finite disjoint union of affine schemes, and hence is affine by Lemma 25.6.8. □

Example 25.11.8. There exists a scheme without closed points. Namely, let R be a local domain whose spectrum looks like $(0) = \mathfrak{p}_0 \subset \mathfrak{p}_1 \subset \mathfrak{p}_2 \subset \ldots \subset \mathfrak{m}$. Then the open subscheme $\mathrm{Spec}(R) \setminus \{\mathfrak{m}\}$ does not have a closed point. To see that such a ring R exists, we use that given any totally ordered group $(\Gamma, \geq)$ there exists a valuation ring A with valuation group $(\Gamma, \geq)$, see [**Kru32**]. See Algebra, Section 10.46 for notation. We take $\Gamma = \mathbf{Z}x_1 \oplus \mathbf{Z}x_2 \oplus \mathbf{Z}x_3 \oplus \ldots$ and we define $\sum_i a_i x_i \geq 0$ if and only if the first nonzero a_i is > 0, or all $a_i = 0$. So $x_1 \geq x_2 \geq x_3 \geq \ldots \geq 0$. The subsets $x_i + \Gamma_{\geq 0}$ are prime ideals of $(\Gamma, \geq)$, see Algebra, notation above Lemma 10.46.15. These together with $\emptyset$ and $\Gamma_{\geq 0}$ are the only prime ideals. Hence A is an example of a ring with the given structure of its spectrum, by Algebra, Lemma 10.46.15.

25.12. Reduced schemes

Definition 25.12.1. Let X be a scheme. We say X is *reduced* if every local ring $\mathcal{O}_{X,x}$ is reduced.

Lemma 25.12.2. *A scheme X is reduced if and only if $\mathcal{O}_X(U)$ is a reduced ring for all $U \subset X$ open.*

Proof. Assume that X is reduced. Let $f \in \mathcal{O}_X(U)$ be a section such that $f^n = 0$. Then the image of f in $\mathcal{O}_{U,u}$ is zero for all $u \in U$. Hence f is zero, see Sheaves, Lemma 6.11.1. Conversely, assume that $\mathcal{O}_X(U)$ is reduced for all opens U. Pick any nonzero element $f \in \mathcal{O}_{X,x}$. Any representative $(U, f \in \mathcal{O}(U))$ of f is nonzero and hence not nilpotent. Hence f is not nilpotent in $\mathcal{O}_{X,x}$. □

Lemma 25.12.3. *An affine scheme* $\mathrm{Spec}(R)$ *is reduced if and only if R is reduced.*

Proof. The direct implication follows immediately from Lemma 25.12.2 above. In the other direction it follows since any localization of a reduced ring is reduced, and in particular the local rings of a reduced ring are reduced. □

Lemma 25.12.4. *Let X be a scheme. Let $T \subset X$ be a closed subset. There exists a unique closed subscheme $Z \subset X$ with the following properties: (a) the underlying topological space of Z is equal to T, and (b) Z is reduced.*

Proof. Let $\mathcal{I} \subset \mathcal{O}_X$ be the sub presheaf defined by the rule

$$\mathcal{I}(U) = \{f \in \mathcal{O}_X(U) \mid f(t) = 0 \text{ for all } t \in T \cap U\}$$

Here we use $f(t)$ to indicate the image of f in the residue field $\kappa(t)$ of X at t. Because of the local nature of the condition it is clear that $\mathcal{I}$ is a sheaf of ideals. Moreover, let $U = \mathrm{Spec}(R)$ be an affine open. We may write $T \cap U = V(I)$ for a unique radical ideal $I \subset R$. Given a prime $\mathfrak{p} \in V(I)$ corresponding to $t \in T \cap U$ and an element $f \in R$ we have $f(t) = 0 \Leftrightarrow f \in \mathfrak{p}$. Hence $\mathcal{I}(U) = \cap_{\mathfrak{p} \in V(I)} \mathfrak{p} = I$ by Algebra, Lemma 10.16.2. Moreover, for any standard open $D(g) \subset \mathrm{Spec}(R) = U$ we have $\mathcal{I}(D(g)) = I_g$ by the same reasoning. Thus $\widetilde{I}$ and $\mathcal{I}|_U$ agree (as ideals) on a basis of opens and hence are equal. Therefore $\mathcal{I}$ is a quasi-coherent sheaf of ideals.

At this point we may define Z as the closed subspace associated to the sheaf of ideals $\mathcal{I}$. For every affine open $U = \mathrm{Spec}(R)$ of X we see that $Z \cap U = \mathrm{Spec}(R/I)$ where I is a radical ideal and hence Z is reduced (by Lemma 25.12.3 above). By

construction the underlying closed subset of Z is T. Hence we have found a closed subscheme with properties (a) and (b).

Let $Z' \subset X$ be a second closed subscheme with properties (a) and (b). For every affine open $U = \mathrm{Spec}(R)$ of X we see that $Z' \cap U = \mathrm{Spec}(R/I')$ for some ideal $I' \subset R$. By Lemma 25.12.3 the ring R/I' is reduced and hence I' is radical. Since $V(I') = T \cap U = V(I)$ we deduced that $I = I'$ by Algebra, Lemma 10.16.2. Hence Z' and Z are defined by the same sheaf of ideals and hence are equal. □

Definition 25.12.5. Let X be a scheme. Let $i : Z \to X$ be the inclusion of a closed subset. A *scheme structure on* Z is given by a closed subscheme Z' of X whose underlying closed is equal to Z. We often say "let $(Z, \mathcal{O}_Z)$ be a scheme structure on Z" to indicate this. The *reduced induced scheme structure* on Z is the one constructed in Lemma 25.12.4. The *reduction* X_{red} *of* X is the reduced induced scheme structure on X itself.

Often when we say "let $Z \subset X$ be an irreducible component of X" we think of Z as a reduced closed subscheme of X using the reduced induced scheme structure.

Lemma 25.12.6. *Let X be a scheme. Let $Z \subset X$ be a closed subscheme. Let Y be a reduced scheme. A morphism $f : Y \to X$ factors through Z if and only if $f(Y) \subset Z$ (set theoretically). In particular, any morphism $Y \to X$ factors as $Y \to X_{red} \to X$.*

Proof. Assume $f(Y) \subset Z$ (set theoretically). Let $I \subset \mathcal{O}_X$ be the ideal sheaf of Z. For any affine opens $U \subset X$, $\mathrm{Spec}(B) = V \subset Y$ with $f(V) \subset U$ and any $g \in \mathcal{I}(U)$ the pullback $b = f^\sharp(g) \in \Gamma(V, \mathcal{O}_Y) = B$ maps to zero in the residue field of any $y \in V$. In other words $g \in \bigcap_{\mathfrak{p} \subset B} \mathfrak{p}$. This implies $b = 0$ as B is reduced (Lemma 25.12.2, and Algebra, Lemma 10.16.2). Hence f factors through Z by Lemma 25.4.6. □

25.13. Points of schemes

Given a scheme X we can define a functor

$$h_X : Sch^{opp} \longrightarrow Sets, \quad T \longmapsto \mathrm{Mor}(T, X).$$

See Categories, Example 4.3.4. This is called the *functor of points of* X. A fun part of scheme theory is to find descriptions of the internal geometry of X in terms of this functor h_X. In this section we find a simple way to describe points of X.

Let X be a scheme. Let R be a local ring with maximal ideal $\mathfrak{m} \subset R$. Suppose that $f : \mathrm{Spec}(R) \to X$ is a morphism of schemes. Let $x \in X$ be the image of the closed point $\mathfrak{m} \in \mathrm{Spec}(R)$. Then we obtain a local homomorphism of local rings

$$f^\sharp : \mathcal{O}_{X,x} \longrightarrow \mathcal{O}_{\mathrm{Spec}(R),\mathfrak{m}} = R.$$

Lemma 25.13.1. *Let X be a scheme. Let R be a local ring. The construction above gives a bijective correspondence between morphisms $\mathrm{Spec}(R) \to X$ and pairs (x, φ) consisting of a point $x \in X$ and a local homomorphism of local rings $\varphi : \mathcal{O}_{X,x} \to R$.*

Proof. Let A be a ring. For any ring homomorphism $\psi : A \to R$ there exists a unique prime ideal $\mathfrak{p} \subset A$ and a factorization $A \to A_\mathfrak{p} \to R$ where the last map is a local homomorphism of local rings. Namely, $\mathfrak{p} = \psi^{-1}(\mathfrak{m})$. Via Lemma 25.6.4 this proves that the lemma holds if X is an affine scheme.

Let X be a general scheme. Any $x \in X$ is contained in an open affine $U \subset X$. By the affine case we conclude that every pair (x, φ) occurs as the end product of the construction above the lemma.

To finish the proof it suffices to show that any morphism $f : \mathrm{Spec}(R) \to X$ has image contained in any affine open containing the image x of the closed point of $\mathrm{Spec}(R)$. In fact, let $x \in V \subset X$ be any open neighbourhood containing x. Then $f^{-1}(V) \subset \mathrm{Spec}(R)$ is an open containing the unique closed point and hence equal to $\mathrm{Spec}(R)$. □

As a special case of the lemma above we obtain for every point x of a scheme X a canonical morphism

$$\mathrm{Spec}(\mathcal{O}_{X,x}) \longrightarrow X \tag{25.13.1.1}$$

corresponding to the identity map on the local ring of X at x. We may reformulate the lemma above as saying that for any morphism $f : \mathrm{Spec}(R) \to X$ there exists a unique point $x \in X$ such that f factors as $\mathrm{Spec}(R) \to \mathrm{Spec}(\mathcal{O}_{X,x}) \to X$ where the first map comes from a local homomorphism $\mathcal{O}_{X,x} \to R$.

In case we have a morphism of schemes $f : X \to S$, and a point x mapping to a point $s \in S$ we obtain a commutative diagram

$$\begin{array}{ccc} \mathrm{Spec}(\mathcal{O}_{X,x}) & \longrightarrow & X \\ \downarrow & & \downarrow \\ \mathrm{Spec}(\mathcal{O}_{S,s}) & \longrightarrow & S \end{array}$$

where the left vertical map corresponds to the local ring map $f_x^\sharp : \mathcal{O}_{X,x} \to \mathcal{O}_{S,s}$.

Lemma 25.13.2. *Let X be a scheme. Let $x, x' \in X$ be points of X. Then $x' \in X$ is a generalization of x if and only if x' is in the image of the canonical morphism $\mathrm{Spec}(\mathcal{O}_{X,x}) \to X$.*

Proof. A continuous map preserves the relation of specialization/generalization. Since every point of $\mathrm{Spec}(\mathcal{O}_{X,x})$ is a generalization of the closed point we see every point in the image of $\mathrm{Spec}(\mathcal{O}_{X,x}) \to X$ is a generalization of x. Conversely, suppose that x' is a generalization of x. Choose an affine open neighbourhood $U = \mathrm{Spec}(R)$ of x. Then $x' \in U$. Say $\mathfrak{p} \subset R$ and $\mathfrak{p}' \subset R$ are the primes corresponding to x and x'. Since x' is a generalization of x we see that $\mathfrak{p}' \subset \mathfrak{p}$. This means that $\mathfrak{p}'$ is in the image of the morphism $\mathrm{Spec}(\mathcal{O}_{X,x}) = \mathrm{Spec}(R_\mathfrak{p}) \to \mathrm{Spec}(R) = U \subset X$ as desired. □

Now, let us discuss morphisms from spectra of fields. Let $(R, \mathfrak{m}, \kappa)$ be a local ring with maximal ideal $\mathfrak{m}$ and residue field κ. Let K be a field. A local homomorphism $R \to K$ by definition factors as $R \to \kappa \to K$, i.e., is the same thing as a morphism $\kappa \to K$. Thus we see that morphisms

$$\mathrm{Spec}(K) \longrightarrow X$$

correspond to pairs $(x, \kappa(x) \to K)$. We may define a partial ordering on morphisms of spectra of fields to X by saying that $\mathrm{Spec}(K) \to X$ dominates $\mathrm{Spec}(L) \to X$ if $\mathrm{Spec}(K) \to X$ factors through $\mathrm{Spec}(L) \to X$. This suggests the following notion:

Let us temporarily say that two morphisms $p : \mathrm{Spec}(K) \to X$ and $q : \mathrm{Spec}(L) \to X$ are *equivalent* if there exists a third field Ω and a commutative diagram

$$\begin{array}{ccc} \mathrm{Spec}(\Omega) & \longrightarrow & \mathrm{Spec}(L) \\ \downarrow & & \downarrow q \\ \mathrm{Spec}(K) & \xrightarrow{p} & X \end{array}$$

Of course this immediately implies that the unique points of all three of the schemes $\mathrm{Spec}(K)$, $\mathrm{Spec}(L)$, and $\mathrm{Spec}(\Omega)$ map to the same $x \in X$. Thus a diagram (by the remarks above) corresponds to a point $x \in X$ and a commutative diagram

$$\begin{array}{ccc} \Omega & \longleftarrow & L \\ \uparrow & & \uparrow \\ K & \longleftarrow & \kappa(x) \end{array}$$

of fields. This defines an equivalence relation, because given any set of extensions $\kappa \subset K_i$ there exists some field extension $\kappa \subset \Omega$ such that all the field extensions K_i are contained in the extension Ω.

Lemma 25.13.3. *Let X be a scheme. Points of X correspond bijectively to equivalence classes of morphisms from spectra of fields into X. Moreover, each equivalence class contains a (unique up to unique isomorphism) smallest element $\mathrm{Spec}(\kappa(x)) \to X$.*

Proof. Follows from the discussion above. □

Of course the morphisms $\mathrm{Spec}(\kappa(x)) \to X$ factor through the canonical morphisms $\mathrm{Spec}(\mathcal{O}_{X,x}) \to X$. And the content of Lemma 25.13.2 is in this setting that the morphism $\mathrm{Spec}(\kappa(x')) \to X$ factors as $\mathrm{Spec}(\kappa(x')) \to \mathrm{Spec}(\mathcal{O}_{X,x}) \to X$ whenever x' is a generalization of x. In case we have a morphism of schemes $f : X \to S$, and a point x mapping to a point $s \in S$ we obtain a commutative diagram

$$\begin{array}{ccccc} \mathrm{Spec}(\kappa(x)) & \longrightarrow & \mathrm{Spec}(\mathcal{O}_{X,x}) & \longrightarrow & X \\ \downarrow & & \downarrow & & \downarrow \\ \mathrm{Spec}(\kappa(s)) & \longrightarrow & \mathrm{Spec}(\mathcal{O}_{S,s}) & \longrightarrow & S. \end{array}$$

25.14. Glueing schemes

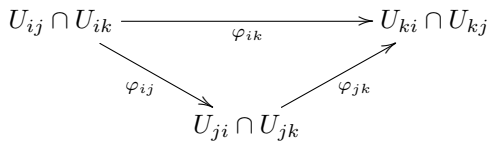
Let I be a set. For each $i \in I$ let $(X_i, \mathcal{O}_i)$ be a locally ringed space. (Actually the construction that follows works equally well for ringed spaces.) For each pair $i, j \in I$ let $U_{ij} \subset X_i$ be an open subspace. For each pair $i, j \in I$, let

$$\varphi_{ij} : U_{ij} \to U_{ji}$$

be an isomorphism of locally ringed spaces. For convenience we assume that $U_{ii} = X_i$ and $\varphi_{ii} = \mathrm{id}_{X_i}$. For each triple $i, j, k \in I$ assume that

(1) we have $\varphi_{ij}^{-1}(U_{ji} \cap U_{jk}) = U_{ij} \cap U_{ik}$, and

(2) the diagram



is commutative.

Let us call a collection $(I, (X_i)_{i\in I}, (U_{ij})_{i,j\in I}, (\varphi_{ij})_{i,j\in I})$ satisfying the conditions above a glueing data.

Lemma 25.14.1. *Given any glueing data of locally ringed spaces there exists a locally ringed space X and open subspaces $U_i \subset X$ together with isomorphisms $\varphi_i : X_i \to U_i$ of locally ringed spaces such that*

(1) $\varphi_i(U_{ij}) = U_i \cap U_j$, *and*
(2) $\varphi_{ij} = \varphi_j^{-1}|_{U_i \cap U_j} \circ \varphi_i|_{U_{ij}}$.

The locally ringed space X is characterized by the following mapping properties: Given a locally ringed space Y we have

$$\begin{aligned}
\mathrm{Mor}(X, Y) &= \{(f_i)_{i\in I} \mid f_i : X_i \to Y,\ f_j \circ \varphi_{ij} = f_i|_{U_{ij}}\} \\
f &\mapsto (f|_{U_i} \circ \varphi_i)_{i\in I} \\
\mathrm{Mor}(Y, X) &= \left\{ \begin{array}{c} \text{open covering } Y = \bigcup_{i\in I} V_i \text{ and } (g_i : V_i \to X_i)_{i\in I} \text{ such that} \\ g_i^{-1}(U_{ij}) = V_i \cap V_j \text{ and } g_j|_{V_i\cap V_j} = \varphi_{ij} \circ g_i|_{V_i\cap V_j} \end{array} \right\} \\
g &\mapsto V_i = g^{-1}(U_i),\ g_i = g|_{V_i}
\end{aligned}$$

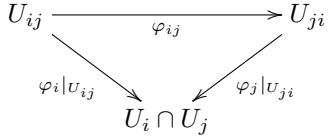
Proof. We construct X in stages. As a set we take

$$X = (\coprod X_i)/\sim .$$

Here given $x \in X_i$ and $x' \in X_j$ we say $x \sim x'$ if and only if $x \in U_{ij}$, $x' \in U_{ji}$ and $\varphi_{ij}(x) = x'$. This is an equivalence relation since if $x \in X_i$, $x' \in X_j$, $x'' \in X_k$, and $x \sim x'$ and $x' \sim x''$, then $x' \in U_{ji} \cap U_{jk}$, hence by condition (1) of a glueing data also $x \in U_{ij} \cap U_{ik}$ and $x'' \in U_{ki} \cap U_{kj}$ and by condition (2) we see that $\varphi_{ik}(x) = x''$. (Reflexivity and symmetry follows from our assumptions that $U_{ii} = X_i$ and $\varphi_{ii} = \mathrm{id}_{X_i}$.) Denote $\varphi_i : X_i \to X$ the natural maps. Denote $U_i = \varphi_i(X_i) \subset X$. Note that $\varphi_i : X_i \to U_i$ is a bijection.

The topology on X is defined by the rule that $U \subset X$ is open if and only if $\varphi_i^{-1}(U)$ is open for all i. We leave it to the reader to verify that this does indeed define a topology. Note that in particular U_i is open since $\varphi_j^{-1}(U_i) = U_{ji}$ which is open in X_j for all j. Moreover, for any open set $W \subset X_i$ the image $\varphi_i(W) \subset U_i$ is open because $\varphi_j^{-1}(\varphi_i(W)) = \varphi_{ji}^{-1}(W \cap U_{ij})$. Therefore $\varphi_i : X_i \to U_i$ is a homeomorphism.

To obtain a locally ringed space we have to construct the sheaf of rings $\mathcal{O}_X$. We do this by glueing the sheaves of rings $\mathcal{O}_{U_i} := \varphi_{i,*}\mathcal{O}_{X_i}$. Namely, in the commutative diagram



the arrow on top is an isomorphism of ringed spaces, and hence we get unique isomorphisms of sheaves of rings

$$\mathcal{O}_{U_i}|_{U_i \cap U_j} \longrightarrow \mathcal{O}_{U_j}|_{U_i \cap U_j}.$$

These satisfy a cocycle condition as in Sheaves, Section 6.33. By the results of that section we obtain a sheaf of rings $\mathcal{O}_X$ on X such that $\mathcal{O}_X|_{U_i}$ is isomorphic to $\mathcal{O}_{U_i}$ compatibly with the glueing maps displayed above. In particular $(X, \mathcal{O}_X)$ is a locally ringed space since the stalks of $\mathcal{O}_X$ are equal to the stalks of $\mathcal{O}_{X_i}$ at corresponding points.

The proof of the mapping properties is omitted. □

Lemma 25.14.2. *In Lemma 25.14.1 above, assume that all X_i are schemes. Then the resulting locally ringed space X is a scheme.*

Proof. This is clear since each of the U_i is a scheme and hence every $x \in X$ has an affine neighbourhood. □

It is customary to think of X_i as an open subspace of X via the isomorphisms φ_i. We will do this in the next two examples.

Example 25.14.3 (Affine space with zero doubled)**.** Let k be a field. Let $n \geq 1$. Let $X_1 = \mathrm{Spec}(k[x_1, \ldots, x_n])$, let $X_2 = \mathrm{Spec}(k[y_1, \ldots, y_n])$. Let $0_1 \in X_1$ be the point corresponding to the maximal ideal $(x_1, \ldots, x_n) \subset k[x_1, \ldots, x_n]$. Let $0_2 \in X_2$ be the point corresponding to the maximal ideal $(y_1, \ldots, y_n) \subset k[y_1, \ldots, y_n]$. Let $U_{12} = X_1 \setminus \{0_1\}$ and let $U_{21} = X_2 \setminus \{0_2\}$. Let $\varphi_{12} : U_{12} \to U_{21}$ be the isomorphism coming from the isomorphism of k-algebras $k[y_1, \ldots, y_n] \to k[x_1, \ldots, x_n]$ mapping y_i to x_i (which induces $X_1 \cong X_2$ mapping 0_1 to 0_2). Let X be the scheme obtained from the glueing data $(X_1, X_2, U_{12}, U_{21}, \varphi_{12}, \varphi_{21} = \varphi_{12}^{-1})$. Via the slight abuse of notation introduced above the example we think of $X_i \subset X$ as open subschemes. There is a morphism $f : X \to \mathrm{Spec}(k[t_1, \ldots, t_n])$ which on X_i corresponds to k algebra map $k[t_1, \ldots, t_n] \to k[x_1, \ldots, x_n]$ (resp. $k[t_1, \ldots, t_n] \to k[y_1, \ldots, y_n]$) mapping t_i to x_i (resp. t_i to y_i). It is easy to see that this morphism identifies $k[t_1, \ldots, t_n]$ with $\Gamma(X, \mathcal{O}_X)$. Since $f(0_1) = f(0_2)$ we see that X is not affine.

Note that X_1 and X_2 are affine opens of X. But, if $n = 2$, then $X_1 \cap X_2$ is the scheme described in Example 25.9.3 and hence not affine. Thus in general the intersection of affine opens of a scheme is not affine. (This fact holds more generally for any $n > 1$.)

Another curious feature of this example is the following. If $n > 1$ there are many irreducible closed subsets $T \subset X$ (take the closure of any non closed point in X_1 for example). But unless $T = \{0_1\}$, or $T = \{0_2\}$ we have $0_1 \in T \Leftrightarrow 0_2 \in T$. Proof omitted.

Example 25.14.4 (Projective line)**.** Let k be a field. Let $X_1 = \mathrm{Spec}(k[x])$, let $X_2 = \mathrm{Spec}(k[y])$. Let $0 \in X_1$ be the point corresponding to the maximal ideal $(x) \subset k[x]$. Let $\infty \in X_2$ be the point corresponding to the maximal ideal $(y) \subset k[y]$. Let $U_{12} = X_1 \setminus \{0\} = D(x) = \mathrm{Spec}(k[x, 1/x])$ and let $U_{21} = X_2 \setminus \{\infty\} = D(y) = \mathrm{Spec}(k[y, 1/y])$. Let $\varphi_{12} : U_{12} \to U_{21}$ be the isomorphism coming from the isomorphism of k-algebras $k[y, 1/y] \to k[x, 1/x]$ mapping y to $1/x$. Let $\mathbf{P}^1_k$ be the scheme obtained from the glueing data $(X_1, X_2, U_{12}, U_{21}, \varphi_{12}, \varphi_{21} = \varphi_{12}^{-1})$. Via the slight abuse of notation introduced above the example we think of $X_i \subset \mathbf{P}^1_k$ as open

subschemes. In this case we see that $\Gamma(\mathbf{P}^1_k, \mathcal{O}) = k$ because the only polynomials $g(x)$ in x such that $g(1/y)$ is also a polynomial in y are constant polynomials. Since $\mathbf{P}^1_k$ is infinite we see that $\mathbf{P}^1_k$ is not affine.

We claim that there exists an affine open $U \subset \mathbf{P}^1_k$ which contains both 0 and ∞. Namely, let $U = \mathbf{P}^1_k \setminus \{1\}$, where 1 is the point of X_1 corresponding to the maximal ideal $(x-1)$ and also the point of X_2 corresponding to the maximal ideal $(y-1)$. Then it is easy to see that $s = 1/(x-1) = y/(1-y) \in \Gamma(U, \mathcal{O}_U)$. In fact you can show that $\Gamma(U, \mathcal{O}_U)$ is equal to the polynomial ring $k[s]$ and that the corresponding morphism $U \to \mathrm{Spec}(k[s])$ is an isomorphism of schemes. Details omitted.

25.15. A representability criterion

In this section we reformulate the glueing lemma of Section 25.14 in terms of functors. We recall some of the material from Categories, Section 4.3. Recall that given a scheme X we can define a functor

$$h_X : Sch^{opp} \longrightarrow Sets, \quad T \longmapsto \mathrm{Mor}(T, X).$$

This is called the *functor of points of* X.

Let F be a contravariant functor from the category of schemes to the category of sets. In a formula

$$F : Sch^{opp} \longrightarrow Sets.$$

We will use the same terminology as in Sites, Section 7.2. Namely, given a scheme T, an element $\xi \in F(T)$, and a morphism $f : T' \to T$ we will denote $f^*\xi$ the element $F(f)(\xi)$, and sometimes we will even use the notation $\xi|_{T'}$

Definition 25.15.1. (See Categories, Definition 4.3.6.) Let F be a contravariant functor from the category of schemes to the category of sets (as above). We say that F is *representable by a scheme* or *representable* if there exists a scheme X such that $h_X \cong F$.

Suppose that F is representable by the scheme X and that $s : h_X \to F$ is an isomorphism. By Categories, Yoneda Lemma 4.3.5 the pair $(X, s : h_X \to F)$ is unique up to unique isomorphism if it exists. Moreover, the Yoneda lemma says that given any contravariant functor F as above and any scheme Y, we have a bijection

$$\mathrm{Mor}_{\mathrm{Fun}(Sch^{opp}, Sets)}(h_Y, F) \longrightarrow F(Y), \quad s \longmapsto s(\mathrm{id}_Y).$$

Here is the reverse construction. Given any $\xi \in F(Y)$ the transformation of functors $s_\xi : h_Y \to F$ associates to any morphism $f : T \to Y$ the element $f^*\xi \in F(T)$.

In particular, in the case that F is representable, there exists a scheme X and an element $\xi \in F(X)$ such that the corresponding morphism $h_X \to F$ is an isomorphism. In this case we also say *the pair* (X, ξ) *represents* F. The element $\xi \in F(X)$ is often called the *"universal family"* for reasons that will become more clear when we talk about algebraic stacks (insert future reference here). For the moment we simply observe that the fact that if the pair (X, ξ) represents F, then every element $\xi' \in F(T)$ for any T is of the form $\xi' = f^*\xi$ for a unique morphism $f : T \to X$.

Example 25.15.2. Consider the rule which associates to every scheme T the set $F(T) = \Gamma(T, \mathcal{O}_T)$. We can turn this into a contravariant functor by using for a morphism $f : T' \to T$ the pullback map $f^\sharp : \Gamma(T, \mathcal{O}_T) \to \Gamma(T', \mathcal{O}_{T'})$. Given a ring

R and an element $t \in R$ there exists a unique ring homomorphism $\mathbf{Z}[x] \to R$ which maps x to t. Thus, using Lemma 25.6.4, we see that

$$\mathrm{Mor}(T, \mathrm{Spec}(\mathbf{Z}[x])) = \mathrm{Hom}(\mathbf{Z}[x], \Gamma(T, \mathcal{O}_T)) = \Gamma(T, \mathcal{O}_T).$$

This does indeed give an isomorphism $h_{\mathrm{Spec}(\mathbf{Z}[x])} \to F$. What is the "universal family" ξ? To get it we have to apply the identifications above to $\mathrm{id}_{\mathrm{Spec}(\mathbf{Z}[x])}$. Clearly under the identifications above this gives that $\xi = x \in \Gamma(\mathrm{Spec}(\mathbf{Z}[x]), \mathcal{O}_{\mathrm{Spec}(\mathbf{Z}[x])}) = \mathbf{Z}[x]$ as expected.

Definition 25.15.3. Let F be a contravariant functor on the category of schemes with values in sets.

(1) We say that F *satisfies the sheaf property for the Zariski topology* if for every scheme T and every open covering $T = \bigcup_{i \in I} U_i$, and for any collection of elements $\xi_i \in F(U_i)$ such that $\xi_i|_{U_i \cap U_j} = \xi_j|_{U_i \cap U_j}$ there exists a unique element $\xi \in F(T)$ such that $\xi_i = \xi|_{U_i}$ in $F(U_i)$.
(2) A *subfunctor* $H \subset F$ is a rule that associates to every scheme T a subset $H(T) \subset F(T)$ such that the maps $F(f) : F(T) \to F(T')$ maps $H(T)$ into $H(T')$ for all morphisms of schemes $f : T' \to T$.
(3) Let $H \subset F$ be a subfunctor. We say that $H \subset F$ is *representable by open immersions* if for all pairs (T, ξ), where T is a scheme and $\xi \in F(T)$ there exists an open subscheme $U_\xi \subset T$ with the following property:
   (∗) A morphism $f : T' \to T$ factors through U_ξ if and only if $f^*\xi \in H(T')$.
(4) Let I be a set. For each $i \in I$ let $H_i \subset F$ be a subfunctor. We say that the collection $(H_i)_{i \in I}$ *covers* F if and only if for every $\xi \in F(T)$ there exists an open covering $T = \bigcup U_i$ such that $\xi|_{U_i} \in H_i(U_i)$.

Lemma 25.15.4. *Let F be a contravariant functor on the category of schemes with values in the category of sets. Suppose that*

(1) *F satisfies the sheaf property for the Zariski topology,*
(2) *there exists a set I and a collection of subfunctors $F_i \subset F$ such that*
   (a) *each F_i is representable,*
   (b) *each $F_i \subset F$ is representable by open immersions, and*
   (c) *the collection $(F_i)_{i \in I}$ covers F.*

Then F is representable.

Proof. Let X_i be a scheme representing F_i and let $\xi_i \in F_i(X_i) \subset F(X_i)$ be the "universal family". Because $F_j \subset F$ is representable by open immersions, there exists an open $U_{ij} \subset X_i$ such that $T \to X_i$ factors through U_{ij} if and only if $\xi_i|_T \in F_j(T)$. In particular $\xi_i|_{U_{ij}} \in F_j(U_{ij})$ and therefore we obtain a canonical morphism $\varphi_{ij} : U_{ij} \to X_j$ such that $\varphi_{ij}^*\xi_j = \xi_i|_{U_{ij}}$. By definition of U_{ji} this implies that φ_{ij} factors through U_{ji}. Since $(\varphi_{ij} \circ \varphi_{ji})^*\xi_j = \varphi_{ji}^*(\varphi_{ij}^*\xi_j) = \varphi_{ji}^*\xi_i = \xi_j$ we conclude that $\varphi_{ij} \circ \varphi_{ji} = \mathrm{id}_{U_{ji}}$ because the pair (X_j, ξ_j) represents F_j. In particular the maps $\varphi_{ij} : U_{ij} \to U_{ji}$ are isomorphisms of schemes. Next we have to show that $\varphi_{ij}^{-1}(U_{ji} \cap U_{jk}) = U_{ij} \cap U_{ik}$. This is true because (a) $U_{ji} \cap U_{jk}$ is the largest open of U_{ji} such that ξ_j restricts to an element of F_k, (b) $U_{ij} \cap U_{ik}$ is the largest open of U_{ij} such that ξ_i restricts to an element of F_k, and (c) $\varphi_{ij}^*\xi_j = \xi_i$. Moreover, the cocycle condition in Section 25.14 follows because both $\varphi_{jk}|_{U_{ji} \cap U_{jk}} \circ \varphi_{ij}|_{U_{ij} \cap U_{ik}}$ and $\varphi_{ik}|_{U_{ij} \cap U_{ik}}$ pullback ξ_k to the element ξ_i. Thus we may apply Lemma 25.14.2 to obtain a scheme X with an open covering $X = \bigcup U_i$ and isomorphisms $\varphi_i : X_i \to U_i$

with properties as in Lemma 25.14.1. Let $\xi_i' = (\varphi_i^{-1})^*\xi_i$. The conditions of Lemma 25.14.1 imply that $\xi_i'|_{U_i \cap U_j} = \xi_j'|_{U_i \cap U_j}$. Therefore, by the condition that F satisfies the sheaf condition in the Zariski topology we see that there exists an element $\xi' \in F(X)$ such that $\xi_i = \varphi_i^*\xi'|_{U_i}$ for all i. Since φ_i is an isomorphism we also get that $(U_i, \xi'|_{U_i})$ represents the functor F_i.

We claim that the pair (X, ξ') represents the functor F. To show this, let T be a scheme and let $\xi \in F(T)$. We will construct a unique morphism $g : T \to X$ such that $g^*\xi' = \xi$. Namely, by the condition that the subfunctors F_i cover T there exists an open covering $T = \bigcup V_i$ such that for each i the restriction $\xi|_{V_i} \in F_i(V_i)$. Moreover, since each of the inclusions $F_i \subset F$ are representable by open immersions we may assume that each $V_i \subset T$ is maximal open with this property. Because, (U_i, ξ'_{U_i}) represents the functor F_i we get a unique morphism $g_i : V_i \to U_i$ such that $g_i^*\xi'|_{U_i} = \xi|_{V_i}$. On the overlaps $V_i \cap V_j$ the morphisms g_i and g_j agree, for example because they both pull back $\xi'|_{U_i \cap U_j} \in F_i(U_i \cap U_j)$ to the same element. Thus the morphisms g_i glue to a unique morphism from $T \to X$ as desired. □

Remark 25.15.5. Suppose the functor F is defined on all locally ringed spaces, and if conditions of Lemma 25.15.4 are replaced by the following:

(1) F satisfies the sheaf property on the category of locally ringed spaces,
(2) there exists a set I and a collection of subfunctors $F_i \subset F$ such that
   (a) each F_i is representable by a scheme,
   (b) each $F_i \subset F$ is representable by open immersions on the category of locally ringed spaces, and
   (c) the collection $(F_i)_{i \in I}$ covers F as a functor on the category of locally ringed spaces.

We leave it to the reader to spell this out further. Then the end result is that the functor F is representable in the category of locally ringed spaces and that the representing object is a scheme.

25.16. Existence of fibre products of schemes

A very basic question is whether or not products and fibre products exist on the category of schemes. We first prove abstractly that products and fibre products exist, and in the next section we show how we may think in a reasonable way about fibre products of schemes.

Lemma 25.16.1. *The category of schemes has a final object, products and fibre products. In other words, the category of schemes has finite limits, see Categories, Lemma 4.18.4.*

Proof. Please skip this proof. It is more important to learn how to work with the fibre product which is explained in the next section.

By Lemma 25.6.4 the scheme $\mathrm{Spec}(\mathbf{Z})$ is a final object in the category of locally ringed spaces. Thus it suffices to prove that fibred products exist.

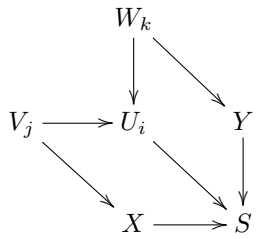
Let $f : X \to S$ and $g : Y \to S$ be morphisms of schemes. We have to show that the functor

$$\begin{aligned} F : Sch^{opp} &\longrightarrow Sets \\ T &\longmapsto \mathrm{Mor}(T, X) \times_{\mathrm{Mor}(T,S)} \mathrm{Mor}(T, Y) \end{aligned}$$

is representable. We claim that Lemma 25.15.4 applies to the functor F. If we prove this then the lemma is proved.

First we show that F satisfies the sheaf property in the Zariski topology. Namely, suppose that T is a scheme, $T = \bigcup_{i \in I} U_i$ is an open covering, and $\xi_i \in F(U_i)$ such that $\xi_i|_{U_i \cap U_j} = \xi_j|_{U_i \cap U_j}$ for all pairs i, j. By definition ξ_i corresponds to a pair (a_i, b_i) where $a_i : U_i \to X$ and $b_i : U_i \to Y$ are morphisms of schemes such that $f \circ a_i = g \circ b_i$. The glueing condition says that $a_i|_{U_i \cap U_j} = a_j|_{U_i \cap U_j}$ and $b_i|_{U_i \cap U_j} = b_j|_{U_i \cap U_j}$. Thus by glueing the morphisms a_i we obtain a morphism of locally ringed spaces (i.e., a morphism of schemes) $a : T \to X$ and similarly $b : T \to Y$ (see for example the mapping property of Lemma 25.14.1). Moreover, on the members of an open covering the compositions $f \circ a$ and $g \circ b$ agree. Therefore $f \circ a = g \circ b$ and the pair (a, b) defines an element of $F(T)$ which restricts to the pairs (a_i, b_i) on each U_i. The sheaf condition is verified.

Next, we construct the family of subfunctors. Choose an open covering by open affines $S = \bigcup_{i \in I} U_i$. For every $i \in I$ choose open coverings by open affines $f^{-1}(U_i) = \bigcup_{j \in J_i} V_j$ and $g^{-1}(U_i) = \bigcup_{k \in K_i} W_k$. Note that $X = \bigcup_{i \in I} \bigcup_{j \in J_i} V_j$ is an open covering and similarly for Y. For any $i \in I$ and each pair $(j, k) \in J_i \times K_i$ we have a commutative diagram



where all the skew arrows are open immersions. For such a triple we get a functor

$$\begin{aligned} F_{i,j,k} : Sch^{opp} &\longrightarrow Sets \\ T &\longmapsto \mathrm{Mor}(T, V_j) \times_{\mathrm{Mor}(T, U_i)} \mathrm{Mor}(T, W_j). \end{aligned}$$

There is an obvious transformation of functors $F_{i,j,k} \to F$ (coming from the huge commutative diagram above) which is injective, so we may think of $F_{i,j,k}$ as a subfunctor of F.

We check condition (2)(a) of Lemma 25.15.4. This follows directly from Lemma 25.6.7. (Note that we use here that the fibre products in the category of affine schemes are also fibre products in the whole category of locally ringed spaces.)

We check condition (2)(b) of Lemma 25.15.4. Let T be a scheme and let $\xi \in F(T)$. In other words, $\xi = (a, b)$ where $a : T \to X$ and $b : T \to Y$ are morphisms of schemes such that $f \circ a = g \circ b$. Set $V_{i,j,k} = a^{-1}(V_j) \cap b^{-1}(W_k)$. For any further morphism $h : T' \to T$ we have $h^*\xi = (a \circ h, b \circ h)$. Hence we see that $h^*\xi \in F_{i,j,k}(T')$ if and only if $a(h(T')) \subset V_j$ and $b(h(T')) \subset W_k$. In other words, if and only if $h(T') \subset V_{i,j,k}$. This proves condition (2)(b).

We check condition (2)(c) of Lemma 25.15.4. Let T be a scheme and let $\xi = (a, b) \in F(T)$ as above. Set $V_{i,j,k} = a^{-1}(V_j) \cap b^{-1}(W_k)$ as above. Condition (2)(c) just means that $T = \bigcup V_{i,j,k}$ which is evident. Thus the lemma is proved and fibre products exist. $\square$

Remark 25.16.2. Using Remark 25.15.5 you can show that the fibre product of morphisms of schemes exists in the category of locally ringed spaces and is a scheme.

25.17. Fibre products of schemes

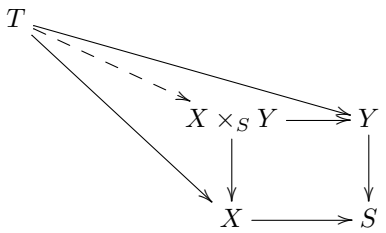
Here is a review of the general definition, even though we have already shown that fibre products of schemes exist.

Definition 25.17.1. Given morphisms of schemes $f : X \to S$ and $g : Y \to S$ the *fibre product* is a scheme $X \times_S Y$ together with projection morphisms $p : X \times_S Y \to X$ and $q : X \times_S Y \to Y$ sitting into the following commutative diagram

$$\begin{array}{ccc} X \times_S Y & \xrightarrow{q} & Y \\ {\scriptstyle p}\downarrow & & \downarrow{\scriptstyle g} \\ X & \xrightarrow{f} & S \end{array}$$

which is universal among all diagrams of this sort, see Categories, Definition 4.6.1.

In other words, given any solid commutative diagram of morphisms of schemes



there exists a unique dotted arrow making the diagram commute. We will prove some lemmas which will tell us how to think about fibre products.

Lemma 25.17.2. *Let $f : X \to S$ and $g : Y \to S$ be morphisms of schemes with the same target. If X, Y, S are all affine then $X \times_S Y$ is affine.*

Proof. Suppose that $X = \mathrm{Spec}(A)$, $Y = \mathrm{Spec}(B)$ and $S = \mathrm{Spec}(R)$. By Lemma 25.6.7 the affine scheme $\mathrm{Spec}(A \otimes_R B)$ is the fibre product $X \times_S Y$ in the category of locally ringed spaces. Hence it is a fortiori the fibre product in the category of schemes. □

Lemma 25.17.3. *Let $f : X \to S$ and $g : Y \to S$ be morphisms of schemes with the same target. Let $X \times_S Y$, p, q be the fibre product. Suppose that $U \subset S$, $V \subset X$, $W \subset Y$ are open subschemes such that $f(V) \subset U$ and $g(W) \subset U$. Then the canonical morphism $V \times_U W \to X \times_S Y$ is an open immersion which identifies $V \times_U W$ with $p^{-1}(V) \cap q^{-1}(W)$.*

Proof. Let T be a scheme Suppose $a : T \to V$ and $b : T \to W$ are morphisms such that $f \circ a = g \circ b$ as morphisms into U. Then they agree as morphisms into S. By the universal property of the fibre product we get a unique morphism $T \to X \times_S Y$. Of course this morphism has image contained in the open $p^{-1}(V) \cap q^{-1}(W)$. Thus $p^{-1}(V) \cap q^{-1}(W)$ is a fibre product of V and W over U. The result follows from the uniqueness of fibre products, see Categories, Section 4.6. □

In particular this shows that $V \times_U W = V \times_S W$ in the situation of the lemma. Moreover, if U, V, W are all affine, then we know that $V \times_U W$ is affine. And of course we may cover $X \times_S Y$ by such affine opens $V \times_U W$. We formulate this as a lemma.

Lemma 25.17.4. *Let $f : X \to S$ and $g : Y \to S$ be morphisms of schemes with the same target. Let $S = \bigcup U_i$ be any affine open covering of S. For each $i \in I$, let $f^{-1}(U_i) = \bigcup_{j \in J_i} V_j$ be an affine open covering of $f^{-1}(U_i)$ and let $g^{-1}(U_i) = \bigcup_{k \in K_i} W_k$ be an affine open covering of $f^{-1}(U_i)$. Then*

$$X \times_S Y = \bigcup\nolimits_{i \in I} \bigcup\nolimits_{j \in J_i,\ k \in K_i} V_j \times_{U_i} W_k$$

is an affine open covering of $X \times_S Y$.

Proof. See discussion above the lemma. □

In other words, we might have used the previous lemma as a way of construction the fibre product directly by glueing the affine schemes. (Which is of course exactly what we did in the proof of Lemma 25.16.1 anyway.) Here is a way to describe the set of points of a fibre product of schemes.

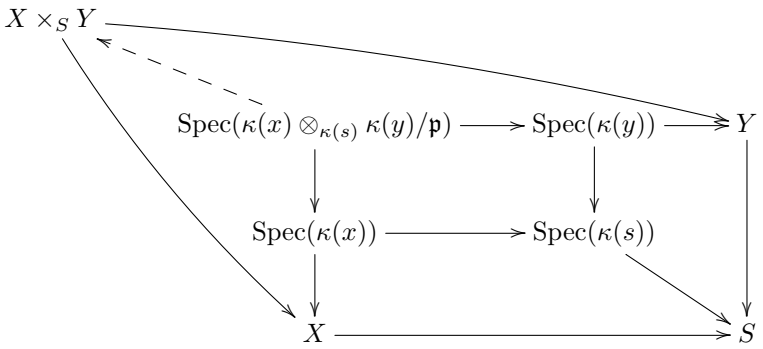
Lemma 25.17.5. *Let $f : X \to S$ and $g : Y \to S$ be morphisms of schemes with the same target. Points z of $X \times_S Y$ are in bijective correspondence to quadruples*

$$(x, y, s, \mathfrak{p})$$

where $x \in X$, $y \in Y$, $s \in S$ are points with $f(x) = s$, $g(y) = s$ and $\mathfrak{p}$ is a prime ideal of the ring $\kappa(x) \otimes_{\kappa(s)} \kappa(y)$. The residue field of z corresponds to the residue field of the prime $\mathfrak{p}$.

Proof. Let z be a point of $X \times_S Y$ and let us construct a triple as above. Recall that we may think of z as a morphism $\mathrm{Spec}(\kappa(z)) \to X \times_S Y$, see Lemma 25.13.3. This morphism corresponds to morphisms $a : \mathrm{Spec}(\kappa(z)) \to X$ and $b : \mathrm{Spec}(\kappa(z)) \to Y$ such that $f \circ a = g \circ b$. By the same lemma again we get points $x \in X$, $y \in Y$ lying over the same point $s \in S$ as well as field maps $\kappa(x) \to \kappa(z)$, $\kappa(y) \to \kappa(z)$ such that the compositions $\kappa(s) \to \kappa(x) \to \kappa(z)$ and $\kappa(s) \to \kappa(y) \to \kappa(z)$ are the same. In other words we get a ring map $\kappa(x) \otimes_{\kappa(s)} \kappa(y) \to \kappa(z)$. We let $\mathfrak{p}$ be the kernel of this map.

Conversely, given a quadruple $(x, y, s, \mathfrak{p})$ we get a commutative solid diagram



see the discussion in Section 25.13. Thus we get the dotted arrow. The corresponding point z of $X \times_S Y$ is the image of the generic point of $\mathrm{Spec}(\kappa(x) \otimes_{\kappa(s)} \kappa(y)/\mathfrak{p})$. We omit the verification that the two constructions are inverse to each other. □

Lemma 25.17.6. *Let $f : X \to S$ and $g : Y \to S$ be morphisms of schemes with the same target.*

(1) *If $f : X \to S$ is a closed immersion, then $X \times_S Y \to Y$ is a closed immersion. Moreover, if $X \to S$ corresponds to the quasi-coherent sheaf of ideals $\mathcal{I} \subset \mathcal{O}_S$, then $X \times_S Y \to Y$ corresponds to the sheaf of ideals $Im(g^*\mathcal{I} \to \mathcal{O}_Y)$.*
(2) *If $f : X \to S$ is an open immersion, then $X \times_S Y \to Y$ is an open immersion.*
(3) *If $f : X \to S$ is an immersion, then $X \times_S Y \to Y$ is an immersion.*

Proof. Assume that $X \to S$ is a closed immersion corresponding to the quasi-coherent sheaf of ideals $\mathcal{I} \subset \mathcal{O}_S$. By Lemma 25.4.7 the closed subspace $Z \subset Y$ defined by the sheaf of ideals $\mathrm{Im}(g^*\mathcal{I} \to \mathcal{O}_Y)$ is the fibre product in the category of locally ringed spaces. By Lemma 25.10.1 Z is a scheme. Hence $Z = X \times_S Y$ and the first statement follows. The second follows from Lemma 25.17.3 for example. The third is a combination of the first two. □

Definition 25.17.7. Let $f : X \to Y$ be a morphism of schemes. Let $Z \subset Y$ be a closed subscheme of Y. The *inverse image $f^{-1}(Z)$ of the closed subscheme Z* is the closed subscheme $Z \times_Y X$ of X. See Lemma 25.17.6 above.

We may occasionally also use this terminology with locally closed and open subschemes.

25.18. Base change in algebraic geometry

One motivation for the introduction of the language of schemes is that it gives a very precise notion of what it means to define a variety over a particular field. For example a variety X over $\mathbf{Q}$ is synonymous (insert future reference here) with $X \to \mathrm{Spec}(\mathbf{Q})$ which is of finite type, separated, irreducible and reduced[1]. In any case, the idea is more generally to work with schemes over a given *base scheme*, often denoted S. We use the language: "let X be a scheme over S" to mean simply that X comes equipped with a morphism $X \to S$. In diagrams we will try to picture the *structure morphism* $X \to S$ as a downward arrow from X to S. We are often more interested in the properties of X relative to S rather than the internal geometry of X. For example, we would like to know things about the fibres of $X \to S$, what happens to X after base change, etc, etc.

We introduce some of the language that is customarily used. Of course this language is just a special case of thinking about the category of objects over a given object in a category, see Categories, Example 4.2.13.

Definition 25.18.1. Let S be a scheme.

(1) We say X is a *scheme over S* to mean that X comes equipped with a morphism of schemes $X \to S$. The morphism $X \to S$ is sometimes called the *structure morphism*.
(2) If R is a ring we say X is a *scheme over R* instead of X is a scheme over $\mathrm{Spec}(R)$.

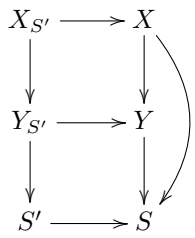
[1] Of course algebraic geometers still quibble over whether one should require X to be geometrically irreducible over $\mathbf{Q}$.

(3) A *morphism $f : X \to Y$ of schemes over S* is a morphism of schemes such that the composition $X \to Y \to S$ of f with the structure morphism of Y is equal to the structure morphism of X.
(4) We denote $\mathrm{Mor}_S(X, Y)$ the set of all morphisms from X to Y over S.
(5) Let X be a scheme over S. Let $S' \to S$ be a morphism of schemes. The *base change* of X is the scheme $X_{S'} = S' \times_S X$ over S'.
(6) Let $f : X \to Y$ be a morphism of schemes over S. Let $S' \to S$ be a morphism of schemes. The *base change* of f is the induced morphism $f' : X_{S'} \to Y_{S'}$ (namely the morphism $\mathrm{id}_{S'} \times_{\mathrm{id}_S} f$).
(7) Let R be a ring. Let X be a scheme over R. Let $R \to R'$ be a ring map. The *base change* $X_{R'}$ is the scheme $\mathrm{Spec}(R') \times_{\mathrm{Spec}(R)} X$ over R'.

Here is a typical result.

Lemma 25.18.2. *Let S be a scheme. Let $f : X \to Y$ be an immersion (resp. closed immersion, resp. open immersion) of schemes over S. Then any base change of f is an immersion (resp. closed immersion, resp. open immersion).*

Proof. We can think of the base change of f via the morphism $S' \to S$ as the top left vertical arrow in the following commutative diagram:



The diagram implies $X_{S'} \cong Y_{S'} \times_Y X$, and the lemma follows from Lemma 25.17.6. □

In fact this type of result is so typical that there is a piece of language to express it. Here it is.

Definition 25.18.3. Properties and base change.

(1) Let $\mathcal{P}$ be a property of schemes over a base. We say that $\mathcal{P}$ is *preserved under arbitrary base change*, or simply that *preserved under base change* if whenever X/S has $\mathcal{P}$, any base change $X_{S'}/S'$ has $\mathcal{P}$.
(2) Let $\mathcal{P}$ be a property of morphisms of schemes over a base. We say that $\mathcal{P}$ is *preserved under arbitrary base change*, or simply that *preserved under base change* if whenever $f : X \to Y$ over S has $\mathcal{P}$, any base change $f' : X_{S'} \to Y_{S'}$ over S' has $\mathcal{P}$.

At this point we can say that "being a closed immersion" is preserved under arbitrary base change.

Definition 25.18.4. Let $f : X \to S$ be a morphism of schemes. Let $s \in S$ be a point. The *scheme theoretic fibre X_s of f over s*, or simply the *fibre of f over s* is

the scheme fitting in the following fibre product diagram

$$\begin{array}{ccc} X_s = \mathrm{Spec}(\kappa(s)) \times_S X & \longrightarrow & X \\ \downarrow & & \downarrow \\ \mathrm{Spec}(\kappa(s)) & \longrightarrow & S \end{array}$$

We think of the fibre X_s always as a scheme over $\kappa(s)$.

Lemma 25.18.5. *Let $f : X \to S$ be a morphism of schemes. Consider the diagrams*

$$\begin{array}{ccc} X_s & \longrightarrow & X \\ \downarrow & & \downarrow \\ \mathrm{Spec}(\kappa(s)) & \longrightarrow & S \end{array} \qquad \begin{array}{ccc} \mathrm{Spec}(\mathcal{O}_{S,s}) \times_S X & \longrightarrow & X \\ \downarrow & & \downarrow \\ \mathrm{Spec}(\mathcal{O}_{S,s}) & \longrightarrow & S \end{array}$$

In both cases the top horizontal arrow is a homeomorphism onto its image.

Proof. Choose an open affine $U \subset S$ that contains s. The bottom horizontal morphisms factor through U, see Lemma 25.13.1 for example. Thus we may assume that S is affine. If X is also affine, then the result follows from Algebra, Remark 10.16.8. In the general case the result follows by covering X by open affines. □

25.19. Quasi-compact morphisms

A scheme is *quasi-compact* if its underlying topological space is quasi-compact. There is a relative notion which is defined as follows.

Definition 25.19.1. A morphism of schemes is called *quasi-compact* if the underlying map of topological spaces is quasi-compact, see Topology, Definition 5.11.1.

Lemma 25.19.2. *Let $f : X \to S$ be a morphism of schemes. The following are equivalent*

(1) *$f : X \to S$ is quasi-compact,*
(2) *the inverse image of every affine open is quasi-compact, and*
(3) *there exists some affine open covering $S = \bigcup_{i \in I} U_i$ such that $f^{-1}(U_i)$ is quasi-compact for all i.*

Proof. Suppose we are given a covering $X = \bigcup_{i \in I} U_i$ as in (3). First, let $U \subset S$ be any affine open. For any $u \in U$ we can find an index $i(u) \in I$ such that $u \in U_{i(u)}$. As standard opens form a basis for the topology on $U_{i(u)}$ we can find $W_u \subset U \cap U_{i(u)}$ which is standard open in $U_{i(u)}$. By compactness we can find finitely many points $u_1, \ldots, u_n \in U$ such that $U = \bigcup_{j=1}^n W_{u_j}$. For each j write $f^{-1}U_{i(u_j)} = \bigcup_{k \in K_j} V_{jk}$ as a finite union of affine opens. Since $W_{u_j} \subset U_{i(u)}$ is a standard open we see that $f^{-1}(W_{u_j}) \cap V_{jk}$ is a standard open of V_{jk}, see Algebra, Lemma 10.16.4. Hence $f^{-1}(W_{u_j}) \cap V_{jk}$ is affine, and so $f^{-1}(W_{u_j})$ is a finite union of affines. This proves that the inverse image of any affine open is a finite union of affine opens.

Next, assume that the inverse image of every affine open is a finite union of affine opens. Let $K \subset X$ be any quasi-compact open. Since X has a basis of the topology consisting of affine opens we see that K is a finite union of affine opens. Hence the inverse image of K is a finite union of affine opens. Hence f is quasi-compact.

Finally, assume that f is quasi-compact. In this case the argument of the previous paragraph shows that the inverse image of any affine is a finite union of affine opens. □

Lemma 25.19.3. *Being quasi-compact is a property of morphisms of schemes over a base which is preserved under arbitrary base change.*

Proof. Omitted. □

Lemma 25.19.4. *The composition of quasi-compact morphisms is quasi-compact.*

Proof. Omitted. □

Lemma 25.19.5. *A closed immersion is quasi-compact.*

Proof. Follows from the definitions and Topology, Lemma 5.11.3. □

Example 25.19.6. An open immersion is in general not quasi-compact. The standard example of this is the open subspace $U \subset X$, where $X = \mathrm{Spec}(k[x_1, x_2, x_3, \ldots])$, where U is $X \setminus \{0\}$, and where 0 is the point of X corresponding to the maximal ideal $(x_1, x_2, x_3, \ldots)$.

Lemma 25.19.7. *Let $f : X \to S$ be a quasi-compact morphism of schemes. The following are equivalent*

(1) *$f(X) \subset S$ is closed, and*
(2) *$f(X) \subset S$ is stable under specialization.*

Proof. We have (1) $\Rightarrow$ (2) by Topology, Lemma 5.18.2. Assume (2). Let $U \subset S$ be an affine open. It suffices to prove that $f(X) \cap U$ is closed. Since $U \cap f(X)$ is stable under specializations, we have reduced to the case where S is affine. Because f is quasi-compact we deduce that $X = f^{-1}(S)$ is quasi-compact as S is affine. Thus we may write $X = \bigcup_{i=1}^n U_i$ with $U_i \subset X$ open affine. Say $S = \mathrm{Spec}(R)$ and $U_i = \mathrm{Spec}(A_i)$ for some R-algebra A_i. Then $f(X) = \mathrm{Im}(\mathrm{Spec}(A_1 \times \ldots \times A_n) \to \mathrm{Spec}(R))$. Thus the lemma follows from Algebra, Lemma 10.38.5. □

Lemma 25.19.8. *Let $f : X \to S$ be a quasi-compact morphism of schemes. Then f is closed if and only if specializations lift along f, see Topology, Definition 5.18.3.*

Proof. According to Topology, Lemma 5.18.6 if f is closed then specializations lift along f. Conversely, suppose that specializations lift along f. Let $Z \subset X$ be a closed subset. We may think of Z as a scheme with the reduced induced scheme structure, see Definition 25.12.5. Since $Z \subset X$ is closed the restriction of f to Z is still quasi-compact. Moreover specializations lift along $Z \to S$ as well, see Topology, Lemma 5.18.4. Hence it suffices to prove $f(X)$ is closed if specializations lift along f. In particular $f(X)$ is stable under specializations, see Topology, Lemma 5.18.5. Thus $f(X)$ is closed by Lemma 25.19.7. □

25.20. Valuative criterion for universal closedness

In Topology, Section 5.16 there is a discussion of proper maps as closed maps of topological spaces all of whose fibres are quasi-compact, or as maps such that all base changes are closed maps. Here is the corresponding notion in algebraic geometry.

Definition 25.20.1. A morphism of schemes $f : X \to S$ is said to be *universally closed* if every base change $f' : X_{S'} \to S'$ is closed.

In fact the adjective "universally" is often used in this way. In other words, given a property $\mathcal{P}$ of morphisms the we say that "$X \to S$ is universally $\mathcal{P}$" if and only if every base change $X_{S'} \to S'$ has $\mathcal{P}$.

Please take a look at Morphisms, Section 28.42 for a more detailed discussion of the properties of universally closed morphisms. In this section we restrict the discussion to the relationship between universal closed morphisms and morphisms satisfying the existence part of the valuative criterion.

Lemma 25.20.2. *Let $f : X \to S$ be a morphism of schemes.*

(1) *If f is universally closed then specializations lift along any base change of f, see Topology, Definition 5.18.3.*
(2) *If f is quasi-compact and specializations lift along any base change of f, then f is universally closed.*

Proof. Part (1) is a direct consequence of Topology, Lemma 5.18.6. Part (2) follows from Lemmas 25.19.8 and 25.19.3. □

Definition 25.20.3. Let $f : X \to S$ be a morphism of schemes. We say f *satisfies the existence part of the valuative criterion* if given any commutative solid diagram

$$\begin{array}{ccc} \mathrm{Spec}(K) & \longrightarrow & X \\ \downarrow & \nearrow & \downarrow \\ \mathrm{Spec}(A) & \longrightarrow & S \end{array}$$

where A is a valuation ring with field of fractions K, the dotted arrow exists. We say f *satisfies the uniqueness part of the valuative criterion* if there is at most one dotted arrow given any diagram as above (without requiring existence of course).

A *valuation ring* is a local domain maximal among the relation of domination in its fraction field, see Algebra, Definition 10.46.1. Hence the spectrum of a valuation ring has a unique generic point η and a unique closed point 0, and of course we have the specialization $\eta \rightsquigarrow 0$. The significance of valuation rings is that any specialization of points in any scheme is the image of $\eta \rightsquigarrow 0$ under some morphism from the spectrum of some valuation ring. Here is the precise result.

Lemma 25.20.4. *Let S be a scheme. Let $s' \rightsquigarrow s$ be a specialization of points of S. Then*

(1) *there exists a valuation ring A and a morphism $\mathrm{Spec}(A) \to S$ such that the generic point η of $\mathrm{Spec}(A)$ maps to s' and the special point maps to s, and*
(2) *given a field extension $\kappa(s') \subset K$ we may arrange it so that the extension $\kappa(s') \subset \kappa(\eta)$ induced by f is isomorphic to the given extension.*

Proof. Let $s' \rightsquigarrow s$ be a specialization in S, and let $\kappa(s') \subset K$ be an extension of fields. By Lemma 25.13.2 and the discussion following Lemma 25.13.3 this leads to ring maps $\mathcal{O}_{S,s} \to \kappa(s') \to K$. Let $A \subset K$ be any valuation ring whose field of fractions is K and which dominates the image of $\mathcal{O}_{S,s} \to K$, see Algebra, Lemma 10.46.2. The ring map $\mathcal{O}_{S,s} \to A$ induces the morphism $f : \mathrm{Spec}(A) \to S$, see Lemma 25.13.1. This morphism has all the desired properties by construction. □

Lemma 25.20.5. *Let $f : X \to S$ be a morphism of schemes. The following are equivalent*

(1) *Specializations lift along any base change of f*
(2) *The morphism f satisfies the existence part of the valuative criterion.*

Proof. Assume (1) holds. Let a solid diagram as in Definition 25.20.3 be given. In order to find the dotted arrow we may replace $X \to S$ by $X_{\mathrm{Spec}(A)} \to \mathrm{Spec}(A)$ since after all the assumption is stable under base change. Thus we may assume $S = \mathrm{Spec}(A)$. Let $x' \in X$ be the image of $\mathrm{Spec}(K) \to X$, so that we have $\kappa(x') \subset K$, see Lemma 25.13.3. By assumption there exists a specialization $x' \rightsquigarrow x$ in X such that x maps to the closed point of $S = \mathrm{Spec}(A)$. We get a local ring map $A \to \mathcal{O}_{X,x}$ and a ring map $\mathcal{O}_{X,x} \to \kappa(x')$, see Lemma 25.13.2 and the discussion following Lemma 25.13.3. The composition $A \to \mathcal{O}_{X,x} \to \kappa(x') \to K$ is the given injection $A \to K$. Since $A \to \mathcal{O}_{X,x}$ is local, the image of $\mathcal{O}_{X,x} \to K$ dominates A and hence is equal to A, by Algebra, Definition 10.46.1. Thus we obtain a ring map $\mathcal{O}_{X,x} \to A$ and hence a morphism $\mathrm{Spec}(A) \to X$ (see Lemma 25.13.1 and discussion following it). This proves (2).

Conversely, assume (2) holds. It is immediate that the existence part of the valuative criterion holds for any base change $X_{S'} \to S'$ of f by considering the following commutative diagram

$$\begin{array}{ccccc} \mathrm{Spec}(K) & \longrightarrow & X_{S'} & \longrightarrow & X \\ \downarrow & \nearrow & \downarrow & & \downarrow \\ \mathrm{Spec}(A) & \longrightarrow & S' & \longrightarrow & S \end{array}$$

Namely, the more horizontal dotted arrow will lead to the other one by definition of the fibre product. OK, so it clearly suffices to show that specializations lift along f. Let $s' \rightsquigarrow s$ be a specialization in S, and let $x' \in X$ be a point lying over s'. Apply Lemma 25.20.4 to $s' \rightsquigarrow s$ and the extension of fields $\kappa(s') \subset \kappa(x') = K$. We get a commutative diagram

$$\begin{array}{ccccc} \mathrm{Spec}(K) & & \longrightarrow & & X \\ \downarrow & & \nearrow & & \downarrow \\ \mathrm{Spec}(A) & \longrightarrow & \mathrm{Spec}(\mathcal{O}_{S,s}) & \longrightarrow & S \end{array}$$

and by condition (2) we get the dotted arrow. The image x of the closed point of $\mathrm{Spec}(A)$ in X will be a solution to our problem, i.e., x is a specialization of x' and maps to s. □

Proposition 25.20.6 (Valuative criterion of universal closedness)**.** *Let f be a quasi-compact morphism of schemes. Then f is universally closed if and only if f satisfies the existence part of the valuative criterion.*

Proof. This is a formal consequence of Lemmas 25.20.2 and 25.20.5 above. □

Example 25.20.7. Let k be a field. Consider the structure morphism $p : \mathbf{P}^1_k \to \mathrm{Spec}(k)$ of the projective line over k, see Example 25.14.4. Let us use the valuative criterion above to prove that p is universally closed. By construction $\mathbf{P}^1_k$ is covered

by two affine opens and hence p is quasi-compact. Let a commutative diagram

$$\begin{array}{ccc} \mathrm{Spec}(K) & \xrightarrow{\xi} & \mathbf{P}^1_k \\ \downarrow & & \downarrow \\ \mathrm{Spec}(A) & \xrightarrow{\varphi} & \mathrm{Spec}(k) \end{array}$$

be given, where A is a valuation ring and K is its field of fractions. Recall that $\mathbf{P}^1_k$ is gotten by glueing $\mathrm{Spec}(k[x])$ to $\mathrm{Spec}(k[y])$ by glueing $D(x)$ to $D(y)$ via $x = y^{-1}$ (or more symmetrically $xy = 1$). To show there is a morphism $\mathrm{Spec}(A) \to \mathbf{P}^1_k$ fitting diagonally into the diagram above we may assume that ξ maps into the open $\mathrm{Spec}(k[x])$ (by symmetry). This gives the following commutative diagram of rings

$$\begin{array}{ccc} K & \xleftarrow{\xi^\sharp} & k[x] \\ \uparrow & & \uparrow \\ A & \xleftarrow{\varphi^\sharp} & k \end{array}$$

By Algebra, Lemma 10.46.3 we see that either $\xi^\sharp(x) \in A$ or $\xi^\sharp(x)^{-1} \in A$. In the first case we get a ring map

$$k[x] \to A,\ \lambda \mapsto \varphi^\sharp(\lambda),\ x \mapsto \xi^\sharp(x)$$

fitting into the diagram of rings above, and we win. In the second case we see that we get a ring map

$$k[y] \to A,\ \lambda \mapsto \varphi^\sharp(\lambda),\ y \mapsto \xi^\sharp(x)^{-1}.$$

This gives a morphism $\mathrm{Spec}(A) \to \mathrm{Spec}(k[y]) \to \mathbf{P}^1_k$ which fits diagonally into the initial commutative diagram of this example (check omitted).

25.21. Separation axioms

A topological space X is Hausdorff if and only if the diagonal $\Delta \subset X \times X$ is a closed subset. The analogue in algebraic geometry is, given a scheme X over a base scheme S, to consider the diagonal morphism

$$\Delta_{X/S} : X \longrightarrow X \times_S X.$$

This is the unique morphism of schemes such that $\mathrm{pr}_1 \circ \Delta_{X/S} = \mathrm{id}_X$ and $\mathrm{pr}_2 \circ \Delta_{X/S} = \mathrm{id}_X$ (it exists in any category with fibre products).

Lemma 25.21.1. *The diagonal morphism of a morphism between affines is closed.*

Proof. The diagonal morphism associated to the morphism $\mathrm{Spec}(S) \to \mathrm{Spec}(R)$ is the morphism on spectra corresponding to the ring map $S \otimes_R S \to S$, $a \otimes b \mapsto ab$. This map is clearly surjective, so $S \cong S \otimes_R S/J$ for some ideal $J \subset S \otimes_R S$. Hence Δ is a closed immersion according to Example 25.8.1 □

Lemma 25.21.2. *Let X be a scheme over S. The diagonal morphism $\Delta_{X/S}$ is an immersion.*

Proof. Recall that if $V \subset X$ is affine open and maps into $U \subset S$ affine open, then $V \times_U V$ is affine open in $X \times_S X$, see Lemmas 25.17.2 and 25.17.3. Consider the open subscheme W of $X \times_S X$ which is the union of these affine opens $V \times_U V$. By Lemma 25.4.2 it is enough to show that each morphism $\Delta^{-1}_{X/S}(V \times_U V) \to V \times_U V$

is a closed immersion. Since $V = \Delta_{X/S}^{-1}(V \times_U V)$ we are just checking that $\Delta_{V/U}$ is a closed immersion, which is Lemma 25.21.1. □

Definition 25.21.3. Let $f : X \to S$ be a morphism of schemes.

(1) We say f is *separated* if the diagonal morphism $\Delta_{X/S}$ is a closed immersion.
(2) We say f is *quasi-separated* if the diagonal morphism $\Delta_{X/S}$ is a quasi-compact morphism.
(3) We say a scheme Y is *separated* if the morphism $Y \to \mathrm{Spec}(\mathbf{Z})$ is separated.
(4) We say a scheme Y is *quasi-separated* if the morphism $Y \to \mathrm{Spec}(\mathbf{Z})$ is quasi-separated.

By Lemmas 25.21.2 and 25.10.4 we see that $\Delta_{X/S}$ is a closed immersion if an only if $\Delta_{X/S}(X) \subset X \times_S X$ is a closed subset. Moreover, by Lemma 25.19.5 we see that a separated morphism is quasi-separated. The reason for introducing quasi-separated morphisms is that nonseparated morphisms come up naturally in studying algebraic varieties (especially when doing moduli, algebraic stacks, etc). But most often they are still quasi-separated.

Example 25.21.4. Here is an example of a non-quasi-separated morphism. Suppose $X = X_1 \cup X_2 \to S = \mathrm{Spec}(k)$ with $X_1 = X_2 = \mathrm{Spec}(k[t_1, t_2, t_3, \ldots])$ glued along the complement of $\{0\} = \{(t_1, t_2, t_3, \ldots)\}$ (glued as in Example 25.14.3). In this case the inverse image of the affine scheme $X_1 \times_S X_2$ under $\Delta_{X/S}$ is the scheme $\mathrm{Spec}(k[t_1, t_2, t_3, \ldots]) \setminus \{0\}$ which is not quasi-compact.

Lemma 25.21.5. *Let X, Y be schemes over S. Let $a, b : X \to Y$ be morphisms of schemes over S. There exists a largest locally closed subscheme $Z \subset X$ such that $a|_Z = b|_Z$. In fact Z is the equalizer of (a, b). Moreover, if Y is separated over S, then Z is a closed subscheme.*

Proof. The equalizer of (a, b) is for categorical reasons the fibre product Z in the following diagram

$$\begin{array}{ccc} Z = Y \times_{(Y \times_S Y)} X & \longrightarrow & X \\ \downarrow & & \downarrow{\scriptstyle (a,b)} \\ Y & \xrightarrow{\Delta_{Y/S}} & Y \times_S Y \end{array}$$

Thus the lemma follows from Lemmas 25.18.2, 25.21.2 and Definition 25.21.3. □

Lemma 25.21.6. *An affine scheme is separated. A morphism of affine schemes is separated.*

Proof. See Lemma 25.21.1. □

Lemma 25.21.7. *Let $f : X \to S$ be a morphism of schemes. The following are equivalent:*

(1) *The morphism f is quasi-separated.*
(2) *For every pair of affine opens $U, V \subset X$ which map into a common affine open of S the intersection $U \cap V$ is a finite union of affine opens of X.*
(3) *There exists an affine open covering $S = \bigcup_{i \in I} U_i$ and for each i an affine open covering $f^{-1}U_i = \bigcup_{j \in I_i} V_j$ such that for each i and each pair $j, j' \in I_i$ the intersection $V_j \cap V_{j'}$ is a finite union of affine opens of X.*

Proof. Let us prove that (3) implies (1). By Lemma 25.17.4 the covering $X \times_S X = \bigcup_i \bigcup_{j,j'} V_j \times_{U_i} V_{j'}$ is an affine open covering of $X \times_S X$. Moreover, $\Delta_{X/S}^{-1}(V_j \times_{U_i} V_{j'}) = V_j \cap V_{j'}$. Hence the implication follows from Lemma 25.19.2.

The implication (1) $\Rightarrow$ (2) follows from the fact that under the hypotheses of (1) the fibre product $U \times_S V$ is an affine open of $X \times_S X$. The implication (2) $\Rightarrow$ (3) is trivial. □

Lemma 25.21.8. *Let $f : X \to S$ be a morphism of schemes.*

(1) *If f is separated then for every pair of affine opens (U, V) of X which map into a common affine open of S we have*
   (a) *the intersection $U \cap V$ is affine.*
   (b) *the ring map $\mathcal{O}_X(U) \otimes_{\mathbf{Z}} \mathcal{O}_X(V) \to \mathcal{O}_X(U \cap V)$ is surjective.*

(2) *If any pair of points $x_1, x_2 \in X$ lying over a common point $s \in S$ are contained in affine opens $x_1 \in U$, $x_2 \in V$ which map into a common affine open of S such that (a), (b) hold, then f is separated.*

Proof. Assume f separated. Suppose (U, V) is a pair as in (1). Let $W = \mathrm{Spec}(R)$ be an affine open of S containing both $f(U)$ and $g(V)$. Write $U = \mathrm{Spec}(A)$ and $V = \mathrm{Spec}(B)$ for R-algebras A and B. By Lemma 25.17.3 we see that $U \times_S V = U \times_W V = \mathrm{Spec}(A \otimes_R B)$ is an affine open of $X \times_S X$. Hence, by Lemma 25.10.1 we see that $\Delta^{-1}(U \times_S V) \to U \times_S V$ can be identified with $\mathrm{Spec}(A \otimes_R B/J)$ for some ideal $J \subset A \otimes_R B$. Thus $U \cap V = \Delta^{-1}(U \times_S V)$ is affine. Assertion (1)(b) holds because $A \otimes_{\mathbf{Z}} B \to (A \otimes_R B)/J$ is surjective.

Assume the hypothesis formulated in (2) holds. Clearly the collection of affine opens $U \times_S V$ for pairs (U, V) as in (2) form an affine open covering of $X \times_S X$ (see e.g. Lemma 25.17.4). Hence it suffices to show that each morphism $U \cap V = \Delta_{X/S}^{-1}(U \times_S V) \to U \times_S V$ is a closed immersion, see Lemma 25.4.2. By assumption (a) we have $U \cap V = \mathrm{Spec}(C)$ for some ring C. After choosing an affine open $W = \mathrm{Spec}(R)$ of S into which both U and V map and writing $U = \mathrm{Spec}(A)$, $V = \mathrm{Spec}(B)$ we see that the assumption (b) means that the composition

$$A \otimes_{\mathbf{Z}} B \to A \otimes_R B \to C$$

is surjective. Hence $A \otimes_R B \to C$ is surjective and we conclude that $\mathrm{Spec}(C) \to \mathrm{Spec}(A \otimes_R B)$ is a closed immersion. □

Example 25.21.9. Let k be a field. Consider the structure morphism $p : \mathbf{P}^1_k \to \mathrm{Spec}(k)$ of the projective line over k, see Example 25.14.4. Let us use the lemma above to prove that p is separated. By construction $\mathbf{P}^1_k$ is covered by two affine opens $U = \mathrm{Spec}(k[x])$ and $V = \mathrm{Spec}(k[y])$ with intersection $U \cap V = \mathrm{Spec}(k[x, y]/(xy-1))$ (using obvious notation). Thus it suffices to check that conditions (2)(a) and (2)(b) of Lemma 25.21.8 hold for the pairs of affine opens (U, U), (U, V), (V, U) and (V, V). For the pairs (U, U) and (V, V) this is trivial. For the pair (U, V) this amounts to proving that $U \cap V$ is affine, which is true, and that the ring map

$$k[x] \otimes_{\mathbf{Z}} k[y] \longrightarrow k[x, y]/(xy - 1)$$

is surjective. This is clear because any element in the right hand side can be written as a sum of a polynomial in x and a polynomial in y.

Lemma 25.21.10. *Let $f : X \to T$ and $g : Y \to T$ be morphisms of schemes with the same target. Let $h : T \to S$ be a morphism of schemes. Then the induced morphism $i : X \times_T Y \to X \times_S Y$ is an immersion. If $T \to S$ is separated, then i is a closed immersion. If $T \to S$ is quasi-separated, then i is a quasi-compact morphism.*

Proof. By general category theory the following diagram

$$\begin{array}{ccc} X \times_T Y & \longrightarrow & X \times_S Y \\ \downarrow & & \downarrow \\ T & \xrightarrow{\Delta_{T/S}} & T \times_S T \end{array}$$

is a fibre product diagram. The lemma follows from Lemmas 25.21.2, 25.17.6 and 25.19.3. □

Lemma 25.21.11. *Let $g : X \to Y$ be a morphism of schemes over S. The morphism $i : X \to X \times_S Y$ is an immersion. If Y is separated over S it is a closed immersion. If Y is quasi-separated over S it is quasi-compact.*

Proof. This is a special case of Lemma 25.21.10 applied to the morphism $X = X \times_Y Y \to X \times_S Y$. □

Lemma 25.21.12. *Let $f : X \to S$ be a morphism of schemes. Let $s : S \to X$ be a section of f (in a formula $f \circ s = id_S$). Then s is an immersion. If f is separated then s is a closed immersion. If f is quasi-separated, then s is quasi-compact.*

Proof. This is a special case of Lemma 25.21.11 applied to $g = s$ so the morphism $i = s : S \to S \times_S X$. □

Lemma 25.21.13. *Permanence properties.*

(1) *A composition of separated morphisms is separated.*
(2) *A composition of quasi-separated morphisms is quasi-separated.*
(3) *The base change of a separated morphism is separated.*
(4) *The base change of a quasi-separated morphism is quasi-separated.*
(5) *A (fibre) product of separated morphisms is separated.*
(6) *A (fibre) product of quasi-separated morphisms is quasi-separated.*

Proof. Let $X \to Y \to Z$ be morphisms. Assume that $X \to Y$ and $Y \to Z$ are separated. The composition

$$X \to X \times_Y X \to X \times_Z X$$

is closed because the first one is by assumption and the second one by Lemma 25.21.10. The same argument works for "quasi-separated" (with the same references).

Let $f : X \to Y$ be a morphism of schemes over a base S. Let $S' \to S$ be a morphism of schemes. Let $f' : X_{S'} \to Y_{S'}$ be the base change of f. Then the diagonal morphism of f' is a morphism

$$\Delta_{f'} : X_{S'} = S' \times_S X \longrightarrow X_{S'} \times_{Y_{S'}} X_{S'} = S' \times_S (X \times_Y X)$$

which is easily seen to be the base change of Δ_f. Thus (3) and (4) follow from the fact that closed immersions and quasi-compact morphisms are preserved under arbitrary base change (Lemmas 25.17.6 and 25.19.3).

If $f : X \to Y$ and $g : U \to V$ are morphisms of schemes over a base S, then $f \times g$ is the composition of $X \times_S U \to X \times_S V$ (a base change of g) and $X \times_S V \to Y \times_S V$ (a base change of f). Hence (5) and (6) follow from (1) – (4). □

Lemma 25.21.14. *Let $f : X \to Y$ and $g : Y \to Z$ be morphisms of schemes. If $g \circ f$ is separated then so is f. If $g \circ f$ is quasi-separated then so is f.*

Proof. Assume that $g \circ f$ is separated. Consider the factorization $X \to X \times_Y X \to X \times_Z X$ of the diagonal morphism of $g \circ f$. By Lemma 25.21.10 the last morphism is an immersion. By assumption the image of X in $X \times_Z X$ is closed. Hence it is also closed in $X \times_Y X$. Thus we see that $X \to X \times_Y X$ is a closed immersion by Lemma 25.10.4.

Assume that $g \circ f$ is quasi-separated. Let $V \subset Y$ be an affine open which maps into an affine open of Z. Let $U_1, U_2 \subset X$ be affine opens which map into V. Then $U_1 \cap U_2$ is a finite union of affine opens because U_1, U_2 map into a common affine open of Z. Since we may cover Y by affine opens like V we deduce the lemma from Lemma 25.21.7. □

Lemma 25.21.15. *Let $f : X \to Y$ and $g : Y \to Z$ be morphisms of schemes. If $g \circ f$ is quasi-compact and g is quasi-separated then f is quasi-compact.*

Proof. This is true because f equals the composition $(1, f) : X \to X \times_Z Y \to Y$. The first map is quasi-compact by Lemma 25.21.12 because it is a section of the quasi-separated morphism $X \times_Z Y \to X$ (a base change of g, see Lemma 25.21.13). The second map is quasi-compact as it is the base change of f, see Lemma 25.19.3. And compositions of quasi-compact morphisms are quasi-compact, see Lemma 25.19.4. □

You may have been wondering whether the condition of only considering pairs of affine opens whose image is contained in an affine open is really necessary to be able to conclude that their intersection is affine. Often it isn't!

Lemma 25.21.16. *Let $f : X \to S$ be a morphism. Assume f is separated and S is a separated scheme. Suppose $U \subset X$ and $V \subset X$ are affine. Then $U \cap V$ is affine (and a closed subscheme of $U \times V$).*

Proof. In this case X is separated by Lemma 25.21.13. Hence $U \cap V$ is affine by applying Lemma 25.21.8 to the morphism $X \to \mathrm{Spec}(\mathbf{Z})$. □

On the other hand, the following example shows that we cannot expect the image of an affine to be contained in an affine.

Example 25.21.17. Consider the nonaffine scheme $U = \mathrm{Spec}(k[x, y]) \setminus \{(x, y)\}$ of Example 25.9.3. On the other hand, consider the scheme

$$\mathbf{GL}_{2,k} = \mathrm{Spec}(k[a, b, c, d, 1/ad - bc]).$$

There is a morphism $\mathbf{GL}_{2,k} \to U$ corresponding to the ring map $x \mapsto a$, $y \mapsto b$. It is easy to see that this is a surjective morphism, and hence the image is not contained in any affine open of U. In fact, the affine scheme $\mathbf{GL}_{2,k}$ also surjects onto $\mathbf{P}^1_k$, and $\mathbf{P}^1_k$ does not even have an immersion into *any* affine scheme.

Remark 25.21.18. The category of quasi-compact and quasi-separated schemes $\mathcal{C}$ has the following properties. If $X, Y \in \mathrm{Ob}(\mathcal{C})$, then any morphism of schemes $f : X \to Y$ is quasi-compact and quasi-separated by Lemmas 25.21.15 and 25.21.14 with $Z = \mathrm{Spec}(\mathbf{Z})$. Moreover, if $X \to Y$ and $Z \to Y$ are morphisms $\mathcal{C}$, then $X \times_Y Z$ is an object of $\mathcal{C}$ too. Namely, the projection $X \times_Y Z \to Z$ is quasi-compact and quasi-separated as a base change of the morphism $Z \to Y$, see Lemmas 25.21.13 and 25.19.3. Hence the composition $X \times_Y Z \to Z \to \mathrm{Spec}(\mathbf{Z})$ is quasi-compact and quasi-separated, see Lemmas 25.21.13 and 25.19.4.

25.22. Valuative criterion of separatedness

Lemma 25.22.1. *Let $f : X \to S$ be a morphism of schemes. If f is separated, then f satisfies the uniqueness part of the valuative criterion.*

Proof. Let a diagram as in Definition 25.20.3 be given. Suppose there are two morphisms $a, b : \mathrm{Spec}(A) \to X$ fitting into the diagram. Let $Z \subset \mathrm{Spec}(A)$ be the equalizer of a and b. By Lemma 25.21.5 this is a closed subscheme of $\mathrm{Spec}(A)$. By assumption it contains the generic point of $\mathrm{Spec}(A)$. Since A is a domain this implies $Z = \mathrm{Spec}(A)$. Hence $a = b$ as desired. □

Lemma 25.22.2 (Valuative criterion separatedness)**.** *Let $f : X \to S$ be a morphism. Assume*

(1) *the morphism f is quasi-separated, and*
(2) *the morphism f satisfies the uniqueness part of the valuative criterion.*

Then f is separated.

Proof. By assumption (1) and Proposition 25.20.6 we see that it suffices to prove the morphism $\Delta_{X/S} : X \to X \times_S X$ satisfies the existence part of the valuative criterion. Let a solid commutative diagram

$$\begin{array}{ccc} \mathrm{Spec}(K) & \longrightarrow & X \\ \downarrow & \nearrow & \downarrow \\ \mathrm{Spec}(A) & \longrightarrow & X \times_S X \end{array}$$

be given. The lower right arrow corresponds to a pair of morphisms $a, b : \mathrm{Spec}(A) \to X$ over S. By (2) we see that $a = b$. Hence using a as the dotted arrow works. □

25.23. Monomorphisms

Definition 25.23.1. A morphism of schemes is called a *monomorphism* if it is a monomorphism in the category of schemes, see Categories, Definition 4.13.1.

Lemma 25.23.2. *Let $j : X \to Y$ be a morphism of schemes. Then j is a monomorphism if and only if the diagonal morphism $\Delta_{X/Y} : X \to X \times_Y X$ is an isomorphism.*

Proof. This is true in any category with fibre products. □

Lemma 25.23.3. *A monomorphism of schemes is separated.*

Proof. This is true because an isomorphism is a closed immersion, and Lemma 25.23.2 above. □

Lemma 25.23.4. *A composition of monomorphisms is a monomorphism.*

Proof. True in any category. □

Lemma 25.23.5. *The base change of a monomorphism is a monomorphism.*

Proof. True in any category with fibre products. □

Lemma 25.23.6. *Let $j : X \to Y$ be a morphism of schemes. If*

(1) *j is injective on points, and*
(2) *for any $x \in X$ the ring map $j_x^\sharp : \mathcal{O}_{Y,j(x)} \to \mathcal{O}_{X,x}$ is surjective,*

then j is a monomorphism.

Proof. Let $a, b : Z \to X$ be two morphisms of schemes such that $j \circ a = j \circ b$. Then (1) implies $a = b$ as underlying maps of topological spaces. For any $z \in Z$ we have $a_z^\sharp \circ j_{a(z)}^\sharp = b_z^\sharp \circ j_{b(z)}^\sharp$ as maps $\mathcal{O}_{Y,j(a(z))} \to \mathcal{O}_{Z,z}$. The surjectivity of the maps $j_x^\sharp$ forces $a_z^\sharp = b_z^\sharp$, $\forall z \in Z$. This implies that $a^\sharp = b^\sharp$. Hence we conclude $a = b$ as morphisms of schemes as desired. □

Lemma 25.23.7. *An immersion of schemes is a monomorphism. In particular, any immersion is separated.*

Proof. We can see this by checking that the criterion of Lemma 25.23.6 applies. More elegantly perhaps, we can use that Lemmas 25.3.5 and 25.4.6 imply that open and closed immersions are monomorphisms and hence any immersion (which is a composition of such) is a monomorphism. □

Lemma 25.23.8. *Let $f : X \to S$ be a separated morphism. Any locally closed subscheme $Z \subset X$ is separated over S.*

Proof. Follows from Lemma 25.23.7 and the fact that a composition of separated morphisms is separated (Lemma 25.21.13). □

Example 25.23.9. The morphism $\mathrm{Spec}(\mathbf{Q}) \to \mathrm{Spec}(\mathbf{Z})$ is a monomorphism. This is true because $\mathbf{Q} \otimes_{\mathbf{Z}} \mathbf{Q} = \mathbf{Q}$. More generally, for any scheme S and any point $s \in S$ the canonical morphism

$$\mathrm{Spec}(\mathcal{O}_{S,s}) \longrightarrow S$$

is a monomorphism.

Lemma 25.23.10. *Let $k_1, \ldots, k_n$ be fields. For any monomorphism of schemes $X \to \mathrm{Spec}(k_1 \times \ldots \times k_n)$ there exists a subset $I \subset \{1, \ldots, n\}$ such that $X \cong \mathrm{Spec}(\prod_{i \in I} k_i)$ as schemes over $\mathrm{Spec}(k_1 \times \ldots \times k_n)$. More generally, if $X = \coprod_{i \in I} \mathrm{Spec}(k_i)$ is a disjoint union of spectra of fields and $Y \to X$ is a monomorphism, then there exists a subset $J \subset I$ such that $Y = \coprod_{i \in J} \mathrm{Spec}(k_i)$.*

Proof. First reduce to the case $n = 1$ (or $\#I = 1$) by taking the inverse images of the open and closed subschemes $\mathrm{Spec}(k_i)$. In this case X has only one point hence is affine. The corresponding algebra problem is this: If $k \to R$ is an algebra map with $R \otimes_k R \cong R$, then $R \cong k$. This holds for dimension reasons. See also Algebra, Lemma 10.102.8 □

25.24. Functoriality for quasi-coherent modules

Let X be a scheme. We denote $QCoh(\mathcal{O}_X)$ the category of quasi-coherent $\mathcal{O}_X$-modules as defined in Modules, Definition 17.10.1. We have seen in Section 25.7 that the category $QCoh(\mathcal{O}_X)$ has a lot of good properties when X is affine. Since the property of being quasi-coherent is local on X, these properties are inherited by the category of quasi-coherent sheaves on any scheme X. We enumerate them here.

(1) A sheaf of $\mathcal{O}_X$-modules $\mathcal{F}$ is quasi-coherent if and only if the restriction of $\mathcal{F}$ to each affine open $U = \mathrm{Spec}(R)$ is of the form $\widetilde{M}$ for some R-module M.
(2) A sheaf of $\mathcal{O}_X$-modules $\mathcal{F}$ is quasi-coherent if and only if the restriction of $\mathcal{F}$ to each of the members of an affine open covering is quasi-coherent.
(3) Any direct sum of quasi-coherent sheaves is quasi-coherent.
(4) Any colimit of quasi-coherent sheaves is quasi-coherent.
(5) The kernel and cokernel of a morphism of quasi-coherent sheaves is quasi-coherent.
(6) Given a short exact sequence of $\mathcal{O}_X$-modules $0 \to \mathcal{F}_1 \to \mathcal{F}_2 \to \mathcal{F}_3 \to 0$ if two out of three are quasi-coherent so is the third.
(7) Given a morphism of schemes $f : Y \to X$ the pullback of a quasi-coherent $\mathcal{O}_X$-module is a quasi-coherent $\mathcal{O}_Y$-module. See Modules, Lemma 17.10.4.
(8) Given two quasi-coherent $\mathcal{O}_X$-modules the tensor product is quasi-coherent, see Modules, Lemma 17.15.5.
(9) Given a quasi-coherent $\mathcal{O}_X$-module $\mathcal{F}$ the tensor, symmetric and exterior algebras on $\mathcal{F}$ are quasi-coherent, see Modules, Lemma 17.18.6.
(10) Given two quasi-coherent $\mathcal{O}_X$-modules $\mathcal{F}, \mathcal{G}$ such that $\mathcal{F}$ is of finite presentation, then the internal hom $\mathcal{H}om_{\mathcal{O}_X}(\mathcal{F}, \mathcal{G})$ is quasi-coherent, see Modules, Lemma 17.19.4 and (5) above.

On the other hand, it is in general not the case that the pushforward of a quasi-coherent module is quasi-coherent. Here is a case where it this does hold.

Lemma 25.24.1. *Let $f : X \to S$ be a morphism of schemes. If f is quasi-compact and quasi-separated then f_* transforms quasi-coherent $\mathcal{O}_X$-modules into quasi-coherent $\mathcal{O}_S$-modules.*

Proof. The question is local on S and hence we may assume that S is affine. Because X is quasi-compact we may write $X = \bigcup_{i=1}^n U_i$ with each U_i open affine. Because f is quasi-separated we may write $U_i \cap U_j = \bigcup_{k=1}^{n_{ij}} U_{ijk}$ for some affine open U_{ijk}, see Lemma 25.21.7. Denote $f_i : U_i \to S$ and $f_{ijk} : U_{ijk} \to S$ the restrictions of f. For any open V of S and any sheaf $\mathcal{F}$ on X we have

$$\begin{aligned} f_*\mathcal{F}(V) &= \mathcal{F}(f^{-1}V) \\ &= \mathrm{Ker}\left(\bigoplus\nolimits_i \mathcal{F}(f^{-1}V \cap U_i) \to \bigoplus\nolimits_{i,j,k} \mathcal{F}(f^{-1}V \cap U_{ijk})\right) \\ &= \mathrm{Ker}\left(\bigoplus\nolimits_i f_{i,*}(\mathcal{F}|_{U_i})(V) \to \bigoplus\nolimits_{i,j,k} f_{ijk,*}(\mathcal{F}|_{U_{ijk}})\right)(V) \\ &= \mathrm{Ker}\left(\bigoplus\nolimits_i f_{i,*}(\mathcal{F}|_{U_i}) \to \bigoplus\nolimits_{i,j,k} f_{ijk,*}(\mathcal{F}|_{U_{ijk}})\right)(V) \end{aligned}$$

In other words there is a short exact sequence of sheaves

$$0 \to f_*\mathcal{F} \to \bigoplus f_{i,*}\mathcal{F}_i \to \bigoplus f_{ijk,*}\mathcal{F}_{ijk}$$

where $\mathcal{F}_i, \mathcal{F}_{ijk}$ denotes the restriction of $\mathcal{F}$ to the corresponding open. If $\mathcal{F}$ is a quasi-coherent $\mathcal{O}_X$-modules then $\mathcal{F}_i$, $\mathcal{F}_{ijk}$ is a quasi-coherent $\mathcal{O}_{U_i}$, $\mathcal{O}_{U_{ijk}}$-module. Hence by Lemma 25.7.3 we see that the second and third term of the exact sequence are quasi-coherent $\mathcal{O}_S$-modules. Thus we conclude that $f_*\mathcal{F}$ is a quasi-coherent $\mathcal{O}_S$-module. □

Using this we can characterize (closed) immersions of schemes as follows.

Lemma 25.24.2. *Let $f : X \to Y$ be a morphism of schemes. Suppose that*

(1) *f induces a homeomorphism of X with a closed subset of Y, and*
(2) *$f^\sharp : \mathcal{O}_Y \to f_*\mathcal{O}_X$ is surjective.*

Then f is a closed immersion of schemes.

Proof. Assume (1) and (2). By (1) the morphism f is quasi-compact (see Topology, Lemma 5.11.3). Conditions (1) and (2) imply conditions (1) and (2) of Lemma 25.23.6. Hence $f : X \to Y$ is a monomorphism. In particular, f is separated, see Lemma 25.23.3. Hence Lemma 25.24.1 above applies and we conclude that $f_*\mathcal{O}_X$ is a quasi-coherent $\mathcal{O}_Y$-module. Therefore the kernel of $\mathcal{O}_Y \to f_*\mathcal{O}_X$ is quasi-coherent by Lemma 25.7.8. Since a quasi-coherent sheaf is locally generated by sections (see Modules, Definition 17.10.1) this implies that f is a closed immersion, see Definition 25.4.1. □

We can use this lemma to prove the following lemma.

Lemma 25.24.3. *A composition of immersions of schemes is an immersion, a composition of closed immersions of schemes is a closed immersion, and a composition of open immersions of schemes is an open immersion.*

Proof. This is clear for the case of open immersions since an open subspace of an open subspace is also an open subspace.

Suppose $a : Z \to Y$ and $b : Y \to X$ are closed immersions of schemes. We will verify that $c = b \circ a$ is also a closed immersion. The assumption implies that a and b are homeomorphisms onto closed subsets, and hence also $c = b \circ a$ is a homeomorphism onto a closed subset. Moreover, the map $\mathcal{O}_X \to c_*\mathcal{O}_Z$ is surjective since it factors as the composition of the surjective maps $\mathcal{O}_X \to b_*\mathcal{O}_Y$ and $b_*\mathcal{O}_Y \to b_*a_*\mathcal{O}_Z$ (surjective as b_* is exact, see Modules, Lemma 17.6.1). Hence by Lemma 25.24.2 above c is a closed immersion.

Finally, we come to the case of immersions. Suppose $a : Z \to Y$ and $b : Y \to X$ are immersions of schemes. This means there exist open subschemes $V \subset Y$ and $U \subset X$ such that $a(Z) \subset V$, $b(Y) \subset U$ and $a : Z \to V$ and $b : Y \to U$ are closed immersions. Since the topology on Y is induced from the topology on U we can find an open $U' \subset U$ such that $V = b^{-1}(U')$. Then we see that $Z \to V = b^{-1}(U') \to U'$ is a composition of closed immersions and hence a closed immersion. This proves that $Z \to X$ is an immersion and we win. □

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CHAPTER 26

Constructions of Schemes

26.1. Introduction

In this chapter we introduce ways of constructing schemes out of others. A basic reference is [**DG67**].

26.2. Relative glueing

The following lemma is relevant in case we are trying to construct a scheme X over S, and we already know how to construct the restriction of X to the affine opens of S. The actual result is completely general and works in the setting of (locally) ringed spaces, although our proof is written in the language of schemes.

Lemma 26.2.1. *Let S be a scheme. Let $\mathcal{B}$ be a basis for the topology of S. Suppose given the following data:*

(1) *For every $U \in \mathcal{B}$ a scheme $f_U : X_U \to U$ over U.*
(2) *For every pair $U, V \in \mathcal{B}$ such that $V \subset U$ a morphism $\rho_V^U : X_V \to X_U$.*

Assume that

(a) *each ρ_V^U induces an isomorphism $X_V \to f_U^{-1}(V)$ of schemes over V,*
(b) *whenever $W, V, U \in \mathcal{B}$, with $W \subset V \subset U$ we have $\rho_W^U = \rho_V^U \circ \rho_W^V$.*

Then there exists a unique scheme $f : X \to S$ over S and isomorphisms $i_U : f^{-1}(U) \to X_U$ over U such that for $V \subset U \subset S$ affine open the composition

$$X_V \xrightarrow{i_V^{-1}} f^{-1}(V) \xrightarrow{inclusion} f^{-1}(U) \xrightarrow{i_U} X_U$$

is the morphism ρ_V^U.

Proof. To prove this we will use Schemes, Lemma 25.15.4. First we define a contravariant functor F from the category of schemes to the category of sets. Namely, for a scheme T we set

$$F(T) = \left\{ \begin{array}{c} (g, \{h_U\}_{U \in \mathcal{B}}),\ g : T \to S,\ h_U : g^{-1}(U) \to X_U, \\ f_U \circ h_U = g|_{g^{-1}(U)},\ h_U|_{g^{-1}(V)} = \rho_V^U \circ h_V\ \forall\ V, U \in \mathcal{B}, V \subset U \end{array} \right\}.$$

The restriction mapping $F(T) \to F(T')$ given a morphism $T' \to T$ is just gotten by composition. For any $W \in \mathcal{B}$ we consider the subfunctor $F_W \subset F$ consisting of those systems $(g, \{h_U\})$ such that $g(T) \subset W$.

First we show F satisfies the sheaf property for the Zariski topology. Suppose that T is a scheme, $T = \bigcup V_i$ is an open covering, and $\xi_i \in F(V_i)$ is an element such that $\xi_i|_{V_i \cap V_j} = \xi_j|_{V_i \cap V_j}$. Say $\xi_i = (g_i, \{f_{i,U}\})$. Then we immediately see that the morphisms g_i glue to a unique global morphism $g : T \to S$. Moreover, it is clear that $g^{-1}(U) = \bigcup g_i^{-1}(U)$. Hence the morphisms $h_{i,U} : g_i^{-1}(U) \to X_U$ glue to a

unique morphism $h_U : U \to X_U$. It is easy to verify that the system $(g, \{f_U\})$ is an element of $F(T)$. Hence F satisfies the sheaf property for the Zariski topology.

Next we verify that each F_W, $W \in \mathcal{B}$ is representable. Namely, we claim that the transformation of functors

$$F_W \longrightarrow \mathrm{Mor}(-, X_W),\ (g, \{h_U\}) \longmapsto h_W$$

is an isomorphism. To see this suppose that T is a scheme and $\alpha : T \to X_W$ is a morphism. Set $g = f_W \circ \alpha$. For any $U \in \mathcal{B}$ such that $U \subset W$ we can define $h_U : g^{-1}(U) \to X_U$ be the composition $(\rho_U^W)^{-1} \circ \alpha|_{g^{-1}(U)}$. This works because the image $\alpha(g^{-1}(U))$ is contained in $f_W^{-1}(U)$ and condition (a) of the lemma. It is clear that $f_U \circ h_U = g|_{g^{-1}(U)}$ for such a U. Moreover, if also $V \in \mathcal{B}$ and $V \subset U \subset W$, then $\rho_V^U \circ h_V = h_U|_{g^{-1}(V)}$ by property (b) of the lemma. We still have to define h_U for an arbitrary element $U \in \mathcal{B}$. Since $\mathcal{B}$ is a basis for the topology on S we can find an open covering $U \cap W = \bigcup U_i$ with $U_i \in \mathcal{B}$. Since g maps into W we have $g^{-1}(U) = g^{-1}(U \cap W) = \bigcup g^{-1}(U_i)$. Consider the morphisms $h_i = \rho_{U_i}^U \circ h_{U_i} : g^{-1}(U_i) \to X_U$. It is a simple matter to use condition (b) of the lemma to prove that $h_i|_{g^{-1}(U_i) \cap g^{-1}(U_j)} = h_j|_{g^{-1}(U_i) \cap g^{-1}(U_j)}$. Hence these morphisms glue to give the desired morphism $h_U : g^{-1}(U) \to X_U$. We omit the (easy) verification that the system $(g, \{h_U\})$ is an element of $F_W(T)$ which maps to α under the displayed arrow above.

Next, we verify each $F_W \subset F$ is representable by open immersions. This is clear from the definitions.

Finally we have to verify the collection $(F_W)_{W \in \mathcal{B}}$ covers F. This is clear by construction and the fact that $\mathcal{B}$ is a basis for the topology of S.

Let X be a scheme representing the functor F. Let $(f, \{i_U\}) \in F(X)$ be a "universal family". Since each F_W is representable by X_W (via the morphism of functors displayed above) we see that $i_W : f^{-1}(W) \to X_W$ is an isomorphism as desired. The lemma is proved. □

Lemma 26.2.2. *Let S be a scheme. Let $\mathcal{B}$ be a basis for the topology of S. Suppose given the following data:*

1. *For every $U \in \mathcal{B}$ a scheme $f_U : X_U \to U$ over U.*
2. *For every $U \in \mathcal{B}$ a quasi-coherent sheaf $\mathcal{F}_U$ over X_U.*
3. *For every pair $U, V \in \mathcal{B}$ such that $V \subset U$ a morphism $\rho_V^U : X_V \to X_U$.*
4. *For every pair $U, V \in \mathcal{B}$ such that $V \subset U$ a morphism $\theta_V^U : (\rho_V^U)^* \mathcal{F}_U \to \mathcal{F}_V$.*

Assume that

(a) *each ρ_V^U induces an isomorphism $X_V \to f_U^{-1}(V)$ of schemes over V,*
(b) *each θ_V^U is an isomorphism,*
(c) *whenever $W, V, U \in \mathcal{B}$, with $W \subset V \subset U$ we have $\rho_W^U = \rho_V^U \circ \rho_W^V$,*
(d) *whenever $W, V, U \in \mathcal{B}$, with $W \subset V \subset U$ we have $\theta_W^U = \theta_W^V \circ (\rho_W^V)^* \theta_V^U$.*

Then there exists a unique scheme $f : X \to S$ over S together with a unique quasi-coherent sheaf $\mathcal{F}$ on X and isomorphisms $i_U : f^{-1}(U) \to X_U$ and $\theta_U : i_U^* \mathcal{F}_U \to \mathcal{F}|_{f^{-1}(U)}$ over U such that for $V \subset U \subset S$ affine open the composition

$$X_V \xrightarrow{i_V^{-1}} f^{-1}(V) \xrightarrow{inclusion} f^{-1}(U) \xrightarrow{i_U} X_U$$

is the morphism ρ_V^U, and the composition

$$(\rho_V^U)^*\mathcal{F}_U = (i_V^{-1})^*((i_U^*\mathcal{F}_U)|_{f^{-1}(V)}) \xrightarrow{\theta_U|_{f^{-1}(V)}} (i_V^{-1})^*(\mathcal{F}|_{f^{-1}(V)}) \xrightarrow{\theta_V^{-1}} \mathcal{F}_V \tag{26.2.2.1}$$

is equal to θ_V^U.

Proof. By Lemma 26.2.1 we get the scheme X over S and the isomorphisms i_U. Set $\mathcal{F}'_U = i_U^*\mathcal{F}_U$ for $U \in \mathcal{B}$. This is a quasi-coherent $\mathcal{O}_{f^{-1}(U)}$-module. The maps

$$\mathcal{F}'_U|_{f^{-1}(V)} = i_U^*\mathcal{F}_U|_{f^{-1}(V)} = i_V^*(\rho_V^U)^*\mathcal{F}_U \xrightarrow{i_V^*\theta_V^U} i_V^*\mathcal{F}_V = \mathcal{F}'_V$$

define isomorphisms $(\theta')_V^U : \mathcal{F}'_U|_{f^{-1}(V)} \to \mathcal{F}'_V$ whenever $V \subset U$ are elements of $\mathcal{B}$. Condition (d) says exactly that this is compatible in case we have a triple of elements $W \subset V \subset U$ of $\mathcal{B}$. This allows us to get well defined isomorphisms

$$\varphi_{12} : \mathcal{F}'_{U_1}|_{f^{-1}(U_1 \cap U_2)} \longrightarrow \mathcal{F}'_{U_2}|_{f^{-1}(U_1 \cap U_2)}$$

whenever $U_1, U_2 \in \mathcal{B}$ by covering the intersection $U_1 \cap U_2 = \bigcup V_j$ by elements V_j of $\mathcal{B}$ and taking

$$\varphi_{12}|_{V_j} = \left((\theta')_{V_j}^{U_2}\right)^{-1} \circ (\theta')_{V_j}^{U_1}.$$

We omit the verification that these maps do indeed glue to a φ_{12} and we omit the verification of the cocycle condition of a glueing datum for sheaves (as in Sheaves, Section 6.33). By Sheaves, Lemma 6.33.2 we get our $\mathcal{F}$ on X. We omit the verification of (26.2.2.1). □

Remark 26.2.3. There is a functoriality property for the constructions explained in Lemmas 26.2.1 and 26.2.2. Namely, suppose given two collections of data $(f_U : X_U \to U, \rho_V^U)$ and $(g_U : Y_U \to U, \sigma_V^U)$ as in Lemma 26.2.1. Suppose for every $U \in \mathcal{B}$ given a morphism $h_U : X_U \to Y_U$ over U compatible with the restrictions ρ_V^U and σ_V^U. Functoriality means that this gives rise to a morphism of schemes $h : X \to Y$ over S restricting back to the morphisms h_U, where $f : X \to S$ is obtained from the datum $(f_U : X_U \to U, \rho_V^U)$ and $g : Y \to S$ is obtained from the datum $(g_U : Y_U \to U, \sigma_V^U)$.

Similarly, suppose given two collections of data $(f_U : X_U \to U, \mathcal{F}_U, \rho_V^U, \theta_V^U)$ and $(g_U : Y_U \to U, \mathcal{G}_U, \sigma_V^U, \eta_V^U)$ as in Lemma 26.2.2. Suppose for every $U \in \mathcal{B}$ given a morphism $h_U : X_U \to Y_U$ over U compatible with the restrictions ρ_V^U and σ_V^U, and a morphism $\tau_U : h_U^*\mathcal{G}_U \to \mathcal{F}_U$ compatible with the maps θ_V^U and η_V^U. Functoriality means that these give rise to a morphism of schemes $h : X \to Y$ over S restricting back to the morphisms h_U, and a morphism $h^*\mathcal{G} \to \mathcal{F}$ restricting back to the maps h_U where $(f : X \to S, \mathcal{F})$ is obtained from the datum $(f_U : X_U \to U, \mathcal{F}_U, \rho_V^U, \theta_V^U)$ and where $(g : Y \to S, \mathcal{G})$ is obtained from the datum $(g_U : Y_U \to U, \mathcal{G}_U, \sigma_V^U, \eta_V^U)$.

We omit the verifications and we omit a suitable formulation of "equivalence of categories" between relative glueing data and relative objects.

26.3. Relative spectrum via glueing

Situation 26.3.1. Here S is a scheme, and $\mathcal{A}$ is a quasi-coherent $\mathcal{O}_S$-algebra.

In this section we outline how to construct a morphism of schemes

$$\underline{\mathrm{Spec}}_S(\mathcal{A}) \longrightarrow S$$

by glueing the spectra $\mathrm{Spec}(\Gamma(U, \mathcal{A}))$ where U ranges over the affine opens of S. We first show that the spectra of the values of $\mathcal{A}$ over affines form a suitable collection of schemes, as in Lemma 26.2.1.

Lemma 26.3.2. *In Situation 26.3.1. Suppose $U \subset U' \subset S$ are affine opens. Let $A = \mathcal{A}(U)$ and $A' = \mathcal{A}(U')$. The map of rings $A' \to A$ induces a morphism $\mathrm{Spec}(A) \to \mathrm{Spec}(A')$, and the diagram*

$$\begin{array}{ccc} \mathrm{Spec}(A) & \longrightarrow & \mathrm{Spec}(A') \\ \downarrow & & \downarrow \\ U & \longrightarrow & U' \end{array}$$

is cartesian.

Proof. Let $R = \mathcal{O}_S(U)$ and $R' = \mathcal{O}_S(U')$. Note that the map $R \otimes_{R'} A' \to A$ is an isomorphism as $\mathcal{A}$ is quasi-coherent (see Schemes, Lemma 25.7.3 for example). The result follows from the description of the fibre product of affine schemes in Schemes, Lemma 25.6.7. □

In particular the morphism $\mathrm{Spec}(A) \to \mathrm{Spec}(A')$ of the lemma is an open immersion.

Lemma 26.3.3. *In Situation 26.3.1. Suppose $U \subset U' \subset U'' \subset S$ are affine opens. Let $A = \mathcal{A}(U)$, $A' = \mathcal{A}(U')$ and $A'' = \mathcal{A}(U'')$. The composition of the morphisms $\mathrm{Spec}(A) \to \mathrm{Spec}(A')$, and $\mathrm{Spec}(A') \to \mathrm{Spec}(A'')$ of Lemma 26.3.2 gives the morphism $\mathrm{Spec}(A) \to \mathrm{Spec}(A'')$ of Lemma 26.3.2.*

Proof. This follows as the map $A'' \to A$ is the composition of $A'' \to A'$ and $A' \to A$ (because $\mathcal{A}$ is a sheaf). □

Lemma 26.3.4. *In Situation 26.3.1. There exists a morphism of schemes*

$$\pi : \underline{\mathrm{Spec}}_S(\mathcal{A}) \longrightarrow S$$

with the following properties:

1. *for every affine open $U \subset S$ there exists an isomorphism $i_U : \pi^{-1}(U) \to \mathrm{Spec}(\mathcal{A}(U))$, and*
2. *for $U \subset U' \subset S$ affine open the composition*

$$\mathrm{Spec}(\mathcal{A}(U)) \xrightarrow{i_U^{-1}} \pi^{-1}(U) \xrightarrow{inclusion} \pi^{-1}(U') \xrightarrow{i_{U'}} \mathrm{Spec}(\mathcal{A}(U'))$$

is the open immersion of Lemma 26.3.2 above.

Proof. Follows immediately from Lemmas 26.2.1, 26.3.2, and 26.3.3. □

26.4. Relative spectrum as a functor

We place ourselves in Situation 26.3.1. So S is a scheme and $\mathcal{A}$ is a quasi-coherent sheaf of $\mathcal{O}_S$-algebras. (This means that $\mathcal{A}$ is a sheaf of $\mathcal{O}_S$-algebras which is quasi-coherent as an $\mathcal{O}_S$-module.)

For any $f : T \to S$ the pullback $f^*\mathcal{A}$ is a quasi-coherent sheaf of $\mathcal{O}_T$-algebras. We are going to consider pairs $(f : T \to S, \varphi)$ where f is a morphism of schemes and $\varphi : f^*\mathcal{A} \to \mathcal{O}_T$ is a morphism of $\mathcal{O}_T$-algebras. Note that this is the same as giving a $f^{-1}\mathcal{O}_S$-algebra homomorphism $\varphi : f^{-1}\mathcal{A} \to \mathcal{O}_T$, see Sheaves, Lemma 6.20.2. This

is also the same as giving a $\mathcal{O}_S$-algebra map $\varphi : \mathcal{A} \to f_*\mathcal{O}_T$, see Sheaves, Lemma 6.24.7. We will use all three ways of thinking about φ, without further mention.

Given such a pair $(f : T \to S, \varphi)$ and a morphism $a : T' \to T$ we get a second pair $(f' = f \circ a, \varphi' = a^*\varphi)$ which we call the pullback of (f, φ). One way to describe $\varphi' = a^*\varphi$ is as the composition $\mathcal{A} \to f_*\mathcal{O}_T \to f'_*\mathcal{O}_{T'}$ where the second map is $f_*a^\sharp$ with $a^\sharp : \mathcal{O}_T \to a_*\mathcal{O}_{T'}$. In this way we have defined a functor

$$\begin{aligned} F : Sch^{opp} &\longrightarrow Sets \\ T &\longmapsto F(T) = \{\text{pairs } (f, \varphi) \text{ as above}\} \end{aligned} \tag{26.4.0.1}$$

Lemma 26.4.1. *In Situation 26.3.1. Let F be the functor associated to $(S, \mathcal{A})$ above. Let $g : S' \to S$ be a morphism of schemes. Set $\mathcal{A}' = g^*\mathcal{A}$. Let F' be the functor associated to $(S', \mathcal{A}')$ above. Then there is a canonical isomorphism*

$$F' \cong h_{S'} \times_{h_S} F$$

of functors.

Proof. A pair $(f' : T \to S', \varphi' : (f')^*\mathcal{A}' \to \mathcal{O}_T)$ is the same as a pair $(f, \varphi : f^*\mathcal{A} \to \mathcal{O}_T)$ together with a factorization of f as $f = g \circ f'$. Namely with this notation we have $(f')^*\mathcal{A}' = (f')^*g^*\mathcal{A} = f^*\mathcal{A}$. Hence the lemma. □

Lemma 26.4.2. *In Situation 26.3.1. Let F be the functor associated to $(S, \mathcal{A})$ above. If S is affine, then F is representable by the affine scheme* $\mathrm{Spec}(\Gamma(S, \mathcal{A}))$.

Proof. Write $S = \mathrm{Spec}(R)$ and $A = \Gamma(S, \mathcal{A})$. Then A is an R-algebra and $\mathcal{A} = \widetilde{A}$. The ring map $R \to A$ gives rise to a canonical map

$$f_{univ} : \mathrm{Spec}(A) \longrightarrow S = \mathrm{Spec}(R).$$

We have $f_{univ}^*\mathcal{A} = \widetilde{A \otimes_R A}$ by Schemes, Lemma 25.7.3. Hence there is a canonical map

$$\varphi_{univ} : f_{univ}^*\mathcal{A} = \widetilde{A \otimes_R A} \longrightarrow \widetilde{A} = \mathcal{O}_{\mathrm{Spec}(A)}$$

coming from the A-module map $A \otimes_R A \to A$, $a \otimes a' \mapsto aa'$. We claim that the pair $(f_{univ}, \varphi_{univ})$ represents F in this case. In other words we claim that for any scheme T the map

$$\mathrm{Mor}(T, \mathrm{Spec}(A)) \longrightarrow \{\text{pairs } (f, \varphi)\}, \quad a \longmapsto (a^*f_{univ}, a^*\varphi)$$

is bijective.

Let us construct the inverse map. For any pair $(f : T \to S, \varphi)$ we get the induced ring map

$$A = \Gamma(S, \mathcal{A}) \xrightarrow{f^*} \Gamma(T, f^*\mathcal{A}) \xrightarrow{\varphi} \Gamma(T, \mathcal{O}_T)$$

This induces a morphism of schemes $T \to \mathrm{Spec}(A)$ by Schemes, Lemma 25.6.4.

The verification that this map is inverse to the map displayed above is omitted. □

Lemma 26.4.3. *In Situation 26.3.1. The functor F is representable by a scheme.*

Proof. We are going to use Schemes, Lemma 25.15.4.

First we check that F satisfies the sheaf property for the Zariski topology. Namely, suppose that T is a scheme, that $T = \bigcup_{i \in I} U_i$ is an open covering, and that $(f_i, \varphi_i) \in F(U_i)$ such that $(f_i, \varphi_i)|_{U_i \cap U_j} = (f_j, \varphi_j)|_{U_i \cap U_j}$. This implies that the morphisms $f_i : U_i \to S$ glue to a morphism of schemes $f : T \to S$ such that $f|_{I_i} = f_i$, see

Schemes, Section 25.14. Thus $f_i^*\mathcal{A} = f^*\mathcal{A}|_{U_i}$ and by assumption the morphisms φ_i agree on $U_i \cap U_j$. Hence by Sheaves, Section 6.33 these glue to a morphism of $\mathcal{O}_T$-algebras $f^*\mathcal{A} \to \mathcal{O}_T$. This proves that F satisfies the sheaf condition with respect to the Zariski topology.

Let $S = \bigcup_{i\in I} U_i$ be an affine open covering. Let $F_i \subset F$ be the subfunctor consisting of those pairs $(f : T \to S, \varphi)$ such that $f(T) \subset U_i$.

We have to show each F_i is representable. This is the case because F_i is identified with the functor associated to U_i equipped with the quasi-coherent $\mathcal{O}_{U_i}$-algebra $\mathcal{A}|_{U_i}$, by Lemma 26.4.1. Thus the result follows from Lemma 26.4.2.

Next we show that $F_i \subset F$ is representable by open immersions. Let $(f : T \to S, \varphi) \in F(T)$. Consider $V_i = f^{-1}(U_i)$. It follows from the definition of F_i that given $a : T' \to T$ we gave $a^*(f, \varphi) \in F_i(T')$ if and only if $a(T') \subset V_i$. This is what we were required to show.

Finally, we have to show that the collection $(F_i)_{i\in I}$ covers F. Let $(f : T \to S, \varphi) \in F(T)$. Consider $V_i = f^{-1}(U_i)$. Since $S = \bigcup_{i\in I} U_i$ is an open covering of S we see that $T = \bigcup_{i\in I} V_i$ is an open covering of T. Moreover $(f, \varphi)|_{V_i} \in F_i(V_i)$. This finishes the proof of the lemma. □

Lemma 26.4.4. *In Situation 26.3.1. The scheme $\pi : \underline{\mathrm{Spec}}_S(\mathcal{A}) \to S$ constructed in Lemma 26.3.4 and the scheme representing the functor F are canonically isomorphic as schemes over S.*

Proof. Let $X \to S$ be the scheme representing the functor F. Consider the sheaf of $\mathcal{O}_S$-algebras $\mathcal{R} = \pi_*\mathcal{O}_{\underline{\mathrm{Spec}}_S(\mathcal{A})}$. By construction of $\underline{\mathrm{Spec}}_S(\mathcal{A})$ we have isomorphisms $\mathcal{A}(U) \to \mathcal{R}(U)$ for every affine open $U \subset S$; this follows from Lemma 26.3.4 part (1). For $U \subset U' \subset S$ open these isomorphisms are compatible with the restriction mappings; this follows from Lemma 26.3.4 part (2). Hence by Sheaves, Lemma 6.30.13 these isomorphisms result from an isomorphism of $\mathcal{O}_S$-algebras $\varphi : \mathcal{A} \to \mathcal{R}$. Hence this gives an element $(\underline{\mathrm{Spec}}_S(\mathcal{A}), \varphi) \in F(\underline{\mathrm{Spec}}_S(\mathcal{A}))$. Since X represents the functor F we get a corresponding morphism of schemes $can : \underline{\mathrm{Spec}}_S(\mathcal{A}) \to X$ over S.

Let $U \subset S$ be any affine open. Let $F_U \subset F$ be the subfunctor of F corresponding to pairs (f, φ) over schemes T with $f(T) \subset U$. Clearly the base change X_U represents F_U. Moreover, F_U is represented by $\mathrm{Spec}(\mathcal{A}(U)) = \pi^{-1}(U)$ according to Lemma 26.4.2. In other words $X_U \cong \pi^{-1}(U)$. We omit the verification that this identification is brought about by the base change of the morphism can to U. □

Definition 26.4.5. Let S be a scheme. Let $\mathcal{A}$ be a quasi-coherent sheaf of $\mathcal{O}_S$-algebras. The *relative spectrum of $\mathcal{A}$ over S*, or simply the *spectrum of $\mathcal{A}$ over S* is the scheme constructed in Lemma 26.3.4 which represents the functor F (26.4.0.1), see Lemma 26.4.4. We denote it $\pi : \underline{\mathrm{Spec}}_S(\mathcal{A}) \to S$. The "universal family" is a morphism of $\mathcal{O}_S$-algebras

$$\mathcal{A} \longrightarrow \pi_*\mathcal{O}_{\underline{\mathrm{Spec}}_S(\mathcal{A})}$$

The following lemma says among other things that forming the relative spectrum commutes with base change.

Lemma 26.4.6. *Let S be a scheme. Let $\mathcal{A}$ be a quasi-coherent sheaf of $\mathcal{O}_S$-algebras. Let $\pi : \underline{\mathrm{Spec}}_S(\mathcal{A}) \to S$ be the relative spectrum of $\mathcal{A}$ over S.*

(1) *For every affine open $U \subset S$ the inverse image $\pi^{-1}(U)$ is affine.*
(2) *For every morphism $g : S' \to S$ we have $S' \times_S \underline{\mathrm{Spec}}_S(\mathcal{A}) = \underline{\mathrm{Spec}}_{S'}(g^*\mathcal{A})$.*
(3) *The universal map*

$$\mathcal{A} \longrightarrow \pi_*\mathcal{O}_{\underline{\mathrm{Spec}}_S(\mathcal{A})}$$

is an isomorphism of $\mathcal{O}_S$-algebras.

Proof. Part (1) comes from the description of the relative spectrum by glueing, see Lemma 26.3.4. Part (2) follows immediately from Lemma 26.4.1. Part (3) follows because it is local on S and it is clear in case S is affine by Lemma 26.4.2 for example. □

Lemma 26.4.7. *Let $f : X \to S$ be a quasi-compact and quasi-separated morphism of schemes. By Schemes, Lemma 25.24.1 the sheaf $f_*\mathcal{O}_X$ is a quasi-coherent sheaf of $\mathcal{O}_S$-algebras. There is a canonical morphism*

$$can : X \longrightarrow \underline{\mathrm{Spec}}_S(f_*\mathcal{O}_X)$$

of schemes over S. For any affine open $U \subset S$ the restriction $can|_{f^{-1}(U)}$ is identified with the canonical morphism

$$f^{-1}(U) \longrightarrow \mathrm{Spec}(\Gamma(f^{-1}(U), \mathcal{O}_X))$$

coming from Schemes, Lemma 25.6.4.

Proof. The morphism comes, via the definition of $\underline{\mathrm{Spec}}$ as the scheme representing the functor F, from the canonical map $\varphi : f^*f_*\mathcal{O}_X \to \mathcal{O}_X$ (which by adjointness of push and pull corresponds to id : $f_*\mathcal{O}_X \to f_*\mathcal{O}_X$). The statement on the restriction to $f^{-1}(U)$ follows from the description of the relative spectrum over affines, see Lemma 26.4.2. □

26.5. Affine n-space

As an application of the relative spectrum we define affine n-space over a base scheme S as follows. For any integer $n \geq 0$ we can consider the quasi-coherent sheaf of $\mathcal{O}_S$-algebras $\mathcal{O}_S[T_1, \ldots, T_n]$. It is quasi-coherent because as a sheaf of $\mathcal{O}_S$-modules it is just the direct sum of copies of $\mathcal{O}_S$ indexed by multi-indices.

Definition 26.5.1. Let S be a scheme and $n \geq 0$. The scheme

$$\mathbf{A}^n_S = \underline{\mathrm{Spec}}_S(\mathcal{O}_S[T_1, \ldots, T_n])$$

over S is called *affine n-space over S*. If $S = \mathrm{Spec}(R)$ is affine then we also call this *affine n-space over R* and we denote it $\mathbf{A}^n_R$.

Note that $\mathbf{A}^n_R = \mathrm{Spec}(R[T_1, \ldots, T_n])$. For any morphism $g : S' \to S$ of schemes we have $g^*\mathcal{O}_S[T_1, \ldots, T_n] = \mathcal{O}_{S'}[T_1, \ldots, T_n]$ and hence $\mathbf{A}^n_{S'} = S' \times_S \mathbf{A}^n_S$ is the base change. Therefore an alternative definition of affine n-space is the formula

$$\mathbf{A}^n_S = S \times_{\mathrm{Spec}(\mathbf{Z})} \mathbf{A}^n_{\mathbf{Z}}.$$

Also, a morphism from an S-scheme $f : X \to S$ to $\mathbf{A}^n_S$ is given by a homomorphism of $\mathcal{O}_S$-algebras $\mathcal{O}_S[T_1, \ldots, T_n] \to f_*\mathcal{O}_X$. This is clearly the same thing as giving the images of the T_i. In other words, a morphism from X to $\mathbf{A}^n_S$ over S is the same as giving n elements $h_1, \ldots, h_n \in \Gamma(X, \mathcal{O}_X)$.

26.6. Vector bundles

Let S be a scheme. Let $\mathcal{E}$ be a quasi-coherent sheaf of $\mathcal{O}_S$-modules. By Modules, Lemma 17.18.6 the symmetric algebra $\text{Sym}(\mathcal{E})$ of $\mathcal{E}$ over $\mathcal{O}_S$ is a quasi-coherent sheaf of $\mathcal{O}_S$-algebras. Hence it makes sense to apply the construction of the previous section to it.

Definition 26.6.1. Let S be a scheme. Let $\mathcal{E}$ be a quasi-coherent $\mathcal{O}_S$-module[1]. The *vector bundle associated to* $\mathcal{E}$ is

$$\mathbf{V}(\mathcal{E}) = \underline{\text{Spec}}_S(\text{Sym}(\mathcal{E})).$$

The vector bundle associated to $\mathcal{E}$ comes with a bit of extra structure. Namely, we have a grading

$$\pi_*\mathcal{O}_{\mathbf{V}(\mathcal{E})} = \bigoplus\nolimits_{n\geq 0} \text{Sym}^n(\mathcal{E}).$$

which turns $\pi_*\mathcal{O}_{\mathbf{V}(\mathcal{E})}$ into a graded $\mathcal{O}_S$-algebra. Conversely, we can recover $\mathcal{E}$ from the degree 1 part of this. Thus we define an abstract vector bundle as follows.

Definition 26.6.2. Let S be a scheme. A *vector bundle* $\pi : V \to S$ *over* S is an affine morphism of schemes such that $\pi_*\mathcal{O}_V$ is endowed with the structure of a graded $\mathcal{O}_S$-algebra $\pi_*\mathcal{O}_V = \bigoplus_{n\geq 0} \mathcal{E}_n$ such that $\mathcal{E}_0 = \mathcal{O}_S$ and such that the maps

$$\text{Sym}^n(\mathcal{E}_1) \longrightarrow \mathcal{E}_n$$

are isomorphisms for all $n \geq 0$. A *morphism of vector bundles over* S is a morphism $f : V \to V'$ such that the induced map

$$f^* : \pi'_*\mathcal{O}_{V'} \longrightarrow \pi_*\mathcal{O}_V$$

is compatible with the given gradings.

An example of a vector bundle over S is affine n-space $\mathbf{A}^n_S$ over S, see Definition 26.5.1. This is true because $\mathcal{O}_S[T_1, \ldots, T_n] = \text{Sym}(\mathcal{O}_S^{\oplus n})$.

Lemma 26.6.3. *The category of vector bundles over a scheme S is anti-equivalent to the category of quasi-coherent $\mathcal{O}_S$-modules.*

Proof. Omitted. Hint: In one direction one uses the functor $\underline{\text{Spec}}_S(-)$ and in the other the functor $(\pi : V \to S) \rightsquigarrow (\pi_*\mathcal{O}_V)_1$ (degree 1 part). □

26.7. Cones

In algebraic geometry cones correspond to graded algebras. By our conventions a graded ring or algebra A comes with a grading $A = \bigoplus_{d\geq 0} A_d$ by the nonnegative integers, see Algebra, Section 10.52.

Definition 26.7.1. Let S be a scheme. Let $\mathcal{A}$ be a quasi-coherent graded $\mathcal{O}_S$-algebra. Assume that $\mathcal{O}_S \to \mathcal{A}_0$ is an isomorphism[2]. The *cone associated to* $\mathcal{A}$ or the *affine cone associated to* $\mathcal{A}$ is

$$C(\mathcal{A}) = \underline{\text{Spec}}_S(\mathcal{A}).$$

[1] The reader may expect here the condition that $\mathcal{E}$ is finite locally free. We do not do so in order to be consistent with [**DG67**, II, Definition 1.7.8].

[2] Often one imposes the assumption that $\mathcal{A}$ is generated by $\mathcal{A}_1$ over $\mathcal{O}_S$. We do not assume this in order to be consistent with [**DG67**, II, (8.3.1)].

The cone associated to a graded sheaf of $\mathcal{O}_S$-algebras comes with a bit of extra structure. Namely, we obtain a grading

$$\pi_*\mathcal{O}_{C(\mathcal{A})} = \bigoplus\nolimits_{n\geq 0} \mathcal{A}_n$$

Thus we can define an abstract cone as follows.

Definition 26.7.2. Let S be a scheme. A *cone* $\pi : C \to S$ *over* S is an affine morphism of schemes such that $\pi_*\mathcal{O}_C$ is endowed with the structure of a graded $\mathcal{O}_S$-algebra $\pi_*\mathcal{O}_C = \bigoplus_{n\geq 0} \mathcal{A}_n$ such that $\mathcal{A}_0 = \mathcal{O}_S$. A *morphism of cones* from $\pi : C \to S$ to $\pi' : C' \to S$ is a morphism $f : C \to C'$ such that the induced map

$$f^* : \pi'_*\mathcal{O}_{C'} \longrightarrow \pi_*\mathcal{O}_C$$

is compatible with the given gradings.

Any vector bundle is an example of a cone. In fact the category of vector bundles over S is a full subcategory of the category of cones over S.

26.8. Proj of a graded ring

Let S be a graded ring. Consider the topological space $\mathrm{Proj}(S)$ associated to S, see Algebra, Section 10.53. We will endow this space with a sheaf of rings $\mathcal{O}_{\mathrm{Proj}(S)}$ such that the resulting pair $(\mathrm{Proj}(S), \mathcal{O}_{\mathrm{Proj}(S)})$ will be a scheme.

Recall that $\mathrm{Proj}(S)$ has a basis of open sets $D_+(f)$, $f \in S_d$, $d \geq 1$ which we call *standard opens*, see Algebra, Section 10.53. This terminology will always imply that f is homogeneous of positive degree even if we forget to mention it. In addition, the intersection of two standard opens is another: $D_+(f) \cap D_+(g) = D_+(fg)$, for $f, g \in S$ homogeneous of positive degree.

Lemma 26.8.1. *Let S be a graded ring. Let $f \in S$ homogeneous of positive degree.*

(1) *If $g \in S$ homogeneous of positive degree and $D_+(g) \subset D_+(f)$, then*
   (a) *f is invertible in S_g, and $f^{\deg(g)}/g^{\deg(f)}$ is invertible in $S_{(g)}$,*
   (b) *$g^e = af$ for some $e \geq 1$ and $a \in S$ homogeneous,*
   (c) *there is a canonical S-algebra map $S_f \to S_g$,*
   (d) *there is a canonical S_0-algebra map $S_{(f)} \to S_{(g)}$ compatible with the map $S_f \to S_g$,*
   (e) *the map $S_{(f)} \to S_{(g)}$ induces an isomorphism*
   $$(S_{(f)})_{g^{\deg(f)}/f^{\deg(g)}} \cong S_{(g)},$$
   (f) *these maps induce a commutative diagram of topological spaces*
   $$\begin{array}{ccccc} D_+(g) & \longleftarrow & \{\mathbf{Z}\text{-graded primes of } S_g\} & \longrightarrow & \mathrm{Spec}(S_{(g)}) \\ \downarrow & & \downarrow & & \downarrow \\ D_+(f) & \longleftarrow & \{\mathbf{Z}\text{-graded primes of } S_f\} & \longrightarrow & \mathrm{Spec}(S_{(f)}) \end{array}$$
   where the horizontal maps are homeomorphisms and the vertical maps are open immersions,
   (g) *there are compatible canonical S_f and $S_{(f)}$-module maps $M_f \to M_g$ and $M_{(f)} \to M_{(g)}$ for any graded S-module M, and*
   (h) *the map $M_{(f)} \to M_{(g)}$ induces an isomorphism*
   $$(M_{(f)})_{g^{\deg(f)}/f^{\deg(g)}} \cong M_{(g)}.$$

(2) *Any open covering of $D_+(f)$ can be refined to a finite open covering of the form $D_+(f) = \bigcup_{i=1}^n D_+(g_i)$.*
(3) *Let $g_1, \ldots, g_n \in S$ be homogeneous of positive degree. Then $D_+(f) \subset \bigcup D_+(g_i)$ if and only if $g_1^{\deg(f)}/f^{\deg(g_1)}, \ldots, g_n^{\deg(f)}/f^{\deg(g_n)}$ generate the unit ideal in $S_{(f)}$.*

Proof. Recall that $D_+(g) = \mathrm{Spec}(S_{(g)})$ with identification given by the ring maps $S \to S_g \leftarrow S_{(g)}$, see Algebra, Lemma 10.53.3. Thus $f^{\deg(g)}/g^{\deg(f)}$ is an element of $S_{(g)}$ which is not contained in any prime ideal, and hence invertible, see Algebra, Lemma 10.16.2. We conclude that (a) holds. Write the inverse of f in S_g as a/g^d. We may replace a by its homogeneous part of degree $d\deg(g) - \deg(f)$. This means $g^d - af$ is annihilated by a power of g, whence $g^e = af$ for some $a \in S$ homogeneous of degree $e\deg(g) - \deg(f)$. This proves (b). For (c), the map $S_f \to S_g$ exists by (a) from the universal property of localization, or we can define it by mapping b/f^n to $a^n b/g^{ne}$. This clearly induces a map of the subrings $S_{(f)} \to S_{(g)}$ of degree zero elements as well. We can similarly define $M_f \to M_g$ and $M_{(f)} \to M_{(g)}$ by mapping x/f^n to $a^n x/g^{ne}$. The statements writing $S_{(g)}$ resp. $M_{(g)}$ as principal localizations of $S_{(f)}$ resp. $M_{(f)}$ are clear from the formulas above. The maps in the commutative diagram of topological spaces correspond to the ring maps given above. The horizontal arrows are homeomorphisms by Algebra, Lemma 10.53.3. The vertical arrows are open immersions since the left one is the inclusion of an open subset.

The open $D_+(f)$ is quasi-compact because it is homeomorphic to $\mathrm{Spec}(S_{(f)})$, see Algebra, Lemma 10.16.10. Hence the second statement follows directly from the fact that the standard opens form a basis for the topology.

The third statement follows directly from Algebra, Lemma 10.16.2. □

In Sheaves, Section 6.30 we defined the notion of a sheaf on a basis, and we showed that it is essentially equivalent to the notion of a sheaf on the space, see Sheaves, Lemmas 6.30.6 and 6.30.9. Moreover, we showed in Sheaves, Lemma 6.30.4 that it is sufficient to check the sheaf condition on a cofinal system of open coverings for each standard open. By the lemma above it suffices to check on the finite coverings by standard opens.

Definition 26.8.2. Let S be a graded ring. Suppose that $D_+(f) \subset \mathrm{Proj}(S)$ is a standard open. A *standard open covering* of $D_+(f)$ is a covering $D_+(f) = \bigcup_{i=1}^n D_+(g_i)$, where $g_1, \ldots, g_n \in S$ are homogeneous of positive degree.

Let S be a graded ring. Let M be a graded S-module. We will define a presheaf $\widetilde{M}$ on the basis of standard opens. Suppose that $U \subset \mathrm{Proj}(S)$ is a standard open. If $f, g \in S$ are homogeneous of positive degree such that $D_+(f) = D_+(g)$, then by Lemma 26.8.1 above there are canonical maps $M_{(f)} \to M_{(g)}$ and $M_{(g)} \to M_{(f)}$ which are mutually inverse. Hence we may choose any f such that $U = D_+(f)$ and define

$$\widetilde{M}(U) = M_{(f)}.$$

Note that if $D_+(g) \subset D_+(f)$, then by Lemma 26.8.1 above we have a canonical map

$$\widetilde{M}(D_+(f)) = M_{(f)} \longrightarrow M_{(g)} = \widetilde{M}(D_+(g)).$$

Clearly, this defines a presheaf of abelian groups on the basis of standard opens. If $M = S$, then $\widetilde{S}$ is a presheaf of rings on the basis of standard opens. And for general M we see that $\widetilde{M}$ is a presheaf of $\widetilde{S}$-modules on the basis of standard opens.

Let us compute the stalk of $\widetilde{M}$ at a point $x \in \text{Proj}(S)$. Suppose that x corresponds to the homogeneous prime ideal $\mathfrak{p} \subset S$. By definition of the stalk we see that

$$\widetilde{M}_x = \text{colim}_{f \in S_d, d > 0, f \notin \mathfrak{p}} M_{(f)}$$

Here the set $\{f \in S_d, d > 0, f \notin \mathfrak{p}\}$ is partially ordered by the rule $f \geq f' \Leftrightarrow D_+(f) \subset D_+(f')$. If $f_1, f_2 \in S \setminus \mathfrak{p}$ are homogeneous of positive degree, then we have $f_1 f_2 \geq f_1$ in this ordering. In Algebra, Section 10.53 we defined $M_{(\mathfrak{p})}$ as the ring whose elements are fractions x/f with x, f homogeneous, $\deg(x) = \deg(f)$, $f \notin \mathfrak{p}$. Since $\mathfrak{p} \in \text{Proj}(S)$ there exists at least one $f_0 \in S$ homogeneous of positive degree with $f_0 \notin \mathfrak{p}$. Hence $x/f = f_0 x / f f_0$ and we see that we may always assume the denominator of an element in $M_{(\mathfrak{p})}$ has positive degree. From these remarks it follows easily that

$$\widetilde{M}_x = M_{(\mathfrak{p})}.$$

Next, we check the sheaf condition for the standard open coverings. If $D_+(f) = \bigcup_{i=1}^n D_+(g_i)$, then the sheaf condition for this covering is equivalent with the exactness of the sequence

$$0 \to M_{(f)} \to \bigoplus M_{(g_i)} \to \bigoplus M_{(g_i g_j)}.$$

Note that $D_+(g_i) = D_+(f g_i)$, and hence we can rewrite this sequence as the sequence

$$0 \to M_{(f)} \to \bigoplus M_{(f g_i)} \to \bigoplus M_{(f g_i g_j)}.$$

By Lemma 26.8.1 we see that $g_1^{\deg(f)}/f^{\deg(g_1)}, \ldots, g_n^{\deg(f)}/f^{\deg(g_n)}$ generate the unit ideal in $S_{(f)}$, and that the modules $M_{(f g_i)}$, $M_{(f g_i g_j)}$ are the principal localizations of the $S_{(f)}$-module $M_{(f)}$ at these elements and their products. Thus we may apply Algebra, Lemma 10.21.2 to the module $M_{(f)}$ over $S_{(f)}$ and the elements $g_1^{\deg(f)}/f^{\deg(g_1)}, \ldots, g_n^{\deg(f)}/f^{\deg(g_n)}$. We conclude that the sequence is exact. By the remarks made above, we see that $\widetilde{M}$ is a sheaf on the basis of standard opens.

Thus we conclude from the material in Sheaves, Section 6.30 that there exists a unique sheaf of rings $\mathcal{O}_{\text{Proj}(S)}$ which agrees with $\widetilde{S}$ on the standard opens. Note that by our computation of stalks above and Algebra, Lemma 10.53.5 the stalks of this sheaf of rings are all local rings.

Similarly, for any graded S-module M there exists a unique sheaf of $\mathcal{O}_{\text{Proj}(S)}$-modules $\mathcal{F}$ which agrees with $\widetilde{M}$ on the standard opens, see Sheaves, Lemma 6.30.12.

Definition 26.8.3. Let S be a graded ring.

(1) The *structure sheaf $\mathcal{O}_{Proj(S)}$ of the homogeneous spectrum of S* is the unique sheaf of rings $\mathcal{O}_{\text{Proj}(S)}$ which agrees with $\widetilde{S}$ on the basis of standard opens.
(2) The locally ringed space $(\text{Proj}(S), \mathcal{O}_{\text{Proj}(S)})$ is called the *homogeneous spectrum* of S and denoted $\text{Proj}(S)$.

(3) The sheaf of $\mathcal{O}_{\mathrm{Proj}(S)}$-modules extending $\widetilde{M}$ to all opens of $\mathrm{Proj}(S)$ is called the sheaf of $\mathcal{O}_{\mathrm{Proj}(S)}$-modules associated to M. This sheaf is denoted $\widetilde{M}$ as well.

We summarize the results obtained so far.

Lemma 26.8.4. *Let S be a graded ring. Let M be a graded S-module. Let $\widetilde{M}$ be the sheaf of $\mathcal{O}_{Proj(S)}$-modules associated to M.*

(1) *For every $f \in S$ homogeneous of positive degree we have*
$$\Gamma(D_+(f), \mathcal{O}_{Proj(S)}) = S_{(f)}.$$
(2) *For every $f \in S$ homogeneous of positive degree we have $\Gamma(D_+(f), \widetilde{M}) = M_{(f)}$ as an $S_{(f)}$-module.*
(3) *Whenever $D_+(g) \subset D_+(f)$ the restriction mappings on $\mathcal{O}_{Proj(S)}$ and $\widetilde{M}$ are the maps $S_{(f)} \to S_{(g)}$ and $M_{(f)} \to M_{(g)}$ from Lemma 26.8.1.*
(4) *Let $\mathfrak{p}$ be a homogeneous prime of S not containing S_+, and let $x \in Proj(S)$ be the corresponding point. We have $\mathcal{O}_{Proj(S),x} = S_{(\mathfrak{p})}$.*
(5) *Let $\mathfrak{p}$ be a homogeneous prime of S not containing S_+, and let $x \in Proj(S)$ be the corresponding point. We have $\mathcal{F}_x = M_{(\mathfrak{p})}$ as an $S_{(\mathfrak{p})}$-module.*
(6) *There is a canonical ring map $S_0 \longrightarrow \Gamma(Proj(S), \widetilde{S})$ and a canonical S_0-module map $M_0 \longrightarrow \Gamma(Proj(S), \widetilde{M})$ compatible with the descriptions of sections over standard opens and stalks above.*

Moreover, all these identifications are functorial in the graded S-module M. In particular, the functor $M \mapsto \widetilde{M}$ is an exact functor from the category of graded S-modules to the category of $\mathcal{O}_{Proj(S)}$-modules.

Proof. Assertions (1) - (5) are clear from the discussion above. We see (6) since there are canonical maps $M_0 \to M_{(f)}$, $x \mapsto x/1$ compatible with the restriction maps described in (3). The exactness of the functor $M \mapsto \widetilde{M}$ follows from the fact that the functor $M \mapsto M_{(\mathfrak{p})}$ is exact (see Algebra, Lemma 10.53.5) and the fact that exactness of short exact sequences may be checked on stalks, see Modules, Lemma 17.3.1. □

Remark 26.8.5. The map from M_0 to the global sections of $\widetilde{M}$ is generally far from being an isomorphism. A trivial example is to take $S = k[x, y, z]$ with $1 = \deg(x) = \deg(y) = \deg(z)$ (or any number of variables) and to take $M = S/(x^{100}, y^{100}, z^{100})$. It is easy to see that $\widetilde{M} = 0$, but $M_0 = k$.

Lemma 26.8.6. *Let S be a graded ring. Let $f \in S$ be homogeneous of positive degree. Suppose that $D(g) \subset \mathrm{Spec}(S_{(f)})$ is a standard open. Then there exists a $h \in S$ homogeneous of positive degree such that $D(g)$ corresponds to $D_+(h) \subset D_+(f)$ via the homeomorphism of Algebra, Lemma 10.53.3. In fact we can take h such that $g = h/f^n$ for some n.*

Proof. Write $g = h/f^n$ for some h homogeneous of positive degree and some $n \geq 1$. If $D_+(h)$ is not contained in $D_+(f)$ then we replace h by hf and n by $n+1$. Then h has the required shape and $D_+(h) \subset D_+(f)$ corresponds to $D(g) \subset \mathrm{Spec}(S_{(f)})$. □

Lemma 26.8.7. *Let S be a graded ring. The locally ringed space $Proj(S)$ is a scheme. The standard opens $D_+(f)$ are affine opens. For any graded S-module M the sheaf $\widetilde{M}$ is a quasi-coherent sheaf of $\mathcal{O}_{Proj(S)}$-modules.*

Proof. Consider a standard open $D_+(f) \subset \mathrm{Proj}(S)$. By Lemmas 26.8.1 and 26.8.4 we have $\Gamma(D_+(f), \mathcal{O}_{\mathrm{Proj}(S)}) = S_{(f)}$, and we have a homeomorphism $\varphi : D_+(f) \to \mathrm{Spec}(S_{(f)})$. For any standard open $D(g) \subset \mathrm{Spec}(S_{(f)})$ we may pick a $h \in S_+$ as in Lemma 26.8.6. Then $\varphi^{-1}(D(g)) = D_+(h)$, and by Lemmas 26.8.4 and 26.8.1 we see

$$\Gamma(D_+(h), \mathcal{O}_{\mathrm{Proj}(S)}) = S_{(h)} = (S_{(f)})_{h^{\deg(f)}/f^{\deg(h)}} = (S_{(f)})_g = \Gamma(D(g), \mathcal{O}_{\mathrm{Spec}(S_{(f)})}).$$

Thus the restriction of $\mathcal{O}_{\mathrm{Proj}(S)}$ to $D_+(f)$ corresponds via the homeomorphism φ exactly to the sheaf $\mathcal{O}_{\mathrm{Spec}(S_{(f)})}$ as defined in Schemes, Section 25.5. We conclude that $D_+(f)$ is an affine scheme isomorphic to $\mathrm{Spec}(S_{(f)})$ via φ and hence that $\mathrm{Proj}(S)$ is a scheme.

In exactly the same way we show that $\widetilde{M}$ is a quasi-coherent sheaf of $\mathcal{O}_{\mathrm{Proj}(S)}$-modules. Namely, the argument above will show that

$$\widetilde{M}|_{D_+(f)} \cong \varphi^* \left(\widetilde{M_{(f)}} \right)$$

which shows that $\widetilde{M}$ is quasi-coherent. □

Lemma 26.8.8. *Let S be a graded ring. The scheme $\mathrm{Proj}(S)$ is separated.*

Proof. We have to show that the canonical morphism $\mathrm{Proj}(S) \to \mathrm{Spec}(\mathbf{Z})$ is separated. We will use Schemes, Lemma 25.21.8. Thus it suffices to show given any pair of standard opens $D_+(f)$ and $D_+(g)$ that $D_+(f) \cap D_+(g) = D_+(fg)$ is affine (clear) and that the ring map

$$S_{(f)} \otimes_{\mathbf{Z}} S_{(g)} \longrightarrow S_{(fg)}$$

is surjective. Any element s in $S_{(fg)}$ is of the form $s = h/(f^n g^m)$ with $h \in S$ homogeneous of degree $n\deg(f) + m\deg(g)$. We may multiply h by a suitable monomial $f^i g^j$ and assume that $n = n'\deg(g)$, and $m = m'\deg(f)$. Then we can rewrite s as $s = h/f^{(n'+m')\deg(g)} \cdot f^{m'\deg(g)}/g^{m'\deg(f)}$. So s is indeed in the image of the displayed arrow. □

Lemma 26.8.9. *Let S be a graded ring. The scheme $\mathrm{Proj}(S)$ is quasi-compact if and only if there exist finitely many homogeneous elements $f_1, \ldots, f_n \in S_+$ such that $S_+ \subset \sqrt{(f_1, \ldots, f_n)}$.*

Proof. Given such a collection of elements the standard affine opens $D_+(f_i)$ cover $\mathrm{Proj}(S)$ by Algebra, Lemma 10.53.3. Conversely, if $\mathrm{Proj}(S)$ is quasi-compact, then we may cover it by finitely many standard opens $D_+(f_i)$, $i = 1, \ldots, n$ and we see that $S_+ \subset \sqrt{(f_1, \ldots, f_n)}$ by the lemma referenced above. □

Lemma 26.8.10. *Let S be a graded ring. The scheme $\mathrm{Proj}(S)$ has a canonical morphism towards the affine scheme $\mathrm{Spec}(S_0)$, agreeing with the map on topological spaces coming from Algebra, Definition 10.53.1.*

Proof. We saw above that our construction of $\widetilde{S}$, resp. $\widetilde{M}$ gives a sheaf of S_0-algebras, resp. S_0-modules. Hence we get a morphism by Schemes, Lemma 25.6.4. This morphism, when restricted to $D_+(f)$ comes from the canonical ring map $S_0 \to S_{(f)}$. The maps $S \to S_f$, $S_{(f)} \to S_f$ are S_0-algebra maps, see Lemma 26.8.1. Hence if the homogeneous prime $\mathfrak{p} \subset S$ corresponds to the $\mathbf{Z}$-graded prime $\mathfrak{p}' \subset S_f$ and the (usual) prime $\mathfrak{p}'' \subset S_{(f)}$, then each of these has the same inverse image in S_0. □

Lemma 26.8.11. *Let S be a graded ring. If S is finitely generated as an algebra over S_0, then the morphism $Proj(S) \to \mathrm{Spec}(S_0)$ satisfies the existence and uniqueness parts of the valuative criterion, see Schemes, Definition 25.20.3.*

Proof. The uniqueness part follows from the fact that $\mathrm{Proj}(S)$ is separated (Lemma 26.8.8 and Schemes, Lemma 25.22.1). Choose $x_i \in S_+$ homogeneous, $i = 1, \ldots, n$ which generate S over S_0. Let $d_i = \deg(x_i)$ and set $d = \mathrm{lcm}\{d_i\}$. Suppose we are given a diagram

$$\begin{array}{ccc} \mathrm{Spec}(K) & \longrightarrow & \mathrm{Proj}(S) \\ \downarrow & & \downarrow \\ \mathrm{Spec}(A) & \longrightarrow & \mathrm{Spec}(S_0) \end{array}$$

as in Schemes, Definition 25.20.3. Denote $v : K^* \to \Gamma$ the valuation of A, see Algebra, Definition 10.46.11. We may choose an $f \in S_+$ homogeneous such that $\mathrm{Spec}(K)$ maps into $D_+(f)$. Then we get a commutative diagram of ring maps

$$\begin{array}{ccc} K & \xleftarrow{\varphi} & S_{(f)} \\ \uparrow & & \uparrow \\ A & \longleftarrow & S_0 \end{array}$$

Let $i_0 \in \{1, \ldots, n\}$ be an index minimizing the valuation $(d/d_i)v(\varphi(x_i^{\deg(f)}/f^{d_i}))$ where we temporarily use the convention that the valuation of zero is bigger than any element of the value group. For convenience set $x_0 = x_{i_0}$ and $d_0 = d_{i_0}$. Since the open sets $D_+(x_i)$ cover $\mathrm{Proj}(S)$ we see that $\varphi(x_0) \neq 0$. This means that the ring map φ factors though a map $\varphi' : S_{(fx_0)} \to K$. We see that

$$\deg(f)v(\varphi'(x_i^{d_0}/x_0^{d_i})) = d_0 v(\varphi(x_i^{\deg(f)}/f^{d_i})) - d_i v(\varphi(x_0^{\deg(f)}/f^{d_0})) \geq 0$$

by our choice of i_0. This implies that the S_0-algebra $S_{(x_0)}$, which is generated by the elements $x_i^{d_0}/x_0^{d_i}$ over S_0, maps into A via φ'. The corresponding morphism of schemes $\mathrm{Spec}(A) \to \mathrm{Spec}(S_{(x_0)}) = D_+(x_0) \subset \mathrm{Proj}(S)$ provides the morphism fitting into the first commutative diagram of this proof. □

We saw in the proof of Lemma 26.8.11 that, under the hypotheses of that lemma, the morphism $\mathrm{Proj}(S) \to \mathrm{Spec}(S_0)$ is quasi-compact as well. Hence (by Schemes, Proposition 25.20.6) we see that $\mathrm{Proj}(S) \to \mathrm{Spec}(S_0)$ is universally closed in the situation of the lemma. We give two examples showing these results do not hold without some assumption on the graded ring S.

Example 26.8.12. Let $\mathbf{C}[X_1, X_2, X_3, \ldots]$ be the graded $\mathbf{C}$-algebra with each X_i in degree 0. Consider the ring map

$$\mathbf{C}[X_1, X_2, X_3, \ldots] \longrightarrow \mathbf{C}[t^\alpha; \alpha \in \mathbf{Q}_{\geq 0}]$$

which maps X_i to $t^{1/i}$. The right hand side becomes a valuation ring A upon localization at the ideal $\mathfrak{m} = (t^\alpha; \alpha > 0)$. This gives a morphism from $\mathrm{Spec}(f.f.(A))$ to $\mathrm{Proj}(\mathbf{C}[X_1, X_2, X_3, \ldots])$ which does not extend to a morphism defined on all of $\mathrm{Spec}(A)$. The reason is that the image of $\mathrm{Spec}(A)$ would be contained in one of the $D_+(X_i)$ but then X_{i+1}/X_i would map to an element of A which it doesn't since it maps to $t^{1/(i+1)-1/i}$.

Example 26.8.13. Let $R = \mathbf{C}[t]$ and

$$S = R[X_1, X_2, X_3, \ldots]/(X_i^2 - tX_{i+1}).$$

The grading is such that $R = S_0$ and $\deg(X_i) = 2^{i-1}$. Note that if $\mathfrak{p} \in \text{Proj}(S)$ then $t \notin \mathfrak{p}$ (otherwise $\mathfrak{p}$ has to contain all of the X_i which is not allowed for an element of the homogeneous spectrum). Thus we see that $D_+(X_i) = D_+(X_{i+1})$ for all i. Hence $\text{Proj}(S)$ is quasi-compact; in fact it is affine since it is equal to $D_+(X_1)$. It is easy to see that the image of $\text{Proj}(S) \to \text{Spec}(R)$ is $D(t)$. Hence the morphism $\text{Proj}(S) \to \text{Spec}(R)$ is not closed. Thus the valuative criterion cannot apply because it would imply that the morphism is closed (see Schemes, Proposition 25.20.6).

Example 26.8.14. Let A be a ring. Let $S = A[T]$ as a graded A algebra with T in degree 1. Then the canonical morphism $\text{Proj}(S) \to \text{Spec}(A)$ (see Lemma 26.8.10) is an isomorphism.

26.9. Quasi-coherent sheaves on Proj

Let S be a graded ring. Let M be a graded S-module. We saw in the previous section how to construct a quasi-coherent sheaf of modules $\widetilde{M}$ on $\text{Proj}(S)$ and a map

$$M_0 \longrightarrow \Gamma(\text{Proj}(S), \widetilde{M})$$

of the degree 0 part of M to the global sections of $\widetilde{M}$. The degree 0 part of the nth twist $M(n)$ of the graded module M (see Algebra, Section 10.52) is equal to M_n. Hence we can get maps

$$M_n \longrightarrow \Gamma(\text{Proj}(S), \widetilde{M(n)}).$$

We would like to be able to perform this operation for any quasi-coherent sheaf $\mathcal{F}$ on $\text{Proj}(S)$. We will do this by tensoring with the nth twist of the structure sheaf, see Definition 26.10.1. In order to relate the two notions we will use the following lemma.

Lemma 26.9.1. *Let S be a graded ring. Let $(X, \mathcal{O}_X) = (Proj(S), \mathcal{O}_{Proj(S)})$ be the scheme of Lemma 26.8.7. Let $f \in S_+$ be homogeneous. Let $x \in X$ be a point corresponding to the homogeneous prime $\mathfrak{p} \subset S$. Let M, N be graded S-modules. There is a canonical map of $\mathcal{O}_{Proj(S)}$-modules*

$$\widetilde{M} \otimes_{\mathcal{O}_X} \widetilde{N} \longrightarrow \widetilde{M \otimes_S N}$$

which induces the canonical map $M_{(f)} \otimes_{S_{(f)}} N_{(f)} \to (M \otimes_S N)_{(f)}$ on sections over $D_+(f)$ and the canonical map $M_{(\mathfrak{p})} \otimes_{S_{(\mathfrak{p})}} N_{(\mathfrak{p})} \to (M \otimes_S N)_{(\mathfrak{p})}$ on stalks at x. Moreover, the following diagram

$$\begin{array}{ccc} M_0 \otimes_{S_0} N_0 & \longrightarrow & (M \otimes_S N)_0 \\ \downarrow & & \downarrow \\ \Gamma(X, \widetilde{M} \otimes_{\mathcal{O}_X} \widetilde{N}) & \longrightarrow & \Gamma(X, \widetilde{M \otimes_R N}) \end{array}$$

is commutative.

Proof. To construct a morphism as displayed is the same as constructing a $\mathcal{O}_X$-bilinear map

$$\widetilde{M} \times \widetilde{N} \longrightarrow \widetilde{M \otimes_R N}$$

see Modules, Section 17.15. It suffices to define this on sections over the opens $D_+(f)$ compatible with restriction mappings. On $D_+(f)$ we use the $S_{(f)}$-bilinear map $M_{(f)} \times N_{(f)} \to (M \otimes_S N)_{(f)}$, $(x/f^n, y/f^m) \mapsto (x \otimes y)/f^{n+m}$. Details omitted. □

Remark 26.9.2. In general the map constructed in Lemma 26.9.1 above is not an isomorphism. Here is an example. Let k be a field. Let $S = k[x, y, z]$ with k in degree 0 and $\deg(x) = 1$, $\deg(y) = 2$, $\deg(z) = 3$. Let $M = S(1)$ and $N = S(2)$, see Algebra, Section 10.52 for notation. Then $M \otimes_S N = S(3)$. Note that

$$\begin{aligned} S_z &= k[x, y, z, 1/z] \\ S_{(z)} &= k[x^3/z, xy/z, y^3/z^2] \cong k[u, v, w]/(uw - v^3) \\ M_{(z)} &= S_{(z)} \cdot x + S_{(z)} \cdot y^2/z \subset S_z \\ N_{(z)} &= S_{(z)} \cdot y + S_{(z)} \cdot x^2 \subset S_z \\ S(3)_{(z)} &= S_{(z)} \cdot z \subset S_z \end{aligned}$$

Consider the maximal ideal $\mathfrak{m} = (u, v, w) \subset S_{(z)}$. It is not hard to see that both $M_{(z)}/\mathfrak{m}M_{(z)}$ and $N_{(z)}/\mathfrak{m}N_{(z)}$ have dimension 2 over $\kappa(\mathfrak{m})$. But $S(3)_{(z)}/\mathfrak{m}S(3)_{(z)}$ has dimension 1. Thus the map $M_{(z)} \otimes N_{(z)} \to S(3)_{(z)}$ is not an isomorphism.

26.10. Invertible sheaves on Proj

Recall from Algebra, Section 10.52 the construction of the twisted module $M(n)$ associated to a graded module over a graded ring.

Definition 26.10.1. Let S be a graded ring. Let $X = \operatorname{Proj}(S)$.

(1) We define $\mathcal{O}_X(n) = \widetilde{S(n)}$. This is called the nth *twist of the structure sheaf of Proj*(S).
(2) For any sheaf of $\mathcal{O}_X$-modules $\mathcal{F}$ we set $\mathcal{F}(n) = \mathcal{F} \otimes_{\mathcal{O}_X} \mathcal{O}_X(n)$.

We are going to use Lemma 26.9.1 to construct some canonical maps. Since $S(n) \otimes_S S(m) = S(n + m)$ we see that there are canonical maps

$$\mathcal{O}_X(n) \otimes_{\mathcal{O}_X} \mathcal{O}_X(m) \longrightarrow \mathcal{O}_X(n + m). \tag{26.10.1.1}$$

These maps are not isomorphisms in general, see the example in Remark 26.9.2. The same example shows that $\mathcal{O}_X(n)$ is *not* an invertible sheaf on X in general. Tensoring with an arbitrary $\mathcal{O}_X$-module $\mathcal{F}$ we get maps

$$\mathcal{O}_X(n) \otimes_{\mathcal{O}_X} \mathcal{F}(m) \longrightarrow \mathcal{F}(n + m). \tag{26.10.1.2}$$

The maps (26.10.1.1) on global sections give a map of graded rings

$$S \longrightarrow \bigoplus\nolimits_{n \geq 0} \Gamma(X, \mathcal{O}_X(n)). \tag{26.10.1.3}$$

And for an arbitrary $\mathcal{O}_X$-module $\mathcal{F}$ the maps (26.10.1.2) give a graded module structure

$$\bigoplus\nolimits_{n \geq 0} \Gamma(X, \mathcal{O}_X(n)) \times \bigoplus\nolimits_{m \in \mathbf{Z}} \Gamma(X, \mathcal{F}(m)) \longrightarrow \bigoplus\nolimits_{m \in \mathbf{Z}} \Gamma(X, \mathcal{F}(m)) \tag{26.10.1.4}$$

and via (26.10.1.3) also a S-module structure. More generally, given any graded S-module M we have $M(n) = M \otimes_S S(n)$. Hence we get maps

$$\widetilde{M}(n) = \widetilde{M} \otimes_{\mathcal{O}_X} \mathcal{O}_X(n) \longrightarrow \widetilde{M(n)}. \tag{26.10.1.5}$$

On global sections we get a map of graded S-modules

$$M \longrightarrow \bigoplus\nolimits_{n \in \mathbf{Z}} \Gamma(X, \widetilde{M(n)}). \tag{26.10.1.6}$$

Here is an important fact which follows basically immediately from the definitions.

Lemma 26.10.2. *Let S be a graded ring. Set $X = Proj(S)$. Let $f \in S$ be homogeneous of degree $d > 0$. The sheaves $\mathcal{O}_X(nd)|_{D_+(f)}$ are invertible, and in fact trivial for all $n \in \mathbf{Z}$ (see Modules, Definition 17.21.1). The maps (26.10.1.1) restricted to $D_+(f)$*

$$\mathcal{O}_X(nd)|_{D_+(f)} \otimes_{\mathcal{O}_{D_+(f)}} \mathcal{O}_X(m)|_{D_+(f)} \longrightarrow \mathcal{O}_X(nd+m)|_{D_+(f)}$$

and the maps (26.10.1.5) restricted to $D_+(f)$

$$\widetilde{M}(nd)|_{D_+(f)} = \widetilde{M}|_{D_+(f)} \otimes_{\mathcal{O}_{D_+(f)}} \mathcal{O}_X(nd)|_{D_+(f)} \longrightarrow \widetilde{M(nd)}|_{D_+(f)}$$

are isomorphisms for all $n, m \in \mathbf{Z}$.

Proof. The (not graded) S-module maps $S \to S(n)$, and $M \to M(n)$, given by $x \mapsto f^{n/d}x$ become isomorphisms after inverting f. The first shows that $S_{(f)} \cong S(n)_{(f)}$ which gives an isomorphism $\mathcal{O}_{D_+(f)} \cong \mathcal{O}_X(n)|_{D_+(f)}$. The second shows that the map $S(n)_{(f)} \otimes_{S_{(f)}} M_{(f)} \to M(n)_{(f)}$ is an isomorphism. □

Lemma 26.10.3. *Let S be a graded ring generated as an S_0-algebra by the elements of S_1. Set $X = Proj(S)$. In this case the sheaves $\mathcal{O}_X(n)$ are all invertible, and all the maps (26.10.1.1) and (26.10.1.5) are isomorphisms. In particular, these maps induce isomorphisms*

$$\mathcal{O}_X(n) \cong \mathcal{O}_X(1)^{\otimes n} \quad \text{and} \quad \widetilde{M(n)} = \widetilde{M} \otimes_{\mathcal{O}_X} \mathcal{O}_X(1)^{\otimes n}.$$

In fact the lemma holds more generally if X is covered by the standard opens $D_+(f)$ with $f \in S_1$.

Proof. Under the assumptions of the lemma X is covered by the open subsets $D_+(f)$ with $f \in S_1$ and the lemma is a consequence of Lemma 26.10.2 above. □

Lemma 26.10.4. *Let S be a graded ring. Set $X = Proj(S)$. Fix $d \geq 1$ an integer. The following open subsets of X are equal:*

1. *The largest open subset $W = W_d \subset X$ such that each $\mathcal{O}_X(dn)|_W$ is invertible and all the multiplication maps $\mathcal{O}_X(nd)|_W \otimes_{\mathcal{O}_W} \mathcal{O}_X(md)|_W \to \mathcal{O}_X(nd+md)|_W$ (see 26.10.1.1) are isomorphisms.*
2. *The union of the open subsets $D_+(fg)$ with $f, g \in S$ homogeneous and $\deg(f) = \deg(g) + d$.*

Moreover, all the maps $\widetilde{M}(nd)|_W = \widetilde{M}|_W \otimes_{\mathcal{O}_W} \mathcal{O}_X(nd)|_W \to \widetilde{M(nd)}|_W$ (see 26.10.1.5) are isomorphisms.

Proof. If $x \in D_+(fg)$ with $\deg(f) = \deg(g) + d$ then on $D_+(fg)$ the sheaves $\mathcal{O}_X(dn)$ are generated by the element $(f/g)^n = f^{2n}/(fg)^n$. This implies x is in the open subset W defined in (1) by arguing as in the proof of Lemma 26.10.2.

Conversely, suppose that $\mathcal{O}_X(d)$ is free of rank 1 in an open neighbourhood V of $x \in X$ and all the multiplication maps $\mathcal{O}_X(nd)|_V \otimes_{\mathcal{O}_V} \mathcal{O}_X(md)|_V \to \mathcal{O}_X(nd+md)|_V$ are isomorphisms. We may choose $h \in S_+$ homogeneous such that $D_+(h) \subset V$. By the definition of the twists of the structure sheaf we conclude there exists an element s of $(S_h)_d$ such that s^n is a basis of $(S_h)_{nd}$ as a module over $S_{(h)}$ for all $n \in \mathbf{Z}$. We may write $s = f/h^m$ for some $m \geq 1$ and $f \in S_{d+m\deg(h)}$. Set $g = h^m$ so $s = f/g$. Note that $x \in D(g)$ by construction. Note that $g^d \in (S_h)_{-d\deg(g)}$. By assumption we can write this as a multiple of $s^{\deg(g)} = f^{\deg(g)}/g^{\deg(g)}$, say $g^d = a/g^e \cdot f^{\deg(g)}/g^{\deg(g)}$. Then we conclude that $g^{d+e+\deg(g)} = af^{\deg(g)}$ and hence also $x \in D_+(f)$. So x is an element of the set defined in (2).

The existence of the generating section $s = f/g$ over the affine open $D_+(fg)$ whose powers freely generate the sheaves of modules $\mathcal{O}_X(nd)$ easily implies that the multiplication maps $\widetilde{M}(nd)|_W = \widetilde{M}|_W \otimes_{\mathcal{O}_W} \mathcal{O}_X(nd)|_W \to \widetilde{M(nd)}|_W$ (see 26.10.1.5) are isomorphisms. Compare with the proof of Lemma 26.10.2. □

Recall from Modules, Lemma 17.21.7 that given an invertible sheaf $\mathcal{L}$ on a locally ringed space X, and given a global section s of $\mathcal{L}$ the set $X_s = \{x \in X \mid s \notin \mathfrak{m}_x\mathcal{L}_x\}$ is open.

Lemma 26.10.5. *Let S be a graded ring. Set $X = Proj(S)$. Fix $d \geq 1$ an integer. Let $W = W_d \subset X$ be the open subscheme defined in Lemma 26.10.4. Let $n \geq 1$ and $f \in S_{nd}$. Denote $s \in \Gamma(W, \mathcal{O}_W(nd))$ the section which is the image of f via (26.10.1.3) restricted to W. Then*

$$W_s = D_+(f) \cap W.$$

Proof. Let $D_+(ab) \subset W$ be a standard affine open with $a, b \in S$ homogeneous and $\deg(a) = \deg(b) + d$. Note that $D_+(ab) \cap D_+(f) = D_+(abf)$. On the other hand the restriction of s to $D_+(ab)$ corresponds to the element $f/1 = b^n f/a^n(a/b)^n \in (S_{ab})_{nd}$. We have seen in the proof of Lemma 26.10.4 that $(a/b)^n$ is a generator for $\mathcal{O}_W(nd)$ over $D_+(ab)$. We conclude that $W_s \cap D_+(ab)$ is the principal open associated to $b^n f/a^n \in \mathcal{O}_X(D_+(ab))$. Thus the result of the lemma is clear. □

The following lemma states the properties that we will later use to characterize schemes with an ample invertible sheaf.

Lemma 26.10.6. *Let S be a graded ring. Let $X = Proj(S)$. Let $Y \subset X$ be a quasi-compact open subscheme. Denote $\mathcal{O}_Y(n)$ the restriction of $\mathcal{O}_X(n)$ to Y. There exists an integer $d \geq 1$ such that*

1. *the subscheme Y is contained in the open W_d defined in Lemma 26.10.4,*
2. *the sheaf $\mathcal{O}_Y(dn)$ is invertible for all $n \in \mathbf{Z}$,*
3. *all the maps $\mathcal{O}_Y(nd) \otimes_{\mathcal{O}_Y} \mathcal{O}_Y(m) \longrightarrow \mathcal{O}_Y(nd+m)$ of Equation (26.10.1.1) are isomorphisms,*
4. *all the maps $\widetilde{M}(nd)|_Y = \widetilde{M}|_Y \otimes_{\mathcal{O}_Y} \mathcal{O}_X(nd)|_Y \to \widetilde{M(n)}|_Y$ (see 26.10.1.5) are isomorphisms,*
5. *given $f \in S_{nd}$ denote $s \in \Gamma(Y, \mathcal{O}_Y(nd))$ the image of f via (26.10.1.3) restricted to Y, then $D_+(f) \cap Y = Y_s$,*
6. *a basis for the topology on Y is given by the collection of opens Y_s, where $s \in \Gamma(Y, \mathcal{O}_Y(nd))$, $n \geq 1$, and*
7. *a basis for the topology of Y is given by those opens $Y_s \subset Y$, for $s \in \Gamma(Y, \mathcal{O}_Y(nd))$, $n \geq 1$ which are affine.*

Proof. Since Y is quasi-compact there exist finitely many homogeneous $f_i \in S_+$, $i = 1, \ldots, n$ such that the standard opens $D_+(f_i)$ give an open covering of Y. Let $d_i = \deg(f_i)$ and set $d = d_1 \ldots d_n$. Note that $D_+(f_i) = D_+(f_i^{d/d_i})$ and hence we see immediately that $Y \subset W_d$, by characterization (2) in Lemma 26.10.4 or by (1) using Lemma 26.10.2. Note that (1) implies (2), (3) and (4) by Lemma 26.10.4. (Note that (3) is a special case of (4).) Assertion (5) follows from Lemma 26.10.5. Assertions (6) and (7) follow because the open subsets $D_+(f)$ form a basis for the topology of X and are affine. □

26.11. Functoriality of Proj

A graded ring map $\psi : A \to B$ does not always give rise to a morphism of associated projective homogeneous spectra. The reason is that the inverse image $\psi^{-1}(\mathfrak{q})$ of a homogeneous prime $\mathfrak{q} \subset B$ may contain the irrelevant prime A_+ even if $\mathfrak{q}$ does not contain B_+. The correct result is stated as follows.

Lemma 26.11.1. *Let A, B be two graded rings. Set $X = Proj(A)$ and $Y = Proj(B)$. Let $\psi : A \to B$ be a graded ring map. Set*

$$U(\psi) = \bigcup\nolimits_{f \in A_+ \text{ homogeneous}} D_+(\psi(f)) \subset Y.$$

Then there is a canonical morphism of schemes

$$r_\psi : U(\psi) \longrightarrow X$$

and a map of $\mathbf{Z}$-graded $\mathcal{O}_{U(\psi)}$-algebras

$$\theta = \theta_\psi : r_\psi^* \left(\bigoplus\nolimits_{d \in \mathbf{Z}} \mathcal{O}_X(d) \right) \longrightarrow \bigoplus\nolimits_{d \in \mathbf{Z}} \mathcal{O}_{U(\psi)}(d).$$

The triple $(U(\psi), r_\psi, \theta)$ is characterized by the following properties:

(1) *For every $d \geq 0$ the diagram*

$$\begin{array}{ccccc} A_d & & \xrightarrow{\qquad\qquad\psi\qquad\qquad} & & B_d \\ \downarrow & & & & \downarrow \\ \Gamma(X, \mathcal{O}_X(d)) & \xrightarrow{\theta} & \Gamma(U(\psi), \mathcal{O}_Y(d)) & \longleftarrow & \Gamma(Y, \mathcal{O}_Y(d)) \end{array}$$

is commutative.

(2) *For any $f \in A_+$ homogeneous we have $r_\psi^{-1}(D_+(f)) = D_+(\psi(f))$ and the restriction of r_ψ to $D_+(\psi(f))$ corresponds to the ring map $A_{(f)} \to B_{(\psi(f))}$ induced by ψ.*

Proof. Clearly condition (2) uniquely determines the morphism of schemes and the open subset $U(\psi)$. Pick $f \in A_d$ with $d \geq 1$. Note that $\mathcal{O}_X(n)|_{D_+(f)}$ corresponds to the $A_{(f)}$-module $(A_f)_n$ and that $\mathcal{O}_Y(n)|_{D_+(\psi(f))}$ corresponds to the $B_{(\psi(f))}$-module $(B_{\psi(f)})_n$. In other words θ when restricted to $D_+(\psi(f))$ corresponds to a map of $\mathbf{Z}$-graded $B_{(\psi(f))}$-algebras

$$A_f \otimes_{A_{(f)}} B_{(\psi(f))} \longrightarrow B_{\psi(f)}$$

Condition (1) determines the images of all elements of A. Since f is an invertible element which is mapped to $\psi(f)$ we see that $1/f^m$ is mapped to $1/\psi(f)^m$. It easily follows from this that θ is uniquely determined, namely it is given by the rule

$$a/f^m \otimes b/\psi(f)^e \longmapsto \psi(a)b/\psi(f)^{m+e}.$$

To show existence we remark that the proof of uniqueness above gave a well defined prescription for the morphism r and the map θ when restricted to every standard open of the form $D_+(\psi(f)) \subset U(\psi)$ into $D_+(f)$. Call these r_f and θ_f. Hence we only need to verify that if $D_+(f) \subset D_+(g)$ for some $f, g \in A_+$ homogeneous, then the restriction of r_g to $D_+(\psi(f))$ matches r_f. This is clear from the formulas given for r and θ above. □

Lemma 26.11.2. *Let A, B, and C be graded rings. Set $X = Proj(A)$, $Y = Proj(B)$ and $Z = Proj(C)$. Let $\varphi : A \to B$, $\psi : B \to C$ be graded ring maps. Then we have*

$$U(\psi \circ \varphi) = r_\varphi^{-1}(U(\psi)) \quad \text{and} \quad r_{\psi\circ\varphi} = r_\varphi \circ r_\psi|_{U(\psi\circ\varphi)}.$$

In addition we have

$$\theta_\psi \circ r_\psi^* \theta_\varphi = \theta_{\psi\circ\varphi}$$

with obvious notation.

Proof. Omitted. □

Lemma 26.11.3. *With hypotheses and notation as in Lemma 26.11.1 above. Assume $A_d \to B_d$ is surjective for all $d \gg 0$. Then*

1. *$U(\psi) = Y$,*
2. *$r_\psi : Y \to X$ is a closed immersion, and*
3. *the maps $\theta : r_\psi^* \mathcal{O}_X(n) \to \mathcal{O}_Y(n)$ are surjective but not isomorphisms in general (even if $A \to B$ is surjective).*

Proof. Part (1) follows from the definition of $U(\psi)$ and the fact that $D_+(f) = D_+(f^n)$ for any $n > 0$. For $f \in A_+$ homogeneous we see that $A_{(f)} \to B_{(\psi(f))}$ is surjective because any element of $B_{(\psi(f))}$ can be represented by a fraction $b/\psi(f)^n$ with n arbitrarily large (which forces the degree of $b \in B$ to be large). This proves (2). The same argument shows the map

$$A_f \to B_{\psi(f)}$$

is surjective which proves the surjectivity of θ. For an example where this map is not an isomorphism consider the graded ring $A = k[x, y]$ where k is a field and $\deg(x) = 1$, $\deg(y) = 2$. Set $I = (x)$, so that $B = k[y]$. Note that $\mathcal{O}_Y(1) = 0$ in this case. But it is easy to see that $r_\psi^* \mathcal{O}_Y(1)$ is not zero. (There are less silly examples.) □

Lemma 26.11.4. *With hypotheses and notation as in Lemma 26.11.1 above. Assume $A_d \to B_d$ is an isomorphism for all $d \gg 0$. Then*

1. *$U(\psi) = Y$,*
2. *$r_\psi : Y \to X$ is an isomorphism, and*
3. *the maps $\theta : r_\psi^* \mathcal{O}_X(n) \to \mathcal{O}_Y(n)$ are isomorphisms.*

Proof. We have (1) by Lemma 26.11.3. Let $f \in A_+$ be homogeneous. The assumption on ψ implies that $A_f \to B_f$ is an isomorphism (details omitted). Thus it is clear that r_ψ and θ restrict to isomorphisms over $D_+(f)$. The lemma follows. □

Lemma 26.11.5. *With hypotheses and notation as in Lemma 26.11.1 above. Assume $A_d \to B_d$ is surjective for $d \gg 0$ and that A is generated by A_1 over A_0. Then*

1. *$U(\psi) = Y$,*
2. *$r_\psi : Y \to X$ is a closed immersion, and*

(3) *the maps* $\theta : r_\psi^*\mathcal{O}_X(n) \to \mathcal{O}_Y(n)$ *are isomorphisms.*

Proof. By Lemmas 26.11.4 and 26.11.2 we may replace B by the image of $A \to B$ without changing X or the sheaves $\mathcal{O}_X(n)$. Thus we may assume that $A \to B$ is surjective. By Lemma 26.11.3 we get (1) and (2) and surjectivity in (3). By Lemma 26.10.3 we see that both $\mathcal{O}_X(n)$ and $\mathcal{O}_Y(n)$ are invertible. Hence θ is an isomorphism. □

Lemma 26.11.6. *With hypotheses and notation as in Lemma 26.11.1 above. Assume there exists a ring map* $R \to A_0$ *and a ring map* $R \to R'$ *such that* $B = R' \otimes_R A$. *Then*

(1) $U(\psi) = Y$,
(2) *the diagram*

$$\begin{array}{ccc} Y = Proj(B) & \xrightarrow{r_\psi} & Proj(A) = X \\ \downarrow & & \downarrow \\ \mathrm{Spec}(R') & \longrightarrow & \mathrm{Spec}(R) \end{array}$$

is a fibre product square, and
(3) *the maps* $\theta : r_\psi^*\mathcal{O}_X(n) \to \mathcal{O}_Y(n)$ *are isomorphisms.*

Proof. This follows immediately by looking at what happens over the standard opens $D_+(f)$ for $f \in A_+$. □

Lemma 26.11.7. *With hypotheses and notation as in Lemma 26.11.1 above. Assume there exists a* $g \in A_0$ *such that* ψ *induces an isomorphism* $A_g \to B$. *Then* $U(\psi) = Y$, $r_\psi : Y \to X$ *is an open immersion which induces an isomorphism of* Y *with the inverse image of* $D(g) \subset \mathrm{Spec}(A_0)$. *Moreover the map* θ *is an isomorphism.*

Proof. This is a special case of Lemma 26.11.6 above. □

26.12. Morphisms into Proj

Let S be a graded ring. Let $X = \mathrm{Proj}(S)$ be the homogeneous spectrum of S. Let $d \geq 1$ be an integer. Consider the open subscheme

$$U_d = \bigcup\nolimits_{f \in S_d} D_+(f) \quad \subset \quad X = \mathrm{Proj}(S) \tag{26.12.0.1}$$

Note that $d|d' \Rightarrow U_d \subset U_{d'}$ and $X = \bigcup_d U_d$. Neither X nor U_d need be quasi-compact, see Algebra, Lemma 10.53.3. Let us write $\mathcal{O}_{U_d}(n) = \mathcal{O}_X(n)|_{U_d}$. By Lemma 26.10.2 we know that $\mathcal{O}_{U_d}(nd)$, $n \in \mathbf{Z}$ is an invertible $\mathcal{O}_{U_d}$-module and that all the multiplication maps $\mathcal{O}_{U_d}(nd) \otimes_{\mathcal{O}_{U_d}} \mathcal{O}_X(m) \to \mathcal{O}_{U_d}(nd+m)$ of (26.10.1.1) are isomorphisms. In particular we have $\mathcal{O}_{U_d}(nd) \cong \mathcal{O}_{U_d}(d)^{\otimes n}$. The graded ring map (26.10.1.3) on global sections combined with restriction to U_d give a homomorphism of graded rings

$$\psi^d : S^{(d)} \longrightarrow \Gamma_*(U_d, \mathcal{O}_{U_d}(d)). \tag{26.12.0.2}$$

For the notation $S^{(d)}$, see Algebra, Section 10.52. For the notation Γ_* see Modules, Definition 17.21.4. Moreover, since U_d is covered by the opens $D_+(f)$, $f \in S_d$ we see that $\mathcal{O}_{U_d}(d)$ is globally generated by the sections in the image of $\psi_1^d : S_1^{(d)} = S_d \to \Gamma(U_d, \mathcal{O}_{U_d}(d))$, see Modules, Definition 17.4.1.

Let Y be a scheme, and let $\varphi : Y \to X$ be a morphism of schemes. Assume the image $\varphi(Y)$ is contained in the open subscheme U_d of X. By the discussion following Modules, Definition 17.21.4 we obtain a homomorphism of graded rings

$$\Gamma_*(U_d, \mathcal{O}_{U_d}(d)) \longrightarrow \Gamma_*(Y, \varphi^*\mathcal{O}_X(d)).$$

The composition of this and ψ^d gives a graded ring homomorphism

$$\psi^d_\varphi : S^{(d)} \longrightarrow \Gamma_*(Y, \varphi^*\mathcal{O}_X(d)) \tag{26.12.0.3}$$

which has the property that the invertible sheaf $\varphi^*\mathcal{O}_X(d)$ is globally generated by the sections in the image of $(S^{(d)})_1 = S_d \to \Gamma(Y, \varphi^*\mathcal{O}_X(d))$.

Lemma 26.12.1. *Let S be a graded ring, and $X = Proj(S)$. Let $d \geq 1$ and $U_d \subset X$ as above. Let Y be a scheme. Let $\mathcal{L}$ be an invertible sheaf on Y. Let $\psi : S^{(d)} \to \Gamma_*(Y, \mathcal{L})$ be a graded ring homomorphism such that $\mathcal{L}$ is generated by the sections in the image of $\psi|_{S_d} : S_d \to \Gamma(Y, \mathcal{L})$. Then there exists a morphism $\varphi : Y \to X$ such that $\varphi(Y) \subset U_d$ and an isomorphism $\alpha : \varphi^*\mathcal{O}_{U_d}(d) \to \mathcal{L}$ such that ψ^d_φ agrees with ψ via α:*

$$\begin{array}{ccccc}
\Gamma_*(Y, \mathcal{L}) & \xleftarrow{\alpha} & \Gamma_*(Y, \varphi^*\mathcal{O}_{U_d}(d)) & \xleftarrow{\varphi^*} & \Gamma_*(U_d, \mathcal{O}_{U_d}(d)) \\
\uparrow{\scriptstyle \psi} & & & \nwarrow{\scriptstyle \psi^d_\varphi} & \uparrow{\scriptstyle \psi^d} \\
S^{(d)} & & \xleftarrow{id} & & S^{(d)}
\end{array}$$

commutes. Moreover, the pair (φ, α) is unique.

Proof. Pick $f \in S_d$. Denote $s = \psi(f) \in \Gamma(Y, \mathcal{L})$. On the open set Y_s where s does not vanish multiplication by s induces an isomorphism $\mathcal{O}_{Y_s} \to \mathcal{L}|_{Y_s}$, see Modules, Lemma 17.21.7. We will denote the inverse of this map $x \mapsto x/s$, and similarly for powers of $\mathcal{L}$. Using this we define a ring map $\psi_{(f)} : S_{(f)} \to \Gamma(Y_s, \mathcal{O})$ by mapping the fraction a/f^n to $\psi(a)/s^n$. By Schemes, Lemma 25.6.4 this corresponds to a morphism $\varphi_f : Y_s \to \mathrm{Spec}(S_{(f)}) = D_+(f)$. We also introduce the isomorphism $\alpha_f : \varphi_f^*\mathcal{O}_{D_+(f)}(d) \to \mathcal{L}|_{Y_s}$ which maps the pullback of the trivializing section f over $D_+(f)$ to the trivializing section s over Y_s. With this choice the commutativity of the diagram in the lemma holds with Y replace by Y_s, φ replaced by φ_f, and α replaced by α_f; verification omitted.

Suppose that $f' \in S_d$ is a second element, and denote $s' = \psi(f') \in \Gamma(Y, \mathcal{L})$. Then $Y_s \cap Y_{s'} = Y_{ss'}$ and similarly $D_+(f) \cap D_+(f') = D_+(ff')$. In Lemma 26.10.6 we saw that $D_+(f') \cap D_+(f)$ is the same as the set of points of $D_+(f)$ where the section of $\mathcal{O}_X(d)$ defined by f' does not vanish. Hence $\varphi_f^{-1}(D_+(f') \cap D_+(f)) = Y_s \cap Y_{s'} = \varphi_{f'}^{-1}(D_+(f') \cap D_+(f))$. On $D_+(f) \cap D_+(f')$ the fraction f/f' is an invertible section of the structure sheaf with inverse f'/f. Note that $\psi_{(f')}(f/f') = \psi(f)/s' = s/s'$ and $\psi_{(f)}(f'/f) = \psi(f')/s = s'/s$. We claim there is a unique ring map $S_{(ff')} \to \Gamma(Y_{ss'}, \mathcal{O})$ making the following diagram commute

$$\begin{array}{ccccc}
\Gamma(Y_s, \mathcal{O}) & \longrightarrow & \Gamma(Y_{ss'}, \mathcal{O}) & \longleftarrow & \Gamma(Y_{s,'}\mathcal{O}) \\
\uparrow{\scriptstyle \psi_{(f)}} & & \uparrow & & \uparrow{\scriptstyle \psi_{(f')}} \\
S_{(f)} & \longrightarrow & S_{(ff')} & \longleftarrow & S_{(f')}
\end{array}$$

It exists because we may use the rule $x/(ff')^n \mapsto \psi(x)/(ss')^n$, which "works" by the formulas above. Uniqueness follows as $\mathrm{Proj}(S)$ is separated, see Lemma 26.8.8

and its proof. This shows that the morphisms φ_f and $\varphi_{f'}$ agree over $Y_s \cap Y_{s'}$. The restrictions of α_f and $\alpha_{f'}$ agree over $Y_s \cap Y_{s'}$ because the regular functions s/s' and $\psi_{(f')}(f)$ agree. This proves that the morphisms ψ_f glue to a global morphism from Y into $U_d \subset X$, and that the maps α_f glue to an isomorphism satisfying the conditions of the lemma.

We still have to show the pair (φ, α) is unique. Suppose (φ', α') is a second such pair. Let $f \in S_d$. By the commutativity of the diagrams in the lemma we have that the inverse images of $D_+(f)$ under both φ and φ' are equal to $Y_{\psi(f)}$. Since the opens $D_+(f)$ are a basis for the topology on X, and since X is a sober topological space (see Schemes, Lemma 25.11.1) this means the maps φ and φ' are the same on underlying topological spaces. Let us use $s = \psi(f)$ to trivialize the invertible sheaf $\mathcal{L}$ over $Y_{\psi(f)}$. By the commutativity of the diagrams we have that $\alpha^{\otimes n}(\psi_\varphi^d(x)) = \psi(x) = (\alpha')^{\otimes n}(\psi_{\varphi'}^d(x))$ for all $x \in S_{nd}$. By construction of ψ_φ^d and $\psi_{\varphi'}^d$ we have $\psi_\varphi^d(x) = \varphi^\sharp(x/f^n)\psi_\varphi^d(f^n)$ over $Y_{\psi(f)}$, and similarly for $\psi_{\varphi'}^d$. by the commutativity of the diagrams of the lemma we deduce that $\varphi^\sharp(x/f^n) = (\varphi')^\sharp(x/f^n)$. This proves that φ and φ' induce the same morphism from $Y_{\psi(f)}$ into the affine scheme $D_+(f) = \mathrm{Spec}(S_{(f)})$. Hence φ and φ' are the same as morphisms. Finally, it remains to show that the commutativity of the diagram of the lemma singles out, given φ, a unique α. We omit the verification. □

We continue the discussion from above the lemma. Let S be a graded ring. Let Y be a scheme. We will consider *triples* $(d, \mathcal{L}, \psi)$ where

(1) $d \geq 1$ is an integer,
(2) $\mathcal{L}$ is an invertible $\mathcal{O}_Y$-module, and
(3) $\psi : S^{(d)} \to \Gamma_*(Y, \mathcal{L})$ is a graded ring homomorphism such that $\mathcal{L}$ is generated by the global sections $\psi(f)$, with $f \in S_d$.

Given a morphism $h : Y' \to Y$ and a triple $(d, \mathcal{L}, \psi)$ over Y we can pull it back to the triple $(d, h^*\mathcal{L}, h^* \circ \psi)$. Given two triples $(d, \mathcal{L}, \psi)$ and $(d, \mathcal{L}', \psi')$ with the same integer d we say they are *strictly equivalent* if there exists an isomorphism $\beta : \mathcal{L} \to \mathcal{L}'$ such that $\beta \circ \psi = \psi'$ as graded ring maps $S^{(d)} \to \Gamma_*(Y, \mathcal{L}')$.

For each integer $d \geq 1$ we define

$$\begin{aligned} F_d : Sch^{opp} &\longrightarrow Sets, \\ Y &\longmapsto \{\text{strict equivalence classes of triples } (d, \mathcal{L}, \psi) \text{ as above}\} \end{aligned}$$

with pullbacks as defined above.

Lemma 26.12.2. *Let S be a graded ring. Let $X = Proj(S)$. The open subscheme $U_d \subset X$ (26.12.0.1) represents the functor F_d and the triple $(d, \mathcal{O}_{U_d}(d), \psi^d)$ defined above is the universal family (see Schemes, Section 25.15).*

Proof. This is a reformulation of Lemma 26.12.1 □

Lemma 26.12.3. *Let S be a graded ring generated as an S_0-algebra by the elements of S_1. In this case the scheme $X = Proj(S)$ represents the functor which associates to a scheme Y the set of pairs $(\mathcal{L}, \psi)$, where*

(1) *$\mathcal{L}$ is an invertible $\mathcal{O}_Y$-module, and*
(2) *$\psi : S \to \Gamma_*(Y, \mathcal{L})$ is a graded ring homomorphism such that $\mathcal{L}$ is generated by the global sections $\psi(f)$, with $f \in S_1$*

up to strict equivalence as above.

Proof. Under the assumptions of the lemma we have $X = U_1$ and the lemma is a reformulation of Lemma 26.12.2 above. □

We end this section with a discussion of a functor corresponding to $\mathrm{Proj}(S)$ for a general graded ring S. We advise the reader to skip the rest of this section.

Fix an arbitrary graded ring S. Let T be a scheme. We will say two triples $(d, \mathcal{L}, \psi)$ and $(d', \mathcal{L}', \psi')$ over T with possibly different integers d, d' are *equivalent* if there exists an isomorphism $\beta : \mathcal{L}^{\otimes d'} \to (\mathcal{L}')^{\otimes d}$ of invertible sheaves over T such that $\beta \circ \psi|_{S^{(dd')}}$ and $\psi'|_{S^{(dd')}}$ agree as graded ring maps $S^{(dd')} \to \Gamma_*(Y, (\mathcal{L}')^{\otimes dd'})$.

Lemma 26.12.4. *Let S be a graded ring. Set $X = Proj(S)$. Let T be a scheme. Let $(d, \mathcal{L}, \psi)$ and $(d', \mathcal{L}', \psi')$ be two triples over T. The following are equivalent:*

1. *Let $n = lcm(d, d')$. Write $n = ad = a'd'$. There exists an isomorphism $\beta : \mathcal{L}^{\otimes a} \to (\mathcal{L}')^{\otimes a'}$ with the property that $\beta \circ \psi|_{S^{(n)}}$ and $\psi'|_{S^{(n)}}$ agree as graded ring maps $S^{(n)} \to \Gamma_*(Y, (\mathcal{L}')^{\otimes n})$.*
2. *The triples $(d, \mathcal{L}, \psi)$ and $(d', \mathcal{L}', \psi')$ are equivalent.*
3. *For some positive integer $n = ad = a'd'$ there exists an isomorphism $\beta : \mathcal{L}^{\otimes a} \to (\mathcal{L}')^{\otimes a'}$ with the property that $\beta \circ \psi|_{S^{(n)}}$ and $\psi'|_{S^{(n)}}$ agree as graded ring maps $S^{(n)} \to \Gamma_*(Y, (\mathcal{L}')^{\otimes n})$.*
4. *The morphisms $\varphi : T \to X$ and $\varphi' : T \to X$ associated to $(d, \mathcal{L}, \psi)$ and $(d', \mathcal{L}', \psi')$ are equal.*

Proof. Clearly (1) implies (2) and (2) implies (3) by restricting to more divisible degrees and powers of invertible sheaves. Also (3) implies (4) by the uniqueness statement in Lemma 26.12.1. Thus we have to prove that (4) implies (1). Assume (4), in other words $\varphi = \varphi'$. Note that this implies that we may write $\mathcal{L} = \varphi^*\mathcal{O}_X(d)$ and $\mathcal{L}' = \varphi^*\mathcal{O}_X(d')$. Moreover, via these identifications we have that the graded ring maps ψ and ψ' correspond to the restriction of the canonical graded ring map

$$S \longrightarrow \bigoplus\nolimits_{n \geq 0} \Gamma(X, \mathcal{O}_X(n))$$

to $S^{(d)}$ and $S^{(d')}$ composed with pullback by φ (by Lemma 26.12.1 again). Hence taking β to be the isomorphism

$$(\varphi^*\mathcal{O}_X(d))^{\otimes a} = \varphi^*\mathcal{O}_X(n) = (\varphi^*\mathcal{O}_X(d'))^{\otimes a'}$$

works. □

Let S be a graded ring. Let $X = \mathrm{Proj}(S)$. Over the open subscheme scheme $U_d \subset X = \mathrm{Proj}(S)$ (26.12.0.1) we have the triple $(d, \mathcal{O}_{U_d}(d), \psi^d)$. Clearly, if $d|d'$ the triples $(d, \mathcal{O}_{U_d}(d), \psi^d)$ and $(d', \mathcal{O}_{U_{d'}}(d'), \psi^{d'})$ are equivalent when restricted to the open U_d (which is a subset of $U_{d'}$). This, combined with Lemma 26.12.1 shows that morphisms $Y \to X$ correspond roughly to equivalence classes of triples over Y. This is not quite true since if Y is not quasi-compact, then there may not be a single triple which works. Thus we have to be slightly careful in defining the corresponding functor.

Here is one possible way to do this. Suppose $d' = ad$. Consider the transformation of functors $F_d \to F_{d'}$ which assigns to the triple $(d, \mathcal{L}, \psi)$ over T the triple $(d', \mathcal{L}^{\otimes a}, \psi|_{S^{(d')}})$. One of the implications of Lemma 26.12.4 is that the transformation $F_d \to F_{d'}$ is injective! For a quasi-compact scheme T we define

$$F(T) = \bigcup\nolimits_{d \in \mathbf{N}} F_d(T)$$

with transition maps as explained above. This clearly defines a contravariant functor on the category of quasi-compact schemes with values in sets. For a general scheme T we define

$$F(T) = \lim_{V \subset T \text{ quasi-compact open}} F(V).$$

In other words, an element ξ of $F(T)$ corresponds to a compatible system of choices of elements $\xi_V \in F(V)$ where V ranges over the quasi-compact opens of T. We omit the definition of the pullback map $F(T) \to F(T')$ for a morphism $T' \to T$ of schemes. Thus we have defined our functor

$$F : Sch^{opp} \longrightarrow Sets$$

Lemma 26.12.5. *Let S be a graded ring. Let $X = Proj(S)$. The functor F defined above is representable by the scheme X.*

Proof. We have seen above that the functor F_d corresponds to the open subscheme $U_d \subset X$. Moreover the transformation of functors $F_d \to F_{d'}$ (if $d|d'$) defined above corresponds to the inclusion morphism $U_d \to U_{d'}$ (see discussion above). Hence to show that F is represented by X it suffices to show that $T \to X$ for a quasi-compact scheme T ends up in some U_d, and that for a general scheme T we have

$$\mathrm{Mor}(T, X) = \lim_{V \subset T \text{ quasi-compact open}} \mathrm{Mor}(V, X).$$

These verifications are omitted. □

26.13. Projective space

Projective space is one of the fundamental objects studied in algebraic geometry. In this section we just give its construction as Proj of a polynomial ring. Later we will discover many of its beautiful properties.

Lemma 26.13.1. *Let $S = \mathbf{Z}[T_0, \ldots, T_n]$ with $\deg(T_i) = 1$. The scheme*

$$\mathbf{P}^n_{\mathbf{Z}} = Proj(S)$$

represents the functor which associates to a scheme Y the pairs $(\mathcal{L}, (s_0, \ldots, s_n))$ where

1. *$\mathcal{L}$ is an invertible $\mathcal{O}_Y$-module, and*
2. *$s_0, \ldots, s_n$ are global sections of $\mathcal{L}$ which generate $\mathcal{L}$*

up to the following equivalence: $(\mathcal{L}, (s_0, \ldots, s_n)) \sim (\mathcal{N}, (t_0, \ldots, t_n)) \Leftrightarrow$ there exists an isomorphism $\beta : \mathcal{L} \to \mathcal{N}$ with $\beta(s_i) = t_i$ for $i = 0, \ldots, n$.

Proof. This is a special case of Lemma 26.12.3 above. Namely, for any graded ring A we have

$$\begin{aligned} \mathrm{Mor}_{gradedrings}(\mathbf{Z}[T_0, \ldots, T_n], A) &= A_1 \times \ldots \times A_1 \\ \psi &\mapsto (\psi(T_0), \ldots, \psi(T_n)) \end{aligned}$$

and the degree 1 part of $\Gamma_*(Y, \mathcal{L})$ is just $\Gamma(Y, \mathcal{L})$. □

Definition 26.13.2. The scheme $\mathbf{P}^n_{\mathbf{Z}} = \mathrm{Proj}(\mathbf{Z}[T_0, \ldots, T_n])$ is called *projective n-space over $\mathbf{Z}$*. Its base change $\mathbf{P}^n_S$ to a scheme S is called *projective n-space over S*. If R is a ring the base change to $\mathrm{Spec}(R)$ is denoted $\mathbf{P}^n_R$ and called *projective n-space over R*.

Given a scheme Y over S and a pair $(\mathcal{L}, (s_0, \ldots, s_n))$ as in Lemma 26.13.1 the induced morphism to $\mathbf{P}^n_S$ is denoted

$$\varphi_{(\mathcal{L},(s_0,\ldots,s_n))} : Y \longrightarrow \mathbf{P}^n_S$$

This makes sense since the pair defines a morphism into $\mathbf{P}^n_{\mathbf{Z}}$ and we already have the structure morphism into S so combined we get a morphism into $\mathbf{P}^n_S = \mathbf{P}^n_{\mathbf{Z}} \times S$. Note that this is the S-morphism characterized by

$$\mathcal{L} = \varphi^*_{(\mathcal{L},(s_0,\ldots,s_n))}\mathcal{O}_{\mathbf{P}^n_R}(1) \quad \text{and} \quad s_i = \varphi^*_{(\mathcal{L},(s_0,\ldots,s_n))}T_i$$

where we think of T_i as a global section of $\mathcal{O}_{\mathbf{P}^n_S}(1)$ via (26.10.1.3).

Lemma 26.13.3. *Projective n-space over $\mathbf{Z}$ is covered by $n+1$ standard opens*

$$\mathbf{P}^n_{\mathbf{Z}} = \bigcup\nolimits_{i=0,\ldots,n} D_+(T_i)$$

where each $D_+(T_i)$ is isomorphic to $\mathbf{A}^n_{\mathbf{Z}}$ affine n-space over $\mathbf{Z}$.

Proof. This is true because $\mathbf{Z}[T_0, \ldots, T_n]_+ = (T_0, \ldots, T_n)$ and since

$$\operatorname{Spec}\left(\mathbf{Z}\left[\frac{T_0}{T_i}, \ldots, \frac{T_n}{T_i}\right]\right) \cong \mathbf{A}^n_{\mathbf{Z}}$$

in an obvious way. □

Lemma 26.13.4. *Let S be a scheme. The structure morphism $\mathbf{P}^n_S \to S$ is*

(1) *separated,*
(2) *quasi-compact,*
(3) *satisfies the existence and uniqueness parts of the valuative criterion, and*
(4) *universally closed.*

Proof. All these properties are stable under base change (this is clear for the last two and for the other two see Schemes, Lemmas 25.21.13 and 25.19.3). Hence it suffices to prove them for the morphism $\mathbf{P}^n_{\mathbf{Z}} \to \operatorname{Spec}(\mathbf{Z})$. Separatedness is Lemma 26.8.8. Quasi-compactness follows from Lemma 26.13.3. Existence and uniqueness of the valuative criterion follow from Lemma 26.8.11. Universally closed follows from the above and Schemes, Proposition 25.20.6. □

Remark 26.13.5. What's missing in the list of properties above? Well to be sure the property of being of finite type. The reason we do not list this here is that we have not yet defined the notion of finite type at this point. (Another property which is missing is "smoothness". And I'm sure there are many more you can think of.)

We finish this section with two simple lemmas. These lemmas are special cases of more general results later, but perhaps it makes sense to prove these directly here now.

Lemma 26.13.6. *Let R be a ring. Let $Z \subset \mathbf{P}^n_R$ be a closed subscheme. Let*

$$I_d = Ker\left(R[T_0, \ldots, T_n]_d \longrightarrow \Gamma(Z, \mathcal{O}_{\mathbf{P}^n_R}(d)|_Z)\right)$$

Then $I = \bigoplus I_d \subset R[T_0, \ldots, T_n]$ is a graded ideal and $Z = Proj(R[T_0, \ldots, T_n]/I)$.

Proof. It is clear that I is a graded ideal. Set $Z' = \text{Proj}(R[T_0, \ldots, T_n]/I)$. By Lemma 26.11.5 we see that Z' is a closed subscheme of $\mathbf{P}^n_R$. To see the equality $Z = Z'$ it suffices to check on an standard affine open $D_+(T_i)$. By renumbering the homogeneous coordinates we may assume $i = 0$. Say $Z \cap D_+(T_0)$, resp. $Z' \cap D_+(T_0)$ is cut out by the ideal J, resp. J' of $R[T_1/T_0, \ldots, T_n/T_0]$. Then J' is the ideal generated by the elements $F/T_0^{\deg(F)}$ where $F \in I$ is homogeneous. Suppose the degree of $F \in I$ is d. Since F vanishes as a section of $\mathcal{O}_{\mathbf{P}^n_R}(d)$ restricted to Z we see that F/T_0^d is an element of J. Thus $J' \subset J$.

Conversely, suppose that $f \in J$. If f has total degree d in $T_1/T_0, \ldots, T_n/T_0$, then we can write $f = F/T_0^d$ for some $F \in R[T_0, \ldots, T_n]_d$. Pick $i \in \{1, \ldots, n\}$. Then $Z \cap D_+(T_i)$ is cut out by some ideal $J_i \subset R[T_0/T_i, \ldots, T_n/T_i]$. Moreover,

$$J \cdot R\left[\frac{T_1}{T_0}, \ldots, \frac{T_n}{T_0}, \frac{T_0}{T_i}, \ldots, \frac{T_n}{T_i}\right] = J_i \cdot R\left[\frac{T_1}{T_0}, \ldots, \frac{T_n}{T_0}, \frac{T_0}{T_i}, \ldots, \frac{T_n}{T_i}\right]$$

The left hand side is the localization of J with respect to the element T_i/T_0 and the right hand side is the localization of J_i with respect to the element T_0/T_i. It follows that $T_0^{d_i}F/T_i^{d+d_i}$ is an element of J_i for some d_i sufficiently large. This proves that $T_0^{\max(d_i)}F$ is an element of I, because its restriction to each standard affine open $D_+(T_i)$ vanishes on the closed subscheme $Z \cap D_+(T_i)$. Hence $f \in J'$ and we conclude $J \subset J'$ as desired. □

The following lemma is a special case of the more general Properties, Lemma 27.26.3.

Lemma 26.13.7. *Let R be a ring. Let $\mathcal{F}$ be a quasi-coherent sheaf on $\mathbf{P}^n_R$. For $d \geq 0$ set*

$$M_d = \Gamma(\mathbf{P}^n_R, \mathcal{F} \otimes_{\mathcal{O}_{\mathbf{P}^n_R}} \mathcal{O}_{\mathbf{P}^n_R}(d)) = \Gamma(\mathbf{P}^n_R, \mathcal{F}(d))$$

Then $M = \bigoplus_{d \geq 0} M_d$ is a graded $R[T_0, \ldots, R_n]$-module and there is a canonical isomorphism $\mathcal{F} = \widetilde{M}$.

Proof. The multiplication maps

$$R[T_0, \ldots, R_n]_e \times M_d \longrightarrow M_{d+e}$$

come from the natural isomorphisms

$$\mathcal{O}_{\mathbf{P}^n_R}(e) \otimes_{\mathcal{O}_{\mathbf{P}^n_R}} \mathcal{F}(d) \longrightarrow \mathcal{F}(e + d)$$

see Equation (26.10.1.4). Let us construct the map $c : \widetilde{M} \to \mathcal{F}$. On each of the standard affines $U_i = D_+(T_i)$ we see that $\Gamma(U_i, \widetilde{M}) = (M[1/T_i])_0$ where the subscript $_0$ means degree 0 part. An element of this can be written as m/T_i^d with $m \in M_d$. Since T_i is a generator of $\mathcal{O}(1)$ over U_i we can always write $m|_{U_i} = m_i \otimes T_i^d$ where $m_i \in \Gamma(U_i, \mathcal{F})$ is a unique section. Thus a natural guess is $c(m/T_i^d) = m_i$. A small argument, which is omitted here, shows that this gives a well defined map $c : \widetilde{M} \to \mathcal{F}$ if we can show that

$$(T_i/T_j)^d m_i|_{U_i \cap U_j} = m_j|_{U_i \cap U_j}$$

in $M[1/T_iT_j]$. But this is clear since on the overlap the generators T_i and T_j of $\mathcal{O}(1)$ differ by the invertible function T_i/T_j.

Injectivity of c. We may check for injectivity over the affine opens U_i. Let $i \in \{0, \ldots, n\}$ and let s be an element $s = m/T_i^d \in \Gamma(U_i, \widetilde{M})$ such that $c(m/T_i^d) = 0$.

By the description of c above this means that $m_i = 0$, hence $m|_{U_i} = 0$. Hence $T_i^e m = 0$ in M for some e. Hence $s = m/T_i^d = T_i^e/T_i^{e+d} = 0$ as desired.

Surjectivity of c. We may check for surjectivity over the affine opens U_i. By renumbering it suffices to check it over U_0. Let $s \in \mathcal{F}(U_0)$. Let us write $\mathcal{F}|_{U_i} = \widetilde{N_i}$ for some $R[T_0/T_i, \ldots, T_0/T_i]$-module N_i, which is possible because $\mathcal{F}$ is quasi-coherent. So s corresponds to an element $x \in N_0$. Then we have that

$$(N_i)_{T_j/T_i} \cong (N_j)_{T_i/T_j}$$

(where the subscripts mean "principal localization at") as modules over the ring

$$R\left[\frac{T_0}{T_i}, \ldots, \frac{T_n}{T_i}, \frac{T_0}{T_j}, \ldots, \frac{T_n}{T_j}\right].$$

This means that for some large integer d there exist elements $s_i \in N_i$, $i = 1, \ldots, n$ such that

$$s = (T_i/T_0)^d s_i$$

on $U_0 \cap U_i$. Next, we look at the difference

$$t_{ij} = s_i - (T_j/T_i)^d s_j$$

on $U_i \cap U_j$, $0 < i < j$. By our choice of s_i we know that $t_{ij}|_{U_0 \cap U_i \cap U_j} = 0$. Hence there exists a large integer e such that $(T_0/T_i)^e t_{ij} = 0$. Set $s_i' = (T_0/T_i)^e s_i$, and $s_0' = s$. Then we will have

$$s_a' = (T_b/T_a)^{e+d} s_b'$$

on $U_a \cap U_b$ for all a, b. This is exactly the condition that the elements s_a' glue to a global section $m \in \Gamma(\mathbf{P}_R^n, \mathcal{F}(e+d))$. And moreover $c(m/T_0^{e+d}) = s$ by construction. Hence c is surjective and we win. □

26.14. Invertible sheaves and morphisms into Proj

Let T be a scheme and let $\mathcal{L}$ be an invertible sheaf on T. For a section $s \in \Gamma(T, \mathcal{L})$ we denote T_s the open subset of points where s does not vanish. See Modules, Lemma 17.21.7. We can view the following lemma as a slight generalization of Lemma 26.12.3. It also is a generalization of Lemma 26.11.1.

Lemma 26.14.1. *Let A be a graded ring. Set $X = Proj(A)$. Let T be a scheme. Let $\mathcal{L}$ be an invertible $\mathcal{O}_T$-module. Let $\psi : A \to \Gamma_*(T, \mathcal{L})$ be a homomorphism of graded rings. Set*

$$U(\psi) = \bigcup\nolimits_{f \in A_+ \text{ homogeneous}} T_{\psi(f)}$$

The morphism ψ induces a canonical morphism of schemes

$$r_{\mathcal{L},\psi} : U(\psi) \longrightarrow X$$

together with a map of $\mathbf{Z}$-graded $\mathcal{O}_T$-algebras

$$\theta : r_{\mathcal{L},\psi}^* \left(\bigoplus\nolimits_{d \in \mathbf{Z}} \mathcal{O}_X(d)\right) \longrightarrow \bigoplus\nolimits_{d \in \mathbf{Z}} \mathcal{L}^{\otimes d}|_{U(\psi)}.$$

The triple $(U(\psi), r_{\mathcal{L},\psi}, \theta)$ is characterized by the following properties:

(1) *For $f \in A_+$ homogeneous we have $r_{\mathcal{L},\psi}^{-1}(D_+(f)) = T_{\psi(f)}$.*

(2) *For every $d \geq 0$ the diagram*

$$\begin{array}{ccc} A_d & \xrightarrow{\psi} & \Gamma(T, \mathcal{L}^{\otimes d}) \\ {\scriptstyle (26.10.1.3)}\downarrow & & \downarrow{\scriptstyle restrict} \\ \Gamma(X, \mathcal{O}_X(d)) & \xrightarrow{\theta} & \Gamma(U(\psi), \mathcal{L}^{\otimes d}) \end{array}$$

is commutative.

Moreover, for any $d \geq 1$ and any open subscheme $V \subset T$ such that the sections in $\psi(A_d)$ generate $\mathcal{L}^{\otimes d}|_V$ the morphism $r_{\mathcal{L},\psi}|_V$ agrees with the morphism $\varphi : V \to Proj(A)$ and the map $\theta|_V$ agrees with the map $\alpha : \varphi^*\mathcal{O}_X(d) \to \mathcal{L}^{\otimes d}|_V$ where (φ, α) is the pair of Lemma 26.12.1 associated to $\psi|_{A^{(d)}} : A^{(d)} \to \Gamma_*(V, \mathcal{L}^{\otimes d})$.

Proof. Suppose that we have two triples $(U, r : U \to X, \theta)$ and $(U', r' : U' \to X, \theta')$ satisfying (1) and (2). Property (1) implies that $U = U' = U(\psi)$ and that $r = r'$ as maps of underlying topological spaces, since the opens $D_+(f)$ form a basis for the topology on X, and since X is a sober topological space (see Algebra, Section 10.53 and Schemes, Lemma 25.11.1). Let $f \in A_+$ be homogeneous. Note that $\Gamma(D_+(f), \bigoplus_{n\in\mathbf{Z}} \mathcal{O}_X(n)) = A_f$ as a $\mathbf{Z}$-graded algebra. Consider the two $\mathbf{Z}$-graded ring maps

$$\theta, \theta' : A_f \longrightarrow \Gamma(T_{\psi(f)}, \bigoplus \mathcal{L}^{\otimes n}).$$

We know that multiplication by f (resp. $\psi(f)$) is an isomorphism on the left (resp. right) hand side. We also know that $\theta(x/1) = \theta'(x/1) = \psi(x)|_{T_{\psi(f)}}$ by (2) for all $x \in A$. Hence we deduce easily that $\theta = \theta'$ as desired. Considering the degree 0 parts we deduce that $r^\sharp = (r')^\sharp$, i.e., that $r = r'$ as morphisms of schemes. This proves the uniqueness.

Now we come to existence. By the uniqueness just proved, it is enough to construct the pair (r, θ) locally on T. Hence we may assume that $T = \mathrm{Spec}(R)$ is affine, that $\mathcal{L} = \mathcal{O}_T$ and that for some $f \in A_+$ homogeneous we have $\psi(f)$ generates $\mathcal{O}_T = \mathcal{O}_T^{\otimes \deg(f)}$. In other words, $\psi(f) = u \in R^*$ is a unit. In this case the map ψ is a graded ring map

$$A \longrightarrow R[x] = \Gamma_*(T, \mathcal{O}_T)$$

which maps f to $ux^{\deg(f)}$. Clearly this extends (uniquely) to a $\mathbf{Z}$-graded ring map $\theta : A_f \to R[x, x^{-1}]$ by mapping $1/f$ to $u^{-1}x^{-\deg(f)}$. This map in degree zero gives the ring map $A_{(f)} \to R$ which gives the morphism $r : T = \mathrm{Spec}(R) \to \mathrm{Spec}(A_{(f)}) = D_+(f) \subset X$. Hence we have constructed (r, θ) in this special case.

Let us show the last statement of the lemma. According to Lemma 26.12.1 the morphism constructed there is the unique one such that the displayed diagram in its statement commutes. The commutativity of the diagram in the lemma implies the commutativity when restricted to V and $A^{(d)}$. Whence the result. □

Remark 26.14.2. Assumptions as in Lemma 26.14.1 above. The image of the morphism $r_{\mathcal{L},\psi}$ need not be contained in the locus where the sheaf $\mathcal{O}_X(1)$ is invertible. Here is an example. Let k be a field. Let $S = k[A, B, C]$ graded by $\deg(A) = 1$, $\deg(B) = 2$, $\deg(C) = 3$. Set $X = \mathrm{Proj}(S)$. Let $T = \mathbf{P}^2_k = \mathrm{Proj}(k[X_0, X_1, X_2])$. Recall that $\mathcal{L} = \mathcal{O}_T(1)$ is invertible and that $\mathcal{O}_T(n) = \mathcal{L}^{\otimes n}$. Consider the composition ψ of the maps

$$S \to k[X_0, X_1, X_2] \to \Gamma_*(T, \mathcal{L}).$$

Here the first map is $A \mapsto X_0^6$, $B \mapsto X_1^3$, $C \mapsto X_2^3$ and the second map is (26.10.1.3). By the lemma this corresponds to a morphism $r_{\mathcal{L},\psi} : T \to X = \text{Proj}(S)$ which is easily seen to be surjective. On the other hand, in Remark 26.9.2 we showed that the sheaf $\mathcal{O}_X(1)$ is not invertible at all points of X.

26.15. Relative Proj via glueing

Situation 26.15.1. Here S is a scheme, and $\mathcal{A}$ is a quasi-coherent graded $\mathcal{O}_S$-algebra.

In this section we outline how to construct a morphism of schemes

$$\underline{\text{Proj}}_S(\mathcal{A}) \longrightarrow S$$

by glueing the homogeneous spectra $\text{Proj}(\Gamma(U, \mathcal{A}))$ where U ranges over the affine opens of S. We first show that the homogeneous spectra of the values of $\mathcal{A}$ over affines form a suitable collection of schemes, as in Lemma 26.2.1.

Lemma 26.15.2. *In Situation 26.15.1. Suppose $U \subset U' \subset S$ are affine opens. Let $A = \mathcal{A}(U)$ and $A' = \mathcal{A}(U')$. The map of graded rings $A' \to A$ induces a morphism $r : Proj(A) \to Proj(A')$, and the diagram*

$$\begin{array}{ccc} Proj(A) & \longrightarrow & Proj(A') \\ \downarrow & & \downarrow \\ U & \longrightarrow & U' \end{array}$$

is cartesian. Moreover there are canonical isomorphisms $\theta : r^*\mathcal{O}_{Proj(A')}(n) \to \mathcal{O}_{Proj(A)}(n)$ compatible with multiplication maps.

Proof. Let $R = \mathcal{O}_S(U)$ and $R' = \mathcal{O}_S(U')$. Note that the map $R \otimes_{R'} A' \to A$ is an isomorphism as $\mathcal{A}$ is quasi-coherent (see Schemes, Lemma 25.7.3 for example). Hence the lemma follows from Lemma 26.11.6. □

In particular the morphism $\text{Proj}(A) \to \text{Proj}(A')$ of the lemma is an open immersion.

Lemma 26.15.3. *In Situation 26.15.1. Suppose $U \subset U' \subset U'' \subset S$ are affine opens. Let $A = \mathcal{A}(U)$, $A' = \mathcal{A}(U')$ and $A'' = \mathcal{A}(U'')$. The composition of the morphisms $r : Proj(A) \to Proj(A')$, and $r' : Proj(A') \to Proj(A'')$ of Lemma 26.15.2 gives the morphism $r'' : Proj(A) \to Proj(A'')$ of Lemma 26.15.2. A similar statement holds for the isomorphisms θ.*

Proof. This follows from Lemma 26.11.2 since the map $A'' \to A$ is the composition of $A'' \to A'$ and $A' \to A$. □

Lemma 26.15.4. *In Situation 26.15.1. There exists a morphism of schemes*

$$\pi : \underline{Proj}_S(\mathcal{A}) \longrightarrow S$$

with the following properties:

1. *for every affine open $U \subset S$ there exists an isomorphism $i_U : \pi^{-1}(U) \to Proj(A)$ with $A = \mathcal{A}(U)$, and*
2. *for $U \subset U' \subset S$ affine open the composition*

$$Proj(A) \xrightarrow{i_U^{-1}} \pi^{-1}(U) \xrightarrow{inclusion} \pi^{-1}(U') \xrightarrow{i_{U'}} Proj(A')$$

with $A = \mathcal{A}(U)$, $A' = \mathcal{A}(U')$ is the open immersion of Lemma 26.15.2 above.

Proof. Follows immediately from Lemmas 26.2.1, 26.15.2, and 26.15.3. □

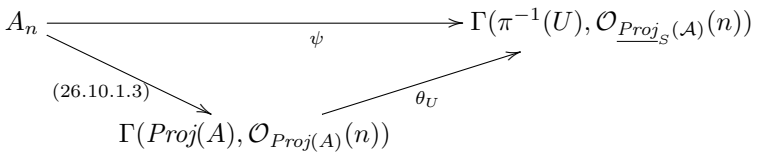
Lemma 26.15.5. *In Situation 26.15.1. The morphism $\pi : \underline{Proj}_S(\mathcal{A}) \to S$ of Lemma 26.15.4 comes with the following additional structure. There exists a quasi-coherent $\mathbf{Z}$-graded sheaf of $\mathcal{O}_{\underline{Proj}_S(\mathcal{A})}$-algebras $\bigoplus_{n\in\mathbf{Z}} \mathcal{O}_{\underline{Proj}_S(\mathcal{A})}(n)$, and a morphism of graded $\mathcal{O}_S$-algebras*

$$\psi : \mathcal{A} \longrightarrow \bigoplus\nolimits_{n\geq 0} \pi_* \left(\mathcal{O}_{\underline{Proj}_S(\mathcal{A})}(n)\right)$$

uniquely determined by the following property: For every affine open $U \subset S$ with $A = \mathcal{A}(U)$ there is an isomorphism

$$\theta_U : i_U^* \left(\bigoplus\nolimits_{n\in\mathbf{Z}} \mathcal{O}_{Proj(A)}(n)\right) \longrightarrow \left(\bigoplus\nolimits_{n\in\mathbf{Z}} \mathcal{O}_{\underline{Proj}_S(\mathcal{A})}(n)\right)|_{\pi^{-1}(U)}$$

of $\mathbf{Z}$-graded $\mathcal{O}_{\pi^{-1}(U)}$-algebras such that



is commutative.

Proof. We are going to use Lemma 26.2.2 to glue the sheaves of $\mathbf{Z}$-graded algebras $\bigoplus_{n\in\mathbf{Z}} \mathcal{O}_{\mathrm{Proj}(A)}(n)$ for $A = \mathcal{A}(U)$, $U \subset S$ affine open over the scheme $\underline{\mathrm{Proj}}_S(\mathcal{A})$. We have constructed the data necessary for this in Lemma 26.15.2 and we have checked condition (d) of Lemma 26.2.2 in Lemma 26.15.3. Hence we get the sheaf of $\mathbf{Z}$-graded $\mathcal{O}_{\underline{\mathrm{Proj}}_S(\mathcal{A})}$-algebras $\bigoplus_{n\in\mathbf{Z}} \mathcal{O}_{\underline{\mathrm{Proj}}_S(\mathcal{A})}(n)$ together with the isomorphisms θ_U for all $U \subset S$ affine open and all $n \in \mathbf{Z}$. For every affine open $U \subset S$ with $A = \mathcal{A}(U)$ we have a map $A \to \Gamma(\mathrm{Proj}(A), \bigoplus_{n\geq 0} \mathcal{O}_{\mathrm{Proj}(A)}(n))$. Hence the map ψ exists by functoriality of relative glueing, see Remark 26.2.3. The diagram of the lemma commutes by construction. This characterizes the sheaf of $\mathbf{Z}$-graded $\mathcal{O}_{\underline{\mathrm{Proj}}_S(\mathcal{A})}$-algebras $\bigoplus \mathcal{O}_{\underline{\mathrm{Proj}}_S(\mathcal{A})}(n)$ because the proof of Lemma 26.11.1 shows that having these diagrams commute uniquely determines the maps θ_U. Some details omitted. □

26.16. Relative Proj as a functor

We place ourselves in Situation 26.15.1. So S is a scheme and $\mathcal{A} = \bigoplus_{d\geq 0} \mathcal{A}_d$ is a quasi-coherent graded $\mathcal{O}_S$-algebra. In this section we relativize the construction of Proj by constructing a functor which the relative homogeneous spectrum will represent. As a result we will construct a morphism of schemes

$$\underline{\mathrm{Proj}}_S(\mathcal{A}) \longrightarrow S$$

which above affine opens of S will look like the homogeneous spectrum of a graded ring. The discussion will be modeled after our discussion of the relative spectrum in Section 26.4. The easier method using glueing schemes of the form $\mathrm{Proj}(A)$, $A = \Gamma(U, \mathcal{A})$, $U \subset S$ affine open, is explained in Section 26.15, and the result in this section will be shown to be isomorphic to that one.

Fix for the moment an integer $d \geq 1$. We denote $\mathcal{A}^{(d)} = \bigoplus_{n \geq 0} \mathcal{A}_{nd}$ similarly to the notation in Algebra, Section 10.52. Let T be a scheme. Let us consider *quadruples* $(d, f : T \to S, \mathcal{L}, \psi)$ *over* T where

(1) d is the integer we fixed above,
(2) $f : T \to S$ is a morphism of schemes,
(3) $\mathcal{L}$ is an invertible $\mathcal{O}_T$-module, and
(4) $\psi : f^*\mathcal{A}^{(d)} \to \bigoplus_{n \geq 0} \mathcal{L}^{\otimes n}$ is a homomorphism of graded $\mathcal{O}_T$-algebras such that $f^*\mathcal{A}_d \to \mathcal{L}$ is surjective.

Given a morphism $h : T' \to T$ and a quadruple $(d, f, \mathcal{L}, \psi)$ over T we can pull it back to the quadruple $(d, f \circ h, h^*\mathcal{L}, h^*\psi)$ over T'. Given two quadruples $(d, f, \mathcal{L}, \psi)$ and $(d, f', \mathcal{L}', \psi')$ over T with the same integer d we say they are *strictly equivalent* if $f = f'$ and there exists an isomorphism $\beta : \mathcal{L} \to \mathcal{L}'$ such that $\beta \circ \psi = \psi'$ as graded $\mathcal{O}_T$-algebra maps $f^*\mathcal{A}^{(d)} \to \bigoplus_{n \geq 0} (\mathcal{L}')^{\otimes n}$.

For each integer $d \geq 1$ we define

$$\begin{aligned} F_d : Sch^{opp} &\longrightarrow Sets, \\ T &\longmapsto \{\text{strict equivalence classes of } (d, f : T \to S, \mathcal{L}, \psi) \text{ as above}\} \end{aligned}$$

with pullbacks as defined above.

Lemma 26.16.1. *In Situation 26.15.1. Let $d \geq 1$. Let F_d be the functor associated to $(S, \mathcal{A})$ above. Let $g : S' \to S$ be a morphism of schemes. Set $\mathcal{A}' = g^*\mathcal{A}$. Let F'_d be the functor associated to $(S', \mathcal{A}')$ above. Then there is a canonical isomorphism*

$$F'_d \cong h_{S'} \times_{h_S} F_d$$

of functors.

Proof. A quadruple $(d, f' : T \to S', \mathcal{L}', \psi' : (f')^*(\mathcal{A}')^{(d)} \to \bigoplus_{n \geq 0} (\mathcal{L}')^{\otimes n})$ is the same as a quadruple $(d, f, \mathcal{L}, \psi : f^*\mathcal{A}^{(d)} \to \bigoplus_{n \geq 0} \mathcal{L}^{\otimes n})$ together with a factorization of f as $f = g \circ f'$. Namely, the correspondence is $f = g \circ f'$, $\mathcal{L} = \mathcal{L}'$ and $\psi = \psi'$ via the identifications $(f')^*(\mathcal{A}')^{(d)} = (f')^*g^*(\mathcal{A}^{(d)}) = f^*\mathcal{A}^{(d)}$. Hence the lemma. □

Lemma 26.16.2. *In Situation 26.15.1. Let F_d be the functor associated to $(d, S, \mathcal{A})$ above. If S is affine, then F_d is representable by the open subscheme U_d (26.12.0.1) of the scheme Proj$(\Gamma(S, \mathcal{A}))$.*

Proof. Write $S = \mathrm{Spec}(R)$ and $A = \Gamma(S, \mathcal{A})$. Then A is a graded R-algebra and $\mathcal{A} = \widetilde{A}$. To prove the lemma we have to identify the functor F_d with the functor $F_d^{triples}$ of triples defined in Section 26.12.

Let $(d, f : T \to S, \mathcal{L}, \psi)$ be a quadruple. We may think of ψ as a $\mathcal{O}_S$-module map $\mathcal{A}^{(d)} \to \bigoplus_{n \geq 0} f_*\mathcal{L}^{\otimes n}$. Since $\mathcal{A}^{(d)}$ is quasi-coherent this is the same thing as an R-linear homomorphism of graded rings $A^{(d)} \to \Gamma(S, \bigoplus_{n \geq 0} f_*\mathcal{L}^{\otimes n})$. Clearly, $\Gamma(S, \bigoplus_{n \geq 0} f_*\mathcal{L}^{\otimes n}) = \Gamma_*(T, \mathcal{L})$. Thus we may associate to the quadruple the triple $(d, \mathcal{L}, \psi)$.

Conversely, let $(d, \mathcal{L}, \psi)$ be a triple. The composition $R \to A_0 \to \Gamma(T, \mathcal{O}_T)$ determines a morphism $f : T \to S = \mathrm{Spec}(R)$, see Schemes, Lemma 25.6.4. With this choice of f the map $A^{(d)} \to \Gamma(S, \bigoplus_{n \geq 0} f_*\mathcal{L}^{\otimes n})$ is R-linear, and hence corresponds to a ψ which we can use for a quadruple $(d, f : T \to S, \mathcal{L}, \psi)$. We omit the verification that this establishes an isomorphism of functors $F_d = F_d^{triples}$. □

Lemma 26.16.3. *In Situation 26.15.1. The functor F_d is representable by a scheme.*

Proof. We are going to use Schemes, Lemma 25.15.4.

First we check that F_d satisfies the sheaf property for the Zariski topology. Namely, suppose that T is a scheme, that $T = \bigcup_{i\in I} U_i$ is an open covering, and that $(d, f_i, \mathcal{L}_i, \psi_i) \in F_d(U_i)$ such that $(d, f_i, \mathcal{L}_i, \psi_i)|_{U_i\cap U_j}$ and $(d, f_j, \mathcal{L}_j, \psi_j)|_{U_i\cap U_j}$ are strictly equivalent. This implies that the morphisms $f_i : U_i \to S$ glue to a morphism of schemes $f : T \to S$ such that $f|_{I_i} = f_i$, see Schemes, Section 25.14. Thus $f_i^*\mathcal{A}^{(d)} = f^*\mathcal{A}^{(d)}|_{U_i}$. It also implies there exist isomorphisms $\beta_{ij} : \mathcal{L}_i|_{U_i\cap U_j} \to \mathcal{L}_j|_{U_i\cap U_j}$ such that $\beta_{ij} \circ \psi_i = \psi_j$ on $U_i \cap U_j$. Note that the isomorphisms β_{ij} are uniquely determined by this requirement because the maps $f_i^*\mathcal{A}_d \to \mathcal{L}_i$ are surjective. In particular we see that $\beta_{jk} \circ \beta_{ij} = \beta_{ik}$ on $U_i \cap U_j \cap U_k$. Hence by Sheaves, Section 6.33 the invertible sheaves $\mathcal{L}_i$ glue to an invertible $\mathcal{O}_T$-module $\mathcal{L}$ and the morphisms ψ_i glue to morphism of $\mathcal{O}_T$-algebras $\psi : f^*\mathcal{A}^{(d)} \to \bigoplus_{n\geq 0} \mathcal{L}^{\otimes n}$. This proves that F_d satisfies the sheaf condition with respect to the Zariski topology.

Let $S = \bigcup_{i\in I} U_i$ be an affine open covering. Let $F_{d,i} \subset F_d$ be the subfunctor consisting of those pairs $(f : T \to S, \varphi)$ such that $f(T) \subset U_i$.

We have to show each $F_{d,i}$ is representable. This is the case because $F_{d,i}$ is identified with the functor associated to U_i equipped with the quasi-coherent graded $\mathcal{O}_{U_i}$-algebra $\mathcal{A}|_{U_i}$) by Lemma 26.16.1. Thus the result follows from Lemma 26.16.2.

Next we show that $F_{d,i} \subset F_d$ is representable by open immersions. Let $(f : T \to S, \varphi) \in F_d(T)$. Consider $V_i = f^{-1}(U_i)$. It follows from the definition of $F_{d,i}$ that given $a : T' \to T$ we gave $a^*(f, \varphi) \in F_{d,i}(T')$ if and only if $a(T') \subset V_i$. This is what we were required to show.

Finally, we have to show that the collection $(F_{d,i})_{i\in I}$ covers F_d. Let $(f : T \to S, \varphi) \in F_d(T)$. Consider $V_i = f^{-1}(U_i)$. Since $S = \bigcup_{i\in I} U_i$ is an open covering of S we see that $T = \bigcup_{i\in I} V_i$ is an open covering of T. Moreover $(f, \varphi)|_{V_i} \in F_{d,i}(V_i)$. This finishes the proof of the lemma. □

At this point we can redo the material at the end of Section 26.12 in the current relative setting and define a functor which is representable by $\underline{\mathrm{Proj}}_S(\mathcal{A})$. To do this we introduce the notion of equivalence between two quadruples $(d, f : T \to S, \mathcal{L}, \psi)$ and $(d', f' : T \to S, \mathcal{L}', \psi')$ with possibly different values of the integers d, d'. Namely, we say these are *equivalent* if $f = f'$, and there exists an isomorphism $\beta : \mathcal{L}^{\otimes d'} \to (\mathcal{L}')^{\otimes d}$ such that $\beta \circ \psi|_{f^*\mathcal{A}^{(dd')}} = \psi'|_{f^*\mathcal{A}^{(dd')}}$. The following lemma implies that this defines an equivalence relation. (This is not a complete triviality.)

Lemma 26.16.4. *In Situation 26.15.1. Let T be a scheme. Let $(d, f, \mathcal{L}, \psi)$, $(d', f', \mathcal{L}', \psi')$ be two quadruples over T. The following are equivalent:*

1. *Let $m = lcm(d, d')$. Write $m = ad = a'd'$. We have $f = f'$ and there exists an isomorphism $\beta : \mathcal{L}^{\otimes a} \to (\mathcal{L}')^{\otimes a'}$ with the property that $\beta \circ \psi|_{f^*\mathcal{A}^{(m)}}$ and $\psi'|_{f^*\mathcal{A}^{(m)}}$ agree as graded ring maps $f^*\mathcal{A}^{(m)} \to \bigoplus_{n\geq 0}(\mathcal{L}')^{\otimes mn}$.*
2. *The quadruples $(d, f, \mathcal{L}, \psi)$ and $(d', f', \mathcal{L}', \psi')$ are equivalent.*
3. *We have $f = f'$ and for some positive integer $m = ad = a'd'$ there exists an isomorphism $\beta : \mathcal{L}^{\otimes a} \to (\mathcal{L}')^{\otimes a'}$ with the property that $\beta \circ \psi|_{f^*\mathcal{A}^{(m)}}$ and $\psi'|_{f^*\mathcal{A}^{(m)}}$ agree as graded ring maps $f^*\mathcal{A}^{(m)} \to \bigoplus_{n\geq 0}(\mathcal{L}')^{\otimes mn}$.*

Proof. Clearly (1) implies (2) and (2) implies (3) by restricting to more divisible degrees and powers of invertible sheaves. Assume (3) for some integer $m = ad = a'd'$. Let $m_0 = \text{lcm}(d, d')$ and write it as $m_0 = a_0 d = a'_0 d'$. We are given an isomorphism $\beta : \mathcal{L}^{\otimes a} \to (\mathcal{L}')^{\otimes a'}$ with the property described in (3). We want to find an isomorphism $\beta_0 : \mathcal{L}^{\otimes a_0} \to (\mathcal{L}')^{\otimes a'_0}$ having that property as well. Since by assumption the maps $\psi : f^*\mathcal{A}_d \to \mathcal{L}$ and $\psi' : (f')^*\mathcal{A}_{d'} \to \mathcal{L}'$ are surjective the same is true for the maps $\psi : f^*\mathcal{A}_{m_0} \to \mathcal{L}^{\otimes a_0}$ and $\psi' : (f')^*\mathcal{A}_{m_0} \to (\mathcal{L}')^{\otimes a_0}$. Hence if β_0 exists it is uniquely determined by the condition that $\beta_0 \circ \psi = \psi'$. This means that we may work locally on T. Hence we may assume that $f = f' : T \to S$ maps into an affine open, in other words we may assume that S is affine. In this case the result follows from the corresponding result for triples (see Lemma 26.12.4) and the fact that triples and quadruples correspond in the affine base case (see proof of Lemma 26.16.2). □

Suppose $d' = ad$. Consider the transformation of functors $F_d \to F_{d'}$ which assigns to the quadruple $(d, f, \mathcal{L}, \psi)$ over T the quadruple $(d', f, \mathcal{L}^{\otimes a}, \psi|_{f^*\mathcal{A}^{(d')}})$. One of the implications of Lemma 26.16.4 is that the transformation $F_d \to F_{d'}$ is injective! For a quasi-compact scheme T we define

$$F(T) = \bigcup\nolimits_{d \in \mathbf{N}} F_d(T)$$

with transition maps as explained above. This clearly defines a contravariant functor on the category of quasi-compact schemes with values in sets. For a general scheme T we define

$$F(T) = \lim\nolimits_{V \subset T \text{ quasi-compact open}} F(V).$$

In other words, an element ξ of $F(T)$ corresponds to a compatible system of choices of elements $\xi_V \in F(V)$ where V ranges over the quasi-compact opens of T. We omit the definition of the pullback map $F(T) \to F(T')$ for a morphism $T' \to T$ of schemes. Thus we have defined our functor

$$F : Sch^{opp} \longrightarrow Sets \tag{26.16.4.1}$$

Lemma 26.16.5. *In Situation 26.15.1. The functor F above is representable by a scheme.*

Proof. Let $U_d \to S$ be the scheme representing the functor F_d defined above. Let $\mathcal{L}_d$, $\psi^d : \pi_d^*\mathcal{A}^{(d)} \to \bigoplus_{n \geq 0} \mathcal{L}_d^{\otimes n}$ be the universal object. If $d|d'$, then we may consider the quadruple $(d', \pi_d, \mathcal{L}_d^{\otimes d'/d}, \psi^d|_{\mathcal{A}^{(d')}})$ which determines a canonical morphism $U_d \to U_{d'}$ over S. By construction this morphism corresponds to the transformation of functors $F_d \to F_{d'}$ defined above.

For every affine open $\text{Spec}(R) = V \subset S$ setting $A = \Gamma(V, \mathcal{A})$ we have a canonical identification of the base change $U_{d,V}$ with the corresponding open subscheme of $\text{Proj}(A)$, see Lemma 26.16.2. Moreover, the morphisms $U_{d,V} \to U_{d',V}$ constructed above correspond to the inclusions of opens in $\text{Proj}(A)$. Thus we conclude that $U_d \to U_{d'}$ is an open immersion.

This allows us to construct X by glueing the schemes U_d along the open immersions $U_d \to U_{d'}$. Technically, it is convenient to choose a sequence $d_1|d_2|d_3|\ldots$ such that every positive integer divides one of the d_i and to simply take $X = \bigcup U_{d_i}$ using the open immersions above. It is then a simple matter to prove that X represents the functor F. □

Lemma 26.16.6. *In Situation 26.15.1. The scheme $\pi : \underline{Proj}_S(\mathcal{A}) \to S$ constructed in Lemma 26.15.4 and the scheme representing the functor F are canonically isomorphic as schemes over S.*

Proof. Let X be the scheme representing the functor F. Note that X is a scheme over S since the functor F comes equipped with a natural transformation $F \to h_S$. Write $Y = \underline{\mathrm{Proj}}_S(\mathcal{A})$. We have to show that $X \cong Y$ as S-schemes. We give two arguments.

The first argument uses the construction of X as the union of the schemes U_d representing F_d in the proof of Lemma 26.16.5. Over each affine open of S we can identify X with the homogeneous spectrum of the sections of $\mathcal{A}$ over that open, since this was true for the opens U_d. Moreover, these identifications are compatible with further restrictions to smaller affine opens. On the other hand, Y was constructed by glueing these homogeneous spectra. Hence we can glue these isomorphisms to an isomorphism between X and $\underline{\mathrm{Proj}}_S(\mathcal{A})$ as desired. Details omitted.

Here is the second argument. Lemma 26.15.5 shows that there exists a morphism of graded algebras

$$\psi : \pi^*\mathcal{A} \longrightarrow \bigoplus\nolimits_{n\geq 0} \mathcal{O}_Y(n)$$

over Y which on sections over affine opens of S agrees with (26.10.1.3). Hence for every $y \in Y$ there exists an open neighbourhood $V \subset Y$ of y and an integer $d \geq 1$ such that for $d|n$ the sheaf $\mathcal{O}_Y(n)|_V$ is invertible and the multiplication maps $\mathcal{O}_Y(n)|_V \otimes_{\mathcal{O}_V} \mathcal{O}_Y(m)|_V \to \mathcal{O}_Y(n+m)|_V$ are isomorphisms. Thus ψ restricted to the sheaf $\pi^*\mathcal{A}^{(d)}|_V$ gives an element of $F_d(V)$. Since the opens V cover Y we see "ψ" gives rise to an element of $F(Y)$. Hence a canonical morphism $Y \to X$ over S. Because this construction is completely canonical to see that it is an isomorphism we may work locally on S. Hence we reduce to the case S affine where the result is clear. □

Definition 26.16.7. Let S be a scheme. Let $\mathcal{A}$ be a quasi-coherent sheaf of graded $\mathcal{O}_S$-algebras. The *relative homogeneous spectrum of $\mathcal{A}$ over S*, or the *homogeneous spectrum of $\mathcal{A}$ over S*, or the *relative Proj of $\mathcal{A}$ over S* is the scheme constructed in Lemma 26.15.4 which represents the functor F (26.16.4.1), see Lemma 26.16.6. We denote it $\pi : \underline{\mathrm{Proj}}_S(\mathcal{A}) \to S$.

The relative Proj comes equipped with a quasi-coherent sheaf of $\mathbf{Z}$-graded algebras $\bigoplus_{n\in\mathbf{Z}} \mathcal{O}_{\underline{\mathrm{Proj}}_S(\mathcal{A})}(n)$ (the twists of the structure sheaf) and a "universal" homomorphism of graded algebras

$$\psi_{univ} : \mathcal{A} \longrightarrow \pi_* \left(\bigoplus\nolimits_{n\geq 0} \mathcal{O}_{\underline{\mathrm{Proj}}_S(\mathcal{A})}(n) \right)$$

see Lemma 26.15.5. We may also think of this as a homomorphism

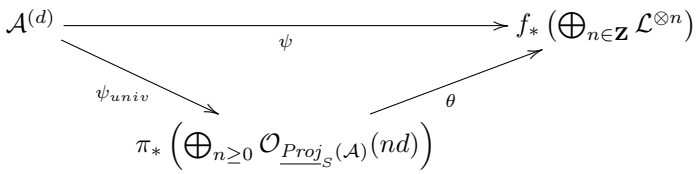
$$\psi_{univ} : \pi^*\mathcal{A} \longrightarrow \bigoplus\nolimits_{n\geq 0} \mathcal{O}_{\underline{\mathrm{Proj}}_S(\mathcal{A})}(n)$$

if we like. The following lemma is a formulation of the universality of this object.

Lemma 26.16.8. *In Situation 26.15.1. Let $(f : T \to S, d, \mathcal{L}, \psi)$ be a quadruple. Let $r_{d,\mathcal{L},\psi} : T \to \underline{Proj}_S(\mathcal{A})$ be the associated S-morphism. There exists an isomorphism of $\mathbf{Z}$-graded $\mathcal{O}_T$-algebras*

$$\theta : r_{d,\mathcal{L},\psi}^* \left(\bigoplus\nolimits_{n\in\mathbf{Z}} \mathcal{O}_{\underline{Proj}_S(\mathcal{A})}(nd) \right) \longrightarrow \bigoplus\nolimits_{n\in\mathbf{Z}} \mathcal{L}^{\otimes n}$$

such that the following diagram commutes



The commutativity of this diagram uniquely determines θ.

Proof. Note that the quadruple $(f : T \to S, d, \mathcal{L}, \psi)$ defines an element of $F_d(T)$. Let $U_d \subset \underline{\mathrm{Proj}}_S(\mathcal{A})$ be the locus where the sheaf $\mathcal{O}_{\underline{\mathrm{Proj}}_S(\mathcal{A})}(d)$ is invertible and generated by the image of $\psi_{univ} : \pi^*\mathcal{A}_d \to \mathcal{O}_{\underline{\mathrm{Proj}}_S(\mathcal{A})}(d)$. Recall that U_d represents the functor F_d, see the proof of Lemma 26.16.5. Hence the result will follow if we can show the quadruple $(U_d \to S, d, \mathcal{O}_{U_d}(d), \psi_{univ}|_{\mathcal{A}^{(d)}})$ is the universal family, i.e., the representing object in $F_d(U_d)$. We may do this after restricting to an affine open of S because (a) the formation of the functors F_d commutes with base change (see Lemma 26.16.1), and (b) the pair $(\bigoplus_{n\in\mathbf{Z}} \mathcal{O}_{\underline{\mathrm{Proj}}_S(\mathcal{A})}(n), \psi_{univ})$ is constructed by glueing over affine opens in S (see Lemma 26.15.5). Hence we may assume that S is affine. In this case the functor of quadruples F_d and the functor of triples F_d agree (see proof of Lemma 26.16.2) and moreover Lemma 26.12.2 shows that $(d, \mathcal{O}_{U_d}(d), \psi^d)$ is the universal triple over U_d. Going backwards through the identifications in the proof of Lemma 26.16.2 shows that $(U_d \to S, d, \mathcal{O}_{U_d}(d), \psi_{univ}|_{\mathcal{A}^{(d)}})$ is the universal quadruple as desired. □

Lemma 26.16.9. *Let S be a scheme and $\mathcal{A}$ be a quasi-coherent sheaf of graded $\mathcal{O}_S$-algebras. The morphism $\pi : \underline{Proj}_S(\mathcal{A}) \to S$ is separated.*

Proof. To prove a morphism is separated we may work locally on the base, see Schemes, Section 25.21. By construction $\underline{\mathrm{Proj}}_S(\mathcal{A})$ is over any affine $U \subset S$ isomorphic to $\mathrm{Proj}(A)$ with $A = \mathcal{A}(U)$. By Lemma 26.8.8 we see that $\mathrm{Proj}(A)$ is separated. Hence $\mathrm{Proj}(A) \to U$ is separated (see Schemes, Lemma 25.21.14) as desired. □

Lemma 26.16.10. *Let S be a scheme and $\mathcal{A}$ be a quasi-coherent sheaf of graded $\mathcal{O}_S$-algebras. Let $g : S' \to S$ be any morphism of schemes. Then there is a canonical isomorphism*

$$r : \underline{Proj}_{S'}(g^*\mathcal{A}) \longrightarrow S' \times_S \underline{Proj}_S(\mathcal{A})$$

as well as a corresponding isomorphism

$$\theta : r^* pr_2^* \left(\bigoplus\nolimits_{d\in\mathbf{Z}} \mathcal{O}_{\underline{Proj}_S(\mathcal{A})}(d)\right) \longrightarrow \bigoplus\nolimits_{d\in\mathbf{Z}} \mathcal{O}_{\underline{Proj}_{S'}(g^*\mathcal{A})}(d)$$

of $\mathbf{Z}$-graded $\mathcal{O}_{\underline{Proj}_{S'}(g^*\mathcal{A})}$-algebras.

Proof. This follows from Lemma 26.16.1 and the construction of $\underline{\mathrm{Proj}}_S(\mathcal{A})$ in Lemma 26.16.5 as the union of the schemes U_d representing the functors F_d. In terms of the construction of relative Proj via glueing this isomorphism is given by the isomorphisms constructed in Lemma 26.11.6 which provides us with the isomorphism θ. Some details omitted. □

Lemma 26.16.11. *Let S be a scheme. Let $\mathcal{A}$ be a quasi-coherent sheaf of graded $\mathcal{O}_S$-modules generated as an $\mathcal{A}_0$-algebra by $\mathcal{A}_1$. In this case the scheme $X =$*

$\underline{Proj}_S(\mathcal{A})$ represents the functor F_1 which associates to a scheme $f : T \to S$ over S the set of pairs $(\mathcal{L}, \psi)$, where

(1) *$\mathcal{L}$ is an invertible $\mathcal{O}_T$-module, and*
(2) *$\psi : f^*\mathcal{A} \to \bigoplus_{n\geq 0} \mathcal{L}^{\otimes n}$ is a graded $\mathcal{O}_T$-algebra homomorphism such that $f^*\mathcal{A}_1 \to \mathcal{L}$ is surjective*

up to strict equivalence as above. Moreover, in this case all the quasi-coherent sheaves $\mathcal{O}_{\underline{Proj}(\mathcal{A})}(n)$ are invertible $\mathcal{O}_{\underline{Proj}(\mathcal{A})}$-modules and the multiplication maps induce isomorphisms $\mathcal{O}_{\underline{Proj}(\mathcal{A})}(n) \otimes_{\mathcal{O}_{\underline{Proj}(\mathcal{A})}} \mathcal{O}_{\underline{Proj}(\mathcal{A})}(m) = \mathcal{O}_{\underline{Proj}(\mathcal{A})}(n+m)$.

Proof. Under the assumptions of the lemma the sheaves $\mathcal{O}_{\underline{\mathrm{Proj}}(\mathcal{A})}(n)$ are invertible and the multiplication maps isomorphisms by Lemma 26.16.5 and Lemma 26.12.3 over affine opens of S. Thus X actually represents the functor F_1, see proof of Lemma 26.16.5. □

26.17. Quasi-coherent sheaves on relative Proj

We briefly discuss how to deal with graded modules in the relative setting.

We place ourselves in Situation 26.15.1. So S is a scheme, and $\mathcal{A}$ is a quasi-coherent graded $\mathcal{O}_S$-algebra. Let $\mathcal{M} = \bigoplus_{n\in\mathbf{Z}} \mathcal{M}_n$ be a graded $\mathcal{A}$-module, quasi-coherent as an $\mathcal{O}_S$-module. We are going to describe the associated quasi-coherent sheaf of modules on $\underline{\mathrm{Proj}}_S(\mathcal{A})$. We first describe the value of this sheaf schemes T mapping into the relative Proj.

Let T be a scheme. Let $(d, f : T \to S, \mathcal{L}, \psi)$ be a quadruple over T, as in Section 26.16. We define a quasi-coherent sheaf $\widetilde{\mathcal{M}}_T$ of $\mathcal{O}_T$-modules as follows

$$\widetilde{\mathcal{M}}_T = \left(f^*\mathcal{M}^{(d)} \otimes_{f^*\mathcal{A}^{(d)}} \left(\bigoplus\nolimits_{n\in\mathbf{Z}} \mathcal{L}^{\otimes n}\right)\right)_0 \tag{26.17.0.1}$$

So $\widetilde{\mathcal{M}}_T$ is the degree 0 part of the tensor product of the graded $f^*\mathcal{A}^{(d)}$-modules $\mathcal{M}^{(d)}$ and $\bigoplus_{n\in\mathbf{Z}} \mathcal{L}^{\otimes n}$. Note that the sheaf $\widetilde{\mathcal{M}}_T$ depends on the quadruple even though we suppressed this in the notation. This construction has the pleasing property that given any morphism $g : T' \to T$ we have $\widetilde{\mathcal{M}}_{T'} = g^*\widetilde{\mathcal{M}}_T$ where $\widetilde{\mathcal{M}}_{T'}$ denotes the quasi-coherent sheaf associated to the pullback quadruple $(d, f \circ g, g^*\mathcal{L}, g^*\psi)$.

Since all sheaves in (26.17.0.1) are quasi-coherent we can spell out the construction over an affine open $\mathrm{Spec}(C) = V \subset T$ which maps into an affine open $\mathrm{Spec}(R) = U \subset S$. Namely, suppose that $\mathcal{A}|_U$ corresponds to the graded R-algebra A, that $\mathcal{M}|_U$ corresponds to the graded A-module M, and that $\mathcal{L}|_V$ corresponds to the invertible C-module L. The map ψ gives rise to a graded R-algebra map $\gamma : A^{(d)} \to \bigoplus_{n\geq 0} L^{\otimes n}$. (Tensor powers of L over C.) Then $(\widetilde{\mathcal{M}}_T)|_V$ is the quasi-coherent sheaf associated to the C-module

$$N_{R,C,A,M,\gamma} = \left(M^{(d)} \otimes_{A^{(d)},\gamma} \left(\bigoplus\nolimits_{n\in\mathbf{Z}} L^{\otimes n}\right)\right)_0$$

By assumption we may even cover T by affine opens V such that there exists some $a \in A_d$ such that $\gamma(a) \in L$ is a C-basis for the module L. In that case any element of $N_{R,C,A,M,\gamma}$ is a sum of pure tensors $\sum m_i \otimes \gamma(a)^{-n_i}$ with $m \in M_{n_i d}$. In fact we may multiply each m_i with a suitable positive power of a and collect terms to see that each element of $N_{R,C,A,M,\gamma}$ can be written as $m \otimes \gamma(a)^{-n}$ with $m \in M_{nd}$ and $n \gg 0$. In other words we see that in this case

$$N_{R,C,A,M,\gamma} = M_{(a)} \otimes_{A_{(a)}} C$$

where the map $A_{(a)} \to C$ is the map $x/a^n \mapsto \gamma(x)/\gamma(a)^n$. In other words, this is the value of $\widetilde{M}$ on $D_+(a) \subset \text{Proj}(A)$ pulled back to $\text{Spec}(C)$ via the morphism $\text{Spec}(C) \to D_+(a)$ coming from γ.

Lemma 26.17.1. *In Situation 26.15.1. For any quasi-coherent sheaf of graded $\mathcal{A}$-modules $\mathcal{M}$ on S, there exists a canonical associated sheaf of $\mathcal{O}_{\underline{Proj}_S(\mathcal{A})}$-modules $\widetilde{\mathcal{M}}$ with the following properties:*

1. *Given a scheme T and a quadruple $(T \to S, d, \mathcal{L}, \psi)$ over T corresponding to a morphism $h : T \to \underline{Proj}_S(\mathcal{A})$ there is a canonical isomorphism $\widetilde{\mathcal{M}}_T = h^*\widetilde{\mathcal{M}}$ where $\widetilde{\mathcal{M}}_T$ is defined by (26.17.0.1).*
2. *The isomorphisms of (1) are compatible with pullbacks.*
3. *There is a canonical map*
$$\pi^*\mathcal{M}_0 \longrightarrow \widetilde{\mathcal{M}}.$$
4. *The construction $\mathcal{M} \mapsto \widetilde{\mathcal{M}}$ is functorial in $\mathcal{M}$.*
5. *The construction $\mathcal{M} \mapsto \widetilde{\mathcal{M}}$ is exact.*
6. *There are canonical maps*
$$\widetilde{\mathcal{M}} \otimes_{\mathcal{O}_{\underline{Proj}_S(\mathcal{A})}} \widetilde{\mathcal{N}} \longrightarrow \widetilde{\mathcal{M} \otimes_{\mathcal{A}} \mathcal{N}}$$
as in Lemma 26.9.1.
7. *There exist canonical maps*
$$\pi^*\mathcal{M} \longrightarrow \bigoplus\nolimits_{n \in \mathbf{Z}} \widetilde{\mathcal{M}(n)}$$
generalizing (26.10.1.6).
8. *The formation of $\widetilde{\mathcal{M}}$ commutes with base change.*

Proof. Omitted. We should split this lemma into parts and prove the parts separately. □

26.18. Functoriality of relative Proj

This section is the analogue of Section 26.11 for the relative Proj. Let S be a scheme. A graded $\mathcal{O}_S$-algebra map $\psi : \mathcal{A} \to \mathcal{B}$ does not always give rise to a morphism of associated relative Proj. The correct result is stated as follows.

Lemma 26.18.1. *Let S be a scheme. Let $\mathcal{A}$, $\mathcal{B}$ be two graded quasi-coherent $\mathcal{O}_S$-algebras. Set $p : X = \underline{Proj}_S(\mathcal{A}) \to S$ and $q : Y = \underline{Proj}_S(\mathcal{B}) \to S$. Let $\psi : \mathcal{A} \to \mathcal{B}$ be a homomorphism of graded $\mathcal{O}_S$-algebras. There is a canonical open $U(\psi) \subset Y$ and a canonical morphism of schemes*
$$r_\psi : U(\psi) \longrightarrow X$$
over S and a map of $\mathbf{Z}$-graded $\mathcal{O}_{U(\psi)}$-algebras
$$\theta = \theta_\psi : r_\psi^* \left(\bigoplus\nolimits_{d \in \mathbf{Z}} \mathcal{O}_X(d) \right) \longrightarrow \bigoplus\nolimits_{d \in \mathbf{Z}} \mathcal{O}_{U(\psi)}(d).$$
The triple $(U(\psi), r_\psi, \theta)$ is characterized by the property that for any affine open $W \subset S$ the triple
$$(U(\psi) \cap p^{-1}W, \quad r_\psi|_{U(\psi)\cap p^{-1}W} : U(\psi) \cap p^{-1}W \to q^{-1}W, \quad \theta|_{U(\psi)\cap p^{-1}W})$$
is equal to the triple associated to $\psi : \mathcal{A}(W) \to \mathcal{B}(W)$ in Lemma 26.11.1 via the identifications $p^{-1}W = Proj(\mathcal{A}(W))$ and $q^{-1}W = Proj(\mathcal{B}(W))$ of Section 26.15.

Proof. This lemma proves itself by glueing the local triples. □

Lemma 26.18.2. *Let S be a scheme. Let $\mathcal{A}$, $\mathcal{B}$, and $\mathcal{C}$ be quasi-coherent graded $\mathcal{O}_S$-algebras. Set $X = \underline{Proj}_S(\mathcal{A})$, $Y = \underline{Proj}_S(\mathcal{B})$ and $Z = \underline{Proj}_S(\mathcal{C})$. Let $\varphi : \mathcal{A} \to \mathcal{B}$, $\psi : \mathcal{B} \to \mathcal{C}$ be graded $\mathcal{O}_S$-algebra maps. Then we have*

$$U(\psi \circ \varphi) = r_\varphi^{-1}(U(\psi)) \quad \text{and} \quad r_{\psi\circ\varphi} = r_\varphi \circ r_\psi|_{U(\psi\circ\varphi)}.$$

In addition we have

$$\theta_\psi \circ r_\psi^* \theta_\varphi = \theta_{\psi\circ\varphi}$$

with obvious notation.

Proof. Omitted. □

Lemma 26.18.3. *With hypotheses and notation as in Lemma 26.18.1 above. Assume $\mathcal{A}_d \to \mathcal{B}_d$ is surjective for $d \gg 0$. Then*

1. *$U(\psi) = Y$,*
2. *$r_\psi : Y \to X$ is a closed immersion, and*
3. *the maps $\theta : r_\psi^* \mathcal{O}_X(n) \to \mathcal{O}_Y(n)$ are surjective but not isomorphisms in general (even if $\mathcal{A} \to \mathcal{B}$ is surjective).*

Proof. Follows on combining Lemma 26.18.1 with Lemma 26.11.3. □

Lemma 26.18.4. *With hypotheses and notation as in Lemma 26.18.1 above. Assume $\mathcal{A}_d \to \mathcal{B}_d$ is an isomorphism for all $d \gg 0$. Then*

1. *$U(\psi) = Y$,*
2. *$r_\psi : Y \to X$ is an isomorphism, and*
3. *the maps $\theta : r_\psi^* \mathcal{O}_X(n) \to \mathcal{O}_Y(n)$ are isomorphisms.*

Proof. Follows on combining Lemma 26.18.1 with Lemma 26.11.4. □

Lemma 26.18.5. *With hypotheses and notation as in Lemma 26.18.1 above. Assume $\mathcal{A}_d \to \mathcal{B}_d$ is surjective for $d \gg 0$ and that $\mathcal{A}$ is generated by $\mathcal{A}_1$ over $\mathcal{A}_0$. Then*

1. *$U(\psi) = Y$,*
2. *$r_\psi : Y \to X$ is a closed immersion, and*
3. *the maps $\theta : r_\psi^* \mathcal{O}_X(n) \to \mathcal{O}_Y(n)$ are isomorphisms.*

Proof. Follows on combining Lemma 26.18.1 with Lemma 26.11.5. □

26.19. Invertible sheaves and morphisms into relative Proj

It seems that we may need the following lemma somewhere. The situation is the following:

1. Let S be a scheme.
2. Let $\mathcal{A}$ be a quasi-coherent graded $\mathcal{O}_S$-algebra.
3. Denote $\pi : \underline{\mathrm{Proj}}_S(\mathcal{A}) \to S$ the relative homogeneous spectrum over S.
4. Let $f : X \to S$ be a morphism of schemes.
5. Let $\mathcal{L}$ be an invertible $\mathcal{O}_X$-module.
6. Let $\psi : f^*\mathcal{A} \to \bigoplus_{d\geq 0} \mathcal{L}^{\otimes d}$ be a homomorphism of graded $\mathcal{O}_X$-algebras.

Given this data set

$$U(\psi) = \bigcup\nolimits_{(U,V,a)} U_{\psi(a)}$$

where (U, V, a) satisfies:

(1) $V \subset S$ affine open,
(2) $U = f^{-1}(V)$, and
(3) $a \in \mathcal{A}(V)_+$ is homogeneous.

Namely, then $\psi(a) \in \Gamma(U, \mathcal{L}^{\otimes \deg(a)})$ and $U_{\psi(a)}$ is the corresponding open (see Modules, Lemma 17.21.7).

Lemma 26.19.1. *With assumptions and notation as above. The morphism ψ induces a canonical morphism of schemes over S*

$$r_{\mathcal{L},\psi} : U(\psi) \longrightarrow \underline{Proj}_S(\mathcal{A})$$

together with a map of graded $\mathcal{O}_{U(\psi)}$-algebras

$$\theta : r_{\mathcal{L},\psi}^* \left(\bigoplus\nolimits_{d \geq 0} \mathcal{O}_{\underline{Proj}_S(\mathcal{A})}(d) \right) \longrightarrow \bigoplus\nolimits_{d \geq 0} \mathcal{L}^{\otimes d}|_{U(\psi)}$$

characterized by the following properties:

(1) *For every open $V \subset S$ and every $d \geq 0$ the diagram*

$$\begin{array}{ccc} \mathcal{A}_d(V) & \xrightarrow{\psi} & \Gamma(f^{-1}(V), \mathcal{L}^{\otimes d}) \\ \downarrow{\scriptstyle \psi} & & \downarrow{\scriptstyle restrict} \\ \Gamma(\pi^{-1}(V), \mathcal{O}_{\underline{Proj}_S(\mathcal{A})}(d)) & \xrightarrow{\theta} & \Gamma(f^{-1}(V) \cap U(\psi), \mathcal{L}^{\otimes d}) \end{array}$$

is commutative.

(2) *For any $d \geq 1$ and any open subscheme $W \subset X$ such that $\psi|_W : f^*\mathcal{A}_d|_W \to \mathcal{L}^{\otimes d}|_W$ is surjective the restriction of the morphism $r_{\mathcal{L},\psi}$ agrees with the morphism $W \to \underline{Proj}_S(\mathcal{A})$ which exists by the construction of the relative homogeneous spectrum, see Definition 26.16.7.*

(3) *For any affine open $V \subset S$, the restriction*

$$(U(\psi) \cap f^{-1}(V), r_{\mathcal{L},\psi}|_{U(\psi)\cap f^{-1}(V)}, \theta|_{U(\psi)\cap f^{-1}(V)})$$

agrees via i_V (see Lemma 26.15.4) with the triple $(U(\psi'), r_{\mathcal{L},\psi'}, \theta')$ of Lemma 26.14.1 associated to the map $\psi' : A = \mathcal{A}(V) \to \Gamma_*(f^{-1}(V), \mathcal{L}|_{f^{-1}(V)})$ induced by ψ.

Proof. Use characterization (3) to construct the morphism $r_{\mathcal{L},\psi}$ and θ locally over S. Use the uniqueness of Lemma 26.14.1 to show that the construction glues. Details omitted. □

26.20. Twisting by invertible sheaves and relative Proj

Let S be a scheme. Let $\mathcal{A} = \bigoplus_{d \geq 0} \mathcal{A}_d$ be a quasi-coherent graded $\mathcal{O}_S$-algebra. Let $\mathcal{L}$ be an invertible sheaf on S. In this situation we obtain another quasi-coherent graded $\mathcal{O}_S$-algebra, namely

$$\mathcal{B} = \bigoplus\nolimits_{d \geq 0} \mathcal{A}_d \otimes_{\mathcal{O}_S} \mathcal{L}^{\otimes d}$$

It turns out that $\mathcal{A}$ and $\mathcal{B}$ have isomorphic relative homogeneous spectra.

Lemma 26.20.1. *With notation S, $\mathcal{A}$, $\mathcal{L}$ and $\mathcal{B}$ as above. There is a canonical isomorphism*

$$\begin{array}{ccc} P = \underline{Proj}_S(\mathcal{A}) & \xrightarrow[g]{} & \underline{Proj}_S(\mathcal{B}) = P' \\ & {\scriptstyle\pi}\searrow \quad \swarrow{\scriptstyle\pi'} & \\ & S & \end{array}$$

with the following properties

(1) *There are isomorphisms $\theta_n : g^*\mathcal{O}_{P'}(n) \to \mathcal{O}_P(n) \otimes \pi^*\mathcal{L}^{\otimes n}$ which fit together to give an isomorphism of $\mathbf{Z}$-graded algebras*

$$\theta : g^*\left(\bigoplus\nolimits_{n\in\mathbf{Z}} \mathcal{O}_{P'}(n)\right) \longrightarrow \bigoplus\nolimits_{n\in\mathbf{Z}} \mathcal{O}_P(n) \otimes \pi^*\mathcal{L}^{\otimes n}$$

(2) *For every open $V \subset S$ the diagrams*

$$\begin{array}{ccc} \mathcal{A}_n(V) \otimes \mathcal{L}^{\otimes n}(V) & \xrightarrow[multiply]{} & \mathcal{B}_n(V) \\ \downarrow{\scriptstyle\psi\otimes\pi^*} & & \\ \Gamma(\pi^{-1}V, \mathcal{O}_P(n)) \otimes \Gamma(\pi^{-1}V, \pi^*\mathcal{L}^{\otimes n}) & & \downarrow{\scriptstyle\psi} \\ \downarrow{\scriptstyle multiply} & & \\ \Gamma(\pi^{-1}V, \mathcal{O}_P(n) \otimes \pi^*\mathcal{L}^{\otimes n}) & \xleftarrow{\theta_n} & \Gamma(\pi'^{-1}V, \mathcal{O}_{P'}(n)) \end{array}$$

are commutative.

(3) *Add more here as necessary.*

Proof. This is the identity map when $\mathcal{L} \cong \mathcal{O}_S$. In general choose an open covering of S such that $\mathcal{L}$ is trivialized over the pieces and glue the corresponding maps. Details omitted. □

26.21. Projective bundles

Let S be a scheme. Let $\mathcal{E}$ be a quasi-coherent sheaf of $\mathcal{O}_S$-modules. By Modules, Lemma 17.18.6 the symmetric algebra $\mathrm{Sym}(\mathcal{E})$ of $\mathcal{E}$ over $\mathcal{O}_S$ is a quasi-coherent sheaf of $\mathcal{O}_S$-algebras. Note that it is generated in degree 1 over $\mathcal{O}_S$. Hence it makes sense to apply the construction of the previous section to it, specifically Lemmas 26.16.5 and 26.16.11.

Definition 26.21.1. Let S be a scheme. Let $\mathcal{E}$ be a quasi-coherent $\mathcal{O}_S$-module[3]. We denote

$$\pi : \mathbf{P}(\mathcal{E}) = \underline{\mathrm{Proj}}_S(\mathrm{Sym}(\mathcal{E})) \longrightarrow S$$

and we call it the *projective bundle associated to* $\mathcal{E}$. The symbol $\mathcal{O}_{\mathbf{P}(\mathcal{E})}(n)$ indicates the invertible $\mathcal{O}_{\mathbf{P}(\mathcal{E})}$-modules introduced in Lemma 26.16.5 and is called the nth *twist of the structure sheaf*.

Note that according to Lemma 26.16.5 there are canonical $\mathcal{O}_S$-module homomorphisms

$$\mathrm{Sym}^n(\mathcal{E}) \longrightarrow \pi_*(\mathcal{O}_{\mathbf{P}(\mathcal{E})}(n))$$

for all $n \geq 0$. This, combined with the fact that $\mathcal{O}_{\mathbf{P}(\mathcal{E})}(1)$ is the canonical relatively ample invertible sheaf on $\mathbf{P}(\mathcal{E})$, is a good way to remember how we have normalized

[3]The reader may expect here the condition that $\mathcal{E}$ is finite locally free. We do not do so in order to be consistent with [**DG67**, II, Definition 4.1.1].

our construction of $\mathbf{P}(\mathcal{E})$. Namely, in some references the space $\mathbf{P}(\mathcal{E})$ is only defined for $\mathcal{E}$ finite locally free on S, and sometimes $\mathbf{P}(\mathcal{E})$ is actually defined as our $\mathbf{P}(\mathcal{E}^\wedge)$ where $\mathcal{E}^\wedge$ is the dual of the sheaf $\mathcal{E}$.

Example 26.21.2. The map $\mathrm{Sym}^n(\mathcal{E}) \to \pi_*(\mathcal{O}_{\mathbf{P}(\mathcal{E})}(n))$ is an isomorphism if $\mathcal{E}$ is locally free, but in general need not be an isomorphism. In fact we will give an example where this map is not injective for $n = 1$. Set $S = \mathrm{Spec}(A)$ with

$$A = k[u, v, s_1, s_2, t_1, t_2]/I$$

where k is a field and

$$I = (-us_1 + vt_1 + ut_2, vs_1 + us_2 - vt_2, vs_2, ut_1).$$

Denote $\overline{u}$ the class of u in A and similarly for the other variables. Let $M = (Ax \oplus Ay)/A(\overline{u}x + \overline{v}y)$ so that

$$\mathrm{Sym}(M) = A[x, y]/(\overline{u}x + \overline{v}y) = k[x, y, u, v, s_1, s_2, t_1, t_2]/J$$

where

$$J = (-us_1 + vt_1 + ut_2, vs_1 + us_2 - vt_2, vs_2, ut_1, ux + vy).$$

In this case the projective bundle associated to the quasi-coherent sheaf $\mathcal{E} = \widetilde{M}$ on $S = \mathrm{Spec}(A)$ is the scheme

$$P = \mathrm{Proj}(\mathrm{Sym}(M)).$$

Note that this scheme as an affine open covering $P = D_+(x) \cup D_+(y)$. Consider the element $m \in M$ which is the image of the element $us_1x + vt_2y$. Note that

$$x(us_1x + vt_2y) = (s_1x + s_2y)(ux + vy) \bmod I$$

and

$$y(us_1x + vt_2y) = (t_1x + t_2y)(ux + vy) \bmod I.$$

The first equation implies that m maps to zero as a section of $\mathcal{O}_P(1)$ on $D_+(x)$ and the second that it maps to zero as a section of $\mathcal{O}_P(1)$ on $D_+(y)$. This shows that m maps to zero in $\Gamma(P, \mathcal{O}_P(1))$. On the other hand we claim that $m \neq 0$, so that m gives an example of a nonzero global section of $\mathcal{E}$ mapping to zero in $\Gamma(P, \mathcal{O}_P(1))$. Assume $m = 0$ to get a contradiction. In this case there exists an element $f \in k[u, v, s_1, s_2, t_1, t_2]$ such that

$$us_1x + vt_2y = f(ux + vy) \bmod I$$

Since I is generated by homogeneous polynomials of degree 2 we may decompose f into its homogeneous components and take the degree 1 component. In other words we may assume that

$$f = au + bv + \alpha_1 s_1 + \alpha_2 s_2 + \beta_1 t_1 + \beta_2 t_2$$

for some $a, b, \alpha_1, \alpha_2, \beta_1, \beta_2 \in k$. The resulting conditions are that

$$us_1 - u(au + bv + \alpha_1 s_1 + \alpha_2 s_2 + \beta_1 t_1 + \beta_2 t_2) \in I$$
$$vt_2 - v(au + bv + \alpha_1 s_1 + \alpha_2 s_2 + \beta_1 t_1 + \beta_2 t_2) \in I$$

There are no terms u^2, uv, v^2 in the generators of I and hence we see $a = b = 0$. Thus we get the relations

$$us_1 - u(\alpha_1 s_1 + \alpha_2 s_2 + \beta_1 t_1 + \beta_2 t_2) \in I$$
$$vt_2 - v(\alpha_1 s_1 + \alpha_2 s_2 + \beta_1 t_1 + \beta_2 t_2) \in I$$

We may use the first generator of I to replace any occurrence of us_1 by $vt_1 + ut_2$, the second generator of I to replace any occurrence of vs_1 by $-us_2 + vt_2$, the third generator to remove occurrences of vs_2 and the third to remove occurences of ut_1. Then we get the relations

$$(1-\alpha_1)vt_1 + (1-\alpha_1)ut_2 - \alpha_2 us_2 - \beta_2 ut_2 = 0$$
$$(1-\alpha_1)vt_2 + \alpha_1 us_2 - \beta_1 vt_1 - \beta_2 vt_2 = 0$$

This implies that α_1 should be both 0 and 1 which is a contradiction as desired.

Lemma 26.21.3. *Let S be a scheme. The structure morphism $\mathbf{P}(\mathcal{E}) \to S$ of a projective bundle over S is separated.*

Proof. Immediate from Lemma 26.16.9. □

Lemma 26.21.4. *Let S be a scheme. Let $n \geq 0$. Then $\mathbf{P}^n_S$ is a projective bundle over S.*

Proof. Note that

$$\mathbf{P}^n_{\mathbf{Z}} = \mathrm{Proj}(\mathbf{Z}[T_0, \ldots, T_n]) = \underline{\mathrm{Proj}}_{\mathrm{Spec}(\mathbf{Z})}\left(\widetilde{\mathbf{Z}[T_0, \ldots, T_n]}\right)$$

where the grading on the ring $\mathbf{Z}[T_0, \ldots, T_n]$ is given by $\deg(T_i) = 1$ and the elements of $\mathbf{Z}$ are in degree 0. Recall that $\mathbf{P}^n_S$ is defined as $\mathbf{P}^n_{\mathbf{Z}} \times_{\mathrm{Spec}(\mathbf{Z})} S$. Moreover, forming the relative homogeneous spectrum commutes with base change, see Lemma 26.16.10. For any scheme $g : S \to \mathrm{Spec}(\mathbf{Z})$ we have $g^*\mathcal{O}_{\mathrm{Spec}(\mathbf{Z})}[T_0, \ldots, T_n] = \mathcal{O}_S[T_0, \ldots, T_n]$. Combining the above we see that

$$\mathbf{P}^n_S = \underline{\mathrm{Proj}}_S(\mathcal{O}_S[T_0, \ldots, T_n]).$$

Finally, note that $\mathcal{O}_S[T_0, \ldots, T_n] = \mathrm{Sym}(\mathcal{O}_S^{\oplus n+1})$. Hence we see that $\mathbf{P}^n_S$ is a projective bundle over S. □

26.22. Grassmannians

In this section we introduce the standard Grassmannian functors and we show that they are represented by schemes. Pick integers k, n with $0 < k < n$. We will construct a functor

$$G(k, n) : Sch \longrightarrow Sets \tag{26.22.0.1}$$

which will loosely speaking parametrize k-dimensional subspaces of n-space. However, for technical reasons it is more convenient to parametrize $(n-k)$-dimensional quotients and this is what we will do.

More precisely, $G(k, n)$ associates to a scheme S the set $G(k, n)(S)$ of isomorphism classes of surjections

$$q : \mathcal{O}_S^{\oplus n} \longrightarrow \mathcal{Q}$$

where $\mathcal{Q}$ is a finite locally free $\mathcal{O}_S$-module of rank $n - k$. Note that this is indeed a set, for example by Modules, Lemma 17.9.8 or by the observation that the isomorphism class of the surjection q is determined by the kernel of q (and given a sheaf there is a set of subsheaves). Given a morphism of schemes $f : T \to S$ we let $G(k, n)(f) : G(k, n)(S) \to G(k, n)(T)$ which sends the isomorphism class of $q : \mathcal{O}_S^{\oplus n} \longrightarrow \mathcal{Q}$ to the isomorphism class of $f^*q : \mathcal{O}_T^{\oplus n} \longrightarrow f^*\mathcal{Q}$. This makes sense since (1) $f^*\mathcal{O}_S = \mathcal{O}_T$, (2) f^* is additive, (3) f^* preserves locally free modules (Modules, Lemma 17.14.3), and (4) f^* is right exact (Modules, Lemma 17.3.3).

Lemma 26.22.1. *Let $0 < k < n$. The functor $G(k, n)$ of (26.22.0.1) is representable by a scheme.*

Proof. Set $F = G(k, n)$. To prove the lemma we will use the criterion of Schemes, Lemma 25.15.4. The reason F satisfies the sheaf property for the Zariski topology is that we can glue sheaves, see Sheaves, Section 6.33 (some details omitted).

The family of subfunctors F_i. Let I be the set of subsets of $\{1, \ldots, n\}$ of cardinality $n - k$. Given a scheme S and $j \in \{1, \ldots, n\}$ we denote e_j the global section

$$e_j = (0, \ldots, 0, 1, 0, \ldots, 0) \quad (1 \text{ in } j\text{th spot})$$

of $\mathcal{O}_S^{\oplus n}$. Of course these sections freely generate $\mathcal{O}_S^{\oplus n}$. Similarly, for $j \in \{1, \ldots, k\}$ we denote f_j the global section of $\mathcal{O}_S^{\oplus k}$ which is zero in all summands except the jth where we put a 1. For $i \in I$ we let

$$s_i : \mathcal{O}_S^{\oplus n-k} \longrightarrow \mathcal{O}_S^{\oplus n}$$

which is the direct sum of the coprojections $\mathcal{O}_S \to \mathcal{O}_S^{\oplus n}$ corresponding to elements of i. More precisely, if $i = \{i_1, \ldots, i_{n-k}\}$ with $i_1 < i_2 < \ldots < i_{n-k}$ then s_i maps f_j to e_{i_j} for $j \in \{1, \ldots, n - k\}$. With this notation we can set

$$F_i(S) = \{q : \mathcal{O}_S^{\oplus n} \to \mathcal{Q} \in F(S) \mid q \circ s_i \text{ is surjective}\} \subset F(S)$$

Given a morphism $f : T \to S$ of schemes the pullback $f^* s_i$ is the corresponding map over T. Since f^* is right exact (Modules, Lemma 17.3.3) we conclude that F_i is a subfunctor of F.

Representability of F_i. To prove this we may assume (after renumbering) that $i = \{1, \ldots, n - k\}$. This means s_i is the inclusion of the first $n - k$ summands. Observe that if $q \circ s_i$ is surjective, then $q \circ s_i$ is an isomorphism as a surjective map between finite locally free modules of the same rank (Modules, Lemma 17.14.5). Thus if $q : \mathcal{O}_S^{\oplus n} \to \mathcal{Q}$ is an element of $F_i(S)$, then we can use $q \circ s_i$ to identify $\mathcal{Q}$ with $\mathcal{O}_S^{\oplus n-k}$. After doing so we obtain

$$q : \mathcal{O}_S^{\oplus n} \longrightarrow \mathcal{O}_S^{\oplus n-k}$$

mapping e_j to f_j (notation as above) for $j = 1, \ldots, n-k$. To determine q completely we have to fix the images $q(e_{n-k+1}), \ldots, q(e_n)$ in $\Gamma(S, \mathcal{O}_S^{\oplus n-k})$. It follows that F_i is isomorphic to the functor

$$S \longmapsto \prod\nolimits_{j=n-k+1,\ldots,n} \Gamma(S, \mathcal{O}_S^{\oplus n-k})$$

This functor is isomorphic to the $k(n - k)$-fold self product of the functor $S \mapsto \Gamma(S, \mathcal{O}_S)$. By Schemes, Example 25.15.2 the latter is representable by $\mathbf{A}^1_{\mathbf{Z}}$. It follows F_i is representable by $\mathbf{A}^{k(n-k)}_{\mathbf{Z}}$ since fibred product over $\mathrm{Spec}(\mathbf{Z})$ is the product in the category of schemes.

The inclusion $F_i \subset F$ is representable by open immersions. Let S be a scheme and let $q : \mathcal{O}_S^{\oplus n} \to \mathcal{Q}$ be an element of $F(S)$. By Modules, Lemma 17.9.4. the set $U_i = \{s \in S \mid (q \circ s_i)_s \text{ surjective}\}$ is open in S. Since $\mathcal{O}_{S,s}$ is a local ring and $\mathcal{Q}_s$ a finite $\mathcal{O}_{S,s}$-module by Nakayama's lemma (Algebra, Lemma 10.18.1) we have

$$s \in U_i \Leftrightarrow \big(\text{the map } \kappa(s)^{\oplus n-k} \to \mathcal{Q}_s/\mathfrak{m}_s\mathcal{Q}_s \text{ induced by } (q \circ s_i)_s \text{ is surjective}\big)$$

Let $f : T \to S$ be a morphism of schemes and let $t \in T$ be a point mapping to $s \in S$. We have $(f^*\mathcal{Q})_t = \mathcal{Q}_s \otimes_{\mathcal{O}_{S,s}} \mathcal{O}_{T,t}$ (Sheaves, Lemma 6.26.4) and so on. Thus the map

$$\kappa(t)^{\oplus n-k} \to (f^*\mathcal{Q})_t/\mathfrak{m}_t(f^*\mathcal{Q})_t$$

induced by $(f^*q \circ f^*s_i)_t$ is the base change of the map $\kappa(s)^{\oplus n-k} \to \mathcal{Q}_s/\mathfrak{m}_s\mathcal{Q}_s$ above by the field extension $\kappa(s) \subset \kappa(t)$. It follows that $s \in U_i$ if and only if t is in the corresponding open for f^*q. In particular $T \to S$ factors through U_i if and only if $f^*q \in F_i(T)$ as desired.

The collection F_i, $i \in I$ covers F. Let $q : \mathcal{O}_S^{\oplus n} \to \mathcal{Q}$ be an element of $F(S)$. We have to show that for every point s of S there exists an $i \in I$ such that s_i is surjective in a neighbourhood of s. Thus we have to show that one of the compositions

$$\kappa(s)^{\oplus n-k} \xrightarrow{s_i} \kappa(s)^{\oplus n} \to \mathcal{Q}_s/\mathfrak{m}_s\mathcal{Q}_s$$

is surjective (see previous paragraph). As $\mathcal{Q}_s/\mathfrak{m}_s\mathcal{Q}_s$ is a vector space of dimension $n-k$ this follows from the theory of vector spaces. □

Definition 26.22.2. Let $0 < k < n$. The scheme $\mathbf{G}(k,n)$ representing the functor $G(k,n)$ is called *Grassmannian over* $\mathbf{Z}$. Its base change $\mathbf{G}(k,n)_S$ to a scheme S is called *Grassmannian over* S. If R is a ring the base change to $\mathrm{Spec}(R)$ is denoted $\mathbf{G}(k,n)_R$ and called *Grassmannian over* R.

The definition makes sense as we've shown in Lemma 26.22.1 that these functors are indeed representable.

Lemma 26.22.3. *Let $n \geq 1$. There is a canonical isomorphism $\mathbf{G}(n,n+1) = \mathbf{P}^n_{\mathbf{Z}}$.*

Proof. According to Lemma 26.13.1 the scheme $\mathbf{P}^n_{\mathbf{Z}}$ represents the functor which assigns to a scheme S the set of isomorphisms classes of pairs $(\mathcal{L}, (s_0, \ldots, s_n))$ consisting of an invertible module $\mathcal{L}$ and an $(n+1)$-tuple of global sections generating $\mathcal{L}$. Given such a pair we obtain a quotient

$$\mathcal{O}_S^{\oplus n+1} \longrightarrow \mathcal{L}, \quad (h_0, \ldots, h_n) \longmapsto \sum h_i s_i.$$

Conversely, given an element $q : \mathcal{O}_S^{\oplus n+1} \to \mathcal{Q}$ of $G(n,n+1)(S)$ we obtain such a pair, namely $(\mathcal{Q}, (q(e_1), \ldots, q(e_{n+1})))$. Here e_i, $i = 1, \ldots, n+1$ are the standard generating sections of the free module $\mathcal{O}_S^{\oplus n+1}$. We omit the verification that these constructions define mutually inverse transformations of functors. □

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CHAPTER 27

Properties of Schemes

27.1. Introduction

In this chapter we introduce some absolute properties of schemes. A foundational reference is [**DG67**].

27.2. Constructible sets

Constructible and locally constructible sets are introduced in Topology, Section 5.14. We may characterize locally constructible subsets of schemes as follows.

Lemma 27.2.1. *Let X be a scheme. A subset E of X is locally constructible in X if and only if $E \cap U$ is constructible in U for every affine open U of X.*

Proof. Assume E is locally constructible. Then there exists an open covering $X = \bigcup U_i$ such that $E \cap U_i$ is constructible in U_i for each i. Let $V \subset X$ be any affine open. We can find a finite open affine covering $V = V_1 \cup \ldots \cup V_m$ such that for each j we have $V_j \subset U_i$ for some $i = i(j)$. By Topology, Lemma 5.14.4 we see that each $E \cap V_j$ is constructible in V_j. Since the inclusions $V_j \to V$ are quasi-compact (see Schemes, Lemma 25.19.2) we conclude that $E \cap V$ is constructible in V by Topology, Lemma 5.14.6. The converse implication is immediate. □

Lemma 27.2.2. *Let X be a quasi-separated scheme. The intersection of any two quasi-compact opens of X is a quasi-compact open of X. Every quasi-compact open of X is retrocompact in X.*

Proof. If U and V are quasi-compact open then $U \cap V = \Delta^{-1}(U \times V)$, where $\Delta : X \to X \times X$ is the diagonal. As X is quasi-separated we see that Δ is quasi-compact. Hence we see that $U \cap V$ is quasi-compact as $U \times V$ is quasi-compact (details omitted; use Schemes, Lemma 25.17.4 to see $U \times V$ is a finite union of affines). The other assertions follow from the first and Topology, Lemma 5.25.1. □

Lemma 27.2.3. *Let X be a quasi-compact and quasi-separated scheme. Then the underlying topological space of X is a spectral space.*

Proof. By Topology, Definition 5.22.1 we have to check that X is sober, quasi-compact, has a basis of quasi-compact opens, and the intersection of any two quasi-compact opens is quasi-compact. This follows from Schemes, Lemma 25.11.1 and 25.11.2 and Lemma 27.2.2 above. □

Lemma 27.2.4. *Let X be a quasi-compact and quasi-separated scheme. Any locally constructible subset of X is constructible.*

Proof. As X is quasi-compact we can choose a finite affine open covering $X = V_1 \cup \ldots \cup V_m$. As X is quasi-separated each V_i is retrocompact in X by Lemma

27.2.2. Hence by Topology, Lemma 5.14.6 we see that $E \subset X$ is constructible in X if and only if $E \cap V_j$ is constructible in V_j. Thus we win by Lemma 27.2.1. □

Lemma 27.2.5. *Let X be a scheme. A subset Z of X is retrocompact in X if and only if $E \cap U$ is quasi-compact for every affine open U of X.*

Proof. Immediate from the fact that every quasi-compact open of X is a finite union of affine opens. □

27.3. Integral, irreducible, and reduced schemes

Definition 27.3.1. Let X be a scheme. We say X is *integral* if it is nonempty and for every nonempty affine open $\mathrm{Spec}(R) = U \subset X$ the ring R is an integral domain.

Lemma 27.3.2. *Let X be a scheme. The following are equivalent.*

1. *The scheme X is reduced, see Schemes, Definition 25.12.1.*
2. *There exists an affine open covering $X = \bigcup U_i$ such that each $\Gamma(U_i, \mathcal{O}_X)$ is reduced.*
3. *For every affine open $U \subset X$ the ring $\mathcal{O}_X(U)$ is reduced.*
4. *For every open $U \subset X$ the ring $\mathcal{O}_X(U)$ is reduced.*

Proof. See Schemes, Lemmas 25.12.2 and 25.12.3. □

Lemma 27.3.3. *Let X be a scheme. The following are equivalent.*

1. *The scheme X is irreducible.*
2. *There exists an affine open covering $X = \bigcup_{i \in I} U_i$ such that I is not empty, U_i is irreducible for all $i \in I$, and $U_i \cap U_j \neq \emptyset$ for all $i, j \in I$.*
3. *The scheme X is nonempty and every nonempty affine open $U \subset X$ is irreducible.*

Proof. Assume (1). By Schemes, Lemma 25.11.1 we see that X has a unique generic point η. Then $X = \overline{\{\eta\}}$. Hence η is an element of every nonempty affine open $U \subset X$. This implies that $U = \overline{\{\eta\}}$ and that any two nonempty affines meet. Thus (1) implies both (2) and (3).

Assume (2). Suppose $X = Z_1 \cup Z_2$ is a union of two closed subsets. For every i we see that either $U_i \subset Z_1$ or $U_i \subset Z_2$. Pick some $i \in I$ and assume $U_i \subset Z_1$ (possibly after renumbering Z_1, Z_2). For any $j \in I$ the open subset $U_i \cap U_j$ is dense in U_j and contained in the closed subset $Z_1 \cap U_j$. We conclude that also $U_j \subset Z_1$. Thus $X = Z_1$ as desired.

Assume (3). Choose an affine open covering $X = \bigcup_{i \in I} U_i$. We may assume that each U_i is nonempty. Since X is nonempty we see that I is not empty. By assumption each U_i is irreducible. Suppose $U_i \cap U_j = \emptyset$ for some pair $i, j \in I$. Then the open $U_i \coprod U_j = U_i \cup U_j$ is affine, see Schemes, Lemma 25.6.8. Hence it is irreducible by assumption which is absurd. We conclude that (3) implies (2). The lemma is proved. □

Lemma 27.3.4. *A scheme X is integral if and only if it is reduced and irreducible.*

Proof. If X is irreducible, then every affine open $\mathrm{Spec}(R) = U \subset X$ is irreducible. If X is reduced, then R is reduced, by Lemma 27.3.2 above. Hence R is reduced and (0) is a prime ideal, i.e., R is an integral domain.

If X is integral, then for every nonempty affine open $\mathrm{Spec}(R) = U \subset X$ the ring R is reduced and hence X is reduced by Lemma 27.3.2. Moreover, every nonempty affine open is irreducible. Hence X is irreducible, see Lemma 27.3.3. □

In Examples, Section 78.3 we construct a connected affine scheme all of whose local rings are domains, but which is not integral.

27.4. Types of schemes defined by properties of rings

In this section we study what properties of rings allow one to define local properties of schemes.

Definition 27.4.1. Let P be a property of rings. We say that P is *local* if the following hold:

(1) For any ring R, and any $f \in R$ we have $P(R) \Rightarrow P(R_f)$.
(2) For any ring R, and $f_i \in R$ such that $(f_1, \ldots, f_n) = R$ then $\forall i, P(R_{f_i}) \Rightarrow P(R)$.

Definition 27.4.2. Let P be a property of rings. Let X be a scheme. We say X is *locally P* if for any $x \in X$ there exists an affine open neighbourhood U of x in X such that $\mathcal{O}_X(U)$ has property P.

This is only a good notion if the property is local. Even if P is a local property we will not automatically use this definition to say that a scheme is "locally P" unless we also explicitly state the definition elsewhere.

Lemma 27.4.3. *Let X be a scheme. Let P be a local property of rings. The following are equivalent:*

(1) *The scheme X is locally P.*
(2) *For every affine open $U \subset X$ the property $P(\mathcal{O}_X(U))$ holds.*
(3) *There exists an affine open covering $X = \bigcup U_i$ such that each $\mathcal{O}_X(U_i)$ satisfies P.*
(4) *There exists an open covering $X = \bigcup X_j$ such that each open subscheme X_j is locally P.*

Moreover, if X is locally P then every open subscheme is locally P.

Proof. Of course (1) $\Leftrightarrow$ (3) and (2) $\Rightarrow$ (1). If (3) $\Rightarrow$ (2), then the final statement of the lemma holds and it follows easily that (4) is also equivalent to (1). Thus we show (3) $\Rightarrow$ (2).

Let $X = \bigcup U_i$ be an affine open covering, say $U_i = \mathrm{Spec}(R_i)$. Assume $P(R_i)$. Let $\mathrm{Spec}(R) = U \subset X$ be an arbitrary affine open. By Schemes, Lemma 25.11.6 there exists a standard covering of $U = \mathrm{Spec}(R)$ by standard opens $D(f_j)$ such that each ring R_{f_j} is a principal localization of one of the rings R_i. By Definition 27.4.1 (1) we get $P(R_{f_j})$. Whereupon $P(R)$ by Definition 27.4.1 (2). □

Here is a sample application.

Lemma 27.4.4. *Let X be a scheme. Then X is reduced if and only if X is "locally reduced" in the sense of Definition 27.4.2.*

Proof. This is clear from Lemma 27.3.2. □

Lemma 27.4.5. *The following properties of a ring R are local.*

(1) *(Cohen-Macaulay.) The ring R is Noetherian and CM, see Algebra, Definition 10.99.6.*
(2) *(Regular.) The ring R is Noetherian and regular, see Algebra, Definition 10.105.6.*
(3) *(Absolutely Noetherian.) The ring R is of finite type over Z.*
(4) *Add more here as needed.*[1]

Proof. Omitted. □

27.5. Noetherian schemes

Recall that a ring R is *Noetherian* if it satisfies the ascending chain condition of ideals. Equivalently every ideal of R is finitely generated.

Definition 27.5.1. Let X be a scheme.

(1) We say X is *locally Noetherian* if every $x \in X$ has an affine open neighbourhood $\mathrm{Spec}(R) = U \subset X$ such that the ring R is Noetherian.
(2) We say X is *Noetherian* if X is locally Noetherian and quasi-compact.

Here is the standard result characterizing locally Noetherian schemes.

Lemma 27.5.2. *Let X be a scheme. The following are equivalent:*

(1) *The scheme X is locally Noetherian.*
(2) *For every affine open $U \subset X$ the ring $\mathcal{O}_X(U)$ is Noetherian.*
(3) *There exists an affine open covering $X = \bigcup U_i$ such that each $\mathcal{O}_X(U_i)$ is Noetherian.*
(4) *There exists an open covering $X = \bigcup X_j$ such that each open subscheme X_j is locally Noetherian.*

Moreover, if X is locally Noetherian then every open subscheme is locally Noetherian.

Proof. To show this it suffices to show that being Noetherian is a local property of rings, see Lemma 27.4.3. Any localization of a Noetherian ring is Noetherian, see Algebra, Lemma 10.29.1. By Algebra, Lemma 10.22.2 we see the second property to Definition 27.4.1. □

Lemma 27.5.3. *Any immersion $Z \to X$ with X locally Noetherian is quasi-compact.*

Proof. A closed immersion is clearly quasi-compact. A composition of quasi-compact morphisms is quasi-compact, see Topology, Lemma 5.11.2. Hence it suffices to show that an open immersion into a locally Noetherian scheme is quasi-compact. Using Schemes, Lemma 25.19.2 we reduce to the case where X is affine. Any open subset of the spectrum of a Noetherian ring is quasi-compact (for example combine Algebra, Lemma 10.29.5 and Topology, Lemmas 5.8.2 and 5.11.13). □

Lemma 27.5.4. *A locally Noetherian scheme is quasi-separated.*

Proof. By Schemes, Lemma 25.21.7 we have to show that the intersection $U \cap V$ of two affine opens of X is quasi-compact. This follows from Lemma 27.5.3 above on considering the open immersion $U \cap V \to U$ for example. (But really it is just because any open of the spectrum of a Noetherian ring is quasi-compact.) □

[1] But we only list those properties here which we have not already dealt with separately somewhere else.

Lemma 27.5.5. *A (locally) Noetherian scheme has a (locally) Noetherian underlying topological space, see Topology, Definition 5.8.1.*

Proof. This is because a Noetherian scheme is a finite union of spectra of Noetherian rings and Algebra, Lemma 10.29.5 and Topology, Lemma 5.8.4. □

Lemma 27.5.6. *Any morphism of schemes $f : X \to Y$ with X Noetherian is quasi-compact.*

Proof. Use Lemma 27.5.5 and use that any subset of a Noetherian topological space is quasi-compact (see Topology, Lemmas Lemmas 5.8.2 and 5.11.13). □

Lemma 27.5.7. *Any locally closed subscheme of a (locally) Noetherian scheme is (locally) Noetherian.*

Proof. Omitted. Hint: Any quotient, and any localization of a Noetherian ring is Noetherian. For the Noetherian case use again that any subset of a Noetherian space is a Noetherian space (with induced topology). □

Here is a fun lemma. It says that every locally Noetherian scheme has plenty of closed points (at least one in every closed subset).

Lemma 27.5.8. *Any locally Noetherian scheme has a closed point. Any closed subset of a locally Noetherian scheme has a closed point. Equivalently, any point of a locally Noetherian scheme specializes to a closed point.*

Proof. The second assertion follows from the first (using Schemes, Lemma 25.12.4 and Lemma 27.5.7). Consider any nonempty affine open $U \subset X$. Let $x \in U$ be a closed point. If x is a closed point of X then we are done. If not, let $y \in \overline{\{x\}}$ be a specialization of x. Note that $y \in X \setminus U$. Consider the local ring $R = \mathcal{O}_{X,y}$. This is a Noetherian local ring. Denote $V \subset \mathrm{Spec}(R)$ the inverse image of U in $\mathrm{Spec}(R)$ by the canonical morphism $\mathrm{Spec}(R) \to X$ (see Schemes, Section 25.13.) By construction V is a singleton with unique point corresponding to x (use Schemes, Lemma 25.13.2). Say $V = \{\mathfrak{q}\}$. Consider the Noetherian local domain $R/\mathfrak{q}$. By Algebra, Lemma 10.58.1 we see that $\dim(R/\mathfrak{q}) = 1$. In other words, we see that y is an immediate specialization of x (see Topology, Definition 5.19.1). In other words, any point $y \neq x$ such that $x \rightsquigarrow y$ is an immediate specialization of x. Clearly each of these points is a closed point, and we win. □

Lemma 27.5.9. *Let X be a locally Noetherian scheme. Let $x' \rightsquigarrow x$ be a specialization of points of X. Then*

1. *there exists a discrete valuation ring R and a morphism $f : \mathrm{Spec}(R) \to X$ such that the generic point η of $\mathrm{Spec}(R)$ maps to x' and the special point maps to x, and*
2. *given a finitely generated field extension $\kappa(x') \subset K$ we may arrange it so that the extension $\kappa(x') \subset \kappa(\eta)$ induced by f is isomorphic to the given one.*

Proof. Let $x' \rightsquigarrow x$ be a specialization in X, and let $\kappa(x') \subset K$ be a finitely generated extension of fields. By Schemes, Lemma 25.13.2 and the discussion following Schemes, Lemma 25.13.3 this leads to ring maps $\mathcal{O}_{X,x} \to \kappa(x') \to K$. Let $R \subset K$ be any discrete valuation ring whose field of fractions is K and which dominates the image of $\mathcal{O}_{X,x} \to K$, see Algebra, Lemma 10.113.12. The ring map $\mathcal{O}_{X,x} \to R$

induces the morphism $f : \mathrm{Spec}(R) \to X$, see Schemes, Lemma 25.13.1. This morphism has all the desired properties by construction. □

27.6. Jacobson schemes

Recall that a space is said to be *Jacobson* if the closed points are dense in every closed subset, see Topology, Section 5.17.

Definition 27.6.1. A scheme S is said to be *Jacobson* if its underlying topological space is Jacobson.

Recall that a ring R is Jacobson if every radical ideal of R is the intersection of maximal ideals, see Algebra, Definition 10.32.1.

Lemma 27.6.2. *An affine scheme* $\mathrm{Spec}(R)$ *is Jacobson if and only if the ring* R *is Jacobson.*

Proof. This is Algebra, Lemma 10.32.4. □

Here is the standard result characterizing Jacobson schemes. Intuitively it claims that Jacobson ⇔ locally Jacobson.

Lemma 27.6.3. *Let* X *be a scheme. The following are equivalent:*

1. *The scheme* X *is Jacobson.*
2. *The scheme* X *is "locally Jacobson" in the sense of Definition 27.4.2.*
3. *For every affine open* $U \subset X$ *the ring* $\mathcal{O}_X(U)$ *is Jacobson.*
4. *There exists an affine open covering* $X = \bigcup U_i$ *such that each* $\mathcal{O}_X(U_i)$ *is Jacobson.*
5. *There exists an open covering* $X = \bigcup X_j$ *such that each open subscheme* X_j *is Jacobson.*

Moreover, if X *is Jacobson then every open subscheme is Jacobson.*

Proof. The final assertion of the lemma holds by Topology, Lemma 5.17.5. The equivalence of (5) and (1) is Topology, Lemma 5.17.4. Hence, using Lemma 27.6.2, we see that (1) ⇔ (2). To finish proving the lemma it suffices to show that "Jacobson" is a local property of rings, see Lemma 27.4.3. Any localization of a Jacobson ring at an element is Jacobson, see Algebra, Lemma 10.32.14. Suppose R is a ring, $f_1, \ldots, f_n \in R$ generate the unit ideal and each R_{f_i} is Jacobson. Then we see that $\mathrm{Spec}(R) = \bigcup D(f_i)$ is a union of open subsets which are all Jacobson, and hence $\mathrm{Spec}(R)$ is Jacobson by Topology, Lemma 5.17.4 again. This proves the second property of Definition 27.4.1. □

Many schemes used commonly in algebraic geometry are Jacobson, see Morphisms, Lemma 28.17.10. We mention here the following interesting case.

Lemma 27.6.4. *Let* R *be a Noetherian local ring with maximal ideal* $\mathfrak{m}$*. In this case the scheme* $S = \mathrm{Spec}(R) \setminus \{\mathfrak{m}\}$ *is Jacobson.*

Proof. Since $\mathrm{Spec}(R)$ is a Noetherian scheme, hence S is a Noetherian scheme (Lemma 27.5.7). Hence S is a sober, Noetherian topological space (use Schemes, Lemma 25.11.1). Assume S is not Jacobson to get a contradiction. By Topology, Lemma 5.17.3 there exists some non-closed point $\xi \in S$ such that $\{\xi\}$ is locally closed. This corresponds to a prime $\mathfrak{p} \subset R$ such that (1) there exists a prime $\mathfrak{q}$, $\mathfrak{p} \subset \mathfrak{q} \subset \mathfrak{m}$ with both inclusions strict, and (2) $\{\mathfrak{p}\}$ is open in $\mathrm{Spec}(R/\mathfrak{p})$. This is impossible by Algebra, Lemma 10.58.1. □

27.7. Normal schemes

Recall that a ring R is said to be normal if all its local rings are normal domains, see Algebra, Definition 10.34.10. A normal domain is a domain which is integrally closed in its field of fractions, see Algebra, Definition 10.34.1. Thus it makes sense to define a normal scheme as follows.

Definition 27.7.1. A scheme X is *normal* if and only if for all $x \in X$ the local ring $\mathcal{O}_{X,x}$ is a normal domain.

This seems to be the definition used in EGA, see [**DG67**, 0, 4.1.4]. Suppose $X = \mathrm{Spec}(A)$, and A is reduced. Then saying that X is normal is not equivalent to saying that A is integrally closed in its total ring of fractions. However, if A is Noetherian then this is the case (see Algebra, Lemma 10.34.14).

Lemma 27.7.2. *Let X be a scheme. The following are equivalent:*

1. *The scheme X is normal.*
2. *For every affine open $U \subset X$ the ring $\mathcal{O}_X(U)$ is normal.*
3. *There exists an affine open covering $X = \bigcup U_i$ such that each $\mathcal{O}_X(U_i)$ is normal.*
4. *There exists an open covering $X = \bigcup X_j$ such that each open subscheme X_j is normal.*

Moreover, if X is normal then every open subscheme is normal.

Proof. This is clear from the definitions. □

Lemma 27.7.3. *A normal scheme is reduced.*

Proof. Immediate from the definitions. □

Lemma 27.7.4. *Let X be an integral scheme. Then X is normal if and only if for every affine open $U \subset X$ the ring $\mathcal{O}_X(U)$ is a normal domain.*

Proof. This follows from Algebra, Lemma 10.34.9. □

Lemma 27.7.5. *Let X be a scheme with a finite number of irreducible components. The following are equivalent:*

1. *X is normal, and*
2. *X is a finite disjoint union of normal integral schemes.*

Proof. It is immediate from the definitions that (2) implies (1). Let X be a normal scheme with a finite number of irreducible components. If X is affine then X satisfies (2) by Algebra, Lemma 10.34.14. For a general X, let $X = \bigcup X_i$ be an affine open covering. Note that also each X_i has but a finite number of irreducible components, and the lemma holds for each X_i. Let $T \subset X$ be an irreducible component. By the affine case each intersection $T \cap X_i$ is open in X_i and an integral normal scheme. Hence $T \subset X$ is open, and an integral normal scheme. This proves that X is the disjoint union of its irreducible components, which are integral normal schemes. There are only finitely many by assumption. □

Lemma 27.7.6. *Let X be a Noetherian scheme. The following are equivalent:*

1. *X is normal, and*
2. *X is a finite disjoint union of normal integral schemes.*

Proof. This is a special case of Lemma 27.7.5 because a Noetherian scheme has a Noetherian underlying topological space (Lemma 27.5.5 and Topology, Lemma 5.8.2. □

Lemma 27.7.7. *Let X be a locally Noetherian scheme. The following are equivalent:*

(1) *X is normal, and*
(2) *X is a disjoint union of integral normal schemes.*

Proof. Omitted. Hint: This is purely topological from Lemma 27.7.6. □

Remark 27.7.8. Let X be a normal scheme. If X is locally Noetherian then we see that X is integral if and only if X is connected, see Lemma 27.7.7. But there exists a connected affine scheme X such that $\mathcal{O}_{X,x}$ is a domain for all $x \in X$, but X is not irreducible, see Examples, Section 78.3. This example is even a normal scheme (proof omitted), so beware!

Lemma 27.7.9. *Let X be an integral normal scheme. Then $\Gamma(X, \mathcal{O}_X)$ is a normal domain.*

Proof. Set $R = \Gamma(X, \mathcal{O}_X)$. It is clear that R is a domain. Suppose $f = a/b$ is an element of its fraction field which is integral over R. Say we have $f^d + \sum_{i=1,\ldots,d} a_i f^i = 0$ with $a_i \in R$. Let $U \subset X$ be affine open. Since $b \in R$ is not zero and since X is integral we see that also $b|_U \in \mathcal{O}_X(U)$ is not zero. Hence a/b is an element of the fraction field of $\mathcal{O}_X(U)$ which is integral over $\mathcal{O}_X(U)$ (because we can use the same polynomial $f^d + \sum_{i=1,\ldots,d} a_i|_U f^i = 0$ on U). Since $\mathcal{O}_X(U)$ is a normal domain (Lemma 27.7.2), we see that $f_U = (a|_U)/(b|_U) \in \mathcal{O}_X(U)$. It is easy to see that $f_U|_V = f_V$ whenever $V \subset U \subset X$ are affine open. Hence the local sections f_U glue to a global section f as desired. □

27.8. Cohen-Macaulay schemes

Recall, see Algebra, Definition 10.99.1, that a local Noetherian ring $(R, \mathfrak{m})$ is said to be Cohen-Macaulay if $\text{depth}_{\mathfrak{m}}(R) = \dim(R)$. Recall that a Noetherian ring R is said to be Cohen-Macaulay if every local ring $R_{\mathfrak{p}}$ of R is Cohen-Macaulay, see Algebra, Definition 10.99.6.

Definition 27.8.1. Let X be a scheme. We say X is *Cohen-Macaulay* if for every $x \in X$ there exists an affine open neighbourhood $U \subset X$ of x such that the ring $\mathcal{O}_X(U)$ is Noetherian and Cohen-Macaulay.

Lemma 27.8.2. *Let X be a scheme. The following are equivalent:*

(1) *X is Cohen-Macaulay,*
(2) *X is locally Noetherian and all of its local rings are Cohen-Macaulay, and*
(3) *X is locally Noetherian and for any closed point $x \in X$ the local ring $\mathcal{O}_{X,x}$ is Cohen-Macaulay.*

Proof. Algebra, Lemma 10.99.5 says that the localization of a Cohen-Macaulay local ring is Cohen-Macaulay. The lemma follows by combining this with Lemma 27.5.2, with the existence of closed points on locally Noetherian schemes (Lemma 27.5.8), and the definitions. □

Lemma 27.8.3. *Let X be a scheme. The following are equivalent:*

(1) *The scheme X is Cohen-Macaulay.*
(2) *For every affine open $U \subset X$ the ring $\mathcal{O}_X(U)$ is Noetherian and Cohen-Macaulay.*
(3) *There exists an affine open covering $X = \bigcup U_i$ such that each $\mathcal{O}_X(U_i)$ is Noetherian and Cohen-Macaulay.*
(4) *There exists an open covering $X = \bigcup X_j$ such that each open subscheme X_j is Cohen-Macaulay.*

Moreover, if X is Cohen-Macaulay then every open subscheme is Cohen-Macaulay.

Proof. Combine Lemmas 27.5.2 and 27.8.2. □

More information on Cohen-Macaulay schemes and depth can be found in Cohomology of Schemes, Section 29.11.

27.9. Regular schemes

Recall, see Algebra, Definition 10.57.9, that a local Noetherian ring $(R, \mathfrak{m})$ is said to be *regular* if $\mathfrak{m}$ can be generated by $\dim(R)$ elements. Recall that a Noetherian ring R is said to be *regular* if every local ring $R_\mathfrak{p}$ of R is regular, see Algebra, Definition 10.105.6.

Definition 27.9.1. Let X be a scheme. We say X is *regular*, or *nonsingular* if for every $x \in X$ there exists an affine open neighbourhood $U \subset X$ of x such that the ring $\mathcal{O}_X(U)$ is Noetherian and regular.

Lemma 27.9.2. *Let X be a scheme. The following are equivalent:*

(1) *X is regular,*
(2) *X is locally Noetherian and all of its local rings are regular, and*
(3) *X is locally Noetherian and for any closed point $x \in X$ the local ring $\mathcal{O}_{X,x}$ is regular.*

Proof. By the discussion in Algebra preceding Algebra, Definition 10.105.6 we know that the localization of a regular local ring is regular. The lemma follows by combining this with Lemma 27.5.2, with the existence of closed points on locally Noetherian schemes (Lemma 27.5.8), and the definitions. □

Lemma 27.9.3. *Let X be a scheme. The following are equivalent:*

(1) *The scheme X is regular.*
(2) *For every affine open $U \subset X$ the ring $\mathcal{O}_X(U)$ is Noetherian and regular.*
(3) *There exists an affine open covering $X = \bigcup U_i$ such that each $\mathcal{O}_X(U_i)$ is Noetherian and regular.*
(4) *There exists an open covering $X = \bigcup X_j$ such that each open subscheme X_j is regular.*

Moreover, if X is regular then every open subscheme is regular.

Proof. Combine Lemmas 27.5.2 and 27.9.2. □

Lemma 27.9.4. *A regular scheme is normal.*

Proof. See Algebra, Lemma 10.144.5. □

27.10. Dimension

The dimension of a scheme is just the dimension of its underlying topological space.

Definition 27.10.1. Let X be a scheme.

(1) The *dimension* of X is just the dimension of X as a topological spaces, see Topology, Definition 5.9.1.
(2) For $x \in X$ we denote $\dim_x(X)$ the dimension of the underlying topological space of X at x as in Topology, Definition 5.9.1. We say $\dim_x(X)$ is the *dimension of X at x*.

As a scheme has a sober underlying topological space (Schemes, Lemma 25.11.1) we may compute the dimension of X as the supremum of the lengths n of chains

$$T_0 \subset T_1 \subset \ldots \subset T_n$$

of irreducible closed subsets of X, or as the supremum of the lengths n of chains of specializations

$$\xi_n \rightsquigarrow \xi_{n-1} \rightsquigarrow \ldots \rightsquigarrow \xi_0$$

of points of X.

Lemma 27.10.2. *Let X be a scheme. The following are equal*

(1) *The dimension of X.*
(2) *The supremum of the dimensions of the local rings of X.*
(3) *The supremum of $\dim_x(X)$ for $x \in X$.*

Proof. Note that given a chain of specializations

$$\xi_n \rightsquigarrow \xi_{n-1} \rightsquigarrow \ldots \rightsquigarrow \xi_0$$

of points of X all of the points ξ_i correspond to prime ideals of the local ring of X at ξ_0 by Schemes, Lemma 25.13.2. Hence we see that the dimension of X is the supremum of the dimensions of its local rings. In particular $\dim_x(X) \geq \dim(\mathcal{O}_{X,x})$ as $\dim_x(X)$ is the minimum of the dimensions of open neighbourhoods of x. Thus $\sup_{x \in X} \dim_x(X) \geq \dim(X)$. On the other hand, it is clear that $\sup_{x \in X} \dim_x(X) \leq \dim(X)$ as $\dim(U) \leq \dim(X)$ for any open subset of X. □

27.11. Catenary schemes

Recall that a topological space X is called *catenary* if for every pair of irreducible closed subsets $T \subset T'$ there exist a maximal chain of irreducible closed subsets

$$T = T_0 \subset T_1 \subset \ldots \subset T_e = T'$$

and every such chain has the same length. See Topology, Definition 5.10.4.

Definition 27.11.1. Let S be a scheme. We say S is *catenary* if the underlying topological space of S is catenary.

Recall that a ring A is called *catenary* if for any pair of prime ideals $\mathfrak{p} \subset \mathfrak{q}$ there exists a maximal chain of primes

$$\mathfrak{p} = \mathfrak{p}_0 \subset \ldots \subset \mathfrak{p}_e = \mathfrak{q}$$

and all of these have the same length. See Algebra, Definition 10.100.1.

Lemma 27.11.2. *Let S be a scheme. The following are equivalent*

(1) *S is catenary,*

(2) *there exists an open covering of S all of whose members are catenary schemes,*
(3) *for every affine open $\mathrm{Spec}(R) = U \subset S$ the ring R is catenary, and*
(4) *there exists an affine open covering $S = \bigcup U_i$ such that each U_i is the spectrum of a catenary ring.*

Moreover, in this case any locally closed subscheme of S is catenary as well.

Proof. Combine Topology, Lemma 5.10.5, and Algebra, Lemma 10.100.2. □

Lemma 27.11.3. *Let S be a locally Noetherian scheme. The following are equivalent:*

(1) *S is catenary, and*
(2) *locally in the Zariski topology there exists a dimension function on S (see Topology, Definition 5.19.1).*

Proof. This follows from Topology, Lemmas 5.10.5, 5.19.2, and 5.19.4, Schemes, Lemma 25.11.1 and finally Lemma 27.5.5. □

Lemma 27.11.4. *Let X be a scheme. Let $Y \subset X$ be an irreducible closed subset. Let $\xi \in Y$ be the generic point. Then*

$$\mathit{codim}(Y, X) = \dim(\mathcal{O}_{X,\xi})$$

where the codimension is as defined in Topology, Definition 5.10.1.

Proof. By Topology, Lemma 5.10.2 we may replace X by an affine open neighbourhood of ξ. In this case the result follows easily from Algebra, Lemma 10.24.3. □

In particular the dimension of a scheme is the supremum of the dimensions of all of its local rings. It turns out that we can use this lemma to characterize a catenary scheme as a scheme all of whose local rings are catenary.

Lemma 27.11.5. *Let X be a scheme. The following are equivalent*

(1) *X is catenary, and*
(2) *for any $x \in X$ the local ring $\mathcal{O}_{X,x}$ is catenary.*

Proof. Assume X is catenary. Let $x \in X$. By Lemma 27.11.2 we may replace X by an affine open neighbourhood of x, and then $\Gamma(X, \mathcal{O}_X)$ is a catenary ring. By Algebra, Lemma 10.100.3 any localization of a catenary ring is catenary. Whence $\mathcal{O}_{X,x}$ is catenary.

Conversely assume all local rings of X are catenary. Let $Y \subset Y'$ be an inclusion of irreducible closed subsets of X. Let $\xi \in Y$ be the generic point. Let $\mathfrak{p} \subset \mathcal{O}_{X,\xi}$ be the prime corresponding to the generic point of Y', see Schemes, Lemma 25.13.2. By that same lemma the irreducible closed subsets of X in between Y and Y' correspond to primes $\mathfrak{q} \subset \mathcal{O}_{X,\xi}$ with $\mathfrak{p} \subset \mathfrak{q} \subset \mathfrak{m}_\xi$. Hence we see all maximal chains of these are finite and have the same length as $\mathcal{O}_{X,\xi}$ is a catenary ring. □

27.12. Serre's conditions

Here are two technical notions that are often useful. See also Cohomology of Schemes, Section 29.11.

Definition 27.12.1. Let X be a locally Noetherian scheme. Let $k \geq 0$.

(1) We say X is *regular in codimension k*, or we say X has property (R_k) if for every $x \in X$ we have
$$\dim(\mathcal{O}_{X,x}) \leq k \Rightarrow \mathcal{O}_{X,x} \text{ is regular}$$
(2) We say X has property (S_k) if for every $x \in X$ we have $\text{depth}(\mathcal{O}_{X,x}) \geq \min(k, \dim(\mathcal{O}_{X,x}))$.

The phrase "regular in codimension k" makes sense since we have seen in Section 27.11 that if $Y \subset X$ is irreducible closed with generic point x, then $\dim(\mathcal{O}_{X,x}) = \text{codim}(Y, X)$. For example condition (R_0) means that for every generic point $\eta \in X$ of an irreducible component of X the local ring $\mathcal{O}_{X,\eta}$ is a field. But for general Noetherian schemes it can happen that the regular locus of X is badly behaved, so care has to be taken.

Lemma 27.12.2. *Let X be a locally Noetherian scheme. Then X is Cohen-Macaulay if and only if X has (S_k) for all $k \geq 0$.*

Proof. By Lemma 27.8.2 we reduce to looking at local rings. Hence the lemma is true because a Noetherian local ring is Cohen-Macaulay if and only if it has depth equal to its dimension. □

Lemma 27.12.3. *Let X be a locally Noetherian scheme. Then X is reduced if and only if X has properties (S_1) and (R_0).*

Proof. This is Algebra, Lemma 10.144.3. □

Lemma 27.12.4. *Let X be a locally Noetherian scheme. Then X is normal if and only if X has properties (S_2) and (R_1).*

Proof. This is Algebra, Lemma 10.144.4. □

27.13. Japanese and Nagata schemes

The notions considered in this section are not prominently defined in EGA. A "universally Japanese scheme" is mentioned and defined in [**DG67**, IV Corollary 5.11.4]. A "Japanese scheme" is mentioned in [**DG67**, IV Remark 10.4.14 (ii)] but no definition is given. A Nagata scheme (as given below) occurs in a few places in the literature (see for example [**Liu02**, Definition 8.2.30] and [**Gre76**, Page 142]).

We briefly recall that a domain R is called *Japanese* if the integral closure of R in any finite extension of its fraction field is finite over R. A ring R is called *universally Japanese* if for any finite type ring map $R \to S$ with S a domain S is Japanese. A ring R is called *Nagata* if it is Noetherian and $R/\mathfrak{p}$ is Japanese for every prime $\mathfrak{p}$ of R.

Definition 27.13.1. Let X be a scheme.

(1) Assume X integral. We say X is *Japanese* if for every $x \in X$ there exists an affine open neighbourhood $x \in U \subset X$ such that the ring $\mathcal{O}_X(U)$ is Japanese (see Algebra, Definition 10.148.1).
(2) We say X is *universally Japanese* if for every $x \in X$ there exists an affine open neighbourhood $x \in U \subset X$ such that the ring $\mathcal{O}_X(U)$ is universally Japanese (see Algebra, Definition 10.148.15).
(3) We say X is *Nagata* if for every $x \in X$ there exists an affine open neighbourhood $x \in U \subset X$ such that the ring $\mathcal{O}_X(U)$ is Nagata (see Algebra, Definition 10.148.15).

Being Nagata is the same thing as being locally Noetherian and universally Japanese, see Lemma 27.13.8.

Remark 27.13.2. In [**Hoo72**] a (locally Noetherian) scheme X is called Japanese if for every $x \in X$ and every associated prime $\mathfrak{p}$ of $\mathcal{O}_{X,x}$ the ring $\mathcal{O}_{X,x}/\mathfrak{p}$ is Japanese. We do not use this definition since it is not clear that this gives the same notion as above for Noetherian integral schemes. In other words, we do not know whether a Noetherian domain all of whose local rings are Japanese is Japanese. If you do please email stacks.project@gmail.com. On the other hand, we could circumvent this problem by calling a scheme X Japanese if for every affine open $\mathrm{Spec}(A) \subset X$ the ring $A/\mathfrak{p}$ is Japanese for every associated prime $\mathfrak{p}$ of A.

Lemma 27.13.3. *A Nagata scheme is locally Noetherian.*

Proof. This is true because a Nagata ring is Noetherian by definition. □

Lemma 27.13.4. *Let X be an integral scheme. The following are equivalent:*

(1) *The scheme X is Japanese.*
(2) *For every affine open $U \subset X$ the domain $\mathcal{O}_X(U)$ is Japanese.*
(3) *There exists an affine open covering $X = \bigcup U_i$ such that each $\mathcal{O}_X(U_i)$ is Japanese.*
(4) *There exists an open covering $X = \bigcup X_j$ such that each open subscheme X_j is Japanese.*

Moreover, if X is Japanese then every open subscheme is Japanese.

Proof. This follows from Lemma 27.4.3 and Algebra, Lemmas 10.148.3 and 10.148.4. □

Lemma 27.13.5. *Let X be a scheme. The following are equivalent:*

(1) *The scheme X is universally Japanese.*
(2) *For every affine open $U \subset X$ the ring $\mathcal{O}_X(U)$ is universally Japanese.*
(3) *There exists an affine open covering $X = \bigcup U_i$ such that each $\mathcal{O}_X(U_i)$ is universally Japanese.*
(4) *There exists an open covering $X = \bigcup X_j$ such that each open subscheme X_j is universally Japanese.*

Moreover, if X is universally Japanese then every open subscheme is universally Japanese.

Proof. This follows from Lemma 27.4.3 and Algebra, Lemmas 10.148.18 and 10.148.21. □

Lemma 27.13.6. *Let X be a scheme. The following are equivalent:*

(1) *The scheme X is Nagata.*
(2) *For every affine open $U \subset X$ the ring $\mathcal{O}_X(U)$ is Nagata.*
(3) *There exists an affine open covering $X = \bigcup U_i$ such that each $\mathcal{O}_X(U_i)$ is Nagata.*
(4) *There exists an open covering $X = \bigcup X_j$ such that each open subscheme X_j is Nagata.*

Moreover, if X is Nagata then every open subscheme is Nagata.

Proof. This follows from Lemma 27.4.3 and Algebra, Lemmas 10.148.20 and 10.148.21. □

Lemma 27.13.7. *Let X be a locally Noetherian scheme. Then X is Nagata if and only if every integral closed subscheme $Z \subset X$ is Japanese.*

Proof. Assume X is Nagata. Let $Z \subset X$ be an integral closed subscheme. Let $z \in Z$. Let $\mathrm{Spec}(A) = U \subset X$ be an affine open containing z such that A is Nagata. Then $Z \cap U \cong \mathrm{Spec}(A/\mathfrak{p})$ for some prime $\mathfrak{p}$, see Schemes, Lemma 25.10.1 (and Definition 27.3.1). By Algebra, Definition 10.148.15 we see that $A/\mathfrak{p}$ is Japanese. Hence Z is Japanese by definition.

Assume every integral closed subscheme of X is Japanese. Let $\mathrm{Spec}(A) = U \subset X$ be any affine open. As X is locally Noetherian we see that A is Noetherian (Lemma 27.5.2). Let $\mathfrak{p} \subset A$ be a prime ideal. We have to show that $A/\mathfrak{p}$ is Japanese. Let $T \subset U$ be the closed subset $V(\mathfrak{p}) \subset \mathrm{Spec}(A)$. Let $\overline{T} \subset X$ be the closure. Then $\overline{T}$ is irreducible as the closure of an irreducible subset. Hence the reduced closed subscheme defined by $\overline{T}$ is an integral closed subscheme (called $\overline{T}$ again), see Schemes, Lemma 25.12.4. In other words, $\mathrm{Spec}(A/\mathfrak{p})$ is an affine open of an integral closed subscheme of X. This subscheme is Japanese by assumption and by Lemma 27.13.4 we see that $A/\mathfrak{p}$ is Japanese. □

Lemma 27.13.8. *Let X be a scheme. The following are equivalent:*

1. *X is Nagata, and*
2. *X is locally Noetherian and universally Japanese.*

Proof. This is Algebra, Proposition 10.148.30. □

This discussion will be continued in Morphisms, Section 28.19.

27.14. The singular locus

Here is the definition.

Definition 27.14.1. Let X be a locally Noetherian scheme. The *regular locus* $\mathrm{Reg}(X)$ of X is the set of $x \in X$ such that $\mathcal{O}_{X,x}$ is a regular local ring. The *singular locus* $\mathrm{Sing}(X)$ is the complement $X \setminus \mathrm{Reg}(X)$, i.e., the set of points $x \in X$ such that $\mathcal{O}_{X,x}$ is not a regular local ring.

The regular locus of a locally Noetherian scheme is stable under generalizations, see the discussion preceding Algebra, Definition 10.105.6. However, for general locally Noetherian schemes the regular locus need not be open. In More on Algebra, Section 15.34 the reader can find some criteria for when this is the case. We will discuss this further in Morphisms, Section 28.20.

27.15. Quasi-affine schemes

Definition 27.15.1. A scheme X is called *quasi-affine* if it is quasi-compact and isomorphic to an open subscheme of an affine scheme.

Lemma 27.15.2. *Let X be a scheme. Let $f \in \Gamma(X, \mathcal{O}_X)$. Denote X_f the maximal open subscheme of X where f is invertible, see Schemes, Lemma 25.6.2 or Modules, Lemma 17.21.7. If X is quasi-compact and quasi-separated, the canonical map*

$$\Gamma(X, \mathcal{O}_X)_f \longrightarrow \Gamma(X_f, \mathcal{O}_X)$$

is an isomorphism. Moreover, if $\mathcal{F}$ is a quasi-coherent sheaf of $\mathcal{O}_X$-modules the map

$$\Gamma(X, \mathcal{F})_f \longrightarrow \Gamma(X_f, \mathcal{F})$$

is an isomorphism.

Proof. Write $R = \Gamma(X, \mathcal{O}_X)$. Consider the canonical morphism

$$\varphi : X \longrightarrow \mathrm{Spec}(R)$$

of schemes, see Schemes, Lemma 25.6.4. Then the inverse image of the standard open $D(f)$ on the right hand side is X_f on the left hand side. Moreover, since X is assumed quasi-compact and quasi-separated the morphism φ is quasi-compact and quasi-separated, see Schemes, Lemma 25.19.2 and 25.21.14. Hence by Schemes, Lemma 25.24.1 we see that $\varphi_*\mathcal{F}$ is quasi-coherent. Hence we see that $\varphi_*\mathcal{F} = \widetilde{M}$ with $M = \Gamma(X, \mathcal{F})$ as an R-module. Thus we see that

$$\Gamma(X_f, \mathcal{F}) = \Gamma(D(f), \varphi_*\mathcal{F}) = \Gamma(D(f), \widetilde{M}) = M_f$$

which is exactly the content of the lemma. The case of $\mathcal{F} = \mathcal{O}_X$ will given the first displayed isomorphism of the lemma. □

Lemma 27.15.3. *Let X be a scheme. Let $f \in \Gamma(X, \mathcal{O}_X)$. Assume X is quasi-compact and quasi-separated and assume that X_f is affine. Then the canonical morphism*

$$j : X \longrightarrow \mathrm{Spec}(\Gamma(X, \mathcal{O}_X))$$

from Schemes, Lemma 25.6.4 induces an isomorphism of $X_f = j^{-1}(D(f))$ onto the standard affine open $D(f) \subset \mathrm{Spec}(\Gamma(X, \mathcal{O}_X))$.

Proof. This is clear as j induces an isomorphism of rings $\Gamma(X, \mathcal{O}_X)_f \to \mathcal{O}_X(X_f)$ by Lemma 27.15.2 above. □

Lemma 27.15.4. *Let X be a scheme. Then X is quasi-affine if and only if the canonical morphism*

$$X \longrightarrow \mathrm{Spec}(\Gamma(X, \mathcal{O}_X))$$

from Schemes, Lemma 25.6.4 is a quasi-compact open immersion.

Proof. If the displayed morphism is a quasi-compact open immersion then X is isomorphic to a quasi-compact open subscheme of $\mathrm{Spec}(\Gamma(X, \mathcal{O}_X))$ and clearly X is quasi-affine.

Assume X is quasi-affine, say $X \subset \mathrm{Spec}(R)$ is quasi-compact open. This in particular implies that X is separated, see Schemes, Lemma 25.23.8. Let $A = \Gamma(X, \mathcal{O}_X)$. Consider the ring map $R \to A$ coming from $R = \Gamma(\mathrm{Spec}(R), \mathcal{O}_{\mathrm{Spec}(R)})$ and the restriction mapping of the sheaf $\mathcal{O}_{\mathrm{Spec}(R)}$. By Schemes, Lemma 25.6.4 we obtain a factorization:

$$X \longrightarrow \mathrm{Spec}(A) \longrightarrow \mathrm{Spec}(R)$$

of the inclusion morphism. Let $x \in X$. Choose $r \in R$ such that $x \in D(r)$ and $D(r) \subset X$. Denote $f \in A$ the image of r in A. The open X_f of Lemma 27.15.2 above is equal to $D(r) \subset X$ and hence $A_f \cong R_r$ by the conclusion of that lemma. Hence $D(r) \to \mathrm{Spec}(A)$ is an isomorphism onto the standard affine open $D(f)$ of $\mathrm{Spec}(A)$. Since X can be covered by such affine opens $D(f)$ we win. □

27.16. Characterizing modules of finite type and finite presentation

Let X be a scheme. Let $\mathcal{F}$ be a quasi-coherent $\mathcal{O}_X$-module. The following lemma implies that $\mathcal{F}$ is of finite type (see Modules, Definition 17.9.1) if and only if $\mathcal{F}$ is on each open affine $\mathrm{Spec}(A) = U \subset X$ of the form $\widetilde{M}$ for some finite type A-module M. Similarly, $\mathcal{F}$ is of finite presentation (see Modules, Definition 17.11.1) if and only if $\mathcal{F}$ is on each open affine $\mathrm{Spec}(A) = U \subset X$ of the form $\widetilde{M}$ for some finitely presented A-module M.

Lemma 27.16.1. *Let $X = \mathrm{Spec}(R)$ be an affine scheme. The quasi-coherent sheaf of $\mathcal{O}_X$-modules $\widetilde{M}$ is a finite type $\mathcal{O}_X$-module if and only if M is a finite R-module.*

Proof. Assume $\widetilde{M}$ is a finite type $\mathcal{O}_X$-module. This means there exists an open covering of X such that $\widetilde{M}$ restricted to the members of this covering is globally generated by finitely many sections. Thus there also exists a standard open covering $X = \bigcup_{i=1,\ldots,n} D(f_i)$ such that $\widetilde{M}|_{D(f_i)}$ is generated by finitely many sections. Thus M_{f_i} is finitely generated for each i. Hence we conclude by Algebra, Lemma 10.22.2. □

Lemma 27.16.2. *Let $X = \mathrm{Spec}(R)$ be an affine scheme. The quasi-coherent sheaf of $\mathcal{O}_X$-modules $\widetilde{M}$ is an $\mathcal{O}_X$-module of finite presentation if and only if M is an R-module of finite presentation.*

Proof. Assume $\widetilde{M}$ is an $\mathcal{O}_X$-module of finite presentation. By Lemma 27.16.1 we see that M is a finite R-module. Choose a surjection $R^n \to M$ with kernel K. By Schemes, Lemma 25.5.4 there is a short exact sequence

$$0 \to \widetilde{K} \to \bigoplus \mathcal{O}_X^{\oplus n} \to \widetilde{M} \to 0$$

By Modules, Lemma 17.11.3 we see that $\widetilde{K}$ is a finite type $\mathcal{O}_X$-module. Hence by Lemma 27.16.1 again we see that K is a finite R-module. Hence M is an R-module of finite presentation. □

27.17. Flat modules

On any ringed space $(X, \mathcal{O}_X)$ we know what it means for an $\mathcal{O}_X$-module to be flat (at a point), see Modules, Definition 17.16.1 (Definition 17.16.3). On an affine scheme this matches the notion defined in the algebra chapter.

Lemma 27.17.1. *Let $X = \mathrm{Spec}(R)$ be an affine scheme. Let $\mathcal{F} = \widetilde{M}$ for some R-module M. The quasi-coherent sheaf $\mathcal{F}$ is a flat $\mathcal{O}_X$-module of if and only if M is a flat R-module.*

Proof. Flatness of $\mathcal{F}$ may be checked on the stalks, see Modules, Lemma 17.16.2. The same is true in the case of modules over a ring, see Algebra, Lemma 10.36.19. And since $\mathcal{F}_x = M_{\mathfrak{p}}$ if x corresponds to $\mathfrak{p}$ the lemma is true. □

27.18. Locally free modules

On any ringed space we know what it means for an $\mathcal{O}_X$-module to be (finite) locally free. On an affine scheme this matches the notion defined in the algebra chapter.

Lemma 27.18.1. *Let $X = \mathrm{Spec}(R)$ be an affine scheme. Let $\mathcal{F} = \widetilde{M}$ for some R-module M. The quasi-coherent sheaf $\mathcal{F}$ is a (finite) locally free $\mathcal{O}_X$-module of if and only if M is a (finite) locally free R-module.*

Proof. Follows from the definitions, see Modules, Definition 17.14.1 and Algebra, Definition 10.73.1. □

We can characterize finite locally free modules in many different ways.

Lemma 27.18.2. *Let X be a scheme. Let $\mathcal{F}$ be a quasi-coherent $\mathcal{O}_X$-module. The following are equivalent:*

1. *$\mathcal{F}$ is a flat $\mathcal{O}_X$-module of finite presentation,*
2. *$\mathcal{F}$ is $\mathcal{O}_X$-module of finite presentation and for all $x \in X$ the stalk $\mathcal{F}_x$ is a free $\mathcal{O}_{X,x}$-module,*
3. *$\mathcal{F}$ is a locally free, finite type $\mathcal{O}_X$-module,*
4. *$\mathcal{F}$ is a finite locally free $\mathcal{O}_X$-module, and*
5. *$\mathcal{F}$ is an $\mathcal{O}_X$-module of finite type, for every $x \in X$ the stalk $\mathcal{F}_x$ is a free $\mathcal{O}_{X,x}$-module, and the function*
$$\rho_{\mathcal{F}} : X \to \mathbf{Z}, \quad x \longmapsto \dim_{\kappa(x)} \mathcal{F}_x \otimes_{\mathcal{O}_{X,x}} \kappa(x)$$
is locally constant in the Zariski topology on X.

Proof. This lemma immediately reduces to the affine case. In this case the lemma is a reformulation of Algebra, Lemma 10.73.2. The translation uses Lemmas 27.16.1, 27.16.2, 27.17.1, and 27.18.1. □

27.19. Locally projective modules

A consequence of the work done in the algebra chapter is that it makes sense to define a locally projective module as follows.

Definition 27.19.1. Let X be a scheme. Let $\mathcal{F}$ be a quasi-coherent $\mathcal{O}_X$-module. We say $\mathcal{F}$ is *locally projective* if for every affine open $U \subset X$ the $\mathcal{O}_X(U)$-module $\mathcal{F}(U)$ is projective.

Lemma 27.19.2. *Let X be a scheme. Let $\mathcal{F}$ be a quasi-coherent $\mathcal{O}_X$-module. The following are equivalent*

1. *$\mathcal{F}$ is locally projective, and*
2. *there exists an affine open covering $X = \bigcup U_i$ such that the $\mathcal{O}_X(U_i)$-module $\mathcal{F}(U_i)$ is projective for every i.*

In particular, if $X = \operatorname{Spec}(A)$ and $\mathcal{F} = \widetilde{M}$ then $\mathcal{F}$ is locally projective if and only if M is a projective A-module.

Proof. First, note that if M is a projective A-module and $A \to B$ is a ring map, then $M \otimes_A B$ is a projective B-module, see Algebra, Lemma 10.89.1. Hence if U is an affine open such that $\mathcal{F}(U)$ is a projective $\mathcal{O}_X(U)$-module, then the standard open $D(f)$ is an affine open such that $\mathcal{F}(D(f))$ is a projective $\mathcal{O}_X(D(f))$-module for all $f \in \mathcal{O}_X(U)$. Assume (2) holds. Let $U \subset X$ be an arbitrary affine open. We can find an open covering $U = \bigcup_{j=1,\ldots,m} D(f_j)$ by finitely many standard opens $D(f_j)$ such that for each j the open $D(f_j)$ is a standard open of some U_i, see Schemes, Lemma 25.11.5. Hence, if we set $A = \mathcal{O}_X(U)$ and if M is an A-module such that $\mathcal{F}|_U$ corresponds to M, then we see that M_{f_j} is a projective A_{f_j}-module. It follows that $A \to B = \prod A_{f_j}$ is a faithfully flat ring map such that $M \times_A B$ is a projective B-module. Hence M is projective by Algebra, Theorem 10.90.5. □

Lemma 27.19.3. *Let $f : X \to Y$ be a morphism of schemes. Let $\mathcal{G}$ be a quasi-coherent $\mathcal{O}_Y$-module. If $\mathcal{G}$ is locally projective on Y, then $f^*\mathcal{G}$ is locally projective on X.*

Proof. See Algebra, Lemma 10.89.1. □

27.20. Extending quasi-coherent sheaves

It is sometimes useful to be able to show that a given quasi-coherent sheaf on an open subscheme extends to the whole scheme.

Lemma 27.20.1. *Let $j : U \to X$ be a quasi-compact open immersion of schemes.*

1. *Any quasi-coherent sheaf on U extends to a quasi-coherent sheaf on X.*
2. *Let $\mathcal{F}$ be a quasi-coherent sheaf on X. Let $\mathcal{G} \subset \mathcal{F}|_U$ be a quasi-coherent subsheaf. There exists a quasi-coherent subsheaf $\mathcal{H}$ of $\mathcal{F}$ such that $\mathcal{H}|_U = \mathcal{G}$ as subsheaves of $\mathcal{F}|_U$.*
3. *Let $\mathcal{F}$ be a quasi-coherent sheaf on X. Let $\mathcal{G}$ be a quasi-coherent sheaf on U. Let $\varphi : \mathcal{G} \to \mathcal{F}|_U$ be a morphism of $\mathcal{O}_U$-modules. There exists a quasi-coherent sheaf $\mathcal{H}$ of $\mathcal{O}_X$-modules and a map $\psi : \mathcal{H} \to \mathcal{F}$ such that $\mathcal{H}|_U = \mathcal{G}$ and that $\psi|_U = \varphi$.*

Proof. An immersion is separated (see Schemes, Lemma 25.23.7) and j is quasi-compact by assumption. Hence for any quasi-coherent sheaf $\mathcal{G}$ on U the sheaf $j_*\mathcal{G}$ is an extension to X. See Schemes, Lemma 25.24.1 and Sheaves, Section 6.31.

Assume $\mathcal{F}$, $\mathcal{G}$ are as in (2). Then $j_*\mathcal{G}$ is a quasi-coherent sheaf on X (see above). It is a subsheaf of $j_*j^*\mathcal{F}$. Hence the kernel

$$\mathcal{H} = \mathrm{Ker}(\mathcal{F} \oplus j_*\mathcal{G} \longrightarrow j_*j^*\mathcal{F})$$

is quasi-coherent as well, see Schemes, Section 25.24. It is formal to check that $\mathcal{H} \subset \mathcal{F}$ and that $\mathcal{H}|_U = \mathcal{G}$ (using the material in Sheaves, Section 6.31 again).

The same proof as above works. Just take $\mathcal{H} = \mathrm{Ker}(\mathcal{F} \oplus j_*\mathcal{G} \to j_*j^*\mathcal{F})$ with its obvious map to $\mathcal{F}$ and its obvious identification with $\mathcal{G}$ over U. □

Lemma 27.20.2. *Let X be a quasi-compact and quasi-separated scheme. Let $U \subset X$ be a quasi-compact open. Let $\mathcal{F}$ be a quasi-coherent $\mathcal{O}_X$-module. Let $\mathcal{G} \subset \mathcal{F}|_U$ be a quasi-coherent $\mathcal{O}_U$-submodule which is of finite type. Then there exists a quasi-coherent submodule $\mathcal{G}' \subset \mathcal{F}$ which is of finite type such that $\mathcal{G}'|_U = \mathcal{G}$.*

Proof. Let n be the minimal number of affine opens $U_i \subset X$, $i = 1, \ldots, n$ such that $X = U \cup \bigcup U_i$. (Here we use that X is quasi-compact.) Suppose we can prove the lemma for the case $n = 1$. Then we can successively extend $\mathcal{G}$ to a $\mathcal{G}_1$ over $U \cup U_1$ to a $\mathcal{G}_2$ over $U \cup U_1 \cup U_2$ to a $\mathcal{G}_3$ over $U \cup U_1 \cup U_2 \cup U_3$, and so on. Thus we reduce to the case $n = 1$.

Thus we may assume that $X = U \cup V$ with V affine. Since X is quasi-separated and U, V are quasi-compact open, we see that $U \cap V$ is a quasi-compact open. It suffices to prove the lemma for the system $(V, U \cap V, \mathcal{F}|_V, \mathcal{G}|_{U \cap V})$ since we can glue the resulting sheaf $\mathcal{G}'$ over V to the given sheaf $\mathcal{G}$ over U along the common value over $U \cap V$. Thus we reduce to the case where X is affine.

Assume $X = \mathrm{Spec}(R)$. Write $\mathcal{F} = \widetilde{M}$ for some R-module M. By Lemma 27.20.1 above we may find a quasi-coherent subsheaf $\mathcal{H} \subset \mathcal{F}$ which restricts to $\mathcal{G}$ over U.

Write $\mathcal{H} = \widetilde{N}$ for some R-module N. For every $u \in U$ there exists an $f \in R$ such that $u \in D(f) \subset U$ and such that N_f is finitely generated, see Lemma 27.16.1. Since U is quasi-compact we can cover it by finitely many $D(f_i)$ such that N_{f_i} is generated by finitely many elements, say $x_{i,1}/f_i^N, \ldots, x_{i,r_i}/f_i^N$. Let $N' \subset N$ be the submodule generated by the elements $x_{i,j}$. Then the subsheaf $\mathcal{G} := \widetilde{N'} \subset \mathcal{H} \subset \mathcal{F}$ works. □

Lemma 27.20.3. *Let X be a quasi-compact and quasi-separated scheme. Any quasi-coherent sheaf of $\mathcal{O}_X$-modules is the directed colimit of its quasi-coherent $\mathcal{O}_X$-submodules which are of finite type.*

Proof. The colimit is direct because if $\mathcal{G}_1$, $\mathcal{G}_2$ are quasi-coherent subsheaves of finite type, then $\mathcal{G}_1 + \mathcal{G}_2 \subset \mathcal{F}$ is a quasi-coherent subsheaf of finite type. Let $U \subset X$ be any affine open, and let $s \in \Gamma(U, \mathcal{F})$ be any section. Let $\mathcal{G} \subset \mathcal{F}|_U$ be the subsheaf generated by s. Then clearly $\mathcal{G}$ is quasi-coherent and has finite type as an $\mathcal{O}_U$-module. By Lemma 27.20.2 we see that $\mathcal{G}$ is the restriction of a quasi-coherent subsheaf $\mathcal{G}' \subset \mathcal{F}$ which has finite type. Since X has a basis for the topology consisting of affine opens we conclude that every local section of $\mathcal{F}$ is locally contained in a quasi-coherent submodule of finite type. Thus we win. □

Lemma 27.20.4. *(Variant of Lemma 27.20.2 dealing with modules of finite presentation.) Let X be a quasi-compact and quasi-separated scheme. Let $\mathcal{F}$ be a quasi-coherent $\mathcal{O}_X$-module. Let $U \subset X$ be a quasi-compact open. Let $\mathcal{G}$ be an $\mathcal{O}_U$-module which of finite presentation. Let $\varphi : \mathcal{G} \to \mathcal{F}|_U$ be a morphism of $\mathcal{O}_U$-modules. Then there exists an $\mathcal{O}_X$-module $\mathcal{G}'$ of finite presentation, and a morphism of $\mathcal{O}_X$-modules $\varphi' : \mathcal{G}' \to \mathcal{F}$ such that $\mathcal{G}'|_U = \mathcal{G}$ and such that $\varphi'|_U = \varphi$.*

Proof. The beginning of the proof is a repeat of the beginning of the proof of Lemma 27.20.2. We write it out carefuly anyway.

Let n be the minimal number of affine opens $U_i \subset X$, $i = 1, \ldots, n$ such that $X = U \cup \bigcup U_i$. (Here we use that X is quasi-compact.) Suppose we can prove the lemma for the case $n = 1$. Then we can successively extend the pair $(\mathcal{G}, \varphi)$ to a pair $(\mathcal{G}_1, \varphi_1)$ over $U \cup U_1$ to a pair $(\mathcal{G}_2, \varphi_2)$ over $U \cup U_1 \cup U_2$ to a pair $(\mathcal{G}_3, \varphi_3)$ over $U \cup U_1 \cup U_2 \cup U_3$, and so on. Thus we reduce to the case $n = 1$.

Thus we may assume that $X = U \cup V$ with V affine. Since X is quasi-separated and U quasi-compact, we see that $U \cap V \subset V$ is quasi-compact. Suppose we prove the lemma for the system $(V, U \cap V, \mathcal{F}|_V, \mathcal{G}|_{U \cap V}, \varphi|_{U \cap V})$ thereby producing $(\mathcal{G}', \varphi')$ over V. Then we can glue $\mathcal{G}'$ over V to the given sheaf $\mathcal{G}$ over U along the common value over $U \cap V$, and similarly we can glue the map φ' to the map φ along the common value over $U \cap V$. Thus we reduce to the case where X is affine.

Assume $X = \mathrm{Spec}(R)$. By Lemma 27.20.1 above we may find a quasi-coherent sheaf $\mathcal{H}$ with a map $\psi : \mathcal{H} \to \mathcal{F}$ over X which restricts to $\mathcal{G}$ and φ over U. By Lemma 27.20.2 we can find a finite type quasi-coherent $\mathcal{O}_X$-submodule $\mathcal{H}' \subset \mathcal{H}$ such that $\mathcal{H}'|_U = \mathcal{G}$. Thus after replacing $\mathcal{H}$ by $\mathcal{H}'$ and ψ by the restriction of ψ to $\mathcal{H}'$ we may assume that $\mathcal{H}$ is of finite type. By Lemma 27.16.2 we conclude that $\mathcal{H} = \widetilde{N}$ with N a finitely generated R-module. Hence there exists a surjection as in the following short exact sequence of quasi-coherent $\mathcal{O}_X$-modules

$$0 \to \mathcal{K} \to \mathcal{O}_X^{\oplus n} \to \mathcal{H} \to 0$$

where $\mathcal{K}$ is defined as the kernel. Since $\mathcal{G}$ is of finite presentation and $\mathcal{H}|_U = \mathcal{G}$ by Modules, Lemma 17.11.3 the restriction $\mathcal{K}|_U$ is an $\mathcal{O}_U$-module of finite type. Hence by Lemma 27.20.2 again we see that there exists a finite type quasi-coherent $\mathcal{O}_X$-submodule $\mathcal{K}' \subset \mathcal{K}$ such that $\mathcal{K}'|_U = \mathcal{K}|_U$. The solution to the problem posed in the lemma is to set

$$\mathcal{G}' = \mathcal{O}_X^{\oplus n}/\mathcal{K}'$$

which is clearly of finite presentation and restricts to give $\mathcal{G}$ on U with φ' equal to the composition

$$\mathcal{G}' = \mathcal{O}_X^{\oplus n}/\mathcal{K}' \to \mathcal{O}_X^{\oplus n}/\mathcal{K} = \mathcal{H} \xrightarrow{\psi} \mathcal{F}.$$

This finishes the proof of the lemma. □

The following lemma says that every quasi-coherent sheaf on a quasi-compact and quasi-separated scheme is a filtered colimit of $\mathcal{O}$-modules of finite presentation. Actually, we reformulate this in (perhaps more familiar) terms of directed colimits over posets in the next lemma.

Lemma 27.20.5. *Let X be a scheme. Assume X is quasi-compact and quasi-separated. Let $\mathcal{F}$ be a quasi-coherent $\mathcal{O}_X$-module. There exist*

1. *a filtered index category $\mathcal{I}$ (see Categories, Definition 4.19.1),*
2. *a diagram $\mathcal{I} \to Mod(\mathcal{O}_X)$ (see Categories, Section 4.14), $i \mapsto \mathcal{F}_i$,*
3. *morphisms of $\mathcal{O}_X$-modules $\varphi_i : \mathcal{F}_i \to \mathcal{F}$*

such that each $\mathcal{F}_i$ is of finite presentation and such that the morphisms φ_i induce an isomorphism

$$\operatorname{colim}_i \mathcal{F}_i = \mathcal{F}.$$

Proof. Choose a set I and for each $i \in I$ an $\mathcal{O}_X$-module of finite presentation and a homomorphism of $\mathcal{O}_X$-modules $\varphi_i : \mathcal{F}_i \to \mathcal{F}$ with the following property: For any $\psi : \mathcal{G} \to \mathcal{F}$ with $\mathcal{G}$ of finite presentation there is an $i \in I$ such that there exists an isomorphism $\alpha : \mathcal{F}_i \to \mathcal{G}$ with $\varphi_i = \psi \circ \alpha$. It is clear from Modules, Lemma 17.9.8 that such a set exists (see also its proof). We denote $\mathcal{I}$ the category with $\mathrm{Ob}(\mathcal{I}) = I$ and given $i, i' \in I$ we set

$$\mathrm{Mor}_{\mathcal{I}}(i, i') = \{\alpha : \mathcal{F}_i \to \mathcal{F}_{i'} \mid \alpha \circ \varphi_{i'} = \varphi_i\}.$$

We claim that $\mathcal{I}$ is a filtered category and that $\mathcal{F} = \operatorname{colim}_i \mathcal{F}_i$.

Let $i, i' \in I$. Then we can consider the morphism

$$\mathcal{F}_i \oplus \mathcal{F}_{i'} \longrightarrow \mathcal{F}$$

which is the direct sum of φ_i and $\varphi_{i'}$. Since a direct sum of finitely presented $\mathcal{O}_X$-modules is finitely presented we see that there exists some $i'' \in I$ such that $\varphi_{i''} : \mathcal{F}_{i''} \to \mathcal{F}$ is isomorphic to the displayed arrow towards $\mathcal{F}$ above. Since there are commutative diagrams

$$\begin{array}{ccc} \mathcal{F}_i & \longrightarrow & \mathcal{F} \\ \downarrow & & \| \\ \mathcal{F}_i \oplus \mathcal{F}_{i'} & \longrightarrow & \mathcal{F} \end{array} \quad \text{and} \quad \begin{array}{ccc} \mathcal{F}_{i'} & \longrightarrow & \mathcal{F} \\ \downarrow & & \| \\ \mathcal{F}_i \oplus \mathcal{F}_{i'} & \longrightarrow & \mathcal{F} \end{array}$$

we see that there are morphisms $i \to i''$ and $i' \to i''$ in $\mathcal{I}$. Next, suppose that we have $i, i' \in I$ and morphisms $\alpha, \beta : i \to i'$ (corresponding to $\mathcal{O}_X$-module maps $\alpha, \beta : \mathcal{F}_i \to \mathcal{F}_{i'}$). In this case consider the coequalizer

$$\mathcal{G} = \mathrm{Coker}(\mathcal{F}_i \xrightarrow{\alpha-\beta} \mathcal{F}_{i'})$$

Note that $\mathcal{G}$ is an $\mathcal{O}_X$-module of finite presentation. Since by definition of morphisms in the category $\mathcal{I}$ we have $\varphi_{i'} \circ \alpha = \varphi_{i'} \circ \beta$ we see that we get an induced map $\psi : \mathcal{G} \to \mathcal{F}$. Hence again the pair $(\mathcal{G}, \psi)$ is isomorphic to the pair $(\mathcal{F}_{i''}, \varphi_{i''})$ for some i''. Hence we see that there exists a morphism $i' \to i''$ in $\mathcal{I}$ which equalizes α and β. Thus we have shown that the category $\mathcal{I}$ is filtered.

We still have to show that the colimit of the diagram is $\mathcal{F}$. By definition of the colimit, and by our definition of the category $\mathcal{I}$ there is a canonical map

$$\varphi : \mathrm{colim}_i \mathcal{F}_i \longrightarrow \mathcal{F}.$$

Pick $x \in X$. Let us show that φ_x is an isomorphism. Recall that

$$(\mathrm{colim}_i \mathcal{F}_i)_x = \mathrm{colim}_i \mathcal{F}_{i,x},$$

see Sheaves, Section 6.29. First we show that the map φ_x is injective. Suppose that $s \in \mathcal{F}_{i,x}$ is an element such that s maps to zero in $\mathcal{F}_x$. Then there exists a quasi-compact open U such that s comes from $s \in \mathcal{F}_i(U)$ and such that $\varphi_i(s) = 0$ in $\mathcal{F}(U)$. By Lemma 27.20.2 we can find a finite type quasi-coherent subsheaf $\mathcal{K} \subset \mathrm{Ker}(\varphi_i)$ which restricts to the quasi-coherent $\mathcal{O}_U$-submodule of $\mathcal{F}_i$ generated by s: $\mathcal{K}|_U = \mathcal{O}_U \cdot s \subset \mathcal{F}_i|_U$. Clearly, $\mathcal{F}_i/\mathcal{K}$ is of finite presentation and the map φ_i factors through the quotient map $\mathcal{F}_i \to \mathcal{F}_i/\mathcal{K}$. Hence we can find an $i' \in I$ and a morphism $\alpha : \mathcal{F}_i \to \mathcal{F}_{i'}$ in $\mathcal{I}$ which can be identified with the quotient map $\mathcal{F}_i \to \mathcal{F}_i/\mathcal{K}$. Then it follows that the section s maps to zero in $\mathcal{F}_{i'}(U)$ and in particular in $(\mathrm{colim}_i \mathcal{F}_i)_x = \mathrm{colim}_i \mathcal{F}_{i,x}$. The injectivity follows. Finally, we show that the map φ_x is surjective. Pick $s \in \mathcal{F}_x$. Choose a quasi-compact open neighbourhood $U \subset X$ of x such that s corresponds to a section $s \in \mathcal{F}(U)$. Consider the map $s : \mathcal{O}_U \to \mathcal{F}$ (multiplication by s). By Lemma 27.20.4 there exists an $\mathcal{O}_X$-module $\mathcal{G}$ of finite presentation and an $\mathcal{O}_X$-module map $\mathcal{G} \to \mathcal{F}$ such that $\mathcal{G}|_U \to \mathcal{F}|_U$ is identified with $s : \mathcal{O}_U \to \mathcal{F}$. Again by definition of $\mathcal{I}$ there exists an $i \in I$ such that $\mathcal{G} \to \mathcal{F}$ is isomorphic to $\varphi_i : \mathcal{F}_i \to \mathcal{F}$. Clearly there exists a section $s' \in \mathcal{F}_i(U)$ mapping to $s \in \mathcal{F}(U)$. This proves surjectivity and the proof of the lemma is complete. □

Lemma 27.20.6. *Let X be a scheme. Assume X is quasi-compact and quasi-separated. Let $\mathcal{F}$ be a quasi-coherent $\mathcal{O}_X$-module. There exist*

(1) *a directed partially ordered set I (see Categories, Definition 4.21.2),*
(2) *a system $(\mathcal{F}_i, \varphi_{ii'})$ over I in Mod$(\mathcal{O}_X)$ (see Categories, Definition 4.21.1)*
(3) *morphisms of $\mathcal{O}_X$-modules $\varphi_i : \mathcal{F}_i \to \mathcal{F}$*

such that each $\mathcal{F}_i$ is of finite presentation and such that the morphisms φ_i induce an isomorphism

$$\mathrm{colim}_i \mathcal{F}_i = \mathcal{F}.$$

Proof. This is a direct consequence of Lemma 27.20.5 and Categories, Lemma 4.21.3 (combined with the fact that colimits exist in the category of sheaves of $\mathcal{O}_X$-modules, see Sheaves, Section 6.29). □

Lemma 27.20.7. *Let X be a scheme. Assume X is quasi-compact and quasi-separated. Let $\mathcal{F}$ be a quasi-coherent $\mathcal{O}_X$-module. Then $\mathcal{F}$ is the directed colimit of its finite type quasi-coherent submodules.*

Proof. If $\mathcal{G}, \mathcal{H} \subset \mathcal{F}$ are finite type quasi-coherent $\mathcal{O}_X$-submodules then the image of $\mathcal{G} \oplus \mathcal{H} \to \mathcal{F}$ is another finite type quasi-coherent $\mathcal{O}_X$-submodule which contains both of them. In this way we see that the system is directed. To show that $\mathcal{F}$ is the colimit of this system, write $\mathcal{F} = \operatorname{colim}_i \mathcal{F}_i$ as a directed colimit of finitely presented quasi-coherent sheaves as in Lemma 27.20.6. Then the images $\mathcal{G}_i = \operatorname{Im}(\mathcal{F}_i \to \mathcal{F})$ are finite type quasi-coherent subsheaves of $\mathcal{F}$. Since $\mathcal{F}$ is the colimit of these the result follows. □

Lemma 27.20.8. *Let X be a scheme. Assume X is quasi-compact and quasi-separated. Let $\mathcal{F}$ be a finite type quasi-coherent $\mathcal{O}_X$-module. Then we can write $\mathcal{F} = \lim \mathcal{F}_i$ with $\mathcal{F}_i$ of finite presentation and all transition maps $\mathcal{F}_i \to \mathcal{F}_{i'}$ surjective.*

Proof. Write $\mathcal{F} = \operatorname{colim} \mathcal{G}_i$ as a filtered colimit of finitely presented $\mathcal{O}_X$-modules (Lemma 27.20.6). We claim that $\mathcal{G}_i \to \mathcal{F}$ is surjective for some i. Namely, choose a finite affine open covering $X = U_1 \cup \ldots \cup U_m$. Choose sections $s_{jl} \in \mathcal{F}(U_j)$ generating $\mathcal{F}|_{U_j}$, see Lemma 27.16.1. By Sheaves, Lemma 6.29.1 we see that s_{jl} is in the image of $\mathcal{G}_i \to \mathcal{F}$ for i large enough. Hence $\mathcal{G}_i \to \mathcal{F}$ is surjective for i large enough. Choose such an i and let $\mathcal{K} \subset \mathcal{G}_i$ be the kernel of the map $\mathcal{G}_i \to \mathcal{F}$. Write $\mathcal{K} = \operatorname{colim} \mathcal{K}_a$ as the filtered colimit of its finite type quasi-coherent submodules (Lemma 27.20.7). Then $\mathcal{F} = \operatorname{colim} \mathcal{G}_i/\mathcal{K}_a$ is a solution to the problem posed by the lemma. □

Lemma 27.20.9. *Let X be a quasi-compact and quasi-separated scheme. Let $\mathcal{F}$ be a finite type quasi-coherent $\mathcal{O}_X$-module. Let $U \subset X$ be a quasi-compact open such that $\mathcal{F}|_U$ is of finite presentation. Then there exists a map of $\mathcal{O}_X$-modules $\varphi : \mathcal{G} \to \mathcal{F}$ with (a) $\mathcal{G}$ of finite presentation, (b) φ is surjective, and (c) $\varphi|_U$ is an isomorphism.*

Proof. Write $\mathcal{F} = \operatorname{colim} \mathcal{F}_i$ as a directed colimit with each $\mathcal{F}_i$ of finite presentation, see Lemma 27.20.6. Choose a finite affine open covering $X = \bigcup V_j$ and choose finitely many sections $s_{jl} \in \mathcal{F}(V_j)$ generating $\mathcal{F}|_{V_j}$, see Lemma 27.16.1. By Sheaves, Lemma 6.29.1 we see that s_{jl} is in the image of $\mathcal{F}_i \to \mathcal{F}$ for i large enough. Hence $\mathcal{F}_i \to \mathcal{F}$ is surjective for i large enough. Choose such an i and let $\mathcal{K} \subset \mathcal{F}_i$ be the kernel of the map $\mathcal{F}_i \to \mathcal{F}$. Since $\mathcal{F}_U$ is of finite presentation, we see that $\mathcal{K}|_U$ is of finite type, see Modules, Lemma 17.11.3. Hence we can find a finite type quasi-coherent submodule $\mathcal{K}' \subset \mathcal{K}$ with $\mathcal{K}'|_U = \mathcal{K}|_U$, see Lemma 27.20.2. Then $\mathcal{G} = \mathcal{F}_i/\mathcal{K}'$ with the given map $\mathcal{G} \to \mathcal{F}$ is a solution. □

Let X be a scheme. In the following lemma we use the notion of a *quasi-coherent $\mathcal{O}_X$-algebra $\mathcal{A}$ of finite presentation*. This means that for every affine open $\operatorname{Spec}(R) \subset X$ we have $\mathcal{A} = \widetilde{A}$ where A is a (commutative) R-algebra which is of finite presentation as an R-algebra.

Lemma 27.20.10. *Let X be a scheme. Assume X is quasi-compact and quasi-separated. Let $\mathcal{A}$ be a quasi-coherent $\mathcal{O}_X$-algebra. There exist*

1. *a directed partially ordered set I (see Categories, Definition 4.21.2),*
2. *a system $(\mathcal{A}_i, \varphi_{ii'})$ over I in the category of $\mathcal{O}_X$-algebras,*
3. *morphisms of $\mathcal{O}_X$-algebras $\varphi_i : \mathcal{A}_i \to \mathcal{A}$*

such that each $\mathcal{A}_i$ is a quasi-coherent $\mathcal{O}_X$-algebra of finite presentation and such that the morphisms φ_i induce an isomorphism

$$\operatorname{colim}_i \mathcal{A}_i = \mathcal{A}.$$

Proof. First we write $\mathcal{A} = \operatorname{colim}_i \mathcal{F}_i$ as a directed colimit of finitely presented quasi-coherent sheaves as in Lemma 27.20.6. For each i let $\mathcal{B}_i = \operatorname{Sym}(\mathcal{F}_i)$ be the symmetric algebra on $\mathcal{F}_i$ over $\mathcal{O}_X$. Write $\mathcal{I}_i = \operatorname{Ker}(\mathcal{B}_i \to \mathcal{A})$. Write $\mathcal{I}_i = \operatorname{colim}_j \mathcal{F}_{i,j}$ where $\mathcal{F}_{i,j}$ is a finite type quasi-coherent submodule of $\mathcal{I}_i$, see Lemma 27.20.7. Set $\mathcal{I}_{i,j} \subset \mathcal{I}_i$ equal to the $\mathcal{B}_i$-ideal generated by $\mathcal{F}_{i,j}$. Set $\mathcal{A}_{i,j} = \mathcal{B}_i/\mathcal{I}_{i,j}$. Then $\mathcal{A}_{i,j}$ is a quasi-coherent finitely presented $\mathcal{O}_X$-algebra. Define $(i,j) \leq (i',j')$ if $i \leq i'$ and the map $\mathcal{B}_i \to \mathcal{B}_{i'}$ maps the ideal $\mathcal{I}_{i,j}$ into the ideal $\mathcal{I}_{i',j'}$. Then it is clear that $\mathcal{A} = \operatorname{colim}_{i,j} \mathcal{A}_{i,j}$. □

Let X be a scheme. In the following lemma we use the notion of a *quasi-coherent $\mathcal{O}_X$-algebra $\mathcal{A}$ of finite type*. This means that for every affine open $\operatorname{Spec}(R) \subset X$ we have $\mathcal{A} = \widetilde{A}$ where A is a (commutative) R-algebra which is of finite type as an R-algebra.

Lemma 27.20.11. *Let X be a scheme. Assume X is quasi-compact and quasi-separated. Let $\mathcal{A}$ be a quasi-coherent $\mathcal{O}_X$-algebra. Then $\mathcal{A}$ is the directed colimit of its finite type quasi-coherent $\mathcal{O}_X$-subalgebras.*

Proof. Omitted. Hint: Compare with the proof of Lemma 27.20.7. □

Let X be a scheme. In the following lemma we use the notion of a *finite (resp. integral) quasi-coherent $\mathcal{O}_X$-algebra $\mathcal{A}$*. This means that for every affine open $\operatorname{Spec}(R) \subset X$ we have $\mathcal{A} = \widetilde{A}$ where A is a (commutative) R-algebra which is finite (resp. integral) as an R-algebra.

Lemma 27.20.12. *Let X be a scheme. Assume X is quasi-compact and quasi-separated. Let $\mathcal{A}$ be a finite quasi-coherent $\mathcal{O}_X$-algebra. Then $\mathcal{A} = \operatorname{colim} \mathcal{A}_i$ is a directed colimit of finite and finitely presented quasi-coherent $\mathcal{O}_X$-algebras such that all transition maps $\mathcal{A}_{i'} \to \mathcal{A}_i$ are surjective.*

Proof. By Lemma 27.20.8 there exists a finitely presented $\mathcal{O}_X$-module $\mathcal{F}$ and a surjection $\mathcal{F} \to \mathcal{A}$. Using the algebra structure we obtain a surjection

$$\operatorname{Sym}^*_{\mathcal{O}_X}(\mathcal{F}) \longrightarrow \mathcal{A}$$

Denote $\mathcal{J}$ the kernel. Write $\mathcal{J} = \operatorname{colim} \mathcal{E}_i$ as a filtered colimit of finite type $\mathcal{O}_X$-submodules $\mathcal{E}_i$ (Lemma 27.20.7). Set

$$\mathcal{A}_i = \operatorname{Sym}^*_{\mathcal{O}_X}(\mathcal{F})/(\mathcal{E}_i)$$

where $(\mathcal{E}_i)$ indicates the ideal sheaf generated by the image of $\mathcal{E}_i \to \operatorname{Sym}^*_{\mathcal{O}_X}(\mathcal{F})$. Then each $\mathcal{A}_i$ is a finitely presented $\mathcal{O}_X$-algebra, the transition maps are surjections, and $\mathcal{A} = \operatorname{colim} \mathcal{A}_i$. To finish the proof we still have to show that $\mathcal{A}_i$ is a finite $\mathcal{O}_X$-algebra for i sufficiently large. To do this we choose an affine open covering $X = U_1 \cup \ldots \cup U_m$. Take generators $f_{j,1}, \ldots, f_{j,N_j} \in \Gamma(U_i, \mathcal{F})$. As $\mathcal{A}(U_j)$ is a finite $\mathcal{O}_X(U_j)$-algebra we see that for each k there exists a monic polynomial $P_{j,k} \in \mathcal{O}(U_j)[T]$ such that $P_{j,k}(f_{j,k})$ is zero in $\mathcal{A}(U_j)$. Since $\mathcal{A} = \operatorname{colim} \mathcal{A}_i$ by construction, we have $P_{j,k}(f_{j,k}) = 0$ in $\mathcal{A}_i(U_j)$ for all sufficiently large i. For such i the algebras $\mathcal{A}_i$ are finite. □

Lemma 27.20.13. *Let X be a scheme. Assume X is quasi-compact and quasi-separated. Let $\mathcal{A}$ be an integral quasi-coherent $\mathcal{O}_X$-algebra. Then*

(1) *$\mathcal{A}$ is the directed colimit of its finite quasi-coherent $\mathcal{O}_X$-subalgebras, and*
(2) *$\mathcal{A}$ is a direct colimit of finite and finitely presented quasi-cohernet $\mathcal{O}_X$-algebras.*

Proof. By Lemma 27.20.11 we have $\mathcal{A} = \operatorname{colim} \mathcal{A}_i$ where $\mathcal{A}_i \subset \mathcal{A}$ runs through the quasi-coherent $\mathcal{O}_X$-algebras of finite type. Any finite type quasi-coherent $\mathcal{O}_X$-subalgebra of $\mathcal{A}$ is finite (apply Algebra, Lemma 10.33.5 to $\mathcal{A}_i(U) \subset \mathcal{A}(U)$ for affine opens U in X). This proves (1).

To prove (2), write $\mathcal{A} = \operatorname{colim} \mathcal{F}_i$ as a colimit of finitely presented $\mathcal{O}_X$-modules using Lemma 27.20.6. For each i, let $\mathcal{J}_i$ be the kernel of the map

$$\operatorname{Sym}^*_{\mathcal{O}_X}(\mathcal{F}_i) \longrightarrow \mathcal{A}$$

For $i' \geq i$ there is an induced map $\mathcal{J}_i \to \mathcal{J}_{i'}$ and we have $\mathcal{A} = \operatorname{colim} \operatorname{Sym}^*_{\mathcal{O}_X}(\mathcal{F}_i)/\mathcal{J}_i$. Moreover, the quasi-coherent $\mathcal{O}_X$-algebras $\operatorname{Sym}^*_{\mathcal{O}_X}(\mathcal{F}_i)/\mathcal{J}_i$ are finite (see above). Write $\mathcal{J}_i = \operatorname{colim} \mathcal{E}_{ik}$ as a colimit of finitely presented $\mathcal{O}_X$-modules. Given $i' \geq i$ and k there exists a k' such that we have a map $\mathcal{E}_{ik} \to \mathcal{E}_{i'k'}$ making

$$\begin{array}{ccc} \mathcal{J}_i & \longrightarrow & \mathcal{J}_{i'} \\ \uparrow & & \uparrow \\ \mathcal{E}_{ik} & \longrightarrow & \mathcal{E}_{i'k'} \end{array}$$

commute. This follows from Modules, Lemma 17.11.6. This induces a map

$$\mathcal{A}_{ik} = \operatorname{Sym}^*_{\mathcal{O}_X}(\mathcal{F}_i)/(\mathcal{E}_{ik}) \longrightarrow \operatorname{Sym}^*_{\mathcal{O}_X}(\mathcal{F}_{i'})/(\mathcal{E}_{i'k'}) = \mathcal{A}_{i'k'}$$

where $(\mathcal{E}_{ik})$ denotes the ideal generated by $\mathcal{E}_{ik}$. The quasi-coherent $\mathcal{O}_X$-algebras $\mathcal{A}_{ki}$ are of finite presentation and finite for k large enough (see proof of Lemma 27.20.12). Finally, we have

$$\operatorname{colim} \mathcal{A}_{ik} = \operatorname{colim} \mathcal{A}_i = \mathcal{A}$$

Namely, the first equality was shown in the proof of Lemma 27.20.12 and the second equality because $\mathcal{A}$ is the colimit of the modules $\mathcal{F}_i$. □

27.21. Gabber's result

In this section we prove a result of Gabber which guarantees that on every scheme there exists a cardinal κ such that every quasi-coherent module $\mathcal{F}$ is the union of its quasi-coherent κ-generated subsheaves. It follows that the category of quasi-coherent sheaves on a scheme is a Grothendieck abelian category having limits and enough injectives[2].

Definition 27.21.1. Let $(X, \mathcal{O}_X)$ be a ringed space. Let κ be an infinite cardinal. We say a sheaf of $\mathcal{O}_X$-modules $\mathcal{F}$ is *κ-generated* if there exists an open covering $X = \bigcup U_i$ such that $\mathcal{F}|_{U_i}$ is generated by a subset $R_i \subset \mathcal{F}(U_i)$ whose cardinality is at most κ.

[2]Nicely explained in a blog post by Akhil Mathew.

Note that a direct sum of at most κ κ-generated modules is again κ-generated because $\kappa \otimes \kappa = \kappa$, see Sets, Section 3.6. In particular this holds for the direct sum of two κ-generated modules. Moreover, a quotient of a κ-generated sheaf is κ-generated. (But the same needn't be true for submodules.)

Lemma 27.21.2. *Let $(X, \mathcal{O}_X)$ be a ringed space. Let κ be a cardinal. There exists a set T and a family $(\mathcal{F}_t)_{t \in T}$ of κ-generated $\mathcal{O}_X$-modules such that every κ-generated $\mathcal{O}_X$-module is isomorphic to one of the $\mathcal{F}_t$.*

Proof. There is a set of coverings of X (provided we disallow repeats). Suppose $X = \bigcup U_i$ is a covering and suppose $\mathcal{F}_i$ is an $\mathcal{O}_{U_i}$-module. Then there is a set of isomorphism classes of $\mathcal{O}_X$-modules $\mathcal{F}$ with the property that $\mathcal{F}|_{U_i} \cong \mathcal{F}_i$ since there is a set of glueing maps. This reduces us to proving there is a set of (isomorphism classes of) quotients $\oplus_{k \in \kappa} \mathcal{O}_X \to \mathcal{F}$ for any ringed space X. This is clear. □

Here is the result the title of this section refers to.

Lemma 27.21.3. *Let X be a scheme. There exists a cardinal κ such that every quasi-coherent module $\mathcal{F}$ is the directed colimit of its quasi-coherent κ-generated quasi-coherent subsheaves.*

Proof. Choose an affine open covering $X = \bigcup_{i \in I} U_i$. For each pair i, j choose an affine open covering $U_i \cap U_j = \bigcup_{k \in I_{ij}} U_{ijk}$. Write $U_i = \mathrm{Spec}(A_i)$ and $U_{ijk} = \mathrm{Spec}(A_{ijk})$. Let κ be any infinite cardinal $\geq$ than the cardinality of any of the sets I, I_{ij}.

Let $\mathcal{F}$ be a quasi-coherent sheaf. Set $M_i = \mathcal{F}(U_i)$ and $M_{ijk} = \mathcal{F}(U_{ijk})$. Note that

$$M_i \otimes_{A_i} A_{ijk} = M_{ijk} = M_j \otimes_{A_j} A_{ijk}.$$

see Schemes, Lemma 25.7.3. Using the axiom of choice we choose a map

$$(i, j, k, m) \mapsto S(i, j, k, m)$$

which associates to every $i, j \in I$, $k \in I_{ij}$ and $m \in M_i$ a finite subset $S(i, j, k, m) \subset M_j$ such that we have

$$m \otimes 1 = \sum\nolimits_{m' \in S(i,j,k,m)} m' \otimes a_{m'}$$

in M_{ijk} for some $a_{m'} \in A_{ijk}$. Moreover, let's agree that $S(i, i, k, m) = \{m\}$ for all $i, j = i, k, m$ as above. Fix such a map.

Given a family $\mathcal{S} = (S_i)_{i \in I}$ of subsets $S_i \subset M_i$ of cardinality at most κ we set $\mathcal{S}' = (S_i')$ where

$$S_j' = \bigcup\nolimits_{(i,j,k,m) \text{ such that } m \in S_i} S(i, j, k, m)$$

Note that $S_i \subset S_i'$. Note that S_i' has cardinality at most κ because it is a union over a set of cardinality at most κ of finite sets. Set $\mathcal{S}^{(0)} = \mathcal{S}$, $\mathcal{S}^{(1)} = \mathcal{S}'$ and by induction $\mathcal{S}^{(n+1)} = (\mathcal{S}^{(n)})'$. Then set $\mathcal{S}^{(\infty)} = \bigcup_{n \geq 0} \mathcal{S}^{(n)}$. Writing $\mathcal{S}^{(\infty)} = (S_i^{(\infty)})$ we see that for any element $m \in S_i^{(\infty)}$ the image of m in M_{ijk} can be written as a finite sum $\sum m' \otimes a_{m'}$ with $m' \in S_j^{(\infty)}$. In this way we see that setting

$$N_i = A_i\text{-submodule of } M_i \text{ generated by } S_i^{(\infty)}$$

we have

$$N_i \otimes_{A_i} A_{ijk} = N_j \otimes_{A_j} A_{ijk}.$$

as submodules of M_{ijk}. Thus there exists a quasi-coherent subsheaf $\mathcal{G} \subset \mathcal{F}$ with $\mathcal{G}(U_i) = N_i$. Moreover, by construction the sheaf $\mathcal{G}$ is κ-generated.

Let $\{\mathcal{G}_t\}_{t \in T}$ be the set of κ-generated quasi-coherent subsheaves. If $t, t' \in T$ then $\mathcal{G}_t + \mathcal{G}_{t'}$ is also a κ-generated quasi-coherent subsheaf as it is the image of the map $\mathcal{G}_t \oplus \mathcal{G}_{t'} \to \mathcal{F}$. Hence the system (ordered by inclusion) is directed. The arguments above show that every section of $\mathcal{F}$ over U_i is in one of the $\mathcal{G}_t$ (because we can start with $\mathcal{S}$ such that the given section is an element of S_i). Hence $\operatorname{colim}_t \mathcal{G}_t \to \mathcal{F}$ is both injective and surjective as desired. □

Proposition 27.21.4. *Let X be a scheme.*

(1) *The category $QCoh(\mathcal{O}_X)$ is a Grothendieck abelian category. Consequently, $QCoh(\mathcal{O}_X)$ has enough injectives and all limits.*
(2) *The inclusion functor $QCoh(\mathcal{O}_X) \to Mod(\mathcal{O}_X)$ has a right adjoint*

$$Q^3 : Mod(\mathcal{O}_X) \longrightarrow QCoh(\mathcal{O}_X)$$

such that for every quasi-coherent sheaf $\mathcal{F}$ the adjunction mapping $Q(\mathcal{F}) \to \mathcal{F}$ is an isomorphism.

Proof. Part (1) means $QCoh(\mathcal{O}_X)$ (a) has all colimits, (b) filtered colimits are exact, and (c) has a generator, see Injectives, Section 19.10. By Schemes, Section 25.24 colimits in $QCoh(\mathcal{O}_X)$ exist and agree with colimits in $Mod(\mathcal{O}_X)$. By Modules, Lemma 17.3.2 filtered colimits are exact. Hence (a) and (b) hold. To construct a generator U, pick a cardinal κ as in Lemma 27.21.3. Pick a collection $(\mathcal{F}_t)_{t \in T}$ of κ-generated quasi-coherent sheaves as in Lemma 27.21.2. Set $U = \bigoplus_{t \in T} \mathcal{F}_t$. Since every object of $QCoh(\mathcal{O}_X)$ is a filtered colimit of κ-generated quasi-coherent modules, i.e., of objects isomorphic to $\mathcal{F}_t$, it is clear that U is a generator. The assertions on limits and injectives hold in any Grothendieck abelian category, see Injectives, Theorem 19.11.6 and Lemma 19.13.2.

Proof of (2). To construct Q we use the following general procedure. Given an object $\mathcal{F}$ of $Mod(\mathcal{O}_X)$ we consider the functor

$$QCoh(\mathcal{O}_X)^{opp} \longrightarrow Sets, \quad \mathcal{G} \longmapsto \operatorname{Hom}_X(\mathcal{G}, \mathcal{F})$$

This functor transforms colimits into limits, hence is representable, see Injectives, Lemma 19.13.1. Thus there exists a quasi-coherent sheaf $Q(\mathcal{F})$ and a functorial isomorphism $\operatorname{Hom}_X(\mathcal{G}, \mathcal{F}) = \operatorname{Hom}_X(\mathcal{G}, Q(\mathcal{F}))$ for $\mathcal{G}$ in $QCoh(\mathcal{O}_X)$. By the Yoneda lemma (Categories, Lemma 4.3.5) the construction $\mathcal{F} \rightsquigarrow Q(\mathcal{F})$ is functorial in $\mathcal{F}$. By construction Q is a right adjoint to the inclusion functor. The fact that $Q(\mathcal{F}) \to \mathcal{F}$ is an isomorphism when $\mathcal{F}$ is quasi-coherent is a formal consequence of the fact that the inclusion functor $QCoh(\mathcal{O}_X) \to Mod(\mathcal{O}_X)$ is fully faithful. □

27.22. Sections with support in a closed subset

Given any topological space X, a closed subset $Z \subset X$, and an abelian sheaf $\mathcal{F}$ you can take the subsheaf of sections whose support is contained in Z. If X is a scheme, Z a closed subscheme, and $\mathcal{F}$ a quasi-coherent module there is a variant where you take sections which are scheme theoretically supported on Z. However, in the scheme setting you have to be careful because the resulting $\mathcal{O}_X$-module may not be quasi-coherent.

[3]This functor is sometimes called the *coherator*.

Lemma 27.22.1. *Let X be a quasi-compact and quasi-separated scheme. Let $U \subset X$ be an open subscheme. The following are equivalent:*

(1) *U is retrocompact in X,*
(2) *U is quasi-compact,*
(3) *U is a finite union of affine opens, and*
(4) *there exists a finite type quasi-coherent sheaf of ideals $\mathcal{I} \subset \mathcal{O}_X$ such that $X \setminus U = V(\mathcal{I})$ (set theoretically).*

Proof. The equivalence of (1), (2), and (3) follows from Lemma 27.2.2. Assume (1), (2), (3). Let $T = X \setminus U$. By Schemes, Lemma 25.12.4 there exists a unique quasi-coherent sheaf of ideals $\mathcal{J}$ cutting out the reduced induced closed subscheme structure on T. Note that $\mathcal{J}|_U = \mathcal{O}_U$ which is an $\mathcal{O}_U$-modules of finite type. By Lemma 27.20.2 there exists a quasi-coherent subsheaf $\mathcal{I} \subset \mathcal{J}$ which is of finite type and has the property that $\mathcal{I}|_U = \mathcal{J}|_U$. Then $X \setminus U = V(\mathcal{I})$ and we obtain (4). Conversely, if $\mathcal{I}$ is as in (4) and $W = \mathrm{Spec}(R) \subset X$ is an affine open, then $\mathcal{I}|_W = \widetilde{I}$ for some finitely generated ideal $I \subset R$, see Lemma 27.16.1. It follows that $U \cap W = \mathrm{Spec}(R) \setminus V(I)$ is quasi-compact, see Algebra, Lemma 10.27.1. Hence $U \subset X$ is retrocompact by Lemma 27.2.5. □

Lemma 27.22.2. *Let X be a scheme. Let $\mathcal{I} \subset \mathcal{O}_X$ be a quasi-coherent sheaf of ideals. Let $\mathcal{F}$ be a quasi-coherent $\mathcal{O}_X$-module. Consider the sheaf of $\mathcal{O}_X$-modules $\mathcal{F}'$ which associates to every open $U \subset X$*

$$\mathcal{F}'(U) = \{s \in \mathcal{F}(U) \mid \mathcal{I}s = 0\}$$

Assume $\mathcal{I}$ is of finite type. Then

(1) *$\mathcal{F}'$ is a quasi-coherent sheaf of $\mathcal{O}_X$-modules,*
(2) *on any affine open $U \subset X$ we have $\mathcal{F}'(U) = \{s \in \mathcal{F}(U) \mid \mathcal{I}(U)s = 0\}$, and*
(3) *$\mathcal{F}'_x = \{s \in \mathcal{F}_x \mid \mathcal{I}_x s = 0\}$.*

Proof. It is clear that the rule defining $\mathcal{F}'$ gives a subsheaf of $\mathcal{F}$ (the sheaf condition is easy to verify). Hence we may work locally on X to verify the other statements. In other words we may assume that $X = \mathrm{Spec}(A)$, $\mathcal{F} = \widetilde{M}$ and $\mathcal{I} = \widetilde{I}$. It is clear that in this case $\mathcal{F}'(U) = \{x \in M \mid Ix = 0\} =: M'$ because $\widetilde{I}$ is generated by its global sections I which proves (2). To show $\mathcal{F}'$ is quasi-coherent it suffices to show that for every $f \in A$ we have $\{x \in M_f \mid I_f x = 0\} = (M')_f$. Write $I = (g_1, \ldots, g_t)$, which is possible because $\mathcal{I}$ is of finite type, see Lemma 27.16.1. If $x = y/f^n$ and $I_f x = 0$, then that means that for every i there exists an $m \geq 0$ such that $f^m g_i x = 0$. We may choose one m which works for all i (and this is where we use that I is finitely generated). Then we see that $f^m x \in M'$ and $x/f^n = f^m x/f^{n+m}$ in $(M')_f$ as desired. The proof of (3) is similar and omitted. □

Definition 27.22.3. Let X be a scheme. Let $\mathcal{I} \subset \mathcal{O}_X$ be a quasi-coherent sheaf of ideals of finite type. Let $\mathcal{F}$ be a quasi-coherent $\mathcal{O}_X$-module. The subsheaf $\mathcal{F}' \subset \mathcal{F}$ defined in Lemma 27.22.2 above is called the *subsheaf of sections annihilated by $\mathcal{I}$*.

Lemma 27.22.4. *Let $f : X \to Y$ be a quasi-compact and quasi-separated morphism of schemes. Let $\mathcal{I} \subset \mathcal{O}_Y$ be a quasi-coherent sheaf of ideals of finite type. Let $\mathcal{F}$ be a quasi-coherent $\mathcal{O}_X$-module. Let $\mathcal{F}' \subset \mathcal{F}$ be the subsheaf of sections annihilated by $f^{-1}\mathcal{I}\mathcal{O}_X$. Then $f_*\mathcal{F}' \subset f_*\mathcal{F}$ is the subsheaf of sections annihilated by $\mathcal{I}$.*

Proof. Omitted. (Hint: The assumption that f is quasi-compact and quasi-separated implies that $f_*\mathcal{F}$ is quasi-coherent so that Lemma 27.22.2 applies to $\mathcal{I}$ and $f_*\mathcal{F}$.) □

For an abelian sheaf on a topological space we have discussed the subsheaf of sections with support in a closed subset in Modules, Lemma 17.6.2. For quasi-coherent modules this submodule isn't always a quasi-coherent module, but if the closed subset has a retrocompact complement, then it is.

Lemma 27.22.5. *Let X be a scheme. Let $Z \subset X$ be a closed subset. Let $\mathcal{F}$ be a quasi-coherent $\mathcal{O}_X$-module. Consider the sheaf of $\mathcal{O}_X$-modules $\mathcal{F}'$ which associates to every open $U \subset X$*

$$\mathcal{F}'(U) = \{s \in \mathcal{F}(U) \mid \text{the support of } s \text{ is contained in } Z \cap U\}$$

If $X \setminus Z$ is a retrocompact open in X, then

1. *for an affine open $U \subset X$ there exist a finitely generated ideal $I \subset \mathcal{O}_X(U)$ such that $Z \cap U = V(I)$,*
2. *for U and I as in (1) we have $\mathcal{F}'(U) = \{x \in \mathcal{F}(U) \mid I^n x = 0 \text{ for some } n\}$,*
3. *$\mathcal{F}'$ is a quasi-coherent sheaf of $\mathcal{O}_X$-modules.*

Proof. Part (1) is Algebra, Lemma 10.27.1. Let $U = \operatorname{Spec}(A)$ and I be as in (1). Then $\mathcal{F}|_U$ is the quasi-coherent sheaf associated to some A-module M. We have

$$\mathcal{F}'(U) = \{x \in M \mid x = 0 \text{ in } M_\mathfrak{p} \text{ for all } \mathfrak{p} \notin Z\}.$$

by Modules, Definition 17.5.1. Thus $x \in \mathcal{F}'(U)$ if and only if $V(\operatorname{Ann}(x)) \subset V(I)$, see Algebra, Lemma 10.37.6. Since I is finitely generated this is equivalent to $I^n x = 0$ for some n. This proves (2).

The rule for $\mathcal{F}'$ indeed defines a submodule of $\mathcal{F}$. Hence we may work locally on X to verify (3). Let U, I and M be as above. Let $\mathcal{I} \subset \mathcal{O}_X$ be the quasi-coherent sheaf of ideals corresponding to I. Part (2) implies sections of $\mathcal{F}'$ over any affine open of U are the sections of $\mathcal{F}$ which are annihilated by some power of $\mathcal{I}$. Hence we see that $\mathcal{F}'|_U = \operatorname{colim} \mathcal{F}_n$, where $\mathcal{F}_n \subset \mathcal{F}|_U$ is the subsheaf of sections annihilated by $\mathcal{I}^n$, see Definition 27.22.3. Thus (3) follows from Lemma 27.22.2 and that colimits of quasi-coherent modules are quasi-coherent, see Schemes, Section 25.24. □

Definition 27.22.6. Let X be a scheme. Let $T \subset X$ be a closed subset whose complement is retrocompact in X. Let $\mathcal{F}$ be a quasi-coherent $\mathcal{O}_X$-module. The quasi-coherent subsheaf $\mathcal{F}' \subset \mathcal{F}$ defined in Lemma 27.22.5 is called the *subsheaf of sections supported on T*.

Lemma 27.22.7. *Let $f : X \to Y$ be a quasi-compact and quasi-separated morphism of schemes. Let $Z \subset Y$ be a closed subset such that $Y \setminus Z$ is retrocompact in Y. Let $\mathcal{F}$ be a quasi-coherent $\mathcal{O}_X$-module. Let $\mathcal{F}' \subset \mathcal{F}$ be the subsheaf of sections supported in $f^{-1}Z$. Then $f_*\mathcal{F}' \subset f_*\mathcal{F}$ is the subsheaf of sections supported in Z.*

Proof. Omitted. (Hint: First show that $X \setminus f^{-1}Z$ is retrocompact in X as $Y \setminus Z$ is retrocompact in Y. Hence Lemma 27.22.5 applies to $f^{-1}Z$ and $\mathcal{F}$. As f is quasi-compact and quasi-separated we see that $f_*\mathcal{F}$ is quasi-coherent. Hence Lemma 27.22.5 applies to Z and $f_*\mathcal{F}$. Finally, match the sheaves directly.) □

27.23. Sections of quasi-coherent sheaves

Here is a computation of sections of a quasi-coherent sheaf on a quasi-compact open of an affine spectrum.

Lemma 27.23.1. *Let A be a ring. Let $I \subset A$ be a finitely generated ideal. Let M be an A-module. Then there is a canonical map*

$$\operatorname{colim}_n \operatorname{Hom}_A(I^n, M) \longrightarrow \Gamma(\operatorname{Spec}(A) \setminus V(I), \widetilde{M}).$$

This map is always injective. If for all $x \in M$ we have $Ix = 0 \Rightarrow x = 0$ then this map is an isomorphism. In general, set $M_n = \{x \in M \mid I^n x = 0\}$, then there is an isomorphism

$$\operatorname{colim}_n \operatorname{Hom}_A(I^n, M/M_n) \longrightarrow \Gamma(\operatorname{Spec}(A) \setminus V(I), \widetilde{M}).$$

Proof. Since $I^n \subset I^{n+1}$ and $M_n \subset M_{n+1}$ we can use composition via these maps to get canonical maps of A-modules

$$\operatorname{Hom}_A(I^n, M) \longrightarrow \operatorname{Hom}_A(I^{n+1}, M)$$

and

$$\operatorname{Hom}_A(I^n, M/M_n) \longrightarrow \operatorname{Hom}_A(I^{n+1}, M/M_{n+1})$$

which we will use as the transition maps in the systems. Given an A-module map $\varphi : I^n \to M$, then we get a map of sheaves $\widetilde{\varphi} : \widetilde{I} \to \widetilde{M}$ which we can restrict to the open $\operatorname{Spec}(A) \setminus V(I)$. Since $\widetilde{I}$ restricted to this open gives the structure sheaf we get an element of $\Gamma(\operatorname{Spec}(A) \setminus V(I), \widetilde{M})$. We omit the verification that this is compatible with the transition maps in the system $\operatorname{Hom}_A(I^n, M)$. This gives the first arrow. To get the second arrow we note that $\widetilde{M}$ and $\widetilde{M/M_n}$ agree over the open $\operatorname{Spec}(A) \setminus V(I)$ since the sheaf $\widetilde{M_n}$ is clearly supported on $V(I)$. Hence we can use the same mechanism as before.

Next, we work out how to define this arrow in terms of algebra. Say $I = (f_1, \ldots, f_t)$. Then $\operatorname{Spec}(A) \setminus V(I) = \bigcup_{i=1,\ldots,t} D(f_i)$. Hence

$$0 \to \Gamma(\operatorname{Spec}(A) \setminus V(I), \widetilde{M}) \to \bigoplus\nolimits_i M_{f_i} \to \bigoplus\nolimits_{i,j} M_{f_i f_j}$$

is exact. Suppose that $\varphi : I^n \to M$ is an A-module map. Consider the vector of elements $\varphi(f_i^n)/f_i^n \in M_{f_i}$. It is easy to see that this vector maps to zero in the second direct sum of the exact sequence above. Whence an element of $\Gamma(\operatorname{Spec}(A) \setminus V(I), \widetilde{M})$. We omit the verification that this description agrees with the one given above.

Let us show that the first arrow is injective using this description. Namely, if φ maps to zero, then for each i the element $\varphi(f_i^n)/f_i^n$ is zero in M_{f_i}. In other words we see that for each i we have $f_i^m \varphi(f_i^n) = 0$ for some $m \geq 0$. We may choose a single m which works for all i. Then we see that $\varphi(f_i^{n+m}) = 0$ for all i. It is easy to see that this means that $\varphi|_{I^{t(n+m-1)+1}} = 0$ in other words that φ maps to zero in the $t(n+m-1)+1$st term of the colimit. Hence injectivity follows.

Note that each $M_n = 0$ in case we have $Ix = 0 \Rightarrow x = 0$ for $x \in M$. Thus to finish the proof of the lemma it suffices to show that the second arrow is an isomorphism.

Let us attempt to construct an inverse of the second map of the lemma. Let $s \in \Gamma(\operatorname{Spec}(A) \setminus V(I), \widetilde{M})$. This corresponds to a vector x_i/f_i^n with $x_i \in M$ of the

first direct sum of the exact sequence above. Hence for each i, j there exists $m \geq 0$ such that $f_i^m f_j^m (f_j^n x_i - f_i^n x_j) = 0$ in M. We may choose a single m which works for all pairs i, j. After replacing x_i by $f_i^m x_i$ and n by $n + m$ we see that we get $f_j^n x_i = f_i^n x_j$ in M for all i, j. Let us introduce

$$K_n = \{x \in M \mid f_1^n x = \ldots = f_t^n x = 0\}$$

We claim there is an A-module map

$$\varphi : I^{t(n-1)+1} \longrightarrow M/K_n$$

which maps the monomial $f_1^{e_1} \ldots f_t^{e_t}$ with $\sum e_i = t(n-1)+1$ to the class modulo K_n of the expression $f_1^{e_1} \ldots f_i^{e_i - n} \ldots f_t^{e_t} x_i$ where i is chosen such that $e_i \geq n$ (note that there is at least one such i). To see that this is indeed the case suppose that

$$\sum\nolimits_{E=(e_1,\ldots,e_t), |E|=t(n-1)+1} a_E f_1^{e_1} \ldots f_t^{e_t} = 0$$

is a relation between the monomials with coefficients a_E in A. Then we would map this to

$$z = \sum\nolimits_{E=(e_1,\ldots,e_t), |E|=t(n-1)+1} a_E f_1^{e_1} \ldots f_{i(E)}^{e_{i(E)} - n} \ldots f_t^{e_t} x_i$$

where for each multiindex E we have chosen a particular $i(E)$ such that $e_{i(E)} \geq n$. Note that if we multiply this by f_j^n for any j, then we get zero, since by the relations $f_j^n x_i = f_i^n x_j$ above we get

$$\begin{aligned} f_j^n z &= \sum\nolimits_{E=(e_1,\ldots,e_t), |E|=t(n-1)+1} a_E f_1^{e_1} \ldots f_j^{e_j + n} \ldots f_{i(E)}^{e_{i(E)} - n} \ldots f_t^{e_t} x_i \\ &= \sum\nolimits_{E=(e_1,\ldots,e_t), |E|=t(n-1)+1} a_E f_1^{e_1} \ldots f_t^{e_t} x_j = 0. \end{aligned}$$

Hence $z \in K_n$ and we see that every relation gets mapped to zero in M/K_n. This proves the claim.

Note that $K_n \subset M_{t(n-1)+1}$. Hence the map φ in particular gives rise to a A-module map $I^{t(n-1)+1} \to M/M_{t(n-1)+1}$. This proves the second arrow of the lemma is surjective. We omit the proof of injectivity. □

Example 27.23.2. Let k be a field. Consider the ring

$$A = k[f, g, x, y, \{a_n, b_n\}_{n \geq 1}]/(fy - gx, \{a_n f^n + b_n g^n\}_{n \geq 1}).$$

Then $x/f \in A_f$ and $y/g \in A_g$ map to the same element of A_{fg}. Hence these define a section s of the structure sheaf of $\mathrm{Spec}(A)$ over $D(f) \cup D(g) = \mathrm{Spec}(A) \setminus V(I)$. Here $I = (f, g) \subset A$. However, there is no $n \geq 0$ such that s comes from an A-module map $\varphi : I^n \to A$ as in the source of the first displayed arrow of Lemma 27.23.1. Namely, given such a module map set $x_n = \varphi(f^n)$ and $y_n = \varphi(g^n)$. Then $f^m x_n = f^{n+m-1} x$ and $g^m y_n = g^{n+m-1} y$ for some $m \geq 0$ (see proof of the lemma). But then we would have $0 = \varphi(0) = \varphi(a_{n+m} f^{n+m} + b_{n+m} g^{n+m}) = a_{n+m} f^{n+m-1} x + b_{n+m} g^{n+m-1} y$ which is not the case in the ring A.

We will improve on the following lemma in the Noetherian case, see Cohomology of Schemes, Lemma 29.10.4.

Lemma 27.23.3. *Let X be a scheme. Let $\mathcal{I} \subset \mathcal{O}_X$ be a quasi-coherent sheaf of ideals. Let $Z \subset X$ be the closed subscheme defined by $\mathcal{I}$ and set $U = X \setminus Z$. Let $\mathcal{F}$ be a quasi-coherent $\mathcal{O}_X$-module. Assume that X is quasi-compact and quasi-separated*

and that $\mathcal{I}$ is of finite type. Let $\mathcal{F}_n \subset \mathcal{F}$ be subsheaf of sections annihilated by $\mathcal{I}^n$. The canonical map

$$\operatorname{colim}_n \operatorname{Hom}_{\mathcal{O}_X}(\mathcal{I}^n, \mathcal{F}) \longrightarrow \Gamma(U, \mathcal{F})$$

is injective and the canonical map

$$\operatorname{colim}_n \operatorname{Hom}_{\mathcal{O}_X}(\mathcal{I}^n, \mathcal{F}/\mathcal{F}_n) \longrightarrow \Gamma(U, \mathcal{F})$$

is an isomorphism.

Proof. Let $\operatorname{Spec}(A) = W \subset X$ be an affine open. Write $\mathcal{F}|_W = \widetilde{M}$ for some A-module M and $\mathcal{I}|_W = \widetilde{I}$ for some ideal $I \subset A$. We omit the verification that $\mathcal{F}_n = \widetilde{M_n}$ where $M_n \subset M$ is defined as in Lemma 27.23.1. This proves (1). It also follows from Lemma 27.23.1 that we have an injection

$$\operatorname{colim}_n \operatorname{Hom}_{\mathcal{O}_W}(\mathcal{I}^n|_W, \mathcal{F}|_W) \longrightarrow \Gamma(U \cap W, \mathcal{F})$$

and a bijection

$$\operatorname{colim}_n \operatorname{Hom}_{\mathcal{O}_W}(\mathcal{I}^n|_W, (\mathcal{F}/\mathcal{F}_n)|_W) \longrightarrow \Gamma(U \cap W, \mathcal{F})$$

for any such affine open W.

To see (2) we choose a finite affine open covering $X = \bigcup_{j=1,\ldots,m} W_j$. The injectivity of the first arrow of (2) follows immediately from the above and the finiteness of the covering. Moreover for each pair j, j' we choose a finite affine open covering

$$W_j \cap W_{j'} = \bigcup\nolimits_{k=1,\ldots,m_{jj'}} W_{jj'k}.$$

Let $s \in \Gamma(U, \mathcal{F})$. As seen above for each j there exists an n_j and a map $\varphi_j : \mathcal{I}^{n_j}|_{W_j} \to (\mathcal{F}/\mathcal{F}_{n_j})|_{W_j}$ which corresponds to $s|_{W_j}$. By the same token for each triple (j, j', k) there exists an integer $n_{jj'k}$ such that the restriction of φ_j and $\varphi_{j'}$ as maps $\mathcal{I}^{n_{jj'k}} \to \mathcal{F}/\mathcal{F}_{n_{jj'k}}$ agree over $W_{jj'l}$. Let $n = \max\{n_j, n_{jj'k}\}$ and we see that the φ_j glue as maps $\mathcal{I}^n \to \mathcal{F}/\mathcal{F}_n$ over X. This proves surjectivity of the map. We omit the proof of injectivity. □

27.24. Ample invertible sheaves

Recall from Modules, Lemma 17.21.7 that given an invertible sheaf $\mathcal{L}$ on a locally ringed space X, and given a global section s of $\mathcal{L}$ the set $X_s = \{x \in X \mid s \notin \mathfrak{m}_x \mathcal{L}_x\}$ is open. A general remark is that $X_s \cap X_{s'} = X_{ss'}$, where ss' denote the section $s \otimes s' \in \Gamma(X, \mathcal{L} \otimes \mathcal{L}')$.

Definition 27.24.1. Let X be a scheme. Let $\mathcal{L}$ be an invertible $\mathcal{O}_X$-module. We say $\mathcal{L}$ is *ample* if

(1) X is quasi-compact, and
(2) for every $x \in X$ there exists an $n \geq 1$ and $s \in \Gamma(X, \mathcal{L}^{\otimes n})$ such that $x \in X_s$ and X_s is affine.

Lemma 27.24.2. *Let X be a scheme. Let $\mathcal{L}$ be an invertible $\mathcal{O}_X$-module. Let $n \geq 1$. Then $\mathcal{L}$ is ample if and only if $\mathcal{L}^{\otimes n}$ is ample.*

Proof. This follows from the fact that $X_{s^n} = X_s$. □

Lemma 27.24.3. *Let X be a scheme. Let $\mathcal{L}$ be an ample invertible $\mathcal{O}_X$-module. For any closed subscheme $Z \subset X$ the restriction of $\mathcal{L}$ to Z is ample.*

Proof. This is clear since a closed subset of a quasi-compact space is quasi-compact and a closed subscheme of an affine scheme is affine (see Schemes, Lemma 25.8.2). □

Lemma 27.24.4. *Let X be a scheme. Let $\mathcal{L}$ be an invertible $\mathcal{O}_X$-module. Let $s \in \Gamma(X, \mathcal{L})$. For any affine $U \subset X$ the intersection $U \cap X_s$ is affine.*

Proof. This translates into the following algebra problem. Let R be a ring. Let N be an invertible R-module (i.e., locally free of rank 1). Let $s \in N$ be an element. Then $U = \{\mathfrak{p} \mid s \notin \mathfrak{p}N\}$ is an affine open subset of $\mathrm{Spec}(R)$. This you can see as follows. Think of s as an R-module map $R \to N$. This gives rise to R-module maps $N^{\otimes k} \to N^{\otimes k+1}$. Consider

$$R' = \mathrm{colim}_n N^{\otimes n}$$

with transition maps as above. Define an R-algebra structure on R' by the rule $x \cdot y = x \otimes y \in N^{\otimes n+m}$ if $x \in N^{\otimes n}$ and $y \in N^{\otimes m}$. We claim that $\mathrm{Spec}(R') \to \mathrm{Spec}(R)$ is an open immersion with image U.

To prove this is a local question on $\mathrm{Spec}(R)$. Let $\mathfrak{p} \in \mathrm{Spec}(R)$. Pick $f \in R$, $f \notin \mathfrak{p}$ such that $N_f \cong R_f$ as a module. Replacing R by R_f, N by N_f and R' by $R'_f = \mathrm{colim}\, N_f^{\otimes n}$ we may assume that $N \cong R$. Say $N = R$. In this case s is an element of R and it is easy to see that $R' \cong R_s$. Thus the lemma follows. □

Lemma 27.24.5. *Let X be a scheme. Let $\mathcal{L}$ and $\mathcal{M}$ be invertible $\mathcal{O}_X$-modules. If*

(1) *$\mathcal{L}$ is ample, and*
(2) *the open sets X_t where $t \in \Gamma(X, \mathcal{M}^{\otimes m})$ for $m > 0$ cover X,*

then $\mathcal{L} \otimes \mathcal{M}$ is ample.

Proof. We check the conditions of Definition 27.24.1. As $\mathcal{L}$ is ample we see that X is quasi-compact. Let $x \in X$. Choose $n \geq 1$, $m \geq 1$, $s \in \Gamma(X, \mathcal{L}^{\otimes n})$, and $t \in \Gamma(X, \mathcal{M}^{\otimes m})$ such that $x \in X_s$, $x \in X_t$ and X_s is affine. Then $s^m t^n \in \Gamma(X, (\mathcal{L} \otimes \mathcal{M})^{\otimes nm})$, $x \in X_{s^m t^n}$, and $X_{s^m t^n}$ is affine by Lemma 27.24.4. □

Recall that given a scheme X and an invertible sheaf $\mathcal{L}$ on X we get a graded ring $\Gamma_*(X, \mathcal{L}) = \bigoplus_{n \geq 0} \Gamma(X, \mathcal{L}^{\otimes n})$, see Modules, Definition 17.21.4. Also, given a sheaf of $\mathcal{O}_X$-modules we have the graded $\Gamma_*(X, \mathcal{L})$-module $\Gamma_*(X, \mathcal{F}) = \Gamma_*(X, \mathcal{L}, \mathcal{F})$.

Lemma 27.24.6. *Let X be a scheme. Let $\mathcal{L}$ be an invertible sheaf on X. Let $s \in \Gamma(X, \mathcal{L})$. If X is quasi-compact and quasi-separated, the canonical map*

$$\Gamma_*(X, \mathcal{L})_{(s)} \longrightarrow \Gamma(X_s, \mathcal{O})$$

which maps a/s^n to $a \otimes s^{-n}$ is an isomorphism. Moreover, if $\mathcal{F}$ is a quasi-coherent $\mathcal{O}_X$-module then the map

$$\Gamma_*(X, \mathcal{L}, \mathcal{F})_{(s)} \longrightarrow \Gamma(X_s, \mathcal{F})$$

is an isomorphism.

Proof. Consider the scheme

$$\pi : L^* = \underline{\mathrm{Spec}}_X \left(\bigoplus\nolimits_{n \in \mathbf{Z}} \mathcal{L}^{\otimes n} \right) \longrightarrow X$$

see Constructions, Section 26.4. Since the inverse image $\pi^{-1}(U)$ of every affine open $U \subset X$ is affine (see Constructions, Lemma 26.4.6), it follows that L^* quasi-compact and separated, since X is assumed quasi-compact and separated (use Schemes,

Lemma 25.21.7). Note that s gives rise to an element $f \in \Gamma(L^*, \mathcal{O})$, via $\pi_*\mathcal{O}_{L^*} = \bigoplus_{n\in\mathbf{Z}} \mathcal{L}^{\otimes n}$. Note that $(L^*)_f = \pi^{-1}(X_s)$. Hence we have

$$\begin{aligned}\left(\bigoplus\nolimits_{n\in\mathbf{Z}} \Gamma(X, \mathcal{L}^{\otimes n})\right)_s &= \Gamma(L^*, \mathcal{O}_{L^*})_f \\ &= \Gamma((L^*)_f, \mathcal{O}_{L^*}) \\ &= \bigoplus\nolimits_{n\in\mathbf{Z}} \Gamma(X_s, \mathcal{L}^{\otimes n})\end{aligned}$$

where the middle "=" is Lemma 27.15.2. The first statement of the lemma follows from this equality by looking at degree zero terms. The second statement also follows from Lemma 27.15.2 applied to the quasi-coherent sheaf of $\mathcal{O}_{L^*}$-modules $\pi^*\mathcal{F}$ using that

$$\pi_*\pi^*\mathcal{F} = \mathcal{F} \otimes_{\mathcal{O}_X} \left(\bigoplus\nolimits_{n\in\mathbf{Z}} \mathcal{L}^{\otimes n}\right) = \bigoplus\nolimits_{n\in\mathbf{Z}} \mathcal{F} \otimes_{\mathcal{O}_X} \mathcal{L}^{\otimes n}$$

which is proved by computing both sides on affine opens of X. □

Lemma 27.24.7. *Let X be a scheme. Let $\mathcal{L}$ be an invertible $\mathcal{O}_X$-module. Assume the open sets X_s, where $s \in \Gamma(X, \mathcal{L}^{\otimes n})$ and $n \geq 1$, form a basis for the topology on X. Then among those opens, the open sets X_s which are affine form a basis for the topology on X.*

Proof. Let $x \in X$. Choose an affine open neighbourhood $\mathrm{Spec}(R) = U \subset X$ of x. By assumption, there exists a $n \geq 1$ and a $s \in \Gamma(X, \mathcal{L}^{\otimes n})$ such that $X_s \subset U$. By Lemma 27.24.4 above the intersection $X_s = U \cap X_s$ is affine. Since U can be chosen arbitrarily small we win. □

Lemma 27.24.8. *Let X be a scheme. Let $\mathcal{L}$ be an invertible $\mathcal{O}_X$-module. Assume for every point x of X there exists $n \geq 1$ and $s \in \Gamma(X, \mathcal{L}^{\otimes n})$ such that $x \in X_s$ and X_s is affine. Then X is separated.*

Proof. By assumption we can find a covering of X by affine opens of the form X_s. To show that X is quasi-separated, by Schemes, Lemma 25.21.7 it suffices to show that $X_s \cap X_{s'}$ is quasi-compact whenever X_s is affine. This is true by Lemma 27.24.4. Finally, to show that X is separated, we can use the valuative criterion, see Schemes, Lemma 25.22.2.

Thus, let A be a valuation ring with fraction field K and consider two morphisms $f, g : \mathrm{Spec}(A) \to X$ such that the two compositions $\mathrm{Spec}(K) \to \mathrm{Spec}(A) \to X$ agree. Then $f^*\mathcal{L}$ corresponds to an A-module M and $g^*\mathcal{L}$ corresponds to an A-module N by our classification of quasi-coherent modules over affine schemes (Schemes, Lemma 25.7.4). The A-modules M and N are locally free of rank 1 (Lemma 27.18.1) and as A is local they are free of rank 1. We are given an isomorphism $N \otimes_A K \cong M \otimes_A K$ because $f|_{\mathrm{Spec}(K)} = g|_{\mathrm{Spec}(K)}$. We fix an isomorphism $M \otimes_A K \cong K \cong N \otimes_A K$ compatible with the given isomorphism above, so that we may think of M and N as A-submodules of K (free of rank 1 over A). Next, choose $s \in \Gamma(X, \mathcal{L}^{\otimes n})$ such that $\mathrm{Im}(f) \subset X_s$ and such that X_s is affine. This is possible by assumption and the fact that A is local, so it suffices to look at the image of the closed point of $\mathrm{Spec}(A)$. Then s corresponds to an element $x \in M^{\otimes n}$ and $y \in N^{\otimes n}$ mapping to the same element of $K^{\otimes n}$ and moreover $x \notin \mathfrak{m}_A M^{\otimes n}$ because $f(\mathrm{Spec}(A)) \subset X_s$. We conclude that $N^{\otimes n} = Ax = Ay \subset M^{\otimes n}$ inside of $K^{\otimes n}$. Thus $N \subset M$. By symmetry we get $M = N$. This in turn implies that

$g(\mathrm{Spec}(A)) \subset X_s$. Then $f = g$ because X_s is affine and hence separated, thereby finishing the proof. □

Lemma 27.24.9. *Let X be a scheme. If there exists an ample invertible sheaf on X then X is separated.*

Proof. Follows immediately from Lemma 27.24.8 and Definition 27.24.1. □

Lemma 27.24.10. *Let X be a scheme. Let $\mathcal{L}$ be an invertible $\mathcal{O}_X$-module. Set $S = \Gamma_*(X, \mathcal{L})$ as a graded ring. If every point of X is contained in one of the open subschemes X_s, for some $s \in S_+$ homogeneous, then there is a canonical morphism of schemes*

$$f : X \longrightarrow Y = Proj(S),$$

to the homogeneous spectrum of S (see Constructions, Section 26.8). This morphism has the following properties

1. *$f^{-1}(D_+(s)) = X_s$ for any $s \in S_+$ homogeneous,*
2. *there are $\mathcal{O}_Y$-module maps $f^*\mathcal{O}_Y(n) \to \mathcal{L}^{\otimes n}$ compatible with multiplication maps, see Constructions, Equation (26.10.1.1),*
3. *the compositions $S_n \to \Gamma(Y, \mathcal{O}_Y(n)) \to \Gamma(X, \mathcal{L}^{\otimes n})$ are equal to the identity maps, and*
4. *for every $x \in X$ there is an integer $d \geq 1$ and an open neighbourhood $U \subset X$ of x such that $f^*\mathcal{O}_Y(dn)|_U \to \mathcal{L}^{\otimes dn}|_U$ is an isomorphism for all $n \in \mathbf{Z}$.*

Proof. Denote $\psi : S \to \Gamma_*(X, \mathcal{L})$ the identity map. We are going to use the triple $(U(\psi), r_{\mathcal{L},\psi}, \theta)$ of Constructions, Lemma 26.14.1. By assumption the open subscheme $U(\psi)$ of equals X. Hence $r_{\mathcal{L},\psi} : U(\psi) \to Y$ is defined on all of X. We set $f = r_{\mathcal{L},\psi}$. The maps in part (2) are the components of θ. Part (3) follows from condition (2) in the lemma cited above. Part (1) follows from (3) combined with condition (1) in the lemma cited above. Part (4) follows from the last statement in Constructions, Lemma 26.14.1 since the map α mentioned there is an isomorphism. □

Lemma 27.24.11. *Let X be a scheme. Let $\mathcal{L}$ be an invertible $\mathcal{O}_X$-module. Set $S = \Gamma_*(X, \mathcal{L})$. Assume (a) every point of X is contained in one of the open subschemes X_s, for some $s \in S_+$ homogeneous, and (b) X is quasi-compact. Then the canonical morphism of schemes $f : X \longrightarrow Proj(S)$ of Lemma 27.24.10 above is quasi-compact.*

Proof. It suffices to show that $f^{-1}(D_+(s))$ is quasi-compact for any $s \in S_+$ homogeneous. Write $X = \bigcup_{i=1,\ldots,n} X_i$ as a finite union of affine opens. By Lemma 27.24.4 each intersection $X_s \cap X_i$ is affine. Hence $X_s = \bigcup_{i=1,\ldots,n} X_s \cap X_i$ is quasi-compact. □

Lemma 27.24.12. *Let X be a scheme. Let $\mathcal{L}$ be an invertible $\mathcal{O}_X$-module. Set $S = \Gamma_*(X, \mathcal{L})$. Assume $\mathcal{L}$ is ample. Then the canonical morphism of schemes $f : X \longrightarrow Proj(S)$ of Lemma 27.24.10 is an open immersion.*

Proof. By Lemma 27.24.8 we see that X is quasi-separated. Choose finitely many $s_1, \ldots, s_n \in S_+$ homogeneous such that X_{s_i} are affine, and $X = \bigcup X_{s_i}$. Say s_i has

degree d_i. The inverse image of $D_+(s_i)$ under f is X_{s_i}, see Lemma 27.24.10. By Lemma 27.24.6 the ring map

$$(S^{(d_i)})_{(s_i)} = \Gamma(D_+(s_i), \mathcal{O}_{\mathrm{Proj}(S)}) \longrightarrow \Gamma(X_{s_i}, \mathcal{O}_X)$$

is an isomorphism. Hence f induces an isomorphism $X_{s_i} \to D_+(s_i)$. Thus f is an isomorphism of X onto the open subscheme $\bigcup_{i=1,\ldots,n} D_+(s_i)$ of $\mathrm{Proj}(S)$. □

Lemma 27.24.13. *Let X be a scheme. Let S be a graded ring. Assume X is quasi-compact, and assume there exists an open immersion*

$$j : X \longrightarrow Y = Proj(S).$$

Then $j^*\mathcal{O}_Y(d)$ is an invertible ample sheaf for some $d > 0$.

Proof. This is Constructions, Lemma 26.10.6. □

Proposition 27.24.14. *Let X be a quasi-compact scheme. Let $\mathcal{L}$ be an invertible sheaf on X. Set $S = \Gamma_*(X, \mathcal{L})$. The following are equivalent:*

1. *$\mathcal{L}$ is ample,*
2. *the open sets X_s, with $s \in S_+$ homogeneous, cover X and the associated morphism $X \to Proj(S)$ is an open immersion,*
3. *the open sets X_s, with $s \in S_+$ homogeneous, form a basis for the topology of X,*
4. *the open sets X_s, with $s \in S_+$ homogeneous, which are affine form a basis for the topology of X,*
5. *for every quasi-coherent sheaf $\mathcal{F}$ on X the sum of the images of the canonical maps*
$$\Gamma(X, \mathcal{F} \otimes_{\mathcal{O}_X} \mathcal{L}^{\otimes n}) \otimes_{\mathbf{Z}} \mathcal{L}^{\otimes -n} \longrightarrow \mathcal{F}$$
with $n \geq 1$ equals $\mathcal{F}$,
6. *same property as (5) with $\mathcal{F}$ ranging over all quasi-coherent sheaves of ideals,*
7. *X is quasi-separated and for every quasi-coherent sheaf $\mathcal{F}$ of finite type on X there exists an integer n_0 such that $\mathcal{F} \otimes_{\mathcal{O}_X} \mathcal{L}^{\otimes n}$ is globally generated for all $n \geq n_0$,*
8. *X is quasi-separated and for every quasi-coherent sheaf $\mathcal{F}$ of finite type on X there exist integers $n > 0$, $k \geq 0$ such that $\mathcal{F}$ is a quotient of a direct sum of k copies of $\mathcal{L}^{\otimes -n}$, and*
9. *same as in (8) with $\mathcal{F}$ ranging over all sheaves of ideals of finite type on X.*

Proof. Lemma 27.24.12 is (1) ⇒ (2). Lemmas 27.24.2 and 27.24.13 provide the implication (1) ⇐ (2). The implications (2) ⇒ (4) ⇒ (3) are clear from Constructions, Section 26.8. Lemma 27.24.7 is (3) ⇒ (1). Thus we see that the first 4 conditions are all equivalent.

Assume the equivalent conditions (1) – (4). Note that in particular X is separated (as an open subscheme of the separated scheme $\mathrm{Proj}(S)$). Let $\mathcal{F}$ be a quasi-coherent sheaf on X. Choose $s \in S_+$ homogeneous such that X_s is affine. We claim that any section $m \in \Gamma(X_s, \mathcal{F})$ is in the image of one of the maps displayed in (5) above. This will imply (5) since these affines X_s cover X. Namely, by Lemma 27.24.6 we may write m as the image of $m' \otimes s^{-n}$ for some $n \geq 1$, some $m' \in \Gamma(X, \mathcal{F} \otimes \mathcal{L}^{\otimes n})$. This proves the claim.

Clearly (5) $\Rightarrow$ (6). Let us assume (6) and prove $\mathcal{L}$ is ample. Pick $x \in X$. Let $U \subset X$ be an affine open which contains x. Set $Z = X \setminus U$. We may think of Z as a reduced closed subscheme, see Schemes, Section 25.12. Let $\mathcal{I} \subset \mathcal{O}_X$ be the quasi-coherent sheaf of ideals corresponding to the closed subscheme Z. By assumption (6), there exists an $n \geq 1$ and a section $s \in \Gamma(X, \mathcal{I} \otimes \mathcal{L}^{\otimes n})$ such that s does not vanish at x (more precisely such that $s \notin \mathfrak{m}_x \mathcal{I}_x \otimes \mathcal{L}_x^{\otimes n}$). We may think of s as a section of $\mathcal{L}^{\otimes n}$. Since it clearly vanishes along Z we see that $X_s \subset U$. Hence X_s is affine, see Lemma 27.24.4. This proves that $\mathcal{L}$ is ample. At this point we have proved that (1) – (6) are equivalent.

Assume the equivalent conditions (1) – (6). In the following we will use the fact that the tensor product of two sheaves of modules which are globally generated is globally generated without further mention (see Modules, Lemma 17.4.3). By (1) we can find elements $s_i \in S_{d_i}$ with $d_i \geq 1$ such that $X = \bigcup_{i=1,\ldots,n} X_{s_i}$. Set $d = d_1 \ldots d_n$. It follows that $\mathcal{L}^{\otimes d}$ is globally generated by

$$s_1^{d/d_1}, \ldots, s_n^{d/d_n}.$$

This means that if $\mathcal{L}^{\otimes j}$ is globally generated then so is $\mathcal{L}^{\otimes j+dn}$ for all $n \geq 0$. Fix a $j \in \{0, \ldots, d-1\}$. For any point $x \in X$ there exists an $n \geq 1$ and a global section s of $\mathcal{L}^{j+dn}$ which does not vanish at x, as follows from (5) applied to $\mathcal{F} = \mathcal{L}^{\otimes j}$ and ample invertible sheaf $\mathcal{L}^{\otimes d}$. Since X is quasi-compact there we may find a finite list of integers n_i and global sections s_i of $\mathcal{L}^{\otimes j+dn_i}$ which do not vanish at any point of X. Since $\mathcal{L}^{\otimes d}$ is globally generated this means that $\mathcal{L}^{\otimes j+dn}$ is globally generated where $n = \max\{n_i\}$. Since we proved this for every congruence class mod d we conclude that there exists an $n_0 = n_0(\mathcal{L})$ such that $\mathcal{L}^{\otimes n}$ is globally generated for all $n \geq n_0$. At this point we see that if $\mathcal{F}$ is globally generated then so is $\mathcal{F} \otimes \mathcal{L}^{\otimes n}$ for all $n \geq n_0$.

We continue to assume the equivalent conditions (1) – (6). Let $\mathcal{F}$ be a quasi-coherent sheaf of $\mathcal{O}_X$-modules of finite type. Denote $\mathcal{F}_n \subset \mathcal{F}$ the image of the canonical map of (5). By construction $\mathcal{F}_n \otimes \mathcal{L}^{\otimes n}$ is globally generated. By (5) we see $\mathcal{F}$ is the sum of the subsheaves $\mathcal{F}_n$, $n \geq 1$. By Modules, Lemma 17.9.7 we see that $\mathcal{F} = \sum_{n=1,\ldots,N} \mathcal{F}_n$ for some $N \geq 1$. It follows that $\mathcal{F} \otimes \mathcal{L}^{\otimes n}$ is globally generated whenever $n \geq N + n_0(\mathcal{L})$ with $n_0(\mathcal{L})$ as above. We conclude that (1) – (6) implies (7).

Assume (7). Let $\mathcal{F}$ be a quasi-coherent sheaf of $\mathcal{O}_X$-modules of finite type. By (7) there exists an integer $n \geq 1$ such that the canonical map

$$\Gamma(X, \mathcal{F} \otimes_{\mathcal{O}_X} \mathcal{L}^{\otimes n}) \otimes_{\mathbf{Z}} \mathcal{L}^{\otimes -n} \longrightarrow \mathcal{F}$$

is surjective. Let I be the set of finite subsets of $\Gamma(X, \mathcal{F} \otimes_{\mathcal{O}_X} \mathcal{L}^{\otimes n})$ partially ordered by inclusion. Then I is a directed partially ordered set. For $i = \{s_1, \ldots, s_{r(i)}\}$ let $\mathcal{F}_i \subset \mathcal{F}$ be the image of the map

$$\bigoplus\nolimits_{j=1,\ldots,r(i)} \mathcal{L}^{\otimes -n} \longrightarrow \mathcal{F}$$

which is multiplication by s_j on the jth factor. The surjectivity above implies that $\mathcal{F} = \operatorname{colim}_{i \in I} \mathcal{F}_i$. Hence Modules, Lemma 17.9.7 applies and we conclude that $\mathcal{F} = \mathcal{F}_i$ for some i. Hence we have proved (8). In other words, (7) $\Rightarrow$ (8).

The implication (8) $\Rightarrow$ (9) is trivial.

Finally, assume (9). Let $\mathcal{I} \subset \mathcal{O}_X$ be a quasi-coherent sheaf of ideals. By Lemma 27.20.3 (this is where we use the condition that X be quasi-separated) we see that $\mathcal{I} = \operatorname{colim}_\alpha I_\alpha$ with each I_α quasi-coherent of finite type. Since by assumption each of the I_α is a quotient of negative tensor powers of $\mathcal{L}$ we conclude the same for $\mathcal{I}$ (but of course without the finiteness or boundedness of the powers). Hence we conclude that (9) implies (6). This ends the proof of the proposition. □

27.25. Affine and quasi-affine schemes

Lemma 27.25.1. *Let X be a scheme. Then X is quasi-affine if and only if $\mathcal{O}_X$ is ample.*

Proof. Suppose that X is quasi-affine. Consider the open immersion

$$j : X \longrightarrow \operatorname{Spec}(\Gamma(X, \mathcal{O}_X))$$

from Lemma 27.15.4. Note that $\operatorname{Spec}(A) = \operatorname{Proj}(A[T])$, see Constructions, Example 26.8.14. Hence we can apply Lemma 27.24.13 to deduce that $\mathcal{O}_X$ is ample.

Suppose that $\mathcal{O}_X$ is ample. Note that $\Gamma_*(X, \mathcal{O}_X) \cong \Gamma(X, \mathcal{O}_X)[T]$ as graded rings. Hence the result follows from Lemmas 27.24.12 and 27.15.4 taking into account that $\operatorname{Spec}(A) = \operatorname{Proj}(A[T])$ for any ring A as seen above. □

Lemma 27.25.2. *Let X be a scheme. Suppose that there exist finitely many elements $f_1, \ldots, f_n \in \Gamma(X, \mathcal{O}_X)$ such that*

(1) *each X_{f_i} is an affine open of X, and*
(2) *the ideal generated by $f_1, \ldots, f_n$ in $\Gamma(X, \mathcal{O}_X)$ is equal to the unit ideal.*

Then X is affine.

Proof. Assume we have $f_1, \ldots, f_n$ as in the lemma. We may write $1 = \sum g_i f_i$ for some $g_j \in \Gamma(X, \mathcal{O}_X)$ and hence it is clear that $X = \bigcup X_{f_i}$. (The f_i's cannot all vanish at a point.) Since each X_{f_i} is quasi-compact (being affine) it follows that X is quasi-compact. Hence we see that X is quasi-affine by Lemma 27.25.1 above. Consider the open immersion

$$j : X \to \operatorname{Spec}(\Gamma(X, \mathcal{O}_X)),$$

see Lemma 27.15.4. The inverse image of the standard open $D(f_i)$ on the right hand side is equal to X_{f_i} on the left hand side and the morphism j induces an isomorphism $X_{f_i} \cong D(f_i)$, see Lemma 27.15.3. Since the f_i generate the unit ideal we see that $\operatorname{Spec}(\Gamma(X, \mathcal{O}_X)) = \bigcup_{i=1,\ldots,n} D(f_i)$. Thus j is an isomorphism. □

27.26. Quasi-coherent sheaves and ample invertible sheaves

Situation 27.26.1. Let X be a scheme. Let $\mathcal{L}$ be an invertible sheaf on X. Assume $\mathcal{L}$ is ample. Set $S = \Gamma_*(X, \mathcal{L})$ as a graded ring. Set $Y = \operatorname{Proj}(S)$. Let $f : X \to Y$ be the canonical morphism of Lemma 27.24.10. It comes equipped with a $\mathbf{Z}$-graded $\mathcal{O}_X$-algebra map $\bigoplus f^*\mathcal{O}_Y(n) \to \bigoplus \mathcal{L}^{\otimes n}$.

The following lemma is really a special case of the next lemma but it seems like a good idea to point out its validity first.

Lemma 27.26.2. *In Situation 27.26.1. The canonical morphism $f : X \to Y$ maps X into the open subscheme $W = W_1 \subset Y$ where $\mathcal{O}_Y(1)$ is invertible and where all multiplication maps $\mathcal{O}_Y(n) \otimes_{\mathcal{O}_Y} \mathcal{O}_Y(m) \to \mathcal{O}_Y(n+m)$ are isomorphisms*

(see Constructions, Lemma 26.10.4). Moreover, the maps $f^*\mathcal{O}_Y(n) \to \mathcal{L}^{\otimes n}$ are all isomorphisms.

Proof. By Proposition 27.24.14 there exists an integer n_0 such that $\mathcal{L}^{\otimes n}$ is globally generated for all $n \geq n_0$. Let $x \in X$ be a point. By the above we can find $a \in S_{n_0}$ and $b \in S_{n_0+1}$ such that a and b do not vanish at x. Hence $f(x) \in D_+(a) \cap D_+(b) = D_+(ab)$. By Constructions, Lemma 26.10.4 we see that $f(x) \in W_1$ as desired. By Constructions, Lemma 26.14.1 which was used in the construction of the map f the maps $f^*\mathcal{O}_Y(n_0) \to \mathcal{L}^{\otimes n_0}$ and $f^*\mathcal{O}_Y(n_0+1) \to \mathcal{L}^{\otimes n_0+1}$ are isomorphisms in a neighbourhood of x. By compatibility with the algebra structure and the fact that f maps into W we conclude all the maps $f^*\mathcal{O}_Y(n) \to \mathcal{L}^{\otimes n}$ are isomorphisms in a neighbourhood of x. Hence we win. □

Recall from Modules, Definition 17.21.4 that given a locally ringed space X, an invertible sheaf $\mathcal{L}$, and a $\mathcal{O}_X$-module $\mathcal{F}$ we have the graded $\Gamma_*(X, \mathcal{L})$-module

$$\Gamma(X, \mathcal{L}, \mathcal{F}) = \bigoplus\nolimits_{n \in \mathbf{Z}} \Gamma(X, \mathcal{F} \otimes_{\mathcal{O}_X} \mathcal{L}^{\otimes n}).$$

The following lemma says that, in Situation 27.26.1, we can recover a quasi-coherent $\mathcal{O}_X$-module $\mathcal{F}$ from this graded module. Take a look also at Constructions, Lemma 26.13.7 where we prove this lemma in the special case $X = \mathbf{P}^n_R$.

Lemma 27.26.3. *In Situation 27.26.1. Let $\mathcal{F}$ be a quasi-coherent sheaf on X. Set $M = \Gamma_*(X, \mathcal{L}, \mathcal{F})$ as a graded S-module. There are isomorphisms*

$$f^*\widetilde{M} \longrightarrow \mathcal{F}$$

functorial in $\mathcal{F}$ such that $M_0 \to \Gamma(Proj(S), \widetilde{M}) \to \Gamma(X, \mathcal{F})$ is the identity map.

Proof. Let $s \in S_+$ be homogeneous such that X_s is affine open in X. Recall that $\widetilde{M}|_{D_+(s)}$ corresponds to the $S_{(s)}$-module $M_{(s)}$, see Constructions, Lemma 26.8.4. Recall that $f^{-1}(D_+(s)) = X_s$. As X carries an ample invertible sheaf it is quasi-compact and quasi-separated, see Section 27.24. By Lemma 27.24.6 there is a canonical isomorphism $M_{(s)} = \Gamma_*(X, \mathcal{L}, \mathcal{F})_{(s)} \to \Gamma(X_s, \mathcal{F})$. Since $\mathcal{F}$ is quasi-coherent this leads to a canonical isomorphism

$$f^*\widetilde{M}|_{X_s} \to \mathcal{F}|_{X_s}$$

Since $\mathcal{L}$ is ample on X we know that X is covered by the affine opens of the form X_s. Hence it suffices to prove that the displayed maps glue on overlaps. Proof of this is omitted. □

Remark 27.26.4. With assumptions and notation of Lemma 27.26.3. Denote the displayed map of the lemma by $\theta_{\mathcal{F}}$. Note that the isomorphism $f^*\mathcal{O}_Y(n) \to \mathcal{L}^{\otimes n}$ of Lemma 27.26.2 is just $\theta_{\mathcal{L}^{\otimes n}}$. Consider the multiplication maps

$$\widetilde{M} \otimes_{\mathcal{O}_Y} \mathcal{O}_Y(n) \longrightarrow \widetilde{M(n)}$$

see Constructions, Equation (26.10.1.5). Pull this back to X and consider

$$\begin{array}{ccc} f^*\widetilde{M} \otimes_{\mathcal{O}_X} f^*\mathcal{O}_Y(n) & \longrightarrow & f^*\widetilde{M(n)} \\ \downarrow{\scriptstyle \theta_{\mathcal{F}} \otimes \theta_{\mathcal{L}^{\otimes n}}} & & \downarrow{\scriptstyle \theta_{\mathcal{F} \otimes \mathcal{L}^{\otimes n}}} \\ \mathcal{F} \otimes \mathcal{L}^{\otimes n} & \xrightarrow{\text{id}} & \mathcal{F} \otimes \mathcal{L}^{\otimes n} \end{array}$$

Here we have used the obvious identification $M(n) = \Gamma_*(X, \mathcal{L}, \mathcal{F} \otimes \mathcal{L}^{\otimes n})$. This diagram commutes. Proof omitted.

27.27. Finding suitable affine opens

In this section we collect some results on the existence of affine opens in more and less general situations.

Lemma 27.27.1. *Let X be a quasi-separated scheme. Let $Z_1, \ldots, Z_n$ be pairwise distinct irreducible components of X, see Topology, Section 5.7. Let $\eta_i \in Z_i$ be their generic points, see Schemes, Lemma 25.11.1. There exist affine open neighbourhoods $\eta_i \in U_i$ such that $U_i \cap U_j = \emptyset$ for all $i \neq j$. In particular, $U = U_1 \cup \ldots \cup U_n$ is an affine open containing all of the points $\eta_1, \ldots, \eta_n$.*

Proof. Let V_i be any affine open containing η_i and disjoint from the closed set $Z_1 \cup \ldots \hat{Z}_i \ldots \cup Z_n$. Since X is quasi-separated for each i the union $W_i = \bigcup_{j,j\neq i} V_i \cap V_j$ is a quasi-compact open of V_i not containing η_i. We can find open neighbourhoods $U_i \subset V_i$ containing η_i and disjoint from W_i by Algebra, Lemma 10.24.5. Finally, U is affine since it is the spectrum of the ring $R_1 \times \ldots \times R_n$ where $R_i = \mathcal{O}_X(U_i)$, see Schemes, Lemma 25.6.8. □

Remark 27.27.2. Lemma 27.27.1 above is false if X is not quasi-separated. Here is an example. Take $R = \mathbf{Q}[x, y_1, y_2, \ldots]/((x-i)y_i)$. Consider the minimal prime ideal $\mathfrak{p} = (y_1, y_2, \ldots)$ of R. Glue two copies of $\mathrm{Spec}(R)$ along the (not quasi-compact) open $\mathrm{Spec}(R) \setminus V(\mathfrak{p})$ to get a scheme X (glueing as in Schemes, Example 25.14.3). Then the two maximal points of X corresponding to $\mathfrak{p}$ are not contained in a common affine open. The reason is that any open of $\mathrm{Spec}(R)$ containing $\mathfrak{p}$ contains infinitely many of the "lines" $x = i$, $y_j = 0$, $j \neq i$ with parameter y_i. Details omitted.

Notwithstanding the example above, for "most" finite sets of irreducible closed subsets one can apply Lemma 27.27.1 above, at least if X is quasi-compact. This is true because X contains a dense open which is separated.

Lemma 27.27.3. *Let X be a quasi-compact scheme. There exists a dense open $V \subset X$ which is separated.*

Proof. Say $X = \bigcup_{i=1,\ldots,n} U_i$ is a union of n affine open subschemes. We will prove the lemma by induction on n. It is trivial for $n = 1$. Let $V' \subset \bigcup_{i=1,\ldots,n-1} U_i$ be a separated dense open subscheme, which exists by induction hypothesis. Consider

$$V = V' \coprod (U_n \setminus \overline{V'}).$$

It is clear that V is separated and a dense open subscheme of X. □

It turns out that, even if X is quasi-separated as well as quasi-compact, there does not exist a separated, quasi-compact dense open, see Examples, Lemma 78.14.2. Here is a slight refinement of Lemma 27.27.1 above.

Lemma 27.27.4. *Let X be a quasi-separated scheme. Let $Z_1, \ldots, Z_n$ be pairwise distinct irreducible components of X. Let $\eta_i \in Z_i$ be their generic points. Let $x \in X$ be arbitrary. There exists an affine open $U \subset X$ containing x and all the η_i.*

Proof. Suppose that $x \in Z_1 \cap \ldots \cap Z_r$ and $x \notin Z_{r+1}, \ldots, Z_n$. Then we may choose an affine open $W \subset X$ such that $x \in W$ and $W \cap Z_i = \emptyset$ for $i = r+1, \ldots, n$. Note that clearly $\eta_i \in W$ for $i = 1, \ldots, r$. By Lemma 27.27.1 we may choose affine opens $U_i \subset X$ which are pairwise disjoint such that $\eta_i \in U_i$ for $i = r+1, \ldots, n$. Since X is quasi-separated the opens $W \cap U_i$ are quasi-compact and do not contain η_i for $i = r+1, \ldots, n$. Hence by Algebra, Lemma 10.24.5 we may shrink U_i such that $W \cap U_i = \emptyset$ for $i = r+1, \ldots, n$. Then the union $U = W \cup \bigcup_{i=r+1,\ldots,n} U_i$ is disjoint and hence (by Schemes, Lemma 25.6.8) a suitable affine open. □

Lemma 27.27.5. *Let X be a scheme. Assume either*

1. *The scheme X is quasi-affine.*
2. *The scheme X is isomorphic to a locally closed subscheme of an affine scheme.*
3. *There exists an ample invertible sheaf on X.*
4. *The scheme X is isomorphic to a locally closed subscheme of Proj(S) for some graded ring S.*

Then for any finite subset $E \subset X$ there exists an affine open $U \subset X$ with $E \subset U$.

Proof. By Properties, Definition 27.15.1 a quasi-affine scheme is a quasi-compact open subscheme of an affine scheme. Any affine scheme $\mathrm{Spec}(R)$ is isomorphic to $\mathrm{Proj}(R[X])$ where $R[X]$ is graded by setting $\deg(X) = 1$. By Proposition 27.24.14 if X has an ample invertible sheaf then X is isomorphic to an open subscheme of $\mathrm{Proj}(S)$ for some graded ring S. Hence, it suffices to prove the lemma in case (4). (We urge the reader to prove case (2) directly for themselves.)

Thus assume $X \subset \mathrm{Proj}(S)$ is a locally closed subscheme where S is some graded ring. Let $T = \overline{X} \setminus X$. Recall that the standard opens $D_+(f)$ form a basis of the topology on $\mathrm{Proj}(S)$. Since E is finite we may choose finitely many homogeneous elements $f_i \in S_+$ such that

$$E \subset D_+(f_1) \cup \ldots \cup D_+(f_n) \subset \mathrm{Proj}(S) \setminus T$$

Suppose that $E = \{\mathfrak{p}_1, \ldots, \mathfrak{p}_m\}$ as a subset of $\mathrm{Proj}(S)$. Consider the ideal $I = (f_1, \ldots, f_n) \subset S$. Since $I \not\subset \mathfrak{p}_j$ for all $j = 1, \ldots, m$ we see from Algebra, Lemma 10.53.6 that there exists a homogeneous element $f \in I$, $f \notin \mathfrak{p}_j$ for all $j = 1, \ldots, m$. Then $E \subset D_+(f) \subset D_+(f_1) \cup \ldots \cup D_+(f_n)$. Since $D_+(f)$ does not meet T we see that $X \cap D_+(f)$ is a closed subscheme of the affine scheme $D_+(f)$, hence is an affine open of X as desired. □

Lemma 27.27.6. *Let X be a scheme. Let $\mathcal{L}$ be an ample invertible sheaf on X. Let $E \subset X$ be a finite subset. Then there exists an $n > 0$ and a section $s \in \Gamma(X, \mathcal{L}^{\otimes n})$ such that X_s is affine and $E \subset X_s$.*

Proof. The reader can modify the proof of Lemma 27.27.5 to prove this lemma; we will instead deduce the lemma from it. By Lemma 27.27.5 we can choose an affine open $U \subset X$ such that $E \subset U$. Consider the graded ring $S = \Gamma_*(X, \mathcal{L}) = \bigoplus_{n \geq 0} \Gamma(X, \mathcal{L}^{\otimes n})$. For each $x \in E$ let $\mathfrak{p}_x \subset S$ be the graded ideal of sections vanishing at x. It is clear that $\mathfrak{p}_x$ is a prime ideal and since some power of $\mathcal{L}$ is globally generated, it is clear that $S_+ \not\subset \mathfrak{p}_x$. Let $I \subset S$ be the graded ideal of sections vanishing on all points of $X \setminus U$. Since the sets X_s form a basis for the topology we see that $I \not\subset \mathfrak{p}_x$ for all $x \in E$. By (graded) prime avoidance (Algebra,

Lemma 10.53.6) we can find $s \in I$ homogeneous with $s \notin \mathfrak{p}_x$ for all $x \in E$. Then $E \subset X_s \subset U$ and X_s is affine by Lemma 27.24.4. □

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CHAPTER 28

Morphisms of Schemes

28.1. Introduction

In this chapter we introduce some types of morphisms of schemes. A basic reference is [**DG67**].

28.2. Closed immersions

In this section we elucidate some of the results obtained previously on closed immersions of schemes. Recall that a morphism of schemes $i : Z \to X$ is defined to be a closed immersion if (a) i induces a homeomorphism onto a closed subset of X, (b) $i^\sharp : \mathcal{O}_X \to i_*\mathcal{O}_Z$ is surjective, and (c) the kernel of $i^\sharp$ is locally generated by sections, see Schemes, Definitions 25.10.2 and 25.4.1. It turns out that, given that Z and X are schemes, there are many different ways of characterizing a closed immersion.

Lemma 28.2.1. *Let $i : Z \to X$ be a morphism of schemes. The following are equivalent:*

1. *The morphism i is a closed immersion.*
2. *For every affine open $\mathrm{Spec}(R) = U \subset X$, there exists an ideal $I \subset R$ such that $i^{-1}(U) = \mathrm{Spec}(R/I)$ as schemes over $U = \mathrm{Spec}(R)$.*
3. *There exists an affine open covering $X = \bigcup_{j \in J} U_j$, $U_j = \mathrm{Spec}(R_j)$ and for every $j \in J$ there exists an ideal $I_j \subset R_j$ such that $i^{-1}(U_j) = \mathrm{Spec}(R_j/I_j)$ as schemes over $U_j = \mathrm{Spec}(R_j)$.*
4. *The morphism i induces a homeomorphism of Z with a closed subset of X and $i^\sharp : \mathcal{O}_X \to i_*\mathcal{O}_Z$ is surjective.*
5. *The morphism i induces a homeomorphism of Z with a closed subset of X, the map $i^\sharp : \mathcal{O}_X \to i_*\mathcal{O}_Z$ is surjective, and the kernel $Ker(i^\sharp) \subset \mathcal{O}_X$ is a quasi-coherent sheaf of ideals.*
6. *The morphism i induces a homeomorphism of Z with a closed subset of X, the map $i^\sharp : \mathcal{O}_X \to i_*\mathcal{O}_Z$ is surjective, and the kernel $Ker(i^\sharp) \subset \mathcal{O}_X$ is a sheaf of ideals which is locally generated by sections.*

Proof. Condition (6) is our definition of a closed immersion, see Schemes, Definitions 25.4.1 and 25.10.2. So (6) $\Leftrightarrow$ (1). We have (1) $\Rightarrow$ (2) by Schemes, Lemma 25.10.1. Trivially (2) $\Rightarrow$ (3).

Assume (3). Each of the morphisms $\mathrm{Spec}(R_j/I_j) \to \mathrm{Spec}(R_j)$ is a closed immersion, see Schemes, Example 25.8.1. Hence $i^{-1}(U_j) \to U_j$ is a homeomorphism onto its image and $i^\sharp|_{U_j}$ is surjective. Hence i is a homeomorphism onto its image and $i^\sharp$ is surjective since this may be checked locally. We conclude that (3) $\Rightarrow$ (4).

The implication (4) $\Rightarrow$ (1) is Schemes, Lemma 25.24.2. The implication (5) $\Rightarrow$ (6) is trivial. And the implication (6) $\Rightarrow$ (5) follows from Schemes, Lemma 25.10.1. $\square$

Lemma 28.2.2. *Let X be a scheme. Suppose $i : Z \to X$ and $i' : Z' \to X$ are closed immersions corresponding to the quasi-coherent ideal sheaves $\mathcal{I} = Ker(i^\sharp)$ and $\mathcal{I}' = Ker((i')^\sharp)$ of $\mathcal{O}_X$.*

1. *The morphism $i : Z \to X$ factors as $Z \to Z' \to X$ for some $a : Z \to Z'$ if and only if $\mathcal{I}' \subset \mathcal{I}$. If this happens, then a is a closed immersion.*
2. *We have $Z \cong Z'$ as schemes over X if and only if $\mathcal{I} = \mathcal{I}'$.*

Proof. This follows from our discussion of closed subspaces in Schemes, Section 25.4 especially Schemes, Lemma 25.4.6. It also follows in a straightforward way from characterization (3) in Lemma 28.2.1 above. □

Lemma 28.2.3. *Let X be a scheme. Let $\mathcal{I} \subset \mathcal{O}_X$ be a sheaf of ideals. The following are equivalent:*

1. *The sheaf of ideals $\mathcal{I}$ is locally generated by sections as a sheaf of $\mathcal{O}_X$ modules.*
2. *The sheaf of ideals $\mathcal{I}$ is quasi-coherent as a sheaf of $\mathcal{O}_X$-modules.*
3. *There exists a closed immersion $i : Z \to X$ whose corresponding sheaf of ideals $Ker(i^\sharp)$ is equal to $\mathcal{I}$.*

Proof. In Schemes, Section 25.4 we constructed the closed subspace associated to a sheaf of ideals locally generated by sections. This closed subspace is a scheme by Schemes, Lemma 25.10.1. Hence we see that (1) $\Rightarrow$ (3) by our definition of a closed immersion of schemes. By Lemma 28.2.1 above we see that (3) $\Rightarrow$ (2). And of course (2) $\Rightarrow$ (1). □

Lemma 28.2.4. *The base change of a closed immersion is a closed immersion.*

Proof. See Schemes, Lemma 25.18.2. □

Lemma 28.2.5. *A composition of closed immersions is a closed immersion.*

Proof. We have seen this in Schemes, Lemma 25.24.3, but here is another proof. Namely, it follows from the characterization (3) of closed immersions in Lemma 28.2.1. Since if $I \subset R$ is an ideal, and $\overline{J} \subset R/I$ is an ideal, then $\overline{J} = J/I$ for some ideal $J \subset R$ which contains I and $(R/I)/\overline{J} = R/J$. □

Lemma 28.2.6. *A closed immersion is quasi-compact.*

Proof. This lemma is a duplicate of Schemes, Lemma 25.19.5. □

Lemma 28.2.7. *A closed immersion is separated.*

Proof. This lemma is a special case of Schemes, Lemma 25.23.7. □

28.3. Immersions

In this section we collect some facts on immersions.

Lemma 28.3.1. *Let $Z \to Y \to X$ be morphisms of schemes.*

1. *If $Z \to X$ is an immersion, then $Z \to Y$ is an immersion.*
2. *If $Z \to X$ is a quasi-compact immersion and $Z \to Y$ is quasi-separated, then $Z \to Y$ is a quasi-compact immersion.*
3. *If $Z \to X$ is a closed immersion and $Y \to X$ is separated, then $Z \to Y$ is a closed immersion.*

Proof. In each case the proof is to contemplate the commutative diagram

$$\begin{array}{ccccc} Z & \longrightarrow & Y \times_X Z & \longrightarrow & Z \\ & \searrow & \downarrow & & \downarrow \\ & & Y & \longrightarrow & X \end{array}$$

where the composition of the top horizontal arrows is the identity. Let us prove (1). The first horizontal arrow is a section of $Y \times_X Z \to Z$, whence an immersion by Schemes, Lemma 25.21.12. The arrow $Y \times_X Z \to Y$ is a base change of $Z \to X$ hence an immersion (Schemes, Lemma 25.18.2). Finally, a composition of immersions is an immersion (Schemes, Lemma 25.24.3). This proves (1). The other two results are proved in exactly the same manner. □

Lemma 28.3.2. *Let $h : Z \to X$ be an immersion. If h is quasi-compact, then we can factor $h = i \circ j$ with $j : Z \to \overline{Z}$ an open immersion and $i : \overline{Z} \to X$ a closed immersion.*

Proof. Note that h is quasi-compact and quasi-separated (see Schemes, Lemma 25.23.7). Hence $h_*\mathcal{O}_Z$ is a quasi-coherent sheaf of $\mathcal{O}_X$-modules by Schemes, Lemma 25.24.1. This implies that $\mathcal{I} = \mathrm{Ker}(\mathcal{O}_X \to h_*\mathcal{O}_Z)$ is a quasi-coherent sheaf of ideals, see Schemes, Section 25.24. Let $\overline{Z} \subset X$ be the closed subscheme corresponding to $\mathcal{I}$, see Lemma 28.2.3. By Schemes, Lemma 25.4.6 the morphism h factors as $h = i \circ j$ where $i : \overline{Z} \to X$ is the inclusion morphism. To see that j is an open immersion, choose an open subscheme $U \subset X$ such that h induces a closed immersion of Z into U. Then it is clear that $\mathcal{I}|_U$ is the sheaf of ideals corresponding to the closed immersion $Z \to U$. Hence we see that $Z = \overline{Z} \cap U$. □

Lemma 28.3.3. *Let $h : Z \to X$ be an immersion. If Z is reduced, then we can factor $h = i \circ j$ with $j : Z \to \overline{Z}$ an open immersion and $i : \overline{Z} \to X$ a closed immersion.*

Proof. Let $\overline{Z} \subset X$ be the closure of $h(Z)$ with the reduced induced closed subscheme structure, see Schemes, Definition 25.12.5. By Schemes, Lemma 25.12.6 the morphism h factors as $h = i \circ j$ with $i : \overline{Z} \to X$ the inclusion morphism and $j : Z \to \overline{Z}$. From the definition of an immersion we see there exists an open subscheme $U \subset X$ such that h factors through a closed immersion into U. Hence $\overline{Z} \cap U$ and $h(Z)$ are reduced closed subschemes of U with the same underlying closed set. Hence by the uniqueness in Schemes, Lemma 25.12.4 we see that $h(Z) \cong \overline{Z} \cap U$. So j induces an isomorphism of Z with $\overline{Z} \cap U$. In other words j is an open immersion. □

Example 28.3.4. Here is an example of an immersion which is not a composition of an open immersion followed by a closed immersion. Let k be a field. Let $X = \mathrm{Spec}(k[x_1, x_2, x_3, \ldots])$. Let $U = \bigcup_{n=1}^{\infty} D(x_n)$. Then $U \to X$ is an open immersion. Consider the ideals

$$I_n = (x_1^n, x_2^n, \ldots, x_{n-1}^n, x_n - 1, x_{n+1}, x_{n+2}, \ldots) \subset k[x_1, x_2, x_3, \ldots][1/x_n].$$

Note that $I_n k[x_1, x_2, x_3, \ldots][1/x_n x_m] = (1)$ for any $m \neq n$. Hence the quasi-coherent ideals $\widetilde{I_n}$ on $D(x_n)$ agree on $D(x_n x_m)$, namely $\widetilde{I_n}|_{D(x_n x_m)} = \mathcal{O}_{D(x_n x_m)}$ if $n \neq m$. Hence these ideals glue to a quasi-coherent sheaf of ideals $\mathcal{I} \subset \mathcal{O}_U$. Let $Z \subset U$ be the closed subscheme corresponding to $\mathcal{I}$. Thus $Z \to X$ is an immersion.

We claim that we cannot factor $Z \to X$ as $Z \to \overline{Z} \to X$, where $\overline{Z} \to X$ is closed and $Z \to \overline{Z}$ is open. Namely, $\overline{Z}$ would have to be defined by an ideal $I \subset k[x_1, x_2, x_3, \ldots]$ such that $I_n = Ik[x_1, x_2, x_3, \ldots][1/x_n]$. But the only element $f \in k[x_1, x_2, x_3, \ldots]$ which ends up in all I_n is 0! Hence I does not exist.

28.4. Closed immersions and quasi-coherent sheaves

The following lemma finally does for quasi-coherent sheaves on schemes what Modules, Lemma 17.6.1 does for abelian sheaves. See also the discussion in Modules, Section 17.13.

Lemma 28.4.1. *Let $i : Z \to X$ be a closed immersion of schemes. Let $\mathcal{I} \subset \mathcal{O}_X$ be the quasi-coherent sheaf of ideals cutting out Z. The functor*

$$i_* : QCoh(\mathcal{O}_Z) \longrightarrow QCoh(\mathcal{O}_X)$$

is exact, fully faithful, with essential image those quasi-coherent $\mathcal{O}_X$-modules $\mathcal{G}$ such that $\mathcal{I}\mathcal{G} = 0$.

Proof. A closed immersion is quasi-compact and separated, see Lemmas 28.2.6 and 28.2.7. Hence Schemes, Lemma 25.24.1 applies and the pushforward of a quasi-coherent sheaf on Z is indeed a quasi-coherent sheaf on X.

By Modules, Lemma 17.13.4 the functor i_* is fully faithful.

Now we turn to the description of the essential image of the functor i_*. It is clear that $\mathcal{I}(i_*\mathcal{F}) = 0$ for any quasi-coherent $\mathcal{O}_Z$-module, for example by our local description above. Next, suppose that $\mathcal{G}$ is any quasi-coherent $\mathcal{O}_X$-module such that $\mathcal{I}\mathcal{G} = 0$. It suffices to show that the canonical map

$$\mathcal{G} \longrightarrow i_* i^* \mathcal{G}$$

is an isomorphism. By exactly the same arguments as above we see that it suffices to prove the following algebraic statement: Given a ring R, an ideal I and an R-module N such that $IN = 0$ the canonical map

$$N \longrightarrow N \otimes_R R/I, \quad n \longmapsto n \otimes 1$$

is an isomorphism of R-modules. Proof of this easy algebra fact is omitted. □

Let $i : Z \to X$ be a closed immersion. Because of the lemma above we often, by abuse of notation, denote $\mathcal{F}$ the sheaf $i_*\mathcal{F}$ on X.

Lemma 28.4.2. *Let X be a scheme. Let $\mathcal{F}$ be a quasi-coherent $\mathcal{O}_X$-module. Let $\mathcal{G} \subset \mathcal{F}$ be a $\mathcal{O}_X$-submodule. There exists a unique quasi-coherent $\mathcal{O}_X$-submodule $\mathcal{G}' \subset \mathcal{G}$ with the following property: For every quasi-coherent $\mathcal{O}_X$-module $\mathcal{H}$ the map*

$$\mathrm{Hom}_{\mathcal{O}_X}(\mathcal{H}, \mathcal{G}') \longrightarrow \mathrm{Hom}_{\mathcal{O}_X}(\mathcal{H}, \mathcal{G})$$

is bijective. In particular $\mathcal{G}'$ is the largest quasi-coherent $\mathcal{O}_X$-submodule of $\mathcal{F}$ contained in $\mathcal{G}$.

Proof. Let $\mathcal{G}_a$, $a \in A$ be the set of quasi-coherent $\mathcal{O}_X$-submodules contained in $\mathcal{G}$. Then the image $\mathcal{G}'$ of

$$\bigoplus\nolimits_{a \in A} \mathcal{G}_a \longrightarrow \mathcal{F}$$

is quasi-coherent as the image of a map of quasi-coherent sheaves on X is quasi-coherent and since a direct sum of quasi-coherent sheaves is quasi-coherent, see

Schemes, Section 25.24. The module $\mathcal{G}'$ is contained in $\mathcal{G}$. Hence this is the largest quasi-coherent $\mathcal{O}_X$-module contained in $\mathcal{G}$.

To prove the formula, let $\mathcal{H}$ be a quasi-coherent $\mathcal{O}_X$-module and let $\alpha : \mathcal{H} \to \mathcal{G}$ be an $\mathcal{O}_X$-module map. The image of the composition $\mathcal{H} \to \mathcal{G} \to \mathcal{F}$ is quasi-coherent as the image of a map of quasi-coherent sheaves. Hence it is contained in $\mathcal{G}'$. Hence α factors through $\mathcal{G}'$ as desired. □

Lemma 28.4.3. *Let $i : Z \to X$ be a closed immersion of schemes. There is a functor[1] $i^! : QCoh(\mathcal{O}_X) \to QCoh(\mathcal{O}_Z)$ which is a right adjoint to i_*. (Compare Modules, Lemma 17.6.3.)*

Proof. Given quasi-coherent $\mathcal{O}_X$-module $\mathcal{G}$ we consider the subsheaf $\mathcal{H}_Z(\mathcal{G})$ of $\mathcal{G}$ of local sections annihilated by $\mathcal{I}$. By Lemma 28.4.2 there is a canonical largest quasi-coherent $\mathcal{O}_X$-submodule $\mathcal{H}_Z(\mathcal{G})'$. By construction we have

$$\mathrm{Hom}_{\mathcal{O}_X}(i_*\mathcal{F}, \mathcal{H}_Z(\mathcal{G})') = \mathrm{Hom}_{\mathcal{O}_X}(i_*\mathcal{F}, \mathcal{G})$$

for any quasi-coherent $\mathcal{O}_Z$-module $\mathcal{F}$. Hence we can set $i^!\mathcal{G} = i^*(\mathcal{H}_Z(\mathcal{G})')$. Details omitted. □

28.5. Supports of modules

In this section we collect some elementary results on supports of quasi-coherent modules on schemes. Recall that the support of a sheaf of modules has been defined in Modules, Section 17.5. On the other hand, the support of a module was defined in Algebra, Section 10.59. These match.

Lemma 28.5.1. *Let X be a scheme. Let $\mathcal{F}$ be a quasi-coherent sheaf on X. Let $\mathrm{Spec}(A) = U \subset X$ be an affine open, and set $M = \Gamma(U, \mathcal{F})$. Let $x \in U$, and let $\mathfrak{p} \subset A$ be the corresponding prime. The following are equivalent*

1. *$\mathfrak{p}$ is in the support of M, and*
2. *x is in the support of $\mathcal{F}$.*

Proof. This follows from the equality $\mathcal{F}_x = M_{\mathfrak{p}}$, see Schemes, Lemma 25.5.4 and the definitions. □

Lemma 28.5.2. *Let X be a scheme. Let $\mathcal{F}$ be a quasi-coherent sheaf on X. The support of $\mathcal{F}$ is closed under specialization.*

Proof. If $x' \rightsquigarrow x$ is a specialization and $\mathcal{F}_x = 0$ then $\mathcal{F}_{x'}$ is zero, as $\mathcal{F}_{x'}$ is a localization of the module $\mathcal{F}_x$. Hence the complement of $\mathrm{Supp}(\mathcal{F})$ is closed under generalization. □

For finite type quasi-coherent modules the support is closed, can be checked on fibres, and commutes with base change.

Lemma 28.5.3. *Let $\mathcal{F}$ be a finite type quasi-coherent module on a scheme X. Then*

1. *The support of $\mathcal{F}$ is closed.*
2. *For $x \in X$ we have*

$$x \in Supp(\mathcal{F}) \Leftrightarrow \mathcal{F}_x \neq 0 \Leftrightarrow \mathcal{F}_x \otimes_{\mathcal{O}_{X,x}} \kappa(x) \neq 0.$$

[1] This is likely nonstandard notation.

(3) *For any morphism of schemes $f : Y \to X$ the pullback $f^*\mathcal{F}$ is of finite type as well and we have $Supp(f^*\mathcal{F}) = f^{-1}(Supp(\mathcal{F}))$.*

Proof. Part (1) is a reformulation of Modules, Lemma 17.9.6. You can also combine Lemma 28.5.1, Properties, Lemma 27.16.1, and Algebra, Lemma 10.37.5 to see this. The first equivalence in (2) is the definition of support, and the second equivalence follows from Nakayama's lemma, see Algebra, Lemma 10.18.1. Let $f : Y \to X$ be a morphism of schemes. Note that $f^*\mathcal{F}$ is of finite type by Modules, Lemma 17.9.2. For the final assertion, let $y \in Y$ with image $x \in X$. Recall that

$$(f^*\mathcal{F})_y = \mathcal{F}_x \otimes_{\mathcal{O}_{X,x}} \mathcal{O}_{Y,y},$$

see Sheaves, Lemma 6.26.4. Hence $(f^*\mathcal{F})_y \otimes \kappa(y)$ is nonzero if and only if $\mathcal{F}_x \otimes \kappa(x)$ is nonzero. By (2) this implies $x \in \text{Supp}(\mathcal{F})$ if and only if $y \in \text{Supp}(f^*\mathcal{F})$, which is the content of assertion (3). □

Lemma 28.5.4. *Let $\mathcal{F}$ be a finite type quasi-coherent module on a scheme X. There exists a smallest closed subscheme $i : Z \to X$ such that there exists a quasi-coherent $\mathcal{O}_Z$-module $\mathcal{G}$ with $i_*\mathcal{G} \cong \mathcal{F}$. Moreover:*

(1) *If $\text{Spec}(A) \subset X$ is any affine open, and $\mathcal{F}|_{\text{Spec}(A)} = \widetilde{M}$ then $Z \cap \text{Spec}(A) = \text{Spec}(A/I)$ where $I = Ann_A(M)$.*
(2) *The quasi-coherent sheaf $\mathcal{G}$ is unique up to unique isomorphism.*
(3) *The quasi-coherent sheaf $\mathcal{G}$ is of finite type.*
(4) *The support of $\mathcal{G}$ and of $\mathcal{F}$ is Z.*

Proof. Suppose that $i' : Z' \to X$ is a closed subscheme which satisfies the description on open affines from the lemma. Then by Lemma 28.4.1 we see that $\mathcal{F} \cong i'_*\mathcal{G}'$ for some unique quasi-coherent sheaf $\mathcal{G}'$ on Z'. Furthermore, it is clear that Z' is the smallest closed subscheme with this property (by the same lemma). Finally, using Properties, Lemma 27.16.1 and Algebra, Lemma 10.5.5 it follows that $\mathcal{G}'$ is of finite type. We have $\text{Supp}(\mathcal{G}') = Z$ by Algebra, Lemma 10.37.5. Hence, in order to prove the lemma it suffices to show that the characterization in (1) actually does define a closed subscheme. And, in order to do this it suffices to prove that the given rule produces a quasi-coherent sheaf of ideals, see Lemma 28.2.3. This comes down to the following algebra fact: If A is a ring, $f \in A$, and M is a finite A-module, then $\text{Ann}_A(M)_f = \text{Ann}_{A_f}(M_f)$. We omit the proof. □

Definition 28.5.5. Let X be a scheme. Let $\mathcal{F}$ be a quasi-coherent $\mathcal{O}_X$-module of finite type. The *scheme theoretic support of $\mathcal{F}$* is the closed subscheme $Z \subset X$ constructed in Lemma 28.5.4.

In this situation we often think of $\mathcal{F}$ as a quasi-coherent sheaf of finite type on Z (via the equivalence of categories of Lemma 28.4.1).

28.6. Scheme theoretic image

Caution: Some of the material in this section is ultra-general and behaves differently from what you might expect.

Lemma 28.6.1. *Let $f : X \to Y$ be a morphism of schemes. There exists a closed subscheme $Z \subset Y$ such that f factors through Z and such that for any other closed subscheme $Z' \subset Y$ such that f factors through Z' we have $Z \subset Z'$.*

Proof. Let $\mathcal{I} = \mathrm{Ker}(\mathcal{O}_Y \to f_*\mathcal{O}_X)$. If $\mathcal{I}$ is quasi-coherent then we just take Z to be the closed subscheme determined by $\mathcal{I}$, see Lemma 28.2.3. This works by Schemes, Lemma 25.4.6. In general the same lemma requires us to show that there exists a largest quasi-coherent sheaf of ideals $\mathcal{I}'$ contained in $\mathcal{I}$. This follows from Lemma 28.4.2. □

Definition 28.6.2. Let $f : X \to Y$ be a morphism of schemes. The *scheme theoretic image* of f is the smallest closed subscheme $Z \subset Y$ through which f factors, see Lemma 28.6.1 above.

We often just denote $f : X \to Z$ the factorization of f. If the morphism f is not quasi-compact, then (in general) the construction of the scheme theoretic image does not commute with restriction to open subschemes to Y. Namely, if f is the immersion $Z \to X$ of Example 28.3.4 above then the scheme theoretic image of $Z \to X$ is X. But clearly the scheme theoretic image of $Z = Z \cap U \to U$ is just Z.

Lemma 28.6.3. *Let $f : X \to Y$ be a morphism of schemes. Let $Z \subset Y$ be the scheme theoretic image of f. If f is quasi-compact then*

1. *the sheaf of ideals $\mathcal{I} = Ker(\mathcal{O}_Y \to f_*\mathcal{O}_X)$ is quasi-coherent,*
2. *the scheme theoretic image Z is the closed subscheme determined by $\mathcal{I}$,*
3. *for any open $U \subset Y$ the scheme theoretic image of $f|_{f^{-1}(U)} : f^{-1}(U) \to U$ is equal to $Z \cap U$, and*
4. *the image $f(X) \subset Z$ is a dense subset of Z, in other words the morphism $X \to Z$ is dominant (see Definition 28.8.1).*

Proof. Part (4) follows from part (3). To show (3) it suffices to prove (1) since the formation of $\mathcal{I}$ commutes with restriction to open subschemes of Y. And if (1) holds then in the proof of Lemma 28.6.1 we showed (2). Thus it suffices to prove that $\mathcal{I}$ is quasi-coherent. Since the property of being quasi-coherent is local we may assume Y is affine. As f is quasi-compact, we can find a finite affine open covering $X = \bigcup_{i=1,\ldots,n} U_i$. Denote f' the composition

$$X' = \coprod U_i \longrightarrow X \longrightarrow Y.$$

Then $f_*\mathcal{O}_X$ is a subsheaf of $f'_*\mathcal{O}_{X'}$, and hence $\mathcal{I} = \mathrm{Ker}(\mathcal{O}_Y \to \mathcal{O}_{X'})$. By Schemes, Lemma 25.24.1 the sheaf $f'_*\mathcal{O}_{X'}$ is quasi-coherent on Y. Hence we win. □

Example 28.6.4. If $A \to B$ is a ring map with kernel I, then the scheme theoretic image of $\mathrm{Spec}(B) \to \mathrm{Spec}(A)$ is the closed subscheme $\mathrm{Spec}(A/I)$ of $\mathrm{Spec}(A)$. This follows from Lemma 28.6.3.

If the morphism is quasi-compact, then the scheme theoretic image only adds points which are specializations of points in the image.

Lemma 28.6.5. *Let $f : X \to Y$ be a quasi-compact morphism. Let Z be the scheme theoretic image of f. Let $z \in Z$. There exists a valuation ring A with fraction field K and a commutative diagram*

$$\begin{array}{ccccc} \mathrm{Spec}(K) & & \longrightarrow & & X \\ \downarrow & & & \swarrow & \downarrow \\ \mathrm{Spec}(A) & \longrightarrow & Z & \longrightarrow & Y \end{array}$$

such that the closed point of $\mathrm{Spec}(A)$ *maps to* z*. In particular any point of* Z *is the specialization of a point of* $f(X)$*.*

Proof. Let $z \in \mathrm{Spec}(R) = V \subset Y$ be an affine open neighbourhood of z. By Lemma 28.6.3 we have $Z \cap V$ is the scheme theoretic closure of $f^{-1}(V) \to V$, and hence we may replace Y by V and assume $Y = \mathrm{Spec}(R)$ is affine. In this case X is quasi-compact as f is quasi-compact. Say $X = U_1 \cup \ldots \cup U_n$ is a finite affine open covering. Write $U_i = \mathrm{Spec}(A_i)$. Let $I = \mathrm{Ker}(R \to A_1 \times \ldots \times A_n)$. By Lemma 28.6.3 again we see that Z corresponds to the closed subscheme $\mathrm{Spec}(R/I)$ of Y. If $\mathfrak{p} \subset R$ is the prime corresponding to z, then we see that $I_{\mathfrak{p}} \subset R_{\mathfrak{p}}$ is not an equality. Hence (as localization is exact, see Algebra, Proposition 10.9.12) we see that $R_{\mathfrak{p}} \to (A_1)_{\mathfrak{p}} \times \ldots \times (A_1)_{\mathfrak{p}}$ is not zero. Hence one of the rings $(A_i)_{\mathfrak{p}}$ is not zero. Hence there exists an i and a prime $\mathfrak{q}_i \subset A_i$ lying over a prime $\mathfrak{p}_i \subset \mathfrak{p}$. By Algebra, Lemma 10.46.2 we can choose a valuation ring $A \subset K = f.f.(A_i/\mathfrak{q}_i)$ dominating the local ring $R_{\mathfrak{p}}/\mathfrak{p}_1 R_{\mathfrak{p}} \subset f.f.(A_i/\mathfrak{q}_i)$. This gives the desired diagram. Some details omitted. □

Lemma 28.6.6. *Let* $f_1 : X \to Y_1$ *and* $Y_1 \to Y_2$ *be morphisms of schemes. Let* $f_2 : X \to Y_2$ *be the composition. Let* $Z_i \subset Y_i$*,* $i = 1, 2$ *be the scheme theoretic image of* f_i*. Then the morphism* $Y_1 \to Y_2$ *induces a morphism* $Z_1 \to Z_2$ *and a commutative diagram*

$$\begin{array}{ccccc} X & \longrightarrow & Z_1 & \longrightarrow & Y_1 \\ & \searrow & \downarrow & & \downarrow \\ & & Z_2 & \longrightarrow & Y_2 \end{array}$$

Proof. See Schemes, Lemma 25.4.6. □

Lemma 28.6.7. *Let* $f : X \to Y$ *be a morphism of schemes. If* X *is reduced, then the scheme theoretic image of* f *is the reduced induced scheme structure on* $\overline{f(X)}$*.*

Proof. This is true because the reduced induced scheme structure on $\overline{f(X)}$ is clearly the smallest closed subscheme of Y through which f factors, see Schemes, Lemma 25.12.6. □

28.7. Scheme theoretic closure and density

We take the following definition from [**DG67**, IV, Definition 11.10.2].

Definition 28.7.1. Let X be a scheme. Let $U \subset X$ be an open subscheme.

(1) The scheme theoretic image of the morphism $U \to X$ is called the *scheme theoretic closure of* U *in* X.
(2) We say U is *scheme theoretically dense in* X if for every open $V \subset X$ the scheme theoretic closure of $U \cap V$ in V is equal to V.

With this definition it is **not** the case that U is scheme theoretically dense in X if and only if the scheme theoretic closure of U is X, see Example 28.7.2. This is somewhat inelegant; but see Lemmas 28.7.3 and 28.7.8 below. On the other hand, with this definition U is scheme theoretically dense in X if and only if for every $V \subset X$ open the ring map $\mathcal{O}_X(V) \to \mathcal{O}_X(U \cap V)$ is injective, see Lemma 28.7.5 below. In particular we see that scheme theoretically dense implies dense which is pleasing.

Example 28.7.2. Here is an example where scheme theoretic closure being X does not imply dense for the underlying topological spaces. Let k be a field. Set $A = k[x, z_1, z_2, \ldots]/(x^n z_n)$ Set $I = (z_1, z_2, \ldots) \subset A$. Consider the affine scheme $X = \mathrm{Spec}(A)$ and the open subscheme $U = X \setminus V(I)$. Since $A \to \prod_n A_{z_n}$ is injective we see that the scheme theoretic closure of U is X. Consider the morphism $X \to \mathrm{Spec}(k[x])$. This morphism is surjective (set all $z_n = 0$ to see this). But the restriction of this morphism to U is not surjective because it maps to the point $x = 0$. Hence U cannot be topologically dense in X.

Lemma 28.7.3. *Let X be a scheme. Let $U \subset X$ be an open subscheme. If the inclusion morphism $U \to X$ is quasi-compact, then U is scheme theoretically dense in X if and only if the scheme theoretic closure of U in X is X.*

Proof. Follows from Lemma 28.6.3 part (3). □

Example 28.7.4. Let A be a ring and $X = \mathrm{Spec}(A)$. Let $f_1, \ldots, f_n \in A$ and let $U = D(f_1) \cup \ldots \cup D(f_n)$. Let $I = \mathrm{Ker}(A \to \prod A_{f_i})$. Then the scheme theoretic closure of U in X is the closed subscheme $\mathrm{Spec}(A/I)$ of X. Note that $U \to X$ is quasi-compact. Hence by Lemma 28.7.3 we see U is scheme theoretically dense in X if and only if $I = 0$.

Lemma 28.7.5. *Let $j : U \to X$ be an open immersion of schemes. Then U is scheme theoretically dense in X if and only if $\mathcal{O}_X \to j_*\mathcal{O}_U$ is injective.*

Proof. If $\mathcal{O}_X \to j_*\mathcal{O}_U$ is injective, then the same is true when restricted to any open V of X. Hence the scheme theoretic closure of $U \cap V$ in V is equal to V, see proof of Lemma 28.6.1. Conversely, suppose that the scheme theoretic closure of $U \cap V$ is equal to V for all opens V. Suppose that $\mathcal{O}_X \to j_*\mathcal{O}_U$ is not injective. Then we can find an affine open, say $\mathrm{Spec}(A) = V \subset X$ and a nonzero element $f \in A$ such that f maps to zero in $\Gamma(V \cap U, \mathcal{O}_X)$. In this case the scheme theoretic closure of $V \cap U$ in V is clearly contained in $\mathrm{Spec}(A/(f))$ a contradiction. □

Lemma 28.7.6. *Let X be a scheme. If U, V are scheme theoretically dense open subschemes of X, then so is $U \cap V$.*

Proof. Let $W \subset X$ be any open. Consider the map $\mathcal{O}_X(W) \to \mathcal{O}_X(W \cap V) \to \mathcal{O}_X(W \cap V \cap U)$. By Lemma 28.7.5 both maps are injective. Hence the composite is injective. Hence by Lemma 28.7.5 $U \cap V$ is scheme theoretically dense in X. □

Lemma 28.7.7. *Let $h : Z \to X$ be an immersion. Assume either h is quasi-compact or Z is reduced. Let $\overline{Z} \subset X$ be the scheme theoretic image of h. Then the morphism $Z \to \overline{Z}$ is an open immersion which identifies Z with a scheme theoretically dense open subscheme of $\overline{Z}$. Moreover, Z is topologically dense in $\overline{Z}$.*

Proof. By Lemma 28.3.2 or Lemma 28.3.3 we can factor $Z \to X$ as $Z \to \overline{Z}_1 \to X$ with $Z \to \overline{Z}_1$ open and $\overline{Z}_1 \to X$ closed. On the other hand, let $Z \to \overline{Z} \subset X$ be the scheme theoretic closure of $Z \to X$. We conclude that $\overline{Z} \subset \overline{Z}_1$. Since Z is an open subscheme of $\overline{Z}_1$ it follows that Z is an open subscheme of $\overline{Z}$ as well. In the case that Z is reduced we know that $Z \subset \overline{Z}_1$ is topologically dense by the construction of $\overline{Z}_1$ in the proof of Lemma 28.3.3. Hence $\overline{Z}_1$ and $\overline{Z}$ have the same underlying topological spaces. Thus $\overline{Z} \subset \overline{Z}_1$ is a closed immersion into a reduced scheme which induces a bijection on underlying topological spaces, and hence it is an isomorphism. In the case that $Z \to X$ is quasi-compact we argue as follows:

The assertion that Z is scheme theoretically dense in $\overline{Z}$ follows from Lemma 28.6.3 part (3). The last assertion follows from Lemma 28.6.3 part (4). □

Lemma 28.7.8. *Let X be a reduced scheme and let $U \subset X$ be an open subscheme. Then the following are equivalent*

(1) *U is topologically dense in X,*
(2) *the scheme theoretic closure of U in X is X, and*
(3) *U is scheme theoretically dense in X.*

Proof. This follows from Lemma 28.7.7 and the fact that the a closed subscheme Z of X whose underlying topological space equals X must be equal to X as a scheme. □

Lemma 28.7.9. *Let X be a scheme and let $U \subset X$ be a reduced open subscheme. Then the following are equivalent*

(1) *the scheme theoretic closure of U in X is X, and*
(2) *U is scheme theoretically dense in X.*

If this holds then X is a reduced scheme.

Proof. This follows from Lemma 28.7.7 and the fact that the scheme theoretic closure of U in X is reduced by Lemma 28.6.7. □

Lemma 28.7.10. *Let S be a scheme. Let X, Y be schemes over S. Let $f, g : X \to Y$ be morphisms of schemes over S. Let $U \subset X$ be an open subscheme such that $f|_U = g|_U$. If the scheme theoretic closure of U in X is X and $Y \to S$ is separated, then $f = g$.*

Proof. Follows from the definitions and Schemes, Lemma 25.21.5. □

28.8. Dominant morphisms

The definition of a morphism of schemes being dominant is a little different from what you might expect if you are used to the notion of a dominant morphism of varieties.

Definition 28.8.1. A morphism $f : X \to S$ of schemes is called *dominant* if the image of f is a dense subset of S.

So for example, if k is an infinite field and $\lambda_1, \lambda_2, \ldots$ is a countable collection of elements of k, then the morphism

$$\coprod_{i=1,2,\ldots} \mathrm{Spec}(k) \longrightarrow \mathrm{Spec}(k[x])$$

with ith factor mapping to the point $x = \lambda_i$ is dominant.

Lemma 28.8.2. *Let $f : X \to S$ be a morphism of schemes. If every generic point of every irreducible component of S is in the image of f, then f is dominant.*

Proof. This is a topological fact which follows directly from the fact that the topological space underlying a scheme is sober, see Schemes, Lemma 25.11.1, and that every point of S is contained in an irreducible component of S, see Topology, Lemma 5.7.3. □

The expectation that morphisms are dominant only if generic points of the target are in the image does hold if the morphism is quasi-compact.

Lemma 28.8.3. *Let $f : X \to S$ be a quasi-compact morphism of schemes. Then f is dominant (if and) only if for every irreducible component $Z \subset S$ the generic point of Z is in the image of f.*

Proof. Let $V \subset S$ be an affine open. Because f is quasi-compact we may choose finitely many affine opens $U_i \subset f^{-1}(V)$, $i = 1, \ldots, n$ covering $f^{-1}(V)$. Consider the morphism of affines

$$f' : \coprod\nolimits_{i=1,\ldots,n} U_i \longrightarrow V.$$

A disjoint union of affines is affine, see Schemes, Lemma 25.6.8. Generic points of irreducible components of V are exactly the generic points of the irreducible components of S that meet V. Also, f is dominant if and only f' is dominant no matter what choices of V, n, U_i we make above. Thus we have reduced the lemma to the case of a morphism of affine schemes. The affine case is Algebra, Lemma 10.28.6. □

Here is a slightly more useful variant of the lemma above.

Lemma 28.8.4. *Let $f : X \to S$ be a quasi-compact morphism of schemes. Let $\eta \in S$ be a generic point of an irreducible component of S. If $\eta \notin f(X)$ then there exists an open neighbourhood $V \subset S$ of η such that $f^{-1}(V) = \emptyset$.*

Proof. Let $Z \subset S$ be the scheme theoretic image of f. We have to show that $\eta \notin Z$. This follows from Lemma 28.6.5 but can also be seen as follows. By Lemma 28.6.3 the morphism $X \to Z$ is dominant, which by Lemma 28.8.3 means all the generic points of all irreducible components of Z are in the image of $X \to Z$. By assumption we see that $\eta \notin Z$ since η would be the generic point of some irreducible component of Z if it were in Z. □

There is another case where dominant is the same as having all generic points of irreducible components in the image.

Lemma 28.8.5. *Let $f : X \to S$ be a morphism of schemes. Suppose that X has finitely many irreducible components. Then f is dominant (if and) only if for every irreducible component $Z \subset S$ the generic point of Z is in the image of f. If so, then S has finitely many irreducible components as well.*

Proof. Assume f is dominant. Say $X = Z_1 \cup Z_2 \cup \ldots \cup Z_n$ is the decomposition of X into irreducible components. Let $\xi_i \in Z_i$ be its generic point, so $Z_i = \overline{\{\xi_i\}}$. Note that $f(Z_i)$ is an irreducible subset of S. Hence

$$S = \overline{f(X)} = \bigcup \overline{f(Z_i)} = \bigcup \overline{\{f(\xi_i)\}}$$

is a finite union of irreducible subsets whose generic points are in the image of f. The lemma follows. □

28.9. Birational morphisms

You may be used to the notion of a birational map of varieties having the property that it is an isomorphism over an open subset of the target. However, in general a birational morphism may not be an isomorphism over any nonempty open, see Example 28.9.3. Here is the formal definition.

Definition 28.9.1. Let X, Y be schemes. Assume X and Y have finitely many irreducible components. We say a morphism $f : X \to Y$ is *birational* if

(1) f induces a bijection between the set of generic points of irreducible components of X and the set of generic points of the irreducible components of Y, and
(2) for every generic point $\eta \in X$ of an irreducible component of X the local ring map $\mathcal{O}_{Y,f(\eta)} \to \mathcal{O}_{X,\eta}$ is an isomorphism.

Lemma 28.9.2. *Let $f : X \to Y$ be a morphism of schemes having finitely many irreducible components. If f is birational then f is dominant.*

Proof. Follows immediately from the definitions. □

Example 28.9.3. Here is an example of a birational morphism which is not an isomorphism over any open of the target. Let k be an infinite field. Let $A = k[x]$. Let $B = k[x, \{y_\alpha\}_{\alpha \in k}]/((x-\alpha)y_\alpha, y_\alpha y_\beta)$. There is an inclusion $A \subset B$ and a retraction $B \to A$ setting all y_α equal to zero. Both the morphism $\mathrm{Spec}(A) \to \mathrm{Spec}(B)$ and the morphism $\mathrm{Spec}(B) \to \mathrm{Spec}(A)$ are birational but not an isomorphism over any open.

28.10. Rational maps

Let X be a scheme. Note that if U, V are dense open in X, then so is $U \cap V$.

Definition 28.10.1. Let X, Y be schemes.

(1) Let $f : U \to Y$, $g : V \to Y$ be morphisms of schemes defined on dense open subsets U, V of X. We say that f is *equivalent* to g if $f|_W = g|_W$ for some $W \subset U \cap V$ dense open in X.
(2) A *rational map from X to Y* is an equivalence class for the equivalence relation defined in (1).
(3) If X, Y are schemes over a base scheme S we say that a rational map from X to Y is an *S-rational map from X to Y* if there exists a representative $f : U \to Y$ of the equivalence class which is an S-morphism.

We say that two morphisms f, g as in (1) of the definition define the same rational map instead of saying that they are equivalent.

Definition 28.10.2. Let X be a scheme. A *rational function on X* is a rational morphism from X to $\mathbf{A}^1_{\mathbf{Z}}$.

See Constructions, Definition 26.5.1 for the definition of the affine line $\mathbf{A}^1$. Let X be a scheme over S. For any open $U \subset X$ a morphism $U \to \mathbf{A}^1_{\mathbf{Z}}$ is the same as a morphism $U \to \mathbf{A}^1_S$ over S. Hence a rational function is also the same as a S-rational map from X into $\mathbf{A}^1_S$.

Recall that we have the canonical identification $\mathrm{Mor}(T, \mathbf{A}^1_{\mathbf{Z}}) = \Gamma(T, \mathcal{O}_T)$ for any scheme T, see Schemes, Example 25.15.2. Hence $\mathbf{A}^1_{\mathbf{Z}}$ is a ring-object in the category of schemes. More precisely, the morphisms

$$\begin{aligned} + : \mathbf{A}^1_{\mathbf{Z}} \times \mathbf{A}^1_{\mathbf{Z}} &\longrightarrow \mathbf{A}^1_{\mathbf{Z}} \\ (f, g) &\longmapsto f + g \\ * : \mathbf{A}^1_{\mathbf{Z}} \times \mathbf{A}^1_{\mathbf{Z}} &\longrightarrow \mathbf{A}^1_{\mathbf{Z}} \\ (f, g) &\longmapsto fg \end{aligned}$$

satisfy all the axioms of the addition and multiplication in a ring (commutative with 1 as always). Hence also the set of rational maps into $\mathbf{A}^1_{\mathbf{Z}}$ has a natural ring structure.

Definition 28.10.3. Let X be a scheme. The *ring of rational functions on* X is the ring $R(X)$ whose elements are rational functions with addition and multiplication as just described.

Lemma 28.10.4. *Let X be an irreducible scheme. Let $\eta \in X$ be the generic point of X. There is a canonical identification $R(X) \cong \mathcal{O}_{X,\eta}$. If X is integral then $R(X) = \kappa(\eta) = \mathcal{O}_{X,\eta}$ is a field.*

Proof. Omitted. □

Definition 28.10.5. Let X be an integral scheme. The *function field*, or the *field of rational functions* of X is the field $R(X)$.

We may occasionally indicate this field $k(X)$ instead of $R(X)$.

Remark 28.10.6. There is a variant of Definition 28.10.1 where we consider only those morphism $U \to Y$ defined on scheme theoretically dense open subschemes $U \subset X$. We use Lemma 28.7.6 to see that we obtain an equivalence relation. An equivalence class of these is called a *pseudo-morphism from X to Y*. If X is reduced the two notions coincide.

Here is a fun application of these notions. Note that by Lemma 28.10.4 on an integral scheme every local ring $\mathcal{O}_{X,x}$ may be viewed as a local subring of $R(X)$.

Lemma 28.10.7. *Let X be an integral separated scheme. Let Z_1, Z_2 be distinct irreducible closed subsets of X. Let η_i be the generic point of Z_i. If $Z_1 \not\subset Z_2$, then $\mathcal{O}_{X,\eta_1} \not\subset \mathcal{O}_{X,\eta_2}$ as subrings of $R(X)$. In particular, if $Z_1 = \{x\}$ consists of one closed point x, there exists a function regular in a neighborhood of x which is not in $\mathcal{O}_{X,\eta_2}$.*

Proof. First observe that under the assumption of X being seperated, there is a unique map of schemes $\mathrm{Spec}(\mathcal{O}_{X,\eta_2}) \to X$ over X such that the composition

$$\mathrm{Spec}(R(X)) \longrightarrow \mathrm{Spec}(\mathcal{O}_{X,\eta_2}) \longrightarrow X$$

is the canonical map $\mathrm{Spec}(R(X)) \to X$. Namely, there is the canonical map $can : \mathrm{Spec}(\mathcal{O}_{X,\eta_2}) \to X$, see Schemes, Equation (25.13.1.1). Given a second morphism a to X, we have that a agrees with can on the generic point of $\mathrm{Spec}(\mathcal{O}_{X,\eta_2})$ by assumption. Now being X being seperated guarantees that the subset in $\mathrm{Spec}(\mathcal{O}_{X,\eta_2})$ where these two maps agree is closed, see Schemes, Lemma 25.21.5. Hence $a = can$ on all of $\mathrm{Spec}(\mathcal{O}_{X,\eta_2})$.

Assume $Z_1 \not\subset Z_2$ and assume on the contrary that $\mathcal{O}_{X,\eta_1} \subset \mathcal{O}_{X,\eta_2}$ as subrings of $R(X)$. Then we would obtain a second morphism

$$\mathrm{Spec}(\mathcal{O}_{X,\eta_2}) \longrightarrow \mathrm{Spec}(\mathcal{O}_{X,\eta_1}) \longrightarrow X.$$

By the above this composition would have to be equal to can. This implies that η_2 specializes to η_1 (see Schemes, Lemma 25.13.2). But this contradicts our assumption $Z_1 \not\subset Z_2$. □

28.11. Surjective morphisms

Definition 28.11.1. A morphism of schemes is said to be *surjective* if it is surjective on underlying topological spaces.

Lemma 28.11.2. *The composition of surjective morphisms is surjective.*

Proof. Omitted. □

Lemma 28.11.3. *Let X and Y be schemes over a base scheme S. Given points $x \in X$ and $y \in Y$, there is a point of $X \times_S Y$ mapping to x and y under the projections if and only if x and y lie above the same point of S.*

Proof. The condition is obviously necessary, and the converse follows from the proof of Schemes, Lemma 25.17.5. □

Lemma 28.11.4. *The base change of a surjective morphism is surjective.*

Proof. Let $f : X \to Y$ be a morphism of schemes over a base scheme S. If $S' \to S$ is a morphism of schemes, let $p : X_{S'} \to X$ and $q : Y_{S'} \to Y$ be the canonical projections. The commutative square

$$\begin{array}{ccc} X_{S'} & \xrightarrow{p} & X \\ {\scriptstyle f_{S'}}\downarrow & & \downarrow{\scriptstyle f} \\ Y_{S'} & \xrightarrow{q} & Y. \end{array}$$

identifies $X_{S'}$ as a fibre product of $X \to Y$ and $Y_{S'} \to Y$. Let Z be a subset of the underlying topological space of X. Then $q^{-1}(f(Z)) = f_{S'}(p^{-1}(Z))$, because $y' \in q^{-1}(f(Z))$ if and only if $q(y') = f(x)$ for some $x \in Z$, if and only if, by Lemma 28.11.3, there exists $x' \in X_{S'}$ such that $f_{S'}(x') = y'$ and $p(x') = x$. In particular taking $Z = X$ we see that if f is surjective so is the base change $f_{S'} : X_{S'} \to Y_{S'}$. □

Example 28.11.5. Bijectivity is not stable under base change, and so neither is injectivity. For example consider the bijection $\mathrm{Spec}(\mathbf{C}) \to \mathrm{Spec}(\mathbf{R})$. The base change $\mathrm{Spec}(\mathbf{C} \otimes_{\mathbf{R}} \mathbf{C}) \to \mathrm{Spec}(\mathbf{C})$ is not injective, since there is an isomorphism $\mathbf{C} \otimes_{\mathbf{R}} \mathbf{C} \cong \mathbf{C} \times \mathbf{C}$ (the decomposition comes from the idempotent $\frac{1 \otimes 1 + i \otimes i}{2}$) and hence $\mathrm{Spec}(\mathbf{C} \otimes_{\mathbf{R}} \mathbf{C})$ has two points.

Lemma 28.11.6. *Let*

$$\begin{array}{ccc} X & \xrightarrow{\quad f \quad} & Y \\ & {\scriptstyle p}\searrow \quad \swarrow{\scriptstyle q} & \\ & Z & \end{array}$$

be a commutative diagram of morphisms of schemes. If f is surjective and p is quasi-compact, then q is quasi-compact.

Proof. Let $W \subset Z$ be a quasi-compact open. By assumption $p^{-1}(W)$ is quasi-compact. Hence by Topology, Lemma 5.11.7 the inverse image $q^{-1}(W) = f(p^{-1}(W))$ is quasi-compact too. This proves the lemma. □

28.12. Radicial and universally injective morphisms

In this section we define what it means for a morphism of schemes to be *radicial* and what it means for a morphism of schemes to be *universally injective*. We then show that these notions agree. The reason for introducing both is that in the case of algebraic spaces there are corresponding notions which may not always agree.

Definition 28.12.1. Let $f : X \to S$ be a morphism.

(1) We say that f is *universally injective* if and only if for any morphism of schemes $S' \to S$ the base change $f' : X_{S'} \to S'$ is injective (on underlying topological spaces).
(2) We say f is *radicial* if f is injective as a map of topological spaces, and for every $x \in X$ the field extension $\kappa(x) \supset \kappa(f(x))$ is purely inseparable.

Lemma 28.12.2. *Let $f : X \to S$ be a morphism of schemes. The following are equivalent:*

(1) *For every field K the induced map $\mathrm{Mor}(\mathrm{Spec}(K), X) \to \mathrm{Mor}(\mathrm{Spec}(K), S)$ is injective.*
(2) *The morphism f is universally injective.*
(3) *The morphism f is radicial.*
(4) *The diagonal morphism $\Delta_{X/S} : X \longrightarrow X \times_S X$ is surjective.*

Proof. Let K be a field, and let $s : \mathrm{Spec}(K) \to S$ be a morphism. Giving a morphism $x : \mathrm{Spec}(K) \to X$ such that $f \circ x = s$ is the same as giving a section of the projection $X_K = \mathrm{Spec}(K) \times_S X \to \mathrm{Spec}(K)$, which in turn is the same as giving a point $x \in X_K$ whose residue field is K. Hence we see that (2) implies (1).

Conversely, suppose that (1) holds. Assume that $x, x' \in X_{S'}$ map to the same point $s' \in S'$. Choose a commutative diagram

$$\begin{array}{ccc} K & \longleftarrow & \kappa(x) \\ \uparrow & & \uparrow \\ \kappa(x') & \longleftarrow & \kappa(s') \end{array}$$

of fields. By Schemes, Lemma 25.13.3 we get two morphisms $a, a' : \mathrm{Spec}(K) \to X_{S'}$. One corresponding to the point x and the embedding $\kappa(x) \subset K$ and the other corresponding to the point x' and the embedding $\kappa(x') \subset K$. Also we have $f' \circ a = f' \circ a'$. Condition (1) now implies that the compositions of a and a' with $X_{S'} \to X$ are equal. Since $X_{S'}$ is the fibre product of S' and X over S we see that $a = a'$. Hence $x = x'$. Thus (1) implies (2).

If there are two different points $x, x' \in X$ mapping to the same point of s then (2) is violated. If for some $s = f(x)$, $x \in X$ the field extension $\kappa(s) \subset \kappa(x)$ is not purely inseparable, then we may find a field extension $\kappa(s) \subset K$ such that $\kappa(x)$ has two $\kappa(s)$-homomorphisms into K. By Schemes, Lemma 25.13.3 this implies that the map $\mathrm{Mor}(\mathrm{Spec}(K), X) \to \mathrm{Mor}(\mathrm{Spec}(K), S)$ is not injective, and hence (1) is violated. Thus we see that the equivalent conditions (1) and (2) imply f is radicial, i.e., they imply (3).

Assume (3). By Schemes, Lemma 25.13.3 a morphism $\mathrm{Spec}(K) \to X$ is given by a pair $(x, \kappa(x) \to K)$. Property (3) says exactly that associating to the pair $(x, \kappa(x) \to K)$ the pair $(s, \kappa(s) \to \kappa(x) \to K)$ is injective. In other words (1) holds. At this point we know that (1), (2) and (3) are all equivalent.

Finally, we prove the equivalence of (4) with (1), (2) and (3). A point of $X \times_S X$ is given by a quadruple $(x_1, x_2, s, \mathfrak{p})$, where $x_1, x_2 \in X$, $f(x_1) = f(x_2) = s$ and $\mathfrak{p} \subset \kappa(x_1) \otimes_{\kappa(s)} \kappa(x_2)$ is a prime ideal, see Schemes, Lemma 25.17.5. If f is universally injective, then by taking $S' = X$ in the definition of universally injective, $\Delta_{X/S}$ must be surjective since it is a section of the injective morphism $X \times_S X \longrightarrow X$. Conversely, if $\Delta_{X/S}$ is surjective, then always $x_1 = x_2 = x$ and there is exactly one

such prime ideal $\mathfrak{p}$, which means that $\kappa(s) \subset \kappa(x)$ is purely inseparable. Hence f is radicial. Alternatively, if $\Delta_{X/S}$ is surjective, then for any $S' \to S$ the base change $\Delta_{X_{S'}/S'}$ is surjective which implies that f is universally injective. This finishes the proof of the lemma. □

Lemma 28.12.3. *A universally injective morphism is separated.*

Proof. Combine Lemma 28.12.2 with the remark that $X \to S$ is separated if and only if the image of $\Delta_{X/S}$ is closed in $X \times_S X$, see Schemes, Definition 25.21.3 and the discussion following it. □

Lemma 28.12.4. *A base change of a universally injective morphism is universally injective.*

Proof. This is formal. □

Lemma 28.12.5. *A composition of radicial morphisms is radicial, and so the same holds for the equivalent condition of being universally injective.*

Proof. Omitted. □

28.13. Affine morphisms

Definition 28.13.1. A morphism of schemes $f : X \to S$ is called *affine* if the inverse image of every affine open of S is an affine open of X.

Lemma 28.13.2. *An affine morphism is separated and quasi-compact.*

Proof. Let $f : X \to S$ be affine. Quasi-compactness is immediate from Schemes, Lemma 25.19.2. We will show f is separated using Schemes, Lemma 25.21.8. Let $x_1, x_2 \in X$ be points of X which map to the same point $s \in S$. Choose any affine open $W \subset S$ containing s. By assumption $f^{-1}(W)$ is affine. Apply the lemma cited with $U = V = f^{-1}(W)$. □

Lemma 28.13.3. *Let $f : X \to S$ be a morphism of schemes. The following are equivalent*

1. *The morphism f is affine.*
2. *There exists an affine open covering $S = \bigcup W_j$ such that each $f^{-1}(W_j)$ is affine.*
3. *There exists a quasi-coherent sheaf of $\mathcal{O}_S$-algebras $\mathcal{A}$ and an isomorphism $X \cong \underline{\mathrm{Spec}}_S(\mathcal{A})$ of schemes over S. See Constructions, Section 26.4 for notation.*

Moreover, in this case $X = \underline{\mathrm{Spec}}_S(f_*\mathcal{O}_X)$.

Proof. It is obvious that (1) implies (2).

Assume $S = \bigcup_{j \in J} W_j$ is an affine open covering such that each $f^{-1}(W_j)$ is affine. By Schemes, Lemma 25.19.2 we see that f is quasi-compact. By Schemes, Lemma 25.21.7 we see the morphism f is quasi-separated. Hence by Schemes, Lemma 25.24.1 the sheaf $\mathcal{A} = f_*\mathcal{O}_X$ is a quasi-coherent sheaf of $\mathcal{O}_X$-algebras. Thus we have the scheme $g : Y = \underline{\mathrm{Spec}}_S(\mathcal{A}) \to S$ over S. The identity map $\mathrm{id} : \mathcal{A} = f_*\mathcal{O}_X \to f_*\mathcal{O}_X$ provides, via the definition of the relative spectrum, a morphism $can : X \to Y$ over S, see Constructions, Lemma 26.4.7. By assumption and the lemma just cited the restriction $can|_{f^{-1}(W_j)} : f^{-1}(W_j) \to g^{-1}(W_j)$ is an isomorphism. Thus can is an isomorphism. We have shown that (2) implies (3).

Assume (3). By Constructions, Lemma 26.4.6 we see that the inverse image of every affine open is affine, and hence the morphism is affine by definition. □

Remark 28.13.4. We can also argue directly that (2) implies (1) in Lemma 28.13.3 above as follows. Assume $S = \bigcup W_j$ is an affine open covering such that each $f^{-1}(W_j)$ is affine. First argue that $\mathcal{A} = f_*\mathcal{O}_X$ is quasi-coherent as in the proof above. Let $\mathrm{Spec}(R) = V \subset S$ be affine open. We have to show that $f^{-1}(V)$ is affine. Set $A = \mathcal{A}(V) = f_*\mathcal{O}_X(V) = \mathcal{O}_X(f^{-1}(V))$. By Schemes, Lemma 25.6.4 there is a canonical morphism $\psi : f^{-1}(V) \to \mathrm{Spec}(A)$ over $\mathrm{Spec}(R) = V$. By Schemes, Lemma 25.11.6 there exists an integer $n \geq 0$, a standard open covering $V = \bigcup_{i=1,\ldots,n} D(h_i)$, $h_i \in R$, and a map $a : \{1, \ldots, n\} \to J$ such that each $D(h_i)$ is also a standard open of the affine scheme $W_{a(i)}$. The inverse image of a standard open under a morphism of affine schemes is standard open, see Algebra, Lemma 10.16.4. Hence we see that $f^{-1}(D(h_i))$ is a standard open of $f^{-1}(W_{a(i)})$, in particular that $f^{-1}(D(h_i))$ is affine. Because $\mathcal{A}$ is quasi-coherent we have $A_{h_i} = \mathcal{A}(D(h_i)) = \mathcal{O}_X(f^{-1}(D(h_i)))$, so $f^{-1}(D(h_i))$ is the spectrum of A_{h_i}. It follows that the morphism ψ induces an isomorphism of the open $f^{-1}(D(h_i))$ with the open $\mathrm{Spec}(A_{h_i})$ of $\mathrm{Spec}(A)$. Since $f^{-1}(V) = \bigcup f^{-1}(D(h_i))$ and $\mathrm{Spec}(A) = \bigcup \mathrm{Spec}(A_{h_i})$ we win.

Lemma 28.13.5. *Let S be a scheme. There is an anti-equivalence of categories*

$$\begin{matrix}\text{Schemes affine} \\ \text{over } S\end{matrix} \longleftrightarrow \begin{matrix}\text{quasi-coherent sheaves} \\ \text{of } \mathcal{O}_S\text{-algebras}\end{matrix}$$

which associates to $f : X \to S$ the sheaf $f_*\mathcal{O}_X$. Moreover, this equivalence if compatible with arbitrary base change.

Proof. The functor from right to left is given by $\underline{\mathrm{Spec}}_S$. The two functors are mutually inverse by Lemma 28.13.3 and Constructions, Lemma 26.4.6 part (3). The final statement is Constructions, Lemma 26.4.6 part (2). □

Lemma 28.13.6. *Let $f : X \to S$ be an affine morphism of schemes. Let $\mathcal{A} = f_*\mathcal{O}_X$. The functor $\mathcal{F} \mapsto f_*\mathcal{F}$ induces an equivalence of categories*

$$\left\{\begin{matrix}\text{category of quasi-coherent} \\ \mathcal{O}_X\text{-modules}\end{matrix}\right\} \longrightarrow \left\{\begin{matrix}\text{category of quasi-coherent} \\ \mathcal{A}\text{-modules}\end{matrix}\right\}$$

Moreover, an $\mathcal{A}$-module is quasi-coherent as an $\mathcal{O}_S$-module if and only if it is quasi-coherent as an $\mathcal{A}$-module.

Proof. Omitted. □

Lemma 28.13.7. *The composition of affine morphisms is affine.*

Proof. Let $f : X \to Y$ and $g : Y \to Z$ be affine morphisms. Let $U \subset Z$ be affine open. Then $g^{-1}(U)$ is affine by assumption on g. Whereupon $f^{-1}(g^{-1}(U))$ is affine by assumption on f. Hence $(g \circ f)^{-1}(U)$ is affine. □

Lemma 28.13.8. *The base change of an affine morphism is affine.*

Proof. Let $f : X \to S$ be an affine morphism. Let $S' \to S$ be any morphism. Denote $f' : X_{S'} = S' \times_S X \to S'$ the base change of f. For every $s' \in S'$ there exists an open affine neighbourhood $s' \in V \subset S'$ which maps into some open affine $U \subset S$. By assumption $f^{-1}(U)$ is affine. By the material in Schemes, Section 25.17 we see that $f^{-1}(U)_V = V \times_U f^{-1}(U)$ is affine and equal to $(f')^{-1}(V)$. This proves

that S' has an open covering by affines whose inverse image under f' is affine. We conclude by Lemma 28.13.3 above. □

Lemma 28.13.9. *A closed immersion is affine.*

Proof. The first indication of this is Schemes, Lemma 25.8.2. See Schemes, Lemma 25.10.1 for a complete statement. □

Lemma 28.13.10. *Let X be a scheme. Let $\mathcal{L}$ be an invertible $\mathcal{O}_X$-module. Let $s \in \Gamma(X, \mathcal{L})$. The inclusion morphism $j : X_s \to X$ is affine.*

Proof. This follows from Properties, Lemma 27.24.4 and the definition. □

Lemma 28.13.11. *Suppose $g : X \to Y$ is a morphism of schemes over S.*

(1) *If X is affine over S and $\Delta : Y \to Y \times_S Y$ is affine, then g is affine.*
(2) *If X is affine over S and Y is separated over S, then g is affine.*
(3) *A morphism from an affine scheme to a scheme with affine diagonal is affine.*
(4) *A morphism from an affine scheme to a separated scheme is affine.*

Proof. Proof of (1). The base change $X \times_S Y \to Y$ is affine by Lemma 28.13.8. The morphism $(1, g) : X \to X \times_S Y$ is the base change of $Y \to Y \times_S Y$ by the morphism $X \times_S Y \to Y \times_S Y$. Hence it is affine by Lemma 28.13.8. The composition of affine morphisms is affine (see Lemma 28.13.7) and (1) follows. Part (2) follows from (1) as a closed immersion is affine (see Lemma 28.13.9) and Y/S separated means Δ is a closed immersion. Parts (3) and (4) are special cases of (1) and (2). □

Lemma 28.13.12. *A morphism between affine schemes is affine.*

Proof. Immediate from Lemma 28.13.11 with $S = \mathrm{Spec}(\mathbf{Z})$. It also follows directly from the equivalence of (1) and (2) in Lemma 28.13.3. □

Lemma 28.13.13. *Let S be a scheme. Let A be an Artinian ring. Any morphism $\mathrm{Spec}(A) \to S$ is affine.*

Proof. Omitted. □

28.14. Quasi-affine morphisms

Recall that a scheme X is called *quasi-affine* if it is quasi-compact and isomorphic to an open subscheme of an affine scheme, see Properties, Definition 27.15.1.

Definition 28.14.1. A morphism of schemes $f : X \to S$ is called *quasi-affine* if the inverse image of every affine open of S is a quasi-affine scheme.

Lemma 28.14.2. *A quasi-affine morphism is separated and quasi-compact.*

Proof. Let $f : X \to S$ be quasi-affine. Quasi-compactness is immediate from Schemes, Lemma 25.19.2. We will show f is separated using Schemes, Lemma 25.21.8. Let $x_1, x_2 \in X$ be points of X which map to the same point $s \in S$. Choose any affine open $W \subset S$ containing s. By assumption $f^{-1}(W)$ is isomorphic to an open subscheme of an affine scheme, say $f^{-1}(W) \to Y$ is such an open immersion. Choose affine open neighbourhoods $x_1 \in U \subset f^{-1}(W)$ and $x_2 \in V \subset f^{-1}(W)$. We may think of U and V as open subschemes of Y and hence we see that $U \cap V$ is affine and that $\mathcal{O}(U) \otimes_{\mathbf{Z}} \mathcal{O}(V) \to \mathcal{O}(U \cap V)$ is surjective (by the lemma cited above applied to U, V in Y). Hence by the lemma cited we conclude that f is separated. □

Lemma 28.14.3. *Let $f : X \to S$ be a morphism of schemes. The following are equivalent*

(1) *The morphism f is quasi-affine.*
(2) *There exists an affine open covering $S = \bigcup W_j$ such that each $f^{-1}(W_j)$ is quasi-affine.*
(3) *There exists a quasi-coherent sheaf of $\mathcal{O}_S$-algebras $\mathcal{A}$ and a quasi-compact open immersion*

$$\begin{array}{ccc} X & \longrightarrow & \underline{\mathrm{Spec}}_S(\mathcal{A}) \\ & \searrow \quad \swarrow & \\ & S & \end{array}$$

over S.
(4) *Same as in (3) but with $\mathcal{A} = f_*\mathcal{O}_X$ and the horizontal arrow the canonical morphism of Constructions, Lemma 26.4.7.*

Proof. It is obvious that (1) implies (2) and that (4) implies (3).

Assume $S = \bigcup_{j \in J} W_j$ is an affine open covering such that each $f^{-1}(W_j)$ is quasi-affine. By Schemes, Lemma 25.19.2 we see that f is quasi-compact. By Schemes, Lemma 25.21.7 we see the morphism f is quasi-separated. Hence by Schemes, Lemma 25.24.1 the sheaf $\mathcal{A} = f_*\mathcal{O}_X$ is a quasi-coherent sheaf of $\mathcal{O}_X$-algebras. Thus we have the scheme $g : Y = \underline{\mathrm{Spec}}_S(\mathcal{A}) \to S$ over S. The identity map $\mathrm{id} : \mathcal{A} = f_*\mathcal{O}_X \to f_*\mathcal{O}_X$ provides, via the definition of the relative spectrum, a morphism $can : X \to Y$ over S, see Constructions, Lemma 26.4.7. By assumption, the lemma just cited, and Properties, Lemma 27.15.4 the restriction $can|_{f^{-1}(W_j)} : f^{-1}(W_j) \to g^{-1}(W_j)$ is a quasi-compact open immersion. Thus can is a quasi-compact open immersion. We have shown that (2) implies (4).

Assume (3). Choose any affine open $U \subset S$. By Constructions, Lemma 26.4.6 we see that the inverse image of U in the relative spectrum is affine. Hence we conclude that $f^{-1}(U)$ is quasi-affine (note that quasi-compactness is encoded in (3) as well). Thus (3) implies (1). □

Lemma 28.14.4. *The composition of quasi-affine morphisms is quasi-affine.*

Proof. Let $f : X \to Y$ and $g : Y \to Z$ be quasi-affine morphisms. Let $U \subset Z$ be affine open. Then $g^{-1}(U)$ is quasi-affine by assumption on g. Let $j : g^{-1}(U) \to V$ be a quasi-compact open immersion into an affine scheme V. By Lemma 28.14.3 above we see that $f^{-1}(g^{-1}(U))$ is a quasi-compact open subscheme of the relative spectrum $\underline{\mathrm{Spec}}_{g^{-1}(U)}(\mathcal{A})$ for some quasi-coherent sheaf of $\mathcal{O}_{g^{-1}(U)}$-algebras $\mathcal{A}$. By Schemes, Lemma 25.24.1 the sheaf $\mathcal{A}' = j_*\mathcal{A}$ is a quasi-coherent sheaf of $\mathcal{O}_V$-algebras with the property that $j^*\mathcal{A}' = \mathcal{A}$. Hence we get a commutative diagram

$$\begin{array}{ccccc} f^{-1}(g^{-1}(U)) & \longrightarrow & \underline{\mathrm{Spec}}_{g^{-1}(U)}(\mathcal{A}) & \longrightarrow & \underline{\mathrm{Spec}}_V(\mathcal{A}') \\ & & \downarrow & & \downarrow \\ & & g^{-1}(U) & \xrightarrow{j} & V \end{array}$$

with the square being a fibre square, see Constructions, Lemma 26.4.6. Note that the upper right corner is an affine scheme. Hence $(g \circ f)^{-1}(U)$ is quasi-affine. □

Lemma 28.14.5. *The base change of a quasi-affine morphism is quasi-affine.*

Proof. Let $f : X \to S$ be a quasi-affine morphism. By Lemma 28.14.3 above we can find a quasi-coherent sheaf of $\mathcal{O}_S$-algebras $\mathcal{A}$ and a quasi-compact open immersion $X \to \underline{\mathrm{Spec}}_S(\mathcal{A})$ over S. Let $g : S' \to S$ be any morphism. Denote $f' : X_{S'} = S' \times_S X \to S'$ the base change of f. Since the base change of a quasi-compact open immersion is a quasi-compact open immersion we see that $X_{S'} \to \underline{\mathrm{Spec}}_{S'}(g^*\mathcal{A})$ is a quasi-compact open immersion (we have used Schemes, Lemmas 25.19.3 and 25.18.2 and Constructions, Lemma 26.4.6). By Lemma 28.14.3 again we conclude that $X_{S'} \to S'$ is quasi-affine. □

Lemma 28.14.6. *A quasi-compact immersion is quasi-affine.*

Proof. Let $X \to S$ be a quasi-compact immersion. We have to show the inverse image of every affine open is quasi-affine. Hence, assuming S is an affine scheme, we have to show X is quasi-affine. By Lemma 28.7.7 the morphism $X \to S$ factors as $X \to Z \to S$ where Z is a closed subscheme of S and $X \subset Z$ is a quasi-compact open. Since S is affine Lemma 28.2.1 implies Z is affine. Hence we win. □

Lemma 28.14.7. *Let S be a scheme. Let X be an affine scheme. A morphism $f : X \to S$ is quasi-affine if and only if it is quasi-compact. In particular any morphism from an affine scheme to a quasi-separated scheme is quasi-affine.*

Proof. Let $V \subset S$ be an affine open. Then $f^{-1}(V)$ is an open subscheme of the affine scheme X, hence quasi-affine if and only if it is quasi-compact. This proves the first assertion. The quasi-compactness of any $f : X \to S$ where X is affine and S quasi-separated follows from Schemes, Lemma 25.21.15 applied to $X \to S \to \mathrm{Spec}(\mathbf{Z})$. □

Lemma 28.14.8. *Suppose $g : X \to Y$ is a morphism of schemes over S. If X is quasi-affine over S and Y is quasi-separated over S, then g is quasi-affine. In particular, any morphism from a quasi-affine scheme to a quasi-separated scheme is quasi-affine.*

Proof. The base change $X \times_S Y \to Y$ is quasi-affine by Lemma 28.14.5. The morphism $X \to X \times_S Y$ is a quasi-compact immersion as $Y \to S$ is quasi-separated, see Schemes, Lemma 25.21.12. A quasi-compact immersion is quasi-affine by Lemma 28.14.6 and the composition of quasi-affine morphisms is quasi-affine (see Lemma 28.14.4). Thus we win. □

28.15. Types of morphisms defined by properties of ring maps

In this section we study what properties of ring maps allow one to define local properties of morphisms of schemes.

Definition 28.15.1. Let P be a property of ring maps.

(1) We say that P is *local* if the following hold:
   (a) For any ring map $R \to A$, and any $f \in R$ we have $P(R \to A) \Rightarrow P(R_f \to A_f)$.
   (b) For any rings R, A, any $f \in R$, $a \in A$, and any ring map $R_f \to A$ we have $P(R_f \to A) \Rightarrow P(R \to A_a)$.
   (c) For any ring map $R \to A$, and $a_i \in A$ such that $(a_1, \ldots, a_n) = A$ then $\forall i, P(R \to A_{a_i}) \Rightarrow P(R \to A)$.

(2) We say that P is *stable under base change* if for any ring maps $R \to A$, $R \to R'$ we have $P(R \to A) \Rightarrow P(R' \to R' \otimes_R A)$.
(3) We say that P is *stable under composition* if for any ring maps $A \to B$, $B \to C$ we have $P(A \to B) \wedge P(B \to C) \Rightarrow P(A \to C)$.

Definition 28.15.2. Let P be a property of ring maps. Let $f : X \to S$ be a morphisms of schemes. We say f is *locally of type* P if for any $x \in X$ there exists an affine open neighbourhood U of x in X which maps into an affine open $V \subset S$ such that the induced ring map $\mathcal{O}_S(V) \to \mathcal{O}_X(U)$ has property P.

This is not a "good" definition unless the property P is a local property. Even if P is a local property we will not automatically use this definition to say that a morphism is "locally of type P" unless we also explicitly state the definition elsewhere.

Lemma 28.15.3. *Let $f : X \to S$ be a morphism of schemes. Let P be a property of ring maps. Let U be an affine open of X, and V an affine open of S such that $f(U) \subset V$. If f is locally of type P and P is local, then $P(\mathcal{O}_S(V) \to \mathcal{O}_X(U))$ holds.*

Proof. As f is locally of type P for every $u \in U$ there exists an affine open $U_u \subset X$ mapping into an affine open $V_u \subset S$ such that $P(\mathcal{O}_S(V_u) \to \mathcal{O}_X(U_u))$ holds. Choose an open neighbourhood $U'_u \subset U \cap U_u$ of u which is standard affine open in both U and U_u, see Schemes, Lemma 25.11.5. By Definition 28.15.1 (1)(b) we see that $P(\mathcal{O}_S(V_u) \to \mathcal{O}_X(U'_u))$ holds. Hence we may assume that $U_u \subset U$ is a standard affine open. Choose an open neighbourhood $V'_u \subset V \cap V_u$ of $f(u)$ which is standard affine open in both V and V_u, see Schemes, Lemma 25.11.5. Then $U'_u = f^{-1}(V'_u) \cap U_u$ is a standard affine open of U_u (hence of U) and we have $P(\mathcal{O}_S(V'_u) \to \mathcal{O}_X(U'_u))$ by Definition 28.15.1 (1)(a). Hence we may assume both $U_u \subset U$ and $V_u \subset V$ are standard affine open. Applying Definition 28.15.1 (1)(b) one more time we conclude that $P(\mathcal{O}_S(V) \to \mathcal{O}_X(U_u))$ holds. Because U is quasi-compact we may choose a finite number of points $u_1, \ldots, u_n \in U$ such that

$$U = U_{u_1} \cup \ldots \cup U_{u_n}.$$

By Definition 28.15.1 (1)(c) we conclude that $P(\mathcal{O}_S(V) \to \mathcal{O}_X(U))$ holds. □

Lemma 28.15.4. *Let P be a local property of ring maps. Let $f : X \to S$ be a morphism of schemes. The following are equivalent*

(1) *The morphism f is locally of type P.*
(2) *For every affine opens $U \subset X$, $V \subset S$ with $f(U) \subset V$ we have $P(\mathcal{O}_S(V) \to \mathcal{O}_X(U))$.*
(3) *There exists an open covering $S = \bigcup_{j \in J} V_j$ and open coverings $f^{-1}(V_j) = \bigcup_{i \in I_j} U_i$ such that each of the morphisms $U_i \to V_j$, $j \in J, i \in I_j$ is locally of type P.*
(4) *There exists an affine open covering $S = \bigcup_{j \in J} V_j$ and affine open coverings $f^{-1}(V_j) = \bigcup_{i \in I_j} U_i$ such that $P(\mathcal{O}_S(V_j) \to \mathcal{O}_X(U_i))$ holds, for all $j \in J, i \in I_j$.*

Moreover, if f is locally of type P then for any open subschemes $U \subset X$, $V \subset S$ with $f(U) \subset V$ the restriction $f|_U : U \to V$ is locally of type P.

Proof. This follows from Lemma 28.15.3 above. □

Lemma 28.15.5. *Let P be a property of ring maps. Assume P is local and stable under composition. The composition of morphisms locally of type P is locally of type P.*

Proof. Let $f : X \to Y$ and $g : Y \to Z$ be morphisms locally of type P. Let $x \in X$. Choose an affine open neighbourhood $W \subset Z$ of $g(f(x))$. Choose an affine open neighbourhood $V \subset g^{-1}(W)$ of $f(x)$. Choose an affine open neighbourhood $U \subset f^{-1}(V)$ of x. By Lemma 28.15.4 the ring maps $\mathcal{O}_Z(W) \to \mathcal{O}_Y(V)$ and $\mathcal{O}_Y(V) \to \mathcal{O}_X(U)$ satisfy P. Hence $\mathcal{O}_Z(W) \to \mathcal{O}_X(U)$ satisfies P as P is assumed stable under composition. □

Lemma 28.15.6. *Let P be a property of ring maps. Assume P is local and stable under base change. The base change of a morphism locally of type P is locally of type P.*

Proof. Let $f : X \to S$ be a morphism locally of type P. Let $S' \to S$ be any morphism. Denote $f' : X_{S'} = S' \times_S X \to S'$ the base change of f. For every $s' \in S'$ there exists an open affine neighbourhood $s' \in V' \subset S'$ which maps into some open affine $V \subset S$. By Lemma 28.15.4 the open $f^{-1}(V)$ is a union of affines U_i such that the ring maps $\mathcal{O}_S(V) \to \mathcal{O}_X(U_i)$ all satisfy P. By the material in Schemes, Section 25.17 we see that $f^{-1}(U)_{V'} = V' \times_V f^{-1}(V)$ is the union of the affine opens $V' \times_V U_i$. Since $\mathcal{O}_{X_{S'}}(V' \times_V U_i) = \mathcal{O}_{S'}(V') \otimes_{\mathcal{O}_S(V)} \mathcal{O}_X(U_i)$ we see that the ring maps $\mathcal{O}_{S'}(V') \to \mathcal{O}_{X_{S'}}(V' \times_V U_i)$ satisfy P as P is assumed stable under base change. □

Lemma 28.15.7. *The following properties of a ring map $R \to A$ are local.*

1. *(Isomorphism on local rings.) For every prime $\mathfrak{q}$ of A lying over $\mathfrak{p} \subset R$ the ring map $R \to A$ induces an isomorphism $R_\mathfrak{p} \to A_\mathfrak{q}$.*
2. *(Open immersion.) For every prime $\mathfrak{q}$ of A there exists an $f \in R$, $\varphi(f) \notin \mathfrak{q}$ such that the ring map $\varphi : R \to A$ induces an isomorphism $R_f \to A_f$.*
3. *(Reduced fibres.) For every prime $\mathfrak{p}$ of R the fibre ring $A \otimes_R \kappa(\mathfrak{p})$ is reduced.*
4. *(Fibres of dimension at most n.) For every prime $\mathfrak{p}$ of R the fibre ring $A \otimes_R \kappa(\mathfrak{p})$ has Krull dimension at most n.*
5. *(Locally Noetherian on the target.) The ring map $R \to A$ has the property that A is Noetherian.*
6. *Add more here as needed[2].*

Proof. Omitted. □

Lemma 28.15.8. *The following properties of ring maps are stable under base change.*

1. *(Isomorphism on local rings.) For every prime $\mathfrak{q}$ of A lying over $\mathfrak{p} \subset R$ the ring map $R \to A$ induces an isomorphism $R_\mathfrak{p} \to A_\mathfrak{q}$.*
2. *(Open immersion.) For every prime $\mathfrak{q}$ of A there exists an $f \in R$, $\varphi(f) \notin \mathfrak{q}$ such that the ring map $\varphi : R \to A$ induces an isomorphism $R_f \to A_f$.*
3. *(Reduced fibres.) For every prime $\mathfrak{p}$ of R the fibre ring $A \otimes_R \kappa(\mathfrak{p})$ is reduced.*
4. *(Fibres of dimension at most n.) For every prime $\mathfrak{p}$ of R the fibre ring $A \otimes_R \kappa(\mathfrak{p})$ has Krull dimension at most n.*

[2]But only those properties that are not already dealt with separately elsewhere.

(5) *Add more here as needed*[3].

Proof. Omitted. □

Lemma 28.15.9. *The following properties of ring maps are stable under composition.*

(1) *(Isomorphism on local rings.) For every prime $\mathfrak{q}$ of A lying over $\mathfrak{p} \subset R$ the ring map $R \to A$ induces an isomorphism $R_\mathfrak{p} \to A_\mathfrak{q}$.*
(2) *(Open immersion.) For every prime $\mathfrak{q}$ of A there exists an $f \in R$, $\varphi(f) \notin \mathfrak{q}$ such that the ring map $\varphi : R \to A$ induces an isomorphism $R_f \to A_f$.*
(3) *(Locally Noetherian on the target.) The ring map $R \to A$ has the property that A is Noetherian.*
(4) *Add more here as needed*[4].

Proof. Omitted. □

28.16. Morphisms of finite type

Recall that a ring map $R \to A$ is said to be of finite type if A is isomorphic to a quotient of $R[x_1, \ldots, x_n]$ as an R-algebra, see Algebra, Definition 10.6.1.

Definition 28.16.1. Let $f : X \to S$ be a morphism of schemes.

(1) We say that f is of *finite type at* $x \in X$ if there exists an affine open neighbourhood $\mathrm{Spec}(A) = U \subset X$ of x and an affine open $\mathrm{Spec}(R) = V \subset S$ with $f(U) \subset V$ such that the induced ring map $R \to A$ is of finite type.
(2) We say that f is *locally of finite type* if it is of finite type at every point of X.
(3) We say that f is of *finite type* if it is locally of finite type and quasi-compact.

Lemma 28.16.2. *Let $f : X \to S$ be a morphism of schemes. The following are equivalent*

(1) *The morphism f is locally of finite type.*
(2) *For every affine opens $U \subset X$, $V \subset S$ with $f(U) \subset V$ the ring map $\mathcal{O}_S(V) \to \mathcal{O}_X(U)$ is of finite type.*
(3) *There exists an open covering $S = \bigcup_{j \in J} V_j$ and open coverings $f^{-1}(V_j) = \bigcup_{i \in I_j} U_i$ such that each of the morphisms $U_i \to V_j$, $j \in J, i \in I_j$ is locally of finite type.*
(4) *There exists an affine open covering $S = \bigcup_{j \in J} V_j$ and affine open coverings $f^{-1}(V_j) = \bigcup_{i \in I_j} U_i$ such that the ring map $\mathcal{O}_S(V_j) \to \mathcal{O}_X(U_i)$ is of finite type, for all $j \in J, i \in I_j$.*

Moreover, if f is locally of finite type then for any open subschemes $U \subset X$, $V \subset S$ with $f(U) \subset V$ the restriction $f|_U : U \to V$ is locally of finite type.

Proof. This follows from Lemma 28.15.3 if we show that the property "$R \to A$ is of finite type" is local. We check conditions (a), (b) and (c) of Definition 28.15.1. By Algebra, Lemma 10.13.2 being of finite type is stable under base change and hence we conclude (a) holds. By the same lemma being of finite type is stable under composition and trivially for any ring R the ring map $R \to R_f$ is of finite type.

[3] But only those properties that are not already dealt with separately elsewhere.
[4] But only those properties that are not already dealt with separately elsewhere.

We conclude (b) holds. Finally, property (c) is true according to Algebra, Lemma 10.22.3. □

Lemma 28.16.3. *The composition of two morphisms which locally of finite type is locally of finite type. The same is true for morphisms of finite type.*

Proof. In the proof of Lemma 28.16.2 we saw that being of finite type is a local property of ring maps. Hence the first statement of the lemma follows from Lemma 28.15.5 combined with the fact that being of finite type is a property of ring maps that is stable under composition, see Algebra, Lemma 10.6.2. By the above and the fact that compositions of quasi-compact morphisms are quasi-compact, see Schemes, Lemma 25.19.4 we see that the composition of morphisms of finite type is of finite type. □

Lemma 28.16.4. *The base change of a morphism which is locally of finite type is locally of finite type. The same is true for morphisms of finite type.*

Proof. In the proof of Lemma 28.16.2 we saw that being of finite type is a local property of ring maps. Hence the first statement of the lemma follows from Lemma 28.15.5 combined with the fact that being of finite type is a property of ring maps that is stable under base change, see Algebra, Lemma 10.13.2. By the above and the fact that a base change of a quasi-compact morphism is quasi-compact, see Schemes, Lemma 25.19.3 we see that the base change of a morphism of finite type is a morphism of finite type. □

Lemma 28.16.5. *A closed immersion is of finite type. An immersion is locally of finite type.*

Proof. This is true because an open immersion is a local isomorphism, and a closed immersion is obviously of finite type. □

Lemma 28.16.6. *Let $f : X \to S$ be a morphism. If S is (locally) Noetherian and f (locally) of finite type then X is (locally) Noetherian.*

Proof. This follows immediately from the fact that a ring of finite type over a Noetherian ring is Noetherian, see Algebra, Lemma 10.29.1. (Also: use the fact that the source of a quasi-compact morphism with quasi-compact target is quasi-compact.) □

Lemma 28.16.7. *Let $f : X \to S$ be locally of finite type with S locally Noetherian. Then f is quasi-separated.*

Proof. In fact, it is true that X is quasi-separated, see Properties, Lemma 27.5.4 and Lemma 28.16.6 above. Then apply Schemes, Lemma 25.21.14 to conclude that f is quasi-separated. □

Lemma 28.16.8. *Let $X \to Y$ be a morphism of schemes over a base scheme S. If X is locally of finite type over S, then $X \to Y$ is locally of finite type.*

Proof. Via Lemma 28.16.2 this translates into the following algebra fact: Given ring maps $A \to B \to C$ such that $A \to C$ is of finite type, then $B \to C$ is of finite type. (See Algebra, Lemma 10.6.2). □

28.17. Points of finite type and Jacobson schemes

Let S be a scheme. A finite type point s of S is a point such that the morphism $\mathrm{Spec}(\kappa(s)) \to S$ is of finite type. The reason for studying this is that finite type points can replace closed points in a certain sense and in certain situations. There are always enough of them for example. Moreover, a scheme is Jacobson if and only if all finite type points are closed points.

Lemma 28.17.1. *Let S be a scheme. Let k be a field. Let $f : \mathrm{Spec}(k) \to S$ be a morphism. The following are equivalent:*

1. *The morphism f is of finite type.*
2. *The morphism f is locally of finite type.*
3. *There exists an affine open $U = \mathrm{Spec}(R)$ of S such that f corresponds to a finite ring map $R \to k$.*
4. *There exists an affine open $U = \mathrm{Spec}(R)$ of S such that the image of f consists of a closed point u in U and the field extension $\kappa(u) \subset k$ is finite.*

Proof. The equivalence of (1) and (2) is obvious as $\mathrm{Spec}(k)$ is a singleton and hence any morphism from it is quasi-compact.

Suppose f is locally of finite type. Choose any affine open $\mathrm{Spec}(R) = U \subset S$ such that the image of f is contained in U, and the ring map $R \to k$ is of finite type. Let $\mathfrak{p} \subset R$ be the kernel. Then $R/\mathfrak{p} \subset k$ is of finite type. By Algebra, Lemma 10.31.2 there exist a $\overline{f} \in R/\mathfrak{p}$ such that $(R/\mathfrak{p})_{\overline{f}}$ is a field and $(R/\mathfrak{p})_{\overline{f}} \to k$ is a finite field extension. If $f \in R$ is a lift of $\overline{f}$, then we see that k is a finite R_f-module. Thus (2) $\Rightarrow$ (3).

Suppose that $\mathrm{Spec}(R) = U \subset S$ is an affine open such that f corresponds to a finite ring map $R \to k$. Then f is locally of finite type by Lemma 28.16.2. Thus (3) $\Rightarrow$ (2).

Suppose $R \to k$ is finite. The image of $R \to k$ is a field over which k is finite by Algebra, Lemma 10.33.16. Hence the kernel of $R \to k$ is a maximal ideal. Thus (3) $\Rightarrow$ (4).

The implication (4) $\Rightarrow$ (3) is immediate. □

Lemma 28.17.2. *Let S be a scheme. Let A be an Artinian local ring with residue field κ. Let $f : \mathrm{Spec}(A) \to S$ be a morphism of schemes. Then f is of finite type if and only if the composition $\mathrm{Spec}(\kappa) \to \mathrm{Spec}(A) \to S$ is of finite type.*

Proof. Since the morphism $\mathrm{Spec}(\kappa) \to \mathrm{Spec}(A)$ is of finite type it is clear that if f is of finite type so is the composition $\mathrm{Spec}(\kappa) \to S$ (see Lemma 28.16.3). For the converse, note that $\mathrm{Spec}(A) \to S$ maps into some affine open $U = \mathrm{Spec}(B)$ of S as $\mathrm{Spec}(A)$ has only one point. To finish apply Algebra, Lemma 10.50.3 to $B \to A$. □

Recall that given a point s of a scheme S there is a canonical morphism $\mathrm{Spec}(\kappa(s)) \to S$, see Schemes, Section 25.13.

Definition 28.17.3. Let S be a scheme. Let us say that a point s of S is a *finite type point* if the canonical morphism $\mathrm{Spec}(\kappa(s)) \to S$ is of finite type. We denote $S_{\text{ft-pts}}$ the set of finite type points of S.

We can describe the set of finite type points as follows.

Lemma 28.17.4. *Let S be a scheme. We have*

$$S_{ft\text{-}pts} = \bigcup\nolimits_{U \subset S \text{ open}} U_0$$

where U_0 is the set of closed points of U. Here we may let U range over all opens or over all affine opens of S.

Proof. Immediate from Lemma 28.17.1. □

Lemma 28.17.5. *Let $f : T \to S$ be a morphism of schemes. If f is locally of finite type, then $f(T_{ft\text{-}pts}) \subset S_{ft\text{-}pts}$.*

Proof. If T is the spectrum of a field this is Lemma 28.17.1. In general it follows since the composition of morphisms locally of finite type is locally of finite type (Lemma 28.16.3). □

Lemma 28.17.6. *Let $f : T \to S$ be a morphism of schemes. If f is locally of finite type and surjective, then $f(T_{ft\text{-}pts}) = S_{ft\text{-}pts}$.*

Proof. We have $f(T_{\text{ft-pts}}) \subset S_{\text{ft-pts}}$ by Lemma 28.17.5. Let $s \in S$ be a finite type point. As f is surjective the scheme $T_s = \text{Spec}(\kappa(s)) \times_S T$ is nonempty, therefore has a finite type point $t \in T_s$ by Lemma 28.17.4. Now $T_s \to T$ is a morphism of finite type as a base change of $s \to S$ (Lemma 28.16.4). Hence the image of t in T is a finite type point by Lemma 28.17.5 which maps to s by construction. □

Lemma 28.17.7. *Let S be a scheme. For any locally closed subset $T \subset S$ we have*

$$T \neq \emptyset \Rightarrow T \cap S_{ft\text{-}pts} \neq \emptyset.$$

In particular, for any closed subset $T \subset S$ we see that $T \cap S_{ft\text{-}pts}$ is dense in T.

Proof. Note that T carries a scheme structure (see Schemes, Lemma 25.12.4) such that $T \to S$ is a locally closed immersion. Any locally closed immersion is locally of finite type, see Lemma 28.16.5. Hence by Lemma 28.17.5 we see $T_{\text{ft-pts}} \subset S_{\text{ft-pts}}$. Finally, any nonempty affine open of T has at least one closed point which is a finite type point of T by Lemma 28.17.4. □

It follows that most of the material from Topology, Section 5.17 goes through with the set of closed points replaced by the set of points of finite type. In fact, if S is Jacobson then we recover the closed points as the finite type points.

Lemma 28.17.8. *Let S be a scheme. The following are equivalent:*

1. *For every finite type morphism $f : \text{Spec}(k) \to S$ with k a field the image consists of a closed point of S. In the terminology introduced above: finite type points of S are closed points of S.*
2. *For every locally finite type morphism $T \to S$ closed points map to closed points.*
3. *For every locally finite type morphism $f : T \to S$ any closed point $t \in T$ maps to a closed point $s \in S$ and $\kappa(s) \subset \kappa(t)$ is finite.*
4. *The scheme S is Jacobson.*

Proof. We have trivially (3) $\Rightarrow$ (2) $\Rightarrow$ (1). The discussion above shows that (1) implies (4). Hence it suffices to show that (4) implies (3). Suppose that $T \to S$ is locally of finite type. Choose $t \in T$ with $s = f(t)$ as in (3). Choose affine open neighbourhoods $\text{Spec}(R) = U \subset S$ of s and $\text{Spec}(A) = V \subset T$ of t with $f(V) \subset U$. The induced ring map $R \to A$ is of finite type (see Lemma 28.16.2) and

R is Jacobson by Properties, Lemma 27.6.3. Thus the result follows from Algebra, Proposition 10.32.18. □

Lemma 28.17.9. *Let S be a Jacobson scheme. Any scheme locally of finite type over S is Jacobson.*

Proof. This is clear from Algebra, Proposition 10.32.18 (and Properties, Lemma 27.6.3 and Lemma 28.16.2). □

Lemma 28.17.10. *The following types of schemes are Jacobson.*

(1) *Any scheme locally of finite type over a field.*
(2) *Any scheme locally of finite type over $\mathbf{Z}$.*
(3) *Any scheme locally of finite type over a 1-dimensional Noetherian domain with infinitely many primes.*
(4) *A scheme of the form $\mathrm{Spec}(R) \setminus \{\mathfrak{m}\}$ where $(R, \mathfrak{m})$ is a Noetherian local ring. Also any scheme locally of finite type over it.*

Proof. We will use Lemma 28.17.9 without mention. The spectrum of a field is clearly Jacobson. The spectrum of $\mathbf{Z}$ is Jacobson, see Algebra, Lemma 10.32.6. For (3) see Algebra, Lemma 10.58.2. For (4) see Properties, Lemma 27.6.4. □

28.18. Universally catenary schemes

Recall that a topological space X is called *catenary* if for every pair of irreducible closed subsets $T \subset T'$ there exist a maximal chain of irreducible closed subsets

$$T = T_0 \subset T_1 \subset \ldots \subset T_e = T'$$

and every such chain has the same length. See Topology, Definition 5.10.4. Recall that a scheme is catenary if its underlying topological space is catenary. See Properties, Definition 27.11.1.

Definition 28.18.1. Let S be a scheme. Assume S is locally Noetherian. We say S is *universally catenary* if for every morphism $X \to S$ locally of finite type the scheme X is catenary.

This is a "better" notion than catenary as there exist Noetherian schemes which are catenary but not universally catenary. See Examples, Section 78.11. Many schemes are universally catenary, see Lemma 28.18.4 below.

Recall that a ring A is called *catenary* if for any pair of prime ideals $\mathfrak{p} \subset \mathfrak{q}$ there exists a maximal chain of primes

$$\mathfrak{p} = \mathfrak{p}_0 \subset \ldots \subset \mathfrak{p}_e = \mathfrak{q}$$

and all of these have the same length. See Algebra, Definition 10.100.1. We have seen the relationship between catenary schemes and catenary rings in Properties, Section 27.11. Recall that a ring A is called *universally catenary* if A is Noetherian and for every finite type ring map $A \to B$ the ring B is catenary. See Algebra, Definition 10.100.5. Many interesting rings which come up in algebraic geometry satisfy this property.

Lemma 28.18.2. *Let S be a locally Noetherian scheme. The following are equivalent*

(1) *S is universally catenary,*

(2) *there exists an open covering of S all of whose members are universally catenary schemes,*
(3) *for every affine open $\mathrm{Spec}(R) = U \subset S$ the ring R is universally catenary, and*
(4) *there exists an affine open covering $S = \bigcup U_i$ such that each U_i is the spectrum of a universally catenary ring.*

Moreover, in this case any scheme locally of finite type over S is universally catenary as well.

Proof. By Lemma 28.16.5 an open immersion is locally of finite type. A composition of morphisms locally of finite type is locally of finite type (Lemma 28.16.3). Thus it is clear that if S is universally catenary then any open and any scheme locally of finite type over S is universally catenary as well. This proves the final statement of the lemma and that (1) implies (2).

If $\mathrm{Spec}(R)$ is a universally catenary scheme, then every scheme $\mathrm{Spec}(A)$ with A a finite type R-algebra is catenary. Hence all these rings A are catenary by Algebra, Lemma 10.100.2. Thus R is universally catenary. Combined with the remarks above we conclude that (1) implies (3), and (2) implies (4). Of course (3) implies (4) trivially.

To finish the proof we show that (4) implies (1). Assume (4) and let $X \to S$ be a morphism locally of finite type. We can find an affine open covering $X = \bigcup V_j$ such that each $V_j \to S$ maps into one of the U_i. By Lemma 28.16.2 the induced ring map $\mathcal{O}(U_i) \to \mathcal{O}(V_j)$ is of finite type. Hence $\mathcal{O}(V_j)$ is catenary. Hence X is catenary by Properties, Lemma 27.11.2. □

Lemma 28.18.3. *Let S be a locally Noetherian scheme. The following are equivalent:*

(1) *S is universally catenary, and*
(2) *all local rings $\mathcal{O}_{S,s}$ of S are universally catenary.*

Proof. Assume that all local rings of S are universally catenary. Let $f : X \to S$ be locally of finite type. We know that X is catenary if and only if $\mathcal{O}_{X,x}$ is catenary for all $x \in X$. If $f(x) = s$, then $\mathcal{O}_{X,x}$ is essentially of finite type over $\mathcal{O}_{S,s}$. Hence $\mathcal{O}_{X,x}$ is catenary by the assumption that $\mathcal{O}_{S,s}$ is universally catenary.

Conversely, assume that S is universally catenary. Let $s \in S$. We may replace S by an affine open neighbourhood of s by Lemma 28.18.2. Say $S = \mathrm{Spec}(R)$ and s corresponds to the prime ideal $\mathfrak{p}$. Any finite type $R_\mathfrak{p}$-algebra A' is of the form $A_\mathfrak{p}$ for some finite type R-algebra A. By assumption (and Lemma 28.18.2 if you like) the ring A is catenary, and hence A' (a localization of A) is catenary. Thus $R_\mathfrak{p}$ is universally catenary. □

Lemma 28.18.4. *The following types of schemes are universally catenary.*

(1) *Any scheme locally of finite type over a field.*
(2) *Any scheme locally of finite type over a Cohen-Macaulay scheme.*
(3) *Any scheme locally of finite type over $\mathbf{Z}$.*
(4) *Any scheme locally of finite type over a 1-dimensional Noetherian domain.*
(5) *And so on.*

Proof. All of these follow from the fact that a Cohen-Macaulay ring is universally catenary, see Algebra, Lemma 10.100.6. Also, use the last assertion of Lemma 28.18.2. Some details omitted. □

28.19. Nagata schemes, reprise

See Properties, Section 27.13 for the definitions and basic properties of Nagata and universally Japanese schemes.

Lemma 28.19.1. *Let $f : X \to S$ be a morphism. If S is Nagata and f locally of finite type then X is Nagata. If S is universally Japanese and f locally of finite type then X is universally Japanese.*

Proof. For "universally Japanese" this follows from Algebra, Lemma 10.148.18. For "Nagata" this follows from Algebra, Proposition 10.148.30. □

Lemma 28.19.2. *The following types of schemes are Nagata.*

(1) *Any scheme locally of finite type over a field.*
(2) *Any scheme locally of finite type over a Noetherian complete local ring.*
(3) *Any scheme locally of finite type over $\mathbf{Z}$.*
(4) *Any scheme locally of finite type over a Dedekind ring of characteristic zero.*
(5) *And so on.*

Proof. By Lemma 28.19.1 we only need to show that the rings mentioned above are Nagata rings. For this see Algebra, Proposition 10.148.31. □

28.20. The singular locus, reprise

We look for a criterion that implies openness of the regular locus for any scheme locally of finite type over the base. Here is the definition.

Definition 28.20.1. Let X be a locally Noetherian scheme. We say X is *J-2* if for every morphism $Y \to X$ which is locally of finite type the regular locus $\mathrm{Reg}(Y)$ is open in Y.

This is the analogue of the corresponding notion for Noetherian rings, see More on Algebra, Definition 15.34.1.

Lemma 28.20.2. *Let X be a locally Noetherian scheme. The following are equivalent*

(1) *X is J-2,*
(2) *there exists an open covering of X all of whose members are J-2 schemes,*
(3) *for every affine open $\mathrm{Spec}(R) = U \subset X$ the ring R is J-2, and*
(4) *there exists an affine open covering $S = \bigcup U_i$ such that each $\mathcal{O}(U_i)$ is J-2 for all i.*

Moreover, in this case any scheme locally of finite type over X is J-2 as well.

Proof. By Lemma 28.16.5 an open immersion is locally of finite type. A composition of morphisms locally of finite type is locally of finite type (Lemma 28.16.3). Thus it is clear that if X is J-2 then any open and any scheme locally of finite type over X is J-2 as well. This proves the final statement of the lemma.

If $\mathrm{Spec}(R)$ is J-2, then for every finite type R-algebra A the regular locus of the scheme $\mathrm{Spec}(A)$ is open. Hence R is J-2, by definition (see More on Algebra,

Definition 15.34.1). Combined with the remarks above we conclude that (1) implies (3), and (2) implies (4). Of course (1) $\Rightarrow$ (2) and (3) $\Rightarrow$ (4) trivially.

To finish the proof we show that (4) implies (1). Assume (4) and let $Y \to X$ be a morphism locally of finite type. We can find an affine open covering $Y = \bigcup V_j$ such that each $V_j \to X$ maps into one of the U_i. By Lemma 28.16.2 the induced ring map $\mathcal{O}(U_i) \to \mathcal{O}(V_j)$ is of finite type. Hence the regular locus of $V_j = \mathrm{Spec}(\mathcal{O}(V_j))$ is open. Since $\mathrm{Reg}(Y) \cap V_j = \mathrm{Reg}(V_j)$ we conclude that $\mathrm{Reg}(Y)$ is open as desired. □

Lemma 28.20.3. *The following types of schemes are J-2.*

(1) *Any scheme locally of finite type over a field.*
(2) *Any scheme locally of finite type over a Noetherian complete local ring.*
(3) *Any scheme locally of finite type over* $\mathbf{Z}$.
(4) *Any scheme locally of finite type over a Dedekind ring of characteristic zero.*
(5) *And so on.*

Proof. By Lemma 28.20.2 we only need to show that the rings mentioned above are J-2. For this see More on Algebra, Proposition 15.35.6. □

28.21. Quasi-finite morphisms

A solid treatment of quasi-finite morphisms is the basis of many developments further down the road. It will lead to various versions of Zariski's Main Theorem, behaviour of dimensions of fibres, descent for étale morphisms, etc, etc. Before reading this section it may be a good idea to take a look at the algebra results in Algebra, Section 10.116.

Recall that a finite type ring map $R \to A$ is quasi-finite at a prime $\mathfrak{q}$ if $\mathfrak{q}$ defines an isolated point of its fibre, see Algebra, Definition 10.116.3.

Definition 28.21.1. Let $f : X \to S$ be a morphism of schemes.

(1) We say that f is *quasi-finite at a point* $x \in X$ if there exist an affine neighbourhood $\mathrm{Spec}(A) = U \subset X$ of x and an affine open $\mathrm{Spec}(R) = V \subset S$ such that $f(U) \subset V$, the ring map $R \to A$ is of finite type, and $R \to A$ is quasi-finite at the prime of A corresponding to x (see above).
(2) We say f is *locally quasi-finite* if f is quasi-finite at every point x of X.
(3) We say that f is *quasi-finite* if f is of finite type and every point x is an isolated point of its fibre.

Trivially, a locally quasi-finite morphism is locally of finite type. We will see below that a morphism f which is locally of finite type is quasi-finite at x if and only if x is isolated in its fibre. Moreover, the set of points at which a morphism is quasi-finite is open; we will see this in Section 28.49 on Zariski's Main Theorem.

Lemma 28.21.2. *Let $f : X \to S$ be a morphism of schemes. Let $x \in X$ be a point. Set $s = f(x)$. If $\kappa(s) \supset \kappa(x)$ is an algebraic field extension, then*

(1) *x is a closed point of its fibre, and*
(2) *if in addition s is a closed point of S, then x is a closed point of X.*

Proof. The second statement follows from the first by elementary topology. According to Schemes, Lemma 25.18.5 to prove the first statement we may replace X by X_s and S by $\mathrm{Spec}(\kappa(s))$. Thus we may assume that $S = \mathrm{Spec}(k)$ is the spectrum

of a field. In this case, let $\mathrm{Spec}(A) = U \subset X$ be any affine open containing x. The point x corresponds to a prime ideal $\mathfrak{q} \subset A$ such that $k \subset \kappa(\mathfrak{q})$ is an algebraic field extension. By Algebra, Lemma 10.32.9 we see that $\mathfrak{q}$ is a maximal ideal, i.e., $x \in U$ is a closed point. Since the affine opens form a basis of the topology of X we conclude that $\{x\}$ is closed. □

The following lemma is a version of the Hilbert Nullstellensatz.

Lemma 28.21.3. *Let $f : X \to S$ be a morphism of schemes. Let $x \in X$ be a point. Set $s = f(x)$. Assume f is locally of finite type. Then x is a closed point of its fibre if and only if $\kappa(s) \subset \kappa(x)$ is a finite field extension.*

Proof. If the extension is finite, then x is a closed point of the fibre by Lemma 28.21.2 above. For the converse, assume that x is a closed point of its fibre. Choose affine opens $\mathrm{Spec}(A) = U \subset X$ and $\mathrm{Spec}(R) = V \subset S$ such that $f(U) \subset V$. By Lemma 28.16.2 the ring map $R \to A$ is of finite type. Let $\mathfrak{q} \subset A$, resp. $\mathfrak{p} \subset R$ be the prime ideal corresponding to x, resp. s. Consider the fibre ring $\overline{A} = A \otimes_R \kappa(\mathfrak{p})$. Let $\overline{\mathfrak{q}}$ be the prime of $\overline{A}$ corresponding to $\mathfrak{q}$. The assumption that x is a closed point of its fibre implies that $\overline{\mathfrak{q}}$ is a maximal ideal of $\overline{A}$. Since $\overline{A}$ is an algebra of finite type over the field $\kappa(\mathfrak{p})$ we see by the Hilbert Nullstellensatz, see Algebra, Theorem 10.31.1, that $\kappa(\overline{\mathfrak{q}})$ is a finite extension of $\kappa(\mathfrak{p})$. Since $\kappa(s) = \kappa(\mathfrak{p})$ and $\kappa(x) = \kappa(\mathfrak{q}) = \kappa(\overline{\mathfrak{q}})$ we win. □

Lemma 28.21.4. *Let $f : X \to S$ be a morphism of schemes which is locally of finite type. Let $g : S' \to S$ be any morphism. Denote $f' : X' \to S'$ the base change. If $x' \in X'$ maps to a point $x \in X$ which is closed in $X_{f(s)}$ then x' is closed in $X'_{f'(x')}$.*

Proof. The residue field $\kappa(x')$ is a quotient of $\kappa(f'(x')) \otimes_{\kappa(f(x))} \kappa(x)$, see Schemes, Lemma 25.17.5. Hence it is a finite extension of $\kappa(f'(x'))$ as $\kappa(x)$ is a finite extension of $\kappa(f(x))$ by Lemma 28.21.3. Thus we see that x' is closed in its fibre by applying that lemma one more time. □

Lemma 28.21.5. *Let $f : X \to S$ be a morphism of schemes. Let $x \in X$ be a point. Set $s = f(x)$. If f is quasi-finite at x, then the residue field extension $\kappa(s) \subset \kappa(x)$ is finite.*

Proof. This is clear from Algebra, Definition 10.116.3. □

Lemma 28.21.6. *Let $f : X \to S$ be a morphism of schemes. Let $x \in X$ be a point. Set $s = f(x)$. Let X_s be the fibre of f at s. Assume f is locally of finite type. The following are equivalent:*

1. *The morphism f is quasi-finite at x.*
2. *The point x is isolated in X_s.*
3. *The point x is closed in X_s and there is no point $x' \in X_s$, $x' \neq x$ which specializes to x.*
4. *For any pair of affine opens $\mathrm{Spec}(A) = U \subset X$, $\mathrm{Spec}(R) = V \subset S$ with $f(U) \subset V$ and $x \in U$ corresponding to $\mathfrak{q} \subset A$ the ring map $R \to A$ is quasi-finite at $\mathfrak{q}$.*

Proof. Assume f is quasi-finite at x. By assumption there exist opens $U \subset X$, $V \subset S$ such that $f(U) \subset V$, $x \in U$ and x an isolated point of U_s. Hence $\{x\} \subset U_s$

is an open subset. Since $U_s = U \cap X_s \subset X_s$ is also open we conclude that $\{x\} \subset X_s$ is an open subset also. Thus we conclude that x is an isolated point of X_s.

Note that X_s is a Jacobson scheme by Lemma 28.17.10 (and Lemma 28.16.4). If x is isolated in X_s, i.e., $\{x\} \subset X_s$ is open, then $\{x\}$ contains a closed point (by the Jacobson property), hence x is closed in X_s. It is clear that there is no point $x' \in X_s$, distinct from x, specializing to x.

Assume that x is closed in X_s and that there is no point $x' \in X_s$, distinct from x, specializing to x. Consider a pair of affine opens $\mathrm{Spec}(A) = U \subset X$, $\mathrm{Spec}(R) = V \subset S$ with $f(U) \subset V$ and $x \in U$. Let $\mathfrak{q} \subset A$ correspond to x and $\mathfrak{p} \subset R$ correspond to s. By Lemma 28.16.2 the ring map $R \to A$ is of finite type. Consider the fibre ring $\overline{A} = A \otimes_R \kappa(\mathfrak{p})$. Let $\overline{\mathfrak{q}}$ be the prime of $\overline{A}$ corresponding to $\mathfrak{q}$. Since $\mathrm{Spec}(\overline{A})$ is an open subscheme of the fibre X_s we see that $\overline{q}$ is a maximal ideal of $\overline{A}$ and that there is no point of $\mathrm{Spec}(\overline{A})$ specializing to $\overline{\mathfrak{q}}$. This implies that $\dim(\overline{A}_{\overline{q}}) = 0$. Hence by Algebra, Definition 10.116.3 we see that $R \to A$ is quasi-finite at $\mathfrak{q}$, i.e., $X \to S$ is quasi-finite at x by definition.

At this point we have shown conditions (1) – (3) are all equivalent. It is clear that (4) implies (1). And it is also clear that (2) implies (4) since if x is an isolated point of X_s then it is also an isolated point of U_s for any open U which contains it. □

Lemma 28.21.7. *Let $f : X \to S$ be a morphism of schemes. Let $s \in S$. Assume that*

(1) *f is locally of finite type, and*
(2) *$f^{-1}(\{s\})$ is a finite set.*

Then X_s is a finite discrete topological space, and f is quasi-finite at each point of X lying over s.

Proof. Suppose T is a scheme which (a) is locally of finite type over a field k, and (b) has finitely many points. Then Lemma 28.17.10 shows T is a Jacobson scheme. A finite sober Jacobson space is discrete, see Topology, Lemma 5.17.6. Apply this remark to the fibre X_s which is locally of finite type over $\mathrm{Spec}(\kappa(s))$ to see the first statement. Finally, apply Lemma 28.21.6 to see the second. □

Lemma 28.21.8. *Let $f : X \to S$ be a morphism of schemes. Assume f is locally of finite type. Then the following are equivalent*

(1) *f is locally quasi-finite,*
(2) *for every $s \in S$ the fibre X_s is a discrete topological space, and*
(3) *for every morphism $\mathrm{Spec}(k) \to S$ where k is a field the base change X_k has an underlying discrete topological space.*

Proof. It is immediate that (3) implies (2). Lemma 28.21.6 shows that (2) is equivalent to (1). Assume (2) and let $\mathrm{Spec}(k) \to S$ be as in (3). Denote $s \in S$ the image of $\mathrm{Spec}(k) \to S$. Then X_k is the base change of X_s via $\mathrm{Spec}(k) \to \mathrm{Spec}(\kappa(s))$. Hence every point of X_k is closed by Lemma 28.21.4. As $X_k \to \mathrm{Spec}(k)$ is locally of finite type (by Lemma 28.16.4), we may apply Lemma 28.21.6 to conclude that every point of X_k is isolated, i.e., X_k has a discrete underlying topological space. □

Lemma 28.21.9. *Let $f : X \to S$ be a morphism of schemes. Then f is quasi-finite if and only if f is locally quasi-finite and quasi-compact.*

Proof. Assume f is quasi-finite. It is quasi-compact by Definition 28.16.1. Let $x \in X$. We see that f is quasi-finite at x by Lemma 28.21.6. Hence f is quasi-compact and locally quasi-finite.

Assume f is quasi-compact and locally quasi-finite. Then f is of finite type. Let $x \in X$ be a point. By Lemma 28.21.6 we see that x is an isolated point of its fibre. The lemma is proved. □

Lemma 28.21.10. *Let $f : X \to S$ be a morphism of schemes. The following are equivalent:*

(1) *f is quasi-finite, and*
(2) *f is locally of finite type, quasi-compact, and has finite fibres.*

Proof. Assume f is quasi-finite. In particular f is locally of finite type and quasi-compact (since it is of finite type). Let $s \in S$. Since every $x \in X_s$ is isolated in X_s we see that $X_s = \bigcup_{x \in X_s} \{x\}$ is an open covering. As f is quasi-compact, the fibre X_s is quasi-compact. Hence we see that X_s is finite.

Conversely, assume f is locally of finite type, quasi-compact and has finite fibres. Then it is locally quasi-finite by Lemma 28.21.7. Hence it is quasi-finite by Lemma 28.21.9. □

Recall that a ring map $R \to A$ is quasi-finite if it is of finite type and quasi-finite at *all* primes of A, see Algebra, Definition 10.116.3.

Lemma 28.21.11. *Let $f : X \to S$ be a morphism of schemes. The following are equivalent*

(1) *The morphism f is locally quasi-finite.*
(2) *For every pair of affine opens $U \subset X$, $V \subset S$ with $f(U) \subset V$ the ring map $\mathcal{O}_S(V) \to \mathcal{O}_X(U)$ is quasi-finite.*
(3) *There exists an open covering $S = \bigcup_{j \in J} V_j$ and open coverings $f^{-1}(V_j) = \bigcup_{i \in I_j} U_i$ such that each of the morphisms $U_i \to V_j$, $j \in J, i \in I_j$ is locally quasi-finite.*
(4) *There exists an affine open covering $S = \bigcup_{j \in J} V_j$ and affine open coverings $f^{-1}(V_j) = \bigcup_{i \in I_j} U_i$ such that the ring map $\mathcal{O}_S(V_j) \to \mathcal{O}_X(U_i)$ is quasi-finite, for all $j \in J, i \in I_j$.*

Moreover, if f is locally quasi-finite then for any open subschemes $U \subset X$, $V \subset S$ with $f(U) \subset V$ the restriction $f|_U : U \to V$ is locally quasi-finite.

Proof. For a ring map $R \to A$ let us define $P(R \to A)$ to mean "$R \to A$ is quasi-finite" (see remark above lemma). We claim that P is a local property of ring maps. We check conditions (a), (b) and (c) of Definition 28.15.1. In the proof of Lemma 28.16.2 we have seen that (a), (b) and (c) hold for the property of being "of finite type". Note that, for a finite type ring map $R \to A$, the property $R \to A$ is quasi-finite at $\mathfrak{q}$ depends only on the local ring $A_\mathfrak{q}$ as an algebra over $R_\mathfrak{p}$ where $\mathfrak{p} = R \cap \mathfrak{q}$ (usual abuse of notation). Using these remarks (a), (b) and (c) of Definition 28.15.1 follow immediately. For example, suppose $R \to A$ is a ring map such that all of the ring maps $R \to A_{a_i}$ are quasi-finite for $a_1, \ldots, a_n \in A$ generating the unit ideal. We conclude that $R \to A$ is of finite type. Also, for any prime $\mathfrak{q} \subset A$ the local ring $A_\mathfrak{q}$ is isomorphic as an R-algebra to the local ring $(A_{a_i})_{\mathfrak{q}_i}$ for some i and some $\mathfrak{q}_i \subset A_{a_i}$. Hence we conclude that $R \to A$ is quasi-finite at $\mathfrak{q}$.

We conclude that Lemma 28.15.3 applies with P as in the previous paragraph. Hence it suffices to prove that f is locally quasi-finite is equivalent to f is locally of type P. Since $P(R \to A)$ is "$R \to A$ is quasi-finite" which means $R \to A$ is quasi-finite at every prime of A, this follows from Lemma 28.21.6. □

Lemma 28.21.12. *The composition of two morphisms which are locally quasi-finite is locally quasi-finite. The same is true for quasi-finite morphisms.*

Proof. In the proof of Lemma 28.21.11 we saw that $P =$"quasi-finite" is a local property of ring maps, and that a morphism of schemes is locally quasi-finite if and only if it is locally of type P as in Definition 28.15.2. Hence the first statement of the lemma follows from Lemma 28.15.5 combined with the fact that being quasi-finite is a property of ring maps that is stable under composition, see Algebra, Lemma 10.116.7. By the above, Lemma 28.21.9 and the fact that compositions of quasi-compact morphisms are quasi-compact, see Schemes, Lemma 25.19.4 we see that the composition of quasi-finite morphisms is quasi-finite. □

We will see later (Lemma 28.49.2) that the set U of the following lemma is open.

Lemma 28.21.13. *Let $f : X \to S$ be a morphism of schemes. Let $g : S' \to S$ be a morphism of schemes. Denote $f' : X_{S'} \to S'$ the base change of f by g and denote $g' : X_{S'} \to X$ the projection. Assume X is locally of finite type over S.*

1. *Let $U \subset X$ (resp. $U' \subset X'$) be the set of points where f (resp. f') is quasi-finite. Then $U' = U_{S'} = (g')^{-1}(U)$.*
2. *The base change of a locally quasi-finite morphism is locally quasi-finite.*
3. *The base change of a quasi-finite morphism is quasi-finite.*

Proof. The first and second assertion follow from the corresponding algebra result, see Algebra, Lemma 10.116.8 (combined with the fact that f' is also locally of finite type by Lemma 28.16.4). By the above, Lemma 28.21.9 and the fact that a base change of a quasi-compact morphism is quasi-compact, see Schemes, Lemma 25.19.3 we see that the base change of a quasi-finite morphism is quasi-finite. □

Lemma 28.21.14. *Any immersion is locally quasi-finite.*

Proof. This is true because an open immersion is a local isomorphism and a closed immersion is clearly quasi-finite. □

Lemma 28.21.15. *Let $X \to Y$ be a morphism of schemes over a base scheme S. Let $x \in X$. If $X \to S$ is quasi-finite at x, then $X \to Y$ is quasi-finite at x. If X is locally quasi-finite over S, then $X \to Y$ is locally quasi-finite.*

Proof. Via Lemma 28.21.11 this translates into the following algebra fact: Given ring maps $A \to B \to C$ such that $A \to C$ is quasi-finite, then $B \to C$ is quasi-finite. This follows from Algebra, Lemma 10.116.6 with $R = A$, $S = S' = C$ and $R' = B$. □

28.22. Morphisms of finite presentation

Recall that a ring map $R \to A$ is of finite presentation if A is isomorphic to $R[x_1, \ldots, x_n]/(f_1, \ldots, f_m)$ as an R-algebra for some n, m and some polynomials f_j, see Algebra, Definition 10.6.1.

Definition 28.22.1. Let $f : X \to S$ be a morphism of schemes.

(1) We say that f is of *finite presentation at* $x \in X$ if there exists a affine open neighbourhood $\mathrm{Spec}(A) = U \subset X$ of x and affine open $\mathrm{Spec}(R) = V \subset S$ with $f(U) \subset V$ such that the induced ring map $R \to A$ is of finite presentation.
(2) We say that f is *locally of finite presentation* if it is of finite presentation at every point of X.
(3) We say that f is of *finite presentation* if it is locally of finite presentation, quasi-compact and quasi-separated.

Note that a morphism of finite presentation is **not** just a quasi-compact morphism which is locally of finite presentation. Later we will characterize morphisms which are locally of finite presentation as those morphisms such that

$$\operatorname{colim} \mathrm{Mor}_S(T_i, X) = \mathrm{Mor}_S(\lim T_i, X)$$

for any directed system of affine schemes T_i over S. See Limits, Proposition 31.5.1. In Limits, Section 31.9 we show that, if $S = \lim_i S_i$ is a limit of affine schemes, any scheme X of finite presentation over S descends to a scheme X_i over S_i for some i.

Lemma 28.22.2. *Let $f : X \to S$ be a morphism of schemes. The following are equivalent*

(1) *The morphism f is locally of finite presentation.*
(2) *For every affine opens $U \subset X$, $V \subset S$ with $f(U) \subset V$ the ring map $\mathcal{O}_S(V) \to \mathcal{O}_X(U)$ is of finite presentation.*
(3) *There exists an open covering $S = \bigcup_{j \in J} V_j$ and open coverings $f^{-1}(V_j) = \bigcup_{i \in I_j} U_i$ such that each of the morphisms $U_i \to V_j$, $j \in J, i \in I_j$ is locally of finite presentation.*
(4) *There exists an affine open covering $S = \bigcup_{j \in J} V_j$ and affine open coverings $f^{-1}(V_j) = \bigcup_{i \in I_j} U_i$ such that the ring map $\mathcal{O}_S(V_j) \to \mathcal{O}_X(U_i)$ is of finite presentation, for all $j \in J, i \in I_j$.*

Moreover, if f is locally of finite presentation then for any open subschemes $U \subset X$, $V \subset S$ with $f(U) \subset V$ the restriction $f|_U : U \to V$ is locally of finite presentation.

Proof. This follows from Lemma 28.15.3 if we show that the property "$R \to A$ is of finite presentation" is local. We check conditions (a), (b) and (c) of Definition 28.15.1. By Algebra, Lemma 10.13.2 being of finite presentation is stable under base change and hence we conclude (a) holds. By the same lemma being of finite presentation is stable under composition and trivially for any ring R the ring map $R \to R_f$ is of finite presentation. We conclude (b) holds. Finally, property (c) is true according to Algebra, Lemma 10.22.3. □

Lemma 28.22.3. *The composition of two morphisms which locally of finite presentation is locally of finite presentation. The same is true for morphisms of finite presentation.*

Proof. In the proof of Lemma 28.22.2 we saw that being of finite presentation is a local property of ring maps. Hence the first statement of the lemma follows from Lemma 28.15.5 combined with the fact that being of finite presentation is a property of ring maps that is stable under composition, see Algebra, Lemma 10.6.2. By the above and the fact that compositions of quasi-compact, quasi-separated morphisms are quasi-compact and quasi-separated, see Schemes, Lemmas 25.19.4

and 25.21.13 we see that the composition of morphisms of finite presentation is of finite presentation. □

Lemma 28.22.4. *The base change of a morphism which is locally of finite presentation is locally of finite presentation. The same is true for morphisms of finite presentation.*

Proof. In the proof of Lemma 28.22.2 we saw that being of finite presentation is a local property of ring maps. Hence the first statement of the lemma follows from Lemma 28.15.5 combined with the fact that being of finite presentation is a property of ring maps that is stable under base change, see Algebra, Lemma 10.13.2. By the above and the fact that a base change of a quasi-compact, quasi-separated morphism is quasi-compact and quasi-separated, see Schemes, Lemmas 25.19.3 and 25.21.13 we see that the base change of a morphism of finite presentation is a morphism of finite presentation. □

Lemma 28.22.5. *Any open immersion is locally of finite presentation.*

Proof. This is true because an open immersion is a local isomorphism. □

Lemma 28.22.6. *Any open immersion is of finite presentation if and only if it is quasi-compact.*

Proof. We have seen (Lemma 28.22.5) that an open immersion is locally of finite presentation. We have see (Schemes, Lemma 25.23.7) that an immersion is separated and hence quasi-separated. From this and Definition 28.22.1 the lemma follows. □

Lemma 28.22.7. *A closed immersion $i : Z \to X$ is of finite presentation if and only if the associated quasi-coherent sheaf of ideals $\mathcal{I} = Ker(\mathcal{O}_X \to i_*\mathcal{O}_Z)$ is of finite type (as an $\mathcal{O}_X$-module).*

Proof. On any affine open $\mathrm{Spec}(R) \subset X$ we have $i^{-1}(\mathrm{Spec}(R)) = \mathrm{Spec}(R/I)$ and $\mathcal{I} = \widetilde{I}$. Moreover, $\mathcal{I}$ is of finite type if and only if I is a finite R-module for every such affine open (see Properties, Lemma 27.16.1). And R/I is of finite presentation over R if and only if I is a finite R-module. Hence we win. □

Lemma 28.22.8. *A morphism which is locally of finite presentation is locally of finite type. A morphism of finite presentation is of finite type.*

Proof. Omitted. □

Lemma 28.22.9. *Let $f : X \to S$ be a morphism.*

(1) *If S is locally Noetherian and f locally of finite type then f is locally of finite presentation.*
(2) *If S is locally Noetherian and f of finite type then f is of finite presentation.*

Proof. The first statement follows from the fact that a ring of finite type over a Noetherian ring is of finite presentation, see Algebra, Lemma 10.29.4. Suppose that f is of finite type and S is locally Noetherian. Then f is quasi-compact and locally of finite presentation by (1). Hence it suffices to prove that f is quasi-separated. This follows from Lemma 28.16.7 (and Lemma 28.22.8). □

Lemma 28.22.10. *Let S be a scheme which is quasi-compact and quasi-separated. If X is of finite presentation over S, then X is quasi-compact and quasi-separated.*

Proof. Omitted. □

Lemma 28.22.11. *Let $f : X \to Y$ be a morphism of schemes over S.*

(1) *If X is locally of finite presentation over S and Y is locally of finite type over S, then f is locally of finite presentation.*
(2) *If X is of finite presentation over S and Y is quasi-separated and locally of finite type over S, then f is of finite presentation.*

Proof. Proof of (1). Via Lemma 28.22.2 this translates into the following algebra fact: Given ring maps $A \to B \to C$ such that $A \to C$ is of finite presentation and $A \to B$ is of finite type, then $B \to C$ is of finite type. See Algebra, Lemma 10.6.2.

Part (2) follows from (1) and Schemes, Lemmas 25.21.14 and 25.21.15. □

Lemma 28.22.12. *Let $f : X \to Y$ be a morphism of schemes with diagonal $\Delta : X \to X \times_Y X$. If f is locally of finite type then Δ is locally of finite presentation. If f is quasi-separated and locally of finite type, then Δ is of finite presentation.*

Proof. Note that Δ is a morphism of schemes over X (via the second projection $X \times_Y X \to X$). Assume f is locally of finite type. Note that X is of finite presentation over X and $X \times_Y X$ is of finite type over X (by Lemma 28.16.4). Thus the first statement holds by Lemma 28.22.11. The second statement follows from the first, the definitions, and the fact that a diagonal morphism is a monomorphism, hence separated (Schemes, Lemma 25.23.3). □

28.23. Constructible sets

Constructible and locally constructible sets of schemes have been discussed in Properties, Section 27.2. In this section we prove some results concerning images and inverse images of (locally) constructible sets. The main result is Chevalley's theorem which states that the image of a locally constructible set under a morphism of finite presentation is locally constructible.

Lemma 28.23.1. *Let $f : X \to Y$ be a morphism of schemes. Let $E \subset Y$ be a subset. If E is (locally) constructible in Y, then $f^{-1}(E)$ is (locally) constructible in X.*

Proof. To show that the inverse image of every constructible subset is constructible it suffices to show that the inverse image of every retrocompact open V of Y is retrocompact in X, see Topology, Lemma 5.14.3. The significance of V being retrocompact in Y is just that the open immersion $V \to Y$ is quasi-compact. Hence the base change $f^{-1}(V) = X \times_Y V \to X$ is quasi-compact too, see Schemes, Lemma 25.19.3. Hence we see $f^{-1}(V)$ is retrocompact in X. Suppose E is locally constructible in Y. Choose $x \in X$. Choose an affine neighbourhood V of $f(x)$ and an affine neighbourhood $U \subset X$ of x such that $f(U) \subset V$. Thus we think of $f|_U : U \to V$ as a morphism into V. By Properties, Lemma 27.2.1 we see that $E \cap V$ is constructible in V. By the constructible case we see that $(f|_U)^{-1}(E \cap V)$ is constructible in U. Since $(f|_U)^{-1}(E \cap V) = f^{-1}(E) \cap U$ we win. □

Lemma 28.23.2. *Let $f : X \to Y$ be a morphism of schemes. Assume*

(1) *f is quasi-compact and locally of finite presentation, and*

(2) *Y is quasi-compact and quasi-separated.*

Then the image of every constructible subset of X is constructible in Y.

Proof. By Properties, Lemma 27.2.4 it suffices to prove this lemma in case Y is affine. In this case X is quasi-compact. Hence we can write $X = U_1 \cup \ldots \cup U_n$ with each U_i affine open in X. If $E \subset X$ is constructible, then each $E \cap U_i$ is constructible too, see Topology, Lemma 5.14.4. Hence, since $f(E) = \bigcup f(E \cap U_i)$ and since finite unions of constructible sets are constructible, this reduces us to the case where X is affine. In this case the result is Algebra, Theorem 10.27.9. □

Theorem 28.23.3 (Chevalley's Theorem)**.** *Let $f : X \to Y$ be a morphism of schemes. Assume f is quasi-compact and locally of finite presentation. Then the image of every locally constructible subset is locally constructible.*

Proof. Let $E \subset X$ be locally constructible. We have to show that $f(E)$ is locally constructible too. We will show that $f(E) \cap V$ is constructible for any affine open $V \subset Y$. Thus we reduce to the case where Y is affine. In this case X is quasi-compact. Hence we can write $X = U_1 \cup \ldots \cup U_n$ with each U_i affine open in X. If $E \subset X$ is locally constructible, then each $E \cap U_i$ is constructible, see Properties, Lemma 27.2.1. Hence, since $f(E) = \bigcup f(E \cap U_i)$ and since finite unions of constructible sets are constructible, this reduces us to the case where X is affine. In this case the result is Algebra, Theorem 10.27.9. □

Lemma 28.23.4. *Let X be a scheme. Let $x \in X$. Let $E \subset X$ be a locally constructible subset. If $\{x' \mid x' \rightsquigarrow x\} \subset E$, then E contains an open neighbourhood of x.*

Proof. Assume $\{x' \mid x' \rightsquigarrow x\} \subset E$. We may assume X is affine. In this case E is constructible, see Properties, Lemma 27.2.1. In particular, also the complement E^c is constructible. By Algebra, Lemma 10.27.3 we can find a morphism of affine schemes $f : Y \to X$ such that $E^c = f(Y)$. Let $Z \subset X$ be the scheme theoretic image of f. By Lemma 28.6.5 and the assumption $\{x' \mid x' \rightsquigarrow x\} \subset E$ we see that $x \notin Z$. Hence $X \setminus Z \subset E$ is an open neighbourhood of x contained in E. □

28.24. Open morphisms

Definition 28.24.1. Let $f : X \to S$ be a morphism.

(1) We say f is *open* if the map on underlying topological spaces is open.
(2) We say f is *universally open* if for any morphism of schemes $S' \to S$ the base change $f' : X_{S'} \to S'$ is open.

According to Topology, Lemma 5.18.6 generalizations lift along certain types of open maps of topological spaces. In fact generalizations lift along any open morphism of schemes (see Lemma 28.24.5). Also, we will see that generalizations lift along flat morphisms of schemes (Lemma 28.26.8). This sometimes in turn implies that the morphism is open.

Lemma 28.24.2. *Let $f : X \to S$ be a morphism.*

(1) *If f is locally of finite presentation and generalizations lift along f, then f is open.*
(2) *If f is locally of finite presentation and generalizations lift along every base change of f, then f is universally open.*

Proof. It suffices to prove the first assertion. This reduces to the case where both X and S are affine. In this case the result follows from Algebra, Lemma 10.38.3 and Proposition 10.38.8. □

See also Lemma 28.26.9 for the case of a morphism flat of finite presentation.

Lemma 28.24.3. *A composition of (universally) open morphisms is (universally) open.*

Proof. Omitted. □

Lemma 28.24.4. *Let k be a field. Let X be a scheme over k. The structure morphism $X \to \mathrm{Spec}(k)$ is universally open.*

Proof. Let $S \to \mathrm{Spec}(k)$ be a morphism. We have to show that the base change $X_S \to S$ is open. The question is local on S and X, hence we may assume that S and X are affine. In this case the result is Algebra, Lemma 10.38.10. □

Lemma 28.24.5. *Let $\varphi : X \to Y$ be a morphism of schemes. If φ is open, then φ is generizing (i.e., generalizations lift along φ). If φ is universally open, then φ is universally generizing.*

Proof. Assume φ is open. Let $y' \rightsquigarrow y$ be a specialization of points of Y. Let $x \in X$ with $\varphi(x) = y$. Choose affine opens $U \subset X$ and $V \subset Y$ such that $\varphi(U) \subset V$ and $x \in U$. Then also $y' \in V$. Hence we may replace X by U and Y by V and assume X, Y affine. The affine case is Algebra, Lemma 10.38.2 (combined with Algebra, Lemma 10.38.3). □

Lemma 28.24.6. *Let $f : X \to Y$ be a morphism of schemes. Let $g : Y' \to Y$ be open and surjective such that the base change $f' : X' \to Y'$ is quasi-compact. Then f is quasi-compact.*

Proof. Let $V \subset Y$ be a quasi-compact open. As g is open and surjective we can find a quasi-compact open $W' \subset W$ such that $g(W') = V$. By assumption $(f')^{-1}(W')$ is quasi-compact. The image of $(f')^{-1}(W')$ in X is equal to $f^{-1}(V)$, see Lemma 28.11.3. Hence $f^{-1}(V)$ is quasi-compact as the image of a quasi-compact space, see Topology, Lemma 5.11.7. Thus f is quasi-compact. □

28.25. Submersive morphisms

Definition 28.25.1. Let $f : X \to Y$ be a morphism of schemes.

(1) We say f is *submersive*[5] if the continuous map of underlying topological spaces is submersive, see Topology, Definition 5.5.3.
(2) We say f is *universally submersive* if for every morphism of schemes $Y' \to Y$ the base change $Y' \times_Y X \to Y'$ is submersive.

We note that a submersive morphism is in particular surjective.

[5]This is very different from the notion of a submersion of differential manifolds.

28.26. Flat morphisms

Flatness is one of the most important technical tools in algebraic geometry. In this section we introduce this notion. We intentionally limit the discussion to straightforward observations, apart from Lemma 28.26.9. A very important class of results, namely criteria for flatness will be discussed (insert future reference here).

Recall that a module M over a ring R is *flat* if the functor $- \otimes_R M : \text{Mod}_R \to \text{Mod}_R$ is exact. A ring map $R \to A$ is said to be *flat* if A is flat as an R-module. See Algebra, Definition 10.36.1.

Definition 28.26.1. Let $f : X \to S$ be a morphism of schemes. Let $\mathcal{F}$ be a quasi-coherent sheaf of $\mathcal{O}_X$-modules.

(1) We say f is *flat at a point* $x \in X$ if the local ring $\mathcal{O}_{X,x}$ is flat over the local ring $\mathcal{O}_{S,f(x)}$.
(2) We say that $\mathcal{F}$ is *flat over* S *at a point* $x \in X$ if the stalk $\mathcal{F}_x$ is a flat $\mathcal{O}_{S,f(x)}$-module.
(3) We say f is *flat* if f is flat at every point of X.
(4) We say that $\mathcal{F}$ is *flat over* S if $\mathcal{F}$ is flat over S at every point x of X.

Thus we see that f is flat if and only if the structure sheaf $\mathcal{O}_X$ is flat over S.

Lemma 28.26.2. *Let $f : X \to S$ be a morphism of schemes. Let $\mathcal{F}$ be a quasi-coherent sheaf of $\mathcal{O}_X$-modules. The following are equivalent*

(1) *The sheaf $\mathcal{F}$ is flat over S.*
(2) *For every affine opens $U \subset X$, $V \subset S$ with $f(U) \subset V$ the $\mathcal{O}_S(V)$-module $\mathcal{F}(U)$ is flat.*
(3) *There exists an open covering $S = \bigcup_{j \in J} V_j$ and open coverings $f^{-1}(V_j) = \bigcup_{i \in I_j} U_i$ such that each of the modules $\mathcal{F}|_{U_i}$ is flat over V_j, for all $j \in J, i \in I_j$.*
(4) *There exists an affine open covering $S = \bigcup_{j \in J} V_j$ and affine open coverings $f^{-1}(V_j) = \bigcup_{i \in I_j} U_i$ such that $\mathcal{F}(U_i)$ is a flat $\mathcal{O}_S(V_j)$-module, for all $j \in J, i \in I_j$.*

Moreover, if $\mathcal{F}$ is flat over S then for any open subschemes $U \subset X$, $V \subset S$ with $f(U) \subset V$ the restriction $\mathcal{F}|_U$ is flat over V.

Proof. Let $R \to A$ be a ring map. Let M be an A-module. If M is R-flat, then for all primes $\mathfrak{q}$ the module $M_\mathfrak{q}$ is flat over $R_\mathfrak{p}$ with $\mathfrak{p}$ the prime of R lying under $\mathfrak{q}$. Conversely, if $M_\mathfrak{q}$ is flat over $R_\mathfrak{p}$ for all primes $\mathfrak{q}$ of A, then M is flat over R. See Algebra, Lemma 10.36.19. This equivalence easily implies the statements of the lemma. □

Lemma 28.26.3. *Let $f : X \to S$ be a morphism of schemes. The following are equivalent*

(1) *The morphism f is flat.*
(2) *For every affine opens $U \subset X$, $V \subset S$ with $f(U) \subset V$ the ring map $\mathcal{O}_S(V) \to \mathcal{O}_X(U)$ is flat.*
(3) *There exists an open covering $S = \bigcup_{j \in J} V_j$ and open coverings $f^{-1}(V_j) = \bigcup_{i \in I_j} U_i$ such that each of the morphisms $U_i \to V_j$, $j \in J, i \in I_j$ is flat.*

(4) *There exists an affine open covering* $S = \bigcup_{j \in J} V_j$ *and affine open coverings* $f^{-1}(V_j) = \bigcup_{i \in I_j} U_i$ *such that* $\mathcal{O}_S(V_j) \to \mathcal{O}_X(U_i)$ *is flat, for all* $j \in J, i \in I_j$.

Moreover, if f *is flat then for any open subschemes* $U \subset X$, $V \subset S$ *with* $f(U) \subset V$ *the restriction* $f|_U : U \to V$ *is flat.*

Proof. This is a special case of Lemma 28.26.2 above. □

Lemma 28.26.4. *Let* $X \to Y \to Z$ *be morphisms of schemes. Let* $\mathcal{F}$ *be a quasi-coherent* $\mathcal{O}_X$*-module. If* $\mathcal{F}$ *is flat over* Y*, and* Y *is flat over* Z*, then* $\mathcal{F}$ *is flat over* Z.

Proof. See Algebra, Lemma 10.36.3. □

Lemma 28.26.5. *The composition of flat morphisms is flat.*

Proof. This is a special case of Lemma 28.26.4. □

Lemma 28.26.6. *Let* $f : X \to S$ *be a morphism of schemes. Let* $\mathcal{F}$ *be a quasi-coherent sheaf of* $\mathcal{O}_X$*-modules. Let* $g : S' \to S$ *be a morphism of schemes. Denote* $g' : X' = X_{S'} \to X$ *the projection. Let* $x' \in X'$ *be a point with image* $x = g(x') \in X$. *If* $\mathcal{F}$ *is flat over* S *at* x*, then* $(g')^*\mathcal{F}$ *is flat over* S' *at* x'*. In particular, if* $\mathcal{F}$ *is flat over* S*, then* $(g')^*\mathcal{F}$ *is flat over* S'.

Proof. See Algebra, Lemma 10.36.6. □

Lemma 28.26.7. *The base change of a flat morphism is flat.*

Proof. This is a special case of Lemma 28.26.6. □

Lemma 28.26.8. *Let* $f : X \to S$ *be a flat morphism of schemes. Then generalizations lift along* f*, see Topology, Definition 5.18.3.*

Proof. See Algebra, Section 10.38. □

Lemma 28.26.9. *A flat morphism locally of finite presentation is universally open.*

Proof. This follows from Lemmas 28.26.8 and Lemma 28.24.2 above. We can also argue directly as follows.

Let $f : X \to S$ be flat locally of finite presentation. To show f is open it suffices to show that we may cover X by open affines $X = \bigcup U_i$ such that $U_i \to S$ is open. By definition we may cover X by affine opens $U_i \subset X$ such that each U_i maps into an affine open $V_i \subset S$ and such that the induced ring map $\mathcal{O}_S(V_i) \to \mathcal{O}_X(U_i)$ is of finite presentation. Thus $U_i \to V_i$ is open by Algebra, Proposition 10.38.8. The lemma follows. □

Lemma 28.26.10. *Let* $f : X \to Y$ *be a quasi-compact, surjective, flat morphism. A subset* $T \subset Y$ *is open (resp. closed) if and only* $f^{-1}(T)$ *is open (resp. closed). In other words,* f *is a submersive morphism.*

Proof. The question is local on Y, hence we may assume that Y is affine. In this case X is quasi-compact as f is quasi-compact. Write $X = X_1 \cup \ldots \cup X_n$ as a finite union of affine opens. Then $f' : X' = X_1 \coprod \ldots \coprod X_n \to Y$ is a surjective flat morphism of affine schemes. Note that for $T \subset Y$ we have $(f')^{-1}(T) = f^{-1}(T) \cap X_1 \coprod \ldots \coprod f^{-1}(T) \cap X_n$. Hence, $f^{-1}(T)$ is open if and only if $(f')^{-1}(T)$ is open. Thus we may assume both X and Y are affine.

Let $f : \mathrm{Spec}(B) \to \mathrm{Spec}(A)$ be a surjective morphism of affine schemes corresponding to a flat ring map $A \to B$. Suppose that $f^{-1}(T)$ is closed, say $f^{-1}(T) = V(I)$ for $I \subset A$ an ideal. Then $T = f(f^{-1}(T)) = f(V(I))$ is the image of $\mathrm{Spec}(A/I) \to \mathrm{Spec}(B)$ (here we use that f is surjective). On the other hand, generalizations lift along f (Lemma 28.26.8). Hence by Topology, Lemma 5.18.5 we see that $Y \setminus T = f(X \setminus f^{-1}(T))$ is stable under generalization. Hence T is stable under specialization (Topology, Lemma 5.18.2). Thus T is closed by Algebra, Lemma 10.38.5. □

Lemma 28.26.11. *Let $h : X \to Y$ be a morphism of schemes over S. Let $\mathcal{G}$ be a quasi-coherent sheaf on Y. Let $x \in X$ with $y = h(x) \in Y$. If h is flat at x, then*

$$\mathcal{G} \text{ flat over } S \text{ at } y \Leftrightarrow h^*\mathcal{G} \text{ flat over } S \text{ at } x.$$

In particular: If h is surjective and flat, then $\mathcal{G}$ is flat over S, if and only if $h^*\mathcal{G}$ is flat over S. If h is surjective and flat, and X is flat over S, then Y is flat over S.

Proof. You can prove this by applying Algebra, Lemma 10.36.8. Here is a direct proof. Let $s \in S$ be the image of y. Consider the local ring maps $\mathcal{O}_{S,s} \to \mathcal{O}_{Y,y} \to \mathcal{O}_{X,x}$. By assumption the ring map $\mathcal{O}_{Y,y} \to \mathcal{O}_{X,x}$ is faithfully flat, see Algebra, Lemma 10.36.16. Let $N = \mathcal{G}_y$. Note that $h^*\mathcal{G}_x = N \otimes_{\mathcal{O}_{Y,y}} \mathcal{O}_{X,x}$, see Sheaves, Lemma 6.26.4. Let $M' \to M$ be an injection of $\mathcal{O}_{S,s}$-modules. By the faithful flatness mentioned above we have

$$\begin{aligned} &\mathrm{Ker}(M' \otimes_{\mathcal{O}_{S,s}} N \to M \otimes_{\mathcal{O}_{S,s}} N) \otimes_{\mathcal{O}_{Y,y}} \mathcal{O}_{X,x} \\ &= \mathrm{Ker}(M' \otimes_{\mathcal{O}_{S,s}} N \otimes_{\mathcal{O}_{Y,y}} \mathcal{O}_{X,x} \to M \otimes_{\mathcal{O}_{S,s}} N \otimes_{\mathcal{O}_{Y,y}} \mathcal{O}_{X,x}) \end{aligned}$$

Hence the equivalence of the lemma follows from the second characterization of flatness in Algebra, Lemma 10.36.4. □

Lemma 28.26.12. *Let $f : Y \to X$ be a morphism of schemes. Let $\mathcal{F}$ be a finite type quasi-coherent $\mathcal{O}_X$-module with scheme theoretic support $Z \subset X$. If f is flat, then $f^{-1}(Z)$ is the scheme theoretic support of $f^*\mathcal{F}$.*

Proof. Using the characterization of scheme theoretic support on affines as given in Lemma 28.5.4 we reduce to Algebra, Lemma 10.37.4. □

Lemma 28.26.13. *Let $f : X \to Y$ be a flat morphism of schemes. Let $V \subset Y$ be a retrocompact open which is scheme theoretically dense. Then $f^{-1}V$ is scheme theoretically dense in X.*

Proof. We will use the characterization of Lemma 28.7.5. We have to show that for any open $U \subset X$ the map $\mathcal{O}_X(U) \to \mathcal{O}_X(U \cap f^{-1}V)$ is injective. It suffices to prove this when U is an affine open which maps into an affine open $W \subset Y$. Say $W = \mathrm{Spec}(A)$ and $U = \mathrm{Spec}(B)$. Then $V \cap W = D(f_1) \cup \ldots \cup D(f_n)$ for some $f_i \in A$, see Algebra, Lemma 10.27.1. Thus we have to show that $B \to B_{f_1} \times \ldots \times B_{f_n}$ is injective. We are given that $A \to A_{f_1} \times \ldots \times A_{f_n}$ is injective and that $A \to B$ is flat. Since $B_{f_i} = A_{f_i} \otimes_A B$ we win. □

Lemma 28.26.14. *Let $f : X \to Y$ be a flat morphism of schemes. Let $g : V \to Y$ be a quasi-compact morphism of schemes. Let $Z \subset Y$ be the scheme theoretic image of g and let $Z' \subset X$ be the scheme theoretic image of the base change $V \times_Y X \to X$. Then $Z' = f^{-1}Z$.*

Proof. Recall that Z is cut out by $\mathcal{I} = \mathrm{Ker}(\mathcal{O}_Y \to g_*\mathcal{O}_V)$ and Z' is cut out by $\mathcal{I}' = \mathrm{Ker}(\mathcal{O}_X \to (V \times_Y X \to X)_*\mathcal{O}_{V \times_Y X})$, see Lemma 28.6.3. Hence the question is local on X and Y and we may assume X and Y affine. Note that we may replace V by $\coprod V_i$ where $V = V_1 \cup \ldots \cup V_n$ is a finite affine open covering. Hence we may assume g is affine. In this case $(V \times_Y X \to X)_*\mathcal{O}_{V \times_Y X}$ is the pullback of $g_*\mathcal{O}_V$ by f. Since f is flat we conclude that $f^*\mathcal{I} = \mathcal{I}'$ and the lemma holds. □

28.27. Flat closed immersions

Connected components of schemes are not always open. But they do always have a canonical scheme structure. We explain this in this section.

Lemma 28.27.1. *Let X be a scheme. The rule which associates to a closed subscheme of X its underlying closed subset defines a bijection*

$$\left\{\begin{array}{c}\text{closed subschemes } Z \subset X \\ \text{such that } Z \to X \text{ is flat}\end{array}\right\} \leftrightarrow \left\{\begin{array}{c}\text{closed subsets } Z \subset X \\ \text{closed under generalizations}\end{array}\right\}$$

Proof. The affine case is Algebra, Lemma 10.103.4. In general the lemma follows by covering X by affines and glueing. Details omitted. □

Lemma 28.27.2. *A flat closed immersion of finite presentation is the open immersion of an open and closed subscheme.*

Proof. The affine case is Algebra, Lemma 10.103.5. In general the lemma follows by covering X by affines. Details omitted. □

Note that a connected component T of a scheme X is a closed subset stable under generalization. Hence the following definition makes sense.

Definition 28.27.3. Let X be a scheme. Let $T \subset X$ be a connected component. The *canonical scheme structure on T* is the unique scheme structure on T such that the closed immersion $T \to X$ is flat, see Lemma 28.27.1.

It turns out that we can determine when every finite flat $\mathcal{O}_X$-module is finite locally free using the previous lemma.

Lemma 28.27.4. *Let X be a scheme. The following are equivalent*

1. *every finite flat quasi-coherent $\mathcal{O}_X$-module is finite locally free, and*
2. *every closed subset $Z \subset X$ which is closed under generalizations is open.*

Proof. In the affine case this is Algebra, Lemma 10.103.6. The scheme case does not follow directly from the affine case, so we simply repeat the arguments.

Assume (1). Consider a closed immersion $i : Z \to X$ such that i is flat. Then $i_*\mathcal{O}_Z$ is quasi-coherent and flat, hence finite locally free by (1). Thus $Z = \mathrm{Supp}(i_*\mathcal{O}_Z)$ is also open and we see that (2) holds. Hence the implication (1) $\Rightarrow$ (2) follows from the characterization of flat closed immersions in Lemma 28.27.1.

For the converse assume that X satisfies (2). Let $\mathcal{F}$ be a finite flat quasi-coherent $\mathcal{O}_X$-module. The support $Z = \mathrm{Supp}(\mathcal{F})$ of $\mathcal{F}$ is closed, see Modules, Lemma 17.9.6. On the other hand, if $x \rightsquigarrow x'$ is a specialization, then by Algebra, Lemma 10.73.4 the module $\mathcal{F}_{x'}$ is free over $\mathcal{O}_{X,x'}$, and

$$\mathcal{F}_x = \mathcal{F}_{x'} \otimes_{\mathcal{O}_{X,x'}} \mathcal{O}_{X,x}.$$

Hence $x' \in \text{Supp}(\mathcal{F}) \Rightarrow x \in \text{Supp}(\mathcal{F})$, in other words, the support is closed under generalization. As X satisfies (2) we see that the support of $\mathcal{F}$ is open and closed. The modules $\wedge^i(\mathcal{F})$, $i = 1, 2, 3, \ldots$ are finite flat quasi-coherent $\mathcal{O}_X$-modules also, see Modules, Section 17.18. Note that $\text{Supp}(\wedge^{i+1}(\mathcal{F})) \subset \text{Supp}(\wedge^i(\mathcal{F}))$. Thus we see that there exists a decomposition

$$X = U_0 \coprod U_1 \coprod U_2 \coprod \ldots$$

by open and closed subsets such that the support of $\wedge^i(\mathcal{F})$ is $U_i \cup U_{i+1} \cup \ldots$ for all i. Let x be a point of X, and say $x \in U_r$. Note that $\wedge^i(\mathcal{F})_x \otimes \kappa(x) = \wedge^i(\mathcal{F}_x \otimes \kappa(x))$. Hence, $x \in U_r$ implies that $\mathcal{F}_x \otimes \kappa(x)$ is a vector space of dimension r. By Nakayama's lemma, see Algebra, Lemma 10.18.1 we can choose an affine open neighbourhood $U \subset U_r \subset X$ of x and sections $s_1, \ldots, s_r \in \mathcal{F}(U)$ such that the induced map

$$\mathcal{O}_U^{\oplus r} \longrightarrow \mathcal{F}|_U, \quad (f_1, \ldots, f_r) \longmapsto \sum f_i s_i$$

is surjective. This means that $\wedge^r(\mathcal{F}|_U)$ is a finite flat quasi-coherent $\mathcal{O}_U$-module whose support is all of U. By the above it is generated by a single element, namely $s_1 \wedge \ldots \wedge s_r$. Hence $\wedge^r(\mathcal{F}|_U) \cong \mathcal{O}_U/\mathcal{I}$ for some quasi-coherent sheaf of ideals $\mathcal{I}$ such that $\mathcal{O}_U/\mathcal{I}$ is flat over $\mathcal{O}_U$ and such that $V(\mathcal{I}) = U$. It follows that $\mathcal{I} = 0$ by applying Lemma 28.27.1. Thus $s_1 \wedge \ldots \wedge s_r$ is a basis for $\wedge^r(\mathcal{F}|_U)$ and it follows that the displayed map is injective as well as surjective. This proves that $\mathcal{F}$ is finite locally free as desired. □

28.28. Generic flatness

A scheme of finite type over an integral base is flat over a dense open of the base. In Algebra, Section 10.112 we proved a Noetherian version, a version for morphisms of finite presentation, and a general version. We only state and prove the general version here. However, it turns out that this will be superseded by Proposition 28.28.2 which shows the result holds if we only assume the base is reduced.

Proposition 28.28.1 (Generic flatness)**.** *Let $f : X \to S$ be a morphism of schemes. Let $\mathcal{F}$ be a quasi-coherent sheaf of $\mathcal{O}_X$-modules. Assume*

(1) *S is integral,*
(2) *f is of finite type, and*
(3) *$\mathcal{F}$ is a finite type $\mathcal{O}_X$-module.*

Then there exists an open dense subscheme $U \subset S$ such that $X_U \to U$ is flat and of finite presentation and such that $\mathcal{F}|_{X_U}$ is flat over U and of finite presentation over $\mathcal{O}_{X_U}$.

Proof. As S is integral it is irreducible (see Properties, Lemma 27.3.4) and any nonempty open is dense. Hence we may replace S by an affine open of S and assume that $S = \text{Spec}(A)$ is affine. As S is integral we see that A is a domain. As f is of finite type, it is quasi-compact, so X is quasi-compact. Hence we can find a finite affine open cover $X = \bigcup_{i=1,\ldots,n} X_i$. Write $X_i = \text{Spec}(B_i)$. Then B_i is a finite type A-algebra, see Lemma 28.16.2. Moreover there are finite type B_i-modules M_i such that $\mathcal{F}|_{X_i}$ is the quasi-coherent sheaf associated to the B_i-module M_i, see Properties, Lemma 27.16.1. Next, for each pair of indices i, j choose an ideal $I_{ij} \subset B_i$ such that $X_i \setminus X_i \cap X_j = V(I_{ij})$ inside $X_i = \text{Spec}(B_i)$. Set $M_{ij} = B_i/I_{ij}$

and think of it as a B_i-module. Then $V(I_{ij}) = \mathrm{Supp}(M_{ij})$ and M_{ij} is a finite B_i-module.

At this point we apply Algebra, Lemma 10.112.3 the pairs $(A \to B_i, M_{ij})$ and to the pairs $(A \to B_i, M_i)$. Thus we obtain nonzero $f_{ij}, f_i \in A$ such that (a) $A_{f_{ij}} \to B_{i,f_{ij}}$ is flat and of finite presentation and $M_{ij,f_{ij}}$ is flat over $A_{f_{ij}}$ and of finite presentation over $B_{i,f_{ij}}$, and (b) B_{i,f_i} is flat and of finite presentation over A_f and M_{i,f_i} is flat and of finite presentation over B_{i,f_i}. Set $f = (\prod f_i)(\prod f_{ij})$. We claim that taking $U = D(f)$ works.

To prove our claim we may replace A by A_f, i.e., perform the base change by $U = \mathrm{Spec}(A_f) \to S$. After this base change we see that each of $A \to B_i$ is flat and of finite presentation and that M_i, M_{ij} are flat over A and of finite presentation over B_i. This already proves that $X \to S$ is quasi-compact, locally of finite presentation, flat, and that $\mathcal{F}$ is flat over S and of finite presentation over $\mathcal{O}_X$, see Lemma 28.22.2 and Properties, Lemma 27.16.2. Since M_{ij} is of finite presentation over B_i we see that $X_i \cap X_j = X_i \setminus \mathrm{Supp}(M_{ij})$ is a quasi-compact open of X_i, see Algebra, Lemma 10.37.7. Hence we see that $X \to S$ is quasi-separated by Schemes, Lemma 25.21.7. This proves the proposition. □

It actually turns out that there is also a version of generic flatness over an arbitrary reduced base. Here it is.

Proposition 28.28.2 (Generic flatness, reduced case)**.** *Let $f : X \to S$ be a morphism of schemes. Let $\mathcal{F}$ be a quasi-coherent sheaf of $\mathcal{O}_X$-modules. Assume*

(1) *S is reduced,*
(2) *f is of finite type, and*
(3) *$\mathcal{F}$ is a finite type $\mathcal{O}_X$-module.*

Then there exists an open dense subscheme $U \subset S$ such that $X_U \to U$ is flat and of finite presentation and such that $\mathcal{F}|_{X_U}$ is flat over U and of finite presentation over $\mathcal{O}_{X_U}$.

Proof. For the impatient reader: This proof is a repeat of the proof of Proposition 28.28.1 using Algebra, Lemma 10.112.7 instead of Algebra, Lemma 10.112.3.

Since being flat and being of finite presentation is local on the base, see Lemmas 28.26.2 and 28.22.2, we may work affine locally on S. Thus we may assume that $S = \mathrm{Spec}(A)$, where A is a reduced ring (see Properties, Lemma 27.3.2). As f is of finite type, it is quasi-compact, so X is quasi-compact. Hence we can find a finite affine open cover $X = \bigcup_{i=1,\ldots,n} X_i$. Write $X_i = \mathrm{Spec}(B_i)$. Then B_i is a finite type A-algebra, see Lemma 28.16.2. Moreover there are finite type B_i-modules M_i such that $\mathcal{F}|_{X_i}$ is the quasi-coherent sheaf associated to the B_i-module M_i, see Properties, Lemma 27.16.1. Next, for each pair of indices i, j choose an ideal $I_{ij} \subset B_i$ such that $X_i \setminus X_i \cap X_j = V(I_{ij})$ inside $X_i = \mathrm{Spec}(B_i)$. Set $M_{ij} = B_i/I_{ij}$ and think of it as a B_i-module. Then $V(I_{ij}) = \mathrm{Supp}(M_{ij})$ and M_{ij} is a finite B_i-module.

At this point we apply Algebra, Lemma 10.112.7 the pairs $(A \to B_i, M_{ij})$ and to the pairs $(A \to B_i, M_i)$. Thus we obtain dense opens $U(A \to B_i, M_{ij}) \subset S$ and dense opens $U(A \to B_i, M_i) \subset S$ with notation as in Algebra, Equation (10.112.3.2). Since a finite intersection of dense opens is dense open, we see that

$$U = \bigcap\nolimits_{i,j} U(A \to B_i, M_{ij}) \quad \cap \quad \bigcap\nolimits_i U(A \to B_i, M_i)$$

is open and dense in S. We claim that U is the desired open.

Pick $u \in U$. By definition of the loci $U(A \to B_i, M_{ij})$ and $U(A \to B, M_i)$ there exist $f_{ij}, f_i \in A$ such that (a) $u \in D(f_i)$ and $u \in D(f_{ij})$, (b) $A_{f_{ij}} \to B_{i,f_{ij}}$ is flat and of finite presentation and $M_{ij,f_{ij}}$ is flat over $A_{f_{ij}}$ and of finite presentation over $B_{i,f_{ij}}$, and (c) B_{i,f_i} is flat and of finite presentation over A_f and M_{i,f_i} is flat and of finite presentation over B_{i,f_i}. Set $f = (\prod f_i)(\prod f_{ij})$. Now it suffices to prove that $X \to S$ is flat and of finite presentation over $D(f)$ and that $\mathcal{F}$ restricted to $X_{D(f)}$ is flat over $D(f)$ and of finite presentation over the structure sheaf of $X_{D(f)}$.

Hence we may replace A by A_f, i.e., perform the base change by $\mathrm{Spec}(A_f) \to S$. After this base change we see that each of $A \to B_i$ is flat and of finite presentation and that M_i, M_{ij} are flat over A and of finite presentation over B_i. This already proves that $X \to S$ is quasi-compact, locally of finite presentation, flat, and that $\mathcal{F}$ is flat over S and of finite presentation over $\mathcal{O}_X$, see Lemma 28.22.2 and Properties, Lemma 27.16.2. Since M_{ij} is of finite presentation over B_i we see that $X_i \cap X_j = X_i \setminus \mathrm{Supp}(M_{ij})$ is a quasi-compact open of X_i, see Algebra, Lemma 10.37.7. Hence we see that $X \to S$ is quasi-separated by Schemes, Lemma 25.21.7. This proves the proposition. □

Remark 28.28.3. The results above are a first step towards more refined flattening techniques for morphisms of schemes. The article [**GR71**] by Raynaud and Gruson contains many wonderful results in this direction.

28.29. Morphisms and dimensions of fibres

Let X be a topological space, and $x \in X$. Recall that we have defined $\dim_x(X)$ as the minimum of the dimensions of the open neighbourhoods of x in X. See Topology, Definition 5.9.1.

Lemma 28.29.1. *Let $f : X \to S$ be a morphism of schemes. Let $x \in X$ and set $s = f(x)$. Assume f is locally of finite type. Then*

$$\dim_x(X_s) = \dim(\mathcal{O}_{X_s,x}) + trdeg_{\kappa(s)}(\kappa(x)).$$

Proof. This immediately reduces to the case $S = s$, and X affine. In this case the result follows from Algebra, Lemma 10.110.3. □

Lemma 28.29.2. *Let $f : X \to Y$ and $g : Y \to S$ be morphisms of schemes. Let $x \in X$ and set $y = f(x)$, $s = g(y)$. Assume f and g locally of finite type. Then*

$$\dim_x(X_s) \leq \dim_x(X_y) + \dim_y(Y_s).$$

Moreover, equality holds if $\mathcal{O}_{X_s,x}$ is flat over $\mathcal{O}_{Y_s,y}$, which holds for example if $\mathcal{O}_{X,x}$ is flat over $\mathcal{O}_{Y,y}$.

Proof. Note that $\mathrm{trdeg}_{\kappa(s)}(\kappa(x)) = \mathrm{trdeg}_{\kappa(y)}(\kappa(x)) + \mathrm{trdeg}_{\kappa(s)}(\kappa(y))$. Thus by Lemma 28.29.1 the statement is equivalent to

$$\dim(\mathcal{O}_{X_s,x}) \leq \dim(\mathcal{O}_{X_y,x}) + \dim(\mathcal{O}_{Y_s,y}).$$

For this see Algebra, Lemma 10.106.6. For the flat case see Algebra, Lemma 10.106.7. □

Lemma 28.29.3. *Let*

$$\begin{array}{ccc} X' & \xrightarrow{g'} & X \\ {\scriptstyle f'}\downarrow & & \downarrow{\scriptstyle f} \\ S' & \xrightarrow{g} & S \end{array}$$

be a fibre product diagram of schemes. Assume f locally of finite type. Suppose that $x' \in X'$, $x = g'(x')$, $s' = f'(x')$ and $s = g(s') = f(x)$. Then $\dim_x(X_s) = \dim_{x'}(X'_{s'})$.

Proof. Follows immediately from Algebra, Lemma 10.110.6. □

The following lemma follows from a nontrivial algebraic result. Namely, the algebraic version of Zariski's main theorem.

Lemma 28.29.4. *Let $f : X \to S$ be a morphism of schemes. Let $n \geq 0$. Assume f is locally of finite type. The set*

$$U_n = \{x \in X \mid \dim_x X_{f(x)} \leq n\}$$

is open in X.

Proof. This is immediate from Algebra, Lemma 10.119.6 □

Lemma 28.29.5. *Let $f : X \to S$ be a morphism of schemes. Let $n \geq 0$. Assume f is locally of finite presentation. The open*

$$U_n = \{x \in X \mid \dim_x X_{f(x)} \leq n\}$$

of Lemma 28.29.4 is retrocompact in X. (See Topology, Definition 5.11.1.)

Proof. The topological space X has a basis for its topology consisting of affine opens $U \subset X$ such that the induced morphism $f|_U : U \to S$ factors through an affine open $V \subset S$. Hence it is enough to show that $U \cap U_n$ is quasi-compact for such a U. Note that $U_n \cap U$ is the same as the open $\{x \in U \mid \dim_x U_{f(x)} \leq n\}$. This reduces us to the case where X and S are affine. In this case the lemma follows from Algebra, Lemma 10.119.8 (and Lemma 28.22.2). □

Lemma 28.29.6. *Let $f : X \to S$ be a morphism of schemes. Let $x \rightsquigarrow x'$ be a nontrivial specialization of points in X lying over the same point $s \in S$. Assume f is locally of finite type. Then*

1. *$\dim_x(X_s) \leq \dim_{x'}(X_s)$,*
2. *$trdeg_{\kappa(s)}(\kappa(x)) > trdeg_{\kappa(s)}(\kappa(x'))$, and*
3. *$\dim(\mathcal{O}_{X_s,x}) < \dim(\mathcal{O}_{X_s,x'})$.*

Proof. The first inequality follows from Lemma 28.29.4. The third inequality follows since $\mathcal{O}_{X_s,x}$ is a localization of $\mathcal{O}_{X_s,x}$ in a prime ideal, hence any chain of prime ideals in $\mathcal{O}_{X_s,x}$ is part of a strictly longer chain of primes in $\mathcal{O}_{X_s,x'}$. The second inequality follows from Algebra, Lemma 10.110.2. □

28.30. Morphisms of given relative dimension

In order to be able to speak comfortably about morphisms of a given relative dimension we introduce the following notion.

Definition 28.30.1. Let $f : X \to S$ be a morphism of schemes. Assume f is locally of finite type.

(1) We say f is of *relative dimension* $\leq d$ *at* x if $\dim_x(X_{f(x)}) \leq d$.
(2) We say f is of *relative dimension* $\leq d$ if $\dim_x(X_{f(x)}) \leq d$ for all $x \in X$.
(3) We say f is of *relative dimension* d if all nonempty fibres X_s are equidimensional of dimension d.

This is not a particularly well behaved notion, but it works well in a number of situations.

Lemma 28.30.2. *Let $f : X \to S$ be a morphism of schemes which is locally of finite type. If f has relative dimension d, then so does any base change of f. Same for relative dimension $\leq d$.*

Proof. This is immediate from Lemma 28.29.3. □

Lemma 28.30.3. *Let $f : X \to Y$, $g : Y \to Z$ be locally of finite type. If f has relative dimension $\leq d$ and g has relative dimension $\leq e$ then $g \circ f$ has relative dimension $\leq d + e$. If*

(1) *f has relative dimension d,*
(2) *g has relative dimension e, and*
(3) *f is flat,*

then $g \circ f$ has relative dimension $d + e$.

Proof. This is immediate from Lemma 28.29.2. □

In general it is not possible to decompose a morphism into its pieces where the relative dimension is a given one. However, it is possible if the morphism has Cohen-Macaulay fibres and is flat of finite presentation.

Lemma 28.30.4. *Let $f : X \to S$ be a morphism of schemes. Assume that*

(1) *f is flat,*
(2) *f is locally of finite presentation, and*
(3) *for all $s \in S$ the fibre X_s is Cohen-Macaulay (Properties, Definition 27.8.1)*

Then there exist open and closed subschemes $X_d \subset X$ such that $X = \coprod_{d \geq 0} X_d$ and $f|_{X_d} : X_d \to S$ has relative dimension d.

Proof. This is immediate from Algebra, Lemma 10.124.8. □

Lemma 28.30.5. *Let $f : X \to S$ be a morphism of schemes. Assume f is locally of finite type. Let $x \in X$ with $s = f(x)$. Then f is quasi-finite at x if and only if $\dim_x(X_s) = 0$. In particular, f is locally quasi-finite if and only if f has relative dimension 0.*

Proof. If f is quasi-finite at x then $\kappa(x)$ is a finite extension of $\kappa(s)$ (by Lemma 28.21.5) and x is isolated in X_s (by Lemma 28.21.6), hence $\dim_x(X_s) = 0$ by Lemma 28.29.1. Conversely, if $\dim_x(X_s) = 0$ then by Lemma 28.29.1 we see $\kappa(s) \subset \kappa(x)$ is algebraic and there are no other points of X_s specializing to x. Hence x is closed in its fibre by Lemma 28.21.2 and by Lemma 28.21.6 (3) we conclude that f is quasi-finite at x. □

28.31. The dimension formula

For morphisms between Noetherian schemes we can say a little more about dimensions of local rings. Here is an important (and not so hard to prove) result. Recall that $R(X)$ denotes the function field of an integral scheme X.

Lemma 28.31.1. *Let S be a scheme. Let $f : X \to S$ be a morphism of schemes. Let $x \in X$, and set $s = f(x)$. Assume*

(1) *S is locally Noetherian,*
(2) *f is locally of finite type,*
(3) *X and S integral, and*
(4) *f dominant.*

We have

$$\dim(\mathcal{O}_{X,x}) \leq \dim(\mathcal{O}_{S,s}) + trdeg_{R(S)}R(X) - trdeg_{\kappa(s)}\kappa(x). \tag{28.31.1.1}$$

Moreover, equality holds if S is universally catenary.

Proof. The corresponding algebra statement is Algebra, Lemma 10.107.1. □

An application is the construction of a dimension function on any scheme of finite type over a universally catenary scheme endowed with a dimension function. For the definition of dimension functions, see Topology, Definition 5.19.1.

Lemma 28.31.2. *Let S be a universally catenary scheme. Let $\delta : S \to \mathbf{Z}$ be a dimension function. Let $f : X \to S$ be a morphism of schemes. Assume f locally of finite type. Then the map*

$$\begin{aligned} \delta = \delta_{X/S} : X &\longrightarrow \mathbf{Z} \\ x &\longmapsto \delta(f(x)) + trdeg_{\kappa(f(x))}\kappa(x) \end{aligned}$$

is a dimension function on X.

Proof. Let $f : X \to S$ be locally of finite type. Let $x \rightsquigarrow y$, $x \neq y$ be a specialization in X. We have to show that $\delta_{X/S}(x) > \delta_{X/S}(y)$ and that $\delta_{X/S}(x) = \delta_{X/S}(y) + 1$ if y is an immediate specialization of x.

Choose an affine open $V \subset S$ containing the image of y and choose an affine open $U \subset X$ mapping into V and containing y. We may clearly replace X by U and S by V. Thus we may assume that $X = \mathrm{Spec}(A)$ and $S = \mathrm{Spec}(R)$ and that f is given by a ring map $R \to A$. The ring R is universally catenary (Lemma 28.18.2) and the map $R \to A$ is of finite type (Lemma 28.16.2).

Let $\mathfrak{q} \subset A$ be the prime ideal corresponding to the point x and let $\mathfrak{p} \subset R$ be the prime ideal corresponding to $f(x)$. The restriction δ' of δ to $S' = \mathrm{Spec}(R/\mathfrak{p}) \subset S$ is a dimension function. The ring $R/\mathfrak{p}$ is universally catenary. The restriction of $\delta_{X/S}$ to $X' = \mathrm{Spec}(A/\mathfrak{q})$ is clearly equal to the function $\delta_{X'/S'}$ constructed using the dimension function δ'. Hence we may assume in addition to the above that $R \subset A$ are domains, in other words that X and S are integral schemes.

Note that $\mathcal{O}_{X,x}$ is a localization of $\mathcal{O}_{X,y}$ at a non-maximal prime (Schemes, Lemma 25.13.2). Hence $\dim(\mathcal{O}_{X,x}) < \dim(\mathcal{O}_{X,y})$ and $\dim(\mathcal{O}_{X,x}) = \dim(\mathcal{O}_{X,y}) - 1$ if y is an immediate specialization of x.

Write $s = f(x) \neq f(y) = s'$. We see, using equality in (28.31.1.1), that

$$\begin{aligned}\delta_{X/S}(x) - \delta_{X/S}(y) = \delta(s) - \delta(s') \\ + \dim(\mathcal{O}_{S,s}) - \dim(\mathcal{O}_{S,s'}) \\ - \dim(\mathcal{O}_{X,x}) + \dim(\mathcal{O}_{X,y}).\end{aligned}$$

Since δ is a dimension function on the scheme S the difference $\delta(s) - \delta(s')$ is equal to $\operatorname{codim}(\overline{\{s'\}}, \overline{\{s\}})$ by Topology, Lemma 5.19.2. As S is integral, catenary this is equal to $\operatorname{codim}(\overline{\{s'\}}, S) - \operatorname{codim}(\overline{\{s\}}, S)$ (Topology, Lemma 5.10.6). And this in turn is equal to $\dim(\mathcal{O}_{S,s'}) - \dim(\mathcal{O}_{S,s})$ by Properties, Lemma 27.11.4. Hence we conclude that

$$\delta_{X/S}(x) - \delta_{X/S}(y) = -\dim(\mathcal{O}_{X,x}) + \dim(\mathcal{O}_{X,y})$$

and hence the lemma follows from our remarks about the dimensions of these local rings above. □

Another application of the dimension formula is that the dimension does not change under "alterations" (to be defined later).

Lemma 28.31.3. *Let $f : X \to Y$ be a morphism of schemes. Assume that*

(1) *Y is locally Noetherian,*
(2) *X and Y are integral schemes,*
(3) *f is dominant, and*
(4) *f is locally of finite type.*

Then we have

$$\dim(X) \leq \dim(Y) + \mathit{trdeg}_{R(Y)} R(X).$$

If f is closed[6] *then equality holds.*

Proof. Let $f : X \to Y$ be as in the lemma. Let $\xi_0 \rightsquigarrow \xi_1 \rightsquigarrow \ldots \rightsquigarrow \xi_e$ be a sequence of specializations in X. We may assume that $x = \xi_e$ is a closed point of X, see Properties, Lemma 27.5.8. In particular, setting $y = f(x)$, we see x is a closed point of its fibre X_y. By the Hilbert Nullstellensatz we see that $\kappa(x)$ is a finite extension of $\kappa(y)$, see Lemma 28.21.3. By the dimension formula, Lemma 28.31.1, we see that

$$\dim(\mathcal{O}_{X,x}) \leq \dim(\mathcal{O}_{Y,y}) + \operatorname{trdeg}_{R(Y)} R(X)$$

Hence we conclude that $e \leq \dim(Y) + \operatorname{trdeg}_{R(Y)} R(X)$ as desired.

Next, assume f is also closed. Say $\overline{\xi}_0 \rightsquigarrow \overline{\xi}_1 \rightsquigarrow \ldots \rightsquigarrow \overline{\xi}_d$ is a sequence of specializations in Y. We want to show that $\dim(X) \geq d + r$. We may assume that $\overline{\xi}_0 = \eta$ is the generic point of Y. The generic fibre X_η is a scheme locally of finite type over $\kappa(\eta) = R(Y)$. It is nonempty as f is dominant. Hence by Lemma 28.17.10 it is a Jacobson scheme. Thus by Lemma 28.17.8 we can find a closed point $\xi_0 \in X_\eta$ and the extension $\kappa(\eta) \subset \kappa(\xi_0)$ is a finite extension. Note that $\mathcal{O}_{X,\xi_0} = \mathcal{O}_{X_\eta,\xi_0}$ because η is the generic point of Y. Hence we see that $\dim(\mathcal{O}_{X,\xi_0}) = r$ by Lemma 28.31.1 applied to the scheme X_η over the universally catenary scheme $\operatorname{Spec}(\kappa(\eta))$ (see Lemma 28.18.4) and the point ξ_0. This means that we can find $\xi_{-r} \rightsquigarrow \ldots \rightsquigarrow \xi_{-1} \rightsquigarrow \xi_0$ in X. On the other hand, as f is closed specializations lift along f, see Topology, Lemma

[6] For example if f is proper, see Definition 28.42.1.

5.18.6. Thus, as ξ_0 lies over $\eta = \overline{\xi}_0$ we can find specializations $\xi_0 \rightsquigarrow \xi_1 \rightsquigarrow \ldots \rightsquigarrow \xi_d$ lying over $\overline{\xi}_0 \rightsquigarrow \overline{\xi}_1 \rightsquigarrow \ldots \rightsquigarrow \overline{\xi}_d$. In other words we have

$$\xi_{-r} \rightsquigarrow \ldots \rightsquigarrow \xi_{-1} \rightsquigarrow \xi_0 \rightsquigarrow \xi_1 \rightsquigarrow \ldots \rightsquigarrow \xi_d$$

which means that $\dim(X) \geq d + r$ as desired. □

28.32. Syntomic morphisms

An algebra A over a field k is called a *global complete intersection over k* if $A \cong k[x_1, \ldots, x_n]/(f_1, \ldots, f_c)$ and $\dim(A) = n - c$. An algebra A over a field k is called a *local complete intersection* if $\mathrm{Spec}(A)$ can be covered by standard opens each of which are global complete intersections over k. See Algebra, Section 10.128. Recall that a ring map $R \to A$ is *syntomic* if it is of finite presentation, flat with local complete intersection rings as fibres, see Algebra, Definition 10.129.1.

Definition 28.32.1. Let $f : X \to S$ be a morphism of schemes.

(1) We say that f is *syntomic at $x \in X$* if there exists a affine open neighbourhood $\mathrm{Spec}(A) = U \subset X$ of x and affine open $\mathrm{Spec}(R) = V \subset S$ with $f(U) \subset V$ such that the induced ring map $R \to A$ is syntomic.
(2) We say that f is *syntomic* if it is syntomic at every point of X.
(3) If $S = \mathrm{Spec}(k)$ and f is syntomic, then we say that X is a *local complete intersection over k*.
(4) A morphism of affine schemes $f : X \to S$ is called *standard syntomic* if there exists a global relative complete intersection $R \to R[x_1, \ldots, x_n]/(f_1, \ldots, f_c)$ (see Algebra, Definition 10.129.5) such that $X \to S$ is isomorphic to

$$\mathrm{Spec}(R[x_1, \ldots, x_n]/(f_1, \ldots, f_c)) \to \mathrm{Spec}(R).$$

In the literature a syntomic morphism is sometimes referred to as a *flat local complete intersection morphism*. It turns out this is a convenient class of morphisms. For example one can define a syntomic topology using these, which is finer than the smooth and étale topologies, but has many of the same formal properties.

A global relative complete intersection (which we used to define standard syntomic ring maps) is in particular flat. In More on Morphisms, Section 36.42 we will consider morphisms $X \to S$ which locally are of the form

$$\mathrm{Spec}(R[x_1, \ldots, x_n]/(f_1, \ldots, f_c)) \to \mathrm{Spec}(R).$$

for some Koszul-regular sequence $f_1, \ldots, f_r$ in $R[x_1, \ldots, x_n]$. Such a morphism will be called a *local complete intersection morphism*. One we have this definition in place it will be the case that a morphism is syntomic if and only if it is a flat, local complete intersection morphism.

Note that there is no separation or quasi-compactness hypotheses in the definition of a syntomic morphism. Hence the question of being syntomic is local in nature on the source. Here is the precise result.

Lemma 28.32.2. *Let $f : X \to S$ be a morphism of schemes. The following are equivalent*

(1) *The morphism f is syntomic.*
(2) *For every affine opens $U \subset X$, $V \subset S$ with $f(U) \subset V$ the ring map $\mathcal{O}_S(V) \to \mathcal{O}_X(U)$ is syntomic.*

(3) *There exists an open covering* $S = \bigcup_{j \in J} V_j$ *and open coverings* $f^{-1}(V_j) = \bigcup_{i \in I_j} U_i$ *such that each of the morphisms* $U_i \to V_j$, $j \in J, i \in I_j$ *is syntomic.*
(4) *There exists an affine open covering* $S = \bigcup_{j \in J} V_j$ *and affine open coverings* $f^{-1}(V_j) = \bigcup_{i \in I_j} U_i$ *such that the ring map* $\mathcal{O}_S(V_j) \to \mathcal{O}_X(U_i)$ *is syntomic, for all* $j \in J, i \in I_j$.

Moreover, if f *is syntomic then for any open subschemes* $U \subset X$, $V \subset S$ *with* $f(U) \subset V$ *the restriction* $f|_U : U \to V$ *is syntomic.*

Proof. This follows from Lemma 28.15.3 if we show that the property "$R \to A$ is syntomic" is local. We check conditions (a), (b) and (c) of Definition 28.15.1. By Algebra, Lemma 10.129.3 being syntomic is stable under base change and hence we conclude (a) holds. By Algebra, Lemma 10.129.17 being syntomic is stable under composition and trivially for any ring R the ring map $R \to R_f$ is syntomic. We conclude (b) holds. Finally, property (c) is true according to Algebra, Lemma 10.129.4. □

Lemma 28.32.3. *The composition of two morphisms which are syntomic is syntomic.*

Proof. In the proof of Lemma 28.32.2 we saw that being syntomic is a local property of ring maps. Hence the first statement of the lemma follows from Lemma 28.15.5 combined with the fact that being syntomic is a property of ring maps that is stable under composition, see Algebra, Lemma 10.129.17. □

Lemma 28.32.4. *The base change of a morphism which is syntomic is syntomic.*

Proof. In the proof of Lemma 28.32.2 we saw that being syntomic is a local property of ring maps. Hence the lemma follows from Lemma 28.15.5 combined with the fact that being syntomic is a property of ring maps that is stable under base change, see Algebra, Lemma 10.129.3. □

Lemma 28.32.5. *Any open immersion is syntomic.*

Proof. This is true because an open immersion is a local isomorphism. □

Lemma 28.32.6. *A syntomic morphism is locally of finite presentation.*

Proof. True because a syntomic ring map is of finite presentation by definition. □

Lemma 28.32.7. *A syntomic morphism is flat.*

Proof. True because a syntomic ring map is flat by definition. □

Lemma 28.32.8. *A syntomic morphism is universally open.*

Proof. Combine Lemmas 28.32.6, 28.32.7, and 28.26.9. □

Let k be a field. Let A be a local k-algebra essentially of finite type over k. Recall that A is called a *complete intersection over* k if we can write $A \cong R/(f_1, \ldots, f_c)$ where R is a regular local ring essentially of finite type over k, and $f_1, \ldots, f_c$ is a regular sequence in R, see Algebra, Definition 10.128.5.

Lemma 28.32.9. *Let* k *be a field. Let* X *be a scheme locally of finite type over* k. *The following are equivalent:*

(1) X *is a local complete intersection over* k,
(2) *for every* $x \in X$ *there exists an affine open* $U = \mathrm{Spec}(R) \subset X$ *neighbourhood of* x *such that* $R \cong k[x_1, \ldots, x_n]/(f_1, \ldots, f_c)$ *is a global complete intersection over* k, *and*
(3) *for every* $x \in X$ *the local ring* $\mathcal{O}_{X,x}$ *is a complete intersection over* k.

Proof. The corresponding algebra results can be found in Algebra, Lemmas 10.128.8 and 10.128.9. □

The following lemma says locally any syntomic morphism is standard syntomic. Hence we can use standard syntomic morphisms as a *local model* for a syntomic morphism. Moreover, it says that a flat morphism of finite presentation is syntomic if and only if the fibres are local complete intersection schemes.

Lemma 28.32.10. *Let* $f : X \to S$ *be a morphism of schemes. Assume* f *locally of finite presentation. Let* $x \in X$ *be a point. Set* $s = f(x)$. *The following are equivalent*

(1) *The morphism* f *is syntomic at* x.
(2) *There exist affine opens* $U \subset X$, *and* $V \subset S$ *such that* $x \in U$, $f(U) \subset V$ *and the induced morphism* $f|_U : U \to V$ *is standard syntomic.*
(3) *The local ring map* $\mathcal{O}_{S,s} \to \mathcal{O}_{X,x}$ *is flat and* $\mathcal{O}_{X,x}/\mathfrak{m}_s\mathcal{O}_{X,x}$ *is a complete intersection over* $\kappa(s)$ *(see Algebra, Definition 10.128.5).*

Proof. Follows from the definitions and Algebra, Lemma 10.129.15. □

Lemma 28.32.11. *Let* $f : X \to S$ *be a morphism of schemes. If* f *is flat, locally of finite presentation, and all fibres* X_s *are local complete intersections, then* f *is syntomic.*

Proof. Clear from Lemmas 28.32.9 and 28.32.10 and the isomorphisms of local rings $\mathcal{O}_{X,x}/\mathfrak{m}_s\mathcal{O}_{X,x} \cong \mathcal{O}_{X_s,x}$. □

Lemma 28.32.12. *Let* $f : X \to S$ *be a morphism of schemes. Assume* f *locally of finite type. Formation of the set*

$$T = \{x \in X \mid \mathcal{O}_{X_{f(x)},x} \text{ is a complete intersection over } \kappa(f(x))\}$$

commutes with arbitrary base change: For any morphism $g : S' \to S$, *consider the base change* $f' : X' \to S'$ *of* f *and the projection* $g' : X' \to X$. *Then the corresponding set* T' *for the morphism* f' *is equal to* $T' = (g')^{-1}(T)$. *In particular, if* f *is assumed flat, and locally of finite presentation then the same holds for the open set of points where* f *is syntomic.*

Proof. Let $s' \in S'$ be a point, and let $s = g(s')$. Then we have

$$X'_{s'} = \mathrm{Spec}(\kappa(s')) \times_{\mathrm{Spec}(\kappa(s))} X_s$$

In other words the fibres of the base change are the base changes of the fibres. Hence the first part is equivalent to Algebra, Lemma 10.128.10. The second part follows from the first because in that case T is the set of points where f is syntomic according to Lemma 28.32.10. □

Lemma 28.32.13. *Let* R *be a ring. Let* $R \to A = R[x_1, \ldots, x_n]/(f_1, \ldots, f_c)$ *be a relative global complete intersection. Set* $S = \mathrm{Spec}(R)$ *and* $X = \mathrm{Spec}(A)$. *Consider the morphism* $f : X \to S$ *associated to the ring map* $R \to A$. *The function* $x \mapsto \dim_x(X_{f(x)})$ *is constant with value* $n - c$.

Proof. By Algebra, Definition 10.129.5 $R \to A$ being a relative global complete intersection means all nonzero fibre rings have dimension $n - c$. Thus for a prime $\mathfrak{p}$ of R the fibre ring $\kappa(\mathfrak{p})[x_1, \ldots, x_n]/(\overline{f}_1, \ldots, \overline{f}_c)$ is either zero or a global complete intersection ring of dimension $n - c$. By the discussion following Algebra, Definition 10.128.1 this implies it is equidimensional of dimension $n - c$. Whence the lemma. □

Lemma 28.32.14. *Let $f : X \to S$ be a syntomic morphism. The function $x \mapsto \dim_x(X_{f(x)})$ is locally constant on X.*

Proof. By Lemma 28.32.10 the morphism f locally looks like a standard syntomic morphism of affines. Hence the result follows from Lemma 28.32.13. □

Lemma 28.32.14 says that the following definition makes sense.

Definition 28.32.15. Let $d \geq 0$ be an integer. We say a morphism of schemes $f : X \to S$ is *syntomic of relative dimension d* if f is syntomic and the function $\dim_x(X_{f(x)}) = d$ for all $x \in X$.

In other words, f is syntomic and the nonempty fibres are equidimensional of dimension d.

Lemma 28.32.16. *Let*

$$\begin{array}{ccc} X & \xrightarrow{\quad f \quad} & Y \\ & {\scriptstyle p} \searrow \quad \swarrow {\scriptstyle q} & \\ & S & \end{array}$$

be a commutative diagram of morphisms of schemes. Assume that

(1) *f is surjective and syntomic,*
(2) *p is syntomic, and*
(3) *q is locally of finite presentation*[7]*.*

Then q is syntomic.

Proof. By Lemma 28.26.11 we see that q is flat. Hence it suffices to show that the fibres of $Y \to S$ are local complete intersections, see Lemma 28.32.11. Let $s \in S$. Consider the morphism $X_s \to Y_s$. This is a base change of the morphism $X \to Y$ and hence surjective, and syntomic (Lemma 28.32.4). For the same reason X_s is syntomic over $\kappa(s)$. Moreover, Y_s is locally of finite type over $\kappa(s)$ (Lemma 28.16.4). In this way we reduce to the case where S is the spectrum of a field.

Assume $S = \mathrm{Spec}(k)$. Let $y \in Y$. Choose an affine open $\mathrm{Spec}(A) \subset Y$ neighbourhood of y. Let $\mathrm{Spec}(B) \subset X$ be an affine open such that $f(\mathrm{Spec}(B)) \subset \mathrm{Spec}(A)$, containing a point $x \in X$ such that $f(x) = y$. Choose a surjection $k[x_1, \ldots, x_n] \to A$ with kernel I. Choose a surjection $A[y_1, \ldots, y_m] \to B$, which gives rise in turn to a surjection $k[x_i, y_j] \to B$ with kernel J. Let $\mathfrak{q} \subset k[x_i, y_j]$ be the prime corresponding to $y \in \mathrm{Spec}(B)$ and let $\mathfrak{p} \subset k[x_i]$ the prime corresponding to $x \in \mathrm{Spec}(A)$. Since x maps to y we have $\mathfrak{p} = \mathfrak{q} \cap k[x_i]$. Consider the following commutative diagram of

[7] In fact, if f is surjective, flat, and of finite presentation and p is sytomic, then both q and f are syntomic, see Descent, Lemma 34.10.7.

local rings:

$$\begin{array}{ccccc} \mathcal{O}_{X,x} & = & B_{\mathfrak{q}} & \longleftarrow & k[x_1, \ldots, x_n, y_1, \ldots, y_m]_{\mathfrak{q}} \\ \uparrow & & \uparrow & & \uparrow \\ \mathcal{O}_{Y,y} & = & A_{\mathfrak{p}} & \longleftarrow & k[x_1, \ldots, x_n]_{\mathfrak{p}} \end{array}$$

We claim that the hypotheses of Algebra, Lemma 10.128.12 are satisfied. Conditions (1) and (2) are trivial. Condition (4) follows as $X \to Y$ is flat. Condition (3) follows as the rings $\mathcal{O}_{Y,y}$ and $\mathcal{O}_{X_y,x} = \mathcal{O}_{X,x}/\mathfrak{m}_y\mathcal{O}_{X,x}$ are complete intersection rings by our assumptions that f and p are syntomic, see Lemma 28.32.10. The output of Algebra, Lemma 10.128.12 is exactly that $\mathcal{O}_{Y,y}$ is a complete intersection ring! Hence by Lemma 28.32.10 again we see that Y is syntomic over k at y as desired. □

28.33. Conormal sheaf of an immersion

Let $i : Z \to X$ be a closed immersion. Let $\mathcal{I} \subset \mathcal{O}_X$ be the corresponding quasi-coherent sheaf of ideals. Consider the short exact sequence

$$0 \to \mathcal{I}^2 \to \mathcal{I} \to \mathcal{I}/\mathcal{I}^2 \to 0$$

of quasi-coherent sheaves on X. Since the sheaf $\mathcal{I}/\mathcal{I}^2$ is annihilated by $\mathcal{I}$ it corresponds to a sheaf on Z by Lemma 28.4.1. This quasi-coherent $\mathcal{O}_Z$-module is called the *conormal sheaf of Z in X* and is often simply denoted $\mathcal{I}/\mathcal{I}^2$ by the abuse of notation mentioned in Section 28.4.

In case $i : Z \to X$ is a (locally closed) immersion we define the conormal sheaf of i as the conormal sheaf of the closed immersion $i : Z \to X \setminus \partial Z$, where $\partial Z = \overline{Z} \setminus Z$. It is often denoted $\mathcal{I}/\mathcal{I}^2$ where $\mathcal{I}$ is the ideal sheaf of the closed immersion $i : Z \to X \setminus \partial Z$.

Definition 28.33.1. Let $i : Z \to X$ be an immersion. The *conormal sheaf $\mathcal{C}_{Z/X}$ of Z in X* or the *conormal sheaf of i* is the quasi-coherent $\mathcal{O}_Z$-module $\mathcal{I}/\mathcal{I}^2$ described above.

In [**DG67**, IV Definition 16.1.2] this sheaf is denoted $\mathcal{N}_{Z/X}$. We will not follow this convention since we would like to reserve the notation $\mathcal{N}_{Z/X}$ for the *normal sheaf of the immersion*. It is defined as

$$\mathcal{N}_{Z/X} = \mathcal{H}om_{\mathcal{O}_Z}(\mathcal{C}_{Z/X}, \mathcal{O}_Z) = \mathcal{H}om_{\mathcal{O}_Z}(\mathcal{I}/\mathcal{I}^2, \mathcal{O}_Z)$$

provided the conormal sheaf is of finite presentation (otherwise the normal sheaf may not even be quasi-coherent). We will come back to the normal sheaf later (insert future reference here).

Lemma 28.33.2. *Let $i : Z \to X$ be an immersion. The conormal sheaf of i has the following properties:*

(1) *Let $U \subset X$ be any open such that $i(Z)$ is a closed subset of U. Let $\mathcal{I} \subset \mathcal{O}_U$ be the sheaf of ideals corresponding to the closed subscheme $i(Z) \subset U$. Then*
$$\mathcal{C}_{Z/X} = i^*\mathcal{I} = i^{-1}(\mathcal{I}/\mathcal{I}^2)$$
(2) *For any affine open $\mathrm{Spec}(R) = U \subset X$ such that $Z \cap U = \mathrm{Spec}(R/I)$ there is a canonical isomorphism $\Gamma(Z \cap U, \mathcal{C}_{Z/X}) = I/I^2$.*

Proof. Mostly clear from the definitions. Note that given a ring R and an ideal I of R we have $I/I^2 = I \otimes_R R/I$. Details omitted. □

Lemma 28.33.3. *Let*

$$\begin{array}{ccc} Z & \xrightarrow{i} & X \\ {\scriptstyle f}\downarrow & & \downarrow{\scriptstyle g} \\ Z' & \xrightarrow{i'} & X' \end{array}$$

be a commutative diagram in the category of schemes. Assume i, i' immersions. There is a canonical map of $\mathcal{O}_Z$-modules

$$f^*\mathcal{C}_{Z'/X'} \longrightarrow \mathcal{C}_{Z/X}$$

characterized by the following property: For every pair of affine opens $(\mathrm{Spec}(R) = U \subset X, \mathrm{Spec}(R') = U' \subset X')$ with $f(U) \subset U'$ such that $Z \cap U = \mathrm{Spec}(R/I)$ and $Z' \cap U' = \mathrm{Spec}(R'/I')$ the induced map

$$\Gamma(Z' \cap U', \mathcal{C}_{Z'/X'}) = I'/I'^2 \longrightarrow I/I^2 = \Gamma(Z \cap U, \mathcal{C}_{Z/X})$$

is the one induced by the ring map $f^\sharp : R' \to R$ which has the property $f^\sharp(I') \subset I$.

Proof. Let $\partial Z' = \overline{Z'} \setminus Z'$ and $\partial Z = \overline{Z} \setminus Z$. These are closed subsets of X' and of X. Replacing X' by $X' \setminus \partial Z'$ and X by $X \setminus \left(g^{-1}(\partial Z') \cup \partial Z\right)$ we see that we may assume that i and i' are closed immersions.

The fact that $g \circ i$ factors through i' implies that $g^*\mathcal{I}'$ maps into $\mathcal{I}$ under the canonical map $g^*\mathcal{I}' \to \mathcal{O}_X$, see Schemes, Lemmas 25.4.6 and 25.4.7. Hence we get an induced map of quasi-coherent sheaves $g^*(\mathcal{I}'/(\mathcal{I}')^2) \to \mathcal{I}/\mathcal{I}^2$. Pulling back by i gives $i^*g^*(\mathcal{I}'/(\mathcal{I}')^2) \to i^*(\mathcal{I}/\mathcal{I}^2)$. Note that $i^*(\mathcal{I}/\mathcal{I}^2) = \mathcal{C}_{Z/X}$. On the other hand, $i^*g^*(\mathcal{I}'/(\mathcal{I}')^2) = f^*(i')^*(\mathcal{I}'/(\mathcal{I}')^2) = f^*\mathcal{C}_{Z'/X'}$. This gives the desired map.

Checking that the map is locally described as the given map $I'/(I')^2 \to I/I^2$ is a matter of unwinding the definitions and is omitted. Another observation is that given any $x \in i(Z)$ there do exist affine open neighbourhoods U, U' with $f(U) \subset U'$ and $Z \cap U$ as well as $U' \cap Z'$ closed such that $x \in U$. Proof omitted. Hence the requirement of the lemma indeed characterizes the map (and could have been used to define it). □

Lemma 28.33.4. *Let*

$$\begin{array}{ccc} Z & \xrightarrow{i} & X \\ {\scriptstyle f}\downarrow & & \downarrow{\scriptstyle g} \\ Z' & \xrightarrow{i'} & X' \end{array}$$

be a fibre product diagram in the category of schemes with i, i' immersions. Then the canonical map $f^*\mathcal{C}_{Z'/X'} \to \mathcal{C}_{Z/X}$ of Lemma 28.33.3 is surjective. If g is flat, then it is an isomorphism.

Proof. Let $R' \to R$ be a ring map, and $I' \subset R'$ an ideal. Set $I = I'R$. Then $I'/(I')^2 \otimes_{R'} R \to I/I^2$ is surjective. If $R' \to R$ is flat, then $I = I' \otimes_{R'} R$ and $I^2 = (I')^2 \otimes_{R'} R$ and we see the map is an isomorphism. □

Lemma 28.33.5. *Let $Z \to Y \to X$ be immersions of schemes. Then there is a canonical exact sequence*

$$i^*\mathcal{C}_{Y/X} \to \mathcal{C}_{Z/X} \to \mathcal{C}_{Z/Y} \to 0$$

where the maps come from Lemma 28.33.3 and $i : Z \to Y$ is the first morphism.

Proof. Via Lemma 28.33.3 this translates into the following algebra fact. Suppose that $C \to B \to A$ are surjective ring maps. Let $I = \mathrm{Ker}(B \to A)$, $J = \mathrm{Ker}(C \to A)$ and $K = \mathrm{Ker}(C \to B)$. Then there is an exact sequence

$$K/K^2 \otimes_B A \to J/J^2 \to I/I^2 \to 0.$$

This follows immediately from the observation that $I = J/K$. □

28.34. Sheaf of differentials of a morphism

We suggest the reader take a look at the corresponding section in the chapter on commutative algebra (Algebra, Section 10.125) and the corresponding section in the chapter on sheaves of modules (Modules, Section 17.23).

Definition 28.34.1. Let $f : X \to S$ be a morphism of schemes. The *sheaf of differentials $\Omega_{X/S}$ of X over S* is the sheaf of differentials of f viewed as a morphism of ringed spaces (Modules, Definition 17.23.10) equipped with its *universal S-derivation*

$$\mathrm{d}_{X/S} : \mathcal{O}_X \longrightarrow \Omega_{X/S}.$$

It turns out that $\Omega_{X/S}$ is a quasi-coherent $\mathcal{O}_X$-module for example as it is isomorphic to the conormal sheaf of the diagonal morphism $\Delta : X \to X \times_S X$ (Lemma 28.34.5). We have defined the module of differentials of X over S using a universal property, namely as the receptacle of the universal derivation. If you have any other construction of the sheaf of relative differentials which satisfies this universal property then, by the Yoneda lemma, it will be canonically isomorphic to the one defined above. For convenience we restate the universal property here.

Lemma 28.34.2. *Let $f : X \to S$ be a morphism of schemes. The map*

$$\mathrm{Hom}_{\mathcal{O}_X}(\Omega_{X/S}, \mathcal{F}) \longrightarrow Der_S(\mathcal{O}_X, \mathcal{F}), \quad \alpha \longmapsto \alpha \circ d_{X/S}$$

is an isomorphism of functors $Mod(\mathcal{O}_X) \to Sets$.

Proof. This is just a restatement of the definition. □

Lemma 28.34.3. *Let $f : X \to S$ be a morphism of schemes. Let $U \subset X$, $V \subset S$ be open subschemes such that $f(U) \subset V$. Then there is a unique isomorphism $\Omega_{X/S}|_U = \Omega_{U/V}$ of $\mathcal{O}_U$-modules such that $d_{X/S}|_U = d_{U/V}$.*

Proof. This is a special case of Modules, Lemma 17.23.5 if we use the canonical identification $f^{-1}\mathcal{O}_S|_U = (f|_U)^{-1}\mathcal{O}_V$. □

From now on we will use these canonical identifications and simply write $\Omega_{U/S}$ or $\Omega_{U/V}$ for the restriction of $\Omega_{X/S}$ to U.

Lemma 28.34.4. *Let $R \to A$ be a ring map. Let $\mathcal{F}$ be a sheaf of $\mathcal{O}_X$-modules on $X = \mathrm{Spec}(A)$. Set $S = \mathrm{Spec}(R)$. The rule which associates to an S-derivation on $\mathcal{F}$ its action on global sections defines a bijection between the set of S-derivations of $\mathcal{F}$ and the set of R-derivations on $M = \Gamma(X, \mathcal{F})$.*

Proof. Let $D : A \to M$ be an R-derivation. We have to show there exists a unique S-derivation on $\mathcal{F}$ which gives rise to D on global sections. Let $U = D(f) \subset X$ be a standard affine open. Any element of $\Gamma(U, \mathcal{O}_X)$ is of the form a/f^n for some $a \in A$ and $n \geq 0$. By the Leibniz rule we have

$$D(a)|_U = a/f^n D(f^n)|_U + f^n D(a/f^n)$$

in $\Gamma(U, \mathcal{F})$. Since f acts invertibly on $\Gamma(U, \mathcal{F})$ this completely determines the value of $D(a/f^n) \in \Gamma(U, \mathcal{F})$. This proves uniqueness. Existence follows by simply defining

$$D(a/f^n) := (1/f^n)D(a)|_U - a/f^{2n}D(f^n)|_U$$

and proving this has all the desired properties (on the basis of standard opens of X). Details omitted. □

Lemma 28.34.5. *Let $f : X \to S$ be a morphism of schemes. For any pair of affine opens $\mathrm{Spec}(A) = U \subset X$, $\mathrm{Spec}(R) = V \subset S$ with $f(U) \subset V$ there is a unique isomorphism*

$$\Gamma(U, \mathcal{O}_{X/S}) = \Omega_{A/R}.$$

compatible with $d_{X/S}$ and $d : A \to \Omega_{A/R}$.

Proof. We claim that the A-module $M = \Gamma(U, \Omega_{X/S}) = \Gamma(U, \Omega_{U/V})$ together with $\mathrm{d}_{X/S} = \mathrm{d}_{U/V} : A \to M$ is a universal R-derivation of A. This follows by combining Lemmas 28.34.4 and 28.34.2 above. The universal property of $\mathrm{d} : A \to \Omega_{A/R}$ (see Algebra, Lemma 10.125.3) and the Yoneda lemma (Categories, Lemma 4.3.5) imply there is a unique isomorphism of A-modules $M \cong \Omega_{A/R}$ compatible with derivations. □

Remark 28.34.6. The lemma above gives a second way of constructing the module of differentials. Namely, let $f : X \to S$ be a morphism of schemes. Consider the collection of all affine opens $U \subset X$ which map into an affine open of S. These form a basis for the topology on X. Thus it suffices to define $\Gamma(U, \Omega_{X/S})$ for such U. We simply set $\Gamma(U, \Omega_{X/S}) = \Omega_{A/R}$ if A, R are as in Lemma 28.34.5 above. This works, but it takes somewhat more algebraic preliminaries to construct the restriction mappings and to verify the sheaf condition with this ansatz.

The following lemma gives yet another way to define the sheaf of differentials and it in particular shows that $\Omega_{X/S}$ is quasi-coherent if X and S are schemes.

Lemma 28.34.7. *Let $f : X \to S$ be a morphism of schemes. There is a canonical isomorphism between $\Omega_{X/S}$ and the conormal sheaf of the diagonal morphism $\Delta_{X/S} : X \longrightarrow X \times_S X$.*

Proof. We first establish the existence of a couple of "global" sheaves and global maps of sheaves, and further down we describe the constructions over some affine opens.

Recall that $\Delta = \Delta_{X/S} : X \to X \times_S X$ is an immersion, see Schemes, Lemma 25.21.2. Let $\mathcal{J}$ be the ideal sheaf of the immersion which lives over some open open subscheme W of $X \times_S X$ such that $\Delta(X) \subset W$ is closed. Let us take the one that was found in the proof of Schemes, Lemma 25.21.2. Note that the sheaf of rings $\mathcal{O}_U/\mathcal{J}^2$ is supported on $\Delta(X)$. Moreover it sits in a short exact sequence of sheaves

$$0 \to \mathcal{J}/\mathcal{J}^2 \to \mathcal{O}_W/\mathcal{J}^2 \to \Delta_*\mathcal{O}_X \to 0.$$

Using Δ^{-1} we can think of this as a surjection of sheaves of $f^{-1}\mathcal{O}_S$-algebras with kernel the conormal sheaf of Δ (see Definition 28.33.1 and Lemma 28.33.2).

$$0 \to \mathcal{C}_{X/X\times_S X} \to \Delta^{-1}(\mathcal{O}_W/\mathcal{J}^2) \to \mathcal{O}_X \to 0$$

This places us in the situation of Modules, Lemma 17.23.11. The projection morphisms $p_i : X \times_S X \to X$, $i = 1, 2$ induce maps of sheaves of rings $(p_i)^\sharp : (p_i)^{-1}\mathcal{O}_X \to \mathcal{O}_{X\times_S X}$. We may restrict to W and quotient by $\mathcal{J}^2$ to get $(p_i)^{-1}\mathcal{O}_X \to \mathcal{O}_W/\mathcal{J}^2$. Since $\Delta^{-1}p_i^{-1}\mathcal{O}_X = \mathcal{O}_X$ we get maps

$$s_i : \mathcal{O}_X \to \Delta^{-1}(\mathcal{O}_W/\mathcal{J}^2).$$

Both s_1 and s_2 are sections to the map $\Delta^{-1}(\mathcal{O}_W/\mathcal{J}^2) \to \mathcal{O}_X$, as in Modules, Lemma 17.23.11. Thus we get an S-derivation $\mathrm{d} = s_2 - s_1 : \mathcal{O}_X \to \mathcal{C}_{X/X\times_S X}$. By the universal property of the module of differentials we find a unique $\mathcal{O}_X$-linear map

$$\Omega_{X/S} \longrightarrow \mathcal{C}_{X/X\times_S X}, \quad f\mathrm{d}g \longmapsto fs_2(g) - fs_1(g)$$

To see the map is an isomorphism, let us work this out over suitable affine opens. We can cover X by affine opens $\mathrm{Spec}(A) = U \subset X$ whose image is contained in an affine open $\mathrm{Spec}(R) = V \subset S$. According to the proof of Schemes, Lemma 25.21.2 $U \times_V U \subset X \times_S X$ is an affine open contained in the open W mentioned above. Also $U \times_V U = \mathrm{Spec}(A \otimes_R A)$. The sheaf $\mathcal{J}$ corresponds to the ideal $J = \mathrm{Ker}(A \otimes_R A \to A)$. The short exact sequence to the short exact sequence of $A \otimes_R A$-modules

$$0 \to J/J^2 \to (A \otimes_R A)/J^2 \to A \to 0$$

The sections s_i correspond to the ring maps

$$A \longrightarrow (A \otimes_R A)/J^2, \quad s_1 : a \mapsto a \otimes 1, \quad s_2 : a \mapsto 1 \otimes a.$$

By Lemma 28.33.2 we have $\Gamma(U, \mathcal{C}_{X/X\times_S X}) = J/J^2$ and by Lemma 28.34.5 we have $\Gamma(U, \Omega_{X/S}) = \Omega_{A/R}$. The map above is the map $a\mathrm{d}b \mapsto a \otimes b - ab \otimes 1$ which is shown to be an isomorphism in Algebra, Lemma 10.125.13. □

Lemma 28.34.8. *Let*

$$\begin{array}{ccc} X' & \xrightarrow{f} & X \\ \downarrow & & \downarrow \\ S' & \longrightarrow & S \end{array}$$

be a commutative diagram of schemes. The canonical map $\mathcal{O}_X \to f_*\mathcal{O}_{X'}$ composed with the map $f_*d_{X'/S'} : f_*\mathcal{O}_{X'} \to f_*\Omega_{X'/S'}$ is a S-derivation. Hence we obtain a canonical map of $\mathcal{O}_X$-modules $\Omega_{X/S} \to f_*\Omega_{X'/S'}$, and by adjointness of f_* and f^* a canonical $\mathcal{O}_{X'}$-module homomorphism

$$c_f : f^*\Omega_{X/S} \longrightarrow \Omega_{X'/S'}.$$

It is uniquely characterized by the property that $f^*d_{X/S}(h)$ mapsto $d_{X'/S'}(f^*h)$ for any local section h of $\mathcal{O}_X$.

Proof. This is a special case of Modules, Lemma 17.23.12. In the case of schemes we can also use the functoriality of the conormal sheaves (see Lemma 28.33.3) and Lemma 28.34.7 to define c_f. Or we can use the characterization in the last line of the lemma to glue maps defined on affine patches (see Algebra, Equation (10.125.5.1)). □

Lemma 28.34.9. *Let $f : X \to Y$, $g : Y \to S$ be morphisms of schemes. Then there is a canonical exact sequence*

$$f^*\Omega_{Y/S} \to \Omega_{X/S} \to \Omega_{X/Y} \to 0$$

where the maps come from applications of Lemma 28.34.8.

Proof. This is the sheafified version of Algebra, Lemma 10.125.7. □

Lemma 28.34.10. *Let $X \to S$ be a morphism of schemes. Let $g : S' \to S$ be a morphism of schemes. Let $X' = X_{S'}$ be the base change of X. Denote $g' : X' \to X$ the projection. Then the map*

$$(g')^*\Omega_{X/S} \to \Omega_{X'/S'}$$

of Lemma 28.34.8 is an isomorphism.

Proof. This is the sheafified version of Algebra, Lemma 10.125.12. □

Lemma 28.34.11. *Let $f : X \to S$ and $g : Y \to S$ be morphisms of schemes with the same target. Let $p : X \times_S Y \to X$ and $q : X \times_S Y \to Y$ be the projection morphisms. The maps from Lemma 28.34.8*

$$p^*\Omega_{X/S} \oplus q^*\Omega_{Y/S} \longrightarrow \Omega_{X \times_S Y/S}$$

give an isomorphism.

Proof. By Lemma 28.34.10 the composition $p^*\Omega_{X/S} \to \Omega_{X \times_S Y/S} \to \Omega_{X \times_S Y/Y}$ is an isomorphism, and similarly for q. Moreover, the cokernel of $p^*\Omega_{X/S} \to \Omega_{X \times_S Y/S}$ is $\Omega_{X \times_S Y/X}$ by Lemma 28.34.9. The result follows. □

Lemma 28.34.12. *Let $f : X \to S$ be a morphism of schemes. If f is locally of finite type, then $\Omega_{X/S}$ is a finite type $\mathcal{O}_X$-module.*

Proof. Immediate from Algebra, Lemma 10.125.16, Lemma 28.34.5, Lemma 28.16.2, and Properties, Lemma 27.16.1. □

Lemma 28.34.13. *Let $f : X \to S$ be a morphism of schemes. If f is locally of finite type, then $\Omega_{X/S}$ is an $\mathcal{O}_X$-module of finite presentation.*

Proof. Immediate from Algebra, Lemma 10.125.15, Lemma 28.34.5, Lemma 28.22.2, and Properties, Lemma 27.16.2. □

Lemma 28.34.14. *If $X \to S$ is an immersion, or more generally a monomorphism, then $\Omega_{X/S}$ is zero.*

Proof. This is true because $\Delta_{X/S}$ is an isomorphism in this case and hence has trivial conormal sheaf. Hence $\Omega_{X/S} = 0$ by Lemma 28.34.7. The algebraic version is Algebra, Lemma 10.125.5. □

Lemma 28.34.15. *Let $i : Z \to X$ be an immersion of schemes over S. There is a canonical exact sequence*

$$\mathcal{C}_{Z/X} \to i^*\Omega_{X/S} \to \Omega_{Z/S} \to 0$$

where the first arrow is induced by $d_{X/S}$ and the second arrow comes from Lemma 28.34.8.

Proof. This is the sheafified version of Algebra, Lemma 10.125.9. However we should make sure we can define the first arrow globally. Hence we explain the meaning of "induced by $\mathrm{d}_{X/S}$" here. Namely, we may assume that i is a closed immersion by shrinking X. Let $\mathcal{I} \subset \mathcal{O}_X$ be the sheaf of ideals corresponding to $Z \subset X$. Then $\mathrm{d}_{X/S} : \mathcal{I} \to \Omega_{X/S}$ maps the subsheaf $\mathcal{I}^2 \subset \mathcal{I}$ to $\mathcal{I}\Omega_{X/S}$. Hence it induces a map $\mathcal{I}/\mathcal{I}^2 \to \Omega_{X/S}/\mathcal{I}\Omega_{X/S}$ which is $\mathcal{O}_X/\mathcal{I}$-linear. By Lemma 28.4.1 this corresponds to a map $\mathcal{C}_{Z/X} \to i^*\Omega_{X/S}$ as desired. □

Lemma 28.34.16. *Let $i : Z \to X$ be an immersion of schemes over S, and assume i (locally) has a left inverse. Then the canonical sequence*

$$0 \to \mathcal{C}_{Z/X} \to i^*\Omega_{X/S} \to \Omega_{Z/S} \to 0$$

of Lemma 28.34.15 is (locally) split exact. In particular, if $s : S \to X$ is a section of the structure morphism $X \to S$ then the map $\mathcal{C}_{S/X} \to s^*\Omega_{X/S}$ induced by $d_{X/S}$ is an isomorphism.

Proof. Follows from Algebra, Lemma 10.125.10. Clarification: if $g : X \to Z$ is a left inverse of i, then i^*c_g is a right inverse of the map $i^*\Omega_{X/S} \to \Omega_{Z/S}$. Also, if s is a section, then it is an immersion $s : Z = S \to X$ over S (see Schemes, Lemma 25.21.12) and in that case $\Omega_{Z/S} = 0$. □

Remark 28.34.17. Let $X \to S$ be a morphism of schemes. According to Lemma 28.34.11 we have

$$\Omega_{X \times_S X/S} = \mathrm{pr}_1^*\Omega_{X/S} \oplus \mathrm{pr}_2^*\Omega_{X/S}$$

On the other hand, the diagonal morphism $\Delta : X \to X \times_S X$ is an immersion, which locally has a left inverse. Hence by Lemma 28.34.16 we obtain a canonical short exact sequence

$$0 \to \mathcal{C}_{X/X \times_S X} \to \Omega_{X/S} \oplus \Omega_{X/S} \to \Omega_{X/S} \to 0$$

Note that the right arrow is $(1, 1)$ which is indeed a split surjection. On the other hand, by Lemma 28.34.7 we have an identification $\Omega_{X/S} = \mathcal{C}_{X/X \times_S X}$. Because we chose $\mathrm{d}_{X/S}(f) = s_2(f) - s_1(f)$ in this identification it turns out that the left arrow is the map $(-1, 1)$[8].

Lemma 28.34.18. *Let*

$$\begin{array}{ccc} Z & \xrightarrow{i} & X \\ & \searrow^{j} & \downarrow \\ & & Y \end{array}$$

be a commutative diagram of schemes where i and j are immersions. Then there is a canonical exact sequence

$$\mathcal{C}_{Z/Y} \to \mathcal{C}_{Z/X} \to i^*\Omega_{X/Y} \to 0$$

where the first arrow comes from Lemma 28.33.3 and the second from Lemma 28.34.15.

Proof. The algebraic version of this is Algebra, Lemma 10.127.7. □

[8] Namely, the local section $\mathrm{d}_{X/S}(f) = 1 \otimes f - f \otimes 1$ of the ideal sheaf of Δ maps via $\mathrm{d}_{X \times_S X/X}$ to the local section $1 \otimes 1 \otimes 1 \otimes f - 1 \otimes f \otimes 1 \otimes 1 - 1 \otimes 1 \otimes f \otimes 1 + f \otimes 1 \otimes 1 \otimes 1 = \mathrm{pr}_2^*\mathrm{d}_{X/S}(f) - \mathrm{pr}_1^*\mathrm{d}_{X/S}(f)$.

28.35. Smooth morphisms

Let $f : X \to Y$ be a map of topological spaces. Consider the following condition:

(*) For every $x \in X$ there exist open neighbourhoods $x \in U \subset X$ and $f(x) \in V \subset Y$, and an integer d such that $f(U) = V$ and such that there is an isomorphism

$$\begin{array}{ccccc} V \times B_d(0,1) & \xrightarrow{\cong} & U & \longrightarrow & X \\ \downarrow & & \downarrow & & \downarrow \\ V & = & V & \longrightarrow & Y \end{array}$$

where $B_d(0,1) \subset \mathbf{R}^d$ is a ball of radius 1 around 0.

Smooth morphisms are the analogue of such morphisms in the category of schemes. See Lemma 28.35.11 and Lemma 28.37.20.

Contrary to expectations (perhaps) the notion of a smooth ring map is not defined solely in terms of the module of differentials. Namely, recall that $R \to A$ is a *smooth ring map* if A is of finite presentation over R and if the naive cotangent complex of A over R is quasi-isomorphic to a projective module placed in degree 0, see Algebra, Definition 10.130.1.

Definition 28.35.1. Let $f : X \to S$ be a morphism of schemes.

(1) We say that f is *smooth at* $x \in X$ if there exists a affine open neighbourhood $\mathrm{Spec}(A) = U \subset X$ of x and affine open $\mathrm{Spec}(R) = V \subset S$ with $f(U) \subset V$ such that the induced ring map $R \to A$ is smooth.
(2) We say that f is *smooth* if it is smooth at every point of X.
(3) A morphism of affine schemes $f : X \to S$ is called *standard smooth* if there exists a standard smooth ring map $R \to R[x_1, \ldots, x_n]/(f_1, \ldots, f_c)$ (see Algebra, Definition 10.130.6) such that $X \to S$ is isomorphic to
$$\mathrm{Spec}(R[x_1, \ldots, x_n]/(f_1, \ldots, f_c)) \to \mathrm{Spec}(R).$$

A pleasing feature of this definition is that the set of points where a morphism is smooth is automatically open.

Note that there is no separation or quasi-compactness hypotheses in the definition. Hence the question of being smooth is local in nature on the source. Here is the precise result.

Lemma 28.35.2. *Let $f : X \to S$ be a morphism of schemes. The following are equivalent*

(1) *The morphism f is smooth.*
(2) *For every affine opens $U \subset X$, $V \subset S$ with $f(U) \subset V$ the ring map $\mathcal{O}_S(V) \to \mathcal{O}_X(U)$ is smooth.*
(3) *There exists an open covering $S = \bigcup_{j \in J} V_j$ and open coverings $f^{-1}(V_j) = \bigcup_{i \in I_j} U_i$ such that each of the morphisms $U_i \to V_j$, $j \in J, i \in I_j$ is smooth.*
(4) *There exists an affine open covering $S = \bigcup_{j \in J} V_j$ and affine open coverings $f^{-1}(V_j) = \bigcup_{i \in I_j} U_i$ such that the ring map $\mathcal{O}_S(V_j) \to \mathcal{O}_X(U_i)$ is smooth, for all $j \in J, i \in I_j$.*

Moreover, if f is smooth then for any open subschemes $U \subset X$, $V \subset S$ with $f(U) \subset V$ the restriction $f|_U : U \to V$ is smooth.

Proof. This follows from Lemma 28.15.3 if we show that the property "$R \to A$ is smooth" is local. We check conditions (a), (b) and (c) of Definition 28.15.1. By Algebra, Lemma 10.130.4 being smooth is stable under base change and hence we conclude (a) holds. By Algebra, Lemma 10.130.14 being smooth is stable under composition and for any ring R the ring map $R \to R_f$ is (standard) smooth. We conclude (b) holds. Finally, property (c) is true according to Algebra, Lemma 10.130.13. □

The following lemma characterizes a smooth morphism as a flat, finitely presented morphism with smooth fibres. Note that schemes smooth over a field are discussed in more detail in Varieties, Section 32.15.

Lemma 28.35.3. *Let $f : X \to S$ be a morphism of schemes. If f is flat, locally of finite presentation, and all fibres X_s are smooth, then f is smooth.*

Proof. Follows from Algebra, Lemma 10.130.16. □

Lemma 28.35.4. *The composition of two morphisms which are smooth is smooth.*

Proof. In the proof of Lemma 28.35.2 we saw that being smooth is a local property of ring maps. Hence the first statement of the lemma follows from Lemma 28.15.5 combined with the fact that being smooth is a property of ring maps that is stable under composition, see Algebra, Lemma 10.130.14. □

Lemma 28.35.5. *The base change of a morphism which is smooth is smooth.*

Proof. In the proof of Lemma 28.35.2 we saw that being smooth is a local property of ring maps. Hence the lemma follows from Lemma 28.15.5 combined with the fact that being smooth is a property of ring maps that is stable under base change, see Algebra, Lemma 10.130.4. □

Lemma 28.35.6. *Any open immersion is smooth.*

Proof. This is true because an open immersion is a local isomorphism. □

Lemma 28.35.7. *A smooth morphism is syntomic.*

Proof. See Algebra, Lemma 10.130.10. □

Lemma 28.35.8. *A smooth morphism is locally of finite presentation.*

Proof. True because a smooth ring map is of finite presentation by definition. □

Lemma 28.35.9. *A smooth morphism is flat.*

Proof. Combine Lemmas 28.32.7 and 28.35.7. □

Lemma 28.35.10. *A smooth morphism is universally open.*

Proof. Combine Lemmas 28.35.9, 28.35.8, and 28.26.9. Or alternatively, combine Lemmas 28.35.7, 28.32.8. □

The following lemma says locally any smooth morphism is standard smooth. Hence we can use standard smooth morphisms as a *local model* for a smooth morphism.

Lemma 28.35.11. *Let $f : X \to S$ be a morphism of schemes. Let $x \in X$ be a point. Set $s = f(x)$. The following are equivalent*

(1) *The morphism f is smooth at x.*

(2) *There exist affine opens $U \subset X$, and $V \subset S$ such that $x \in U$, $f(U) \subset V$ and the induced morphism $f|_U : U \to V$ is standard smooth.*

Proof. Follows from the definitions and Algebra, Lemmas 10.130.7 and 10.130.10. □

Lemma 28.35.12. *Let $f : X \to S$ be a morphism of schemes. Assume f is smooth. Then the module of differentials $\Omega_{X/S}$ of X over S is finite locally free and*

$$rank_x(\Omega_{X/S}) = \dim_x(X_{f(x)})$$

for every $x \in X$.

Proof. The statement is local on X and S. By Lemma 28.35.11 above we may assume that f is a standard smooth morphism of affines. In this case the result follows from Algebra, Lemma 10.130.7 (and the definition of a relative global complete intersection, see Algebra, Definition 10.129.5). □

Lemma 28.35.12 says that the following definition makes sense.

Definition 28.35.13. Let $d \geq 0$ be an integer. We say a morphism of schemes $f : X \to S$ is *smooth of relative dimension d* if f is smooth and $\Omega_{X/S}$ is finite locally free of constant rank d.

In other words, f is smooth and the nonempty fibres are equidimensional of dimension d. By Lemma 28.35.14 below this is also the same as requiring: (a) f is locally of finite presentation, (b) f is flat, (c) all nonempty fibres equidimensional of dimension d, and (d) $\Omega_{X/S}$ finite locally free of rank d. It is not enough to simply assume that f is flat, of finite presentation, and $\Omega_{X/S}$ is finite locally free of rank d. A counter example is given by $\mathrm{Spec}(\mathbf{F}_p[t]) \to \mathrm{Spec}(\mathbf{F}_p[t^p])$.

Here is a differential criterion of smoothness at a point. There are many variants of this result all of which may be useful at some point. We will just add them here as needed.

Lemma 28.35.14. *Let $f : X \to S$ be a morphism of schemes. Let $x \in X$. Set $s = f(x)$. Assume f is locally of finite presentation. The following are equivalent:*

(1) *The morphism f is smooth at x.*
(2) *The local ring map $\mathcal{O}_{S,s} \to \mathcal{O}_{X,x}$ is flat and the $\mathcal{O}_{X,x}$-module $\Omega_{X/S,x}$ can be generated by at most $\dim_x(X_{f(x)})$ elements.*
(3) *The local ring map $\mathcal{O}_{S,s} \to \mathcal{O}_{X,x}$ is flat and the $\kappa(x)$-vector space*

$$\Omega_{X_s/s,x} \otimes_{\mathcal{O}_{X_s,x}} \kappa(x) = \Omega_{X/S,x} \otimes_{\mathcal{O}_{X,x}} \kappa(x)$$

can be generated by at most $\dim_x(X_{f(x)})$ elements.
(4) *There exist affine opens $U \subset X$, and $V \subset S$ such that $x \in U$, $f(U) \subset V$ and the induced morphism $f|_U : U \to V$ is standard smooth.*
(5) *There exist affine opens $\mathrm{Spec}(A) = U \subset X$ and $\mathrm{Spec}(R) = V \subset S$ with $x \in U$ corresponding to $\mathfrak{q} \subset A$, and $f(U) \subset V$ such that there exists a presentation*

$$A = R[x_1, \ldots, x_n]/(f_1, \ldots, f_c)$$

with

$$g = \det \begin{pmatrix} \partial f_1/\partial x_1 & \partial f_2/\partial x_1 & \ldots & \partial f_c/\partial x_1 \\ \partial f_1/\partial x_2 & \partial f_2/\partial x_2 & \ldots & \partial f_c/\partial x_2 \\ \ldots & \ldots & \ldots & \ldots \\ \partial f_1/\partial x_c & \partial f_2/\partial x_c & \ldots & \partial f_c/\partial x_c \end{pmatrix}$$

mapping to an element of A not in $\mathfrak{q}$.

Proof. Note that if f is smooth at x, then we see from Lemma 28.35.11 that (4) holds, and (5) is a slightly weakened version of (4). Moreover, this implies that the ring map $\mathcal{O}_{S,s} \to \mathcal{O}_{X,x}$ is flat (see Lemma 28.35.9) and that $\Omega_{X/S}$ is finite locally free of rank equal to $\dim_x(X_s)$ (see Lemma 28.35.12). This implies (2) and (3).

By Lemma 28.34.10 the module of differentials $\Omega_{X_s/s}$ of the fibre X_s over $\kappa(s)$ is the pullback of the module of differentials $\Omega_{X/S}$ of X over S. Hence the displayed equality in part (3) of the lemma. By Lemma 28.34.12 these modules are of finite type. Hence the minimal number of generators of the modules $\Omega_{X/S,x}$ and $\Omega_{X_s/s,x}$ is the same and equal to the dimension of this $\kappa(x)$-vector space by Nakayama's Lemma (Algebra, Lemma 10.18.1). This in particular shows that (2) and (3) are equivalent.

Combining Algebra, Lemmas 10.130.16 and 10.133.3 shows that (2) and (3) imply (1). Finally, (5) implies (4) see for example Algebra, Example 10.130.8. □

Lemma 28.35.15. *Let $f : X \to S$ be a morphism of schemes. Assume f locally of finite type. Formation of the set*

$$T = \{x \in X \mid X_{f(x)} \text{ is smooth over } \kappa(f(x)) \text{ at } x\}$$

commutes with arbitrary base change: For any morphism $g : S' \to S$, consider the base change $f' : X' \to S'$ of f and the projection $g' : X' \to X$. Then the corresponding set T' for the morphism f' is equal to $T' = (g')^{-1}(T)$. In particular, if f is assumed flat, and locally of finite presentation then the same holds for the open set of points where f is smooth.

Proof. Let $s' \in S'$ be a point, and let $s = g(s')$. Then we have

$$X'_{s'} = \operatorname{Spec}(\kappa(s')) \times_{\operatorname{Spec}(\kappa(s))} X_s$$

In other words the fibres of the base change are the base changes of the fibres. Hence the first part is equivalent to Algebra, Lemma 10.130.18. The second part follows from the first because in that case T is the (open) set of points where f is smooth according to Lemma 28.35.3. □

Here is a lemma that actually uses the vanishing of H^{-1} of the naive cotangent complex for a smooth ring map.

Lemma 28.35.16. *Let $f : X \to Y$, $g : Y \to S$ be morphisms of schemes. Assume f is smooth. Then*

$$0 \to f^*\Omega_{Y/S} \to \Omega_{X/S} \to \Omega_{X/Y} \to 0$$

(see Lemma 28.34.9) is short exact.

Proof. The algebraic version of this lemma is the following: Given ring maps $A \to B \to C$ with $B \to C$ smooth, then the sequence

$$0 \to C \otimes_B \Omega_{B/A} \to \Omega_{C/A} \to \Omega_{C/B} \to 0$$

of Algebra, Lemma 10.125.7 is exact. This is Algebra, Lemma 10.132.1. □

Lemma 28.35.17. *Let $i : Z \to X$ be an immersion of schemes over S. Assume that Z is smooth over S. Then the canonical exact sequence*

$$0 \to \mathcal{C}_{Z/X} \to i^*\Omega_{X/S} \to \Omega_{Z/S} \to 0$$

of Lemma 28.34.15 is short exact.

Proof. The algebraic version of this lemma is the following: Given ring maps $A \to B \to C$ with $A \to C$ smooth and $B \to C$ surjective with kernel J, then the sequence

$$0 \to J/J^2 \to C \otimes_B \Omega_{B/A} \to \Omega_{C/A} \to 0$$

of Algebra, Lemma 10.125.9 is exact. This is Algebra, Lemma 10.132.2. □

Lemma 28.35.18. *Let*

$$\begin{array}{ccc} Z & \xrightarrow{i} & X \\ & \searrow^{j} & \downarrow \\ & & Y \end{array}$$

be a commutative diagram of schemes where i and j are immersions and $X \to Y$ is smooth. Then the canonical exact sequence

$$0 \to \mathcal{C}_{Z/Y} \to \mathcal{C}_{Z/X} \to i^*\Omega_{X/Y} \to 0$$

of Lemma 28.34.18 is exact.

Proof. The algebraic version of this lemma is the following: Given ring maps $A \to B \to C$ with $A \to C$ surjective and $A \to B$ smooth, then the sequence

$$0 \to I/I^2 \to J/J^2 \to C \otimes_B \Omega_{B/A} \to 0$$

of Algebra, Lemma 10.127.7 is exact. This is Algebra, Lemma 10.132.3. □

Lemma 28.35.19. *Let*

$$\begin{array}{ccc} X & \xrightarrow{f} & Y \\ & {\scriptstyle p}\searrow \quad \swarrow{\scriptstyle q} & \\ & S & \end{array}$$

be a commutative diagram of morphisms of schemes. Assume that

(1) *f is surjective, and smooth,*
(2) *p is smooth, and*
(3) *q is locally of finite presentation*[9].

Then q is smooth.

Proof. By Lemma 28.26.11 we see that q is flat. Pick a point $y \in Y$. Pick a point $x \in X$ mapping to y. Suppose f has relative dimension a at x and p has relative dimension b at x. By Lemma 28.35.12 this means that $\Omega_{X/S,x}$ is free of rank b and $\Omega_{X/Y,x}$ is free of rank a. By the short exact sequence of Lemma 28.35.16 this means that $(f^*\Omega_{Y/S})_x$ is free of rank $b - a$. By Nakayama's Lemma this implies that $\Omega_{Y/S,y}$ can be generated by $b - a$ elements. Also, by Lemma 28.29.2 we see that $\dim_y(Y_s) = b - a$. Hence we conclude that $Y \to S$ is smooth at y by Lemma 28.35.14 part (2). □

[9]In fact this is implied by (1) and (2), see Descent, Lemma 34.10.3. Moreover, it suffices to assume f is surjective, flat and locally of finite presentation, see Descent, Lemma 34.10.5.

In the situation of the following lemma the image of σ is locally on X cut out by a regular sequence, see Divisors, Lemma 30.14.7.

Lemma 28.35.20. *Let $f : X \to S$ be a morphism of schemes. Let $\sigma : S \to X$ be a section of f. Let $s \in S$ be a point such that f is smooth at $x = \sigma(s)$. Then there exist affine open neighbourhoods $\mathrm{Spec}(A) = U \subset S$ of s and $\mathrm{Spec}(B) = V \subset X$ of x such that*

1. *$f(V) \subset U$ and $\sigma(U) \subset V$,*
2. *with $I = Ker(\sigma^\# : B \to A)$ the module I/I^2 is a free A-module, and*
3. *$B^\wedge \cong A[[x_1, \ldots, x_d]]$ as A-algebras where $B^\wedge$ denotes the completion of B with respect to I.*

Proof. Pick an affine open $U \subset S$ containing s Pick an affine open $V \subset f^{-1}(U)$ containing x. Pick an affine open $U' \subset \sigma^{-1}(V)$ containing s. Note that $V' = f^{-1}(U') \cap V$ is affine as it is equal to the fibre product $V' = U' \times_U V$. Then U' and V' satisfy (1). Write $U' = \mathrm{Spec}(A')$ and $V' = \mathrm{Spec}(B')$. By Algebra, Lemma 10.132.4 the module $I'/(I')^2$ is finite locally free as a A'-module. Hence after replacing U' by a smaller affine open $U'' \subset U'$ and V' by $V'' = V' \cap f^{-1}(U'')$ we obtain the situation where $I''/(I'')^2$ is free, i.e., (2) holds. In this case (3) holds also by Algebra, Lemma 10.132.4. □

28.36. Unramified morphisms

We briefly discuss unramified morphisms before the (perhaps) more interesting class of étale morphisms. Recall that a ring map $R \to A$ is *unramified* if it is of finite type and $\Omega_{A/R} = 0$ (this is the definition of [**Ray70**]). A ring map $R \to A$ is called *G-unramified* if it is of finite presentation and $\Omega_{A/R} = 0$ (this is the definition of [**DG67**]). See Algebra, Definition 10.142.1.

Definition 28.36.1. Let $f : X \to S$ be a morphism of schemes.

1. We say that f is *unramified at $x \in X$* if there exists a affine open neighbourhood $\mathrm{Spec}(A) = U \subset X$ of x and affine open $\mathrm{Spec}(R) = V \subset S$ with $f(U) \subset V$ such that the induced ring map $R \to A$ is unramified.
2. We say that f is *G-unramified at $x \in X$* if there exists a affine open neighbourhood $\mathrm{Spec}(A) = U \subset X$ of x and affine open $\mathrm{Spec}(R) = V \subset S$ with $f(U) \subset V$ such that the induced ring map $R \to A$ is G-unramified.
3. We say that f is *unramified* if it is unramified at every point of X.
4. We say that f is *G-unramified* if it is G-unramified at every point of X.

Note that a G-unramified morphism is unramified. Hence any result for unramified morphisms implies the corresponding result for G-unramified morphisms. Moreover, if S is locally Noetherian then there is no difference between G-unramified and unramified morphisms, see Lemma 28.36.6. A pleasing feature of this definition is that the set of points where a morphism is unramified (resp. G-unramified) is automatically open.

Lemma 28.36.2. *Let $f : X \to S$ be a morphism of schemes. Then*

1. *f is unramified if and only if f is locally of finite type and $\Omega_{X/S} = 0$, and*
2. *f is G-unramified if and only if f is locally of finite presentation and $\Omega_{X/S} = 0$.*

Proof. By definition a ring map $R \to A$ is unramified (resp. G-unramified) if and only if it is of finite type (resp. finite presentation) and $\Omega_{A/R} = 0$. Hence the lemma follows directly from the definitions and Lemma 28.34.5. □

Note that there is no separation or quasi-compactness hypotheses in the definition. Hence the question of being unramified is local in nature on the source. Here is the precise result.

Lemma 28.36.3. *Let $f : X \to S$ be a morphism of schemes. The following are equivalent*

1. *The morphism f is unramified (resp. G-unramified).*
2. *For every affine opens $U \subset X$, $V \subset S$ with $f(U) \subset V$ the ring map $\mathcal{O}_S(V) \to \mathcal{O}_X(U)$ is unramified (resp. G-unramified).*
3. *There exists an open covering $S = \bigcup_{j \in J} V_j$ and open coverings $f^{-1}(V_j) = \bigcup_{i \in I_j} U_i$ such that each of the morphisms $U_i \to V_j$, $j \in J, i \in I_j$ is unramified (resp. G-unramified).*
4. *There exists an affine open covering $S = \bigcup_{j \in J} V_j$ and affine open coverings $f^{-1}(V_j) = \bigcup_{i \in I_j} U_i$ such that the ring map $\mathcal{O}_S(V_j) \to \mathcal{O}_X(U_i)$ is unramified (resp. G-unramified), for all $j \in J, i \in I_j$.*

Moreover, if f is unramified (resp. G-unramified) then for any open subschemes $U \subset X$, $V \subset S$ with $f(U) \subset V$ the restriction $f|_U : U \to V$ is unramified (resp. G-unramified).

Proof. This follows from Lemma 28.15.3 if we show that the property "$R \to A$ is unramified" is local. We check conditions (a), (b) and (c) of Definition 28.15.1. These properties are proved in Algebra, Lemma 10.142.3. □

Lemma 28.36.4. *The composition of two morphisms which are unramified is unramified. The same holds for G-unramified morphisms.*

Proof. The proof of Lemma 28.36.3 shows that being unramified (resp. G-unramified) is a local property of ring maps. Hence the first statement of the lemma follows from Lemma 28.15.5 combined with the fact that being unramified (resp. G-unramified) is a property of ring maps that is stable under composition, see Algebra, Lemma 10.142.3. □

Lemma 28.36.5. *The base change of a morphism which is unramified is unramified. The same holds for G-unramified morphisms.*

Proof. The proof of Lemma 28.36.3 shows that being unramified (resp. G-unramified) is a local property of ring maps. Hence the lemma follows from Lemma 28.15.5 combined with the fact that being unramified (resp. G-unramified) is a property of ring maps that is stable under base change, see Algebra, Lemma 10.142.3. □

Lemma 28.36.6. *Let $f : X \to S$ be a morphism of schemes. Assume S is locally Noetherian. Then f is unramified if and only if f is G-unramified.*

Proof. Follows from the definitions and Lemma 28.22.9. □

Lemma 28.36.7. *Any open immersion is G-unramified.*

Proof. This is true because an open immersion is a local isomorphism. □

Lemma 28.36.8. *A closed immersion $i : Z \to X$ is unramified. It is G-unramified if and only if the associated quasi-coherent sheaf of ideals $\mathcal{I} = Ker(\mathcal{O}_X \to i_*\mathcal{O}_Z)$ is of finite type (as an $\mathcal{O}_X$-module).*

Proof. Follows from Lemma 28.22.7 and Algebra, Lemma 10.142.3. □

Lemma 28.36.9. *An unramified morphism is locally of finite type. A G-unramified morphism is locally of finite presentation.*

Proof. An unramified ring map is of finite type by definition. A G-unramified ring map is of finite presentation by definition. □

Lemma 28.36.10. *Let $f : X \to S$ be a morphism of schemes. If f is unramified at x then f is quasi-finite at x. In particular, an unramified morphism is locally quasi-finite.*

Proof. See Algebra, Lemma 10.142.6. □

Lemma 28.36.11. *Fibres of unramified morphisms.*

(1) *Let X be a scheme over a field k. The structure morphism $X \to \mathrm{Spec}(k)$ is unramified if and only if X is a disjoint union of spectra of finite separable field extensions of k.*
(2) *If $f : X \to S$ is an unramified morphism then for every $s \in S$ the fibre X_s is a disjoint union of spectra of finite separable field extensions of $\kappa(s)$.*

Proof. Part (2) follows from part (1) and Lemma 28.36.5. Let us prove part (1). We first use Algebra, Lemma 10.142.7. This lemma implies that if X is a disjoint union of spectra of finite separable field extensions of k then $X \to \mathrm{Spec}(k)$ is unramified. Conversely, suppose that $X \to \mathrm{Spec}(k)$ is unramified. By Algebra, Lemma 10.142.5 for every $x \in X$ the residue field extension $k \subset \kappa(x)$ is finite separable. Hence all points of X are closed points (see Lemma 28.21.2 for example). Thus X is a discrete space, in particular the disjoint union of the spectra of its local rings. By Algebra, Lemma 10.142.5 again these local rings are fields, and we win. □

The following lemma characterizes an unramified morphisms as morphisms locally of finite type with unramified fibres.

Lemma 28.36.12. *Let $f : X \to S$ be a morphism of schemes.*

(1) *If f is unramified then for any $x \in X$ the field extension $\kappa(f(x)) \subset \kappa(x)$ is finite separable.*
(2) *If f is locally of finite type, and for every $s \in S$ the fibre X_s is a disjoint union of spectra of finite separable field extensions of $\kappa(s)$ then f is unramified.*
(3) *If f is locally of finite presentation, and for every $s \in S$ the fibre X_s is a disjoint union of spectra of finite separable field extensions of $\kappa(s)$ then f is G-unramified.*

Proof. Follows from Algebra, Lemmas 10.142.5 and 10.142.7. □

Here is a characterization of unramified morphisms in terms of the diagonal morphism.

Lemma 28.36.13. *Let $f : X \to S$ be a morphism.*

(1) *If f is unramified, then the diagonal morphism $\Delta : X \to X \times_S X$ is an open immersion.*
(2) *If f is locally of finite type and Δ is an open immersion, then f is unramified.*
(3) *If f is locally of finite presentation and Δ is an open immersion, then f is G-unramified.*

Proof. The first statement follows from Algebra, Lemma 10.142.4. The second statement from the fact that $\Omega_{X/S}$ is the conormal sheaf of the diagonal morphism (Lemma 28.34.7) and hence clearly zero if Δ is an open immersion. □

Lemma 28.36.14. *Let $f : X \to S$ be a morphism of schemes. Let $x \in X$. Set $s = f(x)$. Assume f is locally of finite type (resp. locally of finite presentation). The following are equivalent:*

(1) *The morphism f is unramified (resp. G-unramified) at x.*
(2) *The fibre X_s is unramified over $\kappa(s)$ at x.*
(3) *The $\mathcal{O}_{X,x}$-module $\Omega_{X/S,x}$ is zero.*
(4) *The $\mathcal{O}_{X_s,x}$-module $\Omega_{X_s/s,x}$ is zero.*
(5) *The $\kappa(x)$-vector space*
$$\Omega_{X_s/s,x} \otimes_{\mathcal{O}_{X_s,x}} \kappa(x) = \Omega_{X/S,x} \otimes_{\mathcal{O}_{X,x}} \kappa(x)$$
is zero.
(6) *We have $\mathfrak{m}_s\mathcal{O}_{X,x} = \mathfrak{m}_x$ and the field extension $\kappa(s) \subset \kappa(x)$ is finite separable.*

Proof. Note that if f is unramified at x, then we see that $\Omega_{X/S} = 0$ in a neighbourhood of x by the definitions and the results on modules of differentials in Section 28.34. Hence (1) implies (3) and the vanishing of the right hand vector space in (5). It also implies (2) because by Lemma 28.34.10 the module of differentials $\Omega_{X_s/s}$ of the fibre X_s over $\kappa(s)$ is the pullback of the module of differentials $\Omega_{X/S}$ of X over S. This fact on modules of differentials also implies the displayed equality of vector spaces in part (4). By Lemma 28.34.12 the modules $\Omega_{X/S,x}$ and $\Omega_{X_s/s,x}$ are of finite type. Hence he modules $\Omega_{X/S,x}$ and $\Omega_{X_s/s,x}$ are zero if and only if the corresponding $\kappa(x)$-vector space in (4) is zero by Nakayama's Lemma (Algebra, Lemma 10.18.1). This in particular shows that (3), (4) and (5) are equivalent. The support of $\Omega_{X/S}$ is closed in X, see Modules, Lemma 17.9.6. Assumption (3) implies that x is not in the support. Hence $\Omega_{X/S}$ is zero in a neighbourhood of x, which implies (1). The equivalence of (1) and (3) applied to $X_s \to s$ implies the equivalence of (2) and (4). At this point we have seen that (1) – (5) are equivalent.

Alternatively you can use Algebra, Lemma 10.142.3 to see the equivalence of (1) – (5) more directly.

The equivalence of (1) and (6) follows from Lemma 28.36.12. It also follows more directly from Algebra, Lemmas 10.142.5 and 10.142.7. □

Lemma 28.36.15. *Let $f : X \to S$ be a morphism of schemes. Assume f locally of finite type. Formation of the open set*
$$\begin{aligned} T &= \{x \in X \mid X_{f(x)} \text{ is unramified over } \kappa(f(x)) \text{ at } x\} \\ &= \{x \in X \mid X \text{ is unramified over } S \text{ at } x\} \end{aligned}$$

commutes with arbitrary base change: For any morphism $g : S' \to S$, consider the base change $f' : X' \to S'$ of f and the projection $g' : X' \to X$. Then the corresponding set T' for the morphism f' is equal to $T' = (g')^{-1}(T)$. If f is assumed locally of finite presentation then the same holds for the open set of points where f is G-unramified.

Proof. Let $s' \in S'$ be a point, and let $s = g(s')$. Then we have

$$X'_{s'} = \mathrm{Spec}(\kappa(s')) \times_{\mathrm{Spec}(\kappa(s))} X_s$$

In other words the fibres of the base change are the base changes of the fibres. In particular

$$\Omega_{X_s/s,x} \otimes_{\mathcal{O}_{X_s,x}} \kappa(x') = \Omega_{X'_{s'}/s',x'} \otimes_{\mathcal{O}_{X'_{s'},x'}} \kappa(x')$$

see Lemma 28.34.10. Whence $x' \in T'$ if and only if $x \in T$ by Lemma 28.36.14. The second part follows from the first because in that case T is the (open) set of points where f is G-unramified according to Lemma 28.36.14. □

Lemma 28.36.16. *Let $f : X \to Y$ be a morphism of schemes over S.*

1. *If X is unramified over S, then f is unramified.*
2. *If X is G-unramified over S and Y of finite type over S, then f is G-unramified.*

Proof. Assume that X is unramified over S. By Lemma 28.16.8 we see that f is locally of finite type. By assumption we have $\Omega_{X/S} = 0$. Hence $\Omega_{X/Y} = 0$ by Lemma 28.34.9. Thus f is unramified. If X is G-unramified over S and Y of finite type over S, then by Lemma 28.22.11 we see that f is locally of finite presentation and we conclude that f is G-unramified. □

Lemma 28.36.17. *Let S be a scheme. Let X, Y be schemes over S. Let $f, g : X \to Y$ be morphisms over S. Let $x \in X$. Assume that*

1. *the structure morphism $Y \to S$ is unramified,*
2. *$f(x) = g(x)$ in Y, say $y = f(x) = g(x)$, and*
3. *the induced maps $f^\sharp, g^\sharp : \kappa(y) \to \kappa(x)$ are equal.*

Then there exists an open neighbourhood of x in X on which f and g are equal.

Proof. Consider the morphism $(f, g) : X \to Y \times_S Y$. By assumption (1) and Lemma 28.36.13 the inverse image of $\Delta_{Y/S}(Y)$ is open in X. And assumptions (2) and (3) imply that x is in this open subset. □

28.37. Étale morphisms

The Zariski topology of a scheme is a very coarse topology. This is particularly clear when looking at varieties over $\mathbf{C}$. It turns out that declaring an étale morphism to be the analogue of a local isomorphism in topology introduces a much finer topology. On varieties over $\mathbf{C}$ this topology gives rise to the "correct" Betti numbers when computing cohomology with finite coefficients. Another observable is that if $f : X \to Y$ is an étale morphism of varieties over $\mathbf{C}$, and if x is a closed point of X, then f induces an isomorphism $\mathcal{O}^\wedge_{Y,f(x)} \to \mathcal{O}^\wedge_{X,x}$ of complete local rings.

In this section we start our study of these matters. In fact we deliberately restrict our discussion to a minimum since we will discuss more interesting results elsewhere. Recall that a ring map $R \to A$ is said to be *étale* if it is smooth and $\Omega_{A/R} = 0$, see Algebra, Definition 10.136.1.

Definition 28.37.1. Let $f : X \to S$ be a morphism of schemes.

(1) We say that f is *étale at* $x \in X$ if there exists a affine open neighbourhood $\mathrm{Spec}(A) = U \subset X$ of x and affine open $\mathrm{Spec}(R) = V \subset S$ with $f(U) \subset V$ such that the induced ring map $R \to A$ is étale.
(2) We say that f is *étale* if it is étale at every point of X.
(3) A morphism of affine schemes $f : X \to S$ is called *standard étale* if $X \to S$ is isomorphic to
$$\mathrm{Spec}(R[x]_g/(f)) \to \mathrm{Spec}(R)$$
where $R \to R[x]_g/(f)$ is a standard étale ring map, see Algebra, Definition 10.136.14, i.e., f is monic and f' invertible in $R[x]_g/(f)$.

A morphism is étale if and only if it is smooth of relative dimension 0 (see Definition 28.35.13). A pleasing feature of the definition is that the set of points where a morphism is étale is automatically open.

Note that there is no separation or quasi-compactness hypotheses in the definition. Hence the question of being étale is local in nature on the source. Here is the precise result.

Lemma 28.37.2. *Let $f : X \to S$ be a morphism of schemes. The following are equivalent*

(1) *The morphism f is étale.*
(2) *For every affine opens $U \subset X$, $V \subset S$ with $f(U) \subset V$ the ring map $\mathcal{O}_S(V) \to \mathcal{O}_X(U)$ is étale.*
(3) *There exists an open covering $S = \bigcup_{j \in J} V_j$ and open coverings $f^{-1}(V_j) = \bigcup_{i \in I_j} U_i$ such that each of the morphisms $U_i \to V_j$, $j \in J, i \in I_j$ is étale.*
(4) *There exists an affine open covering $S = \bigcup_{j \in J} V_j$ and affine open coverings $f^{-1}(V_j) = \bigcup_{i \in I_j} U_i$ such that the ring map $\mathcal{O}_S(V_j) \to \mathcal{O}_X(U_i)$ is étale, for all $j \in J, i \in I_j$.*

Moreover, if f is étale then for any open subschemes $U \subset X$, $V \subset S$ with $f(U) \subset V$ the restriction $f|_U : U \to V$ is étale.

Proof. This follows from Lemma 28.15.3 if we show that the property "$R \to A$ is étale" is local. We check conditions (a), (b) and (c) of Definition 28.15.1. These all follow from Algebra, Lemma 10.136.3. □

Lemma 28.37.3. *The composition of two morphisms which are étale is étale.*

Proof. In the proof of Lemma 28.37.2 we saw that being étale is a local property of ring maps. Hence the first statement of the lemma follows from Lemma 28.15.5 combined with the fact that being étale is a property of ring maps that is stable under composition, see Algebra, Lemma 10.136.3. □

Lemma 28.37.4. *The base change of a morphism which is étale is étale.*

Proof. In the proof of Lemma 28.37.2 we saw that being étale is a local property of ring maps. Hence the lemma follows from Lemma 28.15.5 combined with the fact that being étale is a property of ring maps that is stable under base change, see Algebra, Lemma 10.136.3. □

Lemma 28.37.5. *Let $f : X \to S$ be a morphism of schemes. Let $x \in X$. Then f is étale at x if and only if f is smooth and unramified at x.*

Proof. This follows immediately from the definitions. □

Lemma 28.37.6. *An étale morphism is locally quasi-finite.*

Proof. By Lemma 28.37.5 an étale morphism is unramified. By Lemma 28.36.10 an unramified morphism is locally quasi-finite. □

Lemma 28.37.7. *Fibres of étale morphisms.*

(1) *Let X be a scheme over a field k. The structure morphism $X \to \mathrm{Spec}(k)$ is étale if and only if X is a disjoint union of spectra of finite separable field extensions of k.*
(2) *If $f : X \to S$ is an étale morphism, then for every $s \in S$ the fibre X_s is a disjoint union of spectra of finite separable field extensions of $\kappa(s)$.*

Proof. You can deduce this from Lemma 28.36.11 via Lemma 28.37.5 above. Here is a direct proof.

We will use Algebra, Lemma 10.136.4. Hence it is clear that if X is a disjoint union of spectra of finite separable field extensions of k then $X \to \mathrm{Spec}(k)$ is étale. Conversely, suppose that $X \to \mathrm{Spec}(k)$ is étale. Then for any affine open $U \subset X$ we see that U is a finite disjoint union of spectra of finite separable field extensions of k. Hence all points of X are closed points (see Lemma 28.21.2 for example). Thus X is a discrete space and we win. □

The following lemma characterizes an étale morphism as a flat, finitely presented morphism with "étale fibres".

Lemma 28.37.8. *Let $f : X \to S$ be a morphism of schemes. If f is flat, locally of finite presentation, and for every $s \in S$ the fibre X_s is a disjoint union of spectra of finite separable field extensions of $\kappa(s)$, then f is étale.*

Proof. You can deduce this from Algebra, Lemma 10.136.7. Here is another proof.

By Lemma 28.37.7 a fibre X_s is étale and hence smooth over s. By Lemma 28.35.3 we see that $X \to S$ is smooth. By Lemma 28.36.12 we see that f is unramified. We conclude by Lemma 28.37.5. □

Lemma 28.37.9. *Any open immersion is étale.*

Proof. This is true because an open immersion is a local isomorphism. □

Lemma 28.37.10. *An étale morphism is syntomic.*

Proof. See Algebra, Lemma 10.130.10 and use that an étale morphism is the same as a smooth morphism of relative dimension 0. □

Lemma 28.37.11. *An étale morphism is locally of finite presentation.*

Proof. True because an étale ring map is of finite presentation by definition. □

Lemma 28.37.12. *An étale morphism is flat.*

Proof. Combine Lemmas 28.32.7 and 28.37.10. □

Lemma 28.37.13. *An étale morphism is open.*

Proof. Combine Lemmas 28.37.12, 28.37.11, and 28.26.9. □

The following lemma says locally any étale morphism is standard étale. This is actually kind of a tricky result to prove in complete generality. The tricky parts are hidden in the chapter on commutative algebra. Hence a standard étale morphism is a *local model* for a general étale morphism.

Lemma 28.37.14. *Let $f : X \to S$ be a morphism of schemes. Let $x \in X$ be a point. Set $s = f(x)$. The following are equivalent*

(1) *The morphism f is étale at x.*
(2) *There exist affine opens $U \subset X$, and $V \subset S$ such that $x \in U$, $f(U) \subset V$ and the induced morphism $f|_U : U \to V$ is standard étale (see Definition 28.37.1).*

Proof. Follows from the definitions and Algebra, Proposition 10.136.17. □

Here is a differential criterion of étaleness at a point. There are many variants of this result all of which may be useful at some point. We will just add them here as needed.

Lemma 28.37.15. *Let $f : X \to S$ be a morphism of schemes. Let $x \in X$. Set $s = f(x)$. Assume f is locally of finite presentation. The following are equivalent:*

(1) *The morphism f is étale at x.*
(2) *The local ring map $\mathcal{O}_{S,s} \to \mathcal{O}_{X,x}$ is flat and the $\mathcal{O}_{X,x}$-module $\Omega_{X/S,x}$ is zero.*
(3) *The local ring map $\mathcal{O}_{S,s} \to \mathcal{O}_{X,x}$ is flat and the $\kappa(x)$-vector space*
$$\Omega_{X_s/s,x} \otimes_{\mathcal{O}_{X_s,x}} \kappa(x) = \Omega_{X/S,x} \otimes_{\mathcal{O}_{X,x}} \kappa(x)$$
is zero.
(4) *The local ring map $\mathcal{O}_{S,s} \to \mathcal{O}_{X,x}$ is flat, we have $\mathfrak{m}_s\mathcal{O}_{X,x} = \mathfrak{m}_x$ and the field extension $\kappa(s) \subset \kappa(x)$ is finite separable.*
(5) *There exist affine opens $U \subset X$, and $V \subset S$ such that $x \in U$, $f(U) \subset V$ and the induced morphism $f|_U : U \to V$ is standard smooth of relative dimension 0.*
(6) *There exist affine opens $\mathrm{Spec}(A) = U \subset X$ and $\mathrm{Spec}(R) = V \subset S$ with $x \in U$ corresponding to $\mathfrak{q} \subset A$, and $f(U) \subset V$ such that there exists a presentation*
$$A = R[x_1, \ldots, x_n]/(f_1, \ldots, f_n)$$
with
$$g = \det \begin{pmatrix} \partial f_1/\partial x_1 & \partial f_2/\partial x_1 & \ldots & \partial f_n/\partial x_1 \\ \partial f_1/\partial x_2 & \partial f_2/\partial x_2 & \ldots & \partial f_n/\partial x_2 \\ \ldots & \ldots & \ldots & \ldots \\ \partial f_1/\partial x_n & \partial f_2/\partial x_n & \ldots & \partial f_n/\partial x_n \end{pmatrix}$$
mapping to an element of A not in $\mathfrak{q}$.
(7) *There exist affine opens $U \subset X$, and $V \subset S$ such that $x \in U$, $f(U) \subset V$ and the induced morphism $f|_U : U \to V$ is standard étale.*
(8) *There exist affine opens $\mathrm{Spec}(A) = U \subset X$ and $\mathrm{Spec}(R) = V \subset S$ with $x \in U$ corresponding to $\mathfrak{q} \subset A$, and $f(U) \subset V$ such that there exists a presentation*
$$A = R[x]_Q/(P) = R[x, 1/Q]/(P)$$

with $P, Q \in R[x]$, P monic and $P' = dP/dx$ mapping to an element of A not in $\mathfrak{q}$.

Proof. Use Lemma 28.37.14 and the definitions to see that (1) implies all of the other conditions. For each of the conditions (2) – (7) combine Lemmas 28.35.14 and 28.36.14 to see that (1) holds by showing f is both smooth and unramified at x and applying Lemma 28.37.5. Some details omitted. □

Lemma 28.37.16. *A morphism is étale at a point if and only if it is flat and G-unramified at that point. A morphism is étale if and only if it is flat and G-unramified.*

Proof. This is clear from Lemmas 28.37.15 and 28.36.14. □

Lemma 28.37.17. *Let $f : X \to S$ be a morphism of schemes. Assume f locally of finite type. Formation of the set*

$$T = \{x \in X \mid X_{f(x)} \text{ is étale over } \kappa(f(x)) \text{ at } x\}$$

commutes with arbitrary base change: For any morphism $g : S' \to S$, consider the base change $f' : X' \to S'$ of f and the projection $g' : X' \to X$. Then the corresponding set T' for the morphism f' is equal to $T' = (g')^{-1}(T)$. In particular, if f is assumed locally of finite presentation and flat then the same holds for the open set of points where f is étale.

Proof. Combine Lemmas 28.37.16 and 28.36.15. □

Our proof of the following lemma is somewhat complicated. It uses the "Critère de platitude par fibres" to see that a morphism $X \to Y$ over S between schemes étale over S is automatically flat. The details are in the chapter on commutative algebra.

Lemma 28.37.18. *Let $f : X \to Y$ be a morphism of schemes over S. If X and Y are étale over S, then f is étale.*

Proof. See Algebra, Lemma 10.136.9. □

Lemma 28.37.19. *Let*

$$\begin{array}{ccc} X & \xrightarrow{f} & Y \\ & {\scriptstyle p}\searrow \quad \swarrow{\scriptstyle q} & \\ & S & \end{array}$$

be a commutative diagram of morphisms of schemes. Assume that

(1) *f is surjective, and étale,*
(2) *p is étale, and*
(3) *q is locally of finite presentation*[10].

Then q is étale.

Proof. By Lemma 28.35.19 we see that q is smooth. Thus we only need to see that q has relative dimension 0. This follows from Lemma 28.29.2 and the fact that f and p have relative dimension 0. □

[10] In fact this is implied by (1) and (2), see Descent, Lemma 34.10.3. Moreover, it suffices to assume that f is surjective, flat and locally of finite presentation, see Descent, Lemma 34.10.5.

A final characterization of smooth morphisms is that a smooth morphism $f : X \to S$ is locally the composition of an étale morphism by a projection $\mathbf{A}^d_S \to S$.

Lemma 28.37.20. *Let $\varphi : X \to Y$ be a morphism of schemes. Let $x \in X$. If φ is smooth at x, then there exist exist an integer $d \geq 0$ and affine opens $V \subset Y$ and $U \subset X$ with $x \in U$ and $\varphi(U) \subset V$ such that there exists a commutative diagram*

$$\begin{array}{ccccc} X & \longleftarrow & U & \xrightarrow[\pi]{} & \mathbf{A}^d_V \\ \downarrow & & \downarrow & \swarrow & \\ Y & \longleftarrow & V & & \end{array}$$

where π is étale.

Proof. By Lemma 28.35.11 we can find affine opens U and V as in the lemma such that $\varphi|_U : U \to V$ is standard smooth. Write $U = \mathrm{Spec}(A)$ and $V = \mathrm{Spec}(R)$ so that we can write

$$A = R[x_1, \ldots, x_n]/(f_1, \ldots, f_c)$$

with

$$g = \det \begin{pmatrix} \partial f_1/\partial x_1 & \partial f_2/\partial x_1 & \ldots & \partial f_c/\partial x_1 \\ \partial f_1/\partial x_2 & \partial f_2/\partial x_2 & \ldots & \partial f_c/\partial x_2 \\ \ldots & \ldots & \ldots & \ldots \\ \partial f_1/\partial x_c & \partial f_2/\partial x_c & \ldots & \partial f_c/\partial x_c \end{pmatrix}$$

mapping to an invertible element of A. Then it is clear that $R[x_{c+1}, \ldots, x_n] \to A$ is standard smooth of relative dimension 0. Hence it is smooth of relative dimension 0. In other words the ring map $R[x_{c+1}, \ldots, x_n] \to A$ is étale. As $\mathbf{A}^{n-c}_V = \mathrm{Spec}(R[x_{c+1}, \ldots, x_n])$ the lemma with $d = n - c$. □

28.38. Relatively ample sheaves

Let X be a scheme and $\mathcal{L}$ an invertible sheaf on X. Then $\mathcal{L}$ is ample on X if X is quasi-compact and every point of X is contained in an affine open of the form X_s, where $s \in \Gamma(X, \mathcal{L}^{\otimes n})$ and $n \geq 1$, see Properties, Definition 27.24.1. We relativize this as follows.

Definition 28.38.1. Let $f : X \to S$ be a morphism of schemes. Let $\mathcal{L}$ be an invertible $\mathcal{O}_X$-module. We say $\mathcal{L}$ is *relatively ample*, or *f-relatively ample*, or *ample on X/S*, or *f-ample* if $f : X \to S$ is quasi-compact, and if for every affine open $V \subset S$ the restriction of $\mathcal{L}$ to the open subscheme $f^{-1}(V)$ of X is ample.

We note that the existence of a relatively ample sheaf on X does not force the morphism $X \to S$ to be of finite type.

Lemma 28.38.2. *Let $X \to S$ be a morphism of schemes. Let $\mathcal{L}$ be an invertible $\mathcal{O}_X$-module. Let $n \geq 1$. Then $\mathcal{L}$ is f-ample if and only if $\mathcal{L}^{\otimes n}$ is f-ample.*

Proof. This follows from Properties, Lemma 27.24.2. □

Lemma 28.38.3. *Let $f : X \to S$ be a morphism of schemes. If there exists an f-ample invertible sheaf, then f is separated.*

Proof. Being separated is local on the base (see Schemes, Lemma 25.21.8 for example; it also follows easily from the definition). Hence we may assume S is affine and X has an ample invertible sheaf. In this case the result follows from Properties, Lemma 27.24.9. □

There are many ways to characterize relatively ample invertible sheaves, by relativizing any of the list of equivalent conditions in Properties, Proposition 27.24.14. We will add these here as needed.

Lemma 28.38.4. *Let $f : X \to S$ be a quasi-compact morphism of schemes. Let $\mathcal{L}$ be an invertible sheaf on X. The following are equivalent:*

1. *The invertible sheaf $\mathcal{L}$ is f-ample.*
2. *There exists an open covering $S = \bigcup V_i$ such that each $\mathcal{L}|_{f^{-1}(V_i)}$ is ample relative to $f^{-1}(V_i) \to V_i$.*
3. *There exists an affine open covering $S = \bigcup V_i$ such that each $\mathcal{L}|_{f^{-1}(V_i)}$ is ample.*
4. *There exists a quasi-coherent graded $\mathcal{O}_S$-algebra $\mathcal{A}$ and a map of graded $\mathcal{O}_X$-algebras $\psi : f^*\mathcal{A} \to \bigoplus_{d \geq 0} \mathcal{L}^{\otimes d}$ such that $U(\psi) = X$ and*
$$r_{\mathcal{L},\psi} : X \longrightarrow \underline{Proj}_S(\mathcal{A})$$
is an open immersion (see Constructions, Lemma 26.19.1 for notation).
5. *The morphism f is quasi-separated and part (4) above holds with $\mathcal{A} = f_*(\bigoplus_{d \geq 0} \mathcal{L}^{\otimes d})$ and ψ the adjunction mapping.*
6. *Same as (4) but just requiring $r_{\mathcal{L},\psi}$ to be an immersion.*

Proof. It is immediate from the definition that (1) implies (2) and (2) implies (3). It is clear that (5) implies (4).

Assume (3) holds for the affine open covering $S = \bigcup V_i$. We are going to show (5) holds. Since each $f^{-1}(V_i)$ has an ample invertible sheaf we see that $f^{-1}(V_i)$ is separated (see Properties, Lemma 27.24.12 and Constructions, Lemma 26.8.8). Hence f is separated. By Schemes, Lemma 25.24.1 we see that $\mathcal{A} = f_*(\bigoplus_{d \geq 0} \mathcal{L}^{\otimes d})$ is a quasi-coherent graded $\mathcal{O}_S$-algebra. Denote $\psi : f^*\mathcal{A} \to \bigoplus_{d \geq 0} \mathcal{L}^{\otimes d}$ the adjunction mapping. The description of the open $U(\psi)$ in Constructions, Section 26.19 and the definition of ampleness of $\mathcal{L}|_{f^{-1}(V_i)}$ show that $U(\psi) = X$. Moreover, Constructions, Lemma 26.19.1 part (3) shows that the restriction of $r_{\mathcal{L},\psi}$ to $f^{-1}(V_i)$ is the same as the morphism from Properties, Lemma 27.24.10 which is an open immersion according to Properties, Lemma 27.24.12. Hence (5) holds.

Let us show that (4) implies (1). Assume (4). Denote $\pi : \underline{\mathrm{Proj}}_S(\mathcal{A}) \to S$ the structure morphism. Choose $V \subset S$ affine open. By Constructions, Definition 26.16.7 we see that $\pi^{-1}(V) \subset \underline{\mathrm{Proj}}_S(\mathcal{A})$ is equal to $\mathrm{Proj}(A)$ where $A = \mathcal{A}(V)$ as a graded ring. Hence $r_{\mathcal{L},\psi}$ maps $f^{-1}(V)$ isomorphically onto a quasi-compact open of $\mathrm{Proj}(A)$. Moreover, $\mathcal{L}^{\otimes d}$ is isomorphic to the pullback of $\mathcal{O}_{\mathrm{Proj}(A)}(d)$ for some $d \geq 1$. (See part (3) of Constructions, Lemma 26.19.1 and the final statement of Constructions, Lemma 26.14.1.) This implies that $\mathcal{L}|_{f^{-1}(V)}$ is ample by Properties, Lemmas 27.24.13 and 27.24.2.

Assume (6). By the equivalence of (1) - (5) above we see that the property of being relatively ample on X/S is local on S. Hence we may assume that S is affine, and we have to show that $\mathcal{L}$ is ample on X. In this case the morphism $r_{\mathcal{L},\psi}$ is identified with the morphism, also denoted $r_{\mathcal{L},\psi} : X \to \mathrm{Proj}(A)$ associated to the map $\psi : A = \mathcal{A}(V) \to \Gamma_*(X, \mathcal{L})$. (See references above.) As above we also see that $\mathcal{L}^{\otimes d}$ is the pullback of the sheaf $\mathcal{O}_{\mathrm{Proj}(A)}(d)$ for some $d \geq 1$. Moreover, since X is quasi-compact we see that X gets identified with a closed subscheme of a quasi-compact open subscheme $Y \subset \mathrm{Proj}(A)$. By Constructions, Lemma 26.10.6 (see

also Properties, Lemma 27.24.13) we see that $\mathcal{O}_Y(d')$ is an ample invertible sheaf on Y for some $d' \geq 1$. Since the restriction of an ample sheaf to a closed subscheme is ample, see Properties, Lemma 27.24.3 we conclude that the pullback of $\mathcal{O}_Y^{d'}$ is ample. Combining these results with Properties, Lemma 27.24.2 we conclude that $\mathcal{L}$ is ample as desired. □

Lemma 28.38.5. *Let $f : X \to S$ be a morphism of schemes. Let $\mathcal{L}$ be an invertible $\mathcal{O}_X$-module. Assume S affine. Then $\mathcal{L}$ is f-relatively ample if and only if $\mathcal{L}$ is ample on X.*

Proof. Immediate from Lemma 28.38.4 and the definitions. □

Lemma 28.38.6. *Let $f : X \to S$ be a morphism of schemes. Then f is quasi-affine if and only if $\mathcal{O}_X$ is f-relatively ample.*

Proof. Follows from Properties, Lemma 27.25.1 and the definitions. □

Lemma 28.38.7. *Let $f : X \to Y$ be a morphism of schemes, $\mathcal{M}$ an invertible $\mathcal{O}_Y$-module, and $\mathcal{L}$ an invertible $\mathcal{O}_X$-module.*

(1) *If $\mathcal{L}$ is f-ample and $\mathcal{M}$ is ample, then $\mathcal{L} \otimes f^*\mathcal{M}^{\otimes a}$ is ample for $a \gg 0$.*
(2) *If $\mathcal{M}$ is ample and f quasi-affine, then $f^*\mathcal{M}$ is ample.*

Proof. Assume $\mathcal{L}$ is f-ample and $\mathcal{M}$ ample. By assumption Y and f are quasi-compact (see Definition 28.38.1 and Properties, Definition 27.24.1). Hence X is quasi-compact. Pick $x \in X$. We can choose $m \geq 1$ and $t \in \Gamma(Y, \mathcal{M}^{\otimes m})$ such that Y_t is affine and $f(x) \in Y_t$. Since $\mathcal{L}$ restricts to an ample invertible sheaf on $f^{-1}(Y_t) = X_{f^*t}$ we can choose $n \geq 1$ and $s \in \Gamma(X_{f^*t}, \mathcal{L}^{\otimes n})$ with $x \in (X_{f^*t})_s$ with $(X_{f^*t})_s$ affine. By Properties, Lemma 27.24.6 there exists an integer $e \geq 1$ and a section $s' \in \Gamma(X, \mathcal{L}^{\otimes n} \otimes f^*\mathcal{M}^{\otimes em})$ which restricts to $s(f^*t)^e$ on X_{f^*t}. For any $b > 0$ consider the section $s'' = s'(f^*t)^b$ of $\mathcal{L}^{\otimes n} \otimes f^*\mathcal{M}^{\otimes (e+b)m}$. Then $X_{s''} = (X_{f^*t})_s$ is an affine open of X containing x. Picking b such that n divides $e + b$ we see $\mathcal{L}^{\otimes n} \otimes f^*\mathcal{M}^{\otimes (e+b)m}$ is the nth power of $\mathcal{L} \otimes f^*\mathcal{M}^{\otimes a}$ for some a and we can get any a divisible by m and big enough. Since X is quasi-compact a finite number of these affine opens cover X. We conclude that for some a sufficiently divisible and large enough the invertible sheaf $\mathcal{L} \otimes f^*\mathcal{M}^{\otimes a}$ is ample on X. On the other hand, we know that $\mathcal{M}^{\otimes c}$ (and hence its pullback to X) is globally generated for all $c \gg 0$ by Properties, Proposition 27.24.14. Thus $\mathcal{L} \otimes f^*\mathcal{M}^{\otimes a+c}$ is ample (Properties, Lemma 27.24.5) for $c \gg 0$ and (1) is proved.

Part (2) follows from Lemma 28.38.6, Properties, Lemma 27.24.2, and part (1). □

Lemma 28.38.8. *Let $f : X \to S$ be a morphism of schemes. Let $\mathcal{L}$ be an invertible $\mathcal{O}_X$-module. Let $S' \to S$ be a morphism of schemes. Let $f' : X' \to S'$ be the base change of f and denote $\mathcal{L}'$ the pullback of $\mathcal{L}$ to X'. If $\mathcal{L}$ is f-ample, then $\mathcal{L}'$ is f'-ample.*

Proof. By Lemma 28.38.4 it suffices to find an affine open covering $S' = \bigcup U'_i$ such that $\mathcal{L}'$ restricts to an ample invertible sheaf on $(f')^{-1}(U'_i)$ for all i. We may choose U'_i mapping into an affine open $U_i \subset S$. In this case the morphism $(f')^{-1}(U'_i) \to f^{-1}(U_i)$ is affine as a base change of the affine morphism $U'_i \to U_i$ (Lemma 28.13.8). Thus $\mathcal{L}'|_{(f')^{-1}(U'_i)}$ is ample by Lemma 28.38.7. □

28.39. Very ample sheaves

Recall that given a quasi-coherent sheaf $\mathcal{E}$ on a scheme S the *projective bundle* associated to $\mathcal{E}$ is the morphism $\mathbf{P}(\mathcal{E}) \to S$, where $\mathbf{P}(\mathcal{E}) = \underline{\mathrm{Proj}}_S(\mathrm{Sym}(\mathcal{E}))$, see Constructions, Definition 26.21.1.

Definition 28.39.1. Let $f : X \to S$ be a morphism of schemes. Let $\mathcal{L}$ be an invertible $\mathcal{O}_X$-module. We say $\mathcal{L}$ is *relatively very ample* or more precisely *f-relatively very ample*, or *very ample on X/S*, or *f-very ample* if there exist a quasi-coherent $\mathcal{O}_S$-module $\mathcal{E}$ and an immersion $i : X \to \mathbf{P}(\mathcal{E})$ over S such that $\mathcal{L} \cong i^*\mathcal{O}_{\mathbf{P}(\mathcal{E})}(1)$.

Since there is no assumption of quasi-compactness in this definition it is not true in general that a relatively very ample invertible sheaf is a relatively ample invertible sheaf.

Lemma 28.39.2. *Let $f : X \to S$ be a morphism of schemes. Let $\mathcal{L}$ be an invertible $\mathcal{O}_X$-module. If f is quasi-compact and $\mathcal{L}$ is a relatively very ample invertible sheaf, then $\mathcal{L}$ is a relatively ample invertible sheaf.*

Proof. By definition there exists quasi-coherent $\mathcal{O}_S$-module $\mathcal{E}$ and an immersion $i : X \to \mathbf{P}(\mathcal{E})$ over S such that $\mathcal{L} \cong i^*\mathcal{O}_{\mathbf{P}(\mathcal{E})}(1)$. Set $\mathcal{A} = \mathrm{Sym}(\mathcal{E})$, so $\mathbf{P}(\mathcal{E}) = \underline{\mathrm{Proj}}_S(\mathcal{A})$ by definition. The graded $\mathcal{O}_S$-algebra $\mathcal{A}$ comes equipped with a map

$$\psi : \mathcal{A} \to \bigoplus\nolimits_{n\geq 0} \pi_*\mathcal{O}_{\mathbf{P}(\mathcal{E})}(n) \to \bigoplus\nolimits_{n\geq 0} f_*\mathcal{L}^{\otimes n}$$

where the second arrow uses the identification $\mathcal{L} \cong i^*\mathcal{O}_{\mathbf{P}(\mathcal{E})}(1)$. By adjointness of f_* and f^* we get a morphism $\psi : f^*\mathcal{A} \to \bigoplus_{n\geq 0} \mathcal{L}^{\otimes n}$. We omit the verification that the morphism $r_{\mathcal{L},\psi}$ associated to this map is exactly the immersion i. Hence the result follows from part (6) of Lemma 28.38.4. □

To arrive at the correct converse of this lemma we ask whether given a relatively ample invertible sheaf $\mathcal{L}$ there exists an integer $n \geq 1$ such that $\mathcal{L}^{\otimes n}$ is relatively very ample? In general this is false. There are several things that prevent this from being true:

(1) Even if S is affine, it can happen that no finite integer n works because $X \to S$ is not of finite type, see Example 28.39.4.
(2) The base not being quasi-compact means the result can be prevented from being true even with f finite type. Namely, given a field k there exists a scheme X_d of finite type over k with an ample invertible sheaf $\mathcal{O}_{X_d}(1)$ so that the smallest tensor power of $\mathcal{O}_{X_d}(1)$ which is very ample is the dth power. See Example 28.39.5. Taking f to be the disjoint union of the schemes X_d mapping to the disjoint union of copies of $\mathrm{Spec}(k)$ gives an example.

To see our version of the converse take a look at Lemma 28.40.5 below. We will do some preliminary work before proving it.

Example 28.39.3. Let S be a scheme. Let $\mathcal{A}$ be a quasi-coherent graded $\mathcal{O}_S$-algebra generated by $\mathcal{A}_1$ over $\mathcal{A}_0$. Set $X = \underline{\mathrm{Proj}}_S(\mathcal{A})$. In this case $\mathcal{O}_X(1)$ is a very ample invertible sheaf on X. Namely, the morphism associated to the graded $\mathcal{O}_S$-algebra map

$$\mathrm{Sym}^*_{\mathcal{O}_X}(\mathcal{A}_1) \longrightarrow \mathcal{A}$$

is a closed immersion $X \to \mathbf{P}(\mathcal{A}_1)$ which pulls back $\mathcal{O}_{\mathbf{P}(\mathcal{A}_1)}(1)$ to $\mathcal{O}_X(1)$, see Constructions, Lemma 26.18.5.

Example 28.39.4. Let k be a field. Consider the graded k-algebra

$$A = k[U, V, Z_1, Z_2, Z_3, \ldots]/I \quad \text{with} \quad I = (U^2 - Z_1^2, U^4 - Z_2^2, U^6 - Z_3^2, \ldots)$$

with grading given by $\deg(U) = \deg(V) = \deg(Z_1) = 1$ and $\deg(Z_d) = d$. Note that $X = \text{Proj}(A)$ is covered by $D_+(U)$ and $D_+(V)$. Hence the sheaves $\mathcal{O}_X(n)$ are all invertible and isomorphic to $\mathcal{O}_X(1)^{\otimes n}$. In particular $\mathcal{O}_X(1)$ is ample and f-ample for the morphism $f : X \to \text{Spec}(k)$. We claim that no power of $\mathcal{O}_X(1)$ is f-relatively very ample. Namely, it is easy to see that $\Gamma(X, \mathcal{O}_X(n))$ is the degree n summand of the algebra A. Hence if $\mathcal{O}_X(n)$ were very ample, then X would be a closed subscheme of a projective space over k and hence of finite type over k. On the other hand $D_+(V)$ is the spectrum of $k[t, t_1, t_2, \ldots]/(t^2 - t_1^2, t^4 - t_2^2, t^6 - t_3^2, \ldots)$ which is not of finite type over k.

Example 28.39.5. Let k be an infinite field. Let $\lambda_1, \lambda_2, \lambda_3, \ldots$ be pairwise distinct elements of k^*. (This is not strictly necessary, and in fact the example works perfectly well even if all λ_i are equal to 1.) Consider the graded k-algebra

$$A_d = k[U, V, Z]/I_d \quad \text{with} \quad I_d = (Z^2 - \prod_{i=1}^{2d}(U - \lambda_i V)).$$

with grading given by $\deg(U) = \deg(V) = 1$ and $\deg(Z) = d$. Then $X_d = \text{Proj}(A_d)$ has ample invertible sheaf $\mathcal{O}_{X_d}(1)$. We claim that if $\mathcal{O}_{X_d}(n)$ is very ample, then $n \geq d$. The reason for this is that Z has degree d, and hence $\Gamma(X_d, \mathcal{O}_{X_d}(n)) = k[U, V]_n$ for $n < d$. Details omitted.

Lemma 28.39.6. *Let $f : X \to S$ be a morphism of schemes. Let $\mathcal{L}$ be an invertible sheaf on X. If $\mathcal{L}$ is relatively very ample on X/S then f is separated.*

Proof. Being separated is local on the base (see Schemes, Section 25.21). An immersion is separated (see Schemes, Lemma 25.23.7). Hence the lemma follows since locally X has an immersion into the homogeneous spectrum of a graded ring which is separated, see Constructions, Lemma 26.8.8. □

Lemma 28.39.7. *Let $f : X \to S$ be a morphism of schemes. Let $\mathcal{L}$ be an invertible sheaf on X. Assume f is quasi-compact. The following are equivalent*

1. *$\mathcal{L}$ is relatively very ample on X/S,*
2. *there exists an open covering $S = \bigcup V_j$ such that $\mathcal{L}|_{f^{-1}(V_j)}$ is relatively very ample on $f^{-1}(V_j)/V_j$ for all j,*
3. *there exists a quasi-coherent sheaf of graded $\mathcal{O}_S$-algebras $\mathcal{A}$ generated in degree 1 over $\mathcal{O}_S$ and a map of graded $\mathcal{O}_X$-algebras $\psi : f^*\mathcal{A} \to \bigoplus_{n \geq 0} \mathcal{L}^{\otimes n}$ such that $f^*\mathcal{A}_1 \to \mathcal{L}$ is surjective and the associated morphism $r_{\mathcal{L},\psi} : X \to \underline{Proj}_S(\mathcal{A})$ is an immersion, and*
4. *f is quasi-separated, the canonical map $\psi : f^*f_*\mathcal{L} \to \mathcal{L}$ is surjective, and the associated map $r_{\mathcal{L},\psi} : X \to \mathbf{P}(f_*\mathcal{L})$ is an immersion.*

Proof. It is clear that (1) implies (2). It is also clear that (4) implies (1); the hypothesis of quasi-separation in (4) is used to guarantee that $f_*\mathcal{L}$ is quasi-coherent via Schemes, Lemma 25.24.1.

Assume (2). We will prove (4). Let $S = \bigcup V_j$ be an open covering as in (2). Set $X_j = f^{-1}(V_j)$ and $f_j : X_j \to V_j$ the restriction of f. We see that f is

separated by Lemma 28.39.6 (as being separated is local on the base). Consider the map $\psi : f^*f_*\mathcal{L} \to \mathcal{L}$. On each V_j there exists a quasi-coherent sheaf $\mathcal{E}_j$ and an embedding $i : X_j \to \mathbf{P}(\mathcal{E}_j)$ with $\mathcal{L}_{X_j} \cong i^*\mathcal{O}_{\mathbf{P}(\mathcal{E}_j)}(1)$. In other words there is a map $\mathcal{E}_j \to (f_*\mathcal{L})|_{X_j}$ such that the composition

$$f_j^*\mathcal{E}_j \to (f^*f_*\mathcal{L})|_{X_j} \to \mathcal{L}|_{X_j}$$

is surjective. Hence we conclude that ψ is surjective. Let $r_{\mathcal{L},\psi} : X \to \mathbf{P}(f_*\mathcal{L})$ be the associated morphism. Using the maps $\mathcal{E}_j \to (f_*\mathcal{L})|_{X_j}$ we see that there is a factorization

$$X_j \xrightarrow{r_{\mathcal{L},\psi}} \mathbf{P}(f_*\mathcal{L})|_{V_j} \longrightarrow \mathbf{P}(\mathcal{E}_j)$$

which shows that $r_{\mathcal{L},\psi}$ is an immersion.

At this point we see that (1), (2) and (4) are equivalent. Clearly (4) implies (3). Assume (3). We will prove (1). Let $\mathcal{A}$ be a quasi-coherent sheaf of graded $\mathcal{O}_S$-algebras generated in degree 1 over $\mathcal{O}_S$. Consider the map of graded $\mathcal{O}_S$-algebras $\mathrm{Sym}(\mathcal{A}_1) \to \mathcal{A}$. This is surjective by hypothesis and hence induces a closed immersion

$$\underline{\mathrm{Proj}}_S(\mathcal{A}) \longrightarrow \mathbf{P}(\mathcal{A}_1)$$

which pulls back $\mathcal{O}(1)$ to $\mathcal{O}(1)$, see Constructions, Lemma 26.18.5. Hence it is clear that (3) implies (1). □

28.40. Ample and very ample sheaves relative to finite type morphisms

In fact most of the material in this section is about the notion of a (quasi-)projective morphism which we have not defined yet.

Lemma 28.40.1. *Let $f : X \to S$ be a morphism of schemes. Let $\mathcal{L}$ be an invertible sheaf on X. Assume that*

1. *the invertible sheaf $\mathcal{L}$ is very ample on X/S,*
2. *the morphism $X \to S$ is of finite type, and*
3. *S is affine.*

Then there exists an $n \geq 0$ and an immersion $i : X \to \mathbf{P}^n_S$ over S such that $\mathcal{L} \cong i^*\mathcal{O}_{\mathbf{P}^n_S}(1)$.

Proof. Assume (1), (2) and (3). Condition (3) means $S = \mathrm{Spec}(R)$ for some ring R. Condition (1) means by definition there exists a quasi-coherent $\mathcal{O}_S$-module $\mathcal{E}$ and an immersion $\alpha : X \to \mathbf{P}(\mathcal{E})$ such that $\mathcal{L} = \alpha^*\mathcal{O}_{\mathbf{P}(\mathcal{E})}(1)$. Write $\mathcal{E} = \widetilde{M}$ for some R-module M. Thus we have

$$\mathbf{P}(\mathcal{E}) = \mathrm{Proj}(\mathrm{Sym}_R(M)).$$

Since α is an immersion, and since the topology of $\mathrm{Proj}(\mathrm{Sym}_R(M))$ is generated by the standard opens $D_+(f)$, $f \in \mathrm{Sym}^d_R(M)$, $d \geq 1$, we can find for each $x \in X$ an $f \in \mathrm{Sym}^d_R(M)$, $d \geq 1$, with $\alpha(x) \in D_+(f)$ such that

$$\alpha|_{\alpha^{-1}(D_+(f))} : \alpha^{-1}(D_+(f)) \to D_+(f)$$

is a closed immersion. Condition (2) implies X is quasi-compact. Hence we can find a finite collection of elements $f_j \in \mathrm{Sym}^{d_j}_R(M)$, $d_j \geq 1$ such that for each $f = f_j$ the displayed map above is a closed immersion and such that $\alpha(X) \subset \bigcup D_+(f_j)$. Write $U_j = \alpha^{-1}(D_+(f_j))$. Note that U_j is affine as a closed subscheme of the affine scheme $D_+(f_j)$. Write $U_j = \mathrm{Spec}(A_j)$. Condition (2) also implies that A_j

is of finite type over R, see Lemma 28.16.2. Choose finitely many $x_{j,k} \in A_j$ which generate A_j as a R-algebra. Since $\alpha|_{U_j}$ is a closed immersion we see that $x_{j,k}$ is the image of an element

$$f_{j,k}/f_j^{e_{j,k}} \in \mathrm{Sym}_R(M)_{(f_j)} = \Gamma(D_+(f_j), \mathcal{O}_{\mathrm{Proj}(\mathrm{Sym}_R(M))}).$$

Finally, choose $n \geq 1$ and elements $y_0, \ldots, y_n \in M$ such that each of the polynomials $f_j, f_{j,k} \in \mathrm{Sym}_R(M)$ is a polynomial in the elements y_t with coefficients in R. Consider the graded ring map

$$\psi : R[Y_0, \ldots, Y_n] \longrightarrow \mathrm{Sym}_R(M), \quad Y_i \longmapsto y_i.$$

Denote F_j, $F_{j,k}$ the elements of $R[Y_0, \ldots, Y_n]$ such that $\psi(F_j) = f_j$ and $\psi(F_{j,k}) = f_{j,k}$. By Constructions, Lemma 26.11.1 we obtain an open subscheme

$$U(\psi) \subset \mathrm{Proj}(\mathrm{Sym}_R(M))$$

and a morphism $r_\psi : U(\psi) \to \mathbf{P}^n_R$. This morphism satisfies $r_\psi^{-1}(D_+(F_j)) = D_+(f_j)$, and hence we see that $\alpha(X) \subset U(\psi)$. Moreover, it is clear that

$$i = r_\psi \circ \alpha : X \longrightarrow \mathbf{P}^n_R$$

is still an immersion since $i^\sharp(F_{j,k}/F_j^{e_{j,k}}) = x_{j,k} \in A_j = \Gamma(U_j, \mathcal{O}_X)$ by construction. Moreover, the morphism r_ψ comes equipped with a map $\theta : r_\psi^* \mathcal{O}_{\mathbf{P}^n_R}(1) \to \mathcal{O}_{\mathrm{Proj}(\mathrm{Sym}_R(M))}(1)|_{U(\psi)}$ which is an isomorphism in this case (for construction θ see lemma cited above; some details omitted). Since the original map α was assumed to have the property that $\mathcal{L} = \alpha^* \mathcal{O}_{\mathrm{Proj}(\mathrm{Sym}_R(M))}(1)$ we win. □

Lemma 28.40.2. *Let $\pi : X \to S$ be a morphism of schemes. Assume that X is quasi-affine and that π is locally of finite type. Then there exist $n \geq 0$ and an immersion $i : X \to \mathbf{A}^n_S$ over S.*

Proof. Let $A = \Gamma(X, \mathcal{O}_X)$. By assumption X is quasi-compact and is identified with an open subscheme of $\mathrm{Spec}(A)$, see Properties, Lemma 27.15.4. Moreover, the set of opens X_f, for those $f \in A$ such that X_f is affine, forms a basis for the topology of X, see the proof of Properties, Lemma 27.15.4. Hence we can find a finite number of $f_j \in A$, $j = 1, \ldots, m$ such that $X = \bigcup X_{f_j}$, and such that $\pi(X_{f_j}) \subset V_j$ for some affine open $V_j \subset S$. By Lemma 28.16.2 the ring maps $\mathcal{O}(V_j) \to \mathcal{O}(X_{f_j}) = A_{f_j}$ are of finite type. Thus we may choose $a_1, \ldots, a_N \in A$ such that the elements $a_1, \ldots, a_N, f_1, \ldots, f_m, 1/f_j$ generate A_{f_j} over $\mathcal{O}(V_j)$ for each j. Take $n = N + m$ and let

$$i : X \longrightarrow \mathbf{A}^n_S$$

be the morphism given by the global sections $a_1, \ldots, a_n, f_1, \ldots, f_n$ of the structure sheaf of X. Let $D(x_j) \subset \mathbf{A}^n_S$ be the open subscheme where the jth coordinate function is nonzero. Then it is clear that $i^{-1}(D(x_j))$ is X_{f_j} and that the induced morphism $X_{f_j} \to D(x_j)$ factors through the affine open $\mathrm{Spec}(\mathcal{O}(V_j)[x_1, \ldots, x_n, 1/x_j])$ of $D(x_j)$. Since the ring map $\mathcal{O}(V_j)[x_1, \ldots, x_n, 1/x_j] \to A_{f_j}$ is surjective by construction we conclude that the restriction of i to X_{f_j} is an immersion as desired. □

Lemma 28.40.3. *Let $f : X \to S$ be a morphism of schemes. Let $\mathcal{L}$ be an invertible sheaf on X. Assume that*

1. *the invertible sheaf $\mathcal{L}$ is ample on X, and*
2. *the morphism $X \to S$ is locally of finite type.*

Then there exists a $d_0 \geq 1$ such that for every $d \geq d_0$ there exists an $n \geq 0$ and an immersion $i : X \to \mathbf{P}^n_S$ over S such that $\mathcal{L}^{\otimes d} \cong i^*\mathcal{O}_{\mathbf{P}^n_S}(1)$.

Proof. Let $A = \Gamma_*(X, \mathcal{L}) = \bigoplus_{d \geq 0} \Gamma(X, \mathcal{L}^{\otimes d})$. By Properties, Proposition 27.24.14 the set of affine opens X_a with $a \in A_+$ homogeneous forms a basis for the topology of X. Hence we can find finitely many such elements $a_0, \ldots, a_n \in A_+$ such that

(1) we have $X = \bigcup_{i=0,\ldots,n} X_{a_i}$,
(2) each X_{a_i} is affine, and
(3) each X_{a_i} maps into an affine open $V_i \subset S$.

By Lemma 28.16.2 we see that the ring maps $\mathcal{O}_S(V_i) \to \mathcal{O}_X(X_{a_i})$ are of finite type. Hence we can find finitely many elements $f_{ij} \in \mathcal{O}_X(X_{a_i})$, $j = 1, \ldots, n_i$ which generate $\mathcal{O}_X(X_{a_i})$ as an $\mathcal{O}_S(V_i)$-algebra. By Properties, Lemma 27.24.6 we may write each f_{ij} as $a_{ij}/a_i^{e_{ij}}$ for some $a_{ij} \in A_+$ homogeneous. Let N be a positive integer which is a common multiple of all the degrees of the elements a_i, a_{ij}. Consider the elements

$$a_i^{N/\deg(a_i)},\ a_{ij}a_i^{(N/\deg(a_i))-e_{ij}} \in A_N.$$

By construction these generate the invertible sheaf $\mathcal{L}^{\otimes N}$ over X. Hence they give rise to a morphism

$$j : X \longrightarrow \mathbf{P}^m_S \quad \text{with } m = n + \sum n_i$$

over S, see Constructions, Lemma 26.13.1 and Definition 26.13.2. Moreover, $j^*\mathcal{O}_{\mathbf{P}_S}(1) = \mathcal{L}^{\otimes N}$. We name the homogeneous coordinates $T_0, \ldots, T_n, T_{ij}$ instead of $T_0, \ldots, T_m$. For $i = 0, \ldots, n$ we have $i^{-1}(D_+(T_i)) = X_{a_i}$. Moreover, pulling back the element T_{ij}/T_i via $j^\sharp$ we get the element $f_{ij} \in \mathcal{O}_X(X_{a_i})$. Hence the morphism j restricted to X_{a_i} gives a closed immersion of X_{a_i} into the affine open $D_+(T_i) \cap \mathbf{P}^m_{V_i}$ of $\mathbf{P}^N_S$. Hence we conclude that the morphism j is an immersion. This implies the lemma holds for some d and n which is enough in virtually all applications.

This proves that for one $d_2 \geq 1$ (namely $d_2 = N$ above), some $m \geq 0$ there exists some immersion $j : X \to \mathbf{P}^m_S$ given by global sections $s'_0, \ldots, s'_m \in \Gamma(X, \mathcal{L}^{\otimes d_2})$. By Properties, Proposition 27.24.14 we know there exists an integer d_1 such that $\mathcal{L}^{\otimes d}$ is globally generated for all $d \geq d_1$. Set $d_0 = d_1 + d_2$. We claim that the lemma holds with this value of d_0. Namely, given an integer $d \geq d_0$ we may choose $s''_1, \ldots, s''_t \in \Gamma(X, \mathcal{L}^{\otimes d-d_2})$ which generate $\mathcal{L}^{\otimes d-d_2}$ over X. Set $n = (m+1)t$ and denote $s_0, \ldots, s_n$ the collection of sections $s'_\alpha s''_\beta$, $\alpha = 0, \ldots, m$, $\beta = 1, \ldots, t$. These generate $\mathcal{L}^{\otimes d}$ over X and therefore define a morphism

$$i : X \longrightarrow \mathbf{P}^n_S$$

such that $i^*\mathcal{O}_{\mathbf{P}^n_S}(1) \cong \mathcal{L}^{\otimes d}$. We omit the verification that since j was an immersion also the morphism i so obtained is an immersion also. (Hint: Segre embedding.) □

Lemma 28.40.4. *Let $f : X \to S$ be a morphism of schemes. Let $\mathcal{L}$ be an invertible $\mathcal{O}_X$-module. Assume S affine and f of finite type. The following are equivalent*

(1) *$\mathcal{L}$ is ample on X,*
(2) *$\mathcal{L}$ is f-ample,*
(3) *$\mathcal{L}^{\otimes d}$ is f-very ample for some $d \geq 1$,*
(4) *$\mathcal{L}^{\otimes d}$ is f-very ample for all $d \gg 1$,*
(5) *for some $d \geq 1$ there exist $n \geq 1$ and an immersion $i : X \to \mathbf{P}^n_S$ such that $\mathcal{L}^{\otimes d} \cong i^*\mathcal{O}_{\mathbf{P}^n_S}(1)$, and*

(6) *for all $d \gg 1$ there exist $n \geq 1$ and an immersion $i : X \to \mathbf{P}^n_S$ such that $\mathcal{L}^{\otimes d} \cong i^*\mathcal{O}_{\mathbf{P}^n_S}(1)$.*

Proof. The equivalence of (1) and (2) is Lemma 28.38.5. The implication (2) $\Rightarrow$ (6) is Lemma 28.40.3. Trivially (6) implies (5). As $\mathbf{P}^n_S$ is a projective bundle over S (see Constructions, Lemma 26.21.4) we see that (5) implies (3) and (6) implies (4) from the definition of a relatively very ample sheaf. Trivially (4) implies (3). To finish we have to show that (3) implies (2) which follows from Lemma 28.39.2 and Lemma 28.38.2. □

Lemma 28.40.5. *Let $f : X \to S$ be a morphism of schemes. Let $\mathcal{L}$ be an invertible $\mathcal{O}_X$-module. Assume S quasi-compact and f of finite type. The following are equivalent*

(1) *$\mathcal{L}$ is f-ample,*
(2) *$\mathcal{L}^{\otimes d}$ is f-very ample for some $d \geq 1$,*
(3) *$\mathcal{L}^{\otimes d}$ is f-very ample for all $d \gg 1$.*

Proof. Trivially (3) implies (2). Lemma 28.39.2 guarantees that (2) implies (1) since a morphism of finite type is quasi-compact by definition. Assume that $\mathcal{L}$ is f-ample. Choose a finite affine open covering $S = V_1 \cup \ldots \cup V_m$. Write $X_i = f^{-1}(V_i)$. By Lemma 28.40.4 above we see there exists a d_0 such that $\mathcal{L}^{\otimes d}$ is relatively very ample on X_i/V_i for all $d \geq d_0$. Hence we conclude (1) implies (3) by Lemma 28.39.7. □

The following two lemmas provide the most used and most useful characterizations of relatively very ample and relatively ample invertible sheaves when the morphism is of finite type.

Lemma 28.40.6. *Let $f : X \to S$ be a morphism of schemes. Let $\mathcal{L}$ be an invertible sheaf on X. Assume f is of finite type. The following are equivalent:*

(1) *$\mathcal{L}$ is f-relatively very ample, and*
(2) *there exist an open covering $S = \bigcup V_j$, for each j an integer n_j, and immersions*

$$i_j : X_j = f^{-1}(V_j) = V_j \times_S X \longrightarrow \mathbf{P}^{n_j}_{V_j}$$

over V_j such that $\mathcal{L}|_{X_j} \cong i_j^*\mathcal{O}_{\mathbf{P}^{n_j}_{V_j}}(1)$.

Proof. We see that (1) implies (2) by taking an affine open covering of S and applying Lemma 28.40.1 to each of the restrictions of f and $\mathcal{L}$. We see that (2) implies (1) by Lemma 28.39.7. □

Lemma 28.40.7. *Let $f : X \to S$ be a morphism of schemes. Let $\mathcal{L}$ be an invertible sheaf on X. Assume f is of finite type. The following are equivalent:*

(1) *$\mathcal{L}$ is f-relatively ample, and*
(2) *there exist an open covering $S = \bigcup V_j$, for each j an integers $d_j \geq 1$, $n_j \geq 0$, and immersions*

$$i_j : X_j = f^{-1}(V_j) = V_j \times_S X \longrightarrow \mathbf{P}^{n_j}_{V_j}$$

over V_j such that $\mathcal{L}^{\otimes d_j}|_{X_j} \cong i_j^*\mathcal{O}_{\mathbf{P}^{n_j}_{V_j}}(1)$.

Proof. We see that (1) implies (2) by taking an affine open covering of S and applying Lemma 28.40.4 to each of the restrictions of f and $\mathcal{L}$. We see that (2) implies (1) by Lemma 28.38.4. □

28.41. Quasi-projective morphisms

The discussion in the previous section suggests the following definitions. We take our definition of quasi-projective from [**DG67**]. The version with the letter "H" is the definition in [**Har77**].

Definition 28.41.1. Let $f : X \to S$ be a morphism of schemes.

(1) We say f is *quasi-projective* if f is of finite type and there exists an f-relatively ample invertible $\mathcal{O}_X$-module.
(2) We say f is *H-quasi-projective* if f if there exists a quasi-compact immersion $X \to \mathbf{P}^n_S$ over S for some n.[11]
(3) We say f is *locally quasi-projective* if there exists an open covering $S = \bigcup V_j$ such that each $f^{-1}(V_j) \to V_j$ is quasi-projective.

As this definition suggests the property of being quasi-projective is not local on S.

Lemma 28.41.2. *Let $f : X \to S$ be a morphism of schemes. If f is quasi-projective, or H-quasi-projective or locally quasi-projective, then f is separated of finite type.*

Proof. Omitted. □

Lemma 28.41.3. *A H-quasi-projective morphism is quasi-projective.*

Proof. Omitted. □

Lemma 28.41.4. *Let $f : X \to S$ be a morphism of schemes. The following are equivalent:*

(1) *The morphism f is locally quasi-projective.*
(2) *There exists an open covering $S = \bigcup V_j$ such that each $f^{-1}(V_j) \to V_j$ is H-quasi-projective.*

Proof. By Lemma 28.41.3 we see that (2) implies (1). Assume (1). The question is local on S and hence we may assume S is affine, X of finite type over S and $\mathcal{L}$ is a relatively ample invertible sheaf on X/S. By Lemma 28.40.4 we may assume $\mathcal{L}$ is ample on X. By Lemma 28.40.3 we see that there exists an immersion of X into a projective space over S, i.e., X is H-quasi-projective over S as desired. □

28.42. Proper morphisms

The notion of a proper morphism plays an important role in algebraic geometry. An important example of a proper morphism will be the structure morphism $\mathbf{P}^n_S \to S$ of projective n-space, and this is in fact the motivating example leading to the definition.

Definition 28.42.1. Let $f : X \to S$ be a morphism of schemes. We say f is *proper* if f is separated, finite type, and universally closed.

[11] This is not exactly the same as the definition in Hartshorne. Namely, the definition in Hartshorne (8th corrected printing, 1997) is that f should be the composition of an open immersion followed by a H-projective morphism (see Definition 28.43.1), which does not imply f is quasi-compact. See Lemma 28.43.11 for the implication in the other direction.

The morphism from the affine line with zero doubled to the affine line is of finite type and universally closed, so the separation condition is necessary in the definition above. In the rest of this section we prove some of the basic properties of proper morphisms and of universally closed morphisms.

Lemma 28.42.2. *Let $f : X \to S$ be a morphism of schemes. The following are equivalent:*

(1) *The morphism f is universally closed.*
(2) *There exists an open covering $S = \bigcup V_j$ such that $f^{-1}(V_j) \to V_j$ is universally closed for all indices j.*

Proof. This is clear from the definition. □

Lemma 28.42.3. *Let $f : X \to S$ be a morphism of schemes. The following are equivalent:*

(1) *The morphism f is proper.*
(2) *There exists an open covering $S = \bigcup V_j$ such that $f^{-1}(V_j) \to V_j$ is proper for all indices j.*

Proof. Omitted. □

Lemma 28.42.4. *The composition of proper morphisms is proper. The same is true for universally closed morphisms.*

Proof. A composition of closed morphisms is closed. If $X \to Y \to Z$ are universally closed morphisms and $Z' \to Z$ is any morphism, then we see that $Z' \times_Z X = (Z' \times_Z Y) \times_Y X \to Z' \times_Z Y$ is closed and $Z' \times_Z Y \to Z'$ is closed. Hence the result for universally closed morphisms. We have seen that "separated" and "finite type" are preserved under compositions (Schemes, Lemma 25.21.13 and Lemma 28.16.3). Hence the result for proper morphisms. □

Lemma 28.42.5. *The base change of a proper morphism is proper. The same is true for universally closed morphisms.*

Proof. This is true by definition for universally closed morphisms. It is true for separated morphisms (Schemes, Lemma 25.21.13). It is true for morphisms of finite type (Lemma 28.16.4). Hence it is true for proper morphisms. □

Lemma 28.42.6. *A closed immersion is proper, hence a fortiori universally closed.*

Proof. The base change of a closed immersion is a closed immersion (Schemes, Lemma 25.18.2). Hence it is universally closed. A closed immersion is separated (Schemes, Lemma 25.23.7). A closed immersion is of finite type (Lemma 28.16.5). Hence a closed immersion is proper. □

Lemma 28.42.7. *Suppose given a commutative diagram of schemes*

$$\begin{array}{ccc} X & \longrightarrow & Y \\ & \searrow \quad \swarrow & \\ & S & \end{array}$$

with Y separated over S.

(1) *If $X \to S$ is universally closed, then the morphism $X \to Y$ is universally closed.*

(2) *If X proper over S, then the morphism $X \to Y$ is proper.*

In particular, in both cases the image of X in Y is closed.

Proof. Assume that $X \to S$ is universally closed (resp. proper). We factor the morphism as $X \to X \times_S Y \to Y$. The first morphism is a closed immersion, see Schemes, Lemma 25.21.11. Hence the first morphism is proper (Lemma 28.42.6). The projection $X \times_S Y \to Y$ is the base change of a universally closed (resp. proper) morphism and hence universally closed (resp. proper), see Lemma 28.42.5. Thus $X \to Y$ is universally closed (resp. proper) as the composition of universally closed (resp. proper) morphisms (Lemma 28.42.4). □

The following lemma says that the image of a proper scheme (in a separated scheme of finite type over the base) is proper.

Lemma 28.42.8. *Let S be a scheme. Let $f : X \to Y$ be a morphism of schemes over S. If X is universally closed over S and f is surjective then Y is universally closed over S. In particular, if also Y is separated and of finite type over S, then Y is proper over S.*

Proof. Assume X is universally closed and f surjective. Denote $p : X \to S$, $q : Y \to S$ the structure morphisms. Let $S' \to S$ be a morphism of schemes. The base change $f' : X_{S'} \to Y_{S'}$ is surjective (Lemma 28.11.4), and the base change $p' : X_{S'} \to S'$ is closed. If $T \subset Y_{S'}$ is closed, then $(f')^{-1}(T) \subset X_{S'}$ is closed, hence $p'((f')^{-1}(T)) = q'(T)$ is closed. So q' is closed. □

The proof of the following lemma is due to Bjorn Poonen, see this location.

Lemma 28.42.9. *A universally closed morphism of schemes is quasi-compact.*

Proof. Let $f : X \to S$ be a morphism. Assume that f is not quasi-compact. Our goal is to show that f is not universally closed. By Schemes, Lemma 25.19.2 there exists an affine open $V \subset S$ such that $f^{-1}(V)$ is not quasi-compact. To achieve our goal it suffices to show that $f^{-1}(V) \to V$ is not universally closed, hence we may assume that $S = \mathrm{Spec}(A)$ for some ring A.

Write $X = \bigcup_{i \in I} X_i$ where the X_i are affine open subschemes of X. Let $T = \mathrm{Spec}(A[y_i; i \in I])$. Let $T_i = D(y_i) \subset T$. Let Z be the closed set $(X \times_S T) - \bigcup_{i \in I}(X_i \times_S T_i)$. It suffices to prove that the image $f_T(Z)$ of Z under $f_T : X \times_S T \to T$ is not closed.

There exists a point $s \in S$ such that there is no neighborhood U of s in S such that X_U is quasi-compact. Otherwise we could cover S with finitely many such U and Schemes, Lemma 25.19.2 would imply f quasi-compact. Fix such an $s \in S$.

First we check that $f_T(Z_s) \neq T_s$. Let $t \in T$ be the point lying over s with $\kappa(t) = \kappa(s)$ such that $y_i = 1$ in $\kappa(t)$ for all i. Then $t \in T_i$ for all i, and the fiber of $Z_s \to T_s$ above t is isomorphic to $(X - \bigcup_{i \in I} X_i)_s$, which is empty. Thus $t \in T_s - f_T(Z_s)$.

Assume $f_T(Z)$ is closed in T. Then there exists an element $g \in A[y_i; i \in I]$ with $f_T(Z) \subset V(g)$ but $t \notin V(g)$. Hence the image of g in $\kappa(t)$ is nonzero. In particular some coefficient of g has nonzero image in $\kappa(s)$. Hence this coefficient is invertible on some neighborhood U of s. Let J be the finite set of $j \in I$ such that y_j appears in g. Since X_U is not quasi-compact, we may choose a point $x \in X - \bigcup_{j \in J} X_j$ lying above some $u \in U$. Since g has a coefficient that is invertible on U, we can find a

point $t' \in T$ lying above u such that $t' \notin V(g)$ and $t' \in V(y_i)$ for all $i \notin J$. This is true because $V(y_i; i \in I, i \notin J) = \mathrm{Spec}(A[t_j; j \in J])$ and the set of points of this scheme lying over u is bijective with $\mathrm{Spec}(\kappa(u)[t_j; j \in J])$. In other words $t' \notin T_i$ for each $i \notin J$. By Schemes, Lemma 25.17.5 we can find a point z of $X \times_S T$ mapping to $x \in X$ and to $t' \in T$. Since $x \notin X_j$ for $j \in J$ and $t' \notin T_i$ for $i \in I \setminus J$ we see that $z \in Z$. On the other hand $f_T(z) = t' \notin V(g)$ which contradicts $f_T(Z) \subset V(g)$. Thus the assumption "$f_T(Z)$ closed" is wrong and we conclude indeed that f_T is not closed, as desired. □

The target of a separated scheme under a surjective universally closed morphism is separated.

Lemma 28.42.10. *Let S be a scheme. Let $f : X \to Y$ be a surjective universally closed morphism of schemes over S.*

(1) *If X is quasi-separated, then Y is quasi-separated.*
(2) *If X is separated, then Y is separated.*
(3) *If X is quasi-separated over S, then Y is quasi-separated over S.*
(4) *If X is separated over S, then Y is separated over S.*

Proof. Parts (1) and (2) are a consequence of (3) and (4) for $S = \mathrm{Spec}(\mathbf{Z})$ (see Schemes, Definition 25.21.3). Consider the commutative diagram

$$\begin{array}{ccc} X & \xrightarrow{\Delta_{X/S}} & X \times_S X \\ \downarrow & & \downarrow \\ Y & \xrightarrow{\Delta_{Y/S}} & Y \times_S Y \end{array}$$

The left vertical arrow is surjective (i.e., universally surjective). The right vertical arrow is universally closed as a composition of the universally closed morphisms $X \times_S X \to X \times_S Y \to Y \times_S Y$. Hence it is also quasi-compact, see Lemma 28.42.9.

Assume X is quasi-separated over S, i.e., $\Delta_{X/S}$ is quasi-compact. If $V \subset Y \times_S Y$ is a quasi-compact open, then $V \times_{Y \times_S Y} X \to \Delta_{Y/S}^{-1}(V)$ is surjective and $V \times_{Y \times_S Y} X$ is quasi-compact by our remarks above. We conclude that $\Delta_{Y/S}$ is quasi-compact, i.e., Y is quasi-separated over S.

Assume X is separated over S, i.e., $\Delta_{X/S}$ is a closed immersion. Then $X \to Y \times_S Y$ is closed as a composition of closed morphisms. Since $X \to Y$ is surjective, it follows that $\Delta_{Y/S}(Y)$ is closed in $Y \times_S Y$. Hence Y is separated over S by the discussion following Schemes, Definition 25.21.3. □

Lemma 28.42.11. *Let $f : X \to S$ and $h : U \to X$ be morphisms of schemes. Assume that f and h are quasi-compact and that $h(U)$ is dense in X. If given any commutative solid diagram*

$$\begin{array}{ccccc} \mathrm{Spec}(K) & \longrightarrow & U & \longrightarrow & X \\ \downarrow & & & \nearrow & \downarrow \\ \mathrm{Spec}(A) & & \longrightarrow & & S \end{array}$$

where A is a valuation ring with field of fractions K, there exists a unique dotted arrow making the diagram commute, then f is universally closed.

Proof. We will verify the existence part of the valuative criterion for f which will imply f is universally closed by Schemes, Proposition 25.20.6. To do this, consider a commutative diagram

$$\begin{array}{ccc} \mathrm{Spec}(K) & \longrightarrow & X \\ \downarrow & & \downarrow \\ \mathrm{Spec}(A) & \longrightarrow & S \end{array}$$

where A is a valuation ring and K is the fraction field of A. Note that since valuation rings and fields are reduced, we may replace U, X, and S by their respective reductions by Schemes, Lemma 25.12.6. In this case the assumption that $h(U)$ is dense means that the scheme theoretic image of $h : U \to X$ is X, see Lemma 28.6.7. We may also replace S by an affine open through which the morphism $\mathrm{Spec}(A) \to S$ factors. Thus we may assume that $S = \mathrm{Spec}(R)$.

Let $\mathrm{Spec}(B) \subset X$ be an affine open through which the morphism $\mathrm{Spec}(K) \to X$ factors. Choose a polynomial algebra P over B and a B-algebra surjection $P \to K$. Then $\mathrm{Spec}(P) \to X$ is flat. Hence the scheme theoretic image of the morphism $U \times_X \mathrm{Spec}(P) \to \mathrm{Spec}(P)$ is $\mathrm{Spec}(P)$ by Lemma 28.26.14. By Lemma 28.6.5 we can find a commutative diagram

$$\begin{array}{ccc} \mathrm{Spec}(K') & \longrightarrow & U \times_X \mathrm{Spec}(P) \\ \downarrow & & \downarrow \\ \mathrm{Spec}(A') & \longrightarrow & \mathrm{Spec}(P) \end{array}$$

where A' is a valuation ring and K' is the fraction field of A' such that the closed point of $\mathrm{Spec}(A')$ maps to $\mathrm{Spec}(K) \subset \mathrm{Spec}(P)$. In other words, there is a B-algebra map $\varphi : K \to A'/\mathfrak{m}_{A'}$. Choose a valuation ring $A'' \subset A'/\mathfrak{m}_{A'}$ dominating $\varphi(A)$ with field of fractions $K'' = A'/\mathfrak{m}_{A'}$ (Algebra, Lemma 10.46.2). We set

$$C = \{\lambda \in A' \mid \lambda \bmod \mathfrak{m}_{A'} \in A''\}.$$

which is a valuation ring by Algebra, Lemma 10.46.7. As C is an R-algebra with fraction field K', we obtain a commutative diagram

$$\begin{array}{ccccc} \mathrm{Spec}(K') & \longrightarrow & U & \longrightarrow & X \\ \downarrow & & & \nearrow & \downarrow \\ \mathrm{Spec}(C) & & \longrightarrow & & S \end{array}$$

as in the statement of the lemma. Thus a dotted arrow fitting into the diagram as indicated. By the uniqueness assumption of the lemma the composition $\mathrm{Spec}(A') \to \mathrm{Spec}(C) \to X$ agrees with the given morphism $\mathrm{Spec}(A') \to \mathrm{Spec}(P) \to \mathrm{Spec}(B) \subset X$. Hence the restriction of the morphism to the spectrum of $C/\mathfrak{m}_{A'} = A''$ induces the given morphism $\mathrm{Spec}(K'') = \mathrm{Spec}(A'/\mathfrak{m}_{A'}) \to \mathrm{Spec}(K) \to X$. Let $x \in X$ be the image of the closed point of $\mathrm{Spec}(A'') \to X$. The image of the induced ring map $\mathcal{O}_{X,x} \to A''$ is a local subring which is contained in $K \subset K''$. Since A is maximal for the relation of domination in K and since $A \subset A''$, we have $A = K \cap A''$. We conclude that $\mathcal{O}_{X,x} \to A''$ factors through $A \subset A''$. In this way we obtain our desired arrow $\mathrm{Spec}(A) \to X$. $\square$

Remark 28.42.12. The assumption on uniqueness of the dotted arrows in Lemma 28.42.11 is necessary (details omitted). The uniqueness is guaranteed if f is separated (Schemes, Lemma 25.22.1). Conversely, if h and f satisfy the assumptions of the lemma and f is quasi-separated, then f is separated as well as universally closed (details omitted; apply the lemma to the diagonal morphism as in the proof of Schemes, Lemma 25.22.2).

28.43. Projective morphisms

We will use the definition of a projective morphism from [**DG67**]. The version of the definition with the "H" is the one from [**Har77**]. The resulting definitions are different. Both are useful.

Definition 28.43.1. Let $f : X \to S$ be a morphism of schemes.

(1) We say f is *projective* if X is isomorphic as an S-scheme to a closed subscheme of a projective bundle $\mathbf{P}(\mathcal{E})$ for some quasi-coherent, finite type $\mathcal{O}_S$-module $\mathcal{E}$.
(2) We say f is *H-projective* if there exists and integer n and a closed immersion $X \to \mathbf{P}^n_S$ over S.
(3) We say f is *locally projective* if there exists an open covering $S = \bigcup U_i$ such that each $f^{-1}(U_i) \to U_i$ is projective.

As expected, a projective morphism is quasi-projective, see Lemma 28.43.10. Conversely, quasi-projective morphisms are often compositions of open immersions and projective morphisms, see Lemma 28.43.12.

Example 28.43.2. Let S be a scheme. Let $\mathcal{A}$ be a quasi-coherent graded $\mathcal{O}_S$-algebra generated by $\mathcal{A}_1$ over $\mathcal{A}_0$. Assume furthermore that $\mathcal{A}_1$ is of finite type over $\mathcal{O}_S$. Set $X = \underline{\mathrm{Proj}}_S(\mathcal{A})$. In this case $X \to S$ is projective. Namely, the morphism associated to the graded $\mathcal{O}_S$-algebra map

$$\mathrm{Sym}^*_{\mathcal{O}_X}(\mathcal{A}_1) \longrightarrow \mathcal{A}$$

is a closed immersion $X \to \mathbf{P}(\mathcal{A}_1)$ which pulls back $\mathcal{O}_{\mathbf{P}(\mathcal{A}_1)}(1)$ to $\mathcal{O}_X(1)$, see Constructions, Lemma 26.18.5.

Lemma 28.43.3. *An H-projective morphism is H-quasi-projective. An H-projective morphism is projective.*

Proof. The first statement is immediate from the definitions. The second holds as $\mathbf{P}^n_S$ is a projective bundle over S, see Constructions, Lemma 26.21.4. □

Lemma 28.43.4. *Let $f : X \to S$ be a morphism of schemes. The following are equivalent:*

(1) *The morphism f is locally projective.*
(2) *There exists an open covering $S = \bigcup U_i$ such that each $f^{-1}(U_i) \to U_i$ is H-projective.*

Proof. By Lemma 28.43.3 we see that (2) implies (1). Assume (1). For every point $s \in S$ we can find $\mathrm{Spec}(R) = U \subset S$ an affine open neighbourhood of s such that X_U is isomorphic to a closed subscheme of $\mathbf{P}(\mathcal{E})$ for some finite type, quasi-coherent sheaf of $\mathcal{O}_U$-modules $\mathcal{E}$. Write $\mathcal{E} = \widetilde{M}$ for some finite type R-module M

(see Properties, Lemma 27.16.1). Choose generators $x_0, \ldots, x_n \in M$ of M as an R-module. Consider the surjective graded R-algebra map

$$R[X_0, \ldots, X_n] \longrightarrow \mathrm{Sym}_R(M).$$

According to Constructions, Lemma 26.11.3 the corresponding morphism

$$\mathbf{P}(\mathcal{E}) \to \mathbf{P}^n_R$$

is a closed immersion. Hence we conclude that $f^{-1}(U)$ is isomorphic to a closed subscheme of $\mathbf{P}^n_U$ (as a scheme over U). In other words: (2) holds. □

Lemma 28.43.5. *A locally projective morphism is proper.*

Proof. Let $f : X \to S$ be locally projective. In order to show that f is proper we may work locally on the base, see Lemma 28.42.3. Hence, by Lemma 28.43.4 above we may assume there exists a closed immersion $X \to \mathbf{P}^n_S$. By Lemmas 28.42.4 and 28.42.6 it suffices to prove that $\mathbf{P}^n_S \to S$ is proper. Since $\mathbf{P}^n_S \to S$ is the base change of $\mathbf{P}^n_{\mathbf{Z}} \to \mathrm{Spec}(\mathbf{Z})$ it suffices to show that $\mathbf{P}^n_{\mathbf{Z}} \to \mathrm{Spec}(\mathbf{Z})$ is proper, see Lemma 28.42.5. By Constructions, Lemma 26.8.8 the scheme $\mathbf{P}^n_{\mathbf{Z}}$ is separated. By Constructions, Lemma 26.8.9 the scheme $\mathbf{P}^n_{\mathbf{Z}}$ is quasi-compact. It is clear that $\mathbf{P}^n_{\mathbf{Z}} \to \mathrm{Spec}(\mathbf{Z})$ is locally of finite type since $\mathbf{P}^n_{\mathbf{Z}}$ is covered by the affine opens $D_+(X_i)$ each of which is the spectrum of the finite type $\mathbf{Z}$-algebra

$$\mathbf{Z}[X_0/X_i, \ldots, X_n/X_i].$$

Finally, we have to show that $\mathbf{P}^n_{\mathbf{Z}} \to \mathrm{Spec}(\mathbf{Z})$ is universally closed. This follows from Constructions, Lemma 26.8.11 and the valuative criterion (see Schemes, Proposition 25.20.6). □

Lemma 28.43.6. *Let S be a scheme. There exists a closed immersion*

$$\mathbf{P}^n_S \times_S \mathbf{P}^m_S \longrightarrow \mathbf{P}^{nm+n+m}_S$$

called the Segre embedding.

Proof. It suffices to prove this when $S = \mathrm{Spec}(\mathbf{Z})$. Hence we will drop the index S and work in the absolute setting. Write $\mathbf{P}^n = \mathrm{Proj}(\mathbf{Z}[X_0, \ldots, X_n])$, $\mathbf{P}^m = \mathrm{Proj}(\mathbf{Z}[Y_0, \ldots, Y_m])$, and $\mathbf{P}^{nm+n+m} = \mathrm{Proj}(\mathbf{Z}[Z_0, \ldots, Z_{nm+n+m}])$. In order to map into $\mathbf{P}^{nm+n+m}$ we have to write down an invertible sheaf $\mathcal{L}$ on the left hand side and $(n+1)(m+1)$ sections s_i which generate it. See Constructions, Lemma 26.13.1. The invertible sheaf we take is

$$\mathcal{L} = \mathrm{pr}_1^*\mathcal{O}_{\mathbf{P}^n}(1) \otimes \mathrm{pr}_2^*\mathcal{O}_{\mathbf{P}^m}(1)$$

The sections we take are

$$s_0 = X_0Y_0,\ s_1 = X_1Y_0, \ldots,\ s_n = X_nY_0,\ s_{n+1} = X_0Y_1, \ldots,\ s_{nm+n+m} = X_nY_m.$$

These generate $\mathcal{L}$ since the sections X_i generate $\mathcal{O}_{\mathbf{P}^n}(1)$ and the sections Y_j generate $\mathcal{O}_{\mathbf{P}^m}(1)$. The induced morphism φ has the property that

$$\varphi^{-1}(D_+(Z_{i+(n+1)j})) = D_+(X_i) \times D_+(Y_j).$$

Hence it is an affine morphism. The corresponding ring map in case $(i, j) = (0, 0)$ is the map

$$\mathbf{Z}[Z_1/Z_0, \ldots, Z_{nm+n+m}/Z_0] \longrightarrow \mathbf{Z}[X_1/X_0, \ldots, X_n/X_0, Y_1/Y_0, \ldots, Y_n/Y_0]$$

which maps Z_i/Z_0 to the element X_i/X_0 for $i \leq n$ and the element $Z_{(n+1)j}/Z_0$ to the element Y_j/Y_0. Hence it is surjective. A similar argument works for the other affine open subsets. Hence the morphism φ is a closed immersion. □

Lemma 28.43.7. *A composition of H-projective morphisms is H-projective.*

Proof. Suppose $X \to Y$ and $Y \to Z$ are H-projective. Then there exist closed immersions $X \to \mathbf{P}^n_Y$ over Y, and $Y \to \mathbf{P}^m_Z$ over Z. Consider the following diagram

$$\begin{array}{ccccccc} X & \longrightarrow & \mathbf{P}^n_Y & \longrightarrow & \mathbf{P}^n_{\mathbf{P}^m_Z} & = & \mathbf{P}^n_Z \times_Z \mathbf{P}^m_Z \longrightarrow \mathbf{P}^{nm+n+m}_Z \\ \downarrow & \swarrow & & \swarrow & & & \\ Y & \longrightarrow & \mathbf{P}^m_Z & & & & \\ \downarrow & & & \swarrow & & & \\ Z & & & & & & \end{array}$$

Here the rightmost top horizontal arrow is the Segre embedding, see Lemma 28.43.6. The diagram identifies X as a closed subscheme of $\mathbf{P}^{nm+n+m}_Z$ as desired. □

Lemma 28.43.8. *A base change of a H-projective morphism is H-projective.*

Proof. This is true because the base change of projective space over a scheme is projective space, and the fact that the base change of a closed immersion is a closed immersion, see Schemes, Lemma 25.18.2. □

Lemma 28.43.9. *A base change of a (locally) projective morphism is (locally) projective.*

Proof. This is true because the base change of a projective bundle over a scheme is a projective bundle, the pullback of a finite type $\mathcal{O}$-module is of finite type (Modules, Lemma 17.9.2) and the fact that the base change of a closed immersion is a closed immersion, see Schemes, Lemma 25.18.2. Some details omitted. □

Lemma 28.43.10. *A projective morphism is quasi-projective.*

Proof. Let $f : X \to S$ be a projective morphism. Choose a closed immersion $i : X \to \mathbf{P}(\mathcal{E})$ where $\mathcal{E}$ is a quasi-coherent, finite type $\mathcal{O}_S$-module. Then $\mathcal{L} = i^*\mathcal{O}_{\mathbf{P}(\mathcal{E})}(1)$ is f-very ample. Since f is proper (Lemma 28.43.5) it is quasi-compact. Hence Lemma 28.39.2 implies that $\mathcal{L}$ is f-ample. Since f is proper it is of finite type. Thus we've checked all the defining properties of quasi-projective holds and we win. □

Lemma 28.43.11. *Let $f : X \to S$ be a H-quasi-projective morphism. Then f factors as $X \to X' \to S$ where $X \to X'$ is an open immersion and $X' \to S$ is H-projective.*

Proof. By definition we can factor f as a quasi-compact immersion $i : X \to \mathbf{P}^n_S$ followed by the projection $\mathbf{P}^n_S \to S$. By Lemma 28.7.7 there exists a closed subscheme $X' \subset \mathbf{P}^n_S$ such that i factors through an open immersion $X \to X'$. The lemma follows. □

Lemma 28.43.12. *Let $f : X \to S$ be a quasi-projective morphism with S quasi-compact and quasi-separated. Then f factors as $X \to X' \to S$ where $X \to X'$ is an open immersion and $X' \to S$ is projective.*

Proof. Let $\mathcal{L}$ be f-ample. Since f is of finite type and S is quasi-compact $\mathcal{L}^{\otimes n}$ is f-very ample for some $n > 0$, see Lemma 28.40.5. Replace $\mathcal{L}$ by $\mathcal{L}^{\otimes n}$. Write $\mathcal{F} = f_*\mathcal{L}$. This is a quasi-coherent $\mathcal{O}_S$-module by Schemes, Lemma 25.24.1 (quasi-projective morphisms are quasi-compact and separated, see Lemma 28.41.2). By Properties, Lemma 27.20.6 we can find a directed partially ordered set I and a system of finite type quasi-coherent $\mathcal{O}_S$-modules $\mathcal{E}_i$ over I such that $\mathcal{F} = \operatorname{colim} \mathcal{E}_i$. Consider the compositions $\psi_i : f^*\mathcal{E}_i \to f^*\mathcal{F} \to \mathcal{L}$. Choose a finite affine open covering $S = \bigcup_{j=1,\ldots,m} V_j$. For each j we can choose sections

$$s_{j,0}, \ldots, s_{j,n_j} \in \Gamma(f^{-1}(V_j), \mathcal{L}) = f_*\mathcal{L}(V_j) = \mathcal{F}(V_j)$$

which generate $\mathcal{L}$ over $f^{-1}V_j$ and define an immersion

$$f^{-1}V_j \longrightarrow \mathbf{P}^{n_j}_{V_j},$$

see Lemma 28.40.1. Choose i such that there exist sections $e_{j,t} \in \mathcal{E}_i(V_j)$ mapping to $s_{j,t}$ in $\mathcal{F}$ for all $j = 1, \ldots, m$ and $t = 1, \ldots, n_j$. Then the map ψ_i is surjective as the sections $f^*e_{j,t}$ have the same image as the sections $s_{j,t}$ which generate $\mathcal{L}|_{f^{-1}V_j}$. Whence we obtain a morphism

$$r_{\mathcal{L},\psi_i} : X \longrightarrow \mathbf{P}(\mathcal{E}_i)$$

over S such that over V_j we have a factorization

$$f^{-1}V_j \to \mathbf{P}(\mathcal{E}_i)|_{V_j} \to \mathbf{P}^{n_j}_{V_j}$$

of the immersion given above. It follows that $r_{\mathcal{L},\psi_i}|_{V_j}$ is an immersion, see Lemma 28.3.1. Since $S = \bigcup V_j$ we conclude that $r_{\mathcal{L},\psi_i}$ is an immersion. Note that $r_{\mathcal{L},\psi_i}$ is quasi-compact as $X \to S$ is quasi-compact and $\mathbf{P}(\mathcal{E}_i) \to S$ is separated (see Schemes, Lemma 25.21.15). By Lemma 28.7.7 there exists a closed subscheme $X' \subset \mathbf{P}(\mathcal{E}_i)$ such that i factors through an open immersion $X \to X'$. Then $X' \to S$ is projective by definition and we win. □

Lemma 28.43.13. *Let S be a scheme which admits an ample invertible sheaf. Then*

(1) *any projective morphism $X \to S$ is H-projective, and*
(2) *any quasi-projective morphism $X \to S$ is H-quasi-projective.*

Proof. The assumptions on S imply that S is quasi-compact and separated, see Properties, Definition 27.24.1 and Lemma 27.24.12 and Constructions, Lemma 26.8.8. Hence Lemma 28.43.12 applies and we see that (1) implies (2). Let $\mathcal{E}$ be a finite type quasi-coherent $\mathcal{O}_S$-module. By our definition of projective morphisms it suffices to show that $\mathbf{P}(\mathcal{E}) \to S$ is H-projective. If $\mathcal{E}$ is generated by finitely many global sections, then the corresponding surjection $\mathcal{O}_S^{\oplus n} \to \mathcal{E}$ induces a closed immersion

$$\mathbf{P}(\mathcal{E}) \longrightarrow \mathbf{P}(\mathcal{O}_S^{\oplus n}) = \mathbf{P}^n_S$$

as desired. In general, let $\mathcal{L}$ be an invertible sheaf on S. By Properties, Proposition 27.24.14 there exists an integer n such that $\mathcal{E} \otimes_{\mathcal{O}_S} \mathcal{L}^{\otimes n}$ is globally generated by finitely many sections. Since $\mathbf{P}(\mathcal{E}) = \mathbf{P}(\mathcal{E} \otimes_{\mathcal{O}_S} \mathcal{L}^{\otimes n})$ by Constructions, Lemma 26.20.1 this finishes the proof. □

28.44. Integral and finite morphisms

Recall that a ring map $R \to A$ is said to be *integral* if every element of A satisfies a monic equation with coefficients in R. Recall that a ring map $R \to A$ is said to be finite if A is *finite* as an R-module. See Algebra, Definition 10.33.1.

Definition 28.44.1. Let $f : X \to S$ be a morphism of schemes.

(1) We say that f is *integral* if f is affine and if for every affine open $\mathrm{Spec}(R) = V \subset S$ with inverse image $\mathrm{Spec}(A) = f^{-1}(V) \subset X$ the associated ring map $R \to A$ is integral.
(2) We say that f is *finite* if f is affine and if for every affine open $\mathrm{Spec}(R) = V \subset S$ with inverse image $\mathrm{Spec}(A) = f^{-1}(V) \subset X$ the associated ring map $R \to A$ is finite.

It is clear that integral/finite morphisms are separated and quasi-compact. It is also clear that a finite morphism is a morphism of finite type. Most of the lemmas in this section are completely standard. But note the fun Lemma 28.44.7 at the end of the section.

Lemma 28.44.2. *Let $f : X \to S$ be a morphism of schemes. The following are equivalent:*

(1) *The morphism f is integral.*
(2) *There exists an affine open covering $S = \bigcup U_i$ such that each $f^{-1}(U_i)$ is affine and $\mathcal{O}_S(U_i) \to \mathcal{O}_X(f^{-1}(U_i))$ is integral.*
(3) *There exists an open covering $S = \bigcup S_j$ such that each $f^{-1}(U_i) \to U_i$ is integral.*

Moreover, if f is integral then for every open subscheme $U \subset S$ the morphism $f : f^{-1}(U) \to U$ is integral.

Proof. See Algebra, Lemma 10.33.12. Some details omitted. □

Lemma 28.44.3. *Let $f : X \to S$ be a morphism of schemes. The following are equivalent:*

(1) *The morphism f is finite.*
(2) *There exists an affine open covering $S = \bigcup U_i$ such that each $f^{-1}(U_i)$ is affine and $\mathcal{O}_S(U_i) \to \mathcal{O}_X(f^{-1}(U_i))$ is finite.*
(3) *There exists an open covering $S = \bigcup S_j$ such that each $f^{-1}(U_i) \to U_i$ is finite.*

Moreover, if f is finite then for every open subscheme $U \subset S$ the morphism $f : f^{-1}(U) \to U$ is finite.

Proof. See Algebra, Lemma 10.33.12. Some details omitted. □

Lemma 28.44.4. *A finite morphism is integral. An integral morphism which is locally of finite type is finite.*

Proof. See Algebra, Lemma 10.33.3 and Lemma 10.33.5. □

Lemma 28.44.5. *A composition of finite morphisms is finite. Same is true for integral morphisms.*

Proof. See Algebra, Lemmas 10.7.3 and 10.33.6. □

Lemma 28.44.6. *A base change of a finite morphism is finite. Same is true for integral morphisms.*

Proof. See Algebra, Lemma 10.33.11. □

Lemma 28.44.7. *Let $f : X \to S$ be a morphism of schemes. The following are equivalent*

(1) *f is integral, and*
(2) *f is affine and universally closed.*

Proof. Assume (1). An integral morphism is affine by definition. A base change of an integral morphism is integral so in order to prove (2) it suffices to show that an integral morphism is closed. This follows from Algebra, Lemmas 10.33.20 and 10.38.6.

Assume (2). We may assume f is the morphism $f : \mathrm{Spec}(A) \to \mathrm{Spec}(R)$ coming from a ring map $R \to A$. Let a be an element of A. We have to show that a is integral over R, i.e. that in the kernel I of the map $R[x] \to A$ sending x to a there is a monic polynomial. Consider the ring $B = A[x]/(ax-1)$ and let J be the kernel of the composition $R[x] \to A[x] \to B$. If $f \in J$ there exists $q \in A[x]$ such that $f = (ax-1)q$ in $A[x]$ so if $f = \sum_i f_i x^i$ and $q = \sum_i q_i x^i$, for all $i \geq 0$ we have $f_i = aq_{i-1} - q_i$. For $n \geq \deg q + 1$ the polynomial

$$\sum_{i \geq 0} f_i x^{n-i} = \sum_{i \geq 0} (aq_{i-1} - q_i) x^{n-i} = (a - x) \sum_{i \geq 0} q_i x^{n-i-1}$$

is clearly in I; if $f_0 = 1$ this polynomial is also monic, so we are reduced to prove that J contains a polynomial with constant term 1. We do it by proving $\mathrm{Spec}(R[x]/(J + (x)))$ is empty.

Since f is universally closed the base change $\mathrm{Spec}(A[x]) \to \mathrm{Spec}(R[x])$ is closed. Hence the image of the closed subset $\mathrm{Spec}(B) \subset \mathrm{Spec}(A[x])$ is the closed subset $\mathrm{Spec}(R[x]/J) \subset \mathrm{Spec}(R[x])$, see Example 28.6.4 and Lemma 28.6.3. In particular $\mathrm{Spec}(B) \to \mathrm{Spec}(R[x]/J)$ is surjective. Consider the following diagram where every square is a pullback:

$$\begin{array}{ccccc} \mathrm{Spec}(B) & \xrightarrow{g} & \mathrm{Spec}(R[x]/J) & \longrightarrow & \mathrm{Spec}(R[x]) \\ \uparrow & & \uparrow & & \uparrow{\scriptstyle 0} \\ \emptyset & \longrightarrow & \mathrm{Spec}(R[x]/(J + (x))) & \longrightarrow & \mathrm{Spec}(R) \end{array}$$

The bottom left corner is empty because it is the spectrum of $R \otimes_{R[x]} B$ where the map $R[x] \to B$ sends x to an invertible element and $R[x] \to R$ sends x to 0. Since g is surjective this implies $\mathrm{Spec}(R[x]/(J + (x)))$ is empty, as we wanted to show. □

Lemma 28.44.8. *Let $f : X \to S$ be an integral morphism. Then every point of X is closed in its fibre.*

Proof. See Algebra, Lemma 10.33.18. □

Lemma 28.44.9. *A finite morphism is quasi-finite.*

Proof. This is implied by Algebra, Lemma 10.116.4 and Lemma 28.21.9. Alternatively, all points in fibres are closed points by Lemma 28.44.8 (and the fact that a

finite morphism is integral) and use Lemma 28.21.6 (3) to see that f is quasi-finite at x for all $x \in X$. □

Lemma 28.44.10. *A finite morphism is proper.*

Proof. A finite morphism is integral and hence universally closed by Lemma 28.44.7. It is also clearly separated and of finite type. Hence it is proper by definition. □

Lemma 28.44.11. *A closed immersion is finite (and a fortiori integral).*

Proof. True because a closed immersion is affine (Lemma 28.13.9) and a surjective ring map is finite and integral. □

Lemma 28.44.12. *Let $f : X \to Y$ and $g : Y \to Z$ be morphisms.*

(1) *If $g \circ f$ is finite and g separated then f is finite.*
(2) *If $g \circ f$ is integral and g separated then f is integral.*

Proof. Assume $g \circ f$ is finite (resp. integral) and g separated. The base change $X \times_Z Y \to Y$ is finite (resp. integral) by Lemma 28.44.6. The morphism $X \to X \times_Z Y$ is a closed immersion as $Y \to Z$ is separated, see Schemes, Lemma 25.21.12. A closed immersion is finite (resp. integral), see Lemma 28.44.11. The composition of finite (resp. integral) morphisms is finite (resp. integral), see Lemma 28.44.5. Thus we win. □

Lemma 28.44.13. *Let $f : X \to Y$ be a morphism of schemes. If f is finite and a monomorphism, then f is a closed immersion.*

Proof. This reduces to Algebra, Lemma 10.102.6. □

28.45. Universal homeomorphisms

The following definition is really superfluous since a universal homeomorphism is really just an integral, universally injective and surjective morphism, see Lemma 28.45.3.

Definition 28.45.1. A morphisms $f : X \to Y$ of schemes is called a *universal homeomorphism* if the base change $f' : Y' \times_Y X \to Y'$ is a homeomorphism for every morphism $Y' \to Y$.

Lemma 28.45.2. *Let $f : X \to Y$ be a morphism of schemes. If f is a homeomorphism onto a closed subset of Y then f is affine.*

Proof. Let $y \in Y$ be a point. If $y \notin f(X)$, then there exists an affine neighbourhood of y which is disjoint from $f(X)$. If $y \in f(X)$, let $x \in X$ be the unique point of X mapping to y. Let $y \in V$ be an affine open neighbourhood. Let $U \subset X$ be an affine open neighbourhood of x which maps into V. Since $f(U) \subset V \cap f(X)$ is open in the induced topology by our assumption on f we may choose a $h \in \Gamma(V, \mathcal{O}_Y)$ such that $y \in D(h)$ and $D(h) \cap f(X) \subset f(U)$. Denote $h' \in \Gamma(U, \mathcal{O}_X)$ the restriction of $f^\sharp(h)$ to U. Then we see that $D(h') \subset U$ is equal to $f^{-1}(D(h))$. In other words, every point of Y has an open neighbourhood whose inverse image is affine. Thus f is affine, see Lemma 28.13.3. □

Lemma 28.45.3. *Let $f : X \to Y$ be a morphism of schemes. The following are equivalent:*

(1) *f is a universal homeomorphism, and*
(2) *f is integral, universally injective and surjective.*

Proof. Assume f is a universal homeomorphism. By Lemma 28.45.2 we see that f is affine. Since f is clearly universally closed we see that f is integral by Lemma 28.44.7. It is also clear that f is universally injective and surjective.

Assume f is integral, universally injective and surjective. By Lemma 28.44.7 f is universally closed. Since it is also universally bijective (see Lemma 28.11.4) we see that it is a universal homeomorphism. □

Lemma 28.45.4. *Let X be a scheme. The canonical closed immersion $X_{red} \to X$ (see Schemes, Definition 25.12.5) is a universal homeomorphism.*

Proof. Omitted. □

Lemma 28.45.5. *Let $f : X \to S$ and $S' \to S$ be morphisms of schemes. Assume*

(1) *$S' \to S$ is a closed immersion,*
(2) *$S' \to S$ is bijective on points,*
(3) *$X \times_S S' \to S'$ is a closed immersion, and*
(4) *$X \to S$ is of finite type or $S' \to S$ is of finite presentation.*

Then $f : X \to S$ is a closed immersion.

Proof. Assumptions (1) and (2) imply that $S' \to S$ is a universal homeomorphism (for example because $S_{red} = S'_{red}$ and using Lemma 28.45.4). Hence (3) implies that $X \to S$ is homeomorphism onto a closed subset of S. Then $X \to S$ is affine by Lemma 28.45.2. Let $U \subset S$ be an affine open, say $U = \mathrm{Spec}(A)$. Then $S' = \mathrm{Spec}(A/I)$ by (1) for a locally nilpotent ideal I by (2). As f is affine we see that $f^{-1}(U) = \mathrm{Spec}(B)$. Assumption (4) tells us B is a finite type A-algebra (Lemma 28.16.2) or that I is finitely generated (Lemma 28.22.7). Assumption (3) is that $A/I \to B/IB$ is surjective. From Algebra, Lemma 10.120.8 if $A \to B$ is of finite type or Algebra, Lemma 10.18.1 if I is finitely generated and hence nilpotent we deduce that $A \to B$ is surjective. This means that f is a closed immersion, see Lemma 28.2.1. □

28.46. Finite locally free morphisms

In many papers the authors use finite flat morphisms when they really mean finite locally free morphisms. The reason is that if the base is locally Noetherian then this is the same thing. But in general it is not, see Exercises, Exercise 79.4.3.

Definition 28.46.1. Let $f : X \to S$ be a morphism of schemes. We say f is *finite locally free* if f is affine and $f_*\mathcal{O}_X$ is a finite locally free $\mathcal{O}_S$-module. In this case we say f is has *rank* or *degree* d if the sheaf $f_*\mathcal{O}_X$ is finite locally free of degree d.

Note that if $f : X \to S$ is finite locally free then S is the disjoint union of open and closed subschemes S_d such that $f^{-1}(S_d) \to S_d$ is finite locally free of degree d.

Lemma 28.46.2. *Let $f : X \to S$ be a morphism of schemes. The following are equivalent:*

(1) *f is finite locally free,*
(2) *f is finite, flat, and locally of finite presentation.*

If S is locally Noetherian these are also equivalent to

(3) *f is finite and flat.*

Proof. See Algebra, Lemma 10.73.2. The Noetherian case follows as a finite module over a Noetherian ring is a finitely presented module, see Algebra, Lemma 10.29.4. □

Lemma 28.46.3. *A composition of finite locally free morphisms is finite locally free.*

Proof. Omitted. □

Lemma 28.46.4. *A base change of a finite locally free morphism is finite locally free.*

Proof. Omitted. □

Lemma 28.46.5. *Let $f : X \to S$ be a finite locally free morphism of schemes. There exists a disjoint union decomposition $S = \coprod_{d \geq 0} S_d$ by open and closed subschemes such that setting $X_d = f^{-1}(S_d)$ the restrictions $f|_{X_d}$ are finite locally free morphisms $X_d \to S_d$ of degree d.*

Proof. This is true because a finite locally free sheaf locally has a well defined rank. Details omitted. □

Lemma 28.46.6. *Let $f : Y \to X$ be a finite morphism with X affine. There exists a diagram*

$$\begin{array}{ccccc} Z' & \xleftarrow[i]{} & Y' & \longrightarrow & Y \\ & \searrow & \downarrow & & \downarrow \\ & & X' & \longrightarrow & X \end{array}$$

where

(1) *$Y' \to Y$ and $X' \to X$ are surjective finite locally free,*
(2) *$Y' = X' \times_X Y$,*
(3) *$i : Y' \to Z'$ is a closed immersion,*
(4) *$Z' \to X'$ is finite locally free, and*
(5) *$Z' = \bigcup_{j=1,\ldots,m} Z'_j$ is a (set theoretic) finite union of closed subschemes, each of which maps isomorphically to X'.*

Proof. Write $X = \mathrm{Spec}(A)$ and $Y = \mathrm{Spec}(B)$. See also More on Algebra, Section 15.14. Let $x_1, \ldots, x_n \in B$ be generators of B over A. For each i we can choose a monic polynomial $P_i(T) \in A[T]$ such that $P(x_i) = 0$ in B. By Algebra, Lemma 10.129.9 (applied n times) there exists a finite locally free ring extension $A \subset A'$ such that each P_i splits completely:

$$P_i(T) = \prod\nolimits_{k=1,\ldots,d_i} (T - \alpha_{ik})$$

for certain $\alpha_{ik} \in A'$. Set

$$C = A'[T_1, \ldots, T_n]/(P_1(T_1), \ldots, P_n(T_n))$$

and $B' = A' \otimes_A B$. The map $C \to B'$, $T_i \mapsto 1 \otimes x_i$ is an A'-algebra surjection. Setting $X' = \mathrm{Spec}(A')$, $Y' = \mathrm{Spec}(B')$ and $Z' = \mathrm{Spec}(C)$ we see that (1) – (4)

hold. Part (5) holds because set theoretically $\mathrm{Spec}(C)$ is the union of the closed subschemes cut out by the ideals

$$(T_1 - \alpha_{1k_1}, T_2 - \alpha_{2k_2}, \ldots, T_n - \alpha_{nk_n})$$

for any $1 \leq k_i \leq d_i$. □

The following lemma is stated in the correct generality in Lemma 28.49.4 below.

Lemma 28.46.7. *Let $f : Y \to X$ be a finite morphism of schemes. Let $T \subset Y$ be a closed nowhere dense subset of Y. Then $f(T) \subset X$ is a closed nowhere dense subset of X.*

Proof. By Lemma 28.44.10 we know that $f(T) \subset X$ is closed. Let $X = \bigcup X_i$ be an affine covering. Since T is nowhere dense in Y, we see that also $T \cap f^{-1}(X_i)$ is nowhere dense in $f^{-1}(X_i)$. Hence if we can prove the theorem in the affine case, then we see that $f(T) \cap X_i$ is nowhere dense. This then implies that T is nowhere dense in X by Topology, Lemma 5.20.4.

Assume X is affine. Choose a diagram

$$\begin{array}{ccccc} Z' & \xleftarrow{i} & Y' & \xrightarrow{a} & Y \\ & \searrow & \downarrow{\scriptstyle f'} & & \downarrow{\scriptstyle f} \\ & & X' & \xrightarrow{b} & X \end{array}$$

as in Lemma 28.46.6. The morphisms a, b are open since they are finite locally free (Lemmas 28.46.2 and 28.26.9). Hence $T' = a^{-1}(T)$ is nowhere dense, see Topology, Lemma 5.20.6. The morphism b is surjective and open. Hence, if we can prove $f'(T') = b^{-1}(f(T))$ is nowhere dense, then $f(T)$ is nowhere dense, see Topology, Lemma 5.20.6. As i is a closed immersion, by Topology, Lemma 5.20.5 we see that $i(T') \subset Z'$ is closed and nowhere dense. Thus we have reduced the problem to the case discussed in the following paragraph.

Assume that $Y = \bigcup_{i=1,\ldots,n} Y_i$ is a finite union of closed subsets, each mapping isomorphically to X. Consider $T_i = Y_i \cap T$. If each of the T_i is nowhere dense in Y_i, then each $f(T_i)$ is nowhere dense in X as $Y_i \to X$ is an isomorphism. Hence $f(T) = f(T_i)$ is a finite union of nowhere dense closed subsets of X and we win, see Topology, Lemma 5.20.2. Suppose not, say T_1 contains a nonempty open $V \subset Y_1$. We are going to show this leads to a contradiction. Consider $Y_2 \cap V \subset V$. This is either a proper closed subset, or equal to V. In the first case we replace V by $V \setminus V \cap Y_2$, so $V \subset T_1$ is open in Y_1 and does not meet Y_2. In the second case we have $V \subset Y_1 \cap Y_2$ is open in both Y_1 and Y_2. Repeat sequentially with $i = 3, \ldots, n$. The result is a disjoint union decomposition

$$\{1, \ldots, n\} = I_1 \coprod I_2, \quad 1 \in I_1$$

and an open V of Y_1 contained in T_1 such that $V \subset Y_i$ for $i \in I_1$ and $V \cap Y_i = \emptyset$ for $i \in I_2$. Set $U = f(V)$. This is an open of X since $f|_{Y_1} : Y_1 \to X$ is an isomorphism. Then

$$f^{-1}(U) = V \coprod \bigcup\nolimits_{i \in I_2} (Y_i \cap f^{-1}(U))$$

As $\bigcup_{i \in I_2} Y_i$ is closed, this implies that $V \subset f^{-1}(U)$ is open, hence $V \subset Y$ is open. This contradicts the assumption that T is nowhere dense in Y, as desired. □

28.47. Generically finite morphisms

In this section we characterize maps between schemes which are locally of finite type and which are "generically finite" in some sense.

Lemma 28.47.1. *Let X, Y be schemes. Let $f : X \to Y$ be locally of finite type. Let $\eta \in Y$ be a generic point of an irreducible component of Y. The following are equivalent:*

1. *the set $f^{-1}(\{\eta\})$ is finite,*
2. *there exist affine opens $U_i \subset X$, $i = 1, \ldots, n$ and $V \subset Y$ with $f(U_i) \subset V$, $\eta \in V$ and $f^{-1}(\{\eta\}) \subset \bigcup U_i$ such that each $f|_{U_i} : U_i \to V$ is finite.*

If f is quasi-separated, then these are also equivalent to

3. *there exist affine opens $V \subset Y$, and $U \subset X$ with $f(U) \subset V$, $\eta \in V$ and $f^{-1}(\{\eta\}) \subset U$ such that $f|_U : U \to V$ is finite.*

If f is quasi-compact and quasi-separated, then these are also equivalent to

4. *there exists an affine open $V \subset Y$, $\eta \in V$ such that $f^{-1}(V) \to V$ is finite.*

Proof. The question is local on the base. Hence we may replace Y by an affine neighbourhood of η, and we may and do assume throughout the proof below that Y is affine, say $Y = \mathrm{Spec}(R)$.

It is clear that (2) implies (1). Assume that $f^{-1}(\{\eta\}) = \{\xi_1, \ldots, \xi_n\}$ is finite. Choose affine opens $U_i \subset X$ with $\xi_i \in U_i$. By Algebra, Lemma 10.116.9 we see that after replacing Y by a standard open in Y each of the morphisms $U_i \to Y$ is finite. In other words (2) holds.

It is clear that (3) implies (1). Assume $f^{-1}(\{\eta\}) = \{\xi_1, \ldots, \xi_n\}$ and assume that f is quasi-separated. Since Y is affine this implies that X is quasi-separated. Since each ξ_i maps to a generic point of an irreducible component of Y, we see that each ξ_i is a generic point of an irreducible component of X. By Properties, Lemma 27.27.1 we can find an affine open $U \subset X$ containing each ξ_i. By Algebra, Lemma 10.116.9 we see that after replacing Y by a standard open in Y the morphisms $U \to Y$ is finite. In other words (3) holds.

It is clear that (4) implies all of (1) – (3) with no further assumptions on f. Suppose that f is quasi-compact and quasi-separated. We have to show that the equivalent conditions (1) – (3) imply (4). Let U, V be as in (3). Replace Y by V. Since f is quasi-compact and Y is quasi-compact (being affine) we see that X is quasi-compact. Hence $Z = X \setminus U$ is quasi-compact, hence the morphism $f|_Z : Z \to Y$ is quasi-compact. By construction of Z we see that $\eta \notin f(Z)$. Hence by Lemma 28.8.4 we see that there exists an affine open neighbourhood V' of η in Y such that $f^{-1}(V') \cap Z = \emptyset$. Then we have $f^{-1}(V') \subset U$ and this means that $f^{-1}(V') \to V'$ is finite. □

Example 28.47.2. Let $A = \prod_{n \in \mathbf{N}} \mathbf{F}_2$. Every element of A is an idempotent. Hence every prime ideal is maximal with residue field $\mathbf{F}_2$. Thus the topology on $X = \mathrm{Spec}(A)$ is totally disconnected and quasi-compact. The projection maps $A \to \mathbf{F}_2$ define open points of $\mathrm{Spec}(A)$. It cannot be the case that all the points of X are open since X is quasi-compact. Let $x \in X$ be a closed point which is not open. Then we can form a scheme Y which is two copies of X glued along $X \setminus \{x\}$. In other words, this is X with x doubled, compare Schemes, Example 25.14.3. The morphism $f : Y \to X$ is quasi-compact, finite type and has finite fibres but is not

quasi-separated. The point $x \in X$ is a generic point of an irreducible component of X (since X is totally disconnected). But properties (3) and (4) of Lemma 28.47.1 do not hold. The reason is that for any open neighbourhood $x \in U \subset X$ the inverse image $f^{-1}(U)$ is not affine because functions on $f^{-1}(U)$ cannot separated the two points lying over x (proof omitted; this is a nice exercise). Hence the condition that f is quasi-separated is necessary in parts (3) and (4) of the lemma.

Remark 28.47.3. An alternative to Lemma 28.47.1 is the statement that a quasi-finite morphism is finite over a dense open of the target. This will be shown in More on Morphisms, Section 36.31.

Lemma 28.47.4. *Let X, Y be integral schemes. Let $f : X \to Y$ be locally of finite type. Assume f is dominant. The following are equivalent:*

1. *the extension $R(Y) \subset R(X)$ has transcendence degree 0,*
2. *the extension $R(Y) \subset R(X)$ is finite,*
3. *there exist nonempty affine opens $U \subset X$ and $V \subset Y$ such that $f(U) \subset V$ and $f|_U : U \to V$ is finite, and*
4. *the generic point of X is the only point of X mapping to the generic point of Y.*

If f is separated, or if f is quasi-compact, then these are also equivalent to

5. *there exists a nonempty affine open $V \subset Y$ such that $f^{-1}(V) \to V$ is finite.*

Proof. Choose any affine opens $\mathrm{Spec}(A) = U \subset X$ and $\mathrm{Spec}(R) = V \subset Y$ such that $f(U) \subset V$. Then R and A are domains by definition. The ring map $R \to A$ is of finite type Lemma 28.16.2). Let $K = f.f.(R) = R(Y)$ and $L = f.f.(A) = R(X)$. Then $K \subset L$ is a finitely generated field extension. Hence we see that (1) is equivalent to (2).

Suppose (2) holds. Let $x_1, \ldots, x_n \in A$ be generators of A over R. By assumption there exist nonzero polynomials $P_i(X) \in R[X]$ such that $P_i(x_i) = 0$. Let $f_i \in R$ be the leading coefficient of P_i. Then we conclude that $R_{f_1 \ldots f_n} \to A_{f_1 \ldots f_n}$ is finite, i.e., (3) holds. Note that (3) implies (2). So now we see that (1), (2) and (3) are all equivalent.

Let η be the generic point of X, and let $\eta' \in Y$ be the generic point of Y. Assume (4). Then $\dim_\eta(X_{\eta'}) = 0$ and we see that $R(X) = \kappa(\eta)$ has transcendence degree 0 over $R(Y) = \kappa(\eta')$ by Lemma 28.29.1. In other words (1) holds. Assume the equivalent conditions (1), (2) and (3). Suppose that $x \in X$ is a point mapping to η'. As x is a specialization of η, this gives inclusions $R(Y) \subset \mathcal{O}_{X,x} \subset R(X)$, which implies $\mathcal{O}_{X,x}$ is a field, see Algebra, Lemma 10.33.17. Hence $x = \eta$. Thus we see that (1) – (4) are all equivalent.

It is clear that (5) implies (3) with no additional assumptions on f. What remains is to prove that if f is either separated or quasi-compact, then the equivalent conditions (1) – (4) imply (5).

Assume U, V as in (3) and assume f is separated. Then $U \to f^{-1}(V)$ is a morphism from a scheme proper over V Lemma 28.44.10) into a scheme separated over V. Hence $U \subset f^{-1}(V)$ is closed Lemma 28.42.7. Since X is irreducible we conclude $U = f^{-1}(V)$. This proves (5).

Assume f is quasi-compact. Let U, V be as in (3). Then $f^{-1}(V)$ is quasi-compact. Consider the closed subset $Z = f^{-1}(V) \setminus U$. Since Z does not contain the generic point of X we see that the quasi-compact morphism $f : Z \to V$ is not dominant by Lemma 28.8.3. Hence after shrinking V we may assume that $f^{-1}(V) = U$ which implies that (5) holds. □

Definition 28.47.5. Let X and Y be integral schemes. Let $f : X \to Y$ be locally of finite type and dominant. Assume $[R(X) : R(Y)] < \infty$, or any other of the equivalent conditions (1) – (4) of Lemma 28.47.4. Then the positive integer

$$\deg(X/Y) = [R(X) : R(Y)]$$

is called the *degree of X over Y*.

It is possible to extend this notion to a morphism $f : X \to Y$ if (a) Y is integral with generic point η, (b) f is locally of finite type, and (c) $f^{-1}(\{\eta\})$ is finite. Namely, in this case we can define

$$\deg(X/Y) = \sum\nolimits_{\xi \in X,\ f(\xi) = \eta} \dim_{R(Y)}(\mathcal{O}_{X,\xi}).$$

Namely, given that $R(Y) = \kappa(\eta) = \mathcal{O}_{Y,\eta}$ (Lemma 28.10.4) the dimensions above are finite by Lemma 28.47.1 above. However, for most applications the definition given above is the right one.

Lemma 28.47.6. *Let X, Y, Z be integral schemes. Let $f : X \to Y$ and $g : Y \to Z$ be dominant morphisms locally of finite type. Assume that $[R(X) : R(Y)] < \infty$ and $[R(Y) : R(Z)] < \infty$. Then*

$$\deg(X/Z) = \deg(X/Y) \deg(Y/Z).$$

Proof. This comes from the multiplicativity of degrees in towers of finite extensions of fields, see Fields, Lemma 9.7.6. □

Remark 28.47.7. Let $f : X \to Y$ be a morphism of schemes which is locally of finite type. There are (at least) two properties that we could use to define *generically finite* morphisms. These correspond to whether you want the property to be local on the source or local on the target:

(1) (Local on the target; suggested by Ravi Vakil.) Assume every quasi-compact open of Y has finitely many irreducible components (for example if Y is locally Noetherian). The requirement is that the inverse image of each generic point is finite, see Lemma 28.47.1.
(2) (Local on the source.) The requirement is that there exists a dense open $U \subset X$ such that $U \to Y$ is locally quasi-finite.

In case (1) the requirement can be formulated without the auxiliary condition on Y, but probably doesn't give the right notion for general schemes. Property (2) as formulated doesn't imply that the fibres over generic points are finite; however, if f is quasi-compact and Y is as in (1) then it does.

28.48. Normalization

In this section we construct the *normalization*, and the *normalization of one scheme in another*.

Lemma 28.48.1. *Let X be a scheme. Let $\mathcal{A}$ be a quasi-coherent sheaf of $\mathcal{O}_X$-algebras. The subsheaf $\mathcal{A}' \subset \mathcal{A}$ defined by the rule*

$$U \longmapsto \{f \in \mathcal{A}(U) \mid f_x \in \mathcal{A}_x \text{ integral over } \mathcal{O}_{X,x} \text{ for all } x \in U\}$$

is a quasi-coherent $\mathcal{O}_X$-algebra, and for any affine open $U \subset X$ the ring $\mathcal{A}'(U) \subset \mathcal{A}(U)$ is the integral closure of $\mathcal{O}_X(U)$ in $\mathcal{A}(U)$.

Proof. This is a subsheaf by the local nature of the conditions. It is an $\mathcal{O}_X$-algebra by Algebra, Lemma 10.33.7. Let $U \subset X$ be an affine open. Say $U = \mathrm{Spec}(R)$ and say $\mathcal{A}$ is the quasi-coherent sheaf associated to the R-algebra A. Then according to Algebra, Lemma 10.33.10 the value of $\mathcal{A}'$ over U is given by the integral closure A' of R in A. This proves the last assertion of the lemma. To prove that $\mathcal{A}'$ is quasi-coherent, it suffices to show that $\mathcal{A}'(D(f)) = A'_f$. This follows from the fact that integral closure and localization commute, see Algebra, Lemma 10.33.9. □

Definition 28.48.2. Let X be a scheme. Let $\mathcal{A}$ be a quasi-coherent sheaf of $\mathcal{O}_X$-algebras. The *integral closure of $\mathcal{O}_X$ in $\mathcal{A}$* is the quasi-coherent $\mathcal{O}_X$-subalgebra $\mathcal{A}' \subset \mathcal{A}$ constructed in Lemma 28.48.1 above.

In the setting of the definition above we can consider the morphism of relative spectra

$$\begin{array}{ccc} Y = \underline{\mathrm{Spec}}_X(\mathcal{A}) & \longrightarrow & X' = \underline{\mathrm{Spec}}_X(\mathcal{A}') \\ & \searrow \quad \swarrow & \\ & X & \end{array}$$

see Lemma 28.13.5. The scheme $X' \to X$ will be the normalization of X in the scheme Y. Here is a slightly more general setting. Suppose we have a quasi-compact and quasi-separated morphism $f : Y \to X$ of schemes. In this case the sheaf of $\mathcal{O}_X$-algebras $f_*\mathcal{O}_Y$ is quasi-coherent, see Schemes, Lemma 25.24.1. Taking the integral closure $\mathcal{O}' \subset f_*\mathcal{O}_Y$ we obtain a quasi-coherent sheaf of $\mathcal{O}_X$-algebras whose relative spectrum is the normalization of X in Y. Here is the formal definition.

Definition 28.48.3. Let $f : Y \to X$ be a quasi-compact and quasi-separated morphism of schemes. Let $\mathcal{O}'$ be the integral closure of $\mathcal{O}_X$ in $f_*\mathcal{O}_Y$. The *normalization of X in Y* is the scheme[12]

$$\nu : X' = \underline{\mathrm{Spec}}_X(\mathcal{O}') \to X$$

over X. It comes equipped with a natural factorization

$$Y \xrightarrow{f'} X' \xrightarrow{\nu} X$$

of the initial morphism f.

The factorization is the composition of the canonical morphism $Y \to \underline{\mathrm{Spec}}(f_*\mathcal{O}_Y)$ (see Constructions, Lemma 26.4.7) and the morphism of relative spectra coming from the inclusion map $\mathcal{O}' \to f_*\mathcal{O}_Y$. We can characterize the normalization as follows.

Lemma 28.48.4. *Let $f : Y \to X$ be a quasi-compact and quasi-separated morphism of schemes. The factorization $f = \nu \circ f'$, where $\nu : X' \to X$ is the normalization of X in Y is characterized by the following two properties:*

[12] The scheme X' need not be normal, for example if $Y = X$ and $f = \mathrm{id}_X$, then $X' = X$.

(1) *the morphism ν is integral, and*
(2) *for any factorization $f = \pi \circ g$, with $\pi : Z \to X$ integral, there exists a commutative diagram*

$$\begin{array}{ccc} Y & \xrightarrow{g} & Z \\ {\scriptstyle f'}\downarrow & \nearrow_{h} & \downarrow{\scriptstyle \pi} \\ X' & \xrightarrow{\nu} & X \end{array}$$

for some unique morphism $h : X' \to Z$.

Moreover, in (2) the morphism $h : X' \to Z$ is the normalization of Z in Y.

Proof. Let $\mathcal{O}' \subset f_*\mathcal{O}_Y$ be the integral closure of $\mathcal{O}_X$ as in Definition 28.48.3. The morphism ν is integral by construction, which proves (1). Assume given a factorization $f = \pi \circ g$ with $\pi : Z \to X$ integral as in (2). By Definition 28.44.1 π is affine, and hence Z is the relative spectrum of a quasi-coherent sheaf of $\mathcal{O}_X$-algebras $\mathcal{B}$. The morphism $g : X \to Z$ corresponds to a map of $\mathcal{O}_X$-algebras $\chi : \mathcal{B} \to f_*\mathcal{O}_Y$. Since $\mathcal{B}(U)$ is integral over $\mathcal{O}_X(U)$ for every affine open $U \subset X$ (by Definition 28.44.1) we see from Lemma 28.48.1 that $\chi(\mathcal{B}) \subset \mathcal{O}'$. By the functoriality of the relative spectrum Lemma 28.13.5 this provides us with a unique morphism $h : X' \to Z$. We omit the verification that the diagram commutes.

It is clear that (1) and (2) characterize the factorization $f = \nu \circ f'$ since it characterizes it as an initial object in a category. The morphism h in (2) is integral by Lemma 28.44.12. Given a factorization $g = \pi' \circ g'$ with $\pi' : Z' \to Z$ integral, we get a factorization $f = (\pi \circ \pi') \circ g'$ and we get a morphism $h' : X' \to Z'$. Uniqueness implies that $\pi' \circ h' = h$. Hence the characterization (1), (2) applies to the morphism $h : X' \to Z$ which gives the last statement of the lemma. □

Lemma 28.48.5. *Let*

$$\begin{array}{ccc} Y_2 & \longrightarrow & Y_1 \\ {\scriptstyle f_2}\downarrow & & \downarrow{\scriptstyle f_1} \\ X_2 & \longrightarrow & X_1 \end{array}$$

be a commutative diagram of morphisms of schemes. Assume f_1, f_2 quasi-compact and quasi-separated. Let $f_i = \nu_i \circ f_i'$, $i = 1, 2$ be the canonical factorizations, where $\nu_i : X_i' \to X_i$ is the normalization of X_i in Y_i. Then there exists a canonical commutative diagram

$$\begin{array}{ccc} Y_2 & \longrightarrow & Y_1 \\ {\scriptstyle f_2'}\downarrow & & \downarrow{\scriptstyle f_1'} \\ X_2' & \longrightarrow & X_1' \\ {\scriptstyle \nu_2}\downarrow & & \downarrow{\scriptstyle \nu_1} \\ X_2 & \longrightarrow & X_1 \end{array}$$

Proof. By Lemmas 28.48.4 (1) and 28.44.6 the base change $X_2 \times_{X_1} X_1' \to X_2$ is integral. Note that f_2 factors through this morphism. Hence we get a canonical morphism $X_2' \to X_2 \times_{X_1} X_1'$ from Lemma 28.48.4 (2). This gives the middle horizontal arrow in the last diagram. □

Lemma 28.48.6. *Let $f : Y \to X$ be a quasi-compact and quasi-separated morphism of schemes. Let $U \subset X$ be an open subscheme and set $V = f^{-1}(U)$. Then the normalization of U in V is the inverse image of U in the normalization of X in Y.*

Proof. Clear from the construction. □

Lemma 28.48.7. *Let $f : Y \to X$ be a quasi-compact and quasi-separated morphism of schemes. Suppose that $Y = Y_1 \amalg Y_2$ is a disjoint union of two schemes. Write $f_i = f|_{Y_i}$. Let X_i' be the normalization of X in Y_i. Then $X_1' \amalg X_2'$ is the normalization of X in Y.*

Proof. In terms of integral closures this corresponds to the following fact: Let $A \to B$ be a ring map. Suppose that $B = B_1 \times B_2$. Let A_i' be the integral closure of A in B_i. Then $A_1' \times A_2'$ is the integral closure of A in B. The reason this works is that the elements $(1, 0)$ and $(0, 1)$ of B are idempotents and hence integral over A. Thus the integral closure A' of A in B is a product and it is not hard to see that the factors are the integral closures A_i' as described above (some details omitted). □

Lemma 28.48.8. *Let $f : Y \to X$ be an integral morphism. Then the integral closure of X in Y is equal to Y.*

Proof. Omitted. □

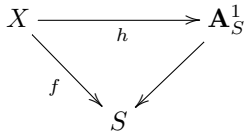
The following lemma is a generalization of the preceding one.

Lemma 28.48.9. *Let $f : X \to S$ be a quasi-compact, quasi-separated and universally closed morphisms of schemes. Then $f_*\mathcal{O}_X$ is integral over $\mathcal{O}_S$. In other words, the normalization of S in X is equal to the factorization*

$$X \longrightarrow \underline{\mathrm{Spec}}_S(f_*\mathcal{O}_X) \longrightarrow S$$

of Constructions, Lemma 26.4.7.

Proof. The question is local on S, hence we may assume $S = \mathrm{Spec}(R)$ is affine. Let $h \in \Gamma(X, \mathcal{O}_X)$. We have to show that h satisfies a monic equation over R. Think of h as a morphism as in the following commutative diagram



Let $Z \subset \mathbf{A}^1_S$ be the scheme theoretic image of h, see Definition 28.6.2. The morphism h is quasi-compact as f is quasi-compact and $\mathbf{A}^1_S \to S$ is separated, see Schemes, Lemma 25.21.15. By Lemma 28.6.3 the morphism $X \to Z$ is dominant. By Lemma 28.42.7 the morphism $X \to Z$ is closed. Hence $h(X) = Z$ (set theoretically). Thus we can use Lemma 28.42.8 to conclude that $Z \to S$ is universally closed (and even proper). Since $Z \subset \mathbf{A}^1_S$, we see that $Z \to S$ is affine and proper, hence integral by Lemma 28.44.7. Writing $\mathbf{A}^1_S = \mathrm{Spec}(R[T])$ we conclude that the ideal $I \subset R[T]$ of Z contains a monic polynomial $P(T) \in R[T]$. Hence $P(h) = 0$ and we win. □

Lemma 28.48.10. *Let $f : Y \to X$ be a quasi-compact and quasi-separated morphism of schemes. Assume*

(1) *Y is a normal scheme,*

(2) *any quasi-compact open $V \subset Y$ has a finite number of irreducible components.*

Then the normalization X' of X in Y is a normal scheme. Moreover, the morphism $Y \to X'$ is dominant and induces a bijection of irreducible components.

Proof. We first prove that X' is normal. Let $U \subset X$ be an affine open. It suffices to prove that the inverse image of U in X' is normal (see Properties, Lemma 27.7.2). By Lemma 28.48.6 we may replace X by U, and hence we may assume $X = \mathrm{Spec}(A)$ affine. In this case Y is quasi-compact, and hence has a finite number of irreducible components by assumption. Hence $Y = \coprod_{i=1,\ldots,n} Y_i$ is a finite disjoint union of normal integral schemes by Properties, Lemma 27.7.5. By Lemma 28.48.7 we see that $X' = \coprod_{i=1,\ldots,n} X'_i$, where X'_i is the normalization of X in Y_i. By Properties, Lemma 27.7.9 we see that $B_i = \Gamma(Y_i, \mathcal{O}_{Y_i})$ is a normal domain. Note that $X'_i = \mathrm{Spec}(A'_i)$, where $A'_i \subset B_i$ is the integral closure of A in B_i, see Lemma 28.48.1. By Algebra, Lemma 10.34.2 we see that $A'_i \subset B_i$ is a normal domain. Hence $X' = \coprod X'_i$ is a finite union of normal schemes and hence is normal.

It is clear from the description of X' above that $Y \to X'$ is dominant and induces a bijection on irreducible components if X is affine. The result in general follows from this by a topological argument (omitted). □

Lemma 28.48.11. *Let $f : X \to S$ be a morphism. Assume that*

(1) *S is a Nagata scheme,*
(2) *f is of finite type*[13]*, and*
(3) *X is reduced.*

Then the normalization $\nu : S' \to S$ of S in X is finite.

Proof. There is an immediate reduction to the case $S = \mathrm{Spec}(R)$ where R is a Nagata ring. In this case we have to show that the integral closure A of R in $\Gamma(X, \mathcal{O}_X)$ is finite over R. Since f is of finite type we can write $X = \bigcup_{i=1,\ldots,n} U_i$ with each U_i affine. Say $U_i = \mathrm{Spec}(B_i)$. Each B_i is a reduced ring of finite type over R (Lemma 28.16.2). Moreover, $\Gamma(X, \mathcal{O}_X) \subset B = B_1 \times \ldots \times B_n$. So A is contained in the integral closure A' of R in B. Note that B is a reduced finite type R-algebra. Since R is Noetherian it suffices to prove that A' is finite over R. This is Algebra, Lemma 10.148.16. □

Next, we come to the normalization of a scheme X. We only define/construct it when X has locally finitely many irreducible components. Let X be a scheme such that every quasi-compact open has finitely many irreducible components. Let $X^{(0)} \subset X$ be the set of generic points of irreducible components of X. Let

$$f : Y = \coprod\nolimits_{\eta \in X^{(0)}} \mathrm{Spec}(\kappa(\eta)) \longrightarrow X \tag{28.48.11.1}$$

be the inclusion of the generic points into X using the canonical maps of Schemes, Section 25.13. Note that this morphism is quasi-compact by assumption and quasi-separated as Y is separated (see Schemes, Section 25.21).

Definition 28.48.12. Let X be a scheme such that every quasi-compact open has finitely many irreducible components. We define the *normalization* of X as the morphism

$$\nu : X^\nu \longrightarrow X$$

[13] The proof shows that the lemma holds if f is quasi-compact and "essentially of finite type".

which is the normalization of X in the morphism $f : Y \to X$ (28.48.11.1) constructed above.

Any locally Noetherian scheme has a locally finite set of irreducible components and the definition applies to it. Usually the normalization is defined only for reduced schemes. With the definition above the normalization of X is the same as the normalization of the reduction X_{red} of X.

Lemma 28.48.13. *Let X be a scheme such that every quasi-compact open has finitely many irreducible components. The normalization morphism ν factors through the reduction X_{red} and $X^\nu \to X_{red}$ is the normalization of X_{red}.*

Proof. Let $f : Y \to X$ be the morphism (28.48.11.1). We get a factorization $Y \to X_{red} \to X$ of f from Schemes, Lemma 25.12.6. By Lemma 28.48.4 we obtain a canonical morphism $X^\nu \to X_{red}$ and that X^ν is the normalization of X_{red} in Y. The lemma follows as $Y \to X_{red}$ is identical to the morphism (28.48.11.1) constructed for X_{red}. □

If X is reduced, then the normalization of X is the same as the relative spectrum of the integral closure of $\mathcal{O}_X$ in the sheaf of meromorphic functions $\mathcal{K}_X$ (see Divisors, Section 30.15). Namely, $\mathcal{K}_X = f_*\mathcal{O}_Y$ in this case, see Divisors, Lemma 30.15.8 and its proof. We describe this here explicitly.

Lemma 28.48.14. *Let X be a reduced scheme such that every quasi-compact open has finitely many irreducible components. Let $\mathrm{Spec}(A) = U \subset X$ be an affine open. Then*

1. *A has finitely many minimal primes $\mathfrak{q}_1, \ldots, \mathfrak{q}_t$,*
2. *the total ring of fractions $Q(A)$ of A is $Q(A/\mathfrak{q}_1) \times \ldots \times Q(A/\mathfrak{q}_t)$,*
3. *the integral closure A' of A in $Q(A)$ is the product of the integral closures of the domains $A/\mathfrak{q}_i$ in the fields $Q(A/\mathfrak{q}_i)$, and*
4. *$\nu^{-1}(U)$ is identified with the spectrum of A'.*

Proof. Minimal primes correspond to irreducible components (Algebra, Lemma 10.24.1), hence we have (1) by assumption. Then $(0) = \mathfrak{q}_1 \cap \ldots \cap \mathfrak{q}_t$ because A is reduced (Algebra, Lemma 10.16.2). Then we have $Q(A) = \prod A_{\mathfrak{q}_i} = \prod \kappa(\mathfrak{q}_i)$ by Algebra, Lemmas 10.23.2 and 10.24.4. This proves (2). Part (3) follows from Algebra, Lemma 10.34.14, or Lemma 28.48.7. Part (4) holds because it is clear that $f^{-1}(U) \to U$ is the morphism

$$\mathrm{Spec}\left(\prod \kappa(\mathfrak{q}_i)\right) \longrightarrow \mathrm{Spec}(A)$$

where $f : Y \to X$ is the morphism (28.48.11.1). □

Lemma 28.48.15. *Let X be a scheme such that every quasi-compact open has finitely many irreducible components.*

1. *The normalization X^ν is a normal scheme.*
2. *The morphism $\nu : X^\nu \to X$ is integral, surjective, and induces a bijection on irreducible components.*
3. *For any integral, birational[14] morphism $X' \to X$ there exists a factorization $X^\nu \to X' \to X$ and $X^\nu \to X'$ is the normalization of X'.*

[14] It suffices if $X'_{red} \to X_{red}$ is birational.

(4) *For any morphism $Z \to X$ with Z a normal scheme such that each irreducible component of Z dominates an irreducible component of X there exists a unique factorization $Z \to X^\nu \to X$.*

Proof. Let $f : Y \to X$ be as in (28.48.11.1). Part (1) follows from Lemma 28.48.10 and the fact that Y is normal. It also follows from the description of the affine opens in Lemma 28.48.14.

The morphism ν is integral by Lemma 28.48.4. By Lemma 28.48.10 the morphism $Y \to X^\nu$ induces a bijection on irreducible components, and by construction of Y this implies that $X^\nu \to X$ induces a bijection on irreducible components. By construction $f : Y \to X$ is dominant, hence also ν is dominant. Since an integral morphism is closed (Lemma 28.44.7) this implies that ν is surjective. This proves (2).

Suppose that $\alpha : X' \to X$ is integral and birational. Any quasi-compact open U' of X' maps to a quasi-compact open of X, hence we see that U' has only finitely many irreducible components. Let $f' : Y' \to X'$ be the morphism (28.48.11.1) constructed starting with X'. As α is birational it is clear that $Y' = Y$ and $f = \alpha \circ f'$. Hence the factorization $X^\nu \to X' \to X$ exists and $X^\nu \to X'$ is the normalization of X' by Lemma 28.48.4. This proves (3).

Let $g : Z \to X$ be a morphism whose domain is a normal scheme and such that every irreducible component dominates an irreducible component of X. By Lemma 28.48.13 we have $X^\nu = X^\nu_{red}$ and by Schemes, Lemma 25.12.6 $Z \to X$ factors through X_{red}. Hence we may replace X by X_{red} and assume X is reduced. Moreover, as the factorization is unique it suffices to construct it locally on Z. Let $W \subset Z$ and $U \subset X$ be affine opens such that $g(W) \subset U$. Write $U = \mathrm{Spec}(A)$ and $W = \mathrm{Spec}(B)$, with $g|_W$ given by $\varphi : A \to B$. We will use the results of Lemma 28.48.14 freely. Let $\mathfrak{p}_1, \ldots, \mathfrak{p}_t$ be the minimal primes of A. As Z is normal, we see that B is a normal ring, in particular reduced. Moreover, by assumption any minimal prime $\mathfrak{q} \subset B$ we have that $\varphi^{-1}(\mathfrak{q})$ is a minimal prime of A. Hence if $x \in A$ is a nonzerodivisor, i.e., $x \notin \bigcup \mathfrak{p}_i$, then $\varphi(x)$ is a nonzerodivisor in B. Thus we obtain a canonical ring map $Q(A) \to Q(B)$. As B is normal it is equal to its integral closure in $Q(B)$ (see Algebra, Lemma 10.34.11). Hence we see that the integral closure $A' \subset Q(A)$ of A maps into B via the canonical map $Q(A) \to Q(B)$. Since $\nu^{-1}(U) = \mathrm{Spec}(A')$ this gives the canonical factorization $W \to \nu^{-1}(U) \to U$ of $\nu|_W$. We omit the verification that it is unique. □

Lemma 28.48.16. *Let X be an integral, Japanese scheme. The normalization $\nu : X^\nu \to X$ is a finite morphism.*

Proof. Follows from the definitions and Lemma 28.48.14. Namely, in this case the lemma says that $\nu^{-1}(\mathrm{Spec}(A))$ is the spectrum of the integral closure of A in its field of fractions. □

Lemma 28.48.17. *Let X be a nagata scheme. The normalization $\nu : X^\nu \to X$ is a finite morphism.*

Proof. Note that a Nagata scheme is locally Noetherian, thus Definition 28.48.12 does apply. Write $X^\nu \to X$ as the composition $X^\nu \to X_{red} \to X$. As $X_{red} \to X$ is a closed immersion it is finite. Hence it suffices to prove the lemma for a reduced Nagata scheme (by Lemma 28.44.5). Let $\mathrm{Spec}(A) = U \subset X$ be an affine open. By

Lemma 28.48.14 we have $\nu^{-1}(U) = \mathrm{Spec}(\prod A'_i)$ where A'_i is the integral closure of $A/\mathfrak{q}_i$ in its fraction field. As A is a Nagata ring (see Properties, Lemma 27.13.6) each of the ring extensions $A/\mathfrak{q}_i \subset A'_i$ are finite. Hence $A \to \prod A'_i$ is a finite ring map and we win. □

28.49. Zariski's Main Theorem (algebraic version)

This is the version you can prove using purely algebraic methods. Before we can prove more powerful versions (for non-affine morphisms) we need to develop more tools. See Cohomology of Schemes, Section 29.19 and More on Morphisms, Section 36.31.

Theorem 28.49.1 (Algebraic version of Zariski's Main Theorem)**.** *Let $f : Y \to X$ be an affine morphism of schemes. Assume f is of finite type. Let X' be the normalization of X in Y. Picture:*

$$\begin{array}{ccc} Y & \longrightarrow & X' \\ & {\scriptstyle f} \searrow \quad \swarrow {\scriptstyle \nu} & \\ & X & \end{array}$$

Then there exists an open subscheme $U' \subset X'$ such that

(1) *$(f')^{-1}(U') \to U'$ is an isomorphism, and*
(2) *$(f')^{-1}(U') \subset Y$ is the set of points at which f is quasi-finite.*

Proof. There is an immediate reduction to the case where X and hence Y are affine. Say $X = \mathrm{Spec}(R)$ and $Y = \mathrm{Spec}(A)$. Then $X' = \mathrm{Spec}(A')$, where A' is the integral closure of R in A, see Definitions 28.48.2 and 28.48.3. By Algebra, Theorem 10.117.13 for every $y \in Y$ at which f is quasi-finite, there exists an open $U'_y \subset X'$ such that $(f')^{-1}(U'_y) \to U'_y$ is an isomorphism. Set $U' = \bigcup U'_y$ where $y \in Y$ ranges over all points where f is quasi-finite. It remains to show that f is quasi-finite at all points of $(f')^{-1}(U')$. If $y \in (f')^{-1}(U')$ with image $x \in X$, then we see that $Y_x \to X'_x$ is an isomorphism in a neighbourhood of y. Hence there is no point of Y_x which specializes to y, since this is true for $f'(y)$ in X'_x, see Lemma 28.44.8. By Lemma 28.21.6 part (3) this implies f is quasi-finite at y. □

We can use the algebraic version of Zariski's Main Theorem to show that the set of points where a morphism is quasi-finite is open.

Lemma 28.49.2. *Let $f : X \to S$ be a morphism of schemes. The set of points of X where f is quasi-finite is an open $U \subset X$. The induced morphism $U \to S$ is locally quasi-finite.*

Proof. Suppose f is quasi-finite at x. Let $x \in U = \mathrm{Spec}(R) \subset X$, $V = \mathrm{Spec}(A) \subset S$ be affine opens as in Definition 28.21.1. By either Theorem 28.49.1 above or Algebra, Lemma 10.117.14, the set of primes $\mathfrak{q}$ at which $R \to A$ is quasi-finite is open in $\mathrm{Spec}(A)$. Since these all correspond to points of X where f is quasi-finite we get the first statement. The second statement is obvious. □

We will improve the following lemma to general quasi-finite separated morphisms later, see More on Morphisms, Lemma 36.31.3.

Lemma 28.49.3. *Let $f : Y \to X$ be a morphism of schemes. Assume*

(1) *X and Y are affine, and*

(2) *f is quasi-finite.*

Then there exists a diagram

$$\begin{array}{ccc} Y & \xrightarrow{j} & Z \\ & {\scriptstyle f}\searrow \quad \swarrow{\scriptstyle \pi} & \\ & X & \end{array}$$

with Z affine, π finite and j an open immersion.

Proof. This is Algebra, Lemma 10.117.15 reformulated in the language of schemes. □

Lemma 28.49.4. *Let $f : Y \to X$ be a quasi-finite morphism of schemes. Let $T \subset Y$ be a closed nowhere dense subset of Y. Then $f(T) \subset X$ is a nowhere dense subset of X.*

Proof. As in the proof of Lemma 28.46.7 this reduces immediately to the case where the base X is affine. In this case $Y = \bigcup_{i=1,\ldots,n} Y_i$ is a finite union of affine opens (as f is quasi-compact). Since each $T \cap Y_i$ is nowhere dense, and since a finite union of nowhere dense sets is nowhere dense (see Topology, Lemma 5.20.2), it suffices to prove that the image $f(T \cap Y_i)$ is nowhere dense in X. This reduces us to the case where both X and Y are affine. At this point we apply Lemma 28.49.3 above to get a diagram

$$\begin{array}{ccc} Y & \xrightarrow{j} & Z \\ & {\scriptstyle f}\searrow \quad \swarrow{\scriptstyle \pi} & \\ & X & \end{array}$$

with Z affine, π finite and j an open immersion. Set $\overline{T} = \overline{j(T)} \subset Z$. By Topology, Lemma 5.20.3 we see $\overline{T}$ is nowhere dense in Z. Since $f(T) \subset \pi(\overline{T})$ the lemma follows from the corresponding result in the finite case, see Lemma 28.46.7. □

28.50. Universally bounded fibres

Let X be a scheme over a field k. If X is finite over k, then $X = \mathrm{Spec}(A)$ where A is a finite k-algebra. Another way to say this is that X is finite locally free over $\mathrm{Spec}(k)$, see Definition 28.46.1. Hence $X \to \mathrm{Spec}(k)$ has a *degree* which is an integer $d \geq 0$, namely $d = \dim_k(A)$. We sometime call this the *degree* of the (finite) scheme X over k.

Definition 28.50.1. Let $f : X \to Y$ be a morphism of schemes.

(1) We say the integer n *bounds the degrees of the fibres of f* if for all $y \in Y$ the fibre X_y is a finite scheme over $\kappa(y)$ whose degree over $\kappa(y)$ is $\leq n$.
(2) We say the *fibres of f are universally bounded*[15] if there exists an integer n which bounds the degrees of the fibres of f.

Note that in particular the number of points in a fibre is bounded by n as well. (The converse does not hold, even if all fibres are finite reduced schemes.)

Lemma 28.50.2. *Let $f : X \to Y$ be a morphism of schemes. Let $n \geq 0$. The following are equivalent:*

[15] This is probably nonstandard notation.

(1) *the integer n bounds the degrees of the fibres of f, and*
(2) *for every morphism $\mathrm{Spec}(k) \to Y$, where k is a field, the fibre product $X_k = \mathrm{Spec}(k) \times_Y X$ is finite over k of degree $\leq n$.*

In this case f is universally bounded and the schemes X_k have at most n points.

Proof. The implication (2) $\Rightarrow$ (1) is trivial. The other implication holds because if the image of $\mathrm{Spec}(k) \to Y$ is y, then $X_k = \mathrm{Spec}(k) \times_{\mathrm{Spec}(\kappa(y))} X_y$. □

Lemma 28.50.3. *A composition of morphisms with universally bounded fibres is a morphism with universally bounded fibres. More precisely, assume that n bounds the degrees of the fibres of $f : X \to Y$ and m bounds the degrees of $g : Y \to Z$. Then nm bounds the degrees of the fibres of $g \circ f : X \to Z$.*

Proof. Let $f : X \to Y$ and $g : Y \to Z$ have universally bounded fibres. Say that $\deg(X_y/\kappa(y)) \leq n$ for all $y \in Y$, and that $\deg(Y_z/\kappa(z)) \leq m$ for all $z \in Z$. Let $z \in Z$ be a point. By assumption the scheme Y_z is finite over $\mathrm{Spec}(\kappa(z))$. In particular, the underlying topological space of Y_z is a finite discrete set. The fibres of the morphism $f_z : X_z \to Y_z$ are the fibres of f at the corresponding points of Y, which are finite discrete sets by the reasoning above. Hence we conclude that the underlying topological space of X_z is a finite discrete set as well. Thus X_z is an affine scheme (this is a nice exercise; it also follows for example from Properties, Lemma 27.27.1 applied to the set of all points of X_z). Write $X_z = \mathrm{Spec}(A)$, $Y_z = \mathrm{Spec}(B)$, and $k = \kappa(z)$. Then $k \to B \to A$ and we know that (a) $\dim_k(B) \leq m$, and (b) for every maximal ideal $\mathfrak{m} \subset B$ we have $\dim_{\kappa(\mathfrak{m})}(A/\mathfrak{m}A) \leq n$. We claim this implies that $\dim_k(A) \leq nm$. Note that B is the product of its localizations $B_\mathfrak{m}$, for example because Y_z is a disjoint union of 1-point schemes, or by Algebra, Lemmas 10.49.2 and 10.49.8. So we see that $\dim_k(B) = \sum_\mathfrak{m}(B_\mathfrak{m})$ and $\dim_k(A) = \sum_\mathfrak{m}(A_\mathfrak{m})$ where in both cases $\mathfrak{m}$ runs over the maximal ideals of B (not of A). By the above, and Nakayama's Lemma (Algebra, Lemma 10.18.1) we see that each $A_\mathfrak{m}$ is a quotient of $B_\mathfrak{m}^{\oplus n}$ as a $B_\mathfrak{m}$-module. Hence $\dim_k(A_\mathfrak{m}) \leq n \dim_k(B_\mathfrak{m})$. Putting everything together we see that

$$\dim_k(A) = \sum\nolimits_\mathfrak{m}(A_\mathfrak{m}) \leq \sum\nolimits_\mathfrak{m} n \dim_k(B_\mathfrak{m}) = n \dim_k(B) \leq nm$$

as desired. □

Lemma 28.50.4. *A base change of a morphism with universally bounded fibres is a morphism with universally bounded fibres. More precisely, if n bounds the degrees of the fibres of $f : X \to Y$ and $Y' \to Y$ is any morphism, then the degrees of the fibres of the base change $f' : Y' \times_Y X \to Y' \to Y'$ is also bounded by n.*

Proof. This is clear from the result of Lemma 28.50.2. □

Lemma 28.50.5. *Let $f : X \to Y$ be a morphism of schemes. Let $Y' \to Y$ be a morphism of schemes, and let $f' : X' = X_{Y'} \to Y'$ be the base change of f. If $Y' \to Y$ is surjective and f' has universally bounded fibres, then f has universally bounded fibres. More precisely, if n bounds the degree of the fibres of f', then also n bounds the degrees of the fibres of f.*

Proof. Let $n \geq 0$ be an integer bounding the degrees of the fibres of f'. We claim that n works for f also. Namely, if $y \in Y$ is a point, then choose a point $y' \in Y'$ lying over y and observe that

$$X'_{y'} = \mathrm{Spec}(\kappa(y')) \times_{\mathrm{Spec}(\kappa(y))} X_y.$$

Since $X'_{y'}$ is assumed finite of degree $\leq n$ over $\kappa(y')$ it follows that also X_y is finite of degree $\leq n$ over $\kappa(y)$. (Some details omitted.) □

Lemma 28.50.6. *An immersion has universally bounded fibres.*

Proof. The integer $n = 1$ works in the definition. □

Lemma 28.50.7. *Let $f : X \to Y$ be an étale morphism of schemes. Let $n \geq 0$. The following are equivalent*

1. *the integer n bounds the degrees of the fibres,*
2. *for every field k and morphism $\mathrm{Spec}(k) \to Y$ the base change $X_k = \mathrm{Spec}(k) \times_Y X$ has at most n points, and*
3. *for every $y \in Y$ and every separable algebraic closure $\kappa(y) \subset \kappa(y)^{sep}$ the scheme $X_{\kappa(y)^{sep}}$ has at most n points.*

Proof. This follows from Lemma 28.50.2 and the fact that the fibres X_y are disjoint unions of spectra of finite separable field extensions of $\kappa(y)$, see Lemma 28.37.7. □

Having universally bounded fibres is an absolute notion and not a relative notion. This is why the condition in the following lemma is that X is quasi-compact, and not that f is quasi-compact.

Lemma 28.50.8. *Let $f : X \to Y$ be a morphism of schemes. Assume that*

1. *f is locally quasi-finite, and*
2. *X is quasi-compact.*

Then f has universally bounded fibres.

Proof. Since X is quasi-compact, there exists a finite affine open covering $X = \bigcup_{i=1,\ldots,n} U_i$ and affine opens $V_i \subset Y$, $i = 1, \ldots, n$ such that $f(U_i) \subset V_i$. Because of the local nature of "local quasi-finiteness" (see Lemma 28.21.6 part (4)) we see that the morphisms $f|_{U_i} : U_i \to V_i$ are locally quasi-finite morphisms of affines, hence quasi-finite, see Lemma 28.21.9. For $y \in Y$ it is clear that $X_y = \bigcup_{y \in V_i} (U_i)_y$ is an open covering. Hence it suffices to prove the lemma for a quasi-finite morphism of affines (namely, if n_i works for the morphism $f|_{U_i} : U_i \to V_i$, then $\sum n_i$ works for f).

Assume $f : X \to Y$ is a quasi-finite morphism of affines. By Lemma 28.49.3 we can find a diagram

$$\begin{array}{ccc} X & \xrightarrow{\quad j \quad} & Z \\ & {\scriptstyle f} \searrow \quad \swarrow {\scriptstyle \pi} & \\ & Y & \end{array}$$

with Z affine, π finite and j an open immersion. Since j has universally bounded fibres (Lemma 28.50.6) this reduces us to showing that π has universally bounded fibres (Lemma 28.50.3).

This reduces us to a morphism of the form $\mathrm{Spec}(B) \to \mathrm{Spec}(A)$ where $A \to B$ is finite. Say B is generated by $x_1, \ldots, x_n$ over A and say $P_i(T) \in A[T]$ is a monic polynomial of degree d_i such that $P_i(x_i) = 0$ in B (a finite ring extension is integral, see Algebra, Lemma 10.33.3). With these notations it is clear that

$$\bigoplus_{0 \leq e_i < d_i, i=1,\ldots n} A \longrightarrow B, \quad (a_{(e_1,\ldots,e_n)}) \longmapsto \sum a_{(e_1,\ldots,e_n)} x_1^{e_1} \ldots x_n^{e_n}$$

is a surjective A-module map. Thus for any prime $\mathfrak{p} \subset A$ this induces a surjective map $\kappa(\mathfrak{p})$-vector spaces

$$\kappa(\mathfrak{p})^{\oplus d_1 \ldots d_n} \longrightarrow B \otimes_A \kappa(\mathfrak{p})$$

In other words, the integer $d_1 \ldots d_n$ works in the definition of a morphism with universally bounded fibres. □

Lemma 28.50.9. *Consider a commutative diagram of morphisms of schemes*

$$\begin{array}{ccc} X & \xrightarrow{\quad f \quad} & Y \\ & {\scriptstyle g}\searrow \quad \swarrow{\scriptstyle h} & \\ & Z & \end{array}$$

If g has universally bounded fibres, and f is surjective and flat, then also h has universally bounded fibres. More precisely, if n bounds the degree of the fibres of g, then also n bounds the degree of the fibres of h.

Proof. Assume g has universally bounded fibres, and f is surjective and flat. Say the degree of the fibres of g is bounded by $n \in \mathbf{N}$. We claim n also works for h. Let $z \in Z$. Consider the morphism of schemes $X_z \to Y_z$. It is flat and surjective. By assumption X_z is a finite scheme over $\kappa(z)$, in particular it is the spectrum of an Artinian ring (by Algebra, Lemma 10.49.2). By Lemma 28.13.13 the morphism $X_z \to Y_z$ is affine in particular quasi-compact. It follows from Lemma 28.26.10 that Y_z is a finite discrete as this holds for X_z. Hence Y_z is an affine scheme (this is a nice exercise; it also follows for example from Properties, Lemma 27.27.1 applied to the set of all points of Y_z). Write $Y_z = \mathrm{Spec}(B)$ and $X_z = \mathrm{Spec}(A)$. Then A is faithfully flat over B, so $B \subset A$. Hence $\dim_k(B) \leq \dim_k(A) \leq n$ as desired. □

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CHAPTER 29

Cohomology of Schemes

29.1. Introduction

In this chapter we first prove a number of results on the cohomology of quasi-coherent sheaves. A fundamental reference is [**DG67**]. Having done this we will elaborate on cohomology of coherent sheaves in the Noetherian setting. See [**Ser55b**].

29.2. Cech cohomology of quasi-coherent sheaves

Let X be a scheme. Let $U \subset X$ be an affine open. Recall that a *standard open covering* of U is a covering of the form $\mathcal{U} : U = \bigcup_{i=1}^n D(f_i)$ where $f_1, \ldots, f_n \in \Gamma(U, \mathcal{O}_X)$ generate the unit ideal, see Schemes, Definition 25.5.2.

Lemma 29.2.1. *Let X be a scheme. Let $\mathcal{F}$ be a quasi-coherent $\mathcal{O}_X$-module. Let $\mathcal{U} : U = \bigcup_{i=1}^n D(f_i)$ be a standard open covering of an affine open of X. Then $\check{H}^p(\mathcal{U}, \mathcal{F}) = 0$ for all $p > 0$.*

Proof. Write $U = \mathrm{Spec}(A)$ for some ring A. In other words, $f_1, \ldots, f_n$ are elements of A which generate the unit ideal of A. Write $\mathcal{F}|_U = \widetilde{M}$ for some A-module M. Clearly the Cech complex $\check{\mathcal{C}}^\bullet(\mathcal{U}, \mathcal{F})$ is identified with the complex

$$\prod_{i_0} M_{f_{i_0}} \to \prod_{i_0 i_1} M_{f_{i_0} f_{i_1}} \to \prod_{i_0 i_1 i_2} M_{f_{i_0} f_{i_1} f_{i_2}} \to \ldots$$

We are asked to show that the extended complex

$$0 \to M \to \prod_{i_0} M_{f_{i_0}} \to \prod_{i_0 i_1} M_{f_{i_0} f_{i_1}} \to \prod_{i_0 i_1 i_2} M_{f_{i_0} f_{i_1} f_{i_2}} \to \ldots \tag{29.2.1.1}$$

(whose truncation we have studied in Algebra, Lemma 10.21.2) is exact. It suffices to show that (29.2.1.1) is exact after localizing at a prime $\mathfrak{p}$, see Algebra, Lemma 10.22.1. In fact we will show that the extended complex localized at $\mathfrak{p}$ is homotopic to zero.

There exists an index i such that $f_i \notin \mathfrak{p}$. Choose and fix such an element i_{fix}. Note that $M_{f_{i_{\mathrm{fix}}}, \mathfrak{p}} = M_{\mathfrak{p}}$. Similarly for a localization at a product $f_{i_0} \ldots f_{i_p}$ and $\mathfrak{p}$ we can drop any f_{i_j} for which $i_j = i_{\mathrm{fix}}$. Let us define a homotopy

$$h : \prod\nolimits_{i_0 \ldots i_{p+1}} M_{f_{i_0} \ldots f_{i_{p+1}}, \mathfrak{p}} \longrightarrow \prod\nolimits_{i_0 \ldots i_p} M_{f_{i_0} \ldots f_{i_p}, \mathfrak{p}}$$

by the rule

$$h(s)_{i_0 \ldots i_p} = s_{i_{\mathrm{fix}} i_0 \ldots i_p}$$

(This is "dual" to the homotopy in the proof of Cohomology, Lemma 20.11.4.) In other words, $h : \prod_{i_0} M_{f_{i_0}, \mathfrak{p}} \to M$ is projection onto the factor $M_{f_{i_{\mathrm{fix}}}, \mathfrak{p}} = M_{\mathfrak{p}}$

and in general the map h equal projection onto the factors $M_{f_{i_{\text{fix}}} f_{i_1} \ldots f_{i_{p+1}}, \mathfrak{p}} = M_{f_{i_1} \ldots f_{i_{p+1}}, \mathfrak{p}}$. We compute

$$\begin{aligned}(dh + hd)(s)_{i_0 \ldots i_p} &= \sum_{j=0}^{p} (-1)^j h(s)_{i_0 \ldots \hat{i}_j \ldots i_p} + d(s)_{i_{\text{fix}} i_0 \ldots i_p} \\ &= \sum_{j=0}^{p} (-1)^j s_{i_{\text{fix}} i_0 \ldots \hat{i}_j \ldots i_p} + s_{i_0 \ldots i_p} + \sum_{j=0}^{p} (-1)^{j+1} s_{i_{\text{fix}} i_0 \ldots \hat{i}_j \ldots i_p} \\ &= s_{i_0 \ldots i_p}\end{aligned}$$

This proves the identity map is homotopic to zero as desired. □

The following lemma says in particular that for any affine scheme X and any quasi-coherent sheaf $\mathcal{F}$ on X we have

$$H^p(X, \mathcal{F}) = 0$$

for all $p > 0$.

Lemma 29.2.2. *Let X be a scheme. Let $\mathcal{F}$ be a quasi-coherent $\mathcal{O}_X$-module. For any affine open $U \subset X$ we have $H^p(U, \mathcal{F}) = 0$ for all $p > 0$.*

Proof. We are going to apply Cohomology, Lemma 20.12.8. As our basis $\mathcal{B}$ for the topology of X we are going to use the affine opens of X. As our set Cov of open coverings we are going to use the standard open coverings of affine opens of X. Next we check that conditions (1), (2) and (3) of Cohomology, Lemma 20.12.8 hold. Note that the intersection of standard opens in an affine is another standard open. Hence property (1) holds. The coverings form a cofinal system of open coverings of any element of $\mathcal{B}$, see Schemes, Lemma 25.5.1. Hence (2) holds. Finally, condition (3) of the lemma follows from Lemma 29.2.1. □

Here is a relative version of the vanishing of cohomology of quasi-coherent sheaves on affines.

Lemma 29.2.3. *Let $f : X \to S$ be a morphism of schemes. Let $\mathcal{F}$ be a quasi-coherent $\mathcal{O}_X$-module. If f is affine then $R^i f_* \mathcal{F} = 0$ for all $i > 0$.*

Proof. According to Cohomology, Lemma 20.7.3 the sheaf $R^i f_* \mathcal{F}$ is the sheaf associated to the presheaf $V \mapsto H^i(f^{-1}(V), \mathcal{F}|_{f^{-1}(V)})$. By assumption, whenever V is affine we have that $f^{-1}(V)$ is affine, see Morphisms, Definition 28.13.1. By Lemma 29.2.2 we conclude that $H^i(f^{-1}(V), \mathcal{F}|_{f^{-1}(V)}) = 0$ whenever V is affine. Since S has a basis consisting of affine opens we win. □

Lemma 29.2.4. *Let $f : X \to S$ be an affine morphism of schemes. Let $\mathcal{F}$ be a quasi-coherent $\mathcal{O}_X$-module. Then $H^i(X, \mathcal{F}) = H^i(S, f_* \mathcal{F})$ for all $i \geq 0$.*

Proof. Follows from Lemma 29.2.3 and the Leray spectral sequence. See Cohomology, Lemma 20.14.6. □

Lemma 29.2.5. *Let X be a scheme. Let $\mathcal{U} : X = \bigcup_{i \in I} U_i$ be an open covering such that $U_{i_0 \ldots i_p}$ is affine open for all $p \geq 0$ and all $i_0, \ldots, i_p \in I$ In this case for any quasi-coherent sheaf $\mathcal{F}$ we have*

$$\check{H}^p(\mathcal{U}, \mathcal{F}) = H^p(X, \mathcal{F})$$

as $\Gamma(X, \mathcal{O}_X)$-modules for all p.

Proof. In view of Lemma 29.2.2 this is a special case of Cohomology, Lemma 20.12.5. □

29.3. Vanishing of cohomology

We have seen that on an affine scheme the higher cohomology groups of any quasi-coherent sheaf vanish (Lemma 29.2.2). It turns out that this also characterizes affine schemes. We give two versions although the first covers all conceivable cases.

Lemma 29.3.1. *Let X be a scheme. Assume that*

(1) *X is quasi-compact,*
(2) *for every quasi-coherent sheaf of ideals $\mathcal{I} \subset \mathcal{O}_X$ we have $H^1(X, \mathcal{I}) = 0$.*

Then X is affine.

Proof. Let $x \in X$ be a closed point. Let $U \subset X$ be an affine open neighbourhood of x. Write $U = \mathrm{Spec}(A)$ and let $\mathfrak{m} \subset A$ be the maximal ideal corresponding to x. Set $Z = X \setminus U$ and $Z' = Z \cup \{x\}$. By Schemes, Lemma 25.12.4 there are quasi-coherent sheaves of ideals $\mathcal{I}$, resp. $\mathcal{I}'$ cutting out the reduced closed subschemes Z, resp. Z'. Consider the short exact sequence

$$0 \to \mathcal{I}' \to \mathcal{I} \to \mathcal{I}/\mathcal{I}' \to 0.$$

Since x is a closed point of X and $x \notin Z$ we see that $\mathcal{I}/\mathcal{I}'$ is supported at x. In fact, the restriction of $\mathcal{I}/\mathcal{I}'$ to U corresponds to the A-module $A/\mathfrak{m}$. Hence we see that $\Gamma(X, \mathcal{I}/\mathcal{I}') = A/\mathfrak{m}$. Since by assumption $H^1(X, \mathcal{I}') = 0$ we see there exists a global section $f \in \Gamma(X, \mathcal{I})$ which maps to the element $1 \in A/\mathfrak{m}$ as a section of $\mathcal{I}/\mathcal{I}'$. Clearly we have $x \in X_f \subset U$. This implies that $X_f = D(f_A)$ where f_A is the image of f in $A = \Gamma(U, \mathcal{O}_X)$. In particular X_f is affine.

Consider the union $W = \bigcup X_f$ over all $f \in \Gamma(X, \mathcal{O}_X)$ such that X_f is affine. Obviously W is open in X. By the arguments above every closed point of X is contained in W. The closed subset $X \setminus W$ of X is also quasi-compact (see Topology, Lemma 5.11.3). Hence it has a closed point if it is nonempty (see Topology, Lemma 5.11.8). This is a would contradict the fact that all closed points are in W. Hence we conclude $X = W$.

Choose finitely many $f_1, \ldots, f_n \in \Gamma(X, \mathcal{O}_X)$ such that $X = X_{f_1} \cup \ldots \cup X_{f_n}$ and such that each X_{f_i} is affine. This is possible as we've seen above. By Properties, Lemma 27.25.2 to finish the proof it suffices to show that $f_1, \ldots, f_n$ generate the unit ideal in $\Gamma(X, \mathcal{O}_X)$. Consider the short exact sequence

$$0 \longrightarrow \mathcal{F} \longrightarrow \mathcal{O}_X^{\oplus n} \xrightarrow{f_1, \ldots, f_n} \mathcal{O}_X \longrightarrow 0$$

The arrow defined by $f_1, \ldots, f_n$ is surjective since the opens X_{f_i} cover X. We let $\mathcal{F}$ be the kernel of this surjective map. Observe that $\mathcal{F}$ has a filtration

$$0 = \mathcal{F}_0 \subset \mathcal{F}_1 \subset \ldots \subset \mathcal{F}_n = \mathcal{F}$$

so that each subquotient $\mathcal{F}_i/\mathcal{F}_{i-1}$ is isomorphic to a quasi-coherent sheaf of ideals. Namely we can take $\mathcal{F}_i$ to be the intersection of $\mathcal{F}$ with the first i direct summands of $\mathcal{O}_X^{\oplus n}$. The assumption of the lemma implies that $H^1(X, \mathcal{F}_i/\mathcal{F}_{i-1}) = 0$ for all i. This implies that $H^1(X, \mathcal{F}_2) = 0$ because it is sandwiched between $H^1(X, \mathcal{F}_1)$ and

$H^1(X, \mathcal{F}_2/\mathcal{F}_1)$. Continuing like this we deduce that $H^1(X, \mathcal{F}) = 0$. Therefore we conclude that the map

$$\bigoplus_{i=1,\ldots,n} \Gamma(X, \mathcal{O}_X) \xrightarrow{f_1,\ldots,f_n} \Gamma(X, \mathcal{O}_X)$$

is surjective as desired. □

Note that if X is a Noetherian scheme then every quasi-coherent sheaf of ideals is automatically a coherent sheaf of ideals and a finite type quasi-coherent sheaf of ideals. Hence the preceding lemma and the next lemma both apply in this case.

Lemma 29.3.2. *Let X be a scheme. Assume that*

(1) *X is quasi-compact,*
(2) *X is quasi-separated, and*
(3) *$H^1(X, \mathcal{I}) = 0$ for every quasi-coherent sheaf of ideals $\mathcal{I}$ of finite type.*

Then X is affine.

Proof. By Properties, Lemma 27.20.3 every quasi-coherent sheaf of ideals is a directed colimit of quasi-coherent sheaves of ideals of finite type. By Cohomology, Lemma 20.20.1 taking cohomology on X commutes with directed colimits. Hence we see that $H^1(X, \mathcal{I}) = 0$ for every quasi-coherent sheaf of ideals on X. In other words we see that Lemma 29.3.1 applies. □

29.4. Quasi-coherence of higher direct images

We have seen that the higher cohomology groups of a quasi-coherent module on an affine is zero. For (quasi-)separated quasi-compact schemes X this implies vanishing of cohomology groups of quasi-coherent sheaves beyond a certain degree. However, it may not be the case that X has finite cohomological dimension, because that is defined in terms of vanishing of cohomology of *all* $\mathcal{O}_X$-modules.

Lemma 29.4.1. *Let X be a quasi-compact and quasi-separated scheme. Let P be a property of the quasi-compact opens of X. Assume that*

(1) *P holds for every affine open of X,*
(2) *if U is quasi-compact open, V affine open, P holds for U, V, and $U \cap V$, then P holds for $U \cup V$.*

Then P holds for every quasi-compact open of X and in particular for X.

Proof. First we argue by induction that P holds for *separated* quasi-compact opens $W \subset X$. Namely, such an open can be written as $W = U_1 \cup \ldots \cup U_n$ and we can do induction on n using property (2) with $U = U_1 \cup \ldots \cup U_{n-1}$ and $V = U_n$. This is allowed because $U \cap V = (U_1 \cap U_n) \cup \ldots \cup (U_{n-1} \cap U_n)$ is also a union of $n-1$ affine open subschemes by Schemes, Lemma 25.21.8 applied to the affine opens U_i and U_n of W. Having said this, for any quasi-compact open $W \subset X$ we can do induction on the number of affine opens needed to cover W using the same trick as before and using that the quasi-compact open $U_i \cap U_n$ is separated as an open subscheme of the affine scheme U_n. □

Lemma 29.4.2. *Let X be a quasi-compact separated scheme. Let $t = t(X)$ be the minimal number of affine opens needed to cover X. Then $H^n(X, \mathcal{F}) = 0$ for all $n \geq t$ and all quasi-coherent sheaves $\mathcal{F}$.*

Proof. First proof. By induction on t. If $t = 1$ the result follows from Lemma 29.2.2. If $t > 1$ write $X = U \cup V$ with V affine open and $U = U_1 \cup \ldots \cup U_{t-1}$ a union of $t-1$ open affines. Note that in this case $U \cap V = (U_1 \cap V) \cup \ldots (U_{t-1} \cap V)$ is also a union of $t-1$ affine open subschemes, see Schemes, Lemma 25.21.8. We apply the Mayer-Vietoris long exact sequence

$$0 \to H^0(X, \mathcal{F}) \to H^0(U, \mathcal{F}) \oplus H^0(V, \mathcal{F}) \to H^0(U \cap V, \mathcal{F}) \to H^1(X, \mathcal{F}) \to \ldots$$

see Cohomology, Lemma 20.9.2. By induction we see that the groups $H^i(U, \mathcal{F})$, $H^i(V, \mathcal{F})$, $H^i(U \cap V, \mathcal{F})$ are zero for $i \geq t-1$. It follows immediately that $H^i(X, \mathcal{F})$ is zero for $i \geq t$.

Second proof. Let $\mathcal{U} : X = \bigcup_{i=1}^t U_i$ be a finite affine open covering. Since X is separated the multiple intersections $U_{i_0 \ldots i_p}$ are all affine, see Schemes, Lemma 25.21.8. By Lemma 29.2.5 the Cech cohomology groups $\check{H}^p(\mathcal{U}, \mathcal{F})$ agree with the cohomology groups. By Cohomology, Lemma 20.22.6 the Cech cohomology groups may be computed using the alternating Cech complex $\check{\mathcal{C}}^\bullet_{alt}(\mathcal{U}, \mathcal{F})$. As the covering consists of t elements we see immediately that $\check{\mathcal{C}}^p_{alt}(\mathcal{U}, \mathcal{F}) = 0$ for all $p \geq t$. Hence the result follows. □

Lemma 29.4.3. *Let X be a quasi-compact quasi-separated scheme. Let $X = U_1 \cup \ldots \cup U_t$ be an affine open covering. Set*

$$d = \max_{I \subset \{1, \ldots, t\}} |I| + t(\bigcap\nolimits_{i \in I} U_i)$$

where $t(U)$ is the minimal number of affines needed to cover the scheme U. Then $H^n(X, \mathcal{F}) = 0$ for all $n \geq d$ and all quasi-coherent sheaves $\mathcal{F}$.

Proof. Note that since X is quasi-separated the numbers $t(\bigcap_{i \in I} U_i)$ are finite. Let $\mathcal{U} : X = \bigcup_{i=1}^t U_i$. By Cohomology, Lemma 20.12.4 there is a spectral sequence

$$E_2^{p,q} = \check{H}^p(\mathcal{U}, \underline{H}^q(\mathcal{F}))$$

converging to $H^{p+q}(U, \mathcal{F})$. By Cohomology, Lemma 20.22.6 we have

$$E_2^{p,q} = H^p(\check{\mathcal{C}}^\bullet_{alt}(\mathcal{U}, \underline{H}^q(\mathcal{F}))$$

The alternating Čech complex with values in the presheaf $\underline{H}^q(\mathcal{F})$ vanishes in high degrees by Lemma 29.4.2, more precisely $E_2^{p,q} = 0$ for $p + q \geq d$. Hence the result follows. □

Lemma 29.4.4. *Let $f : X \to S$ be a morphism of schemes. Assume that f is quasi-separated and quasi-compact.*

1. *For any quasi-coherent $\mathcal{O}_X$-module $\mathcal{F}$ the higher direct images $R^p f_* \mathcal{F}$ are quasi-coherent on S.*
2. *If S is quasi-compact, there exists an integer $n = n(X, S, f)$ such that $R^p f_* \mathcal{F} = 0$ for all $p \geq n$ and any quasi-coherent sheaf $\mathcal{F}$ on X.*
3. *In fact, if S is quasi-compact we can find $n = n(X, S, f)$ such that for every morphism of schemes $S' \to S$ we have $R^p(f')_* \mathcal{F}' = 0$ for $p \geq n$ and any quasi-coherent sheaf $\mathcal{F}'$ on X'. Here $f' : X' = S' \times_S X \to S'$ is the base change of f.*

Proof. We first prove (1). Note that under the hypotheses of the lemma the sheaf $R^0 f_* \mathcal{F} = f_* \mathcal{F}$ is quasi-coherent by Schemes, Lemma 25.24.1. Using Cohomology, Lemma 20.7.4 we see that forming higher direct images commutes with restriction

to open subschemes. Since being quasi-coherent is local on S we may assume S is affine.

Assume S is affine and f quasi-compact and separated. Let $t \geq 1$ be the minimal number of affine opens needed to cover X. We will prove this case of (1) by induction on t. If $t = 1$ then the morphism f is affine by Morphisms, Lemma 28.13.12 and (1) follows from Lemma 29.2.3. If $t > 1$ write $X = U \cup V$ with V affine open and $U = U_1 \cup \ldots \cup U_{t-1}$ a union of $t-1$ open affines. Note that in this case $U \cap V = (U_1 \cap V) \cup \ldots (U_{t-1} \cap V)$ is also a union of $t-1$ affine open subschemes, see Schemes, Lemma 25.21.8. We will apply the relative Mayer-Vietoris sequence

$$0 \to f_*\mathcal{F} \to a_*(\mathcal{F}|_U) \oplus b_*(\mathcal{F}|_V) \to c_*(\mathcal{F}|_{U\cap V}) \to R^1 f_*\mathcal{F} \to \ldots$$

see Cohomology, Lemma 20.9.3. By induction we see that $R^p a_*\mathcal{F}$, $R^p b_*\mathcal{F}$ and $R^p c_*\mathcal{F}$ are all quasi-coherent. This implies that each of the sheaves $R^p f_*\mathcal{F}$ is quasi-coherent since it sits in the middle of a short exact sequence with a cokernel of a map between quasi-coherent sheaves on the left and a kernel of a map between quasi-coherent sheaves on the right. Using the results on quasi-coherent sheaves in Schemes, Section 25.24 we see conclude $R^p f_*\mathcal{F}$ is quasi-coherent.

Assume S is affine and f quasi-compact and quasi-separated. Let $t \geq 1$ be the minimal number of affine opens needed to cover X. We will prove (1) by induction on t. In case $t = 1$ the morphism f is separated and we are back in the previous case (see previous paragraph). If $t > 1$ write $X = U \cup V$ with V affine open and U a union of $t-1$ open affines. Note that in this case $U \cap V$ is an open subscheme of an affine scheme and hence separated (see Schemes, Lemma 25.21.6). We will apply the relative Mayer-Vietoris sequence

$$0 \to f_*\mathcal{F} \to a_*(\mathcal{F}|_U) \oplus b_*(\mathcal{F}|_V) \to c_*(\mathcal{F}|_{U\cap V}) \to R^1 f_*\mathcal{F} \to \ldots$$

see Cohomology, Lemma 20.9.3. By induction and the result of the previous paragraph we see that $R^p a_*\mathcal{F}$, $R^p b_*\mathcal{F}$ and $R^p c_*\mathcal{F}$ are quasi-coherent. As in the previous paragraph this implies each of sheaves $R^p f_*\mathcal{F}$ is quasi-coherent.

Next, we prove (3) and a fortiori (2). Choose a finite affine open covering $S = \bigcup_{j=1,\ldots m} S_j$. For each i choose a finite affine open covering $f^{-1}(S_j) = \bigcup_{i=1,\ldots t_j} U_{ji}$. Let

$$d_j = \max\nolimits_{I \subset \{1,\ldots,t_j\}} |I| + t(\bigcap\nolimits_{i\in I} U_{ji})$$

be the integer found in Lemma 29.4.3. We claim that $n(X, S, f) = \max d_j$ works.

Namely, let $S' \to S$ be a morphism of schemes and let $\mathcal{F}'$ be a quasi-coherent sheaf on $X' = S' \times_S X$. We want to show that $R^p f'_*\mathcal{F}' = 0$ for $p \geq n(X, S, f)$. Since this question is local on S' we may assume that S' is affine and maps into S_j for some j. Then $X' = S' \times_{S_j} f^{-1}(S_j)$ is covered by the open affines $S' \times_{S_j} U_{ji}$, $i = 1, \ldots t_j$ and the intersections

$$\bigcap\nolimits_{i\in I} S' \times_{S_j} U_{ji} = S' \times_{S_j} \bigcap\nolimits_{i\in I} U_{ji}$$

are covered by the same number of affines as before the base change. Applying Lemma 29.4.3 we get $H^p(X', \mathcal{F}') = 0$. By the first part of the proof we already know that each $R^q f'_*\mathcal{F}'$ is quasi-coherent hence has vanishing higher cohomology groups on our affine scheme S', thus we see that $H^0(S', R^p f'_*\mathcal{F}') = H^p(X', \mathcal{F}') = 0$ by Cohomology, Lemma 20.14.6. Since $R^p f'_*\mathcal{F}'$ is quasi-coherent we conclude that $R^p f'_*\mathcal{F}' = 0$. $\square$

Lemma 29.4.5. *Let $f : X \to S$ be a morphism of schemes. Assume that f is quasi-separated and quasi-compact. Assume S is affine. For any quasi-coherent $\mathcal{O}_X$-module $\mathcal{F}$ we have*

$$H^q(X, \mathcal{F}) = H^0(S, R^q f_* \mathcal{F})$$

for all $q \in \mathbf{Z}$.

Proof. Consider the Leray spectral sequence $E_2^{p,q} = H^p(S, R^q f_* \mathcal{F})$ converging to $H^{p+q}(X, \mathcal{F})$, see Cohomology, Lemma 20.14.4. By Lemma 29.4.4 we see that the sheaves $R^q f_* \mathcal{F}$ are quasi-coherent. By Lemma 29.2.2 we see that $E_2^{p,q} = 0$ when $p > 0$. Hence the spectral sequence degenerates at E_2 and we win. See also Cohomology, Lemma 20.14.6 (2) for the general principle. □

29.5. Cohomology and base change, I

Let $f : X \to S$ be a morphism of schemes. Let $\mathcal{F}$ be a quasi-coherent sheaf on X. Suppose further that $g : S' \to S$ is any morphism of schemes. Denote $X' = X_{S'} = S' \times_S X$ the base change of X and denote $f' : X' \to S'$ the base change of f. Also write $g' : X' \to X$ the projection, and set $\mathcal{F}' = (g')^* \mathcal{F}$. Here is a diagram representing the situation:

$$\begin{array}{ccccc} \mathcal{F}' = (g')^*\mathcal{F} & & X' & \xrightarrow[g']{} & X & & \mathcal{F} \\ & & {\scriptstyle f'}\downarrow & & \downarrow{\scriptstyle f} & & \\ R f'_* \mathcal{F}' & & S' & \xrightarrow{g} & S & & R f_* \mathcal{F} \end{array} \tag{29.5.0.1}$$

Here is the simplest case of the base change property we have in mind.

Lemma 29.5.1. *Let $f : X \to S$ be a morphism of schemes. Let $\mathcal{F}$ be a quasi-coherent $\mathcal{O}_X$-module. Assume f is affine. In this case $f_* \mathcal{F} \cong R f_* \mathcal{F}$ is a quasi-coherent sheaf, and for every base change diagram (29.5.0.1) we have*

$$g^* f_* \mathcal{F} = f'_* (g')^* \mathcal{F}.$$

Proof. The vanishing of higher direct images is Lemma 29.2.3. The statement is local on S and S'. Hence we may assume $X = \mathrm{Spec}(A)$, $S = \mathrm{Spec}(R)$, $S' = \mathrm{Spec}(R')$ and $\mathcal{F} = \widetilde{M}$ for some A-module M. We use Schemes, Lemma 25.7.3 to describe pullbacks and pushforwards of $\mathcal{F}$. Namely, $X' = \mathrm{Spec}(R' \otimes_R A)$ and $\mathcal{F}'$ is the quasi-coherent sheaf associated to $(R' \otimes_R A) \otimes_A M$. Thus we see that the lemma boils down to the equality

$$(R' \otimes_R A) \otimes_A M = R' \otimes_R M$$

as R'-modules. □

In many situations it is sufficient to know about the following special case of cohomology and base change. It follows immediately from the stronger results in the next section, but since it is so important it deserves its own proof.

Lemma 29.5.2. *Let $f : X \to S$ be a morphism of schemes. Let $\mathcal{F}$ be a quasi-coherent sheaf on X. Let $g : S' \to S$ be a morphism of schemes. Assume that g is flat and that f is quasi-compact and quasi-separated. Then for any $i \geq 0$ we have*

$$R^i f'_* \mathcal{F}' = g^* R^i f_* \mathcal{F}$$

with notation as in (29.5.0.1). Moreover, the induced isomorphism is the map given by the base change map of Cohomology, Lemma 20.18.1.

Proof. The statement is local on S' and hence we may assume S and S' are affine. Say $S = \mathrm{Spec}(A)$ and $S' = \mathrm{Spec}(B)$. In this case we are really trying to show that the map

$$H^i(X, \mathcal{F}) \otimes_A B \longrightarrow H^i(X_B, \mathcal{F}_B)$$

(given by the reference in the statement of the lemma) is an isomorphism where $X_B = \mathrm{Spec}(B) \times_{\mathrm{Spec}(A)} X$ and $\mathcal{F}_B$ is the pullback of $\mathcal{F}$ to X_B.

In case X is separated, choose an affine open covering $\mathcal{U} : X = U_1 \cup \ldots \cup U_t$ and recall that

$$\check{H}^p(\mathcal{U}, \mathcal{F}) = H^p(X, \mathcal{F}),$$

see Lemma 29.2.5. If $\mathcal{U}_B : X_B = (U_1)_B \cup \ldots \cup (U_t)_B$ we obtain by base change, then it is still the case that each $(U_i)_B$ is affine and that X_B is separated. Thus we obtain

$$\check{H}^p(\mathcal{U}_B, \mathcal{F}_B) = H^p(X_B, \mathcal{F}_B).$$

We have the following relation between the Čech complexes

$$\check{\mathcal{C}}^\bullet(\mathcal{U}_B, \mathcal{F}_B) = \check{\mathcal{C}}^\bullet(\mathcal{U}, \mathcal{F}) \otimes_A B$$

as follows from Lemma 29.5.1. Since $A \to B$ is flat, the same thing remains true on taking cohomology.

In case X is quasi-separated, choose an affine open covering $\mathcal{U} : X = U_1 \cup \ldots \cup U_t$. We will use the Čech-to-cohomology spectral sequence Cohomology, Lemma 20.12.4. The reader who wishes to avoid this spectral sequence can use Majer-Vietoris and induction on t as in the proof of Lemma 29.4.4. The spectral sequence has E_2-page $E_2^{p,q} = \check{H}^p(\mathcal{U}, \underline{H}^q(\mathcal{F}))$ and converges to $H^{p+q}(X, \mathcal{F})$. Similarly, we have a spectral sequence with E_2-page $E_2^{p,q} = \check{H}^p(\mathcal{U}_B, \underline{H}^q(\mathcal{F}_B))$ and converges to $H^{p+q}(X_B, \mathcal{F}_B)$. Since the intersections $U_{i_0 \ldots i_p}$ are quasi-compact and separated, the result of the second paragraph of the proof gives $\check{H}^p(\mathcal{U}_B, \underline{H}^q(\mathcal{F}_B)) = \check{H}^p(\mathcal{U}, \underline{H}^q(\mathcal{F})) \otimes_A B$. Using that $A \to B$ is flat we conclude that $H^i(X, \mathcal{F}) \otimes_A B \to H^i(X_B, \mathcal{F}_B)$ is an isomorphism for all i and we win. □

29.6. Colimits and higher direct images

General results of this nature can be found in Cohomology, Section 20.20, Sheaves, Lemma 6.29.1, and Modules, Lemma 17.11.6.

Lemma 29.6.1. *Let $f : X \to S$ be a quasi-compact and quasi-separated morphism of schemes. Let $\mathcal{F} = \mathrm{colim}\, \mathcal{F}_i$ be a filtered colimit of quasi-coherent sheaves on X. Then for any $p \geq 0$ we have*

$$R^p f_* \mathcal{F} = \mathrm{colim}\, R^p f_* \mathcal{F}_i.$$

Proof. Recall that $R^p f_* \mathcal{F}$ is the sheaf associated to $U \mapsto H^p(f^{-1}U, \mathcal{F})$, see Cohomology, Lemma 20.7.3. Recall that the colimit is the sheaf associated to the presheaf colimit (taking colimits over opens). Hence we can apply Cohomology, Lemma 20.20.1 to $H^p(f^{-1}U, -)$ where U is affine to conclude. (Because the basis of affine opens in $f^{-1}U$ satisfies the assumptions of that lemma.) □

29.7. Cohomology and base change, II

Let $f : X \to S$ be a morphism of schemes and let $\mathcal{F}$ be a quasi-coherent $\mathcal{O}_X$-module. If f is quasi-compact and quasi-separated we would like to represent $Rf_*\mathcal{F}$ by a complex of quasi-coherent sheaves on S. This follows from the fact that the sheaves $R^i f_*\mathcal{F}$ are quasi-coherent if S is quasi-compact and has affine diagonal, using that $D_{QCoh}(S)$ is equivalent to $D(QCoh(\mathcal{O}_S))$, see Derived Categories of Schemes, Proposition 35.6.5.

In this section we will use a different approach which produces an explicit complex having a good base change property. The construction is particularly easy if f and S are separated, or more generally have affine diagonal. Since this is the case which by far the most often used we treat it separately.

Lemma 29.7.1. *Let $f : X \to S$ be a morphism of schemes. Let $\mathcal{F}$ be a quasi-coherent $\mathcal{O}_X$-module. Assume X and S are quasi-compact and have affine diagonal (e.g., if X and S are separated). In this case we can compute $Rf_*\mathcal{F}$ as follows:*

1. *Choose a finite affine open covering $\mathcal{U} : X = \bigcup_{i=1,\ldots,n} U_i$.*
2. *For $i_0, \ldots, i_p \in \{1, \ldots, n\}$ denote $f_{i_0\ldots i_p} : U_{i_0\ldots i_p} \to S$ the restriction of f to the intersection $U_{i_0\ldots i_p} = U_{i_0} \cap \ldots \cap U_{i_p}$.*
3. *Set $\mathcal{F}_{i_0\ldots i_p}$ equal to the restriction of $\mathcal{F}$ to $U_{i_0\ldots i_p}$.*
4. *Set*
$$\check{\mathcal{C}}^p(\mathcal{U}, f, \mathcal{F}) = \bigoplus\nolimits_{i_0\ldots i_p} f_{i_0\ldots i_p *}\mathcal{F}_{i_0\ldots i_p}$$
and define differentials $d : \check{\mathcal{C}}^p(\mathcal{U}, f, \mathcal{F}) \to \check{\mathcal{C}}^{p+1}(\mathcal{U}, f, \mathcal{F})$ as in Cohomology, Equation (20.10.0.1).

Then the complex $\check{\mathcal{C}}^\bullet(\mathcal{U}, f, \mathcal{F})$ is a complex of quasi-coherent sheaves on S which comes equipped with an isomorphism
$$\check{\mathcal{C}}^\bullet(\mathcal{U}, f, \mathcal{F}) \longrightarrow Rf_*\mathcal{F}$$
in $D^+(S)$. This isomorphism is functorial in the quasi-coherent sheaf $\mathcal{F}$.

Proof. Consider the resolution $\mathcal{F} \to \mathfrak{C}^\bullet(\mathcal{U}, \mathcal{F})$ of Cohomology, Lemma 20.23.3. We have an equality of complexes $\check{\mathcal{C}}^\bullet(\mathcal{U}, f, \mathcal{F}) = f_*\mathfrak{C}^\bullet(\mathcal{U}, \mathcal{F})$ of quasi-coherent $\mathcal{O}_S$-modules. The morphisms $j_{i_0\ldots i_p} : U_{i_0\ldots i_p} \to X$ and the morphisms $f_{i_0\ldots i_p} : U_{i_0\ldots i_p} \to S$ are affine by Morphisms, Lemma 28.13.11. Hence $R^q j_{i_0\ldots i_p *}\mathcal{F}_{i_0\ldots i_p}$ as well as $R^q f_{i_0\ldots i_p *}\mathcal{F}_{i_0\ldots i_p}$ are zero for $q > 0$. By Cohomology, Lemma 20.14.8 we conclude that $Rj_{i_0\ldots i_p *}\mathcal{F}_{i_0\ldots i_p} = j_{i_0\ldots i_p *}\mathcal{F}_{i_0\ldots i_p}$ and $Rf_{i_0\ldots i_p *}\mathcal{F}_{i_0\ldots i_p} = f_{i_0\ldots i_p *}\mathcal{F}_{i_0\ldots i_p}$. Using that $Rf_* \circ Rj_{i_0\ldots i_p *} = Rf_{i_0\ldots i_p *}$ (Cohomology, Lemma 20.14.7) we find that $Rf_* j_{i_0\ldots i_p *}\mathcal{F}_{i_0\ldots i_p} = f_{i_0\ldots i_p *}\mathcal{F}_{i_0\ldots i_p}$. Since the terms of the complex $\mathfrak{C}^\bullet(\mathcal{U}, \mathcal{F})$ are finite direct summands of the sheaves $j_{i_0\ldots i_p *}\mathcal{F}_{i_0\ldots i_p}$ we conclude using Leray's acyclicity lemma (Derived Categories, Lemma 13.16.7) that
$$Rf_*\mathcal{F} = Rf_*\mathfrak{C}^\bullet(\mathcal{U}, \mathcal{F}) = f_*\mathfrak{C}^\bullet(\mathcal{U}, \mathcal{F})$$
as desired. □

Next, we are going to consider what happens if we do a base change.

Lemma 29.7.2. *With notation as in diagram (29.5.0.1). Assume $f : X \to S$ and $\mathcal{F}$ satisfy the hypotheses of Lemma 29.7.1. Choose a finite affine open covering $\mathcal{U} : X = \bigcup U_i$ of X. There is a canonical isomorphism*
$$g^*\check{\mathcal{C}}^\bullet(\mathcal{U}, f, \mathcal{F}) \longrightarrow Rf'_*\mathcal{F}'$$

in $D^+(S')$. Moreover, if $S' \to S$ is affine, then in fact

$$g^*\check{\mathcal{C}}^\bullet(\mathcal{U}, f, \mathcal{F}) = \check{\mathcal{C}}^\bullet(\mathcal{U}', f', \mathcal{F}')$$

with $\mathcal{U}' : X' = \bigcup U'_i$ where $U'_i = (g')^{-1}(U_i) = U_{i,S'}$ is also affine.

Proof. In fact we may define $U'_i = (g')^{-1}(U_i) = U_{i,S'}$ no matter whether S' is affine over S or not. Let $\mathcal{U}' : X' = \bigcup U'_i$ be the induced covering of X'. In this case we claim that

$$g^*\check{\mathcal{C}}^\bullet(\mathcal{U}, f, \mathcal{F}) = \check{\mathcal{C}}^\bullet(\mathcal{U}', f', \mathcal{F}')$$

with $\check{\mathcal{C}}^\bullet(\mathcal{U}', f', \mathcal{F}')$ defined in exactly the same manner as in Lemma 29.7.1. This is clear from the case of affine morphisms (Lemma 29.5.1) by working locally on S'. Moreover, exactly as in the proof of Lemma 29.7.1 one sees that there is an isomorphism

$$\check{\mathcal{C}}^\bullet(\mathcal{U}', f', \mathcal{F}') \longrightarrow Rf'_*\mathcal{F}'$$

in $D^+(S')$ since the morphisms $U'_i \to X'$ and $U'_i \to S'$ are still affine (being base changes of affine morphisms). Details omitted. □

The lemma above says that the complex

$$\mathcal{K}^\bullet = \check{\mathcal{C}}^\bullet(\mathcal{U}, f, \mathcal{F})$$

is a bounded below complex of quasi-coherent sheaves on S which *universally* computes the higher direct images of $f : X \to S$. This is something about this particular complex and it is not preserved by replacing $\check{\mathcal{C}}^\bullet(\mathcal{U}, f, \mathcal{F})$ by a quasi-isomorphic complex in general! In other words, this is not a statement that makes sense in the derived category. The reason is that the pullback $g^*\mathcal{K}^\bullet$ is *not* equal to the derived pullback $Lg^*\mathcal{K}^\bullet$ of $\mathcal{K}^\bullet$ in general!

Here is a more general case where we can prove this statement. We remark that the condition of S being separated is harmless in most applications, since this is usually used to prove some local property of the total derived image. The proof is significantly more involved and uses hypercoverings; it is a nice example of how you can use them sometimes.

Lemma 29.7.3. *Let $f : X \to S$ be a morphism of schemes. Let $\mathcal{F}$ be a quasi-coherent sheaf on X. Assume that f is quasi-compact and quasi-separated and that S is quasi-compact and separated. There exists a bounded below complex $\mathcal{K}^\bullet$ of quasi-coherent $\mathcal{O}_S$-modules with the following property: For every morphism $g : S' \to S$ the complex $g^*\mathcal{K}^\bullet$ is a representative for $Rf'_*\mathcal{F}'$ with notation as in diagram (29.5.0.1).*

Proof. (If f is separated as well, please see Lemma 29.7.2.) The assumptions imply in particular that X is quasi-compact and quasi-separated as a scheme. Let $\mathcal{B}$ be the set of affine opens of X. By Hypercoverings, Lemma 24.10.4 we can find a hypercovering $K = (I, \{U_i\})$ such that each I_n is finite and each U_i is an affine open of X. By Hypercoverings, Lemma 24.4.3 there is a spectral sequence with E_2-page

$$E_2^{p,q} = \check{H}^p(K, \underline{H}^q(\mathcal{F}))$$

converging to $H^{p+q}(X, \mathcal{F})$. Note that $\check{H}^p(K, \underline{H}^q(\mathcal{F}))$ is the pth cohomology group of the complex

$$\prod\nolimits_{i \in I_0} H^q(U_i, \mathcal{F}) \to \prod\nolimits_{i \in I_1} H^q(U_i, \mathcal{F}) \to \prod\nolimits_{i \in I_2} H^q(U_i, \mathcal{F}) \to \ldots$$

Since each U_i is affine we see that this is zero unless $q = 0$ in which case we obtain

$$\prod_{i\in I_0} \mathcal{F}(U_i) \to \prod_{i\in I_1} \mathcal{F}(U_i) \to \prod_{i\in I_2} \mathcal{F}(U_i) \to \ldots$$

Thus we conclude that $R\Gamma(X, \mathcal{F})$ is computed by this complex.

For any n and $i \in I_n$ denote $f_i : U_i \to S$ the restriction of f to U_i. As S is separated and U_i is affine this morphism is affine. Consider the complex of quasi-coherent sheaves

$$\mathcal{K}^\bullet = (\prod_{i\in I_0} f_{i,*}\mathcal{F}|_{U_i} \to \prod_{i\in I_1} f_{i,*}\mathcal{F}|_{U_i} \to \prod_{i\in I_2} f_{i,*}\mathcal{F}|_{U_i} \to \ldots)$$

on S. As in Hypercoverings, Lemma 24.4.3 we obtain a map $\mathcal{K}^\bullet \to Rf_*\mathcal{F}$ in $D(\mathcal{O}_S)$ by choosing an injective resolution of $\mathcal{F}$ (details omitted). Consider any affine scheme V and a morphism $g : V \to S$. Then the base change X_V has a hypercovering $K_V = (I, \{U_{i,V}\})$ obtained by base change. Moreover, $g^* f_{i,*}\mathcal{F} = f_{i,V,*}(g')^*\mathcal{F}|_{U_{i,V}}$. Thus the arguments above prove that $\Gamma(V, g^*\mathcal{K}^\bullet)$ computes $R\Gamma(X_V, (g')^*\mathcal{F})$. This finishes the proof of the lemma as it suffices to prove the equality of complexes Zariski locally on S'. □

29.8. Cohomology of projective space

In this section we compute the cohomology of the twists of the structure sheaf on $\mathbf{P}^n_S$ over a scheme S. Recall that $\mathbf{P}^n_S$ was defined as the fibre product $\mathbf{P}^n_S = S \times_{\mathrm{Spec}(\mathbf{Z})} \mathbf{P}^n_{\mathbf{Z}}$ in Constructions, Definition 26.13.2. It was shown to be equal to

$$\mathbf{P}^n_S = \underline{\mathrm{Proj}}_S(\mathcal{O}_S[T_0, \ldots, T_n])$$

in Constructions, Lemma 26.21.4. In particular, projective space is a particular case of a projective bundle. If $S = \mathrm{Spec}(R)$ is affine then we have

$$\mathbf{P}^n_S = \mathbf{P}^n_R = \mathrm{Proj}(R[T_0, \ldots, T_n]).$$

All these identifications are compatible and compatible with the constructions of the twisted structure sheaves $\mathcal{O}_{\mathbf{P}^n_S}(d)$.

Before we state the result we need some notation. Let R be a ring. Recall that $R[T_0, \ldots, T_n]$ is a graded R-algebra where each T_i is homogeneous of degree 1. Denote $(R[T_0, \ldots, T_n])_d$ the degree d summand. It is a finite free R-module of rank $\binom{n+d}{d}$ when $d \geq 0$ and zero else. It has a basis consisting of monomials $T_0^{e_0} \ldots T_n^{e_n}$ with $\sum e_i = d$. We will also use the following notation: $R[\frac{1}{T_0}, \ldots, \frac{1}{T_n}]$ denotes the $\mathbf{Z}$-graded ring with $\frac{1}{T_i}$ in degree -1. In particular the $\mathbf{Z}$-graded $R[\frac{1}{T_0}, \ldots, \frac{1}{T_n}]$ module

$$\frac{1}{T_0 \ldots T_n} R[\frac{1}{T_0}, \ldots, \frac{1}{T_n}]$$

which shows up in the statement below is zero in degrees $\geq -n$, is free on the generator $\frac{1}{T_0 \ldots T_n}$ in degree $-n-1$ and is free of rank $(-1)^n\binom{n+d}{d}$ for $d \leq -n-1$.

Lemma 29.8.1. *Let R be a ring. Let $n \geq 0$ be an integer. We have*

$$H^q(\mathbf{P}^n, \mathcal{O}_{\mathbf{P}^n_R}(d)) = \begin{cases} (R[T_0, \ldots, T_n])_d & \text{if} \quad q = 0 \\ 0 & \text{if} \quad q \neq 0, n \\ \left(\frac{1}{T_0 \ldots T_n} R[\frac{1}{T_0}, \ldots, \frac{1}{T_n}]\right)_d & \text{if} \quad q = n \end{cases}$$

as R-modules.

Proof. We will use the standard affine open covering

$$\mathcal{U} : \mathbf{P}^n_R = \bigcup\nolimits_{i=0}^{n} D_+(T_i)$$

to compute the cohomology using the Cech complex. This is permissible by Lemma 29.2.5 since any intersection of finitely many affine $D_+(T_i)$ is also a standard affine open (see Constructions, Section 26.8). In fact, we can use the alternating or ordered Cech complex according to Cohomology, Lemmas 20.22.3 and 20.22.6.

The ordering we will use on $\{0, \ldots, n\}$ is the usual one. Hence the complex we are looking at has terms

$$\check{\mathcal{C}}^p_{ord}(\mathcal{U}, \mathcal{O}_{\mathbf{P}_R}(d)) = \bigoplus\nolimits_{i_0 < \ldots < i_p} (R[T_0, \ldots, T_n, \frac{1}{T_{i_0} \ldots T_{i_p}}])_d$$

Moreover, the maps are given by the usual formula

$$d(s)_{i_0 \ldots i_{p+1}} = \sum\nolimits_{j=0}^{p+1} (-1)^j s_{i_0 \ldots \hat{i}_j \ldots i_{p+1}}$$

see Cohomology, Section 20.22. Note that each term of this complex has a natural $\mathbf{Z}^{n+1}$-grading. Namely, we get this by declaring a monomial $T_0^{e_0} \ldots T_n^{e_n}$ to be homogeneous with weight $(e_0, \ldots, e_n) \in \mathbf{Z}^{n+1}$. It is clear that the differential given above respects the grading. In a formula we have

$$\check{\mathcal{C}}^\bullet_{ord}(\mathcal{U}, \mathcal{O}_{\mathbf{P}_R}(d)) = \bigoplus\nolimits_{\vec{e} \in \mathbf{Z}^{n+1}} \check{\mathcal{C}}^\bullet(\vec{e})$$

where not all summand on the right hand side occur (see below). Hence in order to compute the cohomology modules of the complex it suffices to compute the cohomology of the graded pieces and take the direct sum at the end.

Fix $\vec{e} = (e_0, \ldots, e_n) \in \mathbf{Z}^{n+1}$. In order for this weight to occur in the complex above we need to assume $e_0 + \ldots + e_n = d$ (if not then it occurs for a different twist of the structure sheaf of course). Assuming this set

$$NEG(\vec{e}) = \{i \in \{0, \ldots, n\} \mid e_i < 0\}.$$

With this notation the weight $\vec{e}$ summand $\check{\mathcal{C}}^\bullet(\vec{e})$ of the Cech complex above has the following terms

$$\check{\mathcal{C}}^p(\vec{e}) = \bigoplus\nolimits_{i_0 < \ldots < i_p,\ NEG(\vec{e}) \subset \{i_0, \ldots, i_p\}} R \cdot T_0^{e_0} \ldots T_n^{e_n}$$

In other words, the terms corresponding to $i_0 < \ldots < i_p$ such that $NEG(\vec{e})$ is not contained in $\{i_0 \ldots i_p\}$ are zero. The differential of the complex $\check{\mathcal{C}}^\bullet(\vec{e})$ is still given by the exact same formula as above.

Suppose that $NEG(\vec{e}) = \{0, \ldots, n\}$, i.e., that all exponents e_i are negative. In this case the complex $\check{\mathcal{C}}^\bullet(\vec{e})$ has only one term, namely $\check{\mathcal{C}}^n(\vec{e}) = R \cdot \frac{1}{T^{-e_0} \ldots T^{-e_n}}$. Hence in this case

$$H^q(\check{\mathcal{C}}^\bullet(\vec{e})) = \begin{cases} R \cdot \frac{1}{T^{-e_0} \ldots T^{-e_n}} & \text{if} \quad q = n \\ 0 & \text{if} \quad \text{else} \end{cases}$$

The direct sum of all of these terms clearly gives the value

$$\left(\frac{1}{T_0 \ldots T_n} R[\frac{1}{T_0}, \ldots, \frac{1}{T_n}] \right)_d$$

in degree n as stated in the lemma. Moreover these terms do not contribute to cohomology in other degrees (also in accordance with the statement of the lemma).

Assume $NEG(\vec{e}) = \emptyset$. In this case the complex $\check{\mathcal{C}}^\bullet(\vec{e})$ has a summand R corresponding to all $i_0 < \ldots < i_p$. Let us compare the complex $\check{\mathcal{C}}^\bullet(\vec{e})$ to another complex. Namely, consider the affine open open covering

$$\mathcal{V} : \mathrm{Spec}(R) = \bigcup\nolimits_{i \in \{0,\ldots,n\}} V_i$$

where $V_i = \mathrm{Spec}(R)$ for all i. Consider the alternating Cech complex

$$\check{\mathcal{C}}^\bullet_{ord}(\mathcal{V}, \mathcal{O}_{\mathrm{Spec}(R)})$$

By the same reasoning as above this computes the cohomology of the structure sheaf on $\mathrm{Spec}(R)$. Hence we see that $H^p(\check{\mathcal{C}}^\bullet_{ord}(\mathcal{V}, \mathcal{O}_{\mathrm{Spec}(R)})) = R$ if $p = 0$ and is 0 whenever $p > 0$. For these facts, see Lemma 29.2.1 and its proof. Note that also $\check{\mathcal{C}}^\bullet_{ord}(\mathcal{V}, \mathcal{O}_{\mathrm{Spec}(R)})$ has a summand R for every $i_0 < \ldots < i_p$ and has exactly the same differential as $\check{\mathcal{C}}^\bullet(\vec{e})$. In other words these complexes are isomorphic complexes and hence have the same cohomology. We conclude that

$$H^q(\check{\mathcal{C}}^\bullet(\vec{e})) = \begin{cases} R \cdot T^{e_0} \ldots T^{e_n} & \text{if} \quad q = 0 \\ 0 & \text{if} \quad \text{else} \end{cases}$$

in the case that $NEG(\vec{e}) = \emptyset$. The direct sum of all of these terms clearly gives the value

$$(R[T_0, \ldots, T_n])_d$$

in degree 0 as stated in the lemma. Moreover these terms do not contribute to cohomology in other degrees (also in accordance with the statement of the lemma).

To finish the proof of the lemma we have to show that the complexes $\check{\mathcal{C}}^\bullet(\vec{e})$ are acyclic when $NEG(\vec{e})$ is neither empty nor equal to $\{0, \ldots, n\}$. Pick an index $i_{\mathrm{fix}} \notin NEG(\vec{e})$ (such an index exists). Consider the map

$$h : \check{\mathcal{C}}^{p+1}(\vec{e}) \to \check{\mathcal{C}}^p(\vec{e})$$

given by the rule

$$h(s)_{i_0 \ldots i_p} = s_{i_{\mathrm{fix}} i_0 \ldots i_p}$$

(compare with the proof of Lemma 29.2.1). It is clear that this is well defined since

$$NEG(\vec{e}) \subset \{i_0, \ldots, i_p\} \Leftrightarrow NEG(\vec{e}) \subset \{i_{\mathrm{fix}}, i_0, \ldots, i_p\}$$

Also $\check{\mathcal{C}}^0(\vec{e}) = 0$ so that this formula does work for all p including $p = -1$. The exact same (combinatorial) computation as in the proof of Lemma 29.2.1 shows that

$$(hd + dh)(s)_{i_0 \ldots i_p} = s_{i_0 \ldots i_p}$$

Hence we see that the identity map of the complex $\check{\mathcal{C}}^\bullet(\vec{e})$ is homotopic to zero which implies that it is acyclic. □

In the following lemma we are going to use the pairing of free R-modules

$$R[T_0, \ldots, T_n] \times \frac{1}{T_0 \ldots T_n} R[\frac{1}{T_0}, \ldots, \frac{1}{T_n}] \longrightarrow R$$

which is defined by the rule

$$(f, g) \longmapsto \text{coefficient of } \frac{1}{T_0 \ldots T_n} \text{ in } fg.$$

In other words, the basis element $T_0^{e_0} \ldots T_n^{e_n}$ pairs with the basis element $T_0^{d_0} \ldots T_n^{d_n}$ to give 1 if and only if $e_i + d_i = -1$ for all i, and pairs to zero in all other cases. Using this pairing we get an identification

$$\left(\frac{1}{T_0 \ldots T_n} R[\frac{1}{T_0}, \ldots, \frac{1}{T_n}]\right)_d = \mathrm{Hom}_R((R[T_0, \ldots, T_n])_{-n-1-d}, R)$$

Thus we can reformulate the result of Lemma 29.8.1 as saying that

$$H^q(\mathbf{P}^n, \mathcal{O}_{\mathbf{P}^n_R}(d)) = \begin{cases} (R[T_0, \ldots, T_n])_d & \text{if} \quad q = 0 \\ 0 & \text{if} \quad q \neq 0, n \\ \mathrm{Hom}_R((R[T_0, \ldots, T_n])_{-n-1-d}, R) & \text{if} \quad q = n \end{cases} \tag{29.8.1.1}$$

Lemma 29.8.2. *The identifications of Equation (29.8.1.1) are compatible with base change w.r.t. ring maps $R \to R'$. Moreover, for any $f \in R[T_0, \ldots, T_n]$ homogeneous of degree m the map multiplication by f*

$$\mathcal{O}_{\mathbf{P}^n_R}(d) \longrightarrow \mathcal{O}_{\mathbf{P}^n_R}(d + m)$$

induces the map on the cohomology group via the identifications of Equation (29.8.1.1) which is multiplication by f for H^0 and the contragredient of multiplication by f

$$(R[T_0, \ldots, T_n])_{-n-1-(d+m)} \longrightarrow (R[T_0, \ldots, T_n])_{-n-1-d}$$

on H^n.

Proof. Suppose that $R \to R'$ is a ring map. Let $\mathcal{U}$ be the standard affine open covering of $\mathbf{P}^n_R$, and let $\mathcal{U}'$ be the standard affine open covering of $\mathbf{P}^n_{R'}$. Note that $\mathcal{U}'$ is the pullback of the covering $\mathcal{U}$ under the canonical morphism $\mathbf{P}^n_{R'} \to \mathbf{P}^n_R$. Hence there is a map of Cech complexes

$$\gamma : \check{\mathcal{C}}^\bullet_{ord}(\mathcal{U}, \mathcal{O}_{\mathbf{P}_R}(d)) \longrightarrow \check{\mathcal{C}}^\bullet_{ord}(\mathcal{U}', \mathcal{O}_{\mathbf{P}_{R'}}(d))$$

which is compatible with the map on cohomology by Cohomology, Lemma 20.16.1. It is clear from the computations in the proof of Lemma 29.8.1 that this map of Cech complexes is compatible with the identifications of the cohomology groups in question. (Namely the basis elements for the Cech complex over R simply map to the corresponding basis elements for the Cech complex over R'.) Whence the first statement of the lemma.

Now fix the ring R and consider two homogeneous polynomials $f, g \in R[T_0, \ldots, T_n]$ both of the same degree m. Since cohomology is an additive functor, it is clear that the map induced by multiplication by $f + g$ is the same as the sum of the maps induced by multiplication by f and the map induced by multiplication by g. Moreover, since cohomology is a functor a similar result holds for multiplication by a product fg where f, g are both homogeneous (but not necessarily of the same degree). Hence to verify the second statement of the lemma it suffices to prove this when $f = x \in R$ or when $f = T_i$. In the case of multiplication by an element $x \in R$ the result follows since every cohomology groups or complex in sight has the structure of an R-module or complex of R-modules. Finally, we consider the case of multiplication by T_i as a $\mathcal{O}_{\mathbf{P}^n_R}$-linear map

$$\mathcal{O}_{\mathbf{P}^n_R}(d) \longrightarrow \mathcal{O}_{\mathbf{P}^n_R}(d + 1)$$

The statement on H^0 is clear. For the statement on H^n consider multiplication by T_i as a map on Cech complexes

$$\check{\mathcal{C}}^\bullet_{ord}(\mathcal{U}, \mathcal{O}_{\mathbf{P}_R}(d)) \longrightarrow \check{\mathcal{C}}^\bullet_{ord}(\mathcal{U}, \mathcal{O}_{\mathbf{P}_R}(d + 1))$$

We are going to use the notation introduced in the proof of Lemma 29.8.1. We consider the effect of multiplication by T_i in terms of the decompositions

$$\check{\mathcal{C}}^\bullet_{ord}(\mathcal{U}, \mathcal{O}_{\mathbf{P}_R}(d)) = \bigoplus\nolimits_{\vec{e}\in\mathbf{Z}^{n+1},\ \sum e_i = d} \check{\mathcal{C}}^\bullet(\vec{e})$$

and

$$\check{\mathcal{C}}^\bullet_{ord}(\mathcal{U}, \mathcal{O}_{\mathbf{P}_R}(d+1)) = \bigoplus\nolimits_{\vec{e}\in\mathbf{Z}^{n+1},\ \sum e_i = d+1} \check{\mathcal{C}}^\bullet(\vec{e})$$

It is clear that it maps the subcomplex $\check{\mathcal{C}}^\bullet(\vec{e})$ to the subcomplex $\check{\mathcal{C}}^\bullet(\vec{e}+\vec{b}_i)$ where $\vec{b}_i = (0, \ldots, 0, 1, 0, \ldots, 0))$ the ith basis vector. In other words, it maps the summand of H^n corresponding to $\vec{e}$ with $e_i < 0$ and $\sum e_i = d$ to the summand of H^n corresponding to $\vec{e} + \vec{b}_i$ (which is zero if $e_i + b_i \geq 0$). It is easy to see that this corresponds exactly to the action of the contragredient of multiplication by T_i as a map

$$(R[T_0, \ldots, T_n])_{-n-1-(d+1)} \longrightarrow (R[T_0, \ldots, T_n])_{-n-1-d}$$

This proves the lemma. □

Before we state the relative version we need some notation. Namely, recall that $\mathcal{O}_S[T_0, \ldots, T_n]$ is a graded $\mathcal{O}_S$-module where each T_i is homogeneous of degree 1. Denote $(\mathcal{O}_S[T_0, \ldots, T_n])_d$ the degree d summand. It is a finite locally free sheaf of rank $\binom{n+d}{d}$ on S.

Lemma 29.8.3. *Let S be a scheme. Let $n \geq 0$ be an integer. Consider the structure morphism*

$$f : \mathbf{P}^n_S \longrightarrow S.$$

We have

$$R^q f_*(\mathcal{O}_{\mathbf{P}^n_S}(d)) = \begin{cases} (\mathcal{O}_S[T_0, \ldots, T_n])_d & \text{if} \quad q = 0 \\ 0 & \text{if} \quad q \neq 0, n \\ \mathcal{H}om_{\mathcal{O}_S}((\mathcal{O}_S[T_0, \ldots, T_n])_{-n-1-d}, \mathcal{O}_S) & \text{if} \quad q = n \end{cases}$$

Proof. Omitted. Hint: This follows since the identifications in (29.8.1.1) are compatible with affine base change by Lemma 29.8.2. □

Next we state the version for projective bundles associated to finite locally free sheaves. Let S be a scheme. Let $\mathcal{E}$ be a finite locally free $\mathcal{O}_S$-module of constant rank $n+1$, see Modules, Section 17.14. In this case we think of $\mathrm{Sym}(\mathcal{E})$ as a graded $\mathcal{O}_S$-module where $\mathcal{E}$ is the graded part of degree 1. And $\mathrm{Sym}^d(\mathcal{E})$ is the degree d summand. It is a finite locally free sheaf of rank $\binom{n+d}{d}$ on S. Recall that our normalization is that

$$\pi : \mathbf{P}(\mathcal{E}) = \underline{\mathrm{Proj}}_S(\mathrm{Sym}(\mathcal{E})) \longrightarrow S$$

and that there are natural maps $\mathrm{Sym}^d(\mathcal{E}) \to \pi_*\mathcal{O}_{\mathbf{P}(\mathcal{E})}(d)$.

Lemma 29.8.4. *Let S be a scheme. Let $n \geq 1$. Let $\mathcal{E}$ be a finite locally free $\mathcal{O}_S$-module of constant rank $n+1$. Consider the structure morphism*

$$\pi : \mathbf{P}(\mathcal{E}) \longrightarrow S.$$

We have

$$R^q\pi_*(\mathcal{O}_{\mathbf{P}(\mathcal{E})}(d)) = \begin{cases} Sym^d(\mathcal{E}) & \text{if} \quad q = 0 \\ 0 & \text{if} \quad q \neq 0, n \\ \mathcal{H}om_{\mathcal{O}_S}(Sym^{-n-1-d}(\mathcal{E}) \otimes_{\mathcal{O}_S} \wedge^{n+1}\mathcal{E}, \mathcal{O}_S) & \text{if} \quad q = n \end{cases}$$

These identifications are compatible with base change and isomorphism between locally free sheaves.

Proof. Consider the canonical map

$$\pi^*\mathcal{E} \longrightarrow \mathcal{O}_{\mathbf{P}(\mathcal{E})}(1)$$

and twist down by 1 to get

$$\pi^*(\mathcal{E})(-1) \longrightarrow \mathcal{O}_{\mathbf{P}(\mathcal{E})}$$

This is a surjective map from a locally free rank $n+1$ sheaf onto the structure sheaf. Hence the corresponding Koszul complex is exact (insert future reference here). In other words there is an exact complex

$$0 \to \pi^*(\wedge^{n+1}\mathcal{E})(-n-1) \to \ldots \to \pi^*(\wedge^i\mathcal{E})(-i) \to \ldots \to \mathcal{E}(-1) \to \mathcal{O}_{\mathbf{P}(\mathcal{E})} \to 0$$

We will think of the term $\pi^*(\wedge^i\mathcal{E})(-i)$ as being in degree $-i$. We are going to compute the higher direct images of this acyclic complex using the first spectral sequence of Derived Categories, Lemma 13.20.3. Namely, we see that there is a spectral sequence with terms

$$E_2^{p,q} = H^p(L^{\bullet,q}) \quad \text{with} \quad L^{-i,q} = R^q\pi_*\left(\pi^*(\wedge^i\mathcal{E})(-i)\right)$$

converging to zero! By the projection formula (Cohomology, Lemma 20.8.2) we have

$$L^{-i,q} = \wedge^i\mathcal{E} \otimes_{\mathcal{O}_S} R^q\pi_*\left(\mathcal{O}_{\mathbf{P}(\mathcal{E})}(-i)\right).$$

Note that locally on S the sheaf $\mathcal{E}$ is trivial, i.e., isomorphic to $\mathcal{O}_S^{\oplus n+1}$, hence locally on S the morphism $\mathbf{P}(\mathcal{E}) \to S$ can be identified with $\mathbf{P}_S^n \to S$. Hence locally on S we can use the result of Lemmas 29.8.1, 29.8.2, or 29.8.3. It follows that $L^{-i,q} = 0$ unless $i = q = 0$ or $i = n+1$ and $q = n$. This in turn implies that $E_2^{p,q} = 0$ unless $(p,q) = (0,0)$ or $(p,q) = (-n-1, n)$, and

$$E_2^{0,0} = \pi_*\mathcal{O}_{\mathbf{P}(\mathcal{E})} = \mathcal{O}_S$$
$$E_2^{-n-1,n} = \wedge^{n+1}\mathcal{E} \otimes_{\mathcal{O}_S} R^n\pi_*\left(\mathcal{O}_{\mathbf{P}(\mathcal{E})}(-n-1)\right).$$

Hence there can only be one nonzero differential in the spectral sequence namely the map d_{n+1} inducing a map

$$d_{n+1}^{0,0} : \mathcal{O}_S \longrightarrow \wedge^{n+1}\mathcal{E} \otimes_{\mathcal{O}_S} R^n\pi_*\left(\mathcal{O}_{\mathbf{P}(\mathcal{E})}(-n-1)\right)$$

which has to be an isomorphism (because the spectral sequence converges to the 0 sheaf). Since $\wedge^{n+1}\mathcal{E}$ is an invertible sheaf, this implies that $R^n\pi_*\mathcal{O}_{\mathbf{P}(\mathcal{E})}(-n-1)$ is invertible as well and canonically isomorphic to the inverse of $\wedge^{n+1}\mathcal{E}$. In other words we have proved the case $d = -n-1$ of the lemma.

Working locally on S we see immediately from the computation of cohomology in Lemmas 29.8.1, 29.8.2, or 29.8.3 the statements on vanishing of the lemma. Moreover the result on $R^0\pi_*$ is clear as well, since there are canonical maps $\mathrm{Sym}^d(\mathcal{E}) \to \pi_*\mathcal{O}_{\mathbf{P}(\mathcal{E})}(d)$ for all d. It remains to show that the description of $R^n\pi_*\mathcal{O}_{\mathbf{P}(\mathcal{E})}(d)$ is correct for $d < -n-1$. In other to do this we consider the map

$$\pi^*(\mathrm{Sym}^{-d+n+1}(\mathcal{E})) \otimes_{\mathcal{O}_{\mathbf{P}(\mathcal{E})}} \mathcal{O}_{\mathbf{P}(\mathcal{E})}(d) \longrightarrow \mathcal{O}_{\mathbf{P}(\mathcal{E})}(-n-1)$$

Applying $R^n\pi_*$ and the projection formula (see above) we get a map

$$\mathrm{Sym}^{-d+n+1}(\mathcal{E}) \otimes_{\mathcal{O}_S} R^n\pi_*(\mathcal{O}_{\mathbf{P}(\mathcal{E})}(d)) \longrightarrow R^n\pi_*\mathcal{O}_{\mathbf{P}(\mathcal{E})}(-n-1) = (\wedge^{n+1}\mathcal{E})^{\otimes -1}$$

(the last equality we have shown above). Again by the local calculations of Lemmas 29.8.1, 29.8.2, or 29.8.3 it follows that this map induces a perfect pairing between $R^n\pi_*(\mathcal{O}_{\mathbf{P}(\mathcal{E})}(d))$ and $\mathrm{Sym}^{-d+n+1}(\mathcal{E}) \otimes \wedge^{n+1}(\mathcal{E})$ as desired. □

29.9. Coherent sheaves on locally Noetherian schemes

We have defined the notion of a coherent module on any ringed space in Modules, Section 17.12. Although it is possible to consider coherent sheaves on non-Noetherian schemes we will always assume the base scheme is locally Noetherian when we consider coherent sheaves. Here is a characterization of coherent sheaves on locally Noetherian schemes.

Lemma 29.9.1. *Let X be a locally Noetherian scheme. Let $\mathcal{F}$ be an $\mathcal{O}_X$-module. The following are equivalent*

1. *$\mathcal{F}$ is coherent,*
2. *$\mathcal{F}$ is a quasi-coherent, finite type $\mathcal{O}_X$-module,*
3. *$\mathcal{F}$ is a finitely presented $\mathcal{O}_X$-module,*
4. *for any affine open $\mathrm{Spec}(A) = U \subset X$ we have $\mathcal{F}|_U = \widetilde{M}$ with M a finite A-module, and*
5. *there exists an affine open covering $X = \bigcup U_i$, $U_i = \mathrm{Spec}(A_i)$ such that each $\mathcal{F}|_{U_i} = \widetilde{M_i}$ with M_i a finite A_i-module.*

In particular $\mathcal{O}_X$ is coherent, any invertible $\mathcal{O}_X$-module is coherent, and more generally any finite locally free $\mathcal{O}_X$-module is coherent.

Proof. The implications (1) ⇒ (2) and (1) ⇒ (3) hold in general, see Modules, Lemma 17.12.2. If $\mathcal{F}$ is finitely presented then $\mathcal{F}$ is quasi-coherent, see Modules, Lemma 17.11.2. Hence also (3) ⇒ (2).

Assume $\mathcal{F}$ is a quasi-coherent, finite type $\mathcal{O}_X$-module. By Properties, Lemma 27.16.1 we see that on any affine open $\mathrm{Spec}(A) = U \subset X$ we have $\mathcal{F}|_U = \widetilde{M}$ with M a finite A-module. Since A is Noetherian we see that M has a finite resolution

$$A^{\oplus m} \to A^{\oplus n} \to M \to 0.$$

Hence $\mathcal{F}$ is of finite presentation by Properties, Lemma 27.16.2. In other words (2) ⇒ (3).

By Modules, Lemma 17.12.5 it suffices to show that $\mathcal{O}_X$ is coherent in order to show that (3) implies (1). Thus we have to show: given any open $U \subset X$ and any finite collection of sections $f_i \in \mathcal{O}_X(U)$, $i = 1, \ldots, n$ the kernel of the map $\bigoplus_{i=1,\ldots,n} \mathcal{O}_U \to \mathcal{O}_U$ is of finite type. Since being of finite type is a local property it suffices to check this in a neighbourhood of any $x \in U$. Thus we may assume $U = \mathrm{Spec}(A)$ is affine. In this case $f_1, \ldots, f_n \in A$ are elements of A. Since A is Noetherian, see Properties, Lemma 27.5.2 the kernel K of the map $\bigoplus_{i=1,\ldots,n} A \to A$ is a finite A-module. See for example Algebra, Lemma 10.47.1. As the functor $\widetilde{}$ is exact, see Schemes, Lemma 25.5.4 we get an exact sequence

$$\widetilde{K} \to \bigoplus\nolimits_{i=1,\ldots,n} \mathcal{O}_U \to \mathcal{O}_U$$

and by Properties, Lemma 27.16.1 again we see that $\widetilde{K}$ is of finite type. We conclude that (1), (2) and (3) are all equivalent.

It follows from Properties, Lemma 27.16.1 that (2) implies (4). It is trivial that (4) implies (5). The discussion in Schemes, Section 25.24 show that (5) implies that $\mathcal{F}$

is quasi-coherent and it is clear that (5) implies that $\mathcal{F}$ is of finite type. Hence (5) implies (2) and we win. □

Lemma 29.9.2. *Let X be a locally Noetherian scheme. The category of coherent $\mathcal{O}_X$-modules is abelian. More precisely, the kernel and cokernel of a map of coherent $\mathcal{O}_X$-modules are coherent. Any extension of coherent sheaves is coherent.*

Proof. This is a restatement of Modules, Lemma 17.12.4 in a particular case. □

The following lemma does not always hold for the category of coherent $\mathcal{O}_X$-modules on a general ringed space X.

Lemma 29.9.3. *Let X be a locally Noetherian scheme. Let $\mathcal{F}$ be a coherent $\mathcal{O}_X$-module. Any quasi-coherent submodule of $\mathcal{F}$ is coherent. Any quasi-coherent quotient module of $\mathcal{F}$ is coherent.*

Proof. We may assume that X is affine, say $X = \mathrm{Spec}(A)$. Properties, Lemma 27.5.2 implies that A is Noetherian. Lemma 29.9.1 turns this into algebra. The algebraic counter part of the lemma is that a quotient, or a submodule of a finite A-module is a finite A-module, see for example Algebra, Lemma 10.47.1. □

Lemma 29.9.4. *Let X be a locally Noetherian scheme. Let $\mathcal{F}$, $\mathcal{G}$ be coherent $\mathcal{O}_X$-modules. The $\mathcal{O}_X$-modules $\mathcal{F} \otimes_{\mathcal{O}_X} \mathcal{G}$ and $\mathcal{H}om_{\mathcal{O}_X}(\mathcal{F}, \mathcal{G})$ are coherent.*

Proof. It is shown in Modules, Lemma 17.19.4 that $\mathcal{H}om_{\mathcal{O}_X}(\mathcal{F}, \mathcal{G})$ is coherent. The result for tensor products is Modules, Lemma 17.15.5 □

Lemma 29.9.5. *Let X be a locally Noetherian scheme. Let $\mathcal{F}$, $\mathcal{G}$ be coherent $\mathcal{O}_X$-modules. Let $\varphi : \mathcal{G} \to \mathcal{F}$ be a homomorphism of $\mathcal{O}_X$-modules. Let $x \in X$.*

1. *If $\mathcal{F}_x = 0$ then there exists an open neighbourhood $U \subset X$ of x such that $\mathcal{F}|_U = 0$.*
2. *If $\varphi_x : \mathcal{G}_x \to \mathcal{F}_x$ is injective, then there exists an open neighbourhood $U \subset X$ of x such that $\varphi|_U$ is injective.*
3. *If $\varphi_x : \mathcal{G}_x \to \mathcal{F}_x$ is surjective, then there exists an open neighbourhood $U \subset X$ of x such that $\varphi|_U$ is surjective.*
4. *If $\varphi_x : \mathcal{G}_x \to \mathcal{F}_x$ is bijective, then there exists an open neighbourhood $U \subset X$ of x such that $\varphi|_U$ is an isomorphism.*

Proof. See Modules, Lemmas 17.9.4, 17.9.5, and 17.12.6. □

Lemma 29.9.6. *Let X be a locally Noetherian scheme. Let $\mathcal{F}$, $\mathcal{G}$ be coherent $\mathcal{O}_X$-modules. Let $x \in X$. Suppose $\psi : \mathcal{G}_x \to \mathcal{F}_x$ is a map of $\mathcal{O}_{X,x}$-modules. Then there exists an open neighbourhood $U \subset X$ of x and a map $\varphi : \mathcal{G}|_U \to \mathcal{F}|_U$ such that $\varphi_x = \psi$.*

Proof. In view of Lemma 29.9.1 this is a reformulation of Modules, Lemma 17.19.3. □

Lemma 29.9.7. *Let X be a locally Noetherian scheme. Let $\mathcal{F}$ be a coherent $\mathcal{O}_X$-module. Then $\mathrm{Supp}(\mathcal{F})$ is closed, and $\mathcal{F}$ comes from a coherent sheaf on the scheme theoretic support of $\mathcal{F}$, see Morphisms, Definition 28.5.5.*

Proof. Let $i : Z \to X$ be the scheme theoretic support of $\mathcal{F}$ and let $\mathcal{G}$ be the finite type quasi-coherent sheaf on Z such that $i_*\mathcal{G} \cong \mathcal{F}$. Since $Z = \mathrm{Supp}(\mathcal{F})$ we see that the support is closed. The scheme Z is locally Noetherian by Morphisms, Lemmas 28.16.5 and 28.16.6. Finally, $\mathcal{G}$ is a coherent $\mathcal{O}_Z$-module by Lemma 29.9.1 □

Lemma 29.9.8. *Let $i : Z \to X$ be a closed immersion of locally Noetherian schemes. Let $\mathcal{I} \subset \mathcal{O}_X$ be the quasi-coherent sheaf of ideals cutting out Z. The functor i_* induces an equivalence between the category of coherent $\mathcal{O}_X$-modules annihilated by $\mathcal{I}$ and the category of coherent $\mathcal{O}_Z$-modules.*

Proof. The functor is fully faithful by Morphisms, Lemma 28.4.1. Let $\mathcal{F}$ be a coherent $\mathcal{O}_X$-module annihilated by $\mathcal{I}$. By Morphisms, Lemma 28.4.1 we can write $\mathcal{F} = i_*\mathcal{G}$ for some quasi-coherent sheaf $\mathcal{G}$ on Z. By Modules, Lemma 17.13.3 we see that $\mathcal{G}$ is of finite type. Hence $\mathcal{G}$ is coherent by Lemma 29.9.1. Thus the functor is also essentially surjective as desired. □

Lemma 29.9.9. *Let $f : X \to Y$ be a morphism of schemes. Let $\mathcal{F}$ be a quasi-coherent $\mathcal{O}_X$-module. Assume f is finite and Y locally Noetherian. Then $R^p f_*\mathcal{F} = 0$ for $p > 0$ and $f_*\mathcal{F}$ is coherent if $\mathcal{F}$ is coherent.*

Proof. The higher direct images vanish by Lemma 29.2.3 and because a finite morphism is affine (by definition). Note that the assumptions imply that also X is locally Noetherian (see Morphisms, Lemma 28.16.6) and hence the statement makes sense. Let $\mathrm{Spec}(A) = V \subset Y$ be an affine open subset. By Morphisms, Definition 28.44.1 we see that $f^{-1}(V) = \mathrm{Spec}(B)$ with $A \to B$ finite. Lemma 29.9.1 turns the statement of the lemma into the following algebra fact: If M is a finite B-module, then M is also finite viewed as a A-module, see Algebra, Lemma 10.7.2. □

In the situation of the lemma also the higher direct images are coherent since they vanish. We will show that this is always the case for a proper morphism between locally Noetherian schemes (insert future reference here).

29.10. Coherent sheaves on Noetherian schemes

In this section we mention some properties of coherent sheaves on Noetherian schemes.

Lemma 29.10.1. *Let X be a Noetherian scheme. Let $\mathcal{F}$ be a coherent $\mathcal{O}_X$-module. The ascending chain condition holds for quasi-coherent submodules of $\mathcal{F}$. In other words, given any sequence*

$$\mathcal{F}_1 \subset \mathcal{F}_2 \subset \ldots \subset \mathcal{F}$$

of quasi-coherent submodules, then $\mathcal{F}_n = \mathcal{F}_{n+1} = \ldots$ for some $n \geq 0$.

Proof. Choose a finite affine open covering. On each member of the covering we get stabilization by Algebra, Lemma 10.47.1. Hence the lemma follows. □

Lemma 29.10.2. *Let X be a Noetherian scheme. Let $\mathcal{F}$ be a coherent sheaf on X. Let $\mathcal{I} \subset \mathcal{O}_X$ be a quasi-coherent sheaf of ideals corresponding to a closed subscheme $Z \subset X$. Then there is some $n \geq 0$ such that $\mathcal{I}^n\mathcal{F} = 0$ if and only if $\mathrm{Supp}(\mathcal{F}) \subset Z$ (set theoretically).*

Proof. This follows immediately from Algebra, Lemma 10.59.4 because X has a finite covering by spectra of Noetherian rings. □

Lemma 29.10.3 (Artin-Rees)**.** *Let X be a Noetherian scheme. Let $\mathcal{F}$ be a coherent sheaf on X. Let $\mathcal{G} \subset \mathcal{F}$ be a quasi-coherent subsheaf. Let $\mathcal{I} \subset \mathcal{O}_X$ be a quasi-coherent sheaf of ideals. Then there exists a $c \geq 0$ such that for all $n \geq c$ we have*

$$\mathcal{I}^{n-c}(\mathcal{I}^c\mathcal{F} \cap \mathcal{G}) = \mathcal{I}^n\mathcal{F}$$

Proof. This follows immediately from Algebra, Lemma 10.47.4 because X has a finite covering by spectra of Noetherian rings. □

Lemma 29.10.4. *Let X be a Noetherian scheme. Let $\mathcal{F}$ be a quasi-coherent $\mathcal{O}_X$-module. Let $\mathcal{G}$ be coherent $\mathcal{O}_X$-module. Let $\mathcal{I} \subset \mathcal{O}_X$ be a quasi-coherent sheaf of ideals. Denote $Z \subset X$ the corresponding closed subscheme and set $U = X \setminus Z$. There is a canonical isomorphism*

$$\operatorname{colim}_n \operatorname{Hom}_{\mathcal{O}_X}(\mathcal{I}^n\mathcal{G}, \mathcal{F}) \longrightarrow \operatorname{Hom}_{\mathcal{O}_U}(\mathcal{G}|_U, \mathcal{F}|_U).$$

In particular we have an isomorphism

$$\operatorname{colim}_n \operatorname{Hom}_{\mathcal{O}_X}(\mathcal{I}^n, \mathcal{F}) \longrightarrow \Gamma(U, \mathcal{F}).$$

Proof. We first prove the second map is an isomorphism. It is injective by Properties, Lemma 27.23.3. Since $\mathcal{F}$ is the union of its coherent submodules, see Properties, Lemma 27.20.3 (and Lemma 29.9.1) we may and do assume that $\mathcal{F}$ is coherent to prove surjectivity. Let $\mathcal{F}_n$ denote the quasi-coherent subsheaf of $\mathcal{F}$ consisting of sections annihilated by $\mathcal{I}^n$, see Properties, Lemma 27.23.3. Since $\mathcal{F}_1 \subset \mathcal{F}_2 \subset \ldots$ we see that $\mathcal{F}_n = \mathcal{F}_{n+1} = \ldots$ for some $n \geq 0$ by Lemma 29.10.1. Set $\mathcal{H} = \mathcal{F}_n$ for this n. By Artin-Rees (Lemma 29.10.3) there exists an $c \geq 0$ such that $\mathcal{I}^m\mathcal{F} \cap \mathcal{H} \subset \mathcal{I}^{m-c}\mathcal{H}$. Picking $m = n + c$ we get $\mathcal{I}^m\mathcal{F} \cap \mathcal{H} \subset \mathcal{I}^n\mathcal{H} = 0$. Thus if we set $\mathcal{F}' = \mathcal{I}^m\mathcal{F}$ then we see that $\mathcal{F}' \cap \mathcal{F}_n = 0$ and $\mathcal{F}'|_U = \mathcal{F}|_U$. Note in particular that the subsheaf $(\mathcal{F}')_N$ of sections annihilated by $\mathcal{I}^N$ is zero for all $N \geq 0$. Hence by Properties, Lemma 27.23.3 we deduce that the top horizontal arrow in the following commutative diagram is a bijection:

$$\begin{array}{ccc} \operatorname{colim}_n \operatorname{Hom}_{\mathcal{O}_X}(\mathcal{I}^n, \mathcal{F}') & \longrightarrow & \Gamma(U, \mathcal{F}') \\ \downarrow & & \downarrow \\ \operatorname{colim}_n \operatorname{Hom}_{\mathcal{O}_X}(\mathcal{I}^n, \mathcal{F}) & \longrightarrow & \Gamma(U, \mathcal{F}) \end{array}$$

Since also the right vertical arrow is a bijection we conclude that the bottom horizontal arrow is surjective as desired.

Next, we prove the first arrow of the lemma is a bijection. By Lemma 29.9.1 the sheaf $\mathcal{G}$ is of finite presentation and hence the sheaf $\mathcal{H} = \mathcal{H}om_{\mathcal{O}_X}(\mathcal{G}, \mathcal{F})$ is quasi-coherent, see Schemes, Section 25.24. By definition we have

$$\mathcal{H}(U) = \operatorname{Hom}_{\mathcal{O}_U}(\mathcal{G}|_U, \mathcal{F}|_U)$$

Pick a ψ in the right hand side of the first arrow of the lemma, i.e., $\psi \in \mathcal{H}(U)$. The result just proved applies to $\mathcal{H}$ and hence there exists an $n \geq 0$ and an $\varphi : \mathcal{I}^n \to \mathcal{H}$ which recovers ψ on restriction to U. By Modules, Lemma 17.19.1 φ corresponds to a map

$$\varphi : \mathcal{I}^{\otimes n} \otimes_{\mathcal{O}_X} \mathcal{G} \longrightarrow \mathcal{F}.$$

This is almost what we want except that the source of the arrow is the tensor product of $\mathcal{I}^n$ and $\mathcal{G}$ and not the product. We will show that, at the cost of increasing n, the difference is irrelevant. Consider the short exact sequence

$$0 \to \mathcal{K} \to \mathcal{I}^n \otimes_{\mathcal{O}_X} \mathcal{G} \to \mathcal{I}^n\mathcal{G} \to 0$$

where $\mathcal{K}$ is defined as the kernel. Note that $\mathcal{I}^n\mathcal{K} = 0$ (proof omitted). By Artin-Rees again we see that

$$\mathcal{K} \cap \mathcal{I}^m(\mathcal{I}^n \otimes_{\mathcal{O}_X} \mathcal{G}) = 0$$

for some m large enough. In other words we see that

$$\mathcal{I}^m(\mathcal{I}^n \otimes_{\mathcal{O}_X} \mathcal{G}) \longrightarrow \mathcal{I}^{n+m}\mathcal{G}$$

is an isomorphism. Let φ' be the restriction of φ to this submodule thought of as a map $\mathcal{I}^{m+n}\mathcal{G} \to \mathcal{F}$. Then φ' gives an element of the left hand side of the first arrow of the lemma which maps to ψ via the arrow. In other words we have prove surjectivity of the arrow. We omit the proof of injectivity. □

29.11. Depth

In this section we talk a little bit about depth and property (S_k) for coherent modules on locally Noetherian schemes. Note that we have already discussed this notion for locally Noetherian schemes in Properties, Section 27.12.

Definition 29.11.1. Let X be a locally Noetherian scheme. Let $\mathcal{F}$ be a coherent $\mathcal{O}_X$-module. Let $k \geq 0$ be an integer.

(1) We say $\mathcal{F}$ has *depth k at a point x* of X if $\text{depth}_{\mathcal{O}_{X,x}}(\mathcal{F}_x) = k$.
(2) We say X has *depth k at a point x* of X if $\text{depth}(\mathcal{O}_{X,x}) = k$.
(3) We say $\mathcal{F}$ has property (S_k) if
$$\text{depth}_{\mathcal{O}_{X,x}}(\mathcal{F}_x) \geq \min(k, \dim(\mathcal{F}_x))$$
for all $x \in X$.
(4) We say X has property (S_k) if $\mathcal{O}_X$ has property (S_k).

Any coherent sheaf satisfies condition (S_0). Condition (S_1) is equivalent to having no embedded associated points, see Divisors, Lemma 30.4.3.

We have seen in Properties, Lemma 27.12.2 that a locally Noetherian scheme is Cohen-Macaulay if and only if (S_k) holds for all k. Thus it makes sense to introduce the following definition, which is equivalent to the condition that all stalks are Cohen-Macaulay modules.

Definition 29.11.2. Let X be a locally Noetherian scheme. Let $\mathcal{F}$ be a coherent $\mathcal{O}_X$-module. We say $\mathcal{F}$ is *Cohen-Macaulay* if and only if (S_k) holds for all $k \geq 0$.

29.12. Devissage of coherent sheaves

Let X be a Noetherian scheme. Consider an integral closed subscheme $i : Z \to X$. It is often convenient to consider coherent sheaves of the form $i_*\mathcal{G}$ where $\mathcal{G}$ is a coherent sheaf on Z. In particular we are interested in these sheaves when $\mathcal{G}$ is a torsion free rank 1 sheaf. For example $\mathcal{G}$ could be a nonzero sheaf of ideals on Z, or even more specifically $\mathcal{G} = \mathcal{O}_Z$.

Throughout this section we will use that a coherent sheaf is the same thing as a finite type quasi-coherent sheaf and that a quasi-coherent subquotient of a coherent sheaf is coherent, see Section 29.9. The support of a coherent sheaf is closed, see Modules, Lemma 17.9.6.

Lemma 29.12.1. *Let X be a Noetherian scheme. Let $\mathcal{F}$ be a coherent sheaf on X. Suppose that $\text{Supp}(\mathcal{F}) = Z \cup Z'$ with Z, Z' closed. Then there exists a short exact sequence of coherent sheaves*

$$0 \to \mathcal{G}' \to \mathcal{F} \to \mathcal{G} \to 0$$

with $\text{Supp}(\mathcal{G}') \subset Z'$ and $\text{Supp}(\mathcal{G}) \subset Z$.

Proof. Throughout the proof we will use that a coherent sheaf is the same thing as a finite type quasi-coherent sheaf and that a quasi-coherent subquotient of a coherent sheaf is coherent, see Section 29.9. The support of a coherent sheaf is closed, see Modules, Lemma 17.9.6. Let $\mathcal{I} \subset \mathcal{O}_X$ be the sheaf of ideals defining the reduced induced closed subscheme structure on Z, see Schemes, Lemma 25.12.4. Consider the subsheaves $\mathcal{G}'_n = \mathcal{I}^n\mathcal{F}$ and the quotients $\mathcal{G}_n = \mathcal{F}/\mathcal{I}^n\mathcal{F}$. For each n we have a short exact sequence

$$0 \to \mathcal{G}'_n \to \mathcal{F} \to \mathcal{G}_n \to 0$$

For every point x of $Z' \setminus Z$ we have $\mathcal{I}_x = \mathcal{O}_{X,x}$ and hence $\mathcal{G}_{n,x} = 0$. Thus we see that $\text{Supp}(\mathcal{G}_n) \subset Z$. Note that $X \setminus Z'$ is a Noetherian scheme. Hence by Lemma 29.10.2 there exists an n such that $\mathcal{G}'_n|_{X\setminus Z'} = \mathcal{I}^n\mathcal{F}|_{X\setminus Z'} = 0$. For such an n we see that $\text{Supp}(\mathcal{G}'_n) \subset Z'$. Thus setting $\mathcal{G}' = \mathcal{G}'_n$ and $\mathcal{G} = \mathcal{G}_n$ works. □

Lemma 29.12.2. *Let X be a Noetherian scheme. Let $i : Z \to X$ be an integral closed subscheme. Let $\xi \in Z$ be the generic point. Let $\mathcal{F}$ be a coherent sheaf on X. Assume that $\mathcal{F}_\xi$ is annihilated by $\mathfrak{m}_\xi$. Then there exists an integer $r \geq 0$ and a sheaf of ideals $\mathcal{I} \subset \mathcal{O}_Z$ and an injective map of coherent sheaves*

$$i_*\left(\mathcal{I}^{\oplus r}\right) \to \mathcal{F}$$

which is an isomorphism in a neighbourhood of ξ.

Proof. Let $\mathcal{J} \subset \mathcal{O}_X$ be the ideal sheaf of Z. Let $\mathcal{F}' \subset \mathcal{F}$ be the subsheaf of local sections of $\mathcal{F}$ which are annihilated by $\mathcal{J}$. It is a quasi-coherent sheaf by Properties, Lemma 27.22.2. Moreover, $\mathcal{F}'_\xi = \mathcal{F}_\xi$ because $\mathcal{J}_\xi = \mathfrak{m}_\xi$ and part (3) of Properties, Lemma 27.22.2. By Lemma 29.9.5 we see that $\mathcal{F}' \to \mathcal{F}$ induces an isomorphism in a neighbourhood of ξ. Hence we may replace $\mathcal{F}$ by $\mathcal{F}'$ and assume that $\mathcal{F}$ is annihilated by $\mathcal{J}$.

Assume $\mathcal{J}\mathcal{F} = 0$. By Lemma 29.9.8 we can write $\mathcal{F} = i_*\mathcal{G}$ for some coherent sheaf $\mathcal{G}$ on Z. Suppose we can find a morphism $\mathcal{I}^{\oplus r} \to \mathcal{G}$ which is an isomorphism in a neighbourhood of the generic point ξ of Z. Then applying i_* (which is left exact) we get the result of the lemma. Hence we have reduced to the case $X = Z$.

Suppose $Z = X$ is an integral Noetherian scheme with generic point ξ. Note that $\mathcal{O}_{X,\xi} = \kappa(\xi)$ is the function field of X in this case. Since $\mathcal{F}_\xi$ is a finite $\mathcal{O}_\xi$-module we see that $r = \dim_{\kappa(\xi)} \mathcal{F}_\xi$ is finite. Hence the sheaves $\mathcal{O}_X^{\oplus r}$ and $\mathcal{F}$ have isomorphic stalks at ξ. By Lemma 29.9.6 there exists a nonempty open $U \subset X$ and a morphism $\psi : \mathcal{O}_X^{\oplus r}|_U \to \mathcal{F}|_U$ which is an isomorphism at ξ, and hence an isomorphism in a neighbourhood of ξ by Lemma 29.9.5. By Schemes, Lemma 25.12.4 there exists a quasi-coherent sheaf of ideals $\mathcal{I} \subset \mathcal{O}_X$ whose associated closed subscheme $Z \subset X$ is the complement of U. By Lemma 29.10.4 there exists an $n \geq 0$ and a morphism $\mathcal{I}^n(\mathcal{O}_X^{\oplus r}) \to \mathcal{F}$ which recovers our ψ over U. Since $\mathcal{I}^n(\mathcal{O}_X^{\oplus r}) = (\mathcal{I}^n)^{\oplus r}$ we get a map as in the lemma. It is injective because X is integral and it is injective at the generic point of X (easy proof omitted). □

Lemma 29.12.3. *Let X be a Noetherian scheme. Let $\mathcal{F}$ be a coherent sheaf on X. There exists a filtration*

$$0 = \mathcal{F}_0 \subset \mathcal{F}_1 \subset \ldots \subset \mathcal{F}_m = \mathcal{F}$$

by coherent subsheaves such that for each $j = 1, \ldots, m$ there exists an integral closed subscheme $Z_j \subset X$ and a sheaf of ideals $\mathcal{I}_j \subset \mathcal{O}_{Z_j}$ such that

$$\mathcal{F}_j/\mathcal{F}_{j-1} \cong (Z_j \to X)_*\mathcal{I}_j$$

Proof. Consider the collection

$$\mathcal{T} = \left\{ \begin{array}{c} Z \subset X \text{ closed such that there exists a coherent sheaf } \mathcal{F} \\ \text{with } \mathrm{Supp}(\mathcal{F}) = Z \text{ for which the lemma is wrong} \end{array} \right\}$$

We are trying to show that $\mathcal{T}$ is empty. If not, then because X is Noetherian we can choose a minimal element $Z \in \mathcal{T}$. This means that there exists a coherent sheaf $\mathcal{F}$ on X whose support is Z and for which the lemma does not hold. Clearly $Z \neq \emptyset$ since the only sheaf whose support is empty is the zero sheaf for which the lemma does hold (with $m = 0$).

If Z is not irreducible, then we can write $Z = Z_1 \cup Z_2$ with Z_1, Z_2 closed and strictly smaller than Z. Then we can apply Lemma 29.12.1 to get a short exact sequence of coherent sheaves

$$0 \to \mathcal{G}_1 \to \mathcal{F} \to \mathcal{G}_2 \to 0$$

with $\mathrm{Supp}(\mathcal{G}_i) \subset Z_i$. By minimality of Z each of $\mathcal{G}_i$ has a filtration as in the statement of the lemma. By considering the induced filtration on $\mathcal{F}$ we arrive at a contradiction. Hence we conclude that Z is irreducible.

Suppose Z is irreducible. Let $\mathcal{J}$ be the sheaf of ideals cutting out the reduced induced closed subscheme structure of Z, see Schemes, Lemma 25.12.4. By Lemma 29.10.2 we see there exists an $n \geq 0$ such that $\mathcal{J}^n\mathcal{F} = 0$. Hence we obtain a filtration

$$0 = \mathcal{I}^n\mathcal{F} \subset \mathcal{I}^{n-1}\mathcal{F} \subset \ldots \subset \mathcal{I}\mathcal{F} \subset \mathcal{F}$$

each of whose successive subquotients is annihilated by $\mathcal{J}$. Hence if each of these subquotients has a filtration as in the statement of the lemma then also $\mathcal{F}$ does. In other words we may assume that $\mathcal{J}$ does annihilate $\mathcal{F}$.

In the case where Z is irreducible and $\mathcal{J}\mathcal{F} = 0$ we can apply Lemma 29.12.2. This gives a short exact sequence

$$0 \to i_*(\mathcal{I}^{\oplus r}) \to \mathcal{F} \to \mathcal{Q} \to 0$$

where $\mathcal{Q}$ is defined as the quotient. Since $\mathcal{Q}$ is zero in a neighbourhood of ξ by the lemma just cited we see that the support of $\mathcal{Q}$ is strictly smaller than Z. Hence we see that $\mathcal{Q}$ has a filtration of the desired type by minimality of Z. But then clearly $\mathcal{F}$ does too, which is our final contradiction. □

Lemma 29.12.4. *Let X be a Noetherian scheme. Let $\mathcal{P}$ be a property of coherent sheaves on X. Assume*

(1) *For any short exact sequence of coherent sheaves*

$$0 \to \mathcal{F}_1 \to \mathcal{F} \to \mathcal{F}_2 \to 0$$

if $\mathcal{F}_i$, $i = 1, 2$ have property $\mathcal{P}$ then so does $\mathcal{F}$.

(2) *For every integral closed subscheme $Z \subset X$ and every quasi-coherent sheaf of ideals $\mathcal{I} \subset \mathcal{O}_Z$ we have $\mathcal{P}$ for $i_*\mathcal{I}$.*

Then property $\mathcal{P}$ holds for every coherent sheaf on X.

Proof. First note that if $\mathcal{F}$ is a coherent sheaf with a filtration

$$0 = \mathcal{F}_0 \subset \mathcal{F}_1 \subset \ldots \subset \mathcal{F}_m = \mathcal{F}$$

by coherent subsheaves such that each of $\mathcal{F}_i/\mathcal{F}_{i-1}$ has property $\mathcal{P}$, then so does $\mathcal{F}$. This follows from the property (1) for $\mathcal{P}$. On the other hand, by Lemma 29.12.3 we can filter any $\mathcal{F}$ with successive subquotients as in (2). Hence the lemma follows. □

Lemma 29.12.5. *Let X be a Noetherian scheme. Let $Z_0 \subset X$ be an irreducible closed subset with generic point ξ. Let $\mathcal{P}$ be a property of coherent sheaves on X such that*

1. *For any short exact sequence of coherent sheaves if two out of three of them have property $\mathcal{P}$ then so does the third.*
2. *For every integral closed subscheme $Z \subset Z_0 \subset X$, $Z \neq Z_0$ and every quasi-coherent sheaf of ideals $\mathcal{I} \subset \mathcal{O}_Z$ we have $\mathcal{P}$ for $(Z \to X)_*\mathcal{I}$.*
3. *There exists some coherent sheaf $\mathcal{G}$ on X such that*
   (a) *$Supp(\mathcal{G}) = Z_0$,*
   (b) *$\mathcal{G}_\xi$ is annihilated by $\mathfrak{m}_\xi$,*
   (c) *$\dim_{\kappa(\xi)} \mathcal{G}_\xi = 1$, and*
   (d) *property $\mathcal{P}$ holds for $\mathcal{G}$.*

Then property $\mathcal{P}$ holds for every coherent sheaf $\mathcal{F}$ on X whose support is contained in Z_0.

Proof. First note that if $\mathcal{F}$ is a coherent sheaf with a filtration

$$0 = \mathcal{F}_0 \subset \mathcal{F}_1 \subset \ldots \subset \mathcal{F}_m = \mathcal{F}$$

by coherent subsheaves such that each of $\mathcal{F}_i/\mathcal{F}_{i-1}$ has property $\mathcal{P}$, then so does $\mathcal{F}$. Or, if $\mathcal{F}$ has property $\mathcal{P}$ and all but one of the $\mathcal{F}_i/\mathcal{F}_{i-1}$ has property $\mathcal{P}$ then so does the last one. This follows from assumption (1).

As a first application we conclude that any coherent sheaf whose support is strictly contained in Z_0 has property $\mathcal{P}$. Namely, such a sheaf has a filtration (see Lemma 29.12.3) whose subquotients have property $\mathcal{P}$ according to (2).

Let $\mathcal{G}$ be as in (3). By Lemma 29.12.2 there exist a sheaf of ideals $\mathcal{I}$ on Z_0, an integer $r \geq 1$, and a short exact sequence

$$0 \to ((Z_0 \to X)_*\mathcal{I})^{\oplus r} \to \mathcal{G} \to \mathcal{Q} \to 0$$

where the support of $\mathcal{Q}$ is strictly contained in Z_0. By (3)(c) we see that $r = 1$. Since $\mathcal{Q}$ has property $\mathcal{P}$ too we conclude that $(Z_0 \to X)_*\mathcal{I}$ has property $\mathcal{P}$.

Next, suppose that $\mathcal{I}'$ is another quasi-coherent sheaf of ideals Z_0. Then we can consider the intersection $\mathcal{I}'' = \mathcal{I}' \cap \mathcal{I}$ and we get two short exact sequences

$$0 \to (Z_0 \to X)_*\mathcal{I}'' \to (Z_0 \to X)_*\mathcal{I} \to \mathcal{Q} \to 0$$

and

$$0 \to (Z_0 \to X)_*\mathcal{I}' \to (Z_0 \to X)_*\mathcal{I} \to \mathcal{Q}' \to 0.$$

Note that the support of the coherent sheaves $\mathcal{Q}$ and $\mathcal{Q}'$ are strictly contained in Z_0. Hence $\mathcal{Q}$ and $\mathcal{Q}$ have property $\mathcal{P}$ (see above). Hence we conclude using (1) that $(Z_0 \to X)_*\mathcal{I}''$ and $(Z_0 \to X)_*\mathcal{I}'$ both have $\mathcal{P}$ as well.

The final step of the proof is to note that any coherent sheaf $\mathcal{F}$ on X whose support is contained in Z_0 has a filtration (see Lemma 29.12.3 again) whose subquotients all have property $\mathcal{P}$ by what we just said. □

Lemma 29.12.6. *Let X be a Noetherian scheme. Let $\mathcal{P}$ be a property of coherent sheaves on X such that*

(1) *For any short exact sequence of coherent sheaves if two out of three of them have property $\mathcal{P}$ then so does the third.*
(2) *For every integral closed subscheme $Z \subset X$ with generic point ξ there exists some coherent sheaf $\mathcal{G}$ such that*
   (a) *$Supp(\mathcal{G}) = Z$,*
   (b) *$\mathcal{G}_\xi$ is annihilated by $\mathfrak{m}_\xi$,*
   (c) *$\dim_{\kappa(\xi)} \mathcal{G}_\xi = 1$, and*
   (d) *property $\mathcal{P}$ holds for $\mathcal{G}$.*

Then property $\mathcal{P}$ holds for every coherent sheaf on X.

Proof. According to Lemma 29.12.4 it suffices to show that given any integral closed subscheme $Z \subset X$ and every quasi-coherent sheaf of ideals $\mathcal{I} \subset \mathcal{O}_Z$ we have $\mathcal{P}$ for $(Z \to X)_*\mathcal{I}$. If this fails, then since X is Noetherian there is a minimal integral closed subscheme $Z_0 \subset X$ such that $\mathcal{P}$ fails for $(Z_0 \to X)_*\mathcal{I}$ for some quasi-coherent sheaf of ideals $\mathcal{I} \subset \mathcal{O}_{Z_0}$. In other words, the result does hold for any integral closed subscheme of Z. According to Lemma 29.12.5 this cannot happen. □

Lemma 29.12.7. *Let X be a Noetherian scheme. Let $Z_0 \subset X$ be an irreducible closed subset with generic point ξ. Let $\mathcal{P}$ be a property of coherent sheaves on X such that*

(1) *For any short exact sequence of coherent sheaves*
$$0 \to \mathcal{F}_1 \to \mathcal{F} \to \mathcal{F}_2 \to 0$$
   if $\mathcal{F}_i$, $i = 1, 2$ have property $\mathcal{P}$ then so does $\mathcal{F}$.
(2) *If $\mathcal{P}$ holds for a direct sum of coherent sheaves then it holds for both.*
(3) *For every integral closed subscheme $Z \subset Z_0 \subset X$, $Z \neq Z_0$ and every quasi-coherent sheaf of ideals $\mathcal{I} \subset \mathcal{O}_Z$ we have $\mathcal{P}$ for $(Z \to X)_*\mathcal{I}$.*
(4) *There exists some coherent sheaf $\mathcal{G}$ such that*
   (a) *$Supp(\mathcal{G}) = Z_0$,*
   (b) *$\mathcal{G}_\xi$ is annihilated by $\mathfrak{m}_\xi$, and*
   (c) *for every quasi-coherent sheaf of ideals $\mathcal{J} \subset \mathcal{O}_X$ such that $\mathcal{J}_\xi = \mathcal{O}_{X,\xi}$ there exists a quasi-coherent subsheaf $\mathcal{G}' \subset \mathcal{J}\mathcal{G}$ with $\mathcal{G}'_\xi = \mathcal{G}_\xi$ and such that $\mathcal{P}$ holds for $\mathcal{G}'$.*

Then property $\mathcal{P}$ holds for every coherent sheaf $\mathcal{F}$ on X whose support is contained in Z_0.

Proof. Note that if $\mathcal{F}$ is a coherent sheaf with a filtration
$$0 = \mathcal{F}_0 \subset \mathcal{F}_1 \subset \ldots \subset \mathcal{F}_m = \mathcal{F}$$
by coherent subsheaves such that each of $\mathcal{F}_i/\mathcal{F}_{i-1}$ has property $\mathcal{P}$, then so does $\mathcal{F}$. This follows from assumption (1).

As a first application we conclude that any coherent sheaf whose support is strictly contained in Z_0 has property $\mathcal{P}$. Namely, such a sheaf has a filtration (see Lemma 29.12.3) whose subquotients have property $\mathcal{P}$ according to (3).

Let us denote $i : Z_0 \to X$ the closed immersion. Consider a coherent sheaf $\mathcal{G}$ as in (4). By Lemma 29.12.2 there exists a sheaf of ideals $\mathcal{I}$ on Z_0 and a short exact

sequence

$$0 \to i_*\mathcal{I}^{\oplus r} \to \mathcal{G} \to \mathcal{Q} \to 0$$

where the support of $\mathcal{Q}$ is strictly contained in Z_0. In particular $r > 0$ and $\mathcal{I}$ is nonzero because the support of $\mathcal{G}$ is equal to Z_0. Let $\mathcal{I}' \subset \mathcal{I}$ be any nonzero quasi-coherent sheaf of ideals on Z contained in $\mathcal{I}$. Then we also get a short exact sequence

$$0 \to i_*(\mathcal{I}')^{\oplus r} \to \mathcal{G} \to \mathcal{Q}' \to 0$$

where $\mathcal{Q}'$ has support properly contained in Z_0. Let $\mathcal{J} \subset \mathcal{O}_X$ be a quasi-coherent sheaf of ideals cutting out the support of $\mathcal{Q}'$ (for example the ideal corresponding to the reduced induced closed subscheme structure on the support of $\mathcal{Q}'$). Then $\mathcal{J}_\xi = \mathcal{O}_{X,\xi}$. By Lemma 29.10.2 we see that $\mathcal{J}^n\mathcal{Q}' = 0$ for some n. Hence $\mathcal{J}^n\mathcal{G} \subset i_*(\mathcal{I}')^{\oplus r}$. By assumption (4)(c) of the lemma we see there exists a quasi-coherent subsheaf $\mathcal{G}' \subset \mathcal{J}^n\mathcal{G}$ with $\mathcal{G}'_\xi = \mathcal{G}_\xi$ for which property $\mathcal{P}$ holds. Hence we get a short exact sequence

$$0 \to \mathcal{G}' \to i_*(\mathcal{I}')^{\oplus r} \to \mathcal{Q}'' \to 0$$

where $\mathcal{Q}''$ has support properly contained in Z_0. Thus by our initial remarks and property (1) of the lemma we conclude that $i_*(\mathcal{I}')^{\oplus r}$ satisfies $\mathcal{P}$. Hence we see that $i_*\mathcal{I}'$ satisfies $\mathcal{P}$ by (2). Finally, for an arbitrary quasi-coherent sheaf of ideals $\mathcal{I}'' \subset \mathcal{O}_{Z_0}$ we can set $\mathcal{I}' = \mathcal{I}'' \cap \mathcal{I}$ and we get a short exact sequence

$$0 \to i_*(\mathcal{I}') \to i_*(\mathcal{I}'') \to \mathcal{Q}''' \to 0$$

where $\mathcal{Q}'''$ has support properly contained in Z_0. Hence we conclude that property $\mathcal{P}$ holds for $i_*\mathcal{I}''$ as well.

The final step of the proof is to note that any coherent sheaf $\mathcal{F}$ on X whose support is contained in Z_0 has a filtration (see Lemma 29.12.3 again) whose subquotients all have property $\mathcal{P}$ by what we just said. □

Lemma 29.12.8. *Let X be a Noetherian scheme. Let $\mathcal{P}$ be a property of coherent sheaves on X such that*

(1) *For any short exact sequence of coherent sheaves*

$$0 \to \mathcal{F}_1 \to \mathcal{F} \to \mathcal{F}_2 \to 0$$

if $\mathcal{F}_i$, $i = 1, 2$ have property $\mathcal{P}$ then so does $\mathcal{F}$.

(2) *If $\mathcal{P}$ holds for a direct sum of coherent sheaves then it holds for both.*

(3) *For every integral closed subscheme $Z \subset X$ with generic point ξ there exists some coherent sheaf $\mathcal{G}$ such that*
   (a) *$\mathrm{Supp}(\mathcal{G}) = Z$,*
   (b) *$\mathcal{G}_\xi$ is annihilated by $\mathfrak{m}_\xi$, and*
   (c) *for every quasi-coherent sheaf of ideals $\mathcal{J} \subset \mathcal{O}_X$ such that $\mathcal{J}_\xi = \mathcal{O}_{X,\xi}$ there exists a quasi-coherent subsheaf $\mathcal{G}' \subset \mathcal{J}\mathcal{G}$ with $\mathcal{G}'_\xi = \mathcal{G}_\xi$ and such that $\mathcal{P}$ holds for $\mathcal{G}'$.*

Then property $\mathcal{P}$ holds for every coherent sheaf on X.

Proof. Follows from Lemma 29.12.7 in exactly the same way that Lemma 29.12.6 follows from Lemma 29.12.5. □

29.13. Finite morphisms and affines

In this section we use the results of the preceding sections to show that the image of a Noetherian affine scheme under a finite morphism is affine. We will see later that this result holds more generally (see Limits, Lemma 31.10.1).

Lemma 29.13.1. *Let $f : Y \to X$ be a morphism of schemes. Assume f is finite, surjective and X locally Noetherian. Let $Z \subset X$ be an integral closed subscheme with generic point ξ. Then there exists a coherent sheaf $\mathcal{F}$ on Y such that the support of $f_*\mathcal{F}$ is equal to Z and $(f_*\mathcal{F})_\xi$ is annihilated by $\mathfrak{m}_\xi$.*

Proof. Note that Y is locally Noetherian by Morphisms, Lemma 28.16.6. Because f is surjective the fibre Y_ξ is not empty. Pick $\xi' \in Y$ mapping to ξ. Let $Z' = \overline{\{\xi'\}}$. We may think of $Z' \subset Y$ as a reduced closed subscheme, see Schemes, Lemma 25.12.4. Hence the sheaf $\mathcal{F} = (Z' \to Y)_*\mathcal{O}_{Z'}$ is a coherent sheaf on Y (see Lemma 29.9.9). Look at the commutative diagram

$$\begin{array}{ccc} Z' & \xrightarrow{i'} & Y \\ {\scriptstyle f'}\downarrow & & \downarrow{\scriptstyle f} \\ Z & \xrightarrow{i} & X \end{array}$$

We see that $f_*\mathcal{F} = i_*f'_*\mathcal{O}_{Z'}$. Hence the stalk of $f_*\mathcal{F}$ at ξ is the stalk of $f'_*\mathcal{O}_{Z'}$ at ξ. Note that since Z' is integral with generic point ξ' we have that ξ' is the only point of Z' lying over ξ, see Algebra, Lemmas 10.33.3 and 10.33.18. Hence the stalk of $f'_*\mathcal{O}_{Z'}$ at ξ equal $\mathcal{O}_{Z',\xi'} = \kappa(\xi')$. In particular the stalk of $f_*\mathcal{F}$ at ξ is not zero. This combined with the fact that $f_*\mathcal{F}$ is of the form $i_*f'_*$(something) implies the lemma. □

Lemma 29.13.2. *Let $f : Y \to X$ be a morphism of schemes. Let $\mathcal{F}$ be a quasi-coherent sheaf on Y. Let $\mathcal{I}$ be a quasi-coherent sheaf of ideals on X. If the morphism f is affine then $\mathcal{I}f_*\mathcal{F} = f_*(f^{-1}\mathcal{I}\mathcal{F})$.*

Proof. The notation means the following. Since f^{-1} is an exact functor we see that $f^{-1}\mathcal{I}$ is a sheaf of ideals of $f^{-1}\mathcal{O}_X$. Via the map $f^\sharp : f^{-1}\mathcal{O}_X \to \mathcal{O}_Y$ this acts on $\mathcal{F}$. Then $f^{-1}\mathcal{I}\mathcal{F}$ is the subsheaf generated by sums of local sections of the form as where a is a local section of $f^{-1}\mathcal{I}$ and s is a local section of $\mathcal{F}$. It is a quasi-coherent $\mathcal{O}_Y$-submodule of $\mathcal{F}$ because it is also the image of a natural map $f^*\mathcal{I} \otimes_{\mathcal{O}_Y} \mathcal{F} \to \mathcal{F}$.

Having said this the proof is straightforward. Namely, the question is local and hence we may assume X is affine. Since f is affine we see that Y is affine too. Thus we may write $Y = \mathrm{Spec}(B)$, $X = \mathrm{Spec}(A)$, $\mathcal{F} = \widetilde{M}$, and $\mathcal{I} = \widetilde{I}$. The assertion of the lemma in this case boils down to the statement that

$$I(M_A) = ((IB)M)_A$$

where M_A indicates the A-module associated to the B-module M. □

Lemma 29.13.3. *Let $f : Y \to X$ be a morphism of schemes. Assume*

(1) *f finite,*
(2) *f surjective,*
(3) *Y affine, and*
(4) *X Noetherian.*

Then X is affine.

Proof. We will prove that under the assumptions of the lemma for any coherent $\mathcal{O}_X$-module $\mathcal{F}$ we have $H^1(X, \mathcal{F}) = 0$. Since this will in particular imply that $H^1(X, \mathcal{I}) = 0$ for every quasi-coherent sheaf of ideals of $\mathcal{O}_X$. Then it will follows that X is affine from either Lemma 29.3.1 or Lemma 29.3.2.

Let $\mathcal{P}$ be the property of coherent sheaves $\mathcal{F}$ on X defined by the rule

$$\mathcal{P}(\mathcal{F}) \Leftrightarrow H^1(X, \mathcal{F}) = 0.$$

We are going to apply Lemma 29.12.8. Thus we have to verify (1), (2) and (3) of that lemma for $\mathcal{P}$. Property (1) follows from the long exact cohomology sequence associated to a short exact sequence of sheaves. Property (2) follows since $H^1(X, -)$ is an additive functor. To see (3) let $Z \subset X$ be an integral closed subscheme with generic point ξ. Let $\mathcal{F}$ be a coherent sheaf on Y such that the support of $f_*\mathcal{F}$ is equal to Z and $(f_*\mathcal{F})_\xi$ is annihilated by $\mathfrak{m}_\xi$, see Lemma 29.13.1. We claim that taking $\mathcal{G} = f_*\mathcal{F}$ works. We only have to verify part (3)(c) of Lemma 29.12.8. Hence assume that $\mathcal{J} \subset \mathcal{O}_X$ is a quasi-coherent sheaf of ideals such that $\mathcal{J}_\xi = \mathcal{O}_{X,\xi}$. A finite morphism is affine hence by Lemma 29.13.2 we see that $\mathcal{J}\mathcal{G} = f_*(f^{-1}\mathcal{J}\mathcal{F})$. Also, as pointed out in the proof of Lemma 29.13.2 the sheaf $f^{-1}\mathcal{J}\mathcal{F}$ is a quasi-coherent $\mathcal{O}_Y$-module. Since Y is affine we see that $H^1(Y, f^{-1}\mathcal{J}\mathcal{F}) = 0$, see Lemma 29.2.2. Since f is finite, hence affine, we see that

$$H^1(X, \mathcal{J}\mathcal{G}) = H^1(X, f_*(f^{-1}\mathcal{J}\mathcal{F})) = H^1(Y, f^{-1}\mathcal{J}\mathcal{F}) = 0$$

by Lemma 29.2.4. Hence the quasi-coherent subsheaf $\mathcal{G}' = \mathcal{J}\mathcal{G}$ satisfies $\mathcal{P}$. This verifies property (3)(c) of Lemma 29.12.8 as desired. □

29.14. Ample invertible sheaves and cohomology

Given a ringed space X, an invertible $\mathcal{O}_X$-module $\mathcal{L}$, a section $s \in \Gamma(X, \mathcal{L})$ and an $\mathcal{O}_X$-module $\mathcal{F}$ we get a map $\mathcal{F} \to \mathcal{F} \otimes_{\mathcal{O}_X} \mathcal{L}$, $t \mapsto t \otimes s$ which we call multiplication by s. We usually denote it $t \mapsto st$.

Lemma 29.14.1. *Let X be a scheme. Let $\mathcal{L}$ be an invertible $\mathcal{O}_X$-module. Let $s \in \Gamma(X, \mathcal{L})$ be a section. Let $\mathcal{F}' \subset \mathcal{F}$ be quasi-coherent $\mathcal{O}_X$-modules. Assume that*

1. *X is quasi-compact,*
2. *$\mathcal{F}$ is of finite type, and*
3. *$\mathcal{F}'|_{X_s} = \mathcal{F}|_{X_s}$.*

Then there exists an $n \geq 0$ such that multiplication by s^n on $\mathcal{F}$ factors through $\mathcal{F}'$.

Proof. In other words we claim that $s^n\mathcal{F} \subset \mathcal{F}' \otimes_{\mathcal{O}_X} \mathcal{L}^{\otimes n}$ for some $n \geq 0$. If this is true for n_0 then it is true for all $n \geq n_0$. Hence it suffices to show there is a finite open covering such that the result holds for each of the members of this open covering. Since X is quasi-compact we may therefore assume that X is affine and that $\mathcal{L} \cong \mathcal{O}_X$. Thus the lemma translates into the following algebra problem (use Properties, Lemma 27.16.1): Let A be a ring. Let $f \in A$. Let $M' \subset M$ be A-modules. Assume M is a finite A-module, and assume that $(M')_f = M_f$. Then there exists an $n \geq 0$ such that $f^n M \subset M'$. The proof of this is omitted. □

Let X be a scheme. Let $\mathcal{L}$ be an invertible $\mathcal{O}_X$-module. Let $s \in \Gamma(X, \mathcal{L})$ be a section. Assume X quasi-compact and quasi-separated. The following lemma

says roughly that the category of finitely presented $\mathcal{O}_{X_s}$-modules is the category of finitely presented $\mathcal{O}_X$-modules where the map multiplication by s has been inverted.

Lemma 29.14.2. *Let X be a scheme. Let $\mathcal{L}$ be an invertible $\mathcal{O}_X$-module. Let $s \in \Gamma(X, \mathcal{L})$ be a section. Let $\mathcal{F}$, $\mathcal{F}'$ be quasi-coherent $\mathcal{O}_X$-modules. Let $\psi : \mathcal{F}|_{X_s} \to \mathcal{F}'|_{X_s}$ be a map of $\mathcal{O}_{X_s}$-modules. Assume that*

(1) *X is quasi-compact and quasi-separated, and*
(2) *$\mathcal{F}$ is of finitely presented.*

Then there exists an $n \geq 0$ and a morphism $\alpha : \mathcal{F} \to \mathcal{F}' \otimes_{\mathcal{O}_X} \mathcal{L}^{\otimes n}$ whose restriction to X_s equals ψ via the identification $\mathcal{L}^{\otimes n}|_{X_s} = \mathcal{O}_{X_s}$ coming from s. Moreover, given a pair of solutions (n, α) and (n', α') there exists an $m \geq \max(n, n')$ such that $s^{m-n}\alpha = s^{m-n'}\alpha'$.

Proof. If the lemma holds for n_0 with map α_0 then it holds for all $n \geq n_0$ simply by taking $\alpha = s^{n-n_0}\alpha_0$. Choose a finite affine open covering $X = \bigcup U_i$ such that $\mathcal{L}|_{U_i}$ is trivial. Choose finite affine open coverings $U_i \cap U_{i'} = \bigcup U_{ii'j}$. Suppose we can prove the lemma when X is affine and $\mathcal{L}$ is trivial. Then we can find $n_i \geq 0$ $\alpha_i : \mathcal{F}|_{U_i} \to \mathcal{F}'|_{U_i} \otimes_{\mathcal{O}_{U_i}} \mathcal{L}^{\otimes n_i}|_{U_i}$ satisfying the relation over U_i. By the uniqueness assertion of the lemma, and the finiteness of the number of affines $U_{ii'j}$ we can find a single large integer m such that the maps $s^{m-n_i}\alpha_i$ and $s^{m-n_{i'}}\alpha_{i'}$ agree over $U_{ii'j}$ and hence over $U_i \cap U_{i'}$. Thus the morphisms $s^{m-n_i}\alpha_i$ glue to give our global map α. Proof of the uniqueness statement is omitted.

Assume X affine and that $\mathcal{L} \cong \mathcal{O}_X$. Then the lemma translates into the following algebra problem (use Properties, Lemma 27.16.2): Let A be a ring. Let $f \in A$. Let $\psi : M_f \to (M')_f$ be a map of A_f-modules. Assume M is a finitely presented A-module. Then there exists an $n \geq 0$ and an A-module map $\alpha : M \to M'$ such that $\alpha \otimes 1_{A_f} = f^n\psi$. Moreover, given any second solution (n', α') there exists an $m \geq \max(n, n')$ such that $f^{m-n}\alpha = f^{m-n'}\alpha'$. The proof of this algebraic fact is omitted. □

Cohomology is functorial. In particular, given a ringed space X, an invertible $\mathcal{O}_X$-module $\mathcal{L}$, a section $s \in \Gamma(X, \mathcal{L})$ we get maps

$$H^p(X, \mathcal{F}) \longrightarrow H^p(X, \mathcal{F} \otimes_{\mathcal{O}_X} \mathcal{L}), \quad \xi \longmapsto s\xi$$

induced by the map $\mathcal{F} \to \mathcal{F} \otimes_{\mathcal{O}_X} \mathcal{L}$ which is multiplication by s. We set $\Gamma_*(X, \mathcal{L}) = \bigoplus_{n \geq 0} \Gamma(X, \mathcal{L}^{\otimes n})$ as a graded ring, see Modules, Definition 17.21.4. Given a sheaf of $\mathcal{O}_X$-modules $\mathcal{F}$ and an integer $p \geq 0$ we set

$$H^p_*(X, \mathcal{L}, \mathcal{F}) = \bigoplus\nolimits_{n \in \mathbf{Z}} H^p(X, \mathcal{F} \otimes_{\mathcal{O}_X} \mathcal{L}^{\otimes n})$$

This is a graded $\Gamma_*(X, \mathcal{L})$-module by the multiplication defined above. Warning: the notation $H^p_*(X, \mathcal{L}, \mathcal{F})$ is nonstandard.

Lemma 29.14.3. *Let X be a scheme. Let $\mathcal{L}$ be an invertible sheaf on X. Let $s \in \Gamma(X, \mathcal{L})$. Let $\mathcal{F}$ be a quasi-coherent $\mathcal{O}_X$-module. If X is quasi-compact and quasi-separated, the canonical map*

$$H^p_*(X, \mathcal{L}, \mathcal{F})_{(s)} \longrightarrow H^p(X_s, \mathcal{F})$$

which maps ξ/s^n to $s^{-n}\xi$ is an isomorphism.

Proof. Note that for $p = 0$ this is Properties, Lemma 27.24.6. We will prove the statement using the induction principle (Lemma 29.4.1) where for $U \subset X$ quasi-compact open we let $P(U)$ be the property: for all $p \geq 0$ the map

$$H^p_*(U, \mathcal{L}, \mathcal{F})_{(s)} \longrightarrow H^p(U_s, \mathcal{F})$$

is an isomorphism.

If U is affine, then both sides of the arrow displayed above are zero for $p > 0$ by Lemma 29.2.2 and Properties, Lemma 27.24.4 and the statement is true. If P is true for U, V, and $U \cap V$, then we can use the Mayer-Vietoris sequences (Cohomology, Lemma 20.9.2) to obtain a map of long exact sequences

$$\begin{array}{ccccc} H^{p-1}_*(U \cap V, \mathcal{L}, \mathcal{F})_{(s)} & \longrightarrow & H^p_*(U \cup V, \mathcal{L}, \mathcal{F})_{(s)} & \longrightarrow & H^p_*(U, \mathcal{L}, \mathcal{F})_{(s)} \oplus H^p_*(V, \mathcal{L}, \mathcal{F})_{(s)} \\ \downarrow & & \downarrow & & \downarrow \\ H^{p-1}(U_s \cap V_s, \mathcal{F}) & \longrightarrow & H^p(U_s \cup V_s, \mathcal{F}) & \longrightarrow & H^p(U_s, \mathcal{F}) \oplus H^p(V_s, \mathcal{F}) \end{array}$$

(only a snippet shown). Observe that $U_s \cap V_s = (U \cap V)_s$ and that $U_s \cup V_s = (U \cup V)_s$. Thus the left and right vertical maps are isomorphisms (as well as one more to the right and one more to the left which are not shown in the diagram). We conclude that $P(U \cup V)$ holds by the 5-lemma (Homology, Lemma 12.5.20). This finishes the proof. □

Lemma 29.14.4. *Let X be a scheme. Let $\mathcal{L}$ be an invertible $\mathcal{O}_X$-module. Let $s \in \Gamma(X, \mathcal{L})$ be a section. Assume that*

(1) *X is quasi-compact and quasi-separated, and*
(2) *X_s is affine.*

Then for every quasi-coherent $\mathcal{O}_X$-module $\mathcal{F}$ and every $p > 0$ and all $\xi \in H^p(X, \mathcal{F})$ there exists an $n \geq 0$ such that $s^n \xi = 0$ in $H^p(X, \mathcal{F} \otimes_{\mathcal{O}_X} \mathcal{L}^{\otimes n})$.

Proof. Recall that $H^p(X_s, \mathcal{G})$ is zero for every quasi-coherent module $\mathcal{G}$ by Lemma 29.2.2. Hence the lemma follows from Lemma 29.14.3. □

For a more general version of the following lemma see Limits, Lemma 31.10.4.

Lemma 29.14.5. *Let $i : Z \to X$ be a closed immersion of Noetherian schemes inducing a homeomorphism of underlying topological spaces. Let $\mathcal{L}$ be an invertible sheaf on X. If $i^*\mathcal{L}$ is ample on Z, then $\mathcal{L}$ is ample on X.*

Proof. Let $\mathcal{I} \subset \mathcal{O}_X$ be the coherent sheaf of ideals cutting out the closed subscheme Z. Since $i(Z) = X$ set theoretically we see that $\mathcal{I}^n = 0$ for some n by Lemma 29.10.2. Consider the sequence

$$X = Z_n \supset Z_{n-1} \supset Z_{n-2} \supset \ldots \supset Z_1 = Z$$

of closed subschemes cut out by $0 = \mathcal{I}^n \subset \mathcal{I}^{n-1} \subset \ldots \subset \mathcal{I}$. Then each of the closed immersions $Z_i \to Z_{i-1}$ is defined by a coherent sheaf of ideals of square zero. In this way we reduce to the case that $\mathcal{I}^2 = 0$.

Consider the short exact sequence

$$0 \to \mathcal{I} \to \mathcal{O}_X \to i_*\mathcal{O}_Z \to 0$$

of quasi-coherent $\mathcal{O}_X$-modules. Tensoring with $\mathcal{L}^{\otimes n}$ we obtain short exact sequences

$$0 \to \mathcal{I} \otimes_{\mathcal{O}_X} \mathcal{L}^{\otimes n} \to \mathcal{L}^{\otimes n} \to i_*i^*\mathcal{L}^{\otimes n} \to 0$$

As $\mathcal{I}^2 = 0$, we can use Morphisms, Lemma 28.4.1 to think of $\mathcal{I}$ as a quasi-coherent $\mathcal{O}_Z$-module and then $\mathcal{I} \otimes_{\mathcal{O}_X} \mathcal{L}^{\otimes n} = \mathcal{I} \otimes_{\mathcal{O}_Z} i^*\mathcal{L}^{\otimes n}$ with obvious abuse of notation. Moreover, the cohomology of this sheaf over Z is canonically the same as the cohomology of this sheaf over X (as i is a homeomorphism).

Let $x \in X$ be a point and denote $z \in Z$ the corresponding point. Because $i^*\mathcal{L}$ is ample there exists an n and a section $s \in \Gamma(Z, i^*\mathcal{L}^{\otimes n})$ with $z \in Z_s$ and with Z_s affine. The obstruction to lifting s to a section of $\mathcal{L}^{\otimes n}$ over X is an element ξ of

$$H^1(X, \mathcal{I} \otimes_{\mathcal{O}_X} \mathcal{L}^{\otimes n}) = H^1(Z, \mathcal{I} \otimes_{\mathcal{O}_Z} i^*\mathcal{L}^{\otimes n})$$

If we replace s by s^{e+1} then ξ is replaced by $s^e\xi$ in $H^1(Z, \mathcal{I} \otimes_{\mathcal{O}_Z} i^*\mathcal{L}^{\otimes (e+1)n})$. Computation omitted. By Lemma 29.14.4 we see that $s^e\xi$ is zero for e large enough. Hence, after replacing s by a power, we can assume s is the image of a section $s' \in \Gamma(X, \mathcal{L}^{\otimes n})$. Then X_s is an open subscheme and $Z_s \to X_s$ is a surjective closed immersion of Noetherian schemes with Z_s affine. Hence X_s is affine by Lemma 29.13.3 and we conclude that $\mathcal{L}$ is ample. □

29.15. Coherent sheaves and projective morphisms

It seems illuminating to formulate an all-in-one result for projective space over a Noetherian ring.

Lemma 29.15.1. *Let R be a Noetherian ring. Let $n \geq 0$ be an integer. For every coherent sheaf $\mathcal{F}$ on $\mathbf{P}^n_R$ we have the following:*

1. *There exists an $r \geq 0$ and $d_1, \ldots, d_r \in \mathbf{Z}$ and a surjection*
$$\bigoplus\nolimits_{j=1,\ldots,r} \mathcal{O}_{\mathbf{P}^n_R}(d_j) \longrightarrow \mathcal{F}.$$
2. *We have $H^i(\mathbf{P}^n_R, \mathcal{F}) = 0$ unless $0 \leq i \leq n$.*
3. *For any i the cohomology group $H^i(\mathbf{P}^n_R, \mathcal{F})$ is a finite R-module.*
4. *If $i > 0$, then $H^i(\mathbf{P}^n_R, \mathcal{F}(d)) = 0$ for all d large enough.*
5. *For any $k \in \mathbf{Z}$ the graded $R[T_0, \ldots, T_n]$-module*
$$\bigoplus\nolimits_{d \geq k} H^0(\mathbf{P}^n_R, \mathcal{F}(d))$$
is a finite $R[T_0, \ldots, T_n]$-module.

Proof. We will use that $\mathcal{O}_{\mathbf{P}^n_R}(1)$ is an ample invertible sheaf on the scheme $\mathbf{P}^n_R$. This follows directly from the definition since $\mathbf{P}^n_R$ covered by the standard affine opens $D_+(T_i)$. Hence by Properties, Proposition 27.24.14 every finite type quasi-coherent $\mathcal{O}_{\mathbf{P}^n_R}$-module is a quotient of a finite direct sum of tensor powers of $\mathcal{O}_{\mathbf{P}^n_R}(1)$. On the other hand a coherent sheaves and finite type quasi-coherent sheaves are the same thing on projective space over R by Lemma 29.9.1. Thus we see (1).

Projective n-space $\mathbf{P}^n_R$ is covered by $n+1$ affines, namely the standard opens $D_+(T_i)$, $i = 0, \ldots, n$, see Constructions, Lemma 26.13.3. Hence we see that for any quasi-coherent sheaf $\mathcal{F}$ on $\mathbf{P}^n_R$ we have $H^i(\mathbf{P}^n_R, \mathcal{F}) = 0$ for $i \geq n+1$, see Lemma 29.4.2. Hence (2) holds.

Let us prove (3) and (4) simultaneously for all coherent sheaves on $\mathbf{P}^n_R$ by descending induction on i. Clearly the result holds for $i \geq n+1$ by (2). Suppose we know the result for $i+1$ and we want to show the result for i. (If $i = 0$, then part (4)

is vacuous.) Let $\mathcal{F}$ be a coherent sheaf on $\mathbf{P}^n_R$. Choose a surjection as in (1) and denote $\mathcal{G}$ the kernel so that we have a short exact sequence

$$0 \to \mathcal{G} \to \bigoplus\nolimits_{j=1,\ldots,r} \mathcal{O}_{\mathbf{P}^n_R}(d_j) \to \mathcal{F} \to 0$$

By Lemma 29.9.2 we see that $\mathcal{G}$ is coherent. The long exact cohomology sequence gives an exact sequence

$$H^i(\mathbf{P}^n_R, \bigoplus\nolimits_{j=1,\ldots,r} \mathcal{O}_{\mathbf{P}^n_R}(d_j)) \to H^i(\mathbf{P}^n_R, \mathcal{F}) \to H^{i+1}(\mathbf{P}^n_R, \mathcal{G}).$$

By induction assumption the right R-module is finite and by Lemma 29.8.1 the left R-module is finite. Since R is Noetherian it follows immediately that $H^i(\mathbf{P}^n_R, \mathcal{F})$ is a finite R-module. This proves the induction step for assertion (3). Since $\mathcal{O}_{\mathbf{P}^n_R}(d)$ is invertible we see that twisting on $\mathbf{P}^n_R$ is an exact functor (since you get it by tensoring with an invertible sheaf, see Constructions, Definition 26.10.1). This means that for all $d \in \mathbf{Z}$ the sequence

$$0 \to \mathcal{G}(d) \to \bigoplus\nolimits_{j=1,\ldots,r} \mathcal{O}_{\mathbf{P}^n_R}(d_j + d) \to \mathcal{F}(d) \to 0$$

is short exact. The resulting cohomology sequence is

$$H^i(\mathbf{P}^n_R, \bigoplus\nolimits_{j=1,\ldots,r} \mathcal{O}_{\mathbf{P}^n_R}(d_j + d)) \to H^i(\mathbf{P}^n_R, \mathcal{F}(d)) \to H^{i+1}(\mathbf{P}^n_R, \mathcal{G}(d)).$$

By induction assumption we see the module on the right is zero for $d \gg 0$ and by the computation in Lemma 29.8.1 the module on the left is zero as soon as $d \geq -\min\{d_j\}$ and $i \geq 1$. Hence the induction step for assertion (4). This concludes the proof of (3) and (4).

In order to prove (5) note that for all sufficiently large d the map

$$H^0(\mathbf{P}^n_R, \bigoplus\nolimits_{j=1,\ldots,r} \mathcal{O}_{\mathbf{P}^n_R}(d_j + d)) \to H^0(\mathbf{P}^n_R, \mathcal{F}(d))$$

is surjective by the vanishing of $H^1(\mathbf{P}^n_R, \mathcal{G}(d))$ we just proved. In other words, the module

$$M_k = \bigoplus\nolimits_{d \geq k} H^0(\mathbf{P}^n_R, \mathcal{F}(d))$$

is for k large enough a quotient of the corresponding module

$$N_k = \bigoplus\nolimits_{d \geq k} H^0(\mathbf{P}^n_R, \bigoplus\nolimits_{j=1,\ldots,r} \mathcal{O}_{\mathbf{P}^n_R}(d_j + d))$$

When k is sufficiently small (e.g. $k < -d_j$ for all j) then

$$N_k = \bigoplus\nolimits_{j=1,\ldots,r} R[T_0, \ldots, T_n](d_j)$$

by our computations in Section 29.8. In particular it is finitely generated. Suppose $k \in \mathbf{Z}$ is arbitrary. Choose $k_- \ll k \ll k_+$. Consider the diagram

$$\begin{array}{ccc} N_{k_-} & \longleftarrow & N_{k_+} \\ & & \downarrow \\ M_k & \longleftarrow & M_{k_+} \end{array}$$

where the vertical arrow is the surjective map above and the horizontal arrows are the obvious inclusion maps. By what was said above we see that N_{k_-} is a finitely generated $R[T_0, \ldots, T_n]$-module. Hence N_{k_+} is a finitely generated $R[T_0, \ldots, T_n]$-module because it is a submodule of a finitely generated module and the ring

$R[T_0, \ldots, T_n]$ is Noetherian. Since the vertical arrow is surjective we conclude that M_{k_+} is a finitely generated $R[T_0, \ldots, T_n]$-module. The quotient M_k/M_{k_+} is finite as an R-module since it is a finite direct sum of the finite R-modules $H^0(\mathbf{P}^n_R, \mathcal{F}(d))$ for $k \leq d < k_+$. Note that we use part (3) for $i = 0$ here. Hence M_k/M_{k_+} is a fortiori a finite $R[T_0, \ldots, T_n]$-module. In other words, we have sandwiched M_k between two finite $R[T_0, \ldots, T_n]$-modules and we win. □

Lemma 29.15.2. *Let $f : X \to S$ be a morphism of schemes. Let $\mathcal{F}$ be a quasi-coherent $\mathcal{O}_X$-module. Let $\mathcal{L}$ be an invertible sheaf on X. Assume that*

(1) *S is Noetherian,*
(2) *f is proper,*
(3) *$\mathcal{F}$ is coherent, and*
(4) *$\mathcal{L}$ is relatively ample on X/S.*

Then there exists an n_0 such that for all $n \geq n_0$ we have

$$R^p f_* \left(\mathcal{F} \otimes_{\mathcal{O}_X} \mathcal{L}^{\otimes n} \right) = 0$$

for all $p > 0$.

Proof. A proper morphism is of finite type by definition. By Morphisms, Lemma 28.40.7 there exists an open covering $S = \bigcup V_j$ and immersions $i_j : X_j \to \mathbf{P}^{n_j}_{V_j}$, where $X_j = f^{-1}(V_j)$ such that $i_j^* \mathcal{O}(1)$ is a power of $\mathcal{L}$. Since S is quasi-compact we may assume the covering is finite. Clearly, if we solve the question for each of the finitely many systems $(X_j \to V_j, \mathcal{L}|_{X_j}, \mathcal{F}|_{V_j})$ then the result follows. Hence we may assume there exists an immersion $i : X \to \mathbf{P}^n_S$ such that $\mathcal{L}^{\otimes d} = i^* \mathcal{O}(1)$ for some $d \geq 1$.

Repeating the argument above with a finite affine open covering of S we see that we may also assume that S is affine. In this case the vanishing of $R^p f_*(\mathcal{F} \otimes \mathcal{L}^{\otimes n})$ is equivalent to the vanishing of $H^p(X, \mathcal{F} \otimes \mathcal{L}^{\otimes n})$, see Lemma 29.4.5.

Since f is proper we see that i is a closed immersion (Morphisms, Lemma 28.42.7). Hence we see that $R^p i_*(\mathcal{F} \otimes_{\mathcal{O}_X} \mathcal{L}^{\otimes n}) = 0$ for all $p \geq 1$ (see Lemma 29.9.9 for example). This implies that

$$H^p(X, \mathcal{F} \otimes \mathcal{L}^{\otimes n}) = H^p(\mathbf{P}^n_S, i_*(\mathcal{F} \otimes \mathcal{L}^{\otimes n}))$$

by the Leray spectral sequence (Cohomology, Lemma 20.14.4). Moreover, by the projection formula (Cohomology, Lemma 20.8.2) we have

$$i_*(\mathcal{F} \otimes_{\mathcal{O}_X} \mathcal{L}^{\otimes n}) = i_*(\mathcal{F} \otimes_{\mathcal{O}_X} \mathcal{L}^{\otimes \langle n \rangle_d}) \otimes_{\mathcal{O}_{\mathbf{P}^n_S}} \mathcal{O}(\lfloor n/d \rfloor)$$

for all $n \in \mathbf{Z}$ where $\langle n \rangle_d \in \{0, 1, \ldots, d-1\}$ is the unique element congruent to n module d. The sheaves $\mathcal{F}_j = i_*(\mathcal{F} \otimes \mathcal{L}^{\otimes j})$, $j \in \{0, 1, \ldots, d-1\}$ are coherent by Lemma 29.9.9. Thus we see that for all n large enough the cohomology groups $H^p(\mathbf{P}^n_S, \mathcal{F}_j(n))$ vanish by Lemma 29.15.1. Putting everything together this implies the lemma. □

29.16. Chow's Lemma

In this section we prove Chow's lemma in the Noetherian case (Lemma 29.16.1). In Limits, Section 31.11 we prove some variants for the non-Noetherian case.

Lemma 29.16.1. *Let S be a Noetherian scheme. Let $f : X \to S$ be a separated morphism of finite type. Then there exists an $n \geq 0$ and a diagram*

$$\begin{array}{ccccc} X & \xleftarrow{\pi} & X' & \longrightarrow & \mathbf{P}^n_S \\ & \searrow & \downarrow & \swarrow & \\ & & S & & \end{array}$$

where $X' \to \mathbf{P}^n_S$ is an immersion, and $\pi : X' \to X$ is proper and surjective. Moreover, we may arrange it such that there exists a dense open subscheme $U \subset X$ such that $\pi^{-1}(U) \to U$ is an isomorphism.

Proof. All of the schemes we will encounter during the rest of the proof are going to be of finite type over the Noetherian scheme S and hence Noetherian (see Morphisms, Lemma 28.16.6). All morphisms between them will automatically be quasi-compact, locally of finite type and quasi-separated, see Morphisms, Lemma 28.16.8 and Properties, Lemmas 27.5.4 and 27.5.6.

The underlying topological space of X is Noetherian (see Properties, Lemma 27.5.5) and we conclude that X has only finitely many irreducible components (see Topology, Lemma 5.8.2). Say $X = X_1 \cup \ldots \cup X_r$ is the decomposition of X into irreducible components. Let $\eta_i \in X_i$ be the generic point. For every point $x \in X$ there exists an affine open $U_x \subset X$ which contains x and each of the generic points η_i. See Properties, Lemma 27.27.4. Since X is quasi-compact, we can find a finite affine open covering $X = U_1 \cup \ldots \cup U_m$ such that each U_i contains $\eta_1, \ldots, \eta_r$. In particular we conclude that the open $U = U_1 \cap \ldots \cap U_m \subset X$ is a dense open. This and the fact that the U_i are affine opens covering X is all that we will use below.

Let $X^* \subset X$ be the scheme theoretic closure of $U \to X$, see Morphisms, Definition 28.6.2. Let $U_i^* = X^* \cap U_i$. Note that U_i^* is a closed subscheme of U_i. Hence U_i^* is affine. Since U is dense in X the morphism $X^* \to X$ is a surjective closed immersion. It is an isomorphism over U. Hence we may replace X by X^* and U_i by U_i^* and assume that U is scheme theoretically dense in X, see Morphisms, Definition 28.7.1.

By Morphisms, Lemma 28.40.3 we can find an immersion $j_i : U_i \to \mathbf{P}^{n_i}_S$ for each i. By Morphisms, Lemma 28.7.7 we can find closed subschemes $Z_i \subset \mathbf{P}^{n_i}_S$ such that $j_i : U_i \to Z_i$ is a scheme theoretically dense open immersion. Note that $Z_i \to S$ is proper, see Morphisms, Lemma 28.43.5. Consider the morphism

$$j = (j_1|_U, \ldots, j_n|_U) : U \longrightarrow \mathbf{P}^{n_1}_S \times_S \ldots \times_S \mathbf{P}^{n_n}_S.$$

By the lemma cited above we can find a closed subscheme Z of $\mathbf{P}^{n_1}_S \times_S \ldots \times_S \mathbf{P}^{n_n}_S$ such that $j : U \to Z$ is an open immersion and such that U is scheme theoretically dense in Z. The morphism $Z \to S$ is proper. Consider the ith projection

$$\mathrm{pr}_i|_Z : Z \longrightarrow \mathbf{P}^{n_i}_S.$$

This morphism factors through Z_i (see Morphisms, Lemma 28.6.6). Denote $p_i : Z \to Z_i$ the induced morphism. This is a proper morphism, see Morphisms, Lemma 28.42.7 for example. At this point we have that $U \subset U_i \subset Z_i$ are scheme theoretically dense open immersions. Moreover, we can think of Z as the scheme theoretic image of the "diagonal" morphism $U \to Z_1 \times_S \ldots \times_S Z_n$.

Set $V_i = p_i^{-1}(U_i)$. Note that $p_i|_{V_i} : V_i \to U_i$ is proper. Set $X' = V_1 \cup \ldots \cup V_n$. By construction X' has an immersion into the scheme $\mathbf{P}_S^{n_1} \times_S \ldots \times_S \mathbf{P}_S^{n_n}$. Thus by the Segre embedding (see Morphisms, Lemma 28.43.6) we see that X' has an immersion into a projective space over S.

We claim that the morphisms $p_i|_{V_i} : V_i \to U_i$ glue to a morphism $X' \to X$. Namely, it is clear that $p_i|_U$ is the identity map from U to U. Since $U \subset X'$ is scheme theoretically dense by construction, it is also scheme theoretically dense in the open subscheme $V_i \cap V_j$. Thus we see that $p_i|_{V_i \cap V_j} = p_j|_{V_i \cap V_j}$ as morphisms into the separated S-scheme X, see Morphisms, Lemma 28.7.10. We denote the resulting morphism $\pi : X' \to X$.

We claim that $\pi^{-1}(U_i) = V_i$. Since $\pi|_{V_i} = p_i|_{V_i}$ it follows that $V_i \subset \pi^{-1}(U_i)$. Consider the diagram

$$\begin{array}{ccc} V_i & \longrightarrow & \pi^{-1}(U_i) \\ & \searrow^{p_i|_{V_i}} & \downarrow \\ & & U_i \end{array}$$

Since $V_i \to U_i$ is proper we see that the image of the horizontal arrow is closed, see Morphisms, Lemma 28.42.7. Since $V_i \subset \pi^{-1}(U_i)$ is scheme theoretically dense (as it contains U) we conclude that $V_i = \pi^{-1}(U_i)$ as claimed.

This shows that $\pi^{-1}(U_i) \to U_i$ is identified with the proper morphism $p_i|_{V_i} : V_i \to U_i$. Hence we see that X has a finite affine covering $X = \bigcup U_i$ such that the restriction of π is proper on each member of the covering. Thus by Morphisms, Lemma 28.42.3 we see that π is proper.

Finally we have to show that $\pi^{-1}(U) = U$. To see this we argue in the same way as above using the diagram

$$\begin{array}{ccc} U & \longrightarrow & \pi^{-1}(U) \\ & \searrow & \downarrow \\ & & U \end{array}$$

and using that $\mathrm{id}_U : U \to U$ is proper and that U is scheme theoretically dense in $\pi^{-1}(U)$. □

Remark 29.16.2. In the situation of Chow's Lemma 29.16.1:

(1) The morphism π is actually H-projective (hence projective, see Morphisms, Lemma 28.43.3) since the morphism $X' \to \mathbf{P}_S^n \times_S X = \mathbf{P}_X^n$ is a closed immersion (use the fact that π is proper, see Morphisms, Lemma 28.42.7).
(2) We may assume that $\pi^{-1}(U)$ is scheme theoretically dense in X'. Namely, we can simply replace X' by the scheme theoretic closure of $\pi^{-1}(U)$. In this case we can think of U as a scheme theoretically dense open subscheme of X'. See Morphisms, Section 28.6.
(3) If X is reduced then we may choose X' reduced. This is clear from (2).

29.17. Higher direct images of coherent sheaves

In this section we prove the fundamental fact that the higher direct images of a coherent sheaf under a proper morphism are coherent.

Lemma 29.17.1. *Let S be a locally Noetherian scheme. Let $f : X \to S$ be a locally projective morphism. Let $\mathcal{F}$ be a coherent $\mathcal{O}_X$-module. Then $R^i f_*\mathcal{F}$ is a coherent $\mathcal{O}_S$-module for all $i \geq 0$.*

Proof. We first remark that a locally projective morphism is proper (Morphisms, Lemma 28.43.5) and hence of finite type. In particular X is locally Noetherian (Morphisms, Lemma 28.16.6) and hence the statement makes sense. Moreover, by Lemma 29.4.4 the sheaves $R^p f_*\mathcal{F}$ are quasi-coherent.

Having said this the statement is local on S (for example by Cohomology, Lemma 20.7.4). Hence we may assume $S = \mathrm{Spec}(R)$ is the spectrum of a Noetherian ring, and X is a closed subscheme of $\mathbf{P}^n_R$ for some n, see Morphisms, Lemma 28.43.4. In this case, the sheaves $R^p f_*\mathcal{F}$ are the quasi-coherent sheaves associated to the R-modules $H^p(X, \mathcal{F})$, see Lemma 29.4.5. Hence it suffices to show that R-modules $H^p(X, \mathcal{F})$ are finite R-modules (Lemma 29.9.1). Denote $i : X \to \mathbf{P}^n_R$ the closed immersion. Note that $R^p i_*\mathcal{F} = 0$ by Lemma 29.9.9. Hence the Leray spectral sequence (Cohomology, Lemma 20.14.4) for $i : X \to \mathbf{P}^n_R$ degenerates, and we see that $H^p(X, \mathcal{F}) = H^p(\mathbf{P}^n_R, i_*\mathcal{F})$. Since the sheaf $i_*\mathcal{F}$ is coherent by Lemma 29.9.9 we see that the lemma follows from Lemma 29.15.1. □

Here is the general statement.

Lemma 29.17.2. *Let S be a locally Noetherian scheme. Let $f : X \to S$ be a proper morphism. Let $\mathcal{F}$ be a coherent $\mathcal{O}_X$-module. Then $R^i f_*\mathcal{F}$ is a coherent $\mathcal{O}_S$-module for all $i \geq 0$.*

Proof. Since the problem is local on S we may assume that S is a Noetherian scheme. Since a proper morphism is of finite type we see that in this case X is a Noetherian scheme also. Consider the property $\mathcal{P}$ of coherent sheaves on X defined by the rule

$$\mathcal{P}(\mathcal{F}) \Leftrightarrow R^p f_*\mathcal{F} \text{ is coherent for all } p \geq 0$$

We are going to use the result of Lemma 29.12.6 to prove that $\mathcal{P}$ holds for every coherent sheaf on X.

Let

$$0 \to \mathcal{F}_1 \to \mathcal{F}_2 \to \mathcal{F}_3 \to 0$$

be a short exact sequence of coherent sheaves on X. Consider the long exact sequence of higher direct images

$$R^{p-1} f_*\mathcal{F}_3 \to R^p f_*\mathcal{F}_1 \to R^p f_*\mathcal{F}_2 \to R^p f_*\mathcal{F}_3 \to R^{p+1} f_*\mathcal{F}_1$$

Then it is clear that if 2-out-of-3 of the sheaves $\mathcal{F}_i$ have property $\mathcal{P}$, then the higher direct images of the third are sandwiched in this exact complex between two coherent sheaves. Hence these higher direct images are also coherent by Lemma 29.9.2 and 29.9.3. Hence property $\mathcal{P}$ holds for the third as well.

Let $Z \subset X$ be an integral closed subscheme. We have to find a coherent sheaf $\mathcal{F}$ on X whose support is contained in Z, whose stalk at the generic point ξ of Z is a 1-dimensional vector space over $\kappa(\xi)$ such that $\mathcal{P}$ holds for $\mathcal{F}$. Denote

$g = f|_Z : Z \to S$ the restriction of f. Suppose we can find a coherent sheaf $\mathcal{G}$ on Z such that (a) $\mathcal{G}_\xi$ is a 1-dimensional vector space over $\kappa(\xi)$, (b) $R^p g_*\mathcal{G} = 0$ for $p > 0$, and (c) $g_*\mathcal{G}$ is coherent. Then we can consider $\mathcal{F} = (Z \to X)_*\mathcal{G}$. As $Z \to X$ is a closed immersion we see that $(Z \to X)_*\mathcal{G}$ is coherent on X and $R^p(Z \to X)_*\mathcal{G} = 0$ for $p > 0$ (Lemma 29.9.9). Hence by the relative Leray spectral sequence (Cohomology, Lemma 20.14.8) we will have $R^p f_*\mathcal{F} = R^p g_*\mathcal{G} = 0$ for $p > 0$ and $f_*\mathcal{F} = g_*\mathcal{G}$ is coherent. Finally $\mathcal{F}_\xi = ((Z \to X)_*\mathcal{G})_\xi = \mathcal{G}_\xi$ which verifies the condition on the stalk at ξ. Hence everything depends on finding a coherent sheaf $\mathcal{G}$ on Z which has properties (a), (b), and (c).

We can apply Chow's Lemma 29.16.1 to the morphism $Z \to S$. Thus we get a diagram

$$\begin{array}{ccccc} Z & \xleftarrow{\pi} & Z' & \xrightarrow{i} & \mathbf{P}^n_S \\ & {\scriptstyle g}\searrow & \downarrow{\scriptstyle g'} & \swarrow & \\ & & S & & \end{array}$$

as in the statement of Chow's lemma. Also, let $U \subset Z$ be the dense open subscheme such that $\pi^{-1}(U) \to U$ is an isomorphism. By the discussion in Remark 29.16.2 we see that $i' = (i, \pi) : \mathbf{P}^n_S \times_S Z' = \mathbf{P}^n_Z$ is a closed immersion. Hence

$$\mathcal{L} = i^*\mathcal{O}_{\mathbf{P}^n_X}(1) \cong (i')^*\mathcal{O}_{\mathbf{P}^n_Z}(1)$$

is g'-relatively ample and π-relatively ample (for example by Morphisms, Lemma 28.40.7). Hence by Lemma 29.15.2 there exists an $n \geq 0$ such that both $R^p\pi_*\mathcal{L}^{\otimes n} = 0$ for all $p > 0$ and $R^p(g')_*\mathcal{L}^{\otimes n} = 0$ for all $p > 0$. Set $\mathcal{G} = \pi_*\mathcal{L}^{\otimes n}$. Property (a) holds because $\pi_*\mathcal{L}^{\otimes}|_U$ is an invertible sheaf (as $\pi^{-1}(U) \to U$ is an isomorphism). Properties (b) and (c) hold because by the relative Leray spectral sequence (Cohomology, Lemma 20.14.8) we have

$$E_2^{p,q} = R^p g_* R^q \pi_* \mathcal{L}^{\otimes n} \Rightarrow R^{p+q}(g')_*\mathcal{L}^{\otimes n}$$

and by choice of n the only nonzero terms in $E_2^{p,q}$ are those with $q = 0$ and the only nonzero terms of $R^{p+q}(g')_*\mathcal{L}^{\otimes n}$ are those with $p = q = 0$. This implies that $R^p g_*\mathcal{G} = 0$ for $p > 0$ and that $g_*\mathcal{G} = (g')_*\mathcal{L}^{\otimes n}$. Finally, applying the previous Lemma 29.17.1 we see that $g_*\mathcal{G} = (g')_*\mathcal{L}^{\otimes n}$ is coherent as desired. □

Remark 29.17.3. Let S be a locally Noetherian scheme. Let $f : X \to S$ be a morphism of schemes which is locally of finite type. Then X is locally Noetherian (Morphisms, Lemma 28.16.6). Let $\mathcal{F}$ be a coherent $\mathcal{O}_X$-module. Assume the scheme theoretic support Z of $\mathcal{F}$ is proper over S. we claim $R^p f_*\mathcal{F}$ is a coherent $\mathcal{O}_S$-module for all $p \geq 0$. Namely, Let $i : Z \to X$ be the closed immersion and write $\mathcal{F} = i_*\mathcal{G}$ for some coherent module $\mathcal{G}$ on Z (Lemma 29.9.7). Denoting $g : Z \to S$ the composition $f \circ i$ we see that $R^p g_*\mathcal{G}$ is coherent on S by Lemma 29.17.2. On the other hand, $R^q i_*\mathcal{G} = 0$ for $q > 0$ (Lemma 29.9.9). By Cohomology, Lemma 20.14.8 we get $R^p f_*\mathcal{F} = R^p g_*\mathcal{G}$ and the claim.

Lemma 29.17.4. *Let $S = \mathrm{Spec}(A)$ with A a Noetherian ring. Let $f : X \to S$ be a proper morphism. Let $\mathcal{F}$ be a coherent $\mathcal{O}_X$-module. Then $H^i(X, \mathcal{F})$ is finite A-module for all $i \geq 0$.*

Proof. This is just the affine case of Lemma 29.17.2. Namely, by Lemmas 29.4.4 and 29.4.5 we know that $R^i f_* \mathcal{F}$ is the quasi-coherent sheaf associated to the A-module $H^i(X, \mathcal{F})$ and by Lemma 29.9.1 this is a coherent sheaf if and only if $H^i(X, \mathcal{F})$ is an A-module of finite type. □

Lemma 29.17.5. *Let A be a Noetherian ring. Let B be a finitely generated graded A-algebra. Let $f : X \to \mathrm{Spec}(A)$ be a proper morphism. Set $\mathcal{B} = f^* \widetilde{B}$. Let $\mathcal{F}$ be a quasi-coherent graded $\mathcal{B}$-module of finite type.*

1. *For every $p \geq 0$ the graded B-module $H^p(X, \mathcal{F})$ is a finite B-module.*
2. *If $\mathcal{L}$ is an ample invertible $\mathcal{O}_X$-module, then there exists an integer d_0 such that $H^p(X, \mathcal{F} \otimes \mathcal{L}^{\otimes d}) = 0$ for all $p > 0$ and $d \geq d_0$.*

Proof. To prove this we consider the fibre product diagram

$$\begin{array}{ccc} X' = \mathrm{Spec}(B) \times_{\mathrm{Spec}(A)} X & \xrightarrow{\pi} & X \\ \downarrow{\scriptstyle f'} & & \downarrow{\scriptstyle f} \\ \mathrm{Spec}(B) & \longrightarrow & \mathrm{Spec}(A) \end{array}$$

Note that f' is a proper morphism, see Morphisms, Lemma 28.42.5. Also, B is a finitely generated A-algebra, and hence Noetherian (Algebra, Lemma 10.29.1). This implies that X' is a Noetherian scheme (Morphisms, Lemma 28.16.6). Note that X' is the relative spectrum of the quasi-coherent $\mathcal{O}_X$-algebra $\mathcal{B}$ by Constructions, Lemma 26.4.6. Since $\mathcal{F}$ is a quasi-coherent $\mathcal{B}$-module we see that there is a unique quasi-coherent $\mathcal{O}_{X'}$-module $\mathcal{F}'$ such that $\pi_* \mathcal{F}' = \mathcal{F}$, see Morphisms, Lemma 28.13.6 Since $\mathcal{F}$ is finite type as a $\mathcal{B}$-module we conclude that $\mathcal{F}'$ is a finite type $\mathcal{O}_{X'}$-module (details omitted). In other words, $\mathcal{F}'$ is a coherent $\mathcal{O}_{X'}$-module (Lemma 29.9.1). Since the morphism $\pi : X' \to X$ is affine we have

$$H^p(X, \mathcal{F}) = H^p(X', \mathcal{F}')$$

by Lemma 29.2.4. Thus (1) follows from Lemma 29.17.4. Given $\mathcal{L}$ as in (2) we set $\mathcal{L}' = \pi^* \mathcal{L}$. Note that $\mathcal{L}'$ is ample on X' by Morphisms, Lemma 28.38.7. By the projection formula (Cohomology, Lemma 20.8.2) we have $\pi_*(\mathcal{F}' \otimes \mathcal{L}') = \mathcal{F} \otimes \mathcal{L}$. Thus part (2) follows by the same reasoning as above from Lemma 29.15.2. □

29.18. The theorem on formal functions

In this section we study the behaviour of cohomology of sequences of sheaves either of the form $\{I^n \mathcal{F}\}_{n \geq 0}$ or of the form $\{\mathcal{F}/I^n \mathcal{F}\}_{n \geq 0}$ as n varies.

Here and below we use the following notation. Given a morphism of schemes $f : X \to Y$, a quasi-coherent sheaf $\mathcal{F}$ on X, and a quasi-coherent sheaf of ideals $\mathcal{I} \subset \mathcal{O}_Y$ we denote $\mathcal{I}^n \mathcal{F}$ the quasi-coherent subsheaf generated by products of local sections of $f^{-1}(\mathcal{I}^n)$ and $\mathcal{F}$. In a formula

$$\mathcal{I}^n \mathcal{F} = \mathrm{Im}\left(f^*(\mathcal{I}^n) \otimes_{\mathcal{O}_X} \mathcal{F} \longrightarrow \mathcal{F}\right).$$

Note that there are natural maps

$$f^{-1}(\mathcal{I}^n) \otimes_{f^{-1}\mathcal{O}_Y} \mathcal{I}^m \mathcal{F} \longrightarrow f^*(\mathcal{I}^n) \otimes_{\mathcal{O}_X} \mathcal{I}^m \mathcal{F} \longrightarrow \mathcal{I}^{n+m} \mathcal{F}$$

Hence a section of $\mathcal{I}^n$ will give rise to a map $R^p f_*(\mathcal{I}^m \mathcal{F}) \to R^p f_*(\mathcal{I}^{n+m} \mathcal{F})$ by functoriality of higher direct images. Localizing and then sheafifying we see that

there are $\mathcal{O}_Y$-module maps

$$\mathcal{I}^n \otimes_{\mathcal{O}_Y} R^p f_*(\mathcal{I}^m \mathcal{F}) \longrightarrow R^p f_*(\mathcal{I}^{n+m}\mathcal{F}).$$

In other words we see that $\bigoplus_{n\geq 0} R^p f_*(\mathcal{I}^n\mathcal{F})$ is a graded $\bigoplus_{n\geq 0}\mathcal{I}^n$-module.

If $Y = \operatorname{Spec}(A)$ and $\mathcal{I} = \widetilde{I}$ we denote $\mathcal{I}^n\mathcal{F}$ simply $I^n\mathcal{F}$. The maps introduced above give $M = \bigoplus H^p(X, I^n\mathcal{F})$ the structure of a graded $S = \bigoplus I^n$-module. If f is proper, A is Noetherian and $\mathcal{F}$ is coherent, then this turns out to be a module of finite type.

Lemma 29.18.1. *Let A be a Noetherian ring. Let $I \subset A$ be an ideal. Set $B = \bigoplus_{n\geq 0} I^n$. Let $f : X \to \operatorname{Spec}(A)$ be a proper morphism. Let $\mathcal{F}$ be a coherent sheaf on X. Then for every $p \geq 0$ the graded B-module $\bigoplus_{n\geq 0} H^p(X, I^n\mathcal{F})$ is a finite B-module.*

Proof. Let $\mathcal{B} = \bigoplus I^n\mathcal{O}_X = f^*\widetilde{B}$. Then $\bigoplus I^n\mathcal{F}$ is a finite type graded $\mathcal{B}$-module. Hence the result follows from Lemma 29.17.5 part (1). □

Lemma 29.18.2. *Given a morphism of schemes $f : X \to Y$, a quasi-coherent sheaf $\mathcal{F}$ on X, and a quasi-coherent sheaf of ideals $\mathcal{I} \subset \mathcal{O}_Y$. Assume Y locally Noetherian, f proper, and $\mathcal{F}$ coherent. Then*

$$\mathcal{M} = \bigoplus\nolimits_{n\geq 0} R^p f_*(\mathcal{I}^n\mathcal{F})$$

is a graded $\mathcal{A} = \bigoplus_{n\geq 0}\mathcal{I}^n$-module which is quasi-coherent and of finite type.

Proof. The statement is local on Y, hence this reduces to the case where Y is affine. In the affine case the result follows from Lemma 29.18.1. Details omitted. □

Lemma 29.18.3. *Let A be a Noetherian ring. Let $I \subset A$ be an ideal. Let $f : X \to \operatorname{Spec}(A)$ be a proper morphism. Let $\mathcal{F}$ be a coherent sheaf on X. Then for every $p \geq 0$ there exists an integer $c \geq 0$ such that*

1. *the multiplication map $I^{n-c} \otimes H^p(X, I^c\mathcal{F}) \to H^p(X, I^n\mathcal{F})$ is surjective for all $n \geq c$, and*
2. *the image of $H^p(X, I^{n+m}\mathcal{F}) \to H^p(X, I^n\mathcal{F})$ is contained in the submodule $I^{m-c}H^p(X, I^n\mathcal{F})$ for all $n \geq 0$, $m \geq c$.*

Proof. By Lemma 29.18.1 we can find $d_1, \ldots, d_t \geq 0$, and $x_i \in H^p(X, I^{d_i}\mathcal{F})$ such that $\bigoplus_{n\geq 0} H^p(X, I^n\mathcal{F})$ is generated by $x_1, \ldots, x_t$ over $S = \bigoplus_{n\geq 0} I^n$. Take $c = \max\{d_i\}$. It is clear that (1) holds. For (2) let $b = \max(0, n-c)$. Consider the commutative diagram of A-modules

$$\begin{array}{ccccc} I^{n+m-c-b} \otimes I^b \otimes H^p(X, I^c\mathcal{F}) & \longrightarrow & I^{n+m-c} \otimes H^p(X, I^c\mathcal{F}) & \longrightarrow & H^p(X, I^{n+m}\mathcal{F}) \\ \downarrow & & & & \downarrow \\ I^{n+m-c-b} \otimes H^p(X, I^n\mathcal{F}) & & \longrightarrow & & H^p(X, I^n\mathcal{F}) \end{array}$$

By part (1) of the lemma the composition of the horizontal arrows is surjective if $n + m \geq c$. On the other hand, it is clear that $n + m - c - b \geq m - c$. Hence part (2). □

In the situation of Lemmas 29.18.1 and 29.18.3 consider the inverse system

$$\mathcal{F}/I\mathcal{F} \leftarrow \mathcal{F}/I^2\mathcal{F} \leftarrow \mathcal{F}/I^3\mathcal{F} \leftarrow \ldots$$

We would like to know what happens to the cohomology groups. Here is a first result.

Lemma 29.18.4. *Let A be a Noetherian ring. Let $I \subset A$ be an ideal. Let $f : X \to \mathrm{Spec}(A)$ be a proper morphism. Let $\mathcal{F}$ be a coherent sheaf on X. Fix $p \geq 0$.*

(1) *There exists a $c_1 \geq 0$ such that for all $n \geq c_1$ we have*
$$Ker(H^p(X, \mathcal{F}) \to H^p(X, \mathcal{F}/I^n\mathcal{F})) \subset I^{n-c_1} H^p(X, \mathcal{F}).$$
(2) *The inverse system*
$$(H^p(X, \mathcal{F}/I^n\mathcal{F}))_{n \in \mathbf{N}}$$
satisfies the Mittag-Leffler condition (see Homology, Definition 12.26.2).
(3) *In fact for any p and n there exists a $c_2(n) \geq n$ such that*
$$Im(H^p(X, \mathcal{F}/I^k\mathcal{F}) \to H^p(X, \mathcal{F}/I^n\mathcal{F})) = Im(H^p(X, \mathcal{F}) \to H^p(X, \mathcal{F}/I^n\mathcal{F}))$$
for all $k \geq c_2(n)$.

Proof. Let $c_1 = \max\{c_p, c_{p+1}\}$, where c_p, c_{p+1} are the integers found in Lemma 29.18.3 for H^p and H^{p+1}. We will use this constant in the proofs of (1), (2) and (3).

Let us prove part (1). Consider the short exact sequence
$$0 \to I^n\mathcal{F} \to \mathcal{F} \to \mathcal{F}/I^n\mathcal{F} \to 0$$
From the long exact cohomology sequence we see that
$$\mathrm{Ker}(H^p(X, \mathcal{F}) \to H^p(X, \mathcal{F}/I^n\mathcal{F})) = \mathrm{Im}(H^p(X, I^n\mathcal{F}) \to H^p(X, \mathcal{F}))$$
Hence by our choice of c_1 we see that this is contained in $I^{n-c_1}H^p(X, \mathcal{F})$ for $n \geq c_1$.

Note that part (3) implies part (2) by definition of the Mittag-Leffler condition.

Let us prove part (3). Fix an n throughout the rest of the proof. Consider the commutative diagram
$$\begin{array}{ccccccccc} 0 & \longrightarrow & I^n\mathcal{F} & \longrightarrow & \mathcal{F} & \longrightarrow & \mathcal{F}/I^n\mathcal{F} & \longrightarrow & 0 \\ & & \uparrow & & \uparrow & & \uparrow & & \\ 0 & \longrightarrow & I^{n+m}\mathcal{F} & \longrightarrow & \mathcal{F} & \longrightarrow & \mathcal{F}/I^{n+m}\mathcal{F} & \longrightarrow & 0 \end{array}$$
This gives rise to the following commutative diagram
$$\begin{array}{ccccccc} H^p(X, I^n\mathcal{F}) & \longrightarrow & H^p(X, \mathcal{F}) & \longrightarrow & H^p(X, \mathcal{F}/I^n\mathcal{F}) & \xrightarrow{\delta} & H^{p+1}(X, I^n\mathcal{F}) \\ \uparrow & & \uparrow 1 & & \uparrow & & \uparrow a \\ H^p(X, I^{n+m}\mathcal{F}) & \longrightarrow & H^p(X, \mathcal{F}) & \longrightarrow & H^p(X, \mathcal{F}/I^{n+m}\mathcal{F}) & \longrightarrow & H^{p+1}(X, I^{n+m}\mathcal{F}) \end{array}$$
If $m \geq c_1$ we see that the image of a is contained in $I^{m-c_1}H^{p+1}(X, I^n\mathcal{F})$. By the Artin-Rees lemma (see Algebra, Lemma 10.47.5) there exists an integer $c_3(n)$ such that
$$I^N H^{p+1}(X, I^n\mathcal{F}) \cap \mathrm{Im}(\delta) \subset \delta\left(I^{N-c_3(n)}H^p(X, \mathcal{F}/I^n\mathcal{F})\right)$$
for all $N \geq c_3(n)$. As $H^p(X, \mathcal{F}/I^n\mathcal{F})$ is annihilated by I^n, we see that if $m \geq c_3(n) + c_1 + n$, then
$$\mathrm{Im}(H^p(X, \mathcal{F}/I^{n+m}\mathcal{F}) \to H^p(X, \mathcal{F}/I^n\mathcal{F})) = \mathrm{Im}(H^p(X, \mathcal{F}) \to H^p(X, \mathcal{F}/I^n\mathcal{F}))$$

In other words, part (3) holds with $c_2(n) = c_3(n) + c_1 + n$. □

Theorem 29.18.5 (Theorem on formal functions)**.** *Let A be a Noetherian ring. Let $I \subset A$ be an ideal. Let $f : X \to \mathrm{Spec}(A)$ be a proper morphism. Let $\mathcal{F}$ be a coherent sheaf on X. Fix $p \geq 0$. The system of maps*

$$H^p(X, \mathcal{F})/I^n H^p(X, \mathcal{F}) \longrightarrow H^p(X, \mathcal{F}/I^n\mathcal{F})$$

define an isomorphism of limits

$$H^p(X, \mathcal{F})^\wedge \longrightarrow \lim_n H^p(X, \mathcal{F}/I^n\mathcal{F})$$

where the left hand side is the completion of the A-module $H^p(X, \mathcal{F})$ with respect to the ideal I, see Algebra, Section 10.91. Moreover, this is in fact a homeomorphism for the limit topologies.

Proof. In fact, this follows immediately from Lemma 29.18.4. We spell out the details. Set $M = H^p(X, \mathcal{F})$ and $M_n = H^p(X, \mathcal{F}/I^n\mathcal{F})$. Denote $N_n = \mathrm{Im}(M \to M_n)$. By the description of the limit in Homology, Section 12.26 we have

$$\lim_n M_n = \{(x_n) \in \prod M_n \mid \varphi_i(x_n) = x_{n-1},\ n = 2, 3, \ldots\}$$

Pick an element $x = (x_n) \in \lim_n M_n$. By Lemma 29.18.4 part (3) we have $x_n \in N_n$ for all n since by definition x_n is the image of some $x_{n+m} \in M_{n+m}$ for all m. By Lemma 29.18.4 part (1) we see that there exists a factorization

$$M \to N_n \to M/I^{n-c_1}M$$

of the reduction map. Denote $y_n \in M/I^{n-c_1}M$ the image of x_n for $n \geq c_1$. Since for $n' \geq n$ the composition $M \to M_{n'} \to M_n$ is the given map $M \to M_n$ we see that $y_{n'}$ maps to y_n under the canonical map $M/I^{n'-c_1}M \to M/I^{n-c_1}M$. Hence $y = (y_{n+c_1})$ defines an element of $\lim_n M/I^nM$. We omit the verification that y maps to x under the map

$$M^\wedge = \lim_n M/I^nM \longrightarrow \lim_n M_n$$

of the lemma. We also omit the verification on topologies. □

Lemma 29.18.6. *Let A be a ring. Let $I \subset A$ be an ideal. Assume A is Noetherian and complete with respect to I. Let $f : X \to \mathrm{Spec}(A)$ be a proper morphism. Let $\mathcal{F}$ be a coherent sheaf on X. Then*

$$H^p(X, \mathcal{F}) = \lim_n H^p(X, \mathcal{F}/I^n\mathcal{F})$$

for all $p \geq 0$.

Proof. This is a reformulation of the theorem on formal functions (Theorem 29.18.5) in the case of a complete Noetherian base ring. Namely, in this case the A-module $H^p(X, \mathcal{F})$ is finite (Lemma 29.17.4) hence I-adically complete (Algebra, Lemma 10.91.2) and we see that completion on the left hand side is not necessary. □

Lemma 29.18.7. *Given a morphism of schemes $f : X \to Y$ and a quasi-coherent sheaf $\mathcal{F}$ on X. Assume*

(1) *Y locally Noetherian,*
(2) *f proper, and*
(3) *$\mathcal{F}$ coherent.*

Let $y \in Y$ be a point. Consider the infinitesimal neighbourhoods

$$\begin{array}{ccc} X_n = \mathrm{Spec}(\mathcal{O}_{Y,y}/\mathfrak{m}_y^n) \times_Y X & \xrightarrow[i_n]{} & X \\ \downarrow{\scriptstyle f_n} & & \downarrow{\scriptstyle f} \\ \mathrm{Spec}(\mathcal{O}_{Y,y}/\mathfrak{m}_y^n) & \xrightarrow{c_n} & Y \end{array}$$

of the fibre $X_1 = X_y$ and set $\mathcal{F}_n = i_n^*\mathcal{F}$. Then we have

$$(R^p f_*\mathcal{F})^\wedge_y \cong \lim_n H^p(X_n, \mathcal{F}_n)$$

as $\mathcal{O}^\wedge_{Y,y}$-modules.

Proof. This is just a reformulation of a special case of the theorem on formal functions, Theorem 29.18.5. Let us spell it out. Note that $\mathcal{O}_{Y,y}$ is a Noetherian local ring. Consider the canonical morphism $c : \mathrm{Spec}(\mathcal{O}_{Y,y}) \to Y$, see Schemes, Equation (25.13.1.1). This is a flat morphism as it identifies local rings. Denote momentarily $f' : X' \to \mathrm{Spec}(\mathcal{O}_{Y,y})$ the base change of f to this local ring. We see that $c^*R^p f_*\mathcal{F} = R^p f'_*\mathcal{F}'$ by Lemma 29.5.2. Moreover, the infinitesimal neighbourhoods of the fibre X_y and X'_y are identified (verification omitted; hint: the morphisms c_n factor through c).

Hence we may assume that $Y = \mathrm{Spec}(A)$ is the spectrum of a Noetherian local ring A with maximal ideal $\mathfrak{m}$ and that $y \in Y$ corresponds to the closed point (i.e., to $\mathfrak{m}$). In particular it follows that

$$(R^p f_*\mathcal{F})_y = \Gamma(Y, R^p f_*\mathcal{F}) = H^p(X, \mathcal{F}).$$

In this case also, the morphisms c_n are each closed immersions. Hence their base changes i_n are closed immersions as well. Note that $i_{n,*}\mathcal{F}_n = i_{n,*}i_n^*\mathcal{F} = \mathcal{F}/\mathfrak{m}^n\mathcal{F}$. By the Leray spectral sequence for i_n, and Lemma 29.9.9 we see that

$$H^p(X_n, \mathcal{F}_n) = H^p(X, i_{n,*}\mathcal{F}) = H^p(X, \mathcal{F}/\mathfrak{m}^n\mathcal{F})$$

Hence we may indeed apply the theorem on formal functions to compute the limit in the statement of the lemma and we win. □

Here is a lemma which we will generalize later to fibres of dimension > 0, namely the next lemma.

Lemma 29.18.8. *Let $f : X \to Y$ be a morphism of schemes. Let $y \in Y$. Assume*

(1) *Y locally Noetherian,*
(2) *f is proper, and*
(3) *$f^{-1}(\{y\})$ is finite.*

Then for any coherent sheaf $\mathcal{F}$ on X we have $(R^p f_*\mathcal{F})_y = 0$ for all $p > 0$.

Proof. The fibre X_y is finite, and by Morphisms, Lemma 28.21.7 it is a finite discrete space. Moreover, the underlying topological space of each infinitesimal neighbourhood X_n is the same. Hence each of the schemes X_n is affine according to Schemes, Lemma 25.11.7. Hence it follows that $H^p(X_n, \mathcal{F}_n) = 0$ for all $p > 0$. Hence we see that $(R^p f_*\mathcal{F})^\wedge_y = 0$ by Lemma 29.18.7. Note that $R^p f_*\mathcal{F}$ is coherent by Lemma 29.17.2 and hence $R^p f_*\mathcal{F}_y$ is a finite $\mathcal{O}_{Y,y}$-module. By Algebra, Lemma 10.91.2 this implies that $(R^p f_*\mathcal{F})_y = 0$. □

Lemma 29.18.9. *Let $f : X \to Y$ be a morphism of schemes. Let $y \in Y$. Assume*

(1) Y *locally Noetherian,*
(2) f *is proper, and*
(3) $\dim(X_y) = d$.

Then for any coherent sheaf $\mathcal{F}$ on X we have $(R^p f_*\mathcal{F})_y = 0$ for all $p > d$.

Proof. The fibre X_y is of finite type over $\mathrm{Spec}(\kappa(y))$. Hence X_y is a Noetherian scheme by Morphisms, Lemma 28.16.6. Hence the underlying topological space of X_y is Noetherian, see Properties, Lemma 27.5.5. Moreover, the underlying topological space of each infinitesimal neighbourhood X_n is the same as that of X_y. Hence $H^p(X_n, \mathcal{F}_n) = 0$ for all $p > d$ by Cohomology, Lemma 20.21.5. Hence we see that $(R^p f_*\mathcal{F})^\wedge_y = 0$ by Lemma 29.18.7 for $p > d$. Note that $R^p f_*\mathcal{F}$ is coherent by Lemma 29.17.2 and hence $R^p f_*\mathcal{F}_y$ is a finite $\mathcal{O}_{Y,y}$-module. By Algebra, Lemma 10.91.2 this implies that $(R^p f_*\mathcal{F})_y = 0$. □

29.19. Applications of the theorem on formal functions

We will add more here as needed. For the moment we need the following characterization of finite morphisms (in the Noetherian case – for a more general version see the chapter More on Morphisms, Section 36.31).

Lemma 29.19.1. *(For a more general version see More on Morphisms, Lemma 36.31.5). Let $f : X \to S$ be a morphism of schemes. Assume S is locally Noetherian. The following are equivalent*

(1) f *is finite, and*
(2) f *is proper with finite fibres.*

Proof. A finite morphism is proper according to Morphisms, Lemma 28.44.10. A finite morphism is quasi-finite according to Morphisms, Lemma 28.44.9. A quasi-finite morphism has finite fibres, see Morphisms, Lemma 28.21.10. Hence a finite morphism is proper and has finite fibres.

Assume f is proper with finite fibres. We want to show f is finite. In fact it suffices to prove f is affine. Namely, if f is affine, then it follows that f is integral by Morphisms, Lemma 28.44.7 whereupon it follows from Morphisms, Lemma 28.44.4 that f is finite.

To show that f is affine we may assume that S is affine, and our goal is to show that X is affine too. Since f is proper we see that X is separated and quasi-compact. Hence we may use the criterion of Lemma 29.3.2 to prove that X is affine. To see this let $\mathcal{I} \subset \mathcal{O}_X$ be a finite type ideal sheaf. In particular $\mathcal{I}$ is a coherent sheaf on X. By Lemma 29.18.8 we conclude that $R^1 f_*\mathcal{I}_s = 0$ for all $s \in S$. In other words, $R^1 f_*\mathcal{I} = 0$. Hence we see from the Leray Spectral Sequence for f that $H^1(X, \mathcal{I}) = H^1(S, f_*\mathcal{I})$. Since S is affine, and $f_*\mathcal{I}$ is quasi-coherent (Schemes, Lemma 25.24.1) we conclude $H^1(S, f_*\mathcal{I}) = 0$ from Lemma 29.2.2 as desired. Hence $H^1(X, \mathcal{I}) = 0$ as desired. □

As a consequence we have the following useful result.

Lemma 29.19.2. *(For a more general version see More on Morphisms, Lemma 36.31.6). Let $f : X \to S$ be a morphism of schemes. Let $s \in S$. Assume*

(1) S *is locally Noetherian,*
(2) f *is proper, and*
(3) $f^{-1}(\{s\})$ *is a finite set.*

Then there exists an open neighbourhood $V \subset S$ of s such that $f|_{f^{-1}(V)} : f^{-1}(V) \to V$ is finite.

Proof. The morphism f is quasi-finite at all the points of $f^{-1}(\{s\})$ by Morphisms, Lemma 28.21.7. By Morphisms, Lemma 28.49.2 the set of points at which f is quasi-finite is an open $U \subset X$. Let $Z = X \setminus U$. Then $s \notin f(Z)$. Since f is proper the set $f(Z) \subset S$ is closed. Choose any open neighbourhood $V \subset S$ of s with $Z \cap V = \emptyset$. Then $f^{-1}(V) \to V$ is locally quasi-finite and proper. Hence it is quasi-finite (Morphisms, Lemma 28.21.9), hence has finite fibres (Morphisms, Lemma 28.21.10), hence is finite by Lemma 29.19.1. □

29.20. Cohomology and base change, III

In this section we prove the simplest case of a very general phenomenon that will be discussed in Derived Categories of Schemes, Section 35.16. Please see Remark 29.20.2 for a translation of the following lemma into algebra.

Lemma 29.20.1. *Let A be a Noetherian ring and set $S = \mathrm{Spec}(A)$. Let $f : X \to S$ be a proper morphism of schemes. Let $\mathcal{F}$ be a coherent $\mathcal{O}_X$-module flat over S. Then*

1. *$R\Gamma(X, \mathcal{F})$ is a perfect object of $D(A)$, and*
2. *for any ring map $A \to A'$ the base change map*

$$R\Gamma(X, \mathcal{F}) \otimes^{\mathbf{L}}_A A' \longrightarrow R\Gamma(X_{A'}, \mathcal{F}_{A'})$$

is an isomorphism.

Proof. Choose a finite affine open covering $X = \bigcup_{i=1,\ldots,n} U_i$. By Lemmas 29.7.1 and 29.7.2 the Čech complex $K^\bullet = \check{C}^\bullet(\mathcal{U}, \mathcal{F})$ satisfies

$$K^\bullet \otimes_A A' = R\Gamma(X_{A'}, \mathcal{F}_{A'})$$

for all ring maps $A \to A'$. Let $K^\bullet_{alt} = \check{C}^\bullet_{alt}(\mathcal{U}, \mathcal{F})$ be the alternating Čech complex. By Cohomology, Lemma 20.22.6 there is a homotopy equivalence $K^\bullet_{alt} \to K^\bullet$ of A-modules. In particular, we have

$$K^\bullet_{alt} \otimes_A A' = R\Gamma(X_{A'}, \mathcal{F}_{A'})$$

as well. Since $\mathcal{F}$ is flat over A we see that each K^n_{alt} is flat over A (see Morphisms, Lemma 28.26.2). Since moreover $K^\bullet_{alt}$ is bounded above (this is why we switched to the alternating Čech complex) $K^\bullet_{alt} \otimes_A A' = K^\bullet_{alt} \otimes^{\mathbf{L}}_A A'$ by the definition of derived tensor products (see More on Algebra, Section 15.43). By Lemma 29.17.4 the cohomology groups $H^i(K^\bullet_{alt})$ are finite A-modules. As $K^\bullet_{alt}$ is bounded, we conclude that $K^\bullet_{alt}$ is pseudo-coherent, see More on Algebra, Lemma 15.48.16. Given any A-module M set $A' = A \oplus M$ where M is a square zero ideal, i.e., $(a, m) \cdot (a', m') = (aa', am' + a'm)$. By the above we see that $K^\bullet_{alt} \otimes^{\mathbf{L}}_A A'$ has cohomology in degrees $0, \ldots, n$. Hence $K^\bullet_{alt} \otimes^{\mathbf{L}}_A M$ has cohomology in degrees $0, \ldots, n$. Hence $K^\bullet_{alt}$ has finite Tor dimension, see More on Algebra, Definition 15.49.1. We win by More on Algebra, Lemma 15.50.2. □

Remark 29.20.2. A consequence of Lemma 29.20.1 is that there exists a finite complex of finite projective A-modules $M^\bullet$ such that we have

$$H^i(X_{A'}, \mathcal{F}_{A'}) = H^i(M^\bullet \otimes_A A')$$

functorially in A'. The condition that $\mathcal{F}$ is flat over A is essential, see [**Har98**].

29.21. Grothendieck's existence theorem, I

In this section we discuss Grothendieck's existence theorem for the projective case. As we do not yet have the theory of formal schemes to our disposal, we temporarily develop a bit of language that replaces the notion of a "coherent module on a Noetherian adic formal scheme". The reader who is familiar with formal schemes is encouraged to read the statement and proof of the theorem in [**DG67**].

Let X be a Noetherian scheme and let $\mathcal{I} \subset \mathcal{O}_X$ be a quasi-coherent sheaf of ideals. Below we will consider inverse systems $(\mathcal{F}_n)$ of coherent $\mathcal{O}_X$-modules such that

(1) $\mathcal{F}_n$ is annihilated by $\mathcal{I}^n$, and
(2) the transition maps induce isomorphisms $\mathcal{F}_{n+1}/\mathcal{I}^n\mathcal{F}_{n+1} \to \mathcal{F}_n$.

A morphism of such inverse systems is defined as usual. Let us denote the category of these inverse systems with $Coh(X, \mathcal{I})$. We are going to proceed by proving a bunch of lemmas about objects in this category. In fact, most of the lemmas that follow are straightforward consequences of the following description of the category in the affine case.

Lemma 29.21.1. *If $X = \mathrm{Spec}(A)$ is the spectrum of a Noetherian ring and $\mathcal{I}$ is the quasi-coherent sheaf of ideals associated to the ideal $I \subset A$, then $Coh(X, \mathcal{I})$ is equivalent to the category of finite $A^\wedge$-modules where $A^\wedge$ is the completion of A with respect to I.*

Proof. Let $\mathrm{Mod}^{fg}_{A,I}$ be the category of inverse systems (M_n) of finite A-modules satisfying: (1) M_n is annihilated by I^n and (2) $M_{n+1}/I^nM_{n+1} = M_n$. By the correspondence between coherent sheaves on X and finite A-modules (Lemma 29.9.1) it suffices to show $\mathrm{Mod}^{fg}_{A,I}$ is equivalent to the category of finite $A^\wedge$-modules. To see this it suffices to prove that given an object (M_n) of $\mathrm{Mod}^{fg}_{A,I}$ the module

$$M = \lim M_n$$

is a finite $A^\wedge$-module and that $M/I^nM = M_n$. As the transition maps are surjective, we see that $M \to M_1$ is surjective. Pick $x_1, \ldots, x_t \in M$ which map to generators of M_1. This induces a map of systems $(A/I^n)^{\oplus t} \to M_n$. By Nakayama's lemma (Algebra, Lemma 10.18.1) these maps are surjective. Let $K_n \subset (A/I^n)^{\oplus t}$ be the kernel. Property (2) implies that $K_{n+1} \to K_n$ is surjective, in particular the system (K_n) satisfies the Mittag-Leffler condition. By Homology, Lemma 12.26.3 we obtain an exact sequence $0 \to K \to (A^\wedge)^{\oplus t} \to M \to 0$ with $K = \lim K_n$. Hence M is a finite $A^\wedge$-module. As $K \to K_n$ is surjective it follows that

$$M/I^nM = \mathrm{Coker}(K \to (A/I^n)^{\oplus t}) = (A/I^n)^{\oplus t}/K_n = M_n$$

as desired. □

Lemma 29.21.2. *Let X be a Noetherian scheme and let $\mathcal{I} \subset \mathcal{O}_X$ be a quasi-coherent sheaf of ideals.*

(1) *The category $Coh(X, \mathcal{I})$ is abelian.*
(2) *For $U \subset X$ open the restriction functor $Coh(X, \mathcal{I}) \to Coh(U, \mathcal{I}|_U)$ is exact.*
(3) *Exactness in $Coh(X, \mathcal{I})$ may be checked by restricting to the members of an open covering of X.*

Proof. Let $\alpha = (\alpha_n) : (\mathcal{F}_n) \to (\mathcal{G}_n)$ be a morphism of $Coh(X, \mathcal{I})$. The cokernel of α is the inverse system $(\mathrm{Coker}(\alpha_n))$ (details omitted). To describe the kernel let

$$\mathcal{K}'_{l,m} = \mathrm{Im}(\mathrm{Ker}(\alpha_l) \to \mathcal{F}_m)$$

for $l \geq m$. We claim:

(a) the inverse system $(\mathcal{K}'_{l,m})_{l \geq m}$ is eventually constant, say with value $\mathcal{K}'_m$,
(b) the system $(\mathcal{K}'_m/\mathcal{I}^n\mathcal{K}'_m)_{m \geq n}$ is eventually constant, say with value $\mathcal{K}_n$,
(c) the system $(\mathcal{K}_n)$ forms an object of $Coh(X, \mathcal{I})$, and
(d) this object is the kernel of α.

To see (a), (b), and (c) we may work affine locally, say $X = \mathrm{Spec}(A)$ and $\mathcal{I}$ corresponds to the ideal $I \subset A$. By Lemma 29.21.1 α corresponds to a map $f : M \to N$ of finite $A^\wedge$-modules. Denote $K = \mathrm{Ker}(f)$. Note that $A^\wedge$ is a Noetherian ring (Algebra, Lemma 10.91.10). Choose an integer $c \geq 0$ such that $K \cap I^n M \subset I^{n-c}K$ for $n \geq c$ (Algebra, Lemma 10.47.4) and which satisfies Algebra, Lemma 10.47.5 for the map f and the ideal $I^\wedge = IA^\wedge$. Then $\mathcal{K}'_{l,m}$ corresponds to the A-module

$$K'_{l,m} = \frac{a^{-1}(I^l N) + I^m M}{I^m M} = \frac{K + I^{l-c} f^{-1}(I^c N) + I^m M}{I^m M} = \frac{K + I^m M}{I^m M}$$

where the last equality holds if $l \geq m + c$. So $\mathcal{K}'_m$ corresponds to the A-module $K/K \cap I^m M$ and $\mathcal{K}'_m/\mathcal{I}^n\mathcal{K}'_m$ corresponds to

$$\frac{K}{K \cap I^m M + I^n K} = \frac{K}{I^n K}$$

for $m \geq n + c$ by our choice of c above. Hence $\mathcal{K}_n$ corresponds to $K/I^n K$.

We prove (d). It is clear from the description on affines above that the composition $(\mathcal{K}_n) \to (\mathcal{F}_n) \to (\mathcal{G}_n)$ is zero. Let $\beta : (\mathcal{H}_n) \to (\mathcal{F}_n)$ be a morphism such that $\alpha \circ \beta = 0$. Then $\mathcal{H}_l \to \mathcal{F}_l$ maps into $\mathrm{Ker}(\alpha_l)$. Since $\mathcal{H}_m = \mathcal{H}_l/\mathcal{I}^m\mathcal{H}_l$ for $l \geq m$ we obtain a system of maps $\mathcal{H}_m \to \mathcal{K}'_{l,m}$. Thus a map $\mathcal{H}_m \to \mathcal{K}'_m$. Since $\mathcal{H}_n = \mathcal{H}_m/\mathcal{I}^n\mathcal{H}_m$ we obtain a system of maps $\mathcal{H}_n \to \mathcal{K}'_m/\mathcal{I}^n\mathcal{K}'_m$ and hence a map $\mathcal{H}_n \to \mathcal{K}_n$ as desired.

To finish the proof of (1) we still have to show that Coim = Im in $Coh(X, \mathcal{I})$. We have seen above that taking kernels and cokernels commutes, over affines, with the description of $Coh(X, \mathcal{I})$ as a category of modules. Since Im = Coim holds in the category of modules this gives Coim = Im in $Coh(X, \mathcal{I})$. Parts (2) and (3) of the lemma are immediate from our construction of kernels and cokernels. □

Lemma 29.21.3. *Let X be a Noetherian scheme and let $\mathcal{I} \subset \mathcal{O}_X$ be a quasi-coherent sheaf of ideals. A map $(\mathcal{F}_n) \to (\mathcal{G}_n)$ is surjective in $Coh(X, \mathcal{I})$ if and only if $\mathcal{F}_1 \to \mathcal{G}_1$ is surjective.*

Proof. Omitted. Hint: Look on affine opens, use Lemma 29.21.1, and use Algebra, Lemma 10.18.1. □

Lemma 29.21.4. *Let X be a Noetherian scheme and let $\mathcal{I} \subset \mathcal{O}_X$ be a quasi-coherent sheaf of ideals. If $(\mathcal{F}_n)$ is an object of $Coh(X, \mathcal{I})$ then $\bigoplus Ker(\mathcal{F}_{n+1} \to \mathcal{F}_n)$ is a finite type, graded, quasi-coherent $\bigoplus \mathcal{I}^n/\mathcal{I}^{n+1}$-module.*

Proof. The question is local on X hence we may assume X is affine, i.e., we have a situation as in Lemma 29.21.1. In this case, if $(\mathcal{F}_n)$ corresponds to the finite $A^\wedge$ module M, then $\bigoplus \mathrm{Ker}(\mathcal{F}_{n+1} \to \mathcal{F}_n)$ corresponds to $\bigoplus I^n M/I^{n+1}M$ which is clearly a finite module over $\bigoplus I^n/I^{n+1}$. □

Let X be a Noetherian scheme and let $\mathcal{I} \subset \mathcal{O}_X$ be a quasi-coherent sheaf of ideals. There is a functor

$$Coh(\mathcal{O}_X) \longrightarrow Coh(X, \mathcal{I}), \quad \mathcal{F} \longmapsto \mathcal{F}^\wedge \tag{29.21.4.1}$$

which associates to the coherent $\mathcal{O}_X$-module $\mathcal{F}$ the object $\mathcal{F}^\wedge = (\mathcal{F}/\mathcal{I}^n\mathcal{F})$ of $Coh(X, \mathcal{I})$.

Lemma 29.21.5. *The functor (29.21.4.1) is exact.*

Proof. It suffices to check this locally on X. Hence we may assume X is affine, i.e., we have a situation as in Lemma 29.21.1. The functor is the functor $\mathrm{Mod}_A^{fg} \to \mathrm{Mod}_{A^\wedge}^{fg}$ which associates to a finite A-module M the completion $M^\wedge$. Thus the result follows from Algebra, Lemma 10.91.3. □

Lemma 29.21.6. *Let X be a Noetherian scheme and let $\mathcal{I} \subset \mathcal{O}_X$ be a quasi-coherent sheaf of ideals. Let $\mathcal{F}$, $\mathcal{G}$ be coherent $\mathcal{O}_X$-modules. Set $\mathcal{H} = \mathcal{H}om_{\mathcal{O}_X}(\mathcal{G}, \mathcal{F})$. Then*

$$\lim H^0(X, \mathcal{H}/\mathcal{I}^n\mathcal{H}) = \mathrm{Mor}_{Coh(X,\mathcal{I})}(\mathcal{G}^\wedge, \mathcal{F}^\wedge).$$

Proof. To prove this we may work affine locally on X. Hence we may assume $X = \mathrm{Spec}(A)$ and $\mathcal{F}$, $\mathcal{G}$ given by finite A-module M and N. Then $\mathcal{H}$ corresponds to the finite A-module $H = \mathrm{Hom}_A(M, N)$. The statement of the lemma becomes the statement

$$H^\wedge = \mathrm{Hom}_{A^\wedge}(M^\wedge, N^\wedge)$$

via the equivalence of Lemma 29.21.1. By Algebra, Lemma 10.91.3 (used 3 times) we have

$$H^\wedge = \mathrm{Hom}_A(M, N) \otimes_A A^\wedge = \mathrm{Hom}_{A^\wedge}(M \otimes_A A^\wedge, N \otimes_A A^\wedge) = \mathrm{Hom}_{A^\wedge}(M^\wedge, N^\wedge)$$

where the second equality uses that $A^\wedge$ is flat over A (see More on Algebra, Remark 15.48.18). The lemma follows. □

Lemma 29.21.7. *Let A be Noetherian ring complete with respect to an ideal I. Let $f : X \to \mathrm{Spec}(A)$ be a proper morphism. Let $\mathcal{I} = I\mathcal{O}_X$. Then the functor (29.21.4.1) is fully faithful.*

Proof. Let $\mathcal{F}$, $\mathcal{G}$ be coherent $\mathcal{O}_X$-modules. Then $\mathcal{H} = \mathcal{H}om_{\mathcal{O}_X}(\mathcal{G}, \mathcal{F})$ is a coherent $\mathcal{O}_X$-module, see Modules, Lemma 17.19.4. By Lemma 29.21.6 the map

$$\lim_n H^0(X, \mathcal{H}/\mathcal{I}^n\mathcal{H}) \to \mathrm{Mor}_{Coh(X,\mathcal{I})}(\mathcal{G}^\wedge, \mathcal{F}^\wedge)$$

is bijective. Hence fully faithfulness of (29.21.4.1) follows from the theorem on formal functions (Lemma 29.18.6) for the coherent sheaf $\mathcal{H}$. □

Lemma 29.21.8. *Let A be Noetherian ring and $I \subset A$ and ideal. Let $f : X \to \mathrm{Spec}(A)$ be a proper morphism and let $\mathcal{L}$ be an f-ample invertible sheaf. Let $\mathcal{I} = I\mathcal{O}_X$. Let $(\mathcal{F}_n)$ be an object of $Coh(X, \mathcal{I})$. Then there exists an integer d_0 such that*

$$H^1(X, Ker(\mathcal{F}_{n+1} \to \mathcal{F}_n) \otimes \mathcal{L}^{\otimes d}) = 0$$

for all $n \geq 0$ and all $d \geq d_0$.

Proof. Set $B = \bigoplus I^n/I^{n+1}$ and $\mathcal{B} = \bigoplus \mathcal{I}^n/\mathcal{I}^{n+1} = f^*\widetilde{B}$. By Lemma 29.21.4 the graded quasi-coherent $\mathcal{B}$-module $\mathcal{G} = \bigoplus \mathrm{Ker}(\mathcal{F}_{n+1} \to \mathcal{F}_n)$ is of finite type. Hence the lemma follows from Lemma 29.17.5 part (2). □

Lemma 29.21.9. *Let A be Noetherian ring complete with respect to an ideal I. Let $f : X \to \mathrm{Spec}(A)$ be a projective morphism. Let $\mathcal{I} = I\mathcal{O}_X$. Then the functor (29.21.4.1) is an equivalence.*

Proof. We have already seen that (29.21.4.1) is fully faithful in Lemma 29.21.7. Thus it suffices to show that the functor is essentially surjective.

We first show that every object $(\mathcal{F}_n)$ of $Coh(X, \mathcal{I})$ is the quotient of an object in the image of (29.21.4.1). Let $\mathcal{L}$ be an f-ample invertible sheaf on X. Choose d_0 as in Lemma 29.21.8. Choose a $d \geq d_0$ such that $\mathcal{F}_1 \otimes \mathcal{L}^{\otimes d}$ is globally generated by some sections $s_{1,1}, \ldots, s_{t,1}$. Since the transition maps of the system

$$H^0(X, \mathcal{F}_{n+1} \otimes \mathcal{L}^{\otimes d}) \longrightarrow H^0(X, \mathcal{F}_n \otimes \mathcal{L}^{\otimes d})$$

are surjective by the vanishing of H^1 we can lift $s_{1,1}, \ldots, s_{t,1}$ to a compatible system of global sections $s_{1,n}, \ldots, s_{t,n}$ of $\mathcal{F}_n \otimes \mathcal{L}^{\otimes d}$. These determine a compatible system of maps

$$(s_{1,n}, \ldots, s_{t,n}) : (\mathcal{L}^{\otimes -d})^{\oplus t} \longrightarrow \mathcal{F}_n$$

Using Lemma 29.21.3 we deduce that we have a surjective map

$$\left((\mathcal{L}^{\otimes -d})^{\oplus t}\right)^\wedge \longrightarrow (\mathcal{F}_n)$$

as desired.

The result of the previous paragraph and the fact that $Coh(X, \mathcal{I})$ is abelian (Lemma 29.21.2) implies that every object of $Coh(X, \mathcal{I})$ is a cokernel of a map between objects coming from $Coh(\mathcal{O}_X)$. As (29.21.4.1) is fully faithful and exact by Lemmas 29.21.7 and 29.21.5 we conclude. □

29.22. Grothendieck's existence theorem, II

In this section we discuss Grothendieck's existence theorem. Before we give the statement and proof, we need to develop a bit more theory regarding the categories $Coh(X, \mathcal{I})$ introduced in Section 29.21.

Lemma 29.22.1. *Let $f : X \to Y$ be a morphism of Noetherian schemes. Let $\mathcal{J} \subset \mathcal{O}_Y$ be a quasi-coherent sheaf of ideals and set $\mathcal{I} = f^{-1}\mathcal{J}\mathcal{O}_X$. Then there is a right exact functor*

$$f^* : Coh(Y, \mathcal{J}) \longrightarrow Coh(X, \mathcal{I})$$

which sends $(\mathcal{G}_n)$ to $(f^*\mathcal{G}_n)$. If f is flat, then f^* is an exact functor.

Proof. Since $f^* : Coh(\mathcal{O}_Y) \to Coh(\mathcal{O}_X)$ is right exact we have

$$f^*\mathcal{G}_n = f^*(\mathcal{G}_{n+1}/\mathcal{I}^n\mathcal{G}_{n+1}) = f^*\mathcal{G}_{n+1}/f^{-1}\mathcal{I}^n f^*\mathcal{G}_{n+1} = f^*\mathcal{G}_{n+1}/\mathcal{J}^n f^*\mathcal{G}_{n+1}$$

hence the pullback of a system is a system. The construction of cokernels in the proof of Lemma 29.21.2 shows that $f^* : Coh(Y, \mathcal{J}) \to Coh(X, \mathcal{I})$ is always right exact. If f is flat, then $f^* : Coh(\mathcal{O}_Y) \to Coh(\mathcal{O}_X)$ is an exact functor. It follows from the construction of kernels in the proof of Lemma 29.21.2 that in this case $f^* : Coh(Y, \mathcal{J}) \to Coh(X, \mathcal{I})$ also transforms kernels into kernels. □

Remark 29.22.2. Let X be a Noetherian scheme and let $\mathcal{I}, \mathcal{K} \subset \mathcal{O}_X$ be quasi-coherent sheaves of ideals. Let $\alpha : (\mathcal{F}_n) \to (\mathcal{G}_n)$ be a morphism of $Coh(X, \mathcal{I})$. Given an affine open $\mathrm{Spec}(A) = U \subset X$ with $\mathcal{I}|_U, \mathcal{K}|_U$ corresponding to ideals $I, K \subset A$ denote $\alpha_U : M \to N$ of finite $A^\wedge$-modules which corresponds to $\alpha|_U$ via Lemma 29.21.1. We claim the following are equivalent

(1) there exists an integer $t \geq 1$ such that $\mathrm{Ker}(\alpha_n)$ and $\mathrm{Coker}(\alpha_n)$ are annihilated by $\mathcal{K}^t$ for all $n \geq 1$,
(2) for any affine open $\mathrm{Spec}(A) = U \subset X$ as above the modules $\mathrm{Ker}(\alpha_U)$ and $\mathrm{Coker}(\alpha_U)$ are annihilated by K^t for some integer $t \geq 1$, and
(3) there exists a finite affine open covering $X = \bigcup U_i$ such that the conclusion of (2) holds for α_{U_i}.

If these equivalent conditions hold we will say that α is a *map whose kernel and cokernel are annihilated by a power of* $\mathcal{K}$. To see the equivalence we use the following commutative algebra fact: suppose given an exact sequence

$$0 \to T \to M \to N \to Q \to 0$$

of A-modules with T and Q annihilated by K^t for some ideal $K \subset A$. Then for every $f, g \in K^t$ there exists a canonical map "fg" : $N \to M$ such that $M \to N \to M$ is equal to multiplication by fg. Namely, for $y \in N$ we can pick $x \in M$ mapping to fy in N and then we can set "fg"$(y) = gx$. Thus it is clear that $\mathrm{Ker}(M/JM \to N/JN)$ and $\mathrm{Coker}(M/JM \to N/JN)$ are annihilated by K^{2t} for any ideal $J \subset A$.

Applying the commutative algebra fact to α_{U_i} and $J = I^n$ we see that (3) implies (1). Conversely, suppose (1) holds and $M \to N$ is equal to α_U. Then there is a $t \geq 1$ such that $\mathrm{Ker}(M/I^nM \to N/I^nN)$ and $\mathrm{Coker}(M/I^nM \to N/I^nN)$ are annihilated by K^t for all n. We obtain maps "fg" : $N/I^nN \to M/I^nM$ which in the limit induce a map $N \to M$ as N and M are I-adically complete. Since the composition with $N \to M \to N$ is multiplication by fg we conclude that fg annihilates T and Q. In other words T and Q are annihilated by K^{2t} as desired.

Lemma 29.22.3. *Let X be a Noetherian scheme and let $\mathcal{I} \subset \mathcal{O}_X$ be a quasi-coherent sheaf of ideals. Let $\mathcal{G}$ be a coherent $\mathcal{O}_X$-module, $(\mathcal{F}_n)$ an object of $Coh(X, \mathcal{I})$, and $\alpha : (\mathcal{F}_n) \to \mathcal{G}^\wedge$ a map whose kernel and cokernel are annihilated by a power of $\mathcal{I}$. Then there exists a unique (up to unique isomorphism) triple $(\mathcal{F}, a, \beta)$ where*

(1) *$\mathcal{F}$ is a coherent $\mathcal{O}_X$-module,*
(2) *$a : \mathcal{F} \to \mathcal{G}$ is an $\mathcal{O}_X$-module map whose kernel and cokernel are annihilated by a power of $\mathcal{I}$,*
(3) *$\beta : (\mathcal{F}_n) \to \mathcal{F}^\wedge$ is an isomorphism, and*
(4) *$\alpha = a^\wedge \circ \beta$.*

Proof. The uniqueness implies it suffices to construct $(\mathcal{F}, a, \beta)$ Zariski locally on X. Thus we may assume $X = \mathrm{Spec}(A)$ and $\mathcal{I}$ corresponds to the ideal $I \subset A$. In this situation Lemma 29.21.1 applies. Let M' be the finite $A^\wedge$-module corresponding to $(\mathcal{F}_n)$. Let N be the finite A-module corresponding to $\mathcal{G}$. Then α corresponds to a map

$$\varphi : M' \longrightarrow N^\wedge$$

whose kernel and cokernel are annihilated by I^t for some t. In particular, if $I = (f_1, \ldots, f_t)$, then it is clear that the localization φ_{f_i} is an isomorphism. Recall that $N^\wedge = N \otimes_A A^\wedge$ (Algebra, Lemma 10.91.2). Thus we see that $(M', N_{f_i}, \varphi_{f_i}, can_{ij})$ is an object of $\mathrm{Glue}(A \to A^\wedge, f_1, \ldots, f_t)$, see More on Algebra, Remark 15.56.15. By More on Algebra, Proposition 15.56.20 we conclude that there exists an A-module M such that $M' = M \otimes_A A^\wedge$ and $N_{f_i} = M_{f_i}$ compatibly with the isomorphisms φ_{f_i} and can_{ij}. There is a morphism

$$(M \otimes_A A^\wedge, M_{f_i}, can_i, can_{ij}) = (M', N_{f_i}, \varphi_{f_i}, can_{ij}) \to (N \otimes_A A^\wedge, N_{f_i}, can_i, can_{ij})$$

of Glue$(A \to A^\wedge, f_1, \ldots, f_t)$ which uses φ in the first component. This corresponds to an A-module map $M \to N$ (by the equivalence of categories of More on Algebra, Proposition 15.56.20). The composition of the completion of $M \to N$ with the isomorphism $M' \cong M \otimes_A A^\wedge$ is φ. Since $A \to \prod A_{f_i} \times A^\wedge$ is faithfully flat, we conclude that M is a finite A-module (Algebra, Lemma 10.78.2). We omit the verification that the triple $(M, N \to M, M^\wedge \to M')$ so obtained is unique up to unique isomorphism. □

Lemma 29.22.4. *Let X be a Noetherian scheme. Let $\mathcal{I}, \mathcal{K} \subset \mathcal{O}_X$ be quasi-coherent sheaves of ideals. Let $X_e \subset X$ be the closed subscheme cut out by $\mathcal{K}^e$. Let $\mathcal{I}_e = \mathcal{I}\mathcal{O}_{X_e}$. Let $(\mathcal{F}_n)$ be an object of $Coh(X, \mathcal{I})$. Assume*

1. *the functor $Coh(\mathcal{O}_{X_e}) \to Coh(X_e, \mathcal{I}_e)$ is an equivalence for all $e \geq 1$, and*
2. *there exists a coherent sheaf $\mathcal{H}$ on X and a map $\alpha : (\mathcal{F}_n) \to \mathcal{H}^\wedge$ whose kernel and cokernel are annihilated by a power of $\mathcal{K}$.*

Then $(\mathcal{F}_n)$ is in the essential image of (29.21.4.1).

Proof. During this proof we will use without further mention that for a closed immersion $i : Z \to X$ the functor i_* gives an equivalence between the category of coherent modules on Z and coherent modules on X annihilated by the ideal sheaf of Z, see Lemma 29.9.8. In particular we may identify $Coh(\mathcal{O}_{X_e})$ with the category of coherent $\mathcal{O}_X$-modules annihilated by $\mathcal{K}^e$ and $Coh(X_e, \mathcal{I}_e)$ as the full subcategory of $Coh(X, \mathcal{I})$ of objects annihilated by $\mathcal{K}^e$. Moreover (1) tells us these two categories are equivalent under the completion functor (29.21.4.1).

Applying this equivalence we get a coherent $\mathcal{O}_X$-module $\mathcal{G}_e$ annihilated by $\mathcal{K}^e$ corresponding to the system $(\mathcal{F}_n/\mathcal{K}^e\mathcal{F}_n)$ of $Coh(X, \mathcal{I})$. The maps $\mathcal{F}_n/\mathcal{K}^{e+1}\mathcal{F}_n \to \mathcal{F}_n/\mathcal{K}^e\mathcal{F}_n$ correspond to canonical maps $\mathcal{G}_{e+1} \to \mathcal{G}_e$ which induce isomorphisms $\mathcal{G}_{e+1}/\mathcal{K}^e\mathcal{G}_{e+1} \to \mathcal{G}_e$. Hence $(\mathcal{G}_e)$ is an object of $Coh(X, \mathcal{K})$. The map α induces a system of maps

$$\mathcal{F}_n/\mathcal{K}^e\mathcal{F}_n \longrightarrow \mathcal{H}/(\mathcal{I}^n + \mathcal{K}^e)\mathcal{H}$$

whence maps $\mathcal{G}_e \to \mathcal{H}/\mathcal{K}^e\mathcal{H}$ (by the equivalence of categories again). Let $t \geq 1$ be an integer, which exists by assumption (2), such that $\mathcal{K}^t$ annihilates the kernel and cokernel of all the maps $\mathcal{F}_n \to \mathcal{H}/\mathcal{I}^n\mathcal{H}$. Then $\mathcal{K}^{2t}$ annihilates the kernel and cokernel of the maps $\mathcal{F}_n/\mathcal{K}^e\mathcal{F}_n \to \mathcal{H}/(\mathcal{I}^n + \mathcal{K}^e)\mathcal{H}$, see Remark 29.22.2. Whereupon we conclude that $\mathcal{K}^{4t}$ annihilates the kernel and the cokernel of the maps

$$\mathcal{G}_e \longrightarrow \mathcal{H}/\mathcal{K}^e\mathcal{H},$$

see Remark 29.22.2. We apply Lemma 29.22.3 to obtain a coherent $\mathcal{O}_X$-module $\mathcal{F}$, a map $a : \mathcal{F} \to \mathcal{H}$ and an isomorphism $\beta : (\mathcal{G}_e) \to (\mathcal{F}/\mathcal{K}^e\mathcal{F})$ in $Coh(X, \mathcal{K})$. Working backwards, for a given n the triple $(\mathcal{F}/\mathcal{I}^n\mathcal{F}, a \bmod \mathcal{I}^n, \beta \bmod \mathcal{I}^n)$ is a triple as in the lemma for the morphism $\alpha_n \bmod \mathcal{K}^e : (\mathcal{F}_n/\mathcal{K}^e\mathcal{F}_n) \to (\mathcal{H}/(\mathcal{I}^n + \mathcal{K}^e)\mathcal{H})$ of $Coh(X, \mathcal{K})$. Thus the uniqueness in Lemma 29.22.3 gives a canonical isomorphism $\mathcal{F}/\mathcal{I}^n\mathcal{F} \to \mathcal{F}_n$ compatible with all the morphisms in sight. This finishes the proof of the lemma. □

Lemma 29.22.5. *Let Y be a Noetherian scheme. Let $\mathcal{J}, \mathcal{K} \subset \mathcal{O}_Y$ be quasi-coherent sheaves of ideals. Let $f : X \to Y$ be a proper morphism which is an isomorphism over $V = Y \setminus V(\mathcal{K})$. Set $\mathcal{I} = f^{-1}\mathcal{J}\mathcal{O}_X$. Let $(\mathcal{G}_n)$ be an object of $Coh(Y, \mathcal{J})$, let $\mathcal{F}$ be*

a coherent $\mathcal{O}_X$-module, and let $\beta : (f^*\mathcal{G}_n) \to \mathcal{F}^\wedge$ be an isomorphism in $Coh(X, \mathcal{I})$. Then there exists a map

$$\alpha : (\mathcal{G}_n) \longrightarrow (f_*\mathcal{F})^\wedge$$

in $Coh(Y, \mathcal{J})$ whose kernel and cokernel are annihilated by a power of $\mathcal{K}$.

Proof. Since f is a proper morphism we see that $f_*\mathcal{F}$ is a coherent $\mathcal{O}_Y$-module (Lemma 29.17.2). Thus the statement of the lemma makes sense. Consider the compositions

$$\gamma_n : \mathcal{G}_n \to f_*f^*\mathcal{G}_n \to f_*(\mathcal{F}/\mathcal{I}^n\mathcal{F}).$$

Here the first map is the adjunction map and the second is $f_*\beta_n$. We claim that there exists a unique α as in the lemma such that the compositions

$$\mathcal{G}_n \xrightarrow{\alpha_n} f_*\mathcal{F}/\mathcal{J}^n f_*\mathcal{F} \to f_*(\mathcal{F}/\mathcal{I}^n\mathcal{F})$$

equal γ_n for all n. Because of the uniqueness we may assume that $Y = \mathrm{Spec}(B)$ is affine. Let $J \subset B$ corresponds to the ideal $\mathcal{J}$. Set

$$M_n = H^0(X, \mathcal{F}/\mathcal{I}^n\mathcal{F}) \quad \text{and} \quad M = H^0(X, \mathcal{F})$$

By Lemma 29.18.4 and Theorem 29.18.5 the inverse limit of the modules M_n equals the completion $M^\wedge = \lim M/J^nM$. Set $N_n = H^0(Y, \mathcal{G}_n)$ and $N = \lim N_n$. Via the equivalence of categories of Lemma 29.21.1 the finite $B^\wedge$ modules N and $M^\wedge$ correspond to $(\mathcal{G}_n)$ and $f_*\mathcal{F}^\wedge$. It follows from this that α has to be the morphism of $Coh(Y, \mathcal{J})$ corresponding to the homomorphism

$$\lim \gamma_n : N = \lim_n N_n \longrightarrow \lim M_n = M^\wedge$$

of finite $B^\wedge$-modules.

We still have to show that the kernel and cokernel of α are annihilated by a power of $\mathcal{K}$. Set $Y' = \mathrm{Spec}(B^\wedge)$ and $X' = Y' \times_Y X$. Let $\mathcal{K}'$, $\mathcal{J}'$, $\mathcal{G}'_n$ and $\mathcal{I}'$, $\mathcal{F}'$ be the pullback of $\mathcal{K}$, $\mathcal{J}$, $\mathcal{G}_n$ and $\mathcal{I}$, $\mathcal{F}$, to Y' and X'. The projection morphism $f' : X' \to Y'$ is the base change of f by $Y' \to Y$. Note that $Y' \to Y$ is a flat morphism of schemes as $B \to B^\wedge$ is flat by Algebra, Lemma 10.91.3. Hence $f'_*\mathcal{F}'$, resp. $f'_*(f')^*\mathcal{G}'_n$ is the pullback of $f_*\mathcal{F}$, resp. $f_*f^*\mathcal{G}_n$ to Y' by Lemma 29.5.2. The uniqueness of our construction shows the pullback of α to Y' is the corresponding map α' constructed for the situation on Y'. Moreover, to check that the kernel and cokernel of α are annihilated by $\mathcal{K}^t$ it suffices to check that the kernel and cokernel of α' are annihilated by $(\mathcal{K}')^t$. Namely, to see this we need to check this for kernels and cokernels of the maps α_n and α'_n (see Remark 29.22.2) and the ring map $B \to B^\wedge$ induces an equivalence of categories between modules annihilated by J^n and $(J')^n$, see More on Algebra, Lemma 15.56.7. Thus we may assume B is complete with respect to J.

Assume $Y = \mathrm{Spec}(B)$ is affine, $\mathcal{J}$ corresponds to the ideal $J \subset B$, and B is complete with respect to J. In this case $(\mathcal{G}_n)$ is in the essential image of the functor $Coh(\mathcal{O}_Y) \to Coh(Y, \mathcal{J})$. Say $\mathcal{G}$ is a coherent $\mathcal{O}_Y$-module such that $(\mathcal{G}_n) = \mathcal{G}^\wedge$. Note that $f^*(\mathcal{G}^\wedge) = (f^*\mathcal{G})^\wedge$. Hence Lemma 29.21.7 tells us that β comes from an isomorphism $b : f^*\mathcal{G} \to \mathcal{F}$ and α is the completion functor applied to

$$\mathcal{G} \to f_*f^*\mathcal{G} \cong f_*\mathcal{F}$$

Hence we are trying to verify that the kernel and cokernel of the adjunction map $c : \mathcal{G} \to f_*f^*\mathcal{G}$ are annihilated by a power of $\mathcal{K}$. However, since the restriction $f|_{f^{-1}(V)} : f^{-1}(V) \to V$ is an isomorphism we see that $c|_V$ is an isomorphism.

Thus the coherent sheaves $\mathrm{Ker}(c)$ and $\mathrm{Coker}(c)$ are supported on $V(\mathcal{K})$ hence are annihilated by a power of $\mathcal{K}$ (Lemma 29.10.2) as desired. □

The following proposition is the form of Grothendieck's existence theorem which is most often used in practice.

Proposition 29.22.6. *Let A be a Noetherian ring complete with respect to an ideal I. Let $f : X \to \mathrm{Spec}(A)$ be a proper morphism of schemes. Set $\mathcal{I} = I\mathcal{O}_X$. Then the functor (29.21.4.1) is an equivalence.*

Proof. We have already seen that (29.21.4.1) is fully faithful in Lemma 29.21.7. Thus it suffices to show that the functor is essentially surjective.

Consider the collection Ξ of quasi-coherent sheaves of ideals $\mathcal{K} \subset \mathcal{O}_X$ such that every object $(\mathcal{F}_n)$ annihilated by $\mathcal{K}$ is in the essential image. We want to show (0) is in Ξ. If not, then since X is Noetherian there exists a maximal quasi-coherent sheaf of ideals $\mathcal{K}$ not in Ξ, see Lemma 29.10.1. After replacing X by the closed subscheme of X corresponding to $\mathcal{K}$ we may assume that every nonzero $\mathcal{K}$ is in Ξ. (This uses the correspondence by coherent modules annihilated by $\mathcal{K}$ and coherent modules on the closed subscheme corresponding to $\mathcal{K}$, see Lemma 29.9.8.) Let $(\mathcal{F}_n)$ be an object of $Coh(X, \mathcal{I})$. We will show that this object is in the essential image of the functor (29.21.4.1), thereby completion the proof of the proposition.

Apply Chow's lemma (Lemma 29.16.1) to find a proper surjective morphism $f : X' \to X$ which is an isomorphism over a dense open $U \subset X$ such that X' is projective over A. Let $\mathcal{K}$ be the quasi-coherent sheaf of ideals cutting out the reduced complement $X \setminus U$. By the projective case of Grothendieck's existence theorem (Lemma 29.21.9) there exists a coherent module $\mathcal{F}'$ on X' such that $(\mathcal{F}')^\wedge \cong (f^*\mathcal{F}_n)$. By Lemma 29.17.2 the $\mathcal{O}_X$-module $\mathcal{H} = f_*\mathcal{F}'$ is coherent and by Lemma 29.22.5 there exists a morphism $(\mathcal{F}_n) \to \mathcal{H}^\wedge$ of $Coh(X, \mathcal{I})$ whose kernel and cokernel are annihilated by a power of $\mathcal{K}$. The powers $\mathcal{K}^e$ are all in Ξ so that (29.21.4.1) is an equivalence for the closed subschemes $X_e = V(\mathcal{K}^e)$. We conclude by Lemma 29.22.4. □

To state the general version of Grothendieck's existence theorem we introduce a bit more notation. Let A be a Noetherian ring complete with respect to an ideal I. Let $f : X \to \mathrm{Spec}(A)$ be a separated finite type morphism of schemes. Set $\mathcal{I} = I\mathcal{O}_X$. In this situation we let

$$Coh_{\text{support proper over } A}(\mathcal{O}_X)$$

be the full subcategory of $Coh(\mathcal{O}_X)$ consisting of those coherent $\mathcal{O}_X$-modules whose scheme theoretic support is proper over $\mathrm{Spec}(A)$. Similarly, we let

$$Coh_{\text{support proper over } A}(X, \mathcal{I})$$

be the full subcategory of $Coh(X, \mathcal{I})$ consisting of those objects $(\mathcal{F}_n)$ such that the scheme theoretic support of $\mathcal{F}_1$ is proper over $\mathrm{Spec}(A)$. Since the support of a quotient module is contained in the support of the module, it follows that (29.21.4.1) induces a functor

$$Coh_{\text{support proper over } A}(\mathcal{O}_X) \longrightarrow Coh_{\text{support proper over } A}(X, \mathcal{I}) \tag{29.22.6.1}$$

We are now ready to state the main theorem of this section.

Theorem 29.22.7 (Grothendieck's existence theorem)**.** *In the situation described above the functor (29.22.6.1) is an equivalence.*

Proof. We will use the equivalence of categories of Lemma 29.9.8 without further mention in the proof of the theorem. Let $Z \subset X$ be a closed subscheme proper over A. By Proposition 29.22.6 we know that the result is true for the functor between coherent modules and systems of coherent modules supported on Z. Hence it suffices to show that every object of $Coh_{\text{support proper over } A}(\mathcal{O}_X)$ and every object of $Coh_{\text{support proper over } A}(X, \mathcal{I})$ is supported on such a closed subscheme $Z \subset X$ proper over A. This holds by definition for objects of $Coh_{\text{support proper over } A}(\mathcal{O}_X)$. We will prove this statement for objects of $Coh_{\text{support proper over } A}(X, \mathcal{I})$ using the method of proof of Proposition 29.22.6. We urge the reader to read that proof first.

Consider the collection Ξ of quasi-coherent sheaves of ideals $\mathcal{K} \subset \mathcal{O}_X$ such that the statement holds for every object $(\mathcal{F}_n)$ of $Coh_{\text{support proper over } A}(X, \mathcal{I})$ annihilated by $\mathcal{K}$. We want to show (0) is in Ξ. If not, then since X is Noetherian there exists a maximal quasi-coherent sheaf of ideals $\mathcal{K}$ not in Ξ, see Lemma 29.10.1. After replacing X by the closed subscheme of X corresponding to $\mathcal{K}$ we may assume that every nonzero $\mathcal{K}$ is in Ξ. Let $(\mathcal{F}_n)$ be an object of $Coh_{\text{support proper over } A}(X, \mathcal{I})$. We will show that this object is supported on a closed subscheme $Z \subset X$ proper over A, thereby completing the proof of the theorem.

Apply Chow's lemma (Lemma 29.16.1) to find a proper surjective morphism $f : Y \to X$ which is an isomorphism over a dense open $U \subset X$ such that Y is H-quasi-projective over A. Choose an open immersion $j : Y \to Y'$ with Y' projective over A, see Morphisms, Lemma 28.43.11. Let T_n be the scheme theoretic support of $\mathcal{F}_n$. Note that $T_n = T_1$ set-theoretically, hence T_n is proper over A for all n (Morphisms, Lemma 28.42.8). Then $f^*\mathcal{F}_n$ is supported on the closed subscheme $f^{-1}T_n$ which is proper over A (by Morphisms, Lemma 28.42.4 and properness of f). In particular, the composition $f^{-1}T_n \to Y \to Y'$ is closed (Morphisms, Lemma 28.42.7). Let $T'_n \subset Y'$ be the corresponding closed subscheme; it is contained in the open subscheme Y and equal to $f^{-1}T_n$ as a closed subscheme of Y. Let $\mathcal{F}'_n$ be the coherent $\mathcal{O}_{Y'}$-module corresponding to $f^*\mathcal{F}_n$ viewed as a coherent module on Y' via the closed immersion $f^{-1}T_n = T'_n \subset Y'$. Then $(\mathcal{F}'_n)$ is an object of $Coh(Y', I\mathcal{O}_{Y'})$. By the projective case of Grothendieck's existence theorem (Lemma 29.21.9) there exists a coherent $\mathcal{O}_{Y'}$-module $\mathcal{F}'$ and an isomorphism $(\mathcal{F}')^\wedge \cong (\mathcal{F}'_n)$ in $Coh(Y', I\mathcal{O}_{Y'})$. Let $Z' \subset Y'$ be the scheme theoretic support of $\mathcal{F}'$. Since $\mathcal{F}'/I\mathcal{F}' = \mathcal{F}'_1$ we see that $Z' \cap V(I\mathcal{O}_{Y'}) = T'_1$ set-theoretically. The structure morphism $p' : Y' \to \mathrm{Spec}(A)$ is proper, hence $p'(Z' \cap (Y' \setminus Y))$ is closed in $\mathrm{Spec}(A)$. If nonempty, then it would contain a point of $V(I)$ as I is contained in the radical of A (Algebra, Lemma 10.91.11). But we've seen above that $Z' \cap (p')^{-1}V(I) = T'_1 \subset Y$ hence we conclude that $Z' \subset Y$. Thus $\mathcal{F}'|_Y$ is supported on a closed subscheme of Y proper over A.

Let $\mathcal{K}$ be the quasi-coherent sheaf of ideals cutting out the reduced complement $X \setminus U$. By Lemma 29.17.2 the $\mathcal{O}_X$-module $\mathcal{H} = f_*\mathcal{F}'$ is coherent and by Lemma 29.22.5 there exists a morphism $\alpha : (\mathcal{F}_n) \to \mathcal{H}^\wedge$ of $Coh(X, \mathcal{I})$ whose kernel and cokernel are annihilated by a power of $\mathcal{K}$. Let $Z_0 \subset X$ be the scheme theoretic support of $\mathcal{H}$. It is clear that $Z_0 \subset f(Z')$ set-theoretically. Hence $Z_0 \to \mathrm{Spec}(A)$ is proper (Morphisms, Lemma 28.42.7). The kernel of α is an object of $Coh_{\text{support proper over } A}(X, \mathcal{I})$ annihilated by a power of $\mathcal{K}^t$ which is in Ξ. Hence the kernel of α are supported on closed subschemes $Z_1 \subset X$ proper over A. Let $\mathcal{K}_i \subset \mathcal{O}_X$ be the quasi-coherent sheaf of ideals cutting out Z_i for $i = 0, 1$. Set

$\mathcal{K} = \mathcal{K}_0\mathcal{K}_1$ and let $Z = V(\mathcal{K}) \subset X$. Then $(\mathcal{F}_n)$ is supported on Z (details omitted). Finally, $Z_0 \amalg Z_1 \to Z$ is surjective, whence Z is proper over A by Morphisms, Lemma 28.42.7. This finishes the proof of the theorem. □

Remark 29.22.8 (Unwinding Grothendieck's existence theorem)**.** Let A be a Noetherian ring complete with respect to an ideal I. Write $S = \mathrm{Spec}(A)$ and $S_n = \mathrm{Spec}(A/I^n)$. Let $X \to S$ be a separated morphism of finite type. For $n \geq 1$ we set $X_n = X \times_S S_n$. Picture:

$$\begin{array}{ccccccccc} X_1 & \xrightarrow[i_1]{} & X_2 & \xrightarrow[i_2]{} & X_3 & \longrightarrow & \ldots & & X \\ \downarrow & & \downarrow & & \downarrow & & & & \downarrow \\ S_1 & \longrightarrow & S_2 & \longrightarrow & S_3 & \longrightarrow & \ldots & & S \end{array}$$

In this situation we consider systems $(\mathcal{F}_n, \varphi_n)$ where

(1) $\mathcal{F}_n$ is a coherent $\mathcal{O}_{X_n}$-module,
(2) $\varphi_n : i_n^*\mathcal{F}_{n+1} \to \mathcal{F}_n$ is an isomorphism, and
(3) $\mathrm{Supp}(\mathcal{F}_1)$ is proper over S_1.

Theorem 29.22.7 says that the completion functor

$$\begin{array}{c} \text{coherent } \mathcal{O}_X\text{-modules } \mathcal{F} \\ \text{with support proper over } A \end{array} \longrightarrow \begin{array}{c} \text{systems } (\mathcal{F}_n) \\ \text{as above} \end{array}$$

is an equivalence of categories. In the special case that X is proper over A we can omit the conditions on the supports.

29.23. Grothendieck's algebraization theorem

Our first result is a translation of Grothendieck's existence theorem in terms of closed subschemes and finite morphisms.

Lemma 29.23.1. *Let A be a Noetherian ring complete with respect to an ideal I. Write $S = \mathrm{Spec}(A)$ and $S_n = \mathrm{Spec}(A/I^n)$. Let $X \to S$ be a separated morphism of finite type. For $n \geq 1$ we set $X_n = X \times_S S_n$. Suppose given a commutative diagram*

$$\begin{array}{ccccccccc} Z_1 & \longrightarrow & Z_2 & \longrightarrow & Z_3 & \longrightarrow & \ldots \\ \downarrow & & \downarrow & & \downarrow & & \\ X_1 & \xrightarrow{i_1} & X_2 & \xrightarrow{i_2} & X_3 & \longrightarrow & \ldots \end{array}$$

of schemes with cartesian squares. Assume that

(1) *$Z_1 \to X_1$ is a closed immersion, and*
(2) *$Z_1 \to S_1$ is proper.*

Then there exists a closed immersion of schemes $Z \to X$ such that $Z_n = Z \times_S S_n$.

Proof. Let's write $j_n : Z_n \to X_n$ for the vertical morphisms. As the squares in the statement are cartesian we see that the base change of j_n to X_1 is j_1. Thus Morphisms, Lemma 28.45.5 shows that j_n is a closed immersion. Set $\mathcal{F}_n = j_{n,*}\mathcal{O}_{Z_n}$, so that $j_n^\sharp$ is a surjection $\mathcal{O}_{X_n} \to \mathcal{F}_n$. Again using that the squares are cartesian we see that the pullback of $\mathcal{F}_{n+1}$ to X_n is $\mathcal{F}_n$. Hence Grothendieck's existence theorem, as reformulated in Remark 29.22.8, tells us there exists a map $\mathcal{O}_X \to \mathcal{F}$ of coherent $\mathcal{O}_X$-modules whose restriction to X_n recovers $\mathcal{O}_{X_n} \to \mathcal{F}_n$. As the completion functor is exact (Lemma 29.21.5) we see that $\mathcal{O}_X \to \mathcal{F}$ is surjective.

Thus $\mathcal{F} = \mathcal{O}_X/\mathcal{J}$ for some quasi-coherent sheaf of ideals $\mathcal{J}$. Setting $Z = V(\mathcal{J})$ finishes the proof. □

In the following lemma it is actually enough to assume that $Y_1 \to X_1$ is finite as it will imply that $Y_n \to X_n$ is finite too (see More on Morphisms, Lemma 36.2.6).

Lemma 29.23.2. *Let A be a Noetherian ring complete with respect to an ideal I. Write $S = \mathrm{Spec}(A)$ and $S_n = \mathrm{Spec}(A/I^n)$. Let $X \to S$ be a separated morphism of finite type. For $n \geq 1$ we set $X_n = X \times_S S_n$. Suppose given a commutative diagram*

$$\begin{array}{ccccccc} Y_1 & \longrightarrow & Y_2 & \longrightarrow & Y_3 & \longrightarrow & \ldots \\ \downarrow & & \downarrow & & \downarrow & & \\ X_1 & \xrightarrow{i_1} & X_2 & \xrightarrow{i_2} & X_3 & \longrightarrow & \ldots \end{array}$$

of schemes with cartesian squares. Assume that

(1) *$Y_n \to X_n$ is a finite morphism, and*
(2) *$Y_1 \to S_1$ is proper.*

Then there exists a finite morphism of schemes $Y \to X$ such that $Y_n = Y \times_S S_n$.

Proof. Let's write $f_n : Y_n \to X_n$ for the vertical morphisms. Set $\mathcal{F}_n = f_{n,*}\mathcal{O}_{Y_n}$. This is a coherent $\mathcal{O}_{X_n}$-module as f_n is finite (Lemma 29.9.9). Using that the squares are cartesian we see that the pullback of $\mathcal{F}_{n+1}$ to X_n is $\mathcal{F}_n$. Hence Grothendieck's existence theorem, as reformulated in Remark 29.22.8, tells us there exists a coherent $\mathcal{O}_X$-module $\mathcal{F}$ whose restriction to X_n recovers $\mathcal{F}_n$. As the completion functor is fuly faithful (Theorem 29.22.7) we see that the multiplication maps $\mathcal{F}_n \otimes_{\mathcal{O}_{X_n}} \mathcal{F}_n \to \mathcal{F}_n$ fit together to give an algebra structure on $\mathcal{F}$. Setting $Y = \underline{\mathrm{Spec}}_X(\mathcal{F})$ finishes the proof. □

In order to prove an "abstract" algebraization theorem we need to assume we have an ample invertible sheaf, as the result is false without such an assumption.

Theorem 29.23.3 (Grothendieck's algebraization theorem)**.** *Let A be a Noetherian ring complete with respect to an ideal I. Set $S = \mathrm{Spec}(A)$ and $S_n = \mathrm{Spec}(A/I^n)$. Consider a commutative diagram*

$$\begin{array}{ccccccc} X_1 & \xrightarrow[i_1]{} & X_2 & \xrightarrow[i_2]{} & X_3 & \longrightarrow & \ldots \\ \downarrow & & \downarrow & & \downarrow & & \\ S_1 & \longrightarrow & S_2 & \longrightarrow & S_3 & \longrightarrow & \ldots \end{array}$$

of schemes with cartesian squares. Suppose given $(\mathcal{L}_n, \varphi_n)$ where each $\mathcal{L}_n$ is an invertible sheaf on X_n and $\varphi_n : i_n^*\mathcal{L}_{n+1} \to \mathcal{L}_n$ is an isomorphism. If

(1) *$X_1 \to S_1$ is proper, and*
(2) *$\mathcal{L}_1$ is ample on X_1*

then there exists a proper morphism of schemes $X \to S$ and an ample invertible $\mathcal{O}_X$-module $\mathcal{L}$ and isomorphisms $X_n \cong X \times_S S_n$ and $\mathcal{L}_n \cong \mathcal{L}|_{X_n}$ compatible with the morphisms i_n and φ_n.

Proof. Since the squares in the diagram are cartesian and since the morphisms $S_n \to S_{n+1}$ are closed immersions, we see that the morphisms i_n are closed immersions too. In particular we may think of X_m as a closed subscheme of X_n for $m < n$. In fact X_m is the closed subscheme cut out by the quasi-coherent sheaf of ideals

$I^m\mathcal{O}_{X_n}$. Moreover, the underlying topological spaces of the schemes $X_1, X_2, X_3, \ldots$ are all identified, hence we may (and do) think of sheaves $\mathcal{O}_{X_n}$ as living on the same underlying topological space; similarly for coherent $\mathcal{O}_{X_n}$-modules. Set

$$\mathcal{F}_n = \mathrm{Ker}(\mathcal{O}_{X_{n+1}} \to \mathcal{O}_{X_n})$$

so that we obtain short exact sequences

$$0 \to \mathcal{F}_n \to \mathcal{O}_{X_{n+1}} \to \mathcal{O}_{X_n} \to 0$$

By the above we have $\mathcal{F}_n = I^n\mathcal{O}_{X_{n+1}}$. It follows $\mathcal{F}_n$ is a coherent sheaf on X_{n+1} annihilated by I, hence we may (and do) think of it as a coherent module $\mathcal{O}_{X_1}$-module. Observe that for $m > n$ the sheaf

$$I^n\mathcal{O}_{X_m}/I^{n+1}\mathcal{O}_{X_m}$$

maps isomorphically to $\mathcal{F}_n$ under the map $\mathcal{O}_{X_m} \to \mathcal{O}_{X_{n+1}}$. Hence given $n_1, n_2 \geq 0$ we can pick an $m > n_1 + n_2$ and consider the multiplication map

$$I^{n_1}\mathcal{O}_{X_m} \times I^{n_2}\mathcal{O}_{X_m} \longrightarrow I^{n_1+n_2}\mathcal{O}_{X_m} \to \mathcal{F}_{n_1+n_2}$$

This induces an $\mathcal{O}_{X_1}$-bilinear map

$$\mathcal{F}_{n_1} \times \mathcal{F}_{n_2} \longrightarrow \mathcal{F}_{n_1+n_2}$$

which in turn defines the structure of a graded $\mathcal{O}_{X_1}$-algebra on $\mathcal{F} = \bigoplus_{n\geq 0}\mathcal{F}_n$.

Set $B = \bigoplus I^n/I^{n+1}$; this is a finitely generated graded A/I-algebra. Set $\mathcal{B} = (X_1 \to S_1)^*\widetilde{B}$. The discussion above provides us with a canonical surjection

$$\mathcal{B} \longrightarrow \mathcal{F}$$

of graded $\mathcal{O}_{X_1}$-algebras. In particular we see that $\mathcal{F}$ is a finite type quasi-coherent graded $\mathcal{B}$-module. By Lemma 29.17.5 we can find an integer d_0 such that $H^1(X_1, \mathcal{F}\otimes \mathcal{L}^{\otimes d}) = 0$ for all $d \geq d_0$. Pick a $d \geq d_0$ such that there exist sections $s_{0,1}, \ldots, s_{N,1} \in \Gamma(X_1, \mathcal{L}_1^{\otimes d})$ which induce an immersion

$$\psi_1 : X_1 \to \mathbf{P}^N_{S_1}$$

over S_1, see Morphisms, Lemma 28.40.4. As X_1 is proper over S_1 we see that ψ_1 is a closed immersion, see Morphisms, Lemma 28.42.7 and Schemes, Lemma 25.10.4. We are going to "lift" ψ_1 to a compatible system of closed immersions of X_n into $\mathbf{P}^N$.

Upon tensoring the short exact sequences of the first paragraph of the proof by $\mathcal{L}_{n+1}^{\otimes d}$ we obtain short exact sequences

$$0 \to \mathcal{F}_n \otimes \mathcal{L}_{n+1}^{\otimes d} \to \mathcal{L}_{n+1}^{\otimes d} \to \mathcal{L}_{n+1}^{\otimes d} \to 0$$

Using the isomorphisms φ_n we obtain isomorphisms $\mathcal{L}_{n+1} \otimes \mathcal{O}_{X_l} = \mathcal{L}_l$ for $l \leq n$. Whence the sequence above becomes

$$0 \to \mathcal{F}_n \otimes \mathcal{L}_1^{\otimes d} \to \mathcal{L}_{n+1}^{\otimes d} \to \mathcal{L}_n^{\otimes d} \to 0$$

The vanishing of $H^1(X, \mathcal{F}_n \otimes \mathcal{L}_1^{\otimes d})$ implies we can inductively lift $s_{0,1}, \ldots, s_{N,1} \in \Gamma(X_1, \mathcal{L}_1^{\otimes d})$ to sections $s_{0,n}, \ldots, s_{N,n} \in \Gamma(X_n, \mathcal{L}_n^{\otimes d})$. Thus we obtain a commutative

diagram

$$\begin{array}{ccccccc} X_1 & \xrightarrow[i_1]{} & X_2 & \xrightarrow[i_2]{} & X_3 & \longrightarrow & \ldots \\ \downarrow{\scriptstyle \psi_1} & & \downarrow{\scriptstyle \psi_2} & & \downarrow{\scriptstyle \psi_3} & & \\ \mathbf{P}^N_{S_1} & \longrightarrow & \mathbf{P}^N_{S_2} & \longrightarrow & \mathbf{P}^N_{S_3} & \longrightarrow & \ldots \end{array}$$

where $\psi_n = \varphi_{(\mathcal{L}_n, (s_{0,n}, \ldots, s_{N,n}))}$ in the notation of Constructions, Section 26.13. As the squares in the statement of the theorem are cartesian we see that the squares in the above diagram are cartesian. We win by applying Lemma 29.23.1. □

29.24. Other chapters

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Schemes

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CHAPTER 30

Divisors

30.1. Introduction

In this chapter we study some very basic questions related to defining divisors, etc. A basic reference is [**DG67**].

30.2. Associated points

Let R be a ring and let M be an R-module. Recall that a prime $\mathfrak{p} \subset R$ is *associated* to M if there exists an element of M whose annihilator is $\mathfrak{p}$. See Algebra, Definition 10.60.1. Here is the definition of associated points for quasi-coherent sheaves on schemes as given in [**DG67**, IV Definition 3.1.1].

Definition 30.2.1. Let X be a scheme. Let $\mathcal{F}$ be a quasi-coherent sheaf on X.

(1) We say $x \in X$ is *associated* to $\mathcal{F}$ if the maximal ideal $\mathfrak{m}_x$ is associated to the $\mathcal{O}_{X,x}$-module $\mathcal{F}_x$.
(2) We denote $\text{Ass}(\mathcal{F})$ or $\text{Ass}_X(\mathcal{F})$ the set of associated points of $\mathcal{F}$.
(3) The *associated points of* X are the associated points of $\mathcal{O}_X$.

These definitions are most useful when X is locally Noetherian and $\mathcal{F}$ of finite type. For example it may happen that a generic point of an irreducible component of X is not associated to X, see Example 30.2.7. In the non-Noetherian case it may be more convenient to use weakly associated points, see Section 30.5. Let us link the scheme theoretic notion with the algebraic notion on affine opens; note that this correspondence works perfectly only for locally Noetherian schemes.

Lemma 30.2.2. *Let X be a scheme. Let $\mathcal{F}$ be a quasi-coherent sheaf on X. Let $\text{Spec}(A) = U \subset X$ be an affine open, and set $M = \Gamma(U, \mathcal{F})$. Let $x \in U$, and let $\mathfrak{p} \subset A$ be the corresponding prime.*

(1) *If $\mathfrak{p}$ is associated to M, then x is associated to $\mathcal{F}$.*
(2) *If $\mathfrak{p}$ is finitely generated, then the converse holds as well.*

In particular, if X is locally Noetherian, then the equivalence

$$\mathfrak{p} \in Ass(M) \Leftrightarrow x \in Ass(\mathcal{F})$$

holds for all pairs $(\mathfrak{p}, x)$ as above.

Proof. This follows from Algebra, Lemma 10.60.14. But we can also argue directly as follows. Suppose $\mathfrak{p}$ is associated to M. Then there exists an $m \in M$ whose annihilator is $\mathfrak{p}$. Since localization is exact we see that $\mathfrak{p}A_\mathfrak{p}$ is the annihilator of $m/1 \in M_\mathfrak{p}$. Since $M_\mathfrak{p} = \mathcal{F}_x$ (Schemes, Lemma 25.5.4) we conclude that x is associated to $\mathcal{F}$.

Conversely, assume that x is associated to $\mathcal{F}$, and $\mathfrak{p}$ is finitely generated. As x is associated to $\mathcal{F}$ there exists an element $m' \in M_\mathfrak{p}$ whose annihilator is $\mathfrak{p}A_\mathfrak{p}$. Write

$m' = m/f$ for some $f \in A$, $f \notin \mathfrak{p}$. The annihilator I of m is an ideal of A such that $IA_\mathfrak{p} = \mathfrak{p}A_\mathfrak{p}$. Hence $I \subset \mathfrak{p}$, and $(\mathfrak{p}/I)_\mathfrak{p} = 0$. Since $\mathfrak{p}$ is finitely generated, there exists a $g \in A$, $g \notin \mathfrak{p}$ such that $g(\mathfrak{p}/I) = 0$. Hence the annihilator of gm is $\mathfrak{p}$ and we win.

If X is locally Noetherian, then A is Noetherian (Properties, Lemma 27.5.2) and $\mathfrak{p}$ is always finitely generated. □

Lemma 30.2.3. *Let X be a scheme. Let $\mathcal{F}$ be a quasi-coherent $\mathcal{O}_X$-module. Then $Ass(\mathcal{F}) \subset Supp(\mathcal{F})$.*

Proof. This is immediate from the definitions. □

Lemma 30.2.4. *Let X be a scheme. Let $0 \to \mathcal{F}_1 \to \mathcal{F}_2 \to \mathcal{F}_3 \to 0$ be a short exact sequence of quasi-coherent sheaves on X. Then $Ass(\mathcal{F}_2) \subset Ass(\mathcal{F}_1) \cup Ass(\mathcal{F}_3)$ and $Ass(\mathcal{F}_1) \subset Ass(\mathcal{F}_2)$.*

Proof. For every point $x \in X$ the sequence of stalks $0 \to \mathcal{F}_{1,x} \to \mathcal{F}_{2,x} \to \mathcal{F}_{3,x} \to 0$ is a short exact sequence of $\mathcal{O}_{X,x}$-modules. Hence the lemma follows from Algebra, Lemma 10.60.3. □

Lemma 30.2.5. *Let X be a locally Noetherian scheme. Let $\mathcal{F}$ be a coherent $\mathcal{O}_X$-module. Then $Ass(\mathcal{F}) \cap U$ is finite for every quasi-compact open $U \subset X$.*

Proof. This is true because the set of associated primes of a finite module over a Noetherian ring is finite, see Algebra, Lemma 10.60.5. To translate from schemes to algebra use that U is a finite union of affine opens, each of these opens is the spectrum of a Noetherian ring (Properties, Lemma 27.5.2), $\mathcal{F}$ corresponds to a finite module over this ring (Cohomology of Schemes, Lemma 29.9.1), and finally use Lemma 30.2.2. □

Lemma 30.2.6. *Let X be a locally Noetherian scheme. Let $\mathcal{F}$ be a quasi-coherent $\mathcal{O}_X$-module. Then*

$$\mathcal{F} = 0 \Leftrightarrow Ass(\mathcal{F}) = \emptyset.$$

Proof. If $\mathcal{F} = 0$, then $\mathrm{Ass}(\mathcal{F}) = \emptyset$ by definition. Conversely, if $\mathrm{Ass}(\mathcal{F}) = \emptyset$, then $\mathcal{F} = 0$ by Algebra, Lemma 10.60.7. To translate from schemes to algebra, restrict to any affine and use Lemma 30.2.2. □

Example 30.2.7. Let k be a field. The ring $R = R[x_1, x_2, x_3, \ldots]/(x_i^2)$ is local with locally nilpotent maximal ideal $\mathfrak{m}$. There exists no element of R which has annihilator $\mathfrak{m}$. Hence $\mathrm{Ass}(R) = \emptyset$, and $X = \mathrm{Spec}(R)$ is an example of a scheme which has no associated points.

Lemma 30.2.8. *Let X be a locally Noetherian scheme. Let $\mathcal{F}$ be a quasi-coherent $\mathcal{O}_X$-module. Let $x \in Supp(\mathcal{F})$ be a point in the support of $\mathcal{F}$ which is not a specialization of another point of $Supp(\mathcal{F})$. Then $x \in Ass(\mathcal{F})$. In particular, any generic point of an irreducible component of X is an associated point of X.*

Proof. Since $x \in \mathrm{Supp}(\mathcal{F})$ the module $\mathcal{F}_x$ is not zero. Hence $\mathrm{Ass}(\mathcal{F}_x) \subset \mathrm{Spec}(\mathcal{O}_{X,x})$ is nonempty by Algebra, Lemma 10.60.7. On the other hand, by assumption $\mathrm{Supp}(\mathcal{F}_x) = \{\mathfrak{m}_x\}$. Since $\mathrm{Ass}(\mathcal{F}_x) \subset \mathrm{Supp}(\mathcal{F}_x)$ (Algebra, Lemma 10.60.2) we see that $\mathfrak{m}_x$ is associated to $\mathcal{F}_x$ and we win. □

30.3. Morphisms and associated points

Lemma 30.3.1. *Let $f : X \to S$ be a morphism of schemes. Let $\mathcal{F}$ be a quasi-coherent sheaf on X which is flat over S. Let $\mathcal{G}$ be a quasi-coherent sheaf on S. Then we have*

$$Ass_X(\mathcal{F} \otimes_{\mathcal{O}_X} f^*\mathcal{G}) \supset \bigcup\nolimits_{s \in Ass_S(\mathcal{G})} Ass_{X_s}(\mathcal{F}_s)$$

and equality holds if S is locally Noetherian.

Proof. Let $x \in X$ and let $s = f(x) \in S$. Set $B = \mathcal{O}_{X,x}$, $A = \mathcal{O}_{S,s}$, $N = \mathcal{F}_x$, and $M = \mathcal{G}_s$. Note that the stalk of $\mathcal{F} \otimes_{\mathcal{O}_X} f^*\mathcal{G}$ at x is equal to the B-module $M \otimes_A N$. Hence $x \in \mathrm{Ass}_X(\mathcal{F} \otimes_{\mathcal{O}_X} f^*\mathcal{G})$ if and only if $\mathfrak{m}_B$ is in $\mathrm{Ass}_B(M \otimes_A N)$. Similarly $s \in \mathrm{Ass}_S(\mathcal{G})$ and $x \in \mathrm{Ass}_{X_s}(\mathcal{F}_s)$ if and only if $\mathfrak{m}_A \in \mathrm{Ass}_A(M)$ and $\mathfrak{m}_B/\mathfrak{m}_A B \in \mathrm{Ass}_{B \otimes \kappa(\mathfrak{m}_A)}(N \otimes \kappa(\mathfrak{m}_A))$. Thus the lemma follows from Algebra, Lemma 10.62.5. □

30.4. Embedded points

Let R be a ring and let M be an R-module. Recall that a prime $\mathfrak{p} \subset R$ is an *embedded associated* to M if it is an associated prime of M which is not minimal among the associated primes of M. See Algebra, Definition 10.64.1. Here is the definition of embedded associated points for quasi-coherent sheaves on schemes as given in [**DG67**, IV Definition 3.1.1].

Definition 30.4.1. Let X be a scheme. Let $\mathcal{F}$ be a quasi-coherent sheaf on X.

(1) An *embedded associated point* of $\mathcal{F}$ is an associated point which is not maximal among the associated points of $\mathcal{F}$, i.e., it is the specialization of another associated point of $\mathcal{F}$.
(2) A point x of X is called an *embedded point* if x is an embedded associated point of $\mathcal{O}_X$.
(3) An *embedded component* of X is an irreducible closed subset $Z = \overline{\{x\}}$ where x is an embedded point of X.

In the Noetherian case when $\mathcal{F}$ is coherent we have the following.

Lemma 30.4.2. *Let X be a locally Noetherian scheme. Let $\mathcal{F}$ be a coherent $\mathcal{O}_X$-module. Then*

(1) *the generic points of irreducible components of Supp($\mathcal{F}$) are associated points of $\mathcal{F}$, and*
(2) *an associated point of $\mathcal{F}$ is embedded if and only if it is not a generic point of an irreducible component of Supp($\mathcal{F}$).*

In particular an embedded point of X is an associated point of X which is not a generic point of an irreducible component of X.

Proof. Recall that in this case $Z = \mathrm{Supp}(\mathcal{F})$ is closed, see Morphisms, Lemma 28.5.3 and that the generic points of irreducible components of Z are associated points of $\mathcal{F}$, see Lemma 30.2.8. Finally, we have $\mathrm{Ass}(\mathcal{F}) \subset Z$, by Lemma 30.2.3. These results, combined with the fact that Z is a sober topological space and hence every point of Z is a specialization of a generic point of Z, imply (1) and (2). □

Lemma 30.4.3. *Let X be a locally Noetherian scheme. Let $\mathcal{F}$ be a coherent sheaf on X. Then the following are equivalent:*

(1) *$\mathcal{F}$ has no embedded associated points, and*

(2) $\mathcal{F}$ *has property* (S_1).

Proof. This is Algebra, Lemma 10.144.2, combined with Lemma 30.2.2 above. □

Lemma 30.4.4. *Let X be a locally Noetherian scheme. Let $U \subset X$ be an open subscheme. The following are equivalent*

(1) *U is scheme theoretically dense in X (Morphisms, Definition 28.7.1),*
(2) *U is dense in X and U contains all embedded points of X.*

Proof. The question is local on X, hence we may assume that $X = \mathrm{Spec}(A)$ where A is a Noetherian ring. Then U is quasi-compact (Properties, Lemma 27.5.3) hence $U = D(f_1) \cup \ldots \cup D(f_n)$ (Algebra, Lemma 10.27.1). In this situation U is scheme theoretically dense in X if and only if $A \to A_{f_1} \times \ldots \times A_{f_n}$ is injective, see Morphisms, Example 28.7.4. Condition (2) translated into algebra means that for every associated prime $\mathfrak{p}$ of A there exists an i with $f_i \notin \mathfrak{p}$.

Assume (1), i.e., $A \to A_{f_1} \times \ldots \times A_{f_n}$ is injective. If $x \in A$ has annihilator a prime $\mathfrak{p}$, then x maps to a nonzero element of A_{f_i} for some i and hence $f_i \notin \mathfrak{p}$. Thus (2) holds. Assume (2), i.e., every associated prime $\mathfrak{p}$ of A corresponds to a prime of A_{f_i} for some i. Then $A \to A_{f_1} \times \ldots \times A_{f_n}$ is injective because $A \to \prod_{\mathfrak{p} \in \mathrm{Ass}(A)} A_{\mathfrak{p}}$ is injective by Algebra, Lemma 10.60.18. □

Lemma 30.4.5. *Let X be a locally Noetherian scheme. Let $\mathcal{F}$ be a coherent sheaf on X. The set of coherent subsheaves*

$$\{\mathcal{K} \subset \mathcal{F} \mid \mathit{Supp}(\mathcal{K}) \text{ is nowhere dense in } \mathit{Supp}(\mathcal{F})\}$$

has a maximal element $\mathcal{K}$. Setting $\mathcal{F}' = \mathcal{F}/\mathcal{K}$ we have the following

(1) $\mathit{Supp}(\mathcal{F}') = \mathit{Supp}(\mathcal{F})$,
(2) *$\mathcal{F}'$ has no embedded associated points, and*
(3) *there exists a dense open $U \subset X$ such that $U \cap \mathit{Supp}(\mathcal{F})$ is dense in $\mathit{Supp}(\mathcal{F})$ and $\mathcal{F}'|_U \cong \mathcal{F}|_U$.*

Proof. This follows from Algebra, Lemmas 10.64.2 and 10.64.3. Note that U can be taken as the complement of the closure of the set of embedded associated points of $\mathcal{F}$. □

Lemma 30.4.6. *Let X be a locally Noetherian scheme. Let $\mathcal{F}$ be a coherent $\mathcal{O}_X$-module without embedded associated points. Set*

$$\mathcal{I} = \mathit{Ker}(\mathcal{O}_X \longrightarrow \mathcal{H}om_{\mathcal{O}_X}(\mathcal{F}, \mathcal{F})).$$

This is a coherent sheaf of ideals which defines a closed subscheme $Z \subset X$ without embedded points. Moreover there exists a coherent sheaf $\mathcal{G}$ on Z such that (a) $\mathcal{F} = (Z \to X)_*\mathcal{G}$, (b) $\mathcal{G}$ has no associated embedded points, and (c) $\mathit{Supp}(\mathcal{G}) = Z$ (as sets).

Proof. Some of the statements we have seen in the proof of Cohomology of Schemes, Lemma 29.9.7. The others follow from Algebra, Lemma 10.64.4. □

30.5. Weakly associated points

Let R be a ring and let M be an R-module. Recall that a prime $\mathfrak{p} \subset R$ is *weakly associated* to M if there exists an element m of M such that $\mathfrak{p}$ is minimal among the primes containing the annihilator of m. See Algebra, Definition 10.63.1. If R is a local ring with maximal ideal $\mathfrak{m}$, then $\mathfrak{m}$ is associated to M if and only if there exists an element $m \in M$ whose annihilator has radical $\mathfrak{m}$, see Algebra, Lemma 10.63.2.

Definition 30.5.1. Let X be a scheme. Let $\mathcal{F}$ be a quasi-coherent sheaf on X.

(1) We say $x \in X$ is *weakly associated* to $\mathcal{F}$ if the maximal ideal $\mathfrak{m}_x$ is weakly associated to the $\mathcal{O}_{X,x}$-module $\mathcal{F}_x$.
(2) We denote $\text{WeakAss}(\mathcal{F})$ the set of weakly associated points of $\mathcal{F}$.
(3) The *weakly associated points of* X are the weakly associated points of $\mathcal{O}_X$.

In this case, on any affine open, this corresponds exactly to the weakly associated primes as defined above. Here is the precise statement.

Lemma 30.5.2. *Let X be a scheme. Let $\mathcal{F}$ be a quasi-coherent sheaf on X. Let $\text{Spec}(A) = U \subset X$ be an affine open, and set $M = \Gamma(U, \mathcal{F})$. Let $x \in U$, and let $\mathfrak{p} \subset A$ be the corresponding prime. The following are equivalent*

(1) *$\mathfrak{p}$ is weakly associated to M, and*
(2) *x is weakly associated to $\mathcal{F}$.*

Proof. This follows from Algebra, Lemma 10.63.2. □

Lemma 30.5.3. *Let X be a scheme. Let $\mathcal{F}$ be a quasi-coherent $\mathcal{O}_X$-module. Then*

$$\mathit{Ass}(\mathcal{F}) \subset \mathit{WeakAss}(\mathcal{F}) \subset \mathit{Supp}(\mathcal{F}).$$

Proof. This is immediate from the definitions. □

Lemma 30.5.4. *Let X be a scheme. Let $0 \to \mathcal{F}_1 \to \mathcal{F}_2 \to \mathcal{F}_3 \to 0$ be a short exact sequence of quasi-coherent sheaves on X. Then $\mathit{WeakAss}(\mathcal{F}_2) \subset \mathit{WeakAss}(\mathcal{F}_1) \cup \mathit{WeakAss}(\mathcal{F}_3)$ and $\mathit{WeakAss}(\mathcal{F}_1) \subset \mathit{WeakAss}(\mathcal{F}_2)$.*

Proof. For every point $x \in X$ the sequence of stalks $0 \to \mathcal{F}_{1,x} \to \mathcal{F}_{2,x} \to \mathcal{F}_{3,x} \to 0$ is a short exact sequence of $\mathcal{O}_{X,x}$-modules. Hence the lemma follows from Algebra, Lemma 10.63.3. □

Lemma 30.5.5. *Let X be a scheme. Let $\mathcal{F}$ be a quasi-coherent $\mathcal{O}_X$-module. Then*

$$\mathcal{F} = (0) \Leftrightarrow \mathit{WeakAss}(\mathcal{F}) = \emptyset$$

Proof. Follows from Lemma 30.5.2 and Algebra, Lemma 10.63.4 □

Lemma 30.5.6. *Let X be a scheme. Let $\mathcal{F}$ be a quasi-coherent $\mathcal{O}_X$-module. Let $x \in \mathit{Supp}(\mathcal{F})$ be a point in the support of $\mathcal{F}$ which is not a specialization of another point of $\mathit{Supp}(\mathcal{F})$. Then $x \in \mathit{WeakAss}(\mathcal{F})$. In particular, any generic point of an irreducible component of X is weakly associated to $\mathcal{O}_X$.*

Proof. Since $x \in \text{Supp}(\mathcal{F})$ the module $\mathcal{F}_x$ is not zero. Hence $\text{WeakAss}(\mathcal{F}_x) \subset \text{Spec}(\mathcal{O}_{X,x})$ is nonempty by Algebra, Lemma 10.63.4. On the other hand, by assumption $\text{Supp}(\mathcal{F}_x) = \{\mathfrak{m}_x\}$. Since $\text{WeakAss}(\mathcal{F}_x) \subset \text{Supp}(\mathcal{F}_x)$ (Algebra, Lemma 10.63.5) we see that $\mathfrak{m}_x$ is weakly associated to $\mathcal{F}_x$ and we win. □

Lemma 30.5.7. *Let X be a scheme. Let $\mathcal{F}$ be a quasi-coherent $\mathcal{O}_X$-module. If $\mathfrak{m}_x$ is a finitely generated ideal of $\mathcal{O}_{X,x}$, then*

$$x \in Ass(\mathcal{F}) \Leftrightarrow x \in WeakAss(\mathcal{F}).$$

In particular, if X is locally Noetherian, then $Ass(\mathcal{F}) = WeakAss(\mathcal{F})$.

Proof. See Algebra, Lemma 10.63.8. □

30.6. Morphisms and weakly associated points

Lemma 30.6.1. *Let $f : X \to S$ be an affine morphism of schemes. Let $\mathcal{F}$ be a quasi-coherent $\mathcal{O}_X$-module. Then we have*

$$WeakAss_S(f_*\mathcal{F}) \subset f(WeakAss_X(\mathcal{F}))$$

Proof. We may assume X and S affine, so $X \to S$ comes from a ring map $A \to B$. Then $\mathcal{F} = \widetilde{M}$ for some B-module M. By Lemma 30.5.2 the weakly associated points of $\mathcal{F}$ correspond exactly to the weakly associated primes of M. Similarly, the weakly associated points of $f_*\mathcal{F}$ correspond exactly to the weakly associated primes of M as an A-module. Hence the lemma follows from Algebra, Lemma 10.63.10. □

Lemma 30.6.2. *Let $f : X \to S$ be an affine morphism of schemes. Let $\mathcal{F}$ be a quasi-coherent $\mathcal{O}_X$-module. If X is locally Noetherian, then we have*

$$f(Ass_X(\mathcal{F})) = Ass_S(f_*\mathcal{F}) = WeakAss_S(f_*\mathcal{F}) = f(WeakAss_X(\mathcal{F}))$$

Proof. We may assume X and S affine, so $X \to S$ comes from a ring map $A \to B$. As X is locally Noetherian the ring B is Noetherian, see Properties, Lemma 27.5.2. Write $\mathcal{F} = \widetilde{M}$ for some B-module M. By Lemma 30.2.2 the associated points of $\mathcal{F}$ correspond exactly to the associated primes of M, and any associated prime of M as an A-module is an associated points of $f_*\mathcal{F}$. Hence the inclusion

$$f(\mathrm{Ass}_X(\mathcal{F})) \subset \mathrm{Ass}_S(f_*\mathcal{F})$$

follows from Algebra, Lemma 10.60.12. We have the inclusion

$$\mathrm{Ass}_S(f_*\mathcal{F}) \subset \mathrm{WeakAss}_S(f_*\mathcal{F})$$

by Lemma 30.5.3. We have the inclusion

$$\mathrm{WeakAss}_S(f_*\mathcal{F}) \subset f(\mathrm{WeakAss}_X(\mathcal{F}))$$

by Lemma 30.6.1. The outer sets are equal by Lemma 30.5.7 hence we have equality everywhere. □

Lemma 30.6.3. *Let $f : X \to S$ be a finite morphism of schemes. Let $\mathcal{F}$ be a quasi-coherent $\mathcal{O}_X$-module. Then $WeakAss(f_*\mathcal{F}) = f(WeakAss(\mathcal{F}))$.*

Proof. We may assume X and S affine, so $X \to S$ comes from a finite ring map $A \to B$. Write $\mathcal{F} = \widetilde{M}$ for some B-module M. By Lemma 30.5.2 the weakly associated points of $\mathcal{F}$ correspond exactly to the weakly associated primes of M. Similarly, the weakly associated points of $f_*\mathcal{F}$ correspond exactly to the weakly associated primes of M as an A-module. Hence the lemma follows from Algebra, Lemma 10.63.12. □

Lemma 30.6.4. *Let $f : X \to S$ be a morphism of schemes. Let $\mathcal{G}$ be a quasi-coherent $\mathcal{O}_S$-module. Let $x \in X$ with $s = f(x)$. If f is flat at x, the point x is a generic point of the fibre X_s, and $s \in WeakAss_S(\mathcal{G})$, then $x \in WeakAss(f^*\mathcal{G})$.*

Proof. Let $A = \mathcal{O}_{S,s}$, $B = \mathcal{O}_{X,x}$, and $M = \mathcal{G}_s$. Let $m \in M$ be an element whose annihilator $I = \{a \in A \mid am = 0\}$ has radical $\mathfrak{m}_A$. Then $m \otimes 1$ has annihilator IB as $A \to B$ is faithfully flat. Thus it suffices to see that $\sqrt{IB} = \mathfrak{m}_B$. This follows from the fact that the maximal ideal of $B/\mathfrak{m}_A B$ is locally nilpotent (see Algebra, Lemma 10.24.4) and the assumption that $\sqrt{I} = \mathfrak{m}_A$. Some details omitted. □

30.7. Relative assassin

Definition 30.7.1. Let $f : X \to S$ be a morphism of schemes. Let $\mathcal{F}$ be a quasi-coherent $\mathcal{O}_X$-module. The *relative assassin of $\mathcal{F}$ in X over S* is the set

$$\mathrm{Ass}_{X/S}(\mathcal{F}) = \bigcup\nolimits_{s \in S} \mathrm{Ass}_{X_s}(\mathcal{F}_s)$$

where $\mathcal{F}_s = (X_s \to X)^*\mathcal{F}$ is the restriction of $\mathcal{F}$ to the fibre of f at s.

Again there is a caveat that this is best used when the fibres of f are locally Noetherian and $\mathcal{F}$ is of finite type. In the general case we should probably use the relative weak assassin (defined in the next section).

Lemma 30.7.2. *Let $f : X \to S$ be a morphism of schemes. Let $\mathcal{F}$ be a quasi-coherent $\mathcal{O}_X$-module. Let $g : S' \to S$ be a morphism of schemes. Consider the base change diagram*

$$\begin{array}{ccc} X' & \xrightarrow{g'} & X \\ \downarrow & & \downarrow \\ S' & \xrightarrow{g} & S \end{array}$$

and set $\mathcal{F}' = (g')^*\mathcal{F}$. Let $x' \in X'$ be a point with images $x \in X$, $s' \in S'$ and $s \in S$. Assume f locally of finite type. Then $x' \in Ass_{X'/S'}(\mathcal{F}')$ if and only if $x \in Ass_{X/S}(\mathcal{F})$ and x' corresponds to a generic point of an irreducible component of $\mathrm{Spec}(\kappa(s') \otimes_{\kappa(s)} \kappa(x))$.

Proof. Consider the morphism $X'_{s'} \to X_s$ of fibres. As $X_{s'} = X_s \times_{\mathrm{Spec}(\kappa(s))} \mathrm{Spec}(\kappa(s'))$ this is a flat morphism. Moreover $\mathcal{F}'_{s'}$ is the pullback of $\mathcal{F}_s$ via this morphism. As X_s is locally of finite type over the Noetherian scheme $\mathrm{Spec}(\kappa(s))$ we have that X_s is locally Noetherian, see Morphisms, Lemma 28.16.6. Thus we may apply Lemma 30.3.1 and we see that

$$\mathrm{Ass}_{X'_{s'}}(\mathcal{F}'_{s'}) = \bigcup\nolimits_{x \in \mathrm{Ass}(\mathcal{F}_s)} \mathrm{Ass}((X'_{s'})_x).$$

Thus to prove the lemma it suffices to show that the associated points of the fibre $(X'_{s'})_x$ of the morphism $X'_{s'} \to X_s$ over x are its generic points. Note that $(X'_{s'})_x = \mathrm{Spec}(\kappa(s') \otimes_{\kappa(s)} \kappa(x))$ as schemes. By Algebra, Lemma 10.153.1 the ring $\kappa(s') \otimes_{\kappa(s)} \kappa(x)$ is a Noetherian Cohen-Macaulay ring. Hence its associated primes are its minimal primes, see Algebra, Proposition 10.60.6 (minimal primes are associated) and Algebra, Lemma 10.144.2 (no embedded primes). □

Remark 30.7.3. With notation and assumptions as in Lemma 30.7.2 we see that it is always the case that $(g')^{-1}(\mathrm{Ass}_{X/S}(\mathcal{F})) \supset \mathrm{Ass}_{X'/S'}(\mathcal{F}')$. If the morphism $S' \to S$ is locally quasi-finite, then we actually have

$$(g')^{-1}(\mathrm{Ass}_{X/S}(\mathcal{F})) = \mathrm{Ass}_{X'/S'}(\mathcal{F}')$$

because in this case the field extensions $\kappa(s) \subset \kappa(s')$ are always finite. In fact, this holds more generally for any morphism $g : S' \to S$ such that all the field extensions $\kappa(s) \subset \kappa(s')$ are algebraic, because in this case all prime ideals of $\kappa(s') \otimes_{\kappa(s)} \kappa(x)$ are maximal (and minimal) primes, see Algebra, Lemma 10.33.17.

30.8. Relative weak assassin

Definition 30.8.1. Let $f : X \to S$ be a morphism of schemes. Let $\mathcal{F}$ be a quasi-coherent $\mathcal{O}_X$-module. The *relative weak assassin of $\mathcal{F}$ in X over S* is the set

$$\mathrm{WeakAss}_{X/S}(\mathcal{F}) = \bigcup\nolimits_{s \in S} \mathrm{WeakAss}(\mathcal{F}_s)$$

where $\mathcal{F}_s = (X_s \to X)^*\mathcal{F}$ is the restriction of $\mathcal{F}$ to the fibre of f at s.

Lemma 30.8.2. *Let $f : X \to S$ be a morphism of schemes which is locally of finite type. Let $\mathcal{F}$ be a quasi-coherent $\mathcal{O}_X$-module. Then $\mathrm{WeakAss}_{X/S}(\mathcal{F}) = \mathrm{Ass}_{X/S}(\mathcal{F})$.*

Proof. This is true bacause the fibres of f are locally Noetherian schemes, and associated and weakly associated points agree on locally Noetherian schemes, see Lemma 30.5.7. □

30.9. Effective Cartier divisors

For some reason it seem convenient to define the notion of an effective Cartier divisor before anything else.

Definition 30.9.1. Let S be a scheme.

(1) A *locally principal closed subscheme* of S is a closed subscheme whose sheaf of ideals is locally generated by a single element.
(2) An *effective Cartier divisor* on S is a closed subscheme $D \subset S$ such that the ideal sheaf $\mathcal{I}_D \subset \mathcal{O}_X$ is an invertible $\mathcal{O}_X$-module.

Thus an effective Cartier divisor is a locally principal closed subscheme, but the converse is not always true. Effective Cartier divisors are closed subschemes of pure codimension 1 in the strongest possible sense. Namely they are locally cut out by a single element which is not a zerodivisor. In particular they are nowhere dense.

Lemma 30.9.2. *Let S be a scheme. Let $D \subset S$ be a closed subscheme. The following are equivalent:*

(1) *The subscheme D is an effective Cartier divisor on S.*
(2) *For every $x \in D$ there exists an affine open neighbourhood $\mathrm{Spec}(A) = U \subset X$ of x such that $U \cap D = \mathrm{Spec}(A/(f))$ with $f \in A$ not a zerodivisor.*

Proof. Assume (1). For every $x \in D$ there exists an affine open neighbourhood $\mathrm{Spec}(A) = U \subset X$ of x such that $\mathcal{I}_D|_U \cong \mathcal{O}_U$. In other words, there exists a section $f \in \Gamma(U, \mathcal{I}_D)$ which freely generates the restriction $\mathcal{I}_D|_U$. Hence $f \in A$, and the multiplication map $f : A \to A$ is injective. Also, since $\mathcal{I}_D$ is quasi-coherent we see that $D \cap U = \mathrm{Spec}(A/(f))$.

Assume (2). Let $x \in D$. By assumption there exists an affine open neighbourhood $\mathrm{Spec}(A) = U \subset X$ of x such that $U \cap D = \mathrm{Spec}(A/(f))$ with $f \in A$ not a zerodivisor. Then $\mathcal{I}_D|_U \cong \mathcal{O}_U$ since it is equal to $\widetilde{(f)} \cong \widetilde{A} \cong \mathcal{O}_U$. Of course $\mathcal{I}_D$ restricted to the open subscheme $S \setminus D$ is isomorphic to $\mathcal{O}_{X \setminus D}$. Hence $\mathcal{I}_D$ is an invertible $\mathcal{O}_S$-module. □

Lemma 30.9.3. *Let S be a scheme. Let $Z \subset S$ be a locally principal closed subscheme. Let $U = S \setminus Z$. Then $U \to S$ is an affine morphism.*

Proof. The question is local on S, see Morphisms, Lemmas 28.13.3. Thus we may assume $S = \mathrm{Spec}(A)$ and $Z = V(f)$ for some $f \in A$. In this case $U = D(f) = \mathrm{Spec}(A_f)$ is affine hence $U \to S$ is affine. □

Lemma 30.9.4. *Let S be a scheme. Let $D \subset S$ be an effective Cartier divisor. Let $U = S \setminus D$. Then $U \to S$ is an affine morphism and U is scheme theoretically dense in S.*

Proof. Affineness is Lemma 30.9.3. The density question is local on S, see Morphisms, Lemma 28.7.5. Thus we may assume $S = \mathrm{Spec}(A)$ and D corresponding to the nonzerodivisor $f \in A$, see Lemma 30.9.2. Thus $A \subset A_f$ which implies that $U \subset S$ is scheme theoretically dense, see Morphisms, Example 28.7.4. □

Lemma 30.9.5. *Let S be a scheme. Let $D \subset S$ be an effective Cartier divisor. Let $s \in D$. If $\dim_s(S) < \infty$, then $\dim_s(D) < \dim_s(S)$.*

Proof. Assume $\dim_s(S) < \infty$. Let $U = \mathrm{Spec}(A) \subset S$ be an affine open neighbourhood of X such that $\dim(U) = \dim_s(S)$ and such that $D = V(f)$ for some nonzerodivisor $f \in A$ (see Lemma 30.9.2). Recall that $\dim(U)$ is the Krull dimension of the ring A and that $\dim(U \cap D)$ is the Krull dimension of the ring $A/(f)$. Then f is not contained in any minimal prime of A. Hence any maximal chain of primes in $A/(f)$, viewed as a chain of primes in A, can be extended by adding a minimal prime. □

Definition 30.9.6. Let S be a scheme. Given effective Cartier divisors D_1, D_2 on S we set $D = D_1 + D_2$ equal to the closed subscheme of S corresponding to the quasi-coherent sheaf of ideals $\mathcal{I}_{D_1}\mathcal{I}_{D_2} \subset \mathcal{O}_S$. We call this the *sum of the effective Cartier divisors D_1 and D_2*.

It is clear that we may define the sum $\sum n_i D_i$ given finitely many effective Cartier divisors D_i on X and nonnegative integers n_i.

Lemma 30.9.7. *The sum of two effective Cartier divisors is an effective Cartier divisor.*

Proof. Omitted. Locally $f_1, f_2 \in A$ are nonzerodivisors, then also $f_1 f_2 \in A$ is a nonzerodivisor. □

Lemma 30.9.8. *Let X be a scheme. Let D, D' be two effective Cartier divisors on X. If $D \subset D'$ (as closed subschemes of X), then there exists an effective Cartier divisor D'' such that $D' = D + D''$.*

Proof. Omitted. □

Lemma 30.9.9. *Let X be a scheme. Let Z, Y be two closed subschemes of X with ideal sheaves $\mathcal{I}$ and $\mathcal{J}$. If $\mathcal{I}\mathcal{J}$ defines an effective Cartier divisor $D \subset X$, then Z and Y are effective Cartier divisors and $D = Z + Y$.*

Proof. Applying Lemma 30.9.2 we obtain the following algebra situation: A is a ring, $I, J \subset A$ ideals and $f \in A$ a nonzerodivisor such that $IJ = (f)$. Thus the result follows from Algebra, Lemma 10.114.9. □

Recall that we have defined the inverse image of a closed subscheme under any morphism of schemes in Schemes, Definition 25.17.7.

Lemma 30.9.10. *Let $f : S' \to S$ be a morphism of schemes. Let $Z \subset S$ be a locally principal closed subscheme. Then the inverse image $f^{-1}(Z)$ is a locally principal closed subscheme of S'.*

Proof. Omitted. □

Definition 30.9.11. Let $f : S' \to S$ be a morphism of schemes. Let $D \subset S$ be an effective Cartier divisor. We say the *pullback of D by f is defined* if the closed subscheme $f^{-1}(D) \subset S'$ is an effective Cartier divisor. In this case we denote it either f^*D or $f^{-1}(D)$ and we call it the *pullback of the effective Cartier divisor*.

The condition that $f^{-1}(D)$ is an effective Cartier divisor is often satisfied in practice. Here is an example lemma.

Lemma 30.9.12. *Let $f : X \to Y$ be a morphism of schemes. Let $D \subset Y$ be an effective Cartier divisor. The pullback of D by f is defined in each of the following cases:*

1. *X, Y integral and f dominant,*
2. *X reduced, and for any generic point ξ of any irreducible component of X we have $f(\xi) \notin D$,*
3. *X is locally Noetherian and for any associated point x of X we have $f(x) \notin D$,*
4. *X is locally Noetherian, has no embedded points, and for any generic point ξ of any irreducible component of X we have $f(\xi) \notin D$,*
5. *f is flat, and*
6. *add more here as needed.*

Proof. The question is local on X, and hence we reduce to the case where $X = \mathrm{Spec}(A)$, $Y = \mathrm{Spec}(R)$, f is given by $\varphi : R \to A$ and $D = \mathrm{Spec}(R/(t))$ where $t \in R$ is not a zerodivisor. The goal in each case is to show that $\varphi(t) \in A$ is not a zerodivisor.

In case (2) this follows as the intersection of all minimal primes of a ring is the nilradical of the ring, see Algebra, Lemma 10.16.2.

Let us prove (3). By Lemma 30.2.2 the associated points of X correspond to the primes $\mathfrak{p} \in \mathrm{Ass}(A)$. By Algebra, Lemma 10.60.9 we have $\bigcup_{\mathfrak{p} \in \mathrm{Ass}(A)} \mathfrak{p}$ is the set of zerodivisors of A. The hypothesis of (3) is that $\varphi(t) \notin \mathfrak{p}$ for all $\mathfrak{p} \in \mathrm{Ass}(A)$. Hence $\varphi(t)$ is a nonzerodivisor as desired.

Part (4) follows from (3) and the definitions. □

Lemma 30.9.13. *Let $f : S' \to S$ be a morphism of schemes. Let D_1, D_2 be effective Cartier divisors on S. If the pullbacks of D_1 and D_2 are defined then the pullback of $D = D_1 + D_2$ is defined and $f^*D = f^*D_1 + f^*D_2$.*

Proof. Omitted. □

Definition 30.9.14. Let S be a scheme and let D be an effective Cartier divisor. The *invertible sheaf* $\mathcal{O}_S(D)$ *associated to* D is given by

$$\mathcal{O}_S(D) := \mathcal{H}om_{\mathcal{O}_S}(\mathcal{I}_D, \mathcal{O}_S) = \mathcal{I}_D^{\otimes -1}.$$

The canonical section, usually denoted 1 or 1_D, is the global section of $\mathcal{O}_S(D)$ corresponding to the inclusion mapping $\mathcal{I}_D \to \mathcal{O}_S$.

Lemma 30.9.15. *Let S be a scheme. Let D_1, D_2 be effective Cartier divisors on S. Let $D = D_1 + D_2$. Then there is a unique isomorphism*

$$\mathcal{O}_S(D_1) \otimes_{\mathcal{O}_S} \mathcal{O}_S(D_2) \longrightarrow \mathcal{O}_S(D)$$

which maps $1_{D_1} \otimes 1_{D_2}$ to 1_D.

Proof. Omitted. □

Definition 30.9.16. Let $(X, \mathcal{O}_X)$ be a locally ringed space. Let $\mathcal{L}$ be an invertible sheaf on X. A global section $s \in \Gamma(X, \mathcal{L})$ is called a *regular section* if the map $\mathcal{O}_X \to \mathcal{L}$, $f \mapsto fs$ is injective.

Lemma 30.9.17. *Let X be a locally ringed space. Let $f \in \Gamma(X, \mathcal{O}_X)$. The following are equivalent:*

1. *f is a regular section, and*
2. *for any $x \in X$ the image $f \in \mathcal{O}_{X,x}$ is not a zerodivisor.*

If X is a scheme these are also equivalent to

3. *for any affine open $\mathrm{Spec}(A) = U \subset X$ the image $f \in A$ is not a zerodivisor, and*
4. *there exists an affine open covering $X = \bigcup \mathrm{Spec}(A_i)$ such that the image of f in A_i is not a zerodivisor for all i.*

Proof. Omitted. □

Note that a global section s of an invertible $\mathcal{O}_X$-module $\mathcal{L}$ may be seen as an $\mathcal{O}_X$-module map $s : \mathcal{O}_X \to \mathcal{L}$. Its dual is therefore a map $s : \mathcal{L}^{\otimes -1} \to \mathcal{O}_X$. (See Modules, Definition 17.21.3 for the definition of the dual invertible sheaf.)

Definition 30.9.18. Let X be a scheme. Let $\mathcal{L}$ be an invertible sheaf. Let $s \in \Gamma(X, \mathcal{L})$. The *zero scheme* of s is the closed subscheme $Z(s) \subset X$ defined by the quasi-coherent sheaf of ideals $\mathcal{I} \subset \mathcal{O}_X$ which is the image of the map $s : \mathcal{L}^{\otimes -1} \to \mathcal{O}_X$.

Lemma 30.9.19. *Let X be a scheme. Let $\mathcal{L}$ be an invertible sheaf. Let $s \in \Gamma(X, \mathcal{L})$.*

1. *Consider closed immersions $i : Z \to X$ such that $i^*s \in \Gamma(Z, i^*\mathcal{L}))$ is zero ordered by inclusion. The zero scheme $Z(s)$ is the maximal element of this ordered set.*
2. *For any morphism of schemes $f : Y \to X$ we have $f^*s = 0$ in $\Gamma(Y, f^*\mathcal{L})$ if and only if f factors through $Z(s)$.*
3. *The zero scheme $Z(s)$ is a locally principal closed subscheme.*
4. *The zero scheme $Z(s)$ is an effective Cartier divisor if and only if s is a regular section of $\mathcal{L}$.*

Proof. Omitted. □

Lemma 30.9.20. *Let X be a scheme.*

(1) *If $D \subset X$ is an effective Cartier divisor, then the canonical section 1_D of $\mathcal{O}_X(D)$ is regular.*
(2) *Conversely, if s is a regular section of the invertible sheaf $\mathcal{L}$, then there exists a unique effective Cartier divisor $D = Z(s) \subset X$ and a unique isomorphism $\mathcal{O}_X(D) \to \mathcal{L}$ which maps 1_D to s.*

The constructions $D \mapsto (\mathcal{O}_X(D), 1_D)$ and $(\mathcal{L}, s) \mapsto Z(s)$ give mutually inverse maps

$$\left\{\text{effective Cartier divisors on } X\right\} \leftrightarrow \left\{\begin{array}{c}\text{pairs } (\mathcal{L}, s) \text{ consisting of an invertible} \\ \mathcal{O}_X\text{-module and a regular global section}\end{array}\right\}$$

Proof. Omitted. □

30.10. Relative effective Cartier divisors

The following lemma shows that an effective Cartier divisor which is flat over the base is really a "family of effective Cartier divisors" over the base. For example the restriction to any fibre is an effective Cartier divisor.

Lemma 30.10.1. *Let $f : X \to S$ be a morphism of schemes. Let $D \subset X$ be a closed subscheme. Assume*

(1) *D is an effective Cartier divisor, and*
(2) *$D \to S$ is a flat morphism.*

Then for every morphism of schemes $g : S' \to S$ the pullback $(g')^{-1}D$ is an effective Cartier divisor on $X' = S' \times_S X$.

Proof. Using Lemma 30.9.2 we translate this as follows into algebra. Let $A \to B$ be a ring map and $h \in B$. Assume h is a nonzerodivisor and that B/hB is flat over A. Then

$$0 \to B \xrightarrow{h} B \to B/hB \to 0$$

is a short exact sequence of A-modules with B/hB flat over A. By Algebra, Lemma 10.36.11 this sequence remains exact on tensoring over A with any module, in particular with any A-algebra A'. □

This lemma is the motivation for the following definition.

Definition 30.10.2. Let $f : X \to S$ be a morphism of schemes. A *relative effective Cartier divisor* on X/S is an effective Cartier divisor $D \subset X$ such that $D \to S$ is a flat morphism of schemes.

We warn the reader that this may be nonstandard notation. In particular, in [**DG67**, IV, Section 21.15] the notion of a relative divisor is discussed only when $X \to S$ is flat and locally of finite presentation. Our definition is a bit more general. However, it turns out that if $x \in D$ then $X \to S$ is flat at x in many cases (but not always).

Lemma 30.10.3. *Let $f : X \to S$ be a morphism of schemes. Let $D \subset X$ be a relative effective Cartier divisor on X/S. If $x \in D$ and $\mathcal{O}_{X,x}$ is Noetherian, then f is flat at x.*

Proof. Set $A = \mathcal{O}_{S,f(x)}$ and $B = \mathcal{O}_{X,x}$. Let $h \in B$ be an element which generates the ideal of D. Then h is a nonzerodivisor in B such that B/hB is a flat local

A-algebra. Let $I \subset A$ be a finitely generated ideal. Consider the commutative diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & B & \xrightarrow{h} & B & \longrightarrow & B/hB & \longrightarrow & 0 \\ & & \uparrow & & \uparrow & & \uparrow & & \\ 0 & \longrightarrow & B \otimes_A I & \xrightarrow{h} & B \otimes_A I & \longrightarrow & B/hB \otimes_A I & \longrightarrow & 0 \end{array}$$

The lower sequence is short exact as B/hB is flat over A, see Algebra, Lemma 10.36.11. The right vertical arrow is injective as B/hB is flat over A, see Algebra, Lemma 10.36.4. Hence multiplication by h is surjective on the kernel K of the middle vertical arrow. By Nakayama's lemma, see Algebra, Lemma 10.18.1 we conclude that $K = 0$. Hence B is flat over A, see Algebra, Lemma 10.36.4. □

The following lemma relies on the algebraic version of openness of the flat locus. The scheme theoretic version can be found in More on Morphisms, Section 36.12.

Lemma 30.10.4. *Let $f : X \to S$ be a morphism of schemes. Let $D \subset X$ be a relative effective Cartier divisor. If f is locally of finite presentation, then there exists an open subscheme $U \subset X$ such that $D \subset U$ and such that $f|_U : U \to S$ is flat.*

Proof. Pick $x \in D$. It suffices to find an open neighbourhood $U \subset X$ of x such that $f|_U$ is flat. Hence the lemma reduces to the case that $X = \mathrm{Spec}(B)$ and $S = \mathrm{Spec}(A)$ are affine and that D is given by a nonzerodivisor $h \in B$. By assumption B is a finitely presented A-algebra and B/hB is a flat A-algebra. We are going to use absolute Noetherian approximation.

Write $B = A[x_1, \ldots, x_n]/(g_1, \ldots, g_m)$. Assume h is the image of $h' \in A[x_1, \ldots, x_n]$. Choose a finite type $\mathbf{Z}$-subalgebra $A_0 \subset A$ such that all the coefficients of the polynomials $h', g_1, \ldots, g_m$ are in A_0. Then we can set $B_0 = A_0[x_1, \ldots, x_n]/(g_1, \ldots, g_m)$ and h_0 the image of h' in B_0. Then $B = B_0 \otimes_{A_0} A$ and $B/hB = B_0/h_0B_0 \otimes_{A_0} A$. By Algebra, Lemma 10.154.1 we may, after enlarging A_0, assume that B_0/h_0B_0 is flat over A_0. Let $K_0 = \mathrm{Ker}(h_0 : B_0 \to B_0)$. As B_0 is of finite type over $\mathbf{Z}$ we see that K_0 is a finitely generated ideal. Let $A_1 \subset A$ be a finite type $\mathbf{Z}$-subalgebra containing A_0 and denote B_1, h_1, K_1 the corresponding objects over A_1. By More on Algebra, Lemma 15.20.15 the map $K_0 \otimes_{A_0} A_1 \to K_1$ is surjective. On the other hand, the kernel of $h : B \to B$ is zero by assumption. Hence every element of K_0 maps to zero in K_1 for sufficiently large subrings $A_1 \subset A$. Since K_0 is finitely generated, we conclude that $K_1 = 0$ for a suitable choice of A_1.

Set $f_1 : X_1 \to S_1$ equal to Spec of the ring map $A_1 \to B_1$. Set $D_1 = \mathrm{Spec}(B_1/h_1B_1)$. Since $B = B_1 \otimes_{A_1} A$, i.e., $X = X_1 \times_{S_1} S$, it now suffices to prove the lemma for $X_1 \to S_1$ and the relative effective Cartier divisor D_1, see Morphisms, Lemma 28.26.6. Hence we have reduced to the case where A is a Noetherian ring. In this case we know that the ring map $A \to B$ is flat at every prime $\mathfrak{q}$ of $V(h)$ by Lemma 30.10.3. Combined with the fact that the flat locus is open in this case, see Algebra, Theorem 10.123.4 we win. □

There is also the following lemma (whose idea is apparently due to Michael Artin, see [**Nob77**]) which needs no finiteness assumptions at all.

Lemma 30.10.5. *Let $f : X \to S$ be a morphism of schemes. Let $D \subset X$ be a relative effective Cartier divisor on X/S. If f is flat at all points of $X \setminus D$, then f is flat.*

Proof. This translates into the following algebra fact: Let $A \to B$ be a ring map and $h \in B$. Assume h is a nonzerodivisor, that B/hB is flat over A, and that the localization B_h is flat over A. Then B is flat over A. The reason is that we have a short exact sequence

$$0 \to B \to B_h \to \operatorname{colim}_n (1/h^n)B/B \to 0$$

and that the second and third terms are flat over A, which implies that B is flat over A (see Algebra, Lemma 10.36.12). Note that a filtered colimit of flat modules is flat (see Algebra, Lemma 10.36.2) and that by induction on n each $(1/h^n)B/B \cong B/h^nB$ is flat over A since it fits into the short exact sequence

$$0 \to B/h^{n-1}B \xrightarrow{h} B/h^nB \to B/hB \to 0$$

Some details omitted. □

Example 30.10.6. Here is an example of a relative effective Cartier divisor D where the ambient scheme is not flat in a neighbourhood of D. Namely, let $A = k[t]$ and

$$B = k[t, x, y, x^{-1}y, x^{-2}y, \ldots]/(ty, tx^{-1}y, tx^{-2}y, \ldots)$$

Then B is not flat over A but $B/xB \cong A$ is flat over A. Moreover x is a nonzerodivisor and hence defines a relative effective Cartier divisor in $\operatorname{Spec}(B)$ over $\operatorname{Spec}(A)$.

If the ambient scheme is flat and locally of finite presentation over the base, then we can characterize a relative effective Cartier divisor in terms of its fibres. See also More on Morphisms, Lemma 36.18.1 for a slightly different take on this lemma.

Lemma 30.10.7. *Let $\varphi : X \to S$ be a flat morphism which is locally of finite presentation. Let $Z \subset X$ be a closed subscheme. Let $x \in Z$ with image $s \in S$.*

1. *If $Z_s \subset X_s$ is a Cartier divisor in a neighbourhood of x, then there exists an open $U \subset X$ and a relative effective Cartier divisor $D \subset U$ such that $Z \cap U \subset D$.*
2. *If $Z_s \subset X_s$ is a Cartier divisor in a neighbourhood of x, the morphism $Z \to X$ is of finite presentation, and $Z \to S$ is flat at x, then we can choose U and D such that $Z \cap U = D$.*
3. *If $Z_s \subset X_s$ is a Cartier divisor in a neighbourhood of x and Z is a locally principal closed subscheme of X in a neighbourhood of x, then we can choose U and D such that $Z \cap U = D$.*

In particular, if $Z \to S$ is locally of finite presentation and flat and all fibres $Z_s \subset X_s$ are effective Cartier divisors, then Z is a relative effective Cartier divisor. Similarly, if Z is a locally principal closed subscheme of X such that all fibres $Z_s \subset X_s$ are effective Cartier divisors, then Z is a relative effective Cartier divisor.

Proof. Choose affine open neighbourhoods $\operatorname{Spec}(A)$ of s and $\operatorname{Spec}(B)$ of x such that $\varphi(\operatorname{Spec}(B)) \subset \operatorname{Spec}(A)$. Let $\mathfrak{p} \subset A$ be the prime ideal corresponding to s. Let $\mathfrak{q} \subset B$ be the prime ideal corresponding to x. Let $I \subset B$ be the ideal corresponding to Z. By the initial assumption of the lemma we know that $A \to B$ is flat and of finite presentation. The assumption in (1) means that, after shrinking $\operatorname{Spec}(B)$, we may assume $I(B \otimes_A \kappa(\mathfrak{p}))$ is generated by a single element which is a nonzerodivisor

in $B \otimes_A \kappa(\mathfrak{p})$. Say $f \in I$ maps to this generator. We claim that after inverting an element $g \in B$, $g \notin \mathfrak{q}$ the closed subscheme $D = V(f) \subset \mathrm{Spec}(B_g)$ is a relative effective Cartier divisor.

By Algebra, Lemma 10.154.1 we can find a flat finite type ring map $A_0 \to B_0$ of Noetherian rings, an element $f_0 \in B_0$, a ring map $A_0 \to A$ and an isomorphism $A \otimes_{A_0} B_0 \cong B$. If $\mathfrak{p}_0 = A_0 \cap \mathfrak{p}$ then we see that

$$B \otimes_A \kappa(\mathfrak{p}) = (B_0 \otimes_{A_0} \kappa(\mathfrak{p}_0)) \otimes_{\kappa(\mathfrak{p}_0))} \kappa(\mathfrak{p})$$

hence f_0 is a nonzerodivisor in $B_0 \otimes_{A_0} \kappa(\mathfrak{p}_0)$. By Algebra, Lemma 10.93.2 we see that f_0 is a nonzerodivisor in $(B_0)_{\mathfrak{q}_0}$ where $\mathfrak{q}_0 = B_0 \cap \mathfrak{q}$ and that $(B_0/f_0B_0)_{\mathfrak{q}_0}$ is flat over A_0. Hence by Algebra, Lemma 10.65.8 and Algebra, Theorem 10.123.4 there exists a $g_0 \in B_0$, $g_0 \notin \mathfrak{q}_0$ such that f_0 is a nonzerodivisor in $(B_0)_{g_0}$ and such that $(B_0/f_0B_0)_{g_0}$ is flat over A_0. Hence we see that $D_0 = V(f_0) \subset \mathrm{Spec}((B_0)_{g_0})$ is a relative effective Cartier divisor. Since we know that this property is preserved under base change, see Lemma 30.10.1, we obtain the claim mentioned above with g equal to the image of g_0 in B.

At this point we have proved (1). To see (2) consider the closed immersion $Z \to D$. The surjective ring map $u : \mathcal{O}_{D,x} \to \mathcal{O}_{Z,x}$ is a map of flat local $\mathcal{O}_{S,s}$-algebras which are essentially of finite presentation, and which becomes an isomorphisms after dividing by $\mathfrak{m}_s$. Hence it is an isomorphism, see Algebra, Lemma 10.122.4. It follows that $Z \to D$ is an isomorphism in a neighbourhood of x, see Algebra, Lemma 10.120.6. To see (3), after possibly shrinking U we may assume that the ideal of D is generated by a single nonzerodivisor f and the ideal of Z is generated by an element g. Then $f = gh$. But $g|_{U_s}$ and $f|_{U_s}$ cut out the same effective Cartier divisor in a neighbourhood of x. Hence $h|_{X_s}$ is a unit in $\mathcal{O}_{X_s,x}$, hence h is a unit in $\mathcal{O}_{X,x}$ hence h is a unit in an open neighbourhood of x. I.e., $Z \cap U = D$ after shrinking U.

The final statements of the lemma follow immediately from parts (2) and (3), combined with the fact that $Z \to S$ is locally of finite presentation if and only if $Z \to X$ is of finite presentation, see Morphisms, Lemmas 28.22.3 and 28.22.11. □

30.11. The normal cone of an immersion

Let $i : Z \to X$ be a closed immersion. Let $\mathcal{I} \subset \mathcal{O}_X$ be the corresponding quasi-coherent sheaf of ideals. Consider the quasi-coherent sheaf of graded $\mathcal{O}_X$-algebras $\bigoplus_{n \geq 0} \mathcal{I}^n/\mathcal{I}^{n+1}$. Since the sheaves $\mathcal{I}^n/\mathcal{I}^{n+1}$ are each annihilated by $\mathcal{I}$ this graded algebra corresponds to a quasi-coherent sheaf of graded $\mathcal{O}_Z$-algebras by Morphisms, Lemma 28.4.1. This quasi-coherent graded $\mathcal{O}_Z$-algebra is called the *conormal algebra of Z in X* and is often simply denoted $\bigoplus_{n \geq 0} \mathcal{I}^n/\mathcal{I}^{n+1}$ by the abuse of notation mentioned in Morphisms, Section 28.4.

Let $f : Z \to X$ be an immersion. We define the conormal algebra of f as the conormal sheaf of the closed immersion $i : Z \to X \setminus \partial Z$, where $\partial Z = \overline{Z} \setminus Z$. It is often denoted $\bigoplus_{n \geq 0} \mathcal{I}^n/\mathcal{I}^{n+1}$ where $\mathcal{I}$ is the ideal sheaf of the closed immersion $i : Z \to X \setminus \partial Z$.

Definition 30.11.1. Let $f : Z \to X$ be an immersion. The *conormal algebra $\mathcal{C}_{Z/X,*}$ of Z in X* or the *conormal algebra of f* is the quasi-coherent sheaf of graded $\mathcal{O}_Z$-algebras $\bigoplus_{n \geq 0} \mathcal{I}^n/\mathcal{I}^{n+1}$ described above.

Thus $\mathcal{C}_{Z/X,1} = \mathcal{C}_{Z/X}$ is the conormal sheaf of the immersion. Also $\mathcal{C}_{Z/X,0} = \mathcal{O}_Z$ and $\mathcal{C}_{Z/X,n}$ is a quasi-coherent $\mathcal{O}_Z$-module characterized by the property

$$i_*\mathcal{C}_{Z/X,n} = \mathcal{I}^n/\mathcal{I}^{n+1} \tag{30.11.1.1}$$

where $i : Z \to X \setminus \partial Z$ and $\mathcal{I}$ is the ideal sheaf of i as above. Finally, note that there is a canonical surjective map

$$\mathrm{Sym}^*(\mathcal{C}_{Z/X}) \longrightarrow \mathcal{C}_{Z/X,*} \tag{30.11.1.2}$$

of quasi-coherent graded $\mathcal{O}_Z$-algebras which is an isomorphism in degrees 0 and 1.

Lemma 30.11.2. *Let $i : Z \to X$ be an immersion. The conormal algebra of i has the following properties:*

1. *Let $U \subset X$ be any open such that $i(Z)$ is a closed subset of U. Let $\mathcal{I} \subset \mathcal{O}_U$ be the sheaf of ideals corresponding to the closed subscheme $i(Z) \subset U$. Then*
$$\mathcal{C}_{Z/X,*} = i^*\left(\bigoplus\nolimits_{n\geq 0} \mathcal{I}^n\right) = i^{-1}\left(\bigoplus\nolimits_{n\geq 0} \mathcal{I}^n/\mathcal{I}^{n+1}\right)$$
2. *For any affine open $\mathrm{Spec}(R) = U \subset X$ such that $Z \cap U = \mathrm{Spec}(R/I)$ there is a canonical isomorphism $\Gamma(Z \cap U, \mathcal{C}_{Z/X,*}) = \bigoplus_{n\geq 0} I^n/I^{n+1}$.*

Proof. Mostly clear from the definitions. Note that given a ring R and an ideal I of R we have $I^n/I^{n+1} = I^n \otimes_R R/I$. Details omitted. □

Lemma 30.11.3. *Let*

$$\begin{array}{ccc} Z & \xrightarrow{i} & X \\ {\scriptstyle f}\downarrow & & \downarrow{\scriptstyle g} \\ Z' & \xrightarrow{i'} & X' \end{array}$$

be a commutative diagram in the category of schemes. Assume i, i' immersions. There is a canonical map of graded $\mathcal{O}_Z$-algebras

$$f^*\mathcal{C}_{Z'/X',*} \longrightarrow \mathcal{C}_{Z/X,*}$$

characterized by the following property: For every pair of affine opens $(\mathrm{Spec}(R) = U \subset X, \mathrm{Spec}(R') = U' \subset X')$ with $f(U) \subset U'$ such that $Z \cap U = \mathrm{Spec}(R/I)$ and $Z' \cap U' = \mathrm{Spec}(R'/I')$ the induced map

$$\Gamma(Z' \cap U', \mathcal{C}_{Z'/X',*}) = \bigoplus (I')^n/(I')^{n+1} \longrightarrow \bigoplus\nolimits_{n\geq 0} I^n/I^{n+1} = \Gamma(Z \cap U, \mathcal{C}_{Z/X,*})$$

is the one induced by the ring map $f^\sharp : R' \to R$ which has the property $f^\sharp(I') \subset I$.

Proof. Let $\partial Z' = \overline{Z'} \setminus Z'$ and $\partial Z = \overline{Z} \setminus Z$. These are closed subsets of X' and of X. Replacing X' by $X' \setminus \partial Z'$ and X by $X \setminus \left(g^{-1}(\partial Z') \cup \partial Z\right)$ we see that we may assume that i and i' are closed immersions.

The fact that $g \circ i$ factors through i' implies that $g^*\mathcal{I}'$ maps into $\mathcal{I}$ under the canonical map $g^*\mathcal{I}' \to \mathcal{O}_X$, see Schemes, Lemmas 25.4.6 and 25.4.7. Hence we get an induced map of quasi-coherent sheaves $g^*((\mathcal{I}')^n/(\mathcal{I}')^{n+1}) \to \mathcal{I}^n/\mathcal{I}^{n+1}$. Pulling back by i gives $i^*g^*((\mathcal{I}')^n/(\mathcal{I}')^{n+1}) \to i^*(\mathcal{I}^n/\mathcal{I}^{n+1})$. Note that $i^*(\mathcal{I}^n/\mathcal{I}^{n+1}) = \mathcal{C}_{Z/X,n}$. On the other hand, $i^*g^*((\mathcal{I}')^n/(\mathcal{I}')^{n+1}) = f^*(i')^*((\mathcal{I}')^n/(\mathcal{I}')^{n+1}) = f^*\mathcal{C}_{Z'/X',n}$. This gives the desired map.

Checking that the map is locally described as the given map $(I')^n/(I')^{n+1} \to I^n/I^{n+1}$ is a matter of unwinding the definitions and is omitted. Another observation is that given any $x \in i(Z)$ there do exist affine open neighbourhoods U, U' with $f(U) \subset U'$ and $Z \cap U$ as well as $U' \cap Z'$ closed such that $x \in U$. Proof omitted. Hence the requirement of the lemma indeed characterizes the map (and could have been used to define it). □

Lemma 30.11.4. *Let*

$$\begin{array}{ccc} Z & \xrightarrow{i} & X \\ {\scriptstyle f}\downarrow & & \downarrow{\scriptstyle g} \\ Z' & \xrightarrow{i'} & X' \end{array}$$

be a fibre product diagram in the category of schemes with i, i' immersions. Then the canonical map $f^*\mathcal{C}_{Z'/X',*} \to \mathcal{C}_{Z/X,*}$ of Lemma 30.11.3 is surjective. If g is flat, then it is an isomorphism.

Proof. Let $R' \to R$ be a ring map, and $I' \subset R'$ an ideal. Set $I = I'R$. Then $(I')^n/(I')^{n+1} \otimes_{R'} R \to I^n/I^{n+1}$ is surjective. If $R' \to R$ is flat, then $I^n = (I')^n \otimes_{R'} R$ and we see the map is an isomorphism. □

Definition 30.11.5. Let $i : Z \to X$ be an immersion of schemes. The *normal cone* $C_Z X$ of Z in X is

$$C_Z X = \underline{\mathrm{Spec}}_Z(\mathcal{C}_{Z/X,*})$$

see Constructions, Definitions 26.7.1 and 26.7.2. The *normal bundle* of Z in X is the vector bundle

$$N_Z X = \underline{\mathrm{Spec}}_Z(\mathrm{Sym}(\mathcal{C}_{Z/X}))$$

see Constructions, Definitions 26.6.1 and 26.6.2.

Thus $C_Z X \to Z$ is a cone over Z and $N_Z X \to Z$ is a vector bundle over Z (recall that in our terminology this does not imply that the conormal sheaf is a finite locally free sheaf). Moreover, the canonical surjection (30.11.1.2) of graded algebras defines a canonical closed immersion

$$C_Z X \longrightarrow N_Z X \tag{30.11.5.1}$$

of cones over Z.

30.12. Regular ideal sheaves

In this section we generalize the notion of an effective Cartier divisor to higher codimension. Recall that a sequence of elements $f_1, \ldots, f_r$ of a ring R is a *regular sequence* if for each $i = 1, \ldots, r$ the element f_i is a nonzerodivisor on $R/(f_1, \ldots, f_{i-1})$ and $R/(f_1, \ldots, f_r) \neq 0$, see Algebra, Definition 10.65.1. There are three closely related weaker conditions that we can impose. The first is to assume that $f_1, \ldots, f_r$ is a *Koszul-regular sequence*, i.e., that $H_i(K_\bullet(f_1, \ldots, f_r)) = 0$ for $i > 0$, see More on Algebra, Definition 15.20.1. The sequence is called an *H_1-regular sequence* if $H_1(K_\bullet(f_1, \ldots, f_r)) = 0$. Another condition we can impose is that with $J = (f_1, \ldots, f_r)$, the map

$$R/J[T_1, \ldots, T_r] \longrightarrow \bigoplus\nolimits_{n \geq 0} J^n/J^{n+1}$$

which maps T_i to f_i mod J^2 is an isomorphism. In this case we say that $f_1, \ldots, f_r$ is a *quasi-regular sequence*, see Algebra, Definition 10.66.1. Given an R-module M there is also a notion of M-regular and M-quasi-regular sequence.

We can generalize this to the case of ringed spaces as follows. Let X be a ringed space and let $f_1, \ldots, f_r \in \Gamma(X, \mathcal{O}_X)$. We say that $f_1, \ldots, f_r$ is a *regular sequence* if for each $i = 1, \ldots, r$ the map

$$f_i : \mathcal{O}_X/(f_1, \ldots, f_{i-1}) \longrightarrow \mathcal{O}_X/(f_1, \ldots, f_{i-1}) \tag{30.12.0.2}$$

is an injective map of sheaves. We say that $f_1, \ldots, f_r$ is a *Koszul-regular sequence* if the Koszul complex

$$K_\bullet(\mathcal{O}_X, f_\bullet), \tag{30.12.0.3}$$

see Modules, Definition 17.20.2, is acyclic in degrees > 0. We say that $f_1, \ldots, f_r$ is a *H_1-regular sequence* if the Koszul complex $K_\bullet(\mathcal{O}_X, f_\bullet)$ is exact in degree 1. Finally, we say that $f_1, \ldots, f_r$ is a *quasi-regular* sequence if the map

$$\mathcal{O}_X/\mathcal{J}[T_1, \ldots, T_r] \longrightarrow \bigoplus\nolimits_{d \geq 0} \mathcal{J}^d/\mathcal{J}^{d+1} \tag{30.12.0.4}$$

is an isomorphism of sheaves where $\mathcal{J} \subset \mathcal{O}_X$ is the sheaf of ideals generated by $f_1, \ldots, f_r$. (There is also a notion of $\mathcal{F}$-regular and $\mathcal{F}$-quasi-regular sequence for a given $\mathcal{O}_X$-module $\mathcal{F}$ which we will introduce here if we ever need it.)

Lemma 30.12.1. *Let X be a ringed space. Let $f_1, \ldots, f_r \in \Gamma(X, \mathcal{O}_X)$. We have the following implications $f_1, \ldots, f_r$ is a regular sequence $\Rightarrow$ $f_1, \ldots, f_r$ is a Koszul-regular sequence $\Rightarrow$ $f_1, \ldots, f_r$ is an H_1-regular sequence $\Rightarrow$ $f_1, \ldots, f_r$ is a quasi-regular sequence.*

Proof. Since we may check exactness at stalks, a sequence $f_1, \ldots, f_r$ is a regular sequence if and only if the maps

$$f_i : \mathcal{O}_{X,x}/(f_1, \ldots, f_{i-1}) \longrightarrow \mathcal{O}_{X,x}/(f_1, \ldots, f_{i-1})$$

are injective for all $x \in X$. In other words, the image of the sequence $f_1, \ldots, f_r$ in the ring $\mathcal{O}_{X,x}$ is a regular sequence for all $x \in X$. The other types of regularity can be checked stalkwise as well (details omitted). Hence the implications follow from More on Algebra, Lemmas 15.20.2 and 15.20.5. □

Definition 30.12.2. Let X be a ringed space. Let $\mathcal{J} \subset \mathcal{O}_X$ be a sheaf of ideals.

1. We say $\mathcal{J}$ is *regular* if for every $x \in \mathrm{Supp}(\mathcal{O}_X/\mathcal{J})$ there exists an open neighbourhood $x \in U \subset X$ and a regular sequence $f_1, \ldots, f_r \in \mathcal{O}_X(U)$ such that $\mathcal{J}|_U$ is generated by $f_1, \ldots, f_r$.
2. We say $\mathcal{J}$ is *Koszul-regular* if for every $x \in \mathrm{Supp}(\mathcal{O}_X/\mathcal{J})$ there exists an open neighbourhood $x \in U \subset X$ and a Koszul-regular sequence $f_1, \ldots, f_r \in \mathcal{O}_X(U)$ such that $\mathcal{J}|_U$ is generated by $f_1, \ldots, f_r$.
3. We say $\mathcal{J}$ is *H_1-regular* if for every $x \in \mathrm{Supp}(\mathcal{O}_X/\mathcal{J})$ there exists an open neighbourhood $x \in U \subset X$ and a H_1-regular sequence $f_1, \ldots, f_r \in \mathcal{O}_X(U)$ such that $\mathcal{J}|_U$ is generated by $f_1, \ldots, f_r$.
4. We say $\mathcal{J}$ is *quasi-regular* if for every $x \in \mathrm{Supp}(\mathcal{O}_X/\mathcal{J})$ there exists an open neighbourhood $x \in U \subset X$ and a quasi-regular sequence $f_1, \ldots, f_r \in \mathcal{O}_X(U)$ such that $\mathcal{J}|_U$ is generated by $f_1, \ldots, f_r$.

Many properties of this notion immediately follow from the corresponding notions for regular and quasi-regular sequences in rings.

Lemma 30.12.3. *Let X be a ringed space. Let $\mathcal{J}$ be a sheaf of ideals. We have the following implications: $\mathcal{J}$ is regular $\Rightarrow$ $\mathcal{J}$ is Koszul-regular $\Rightarrow$ $\mathcal{J}$ is H_1-regular $\Rightarrow$ $\mathcal{J}$ is quasi-regular.*

Proof. The lemma immediately reduces to Lemma 30.12.1. □

Lemma 30.12.4. *Let X be a locally ringed space. Let $\mathcal{J} \subset \mathcal{O}_X$ be a sheaf of ideals. Then $\mathcal{J}$ is quasi-regular if and only if the following conditions are satisfied:*

(1) *$\mathcal{J}$ is an $\mathcal{O}_X$-module of finite type,*
(2) *$\mathcal{J}/\mathcal{J}^2$ is a finite locally free $\mathcal{O}_X/\mathcal{J}$-module, and*
(3) *the canonical maps*
$$Sym^n_{\mathcal{O}_X/\mathcal{J}}(\mathcal{J}/\mathcal{J}^2) \longrightarrow \mathcal{J}^n/\mathcal{J}^{n+1}$$
are isomorphisms for all $n \geq 0$.

Proof. It is clear that if $U \subset X$ is an open such that $\mathcal{J}|_U$ is generated by a quasi-regular sequence $f_1, \ldots, f_r \in \mathcal{O}_X(U)$ then $\mathcal{J}|_U$ is of finite type, $\mathcal{J}|_U/\mathcal{J}^2|_U$ is free with basis $f_1, \ldots, f_r$, and the maps in (3) are isomorphisms because they are coordinate free formulation of the degree n part of (30.12.0.4). Hence it is clear that being quasi-regular implies conditions (1), (2), and (3).

Conversely, suppose that (1), (2), and (3) hold. Pick a point $x \in \mathrm{Supp}(\mathcal{O}_X/\mathcal{J})$. Then there exists a neighbourhood $U \subset X$ of x such that $\mathcal{J}|_U/\mathcal{J}^2|_U$ is free of rank r over $\mathcal{O}_U/\mathcal{J}|_U$. After possibly shrinking U we may assume there exist $f_1, \ldots, f_r \in \mathcal{J}(U)$ which map to a basis of $\mathcal{J}|_U/\mathcal{J}^2|_U$ as an $\mathcal{O}_U/\mathcal{J}|_U$-module. In particular we see that the images of $f_1, \ldots, f_r$ in $\mathcal{J}_x/\mathcal{J}_x^2$ generate. Hence by Nakayama's lemma (Algebra, Lemma 10.18.1) we see that $f_1, \ldots, f_r$ generate the stalk $\mathcal{J}_x$. Hence, since $\mathcal{J}$ is of finite type, by Modules, Lemma 17.9.4 after shrinking U we may assume that $f_1, \ldots, f_r$ generate $\mathcal{J}$. Finally, from (3) and the isomorphism $\mathcal{J}|_U/\mathcal{J}^2|_U = \bigoplus \mathcal{O}_U/\mathcal{J}|_U f_i$ it is clear that $f_1, \ldots, f_r \in \mathcal{O}_X(U)$ is a quasi-regular sequence. □

Lemma 30.12.5. *Let $(X, \mathcal{O}_X)$ be a locally ringed space. Let $\mathcal{J} \subset \mathcal{O}_X$ be a sheaf of ideals. Let $x \in X$ and $f_1, \ldots, f_r \in \mathcal{J}_x$ whose images give a basis for the $\kappa(x)$-vector space $\mathcal{J}_x/\mathfrak{m}_x\mathcal{J}_x$.*

(1) *If $\mathcal{J}$ is quasi-regular, then there exists an open neighbourhood such that $f_1, \ldots, f_r \in \mathcal{O}_X(U)$ form a quasi-regular sequence generating $\mathcal{J}|_U$.*
(2) *If $\mathcal{J}$ is H_1-regular, then there exists an open neighbourhood such that $f_1, \ldots, f_r \in \mathcal{O}_X(U)$ form an H_1-regular sequence generating $\mathcal{J}|_U$.*
(3) *If $\mathcal{J}$ is Koszul-regular, then there exists an open neighbourhood such that $f_1, \ldots, f_r \in \mathcal{O}_X(U)$ form an Koszul-regular sequence generating $\mathcal{J}|_U$.*

Proof. First assume that $\mathcal{J}$ is quasi-regular. We may choose an open neighbourhood $U \subset X$ of x and a quasi-regular sequence $g_1, \ldots, g_s \in \mathcal{O}_X(U)$ which generates $\mathcal{J}|_U$. Note that this implies that $\mathcal{J}/\mathcal{J}^2$ is free of rank s over $\mathcal{O}_U/\mathcal{J}|_U$ (see Lemma 30.12.4 and its proof) and hence $r = s$. We may shrink U and assume $f_1, \ldots, f_r \in \mathcal{J}(U)$. Thus we may write
$$f_i = \sum a_{ij} g_j$$
for some $a_{ij} \in \mathcal{O}_X(U)$. By assumption the matrix $A = (a_{ij})$ maps to an invertible matrix over $\kappa(x)$. Hence, after shrinking U once more, we may assume that (a_{ij})

is invertible. Thus we see that $f_1, \ldots, f_r$ give a basis for $(\mathcal{J}/\mathcal{J}^2)|_U$ which proves that $f_1, \ldots, f_r$ is a quasi-regular sequence over U.

Note that in order to prove (2) and (3) we may, because the assumptions of (2) and (3) are stronger than the assumption in (1), already assume that $f_1, \ldots, f_r \in \mathcal{J}(U)$ and $f_i = \sum a_{ij} g_j$ with (a_{ij}) invertible as above, where now $g_1, \ldots, g_r$ is a H_1-regular or Koszul-regular sequence. Since the Koszul complex on $f_1, \ldots, f_r$ is isomorphic to the Koszul complex on $g_1, \ldots, g_r$ via the matrix (a_{ij}) (see More on Algebra, Lemma 15.19.4) we conclude that $f_1, \ldots, f_r$ is H_1-regular or Koszul-regular as desired. □

Lemma 30.12.6. *Any regular, Koszul-regular, H_1-regular, or quasi-regular sheaf of ideals on a scheme is a finite type quasi-coherent sheaf of ideals.*

Proof. This follows as such a sheaf of ideals is locally generated by finitely many sections. And any sheaf of ideals locally generated by sections on a scheme is quasi-coherent, see Schemes, Lemma 25.10.1. □

Lemma 30.12.7. *Let X be a scheme. Let $\mathcal{J}$ be a sheaf of ideals. Then $\mathcal{J}$ is regular (resp. Koszul-regular, H_1-regular, quasi-regular) if and only if for every $x \in \mathit{Supp}(\mathcal{O}_X/\mathcal{J})$ there exists an affine open neighbourhood $x \in U \subset X$, $U = \mathrm{Spec}(A)$ such that $\mathcal{J}|_U = \widetilde{I}$ and such that I is generated by a regular (resp. Koszul-regular, H_1-regular, quasi-regular) sequence $f_1, \ldots, f_r \in A$.*

Proof. By assumption we can find an open neighbourhood U of x over which $\mathcal{J}$ is generated by a regular (resp. Koszul-regular, H_1-regular, quasi-regular) sequence $f_1, \ldots, f_r \in \mathcal{O}_X(U)$. After shrinking U we may assume that U is affine, say $U = \mathrm{Spec}(A)$. Since $\mathcal{J}$ is quasi-coherent by Lemma 30.12.6 we see that $\mathcal{J}|_U = \widetilde{I}$ for some ideal $I \subset A$. Now we can use the fact that

$$\widetilde{\ } : \mathrm{Mod}_A \longrightarrow \mathit{QCoh}(\mathcal{O}_U)$$

is an equivalence of categories which preserves exactness. For example the fact that the functions f_i generate $\mathcal{J}$ means that the f_i, seen as elements of A generate I. The fact that (30.12.0.2) is injective (resp. (30.12.0.3) is exact, (30.12.0.3) is exact in degree 1, (30.12.0.4) is an isomorphism) implies the corresponding property of the map $A/(f_1, \ldots, f_{i-1}) \to A/(f_1, \ldots, f_{i-1})$ (resp. the complex $K_\bullet(A, f_1, \ldots, f_r)$, the map $A/I[T_1, \ldots, T_r] \to \bigoplus I^n/I^{n+1}$). Thus $f_1, \ldots, f_r \in A$ is a regular (resp. Koszul-regular, H_1-regular, quasi-regular) sequence of the ring A. □

Lemma 30.12.8. *Let X be a locally Noetherian scheme. Let $\mathcal{J} \subset \mathcal{O}_X$ be a quasi-coherent sheaf of ideals. Let x be a point of the support of $\mathcal{O}_X/\mathcal{J}$. The following are equivalent*

1. *$\mathcal{J}_x$ is generated by a regular sequence in $\mathcal{O}_{X,x}$,*
2. *$\mathcal{J}_x$ is generated by a Koszul-regular sequence in $\mathcal{O}_{X,x}$,*
3. *$\mathcal{J}_x$ is generated by an H_1-regular sequence in $\mathcal{O}_{X,x}$,*
4. *$\mathcal{J}_x$ is generated by a quasi-regular sequence in $\mathcal{O}_{X,x}$,*
5. *there exists an affine neighbourhood $U = \mathrm{Spec}(A)$ of x such that $\mathcal{J}|_U = \widetilde{I}$ and I is generated by a regular sequence in A, and*
6. *there exists an affine neighbourhood $U = \mathrm{Spec}(A)$ of x such that $\mathcal{J}|_U = \widetilde{I}$ and I is generated by a Koszul-regular sequence in A, and*
7. *there exists an affine neighbourhood $U = \mathrm{Spec}(A)$ of x such that $\mathcal{J}|_U = \widetilde{I}$ and I is generated by an H_1-regular sequence in A, and*

(8) *there exists an affine neighbourhood* $U = \mathrm{Spec}(A)$ *of* x *such that* $\mathcal{J}|_U = \widetilde{I}$ *and* I *is generated by a quasi-regular sequence in* A,
(9) *there exists a neighbourhood* U *of* x *such that* $\mathcal{J}|_U$ *is regular, and*
(10) *there exists a neighbourhood* U *of* x *such that* $\mathcal{J}|_U$ *is Koszul-regular, and*
(11) *there exists a neighbourhood* U *of* x *such that* $\mathcal{J}|_U$ *is* H_1*-regular, and*
(12) *there exists a neighbourhood* U *of* x *such that* $\mathcal{J}|_U$ *is quasi-regular.*

In particular, on a locally Noetherian scheme the notions of regular, Koszul-regular, H_1*-regular, or quasi-regular ideal sheaf all agree.*

Proof. It follows from Lemma 30.12.7 that (5) $\Leftrightarrow$ (9), (6) $\Leftrightarrow$ (10), (7) $\Leftrightarrow$ (11), and (8) $\Leftrightarrow$ (12). It is clear that (5) $\Rightarrow$ (1), (6) $\Rightarrow$ (2), (7) $\Rightarrow$ (3), and (8) $\Rightarrow$ (4). We have (1) $\Rightarrow$ (5) by Algebra, Lemma 10.65.8. We have (9) $\Rightarrow$ (10) $\Rightarrow$ (11) $\Rightarrow$ (12) by Lemma 30.12.3. Finally, (4) $\Rightarrow$ (1) by Algebra, Lemma 10.66.6. Now all 12 statements are equivalent. □

30.13. Regular immersions

Let $i : Z \to X$ be an immersion of schemes. By definition this means there exists an open subscheme $U \subset X$ such that Z is identified with a closed subscheme of U. Let $\mathcal{I} \subset \mathcal{O}_U$ be the corresponding quasi-coherent sheaf of ideals. Suppose $U' \subset X$ is a second such open subscheme, and denote $\mathcal{I}' \subset \mathcal{O}_{U'}$ the corresponding quasi-coherent sheaf of ideals. Then $\mathcal{I}|_{U\cap U'} = \mathcal{I}'|_{U\cap U'}$. Moreover, the support of $\mathcal{O}_U/\mathcal{I}$ is Z which is contained in $U \cap U'$ and is also the support of $\mathcal{O}_{U'}/\mathcal{I}'$. Hence it follows from Definition 30.12.2 that $\mathcal{I}$ is a regular ideal if and only if $\mathcal{I}'$ is a regular ideal. Similarly for being Koszul-regular, H_1-regular, or quasi-regular.

Definition 30.13.1. Let $i : Z \to X$ be an immersion of schemes. Choose an open subscheme $U \subset X$ such that i identifies Z with a closed subscheme of U and denote $\mathcal{I} \subset \mathcal{O}_U$ the corresponding quasi-coherent sheaf of ideals.

(1) We say i is a *regular immersion* if $\mathcal{I}$ is regular.
(2) We say i is a *Koszul-regular immersion* if $\mathcal{I}$ is Koszul-regular.
(3) We say i is a H_1*-regular immersion* if $\mathcal{I}$ is H_1-regular.
(4) We say i is a *quasi-regular immersion* if $\mathcal{I}$ is quasi-regular.

The discussion above shows that this is independent of the choice of U. The conditions are listed in decreasing order of strength, see Lemma 30.13.2. A Koszul-regular closed immersion is smooth locally a regular immersion, see Lemma 30.13.11. In the locally Noetherian case all four notions agree, see Lemma 30.12.8.

Lemma 30.13.2. *Let* $i : Z \to X$ *be an immersion of schemes. We have the following implications:* i *is regular* $\Rightarrow$ i *is Koszul-regular* $\Rightarrow$ i *is* H_1*-regular* $\Rightarrow$ i *is quasi-regular.*

Proof. The lemma immediately reduces to Lemma 30.12.3. □

Lemma 30.13.3. *Let* $i : Z \to X$ *be an immersion of schemes. Assume* X *is locally Noetherian. Then* i *is regular* $\Leftrightarrow$ i *is Koszul-regular* $\Leftrightarrow$ i *is* H_1*-regular* $\Leftrightarrow$ i *is quasi-regular.*

Proof. Follows immediately from Lemma 30.13.2 and Lemma 30.12.8. □

Lemma 30.13.4. *Let* $i : Z \to X$ *be a regular (resp. Koszul-regular,* H_1*-regular, quasi-regular) immersion. Let* $X' \to X$ *be a flat morphism. Then the base change*

$i' : Z \times_X X' \to X'$ is a regular (resp. Koszul-regular, H_1-regular, quasi-regular) immersion.

Proof. Via Lemma 30.12.7 this translates into the algebraic statements in Algebra, Lemmas 10.65.7 and 10.66.3 and More on Algebra, Lemma 15.20.4. □

Lemma 30.13.5. *Let $i : Z \to X$ be an immersion of schemes. Then i is a quasi-regular immersion if and only if the following conditions are satisfied*

(1) *i is locally of finite presentation,*
(2) *the conormal sheaf $\mathcal{C}_{Z/X}$ is finite locally free, and*
(3) *the map (30.11.1.2) is an isomorphism.*

Proof. An open immersion is locally of finite presentation. Hence we may replace X by an open subscheme $U \subset X$ such that i identifies Z with a closed subscheme of U, i.e., we may assume that i is a closed immersion. Let $\mathcal{I} \subset \mathcal{O}_X$ be the corresponding quasi-coherent sheaf of ideals. Recall, see Morphisms, Lemma 28.22.7 that $\mathcal{I}$ is of finite type if and only if i is locally of finite presentation. Hence the equivalence follows from Lemma 30.12.4 and unwinding the definitions. □

Lemma 30.13.6. *Let $Z \to Y \to X$ be immersions of schemes. Assume that $Z \to Y$ is H_1-regular. Then the canonical sequence of Morphisms, Lemma 28.33.5*

$$0 \to i^*\mathcal{C}_{Y/X} \to \mathcal{C}_{Z/X} \to \mathcal{C}_{Z/Y} \to 0$$

is exact and locally split.

Proof. Since $\mathcal{C}_{Z/Y}$ is finite locally free (see Lemma 30.13.5 and Lemma 30.12.3) it suffices to prove that the sequence is exact. By what was proven in Morphisms, Lemma 28.33.5 it suffices to show that the first map is injective. Working affine locally this reduces to the following question: Suppose that we have a ring A and ideals $I \subset J \subset A$. Assume that $J/I \subset A/I$ is generated by an H_1-regular sequence. Does this imply that $I/I^2 \otimes_A A/J \to J/J^2$ is injective? Note that $I/I^2 \otimes_A A/J = I/IJ$. Hence we are trying to prove that $I \cap J^2 = IJ$. This is the result of More on Algebra, Lemma 15.20.8. □

A composition of quasi-regular immersions may not be quasi-regular, see Algebra, Remark 10.66.8. The other types of regular immersions are preserved under composition.

Lemma 30.13.7. *Let $i : Z \to Y$ and $j : Y \to X$ be immersions of schemes.*

(1) *If i and j are regular immersions, so is $j \circ i$.*
(2) *If i and j are Koszul-regular immersions, so is $j \circ i$.*
(3) *If i and j are H_1-regular immersions, so is $j \circ i$.*
(4) *If i is an H_1-regular immersion and j is a quasi-regular immersion, then $j \circ i$ is a quasi-regular immersion.*

Proof. The algebraic version of (1) is Algebra, Lemma 10.65.9. The algebraic version of (2) is More on Algebra, Lemma 15.20.12. The algebraic version of (3) is More on Algebra, Lemma 15.20.10. The algebraic version of (4) is More on Algebra, Lemma 15.20.9. □

Lemma 30.13.8. *Let $i : Z \to Y$ and $j : Y \to X$ be immersions of schemes. Assume that the sequence*

$$0 \to i^*\mathcal{C}_{Y/X} \to \mathcal{C}_{Z/X} \to \mathcal{C}_{Z/Y} \to 0$$

of Morphisms, Lemma 28.33.5 is exact and locally split.

(1) *If $j \circ i$ is a quasi-regular immersion, so is i.*
(2) *If $j \circ i$ is a H_1-regular immersion, so is i.*
(3) *If both j and $j \circ i$ are Koszul-regular immersions, so is i.*

Proof. After shrinking Y and X we may assume that i and j are closed immersions. Denote $\mathcal{I} \subset \mathcal{O}_X$ the ideal sheaf of Y and $\mathcal{J} \subset \mathcal{O}_X$ the ideal sheaf of Z. The conormal sequence is $0 \to \mathcal{I}/\mathcal{I}\mathcal{J} \to \mathcal{J}/\mathcal{J}^2 \to \mathcal{J}/(\mathcal{I} + \mathcal{J}^2) \to 0$. Let $z \in Z$ and set $y = i(z)$, $x = j(y) = j(i(z))$. Choose $f_1, \ldots, f_n \in \mathcal{I}_x$ which map to a basis of $\mathcal{I}_x/\mathfrak{m}_z\mathcal{I}_x$. Extend this to $f_1, \ldots, f_n, g_1, \ldots, g_m \in \mathcal{J}_x$ which map to a basis of $\mathcal{J}_x/\mathfrak{m}_z\mathcal{J}_x$. This is possible as we have assumed that the sequence of conormal sheaves is split in a neighbourhood of z, hence $\mathcal{I}_x/\mathfrak{m}_x\mathcal{I}_x \to \mathcal{J}_x/\mathfrak{m}_x\mathcal{J}_x$ is injective.

Proof of (1). By Lemma 30.12.5 we can find an affine open neighbourhood U of x such that $f_1, \ldots, f_n, g_1, \ldots, g_m$ forms a quasi-regular sequence generating $\mathcal{J}$. Hence by Algebra, Lemma 10.66.5 we see that $g_1, \ldots, g_m$ induces a quasi-regular sequence on $Y \cap U$ cutting out Z.

Proof of (2). Exactly the same as the proof of (1) except using More on Algebra, Lemma 15.20.11.

Proof of (3). By Lemma 30.12.5 (applied twice) we can find an affine open neighbourhood U of x such that $f_1, \ldots, f_n$ forms a Koszul-regular sequence generating $\mathcal{I}$ and $f_1, \ldots, f_n, g_1, \ldots, g_m$ forms a Koszul-regular sequence generating $\mathcal{J}$. Hence by More on Algebra, Lemma 15.20.13 we see that $g_1, \ldots, g_m$ induces a Koszul-regular sequence on $Y \cap U$ cutting out Z. □

Lemma 30.13.9. *Let $i : Z \to Y$ and $j : Y \to X$ be immersions of schemes. Pick $z \in Z$ and denote $y \in Y$, $x \in X$ the corresponding points. Assume X is locally Noetherian. The following are equivalent*

(1) *i is a regular immersion in a neighbourhood of z and j is a regular immersion in a neighbourhood of y,*
(2) *i and $j \circ i$ are regular immersions in a neighbourhood of z,*
(3) *$j \circ i$ is a regular immersion in a neighbourhood of z and the conormal sequence*
$$0 \to i^*\mathcal{C}_{Y/X} \to \mathcal{C}_{Z/X} \to \mathcal{C}_{Z/Y} \to 0$$
is split exact in a neighbourhood of z.

Proof. Since X (and hence Y) is locally Noetherian all 4 types of regular immersions agree, and moreover we may check whether a morphism is a regular immersion on the level of local rings, see Lemma 30.12.8. The implication (1) ⇒ (2) is Lemma 30.13.7. The implication (2) ⇒ (3) is Lemma 30.13.6. Thus it suffices to prove that (3) implies (1).

Assume (3). Set $A = \mathcal{O}_{X,x}$. Denote $I \subset A$ the kernel of the surjective map $\mathcal{O}_{X,x} \to \mathcal{O}_{Y,y}$ and denote $J \subset A$ the kernel of the surjective map $\mathcal{O}_{X,x} \to \mathcal{O}_{Z,z}$. Note that any minimal sequence of elements generating J in A is a quasi-regular hence regular sequence, see Lemma 30.12.5. By assumption the conormal sequence
$$0 \to I/IJ \to J/J^2 \to J/(I + J^2) \to 0$$
is split exact as a sequence of A/J-modules. Hence we can pick a minimal system of generators $f_1, \ldots, f_n, g_1, \ldots, g_m$ of J with $f_1, \ldots, f_n \in I$ a minimal system of

generators of I. As pointed out above $f_1, \ldots, f_n, g_1, \ldots, g_m$ is a regular sequence in A. It follows directly from the definition of a regular sequence that $f_1, \ldots, f_n$ is a regular sequence in A and $\overline{g}_1, \ldots, \overline{g}_m$ is a regular sequence in A/I. Thus j is a regular immersion at y and i is a regular immersion at z. □

Remark 30.13.10. In the situation of Lemma 30.13.9 parts (1), (2), (3) are **not** equivalent to "$j \circ i$ and j are regular immersions at z and y". An example is $X = \mathbf{A}^1_k = \mathrm{Spec}(k[x])$, $Y = \mathrm{Spec}(k[x]/(x^2))$ and $Z = \mathrm{Spec}(k[x]/(x))$.

Lemma 30.13.11. *Let $i : Z \to X$ be a Koszul regular closed immersion. Then there exists a surjective smooth morphism $X' \to X$ such that the base change $i' : Z \times_X X' \to X'$ of i is a regular immersion.*

Proof. We may assume that X is affine and the ideal of Z generated by a Koszul-regular sequence by replacing X by the members of a suitable affine open covering (affine opens as in Lemma 30.12.7). The affine case is More on Algebra, Lemma 15.20.17. □

30.14. Relative regular immersions

In this section we consider the base change property for regular immersions. The following lemma does not hold for regular immersions or for Koszul immersions, see Examples, Lemma 78.8.2.

Lemma 30.14.1. *Let $f : X \to S$ be a morphism of schemes. Let $i : Z \subset X$ be an immersion. Assume*

(1) *i is an H_1-regular (resp. quasi-regular) immersion, and*
(2) *$Z \to S$ is a flat morphism.*

Then for every morphism of schemes $g : S' \to S$ the base change $Z' = S' \times_S Z \to X' = S' \times_S X$ is an H_1-regular (resp. quasi-regular) immersion.

Proof. Unwinding the definitions and using Lemma 30.12.7 we translate this into algebra as follows. Let $A \to B$ be a ring map and $f_1, \ldots, f_r \in B$. Assume $B/(f_1, \ldots, f_r)B$ is flat over A. Consider a ring map $A \to A'$. Set $B' = B \otimes_A A'$ and $J' = JB'$.

Case I: $f_1, \ldots, f_r$ is quasi-regular. Set $J = (f_1, \ldots, f_r)$. By assumption J^n/J^{n+1} is isomorphic to a direct sum of copies of B/J hence flat over A. By induction and Algebra, Lemma 10.36.12 we conclude that B/J^n is flat over A. The ideal $(J')^n$ is equal to $J^n \otimes_A A'$, see Algebra, Lemma 10.36.11. Hence $(J')^n/(J')^{n+1} = J^n/J^{n+1} \otimes_A A'$ which clearly implies that $f_1, \ldots, f_r$ is a quasi-regular sequence in B'.

Case II: $f_1, \ldots, f_r$ is H_1-regular. By More on Algebra, Lemma 15.20.15 the vanishing of the Koszul homology group $H_1(K_\bullet(B, f_1, \ldots, f_r))$ implies the vanishing of $H_1(K_\bullet(B', f'_1, \ldots, f'_r))$ and we win. □

This lemma is the motivation for the following definition.

Definition 30.14.2. Let $f : X \to S$ be a morphism of schemes. Let $i : Z \to X$ be an immersion.

(1) We say i is a *relative quasi-regular immersion* if $Z \to S$ is flat and i is a quasi-regular immersion.

(2) We say i is a *relative H_1-regular immersion* if $Z \to S$ is flat and i is an H_1-regular immersion.

We warn the reader that this may be nonstandard notation. Lemma 30.14.1 guarantees that relative quasi-regular (resp. H_1-regular) immersions are preserved under any base change. A relative H_1-regular immersion is a relative quasi-regular immersion, see Lemma 30.13.2. Please take a look at Lemma 30.14.5 (or Lemma 30.14.4) which shows that if $Z \to X$ is a relative H_1-regular (or quasi-regular) immersion and the ambient scheme is (flat and) locally of finite presentation over S, then $Z \to X$ is actually a regular immersion and the same remains true after any base change.

Lemma 30.14.3. *Let $f : X \to S$ be a morphism of schemes. Let $Z \to X$ be a relative quasi-regular immersion. If $x \in Z$ and $\mathcal{O}_{X,x}$ is Noetherian, then f is flat at x.*

Proof. Let $f_1, \ldots, f_r \in \mathcal{O}_{X,x}$ be a quasi-regular sequence cutting out the ideal of Z at x. By Algebra, Lemma 10.66.6 we know that $f_1, \ldots, f_r$ is a regular sequence. Hence f_r is a nonzerodivisor on $\mathcal{O}_{X,x}/(f_1, \ldots, f_{r-1})$ such that the quotient is a flat $\mathcal{O}_{S,f(x)}$-module. By Lemma 30.10.3 we conclude that $\mathcal{O}_{X,x}/(f_1, \ldots, f_{r-1})$ is a flat $\mathcal{O}_{S,f(x)}$-module. Continuing by induction we find that $\mathcal{O}_{X,x}$ is a flat $\mathcal{O}_{S,s}$-module. □

Lemma 30.14.4. *Let $X \to S$ be a morphism of schemes. Let $Z \to X$ be an immersion. Assume*

(1) *$X \to S$ is flat and locally of finite presentation,*
(2) *$Z \to X$ is a relative quasi-regular immersion.*

Then $Z \to X$ is a regular immersion and the same remains true after any base change.

Proof. Pick $x \in Z$ with image $s \in S$. To prove this it suffices to find an affine neighbourhood of x contained in U such that the result holds on that affine open. Hence we may assume that X is affine and there exist a quasi-regular sequence $f_1, \ldots, f_r \in \Gamma(X, \mathcal{O}_X)$ such that $Z = V(f_1, \ldots, f_r)$. By Lemma 30.14.1 and its proof the sequence $f_1|_{X_s}, \ldots, f_r|_{X_s}$ is a quasi-regular sequence in $\Gamma(X_s, \mathcal{O}_{X_s})$. Since X_s is Noetherian, this implies, possibly after shrinking X a bit, that $f_1|_{X_s}, \ldots, f_r|_{X_s}$ is a regular sequence, see Algebra, Lemmas 10.66.6 and 10.65.8. By Lemma 30.10.7 it follows that $Z_1 = V(f_1) \subset X$ is a relative effective Cartier divisor, again after possibly shrinking X a bit. Applying the same lemma again, but now to $Z_2 = V(f_1, f_2) \subset Z_1$ we see that $Z_2 \subset Z_1$ is a relative effective Cartier divisor. And so on until on reaches $Z = Z_n = V(f_1, \ldots, f_n)$. Since being a relative effective Cartier divisor is preserved under arbitrary base change, see Lemma 30.10.1, we also see that the final statement of the lemma holds. □

Lemma 30.14.5. *Let $X \to S$ be a morphism of schemes. Let $Z \to X$ be a relative H_1-regular immersion. Assume $X \to S$ is locally of finite presentation. Then*

(1) *there exists an open subscheme $U \subset X$ such that $Z \subset U$ and such that $U \to S$ is flat, and*
(2) *$Z \to X$ is a regular immersion and the same remains true after any base change.*

Proof. Pick $x \in Z$. To prove (1) suffices to find an open neighbourhood $U \subset X$ of x such that $U \to S$ is flat. Hence the lemma reduces to the case that $X = \mathrm{Spec}(B)$ and $S = \mathrm{Spec}(A)$ are affine and that Z is given by an H_1-regular sequence $f_1, \ldots, f_r \in B$. By assumption B is a finitely presented A-algebra and $B/(f_1, \ldots, f_r)B$ is a flat A-algebra. We are going to use absolute Noetherian approximation.

Write $B = A[x_1, \ldots, x_n]/(g_1, \ldots, g_m)$. Assume f_i is the image of $f_i' \in A[x_1, \ldots, x_n]$. Choose a finite type $\mathbf{Z}$-subalgebra $A_0 \subset A$ such that all the coefficients of the polynomials $f_1', \ldots, f_r', g_1, \ldots, g_m$ are in A_0. We set $B_0 = A_0[x_1, \ldots, x_n]/(g_1, \ldots, g_m)$ and we denote $f_{i,0}$ the image of f_i' in B_0. Then $B = B_0 \otimes_{A_0} A$ and

$$B/(f_1, \ldots, f_r) = B_0/(f_{0,1}, \ldots, f_{0,r}) \otimes_{A_0} A.$$

By Algebra, Lemma 10.154.1 we may, after enlarging A_0, assume that $B_0/(f_{0,1}, \ldots, f_{0,r})$ is flat over A_0. It may not be the case at this point that the Koszul cohomology group $H_1(K_\bullet(B_0, f_{0,1}, \ldots, f_{0,r}))$ is zero. On the other hand, as B_0 is Noetherian, it is a finitely generated B_0-module. Let $\xi_1, \ldots, \xi_n \in H_1(K_\bullet(B_0, f_{0,1}, \ldots, f_{0,r}))$ be generators. Let $A_0 \subset A_1 \subset A$ be a larger finite type $\mathbf{Z}$-subalgebra of A. Denote $f_{1,i}$ the image of $f_{0,i}$ in $B_1 = B_0 \otimes_{A_0} A_1$. By More on Algebra, Lemma 15.20.15 the map

$$H_1(K_\bullet(B_0, f_{0,1}, \ldots, f_{0,r})) \otimes_{A_0} A_1 \longrightarrow H_1(K_\bullet(B_1, f_{1,1}, \ldots, f_{1,r}))$$

is surjective. Furthermore, it is clear that the colimit (over all choices of A_1 as above) of the complexes $K_\bullet(B_1, f_{1,1}, \ldots, f_{1,r})$ is the complex $K_\bullet(B, f_1, \ldots, f_r)$ which is acyclic in degree 1. Hence

$$\mathrm{colim}_{A_0 \subset A_1 \subset A} H_1(K_\bullet(B_1, f_{1,1}, \ldots, f_{1,r})) = 0$$

by Algebra, Lemma 10.8.9. Thus we can find a choice of A_1 such that $\xi_1, \ldots, \xi_n$ all map to zero in $H_1(K_\bullet(B_1, f_{1,1}, \ldots, f_{1,r}))$. In other words, the Koszul cohomology group $H_1(K_\bullet(B_1, f_{1,1}, \ldots, f_{1,r}))$ is zero.

Consider the morphism of affine schemes $X_1 \to S_1$ equal to Spec of the ring map $A_1 \to B_1$ and $Z_1 = \mathrm{Spec}(B_1/(f_{1,1}, \ldots, f_{1,r}))$. Since $B = B_1 \otimes_{A_1} A$, i.e., $X = X_1 \times_{S_1} S$, and similarly $Z = Z_1 \times_S S_1$, it now suffices to prove (1) for $X_1 \to S_1$ and the relative H_1-regular immersion $Z_1 \to X_1$, see Morphisms, Lemma 28.26.6. Hence we have reduced to the case where $X \to S$ is a finite type morphism of Noetherian schemes. In this case we know that $X \to S$ is flat at every point of Z by Lemma 30.14.3. Combined with the fact that the flat locus is open in this case, see Algebra, Theorem 10.123.4 we see that (1) holds. Part (2) then follows from an application of Lemma 30.14.4. □

If the ambient scheme is flat and locally of finite presentation over the base, then we can characterize a relative quasi-regular immersion in terms of its fibres.

Lemma 30.14.6. *Let $\varphi : X \to S$ be a flat morphism which is locally of finite presentation. Let $T \subset X$ be a closed subscheme. Let $x \in T$ with image $s \in S$.*

1. *If $T_s \subset X_s$ is a quasi-regular immersion in a neighbourhood of x, then there exists an open $U \subset X$ and a relative quasi-regular immersion $Z \subset U$ such that $Z_s = T_s \cap U_s$ and $T \cap U \subset Z$.*
2. *If $T_s \subset X_s$ is a quasi-regular immersion in a neighbourhood of x, the morphism $T \to X$ is of finite presentation, and $T \to S$ is flat at x, then we can choose U and Z as in (1) such that $T \cap U = Z$.*

(3) *If $T_s \subset X_s$ is a quasi-regular immersion in a neighbourhood of x, and T is cut out by c equations in a neighbourhood of x, where $c = \dim_x(X_s) - \dim_x(T_s)$, then we can choose U and Z as in (1) such that $T \cap U = Z$.*

In each case $Z \to U$ is a regular immersion by Lemma 30.14.4. In particular, if $T \to S$ is locally of finite presentation and flat and all fibres $T_s \subset X_s$ are quasi-regular immersions, then $T \to X$ is a relative quasi-regular immersion.

Proof. Choose affine open neighbourhoods $\mathrm{Spec}(A)$ of s and $\mathrm{Spec}(B)$ of x such that $\varphi(\mathrm{Spec}(B)) \subset \mathrm{Spec}(A)$. Let $\mathfrak{p} \subset A$ be the prime ideal corresponding to s. Let $\mathfrak{q} \subset B$ be the prime ideal corresponding to x. Let $I \subset B$ be the ideal corresponding to T. By the initial assumption of the lemma we know that $A \to B$ is flat and of finite presentation. The assumption in (1) means that, after shrinking $\mathrm{Spec}(B)$, we may assume $I(B \otimes_A \kappa(\mathfrak{p}))$ is generated by a quasi-regular sequence of elements. After possibly localizing B at some $g \in B$, $g \notin \mathfrak{q}$ we may assume there exist $f_1, \ldots, f_r \in I$ which map to a quasi-regular sequence in $B \otimes_A \kappa(\mathfrak{p})$ which generates $I(B \otimes_A \kappa(\mathfrak{p}))$. By Algebra, Lemmas 10.66.6 and 10.65.8 we may assume after another localization that $f_1, \ldots, f_r \in I$ form a regular sequence in $B \otimes_A \kappa(\mathfrak{p})$. By Lemma 30.10.7 it follows that $Z_1 = V(f_1) \subset \mathrm{Spec}(B)$ is a relative effective Cartier divisor, again after possibly localizing B. Applying the same lemma again, but now to $Z_2 = V(f_1, f_2) \subset Z_1$ we see that $Z_2 \subset Z_1$ is a relative effective Cartier divisor. And so on until one reaches $Z = Z_n = V(f_1, \ldots, f_n)$. Then $Z \to \mathrm{Spec}(B)$ is a regular immersion and Z is flat over S, in particular $Z \to \mathrm{Spec}(B)$ is a relative quasi-regular immersion over $\mathrm{Spec}(A)$. This proves (1).

To see (2) consider the closed immersion $Z \to D$. The surjective ring map $u : \mathcal{O}_{D,x} \to \mathcal{O}_{Z,x}$ is a map of flat local $\mathcal{O}_{S,s}$-algebras which are essentially of finite presentation, and which becomes an isomorphisms after dividing by $\mathfrak{m}_s$. Hence it is an isomorphism, see Algebra, Lemma 10.122.4. It follows that $Z \to D$ is an isomorphism in a neighbourhood of x, see Algebra, Lemma 10.120.6.

To see (3), after possibly shrinking U we may assume that the ideal of Z is generated by a regular sequence $f_1, \ldots, f_r$ (see our construction of Z above) and the ideal of T is generated by $g_1, \ldots, g_c$. We claim that $c = r$. Namely,

$$\begin{aligned} \dim_x(X_s) &= \dim(\mathcal{O}_{X_s,x}) + \mathrm{trdeg}_{\kappa(s)}(\kappa(x)), \\ \dim_x(T_s) &= \dim(\mathcal{O}_{T_s,x}) + \mathrm{trdeg}_{\kappa(s)}(\kappa(x)), \\ \dim(\mathcal{O}_{X_s,x}) &= \dim(\mathcal{O}_{T_s,x}) + r \end{aligned}$$

the first two equalities by Algebra, Lemma 10.110.3 and the second by r times applying Algebra, Lemma 10.57.11. As $T \subset Z$ we see that $f_i = \sum b_{ij} g_j$. But the ideals of Z and T cut out the same quasi-regular closed subscheme of X_s in a neighbourhood of x. Hence the matrix (b_{ij}) mod $\mathfrak{m}_x$ is invertible (some details omitted). Hence (b_{ij}) is invertible in an open neighbourhood of x. In other words, $T \cap U = Z$ after shrinking U.

The final statements of the lemma follow immediately from part (2), combined with the fact that $Z \to S$ is locally of finite presentation if and only if $Z \to X$ is of finite presentation, see Morphisms, Lemmas 28.22.3 and 28.22.11. □

The following lemma is an enhancement of Morphisms, Lemma 28.35.20.

Lemma 30.14.7. *Let $f : X \to S$ be a smooth morphism of schemes. Let $\sigma : S \to X$ be a section of f. Then σ is a regular immersion.*

Proof. By Schemes, Lemma 25.21.11 the morphism σ is an immersion. After replacing X by an open neighbourhood of $\sigma(S)$ we may assume that σ is a closed immersion. Let $T = \sigma(S)$ be the corresponding closed subscheme of X. Since $T \to S$ is an isomorphism it is flat and of finite presentation. Also a smooth morphism is flat and locally of finite presentation, see Morphisms, Lemmas 28.35.9 and 28.35.8. Thus, according to Lemma 30.14.6, it suffices to show that $T_s \subset X_s$ is a quasi-regular closed subscheme. This follows immediately from Morphisms, Lemma 28.35.20 but we can also see it directly as follows. Let k be a field and let A be a smooth k-algebra. Let $\mathfrak{m} \subset A$ be a maximal ideal whose residue field is k. Then $\mathfrak{m}$ is generated by a quasi-regular sequence, possibly after replacing A by A_g for some $g \in A$, $g \notin \mathfrak{m}$. In Algebra, Lemma 10.133.3 we proved that $A_\mathfrak{m}$ is a regular local ring, hence $\mathfrak{m}A_\mathfrak{m}$ is generated by a regular sequence. This does indeed imply that $\mathfrak{m}$ is generated by a regular sequence (after replacing A by A_g for some $g \in A$, $g \notin \mathfrak{m}$), see Algebra, Lemma 10.65.8. □

The following lemma has a kind of converse, see Lemma 30.14.11.

Lemma 30.14.8. *Let*

$$\begin{array}{ccc} Y & \xrightarrow{i} & X \\ & {\scriptstyle j}\searrow \quad \swarrow & \\ & S & \end{array}$$

be a commutative diagram of morphisms of schemes. Assume $X \to S$ smooth, and i, j immersions. If j is a regular (resp. Koszul-regular, H_1-regular, quasi-regular) immersion, then so is i.

Proof. We can write i as the composition

$$Y \to Y \times_S X \to X$$

By Lemma 30.14.7 the first arrow is a regular immersion. The second arrow is a flat base change of $Y \to S$, hence is a regular (resp. Koszul-regular, H_1-regular, quasi-regular) immersion, see Lemma 30.13.4. We conclude by an application of Lemma 30.13.7. □

Lemma 30.14.9. *Let*

$$\begin{array}{ccc} Y & \xrightarrow{i} & X \\ & \searrow \quad \swarrow & \\ & S & \end{array}$$

be a commutative diagram of morphisms of schemes. Assume that $Y \to S$ is syntomic, $X \to S$ smooth, and i an immersion. Then i is a regular immersion.

Proof. After replacing X by an open neighbourhood of $i(Y)$ we may assume that i is a closed immersion. Let $T = i(Y)$ be the corresponding closed subscheme of X. Since $T \cong Y$ the morphism $T \to S$ is flat and of finite presentation (Morphisms, Lemmas 28.32.6 and 28.32.7). Also a smooth morphism is flat and locally of finite presentation (Morphisms, Lemmas 28.35.9 and 28.35.8). Thus, according to Lemma 30.14.6, it suffices to show that $T_s \subset X_s$ is a quasi-regular closed subscheme. As X_s is locally of finite type over a field, it is Noetherian (Morphisms, Lemma 28.16.6). Thus we can check that $T_s \subset X_s$ is a quasi-regular immersion at points, see Lemma 30.12.8. Take $t \in T_s$. By Morphisms, Lemma 28.32.9 the local ring $\mathcal{O}_{T_s,t}$ is a local

complete intersection over $\kappa(s)$. The local ring $\mathcal{O}_{X_s,t}$ is regular, see Algebra, Lemma 10.133.3. By Algebra, Lemma 10.128.7 we see that the kernel of the surjection $\mathcal{O}_{X_s,t} \to \mathcal{O}_{T_s,t}$ is generated by a regular sequence, which is what we had to show. □

Lemma 30.14.10. *Let*

$$\begin{array}{ccc} Y & \xrightarrow{i} & X \\ & \searrow \quad \swarrow & \\ & S & \end{array}$$

be a commutative diagram of morphisms of schemes. Assume that $Y \to S$ is smooth, $X \to S$ smooth, and i an immersion. Then i is a regular immersion.

Proof. This is a special case of Lemma 30.14.9 because a smooth morphism is syntomic, see Morphisms, Lemma 28.35.7. □

Lemma 30.14.11. *Let*

$$\begin{array}{ccc} Y & \xrightarrow{i} & X \\ & {\scriptstyle j}\searrow \quad \swarrow & \\ & S & \end{array}$$

be a commutative diagram of morphisms of schemes. Assume $X \to S$ smooth, and i, j immersions. If i is a Koszul-regular (resp. H_1-regular, quasi-regular) immersion, then so is j.

Proof. Let $y \in Y$ be any point. Set $x = i(y)$ and set $s = j(y)$. It suffices to prove the result after replacing X, S by open neighbourhoods U, V of x, s and Y by an open neighbourhood of y in $i^{-1}(U) \cap j^{-1}(V)$. Hence we may assume that Y, X and S are affine. In this case we can choose a closed immersion $h : X \to \mathbf{A}^n_S$ over S for some n. Note that h is a regular immersion by Lemma 30.14.10. Hence $h \circ i$ is a Koszul-regular (resp. H_1-regular, quasi-regular) immersion, see Lemmas 30.13.7 and 30.13.2. In this way we reduce to the case $X = \mathbf{A}^n_S$ and S affine.

After replacing S by an affine open V and replacing Y by $j^{-1}(V)$ we may assume that i is a closed immersion and S affine. Write $S = \mathrm{Spec}(A)$. Then $j : Y \to S$ defines an isomorphism of Y to the closed subscheme $\mathrm{Spec}(A/I)$ for some ideal $I \subset A$. The map $i : Y = \mathrm{Spec}(A/I) \to \mathbf{A}^n_S = \mathrm{Spec}(A[x_1, \ldots, x_n])$ corresponds to an A-algebra homomorphism $i^\sharp : A[x_1, \ldots, x_n] \to A/I$. Choose $a_i \in A$ which map to $i^\sharp(x_i)$ in A/I. Observe that the ideal of the closed immersion i is

$$J = (x_1 - a_1, \ldots, x_n - a_n) + IA[x_1, \ldots, x_n].$$

Set $K = (x_1 - a_1, \ldots, x_n - a_n)$. We claim the sequence

$$0 \to K/KJ \to J/J^2 \to J/(K + J^2) \to 0$$

is split exact. To see this note that K/K^2 is free with basis $x_i - a_i$ over the ring $A[x_1, \ldots, x_n]/K \cong A$. Hence K/KJ is free with the same basis over the ring $A[x_1, \ldots, x_n]/J \cong A/I$. On the other hand, taking derivatives gives a map

$$\mathrm{d}_{A[x_1,\ldots,x_n]/A} : J/J^2 \longrightarrow \Omega_{A[x_1,\ldots,x_n]/A} \otimes_{A[x_1,\ldots,x_n]} A[x_1, \ldots, x_n]/J$$

which maps the generators $x_i - a_i$ to the basis elements $\mathrm{d}x_i$ of the free module on the right. The claim follows. Moreover, note that $x_1 - a_1, \ldots, x_n - a_n$ is a regular

sequence in $A[x_1, \ldots, x_n]$ with quotient ring $A[x_1, \ldots, x_n]/(x_1 - a_1, \ldots, x_n - a_n) \cong A$. Thus we have a factorization

$$Y \to V(x_1 - a_1, \ldots, x_n - a_n) \to \mathbf{A}^n_S$$

of our closed immersion i where the composition is Koszul-regular (resp. H_1-regular, quasi-regular), the second arrow is a regular immersion, and the associated conormal sequence is split. Now the result follows from Lemma 30.13.8. □

30.15. Meromorphic functions and sections

See [**Kle79**] for some possible pitfalls[1].

Let $(X, \mathcal{O}_X)$ be a locally ringed space. For any open $U \subset X$ we have defined the set $\mathcal{S}(U) \subset \mathcal{O}_X(U)$ of regular sections of $\mathcal{O}_X$ over U, see Definition 30.9.16. The restriction of a regular section to a smaller open is regular. Hence $\mathcal{S} : U \mapsto \mathcal{S}(U)$ is a subsheaf (of sets) of $\mathcal{O}_X$. We sometimes denote $\mathcal{S} = \mathcal{S}_X$ if we want to indicate the dependence on X. Moreover, $\mathcal{S}(U)$ is a multiplicative subset of the ring $\mathcal{O}_X(U)$ for each U. Hence we may consider the presheaf of rings

$$U \longmapsto \mathcal{S}(U)^{-1}\mathcal{O}_X(U),$$

see Modules, Lemma 17.22.1.

Definition 30.15.1. Let $(X, \mathcal{O}_X)$ be a locally ringed space. The *sheaf of meromorphic functions on* X is the sheaf $\mathcal{K}_X$ associated to the presheaf displayed above. A *meromorphic function* on X is a global section of $\mathcal{K}_X$.

Since each element of each $\mathcal{S}(U)$ is a nonzerodivisor on $\mathcal{O}_X(U)$ we see that the natural map of sheaves of rings $\mathcal{O}_X \to \mathcal{K}_X$ is injective.

Example 30.15.2. Let $A = \mathbf{C}[x, \{y_\alpha\}_{\alpha \in \mathbf{C}}]/((x - \alpha)y_\alpha, y_\alpha y_\beta)$. Any element of A can be written uniquely as $f(x) + \sum \lambda_\alpha y_\alpha$ with $f(x) \in \mathbf{C}[x]$ and $\lambda_\alpha \in \mathbf{C}$. Let $X = \mathrm{Spec}(A)$. In this case $\mathcal{O}_X = \mathcal{K}_X$, since on any affine open $D(f)$ the ring A_f any nonzerodivisor is a unit (proof omitted).

Definition 30.15.3. Let $f : (X, \mathcal{O}_X) \to (Y, \mathcal{O}_Y)$ be a morphism of locally ringed spaces. We say that *pullbacks of meromorphic functions are defined for* f if for every pair of open $U \subset X$, $V \subset Y$ such that $f(U) \subset V$, and any section $s \in \Gamma(V, \mathcal{S}_Y)$ the pullback $f^\sharp(s) \in \Gamma(U, \mathcal{O}_X)$ is an element of $\Gamma(U, \mathcal{S}_X)$.

In this case there is an induced map $f^\sharp : f^{-1}\mathcal{K}_Y \to \mathcal{K}_X$, in other words we obtain a commutative diagram of morphisms of ringed spaces

$$\begin{array}{ccc} (X, \mathcal{K}_X) & \longrightarrow & (X, \mathcal{O}_X) \\ \downarrow f & & \downarrow f \\ (Y, \mathcal{K}_Y) & \longrightarrow & (Y, \mathcal{O}_X) \end{array}$$

We sometimes denote $f^*(s) = f^\sharp(s)$ for a section $s \in \Gamma(Y, \mathcal{K}_Y)$.

Lemma 30.15.4. *Let $f : X \to Y$ be a morphism of schemes. In each of the following cases pullbacks of meromorphic sections are defined.*

(1) *X, Y are integral and f is dominant,*

[1] Danger, Will Robinson!

(2) *X is integral and the generic point of X maps to a generic point of an irreducible component of Y,*
(3) *X is reduced and every generic point of every irreducible component of X maps to the generic point of an irreducible component of Y,*
(4) *X is locally Noetherian, and any associated point of X maps to a generic point of an irreducible component of Y, and*
(5) *X is locally Noetherian, has no embedded points and any generic point of an irreducible component of X maps to the generic point of an irreducible component of Y.*

Proof. Omitted. Hint: Similar to the proof of Lemma 30.9.12, using the following fact (on Y): if an element $x \in R$ maps to a nonzerodivisor in $R_{\mathfrak{p}}$ for a minimal prime $\mathfrak{p}$ of R, then $x \notin \mathfrak{p}$. See Algebra, Lemma 10.24.4. □

Let $(X, \mathcal{O}_X)$ be a locally ringed space. Let $\mathcal{F}$ be a sheaf of $\mathcal{O}_X$-modules. Consider the presheaf $U \mapsto \mathcal{S}(U)^{-1}\mathcal{F}(U)$. Its sheafification is the sheaf $\mathcal{F} \otimes_{\mathcal{O}_X} \mathcal{K}_X$, see Modules, Lemma 17.22.2.

Definition 30.15.5. Let X be a locally ringed space. Let $\mathcal{F}$ be a sheaf of $\mathcal{O}_X$-modules.

(1) We denote $\mathcal{K}_X(\mathcal{F})$ the sheaf of $\mathcal{K}_X$-modules which is the sheafification of the presheaf $U \mapsto \mathcal{S}(U)^{-1}\mathcal{F}(U)$. Equivalently $\mathcal{K}_X(\mathcal{F}) = \mathcal{F} \otimes_{\mathcal{O}_X} \mathcal{K}_X$ (see above).
(2) A *meromorphic section of $\mathcal{F}$* is a global section of $\mathcal{K}_X(\mathcal{F})$.

In particular we have

$$\mathcal{K}_X(\mathcal{F})_x = \mathcal{F}_x \otimes_{\mathcal{O}_{X,x}} \mathcal{K}_{X,x} = \mathcal{S}_x^{-1}\mathcal{F}_x$$

for any point $x \in X$. However, one has to be careful since it may not be the case that $\mathcal{S}_x$ is the set of nonzerodivisors in the local ring $\mathcal{O}_{X,x}$. Namely, there is always an injective map

$$\mathcal{K}_{X,x} \longrightarrow Q(\mathcal{O}_{X,x})$$

to the total quotient ring. It is also surjective if and only if $\mathcal{S}_x$ is the set of nonzerodivisors in $\mathcal{O}_{X,x}$. The sheaves of meromorphic sections aren't quasi-coherent modules in general, but they do have some properties in common with quasi-coherent modules.

Lemma 30.15.6. *Let X be a quasi-compact scheme. Let $h \in \Gamma(X, \mathcal{O}_X)$ and $f \in \Gamma(X, \mathcal{K}_X)$ such that f restricts to zero on X_h. Then $h^n f = 0$ for some $n \gg 0$.*

Proof. We can find a covering of X by affine opens U such that $f|_U = s^{-1}a$ with $a \in \mathcal{O}_X(U)$ and $s \in \mathcal{S}(U)$. Since X is quasi-compact we can cover it by finitely many affine opens of this form. Thus it suffices to prove the lemma when $X = \mathrm{Spec}(A)$ and $f = s^{-1}a$. Note that $s \in A$ is a nonzerodivisor hence it suffices to prove the result when $f = a$. The condition $f|_{X_h} = 0$ implies that a maps to zero in $A_h = \mathcal{O}_X(X_h)$ as $\mathcal{O}_X \subset \mathcal{K}_X$. Thus $h^n a = 0$ for some $n > 0$ as desired. □

Lemma 30.15.7. *Let X be a locally Noetherian scheme.*

(1) *For any $x \in X$ we have $\mathcal{S}_x \subset \mathcal{O}_{X,x}$ is the set of nonzerodivisors, and hence $\mathcal{K}_{X,x}$ is the total quotient ring of $\mathcal{O}_{X,x}$.*
(2) *For any affine open $U \subset X$ the ring $\mathcal{K}_X(U)$ equals the total quotient ring of $\mathcal{O}_X(U)$.*

Proof. To prove this lemma we may assume X is the spectrum of a Noetherian ring A. Say $x \in X$ corresponds to $\mathfrak{p} \subset A$.

Proof of (1). It is clear that $\mathcal{S}_x$ is contained in the set of nonzero divisors of $\mathcal{O}_{X,x} = A_\mathfrak{p}$. For the converse, let $f, g \in A$, $g \notin \mathfrak{p}$ and assume f/g is a nonzerodivisor in $A_\mathfrak{p}$. Let $I = \{a \in A \mid af = 0\}$. Then we see that $I_\mathfrak{p} = 0$ by exactness of localization. Since A is Noetherian we see that I is finitely generated and hence that $g'I = 0$ for some $g' \in A$, $g' \notin \mathfrak{p}$. Hence f is a nonzerodivisor in $A_{g'}$, i.e., in a Zariski open neighbourhood of $\mathfrak{p}$. Thus f/g is an element of $\mathcal{S}_x$.

Proof of (2). Let $f \in \Gamma(X, \mathcal{K}_X)$ be a meromorphic function. Set $I = \{a \in A \mid af \in A\}$. Fix a prime $\mathfrak{p} \subset A$ corresponding to the point $x \in X$. By (1) we can write the image of f in the stalk at $\mathfrak{p}$ as a/b, $a, b \in A_\mathfrak{p}$ with $b \in A_\mathfrak{p}$ not a zerodivisor. Write $b = c/d$ with $c, d \in A$, $d \notin \mathfrak{p}$. Then $ad - cf$ is a section of $\mathcal{K}_X$ which vanishes in an open neighbourhood of x. Say it vanishes on $D(e)$ with $e \in A$, $e \notin \mathfrak{p}$. Then $e^n(ad - cf) = 0$ for some $n \gg 0$ by Lemma 30.15.6. Thus $e^n c \in I$ and $e^n c$ maps to a nonzerodivisor in $A_\mathfrak{p}$. Let $\mathrm{Ass}(A) = \{\mathfrak{q}_1, \ldots, \mathfrak{q}_t\}$ be the associated primes of A. By looking at $IA_{\mathfrak{q}_i}$ and using Algebra, Lemma 10.60.14 the above says that $I \not\subset \mathfrak{q}_i$ for each i. By Algebra, Lemma 10.14.3 there exists an element $x \in I$, $x \notin \bigcup \mathfrak{q}_i$. By Algebra, Lemma 10.60.9 we see that x is not a zerodivisor on A. Hence $f = (xf)/x$ is an element of the total ring of fractions of A. This proves (2). □

Lemma 30.15.8. *Let X be a scheme. Assume X is reduced and any quasi-compact open $U \subset X$ has a finite number of irreducible components.*

(1) *The sheaf $\mathcal{K}_X$ is a quasi-coherent sheaf of $\mathcal{O}_X$-algebras.*
(2) *For any $x \in X$ we have $\mathcal{S}_x \subset \mathcal{O}_{X,x}$ is the set of nonzerodivisors. In particular $\mathcal{K}_{X,x}$ is the total quotient ring of $\mathcal{O}_{X,x}$.*
(3) *For any affine open $\mathrm{Spec}(A) = U \subset X$ we have that $\mathcal{K}_X(U)$ equals the total quotient ring of A.*

Proof. Let X be as in the lemma. Let $X^{(0)} \subset X$ be the set of generic points of irreducible components of X. Let

$$f : Y = \coprod\nolimits_{\eta \in X^{(0)}} \mathrm{Spec}(\kappa(\eta)) \longrightarrow X$$

be the inclusion of the generic points into X using the canonical maps of Schemes, Section 25.13. (This morphism was used in Morphisms, Definition 28.48.12 to define the normalization of X.) We claim that $\mathcal{K}_X = f_*\mathcal{O}_Y$. First note that $\mathcal{K}_Y = \mathcal{O}_Y$ as Y is a disjoint union of spectra of field. Next, note that pullbacks of meromorphic functions are defined for f, by Lemma 30.15.4. This gives a map

$$\mathcal{K}_X \longrightarrow f_*\mathcal{O}_Y.$$

Let $\mathrm{Spec}(A) = U \subset X$ be an affine open. Then A is a reduced ring with finitely many minimal primes $\mathfrak{q}_1, \ldots, \mathfrak{q}_t$. Then we have $Q(A) = \prod A_{\mathfrak{q}_i} = \prod \kappa(\mathfrak{q}_i)$ by Algebra, Lemmas 10.23.2 and 10.24.4. In other words, already the value of the presheaf $U \mapsto \mathcal{S}(U)^{-1}\mathcal{O}_X(U)$ agrees with $f_*\mathcal{O}_Y(U)$ on our affine open U. Hence the displayed map is an isomorphism.

Now we are ready to prove (1), (2) and (3). The morphism f is quasi-compact by our assumption that the set of irreducible components of X is locally finite. Hence

f is quasi-compact and quasi-separated (as Y is separated). By Schemes, Lemma 25.24.1 $f_*\mathcal{O}_Y$ is quasi-coherent. This proves (1). Let $x \in X$. Then

$$(f_*\mathcal{O}_Y)_x = \prod\nolimits_{\eta \in X^{(0)},\ x \in \overline{\{\eta\}}} \kappa(\eta)$$

On the other hand, $\mathcal{O}_{X,x}$ is reduced and has finitely minimal primes $\mathfrak{q}_i$ corresponding exactly to those $\eta \in X^{(0)}$ such that $x \in \overline{\{\eta\}}\kappa(\eta)$. Hence by Algebra, Lemmas 10.23.2 and 10.24.4 again we see that $Q(\mathcal{O}_{X,x}) = \prod \kappa(\mathfrak{q}_i)$ is the same as $(f_*\mathcal{O}_Y)_x$. This proves (2). Part (3) we saw during the course of the proof that $\mathcal{K}_X = f_*\mathcal{O}_Y$. □

Lemma 30.15.9. *Let X be a scheme. Assume X is reduced and any quasi-compact open $U \subset X$ has a finite number of irreducible components. Then the normalization morphism $\nu : X^\nu \to X$ is the morphism*

$$\underline{\mathrm{Spec}}_X(\mathcal{O}') \longrightarrow X$$

where $\mathcal{O}' \subset \mathcal{K}_X$ is the integral closure of $\mathcal{O}_X$ in the sheaf of meromorphic functions.

Proof. Compare the definition of the normalization morphism $\nu : X^\nu \to X$ (see Morphisms, Definition 28.48.12) with the result $\mathcal{K}_X = f_*\mathcal{O}_Y$ obtained in the proof of Lemma 30.15.8 above. □

Lemma 30.15.10. *Let X be an integral scheme with generic point η. We have*

1. *the sheaf of meromorphic functions is isomorphic to the constant sheaf with value the function field (see Morphisms, Definition 28.10.5) of X.*
2. *for any quasi-coherent sheaf $\mathcal{F}$ on X the sheaf $\mathcal{K}_X(\mathcal{F})$ is isomorphic to the constant sheaf with value $\mathcal{F}_\eta$.*

Proof. Omitted. □

Definition 30.15.11. Let X be a locally ringed space. Let $\mathcal{L}$ be an invertible $\mathcal{O}_X$-module. A meromorphic section s of $\mathcal{L}$ is said to be *regular* if the induced map $\mathcal{K}_X \to \mathcal{K}_X(\mathcal{L})$ is injective. (In other words, this means that s is a regular section of the invertible $\mathcal{K}_X$-module $\mathcal{K}_X(\mathcal{L})$. See Definition 30.9.16.)

First we spell out when (regular) meromorphic sections can be pulled back. After that we discuss the existence of regular meromorphic sections and consequences.

Lemma 30.15.12. *Let $f : X \to Y$ be a morphism of locally ringed spaces. Assume that pullbacks of meromorphic functions are defined for f (see Definition 30.15.3).*

1. *Let $\mathcal{F}$ be a sheaf of $\mathcal{O}_Y$-modules. There is a canonical pullback map $f^* : \Gamma(Y, \mathcal{K}_Y(\mathcal{F})) \to \Gamma(X, \mathcal{K}_X(f^*\mathcal{F}))$ for meromorphic sections of $\mathcal{F}$.*
2. *Let $\mathcal{L}$ be an invertible $\mathcal{O}_X$-module. A regular meromorphic section s of $\mathcal{L}$ pulls back to a regular meromorphic section f^*s of $f^*\mathcal{L}$.*

Proof. Omitted. □

In some cases we can show regular meromorphic sections exist.

Lemma 30.15.13. *Let X be a scheme. Let $\mathcal{L}$ be an invertible $\mathcal{O}_X$-module. In each of the following cases $\mathcal{L}$ has a regular meromorphic section:*

1. *X is integral,*
2. *X is reduced and any quasi-compact open has a finite number of irreducible components, and*

(3) *X is locally Noetherian and has no embedded points.*

Proof. In case (1) we have seen in Lemma 30.15.10 that $\mathcal{K}_X(\mathcal{L})$ is a constant sheaf with value $\mathcal{L}_\eta$, and hence the result is clear.

Suppose X is a scheme. Let $G \subset X$ be the set of generic points of irreducible components of X. For each $\eta \in G$ denote $j_\eta : \eta \to X$ the canonical morphism of $\eta = \mathrm{Spec}(\kappa(\eta))$ into X (see Schemes, Lemma 25.13.3). Consider the sheaf

$$\mathcal{G}_X(\mathcal{L}) = \prod\nolimits_{\eta \in G} j_{\eta,*}(\mathcal{L}_\eta).$$

There is a canonical map

$$\varphi : \mathcal{K}_X(\mathcal{L}) \longrightarrow \mathcal{G}_X(\mathcal{L})$$

coming from the maps $\mathcal{K}_X(\mathcal{L})_\eta \to \mathcal{L}_\eta$ and adjunction (see Sheaves, Lemma 6.27.3).

We claim that in cases (2) and (3) the map φ is an isomorphism for any invertible sheaf $\mathcal{L}$. Before proving this let us show that cases (2) and (3) follow from this. Namely, we can choose $s_\eta \in \mathcal{L}_\eta$ which generate $\mathcal{L}_\eta$, i.e., such that $\mathcal{L}_\eta = \mathcal{O}_{X,\eta} s_\eta$. Since the claim applied to $\mathcal{O}_X$ gives $\mathcal{K}_X = \mathcal{G}_X(\mathcal{O}_X)$ it is clear that the global section $s = \prod_{\eta \in G} s_\eta$ is regular as desired.

To prove that φ is an isomorphism we may work locally on X. For example it suffices to show that sections of $\mathcal{K}_X(\mathcal{L})$ and $\mathcal{G}_X(\mathcal{L})$ agree over small affine opens U. Say $U = \mathrm{Spec}(A)$ and $\mathcal{L}|_U \cong \mathcal{O}_U$. By Lemmas 30.15.7 and 30.15.8 we see that $\Gamma(U, \mathcal{K}_X) = Q(A)$ is the total ring of fractions of A. On the other hand, $\Gamma(U, \mathcal{G}_X(\mathcal{O}_X)) = \prod_{\mathfrak{q} \subset A \text{ minimal}} A_\mathfrak{q}$. In both cases we see that the set of minimal primes of A is finite, say $\mathfrak{q}_1, \ldots, \mathfrak{q}_t$, and that the set of zerodivisors of A is equal to $\mathfrak{q}_1 \cup \ldots \cup \mathfrak{q}_t$ (see Algebra, Lemma 10.60.9). Hence the result follows from Algebra, Lemma 10.23.2. □

Lemma 30.15.14. *Let X be a scheme. Let $\mathcal{L}$ be an invertible $\mathcal{O}_X$-module. Let s be a regular meromorphic section of $\mathcal{L}$. Let us denote $\mathcal{I} \subset \mathcal{O}_X$ the sheaf of ideals defined by the rule*

$$\mathcal{I}(V) = \{f \in \mathcal{O}_Z(V) \mid fs \in \mathcal{L}(V)\}.$$

The formula makes sense since $\mathcal{L}(V) \subset \mathcal{K}_X(\mathcal{L})(V)$. Then $\mathcal{I}$ is a quasi-coherent sheaf of ideals and we have injective maps

$$1 : \mathcal{I} \longrightarrow \mathcal{O}_X, \quad s : \mathcal{I} \longrightarrow \mathcal{L}$$

whose cokernels are supported on closed nowhere dense subsets of X.

Proof. The question is local on X. Hence we may assume that $X = \mathrm{Spec}(A)$, and $\mathcal{L} = \mathcal{O}_X$. After shrinking further we may assume that $s = x/y$ with $a, b \in A$ *both* nonzerodivisors in A. Set $I = \{x \in A \mid x(a/b) \in A\}$.

To show that $\mathcal{I}$ is quasi-coherent we have to show that $I_f = \{x \in A_f \mid x(a/b) \in A_f\}$ for every $f \in A$. If $c/f^n \in A_f$, $(c/f^n)(a/b) \in A_f$, then we see that $f^m c(a/b) \in A$ for some m, hence $c/f^n \in I_f$. Conversely it is easy to see that I_f is contained in $\{x \in A_f \mid x(a/b) \in A_f\}$. This proves quasi-coherence.

Let us prove the final statement. It is clear that $(b) \subset I$. Hence $V(I) \subset V(b)$ is a nowhere dense subset as b is a nonzerodivisor. Thus the cokernel of 1 is supported in a nowhere dense closed set. The same argument works for the cokernel of s since $s(b) = (a) \subset sI \subset A$. □

Definition 30.15.15. Let X be a scheme. Let $\mathcal{L}$ be an invertible $\mathcal{O}_X$-module. Let s be a regular meromorphic section of $\mathcal{L}$. The sheaf of ideals $\mathcal{I}$ constructed in Lemma 30.15.14 is called the *ideal sheaf of denominators of* s.

Here is a lemma which will be used later.

Lemma 30.15.16. *Suppose given*

(1) *X a locally Noetherian scheme,*
(2) *$\mathcal{L}$ an invertible $\mathcal{O}_X$-module,*
(3) *s a regular meromorphic section of $\mathcal{L}$, and*
(4) *$\mathcal{F}$ coherent on X without embedded associated points and $Supp(\mathcal{F}) = X$.*

Let $\mathcal{I} \subset \mathcal{O}_X$ be the ideal of denominators of s. Let $T \subset X$ be the union of the supports of $\mathcal{O}_X/\mathcal{I}$ and $\mathcal{L}/s(\mathcal{I})$ which is a nowhere dense closed subset $T \subset X$ according to Lemma 30.15.14. Then there are canonical injective maps

$$1 : \mathcal{I}\mathcal{F} \to \mathcal{F}, \quad s : \mathcal{I}\mathcal{F} \to \mathcal{F} \otimes_{\mathcal{O}_X} \mathcal{L}$$

whose cokernels are supported on T.

Proof. Reduce to the affine case with $\mathcal{L} \cong \mathcal{O}_X$, and $s = a/b$ with $a, b \in A$ both nonzerodivisors. Proof of reduction step omitted. Write $\mathcal{F} = \widetilde{M}$. Let $I = \{x \in A \mid x(a/b) \in A\}$ so that $\mathcal{I} = \widetilde{I}$ (see proof of Lemma 30.15.14). Note that $T = V(I) \cup V((a/b)I)$. For any A-module M consider the map $1 : IM \to M$; this is the map that gives rise to the map 1 of the lemma. Consider on the other hand the map $\sigma : IM \to M_b, x \mapsto ax/b$. Since b is not a zerodivisor in A, and since M has support $\mathrm{Spec}(A)$ and no embedded primes we see that b is a nonzerodivisor on M also. Hence $M \subset M_b$. By definition of I we have $\sigma(IM) \subset M$ as submodules of M_b. Hence we get an A-module map $s : IM \to M$ (namely the unique map such that $s(z)/1 = \sigma(z)$ in M_b for all $z \in IM$). It is injective because a is a nonzerodivisor also (on both A and M). It is clear that M/IM is annihilated by I and that $M/s(IM)$ is annihilated by $(a/b)I$. Thus the lemma follows. □

30.16. Relative Proj

Some results on relative Proj. First some very basic results. Recall that a relative Proj is always separated over the base, see Constructions, Lemma 26.16.9.

Lemma 30.16.1. *Let S be a scheme. Let $\mathcal{A}$ be a quasi-coherent graded $\mathcal{O}_S$-algebra. Let $p : X = \underline{Proj}_S(\mathcal{A}) \to S$ be the relative Proj of $\mathcal{A}$. If one of the following holds*

(1) *$\mathcal{A}$ is of finite type as a sheaf of $\mathcal{A}_0$-algebras,*
(2) *$\mathcal{A}$ is generated by $\mathcal{A}_1$ as an $\mathcal{A}_0$-algebra and $\mathcal{A}_1$ is a finite type $\mathcal{A}_0$-module,*
(3) *there exists a finite type quasi-coherent $\mathcal{A}_0$-submodule $\mathcal{F} \subset \mathcal{A}_+$ such that $\mathcal{A}_+/\mathcal{F}\mathcal{A}$ is a locally nilpotent sheaf of ideals of $\mathcal{A}/\mathcal{F}\mathcal{A}$,*

then p is quasi-compact.

Proof. The question is local on the base, see Schemes, Lemma 25.19.2. Thus we may assume S is affine. Say $S = \mathrm{Spec}(R)$ and $\mathcal{A}$ corresponds to the graded R-algebra A. Then $X = \mathrm{Proj}(A)$, see Constructions, Section 26.15. In case (1) we may after possibly localizing more assume that A is generated by homogeneous elements $f_1, \ldots, f_n \in A_+$ over A_0. Then $A_+ = (f_1, \ldots, f_n)$ by Algebra, Lemma 10.55.1. In case (3) we see that $\mathcal{F} = \widetilde{M}$ for some finite type A_0-module $M \subset A_+$. Say $M = \sum A_0 f_i$. Say $f_i = \sum f_{i,j}$ is the decomposition into homogeneous pieces.

The condition in (2) signifies that $A_+ \subset \sqrt{(f_{i,j})}$. Thus in both cases we conclude that $\text{Proj}(A)$ is quasi-compact by Constructions, Lemma 26.8.9. Finally, (2) follows from (1). □

Lemma 30.16.2. *Let S be a scheme. Let $\mathcal{A}$ be a quasi-coherent graded $\mathcal{O}_S$-algebra. Let $p : X = \underline{Proj}_S(\mathcal{A}) \to S$ be the relative Proj of $\mathcal{A}$. If $\mathcal{A}$ is of finite type as a sheaf of $\mathcal{O}_S$-algebras, then p is of finite type.*

Proof. The assumption implies that p is quasi-compact, see Lemma 30.16.1. Hence it suffices to show that p is locally of finite type. Thus the question is local on the base and target, see Morphisms, Lemma 28.16.2. Say $S = \text{Spec}(R)$ and $\mathcal{A}$ corresponds to the graded R-algebra A. After further localizing on S we may assume that A is a finite type R-algebra. The scheme X is constructed out of glueing the spectra of the rings $A_{(f)}$ for $f \in A_+$ homogeneous. Each of these is of finite type over R by Algebra, Lemma 10.53.9. Thus $\text{Proj}(A)$ is of finite type over R. □

Lemma 30.16.3. *Let S be a scheme. Let $\mathcal{A}$ be a quasi-coherent graded $\mathcal{O}_S$-algebra. Let $p : X = \underline{Proj}_S(\mathcal{A}) \to S$ be the relative Proj of $\mathcal{A}$. If $\mathcal{O}_S \to \mathcal{A}_0$ is an integral algebra map[2] and $\mathcal{A}$ is of finite type as an $\mathcal{A}_0$-algebra, then p is universally closed.*

Proof. The question is local on the base. Thus we may assume that $X = \text{Spec}(R)$ is affine. Let $\mathcal{A}$ be the quasi-coherent $\mathcal{O}_X$-algebra associated to the graded R-algebra A. The assumption is that $R \to A_0$ is integral and A is of finite type over A_0. Write $X \to \text{Spec}(R)$ as the composition $X \to \text{Spec}(A_0) \to \text{Spec}(R)$. Since $R \to A_0$ is an integral ring map, we see that $\text{Spec}(A_0) \to \text{Spec}(R)$ is universally closed, see Morphisms, Lemma 28.44.7. The quasi-compact (see Constructions, Lemma 26.8.9) morphism

$$\text{Proj}(A) \to \text{Proj}(A_0)$$

satisfies the existence part of the valuative criterion by Constructions, Lemma 26.8.11 and hence it is universally closed by Schemes, Proposition 25.20.6. Thus $X \to \text{Spec}(R)$ is universally closed as a composition of universally closed morphisms. □

Lemma 30.16.4. *Let S be a scheme. Let $\mathcal{A}$ be a quasi-coherent graded $\mathcal{O}_S$-algebra. Let $p : X = \underline{Proj}_S(\mathcal{A}) \to S$ be the relative Proj of $\mathcal{A}$. The following conditions are equivalent*

(1) *$\mathcal{A}_0$ is a finite type $\mathcal{O}_S$-module and $\mathcal{A}$ is of finite type as an $\mathcal{A}_0$-algebra,*
(2) *$\mathcal{A}_0$ is a finite type $\mathcal{O}_S$-module and $\mathcal{A}$ is of finite type as an $\mathcal{O}_S$-algebra*

If these conditions hold, then p is locally projective and in particular proper.

Proof. Assume that $\mathcal{A}_0$ is a finite type $\mathcal{O}_S$-module. Choose an affine open $U = \text{Spec}(R) \subset X$ such that $\mathcal{A}$ corresponds to a graded R-algebra A with A_0 a finite R-module. Condition (1) means that (after possibly localizing further on S) that A is a finite type A_0-algebra and condition (2) means that (after possibly localizing further on S) that A is a finite type R-algebra. Thus these conditions imply each other by Algebra, Lemma 10.6.2.

[2] In other words, the integral closure of $\mathcal{O}_S$ in $\mathcal{A}_0$, see Morphisms, Definition 28.48.2, equals $\mathcal{A}_0$.

A locally projective morphism is proper, see Morphisms, Lemma 28.43.5. Thus we may now assume that $S = \mathrm{Spec}(R)$ and $X = \mathrm{Proj}(A)$ and that A_0 is finite over R and A of finite type over R. We will show that $X = \mathrm{Proj}(A) \to \mathrm{Spec}(R)$ is projective. We urge the reader to prove this for themselves, by directly constructing a closed immersion of X into a projective space over R, instead of reading the argument we give below.

By Lemma 30.16.2 we see that X is of finite type over $\mathrm{Spec}(R)$. Constructions, Lemma 26.10.6 tells us that $\mathcal{O}_X(d)$ is ample on X for some $d \geq 1$ (see Properties, Section 27.24). Hence $X \to \mathrm{Spec}(R)$ is quasi-projective (by Morphisms, Definition 28.41.1). By Morphisms, Lemma 28.43.12 we conclude that X is isomorphic to an open subscheme of a scheme projective over $\mathrm{Spec}(R)$. Therefore, to finish the proof, it suffices to show that $X \to \mathrm{Spec}(R)$ is universally closed (use Morphisms, Lemma 28.42.7). This follows from Lemma 30.16.3. □

30.17. Closed subschemes of relative proj

Some auxiliary lemmas about closed subschemes of relative proj.

Lemma 30.17.1. *Let S be a scheme. Let $\mathcal{A}$ be a quasi-coherent graded $\mathcal{O}_S$-algebra. Let $p : X = \underline{Proj}_S(\mathcal{A}) \to S$ be the relative Proj of $\mathcal{A}$. Let $i : Z \to X$ be a closed subscheme. Denote $\mathcal{I} \subset \mathcal{A}$ the kernel of the canonical map*

$$\mathcal{A} \longrightarrow \bigoplus\nolimits_{d \geq 0} p_* \left((i_* \mathcal{O}_Z)(d) \right)$$

If p is quasi-compact, then there is an isomorphism $Z = \underline{Proj}_S(\mathcal{A}/\mathcal{I})$.

Proof. The morphism p is separated by Constructions, Lemma 26.16.9. As p is quasi-compact, p_* transforms quasi-coherent modules into quasi-coherent modules, see Schemes, Lemma 25.24.1. Hence $\mathcal{I}$ is a quasi-coherent $\mathcal{O}_S$-module. In particular, $\mathcal{B} = \mathcal{A}/\mathcal{I}$ is a quasi-coherent graded $\mathcal{O}_S$-algebra. The functoriality morphism $Z' = \underline{\mathrm{Proj}}_S(\mathcal{B}) \to \underline{\mathrm{Proj}}_S(\mathcal{A})$ is everywhere defined and a closed immersion, see Constructions, Lemma 26.18.3. Hence it suffices to prove $Z = Z'$ as closed subschemes of X.

Having said this, the question is local on the base and we may assume that $S = \mathrm{Spec}(R)$ and that $X = \mathrm{Proj}(A)$ for some graded R-algebra A. Assume $\mathcal{I} = \widetilde{I}$ for $I \subset A$ a graded ideal. By Constructions, Lemma 26.8.9 there exist $f_0, \ldots, f_n \in A_+$ such that $A_+ \subset \sqrt{(f_0, \ldots, f_n)}$ in other words $X = \bigcup D_+(f_i)$. Therefore, it suffices to check that $Z \cap D_+(f_i) = Z' \cap D_+(f_i)$ for each i. By renumbering we may assume $i = 0$. Say $Z \cap D_+(f_0)$, resp. $Z' \cap D_+(f_0)$ is cut out by the ideal J, resp. J' of $A_{(f_0)}$.

The inclusion $J' \subset J$. Let d be the least common multiple of $\deg(f_0), \ldots, \deg(f_n)$. Note that each of the twists $\mathcal{O}_X(nd)$ is invertible, trivialized by $f_i^{nd/\deg(f_i)}$ over $D_+(f_i)$, and that for any quasi-coherent module $\mathcal{F}$ on X the multiplication maps $\mathcal{O}_X(nd) \otimes_{\mathcal{O}_X} \mathcal{F}(m) \to \mathcal{F}(nd + m)$ are isomorphisms, see Constructions, Lemma 26.10.2. Observe that J' is the ideal generated by the elements g/f_0^e where $g \in I$ is homogeneous of degree $e \deg(f_0)$ (see proof of Constructions, Lemma 26.11.3). Of course, by replacing g by $f_0^l g$ for suitable l we may always assume that $d|e$. Then, since g vanishes as a section of $\mathcal{O}_X(e \deg(f_0))$ restricted to Z we see that g/f_0^d is an element of J. Thus $J' \subset J$.

Conversely, suppose that $g/f_0^e \in J$. Again we may assume $d|e$. Pick $i \in \{1, \ldots, n\}$. Then $Z \cap D_+(f_i)$ is cut out by some ideal $J_i \subset A_{(f_i)}$. Moreover,

$$J \cdot A_{(f_0 f_i)} = J_i \cdot A_{(f_0 f_i)}$$

The right hand side is the localization of J_i with respect to $f_0^{\deg(f_i)}/f_i^{\deg(f_0)}$. It follows that

$$f_0^{e_i} g / f_i^{(e_i + e)\deg(f_0)/\deg(f_i)} \in J_i$$

for some $e_i \gg 0$ sufficiently divisible. This proves that $f_0^{\max(e_i)} g$ is an element of I, because its restriction to each affine open $D_+(f_i)$ vanishes on the closed subscheme $Z \cap D_+(f_i)$. Hence $g \in J'$ and we conclude $J \subset J'$ as desired. □

In case the closed subscheme is locally cut out by finitely many equations we can define it by a finite type ideal sheaf of $\mathcal{A}$.

Lemma 30.17.2. *Let S be a quasi-compact and quasi-separated scheme. Let $\mathcal{A}$ be a quasi-coherent graded $\mathcal{O}_S$-algebra. Let $p : X = \underline{Proj}_S(\mathcal{A}) \to S$ be the relative Proj of $\mathcal{A}$. Let $i : Z \to X$ be a closed subscheme. If p is quasi-compact and i of finite presentation, then there exists a $d > 0$ and a quasi-coherent finite type $\mathcal{O}_S$-submodule $\mathcal{F} \subset \mathcal{A}_d$ such that $Z = \underline{Proj}_S(\mathcal{A}/\mathcal{F}\mathcal{A})$.*

Proof. By Lemma 30.17.1 we know there exists a quasi-coherent graded sheaf of ideals $\mathcal{I} \subset \mathcal{A}$ such that $Z = \underline{\mathrm{Proj}}(\mathcal{A}/\mathcal{I})$. Since S is quasi-compact we can choose a finite affine open covering $S = U_1 \cup \ldots \cup U_n$. Say $U_i = \mathrm{Spec}(R_i)$. Let $\mathcal{A}|_{U_i}$ correspond to the graded R_i-algebra A_i and $\mathcal{I}|_{U_i}$ to the graded ideal $I_i \subset A_i$. Note that $p^{-1}(U_i) = \mathrm{Proj}(A_i)$ as schemes over R_i. Since p is quasi-compact we can choose finitely many homogeneous elements $f_{i,j} \in A_{i,+}$ such that $p^{-1}(U_i) = D_+(f_{i,j})$. The condition on $Z \to X$ means that the ideal sheaf of Z in $\mathcal{O}_X$ is of finite type, see Morphisms, Lemma 28.22.7. Hence we can find finitely many homogeneous elements $h_{i,j,k} \in I_i \cap A_{i,+}$ such that the ideal of $Z \cap D_+(f_{i,j})$ is generated by the elements $h_{i,j,k}/f_{i,j}^{e_{i,j,k}}$. Choose $d > 0$ to be a common multiple of all the integers $\deg(f_{i,j})$ and $\deg(h_{i,j,k})$. By Properties, Lemma 27.20.7 there exists a finite type $\mathcal{F} \subset \mathcal{I}_d$ such that all the local sections

$$h_{i,j,k} f_{i,j}^{(d - \deg(h_{i,j,k}))/\deg(f_{i,j})}$$

are sections of $\mathcal{F}$. By construction $\mathcal{F}$ is a solution. □

The following version of Lemma 30.17.2 will be used in the proof of Lemma 30.20.2.

Lemma 30.17.3. *Let S be a quasi-compact and quasi-separated scheme. Let $\mathcal{A}$ be a quasi-coherent graded $\mathcal{O}_S$-algebra. Let $p : X = \underline{Proj}_S(\mathcal{A}) \to S$ be the relative Proj of $\mathcal{A}$. Let $i : Z \to X$ be a closed subscheme. Let $U \subset X$ be an open. Assume that*

1. *p is quasi-compact,*
2. *i of finite presentation,*
3. *$U \cap p(i(Z)) = \emptyset$,*
4. *U is quasi-compact,*
5. *$\mathcal{A}_n$ is a finite type $\mathcal{O}_S$-module for all n.*

Then there exists a $d > 0$ and a quasi-coherent finite type $\mathcal{O}_S$-submodule $\mathcal{F} \subset \mathcal{A}_d$ with (a) $Z = \underline{Proj}_S(\mathcal{A}/\mathcal{F}\mathcal{A})$ and (b) the support of $\mathcal{A}_d/\mathcal{F}$ is disjoint from U.

Proof. Let $\mathcal{I} \subset \mathcal{A}$ be the sheaf of quasi-coherent graded ideals constructed in Lemma 30.17.1. Let U_i, R_i, A_i, I_i, $f_{i,j}$, $h_{i,j,k}$, and d be as constructed in the proof of Lemma 30.17.2. Since $U \cap p(i(Z)) = \emptyset$ we see that $\mathcal{I}_d|_U = \mathcal{A}_d|_U$ (by our construction of $\mathcal{I}$ as a kernel). Since U is quasi-compact we can choose a finite affine open covering $U = W_1 \cup \ldots \cup W_m$. Since $\mathcal{A}_d$ is of finite type we can find finitely many sections $g_{t,s} \in \mathcal{A}_d(W_t)$ which generate $\mathcal{A}_d|_{W_t} = \mathcal{I}_d|_{W_t}$ as an $\mathcal{O}_{W_t}$-module. To finish the proof, note that by Properties, Lemma 27.20.7 there exists a finite type $\mathcal{F} \subset \mathcal{I}_d$ such that all the local sections

$$h_{i,j,k} f_{i,j}^{(d-\deg(h_{i,j,k}))/\deg(f_{i,j})} \quad \text{and} \quad g_{t,s}$$

are sections of $\mathcal{F}$. By construction $\mathcal{F}$ is a solution. □

30.18. Blowing up

Blowing up is an important tool in algebraic geometry.

Definition 30.18.1. Let X be a scheme. Let $\mathcal{I} \subset \mathcal{O}_X$ be a quasi-coherent sheaf of ideals, and let $Z \subset X$ be the closed subscheme corresponding to $\mathcal{I}$, see Schemes, Definition 25.10.2. The *blowing up of X along Z*, or the *blowing up of X in the ideal sheaf $\mathcal{I}$* is the morphism

$$b : \underline{\mathrm{Proj}}_X \left(\bigoplus\nolimits_{n \geq 0} \mathcal{I}^n \right) \longrightarrow X$$

The *exceptional divisor* of the blow up is the inverse image $b^{-1}(Z)$. Sometimes Z is called the *center* of the blowup.

We will see later that the exceptional divisor is an effective Cartier divisor. Moreover, the blowing up is characterized as the "smallest" scheme over X such that the inverse image of Z is an effective Cartier divisor.

If $b : X' \to X$ is the blow up of X in Z, then we often denote $\mathcal{O}_{X'}(n)$ the twists of the structure sheaf. Note that these are invertible $\mathcal{O}_{X'}$-modules and that $\mathcal{O}_{X'}(n) = \mathcal{O}_{X'}(1)^{\otimes n}$ because X' is the relative Proj of a quasi-coherent graded $\mathcal{O}_X$-algebra which is generated in degree 1, see Constructions, Lemma 26.16.11. Note that $\mathcal{O}_{X'}(1)$ is b-relatively very ample, even though b need not be of finite type or even quasi-compact, because X' comes equipped with a closed immersion into $\mathbf{P}(\mathcal{I})$, see Morphisms, Example 28.39.3.

Lemma 30.18.2. *Let X be a scheme. Let $\mathcal{I} \subset \mathcal{O}_X$ be a quasi-coherent sheaf of ideals. Let $U = \mathrm{Spec}(A)$ be an affine open subscheme of X and let $I \subset A$ be the ideal corresponding to $\mathcal{I}|_U$. If $b : X' \to X$ is the blow up of X in $\mathcal{I}$, then there is a canonical isomorphism*

$$b^{-1}(U) = Proj(\bigoplus\nolimits_{d \geq 0} I^d)$$

of $b^{-1}(U)$ with the homogeneous spectrum of the Rees algebra of I in A. Moreover, $b^{-1}(U)$ has an affine open covering by spectra of the affine blowup algebras $A[\frac{I}{a}]$.

Proof. The first statement is clear from the construction of the relative Proj via glueing, see Constructions, Section 26.15. For $a \in I$ denote $a^{(1)}$ the element a seen as an element of degree 1 in the Rees algebra $\bigoplus_{n \geq 0} I^n$. Since these elements generate the Rees algebra over A we see that $\mathrm{Proj}(\bigoplus_{d \geq 0} I^d)$ is covered by the affine opens $D_+(a^{(1)})$. The affine scheme $D_+(a^{(1)})$ is the spectrum of the affine blowup algebra $A' = A[\frac{I}{a}]$, see Algebra, Definition 10.54.1. This finishes the proof. □

Lemma 30.18.3. *Let $X_1 \to X_2$ be a flat morphism of schemes. Let $Z_2 \subset X_2$ be a closed subscheme. Let Z_1 be the inverse image of Z_2 in X_1. Let X_i' be the blow up of Z_i in X_i. Then there exists a cartesian diagram*

$$\begin{array}{ccc} X_1' & \longrightarrow & X_2' \\ \downarrow & & \downarrow \\ X_1 & \longrightarrow & X_2 \end{array}$$

of schemes.

Proof. Let $\mathcal{I}_2$ be the ideal sheaf of Z_2 in X_2. Denote $g : X_1 \to X_2$ the given morphism. Then the ideal sheaf $\mathcal{I}_1$ of Z_1 is the image of $g^*\mathcal{I}_2 \to \mathcal{O}_{X_1}$ (by definition of the inverse image, see Schemes, Definition 25.17.7). By Constructions, Lemma 26.16.10 we see that $X_1 \times_{X_2} X_2'$ is the relative Proj of $\bigoplus_{n\geq 0} g^*\mathcal{I}_2^n$. Because g is flat the map $g^*\mathcal{I}_2^n \to \mathcal{O}_{X_1}$ is injective with image $\mathcal{I}_1^n$. Thus we see that $X_1 \times_{X_2} X_2' = X_1'$. □

Lemma 30.18.4. *Let X be a scheme. Let $Z \subset X$ be a closed subscheme. The blowing up $b : X' \to X$ of Z in X has the following properties:*

1. *$b|_{b^{-1}(X\setminus Z)} : b^{-1}(X \setminus Z) \to X \setminus Z$ is an isomorphism,*
2. *the exceptional divisor $E = b^{-1}(Z)$ is an effective Cartier divisor on X',*
3. *there is a canonical isomorphism $\mathcal{O}_{X'}(-1) = \mathcal{O}_{X'}(E)$*

Proof. As blowing up commutes with restrictions to open subschemes (Lemma 30.18.3) the first statement just means that $X' = X$ if $Z = \emptyset$. In this case we are blowing up in the ideal sheaf $\mathcal{I} = \mathcal{O}_X$ and the result follows from Constructions, Example 26.8.14.

The second statement is local on X, hence we may assume X affine. Say $X = \mathrm{Spec}(A)$ and $Z = \mathrm{Spec}(A/I)$. By Lemma 30.18.2 we see that X' is covered by the spectra of the affine blowup algebras $A' = A[\frac{I}{a}]$. Then $IA' = aA'$ and a maps to a nonzerodivisor in A' according to Algebra, Lemma 10.54.2. This proves the lemma as the inverse image of Z in $\mathrm{Spec}(A')$ corresponds to $\mathrm{Spec}(A'/IA') \subset \mathrm{Spec}(A')$.

Consider the canonical map $\psi_{univ,1} : b^*\mathcal{I} \to \mathcal{O}_{X'}(1)$, see discussion following Constructions, Definition 26.16.7. We claim that this factors through an isomorphism $\mathcal{I}_E \to \mathcal{O}_{X'}(1)$ (which proves the final assertion). Namely, on the affine open corresponding to the blowup algebra $A' = A[\frac{I}{a}]$ mentioned above $\psi_{univ,1}$ corresponds to the A'-module map

$$I \otimes_A A' \longrightarrow \Big(\Big(\bigoplus\nolimits_{d\geq 0} I^d\Big)_{a^{(1)}}\Big)_1$$

where $a^{(1)}$ is as in Algebra, Definition 10.54.1. We omit the verification that this is the map $I \otimes_A A' \to IA' = aA'$. □

Lemma 30.18.5 (Universal property blowing up)**.** *Let X be a scheme. Let $Z \subset X$ be a closed subscheme. Let $\mathcal{C}$ be the full subcategory of (Sch/X) consisting of $Y \to X$ such that the inverse image of Z is an effective Cartier divisor on Y. Then the blowing up $b : X' \to X$ of Z in X is a final object of $\mathcal{C}$.*

Proof. We see that $b : X' \to X$ is an object of $\mathcal{C}$ according to Lemma 30.18.4. Let $f : Y \to X$ be an object of $\mathcal{C}$. We have to show there exists a unique morphism $Y \to X'$ over X. Let $D = f^{-1}(Z)$. Let $\mathcal{I} \subset \mathcal{O}_X$ be the ideal sheaf of Z and

let $\mathcal{I}_D$ be the ideal sheaf of D. Then $f^*\mathcal{I} \to \mathcal{I}_D$ is a surjection to an invertible $\mathcal{O}_Y$-module. This extends to a map $\psi : \bigoplus f^*\mathcal{I}^d \to \bigoplus \mathcal{I}_D^d$ of graded $\mathcal{O}_Y$-algebras. (We observe that $\mathcal{I}_D^d = \mathcal{I}_D^{\otimes d}$ as D is an effective Cartier divisor.) By the material in Constructions, Section 26.16 the triple $(1, f : Y \to X, \psi)$ defines a morphism $Y \to X'$ over X. The restriction

$$Y \setminus D \longrightarrow X' \setminus b^{-1}(Z) = X \setminus Z$$

is unique. The open $Y \setminus D$ is scheme theoretically dense in Y according to Lemma 30.9.4. Thus the morphism $Y \to X'$ is unique by Morphisms, Lemma 28.7.10 (also b is separated by Constructions, Lemma 26.16.9). □

Lemma 30.18.6. *Let X be a scheme. Let $Z \subset X$ be an effective Cartier divisor. The blowup of X in Z is the identity morphism of X.*

Proof. Immediate from the universal property of blowups (Lemma 30.18.5). □

Lemma 30.18.7. *Let X be a scheme. Let $\mathcal{I} \subset \mathcal{O}_X$ be a quasi-coherent sheaf of ideals. If X is integral, then the blow up X' of X in $\mathcal{I}$ is integral.*

Proof. Combine Lemma 30.18.2 with Algebra, Lemma 10.54.4. □

Lemma 30.18.8. *Let X be a scheme. Let $\mathcal{I} \subset \mathcal{O}_X$ be a quasi-coherent sheaf of ideals. If X is reduced, then the blow up X' of X in $\mathcal{I}$ is reduced.*

Proof. Combine Lemma 30.18.2 with Algebra, Lemma 10.54.5. □

Lemma 30.18.9. *Let X be a scheme. Let $b : X' \to X$ be a blow up of X in a closed subscheme. For any effective Cartier divisor D on X the pullback $b^{-1}D$ is defined (see Definition 30.9.11).*

Proof. By Lemmas 30.18.2 and 30.9.2 this reduces to the following algebra fact: Let A be a ring, $I \subset A$ an ideal, $a \in I$, and $x \in A$ a nonzerodivisor. Then the image of x in $A[\frac{I}{a}]$ is a nonzerodivisor. Namely, suppose that $x(y/a^n) = 0$ in $A[\frac{I}{a}]$. Then $a^m xy = 0$ in A for some m. Hence $a^m y = 0$ as x is a nonzerodivisor. Whence y/a^n is zero in $A[\frac{I}{a}]$ as desired. □

Lemma 30.18.10. *Let X be a scheme. Let $\mathcal{I} \subset \mathcal{O}_X$ and $\mathcal{J}$ be quasi-coherent sheaves of ideals. Let $b : X' \to X$ be the blowing up of X in $\mathcal{I}$. Let $b' : X'' \to X'$ be the blowing up of X' in $b^{-1}\mathcal{J}\mathcal{O}_{X'}$. Then $X'' \to X$ is canonically isomorphic to the blowing up of X in $\mathcal{I}\mathcal{J}$.*

Proof. Let $E \subset X'$ be the exceptional divisor of b which is an effective Cartier divisor by Lemma 30.18.4. Then $(b')^{-1}E$ is an effective Cartier divisor on X'' by Lemma 30.18.9. Let $E' \subset X''$ be the exceptional divisor of b' (also an effective Cartier divisor). Consider the effective Cartier divisor $E'' = E' + (b')^{-1}E$. By construction the ideal of E'' is $(b \circ b')^{-1}\mathcal{I}(b \circ b')^{-1}\mathcal{J}\mathcal{O}_{X''}$. Hence according to Lemma 30.18.5 there is a canonical morphism from X'' to the blowup $c : Y \to X$ of X in $\mathcal{I}\mathcal{J}$. Conversely, as $\mathcal{I}\mathcal{J}$ pulls back to an invertible ideal we see that $c^{-1}\mathcal{I}\mathcal{O}_Y$ defines an effective Cartier divisor, see Lemma 30.9.9. Thus a morphism $c' : Y \to X'$ over X by Lemma 30.18.5. Then $(c')^{-1}b^{-1}\mathcal{J}\mathcal{O}_Y = c^{-1}\mathcal{J}\mathcal{O}_Y$ which also defines an effective Cartier divisor. Thus a morphism $c'' : Y \to X''$ over X'. We omit the verification that this morphism is inverse to the morphism $X'' \to Y$ constructed earlier. □

Lemma 30.18.11. *Let X be a scheme. Let $\mathcal{I} \subset \mathcal{O}_X$ be a quasi-coherent sheaf of ideals. Let $b : X' \to X$ be the blowing up of X in the ideal sheaf $\mathcal{I}$ If $\mathcal{I}$ is of finite type, then*

(1) *$b : X' \to X$ is a projective morphism, and*
(2) *$\mathcal{O}_{X'}(1)$ is a b-relatively ample invertible sheaf.*

Proof. The surjection of graded $\mathcal{O}_X$-algebras

$$\mathrm{Sym}^*_{\mathcal{O}_X}(\mathcal{I}) \longrightarrow \bigoplus\nolimits_{d \geq 0} \mathcal{I}^d$$

defines via Constructions, Lemma 26.18.5 a closed immersion

$$X' = \underline{\mathrm{Proj}}_X(\bigoplus\nolimits_{d \geq 0} \mathcal{I}^d) \longrightarrow \mathbf{P}(\mathcal{I}).$$

Hence b is projective, see Morphisms, Definition 28.43.1. The second statement follows for example from the characterization of relatively ample invertible sheaves in Morphisms, Lemma 28.38.4. Some details omitted. □

Lemma 30.18.12. *Let X be a quasi-compact and quasi-separated scheme. Let $Z \subset X$ be a closed subscheme of finite presentation. Let $b : X' \to X$ be the blowing up with center Z. Let $Z' \subset X'$ be a closed subscheme of finite presentation. Let $X'' \to X'$ be the blowing up with center Z'. There exists a closed subscheme $Y \subset X$ of finite presentation, such that*

(1) *$Y = Z \cup b(Z')$ set theoretically, and*
(2) *the composition $X'' \to X$ is isomorphic to the blowing up of X in Y.*

Proof. The condition that $Z \to X$ is of finite presentation means that Z is cut out by a finite type quasi-coherent sheaf of ideals $\mathcal{I} \subset \mathcal{O}_X$, see Morphisms, Lemma 28.22.7. Write $\mathcal{A} = \bigoplus_{n \geq 0} \mathcal{I}^n$ so that $X' = \underline{\mathrm{Proj}}(\mathcal{A})$. Note that $X \setminus Z$ is a quasi-compact open of X by Properties, Lemma 27.22.1. Since $b^{-1}(X \setminus Z) \to X \setminus Z$ is an isomorphism (Lemma 30.18.4) the same result shows that $b^{-1}(X \setminus Z) \setminus Z'$ is quasi-compact open in X'. Hence $U = X \setminus (Z \cup b(Z'))$ is quasi-compact open in X. By Lemma 30.17.3 there exist a $d > 0$ and a finite type $\mathcal{O}_X$-submodule $\mathcal{F} \subset \mathcal{I}^d$ such that $Z' = \underline{\mathrm{Proj}}(\mathcal{A}/\mathcal{F}\mathcal{A})$ and such that the support of $\mathcal{I}^d/\mathcal{F}$ is contained in $X \setminus U$.

Since $\mathcal{F} \subset \mathcal{I}^d$ is an $\mathcal{O}_X$-submodule we may think of $\mathcal{F} \subset \mathcal{I}^d \subset \mathcal{O}_X$ as a finite type quasi-coherent sheaf of ideals on X. Let's denote this $\mathcal{J} \subset \mathcal{O}_X$ to prevent confusion. Since $\mathcal{I}^d/\mathcal{J}$ and $\mathcal{O}/\mathcal{I}^d$ are supported on $X \setminus U$ we see that $V(\mathcal{J})$ is contained in $X \setminus U$. Conversely, as $\mathcal{J} \subset \mathcal{I}^d$ we see that $Z \subset V(\mathcal{J})$. Over $X \setminus Z \cong X' \setminus b^{-1}(Z)$ the sheaf of ideals $\mathcal{J}$ cuts out Z' (see displayed formula below). Hence $V(\mathcal{J})$ equals $Z \cup b(Z')$. It follows that also $V(\mathcal{I}\mathcal{J}) = Z \cup b(Z')$ set theoretically. Moreover, $\mathcal{I}\mathcal{J}$ is an ideal of finite type as a product of two such. We claim that $X'' \to X$ is isomorphic to the blowing up of X in $\mathcal{I}\mathcal{J}$ which finishes the proof of the lemma by setting $Y = V(\mathcal{I}\mathcal{J})$.

First, recall that the blow up of X in $\mathcal{I}\mathcal{J}$ is the same as the blow up of X' in $b^{-1}\mathcal{J}\mathcal{O}_{X'}$, see Lemma 30.18.10. Hence it suffices to show that the blow up of X' in $b^{-1}\mathcal{J}\mathcal{O}_{X'}$ agrees with the blow up of X' in Z'. We will show that

$$b^{-1}\mathcal{J}\mathcal{O}_{X'} = \mathcal{I}_E^d \mathcal{I}_{Z'}$$

as ideal sheaves on X''. This will prove what we want as $\mathcal{I}_E^d$ cuts out the effective Cartier divisor dE and we can use Lemmas 30.18.6 and 30.18.10.

To see the displayed equality of the ideals we may work locally. With notation A, I, $a \in I$ as in Lemma 30.18.2 we see that $\mathcal{F}$ corresponds to an R-submodule $M \subset I^d$ mapping isomorphically to an ideal $J \subset R$. The condition $Z' = \underline{\mathrm{Proj}}(\mathcal{A}/\mathcal{F}\mathcal{A})$ means that $Z' \cap \mathrm{Spec}(A[\frac{I}{a}])$ is cut out by the ideal generated by the elements m/a^d, $m \in M$. Say the element $m \in M$ corresponds to the function $f \in J$. Then in the affine blowup algebra $A' = A[\frac{I}{a}]$ we see that $f = (a^d m)/a^d = a^d(m/a^d)$. Thus the equality holds. □

30.19. Strict transform

In this section we briefly discuss strict transform under blowing up. Let S be a scheme and let $Z \subset S$ be a closed subscheme. Let $b : S' \to S$ be the blowing up of S in Z and denote $E \subset S'$ the exceptional divisor $E = b^{-1}Z$. In the following we will often consider a scheme X over S and form the cartesian diagram

$$\begin{array}{ccccc} \mathrm{pr}_{S'}^{-1}E & \longrightarrow & X \times_S S' & \xrightarrow[\mathrm{pr}_X]{} & X \\ \downarrow & & \downarrow{\scriptstyle \mathrm{pr}_{S'}} & & \downarrow{\scriptstyle f} \\ E & \longrightarrow & S' & \longrightarrow & S \end{array}$$

Since E is an effective Cartier divisor (Lemma 30.18.4) we see that $\mathrm{pr}_{S'}^{-1}E \subset X \times_S S'$ is locally principal (Lemma 30.9.10). Thus the complement of $\mathrm{pr}_{S'}^{-1}E$ in $X \times_S S'$ is retrocompact (Lemma 30.9.3). Consequently, for a quasi-coherent $\mathcal{O}_{X \times_S S'}$-module $\mathcal{G}$ the subsheaf of sections supported on $\mathrm{pr}_{S'}^{-1}E$ is a quasi-coherent submodule, see Properties, Lemma 27.22.5. If $\mathcal{G}$ is a quasi-coherent sheaf of algebras, e.g., $\mathcal{G} = \mathcal{O}_{X \times_S S'}$, then this subsheaf is an ideal of $\mathcal{G}$.

Definition 30.19.1. With $Z \subset S$ and $f : X \to S$ as above.

(1) Given a quasi-coherent $\mathcal{O}_X$-module $\mathcal{F}$ the *strict transform* of $\mathcal{F}$ with respect to the blowup of S in Z is the quotient $\mathcal{F}'$ of $\mathrm{pr}_X^*\mathcal{F}$ by the submodule of sections supported on $\mathrm{pr}_{S'}^{-1}E$.
(2) The *strict transform* of X is the closed subscheme $X' \subset X \times_S S'$ cut out by the quasi-coherent ideal of sections of $\mathcal{O}_{X \times_S S'}$ supported on $\mathrm{pr}_{S'}^{-1}E$.

Note that taking the strict transform along a blowup depends on the closed subscheme used for the blowup (and not just on the morphism $S' \to S$). This notion is often used for closed subschemes of S. It turns out that the strict transform of X is a blowup of X.

Lemma 30.19.2. *In the situation of Definition 30.19.1.*

(1) *The strict transform X' of X is the blowup of X in the closed subscheme $f^{-1}Z$ of X.*
(2) *For a quasi-coherent $\mathcal{O}_X$-module $\mathcal{F}$ the strict transform $\mathcal{F}'$ is canonically isomorphic to the pushforward along $X' \to X \times_S S'$ of the strict transform of $\mathcal{F}$ relative to the blowing up $X' \to X$.*

Proof. Let $X'' \to X$ be the blowup of X in $f^{-1}Z$. By the universal property of blowing up (Lemma 30.18.5) there exists a commutative diagram

$$\begin{array}{ccc} X'' & \longrightarrow & X \\ \downarrow & & \downarrow \\ S' & \longrightarrow & S \end{array}$$

whence a morphism $X'' \to X \times_S S'$. Thus the first assertion is that this morphism is a closed immersion with image X'. The question is local on X. Thus we may assume X and S are affine. Say that $S = \mathrm{Spec}(A)$, $X = \mathrm{Spec}(B)$, and Z is cut out by the ideal $I \subset A$. Set $J = IB$. The map $B \otimes_A \bigoplus_{n \geq 0} I^n \to \bigoplus_{n \geq 0} J^n$ defines a closed immersion $X'' \to X \times_S S'$, see Constructions, Lemmas 26.11.6 and 26.11.5. We omit the verification that this morphism is the same as the one constructed above from the universal property. Pick $a \in I$ corresponding to the affine open $\mathrm{Spec}(A[\frac{I}{a}]) \subset S'$, see Lemma 30.18.2. The inverse image of $\mathrm{Spec}(A[\frac{I}{a}])$ in the strict transform X' of X is the spectrum of

$$B' = (B \otimes_A A[\tfrac{I}{a}])/a\text{-power-torsion}$$

On the other hand, letting $b \in J$ be the image of a we see that $\mathrm{Spec}(B[\frac{J}{b}])$ is the inverse image of $\mathrm{Spec}(A[\frac{I}{a}])$ in X''. The ring map

$$B \otimes_A A[\tfrac{I}{a}] \longrightarrow B[\tfrac{J}{b}]$$

see Properties, Lemma 27.22.5. is surjective and annihilates a-power torsion as b is a nonzerodivisor in $B[\frac{J}{b}]$. Hence we obtain a surjective map $B' \to B[\frac{J}{b}]$. To see that the kernel is trivial, we construct an inverse map. Namely, let $z = y/b^n$ be an element of $B[\frac{J}{b}]$, i.e., $y \in J^n$. Write $y = \sum x_i b_i$ with $x_i \in I^n$ and $b_i \in B$. We map z to the class of $\sum b_i \otimes x_i/a^n$ in B'. This is well defined because an element of the kernel of the map $B \otimes_A I^n \to J^n$ is annihilated by a^n, hence maps to zero in B'. This shows that the open $\mathrm{Spec}(B[\frac{J}{b}])$ maps isomorphically to the open subscheme $\mathrm{pr}_{S'}^{-1}(\mathrm{Spec}(A[\frac{I}{a}]))$ of X'. Thus $X'' \to X'$ is an isomorphism.

In the notation above, let $\mathcal{F}$ correspond to the B-module N. The strict transform of $\mathcal{F}$ corresponds to the $B \otimes_A A[\frac{I}{a}]$-module

$$N' = (N \otimes_A A[\tfrac{I}{a}])/a\text{-power-torsion}$$

see Properties, Lemma 27.22.5. The strict transform of $\mathcal{F}$ relative to the blowup of X in $f^{-1}Z$ corresponds to the $B[\frac{J}{b}]$-module $N \otimes_B B[\frac{J}{b}]/b$-power-torsion. In exactly the same way as above one proves that these two modules are isomorphic. Details omitted. □

Lemma 30.19.3. *In the situation of Definition 30.19.1.*

(1) *If X is flat over S at all points lying over Z, then the strict transform of X is equal to the base change $X \times_S S'$.*
(2) *Let $\mathcal{F}$ be a quasi-coherent $\mathcal{O}_X$-module. If $\mathcal{F}$ is flat over S at all points lying over Z, then the strict transform $\mathcal{F}'$ of $\mathcal{F}$ is equal to the pullback $pr_X^*\mathcal{F}$.*

Proof. We will prove part (2) as it implies part (1) by the definition of the strict transform of a scheme over S. The question is local on X. Thus we may assume

that $S = \mathrm{Spec}(A)$, $X = \mathrm{Spec}(B)$, and that $\mathcal{F}$ corresponds to the B-module N. Then $\mathcal{F}'$ over the open $\mathrm{Spec}(B \otimes_A A[\frac{I}{a}])$ of $X \times_S S'$ corresponds to the module

$$N' = (N \otimes_A A[\tfrac{I}{a}])/a\text{-power-torsion}$$

see Properties, Lemma 27.22.5. Thus we have to show that the a-power-torsion of $N \otimes_A A[\frac{I}{a}]$ is zero. Let $y \in N \otimes_A A[\frac{I}{a}]$ with $a^n y = 0$. If $\mathfrak{q} \subset B$ is a prime and $a \notin \mathfrak{q}$, then y maps to zero in $(N \otimes_A A[\frac{I}{a}])_\mathfrak{q}$. on the other hand, if $a \in \mathfrak{q}$, then $N_\mathfrak{q}$ is a flat A-module and we see that $N_\mathfrak{q} \otimes_A A[\frac{I}{a}] = (N \otimes_A A[\frac{I}{a}])_\mathfrak{q}$ has no a-power torsion (as $A[\frac{I}{a}]$ doesn't). Hence y maps to zero in this localization as well. We conclude that y is zero by Algebra, Lemma 10.22.1. □

Lemma 30.19.4. *Let S be a scheme. Let $Z \subset S$ be a closed subscheme. Let $b : S' \to S$ be the blowing up of Z in S. Let $g : X \to Y$ be an affine morphism of schemes over S. Let $\mathcal{F}$ be a quasi-coherent sheaf on X. Let $g' : X \times_S S' \to Y \times_S S'$ be the base change of g. Let $\mathcal{F}'$ be the strict transform of $\mathcal{F}$ relative to b. Then $g'_*\mathcal{F}'$ is the strict transform of $g_*\mathcal{F}$.*

Proof. Observe that $g'_*\mathrm{pr}_X^*\mathcal{F} = \mathrm{pr}_Y^* g_*\mathcal{F}$ by Cohomology of Schemes, Lemma 29.5.1. Let $\mathcal{K} \subset \mathrm{pr}_X^*\mathcal{F}$ be the subsheaf of sections supported in the inverse image of Z in $X \times_S S'$. By Properties, Lemma 27.22.7 the pushforward $g'_*\mathcal{K}$ is the subsheaf of sections of $\mathrm{pr}_Y^* g_*\mathcal{F}$ supported in the inverse image of Z in $Y \times_S S'$. As g' is affine (Morphisms, Lemma 28.13.8) we see that g'_* is exact, hence we conclude. □

Lemma 30.19.5. *Let S be a scheme. Let $Z \subset S$ be a closed subscheme. Let $D \subset S$ be an effective Cartier divisor. Let $Z' \subset S$ be the closed subscheme cut out by the product of the ideal sheaves of Z and D. Let $S' \to S$ be the blowup of S in Z.*

1. *The blowup of S in Z' is isomorphic to $S' \to S$.*
2. *Let $f : X \to S$ be a morphism of schemes and let $\mathcal{F}$ be a quasi-coherent $\mathcal{O}_X$-module. If $\mathcal{F}$ has no nonzero local sections supported in $f^{-1}D$, then the strict transform of $\mathcal{F}$ relative to the blowing up in Z agrees with the strict transform of $\mathcal{F}$ relative to the blowing up of S in Z'.*

Proof. The first statement follows on combining Lemmas 30.18.10 and 30.18.6. Using Lemma 30.18.2 the second statement translates into the following algebra problem. Let A be a ring, $I \subset A$ an ideal, $x \in A$ a nonzerodivisor, and $a \in I$. Let M be an A-module whose x-torsion is zero. To show: the a-power torsion in $M \otimes_A A[\frac{I}{a}]$ is equal to the xa-power torsion. The reason for this is that the kernel and cokernel of the map $A \to A[\frac{I}{a}]$ is a-power torsion, so this map becomes an isomorphism after inverting a. Hence the kernel and cokernel of $M \to M \otimes_A A[\frac{I}{a}]$ are a-power torsion too. This implies the result. □

Lemma 30.19.6. *Let S be a scheme. Let $Z \subset S$ be a closed subscheme. Let $b : S' \to S$ be the blowing up with center Z. Let $Z' \subset S'$ be a closed subscheme. Let $S'' \to S'$ be the blowing up with center Z'. Let $Y \subset S$ be a closed subscheme such that $Y = Z \cup b(Z')$ set theoretically and the composition $S'' \to S$ is isomorphic to the blowing up of S in Y. In this situation, given any scheme X over S and $\mathcal{F} \in QCoh(\mathcal{O}_X)$ we have*

1. *the strict transform of $\mathcal{F}$ with respect to the blowing up of S in Y is equal to the strict transform with respect to the blowup $S'' \to S'$ in Z' of the strict transform of $\mathcal{F}$ with respect to the blowup $S' \to S$ of S in Z, and*

(2) *the strict transform of X with respect to the blowing up of S in Y is equal to the strict transform with respect to the blowup $S'' \to S'$ in Z' of the strict transform of X with respect to the blowup $S' \to S$ of S in Z.*

Proof. Let $\mathcal{F}'$ be the strict transform of $\mathcal{F}$ with respect to the blowup $S' \to S$ of S in Z. Let $\mathcal{F}''$ be the strict transform of $\mathcal{F}'$ with respect to the blowup $S'' \to S'$ of S' in Z'. Let $\mathcal{G}$ be the strict transform of $\mathcal{F}$ with respect to the blowup $S'' \to S$ of S in Y. We also label the morphisms

$$\begin{array}{ccccc} X \times_S S'' & \xrightarrow{q} & X \times_S S' & \xrightarrow{p} & X \\ \downarrow{\scriptstyle f''} & & \downarrow{\scriptstyle f'} & & \downarrow{\scriptstyle f} \\ S'' & \longrightarrow & S' & \longrightarrow & S \end{array}$$

By definition there is a surjection $p^*\mathcal{F} \to \mathcal{F}'$ and a surjection $q^*\mathcal{F}' \to \mathcal{F}''$ which combine by right exactness of q^* to a surjection $(p \circ q)^*\mathcal{F} \to \mathcal{F}''$. Also we have the surjection $(p \circ q)^*\mathcal{F} \to \mathcal{G}$. Thus it suffices to prove that these two surjections have the same kernel.

The kernel of the surjection $p^*\mathcal{F} \to \mathcal{F}'$ is supported on $(f \circ p)^{-1}Z$, so this map is an isomorphism at points in the complement. Hence the kernel of $q^*p^*\mathcal{F} \to q^*\mathcal{F}'$ is supported on $(f \circ p \circ q)^{-1}Z$. The kernel of $q^*\mathcal{F}' \to \mathcal{F}''$ is supported on $(f' \circ q)^{-1}Z'$. Combined we see that the kernel of $(p \circ q)^*\mathcal{F} \to \mathcal{F}''$ is supported on $(f \circ p \circ q)^{-1}Z \cup (f' \circ q)^{-1}Z' = (f \circ p \circ q)^{-1}Y$. By construction of $\mathcal{G}$ we see that we obtain a factorization $(p \circ q)^*\mathcal{F} \to \mathcal{F}'' \to \mathcal{G}$. To finish the proof it suffices to show that $\mathcal{F}''$ has no nonzero (local) sections supported on $(f \circ p \circ q)^{-1}(Y) = (f \circ p \circ q)^{-1}Z \cup (f' \circ q)^{-1}Z'$. This follows from Lemma 30.19.5 applied to $\mathcal{F}'$ on $X \times_S S'$ over S', the closed subscheme Z' and the effective Cartier divisor $b^{-1}Z$. □

Lemma 30.19.7. *In the situation of Definition 30.19.1. Suppose that*

$$0 \to \mathcal{F}_1 \to \mathcal{F}_2 \to \mathcal{F}_3 \to 0$$

is an exact sequence of quasi-coherent sheaves on X which remains exact after any base change $T \to S$. Then the strict transforms of $\mathcal{F}_i'$ relative to any blowup $S' \to S$ form a short exact sequence $0 \to \mathcal{F}_1' \to \mathcal{F}_2' \to \mathcal{F}_3' \to 0$ too.

Proof. We may localize on S and X and assume both are affine. Then we may push $\mathcal{F}_i$ to S, see Lemma 30.19.4. We may assume that our blowup is the morphism $1 : S \to S$ associated to an effective Cartier divisor $D \subset S$. Then the translation into algebra is the following: Suppose that A is a ring and $0 \to M_1 \to M_2 \to M_3 \to 0$ is a universally exact sequence of A-modules. Let $a \in A$. Then the sequence

$$0 \to M_1/a\text{-power torsion} \to M_2/a\text{-power torsion} \to M_3/a\text{-power torsion} \to 0$$

is exact too. Namely, surjectivity of the last map and injectivity of the first map are immediate. The problem is exactness in the middle. Suppose that $x \in M_2$ maps to zero in M_3/a-power torsion. Then $y = a^n x \in M_1$ for some n. Then y maps to zero in $M_2/a^n M_2$. Since $M_1 \to M_2$ is universally injective we see that y maps to zero in $M_1/a^n M_1$. Thus $y = a^n z$ for some $z \in M_1$. Thus $a^n(x - y) = 0$. Hence y maps to the class of x in M_2/a-power torsion as desired. □

30.20. Admissible blowups

To have a bit more control over our blowups we introduce the following standard terminology.

Definition 30.20.1. Let X be a scheme. Let $U \subset X$ be an open subscheme. A morphism $X' \to X$ is called a *U-admissible blowup* if there exists a closed immersion $Z \to X$ of finite presentation with Z disjoint from U such that X' is isomorphic to the blow up of X in Z.

We recall that $Z \to X$ is of finite presentation if and only if the ideal sheaf $\mathcal{I}_Z \subset \mathcal{O}_X$ is of finite type, see Morphisms, Lemma 28.22.7. In particular, a U-admissible blowup is a projective morphism, see Lemma 30.18.11. Note that there can be multiple centers which give rise to the same morphism. Hence the requirement is just the existence of some center disjoint from U which produces X'. Finally, as the morphism $b : X' \to X$ is an isomorphism over U (see Lemma 30.18.4) we will often abuse notation and think of U as an open subscheme of X' as well.

Lemma 30.20.2. *Let X be a quasi-compact and quasi-separated scheme. Let $U \subset X$ be a quasi-compact open subscheme. Let $b : X' \to X$ be a U-admissible blowup. Let $X'' \to X'$ be a U-admissible blowup. Then the composition $X'' \to X$ is a U-admissible blowup.*

Proof. Immediate from the more precise Lemma 30.18.12. □

Lemma 30.20.3. *Let X be a quasi-compact and quasi-separated scheme. Let $U, V \subset X$ be quasi-compact open subschemes. Let $b : V' \to V$ be a $U \cap V$-admissible blowup. Then there exists a U-admissible blowup $X' \to X$ whose restriction to V is V'.*

Proof. Let $\mathcal{I} \subset \mathcal{O}_V$ be the finite type quasi-coherent sheaf of ideals such that $V(\mathcal{I})$ is disjoint from $U \cap V$ and such that V' is isomorphic to the blow up of V in $\mathcal{I}$. Let $\mathcal{I}' \subset \mathcal{O}_{U \cup V}$ be the quasi-coherent sheaf of ideals whose restriction to U is $\mathcal{O}_U$ and whose restriction to V is $\mathcal{I}$ (see Sheaves, Section 6.33). By Properties, Lemma 27.20.2 there exists a finite type quasi-coherent sheaf of ideals $\mathcal{J} \subset \mathcal{O}_X$ whose restriction to $U \cup V$ is $\mathcal{I}'$. The lemma follows. □

Lemma 30.20.4. *Let X be a quasi-compact and quasi-separated scheme. Let $U \subset X$ be a quasi-compact open subscheme. Let $b_i : X_i \to X$, $i = 1, \ldots, n$ be U-admissible blowups. There exists a U-admissible blowup $b : X' \to X$ such that (a) b factors as $X' \to X_i \to X$ for $i = 1, \ldots, n$ and (b) each of the morphisms $X' \to X_i$ is a U-admissible blowup.*

Proof. Let $\mathcal{I}_i \subset \mathcal{O}_X$ be the finite type quasi-coherent sheaf of ideals such that $V(\mathcal{I}_i)$ is disjoint from U and such that X_i is isomorphic to the blow up of X in $\mathcal{I}_i$. Set $\mathcal{I} = \mathcal{I}_1 \cdot \ldots \cdot \mathcal{I}_n$ and let X' be the blowup of X in $\mathcal{I}$. Then $X' \to X$ factors through b_i by Lemma 30.18.10. □

Lemma 30.20.5. *Let X be a quasi-compact and quasi-separated scheme. Let U, V be quasi-compact disjoint open subschemes of X. Then there exist a $U \cup V$-admissible blowup $b : X' \to X$ such that X' is a disjoint union of open subschemes $X' = X'_1 \amalg X'_2$ with $b^{-1}(U) \subset X'_1$ and $b^{-1}(V) \subset X'_2$.*

Proof. Choose a finite type quasi-coherent sheaf of ideals $\mathcal{I}$, resp. $\mathcal{J}$ such that $X \setminus U = V(\mathcal{I})$, resp. $X \setminus V = V(\mathcal{J})$, see Properties, Lemma 27.22.1. Then $V(\mathcal{I}\mathcal{J}) = X$ set theoretically, hence $\mathcal{I}\mathcal{J}$ is a locally nilpotent sheaf of ideals. Since $\mathcal{I}$ and $\mathcal{J}$ are of finite type and X is quasi-compact there exists an $n > 0$ such that $\mathcal{I}^n\mathcal{J}^n = 0$. We may and do replace $\mathcal{I}$ by $\mathcal{I}^n$ and $\mathcal{J}$ by $\mathcal{J}^n$. Whence $\mathcal{I}\mathcal{J} = 0$. Let $b : X' \to X$ be the blowing up in $\mathcal{I} + \mathcal{J}$. This is $U \cup V$-admissible as $V(\mathcal{I} + \mathcal{J}) = X \setminus U \cup V$. We will show that X' is a disjoint union of open subschemes $X' = X'_1 \amalg X'_2$ such that $b^{-1}\mathcal{I}|_{X'_2} = 0$ and $b^{-1}\mathcal{J}|_{X'_1} = 0$ which will prove the lemma.

We will use the description of the blowing up in Lemma 30.18.2. Suppose that $U = \mathrm{Spec}(A) \subset X$ is an affine open such that $\mathcal{I}|_U$, resp. $\mathcal{J}|_U$ corresponds to the finitely generated ideal $I \subset A$, resp. $J \subset A$. Then

$$b^{-1}(U) = \mathrm{Proj}(A \oplus (I + J) \oplus (I + J)^2 \oplus \ldots)$$

This is covered by the affine open subsets $A[\frac{I+J}{x}]$ and $A[\frac{I+J}{y}]$ with $x \in I$ and $y \in J$. Since $x \in I$ is a nonzerodivisor in $A[\frac{I+J}{x}]$ and $IJ = 0$ we see that $JA[\frac{I+J}{x}] = 0$. Since $y \in J$ is a nonzerodivisor in $A[\frac{I+J}{y}]$ and $IJ = 0$ we see that $IA[\frac{I+J}{y}] = 0$. Moreover,

$$\mathrm{Spec}(A[\tfrac{I+J}{x}]) \cap \mathrm{Spec}(A[\tfrac{I+J}{y}]) = \mathrm{Spec}(A[\tfrac{I+J}{xy}]) = \emptyset$$

because xy is both a nonzerodivisor and zero. Thus $b^{-1}(U)$ is the disjoint union of the open subscheme U_1 defined as the union of the standard opens $\mathrm{Spec}(A[\frac{I+J}{x}])$ for $x \in I$ and the open subscheme U_2 which is the union of the affine opens $\mathrm{Spec}(A[\frac{I+J}{y}])$ for $y \in J$. We have seen that $b^{-1}\mathcal{I}\mathcal{O}_{X'}$ restricts to zero on U_2 and $b^{-1}\mathcal{I}\mathcal{O}_{X'}$ restricts to zero on U_1. We omit the verification that these open subschemes glue to global open subschemes X'_1 and X'_2. □

30.21. Other chapters

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CHAPTER 31

Limits of Schemes

31.1. Introduction

In this chapter we put material related to limits of schemes. We mostly study limits of inverse systems over directed partially ordered sets with affine transition maps. We discuss absolute Noetherian approximation. We characterize schemes locally of finite presentation over a base as those whose associated functor of points is limit preserving. As an application of absolute Noetherian approximation we prove that the image of an affine under an integral morphism is affine. Moreover, we prove some very general variants of Chow's lemma. A basic reference is [**DG67**].

31.2. Directed limits of schemes with affine transition maps

In this section we construct the limit.

Lemma 31.2.1. *Let I be a directed partially ordered set. Let $(S_i, f_{ii'})$ be an inverse system of schemes over I. If all the schemes S_i are affine, then the limit $S = \lim_i S_i$ exists in the category of schemes. In fact S is affine and $S = \mathrm{Spec}(\mathrm{colim}_i R_i)$ with $R_i = \Gamma(S_i, \mathcal{O})$.*

Proof. Just define $S = \mathrm{Spec}(\mathrm{colim}_i R_i)$. It follows from Schemes, Lemma 25.6.4 that S is the limit even in the category of locally ringed spaces. □

Lemma 31.2.2. *Let I be a directed partially ordered set. Let $(S_i, f_{ii'})$ be an inverse system of schemes over I. If all the morphisms $f_{ii'} : S_i \to S_{i'}$ are affine, then the limit $S = \lim_i S_i$ exists in the category of schemes. Moreover,*

1. *each of the morphisms $f_i : S \to S_i$ is affine,*
2. *for an element $0 \in I$ and any open subscheme $U_0 \subset S_0$ we have*
$$f_0^{-1}(U_0) = \lim_{i \geq 0} f_{i0}^{-1}(U_0)$$
in the category of schemes.

Proof. Choose an element $0 \in I$. Note that I is nonempty as the limit is directed. For every $i \geq 0$ consider the quasi-coherent sheaf of $\mathcal{O}_{S_0}$-algebras $\mathcal{A}_i = f_{i0,*}\mathcal{O}_{S_i}$. Recall that $S_i = \underline{\mathrm{Spec}}_{S_0}(\mathcal{A}_i)$, see Morphisms, Lemma 28.13.3. Set $\mathcal{A} = \mathrm{colim}_{i \geq 0} \mathcal{A}_i$. This is a quasi-coherent sheaf of $\mathcal{O}_{S_0}$-algebras, see Schemes, Section 25.24. Set $S = \underline{\mathrm{Spec}}_{S_0}(\mathcal{A})$. By Morphisms, Lemma 28.13.5 we get for $i \geq 0$ morphisms $f_i : S \to S_i$ compatible with the transition morphisms. Note that the morphisms f_i are affine by Morphisms, Lemma 28.13.11 for example. By Lemma 31.2.1 above we see that for any affine open $U_0 \subset S_0$ the inverse image $U = f_0^{-1}(U_0) \subset S$ is the limit of the system of opens $U_i = f_{i0}^{-1}(U_0)$, $i \geq 0$ in the category of schemes.

Let T be a scheme. Let $g_i : T \to S_i$ be a compatible system of morphisms. To show that $S = \lim_i S_i$ we have to prove there is a unique morphism $g : T \to S$

with $g_i = f_i \circ g$ for all $i \in I$. For every $t \in T$ there exists an affine open $U_0 \subset S_0$ containing $g_0(t)$. Let $V \subset g_0^{-1}(U_0)$ be an affine open neighbourhood containing t. By the remarks above we obtain a unique morphism $g_V : V \to U = f_0^{-1}(U_0)$ such that $f_i \circ g_V = g_i|_{U_i}$ for all i. The open sets $V \subset T$ so constructed form a basis for the topology of T. The morphisms g_V glue to a morphism $g : T \to S$ because of the uniqueness property. This gives the desired morphism $g : T \to S$.

The final statement is clear from the construction of the limit above. □

Lemma 31.2.3. *Let I be a directed partially ordered set. Let $(S_i, f_{ii'})$ be an inverse system of schemes over I. Assume all the morphisms $f_{ii'} : S_i \to S_{i'}$ are affine, Let $S = \lim_i S_i$. Let $0 \in I$. Suppose that T is a scheme over S_0. Then*

$$T \times_{S_0} S = \lim_{i \geq 0} T \times_{S_0} S_i$$

Proof. The right hand side is a scheme by Lemma 31.2.2. The equality is formal, see Categories, Lemma 4.14.9. □

31.3. Descending properties

In this section we work in the following situation.

Situation 31.3.1. Let $S = \lim_{i \in I} S_i$ be the limit of a directed system of schemes with affine transition morphisms $f_{i'i} : S_{i'} \to S_i$ (Lemma 31.2.2). We assume that S_i is quasi-compact and quasi-separated for all $i \in I$. We denote $f_i : S \to S_i$ the projection. We also choose an element $0 \in I$.

The type of result we are looking for is the following: If we have an object over S, then for some i there is a similar object over S_i.

Lemma 31.3.2. *In Situation 31.3.1.*

1. *We have $S_{set} = \lim_i S_{i,set}$ where S_{set} indicates the underlying set of the scheme S.*
2. *If $s, s' \in S$ and s' is not a specialization of s then for some $i \in I$ the image $s'_i \in S_i$ of s' is not a specialization of the image $s_i \in S_i$ of s.*
3. *Add more easy facts on topology of S here. (Requirement: whatever is added should be easy in the affine case.)*

Proof. Proof of (1). Pick $i \in I$. Take $U_i \subset S_i$ an affine open. Denote $U_{i'} = f_{i'i}^{-1}(U_i)$ and $U = f_i^{-1}(U_i)$. Suppose we can show that $U_{set} = \lim_{i' \geq i} U_{i',set}$. Then assertion (1) follows by a simple argument using an affine covering of S_i. Hence we may assume all S_i and S affine. This reduces us to the following algebra question: Suppose given a system of rings $(A_i, \varphi_{ii'})$ over I. Set $A = \operatorname{colim}_i A_i$ with canonical maps $\varphi_i : A_i \to A$. Then

$$\operatorname{Spec}(A) = \lim_i \operatorname{Spec}(A_i)$$

Namely, suppose that we are given primes $\mathfrak{p}_i \subset A_i$ such that $\mathfrak{p}_i = \varphi_{ii'}^{-1}(\mathfrak{p}_{i'})$ for all $i' \geq i$. Then we simply set

$$\mathfrak{p} = \{x \in A \mid \exists i, x_i \in \mathfrak{p}_i \text{ with } \varphi_i(x_i) = x\}$$

It is clear that this is an ideal and has the property that $\varphi_i^{-1}(\mathfrak{p}) = \mathfrak{p}_i$. Then it follows easily that it is a prime ideal as well. This proves (1).

Proof of (2). Pick $i \in I$. Pick an affine open $U_i \subset S_i$ containing $f_i(s')$. If $f_i(s) \notin S_i$ then we are done. Hence reduce to the affine case by considering the inverse images

of U_i as above. This reduces us to the following algebra question: Suppose given a system of rings $(A_i, \varphi_{ii'})$ over I. Set $A = \operatorname{colim}_i A_i$ with canonical maps $\varphi_i : A_i \to A$. Suppose given primes $\mathfrak{p}$, $\mathfrak{p}'$ of A. Suppose that $\mathfrak{p} \not\subset \mathfrak{p}'$. Then for some i we have $\varphi_i^{-1}(\mathfrak{p}) \not\subset \varphi_i^{-1}(\mathfrak{p}')$. This is clear. □

Lemma 31.3.3. *In Situation 31.3.1. Suppose that $\mathcal{F}_0$ is a quasi-coherent sheaf on S_0. Set $\mathcal{F}_i = f_{i0}^*\mathcal{F}_0$ for $i \geq 0$ and set $\mathcal{F} = f_0^*\mathcal{F}_0$. Then*

$$\Gamma(S, \mathcal{F}) = \operatorname{colim}_{i\geq 0} \Gamma(S_i, \mathcal{F}_i)$$

Proof. Write $\mathcal{A}_j = f_{i0,*}\mathcal{O}_{S_i}$. This is a quasi-coherent sheaf of $\mathcal{O}_{S_0}$-algebras (see Morphisms, Lemma 28.13.5) and S_i is the relative spectrum of $\mathcal{A}_i$ over S_0. In the proof of Lemma 31.2.2 we constructed S as the relative spectrum of $\mathcal{A} = \operatorname{colim}_{i\geq 0} \mathcal{A}_i$ over S_0. Set

$$\mathcal{M}_i = \mathcal{F}_0 \otimes_{\mathcal{O}_{S_0}} \mathcal{A}_i$$

and

$$\mathcal{M} = \mathcal{F}_0 \otimes_{\mathcal{O}_{S_0}} \mathcal{A}.$$

Then we have $f_{i0,*}\mathcal{F}_i = \mathcal{M}_i$ and $f_{0,*}\mathcal{F} = \mathcal{M}$. Since $\mathcal{A}$ is the colimit of the sheaves $\mathcal{A}_i$ and since tensor product commutes with directed colimits, we conclude that $\mathcal{M} = \operatorname{colim}_{i\geq 0} \mathcal{M}_i$. Since S_0 is quasi-compact and quasi-separated we see that

$$\begin{aligned} \Gamma(S, \mathcal{F}) &= \Gamma(S_0, \mathcal{M}) \\ &= \Gamma(S_0, \operatorname{colim}_{i\geq 0} \mathcal{M}_i) \\ &= \operatorname{colim}_{i\geq 0} \Gamma(S_0, \mathcal{M}_i) \\ &= \operatorname{colim}_{i\geq 0} \Gamma(S_i, \mathcal{F}_i) \end{aligned}$$

see Sheaves, Lemma 6.29.1 and Topology, Lemma 5.25.1 for the middle equality. □

Lemma 31.3.4. *In Situation 31.3.1. If all the schemes S_i are nonempty, then the limit $S = \lim_i S_i$ is nonempty.*

Proof. Choose $i_0 \in I$. Note that I is nonempty as the limit is directed. For convenience write $S_0 = S_{i_0}$ and $i_0 = 0$. Choose an affine open covering $S_0 = \bigcup_{j=1,\ldots,m} U_j$. Since I is directed there exists a $j \in \{1, \ldots, m\}$ such that $f_{i0}^{-1}(U_j) \neq \emptyset$ for all $i \geq 0$. Hence $\lim_{i\geq 0} f_{i0}^{-1}(U_j)$ is not empty since a directed colimit of nonzero rings is nonzero (because $1 \neq 0$). As $\lim_{i\geq 0} f_{i0}^{-1}(U_j)$ is an open subscheme of the limit we win. □

Lemma 31.3.5. *In Situation 31.3.1. Suppose for each i we are given a nonempty closed subset $Z_i \subset S_i$ with $f_{ii'}(Z_i) \subset Z_{i'}$. Then there exists a point $s \in S$ with $f_i(s) \in Z_i$ for all i.*

Proof. Let $Z_i \subset S_i$ also denote the reduced closed subscheme associated to Z_i, see Schemes, Definition 25.12.5. A closed immersion is affine, and a composition of affine morphisms is affine (see Morphisms, Lemmas 28.13.9 and 28.13.7), and hence $Z_i \to S_{i'}$ is affine when $i \geq i'$. We conclude that the morphism $f_{ii'} : Z_i \to Z_{i'}$ is affine by Morphisms, Lemma 28.13.11. Each of the schemes Z_i is quasi-compact as a closed subscheme of a quasi-compact scheme. Hence we may apply Lemma 31.3.4 to see that $Z = \lim_i Z_i$ is nonempty. Since there is a canonical morphism $Z \to S$ we win. □

Lemma 31.3.6. *In Situation 31.3.1. Suppose we are given an i and a morphism $T \to S_i$ such that*

(1) $T \times_{S_i} S = \emptyset$, *and*
(2) T *is quasi-compact.*

Then $T \times_{S_i} S_{i'} = \emptyset$ *for all sufficiently large* i'.

Proof. By Lemma 31.2.3 we see that $T \times_{S_i} S = \lim_{i' \geq i} T \times_{S_i} S_{i'}$. Hence the result follows from Lemma 31.3.4. □

Lemma 31.3.7. *In Situation 31.3.1. Suppose we are given an* i *and a locally constructible subset* $E \subset S_i$ *such that* $f_i(S) \subset E$. *Then* $f_{ii'}(S_{i'}) \subset E$ *for all sufficiently large* i'.

Proof. Writing S_i as a finite union of open affine subschemes reduces the question to the case that S_i is affine and E is constructible, see Lemma 31.2.2 and Properties, Lemma 27.2.1. In this case the complement $S_i \setminus E$ is constructible too. Hence there exists an affine scheme T and a morphism $T \to S_i$ whose image is $S_i \setminus E$, see Algebra, Lemma 10.27.3. By Lemma 31.3.6 we see that $T \times_{S_i} S_{i'}$ is empty for all sufficiently large i', and hence $f_{ii'}(S_{i'}) \subset E$ for all sufficiently large i'. □

Lemma 31.3.8. *In Situation 31.3.1 we have the following:*

(1) *Given any quasi-compact open* $V \subset S = \lim_i S_i$ *there exists an* $i \in I$ *and a quasi-compact open* $V_i \subset S_i$ *such that* $f_i^{-1}(V_i) = V$.
(2) *Given* $V_i \subset S_i$ *and* $V_{i'} \subset S_{i'}$ *quasi-compact opens such that* $f_i^{-1}(V_i) = f_{i'}^{-1}(V_{i'})$ *there exists an index* $i'' \geq i, i'$ *such that* $f_{i''i}^{-1}(V_i) = f_{i''i'}^{-1}(V_{i'})$.
(3) *If* $V_{1,i}, \ldots, V_{n,i} \subset S_i$ *are quasi-compact opens and* $S = f_i^{-1}(V_{1,i}) \cup \ldots \cup f_i^{-1}(V_{n,i})$ *then* $S_{i'} = f_{i'i}^{-1}(V_{1,i}) \cup \ldots \cup f_{i'i}^{-1}(V_{n,i})$ *for some* $i' \geq i$.

Proof. Choose $i_0 \in I$. Note that I is nonempty as the limit is directed. For convenience we write $S_0 = S_{i_0}$ and $i_0 = 0$. Choose an affine open covering $S_0 = U_{1,0} \cup \ldots \cup U_{m,0}$. Denote $U_{j,i} \subset S_i$ the inverse image of $U_{j,0}$ under the transition morphism for $i \geq 0$. Denote U_j the inverse image of $U_{j,0}$ in S. Note that $U_j = \lim_i U_{j,i}$ is a limit of affine schemes.

We first prove the uniqueness statement: Let $V_i \subset S_i$ and $V_{i'} \subset S_{i'}$ quasi-compact opens such that $f_i^{-1}(V_i) = f_{i'}^{-1}(V_{i'})$. It suffices to show that $f_{i''i}^{-1}(V_i \cap U_{j,i''})$ and $f_{i''i'}^{-1}(V_{i'} \cap U_{j,i''})$ become equal for i'' large enough. Hence we reduce to the case of a limit of affine schemes. In this case write $S = \mathrm{Spec}(R)$ and $S_i = \mathrm{Spec}(R_i)$ for all $i \in I$. We may write $V_i = S_i \setminus V(h_1, \ldots, h_m)$ and $V_{i'} = S_{i'} \setminus V(g_1, \ldots, g_n)$. The assumption means that the ideals $\sum g_j R$ and $\sum h_j R$ have the same radical in R. This means that $g_j^N = \sum a_{jj'} h_{j'}$ and $h_j^N = \sum b_{jj'} g_{j'}$ for some $N \gg 0$ and $a_{jj'}$ and $b_{jj'}$ in R. Since $R = \mathrm{colim}_i R_i$ we can chose an index $i'' \geq i$ such that the equations $g_j^N = \sum a_{jj'} h_{j'}$ and $h_j^N = \sum b_{jj'} g_{j'}$ hold in $R_{i''}$ for some $a_{jj'}$ and $b_{jj'}$ in $R_{i''}$. This implies that the ideals $\sum g_j R_{i''}$ and $\sum h_j R_{i''}$ have the same radical in $R_{i''}$ as desired.

We prove existence. We may apply the uniqueness statement to the limit of schemes $U_{j_1} \cap U_{j_2} = \lim_i U_{j_1,i} \cap U_{j_2,i}$ since these are still quasi-compact due to the fact that the S_i were assumed quasi-separated. Hence it is enough to prove existence in the affine case. In this case write $S = \mathrm{Spec}(R)$ and $S_i = \mathrm{Spec}(R_i)$ for all $i \in I$. Then $V = S \setminus V(g_1, \ldots, g_n)$ for some $g_1, \ldots, g_n \in R$. Choose any i large enough so that each of the g_j comes from an element $g_{j,i} \in R_i$ and take $V_i = S_i \setminus V(g_{1,i}, \ldots, g_{n,i})$.

The statement on coverings follows from the uniqueness statement for the opens $V_{1,i} \cup \ldots \cup V_{n,i}$ and S_i of S_i. □

Lemma 31.3.9. *In Situation 31.3.1 if S is quasi-affine, then for some $i_0 \in I$ the schemes S_i for $i \geq i_0$ are quasi-affine.*

Proof. Choose $i_0 \in I$. Note that I is nonempty as the limit is directed. For convenience we write $S_0 = S_{i_0}$ and $i_0 = 0$. Let $s \in S$. We may choose an affine open $U_0 \subset S_0$ containing $f_0(s)$. Since S is quasi-affine we may choose an element $a \in \Gamma(S, \mathcal{O}_S)$ such that $s \in D(a) \subset f_0^{-1}(U_0)$, and such that $D(a)$ is affine. By Lemma 31.3.3 there exists an $i \geq 0$ such that a comes from an element $a_i \in \Gamma(S_i, \mathcal{O}_{S_i})$. For any index $j \geq i$ we denote a_j the image of a_i in the global sections of the structure sheaf of S_j. Consider the opens $D(a_j) \subset S_j$ and $U_j = f_{j0}^{-1}(U_0)$. Note that U_j is affine and $D(a_j)$ is a quasi-compact open of S_j, see Properties, Lemma 27.24.4 for example. Hence we may apply Lemma 31.3.8 to the opens U_j and $U_j \cup D(a_j)$ to conclude that $D(a_j) \subset U_j$ for some $j \geq i$. For such an index j we see that $D(a_j) \subset S_j$ is an affine open (because $D(a_j)$ is a standard affine open of the affine open U_j) containing the image $f_j(s)$.

We conclude that for every $s \in S$ there exist an index $i \in I$, and a global section $a \in \Gamma(S_i, \mathcal{O}_{S_i})$ such that $D(a) \subset S_i$ is an affine open containing $f_i(s)$. Because S is quasi-compact we may choose a single index $i \in I$ and global sections $a_1, \ldots, a_m \in \Gamma(S_i, \mathcal{O}_{S_i})$ such that each $D(a_j) \subset S_i$ is affine open and such that $f_i : S \to S_i$ has image contained in the union $W_i = \bigcup_{j=1,\ldots,m} D(a_j)$. For $i' \geq i$ set $W_{i'} = f_{i'i}^{-1}(W_i)$. Since $f_i^{-1}(W_i)$ is all of S we see (by Lemma 31.3.8 again) that for a suitable $i' \geq i$ we have $S_{i'} = W_{i'}$. Thus we may replace i by i' and assume that $S_i = \bigcup_{j=1,\ldots,m} D(a_j)$. This implies that $\mathcal{O}_{S_i}$ is an ample invertible sheaf on S_i (see Properties, Definition 27.24.1) and hence that S_i is quasi-affine, see Properties, Lemma 27.25.1. Hence we win. □

Lemma 31.3.10. *In Situation 31.3.1 if S is affine, then for some $i_0 \in I$ the schemes S_i for $i \geq i_0$ are affine.*

Proof. By Lemma 31.3.9 we may assume that S_0 is quasi-affine for some $0 \in I$. Set $R_0 = \Gamma(S_0, \mathcal{O}_{S_0})$. Then S_0 is a quasi-compact open of $T_0 = \mathrm{Spec}(R_0)$. Denote $j_0 : S_0 \to T_0$ the corresponding quasi-compact open immersion. For $i \geq 0$ set $\mathcal{A}_i = f_{0i,*}\mathcal{O}_{S_i}$. Since f_{0i} is affine we see that $S_i = \underline{\mathrm{Spec}}_{S_0}(\mathcal{A}_i)$. Set $T_i = \underline{\mathrm{Spec}}_{T_0}(j_{0,*}\mathcal{A}_i)$. Then $T_i \to T_0$ is affine, hence T_i is affine. Thus T_i is the spectrum of

$$R_i = \Gamma(T_0, j_{0,*}\mathcal{A}_i) = \Gamma(S_0, \mathcal{A}_i) = \Gamma(S_i, \mathcal{O}_{S_i}).$$

Write $S = \mathrm{Spec}(R)$. We have $R = \mathrm{colim}_i R_i$ by Lemma 31.3.3. Hence also $S = \lim_i T_i$. As formation of the relative spectrum commutes with base change, the inverse image of the open $S_0 \subset T_0$ in T_i is S_i. Let $Z_0 = T_0 \setminus S_0$ and let $Z_i \subset T_i$ be the inverse image of Z_0. As $S_i = T_i \setminus Z_i$, it suffices to show that Z_i is empty for some i. Assume Z_i is nonempty for all i to get a contradiction. By Lemma 31.3.5 there exists a point s of $S = \lim T_i$ which maps to a point of Z_i for every i. But $S = \lim_i S_i$, and hence we arrive at a contradiction by Lemma 31.3.2. □

Lemma 31.3.11. *In Situation 31.3.1 if S is separated, then for some $i_0 \in I$ the schemes S_i for $i \geq i_0$ are separated.*

Proof. Choose a finite affine open covering $S_0 = U_{0,1} \cup \ldots \cup U_{0,m}$. Set $U_{i,j} \subset S_i$ and $U_j \subset S$ equal to the inverse image of $U_{0,j}$. Note that $U_{i,j}$ and U_j are affine. As S is separated the intersections $U_{j_1} \cap U_{j_2}$ are affine. Since $U_{j_1} \cap U_{j_2} = \lim_{i \geq 0} U_{i,j_1} \cap U_{i,j_2}$

we see that $U_{i,j_1} \cap U_{i,j_2}$ is affine for large i by Lemma 31.3.10. To show that S_i is affine for large i it now suffices to show that

$$\mathcal{O}_{S_i}(V_{i,j_1}) \otimes_{\mathcal{O}_S(S)} \mathcal{O}_{S_i}(V_{i,j_2}) \longrightarrow \mathcal{O}_{S_i}(V_{i,j_1} \cap V_{i,j_2})$$

is surjective for large i (Schemes, Lemma 25.21.8).

To get rid of the annoying indices, assume we have affine opens $U, V \subset S_0$ such that $U \cap V$ is affine too. Let $U_i, V_i \subset S_i$, resp. $U, V \subset S$ be the inverse images. We have to show that $\mathcal{O}(U_i) \otimes \mathcal{O}(V_i) \to \mathcal{O}(U_i \cap V_i)$ is surjective for i large enough and we know that $\mathcal{O}(U_) \otimes \mathcal{O}(V) \to \mathcal{O}(U \cap V)$ is surjective. Note that $\mathcal{O}(U_0) \otimes \mathcal{O}(V_0) \to \mathcal{O}(U_0 \cap V_0)$ is of finite type, as the diagonal morphism $S_i \to S_i \times S_i$ is an immersion (Schemes, Lemma 25.21.2) hence locally of finite type (Morphisms, Lemmas 28.16.2 and 28.16.5). Thus we can choose elements $f_{0,1}, \ldots, f_{0,n} \in \mathcal{O}(U_0 \cap V_0)$ which generate $\mathcal{O}(U_0 \cap V_0)$ over $\mathcal{O}(U_0) \otimes \mathcal{O}(V_0)$. Observe that for $i \geq 0$ the diagram of schemes

$$\begin{array}{ccc} U_i \cap V_i & \longrightarrow & U_i \\ \downarrow & & \downarrow \\ U_0 \cap V_0 & \longrightarrow & U_0 \end{array}$$

is cartesian. Thus we see that the images $f_{i,1}, \ldots, f_{i,n} \in \mathcal{O}(U_i \cap V_i)$ generate $\mathcal{O}(U_i \cap V_i)$ over $\mathcal{O}(U_i) \otimes \mathcal{O}(V_0)$ and a fortiori over $\mathcal{O}(U_i) \otimes \mathcal{O}(V_i)$. By assumption the images $f_1, \ldots, f_n \in \mathcal{O}(U \otimes V)$ are in the image of the map $\mathcal{O}(U) \otimes \mathcal{O}(V) \to \mathcal{O}(U \cap V)$. Since $\mathcal{O}(U) \otimes \mathcal{O}(V) = \operatorname{colim} \mathcal{O}(U_i) \otimes \mathcal{O}(V_i)$ we see that they are in the image of the map at some finite level and the lemma is proved. □

Lemma 31.3.12. *In Situation 31.3.1 let $\mathcal{L}_0$ be an invertible sheaf of modules on S_0. If the pullback $\mathcal{L}$ to S is ample, then for some $i \in I$ the pullback $\mathcal{L}_i$ to S_i is ample.*

Proof. The assumption means there are finitely many sections $s_1, \ldots, s_m \in \Gamma(S, \mathcal{L})$ such that S_{s_j} is affine and such that $S = \bigcup S_{s_j}$, see Properties, Definition 27.24.1. By Lemma 31.3.3 we can find an $i \in I$ and sections $s_{i,j} \in \Gamma(S_i, \mathcal{L}_i)$ mapping to s_j. By Lemma 31.3.10 we may, after increasing i, assume that $(S_i)_{s_{i,j}}$ is affine for $j = 1, \ldots, m$. By Lemma 31.3.8 we may, after increasing i a last time, assume that $S_i = \bigcup (S_i)_{s_{i,j}}$. Then $\mathcal{L}_i$ is ample by definition. □

Lemma 31.3.13. *Let S be a scheme. Let $X = \lim X_i$ be a directed limit of schemes over S with affine transition morphisms. Let $Y \to X$ be a morphism of schemes over S.*

1. *If $Y \to X$ is a closed immersion, X_i quasi-compact, and Y locally of finite type over S, then $Y \to X_i$ is a closed immersion for i large enough.*
2. *If $Y \to X$ is an immersion, X_i quasi-separated, $Y \to S$ locally of finite type, and Y quasi-compact, then $Y \to X_i$ is an immersion for i large enough.*

Proof. Proof of (1). Choose $0 \in I$ and a finite affine open covering $X_0 = U_{0,1} \cup \ldots \cup U_{0,m}$ with the property that $U_{0,j}$ maps into an affine open $W_j \subset S$. Let $V_j \subset Y$, resp. $U_{i,j} \subset X_i$, $i \geq 0$, resp. $U_j \subset X$ be the inverse image of $U_{0,j}$. It suffices to prove that $V_j \to U_{i,j}$ is a closed immersion for i sufficiently large and we know that $V_j \to U_j$ is a closed immersion. Thus we reduce to the following algebra fact: If $A = \operatorname{colim} A_i$ is a directed colimit of R-algebras, $A \to B$ is a surjection of

R-algebras, and B is a finitely generated R-algebra, then $A_i \to B$ is surjective for i sufficiently large.

Proof of (2). Choose $0 \in I$. Choose a quasi-compact open $X'_0 \subset X_0$ such that $Y \to X_0$ factors through X'_0. After replacing X_i by the inverse image of X'_0 for $i \geq 0$ we may assume all X'_i are quasi-compact and quasi-separated. Let $U \subset X$ be a quasi-compact open such that $Y \to X$ factors through a closed immersion $Y \to U$ (U exists as Y is quasi-compact). By Lemma 31.3.8 we may assume that $U = \lim U_i$ with $U_i \subset X_i$ quasi-compact open. By part (1) we see that $Y \to U_i$ is a closed immersion for some i. Thus (2) holds. □

Lemma 31.3.14. *Let S be a scheme. Let $X = \lim X_i$ be a directed limit of schemes over S with affine transition morphisms. Assume*

(1) *S quasi-separated,*
(2) *X_i quasi-compact and quasi-separated,*
(3) *$X \to S$ separated.*

Then $X_i \to S$ is separated for all i large enough.

Proof. Let $0 \in I$. Note that I is nonempty as the limit is directed. As X_0 is quasi-compact we can find finitely many affine opens $U_1, \ldots, U_n \subset S$ such that $X_0 \to S$ maps into $U_1 \cup \ldots \cup U_n$. Denote $h_i : X_i \to S$ the structure morphism. It suffices to check that for some $i \geq 0$ the morphisms $h_i^{-1}(U_j) \to U_j$ are separated for $j = 1, \ldots, n$. Since S is quasi-separated the morphisms $U_j \to S$ are quasi-compact. Hence $h_i^{-1}(U_j)$ is quasi-compact and quasi-separated. In this way we reduce to the case S affine. In this case we have to show that X_i is separated and we know that X is separated. Thus the lemma follows from Lemma 31.3.11. □

Lemma 31.3.15. *Let S be a scheme. Let $X = \lim X_i$ be a directed limit of schemes over S with affine transition morphisms. Assume*

(1) *S quasi-separated,*
(2) *X_i quasi-compact and quasi-separated,*
(3) *$X \to S$ affine.*

Then $X_i \to S$ is affine for i large enough.

Proof. Choose a finite affine open covering $S = \bigcup_{j=1,\ldots,n} V_j$. Denote $f : X \to S$ and $f_i : X_i \to S$ the structure morphisms. For each j the scheme $f^{-1}(V_j) = \lim_i f_i^{-1}(V_j)$ is affine (as a finite morphism is affine by definition). Hence by Lemma 31.3.10 there exists an $i \in I$ such that each $f_i^{-1}(V_j)$ is affine. In other words, $f_i : X_i \to S$ is affine for i large enough, see Morphisms, Lemma 28.13.3. □

Lemma 31.3.16. *Let S be a scheme. Let $X = \lim X_i$ be a directed limit of schemes over S with affine transition morphisms. Assume*

(1) *S quasi-separated,*
(2) *X_i quasi-compact and quasi-separated,*
(3) *the transition morphisms $X_{i'} \to X_i$ are finite,*
(4) *$X_i \to S$ locally of finite type*
(5) *$X \to S$ integral.*

Then $X_i \to S$ is finite for i large enough.

Proof. By Lemma 31.3.15 we may assume $X_i \to S$ is affine for all i. Choose a finite affine open covering $S = \bigcup_{j=1,\ldots,n} V_j$. Denote $f : X \to S$ and $f_i : X_i \to S$ the structure morphisms. It suffices to show that there exists an i such that $f_i^{-1}(V_j)$ is finite over V_j for $j = 1, \ldots, m$ (Morphisms, Lemma 28.44.3). Namely, for $i' \geq i$ the composition $X_{i'} \to X_i \to S$ will be finite as a composition of finite morphisms (Morphisms, Lemma 28.44.5). This reduces us to the affine case: Let R be a ring and $A = \operatorname{colim} A_i$ with $R \to A$ integral and $A_i \to A_{i'}$ finite for all $i \leq i'$. Moreover $R \to A_i$ is of finite type for all i. Goal: Show that A_i is finite over R for some i. To prove this choose an $i \in I$ and pick generators $x_1, \ldots, x_m \in A_i$ of A_i as an R-algebra. Since A is integral over R we can find monic polynomials $P_j \in R[T]$ such that $P_j(x_j) = 0$ in A. Thus there exists an $i' \geq i$ such that $P_j(x_j) = 0$ in $A_{i'}$ for $j = 1, \ldots, m$. Then the image A_i' of A_i in $A_{i'}$ is finite over R by Algebra, Lemma 10.33.5. Since $A_i' \subset A_{i'}$ is finite too we conclude that $A_{i'}$ is finite over R by Algebra, Lemma 10.7.3. □

31.4. Absolute Noetherian Approximation

A nice reference for this section is Appendix C of the article by Thomason and Trobaugh [**TT90**]. See Categories, Section 4.21 for our conventions regarding directed systems. We will use the existence result and properties of the limit from Section 31.2 without further mention.

Lemma 31.4.1. *Let W be a quasi-affine scheme of finite type over $\mathbf{Z}$. Suppose $W \to \operatorname{Spec}(R)$ is an open immersion into an affine scheme. There exists a finite type $\mathbf{Z}$-algebra $A \subset R$ which induces an open immersion $W \to \operatorname{Spec}(A)$. Moreover, R is the directed colimit of such subalgebras.*

Proof. Choose an affine open covering $W = \bigcup_{i=1,\ldots,n} W_i$ such that each W_i is a standard affine open in $\operatorname{Spec}(R)$. In other words, if we write $W_i = \operatorname{Spec}(R_i)$ then $R_i = R_{f_i}$ for some $f_i \in R$. Choose finitely many $x_{ij} \in R_i$ which generate R_i over $\mathbf{Z}$. Pick an $N \gg 0$ such that each $f_i^N x_{ij}$ comes from an element of R, say $y_{ij} \in R$. Set A equal to the $\mathbf{Z}$-algebra generated by the f_i and the y_{ij} and (optionally) finitely many additional elements of R. Then A works. Details omitted. □

Lemma 31.4.2. *Suppose given a cartesian diagram of rings*

$$\begin{array}{ccc} B & \xrightarrow{s} & R \\ \uparrow & & \uparrow{\scriptstyle t} \\ B' & \longrightarrow & R' \end{array}$$

Let $W' \subset \operatorname{Spec}(R')$ be an open of the form $W' = D(f_1) \cup \ldots \cup D(f_n)$ such that $t(f_i) = s(g_i)$ for some $g_i \in B$ and $B_{g_i} \cong R_{s(g_i)}$. Then $B' \to R'$ induces an open immersion of W' into $\operatorname{Spec}(B')$.

Proof. Set $h_i = (g_i, f_i) \in B'$. More on Algebra, Lemma 15.4.3 shows that $(B')_{h_i} \cong (R')_{f_i}$ as desired. □

The following lemma is a precise statement of Noetherian approximation.

Lemma 31.4.3. *Let S be a quasi-compact and quasi-separated scheme. Let $V \subset S$ be a quasi-compact open. Let I be a directed partially ordered set and let $(V_i, f_{ii'})$ be an inverse system of schemes over I with affine transition maps, with each V_i of finite type over $\mathbf{Z}$, and with $V = \lim V_i$. Then there exist*

(1) *a directed partially ordered set* J,
(2) *an inverse system of schemes* $(S_j, g_{jj'})$ *over* J,
(3) *an order preserving map* $\alpha : J \to I$,
(4) *open subschemes* $V'_j \subset S_j$, *and*
(5) *isomorphisms* $V'_j \to V_{\alpha(j)}$

such that

(1) *the transition morphisms* $g_{jj'} : S_j \to S_{j'}$ *are affine,*
(2) *each* S_j *is of finite type over* $\mathbf{Z}$,
(3) $g_{jj'}^{-1}(V_{j'}) = V_j$,
(4) $S = \lim S_j$ *and* $V = \lim V_j$, *and*
(5) *the diagrams*

$$\begin{array}{ccc} V & & \\ \downarrow & \searrow & \\ V'_j & \longrightarrow & V_{\alpha(j)} \end{array} \quad \text{and} \quad \begin{array}{ccc} V_j & \longrightarrow & V_{\alpha(j)} \\ \downarrow & & \downarrow \\ V_{j'} & \longrightarrow & V_{\alpha(j')} \end{array}$$

are commutative.

Proof. Set $Z = S \setminus V$. Choose affine opens $U_1, \ldots, U_m \subset S$ such that $Z \subset \bigcup_{l=1,\ldots,m} U_l$. Consider the opens

$$V \subset V \cup U_1 \subset V \cup U_1 \cup U_2 \subset \ldots \subset V \cup \bigcup\nolimits_{l=1,\ldots,m} U_l = S$$

If we can prove the lemma successively for each of the cases

$$V \cup U_1 \cup \ldots \cup U_l \subset V \cup U_1 \cup \ldots \cup U_{l+1}$$

then the lemma will follow for $V \subset S$. In each case we are adding one affine open. Thus we may assume

(1) $S = U \cup V$,
(2) U affine open in S,
(3) V quasi-compact open in S, and
(4) $V = \lim_i V_i$ with $(V_i, f_{ii'})$ an inverse system over a directed set I, each $f_{ii'}$ affine and each V_i of finite type over $\mathbf{Z}$.

Set $W = U \cap V$. As S is quasi-separated, this is a quasi-compact open of V. By Lemma 31.3.8 (and after shrinking I) we may assume that there exist opens $W_i \subset V_i$ such that $f_{ij}^{-1}(W_j) = W_i$ and such that $f_i^{-1}(W_i) = W$. Since W is a quasi-compact open of U it is quasi-affine. Hence we may assume (after shrinking I again) that W_i is quasi-affine for all i, see Lemma 31.3.9.

Write $U = \mathrm{Spec}(B)$. Set $R = \Gamma(W, \mathcal{O}_W)$, and $R_i = \Gamma(W_i, \mathcal{O}_{W_i})$. By Lemma 31.3.3 we have $R = \mathrm{colim}_i R_i$. Now we have the maps of rings

$$\begin{array}{ccc} B & \xrightarrow[s]{} & R \\ & & \uparrow{\scriptstyle t_i} \\ & & R_i \end{array}$$

We set $B_i = \{(b, r) \in B \times R_i \mid s(b) = t_i(t)\}$ so that we have a cartesian diagram

$$\begin{array}{ccc} B & \xrightarrow{s} & R \\ \uparrow & & \uparrow{\scriptstyle t_i} \\ B_i & \longrightarrow & R_i \end{array}$$

for each i. The transition maps $R_i \to R_{i'}$ induce maps $B_i \to B_{i'}$. It is clear that $B = \operatorname{colim}_i B_i$. In the next paragraph we show that for all sufficiently large i the composition $W_i \to \operatorname{Spec}(R_i) \to \operatorname{Spec}(B_i)$ is an open immersion.

As W is a quasi-compact open of $U = \operatorname{Spec}(B)$ we can find a finitely many elements $g_l \in B$, $l = 1, \ldots, m$ such that $D(g_l) \subset W$ and such that $W = \bigcup_{l=1,\ldots,m} D(g_l)$. Note that this implies $D(g_l) = W_{s(g_l)}$ as open subsets of U, where $W_{s(g_l)}$ denotes the largest open subset of W on which $s(g_l)$ is invertible. Hence

$$B_{g_l} = \Gamma(D(g_l), \mathcal{O}_U) = \Gamma(W_{s(g_l)}, \mathcal{O}_W) = R_{s(g_l)},$$

where the last equality is Properties, Lemma 27.15.2. Since $W_{s(g_l)}$ is affine this also implies that $D(s(g_l)) = W_{s(g_l)}$ as open subsets of $\operatorname{Spec}(R)$. Since $R = \operatorname{colim}_i R_i$ we can (after shrinking I) assume there exist $g_{l,i} \in R_i$ for all $i \in I$ such that $s(g_l) = t_i(g_{l,i})$. Of course we choose the $g_{l,i}$ such that $g_{l,i}$ maps to $g_{l,i'}$ under the transition maps $R_i \to R_{i'}$. Then, by Lemma 31.3.8 we can (after shrinking I again) assume the corresponding opens $D(g_{l,i}) \subset \operatorname{Spec}(R_i)$ are contained in W_i, $j = 1, \ldots, m$ and cover W_i. We conclude that the morphism $W_i \to \operatorname{Spec}(R_i) \to \operatorname{Spec}(B_i)$ is an open immersion, see Lemma 31.4.2

By Lemma 31.4.1 we can write B_i as a directed colimit of subalgebras $A_{i,p} \subset B_i$, $p \in P_i$ each of finite type over $\mathbf{Z}$ and such that W_i is identified with an open subscheme of $\operatorname{Spec}(A_{i,p})$. Let $S_{i,p}$ be the scheme obtained by glueing V_i and $\operatorname{Spec}(A_{i,p})$ along the open W_i, see Schemes, Section 25.14. Here is the resulting commutative diagram of schemes:

$$\begin{array}{ccccc} & & V & \longleftarrow & W \\ & \swarrow & \downarrow & \swarrow & \downarrow \\ V_i & \longleftarrow & W_i & & S \longleftarrow U \\ \downarrow & & \downarrow & \swarrow & \swarrow \\ S_{i,p} & \longleftarrow & \operatorname{Spec}(A_{i,p}) & & \end{array}$$

The morphism $S \to S_{i,p}$ arises because the upper right square is a pushout in the category of schemes. Note that $S_{i,p}$ is of finite type over $\mathbf{Z}$ since it has a finite affine open covering whose members are spectra of finite type $\mathbf{Z}$-algebras. We define a partial ordering on $J = \coprod_{i \in I} P_i$ by the rule $(i', p') \geq (i, p)$ if and only if $i' \geq i$ and the map $B_i \to B_{i'}$ maps $A_{i,p}$ into $A_{i',p'}$. This is exactly the condition needed to define a morphism $S_{i',p'} \to S_{i,p}$: namely make a commutative diagram as above using the transition morphisms $V_{i'} \to V_i$ and $W_{i'} \to W_i$ and the morphism $\operatorname{Spec}(A_{i',p'}) \to \operatorname{Spec}(A_{i,p})$ induced by the ring map $A_{i,p} \to A_{i',p'}$. The relevant commutativities have been built into the constructions. We claim that S is the directed limit of the schemes $S_{i,p}$. Since by construction the schemes V_i have limit V this boils down to the fact that B is the limit of the rings $A_{i,p}$ which is true by construction. The map $\alpha : J \to I$ is given by the rule $j = (i, p) \mapsto i$. The

open subscheme V_j' is just the image of $V_i \to S_{i,p}$ above. The commutativity of the diagrams in (5) is clear from the construction. This finishes the proof of the lemma. □

Proposition 31.4.4. *Let S be a quasi-compact and quasi-separated scheme. There exist a directed partially ordered set I and an inverse system of schemes $(S_i, f_{ii'})$ over I such that*

(1) *the transition morphisms $f_{ii'}$ are affine*
(2) *each S_i is of finite type over $\mathbf{Z}$, and*
(3) *$S = \lim_i S_i$.*

Proof. This is a special case of Lemma 31.4.3 with $V = \emptyset$. □

31.5. Limits and morphisms of finite presentation

The following is a generalization of Algebra, Lemma 10.121.2.

Proposition 31.5.1. *Let $f : X \to S$ be a morphism of schemes. The following are equivalent:*

(1) *The morphism f is locally of finite presentation.*
(2) *For any directed partially ordered set I, and any inverse system $(T_i, f_{ii'})$ of S-schemes over I with each T_i affine, we have*
$$\mathrm{Mor}_S(\lim_i T_i, X) = \mathrm{colim}_i \mathrm{Mor}_S(T_i, X)$$
(3) *For any directed partially ordered set I, and any inverse system $(T_i, f_{ii'})$ of S-schemes over I with each $f_{ii'}$ affine and every T_i quasi-compact and quasi-separated as a scheme, we have*
$$\mathrm{Mor}_S(\lim_i T_i, X) = \mathrm{colim}_i \mathrm{Mor}_S(T_i, X)$$

Proof. It is clear that (3) implies (2).

Let us prove that (2) implies (1). Assume (2). Choose any affine opens $U \subset X$ and $V \subset S$ such that $f(U) \subset V$. We have to show that $\mathcal{O}_S(V) \to \mathcal{O}_X(U)$ is of finite presentation. Let $(A_i, \varphi_{ii'})$ be a directed system of $\mathcal{O}_S(V)$-algebras. Set $A = \mathrm{colim}_i A_i$. According to Algebra, Lemma 10.121.2 we have to show that
$$\mathrm{Hom}_{\mathcal{O}_S(V)}(\mathcal{O}_X(U), A) = \mathrm{colim}_i \mathrm{Hom}_{\mathcal{O}_S(V)}(\mathcal{O}_X(U), A_i)$$
Consider the schemes $T_i = \mathrm{Spec}(A_i)$. They form an inverse system of V-schemes over I with transition morphisms $f_{ii'} : T_i \to T_{i'}$ induced by the $\mathcal{O}_S(V)$-algebra maps $\varphi_{i'i}$. Set $T := \mathrm{Spec}(A) = \lim_i T_i$. The formula above becomes in terms of morphism sets of schemes
$$\mathrm{Mor}_V(\lim_i T_i, U) = \mathrm{colim}_i \mathrm{Mor}_V(T_i, U).$$
We first observe that $\mathrm{Mor}_V(T_i, U) = \mathrm{Mor}_S(T_i, U)$ and $\mathrm{Mor}_V(T, U) = \mathrm{Mor}_S(T, U)$. Hence we have to show that
$$\mathrm{Mor}_S(\lim_i T_i, U) = \mathrm{colim}_i \mathrm{Mor}_S(T_i, U)$$
and we are given that
$$\mathrm{Mor}_S(\lim_i T_i, X) = \mathrm{colim}_i \mathrm{Mor}_S(T_i, X).$$
Hence it suffices to prove that given a morphism $g_i : T_i \to X$ over S such that the composition $T \to T_i \to X$ ends up in U there exists some $i' \geq i$ such that the composition $g_{i'} : T_{i'} \to T_i \to X$ ends up in U. Denote $Z_{i'} = g_{i'}^{-1}(X \setminus U)$. Assume

each $Z_{i'}$ is nonempty to get a contradiction. By Lemma 31.3.5 there exists a point t of T which is mapped into $Z_{i'}$ for all $i' \geq i$. Such a point is not mapped into U. A contradiction.

Finally, let us prove that (1) implies (3). Assume (1). Let an inverse directed system $(T_i, f_{ii'})$ of S-schemes be given. Assume the morphisms $f_{ii'}$ are affine and each T_i is quasi-compact and quasi-separated as a scheme. Let $T = \lim_i T_i$. Denote $f_i : T \to T_i$ the projection morphisms. We have to show:

(a) Given morphisms $g_i, g_i' : T_i \to X$ over S such that $g_i \circ f_i = g_i' \circ f_i$, then there exists an $i' \geq i$ such that $g_i \circ f_{i'i} = g_i' \circ f_{i'i}$.
(b) Given any morphism $g : T \to X$ over S there exists an $i \in I$ and a morphism $g_i : T_i \to X$ such that $g = f_i \circ g_i$.

First let us prove the uniqueness part (a). Let $g_i, g_i' : T_i \to X$ be morphisms such that $g_i \circ f_i = g_i' \circ f_i$. For any $i' \geq i$ we set $g_{i'} = g_i \circ f_{i'i}$ and $g_{i'}' = g_i' \circ f_{i'i}$. We also set $g = g_i \circ f_i = g_i' \circ f_i$. Consider the morphism $(g_i, g_i') : T_i \to X \times_S X$. Set

$$W = \bigcup\nolimits_{U \subset X \text{ affine open}, V \subset S \text{ affine open}, f(U) \subset V} U \times_V U.$$

This is an open in $X \times_S X$, with the property that the morphism $\Delta_{X/S}$ factors through a closed immersion into W, see the proof of Schemes, Lemma 25.21.2. Note that the composition $(g_i, g_i') \circ f_i : T \to X \times_S X$ is a morphism into W because it factors through the diagonal by assumption. Set $Z_{i'} = (g_{i'}, g_{i'}')^{-1}(X \times_S X \setminus W)$. If each $Z_{i'}$ is nonempty, then by Lemma 31.3.5 there exists a point $t \in T$ which maps to $Z_{i'}$ for all $i' \geq i$. This is a contradiction with the fact that T maps into W. Hence we may increase i and assume that $(g_i, g_i') : T_i \to X \times_S X$ is a morphism into W. By construction of W, and since T_i is quasi-compact we can find a finite affine open covering $T_i = T_{1,i} \cup \ldots \cup T_{n,i}$ such that $(g_i, g_i')|_{T_{j,i}}$ is a morphism into $U \times_V U$ for some pair (U, V) as in the definition of W above. Since it suffices to prove that $g_{i'}$ and $g_{i'}'$ agree on each of the $f_{i'i}^{-1}(T_{j,i})$ this reduces us to the affine case. The affine case follows from Algebra, Lemma 10.121.2 and the fact that the ring map $\mathcal{O}_S(V) \to \mathcal{O}_X(U)$ is of finite presentation (see Morphisms, Lemma 28.22.2).

Finally, we prove the existence part (b). Let $g : T \to X$ be a morphism of schemes over S. We can find a finite affine open covering $T = W_1 \cup \ldots \cup W_n$ such that for each $j \in \{1, \ldots, n\}$ there exist affine opens $U_j \subset X$ and $V_j \subset S$ with $f(U_j) \subset V_j$ and $g(W_j) \subset U_j$. By Lemmas 31.3.8 and 31.3.10 (after possibly shrinking I) we may assume that there exist affine open coverings $T_i = W_{1,i} \cup \ldots \cup W_{n,i}$ compatible with transition maps such that $W_j = \lim_i W_{j,i}$. We apply Algebra, Lemma 10.121.2 to the rings corresponding to the affine schemes U_j, V_j, $W_{j,i}$ and W_j using that $\mathcal{O}_S(V_j) \to \mathcal{O}_X(U_j)$ is of finite presentation (see Morphisms, Lemma 28.22.2). Thus we can find for each j an index $i_j \in I$ and a morphism $g_{j,i_j} : W_{j,i_j} \to X$ such that $g_{j,i_j} \circ f_i|_{W_j} : W_j \to W_{j,i} \to X$ equals $g|_{W_j}$. By part (a) proved above, using the quasi-compactness of $W_{j_1,i} \cap W_{j_2,i}$ which follows as T_i is quasi-separated, we can find an index $i' \in I$ larger than all i_j such that

$$g_{j_1,i_{j_1}} \circ f_{i'i_{j_1}}|_{W_{j_1,i'} \cap W_{j_2,i'}} = g_{j_2,i_{j_2}} \circ f_{i'i_{j_2}}|_{W_{j_1,i'} \cap W_{j_2,i'}}$$

for all $j_1, j_2 \in \{1, \ldots, n\}$. Hence the morphisms $g_{j,i_j} \circ f_{i'i_j}|_{W_{j,i'}}$ glue to given the desired morphism $T_{i'} \to X$. □

Remark 31.5.2. Let S be a scheme. Let us say that a functor $F : (Sch/S)^{opp} \to Sets$ is *limit preserving* if for every directed inverse system $\{T_i\}_{i \in I}$ of affine schemes

with limit T we have $F(T) = \operatorname{colim}_i F(T_i)$. Let X be a scheme over S, and let $h_X : (Sch/S)^{opp} \to Sets$ be its functor of points, see Schemes, Section 25.15. In this terminology Proposition 31.5.1 says that a scheme X is locally of finite presentation over S if and only if h_X is limit preserving.

31.6. Relative approximation

The title of this section refers to results of the following type.

Lemma 31.6.1. *Let $f : X \to S$ be a morphism of schemes. Assume that*

(1) *X is quasi-compact and quasi-separated, and*
(2) *S is quasi-separated.*

Then $X = \lim X_i$ is a limit of a directed system of schemes X_i of finite presentation over S with affine transition morphisms over S.

Proof. Since $f(X)$ is quasi-compact we may replace S by a quasi-compact open containing $f(X)$. Hence we may assume S is quasi-compact as well. Write $X = \lim X_a$ and $S = \lim S_b$ as in Proposition 31.4.4, i.e., with X_a and S_b of finite type over $\mathbf{Z}$ and with affine transition morphisms. By Proposition 31.5.1 we find that for each b there exists an a and a morphism $f_{a,b} : X_a \to S_b$ making the diagram

$$\begin{array}{ccc} X & \longrightarrow & S \\ \downarrow & & \downarrow \\ X_a & \longrightarrow & S_b \end{array}$$

commute. Moreover the same proposition implies that, given a second triple $(a', b', f_{a',b'})$, there exists an $a'' \geq a'$ such that the compositions $X_{a''} \to X_a \to X_b$ and $X_{a''} \to X_{a'} \to X_{b'} \to X_b$ are equal. Consider the set of triples $(a, b, f_{a,b})$ endowed with the partial ordering

$$(a, b, f_{a,b}) \geq (a', b', f_{a',b'}) \Leftrightarrow a \geq a',\ b' \geq b, \text{ and } f_{a',b'} \circ h_{a,a'} = g_{b',b} \circ f_{a,b}$$

where $h_{a,a'} : X_a \to X_{a'}$ and $g_{b',b} : S_{b'} \to S_b$ are the transition morphisms. The remarks above show that this system is directed. It follows formally from the equalities $X = \lim X_a$ and $S = \lim S_b$ that

$$X = \lim_{(a,b,f_{a,b})} X_a \times_{f_{a,b}, S_b} S.$$

where the limit is over our directed system above. The transition morphisms $X_a \times_{S_b} S \to X_{a'} \times_{S_{b'}} S$ are affine as the composition

$$X_a \times_{S_b} S \to X_a \times_{S_{b'}} S \to X_{a'} \times_{S_{b'}} S$$

where the first morphism is a closed immersion (by Schemes, Lemma 25.21.10) and the second is a base change of an affine morphism (Morphisms, Lemma 28.13.8) and the composition of affine morphisms is affine (Morphisms, Lemma 28.13.7). The morphisms $f_{a,b}$ are of finite presentation (Morphisms, Lemmas 28.22.9 and 28.22.11) and hence the base changes $X_a \times_{f_{a,b}, S_b} S \to S$ are of finite presentation (Morphisms, Lemma 28.22.4). □

Lemma 31.6.2. *Let $X \to S$ be an integral morphism with S quasi-compact and quasi-separated. Then $X = \lim X_i$ with $X_i \to S$ finite and of finite presentation.*

Proof. Consider the sheaf $\mathcal{A} = f_*\mathcal{O}_X$. This is a quasi-coherent sheaf of $\mathcal{O}_S$-algebras, see Schemes, Lemma 25.24.1. Combining Properties, Lemma 27.20.13 we can write $\mathcal{A} = \operatorname{colim}_i \mathcal{A}_i$ as a filtered colimit of finite and finitely presented $\mathcal{O}_S$-algebras. Then

$$X_i = \underline{\operatorname{Spec}}_S(\mathcal{A}_i) \longrightarrow S$$

is a finite and finitely presented morphism of schemes. By construction $X = \lim_i X_i$ which proves the lemma. □

31.7. Descending properties of morphisms

This section is the analogue of Section 31.3 for properties of morphisms over S. We will work in the following situation.

Situation 31.7.1. Let $S = \lim S_i$ be a limit of a directed system of schemes with affine transition morphisms (Lemma 31.2.2). Let $0 \in I$ and let $f_0 : X_0 \to Y_0$ be a morphism of schemes over S_0. Assume S_0, X_0, Y_0 are quasi-compact and quasi-separated. Let $f_i : X_i \to Y_i$ be the base change of f_0 to S_i and let $f : X \to Y$ be the base change of f_0 to S.

Lemma 31.7.2. *Notation and assumptions as in Situation 31.7.1. If f is affine, then there exists an index $i \geq 0$ such that f_i is affine.*

Proof. Let $Y_0 = \bigcup_{j=1,\ldots,m} V_{j,0}$ be a finite affine open covering. Set $U_{j,0} = f_0^{-1}(V_{j,0})$. For $i \geq 0$ we denote $V_{j,i}$ the inverse image of $V_{j,0}$ in Y_i and $U_{j,i} = f_i^{-1}(V_{j,i})$. Similarly we have $U_j = f^{-1}(V_j)$. Then $U_j = \lim_{i \geq 0} U_{j,i}$ (see Lemma 31.2.2). Since U_j is affine by assumption we see that each $U_{j,i}$ is affine for i large enough, see Lemma 31.3.10. As there are finitely many j we can pick an i which works for all j. Thus f_i is affine for i large enough, see Morphisms, Lemma 28.13.3. □

Lemma 31.7.3. *Notation and assumptions as in Situation 31.7.1. If*

(1) *f is a finite morphism, and*
(2) *f_0 is locally of finite type,*

then there exists an $i \geq 0$ such that f_i is finite.

Proof. A finite morphism is affine, see Morphisms, Definition 28.44.1. Hence by Lemma 31.7.2 above after increasing 0 we may assume that f_0 is affine. By writing Y_0 as a finite union of affines we reduce to proving the result when X_0 and Y_0 are affine and map into a common affine $W \subset S_0$. The corresponding algebra statement follows from Algebra, Lemma 10.154.3. □

Lemma 31.7.4. *Notation and assumptions as in Situation 31.7.1. If*

(1) *f is a closed immersion, and*
(2) *f_0 is locally of finite type,*

then there exists an $i \geq 0$ such that f_i is a closed immersion.

Proof. A closed immersion is affine, see Morphisms, Lemma 28.13.9. Hence by Lemma 31.7.2 above after increasing 0 we may assume that f_0 is affine. By writing Y_0 as a finite union of affines we reduce to proving the result when X_0 and Y_0 are affine and map into a common affine $W \subset S_0$. The corresponding algebra statement is a consequence of Algebra, Lemma 10.154.4. □

Lemma 31.7.5. *Notation and assumptions as in Situation 31.7.1. If f is separated, then f_i is separated for some $i \geq 0$.*

Proof. Apply Lemma 31.7.4 to the diagonal morphism $\Delta_{X_0/S_0} : X_0 \to X_0 \times_{S_0} X_0$. (This is permissible as diagonal morphisms are locally of finite type and the fibre product $X_0 \times_{S_0} X_0$ is quasi-compact and quasi-separated, see Schemes, Lemma 25.21.2, Morphisms, Lemma 28.16.5, and Schemes, Remark 25.21.18. □

Lemma 31.7.6. *Notation and assumptions as in Situation 31.7.1. If*

(1) *f is flat,*
(2) *f_0 is locally of finite presentation,*

then f_i is flat for some $i \geq 0$.

Proof. Choose a finite affine open covering $Y_0 = \bigcup_{j=1,\ldots,m} Y_{j,0}$ such that each $Y_{j,0}$ maps into an affine open $S_{j,0} \subset S_0$. For each j let $f_0^{-1}Y_{j,0} = \bigcup_{k=1,\ldots,n_j} X_{k,0}$ be a finite affine open covering. Since the property of being flat is local we see that it suffices to prove the lemma for the morphisms of affines $X_{k,i} \to Y_{j,i} \to S_{j,i}$ which are the base changes of $X_{k,0} \to Y_{j,0} \to S_{j,0}$ to S_i. Thus we reduce to the case that X_0, Y_0, S_0 are affine

In the affine case we reduce to the following algebra result. Suppose that $R = \operatorname{colim}_{i \in I} R_i$. For some $0 \in I$ suppose given an R_0-algebra map $A_i \to B_i$ of finite presentation. If $R \otimes_{R_0} A_0 \to R \otimes_{R_0} B_0$ is flat, then for some $i \geq 0$ the map $R_i \otimes_{R_0} A_0 \to R_i \otimes_{R_0} B_0$ is flat. This follows from Algebra, Lemma 10.154.1 part (3). □

Lemma 31.7.7. *Notation and assumptions as in Situation 31.7.1. If*

(1) *f is finite locally free (of degree d),*
(2) *f_0 is locally of finite presentation,*

then f_i is finite locally free (of degree d) for some $i \geq 0$.

Proof. By Lemmas 31.7.6 and 31.7.3 we find an i such that f_i is flat and finite. On the other hand, f_i is locally of finite presentation. Hence f_i is finite locally free by Morphisms, Lemma 28.46.2. If moreover f is finite locally free of degree d, then the image of $Y \to Y_i$ is contained in the open and closed locus $W_d \subset Y_i$ over which f_i has degree d. By Lemma 31.3.7 we see that for some $i' \geq i$ the image of $Y_{i'} \to Y_i$ is contained in W_d. Then $f_{i'}$ will be finite locally free of degree d. □

Lemma 31.7.8. *Notation and assumptions as in Situation 31.7.1. If*

(1) *f is étale,*
(2) *f_0 is locally of finite presentation,*

then f_i is étale for some $i \geq 0$.

Proof. Being étale is local on the source and the target (Morphisms, Lemma 28.37.2) hence we may assume S_0, X_0, Y_0 affine (details omitted). The corresponding algebra fact is Algebra, Lemma 10.154.5. □

Lemma 31.7.9. *Notation and assumptions as in Situation 31.7.1. If*

(1) *f is an isomorphism, and*
(2) *f_0 is locally of finite presentation,*

then f_i is an isomorphism for some $i \geq 0$.

Proof. By Lemmas 31.7.8 and 31.7.4 we can find an i such that f_i is flat and a closed immersion. Then f_i identifies X_i with an open and closed subscheme of Y_i, see Morphisms, Lemma 28.27.2. By assumption the image of $Y \to Y_i$ maps into $f_i(X_i)$. Thus by Lemma 31.3.7 we find that $Y_{i'}$ maps into $f_i(X_i)$ for some $i' \geq i$. It follows that $X_{i'} \to Y_{i'}$ is surjective and we win. □

Lemma 31.7.10. *Notation and assumptions as in Situation 31.7.1. If*

(1) *f is a monomorphism, and*
(2) *f_0 is locally of finite type,*

then f_i is a monomorphism for some $i \geq 0$.

Proof. Recall that a morphism of schemes $V \to W$ is a monomorphism if and only if the diagonal $V \to V \times_W V$ is an isomorphism (Schemes, Lemma 25.23.2). The morphism $X_0 \to X_0 \times_{Y_0} X_0$ is locally of finite presentation by Morphisms, Lemma 28.22.12. Since $X_0 \times_{Y_0} X_0$ is quasi-compact and quasi-separated (Schemes, Remark 25.21.18) we conclude from Lemma 31.7.9 that $\Delta_i : X_i \to X_i \times_{Y_i} X_i$ is an isomorphism for some $i \geq 0$. For this i the morphism f_i is a monomorphism. □

Lemma 31.7.11. *Notation and assumptions as in Situation 31.7.1. If*

(1) *f is surjective, and*
(2) *f_0 is locally of finite presentation,*

then there exists an $i \geq 0$ such that f_i is surjective.

Proof. The morphism f_0 is of finite presentation. Hence $E = f_0(X_0)$ is a constructible subset of Y_0, see Morphisms, Lemma 28.23.2. Since f_i is the base change of f_0 by $Y_i \to Y_0$ we see that the image of f_i is the inverse image of E in Y_i. Moreover, we know that $Y \to Y_0$ maps into E. Hence we win by Lemma 31.3.7. □

31.8. Finite type closed in finite presentation

A result of this type is [**Kie72**, Satz 2.10]. Another reference is [**Con07**].

Lemma 31.8.1. *Let $f : X \to S$ be a morphism of schemes. Assume:*

(1) *The morphism f is locally of finite type.*
(2) *The scheme X is quasi-compact and quasi-separated.*

Then there exists a morphism of finite presentation $f' : X' \to S$ and an immersion $X \to X'$ of schemes over S.

Proof. By Proposition 31.4.4 we can write $X = \lim_i X_i$ with each X_i of finite type over $\mathbf{Z}$ and with transition morphisms $f_{ii'} : X_i \to X_{i'}$ affine. Consider the commutative diagram

$$\begin{array}{ccccc} X & \longrightarrow & X_{i,S} & \longrightarrow & X_i \\ & \searrow & \downarrow & & \downarrow \\ & & S & \longrightarrow & \mathrm{Spec}(\mathbf{Z}) \end{array}$$

Note that X_i is of finite presentation over $\mathrm{Spec}(\mathbf{Z})$, see Morphisms, Lemma 28.22.9. Hence the base change $X_{i,S} \to S$ is of finite presentation by Morphisms, Lemma 28.22.4. Thus it suffices to show that the arrow $X \to X_{i,S}$ is an immersion for i sufficiently large.

To do this we choose a finite affine open covering $X = V_1 \cup \ldots \cup V_n$ such that f maps each V_j into an affine open $U_j \subset S$. Let $h_{j,a} \in \mathcal{O}_X(V_j)$ be a finite set of elements which generate $\mathcal{O}_X(V_j)$ as an $\mathcal{O}_S(U_j)$-algebra, see Morphisms, Lemma 28.16.2. By Lemmas 31.3.8 and 31.3.10 (after possibly shrinking I) we may assume that there exist affine open coverings $X_i = V_{1,i} \cup \ldots \cup V_{n,i}$ compatible with transition maps such that $V_j = \lim_i V_{j,i}$. By Lemma 31.3.3 we can choose i so large that each $h_{j,a}$ comes from an element $h_{j,a,i} \in \mathcal{O}_{X_i}(V_{j,i})$. Thus the arrow in

$$V_j \longrightarrow U_j \times_{\mathrm{Spec}(\mathbf{Z})} V_{j,i} = (V_{j,i})_{U_j} \subset (V_{j,i})_S \subset X_{i,S}$$

is a closed immersion. Since $\bigcup (V_{j,i})_{U_j}$ forms an open of $X_{i,S}$ and since the inverse image of $(V_{j,i})_{U_j}$ in X is V_j it follows that $X \to X_{i,S}$ is an immersion. □

Remark 31.8.2. We cannot do better than this if we do not assume more on S and the morphism $f : X \to S$. For example, in general it will not be possible to find a *closed* immersion $X \to X'$ as in the lemma. The reason is that this would imply that f is quasi-compact which may not be the case. An example is to take S to be infinite dimensional affine space with 0 doubled and X to be one of the two infinite dimensional affine spaces.

Lemma 31.8.3. *Let $f : X \to S$ be a morphism of schemes. Assume:*

(1) *The morphism f is of locally of finite type.*
(2) *The scheme X is quasi-compact and quasi-separated, and*
(3) *The scheme S is quasi-separated.*

Then there exists a morphism of finite presentation $f' : X' \to S$ and a closed immersion $X \to X'$ of schemes over S.

Proof. By Lemma 31.8.1 above there exists a morphism $Y \to S$ of finite presentation and an immersion $i : X \to Y$ of schemes over S. For every point $x \in X$, there exists an affine open $V_x \subset Y$ such that $i^{-1}(V_x) \to V_x$ is a closed immersion. Since X is quasi-compact we can find finitely may affine opens $V_1, \ldots, V_n \subset Y$ such that $i(X) \subset V_1 \cup \ldots \cup V_n$ and $i^{-1}(V_j) \to V_j$ is a closed immersion. In other words such that $i : X \to X' = V_1 \cup \ldots \cup V_n$ is a closed immersion of schemes over S. Since S is quasi-separated and Y is quasi-separated over S we deduce that Y is quasi-separated, see Schemes, Lemma 25.21.13. Hence the open immersion $X' = V_1 \cup \ldots \cup V_n \to Y$ is quasi-compact. This implies that $X' \to Y$ is of finite presentation, see Morphisms, Lemma 28.22.6. We conclude since then $X' \to Y \to S$ is a composition of morphisms of finite presentation, and hence of finite presentation (see Morphisms, Lemma 28.22.3). □

Lemma 31.8.4. *Let $X \to Y$ be a closed immersion of schemes. Assume Y quasi-compact and quasi-separated. Then X can be written as a directed limit $X = \lim X_i$ of schemes over Y where $X_i \to Y$ is a closed immersion of finite presentation.*

Proof. Let $\mathcal{I} \subset \mathcal{O}_Y$ be the quasi-coherent sheaf of ideals defining X as a closed subscheme of Y. By Properties, Lemma 27.20.3 we can write $\mathcal{I}$ as a directed colimit $\mathcal{I} = \mathrm{colim}_{i \in I} \mathcal{I}_i$ of its quasi-coherent sheaves of ideals of finite type. Let $X_i \subset Y$ be the closed subscheme defined by $\mathcal{I}_i$. These form an inverse system of schemes indexed by I. The transition morphisms $X_i \to X_{i'}$ are affine because they are closed immersions. Each X_i is quasi-compact and quasi-separated since it is a closed subscheme of Y and Y is quasi-compact and quasi-separated by our assumptions.

We have $X = \lim_i X_i$ as follows directly from the fact that $\mathcal{I} = \operatorname{colim}_{i \in I} \mathcal{I}_a$. Each of the morphisms $X_i \to Y$ is of finite presentation, see Morphisms, Lemma 28.22.7. □

Lemma 31.8.5. *Let $f : X \to S$ be a morphism of schemes. Assume*

(1) *The morphism f is of locally of finite type.*
(2) *The scheme X is quasi-compact and quasi-separated, and*
(3) *The scheme S is quasi-separated.*

Then $X = \lim X_i$ where the $X_i \to S$ are of finite presentation, the X_i are quasi-compact and quasi-separated, and the transition morphisms $X_{i'} \to X_i$ are closed immersions (which implies that $X \to X_i$ are closed immersions for all i).

Proof. By Lemma 31.8.3 there is a closed immersion $X \to Y$ with $Y \to S$ of finite presentation. Then Y is quasi-separated by Schemes, Lemma 25.21.13. Since X is quasi-compact, we may assume Y is quasi-compact by replacing Y with a quasi-compact open containing X. We see that $X = \lim X_i$ with $X_i \to Y$ a closed immersion of finite presentation by Lemma 31.8.4. The morphisms $X_i \to S$ are of finite presentation by Morphisms, Lemma 28.22.3. □

Proposition 31.8.6. *Let $f : X \to S$ be a morphism of schemes. Assume*

(1) *f is of finite type and separated, and*
(2) *S is quasi-compact and quasi-separated.*

Then there exists a separated morphism of finite presentation $f' : X' \to S$ and a closed immersion $X \to X'$ of schemes over S.

Proof. Apply Lemma 31.8.5 and note that $X_i \to S$ is separated for large i by Lemma 31.3.14 as we have assumed that $X \to S$ is separated. □

Lemma 31.8.7. *Let $f : X \to S$ be a morphism of schemes. Assume*

(1) *f is finite, and*
(2) *S is quasi-compact and quasi-separated.*

Then there exists a morphism which is finite and of finite presentation $f' : X' \to S$ and a closed immersion $X \to X'$ of schemes over S.

Proof. We may write $X = \lim X_i$ as in Lemma 31.8.5. Applying Lemma 31.3.16 we see that $X_i \to S$ is finite for large enough i. □

Lemma 31.8.8. *Let $f : X \to S$ be a morphism of schemes. Assume*

(1) *f is finite, and*
(2) *S quasi-compact and quasi-separated.*

Then X is a directed limit $X = \lim X_i$ where the transition maps are closed immersions and the objects X_i are finite and of finite presentation over S.

Proof. We may write $X = \lim X_i$ as in Lemma 31.8.5. Applying Lemma 31.3.16 we see that $X_i \to S$ is finite for large enough i. □

31.9. Descending relative objects

The following lemma is typical of the type of results in this section. We write out the "standard" proof completely. It may be faster to convince yourself that the result is true than to read this proof.

Lemma 31.9.1. *Let I be a directed partially ordered set. Let $(S_i, f_{ii'})$ be an inverse system of schemes over I. Assume*

(1) *the morphisms $f_{ii'} : S_i \to S_{i'}$ are affine,*
(2) *the schemes S_i are quasi-compact and quasi-separated.*

Let $S = \lim_i S_i$. Then we have the following:

(1) *For any morphism of finite presentation $X \to S$ there exists an index $i \in I$ and a morphism of finite presentation $X_i \to S_i$ such that $X \cong X_{i,S}$ as schemes over S.*
(2) *Given an index $i \in I$, schemes X_i, Y_i of finite presentation over S_i, and a morphism $\varphi : X_{i,S} \to Y_{i,S}$ over S, there exists an index $i' \geq i$ and a morphism $\varphi_{i'} : X_{i,S_{i'}} \to Y_{i,S_{i'}}$ whose base change to S is φ.*
(3) *Given an index $i \in I$, schemes X_i, Y_i of finite presentation over S_i and a pair of morphisms $\varphi_i, \psi_i : X_i \to Y_i$ whose base changes $\varphi_{i,S} = \psi_{i,S}$ are equal, there exists an index $i' \geq i$ such that $\varphi_{i,S_{i'}} = \psi_{i,S_{i'}}$.*

In other words, the category of schemes of finite presentation over S is the colimit over I of the categories of schemes of finite presentation over S_i.

Proof. In case each of the schemes S_i is affine, and we consider only affine schemes of finite presentation over S_i, resp. S this lemma is equivalent to Algebra, Lemma 10.121.6. We claim that the affine case implies the lemma in general.

Let us prove (3). Suppose given an index $i \in I$, schemes X_i, Y_i of finite presentation over S_i and a pair of morphisms $\varphi_i, \psi_i : X_i \to Y_i$. Assume that the base changes are equal: $\varphi_{i,S} = \psi_{i,S}$. We will use the notation $X_{i'} = X_{i,S_{i'}}$ and $Y_{i'} = Y_{i,S_{i'}}$ for $i' \geq i$. We also set $X = X_{i,S}$ and $Y = Y_{i,S}$. Note that according to Lemma 31.2.3 we have $X = \lim_{i' \geq i} X_{i'}$ and similarly for Y. Additionally we denote $\varphi_{i'}$ and $\psi_{i'}$ (resp. φ and ψ) the base change of φ_i and ψ_i to $S_{i'}$ (resp. S). So our assumption means that $\varphi = \psi$. Since Y_i and X_i are of finite presentation over S_i, and since S_i is quasi-compact and quasi-separated, also X_i and Y_i are quasi-compact and quasi-separated (see Morphisms, Lemma 28.22.10). Hence we may choose a finite affine open covering $Y_i = \bigcup V_{j,i}$ such that each $V_{j,i}$ maps into an affine open of S. As above, denote $V_{j,i'}$ the inverse image of $V_{j,i}$ in $Y_{i'}$ and V_j the inverse image in Y. The immersions $V_{j,i'} \to Y_{i'}$ are quasi-compact, and the inverse images $U_{j,i'} = \varphi_i^{-1}(V_{j,i'})$ and $U'_{j,i'} = \psi_i^{-1}(V_{j,i'})$ are quasi-compact opens of $X_{i'}$. By assumption the inverse images of V_j under φ and ψ in X are equal. Hence by Lemma 31.3.8 there exists an index $i' \geq i$ such that of $U_{j,i'} = U'_{j,i'}$ in $X_{i'}$. Choose an finite affine open covering $U_{j,i'} = U'_{j,i'} = \bigcup W_{j,k,i'}$ which induce coverings $U_{j,i''} = U'_{j,i''} = \bigcup W_{j,k,i''}$ for all $i'' \geq i'$. By the affine case there exists an index i'' such that $\varphi_{i''}|_{W_{j,k,i''}} = \psi_{i''}|_{W_{j,k,i''}}$ for all j, k. Then i'' is an index such that $\varphi_{i''} = \psi_{i''}$ and (3) is proved.

Let us prove (2). Suppose given an index $i \in I$, schemes X_i, Y_i of finite presentation over S_i and a morphism $\varphi : X_{i,S} \to Y_{i,S}$. We will use the notation $X_{i'} = X_{i,S_{i'}}$ and $Y_{i'} = Y_{i,S_{i'}}$ for $i' \geq i$. We also set $X = X_{i,S}$ and $Y = Y_{i,S}$. Note that according to Lemma 31.2.3 we have $X = \lim_{i' \geq i} X_{i'}$ and similarly for Y. Since Y_i and X_i are of finite presentation over S_i, and since S_i is quasi-compact and quasi-separated, also X_i and Y_i are quasi-compact and quasi-separated (see Morphisms, Lemma 28.22.10). Hence we may choose a finite affine open covering $Y_i = \bigcup V_{j,i}$ such that each $V_{j,i}$ maps into an affine open of S. As above, denote $V_{j,i'}$ the inverse image of $V_{j,i}$ in $Y_{i'}$ and V_j the inverse image in Y. The immersions $V_j \to Y$ are quasi-compact, and the inverse images $U_j = \varphi^{-1}(V_j)$ are quasi-compact opens of

X. Hence by Lemma 31.3.8 there exists an index $i' \geq i$ and quasi-compact opens $U_{j,i'}$ of $X_{i'}$ whose inverse image in X is U_j. Choose an finite affine open covering $U_{j,i'} = \bigcup W_{j,k,i'}$ which induce affine open coverings $U_{j,i''} = \bigcup W_{j,k,i''}$ for all $i'' \geq i'$ and an affine open covering $U_j = \bigcup W_{j,k}$. By the affine case there exists an index i'' and morphisms $\varphi_{j,k,i''} : W_{j,k,i''} \to V_{j,i''}$ such that $\varphi|_{W_{j,k}} = \varphi_{j,k,i'',S}$ for all j, k. By part (3) proved above, there is a further index $i''' \geq i''$ such that

$$\varphi_{j_1,k_1,i'',S_{i'''}}|_{W_{j_1,k_1,i'''} \cap W_{j_2,k_2,i'''}} = \varphi_{j_2,k_2,i'',S_{i'''}}|_{W_{j_1,k_1,i'''} \cap W_{j_2,k_2,i'''}}$$

for all j_1, j_2, k_1, k_2. Then i''' is an index such that there exists a morphism $\varphi_{i'''} : X_{i'''} \to Y_{i'''}$ whose base change to S gives φ. Hence (2) holds.

Let us prove (1). Suppose given a scheme X of finite presentation over S. Since X is of finite presentation over S, and since S is quasi-compact and quasi-separated, also X is quasi-compact and quasi-separated (see Morphisms, Lemma 28.22.10). Choose a finite affine open covering $X = \bigcup U_j$ such that each U_j maps into an affine open $V_j \subset S$. Denote $U_{j_1 j_2} = U_{j_1} \cap U_{j_2}$ and $U_{j_1 j_2 j_3} = U_{j_1} \cap U_{j_2} \cap U_{j_3}$. By Lemmas 31.3.8 and 31.3.10 we can find an index i_1 and affine opens $V_{j,i_1} \subset S_{i_1}$ such that each V_j is the inverse of this in S. Let $V_{j,i}$ be the inverse image of V_{j,i_1} in S_i for $i \geq i_1$. By the affine case we may find an index $i_2 \geq i_1$ and affine schemes $U_{j,i_2} \to V_{j,i_2}$ such that $U_j = S \times_{S_{i_2}} U_{j,i_2}$ is the base change. Denote $U_{j,i} = S_i \times_{S_{i_2}} U_{j,i_2}$ for $i \geq i_2$. By Lemma 31.3.8 there exists an index $i_3 \geq i_2$ and open subschemes $W_{j_1,j_2,i_3} \subset U_{j_1,i_3}$ whose base change to S is equal to $U_{j_1 j_2}$. Denote $W_{j_1,j_2,i} = S_i \times_{S_{i_3}} W_{j_1,j_2,i_3}$ for $i \geq i_3$. By part (2) shown above there exists an index $i_4 \geq i_3$ and morphisms $\varphi_{j_1,j_2,i_4} : W_{j_1,j_2,i_4} \to W_{j_2,j_1,i_4}$ whose base change to S gives the identity morphism $U_{j_1 j_2} = U_{j_2 j_1}$ for all j_1, j_2. For all $i \geq i_4$ denote $\varphi_{j_1,j_2,i} = \mathrm{id}_S \times \varphi_{j_1,j_2,i_4}$ the base change. We claim that for some $i_5 \geq i_4$ the system $((U_{j,i_5})_j, (W_{j_1,j_2,i_5})_{j_1,j_2}, (\varphi_{j_1,j_2,i_5})_{j_1,j_2})$ forms a glueing datum as in Schemes, Section 25.14. In order to see this we have to verify that for i large enough we have

$$\varphi_{j_1,j_2,i}^{-1}(W_{j_1,j_2,i} \cap W_{j_1,j_3,i}) = W_{j_1,j_2,i} \cap W_{j_1,j_3,i}$$

and that for large enough i the cocycle condition holds. The first condition follows from Lemma 31.3.8 and the fact that $U_{j_2 j_1 j_3} = U_{j_1 j_2 j_3}$. The second from part (1) of the lemma proved above and the fact that the cocycle condition holds for the maps $\mathrm{id} : U_{j_1 j_2} \to U_{j_2 j_1}$. Ok, so now we can use Schemes, Lemma 25.14.2 to glue the system $((U_{j,i_5})_j, (W_{j_1,j_2,i_5})_{j_1,j_2}, (\varphi_{j_1,j_2,i_5})_{j_1,j_2})$ to get a scheme $X_{i_5} \to S_{i_5}$. By construction the base change of X_{i_5} to S is formed by glueing the open affines U_j along the opens $U_{j_1} \leftarrow U_{j_1 j_2} \to U_{j_2}$. Hence $S \times_{S_{i_5}} X_{i_5} \cong X$ as desired. □

Lemma 31.9.2. *Let I be a directed partially ordered set. Let $(S_i, f_{ii'})$ be an inverse system of schemes over I. Assume*

(1) *all the morphisms $f_{ii'} : S_i \to S_{i'}$ are affine,*
(2) *all the schemes S_i are quasi-compact and quasi-separated.*

Let $S = \lim_i S_i$. Then we have the following:

(1) *For any sheaf of $\mathcal{O}_S$-modules $\mathcal{F}$ of finite presentation there exists an index $i \in I$ and a sheaf of $\mathcal{O}_{S_i}$-modules of finite presentation $\mathcal{F}_i$ such that $\mathcal{F} \cong f_i^* \mathcal{I}_i$.*
(2) *Suppose given an index $i \in I$, sheaves of $\mathcal{O}_{S_i}$-modules $\mathcal{F}_i$, $\mathcal{G}_i$ of finite presentation and a morphism $\varphi : f_i^* \mathcal{F}_i \to f_i^* \mathcal{G}_i$ over S. Then there exists*

an index $i' \geq i$ and a morphism $\varphi_{i'} : f_{i'i}^* \mathcal{F}_i \to f_{i'i}^* \mathcal{G}_i$ whose base change to S is φ.

(3) *Suppose given an index $i \in I$, sheaves of $\mathcal{O}_{S_i}$-modules $\mathcal{F}_i$, $\mathcal{G}_i$ of finite presentation and a pair of morphisms $\varphi_i, \psi_i : \mathcal{F}_i \to \mathcal{G}_i$. Assume that the base changes are equal: $f_i^* \varphi_i = f_i^* \psi_i$. Then there exists an index $i' \geq i$ such that $f_{i'i}^* \varphi_i = f_{i'i}^* \psi_i$.*

In other words, the category of modules of finite presentation over S is the colimit over I of the categories modules of finite presentation over S_i.

Proof. Omitted. Since we have written out completely the proof of Lemma 31.9.1 above it seems wise to use this here and not completely write this proof out also. For example we can use:

(1) there is an equivalence of categories between quasi-coherent $\mathcal{O}_S$-modules and vector bundles over S, see Constructions, Section 26.6.
(2) a vector bundle $\mathbf{V}(\mathcal{F}) \to S$ is of finite presentation over S if and only if $\mathcal{F}$ is an $\mathcal{O}_S$-module of finite presentation.

Then you can descend morphisms in terms of morphisms of the associated vector-bundles. Similarly for objects. □

Lemma 31.9.3. *With notation and assumptions as in Lemma 31.9.1. Let $i \in I$. Suppose that $\varphi_i : X_i \to Y_i$ is a morphism of schemes of finite presentation over S_i and that $\mathcal{F}_i$ is a quasi-coherent $\mathcal{O}_{X_i}$-module of finite presentation. If the pullback of $\mathcal{F}_i$ to $X_i \times_{S_i} S$ is flat over $Y_i \times_{S_i} S$, then there exists an index $i' \geq i$ such that the pullback of $\mathcal{F}_i$ to $X_i \times_{S_i} S_{i'}$ is flat over $Y_i \times_{S_i} S_{i'}$.*

Proof. (This lemma is the analogue of Lemma 31.7.6 for modules.) For $i' \geq i$ denote $X_{i'} = S_{i'} \times_{S_i} X_i$, $\mathcal{F}_{i'} = (X_{i'} \to X_i)^* \mathcal{F}_i$ and similarly for $Y_{i'}$. Denote $\varphi_{i'}$ the base change of φ_i to $S_{i'}$. Also set $X = S \times_{S_i} X_i$, $Y = S \times_{S_i} X_i$, $\mathcal{F} = (X \to X_i)^* \mathcal{F}_i$ and φ the base change of φ_i to S. Let $Y_i = \bigcup_{j=1,\ldots,m} V_{j,i}$ be a finite affine open covering such that each $V_{j,i}$ maps into some affine open of S_i. For each $j = 1, \ldots m$ let $\varphi_i^{-1}(V_{j,i}) = \bigcup_{k=1,\ldots,m(j)} U_{k,j,i}$ be a finite affine open covering. For $i' \geq i$ we denote $V_{j,i'}$ the inverse image of $V_{j,i}$ in $Y_{i'}$ and $U_{k,j,i'}$ the inverse image of $U_{k,j,i}$ in $X_{i'}$. Similarly we have $U_{k,j} \subset X$ and $V_j \subset Y$. Then $U_{k,j} = \lim_{i' \geq i} U_{k,j,i'}$ and $V_j = \lim_{i' \geq i} V_j$ (see Lemma 31.2.2). Since $X_{i'} = \bigcup_{k,j} U_{k,j,i'}$ is a finite open covering it suffices to prove the lemma for each of the morphisms $U_{k,j,i} \to V_{j,i}$ and the sheaf $\mathcal{F}_i|_{U_{k,j,i}}$. Hence we see that the lemma reduces to the case that X_i and Y_i are affine and map into an affine open of S_i, i.e., we may also assume that S is affine.

In the affine case we reduce to the following algebra result. Suppose that $R = \operatorname{colim}_{i \in I} R_i$. For some $i \in I$ suppose given a map $A_i \to B_i$ of finitely presented R_i-algebras. Let N_i be a finitely presented B_i-module. Then, if $R \otimes_{R_i} N_i$ is flat over $R \otimes_{R_i} A_i$, then for some $i' \geq i$ the module $R_{i'} \otimes_{R_i} N_i$ is flat over $R_{i'} \otimes_{R_i} A$. This is exactly the result proved in Algebra, Lemma 10.154.1 part (3). □

31.10. Characterizing affine schemes

If $f : X \to S$ is a surjective integral morphism of schemes such that X is an affine scheme then S is affine too. See [**Con07**, A.2]. Our proof relies on the Noetherian case which we stated and proved in Cohomology of Schemes, Lemma 29.13.3. See also [**DG67**, II 6.7.1].

Lemma 31.10.1. *Let $f : X \to S$ be a morphism of schemes. Assume that f is surjective and finite, and assume that X is affine. Then S is affine.*

Proof. Since f is surjective and X is quasi-compact we see that S is quasi-compact. Since X is separated and f is surjective and universally closed (Morphisms, Lemma 28.44.7), we see that S is separated (Morphisms, Lemma 28.42.10).

By Lemma 31.8.8 we can write $X = \lim_a X_a$ with $X_a \to S$ finite and of finite presentation. By Lemma 31.3.10 we see that X_a is affine for some $a \in A$. Replacing X by X_a we may assume that $X \to S$ is surjective, finite, of finite presentation and that X is affine.

By Proposition 31.4.4 we may write $S = \lim_{i \in I} S_i$ as a directed limits as schemes of finite type over $\mathbf{Z}$. By Lemma 31.9.1 we can after shrinking I assume there exist schemes $X_i \to S_i$ of finite presentation such that $X_{i'} = X_i \times_S S_{i'}$ for $i' \geq i$ and such that $X = \lim_i X_i$. By Lemma 31.7.3 we may assume that $X_i \to S_i$ is finite for all $i \in I$ as well. By Lemma 31.3.10 once again we may assume that X_i is affine for all $i \in I$. Hence the result follows from the Noetherian case, see Cohomology of Schemes, Lemma 29.13.3. □

Proposition 31.10.2. *Let $f : X \to S$ be a morphism of schemes. Assume that f is surjective and integral, and assume that X is affine. Then S is affine.*

Proof. Since f is surjective and X is quasi-compact we see that S is quasi-compact. Since X is separated and f is surjective and universally closed (Morphisms, Lemma 28.44.7), we see that S is separated (Morphisms, Lemma 28.42.10).

By Lemma 31.6.2 we can write $X = \lim_i X_i$ with $X_i \to S$ finite. By Lemma 31.3.10 we see that for i sufficiently large the scheme X_i is affine. Moreover, since $X \to S$ factors through each X_i we see that $X_i \to S$ is surjective. Hence we conclude that S is affine by Lemma 31.10.1. □

Lemma 31.10.3. *Let X be a scheme which is set theoretically the union of finitely many affine closed subschemes. Then X is affine.*

Proof. Let $Z_i \subset X$, $i = 1, \ldots, n$ be affine closed subschemes such that $X = \bigcup Z_i$ set theoretically. Then $\coprod Z_i \to X$ is surjective and integral with affine source. Hence X is affine by Proposition 31.10.2. □

Lemma 31.10.4. *Let $i : Z \to X$ be a closed immersion of schemes inducing a homeomorphism of underlying topological spaces. Let $\mathcal{L}$ be an invertible sheaf on X. If $i^*\mathcal{L}$ is ample on Z, then $\mathcal{L}$ is ample on X.*

Proof. Since $i^*\mathcal{L}$ is ample we see that Z is quasi-compact (Properties, Definition 27.24.1) and separated (Properties, Lemma 27.24.9). Since i is surjective, we see that X is quasi-compact. Since i is universally closed and surjective, we see that X is separated (Morphisms, Lemma 28.42.10).

By Proposition 31.4.4 we can write $X = \lim X_i$ as a directed limit of finite type schemes over $\mathbf{Z}$ with affine transition morphisms. We can find an i and an invertible sheaf $\mathcal{L}_i$ on X_i whose pullback to X is isomorphic to $\mathcal{L}$, see Lemma 31.9.2.

For each i let $Z_i \subset X_i$ be the scheme theoretic image of the morphism $Z \to X$. If $\mathrm{Spec}(A_i) \subset X_i$ is an affine open subscheme with inverse image $\mathrm{Spec}(A)$ in X and if $Z \cap \mathrm{Spec}(A)$ is defined by the ideal $I \subset A$, then $Z_i \cap \mathrm{Spec}(A_i)$ is defined by the

ideal $I_i \subset A_i$ which is the inverse image of I in A_i under the ring map $A_i \to A$, see Morphisms, Example 28.6.4. Since $\operatorname{colim} A_i/I_i = A/I$ it follows that $\lim Z_i = Z$. By Lemma 31.3.12 we see that $\mathcal{L}_i|_{Z_i}$ is ample for some i. Since Z and hence X maps into Z_i set theoretically, we see that $X_{i'} \to X_i$ maps into Z_i set theoretically for some $i' \geq i$, see Lemma 31.3.7. (Observe that since X_i is Noetherian, every closed subset of X_i is constructible.) Let $T \subset X_{i'}$ be the scheme theoretic inverse image of Z_i in $X_{i'}$. Observe that $\mathcal{L}_{i'}|_T$ is the pullback of $\mathcal{L}_i|_{Z_i}$ and hence ample by Morphisms, Lemma 28.38.7 and the fact that $T \to Z_i$ is an affine morphism. Thus we see that $\mathcal{L}_{i'}$ is ample on $X_{i'}$ by Cohomology of Schemes, Lemma 29.14.5. Pulling back to X (using the same lemma as above) we find that $\mathcal{L}$ is ample. □

31.11. Variants of Chow's Lemma

In this section we prove a number of variants of Chow's lemma. The most interesting version is probably just the Noetherian case, which we stated and proved in Cohomology of Schemes, Section 29.16.

Lemma 31.11.1. *Let S be a quasi-compact and quasi-separated scheme. Let $f : X \to S$ be a separated morphism of finite type. Then there exists an $n \geq 0$ and a diagram*

$$\begin{array}{ccccc} X & \xleftarrow{\pi} & X' & \longrightarrow & \mathbf{P}^n_S \\ & \searrow & \downarrow & \swarrow & \\ & & S & & \end{array}$$

where $X' \to \mathbf{P}^n_S$ is an immersion, and $\pi : X' \to X$ is proper and surjective.

Proof. By Proposition 31.8.6 we can find a closed immersion $X \to Y$ where Y is separated and of finite presentation over S. Clearly, if we prove the assertion for Y, then the result follows for X. Hence we may assume that X is of finite presentation over S.

Write $S = \lim_i S_i$ as a directed limit of Noetherian schemes, see Proposition 31.4.4. By Lemma 31.9.1 we can find an index $i \in I$ and a scheme $X_i \to S_i$ of finite presentation so that $X = S \times_{S_i} X_i$. By Lemma 31.7.5 we may assume that $X_i \to S_i$ is separated. Clearly, if we prove the assertion for X_i over S_i, then the assertion holds for X. The case $X_i \to S_i$ is treated by Cohomology of Schemes, Lemma 29.16.1. □

Here is a variant of Chow's lemma where we assume the scheme on top has finitely many irreducible components.

Lemma 31.11.2. *Let S be a quasi-compact and quasi-separated scheme. Let $f : X \to S$ be a separated morphism of finite type. Assume that X has finitely many irreducible components. Then there exists an $n \geq 0$ and a diagram*

$$\begin{array}{ccccc} X & \xleftarrow{\pi} & X' & \longrightarrow & \mathbf{P}^n_S \\ & \searrow & \downarrow & \swarrow & \\ & & S & & \end{array}$$

where $X' \to \mathbf{P}^n_S$ is an immersion, and $\pi : X' \to X$ is proper and surjective. Moreover, there exists an open dense subscheme $U \subset X$ such that $\pi^{-1}(U) \to U$ is an isomorphism of schemes.

Proof. Let $X = Z_1 \cup \ldots \cup Z_n$ be the decomposition of X into irreducible components. Let $\eta_j \in Z_j$ be the generic point.

There are (at least) two ways to proceed with the proof. The first is to redo the proof of Cohomology of Schemes, Lemma 29.16.1 using the general Properties, Lemma 27.27.4 to find suitable affine opens in X. (This is the "standard" proof.) The second is to use absolute Noetherian approximation as in the proof of Lemma 31.11.1 above. This is what we will do here.

By Proposition 31.8.6 we can find a closed immersion $X \to Y$ where Y is separated and of finite presentation over S. Write $S = \lim_i S_i$ as a directed limit of Noetherian schemes, see Proposition 31.4.4. By Lemma 31.9.1 we can find an index $i \in I$ and a scheme $Y_i \to S_i$ of finite presentation so that $Y = S \times_{S_i} Y_i$. By Lemma 31.7.5 we may assume that $Y_i \to S_i$ is separated. We have the following diagram

$$\begin{array}{ccccccc} \eta_j \in Z_j & \longrightarrow & X & \longrightarrow & Y & \longrightarrow & Y_i \\ & & & \searrow & \downarrow & & \downarrow \\ & & & & S & \longrightarrow & S_i \end{array}$$

Denote $h : X \to Y_i$ the composition.

For $i' \geq i$ write $Y_{i'} = S_{i'} \times_{S_i} Y_i$. Then $Y = \lim_{i' \geq i} Y_{i'}$, see Lemma 31.2.3. Choose $j, j' \in \{1, \ldots, n\}$, $j \neq j'$. Note that η_j is not a specialization of $\eta_{j'}$. By Lemma 31.3.2 we can replace i by a bigger index and assume that $h(\eta_j)$ is not a specialization of $h(\eta_{j'})$ for all pairs (j, j') as above. For such an index, let $Y' \subset Y_i$ be the scheme theoretic image of $h : X \to Y_i$, see Morphisms, Definition 28.6.2. The morphism h is quasi-compact as the composition of the quasi-compact morphisms $X \to Y$ and $Y \to Y_i$ (which is affine). Hence by Morphisms, Lemma 28.6.3 the morphism $X \to Y'$ is dominant. Thus the generic points of Y' are all contained in the set $\{h(\eta_1), \ldots, h(\eta_n)\}$, see Morphisms, Lemma 28.8.3. Since none of the $h(\eta_j)$ is the specialization of another we see that the points $h(\eta_1), \ldots, h(\eta_n)$ are pairwise distinct and are each a generic point of Y'.

We apply Cohomology of Schemes, Lemma 29.16.1 above to the morphism $Y' \to S_i$. This gives a diagram

$$\begin{array}{ccccc} Y' & \xleftarrow{\pi} & Y^* & \longrightarrow & \mathbf{P}^n_{S_i} \\ & \searrow & \downarrow & \swarrow & \\ & & S_i & & \end{array}$$

such that π is proper and surjective and an isomorphism over a dense open subscheme $V \subset Y'$. By our choice of i above we know that $h(\eta_1), \ldots, h(\eta_n) \in V$. Consider the commutative diagram

$$\begin{array}{ccccccc} X' & = & X \times_{Y'} Y^* & \longrightarrow & Y^* & \longrightarrow & \mathbf{P}^n_{S_i} \\ & & \downarrow & & \downarrow & \nearrow & \\ & & X & \longrightarrow & Y' & & \\ & & \downarrow & & \downarrow & & \\ & & S & \longrightarrow & S_i & & \end{array}$$

Note that $X' \to X$ is an isomorphism over the open subscheme $U = h^{-1}(V)$ which contains each of the η_j and hence is dense in X. We conclude $X \leftarrow X' \to \mathbf{P}^n_S$ is a solution to the problem posed in the lemma. □

31.12. Applications of Chow's lemma

We can use Chow's lemma to investigate the notions of proper and separated morphisms. As a first application we have the following.

Lemma 31.12.1. *Let S be a scheme. Let $f : X \to S$ be a separated morphism of finite type. The following are equivalent:*

1. *The morphism f is proper.*
2. *For any morphism $S' \to S$ which is locally of finite type the base change $X_{S'} \to S'$ is closed.*
3. *For every $n \geq 0$ the morphism $\mathbf{A}^n \times X \to \mathbf{A}^n \times S$ is closed.*

Proof. Clearly (1) implies (2), and (2) implies (3), so we just need to show (3) implies (1). First we reduce to the case when S is affine. Assume that (3) implies (1) when the base is affine. Now let $f : X \to S$ be a separated morphism of finite type. Being proper is local on the base (see Morphisms, Lemma 28.42.3), so if $S = \bigcup_\alpha S_\alpha$ is an open affine cover, and if we denote $X_\alpha := f^{-1}(S_\alpha)$, then it is enough to show that $f|_{X_\alpha} : X_\alpha \to S_\alpha$ is proper for all α. Since S_α is affine, if the map $f|_{X_\alpha}$ satisfies (3), then it will satisfy (1) by assumption, and will be proper. To finish the reduction to the case S is affine, we must show that if $f : X \to S$ is separated of finite type satisfying (3), then $f|_{X_\alpha} : X_\alpha \to S_\alpha$ is separated of finite type satisfying (3). Separatedness and finite type are clear. To see (3), notice that $\mathbf{A}^n \times X_\alpha$ is the open preimage of $\mathbf{A}^n \times S_\alpha$ under the map $1 \times f$. Fix a closed set $Z \subset \mathbf{A}^n \times X_\alpha$. Let $\bar{Z}$ denote the closure of Z in $\mathbf{A}^n \times X$. Then for topological reasons,

$$1 \times f(\bar{Z}) \cap \mathbf{A}^n \times S_\alpha = 1 \times f(Z).$$

Hence $1 \times f(Z)$ is closed, and we have reduced the proof of (3) $\Rightarrow$ (1) to the affine case.

Assume S affine, and $f : X \to S$ separated of finite type. We can apply Chow's Lemma 31.11.1 to get $\pi : X' \to X$ proper surjective and $X' \to \mathbf{P}^n_S$ an immersion. If X is proper over S, then $X' \to S$ is proper (Morphisms, Lemma 28.42.4). Since $\mathbf{P}^n_S \to S$ is separated, we conclude that $X' \to \mathbf{P}^n_S$ is proper (Morphisms, Lemma 28.42.7) and hence a closed immersion (Schemes, Lemma 25.10.4). Conversely, assume $X' \to \mathbf{P}^n_S$ is a closed immersion. Consider the diagram:

$$\begin{array}{ccc} X' & \longrightarrow & \mathbf{P}^n_S \\ {\scriptstyle\pi}\downarrow & & \downarrow \\ X & \xrightarrow{f} & S \end{array} \tag{31.12.1.1}$$

All maps are a priori proper except for $X \to S$. Hence we conclude that $X \to S$ is proper by Morphisms, Lemma 28.42.8. Therefore, we have shown that $X \to S$ is proper if and only if $X' \to \mathbf{P}^n_S$ is a closed immersion.

Assume S is affine and (3) holds, and let n, X', π be as above. Since being a closed morphism is local on the base, the map $X \times \mathbf{P}^n \to S \times \mathbf{P}^n$ is closed since by (3) $X \times \mathbf{A}^n \to S \times \mathbf{A}^n$ is closed and since projective space is covered by copies of affine

n-space, see Constructions, Lemma 26.13.3. By Morphisms, Lemma 28.42.5 the morphism

$$X' \times_S \mathbf{P}^n_S \to X \times_S \mathbf{P}^n_S = X \times \mathbf{P}^n$$

is proper. Since $\mathbf{P}^n$ is separated, the projection

$$X' \times_S \mathbf{P}^n_S = \mathbf{P}^n_{X'} \to X'$$

will be separated as it is just a base change of a separated morphism. Therefore, the map $X' \to X' \times_S \mathbf{P}^n_S$ is proper, since it is a section to a separated map (see Schemes, Lemma 25.21.12). Composing all these proper morphisms

$$X' \to X' \times_S \mathbf{P}^n_S \to X \times_S \mathbf{P}^n_S = X \times \mathbf{P}^n \to S \times \mathbf{P}^n = \mathbf{P}^n_S$$

we see that the map $X' \to \mathbf{P}^n_S$ is proper, and hence a closed immersion. □

If the base is Noetherian we can show that the valuative criterion holds using only discrete valuation rings. First we state the result concerning separation. We will often use solid commutative diagrams of morphisms of schemes having the following shape

(31.12.1.2)
$$\begin{array}{ccc} \mathrm{Spec}(K) & \longrightarrow & X \\ \downarrow & \nearrow & \downarrow \\ \mathrm{Spec}(A) & \longrightarrow & S \end{array}$$

with A a valuation ring and K its field of fractions.

Lemma 31.12.2. *Let S be a locally Noetherian scheme. Let $f : X \to S$ be a morphism of schemes. Assume f is locally of finite type. The following are equivalent:*

1. *The morphism f is separated.*
2. *For any diagram (31.12.1.2) there is at most one dotted arrow.*
3. *For all diagrams (31.12.1.2) with A a discrete valuation ring there is at most one dotted arrow.*
4. *For any irreducible component X_0 of X with generic point $\eta \in X_0$, for any discrete valuation ring $A \subset K = \kappa(\eta)$ with fraction field K and any diagram (31.12.1.2) such that the morphism $\mathrm{Spec}(K) \to X$ is the canonical one (see Schemes, Section 25.13) there is at most one dotted arrow.*

Proof. Clearly (1) implies (2), (2) implies (3), and (3) implies (4). It remains to show (4) implies (1). Assume (4). We begin by reducing to S affine. Being separated is a local on the base (see Schemes, Lemma 25.21.8). Hence, as in the proof of Lemma 31.12.1, if we can show that whenever $X \to S$ has (4) that the restriction $X_\alpha \to S_\alpha$ has (4) where $S_\alpha \subset S$ is an (affine) open subset and $X_\alpha := f^{-1}(S_\alpha)$, then we will be done. The generic points of the irreducible components of X_α will be the generic points of irreducible components of X, since X_α is open in X. Therefore, any two distinct dotted arrows in the diagram

(31.12.2.1)
$$\begin{array}{ccc} \mathrm{Spec}(K) & \longrightarrow & X_\alpha \\ \downarrow & \nearrow & \downarrow \\ \mathrm{Spec}(A) & \longrightarrow & S_\alpha \end{array}$$

would then give two distinct arrows in diagram (31.12.1.2) via the maps $X_\alpha \to X$ and $S_\alpha \to S$, which is a contradiction. Thus we have reduced to the case S is affine. We remark that in the course of this reduction, we prove that if $X \to S$ has (4) then the restriction $U \to V$ has (4) for opens $U \subset X$ and $V \subset S$ with $f(U) \subset V$.

We next wish to reduce to the case $X \to S$ is finite type. Assume that we know (4) implies (1) when X is finite type. Since S is Noetherian and X is locally of finite type over S we see X is locally Noetherian as well (see Morphisms, Lemma 28.16.6). Thus, $X \to S$ is quasi-separated (see Properties, Lemma 27.5.4), and therefore we may apply the valuative criterion to check whether X is separated (see Schemes, Lemma 25.22.2). Let $X = \bigcup_\alpha X_\alpha$ be an affine open cover of X. Given any two dotted arrows, in a diagram (31.12.1.2), the image of the closed points of Spec A will fall in two sets X_α and X_β. Since $X_\alpha \cup X_\beta$ is open, for topological reasons it must contain the image of Spec(A) under both maps. Therefore, the two dotted arrows factor through $X_\alpha \cup X_\beta \to X$, which is a scheme of finite type over S. Since $X_\alpha \cup X_\beta$ is an open subset of X, by our previous remark, $X_\alpha \cup X_\beta$ satisfies (4), so by assumption, is separated. This implies the two given dotted arrows are the same. Therefore, we have reduced to $X \to S$ is finite type.

Assume $X \to S$ of finite type and assume (4). Since $X \to S$ is finite type, and S is an affine Noetherian scheme, X is also Noetherian (see Morphisms, Lemma 28.16.6). Therefore, $X \to X \times_S X$ will be a quasi-compact immersion of Noetherian schemes. We proceed by contradiction. Assume that $X \to X \times_S X$ is not closed. Then, there is some $y \in X \times_S X$ in the closure of the image that is not in the image. As X is Noetherian it has finitely many irreducible components. Therefore, y is in the closure of the image of one of the irreducible components $X_0 \subset X$. Give X_0 the reduced induced structure. The composition $X_0 \to X \to X \times_S X$ factors through the closed subscheme $X_0 \times_S X_0 \subset X \times_S X$. Denote the closure of $\Delta(X_0)$ in $X_0 \times_S X_0$ by $\bar{X}_0$ (again as a reduced closed subscheme). Thus $y \in \bar{X}_0$. Since $X_0 \to X_0 \times_S X_0$ is an immersion, the image of X_0 will be open in $\bar{X}_0$. Hence X_0 and $\bar{X}_0$ are birational. Since $\bar{X}_0$ is a closed subscheme of a Noetherian scheme, it is Noetherian. Thus, the local ring $\mathcal{O}_{\bar{X}_0,y}$ is a local Noetherian domain with fraction field K equal to the function field of X_0. By the Krull-Akizuki theorem (see Algebra, Lemma 10.113.12), there exists a discrete valuation ring A dominating $\mathcal{O}_{\bar{X}_0,y}$ with fraction field K. This allows to to construct a diagram:

(31.12.2.2)
$$\begin{array}{ccc} \operatorname{Spec} K & \longrightarrow & X_0 \\ \downarrow & \nearrow & \downarrow \Delta \\ A & \longrightarrow & X_0 \times_S X_0 \end{array}$$

which sends Spec K to the generic point of $\Delta(X_0)$ and the closed point of A to $y \in X_0 \times_S X_0$ (use the material in Schemes, Section 25.13 to construct the arrows). There cannot even exist a set theoretic dotted arrow, since y is not in the image of Δ by our choice of y. By categorical means, the existence of the dotted arrow in the above diagram is equivalent to the uniqueness of the dotted arrow in the

following diagram:

(31.12.2.3)
$$\begin{array}{ccc} \operatorname{Spec} K & \longrightarrow & X_0 \\ \downarrow & \nearrow & \downarrow \\ A & \longrightarrow & S \end{array}$$

Therefore, we have non-uniqueness in this latter diagram by the nonexistence in the first. Therefore, X_0 does not satisfy uniqueness for discrete valuation rings, and since X_0 is an irreducible component of X, we have that $X \to S$ does not satisfy (4). Therefore, we have shown (4) implies (1). □

Lemma 31.12.3. *Let S be a locally Noetherian scheme. Let $f : X \to S$ be a morphism of finite type. The following are equivalent:*

1. *The morphism f is proper.*
2. *For any diagram (31.12.1.2) there exists exactly one dotted arrow.*
3. *For all diagrams (31.12.1.2) with A a discrete valuation ring there exists exactly one dotted arrow.*
4. *For any irreducible component X_0 of X with generic point $\eta \in X_0$, for any discrete valuation ring $A \subset K = \kappa(\eta)$ with fraction field K and any diagram (31.12.1.2) such that the morphism $\operatorname{Spec}(K) \to X$ is the canonical one (see Schemes, Section 25.13) there exists exactly one dotted arrow.*

Proof. (1) implies (2) implies (3) implies (4). We will now show (4) implies (1). As in the proof of Lemma 31.12.2, we can reduce to the case S is affine, since properness is local on the base, and if $X \to S$ satisfies (4), then $X_\alpha \to S_\alpha$ does as well for open $S_\alpha \subset S$ and $X_\alpha = f^{-1}(S_\alpha)$.

Now S is a Noetherian scheme, and so X is as well, since $X \to S$ is of finite type. Now we may use Chow's lemma (Cohomology of Schemes, Lemma 29.16.1) to get a surjective, proper, birational $X' \to X$ and an immersion $X' \to \mathbf{P}^n_S$. We wish to show $X \to S$ is universally closed. As in the proof of Lemma 31.12.1, it is enough to check that $X' \to \mathbf{P}^n_S$ is a closed immersion. For the sake of contradiction, assume that $X' \to \mathbf{P}^n_S$ is not a closed immersion. Then there is some $y \in \mathbf{P}^n_S$ that is in the closure of the image of X', but is not in the image. So y is in the closure of the image of an irreducible component X'_0 of X', but not in the image. Let $\bar{X}'_0 \subset \mathbf{P}^n_S$ be the closure of the image of X'_0. As $X' \to \mathbf{P}^n_S$ is an immersion of Noetherian schemes, the morphism $X'_0 \to \bar{X}'_0$ is open and dense. By Algebra, Lemma 10.113.12 or Properties, Lemma 27.5.9 we can find a discrete valuation ring A dominating $\mathcal{O}_{\bar{X}'_0, y}$ and with identical field of fractions K. It is clear that K is the residue field at the generic point of X'_0. Thus the solid commutative diagram

(31.12.3.1)
$$\begin{array}{ccccc} \operatorname{Spec} K & \longrightarrow & X' & \longrightarrow & \mathbf{P}^n_S \\ \downarrow & \nearrow & \downarrow & \nearrow & \downarrow \\ \operatorname{Spec} A & \dashrightarrow & X & \longrightarrow & S \end{array}$$

Note that the closed point of A maps to $y \in \mathbf{P}^n_S$. By construction, there does not exist a set theoretic lift to X'. As $X' \to X$ is birational, the image of X'_0 in X is an irreducible component X_0 of X and K is also identified with the function field of X_0. Hence, as $X \to S$ is assumed to satisfy (4), the dotted arrow $\operatorname{Spec}(A) \to$

X exists. Since $X' \to X$ is proper, the dotted arrow lifts to the dotted arrow $\mathrm{Spec}(A) \to X'$ (use Schemes, Proposition 25.20.6). We can compose this with the immersion $X' \to \mathbf{P}^n_S$ to obtain another morphism (not depicted in the diagram) from $\mathrm{Spec}(A) \to \mathbf{P}^n_S$. Since $\mathbf{P}^n_S$ is proper over S, it satisfies (2), and so these two morphisms agree. This is a contradiction, for we have constructed the forbidden lift of our original map $\mathrm{Spec}(A) \to \mathbf{P}^n_S$ to X'. □

Here is an application of Chow's lemma which goes in a slightly different direction.

Lemma 31.12.4. *Assumptions and notation as in Situation 31.7.1. If*

(1) *f is proper, and*
(2) *f_0 is locally of finite type,*

then there exists an i such that f_i is proper.

Proof. By Lemma 31.7.5 we see that f_i is separated for some $i \geq 0$. Replacing 0 by i we may assume that f_0 is separated. Observe that f_0 is quasi-compact, see Schemes, Lemma 25.21.15. By Lemma 31.11.1 we can choose a diagram

$$\begin{array}{ccccc} X_0 & \xleftarrow{\pi} & X_0' & \longrightarrow & \mathbf{P}^n_{Y_0} \\ & \searrow & \downarrow & \swarrow & \\ & & Y_0 & & \end{array}$$

where $X_0' \to \mathbf{P}^n_{Y_0}$ is an immersion, and $\pi : X_0' \to X_0$ is proper and surjective. Introduce $X' = X_0' \times_{Y_0} Y$ and $X_i' = X_0' \times_{Y_0} Y_i$. By Morphisms, Lemmas 28.42.4 and 28.42.5 we see that $X' \to Y$ is proper. Hence $X' \to \mathbf{P}^n_Y$ is a closed immersion (Morphisms, Lemma 28.42.7). By Morphisms, Lemma 28.42.8 it suffices to prove that $X_i' \to Y_i$ is proper for some i. By Lemma 31.7.4 we find that $X_i' \to \mathbf{P}^n_{Y_i}$ is a closed immersion for i large enough. Then $X_i' \to Y_i$ is proper and we win. □

Lemma 31.12.5. *Let $f : X \to S$ be a proper morphism with S quasi-compact and quasi-separated. Then $X = \lim X_i$ with $X_i \to S$ proper and of finite presentation.*

Proof. By Proposition 31.8.6 we can find a closed immersion $X \to Y$ with Y separated and of finite presentation over S. By Lemma 31.11.1 we can find a diagram

$$\begin{array}{ccccc} Y & \xleftarrow{\pi} & Y' & \longrightarrow & \mathbf{P}^n_S \\ & \searrow & \downarrow & \swarrow & \\ & & S & & \end{array}$$

where $Y' \to \mathbf{P}^n_S$ is an immersion, and $\pi : Y' \to Y$ is proper and surjective. By Lemma 31.8.4 we can write $X = \lim X_i$ with $X_i \to Y$ a closed immersion of finite presentation. Denote $X_i' \subset Y'$, resp. $X' \subset Y'$ the scheme theoretic inverse image of $X_i \subset Y$, resp. $X \subset Y$. Then $\lim X_i' = X'$. Since $X' \to S$ is proper (Morphisms, Lemmas 28.42.4), we see that $X' \to \mathbf{P}^n_S$ is a closed immersion (Morphisms, Lemma 28.42.7). Hence for i large enough we find that $X_i' \to \mathbf{P}^n_S$ is a closed immersion by Lemma 31.7.4. Thus X_i' is proper over S. For such i the morphism $X_i \to S$ is proper by Morphisms, Lemma 28.42.8. □

Recall the scheme theoretic support of a finite type quasi-coherent module, see Morphisms, Definition 28.5.5.

Lemma 31.12.6. *Assumptions and notation as in Situation 31.7.1. Let $\mathcal{F}_0$ be a quasi-coherent $\mathcal{O}_{X_0}$-module. Denote $\mathcal{F}$ and $\mathcal{F}_i$ the pullbacks of $\mathcal{F}_0$ to X and X_i. Assume*

(1) *f_0 is locally of finite type,*
(2) *$\mathcal{F}_0$ is of finite type,*
(3) *the scheme theoretic support of $\mathcal{F}$ is proper over Y.*

Then the scheme theoretic support of $\mathcal{F}_i$ is proper over Y_i for some i.

Proof. We may replace X_0 by the scheme theoretic support of $\mathcal{F}_0$. By Morphisms, Lemma 28.5.3 this guarantees that X_i is the support of $\mathcal{F}_i$ and X is the support of $\mathcal{F}$. Then, if $Z \subset X$ denotes the scheme theoretic support of $\mathcal{F}$, we see that $Z \to X$ is a universal homeomorphism. We conclude that $X \to Y$ is proper as this is true for $Z \to Y$ by assumption, see Morphisms, Lemma 28.42.8. By Lemma 31.12.4 we see that $X_i \to Y$ is proper for some i. Then it follows that the scheme theoretic support Z_i of $\mathcal{F}_i$ is proper over Y by Morphisms, Lemmas 28.42.6 and 28.42.4. □

31.13. Universally closed morphisms

In this section we discuss when a quasi-compact but not necessarily separated morphism is universally closed. We first prove a lemma which will allow us to check universal closedness after a base change which is locally of finite presentation.

Lemma 31.13.1. *Let $f : X \to S$ be a quasi-compact morphism of schemes. Let $g : T \to S$ be a morphism of schemes. Let $t \in T$ be a point and $Z \subset X_T$ be a closed subscheme such that $Z \cap X_t = \emptyset$. Then there exists an open neighbourhood $V \subset T$ of t, a commutative diagram*

$$\begin{array}{ccc} V & \xrightarrow{a} & T' \\ \downarrow & & \downarrow b \\ T & \xrightarrow{g} & S, \end{array}$$

and a closed subscheme $Z' \subset X_{T'}$ such that

(1) *the morphism $b : T' \to S$ is locally of finite presentation,*
(2) *with $t' = a(t)$ we have $Z' \cap X_{t'} = \emptyset$, and*
(3) *$Z \cap X_V$ maps into Z' via the morphism $X_V \to X_{T'}$.*

Proof. Let $s = g(t)$. During the proof we may always replace T by an open neighbourhood of t. Hence we may also replace S by an open neighbourhood of s. Thus we may and do assume that T and S are affine. Say $S = \mathrm{Spec}(A)$, $T = \mathrm{Spec}(B)$, g is given by the ring map $A \to B$, and t correspond to the prime ideal $\mathfrak{q} \subset B$.

As $X \to S$ is quasi-compact and S is affine we may write $X = \bigcup_{i=1,\ldots,n} U_i$ as a finite union of affine opens. Write $U_i = \mathrm{Spec}(C_i)$. In particular we have $X_T = \bigcup_{i=1,\ldots,n} U_{i,T} = \bigcup_{i=1,\ldots,n} \mathrm{Spec}(C_i \otimes_A B)$. Let $I_i \subset C_i \otimes_A B$ be the ideal corresponding to the closed subscheme $Z \cap U_{i,T}$. The condition that $Z \cap X_t = \emptyset$ signifies that I_i generates the unit ideal in the ring

$$C_i \otimes_A \kappa(\mathfrak{q}) = (B \setminus \mathfrak{q})^{-1} (C_i \otimes_A B/\mathfrak{q}C_i \otimes_A B)$$

Since $I_i(B \setminus \mathfrak{q})^{-1}(C_i \otimes_A B) = (B \setminus \mathfrak{q})^{-1} I_i$ this means that $1 = x_i/g_i$ for some $x_i \in I_i$ and $g_i \in B$, $g_i \notin \mathfrak{q}$. Thus, clearing denominators we can find a relation of the form

$$x_i + \sum\nolimits_j f_{i,j} c_{i,j} = g_i$$

with $x_i \in I_i$, $f_{i,j} \in \mathfrak{q}$, $c_{i,j} \in C_i \otimes_A B$, and $g_i \in B$, $g_i \notin \mathfrak{q}$. After replacing B by $B_{g_1 \ldots g_n}$, i.e., after replacing T by a smaller affine neighbourhood of t, we may assume the equations read

$$x_i + \sum\nolimits_j f_{i,j} c_{i,j} = 1$$

with $x_i \in I_i$, $f_{i,j} \in \mathfrak{q}$, $c_{i,j} \in C_i \otimes_A B$.

To finish the argument write B as a colimit of finitely presented A-algebras B_λ over a directed partially ordered set Λ. For each λ set $\mathfrak{q}_\lambda = (B_\lambda \to B)^{-1}(\mathfrak{q})$. For sufficiently large $\lambda \in \Lambda$ we can find

(1) an element $x_{i,\lambda} \in C_i \otimes_A B_\lambda$ which maps to x_i,
(2) elements $f_{i,j,\lambda} \in \mathfrak{q}_{i,\lambda}$ mapping to $f_{i,j}$, and
(3) elements $c_{i,j,\lambda} \in C_i \otimes_A B_\lambda$ mapping to $c_{i,j}$.

After increasing λ a bit more the equation

$$x_{i,\lambda} + \sum\nolimits_j f_{i,j,\lambda} c_{i,j,\lambda} = 1$$

will hold. Fix such a λ and set $T' = \operatorname{Spec}(B_\lambda)$. Then $t' \in T'$ is the point corresponding to the prime $\mathfrak{q}_\lambda$. Finally, let $Z' \subset X_{T'}$ be the scheme theoretic closure of $Z \to X_T \to X_{T'}$. As $X_T \to X_{T'}$ is affine, we can compute Z' on the affine open pieces $U_{i,T'}$ as the closed subscheme associated to $\operatorname{Ker}(C_i \otimes_A B_\lambda \to C_i \otimes_A B/I_i)$, see Morphisms, Example 28.6.4. Hence $x_{i,\lambda}$ is in the ideal defining Z'. Thus the last displayed equation shows that $Z' \cap X_{t'}$ is empty. □

Lemma 31.13.2. *Let $f : X \to S$ be a quasi-compact morphism of schemes. The following are equivalent*

(1) *f is universally closed,*
(2) *for every morphism $S' \to S$ which is locally of finite presentation the base change $X_{S'} \to S'$ is closed, and*
(3) *for every n the morphism $\mathbf{A}^n \times X \to \mathbf{A}^n \times S$ is closed.*

Proof. It is clear that (1) implies (2). Let us prove that (2) implies (1). Suppose that the base change $X_T \to T$ is not closed for some scheme T over S. By Schemes, Lemma 25.19.8 this means that there exists some specialization $t_1 \rightsquigarrow t$ in T and a point $\xi \in X_T$ mapping to t_1 such that ξ does not specialize to a point in the fibre over t. Set $Z = \overline{\{\xi\}} \subset X_T$. Then $Z \cap X_t = \emptyset$. Apply Lemma 31.13.1. We find an open neighbourhood $V \subset T$ of t, a commutative diagram

$$\begin{array}{ccc} V & \xrightarrow{a} & T' \\ \downarrow & & \downarrow b \\ T & \xrightarrow{g} & S, \end{array}$$

and a closed subscheme $Z' \subset X_{T'}$ such that

(1) the morphism $b : T' \to S$ is locally of finite presentation,
(2) with $t' = a(t)$ we have $Z' \cap X_{t'} = \emptyset$, and
(3) $Z \cap X_V$ maps into Z' via the morphism $X_V \to X_{T'}$.

Clearly this means that $X_{T'} \to T'$ maps the closed subset Z' to a subset of T' which contains $a(t_1)$ but not $t' = a(t)$. Since $a(t_1) \rightsquigarrow a(t) = t'$ we conclude that $X_{T'} \to T'$ is not closed. Hence we have shown that $X \to S$ not universally closed implies that $X_{T'} \to T'$ is not closed for some $T' \to S$ which is locally of finite presentation. In order words (2) implies (1).

Assume that $\mathbf{A}^n \times X \to \mathbf{A}^n \times S$ is closed for every integer n. We want to prove that $X_T \to T$ is closed for every scheme T which is locally of finite presentation over S. We may of course assume that T is affine and maps into an affine open V of S (since $X_T \to T$ being a closed is local on T). In this case there exists a closed immersion $T \to \mathbf{A}^n \times V$ because $\mathcal{O}_T(T)$ is a finitely presented $\mathcal{O}_S(V)$-algebra, see Morphisms, Lemma 28.22.2. Then $T \to \mathbf{A}^n \times S$ is a locally closed immersion. Hence we get a cartesian diagram

$$\begin{array}{ccc} X_T & \longrightarrow & \mathbf{A}^n \times X \\ {\scriptstyle f_T}\downarrow & & \downarrow{\scriptstyle f_n} \\ T & \longrightarrow & \mathbf{A}^n \times S \end{array}$$

of schemes where the horizontal arrows are locally closed immersions. Hence any closed subset $Z \subset X_T$ can be written as $X_T \cap Z'$ for some closed subset $Z' \subset \mathbf{A}^n \times X$. Then $f_T(Z) = T \cap f_n(Z')$ and we see that if f_n is closed, then also f_T is closed. □

Lemma 31.13.3. *Let $f : X \to S$ be a finite type morphism of schemes. Assume S is locally Noetherian. Then the following are equivalent*

(1) *f is universally closed,*
(2) *for every n the morphism $\mathbf{A}^n \times X \to \mathbf{A}^n \times S$ is closed,*
(3) *for any diagram (31.12.1.2) there exists some dotted arrow,*
(4) *for all diagrams (31.12.1.2) with A a discrete valuation ring there exists some dotted arrow.*

Proof. The equivalence of (1) and (2) is a special case of Lemma 31.13.2. The equivalence of (1) and (3) is a special case of Schemes, Proposition 25.20.6. Trivially (3) implies (4). Thus all we have to do is prove that (4) implies (2). We will prove that $\mathbf{A}^n \times X \to \mathbf{A}^n \times S$ is closed by the criterion of Schemes, Lemma 25.19.8. Pick n and a specialization $z \rightsquigarrow z'$ of points in $\mathbf{A}^n \times S$ and a point $y \in \mathbf{A}^n \times X$ lying over z. Note that $\kappa(y)$ is a finitely generated field extension of $\kappa(z)$ as $\mathbf{A}^n \times X \to \mathbf{A}^n \times S$ is of finite type. Hence by Properties, Lemma 27.5.9 or Algebra, Lemma 10.113.12 implies that there exists a discrete valuation ring $A \subset \kappa(y)$ with fraction field $\kappa(z)$ dominating the image of $\mathcal{O}_{\mathbf{A}^n \times S, z'}$ in $\kappa(z)$. This gives a commutative diagram

$$\begin{array}{ccccc} \mathrm{Spec}(\kappa(y)) & \longrightarrow & \mathbf{A}^n \times X & \longrightarrow & X \\ \downarrow & & \downarrow & & \downarrow \\ \mathrm{Spec}(A) & \longrightarrow & \mathbf{A}^n \times S & \longrightarrow & S \end{array}$$

Now property (4) implies that there exists a morphism $\mathrm{Spec}(A) \to X$ which fits into this diagram. Since we already have the morphism $\mathrm{Spec}(A) \to \mathbf{A}^n$ from the left lower horizontal arrow we also get a morphism $\mathrm{Spec}(A) \to \mathbf{A}^n \times X$ fitting into the left square. Thus the image $y' \in \mathbf{A}^n \times X$ of the closed point is a specialization of y lying over z'. This proves that specializations lift along $\mathbf{A}^n \times X \to \mathbf{A}^n \times S$ and we win. □

31.14. Limits and dimensions of fibres

The following lemma is most often used in the situation of Lemma 31.9.1 to assure that if the fibres of the limit have dimension $\leq d$, then the fibres at some finite stage have dimension $\leq d$.

Lemma 31.14.1. *Let I be a directed partially ordered set. Let $(f_i : X_i \to S_i)$ be an inverse system of morphisms of schemes over I. Assume*

(1) *all the morphisms $S_{i'} \to S_i$ are affine,*
(2) *all the schemes S_i are quasi-compact and quasi-separated,*
(3) *the morphisms f_i are of finite type, and*
(4) *the morphisms $X_{i'} \to X_i \times_{S_i} S_{i'}$ are closed immersions.*

Let $f : X = \lim_i X_i \to S = \lim_i S_i$ be the limit. Let $d \geq 0$. If every fibre of f has dimension $\leq d$, then for some i every fibre of f_i has dimension $\leq d$.

Proof. For each i let $U_i = \{x \in X_i \mid \dim_x((X_i)_{f_i(x)}) \leq d\}$. This is an open subset of X_i, see Morphisms, Lemma 28.29.4. Set $Z_i = X_i \setminus U_i$ (with reduced induced scheme structure). We have to show that $Z_i = \emptyset$ for some i. If not, then $Z = \lim Z_i \neq \emptyset$, see Lemma 31.3.4. Say $z \in Z$ is a point. Note that $Z \subset X$ is a closed subscheme. Set $s = f(z)$. For each i let $s_i \in S_i$ be the image of s. We remark that Z_s is the limit of the schemes $(Z_i)_{s_i}$ and Z_s is also the limit of the schemes $(Z_i)_{s_i}$ base changed to $\kappa(s)$. Moreover, all the morphisms

$$Z_s \longrightarrow (Z_{i'})_{s_{i'}} \times_{\mathrm{Spec}(\kappa(s_{i'}))} \mathrm{Spec}(\kappa(s)) \longrightarrow (Z_i)_{s_i} \times_{\mathrm{Spec}(\kappa(s_i))} \mathrm{Spec}(\kappa(s)) \longrightarrow X_s$$

are closed immersions by assumption (4). Hence Z_s is the scheme theoretic intersection of the closed subschemes $(Z_i)_{s_i} \times_{\mathrm{Spec}(\kappa(s_i))} \mathrm{Spec}(\kappa(s))$ in X_s. Since all the irreducible components of the schemes $(Z_i)_{s_i} \times_{\mathrm{Spec}(\kappa(s_i))} \mathrm{Spec}(\kappa(s))$ have dimension $> d$ and contain z we conclude that Z_s contains an irreducible component of dimension $> d$ passing through z which contradicts the fact that $Z_s \subset X_s$ and $\dim(X_s) \leq d$. □

Lemma 31.14.2. *Notation and assumptions as in Situation 31.7.1. If*

(1) *f is a quasi-finite morphism, and*
(2) *f_0 is locally of finite type,*

then there exists an $i \geq 0$ such that f_i is quasi-finite.

Proof. Follows immediately from Lemma 31.14.1. □

Lemma 31.14.3. *Let S be a quasi-compact and quasi-separated scheme. Let $f : X \to S$ be a morphism of finite presentation. Let $d \geq 0$ be an integer. If $Z \subset X$ be a closed subscheme such that $\dim(Z_s) \leq d$ for all $s \in S$, then there exists a closed subscheme $Z' \subset X$ such that*

(1) *$Z \subset Z'$,*
(2) *$Z' \to X$ is of finite presentation, and*
(3) *$\dim(Z'_s) \leq d$ for all $s \in S$.*

Proof. By Proposition 31.4.4 we can write $S = \lim S_i$ as the limit of a directed inverse system of Noetherian schemes with affine transition maps. By Lemma 31.9.1 we may assume that there exist a system of morphisms $f_i : X_i \to S_i$ of finite presentation such that $X_{i'} = X_i \times_{S_i} S_{i'}$ for all $i' \geq i$ and such that $X = X_i \times_{S_i} S$. Let $Z_i \subset X_i$ be the scheme theoretic image of $Z \to X \to X_i$. Then for $i' \geq i$ the

morphism $X_{i'} \to X_i$ maps $Z_{i'}$ into Z_i and the induced morphism $Z_{i'} \to Z_i \times_{S_i} S_{i'}$ is a closed immersion. By Lemma 31.14.1 we see that the dimension of the fibres of $Z_i \to S_i$ all have dimension $\leq d$ for a suitable $i \in I$. Fix such an i and set $Z' = Z_i \times_{S_i} S \subset X$. Since S_i is Noetherian, we see that X_i is Noetherian, and hence the morphism $Z_i \to X_i$ is of finite presentation. Therefore also the base change $Z' \to X$ is of finite presentation. Moreover, the fibres of $Z' \to S$ are base changes of the fibres of $Z_i \to S_i$ and hence have dimension $\leq d$. □

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CHAPTER 32

Varieties

32.1. Introduction

In this chapter we start studying varieties and more generally schemes over a field. A fundamental reference is [**DG67**].

32.2. Notation

Throughout this chapter we use the letter k to denote the ground field.

32.3. Varieties

In the stacks project we will use the following as our definition of a variety.

Definition 32.3.1. Let k be a field. A *variety* is a scheme X over k such that X is integral and the structure morphism $X \to \mathrm{Spec}(k)$ is separated and of finite type.

This definition has the following drawback. Suppose that $k \subset k'$ is an extension of fields. Suppose that X is a variety over k. Then the base change $X_{k'} = X \times_{\mathrm{Spec}(k)} \mathrm{Spec}(k')$ is not necessarily a variety over k'. This phenomenon (in greater generality) will be discussed in detail in the following sections. The product of two varieties need not be a variety (this is really the same phenomenon). Here is an example.

Example 32.3.2. Let $k = \mathbf{Q}$. Let $X = \mathrm{Spec}(\mathbf{Q}(i))$ and $Y = \mathrm{Spec}(\mathbf{Q}(i))$. Then the product $X \times_{\mathrm{Spec}(k)} Y$ of the varieties X and Y is not a variety, since it is reducible. (It is isomorphic to the disjoint union of two copies of X.)

If the ground field is algebraically closed however, then the product of varieties is a variety. This follows from the results in the algebra chapter, but there we treat much more general situations. There is also a simple direct proof of it which we present here.

Lemma 32.3.3. *Let k be an algebraically closed field. Let X, Y be varieties over k. Then $X \times_{\mathrm{Spec}(k)} Y$ is a variety over k.*

Proof. The morphism $X \times_{\mathrm{Spec}(k)} Y \to \mathrm{Spec}(k)$ is of finite type and separated because it is the composition of the morphisms $X \times_{\mathrm{Spec}(k)} Y \to Y \to \mathrm{Spec}(k)$ which are separated and of finite type, see Morphisms, Lemmas 28.16.4 and 28.16.3 and Schemes, Lemma 25.21.13. To finish the proof it suffices to show that $X \times_{\mathrm{Spec}(k)} Y$ is integral. Let $X = \bigcup_{i=1,\ldots,n} U_i$, $Y = \bigcup_{j=1,\ldots,m} V_j$ be finite affine open coverings. If we can show that each $U_i \times_{\mathrm{Spec}(k)} V_j$ is integral, then we are done by Properties, Lemmas 27.3.2, 27.3.3, and 27.3.4. This reduces us to the affine case.

The affine case translates into the following algebra statement: Suppose that A, B are integral domains and finitely generated k-algebras. Then $A \otimes_k B$ is an integral domain. To get a contradiction suppose that

$$(\sum\nolimits_{i=1,\ldots,n} a_i \otimes b_i)(\sum\nolimits_{j=1,\ldots,m} c_j \otimes d_j) = 0$$

in $A \otimes_k B$ with both factors nonzero in $A \otimes_k B$. We may assume that $b_1, \ldots, b_n$ are k-linearly independent in B, and that $d_1, \ldots, d_m$ are k-linearly independent in B. Of course we may also assume that a_1 and c_1 are nonzero in A. Hence $D(a_1c_1) \subset \mathrm{Spec}(A)$ is nonempty. By the Hilbert Nullstellensatz (Algebra, Theorem 10.31.1) we can find a maximal ideal $\mathfrak{m} \subset A$ contained in $D(a_1c_1)$ and $A/\mathfrak{m} = k$ as k is algebraically closed. Denote $\overline{a}_i, \overline{c}_j$ the residue classes of a_i, c_j in $A/\mathfrak{m} = k$. Then equation above becomes

$$(\sum\nolimits_{i=1,\ldots,n} \overline{a}_i b_i)(\sum\nolimits_{j=1,\ldots,m} \overline{c}_j d_j) = 0$$

which is a contradiction with $\mathfrak{m} \in D(a_1c_1)$, the linear independence of $b_1, \ldots, b_n$ and $d_1, \ldots, d_m$, and the fact that B is a domain. □

32.4. Geometrically reduced schemes

If X is a reduced scheme over a field, then it can happen that X becomes nonreduced after extending the ground field. This does not happen for geometrically reduced schemes.

Definition 32.4.1. Let k be a field. Let X be a scheme over k. Let $x \in X$ be a point.

(1) Let $x \in X$ be a point. We say X is *geometrically reduced at* x if for any field extension $k \subset k'$ and any point $x' \in X_{k'}$ lying over x the local ring $\mathcal{O}_{X_{k'},x'}$ is reduced.
(2) We say X is *geometrically reduced* over k if X is geometrically reduced at every point of X.

This may seem a little mysterious at first, but it is really the same thing as the notion discussed in the algebra chapter. Here are some basic results explaining the connection.

Lemma 32.4.2. *Let k be a field. Let X be a scheme over k. Let $x \in X$. The following are equivalent*

(1) *X is geometrically reduced at x, and*
(2) *the ring $\mathcal{O}_{X,x}$ is geometrically reduced over k (see Algebra, Definition 10.40.1).*

Proof. Assume (1). This in particular implies that $\mathcal{O}_{X,x}$ is reduced. Let $k \subset k'$ be a finite purely inseparable field extension. Consider the ring $\mathcal{O}_{X,x} \otimes_k k'$. By Algebra, Lemma 10.43.2 its spectrum is the same as the spectrum of $\mathcal{O}_{X,x}$. Hence it is a local ring also (Algebra, Lemma 10.17.2). Therefore there is a unique point $x' \in X_{k'}$ lying over x and $\mathcal{O}_{X_{k'},x'} \cong \mathcal{O}_{X,x} \otimes_k k'$. By assumption this is a reduced ring. Hence we deduce (2) by Algebra, Lemma 10.41.3.

Assume (2). Let $k \subset k'$ be a field extension. Since $\mathrm{Spec}(k') \to \mathrm{Spec}(k)$ is surjective, also $X_{k'} \to X$ is surjective (Morphisms, Lemma 28.11.4). Let $x' \in X_{k'}$ be any point lying over x. The local ring $\mathcal{O}_{X_{k'},x'}$ is a localization of the ring $\mathcal{O}_{X,x} \otimes_k k'$. Hence it is reduced by assumption and (1) is proved. □

The notion isn't interesting in characteristic zero.

Lemma 32.4.3. *Let X be a scheme over a perfect field k (e.g. k has characteristic zero). Let $x \in X$. If $\mathcal{O}_{X,x}$ is reduced, then X is geometrically reduced at x. If X is reduced, then X is geometrically reduced over k.*

Proof. The first statement follows from Lemma 32.4.2 and Algebra, Lemma 10.40.6 and the definition of a perfect field (Algebra, Definition 10.42.1). The second statement follows from the first. □

Lemma 32.4.4. *Let k be a field of characteristic $p > 0$. Let X be a scheme over k. The following are equivalent*

1. *X is geometrically reduced,*
2. *$X_{k'}$ is reduced for every field extension $k \subset k'$,*
3. *$X_{k'}$ is reduced for every finite purely inseparable field extension $k \subset k'$,*
4. *$X_{k^{1/p}}$ is reduced,*
5. *$X_{k^{perf}}$ is reduced,*
6. *$X_{\bar{k}}$ is reduced,*
7. *for every affine open $U \subset X$ the ring $\mathcal{O}_X(U)$ is geometrically reduced (see Algebra, Definition 10.40.1).*

Proof. Assume (1). Then for every field extension $k \subset k'$ and every point $x' \in X_{k'}$ the local ring of $X_{k'}$ at x' is reduced. In other words $X_{k'}$ is reduced. Hence (2).

Assume (2). Let $U \subset X$ be an affine open. Then for every field extension $k \subset k'$ the scheme $X_{k'}$ is reduced, hence $U_{k'} = \mathrm{Spec}(\mathcal{O}(U) \otimes_k k')$ is reduced, hence $\mathcal{O}(U) \otimes_k k'$ is reduced (see Properties, Section 27.3). In other words $\mathcal{O}(U)$ is geometrically reduced, so (7) holds.

Assume (7). For any field extension $k \subset k'$ the base change $X_{k'}$ is gotten by gluing the spectra of the rings $\mathcal{O}_X(U) \otimes_k k'$ where U is affine open in X (see Schemes, Section 25.17). Hence $X_{k'}$ is reduced. So (1) holds.

This proves that (1), (2), and (7) are equivalent. These are equivalent to (3), (4), (5), and (6) because we can apply Algebra, Lemma 10.41.3 to $\mathcal{O}_X(U)$ for $U \subset X$ affine open. □

Lemma 32.4.5. *Let k be a field of characteristic $p > 0$. Let X be a scheme over k. Let $x \in X$. The following are equivalent*

1. *X is geometrically reduced at x,*
2. *$\mathcal{O}_{X_{k'},x'}$ is reduced for every finite purely inseparable field extension k' of k and $x' \in X_{k'}$ the unique point lying over x,*
3. *$\mathcal{O}_{X_{k^{1/p}},x'}$ is reduced for $x' \in X_{k'}$ the unique point lying over x, and*
4. *$\mathcal{O}_{X_{k^{perf}},x'}$ is reduced for $x' \in X_{k^{perf}}$ the unique point lying over x.*

Proof. Note that if $k \subset k'$ is purely inseparable, then $X_{k'} \to X$ induces a homeomorphism on underlying topological spaces, see Algebra, Lemma 10.43.2. Whence the uniqueness of x' lying over x mentioned in the statement. Moreover, in this case $\mathcal{O}_{X_{k'},x'} = \mathcal{O}_{X,x} \otimes_k k'$. Hence the lemma follows from Lemma 32.4.2 above and Algebra, Lemma 10.41.3. □

Lemma 32.4.6. *Let k be a field. Let X be a scheme over k. Let k'/k be a field extension. Let $x \in X$ be a point, and let $x' \in X_{k'}$ be a point lying over x. The following are equivalent*

(1) X *is geometrically reduced at* x,
(2) $X_{k'}$ *is geometrically reduced at* x'.

In particular, X *is geometrically reduced over* k *if and only if* $X_{k'}$ *is geometrically reduced over* k'.

Proof. It is clear that (1) implies (2). Assume (2). Let $k \subset k''$ be a finite purely inseparable field extension and let $x'' \in X_{k''}$ be a point lying over x (actually it is unique). We can find a common field extension $k \subset k'''$ (i.e. with both $k' \subset k'''$ and $k'' \subset k'''$) and a point $x''' \in X_{k'''}$ lying over both x' and x''. Consider the map of local rings

$$\mathcal{O}_{X_{k''},x''} \longrightarrow \mathcal{O}_{X_{k'''},x'''}.$$

This is a flat local ring homomorphism and hence faithfully flat. By (2) we see that the local ring on the right is reduced. Thus by Algebra, Lemma 10.150.2 we conclude that $\mathcal{O}_{X_{k''},x''}$ is reduced. Thus by Lemma 32.4.5 we conclude that X is geometrically reduced at x. □

Lemma 32.4.7. *Let* k *be a field. Let* X, Y *be schemes over* k.

(1) *If* X *is geometrically reduced at* x, *and* Y *reduced, then* $X \times_k Y$ *is reduced at every point lying over* x.
(2) *If* X *geometrically reduced over* k *and* Y *reduced. Then* $X \times_k Y$ *is reduced.*

Proof. Combine, Lemmas 32.4.2 and 32.4.4 and Algebra, Lemma 10.40.5. □

Lemma 32.4.8. *Let* k *be a field. Let* X *be a scheme over* k.

(1) *If* $x' \rightsquigarrow x$ *is a specialization and* X *is geometrically reduced at* x, *then* X *is geometrically reduced at* x'.
(2) *If* $x \in X$ *such that (a)* $\mathcal{O}_{X,x}$ *is reduced, and (b) for each specialization* $x' \rightsquigarrow x$ *where* x' *is a generic point of an irreducible component of* X *the scheme* X *is geometrically reduced at* x', *then* X *is geometrically reduced at* x.
(3) *If* X *is reduced and geometrically reduced at all generic points of irreducible components of* X, *then* X *is geometrically reduced.*

Proof. Part (1) follows from Lemma 32.4.2 and the fact that if A is a geometrically reduced k-algebra, then $S^{-1}A$ is a geometrically reduced k-algebra for any multiplicative subset S of A, see Algebra, Lemma 10.40.3.

Let $A = \mathcal{O}_{X,x}$. The assumptions (a) and (b) of (2) imply that A is reduced, and that $A_{\mathfrak{q}}$ is geometrically reduced over k for every minimal prime $\mathfrak{q}$ of A. Hence A is geometrically reduced over k, see Algebra, Lemma 10.40.7. Thus X is geometrically reduced at x, see Lemma 32.4.2.

Part (3) follows trivially from part (2). □

Lemma 32.4.9. *Let* k *be a field. Let* X *be a scheme over* k. *Let* $x \in X$. *Assume* X *locally Noetherian and geometrically reduced at* x. *Then there exists an open neighbourhood* $U \subset X$ *of* x *which is geometrically reduced over* k.

Proof. Let R be a Noetherian k-algebra. Let $\mathfrak{p} \subset R$ be a prime. Let $I = \mathrm{Ker}(R \to R_{\mathfrak{p}}$. Since $IR_{\mathfrak{p}} = R_{\mathfrak{p}}$ and I is finitely generated there exists an $f \in R$, $f \notin \mathfrak{p}$ such that $fI = 0$. Hence $R_f \subset R_{\mathfrak{p}}$.

Assume X locally Noetherian and geometrically reduced at x. If we apply the above to $R = \mathcal{O}_X(U)$ for some affine open neighbourhood of x, and $\mathfrak{p} \subset R$ the prime corresponding to x, then we see that after shrinking U we may assume $R \subset R_{\mathfrak{p}}$. By Lemma 32.4.2 the assumption means that $R_{\mathfrak{p}}$ is geometrically reduced over k. By Algebra, Lemma 10.40.2 this implies that R is geometrically reduced over k, which in turn implies that U is geometrically reduced. □

Example 32.4.10. Let $k = \mathbf{F}_p(s,t)$, i.e., a purely transcendental extension of the prime field. Consider the variety $X = \mathrm{Spec}(k[x,y]/(1 + sx^p + ty^p))$. Let $k \subset k'$ be any extension such that both s and t have a pth root in k'. Then the base change $X_{k'}$ is not reduced. Namely, the ring $k'[x,y]/(1 + sx^p + ty^p)$ contains the element $1 + s^{1/p}x + t^{1/p}y$ whose pth power is zero but which is not zero (since the ideal $(1 + sx^p + ty^p)$ certainly does not contain any nonzero element of degree $< p$).

Lemma 32.4.11. *Let k be a field. Let $X \to \mathrm{Spec}(k)$ be locally of finite type. Assume X has finitely many irreducible components. Then there exists a finite purely inseparable extension $k \subset k'$ such that $(X_{k'})_{red}$ is geometrically reduced over k'.*

Proof. To prove this lemma we may replace X by its reduction X_{red}. Hence we may assume that X is reduced and locally of finite type over k. Let $x_1, \ldots, x_n \in X$ be the generic points of the irreducible components of X. Note that for every purely inseparable algebraic extension $k \subset k'$ the morphism $(X_{k'})_{red} \to X$ is a homeomorphism, see Algebra, Lemma 10.43.2. Hence the points $x'_1, \ldots, x'_n$ lying over $x_1, \ldots, x_n$ are the generic points of the irreducible components of $(X_{k'})_{red}$. As X is reduced the local rings $K_i = \mathcal{O}_{X,x_i}$ are fields, see Algebra, Lemma 10.24.4. As X is locally of finite type over k the field extensions $k \subset K_i$ are finitely generated field extensions. Finally, the local rings $\mathcal{O}_{X_{k'},x'_i}$ are the fields $(K_i \otimes_k k')_{red}$. By Algebra, Lemma 10.42.3 we can find a finite purely inseparable extension $k \subset k'$ such that $(K_i \otimes_k k')_{red}$ are separable field extensions of k'. In particular each $(K_i \otimes_k k')_{red}$ is geometrically reduced over k' by Algebra, Lemma 10.41.1. At this point Lemma 32.4.8 part (3) implies that $(X_{k'})_{red}$ is geometrically reduced. □

32.5. Geometrically connected schemes

If X is a connected scheme over a field, then it can happen that X becomes disconnected after extending the ground field. This does not happen for geometrically connected schemes.

Definition 32.5.1. Let X be a scheme over the field k. We say X is *geometrically connected* over k if the scheme $X_{k'}$ is connected for every field extension k' of k.

By convention a connected topological space is nonempty; hence a fortiori geometrically connected schemes are nonempty. Here is an example of a variety which is not geometrically connected.

Example 32.5.2. Let $k = \mathbf{Q}$. The scheme $X = \mathrm{Spec}(\mathbf{Q}(i))$ is a variety over $\mathrm{Spec}(\mathbf{Q})$. But the base change $X_{\mathbf{C}}$ is the spectrum of $\mathbf{C} \otimes_{\mathbf{Q}} \mathbf{Q}(i) \cong \mathbf{C} \times \mathbf{C}$ which is the disjoint union of two copies of $\mathrm{Spec}(\mathbf{C})$. So in fact, this is an example of a non-geometrically connected variety.

Lemma 32.5.3. *Let X be a scheme over the field k. Let $k \subset k'$ be a field extension. Then X is geometrically connected over k if and only if $X_{k'}$ is geometrically connected over k'.*

Proof. If X is geometrically connected over k, then it is clear that $X_{k'}$ is geometrically connected over k'. For the converse, note that for any field extension $k \subset k''$ there exists a common field extension $k' \subset k'''$ and $k'' \subset k'''$. As the morphism $X_{k'''} \to X_{k''}$ is surjective (as a base change of a surjective morphism between spectra of fields) we see that the connectedness of $X_{k'''}$ implies the connectedness of $X_{k''}$. Thus if $X_{k'}$ is geometrically connected over k' then X is geometrically connected over k. □

Lemma 32.5.4. *Let k be a field. Let X, Y be schemes over k. Assume X is geometrically connected over k. Then the projection morphism*

$$p : X \times_k Y \longrightarrow Y$$

induces a bijection between connected components.

Proof. The scheme theoretic fibres of p are connected, since they are base changes of the geometrically connected scheme X by field extensions. Moreover the scheme theoretic fibres are homeomorphic to the set theoretic fibres, see Schemes, Lemma 25.18.5. By Morphisms, Lemma 28.24.4 the map p is open. Thus we may apply Topology, Lemma 5.6.5 to conclude. □

Lemma 32.5.5. *Let k be a field. Let A be a k-algebra. Then $X = \mathrm{Spec}(A)$ is geometrically connected over k if and only if A is geometrically connected over k (see Algebra, Definition 10.44.3).*

Proof. Immediate from the definitions. □

Lemma 32.5.6. *Let $k \subset k'$ be an extension of fields. Let X be a scheme over k. Assume k separably algebraically closed. Then the morphism $X_{k'} \to X$ induces a bijection of connected components. In particular, X is geometrically connected over k if and only if X is connected.*

Proof. Since k is separably algebraically closed we see that k' is geometrically connected over k, see Algebra, Lemma 10.44.4. Hence $Z = \mathrm{Spec}(k')$ is geometrically connected over k by Lemma 32.5.5 above. Since $X_{k'} = Z \times_k X$ the result is a special case of Lemma 32.5.4. □

Lemma 32.5.7. *Let k be a field. Let X be a scheme over k. Let $\overline{k}$ be a separable algebraic closure of k. Then X is geometrically connected if and only if the base change $X_{\overline{k}}$ is connected.*

Proof. Assume $X_{\overline{k}}$ is connected. Let $k \subset k'$ be a field extension. There exists a field extension $\overline{k} \subset \overline{k}'$ such that k' embeds into $\overline{k}'$ as an extension of k. By Lemma 32.5.6 we see that $X_{\overline{k}'}$ is connected. Since $X_{\overline{k}'} \to X_{k'}$ is surjective we conclude that $X_{k'}$ is connected as desired. □

Lemma 32.5.8. *Let k be a field. Let X be a scheme over k. Let A be a k-algebra. Let $V \subset X_A$ be a quasi-compact open. Then there exists a finitely generated k-subalgebra $A' \subset A$ and a quasi-compact open $V' \subset X_{A'}$ such that $V = V'_A$.*

Proof. We remark that if X is also quasi-separated this follows from Limits, Lemma 31.3.8. Let $U_1, \ldots, U_n$ be finitely many affine opens of X such that $V \subset \bigcup U_{i,A}$. Say $U_i = \mathrm{Spec}(R_i)$. Since V is quasi-compact we can find finitely many $f_{ij} \in R_i \otimes_k A$, $j = 1, \ldots, n_i$ such that $V = \bigcup_i \bigcup_{j=1,\ldots,n_i} D(f_{ij})$ where $D(f_{ij}) \subset U_{i,A}$ is the corresponding standard open. (We do not claim that $V \cap U_{i,A}$ is the union

of the $D(f_{ij})$, $j = 1, \ldots, n_i$.) It is clear that we can find a finitely generated k-subalgebra $A' \subset A$ such that f_{ij} is the image of some $f'_{ij} \in R_i \otimes_k A'$. Set $V' = \bigcup D(f'_{ij})$ which is a quasi-compact open of $X_{A'}$. Denote $\pi : X_A \to X_{A'}$ the canonical morphism. We have $\pi(V) \subset V'$ as $\pi(D(f_{ij})) \subset D(f'_{ij})$. If $x \in X_A$ with $\pi(x) \in V'$, then $\pi(x) \in D(f'_{ij})$ for some i, j and we see that $x \in D(f_{ij})$ as f'_{ij} maps to f_{ij}. Thus we see that $V = \pi^{-1}(V')$ as desired. □

Let k be a field. Let $k \subset \overline{k}$ be a (possibly infinite) Galois extension. For example $\overline{k}$ could be the separable algebraic closure of k. For any $\sigma \in \mathrm{Gal}(\overline{k}/k)$ we get a corresponding automorphism $\mathrm{Spec}(\sigma) : \mathrm{Spec}(\overline{k}) \longrightarrow \mathrm{Spec}(\overline{k})$. Note that $\mathrm{Spec}(\sigma) \circ \mathrm{Spec}(\tau) = \mathrm{Spec}(\tau \circ \sigma)$. Hence we get an action

$$\mathrm{Gal}(\overline{k}/k)^{opp} \times \mathrm{Spec}(\overline{k}) \longrightarrow \mathrm{Spec}(\overline{k})$$

of the opposite group on the scheme $\mathrm{Spec}(\overline{k})$. Let X be a scheme over k. Since $X_{\overline{k}} = \mathrm{Spec}(\overline{k}) \times_{\mathrm{Spec}(k)} X$ by definition we see that the action above induces a canonical action

$$\mathrm{Gal}(\overline{k}/k)^{opp} \times X_{\overline{k}} \longrightarrow X_{\overline{k}}. \tag{32.5.8.1}$$

Lemma 32.5.9. *Let k be a field. Let X be a scheme over k. Let $\overline{k}$ be a (possibly infinite) Galois extension of k. Let $V \subset X_{\overline{k}}$ be a quasi-compact open. Then*

1. *there exists a finite subextension $k \subset k' \subset \overline{k}$ and a quasi-compact open $V' \subset X_{k'}$ such that $V = (V')_{\overline{k}}$,*
2. *there exists an open subgroup $H \subset Gal(\overline{k}/k)$ such that $\sigma(V) = V$ for all $\sigma \in H$.*

Proof. By Lemma 32.5.8 there exists a finite subextension $k \subset k' \subset \overline{k}$ and an open $V' \subset X_{k'}$ which pulls back to V. This proves (1). Since $\mathrm{Gal}(\overline{k}/k')$ is open in $\mathrm{Gal}(\overline{k}/k)$ part (2) is clear as well. □

Lemma 32.5.10. *Let k be a field. Let $k \subset \overline{k}$ be a (possibly infinite) Galois extension. Let X be a scheme over k. Let $\overline{T} \subset X_{\overline{k}}$ have the following properties*

1. *$\overline{T}$ is a closed subset of $X_{\overline{k}}$,*
2. *for every $\sigma \in Gal(\overline{k}/k)$ we have $\sigma(\overline{T}) = \overline{T}$.*

Then there exists a closed subset $T \subset X$ whose inverse image in $X_{k'}$ is $\overline{T}$.

Proof. This lemma immediately reduces to the case where $X = \mathrm{Spec}(A)$ is affine. In this case, let $\overline{I} \subset A \otimes_k \overline{k}$ be the radical ideal corresponding to $\overline{T}$. Assumption (2) implies that $\sigma(\overline{I}) = \overline{I}$ for all $\sigma \in \mathrm{Gal}(\overline{k}/k)$. Pick $x \in \overline{I}$. There exists a finite Galois extension $k \subset k'$ contained in $\overline{k}$ such that $x \in A \otimes_k k'$. Set $G = \mathrm{Gal}(k'/k)$. Set

$$P(T) = \prod\nolimits_{\sigma \in G} (T - \sigma(x)) \in (A \otimes_k k')[T]$$

It is clear that $P(T)$ is monic and is actually an element of $(A \otimes_k k')^G[T] = A[T]$ (by basic Galois theory). Moreover, if we write $P(T) = T^d + a_1 T^{d-1} + \ldots + a_0$ the we see that $a_i \in I := A \cap \overline{I}$. By Algebra, Lemma 10.35.5 we see that x is contained in the radical of $I(A \otimes_k \overline{k})$. Hence $\overline{I}$ is the radical of $I(A \otimes_k \overline{k})$ and setting $T = V(I)$ is a solution. □

Lemma 32.5.11. *Let k be a field. Let X be a scheme over k. The following are equivalent*

(1) *X is geometrically connected,*
(2) *for every finite separable field extension $k \subset k'$ the scheme $X_{k'}$ is connected.*

Proof. It follows immediately from the definition that (1) implies (2). Assume that X is not geometrically connected. Let $k \subset \overline{k}$ be a separable algebraic closure of k. By Lemma 32.5.7 it follows that $X_{\overline{k}}$ is disconnected. Say $X_{\overline{k}} = \overline{U} \amalg \overline{V}$ with $\overline{U}$ and $\overline{V}$ open, closed, and nonempty.

Suppose that $W \subset X$ is any quasi-compact open. Then $W_{\overline{k}} \cap \overline{U}$ and $W_{\overline{k}} \cap \overline{V}$ are open and closed in $W_{\overline{k}}$. In particular $W_{\overline{k}} \cap \overline{U}$ and $W_{\overline{k}} \cap \overline{V}$ are quasi-compact, and by Lemma 32.5.9 both $W_{\overline{k}} \cap \overline{U}$ and $W_{\overline{k}} \cap \overline{V}$ are defined over a finite subextension and invariant under an open subgroup of $\mathrm{Gal}(\overline{k}/k)$. We will use this without further mention in the following.

Pick $W_0 \subset X$ quasi-compact open such that both $W_{0,\overline{k}} \cap \overline{U}$ and $W_{0,\overline{k}} \cap \overline{V}$ are nonempty. Choose a finite subextension $k \subset k' \subset \overline{k}$ and a decomposition $W_{0,k'} = U'_0 \amalg V'_0$ into open and closed subsets such that $W_{0,\overline{k}} \cap \overline{U} = (U'_0)_{\overline{k}}$ and $W_{0,\overline{k}} \cap \overline{V} = (V'_0)_{\overline{k}}$. Let $H = \mathrm{Gal}(\overline{k}/k') \subset \mathrm{Gal}(\overline{k}/k)$. In particular $\sigma(W_{0,\overline{k}} \cap \overline{U}) = W_{0,\overline{k}} \cap \overline{U}$ and similarly for $\overline{V}$.

Having chosen W_0, k' as above, for every quasi-compact open $W \subset X$ we set

$$U_W = \bigcap\nolimits_{\sigma \in H} \sigma(W_{\overline{k}} \cap \overline{U}), \quad V_W = \bigcup\nolimits_{\sigma \in H} \sigma(W_{\overline{k}} \cap \overline{V}).$$

Now, since $W_{\overline{k}} \cap \overline{U}$ and $W_{\overline{k}} \cap \overline{V}$ are fixed by an open subgroup of $\mathrm{Gal}(\overline{k}/k)$ we see that the union and intersection above are finite. Hence U_W and V_W are both open and closed. Also, by construction $W_{\overline{k}} = U_W \amalg V_W$.

We claim that if $W \subset W' \subset X$ are quasi-compact open, then $W_{\overline{k}} \cap U_{W'} = U_W$ and $W_{\overline{k}} \cap V_{W'} = V_W$. Verification omitted. Hence we see that upon defining $U = \bigcup_{W \subset X} U_W$ and $V = \bigcup_{W \subset X} V_W$ we obtain $X_{\overline{k}} = U \amalg V$ is a disjoint union of open and closed subsets. It is clear that V is nonempty as it is constructed by taking unions (locally). On the other hand, U is nonempty since it contains $W_0 \cap \overline{U}$ by construction. Finally, $U, V \subset X_{\overline{k}}$ are closed and H-invariant by construction. Hence by Lemma 32.5.10 we have $U = (U')_{\overline{k}}$, and $V = (V')_{\overline{k}}$ for some closed $U', V' \subset X_{k'}$. Clearly $X_{k'} = U' \amalg V'$ and we see that $X_{k'}$ is disconnected as desired. □

Lemma 32.5.12. *Let k be a field. Let $k \subset \overline{k}$ be a (possibly infinite) Galois extension. Let $f : T \to X$ be a morphism of schemes over k. Assume $T_{\overline{k}}$ connected and $X_{\overline{k}}$ disconnected. Then X is disconnected.*

Proof. Write $X_{\overline{k}} = \overline{U} \coprod \overline{V}$ with $\overline{U}$ and $\overline{V}$ open and closed. Denote $\overline{f} : T_{\overline{k}} \to X_{\overline{k}}$ the base change of f. Since $T_{\overline{k}}$ is connected we see that $T_{\overline{k}}$ is contained in either $\overline{f}^{-1}(\overline{U})$ or $\overline{f}^{-1}(\overline{V})$. Say $T_{\overline{k}} \subset \overline{f}^{-1}(\overline{U})$.

Fix a quasi-compact open $W \subset X$. There exists a finite Galois subextension $k \subset k' \subset \overline{k}$ such that $\overline{U} \cap W_{\overline{k}}$ and $\overline{V} \cap W_{\overline{k}}$ come from quasi-compact opens $U', V' \subset W_{k'}$. Then also $W_{k'} = U' \coprod V'$. Consider

$$U'' = \bigcap\nolimits_{\sigma \in \mathrm{Gal}(k'/k)} \sigma(U'), \quad V'' = \bigcup\nolimits_{\sigma \in \mathrm{Gal}(k'/k)} \sigma(V').$$

These are Galois invariant, open and closed, and $W_{k'} = U'' \coprod V''$. By Lemma 32.5.10 we get open and closed subsets $U_W, V_W \subset W$ such that $U'' = (U_W)_{k'}$, $V'' = (V_W)_{k'}$ and $W = U_W \coprod V_W$.

We claim that if $W \subset W' \subset X$ are quasi-compact open, then $W \cap U_{W'} = U_W$ and $W \cap V_{W'} = V_W$. Verification omitted. Hence we see that upon defining $U = \bigcup_{W \subset X} U_W$ and $V = \bigcup_{W \subset X} V_W$ we obtain $X = U \coprod V$. It is clear that V is nonempty as it is constructed by taking unions (locally). On the other hand, U is nonempty since it contains $f(T)$ by construction. □

Lemma 32.5.13. *Let k be a field. Let $T \to X$ be a morphism of schemes over k. Assume T is geometrically connected and X connected. Then X is geometrically connected.*

Proof. This is a reformulation of Lemma 32.5.12. □

Lemma 32.5.14. *Let k be a field. Let X be a scheme over k. Assume X is connected and has a point x such that k is algebraically closed in $\kappa(x)$. Then X is geometrically connected. In particular, if X has a k-rational point and X is connected, then X is geometrically connected.*

Proof. Set $T = \mathrm{Spec}(\kappa(x))$. Let $k \subset \overline{k}$ be a separable algebraic closure of k. The assumption on $k \subset \kappa(x)$ implies that $T_{\overline{k}}$ is irreducible, see Algebra, Lemma 10.43.10. Hence by Lemma 32.5.13 we see that $X_{\overline{k}}$ is connected. By Lemma 32.5.7 we conclude that X is geometrically connected. □

Lemma 32.5.15. *Let $k \subset K$ be an extension of fields. Let X be a scheme over k. For every connected component T of X the inverse image $T_K \subset X_K$ is a union of connected components of X_K.*

Proof. This is a purely topological statement. Denote $p : X_K \to X$ the projection morphism. Let $T \subset X$ be a connected component of X. Let $t \in T_K = p^{-1}(T)$. Let $C \subset X_K$ be a connected component containing t. Then $p(C)$ is a connected subset of X which meets T, hence $p(C) \subset T$. Hence $C \subset T_K$. □

Lemma 32.5.16. *Let $k \subset K$ be a finite extension of fields and let X be a scheme over k. Denote by $p : X_K \to X$ the projection morphism. For every connected component T of X_K the image $p(T)$ is a connected component of X.*

Proof. The image $p(T)$ is contained in some connected component X' of X. Consider X' as a closed subscheme of X in any way. Then T is also a connected component of $X'_K = p^{-1}(X')$ and we may therefore assume that X is connected. The morphism p is open (Morphisms, Lemma 28.24.4), closed (Morphisms, Lemma 28.44.7) and the fibers of p are finite sets (Morphisms, Lemma 28.44.9). Thus we may apply Topology, Lemma 5.6.6 to conclude. □

Remark 32.5.17. Let $k \subset K$ be an extension of fields. Let X be a scheme over k. Denote $p : X_K \to X$ the projection morphism. Let $\overline{T} \subset X_K$ be a connected component. Is it true that $p(\overline{T})$ is a connected component of X? When $k \subset K$ is finite Lemma 32.5.16 tells us the answer is "yes". In general we do not know the answer. If you do, or if you have a reference, please email stacks.project@gmail.com.

Let X be a scheme. We denote $\pi_0(X)$ the set of connected components of X.

Lemma 32.5.18. *Let k be a field, with separable algebraic closure $\overline{k}$. Let X be a scheme over k. There is an action*

$$Gal(\overline{k}/k)^{opp} \times \pi_0(X_{\overline{k}}) \longrightarrow \pi_0(X_{\overline{k}})$$

with the following properties:

(1) *An element $\overline{T} \in \pi_0(X_{\overline{k}})$ is fixed by the action if and only if there exists a connected component $T \subset X$, which is geometrically connected over k, such that $T_{\overline{k}} = \overline{T}$.*
(2) *For any field extension $k \subset k'$ with separable algebraic closure $\overline{k}'$ the diagram*

$$\begin{array}{ccc} Gal(\overline{k}'/k') \times \pi_0(X_{\overline{k}'}) & \longrightarrow & \pi_0(X_{\overline{k}'}) \\ \downarrow & & \downarrow \\ Gal(\overline{k}/k) \times \pi_0(X_{\overline{k}}) & \longrightarrow & \pi_0(X_{\overline{k}}) \end{array}$$

is commutative (where the right vertical arrow is a bijection according to Lemma 32.5.6).

Proof. The action (32.5.8.1) of $\mathrm{Gal}(\overline{k}/k)$ on $X_{\overline{k}}$ induces an action on its connected components. Connected components are always closed (Topology, Lemma 5.6.3). Hence if $\overline{T}$ is as in (1), then by Lemma 32.5.10 there exists a closed subset $T \subset X$ such that $\overline{T} = T_{\overline{k}}$. Note that T is geometrically connected over k, see Lemma 32.5.7. To see that T is a connected component of X, suppose that $T \subset T'$, $T \neq T'$ where T' is a connected component of X. In this case $T'_{k'}$ strictly contains $\overline{T}$ and hence is disconnected. By Lemma 32.5.12 this means that T' is disconnected! Contradiction.

We omit the proof of the functoriality in (2). □

Lemma 32.5.19. *Let k be a field, with separable algebraic closure $\overline{k}$. Let X be a scheme over k. Assume*

(1) *X is quasi-compact, and*
(2) *the connected components of $X_{\overline{k}}$ are open.*

Then

(a) *$\pi_0(X_{\overline{k}})$ is finite, and*
(b) *the action of $Gal(\overline{k}/k)$ on $\pi_0(X_{\overline{k}})$ is continuous.*

Moreover, assumptions (1) and (2) are satisfied when X is of finite type over k.

Proof. Since the connected components are open, cover $X_{\overline{k}}$ (Topology, Lemma 5.6.3) and $X_{\overline{k}}$ is quasi-compact, we conclude that there are only finitely many of them. Thus (a) holds. By Lemma 32.5.8 these connected components are each defined over a finite subextension of $k \subset \overline{k}$ and we get (b). If X is of finite type over k, then $X_{\overline{k}}$ is of finite type over $\overline{k}$ (Morphisms, Lemma 28.16.4). Hence $X_{\overline{k}}$ is a Noetherian scheme (Morphisms, Lemma 28.16.6) and has an underlying Noetherian topological space (Properties, Lemma 27.5.5). Thus $X_{\overline{k}}$ has finitely many irreducible components (Topology, Lemma 5.8.2) and a fortiori finitely many connected components (which are therefore open). □

32.6. Geometrically irreducible schemes

If X is an irreducible scheme over a field, then it can happen that X becomes reducible after extending the ground field. This does not happen for geometrically irreducible schemes.

Definition 32.6.1. Let X be a scheme over the field k. We say X is *geometrically irreducible* over k if the scheme $X_{k'}$ is irreducible[1] for any field extension k' of k.

Lemma 32.6.2. *Let X be a scheme over the field k. Let $k \subset k'$ be a field extension. Then X is geometrically irreducible over k if and only if $X_{k'}$ is geometrically irreducible over k'.*

Proof. If X is geometrically irreducible over k, then it is clear that $X_{k'}$ is geometrically irreducible over k'. For the converse, note that for any field extension $k \subset k''$ there exists a common field extension $k' \subset k'''$ and $k'' \subset k'''$. As the morphism $X_{k'''} \to X_{k''}$ is surjective (as a base change of a surjective morphism between spectra of fields) we see that the irreducibility of $X_{k'''}$ implies the irreducibility of $X_{k''}$. Thus if $X_{k'}$ is geometrically irreducible over k' then X is geometrically irreducible over k. □

Lemma 32.6.3. *Let X be a scheme over a separably closed field k. If X is irreducible, then X_K is irreducible for any field extension $k \subset K$. I.e., X is geometrically irreducible over k.*

Proof. Use Properties, Lemma 27.3.3 and Algebra, Lemma 10.43.4. □

Lemma 32.6.4. *Let k be a field. Let X, Y be schemes over k. Assume X is geometrically irreducible over k. Then the projection morphism*

$$p : X \times_k Y \longrightarrow Y$$

induces a bijection between irreducible components.

Proof. First, note that the scheme theoretic fibres of p are irreducible, since they are base changes of the geometrically irreducible scheme X by field extensions. Moreover the scheme theoretic fibres are homeomorphic to the set theoretic fibres, see Schemes, Lemma 25.18.5. By Morphisms, Lemma 28.24.4 the map p is open. Thus we may apply Topology, Lemma 5.7.8 to conclude. □

Lemma 32.6.5. *Let k be a field. Let X be a scheme over k. The following are equivalent*

1. *X is geometrically irreducible over k,*
2. *for every affine open U the k-algebra $\mathcal{O}_X(U)$ is geometrically irreducible over k (see Algebra, Definition 10.43.6),*
3. *X is irreducible and there exists an affine open covering $X = \bigcup U_i$ such that each k-algebra $\mathcal{O}_X(U_i)$ is geometrically irreducible, and*
4. *there exists an open covering $X = \bigcup_{i \in I} X_i$ such that X_i is geometrically irreducible for each i and such that $X_i \cap X_j \neq \emptyset$ for all $i, j \in I$.*

Moreover, if X is geometrically irreducible so is every open subscheme of X.

[1] An irreducible space is nonempty.

Proof. An affine scheme $\mathrm{Spec}(A)$ over k is geometrically irreducible if and only if A is geometrically irreducible over k; this is immediate from the definitions. Recall that if a scheme is irreducible so is every nonempty open subscheme of X, any two nonempty open subsets have a nonempty intersection. Also, if every affine open is irreducible then the scheme is irreducible, see Properties, Lemma 27.3.3. Hence the final statement of the lemma is clear, as well as the implications (1) $\Rightarrow$ (2), (2) $\Rightarrow$ (3), and (3) $\Rightarrow$ (4). If (4) holds, then for any field extension k'/k the scheme $X_{k'}$ has a covering by irreducible opens which pairwise intersect. Hence $X_{k'}$ is irreducible. Hence (4) implies (1). □

Lemma 32.6.6. *Let X be a geometrically irreducible scheme over the field k. Let $\xi \in X$ be its generic point. Then $\kappa(\xi)$ is a geometrically irreducible over k.*

Proof. Combining Lemma 32.6.5 and Algebra, Lemma 10.43.8 we see that $\mathcal{O}_{X,\xi}$ is geometrically irreducible over k. Since $\mathcal{O}_{X,\xi} \to \kappa(\xi)$ is a surjection with locally nilpotent kernel (see Algebra, Lemma 10.24.4) it follows that $\kappa(\xi)$ is geometrically irreducible, see Algebra, Lemma 10.43.2. □

Lemma 32.6.7. *Let $k \subset k'$ be an extension of fields. Let X be a scheme over k. Set $X' = X_{k'}$. Assume k separably algebraically closed. Then the morphism $X' \to X$ induces a bijection of irreducible components.*

Proof. Since k is separably algebraically closed we see that k' is geometrically irreducible over k, see Algebra, Lemma 10.43.7. Hence $Z = \mathrm{Spec}(k')$ is geometrically irreducible over k. by Lemma 32.6.5 above. Since $X' = Z \times_k X$ the result is a special case of Lemma 32.6.4. □

Lemma 32.6.8. *Let k be a field. Let X be a scheme over k. Assume X is quasi-compact. The following are equivalent:*

1. *X is geometrically irreducible over k,*
2. *for every finite separable field extension $k \subset k'$ the scheme $X_{k'}$ is irreducible, and*
3. *$X_{\overline{k}}$ is irreducible, where $k \subset \overline{k}$ is a separable algebraic closure of k.*

Proof. Assume $X_{\overline{k}}$ is irreducible, i.e., assume (3). Let $k \subset k'$ be a field extension. There exists a field extension $\overline{k} \subset \overline{k}'$ such that k' embeds into $\overline{k}'$ as an extension of k. By Lemma 32.6.7 we see that $X_{\overline{k}'}$ is irreducible. Since $X_{\overline{k}'} \to X_{k'}$ is surjective we conclude that $X_{k'}$ is irreducible. Hence (1) holds.

Let $k \subset \overline{k}$ be a separable algebraic closure of k. Assume not (3), i.e., assume $X_{\overline{k}}$ is reducible. Our goal is to show that also $X_{k'}$ is reducible for some finite subextension $k \subset k' \subset \overline{k}$. Let $X = \bigcup_{i \in I} U_i$ be an affine open covering with U_i not empty. If for some i the scheme U_i is reducible, or if for some pair $i \neq j$ the intersection $U_i \cap U_j$ is empty, then X is reducible (Properties, Lemma 27.3.3) and we are done. In particular we may assume that $U_{i,\overline{k}} \cap U_{j,\overline{k}}$ for all $i, j \in I$ is nonempty and we conclude that $U_{i,\overline{k}}$ has to be reducible for some i. According to Algebra, Lemma 10.43.5 this means that $U_{i,k'}$ is reducible for some finite separable field extension $k \subset k'$. Hence also $X_{k'}$ is reducible. Thus we see that (2) implies (3).

The implication (1) $\Rightarrow$ (2) is immediate. This proves the lemma. □

Lemma 32.6.9. *Let $k \subset K$ be an extension of fields. Let X be a scheme over k. For every irreducible component T of X the inverse image $T_K \subset X_K$ is a union of irreducible components of X_K.*

Proof. Let $T \subset X$ be an irreducible component of X. The morphism $T_K \to T$ is flat, so generalizations lift along $T_K \to T$. Hence every $\xi \in T_K$ which is a generic point of an irreducible component of T_K maps to the generic point η of T. If $\xi' \rightsquigarrow \xi$ is a specialization in X_K then ξ' maps to η since there are no points specializing to η in X. Hence $\xi' \in T_K$ and we conclude that $\xi = \xi'$. In other words ξ is the generic point of an irreducible component of X_K. This means that the irreducible components of T_K are all irreducible components of X_K. □

For a scheme X we denote $\text{IrredComp}(X)$ the set of irreducible components of X.

Lemma 32.6.10. *Let $k \subset K$ be an extension of fields. Let X be a scheme over k. For every irreducible component $\overline{T} \subset X_K$ the image of $\overline{T}$ in X is an irreducible component in X. This defines a canonical map*

$$\mathit{IrredComp}(X_K) \longrightarrow \mathit{IrredComp}(X)$$

which is surjective.

Proof. Consider the diagram

$$\begin{array}{ccc} X_K & \longleftarrow & X_{\overline{K}} \\ \downarrow & & \downarrow \\ X & \longleftarrow & X_{\overline{k}} \end{array}$$

where $\overline{K}$ is the separable algebraic closure of K, and where $\overline{k}$ is the separable algebraic closure of k. By Lemma 32.6.7 the morphism $X_{\overline{K}} \to X_{\overline{k}}$ induces a bijection between irreducible components. Hence it suffices to show the lemma for the morphisms $X_{\overline{k}} \to X$ and $X_{\overline{K}} \to X_K$. In other words we may assume that $K = \overline{k}$.

The morphism $p : X_{\overline{k}} \to X$ is integral, flat and surjective. Flatness implies that generalizations lift along p, see Morphisms, Lemma 28.26.8. Hence generic points of irreducible components of $X_{\overline{k}}$ map to generic points of irreducible components of X. Integrality implies that p is universally closed, see Morphisms, Lemma 28.44.7. Hence we conclude that the image $p(\overline{T})$ of an irreducible component is a closed irreducible subset which contains a generic point of an irreducible component of X, hence $p(\overline{T})$ is an irreducible component of X. This proves the first assertion. If $T \subset X$ is an irreducible component, then $p^{-1}(T) = T_K$ is a nonempty union of irreducible components, see Lemma 32.6.9. Each of these necessarily maps onto T by the first part. Hence the map is surjective. □

Lemma 32.6.11. *Let k be a field, with separable algebraic closure $\overline{k}$. Let X be a scheme over k. There is an action*

$$\mathit{Gal}(\overline{k}/k)^{opp} \times \mathit{IrredComp}(X_{\overline{k}}) \longrightarrow \mathit{IrredComp}(X_{\overline{k}})$$

with the following properties:

(1) *An element $\overline{T} \in \mathit{IrredComp}(X_{\overline{k}})$ is fixed by the action if and only if there exists an irreducible component $T \subset X$, which is geometrically irreducible over k, such that $T_{\overline{k}} = \overline{T}$.*

(2) *For any field extension $k \subset k'$ with separable algebraic closure $\overline{k}'$ the diagram*

$$\begin{array}{ccc} Gal(\overline{k}'/k') \times IrredComp(X_{\overline{k}'}) & \longrightarrow & IrredComp(X_{\overline{k}'}) \\ \downarrow & & \downarrow \\ Gal(\overline{k}/k) \times IrredComp(X_{\overline{k}}) & \longrightarrow & IrredComp(X_{\overline{k}}) \end{array}$$

is commutative (where the right vertical arrow is a bijection according to Lemma 32.6.7).

Proof. The action (32.5.8.1) of $\mathrm{Gal}(\overline{k}/k)$ on $X_{\overline{k}}$ induces an action on its irreducible components. Irreducible components are always closed (Topology, Lemma 5.6.3). Hence if $\overline{T}$ is as in (1), then by Lemma 32.5.10 there exists a closed subset $T \subset X$ such that $\overline{T} = T_{\overline{k}}$. Note that T is geometrically irreducible over k, see Lemma 32.6.8. To see that T is an irreducible component of X, suppose that $T \subset T'$, $T \neq T'$ where T' is an irreducible component of X. Let $\overline{\eta}$ be the generic point of $\overline{T}$. It maps to the generic point η of T. Then the generic point $\xi \in T'$ specializes to η. As $X_{\overline{k}} \to X$ is flat there exists a point $\overline{\xi} \in X_{\overline{k}}$ which maps to ξ and specializes to $\overline{\eta}$. It follows that the closure of the singleton $\{\overline{\xi}\}$ is an irreducible closed subset of $X_{\overline{\xi}}$ which strictly contains $\overline{T}$. This is the desired contradiction.

We omit the proof of the functoriality in (2). □

Lemma 32.6.12. *Let k be a field, with separable algebraic closure $\overline{k}$. Let X be a scheme over k. The fibres of the map*

$$IrredComp(X_{\overline{k}}) \longrightarrow IrredComp(X)$$

of Lemma 32.6.10 are exactly the orbits of $Gal(\overline{k}/k)$ under the action of Lemma 32.6.11.

Proof. Let $T \subset X$ be an irreducible component of X. Let $\eta \in T$ be its generic point. By Lemmas 32.6.9 and 32.6.10 the generic points of irreducible components of $\overline{T}$ which map into T map to η. By Algebra, Lemma 10.43.12 the Galois group acts transitively on all of the points of $X_{\overline{k}}$ mapping to η. Hence the lemma follows. □

Lemma 32.6.13. *Let k be a field. Assume $X \to \mathrm{Spec}(k)$ locally of finite type. In this case*

(1) *the action*

$$Gal(\overline{k}/k)^{opp} \times IrredComp(X_{\overline{k}}) \longrightarrow IrredComp(X_{\overline{k}})$$

is continuous if we give $IrredComp(X_{\overline{k}})$ the discrete topology,

(2) *every irreducible component of $X_{\overline{k}}$ can be defined over a finite extension of k, and*

(3) *given any irreducible component $T \subset X$ the scheme $T_{\overline{k}}$ is a finite union of irreducible components of $X_{\overline{k}}$ which are all in the same $Gal(\overline{k}/k)$-orbit.*

Proof. Let $\overline{T}$ be an irreducible component of $X_{\overline{k}}$. We may choose an affine open $U \subset X$ such that $\overline{T} \cap U_{\overline{k}}$ is not empty. Write $U = \mathrm{Spec}(A)$, so A is a finite type k-algebra, see Morphisms, Lemma 28.16.2. Hence $A_{\overline{k}}$ is a finite type $\overline{k}$-algebra, and in particular Noetherian. Let $\mathfrak{p} = (f_1, \ldots, f_n)$ be the prime ideal corresponding to

$\overline{T} \cap U_{\overline{k}}$. Since $A_{\overline{k}} = A \otimes_k \overline{k}$ we see that there exists a finite subextension $k \subset k' \subset \overline{k}$ such that each $f_i \in A_{k'}$. It is clear that $\mathrm{Gal}(\overline{k}/k')$ fixes $\overline{T}$, which proves (1).

Part (2) follows by applying Lemma 32.6.11 (1) to the situation over k' which implies the irreducible component $\overline{T}$ is of the form $T'_{\overline{k}}$ for some irreducible $T' \subset X_{k'}$.

To prove (3), let $T \subset X$ be an irreducible component. Choose an irreducible component $\overline{T} \subset X_{\overline{k}}$ which maps to T, see Lemma 32.6.10. By the above the orbit of $\overline{T}$ is finite, say it is $\overline{T}_1, \ldots, \overline{T}_n$. Then $\overline{T}_1 \cup \ldots \cup \overline{T}_n$ is a $\mathrm{Gal}(\overline{k}/k)$-invariant closed subset of $X_{\overline{k}}$ hence of the form $W_{\overline{k}}$ for some $W \subset X$ closed by Lemma 32.5.10. Clearly $W = T$ and we win. □

Lemma 32.6.14. *Let k be a field. Let $X \to \mathrm{Spec}(k)$ be locally of finite type. Assume X has finitely many irreducible components. Then there exists a finite separable extension $k \subset k'$ such that every irreducible component of $X_{k'}$ is geometrically irreducible over k'.*

Proof. Let $\overline{k}$ be a separable algebraic closure of k. The assumption that X has finitely many irreducible components combined with Lemma 32.6.13 (3) shows that $X_{\overline{k}}$ has finitely many irreducible components $\overline{T}_1, \ldots, \overline{T}_n$. By Lemma 32.6.13 (2) there exists a finite extension $k \subset k' \subset \overline{k}$ and irreducible components $T_i \subset X_{k'}$ such that $\overline{T}_i = T_{i,\overline{k}}$ and we win. □

Lemma 32.6.15. *Let X be a scheme over the field k. Assume X has finitely many irreducible components which are all geometrically irreducible. Then X has finitely many connected components each of which is geometrically connected.*

Proof. This is clear because a connected component is a union of irreducible components. Details omitted. □

32.7. Geometrically integral schemes

If X is an irreducible scheme over a field, then it can happen that X becomes reducible after extending the ground field. This does not happen for geometrically irreducible schemes.

Definition 32.7.1. Let X be a scheme over the field k.

(1) Let $x \in X$. We say X is *geometrically pointwise integral at x* if for every field extension $k \subset k'$ and every $x' \in X_{k'}$ lying over x the local ring $\mathcal{O}_{X_{k'},x'}$ is integral.
(2) We say X is *geometrically pointwise integral* if X is geometrically pointwise integral at every point.
(3) We say X is *geometrically integral* over k if the scheme $X_{k'}$ is integral for every field extension k' of k.

The distinction between notions (2) and (3) is necessary. For example if $k = \mathbf{R}$ and $X = \mathrm{Spec}(\mathbf{C}[x])$, then X is geometrically pointwise integral over $\mathbf{R}$ but of course not geometrically integral.

Lemma 32.7.2. *Let k be a field. Let X be a scheme over k. Then X is geometrically integral over k if and only if X is both geometrically reduced and geometrically irreducible over k.*

Proof. See Properties, Lemma 27.3.4. □

32.8. Geometrically normal schemes

In Properties, Definition 27.7.1 we have defined the notion of a normal scheme. This notion is defined even for non-Noetherian schemes. Hence, contrary to our discussion of "geometrically regular" schemes we consider all field extensions of the ground field.

Definition 32.8.1. Let X be a scheme over the field k.

(1) Let $x \in X$. We say X is *geometrically normal at* x if for every field extension $k \subset k'$ and every $x' \in X_{k'}$ lying over x the local ring $\mathcal{O}_{X_{k'},x'}$ is normal.
(2) We say X is *geometrically normal* over k if X is geometrically normal at every $x \in X$.

Lemma 32.8.2. *Let k be a field. Let X be a scheme over k. Let $x \in X$. The following are equivalent*

(1) *X is geometrically normal at x,*
(2) *for every finite purely inseparable field extension k' of k and $x' \in X_{k'}$ lying over over x the local ring $\mathcal{O}_{X_{k'},x'}$ is normal, and*
(3) *the ring $\mathcal{O}_{X,x}$ is geometrically normal over k (see Algebra, Definition 10.151.2).*

Proof. It is clear that (1) implies (2). Assume (2). Let $k \subset k'$ be a finite purely inseparable field extension (for example $k = k'$). Consider the ring $\mathcal{O}_{X,x} \otimes_k k'$. By Algebra, Lemma 10.43.2 its spectrum is the same as the spectrum of $\mathcal{O}_{X,x}$. Hence it is a local ring also (Algebra, Lemma 10.17.2). Therefore there is a unique point $x' \in X_{k'}$ lying over x and $\mathcal{O}_{X_{k'},x'} \cong \mathcal{O}_{X,x} \otimes_k k'$. By assumption this is a normal ring. Hence we deduce (3) by Algebra, Lemma 10.151.1.

Assume (3). Let $k \subset k'$ be a field extension. Since $\mathrm{Spec}(k') \to \mathrm{Spec}(k)$ is surjective, also $X_{k'} \to X$ is surjective (Morphisms, Lemma 28.11.4). Let $x' \in X_{k'}$ be any point lying over x. The local ring $\mathcal{O}_{X_{k'},x'}$ is a localization of the ring $\mathcal{O}_{X,x} \otimes_k k'$. Hence it is normal by assumption and (1) is proved. □

Lemma 32.8.3. *Let k be a field. Let X be a scheme over k. The following are equivalent*

(1) *X is geometrically normal,*
(2) *$X_{k'}$ is a normal scheme for every field extension $k \subset k'$,*
(3) *$X_{k'}$ is a normal scheme for every finitely generated field extension $k \subset k'$,*
(4) *$X_{k'}$ is a normal scheme for every finite purely inseparable field extension $k \subset k'$, and*
(5) *for every affine open $U \subset X$ the ring $\mathcal{O}_X(U)$ is geometrically normal (see Algebra, Definition 10.151.2).*

Proof. Assume (1). Then for every field extension $k \subset k'$ and every point $x' \in X_{k'}$ the local ring of $X_{k'}$ at x' is normal. By definition this means that $X_{k'}$ is normal. Hence (2).

It is clear that (2) implies (3) implies (4).

Assume (4) and let $U \subset X$ be an affine open subscheme. Then $U_{k'}$ is a normal scheme for any finite purely inseparable extension $k \subset k'$ (including $k = k'$). This

means that $k' \otimes_k \mathcal{O}(U)$ is a normal ring for all finite purely inseparable extensions $k \subset k'$. Hence $\mathcal{O}(U)$ is a geometrically normal k-algebra by definition.

Assume (5). For any field extension $k \subset k'$ the base change $X_{k'}$ is gotten by gluing the spectra of the rings $\mathcal{O}_X(U) \otimes_k k'$ where U is affine open in X (see Schemes, Section 25.17). Hence $X_{k'}$ is normal. So (1) holds. □

Lemma 32.8.4. *Let k be a field. Let X be a scheme over k. Let k'/k be a field extension. Let $x \in X$ be a point, and let $x' \in X_{k'}$ be a point lying over x. The following are equivalent*

(1) *X is geometrically normal at x,*
(2) *$X_{k'}$ is geometrically normal at x'.*

In particular, X is geometrically normal over k if and only if $X_{k'}$ is geometrically normal over k'.

Proof. It is clear that (1) implies (2). Assume (2). Let $k \subset k''$ be a finite purely inseparable field extension and let $x'' \in X_{k''}$ be a point lying over x (actually it is unique). We can find a common field extension $k \subset k'''$ (i.e. with both $k' \subset k'''$ and $k'' \subset k'''$) and a point $x''' \in X_{k'''}$ lying over both x' and x''. Consider the map of local rings

$$\mathcal{O}_{X_{k''},x''} \longrightarrow \mathcal{O}_{X_{k'''},x'''}.$$

This is a flat local ring homomorphism and hence faithfully flat. By (2) we see that the local ring on the right is normal. Thus by Algebra, Lemma 10.150.3 we conclude that $\mathcal{O}_{X_{k''},x''}$ is normal. By Lemma 32.8.2 we see that X is geometrically normal at x. □

Lemma 32.8.5. *Let k be a field. Let X be a geometrically normal scheme over k and let Y be a normal scheme over k. Then $X \times_k Y$ is a normal scheme.*

Proof. This reduces to Algebra, Lemma 10.151.4 by Lemma 32.8.3. □

32.9. Change of fields and locally Noetherian schemes

Let X a locally Noetherian scheme over a field k. It is not always that case that $X_{k'}$ is locally Noetherian too. For example if $X = \mathrm{Spec}(\overline{\mathbf{Q}})$ and $k = \mathbf{Q}$, then $X_{\overline{\mathbf{Q}}}$ is the spectrum of $\overline{\mathbf{Q}} \otimes_{\mathbf{Q}} \overline{\mathbf{Q}}$ which is not Noetherian. (Hint: It has too many idempotents). But if we only base change using finitely generated field extensions then the Noetherian property is preserved. (Or if X is locally of finite type over k, since this property is preserved under base change.)

Lemma 32.9.1. *Let k be a field. Let X be a scheme over k. Let $k \subset k'$ be a finitely generated field extension. Then X is locally Noetherian if and only if $X_{k'}$ is locally Noetherian.*

Proof. Using Properties, Lemma 27.5.2 we reduce to the case where X is affine, say $X = \mathrm{Spec}(A)$. In this case we have to prove that A is Noetherian if and only if $A_{k'}$ is Noetherian. Since $A \to A_{k'} = k' \otimes_k A$ is faithfully flat, we see that if $A_{k'}$ is Noetherian, then so is A, by Algebra, Lemma 10.150.1. Conversely, if A is Noetherian then $A_{k'}$ is Noetherian by Algebra, Lemma 10.29.7. □

32.10. Geometrically regular schemes

A geometrically regular scheme over a field k is a locally Noetherian scheme over k which remains regular upon suitable changes of base field. A finite type scheme over k is geometrically regular if and only if it is smooth over k (see Lemma 32.10.6). The notion of geometric regularity is most interesting in situations where smoothness cannot be used such as formal fibres (insert future reference here).

In the following definition we restrict ourselves to locally Noetherian schemes, since the property of being a regular local ring is only defined for Noetherian local rings. By Lemma 32.8.3 above, if we restrict ourselves to finitely generated field extensions then this property is preserved under change of base field. This comment will be used without further reference in this section. In particular the following definition makes sense.

Definition 32.10.1. Let k be a field. Let X be a locally Noetherian scheme over k.

(1) Let $x \in X$. We say X is *geometrically regular at* x over k if for every finitely generated field extension $k \subset k'$ and any $x' \in X_{k'}$ lying over x the local ring $\mathcal{O}_{X_{k'},x'}$ is regular.
(2) We say X is *geometrically regular over* k if X is geometrically regular at all of its points.

A similar definition works to define geometrically Cohen-Macaulay, (R_k), and (S_k) schemes over a field. We will add a section for these separately as needed.

Lemma 32.10.2. *Let k be a field. Let X be a locally Noetherian scheme over k. Let $x \in X$. The following are equivalent*

(1) *X is geometrically regular at x,*
(2) *for every finite purely inseparable field extension k' of k and $x' \in X_{k'}$ lying over over x the local ring $\mathcal{O}_{X_{k'},x'}$ is regular, and*
(3) *the ring $\mathcal{O}_{X,x}$ is geometrically regular over k (see Algebra, Definition 10.152.2).*

Proof. It is clear that (1) implies (2). Assume (2). This in particular implies that $\mathcal{O}_{X,x}$ is a regular local ring. Let $k \subset k'$ be a finite purely inseparable field extension. Consider the ring $\mathcal{O}_{X,x} \otimes_k k'$. By Algebra, Lemma 10.43.2 its spectrum is the same as the spectrum of $\mathcal{O}_{X,x}$. Hence it is a local ring also (Algebra, Lemma 10.17.2). Therefore there is a unique point $x' \in X_{k'}$ lying over x and $\mathcal{O}_{X_{k'},x'} \cong \mathcal{O}_{X,x} \otimes_k k'$. By assumption this is a regular ring. Hence we deduce (3) from the definition of a geometrically regular ring.

Assume (3). Let $k \subset k'$ be a field extension. Since $\mathrm{Spec}(k') \to \mathrm{Spec}(k)$ is surjective, also $X_{k'} \to X$ is surjective (Morphisms, Lemma 28.11.4). Let $x' \in X_{k'}$ be any point lying over x. The local ring $\mathcal{O}_{X_{k'},x'}$ is a localization of the ring $\mathcal{O}_{X,x} \otimes_k k'$. Hence it is regular by assumption and (1) is proved. □

Lemma 32.10.3. *Let k be a field. Let X be a locally Noetherian scheme over k. The following are equivalent*

(1) *X is geometrically regular,*
(2) *$X_{k'}$ is a regular scheme for every finitely generated field extension $k \subset k'$,*
(3) *$X_{k'}$ is a regular scheme for every finite purely inseparable field extension $k \subset k'$,*

(4) *for every affine open $U \subset X$ the ring $\mathcal{O}_X(U)$ is geometrically regular (see Algebra, Definition 10.152.2), and*
(5) *there exists an affine open covering $X = \bigcup U_i$ such that each $\mathcal{O}_X(U_i)$ is geometrically regular over k.*

Proof. Assume (1). Then for every finitely generated field extension $k \subset k'$ and every point $x' \in X_{k'}$ the local ring of $X_{k'}$ at x' is regular. By Properties, Lemma 27.9.2 this means that $X_{k'}$ is regular. Hence (2).

It is clear that (2) implies (3).

Assume (3) and let $U \subset X$ be an affine open subscheme. Then $U_{k'}$ is a regular scheme for any finite purely inseparable extension $k \subset k'$ (including $k = k'$). This means that $k' \otimes_k \mathcal{O}(U)$ is a regular ring for all finite purely inseparable extensions $k \subset k'$. Hence $\mathcal{O}(U)$ is a geometrically regular k-algebra and we see that (4) holds.

It is clear that (4) implies (5). Let $X = \bigcup U_i$ be an affine open covering as in (5). For any field extension $k \subset k'$ the base change $X_{k'}$ is gotten by gluing the spectra of the rings $\mathcal{O}_X(U_i) \otimes_k k'$ (see Schemes, Section 25.17). Hence $X_{k'}$ is regular. So (1) holds. □

Lemma 32.10.4. *Let k be a field. Let X be a scheme over k. Let k'/k be a finitely generated field extension. Let $x \in X$ be a point, and let $x' \in X_{k'}$ be a point lying over x. The following are equivalent*

(1) *X is geometrically regular at x,*
(2) *$X_{k'}$ is geometrically regular at x'.*

In particular, X is geometrically regular over k if and only if $X_{k'}$ is geometrically regular over k'.

Proof. It is clear that (1) implies (2). Assume (2). Let $k \subset k''$ be a finite purely inseparable field extension and let $x'' \in X_{k''}$ be a point lying over x (actually it is unique). We can find a common, finitely generated, field extension $k \subset k'''$ (i.e. with both $k' \subset k'''$ and $k'' \subset k'''$) and a point $x''' \in X_{k'''}$ lying over both x' and x''. Consider the map of local rings

$$\mathcal{O}_{X_{k''},x''} \longrightarrow \mathcal{O}_{X_{k'''},x'''}.$$

This is a flat local ring homomorphism of Noetherian local rings and hence faithfully flat. By (2) we see that the local ring on the right is regular. Thus by Algebra, Lemma 10.105.8 we conclude that $\mathcal{O}_{X_{k''},x''}$ is regular. By Lemma 32.10.2 we see that X is geometrically regular at x. □

The following lemma is a geometric variant of Algebra, Lemma 10.152.3.

Lemma 32.10.5. *Let k be a field. Let $f : X \to Y$ be a morphism of locally Noetherian schemes over k. Let $x \in X$ be a point and set $y = f(x)$. If X is geometrically regular at x and f is flat at x then Y is geometrically regular at y. In particular, if X is geometrically regular over k and f is flat and surjective, then Y is geometrically regular over k.*

Proof. Let k' be finite purely inseparable extension of k. Let $f' : X_{k'} \to Y_{k'}$ be the base change of f. Let $x' \in X_{k'}$ be the unique point lying over x. If we show that $Y_{k'}$ is regular at $y' = f'(x')$, then Y is geometrically regular over k at y', see

Lemma 32.10.3. By Morphisms, Lemma 28.26.6 the morphism $X_{k'} \to Y_{k'}$ is flat at x'. Hence the ring map

$$\mathcal{O}_{Y_{k'},y'} \longrightarrow \mathcal{O}_{X_{k'},x'}$$

is a flat local homomorphism of local Noetherian rings with right hand side regular by assumption. Hence the left hand side is a regular local ring by Algebra, Lemma 10.105.8. □

Lemma 32.10.6. *Let k be a field. Let X be a scheme of finite type over k. Let $x \in X$. Then X is geometrically regular at x if and only if $X \to \mathrm{Spec}(k)$ is smooth at x (Morphisms, Definition 28.35.1).*

Proof. The question is local around x, hence we may assume that $X = \mathrm{Spec}(A)$ for some finite type k-algebra. Let x correspond to the prime $\mathfrak{p}$.

If A is smooth over k at $\mathfrak{p}$, then we may localize A and assume that A is smooth over k. In this case $k' \otimes_k A$ is smooth over k' for all extension fields k'/k, and each of these Noetherian rings is regular by Algebra, Lemma 10.133.3.

Assume X is geometrically regular at x. Consider the residue field $K := \kappa(x) = \kappa(\mathfrak{p})$ of x. It is a finitely generated extension of k. By Algebra, Lemma 10.42.3 there exists a finite purely inseparable extension $k \subset k'$ such that the compositum $k'K$ is a separable field extension of k'. Let $\mathfrak{p}' \subset A' = k' \otimes_k A$ be a prime ideal lying over $\mathfrak{p}$. It is the unique prime lying over $\mathfrak{p}$, see Algebra, Lemma 10.43.2. Hence the residue field $K' := \kappa(\mathfrak{p}')$ is the compositum $k'K$. By assumption the local ring $(A')_{\mathfrak{p}'}$ is regular. Hence by Algebra, Lemma 10.133.5 we see that $k' \to A'$ is smooth at $\mathfrak{p}'$. This in turn implies that $k \to A$ is smooth at $\mathfrak{p}$ by Algebra, Lemma 10.130.18. The lemma is proved. □

Example 32.10.7. Let $k = \mathbf{F}_p(t)$. It is quite easy to give an example of a regular variety V over k which is not geometrically reduced. For example we can take $\mathrm{Spec}(k[x]/(x^p - t))$. In fact, there exists an example of a regular variety V which is geometrically reduced, but not even geometrically normal. Namely, take for $p > 2$ the scheme $V = \mathrm{Spec}(k[x,y]/(y^2 - x^p + t))$. This is a variety as the polynomial $y^2 - x^p + t \in k[x,y]$ is irreducible. The morphism $V \to \mathrm{Spec}(k)$ is smooth at all points except at the point $v_0 \in V$ corresponding to the maximal ideal $(y, x^p - t)$ (because $2y$ is invertible). In particular we see that V is (geometrically) regular at all points, except possibly v_0. The local ring

$$\mathcal{O}_{V,v_0} = \big(k[x,y]/(y^2 - x^p + t)\big)_{(y,x^p-t)}$$

is a domain of dimension 1. Its maximal ideal is generated by 1 element, namely y. Hence it is a discrete valuation ring and regular. Let $k' = k[t^{1/p}]$. Denote $t' = t^{1/p} \in k'$, $V' = V_{k'}$, $v'_0 \in V'$ the unique point lying over v_0. Over k' we can write $x^p - t = (x - t')^p$, but the polynomial $y^2 - (x - t')^p$ is still irreducible and V' is still a variety. But the element

$$\frac{y}{x - t'} \in f.f.(\mathcal{O}_{V',v'_0})$$

is integral over $\mathcal{O}_{V',v'_0}$ (just compute its square) and not contained in it, so V' is not normal at v'_0. This concludes the example.

32.11. Change of fields and the Cohen-Macaulay property

The following lemma says that it does not make sense to define geometrically Cohen-Macaulay schemes, since these would be the same as Cohen-Macaulay schemes.

Lemma 32.11.1. *Let X be a locally Noetherian scheme over the field k. Let $k \subset k'$ be a finitely generated field extension. Let $x \in X$ be a point, and let $x' \in X_{k'}$ be a point lying over x. Then we have*

$$\mathcal{O}_{X,x} \text{ is Cohen-Macaulay} \Leftrightarrow \mathcal{O}_{X_{k'},x'} \text{ is Cohen-Macaulay}$$

If X is locally of finite type over k, the same holds for any field extension $k \subset k'$.

Proof. The first case of the lemma follows from Algebra, Lemma 10.153.2. The second case of the lemma is equivalent to Algebra, Lemma 10.124.6. □

32.12. Change of fields and the Jacobson property

A scheme locally of finite type over a field has plenty of closed points, namely it is Jacobson. Moreover, the residue fields are finite extensions of the ground field.

Lemma 32.12.1. *Let X be a scheme which is locally of finite type over k. Then*

1. *for any closed point $x \in X$ the extension $k \subset \kappa(x)$ is algebraic, and*
2. *X is a Jacobson scheme (Properties, Definition 27.6.1).*

Proof. A scheme is Jacobson if and only if it has an affine open covering by Jacobson schemes, see Properties, Lemma 27.6.3. The property on residue fields at closed points is also local on X. Hence we may assume that X is affine. In this case the result is a consequence of the Hilbert Nullstellensatz, see Algebra, Theorem 10.31.1. It also follows from a combination of Morphisms, Lemmas 28.17.8, 28.17.9, and 28.17.10. □

It turns out that if X is not locally of finite type, then we can achieve the same result after making a suitably large base field extension.

Lemma 32.12.2. *Let X be a scheme over a field k. For any field extension $k \subset K$ whose cardinality is large enough we have*

1. *for any closed point $x \in X_K$ the extension $K \subset \kappa(x)$ is algebraic, and*
2. *X_K is a Jacobson scheme (Properties, Definition 27.6.1).*

Proof. Choose an affine open covering $X = \bigcup U_i$. By Algebra, Lemma 10.32.12 and Properties, Lemma 27.6.2 there exist cardinals κ_i such that $U_{i,K}$ has the desired properties over K if $\#(K) \geq \kappa_i$. Set $\kappa = \max\{\kappa_i\}$. Then if the cardinality of K is larger than κ we see that each $U_{i,K}$ satisfies the conclusions of the lemma. Hence X_K is Jacobson by Properties, Lemma 27.6.3. The statement on residue fields at closed points of X_K follows from the corresponding statements for residue fields of closed points of the $U_{i,K}$. □

32.13. Algebraic schemes

The following definition is taken from [**DG67**, I Definition 6.4.1].

Definition 32.13.1. Let k be a field. An *algebraic k-scheme* is a scheme X over k such that the structure morphism $X \to \mathrm{Spec}(k)$ is of finite type. A *locally algebraic k-scheme* is a scheme X over k such that the structure morphism $X \to \mathrm{Spec}(k)$ is locally of finite type.

Note that every (locally) algebraic k-scheme is (locally) Noetherian, see Morphisms, Lemma 28.16.6. The category of algebraic k-schemes has all products and fibre products (unlike the category of varieties over k). Similarly for the category of locally algebraic k-schemes.

Lemma 32.13.2. *Let k be a field. Let X be a locally algebraic k-scheme of dimension 0. Then X is a disjoint union of spectra of local Artinian k-algebras A with $\dim_k(A) < \infty$. If X is an algebraic k-scheme of dimension 0, then in addition X is affine and the morphism $X \to \mathrm{Spec}(k)$ is finite.*

Proof. Let X be a locally algebraic k-scheme of dimension 0. Let $U = \mathrm{Spec}(A) \subset X$ be an affine open subscheme. Since $\dim(X) = 0$ we see that $\dim(A) = 0$. By Noether normalization, see Algebra, Lemma 10.109.4 we see that there exists a finite injection $k \to A$, i.e., $\dim_k(A) < \infty$. Hence A is Artinian, see Algebra, Lemma 10.49.2. This implies that $A = A_1 \times \ldots \times A_r$ is a product of finitely many Artinian local rings, see Algebra, Lemma 10.49.8. Of course $\dim_k(A_i) < \infty$ for each i as the sum of these dimensions equals $\dim_k(A)$.

The arguments above show that X has an open covering whose members are finite discrete topological spaces. Hence X is a discrete topological space. It follows that X is isomorphic to the disjoint union of its connected components each of which is a singleton. Since a singleton scheme is affine we conclude (by the results of the paragraph above) that each of these singletons is the spectrum of a local Artinian k-algebra A with $\dim_k(A) < \infty$.

Finally, if X is an algebraic k-scheme of dimension 0, then X is quasi-compact hence is a finite disjoint union $X = \mathrm{Spec}(A_1) \amalg \ldots \amalg \mathrm{Spec}(A_r)$ hence affine (see Schemes, Lemma 25.6.8) and we have seen the finiteness of $X \to \mathrm{Spec}(k)$ in the first paragraph of the proof. □

32.14. Closures of products

Some results on the relation between closure and products.

Lemma 32.14.1. *Let k be a field. Let X, Y be schemes over k, and let $A \subset X$, $B \subset Y$ be subsets. Set*

$$AB = \{z \in X \times_k Y \mid pr_X(\gamma) \in A,\ pr_Y(\gamma) \in B\} \subset X \times_k Y$$

Then set theoretically we have

$$\overline{A} \times_k \overline{B} = \overline{AB}$$

Proof. The inclusion $\overline{AB} \subset \overline{A} \times_k \overline{B}$ is immediate. We may replace X and Y by the reduced closed subschemes $\overline{A}$ and $\overline{B}$. Let $W \subset X \times_k Y$ be a nonempty open subset. By Morphisms, Lemma 28.24.4 the subset $U = \mathrm{pr}_X(W)$ is nonempty open in X. Hence $A \cap U$ is nonempty. Pick $a \in A \cap U$. Denote $Y_{\kappa(a)} = \{a\} \times_k Y$ the fibre of $\mathrm{pr}_X : X \times_k Y \to X$ over a. By Morphisms, Lemma 28.24.4 again the morphism $Y_a \to Y$ is open as $\mathrm{Spec}(\kappa(a)) \to \mathrm{Spec}(k)$ is universally open. Hence the nonempty open subset $W_a = W \times_{X \times_k Y} Y_a$ maps to a nonempty open subset of Y. We conclude there exists a $b \in B$ in the image. Hence $AB \cap W \neq \emptyset$ as desired. □

Lemma 32.14.2. *Let k be a field. Let $f : A \to X$, $g : B \to Y$ be morphisms of schemes over k. Then set theoretically we have*

$$\overline{f(A)} \times_k \overline{g(B)} = \overline{(f \times g)(A \times_k B)}$$

Proof. This follows from Lemma 32.14.1 as the image of $f \times g$ is $f(A)g(B)$ in the notation of that lemma. □

Lemma 32.14.3. *Let k be a field. Let $f : A \to X$, $g : B \to Y$ be quasi-compact morphisms of schemes over k. Let $Z \subset X$ be the scheme theoretic image of f, see Morphisms, Definition 28.6.2. Similarly, let $Z' \subset Y$ be the scheme theoretic image of g. Then $Z \times_k Z'$ is the scheme theoretic image of $f \times g$.*

Proof. Recall that Z is the smallest closed subscheme of X through which f factors. Similarly for Z'. Let $W \subset X \times_k Y$ be the scheme theoretic image of $f \times g$. As $f \times g$ factors through $Z \times_k Z'$ we see that $W \subset Z \times_k Z'$.

To prove the other inclusion let $U \subset X$ and $V \subset Y$ be affine opens. By Morphisms, Lemma 28.6.3 the scheme $Z \cap U$ is the scheme theoretic image of $f|_{f^{-1}(U)} : f^{-1}(U) \to U$, and similarly for $Z' \cap V$ and $W \cap U \times_k V$. Hence we may assume X and Y affine. As f and g are quasi-compact this implies that $A = \bigcup U_i$ is a finite union of affines and $B = \bigcup V_j$ is a finite union of affines. Then we may replace A by $\coprod U_i$ and B by $\coprod V_j$, i.e., we may assume that A and B are affine as well. In this case Z is cut out by $\mathrm{Ker}(\Gamma(X, \mathcal{O}_X) \to \Gamma(A, \mathcal{O}_A))$ and similarly for Z' and W. Hence the result follows from the equality

$$\Gamma(A \times_k B, \mathcal{O}_{A \times_k B}) = \Gamma(A, \mathcal{O}_A) \otimes_k \Gamma(B, \mathcal{O}_B)$$

which holds as A and B are affine. Details omitted. □

32.15. Schemes smooth over fields

Here are two lemmas characterizing smooth schemes over fields.

Lemma 32.15.1. *Let k be a field. Let X be a scheme over k. Assume*

1. *X is locally of finite type over k,*
2. *$\Omega_{X/k}$ is locally free, and*
3. *k has characteristic zero.*

Then the structure morphism $X \to \mathrm{Spec}(k)$ is smooth.

Proof. This follows from Algebra, Lemma 10.133.7. □

In positive characteristic there exist nonreduced schemes of finite type whose sheaf of differentials is free, for example $\mathrm{Spec}(\mathbf{F}_p[t]/(t^p))$ over $\mathrm{Spec}(\mathbf{F}_p)$. If the ground field k is nonperfect of characteristic p, there exist reduced schemes X/k with free $\Omega_{X/k}$ which are nonsmooth, for example $\mathrm{Spec}(k[t]/(t^p - a)$ where $a \in k$ is not a pth power.

Lemma 32.15.2. *Let k be a field. Let X be a scheme over k. Assume*

1. *X is locally of finite type over k,*
2. *$\Omega_{X/k}$ is locally free,*
3. *X is reduced, and*
4. *k is perfect.*

Then the structure morphism $X \to \mathrm{Spec}(k)$ is smooth.

Proof. Let $x \in X$ be a point. As X is locally Noetherian (see Morphisms, Lemma 28.16.6) there are finitely many irreducible components $X_1, \ldots, X_n$ passing through x (see Properties, Lemma 27.5.5 and Topology, Lemma 5.8.2). Let $\eta_i \in X_i$ be the generic point. As X is reduced we have $\mathcal{O}_{X,\eta_i} = \kappa(\eta_i)$, see Algebra, Lemma

10.24.4. Moreover, $\kappa(\eta_i)$ is a finitely generated field extension of the perfect field k hence separably generated over k (see Algebra, Section 10.39). It follows that $\Omega_{X/k,\eta_i} = \Omega_{\kappa(\eta_i)/k}$ is free of rank the transcendence degree of $\kappa(\eta_i)$ over k. By Morphisms, Lemma 28.29.1 we conclude that $\dim_{\eta_i}(X_i) = \text{rank}_{\eta_i}(\Omega_{X/k})$. Since $x \in X_1 \cap \ldots \cap X_n$ we see that

$$\text{rank}_x(\Omega_{X/k}) = \text{rank}_{\eta_i}(\Omega_{X/k}) = \dim(X_i).$$

Therefore $\dim_x(X) = \text{rank}_x(\Omega_{X/k})$, see Algebra, Lemma 10.108.5. It follows that $X \to \text{Spec}(k)$ is smooth at x for example by Algebra, Lemma 10.133.3. □

Lemma 32.15.3. *Let $X \to \text{Spec}(k)$ be a smooth morphism where k is a field. Then X is a regular scheme.*

Proof. (See also Lemma 32.10.6.) By Algebra, Lemma 10.133.3 every local ring $\mathcal{O}_{X,x}$ is regular. And because X is locally of finite type over k it is locally Noetherian. Hence X is regular by Properties, Lemma 27.9.2. □

Lemma 32.15.4. *Let $X \to \text{Spec}(k)$ be a smooth morphism where k is a field. Then X is geometrically regular, geometrically normal, and geometrically reduced over k.*

Proof. (See also Lemma 32.10.6.) Let k' be a finite purely inseparable extension of k. It suffices to prove that $X_{k'}$ is regular, normal, reduced, see Lemmas 32.10.3, 32.8.3, and 32.4.5. By Morphisms, Lemma 28.35.5 the morphism $X_{k'} \to \text{Spec}(k')$ is smooth too. Hence it suffices to show that a scheme X smooth over a field is regular, normal, and reduced. We see that X is regular by Lemma 32.15.3. Hence Properties, Lemma 27.9.4 guarantees that X is normal. □

Lemma 32.15.5. *Let k be a field. Let $d \geq 0$. Let $W \subset \mathbf{A}^d_k$ be nonempty open. Then there exists a closed point $w \in W$ such that $k \subset \kappa(w)$ is finite separable.*

Proof. After possible shrinking W we may assume that $W = \mathbf{A}^d_k \setminus V(f)$ for some $f \in k[x_1, \ldots, x_n]$. If the lemma is wrong then $f(a_1, \ldots, a_n) = 0$ for all $(a_1, \ldots, a_n) \in (k^{sep})^n$. This is absurd as k^{sep} is an infinite field. □

Lemma 32.15.6. *Let k be a field. If X is smooth over $\text{Spec}(k)$ then the set*

$$\{x \in X \text{ closed such that } k \subset \kappa(x) \text{ is finite separable}\}$$

is dense in X.

Proof. It suffices to show that given a nonempty smooth X over k there exists at least one closed point whose residue field is finite separable over k. To see this, choose a diagram

$$X \leftarrow U \xrightarrow{\pi} \mathbf{A}^d_k$$

with π étale, see Morphisms, Lemma 28.37.20. The morphism $\pi : U \to \mathbf{A}^d_k$ is open, see Morphisms, Lemma 28.37.13. By Lemma 32.15.5 we may choose a closed point $w \in \pi(V)$ whose residue field is finite separable over k. Pick any $x \in V$ with $\pi(x) = w$. By Morphisms, Lemma 28.37.7 the field extension $\kappa(w) \subset \kappa(x)$ is finite separable. Hence $k \subset \kappa(x)$ is finite separable. The point x is a closed point of X by Morphisms, Lemma 28.21.2. □

Lemma 32.15.7. *Let X be a scheme over a field k. If X is locally of finite type and geometrically reduced over k then X contains a dense open which is smooth over k.*

Proof. The problem is local on X, hence we may assume X is quasi-compact. Let $X = X_1 \cup \ldots \cup X_n$ be the irreducible components of X. Then $Z = \bigcup_{i \neq j} X_i \cap X_j$ is nowhere dense in X. Hence we may replace X by $X \setminus Z$. As $X \setminus Z$ is a disjoint union of irreducible schemes, this reduces us to the case where X is irreducible. As X is irreducible and reduced, it is integral, see Properties, Lemma 27.3.4. Let $\eta \in X$ be its generic point. Then the function field $K = k(X) = \kappa(\eta)$ is geometrically reduced over k, hence separable over k, see Algebra, Lemma 10.41.1. Let $U = \mathrm{Spec}(A) \subset X$ be any nonempty affine open so that $K = f.f.(A) = A_{(0)}$. Apply Algebra, Lemma 10.133.5 to conclude that A is smooth at (0) over k. By definition this means that some principal localization of A is smooth over k and we win. □

Lemma 32.15.8. *Let k be a field. Let $f : X \to Y$ be a morphism of schemes locally of finite type over k. Let $x \in X$ be a point and set $y = f(x)$. If $X \to \mathrm{Spec}(k)$ is smooth at x and f is flat at x then $Y \to \mathrm{Spec}(k)$ is smooth at y. In particular, if X is smooth over k and f is flat and surjective, then Y is smooth over k.*

Proof. It suffices to show that Y is geometrically regular at y, see Lemma 32.10.6. This follows from Lemma 32.10.5 (and Lemma 32.10.6 applied to (X, x)). □

32.16. Types of varieties

Short section discussion some elementary global properties of varieties.

Definition 32.16.1. Let k be a field. Let X be a variety over k.

(1) We say X is an *affine variety* if X is an affine scheme. This is equivalent to requiring X it be isomorphic to a closed subscheme of $\mathbf{A}^n_k$ for some n.
(2) We say X is a *projective variety* if the structure morphism $X \to \mathrm{Spec}(k)$ is projective. By Morphisms, Lemma 28.43.4 this is true if and only if X is isomorphic to a closed subscheme of $\mathbf{P}^n_k$ for some n.
(3) We say X is a *quasi-projective variety* if the structure morphism $X \to \mathrm{Spec}(k)$ is quasi-projective. By Morphisms, Lemma 28.41.4 this is true if and only if X is isomorphic to a locally closed subscheme of $\mathbf{P}^n_k$ for some n.
(4) A *proper variety* is a variety such that the morphism $X \to \mathrm{Spec}(k)$ is proper.

Note that a projective variety is a proper variety, see Morphisms, Lemma 28.43.5. Also, an affine variety is quasi-projective as $\mathbf{A}^n_k$ is isomorphic to an open subscheme of $\mathbf{P}^n_k$, see Constructions, Lemma 26.13.3.

Lemma 32.16.2. *Let X be a proper variety over k. Then $\Gamma(X, \mathcal{O}_X)$ is a field which is a finite extension of the field k.*

Proof. By Cohomology of Schemes, Lemma 29.17.2 we see that $\Gamma(X, \mathcal{O}_X)$ is a finite dimensional k-vector space. It is also a k-algebra without zero-divisors. Hence it is a field, see Algebra, Lemma 10.33.17. □

32.17. Groups of invertible functions

It is often (but not always) the case that $\mathcal{O}^*(X)/k^*$ is a finitely generated abelian group if X is a variety over k. We show this by a series of lemmas. Everything rests on the following special case.

Lemma 32.17.1. *Let k be an algebraically closed field. Let $\overline{X}$ be a proper variety over k. Let $X \subset \overline{X}$ be an open subscheme. Assume X is normal. Then $\mathcal{O}^*(X)/k^*$ is a finitely generated abelian group.*

Proof. We will use without further mention that for any affine open U of $\overline{X}$ the ring $\mathcal{O}(U)$ is a finitely generated k-algebra, which is Noetherian, a domain and normal, see Algebra, Lemma 10.29.1, Properties, Definition 27.3.1, Properties, Lemmas 27.5.2 and 27.7.2, Morphisms, Lemma 28.16.2.

Let $\xi_1, \ldots, \xi_r$ be the generic points of the complement of X in $\overline{X}$. There are finitely many since $\overline{X}$ has a Noetherian underlying topological space (see Morphisms, Lemma 28.16.6, Properties, Lemma 27.5.5, and Topology, Lemma 5.8.2). For each i the local ring $\mathcal{O}_i = \mathcal{O}_{X,\xi_i}$ is a normal Noetherian local domain (as a localization of a Noetherian normal domain). Let $J \subset \{1, \ldots, r\}$ be the set of indices i such that $\dim(\mathcal{O}_i) = 1$. For $j \in J$ the local ring $\mathcal{O}_j$ is a discrete valuation ring, see Algebra, Lemma 10.113.6. Hence we obtain a valuation

$$v_j : k(\overline{X})^* \longrightarrow \mathbf{Z}$$

with the property that $v_j(f) \geq 0 \Leftrightarrow f \in \mathcal{O}_j$.

Think of $\mathcal{O}(X)$ as a sub k-algebra of $k(X) = k(\overline{X})$. We claim that the kernel of the map

$$\mathcal{O}(X)^* \longrightarrow \prod\nolimits_{j \in J} \mathbf{Z}, \quad f \longmapsto \prod v_j(f)$$

is k^*. It is clear that this claim proves the lemma. Namely, suppose that $f \in \mathcal{O}(X)$ is an element of the kernel. Let $U = \operatorname{Spec}(B) \subset \overline{X}$ be any affine open. Then B is a Noetherian normal domain. For every height one prime $\mathfrak{q} \subset B$ with corresponding point $\xi \in X$ we see that either $\xi = \xi_j$ for some $j \in J$ or that $\xi \in X$. The reason is that $\operatorname{codim}(\overline{\{\xi\}}, \overline{X}) = 1$ by Properties, Lemma 27.11.4 and hence if $\xi \in \overline{X} \setminus X$ it must be a generic point of $\overline{X} \setminus X$, hence equal to some ξ_j, $j \in J$. We conclude that $f \in \mathcal{O}_{X,\xi} = B_\mathfrak{q}$ in either case as f is in the kernel of the map. Thus $f \in \bigcap_{\operatorname{ht}(\mathfrak{q})=1} B_\mathfrak{q} = B$, see Algebra, Lemma 10.144.6. In other words, we see that $f \in \Gamma(\overline{X}, \mathcal{O}_{\overline{X}})$. But since k is algebraically closed we conclude that $f \in k$ by Lemma 32.16.2. □

Next, we generalize the case above by some elementary arguments, still keeping the field algebraically closed.

Lemma 32.17.2. *Let k be an algebraically closed field. Let X be an integral scheme locally of finite type over k. Then $\mathcal{O}^*(X)/k^*$ is a finitely generated abelian group.*

Proof. As X is integral the restriction mapping $\mathcal{O}(X) \to \mathcal{O}(U)$ is injective for any nonempty open subscheme $U \subset X$. Hence we may assume that X is affine. Choose a closed immersion $X \to \mathbf{A}^n_k$ and denote $\overline{X}$ the closure of X in $\mathbf{P}^n_k$ via the usual immersion $\mathbf{A}^n_k \to \mathbf{P}^n_k$. Thus we may assume that X is an affine open of a projective variety $\overline{X}$.

Let $\nu : \overline{X}^\nu \to \overline{X}$ be the normalization morphism, see Morphisms, Definition 28.48.12. We know that ν is finite, dominant, and that $\overline{X}^\nu$ is a normal irreducible scheme, see Morphisms, Lemmas 28.48.15, 28.48.16, and 28.19.2. It follows that $\overline{X}^\nu$ is a proper variety, because $\overline{X} \to \mathrm{Spec}(k)$ is proper as a composition of a finite and a proper morphism (see results in Morphisms, Sections 28.42 and 28.44). It also follows that ν is a surjective morphism, because the image of ν is closed and contains the generic point of $\overline{X}$. Hence setting $X^\nu = \nu^{-1}(X)$ we see that it suffices to prove the result for X^ν. In other words, we may assume that X is a nonempty open of a normal proper variety $\overline{X}$. This case is handled by Lemma 32.17.1. □

The preceding lemma implies the following slight generalization.

Lemma 32.17.3. *Let k be an algebraically closed field. Let X be a connected reduced scheme which is locally of finite type over k with finitely many irreducible components. Then $\mathcal{O}^*(X)/k^*$ is a finitely generated abelian group.*

Proof. Let $X = \bigcup X_i$ be the irreducible components. By Lemma 32.17.2 we see that $\mathcal{O}(X_i)^*/k^*$ is a finitely generated abelian group. Let $f \in \mathcal{O}(X)^*$ be in the kernel of the map

$$\mathcal{O}(X)^* \longrightarrow \prod \mathcal{O}(X_i)^*/k^*.$$

Then for each i there exists an element $\lambda_i \in k$ such that $f|_{X_i} = \lambda_i$. By restricting to $X_i \cap X_j$ we conclude that $\lambda_i = \lambda_j$ if $X_i \cap X_j \neq \emptyset$. Since X is connected we conclude that all λ_i agree and hence that $f \in k^*$. This proves that

$$\mathcal{O}(X)^*/k^* \subset \prod \mathcal{O}(X_i)^*/k^*$$

and the lemma follows as on the right we have a product of finitely many finitely generated abelian groups. □

Lemma 32.17.4. *Let k be a field. Let X be a scheme over k which is connected and reduced. Then the integral closure of k in $\Gamma(X, \mathcal{O}_X)$ is a field.*

Proof. Let $k' \subset \Gamma(X, \mathcal{O}_X)$ be the integral closure of k. Then $X \to \mathrm{Spec}(k)$ factors through $\mathrm{Spec}(k')$, see Schemes, Lemma 25.6.4. As X is reduced we see that k' has no nonzero nilpotent elements. As $k \to k'$ is integral we see that every prime ideal of k' is both a maximal ideal and a minimal prime, and $\mathrm{Spec}(k')$ is totally disconnected, see Algebra, Lemmas 10.33.18 and 10.24.6. As X is connected the morphism $X \to \mathrm{Spec}(k')$ is constant, say with image the point corresponding to $\mathfrak{p} \subset k'$. Then any $f \in k'$, $f \notin \mathfrak{p}$ maps to an invertible element of $\mathcal{O}_X$. By definition of k' this then forces f to be a unit of k'. Hence we see that k' is local with maximal ideal $\mathfrak{p}$, see Algebra, Lemma 10.17.2. Since we've already seen that k' is reduced this implies that k' is a field, see Algebra, Lemma 10.24.4. □

Proposition 32.17.5. *Let k be a field. Let X be a scheme over k. Assume that X is locally of finite type over k, connected, reduced, and has finitely many irreducible components. Then $\mathcal{O}(X)^*/k^*$ is a finitely generated abelian group if in addition to the conditions above at least one of the following conditions is satisfied:*

1. *the integral closure of k in $\Gamma(X, \mathcal{O}_X)$ is k,*
2. *X has a k-rational point, or*
3. *X is geometrically integral.*

Proof. Let $\overline{k}$ be an algebraic closure of k. Let Y be a connected component of $(X_{\overline{k}})_{red}$. Note that the canonical morphism $p : Y \to X$ is open (by Morphisms, Lemma 28.24.4) and closed (by Morphisms, Lemma 28.44.7). Hence $p(Y) = X$ as X was assumed connected. In particular, as X is reduced this implies $\mathcal{O}(X) \subset \mathcal{O}(Y)$. By Lemma 32.6.13 we see that Y has finitely many irreducible components. Thus Lemma 32.17.3 applies to Y. This implies that if $\mathcal{O}(X)^*/k^*$ is not a finitely generated abelian group, then there exist elements $f \in \mathcal{O}(X)$, $f \notin k$ which map to an element of $\overline{k}$ via the map $\mathcal{O}(X) \to \mathcal{O}(Y)$. In this case f is algebraic over k, hence integral over k. Thus, if condition (1) holds, then this cannot happen. To finish the proof we show that conditions (2) and (3) imply (1).

Let $k \subset k' \subset \Gamma(X, \mathcal{O}_X)$ be the integral closure of k in $\Gamma(X, \mathcal{O}_X)$. By Lemma 32.17.4 we see that k' is a field. If $e : \mathrm{Spec}(k) \to X$ is a k-rational point, then $e^\sharp : \Gamma(X, \mathcal{O}_X) \to k$ is a section to the inclusion map $k \to \Gamma(X, \mathcal{O}_X)$. In particular the restriction of $e^\sharp$ to k' is a field map $k' \to k$ over k, which clearly shows that (2) implies (1).

If the integral closure k' of k in $\Gamma(X, \mathcal{O}_X)$ is not trivial, then we see that X is either not geometrically connected (if $k \subset k'$ is not purely inseparable) or that X is not geometrically reduced (if $k \subset k'$ is nontrivial purely inseparable). Details omitted. Hence (3) implies (1). □

Lemma 32.17.6. *Let k be a field. Let X be a variety over k. The group $\mathcal{O}(X)^*/k^*$ is a finitely generated abelian group provided at least one of the following conditions holds:*

1. *k is integrally closed in $\Gamma(X, \mathcal{O}_X)$,*
2. *k is algebraically closed in $k(X)$,*
3. *X is geometrically integral over k, or*
4. *k is the "intersection" of the field extensions $k \subset \kappa(x)$ where x runs over the closed points of x.*

Proof. We see that (1) is enough by Proposition 32.17.5. We omit the verification that each of (2), (3), (4) implies (1). □

32.18. Uniqueness of base field

The phrase "let X be a scheme over k" means that X is a scheme which comes equipped with a morphism $X \to \mathrm{Spec}(k)$. Now we can ask whether the field k is uniquely determined by the scheme X. Of course this is not the case, since for example $\mathbf{A}^1_{\mathbf{C}}$ which we ordinarily consider as a scheme over the field $\mathbf{C}$ of complex numbers, could also be considered as a scheme over $\mathbf{Q}$. But what if we ask that the morphism $X \to \mathrm{Spec}(k)$ does not factor as $X \to \mathrm{Spec}(k') \to \mathrm{Spec}(k)$ for any nontrivial field extension $k \subset k'$? In other words we ask that k is somehow maximal such that X lives over k.

An example to show that this still does not guarantee uniqueness of k is the scheme

$$X = \mathrm{Spec}\left(\mathbf{Q}(x)[y]\left[\frac{1}{P(y)}, P \in \mathbf{Q}[y], P \neq 0\right]\right)$$

At first sight this seems to be a scheme over $\mathbf{Q}(x)$, but on a second look it is clear that it is also a scheme over $\mathbf{Q}(y)$. Moreover, the fields $\mathbf{Q}(x)$ and $\mathbf{Q}(y)$ are subfields of $R = \Gamma(X, \mathcal{O}_X)$ which are maximal among the subfields of R (details omitted).

In particular, both $\mathbf{Q}(x)$ and $\mathbf{Q}(y)$ are maximal in the sense above. Note that both morphisms $X \to \mathrm{Spec}(\mathbf{Q}(x))$ and $X \to \mathrm{Spec}(\mathbf{Q}(y))$ are "essentially of finite type" (i.e., the corresponding ring map is essentially of finite type). Hence X is a Noetherian scheme of finite dimension, i.e., it is not completely pathological.

Another issue that can prevent uniqueness is that the scheme X may be nonreduced. In that case there can be many different morphisms from X to the spectrum of a given field. As an explicit example consider the dual numbers $D = \mathbf{C}[y]/(y^2) = \mathbf{C} \oplus \epsilon\mathbf{C}$. Given any derivation $\theta : \mathbf{C} \to \mathbf{C}$ over $\mathbf{Q}$ we get a ring map

$$\mathbf{C} \longrightarrow D, \quad c \longmapsto c + \epsilon\theta(c).$$

The subfield of $\mathbf{C}$ on which all of these maps are the same is the algebraic closure of $\mathbf{Q}$. This means that taking the intersection of all the fields that X can live over may end up being a very small field if X is nonreduced.

One observation in this regard is the following: given a field k and two subfields k_1, k_2 of k such that k is finite over k_1 and over k_2, then in general it is *not* the case that k is finite over $k_1 \cap k_2$. An example is the field $k = \mathbf{Q}(t)$ and its subfields $k_1 = \mathbf{Q}(t^2)$ and $\mathbf{Q}((t+1)^2)$. Namely we have $k_1 \cap k_2 = \mathbf{Q}$ in this case. So in the following we have to be careful when taking intersections of fields.

Having said all of this we now show that if X is locally of finite type over a field, then some uniqueness holds. Here is the precise result.

Proposition 32.18.1. *Let X be a scheme. Let $a : X \to \mathrm{Spec}(k_1)$ and $b : X \to \mathrm{Spec}(k_2)$ be morphisms from X to spectra of fields. Assume a, b are locally of finite type, and X is reduced, and connected. Then we have $k_1' = k_2'$, where $k_i' \subset \Gamma(X, \mathcal{O}_X)$ is the integral closure of k_i in $\Gamma(X, \mathcal{O}_X)$.*

Proof. First, assume the lemma holds in case X is quasi-compact (we will do the quasi-compact case below). As X is locally of finite type over a field, it is locally Noetherian, see Morphisms, Lemma 28.16.6. In particular this means that it is locally connected, connected components of open subsets are open, and intersections of quasi-compact opens are quasi-compact, see Properties, Lemma 27.5.5, Topology, Lemma 5.6.10, Topology, Section 5.8, and Topology, Lemma 5.15.1. Pick an open covering $X = \bigcup_{i \in I} U_i$ such that each U_i is quasi-compact and connected. For each i let $K_i \subset \mathcal{O}_X(U_i)$ be the integral closure of k_1 and of k_2. For each pair $i, j \in I$ we decompose

$$U_i \cap U_j = \coprod U_{i,j,l}$$

into its finitely many connected components. Write $K_{i,j,l} \subset \mathcal{O}(U_{i,j,l})$ for the integral closure of k_1 and of k_2. By Lemma 32.17.4 the rings K_i and $K_{i,j,l}$ are fields. Now we claim that k_1' and k_2' both equal the kernel of the map

$$\prod K_i \longrightarrow \prod K_{i,j,l}, \quad (x_i)_i \longmapsto x_i|_{U_{i,j,l}} - x_j|_{U_{i,j,l}}$$

which proves what we want. Namely, it is clear that k_1' is contained in this kernel. On the other hand, suppose that $(x_i)_i$ is in the kernel. By the sheaf condition $(x_i)_i$ corresponds to $f \in \mathcal{O}(X)$. Pick some $i_0 \in I$ and let $P(T) \in k_1[T]$ be a monic polynomial with $P(x_{i_0}) = 0$. Then we claim that $P(f) = 0$ which proves that $f \in k_1$. To prove this we have to show that $P(x_i) = 0$ for all i. Pick $i \in I$. As X is connected there exists a sequence $i_0, i_1, \ldots, i_n = i \in I$ such that $U_{i_t} \cap U_{i_{t+1}} \neq \emptyset$. Now this means that for each t there exists an l_t such that x_{i_t} and $x_{i_{t+1}}$ map to

the same element of the field $K_{i,j,l}$. Hence if $P(x_{i_t}) = 0$, then $P(x_{i_{t+1}}) = 0$. By induction, starting with $P(x_{i_0}) = 0$ we deduce that $P(x_i) = 0$ as desired.

To finish the proof of the lemma we prove the lemma under the additional hypothesis that X is quasi-compact. By Lemma 32.17.4 after replacing k_i by k_i' we may assume that k_i is integrally closed in $\Gamma(X, \mathcal{O}_X)$. This implies that $\mathcal{O}(X)^*/k_i^*$ is a finitely generated abelian group, see Proposition 32.17.5. Let $k_{12} = k_1 \cap k_2$ as a subring of $\mathcal{O}(X)$. Note that k_{12} is a field. Since

$$k_1^*/k_{12}^* \longrightarrow \mathcal{O}(X)^*/k_2^*$$

we see that k_1^*/k_{12}^* is a finitely generated abelian group as well. Hence there exist $\alpha_1, \ldots, \alpha_n \in k_1^*$ such that every element $\lambda \in k_1$ has the form

$$\lambda = c\alpha_1^{e_1} \ldots \alpha_n^{e_n}$$

for some $e_i \in \mathbf{Z}$ and $c \in k_{12}$. In particular, the ring map

$$k_{12}[x_1, \ldots, x_n, \frac{1}{x_1 \ldots x_n}] \longrightarrow k_1, \quad x_i \longmapsto \alpha_i$$

is surjective. By the Hilbert Nullstellensatz, Algebra, Theorem 10.31.1 we conclude that k_1 is a finite extension of k_{12}. In the same way we conclude that k_2 is a finite extension of k_{12}. In particular both k_1 and k_2 are contained in the integral closure k_{12}' of k_{12} in $\Gamma(X, \mathcal{O}_X)$. But since k_{12}' is a field by Lemma 32.17.4 and since we chose k_i to be integrally closed in $\Gamma(X, \mathcal{O}_X)$ we conclude that $k_1 = k_{12} = k_2$ as desired. □

32.19. Coherent sheaves on projective space

In this section we prove some results on the cohomology of coherent sheaves on $\mathbf{P}^n$ over a field which can be found in [**Mum66**]. These will be useful later when discussing Quot and Hilbert schemes.

32.19.1. Preliminaries. Let k be a field, $n \geq 1$, $d \geq 1$, and let $s \in \Gamma(\mathbf{P}_k^n, \mathcal{O}(d))$ be a nonzero section. In this section we will write $\mathcal{O}(d)$ for the dth twist of the structure sheaf on projective space (Constructions, Definitions 26.10.1 and 26.13.2). Since $\mathbf{P}_k^n$ is a variety this section is regular, hence s is a regular section of $\mathcal{O}(d)$ and defines an effective Cartier divisor $H = Z(s) \subset \mathbf{P}_k^n$, see Divisors, Section 30.9. Such a divisor H is called a *hypersurface* and if $d = 1$ it is called a *hyperplane*.

Lemma 32.19.2. *Let k be a field. Let $n \geq 1$. Let $i : H \to \mathbf{P}_k^n$ be a hyperplane. Then there exists an isomorphism*

$$\varphi : \mathbf{P}_k^{n-1} \longrightarrow H$$

such that $i^*\mathcal{O}(1)$ pulls back to $\mathcal{O}(1)$.

Proof. We have $\mathbf{P}_k^n = \mathrm{Proj}(k[T_0, \ldots, T_n])$. The section s corresponds to a homogeneous form in $T_0, \ldots, T_n$ of degree 1, see Cohomology of Schemes, Section 29.8. Say $s = \sum a_i T_i$. Constructions, Lemma 26.13.6 gives that $H = \mathrm{Proj}(k[T_0, \ldots, T_n]/I)$ for the graded ideal I defined by setting I_d equal to the kernel of the map $\Gamma(\mathbf{P}_k^n, \mathcal{O}(d)) \to \Gamma(H, i^*\mathcal{O}(d))$. By our construction of $Z(s)$ in Divisors, Definition 30.9.18 we see that on $D_+(T_j)$ the ideal of H is generated by $\sum a_i T_i/T_j$ in the polynomial ring $k[T_0/T_j, \ldots, T_n/T_j]$. Thus it is clear that I is the ideal generated by $\sum a_i T_i$. Note that

$$k[T_0, \ldots, T_n]/I = k[T_0, \ldots, T_n]/(\sum a_i T_i) \cong k[S_0, \ldots, S_{n-1}]$$

as graded rings. For example, if $a_n \neq 0$, then mapping S_i equal to the class of T_i works. We obtain the desired isomorphism by functoriality of Proj. Equality of twists of structure sheaves follows for example from Constructions, Lemma 26.11.5. □

Lemma 32.19.3. *Let k be an infinite field. Let $n \geq 1$. Let $\mathcal{F}$ be a coherent module on $\mathbf{P}^n_k$. Then there exist a nonzero section $s \in \Gamma(\mathbf{P}^n_k, \mathcal{O}(1))$ and a short exact sequence*

$$0 \to \mathcal{F}(-1) \to \mathcal{F} \to i_*\mathcal{G} \to 0$$

where $i : H \to \mathbf{P}^n_k$ is the hyperplane H associated to s and $\mathcal{G} = i^*\mathcal{F}$.

Proof. The map $\mathcal{F}(-1) \to \mathcal{F}$ comes from Constructions, Equation (26.10.1.2) with $n = 1$, $m = -1$ and the section s of $\mathcal{O}(1)$. Let's work out what this map looks like if we restrict it to $D_+(T_0)$. Write $D_+(T_0) = \mathrm{Spec}(k[x_1, \ldots, x_n])$ with $x_i = T_i/T_0$. Identify $\mathcal{O}(1)|_{D_+(T_0)}$ with $\mathcal{O}$ using the section T_0. Hence if $s = \sum a_i T_i$ then $s|_{D_+(T_0)} = a_0 + \sum a_i x_i$ with the identification chosen above. Furthermore, suppose $\mathcal{F}|_{D_+(T_0)}$ corresponds to the finite $k[x_1, \ldots, x_n]$-module M. Via the identification $\mathcal{F}(-1) = \mathcal{F} \otimes \mathcal{O}(-1)$ and our chosen trivialization of $\mathcal{O}(1)$ we see that $\mathcal{F}(-1)$ corresponds to M as well. Thus restricting $\mathcal{F}(-1) \to \mathcal{F}$ to $D_+(T_0)$ gives the map

$$M \xrightarrow{a_0 + \sum a_i x_i} M$$

To see that the arrow is injective, it suffices to pick $a_0 + \sum a_i x_i$ outside any of the associated primes of M, see Algebra, Lemma 10.60.9. By Algebra, Lemma 10.60.5 the set $\mathrm{Ass}(M)$ of associated primes of M is finite. Note that for $\mathfrak{p} \in \mathrm{Ass}(M)$ the intersection $\mathfrak{p} \cap \{a_0 + \sum a_i x_i\}$ is a proper k-subvector space. We conclude that there is a finite family of proper sub vector spaces $V_1, \ldots, V_m \subset \Gamma(\mathbf{P}^n_k, \mathcal{O}(1))$ such that if we take s outside of $\bigcup V_i$, then multiplication by s is injective over $D_+(T_0)$. Similarly for the restriction to $D_+(T_j)$ for $j = 1, \ldots, n$. Since k is infinite, a finite union of proper sub vector spaces is never equal to the whole space, hence we may choose s such that the map is injective. The cokernel of $\mathcal{F}(-1) \to \mathcal{F}$ is annihilated by $\mathrm{Im}(s : \mathcal{O}(-1) \to \mathcal{O})$ which is the ideal sheaf of H by Divisors, Definition 30.9.18. Hence we obtain $\mathcal{G}$ on H using Cohomology of Schemes, Lemma 29.9.8. □

Remark 32.19.4. Let k be an infinite field. Let $n \geq 1$. Given a finite number of coherent modules $\mathcal{F}_i$ on $\mathbf{P}^n_k$ we can choose a single $s \in \Gamma(\mathbf{P}^n_k, \mathcal{O}(1))$ such that the statement of Lemma 32.19.3 works for each of them. To prove this, just apply the lemma to $\bigoplus \mathcal{F}_i$.

32.19.5. Regularity.

Definition 32.19.6. Let k be a field. Let $n \geq 0$. Let $\mathcal{F}$ be a coherent sheaf on $\mathbf{P}^n_k$. We say $\mathcal{F}$ is *m-regular* if

$$H^i(\mathbf{P}^n_k, \mathcal{F}(m - i)) = 0$$

for $i = 1, \ldots, n$.

Note that $\mathcal{F} = \mathcal{O}(d)$ is m-regular if and only if $d \geq m$. This follows from the computation of cohomology groups in Cohomology of Schemes, Equation (29.8.1.1). Namely, we see that $H^n(\mathbf{P}^n_k, \mathcal{O}(d)) = 0$ if and only if $d \geq -n$.

Lemma 32.19.7. *Let $k \subset k'$ be an extension of fields. Let $n \geq 0$. Let $\mathcal{F}$ be a coherent sheaf on $\mathbf{P}^n_k$. Let $\mathcal{F}'$ be the pullback of $\mathcal{F}$ to $\mathbf{P}^n_{k'}$. Then $\mathcal{F}$ is m-regular if and only if $\mathcal{F}'$ is m-regular.*

Proof. This is true because

$$H^i(\mathbf{P}^n_{k'}, \mathcal{F}') = H^i(\mathbf{P}^n_k, \mathcal{F}) \otimes_k k'$$

by flat base change, see Cohomology of Schemes, Lemma 29.5.2. □

Lemma 32.19.8. *In the situation of Lemma 32.19.3, if $\mathcal{F}$ is m-regular, then $\mathcal{G}$ is m-regular on $H \cong \mathbf{P}^{n-1}_k$.*

Proof. Recall that $H^i(\mathbf{P}^n_k, i_*\mathcal{G}) = H^i(H, \mathcal{G})$ by Cohomology of Schemes, Lemma 29.2.4. Hence we see that for $i \geq 1$ we get

$$H^i(\mathbf{P}^n_k, \mathcal{F}(m-i)) \to H^i(H, \mathcal{G}(m-i)) \to H^{i+1}(\mathbf{P}^n_k, \mathcal{F}(m-1-i))$$

as part of the long exact sequence associated to the short exact sequence $0 \to \mathcal{F}(m-1-i) \to \mathcal{F}(m-i) \to i_*\mathcal{G}(m-i) \to 0$ we obtain from the exact sequence of Lemma 32.19.3 by tensoring with the invertible sheaf $\mathcal{O}(m-i)$. The lemma follows. □

Lemma 32.19.9. *Let k be a field. Let $n \geq 0$. Let $\mathcal{F}$ be a coherent sheaf on $\mathbf{P}^n_k$. If $\mathcal{F}$ is m-regular, then $\mathcal{F}$ is $(m+1)$-regular.*

Proof. We prove this by induction on n. If $n = 0$ every sheaf is m-regular for all m and there is nothing to prove. By Lemma 32.19.7 we may replace k by an infinite overfield and assume k is infinite. Thus we may apply Lemma 32.19.3. By Lemma 32.19.8 we know that $\mathcal{G}$ is m-regular. By induction on n we see that $\mathcal{G}$ is $(m+1)$-regular. Considering the long exact cohomology sequence associated to the sequence

$$0 \to \mathcal{F}(m-i) \to \mathcal{F}(m+1-i) \to i_*\mathcal{G}(m+1-i) \to 0$$

the reader easily deduces for $i \geq 1$ the vanishing of $H^i(\mathbf{P}^n_k, \mathcal{F}(m+1-i))$ from the (known) vanishing of $H^i(\mathbf{P}^n_k, \mathcal{F}(m-i))$ and $H^i(\mathbf{P}^n_k, \mathcal{G}(m+1-i))$. □

Lemma 32.19.10. *Let k be a field. Let $n \geq 0$. Let $\mathcal{F}$ be a coherent sheaf on $\mathbf{P}^n_k$. If $\mathcal{F}$ is m-regular, then the multiplication map*

$$H^0(\mathbf{P}^n_k, \mathcal{F}(m)) \otimes_k H^0(\mathbf{P}^n_k, \mathcal{O}(1)) \longrightarrow H^0(\mathbf{P}^n_k, \mathcal{F}(m+1))$$

is surjective.

Proof. Let $k \subset k'$ be an extension of fields. Let $\mathcal{F}'$ be as in Lemma 32.19.7. By Cohomology of Schemes, Lemma 29.5.2 the base change of the linear map of the lemma to k' is the same linear map for the sheaf $\mathcal{F}'$. Since $k \to k'$ is faithfully flat it suffices to prove the lemma over k', i.e., we may assume k is infinite.

Assume k is infinite. We prove the lemma by induction on n. The case $n = 0$ is trivial as $\mathcal{O}(1) \cong \mathcal{O}$ is generated by T_0. For $n > 0$ apply Lemma 32.19.3 and tensor the sequence by $\mathcal{O}(m+1)$ to get

$$0 \to \mathcal{F}(m) \xrightarrow{s} \mathcal{F}(m+1) \to i_*\mathcal{G}(m+1) \to 0$$

Let $t \in H^0(\mathbf{P}^n_k, \mathcal{F}(m+1))$. By induction the image $\overline{t} \in H^0(H, \mathcal{G}(m+1))$ is the image of $\sum g_i \otimes \overline{s}_i$ with $\overline{s}_i \in \Gamma(H, \mathcal{O}(1))$ and $g_i \in H^0(H, \mathcal{G}(m))$. Since $\mathcal{F}$ is m-regular we have $H^1(\mathbf{P}^n_k, \mathcal{F}(m-1)) = 0$, hence long exact cohomology sequence associated to the short exact sequence

$$0 \to \mathcal{F}(m-1) \xrightarrow{s} \mathcal{F}(m) \to i_*\mathcal{G}(m) \to 0$$

shows we can lift g_i to $f_i \in H^0(\mathbf{P}^n_k, \mathcal{F}(m))$. We can also lift $\overline{s}_i$ to $s_i \in H^0(\mathbf{P}^n_k, \mathcal{O}(1))$ (see proof of Lemma 32.19.2 for example). After substracting the image of $\sum f_i \otimes s_i$ from t we see that we may assume $\overline{t} = 0$. But this exactly means that t is the image of $f \otimes s$ for some $f \in H^0(\mathbf{P}^n_k, \mathcal{F}(m))$ as desired. □

Lemma 32.19.11. *Let k be a field. Let $n \geq 0$. Let $\mathcal{F}$ be a coherent sheaf on $\mathbf{P}^n_k$. If $\mathcal{F}$ is m-regular, then $\mathcal{F}(m)$ is globally generated.*

Proof. For all $d \gg 0$ the sheaf $\mathcal{F}(d)$ is globally generated. This follows for example from the first part of Cohomology of Schemes, Lemma 29.15.1. Pick $d \geq m$ such that $\mathcal{F}(d)$ is globally generated. Choose a basis $f_1, \ldots, f_r \in H^0(\mathbf{P}^n_k, \mathcal{F})$. By Lemma 32.19.10 every element $f \in H^0(\mathbf{P}^n_k, \mathcal{F}(d))$ can be written as $f = \sum P_i f_i$ for some $P_i \in k[T_0, \ldots, T_n]$ homogeneous of degree $d - m$. Since the sections f generate $\mathcal{F}(d)$ it follows that the sections f_i generate $\mathcal{F}(m)$. □

32.19.12. Hilbert polynomials. Let k be a field. Let X be a proper scheme over k. Let $\mathcal{F}$ be a coherent $\mathcal{O}_X$-module. In this situation the *Euler characteristic of $\mathcal{F}$* is the integer

$$\chi(X, \mathcal{F}) = \sum (-1)^i \dim_k H^i(X, \mathcal{F}).$$

Note that only a finite number of the vector spaces $H^i(X, \mathcal{F})$ are nonzero (Cohomology of Schemes, Lemma 29.4.4) and that each of these spaces is finite dimensional (Cohomology of Schemes, Lemma 29.17.4). Thus $\chi(X, \mathcal{F}) \in \mathbf{Z}$ is well defined. Observe that this definition depends on the field k and not just on the pair $(X, \mathcal{F})$.

Lemma 32.19.13. *Let k be a field. Let X be a proper scheme over k. Let $0 \to \mathcal{F}_1 \to \mathcal{F}_2 \to \mathcal{F}_3 \to 0$ be a short exact sequence of coherent modules on X. Then*

$$\chi(X, \mathcal{F}_2) = \chi(X, \mathcal{F}_1) + \chi(X, \mathcal{F}_3)$$

Proof. Consider the long exact sequence of cohomology

$$0 \to H^0(X, \mathcal{F}_1) \to H^0(X, \mathcal{F}_2) \to H^0(X, \mathcal{F}_3) \to H^1(X, \mathcal{F}_1) \to \ldots$$

associated to the short exact sequence of the lemma. The rank-nullity theorem in linear algebra shows that

$$0 = \dim H^0(X, \mathcal{F}_1) - \dim H^0(X, \mathcal{F}_2) + \dim H^0(X, \mathcal{F}_3) - \dim H^1(X, \mathcal{F}_1) + \ldots$$

This immediately implies the lemma. □

Lemma 32.19.14. *Let $k \subset k'$ be an extension of fields. Let X be a proper scheme over k. Let $\mathcal{F}$ be a coherent sheaf on X. Let $\mathcal{F}'$ be the pullback of $\mathcal{F}$ to $X_{k'}$. Then $\chi(X, \mathcal{F}) = \chi(X', \mathcal{F}')$.*

Proof. This is true because

$$H^i(X_{k'}, \mathcal{F}') = H^i(X, \mathcal{F}) \otimes_k k'$$

by flat base change, see Cohomology of Schemes, Lemma 29.5.2. □

Lemma 32.19.15. *Let k be a field. Let $n \geq 0$. Let $\mathcal{F}$ be a coherent sheaf on $\mathbf{P}^n_k$. The function*

$$d \longmapsto \chi(\mathbf{P}^n_k, \mathcal{F}(d))$$

is a polynomial.

Proof. We prove this by induction on n. If $n = 0$, then $\mathbf{P}^n_k = \mathrm{Spec}(k)$ and $\mathcal{F}(d) = \mathcal{F}$. Hence in this case the function is constant, i.e., a polynomial of degree 0. Assume $n > 0$. By Lemma 32.19.14 we may assume k is infinite. Apply Lemma 32.19.3. Applying Lemma 32.19.13 to the twisted sequences $0 \to \mathcal{F}(d-1) \to \mathcal{F}(d) \to i_*\mathcal{G}(d) \to 0$ we obtain

$$\chi(\mathbf{P}^n_k, \mathcal{F}(d)) - \chi(\mathbf{P}^n_k, \mathcal{F}(d-1)) = \chi(H, \mathcal{G}(d))$$

(this also uses the identification of the cohomology of $i_*\mathcal{G}$ with the cohomology of $\mathcal{G}$, see Cohomology of Schemes, Lemma 29.2.4). Since $H \cong \mathbf{P}^{n-1}_k$ (Lemma 32.19.2) by induction the right hand side is a polynomial. The lemma is finished by noting that any function $f : \mathbf{Z} \to \mathbf{Z}$ with the property that the map $d \mapsto f(d) - f(d-1)$ is a polynomial, is itself a polynomial. We omit the proof of this fact (hint: compare with Algebra, Lemma 10.55.5). □

Definition 32.19.16. Let k be a field. Let $n \geq 0$. Let $\mathcal{F}$ be a coherent sheaf on $\mathbf{P}^n_k$. The function $d \mapsto \chi(\mathbf{P}^n_k, \mathcal{F}(d))$ is called the *Hilbert polynomial* of $\mathcal{F}$.

The Hilbert polynomial has coefficients in $\mathbf{Q}$ and not in general in $\mathbf{Z}$. For example the Hilbert polynomial of $\mathcal{O}_{\mathbf{P}^n_k}$ is

$$d \longmapsto \binom{d+n}{n} = \frac{d^n}{n!} + \ldots$$

This follows from the following lemma and the fact that

$$H^0(\mathbf{P}^n_k, \mathcal{O}_{\mathbf{P}^n_k}) = k[T_0, \ldots, T_n]_d$$

(degree d part) whose dimension over k is $\binom{d+n}{n}$.

Lemma 32.19.17. *Let k be a field. Let $n \geq 0$. Let $\mathcal{F}$ be a coherent sheaf on $\mathbf{P}^n_k$ with Hilbert polynomial $P \in \mathbf{Q}[t]$. Then*

$$P(d) = \dim_k H^0(\mathbf{P}^n_k, \mathcal{F}(d))$$

for all $d \gg 0$.

Proof. This follows from the vanishing of cohomology of high enough twists of $\mathcal{F}$. See Cohomology of Schemes, Lemma 29.15.1. □

32.19.18. Boundedness of quotients. In this subsection we bound the regularity of quotients of a given coherent sheaf on $\mathbf{P}^n$ in terms of the Hilbert polynomial.

Lemma 32.19.19. *Let k be a field. Let $n \geq 0$. Let $r \geq 1$. Let $P \in \mathbf{Q}[t]$. There exists an integer m depending on n, r, and P with the following property: if*

$$0 \to \mathcal{K} \to \mathcal{O}^{\oplus r} \to \mathcal{F} \to 0$$

is a short exact sequence of coherent sheaves on $\mathbf{P}^n_k$ and $\mathcal{F}$ has Hilbert polynomial P, then $\mathcal{K}$ is m-regular.

Proof. We prove this by induction on n. If $n = 0$, then $\mathbf{P}^n_k = \mathrm{Spec}(k)$ and any coherent module is 0-regular and any surjective map is surjective on global sections. Assume $n > 0$. Consider an exact sequence as in the lemma. Let $P' \in \mathbf{Q}[t]$ be the polynomial $P'(t) = P(t) - P(t-1)$. Let m' be the integer which works for $n-1$, r, and P'. By Lemmas 32.19.7 and 32.19.14 we may replace k by a field extension, hence we may assume k is infinite. Apply Lemma 32.19.3 to the coherent

sheaf $\mathcal{F}$. The Hilbert polynomial of $\mathcal{F}' = i^*\mathcal{F}$ is P' (see proof of Lemma 32.19.15). Since i^* is right exact we see that $\mathcal{F}'$ is a quotient of $\mathcal{O}_H^{\oplus r} = i^*\mathcal{O}^{\oplus r}$. Thus the induction hypothesis applies to $\mathcal{F}'$ on $H \cong \mathbf{P}_k^{n-1}$ (Lemma 32.19.2). Note that the map $\mathcal{K}(-1) \to \mathcal{K}$ is injective as $\mathcal{K} \subset \mathcal{O}^{\oplus r}$ and has cokernel $i_*\mathcal{H}$ where $\mathcal{H} = i^*\mathcal{K}$. By the snake lemma (Homology, Lemma 12.5.17) we obtain a commutative diagram with exact columns and rows

$$\begin{array}{ccccccccc}
 & & 0 & & 0 & & 0 & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & \mathcal{K}(-1) & \longrightarrow & \mathcal{O}^{\oplus r}(-1) & \longrightarrow & \mathcal{F}(-1) & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & \mathcal{K} & \longrightarrow & \mathcal{O}^{\oplus r} & \longrightarrow & \mathcal{F} & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & i_*\mathcal{H} & \longrightarrow & i_*\mathcal{O}_H^{\oplus r} & \longrightarrow & i_*\mathcal{F}' & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
 & & 0 & & 0 & & 0 & &
\end{array}$$

Thus the induction hypothesis applies to the exact sequence $0 \to \mathcal{H} \to \mathcal{O}_H^{\oplus r} \to \mathcal{F}' \to 0$ on $H \cong \mathbf{P}_k^{n-1}$ (Lemma 32.19.2) and $\mathcal{H}$ is m'-regular. Recall that this implies that $\mathcal{H}$ is d-regular for all $d \geq m'$ (Lemma 32.19.9).

Let $i \geq 2$ and $d \geq m'$. It follows from the long exact cohomology sequence associated to the left column of the diagram above and the vanishing of $H^{i-1}(H, \mathcal{H}(d))$ that the map

$$H^i(\mathbf{P}_k^n, \mathcal{K}(d-1)) \longrightarrow H^i(\mathbf{P}_k^n, \mathcal{K}(d))$$

is injective. As these groups are zero for $d \gg 0$ (Cohomology of Schemes, Lemma 29.15.1) we conclude $H^i(\mathbf{P}_k^n, \mathcal{K}(d))$ are zero for all $d \geq m'$ and $i \geq 2$.

We still have to control H^1. First we observe that all the maps

$$H^1(\mathbf{P}_k^n, \mathcal{K}(m'-1)) \to H^1(\mathbf{P}_k^n, \mathcal{K}(m')) \to H^1(\mathbf{P}_k^n, \mathcal{K}(m'+1)) \to \ldots$$

are surjective by the vanishing of $H^1(H, \mathcal{H}(d))$ for $d \geq m'$. Suppose $d > m'$ is such that

$$H^1(\mathbf{P}_k^n, \mathcal{K}(d-1)) \longrightarrow H^1(\mathbf{P}_k^n, \mathcal{K}(d))$$

is injective. Then $H^0(\mathbf{P}_k^n, \mathcal{K}(d)) \to H^0(H, \mathcal{H}(d))$ is surjective. Consider the commutative diagram

$$\begin{array}{ccc}
H^0(\mathbf{P}_k^n, \mathcal{K}(d)) \otimes_k H^0(\mathbf{P}_k^n, \mathcal{O}(1)) & \longrightarrow & H^0(\mathbf{P}_k^n, \mathcal{K}(d+1)) \\
\downarrow & & \downarrow \\
H^0(H, \mathcal{H}(d)) \otimes_k H^0(H, \mathcal{O}_H(1)) & \longrightarrow & H^0(H, \mathcal{H}(d+1))
\end{array}$$

By Lemma 32.19.10 we see that the bottom horizontal arrow is surjective. Hence the right vertical arrow is surjective. We conclude that

$$H^1(\mathbf{P}_k^n, \mathcal{K}(d)) \longrightarrow H^1(\mathbf{P}_k^n, \mathcal{K}(d+1))$$

is injective. By induction we see that

$$H^1(\mathbf{P}^n_k, \mathcal{K}(d-1)) \to H^1(\mathbf{P}^n_k, \mathcal{K}(d)) \to H^1(\mathbf{P}^n_k, \mathcal{K}(d+1)) \to \ldots$$

are all injective and we conclude that $H^1(\mathbf{P}^n_k, \mathcal{K}(d-1)) = 0$ because of the eventual vanishing of these groups. Thus the dimensions of the groups $H^1(\mathbf{P}^n_k, \mathcal{K}(d))$ for $d \geq m'$ are strictly decreasing until they become zero. It follows that the regularity of $\mathcal{K}$ is bounded by $m' + \dim_k H^1(\mathbf{P}^n_k, \mathcal{K}(m'))$. On the other hand, by the vanishing of the higher cohomology groups we have

$$\dim_k H^1(\mathbf{P}^n_k, \mathcal{K}(m')) = -\chi(\mathbf{P}^n_k, \mathcal{K}(m')) + \dim_k H^0(\mathbf{P}^n_k, \mathcal{K}(m'))$$

Note that the H^0 has dimension bounded by the dimension of $H^0(\mathbf{P}^n_k, \mathcal{O}^{\oplus r}(m'))$ which is at most $r\binom{n+m'}{n}$ if $m' > 0$ and zero if not. Finally, the term $\chi(\mathbf{P}^n_k, \mathcal{K}(m'))$ is equal to $r\binom{n+m'}{n} - P(m')$. This gives a bound of the desired type finishing the proof of the lemma. □

32.20. Glueing dimension one rings

This section contains some algebraic preliminaries to proving that a finite set of codimension 1 points of a separated scheme is contained in an affine open.

Situation 32.20.1. Here we are given a commutative diagram of rings

$$\begin{array}{ccc} A & \longrightarrow & K \\ \uparrow & & \uparrow \\ R & \longrightarrow & B \end{array}$$

where K is a field and A, B are subrings of K with fraction field K. Finally, $R = A \times_K B = A \cap B$.

Lemma 32.20.2. *In Situation 32.20.1 assume that B is a valuation ring. Then for every unit u of A either $u \in R$ or $u^{-1} \in R$.*

Proof. Namely, if the image c of u in K is in B, then $u \in R$. Otherwise, $c^{-1} \in B$ (Algebra, Lemma 10.46.3) and $u^{-1} \in R$. □

The following lemma explains the meaning of the condition "$A \otimes B \to K$ is surjective" which comes up quite a bit in the following.

Lemma 32.20.3. *In Situation 32.20.1 assume A is a Noetherian ring of dimension 1. The following are equivalent*

1. *$A \otimes B \to K$ is not surjective,*
2. *there exists a discrete valuation ring $\mathcal{O} \subset K$ containing both A and B.*

Proof. It is clear that (2) implies (1). On the other hand, if $A \otimes B \to K$ is not surjective, then the image $C \subset K$ is not a field hence C has a nonzero maximal ideal $\mathfrak{m}$. Choose a valuation ring $\mathcal{O} \subset K$ dominating $C_{\mathfrak{m}}$. By Algebra, Lemma 10.113.11 applied to $A \subset \mathcal{O}$ the ring $\mathcal{O}$ is Noetherian. Hence $\mathcal{O}$ is a valuation ring by Algebra, Lemma 10.46.16. □

Lemma 32.20.4. *In Situation 32.20.1 assume*

1. *A is a Noetherian semi-local domain of dimension 1,*
2. *B is a discrete valuation ring,*

Then we have the following two possibilities

(a) *If A^* is not contained in R, then* $\mathrm{Spec}(A) \to \mathrm{Spec}(R)$ *and* $\mathrm{Spec}(B) \to \mathrm{Spec}(R)$ *are open immersions and $K = A \otimes_R B$.*
(b) *If A^* is contained in R, then B dominates one of the local rings of A at a maximal ideal and $A \otimes B \to K$ is not surjective.*

Proof. Assumption (a) implies there is a unit of A whose image in K lies in the maximal ideal of B. Then u is a nonzerodivisor of R and for every $a \in A$ there exists an n such that $u^n a \in R$. It follows that $A = R_u$.

Let $\mathfrak{m}_A$ be the radical of A. Let $x \in \mathfrak{m}_A$ be a nonzero element. Since $\dim(A) = 1$ we see that $K = A_x$. After replacing x by $x^n u^m$ for some $n \geq 1$ and $m \in \mathbf{Z}$ we may assume x maps to a unit of B. We see that for every $b \in B$ we have that $x^n b$ in the image of R for some n. Thus $B = R_x$.

Let $z \in R$. If $z \notin \mathfrak{m}_A$ and z does not map to an element of $\mathfrak{m}_B$, then z is invertible. Thus $x + u$ is invertible in R. Hence $\mathrm{Spec}(R) = D(x) \cup D(u)$. We have seen above that $D(u) = \mathrm{Spec}(A)$ and $D(x) = \mathrm{Spec}(B)$.

Case (b). If $x \in \mathfrak{m}_A$, then $1 + x$ is a unit and hence $1 + x \in R$, i.e, $x \in R$. Thus we see that $\mathfrak{m}_A \subset R \subset A$. In fact, in this case A is integral over R. Namely, write $A/\mathfrak{m}_A = \kappa_1 \times \ldots \times \kappa_n$ as a product of fields. Say $x = (c_1, \ldots, c_r, 0, \ldots, 0)$ is an element with $c_i \neq 0$. Then

$$x^2 - x(c_1, \ldots, c_r, 1, \ldots, 1) = 0$$

Since R contains all units we see that $A/\mathfrak{m}_A$ is integral over the image of R in it, and hence A is integral over R. It follows that $R \subset A \subset B$ as B is integrally closed. Moreover, if $x \in \mathfrak{m}_A$ is nonzero, then $K = A_x = \bigcup x^{-n} A = \bigcup x^{-n} R$. Hence $x^{-1} \notin B$, i.e., $x \in \mathfrak{m}_B$. We conclude $\mathfrak{m}_A \subset \mathfrak{m}_B$. Thus $A \cap \mathfrak{m}_B$ is a maximal ideal of A thereby finishing the proof. □

Lemma 32.20.5. *Let B be a semi-local Noetherian domain of dimension 1. Let B' be the integral closure of B in its fraction field. Then B' is a semi-local Dedekind domain. Let x be a nonzero element of the radical of B'. Then for every $y \in B'$ there exists an n such that $x^n y \in B$.*

Proof. Let $\mathfrak{m}_B$ be the radical of B. The structure of B' results from Algebra, Lemma 10.114.11. Given $x, y \in B'$ as in the statement of the lemma consider the subring $B \subset A \subset B'$ generated by x and y. Then A is finite over B (Algebra, Lemma 10.33.5). Since the fraction fields of B and A are the same we see that the finite module A/B is supported on the set of closed points of B. Thus $\mathfrak{m}_B^n A \subset B$ for a suitable n. Moreover, $\mathrm{Spec}(B') \to \mathrm{Spec}(A)$ is surjective (Algebra, Lemma 10.33.15), hence A is semi-local as well. It also follows that x is in the radical $\mathfrak{m}_A$ of A. Note that $\mathfrak{m}_A = \sqrt{\mathfrak{m}_B A}$. Thus $x^m y \in \mathfrak{m}_B A$ for some m. Then $x^{nm} y \in B$. □

Lemma 32.20.6. *In Situation 32.20.1 assume*

(1) *A is a Noetherian semi-local domain of dimension 1,*
(2) *B is a Noetherian semi-local domain of dimension 1,*
(3) *$A \otimes B \to K$ is surjective.*

Then $\mathrm{Spec}(A) \to \mathrm{Spec}(R)$ *and* $\mathrm{Spec}(B) \to \mathrm{Spec}(R)$ *are open immersions and $K = A \otimes_R B$.*

Proof. Special case: B is integrally closed in K. This means that B is a Dedekind domain (Algebra, Lemma 10.114.10) whence all of its localizations at maximal ideals are discrete valuation rings. Let $\mathfrak{m}_1, \ldots, \mathfrak{m}_r$ be the maximal ideals of B. We set

$$R_1 = A \times_K B_{\mathfrak{m}_1}$$

Observing that $A \otimes_{R_1} B_{\mathfrak{m}_1} \to K$ is surjective we conclude from Lemma 32.20.4 that A and $B_{\mathfrak{m}_1}$ define open subschemes covering $\mathrm{Spec}(R_1)$ and that $K = A \otimes_{R_1} B_{\mathfrak{m}_1}$. In particular R_1 is a semi-local Noetherian ring of dimension 1. By induction we define

$$R_{i+1} = R_i \times_K B_{\mathfrak{m}_{i+1}}$$

for $i = 1, \ldots, r-1$. Observe that $R = R_n$ because $B = B_{\mathfrak{m}_1} \cap \ldots \cap B_{\mathfrak{m}_r}$ (see Algebra, Lemma 10.144.6). It follows from the inductive procedure that $R \to A$ defines an open immersion $\mathrm{Spec}(A) \to \mathrm{Spec}(R)$. On the other hand, the maximal ideals $\mathfrak{n}_i$ of R not in this open correspond to the maximal ideals $\mathfrak{m}_i$ of B and in fact the ring map $R \to B$ defines an isomorphisms $R_{\mathfrak{n}_i} \to B_{\mathfrak{m}_i}$ (details omitted; hint: in each step we added exactly one maximal ideal to $\mathrm{Spec}(R_i)$). It follows that $\mathrm{Spec}(B) \to \mathrm{Spec}(R)$ is an open immersion as desired.

General case. Let $B' \subset K$ be the integral closure of B. See Lemma 32.20.5. Then the special case applies to $R' = A \times_K B'$. Pick $x \in R'$ which is not contained in the maximal ideals of A and is contained in the maximal ideals of B' (see Algebra, Lemma 10.14.4). By Lemma 32.20.5 there exists an integer n such that $x^n \in R = A \times_K B$. Replace x by x^n so $x \in R$. For every $y \in R'$ there exists an integer n such that $x^n y \in R$. On the other hand, it is clear that $R'_x = A$. Thus $R_x = A$. Exchanging the roles of A and B we also find an $y \in R$ such that $B = R_y$. Note that inverting both x and y leaves no primes except (0). Thus $K = R_{xy} = R_x \otimes_R R_y$. This finishes the proof. □

Lemma 32.20.7. *Let K be a field. Let $A_1, \ldots, A_r \subset K$ be Noetherian semi-local rings of dimension 1 with fraction field K. If $A_i \otimes A_j \to K$ is surjective for all $i \neq j$, then there exists a Noetherian semi-local domain $A \subset K$ of dimension 1 containing $A_1, \ldots, A_r$ such that*

(1) *$A \to A_i$ induces an open immersion $j_i : \mathrm{Spec}(A_i) \to \mathrm{Spec}(A)$,*
(2) *$\mathrm{Spec}(A)$ is the union of the opens $j_i(\mathrm{Spec}(A_i))$,*
(3) *each closed point of $\mathrm{Spec}(A)$ lies in exactly one of these opens.*

Proof. Namely, we can take $A = A_1 \cap \ldots \cap A_r$. First we note that (3), once (1) and (2) have been proven, follows from the assumption that $A_i \otimes A_j \to K$ is surjective since if $\mathfrak{m} \in j_i(\mathrm{Spec}(A_i)) \cap j_j(\mathrm{Spec}(A_j))$, then $A_i \otimes A_j \to K$ ends up in $A_{\mathfrak{m}}$. To prove (1) and (2) we argue by induction on r. If $r > 1$ by induction we have the results (1) and (2) for $B = A_2 \cap \ldots \cap A_r$. Then we apply Lemma 32.20.6 to see they hold for $A = A_1 \cap B$. □

Lemma 32.20.8. *Let A be a domain with fraction field K. Let $B_1, \ldots, B_r \subset K$ be Noetherian 1-dimensional semi-local rings whose fraction fields are K. If $A \otimes B_i \to K$ are surjective for $i = 1, \ldots, r$, then there exists an $x \in A$ such that x^{-1} is in the radical of B_i for $i = 1, \ldots, r$.*

Proof. Let B'_i be the integral closure of B_i in K. Suppose we find a nonzero $x \in A$ such that x^{-1} is in the radical of B'_i for $i = 1, \ldots, r$. Then by Lemma 32.20.5, after replacing x by a power we get $x^{-1} \in B_i$. Since $\mathrm{Spec}(B'_i) \to \mathrm{Spec}(B_i)$ is surjective

we see that x^{-1} is then also in the radical of B_i. Thus we may assume that each B_i is a semi-local Dedekind domain.

If B_i is not local, then remove B_i from the list and add back the finite collection of local rings $(B_i)_{\mathfrak{m}}$. Thus we may assume that B_i is a discrete valuation ring for $i = 1, \ldots, r$.

Let $v_i : K \to \mathbf{Z}$, $i = 1, \ldots, r$ be the corresponding discrete valuations (see Algebra, Lemma 10.114.10). We are looking for a nonzero $x \in A$ with $v_i(x) < 0$ for $i = 1, \ldots, r$. We will prove this by induction on r.

If $r = 1$ and the result is wrong, then $A \subset B$ and the map $A \otimes B \to K$ is not surjective, contradiction.

If $r > 1$, then by induction we can find a nonzero $x \in A$ such that $v_i(x) < 0$ for $i = 1, \ldots, r-1$. If $v_r(x) < 0$ then we are done, so we may assume $v_r(x) \geq 0$. By the base case we can find $y \in A$ nonzero such that $v_r(y) < 0$. After replacing x by a power we may assume that $v_i(x) < v_i(y)$ for $i = 1, \ldots, r-1$. Then $x + y$ is the element we are looking for. □

Lemma 32.20.9. *Let A be a ring. Let $\mathfrak{p}_1, \ldots, \mathfrak{p}_r$ be a finite set of a primes of A. Let $S = A \setminus \bigcup \mathfrak{p}_i$. Then S is a multiplicative system and $S^{-1}A$ is a semi-local ring whose maximal ideals correspond to the maximal elements of the set $\{\mathfrak{p}_i\}$.*

Proof. If $a, b \in A$ and $a, b \in S$, then $a, b \notin \mathfrak{p}_i$ hence $ab \notin \mathfrak{p}_i$, hence $ab \in S$. Also $1 \in S$. Thus S is a multiplicative subset of A. By the description of $\mathrm{Spec}(S^{-1}A)$ in Algebra, Lemma 10.16.5 and by Algebra, Lemma 10.14.3 we see that the primes of $S^{-1}A$ correspond to the primes of A contained in one of the $\mathfrak{p}_i$. Hence the maximal ideals of $S^{-1}A$ correspond one-to-one with the maximal (w.r.t. inclusion) elements of the set $\{\mathfrak{p}_1, \ldots, \mathfrak{p}_r\}$. □

32.21. One dimensional Noetherian schemes

Some material leading up to a discussion of algebraic curves.

Lemma 32.21.1. *Let X be a scheme all of whose local rings are Noetherian of dimension ≤ 1. Let $U \subset X$ be a retrocompact open. Denote $j : U \to X$ the inclusion morphism. Then $R^p j_* \mathcal{F} = 0$, $p > 0$ for every quasi-coherent $\mathcal{O}_U$-module $\mathcal{F}$.*

Proof. We may check the vanishing of $R^p j_* \mathcal{F}$ at stalks. Formation of $R^q j_*$ commutes with flat base change, see Cohomology of Schemes, Lemma 29.5.2. Thus we may assume that X is the spectrum of a Noetherian local ring of dimension ≤ 1. In this case X has a closed points x and finitely many other points $x_1, \ldots, x_n$ which specialize to x but not each other (see Algebra, Lemma 10.29.6). If $x \in U$, then $U = X$ and the result is clear. If not, then $U = \{x_1, \ldots, x_r\}$ for some r after possibly renumbering the points. Then U is affine (Schemes, Lemma 25.11.7). Thus the result by Cohomology of Schemes, Lemma 29.2.3. □

Lemma 32.21.2. *Let X be an affine scheme all of whose local rings are Noetherian of dimension ≤ 1. Then any quasi-compact open $U \subset X$ is affine.*

Proof. Denote $j : U \to X$ the inclusion morphism. Let $\mathcal{F}$ be a quasi-coherent $\mathcal{O}_U$-module. By Lemma 32.21.1 the higher direct images $R^p j_* \mathcal{F}$ are zero. The $\mathcal{O}_X$-module $j_* \mathcal{F}$ is quasi-coherent (Schemes, Lemma 25.24.1). Hence it has vanishing

higher cohomology groups by Cohomology of Schemes, Lemma 29.2.2. By the Leray spectral sequence Cohomology, Lemma 20.14.6 we have $H^p(U, \mathcal{F}) = 0$ for all $p > 0$. Thus U is affine, for example by Cohomology of Schemes, Lemma 29.3.1. □

Lemma 32.21.3. *Let X be a scheme. Let $U \subset X$ be an open. Assume*

(1) *U is a retrocompact open of X,*
(2) *$X \setminus U$ is discrete, and*
(3) *for $x \in X \setminus U$ the local ring $\mathcal{O}_{X,x}$ is Noetherian of dimension ≤ 1.*

Then (1) there exists an invertible $\mathcal{O}_X$-module $\mathcal{L}$ and a section s such that $U = X_s$ and (2) the map $Pic(X) \to Pic(U)$ is surjective.

Proof. Let $X \setminus U = \{x_i; i \in I\}$. Choose affine opens $U_i \subset X$ with $x_i \in X$ and $x_j \notin U_i$ for $j \neq i$. This is possible by condition (2). Say $U_i = \mathrm{Spec}(A_i)$. Let $\mathfrak{m}_i \subset A_i$ be the maximal ideal corresponding to x_i. By our assumption on the local rings there are only a finite number of prime ideals $\mathfrak{q} \subset \mathfrak{m}_i$, $\mathfrak{q} \neq \mathfrak{m}_i$ (see Algebra, Lemma 10.29.6). Thus by prime avoidance (Algebra, Lemma 10.14.3) we can find $f_i \in \mathfrak{m}_i$ not contained in any of those primes. Then $V(f_i) = \{\mathfrak{m}_i\} \amalg Z_i$ for some closed subset $Z_i \subset U_i$ because Z_i is a retrocompact open subset of $V(f_i)$ closed under specialization, see Algebra, Lemma 10.38.7. After shrinking U_i we may assume $V(f_i) = \{x_i\}$. Then

$$\mathcal{U} : X = U \cup \bigcup U_i$$

is an open covering of X. Consider the 2-cocycle with values in $\mathcal{O}_X^*$ given by f_i on $U \cap U_i$ and by f_i/f_j on $U_i \cap U_j$. This defines a line bundle $\mathcal{L}$ such that the section s defined by 1 on U and f_i on U_i is as in the statement of the lemma.

Let $\mathcal{N}$ be an invertible $\mathcal{O}_U$-module. Let N_i be the invertible $(A_i)_{f_i}$ module such that $\mathcal{N}|_{U \cap U_i}$ is equal to $\tilde{N}_i$. Observe that $(A_{\mathfrak{m}_i})_{f_i}$ is an Artinian ring (as a dimension zero Noetherian ring, see Algebra, Lemma 10.57.4). Thus it is a product of local rings (Algebra, Lemma 10.49.8) and hence has trivial Picard group. Thus, after shrinking U_i (i.e., after replacing A_i by $(A_i)_g$ for some $g \in A_i$, $g \notin \mathfrak{m}_i$) we can assume that $N_i = (A_i)_{f_i}$, i.e., that $\mathcal{N}|_{U \cap U_i}$ is trivial. In this case it is clear how to extend $\mathcal{N}$ to an invertible sheaf over X (by extending it by a trivial invertible module over each U_i). □

Lemma 32.21.4. *Let X be an integral separated scheme. Let $U \subset X$ be a nonempty affine open such that $X \setminus U$ is a finite set of points $x_1, \ldots, x_r$ with $\mathcal{O}_{X,x_i}$ Noetherian of dimension 1. Then there exists a globally generated invertible $\mathcal{O}_X$-module $\mathcal{L}$ and a section s such that $U = X_s$.*

Proof. Say $U = \mathrm{Spec}(A)$ and let K be the fraction field of X. Write $B_i = \mathcal{O}_{X,x_i}$ and $\mathfrak{m}_i = \mathfrak{m}_{x_i}$. Since $x_i \notin U$ we see that the open $U \times_X \mathrm{Spec}(B_i)$ of $\mathrm{Spec}(B_i)$ has only one point, i.e., $U \times_X \mathrm{Spec}(B_i) = \mathrm{Spec}(K)$. Since X is separated, we find that $\mathrm{Spec}(K)$ is a closed subscheme of $U \times \mathrm{Spec}(B_i)$, i.e., the map $A \otimes B_i \to K$ is a surjection. By Lemma 32.20.8 we can find a nonzero $f \in A$ such that $f^{-1} \in \mathfrak{m}_i$ for $i = 1, \ldots, r$. Pick opens $x_i \in U_i \subset X$ such that $f^{-1} \in \mathcal{O}(U_i)$. Then

$$\mathcal{U} : X = U \cup \bigcup U_i$$

is an open covering of X. Consider the 2-cocycle with values in $\mathcal{O}_X^*$ given by f on $U \cap U_i$ and by 1 on $U_i \cap U_j$. This defines a line bundle $\mathcal{L}$ with two sections:

(1) a section s defined by 1 on U and f^{-1} on U_i is as in the statement of the lemma, and
(2) a section t defined by f on U and 1 on U_i.

Note that $X_t \supset U_1 \cup \ldots \cup U_r$. Hence s, t generate $\mathcal{L}$ and the lemma is proved. □

Lemma 32.21.5. *Let X be a quasi-compact scheme. If for every $x \in X$ there exists a pair $(\mathcal{L}, s)$ consisting of a globally generated invertible sheaf $\mathcal{L}$ and a global section s such that $x \in X_s$ and X_s is affine, then X has an ample invertible sheaf.*

Proof. Since X is quasi-compact we can find a finite collection $(\mathcal{L}_i, s_i)$, $i = 1, \ldots, n$ of pairs such that X_{s_i} is affine and $X = \bigcup X_{s_i}$. Again because X is quasi-compact we can find, for each i, a finite collection of sections $t_{i,j}$, $j = 1, \ldots, m_i$ such that $X = \bigcup X_{t_{i,j}}$. Set $t_{i,0} = s_i$. Consider the invertible sheaf

$$\mathcal{L} = \mathcal{L}_1 \otimes_{\mathcal{O}_X} \ldots \otimes_{\mathcal{O}_X} \mathcal{L}_n$$

and the global sections

$$\tau_J = t_{1,j_1} \otimes \ldots \otimes t_{n,j_n}$$

By Properties, Lemma 27.24.4 the open X_{τ_J} is affine as soon as $j_i = 0$ for some i. It is a simple matter to see that these opens cover X. Hence $\mathcal{L}$ is ample by definition. □

Lemma 32.21.6. *Let X be a Noetherian integral separated scheme of dimension 1. Then X has an ample invertible sheaf.*

Proof. Choose an affine open covering $X = U_1 \cup \ldots \cup U_n$. Since X is Noetherian, each of the sets $X \setminus U_i$ is finite. Thus by Lemma 32.21.4 we can find a pair $(\mathcal{L}_i, s_i)$ consisting of a globally generated invertible sheaf $\mathcal{L}_i$ and a global section s_i such that $U_i = X_{s_i}$. We conclude that X has an ample invertible sheaf by Lemma 32.21.5. □

Lemma 32.21.7. *Let X be a scheme. Let $Z_1, \ldots, Z_n \subset X$ be closed subschemes. Let $\mathcal{L}_i$ be an invertible sheaf on Z_i. Assume that*

(1) *X is reduced,*
(2) *$X = \bigcup Z_i$ set theoretically, and*
(3) *$Z_i \cap Z_j$ is a discrete topological space for $i \neq j$.*

Then there exists an invertible sheaf $\mathcal{L}$ on X whose restriction to Z_i is $\mathcal{L}_i$. Moreover, if we are given sections $s_i \in \Gamma(Z_i, \mathcal{L}_i)$ which are nonvanishing at the points of $Z_i \cap Z_j$, then we can choose $\mathcal{L}$ such that there exists a $s \in \Gamma(X, \mathcal{L})$ with $s|_{Z_i} = s_i$ for all i.

Proof. Set $T = \bigcup_{i \neq j} Z_i \cap Z_j$. As X is reduced we have

$$X \setminus T = \bigcup (Z_i \setminus T)$$

as schemes. Assumption (3) implies T is a discrete subset of X. Thus for each $t \in T$ we can find an open $U_t \subset X$ with $t \in U_t$ but $t' \notin U_t$ for $t' \in T$, $t' \neq t$. By shrinking U_t if necessary, we may assume that there exist isomorphisms $\varphi_{t,i} : \mathcal{L}_i|_{U_t \cap Z_i} \to \mathcal{O}_{U_t \cap Z_i}$. Furthermore, for each i choose an open covering

$$Z_i \setminus T = \bigcup\nolimits_j U_{ij}$$

such that there exist isomorphisms $\varphi_{i,j} : \mathcal{L}_i|_{U_{ij}} \cong \mathcal{O}_{U_{ij}}$. Observe that

$$\mathcal{U} : X = \bigcup U_t \cup \bigcup U_{ij}$$

is an open covering of X. We claim that we can use the isomorphisms $\varphi_{t,i}$ and $\varphi_{i,j}$ to define a 2-cocycle with values in $\mathcal{O}_X^*$ for this covering that defines $\mathcal{L}$ as in the statement of the lemma.

Namely, if $i \neq i'$, then $U_{i,j} \cap U_{i',j'} = \emptyset$ and there is nothing to do. For $U_{i,j} \cap U_{i,j'}$ we have $\mathcal{O}_X(U_{i,j} \cap U_{i,j'}) = \mathcal{O}_{Z_i}(U_{i,j} \cap U_{i,j'})$ by the first remark of the proof. Thus the transition function for $\mathcal{L}_i$ (more precisely $\varphi_{i,j} \circ \varphi_{i,j'}^{-1}$) defines the value of our cocycle on this intersection. For $U_t \cap U_{i,j}$ we can do the same thing as before. Finally, for $t \neq t'$ we have

$$U_t \cap U_{t'} = \coprod (U_t \cap U_{t'}) \cap Z_i$$

and moreover the intersections $U_t \cap U_{t'} \cap Z_i$ is contained in $Z_t i \setminus T$. Hence by the same reasoning as before we see that

$$\mathcal{O}_X(U_t \cap U_{t'}) = \prod \mathcal{O}_{Z_i}(U_t \cap U_{t'} \cap Z_i)$$

and we can use the transition functions fpr $\mathcal{L}_i$ (more precisely $\varphi_{t,i} \circ \varphi_{t',i}^{-1}$) to define the value of our cocycle on $U_t \cap U_{t'}$. This finishes the proof of existence of $\mathcal{L}$.

Given sections s_i as in the last assertion of the lemma, in the argument above, we choose U_t such that $s_i|_{U_t \cap Z_i}$ is nonvanishing and we choose $\varphi_{t,i}$ such that $\varphi_{t,i}(s_i|_{U_t \cap Z_i}) = 1$. Then using 1 over U_t and $\varphi_{i,j}(s_i|_{U_{i,j}})$ over $U_{i,j}$ will define a section of $\mathcal{L}$ which restricts to s_i over Z_i. □

Remark 32.21.8. Let A be a reduced ring. Let I, J be ideals of A such that $V(I) \cup V(J) = \mathrm{Spec}(A)$. Set $B = A/J$. Then $I \to IB$ is an isomorphism of A-modules. Namely, we have $IB = I + J/J = I/(I \cap J)$ and $I \cap J$ is zero because A is reduced and $\mathrm{Spec}(A) = V(I) \cup V(J) = V(I \cap J)$. Thus for any projective A-module P we also have $IP = I(P/JP)$.

Lemma 32.21.9. *Let X be a Noetherian reduced separated scheme of dimension 1. Then X has an ample invertible sheaf.*

Proof. Let Z_i, $i = 1, \ldots, n$ be the irreducible components of X. We view these as reduced closed subschemes of X. By Lemma 32.21.6 there exist ample invertible sheaves $\mathcal{L}_i$ on Z_i. Set $T = \bigcup_{i \neq j} Z_i \cap Z_j$. As X is Noetherian of dimension 1, the set T is finite and consists of closed points of X. For each i we may, possibly after replacing $\mathcal{L}_i$ by a power, choose $s_i \in \Gamma(Z_i, \mathcal{L}_i)$ such that $(Z_i)_{s_i}$ is affine and contains $T \cap Z_i$, see Properties, Lemma 27.27.6.

By Lemma 32.21.7 we can find an invertible sheaf $\mathcal{L}$ on X and $s \in \Gamma(X, \mathcal{L})$ such that $(\mathcal{L}, s)|_{Z_i} = (\mathcal{L}_i, s_i)$. Observe that X_s contains T and is set theoretically equal to the affine closed subschemes $(Z_i)_{s_i}$. Thus it is affine by Limits, Lemma 31.10.3. To finish the proof, it suffices to find for every $x \in X$, $x \notin T$ an integer $m > 0$ and a section $t \in \Gamma(X, \mathcal{L}^{\otimes m})$ such that X_t is affine and $x \in X_t$. Since $x \notin T$ we see that $x \in Z_i$ for some unique i, say $i = 1$. Let $Z \subset X$ be the reduced closed subscheme whose underlying topological space is $Z_2 \cup \ldots \cup Z_n$. Let $\mathcal{I} \subset \mathcal{O}_X$ be the ideal sheaf of Z. Denote that $\mathcal{I}_1 \subset \mathcal{O}_{Z_1}$ the inverse image of this ideal sheaf under the inclusion morphism $Z_1 \to X$. Observe that

$$\Gamma(X, \mathcal{I}\mathcal{L}^{\otimes m}) = \Gamma(Z_1, \mathcal{I}_1 \mathcal{L}_1^{\otimes m})$$

see Remark 32.21.8. Thus it suffices to find $m > 0$ and $t \in \Gamma(Z_1, \mathcal{I}_1\mathcal{L}_1^{\otimes m})$ with $x \in (Z_1)_t$ affine. Since $\mathcal{L}_1$ is ample and since x is not in $Z_1 \cap T = V(\mathcal{I}_1)$ we can find a section $t_1 \in \Gamma(Z_1, \mathcal{I}_1\mathcal{L}_1^{\otimes m_1})$ with $x \in (Z_1)_{t_1}$, see Properties, Proposition 27.24.14. Since $\mathcal{L}_1$ is ample we can find a section $t_2 \in \Gamma(Z_1, \mathcal{L}_1^{\otimes m_2})$ with $x \in (Z_1)_{t_2}$ and $(Z_1)_{t_2}$ affine, see Properties, Definition 27.24.1. Set $m = m_1 + m_2$ and $t = t_1t_2$. Then $t \in \Gamma(Z_1, \mathcal{I}_1\mathcal{L}_1^{\otimes m})$ with $x \in (Z_1)_t$ by construction and $(Z_1)_t$ is affine by Properties, Lemma 27.24.4. □

Lemma 32.21.10. *Let $i : Z \to X$ be a closed immersion of schemes inducing a homeomorphism on underlying topological spaces. If the underlying topological space of X is Noetherian and $\dim(X) \leq 1$, then $Pic(X) \to Pic(Z)$ is surjective.*

Proof. Consider the short exact sequence

$$0 \to (1 + \mathcal{I})^* \to \mathcal{O}_X^* \to \mathcal{O}_Z^* \to 0$$

of sheaves of abelian groups on X. Since $\dim(X) \leq 1$ we see that $H^2(X, \mathcal{F}) = 0$ for any abelian sheaf $\mathcal{F}$, see Cohomology, Lemma 20.21.5. Hence the map $H^1(X, \mathcal{O}_X^*) \to H^1(Z, \mathcal{O}_Z^*)$ is surjective. This proves the lemma by Cohomology, Lemma 20.6.1. □

Proposition 32.21.11. *Let X be a Noetherian separated scheme of dimension 1. Then X has an ample invertible sheaf.*

Proof. Let $Z \subset X$ be the reduction of X. By Lemma 32.21.9 the scheme Z has an ample invertible sheaf. Thus by Lemma 32.21.10 there exists an invertible $\mathcal{O}_X$-module $\mathcal{L}$ on X whose restriction to Z is ample. Then $\mathcal{L}$ is ample by an application of Cohomology of Schemes, Lemma 29.14.5. □

Remark 32.21.12. In fact, if X is a scheme whose reduction is a Noetherian separated scheme of dimension 1, then X has an ample invertible sheaf. The argument to prove this is the same as the proof of Proposition 32.21.11 except one uses Limits, Lemma 31.10.4 instead of Cohomology of Schemes, Lemma 29.14.5.

32.22. Finding affine opens

We continue the discussion started in Properties, Section 27.27. It turns out that we can find affines containing a finite given set of codimension 1 points on a separated scheme. See Proposition 32.22.7.

We will improve on the following lemma in Descent, Lemma 34.21.4.

Lemma 32.22.1. *Let $f : X \to Y$ be a morphism of schemes. Let X^0 denote the set of generic points of irreducible components of X. If*

1. *f is separated,*
2. *there is an open covering $X = \bigcup U_i$ such that $f|_{U_i} : U_i \to X$ is an open immersion, and*
3. *if $\xi, \xi' \in X^0$, $\xi \neq \xi'$, then $f(\xi) \neq f(\xi')$,*

then f is an open immersion.

Proof. Suppose that $y = f(x) = f(x')$. Pick a specialization $y_0 \rightsquigarrow y$ where y_0 is a generic point of an irreducible component of Y. Since f is locally on the source an isomorphism we can pick specializations $x_0 \rightsquigarrow x$ and $x_0' \rightsquigarrow x'$ mapping to $y_0 \rightsquigarrow y$. Note that $x_0, x_0' \in X^0$. Hence $x_0 = x_0'$ by assumption (3). As f is separated we conclude that $x = x'$. Thus f is an open immersion. □

Lemma 32.22.2. *Let $X \to S$ be a morphism of schemes. Let $x \in X$ be a point with image $s \in S$. If*

(1) $\mathcal{O}_{X,x} = \mathcal{O}_{S,s}$,
(2) *X is reduced,*
(3) *$X \to S$ is of finite type, and*
(4) *S has finitely many irreducible components,*

then there exists an open neighbourhood U of x such that $f|_U$ is an open immersion.

Proof. We may remove the (finitely many) irreducible components of S which do not contain s. We may replace S by an affine open neighbourhood of s. We may replace X by an affine open neighbourhood of x. Say $S = \mathrm{Spec}(A)$ and $X = \mathrm{Spec}(B)$. Let $\mathfrak{q} \subset B$, resp. $\mathfrak{p} \subset A$ be the prime ideal corresponding to x, resp. s. As A is a reduced and all of the minimal primes of A are contained in $\mathfrak{p}$ we see that $A \subset A_\mathfrak{p}$. As $X \to S$ is of finite type, B is of finite type over A. Let $b_1, \ldots, b_n \in B$ be elements which generate B over A Since $A_\mathfrak{p} = B_\mathfrak{q}$ we can find $f \in A$, $f \notin \mathfrak{p}$ and $a_i \in A$ such that b_i and a_i/f have the same image in $B_\mathfrak{q}$. Thus we can find $g \in B$, $g \notin \mathfrak{q}$ such that $g(fb_i - a_i) = 0$ in B. It follows that the image of $A_f \to B_{fg}$ contains the images of $b_1, \ldots, b_n$, in particular also the image of g. Choose $n \geq 0$ and $f' \in A$ such that f'/f^n maps to the image of g in B_{fg}. Since $A_\mathfrak{p} = B_\mathfrak{q}$ we see that $f' \notin \mathfrak{p}$. We conclude that $A_{ff'} \to B_{fg}$ is surjective. Finally, as $A_{ff'} \subset A_\mathfrak{p} = B_\mathfrak{q}$ (see above) the map $A_{ff'} \to B_{fg}$ is injective, hence an isomorphism. □

Lemma 32.22.3. *Let $f : T \to X$ be a morphism of schemes. Let X^0, resp. T^0 denote the sets of generic points of irreducible components. Let $t_1, \ldots, t_m \in T$ be a finite set of points with images $x_j = f(t_j)$. If*

(1) *T is affine,*
(2) *X is quasi-separated,*
(3) *X^0 is finite*
(4) *$f(T^0) \subset X^0$ and $f : T^0 \to X^0$ is injective, and*
(5) *$\mathcal{O}_{X,x_j} = \mathcal{O}_{T,t_j}$,*

then there exists an affine open of X containing $x_1, \ldots, x_r$.

Proof. Using Limits, Proposition 31.10.2 there is an immediate reduction to the case where X and T are reduced. Details omitted.

Assume X and T are reduced. We may write $T = \lim_{i \in I} T_i$ as a directed limit of schemes of finite presentation over X with affine transition morphisms, see Limits, Lemma 31.6.1. Pick $i \in I$ such that T_i is affine, see Limits, Lemma 31.3.10. Say $T_i = \mathrm{Spec}(R_i)$ and $T = \mathrm{Spec}(R)$. Let $R' \subset R$ be the image of $R_i \to R$. Then $T' = \mathrm{Spec}(R')$ is affine, reduced, of finite type over X, and $T \to T'$ dominant. For $j = 1, \ldots, r$ let $t'_j \in T'$ be the image of t_j. Consider the local ring maps

$$\mathcal{O}_{X,x_j} \to \mathcal{O}_{T',t'_j} \to \mathcal{O}_{T,t_j}$$

Denote $(T')^0$ the set of generic points of irreducible components of T'. Let $\xi \rightsquigarrow t'_j$ be a specialization with $\xi \in (T')^0$. As $T \to T'$ is dominant we can choose $\eta \in T^0$ mapping to ξ (warning: a priori we do not know that η specializes to t_j). Assumption (3) applied to η tells us that the image θ of ξ in X corresponds to a minimal prime of $\mathcal{O}_{X,x_j}$. Lifting ξ via the isomorphism of (5) we obtain a specialization $\eta' \rightsquigarrow t_j$ with $\eta' \in X^0$ mapping to $\theta \rightsquigarrow x_j$. The injectivity of (4)

shows that $\eta = \eta'$. Thus every minimal prime of $\mathcal{O}_{T',t'_j}$ lies below a minimal prime of $\mathcal{O}_{T,t_j}$. We conclude that $\mathcal{O}_{T',t'_j} \to \mathcal{O}_{T,t_j}$ is injective, hence both maps above are isomorphisms.

By Lemma 32.22.2 there exists an open $U \subset T'$ containing all the points t'_j such that $U \to X$ is a local isomorphism as in Lemma 32.22.1. By that lemma we see that $U \to X$ is an open immersion. Finally, by Properties, Lemma 27.27.5 we can find an open $W \subset U \subset T'$ containing all the t'_j. The image of W in X is the desired affine open. □

Lemma 32.22.4. *Let X be an integral separated scheme. Let $x_1, \ldots, x_r \in X$ be a finite set of points such that $\mathcal{O}_{X,x_i}$ is Noetherian of dimension ≤ 1. Then there exists an affine open subscheme of X containing all of $x_1, \ldots, x_r$.*

Proof. Let K be the field of rational functions of X. Set $A_i = \mathcal{O}_{X,x_i}$. Then $A_i \subset K$ and K is the fraction field of A_i. Since X is separated, and $x_i \neq x_j$ there cannot be a valuation ring $\mathcal{O} \subset K$ dominating both A_i and A_j. Namely, considering the diagram

$$\begin{array}{ccc} \mathrm{Spec}(\mathcal{O}) & \longrightarrow & \mathrm{Spec}(A_1) \\ \downarrow & & \downarrow \\ \mathrm{Spec}(A_2) & \longrightarrow & X \end{array}$$

and applying the valuative criterion of separatedness (Schemes, Lemma 25.22.1) we would get $x_i = x_j$. Thus we see by Lemma 32.20.3 that $A_i \otimes A_j \to K$ is surjective for all $i \neq j$. By Lemma 32.20.7 we see that $A = A_1 \cap \ldots \cap A_r$ is a Noetherian semi-local rings with exactly r maximal ideals $\mathfrak{m}_1, \ldots, \mathfrak{m}_r$ such that $A_i = A_{\mathfrak{m}_i}$. Moreover,

$$\mathrm{Spec}(A) = \mathrm{Spec}(A_1) \cup \ldots \cup \mathrm{Spec}(A_r)$$

is an open covering and the intersection of any two pieces of this covering is $\mathrm{Spec}(K)$. Thus the given morphisms $\mathrm{Spec}(A_i) \to X$ glue to a morphism of schemes

$$\mathrm{Spec}(A) \longrightarrow X$$

mapping $\mathfrak{m}_i$ to x_i and inducing isomorphisms of local rings. Thus the result follows from Lemma 32.22.3. □

Lemma 32.22.5. *Let A be a ring, $I \subset A$ an ideal, $\mathfrak{p}_1, \ldots, \mathfrak{p}_r$ primes of A, and $\overline{f} \in A/I$ an element. If $I \not\subset \mathfrak{p}_i$ for all i, then there exists an $f \in A$, $f \notin \mathfrak{p}_i$ which maps to $\overline{f}$ in A/I.*

Proof. We may assume there are no inclusion relations among the $\mathfrak{p}_i$ (by removing the smaller primes). First pick any $f \in A$ lifting $\overline{f}$. Let S be the set $s \in \{1, \ldots, r\}$ such that $f \in \mathfrak{p}_s$. If S is empty we are done. If not, consider the ideal $J = I \prod_{i \notin S} \mathfrak{p}_i$. Note that J is not contained in $\mathfrak{p}_s$ for $s \in S$ because there are no inclusions among the $\mathfrak{p}_i$ and because I is not contained in any $\mathfrak{p}_i$. Hence we can choose $g \in J$, $g \notin \mathfrak{p}_s$ for $s \in S$ by Algebra, Lemma 10.14.3. Then $f + g$ is a solution to the problem posed by the lemma. □

Lemma 32.22.6. *Let X be a scheme. Let $T \subset X$ be finite set of points. Assume*

1. *X has finitely many irreducible components $Z_1, \ldots, Z_t$, and*
2. *$Z_i \cap T$ is contained in an affine open of the reduced induced subscheme corresponding to Z_i.*

Then there exists an affine open subscheme of X containing T.

Proof. Using Limits, Proposition 31.10.2 there is an immediate reduction to the case where X is reduced. Details omitted. In the rest of the proof we endow every closed subset of X with the induced reduced closed subscheme structure.

We argue by induction that we can find an affine open $U \subset Z_1 \cup \ldots \cup Z_r$ containing $T \cap (Z_1 \cup \ldots \cup Z_r)$. For $r = 1$ this holds by assumption. Say $r > 1$ and let $U \subset Z_1 \cup \ldots \cup Z_{r-1}$ be an affine open containing $T \cap (Z_1 \cup \ldots \cup Z_{r-1})$. Let $V \subset X_r$ be an affine open containing $T \cap Z_r$ (exists by assumption). Then $U \cap V$ contains $T \cap (Z_1 \cup \ldots \cup Z_{r-1}) \cap Z_r$. Hence

$$\Delta = (U \cap Z_r) \setminus (U \cap V)$$

does not contain any element of T. Note that Δ is a closed subset of U. By prime avoidance (Algebra, Lemma 10.14.3), we can find a standard open U' of U containing $T \cap U$ and avoiding Δ, i.e., $U' \cap Z_r \subset U \cap V$. After replacing U by U' we may assume that $U \cap V$ is closed in U.

Using that by the same arguments as above also the set $\Delta' = (U \cap (Z_1 \cup \ldots \cup Z_{r-1})) \setminus (U \cap V)$ does not contain any element of T we find a $h \in \mathcal{O}(V)$ such that $D(h) \subset V$ contains $T \cap V$ and such that $U \cap D(h) \subset U \cap V$. Using that $U \cap V$ is closed in U we can use Lemma 32.22.5 to find an element $g \in \mathcal{O}(U)$ whose restriction to $U \cap V$ equals the restriction of h to $U \cap V$ and such that $T \cap U \subset D(g)$. Then we can replace U by $D(g)$ and V by $D(h)$ to reach the situation where $U \cap V$ is closed in both U and V. In this case the scheme $U \cup V$ is affine by Limits, Lemma 31.10.3. This proves the induction step and thereby the lemma. □

Here is a conclusion we can draw from the material above.

Proposition 32.22.7. *Let X be a separated scheme such that every quasi-compact open has a finite number of irreducible components. Let $x_1, \ldots, x_r \in X$ be points such that $\mathcal{O}_{X,x_i}$ is Noetherian of dimension ≤ 1. Then there exists an affine open subscheme of X containing all of $x_1, \ldots, x_r$.*

Proof. We can replace X by a quasi-compact open containing $x_1, \ldots, x_r$ hence we may assume that X has finitely many irreducible components. By Lemma 32.22.6 we reduce to the case where X is integral. This case is Lemma 32.22.4. □

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CHAPTER 33

Topologies on Schemes

33.1. Introduction

In this document we explain what the different topologies on the category of schemes are. Some references are [**Gro71**] and [**BLR90**]. Before doing so we would like to point out that there are many different choices of sites (as defined in Sites, Definition 7.6.2) which give rise to the same notion of sheaf on the underlying category. Hence our choices may be slightly different from those in the references but ultimately lead to the same cohomology groups, etc.

33.2. The general procedure

In this section we explain a general procedure for producing the sites we will be working with. Suppose we want to study sheaves over schemes with respect to some topology τ. In order to get a site, as in Sites, Definition 7.6.2, of schemes with that topology we have to do some work. Namely, we cannot simply say "consider all schemes with the Zariski topology" since that would give a "big" category. Instead, in each section of this chapter we will proceed as follows:

(1) We define a class Cov_τ of coverings of schemes satisfying the axioms of Sites, Definition 7.6.2. It will always be the case that a Zariski open covering of a scheme is a covering for τ.
(2) We single out a notion of standard τ-covering within the category of affine schemes.
(3) We define what is an "absolute" big τ-site Sch_τ. These are the sites one gets by appropriately choosing a set of schemes and a set of coverings.
(4) For any object S of Sch_τ we define the big τ-site $(Sch/S)_\tau$ and for suitable τ the small[1] τ-site S_τ.
(5) In addition there is a site $(Aff/S)_\tau$ using the notion of standard τ-covering of affines whose category of sheaves is equivalent to the category of sheaves on $(Sch/S)_\tau$.

The above is a little clumsy in that we do not end up with a canonical choice for the big τ-site of a scheme, or even the small τ-site of a scheme. If you are willing to ignore set theoretic difficulties, then you can work with classes and end up with canonical big and small sites...

[1] The words big and small here do not relate to bigness/smallness of the corresponding categories.

33.3. The Zariski topology

Definition 33.3.1. Let T be a scheme. A *Zariski covering of T* is a family of morphisms $\{f_i : T_i \to T\}_{i \in I}$ of schemes such that each f_i is an open immersion and such that $T = \bigcup f_i(T_i)$.

This defines a (proper) class of coverings. Next, we show that this notion satisfies the conditions of Sites, Definition 7.6.2.

Lemma 33.3.2. *Let T be a scheme.*

1. *If $T' \to T$ is an isomorphism then $\{T' \to T\}$ is a Zariski covering of T.*
2. *If $\{T_i \to T\}_{i \in I}$ is a Zariski covering and for each i we have a Zariski covering $\{T_{ij} \to T_i\}_{j \in J_i}$, then $\{T_{ij} \to T\}_{i \in I, j \in J_i}$ is a Zariski covering.*
3. *If $\{T_i \to T\}_{i \in I}$ is a Zariski covering and $T' \to T$ is a morphism of schemes then $\{T' \times_T T_i \to T'\}_{i \in I}$ is a Zariski covering.*

Proof. Omitted. □

Lemma 33.3.3. *Let T be an affine scheme. Let $\{T_i \to T\}_{i \in I}$ be a Zariski covering of T. Then there exists a Zariski covering $\{U_j \to T\}_{j=1,\ldots,m}$ which is a refinement of $\{T_i \to T\}_{i \in I}$ such that each U_j is a standard open of T, see Schemes, Definition 25.5.2. Moreover, we may choose each U_j to be an open of one of the T_i.*

Proof. Follows as T is quasi-compact and standard opens form a basis for its topology. This is also proved in Schemes, Lemma 25.5.1. □

Thus we define the corresponding standard coverings of affines as follows.

Definition 33.3.4. Compare Schemes, Definition 25.5.2. Let T be an affine scheme. A *standard Zariski covering* of T is a a Zariski covering $\{U_j \to T\}_{j=1,\ldots,m}$ with each $U_j \to T$ inducing an isomorphism with a standard affine open of T.

Definition 33.3.5. A *big Zariski site* is any site Sch_{Zar} as in Sites, Definition 7.6.2 constructed as follows:

1. Choose any set of schemes S_0, and any set of Zariski coverings Cov_0 among these schemes.
2. As underlying category of Sch_{Zar} take any category Sch_α constructed as in Sets, Lemma 3.9.2 starting with the set S_0.
3. As coverings of Sch_{Zar} choose any set of coverings as in Sets, Lemma 3.11.1 starting with the category Sch_α and the class of Zariski coverings, and the set Cov_0 chosen above.

It is shown in Sites, Lemma 7.8.6 that, after having chosen the category Sch_α, the category of sheaves on Sch_α does not depend on the choice of coverings chosen in (3) above. In other words, the topos $Sh(Sch_{Zar})$ only depends on the choice of the category Sch_α. It is shown in Sets, Lemma 3.9.9 that these categories are closed under many constructions of algebraic geometry, e.g., fibre products and taking open and closed subschemes. We can also show that the exact choice of Sch_α does not matter too much, see Section 33.10.

Another approach would be to assume the existence of a strongly inaccessible cardinal and to define Sch_{Zar} to be the category of schemes contained in a chosen universe with set of coverings the Zariski coverings contained in that same universe.

Before we continue with the introduction of the big Zariski site of a scheme S, let us point out that the topology on a big Zariski site Sch_{Zar} is in some sense induced from the Zariski topology on the category of all schemes.

Lemma 33.3.6. *Let Sch_{Zar} be a big Zariski site as in Definition 33.3.5. Let $T \in \mathrm{Ob}(Sch_{Zar})$. Let $\{T_i \to T\}_{i\in I}$ be an arbitrary Zariski covering of T. There exists a covering $\{U_j \to T\}_{j\in J}$ of T in the site Sch_{Zar} which is tautologically equivalent (see Sites, Definition 7.8.2) to $\{T_i \to T\}_{i\in I}$.*

Proof. Since each $T_i \to T$ is an open immersion, we see by Sets, Lemma 3.9.9 that each T_i is isomorphic to an object V_i of Sch_{Zar}. The covering $\{V_i \to T\}_{i\in I}$ is tautologically equivalent to $\{T_i \to T\}_{i\in I}$ (using the identity map on I both ways). Moreover, $\{V_i \to T\}_{i\in I}$ is combinatorially equivalent to a covering $\{U_j \to T\}_{j\in J}$ of T in the site Sch_{Zar} by Sets, Lemma 3.11.1. □

Definition 33.3.7. Let S be a scheme. Let Sch_{Zar} be a big Zariski site containing S.

(1) The *big Zariski site of S*, denoted $(Sch/S)_{Zar}$, is the site Sch_{Zar}/S introduced in Sites, Section 7.24.
(2) The *small Zariski site of S*, which we denote S_{Zar}, is the full subcategory of $(Sch/S)_{Zar}$ whose objects are those U/S such that $U \to S$ is an open immersion. A covering of S_{Zar} is any covering $\{U_i \to U\}$ of $(Sch/S)_{Zar}$ with $U \in \mathrm{Ob}(S_{Zar})$.
(3) The *big affine Zariski site of S*, denoted $(Aff/S)_{Zar}$, is the full subcategory of $(Sch/S)_{Zar}$ whose objects are affine U/S. A covering of $(Aff/S)_{Zar}$ is any covering $\{U_i \to U\}$ of $(Sch/S)_{Zar}$ which is a standard Zariski covering.

It is not completely clear that the small Zariski site and the big affine Zariski site are sites. We check this now.

Lemma 33.3.8. *Let S be a scheme. Let Sch_{Zar} be a big Zariski site containing S. Both S_{Zar} and $(Aff/S)_{Zar}$ are sites.*

Proof. Let us show that S_{Zar} is a site. It is a category with a given set of families of morphisms with fixed target. Thus we have to show properties (1), (2) and (3) of Sites, Definition 7.6.2. Since $(Sch/S)_{Zar}$ is a site, it suffices to prove that given any covering $\{U_i \to U\}$ of $(Sch/S)_{Zar}$ with $U \in \mathrm{Ob}(S_{Zar})$ we also have $U_i \in \mathrm{Ob}(S_{Zar})$. This follows from the definitions as the composition of open immersions is an open immersion.

Let us show that $(Aff/S)_{Zar}$ is a site. Reasoning as above, it suffices to show that the collection of standard Zariski coverings of affines satisfies properties (1), (2) and (3) of Sites, Definition 7.6.2. Let R be a ring. Let $f_1, \ldots, f_n \in R$ generate the unit ideal. For each $i \in \{1, \ldots, n\}$ let $g_{i1}, \ldots, g_{in_i} \in R_{f_i}$ be elements generating the unit ideal of R_{f_i}. Write $g_{ij} = f_{ij}/f_i^{e_{ij}}$ which is possible. After replacing f_{ij} by $f_i f_{ij}$ if necessary, we have that $D(f_{ij}) \subset D(f_i) \cong \mathrm{Spec}(R_{f_i})$ is equal to $D(g_{ij}) \subset \mathrm{Spec}(R_{f_i})$. Hence we see that the family of morphisms $\{D(g_{ij}) \to \mathrm{Spec}(R)\}$ is a standard Zariski covering. From these considerations it follows that (2) holds for standard Zariski coverings. We omit the verification of (1) and (3). □

Lemma 33.3.9. *Let S be a scheme. Let Sch_{Zar} be a big Zariski site containing S. The underlying categories of the sites Sch_{Zar}, $(Sch/S)_{Zar}$, S_{Zar}, and $(Aff/S)_{Zar}$ have fibre products. In each case the obvious functor into the category Sch of all*

schemes commutes with taking fibre products. The categories $(Sch/S)_{Zar}$, and S_{Zar} both have a final object, namely S/S.

Proof. For Sch_{Zar} it is true by construction, see Sets, Lemma 3.9.9. Suppose we have $U \to S$, $V \to U$, $W \to U$ morphisms of schemes with $U, V, W \in \mathrm{Ob}(Sch_{Zar})$. The fibre product $V \times_U W$ in Sch_{Zar} is a fibre product in Sch and is the fibre product of V/S with W/S over U/S in the category of all schemes over S, and hence also a fibre product in $(Sch/S)_{Zar}$. This proves the result for $(Sch/S)_{Zar}$. If $U \to S$, $V \to U$ and $W \to U$ are open immersions then so is $V \times_U W \to S$ and hence we get the result for S_{Zar}. If U, V, W are affine, so is $V \times_U W$ and hence the result for $(Aff/S)_{Zar}$. □

Next, we check that the big affine site defines the same topos as the big site.

Lemma 33.3.10. *Let S be a scheme. Let Sch_{Zar} be a big Zariski site containing S. The functor $(Aff/S)_{Zar} \to (Sch/S)_{Zar}$ is a special cocontinuous functor. Hence it induces an equivalence of topoi from $Sh((Aff/S)_{Zar})$ to $Sh((Sch/S)_{Zar})$.*

Proof. The notion of a special cocontinuous functor is introduced in Sites, Definition 7.28.2. Thus we have to verify assumptions (1) – (5) of Sites, Lemma 7.28.1. Denote the inclusion functor $u : (Aff/S)_{Zar} \to (Sch/S)_{Zar}$. Being cocontinuous just means that any Zariski covering of T/S, T affine, can be refined by a standard Zariski covering of T. This is the content of Lemma 33.3.3. Hence (1) holds. We see u is continuous simply because a standard Zariski covering is a Zariski covering. Hence (2) holds. Parts (3) and (4) follow immediately from the fact that u is fully faithful. And finally condition (5) follows from the fact that every scheme has an affine open covering. □

Let us check that the notion of a sheaf on the small Zariski site corresponds to notion of a sheaf on S.

Lemma 33.3.11. *The category of sheaves on S_{Zar} is equivalent to the category of sheaves on the underlying topological space of S.*

Proof. We will use repeatedly that for any object U/S of S_{Zar} the morphism $U \to S$ is an isomorphism onto an open subscheme. Let $\mathcal{F}$ be a sheaf on S. Then we define a sheaf on S_{Zar} by the rule $\mathcal{F}'(U/S) = \mathcal{F}(\mathrm{Im}(U \to S))$. For the converse, we choose for every open subscheme $U \subset S$ an object $U'/S \in \mathrm{Ob}(S_{Zar})$ with $\mathrm{Im}(U' \to S) = U$ (here you have to use Sets, Lemma 3.9.9). Given a sheaf $\mathcal{G}$ on S_{Zar} we define a sheaf on S by setting $\mathcal{G}(U) = \mathcal{G}(U'/S)$. To see that $\mathcal{G}'$ is a sheaf we use that for any open covering $U = \bigcup_{i \in I} U_i$ the covering $\{U_i \to U\}_{i \in I}$ is combinatorially equivalent to a covering $\{U'_j \to U'\}_{j \in J}$ in S_{Zar} by Sets, Lemma 3.11.1, and we use Sites, Lemma 7.8.4. Details omitted. □

From now on we will not make any distinction between a sheaf on S_{Zar} or a sheaf on S. We will always use the procedures of the proof of the lemma to go between the two notions. Next, we establish some relationships between the topoi associated to these sites.

Lemma 33.3.12. *Let Sch_{Zar} be a big Zariski site. Let $f : T \to S$ be a morphism in Sch_{Zar}. The functor $T_{Zar} \to (Sch/S)_{Zar}$ is cocontinuous and induces a morphism of topoi*

$$i_f : Sh(T_{Zar}) \longrightarrow Sh((Sch/S)_{Zar})$$

For a sheaf $\mathcal{G}$ on $(Sch/S)_{Zar}$ we have the formula $(i_f^{-1}\mathcal{G})(U/T) = \mathcal{G}(U/S)$. The functor i_f^{-1} also has a left adjoint $i_{f,!}$ which commutes with fibre products and equalizers.

Proof. Denote the functor $u : T_{Zar} \to (Sch/S)_{Zar}$. In other words, given and open immersion $j : U \to T$ corresponding to an object of T_{Zar} we set $u(U \to T) = (f \circ j : U \to S)$. This functor commutes with fibre products, see Lemma 33.3.9. Moreover, T_{Zar} has equalizers (as any two morphisms with the same source and target are the same) and u commutes with them. It is clearly cocontinuous. It is also continuous as u transforms coverings to coverings and commutes with fibre products. Hence the lemma follows from Sites, Lemmas 7.20.5 and 7.20.6. □

Lemma 33.3.13. *Let S be a scheme. Let Sch_{Zar} be a big Zariski site containing S. The inclusion functor $S_{Zar} \to (Sch/S)_{Zar}$ satisfies the hypotheses of Sites, Lemma 7.20.8 and hence induces a morphism of sites*

$$\pi_S : (Sch/S)_{Zar} \longrightarrow S_{Zar}$$

and a morphism of topoi

$$i_S : Sh(S_{Zar}) \longrightarrow Sh((Sch/S)_{Zar})$$

such that $\pi_S \circ i_S = id$. Moreover, $i_S = i_{id_S}$ with i_{id_S} as in Lemma 33.3.12. In particular the functor $i_S^{-1} = \pi_{S,*}$ is described by the rule $i_S^{-1}(\mathcal{G})(U/S) = \mathcal{G}(U/S)$.

Proof. In this case the functor $u : S_{Zar} \to (Sch/S)_{Zar}$, in addition to the properties seen in the proof of Lemma 33.3.12 above, also is fully faithful and transforms the final object into the final object. The lemma follows. □

Definition 33.3.14. In the situation of Lemma 33.3.13 the functor $i_S^{-1} = \pi_{S,*}$ is often called the *restriction to the small Zariski site*, and for a sheaf $\mathcal{F}$ on the big Zariski site we denote $\mathcal{F}|_{S_{Zar}}$ this restriction.

With this notation in place we have for a sheaf $\mathcal{F}$ on the big site and a sheaf $\mathcal{G}$ on the big site that

$$\begin{aligned}\mathrm{Mor}_{Sh(S_{Zar})}(\mathcal{F}|_{S_{Zar}}, \mathcal{G}) &= \mathrm{Mor}_{Sh((Sch/S)_{Zar})}(\mathcal{F}, i_{S,*}\mathcal{G}) \\ \mathrm{Mor}_{Sh(S_{Zar})}(\mathcal{G}, \mathcal{F}|_{S_{Zar}}) &= \mathrm{Mor}_{Sh((Sch/S)_{Zar})}(\pi_S^{-1}\mathcal{G}, \mathcal{F})\end{aligned}$$

Moreover, we have $(i_{S,*}\mathcal{G})|_{S_{Zar}} = \mathcal{G}$ and we have $(\pi_S^{-1}\mathcal{G})|_{S_{Zar}} = \mathcal{G}$.

Lemma 33.3.15. *Let Sch_{Zar} be a big Zariski site. Let $f : T \to S$ be a morphism in Sch_{Zar}. The functor*

$$u : (Sch/T)_{Zar} \longrightarrow (Sch/S)_{Zar}, \quad V/T \longmapsto V/S$$

is cocontinuous, and has a continuous right adjoint

$$v : (Sch/S)_{Zar} \longrightarrow (Sch/T)_{Zar}, \quad (U \to S) \longmapsto (U \times_S T \to T).$$

They induce the same morphism of topoi

$$f_{big} : Sh((Sch/T)_{Zar}) \longrightarrow Sh((Sch/S)_{Zar})$$

We have $f_{big}^{-1}(\mathcal{G})(U/T) = \mathcal{G}(U/S)$. We have $f_{big,*}(\mathcal{F})(U/S) = \mathcal{F}(U \times_S T/T)$. Also, f_{big}^{-1} has a left adjoint $f_{big!}$ which commutes with fibre products and equalizers.

Proof. The functor u is cocontinuous, continuous, and commutes with fibre products and equalizers (details omitted; compare with proof of Lemma 33.3.12). Hence Sites, Lemmas 7.20.5 and 7.20.6 apply and we deduce the formula for f_{big}^{-1} and the existence of $f_{big!}$. Moreover, the functor v is a right adjoint because given U/T and V/S we have $\mathrm{Mor}_S(u(U), V) = \mathrm{Mor}_T(U, V \times_S T)$ as desired. Thus we may apply Sites, Lemmas 7.21.1 and 7.21.2 to get the formula for $f_{big,*}$. □

Lemma 33.3.16. *Let Sch_{Zar} be a big Zariski site. Let $f : T \to S$ be a morphism in Sch_{Zar}.*

(1) *We have $i_f = f_{big} \circ i_T$ with i_f as in Lemma 33.3.12 and i_T as in Lemma 33.3.13.*
(2) *The functor $S_{Zar} \to T_{Zar}$, $(U \to S) \mapsto (U \times_S T \to T)$ is continuous and induces a morphism of topoi*
$$f_{small} : Sh(T_{Zar}) \longrightarrow Sh(S_{Zar}).$$
The functors f_{small}^{-1} and $f_{small,*}$ agree with the usual notions f^{-1} and f_* is we identify sheaves on T_{Zar}, resp. S_{Zar} with sheaves on T, resp. S via Lemma 33.3.11.
(3) *We have a commutative diagram of morphisms of sites*
$$\begin{array}{ccc} T_{Zar} & \xleftarrow{\pi_T} & (Sch/T)_{Zar} \\ {\scriptstyle f_{small}}\downarrow & & \downarrow{\scriptstyle f_{big}} \\ S_{Zar} & \xleftarrow{\pi_S} & (Sch/S)_{Zar} \end{array}$$
so that $f_{small} \circ \pi_T = \pi_S \circ f_{big}$ as morphisms of topoi.
(4) *We have $f_{small} = \pi_S \circ f_{big} \circ i_T = \pi_S \circ i_f$.*

Proof. The equality $i_f = f_{big} \circ i_T$ follows from the equality $i_f^{-1} = i_T^{-1} \circ f_{big}^{-1}$ which is clear from the descriptions of these functors above. Thus we see (1).

Statement (2): See Sites, Example 7.15.2.

Part (3) follows because π_S and π_T are given by the inclusion functors and f_{small} and f_{big} by the base change functor $U \mapsto U \times_S T$.

Statement (4) follows from (3) by precomposing with i_T. □

In the situation of the lemma, using the terminology of Definition 33.3.14 we have: for $\mathcal{F}$ a sheaf on the big Zariski site of T
$$(f_{big,*}\mathcal{F})|_{S_{Zar}} = f_{small,*}(\mathcal{F}|_{T_{Zar}}),$$
This equality is clear from the commutativity of the diagram of sites of the lemma, since restriction to the small Zariski site of T, resp. S is given by $\pi_{T,*}$, resp. $\pi_{S,*}$. A similar formula involving pullbacks and restrictions is false.

Lemma 33.3.17. *Given schemes X, Y, Y in $(Sch/S)_{Zar}$ and morphisms $f : X \to Y$, $g : Y \to Z$ we have $g_{big} \circ f_{big} = (g \circ f)_{big}$ and $g_{small} \circ f_{small} = (g \circ f)_{small}$.*

Proof. This follows from the simple description of pushforward and pullback for the functors on the big sites from Lemma 33.3.15. For the functors on the small sites this is Sheaves, Lemma 6.21.2 via the identification of Lemma 33.3.11. □

We can think about a sheaf on the big Zariski site of S as a collection of "usual" sheaves on all schemes over S.

Lemma 33.3.18. *Let S be a scheme contained in a big Zariski site Sch_{Zar}. A sheaf $\mathcal{F}$ on the big Zariski site $(Sch/S)_{Zar}$ is given by the following data:*

(1) *for every $T/S \in \mathrm{Ob}((Sch/S)_{Zar})$ a sheaf $\mathcal{F}_T$ on T,*
(2) *for every $f : T' \to T$ in $(Sch/S)_{Zar}$ a map $c_f : f^{-1}\mathcal{F}_T \to \mathcal{F}_{T'}$.*

These data are subject to the following conditions:

(i) *given any $f : T' \to T$ and $g : T'' \to T'$ in $(Sch/S)_{Zar}$ the composition $g^{-1}c_f \circ c_g$ is equal to $c_{f\circ g}$, and*
(ii) *if $f : T' \to T$ in $(Sch/S)_{Zar}$ is an open immersion then c_f is an isomorphism.*

Proof. Given a sheaf $\mathcal{F}$ on $Sh((Sch/S)_{Zar})$ we set $\mathcal{F}_T = i_p^{-1}\mathcal{F}$ where $p : T \to S$ is the structure morphism. Note that $\mathcal{F}_T(U) = \mathcal{F}(U'/S)$ for any open $U \subset T$, and $U' \to T$ an open immersion in $(Sch/T)_{Zar}$ with image U, see Lemmas 33.3.11 and 33.3.12. Hence given $f : T' \to T$ over S and $U, U' \to T$ we get a canonical map $\mathcal{F}_T(U) = \mathcal{F}(U'/S) \to \mathcal{F}(U' \times_T T'/S) = \mathcal{F}_{T'}(f^{-1}(U))$ where the middle is the restriction map of $\mathcal{F}$ with respect to the morphism $U' \times_T T' \to U'$ over S. The collection of these maps are compatible with restrictions, and hence define an f-map c_f from $\mathcal{F}_T$ to $\mathcal{F}_{T'}$, see Sheaves, Definition 6.21.7 and the discussion surrounding it. It is clear that $c_{f\circ g}$ is the composition of c_f and c_g, since composition of restriction maps of $\mathcal{F}$ gives restriction maps.

Conversely, given a system $(\mathcal{F}_T, c_f)$ as in the lemma we may define a presheaf $\mathcal{F}$ on $Sh((Sch/S)_{Zar})$ by simply setting $\mathcal{F}(T/S) = \mathcal{F}_T(T)$. As restriction mapping, given $f : T' \to T$ we set for $s \in \mathcal{F}(T)$ the pullback $f^*(s)$ equal to $c_f(s)$ (where we think of c_f as an f-map again). The condition on the c_f guarantees that pullbacks satisfy the required functoriality property. We omit the verification that this is a sheaf. It is clear that the constructions so defined are mutually inverse. □

33.4. The étale topology

Let S be a scheme. We would like to define the étale-topology on the category of schemes over S. According to our general principle we first introduce the notion of an étale covering.

Definition 33.4.1. Let T be a scheme. An *étale covering of T* is a family of morphisms $\{f_i : T_i \to T\}_{i\in I}$ of schemes such that each f_i is étale and such that $T = \bigcup f_i(T_i)$.

Lemma 33.4.2. *Any Zariski covering is an étale covering.*

Proof. This is clear from the definitions and the fact that an open immersion is an étale morphism, see Morphisms, Lemma 28.37.9. □

Next, we show that this notion satisfies the conditions of Sites, Definition 7.6.2.

Lemma 33.4.3. *Let T be a scheme.*

(1) *If $T' \to T$ is an isomorphism then $\{T' \to T\}$ is an étale covering of T.*
(2) *If $\{T_i \to T\}_{i\in I}$ is an étale covering and for each i we have an étale covering $\{T_{ij} \to T_i\}_{j\in J_i}$, then $\{T_{ij} \to T\}_{i\in I, j\in J_i}$ is an étale covering.*

(3) *If $\{T_i \to T\}_{i\in I}$ is an étale covering and $T' \to T$ is a morphism of schemes then $\{T' \times_T T_i \to T'\}_{i\in I}$ is an étale covering.*

Proof. Omitted. □

Lemma 33.4.4. *Let T be an affine scheme. Let $\{T_i \to T\}_{i\in I}$ be an étale covering of T. Then there exists an étale covering $\{U_j \to T\}_{j=1,\ldots,m}$ which is a refinement of $\{T_i \to T\}_{i\in I}$ such that each U_j is an affine scheme. Moreover, we may choose each U_j to be open affine in one of the T_i.*

Proof. Omitted. □

Thus we define the corresponding standard coverings of affines as follows.

Definition 33.4.5. Let T be an affine scheme. A *standard étale covering* of T is a family $\{f_j : U_j \to T\}_{j=1,\ldots,m}$ with each U_j is affine and étale over T and $T = \bigcup f_j(U_j)$.

In the definition above we do **not** assume the morphisms f_j are standard étale. The reason is that if we did then the standard étale coverings would not define a site on Aff/S, for example because of Algebra, Lemma 10.136.15 part (4). On the other hand, an étale morphism of affines is automatically standard smooth, see Algebra, Lemma 10.136.2. Hence a standard étale covering is a standard smooth covering and a standard syntomic covering.

Definition 33.4.6. A *big étale site* is any site $Sch_{étale}$ as in Sites, Definition 7.6.2 constructed as follows:

(1) Choose any set of schemes S_0, and any set of étale coverings Cov_0 among these schemes.
(2) As underlying category take any category Sch_α constructed as in Sets, Lemma 3.9.2 starting with the set S_0.
(3) Choose any set of coverings as in Sets, Lemma 3.11.1 starting with the category Sch_α and the class of étale coverings, and the set Cov_0 chosen above.

See the remarks following Definition 33.3.5 for motivation and explanation regarding the definition of big sites.

Before we continue with the introduction of the big étale site of a scheme S, let us point out that the topology on a big étale site $Sch_{étale}$ is in some sense induced from the étale topology on the category of all schemes.

Lemma 33.4.7. *Let $Sch_{étale}$ be a big étale site as in Definition 33.4.6. Let $T \in \mathrm{Ob}(Sch_{étale})$. Let $\{T_i \to T\}_{i\in I}$ be an arbitrary étale covering of T.*

(1) *There exists a covering $\{U_j \to T\}_{j\in J}$ of T in the site $Sch_{étale}$ which refines $\{T_i \to T\}_{i\in I}$.*
(2) *If $\{T_i \to T\}_{i\in I}$ is a standard étale covering, then it is tautologically equivalent to a covering in $Sch_{étale}$.*
(3) *If $\{T_i \to T\}_{i\in I}$ is a Zariski covering, then it is tautologically equivalent to a covering in $Sch_{étale}$.*

Proof. For each i choose an affine open covering $T_i = \bigcup_{j\in J_i} T_{ij}$ such that each T_{ij} maps into an affine open subscheme of T. By Lemma 33.4.3 the refinement $\{T_{ij} \to T\}_{i\in I, j\in J_i}$ is an étale covering of T as well. Hence we may assume each

T_i is affine, and maps into an affine open W_i of T. Applying Sets, Lemma 3.9.9 we see that W_i is isomorphic to an object of Sch_{Zar}. But then T_i as a finite type scheme over W_i is isomorphic to an object V_i of Sch_{Zar} by a second application of Sets, Lemma 3.9.9. The covering $\{V_i \to T\}_{i\in I}$ refines $\{T_i \to T\}_{i\in I}$ (because they are isomorphic). Moreover, $\{V_i \to T\}_{i\in I}$ is combinatorially equivalent to a covering $\{U_j \to T\}_{j\in J}$ of T in the site Sch_{Zar} by Sets, Lemma 3.9.9. The covering $\{U_j \to T\}_{j\in J}$ is a refinement as in (1). In the situation of (2), (3) each of the schemes T_i is isomorphic to an object of $Sch_{étale}$ by Sets, Lemma 3.9.9, and another application of Sets, Lemma 3.11.1 gives what we want. □

Definition 33.4.8. Let S be a scheme. Let $Sch_{étale}$ be a big étale site containing S.

(1) The *big étale site of* S, denoted $(Sch/S)_{étale}$, is the site $Sch_{étale}/S$ introduced in Sites, Section 7.24.
(2) The *small étale site of* S, which we denote $S_{étale}$, is the full subcategory of $(Sch/S)_{étale}$ whose objects are those U/S such that $U \to S$ is étale. A covering of $S_{étale}$ is any covering $\{U_i \to U\}$ of $(Sch/S)_{étale}$ with $U \in \mathrm{Ob}(S_{étale})$.
(3) The *big affine étale site of* S, denoted $(Aff/S)_{étale}$, is the full subcategory of $(Sch/S)_{étale}$ whose objects are affine U/S. A covering of $(Aff/S)_{étale}$ is any covering $\{U_i \to U\}$ of $(Sch/S)_{étale}$ which is a standard étale covering.

It is not completely clear that the big affine étale site or the small étale site are sites. We check this now.

Lemma 33.4.9. *Let S be a scheme. Let $Sch_{étale}$ be a big étale site containing S. Both $S_{étale}$ and $(Aff/S)_{étale}$ are sites.*

Proof. Let us show that $S_{étale}$ is a site. It is a category with a given set of families of morphisms with fixed target. Thus we have to show properties (1), (2) and (3) of Sites, Definition 7.6.2. Since $(Sch/S)_{étale}$ is a site, it suffices to prove that given any covering $\{U_i \to U\}$ of $(Sch/S)_{étale}$ with $U \in \mathrm{Ob}(S_{étale})$ we also have $U_i \in \mathrm{Ob}(S_{étale})$. This follows from the definitions as the composition of étale morphisms is an étale morphism.

Let us show that $(Aff/S)_{étale}$ is a site. Reasoning as above, it suffices to show that the collection of standard étale coverings of affines satisfies properties (1), (2) and (3) of Sites, Definition 7.6.2. This is clear since for example, given a standard étale covering $\{T_i \to T\}_{i\in I}$ and for each i we have a standard étale covering $\{T_{ij} \to T_i\}_{j\in J_i}$, then $\{T_{ij} \to T\}_{i\in I, j\in J_i}$ is a standard étale covering because $\bigcup_{i\in I} J_i$ is finite and each T_{ij} is affine. □

Lemma 33.4.10. *Let S be a scheme. Let $Sch_{étale}$ be a big étale site containing S. The underlying categories of the sites $Sch_{étale}$, $(Sch/S)_{étale}$, $S_{étale}$, and $(Aff/S)_{étale}$ have fibre products. In each case the obvious functor into the category Sch of all schemes commutes with taking fibre products. The categories $(Sch/S)_{étale}$, and $S_{étale}$ both have a final object, namely S/S.*

Proof. For $Sch_{étale}$ it is true by construction, see Sets, Lemma 3.9.9. Suppose we have $U \to S$, $V \to U$, $W \to U$ morphisms of schemes with $U, V, W \in \mathrm{Ob}(Sch_{étale})$. The fibre product $V \times_U W$ in $Sch_{étale}$ is a fibre product in Sch and is the fibre product of V/S with W/S over U/S in the category of all schemes over S, and

hence also a fibre product in $(Sch/S)_{étale}$. This proves the result for $(Sch/S)_{étale}$. If $U \to S$, $V \to U$ and $W \to U$ are étale then so is $V \times_U W \to S$ and hence we get the result for $S_{étale}$. If U, V, W are affine, so is $V \times_U W$ and hence the result for $(Aff/S)_{étale}$. □

Next, we check that the big affine site defines the same topos as the big site.

Lemma 33.4.11. *Let S be a scheme. Let $Sch_{étale}$ be a big étale site containing S. The functor $(Aff/S)_{étale} \to (Sch/S)_{étale}$ is special cocontinuous and induces an equivalence of topoi from $Sh((Aff/S)_{étale})$ to $Sh((Sch/S)_{étale})$.*

Proof. The notion of a special cocontinuous functor is introduced in Sites, Definition 7.28.2. Thus we have to verify assumptions (1) – (5) of Sites, Lemma 7.28.1. Denote the inclusion functor $u : (Aff/S)_{étale} \to (Sch/S)_{étale}$. Being cocontinuous just means that any étale covering of T/S, T affine, can be refined by a standard étale covering of T. This is the content of Lemma 33.4.4. Hence (1) holds. We see u is continuous simply because a standard étale covering is a étale covering. Hence (2) holds. Parts (3) and (4) follow immediately from the fact that u is fully faithful. And finally condition (5) follows from the fact that every scheme has an affine open covering. □

Next, we establish some relationships between the topoi associated to these sites.

Lemma 33.4.12. *Let $Sch_{étale}$ be a big étale site. Let $f : T \to S$ be a morphism in $Sch_{étale}$. The functor $T_{étale} \to (Sch/S)_{étale}$ is cocontinuous and induces a morphism of topoi*

$$i_f : Sh(T_{étale}) \longrightarrow Sh((Sch/S)_{étale})$$

For a sheaf $\mathcal{G}$ on $(Sch/S)_{étale}$ we have the formula $(i_f^{-1}\mathcal{G})(U/T) = \mathcal{G}(U/S)$. The functor i_f^{-1} also has a left adjoint $i_{f,!}$ which commutes with fibre products and equalizers.

Proof. Denote the functor $u : T_{étale} \to (Sch/S)_{étale}$. In other words, given an étale morphism $j : U \to T$ corresponding to an object of $T_{étale}$ we set $u(U \to T) = (f \circ j : U \to S)$. This functor commutes with fibre products, see Lemma 33.4.10. Let $a, b : U \to V$ be two morphisms in $T_{étale}$. In this case the equalizer of a and b (in the category of schemes) is

$$V \times_{\Delta_{V/T}, V \times_T V, (a,b)} U \times_T U$$

which is a fibre product of schemes étale over T, hence étale over T. Thus $T_{étale}$ has equalizers and u commutes with them. It is clearly cocontinuous. It is also continuous as u transforms coverings to coverings and commutes with fibre products. Hence the Lemma follows from Sites, Lemmas 7.20.5 and 7.20.6. □

Lemma 33.4.13. *Let S be a scheme. Let $Sch_{étale}$ be a big étale site containing S. The inclusion functor $S_{étale} \to (Sch/S)_{étale}$ satisfies the hypotheses of Sites, Lemma 7.20.8 and hence induces a morphism of sites*

$$\pi_S : (Sch/S)_{étale} \longrightarrow S_{étale}$$

and a morphism of topoi

$$i_S : Sh(S_{étale}) \longrightarrow Sh((Sch/S)_{étale})$$

such that $\pi_S \circ i_S = id$. Moreover, $i_S = i_{id_S}$ with i_{id_S} as in Lemma 33.4.12. In particular the functor $i_S^{-1} = \pi_{S,*}$ is described by the rule $i_S^{-1}(\mathcal{G})(U/S) = \mathcal{G}(U/S)$.

Proof. In this case the functor $u : S_{étale} \to (Sch/S)_{étale}$, in addition to the properties seen in the proof of Lemma 33.4.12 above, also is fully faithful and transforms the final object into the final object. The lemma follows from Sites, Lemma 7.20.8. □

Definition 33.4.14. In the situation of Lemma 33.4.13 the functor $i_S^{-1} = \pi_{S,*}$ is often called the *restriction to the small étale site*, and for a sheaf $\mathcal{F}$ on the big étale site we denote $\mathcal{F}|_{S_{étale}}$ this restriction.

With this notation in place we have for a sheaf $\mathcal{F}$ on the big site and a sheaf $\mathcal{G}$ on the big site that

$$\begin{aligned}\mathrm{Mor}_{Sh(S_{étale})}(\mathcal{F}|_{S_{étale}}, \mathcal{G}) &= \mathrm{Mor}_{Sh((Sch/S)_{étale})}(\mathcal{F}, i_{S,*}\mathcal{G}) \\ \mathrm{Mor}_{Sh(S_{étale})}(\mathcal{G}, \mathcal{F}|_{S_{étale}}) &= \mathrm{Mor}_{Sh((Sch/S)_{étale})}(\pi_S^{-1}\mathcal{G}, \mathcal{F})\end{aligned}$$

Moreover, we have $(i_{S,*}\mathcal{G})|_{S_{étale}} = \mathcal{G}$ and we have $(\pi_S^{-1}\mathcal{G})|_{S_{étale}} = \mathcal{G}$.

Lemma 33.4.15. *Let $Sch_{étale}$ be a big étale site. Let $f : T \to S$ be a morphism in $Sch_{étale}$. The functor*

$$u : (Sch/T)_{étale} \longrightarrow (Sch/S)_{étale}, \quad V/T \longmapsto V/S$$

is cocontinuous, and has a continuous right adjoint

$$v : (Sch/S)_{étale} \longrightarrow (Sch/T)_{étale}, \quad (U \to S) \longmapsto (U \times_S T \to T).$$

They induce the same morphism of topoi

$$f_{big} : Sh((Sch/T)_{étale}) \longrightarrow Sh((Sch/S)_{étale})$$

We have $f_{big}^{-1}(\mathcal{G})(U/T) = \mathcal{G}(U/S)$. We have $f_{big,*}(\mathcal{F})(U/S) = \mathcal{F}(U \times_S T/T)$. Also, f_{big}^{-1} has a left adjoint $f_{big!}$ which commutes with fibre products and equalizers.

Proof. The functor u is cocontinuous, continuous and commutes with fibre products and equalizers (details omitted; compare with the proof of Lemma 33.4.12). Hence Sites, Lemmas 7.20.5 and 7.20.6 apply and we deduce the formula for f_{big}^{-1} and the existence of $f_{big!}$. Moreover, the functor v is a right adjoint because given U/T and V/S we have $\mathrm{Mor}_S(u(U), V) = \mathrm{Mor}_T(U, V \times_S T)$ as desired. Thus we may apply Sites, Lemmas 7.21.1 and 7.21.2 to get the formula for $f_{big,*}$. □

Lemma 33.4.16. *Let $Sch_{étale}$ be a big étale site. Let $f : T \to S$ be a morphism in $Sch_{étale}$.*

1. *We have $i_f = f_{big} \circ i_T$ with i_f as in Lemma 33.4.12 and i_T as in Lemma 33.4.13.*
2. *The functor $S_{étale} \to T_{étale}$, $(U \to S) \mapsto (U \times_S T \to T)$ is continuous and induces a morphism of topoi*
$$f_{small} : Sh(T_{étale}) \longrightarrow Sh(S_{étale}).$$
We have $f_{small,*}(\mathcal{F})(U/S) = \mathcal{F}(U \times_S T/T)$.
3. *We have a commutative diagram of morphisms of sites*
$$\begin{array}{ccc} T_{étale} & \xleftarrow{\pi_T} & (Sch/T)_{étale} \\ {\scriptstyle f_{small}}\downarrow & & \downarrow{\scriptstyle f_{big}} \\ S_{étale} & \xleftarrow{\pi_S} & (Sch/S)_{étale} \end{array}$$

so that $f_{small} \circ \pi_T = \pi_S \circ f_{big}$ *as morphisms of topoi.*

(4) *We have* $f_{small} = \pi_S \circ f_{big} \circ i_T = \pi_S \circ i_f$.

Proof. The equality $i_f = f_{big} \circ i_T$ follows from the equality $i_f^{-1} = i_T^{-1} \circ f_{big}^{-1}$ which is clear from the descriptions of these functors above. Thus we see (1).

The functor $u : S_{étale} \to T_{étale}$, $u(U \to S) = (U \times_S T \to T)$ transforms coverings into coverings and commutes with fibre products, see Lemma 33.4.3 (3) and 33.4.10. Moreover, both $S_{étale}, T_{étale}$ have final objects, namely S/S and T/T and $u(S/S) = T/T$. Hence by Sites, Proposition 7.15.6 the functor u corresponds to a morphism of sites $T_{étale} \to S_{étale}$. This in turn gives rise to the morphism of topoi, see Sites, Lemma 7.16.2. The description of the pushforward is clear from these references.

Part (3) follows because π_S and π_T are given by the inclusion functors and f_{small} and f_{big} by the base change functors $U \mapsto U \times_S T$.

Statement (4) follows from (3) by precomposing with i_T. □

In the situation of the lemma, using the terminology of Definition 33.4.14 we have: for $\mathcal{F}$ a sheaf on the big étale site of T

$$(f_{big,*}\mathcal{F})|_{S_{étale}} = f_{small,*}(\mathcal{F}|_{T_{étale}}),$$

This equality is clear from the commutativity of the diagram of sites of the lemma, since restriction to the small étale site of T, resp. S is given by $\pi_{T,*}$, resp. $\pi_{S,*}$. A similar formula involving pullbacks and restrictions is false.

Lemma 33.4.17. *Given schemes* X, Y, Y *in* $Sch_{étale}$ *and morphisms* $f : X \to Y$, $g : Y \to Z$ *we have* $g_{big} \circ f_{big} = (g \circ f)_{big}$ *and* $g_{small} \circ f_{small} = (g \circ f)_{small}$.

Proof. This follows from the simple description of pushforward and pullback for the functors on the big sites from Lemma 33.4.15. For the functors on the small sites this follows from the description of the pushforward functors in Lemma 33.4.16. □

We can think about a sheaf on the big étale site of S as a collection of "usual" sheaves on all schemes over S.

Lemma 33.4.18. *Let* S *be a scheme contained in a big étale site* $Sch_{étale}$. *A sheaf* $\mathcal{F}$ *on the big étale site* $(Sch/S)_{étale}$ *is given by the following data:*

(1) *for every* $T/S \in \mathrm{Ob}((Sch/S)_{étale})$ *a sheaf* $\mathcal{F}_T$ *on* $T_{étale}$,

(2) *for every* $f : T' \to T$ *in* $(Sch/S)_{étale}$ *a map* $c_f : f_{small}^{-1}\mathcal{F}_T \to \mathcal{F}_{T'}$.

These data are subject to the following conditions:

(i) *given any* $f : T' \to T$ *and* $g : T'' \to T'$ *in* $(Sch/S)_{étale}$ *the composition* $g_{small}^{-1}c_f \circ c_g$ *is equal to* $c_{f \circ g}$, *and*

(ii) *if* $f : T' \to T$ *in* $(Sch/S)_{étale}$ *is étale then* c_f *is an isomorphism.*

Proof. Given a sheaf $\mathcal{F}$ on $Sh((Sch/S)_{étale})$ we set $\mathcal{F}_T = i_p^{-1}\mathcal{F}$ where $p : T \to S$ is the structure morphism. Note that $\mathcal{F}_T(U) = \mathcal{F}(U/S)$ for any $U \to T$ in $T_{étale}$ see Lemma 33.4.12. Hence given $f : T' \to T$ over S and $U \to T$ we get a canonical map $\mathcal{F}_T(U) = \mathcal{F}(U/S) \to \mathcal{F}(U \times_T T'/S) = \mathcal{F}_{T'}(U \times_T T')$ where the middle is the restriction map of $\mathcal{F}$ with respect to the morphism $U \times_T T' \to U$ over S. The collection of these maps are compatible with restrictions, and hence define a map $c'_f : \mathcal{F}_T \to f_{small,*}\mathcal{F}_{T'}$ where $u : T_{étale} \to T'_{étale}$ is the base change functor associated to f. By adjunction of $f_{small,*}$ (see Sites, Section 7.14) with f_{small}^{-1} this

is the same as a map $c_f : f_{small}^{-1}\mathcal{F}_T \to \mathcal{F}_{T'}$. It is clear that $c'_{f\circ g}$ is the composition of c'_f and $f_{small,*}c'_g$, since composition of restriction maps of $\mathcal{F}$ gives restriction maps, and this gives the desired relationship among c_f, c_g and $c_{f\circ g}$.

Conversely, given a system $(\mathcal{F}_T, c_f)$ as in the lemma we may define a presheaf $\mathcal{F}$ on $Sh((Sch/S)_{étale})$ by simply setting $\mathcal{F}(T/S) = \mathcal{F}_T(T)$. As restriction mapping, given $f : T' \to T$ we set for $s \in \mathcal{F}(T)$ the pullback $f^*(s)$ equal to $c_f(s)$ where we think of c_f as a map $\mathcal{F}_T \to f_{small,*}\mathcal{F}_{T'}$ again. The condition on the c_f guarantees that pullbacks satisfy the required functoriality property. We omit the verification that this is a sheaf. It is clear that the constructions so defined are mutually inverse. □

33.5. The smooth topology

In this section we define the smooth topology. This is a bit pointless as it will turn out later (see More on Morphisms, Section 36.28) that this topology defines the same topos as the étale topology. But still it makes sense and it is used occasionally.

Definition 33.5.1. Let T be a scheme. An *smooth covering of* T is a family of morphisms $\{f_i : T_i \to T\}_{i\in I}$ of schemes such that each f_i is smooth and such that $T = \bigcup f_i(T_i)$.

Lemma 33.5.2. *Any étale covering is a smooth covering, and a fortiori, any Zariski covering is a smooth covering.*

Proof. This is clear from the definitions, the fact that an étale morphism is smooth see Morphisms, Definition 28.37.1 and Lemma 33.4.2. □

Next, we show that this notion satisfies the conditions of Sites, Definition 7.6.2.

Lemma 33.5.3. *Let T be a scheme.*

1. *If $T' \to T$ is an isomorphism then $\{T' \to T\}$ is an smooth covering of T.*
2. *If $\{T_i \to T\}_{i\in I}$ is a smooth covering and for each i we have a smooth covering $\{T_{ij} \to T_i\}_{j\in J_i}$, then $\{T_{ij} \to T\}_{i\in I, j\in J_i}$ is a smooth covering.*
3. *If $\{T_i \to T\}_{i\in I}$ is a smooth covering and $T' \to T$ is a morphism of schemes then $\{T' \times_T T_i \to T'\}_{i\in I}$ is a smooth covering.*

Proof. Omitted. □

Lemma 33.5.4. *Let T be an affine scheme. Let $\{T_i \to T\}_{i\in I}$ be a smooth covering of T. Then there exists a smooth covering $\{U_j \to T\}_{j=1,\ldots,m}$ which is a refinement of $\{T_i \to T\}_{i\in I}$ such that each U_j is an affine scheme, and such that each morphism $U_j \to T$ is standard smooth, see Morphisms, Definition 28.35.1. Moreover, we may choose each U_j to be open affine in one of the T_i.*

Proof. Omitted, but see Algebra, Lemma 10.130.10. □

Thus we define the corresponding standard coverings of affines as follows.

Definition 33.5.5. Let T be an affine scheme. A *standard smooth covering* of T is a family $\{f_j : U_j \to T\}_{j=1,\ldots,m}$ with each U_j is affine, $U_j \to T$ standard smooth and $T = \bigcup f_j(U_j)$.

Definition 33.5.6. A *big smooth site* is any site Sch_{smooth} as in Sites, Definition 7.6.2 constructed as follows:

(1) Choose any set of schemes S_0, and any set of smooth coverings Cov_0 among these schemes.
(2) As underlying category take any category Sch_α constructed as in Sets, Lemma 3.9.2 starting with the set S_0.
(3) Choose any set of coverings as in Sets, Lemma 3.11.1 starting with the category Sch_α and the class of smooth coverings, and the set Cov_0 chosen above.

See the remarks following Definition 33.3.5 for motivation and explanation regarding the definition of big sites.

Before we continue with the introduction of the big smooth site of a scheme S, let us point out that the topology on a big smooth site Sch_{smooth} is in some sense induced from the smooth topology on the category of all schemes.

Lemma 33.5.7. *Let Sch_{smooth} be a big smooth site as in Definition 33.5.6. Let $T \in \mathrm{Ob}(Sch_{smooth})$. Let $\{T_i \to T\}_{i\in I}$ be an arbitrary smooth covering of T.*

(1) *There exists a covering $\{U_j \to T\}_{j\in J}$ of T in the site Sch_{smooth} which refines $\{T_i \to T\}_{i\in I}$.*
(2) *If $\{T_i \to T\}_{i\in I}$ is a standard smooth covering, then it is tautologically equivalent to a covering of Sch_{smooth}.*
(3) *If $\{T_i \to T\}_{i\in I}$ is a Zariski covering, then it is tautologically equivalent to a covering of Sch_{smooth}.*

Proof. For each i choose an affine open covering $T_i = \bigcup_{j\in J_i} T_{ij}$ such that each T_{ij} maps into an affine open subscheme of T. By Lemma 33.5.3 the refinement $\{T_{ij} \to T\}_{i\in I, j\in J_i}$ is an smooth covering of T as well. Hence we may assume each T_i is affine, and maps into an affine open W_i of T. Applying Sets, Lemma 3.9.9 we see that W_i is isomorphic to an object of Sch_{Zar}. But then T_i as a finite type scheme over W_i is isomorphic to an object V_i of Sch_{Zar} by a second application of Sets, Lemma 3.9.9. The covering $\{V_i \to T\}_{i\in I}$ refines $\{T_i \to T\}_{i\in I}$ (because they are isomorphic). Moreover, $\{V_i \to T\}_{i\in I}$ is combinatorially equivalent to a covering $\{U_j \to T\}_{j\in J}$ of T in the site Sch_{Zar} by Sets, Lemma 3.9.9. The covering $\{U_j \to T\}_{j\in J}$ is a refinement as in (1). In the situation of (2), (3) each of the schemes T_i is isomorphic to an object of Sch_{smooth} by Sets, Lemma 3.9.9, and another application of Sets, Lemma 3.11.1 gives what we want. □

Definition 33.5.8. Let S be a scheme. Let Sch_{smooth} be a big smooth site containing S.

(1) The *big smooth site of S*, denoted $(Sch/S)_{smooth}$, is the site Sch_{smooth}/S introduced in Sites, Section 7.24.
(2) The *big affine smooth site of S*, denoted $(Aff/S)_{smooth}$, is the full subcategory of $(Sch/S)_{smooth}$ whose objects are affine U/S. A covering of $(Aff/S)_{smooth}$ is any covering $\{U_i \to U\}$ of $(Sch/S)_{smooth}$ which is a standard smooth covering.

Next, we check that the big affine site defines the same topos as the big site.

Lemma 33.5.9. *Let S be a scheme. Let $Sch_{étale}$ be a big smooth site containing S. The functor $(Aff/S)_{smooth} \to (Sch/S)_{smooth}$ is special cocontinuous and induces an equivalence of topoi from $Sh((Aff/S)_{smooth})$ to $Sh((Sch/S)_{smooth})$.*

Proof. The notion of a special cocontinuous functor is introduced in Sites, Definition 7.28.2. Thus we have to verify assumptions (1) – (5) of Sites, Lemma 7.28.1. Denote the inclusion functor $u : (Aff/S)_{smooth} \to (Sch/S)_{smooth}$. Being cocontinuous just means that any smooth covering of T/S, T affine, can be refined by a standard smooth covering of T. This is the content of Lemma 33.5.4. Hence (1) holds. We see u is continuous simply because a standard smooth covering is a smooth covering. Hence (2) holds. Parts (3) and (4) follow immediately from the fact that u is fully faithful. And finally condition (5) follows from the fact that every scheme has an affine open covering. □

To be continued...

Lemma 33.5.10. *Let Sch_{smooth} be a big smooth site. Let $f : T \to S$ be a morphism in Sch_{smooth}. The functor*

$$u : (Sch/T)_{smooth} \longrightarrow (Sch/S)_{smooth}, \quad V/T \longmapsto V/S$$

is cocontinuous, and has a continuous right adjoint

$$v : (Sch/S)_{smooth} \longrightarrow (Sch/T)_{smooth}, \quad (U \to S) \longmapsto (U \times_S T \to T).$$

They induce the same morphism of topoi

$$f_{big} : Sh((Sch/T)_{smooth}) \longrightarrow Sh((Sch/S)_{smooth})$$

We have $f_{big}^{-1}(\mathcal{G})(U/T) = \mathcal{G}(U/S)$. We have $f_{big,*}(\mathcal{F})(U/S) = \mathcal{F}(U \times_S T/T)$. Also, f_{big}^{-1} has a left adjoint $f_{big!}$ which commutes with fibre products and equalizers.

Proof. The functor u is cocontinuous, continuous, and commutes with fibre products and equalizers. Hence Sites, Lemmas 7.20.5 and 7.20.6 apply and we deduce the formula for f_{big}^{-1} and the existence of $f_{big!}$. Moreover, the functor v is a right adjoint because given U/T and V/S we have $\mathrm{Mor}_S(u(U), V) = \mathrm{Mor}_T(U, V \times_S T)$ as desired. Thus we may apply Sites, Lemmas 7.21.1 and 7.21.2 to get the formula for $f_{big,*}$. □

33.6. The syntomic topology

In this section we define the syntomic topology. This topology is quite interesting in that it often has the same cohomology groups as the fppf topology but is technically easier to deal with.

Definition 33.6.1. Let T be a scheme. An *syntomic covering of T* is a family of morphisms $\{f_i : T_i \to T\}_{i \in I}$ of schemes such that each f_i is syntomic and such that $T = \bigcup f_i(T_i)$.

Lemma 33.6.2. *Any smooth covering is a syntomic covering, and a fortiori, any étale or Zariski covering is a syntomic covering.*

Proof. This is clear from the definitions and the fact that a smooth morphism is syntomic, see Morphisms, Lemma 28.35.7 and Lemma 33.5.2. □

Next, we show that this notion satisfies the conditions of Sites, Definition 7.6.2.

Lemma 33.6.3. *Let T be a scheme.*

(1) *If $T' \to T$ is an isomorphism then $\{T' \to T\}$ is an syntomic covering of T.*

(2) *If $\{T_i \to T\}_{i\in I}$ is a syntomic covering and for each i we have a syntomic covering $\{T_{ij} \to T_i\}_{j\in J_i}$, then $\{T_{ij} \to T\}_{i\in I, j\in J_i}$ is a syntomic covering.*
(3) *If $\{T_i \to T\}_{i\in I}$ is a syntomic covering and $T' \to T$ is a morphism of schemes then $\{T' \times_T T_i \to T'\}_{i\in I}$ is a syntomic covering.*

Proof. Omitted. □

Lemma 33.6.4. *Let T be an affine scheme. Let $\{T_i \to T\}_{i\in I}$ be a syntomic covering of T. Then there exists a syntomic covering $\{U_j \to T\}_{j=1,\ldots,m}$ which is a refinement of $\{T_i \to T\}_{i\in I}$ such that each U_j is an affine scheme, and such that each morphism $U_j \to T$ is standard syntomic, see Morphisms, Definition 28.32.1. Moreover, we may choose each U_j to be open affine in one of the T_i.*

Proof. Omitted, but see Algebra, Lemma 10.129.15. □

Thus we define the corresponding standard coverings of affines as follows.

Definition 33.6.5. Let T be an affine scheme. A *standard syntomic covering* of T is a family $\{f_j : U_j \to T\}_{j=1,\ldots,m}$ with each U_j is affine, $U_j \to T$ standard syntomic and $T = \bigcup f_j(U_j)$.

Definition 33.6.6. A *big syntomic site* is any site $Sch_{syntomic}$ as in Sites, Definition 7.6.2 constructed as follows:

(1) Choose any set of schemes S_0, and any set of syntomic coverings Cov_0 among these schemes.
(2) As underlying category take any category Sch_α constructed as in Sets, Lemma 3.9.2 starting with the set S_0.
(3) Choose any set of coverings as in Sets, Lemma 3.11.1 starting with the category Sch_α and the class of syntomic coverings, and the set Cov_0 chosen above.

See the remarks following Definition 33.3.5 for motivation and explanation regarding the definition of big sites.

Before we continue with the introduction of the big syntomic site of a scheme S, let us point out that the topology on a big syntomic site $Sch_{syntomic}$ is in some sense induced from the syntomic topology on the category of all schemes.

Lemma 33.6.7. *Let $Sch_{syntomic}$ be a big syntomic site as in Definition 33.6.6. Let $T \in \mathrm{Ob}(Sch_{syntomic})$. Let $\{T_i \to T\}_{i\in I}$ be an arbitrary syntomic covering of T.*

(1) *There exists a covering $\{U_j \to T\}_{j\in J}$ of T in the site $Sch_{syntomic}$ which refines $\{T_i \to T\}_{i\in I}$.*
(2) *If $\{T_i \to T\}_{i\in I}$ is a standard syntomic covering, then it is tautologically equivalent to a covering in $Sch_{syntomic}$.*
(3) *If $\{T_i \to T\}_{i\in I}$ is a Zariski covering, then it is tautologically equivalent to a covering in $Sch_{syntomic}$.*

Proof. For each i choose an affine open covering $T_i = \bigcup_{j\in J_i} T_{ij}$ such that each T_{ij} maps into an affine open subscheme of T. By Lemma 33.6.3 the refinement $\{T_{ij} \to T\}_{i\in I, j\in J_i}$ is an syntomic covering of T as well. Hence we may assume each T_i is affine, and maps into an affine open W_i of T. Applying Sets, Lemma 3.9.9 we see that W_i is isomorphic to an object of Sch_{Zar}. But then T_i as a finite type scheme over W_i is isomorphic to an object V_i of Sch_{Zar} by a second application

of Sets, Lemma 3.9.9. The covering $\{V_i \to T\}_{i\in I}$ refines $\{T_i \to T\}_{i\in I}$ (because they are isomorphic). Moreover, $\{V_i \to T\}_{i\in I}$ is combinatorially equivalent to a covering $\{U_j \to T\}_{j\in J}$ of T in the site Sch_{Zar} by Sets, Lemma 3.9.9. The covering $\{U_j \to T\}_{j\in J}$ is a covering as in (1). In the situation of (2), (3) each of the schemes T_i is isomorphic to an object of Sch_{Zar} by Sets, Lemma 3.9.9, and another application of Sets, Lemma 3.11.1 gives what we want. □

Definition 33.6.8. Let S be a scheme. Let $Sch_{syntomic}$ be a big syntomic site containing S.

(1) The *big syntomic site of* S, denoted $(Sch/S)_{syntomic}$, is the site $Sch_{syntomic}/S$ introduced in Sites, Section 7.24.
(2) The *big affine syntomic site of* S, denoted $(Aff/S)_{syntomic}$, is the full subcategory of $(Sch/S)_{syntomic}$ whose objects are affine U/S. A covering of $(Aff/S)_{syntomic}$ is any covering $\{U_i \to U\}$ of $(Sch/S)_{syntomic}$ which is a standard syntomic covering.

Next, we check that the big affine site defines the same topos as the big site.

Lemma 33.6.9. *Let S be a scheme. Let $Sch_{syntomic}$ be a big syntomic site containing S. The functor $(Aff/S)_{syntomic} \to (Sch/S)_{syntomic}$ is special cocontinuous and induces an equivalence of topoi from $Sh((Aff/S)_{syntomic})$ to $Sh((Sch/S)_{syntomic})$.*

Proof. The notion of a special cocontinuous functor is introduced in Sites, Definition 7.28.2. Thus we have to verify assumptions (1) – (5) of Sites, Lemma 7.28.1. Denote the inclusion functor $u : (Aff/S)_{syntomic} \to (Sch/S)_{syntomic}$. Being cocontinuous just means that any syntomic covering of T/S, T affine, can be refined by a standard syntomic covering of T. This is the content of Lemma 33.6.4. Hence (1) holds. We see u is continuous simply because a standard syntomic covering is a syntomic covering. Hence (2) holds. Parts (3) and (4) follow immediately from the fact that u is fully faithful. And finally condition (5) follows from the fact that every scheme has an affine open covering. □

To be continued...

Lemma 33.6.10. *Let $Sch_{syntomic}$ be a big syntomic site. Let $f : T \to S$ be a morphism in $Sch_{syntomic}$. The functor*

$$u : (Sch/T)_{syntomic} \longrightarrow (Sch/S)_{syntomic}, \quad V/T \longmapsto V/S$$

is cocontinuous, and has a continuous right adjoint

$$v : (Sch/S)_{syntomic} \longrightarrow (Sch/T)_{syntomic}, \quad (U \to S) \longmapsto (U \times_S T \to T).$$

They induce the same morphism of topoi

$$f_{big} : Sh((Sch/T)_{syntomic}) \longrightarrow Sh((Sch/S)_{syntomic})$$

We have $f_{big}^{-1}(\mathcal{G})(U/T) = \mathcal{G}(U/S)$. We have $f_{big,*}(\mathcal{F})(U/S) = \mathcal{F}(U \times_S T/T)$. Also, f_{big}^{-1} has a left adjoint $f_{big!}$ which commutes with fibre products and equalizers.

Proof. The functor u is cocontinuous, continuous, and commutes with fibre products and equalizers. Hence Sites, Lemmas 7.20.5 and 7.20.6 apply and we deduce the formula for f_{big}^{-1} and the existence of $f_{big!}$. Moreover, the functor v is a right adjoint because given U/T and V/S we have $\mathrm{Mor}_S(u(U), V) = \mathrm{Mor}_T(U, V \times_S T)$ as desired. Thus we may apply Sites, Lemmas 7.21.1 and 7.21.2 to get the formula for $f_{big,*}$. □

33.7. The fppf topology

Let S be a scheme. We would like to define the fppf-topology[2] on the category of schemes over S. According to our general principle we first introduce the notion of an fppf-covering.

Definition 33.7.1. Let T be a scheme. An *fppf covering of T* is a family of morphisms $\{f_i : T_i \to T\}_{i \in I}$ of schemes such that each f_i is flat, locally of finite presentation and such that $T = \bigcup f_i(T_i)$.

Lemma 33.7.2. *Any syntomic covering is an fppf covering, and a fortiori, any smooth, étale, or Zariski covering is an fppf covering.*

Proof. This is clear from the definitions, the fact that a syntomic morphism is flat and locally of finite presentation, see Morphisms, Lemmas 28.32.6 and 28.32.7, and Lemma 33.6.2. □

Next, we show that this notion satisfies the conditions of Sites, Definition 7.6.2.

Lemma 33.7.3. *Let T be a scheme.*

(1) *If $T' \to T$ is an isomorphism then $\{T' \to T\}$ is an fppf covering of T.*
(2) *If $\{T_i \to T\}_{i \in I}$ is an fppf covering and for each i we have an fppf covering $\{T_{ij} \to T_i\}_{j \in J_i}$, then $\{T_{ij} \to T\}_{i \in I, j \in J_i}$ is an fppf covering.*
(3) *If $\{T_i \to T\}_{i \in I}$ is an fppf covering and $T' \to T$ is a morphism of schemes then $\{T' \times_T T_i \to T'\}_{i \in I}$ is an fppf covering.*

Proof. The first assertion is clear. The second follows as the composition of flat morphisms is flat (see Morphisms, Lemma 28.26.5) and the composition of morphisms of finite presentation is of finite presentation (see Morphisms, Lemma 28.22.3). The third follows as the base change of a flat morphism is flat (see Morphisms, Lemma 28.26.7) and the base change of a morphism of finite presentation is of finite presentation (see Morphisms, Lemma 28.22.4). Moreover, the base change of a surjective family of morphisms is surjective (proof omitted). □

Lemma 33.7.4. *Let T be an affine scheme. Let $\{T_i \to T\}_{i \in I}$ be an fppf covering of T. Then there exists an fppf covering $\{U_j \to T\}_{j=1,\ldots,m}$ which is a refinement of $\{T_i \to T\}_{i \in I}$ such that each U_j is an affine scheme. Moreover, we may choose each U_j to be open affine in one of the T_i.*

Proof. This follows directly from the definitions using that a morphism which is flat and locally of finite presentation is open, see Morphisms, Lemma 28.26.9. □

Thus we define the corresponding standard coverings of affines as follows.

Definition 33.7.5. Let T be an affine scheme. A *standard fppf covering* of T is a family $\{f_j : U_j \to T\}_{j=1,\ldots,m}$ with each U_j is affine, flat and of finite presentation over T and $T = \bigcup f_j(U_j)$.

Definition 33.7.6. A *big fppf site* is any site Sch_{fppf} as in Sites, Definition 7.6.2 constructed as follows:

(1) Choose any set of schemes S_0, and any set of fppf coverings Cov_0 among these schemes.

[2] The letters fppf stand for "fidèlement plat de présentation finie".

(2) As underlying category take any category Sch_α constructed as in Sets, Lemma 3.9.2 starting with the set S_0.
(3) Choose any set of coverings as in Sets, Lemma 3.11.1 starting with the category Sch_α and the class of fppf coverings, and the set Cov_0 chosen above.

See the remarks following Definition 33.3.5 for motivation and explanation regarding the definition of big sites.

Before we continue with the introduction of the big fppf site of a scheme S, let us point out that the topology on a big fppf site Sch_{fppf} is in some sense induced from the fppf topology on the category of all schemes.

Lemma 33.7.7. *Let Sch_{fppf} be a big fppf site as in Definition 33.7.6. Let $T \in \mathrm{Ob}(Sch_{fppf})$. Let $\{T_i \to T\}_{i\in I}$ be an arbitrary fppf covering of T.*

(1) *There exists a covering $\{U_j \to T\}_{j\in J}$ of T in the site Sch_{fppf} which refines $\{T_i \to T\}_{i\in I}$.*
(2) *If $\{T_i \to T\}_{i\in I}$ is a standard fppf covering, then it is tautologically equivalent to a covering of Sch_{fppf}.*
(3) *If $\{T_i \to T\}_{i\in I}$ is a Zariski covering, then it is tautologically equivalent to a covering of Sch_{fppf}.*

Proof. For each i choose an affine open covering $T_i = \bigcup_{j\in J_i} T_{ij}$ such that each T_{ij} maps into an affine open subscheme of T. By Lemma 33.7.3 the refinement $\{T_{ij} \to T\}_{i\in I, j\in J_i}$ is an fppf covering of T as well. Hence we may assume each T_i is affine, and maps into an affine open W_i of T. Applying Sets, Lemma 3.9.9 we see that W_i is isomorphic to an object of Sch_{Zar}. But then T_i as a finite type scheme over W_i is isomorphic to an object V_i of Sch_{Zar} by a second application of Sets, Lemma 3.9.9. The covering $\{V_i \to T\}_{i\in I}$ refines $\{T_i \to T\}_{i\in I}$ (because they are isomorphic). Moreover, $\{V_i \to T\}_{i\in I}$ is combinatorially equivalent to a covering $\{U_j \to T\}_{j\in J}$ of T in the site Sch_{Zar} by Sets, Lemma 3.9.9. The covering $\{U_j \to T\}_{j\in J}$ is a refinement as in (1). In the situation of (2), (3) each of the schemes T_i is isomorphic to an object of Sch_{fppf} by Sets, Lemma 3.9.9, and another application of Sets, Lemma 3.11.1 gives what we want. □

Definition 33.7.8. Let S be a scheme. Let Sch_{fppf} be a big fppf site containing S.

(1) The *big fppf site of S*, denoted $(Sch/S)_{fppf}$, is the site Sch_{fppf}/S introduced in Sites, Section 7.24.
(2) The *big affine fppf site of S*, denoted $(Aff/S)_{fppf}$, is the full subcategory of $(Sch/S)_{fppf}$ whose objects are affine U/S. A covering of $(Aff/S)_{fppf}$ is any covering $\{U_i \to U\}$ of $(Sch/S)_{fppf}$ which is a standard fppf covering.

It is not completely clear that the big affine fppf site is a site. We check this now.

Lemma 33.7.9. *Let S be a scheme. Let Sch_{fppf} be a big fppf site containing S. Then $(Aff/S)_{fppf}$ is a site.*

Proof. Let us show that $(Aff/S)_{fppf}$ is a site. Reasoning as in the proof of Lemma 33.4.9 it suffices to show that the collection of standard fppf coverings of affines satisfies properties (1), (2) and (3) of Sites, Definition 7.6.2. This is clear since for example, given a standard fppf covering $\{T_i \to T\}_{i\in I}$ and for each i we have a

standard fppf covering $\{T_{ij} \to T_i\}_{j \in J_i}$, then $\{T_{ij} \to T\}_{i \in I, j \in J_i}$ is a standard fppf covering because $\bigcup_{i \in I} J_i$ is finite and each T_{ij} is affine. □

Lemma 33.7.10. *Let S be a scheme. Let Sch_{fppf} be a big fppf site containing S. The underlying categories of the sites Sch_{fppf}, $(Sch/S)_{fppf}$, and $(Aff/S)_{fppf}$ have fibre products. In each case the obvious functor into the category Sch of all schemes commutes with taking fibre products. The category $(Sch/S)_{fppf}$ has a final object, namely S/S.*

Proof. For Sch_{fppf} it is true by construction, see Sets, Lemma 3.9.9. Suppose we have $U \to S$, $V \to U$, $W \to U$ morphisms of schemes with $U, V, W \in \mathrm{Ob}(Sch_{fppf})$. The fibre product $V \times_U W$ in Sch_{fppf} is a fibre product in Sch and is the fibre product of V/S with W/S over U/S in the category of all schemes over S, and hence also a fibre product in $(Sch/S)_{fppf}$. This proves the result for $(Sch/S)_{fppf}$. If U, V, W are affine, so is $V \times_U W$ and hence the result for $(Aff/S)_{fppf}$. □

Next, we check that the big affine site defines the same topos as the big site.

Lemma 33.7.11. *Let S be a scheme. Let Sch_{fppf} be a big fppf site containing S. The functor $(Aff/S)_{fppf} \to (Sch/S)_{fppf}$ is cocontinuous and induces an equivalence of topoi from $Sh((Aff/S)_{fppf})$ to $Sh((Sch/S)_{fppf})$.*

Proof. The notion of a special cocontinuous functor is introduced in Sites, Definition 7.28.2. Thus we have to verify assumptions (1) – (5) of Sites, Lemma 7.28.1. Denote the inclusion functor $u : (Aff/S)_{fppf} \to (Sch/S)_{fppf}$. Being cocontinuous just means that any fppf covering of T/S, T affine, can be refined by a standard fppf covering of T. This is the content of Lemma 33.7.4. Hence (1) holds. We see u is continuous simply because a standard fppf covering is a fppf covering. Hence (2) holds. Parts (3) and (4) follow immediately from the fact that u is fully faithful. And finally condition (5) follows from the fact that every scheme has an affine open covering. □

Next, we establish some relationships between the topoi associated to these sites.

Lemma 33.7.12. *Let Sch_{fppf} be a big fppf site. Let $f : T \to S$ be a morphism in Sch_{fppf}. The functor*

$$u : (Sch/T)_{fppf} \longrightarrow (Sch/S)_{fppf}, \quad V/T \longmapsto V/S$$

is cocontinuous, and has a continuous right adjoint

$$v : (Sch/S)_{fppf} \longrightarrow (Sch/T)_{fppf}, \quad (U \to S) \longmapsto (U \times_S T \to T).$$

They induce the same morphism of topoi

$$f_{big} : Sh((Sch/T)_{fppf}) \longrightarrow Sh((Sch/S)_{fppf})$$

We have $f_{big}^{-1}(\mathcal{G})(U/T) = \mathcal{G}(U/S)$. We have $f_{big,*}(\mathcal{F})(U/S) = \mathcal{F}(U \times_S T/T)$. Also, f_{big}^{-1} has a left adjoint $f_{big!}$ which commutes with fibre products and equalizers.

Proof. The functor u is cocontinuous, continuous, and commutes with fibre products and equalizers. Hence Sites, Lemmas 7.20.5 and 7.20.6 apply and we deduce the formula for f_{big}^{-1} and the existence of $f_{big!}$. Moreover, the functor v is a right adjoint because given U/T and V/S we have $\mathrm{Mor}_S(u(U), V) = \mathrm{Mor}_T(U, V \times_S T)$ as desired. Thus we may apply Sites, Lemmas 7.21.1 and 7.21.2 to get the formula for $f_{big,*}$. □

Lemma 33.7.13. *Given schemes X, Y, Y in $(Sch/S)_{fppf}$ and morphisms $f : X \to Y$, $g : Y \to Z$ we have $g_{big} \circ f_{big} = (g \circ f)_{big}$.*

Proof. This follows from the simple description of pushforward and pullback for the functors on the big sites from Lemma 33.7.12. □

33.8. The fpqc topology

Definition 33.8.1. Let T be a scheme. An *fpqc covering of T* is a family of morphisms $\{f_i : T_i \to T\}_{i \in I}$ of schemes such that each f_i is flat and such that for every affine open $U \subset T$ there exists $n \geq 0$, a map $a : \{1, \ldots, n\} \to I$ and affine opens $V_j \subset T_{a(j)}$, $j = 1, \ldots, n$ with $\bigcup_{j=1}^{n} f_{a(j)}(V_j) = U$.

To be sure this condition implies that $T = \bigcup f_i(T_i)$. It is slightly harder to recognize an fpqc covering, hence we provide some lemmas to do so.

Lemma 33.8.2. *Let T be a scheme. Let $\{f_i : T_i \to T\}_{i \in I}$ be a family of morphisms of schemes with target T. The following are equivalent*

1. *$\{f_i : T_i \to T\}_{i \in I}$ is an fpqc covering,*
2. *each f_i is flat and for every affine open $U \subset T$ there exist quasi-compact opens $U_i \subset T_i$ which are almost all empty, such that $U = \bigcup f_i(U_i)$,*
3. *each f_i is flat and there exists an affine open covering $T = \bigcup_{\alpha \in A} U_\alpha$ and for each $\alpha \in A$ there exist $i_{\alpha,1}, \ldots, i_{\alpha,n(\alpha)} \in I$ and quasi-compact opens $U_{\alpha,j} \subset T_{i_{\alpha,j}}$ such that $U_\alpha = \bigcup_{j=1,\ldots,n(\alpha)} f_{i_{\alpha,j}}(U_{\alpha,j})$.*

If T is quasi-separated, these are also equivalent to

4. *each f_i is flat, and for every $t \in T$ there exist $i_1, \ldots, i_n \in I$ and quasi-compact opens $U_j \subset T_{i_j}$ such that $\bigcup_{j=1,\ldots,n} f_{i_j}(U_j)$ is a (not necessarily open) neighbourhood of t in T.*

Proof. We omit the proof of theequivalence of (1), (2), and (3). From now on assume T is quasi-separated. We prove (4) implies (2). Let $U \subset T$ be an affine open. To prove (2) it suffices to show that for every $t \in U$ there exist finitely many quasi-compact opens $U_j \subset T_{i_j}$ such that $f_{i_j}(U_j) \subset U$ and such that $\bigcup f_{i_j}(U_j)$ is a neighbourhood of t in U. By assumption there do exist finitely many quasi-compact opens $U'_j \subset T_{i_j}$ such that such that $\bigcup f_{i_j}(U'_j)$ is a neighbourhood of t in T. Since T is quasi-separated we see that $U_j = U'_j \cap f_j^{-1}(U)$ is quasi-compact open as desired. Since it is clear that (2) implies (4) the proof is finished. □

Lemma 33.8.3. *Let T be a scheme. Let $\{f_i : T_i \to T\}_{i \in I}$ be a family of morphisms of schemes with target T. The following are equivalent*

1. *$\{f_i : T_i \to T\}_{i \in I}$ is an fpqc covering, and*
2. *setting $T' = \coprod_{i \in I} T_i$, and $f = \coprod_{i \in I} f_i$ the family $\{f : T' \to T\}$ is an fpqc covering.*

Proof. Suppose that $U \subset T$ is an affine open. If (1) holds, then we find $i_1, \ldots, i_n \in I$ and affine opens $U_j \subset T_{i_j}$ such that $U = \bigcup_{j=1,\ldots,n} f_{i_j}(U_j)$. Then $U_1 \amalg \ldots \amalg U_n \subset T'$ is a quasi-compact open surjecting onto U. Thus $\{f : T' \to T\}$ is an fpqc covering by Lemma 33.8.2. Conversely, if (2) holds then there exists a quasi-compact open $U' \subset T'$ with $U = f(U')$. Then $U_j = U' \cap T_j$ is quasi-compact open in T_j and empty for almost all j. By Lemma 33.8.2 we see that (1) holds. □

Lemma 33.8.4. *Let T be a scheme. Let $\{f_i : T_i \to T\}_{i \in I}$ be a family of morphisms of schemes with target T. Assume that*

(1) *each f_i is flat, and*
(2) *the family $\{f_i : T_i \to T\}_{i \in I}$ can be refined by a fpqc covering of T.*

Then $\{f_i : T_i \to T\}_{i \in I}$ is a fpqc covering of T.

Proof. Let $\{g_j : X_j \to T\}_{j \in J}$ be an fpqc covering refining $\{f_i : T_i \to T\}$. Suppose that $U \subset T$ is affine open. Choose $j_1, \ldots, j_m \in J$ and $V_k \subset X_{j_k}$ affine open such that $U = \bigcup g_{j_k}(V_k)$. For each j pick $i_j \in I$ and a morphism $h_j : X_j \to T_{i_j}$ such that $g_j = f_{i_j} \circ h_j$. Since $h_{j_k}(V_k)$ is quasi-compact we can find a quasi-compact open $h_{j_k}(V_k) \subset U_k \subset f_{i_{j_k}}^{-1}(U)$. Then $U = \bigcup f_{i_{j_k}}(U_k)$. We conclude that $\{f_i : T_i \to T\}_{i \in I}$ is an fpqc covering by Lemma 33.8.2. □

Lemma 33.8.5. *Let T be a scheme. Let $\{f_i : T_i \to T\}_{i \in I}$ be a family of morphisms of schemes with target T. Assume that*

(1) *each f_i is flat, and*
(2) *there exists an fpqc covering $\{g_j : S_j \to T\}_{j \in J}$ such that each $\{S_j \times_T T_i \to S_j\}_{i \in I}$ is an fpqc covering.*

Then $\{f_i : T_i \to T\}_{i \in I}$ is a fpqc covering of T.

Proof. We will use Lemma 33.8.2 without further mention. Let $U \subset T$ be an affine open. By (2) we can find quasi-compact opens $V_j \subset S_j$ for $j \in J$, almost all empty, such that $U = \bigcup g_j(V_j)$. Then for each j we can choose quasi-compact opens $W_{ij} \subset S_j \times_T T_i$ for $i \in I$, almost all empty, with $V_j = \bigcup_i \mathrm{pr}_1(W_{ij})$. Thus $\{S_j \times_T T_i \to T\}$ is an fpqc covering. Since this covering refines $\{f_i : T_i \to T\}$ we conclude by Lemma 33.8.4. □

Lemma 33.8.6. *Any fppf covering is an fpqc covering, and a fortiori, any syntomic, smooth, étale or Zariski covering is an fpqc covering.*

Proof. We will show that an fppf covering is an fpqc covering, and then the rest follows from Lemma 33.7.2. Let $\{f_i : U_i \to U\}_{i \in I}$ be an fppf covering. By definition this means that the f_i are flat which checks the first condition of Definition 33.8.1. To check the second, let $V \subset U$ be an affine open subset. Write $f_i^{-1}(V) = \bigcup_{j \in J_i} V_{ij}$ for some affine opens $V_{ij} \subset U_i$. Since each f_i is open (Morphisms, Lemma 28.26.9), we see that $V = \bigcup_{i \in I} \bigcup_{j \in J_i} f_i(V_{ij})$ is an open covering of V. Since V is quasi-compact, this covering has a finite refinement. This finishes the proof. □

The fpqc[3] topology cannot be treated in the same way as the fppf topology[4]. Namely, suppose that R is a nonzero ring. For any faithfully flat ring map $R \to R'$ the morphism $\mathrm{Spec}(R') \to \mathrm{Spec}(R)$ is an fpqc-covering. We claim that there does not exist a set A of fpqc-coverings of $\mathrm{Spec}(R)$ such that every fpqc-covering can be refined by an element of A. For example, if $R = k$ is a field, then for any set I we can consider the purely transcendental field extension $k \subset k(\{t_i\}_{i \in I})$. We leave it to the reader to show that there does not exist a set of morphisms of schemes $\{S_j \to \mathrm{Spec}(k)\}_{j \in J}$ such that every morphism $\mathrm{Spec}(k(\{t_i\}_{i \in I}))$ is dominated by one of the schemes S_j.

[3] The letters fpqc stand for "fidèlement plat quasi-compacte".

[4] A more precise statement would be that the analogue of Lemma 33.7.7 for the fpqc topology does not hold.

A mildly interesting option is to consider only those faithfully flat ring extensions $R \to R'$ where the cardinality of R' is suitably bounded. (And if you consider all schemes in a fixed universe as in SGA4 then you are bounding the cardinality by a strongly inaccessible cardinal.) However, it is not so clear what happens if you change the cardinal to a bigger one.

For these reasons we do not introduce fpqc sites and we will not consider cohomology with respect to the fpqc-topology.

On the other hand, given a contravariant functor $F : Sch^{opp} \to Sets$ it does make sense to ask whether F satisfies the sheaf property for the fpqc topology, see below. Moreover, we can wonder about descent of object in the fpqc topology, etc. Simply put, for certain results the correct generality is to work with fpqc coverings.

Lemma 33.8.7. *Let T be a scheme.*

(1) *If $T' \to T$ is an isomorphism then $\{T' \to T\}$ is an fpqc covering of T.*
(2) *If $\{T_i \to T\}_{i \in I}$ is an fpqc covering and for each i we have an fpqc covering $\{T_{ij} \to T_i\}_{j \in J_i}$, then $\{T_{ij} \to T\}_{i \in I, j \in J_i}$ is an fpqc covering.*
(3) *If $\{T_i \to T\}_{i \in I}$ is an fpqc covering and $T' \to T$ is a morphism of schemes then $\{T' \times_T T_i \to T'\}_{i \in I}$ is an fpqc covering.*

Proof. Part (1) is immediate. Recall that the composition of flat morphisms is flat and that the base change of a flat morphism is flat (Morphisms, Lemmas 28.26.7 and 28.26.5). Thus we can apply Lemma 33.8.2 in each case to check that our families of morphisms are fpqc coverings.

Proof of (2). Assume $\{T_i \to T\}_{i \in I}$ is an fpqc covering and for each i we have an fpqc covering $\{f_{ij} : T_{ij} \to T_i\}_{j \in J_i}$. Let $U \subset T$ be an affine open. We can find quasi-compact opens $U_i \subset T_i$ for $i \in I$, almost all empty, such that $U = \bigcup f_i(U_i)$. Then for each i we can choose quasi-compact opens $W_{ij} \subset T_{ij}$ for $j \in J_i$, almost all empty, with $U_i = \bigcup_j f_{ij}(U_{ij})$. Thus $\{T_{ij} \to T\}$ is an fpqc covering.

Proof of (3). Assume $\{T_i \to T\}_{i \in I}$ is an fpqc covering and $T' \to T$ is a morphism of schemes. Let $U' \subset T'$ be an affine open which maps into the affine open $U \subset T$. Choose quasi-compact opens $U_i \subset T_i$, almost all empty, such that $U = \bigcup f_i(U_i)$. Then $U' \times_U U_i$ is a quasi-compact open of $T' \times_T T_i$ and $U' = \bigcup \mathrm{pr}_1(U' \times_U U_i)$. Since T' can be covered by such affine opens $U' \subset T'$ we see that $\{T' \times_T T_i \to T'\}_{i \in I}$ is an fpqc covering by Lemma 33.8.2 □

Lemma 33.8.8. *Let T be an affine scheme. Let $\{T_i \to T\}_{i \in I}$ be an fpqc covering of T. Then there exists an fpqc covering $\{U_j \to T\}_{j=1,\ldots,n}$ which is a refinement of $\{T_i \to T\}_{i \in I}$ such that each U_j is an affine scheme. Moreover, we may choose each U_j to be open affine in one of the T_i.*

Proof. This follows directly from the definition. □

Definition 33.8.9. Let T be an affine scheme. A *standard fpqc covering* of T is a family $\{f_j : U_j \to T\}_{j=1,\ldots,n}$ with each U_j is affine, flat over T and $T = \bigcup f_j(U_j)$.

Since we do not introduce the affine site we have to show directly that the collection of all standard fpqc coverings satisfies the axioms.

Lemma 33.8.10. *Let T be an affine scheme.*

(1) *If $T' \to T$ is an isomorphism then $\{T' \to T\}$ is a standard fpqc covering of T.*
(2) *If $\{T_i \to T\}_{i \in I}$ is a standard fpqc covering and for each i we have a standard fpqc covering $\{T_{ij} \to T_i\}_{j \in J_i}$, then $\{T_{ij} \to T\}_{i \in I, j \in J_i}$ is a standard fpqc covering.*
(3) *If $\{T_i \to T\}_{i \in I}$ is a standard fpqc covering and $T' \to T$ is a morphism of affine schemes then $\{T' \times_T T_i \to T'\}_{i \in I}$ is a standard fpqc covering.*

Proof. This follows formally from the fact that compositions and base changes of flat morphisms are flat (Morphisms, Lemmas 28.26.7 and 28.26.5) and that fibre products of affine schemes are affine (Schemes, Lemma 25.17.2). □

Lemma 33.8.11. *Let T be a scheme. Let $\{f_i : T_i \to T\}_{i \in I}$ be a family of morphisms of schemes with target T. Assume that*

(1) *each f_i is flat, and*
(2) *every affine scheme Z and morphism $h : Z \to T$ there exists a standard fpqc covering $\{Z_j \to Z\}_{j=1,\ldots,n}$ which refines the family $\{T_i \times_T Z \to Z\}_{i \in I}$.*

Then $\{f_i : T_i \to T\}_{i \in I}$ is a fpqc covering of T.

Proof. Let $T = \bigcup U_\alpha$ be an affine open covering. For each α the pullback family $\{T_i \times_T U_\alpha \to U_\alpha\}$ can be refined by a standard fpqc covering, hence is an fpqc covering by Lemma 33.8.4. As $\{U_\alpha \to T\}$ is an fpqc covering we conclude that $\{T_i \to T\}$ is an fpqc covering by Lemma 33.8.5. □

Definition 33.8.12. Let F be a contravariant functor on the category of schemes with values in sets.

(1) Let $\{U_i \to T\}_{i \in I}$ be a family of morphisms of schemes with fixed target. We say that F *satisfies the sheaf property for the given family* if for any collection of elements $\xi_i \in F(U_i)$ such that $\xi_i|_{U_i \times_T U_j} = \xi_j|_{U_i \times_T U_j}$ there exists a unique element $\xi \in F(T)$ such that $\xi_i = \xi|_{U_i}$ in $F(U_i)$.
(2) We say that F *satisfies the sheaf property for the fpqc topology* if it satisfies the sheaf property for any fpqc covering.

We try to avoid using the terminology "F is a sheaf" in this situation since we are not defining a category of fpqc sheaves as we explained above.

Lemma 33.8.13. *Let F be a contravariant functor on the category of schemes with values in sets. Then F satisfies the sheaf property for the fpqc topology if and only if it satisfies*

(1) *the sheaf property for every Zariski covering, and*
(2) *the sheaf property for any standard fpqc covering.*

Moreover, in the presence of (1) property (2) is equivalent to property

(2') *the sheaf property for $\{V \to U\}$ with V, U affine and $V \to U$ faithfully flat.*

Proof. Assume (1) and (2) hold. Let $\{f_i : T_i \to T\}_{i \in I}$ be an fpqc covering. Let $s_i \in F(T_i)$ be a family of elements such that s_i and s_j map to the same element of $F(T_i \times_T T_j)$. Let $W \subset T$ be the maximal open subset such that there exists a unique $s \in F(W)$ with $s|_{f_i^{-1}(W)} = s_i|_{f_i^{-1}(W)}$ for all i. Such a maximal open exists because F satisfies the sheaf property for Zariski coverings; in fact W is the union of all opens with this property. Let $t \in T$. We will show $t \in W$. To do this we

pick an affine open $t \in U \subset T$ and we will show there is a unique $s \in F(U)$ with $s|_{f_i^{-1}(U)} = s_i|_{f_i^{-1}(U)}$ for all i.

By Lemma 33.8.8 we can find a standard fpqc covering $\{U_j \to U\}_{j=1,\ldots,n}$ refining $\{U \times_T T_i \to U\}$, say by morphisms $h_j : U_j \to T_{i_j}$. By (2) we obtain a unique element $s \in F(U)$ such that $s|_{U_j} = F(h_j)(s_{i_j})$. Note that for any scheme $V \to U$ over U there is a unique section $s_V \in F(V)$ which restricts to $F(h_j \circ \mathrm{pr}_2)(s_{i_j})$ on $V \times_U U_j$ for $j = 1, \ldots, n$. Namely, this is true if V is affine by (2) as $\{V \times_U U_j \to V\}$ is a standard fpqc covering and in general this follows from (1) and the affine case by choosing an affine open covering of V. In particular, $s_V = s|_V$. Now, taking $V = U \times_T T_i$ and using that $s_{i_j}|_{T_{i_j} \times_T T_i} = s_i|_{T_{i_j} \times_T T_i}$ we conclude that $s|_{U \times_T T_i} = s_V = s_i|_{U \times_T T_i}$ which is what we had to show.

Proof of the equivalence of (2) and (2') in the presence of (1). Suppose $\{T_i \to T\}$ is a standard fpqc covering, then $\coprod T_i \to T$ is a faithfully flat morphism of affine schemes. In the presence of (1) we have $F(\coprod T_i) = \prod F(T_i)$ and similarly $F((\coprod T_i) \times_T (\coprod T_i)) = \prod F(T_i \times_T T_{i'})$. Thus the sheaf condition for $\{T_i \to T\}$ and $\{\coprod T_i \to T\}$ is the same. □

33.9. Change of topologies

Let $f : X \to Y$ be a morphism of schemes over a base change S. In this case we have the following morphisms of sites (with suitable choices of sites as in Remark 33.9.1 below):

(1) $(Sch/X)_{fppf} \longrightarrow (Sch/Y)_{fppf}$,
(2) $(Sch/X)_{fppf} \longrightarrow (Sch/Y)_{syntomic}$,
(3) $(Sch/X)_{fppf} \longrightarrow (Sch/Y)_{smooth}$,
(4) $(Sch/X)_{fppf} \longrightarrow (Sch/Y)_{étale}$,
(5) $(Sch/X)_{fppf} \longrightarrow (Sch/Y)_{Zar}$,
(6) $(Sch/X)_{syntomic} \longrightarrow (Sch/Y)_{syntomic}$,
(7) $(Sch/X)_{syntomic} \longrightarrow (Sch/Y)_{smooth}$,
(8) $(Sch/X)_{syntomic} \longrightarrow (Sch/Y)_{étale}$,
(9) $(Sch/X)_{syntomic} \longrightarrow (Sch/Y)_{Zar}$,
(10) $(Sch/X)_{smooth} \longrightarrow (Sch/Y)_{smooth}$,
(11) $(Sch/X)_{smooth} \longrightarrow (Sch/Y)_{étale}$,
(12) $(Sch/X)_{smooth} \longrightarrow (Sch/Y)_{Zar}$,
(13) $(Sch/X)_{étale} \longrightarrow (Sch/Y)_{étale}$,
(14) $(Sch/X)_{étale} \longrightarrow (Sch/Y)_{Zar}$,
(15) $(Sch/X)_{Zar} \longrightarrow (Sch/Y)_{Zar}$,
(16) $(Sch/X)_{fppf} \longrightarrow Y_{étale}$,
(17) $(Sch/X)_{syntomic} \longrightarrow Y_{étale}$,
(18) $(Sch/X)_{smooth} \longrightarrow Y_{étale}$,
(19) $(Sch/X)_{étale} \longrightarrow Y_{étale}$,
(20) $(Sch/X)_{fppf} \longrightarrow Y_{Zar}$,
(21) $(Sch/X)_{syntomic} \longrightarrow Y_{Zar}$,
(22) $(Sch/X)_{smooth} \longrightarrow Y_{Zar}$,
(23) $(Sch/X)_{étale} \longrightarrow Y_{Zar}$,
(24) $(Sch/X)_{Zar} \longrightarrow Y_{Zar}$,
(25) $X_{étale} \longrightarrow Y_{étale}$,
(26) $X_{étale} \longrightarrow Y_{Zar}$,

(27) $X_{Zar} \longrightarrow Y_{Zar}$,

In each case the underlying continuous functor $Sch/Y \to Sch/X$, or $Y_\tau \to Sch/X$ is the functor $Y'/Y \mapsto X \times_Y Y'/Y$. Namely, in the sections above we have seen the morphisms $f_{big} : (Sch/X)_\tau \to (Sch/Y)_\tau$ and $f_{small} : X_\tau \to Y_\tau$ for τ as above. We also have seen the morphisms of sites $\pi_Y : (Sch/Y)_\tau \to Y_\tau$ for $\tau \in \{étale, Zariski\}$. On the other hand, it is clear that the identity functor $(Sch/X)_\tau \to (Sch/X)_{\tau'}$ defines a morphism of sites when τ is a stronger topology than τ'. Hence composing these gives the list of possible morphisms above.

Because of the simple description of the underlying functor it is clear that given morphisms of schemes $X \to Y \to Z$ the composition of two of the morphisms of sites above, e.g.,

$$(Sch/X)_{\tau_0} \longrightarrow (Sch/Y)_{\tau_1} \longrightarrow (Sch/Z)_{\tau_2}$$

is the corresponding morphism of sites associated to the morphism of schemes $X \to Z$.

Remark 33.9.1. Take any category Sch_α constructed as in Sets, Lemma 3.9.2 starting with the set of schemes $\{X, Y, S\}$. Choose any set of coverings Cov_{fppf} on Sch_α as in Sets, Lemma 3.11.1 starting with the category Sch_α and the class of fppf coverings. Let Sch_{fppf} denote the big fppf site so obtained. Next, for $\tau \in \{Zariski, étale, smooth, syntomic\}$ let Sch_τ have the same underlying category as Sch_{fppf} with coverings $\mathrm{Cov}_\tau \subset \mathrm{Cov}_{fppf}$ simply the subset of τ-coverings. It is straightforward to check that this gives rise to a big site Sch_τ.

33.10. Change of big sites

In this section we explain what happens on changing the big Zariski/fppf/étale sites.

Let $\tau, \tau' \in \{Zariski, étale, smooth, syntomic, fppf\}$. Given two big sites Sch_τ and $Sch'_{\tau'}$ we say that *Sch_τ is contained in $Sch'_{\tau'}$* if $\mathrm{Ob}(Sch_\tau) \subset \mathrm{Ob}(Sch'_{\tau'})$ and $\mathrm{Cov}(Sch_\tau) \subset \mathrm{Cov}(Sch'_{\tau'})$. In this case τ is stronger than τ', for example, no fppf site can be contained in an étale site.

Lemma 33.10.1. *Any set of big Zariski sites is contained in a common big Zariski site. The same is true, mutatis mutandis, for big fppf and big étale sites.*

Proof. This is true because the union of a set of sets is a set, and the constructions in the chapter on sets. □

Lemma 33.10.2. *Let $\tau \in \{Zariski, étale, smooth, syntomic, fppf\}$. Suppose given big sites Sch_τ and Sch'_τ. Assume that Sch_τ is contained in Sch'_τ. The inclusion functor $Sch_\tau \to Sch'_\tau$ satisfies the assumptions of Sites, Lemma 7.20.8. There are morphisms of topoi*

$$\begin{aligned} g : Sh(Sch_\tau) &\longrightarrow Sh(Sch'_\tau) \\ f : Sh(Sch'_\tau) &\longrightarrow Sh(Sch_\tau) \end{aligned}$$

such that $f \circ g \cong id$. For any object S of Sch_τ the inclusion functor $(Sch/S)_\tau \to (Sch'/S)_\tau$ satisfies the assumptions of Sites, Lemma 7.20.8 also. Hence similarly we obtain morphisms

$$\begin{aligned} g : Sh((Sch/S)_\tau) &\longrightarrow Sh((Sch'/S)_\tau) \\ f : Sh((Sch'/S)_\tau) &\longrightarrow Sh((Sch/S)_\tau) \end{aligned}$$

with $f \circ g \cong id$.

Proof. Assumptions (b), (c), and (e) of Sites, Lemma 7.20.8 are immediate for the functors $Sch_\tau \to Sch'_\tau$ and $(Sch/S)_\tau \to (Sch'/S)_\tau$. Property (a) holds by Lemma 33.3.6, 33.4.7, 33.5.7, 33.6.7, or 33.7.7. Property (d) holds because fibre products in the categories Sch_τ, Sch'_τ exist and are compatible with fibre products in the category of schemes. □

Discussion: The functor $g^{-1} = f_*$ is simply the restriction functor which associates to a sheaf $\mathcal{G}$ on Sch'_τ the restriction $\mathcal{G}|_{Sch_\tau}$. Hence this lemma simply says that given any sheaf of sets $\mathcal{F}$ on Sch_τ there exists a canonical sheaf $\mathcal{F}'$ on Sch'_τ such that $\mathcal{F}|_{Sch'_\tau} = \mathcal{F}'$. In fact the sheaf $\mathcal{F}'$ has the following description: it is the sheafification of the presheaf

$$Sch'_\tau \longrightarrow Sets, \quad V \longmapsto \operatorname{colim}_{V \to U} \mathcal{F}(U)$$

where U is an object of Sch_τ. This is true because $\mathcal{F}' = f^{-1}\mathcal{F} = (u_p\mathcal{F})^\#$ according to Sites, Lemmas 7.20.5 and 7.20.8.

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CHAPTER 34

Descent

34.1. Introduction

In the chapter on topologies on schemes (see Topologies, Section 33.1) we introduced Zariski, étale, fppf, smooth, syntomic and fpqc coverings of schemes. In this chapter we discuss what kind of structures over schemes can be descended through such coverings. See for example [**Gro95a**], [**Gro95b**], [**Gro95e**], [**Gro95f**], [**Gro95c**], and [**Gro95d**]. This is also meant to introduce the notions of descent, descent data, effective descent data, in the less formal setting of descent questions for quasi-coherent sheaves, schemes, etc. The formal notion, that of a stack over a site, is discussed in the chapter on stacks (see Stacks, Section 8.1).

34.2. Descent data for quasi-coherent sheaves

In this chapter we will use the convention where the projection maps $\mathrm{pr}_i : X \times \ldots \times X \to X$ are labeled starting with $i = 0$. Hence we have $\mathrm{pr}_0, \mathrm{pr}_1 : X \times X \to X$, $\mathrm{pr}_0, \mathrm{pr}_1, \mathrm{pr}_2 : X \times X \times X \to X$, etc.

Definition 34.2.1. Let S be a scheme. Let $\{f_i : S_i \to S\}_{i \in I}$ be a family of morphisms with target S.

(1) A *descent datum* $(\mathcal{F}_i, \varphi_{ij})$ *for quasi-coherent sheaves* with respect to the given family is given by a quasi-coherent sheaf $\mathcal{F}_i$ on S_i for each $i \in I$, an isomorphism of quasi-coherent $\mathcal{O}_{S_i \times_S S_j}$-modules $\varphi_{ij} : \mathrm{pr}_0^* \mathcal{F}_i \to \mathrm{pr}_1^* \mathcal{F}_j$ for each pair $(i, j) \in I^2$ such that for every triple of indices $(i, j, k) \in I^3$ the diagram

$$\begin{array}{ccc} \mathrm{pr}_0^* \mathcal{F}_i & \xrightarrow{\mathrm{pr}_{02}^* \varphi_{ik}} & \mathrm{pr}_2^* \mathcal{F}_k \\ & {\scriptstyle \mathrm{pr}_{01}^* \varphi_{ij}} \searrow \quad \nearrow {\scriptstyle \mathrm{pr}_{12}^* \varphi_{jk}} & \\ & \mathrm{pr}_1^* \mathcal{F}_j & \end{array}$$

of $\mathcal{O}_{S_i \times_S S_j \times_S S_k}$-modules commutes. This is called the *cocycle condition*.

(2) A *morphism* $\psi : (\mathcal{F}_i, \varphi_{ij}) \to (\mathcal{F}'_i, \varphi'_{ij})$ *of descent data* is given by a family $\psi = (\psi_i)_{i \in I}$ of morphisms of $\mathcal{O}_{S_i}$-modules $\psi_i : \mathcal{F}_i \to \mathcal{F}'_i$ such that all the diagrams

$$\begin{array}{ccc} \mathrm{pr}_0^* \mathcal{F}_i & \xrightarrow{\varphi_{ij}} & \mathrm{pr}_1^* \mathcal{F}_j \\ {\scriptstyle \mathrm{pr}_0^* \psi_i} \downarrow & & \downarrow {\scriptstyle \mathrm{pr}_1^* \psi_j} \\ \mathrm{pr}_0^* \mathcal{F}'_i & \xrightarrow{\varphi'_{ij}} & \mathrm{pr}_1^* \mathcal{F}'_j \end{array}$$

commute.

A good example to keep in mind is the following. Suppose that $S = \bigcup S_i$ is an open covering. In that case we have seen descent data for sheaves of sets in Sheaves,

Section 6.33 where we called them "glueing data for sheaves of sets with respect to the given covering". Moreover, we proved that the category of glueing data is equivalent to the category of sheaves on S. We will show the analogue in the setting above when $\{S_i \to S\}_{i\in I}$ is an fpqc covering.

In the extreme case where the covering $\{S \to S\}$ is given by id_S a descent datum is necessarily of the form $(\mathcal{F}, \mathrm{id}_{\mathcal{F}})$. The cocycle condition guarantees that the identity on $\mathcal{F}$ is the only permitted map in this case. The following lemma shows in particular that to every quasi-coherent sheaf of $\mathcal{O}_S$-modules there is associated a unique descent datum with respect to any given family.

Lemma 34.2.2. *Let $\mathcal{U} = \{U_i \to U\}_{i\in I}$ and $\mathcal{V} = \{V_j \to V\}_{j\in J}$ be families of morphisms of schemes with fixed target. Let $(g, \alpha : I \to J, (g_i)) : \mathcal{U} \to \mathcal{V}$ be a morphism of families of maps with fixed target, see Sites, Definition 7.8.1. Let $(\mathcal{F}_j, \varphi_{jj'})$ be a descent datum for quasi-coherent sheaves with respect to the family $\{V_j \to V\}_{j\in J}$. Then*

1. *The system*
$$\left(g_i^*\mathcal{F}_{\alpha(i)}, (g_i \times g_{i'})^*\varphi_{\alpha(i)\alpha(i')}\right)$$
is a descent datum with respect to the family $\{U_i \to U\}_{i\in I}$.
2. *This construction is functorial in the descent datum $(\mathcal{F}_j, \varphi_{jj'})$.*
3. *Given a second morphism $(g', \alpha' : I \to J, (g_i'))$ of families of maps with fixed target with $g = g'$ there exists a functorial isomorphism of descent data*
$$(g_i^*\mathcal{F}_{\alpha(i)}, (g_i \times g_{i'})^*\varphi_{\alpha(i)\alpha(i')}) \cong ((g_i')^*\mathcal{F}_{\alpha'(i)}, (g_i' \times g_{i'}')^*\varphi_{\alpha'(i)\alpha'(i')}).$$

Proof. Omitted. Hint: The maps $g_i^*\mathcal{F}_{\alpha(i)} \to (g_i')^*\mathcal{F}_{\alpha'(i)}$ which give the isomorphism of descent data in part (3) are the pullbacks of the maps $\varphi_{\alpha(i)\alpha'(i)}$ by the morphisms $(g_i, g_i') : U_i \to V_{\alpha(i)} \times_V V_{\alpha'(i)}$. □

Any family $\mathcal{U} = \{S_i \to S\}_{i\in I}$ is a refinement of the trivial covering $\{S \to S\}$ in a unique way. For a quasi-coherent sheaf $\mathcal{F}$ on S we denote simply $(\mathcal{F}|_{S_i}, can)$ the descent datum with respect to $\mathcal{U}$ obtained by the procedure above.

Definition 34.2.3. Let S be a scheme. Let $\{S_i \to S\}_{i\in I}$ be a family of morphisms with target S.

1. Let $\mathcal{F}$ be a quasi-coherent $\mathcal{O}_S$-module. We call the unique descent on $\mathcal{F}$ datum with respect to the covering $\{S \to S\}$ the *trivial descent datum*.
2. The pullback of the trivial descent datum to $\{S_i \to S\}$ is called the *canonical descent datum*. Notation: $(\mathcal{F}|_{S_i}, can)$.
3. A descent datum $(\mathcal{F}_i, \varphi_{ij})$ for quasi-coherent sheaves with respect to the given covering is said to be *effective* if there exists a quasi-coherent sheaf $\mathcal{F}$ on S such that $(\mathcal{F}_i, \varphi_{ij})$ is isomorphic to $(\mathcal{F}|_{S_i}, can)$.

Lemma 34.2.4. *Let S be a scheme. Let $S = \bigcup U_i$ be an open covering. Any descent datum on quasi-coherent sheaves for the family $\mathcal{U} = \{U_i \to S\}$ is effective. Moreover, the functor from the category of quasi-coherent $\mathcal{O}_S$-modules to the category of descent data with respect to $\mathcal{U}$ is fully faithful.*

Proof. This follows immediately from Sheaves, Section 6.33 and the fact that being quasi-coherent is a local property, see Modules, Definition 17.10.1. □

To prove more we first need to study the case of modules over rings.

34.3. Descent for modules

Let $R \to A$ be a ring map. By Simplicial, Example 14.5.5 this gives rise to a cosimplicial R-algebra

$$A \rightleftarrows A \otimes_R A \;\substack{\rightarrow\\ \leftarrow\\ \rightarrow\\ \leftarrow\\ \rightarrow}\; A \otimes_R A \otimes_R A$$

Let us denote this $(A/R)_\bullet$ so that $(A/R)_n$ is the $(n+1)$-fold tensor product of A over R. Given a map $\varphi : [n] \to [m]$ the R-algebra map $(A/R)_\bullet(\varphi)$ is the map

$$a_0 \otimes \ldots \otimes a_n \longmapsto \prod\nolimits_{\varphi(i)=0} a_i \otimes \prod\nolimits_{\varphi(i)=1} a_i \otimes \ldots \otimes \prod\nolimits_{\varphi(i)=m} a_i$$

where we use the convention that the empty product is 1. Thus the first few maps, notation as in Simplicial, Section 14.5, are

$$\begin{array}{lcccc}
\delta^1_0 & : & a_0 & \mapsto & 1 \otimes a_0 \\
\delta^1_1 & : & a_0 & \mapsto & a_0 \otimes 1 \\
\sigma^0_0 & : & a_0 \otimes a_1 & \mapsto & a_0 a_1 \\
\delta^2_0 & : & a_0 \otimes a_1 & \mapsto & 1 \otimes a_0 \otimes a_1 \\
\delta^2_1 & : & a_0 \otimes a_1 & \mapsto & a_0 \otimes 1 \otimes a_1 \\
\delta^2_2 & : & a_0 \otimes a_1 & \mapsto & a_0 \otimes a_1 \otimes 1 \\
\sigma^1_0 & : & a_0 \otimes a_1 \otimes a_2 & \mapsto & a_0 a_1 \otimes a_2 \\
\sigma^1_1 & : & a_0 \otimes a_1 \otimes a_2 & \mapsto & a_0 \otimes a_1 a_2
\end{array}$$

and so on.

An R-module M gives rise to a cosimplicial $(A/R)_\bullet$-module $(A/R)_\bullet \otimes_R M$. In other words $M_n = (A/R)_n \otimes_R M$ and using the R-algebra maps $(A/R)_n \to (A/R)_m$ to define the corresponding maps on $M \otimes_R (A/R)_\bullet$.

The analogue to a descent datum for quasi-coherent sheaves in the setting of modules is the following.

Definition 34.3.1. Let $R \to A$ be a ring map.

(1) A *descent datum* (N, φ) *for modules with respect to* $R \to A$ is given by an A-module N and a isomorphism of $A \otimes_R A$-modules

$$\varphi : N \otimes_R A \to A \otimes_R N$$

such that the *cocycle condition* holds: the diagram of $A \otimes_R A \otimes_R A$-module maps

$$\begin{array}{ccc}
N \otimes_R A \otimes_R A & \xrightarrow{\varphi_{02}} & A \otimes_R A \otimes_R N \\
& {\scriptstyle \varphi_{01}} \searrow \quad \nearrow {\scriptstyle \varphi_{12}} & \\
& A \otimes_R N \otimes_R A &
\end{array}$$

commutes (see below for notation).

(2) A *morphism* $(N, \varphi) \to (N', \varphi')$ *of descent data* is a morphism of A-modules $\psi : N \to N'$ such that the diagram

$$\begin{array}{ccc}
N \otimes_R A & \xrightarrow{\varphi} & A \otimes_R N \\
{\scriptstyle \psi \otimes \mathrm{id}_A} \downarrow & & \downarrow {\scriptstyle \mathrm{id}_A \otimes \psi} \\
N' \otimes_R A & \xrightarrow{\varphi'} & A \otimes_R N'
\end{array}$$

is commutative.

In the definition we use the notation that $\varphi_{01} = \varphi \otimes \mathrm{id}_A$, $\varphi_{12} = \mathrm{id}_A \otimes \varphi$, and $\varphi_{02}(n \otimes 1 \otimes 1) = \sum a_i \otimes 1 \otimes n_i$ if $\varphi(n) = \sum a_i \otimes n_i$. All three are $A \otimes_R A \otimes_R A$-module homomorphisms. Equivalently we have

$$\varphi_{ij} = \varphi \otimes_{(A/R)_1,\ (A/R)_\bullet(\tau^2_{ij})} (A/R)_2$$

where $\tau^2_{ij} : [1] \to [2]$ is the map $0 \mapsto i$, $1 \mapsto j$. Namely, $(A/R)_\bullet(\tau^2_{02})(a_0 \otimes a_1) = a_0 \otimes 1 \otimes a_1$, and similarly for the others[1].

We need some more notation to be able to state the next lemma. Let (N, φ) be a descent datum with respect to a ring map $R \to A$. For $n \geq 0$ and $i \in [n]$ we set

$$N_{n,i} = A \otimes_R \ldots \otimes_R A \otimes_R N \otimes_R A \otimes_R \ldots \otimes_R A$$

with the factor N in the ith spot. It is an $(A/R)_n$-module. If we introduce the maps $\tau^n_i : [0] \to [n]$, $0 \mapsto i$ then we see that

$$N_{n,i} = N \otimes_{(A/R)_0,\ (A/R)_\bullet(\tau^n_i)} (A/R)_n$$

For $0 \leq i \leq j \leq n$ we let $\tau^n_{ij} : [1] \to [n]$ be the map such that 0 maps to i and 1 to j. Similarly to the above the homomorphism φ induces isomorphisms

$$\varphi^n_{ij} = \varphi \otimes_{(A/R)_1,\ (A/R)_\bullet(\tau^n_{ij})} (A/R)_n : N_{n,i} \longrightarrow N_{n,j}$$

of $(A/R)_n$-modules when $i < j$. If $i = j$ we set $\varphi^n_{ij} = \mathrm{id}$. Since these are all isomorphisms they allow us to move the factor N to any spot we like. And the cocycle condition exactly means that it does not matter how we do this (e.g., as a composition of two of these or at once). Finally, for any $\beta : [n] \to [m]$ we define the morphism

$$N_{\beta,i} : N_{n,i} \to N_{m,\beta(i)}$$

as the unique $(A/R)_\bullet(\beta)$-semi linear map such that

$$N_{\beta,i}(1 \otimes \ldots \otimes n \otimes \ldots \otimes 1) = 1 \otimes \ldots \otimes n \otimes \ldots \otimes 1$$

for all $n \in N$. This hints at the following lemma.

Lemma 34.3.2. *Let $R \to A$ be a ring map. Given a descent datum (N, φ) we can associate to it a cosimplicial $(A/R)_\bullet$-module $N_\bullet$[2] by the rules $N_n = N_{n,n}$ and given $\beta : [n] \to [m]$ setting we define*

$$N_\bullet(\beta) = (\varphi^m_{\beta(n)m}) \circ N_{\beta,n} : N_{n,n} \longrightarrow N_{m,m}.$$

This procedure is functorial in the descent datum.

Proof. Here are the first few maps where $\varphi(n \otimes 1) = \sum \alpha_i \otimes x_i$

$$\begin{array}{llcccccc}
\delta^1_0 & : & N & \to & A \otimes N & n & \mapsto & 1 \otimes n \\
\delta^1_1 & : & N & \to & A \otimes N & n & \mapsto & \sum \alpha_i \otimes x_i \\
\sigma^0_0 & : & A \otimes N & \to & N & a_0 \otimes n & \mapsto & a_0 n \\
\delta^2_0 & : & A \otimes N & \to & A \otimes A \otimes N & a_0 \otimes n & \mapsto & 1 \otimes a_0 \otimes n \\
\delta^2_1 & : & A \otimes N & \to & A \otimes A \otimes N & a_0 \otimes n & \mapsto & a_0 \otimes 1 \otimes n \\
\delta^2_2 & : & A \otimes N & \to & A \otimes A \otimes N & a_0 \otimes n & \mapsto & \sum a_0 \otimes \alpha_i \otimes x_i \\
\sigma^1_0 & : & A \otimes A \otimes N & \to & A \otimes N & a_0 \otimes a_1 \otimes n & \mapsto & a_0 a_1 \otimes n \\
\sigma^1_1 & : & A \otimes A \otimes N & \to & A \otimes N & a_0 \otimes a_1 \otimes n & \mapsto & a_0 \otimes a_1 n
\end{array}$$

[1] Note that $\tau^2_{ij} = \delta^2_k$, if $\{i, j, k\} = [2] = \{0, 1, 2\}$, see Simplicial, Definition 14.2.1.

[2] We should really write $(N, \varphi)_\bullet$.

with notation as in Simplicial, Section 14.5. We first verify the two properties $\sigma_0^0 \circ \delta_0^1 = \mathrm{id}$ and $\sigma_0^0 \circ \delta_1^1 = \mathrm{id}$. The first one, $\sigma_0^0 \circ \delta_0^1 = \mathrm{id}$, is clear from the explicit description of the morphisms above. To prove the second relation we have to use the cocycle condition (because it does not holds for an arbitrary isomorphism $\varphi : N \otimes_R A \to A \otimes_R N$). Write $p = \sigma_0^0 \circ \delta_1^1 : N \to N$. By the description of the maps above we deduce that p is also equal to

$$p = \varphi \otimes \mathrm{id} : N = (N \otimes_R A) \otimes_{(A \otimes_R A)} A \longrightarrow (A \otimes_R N) \otimes_{(A \otimes_R A)} A = N$$

Since φ is an isomorphism we see that p is an isomorphism. Write $\varphi(n \otimes 1) = \sum \alpha_i \otimes x_i$ for certain $\alpha_i \in A$ and $x_i \in N$. Then $p(n) = \sum \alpha_i x_i$. Next, write $\varphi(x_i \otimes 1) = \sum \alpha_{ij} \otimes y_j$ for certain $\alpha_{ij} \in A$ and $y_j \in N$. Then the cocycle condition says that

$$\sum \alpha_i \otimes \alpha_{ij} \otimes y_j = \sum \alpha_i \otimes 1 \otimes x_i.$$

This means that $p(n) = \sum \alpha_i x_i = \sum \alpha_i \alpha_{ij} y_j = \sum \alpha_i p(x_i) = p(p(n))$. Thus p is a projector, and since it is an isomorphism it is the identity.

To prove fully that $N_\bullet$ is a cosimplicial module we have to check all 5 types of relations of Simplicial, Remark 14.5.3. The relations on composing σ's are obvious. The relations on composing δ's come down to the cocycle condition for φ. In exactly the same way as above one checks the relations $\sigma_j \circ \delta_j = \sigma_j \circ \delta_{j+1} = \mathrm{id}$. Finally, the other relations on compositions of δ's and σ's hold for any φ whatsoever. □

Note that to an R-module M we can associate a canonical descent datum, namely $(M \otimes_R A, can)$ where $can : (M \otimes_R A) \otimes_R A \to A \otimes_R (M \otimes_R A)$ is the obvious map: $(m \otimes a) \otimes a' \mapsto a \otimes (m \otimes a')$.

Lemma 34.3.3. *Let $R \to A$ be a ring map. Let M be an R-module. The cosimplicial $(A/R)_\bullet$-module associated to the canonical descent datum is isomorphic to the cosimplicial module $(A/R)_\bullet \otimes_R M$.*

Proof. Omitted. □

Definition 34.3.4. Let $R \to A$ be a ring map. We say a descent datum (N, φ) is *effective* if there exists an R-module M and an isomorphism of descent data from $(M \otimes_R A, can)$ to (N, φ).

Let $R \to A$ be a ring map. Let (N, φ) be a descent datum. We may take the cochain complex $s(N_\bullet)$ associated with $N_\bullet$ (see Simplicial, Section 14.23). It has the following shape:

$$N \to A \otimes_R N \to A \otimes_R A \otimes_R N \to \ldots$$

We can describe the maps. The first map is the map

$$n \longmapsto 1 \otimes n - \varphi(n \otimes 1).$$

The second map on pure tensors has the values

$$a \otimes n \longmapsto 1 \otimes a \otimes n - a \otimes 1 \otimes n + a \otimes \varphi(n \otimes 1).$$

It is clear how the pattern continues.

In the special case where $N = A \otimes_R M$ we see that for any $m \in M$ the element $1 \otimes m$ is in the kernel of the first map of the cochain complex associated to the cosimplicial module $(A/R)_\bullet \otimes_R M$. Hence we get an extended cochain complex

$$0 \to M \to A \otimes_R M \to A \otimes_R A \otimes_R M \to \ldots \tag{34.3.4.1}$$

Here we think of the 0 as being in degree -2, the module M in degree -1, the module $A \otimes_R M$ in degree 0, etc. Note that this complex has the shape

$$0 \to R \to A \to A \otimes_R A \to A \otimes_R A \otimes_R A \to \ldots$$

when $M = R$.

Lemma 34.3.5. *Suppose that $R \to A$ has a section. Then for any R-module M the extended cochain complex (34.3.4.1) is exact.*

Proof. By Simplicial, Lemma 14.26.4 the map $R \to (A/R)_\bullet$ is a homotopy equivalence of cosimplicial R-algebras (here R denotes the constant cosimplicial R-algebra). Hence $M \to (A/R)_\bullet \otimes_R M$ is a homotopy equivalence in the category of cosimplicial R-modules, because $\otimes_R M$ is a functor from the category of R-algebras to the category of R-modules, see Simplicial, Lemma 14.26.3. This implies that the induced map of associated complexes is a homotopy equivalence, see Simplicial, Lemma 14.26.5. Since the complex associated to the constant cosimplicial R-module M is the complex

$$M \xrightarrow{0} M \xrightarrow{1} M \xrightarrow{0} M \xrightarrow{1} M \ldots$$

we win (since the extended version simply puts an extra M at the beginning). □

Lemma 34.3.6. *Suppose that $R \to A$ is faithfully flat, see Algebra, Definition 10.36.1. Then for any R-module M the extended cochain complex (34.3.4.1) is exact.*

Proof. Suppose we can show there exists a faithfully flat ring map $R \to R'$ such that the result holds for the ring map $R' \to A' = R' \otimes_R A$. Then the result follows for $R \to A$. Namely, for any R-module M the cosimplicial module $(M \otimes_R R') \otimes_{R'} (A'/R')_\bullet$ is just the cosimplicial module $R' \otimes_R (M \otimes_R (A/R)_\bullet)$. Hence the vanishing of cohomology of the complex associated to $(M \otimes_R R') \otimes_{R'} (A'/R')_\bullet$ implies the vanishing of the cohomology of the complex associated to $M \otimes_R (A/R)_\bullet$ by faithful flatness of $R \to R'$. Similarly for the vanishing of cohomology groups in degrees -1 and 0 of the extended complex (proof omitted).

But we have such a faithful flat extension. Namely $R' = A$ works because the ring map $R' = A \to A' = A \otimes_A A$ has a section $a \otimes a' \mapsto aa'$ and Lemma 34.3.5 applies. □

Here is how the complex relates to the question of effectivity.

Lemma 34.3.7. *Let $R \to A$ be a faithfully flat ring map. Let (N, φ) be a descent datum. Then (N, φ) is effective if and only if the canonical map*

$$A \otimes_R H^0(s(N_\bullet)) \longrightarrow N$$

is an isomorphism.

Proof. If (N, φ) is effective, then we may write $N = A \otimes_R M$ with $\varphi = can$. It follows that $H^0(s(N_\bullet)) = M$ by Lemmas 34.3.3 and 34.3.6. Conversely, suppose the map of the lemma is an isomorphism. In this case set $M = H^0(s(N_\bullet))$. This is an R-submodule of N, namely $M = \{n \in N \mid 1 \otimes n = \varphi(n \otimes 1)\}$. The only thing to check is that via the isomorphism $A \otimes_R M \to N$ the canonical descent data agrees with φ. We omit the verification. □

Lemma 34.3.8. *Let $R \to A$ be a ring map, and let $R \to R'$ be faithfully flat. Set $A' = R' \otimes_R A$. If all descent data for $R' \to A'$ are effective, then so are all descent data for $R \to A$.*

Proof. Let (N, φ) be a descent datum for $R \to A$. Set $N' = R' \otimes_R N = A' \otimes_A N$, and denote $\varphi' = \mathrm{id}_{R'} \otimes \varphi$ the base change of the descend datum φ. Then (N', φ') is a descent datum for $R' \to A'$ and $H^0(s(N'_\bullet)) = R' \otimes_R H^0(s(N_\bullet))$. Moreover, the map $A' \otimes_{R'} H^0(s(N'_\bullet)) \to N'$ is identified with the base change of the A-module map $A \otimes_R H^0(s(N)) \to N$ via the faithfully flat map $A \to A'$. Hence we conclude by Lemma 34.3.7. □

Here is the main result of this section. Its proof may seem a little clumsy; for a more highbrow approach see Remark 34.3.11 below.

Proposition 34.3.9. *Let $R \to A$ be a faithfully flat ring map. Then*

1. *any descent datum on modules with respect to $R \to A$ is effective,*
2. *the functor $M \mapsto (A \otimes_R M, can)$ from R-modules to the category of descent data is an equivalence, and*
3. *the inverse functor is given by $(N, \varphi) \mapsto H^0(s(N_\bullet))$.*

Proof. We only prove (1) and omit the proofs of (2) and (3). As $R \to A$ is faithfully flat, there exists a faithfully flat base change $R \to R'$ such that $R' \to A' = R' \otimes_R A$ has a section (namely take $R' = A$ as in the proof of Lemma 34.3.6). Hence, using Lemma 34.3.8 we may assume that $R \to A$ as a section, say $\sigma : A \to R$. Let (N, φ) be a descent datum relative to $R \to A$. Set

$$M = H^0(s(N_\bullet)) = \{n \in N \mid 1 \otimes n = \varphi(n \otimes 1)\} \subset N$$

By Lemma 34.3.7 it suffices to show that $A \otimes_R M \to N$ is an isomorphism.

Take an element $n \in N$. Write $\varphi(n \otimes 1) = \sum a_i \otimes x_i$ for certain $a_i \in A$ and $x_i \in N$. By Lemma 34.3.2 we have $n = \sum a_i x_i$ in N (because $\sigma_0^0 \circ \delta_0^1 = \mathrm{id}$ in any cosimplicial object). Next, write $\varphi(x_i \otimes 1) = \sum a_{ij} \otimes y_j$ for certain $a_{ij} \in A$ and $y_j \in N$. The cocycle condition means that

$$\sum a_i \otimes a_{ij} \otimes y_j = \sum a_i \otimes 1 \otimes x_i$$

in $A \otimes_R A \otimes_R N$. We conclude two things from this. First, by applying σ to the first A we conclude that $\sum \sigma(a_i)\varphi(x_i \otimes 1) = \sum \sigma(a_i) \otimes x_i$ which means that $\sum \sigma(a_i)x_i \in M$. Next, by applying σ to the middle A and multiplying out we conclude that $\sum_i a_i(\sum_j \sigma(a_{ij})y_j) = \sum a_i x_i = n$. Hence by the first conclusion we see that $A \otimes_R M \to N$ is surjective. Finally, suppose that $m_i \in M$ and $\sum a_i m_i = 0$. Then we see by applying φ to $\sum a_i m_i \otimes 1$ that $\sum a_i \otimes m_i = 0$. In other words $A \otimes_R M \to N$ is injective and we win. □

Remark 34.3.10. Let R be a ring. Let $f_1, \ldots, f_n \in R$ generate the unit ideal. The ring $A = \prod_i R_{f_i}$ is a faithfully flat R-algebra. We remark that the cosimplicial ring $(A/R)_\bullet$ has the following ring in degree n:

$$\prod_{i_0, \ldots, i_n} R_{f_{i_0} \ldots f_{i_n}}$$

Hence the results above recover Algebra, Lemmas 10.21.1, 10.21.2 and 10.22.4. But the results above actually say more because of exactness in higher degrees. Namely, it implies that Cech cohomology of quasi-coherent sheaves on affines is trivial, see (insert future reference here).

Remark 34.3.11. Let R be a ring. Let $A_\bullet$ be a cosimplicial R-algebra. In this setting a descent datum corresponds to an cosimplicial $A_\bullet$-module $M_\bullet$ with the property that for every $n, m \geq 0$ and every $\varphi : [n] \to [m]$ the map $M(\varphi) : M_n \to M_m$ induces an isomorphism

$$M_n \otimes_{A_n, A(\varphi)} A_m \longrightarrow M_m.$$

Let us call such a cosimplicial module a *cartesian module*. In this setting, the proof of Proposition 34.3.9 can be split in the following steps

(1) If $R \to R'$ is faithfully flat, $R \to A$ any ring map, then descent data for A/R are effective if descent data for $(R' \otimes_R A)/R'$ are effective.
(2) Let A be an R-algebra. Descent data for A/R correspond to cartesian $(A/R)_\bullet$-modules.
(3) If $R \to A$ has a section then $(A/R)_\bullet$ is homotopy equivalent to R, the constant cosimplicial R-algebra with value R.
(4) If $A_\bullet \to B_\bullet$ is a homotopy equivalence of cosimplicial R-algebras then the functor $M_\bullet \mapsto M_\bullet \otimes_{A_\bullet} B_\bullet$ induces an equivalence of categories between cartesian $A_\bullet$-modules and cartesian $B_\bullet$-modules.

For (1) see Lemma 34.3.8. Part (2) uses Lemma 34.3.2. Part (3) we have seen in the proof of Lemma 34.3.5 (it relies on Simplicial, Lemma 14.26.4). Moreover, part (4) is a triviality if you think about it right!

34.4. Descent for universally injective morphisms

Numerous constructions in algebraic geometry are made using techniques of *descent*, such as constructing objects over a given space by first working over a somewhat larger space which projects down to the given space, or verifying a property of a space or a morphism by pulling back along a covering map. The utility of such techniques is of course dependent on identification of a wide class of *effective descent morphisms*. Early in the Grothendieckian development of modern algebraic geometry, the class of morphisms which are *quasi-compact* and *faithfully flat* was shown to be effective for descending objects, morphisms, and many properties thereof.

As usual, this statement comes down to a property of rings and modules. For a homomorphism $f : R \to S$ to be an effective descent morphism for modules, Grothendieck showed that it is sufficient for f to be faithfully flat. However, this excludes many natural examples: for instance, any split ring homomorphism is an effective descent morphism. One natural example of this even arises in the proof of faithfully flat descent: for $f : R \to S$ any ring homomorphism, $1_S \otimes f : S \to S \otimes_R S$ is split by the multiplication map whether or not it is flat.

One may then ask whether one there is a natural ring-theoretic condition implying effective descent for modules which includes both the case of a faithfully flat morphism and that of a split ring homomorphism. It may surprise the reader (at least it surprised this author) to learn that a complete answer to this question has been known since around 1970! Namely, it is not hard to check that a necessary condition for $f : R \to S$ to be an effective descent morphism for modules is that f must be *universally injective* in the category of R-modules, that is, for any R-module M, the map $1_M \otimes f : M \to M \otimes_R S$ must be injective. This then turns out to be a sufficient condition as well. For example, if f is split in the category

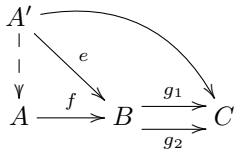
of R-modules (but not necessarily in the category of rings), then f is an effective descent morphism for modules.

The history of this result is a bit involved: it was originally asserted by Olivier [**Oli70**], who called universally injective morphisms *pure*, but without a clear indication of proof. One can extract the result from the work of Joyal and Tierney [**JT84**], but to the best of our knowledge, the first free-standing proof to appear in the literature is that of Mesablishvili [**Mes00**]. The first purpose of this section is to expose Mesablishvili's proof; this requires little modification of his original presentation aside from correcting typos, with the one exception that we make explicit the relationship between the customary definition of a descent datum in algebraic geometry and the one used in [**Mes00**]. The proof turns to be entirely category-theoretic, and consequently can be put in the language of monads (and thus applied in other contexts); see [**JT04**].

The second purpose of this section is to collect some information about which properties of modules, algebras, and morphisms can be descended along universally injective ring homomorphisms. The cases of finite modules and flat modules were treated by Mesablishvili [**Mes02**].

34.4.1. Category-theoretic preliminaries. We start by recalling a few basic notions from category theory which will simplify the exposition. In this subsection, fix an ambient category.

For two morphisms $g_1, g_2 : B \to C$, recall that an *equalizer* of g_1 and g_2 is a morphism $f : A \to B$ which satisfies $g_1 \circ f = g_2 \circ f$ and is universal for this property. This second statement means that any commutative diagram



without the dashed arrow can be uniquely completed. We also say in this situation that the diagram

$$A \xrightarrow{f} B \underset{g_2}{\overset{g_1}{\rightrightarrows}} C \tag{34.4.1.1}$$

is an equalizer. Reversing arrows gives the definition of a *coequalizer*. See Categories, Sections 4.10 and 4.11.

Since it involves a universal property, the property of being an equalizer is typically not stable under applying a covariant functor. Just as for monomorphisms and epimorphisms, one can get around this in some cases by exhibiting splittings.

Definition 34.4.2. A *split equalizer* is a diagram (34.4.1.1) with $g_1 \circ f = g_2 \circ f$ for which there exist auxiliary morphisms $h : B \to A$ and $i : C \to B$ such that

$$h \circ f = 1_A, \quad f \circ h = i \circ g_1, \quad i \circ g_2 = 1_B. \tag{34.4.2.1}$$

The point is that the equalities among arrows force (34.4.1.1) to be an equalizer: the map e factors uniquely through f by writing $e = f \circ (h \circ e)$. Consequently, applying a covariant functor to a split equalizer gives a split equalizer; applying a contravariant functor gives a *split coequalizer*, whose definition is apparent.

34.4.3. Universally injective morphisms. Recall that *Rings* denotes the category of commutative rings with 1. For an object R of *Rings* we denote Mod_R the category of R-modules.

Remark 34.4.4. Any functor $F : \mathcal{A} \to \mathcal{B}$ of abelian categories which is exact and takes nonzero objects to nonzero objects reflects injections and surjections. Namely, exactness implies that F preserves kernels and cokernels (compare with Homology, Section 12.7). For example, if $f : R \to S$ is a faithfully flat ring homomorphism, then $\bullet \otimes_R S : \mathrm{Mod}_R \to \mathrm{Mod}_S$ has these properties.

Let R be a ring. Recall that a morphism $f : M \to N$ in Mod_R is *universally injective* if for all $P \in \mathrm{Mod}_R$, the morphism $f \otimes 1_P : M \otimes_R P \to N \otimes_R P$ is injective. See Algebra, Definition 10.77.1.

Definition 34.4.5. A ring map $f : R \to S$ is *universally injective* if it is universally injective as a morphism in Mod_R.

Example 34.4.6. Any split injection in Mod_R is universally injective. In particular, any split injection in *Rings* is universally injective.

Example 34.4.7. For a ring R and $f_1, \ldots, f_n \in R$ generating the unit ideal, the morphism $R \to R_{f_1} \oplus \cdots \oplus R_{f_n}$ is universally injective. Although this is immediate from Lemma 34.4.8, it is instructive to check it directly: we immediately reduce to the case where R is local, in which case some f_i must be a unit and so the map $R \to R_{f_i}$ is an isomorphism.

Lemma 34.4.8. *Any faithfully flat ring map is universally injective.*

Proof. This is a reformulation of Algebra, Lemma 10.77.11. □

The key observation from [**Mes00**] is that universal injectivity can be usefully reformulated in terms of a splitting, using the usual construction of an injective cogenerator in Mod_R.

Definition 34.4.9. Let R be a ring. Define the contravariant functor $C : \mathrm{Mod}_R \to \mathrm{Mod}_R$ by setting

$$C(M) = \mathrm{Hom}_{Ab}(M, \mathbf{Q}/\mathbf{Z}),$$

with the R-action on $C(M)$ given by $rf(s) = f(rs)$.

This functor was denoted $M \mapsto M^\vee$ in More on Algebra, Section 15.40.

Lemma 34.4.10. *For a ring R, the functor $C : Mod_R \to Mod_R$ is exact and reflects injections and surjections.*

Proof. Exactness is More on Algebra, Lemma 15.40.2 and the other properties follow from this, see Remark 34.4.4. □

Remark 34.4.11. We will use frequently the standard adjunction between Hom and tensor product, in the form of the natural isomorphism of contravariant functors

$$C(\bullet_1 \otimes_R \bullet_2) \cong \mathrm{Hom}_R(\bullet_1, C(\bullet_2)) : \mathrm{Mod}_R \times \mathrm{Mod}_R \to \mathrm{Mod}_R \tag{34.4.11.1}$$

taking $f : M_1 \otimes_R M_2 \to \mathbf{Q}/\mathbf{Z}$ to the map $m_1 \mapsto (m_2 \mapsto f(m_1 \otimes m_2))$. See Algebra, Lemma 10.13.5. A corollary of this observation is that if

$$C(M) \rightrightarrows C(N) \longrightarrow C(P)$$

is a split coequalizer diagram in Mod_R, then so is

$$C(M \otimes_R Q) \rightrightarrows C(N \otimes_R Q) \longrightarrow C(P \otimes_R Q)$$

for any $Q \in \mathrm{Mod}_R$.

Lemma 34.4.12. *Let R be a ring. A morphism $f : M \to N$ in Mod_R is universally injective if and only if $C(f) : C(N) \to C(M)$ is a split surjection.*

Proof. By (34.4.11.1), for any $P \in \mathrm{Mod}_R$ we have a commutative diagram

$$\begin{array}{ccc} \mathrm{Hom}_R(P, C(N)) & \xrightarrow{\mathrm{Hom}_R(P, C(f))} & \mathrm{Hom}_R(P, C(M)) \\ \downarrow \cong & & \downarrow \cong \\ C(P \otimes_R N) & \xrightarrow{C(1_P \otimes f)} & C(P \otimes_R M). \end{array}$$

If f is universally injective, then $1_{C(M)} \otimes f : C(M) \otimes_R M \to C(M) \otimes_R N$ is injective, so both rows in the above diagram are surjective for $P = C(M)$. We may thus lift $1_{C(M)} \in \mathrm{Hom}_R(C(M), C(M))$ to some $g \in \mathrm{Hom}_R(C(N), C(M))$ splitting $C(f)$. Conversely, if $C(f)$ is a split surjection, then both rows in the above diagram are surjective, so by Lemma 34.4.10, $1_P \otimes f$ is injective. □

Remark 34.4.13. Let $f : M \to N$ be a universally injective morphism in Mod_R. By choosing a splitting g of $C(f)$, we may construct a functorial splitting of $C(1_P \otimes f)$ for each $P \in \mathrm{Mod}_R$. Namely, by (34.4.11.1) this amounts to splitting $\mathrm{Hom}_R(P, C(f))$ functorially in P, and this is achieved by the map $g \circ \bullet$.

34.4.14. Descent for modules and their morphisms. Throughout this subsection, fix a ring map $f : R \to S$. As seen in Section 34.3 we can use the language of cosimplicial algebras to talk about descent data for modules, but in this subsection we prefer a more down to earth terminology.

For $i = 1, 2, 3$, let S_i be the i-fold tensor product of S over R. Define the ring homomorphisms $\delta^1_0, \delta^1_1 : S_1 \to S_2$, $\delta^1_{01}, \delta^1_{02}, \delta^1_{12} : S_1 \to S_3$, and $\delta^2_0, \delta^2_1, \delta^2_2 : S_2 \to S_3$ by the formulas

$$\begin{aligned} \delta^1_0(a_0) &= 1 \otimes a_0 \\ \delta^1_1(a_0) &= a_0 \otimes 1 \\ \delta^2_0(a_0 \otimes a_1) &= 1 \otimes a_0 \otimes a_1 \\ \delta^2_1(a_0 \otimes a_1) &= a_0 \otimes 1 \otimes a_1 \\ \delta^2_2(a_0 \otimes a_1) &= a_0 \otimes a_1 \otimes 1 \\ \delta^1_{01}(a_0) &= 1 \otimes 1 \otimes a_0 \\ \delta^1_{02}(a_0) &= 1 \otimes a_0 \otimes 1 \\ \delta^1_{12}(a_0) &= a_0 \otimes 1 \otimes 1. \end{aligned}$$

In other words, the upper index indicates the source ring, while the lower index indicates where to insert factors of 1. (This notation is compatible with the notation introduced in Section 34.3.)

Recall[3] from Definition 34.3.1 that for $M \in \mathrm{Mod}_S$, a *descent datum* on M relative to f is an isomorphism

$$\theta : M \otimes_{S,\delta_0^1} S_2 \longrightarrow M \otimes_{S,\delta_1^1} S_2$$

of S_2-modules satisfying the *cocycle condition*

$$(\theta \otimes \delta_2^2) \circ (\theta \otimes \delta_2^0) = (\theta \otimes \delta_2^1) : M \otimes_{S,\delta_{01}^1} S_3 \to M \otimes_{S,\delta_{12}^1} S_3. \tag{34.4.14.1}$$

Let $DD_{S/R}$ be the category of S-modules equipped with descent data relative to f.

For example, for $M_0 \in \mathrm{Mod}_R$ and a choice of isomorphism $M \cong M_0 \otimes_R S$ gives rise to a descent datum by identifying $M \otimes_{S,\delta_0^1} S_2$ and $M \otimes_{S,\delta_1^1} S_2$ naturally with $M_0 \otimes_R S_2$. This construction in particular defines a functor $f^* : \mathrm{Mod}_R \to DD_{S/R}$.

Definition 34.4.15. The functor $f^* : \mathrm{Mod}_R \to DD_{S/R}$ is called *base extension along* f. We say that f is a *descent morphism for modules* if f^* is fully faithful. We say that f is an *effective descent morphism for modules* if f^* is an equivalence of categories.

Our goal is to show that for f universally injective, we can use θ to locate M_0 within M. This process makes crucial use of some equalizer diagrams.

Lemma 34.4.16. *For $(M, \theta) \in DD_{S/R}$, the diagram*
(34.4.16.1)

$$M \xrightarrow{\theta \circ (1_M \otimes \delta_0^1)} M \otimes_{S,\delta_1^1} S_2 \underset{1_{M \otimes S_2} \otimes \delta_1^2}{\overset{(\theta \otimes \delta_2^2) \circ (1_M \otimes \delta_0^2)}{\rightrightarrows}} M \otimes_{S,\delta_{12}^1} S_3$$

is a split equalizer.

Proof. Define the ring homomorphisms $\sigma_0^0 : S_2 \to S_1$ and $\sigma_0^1, \sigma_1^1 : S_3 \to S_2$ by the formulas

$$\begin{aligned} \sigma_0^0(a_0 \otimes a_1) &= a_0 a_1 \\ \sigma_0^1(a_0 \otimes a_1 \otimes a_2) &= a_0 a_1 \otimes a_2 \\ \sigma_1^1(a_0 \otimes a_1 \otimes a_2) &= a_0 \otimes a_1 a_2. \end{aligned}$$

We then take the auxiliary morphisms to be $1_M \otimes \sigma_0^0 : M \otimes_{S,\delta_1^1} S_2 \to M$ and $1_M \otimes \sigma_0^1 : M \otimes_{S,\delta_{12}^1} S_3 \to M \otimes_{S,\delta_1^1} S_2$. Of the compatibilities required in (34.4.2.1), the first follows from tensoring the cocycle condition (34.4.14.1) with σ_1^1 and the others are immediate. □

Lemma 34.4.17. *For $(M, \theta) \in DD_{S/R}$, the diagram*
(34.4.17.1)

$$C(M \otimes_{S,\delta_{12}^1} S_3) \underset{C(1_{M \otimes S_2} \otimes \delta_1^2)}{\overset{C((\theta \otimes \delta_2^2) \circ (1_M \otimes \delta_0^2))}{\rightrightarrows}} C(M \otimes_{S,\delta_1^1} S_2) \xrightarrow{C(\theta \circ (1_M \otimes \delta_0^1))} C(M).$$

obtained by applying C to (34.4.16.1) is a split coequalizer.

Proof. Omitted. □

[3]To be precise, our θ here is the inverse of φ from Definition 34.3.1.

Lemma 34.4.18. *The diagram*

$$S_1 \xrightarrow{\delta_1^1} S_2 \underset{\delta_1^2}{\overset{\delta_2^2}{\rightrightarrows}} S_3 \tag{34.4.18.1}$$

is a split equalizer.

Proof. In Lemma 34.4.16, take $(M, \theta) = f^*(S)$. □

This suggests a definition of a potential quasi-inverse functor for f^*.

Definition 34.4.19. Define the functor $f_* : DD_{S/R} \to \mathrm{Mod}_R$ by taking $f_*(M, \theta)$ to be the R-submodule of M for which the diagram
(34.4.19.1)

$$f_*(M,\theta) \longrightarrow M \underset{1_M \otimes \delta_1^1}{\overset{\theta \circ (1_M \otimes \delta_0^1)}{\rightrightarrows}} M \otimes_{S,\delta_1^1} S_2 \tag{34.4.19.1}$$

is an equalizer.

Using Lemma 34.4.16 and the fact that the restriction functor $\mathrm{Mod}_S \to \mathrm{Mod}_R$ is right adjoint to the base extension functor $\bullet \otimes_R S : \mathrm{Mod}_R \to \mathrm{Mod}_S$, we deduce that f_* is right adjoint to f^*.

We are ready for the key lemma. In the faithfully flat case this is a triviality (see Remark 34.4.21), but in the general case some argument is needed.

Lemma 34.4.20. *If f is universally injective, then the diagram*
(34.4.20.1)

$$f_*(M,\theta) \otimes_R S \xrightarrow{\theta \circ (1_M \otimes \delta_0^1)} M \otimes_{S,\delta_1^1} S_2 \underset{1_{M \otimes S_2} \otimes \delta_1^2}{\overset{(\theta \otimes \delta_2^2) \circ (1_M \otimes \delta_0^2)}{\rightrightarrows}} M \otimes_{S,\delta_{12}^1} S_3 \tag{34.4.20.1}$$

obtained by tensoring (34.4.19.1) over R with S is an equalizer.

Proof. By Lemma 34.4.12 and Remark 34.4.13, the map $C(1_N \otimes f) : C(N \otimes_R S) \to C(N)$ can be split functorially in N. This gives the upper vertical arrows in the commutative diagram

$$\begin{array}{ccccc}
C(M \otimes_{S,\delta_1^1} S_2) & \underset{C(1_M \otimes \delta_1^1)}{\overset{C(\theta \circ (1_M \otimes \delta_0^1))}{\rightrightarrows}} & C(M) & \longrightarrow & C(f_*(M,\theta)) \\
\downarrow & & \downarrow & & \vdots \\
C(M \otimes_{S,\delta_{12}^1} S_3) & \underset{C(1_{M \otimes S_2} \otimes \delta_1^2)}{\overset{C((\theta \otimes \delta_2^2) \circ (1_M \otimes \delta_0^2))}{\rightrightarrows}} & C(M \otimes_{S,\delta_1^1} S_2) & \xrightarrow{C(\theta \circ (1_M \otimes \delta_0^1))} & C(M) \\
\downarrow & & \downarrow C(1_M \otimes \delta_1^1) & \nearrow & \downarrow \\
C(M \otimes_{S,\delta_1^1} S_2) & \underset{C(1_M \otimes \delta_1^1)}{\overset{C(\theta \circ (1_M \otimes \delta_0^1))}{\rightrightarrows}} & C(M) & \longrightarrow & C(f_*(M,\theta))
\end{array}$$

in which the compositions along the columns are identity morphisms. The second row is the coequalizer diagram (34.4.17.1); this produces the dashed arrow. From the top right square, we obtain auxiliary morphisms $C(f_*(M, \theta)) \to C(M)$ and $C(M) \to C(M \otimes_{S,\delta_1^1} S_2)$ which imply that the first row is a split coequalizer diagram.

By Remark 34.4.11, we may tensor with S inside C to obtain the split coequalizer diagram

$$C(M \otimes_{S,\delta_2^2\circ\delta_1^1} S_3) \underset{C(1_{M\otimes S_2}\otimes\delta_1^2)}{\overset{C((\theta\otimes\delta_2^2)\circ(1_M\otimes\delta_0^2))}{\rightrightarrows}} C(M \otimes_{S,\delta_1^1} S_2) \xrightarrow{C(\theta\circ(1_M\otimes\delta_0^1))} C(f_*(M,\theta)\otimes_R S).$$

By Lemma 34.4.10, we conclude (34.4.20.1) must also be an equalizer. □

Remark 34.4.21. If f is a split injection in Mod_R, one can simplify the argument by splitting f directly, without using C. Things are even simpler if f is faithfully flat; in this case, the conclusion of Lemma 34.4.20 is immediate because tensoring over R with S preserves all equalizers.

Theorem 34.4.22. *The following conditions are equivalent.*

(a) *The morphism f is a descent morphism for modules.*
(b) *The morphism f is an effective descent morphism for modules.*
(c) *The morphism f is universally injective.*

Proof. It is clear that (b) implies (a). We now check that (a) implies (c). If f is not universally injective, we can find $M \in \mathrm{Mod}_R$ such that the map $1_M \otimes f : M \to M \otimes_R S$ has nontrivial kernel N. The natural projection $M \to M/N$ is not an isomorphism, but its image in $DD_{S/R}$ is an isomorphism. Hence f^* is not fully faithful.

We finally check that (c) implies (b). By Lemma 34.4.20, for $(M,\theta) \in DD_{S/R}$, the natural map $f^*f_*(M,\theta) \to M$ is an isomorphism of S-modules. On the other hand, for $M_0 \in \mathrm{Mod}_R$, we may tensor (34.4.18.1) with M_0 over R to obtain an equalizer sequence, so $M_0 \to f_*f^*M$ is an isomorphism. Consequently, f_* and f^* are quasi-inverse functors, proving the claim. □

34.4.23. Descent for properties of modules. Throughout this subsection, fix a universally injective ring map $f : R \to S$, an object $M \in \mathrm{Mod}_R$, and a ring map $R \to A$. We now investigate the question of which properties of M or A can be checked after base extension along f. We start with some results from [**Mes02**].

Lemma 34.4.24. *If $M \in Mod_R$ is flat, then $C(M)$ is an injective R-module.*

Proof. Let $0 \to N \to P \to Q \to 0$ be an exact sequence in Mod_R. Since M is flat,

$$0 \to N \otimes_R M \to P \otimes_R M \to Q \otimes_R M \to 0$$

is exact. By Lemma 34.4.10,

$$0 \to C(Q \otimes_R M) \to C(P \otimes_R M) \to C(N \otimes_R M) \to 0$$

is exact. By (34.4.11.1), this last sequence can be rewritten as

$$0 \to \mathrm{Hom}_R(Q, C(M)) \to \mathrm{Hom}_R(P, C(M)) \to \mathrm{Hom}_R(N, C(M)) \to 0.$$

Hence $C(M)$ is an injective object of Mod_R. □

Theorem 34.4.25. *If $M \otimes_R S$ has one of the following properties as an S-module*

(a) *finitely generated;*
(b) *finitely presented;*
(c) *flat;*
(d) *faithfully flat;*

(e) *finite projective;*

then so does M as an R-module (and conversely).

Proof. To prove (a), choose a finite set $\{n_i\}$ of generators of $M \otimes_R S$ in Mod_S. Write each n_i as $\sum_j m_{ij} \otimes s_{ij}$ with $m_{ij} \in M$ and $s_{ij} \in S$. Let F be the finite free R-module with basis e_{ij} and let $F \to M$ be the R-module map sending e_{ij} to m_{ij}. Then $F \otimes_R S \to M \otimes_R S$ is surjective, so $\text{Coker}(F \to M) \otimes_R S$ is zero and hence $\text{Coker}(F \to M)$ is zero. This proves (a).

To see (b) assume $M \otimes_R S$ is finitely presented. Then M is finitely generated by (a). Choose a surjection $R^{\oplus n} \to M$ with kernel K. Then $K \otimes_R S \to S^{\oplus r} \to M \otimes_R S \to 0$ is exact. By Algebra, Lemma 10.5.3 the kernel of $S^{\oplus r} \to M \otimes_R S$ is a finite S-module. Thus we can find finitely many elements $k_1, \ldots, k_t \in K$ such that the images of $k_i \otimes 1$ in $S^{\oplus r}$ generate the kernel of $S^{\oplus r} \to M \otimes_R S$. Let $K' \subset K$ be the submodule generated by $k_1, \ldots, k_t$. Then $M' = R^{\oplus r}/K'$ is a finitely presented R-module with a morphism $M' \to M$ such that $M' \otimes_R S \to M \otimes_R S$ is an isomorphism. Thus $M' \cong M$ as desired.

To prove (c), let $0 \to M' \to M'' \to M \to 0$ be a short exact sequence in Mod_R. Since $\bullet \otimes_R S$ is a right exact functor, $M'' \otimes_R S \to M \otimes_R S$ is surjective. So by Lemma 34.4.10 the map $C(M \otimes_R S) \to C(M'' \otimes_R S)$ is injective. If $M \otimes_R S$ is flat, then Lemma 34.4.24 shows $C(M \otimes_R S)$ is an injective object of Mod_S, so the injection $C(M \otimes_R S) \to C(M'' \otimes_R S)$ is split in Mod_S and hence also in Mod_R. Since $C(M \otimes_R S) \to C(M)$ is a split surjection by Lemma 34.4.12, it follows that $C(M) \to C(M'')$ is a split injection in Mod_R. That is, the sequence

$$0 \to C(M) \to C(M'') \to C(M') \to 0$$

is split exact. For $N \in \text{Mod}_R$, by (34.4.11.1) we see that

$$0 \to C(M \otimes_R N) \to C(M'' \otimes_R N) \to C(M' \otimes_R N) \to 0$$

is split exact. By Lemma 34.4.10,

$$0 \to M' \otimes_R N \to M'' \otimes_R N \to M \otimes_R N \to 0$$

is exact. This implies M is flat over R. Namely, taking M' a free module surjecting onto M we conclude that $\text{Tor}_1^R(M, N) = 0$ for all modules N and we can use Algebra, Lemma 10.70.7. This proves (c).

To deduce (d) from (c), note that if $N \in \text{Mod}_R$ and $M \otimes_R N$ is zero, then $M \otimes_R S \otimes_S (N \otimes_R S) \cong (M \otimes_R N) \otimes_R S$ is zero, so $N \otimes_R S$ is zero and hence N is zero.

To deduce (e) at this point, it suffices to recall that M is finitely generated and projective if and only if it is finitely presented and flat. See Algebra, Lemma 10.73.2. □

There is a variant for R-algebras.

Theorem 34.4.26. *If $A \otimes_R S$ has one of the following properties as an S-algebra*

(a) *of finite type;*
(b) *of finite presentation;*
(c) *formally unramified;*
(d) *unramified;*
(e) *étale;*

then so does A as an R-algebra (and of course conversely).

Proof. To prove (a), choose a finite set $\{x_i\}$ of generators of $A \otimes_R S$ over S. Write each x_i as $\sum_j y_{ij} \otimes s_{ij}$ with $y_{ij} \in A$ and $s_{ij} \in S$. Let F be the polynomial R-algebra on variables e_{ij} and let $F \to M$ be the R-algebra map sending e_{ij} to y_{ij}. Then $F \otimes_R S \to A \otimes_R S$ is surjective, so $\mathrm{Coker}(F \to A) \otimes_R S$ is zero and hence $\mathrm{Coker}(F \to A)$ is zero. This proves (a).

To see (b) assume $A \otimes_R S$ is a finitely presented S-algebra. Then A is finite type over R by (a). Choose a surjection $R[x_1, \ldots, x_n] \to A$ with kernel I. Then $I \otimes_R S \to S[x_1, \ldots, x_n] \to A \otimes_R S \to 0$ is exact. By Algebra, Lemma 10.6.3 the kernel of $S[x_1, \ldots, x_n] \to A \otimes_R S$ is a finitely generated ideal. Thus we can find finitely many elements $y_1, \ldots, y_t \in I$ such that the images of $y_i \otimes 1$ in $S[x_1, \ldots, x_n]$ generate the kernel of $S[x_1, \ldots, x_n] \to A \otimes_R S$. Let $I' \subset I$ be the ideal generated by $y_1, \ldots, y_t$. Then $A' = R[x_1, \ldots, x_n]/I'$ is a finitely presented R-algebra with a morphism $A' \to A$ such that $A' \otimes_R S \to A \otimes_R S$ is an isomorphism. Thus $A' \cong A$ as desired.

To prove (c), recall that A is formally unramified over R if and only if the module of relative differentials $\Omega_{A/R}$ vanishes, see Algebra, Lemma 10.139.2 or [**GD67**, Proposition 17.2.1]. Since $\Omega_{(A \otimes_R S)/S} = \Omega_{A/R} \otimes_R S$, the vanishing descends by Theorem 34.4.22.

To deduce (d) from the previous cases, recall that A is unramified over R if and only if A is formally unramified and of finite type over R, see Algebra, Lemma 10.142.2.

To prove (e), recall that by Algebra, Lemma 10.136.8 or [**GD67**, Théorème 17.6.1] the algebra A is étale over R if and only if A is flat, unramified, and of finite presentation over R. □

Remark 34.4.27. It would make things easier to have a faithfully flat ring homomorphism $g : R \to T$ for which $T \to S \otimes_R T$ has some extra structure. For instance, if one could ensure that $T \to S \otimes_R T$ is split in *Rings*, then it would follow that every property of a module or algebra which is stable under base extension and which descends along faithfully flat morphisms also descends along universally injective morphisms. An obvious guess would be to find g for which T is not only faithfully flat but also injective in Mod_R, but even for $R = \mathbf{Z}$ no such homomorphism can exist.

34.5. Fpqc descent of quasi-coherent sheaves

The main application of flat descent for modules is the corresponding descent statement for quasi-coherent sheaves with respect to fpqc-coverings.

Lemma 34.5.1. *Let S be an affine scheme. Let $\mathcal{U} = \{f_i : U_i \to S\}_{i=1,\ldots,n}$ be a standard fpqc covering of S, see Topologies, Definition 33.8.1. Any descent datum on quasi-coherent sheaves for $\mathcal{U} = \{U_i \to S\}$ is effective. Moreover, the functor from the category of quasi-coherent $\mathcal{O}_S$-modules to the category of descent data with respect to $\mathcal{U}$ is fully faithful.*

Proof. This is a restatement of Proposition 34.3.9 in terms of schemes. First, note that a descent datum ξ for quasi-coherent sheaves with respect to $\mathcal{U}$ is exactly the same as a descent datum ξ' for quasi-coherent sheaves with respect to the covering $\mathcal{U}' = \{\coprod_{i=1,\ldots,n} U_i \to S\}$. Moreover, effectivity for ξ is the same as effectivity for

ξ'. Hence we may assume $n = 1$, i.e., $\mathcal{U} = \{U \to S\}$ where U and S are affine. In this case descent data correspond to descent data on modules with respect to the ring map

$$\Gamma(S, \mathcal{O}) \longrightarrow \Gamma(U, \mathcal{O}).$$

Since $U \to S$ is surjective and flat, we see that this ring map is faithfully flat. In other words, Proposition 34.3.9 applies and we win. □

Proposition 34.5.2. *Let S be a scheme. Let $\mathcal{U} = \{\varphi_i : U_i \to S\}$ be an fpqc covering, see Topologies, Definition 33.8.1. Any descent datum on quasi-coherent sheaves for $\mathcal{U} = \{U_i \to S\}$ is effective. Moreover, the functor from the category of quasi-coherent $\mathcal{O}_S$-modules to the category of descent data with respect to $\mathcal{U}$ is fully faithful.*

Proof. Let $S = \bigcup_{j \in J} V_j$ be an affine open covering. For $j, j' \in J$ we denote $V_{jj'} = V_j \cap V_{j'}$ the intersection (which need not be affine). For $V \subset S$ open we denote $\mathcal{U}_V = \{V \times_S U_i \to V\}_{i \in I}$ which is a fpqc-covering (Topologies, Lemma 33.8.7). By definition of an fpqc covering, we can find for each $j \in J$ a finite set K_j, a map $\underline{i} : K_j \to I$, affine opens $U_{\underline{i}(k),k} \subset U_{\underline{i}(k)}$, $k \in K_j$ such that $\mathcal{V}_j = \{U_{\underline{i}(k),k} \to V_j\}_{k \in K_j}$ is a standard fpqc covering of V_j. And of course, $\mathcal{V}_j$ is a refinement of $\mathcal{U}_{V_j}$. Picture

$$\begin{array}{ccccc} \mathcal{V}_j & \longrightarrow & \mathcal{U}_{V_j} & \longrightarrow & \mathcal{U} \\ \downarrow & & \downarrow & & \downarrow \\ V_j & = & V_j & \longrightarrow & S \end{array}$$

where the top horizontal arrows are morphisms of families of morphisms with fixed target (see Sites, Definition 7.8.1).

To prove the proposition you show successively the faithfulness, fullyness, and essential surjectivity of the functor from quasi-coherent sheaves to descent data.

Faithfulness. Let $\mathcal{F}$, $\mathcal{G}$ be quasi-coherent sheaves on S and let $a, b : \mathcal{F} \to \mathcal{G}$ be homomorphisms of $\mathcal{O}_S$-modules. Suppose $\varphi_i^*(a) = \varphi^*(b)$ for all i. Pick $s \in S$. Then $s = \varphi_i(u)$ for some $i \in I$ and $u \in U_i$. Since $\mathcal{O}_{S,s} \to \mathcal{O}_{U_i,u}$ is flat, hence faithfully flat (Algebra, Lemma 10.36.16) we see that $a_s = b_s : \mathcal{F}_s \to \mathcal{G}_s$. Hence $a = b$.

Fully faithfulness. Let $\mathcal{F}$, $\mathcal{G}$ be quasi-coherent sheaves on S and let $a_i : \varphi_i^*\mathcal{F} \to \varphi_i^*\mathcal{G}$ be homomorphisms of $\mathcal{O}_{U_i}$-modules such that $\mathrm{pr}_0^* a_i = \mathrm{pr}_1^* a_j$ on $U_i \times_U U_j$. We can pull back these morphisms to get morphisms

$$a_k : \mathcal{F}|_{U_{\underline{i}(k),k}} \longrightarrow \mathcal{G}|_{U_{\underline{i}(k),k}}$$

$k \in K_j$ with notation as above. Moreover, Lemma 34.2.2 assures us that these define a morphism between (canonical) descent data on $\mathcal{V}_j$. Hence, by Lemma 34.5.1, we get correspondingly unique morphisms $a_j : \mathcal{F}|_{V_j} \to \mathcal{G}|_{V_j}$. To see that $a_j|_{V_{jj'}} = a_{j'}|_{V_{jj'}}$ we use that both a_j and $a_{j'}$ agree with the pullback of the morphism $(a_i)_{i \in I}$ of (canonical) descent data to any covering refining both $\mathcal{V}_{j,V_{jj'}}$ and $\mathcal{V}_{j',V_{jj'}}$, and using the faithfulness already shown. For example the covering $\mathcal{V}_{jj'} = \{V_k \times_S V_{k'} \to V_{jj'}\}_{k \in K_j, k' \in K_{j'}}$ will do.

Essential surjectivity. Let $\xi = (\mathcal{F}_i, \varphi_{ii'})$ be a descent datum for quasi-coherent sheaves relative to the covering $\mathcal{U}$. Pull back this descent datum to get descent

data ξ_j for quasi-coherent sheaves relative to the coverings $\mathcal{V}_j$ of V_j. By Lemma 34.5.1 once again there exist quasi-coherent sheaves $\mathcal{F}_j$ on V_j whose associated canonical descent datum is isomorphic to ξ_j. By fully faithfulness (proved above) we see there are isomorphisms

$$\phi_{jj'} : \mathcal{F}_j|_{V_{jj'}} \longrightarrow \mathcal{F}_{j'}|_{V_{jj'}}$$

corresponding to the isomorphism of descent data between the pullback of ξ_j and $\xi_{j'}$ to $\mathcal{V}_{jj'}$. To see that these maps $\phi_{jj'}$ satisfy the cocycle condition we use faithfulness (proved above) over the triple intersections $V_{jj'j''}$. Hence, by Lemma 34.2.4 we see that the sheaves $\mathcal{F}_j$ glue to a quasi-coherent sheaf $\mathcal{F}$ as desired. We still have to verify that the canonical descent datum relative to $\mathcal{U}$ associated to $\mathcal{F}$ is isomorphic to the descent datum we started out with. This verification is omitted. □

34.6. Descent of finiteness properties of modules

In this section we prove that one can check quasi-coherent module has a certain finiteness conditions by checking on the members of a covering.

Lemma 34.6.1. *Let X be a scheme. Let $\mathcal{F}$ be a quasi-coherent $\mathcal{O}_X$-module. Let $\{f_i : X_i \to X\}_{i\in I}$ be an fpqc covering such that each $f_i^*\mathcal{F}$ is a finite type $\mathcal{O}_{X_i}$-module. Then $\mathcal{F}$ is a finite type $\mathcal{O}_X$-module.*

Proof. Omitted. For the affine case, see Algebra, Lemma 10.78.2. □

Lemma 34.6.2. *Let $f : (X, \mathcal{O}_X) \to (Y, \mathcal{O}_Y)$ be a morphism of locally ringed spaces. Let $\mathcal{F}$ be a sheaf of $\mathcal{O}_Y$-modules. If*

1. *f is open as a map of topological spaces,*
2. *f is surjective and flat, and*
3. *$f^*\mathcal{F}$ is of finite type,*

then $\mathcal{F}$ is of finite type.

Proof. Let $y \in Y$ be a point. Choose a point $x \in X$ mapping to y. Choose an open $x \in U \subset X$ and elements $s_1, \ldots, s_n$ of $f^*\mathcal{F}(U)$ which generate $f^*\mathcal{F}$ over U. Since $f^*\mathcal{F} = f^{-1}\mathcal{F} \otimes_{f^{-1}\mathcal{O}_Y} \mathcal{O}_X$ we can after shrinking U assume $s_i = \sum t_{ij} \otimes a_{ij}$ with $t_{ij} \in f^{-1}\mathcal{F}(U)$ and $a_{ij} \in \mathcal{O}_X(U)$. After shrinking U further we may assume that t_{ij} comes from a section $s_{ij} \in \mathcal{F}(V)$ for some $V \subset Y$ open with $f(U) \subset V$. Let N be the number of sections s_{ij} and consider the map

$$\sigma = (s_{ij}) : \mathcal{O}_V^{\oplus N} \to \mathcal{F}|_V$$

By our choice of the sections we see that $f^*\sigma|_U$ is surjective. Hence for every $u \in U$ the map

$$\sigma_{f(u)} \otimes_{\mathcal{O}_{Y,f(u)}} \mathcal{O}_{X,u} : \mathcal{O}_{X,u}^{\oplus N} \longrightarrow \mathcal{F}_{f(u)} \otimes_{\mathcal{O}_{Y,f(u)}} \mathcal{O}_{X,u}$$

is surjective. As f is flat, the local ring map $\mathcal{O}_{Y,f(u)} \to \mathcal{O}_{X,u}$ is flat, hence faithfully flat (Algebra, Lemma 10.36.16). Hence $\sigma_{f(u)}$ is surjective. Since f is open, $f(U)$ is an open neighbourhood of y and the proof is done. □

Lemma 34.6.3. *Let X be a scheme. Let $\mathcal{F}$ be a quasi-coherent $\mathcal{O}_X$-module. Let $\{f_i : X_i \to X\}_{i\in I}$ be an fpqc covering such that each $f_i^*\mathcal{F}$ is an $\mathcal{O}_{X_i}$-module of finite presentation. Then $\mathcal{F}$ is an $\mathcal{O}_X$-module of finite presentation.*

Proof. Omitted. For the affine case, see Algebra, Lemma 10.78.2. □

Lemma 34.6.4. *Let X be a scheme. Let $\mathcal{F}$ be a quasi-coherent $\mathcal{O}_X$-module. Let $\{f_i : X_i \to X\}_{i\in I}$ be an fpqc covering such that each $f_i^*\mathcal{F}$ is locally generated by r sections as an $\mathcal{O}_{X_i}$-module. Then $\mathcal{F}$ is locally generated by r sections as an $\mathcal{O}_X$-module.*

Proof. By Lemma 34.6.1 we see that $\mathcal{F}$ is of finite type. Hence Nakayama's lemma (Algebra, Lemma 10.18.1) implies that $\mathcal{F}$ is generated by r sections in the neighbourhood of a point $x \in X$ if and only if $\dim_{\kappa(x)} \mathcal{F}_x \otimes \kappa(x) \leq r$. Choose an i and a point $x_i \in X_i$ mapping to x. Then $\dim_{\kappa(x)} \mathcal{F}_x \otimes \kappa(x) = \dim_{\kappa(x_i)} (f_i^*\mathcal{F})_{x_i} \otimes \kappa(x_i)$ which is $\leq r$ as $f_i^*\mathcal{F}$ is locally generated by r sections. □

Lemma 34.6.5. *Let X be a scheme. Let $\mathcal{F}$ be a quasi-coherent $\mathcal{O}_X$-module. Let $\{f_i : X_i \to X\}_{i\in I}$ be an fpqc covering such that each $f_i^*\mathcal{F}$ is a flat $\mathcal{O}_{X_i}$-module. Then $\mathcal{F}$ is a flat $\mathcal{O}_X$-module.*

Proof. Omitted. For the affine case, see Algebra, Lemma 10.78.2. □

Lemma 34.6.6. *Let X be a scheme. Let $\mathcal{F}$ be a quasi-coherent $\mathcal{O}_X$-module. Let $\{f_i : X_i \to X\}_{i\in I}$ be an fpqc covering such that each $f_i^*\mathcal{F}$ is a finite locally free $\mathcal{O}_{X_i}$-module. Then $\mathcal{F}$ is a finite locally free $\mathcal{O}_X$-module.*

Proof. This follows from the fact that a quasi-coherent sheaf is finite locally free if and only if it is of finite presentation and flat, see Algebra, Lemma 10.73.2. Namely, if each $f_i^*\mathcal{F}$ is flat and of finite presentation, then so is $\mathcal{F}$ by Lemmas 34.6.5 and 34.6.3. □

The definition of a locally projective quasi-coherent sheaf can be found in Properties, Section 27.19.

Lemma 34.6.7. *Let X be a scheme. Let $\mathcal{F}$ be a quasi-coherent $\mathcal{O}_X$-module. Let $\{f_i : X_i \to X\}_{i\in I}$ be an fpqc covering such that each $f_i^*\mathcal{F}$ is a locally projective $\mathcal{O}_{X_i}$-module. Then $\mathcal{F}$ is a locally projective $\mathcal{O}_X$-module.*

Proof. Omitted. For Zariski coverings this is Properties, Lemma 27.19.2. For the affine case this is Algebra, Theorem 10.90.5. □

Remark 34.6.8. Being locally free is a property of quasi-coherent modules which does not descend in the fpqc topology. Namely, suppose that R is a ring and that M is a projective R-module which is a countable direct sum $M = \bigoplus L_n$ of rank 1 locally free modules, but not locally free, see Examples, Lemma 78.20.5. Then M becomes free on making the faithfully flat base change

$$R \longrightarrow \bigoplus\nolimits_{m\geq 1} \bigoplus\nolimits_{(i_1,\ldots,i_m)\in \mathbf{Z}^{\oplus m}} L_1^{\otimes i_1} \otimes_R \ldots \otimes_R L_m^{\otimes i_m}$$

But we don't know what happens for fppf coverings. In other words, we don't know the answer to the following question: Suppose $A \to B$ is a faithfully flat ring map of finite presentation. Let M be an A-module such that $M \otimes_A B$ is free. Is M a locally free A-module? It turns out that if A is Noetherian, then the answer is yes. This follows from the results of [**Bas63**]. But in general we don't know the answer. If you know the answer, or have a reference, please email stacks.project@gmail.com.

We also add here two results which are related to the results above, but are of a slightly different nature.

Lemma 34.6.9. *Let $f : X \to Y$ be a morphism of schemes. Let $\mathcal{F}$ be a quasi-coherent $\mathcal{O}_X$-module. Assume f is a finite morphism. Then $\mathcal{F}$ is an $\mathcal{O}_X$-module of finite type if and only if $f_*\mathcal{F}$ is an $\mathcal{O}_Y$-module of finite type.*

Proof. As f is finite it is affine. This reduces us to the case where f is the morphism $\mathrm{Spec}(B) \to \mathrm{Spec}(A)$ given by a finite ring map $A \to B$. Moreover, then $\mathcal{F} = \widetilde{M}$ is the sheaf of modules associated to the B-module M. Note that M is finite as a B-module if and only if M is finite as an A-module, see Algebra, Lemma 10.7.2. Combined with Properties, Lemma 27.16.1 this proves the lemma. □

Lemma 34.6.10. *Let $f : X \to Y$ be a morphism of schemes. Let $\mathcal{F}$ be a quasi-coherent $\mathcal{O}_X$-module. Assume f is finite and of finite presentation. Then $\mathcal{F}$ is an $\mathcal{O}_X$-module of finite presentation if and only if $f_*\mathcal{F}$ is an $\mathcal{O}_Y$-module of finite presentation.*

Proof. As f is finite it is affine. This reduces us to the case where f is the morphism $\mathrm{Spec}(B) \to \mathrm{Spec}(A)$ given by a finite and finitely presented ring map $A \to B$. Moreover, then $\mathcal{F} = \widetilde{M}$ is the sheaf of modules associated to the B-module M. Note that M is finitely presented as a B-module if and only if M is finitely presented as an A-module, see Algebra, Lemma 10.7.4. Combined with Properties, Lemma 27.16.2 this proves the lemma. □

34.7. Quasi-coherent sheaves and topologies

Let S be a scheme. Let $\mathcal{F}$ be a quasi-coherent $\mathcal{O}_S$-module. Consider the functor

$$(Sch/S)^{opp} \longrightarrow Ab, \quad (f : T \to S) \longmapsto \Gamma(T, f^*\mathcal{F}). \tag{34.7.0.1}$$

Lemma 34.7.1. *Let S be a scheme. Let $\mathcal{F}$ be a quasi-coherent $\mathcal{O}_S$-module. Let $\tau \in \{Zariski, fpqc, fppf, étale, smooth, syntomic\}$. The functor defined in (34.7.0.1) satisfies the sheaf condition with respect to any τ-covering $\{T_i \to T\}_{i\in I}$ of any scheme T over S.*

Proof. For $\tau \in \{Zariski, fppf, étale, smooth, syntomic\}$ a τ-covering is also a fpqc-covering, see the results in Topologies, Lemmas 33.4.2, 33.5.2, 33.6.2, 33.7.2, and 33.8.6. Hence it suffices to prove the theorem for a fpqc covering. Assume that $\{f_i : T_i \to T\}_{i\in I}$ is an fpqc covering where $f : T \to S$ is given. Suppose that we have a family of sections $s_i \in \Gamma(T_i, f_i^* f^*\mathcal{F})$ such that $s_i|_{T_i\times_T T_j} = s_j|_{T_i\times_T T_j}$. We have to find the correspond section $s \in \Gamma(T, f^*\mathcal{F})$. We can reinterpret the s_i as a family of maps $\varphi_i : f_i^*\mathcal{O}_T = \mathcal{O}_{T_i} \to f_i^* f^*\mathcal{F}$ compatible with the canonical descent data associated to the quasi-coherent sheaves $\mathcal{O}_T$ and $f^*\mathcal{F}$ on T. Hence by Proposition 34.5.2 we see that we may (uniquely) descend these to a map $\mathcal{O}_T \to f^*\mathcal{F}$ which gives us our section s. □

We may in particular make the following definition.

Definition 34.7.2. Let $\tau \in \{Zariski, fppf, étale, smooth, syntomic\}$. Let S be a scheme. Let Sch_τ be a big site containing S. Let $\mathcal{F}$ be a quasi-coherent $\mathcal{O}_S$-module.

1. The *structure sheaf of the big site* $(Sch/S)_\tau$ is the sheaf of rings $T/S \mapsto \Gamma(T, \mathcal{O}_T)$ which is denoted $\mathcal{O}$ or $\mathcal{O}_S$.
2. If $\tau = étale$ the structure sheaf of the small site $S_{étale}$ is the sheaf of rings $T/S \mapsto \Gamma(T, \mathcal{O}_T)$ which is denoted $\mathcal{O}$ or $\mathcal{O}_S$.

(3) The *sheaf of $\mathcal{O}$-modules associated to $\mathcal{F}$* on the big site $(Sch/S)_\tau$ is the sheaf of $\mathcal{O}$-modules $(f : T \to S) \mapsto \Gamma(T, f^*\mathcal{F})$ which is denoted $\mathcal{F}^a$ (and often simply $\mathcal{F}$).
(4) Let $\tau = étale$ (resp. $\tau = Zariski$). The *sheaf of $\mathcal{O}$-modules associated to $\mathcal{F}$* on the small site $S_{étale}$ (resp. S_{Zar}) is the sheaf of $\mathcal{O}$-modules $(f : T \to S) \mapsto \Gamma(T, f^*\mathcal{F})$ which is denoted $\mathcal{F}^a$ (and often simply $\mathcal{F}$).

Note how we use the same notation $\mathcal{F}^a$ in each case. No confusion can really arise from this as by definition the rule that defines the sheaf $\mathcal{F}^a$ is independent of the site we choose to look at.

Remark 34.7.3. In Topologies, Lemma 33.3.11 we have seen that the small Zariski site of a scheme S is equivalent to S as a topological space in the sense that the category of sheaves are naturally equivalent. Now that S_{Zar} is also endowed with a structure sheaf $\mathcal{O}$ we see that sheaves of modules on the ringed site $(S_{Zar}, \mathcal{O})$ agree with sheaves of modules on the ringed space $(S, \mathcal{O}_S)$.

Remark 34.7.4. Let $f : T \to S$ be a morphism of schemes. Each of the morphisms of sites f_{sites} listed in Topologies, Section 33.9 becomes a morphism of ringed sites. Namely, each of these morphisms of sites $f_{sites} : (Sch/T)_\tau \to (Sch/S)_{\tau'}$, or $f_{sites} : (Sch/S)_\tau \to S_{\tau'}$ is given by the continuous functor $S'/S \mapsto T \times_S S'/S$. Hence, given S'/S we let

$$f^\sharp_{sites} : \mathcal{O}(S'/S) \longrightarrow f_{sites,*}\mathcal{O}(S'/S) = \mathcal{O}(S \times_S S'/T)$$

be the usual map $\mathrm{pr}^\sharp_{S'} : \mathcal{O}(S') \to \mathcal{O}(T \times_S S')$. Similarly, the morphism $i_f : Sh(T_\tau) \to Sh((Sch/S)_\tau)$ for $\tau \in \{Zar, étale\}$, see Topologies, Lemmas 33.3.12 and 33.4.12, becomes a morphism of ringed topoi because $i_f^{-1}\mathcal{O} = \mathcal{O}$. Here are some special cases:

(1) The morphism of big sites $f_{big} : (Sch/X)_{fppf} \to (Sch/Y)_{fppf}$, becomes a morphism of ringed sites
$$(f_{big}, f^\sharp_{big}) : ((Sch/X)_{fppf}, \mathcal{O}_X) \longrightarrow ((Sch/Y)_{fppf}, \mathcal{O}_Y)$$
as in Modules on Sites, Definition 18.6.1. Similarly for the big syntomic, smooth, étale and Zariski sites.
(2) The morphism of small sites $f_{small} : X_{étale} \to Y_{étale}$ becomes a morphism of ringed sites
$$(f_{small}, f^\sharp_{small}) : (X_{étale}, \mathcal{O}_X) \longrightarrow (Y_{étale}, \mathcal{O}_Y)$$
as in Modules on Sites, Definition 18.6.1. Similarly for the small Zariski site.

Let S be a scheme. It is clear that given an $\mathcal{O}$-module on (say) $(Sch/S)_{Zar}$ the pullback to (say) $(Sch/S)_{fppf}$ is just the fppf-sheafification. To see what happens when comparing big and small sites we have the following.

Lemma 34.7.5. *Let S be a scheme. Denote*

$$\begin{array}{rcll} id_{\tau,Zar} & : & (Sch/S)_\tau \to S_{Zar}, & \tau \in \{Zar, étale, smooth, syntomic, fppf\} \\ id_{\tau,étale} & : & (Sch/S)_\tau \to S_{étale}, & \tau \in \{étale, smooth, syntomic, fppf\} \\ id_{small,étale,Zar} & : & S_{étale} \to S_{Zar}, & \end{array}$$

the morphisms of ringed sites of Remark 34.7.4. Let $\mathcal{F}$ be a sheaf of $\mathcal{O}_S$-modules which we view a sheaf of $\mathcal{O}$-modules on S_{Zar}. Then

(1) $(id_{\tau,Zar})^*\mathcal{F}$ *is the* τ*-sheafification of the Zariski sheaf*
$$(f : T \to S) \longmapsto \Gamma(T, f^*\mathcal{F})$$
on $(Sch/S)_\tau$*, and*
(2) $(id_{small,étale,Zar})^*\mathcal{F}$ *is the étale sheafification of the Zariski sheaf*
$$(f : T \to S) \longmapsto \Gamma(T, f^*\mathcal{F})$$
on $S_{étale}$*.*

Let $\mathcal{G}$ *be a sheaf of* $\mathcal{O}$*-modules on* $S_{étale}$*. Then*
(3) $(id_{\tau,étale})^*\mathcal{G}$ *is the* τ*-sheafification of the étale sheaf*
$$(f : T \to S) \longmapsto \Gamma(T, f^*_{small}\mathcal{G})$$
where $f_{small} : T_{étale} \to S_{étale}$ *is the morphism of ringed small étale sites of Remark 34.7.4.*

Proof. Proof of (1). We first note that the result is true when $\tau = Zar$ because in that case we have the morphism of topoi $i_f : Sh(T_{Zar}) \to Sh(Sch/S)_{Zar})$ such that $\mathrm{id}_{\tau,Zar} \circ i_f = f_{small}$ as morphisms $T_{Zar} \to S_{Zar}$, see Topologies, Lemmas 33.3.12 and 33.3.16. Since pullback is transitive (see Modules on Sites, Lemma 18.13.3) we see that $i_f^*(\mathrm{id}_{\tau,Zar})^*\mathcal{F} = f^*_{small}\mathcal{F}$ as desired. Hence, by the remark preceding this lemma we see that $(\mathrm{id}_{\tau,Zar})^*\mathcal{F}$ is the τ-sheafification of the presheaf $T \mapsto \Gamma(T, f^*\mathcal{F})$.

The proof of (3) is exactly the same as the proof of (1), except that it uses Topologies, Lemmas 33.4.12 and 33.4.16. We omit the proof of (2). □

Remark 34.7.6. Remark 34.7.4 and Lemma 34.7.5 have the following applications:
(1) Let S be a scheme. The construction $\mathcal{F} \mapsto \mathcal{F}^a$ is the pullback under the morphism of ringed sites $\mathrm{id}_{\tau,Zar} : ((Sch/S)_\tau, \mathcal{O}) \to (S_{Zar}, \mathcal{O})$ or the morphism $\mathrm{id}_{small,étale,Zar} : (S_{étale}, \mathcal{O}) \to (S_{Zar}, \mathcal{O})$.
(2) Let $f : X \to Y$ be a morphism of schemes. For any of the morphisms f_{sites} of ringed sites of Remark 34.7.4 we have
$$(f^*\mathcal{F})^a = f^*_{sites}\mathcal{F}^a.$$
This follows from (1) and the fact that pullbacks are compatible with compositions of morphisms of ringed sites, see Modules on Sites, Lemma 18.13.3.

Lemma 34.7.7. *Let* S *be a scheme. Let* $\mathcal{F}$ *be a quasi-coherent* $\mathcal{O}_S$*-module. Let* $\tau \in \{Zariski, fppf, étale, smooth, syntomic\}$.
(1) *The sheaf* $\mathcal{F}^a$ *is a quasi-coherent* $\mathcal{O}$*-module on* $(Sch/S)_\tau$*, as defined in Modules on Sites, Definition 18.23.1.*
(2) *If* $\tau = étale$ *(resp.* $\tau = Zariski$*), then the sheaf* $\mathcal{F}^a$ *is a quasi-coherent* $\mathcal{O}$*-module on* $S_{étale}$ *(resp.* S_{Zar}*) as defined in Modules on Sites, Definition 18.23.1.*

Proof. Let $\{S_i \to S\}$ be a Zariski covering such that we have exact sequences
$$\bigoplus\nolimits_{k\in K_i} \mathcal{O}_{S_i} \longrightarrow \bigoplus\nolimits_{j\in J_i} \mathcal{O}_{S_i} \longrightarrow \mathcal{F} \longrightarrow 0$$
for some index sets K_i and J_i. This is possible by the definition of a quasi-coherent sheaf on a ringed space (See Modules, Definition 17.10.1).

Proof of (1). Let $\tau \in \{Zariski, fppf, étale, smooth, syntomic\}$. It is clear that $\mathcal{F}^a|_{(Sch/S_i)_\tau}$ also sits in an exact sequence

$$\bigoplus\nolimits_{k\in K_i} \mathcal{O}|_{(Sch/S_i)_\tau} \longrightarrow \bigoplus\nolimits_{j\in J_i} \mathcal{O}|_{(Sch/S_i)_\tau} \longrightarrow \mathcal{F}^a|_{(Sch/S_i)_\tau} \longrightarrow 0$$

Hence $\mathcal{F}^a$ is quasi-coherent by Modules on Sites, Lemma 18.23.3.

Proof of (2). Let $\tau = étale$. It is clear that $\mathcal{F}^a|_{(S_i)_{étale}}$ also sits in an exact sequence

$$\bigoplus\nolimits_{k\in K_i} \mathcal{O}|_{(S_i)_{étale}} \longrightarrow \bigoplus\nolimits_{j\in J_i} \mathcal{O}|_{(S_i)_{étale}} \longrightarrow \mathcal{F}^a|_{(S_i)_{étale}} \longrightarrow 0$$

Hence $\mathcal{F}^a$ is quasi-coherent by Modules on Sites, Lemma 18.23.3. The case $\tau = Zariski$ is similar (actually, it is really tautological since the corresponding ringed topoi agree). □

Lemma 34.7.8. *Let S be a scheme. Let*

(a) *$\tau \in \{Zariski, fppf, étale, smooth, syntomic\}$ and $\mathcal{C} = (Sch/S)_\tau$, or*
(b) *let $\tau = étale$ and $\mathcal{C} = S_{étale}$, or*
(c) *let $\tau = Zariski$ and $\mathcal{C} = S_{Zar}$.*

Let $\mathcal{F}$ be an abelian sheaf on $\mathcal{C}$. Let $U \in \mathrm{Ob}(\mathcal{C})$ be affine. Let $\{U_i \to U\}_{i=1,\ldots,n}$ be a standard affine τ-covering in $\mathcal{C}$. Then

(1) *$\mathcal{V} = \{\coprod_{i=1,\ldots,n} U_i \to U\}$ is a τ-covering of U,*
(2) *$\mathcal{U}$ is a refinement of $\mathcal{V}$, and*
(3) *the induced map on Cech complexes (Cohomology on Sites, Equation (21.9.2.1))*

$$\check{\mathcal{C}}^\bullet(\mathcal{V}, \mathcal{F}) \longrightarrow \check{\mathcal{C}}^\bullet(\mathcal{U}, \mathcal{F})$$

is an isomorphism of complexes.

Proof. This follows because

$$(\coprod\nolimits_{i_0=1,\ldots,n} U_{i_0}) \times_U \ldots \times_U (\coprod\nolimits_{i_p=1,\ldots,n} U_{i_p}) = \coprod\nolimits_{i_0,\ldots,i_p\in\{1,\ldots,n\}} U_{i_0} \times_U \ldots \times_U U_{i_p}$$

and the fact that $\mathcal{F}(\coprod_a V_a) = \prod_a \mathcal{F}(V_a)$ since disjoint unions are τ-coverings. □

Lemma 34.7.9. *Let S be a scheme. Let $\mathcal{F}$ be a quasi-coherent sheaf on S. Let τ, $\mathcal{C}$, U, $\mathcal{U}$ be as in Lemma 34.7.8. Then there is an isomorphism of complexes*

$$\check{\mathcal{C}}^\bullet(\mathcal{U}, \mathcal{F}^a) \cong s((A/R)_\bullet \otimes_R M)$$

(see Section 34.3) where $R = \Gamma(U, \mathcal{O}_U)$, $M = \Gamma(U, \mathcal{F}^a)$ and $R \to A$ is a faithfully flat ring map. In particular

$$\check{H}^p(\mathcal{U}, \mathcal{F}^a) = 0$$

for all $p \geq 1$.

Proof. By Lemma 34.7.8 we see that $\check{\mathcal{C}}^\bullet(\mathcal{U}, \mathcal{F}^a)$ is isomorphic to $\check{\mathcal{C}}^\bullet(\mathcal{V}, \mathcal{F}^a)$ where $\mathcal{V} = \{V \to U\}$ with $V = \coprod_{i=1,\ldots n} U_i$ affine also. Set $A = \Gamma(V, \mathcal{O}_V)$. Since $\{V \to U\}$ is a τ-covering we see that $R \to A$ is faithfully flat. On the other hand, by definition of $\mathcal{F}^a$ we have that the degree p term $\check{\mathcal{C}}^p(\mathcal{V}, \mathcal{F}^a)$ is

$$\Gamma(V \times_U \ldots \times_U V, \mathcal{F}^a) = \Gamma(\mathrm{Spec}(A \otimes_R \ldots \otimes_R A), \mathcal{F}^a) = A \otimes_R \ldots \otimes_R A \otimes_R M$$

We omit the verification that the maps of the chech complex agree with the maps in the complex $s((A/R)_\bullet \otimes_R M)$. The vanishing of cohomology is Lemma 34.3.6. □

Proposition 34.7.10. *Let S be a scheme. Let $\mathcal{F}$ be a quasi-coherent sheaf on S. Let $\tau \in \{Zariski, fppf, étale, smooth, syntomic\}$.*

(1) *There is a canonical isomorphism*

$$H^q(S, \mathcal{F}) = H^q((Sch/S)_\tau, \mathcal{F}^a).$$

(2) *There are canonical isomorphisms*

$$H^q(S, \mathcal{F}) = H^q(S_{Zar}, \mathcal{F}^a) = H^q(S_{étale}, \mathcal{F}^a).$$

Proof. The result for $q = 0$ is clear from the definition of $\mathcal{F}^a$. Let $\mathcal{C} = (Sch/S)_\tau$, or $\mathcal{C} = S_{étale}$, or $\mathcal{C} = S_{Zar}$.

We are going to apply Cohomology on Sites, Lemma 21.11.8 with $\mathcal{F} = \mathcal{F}^a$, $\mathcal{B} \subset \mathrm{Ob}(\mathcal{C})$ the set of affine schemes in $\mathcal{C}$, and $\mathrm{Cov} \subset \mathrm{Cov}_\mathcal{C}$ the set of standard affine τ-coverings. Assumption (3) of the lemma is satisfied by Lemma 34.7.9. Hence we conclude that $H^p(U, \mathcal{F}^a) = 0$ for every affine object U of $\mathcal{C}$.

Next, let $U \in \mathrm{Ob}(\mathcal{C})$ be any separated object. Denote $f : U \to S$ the structure morphism. Let $U = \bigcup U_i$ be an affine open covering. We may also think of this as a τ-covering $\mathcal{U} = \{U_i \to U\}$ of U in $\mathcal{C}$. Note that $U_{i_0} \times_U \ldots \times_U U_{i_p} = U_{i_0} \cap \ldots \cap U_{i_p}$ is affine as we assumed U separated. By Cohomology on Sites, Lemma 21.11.6 and the result above we see that

$$H^p(U, \mathcal{F}^a) = \check{H}^p(\mathcal{U}, \mathcal{F}^a) = H^p(U, f^*\mathcal{F})$$

the last equality by Cohomology of Schemes, Lemma 29.2.5. In particular, if S is separated we can take $U = S$ and $f = \mathrm{id}_S$ and the proposition is proved. We suggest the reader skip the rest of the proof (or rewrite it to give a clearer exposition).

Choose an injective resolution $\mathcal{F} \to \mathcal{I}^\bullet$ on S. Choose an injective resolution $\mathcal{F}^a \to \mathcal{J}^\bullet$ on $\mathcal{C}$. Denote $\mathcal{J}^n|_S$ the restriction of $\mathcal{J}^n$ to opens of S; this is a sheaf on the topological space S as open coverings are τ-coverings. We get a complex

$$0 \to \mathcal{F} \to \mathcal{J}^0|_S \to \mathcal{J}^1|_S \to \ldots$$

which is exact since its sections over any affine open $U \subset S$ is exact (by the vanishing of $H^p(U, \mathcal{F}^a)$, $p > 0$ seen above). Hence by Derived Categories, Lemma 13.17.6 there exists map of complexes $\mathcal{J}^\bullet|_S \to \mathcal{I}^\bullet$ which in particular induces a map

$$R\Gamma(\mathcal{C}, \mathcal{F}^a) = \Gamma(S, \mathcal{J}^\bullet) \longrightarrow \Gamma(S, \mathcal{I}^\bullet) = R\Gamma(S, \mathcal{F}).$$

Taking cohomology gives the map $H^n(\mathcal{C}, \mathcal{F}^a) \to H^n(S, \mathcal{F})$ which we have to prove is an isomorphism. Let $\mathcal{U} : S = \bigcup U_i$ be an affine open covering which we may think of as a τ-covering also. By the above we get a map of double complexes

$$\check{\mathcal{C}}^\bullet(\mathcal{U}, \mathcal{J}) = \check{\mathcal{C}}^\bullet(\mathcal{U}, \mathcal{J}|_S) \longrightarrow \check{\mathcal{C}}^\bullet(\mathcal{U}, \mathcal{I}).$$

This map induces a map of spectral sequences

$${}^\tau E_2^{p,q} = \check{H}^p(\mathcal{U}, \underline{H}^q(\mathcal{F}^a)) \longrightarrow E_2^{p,q} = \check{H}^p(\mathcal{U}, \underline{H}^q(\mathcal{F}))$$

The first spectral sequence converges to $H^{p+q}(\mathcal{C}, \mathcal{F})$ and the second to $H^{p+q}(S, \mathcal{F})$. On the other hand, we have seen that the induced maps ${}^\tau E_2^{p,q} \to E_2^{p,q}$ are bijections (as all the intersections are separated being opens in affines). Whence also the maps $H^n(\mathcal{C}, \mathcal{F}^a) \to H^n(S, \mathcal{F})$ are isomorphisms, and we win. □

Proposition 34.7.11. *Let S be a scheme. Let $\tau \in \{Zariski, fppf, étale, smooth, syntomic\}$.*

(1) *The functor $\mathcal{F} \mapsto \mathcal{F}^a$ defines an equivalence of categories*

$$QCoh(\mathcal{O}_S) \longrightarrow QCoh((Sch/S)_\tau, \mathcal{O})$$

between the category of quasi-coherent sheaves on S and the category of quasi-coherent $\mathcal{O}$-modules on the big τ site of S.

(2) *Let $\tau = étale$, or $\tau = Zariski$. The functor $\mathcal{F} \mapsto \mathcal{F}^a$ defines an equivalence of categories*

$$QCoh(\mathcal{O}_S) \longrightarrow QCoh(S_\tau, \mathcal{O})$$

between the category of quasi-coherent sheaves on S and the category of quasi-coherent $\mathcal{O}$-modules on the small τ site of S.

Proof. We have seen in Lemma 34.7.7 that the functor is well defined. It is straightforward to show that the functor is fully faithful (we omit the verification). To finish the proof we will show that a quasi-coherent $\mathcal{O}$-module on $(Sch/S)_\tau$ gives rise to a descent datum for quasi-coherent sheaves relative to a τ-covering of S. Having produced this descent datum we will appeal to Proposition 34.5.2 to get the corresponding quasi-coherent sheaf on S.

Let $\mathcal{G}$ be a quasi-coherent $\mathcal{O}$-modules on the big τ site of S. By Modules on Sites, Definition 18.23.1 there exists a τ-covering $\{S_i \to S\}_{i \in I}$ of S such that each of the restrictions $\mathcal{G}|_{(Sch/S_i)_\tau}$ has a global presentation

$$\bigoplus\nolimits_{k \in K_i} \mathcal{O}|_{(Sch/S_i)_\tau} \longrightarrow \bigoplus\nolimits_{j \in J_i} \mathcal{O}|_{(Sch/S_i)_\tau} \longrightarrow \mathcal{G}|_{(Sch/S_i)_\tau} \longrightarrow 0$$

for some index sets J_i and K_i. We claim that this implies that $\mathcal{G}|_{(Sch/S_i)_\tau}$ is $\mathcal{F}_i^a$ for some quasi-coherent sheaf $\mathcal{F}_i$ on S_i. Namely, this is clear for the direct sums $\bigoplus_{k \in K_i} \mathcal{O}|_{(Sch/S_i)_\tau}$ and $\bigoplus_{j \in J_i} \mathcal{O}|_{(Sch/S_i)_\tau}$. Hence we see that $\mathcal{G}|_{(Sch/S_i)_\tau}$ is a cokernel of a map $\varphi : \mathcal{K}_i^a \to \mathcal{L}_i^a$ for some quasi-coherent sheaves $\mathcal{K}_i$, $\mathcal{L}_i$ on S_i. By the fully faithfulness of $(\ )^a$ we see that $\varphi = \phi^a$ for some map of quasi-coherent sheaves $\phi : \mathcal{K}_i \to \mathcal{L}_i$ on S_i. Then it is clear that $\mathcal{G}|_{(Sch/S_i)_\tau} \cong \mathrm{Coker}(\phi)^a$ as claimed.

Since $\mathcal{G}$ lives on all of the category $(Sch/S_i)_\tau$ we see that

$$(\mathrm{pr}_0^* \mathcal{F}_i)^a \cong \mathcal{G}|_{(Sch/(S_i \times_S S_j))_\tau} \cong (\mathrm{pr}_1^* \mathcal{F})^a$$

as $\mathcal{O}$-modules on $(Sch/(S_i \times_S S_j))_\tau$. Hence, using fully faithfulness again we get canonical isomorphisms

$$\phi_{ij} : \mathrm{pr}_0^* \mathcal{F}_i \longrightarrow \mathrm{pr}_1^* \mathcal{F}_j$$

of quasi-coherent modules over $S_i \times_S S_j$. We omit the verification that these satisfy the cocycle condition. Since they do we see by effectivity of descent for quasi-coherent sheaves and the covering $\{S_i \to S\}$ (Proposition 34.5.2) that there exists a quasi-coherent sheaf $\mathcal{F}$ on S with $\mathcal{F}|_{S_i} \cong \mathcal{F}_i$ compatible with the given descent data. In other words we are given $\mathcal{O}$-module isomorphisms

$$\phi_i : \mathcal{F}^a|_{(Sch/S_i)_\tau} \longrightarrow \mathcal{G}|_{(Sch/S_i)_\tau}$$

which agree over $S_i \times_S S_j$. Hence, since $\mathcal{H}om_{\mathcal{O}}(\mathcal{F}^a, \mathcal{G})$ is a sheaf (Modules on Sites, Lemma 18.27.1), we conclude that there is a morphism of $\mathcal{O}$-modules $\mathcal{F}^a \to \mathcal{G}$ recovering the isomorphisms ϕ_i above. Hence this is an isomorphism and we win.

The case of the sites $S_{étale}$ and S_{Zar} is proved in the exact same manner. □

Lemma 34.7.12. *Let S be a scheme. Let $\tau \in \{Zariski, fppf, étale, smooth, syntomic\}$. Let $\mathcal{P}$ be one of the properties of modules[4] defined in Modules on Sites, Definitions 18.17.1, 18.23.1, and 18.28.1. The equivalences of categories*

$$QCoh(\mathcal{O}_S) \longrightarrow QCoh((Sch/S)_\tau, \mathcal{O}) \quad and \quad QCoh(\mathcal{O}_S) \longrightarrow QCoh(S_\tau, \mathcal{O})$$

defined by the rule $\mathcal{F} \mapsto \mathcal{F}^a$ seen in Proposition 34.7.11 have the property

$$\mathcal{F} \text{ has } \mathcal{P} \Leftrightarrow \mathcal{F}^a \text{ has } \mathcal{P} \text{ as an } \mathcal{O}\text{-module}$$

except (possibly) when $\mathcal{P}$ is "locally free" or "coherent". If $\mathcal{P}$ = "coherent" the equivalence holds for $QCoh(\mathcal{O}_S) \to QCoh(S_\tau, \mathcal{O})$ when S is locally Noetherian and τ is Zariski or étale.

Proof. This is immediate for the global properties, i.e., those defined in Modules on Sites, Definition 18.17.1. For the local properties we can use Modules on Sites, Lemma 18.23.3 to translate "$\mathcal{F}^a$ has $\mathcal{P}$" into a property on the members of a covering of X. Hence the result follows from Lemmas 34.6.1, 34.6.3, 34.6.4, 34.6.5, and 34.6.6. Being coherent for a quasi-coherent module is the same as being of finite type over a locally Noetherian scheme (see Cohomology of Schemes, Lemma 29.9.1) hence this reduces to the case of finite type modules (details omitted). □

Lemma 34.7.13. *Let S be a scheme. Let $\tau \in \{Zariski, fppf, étale, smooth, syntomic\}$. The functors*

$$QCoh(\mathcal{O}_S) \longrightarrow Mod((Sch/S)_\tau, \mathcal{O}) \quad and \quad QCoh(\mathcal{O}_S) \longrightarrow Mod(S_\tau, \mathcal{O})$$

defined by the rule $\mathcal{F} \mapsto \mathcal{F}^a$ seen in Proposition 34.7.11 are

(1) *fully faithful,*
(2) *compatible with direct sums,*
(3) *compatible with colimits,*
(4) *right exact,*
(5) *exact as a functor $QCoh(\mathcal{O}_S) \to Mod(S_\tau, \mathcal{O})$,*
(6) **not** *exact as a functor $QCoh(\mathcal{O}_S) \to Mod((Sch/S)_\tau, \mathcal{O})$ in general,*
(7) *given two quasi-coherent $\mathcal{O}_S$-modules $\mathcal{F}$, $\mathcal{G}$ we have $(\mathcal{F} \otimes_{\mathcal{O}_S} \mathcal{G})^a = \mathcal{F}^a \otimes_{\mathcal{O}} \mathcal{G}^a$,*
(8) *given two quasi-coherent $\mathcal{O}_S$-modules $\mathcal{F}$, $\mathcal{G}$ such that $\mathcal{F}$ is of finite presentation we have $(\mathcal{H}om_{\mathcal{O}_S}(\mathcal{F}, \mathcal{G}))^a = \mathcal{H}om_{\mathcal{O}}(\mathcal{F}^a, \mathcal{G}^a)$, and*
(9) *given a short exact sequence $0 \to \mathcal{F}_1^a \to \mathcal{E} \to \mathcal{F}_2^a \to 0$ of $\mathcal{O}$-modules then $\mathcal{E}$ is quasi-coherent[5], i.e., $\mathcal{E}$ is in the essential image of the functor.*

Proof. Part (1) we saw in Proposition 34.7.11.

We have seen in Schemes, Section 25.24 that a colimit of quasi-coherent sheaves on a scheme is a quasi-coherent sheaf. Moreover, in Remark 34.7.6 we saw that $\mathcal{F} \mapsto \mathcal{F}^a$ is the pullback functor for a morphism of ringed sites, hence commutes with all colimits, see Modules on Sites, Lemma 18.14.3. Thus (3) and its special case (3) hold.

[4]The list is: free, finite free, generated by global sections, generated by r global sections, generated by finitely many global sections, having a global presentation, having a global finite presentation, locally free, finite locally free, locally generated by sections, locally generated by r sections, finite type, of finite presentation, coherent, or flat.

[5]Warning: This is misleading. See part (6).

This also shows that the functor is right exact (i.e., commutes with finite colimits), hence (4).

The functor $QCoh(\mathcal{O}_S) \to QCoh(S_{étale}, \mathcal{O})$, $\mathcal{F} \mapsto \mathcal{F}^a$ is left exact because an étale morphism is flat, see Morphisms, Lemma 28.37.12. This proves (5).

To see (6), suppose that $S = \mathrm{Spec}(\mathbf{Z})$. Then $2 : \mathcal{O}_S \to \mathcal{O}_S$ is injective but the associated map of $\mathcal{O}$-modules on $(Sch/S)_\tau$ isn't injective because $2 : \mathbf{F}_2 \to \mathbf{F}_2$ isn't injective and $\mathrm{Spec}(\mathbf{F}_2)$ is an object of $(Sch/S)_\tau$.

We omit the proofs of (7) and (8).

Let $0 \to \mathcal{F}_1^a \to \mathcal{E} \to \mathcal{F}_2^a \to 0$ be a short exact sequence of $\mathcal{O}$-modules with $\mathcal{F}_1$ and $\mathcal{F}_2$ quasi-coherent on S. Consider the restriction

$$0 \to \mathcal{F}_1 \to \mathcal{E}|_{S_{Zar}} \to \mathcal{F}_2$$

to S_{Zar}. By Proposition 34.7.10 we see that on any affine $U \subset S$ we have $H^1(U, \mathcal{F}_1^a) = H^1(U, \mathcal{F}_1) = 0$. Hence the sequence above is also exact on the right. By Schemes, Section 25.24 we conclude that $\mathcal{F} = \mathcal{E}|_{S_{Zar}}$ is quasi-coherent. Thus we obtain a commutative diagram

$$\begin{array}{ccccccccc} & & \mathcal{F}_1^a & \longrightarrow & \mathcal{F}^a & \longrightarrow & \mathcal{F}_2^a & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & \mathcal{F}_1^a & \longrightarrow & \mathcal{E} & \longrightarrow & \mathcal{F}_2^a & \longrightarrow & 0 \end{array}$$

To finish the proof it suffices to show that the top row is also right exact. To do this, denote once more $U = \mathrm{Spec}(A) \subset S$ an affine open of S. We have seen above that $0 \to \mathcal{F}_1(U) \to \mathcal{E}(U) \to \mathcal{F}_2(U) \to 0$ is exact. For any affine scheme V/U, $V = \mathrm{Spec}(B)$ the map $\mathcal{F}_1^a(V) \to \mathcal{E}(V)$ is injective. We have $\mathcal{F}_1^a(V) = \mathcal{F}_1(U) \otimes_A B$ by definition. The injection $\mathcal{F}_1^a(V) \to \mathcal{E}(V)$ factors as

$$\mathcal{F}_1(U) \otimes_A B \to \mathcal{E}(U) \otimes_A B \to \mathcal{E}(U)$$

Considering A-algebras B of the form $B = A \oplus M$ we see that $\mathcal{F}_1(U) \to \mathcal{E}(U)$ is universally injective (see Algebra, Definition 10.77.1). Since $\mathcal{E}(U) = \mathcal{F}(U)$ we conclude that $\mathcal{F}_1 \to \mathcal{F}$ remains injective after any base change, or equivalently that $\mathcal{F}_1^a \to \mathcal{F}^a$ is injective. □

Proposition 34.7.14. *Let $f : T \to S$ be a morphism of schemes.*

1. *The equivalences of categories of Proposition 34.7.11 are compatible with pullback. More precisely, we have $f^*(\mathcal{G}^a) = (f^*\mathcal{G})^a$ for any quasi-coherent sheaf $\mathcal{G}$ on S.*
2. *The equivalences of categories of Proposition 34.7.11 part (1) are* **not** *compatible with pushforward in general.*
3. *If f is quasi-compact and quasi-separated, and $\tau \in \{Zariski, étale\}$ then f_* and $f_{small,*}$ preserve quasi-coherent sheaves and the diagram*

$$\begin{array}{ccc} QCoh(\mathcal{O}_T) & \xrightarrow{f_*} & QCoh(\mathcal{O}_S) \\ {\scriptstyle \mathcal{F} \mapsto \mathcal{F}^a}\downarrow & & \downarrow{\scriptstyle \mathcal{G} \mapsto \mathcal{G}^a} \\ QCoh(T_\tau, \mathcal{O}) & \xrightarrow{f_{small,*}} & QCoh(S_\tau, \mathcal{O}) \end{array}$$

is commutative, i.e., $f_{small,*}(\mathcal{F}^a) = (f_*\mathcal{F})^a$.

Proof. Part (1) follows from the discussion in Remark 34.7.6. Part (2) is just a warning, and can be explained in the following way: First the statement cannot be made precise since f_* does not transform quasi-coherent sheaves into quasi-coherent sheaves in general. Even if this is the case for f (and any base change of f), then the compatibility over the big sites would mean that formation of $f_*\mathcal{F}$ commutes with any base change, which does not hold in general. An explicit example is the quasi-compact open immersion $j : X = \mathbf{A}^2_k \setminus \{0\} \to \mathbf{A}^2_k = Y$ where k is a field. We have $j_*\mathcal{O}_X = \mathcal{O}_Y$ but after base change to $\mathrm{Spec}(k)$ by the 0 map we see that the pushforward is zero.

Let us prove (3) in case $\tau = \textit{étale}$. Note that f, and any base change of f, transforms quasi-coherent sheaves into quasi-coherent sheaves, see Schemes, Lemma 25.24.1. The equality $f_{small,*}(\mathcal{F}^a) = (f_*\mathcal{F})^a$ means that for any étale morphism $g : U \to S$ we have $\Gamma(U, g^*f_*\mathcal{F}) = \Gamma(U \times_S T, (g')^*\mathcal{F})$ where $g' : U \times_S T \to T$ is the projection. This is true by Cohomology of Schemes, Lemma 29.5.2. □

Lemma 34.7.15. *Let $f : T \to S$ be a quasi-compact and quasi-separated morphism of schemes. Let $\mathcal{F}$ be a quasi-coherent sheaf on T. For either the étale or Zariski topology, there are canonical isomorphisms $R^i f_{small,*}(\mathcal{F}^a) = (R^i f_*\mathcal{F})^a$.*

Proof. We prove this for the étale topology; we omit the proof in the case of the Zariski topology. By Cohomology of Schemes, Lemma 29.4.4 the sheaves $R^i f_*\mathcal{F}$ are quasi-coherent so that the assertion makes sense. The sheaf $R^i f_{small,*}\mathcal{F}^a$ is the sheaf associated to the presheaf

$$U \longmapsto H^i(U \times_S T, \mathcal{F}^a)$$

where $g : U \to S$ is an object of $S_{\textit{étale}}$, see Cohomology on Sites, Lemma 21.8.4. By our conventions the right hand side is the étale cohomology of the restriction of $\mathcal{F}^a$ to the localization $T_{\textit{étale}}/U \times_S T$ which equals $(U \times_S T)_{\textit{étale}}$. By Proposition 34.7.10 this is presheaf the same as the presheaf

$$U \longmapsto H^i(U \times_S T, (g')^*\mathcal{F}),$$

where $g' : U \times_S T \to T$ is the projection. If U is affine then this is the same as $H^0(U, R^i f'_*(g')^*\mathcal{F})$, see Cohomology of Schemes, Lemma 29.4.5. By Cohomology of Schemes, Lemma 29.5.2 this is equal to $H^0(U, g^*R^i f_*\mathcal{F})$ which is the value of $(R^i f_*\mathcal{F})^a$ on U. Thus the values of the sheaves of modules $R^i f_{small,*}(\mathcal{F}^a)$ and $(R^i f_*\mathcal{F})^a$ on every affine object of $S_{\textit{étale}}$ are canonically isomorphic which implies they are canonically isomorphic. □

The results in this section say there is virtually no difference between quasi-coherent sheaves on S and quasi-coherent sheaves on any of the sites associated to S in the chapter on topologies. Hence one often sees statements on quasi-coherent sheaves formulated in either language, without restatements in the other.

34.8. Parasitic modules

Parasitic modules are those which are zero when restricted to schemes flat over the base scheme. Here is the formal definition.

Definition 34.8.1. Let S be a scheme. Let $\tau \in \{Zar, \textit{étale}, smooth, syntomic, fppf\}$. Let $\mathcal{F}$ be a presheaf of $\mathcal{O}$-modules on $(Sch/S)_\tau$.

(1) $\mathcal{F}$ is called *parasitic*[6] if for every flat morphism $U \to S$ we have $\mathcal{F}(U) = 0$.
(2) $\mathcal{F}$ is called *parasitic for the τ-topology* if for every τ-covering $\{U_i \to S\}_{i \in I}$ we have $\mathcal{F}(U_i) = 0$ for all i.

If $\tau = fppf$ this means that $\mathcal{F}|_{U_{Zar}} = 0$ whenever $U \to S$ is flat and locally of finite presentation; similar for the other cases.

Lemma 34.8.2. *Let S be a scheme. Let $\tau \in \{Zar, étale, smooth, syntomic, fppf\}$. Let $\mathcal{G}$ be a presheaf of $\mathcal{O}$-modules on $(Sch/S)_\tau$.*

(1) *If $\mathcal{G}$ is parasitic for the τ-topology, then $H^p_\tau(U, \mathcal{G}) = 0$ for every U open in S, resp. étale over S, resp. smooth over S, resp. syntomic over S, resp. flat and locally of finite presentation over S.*
(2) *If $\mathcal{G}$ is parasitic then $H^p_\tau(U, \mathcal{G}) = 0$ for every U flat over S.*

Proof. Proof in case $\tau = fppf$; the other cases are proved in the exact same way. The assumption means that $\mathcal{G}(U) = 0$ for any $U \to S$ flat and locally of finite presentation. Apply Cohomology on Sites, Lemma 21.11.8 to the subset $\mathcal{B} \subset \mathrm{Ob}((Sch/S)_{fppf})$ consisting of $U \to S$ flat and locally of finite presentation and the collection Cov of all fppf coverings of elements of $\mathcal{B}$. □

Lemma 34.8.3. *Let $f : T \to S$ be a morphism of schemes. For any parasitic $\mathcal{O}$-module on $(Sch/T)_\tau$ the pushforward $f_*\mathcal{F}$ and the higher direct images $R^i f_*\mathcal{F}$ are parasitic $\mathcal{O}$-modules on $(Sch/S)_\tau$.*

Proof. Recall that $R^i f_*\mathcal{F}$ is the sheaf associated to the presheaf

$$U \mapsto H^i((Sch/U \times_S T)_\tau, \mathcal{F})$$

see Cohomology on Sites, Lemma 21.8.4. If $U \to S$ is flat, then $U \times_S T \to T$ is flat as a base change. Hence the displayed group is zero by Lemma 34.8.2. If $\{U_i \to U\}$ is a τ-covering then $U_i \times_S T \to T$ is also flat. Hence it is clear that the sheafification of the displayed presheaf is zero on schemes U flat over S. □

Lemma 34.8.4. *Let S be a scheme. Let $\tau \in \{Zar, étale\}$. Let $\mathcal{G}$ be a sheaf of $\mathcal{O}$-modules on $(Sch/S)_{fppf}$ such that*

(1) *$\mathcal{G}|_{S_\tau}$ is quasi-coherent, and*
(2) *for every flat, locally finitely presented morphism $g : U \to S$ the canonical map $g^*_{\tau,small}(\mathcal{G}|_{S_\tau}) \to \mathcal{G}|_{U_\tau}$ is an isomorphism.*

Then $H^p(U, \mathcal{G}) = H^p(U, \mathcal{G}|_{U_\tau})$ for every U flat and locally of finite presentation over S.

Proof. Let $\mathcal{F}$ be the pullback of $\mathcal{G}|_{S_\tau}$ to the big fppf site $(Sch/S)_{fppf}$. Note that $\mathcal{F}$ is quasi-coherent. There is a canonical comparison map $\varphi : \mathcal{F} \to \mathcal{G}$ which by assumptions (1) and (2) induces an isomorphism $\mathcal{F}|_{U_\tau} \to \mathcal{G}|_{U_\tau}$ for all $g : U \to S$ flat and locally of finite presentation. Hence in the short exact sequences

$$0 \to \mathrm{Ker}(\varphi) \to \mathcal{F} \to \mathrm{Im}(\varphi) \to 0$$

and

$$0 \to \mathrm{Im}(\varphi) \to \mathcal{G} \to \mathrm{Coker}(\varphi) \to 0$$

the sheaves $\mathrm{Ker}(\varphi)$ and $\mathrm{Coker}(\varphi)$ are parasitic for the fppf topology. By Lemma 34.8.2 we conclude that $H^p(U, \mathcal{F}) \to H^p(U, \mathcal{G})$ is an isomorphism for $g : U \to S$

[6] This may be nonstandard notation.

flat and locally of finite presentation. Since the result holds for $\mathcal{F}$ by Proposition 34.7.10 we win. □

34.9. Fpqc coverings are universal effective epimorphisms

We apply the material above to prove an interesting result, namely Lemma 34.9.3. By Sites, Section 7.13 this lemma implies that the representable presheaves on any of the sites $(Sch/S)_\tau$ are sheaves for $\tau \in \{Zariski, fppf, étale, smooth, syntomic\}$. First we prove a helper lemma.

Lemma 34.9.1. *For a scheme X denote $|X|$ the underlying set. Let $f : X \to S$ be a morphism of schemes. Then*

$$|X \times_S X| \to |X| \times_{|S|} |X|$$

is surjective.

Proof. Follows immediately from the description of points on the fibre product in Schemes, Lemma 25.17.5. □

Lemma 34.9.2. *Let $\{f_i : T_i \to T\}_{i \in I}$ be a fpqc covering. Suppose that for each i we have an open subset $W_i \subset T_i$ such that for all $i, j \in I$ we have $pr_0^{-1}(W_i) = pr_1^{-1}(W_j)$ as open subsets of $T_i \times_T T_j$. Then there exists a unique open subset $W \subset T$ such that $W_i = f_i^{-1}(W)$ for each i.*

Proof. Apply Lemma 34.9.1 to the map $\coprod_{i \in I} T_i \to T$. It implies there exists a subset $W \subset T$ such that $W_i = f_i^{-1}(W)$ for each i, namely $W = \bigcup f_i(W_i)$. To see that W is open we may work Zariski locally on T. Hence we may assume that T is affine. Using the definition of a fpqc covering, this reduces us to the case where $\{f_i : T_i \to T\}$ is a standard fpqc covering. In this case we may apply Morphisms, Lemma 28.26.10 to the morphism $\coprod T_i \to T$ to conclude that W is open. □

Lemma 34.9.3. *Let $\{T_i \to T\}$ be an fpqc covering, see Topologies, Definition 33.8.1. Then $\{T_i \to T\}$ is a universal effective epimorphism in the category of schemes, see Sites, Definition 7.13.1. In other words, every representable functor on the category of schemes satisfies the sheaf condition for the fpqc topology, see Topologies, Definition 33.8.12.*

Proof. Let S be a scheme. We have to show the following: Given morphisms $\varphi_i : T_i \to S$ such that $\varphi_i|_{T_i \times_T T_j} = \varphi_j|_{T_i \times_T T_j}$ there exists a unique morphism $T \to S$ which restricts to φ_i on each T_i. In other words, we have to show that the functor $h_S = \mathrm{Mor}_{Sch}(-, S)$ satisfies the sheaf property for the fpqc topology.

Thus Topologies, Lemma 33.8.13 reduces us to the case of a Zariski covering and a covering $\{\mathrm{Spec}(A) \to \mathrm{Spec}(R)\}$ with $R \to A$ faithfully flat. The case of a Zariski covering follows from Schemes, Lemma 25.14.1.

Suppose that $R \to A$ is a faithfully flat ring map. Denote $\pi : \mathrm{Spec}(A) \to \mathrm{Spec}(R)$ the corresponding morphism of schemes. It is surjective and flat. Let $f : \mathrm{Spec}(A) \to S$ be a morphism such that $f \circ \mathrm{pr}_1 = f \circ \mathrm{pr}_2$ as maps $\mathrm{Spec}(A \otimes_R A) \to S$. By Lemma 34.9.1 we see that as a map on the underlying sets f is of the form $f = g \circ \pi$ for some (set theoretic) map $g : \mathrm{Spec}(R) \to S$. By Morphisms, Lemma 28.26.10 and the fact that f is continuous we see that g is continuous.

Pick $x \in \mathrm{Spec}(R)$. Choose $U \subset S$ affine open containing $g(x)$. Say $U = \mathrm{Spec}(B)$. By the above we may choose an $r \in R$ such that $x \in D(r) \subset g^{-1}(U)$. The

restriction of f to $\pi^{-1}(D(r))$ into U corresponds to a ring map $B \to A_r$. The two induced ring maps $B \to A_r \otimes_{R_r} A_r = (A \otimes_R A)_r$ are equal by assumption on f. Note that $R_r \to A_r$ is faithfully flat. By Lemma 34.3.6 the equalizer of the two arrows $A_r \to A_r \otimes_{R_r} A_r$ is R_r. We conclude that $B \to A_r$ factors uniquely through a map $B \to R_r$. This map in turn gives a morphism of schemes $D(r) \to U \to S$, see Schemes, Lemma 25.6.4.

What have we proved so far? We have shown that for any prime $\mathfrak{p} \subset R$, there exists a standard affine open $D(r) \subset \mathrm{Spec}(R)$ such that the morphism $f|_{\pi^{-1}(D(r))} : \pi^{-1}(D(r)) \to S$ factors uniquely though some morphism of schemes $D(r) \to S$. We omit the verification that these morphisms glue to the desired morphism $\mathrm{Spec}(R) \to S$. □

34.10. Descent of finiteness properties of morphisms

Another application of flat descent for modules is the following amusing and useful result. There is an algebraic version and a scheme theoretic version. (The "Noetherian" reader should consult Lemma 34.10.2 instead of the next lemma.)

Lemma 34.10.1. *Let $R \to A \to B$ be ring maps. Assume $R \to B$ is of finite presentation and $A \to B$ faithfully flat and of finite presentation. Then $R \to A$ is of finite presentation.*

Proof. Consider the algebra $C = B \otimes_A B$ together with the pair of maps $p, q : B \to C$ given by $p(b) = b \otimes 1$ and $q(b) = 1 \otimes b$. Of course the two compositions $A \to B \to C$ are the same. Note that as $p : B \to C$ is flat and of finite presentation (base change of $A \to B$), the ring map $R \to C$ is of finite presentation (as the composite of $R \to B \to C$).

We are going to use the criterion Algebra, Lemma 10.121.2 to show that $R \to A$ is of finite presentation. Let S be any R-algebra, and suppose that $S = \mathrm{colim}_{\lambda \in \Lambda} S_\lambda$ is written as a directed colimit of R-algebras. Let $A \to S$ be an R-algebra homomorphism. We have to show that $A \to S$ factors through one of the S_λ. Consider the rings $B' = S \otimes_A B$ and $C' = S \otimes_A C = B' \otimes_S B'$. As B is faithfully flat of finite presentation over A, also B' is faithfully flat of finite presentation over S. By Algebra, Lemma 10.154.1 part (2) applied to the pair $(S \to B', B')$ and the system (S_λ) there exists a $\lambda_0 \in \Lambda$ and a flat, finitely presented S_{λ_0}-algebra B_{λ_0} such that $B' = S \otimes_{S_{\lambda_0}} B_{\lambda_0}$. For $\lambda \geq \lambda_0$ set $B_\lambda = S_\lambda \otimes_{S_{\lambda_0}} B_{\lambda_0}$ and $C_\lambda = B_\lambda \otimes_{S_\lambda} B_\lambda$.

We interrupt the flow of the argument to show that $S_\lambda \to B_\lambda$ is faithfully flat for λ large enough. (This should really be a separate lemma somewhere else, maybe in the chapter on limits.) Since $\mathrm{Spec}(B_{\lambda_0}) \to \mathrm{Spec}(S_{\lambda_0})$ is flat and of finite presentation it is open (see Morphisms, Lemma 28.26.9). Let $I \subset S_{\lambda_0}$ be an ideal such that $V(I) \subset \mathrm{Spec}(S_{\lambda_0})$ is the complement of the image. Note that formation of the image commutes with base change. Hence, since $\mathrm{Spec}(B') \to \mathrm{Spec}(S)$ is surjective, and $B' = B_{\lambda_0} \otimes_{S_{\lambda_0}} S$ we see that $IS = S$. Thus for some $\lambda \geq \lambda_0$ we have $IS_\lambda = S_\lambda$. For this and all greater λ the morphism $\mathrm{Spec}(B_\lambda) \to \mathrm{Spec}(S_\lambda)$ is surjective.

By analogy with the notation in the first paragraph of the proof denote $p_\lambda, q_\lambda : B_\lambda \to C_\lambda$ the two canonical maps. Then $B' = \mathrm{colim}_{\lambda \geq \lambda_0} B_\lambda$ and $C' = \mathrm{colim}_{\lambda \geq \lambda_0} C_\lambda$. Since B and C are finitely presented over R there exist (by Algebra, Lemma 10.121.2

applied several times) a $\lambda \geq \lambda_0$ and an R-algebra maps $B \to B_\lambda$, $C \to C_\lambda$ such that the diagram

$$\begin{array}{ccc} C & \longrightarrow & C_\lambda \\ {\scriptstyle p}\uparrow\uparrow{\scriptstyle q} & & {\scriptstyle p_\lambda}\uparrow\uparrow{\scriptstyle q_\lambda} \\ B & \longrightarrow & B_\lambda \end{array}$$

is commutative. OK, and this means that $A \to B \to B_\lambda$ maps into the equalizer of p_λ and q_λ. By By Lemma 34.3.6 we see that S_λ is the equalizer of p_λ and q_λ. Thus we get the desired ring map $A \to S_\lambda$ and we win. □

Here is an easier version of this dealing with the property of being of finite type.

Lemma 34.10.2. *Let $R \to A \to B$ be ring maps. Assume $R \to B$ is of finite type and $A \to B$ faithfully flat and of finite presentation. Then $R \to A$ is of finite type.*

Proof. By Algebra, Lemma 10.154.2 there exists a commutative diagram

$$\begin{array}{ccccc} R & \longrightarrow & A_0 & \longrightarrow & B_0 \\ \| & & \downarrow & & \downarrow \\ R & \longrightarrow & A & \longrightarrow & B \end{array}$$

with $R \to A_0$ of finite presentation, $A_0 \to B_0$ faithfully flat of finite presentation and $B = A \otimes_{A_0} B_0$. Since $R \to B$ is of finite type by assumption, we may add some elements to A_0 and assume that the map $B_0 \to B$ is surjective! In this case, since $A_0 \to B_0$ is faithfully flat, we see that as

$$(A_0 \to A) \otimes_{A_0} B_0 \cong (B_0 \to B)$$

is surjective, also $A_0 \to A$ is surjective. Hence we win. □

Lemma 34.10.3. *Let*

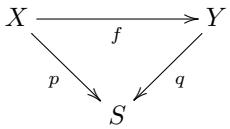
$$\begin{array}{ccc} X & \xrightarrow{f} & Y \\ & {\scriptstyle p}\searrow \quad \swarrow{\scriptstyle q} & \\ & S & \end{array}$$

be a commutative diagram of morphisms of schemes. Assume that f is surjective, flat and locally of finite presentation and assume that p is locally of finite presentation (resp. locally of finite type). Then q is locally of finite presentation (resp. locally of finite type).

Proof. The problem is local on S and Y. Hence we may assume that S and Y are affine. Since f is flat and locally of finite presentation, we see that f is open (Morphisms, Lemma 28.26.9). Hence, since Y is quasi-compact, there exist finitely many affine opens $X_i \subset X$ such that $Y = \bigcup f(X_i)$. Clearly we may replace X by $\coprod X_i$, and hence we may assume X is affine as well. In this case the lemma is equivalent to Lemma 34.10.1 (resp. Lemma 34.10.2) above. □

We use this to improve some of the results on morphisms obtained earlier.

Lemma 34.10.4. *Let*



be a commutative diagram of morphisms of schemes. Assume that

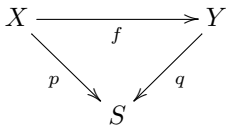
(1) *f is surjective, and syntomic (resp. smooth, resp. étale),*
(2) *p is syntomic (resp. smooth, resp. étale).*

Then q is syntomic (resp. smooth, resp. étale).

Proof. Combine Morphisms, Lemmas 28.32.16, 28.35.19, and 28.37.19 with Lemma 34.10.3 above. □

Actually we can strengthen this result as follows.

Lemma 34.10.5. *Let*



be a commutative diagram of morphisms of schemes. Assume that

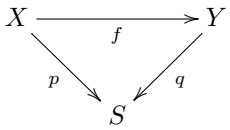
(1) *f is surjective, flat, and locally of finite presentation,*
(2) *p is smooth (resp. étale).*

Then q is smooth (resp. étale).

Proof. Assume (1) and that p is smooth. By Lemma 34.10.3 we see that q is locally of finite presentation. By Morphisms, Lemma 28.26.11 we see that q is flat. Hence now it suffices to show that the fibres of q are smooth, see Morphisms, Lemma 28.35.3. Apply Varieties, Lemma 32.15.8 to the flat surjective morphisms $X_s \to Y_s$ for $s \in S$ to conclude. We omit the proof of the étale case. □

Remark 34.10.6. With the assumptions (1) and p smooth in Lemma 34.10.5 it is not automatically the case that $X \to Y$ is smooth. A counter example is $S = \mathrm{Spec}(k)$, $X = \mathrm{Spec}(k[s])$, $Y = \mathrm{Spec}(k[t])$ and f given by $t \mapsto s^2$. But see also Lemma 34.10.7 for some information on the structure of f.

Lemma 34.10.7. *Let*



be a commutative diagram of morphisms of schemes. Assume that

(1) *f is surjective, flat, and locally of finite presentation,*
(2) *p is syntomic.*

Then both q and f are syntomic.

Proof. By Lemma 34.10.3 we see that q is of finite presentation. By Morphisms, Lemma 28.26.11 we see that q is flat. By Morphisms, Lemma 28.32.10 it now suffices to show that the local rings of the fibres of $Y \to S$ and the fibres of $X \to Y$ are local complete intersection rings. To do this we may take the fibre of $X \to Y \to S$

at a point $s \in S$, i.e., we may assume S is the spectrum of a field. Pick a point $x \in X$ with image $y \in Y$ and consider the ring map

$$\mathcal{O}_{Y,y} \longrightarrow \mathcal{O}_{X,x}$$

This is a flat local homomorphism of local Noetherian rings. The local ring $\mathcal{O}_{X,x}$ is a complete intersection. Thus may use Avramov's result, see Divided Power Algebra, Lemma 23.8.9, to conclude that both $\mathcal{O}_{Y,y}$ and $\mathcal{O}_{X,x}/\mathfrak{m}_y\mathcal{O}_{X,x}$ are complete intersection rings. □

The following type of lemma is occasionally useful.

Lemma 34.10.8. *Let $X \to Y \to Z$ be morphism of schemes. Let P be one of the following properties of morphisms of schemes: flat, locally finite type, locally finite presentation. Assume that $X \to Z$ has P and that $\{X \to Y\}$ can be refined by an fppf covering of Y. Then $Y \to Z$ is P.*

Proof. Let $\mathrm{Spec}(C) \subset Z$ be an affine open and let $\mathrm{Spec}(B) \subset Y$ be an affine open which maps into $\mathrm{Spec}(C)$. The assumption on $X \to Y$ implies we can find a standard affine fppf covering $\{\mathrm{Spec}(B_j) \to \mathrm{Spec}(B)\}$ and lifts $x_j : \mathrm{Spec}(B_j) \to X$. Since $\mathrm{Spec}(B_j)$ is quasi-compact we can find finitely many affine opens $\mathrm{Spec}(A_i) \subset X$ lying over $\mathrm{Spec}(B)$ such that the image of each x_j is contained in the union $\bigcup \mathrm{Spec}(A_i)$. Hence after replacing each $\mathrm{Spec}(B_j)$ by a standard affine Zariski coverings of itself we may assume we have a standard affine fppf covering $\{\mathrm{Spec}(B_i) \to \mathrm{Spec}(B)\}$ such that each $\mathrm{Spec}(B_i) \to Y$ factors through an affine open $\mathrm{Spec}(A_i) \subset X$ lying over $\mathrm{Spec}(B)$. In other words, we have ring maps $C \to B \to A_i \to B_i$ for each i. Note that we can also consider

$$C \to B \to A = \prod A_i \to B' = \prod B_i$$

and that the ring map $B \to \prod B_i$ is faithfully flat and of finite presentation.

The case $P = flat$. In this case we know that $C \to A$ is flat and we have to prove that $C \to B$ is flat. Suppose that $N \to N' \to N''$ is an exact sequence of C-modules. We want to show that $N \otimes_C B \to N' \otimes_C B \to N'' \otimes_C B$ is exact. Let H be its cohomology and let H' be the cohomology of $N \otimes_C B' \to N' \otimes_C B' \to N'' \otimes_C B'$. As $B \to B'$ is flat we know that $H' = H \otimes_B B'$. On the other hand $N \otimes_C A \to N' \otimes_C A \to N'' \otimes_C A$ is exact hence has zero cohomology. Hence the map $H \to H'$ is zero (as it factors through the zero module). Thus $H' = 0$. As $B \to B'$ is faithfully flat we conclude that $H = 0$ as desired.

The case $P =$ *locally finite type*. In this case we know that $C \to A$ is of finite type and we have to prove that $C \to B$ is of finite type. Because $B \to B'$ is of finite presentation (hence of finite type) we see that $A \to B'$ is of finite type, see Algebra, Lemma 10.6.2. Therefore $C \to B'$ is of finite type and we conclude by Lemma 34.10.2.

The case $P =$ *locally finite presentation*. In this case we know that $C \to A$ is of finite presentation and we have to prove that $C \to B$ is of finite presentation. Because $B \to B'$ is of finite presentation and $B \to A$ of finite type we see that $A \to B'$ is of finite presentation, see Algebra, Lemma 10.6.2. Therefore $C \to B'$ is of finite presentation and we conclude by Lemma 34.10.1. □

34.11. Local properties of schemes

It often happens one can prove the members of a covering of a scheme have a certain property. In many cases this implies the scheme has the property too. For example, if S is a scheme, and $f : S' \to S$ is a surjective flat morphism such that S' is a reduced scheme, then S is reduced. You can prove this by looking at local rings and using Algebra, Lemma 10.150.2. We say that the property of being reduced *descends through flat surjective morphisms*. Some results of this type are collected in Algebra, Section 10.150.

On the other hand, there are examples of surjective flat morphisms $f : S' \to S$ with S reduced and S' not, for example the morphism $\mathrm{Spec}(k[x]/(x^2)) \to \mathrm{Spec}(k)$. Hence the property of being reduced does not *ascend along flat morphisms*. Having infinite residue fields is a property which does ascend along flat morphisms (but does not descend along surjective flat morphisms of course). Some results of this type are collected in Algebra, Section 10.149.

Finally, we say that a property is *local for the flat topology* if it ascends along flat morphisms and descends along flat surjective morphisms. A somewhat silly example is the property of having residue fields of a given characteristic. To be more precise, and to tie this in with the various topologies on schemes, we make the following formal definition.

Definition 34.11.1. Let $\mathcal{P}$ be a property of schemes. Let $\tau \in \{fpqc, fppf, syntomic, smooth, étale, Zariski\}$. We say $\mathcal{P}$ is *local in the τ-topology* if for any τ-covering $\{S_i \to S\}_{i \in I}$ (see Topologies, Section 33.2) we have

$$S \text{ has } \mathcal{P} \Leftrightarrow \text{each } S_i \text{ has } \mathcal{P}.$$

To be sure, since isomorphisms are always coverings we see (or require) that property $\mathcal{P}$ holds for S if and only if it holds for any scheme S' isomorphic to S. In fact, if $\tau = fpqc, fppf, syntomic, smooth, étale$, or $Zariski$, then if S has $\mathcal{P}$ and $S' \to S$ is flat, flat and locally of finite presentation, syntomic, smooth, étale, or an open immersion, then S' has $\mathcal{P}$. This is true because we can always extend $\{S' \to S\}$ to a τ-covering.

We have the following implications: $\mathcal{P}$ is local in the fpqc topology $\Rightarrow$ $\mathcal{P}$ is local in the fppf topology $\Rightarrow$ $\mathcal{P}$ is local in the syntomic topology $\Rightarrow$ $\mathcal{P}$ is local in the smooth topology $\Rightarrow$ $\mathcal{P}$ is local in the étale topology $\Rightarrow$ $\mathcal{P}$ is local in the Zariski topology. This follows from Topologies, Lemmas 33.4.2, 33.5.2, 33.6.2, 33.7.2, and 33.8.6.

Lemma 34.11.2. *Let $\mathcal{P}$ be a property of schemes. Let $\tau \in \{fpqc, fppf, étale, smooth, syntomic\}$. Assume that*

1. *the property is local in the Zariski topology,*
2. *for any morphism of affine schemes $S' \to S$ which is flat, flat of finite presentation, étale, smooth or syntomic depending on whether τ is fpqc, fppf, étale, smooth, or syntomic, property $\mathcal{P}$ holds for S' if property $\mathcal{P}$ holds for S, and*
3. *for any surjective morphism of affine schemes $S' \to S$ which is flat, flat of finite presentation, étale, smooth or syntomic depending on whether τ is fpqc, fppf, étale, smooth, or syntomic, property $\mathcal{P}$ holds for S if property $\mathcal{P}$ holds for S'.*

Then $\mathcal{P}$ is τ local on the base.

Proof. This follows almost immediately from the definition of a τ-covering, see Topologies, Definition 33.8.1 33.7.1 33.4.1 33.5.1, or 33.6.1 and Topologies, Lemma 33.8.8, 33.7.4, 33.4.4, 33.5.4, or 33.6.4. Details omitted. □

Remark 34.11.3. In Lemma 34.11.2 above if $\tau = smooth$ then in condition (3) we may assume that the morphism is a (surjective) standard smooth morphism. Similarly, when $\tau = syntomic$ or $\tau = étale$.

34.12. Properties of schemes local in the fppf topology

In this section we find some properties of schemes which are local on the base in the fppf topology.

Lemma 34.12.1. *The property $\mathcal{P}(S) =$ "S is locally Noetherian" is local in the fppf topology.*

Proof. We will use Lemma 34.11.2. First we note that "being locally Noetherian" is local in the Zariski topology. This is clear from the definition, see Properties, Definition 27.5.1. Next, we show that if $S' \to S$ is a flat, finitely presented morphism of affines and S is locally Noetherian, then S' is locally Noetherian. This is Morphisms, Lemma 28.16.6. Finally, we have to show that if $S' \to S$ is a surjective flat, finitely presented morphism of affines and S' is locally Noetherian, then S is locally Noetherian. This follows from Algebra, Lemma 10.150.1. Thus (1), (2) and (3) of Lemma 34.11.2 hold and we win. □

Lemma 34.12.2. *The property $\mathcal{P}(S) =$ "S is Jacobson" is local in the fppf topology.*

Proof. We will use Lemma 34.11.2. First we note that "being Jacobson" is local in the Zariski topology. This is Properties, Lemma 27.6.3. Next, we show that if $S' \to S$ is a flat, finitely presented morphism of affines and S is Jacobson, then S' is Jacobson. This is Morphisms, Lemma 28.17.9. Finally, we have to show that if $f : S' \to S$ is a surjective flat, finitely presented morphism of affines and S' is Jacobson, then S is Jacobson. Say $S = \mathrm{Spec}(A)$ and $S' = \mathrm{Spec}(B)$ and $S' \to S$ given by $A \to B$. Then $A \to B$ is finitely presented and faithfully flat. Moreover, the ring B is Jacobson, see Properties, Lemma 27.6.3.

By Algebra, Lemma 10.154.6 there exists a diagram

$$\begin{array}{ccc} B & \longrightarrow & B' \\ & \nwarrow \quad \nearrow & \\ & A & \end{array}$$

with $A \to B'$ finitely presented, faithfully flat and quasi-finite. In particular, $B \to B'$ is finite type, and we see from Algebra, Proposition 10.32.18 that B' is Jacobson. Hence we may assume that $A \to B$ is quasi-finite as well as faithfully flat and of finite presentation.

Assume A is not Jacobson to get a contradiction. According to Algebra, Lemma 10.32.5 there exists a nonmaximal prime $\mathfrak{p} \subset A$ and an element $f \in A$, $f \notin \mathfrak{p}$ such that $V(\mathfrak{p}) \cap D(f) = \{\mathfrak{p}\}$.

This leads to a contradiction as follows. First let $\mathfrak{p} \subset \mathfrak{m}$ be a maximal ideal of A. Pick a prime $\mathfrak{m}' \subset B$ lying over $\mathfrak{m}$ (exists because $A \to B$ is faithfully flat, see Algebra, Lemma 10.36.15). As $A \to B$ is flat, by going down see Algebra, Lemma

10.36.17, we can find a prime $\mathfrak{q} \subset \mathfrak{m}'$ lying over $\mathfrak{p}$. In particular we see that $\mathfrak{q}$ is not maximal. Hence according to Algebra, Lemma 10.32.5 again the set $V(\mathfrak{q}) \cap D(f)$ is infinite (here we finally use that B is Jacobson). All points of $V(\mathfrak{q}) \cap D(f)$ map to $V(\mathfrak{p}) \cap D(f) = \{\mathfrak{p}\}$. Hence the fibre over $\mathfrak{p}$ is infinite. This contradicts the fact that $A \to B$ is quasi-finite (see Algebra, Lemma 10.116.4 or more explicitly Morphisms, Lemma 28.21.10). Thus the lemma is proved. □

34.13. Properties of schemes local in the syntomic topology

In this section we find some properties of schemes which are local on the base in the syntomic topology.

Lemma 34.13.1. *The property $\mathcal{P}(S) =$ "S is locally Noetherian and (S_k)" is local in the syntomic topology.*

Proof. We will check (1), (2) and (3) of Lemma 34.11.2. As a syntomic morphism is flat of finite presentation (Morphisms, Lemmas 28.32.7 and 28.32.6) we have already checked this for "being locally Noetherian" in the proof of Lemma 34.12.1. We will use this without further mention in the proof. First we note that $\mathcal{P}$ is local in the Zariski topology. This is clear from the definition, see Cohomology of Schemes, Definition 29.11.1. Next, we show that if $S' \to S$ is a syntomic morphism of affines and S has $\mathcal{P}$, then S' has $\mathcal{P}$. This is Algebra, Lemma 10.149.4 (use Morphisms, Lemma 28.32.2 and Algebra, Definition 10.129.1 and Lemma 10.128.3). Finally, we show that if $S' \to S$ is a surjective syntomic morphism of affines and S' has $\mathcal{P}$, then S has $\mathcal{P}$. This is Algebra, Lemma 10.150.5. Thus (1), (2) and (3) of Lemma 34.11.2 hold and we win. □

Lemma 34.13.2. *The property $\mathcal{P}(S) =$ "S is Cohen-Macaulay" is local in the syntomic topology.*

Proof. This is clear from Lemma 34.13.1 above since a scheme is Cohen-Macaulay if and only if it is locally Noetherian and (S_k) for all $k \geq 0$, see Properties, Lemma 27.12.2. □

34.14. Properties of schemes local in the smooth topology

In this section we find some properties of schemes which are local on the base in the smooth topology.

Lemma 34.14.1. *The property $\mathcal{P}(S) =$ "S is reduced" is local in the smooth topology.*

Proof. We will use Lemma 34.11.2. First we note that "being reduced" is local in the Zariski topology. This is clear from the definition, see Schemes, Definition 25.12.1. Next, we show that if $S' \to S$ is a smooth morphism of affines and S is reduced, then S' is reduced. This is Algebra, Lemma 10.149.6. Finally, we show that if $S' \to S$ is a surjective smooth morphism of affines and S' is reduced, then S is reduced. This is Algebra, Lemma 10.150.2. Thus (1), (2) and (3) of Lemma 34.11.2 hold and we win. □

Lemma 34.14.2. *The property $\mathcal{P}(S) =$ "S is normal" is local in the smooth topology.*

Proof. We will use Lemma 34.11.2. First we show "being normal" is local in the Zariski topology. This is clear from the definition, see Properties, Definition 27.7.1. Next, we show that if $S' \to S$ is a smooth morphism of affines and S is normal, then S' is normal. This is Algebra, Lemma 10.149.7. Finally, we show that if $S' \to S$ is a surjective smooth morphism of affines and S' is normal, then S is normal. This is Algebra, Lemma 10.150.3. Thus (1), (2) and (3) of Lemma 34.11.2 hold and we win. □

Lemma 34.14.3. *The property $\mathcal{P}(S) =$ "S is locally Noetherian and (R_k)" is local in the smooth topology.*

Proof. We will check (1), (2) and (3) of Lemma 34.11.2. As a smooth morphism is flat of finite presentation (Morphisms, Lemmas 28.35.9 and 28.35.8) we have already checked this for "being locally Noetherian" in the proof of Lemma 34.12.1. We will use this without further mention in the proof. First we note that $\mathcal{P}$ is local in the Zariski topology. This is clear from the definition, see Properties, Definition 27.12.1. Next, we show that if $S' \to S$ is a smooth morphism of affines and S has $\mathcal{P}$, then S' has $\mathcal{P}$. This is Algebra, Lemmas 10.149.5 (use Morphisms, Lemma 28.35.2, Algebra, Lemmas 10.130.4 and 10.133.3). Finally, we show that if $S' \to S$ is a surjective smooth morphism of affines and S' has $\mathcal{P}$, then S has $\mathcal{P}$. This is Algebra, Lemma 10.150.5. Thus (1), (2) and (3) of Lemma 34.11.2 hold and we win. □

Lemma 34.14.4. *The property $\mathcal{P}(S) =$ "S is regular" is local in the smooth topology.*

Proof. This is clear from Lemma 34.14.3 above since a locally Noetherian scheme is regular if and only if it is locally Noetherian and (R_k) for all $k \geq 0$. □

Lemma 34.14.5. *The property $\mathcal{P}(S) =$ "S is Nagata" is local in the smooth topology.*

Proof. We will check (1), (2) and (3) of Lemma 34.11.2. First we note that being Nagata is local in the Zariski topology. This is Properties, Lemma 27.13.6. Next, we show that if $S' \to S$ is a smooth morphism of affines and S is Nagata, then S' is Nagata. This is Morphisms, Lemma 28.19.1. Finally, we show that if $S' \to S$ is a surjective smooth morphism of affines and S' is Nagata, then S is Nagata. This is Algebra, Lemma 10.150.7. Thus (1), (2) and (3) of Lemma 34.11.2 hold and we win. □

34.15. Variants on descending properties

Sometimes one can descend properties, which are not local. We put results of this kind in this section.

Lemma 34.15.1. *If $f : X \to Y$ is a flat and surjective morphism of schemes and X is reduced, then Y is reduced.*

Proof. The result follows by looking at local rings (Schemes, Definition 25.12.1) and Algebra, Lemma 10.150.2. □

Lemma 34.15.2. *Let $f : X \to Y$ be a morphism of algebraic spaces. If f is locally of finite presentation, flat, and surjective and X is regular, then Y is regular.*

Proof. This lemma reduces to the following algebra statement: If $A \to B$ is a faithfully flat, finitely presented ring homomorphism with B Noetherian and regular, then A is Noetherian and regular. We see that A is Noetherian by Algebra, Lemma 10.150.1 and regular by Algebra, Lemma 10.105.8. □

34.16. Germs of schemes

Definition 34.16.1. Germs of schemes.

(1) A pair (X, x) consisting of a scheme X and a point $x \in X$ is called the *germ of X at x*.
(2) A *morphism of germs* $f : (X, x) \to (S, s)$ is an equivalence class of morphisms of schemes $f : U \to S$ with $f(x) = s$ where $U \subset X$ is an open neighbourhood of x. Two such f, f' are said to be equivalent if and only if f and f' agree in some open neighbourhood of x.
(3) We define the *composition of morphisms of germs* by composing representatives (this is well defined).

Before we continue we need one more definition.

Definition 34.16.2. Let $f : (X, x) \to (S, s)$ be a morphism of germs. We say f is *étale* (resp. *smooth*) if there exists a representative $f : U \to S$ of f which is an étale morphism (resp. a smooth morphism) of schemes.

34.17. Local properties of germs

Definition 34.17.1. Let $\mathcal{P}$ be a property of germs of schemes. We say that $\mathcal{P}$ is *étale local* (resp. *smooth local*) if for any étale (resp. smooth) morphism of germs $(U', u') \to (U, u)$ we have $\mathcal{P}(U, u) \Leftrightarrow \mathcal{P}(U', u')$.

Let (X, x) be a germ of a scheme. The dimension of X at x is the minimum of the dimensions of open neighbourhoods of x in X, and any small enough open neighbourhood has this dimension. Hence this is an invariant of the isomorphism class of the germ. We denote this simply $\dim_x(X)$. The following lemma tells us that the assertion $\dim_x(X) = d$ is an étale local property of germs.

Lemma 34.17.2. *Let $f : U \to V$ be an étale morphism of schemes. Let $u \in U$ and $v = f(u)$. Then $\dim_u(U) = \dim_v(V)$.*

Proof. In the statement $\dim_u(U)$ is the dimension of U at u as defined in Topology, Definition 5.9.1 as the minimum of the Krull dimensions of open neighbourhoods of u in U. Similarly for $\dim_v(V)$.

Let us show that $\dim_v(V) \geq \dim_u(U)$. Let V' be an open neighbourhood of v in V. Then there exists an open neighbourhood U' of u in U contained in $f^{-1}(V')$ such that $\dim_u(U) = \dim(U')$. Suppose that $Z_0 \subset Z_1 \subset \ldots \subset Z_n$ is a chain of irreducible closed subschemes of U'. If $\xi_i \in Z_i$ is the generic point then we have specializations $\xi_n \rightsquigarrow \xi_{n-1} \rightsquigarrow \ldots \rightsquigarrow \xi_0$. This gives specializations $f(\xi_n) \rightsquigarrow f(\xi_{n-1}) \rightsquigarrow \ldots \rightsquigarrow f(\xi_0)$ in V'. Note that $f(\xi_j) \neq f(\xi_i)$ if $i \neq j$ as the fibres of f are discrete (see Morphisms, Lemma 28.37.7). Hence we see that $\dim(V') \geq n$. The inequality $\dim_v(V) \geq \dim_u(U)$ follows formally.

Let us show that $\dim_u(U) \geq \dim_v(V)$. Let U' be an open neighbourhood of u in U. Note that $V' = f(U')$ is an open neighbourhood of v by Morphisms, Lemma 28.26.9. Hence $\dim(V') \geq \dim_v(V)$. Pick a chain $Z_0 \subset Z_1 \subset \ldots \subset Z_n$ of irreducible closed

subschemes of V'. Let $\xi_i \in Z_i$ be the generic point, so we have specializations $\xi_n \rightsquigarrow \xi_{n-1} \rightsquigarrow \ldots \rightsquigarrow \xi_0$. Since $\xi_0 \in f(U')$ we can find a point $\eta_0 \in U'$ with $f(\eta_0) = \xi_0$. Consider the map of local rings

$$\mathcal{O}_{V',\xi_0} \longrightarrow \mathcal{O}_{U',\eta_0}$$

which is a flat local ring map by Morphisms, Lemma 28.37.12. Note that the points ξ_i correspond to primes of the ring on the left by Schemes, Lemma 25.13.2. Hence by going down (see Algebra, Section 10.38) for the displayed ring map we can find a sequence of specializations $\eta_n \rightsquigarrow \eta_{n-1} \rightsquigarrow \ldots \rightsquigarrow \eta_0$ in U' mapping to the sequence $\xi_n \rightsquigarrow \xi_{n-1} \rightsquigarrow \ldots \rightsquigarrow \xi_0$ under f. This implies that $\dim_u(U) \geq \dim_v(V)$. □

Let (X, x) be a germ of a scheme. The isomorphism class of the local ring $\mathcal{O}_{X,x}$ is an invariant of the germ. The following lemma says that the property $\dim(\mathcal{O}_{X,x}) = d$ is an étale local property of germs.

Lemma 34.17.3. *Let $f : U \to V$ be an étale morphism of schemes. Let $u \in U$ and $v = f(u)$. Then $\dim(\mathcal{O}_{U,u}) = \dim(\mathcal{O}_{V,v})$.*

Proof. The algebraic statement we are asked to prove is the following: If $A \to B$ is an étale ring map and $\mathfrak{q}$ is a prime of B lying over $\mathfrak{p} \subset A$, then $\dim(A_\mathfrak{p}) = \dim(B_\mathfrak{q})$. This is More on Algebra, Lemma 15.32.6. □

34.18. Properties of morphisms local on the target

Suppose that $f : X \to Y$ is a morphism of schemes. Let $g : Y' \to Y$ be a morphism of schemes. Let $f' : X' \to Y'$ be the base change of f by g:

$$\begin{array}{ccc} X' & \xrightarrow{g'} & X \\ {\scriptstyle f'}\downarrow & & \downarrow{\scriptstyle f} \\ Y' & \xrightarrow{g} & Y \end{array}$$

Let $\mathcal{P}$ be a property of morphisms of schemes. Then we can wonder if (a) $\mathcal{P}(f) \Rightarrow \mathcal{P}(f')$, and also whether the converse (b) $\mathcal{P}(f') \Rightarrow \mathcal{P}(f)$ is true. If (a) holds whenever g is flat, then we say $\mathcal{P}$ is preserved under flat base change. If (b) holds whenever g is surjective and flat, then we say $\mathcal{P}$ descends through flat surjective base changes. If $\mathcal{P}$ is preserved under flat base changes and descends through flat surjective base changes, then we say $\mathcal{P}$ is flat local on the target. Compare with the discussion in Section 34.11. This turns out to be a very important notion which we formalize in the following definition.

Definition 34.18.1. Let $\mathcal{P}$ be a property of morphisms of schemes over a base. Let $\tau \in \{fpqc, fppf, syntomic, smooth, étale, Zariski\}$. We say $\mathcal{P}$ is *τ local on the base*, or *τ local on the target*, or *local on the base for the τ-topology* if for any τ-covering $\{Y_i \to Y\}_{i \in I}$ (see Topologies, Section 33.2) and any morphism of schemes $f : X \to Y$ over S we have

$$f \text{ has } \mathcal{P} \Leftrightarrow \text{each } Y_i \times_Y X \to Y_i \text{ has } \mathcal{P}.$$

To be sure, since isomorphisms are always coverings we see (or require) that property $\mathcal{P}$ holds for $X \to Y$ if and only if it holds for any arrow $X' \to Y'$ isomorphic to $X \to Y$. If a property is τ-local on the target then it is preserved by base changes by morphisms which occur in τ-coverings. Here is a formal statement.

Lemma 34.18.2. *Let $\tau \in \{fpqc, fppf, syntomic, smooth, étale, Zariski\}$. Let $\mathcal{P}$ be a property of morphisms which is τ local on the target. Let $f : X \to Y$ have property $\mathcal{P}$. For any morphism $Y' \to Y$ which is flat, resp. flat and locally of finite presentation, resp. syntomic, resp. étale, resp. an open immersion, the base change $f' : Y' \times_Y X \to Y'$ of f has property $\mathcal{P}$.*

Proof. This is true because we can fit $Y' \to Y$ into a family of morphisms which forms a τ-covering. □

A simple often used consequence of the above is that if $f : X \to Y$ has property $\mathcal{P}$ which is τ-local on the target and $f(X) \subset V$ for some open subscheme $V \subset Y$, then also the induced morphism $X \to V$ has $\mathcal{P}$. Proof: The base change f by $V \to Y$ gives $X \to V$.

Lemma 34.18.3. *Let $\tau \in \{fppf, syntomic, smooth, étale\}$. Let $\mathcal{P}$ be a property of morphisms which is τ local on the target. For any morphism of schemes $f : X \to Y$ there exists a largest open $W(f) \subset Y$ such that the restriction $X_{W(f)} \to W(f)$ has $\mathcal{P}$. Moreover,*

1. *if $g : Y' \to Y$ is flat and locally of finite presentation, syntomic, smooth, or étale and the base change $f' : X_{Y'} \to Y'$ has $\mathcal{P}$, then $g(Y') \subset W(f)$,*
2. *if $g : Y' \to Y$ is flat and locally of finite presentation, syntomic, smooth, or étale, then $W(f') = g^{-1}(W(f))$, and*
3. *if $\{g_i : Y_i \to Y\}$ is a τ-covering, then $g_i^{-1}(W(f)) = W(f_i)$, where f_i is the base change of f by $Y_i \to Y$.*

Proof. Consider the union W of the images $g(Y') \subset Y$ of morphisms $g : Y' \to Y$ with the properties:

1. g is flat and locally of finite presentation, syntomic, smooth, or étale, and
2. the base change $Y' \times_{g,Y} X \to Y'$ has property $\mathcal{P}$.

Since such a morphism g is open (see Morphisms, Lemma 28.26.9) we see that $W \subset Y$ is an open subset of Y. Since $\mathcal{P}$ is local in the τ topology the restriction $X_W \to W$ has property $\mathcal{P}$ because we are given a covering $\{Y' \to W\}$ of W such that the pullbacks have $\mathcal{P}$. This proves the existence and proves that $W(f)$ has property (1). To see property (2) note that $W(f') \supset g^{-1}(W(f))$ because $\mathcal{P}$ is stable under base change by flat and locally of finite presentation, syntomic, smooth, or étale morphisms, see Lemma 34.18.2. On the other hand, if $Y'' \subset Y'$ is an open such that $X_{Y''} \to Y''$ has property $\mathcal{P}$, then $Y'' \to Y$ factors through W by construction, i.e., $Y'' \subset g^{-1}(W(f))$. This proves (2). Assertion (3) follows from (2) because each morphism $Y_i \to Y$ is flat and locally of finite presentation, syntomic, smooth, or étale by our definition of a τ-covering. □

Lemma 34.18.4. *Let $\mathcal{P}$ be a property of morphisms of schemes over a base. Let $\tau \in \{fpqc, fppf, étale, smooth, syntomic\}$. Assume that*

1. *the property is preserved under flat, flat and locally of finite presentation, étale, smooth, or syntomic base change depending on whether τ is fpqc, fppf, étale, smooth, or syntomic (compare with Schemes, Definition 25.18.3),*
2. *the property is Zariski local on the base.*
3. *for any surjective morphism of affine schemes $S' \to S$ which is flat, flat of finite presentation, étale, smooth or syntomic depending on whether τ*

is fpqc, fppf, étale, smooth, or syntomic, and any morphism of schemes $f : X \to S$ property $\mathcal{P}$ holds for f if property $\mathcal{P}$ holds for the base change $f' : X' = S' \times_S X \to S'$.

Then $\mathcal{P}$ is τ local on the base.

Proof. This follows almost immediately from the definition of a τ-covering, see Topologies, Definition 33.8.1 33.7.1 33.4.1 33.5.1, or 33.6.1 and Topologies, Lemma 33.8.8, 33.7.4, 33.4.4, 33.5.4, or 33.6.4. Details omitted. □

Remark 34.18.5. (This is a repeat of Remark 34.11.3 above.) In Lemma 34.18.4 above if $\tau = \textit{smooth}$ then in condition (3) we may assume that the morphism is a (surjective) standard smooth morphism. Similarly, when $\tau = \textit{syntomic}$ or $\tau = \textit{étale}$.

34.19. Properties of morphisms local in the fpqc topology on the target

In this section we find a large number of properties of morphisms of schemes which are local on the base in the fpqc topology. By contrast, in Examples, Section 78.49 we will show that the properties "projective" and "quasi-projective" are not local on the base even in the Zariski topology.

Lemma 34.19.1. *The property $\mathcal{P}(f) =$"f is quasi-compact" is fpqc local on the base.*

Proof. A base change of a quasi-compact morphism is quasi-compact, see Schemes, Lemma 25.19.3. Being quasi-compact is Zariski local on the base, see Schemes, Lemma 25.19.2. Finally, let $S' \to S$ be a flat surjective morphism of affine schemes, and let $f : X \to S$ be a morphism. Assume that the base change $f' : X' \to S'$ is quasi-compact. Then X' is quasi-compact, and $X' \to X$ is surjective. Hence X is quasi-compact. This implies that f is quasi-compact. Therefore Lemma 34.18.4 applies and we win. □

Lemma 34.19.2. *The property $\mathcal{P}(f) =$"f is quasi-separated" is fpqc local on the base.*

Proof. Any base change of a quasi-separated morphism is quasi-separated, see Schemes, Lemma 25.21.13. Being quasi-separated is Zariski local on the base (from the definition or by Schemes, Lemma 25.21.7). Finally, let $S' \to S$ be a flat surjective morphism of affine schemes, and let $f : X \to S$ be a morphism. Assume that the base change $f' : X' \to S'$ is quasi-separated. This means that $\Delta' : X' \to X' \times_{S'} X'$ is quasi-compact. Note that Δ' is the base change of $\Delta : X \to X \times_S X$ via $S' \to S$. By Lemma 34.19.1 this implies Δ is quasi-compact, and hence f is quasi-separated. Therefore Lemma 34.18.4 applies and we win. □

Lemma 34.19.3. *The property $\mathcal{P}(f) =$"f is universally closed" is fpqc local on the base.*

Proof. A base change of a universally closed morphism is universally closed by definition. Being universally closed is Zariski local on the base (from the definition or by Morphisms, Lemma 28.42.2). Finally, let $S' \to S$ be a flat surjective morphism of affine schemes, and let $f : X \to S$ be a morphism. Assume that the base change

$f' : X' \to S'$ is universally closed. Let $T \to S$ be any morphism. Consider the diagram

$$\begin{array}{ccccc} X' & \longleftarrow & S' \times_S T \times_S X & \longrightarrow & T \times_S X \\ \downarrow & & \downarrow & & \downarrow \\ S' & \longleftarrow & S' \times_S T & \longrightarrow & T \end{array}$$

in which both squares are cartesian. Thus the assumption implies that the middle vertical arrow is closed. The right horizontal arrows are flat, quasi-compact and surjective (as base changes of $S' \to S$). Hence a subset of T is closed if and only if its inverse image in $S' \times_S T$ is closed, see Morphisms, Lemma 28.26.10. An easy diagram chase shows that the right vertical arrow is closed too, and we conclude $X \to S$ is universally closed. Therefore Lemma 34.18.4 applies and we win. □

Lemma 34.19.4. *The property $\mathcal{P}(f) =$ "f is universally open" is fpqc local on the base.*

Proof. The proof is the same as the proof of Lemma 34.19.3. □

Lemma 34.19.5. *The property $\mathcal{P}(f) =$ "f is separated" is fpqc local on the base.*

Proof. A base change of a separated morphism is separated, see Schemes, Lemma 25.21.13. Being separated is Zariski local on the base (from the definition or by Schemes, Lemma 25.21.8). Finally, let $S' \to S$ be a flat surjective morphism of affine schemes, and let $f : X \to S$ be a morphism. Assume that the base change $f' : X' \to S'$ is separated. This means that $\Delta' : X' \to X' \times_{S'} X'$ is a closed immersion, hence universally closed. Note that Δ' is the base change of $\Delta : X \to X \times_S X$ via $S' \to S$. By Lemma 34.19.3 this implies Δ is universally closed. Since it is an immersion (Schemes, Lemma 25.21.2) we conclude Δ is a closed immersion. Hence f is separated. Therefore Lemma 34.18.4 applies and we win. □

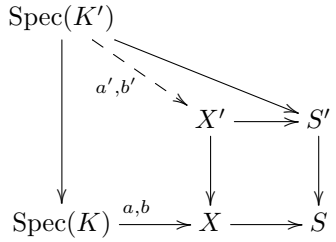
Lemma 34.19.6. *The property $\mathcal{P}(f) =$ "f is surjective" is fpqc local on the base.*

Proof. This is clear. □

Lemma 34.19.7. *The property $\mathcal{P}(f) =$ "f is universally injective" is fpqc local on the base.*

Proof. A base change of a universally injective morphism is universally injective (this is formal). Being universally injective is Zariski local on the base; this is clear from the definition. Finally, let $S' \to S$ be a flat surjective morphism of affine schemes, and let $f : X \to S$ be a morphism. Assume that the base change $f' : X' \to S'$ is universally injective. Let K be a field, and let $a, b : \mathrm{Spec}(K) \to X$ be two morphisms such that $f \circ a = f \circ b$. As $S' \to S$ is surjective and by the discussion in Schemes, Section 25.13 there exists a field extension $K \subset K'$ and a

morphism $\mathrm{Spec}(K') \to S'$ such that the following solid diagram commutes



As the square is cartesian we get the two dotted arrows a', b' making the diagram commute. Since $X' \to S'$ is universally injective we get $a' = b'$, by Morphisms, Lemma 28.12.2. Clearly this forces $a = b$ (by the discussion in Schemes, Section 25.13). Therefore Lemma 34.18.4 applies and we win.

An alternative proof would be to use the characterization of a universally injective morphism as one whose diagonal is surjective, see Morphisms, Lemma 28.12.2. The lemma then follows from the fact that the property of being surjective is fpqc local on the base, see Lemma 34.19.6. (Hint: use that the base change of the diagonal is the diagonal of the base change.) □

Lemma 34.19.8. *The property $\mathcal{P}(f) =$ "f is locally of finite type" is fpqc local on the base.*

Proof. Being locally of finite type is preserved under base change, see Morphisms, Lemma 28.16.4. Being locally of finite type is Zariski local on the base, see Morphisms, Lemma 28.16.2. Finally, let $S' \to S$ be a flat surjective morphism of affine schemes, and let $f : X \to S$ be a morphism. Assume that the base change $f' : X' \to S'$ is locally of finite type. Let $U \subset X$ be an affine open. Then $U' = S' \times_S U$ is affine and of finite type over S'. Write $S = \mathrm{Spec}(R)$, $S' = \mathrm{Spec}(R')$, $U = \mathrm{Spec}(A)$, and $U' = \mathrm{Spec}(A')$. We know that $R \to R'$ is faithfully flat, $A' = R' \otimes_R A$ and $R' \to A'$ is of finite type. We have to show that $R \to A$ is of finite type. This is the result of Algebra, Lemma 10.120.1. It follows that f is locally of finite type. Therefore Lemma 34.18.4 applies and we win. □

Lemma 34.19.9. *The property $\mathcal{P}(f) =$ "f is locally of finite presentation" is fpqc local on the base.*

Proof. Being locally of finite presentation is preserved under base change, see Morphisms, Lemma 28.22.4. Being locally of finite type is Zariski local on the base, see Morphisms, Lemma 28.22.2. Finally, let $S' \to S$ be a flat surjective morphism of affine schemes, and let $f : X \to S$ be a morphism. Assume that the base change $f' : X' \to S'$ is locally of finite presentation. Let $U \subset X$ be an affine open. Then $U' = S' \times_S U$ is affine and of finite type over S'. Write $S = \mathrm{Spec}(R)$, $S' = \mathrm{Spec}(R')$, $U = \mathrm{Spec}(A)$, and $U' = \mathrm{Spec}(A')$. We know that $R \to R'$ is faithfully flat, $A' = R' \otimes_R A$ and $R' \to A'$ is of finite presentation. We have to show that $R \to A$ is of finite presentation. This is the result of Algebra, Lemma 10.120.2. It follows that f is locally of finite presentation. Therefore Lemma 34.18.4 applies and we win. □

Lemma 34.19.10. *The property $\mathcal{P}(f) =$ "f is of finite type" is fpqc local on the base.*

Proof. Combine Lemmas 34.19.1 and 34.19.8. □

Lemma 34.19.11. *The property $\mathcal{P}(f) =$ "f is of finite presentation" is fpqc local on the base.*

Proof. Combine Lemmas 34.19.1, 34.19.2 and 34.19.9. □

Lemma 34.19.12. *The property $\mathcal{P}(f) =$ "f is proper" is fpqc local on the base.*

Proof. The lemma follows by combining Lemmas 34.19.3, 34.19.5 and 34.19.10. □

Lemma 34.19.13. *The property $\mathcal{P}(f) =$ "f is flat" is fpqc local on the base.*

Proof. Being flat is preserved under arbitrary base change, see Morphisms, Lemma 28.26.7. Being flat is Zariski local on the base by definition. Finally, let $S' \to S$ be a flat surjective morphism of affine schemes, and let $f : X \to S$ be a morphism. Assume that the base change $f' : X' \to S'$ is flat. Let $U \subset X$ be an affine open. Then $U' = S' \times_S U$ is affine. Write $S = \mathrm{Spec}(R)$, $S' = \mathrm{Spec}(R')$, $U = \mathrm{Spec}(A)$, and $U' = \mathrm{Spec}(A')$. We know that $R \to R'$ is faithfully flat, $A' = R' \otimes_R A$ and $R' \to A'$ is flat. Goal: Show that $R \to A$ is flat. This follows immediately from Algebra, Lemma 10.36.7. Hence f is flat. Therefore Lemma 34.18.4 applies and we win. □

Lemma 34.19.14. *The property $\mathcal{P}(f) =$ "f is an open immersion" is fpqc local on the base.*

Proof. The property of being an open immersion is stable under base change, see Schemes, Lemma 25.18.2. The property of being an open immersion is Zariski local on the base (this is obvious). Finally, let $S' \to S$ be a flat surjective morphism of affine schemes, and let $f : X \to S$ be a morphism. Assume that the base change $f' : X' \to S'$ is an open immersion. Then f' is universally open, and universally injective. Hence we conclude that f is universally open by Lemma 34.19.4, and universally injective by Lemma 34.19.7. In particular $f(X) \subset S$ is open, and we may replace S by $f(S)$ and assume that f is surjective. This implies that f' is an isomorphism and we have to show that f is an isomorphism also. Since f is universally injective we see that f is bijective. Hence f is a homeomorphism. Let $x \in X$ and choose $U \subset X$ an affine open neighbourhood of x. Since $f(U) \subset S$ is open, and S is affine we may choose a standard open $D(g) \subset f(U)$ containing $f(x)$ where $g \in \Gamma(S, \mathcal{O}_S)$. It is clear that $U \cap f^{-1}(D(g))$ is still affine and still an open neighbourhood of x. Replace U by $U \cap f^{-1}(D(g))$ and write $V = D(g) \subset S$ and V' the inverse image of V in S'. Note that V' is a standard open of S' as well and in particular that V' is affine. Since f' is an isomorphism we have $V' \times_V U \to V'$ is an isomorphism. In terms of rings this means that

$$\mathcal{O}(V') \longrightarrow \mathcal{O}(V') \otimes_{\mathcal{O}(V)} \mathcal{O}(U)$$

is an isomorphism. Since $\mathcal{O}(V) \to \mathcal{O}(V')$ is faithfully flat this implies that $\mathcal{O}(V) \to \mathcal{O}(U)$ is an isomorphism. Hence $U \cong V$ and we see that f is an isomorphism. Therefore Lemma 34.18.4 applies and we win. □

Lemma 34.19.15. *The property $\mathcal{P}(f) =$ "f is an isomorphism" is fpqc local on the base.*

Proof. Combine Lemmas 34.19.6 and 34.19.14. □

Lemma 34.19.16. *The property $\mathcal{P}(f) =$ "f is affine" is fpqc local on the base.*

Proof. A base change of an affine morphism is affine, see Morphisms, Lemma 28.13.8. Being affine is Zariski local on the base, see Morphisms, Lemma 28.13.3. Finally, let $g : S' \to S$ be a flat surjective morphism of affine schemes, and let $f : X \to S$ be a morphism. Assume that the base change $f' : X' \to S'$ is affine. In other words, X' is affine, say $X' = \mathrm{Spec}(A')$. Also write $S = \mathrm{Spec}(R)$ and $S' = \mathrm{Spec}(R')$. We have to show that X is affine.

By Lemmas 34.19.1 and 34.19.5 we see that $X \to S$ is separated and quasi-compact. Thus $f_*\mathcal{O}_X$ is a quasi-coherent sheaf of $\mathcal{O}_S$-algebras, see Schemes, Lemma 25.24.1. Hence $f_*\mathcal{O}_X = \widetilde{A}$ for some R-algebra A. In fact $A = \Gamma(X, \mathcal{O}_X)$ of course. Also, by flat base change (see for example Cohomology of Schemes, Lemma 29.5.2) we have $g^*f_*\mathcal{O}_X = f'_*\mathcal{O}_{X'}$. In other words, we have $A' = R' \otimes_R A$. Consider the canonical morphism

$$X \longrightarrow \mathrm{Spec}(A)$$

over S from Schemes, Lemma 25.6.4. By the above the base change of this morphism to S' is an isomorphism. Hence it is an isomorphism by Lemma 34.19.15. Therefore Lemma 34.18.4 applies and we win. □

Lemma 34.19.17. *The property $\mathcal{P}(f) =$ "f is a closed immersion" is fpqc local on the base.*

Proof. Let $f : X \to Y$ be a morphism of schemes. Let $\{Y_i \to Y\}$ be an fpqc covering. Assume that each $f_i : Y_i \times_Y X \to Y_i$ is a closed immersion. This implies that each f_i is affine, see Morphisms, Lemma 28.13.9. By Lemma 34.19.16 we conclude that f is affine. It remains to show that $\mathcal{O}_Y \to f_*\mathcal{O}_X$ is surjective. For every $y \in Y$ there exists an i and a point $y_i \in Y_i$ mapping to y. By Cohomology of Schemes, Lemma 29.5.2 the sheaf $f_{i,*}(\mathcal{O}_{Y_i \times_Y X})$ is the pullback of $f_*\mathcal{O}_X$. By assumption it is a quotient of $\mathcal{O}_{Y_i}$. Hence we see that

$$\Big(\mathcal{O}_{Y,y} \longrightarrow (f_*\mathcal{O}_X)_y\Big) \otimes_{\mathcal{O}_{Y,y}} \mathcal{O}_{Y_i,y_i}$$

is surjective. Since $\mathcal{O}_{Y_i,y_i}$ is faithfully flat over $\mathcal{O}_{Y,y}$ this implies the surjectivity of $\mathcal{O}_{Y,y} \longrightarrow (f_*\mathcal{O}_X)_y$ as desired. □

Lemma 34.19.18. *The property $\mathcal{P}(f) =$ "f is quasi-affine" is fpqc local on the base.*

Proof. Let $f : X \to Y$ be a morphism of schemes. Let $\{g_i : Y_i \to Y\}$ be an fpqc covering. Assume that each $f_i : Y_i \times_Y X \to Y_i$ is quasi-affine. This implies that each f_i is quasi-compact and separated. By Lemmas 34.19.1 and 34.19.5 this implies that f is quasi-compact and separated. Consider the sheaf of $\mathcal{O}_Y$-algebras $\mathcal{A} = f_*\mathcal{O}_X$. By Schemes, Lemma 25.24.1 it is a quasi-coherent $\mathcal{O}_Y$-algebra. Consider the canonical morphism

$$j : X \longrightarrow \underline{\mathrm{Spec}}_Y(\mathcal{A})$$

see Constructions, Lemma 26.4.7. By flat base change (see for example Cohomology of Schemes, Lemma 29.5.2) we have $g_i^*f_*\mathcal{O}_X = f_{i,*}\mathcal{O}_{X'}$ where $g_i : Y_i \to Y$ are the given flat maps. Hence the base change j_i of j by g_i is the canonical morphism of Constructions, Lemma 26.4.7 for the morphism f_i. By assumption and Morphisms, Lemma 28.14.3 all of these morphisms j_i are quasi-compact open immersions. Hence, by Lemmas 34.19.1 and 34.19.14 we see that j is a quasi-compact

open immersion. Hence by Morphisms, Lemma 28.14.3 again we conclude that f is quasi-affine. □

Lemma 34.19.19. *The property $\mathcal{P}(f) =$ "f is a quasi-compact immersion" is fpqc local on the base.*

Proof. Let $f : X \to Y$ be a morphism of schemes. Let $\{Y_i \to Y\}$ be an fpqc covering. Write $X_i = Y_i \times_Y X$ and $f_i : X_i \to Y_i$ the base change of f. Also denote $q_i : Y_i \to Y$ the given flat morphisms. Assume each f_i is a quasi-compact immersion. By Schemes, Lemma 25.23.7 each f_i is separated. By Lemmas 34.19.1 and 34.19.5 this implies that f is quasi-compact and separated. Let $X \to Z \to Y$ be the factorization of f through its scheme theoretic image. By Morphisms, Lemma 28.6.3 the closed subscheme $Z \subset Y$ is cut out by the quasi-coherent sheaf of ideals $\mathcal{I} = \mathrm{Ker}(\mathcal{O}_Y \to f_*\mathcal{O}_X)$ as f is quasi-compact. By flat base change (see for example Cohomology of Schemes, Lemma 29.5.2; here we use f is separated) we see $f_{i,*}\mathcal{O}_{X_i}$ is the pullback $q_i^* f_*\mathcal{O}_X$. Hence $Y_i \times_Y Z$ is cut out by the quasi-coherent sheaf of ideals $q_i^*\mathcal{I} = \mathrm{Ker}(\mathcal{O}_{Y_i} \to f_{i,*}\mathcal{O}_{X_i})$. By Morphisms, Lemma 28.7.7 the morphisms $X_i \to Y_i \times_Y Z$ are open immersions. Hence by Lemma 34.19.14 we see that $X \to Z$ is an open immersion and hence f is a immersion as desired (we already saw it was quasi-compact). □

Lemma 34.19.20. *The property $\mathcal{P}(f) =$ "f is integral" is fpqc local on the base.*

Proof. An integral morphism is the same thing as an affine, universally closed morphism. See Morphisms, Lemma 28.44.7. Hence the lemma follows on combining Lemmas 34.19.3 and 34.19.16. □

Lemma 34.19.21. *The property $\mathcal{P}(f) =$ "f is finite" is fpqc local on the base.*

Proof. An finite morphism is the same thing as an integral morphism which is locally of finite type. See Morphisms, Lemma 28.44.4. Hence the lemma follows on combining Lemmas 34.19.8 and 34.19.20. □

Lemma 34.19.22. *The properties $\mathcal{P}(f) =$ "f is locally quasi-finite" and $\mathcal{P}(f) =$ "f is quasi-finite" are fpqc local on the base.*

Proof. Let $f : X \to S$ be a morphism of schemes, and let $\{S_i \to S\}$ be an fpqc covering such that each base change $f_i : X_i \to S_i$ is locally quasi-finite. We have already seen (Lemma 34.19.8) that "locally of finite type" is fpqc local on the base, and hence we see that f is locally of finite type. Then it follows from Morphisms, Lemma 28.21.13 that f is locally quasi-finite. The quasi-finite case follows as we have already seen that "quasi-compact" is fpqc local on the base (Lemma 34.19.1). □

Lemma 34.19.23. *The property $\mathcal{P}(f) =$ "f is locally of finite type of relative dimension d" is fpqc local on the base.*

Proof. This follows immediately from the fact that being locally of finite type is fpqc local on the base and Morphisms, Lemma 28.29.3. □

Lemma 34.19.24. *The property $\mathcal{P}(f) =$ "f is syntomic" is fpqc local on the base.*

Proof. A morphism is syntomic if and only if it is locally of finite presentation, flat, and has locally complete intersections as fibres. We have seen already that being flat and locally of finite presentation are fpqc local on the base (Lemmas 34.19.13, and 34.19.9). Hence the result follows for syntomic from Morphisms, Lemma 28.32.12. □

Lemma 34.19.25. *The property $\mathcal{P}(f) =$ "f is smooth" is fpqc local on the base.*

Proof. A morphism is smooth if and only if it is locally of finite presentation, flat, and has smooth fibres. We have seen already that being flat and locally of finite presentation are fpqc local on the base (Lemmas 34.19.13, and 34.19.9). Hence the result follows for smooth from Morphisms, Lemma 28.35.15. □

Lemma 34.19.26. *The property $\mathcal{P}(f) =$ "f is unramified" is fpqc local on the base. The property $\mathcal{P}(f) =$ "f is G-unramified" is fpqc local on the base.*

Proof. A morphism is unramified (resp. G-unramified) if and only if it is locally of finite type (resp. finite presentation) and its diagonal morphism is an open immersion (see Morphisms, Lemma 28.36.13). We have seen already that being locally of finite type (resp. locally of finite presentation) and an open immersion is fpqc local on the base (Lemmas 34.19.9, 34.19.8, and 34.19.14). Hence the result follows formally. □

Lemma 34.19.27. *The property $\mathcal{P}(f) =$ "f is étale" is fpqc local on the base.*

Proof. A morphism is étale if and only if it flat and G-unramified. See Morphisms, Lemma 28.37.16. We have seen already that being flat and G-unramified are fpqc local on the base (Lemmas 34.19.13, and 34.19.26). Hence the result follows. □

Lemma 34.19.28. *The property $\mathcal{P}(f) =$ "f is finite locally free" is fpqc local on the base. Let $d \geq 0$. The property $\mathcal{P}(f) =$ "f is finite locally free of degree d" is fpqc local on the base.*

Proof. Being finite locally free is equivalent to being finite, flat and locally of finite presentation (Morphisms, Lemma 28.46.2). Hence this follows from Lemmas 34.19.21, 34.19.13, and 34.19.9. If $f : Z \to U$ is finite locally free, and $\{U_i \to U\}$ is a surjective family of morphisms such that each pullback $Z \times_U U_i \to U_i$ has degree d, then $Z \to U$ has degree d, for example because we can read off the degree in a point $u \in U$ from the fibre $(f_*\mathcal{O}_Z)_u \otimes_{\mathcal{O}_{U,u}} \kappa(u)$. □

Lemma 34.19.29. *The property $\mathcal{P}(f) =$ "f is a monomorphism" is fpqc local on the base.*

Proof. Let $f : X \to S$ be a morphism of schemes. Let $\{S_i \to S\}$ be an fpqc covering, and assume each of the base changes $f_i : X_i \to S_i$ of f is a monomorphism. Let $a, b : T \to X$ be two morphisms such that $f \circ a = f \circ b$. We have to show that $a = b$. Since f_i is a monomorphism we see that $a_i = b_i$, where $a_i, b_i : S_i \times_S T \to X_i$ are the base changes. In particular the compositions $S_i \times_S T \to T \to X$ are equal. Since $\coprod S_i \times_S T \to T$ is an epimorphism (see e.g. Lemma 34.9.3) we conclude $a = b$. □

Lemma 34.19.30. *The properties*

$\mathcal{P}(f) =$ "f is a Koszul-regular immersion",
$\mathcal{P}(f) =$ "f is an H_1-regular immersion", *and*

$\mathcal{P}(f) =$ "f *is a quasi-regular immersion*"

are fpqc local on the base.

Proof. We will use the criterion of Lemma 34.18.4 to prove this. By Divisors, Definition 30.13.1 being a Koszul-regular (resp. H_1-regular, quasi-regular) immersion is Zariski local on the base. By Divisors, Lemma 30.13.4 being a Koszul-regular (resp. H_1-regular, quasi-regular) immersion is preserved under flat base change. The final hypothesis (3) of Lemma 34.18.4 translates into the following algebra statement: Let $A \to B$ be a faithfully flat ring map. Let $I \subset A$ be an ideal. If IB is locally on $\mathrm{Spec}(B)$ generated by a Koszul-regular (resp. H_1-regular, quasi-regular) sequence in B, then $I \subset A$ is locally on $\mathrm{Spec}(A)$ generated by a Koszul-regular (resp. H_1-regular, quasi-regular) sequence in A. This is More on Algebra, Lemma 15.21.4. □

34.20. Properties of morphisms local in the fppf topology on the target

In this section we find some properties of morphisms of schemes for which we could not (yet) show they are local on the base in the fpqc topology which, however, are local on the base in the fppf topology.

Lemma 34.20.1. *The property $\mathcal{P}(f) =$ "f is an immersion" is fppf local on the base.*

Proof. The property of being an immersion is stable under base change, see Schemes, Lemma 25.18.2. The property of being an immersion is Zariski local on the base. Finally, let $\pi : S' \to S$ be a surjective morphism of affine schemes, which is flat and locally of finite presentation. Note that $\pi : S' \to S$ is open by Morphisms, Lemma 28.26.9. Let $f : X \to S$ be a morphism. Assume that the base change $f' : X' \to S'$ is an immersion. In particular we see that $f'(X') = \pi^{-1}(f(X))$ is locally closed. Hence by Topology, Lemma 5.5.4 we see that $f(X) \subset S$ is locally closed. Let $Z \subset S$ be the closed subset $Z = \overline{f(X)} \setminus f(X)$. By Topology, Lemma 5.5.4 again we see that $f'(X')$ is closed in $S' \setminus Z'$. Hence we may apply Lemma 34.19.17 to the fpqc covering $\{S' \setminus Z' \to S \setminus Z\}$ and conclude that $f : X \to S \setminus Z$ is a closed immersion. In other words, f is an immersion. Therefore Lemma 34.18.4 applies and we win. □

34.21. Application of fpqc descent of properties of morphisms

The following lemma may seem a bit frivolous but turns out is a useful tool in studying étale and unramified morphisms.

Lemma 34.21.1. *Let $f : X \to Y$ be a flat, quasi-compact, surjective monomorphism. Then f is an isomorphism.*

Proof. As f is a flat, quasi-compact, surjective morphism we see $\{X \to Y\}$ is an fpqc covering of Y. The diagonal $\Delta : X \to X \times_Y X$ is an isomorphism. This implies that the base change of f by f is an isomorphism. Hence we see f is an isomorphism by Lemma 34.19.15. □

We can use this lemma to show the following important result. We will discuss this and related results in more detail in Étale Morphisms, Section 40.14.

Lemma 34.21.2. *A universally injective étale morphism is an open immersion.*

Proof. Let $f : X \to Y$ be an étale morphism which is universally injective. Then f is open (Morphisms, Lemma 28.37.13) hence we can replace Y by $f(X)$ and we may assume that f is surjective. Then f is bijective and open hence a homeomorphism. Hence f is quasi-compact. Thus by Lemma 34.21.1 it suffices to show that f is a monomorphism. As $X \to Y$ is étale the morphism $\Delta_{X/Y} : X \to X \times_Y X$ is an open immersion by Morphisms, Lemma 28.36.13 (and Morphisms, Lemma 28.37.16). As f is universally injective $\Delta_{X/Y}$ is also surjective, see Morphisms, Lemma 28.12.2. Hence $\Delta_{X/Y}$ is an isomorphism, i.e., $X \to Y$ is a monomorphism. □

We can reformulate the hypotheses in the lemma above a bit by using the following characterization of flat universally injective morphisms.

Lemma 34.21.3. *Let $f : X \to Y$ be a morphism of schemes. Let X^0 denote the set of generic points of irreducible components of X. If*

(1) *f is flat and separated,*
(2) *for $\xi \in X^0$ we have $\kappa(f(\xi)) = \kappa(\xi)$, and*
(3) *if $\xi, \xi' \in X^0$, $\xi \neq \xi'$, then $f(\xi) \neq f(\xi')$,*

then f is universally injective.

Proof. We have to show that $\Delta : X \to X \times_Y X$ is surjective, see Morphisms, Lemma 28.12.2. As $X \to Y$ is separated, the image of Δ is closed. Thus if Δ is not surjective, we can find a generic point $\eta \in X \times_S X$ of an irreducible component of $X \times_S X$ which is not in the image of Δ. The projection $\mathrm{pr}_1 : X \times_Y X \to X$ is flat as a base change of the flat morphism $X \to Y$, see Morphisms, Lemma 28.26.7. Hence generalizations lift along pr_1, see Morphisms, Lemma 28.26.8. We conclude that $\xi = \mathrm{pr}_1(\eta) \in X^0$. However, assumptions (2) and (3) guarantee that the scheme $(X \times_Y X)_{f(\xi)}$ has at most one point for every $\xi \in X^0$. In other words, we have $\Delta(\xi) = \eta$ a contradiction. □

Thus we can reformulate Lemma 34.21.2 as follows.

Lemma 34.21.4. *Let $f : X \to Y$ be a morphism of schemes. Let X^0 denote the set of generic points of irreducible components of X. If*

(1) *f is étale and separated,*
(2) *for $\xi \in X^0$ we have $\kappa(f(\xi)) = \kappa(\xi)$, and*
(3) *if $\xi, \xi' \in X^0$, $\xi \neq \xi'$, then $f(\xi) \neq f(\xi')$,*

then f is an open immersion.

Proof. Immediate from Lemmas 34.21.3 and 34.21.2. □

34.22. Properties of morphisms local on the source

It often happens one can prove a morphism has a certain property after precomposing with some other morphism. In many cases this implies the morphism has the property too. We formalize this in the following definition.

Definition 34.22.1. Let $\mathcal{P}$ be a property of morphisms of schemes. Let $\tau \in \{Zariski, fpqc, fppf, étale, smooth, syntomic\}$. We say $\mathcal{P}$ is *τ local on the source*, or *local on the source for the τ-topology* if for any morphism of schemes $f : X \to Y$ over S, and any τ-covering $\{X_i \to X\}_{i \in I}$ we have

$$f \text{ has } \mathcal{P} \Leftrightarrow \text{each } X_i \to Y \text{ has } \mathcal{P}.$$

To be sure, since isomorphisms are always coverings we see (or require) that property $\mathcal{P}$ holds for $X \to Y$ if and only if it holds for any arrow $X' \to Y'$ isomorphic to $X \to Y$. If a property is τ-local on the source then it is preserved by precomposing with morphisms which occur in τ-coverings. Here is a formal statement.

Lemma 34.22.2. *Let $\tau \in \{fpqc, fppf, syntomic, smooth, étale, Zariski\}$. Let $\mathcal{P}$ be a property of morphisms which is τ local on the source. Let $f : X \to Y$ have property $\mathcal{P}$. For any morphism $a : X' \to X$ which is flat, resp. flat and locally of finite presentation, resp. syntomic, resp. étale, resp. an open immersion, the composition $f \circ a : X' \to Y$ has property $\mathcal{P}$.*

Proof. This is true because we can fit $X' \to X$ into a family of morphisms which forms a τ-covering. □

Lemma 34.22.3. *Let $\mathcal{P}$ be a property of morphisms of schemes. Let $\tau \in \{fpqc, fppf, étale, smooth, syntomic\}$. Assume that*

1. *the property is preserved under precomposing with flat, flat locally of finite presentation, étale, smooth or syntomic morphisms depending on whether τ is fpqc, fppf, étale, smooth, or syntomic,*
2. *the property is Zariski local on the source,*
3. *the property is Zariski local on the target,*
4. *for any morphism of affine schemes $X \to Y$, and any surjective morphism of affine schemes $X' \to X$ which is flat, flat of finite presentation, étale, smooth or syntomic depending on whether τ is fpqc, fppf, étale, smooth, or syntomic, property $\mathcal{P}$ holds for f if property $\mathcal{P}$ holds for the composition $f' : X' \to Y$.*

Then $\mathcal{P}$ is τ local on the source.

Proof. This follows almost immediately from the definition of a τ-covering, see Topologies, Definition 33.8.1 33.7.1 33.4.1 33.5.1, or 33.6.1 and Topologies, Lemma 33.8.8, 33.7.4, 33.4.4, 33.5.4, or 33.6.4. Details omitted. (Hint: Use locality on the source and target to reduce the verification of property $\mathcal{P}$ to the case of a morphism between affines. Then apply (1) and (4).) □

Remark 34.22.4. (This is a repeat of Remarks 34.11.3 and 34.18.5 above.) In Lemma 34.22.3 above if $\tau = smooth$ then in condition (4) we may assume that the morphism is a (surjective) standard smooth morphism. Similarly, when $\tau = syntomic$ or $\tau = étale$.

34.23. Properties of morphisms local in the fpqc topology on the source

Here are some properties of morphisms that are fpqc local on the source.

Lemma 34.23.1. *The property $\mathcal{P}(f) =$ "f is flat" is fpqc local on the source.*

Proof. Since flatness is defined in terms of the maps of local rings (Morphisms, Definition 28.26.1) what has to be shown is the following algebraic fact: Suppose $A \to B \to C$ are local homomorphisms of local rings, and assume $B \to C$ are flat. Then $A \to B$ is flat if and only if $A \to C$ is flat. If $A \to B$ is flat, then $A \to C$ is flat by Algebra, Lemma 10.36.3. Conversely, assume $A \to C$ is flat. Note that $B \to C$ is faithfully flat, see Algebra, Lemma 10.36.16. Hence $A \to B$ is flat by Algebra, Lemma 10.36.9. (Also see Morphisms, Lemma 28.26.11 for a direct proof.) □

Lemma 34.23.2. *Then property $\mathcal{P}(f : X \to Y) =$ "for every $x \in X$ the map of local rings $\mathcal{O}_{Y,f(x)} \to \mathcal{O}_{X,x}$ is injective" is fpqc local on the source.*

Proof. Omitted. This is just a (probably misguided) attempt to be playful. □

34.24. Properties of morphisms local in the fppf topology on the source

Here are some properties of morphisms that are fppf local on the source.

Lemma 34.24.1. *The property $\mathcal{P}(f) =$ "f is locally of finite presentation" is fppf local on the source.*

Proof. Being locally of finite presentation is Zariski local on the source and the target, see Morphisms, Lemma 28.22.2. It is a property which is preserved under composition, see Morphisms, Lemma 28.22.3. This proves (1), (2) and (3) of Lemma 34.22.3. The final condition (4) is Lemma 34.10.1. Hence we win. □

Lemma 34.24.2. *The property $\mathcal{P}(f) =$ "f is locally of finite type" is fppf local on the source.*

Proof. Being locally of finite type is Zariski local on the source and the target, see Morphisms, Lemma 28.16.2. It is a property which is preserved under composition, see Morphisms, Lemma 28.16.3, and a flat morphism locally of finite presentation is locally of finite type, see Morphisms, Lemma 28.22.8. This proves (1), (2) and (3) of Lemma 34.22.3. The final condition (4) is Lemma 34.10.2. Hence we win. □

Lemma 34.24.3. *The property $\mathcal{P}(f) =$ "f is open" is fppf local on the source.*

Proof. Being an open morphism is clearly Zariski local on the source and the target. It is a property which is preserved under composition, see Morphisms, Lemma 28.24.3, and a flat morphism of finite presentation is open, see Morphisms, Lemma 28.26.9 This proves (1), (2) and (3) of Lemma 34.22.3. The final condition (4) follows from Morphisms, Lemma 28.26.10. Hence we win. □

Lemma 34.24.4. *The property $\mathcal{P}(f) =$ "f is universally open" is fppf local on the source.*

Proof. Let $f : X \to Y$ be a morphism of schemes. Let $\{X_i \to X\}_{i \in I}$ be an fppf covering. Denote $f_i : X_i \to X$ the compositions. We have to show that f is universally open if and only if each f_i is universally open. If f is universally open, then also each f_i is universally open since the maps $X_i \to X$ are universally open and compositions of universally open morphisms are universally open (Morphisms, Lemmas 28.26.9 and 28.24.3). Conversely, assume each f_i is universally open. Let $Y' \to Y$ be a morphism of schemes. Denote $X' = Y' \times_Y X$ and $X'_i = Y' \times_Y X_i$. Note that $\{X'_i \to X'\}_{i \in I}$ is an fppf covering also. The morphisms $f'_i : X'_i \to Y'$ are open by assumption. Hence by the Lemma 34.24.3 above we conclude that $f' : X' \to Y'$ is open as desired. □

34.25. Properties of morphisms local in the syntomic topology on the source

Here are some properties of morphisms that are syntomic local on the source.

Lemma 34.25.1. *The property $\mathcal{P}(f) =$ "f is syntomic" is syntomic local on the source.*

Proof. Combine Lemma 34.22.3 with Morphisms, Lemma 28.32.2 (local for Zariski on source and target), Morphisms, Lemma 28.32.3 (pre-composing), and Lemma 34.10.4 (part (4)). □

34.26. Properties of morphisms local in the smooth topology on the source

Here are some properties of morphisms that are smooth local on the source.

Lemma 34.26.1. *The property $\mathcal{P}(f) =$ "f is smooth" is smooth local on the source.*

Proof. Combine Lemma 34.22.3 with Morphisms, Lemma 28.35.2 (local for Zariski on source and target), Morphisms, Lemma 28.35.4 (pre-composing), and Lemma 34.10.4 (part (4)). □

34.27. Properties of morphisms local in the étale topology on the source

Here are some properties of morphisms that are étale local on the source.

Lemma 34.27.1. *The property $\mathcal{P}(f) =$ "f is étale" is étale local on the source.*

Proof. Combine Lemma 34.22.3 with Morphisms, Lemma 28.37.2 (local for Zariski on source and target), Morphisms, Lemma 28.37.3 (pre-composing), and Lemma 34.10.4 (part (4)). □

Lemma 34.27.2. *The property $\mathcal{P}(f) =$ "f is locally quasi-finite" is étale local on the source.*

Proof. We are going to use Lemma 34.22.3. By Morphisms, Lemma 28.21.11 the property of being locally quasi-finite is local for Zariski on source and target. By Morphisms, Lemmas 28.21.12 and 28.37.6 we see the precomposition of a locally quasi-finite morphism by an étale morphism is locally quasi-finite. Finally, suppose that $X \to Y$ is a morphism of affine schemes and that $X' \to X$ is a surjective étale morphism of affine schemes such that $X' \to Y$ is locally quasi-finite. Then $X' \to Y$ is of finite type, and by Lemma 34.10.2 we see that $X \to Y$ is of finite type also. Moreover, by assumption $X' \to Y$ has finite fibres, and hence $X \to Y$ has finite fibres also. We conclude that $X \to Y$ is quasi-finite by Morphisms, Lemma 28.21.10. This proves the last assumption of Lemma 34.22.3 and finishes the proof. □

Lemma 34.27.3. *The property $\mathcal{P}(f) =$ "f is unramified" is étale local on the source. The property $\mathcal{P}(f) =$ "f is G-unramified" is étale local on the source.*

Proof. We are going to use Lemma 34.22.3. By Morphisms, Lemma 28.36.3 the property of being unramified (resp. G-unramified) is local for Zariski on source and target. By Morphisms, Lemmas 28.36.4 and 28.37.5 we see the precomposition of an unramified (resp. G-unramified) morphism by an étale morphism is unramified (resp. G-unramified). Finally, suppose that $X \to Y$ is a morphism of affine schemes and that $f : X' \to X$ is a surjective étale morphism of affine schemes such that $X' \to Y$ is unramified (resp. G-unramified). Then $X' \to Y$ is of finite type (resp. finite presentation), and by Lemma 34.10.2 (resp. Lemma 34.10.1) we see that

$X \to Y$ is of finite type (resp. finite presentation) also. By Morphisms, Lemma 28.35.16 we have a short exact sequence

$$0 \to f^*\Omega_{X/Y} \to \Omega_{X'/Y} \to \Omega_{X'/X} \to 0.$$

As $X' \to Y$ is unramified we see that the middle term is zero. Hence, as f is faithfully flat we see that $\Omega_{X/Y} = 0$. Hence $X \to Y$ is unramified (resp. G-unramified), see Morphisms, Lemma 28.36.2. This proves the last assumption of Lemma 34.22.3 and finishes the proof. □

34.28. Properties of morphisms étale local on source-and-target

Let $\mathcal{P}$ be a property of morphisms of schemes. There is an intuitive meaning to the phrase "$\mathcal{P}$ is étale local on the source and target". However, it turns out that this notion is not the same as asking $\mathcal{P}$ to be both étale local on the source and étale local on the target. Before we discuss this further we give two silly examples.

Example 34.28.1. Consider the property $\mathcal{P}$ of morphisms of schemes defined by the rule $\mathcal{P}(X \to Y) =$"Y is locally Noetherian". The reader can verify that this is étale local on the source and étale local on the target (omitted, see Lemma 34.12.1). But it is **not** true that if $f : X \to Y$ has $\mathcal{P}$ and $g : Y \to Z$ is étale, then $g \circ f$ has $\mathcal{P}$. Namely, f could be the identity on Y and g could be an open immersion of a locally Noetherian scheme Y into a non locally Noetherian scheme Z.

The following example is in some sense worse.

Example 34.28.2. Consider the property $\mathcal{P}$ of morphisms of schemes defined by the rule $\mathcal{P}(f : X \to Y) =$"for every $y \in Y$ which is a specialization of some $f(x)$, $x \in X$ the local ring $\mathcal{O}_{Y,y}$ is Noetherian". Let us verify that this is étale local on the source and étale local on the target. We will freely use Schemes, Lemma 25.13.2.

Local on the target: Let $\{g_i : Y_i \to Y\}$ be an étale covering. Let $f_i : X_i \to Y_i$ be the base change of f, and denote $h_i : X_i \to X$ the projection. Assume $\mathcal{P}(f)$. Let $f(x_i) \rightsquigarrow y_i$ be a specialization. Then $f(h_i(x_i)) \rightsquigarrow g_i(y_i)$ so $\mathcal{P}(f)$ implies $\mathcal{O}_{Y,g_i(y_i)}$ is Noetherian. Also $\mathcal{O}_{Y,g_i(y_i)} \to \mathcal{O}_{Y_i,y_i}$ is a localization of an étale ring map. Hence $\mathcal{O}_{Y_i,y_i}$ is Noetherian by Algebra, Lemma 10.29.1. Conversely, assume $\mathcal{P}(f_i)$ for all i. Let $f(x) \rightsquigarrow y$ be a specialization. Choose an i and $y_i \in Y_i$ mapping to y. Since x can be viewed as a point of $\mathrm{Spec}(\mathcal{O}_{Y,y}) \times_Y X$ and $\mathcal{O}_{Y,y} \to \mathcal{O}_{Y_i,y_i}$ is faithfully flat, there exists a point $x_i \in \mathrm{Spec}(\mathcal{O}_{Y_i,y_i}) \times_Y X$ mapping to x. Then $x_i \in X_i$, and $f_i(x_i)$ specializes to y_i. Thus we see that $\mathcal{O}_{Y_i,y_i}$ is Noetherian by $\mathcal{P}(f_i)$ which implies that $\mathcal{O}_{Y,y}$ is Noetherian by Algebra, Lemma 10.150.1.

Local on the source: Let $\{h_i : X_i \to X\}$ be an étale covering. Let $f_i : X_i \to Y$ be the composition $f \circ h_i$. Assume $\mathcal{P}(f)$. Let $f(x_i) \rightsquigarrow y$ be a specialization. Then $f(h_i(x_i)) \rightsquigarrow y$ so $\mathcal{P}(f)$ implies $\mathcal{O}_{Y,y}$ is Noetherian. Thus $\mathcal{P}(f_i)$ holds. Conversely, assume $\mathcal{P}(f_i)$ for all i. Let $f(x) \rightsquigarrow y$ be a specialization. Choose an i and $x_i \in X_i$ mapping to x. Then y is a specialization of $f_i(x_i) = f(x)$. Hence $\mathcal{P}(f_i)$ implies $\mathcal{O}_{Y,y}$ is Noetherian as desired.

We claim that there exists a commutative diagram

$$\begin{array}{ccc} U & \xrightarrow{h} & V \\ {\scriptstyle a}\downarrow & & \downarrow{\scriptstyle b} \\ X & \xrightarrow{f} & Y \end{array}$$

with surjective étale vertical arrows, such that h has $\mathcal{P}$ and f does not have $\mathcal{P}$. Namely, let

$$Y = \text{Spec}\left(\mathbf{C}[x_n; n \in \mathbf{Z}]/(x_n x_m; n \neq m)\right)$$

and let $X \subset Y$ be the open subscheme which is the complement of the point all of whose coordinates $x_n = 0$. Let $U = X$, let $V = X \coprod Y$, let a, b the obvious map, and let $h : U \to V$ be the inclusion of $U = X$ into the first summand of V. The claim above holds because U is locally Noetherian, but Y is not.

What should be the correct notion of a property which is étale local on the source-and-target? We think that, by analogy with Morphisms, Definition 28.15.1 it should be the following.

Definition 34.28.3. Let $\mathcal{P}$ be a property of morphisms of schemes. We say $\mathcal{P}$ is *étale local on source-and-target* if

(1) (stable under precomposing with étale maps) if $f : X \to Y$ is étale and $g : Y \to Z$ has $\mathcal{P}$, then $g \circ f$ has $\mathcal{P}$,
(2) (stable under étale base change) if $f : X \to Y$ has $\mathcal{P}$ and $Y' \to Y$ is étale, then the base change $f' : Y' \times_Y X \to Y'$ has $\mathcal{P}$, and
(3) (locality) given a morphism $f : X \to Y$ the following are equivalent
   (a) f has $\mathcal{P}$,
   (b) for every $x \in X$ there exists a commutative diagram

$$\begin{array}{ccc} U & \xrightarrow{h} & V \\ {\scriptstyle a}\downarrow & & \downarrow{\scriptstyle b} \\ X & \xrightarrow{f} & Y \end{array}$$

   with étale vertical arrows and $u \in U$ with $a(u) = x$ such that h has $\mathcal{P}$.

It turns out this definition excludes the behavior seen in Examples 34.28.1 and 34.28.2. We will compare this to the definition in the paper [**DM69**] by Deligne and Mumford in Remark 34.28.8. Moreover, a property which is étale local on the source-and-target is étale local on the source and étale local on the target. Finally, the converse is almost true as we will see in Lemma 34.28.6.

Lemma 34.28.4. *Let $\mathcal{P}$ be a property of morphisms of schemes which is étale local on source-and-target. Then*

(1) *$\mathcal{P}$ is étale local on the source,*
(2) *$\mathcal{P}$ is étale local on the target,*
(3) *$\mathcal{P}$ is stable under postcomposing with étale morphisms: if $f : X \to Y$ has $\mathcal{P}$ and $g : Y \to Z$ is étale, then $g \circ f$ has $\mathcal{P}$, and*
(4) *$\mathcal{P}$ has a permanence property: given $f : X \to Y$ and $g : Y \to Z$ étale such that $g \circ f$ has $\mathcal{P}$, then f has $\mathcal{P}$.*

Proof. We write everything out completely.

Proof of (1). Let $f : X \to Y$ be a morphism of schemes. Let $\{X_i \to X\}_{i \in I}$ be an étale covering of X. If each composition $h_i : X_i \to Y$ has $\mathcal{P}$, then for each $x \in X$ we can find an $i \in I$ and a point $x_i \in X_i$ mapping to x. Then $(X_i, x_i) \to (X, x)$ is an étale morphism of germs, and $\text{id}_Y : Y \to Y$ is an étale morphism, and h_i is as

in part (3) of Definition 34.28.3. Thus we see that f has $\mathcal{P}$. Conversely, if f has $\mathcal{P}$ then each $X_i \to Y$ has $\mathcal{P}$ by Definition 34.28.3 part (1).

Proof of (2). Let $f : X \to Y$ be a morphism of schemes. Let $\{Y_i \to Y\}_{i \in I}$ be an étale covering of Y. Write $X_i = Y_i \times_Y X$ and $h_i : X_i \to Y_i$ for the base change of f. If each $h_i : X_i \to Y_i$ has $\mathcal{P}$, then for each $x \in X$ we pick an $i \in I$ and a point $x_i \in X_i$ mapping to x. Then $(X_i, x_i) \to (X, x)$ is an étale morphism of germs, $Y_i \to Y$ is étale, and h_i is as in part (3) of Definition 34.28.3. Thus we see that f has $\mathcal{P}$. Conversely, if f has $\mathcal{P}$, then each $X_i \to Y_i$ has $\mathcal{P}$ by Definition 34.28.3 part (2).

Proof of (3). Assume $f : X \to Y$ has $\mathcal{P}$ and $g : Y \to Z$ is étale. For every $x \in X$ we can think of $(X, x) \to (X, x)$ as an étale morphism of germs, $Y \to Z$ is an étale morphism, and $h = f$ is as in part (3) of Definition 34.28.3. Thus we see that $g \circ f$ has $\mathcal{P}$.

Proof of (4). Let $f : X \to Y$ be a morphism and $g : Y \to Z$ étale such that $g \circ f$ has $\mathcal{P}$. Then by Definition 34.28.3 part (2) we see that $\mathrm{pr}_Y : Y \times_Z X \to Y$ has $\mathcal{P}$. But the morphism $(f, 1) : X \to Y \times_Z X$ is étale as a section to the étale projection $\mathrm{pr}_X : Y \times_Z X \to X$, see Morphisms, Lemma 28.37.18. Hence $f = \mathrm{pr}_Y \circ (f, 1)$ has $\mathcal{P}$ by Definition 34.28.3 part (1). □

The following lemma is the analogue of Morphisms, Lemma 28.15.4.

Lemma 34.28.5. *Let $\mathcal{P}$ be a property of morphisms of schemes which is étale local on source-and-target. Let $f : X \to Y$ be a morphism of schemes. The following are equivalent:*

(a) *f has property $\mathcal{P}$,*
(b) *for every $x \in X$ there exists an étale morphism of germs $a : (U, u) \to (X, x)$, an étale morphism $b : V \to Y$, and a morphism $h : U \to V$ such that $f \circ a = b \circ h$ and h has $\mathcal{P}$,*
(c) *for any commutative diagram*

$$\begin{array}{ccc} U & \xrightarrow{h} & V \\ {\scriptstyle a}\downarrow & & \downarrow{\scriptstyle b} \\ X & \xrightarrow{f} & Y \end{array}$$

with a, b étale the morphism h has $\mathcal{P}$,
(d) *for some diagram as in (c) with $a : U \to X$ surjective h has $\mathcal{P}$,*
(e) *there exists an étale covering $\{Y_i \to Y\}_{i \in I}$ such that each base change $Y_i \times_Y X \to Y_i$ has $\mathcal{P}$,*
(f) *there exists an étale covering $\{X_i \to X\}_{i \in I}$ such that each composition $X_i \to Y$ has $\mathcal{P}$,*
(g) *there exists an étale covering $\{Y_i \to Y\}_{i \in I}$ and for each $i \in I$ an étale covering $\{X_{ij} \to Y_i \times_Y X\}_{j \in J_i}$ such that each morphism $X_{ij} \to Y_i$ has $\mathcal{P}$.*

Proof. The equivalence of (a) and (b) is part of Definition 34.28.3. The equivalence of (a) and (e) is Lemma 34.28.4 part (2). The equivalence of (a) and (f) is Lemma 34.28.4 part (1). As (a) is now equivalent to (e) and (f) it follows that (a) equivalent to (g).

It is clear that (c) implies (a). If (a) holds, then for any diagram as in (c) the morphism $f \circ a$ has $\mathcal{P}$ by Definition 34.28.3 part (1), whereupon h has $\mathcal{P}$ by Lemma

34.28.4 part (4). Thus (a) and (c) are equivalent. It is clear that (c) implies (d). To see that (d) implies (a) assume we have a diagram as in (c) with $a : U \to X$ surjective and h having $\mathcal{P}$. Then $b \circ h$ has $\mathcal{P}$ by Lemma 34.28.4 part (3). Since $\{a : U \to X\}$ is an étale covering we conclude that f has $\mathcal{P}$ by Lemma 34.28.4 part (1). □

It seems that the result of the following lemma is not a formality, i.e., it actually uses something about the geometry of étale morphisms.

Lemma 34.28.6. *Let $\mathcal{P}$ be a property of morphisms of schemes. Assume*

(1) *$\mathcal{P}$ is étale local on the source,*
(2) *$\mathcal{P}$ is étale local on the target, and*
(3) *$\mathcal{P}$ is stable under postcomposing with open immersions: if $f : X \to Y$ has $\mathcal{P}$ and $Y \subset Z$ is an open subscheme then $X \to Z$ has $\mathcal{P}$.*

Then $\mathcal{P}$ is étale local on the source-and-target.

Proof. Let $\mathcal{P}$ be a property of morphisms of schemes which satisfies conditions (1), (2) and (3) of the lemma. By Lemma 34.22.2 we see that $\mathcal{P}$ is stable under precomposing with étale morphisms. By Lemma 34.18.2 we see that $\mathcal{P}$ is stable under étale base change. Hence it suffices to prove part (3) of Definition 34.28.3 holds.

More precisely, suppose that $f : X \to Y$ is a morphism of schemes which satisfies Definition 34.28.3 part (3)(b). In other words, for every $x \in X$ there exists an étale morphism $a_x : U_x \to X$, a point $u_x \in U_x$ mapping to x, an étale morphism $b_x : V_x \to Y$, and a morphism $h_x : U_x \to V_x$ such that $f \circ a_x = b_x \circ h_x$ and h_x has $\mathcal{P}$. The proof of the lemma is complete once we show that f has $\mathcal{P}$. Set $U = \coprod U_x$, $a = \coprod a_x$, $V = \coprod V_x$, $b = \coprod b_x$, and $h = \coprod h_x$. We obtain a commutative diagram

$$\begin{array}{ccc} U & \xrightarrow{h} & V \\ {\scriptstyle a}\downarrow & & \downarrow{\scriptstyle b} \\ X & \xrightarrow{f} & Y \end{array}$$

with a, b étale, a surjective. Note that h has $\mathcal{P}$ as each h_x does and $\mathcal{P}$ is étale local on the target. Because a is surjective and $\mathcal{P}$ is étale local on the source, it suffices to prove that $b \circ h$ has $\mathcal{P}$. This reduces the lemma to proving that $\mathcal{P}$ is stable under postcomposing with an étale morphism.

During the rest of the proof we let $f : X \to Y$ be a morphism with property $\mathcal{P}$ and $g : Y \to Z$ is an étale morphism. Consider the following statements:

(∅) With no additional assumptions $g \circ f$ has property $\mathcal{P}$.
(A) Whenever Z is affine $g \circ f$ has property $\mathcal{P}$.
(AA) Whenever X and Z are affine $g \circ f$ has property $\mathcal{P}$.
(AAA) Whenever X, Y, and Z are affine $g \circ f$ has property $\mathcal{P}$.

Once we have proved (∅) the proof of the lemma will be complete.

Claim 1: (AAA) ⇒ (AA). Namely, let $f : X \to Y$, $g : Y \to Z$ be as above with X, Z affine. As X is affine hence quasi-compact we can find finitely many affine open $Y_i \subset Y$, $i = 1, \ldots, n$ such that $X = \bigcup_{i=1,\ldots,n} f^{-1}(Y_i)$. Set $X_i = f^{-1}(Y_i)$. By Lemma 34.18.2 each of the morphisms $X_i \to Y_i$ has $\mathcal{P}$. Hence $\coprod_{i=1,\ldots,n} X_i \to \coprod_{i=1,\ldots,n} Y_i$ has $\mathcal{P}$ as $\mathcal{P}$ is étale local on the target. By (AAA) applied to $\coprod_{i=1,\ldots,n} X_i \to$

$\coprod_{i=1,\ldots,n} Y_i$ and the étale morphism $\coprod_{i=1,\ldots,n} Y_i \to Z$ we see that $\coprod_{i=1,\ldots,n} X_i \to Z$ has $\mathcal{P}$. Now $\{\coprod_{i=1,\ldots,n} X_i \to X\}$ is an étale covering, hence as $\mathcal{P}$ is étale local on the source we conclude that $X \to Z$ has $\mathcal{P}$ as desired.

Claim 2: (AAA) $\Rightarrow$ (A). Namely, let $f : X \to Y$, $g : Y \to Z$ be as above with Z affine. Choose an affine open covering $X = \bigcup X_i$. As $\mathcal{P}$ is étale local on the source we see that each $f|_{X_i} : X_i \to Y$ has $\mathcal{P}$. By (AA), which follows from (AAA) according to Claim 1, we see that $X_i \to Z$ has $\mathcal{P}$ for each i. Since $\{X_i \to X\}$ is an étale covering and $\mathcal{P}$ is étale local on the source we conclude that $X \to Z$ has $\mathcal{P}$.

Claim 3: (AAA) $\Rightarrow$ ($\emptyset$). Namely, let $f : X \to Y$, $g : Y \to Z$ be as above. Choose an affine open covering $Z = \bigcup Z_i$. Set $Y_i = g^{-1}(Z_i)$ and $X_i = f^{-1}(Y_i)$. By Lemma 34.18.2 each of the morphisms $X_i \to Y_i$ has $\mathcal{P}$. By (A), which follows from (AAA) according to Claim 2, we see that $X_i \to Z_i$ has $\mathcal{P}$ for each i. Since $\mathcal{P}$ is local on the target and $X_i = (g \circ f)^{-1}(Z_i)$ we conclude that $X \to Z$ has $\mathcal{P}$.

Thus to prove the lemma it suffices to prove (AAA). Let $f : X \to Y$ and $g : Y \to Z$ be as above X, Y, Z affine. Note that an étale morphism of affines has universally bounded fibres, see Morphisms, Lemma 28.37.6 and Lemma 28.50.8. Hence we can do induction on the integer n bounding the degree of the fibres of $Y \to Z$. See Morphisms, Lemma 28.50.7 for a description of this integer in the case of an étale morphism. If $n = 1$, then $Y \to Z$ is an open immersion, see Lemma 34.21.2, and the result follows from assumption (3) of the lemma. Assume $n > 1$.

Consider the following commutative diagram

$$\begin{array}{ccccc} X \times_Z Y & \xrightarrow[f_Y]{} & Y \times_Z Y & \xrightarrow[\text{pr}]{} & Y \\ \downarrow & & \downarrow & & \downarrow \\ X & \xrightarrow{f} & Y & \xrightarrow{g} & Z \end{array}$$

Note that we have a decomposition into open and closed subschemes $Y \times_Z Y = \Delta_{Y/Z}(Y) \amalg Y'$, see Morphisms, Lemma 28.36.13. As a base change the degrees of the fibres of the second projection $\text{pr} : Y \times_Z Y \to Y$ are bounded by n, see Morphisms, Lemma 28.50.4. On the other hand, $\text{pr}|_{\Delta(Y)} : \Delta(Y) \to Y$ is an isomorphism and every fibre has exactly one point. Thus, on applying Morphisms, Lemma 28.50.7 we conclude the degrees of the fibres of the restriction $\text{pr}|_{Y'} : Y' \to Y$ are bounded by $n - 1$. Set $X' = f_Y^{-1}(Y')$. Picture

$$\begin{array}{ccccc} X \amalg X' & \xrightarrow[f \amalg f']{} & \Delta(Y) \amalg Y' & \longrightarrow & Y \\ \| & & \| & & \| \\ X \times_Z Y & \xrightarrow{f_Y} & Y \times_Z Y & \xrightarrow{\text{pr}} & Y \end{array}$$

As $\mathcal{P}$ is étale local on the target and hence stable under étale base change (see Lemma 34.18.2) we see that f_Y has $\mathcal{P}$. Hence, as $\mathcal{P}$ is étale local on the source, $f' = f_Y|_{X'}$ has $\mathcal{P}$. By induction hypothesis we see that $X' \to Y$ has $\mathcal{P}$. As $\mathcal{P}$ is local on the source, and $\{X \to X \times_Z Y, X' \to X \times_Y Z\}$ is an étale covering, we conclude that $\text{pr} \circ f_Y$ has $\mathcal{P}$. Note that $g \circ f$ can be viewed as a morphism $g \circ f : X \to g(Y)$. As $\text{pr} \circ f_Y$ is the pullback of $g \circ f : X \to g(Y)$ via the étale covering $\{Y \to g(Y)\}$, and as $\mathcal{P}$ is étale local on the target, we conclude that $g \circ f : X \to g(Y)$ has property $\mathcal{P}$. Finally, applying assumption (3) of the lemma once more we conclude that $g \circ f : X \to Z$ has property $\mathcal{P}$. $\square$

Remark 34.28.7. Using Lemma 34.28.6 and the work done in the earlier sections of this chapter it is easy to make a list of types of morphisms which are étale local on the source-and-target. In each case we list the lemma which implies the property is étale local on the source and the lemma which implies the property is étale local on the target. In each case the third assumption of Lemma 34.28.6 is trivial to check, and we omit it. Here is the list:

(1) flat, see Lemmas 34.23.1 and 34.19.13,
(2) locally of finite presentation, see Lemmas 34.24.1 and 34.19.9,
(3) locally finite type, see Lemmas 34.24.2 and 34.19.8,
(4) universally open, see Lemmas 34.24.4 and 34.19.4,
(5) syntomic, see Lemmas 34.25.1 and 34.19.24,
(6) smooth, see Lemmas 34.26.1 and 34.19.25,
(7) étale, see Lemmas 34.27.1 and 34.19.27,
(8) locally quasi-finite, see Lemmas 34.27.2 and 34.19.22,
(9) unramified, see Lemmas 34.27.3 and 34.19.26,
(10) G-unramified, see Lemmas 34.27.3 and 34.19.26, and
(11) add more here as needed.

Remark 34.28.8. At this point we have three possible definitions of what it means for a property $\mathcal{P}$ of morphisms to be "étale local on the source and target":

(ST) $\mathcal{P}$ is étale local on the source and $\mathcal{P}$ is étale local on the target,
(DM) (the definition in the paper [**DM69**, Page 100] by Deligne and Mumford) for every diagram

$$\begin{array}{ccc} U & \xrightarrow{h} & V \\ {\scriptstyle a}\downarrow & & \downarrow{\scriptstyle b} \\ X & \xrightarrow{f} & Y \end{array}$$

with surjective étale vertical arrows we have $\mathcal{P}(h) \Leftrightarrow \mathcal{P}(f)$, and
(SP) $\mathcal{P}$ is étale local on the source-and-target.

In this section we have seen that (SP) $\Rightarrow$ (DM) $\Rightarrow$ (ST). The Examples 34.28.1 and 34.28.2 show that neither implication can be reversed. Finally, Lemma 34.28.6 shows that the difference disappears when looking at properties of morphisms which are stable under postcomposing with open immersions, which in practice will always be the case.

34.29. Properties of morphisms of germs local on source-and-target

In this section we discuss the analogue of the material in Section 34.28 for morphisms of germs of schemes.

Definition 34.29.1. Let $\mathcal{Q}$ be a property of morphisms of germs of schemes. We say $\mathcal{Q}$ is *étale local on the source-and-target* if for any commutative diagram

$$\begin{array}{ccc} (U', u') & \xrightarrow{h'} & (V', v') \\ {\scriptstyle a}\downarrow & & \downarrow{\scriptstyle b} \\ (U, u) & \xrightarrow{h} & (V, v) \end{array}$$

with étale vertical arrows we have $\mathcal{Q}(h) \Leftrightarrow \mathcal{Q}(h')$.

Lemma 34.29.2. *Let $\mathcal{P}$ be a property of morphisms of schemes which is étale local on the source-and-target. Consider the property $\mathcal{Q}$ of morphisms of germs defined by the rule*

$$\mathcal{Q}((X, x) \to (S, s)) \Leftrightarrow \text{there exists a representative } U \to S \text{ which has } \mathcal{P}$$

Then $\mathcal{Q}$ is étale local on the source-and-target as in Definition 34.29.1.

Proof. We first remark that as $\mathcal{P}$ is étale local on the source, see Lemma 34.28.4, if $(X, x) \to (S, s)$ has $\mathcal{P}$, then there are arbitrarily small neighbourhoods U of x in X such that a representative $U \to S$ of $(X, x) \to (S, s)$ has $\mathcal{P}$. We will use this without further mention. Let

$$\begin{array}{ccc} (U', u') & \xrightarrow[h']{} & (V', v') \\ {\scriptstyle a}\downarrow & & \downarrow{\scriptstyle b} \\ (U, u) & \xrightarrow{h} & (V, v) \end{array}$$

be as in Definition 34.29.1. We will use a rather pedantic notation in order to distinguish between morphisms of germs and their representatives in this proof.

If $\mathcal{P}(h)$ holds, then $\mathcal{P}$ holds for a representative $h_1 : U_1 \to V$ of h. Let $a_1 : U_1' \to U$ be a representative of a which is étale with $a_1(U_1') \subset U_1$. As $\mathcal{P}$ is stable under precomposing with étale morphisms we see that $h_1 \circ a_1 : U_1' \to V$ has $\mathcal{P}$. Moreover, $h_1 \circ a_1 : U_1' \to V$ is a representative of $b \circ h'$ by the commutativity of the diagram. Choose a representative $b_1 : V_1' \to V$ of b. Choose a representative $h_1' : U_2' \to V'$ with $h_1'(U_1') \subset V_1'$, $U_2' \subset U_1'$, and $(h_1 \circ a_1)|_{U_2'} = b_1 \circ h_1'$. Then we see that $b_1 \circ h_1'$ has $\mathcal{P}$. Hence h' has $\mathcal{P}$ by Lemma 34.28.4 part (4).

Conversely, suppose $\mathcal{P}(h')$ holds. Choose a representative $b_1 : V_1' \to V$ of b. Choose a representative $h_1' : U_1' \to V'$ with $\mathcal{P}$ and with $h_1'(U_1') \subset V_1'$. Then $b_1 \circ h_1'$ has $\mathcal{P}$ by Lemma 34.28.4 part (3). Moreover, $b_1 \circ h_1' : U_1' \to V$ is a representative of $h \circ a$ by the commutativity of the diagram. Choose a representative $h_1 : U_1 \to V$ of h. Choose a representative $a_1 : U_2' \to U$ with $a_1(U_2') \subset U_1$, $U_2' \subset U_1'$, and $h_1 \circ a_1 = (b_1 \circ h_1')|_{U_2'}$. The we see that $h_1 \circ a_1$ has $\mathcal{P}$. As $\mathcal{P}$ is étale local on the source we conclude that $h_1|_{a_1(U_2')}$ has $\mathcal{P}$ and we win. □

Lemma 34.29.3. *Let $\mathcal{P}$ be a property of morphisms of schemes which is étale local on source-and-target. Let Q be the associated property of morphisms of germs, see Lemma 34.29.2. Let $f : X \to Y$ be a morphism of schemes. The following are equivalent:*

1. *f has property $\mathcal{P}$, and*
2. *for every $x \in X$ the morphism of germs $(X, x) \to (Y, f(x))$ has property $\mathcal{Q}$.*

Proof. The implication (1) ⇒ (2) is direct from the definitions. The implication (2) ⇒ (1) also follows from part (3) of Definition 34.28.3. □

A morphism of germs $(X, x) \to (S, s)$ determines a well defined map of local rings. Hence the following lemma makes sense.

Lemma 34.29.4. *The property of morphisms of germs*

$$\mathcal{P}((X, x) \to (S, s)) = \mathcal{O}_{S,s} \to \mathcal{O}_{X,x} \text{ is flat}$$

is étale local on the source-and-target.

Proof. Given a diagram as in Definition 34.29.1 we obtain the following diagram of local homomorphisms of local rings

$$\begin{array}{ccc} \mathcal{O}_{U',u'} & \longleftarrow & \mathcal{O}_{V',v'} \\ \uparrow & & \uparrow \\ \mathcal{O}_{U,u} & \longleftarrow & \mathcal{O}_{V,v} \end{array}$$

Note that the vertical arrows are localizations of étale ring maps, in particular they are essentially of finite presentation, flat, and unramified (see Algebra, Section 10.136). In particular the vertical maps are faithfully flat, see Algebra, Lemma 10.36.16. Now, if the upper horizontal arrow is flat, then the lower horizontal arrow is flat by an application of Algebra, Lemma 10.36.9 with $R = \mathcal{O}_{V,v}$, $S = \mathcal{O}_{U,u}$ and $M = \mathcal{O}_{U',u'}$. If the lower horizontal arrow is flat, then the ring map

$$\mathcal{O}_{V',v'} \otimes_{\mathcal{O}_{V,v}} \mathcal{O}_{U,u} \longleftarrow \mathcal{O}_{V',v'}$$

is flat by Algebra, Lemma 10.36.6. And the ring map

$$\mathcal{O}_{U',u'} \longleftarrow \mathcal{O}_{V',v'} \otimes_{\mathcal{O}_{V,v}} \mathcal{O}_{U,u}$$

is a localization of a map between étale ring extensions of $\mathcal{O}_{U,u}$, hence flat by Algebra, Lemma 10.136.9. □

Lemma 34.29.5. *Consider a commutative diagram of morphisms of schemes*

$$\begin{array}{ccc} U' & \longrightarrow & V' \\ \downarrow & & \downarrow \\ U & \longrightarrow & V \end{array}$$

with étale vertical arrows and a point $v' \in U'$ mapping to $v \in U$. Then the morphism of fibres $U'_{v'} \to U_v$ is étale.

Proof. Note that $U'_v \to U_v$ is étale as a base change of the étale morphism $U' \to U$. The scheme U'_v is a scheme over V'_v. By Morphisms, Lemma 28.37.7 the scheme V'_v is a disjoint union of spectra of finite separable field extensions of $\kappa(v)$. One of these is $v' = \mathrm{Spec}(\kappa(v'))$. Hence $U'_{v'}$ is an open and closed subscheme of U'_v and it follows that $U'_{v'} \to U'_v \to U_v$ is étale (as a composition of an open immersion and an étale morphism, see Morphisms, Section 28.37). □

Given a morphism of germs of schemes $(X, x) \to (S, s)$ we can define the *fibre* as the isomorphism class of germs (U_s, x) where $U \to S$ is any representative. We will often abuse notation and just write (X_s, x).

Lemma 34.29.6. *Let $d \in \{0, 1, 2, \ldots, \infty\}$. The property of morphisms of germs*

$$\mathcal{P}_d((X, x) \to (S, s)) = \textit{the local ring } \mathcal{O}_{X_s,x} \textit{ of the fibre has dimension } d$$

is étale local on the source-and-target.

Proof. Given a diagram as in Definition 34.29.1 we obtain an étale morphism of fibres $U'_{v'} \to U_v$ mapping u' to u, see Lemma 34.29.5. Hence the result follows from Lemma 34.17.3. □

Lemma 34.29.7. *Let $r \in \{0, 1, 2, \ldots, \infty\}$. The property of morphisms of germs*

$$\mathcal{P}_r((X, x) \to (S, s)) \Leftrightarrow trdeg_{\kappa(s)}\kappa(x) = r$$

is étale local on the source-and-target.

Proof. Given a diagram as in Definition 34.29.1 we obtain the following diagram of local homomorphisms of local rings

$$\begin{array}{ccc} \mathcal{O}_{U',u'} & \longleftarrow & \mathcal{O}_{V',v'} \\ \uparrow & & \uparrow \\ \mathcal{O}_{U,u} & \longleftarrow & \mathcal{O}_{V,v} \end{array}$$

Note that the vertical arrows are localizations of étale ring maps, in particular they are unramified (see Algebra, Section 10.136). Hence $\kappa(u) \subset \kappa(u')$ and $\kappa(v) \subset \kappa(v')$ are finite separable field extensions. Thus we have $\mathrm{trdeg}_{\kappa(v)}\kappa(u) = \mathrm{trdeg}_{\kappa(v')}\kappa(u)$ which proves the lemma. □

Let (X, x) be a germ of a scheme. The dimension of X at x is the minimum of the dimensions of open neighbourhoods of x in X, and any small enough open neighbourhood has this dimension. Hence this is an invariant of the isomorphism class of the germ. We denote this simply $\dim_x(X)$.

Lemma 34.29.8. *Let $d \in \{0, 1, 2, \ldots, \infty\}$. The property of morphisms of germs*

$$\mathcal{P}_d((X, x) \to (S, s)) \Leftrightarrow \dim_x(X_s) = d$$

is étale local on the source-and-target.

Proof. Given a diagram as in Definition 34.29.1 we obtain an étale morphism of fibres $U'_{v'} \to U_v$ mapping u' to u, see Lemma 34.29.5. Hence now the equality $\dim_u(U_v) = \dim_{u'}(U'_{v'})$ follows from Lemma 34.17.2. □

34.30. Descent data for schemes over schemes

Most of the arguments in this section are formal relying only on the definition of a descent datum. In Simplicial Spaces, Section 63.8 we will examine the relationship with simplicial schemes which will somewhat clarify the situation.

Definition 34.30.1. Let $f : X \to S$ be a morphism of schemes.

(1) Let $V \to X$ be a scheme over X. A *descent datum for $V/X/S$* is an isomorphism $\varphi : V \times_S X \to X \times_S V$ of schemes over $X \times_S X$ satisfying the *cocycle condition* that the diagram

$$\begin{array}{ccc} V \times_S X \times_S X & \xrightarrow{\varphi_{02}} & X \times_S X \times_S V \\ & {\scriptstyle \varphi_{01}} \searrow \quad \nearrow {\scriptstyle \varphi_{12}} & \\ & X \times_S V \times_S X & \end{array}$$

commutes (with obvious notation).

(2) We also say that the pair $(V/X, \varphi)$ is a *descent datum relative to $X \to S$*.

(3) A *morphism* $f : (V/X, \varphi) \to (V'/X, \varphi')$ *of descent data relative to* $X \to S$ is a morphism $f : V \to V'$ of schemes over X such that the diagram

$$\begin{array}{ccc} V \times_S X & \xrightarrow{\varphi} & X \times_S V \\ {\scriptstyle f\times \mathrm{id}_X}\downarrow & & \downarrow{\scriptstyle \mathrm{id}_X \times f} \\ V' \times_S X & \xrightarrow{\varphi'} & X \times_S V' \end{array}$$

commutes.

There are all kinds of "miraculous" identities which arise out of the definition above. For example the pullback of φ via the diagonal morphism $\Delta : X \to X \times_S X$ can be seen as a morphism $\Delta^*\varphi : V \to V$. This because $X \times_{\Delta, X\times_S X} (V \times_S X) = V$ and also $X \times_{\Delta, X\times_S X} (X \times_S V) = V$. In fact, $\Delta^*\varphi$ is equal to the identity. This is a good exercise if you are unfamiliar with this material.

Remark 34.30.2. Let $X \to S$ be a morphism of schemes. Let $(V/X, \varphi)$ be a descent datum relative to $X \to S$. We may think of the isomorphism φ as an isomorphism

$$(X \times_S X) \times_{\mathrm{pr}_0, X} V \longrightarrow (X \times_S X) \times_{\mathrm{pr}_1, X} V$$

of schemes over $X \times_S X$. So loosely speaking one may think of φ as a map $\varphi : \mathrm{pr}_0^* V \to \mathrm{pr}_1^* V$[7]. The cocycle condition then says that $\mathrm{pr}_{02}^*\varphi = \mathrm{pr}_{12}^*\varphi \circ \mathrm{pr}_{01}^*\varphi$. In this way it is very similar to the case of a descent datum on quasi-coherent sheaves.

Here is the definition in case you have a family of morphisms with fixed target.

Definition 34.30.3. Let S be a scheme. Let $\{X_i \to S\}_{i\in I}$ be a family of morphisms with target S.

(1) A *descent datum* (V_i, φ_{ij}) *relative to the family* $\{X_i \to S\}$ is given by a scheme V_i over X_i for each $i \in I$, an isomorphism $\varphi_{ij} : V_i \times_S X_j \to X_i \times_S V_j$ of schemes over $X_i \times_S X_j$ for each pair $(i, j) \in I^2$ such that for every triple of indices $(i, j, k) \in I^3$ the diagram

$$\begin{array}{ccccc} V_i \times_S X_j \times_S X_k & & \xrightarrow{\quad\mathrm{pr}_{02}^*\varphi_{ik}\quad} & & X_i \times_S X_j \times_S V_k \\ & {\scriptstyle \mathrm{pr}_{01}^*\varphi_{ij}}\searrow & & \nearrow{\scriptstyle \mathrm{pr}_{12}^*\varphi_{jk}} & \\ & & X_i \times_S V_j \times_S X_k & & \end{array}$$

of schemes over $X_i \times_S X_j \times_S X_k$ commutes (with obvious notation).

(2) A *morphism* $\psi : (V_i, \varphi_{ij}) \to (V_i', \varphi_{ij}')$ *of descent data* is given by a family $\psi = (\psi_i)_{i\in I}$ of morphisms of X_i-schemes $\psi_i : V_i \to V_i'$ such that all the diagrams

$$\begin{array}{ccc} V_i \times_S X_j & \xrightarrow{\varphi_{ij}} & X_i \times_S V_j \\ {\scriptstyle \psi_i\times \mathrm{id}}\downarrow & & \downarrow{\scriptstyle \mathrm{id}\times \psi_j} \\ V_i' \times_S X_j & \xrightarrow{\varphi_{ij}'} & X_i \times_S V_j' \end{array}$$

commute.

[7] Unfortunately, we have chosen the "wrong" direction for our arrow here. In Definitions 34.30.1 and 34.30.3 we should have the opposite direction to what was done in Definition 34.2.1 by the general principle that "functions" and "spaces" are dual.

This is the notion that comes up naturally for example when the question arises whether the fibred category of relative curves is a stack in the fpqc topology (it isn't – at least not if you stick to schemes).

Remark 34.30.4. Let S be a scheme. Let $\{X_i \to S\}_{i\in I}$ be a family of morphisms with target S. Let (V_i, φ_{ij}) be a descent datum relative to $\{X_i \to S\}$. We may think of the isomorphisms φ_{ij} as isomorphisms

$$(X_i \times_S X_j) \times_{\mathrm{pr}_0, X_i} V_i \longrightarrow (X_i \times_S X_j) \times_{\mathrm{pr}_1, X_j} V_j$$

of schemes over $X_i \times_S X_j$. So loosely speaking one may think of φ_{ij} as an isomorphism $\mathrm{pr}_0^* V_i \to \mathrm{pr}_1^* V_j$ over $X_i \times_S X_j$. The cocycle condition then says that $\mathrm{pr}_{02}^* \varphi_{ik} = \mathrm{pr}_{12}^* \varphi_{jk} \circ \mathrm{pr}_{01}^* \varphi_{ij}$. In this way it is very similar to the case of a descent datum on quasi-coherent sheaves.

The reason we will usually work with the version of a family consisting of a single morphism is the following lemma.

Lemma 34.30.5. *Let S be a scheme. Let $\{X_i \to S\}_{i\in I}$ be a family of morphisms with target S. Set $X = \coprod_{i\in I} X_i$, and consider it as an S-scheme. There is a canonical equivalence of categories*

$$\begin{array}{c}\text{category of descent data} \\ \text{relative to the family } \{X_i \to S\}_{i\in I}\end{array} \longrightarrow \begin{array}{c}\text{category of descent data} \\ \text{relative to } X/S\end{array}$$

which maps (V_i, φ_{ij}) to (V, φ) with $V = \coprod_{i\in I} V_i$ and $\varphi = \coprod \varphi_{ij}$.

Proof. Observe that $X \times_S X = \coprod_{ij} X_i \times_S X_j$ and similarly for higher fibre products. Giving a morphism $V \to X$ is exactly the same as giving a family $V_i \to X_i$. And giving a descent datum φ is exactly the same as giving a family φ_{ij}. □

Lemma 34.30.6. *Pullback of descent data for schemes over schemes.*

(1) *Let*

$$\begin{array}{ccc} X' & \xrightarrow{f} & X \\ {\scriptstyle a'}\downarrow & & \downarrow{\scriptstyle a} \\ S' & \xrightarrow{h} & S \end{array}$$

be a commutative diagram of morphisms of schemes. The construction

$$(V \to X, \varphi) \longmapsto f^*(V \to X, \varphi) = (V' \to X', \varphi')$$

where $V' = X' \times_X V$ and where φ' is defined as the composition

$$\begin{array}{ccccc} V' \times_{S'} X' & = & (X' \times_X V) \times_{S'} X' & = & (X' \times_{S'} X') \times_{X\times_S X} (V \times_S X) \\ & & & & \downarrow{\scriptstyle id\times\varphi} \\ X' \times_{S'} V' & = & X' \times_{S'} (X' \times_X V) & = & (X' \times_S X') \times_{X\times_S X} (X \times_S V) \end{array}$$

defines a functor from the category of descent data relative to $X \to S$ to the category of descent data relative to $X' \to S$.

(2) *Given two morphisms $f_i : X' \to X$, $i = 0, 1$ making the diagram commute the functors f_0^* and f_1^* are canonically isomorphic.*

Proof. We omit the proof of (1), but we remark that the morphism φ' is the morphism $(f' \times f')^*\varphi$ in the notation introduced in Remark 34.30.2. For (2) we indicate which morphism $f_0^*V \to f_1^*V$ gives the functorial isomorphism. Namely, since f_0 and f_1 both fit into the commutative diagram we see there is a unique morphism $r : X' \to X \times_S X$ with $f_i = \mathrm{pr}_i \circ r$. Then we take

$$\begin{aligned} f_0^*V &= X' \times_{f_0, X} V \\ &= X' \times_{\mathrm{pr}_0 \circ r, X} V \\ &= X' \times_{r, X \times_S X} (X \times_S X) \times_{\mathrm{pr}_0 X} V \\ &\xrightarrow{\varphi} X' \times_{r, X \times_S X} (X \times_S X) \times_{\mathrm{pr}_1 X} V \\ &= X' \times_{\mathrm{pr}_1 \circ r, X} V \\ &= X' \times_{f_1, X} V \\ &= f_1^*V \end{aligned}$$

We omit the verification that this works. □

Definition 34.30.7. With $S, S', X, X', f, a, a', h$ as in Lemma 34.30.6 the functor

$$(V, \varphi) \longmapsto f^*(V, \varphi)$$

constructed in that lemma is called the *pullback functor* on descent data.

Lemma 34.30.8 (Pullback of descent data for schemes over families)**.** *Let $\mathcal{U} = \{U_i \to S'\}_{i \in I}$ and $\mathcal{V} = \{V_j \to S\}_{j \in J}$ be families of morphisms with fixed target. Let $\alpha : I \to J$, $h : S' \to S$ and $g_i : U_i \to V_{\alpha(i)}$ be a morphism of families of maps with fixed target, see Sites, Definition 7.8.1.*

(1) *Let $(Y_j, \varphi_{jj'})$ be a descent datum relative to the family $\{V_j \to S'\}$. The system*
$$\big(g_i^* Y_{\alpha(i)}, (g_i \times g_{i'})^* \varphi_{\alpha(i)\alpha(i')}\big)$$
(with notation as in Remark 34.30.4) is a descent datum relative to $\mathcal{V}$.

(2) *This construction defines a functor between descent data relative to $\mathcal{U}$ and descent data relative to $\mathcal{V}$.*

(3) *Given a second $\alpha' : I \to J$, $h' : S' \to S$ and $g_i' : U_i \to V_{\alpha'(i)}$ morphism of families of maps with fixed target, then if $h = h'$ the two resulting functors between descent data are canonically isomorphic.*

(4) *These functors agree, via Lemma 34.30.5, with the pullback functors constructed in Lemma 34.30.6.*

Proof. This follows from Lemma 34.30.6 via the correspondence of Lemma 34.30.5. □

Definition 34.30.9. With $\mathcal{U} = \{U_i \to S'\}_{i \in I}$, $\mathcal{V} = \{V_j \to S\}_{j \in J}$, $\alpha : I \to J$, $h : S' \to S$, and $g_i : U_i \to V_{\alpha(i)}$ as in Lemma 34.30.8 the functor

$$(Y_j, \varphi_{jj'}) \longmapsto (g_i^* Y_{\alpha(i)}, (g_i \times g_{i'})^* \varphi_{\alpha(i)\alpha(i')})$$

constructed in that lemma is called the *pullback functor* on descent data.

If $\mathcal{U}$ and $\mathcal{V}$ have the same target S, and if $\mathcal{U}$ refines $\mathcal{V}$ (see Sites, Definition 7.8.1) but no explicit pair (α, g_i) is given, then we can still talk about the pullback functor since we have seen in Lemma 34.30.8 that the choice of the pair does not matter (up to a canonical isomorphism).

Definition 34.30.10. Let S be a scheme. Let $f : X \to S$ be a morphism of schemes.

(1) Given a scheme U over S we have the *trivial descent datum* of U relative to $\mathrm{id} : S \to S$, namely the identity morphism on U.
(2) By Lemma 34.30.6 we get a *canonical descent datum* on $X \times_S U$ relative to $X \to S$ by pulling back the trivial descent datum via f. We often denote $(X \times_S U, can)$ this descent datum.
(3) Let $f : X \to S$ be a morphism of schemes. A descent datum (V, φ) relative to X/S is is called *effective* if (V, φ) is isomorphic to the canonical descent datum $(X \times_S U, can)$ for some scheme U over S.

Thus being effective means there exists a scheme U over S and an isomorphism $\psi : V \to X \times_S U$ of X-schemes such that φ is equal to the composition

$$V \times_S X \xrightarrow{\psi \times \mathrm{id}_X} X \times_S U \times_S X = X \times_S X \times_S U \xrightarrow{\mathrm{id}_X \times \psi^{-1}} X \times_S V$$

Definition 34.30.11. Let S be a scheme. Let $\{X_i \to S\}$ be a family of morphisms with target S.

(1) Given a scheme U over S we have a *canonical descent datum* on the family of schemes $X_i \times_S U$ by pulling back the trivial descent datum for U relative to $\{\mathrm{id} : S \to S\}$. We denote this descent datum $(X_i \times_S U, can)$.
(2) A descent datum (V_i, φ_{ij}) relative to $\{X_i \to S\}$ is called *effective* if there exists a scheme U over S such that (V_i, φ_{ij}) is isomorphic to $(X_i \times_S U, can)$.

34.31. Fully faithfulness of the pullback functors

It turns out that the pullback functor between descent data for fpqc-coverings is fully faithful. In other words, morphisms of schemes satisfy fpqc descent. The goal of this section is to prove this. The reader is encouraged instead to prove this him/herself. The key is to use Lemma 34.9.3.

Lemma 34.31.1. *A surjective and flat morphism is an epimorphism in the category of schemes.*

Proof. Suppose we have $h : X' \to X$ surjective and flat and $a, b : X \to Y$ morphisms such that $a \circ h = b \circ h$. As h is surjective we see that a and b agree on underlying topological spaces. Pick $x' \in X'$ and set $x = h(x')$ and $y = a(x) = b(x)$. Consider the local ring maps

$$a_x^\sharp, b_x^\sharp : \mathcal{O}_{Y,y} \to \mathcal{O}_{X,x}$$

These become equal when composed with the flat local homomorphism $h_{x'}^\sharp : \mathcal{O}_{X,x} \to \mathcal{O}_{X',x'}$. Since a flat local homomorphism is faithfully flat (Algebra, Lemma 10.36.16) we conclude that $h_{x'}^\sharp$ is injective. Hence $a_x^\sharp = b_x^\sharp$ which implies $a = b$ as desired. □

Lemma 34.31.2. *Let $h : S' \to S$ be a surjective, flat morphism of schemes. The base change functor*

$$Sch/S \longrightarrow Sch/S', \quad X \longmapsto S' \times_S X$$

is faithful.

Proof. Let X_1, X_2 be schemes over S. Let $\alpha, \beta : X_2 \to X_1$ be morphisms over S. If α, β base change to the same morphism then we get a commutative diagram as follows

$$\begin{array}{ccccc} X_2 & \longleftarrow & S' \times_S X_2 & \longrightarrow & X_2 \\ \downarrow{\scriptstyle \alpha} & & \downarrow & & \downarrow{\scriptstyle \beta} \\ X_1 & \longleftarrow & S' \times_S X_1 & \longrightarrow & X_1 \end{array}$$

Hence it suffices to show that $S' \times_S X_2 \to X_2$ is an epimorphism. As the base change of a surjective and flat morphism it is surjective and flat (see Morphisms, Lemmas 28.11.4 and 28.26.7). Hence the lemma follows from Lemma 34.31.1. □

Lemma 34.31.3. *In the situation of Lemma 34.30.6 assume that $f : X' \to X$ is surjective and flat. Then the pullback functor is faithful.*

Proof. Let (V_i, φ_i), $i = 1, 2$ be descend data for $X \to S$. Let $\alpha, \beta : V_1 \to V_2$ be morphisms of descent data. Suppose that $f^*\alpha = f^*\beta$. Our task is to show that $\alpha = \beta$. Note that α, β are morphisms of schemes over X, and that $f^*\alpha$, $f^*\beta$ are simply the base changes of α, β to morphisms over X'. Hence the lemma follows from Lemma 34.31.2. □

Here is the key lemma of this section.

Lemma 34.31.4. *In the situation of Lemma 34.30.6 assume*

(1) *$\{f : X' \to X\}$ is an fpqc covering (for example if f is surjective, flat, and quasi-compact), and*
(2) *$f \times f : X' \times_{S'} X' \to X \times_S X$ is surjective and flat*[8].

Then the pullback functor is fully faithful.

Proof. Assumption (1) implies that f is surjective and flat. Hence the pullback functor is faithful by Lemma 34.31.3. Let (V, φ) and (W, ψ) be two descent data relative to $X \to S$. Set $(V', \varphi') = f^*(V, \varphi)$ and $(W', \psi') = f^*(W, \psi)$. Let $\alpha' : V' \to W'$ be a morphism of descent data for X' over S'. We have to show there exists a morphism $\alpha : V \to W$ of descent data for X over S whose pullback is α'.

Recall that V' is the base change of V by f and that φ' is the base change of φ by $f \times f$ (see Remark 34.30.2). By assumption the diagram

$$\begin{array}{ccc} V' \times_{S'} X' & \xrightarrow[\varphi']{} & X' \times_{S'} V' \\ {\scriptstyle \alpha' \times \mathrm{id}}\downarrow & & \downarrow{\scriptstyle \mathrm{id} \times \alpha'} \\ W' \times_{S'} X' & \xrightarrow{\psi'} & X' \times_{S'} W' \end{array}$$

commutes. We claim the two compositions

$$V' \times_V V' \xrightarrow{\mathrm{pr}_i} V' \xrightarrow{\alpha'} W' \longrightarrow W \ , \quad i = 0, 1$$

are the same. The reader is advised to prove this themselves rather than read the rest of this paragraph. (Please email if you find a nice clean argument.) Let v_0, v_1 be points of V' which map to the same point $v \in V$. Let $x_i \in X'$ be the image of v_i, and let x be the point of X which is the image of v in X. In other words, $v_i = (x_i, v)$ in $V' = X' \times_X V$. Write $\varphi(v, x) = (x, v')$ for some point v' of V. This is possible

[8] This follows from (1) if $S = S'$.

because φ is a morphism over $X \times_S X$. Denote $v'_i = (x_i, v')$ which is a point of V'. Then a calculation (using the definition of φ') shows that $\varphi'(v_i, x_j) = (x_i, v'_j)$. Denote $w_i = \alpha'(v_i)$ and $w'_i = \alpha'(v'_i)$. Now we may write $w_i = (x_i, u_i)$ for some point u_i of W, and $w'_i = (x_i, u'_i)$ for some point u'_i of W. The claim is equivalent to the assertion: $u_0 = u_1$. A formal calculation using the definition of ψ' (see Lemma 34.30.6) shows that the commutativity of the diagram displayed above says that

$$((x_i, x_j), \psi(u_i, x)) = ((x_i, x_j), (x, u'_j))$$

as points of $(X' \times_{S'} X') \times_{X \times_S X} (X \times_S W)$ for all $i, j \in \{0, 1\}$. This shows that $\psi(u_0, x) = \psi(u_1, x)$ and hence $u_0 = u_1$ by taking ψ^{-1}. This proves the claim because the argument above was formal and we can take scheme points (in other words, we may take $(v_0, v_1) = \mathrm{id}_{V' \times_V V'}$).

At this point we can use Lemma 34.9.3. Namely, $\{V' \to V\}$ is a fpqc covering as the base change of the morphism $f : X' \to X$. Hence, by Lemma 34.9.3 the morphism $\alpha' : V' \to W' \to W$ factors through a unique morphism $\alpha : V \to W$ whose base change is necessarily α'. Finally, we see the diagram

$$\begin{array}{ccc} V \times_S X & \xrightarrow{\varphi} & X \times_S V \\ {\scriptstyle \alpha \times \mathrm{id}} \downarrow & & \downarrow {\scriptstyle \mathrm{id} \times \alpha} \\ W \times_S X & \xrightarrow{\psi} & X \times_S W \end{array}$$

commutes because its base change to $X' \times_{S'} X'$ commutes and the morphism $X' \times_{S'} X' \to X \times_S X$ is surjective and flat (use Lemma 34.31.2). Hence α is a morphism of descent data $(V, \varphi) \to (W, \psi)$ as desired. □

The following two lemmas have been obsoleted by the improved exposition of the previous material. But they are still true!

Lemma 34.31.5. *Let $X \to S$ be a morphism of schemes. Let $f : X \to X$ be a selfmap of X over S. In this case pullback by f is isomorphic to the identity functor on the category of descent data relative to $X \to S$.*

Proof. This is clear from Lemma 34.30.6 since it tells us that $f^* \cong \mathrm{id}^*$. □

Lemma 34.31.6. *Let $f : X' \to X$ be a morphism of schemes over a base scheme S. Assume there exists a morphism $g : X \to X'$ over S, for example if f has a section. Then the pullback functor of Lemma 34.30.6 defines an equivalence of categories between the category of descent data relative to X/S and X'/S.*

Proof. Let $g : X \to X'$ be a morphism over S. Lemma 34.31.5 above shows that the functors $f^* \circ g^* = (g \circ f)^*$ and $g^* \circ f^* = (f \circ g)^*$ are isomorphic to the respective identity functors as desired. □

Lemma 34.31.7. *Let $f : X \to X'$ be a morphism of schemes over a base scheme S. Assume $X \to S$ is surjective and flat. Then the pullback functor of of Lemma 34.30.6 is a faithful functor from the category of descent data relative to X'/S to the category of descent data relative to X/S.*

Proof. We may factor $X \to X'$ as $X \to X \times_S X' \to X'$. The first morphism has a section, hence induces an equivalence of categories of descent data by Lemma 34.31.6. The second morphism is surjective and flat, hence induces a faithful functor by Lemma 34.31.3. □

Lemma 34.31.8. *Let $f : X \to X'$ be a morphism of schemes over a base scheme S. Assume $\{X \to S\}$ is an fpqc covering (for example if f is surjective, flat and quasi-compact). Then the pullback functor of of Lemma 34.30.6 is a fully faithful functor from the category of descent data relative to X'/S to the category of descent data relative to X/S.*

Proof. We may factor $X \to X'$ as $X \to X \times_S X' \to X'$. The first morphism has a section, hence induces an equivalence of categories of descent data by Lemma 34.31.6. The second morphism is an fpqc covering hence induces a fully faithful functor by Lemma 34.31.4. □

Lemma 34.31.9. *Let S be a scheme. Let $\mathcal{U} = \{U_i \to S\}_{i \in I}$, and $\mathcal{V} = \{V_j \to S\}_{j \in J}$, be families of morphisms with target S. Let $\alpha : I \to J$, $id : S \to S$ and $g_i : U_i \to V_{\alpha(i)}$ be a morphism of families of maps with fixed target, see Sites, Definition 7.8.1. Assume that for each $j \in J$ the family $\{g_i : U_i \to V_j\}_{\alpha(i)=j}$ is an fpqc covering of V_j. Then the pullback functor*

$$\textit{descent data relative to } \mathcal{V} \longrightarrow \textit{descent data relative to } \mathcal{U}$$

of Lemma 34.30.8 is fully faithful.

Proof. Consider the morphism of schemes

$$g : X = \coprod\nolimits_{i \in I} U_i \longrightarrow Y = \coprod\nolimits_{j \in J} V_j$$

over S which on the ith component maps into the $\alpha(i)$th component via the morphism $g_{\alpha(i)}$. We claim that $\{g : X \to Y\}$ is an fpqc covering of schemes. Namely, by Topologies, Lemma 33.8.3 for each j the morphism $\{\coprod_{\alpha(i)=j} U_i \to V_j\}$ is an fpqc covering. Thus for every affine open $V \subset V_j$ (which we may think of as an affine open of Y) we can find finitely many affine opens $W_1, \ldots, W_n \subset \coprod_{\alpha(i)=j} U_i$ (which we may think of as affine opens of X) such that $V = \bigcup_{i=1,\ldots,n} g(W_i)$. This provides enough affine opens of Y which can be covered by finitely many affine opens of X so that Topologies, Lemma 33.8.2 part (3) applies, and the claim follows. Let us write $DD(X/S)$, resp. $DD(\mathcal{U})$ for the category of descent data with respect to X/S, resp. $\mathcal{U}$, and similarly for Y/S and $\mathcal{V}$. Consider the diagram

$$\begin{array}{ccc} DD(Y/S) & \longrightarrow & DD(X/S) \\ \uparrow{\scriptstyle \text{Lemma 34.30.5}} & & \uparrow{\scriptstyle \text{Lemma 34.30.5}} \\ DD(\mathcal{V}) & \longrightarrow & DD(\mathcal{U}) \end{array}$$

This diagram is commutative, see the proof of Lemma 34.30.8. The vertical arrows are equivalences. Hence the lemma follows from Lemma 34.31.4 which shows the top horizontal arrow of the diagram is fully faithful. □

The next lemma shows that, in order to check effectiveness, we may always Zariski refine the given family of morphisms with target S.

Lemma 34.31.10. *Let S be a scheme. Let $\mathcal{U} = \{U_i \to S\}_{i \in I}$, and $\mathcal{V} = \{V_j \to S\}_{j \in J}$, be families of morphisms with target S. Let $\alpha : I \to J$, $id : S \to S$ and $g_i : U_i \to V_{\alpha(i)}$ be a morphism of families of maps with fixed target, see Sites, Definition 7.8.1. Assume that for each $j \in J$ the family $\{g_i : U_i \to V_j\}_{\alpha(i)=j}$ is a Zariski covering (see Topologies, Definition 33.3.1) of V_j. Then the pullback functor*

$$\textit{descent data relative to } \mathcal{V} \longrightarrow \textit{descent data relative to } \mathcal{U}$$

of Lemma 34.30.8 is an equivalence of categories. In particular, the category of schemes over S is equivalent to the category of descent data relative to any Zariski covering of S.

Proof. The functor is faithful and fully faithful by Lemma 34.31.9. Let us indicate how to prove that it is essentially surjective. Let $(X_i, \varphi_{ii'})$ be a descend datum relative to $\mathcal{U}$. Fix $j \in J$ and set $I_j = \{i \in I \mid \alpha(i) = j\}$. For $i, i' \in I_j$ note that there is a canonical morphism

$$c_{ii'} : U_i \times_{g_i, V_j, g_{i'}} U_{i'} \to U_i \times_S U_{i'}.$$

Hence we can pullback $\varphi_{ii'}$ by this morphism and set $\psi_{ii'} = c_{ii'}^* \varphi_{ii'}$ for $i, i' \in I_j$. In this way we obtain a descent datum $(X_i, \psi_{ii'})$ relative to the Zariski covering $\{g_i : U_i \to V_i\}_{i \in I_j}$. Note that $\psi_{ii'}$ is an isomorphism from the open $X_{i, U_i \times_{V_j} U_{i'}}$ of X_i to the corresponding open of $X_{i'}$. It follows from Schemes, Section 25.14 that we may glue $(X_i, \psi_{ii'})$ into a scheme Y_j over V_j. Moreover, the morphisms $\varphi_{ii'}$ for $i \in I_j$ and $i' \in I_{j'}$ glue to a morphism $\varphi_{jj'} : Y_j \times_S V_{j'} \to V_j \times_S Y_{j'}$ satisfying the cocycle condition (details omitted). Hence we obtain the desired descent datum $(Y_j, \varphi_{jj'})$ relative to $\mathcal{V}$. □

Lemma 34.31.11. *Let S be a scheme. Let $\mathcal{U} = \{U_i \to S\}_{i \in I}$, and $\mathcal{V} = \{V_j \to S\}_{j \in J}$, be fpqc-coverings of S. If $\mathcal{U}$ is a refinement of $\mathcal{V}$, then the pullback functor*

$$\textit{descent data relative to } \mathcal{V} \longrightarrow \textit{descent data relative to } \mathcal{U}$$

is fully faithfull. In particular, the category of schemes over S is identified with a full subcategory of the category of descent data relative to any fpqc-covering of S.

Proof. Consider the fpqc-covering $\mathcal{W} = \{U_i \times_S V_j \to S\}_{(i,j) \in I \times J}$ of S. It is a refinement of both $\mathcal{U}$ and $\mathcal{V}$. Hence we have a 2-commutative diagram of functors and categories

$$\begin{array}{ccc} DD(\mathcal{V}) & \longrightarrow & DD(\mathcal{U}) \\ & \searrow \quad \swarrow & \\ & DD(\mathcal{W}) & \end{array}$$

Notation as in the proof of Lemma 34.31.9 and commutativity by Lemma 34.30.8 part (3). Hence clearly it suffices to prove the functors $DD(\mathcal{V}) \to DD(\mathcal{W})$ and $DD(\mathcal{U}) \to DD(\mathcal{W})$ are fully faithful. This follows from Lemma 34.31.9 as desired. □

Remark 34.31.12. Lemma 34.31.11 says that morphisms of schemes satisfy fpqc descent. In other words, given a scheme S and schemes X, Y over S the functor

$$(Sch/S)^{opp} \longrightarrow Sets, \quad T \longmapsto \mathrm{Mor}_T(X_T, Y_T)$$

satisfies the sheaf condition for the fpqc topology. The simplest case of this is the following. Suppose that $T \to S$ is a surjective flat morphism of affines. Let $\psi_0 : X_T \to Y_T$ be a morphism of schemes over T which is compatible with the canonical descent data. Then there exists a unique morphism $\psi : X \to Y$ whose base change to T is ψ_0. In fact this special case follows in a straightforward manner from Lemma 34.31.4. And, in turn, that lemma is a formal consequence of the following two facts: (a) the base change functor by a faithfully flat morphism is faithful, see Lemma 34.31.2 and (b) a scheme satisfies the sheaf condition for the fpqc topology, see Lemma 34.9.3.

34.32. Descending types of morphisms

In the following we study the question as to whether descent data for schemes relative to a fpqc-covering are effective. The first remark to make is that this is not always the case. We will see this in Algebraic Spaces, Example 47.14.2. Even projective morphisms do not always satisfy descent for fpqc-coverings, by Examples, Lemma 78.50.1.

On the other hand, if the schemes we are trying to descend are particularly simple, then it is sometime the case that for whole classes of schemes descent data are effective. We will introduce terminology here that describes this phenomenon abstractly, even though it may lead to confusion if not used correctly later on.

Definition 34.32.1. Let $\mathcal{P}$ be a property of morphisms of schemes over a base. Let $\tau \in \{Zariski, fpqc, fppf, étale, smooth, syntomic\}$. We say *morphisms of type $\mathcal{P}$ satisfy descent for τ-coverings* if for any τ-covering $\mathcal{U} : \{U_i \to S\}_{i \in I}$ (see Topologies, Section 33.2), any descent datum (X_i, φ_{ij}) relative to $\mathcal{U}$ such that each morphism $X_i \to U_i$ has property $\mathcal{P}$ is effective.

Note that in each of the cases we have already seen that the functor from schemes over S to descent data over $\mathcal{U}$ is fully faithful (Lemma 34.31.11 combined with the results in Topologies that any τ-covering is also a fpqc-covering). We have also seen that descent data are always effective with respect to Zariski coverings (Lemma 34.31.10). It may be prudent to only study the notion just introduced when $\mathcal{P}$ is either stable under any base change or at least local on the base in the τ-topology (see Definition 34.18.1) in order to avoid erroneous arguments (relying on $\mathcal{P}$ when descending halfway).

Here is the obligatory lemma reducing this question to the case of a covering given by a single morphism of affines.

Lemma 34.32.2. *Let $\mathcal{P}$ be a property of morphisms of schemes over a base. Let $\tau \in \{fpqc, fppf, étale, smooth, syntomic\}$. Suppose that*

1. *$\mathcal{P}$ is stable under any base change (see Schemes, Definition 25.18.3), and*
2. *for any surjective morphism of affines $X \to S$ which is flat, flat of finite presentation, étale, smooth or syntomic depending on whether τ is fpqc, fppf, étale, smooth, or syntomic, any descent datum (V, φ) relative to X over S such that $\mathcal{P}$ holds for $V \to X$ is effective.*

Then morphisms of type $\mathcal{P}$ satisfy descent for τ-coverings.

Proof. Let S be a scheme. Let $\mathcal{U} = \{\varphi_i : U_i \to S\}_{i \in I}$ be a τ-covering of S. Let $(X_i, \varphi_{ii'})$ be a descent datum relative to $\mathcal{U}$ and assume that each morphism $X_i \to U_i$ has property $\mathcal{P}$. We have to show there exists a scheme $X \to S$ such that $(X_i, \varphi_{ii'}) \cong (U_i \times_S X, can)$.

Before we start the proof proper we remark that for any family of morphisms $\mathcal{V} : \{V_j \to S\}$ and any morphism of families $\mathcal{V} \to \mathcal{U}$, if we pullback the descent datum $(X_i, \varphi_{ii'})$ to a descent datum $(Y_j, \varphi_{jj'})$ over $\mathcal{V}$, then each of the morphisms $Y_j \to V_j$ has property $\mathcal{P}$ also. This is true because we assumed that $\mathcal{P}$ is stable under any base change and the definition of pullback (see Definition 34.30.9). We will use this without further mention.

First, let us prove the lemma when S is affine. By Topologies, Lemma 33.8.8, 33.7.4, 33.4.4, 33.5.4, or 33.6.4 there exists a standard τ-covering $\mathcal{V} : \{V_j \to S\}_{j=1,\ldots,m}$

which refines $\mathcal{U}$. The pullback functor $DD(\mathcal{U}) \to DD(\mathcal{V})$ between categories of descent data is fully faithful by Lemma 34.31.11. Hence it suffices to prove that the descend datum over the standard τ-covering $\mathcal{V}$ is effective. By Lemma 34.30.5 this reduces to the covering $\{\coprod_{j=1,\ldots,m} V_j \to S\}$ for which we have assumed the result in property (2) of the lemma. Hence the lemma holds when S is affine.

Assume S is general. Let $V \subset S$ be an affine open. By the properties of site the family $\mathcal{U}_V = \{V \times_S U_i \to V\}_{i \in I}$ is a τ-covering of V. Denote $(X_i, \varphi_{ii'})_V$ the restriction (or pullback) of the given descent datum to $\mathcal{U}_V$. Hence by what we just saw we obtain a scheme X_V over V whose canonical descent datum with respect to $\mathcal{U}_V$ is isomorphic to $(X_i, \varphi_{ii'})_V$. Suppose that $V' \subset V$ is an affine open of V. Then both $X_{V'}$ and $V' \times_V X_V$ have canonical descent data isomorphic to $(X_i, \varphi_{ii'})_{V'}$. Hence, by Lemma 34.31.11 again we obtain a canonical morphism $\rho^V_{V'} : X_{V'} \to X_V$ over S which identifies $X_{V'}$ with the inverse image of V' in X_V. We omit the verification that given affine opens $V'' \subset V' \subset V$ of S we have $\rho^V_{V''} = \rho^V_{V'} \circ \rho^{V'}_{V''}$.

By Constructions, Lemma 26.2.1 the data $(X_V, \rho^V_{V'})$ glue to a scheme $X \to S$. Moreover, we are given isomorphisms $V \times_S X \to X_V$ which recover the maps $\rho^V_{V'}$. Unwinding the construction of the schemes X_V we obtain isomorphisms

$$V \times_S U_i \times_S X \longrightarrow V \times_S X_i$$

compatible with the maps $\varphi_{ii'}$ and compatible with restricting to smaller affine opens in X. This implies that the canonical descent datum on $U_i \times_S X$ is isomorphic to the given descent datum and we win. □

34.33. Descending affine morphisms

In this section we show that "affine morphisms satisfy descent for fpqc-coverings". Here is the formal statement.

Lemma 34.33.1. *Let S be a scheme. Let $\{X_i \to S\}_{i \in I}$ be an fpqc covering, see Topologies, Definition 33.8.1. Let $(V_i/X_i, \varphi_{ij})$ be a descent datum relative to $\{X_i \to S\}$. If each morphism $V_i \to X_i$ is affine, then the descent datum is effective.*

Proof. Being affine is a property of morphisms of schemes which is preserved under any base change, see Morphisms, Lemma 28.13.8. Hence Lemma 34.32.2 applies and it suffices to prove the statement of the lemma in case the fpqc-covering is given by a single $\{X \to S\}$ flat surjective morphism of affines. Say $X = \mathrm{Spec}(A)$ and $S = \mathrm{Spec}(R)$ so that $R \to A$ is a faithfully flat ring map. Let (V, φ) be a descent datum relative to X over S and assume that $V \to X$ is affine. Then $V \to X$ being affine implies that $V = \mathrm{Spec}(B)$ for some A-algebra B (see Morphisms, Definition 28.13.1). The isomorphism φ corresponds to an isomorphism of rings

$$\varphi^\sharp : B \otimes_R A \longleftarrow A \otimes_R B$$

as $A \otimes_R A$-algebras. The cocycle condition on φ says that

$$\begin{array}{ccc} B \otimes_R A \otimes_R A & \longleftarrow & A \otimes_R A \otimes_R B \\ & \nwarrow \quad \swarrow & \\ & A \otimes_R B \otimes_R A & \end{array}$$

is commutative. Inverting these arrows we see that we have a descent datum for modules with respect to $R \to A$ as in Definition 34.3.1. Hence we may apply Proposition 34.3.9 to obtain an R-module $C = \mathrm{Ker}(B \to A \otimes_R B)$ and an isomorphism $A \otimes_R C \cong B$ respecting descent data. Given any pair $c, c' \in C$ the product cc' in B lies in C since the map φ is an algebra homomorphism. Hence C is an R-algebra whose base change to A is isomorphic to B compatibly with descent data. Applying Spec we obtain a scheme U over S such that $(V, \varphi) \cong (X \times_S U, can)$ as desired. □

Lemma 34.33.2. *Let S be a scheme. Let $\{X_i \to S\}_{i \in I}$ be an fpqc covering, see Topologies, Definition 33.8.1. Let $(V_i/X_i, \varphi_{ij})$ be a descent datum relative to $\{X_i \to S\}$. If each morphism $V_i \to X_i$ is a closed immersion, then the descent datum is effective.*

Proof. This is true because a closed immersion is an affine morphism (Morphisms, Lemma 28.13.9), and hence Lemma 34.33.1 applies. □

34.34. Descending quasi-affine morphisms

In this section we show that "quasi-affine morphisms satisfy descent for fpqc-coverings". Here is the formal statement.

Lemma 34.34.1. *Let S be a scheme. Let $\{X_i \to S\}_{i \in I}$ be an fpqc covering, see Topologies, Definition 33.8.1. Let $(V_i/X_i, \varphi_{ij})$ be a descent datum relative to $\{X_i \to S\}$. If each morphism $V_i \to X_i$ is quasi-affine, then the descent datum is effective.*

Proof. Being quasi-affine is a property of morphisms of schemes which is preserved under any base change, see Morphisms, Lemma 28.14.5. Hence Lemma 34.32.2 applies and it suffices to prove the statement of the lemma in case the fpqc-covering is given by a single $\{X \to S\}$ flat surjective morphism of affines. Say $X = \mathrm{Spec}(A)$ and $S = \mathrm{Spec}(R)$ so that $R \to A$ is a faithfully flat ring map. Let (V, φ) be a descent datum relative to X over S and assume that $\pi : V \to X$ is quasi-affine.

According to Morphisms, Lemma 28.14.3 this means that

$$V \longrightarrow \underline{\mathrm{Spec}}_X(\pi_*\mathcal{O}_V) = W$$

is a quasi-compact open immersion of schemes over X. The projections $\mathrm{pr}_i : X \times_S X \to X$ are flat and hence we have

$$\mathrm{pr}_0^*\pi_*\mathcal{O}_V = (\pi \times \mathrm{id}_X)_*\mathcal{O}_{V \times_S X}, \quad \mathrm{pr}_1^*\pi_*\mathcal{O}_V = (\mathrm{id}_X \times \pi)_*\mathcal{O}_{X \times_S V}$$

by flat base change (Cohomology of Schemes, Lemma 29.5.2). Thus the isomorphism $\varphi : V \times_S X \to X \times_S V$ (which is an isomorphism over $X \times_S X$) induces an isomorphism of quasi-coherent sheaves of algebras

$$\varphi^\sharp : \mathrm{pr}_0^*\pi_*\mathcal{O}_V \longrightarrow \mathrm{pr}_1^*\pi_*\mathcal{O}_V$$

on $X \times_S X$. The cocycle condition for φ implies the cocycle condition for $\varphi^\sharp$. Another way to say this is that it produces a descent datum φ' on the affine scheme W relative to X over S, which moreover has the property that the morphism $V \to W$ is a morphism of descent data. Hence by Lemma 34.33.1 (or by effectivity of descent for quasi-coherent algebras) we obtain a scheme $U' \to S$ with an isomorphism $(W, \varphi') \cong (X \times_S U', can)$ of descent data. We note in passing that U' is affine by Lemma 34.19.16.

And now we can think of V as a (quasi-compact) open $V \subset X \times_S U'$ with the property that it is stable under the descent datum

$$can : X \times_S U' \times_S X \to X \times_S X \times_S U', (x_0, u', x_1) \mapsto (x_0, x_1, u').$$

In other words $(x_0, u') \in V \Rightarrow (x_1, u') \in V$ for any x_0, x_1, u' mapping to the same point of S. Because $X \to S$ is surjective we immediately find that V is the inverse image of a subset $U \subset U'$ under the morphism $X \times_S U' \to U'$. Because $X \to S$ is quasi-compact, flat and surjective also $X \times_S U' \to U'$ is quasi-compact flat and surjective. Hence by Morphisms, Lemma 28.26.10 this subset $U \subset U'$ is open and we win. □

34.35. Descent data in terms of sheaves

Here is another way to think about descent data in case of a covering on a site.

Lemma 34.35.1. *Let $\tau \in \{Zariski, fppf, étale, smooth, syntomic$[9]$\}$. Let Sch_τ be a big τ-site. Let $S \in \mathrm{Ob}(Sch_\tau)$. Let $\{S_i \to S\}_{i \in I}$ be a covering in the site $(Sch/S)_\tau$. There is an equivalence of categories*

$$\left\{ \begin{array}{c} \text{descent data } (X_i, \varphi_{ii'}) \text{ such that} \\ \text{each } X_i \in \mathrm{Ob}((Sch/S)_\tau) \end{array} \right\} \leftrightarrow \left\{ \begin{array}{c} \text{sheaves } F \text{ on } (Sch/S)_\tau \text{ such that} \\ \text{each } h_{S_i} \times F \text{ is representable} \end{array} \right\}.$$

Moreover,

1. *the objects representing $h_{S_i} \times F$ on the right hand side correspond to the schemes X_i on the left hand side, and*
2. *the sheaf F is representable if and only if the corresponding descent datum $(X_i, \varphi_{ii'})$ is effective.*

Proof. We have seen in Section 34.9 that representable presheaves are sheaves on the site $(Sch/S)_\tau$. Moreover, the Yonea lemma (Categories, Lemma 4.3.5) guarantees that maps between representable sheaves correspond one to one with maps between the representing objects. We will use these remarks without further mention during the proof.

Let us construct the functor from right to left. Let F be a sheaf on $(Sch/S)_\tau$ such that each $h_{S_i} \times F$ is representable. In this case let X_i be a representing object in $(Sch/S)_\tau$. It comes equipped with a morphism $X_i \to S_i$. Then both $X_i \times_S S_{i'}$ and $S_i \times_S X_{i'}$ represent the sheaf $h_{S_i} \times F \times h_{S_{i'}}$ and hence we obtain an isomorphism

$$\varphi_{ii'} : X_i \times_S S_{i'} \to S_i \times_S X_{i'}$$

It is straightforward to see that the maps $\varphi_{ii'}$ are morphisms over $S_i \times_S S_{i'}$ and satisfy the cocycle condition. The functor from right to left is given by this construction $F \mapsto (X_i, \varphi_{ii'})$.

Let us construct a functor from left to right. For each i denote F_i the sheaf h_{X_i}. The isomorphisms $\varphi_{ii'}$ give isomorphisms

$$\varphi_{ii'} : F_i \times h_{S_{i'}} \longrightarrow h_{S_i} \times F_{i'}$$

over $h_{S_i} \times h_{S_{i'}}$. Set F equal to the coequalizer in the following diagram

$$\coprod_{i,i'} F_i \times h_{S_{i'}} \xrightarrow[\mathrm{pr}_1 \circ \varphi_{ii'}]{\mathrm{pr}_0} \coprod_i F_i \longrightarrow F$$

[9] The fact that fpqc is missing is not a typo. See discussion in Topologies, Section 33.8.

The cocycle condition guarantees that $h_{S_i} \times F$ is isomorphic to F_i and hence representable. The functor from left to right is given by this construction $(X_i, \varphi_{ii'}) \mapsto F$.

We omit the verification that these constructions are mutually quasi-inverse functors. The final statements (1) and (2) follow from the constructions. □

Remark 34.35.2. In the statement of Lemma 34.35.1 the condition that $h_{S_i} \times F$ is representable is equivalent to the condition that the restriction of F to $(Sch/S_i)_\tau$ is representable.

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CHAPTER 35

Derived Categories of Schemes

35.1. Introduction

In this chapter we discuss derived categories of modules on schemes. Most of the material discussed here can be found in [**TT90**], [**BN93**], [**BV03**], and [**LN07**]. Of course there are many other references.

35.2. Conventions

If $\mathcal{A}$ is an abelian category and M is an object of $\mathcal{A}$ then we also denote M the object of $K(\mathcal{A})$ and/or $D(\mathcal{A})$ corresponding to the complex which has M in degree 0 and is zero in all other degrees.

If we have a ring A, then $K(A)$ denotes the homotopy category of complexes of A-modules and $D(A)$ the associated derived category. Similarly, if we have a ringed space $(X, \mathcal{O}_X)$ the symbol $K(\mathcal{O}_X)$ denotes the homotopy category of complexes of $\mathcal{O}_X$-modules and $D(\mathcal{O}_X)$ the associated derived category.

35.3. Derived category of quasi-coherent modules

In this section we discuss the relationship between quasi-coherent modules and all modules on a scheme X. A reference is [**TT90**, Appendix B]. By the discussion in Schemes, Section 25.24 the embedding $QCoh(\mathcal{O}_X) \subset Mod(\mathcal{O}_X)$ exhibits $QCoh(\mathcal{O}_X)$ as a weak Serre subcategory of the category of $\mathcal{O}_X$-modules. Denote

$$D_{QCoh}(\mathcal{O}_X) \subset D(\mathcal{O}_X)$$

the subcategory of complexes whose cohomology sheaves are quasi-coherent, see Derived Categories, Section 13.12. Thus we obtain a canonical functor

$$D(QCoh(\mathcal{O}_X)) \longrightarrow D_{QCoh}(\mathcal{O}_X) \tag{35.3.0.1}$$

see Derived Categories, Equation (13.12.1.1).

Lemma 35.3.1. *Let X be a scheme. Then $D_{QCoh}(\mathcal{O}_X)$ has direct sums.*

Proof. By Injectives, Lemma 19.13.4 the derived category $D(\mathcal{O}_X)$ has direct sums and they are computed by taking termwise direct sums of any representatives. Thus it is clear that the cohomology sheaf of a direct sum is the direct sum of the cohomology sheaves as taking direct sums is an exact functor (in any grothendieck abelian category). The lemma follows as the direct sum of quasi-coherent sheaves is quasi-coherent, see Schemes, Section 25.24. □

The following lemma will help us to "compute" a right derived functor on an object of $D_{QCoh}(\mathcal{O}_X)$.

Lemma 35.3.2. *Let X be a scheme. Let E be an object of $D_{QCoh}(\mathcal{O}_X)$. Then there exists an inverse system $\mathcal{I}_n^\bullet$ of complexes of $\mathcal{O}_X$-modules such that*

(1) *$\mathcal{I}^\bullet = \lim_n \mathcal{I}_n^\bullet$ represents E,*
(2) *$\mathcal{I}_n^\bullet$ is a bounded below complex of injectives,*
(3) *$\mathcal{I}^\bullet \to \mathcal{I}_n^\bullet$ induces an identification $\tau_{\geq -n}E \to \mathcal{I}_n^\bullet$ in $D(\mathcal{O}_X)$,*
(4) *the transition maps $\mathcal{I}_{n+1}^\bullet \to \mathcal{I}_n^\bullet$ are termwise split surjections, and*
(5) *$\mathcal{I}^\bullet$ is a K-injective complex of $\mathcal{O}_X$-modules.*

Moreover, E is the derived limit of the inverse system of its canonical truncations $\tau_{\geq -n}E$.

Proof. Denote $\mathcal{H}^i = H^i(E)$ the ith cohomology sheaf of E. Let $\mathcal{B}$ be the set of affine open subsets of X. Then $H^p(U, \mathcal{H}^i) = 0$ for all $p > 0$, all $i \in \mathbf{Z}$, and all $U \in \mathcal{B}$, see Cohomology of Schemes, Lemma 29.2.2. Thus the lemma follows from Cohomology, Lemmas 20.29.2 and 20.29.3. □

Lemma 35.3.3. *Let X be a scheme. Let $F : Mod(\mathcal{O}_X) \to Ab$ be an additive functor and $N \geq 0$ an integer. Assume that*

(1) *F commutes with countable direct products,*
(2) *$R^pF(\mathcal{F}) = 0$ for all $p \geq N$ and $\mathcal{F}$ quasi-coherent.*

Then for $E \in D_{QCoh}(\mathcal{O}_X)$ the maps $R^pF(E) \to R^pF(\tau_{\geq p-N+1}E)$ are isomorphisms.

Proof. By shifting the complex we see it suffices to prove the assertion for $p = 0$. Write $E_n = \tau_{\geq -n}E$. We have $E = R\lim E_n$, see Lemma 35.3.2. Thus $RF(E) = R\lim RF(E_n)$ in $D(Ab)$ by Injectives, Lemma 19.13.6. Thus we have a short exact sequence

$$0 \to R^1 \lim R^{-1}F(E_n) \to R^0F(E) \to \lim R^0F(E_n) \to 0$$

see More on Algebra, Remark 15.55.16. To finish the proof we will show that the term on the left is zero and that the term on the right equals $R^0F(E_{N-1})$.

We have a distinguished triangle

$$H^{-n}(E)[n] \to E_n \to E_{n-1} \to H^{-n}(E)[n+1]$$

(Derived Categories, Remark 13.11.4) in $D(\mathcal{O}_X)$. Since $H^{-n}(E)$ is quasi-coherent we have

$$R^pF(H^{-n}(E)[n]) = R^{p+n}F(H^{-n}(E)) = 0$$

for $p + n \geq N$ and

$$R^pF(H^{-n}(E)[n+1]) = R^{p+n+1}F(H^{-n}(E)) = 0$$

for $p + n + 1 \geq N$. We conclude that

$$R^pF(E_n) \to R^pF(E_{n-1})$$

is an isomorphism for all $n \gg p$ and an isomorphism for $n \geq N$ for $p = 0$. Thus the systems $R^pF(E_n)$ all satisfy the ML condition and $R^1 \lim$ gives zero (see discussion in More on Algebra, Section 15.55). Moreover, the system $R^0F(\tau_{\geq -n}E)$ is constant starting with $n = N - 1$ as desired. □

The following lemma is the key ingredient to many of the results in this chapter.

Lemma 35.3.4. *Let $X = \mathrm{Spec}(A)$ be an affine scheme. All the functors in the diagram*

$$\begin{array}{ccc} D(QCoh(\mathcal{O}_X)) & \xrightarrow{\quad (35.3.0.1) \quad} & D_{QCoh}(\mathcal{O}_X) \\ & {\scriptstyle \sim} \nwarrow \quad \swarrow {\scriptstyle R\Gamma(X,-)} & \\ & D(A) & \end{array}$$

are equivalences of triangulated categories. Moreover, for E in $D_{QCoh}(\mathcal{O}_X)$ we have $H^0(X, E) = H^0(X, H^0(E))$.

Proof. The functor $R\Gamma(X, -)$ gives a functor $D(\mathcal{O}_X) \to D(A)$ and hence by restriction a functor

$$R\Gamma(X, -) : D_{QCoh}(\mathcal{O}_X) \longrightarrow D(A). \tag{35.3.4.1}$$

We will show this functor is quasi-inverse to (35.3.0.1) via the equivalence between quasi-coherent modules on X and the category of A-modules.

Elucidation. Denote $(Y, \mathcal{O}_Y)$ the one point space with sheaf of rings given by A. Denote $\pi : (X, \mathcal{O}_X) \to (Y, \mathcal{O}_Y)$ the obvious morphism of ringed spaces. Then $R\Gamma(X, -)$ can be identified with $R\pi_*$ and the functor (35.3.0.1) via the equivalence $Mod(\mathcal{O}_Y) = \mathrm{Mod}_A = QCoh(\mathcal{O}_X)$ can be identified with $L\pi^* = \pi^* = {}^\sim$(see Modules, Lemma 17.10.5 and Schemes, Lemmas 25.7.1 and 25.7.5). Thus the functors

$$D(A) \rightleftarrows D_{QCoh}(\mathcal{O}_X)$$

are adjoint (by Cohomology, Lemma 20.27.1). In particular we obtain canonical adjunction mappings

$$a : R\widetilde{\Gamma(X, E)} \longrightarrow E$$

for E in $D(\mathcal{O}_X)$ and

$$b : M^\bullet \longrightarrow R\Gamma(X, \widetilde{M^\bullet})$$

for $M^\bullet$ a complex of A-modules.

Let E be an object of $D_{QCoh}(\mathcal{O}_X)$. We may apply Lemma 35.3.3 to the functor $F(-) = \Gamma(X, -)$ with $N = 1$ by Cohomology of Schemes, Lemma 29.2.2. Hence

$$R^0\Gamma(X, E) = R^0\Gamma(X, \tau_{\geq 0}E) = \Gamma(X, H^0(E))$$

(the last equality by definition of the canonical truncation). Using this we will show that the adjunction mappings a and b induce isomorphisms $H^0(a)$ and $H^0(b)$. Thus a and b are quasi-isomorphisms (as the statement is invariant under shifts) and the lemma is proved.

In both cases we use that ${}^\sim$ is an exact functor (Schemes, Lemma 25.5.4). Namely, this implies that

$$H^0\left(R\widetilde{\Gamma(X, E)}\right) = R^0\widetilde{\Gamma(X, E)} = \Gamma(X, \widetilde{H^0(E)})$$

which is equal to $H^0(E)$ because $H^0(E)$ is quasi-coherent. Thus $H^0(a)$ is an isomorphism. For the other direction we have

$$H^0(R\Gamma(X, \widetilde{M^\bullet})) = R^0\Gamma(X, \widetilde{M^\bullet}) = \Gamma(X, H^0(\widetilde{M^\bullet})) = \Gamma(X, \widetilde{H^0(M^\bullet)}) = H^0(M^\bullet)$$

which proves that $H^0(b)$ is an isomorphism. □

Lemma 35.3.5. *Let $X = \mathrm{Spec}(A)$ be an affine scheme. If $K^\bullet$ is a K-flat complex of A-modules, then $\widetilde{K^\bullet}$ is a K-flat complex of $\mathcal{O}_X$-modules.*

Proof. By More on Algebra, Lemma 15.43.5 we see that $K^\bullet \otimes_A A_\mathfrak{p}$ is a K-flat complex of $A_\mathfrak{p}$-modules for every $\mathfrak{p} \in \mathrm{Spec}(A)$. Hence we conclude from Cohomology, Lemma 20.25.4 (and Schemes, Lemma 25.5.4) that $\widetilde{K^\bullet}$ is K-flat. □

Lemma 35.3.6. *Let $f : Y \to X$ be a morphism of schemes.*

(1) *The functor Lf^* sends $D_{QCoh}(\mathcal{O}_X)$ into $D_{QCoh}(\mathcal{O}_Y)$.*
(2) *If X and Y are affine and f is given by the ring map $A \to B$, then the diagram*

$$\begin{array}{ccc} D(B) & \longrightarrow & D_{QCoh}(\mathcal{O}_Y) \\ {\scriptstyle -\otimes^{\mathbf{L}}_A B}\uparrow & & \uparrow{\scriptstyle Lf^*} \\ D(A) & \longrightarrow & D_{QCoh}(\mathcal{O}_X) \end{array}$$

commutes.

Proof. We first prove the diagram

$$\begin{array}{ccc} D(B) & \longrightarrow & D(\mathcal{O}_Y) \\ {\scriptstyle -\otimes^{\mathbf{L}}_A B}\uparrow & & \uparrow{\scriptstyle Lf^*} \\ D(A) & \longrightarrow & D(\mathcal{O}_X) \end{array}$$

commutes. This is clear from Lemma 35.3.5 and the constructions of the functors in question. To see (1) let E be an object of $D_{QCoh}(\mathcal{O}_X)$. To see that Lf^*E has quasi-coherent cohomology sheaves we may work locally on X. Note that Lf^* is compatible with restricting to open subschemes. Hence we can assume that f is a morphism of affine schemes as in (2). Then we can apply Lemma 35.3.4 to see that E comes from a complex of A-modules. By the commutativity of the first diagram of the proof the same holds for Lf^*E and we conclude (1) is true. □

Lemma 35.3.7. *Let X be a scheme.*

(1) *For objects K, L of $D_{QCoh}(\mathcal{O}_X)$ the derived tensor product $K \otimes^{\mathbf{L}}_{\mathcal{O}_X} L$ is in $D_{QCoh}(\mathcal{O}_X)$.*
(2) *If $X = \mathrm{Spec}(A)$ is affine then*

$$\widetilde{M^\bullet} \otimes^{\mathbf{L}}_{\mathcal{O}_X} \widetilde{K^\bullet} = \widetilde{M^\bullet \otimes^{\mathbf{L}}_A K^\bullet}$$

for any pair of complexes of A-modules $K^\bullet$, $M^\bullet$.

Proof. The equality of (2) follows immediately from Lemma 35.3.5 and the construction of the derived tensor product. To see (1) let K, L be objects of $D_{QCoh}(\mathcal{O}_X)$. To check that $K \otimes^{\mathbf{L}} L$ is in $D_{QCoh}(\mathcal{O}_X)$ we may work locally on X, hence we may assume $X = \mathrm{Spec}(A)$ is affine. By Lemma 35.3.4 we may represent K and L by complexes of A-modules. Then part (2) implies the result. □

35.4. Total direct image

The following lemma is the analogue of Cohomology of Schemes, Lemma 29.4.4.

Lemma 35.4.1. *Let $f : X \to S$ be a morphism of schemes. Assume that f is quasi-separated and quasi-compact.*

1. *The functor Rf_* sends $D_{QCoh}(\mathcal{O}_X)$ into $D_{QCoh}(\mathcal{O}_S)$.*
2. *If S is quasi-compact, there exists an integer $N = N(X, S, f)$ such that for an object E of $D_{QCoh}(\mathcal{O}_X)$ with $H^m(E) = 0$ for $m > 0$ we have $H^m(Rf_*E) = 0$ for $m > N$.*
3. *In fact, if S is quasi-compact we can find $N = N(X, S, f)$ such that for every morphism of schemes $S' \to S$ the same conclusion holds for the functor $R(f')_*$ where $f' : X' \to S'$ is the base change of f.*

Proof. Let E be an object of $D_{QCoh}(\mathcal{O}_X)$. To prove (1) we have to show that Rf_*E has quasi-coherent cohomology sheaves. This question is local on S, hence we may assume S is quasi-compact. Pick $N = N(X, S, f)$ as in Cohomology of Schemes, Lemma 29.4.4. Thus $R^p f_*\mathcal{F} = 0$ for all quasi-coherent $\mathcal{O}_X$-modules $\mathcal{F}$ and all $p \geq N$. In particular, for any affine open $U \subset S$ we have $H^p(f^{-1}(U), \mathcal{F}) = 0$ for $p \geq N$, see Cohomology of Schemes, Lemma 29.4.5.

Let E be an object of $D_{QCoh}(\mathcal{O}_X)$. Choose $\mathcal{I}^\bullet = \lim \mathcal{I}_n^\bullet$ as in Lemma 35.3.2. As $\mathcal{I}^\bullet$ is K-injective Rf_*E is represented by $f_*\mathcal{I}^\bullet = \lim f_*\mathcal{I}_n^\bullet$. Let $U \subset S$ be any affine open. The cohomology $H^m(f_*\mathcal{I}_n^\bullet(U))$ of

$$f_*\mathcal{I}_n^{m-1}(U) \to f_*\mathcal{I}_n^m(U) \to f_*\mathcal{I}_n^{m+1}(U)$$

is equal to $H^m(f^{-1}(U), \tau_{\geq -n}E)$ because $\mathcal{I}_n^\bullet$ is a bounded below complex of injectives representing $\tau_{\geq -n}E$. We have a distinguished triangle

$$H^{-n}(E)[n] \to \tau_{\geq -n}E \to \tau_{\geq -n+1}E \to H^{-n}(E)[n+1]$$

(Derived Categories, Remark 13.11.4) in $D(\mathcal{O}_X)$. Since $H^{-n}(E)$ is quasi-coherent we have $H^m(f^{-1}(U), H^{-n}(E)[n]) = 0$ for $n+m \geq N$ by our choice of N. Similarly, $H^m(f^{-1}(U), H^{-n}(E)[n+1]) = 0$ for $n+m+1 \geq N$. We conclude that

$$H^m(f_*\mathcal{I}_n^\bullet(U)) \to H^m(f_*\mathcal{I}_{n-1}^\bullet(U))$$

is an isomorphism for all $n \geq N - m$. Thus Cohomology, Lemma 20.29.1 applies to show that the mth cohomology sheaf of $\lim f_*\mathcal{I}_n^\bullet$ agrees with the mth cohomology sheaf of $f_*\mathcal{I}_n^\bullet$ for $n \geq N - m$. Since these cohomology sheaves are quasi-coherent by Cohomology of Schemes, Lemma 29.4.4 we get (1).

Finally, we show that (2) and (3) hold with our choice of N. Namely, the stabilization proven above gives that $H^m(Rf_*E)$ is equal to $H^m(Rf_*(\tau_{\geq -n}E))$ for all n large enough which means we can work with objects in $D^+(\mathcal{O}_X)$ in order to prove (2) and (3). In this case we can for example use the spectral sequence

$$R^p f_* H^q(E) \Rightarrow R^{p+q} f_* E$$

(Derived Categories, Lemma 13.20.3) and the vanishing of $R^p f_* H^q(E)$ for $p \geq N$ to conclude. Some details omitted. □

Lemma 35.4.2. *Let $f : X \to S$ be a quasi-separated and quasi-compact morphism of schemes. Then $Rf_* : D_{QCoh}(\mathcal{O}_X) \to D_{QCoh}(\mathcal{O}_S)$ commutes with direct sums.*

Proof. Let E_i be a family of objects of $D_{QCoh}(\mathcal{O}_X)$ and set $E = \bigoplus E_i$. We want to show that the map

$$\bigoplus Rf_*E_i \longrightarrow Rf_*E$$

is an isomorphism. We will show it induces an isomorphism on cohomology sheaves in degree 0 which will imply the lemma. Choose an integer N as in Lemma 35.4.1. Then $R^0f_*E = R^0f_*\tau_{\geq -N}E$ and $R^0f_*E_i = R^0f_*\tau_{\geq -N}E_i$ by the lemma cited. Observe that $\tau_{\geq -N}E = \bigoplus \tau_{\geq -N}E_i$. Thus we may assume all of the E_i have vanishing cohomology sheaves in degrees $< -N$. Next we use the spectral sequences

$$R^pf_*H^q(E) \Rightarrow R^{p+q}f_*E \quad \text{and} \quad R^pf_*H^q(E_i) \Rightarrow R^{p+q}f_*E_i$$

(Derived Categories, Lemma 13.20.3) to reduce to the case of a direct sum of quasi-coherent sheaves. This case is handled by Cohomology of Schemes, Lemma 29.6.1. □

Lemma 35.4.3. *Let $f : X \to S$ be an affine morphism of schemes. Then $Rf_* : D_{QCoh}(\mathcal{O}_X) \to D_{QCoh}(\mathcal{O}_S)$ reflects isomorphisms.*

Proof. The statement means that a morphism $\alpha : E \to F$ of $D_{QCoh}(\mathcal{O}_X)$ is an isomorphism if $Rf_*\alpha$ is an isomorphism. We may check this on cohomology sheaves. In particular, the question is local on S. Hence we may assume S and therefore X is affine. In this case the statement is clear from the description of the derived categories $D_{QCoh}(\mathcal{O}_X)$ and $D_{QCoh}(\mathcal{O}_S)$ given in Lemma 35.3.4. Some details omitted. □

Lemma 35.4.4. *Let $f : X \to S$ be an affine morphism of schemes. For E in $D_{QCoh}(\mathcal{O}_S)$ we have $Rf_*Lf^*E = E \otimes^{\mathbf{L}}_{\mathcal{O}_S} f_*\mathcal{O}_X$.*

Proof. Since f is affine the map $f_*\mathcal{O}_X \to Rf_*\mathcal{O}_X$ is an isomorphism (Cohomology of Schemes, Lemma 29.2.3). There is a canonical map $E \otimes^{\mathbf{L}} f_*\mathcal{O}_X = E \otimes^{\mathbf{L}} Rf_*\mathcal{O}_X \to Rf_*Lf^*E$ adjoint to the map

$$Lf^*(E \otimes^{\mathbf{L}} Rf_*\mathcal{O}_X) = Lf^*E \otimes^{\mathbf{L}} Lf^*Rf_*\mathcal{O}_X \longrightarrow Lf^*E \otimes^{\mathbf{L}} \mathcal{O}_X = Lf^*E$$

coming from $1 : Lf^*E \to Lf^*E$ and the canonical map $Lf^*Rf_*\mathcal{O}_X \to \mathcal{O}_X$. To check the map so constructed is an isomorphism we may work locally on S. Hence we may assume S and therefore X is affine. In this case the statement is clear from the description of the derived categories $D_{QCoh}(\mathcal{O}_X)$ and $D_{QCoh}(\mathcal{O}_S)$ and the functor Lf^* given in Lemmas 35.3.4 and 35.3.6. Some details omitted. □

35.5. Derived category of coherent modules

Let X be a locally Noetherian scheme. In this case the category $Coh(\mathcal{O}_X) \subset Mod(\mathcal{O}_X)$ of coherent $\mathcal{O}_X$-modules is a weak Serre subcategory, see Homology, Section 12.9 and Cohomology of Schemes, Lemma 29.9.2. Denote

$$D_{Coh}(\mathcal{O}_X) \subset D(\mathcal{O}_X)$$

the subcategory of complexes whose cohomology sheaves are coherent, see Derived Categories, Section 13.12. Thus we obtain a canonical functor

$$D(Coh(\mathcal{O}_X)) \longrightarrow D_{Coh}(\mathcal{O}_X) \tag{35.5.0.1}$$

see Derived Categories, Equation (13.12.1.1).

Lemma 35.5.1. *Let S be a Noetherian scheme. Let $f : X \to S$ be a morphism of schemes which is locally of finite type. Let E be an object of $D^b_{Coh}(\mathcal{O}_X)$ such that the scheme theoretic support of $H^i(E)$ is proper over S for all i. Then Rf_*E is an object of $D^b_{Coh}(\mathcal{O}_S)$.*

Proof. Consider the spectral sequence

$$R^p f_* H^q(E) \Rightarrow R^{p+q} f_* E$$

see Derived Categories, Lemma 13.20.3. By assumption and Cohomology of Schemes, Remark 29.17.3 the sheaves $R^p f_* H^q(E)$ are coherent. Hence $R^{p+q} f_* E$ is coherent, i.e., $E \in D_{Coh}(\mathcal{O}_S)$. Boundedness from below is trivial. Boundedness from above follows from Cohomology of Schemes, Lemma 29.4.4 or from Lemma 35.4.1. □

35.6. The coherator

Let X be a scheme. The *coherator* is a functor

$$Q_X : \mathit{Mod}(\mathcal{O}_X) \longrightarrow \mathit{QCoh}(\mathcal{O}_X)$$

which is right adjoint to the inclusion functor $\mathit{QCoh}(\mathcal{O}_X) \to \mathit{Mod}(\mathcal{O}_X)$. It exists for any scheme X and moreover the adjunction mapping $Q_X(\mathcal{F}) \to \mathcal{F}$ is an isomorphism for every quasi-coherent module $\mathcal{F}$, see Properties, Proposition 27.21.4. Since Q_X is left exact (as a right adjoint) we can consider its right derived extension

$$RQ_X : D(\mathcal{O}_X) \longrightarrow D(\mathit{QCoh}(\mathcal{O}_X)).$$

As this functor is constructed by applying Q_X to a K-injective replacement we see that RQ_X is a right adjoint to the canonical functor $D(\mathit{QCoh}(\mathcal{O}_X)) \to D(\mathcal{O}_X)$.

Lemma 35.6.1. *Let $f : X \to Y$ be an affine morphism of schemes. Then f_* defines a derived functor $f_* : D(\mathit{QCoh}(\mathcal{O}_X)) \to D(\mathit{QCoh}(\mathcal{O}_Y))$. This functor has the property that*

$$\begin{array}{ccc} D(\mathit{QCoh}(\mathcal{O}_X)) & \longrightarrow & D_{\mathit{QCoh}}(\mathcal{O}_X) \\ {\scriptstyle f_*}\downarrow & & \downarrow{\scriptstyle Rf_*} \\ D(\mathit{QCoh}(\mathcal{O}_Y)) & \longrightarrow & D_{\mathit{QCoh}}(\mathcal{O}_Y) \end{array}$$

commutes.

Proof. The functor $f_* : \mathit{QCoh}(\mathcal{O}_X) \to \mathit{QCoh}(\mathcal{O}_Y)$ is exact, see Cohomology of Schemes, Lemma 29.2.3. Hence f_* defines a derived functor $f_* : D(\mathit{QCoh}(\mathcal{O}_X)) \to D(\mathit{QCoh}(\mathcal{O}_Y))$ by simply applying f_* to any representative complex, see Derived Categories, Lemma 13.16.8. For any complex of $\mathcal{O}_X$-modules $\mathcal{F}^\bullet$ there is a canonical map $f_*\mathcal{F}^\bullet \to Rf_*\mathcal{F}^\bullet$. To finish the proof we show this is a quasi-isomorphism when $\mathcal{F}^\bullet$ is a complex with each $\mathcal{F}^n$ quasi-coherent. As the statement is invariant under shifts it suffices to show that $H^0(f_*(\mathcal{F}^\bullet)) \to R^0 f_* \mathcal{F}^\bullet$ is an isomorphism. The statement is local on Y hence we may assume Y affine. By Lemma 35.4.1 we have $R^0 f_* \mathcal{F}^\bullet = R^0 f_* \tau_{\geq -n} \mathcal{F}^\bullet$ for all sufficiently large n. Thus we may assume $\mathcal{F}^\bullet$ bounded below. As each $\mathcal{F}^n$ is f_*-acyclic by Cohomology of Schemes, Lemma 29.2.3 we see that $f_*\mathcal{F}^\bullet \to Rf_*\mathcal{F}^\bullet$ is a quasi-isomorphism by Leray's acyclicity lemma (Derived Categories, Lemma 13.16.7). □

Lemma 35.6.2. *Let $f : X \to Y$ be a morphism of schemes. Assume that*

(1) *f is quasi-compact, quasi-separated, and flat, and*

(2) *denoting*

$$\Phi : D(QCoh(\mathcal{O}_X)) \to D(QCoh(\mathcal{O}_Y))$$

the right derived functor of $f_* : QCoh(\mathcal{O}_X) \to QCoh(\mathcal{O}_Y)$ *the diagram*

$$\begin{array}{ccc} D(QCoh(\mathcal{O}_X)) & \longrightarrow & D_{QCoh}(\mathcal{O}_X) \\ \downarrow{\scriptstyle \Phi} & & \downarrow{\scriptstyle Rf_*} \\ D(QCoh(\mathcal{O}_Y)) & \longrightarrow & D_{QCoh}(\mathcal{O}_Y) \end{array}$$

commutes.

Then $RQ_Y \circ Rf_* = \Phi \circ RQ_X$.

Proof. Since f is quasi-compact and quasi-separated, we see that f_* preserve quasi-coherence, see Schemes, Lemma 25.24.1. Recall that $QCoh(\mathcal{O}_X)$ is a Grothendieck abelian category (Properties, Proposition 27.21.4). Hence any K in $D(QCoh(\mathcal{O}_X))$ can be represented by a K-injective complex $\mathcal{I}^\bullet$ of $QCoh(\mathcal{O}_X)$, see Injectives, Theorem 19.12.6. Then we can define $\Phi(K) = f_*\mathcal{I}^\bullet$.

Since f is flat, the functor f^* is exact. Hence f^* defines $f^* : D(\mathcal{O}_Y) \to D(\mathcal{O}_X)$ and also $f^* : D(QCoh(\mathcal{O}_Y)) \to D(QCoh(\mathcal{O}_X))$. The functor $f^* = Lf^* : D(\mathcal{O}_Y) \to D(\mathcal{O}_X)$ is left adjoint to $Rf_* : D(\mathcal{O}_X) \to D(\mathcal{O}_Y)$, see Cohomology, Lemma 20.27.1. Similarly, the functor $f^* : D(QCoh(\mathcal{O}_Y)) \to D(QCoh(\mathcal{O}_X))$ is left adjoint to $\Phi : D(QCoh(\mathcal{O}_X)) \to D(QCoh(\mathcal{O}_Y))$ by Derived Categories, Lemma 13.27.4.

Let A be an object of $D(QCoh(\mathcal{O}_Y))$ and E an object of $D(\mathcal{O}_X)$. Then

$$\begin{aligned} \mathrm{Hom}_{D(QCoh(\mathcal{O}_Y))}(A, RQ_Y(Rf_*E)) &= \mathrm{Hom}_{D(\mathcal{O}_Y)}(A, Rf_*E) \\ &= \mathrm{Hom}_{D(\mathcal{O}_X)}(f^*A, E) \\ &= \mathrm{Hom}_{D(QCoh(\mathcal{O}_X))}(f^*A, RQ_X(E)) \\ &= \mathrm{Hom}_{D(QCoh(\mathcal{O}_Y))}(A, \Phi(RQ_X(E))) \end{aligned}$$

This implies what we want. □

Lemma 35.6.3. *Let* $X = \mathrm{Spec}(A)$ *be an affine scheme. Then*

(1) $Q_X : Mod(\mathcal{O}_X) \to QCoh(\mathcal{O}_X)$ *is the functor which sends* $\mathcal{F}$ *to the quasi-coherent* $\mathcal{O}_X$*-module associated to the* A*-module* $\Gamma(X, \mathcal{F})$*,*

(2) $RQ_X : D(\mathcal{O}_X) \to D(QCoh(\mathcal{O}_X))$ *is the functor which sends* E *to the complex of quasi-coherent* $\mathcal{O}_X$*-modules associated to the object* $R\Gamma(X, E)$ *of* $D(A)$*,*

(3) *restricted to* $D_{QCoh}(\mathcal{O}_X)$ *the functor* RQ_X *defines a quasi-inverse to (35.3.0.1).*

Proof. The functor Q_X is the functor

$$\mathcal{F} \mapsto \widetilde{\Gamma(X, \mathcal{F})}$$

by Schemes, Lemma 25.7.1. This immediately implies (1) and (2). The third assertion follows from (the proof of) Lemma 35.3.4. □

Definition 35.6.4. Let X be a scheme. Let E be an object of $D(\mathcal{O}_X)$. Let $T \subset X$ be a closed subset. We say E is *supported on* T if the cohomology sheaves $H^i(E)$ are supported on T.

Proposition 35.6.5. *Let X be a quasi-compact scheme with affine diagonal. Then the functor (35.3.0.1)*

$$D(QCoh(\mathcal{O}_X)) \longrightarrow D_{QCoh}(\mathcal{O}_X)$$

is an equivalence with quasi-inverse given by RQ_X.

Proof. In this proof we will denote $i_X : D(QCoh(\mathcal{O}_X)) \to D_{QCoh}(\mathcal{O}_X)$ the functor of the lemma. Let E be an object of $D_{QCoh}(\mathcal{O}_X)$ and let A be an object of $D(QCoh(\mathcal{O}_X))$. We have to show that the adjunction maps

$$RQ_X(i_X(A)) \to A \quad \text{and} \quad E \to i_X(RQ_X(E))$$

are isomorphisms. We will prove this by induction on n: the smallest integer $n \geq 0$ such that E and $i_X(A)$ are supported on a closed subset of X which is contained in the union of n affine opens of X.

Base case: $n = 0$. In this case $E = 0$, hence the map $E \to i_X(RQ_X(E))$ is an isomorphism. Similarly $i_X(A) = 0$. Thus the cohomology sheaves of $i_X(A)$ are zero. Since the inclusion functor $QCoh(\mathcal{O}_X) \to Mod(\mathcal{O}_X)$ is fully faithful and exact, we conclude that the cohomology objects of A are zero, i.e., $A = 0$ and $RQ_X(i_X(A)) \to A$ is an isomorphism as well.

Induction step. Suppose that E and $i_X(A)$ are supported on a closed subset T of X contained in $U_1 \cup \ldots \cup U_n$ with $U_i \subset X$ affine open. Set $U = U_n$. The inclusion morphism $j : U \to X$ is flat and affine (Morphisms, Lemma 28.13.11). Consider the distinguished triangles

$$A \to j_*(A|_U) \to A' \to A[1] \quad \text{and} \quad E \to Rj_*(E|_U) \to E' \to E[1]$$

where j_* is as in Lemma 35.6.1. Note that $E \to Rj_*(E|_U)$ is a quasi-isomorphism over $U = U_n$. Since $i_X \circ j_* = Rj_* \circ i_U$ by Lemma 35.6.1 and since $i_X(A)|_U = i_U(A|_U)$ we see that $i_X(A) \to i_X(j_*(A|_U))$ is a quasi-isomorphism over U. Hence $i_X(A')$ and E' are supported on the closed subset $T \backslash U$ of X which is contained in $U_1 \cup \ldots \cup U_{n-1}$. By induction hypothesis the statement is true for A' and E'. By Derived Categories, Lemma 13.4.3 it suffices to prove the maps

$$RQ_X(i_X(j_*(A|_U))) \to j_*(A|_U) \quad \text{and} \quad Rj_*(E|_U) \to i_X(RQ_X(Rj_*E|_U))$$

are isomorphisms. By Lemmas 35.6.1 and 35.6.2 we have

$$RQ_X(i_X(j_*(A|_U))) = RQ_X(Rj_*(i_U(A|_U))) = j_*RQ_U(i_U(A|_U))$$

and

$$i_X(RQ_X(Rj_*(E|_U))) = i_X(j_*RQ_U(E|_U)) = Rj_*(i_U(RQ_U(E|_U)))$$

Finally, the maps

$$RQ_U(i_U(A|_U)) \to A|_U \quad \text{and} \quad E|_U \to i_U(RQ_U(E|_U))$$

are isomorphisms by Lemma 35.6.3. The result follows. □

Remark 35.6.6. Analyzing the proof of Proposition 35.6.5 we see that we have shown the following. Let X be a quasi-compact and quasi-separated scheme. Suppose that for every affine open $U \subset X$ the right derived functor

$$\Phi : D(QCoh(\mathcal{O}_U)) \to D(QCoh(\mathcal{O}_X))$$

of the left exact functor $j_* : QCoh(\mathcal{O}_U) \to QCoh(\mathcal{O}_X)$ fits into a commutative diagram

$$\begin{array}{ccc} D(QCoh(\mathcal{O}_U)) & \xrightarrow[i_U]{} & D_{QCoh}(\mathcal{O}_U) \\ \Phi \downarrow & & \downarrow Rj_* \\ D(QCoh(\mathcal{O}_X)) & \xrightarrow{i_X} & D_{QCoh}(\mathcal{O}_X) \end{array}$$

Then the functor (35.3.0.1)

$$D(QCoh(\mathcal{O}_X)) \longrightarrow D_{QCoh}(\mathcal{O}_X)$$

is an equivalence with quasi-inverse given by RQ_X.

35.7. The coherator for Noetherian schemes

In the case of Noetherian schemes we can use the following lemma.

Lemma 35.7.1. *Let X be a Noetherian scheme. Let $\mathcal{J}$ be an injective object of $QCoh(\mathcal{O}_X)$. Then $\mathcal{J}$ is a flasque sheaf of $\mathcal{O}_X$-modules.*

Proof. Let $U \subset X$ be an open subset and let $s \in \mathcal{J}(U)$ be a section. Let $\mathcal{I} \subset X$ be the quasi-coherent sheaf of ideals defining the reduced induced scheme structure on $X \setminus U$ (see Schemes, Definition 25.12.5). By Cohomology of Schemes, Lemma 29.10.4 the section s corresponds to a map $\sigma : \mathcal{I}^n \to \mathcal{J}$ for some n. As $\mathcal{J}$ is an injective object of $QCoh(\mathcal{O}_X)$ we can extend σ to a map $\tilde{s} : \mathcal{O}_X \to \mathcal{J}$. Then $\tilde{s}$ corresponds to a global section of $\mathcal{J}$ restricting to s. □

Lemma 35.7.2. *Let $f : X \to Y$ be a morphism of Noetherian schemes. Then f_* on quasi-coherent sheaves has a right derived extension $\Phi : D(QCoh(\mathcal{O}_X)) \to D(QCoh(\mathcal{O}_Y))$ such that the diagram*

$$\begin{array}{ccc} D(QCoh(\mathcal{O}_X)) & \longrightarrow & D_{QCoh}(\mathcal{O}_X) \\ \Phi \downarrow & & \downarrow Rf_* \\ D(QCoh(\mathcal{O}_Y)) & \longrightarrow & D_{QCoh}(\mathcal{O}_Y) \end{array}$$

commutes.

Proof. Since X and Y are Noetherian schemes the morphism is quasi-compact and quasi-separated (see Properties, Lemma 27.5.4 and Schemes, Remark 25.21.18). Thus f_* preserve quasi-coherence, see Schemes, Lemma 25.24.1. Next, Let K be an object of $D(QCoh(\mathcal{O}_X))$. Since $QCoh(\mathcal{O}_X)$ is a Grothendieck abelian category (Properties, Proposition 27.21.4), we can represent K by a K-injective complex $\mathcal{I}^\bullet$ such that each $\mathcal{I}^n$ is an injective object of $QCoh(\mathcal{O}_X)$, see Injectives, Theorem 19.12.6. Thus we see that the functor Φ is defined by setting

$$\Phi(K) = f_*\mathcal{I}^\bullet$$

where the right hand side is viewed as an object of $D(QCoh(\mathcal{O}_Y))$. To finish the proof of the lemma it suffices to show that the canonical map

$$f_*\mathcal{I}^\bullet \longrightarrow Rf_*\mathcal{I}^\bullet$$

is an isomorphism in $D(\mathcal{O}_Y)$. To see this it suffices to prove the map induces an isomorphism on cohomology sheaves. Pick any $m \in \mathbf{Z}$. Let $N = N(X, Y, f)$ be as in Lemma 35.4.1. Consider the short exact sequence

$$0 \to \sigma_{\geq m-N-1}\mathcal{I}^\bullet \to \mathcal{I}^\bullet \to \sigma_{\leq m-N-2}\mathcal{I}^\bullet \to 0$$

of complexes of quasi-coherent sheaves on X. By Lemma 35.4.1 we see that the cohomology sheaves of $Rf_*\sigma_{\leq m-N-2}\mathcal{I}^\bullet$ are zero in degrees $\geq m-1$. Thus we see that $R^m f_*\mathcal{I}^\bullet$ is isomorphic to $R^m f_*\sigma_{\geq m-N-1}\mathcal{I}^\bullet$. In other words, we may assume that $\mathcal{I}^\bullet$ is a bounded below complex of injective objects of $QCoh(\mathcal{O}_X)$. This follows from Leray's acyclicity lemma (Derived Categories, Lemma 13.16.7) via Cohomology, Lemma 20.13.5 and Lemma 35.7.1. □

Proposition 35.7.3. *Let X be a Noetherian scheme. Then the functor (35.3.0.1)*

$$D(QCoh(\mathcal{O}_X)) \longrightarrow D_{QCoh}(\mathcal{O}_X)$$

is an equivalence with quasi-inverse given by RQ_X.

Proof. This follows using the exact same argument as in the proof of Proposition 35.6.5 using Lemma 35.7.2. See discussion in Remark 35.6.6. □

35.8. Koszul complexes

Let A be a ring and let $f_1, \ldots, f_r$ be a sequence of elements of A. We have defined the Koszul complex $K_\bullet(f_1, \ldots, f_r)$ in More on Algebra, Definition 15.19.2. It is a chain complex sitting in degrees $r, \ldots, 0$. We turn this into a cochain complex $K^\bullet(f_1, \ldots, f_r)$ by setting $K^{-n}(f_1, \ldots, f_r) = K_n(f_1, \ldots, f_r)$ and using the same differentials. In the rest of this section all the complexes will be cochain complexes.

We define a complex $I^\bullet(f_1, \ldots, f_r)$ such that we have a distinguished triangle

$$I^\bullet(f_1, \ldots, f_r) \to A \to K^\bullet(f_1, \ldots, f_r) \to I^\bullet(f_1, \ldots, f_r)[1]$$

in $K(A)$. In other words, we set

$$I^i(f_1, \ldots, f_r) = \begin{cases} K^{i-1}(f_1, \ldots, f_r) & \text{if } i \leq 0 \\ 0 & \text{else} \end{cases}$$

and we use the negative of the differential on $K^\bullet(f_1, \ldots, f_r)$. The maps in the distinguished triangle are the obvious ones. Note that $I^0(f_1, \ldots, f_r) = A^{\oplus r} \to A$ is given by multiplication by f_i on the ith factor. Hence $I^\bullet(f_1, \ldots, f_r) \to A$ factors as

$$I^\bullet(f_1, \ldots, f_r) \to I \to A$$

where $I = (f_1, \ldots, f_r)$. In fact, there is a short exact sequence

$$0 \to H^{-1}(K^\bullet(f_1, \ldots, f_s)) \to H^0(I^\bullet(f_1, \ldots, f_s)) \to I \to 0$$

and for every $i < 0$ we have $H^i(I^\bullet(f_1, \ldots, f_r)) = H^{i-1}(K^\bullet(f_1, \ldots, f_r)$. Observe that given a second sequence $g_1, \ldots, g_r$ of elements of A there are canonical maps

$$I^\bullet(f_1 g_1, \ldots, f_r g_r) \to I^\bullet(f_1, \ldots, f_r) \quad \text{and} \quad K^\bullet(f_1 g_1, \ldots, f_r g_r) \to K^\bullet(f_1, \ldots, f_r)$$

compatible with the maps described above. The first of these maps is given by multiplication by g_i on the ith summand of $I^0(f_1 g_1, \ldots, f_r g_r) = A^{\oplus r}$. In particular, given $f_1, \ldots, f_r$ we obtain an inverse system of complexes

$$I^\bullet(f_1, \ldots, f_r) \leftarrow I^\bullet(f_1^2, \ldots, f_r^2) \leftarrow I^\bullet(f_1^3, \ldots, f_r^3) \leftarrow \ldots \tag{35.8.0.1}$$

which will play an important role in that which is to follow. To easily formulate the following lemmas we fix some notation.

Situation 35.8.1. Here A is a ring and $f_1, \ldots, f_r$ is a sequence of elements of A. We set $X = \mathrm{Spec}(A)$ and $U = D(f_1) \cup \ldots \cup D(f_r) \subset X$. We denote $\mathcal{U} : U = \bigcup_{i=1,\ldots,r} D(f_i)$ the given open covering of U.

Our first lemma is that the complexes above can be used to compute the cohomology of quasi-coherent sheaves on U. Suppose given a complex $I^\bullet$ of A-modules and an A-module M. Then we define $\mathrm{Hom}_A(I^\bullet, M)$ to be the complex with nth term $\mathrm{Hom}_A(I^{-n}, M)$ and differentials given as the contragredients of the differentials on $I^\bullet$.

Lemma 35.8.2. *In Situation 35.8.1. Let M be an A-module and denote $\mathcal{F}$ the associated $\mathcal{O}_X$-module. Then there is a canonical isomorphism of complexes*

$$\mathrm{colim}_e \mathrm{Hom}_A(I^\bullet(f_1^e, \ldots, f_r^e), M) \longrightarrow \check{\mathcal{C}}^\bullet_{alt}(\mathcal{U}, \mathcal{F})$$

functorial in M.

Proof. Recall that the alternating Čech complex is the subcomplex of the usual Čech complex given by alternating cochains, see Cohomology, Section 20.22. As usual we view a p-cochain in $\check{\mathcal{C}}^\bullet_{alt}(\mathcal{U}, \mathcal{F})$ as an alternating function s on $\{1, \ldots, r\}^{p+1}$ whose value $s_{i_0 \ldots i_p}$ at $(i_0, \ldots, i_p)$ lies in $M_{f_{i_0} \ldots f_{i_p}} = \mathcal{F}(U_{i_0 \ldots i_p})$. On the other hand, a p-cochain t in $\mathrm{Hom}_A(I^\bullet(f_1^e, \ldots, f_r^e), M)$ is given by a map $t : \wedge^{p+1}(A^{\oplus r}) \to M$. Write $[i] \in A^{\oplus r}$ for the ith basis element and write

$$[i_0, \ldots, i_p] = [i_0] \wedge \ldots \wedge [i_p] \in \wedge^{p+1}(A^{\oplus r})$$

Then we send t as above to s with

$$s_{i_0 \ldots i_p} = \frac{t([i_0, \ldots, i_p])}{f_{i_0}^e \ldots f_{i_p}^e}$$

It is clear that s so defined is an alternating cochain. The construction of this map is compatible with the transition maps of the system as the transition map

$$I^\bullet(f_1^e, \ldots, f_r^e) \leftarrow I^\bullet(f_1^{e+1}, \ldots, f_r^{e+1}),$$

of the (35.8.0.1) sends $[i_0, \ldots, i_p]$ to $f_{i_0} \ldots f_{i_p}[i_0, \ldots, i_p]$. It is clear from the description of the localizations $M_{f_{i_0} \ldots f_{i_p}}$ in Algebra, Lemma 10.9.9 that these maps define an isomorphism of cochain modules in degree p in the limit. To finish the proof we have to show that the map is compatible with differentials. To see this recall that

$$\begin{aligned} d(s)_{i_0 \ldots i_{p+1}} &= \sum\nolimits_{j=0}^{p+1} (-1)^j s_{i_0 \ldots \hat{i}_j \ldots i_p} \\ &= \sum\nolimits_{j=0}^{p+1} (-1)^j \frac{t([i_0, \ldots, \hat{i}_j, \ldots i_{p+1}])}{f_{i_0}^e \ldots \hat{f}_{i_j}^e \ldots f_{i_{p+1}}^e} \end{aligned}$$

On the other hand, we have

$$\begin{aligned} \frac{d(t)([i_0, \ldots, i_{p+1}])}{f_{i_0}^e \ldots f_{i_{p+1}}^e} &= \frac{t(d[i_0, \ldots, i_{p+1}])}{f_{i_0}^e \ldots f_{i_{p+1}}^e} \\ &= \frac{\sum_j (-1)^j f_{i_j}^e t([i_0, \ldots, \hat{i}_j, \ldots i_{p+1}])}{f_{i_0}^e \ldots f_{i_{p+1}}^e} \end{aligned}$$

The two formulas agree by inspection. □

Suppose given a finite complex $I^\bullet$ of A-modules and a complex of A-modules $M^\bullet$. We obtain a double complex $H^{\bullet,\bullet} = \mathrm{Hom}_A(I^\bullet, M^\bullet)$ where $H^{p,q} = \mathrm{Hom}_A(I^p, M^q)$. The first differential comes from the differential on $\mathrm{Hom}_A(I^\bullet, M^q)$ and the second from the differential on $M^\bullet$. Associated to this double complex is the total complex with degree n term given by

$$\bigoplus\nolimits_{p+q=n} \mathrm{Hom}_A(I^p, M^q)$$

and differential as in Homology, Definition 12.22.2. As our complex $I^\bullet$ has only finitely many nonzero terms, the direct sum displayed above is finite. The conventions for taking the total complex associated to a Čech complex of a complex are as in Cohomology, Section 20.24.

Lemma 35.8.3. *In Situation 35.8.1. Let $M^\bullet$ be a complex of A-modules and denote $\mathcal{F}^\bullet$ the associated complex of $\mathcal{O}_X$-modules. Then there is a canonical isomorphism of complexes*

$$\mathrm{colim}_e \mathit{Tot}(\mathrm{Hom}_A(I^\bullet(f_1^e, \ldots, f_r^e), M^\bullet)) \longrightarrow \mathit{Tot}(\check{\mathcal{C}}^\bullet_{alt}(\mathcal{U}, \mathcal{F}^\bullet))$$

functorial in $M^\bullet$.

Proof. Immediate from Lemma 35.8.2 and our conventions for taking associated total complexes. □

Lemma 35.8.4. *In Situation 35.8.1. Let $\mathcal{F}^\bullet$ be a complex of quasi-coherent $\mathcal{O}_X$-modules. Then there is a canonical isomorphism*

$$\mathit{Tot}(\check{\mathcal{C}}^\bullet_{alt}(\mathcal{U}, \mathcal{F}^\bullet)) \longrightarrow R\Gamma(U, \mathcal{F}^\bullet)$$

in $D(A)$ functorial in $\mathcal{F}^\bullet$.

Proof. Let $\mathcal{B}$ be the set of affine opens of U. Since the higher cohomology groups of a quasi-coherent module on an affine scheme are zero (Cohomology of Schemes, Lemma 29.2.2) this is a special case of Cohomology, Lemma 20.30.2. □

In Situation 35.8.1 denote I_e the object of $D(\mathcal{O}_X)$ corresponding to the complex of A-modules $I^\bullet(f_1^e, \ldots, f_r^e)$ via the equivalence of Lemma 35.3.4. The maps (35.8.0.1) give a system

$$I_1 \leftarrow I_2 \leftarrow I_3 \leftarrow \ldots$$

Moreover, there is a compatible system of maps $I_e \to \mathcal{O}_X$ which become isomorphisms when restricted to U. Thus we see that for every object E of $D(\mathcal{O}_X)$ there is a canonical map

$$\mathrm{colim}_e \mathrm{Hom}_{D(\mathcal{O}_X)}(I_e, E) \longrightarrow H^0(U, E) \tag{35.8.4.1}$$

constructed by sending a map $I_e \to E$ to its restriction to U and using that $\mathrm{Hom}_{D(\mathcal{O}_U)}(\mathcal{O}_U, E|_U) = H^0(U, E)$.

Proposition 35.8.5. *In Situation 35.8.1. For every object E of $D_{QCoh}(\mathcal{O}_X)$ the map (35.8.4.1) is an isomorphism.*

Proof. By Lemma 35.3.4 we may assume that E is given by a complex of quasi-coherent sheaves $\mathcal{F}^\bullet$. Let $M^\bullet = \Gamma(X, \mathcal{F}^\bullet)$ be the corresponding complex of A-modules. By Lemmas 35.8.3 and 35.8.4 we have quasi-isomorphisms

$$\mathrm{colim}_e \mathrm{Tot}(\mathrm{Hom}_A(I^\bullet(f_1^e, \ldots, f_r^e), M^\bullet)) \longrightarrow \mathrm{Tot}(\check{\mathcal{C}}^\bullet_{alt}(\mathcal{U}, \mathcal{F}^\bullet)) \longrightarrow R\Gamma(U, \mathcal{F}^\bullet)$$

Taking H^0 on both sides we obtain

$$\operatorname{colim}_e \operatorname{Hom}_{D(A)}(I^\bullet(f_1^e, \ldots, f_r^e), M^\bullet) = H^0(U, E)$$

Since $\operatorname{Hom}_{D(A)}(I^\bullet(f_1^e, \ldots, f_r^e), M^\bullet) = \operatorname{Hom}_{D(\mathcal{O}_X)}(I_e, E)$ by Lemma 35.3.4 the lemma follows. □

In Situation 35.8.1 denote K_e the object of $D(\mathcal{O}_X)$ corresponding to the complex of A-modules $K^\bullet(f_1^e, \ldots, f_r^e)$ via the equivalence of Lemma 35.3.4. Thus we have distinguished triangles

$$I_e \to \mathcal{O}_X \to K_e \to I_e[1]$$

and a system

$$K_1 \leftarrow K_2 \leftarrow K_3 \leftarrow \ldots$$

compatible with the system (I_e). Moreover, there is a compatible system of maps

$$K_e \to H^0(K_e) = \mathcal{O}_X/(f_1^e, \ldots, f_r^e)$$

Lemma 35.8.6. *In Situation 35.8.1. Let E be an object of $D_{QCoh}(\mathcal{O}_X)$. Assume that $H^i(E)|_U = 0$ for $i = -r+1, \ldots, 0$. Then given $s \in H^0(X, E)$ there exists an $e \geq 0$ and a morphism $K_e \to E$ such that s is in the image of $H^0(X, K_e) \to H^0(X, E)$.*

Proof. Since U is covered by r affine opens we have $H^j(U, \mathcal{F}) = 0$ for $j \geq r$ and any quasi-coherent module (Cohomology of Schemes, Lemma 29.4.2). By Lemma 35.3.3 we see that $H^0(U, E)$ is equal to $H^0(U, \tau_{\geq -r+1}E)$. There is a spectral sequence

$$H^j(U, H^i(\tau_{\geq -r+1}E)) \Rightarrow H^{i+j}(U, \tau_{\geq -N}E)$$

see Derived Categories, Lemma 13.20.3. Hence $H^0(U, E) = 0$ by our assumed vanishing of cohomology sheaves of E. We conclude that $s|_U = 0$. Think of s as a morphism $\mathcal{O}_X \to E$ in $D(\mathcal{O}_X)$. By Proposition 35.8.5 the composition $I_e \to \mathcal{O}_X \to E$ is zero for some e. By the distinguished triangle $I_e \to \mathcal{O}_X \to K_e \to I_e[1]$ we obtain a morphism $K_e \to E$ such that s is the composition $\mathcal{O}_X \to K_e \to E$. □

35.9. Pseudo-coherent and perfect complexes

In this section we make the connection between the general notions defined in Cohomology, Sections 20.32, 20.33, 20.34, and 20.35 and the corresponding notions for complexes of modules in More on Algebra, Sections 15.48, 15.49, and 15.50.

Lemma 35.9.1. *Let X be a scheme. If E is an m-pseudo-coherent object of $D(\mathcal{O}_X)$, then $H^i(E)$ is a quasi-coherent $\mathcal{O}_X$-module for $i > m$. If E is pseudo-coherent, then E is an object of $D_{QCoh}(\mathcal{O}_X)$.*

Proof. Locally $H^i(E)$ is isomorphic to $H^i(\mathcal{E}^\bullet)$ with $\mathcal{E}^\bullet$ strictly perfect. The sheaves $\mathcal{E}^i$ are direct summands of finite free modules, hence quasi-coherent. The lemma follows. □

Lemma 35.9.2. *Let X be a locally ringed space. A direct summand of a free $\mathcal{O}_X$-module is finite locally free.*

Proof. Omitted. □

Lemma 35.9.3. *Let $X = \operatorname{Spec}(A)$ be an affine scheme. Let $M^\bullet$ be a complex of A-modules and let E be the corresponding object of $D(\mathcal{O}_X)$. Then E is an m-pseudo-coherent (resp. pseudo-coherent) as an object of $D(\mathcal{O}_X)$ if and only if $M^\bullet$ is m-pseudo-coherent (resp. pseudo-coherent) as a complex of A-modules.*

Proof. It is immediate from the definitions that if $M^\bullet$ is m-pseudo-coherent, so is E. To prove the converse, assume E is m-pseudo-coherent. As $X = \mathrm{Spec}(A)$ is quasi-compact with a basis for the topology given by standard opens, we can find a standard open covering $X = D(f_1) \cup \ldots \cup D(f_n)$ and strictly perfect complexes $\mathcal{E}_i^\bullet$ on $D(f_i)$ and maps $\alpha_i : \mathcal{E}_i^\bullet \to E|_{U_i}$ inducing isomorphisms on H^j for $j > m$ and surjections on H^m. By Cohomology, Lemma 20.32.8 after refining the open covering we may assume α_i is given by a map of complexes $\mathcal{E}_i^\bullet \to \widetilde{M^\bullet}|_{U_i}$ for each i. By Lemma 35.9.2 the terms $\mathcal{E}_i^n$ are finite locally free modules. Hence after refining the open covering we may assume each $\mathcal{E}_i^n$ is a finite free $\mathcal{O}_{U_i}$-module. From the definition it follows that $M_{f_i}^\bullet$ is an m-pseudo-coherent complex of A_{f_i}-modules. We conclude by applying More on Algebra, Lemma 15.48.14.

The case "pseudo-coherent" follows from the fact that E is pseudo-coherent if and only if E is m-pseudo-coherent for all m (by definition) and the same is true for $M^\bullet$ by More on Algebra, Lemma 15.48.5. □

Lemma 35.9.4. *Let X be a Noetherian scheme. Let E be an object of $D_{QCoh}(\mathcal{O}_X)$. For $m \in \mathbf{Z}$ the following are equivalent*

1. *$H^i(E)$ is coherent for $i \geq m$ and zero for $i \gg 0$, and*
2. *E is m-pseudo-coherent.*

In particular, E is pseudo-coherent if and only if E is an object of $D^-_{Coh}(\mathcal{O}_X)$.

Proof. As X is quasi-compact we see that in both (1) and (2) the object E is bounded above. Thus the question is local on X and we may assume X is affine. Say $X = \mathrm{Spec}(A)$ for some Noetherian ring A. In this case E corresponds to a complex of A-modules $M^\bullet$ by Lemma 35.3.4. By Lemma 35.9.3 we see that E is m-pseudo-coherent if and only if $M^\bullet$ is m-pseudo-coherent. On the other hand, $H^i(E)$ is coherent if and only if $H^i(M^\bullet)$ is a finite A-module (Properties, Lemma 27.16.1). Thus the result follows from More on Algebra, Lemma 15.48.16. □

Lemma 35.9.5. *Let $X = \mathrm{Spec}(A)$ be an affine scheme. Let $M^\bullet$ be a complex of A-modules and let E be the corresponding object of $D(\mathcal{O}_X)$. Then*

1. *E has tor amplitude in $[a, b]$ if and only if $M^\bullet$ has tor amplitude in $[a, b]$.*
2. *E has finite tor dimension if and only if $M^\bullet$ has finite tor dimension.*

Proof. Part (2) follows trivially from part (1). In the proof of (1) we will use the equivalence $D(A) = D_{QCoh}(X)$ of Lemma 35.3.4 without further mention. Assume $M^\bullet$ has tor amplitude in $[a, b]$. Then $K^\bullet$ is isomorphic in $D(A)$ to a complex $K^\bullet$ of flat A-modules with $K^i = 0$ for $i \notin [a, b]$, see More on Algebra, Lemma 15.49.3. Then E is isomorphic to $\widetilde{K^\bullet}$. Since each $\widetilde{K^i}$ is a flat $\mathcal{O}_X$-module, we see that E has tor amplitude in $[a, b]$ by Cohomology, Lemma 20.34.3.

Assume that E has tor amplitude in $[a, b]$. Then E is bounded whence $M^\bullet$ is in $K^-(A)$. Thus we may replace $M^\bullet$ by a bounded above complex of A-modules. We may even choose a projective resolution and assume that $M^\bullet$ is a bounded above complex of free A-modules. Then for any A-module N we have

$$E \otimes^{\mathbf{L}}_{\mathcal{O}_X} \widetilde{N} \cong \widetilde{M^\bullet} \otimes^{\mathbf{L}}_{\mathcal{O}_X} \widetilde{N} \cong \widetilde{M^\bullet \otimes_A N}$$

in $D(\mathcal{O}_X)$. Thus the vanishing of cohomology sheaves of the left hand side implies $M^\bullet$ has tor amplitude in $[a, b]$. □

Lemma 35.9.6. *Let X be a quasi-separated scheme. Let E be an object of $D_{QCoh}(\mathcal{O}_X)$. Let $a \leq b$. The following are equivalent*

(1) *E has tor amplitude in $[a, b]$, and*
(2) *for all $\mathcal{F}$ in $QCoh(\mathcal{O}_X)$ we have $H^i(E \otimes^{\mathbf{L}}_{\mathcal{O}_X} \mathcal{F}) = 0$ for $i \notin [a, b]$.*

Proof. It is clear that (1) implies (2). Assume (2). Let $U \subset X$ be an affine open. As X is quasi-separated the morphism $j : U \to X$ is quasi-compact and separated, hence j_* transforms quasi-coherent modules into quasi-coherent modules (Schemes, Lemma 25.24.1). Thus the functor $QCoh(\mathcal{O}_X) \to QCoh(\mathcal{O}_U)$ is essentially surjective. It follows that condition (2) implies the vanishing of $H^i(E|_U \otimes^{\mathbf{L}}_{\mathcal{O}_U} \mathcal{G})$ for $i \notin [a, b]$ for all quasi-coherent $\mathcal{O}_U$-modules $\mathcal{G}$. Write $U = \mathrm{Spec}(A)$ and let $M^\bullet$ be the complex of A-modules corresponding to $E|_U$ by Lemma 35.3.4. We have just shown that $M^\bullet \otimes^{\mathbf{L}}_A N$ has vanishing cohomology groups outside the range $[a, b]$, in other words $M^\bullet$ has tor amplitude in $[a, b]$. By Lemma 35.9.5 we conclude that $E|_U$ has tor amplitude in $[a, b]$. This proves the lemma. □

Lemma 35.9.7. *Let $X = \mathrm{Spec}(A)$ be an affine scheme. Let $M^\bullet$ be a complex of A-modules and let E be the corresponding object of $D(\mathcal{O}_X)$. Then E is a perfect object of $D(\mathcal{O}_X)$ if and only if $M^\bullet$ is perfect as an object of $D(A)$.*

Proof. This is a logical consequence of Lemmas 35.9.3 and 35.9.5, Cohomology, Lemma 20.35.4, and More on Algebra, Lemma 15.50.2. □

35.10. Descent finiteness properties of complexes

This section is the analogue of Descent, Section 34.6 for objects of the derived category of a scheme. The easiest such result is probably the following.

Lemma 35.10.1. *Let $f : X \to Y$ be a surjective flat morphism of schemes (or more generally locally ringed spaces). Let $E \in D(\mathcal{O}_Y)$. Let $a, b \in \mathbf{Z}$. Then E has tor-amplitude in $[a, b]$ if and only if Lf^*E has tor-amplitude in $[a, b]$.*

Proof. Pullback always preserves tor-amplitude, see Cohomology, Lemma 20.34.4. We may check tor-amplitude in $[a, b]$ on stalks, see Cohomology, Lemma 20.34.5. A flat local ring homomorphism is faithfully flat by Algebra, Lemma 10.36.16. Thus the result follows from More on Algebra, Lemma 15.49.14. □

Lemma 35.10.2. *Let $\{f_i : X_i \to X\}$ be an fpqc covering of schemes. Let $E \in D_{QCoh}(\mathcal{O}_X)$. Let $m \in \mathbf{Z}$. Then E is m-pseudo-coherent if and only if each Lf_i^*E is m-pseudo-coherent.*

Proof. Pullback always preserves m-pseudo-coherence, see Cohomology, Lemma 20.33.3. Conversely, assume that Lf_i^*E is m-pseudo-coherent for all i. Let $U \subset X$ be an affine open. It suffices to prove that $E|_U$ is m-pseudo-coherent. Since $\{f_i : X_i \to X\}$ is an fpqc covering, we can find finitely many affine open $V_j \subset X_{a(j)}$ such that $f_{a(j)}(V_j) \subset U$ and $U = \bigcup f_{a(j)}(V_j)$. Set $V = \coprod V_i$. Thus we may replace X by U and $\{f_i : X_i \to X\}$ by $\{V \to U\}$ and assume that X is affine and our covering is given by a single surjective flat morphism $\{f : Y \to X\}$ of affine schemes. In this case the result follows from More on Algebra, Lemma 15.48.15 via Lemmas 35.3.4 and 35.9.3. □

Lemma 35.10.3. *Let $\{f_i : X_i \to X\}$ be an fppf covering of schemes. Let $E \in D(\mathcal{O}_X)$. Let $m \in \mathbf{Z}$. Then E is m-pseudo-coherent if and only if each Lf_i^*E is m-pseudo-coherent.*

Proof. Pullback always preserves m-pseudo-coherence, see Cohomology, Lemma 20.33.3. Conversely, assume that Lf_i^*E is m-pseudo-coherent for all i. Let $U \subset X$ be an affine open. It suffices to prove that $E|_U$ is m-pseudo-coherent. Since $\{f_i : X_i \to X\}$ is an fppf covering, we can find finitely many affine open $V_j \subset X_{a(j)}$ such that $f_{a(j)}(V_j) \subset U$ and $U = \bigcup f_{a(j)}(V_j)$. Set $V = \coprod V_i$. Thus we may replace X by U and $\{f_i : X_i \to X\}$ by $\{V \to U\}$ and assume that X is affine and our covering is given by a single surjective flat morphism $\{f : Y \to X\}$ of finite presentation.

Since f is flat the derived functor Lf^* is just given by f^* and f^* is exact. Hence $H^i(Lf^*E) = f^*H^i(E)$. Since Lf^*E is m-pseudo-coherent, we see that $Lf^*E \in D^-(\mathcal{O}_Y)$. Since f is surjective and flat, we see that $E \in D^-(\mathcal{O}_X)$. Let $i \in \mathbf{Z}$ be the largest integer such that $H^i(E)$ is nonzero. If $i < m$, then we are done. Otherwise, $f^*H^i(E)$ is a finite type $\mathcal{O}_Y$-module by Cohomology, Lemma 20.33.9. Then by Descent, Lemma 34.6.2 the $\mathcal{O}_X$-module $H^i(E)$ is of finite type. Thus, after replacing X by the members of a finite affine open covering, we may assume there exists a map

$$\alpha : \mathcal{O}_X^{\oplus n}[-i] \longrightarrow E$$

such that $H^i(\alpha)$ is a surjection. Let C be the cone of α in $D(\mathcal{O}_X)$. Pulling back to Y and using Cohomology, Lemma 20.33.4 we find that Lf^*C is m-pseudo-coherent. Moreover $H^j(C) = 0$ for $j \geq i$. Thus by induction on i we see that C is m-pseudo-coherent. Using Cohomology, Lemma 20.33.4 again we conclude. □

Lemma 35.10.4. *Let $\{f_i : X_i \to X\}$ be an fpqc covering of schemes. Let $E \in D(\mathcal{O}_X)$. Then E is perfect if and only if each Lf_i^*E is perfect.*

Proof. Pullback always preserves perfect complexes, see Cohomology, Lemma 20.35.5. Conversely, assume that Lf_i^*E is perfect for all i. Then the cohomology sheaves of each Lf_i^*E are quasi-coherent, see Lemma 35.9.1 and Cohomology, Lemma 20.35.4. Since the morphisms f_i is flat we see that $H^p(Lf_i^*E) = f_i^*H^p(E)$. Thus the cohomology sheaves of E are quasi-coherent by Descent, Proposition 34.5.2. Having said this the lemma follows formally from Cohomology, Lemma 20.35.4 and Lemmas 35.10.1 and 35.10.2. □

Lemma 35.10.5. *Let $i : Z \to X$ be a morphism of ringed spaces such that i is a closed immersion of underlying topological spaces and such that $i_*\mathcal{O}_Z$ is pseudo-coherent as an $\mathcal{O}_X$-module. Let $E \in D(\mathcal{O}_X)$. Then E is m-pseudo-coherent if and only if Ri_*E is m-pseudo-coherent.*

Proof. Throughout this proof we will use that i_* is an exact functor, and hence that $Ri_* = i_*$, see Modules, Lemma 17.6.1.

Assume E is m-pseudo-coherent. Let $x \in X$. We will find a neighbourhood of x such that i_*E is m-peudo-coherent on it. If $x \notin Z$ then this is clear. Thus we may assume $x \in Z$. We will use that $U \cap Z$ for $x \in U \subset X$ open form a fundamental system of neighbourhoods of x in Z. After shrinking X we may assume E is bounded above. We will argue by induction on the largest integer p such that $H^p(E)$ is nonzero. If $p < m$, then there is nothing to prove. If $p \geq m$, then $H^p(E)$ is an $\mathcal{O}_Z$-module of finite type, see Cohomology, Lemma 20.33.9. Thus we may choose, after shrinking X, a map $\mathcal{O}_Z^{\oplus n}[-p] \to E$ which induces a surjection $\mathcal{O}_Z^{\oplus n} \to H^p(E)$. Choose a distinguished triangle

$$\mathcal{O}_Z^{\oplus n}[-p] \to E \to C \to \mathcal{O}_Z^{\oplus n}[-p+1]$$

We see that $H^j(C) = 0$ for $j \geq p$ and that C is m-pseudo-coherent by Cohomology, Lemma 20.33.4. By induction we see that i_*C is m-pseudo-coherent on X. Since $i_*\mathcal{O}_Z$ is m-pseudo-coherent on X as well, we conclude from the distinguished triangle

$$i_*\mathcal{O}_Z^{\oplus n}[-p] \to i_*E \to i_*C \to i_*\mathcal{O}_Z^{\oplus n}[-p+1]$$

and Cohomology, Lemma 20.33.4 that i_*E is m-pseudo-coherent.

Assume that i_*E is m-pseudo-coherent. Let $z \in Z$. We will find a neighbourhood of z such that E is m-peudo-coherent on it. We will use that $U \cap Z$ for $z \in U \subset X$ open form a fundamental system of neighbourhoods of z in Z. After shrinking X we may assume i_*E and hence E is bounded above. We will argue by induction on the largest integer p such that $H^p(E)$ is nonzero. If $p < m$, then there is nothing to prove. If $p \geq m$, then $H^p(i_*E) = i_*H^p(E)$ is an $\mathcal{O}_X$-module of finite type, see Cohomology, Lemma 20.33.9. Choose a complex $\mathcal{E}^\bullet$ of $\mathcal{O}_Z$-modules representing E. We may choose, after shrinking X, a map $\alpha : \mathcal{O}_X^{\oplus n}[-p] \to i_*\mathcal{E}^\bullet$ which induces a surjection $\mathcal{O}_X^{\oplus n} \to i_*H^p(\mathcal{E}^\bullet)$. By adjunction we find a map $\alpha : \mathcal{O}_Z^{\oplus n}[-p] \to \mathcal{E}^\bullet$ which induces a surjection $\mathcal{O}_Z^{\oplus n} \to H^p(\mathcal{E}^\bullet)$. Choose a distinguished triangle

$$\mathcal{O}_Z^{\oplus n}[-p] \to E \to C \to \mathcal{O}_Z^{\oplus n}[-p+1]$$

We see that $H^j(C) = 0$ for $j \geq p$. From the distinguished triangle

$$i_*\mathcal{O}_Z^{\oplus n}[-p] \to i_*E \to i_*C \to i_*\mathcal{O}_Z^{\oplus n}[-p+1]$$

the fact that $i_*\mathcal{O}_Z$ is pseudo-coherent and Cohomology, Lemma 20.33.4 we conclude that i_*C is m-pseudo-coherent. By induction we conclude that C is m-pseudo-coherent. By Cohomology, Lemma 20.33.4 again we conclude that E is m-pseudo-coherent. □

Lemma 35.10.6. *Let $f : X \to Y$ be a finite morphism of schemes such that $f_*\mathcal{O}_X$ is pseudo-coherent as an $\mathcal{O}_Y$-module[1]. Let $E \in D_{QCoh}(\mathcal{O}_X)$. Then E is m-pseudo-coherent if and only if Rf_*E is m-pseudo-coherent.*

Proof. This is a translation of More on Algebra, Lemma 15.48.11 into the language of schemes. To do the translation, use Lemmas 35.3.4 and 35.9.3. □

35.11. Lifting complexes

Let $U \subset X$ be an open subspace of a ringed space and denote $j : U \to X$ the inclusion morphism. The functor $D(\mathcal{O}_X) \to D(\mathcal{O}_U)$ is essentially surjective as Rj_* is a right inverse to restriction. In this section we extend this to complexes with quasi-coherent cohomology sheaves, etc.

Lemma 35.11.1. *Let X be a scheme and let $j : U \to X$ be a quasi-compact open immersion. The functors*

$$D_{QCoh}(\mathcal{O}_X) \to D_{QCoh}(\mathcal{O}_U) \quad \text{and} \quad D^+_{QCoh}(\mathcal{O}_X) \to D^+_{QCoh}(\mathcal{O}_U)$$

are essentially surjective. If X is quasi-compact, then the functors

$$D^-_{QCoh}(\mathcal{O}_X) \to D^-_{QCoh}(\mathcal{O}_U) \quad \text{and} \quad D^b_{QCoh}(\mathcal{O}_X) \to D^b_{QCoh}(\mathcal{O}_U)$$

are essentially surjective.

[1]This means that f is pseudo-coherent, see More on Morphisms, Definition 36.40.2.

Proof. The argument preceding the lemma applies for the first case because Rj_* maps $D_{QCoh}(\mathcal{O}_U)$ into $D_{QCoh}(\mathcal{O}_X)$ by Lemma 35.4.1. It is clear that Rj_* maps $D^+_{QCoh}(\mathcal{O}_U)$ into $D^+_{QCoh}(\mathcal{O}_X)$ which implies the statement on bounded below complexes. Finally, Lemma 35.4.1 guarantees that Rj_* maps $D^-_{QCoh}(\mathcal{O}_U)$ into $D^-_{QCoh}(\mathcal{O}_X)$ if X is quasi-compact. Combining these two we obtain the last statement. $\square$

Lemma 35.11.2. *Let X be an affine scheme and let $U \subset X$ be a quasi-compact open subscheme. For any pseudo-coherent object E of $D(\mathcal{O}_U)$ there exists a bounded above complex of finite free $\mathcal{O}_X$-modules whose restriction to U is isomorphic to E.*

Proof. By Lemma 35.9.1 we see that E is an object of $D_{QCoh}(\mathcal{O}_U)$. By Lemma 35.11.1 we may assume $E = E'|U$ for some object E' of $D_{QCoh}(\mathcal{O}_X)$. Write $X = \mathrm{Spec}(A)$. By Lemma 35.3.4 we can find a complex $M^\bullet$ of A-modules whose associated complex of $\mathcal{O}_X$-modules is a representative of E'.

Choose $f_1, \ldots, f_r \in A$ such that $U = D(f_1) \cup \ldots \cup D(f_r)$. By Lemma 35.9.3 the complexes $M^\bullet_{f_j}$ are pseudo-coherent complexes of A_{f_j}-modules. Let n be an integer. Assume we have a map of complexes $\alpha : F^\bullet \to M^\bullet$ where $F^\bullet$ is bounded above, $F^i = 0$ for $i < n$, each F^i is a finite free R-module, such that

$$H^i(\alpha_{f_j}) : H^i(F^\bullet_{f_j}) \to H^i(M^\bullet_{f_j})$$

is an isomorphism for $i > n$ and surjective for $i = n$. Picture

$$\begin{array}{ccccccc} & & F^n & \longrightarrow & F^{n+1} & \longrightarrow & \ldots \\ & & \downarrow{\scriptstyle \alpha} & & \downarrow{\scriptstyle \alpha} & & \\ M^{n-1} & \longrightarrow & M^n & \longrightarrow & M^{n+1} & \longrightarrow & \ldots \end{array}$$

Since each $M^\bullet_{f_j}$ has vanishing cohomology in large degrees we can find such a map for $n \gg 0$. By induction on n we are going to extend this to a map of complexes $F^\bullet \to M^\bullet$ such that $H^i(\alpha_{f_j})$ is an isomorphism for all i. The lemma will follow by taking $\widetilde{F^\bullet}$.

The induction step will be to extend the diagram above by adding F^{n-1}. Let $C^\bullet$ be the cone on α (Derived Categories, Definition 13.8.1). The long exact sequence of cohomology shows that $H^i(C^\bullet_{f_j}) = 0$ for $i \geq n$. By More on Algebra, Lemma 15.48.2 we see that $C^\bullet_{f_j}$ is $(n-1)$-pseudo-coherent. By More on Algebra, Lemma 15.48.3 we see that $H^{-1}(C^\bullet_{f_j})$ is a finite A_{f_j}-module. Choose a finite free A-module F^{n-1} and an A-module $\beta : F^{n-1} \to C^{-1}$ such that the composition $F^{n-1} \to C^{n-1} \to C^n$ is zero and such that $F^{n-1}_{f_j}$ surjects onto $H^{n-1}(C^\bullet_{f_j})$. (Some details omitted; hint: clear denominators.) Since $C^{n-1} = M^{n-1} \oplus F^n$ we can write $\beta = (\alpha^{n-1}, -d^{n-1})$. The vanishing of the composition $F^{n-1} \to C^{n-1} \to C^n$ implies these maps fit into a morphism of complexes

$$\begin{array}{ccccccccc} & & F^{n-1} & \xrightarrow{d^{n-1}} & F^n & \longrightarrow & F^{n+1} & \longrightarrow & \ldots \\ & & \downarrow{\scriptstyle \alpha^{n-1}} & & \downarrow{\scriptstyle \alpha} & & \downarrow{\scriptstyle \alpha} & & \\ \ldots & \longrightarrow & M^{n-1} & \longrightarrow & M^n & \longrightarrow & M^{n+1} & \longrightarrow & \ldots \end{array}$$

Moreover, these maps define a morphism of distinguished triangles

$$\begin{array}{ccccccc} (F^n \to \ldots) & \longrightarrow & (F^{n-1} \to \ldots) & \longrightarrow & F^{n-1} & \longrightarrow & (F^n \to \ldots)[1] \\ \downarrow & & \downarrow & & \downarrow{\scriptstyle \beta} & & \downarrow \\ (F^n \to \ldots) & \longrightarrow & M^\bullet & \longrightarrow & C^\bullet & \longrightarrow & (F^n \to \ldots)[1] \end{array}$$

Hence our choice of β implies that the map of complexes $(F^{-1} \to \ldots) \to M^\bullet$ induces an isomorphism on cohomology localized at f_j in degrees $\geq n$ and a surjection in degree -1. This finishes the proof of the lemma. □

Lemma 35.11.3. *Let X be a quasi-compact and quasi-separated scheme. Let $E \in D^b_{QCoh}(\mathcal{O}_X)$. There exists an integer $n_0 > 0$ such that $Ext^n_{D(\mathcal{O}_X)}(\mathcal{E}, E) = 0$ for every finite locally free $\mathcal{O}_X$-module $\mathcal{E}$ and every $n \geq n_0$.*

Proof. Recall that $\mathrm{Ext}^n_{D(\mathcal{O}_X)}(\mathcal{E}, E) = \mathrm{Hom}_{D(\mathcal{O}_X)}(\mathcal{E}, E[n])$. We have Mayer-Vietoris for morphisms in the derived category, see Cohomology, Lemma 20.28.6. Thus if $X = U \cup V$ and the result of the lemma holds for $E|_U$, $E|_V$, and $E|_{U \cap V}$ for some bound n_0, then the result holds for E with bound $n_0 + 1$. Thus it suffices to prove the lemma when X is affine, see Cohomology of Schemes, Lemma 29.4.1.

Assume $X = \mathrm{Spec}(A)$ is affine. Choose a complex of A-modules $M^\bullet$ whose associated complex of quasi-coherent modules represents E, see Lemma 35.3.4. Write $\mathcal{E} = \widetilde{P}$ for some A-module P. Since $\mathcal{E}$ is finite locally free, we see that P is a finite projective A-module. We have

$$\begin{aligned} \mathrm{Hom}_{D(\mathcal{O}_X)}(\mathcal{E}, E[n]) &= \mathrm{Hom}_{D(A)}(P, M^\bullet[n]) \\ &= \mathrm{Hom}_{K(A)}(P, M^\bullet[n]) \\ &= \mathrm{Hom}_A(P, H^n(M^\bullet)) \end{aligned}$$

The first equality by Lemma 35.3.4, the second equality by Derived Categories, Lemma 13.18.8, and the final equality because $\mathrm{Hom}_A(P, -)$ is an exact functor. As E and hence $M^\bullet$ is bounded we get zero for all sufficiently large n. □

Lemma 35.11.4. *Let X be an affine scheme. Let $U \subset X$ be a quasi-compact open. For every perfect object E of $D(\mathcal{O}_U)$ there exists an integer r and a finite locally free sheaf $\mathcal{F}$ on U such that $\mathcal{F}[-r] \oplus E$ is the restriction of a perfect object of $D(\mathcal{O}_X)$.*

Proof. Say $X = \mathrm{Spec}(A)$. Recall that a perfect complex is pseudo-coherent, see Cohomology, Lemma 20.35.4. By Lemma 35.11.2 we can find a bounded above complex $\mathcal{F}^\bullet$ of finite free A-modules such that E is isomorphic to $\mathcal{F}^\bullet|_U$ in $D(\mathcal{O}_U)$. By Cohomology, Lemma 20.35.4 and since U is quasi-compact, we see that E has finite tor dimension, say E has tor amplitude in $[a, b]$. Pick $r < a$ and set

$$\mathcal{F} = \mathrm{Ker}(\mathcal{F}^r \to \mathcal{F}^{r+1}) = \mathrm{Im}(\mathcal{F}^{r-1} \to \mathcal{F}^r).$$

Since E has tor amplitude in $[a, b]$ we see that $\mathcal{F}|_U$ is flat (Cohomology, Lemma 20.34.2). Hence $\mathcal{F}|_U$ is flat and of finite presentation, thus finite locally free (Properties, Lemma 27.18.2). It follows that

$$(\mathcal{F} \to \mathcal{F}^r \to \mathcal{F}^{r+1} \to \ldots)|_U$$

is a strictly perfect complex on U representing E. We obtain a distinguished triangle

$$\mathcal{F}|_U[-r-1] \to E \to (\mathcal{F}^r \to \mathcal{F}^{r+1} \to \ldots)|_U \to \mathcal{F}|_U[-r]$$

Note that $(\mathcal{F}^r \to \mathcal{F}^{r+1} \to \ldots)$ is a perfect complex on X. To finish the proof it suffices to pick r such that the map $\mathcal{F}|_U[-r-1] \to E$ is zero in $D(\mathcal{O}_U)$, see Derived Categories, Lemma 13.4.10. By Lemma 35.11.3 this holds if $r \ll 0$. □

Lemma 35.11.5. *Let X be an affine scheme. Let $U \subset X$ be a quasi-compact open. Let E, E' be objects of $D_{QCoh}(\mathcal{O}_X)$ with E perfect. For every map $\alpha : E|_U \to E'|_U$ there exist maps*

$$E \xleftarrow{\beta} E_1 \xrightarrow{\gamma} E'$$

of perfect complexes on X such that $\beta : E_1 \to E$ restricts to an isomorphism on U and such that $\alpha = \gamma|_U \circ \beta|_U^{-1}$. Moreover we can assume $E_1 = E \otimes^{\mathbf{L}}_{\mathcal{O}_X} I$ for some perfect complex I on X.

Proof. Write $X = \mathrm{Spec}(A)$. Write $U = D(f_1) \cup \ldots \cup D(f_r)$. Choose finite complex of finite projective A-modules $M^\bullet$ representing E (Lemma 35.9.7). Choose a complex of A-modules $(M')^\bullet$ representing E' (Lemma 35.3.4). In this case the complex $H^\bullet = \mathrm{Hom}_A(M^\bullet, (M')^\bullet)$ is a complex of A-modules whose associated complex of quasi-coherent $\mathcal{O}_X$-modules represents $R\mathcal{H}om(E, E')$, see Cohomology, Lemma 20.32.9. Then α determines an element s of $H^0(U, R\mathcal{H}om(E, E'))$, see Cohomology, Lemma 20.31.2. There exists an e and a map

$$\xi : I^\bullet(f_1^e, \ldots, f_r^e) \to \mathrm{Hom}_A(M^\bullet, (M')^\bullet)$$

corresponding to s, see Proposition 35.8.5. Letting E_1 be the object corresponding to complex of quasi-coherent $\mathcal{O}_X$-modules associated to

$$\mathrm{Tot}(I^\bullet(f_1^e, \ldots, f_r^e) \otimes_A M^\bullet)$$

we obtain $E_1 \to E$ using the canonical map $I^\bullet(f_1^e, \ldots, f_r^e) \to A$ and $E_1 \to E'$ using ξ and Cohomology, Lemma 20.31.2. □

Lemma 35.11.6. *Let X be an affine scheme. Let $U \subset X$ be a quasi-compact open. For every perfect object F of $D(\mathcal{O}_U)$ the object $F \oplus F[1]$ is the restriction of a perfect object of $D(\mathcal{O}_X)$.*

Proof. By Lemma 35.11.4 we can find a perfect object E of $D(\mathcal{O}_X)$ such that $E|_U = \mathcal{F}[r] \oplus F$ for some finite locally free $\mathcal{O}_U$-module $\mathcal{F}$. By Lemma 35.11.5 we can find a morphism of perfect complexes $\alpha : E_1 \to E$ such that $(E_1)|_U \cong E|_U$ and such that $\alpha|_U$ is the map

$$\begin{pmatrix} \mathrm{id}_{\mathcal{F}[r]} & 0 \\ 0 & 0 \end{pmatrix} : \mathcal{F}[r] \oplus F \to \mathcal{F}[r] \oplus F$$

Then the cone on α is a solution. □

Lemma 35.11.7. *Let X be a quasi-compact and quasi-separated scheme. Let $f \in \Gamma(X, \mathcal{O}_X)$. For any morphism $\alpha : E \to E'$ in $D_{QCoh}(\mathcal{O}_X)$ such that*

1. *E is perfect, and*
2. *E' is supported on $T = V(f)$*

there exists an $n \geq 0$ such that $f^n\alpha = 0$.

Proof. We have Mayer-Vietoris for morphisms in the derived category, see Cohomology, Lemma 20.28.6. Thus if $X = U \cup V$ and the result of the lemma holds for $f|_U$, $f|_V$, and $f|_{U \cap V}$, then the result holds for f. Thus it suffices to prove the lemma when X is affine, see Cohomology of Schemes, Lemma 29.4.1.

Let $X = \mathrm{Spec}(A)$. Then $f \in A$. We will use the equivalence $D(A) = D_{QCoh}(X)$ of Lemma 35.3.4 without further mention. Represent E by a finite complex of finite projective A-modules $P^\bullet$. This is possible by Lemma 35.9.7. Let t be the largest integer such that P^t is nonzero. The distinguished triangle

$$P^t[-t] \to P^\bullet \to \sigma_{\leq t-1} P^\bullet \to P^t[-t+1]$$

shows that by induction on the length of the complex $P^\bullet$ we can reduce to the case where $P^\bullet$ has a single nonzero term. This and the shift functor reduces us to the case where $P^\bullet$ consists of a single finite projective A-module P in degree 0. Represent E' by a complex $M^\bullet$ of A-modules. Then α corresponds to a map $P \to H^0(M^\bullet)$. Since the module $H^0(M^\bullet)$ is supported on $V(f)$ by assumption (2) we see that every element of $H^0(M^\bullet)$ is annihilated by a power of f. Since P is a finite A-module the map $f^n\alpha : P \to H^0(M^\bullet)$ is zero for some n as desired. □

Lemma 35.11.8. *Let X be an affine scheme. Let $T \subset X$ be a closed subset such that $X \setminus T$ is quasi-compact. Let $U \subset X$ be a quasi-compact open. For every perfect object F of $D(\mathcal{O}_U)$ supported on $T \cap U$ the object $F \oplus F[1]$ is the restriction of a perfect object E of $D(\mathcal{O}_X)$ supported in T.*

Proof. Say $T = V(g_1, \ldots, g_s)$. After replacing g_j by a power we may assume multiplication by g_j is zero on F, see Lemma 35.11.7. Choose E as in Lemma 35.11.6. Note that $g_j : E \to E$ restricts to zero on U. Choose a distinguished triangle

$$E \xrightarrow{g_1} E \to C_1 \to E[1]$$

By Derived Categories, Lemma 13.4.10 the object C_1 restricts to $F \oplus F[1] \oplus F[1] \oplus F[2]$ on U. Moreover, $g_1 : C_1 \to C_1$ has square zero by Derived Categories, Lemma 13.4.5. Namely, the diagram

$$\begin{array}{ccccc} E & \longrightarrow & C_1 & \longrightarrow & E[1] \\ {\scriptstyle 0}\downarrow & & {\scriptstyle g_1}\downarrow & & {\scriptstyle 0}\downarrow \\ E & \longrightarrow & C_1 & \longrightarrow & E[1] \end{array}$$

is commutative since the compositions $E \xrightarrow{g_1} E \to C_1$ and $C_1 \to E[1] \xrightarrow{g_1} E[1]$ are zero. Continuing, setting C_{i+1} equal to the cone of the map $g_i : C_i \to C_i$ we obtain a perfect complex C_s on X supported on T whose restriction to U gives

$$F \oplus F[1]^{\oplus s} \oplus F[2]^{\oplus \binom{s}{2}} \oplus \ldots \oplus F[s]$$

Choose a morphisms of perfect complexes $\beta : C' \to C_s$ and $\gamma : C' \to C_s$ as in Lemma 35.11.5 such that $\beta|_U$ is an isomorphism and such that $\gamma|_U \circ \beta|_U^{-1}$ is the morphism

$$F \oplus F[1]^{\oplus s} \oplus F[2]^{\oplus \binom{s}{2}} \oplus \ldots \oplus F[s] \to F \oplus F[1]^{\oplus s} \oplus F[2]^{\oplus \binom{s}{2}} \oplus \ldots \oplus F[s]$$

which is the identity on all summands except for F where it is zero. By Lemma 35.11.5 we also have $C' = C_s \otimes^{\mathbf{L}} I$ for some perfect complex I on X. Hence the nullity of $g_j^2 \mathrm{id}_{C_s}$ implies the same thing for C'. Thus C' is supported on T as well. Then $\mathrm{Cone}(\gamma)$ is a solution. □

A special case of the following lemma can be found in [**Nee96**].

Lemma 35.11.9. *Let X be a quasi-compact and quasi-separated scheme. Let $U \subset X$ be a quasi-compact open. Let $T \subset X$ be a closed subset with $X \setminus T$ retro-compact in X. Let E be an object of $D_{QCoh}(\mathcal{O}_X)$. Let $\alpha : P \to E|_U$ be a map where P is a perfect object of $D(\mathcal{O}_U)$ supported on $T \cap U$. Then there exists a map $\beta : R \to E$ where R is a perfect object of $D(\mathcal{O}_X)$ supported on T such that P is a direct summand of $R|_U$ in $D(\mathcal{O}_U)$ compatible α and $\beta|_U$.*

Proof. Since X is quasi-compact there exists an integer m such that $X = U \cup V_1 \cup \ldots \cup V_m$ for some affine opens V_j of X. Arguing by induction on m we see that we may assume $m = 1$. In other words, we may assume that $X = U \cup V$ with V affine. By Lemma 35.11.8 we can choose a perfect object Q in $D(\mathcal{O}_V)$ supported on $T \cap V$ and an isomorphism $Q|_{U\cap V} \to (P \oplus P[1])|_{U\cap V}$. By Lemma 35.11.5 we can replace Q by $Q \otimes^{\mathbf{L}} I$ (still supported on $T \cap V$) and assume that the map

$$Q|_{U\cap V} \to (P \oplus P[1])|_{U\cap V} \longrightarrow P|_{U\cap V} \longrightarrow E|_{U\cap V}$$

lifts to $Q \to E|_V$. By Cohomology, Lemma 20.28.10 we find an morphism $a : R \to E$ of $D(\mathcal{O}_X)$ such that $a|_U$ is isomorphic to $P \oplus P[1] \to E|_U$ and $a|_V$ isomorphic to $Q \to E|_V$. Thus R is perfect and supported on T as desired. □

Remark 35.11.10. The proof of Lemma 35.11.9 shows that

$$R|_U = P \oplus P^{\oplus n_1}[1] \oplus \ldots \oplus P^{\oplus n_m}[m]$$

for some $m \geq 0$ and $n_j \geq 0$. Thus the highest degree cohomology sheaf of $R|_U$ equals that of P. By repeating the construction for the map $P^{\oplus n_1}[1] \oplus \ldots \oplus P^{\oplus n_m}[m] \to R|_U$, taking cones, and using induction we can achieve equality of cohomology sheaves of $R|_U$ and P above any given degree.

35.12. Approximation by perfect complexes

In this section we discuss the observation, due to Neeman and Lipman, that a pseudo-coherent complex can be "approximated" by perfect complexes.

Definition 35.12.1. Let X be a scheme. Consider triples (T, E, m) where

(1) $T \subset X$ is a closed subset,
(2) E is an object of $D_{QCoh}(\mathcal{O}_X)$, and
(3) $m \in \mathbf{Z}$.

We say *approximation holds for the triple* (T, E, m) if there exists a perfect object P of $D(\mathcal{O}_X)$ supported on T and a map $\alpha : P \to E$ which induces isomorphisms $H^i(P) \to H^i(E)$ for $i > m$ and a surjection $H^m(P) \to H^m(E)$.

Approximation cannot hold for every triple. Namely, it is clear that if approximation holds for the triple (T, E, m), then

(1) E is m-pseudo-coherent, see Cohomology, Definition 20.33.1, and
(2) the cohomology sheaves $H^i(E)$ are supported on T for $i \geq m$.

Moreover, the "support" of a perfect complex is a closed subscheme whose complement is retrocompact in X (details omitted). Hence we cannot expect approximation to hold without this assumption on T. This partly explains the conditions in the following definition.

Definition 35.12.2. Let X be a scheme. We say *approximation by perfect complexes holds* on X if for any closed subset $T \subset X$ with $X \setminus T$ retro-compact in X

there exists an integer r such that for every triple (T, E, m) as in Definition 35.12.1 with

(1) E is $(m - r)$-pseudo-coherent, and
(2) $H^i(E)$ is supported on T for $i \geq m - r$

approximation holds.

We will prove that approximation by perfect complexes holds for quasi-compact and quasi-separated schemes. It seems that the second condition is necessary for our method of proof. It is possible that the first condition may be weakened to "E is m-pseudo-coherent" by carefuly analyzing the arguments below.

Lemma 35.12.3. *Let X be a scheme. Let $U \subset X$ be an open subscheme. Let (T, E, m) be a triple as in Definition 35.12.1. If*

(1) *$T \subset U$,*
(2) *approximation holds for $(T, E|_U, m)$, and*
(3) *the sheaves $H^i(E)$ for $i \geq m$ are supported on T,*

then approximation holds for (T, E, m).

Proof. Let $j : U \to X$ be the inclusion morphism. If $P \to E|_U$ is an approximation of the triple $(T, E|_U, m)$ over U, then $j_!P = Rj_*P \to j_!(E|_U) \to E$ is an approximation of (T, E, m) over X. See Cohomology, Lemmas 20.28.9 and 20.35.9. □

Lemma 35.12.4. *Let X be an affine scheme. Then approximation holds for every triple (T, E, m) as in Definition 35.12.1 such that there exists an integer $r \geq 0$ with*

(1) *E is m-pseudo-coherent,*
(2) *$H^i(E)$ is supported on T for $i \geq m - r + 1$,*
(3) *$X \setminus T$ is the union of r affine opens.*

In particular, approximation by perfect complexes holds for affine schemes.

Proof. Say $X = \mathrm{Spec}(A)$. Write $T = V(f_1, \ldots, f_r)$. (The case $r = 0$, i.e., $T = X$ follows immediately from Lemma 35.9.3 and the definitions.) Let (T, E, m) be a triple as in the lemma. Let t be the largest integer such that $H^t(E)$ is nonzero. We will proceed by induction on t. The base case is $t < m$; in this case the result is trivial. Now suppose that $t \geq m$. By Cohomology, Lemma 20.33.9 the sheaf $H^t(E)$ is of finite type. Since it is quasi-coherent it is generated by finitely many sections (Properties, Lemma 27.16.1). For every $s \in \Gamma(X, H^t(E)) = H^t(X, E)$ (see proof of Lemma 35.3.4) we can find an $e > 0$ and a morphism $K_e[-t] \to E$ such that s is in the image of $H^0(K_e) = H^t(K_e[-t]) \to H^t(E)$, see Lemma 35.8.6. Taking a finite direct sum of these maps we obtain a map $P \to E$ where P is a perfect complex supported on T, where $H^i(P) = 0$ for $i > t$, and where $H^t(P) \to E$ is surjective. Choose a distinguished triangle

$$P \to E \to E' \to P[1]$$

Then E' is m-pseudo-coherent (Cohomology, Lemma 20.33.4), $H^i(E') = 0$ for $i \geq t$, and $H^i(E')$ is supported on T for $i \geq m - r + 1$. By induction we find an approximation $P' \to E'$ of (T, E', m). Fit the composition $P' \to E' \to P[1]$ into a distringuished triangle $P \to P'' \to P' \to P[1]$ and extend the morphisms $P' \to E'$

and $P[1] \to P[1]$ into a morphism of distinguished triangles

$$\begin{array}{ccccccc} P & \longrightarrow & P'' & \longrightarrow & P' & \longrightarrow & P[1] \\ \downarrow & & \downarrow & & \downarrow & & \downarrow \\ P & \longrightarrow & E & \longrightarrow & E' & \longrightarrow & P[1] \end{array}$$

using TR3. Then P'' is a perfect complex (Cohomology, Lemma 20.35.6) supported on T. An easy diagram chase shows that $P'' \to E$ is the desired approximation. $\square$

Lemma 35.12.5. *Let X be a scheme. Let $X = U \cup V$ be an open covering with U quasi-compact, V affine, and $U \cap V$ quasi-compact. If approximation by perfect complexes holds on U, then approximation holds on X.*

Proof. Let $T \subset X$ be a closed subset with $X \setminus T$ retro-compact in X. Let r_U be the integer of Definition 35.12.2 adapted to the pair $(U, T \cap U)$. Set $T' = T \setminus U$. Note that $T' \subset V$ and that $V \setminus T' = (X \setminus T) \cap U \cap V$ is quasi-compact by our assumption on T. Let r' be the number of affines needed to cover $V \setminus T'$. We claim that $r = \max(r_U, r')$ works for the pair (X, T).

To see this choose a triple (T, E, m) such that E is $(m - r)$-pseudo-coherent and $H^i(E)$ is supported on T for $i \geq m - r$. Let t be the largest integer such that $H^t(E)|_U$ is nonzero. (Such an integer exists as U is quasi-compact and $E|_U$ is $(m-r)$-pseudo-coherent.) We will prove that E can be approximated by induction on t.

Base case: $t \leq m - r'$. This means that $H^i(E)$ is supported on T' for $i \geq m - r'$. Hence Lemma 35.12.4 guarantees the existence of an approximation $P \to E|_V$ of $(T', E|_V, m)$ on V. Applying Lemma 35.12.3 we see that (T', E, m) can be approximated. Such an approximation is also an approximation of (T, E, m).

Induction step. Choose an approximation $P \to E|_U$ of $(T \cap U, E|_U, m)$. This in particular gives a surjection $H^t(P) \to H^t(E|_U)$. By Lemma 35.11.8 we can choose a perfect object Q in $D(\mathcal{O}_V)$ supported on $T \cap V$ and an isomorphism $Q|_{U \cap V} \to (P \oplus P[1])|_{U \cap V}$. By Lemma 35.11.5 we can replace Q by $Q \otimes^{\mathbf{L}} I$ and assume that the map

$$Q|_{U \cap V} \to (P \oplus P[1])|_{U \cap V} \longrightarrow P|_{U \cap V} \longrightarrow E|_{U \cap V}$$

lifts to $Q \to E|_V$. By Cohomology, Lemma 20.28.10 we find an morphism $a : R \to E$ of $D(\mathcal{O}_X)$ such that $a|_U$ is isomorphic to $P \oplus P[1] \to E|_U$ and $a|_V$ isomorphic to $Q \to E|_V$. Thus R is perfect and supported on T and the map $H^t(R) \to H^t(E)$ is surjective on restriction to U. Choose a distinguised triangle

$$R \to E \to E' \to R[1]$$

Then E' is $(m - r)$-pseudo-coherent (Cohomology, Lemma 20.33.4), $H^i(E')|_U = 0$ for $i \geq t$, and $H^i(E')$ is supported on T for $i \geq m - r$. By induction we find an approximation $R' \to E'$ of (T, E', m). Fit the composition $R' \to E' \to R[1]$ into a distringuished triangle $R \to R'' \to R' \to R[1]$ and extend the morphisms $R' \to E'$

and $R[1] \to R[1]$ into a morphism of distinguished triangles

$$\begin{array}{ccccccc} R & \longrightarrow & R'' & \longrightarrow & R' & \longrightarrow & R[1] \\ \downarrow & & \downarrow & & \downarrow & & \downarrow \\ R & \longrightarrow & E & \longrightarrow & E' & \longrightarrow & R[1] \end{array}$$

using TR3. Then R'' is a perfect complex (Cohomology, Lemma 20.35.6) supported on T. An easy diagram chase shows that $R'' \to E$ is the desired approximation. □

Theorem 35.12.6. *Let X be a quasi-compact and quasi-separated scheme. Then approximation by perfect complexes holds on X.*

Proof. This follows from the induction principle of Cohomology of Schemes, Lemma 29.4.1 and Lemmas 35.12.5 and 35.12.4. □

35.13. Generating derived categories

In this section we prove that the derived category $D_{QCoh}(\mathcal{O}_X)$ of a quasi-compact and quasi-separated scheme can be generated by a single perfect object. We urge the reader to read the proof of this result in the wonderful paper by Bondal and van den Bergh, see [**BV03**].

Lemma 35.13.1. *Let X be a quasi-compact and quasi-separated scheme. Let U be a quasi-compact open subscheme. Let P be a perfect object of $D(\mathcal{O}_U)$. Then P is a direct summand of the restriction of a perfect object of $D(\mathcal{O}_X)$.*

Proof. Special case of Lemma 35.11.9. □

Lemma 35.13.2. *In Situation 35.8.1 denote $j : U \to X$ the open immersion and let K be the perfect object of $D(\mathcal{O}_X)$ corresponding to the Koszul complex on $f_1, \ldots, f_r$ over A. For $E \in D_{QCoh}(\mathcal{O}_X)$ the following are equivalent*

1. *$E = Rj_*(E|_U)$, and*
2. *$\mathrm{Hom}_{D(\mathcal{O}_X)}(K[n], E) = 0$ for all $n \in \mathbf{Z}$.*

Proof. Choose a distinguished triangle $E \to Rj_*(E|_U) \to N \to E[1]$. Observe that

$$\mathrm{Hom}_{D(\mathcal{O}_X)}(K[n], Rj_*(E|_U)) = \mathrm{Hom}_{D(\mathcal{O}_U)}(K|_U[n], E) = 0$$

for all n as $K|_U = 0$. Thus it suffices to prove the result for N. In other words, we may assume that E restricts to zero on U. Observe that there are distinguished triangles

$$K^\bullet(f_1^{e_1}, \ldots, f_i^{e_i'}, \ldots, f_r^{e_r}) \to K^\bullet(f_1^{e_1}, \ldots, f_i^{e_i'+e_i''}, \ldots, f_r^{e_r}) \to K^\bullet(f_1^{e_1}, \ldots, f_i^{e_i''}, \ldots, f_r^{e_r}) \to \ldots$$

of Koszul complexes, see More on Algebra, Lemma 15.19.11. Hence if $\mathrm{Hom}_{D(\mathcal{O}_X)}(K[n], E) = 0$ for all $n \in \mathbf{Z}$ then the same thing is true for the K replaced by K_e as in Lemma 35.8.6. Thus our lemma follows immediately from that one and the fact that E is determined by the complex of A-modules $R\Gamma(X, E)$, see Lemma 35.3.4. □

Theorem 35.13.3. *Let X be a quasi-compact and quasi-separated scheme. The category $D_{QCoh}(\mathcal{O}_X)$ can be generated by a single perfect object. More precisely, there exists a perfect object P of $D(\mathcal{O}_X)$ such that for $E \in D_{QCoh}(\mathcal{O}_X)$ the following are equivalent*

1. *$E = 0$, and*

(2) $\mathrm{Hom}_{D(\mathcal{O}_X)}(P[n], E) = 0$ *for all* $n \in \mathbf{Z}$.

Proof. We will prove this using the induction principle of Cohomology of Schemes, Lemma 29.4.1

If X is affine, then $\mathcal{O}_X$ is a perfect generator. This follows from Lemma 35.3.4.

Assume that $X = U \cup V$ is an open covering with U quasi-compact such that the theorem holds for U and V is an affine open. Let P be a perfect object of $D(\mathcal{O}_U)$ which is a generator for $D_{QCoh}(\mathcal{O}_U)$. Using Lemma 35.13.1 we may choose a perfect object Q of $D(\mathcal{O}_X)$ whose restriction to U is a direct sum one of whose summands is P. Say $V = \mathrm{Spec}(A)$. Let $Z = X \setminus U$. This is a retrocompact closed subset of V. Choose $f_1, \ldots, f_r \in A$ such that $Z = V(f_1, \ldots, f_r)$. Let $K \in D(\mathcal{O}_V)$ be the perfect object corresponding to the Koszul complex on $f_1, \ldots, f_r$ over A. Note that since K is supported on $Z \subset V$ closed, the pushforward $K' = R(V \to X)_* K$ is a perfect object of $D(\mathcal{O}_X)$ whose restriction to V is K (see Cohomology, Lemma 20.35.9). We claim that $Q \oplus K'$ is a generator for $D_{QCoh}(\mathcal{O}_X)$.

Namely, by Cohomology, Lemma 20.28.9 we have $K' = R(V \to X)_! K$ and hence

$$\mathrm{Hom}_{D(\mathcal{O}_X)}(K'[n], E) = \mathrm{Hom}_{D(\mathcal{O}_V)}(K[n], E|_V)$$

Thus by Lemma 35.13.2 the vanishing of these groups implies that $E|_V$ is isomorphic to $R(U \cap V \to V)_* E|_{U \cap V}$. This implies that $E = R(U \to X)_* E|_U$ (small detail omitted). If this is the case then

$$\mathrm{Hom}_{D(\mathcal{O}_X)}(Q[n], E) = \mathrm{Hom}_{D(\mathcal{O}_U)}(Q|_U[n], E|_U)$$

which contains $\mathrm{Hom}_{D(\mathcal{O}_U)}(P[n], E|_U)$ as a direct summand. Thus by our choice of P the vanishing of these groups implies that $E|_U$ is zero. Whence E is zero. □

35.14. Compact and perfect objects

Let X be a Noetherian scheme of finite dimension. By Cohomology, Lemma 20.21.5 and Cohomology on Sites, Lemma 21.38.6 the sheaves of modules $j_!\mathcal{O}_U$ are compact objects of $D(\mathcal{O}_X)$ for all opens $U \subset X$. These sheaves are typically not quasi-coherent, hence these do not give perfect object of the derived category $D(\mathcal{O}_X)$. However, if we restrict ourselves to complexes with quasi-coherent cohomology sheaves, then this does not happen. Here is the precise statement.

Proposition 35.14.1. *Let X be a quasi-compact and quasi-separated scheme. An object of $D_{QCoh}(\mathcal{O}_X)$ is compact if and only if it is perfect.*

Proof. By Cohomology, Lemma 20.36.1 the perfect objects define compact objects of $D(\mathcal{O}_X)$. Conversely, let K be a compact object of $D_{QCoh}(\mathcal{O}_X)$. To show that K is perfect, it suffices to show that $K|_U$ is perfect for every affine open $U \subset X$, see Cohomology, Lemma 20.35.2. Observe that $j : U \to X$ is a quasi-compact and separated morphism. Hence $Rj_* : D_{QCoh}(\mathcal{O}_U) \to D_{QCoh}(\mathcal{O}_X)$ commutes with direct sums, see Lemma 35.4.2. Thus the adjointness of restriction to U and Rj_* implies that $K|_U$ is a perfect object of $D_{QCoh}(\mathcal{O}_U)$. Hence we reduce to the case that X is affine.

Assume $X = \mathrm{Spec}(A)$ is affine. By Lemma 35.3.4 the problem is translated into the same problem for $D(A)$. For $D(A)$ the result is More on Algebra, Proposition 15.51.2. □

35.15. Derived categories as module categories

In this section we draw some conclusions of what has gone before. Before we do so we need a couple more lemmas.

Lemma 35.15.1. *Let X be a scheme. Let $K^\bullet$ be a complex of $\mathcal{O}_X$-modules whose cohomology sheaves are quasi-coherent. Let $(E, d) = \mathrm{Hom}_{Comp^{dg}(\mathcal{O}_X)}(K^\bullet, K^\bullet)$ be the endomorphism differential graded algebra. Then the functor*

$$- \otimes^{\mathbf{L}}_E K^\bullet : D(E, d) \longrightarrow D(\mathcal{O}_X)$$

of Differential Graded Algebra, Lemma 22.25.3 has image contained in $D_{QCoh}(\mathcal{O}_X)$.

Proof. Let P be a differential graded E-module with property (P) and let $F_\bullet$ be a filtration on P as in Differential Graded Algebra, Section 22.13. Then we have

$$P \otimes_E K^\bullet = \mathrm{hocolim}\ F_i P \otimes_E K^\bullet$$

Each of the $F_i P$ has a finite filtration whose graded pieces are direct sums of $E[k]$. The result follows easily. □

The following lemma can be strengthened (there is a uniformity in the vanishing over all L with nonzero cohomology sheaves only in a fixed range).

Lemma 35.15.2. *Let X be a quasi-compact and quasi-separated scheme. Let K, L be objects of $D(\mathcal{O}_X)$ with K perfect and L in $D^b_{QCoh}(\mathcal{O}_X)$. Then $Ext^n_{D(\mathcal{O}_X)}(K, L)$ is nonzero for only a finite number of n.*

Proof. Since K is perfect we have

$$\mathrm{Ext}^i_{D(\mathcal{O}_X)}(K, L) = H^i(X, K^\vee \otimes^{\mathbf{L}}_{\mathcal{O}_X} L)$$

where $K^\vee$ is the "dual" perfect complex to K, see Cohomology, Lemma 20.35.10. Note that $P = K^\vee \otimes^{\mathbf{L}}_{\mathcal{O}_X} L$ is in $D_{QCoh}(X)$ by Lemmas 35.3.7 and 35.9.1 (to see that a perfect complex has quasi-coherent cohomology sheaves). On the other hand, the spectral sequence

$$E_1^{p,q} = H^p(K^\vee \otimes^{\mathbf{L}}_{\mathcal{O}_X} H^q(L)) \Rightarrow H^{p+q}(K^\vee \otimes^{\mathbf{L}}_{\mathcal{O}_X} L) = H^{p+q}(P),$$

the boundedness of L, and the finite tor amplitude of $K^\vee$ show that P has only finitely many nonzero cohomology sheaves. It follows that $H^n(X, P) = 0$ for $n \ll 0$. But also $H^n(X, P) = 0$ for $n \gg 0$ by Cohomology of Schemes, Lemma 29.4.3 and the spectral sequence expressing $H^n(X, P^\bullet)$ in terms of $H^p(X, H^q(P^\bullet))$ using that the cohomology sheaves of P are quasi-coherent. □

The following result is taken from [**BV03**].

Theorem 35.15.3. *Let X be a quasi-compact and quasi-separated scheme. Then there exist a differential graded algebra (E, d) with only a finite number of nonzero cohomology groups $H^i(E)$ such that $D_{QCoh}(\mathcal{O}_X)$ is equivalent to $D(E, d)$.*

Proof. Let $K^\bullet$ be a K-injective complex of $\mathcal{O}$-modules which is perfect and generates $D_{QCoh}(\mathcal{O}_X)$. Such a thing exists by Theorem 35.13.3 and the existence of K-injective resolutions. We will show the theorem holds with

$$(E, \mathrm{d}) = \mathrm{Hom}_{\mathrm{Comp}^{dg}(\mathcal{O}_X)}(K^\bullet, K^\bullet)$$

where $\mathrm{Comp}^{dg}(\mathcal{O}_X)$ is the differential graded category of complexes of $\mathcal{O}$-modules. Please see Differential Graded Algebra, Section 22.25. Since $K^\bullet$ is K-injective we have

$$H^n(E) = \mathrm{Ext}^n_{D(\mathcal{O}_X)}(K^\bullet, K^\bullet) \tag{35.15.3.1}$$

for all $n \in \mathbf{Z}$. Only a finite number of these Exts are nonzero by Lemma 35.15.2. Consider the functor

$$- \otimes^{\mathbf{L}}_E K^\bullet : D(E, \mathrm{d}) \longrightarrow D(\mathcal{O}_X)$$

of Differential Graded Algebra, Lemma 22.25.3. Since $K^\bullet$ is perfect, it defines a compact object of $D(\mathcal{O}_X)$, see Proposition 35.14.1. Combined with (35.15.3.1) the functor above is fully faithful as follows from Differential Graded Algebra, Lemmas 22.25.5. It has a right adjoint

$$R\,\mathrm{Hom}(K^\bullet, -) : D(\mathcal{O}_X) \longrightarrow D(E, \mathrm{d})$$

by Differential Graded Algebra, Lemmas 22.25.4 which is a left quasi-inverse functor by generalities on adjoint functors. On the other hand, it follows from Lemma 35.15.1 that we obtain

$$- \otimes^{\mathbf{L}}_E K^\bullet : D(E, \mathrm{d}) \longrightarrow D_{QCoh}(\mathcal{O}_X)$$

and by our choice of $K^\bullet$ as a generator of $D_{QCoh}(\mathcal{O}_X)$ the kernel of the adjoint restricted to $D_{QCoh}(\mathcal{O}_X)$ is zero. A formal argument shows that we obtain the desired equivalence, see Derived Categories, Lemma 13.6.12. □

Remark 35.15.4. Let X be a quasi-compact and quasi-separated scheme over a ring R. By the construction of the proof of Theorem 35.15.3 there exists a differential graded algebra (A, d) over R such that $D_{QCoh}(X)$ is R-linearly equivalent to $D(A, \mathrm{d})$ as a triangulated category. One may ask: how unique is (A, d)? The answer is (only) slightly better than just saying that (A, d) is well defined up to derived equivalence. Namely, suppose that (B, d) is a second such pair. Then we have

$$(A, \mathrm{d}) = \mathrm{Hom}_{\mathrm{Comp}^{dg}(\mathcal{O}_X)}(K^\bullet, K^\bullet)$$

and

$$(B, \mathrm{d}) = \mathrm{Hom}_{\mathrm{Comp}^{dg}(\mathcal{O}_X)}(L^\bullet, L^\bullet)$$

for some K-injective complexes $K^\bullet$ and $L^\bullet$ of $\mathcal{O}_X$-modules corresponding to perfect generators of $D_{QCoh}(\mathcal{O}_X)$. Set

$$\Omega = \mathrm{Hom}_{\mathrm{Comp}^{dg}(\mathcal{O}_X)}(K^\bullet, L^\bullet) \quad \Omega' = \mathrm{Hom}_{\mathrm{Comp}^{dg}(\mathcal{O}_X)}(L^\bullet, K^\bullet)$$

Then Ω is a differential graded $B^{opp} \otimes_R A$-module and Ω' is a differential graded $A^{opp} \otimes_R B$-module. Moreover, the equivalence

$$D(A, \mathrm{d}) \to D_{QCoh}(\mathcal{O}_X) \to D(B, \mathrm{d})$$

is given by the functor $- \otimes^{\mathbf{L}}_A \Omega'$ and similarly for the quasi-inverse. Thus we are in the situation of Differential Graded Algebra, Remark 22.27.10. If we ever need this remark we will provide a precise statement with a detailed proof here.

35.16. Cohomology and base change, IV

This section continues the discussion of Cohomology of Schemes, Section 29.20.

Lemma 35.16.1. *Let $f : X \to Y$ be a quasi-compact and quasi-separated morphism of schemes. For E in $D_{QCoh}(\mathcal{O}_X)$ and K in $D_{QCoh}(\mathcal{O}_Y)$ we have*

$$Rf_*(E) \otimes^{\mathbf{L}}_{\mathcal{O}_Y} K = Rf_*(E \otimes^{\mathbf{L}}_{\mathcal{O}_X} Lf^*K)$$

Proof. Without any assumptions there is a map $Rf_*(E) \otimes^{\mathbf{L}}_{\mathcal{O}_Y} K \to Rf_*(E \otimes^{\mathbf{L}}_{\mathcal{O}_X} Lf^*K)$. Namely, it is the adjoint to the canonical map

$$Lf^*(Rf_*(E) \otimes^{\mathbf{L}}_{\mathcal{O}_Y} K) = Lf^*(Rf_*(E)) \otimes^{\mathbf{L}}_{\mathcal{O}_X} Lf^*K \longrightarrow E \otimes^{\mathbf{L}}_{\mathcal{O}_X} Lf^*K$$

coming from the map $Lf^*Rf_*E \to E$. See Cohomology, Lemmas 20.26.2 and 20.27.1. To check it is an isomorphism we may work locally on Y. Hence we reduce to the case that Y is affine.

Suppose that $K = \bigoplus K_i$ is a direct sum of some complexes $K_i \in D_{QCoh}(\mathcal{O}_Y)$. If the statement holds for each K_i, then it holds for K. Namely, the functors Lf^* and $\otimes^{\mathbf{L}}$ preserve direct sums by construction and Rf_* commutes with direct sums (for complexes with quasi-coherent cohomology sheaves) by Lemma 35.4.2. Moreover, suppose that $K \to L \to M \to K[1]$ is a distinguished triangle in $D_{QCoh}(Y)$. Then if the statement of the lemma holds for two of K, L, M, then it holds for the third (as the functors involved are exact functors of triangulated categories).

Assume Y affine, say $Y = \mathrm{Spec}(A)$. The functor $^\sim : D(A) \to D_{QCoh}(\mathcal{O}_Y)$ is an equivalence (Lemma 35.3.4). Let T be the property for $K \in D(A)$ that the statement of the lemma holds for $\widetilde{K}$. The discussion above and More on Algebra, Remark 15.43.11 shows that it suffices to prove T holds for $A[k]$. This finishes the proof, as the statement of the lemma is clear for shifts of the structure sheaf. □

Definition 35.16.2. Let S be a scheme. Let X, Y be schemes over S. We say X and Y are *Tor independent over S* if for every $x \in X$ and $y \in Y$ mapping to the same point $s \in S$ the rings $\mathcal{O}_{X,x}$ and $\mathcal{O}_{Y,y}$ are Tor independent over $\mathcal{O}_{S,s}$ (see More on Algebra, Definition 15.45.1).

Lemma 35.16.3. *Let $g : S' \to S$ be a morphism of schemes. Let $f : X \to S$ be quasi-compact and quasi-separated. Consider the base change diagram*

$$\begin{array}{ccc} X' & \xrightarrow{h} & X \\ {\scriptstyle f'}\downarrow & & \downarrow{\scriptstyle f} \\ S' & \xrightarrow{g} & S \end{array}$$

If X and S' are Tor independent over S, then for all $E \in D_{QCoh}(\mathcal{O}_X)$ we have $Rf'_*Lh^*E = Lg^*Rf_*E$.

Proof. For any object E of $D(\mathcal{O}_X)$ we can use Cohomology, Remark 20.27.2 to get a canonical base change map $Lg^*Rf_*E \to Rf'_*Lh^*E$. To check this is an isomorphism we may work locally on S'. Hence we may assume $g : S' \to S$ is a morphism of affine schemes. In particular, g is affine and it suffices to show that

$$Rg_*Lg^*Rf_*E \to Rg_*Rf'_*Lh^*E = Rf_*(Rh_*Lh^*E)$$

is an isomorphism, see Lemma 35.4.3 (and use Lemmas 35.3.6, 35.3.7, and 35.4.1 to see that the objects Rf'_*Lh^*E and Lg^*Rf_*E have quasi-coherent cohomology

sheaves). Note that h is affine as well (Morphisms, Lemma 28.13.8). By Lemma 35.4.4 the map becomes a map

$$Rf_*E \otimes^{\mathbf{L}}_{\mathcal{O}_S} g_*\mathcal{O}_{S'} \longrightarrow Rf_*(E \otimes^{\mathbf{L}}_{\mathcal{O}_X} h_*\mathcal{O}_{X'})$$

Observe that $h_*\mathcal{O}_{X'} = f^*g_*\mathcal{O}_{S'}$. Thus by Lemma 35.16.1 it suffices to prove that $Lf^*g_*\mathcal{O}_{S'} = f^*g_*\mathcal{O}_{S'}$. This follows from our assumption that X and S' are Tor independent over S. Namely, to check it we may work locally on X, hence we may also assume X is affine. Say $X = \mathrm{Spec}(A)$, $S = \mathrm{Spec}(R)$ and $S' = \mathrm{Spec}(R')$. Our assumption implies that A and R' are Tor independent over R (More on Algebra, Lemma 15.45.4), i.e., $\mathrm{Tor}^R_i(A, R') = 0$ for $i > 0$. In other words $A \otimes^{\mathbf{L}}_R R' = A \otimes_R R'$ which exactly means that $Lf^*g_*\mathcal{O}_{S'} = f^*g_*\mathcal{O}_{S'}$ (use Lemma 35.3.6). □

Lemma 35.16.4. *Let S be a Noetherian scheme. Let $f : X \to S$ be a morphism of schemes which is locally of finite type. Let $E \in D(\mathcal{O}_X)$ such that*

1. *$E \in D^b_{Coh}(\mathcal{O}_X)$,*
2. *the scheme theoretic support of $H^i(E)$ is proper over S for all i,*
3. *E has finite tor dimension as an object of $D(f^{-1}\mathcal{O}_S)$.*

Then Rf_*E is a perfect object of $D(\mathcal{O}_S)$.

Proof. By Lemma 35.5.1 we see that Rf_*E is an object of $D^b_{Coh}(\mathcal{O}_S)$. Hence Rf_*E is pseudo-coherent (Lemma 35.9.4). Hence it suffices to show that Rf_*E has finite tor dimension, see Cohomology, Lemma 20.35.4. By Lemma 35.9.6 it suffices to check that $Rf_*(E) \otimes^{\mathbf{L}}_{\mathcal{O}_S} \mathcal{F}$ has universally bounded cohomology for all quasi-coherent sheaves $\mathcal{F}$ on S. Bounded from above is clear as $Rf_*(E)$ is bounded from above. Let $T \subset X$ be the union of the supports of $H^i(E)$ for all i. Then T is proper over S by assumptions (1) and (2). In particular there exists a quasi-compact open $X' \subset X$ containing T. Setting $f' = f|_{X'}$ we have $Rf_*(E) = Rf'_*(E|_{X'})$ because E restricts to zero on $X \setminus T$. Thus we may replace X by X' and assume f is quasi-compact. Moreover, f is quasi-separated by Morphisms, Lemma 28.16.7. Now

$$Rf_*(E) \otimes^{\mathbf{L}}_{\mathcal{O}_S} \mathcal{F} = Rf_*\left(E \otimes^{\mathbf{L}}_{\mathcal{O}_X} Lf^*\mathcal{F}\right) = Rf_*\left(E \otimes^{\mathbf{L}}_{f^{-1}\mathcal{O}_S} f^{-1}\mathcal{F}\right)$$

by Lemma 35.16.1 and Cohomology, Lemma 20.26.3. By assumption (3) the complex $E \otimes^{\mathbf{L}}_{f^{-1}\mathcal{O}_S} f^{-1}\mathcal{F}$ has cohomology sheaves in a given finite range, say $[a, b]$. Then Rf_* of it has cohomology in the range $[a, \infty)$ and we win. □

35.17. Computing Ext groups and base change

The results in this section will be used to verify one of Artin's criteria for Quot functors, Hilbert schemes, and other moduli problems.

Lemma 35.17.1. *Let S be a Noetherian scheme. Let $f : X \to S$ be a morphism of schemes which is locally of finite type. Let $E \in D(\mathcal{O}_X)$ and $\mathcal{G}$ an $\mathcal{O}_X$-module. Assume*

1. *E is perfect, and*
2. *$\mathcal{G}$ is a coherent $\mathcal{O}_X$-module flat over S with scheme theoretic support proper over S.*

Then there exists a perfect object K of $D(\mathcal{O}_S)$ and functorial isomorphisms

$$H^i(S, K \otimes^{\mathbf{L}}_{\mathcal{O}_S} \mathcal{F}) \longrightarrow Ext^i_{\mathcal{O}_X}(E, \mathcal{G} \otimes_{\mathcal{O}_X} f^*\mathcal{F})$$

for $\mathcal{F}$ quasi-coherent on S compatible with boundary maps (see proof).

Proof. Since E is a perfect complex there exists a dual perfect complex E', see Cohomology, Lemma 20.35.10. Observe that

$$\mathrm{Ext}^i_{\mathcal{O}_X}(E, \mathcal{G} \otimes_{\mathcal{O}_X} f^*\mathcal{F}) = H^i(X, E' \otimes^{\mathbf{L}}_{\mathcal{O}_X} (\mathcal{G} \otimes_{\mathcal{O}_X} f^*\mathcal{F}))$$

by construction of E'. We have

$$\mathcal{G} \otimes^{\mathbf{L}}_{\mathcal{O}_X} Lf^*\mathcal{F} = \mathcal{G} \otimes^{\mathbf{L}}_{f^{-1}\mathcal{O}_S} f^{-1}\mathcal{F} = \mathcal{G} \otimes_{f^{-1}\mathcal{O}_S} f^{-1}\mathcal{F} = \mathcal{G} \otimes_{\mathcal{O}_X} f^*\mathcal{F}$$

the first equality by Cohomology, Lemma 20.26.3, the second as $\mathcal{G}$ is a flat $f^{-1}\mathcal{O}_S$-module, and the third by definition of pullbacks. Hence we obtain

$$\begin{aligned} H^i(X, E' \otimes^{\mathbf{L}}_{\mathcal{O}_X} (\mathcal{G} \otimes_{\mathcal{O}_X} f^*\mathcal{F})) &= H^i(X, E' \otimes^{\mathbf{L}}_{\mathcal{O}_X} \mathcal{G} \otimes^{\mathbf{L}}_{\mathcal{O}_X} Lf^*\mathcal{F}) \\ &= H^i(S, Rf_*(E' \otimes^{\mathbf{L}}_{\mathcal{O}_X} \mathcal{G} \otimes^{\mathbf{L}}_{\mathcal{O}_X} Lf^*\mathcal{F})) \\ &= H^i(S, Rf_*(E' \otimes^{\mathbf{L}}_{\mathcal{O}_X} \mathcal{G}) \otimes^{\mathbf{L}}_{\mathcal{O}_S} \mathcal{F}) \end{aligned}$$

The first equality by the above, the second by Leray (Cohomology, Lemma 20.14.1), and the third equality by Lemma 35.16.1. Finally, the object $K = Rf_*(E' \otimes^{\mathbf{L}}_{\mathcal{O}_X} \mathcal{G})$ is perfect by Lemma 35.16.4. (The lemma applies: the hypotheses on the cohomology sheaves $H^i(E' \otimes^{\mathbf{L}}_{\mathcal{O}_X} \mathcal{G})$ are clear and the hypothesis on finite tor dimension follows as $\mathcal{G}$ is flat over $f^{-1}\mathcal{O}_S$ and locally E' is isomorphic to a finite complex of free $\mathcal{O}_X$-modules.)

The statement on boundary maps means the following: Given a short exact sequence $0 \to \mathcal{F}_1 \to \mathcal{F}_2 \to \mathcal{F}_3 \to 0$ then the isomorphisms fit into commutative diagrams

$$\begin{array}{ccc} H^i(S, K \otimes^{\mathbf{L}}_{\mathcal{O}_S} \mathcal{F}_3) & \longrightarrow & \mathrm{Ext}^i_{\mathcal{O}_X}(E, \mathcal{G} \otimes_{\mathcal{O}_X} f^*\mathcal{F}_3) \\ \downarrow{\scriptstyle \delta} & & \downarrow{\scriptstyle \delta} \\ H^{i+1}(S, K \otimes^{\mathbf{L}}_{\mathcal{O}_S} \mathcal{F}_1) & \longrightarrow & \mathrm{Ext}^{i+1}_{\mathcal{O}_X}(E, \mathcal{G} \otimes_{\mathcal{O}_X} f^*\mathcal{F}_1) \end{array}$$

where the boundary maps come from the distinguished triangle

$$K \otimes^{\mathbf{L}}_{\mathcal{O}_S} \mathcal{F}_1 \to K \otimes^{\mathbf{L}}_{\mathcal{O}_S} \mathcal{F}_2 \to K \otimes^{\mathbf{L}}_{\mathcal{O}_S} \mathcal{F}_3 \to K \otimes^{\mathbf{L}}_{\mathcal{O}_S} \mathcal{F}_1[1]$$

and the distinguished triangle in $D(\mathcal{O}_X)$ associated to the short exact sequence

$$0 \to \mathcal{G} \otimes_{\mathcal{O}_X} f^*\mathcal{F}_1 \to \mathcal{G} \otimes_{\mathcal{O}_X} f^*\mathcal{F}_2 \to \mathcal{G} \otimes_{\mathcal{O}_X} f^*\mathcal{F}_3 \to 0$$

This sequence is exact because $\mathcal{G}$ is flat over S. We omit the verification of the commutativity of the displayed diagram. □

Lemma 35.17.2. *Let $f : X \to S$ be a quasi-compact and quasi-separated morphism of schemes. Let $E \in D(\mathcal{O}_X)$ be pseudo-coherent. Let $\mathcal{G}$ be a quasi-coherent $\mathcal{O}_X$-module flat over S. Then formation of*

$$Rf_*R\mathcal{H}om(E, \mathcal{G})$$

commutes with arbitrary base change (see proof for precise statement).

Proof. The statement means the following. Let $g : S' \to S$ be a morphism of schemes and consider the base change diagram

$$\begin{array}{ccc} X' & \xrightarrow[h]{} & X \\ {\scriptstyle f'}\downarrow & & \downarrow{\scriptstyle f} \\ S' & \xrightarrow{g} & S \end{array}$$

in other words $X' = S' \times_S X$. Set $E' = Lh^*E$ and $\mathcal{G}' = h^*\mathcal{G}$ (here we do **not** use the derived pullback). The lemma asserts that we have

$$Lg^*Rf_*R\mathcal{H}om(E, \mathcal{G}) = Rf'_*R\mathcal{H}om(E', \mathcal{G}')$$

To prove this, note that in Cohomology, Remark 20.31.7 we have constructed an arrow

$$Lg^*Rf_*R\mathcal{H}om(E, \mathcal{G}) \longrightarrow R(f')_*R\mathcal{H}om(Lh^*E, Lh^*\mathcal{G})$$

which we can compose with the map $Lh^*\mathcal{G} \to h^*\mathcal{G}$ to get a canonical map

$$Lg^*Rf_*R\mathcal{H}om(E, \mathcal{G}) \to Rf'_*R\mathcal{H}om(E', \mathcal{G}')$$

To check this map is an isomorphism we may work locally on S'. Hence we may assume $g : S' \to S$ is a morphism of affine schemes. In this case, we will use the induction principle to prove this map is always an isomorphism for any quasi-compact and quasi-separated X over S (Cohomology of Schemes, Lemma 29.4.1).

If X is affine, then we can represent E by a bounded above complex $\mathcal{E}^\bullet$ of finite free $\mathcal{O}_X$-modules (Lemma 35.11.2). By Cohomology, Lemma 20.32.10 we see that $R\mathcal{H}om(E, \mathcal{G})$ is represented by the complex with terms $\mathcal{H}om_{\mathcal{O}_X}(\mathcal{E}^p, \mathcal{G})$. Similarly, $R\mathcal{H}om(E', \mathcal{G}')$ is computed by the complex with terms $\mathcal{H}om_{\mathcal{O}_{X'}}(h^*\mathcal{E}^p, h^*\mathcal{G})$. Since f and f' are affine, the values of Rf_* and Rf'_* on complexes of quasi-coherent modules are computed by applying f_* and f'_* to the given complex (Lemma 35.6.1). Moreover, as $\mathcal{G}$ is flat over S and f is affine we see that $f_*\mathcal{H}om_{\mathcal{O}_X}(\mathcal{E}^p, \mathcal{G})$ is a flat $\mathcal{O}_S$-module. Thus $Lg^*f_*\mathcal{H}om_{\mathcal{O}_X}(\mathcal{E}^\bullet, \mathcal{G})$ is equal to $g^*f_*\mathcal{H}om_{\mathcal{O}_X}(\mathcal{E}^\bullet, \mathcal{G})$. Hence in the affine case we have to check that the canonical map

$$g^*f_*\mathcal{H}om_{\mathcal{O}_X}(\mathcal{E}, \mathcal{G}) \longrightarrow f'_*\mathcal{H}om_{\mathcal{O}_{X'}}(h^*\mathcal{E}, h^*\mathcal{G})$$

is an isomorphism for any finite free $\mathcal{O}_X$-module $\mathcal{E}$. This reduces to the case $\mathcal{E} = \mathcal{O}_X$. In this case we are saying that $g^*f_*\mathcal{G} = f'_*h^*\mathcal{G}$ which is Cohomology of Schemes, Lemma 29.5.1.

The induction step. Suppose that $X = U \cup V$ is an open covering with U, V, $U \cap V$ quasi-compact such that the result holds for the restriction of E and $\mathcal{G}$ to U, V, and $U \cap V$. Denote $a = f|_U$, $b = f|_V$ and $c = f|_{U\cap V}$. Let $a' : U' \to S'$, $b' : V' \to S'$ and $c' : U' \cap V' \to S'$ be the base changes of a, b, and c. Note that formation of $R\mathcal{H}om$ commutes with restriction to opens (Cohomology, Lemma 20.31.3). Set $H = R\mathcal{H}om(E, \mathcal{G})$ and $H' = R\mathcal{H}om(E', \mathcal{G}')$. Using the distinguished triangles from relative Mayer-Vietoris (Cohomology, Lemma 20.28.8) we obtain a commutative diagram

$$\begin{array}{ccc}
Lg^*Rf_*H & \longrightarrow & Rf'_*H' \\
\downarrow & & \downarrow \\
Lg^*Ra_*H|_U \oplus Lg^*Rb_*H|_V & \longrightarrow & Ra'_*H'|_{U'} \oplus Rb'_*H'|_{V'} \\
\downarrow & & \downarrow \\
Lg^*Rc_*H|_{U\cap V} & \longrightarrow & Rc'_*H'|_{U'\cap V'} \\
\downarrow & & \downarrow \\
Lg^*Rf_*H[1] & \longrightarrow & Rf'_*H'[1]
\end{array}$$

Since the 2nd and 3rd horizontal arrows are isomorphisms so is the first (Derived Categories, Lemma 13.4.3) and the proof of the lemma is finished. □

Lemma 35.17.3. *Let S be a Noetherian scheme. Let $f : X \to S$ be a morphism of schemes which is locally of finite type. Let $E \in D(\mathcal{O}_X)$ and $\mathcal{G}$ an $\mathcal{O}_X$-module. Assume*

(1) *$E \in D^-_{Coh}(\mathcal{O}_X)$, and*
(2) *$\mathcal{G}$ is a coherent $\mathcal{O}_X$-module flat over S with scheme theoretic support is proper over S.*

Then for every $m \in \mathbf{Z}$ there exists a perfect object K of $D(\mathcal{O}_S)$ and functorial maps

$$\alpha^i_{\mathcal{F}} : Ext^i_{\mathcal{O}_X}(E, \mathcal{G} \otimes_{\mathcal{O}_X} f^*\mathcal{F}) \longrightarrow H^i(S, K \otimes^{\mathbf{L}}_{\mathcal{O}_S} \mathcal{F})$$

for $\mathcal{F}$ quasi-coherent on S compatible with boundary maps (see proof) such that $\alpha^i_{\mathcal{F}}$ is an isomorphism for $i \leq m$.

Proof. We may replace X by a quasi-compact open neighbourhood of the support of $\mathcal{G}$, hence we may assume X is Noetherian. In this case X and f are quasi-compact and quasi-separated. Choose an approximation $P \to E$ by a perfect complex P of $(X, E, -m-1)$ (possible by Theorem 35.12.6). Then the induced map

$$\mathrm{Ext}^i_{\mathcal{O}_X}(E, \mathcal{G} \otimes_{\mathcal{O}_X} f^*\mathcal{F}) \longrightarrow \mathrm{Ext}^i_{\mathcal{O}_X}(P, \mathcal{G} \otimes_{\mathcal{O}_X} f^*\mathcal{F})$$

is an isomorphism for $i \leq m$. Namely, the kernel, resp. cokernel of this map is a quotient, resp. submodule of

$$\mathrm{Ext}^i_{\mathcal{O}_X}(C, \mathcal{G} \otimes_{\mathcal{O}_X} f^*\mathcal{F}) \quad \text{resp.} \quad \mathrm{Ext}^{i+1}_{\mathcal{O}_X}(C, \mathcal{G} \otimes_{\mathcal{O}_X} f^*\mathcal{F})$$

where C is the cone of $P \to E$. Since C has vanishing cohomology sheaves in degrees $\geq -m-1$ these Ext-groups are zero for $i \leq m+1$ by Derived Categories, Lemma 13.26.3. This reduces us to the case that E is a perfect complex which is Lemma 35.17.1.

The statement on boundaries is explained in the proof of Lemma 35.17.1. □

35.18. Limits and derived categories

In this section we collect some results about the derived category of a scheme which is the limit of an inverse system of schemes. More precisely, we will work in the following setting.

Situation 35.18.1. Let $S = \lim_{i \in I} S_i$ be a limit of a directed system of schemes over S with affine transition morphisms $f_{i'i} : S_{i'} \to S_i$. We assume that S_i is quasi-compact and quasi-separated for all $i \in I$. We denote $f_i : S \to S_i$ the projection. We also fix an element $0 \in I$.

Lemma 35.18.2. *In Situation 35.18.1. Let E_0 and K_0 be objects of $D(\mathcal{O}_{S_0})$. Set $E_i = Lf_{i0}^*E_0$ and $K_i = Lf_{i0}^*K_0$ for $i \geq 0$ and set $E = Lf_0^*E_0$ and $K = Lf_0^*K_0$. Then the map*

$$\mathrm{colim}_{i \geq 0} \mathrm{Hom}_{D(\mathcal{O}_{S_i})}(E_i, K_i) \longrightarrow \mathrm{Hom}_{D(\mathcal{O}_S)}(E, K)$$

is an isomorphism if either

(1) *E_0 is perfect and $K_0 \in D_{QCoh}(\mathcal{O}_{S_0})$, or*
(2) *E_0 is pseudo-coherent and $K_0 \in D_{QCoh}(\mathcal{O}_{S_0})$ has finite tor dimension.*

Proof. For every open $U_0 \subset S_0$ consider the condition P that the canonical map

$$\operatorname{colim}_{i \geq 0} \operatorname{Hom}_{D(\mathcal{O}_{U_i})}(E_i|_{U_i}, K_i|_{U_i}) \longrightarrow \operatorname{Hom}_{D(\mathcal{O}_U)}(E|_U, K|_U)$$

is an isomorphism, where $U = f_0^{-1}(U_0)$ and $U_i = f_{i0}^{-1}(U_0)$. We will prove P holds for all quasi-compact opens U_0 by the induction principle of Cohomology of Schemes, Lemma 29.4.1. Condition (2) of this lemma follows immediately from Mayer-Vietoris for hom in the derived category, see Cohomology, Lemma 20.28.6. Thus it suffices to prove the lemma when S_0 is affine.

Assume S_0 is affine. Say $S_0 = \operatorname{Spec}(A_0)$, $S_i = \operatorname{Spec}(A_i)$, and $S = \operatorname{Spec}(A)$. We will use Lemma 35.3.4 without further mention.

In case (1) the object $E_0^\bullet$ corresponds to a finite complex of finite projective A_0-modules, see Lemma 35.9.7. We may represent the object K_0 by a K-flat complex $K_0^\bullet$ of A_0-modules. In this situation we are trying to prove

$$\operatorname{colim}_{i \geq 0} \operatorname{Hom}_{D(A_i)}(E_0^\bullet \otimes_{A_0} A_i, K_0^\bullet \otimes_{A_0} A_i) \longrightarrow \operatorname{Hom}_{D(A)}(E_0^\bullet \otimes_{A_0} A, K_0^\bullet \otimes_{A_0} A)$$

Because $E_0^\bullet$ is a bounded above complex of projective modules we can rewrite this as

$$\operatorname{colim}_{i \geq 0} \operatorname{Hom}_{K(A_0)}(E_0^\bullet, K_0^\bullet \otimes_{A_0} A_i) \longrightarrow \operatorname{Hom}_{K(A_0)}(E_0^\bullet, K_0^\bullet \otimes_{A_0} A)$$

Since there are only a finite number of nonzero modules E_0^n and since these are all finitely presented modules, this map is an isomorphism.

In case (2) the object E_0 corresponds to a bounded above complex $E_0^\bullet$ of finite free A_0-modules, see Lemma 35.9.3. We may represent K_0 by a finite complex $K_0^\bullet$ of flat A_0-modules, see Lemma 35.9.5 and More on Algebra, Lemma 15.49.3. In particular $K_0^\bullet$ is K-flat and we can argue as before to arrive at the map

$$\operatorname{colim}_{i \geq 0} \operatorname{Hom}_{K(A_0)}(E_0^\bullet, K_0^\bullet \otimes_{A_0} A_i) \longrightarrow \operatorname{Hom}_{K(A_0)}(E_0^\bullet, K_0^\bullet \otimes_{A_0} A)$$

It is clear that this map is an isomorphism (only a finite number of terms are involved since $K_0^\bullet$ is bounded). □

Lemma 35.18.3. *In Situation 35.18.1 the category of perfect objects of $D(\mathcal{O}_S)$ is the colimit of the categories of perfect objects of $D(\mathcal{O}_{S_i})$.*

Proof. For every open $U_0 \subset S_0$ consider the condition P that the functor

$$\operatorname{colim}_{i \geq 0} D_{perf}(\mathcal{O}_{U_i}) \longrightarrow D_{perf}(\mathcal{O}_U)$$

is an equivalence where $_{perf}$ indicates the full subcategory of perfect objects and where $U = f_0^{-1}(U_0)$ and $U_i = f_{i0}^{-1}(U_0)$. We will prove P holds for all quasi-compact opens U_0 by the induction principle of Cohomology of Schemes, Lemma 29.4.1. First, we observe that we already know the functor is fully faithful by Lemma 35.18.2. Thus it suffices to prove essential surjectivity.

We first check condition (2) of the induction principle. Thus suppose that we have $S_0 = U_0 \cup V_0$ and that P holds for U_0, V_0, and $U_0 \cap V_0$. Let E be a perfect object of $D(\mathcal{O}_S)$. We can find $i \geq 0$ and $E_{U,i}$ perfect on U_i and $E_{V,i}$ perfect on V_i whose pullback to U and V are isomorphic to $E|_U$ and $E|_V$. Denote

$$a : E_{U,i} \to (Rf_{i,*}E)|_{U_i} \quad \text{and} \quad b : E_{V,i} \to (Rf_{i,*}E)|_{V_i}$$

the maps adjoint to the isomorphisms $Lf_i^*E_{U,i} \to E|_U$ and $Lf_i^*E_{V,i} \to E|_V$. By fully faithfulness, after increasing i, we can find an isomorphism $c : E_{U,i}|_{U_i \cap V_i} \to E_{V,i}|_{U_i \cap V_i}$ which pulls back to the identifications

$$Lf_i^*E_{U,i}|_{U \cap V} \to E|_{U \cap V} \to Lf_i^*E_{V,i}|_{U \cap V}.$$

Apply Cohomology, Lemma 20.28.10 to get an object E_i on S_i and a map $d : E_i \to Rf_{i,*}E$ which restricts to the maps a and b over U_i and V_i. Then it is clear that E_i is perfect and that d is adjoint to an isomorphism $Lf_i^*E_i \to E$.

Finally, we check condition (1) of the induction principle, in other words, we check the lemma holds when S_0 is affine. Say $S_0 = \mathrm{Spec}(A_0)$, $S_i = \mathrm{Spec}(A_i)$, and $S = \mathrm{Spec}(A)$. Using Lemmas 35.3.4 and 35.9.7 we see that we have to show that

$$D_{perf}(A) = \mathrm{colim}\, D_{perf}(A_i)$$

This is clear from the fact that perfect complexes over rings are given by finite complexes of finite projective (hence finitely presented) modules. □

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CHAPTER 36

More on Morphisms

36.1. Introduction

In this chapter we continue our study of properties of morphisms of schemes. A fundamental reference is [**DG67**].

36.2. Thickenings

The following terminology may not be completely standard, but it is convenient.

Definition 36.2.1. Thickenings.

(1) We say a scheme X' is a *thickening* of a scheme X if X is a closed subscheme of X' and the underlying topological spaces are equal.
(2) We say a scheme X' is a *first order thickening* of a scheme X if X is a closed subscheme of X' and the quasi-coherent sheaf of ideals $\mathcal{I} \subset \mathcal{O}_{X'}$ defining X has square zero.
(3) Given two thickenings $X \subset X'$ and $Y \subset Y'$ a *morphism of thickenings* is a morphism $f' : X' \to Y'$ such that $f(X) \subset Y$, i.e., such that $f'|_X$ factors through the closed subscheme Y. In this situation we set $f = f'|_X : X \to Y$ and we say that $(f, f') : (X \subset X') \to (Y \subset Y')$ is a morphism of thickenings.
(4) Let S be a scheme. We similarly define *thickenings over* S, and *morphisms of thickenings over* S. This means that the schemes X, X', Y, Y' above are schemes over S, and that the morphisms $X \to X'$, $Y \to Y'$ and $f' : X' \to Y'$ are morphisms over S.

Finite order thickenings. Let $i_X : X \to X'$ be a thickening. Any local section of the kernel $\mathcal{I} = \mathrm{Ker}(i_X^\sharp)$ is locally nilpotent. Let us say that $X \subset X'$ is a *finite order thickening* if the ideal sheaf $\mathcal{I}$ is "globally" nilpotent, i.e., if there exists an $n \geq 0$ such that $\mathcal{I}^{n+1} = 0$. Technically the class of finite order thickenings $X \subset X'$ is much easier to handle than the general case. Namely, in this case we have a filtration

$$0 \subset \mathcal{I}^n \subset \mathcal{I}^{n-1} \subset \ldots \subset \mathcal{I} \subset \mathcal{O}_{X'}$$

and we see that X' is filtered by closed subspaces

$$X = X_0 \subset X_1 \subset \ldots \subset X_{n-1} \subset X_{n+1} = X'$$

such that each pair $X_i \subset X_{i+1}$ is a first order thickening over B. Using simple induction arguments many results proved for first order thickenings can be rephrased as results on finite order thickenings.

First order thickening are described as follows (see Modules, Lemma 17.23.11).

Lemma 36.2.2. *Let X be a scheme over a base S. Consider a short exact sequence*

$$0 \to \mathcal{I} \to \mathcal{A} \to \mathcal{O}_X \to 0$$

of sheaves on X where $\mathcal{A}$ is a sheaf of $f^{-1}\mathcal{O}_S$-algebras, $\mathcal{A} \to \mathcal{O}_X$ is a surjection of sheaves of $f^{-1}\mathcal{O}_S$-algebras, and $\mathcal{I}$ is its kernel. If

(1) *$\mathcal{I}$ is an ideal of square zero in $\mathcal{A}$, and*
(2) *$\mathcal{I}$ is quasi-coherent as an $\mathcal{O}_X$-module*

then $X' = (X, \mathcal{A})$ is a scheme and $X \to X'$ is a first order thickening over S. Moreover, any first order thickening over S is of this form.

Proof. It is clear that X' is a locally ringed space. Let $U = \mathrm{Spec}(B)$ be an affine open of X. Set $A = \Gamma(U, \mathcal{A})$. Note that since $H^1(U, \mathcal{I}) = 0$ (see Cohomology of Schemes, Lemma 29.2.2) the map $A \to B$ is surjective. By assumption the kernel $I = \mathcal{I}(U)$ is an ideal of square zero in the ring A. By Schemes, Lemma 25.6.4 there is a canonical morphism of locally ringed spaces

$$(U, \mathcal{A}|_U) \longrightarrow \mathrm{Spec}(A)$$

coming from the map $B \to \Gamma(U, \mathcal{A})$. Since this morphism fits into the commutative diagram

$$\begin{array}{ccc} (U, \mathcal{O}_X|_U) & \longrightarrow & \mathrm{Spec}(B) \\ \downarrow & & \downarrow \\ (U, \mathcal{A}|_U) & \longrightarrow & \mathrm{Spec}(A) \end{array}$$

we see that it is a homeomorphism on underlying topological spaces. Thus to see that it is an isomorphism, it suffices to check it induces an isomorphism on the local rings. For $u \in U$ corresponding to the prime $\mathfrak{p} \subset A$ we obtain a commutative diagram of short exact sequences

$$\begin{array}{ccccccccc} 0 & \longrightarrow & I_\mathfrak{p} & \longrightarrow & A_\mathfrak{p} & \longrightarrow & B_\mathfrak{p} & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & \mathcal{I}_u & \longrightarrow & \mathcal{A}_u & \longrightarrow & \mathcal{O}_{X,u} & \longrightarrow & 0. \end{array}$$

The left and right vertical arrows are isomorphisms because $\mathcal{I}$ and $\mathcal{O}_X$ are quasi-coherent sheaves. Hence also the middle map is an isomorphism. Hence every point of $X' = (X, \mathcal{A})$ has an affine neighbourhood and X' is a scheme as desired. □

Lemma 36.2.3. *Any thickening of an affine scheme is affine.*

Proof. This is a special case of Limits, Proposition 31.10.2. □

Lemma 36.2.4. *Let $S \subset S'$ be a thickening of schemes. Let $X' \to S'$ be a morphism and set $X = S \times_{S'} X'$. Then $(X \subset X') \to (S \subset S')$ is a morphism of thickenings. If $S \subset S'$ is a first (resp. finite order) thickening, then $X \subset X'$ is a first (resp. finite order) thickening.*

Proof. Omitted. □

Lemma 36.2.5. *Let $(f, f') : (X \subset X') \to (S \subset S')$ be a morphism of thickenings. Then*

(1) *f is an affine morphism if and only if f' is an affine morphism,*
(2) *f is a surjective morphism if and only if f' is a surjective morphism,*

(3) *f is quasi-compact if and only if f' quasi-compact,*
(4) *f is universally closed if and only if f' is universally closed,*
(5) *f is integral if and only if f' is integral,*
(6) *f is (quasi-)separated if and only if f' is (quasi-)separated,*
(7) *f is universally injective if and only if f' is universally injective,*
(8) *f is universally open if and only if f' is universally open, and*
(9) *add more here.*

Proof. Observe that $S \to S'$ and $X \to X'$ are universal homeomorphisms (see for example Morphisms, Lemma 28.45.4). This immediately implies parts (2), (3), (4), (7), and (8). Part (1) follows from Lemma 36.2.3 which tells us that there is a 1-to-1 correspondence between affine opens of S and S' and between affine opens of X and X'. Part (5) follows from (1) and (4) by Morphisms, Lemma 28.44.7. Finally, note that

$$S \times_X S = S \times_{X'} S \to S \times_{X'} S' \to S' \times_{X'} S'$$

is a thickening (the two arrows are thickenings by Lemma 36.2.4). Hence applying (3) and (4) to the morphism $(S \subset S') \to (S \times_X S \to S' \times_{X'} S')$ we obtain (6). □

Lemma 36.2.6. *Let $(f, f') : (X \subset X') \to (S \subset S')$ be a morphism of thickenings such that $X = S \times_{S'} X'$. If $S \subset S'$ is a finite order thickening, then*

(1) *f is a closed immersion if and only if f' is a closed immersion,*
(2) *f is locally of finite type if and only if f' is locally of finite type,*
(3) *f is locally quasi-finite if and only if f' is locally quasi-finite,*
(4) *f is locally of finite type of relative dimension d if and only if f' is locally of finite type of relative dimension d,*
(5) *f is unramified if and only if f' is unramified,*
(6) *f is proper if and only if f' is proper,*
(7) *f is a finite morphism if and only if f' is an finite morphism, and*
(8) *add more here.*

Proof. The properties $\mathcal{P}$ listed in the lemma are all stable under base change, hence if f' has property $\mathcal{P}$, then so does f. See Schemes, Lemma 25.18.2, and Morphisms, Lemmas 28.16.4, 28.21.13, 28.30.2, 28.36.5, 28.42.5, 28.44.6.

The interesting direction in each case is therefore to assume that f has the property and deduce that f' has it too. By induction on the order of the thickening we may assume that $S \subset S'$ is a first order thickening, see discussion immediately following Definition 36.2.1.

Most of the proofs will use a reduction to the affine case. Let $U' \subset S'$ be an affine open and let $V' \subset X'$ be an affine open lying over U'. Let $U' = \mathrm{Spec}(A')$ and denote $I \subset A'$ be the ideal defining the closed subscheme $U' \cap S$. Say $V' = \mathrm{Spec}(B')$. Then $V' \cap X = \mathrm{Spec}(B'/IB')$. Setting $A = A'/I$ and $B = B'/IB'$ we get a commutative diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & IB' & \longrightarrow & B' & \longrightarrow & B & \longrightarrow & 0 \\ & & \uparrow & & \uparrow & & \uparrow & & \\ 0 & \longrightarrow & IA' & \longrightarrow & A' & \longrightarrow & A & \longrightarrow & 0 \end{array}$$

with exact rows and $I^2 = 0$.

The translation of (1) into algebra: If $A \to B$ is surjective, then $A' \to B'$ is surjective. This follows from Nakayama's lemma (Algebra, Lemma 10.18.1).

The translation of (2) into algebra: If $A \to B$ is a finite type ring map, then $A' \to B'$ is a finite type ring map. This follows from Nakayama's lemma (Algebra, Lemma 10.18.1) applied to a map $A'[x_1, \ldots, x_n] \to B'$ such that $A[x_1, \ldots, x_n] \to B$ is surjective.

Proof of (3). Follows from (2) and that quasi-finiteness of a morphism which is locally of finite type can be checked on fibres, see Morphisms, Lemma 28.21.6.

Proof of (4). Follows from (2) and that the additional property of "being of relative dimension d" can be checked on fibres (by definition, see Morphisms, Definition 28.30.1.

The translation of (5) into algebra: If $A \to B$ is unramified map, then $A' \to B'$ is unramified. Since $A \to B$ is of finite type we see that $A' \to B'$ is of finite type by (2) above. Since $A \to B$ is unramified we have $\Omega_{B/A} = 0$. By Algebra, Lemma 10.125.12 we have $0 = \Omega_{B/A} = \Omega_{B'/A'}/I\Omega_{B'/A'}$. Hence $\Omega_{B'/A'} = 0$ by Nakayama's lemma (Algebra, Lemma 10.18.1). Thus $A' \to B'$ is unramified.

Proof of (6). Follows by combining (2) with results of Lemma 36.2.5 and the fact that proper equals quasi-compact + separated + locally of finite type + universally closed.

Proof of (7). Follows by combining (2) with results of Lemma 36.2.5 and using the fact that finite equals integral + locally of finite type (Morphisms, Lemma 28.44.4). □

36.3. First order infinitesimal neighbourhood

A natural construction of first order thickenings is the following. Suppose that $i : Z \to X$ be an immersion of schemes. Choose an open subscheme $U \subset X$ such that i identifies Z with a closed subscheme $Z \subset U$. Let $\mathcal{I} \subset \mathcal{O}_U$ be the quasi-coherent sheaf of ideals defining Z in U. Then we can consider the closed subscheme $Z' \subset U$ defined by the quasi-coherent sheaf of ideals $\mathcal{I}^2$.

Definition 36.3.1. Let $i : Z \to X$ be an immersion of schemes. The *first order infinitesimal neighbourhood* of Z in X is the first order thickening $Z \subset Z'$ over X described above.

This thickening has the following universal property (which will assuage any fears that the construction above depends on the choice of the open U).

Lemma 36.3.2. *Let $i : Z \to X$ be an immersion of schemes. The first order infinitesimal neighbourhood Z' of Z in X has the following universal property: Given any commutative diagram*

$$\begin{array}{ccc} Z & \xleftarrow{a} & T \\ {\scriptstyle i}\downarrow & & \downarrow \\ X & \xleftarrow{b} & T' \end{array}$$

where $T \subset T'$ is a first order thickening over X, there exists a unique morphism $(a', a) : (T \subset T') \to (Z \subset Z')$ of thickenings over X.

Proof. Let $U \subset X$ be the open used in the construction of Z', i.e., an open such that Z is identified with a closed subscheme of U cut out by the quasi-coherent sheaf of ideals $\mathcal{I}$. Since $|T| = |T'|$ we see that $b(T') \subset U$. Hence we can think of b as a morphism into U. Let $\mathcal{J} \subset \mathcal{O}_{T'}$ be the ideal cutting out T. Since $b(T) \subset Z$ by the diagram above we see that $b^\sharp(b^{-1}\mathcal{I}) \subset \mathcal{J}$. As T' is a first order thickening of T we see that $\mathcal{J}^2 = 0$ hence $b^\sharp(b^{-1}(\mathcal{I}^2)) = 0$. By Schemes, Lemma 25.4.6 this implies that b factors through Z'. Denote $a' : T' \to Z'$ this factorization and everything is clear. □

Lemma 36.3.3. *Let $i : Z \to X$ be an immersion of schemes. Let $Z \subset Z'$ be the first order infinitesimal neighbourhood of Z in X. Then the diagram*

$$\begin{array}{ccc} Z & \longrightarrow & Z' \\ \downarrow & & \downarrow \\ Z & \longrightarrow & X \end{array}$$

induces a map of conormal sheaves $\mathcal{C}_{Z/X} \to \mathcal{C}_{Z/Z'}$ by Morphisms, Lemma 28.33.3. This map is an isomorphism.

Proof. This is clear from the construction of Z' above. □

36.4. Formally unramified morphisms

Recall that a ring map $R \to A$ is called *formally unramified* (see Algebra, Definition 10.139.1) if for every commutative solid diagram

$$\begin{array}{ccc} A & \longrightarrow & B/I \\ \uparrow & \nearrow & \uparrow \\ R & \longrightarrow & B \end{array}$$

where $I \subset B$ is an ideal of square zero, at most one dotted arrow exists which makes the diagram commute. This motivates the following analogue for morphisms of schemes.

Definition 36.4.1. Let $f : X \to S$ be a morphism of schemes. We say f is *formally unramified* if given any solid commutative diagram

$$\begin{array}{ccc} X & \longleftarrow & T \\ {\scriptstyle f}\downarrow & \nwarrow & \downarrow{\scriptstyle i} \\ S & \longleftarrow & T' \end{array}$$

where $T \subset T'$ is a first order thickening of affine schemes over S there exists at most one dotted arrow making the diagram commute.

We first prove some formal lemmas, i.e., lemmas which can be proved by drawing the corresponding diagrams.

Lemma 36.4.2. *If $f : X \to S$ is a formally unramified morphism, then given any solid commutative diagram*

$$\begin{array}{ccc} X & \longleftarrow & T \\ {\scriptstyle f}\downarrow & \nwarrow & \downarrow{\scriptstyle i} \\ S & \longleftarrow & T' \end{array}$$

where $T \subset T'$ is a first order thickening of schemes over S there exists at most one dotted arrow making the diagram commute. In other words, in Definition 36.4.1 the condition that T be affine may be dropped.

Proof. This is true because a morphism is determined by its restrictions to affine opens. □

Lemma 36.4.3. *A composition of formally unramified morphisms is formally unramified.*

Proof. This is formal. □

Lemma 36.4.4. *A base change of a formally unramified morphism is formally unramified.*

Proof. This is formal. □

Lemma 36.4.5. *Let $f : X \to S$ be a morphism of schemes. Let $U \subset X$ and $V \subset S$ be open such that $f(U) \subset V$. If f is formally unramified, so is $f|_U : U \to V$.*

Proof. Consider a solid diagram

$$\begin{array}{ccc} U & \xleftarrow{a} & T \\ {\scriptstyle f|_U}\downarrow & \nearrow & \downarrow{\scriptstyle i} \\ V & \longleftarrow & T' \end{array}$$

as in Definition 36.4.1. If f is formally ramified, then there exists at most one S-morphism $a' : T' \to X$ such that $a'|_T = a$. Hence clearly there exists at most one such morphism into U. □

Lemma 36.4.6. *Let $f : X \to S$ be a morphism of schemes. Assume X and S are affine. Then f is formally unramified if and only if $\mathcal{O}_S(S) \to \mathcal{O}_X(X)$ is a formally unramified ring map.*

Proof. This is immediate from the definitions (Definition 36.4.1 and Algebra, Definition 10.139.1) by the equivalence of categories of rings and affine schemes, see Schemes, Lemma 25.6.5. □

Here is a characterization in terms of the sheaf of differentials.

Lemma 36.4.7. *Let $f : X \to S$ be a morphism of schemes. Then f is formally unramified if and only if $\Omega_{X/S} = 0$.*

Proof. We give two proofs.

First proof. It suffices to show that $\Omega_{X/S}$ is zero on the members of an affine open covering of X. Choose an affine open $U \subset X$ with $f(U) \subset V$ where $V \subset S$ is an affine open of S. By Lemma 36.4.5 the restriction $f_U : U \to V$ is formally unramified. By Morphisms, Lemma 28.34.5 we see that $\Omega_{X/S}|_U$ is the quasi-coherent sheaf associated to the $\mathcal{O}_X(U)$-module $\Omega_{\mathcal{O}_X(U)/\mathcal{O}_S(V)}$. By Lemma 36.4.6 we see that $\mathcal{O}_S(V) \to \mathcal{O}_X(U)$ is a formally unramified ring map. Hence by Algebra, Lemma 10.139.2 we conclude that $\Omega_{X/S}|_U = 0$ as desired.

Second proof. We recall some of the arguments of the proof of Morphisms, Lemma 28.34.5. Let $W \subset X \times_S X$ be an open such that $\Delta : X \to X \times_S X$ induces a closed immersion into W. Let $\mathcal{J} \subset \mathcal{O}_W$ be the ideal sheaf of this closed immersion.

Let $X' \subset W$ be the closed subscheme defined by the quasi-coherent sheaf of ideals $\mathcal{J}^2$. Consider the two morphisms $p_1, p_2 : X' \to X$ induced by the two projections $X \times_S X \to X$. Note that p_1 and p_2 agree when composed with $\Delta : X \to X'$ and that $X \to X'$ is a closed immersion defined by a an ideal whose square is zero. Moreover there is a short exact sequence

$$0 \to \mathcal{J}/\mathcal{J}^2 \to \mathcal{O}_{X'} \to \mathcal{O}_X \to 0$$

and $\Omega_{X/S} = \mathcal{J}/\mathcal{J}^2$. Moreover, $\mathcal{J}/\mathcal{J}^2$ is generated by the local sections $p_1^\sharp(f) - p_2^\sharp(f)$ for f a local section of $\mathcal{O}_X$.

Suppose that $f : X \to S$ is formally unramified. By assumption this means that $p_1 = p_2$ when restricted to any affine open $T' \subset X'$. Hence $p_1 = p_2$. By what was said above we conclude that $\Omega_{X/S} = \mathcal{J}/\mathcal{J}^2 = 0$.

Conversely, suppose that $\Omega_{X/S} = 0$. Then $X' = X$. Take any pair of morphisms $f_1', f_2' : T' \to X$ fitting as dotted arrows in the diagram of Definition 36.4.1. This gives a morphism $(f_1', f_2') : T' \to X \times_S X$. Since $f_1'|_T = f_2'|_T$ and $|T| = |T'|$ we see that the image of T' under (f_1', f_2') is contained in the open W chosen above. Since $(f_1', f_2')(T) \subset \Delta(X)$ and since T is defined by an ideal of square zero in T' we see that (f_1', f_2') factors through X'. As $X' = X$ we conclude $f_1' = f_2'$ as desired. □

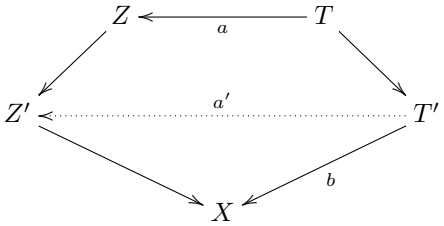
Lemma 36.4.8. *Let $f : X \to S$ be a morphism of schemes. The following are equivalent:*

(1) *The morphism f is unramified (resp. G-unramified), and*
(2) *the morphism f is locally of finite type (resp. locally of finite presentation) and formally unramified.*

Proof. Use Lemma 36.4.7 and Morphisms, Lemma 28.36.2. □

36.5. Universal first order thickenings

Let $h : Z \to X$ be a morphism of schemes. A *universal first order thickening* of Z over X is a first order thickening $Z \subset Z'$ over X such that given any first order thickening $T \subset T'$ over X and a solid commutative diagram



there exists a unique dotted arrow making the diagram commute. Note that in this situation $(a, a') : (T \subset T') \to (Z \subset Z')$ is a morphism of thickenings over X. Thus if a universal first order thickening exists, then it is unique up to unique isomorphism. In general a universal first order thickening does not exist, but if h is formally unramified then it does.

Lemma 36.5.1. *Let $h : Z \to X$ be a formally unramified morphism of schemes. There exists a universal first order thickening $Z \subset Z'$ of Z over X.*

Proof. During this proof we will say $Z \subset Z'$ is a universal first order thickening of Z over X if it satisfies the condition of the lemma. We will construct the universal first order thickening $Z \subset Z'$ over X by glueing, starting with the affine case which is Algebra, Lemma 10.140.1. We begin with some general remarks.

If a universal first order thickening of Z over X exists, then it is unique up to unique isomorphism. Moreover, suppose that $V \subset Z$ and $U \subset X$ are open subschemes such that $h(V) \subset U$. Let $Z \subset Z'$ be a universal first order thickening of Z over X. Let $V' \subset Z'$ be the open subscheme such that $V = Z \cap V'$. Then we claim that $V \subset V'$ is the universal first order thickening of V over U. Namely, suppose given any diagram

$$\begin{array}{ccc} V & \xleftarrow{a} & T \\ {\scriptstyle h}\downarrow & & \downarrow \\ U & \xleftarrow{b} & T' \end{array}$$

where $T \subset T'$ is a first order thickening over U. By the universal property of Z' we obtain $(a, a') : (T \subset T') \to (Z \subset Z')$. But since we have equality $|T| = |T'|$ of underlying topological spaces we see that $a'(T') \subset V'$. Hence we may think of (a, a') as a morphism of thickenings $(a, a') : (T \subset T') \to (V \subset V')$ over U. Uniqueness is clear also. In a completely similar manner one proves that if $h(Z) \subset U$ and $Z \subset Z'$ is a universal first order thickening over U, then $Z \subset Z'$ is a universal first order thickening over X.

Before we glue affine pieces let us show that the lemma holds if Z and X are affine. Say $X = \mathrm{Spec}(R)$ and $Z = \mathrm{Spec}(S)$. By Algebra, Lemma 10.140.1 there exists a first order thickening $Z \subset Z'$ over X which has the universal property of the lemma for diagrams

$$\begin{array}{ccc} Z & \xleftarrow{a} & T \\ {\scriptstyle h}\downarrow & & \downarrow \\ X & \xleftarrow{b} & T' \end{array}$$

where T, T' are affine. Given a general diagram we can choose an affine open covering $T' = \bigcup T'_i$ and we obtain morphisms $a'_i : T'_i \to Z'$ over X such that $a'_i|_{T_i} = a|_{T_i}$. By uniqueness we see that a'_i and a'_j agree on any affine open of $T'_i \cap T'_j$. Hence the morphisms a'_i glue to a global morphism $a' : T' \to Z'$ over X as desired. Thus the lemma holds if X and Z are affine.

Choose an affine open covering $Z = \bigcup Z_i$ such that each Z_i maps into an affine open U_i of X. By Lemma 36.4.5 the morphisms $Z_i \to U_i$ are formally unramified. Hence by the affine case we obtain universal first order thickenings $Z_i \subset Z'_i$ over U_i. By the general remarks above $Z_i \subset Z'_i$ is also a universal first order thickening of Z_i over X. Let $Z'_{i,j} \subset Z'_i$ be the open subscheme such that $Z_i \cap Z_j = Z'_{i,j} \cap Z_i$. By the general remarks we see that both $Z'_{i,j}$ and $Z'_{j,i}$ are universal first order thickenings of $Z_i \cap Z_j$ over X. Thus, by the first of our general remarks, we see that there is a canonical isomorphism $\varphi_{ij} : Z'_{i,j} \to Z'_{j,i}$ inducing the identity on $Z_i \cap Z_j$. We claim that these morphisms satisfy the cocycle condition of Schemes, Section 25.14. (Verification omitted. Hint: Use that $Z'_{i,j} \cap Z'_{i,k}$ is the universal first order thickening of $Z_i \cap Z_j \cap Z_k$ which determines it up to unique isomorphism by what was said above.) Hence we can use the results of Schemes, Section 25.14 to get a first order thickening $Z \subset Z'$ over X which the property that the open

subscheme $Z_i' \subset Z'$ with $Z_i = Z_i' \cap Z$ is a universal first order thickening of Z_i over X.

It turns out that this implies formally that Z' is a universal first order thickening of Z over X. Namely, we have the universal property for any diagram

$$\begin{array}{ccc} Z & \xleftarrow{a} & T \\ {\scriptstyle h}\downarrow & & \downarrow \\ X & \xleftarrow{b} & T' \end{array}$$

where $a(T)$ is contained in some Z_i. Given a general diagram we can choose an open covering $T' = \bigcup T_i'$ such that $a(T_i) \subset Z_i$. We obtain morphisms $a_i' : T_i' \to Z'$ over X such that $a_i'|_{T_i} = a|_{T_i}$. We see that a_i' and a_j' necessarily agree on $T_i' \cap T_j'$ since both $a_i'|_{T_i' \cap T_j'}$ and $a_j'|_{T_i' \cap T_j'}$ are solutions of the problem of mapping into the universal first oder thickening $Z_i' \cap Z_j'$ of $Z_i \cap Z_j$ over X. Hence the morphisms a_i' glue to a global morphism $a' : T' \to Z'$ over X as desired. This finishes the proof. □

Definition 36.5.2. Let $h : Z \to X$ be a formally unramified morphism of schemes.

(1) The *universal first order thickening* of Z over X is the thickening $Z \subset Z'$ constructed in Lemma 36.5.1.
(2) The *conormal sheaf of Z over X* is the conormal sheaf of Z in its universal first order thickening Z' over X.

We often denote the conormal sheaf $\mathcal{C}_{Z/X}$ in this situation.

Thus we see that there is a short exact sequence of sheaves

$$0 \to \mathcal{C}_{Z/X} \to \mathcal{O}_{Z'} \to \mathcal{O}_Z \to 0$$

on Z. The following lemma proves that there is no conflict between this definition and the definition in case $Z \to X$ is an immersion.

Lemma 36.5.3. *Let $i : Z \to X$ be an immersion of schemes. Then*

(1) *i is formally unramified,*
(2) *the universal first order thickening of Z over X is the first order infinitesimal neighbourhood of Z in X of Definition 36.3.1, and*
(3) *the conormal sheaf of i in the sense of Morphisms, Definition 28.33.1 agrees with the conormal sheaf of i in the sense of Definition 36.5.2.*

Proof. By Morphisms, Lemmas 28.36.7 and 28.36.8 an immersion is unramified, hence formally unramified by Lemma 36.4.8. The other assertions follow by combining Lemmas 36.3.2 and 36.3.3 and the definitions. □

Lemma 36.5.4. *Let $Z \to X$ be a formally unramified morphism of schemes. Then the universal first order thickening Z' is formally unramified over X.*

Proof. There are two proofs. The first is to show that $\Omega_{Z'/X} = 0$ by working affine locally and applying Algebra, Lemma 10.140.5. Then Lemma 36.4.7 implies what we want. The second is a direct argument as follows.

Let $T \subset T'$ be a first order thickening. Let

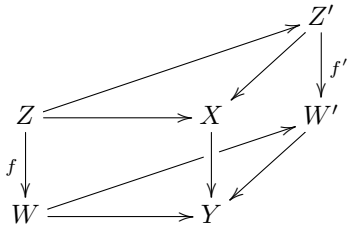
$$\begin{array}{ccc} Z' & \xleftarrow{c} & T \\ \downarrow & \nwarrow^{a,b} & \downarrow \\ X & \longleftarrow & T' \end{array}$$

be a commutative diagram. Consider two morphisms $a, b : T' \to Z'$ fitting into the diagram. Set $T_0 = c^{-1}(Z) \subset T$ and $T'_a = a^{-1}(Z)$ (scheme theoretically). Since Z' is a first order thickening of Z, we see that T' is a first order thickening of T'_a. Moreover, since $c = a|_T$ we see that $T_0 = T \cap T'_a$ (scheme theoretically). As T' is a first order thickening of T it follows that T'_a is a first order thickening of T_0. Now $a|_{T'_a}$ and $b|_{T'_a}$ are morphisms of T'_a into Z' over X which agree on T_0 as morphisms into Z. Hence by the universal property of Z' we conclude that $a|_{T'_a} = b|_{T'_a}$. Thus a and b are morphism from the first order thickening T' of T'_a whose restrictions to T'_a agree as morphisms into Z. Thus using the universal property of Z' once more we conclude that $a = b$. In other words, the defining property of a formally unramified morphism holds for $Z' \to X$ as desired. □

Lemma 36.5.5. *Consider a commutative diagram of schemes*

$$\begin{array}{ccc} Z & \xrightarrow{h} & X \\ {\scriptstyle f}\downarrow & & \downarrow{\scriptstyle g} \\ W & \xrightarrow{h'} & Y \end{array}$$

with h and h' formally unramified. Let $Z \subset Z'$ be the universal first order thickening of Z over X. Let $W \subset W'$ be the universal first order thickening of W over Y. There exists a canonical morphism $(f, f') : (Z, Z') \to (W, W')$ of thickenings over Y which fits into the following commutative diagram



In particular the morphism (f, f') of thickenings induces a morphism of conormal sheaves $f^*\mathcal{C}_{W/Y} \to \mathcal{C}_{Z/X}$.

Proof. The first assertion is clear from the universal property of W'. The induced map on conormal sheaves is the map of Morphisms, Lemma 28.33.3 applied to $(Z \subset Z') \to (W \subset W')$. □

Lemma 36.5.6. *Let*

$$\begin{array}{ccc} Z & \xrightarrow{h} & X \\ {\scriptstyle f}\downarrow & & \downarrow{\scriptstyle g} \\ W & \xrightarrow{h'} & Y \end{array}$$

be a fibre product diagram in the category of schemes with h' formally unramified. Then h is formally unramified and if $W \subset W'$ is the universal first order thickening

of W over Y, then $Z = X \times_Y W \subset X \times_Y W'$ is the universal first order thickening of Z over X. In particular the canonical map $f^*\mathcal{C}_{W/Y} \to \mathcal{C}_{Z/X}$ of Lemma 36.5.5 is surjective.

Proof. The morphism h is formally unramified by Lemma 36.4.4. It is clear that $X \times_Y W'$ is a first order thickening. It is straightforward to check that it has the universal property because W' has the universal property (by mapping properties of fibre products). See Morphisms, Lemma 28.33.4 for why this implies that the map of conormal sheaves is surjective. □

Lemma 36.5.7. *Let*

$$\begin{array}{ccc} Z & \xrightarrow{h} & X \\ {\scriptstyle f}\downarrow & & \downarrow{\scriptstyle g} \\ W & \xrightarrow{h'} & Y \end{array}$$

be a fibre product diagram in the category of schemes with h' formally unramified and g flat. In this case the corresponding map $Z' \to W'$ of universal first order thickenings is flat, and $f^*\mathcal{C}_{W/Y} \to \mathcal{C}_{Z/X}$ is an isomorphism.

Proof. Flatness is preserved under base change, see Morphisms, Lemma 28.26.7. Hence the first statement follows from the description of W' in Lemma 36.5.6. It is clear that $X \times_Y W'$ is a first order thickening. It is straightforward to check that it has the universal property because W' has the universal property (by mapping properties of fibre products). See Morphisms, Lemma 28.33.4 for why this implies that the map of conormal sheaves is an isomorphism. □

Lemma 36.5.8. *Taking the universal first order thickenings commutes with taking opens. More precisely, let $h : Z \to X$ be a formally unramified morphism of schemes. Let $V \subset Z$, $U \subset X$ be opens such that $h(V) \subset U$. Let Z' be the universal first order thickening of Z over X. Then $h|_V : V \to U$ is formally unramified and the universal first order thickening of V over U is the open subscheme $V' \subset Z'$ such that $V = Z \cap V'$. In particular, $\mathcal{C}_{Z/X}|_V = \mathcal{C}_{V/U}$.*

Proof. The first statement is Lemma 36.4.5. The compatibility of universal thickenings can be deduced from the proof of Lemma 36.5.1, or from Algebra, Lemma 10.140.4 or deduced from Lemma 36.5.7. □

Lemma 36.5.9. *Let $h : Z \to X$ be a formally unramified morphism of schemes over S. Let $Z \subset Z'$ be the universal first order thickening of Z over X with structure morphism $h' : Z' \to X$. The canonical map*

$$c_{h'} : (h')^*\Omega_{X/S} \longrightarrow \Omega_{Z'/S}$$

induces an isomorphism $h^*\Omega_{X/S} \to \Omega_{Z'/S} \otimes \mathcal{O}_Z$.

Proof. The map $c_{h'}$ is the map defined in Morphisms, Lemma 28.34.8. If $i : Z \to Z'$ is the given closed immersion, then $i^*c_{h'}$ is a map $h^*\Omega_{X/S} \to \Omega_{Z'/S} \otimes \mathcal{O}_Z$. Checking that it is an isomorphism reduces to the affine case by localization, see Lemma 36.5.8 and Morphisms, Lemma 28.34.3. In this case the result is Algebra, Lemma 10.140.5. □

Lemma 36.5.10. *Let $h : Z \to X$ be a formally unramified morphism of schemes over S. There is a canonical exact sequence*

$$\mathcal{C}_{Z/X} \to h^*\Omega_{X/S} \to \Omega_{Z/S} \to 0.$$

The first arrow is induced by $d_{Z'/S}$ where Z' is the universal first order neighbourhood of Z over X.

Proof. We know that there is a canonical exact sequence

$$\mathcal{C}_{Z/Z'} \to \Omega_{Z'/S} \otimes \mathcal{O}_Z \to \Omega_{Z/S} \to 0.$$

see Morphisms, Lemma 28.34.15. Hence the result follows on applying Lemma 36.5.9. □

Lemma 36.5.11. *Let*

$$\begin{array}{ccc} Z & \xrightarrow{i} & X \\ & \searrow^{j} & \downarrow \\ & & Y \end{array}$$

be a commutative diagram of schemes where i and j are formally unramified. Then there is a canonical exact sequence

$$\mathcal{C}_{Z/Y} \to \mathcal{C}_{Z/X} \to i^*\Omega_{X/Y} \to 0$$

where the first arrow comes from Lemma 36.5.5 and the second from Lemma 36.5.10.

Proof. Denote $Z \to Z'$ the universal first order thickening of Z over X. Denote $Z \to Z''$ the universal first order thickening of Z over Y. By Lemma 36.5.10 here is a canonical morphism $Z' \to Z''$ so that we have a commutative diagram

$$\begin{array}{ccccc} Z & \xrightarrow{i'} & Z' & \longrightarrow & X \\ & \searrow^{j'} & \downarrow & & \downarrow \\ & & Z'' & \longrightarrow & Y \end{array}$$

Apply Morphisms, Lemma 28.34.18 to the left triangle to get an exact sequence

$$\mathcal{C}_{Z/Z''} \to \mathcal{C}_{Z/Z'} \to (i')^*\Omega_{Z'/Z''} \to 0$$

As Z'' is formally unramified over Y (see Lemma 36.5.4) we have $\Omega_{Z'/Z''} = \Omega_{Z/Y}$ (by combining Lemma 36.4.7 and Morphisms, Lemma 28.34.9). Then we have $(i')^*\Omega_{Z'/Y} = i^*\Omega_{X/Y}$ by Lemma 36.5.9. □

Lemma 36.5.12. *Let $Z \to Y \to X$ be formally unramified morphisms of schemes.*

(1) *If $Z \subset Z'$ is the universal first order thickening of Z over X and $Y \subset Y'$ is the universal first order thickening of Y over X, then there is a morphism $Z' \to Y'$ and $Y \times_{Y'} Z'$ is the universal first order thickening of Z over Y.*

(2) *There is a canonical exact sequence*

$$i^*\mathcal{C}_{Y/X} \to \mathcal{C}_{Z/X} \to \mathcal{C}_{Z/Y} \to 0$$

where the maps come from Lemma 36.5.5 and $i : Z \to Y$ is the first morphism.

Proof. The map $h : Z' \to Y'$ in (1) comes from Lemma 36.5.5. The assertion that $Y \times_{Y'} Z'$ is the universal first order thickening of Z over Y is clear from the universal properties of Z' and Y'. By Morphisms, Lemma 28.33.5 we have an exact sequence

$$(i')^*\mathcal{C}_{Y \times_{Y'} Z'/Z'} \to \mathcal{C}_{Z/Z'} \to \mathcal{C}_{Z/Y \times_{Y'} Z'} \to 0$$

where $i' : Z \to Y \times_{Y'} Z'$ is the given morphism. By Morphisms, Lemma 28.33.4 there exists a surjection $h^*\mathcal{C}_{Y/Y'} \to \mathcal{C}_{Y \times_{Y'} Z'/Z'}$. Combined with the equalities $\mathcal{C}_{Y/Y'} = \mathcal{C}_{Y/X}$, $\mathcal{C}_{Z/Z'} = \mathcal{C}_{Z/X}$, and $\mathcal{C}_{Z/Y \times_{Y'} Z'} = \mathcal{C}_{Z/Y}$ this proves the lemma. □

36.6. Formally étale morphisms

Recall that a ring map $R \to A$ is called *formally étale* (see Algebra, Definition 10.141.1) if for every commutative solid diagram

$$\begin{array}{ccc} A & \longrightarrow & B/I \\ \uparrow & \nearrow & \uparrow \\ R & \longrightarrow & B \end{array}$$

where $I \subset B$ is an ideal of square zero, there exists exactly one dotted arrow which makes the diagram commute. This motivates the following analogue for morphisms of schemes.

Definition 36.6.1. Let $f : X \to S$ be a morphism of schemes. We say f is *formally étale* if given any solid commutative diagram

$$\begin{array}{ccc} X & \longleftarrow & T \\ {\scriptstyle f}\downarrow & \nwarrow & \downarrow{\scriptstyle i} \\ S & \longleftarrow & T' \end{array}$$

where $T \subset T'$ is a first order thickening of affine schemes over S there exists exactly one dotted arrow making the diagram commute.

It is clear that a formally étale morphism is formally unramified. Hence if $f : X \to S$ is formally étale, then $\Omega_{X/S}$ is zero, see Lemma 36.4.7.

Lemma 36.6.2. *If $f : X \to S$ is a formally étale morphism, then given any solid commutative diagram*

$$\begin{array}{ccc} X & \longleftarrow & T \\ {\scriptstyle f}\downarrow & \nwarrow & \downarrow{\scriptstyle i} \\ S & \longleftarrow & T' \end{array}$$

where $T \subset T'$ is a first order thickening of schemes over S there exists exactly one dotted arrow making the diagram commute. In other words, in Definition 36.6.1 the condition that T be affine may be dropped.

Proof. Let $T' = \bigcup T'_i$ be an affine open covering, and let $T_i = T \cap T'_i$. Then we get morphisms $a'_i : T'_i \to X$ fitting into the diagram. By uniqueness we see that a'_i and a'_j agree on any affine open subscheme of $T'_i \cap T'_j$. Hence a'_i and a'_j agree on $T'_i \cap T'_j$. Thus we see that the morphisms a'_i glue to a global morphism $a' : T' \to X$. The uniqueness of a' we have seen in Lemma 36.4.2. □

Lemma 36.6.3. *A composition of formally étale morphisms is formally étale.*

Proof. This is formal. □

Lemma 36.6.4. *A base change of a formally étale morphism is formally étale.*

Proof. This is formal. □

Lemma 36.6.5. *Let $f : X \to S$ be a morphism of schemes. Let $U \subset X$ and $V \subset S$ be open subschemes such that $f(U) \subset V$. If f is formally étale, so is $f|_U : U \to V$.*

Proof. Consider a solid diagram

$$\begin{array}{ccc} U & \xleftarrow{a} & T \\ {\scriptstyle f|_U}\downarrow & \nwarrow & \downarrow{\scriptstyle i} \\ V & \xleftarrow{} & T' \end{array}$$

as in Definition 36.6.1. If f is formally ramified, then there exists exactly one S-morphism $a' : T' \to X$ such that $a'|_T = a$. Since $|T'| = |T|$ we conclude that $a'(T') \subset U$ which gives our unique morphism from T' into U. □

Lemma 36.6.6. *Let $f : X \to S$ be a morphism of schemes. The following are equivalent:*

1. *f is formally étale,*
2. *f is formally unramified and the universal first order thickening of X over S is equal to X,*
3. *f is formally unramified and $\mathcal{C}_{X/S} = 0$, and*
4. *$\Omega_{X/S} = 0$ and $\mathcal{C}_{X/S} = 0$.*

Proof. Actually, the last assertion only make sense because $\Omega_{X/S} = 0$ implies that $\mathcal{C}_{X/S}$ is defined via Lemma 36.4.7 and Definition 36.5.2. This also makes it clear that (3) and (4) are equivalent.

Either of the assumptions (1), (2), and (3) imply that f is formally unramified. Hence we may assume f is formally unramified. The equivalence of (1), (2), and (3) follow from the universal property of the universal first order thickening X' of X over S and the fact that $X = X' \Leftrightarrow \mathcal{C}_{X/S} = 0$ since after all by definition $\mathcal{C}_{X/S} = \mathcal{C}_{X/X'}$ is the ideal sheaf of X in X'. □

Lemma 36.6.7. *An unramified flat morphism is formally étale.*

Proof. Say $X \to S$ is unramified and flat. Then $\Delta : X \to X \times_S X$ is an open immersion, see Morphisms, Lemma 28.36.13. We have to show that $\mathcal{C}_{X/S}$ is zero. Consider the two projections $p, q : X \times_S X \to X$. As f is formally unramified (see Lemma 36.4.8), q is formally unramified (see Lemma 36.4.4). As f is flat, p is flat, see Morphisms, Lemma 28.26.7. Hence $p^*\mathcal{C}_{X/S} = \mathcal{C}_q$ by Lemma 36.5.7 where $\mathcal{C}_q$ denotes the conormal sheaf of the formally unramified morphism $q : X \times_S X \to X$. But $\Delta(X) \subset X \times_S X$ is an open subscheme which maps isomorphically to X via q. Hence by Lemma 36.5.8 we see that $\mathcal{C}_q|_{\Delta(X)} = \mathcal{C}_{X/X} = 0$. In other words, the pullback of $\mathcal{C}_{X/S}$ to X via the identity morphism is zero, i.e., $\mathcal{C}_{X/S} = 0$. □

Lemma 36.6.8. *Let $f : X \to S$ be a morphism of schemes. Assume X and S are affine. Then f is formally étale if and only if $\mathcal{O}_S(S) \to \mathcal{O}_X(X)$ is a formally étale ring map.*

Proof. This is immediate from the definitions (Definition 36.6.1 and Algebra, Definition 10.141.1) by the equivalence of categories of rings and affine schemes, see Schemes, Lemma 25.6.5. □

Lemma 36.6.9. *Let $f : X \to S$ be a morphism of schemes. The following are equivalent:*

(1) *The morphism f is étale, and*
(2) *the morphism f is locally of finite presentation and formally étale.*

Proof. Assume f is étale. An étale morphism is locally of finite presentation, flat and unramified, see Morphisms, Section 28.37. Hence f is locally of finite presentation and formally étale, see Lemma 36.6.7.

Conversely, suppose that f is locally of finite presentation and formally étale. Being étale is local in the Zariski topology on X and S, see Morphisms, Lemma 28.37.2. By Lemma 36.6.5 we can cover X by affine opens U which map into affine opens V such that $U \to V$ is formally étale (and of finite presentation, see Morphisms, Lemma 28.22.2). By Lemma 36.6.8 we see that the ring maps $\mathcal{O}(V) \to \mathcal{O}(U)$ are formally étale (and of finite presentation). We win by Algebra, Lemma 10.141.2. (We will give another proof of this implication when we discuss formally smooth morphisms.) □

36.7. Infinitesimal deformations of maps

In this section we explain how a derivation can be used to infinitesimally move a map. Throughout this section we use that a sheaf on a thickening X' of X can be seen as a sheaf on X.

Lemma 36.7.1. *Let S be a scheme. Let $X \subset X'$ and $Y \subset Y'$ be two first order thickenings over S. Let $(a, a'), (b, b') : (X \subset X') \to (Y \subset Y')$ be two morphisms of thickenings over S. Assume that*

(1) *$a = b$, and*
(2) *the two maps $a^*\mathcal{C}_{Y/Y'} \to \mathcal{C}_{X/X'}$ (Morphisms, Lemma 28.33.3) are equal.*

Then the map $(a')^\sharp - (b')^\sharp$ factors as

$$\mathcal{O}_{Y'} \to \mathcal{O}_Y \xrightarrow{D} a_*\mathcal{C}_{X/X'} \to a_*\mathcal{O}_{X'}$$

where D is an $\mathcal{O}_S$-derivation.

Proof. Instead of working on Y we work on X. The advantage is that the pullback functor a^{-1} is exact. Using (1) and (2) we obtain a commutive diagram with exact rows

$$\begin{array}{ccccccccc}
0 & \longrightarrow & \mathcal{C}_{X/X'} & \longrightarrow & \mathcal{O}_{X'} & \longrightarrow & \mathcal{O}_X & \longrightarrow & 0 \\
 & & \uparrow & & \uparrow{\scriptstyle (a')^\sharp} \uparrow{\scriptstyle (b')^\sharp} & & \uparrow & & \\
0 & \longrightarrow & a^{-1}\mathcal{C}_{Y/Y'} & \longrightarrow & a^{-1}\mathcal{O}_{Y'} & \longrightarrow & a^{-1}\mathcal{O}_Y & \longrightarrow & 0
\end{array}$$

Now it is a general fact that in such a situation the difference of the $\mathcal{O}_S$-algebra maps $(a')^\sharp$ and $(b')^\sharp$ is an $\mathcal{O}_S$-derivation from $a^{-1}\mathcal{O}_Y$ to $\mathcal{C}_{X/X'}$. By adjointness of the functors a^{-1} and a_* this is the same thing as an $\mathcal{O}_S$-derivation from $\mathcal{O}_Y$ into $a_*\mathcal{C}_{X/X'}$. Some details omitted. □

Note that in the situation of the lemma above we may write D as

$$D = \mathrm{d}_{Y/S} \circ \theta \tag{36.7.1.1}$$

where θ is an $\mathcal{O}_Y$-linear map $\theta : \Omega_{Y/S} \to a_*\mathcal{C}_{X/X'}$. Of course, then by adjunction again we may view θ as an $\mathcal{O}_X$-linear map $\theta : a^*\Omega_{Y/S} \to \mathcal{C}_{X/X'}$.

Lemma 36.7.2. *Let S be a scheme. Let $(a, a') : (X \subset X') \to (Y \subset Y')$ be a morphism of first order thickenings over S. Let*

$$\theta : a^*\Omega_{Y/S} \to \mathcal{C}_{X/X'}$$

be an $\mathcal{O}_X$-linear map. Then there exists a unique morphism of pairs $(b, b') : (X \subset X') \to (Y \subset Y')$ such that (1) and (2) of Lemma 36.7.1 hold and the derivation D and θ are related by Equation (36.7.1.1).

Proof. We simply set $b = a$ and we define $(b')^\sharp$ to be the map

$$(a')^\sharp + D : a^{-1}\mathcal{O}_{Y'} \to \mathcal{O}_{X'}$$

where D is as in Equation (36.7.1.1). We omit the verification that $(b')^\sharp$ is a map of sheaves of $\mathcal{O}_S$-algebras and that (1) and (2) of Lemma 36.7.1 hold. Equation (36.7.1.1) holds by construction. □

Lemma 36.7.3. *Let S be a scheme. Let $X \subset X'$ and $Y \subset Y'$ be first order thickenings over S. Assume given a morphism $a : X \to Y$ and a map $A : a^*\mathcal{C}_{Y/Y'} \to \mathcal{C}_{X/X'}$ of $\mathcal{O}_X$-modules. For an open subscheme $U' \subset X'$ consider morphisms $a' : U' \to Y'$ such that*

1. *a' is a morphism over S,*
2. *$a'|_U = a|_U$, and*
3. *the induced map $a^*\mathcal{C}_{Y/Y'}|_U \to \mathcal{C}_{X/X'}|_U$ is the restriction of A to U.*

Here $U = X \cap U'$. Then the rule

$$U' \mapsto \{a' : U' \to Y' \text{ such that (1), (2), (3) hold.}\} \tag{36.7.3.1}$$

defines a sheaf of sets on X'.

Proof. Denote $\mathcal{F}$ the rule of the lemma. The restriction mapping $\mathcal{F}(U') \to \mathcal{F}(V')$ for $V' \subset U' \subset X'$ of $\mathcal{F}$ is really the restriction map $a' \mapsto a'|_{V'}$. With this definition in place it is clear that $\mathcal{F}$ is a sheaf since morphisms are defined locally. □

In the following lemma we identify sheaves on X and any thickening of X.

Lemma 36.7.4. *Same notation and assumptions as in Lemma 36.7.3. There is an action of the sheaf*

$$\mathcal{H}om_{\mathcal{O}_X}(a^*\Omega_{Y/S}, \mathcal{C}_{X/X'})$$

on the sheaf (36.7.3.1). Moreover, the action is simply transitive for any open $U' \subset X'$ over which the sheaf (36.7.3.1) has a section.

Proof. This is a combination of Lemmas 36.7.1, 36.7.2, and 36.7.3. □

Remark 36.7.5. A special case of Lemmas 36.7.1, 36.7.2, 36.7.3, and 36.7.4 is where $Y = Y'$. In this case the map A is always zero. The sheaf of Lemma 36.7.3 is just given by the rule

$$U' \mapsto \{a' : U' \to Y \text{ over } S \text{ with } a'|_U = a|_U\}$$

and we act on this by the sheaf $\mathcal{H}om_{\mathcal{O}_X}(a^*\Omega_{Y/S}, \mathcal{C}_{X/X'})$. The action of a local section θ on a' is sometimes indicated by $\theta \cdot a'$. Note that this means nothing else than the fact that $(a')^\sharp$ and $(\theta \cdot a')^\sharp$ differ by a derivation D which is related to θ by Equation (36.7.1.1).

Lemma 36.7.6. *Let S be a scheme. Let $X \subset X'$ be a first order thickening over S. Let Y be a scheme over S. Let $a', b' : X' \to Y$ be two morphisms over S with $a = a'|_X = b'|_X$. This gives rise to a commutative diagram*

$$\begin{array}{ccc} X & \longrightarrow & X' \\ {\scriptstyle a}\downarrow & & \downarrow{\scriptstyle (b',a')} \\ Y & \xrightarrow{\Delta_{Y/S}} & Y \times_S Y \end{array}$$

Since the horizontal arrows are immersions with conormal sheaves $\mathcal{C}_{X/X'}$ and $\Omega_{Y/S}$, by Morphisms, Lemma 28.33.3, we obtain a map $\theta : a^*\Omega_{Y/S} \to \mathcal{C}_{X/X'}$. Then this θ and the derivation D of Lemma 36.7.1 are related by Equation (36.7.1.1).

Proof. Omitted. Hint: The equality may be checked on affine opens where it comes from the following computation. If f is a local section of $\mathcal{O}_Y$, then $1 \otimes f - f \otimes 1$ is a local section of $\mathcal{C}_{Y/(Y \times_S Y)}$ corresponding to $\mathrm{d}_{Y/S}(f)$. It is mapped to the local section $(a')^\sharp(f) - (b')^\sharp(f) = D(f)$ of $\mathcal{C}_{X/X'}$. In other words, $\theta(\mathrm{d}_{Y/S}(f)) = D(f)$. □

For later purposes we need a result that roughly states that the construction of Lemma 36.7.2 is compatible with étale localization.

Lemma 36.7.7. *Let*

$$\begin{array}{ccc} X_1 & \xleftarrow[f]{} & X_2 \\ \downarrow & & \downarrow \\ S_1 & \longleftarrow & S_2 \end{array}$$

be a commutative diagram of schemes with $X_2 \to X_1$ and $S_2 \to S_1$ étale. Then the map $c_f : f^*\Omega_{X_1/S_1} \to \Omega_{X_2/S_2}$ of Morphisms, Lemma 28.34.8 is an isomorphism.

Proof. We recall that an étale morphism $U \to V$ is a smooth morphism with $\Omega_{U/V} = 0$. Using this we see that Morphisms, Lemma 28.34.9 implies $\Omega_{X_2/S_2} = \Omega_{X_2/S_1}$ and Morphisms, Lemma 28.35.16 implies that the map $f^*\Omega_{X_1/S_1} \to \Omega_{X_2/S_1}$ (for the morphism f seen as a morphism over S_1) is an isomorphism. Hence the lemma follows. □

Lemma 36.7.8. *Consider a commutative diagram of schemes*

$$\begin{array}{ccccc} T_2 & \longrightarrow & T_2' & \xrightarrow[a_2']{} & X_2 \\ {\scriptstyle h}\downarrow & & \downarrow{\scriptstyle h'} & \swarrow{\scriptstyle f} & \\ T_1 & \longrightarrow & T_1' & \xrightarrow[a_1']{} & X_1 \\ & & & & \downarrow \\ & & & & S_1 \end{array} \qquad \begin{array}{ccc} & & X_2 \\ & & \downarrow \\ & \longleftarrow & S_2 \end{array}$$

and assume that

(1) *$i_1 : T_1 \to T_1'$ is a first order thickening,*
(2) *$i_2 : T_2 \to T_2'$ is a first order thickening, and*
(3) *$X_2 \to X_1$ and $S_2 \to S_1$ are étale.*

Write $a_i = a'_i \circ i_k$ for $k = 1, 2$. For any $\mathcal{O}_{T_1}$-linear map $\theta_1 : a_1^*\Omega_{X_1/S_1} \to \mathcal{C}_{T_1/T'_1}$ let θ_2 be the composition

$$a_2^*\Omega_{X_2/S_2} = h^*a_1^*\Omega_{X_1/S_1} \xrightarrow{h^*\theta_1} h^*\mathcal{C}_{T_1/T'_1} \longrightarrow \mathcal{C}_{T_2/T'_2}$$

(equality sign is explained in the proof). Then the diagram

$$\begin{array}{ccc} T'_2 & \xrightarrow[\theta_2 \cdot a'_2]{} & X_2 \\ \downarrow & & \downarrow \\ T'_1 & \xrightarrow[\theta_1 \cdot a'_1]{} & X_1 \end{array}$$

commutes where the actions $\theta_2 \cdot a'_2$ and $\theta_1 \cdot a'_1$ are as in Remark 36.7.5.

Proof. The equality sign comes from the identification $f^*\Omega_{X_1/S_1} = \Omega_{X_2/S_2}$ of Lemma 36.7.7. Namely, using this we have $a_2^*\Omega_{X_2/S_2} = a_2^*f^*\Omega_{X_1/S_1} = h^*a_1^*\Omega_{X_1/S_1}$ because $f \circ a_2 = a_1 \circ h$. Having said this, the commutativity of the diagram may be checked on affine opens. Hence we may assume the schemes in the initial big diagram are affine. Thus we obtain a commutative diagram of rings

$$\begin{array}{ccccc} B_2/I_2 & \leftarrow & B_2 & \xleftarrow[a'_2]{} & A_2 \\ \uparrow & & \uparrow{\scriptstyle h'} & \nearrow{\scriptstyle f} & \uparrow \\ B_1/I_1 & \leftarrow & B_1 & \xleftarrow[a'_1]{} & A_1 \\ & & & & \uparrow \\ & & & & R_1 \longrightarrow R_2 \end{array}$$

with $I_1^2 = 0$ and $I_2^2 = 0$ and moreover with the property that $A_2 \otimes_{A_1} \Omega_{A_1/R_1} \to \Omega_{A_2/R_2}$ is an isomorphism. Then $\theta_1 : B_1/I_1 \otimes_{A_1} \Omega_{A_1/R_1} \to I_1$ is B_1-linear. This gives an R_1-derivation $D_1 = \theta_1 \circ \mathrm{d}_{A_1/R_1} : A_1 \to I_1$. In a similar way we see that $\theta_2 : B_2/I_2 \otimes_{A_2} \Omega_{A_2/R_2} \to I_2$ gives rise to a R_2-derivation $D_2 = \theta_2 \circ \mathrm{d}_{A_2/R_2} : A_2 \to I_2$. The construction of θ_2 implies the following compatibility between θ_1 and θ_2: for every $x \in A_1$ we have

$$h'(D_1(x)) = D_2(f(x))$$

as elements of I_2. Now by the construction of the action in Lemma 36.7.2 and Remark 36.7.5 we know that $\theta_1 \cdot a'_1$ corresponds to the ring map $a'_1 + D_1 : A_1 \to B_1$ and $\theta_2 \cdot a'_2$ corresponds to the ring map $a'_2 + D_2 : A_2 \to B_2$. By the displayed equality above we obtain that $h' \circ (a'_1 + D_1) = (a'_2 + D_2) \circ f$ as desired. □

Remark 36.7.9. Lemma 36.7.8 can be improved in the following way. Suppose that we have a commutative diagram of schemes as in Lemma 36.7.8 but we do not assume that $X_2 \to X_1$ and $S_2 \to S_1$ are étale. Next, suppose we have $\theta_1 : a_1^*\Omega_{X_1/S_1} \to \mathcal{I}_1$ and $\theta_2 : a_2^*\Omega_{X_2/S_2} \to \mathcal{I}_2$ such that for a local section t of $\mathcal{O}_{X_1}$ we have $(h')^*\theta_1(a_1^*(\mathrm{d}_{X_1/S_1}(t))) = \theta_2(a_2^*(\mathrm{d}_{X_2/S_2}(f^*t)))$, i.e., such that

$$\begin{array}{ccc} f_*\mathcal{O}_{X_2} & \xrightarrow[f_*D_2]{} & f_*a_{2,*}\mathcal{C}_{T_2/T'_2} \\ {\scriptstyle f^\sharp}\uparrow & & \uparrow{\scriptstyle \text{induced by } (h')^\sharp} \\ \mathcal{O}_{X_1} & \xrightarrow{D_1} & a_{1,*}\mathcal{C}_{T_1/T'_1} \end{array}$$

is commutative where D_i corresponds to θ_i as in Equation (36.7.1.1). Then we have the conclusion of Lemma 36.7.8. The importance of the condition that both $X_2 \to X_1$ and $S_2 \to S_1$ are étale is that it allows us to construct a θ_2 from θ_1.

36.8. Infinitesimal deformations of schemes

The following simple lemma is often a convenient tool to check whether an infinitesimal deformation of a map is flat.

Lemma 36.8.1. *Let $(f, f') : (X \subset X') \to (S \subset S')$ be a morphism of first order thickenings. Assume that f is flat. Then the following are equivalent*

(1) *f' is flat and $X = S \times_{S'} X'$, and*
(2) *the canonical map $f^*\mathcal{C}_{S/S'} \to \mathcal{C}_{X/X'}$ is an isomorphism.*

Proof. As the problem is local on X' we may assume that X, X', S, S' are affine schemes. Say $S' = \mathrm{Spec}(A')$, $X' = \mathrm{Spec}(B')$, $S = \mathrm{Spec}(A)$, $X = \mathrm{Spec}(B)$ with $A = A'/I$ and $B = B'/J$ for some square zero ideals. Then we obtain the following commutative diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & J & \longrightarrow & B' & \longrightarrow & B & \longrightarrow & 0 \\ & & \uparrow & & \uparrow & & \uparrow & & \\ 0 & \longrightarrow & I & \longrightarrow & A' & \longrightarrow & A & \longrightarrow & 0 \end{array}$$

with exact rows. The canonical map of the lemma is the map

$$I \otimes_A B = I \otimes_{A'} B' \longrightarrow J.$$

The assumption that f is flat signifies that $A \to B$ is flat.

Assume (1). Then $A' \to B'$ is flat and $J = IB'$. Flatness implies $\mathrm{Tor}_1^{A'}(B', A) = 0$ (see Algebra, Lemma 10.70.7). This means $I \otimes_{A'} B' \to B'$ is injective (see Algebra, Remark 10.70.8). Hence we see that $I \otimes_A B \to J$ is an isomorphism.

Assume (2). Then it follows that $J = IB'$, so that $X = S \times_{S'} X'$. Moreover, we get $\mathrm{Tor}_1^{A'}(B', A'/I) = 0$ by reversing the implications in the previous paragraph. Hence B' is flat over A' by Algebra, Lemma 10.93.8. □

The following lemma is the "nilpotent" version of the "critère de platitude par fibres", see Section 36.13.

Lemma 36.8.2. *Consider a commutative diagram*

$$\begin{array}{ccc} (X \subset X') & \xrightarrow[(f,f')]{} & (Y \subset Y') \\ & \searrow \quad \swarrow & \\ & (S \subset S') & \end{array}$$

of thickenings. Assume

(1) *X' is flat over S',*
(2) *f is flat,*
(3) *$S \subset S'$ is a finite order thickening, and*
(4) *$X = S \times_{S'} X'$ and $Y = S \times_{S'} Y'$.*

Then f' is flat and Y' is flat over S' at all points in the image of f'.

Proof. Immediate consequence of Algebra, Lemma 10.95.8. □

Many properties of morphisms of schemes are preserved under flat deformations.

Lemma 36.8.3. *Consider a commutative diagram*

$$\begin{array}{ccc} (X \subset X') & \xrightarrow[(f,f')]{} & (Y \subset Y') \\ & \searrow \quad \swarrow & \\ & (S \subset S') & \end{array}$$

of thickenings. Assume $S \subset S'$ is a finite order thickening, X' and Y' flat over S' and $X = S \times_{S'} X'$ and $Y = S \times_{S'} Y'$. Then

(1) *f is flat if and only if f' is flat,*
(2) *f is an isomorphism if and only if f' is an isomorphism,*
(3) *f is an open immersion if and only if f' is an open immersion,*
(4) *f is quasi-compact if and only if f' is quasi-compact,*
(5) *f is universally closed if and only if f' is universally closed,*
(6) *f is (quasi-)separated if and only if f' is (quasi-)separated,*
(7) *f is a monomorphism if and only if f' is a monomorphism,*
(8) *f is surjective if and only if f' is surjective,*
(9) *f is universally injective if and only if f' is universally injective,*
(10) *f is affine if and only if f' is affine,*
(11) *f is locally of finite type if and only if f' is locally of finite type,*
(12) *f is quasi-finite if and only if f' is quasi-finite,*
(13) *f is locally of finite presentation if and only if f' is locally of finite presentation,*
(14) *f is locally of finite type of relative dimension d if and only if f' is locally of finite type of relative dimension d,*
(15) *f is universally open if and only if f' is universally open,*
(16) *f is syntomic if and only if f' is syntomic,*
(17) *f is smooth if and only if f' is smooth,*
(18) *f is unramified if and only if f' is unramified,*
(19) *f is étale if and only if f' is étale,*
(20) *f is proper if and only if f' is proper,*
(21) *f is integral if and only if f' is integral,*
(22) *f is finite if and only if f' is finite,*
(23) *f is finite locally free (of rank d) if and only if f' is finite locally free (of rank d), and*
(24) *add more here.*

Proof. The assumptions on X and Y mean that f is the base change of f' by $X \to X'$. The properties $\mathcal{P}$ listed in (1) – (23) above are all stable under base change, hence if f' has property $\mathcal{P}$, then so does f. See Schemes, Lemmas 25.18.2, 25.19.3, 25.21.13, and 25.23.5 and Morphisms, Lemmas 28.11.4, 28.12.4, 28.13.8, 28.16.4, 28.21.13, 28.22.4, 28.30.2, 28.32.4, 28.35.5, 28.36.5, 28.37.4, 28.42.5, 28.44.6, and 28.46.4.

The interesting direction in each case is therefore to assume that f has the property and deduce that f' has it too. By induction on the order of the thickening we may assume that $S \subset S'$ is a first order thickening, see discussion immediately following Definition 36.2.1. We make a couple of general remarks which we will use without further mention in the arguments below. (I) Let $W' \subset S'$ be an affine open and

let $U' \subset X'$ and $V' \subset Y'$ be affine opens lying over W' with $f'(U') \subset V'$. Let $W' = \mathrm{Spec}(R')$ and denote $I \subset R'$ be the ideal defining the closed subscheme $W' \cap S$. Say $U' = \mathrm{Spec}(B')$ and $V' = \mathrm{Spec}(A')$. Then we get a commutative diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & IB' & \longrightarrow & B' & \longrightarrow & B & \longrightarrow & 0 \\ & & \uparrow & & \uparrow & & \uparrow & & \\ 0 & \longrightarrow & IA' & \longrightarrow & A' & \longrightarrow & A & \longrightarrow & 0 \end{array}$$

with exact rows. Moreover, $IA' \cong I \otimes_R A$ and $IB' \cong I \otimes_R B$, see proof of Lemma 36.8.1. (II) The morphisms $X \to X'$ and $Y \to Y'$ are universal homeomorphisms. Hence the topology of the maps f and f' (after any base change) is identical. (III) If f is flat, then f' is flat, see Lemma 36.8.2.

Ad (1). This is general remark (III).

Ad (2). Assume f is an isomorphism. Choose an affine open $V' \subset Y'$ and set $U' = (f')^{-1}(V')$. Then $V = Y \cap V'$ is affine which implies that $V \cong f^{-1}(V) = U = Y \times_{Y'} U'$ is affine. By Lemma 36.2.3 we see that U' is affine. Hence $IB' \cong I \otimes_R B \cong I \otimes_R A \cong IA'$ and $A \cong B$. By the exactness of the rows in the diagram above we see that $A' \cong B'$, i.e., $U' \cong V'$. Thus f' is an isomorphism.

Ad (3). Assume f is an open immersion. Then f is an isomorphism of X with an open subscheme $V \subset Y$. Let $V' \subset Y'$ be the open subscheme whose underlying topological space is V. Then f' is a map from X' to V' which is an isomorphism by (2). Hence f' is an open immersion.

Ad (4). Immediate from remark (II). See also Lemma 36.2.5 for a more general statement.

Ad (5). Immediate from remark (II). See also Lemma 36.2.5 for a more general statement.

Ad (6). Note that $X \times_Y X = Y \times_{Y'} (X' \times_{Y'} X')$ so that $X' \times_{Y'} X'$ is a thickening of $X \times_Y X$. Hence the topology of the maps $\Delta_{X/Y}$ and $\Delta_{X'/Y'}$ matches and we win. See also Lemma 36.2.5 for a more general statement.

Ad (7). Assume f is a monomorphism. Consider the diagonal morphism $\Delta_{X'/Y'} : X' \to X' \times_{Y'} X'$. Because f is a monomorphism and because $X' \times_{Y'} X'$ is a thickening of $X \times_Y X$ we see that $\Delta_{X'/Y'}$ is surjective. Hence Lemma 36.8.2 implies that $X' \times_{Y'} X'$ is flat over S'. Then (2) shows that $\Delta_{X'/Y'}$ is an isomorphism.

Ad (8). This is clear. See also Lemma 36.2.5 for a more general statement.

Ad (9). Immediate from remark (II). See also Lemma 36.2.5 for a more general statement.

Ad (10). Assume f is affine. Choose an affine open $V' \subset Y'$ and set $U' = (f')^{-1}(V')$. Then $V = Y \cap V'$ is affine which implies that $U = Y \times_{Y'} U'$ is affine. By Lemma 36.2.3 we see that U' is affine. Hence f' is affine. See also Lemma 36.2.5 for a more general statement.

Ad (11). Via remark (I) comes down to proving $A' \to B'$ is of finite type if $A \to B$ is of finite type. Suppose that $x_1, \ldots, x_n \in B'$ are elements whose images in B generate B as an A-algebra. Then $A'[x_1, \ldots, x_n] \to B$ is surjective as both

$A'[x_1, \ldots, x_n] \to B$ is surjective and $I \otimes_R A[x_1, \ldots, x_n] \to I \otimes_R B$ is surjective. See also Lemma 36.2.6 for a more general statement.

Ad (12). Follows from (11) and that quasi-finiteness of a morphism of finite type can be checked on fibres, see Morphisms, Lemma 28.21.6. See also Lemma 36.2.6 for a more general statement.

Ad (13). Via remark (I) comes down to proving $A' \to B'$ is of finite presentation if $A \to B$ is of finite presentation. We may assume that $B' = A'[x_1, \ldots, x_n]/K'$ for some ideal K' by (11). We get a short exact sequence

$$0 \to K' \to A'[x_1, \ldots, x_n] \to B' \to 0$$

As B' is flat over R' we see that $K' \otimes_{R'} R$ is the kernel of the surjection $A[x_1, \ldots, x_n] \to B$. By assumption on $A \to B$ there exist finitely many $f'_1, \ldots, f'_m \in K'$ whose images in $A[x_1, \ldots, x_n]$ generate this kernel. Since I is nilpotent we see that $f'_1, \ldots, f'_m$ generate K' by Nakayama's lemma, see Algebra, Lemma 10.18.1.

Ad (14). Follows from (11) and general remark (II). See also Lemma 36.2.6 for a more general statement.

Ad (15). Immediate from general remark (II). See also Lemma 36.2.5 for a more general statement.

Ad (16). Assume f is syntomic. By (13) f' is locally of finite presentation, by general remark (III) f' is flat and the fibres of f' are the fibres of f. Hence f' is syntomic by Morphisms, Lemma 28.32.11.

Ad (17). Assume f is smooth. By (13) f' is locally of finite presentation, by general remark (III) f' is flat, and the fibres of f' are the fibres of f. Hence f' is smooth by Morphisms, Lemma 28.35.3.

Ad (18). Assume f unramified. By (11) f' is locally of finite type and the fibres of f' are the fibres of f. Hence f' is unramified by Morphisms, Lemma 28.36.12. See also Lemma 36.2.6 for a more general statement.

Ad (19). Assume f étale. By (13) f' is locally of finite presentation, by general remark (III) f' is flat, and the fibres of f' are the fibres of f. Hence f' is étale by Morphisms, Lemma 28.37.8.

Ad (20). This follows from a combination of (6), (11), (4), and (5). See also Lemma 36.2.6 for a more general statement.

Ad (21). Combine (5) and (10) with Morphisms, Lemma 28.44.7. See also Lemma 36.2.5 for a more general statement.

Ad (22). Combine (21), and (11) with Morphisms, Lemma 28.44.4. See also Lemma 36.2.6 for a more general statement.

Ad (23). Assume f finite locally free. By (22) we see that f' is finite, by general remark (III) f' is flat, and by (13) f' is locally of finite presentation. Hence f' is finite locally free by Morphisms, Lemma 28.46.2. □

36.9. Formally smooth morphisms

Michael Artin's position on differential criteria of smoothness (e.g., Morphisms, Lemma 28.35.14) is that they are basically useless (in practice). In this section we introduce the notion of a formally smooth morphism $X \to S$. Such a morphism is characterized by the property that T-valued points of X lift to infinitesimal thickenings of T provided T is affine. The main result is that a morphism which is formally smooth and locally of finite presentation is smooth, see Lemma 36.9.7. It turns out that this criterion is often easier to use than the differential criteria mentioned above.

Recall that a ring map $R \to A$ is called *formally smooth* (see Algebra, Definition 10.131.1) if for every commutative solid diagram

$$\begin{array}{ccc} A & \longrightarrow & B/I \\ \uparrow & \nearrow & \uparrow \\ R & \longrightarrow & B \end{array}$$

where $I \subset B$ is an ideal of square zero, a dotted arrow exists which makes the diagram commute. This motivates the following analogue for morphisms of schemes.

Definition 36.9.1. Let $f : X \to S$ be a morphism of schemes. We say f is *formally smooth* if given any solid commutative diagram

$$\begin{array}{ccc} X & \longleftarrow & T \\ {\scriptstyle f}\downarrow & \nwarrow & \downarrow{\scriptstyle i} \\ S & \longleftarrow & T' \end{array}$$

where $T \subset T'$ is a first order thickening of affine schemes over S there exists a dotted arrow making the diagram commute.

In the cases of formally unramified and formally étale morphisms the condition that T' be affine could be dropped, see Lemmas 36.4.2 and 36.6.2. This is no longer true in the case of formally smooth morphisms. In fact, a slightly more natural condition would be that we should be able to fill in the dotted arrow Zariski locally on T'. In fact, analyzing the proof of Lemma 36.9.7 shows that this would be equivalent to the definition as it currently stands.

Lemma 36.9.2. *A composition of formally smooth morphisms is formally smooth.*

Proof. Omitted. □

Lemma 36.9.3. *A base change of a formally smooth morphism is formally smooth.*

Proof. Omitted, but see Algebra, Lemma 10.131.2 for the algebraic version. □

Lemma 36.9.4. *Let $f : X \to S$ be a morphism of schemes. Then f is formally étale if and only if f is formally smooth and formally unramified.*

Proof. Omitted. □

Lemma 36.9.5. *Let $f : X \to S$ be a morphism of schemes. Let $U \subset X$ and $V \subset S$ be open subschemes such that $f(U) \subset V$. If f is formally smooth, so is $f|_U : U \to V$.*

Proof. Consider a solid diagram

$$\begin{array}{ccc} U & \xleftarrow{a} & T \\ {\scriptstyle f|_U}\downarrow & \nwarrow & \downarrow{\scriptstyle i} \\ V & \longleftarrow & T' \end{array}$$

as in Definition 36.9.1. If f is formally smooth, then there exists an S-morphism $a' : T' \to X$ such that $a'|_T = a$. Since the underlying sets of T and T' are the same we see that a' is a morphism into U (see Schemes, Section 25.3). And it clearly is a V-morphism as well. Hence the dotted arrow above as desired. □

Lemma 36.9.6. *Let $f : X \to S$ be a morphism of schemes. Assume X and S are affine. Then f is formally smooth if and only if $\mathcal{O}_S(S) \to \mathcal{O}_X(X)$ is a formally smooth ring map.*

Proof. This is immediate from the definitions (Definition 36.9.1 and Algebra, Definition 10.131.1) by the equivalence of categories of rings and affine schemes, see Schemes, Lemma 25.6.5. □

The following lemma is the main result of this section. It is a victory of the functorial point of view in that it implies (combined with Limits, Proposition 31.5.1) that we can recognize whether a morphism $f : X \to S$ is smooth in terms of "simple" properties of the functor $h_X : Sch/S \to Sets$.

Lemma 36.9.7 (Infinitesimal lifting criterion)**.** *Let $f : X \to S$ be a morphism of schemes. The following are equivalent:*

(1) *The morphism f is smooth, and*
(2) *the morphism f is locally of finite presentation and formally smooth.*

Proof. Assume $f : X \to S$ is locally of finite presentation and formally smooth. Consider a pair of affine opens $\mathrm{Spec}(A) = U \subset X$ and $\mathrm{Spec}(R) = V \subset S$ such that $f(U) \subset V$. By Lemma 36.9.5 we see that $U \to V$ is formally smooth. By Lemma 36.9.6 we see that $R \to A$ is formally smooth. By Morphisms, Lemma 28.22.2 we see that $R \to A$ is of finite presentation. By Algebra, Proposition 10.131.13 we see that $R \to A$ is smooth. Hence by the definition of a smooth morphism we see that $X \to S$ is smooth.

Conversely, assume that $f : X \to S$ is smooth. Consider a solid commutative diagram

$$\begin{array}{ccc} X & \xleftarrow{a} & T \\ {\scriptstyle f}\downarrow & \nwarrow & \downarrow{\scriptstyle i} \\ S & \longleftarrow & T' \end{array}$$

as in Definition 36.9.1. We will show the dotted arrow exists thereby proving that f is formally smooth.

Let $\mathcal{F}$ be the sheaf of sets on T' of Lemma 36.7.3, see also Remark 36.7.5. Let

$$\mathcal{H} = \mathcal{H}om_{\mathcal{O}_T}(a^*\Omega_{X/S}, \mathcal{C}_{T/T'})$$

be the sheaf of $\mathcal{O}_T$-modules on T introduced in Lemma 36.7.4. Our goal is simply to show that $\mathcal{F}(T) \neq \emptyset$. In other words we are trying to show that $\mathcal{F}$ is a trivial $\mathcal{H}$-torsor on T (see Cohomology, Section 20.5). There are two steps: (I) To show

that $\mathcal{F}$ is a torsor we have to show that $\mathcal{F}_t \neq \emptyset$ for all $t \in T$ (see Cohomology, Definition 20.5.1). (II) To show that $\mathcal{F}$ is the trivial torsor it suffices to show that $H^1(T, \mathcal{H}) = 0$ (see Cohomology, Lemma 20.5.3 – we may use either cohomology of $\mathcal{H}$ as an abelian sheaf or as an $\mathcal{O}_T$-module, see Cohomology, Lemma 20.14.3).

First we prove (I). To see this, for every $t \in T$ we can choose an affine open $U \subset T$ neighbourhood of t such that $a(U)$ is contained in an affine open $\mathrm{Spec}(A) = W \subset X$ which maps to an affine open $\mathrm{Spec}(R) = V \subset S$. By Morphisms, Lemma 28.35.2 the ring map $R \to A$ is smooth. Hence by Algebra, Proposition 10.131.13 the ring map $R \to A$ is formally smooth. Lemma 36.9.6 in turn implies that $W \to V$ is formally smooth. Hence we can lift $a|_U : U \to W$ to a V-morphism $a' : U' \to W \subset X$ showing that $\mathcal{F}(U) \neq \emptyset$.

Finally we prove (II). By Morphisms, Lemma 28.34.13 we see that $\Omega_{X/S}$ is of finite presentation (it is even finite locally free by Morphisms, Lemma 28.35.12). Hence $a^*\Omega_{X/S}$ is of finite presentation (see Modules, Lemma 17.11.4). Hence the sheaf $\mathcal{H} = \mathcal{H}om_{\mathcal{O}_T}(a^*\Omega_{X/S}, \mathcal{C}_{T/T'})$ is quasi-coherent by the discussion in Schemes, Section 25.24. Thus by Cohomology of Schemes, Lemma 29.2.2 we have $H^1(T, \mathcal{H}) = 0$ as desired. □

Locally projective quasi-coherent modules are defined in Properties, Section 27.19.

Lemma 36.9.8. *Let $f : X \to Y$ be a formally smooth morphism of schemes. Then $\Omega_{X/Y}$ is locally projective on X.*

Proof. Choose $U \subset X$ and $V \subset Y$ affine open such that $f(U) \subset V$. By Lemma 36.9.5 $f|_U : U \to V$ is formally smooth. Hence $\Gamma(V, \mathcal{O}_V) \to \Gamma(U, \mathcal{O}_U)$ is a formally smooth ring map, see Lemma 36.9.6. Hence by Algebra, Lemma 10.131.7 the $\Gamma(U, \mathcal{O}_U)$-module $\Omega_{\Gamma(U,\mathcal{O}_U)/\Gamma(V,\mathcal{O}_V)}$ is projective. Hence $\Omega_{U/V}$ is locally projective, see Properties, Section 27.19. □

Lemma 36.9.9. *Let $f : X \to Y$, $g : Y \to S$ be morphisms of schemes. Assume f is formally smooth. Then*

$$0 \to f^*\Omega_{Y/S} \to \Omega_{X/S} \to \Omega_{X/Y} \to 0$$

(see Morphisms, Lemma 28.34.9) is short exact.

Proof. The algebraic version of this lemma is the following: Given ring maps $A \to B \to C$ with $B \to C$ formally smooth, then the sequence

$$0 \to C \otimes_B \Omega_{B/A} \to \Omega_{C/A} \to \Omega_{C/B} \to 0$$

of Algebra, Lemma 10.125.7 is exact. This is Algebra, Lemma 10.131.9. □

Lemma 36.9.10. *Let $h : Z \to X$ be a formally unramified morphism of schemes over S. Assume that Z is formally smooth over S. Then the canonical exact sequence*

$$0 \to \mathcal{C}_{Z/X} \to i^*\Omega_{X/S} \to \Omega_{Z/S} \to 0$$

of Lemma 36.5.10 is short exact.

Proof. Let $Z \to Z'$ be the universal first order thickening of Z over X. From the proof of Lemma 36.5.10 we see that our sequence is identified with the sequence

$$\mathcal{C}_{Z/Z'} \to \Omega_{Z'/S} \otimes \mathcal{O}_Z \to \Omega_{Z/S} \to 0.$$

Since $Z \to S$ is formally smooth we can locally on Z' find a left inverse $Z' \to Z$ over S to the inclusion map $Z \to Z'$. Thus the sequence is locally split, see Morphisms, Lemma 28.34.16. □

Lemma 36.9.11. *Let*

$$\begin{array}{ccc} Z & \xrightarrow{i} & X \\ & \searrow^{j} & \downarrow f \\ & & Y \end{array}$$

be a commutative diagram of schemes where i and j are formally unramified and f is formally smooth. Then the canonical exact sequence

$$0 \to \mathcal{C}_{Z/Y} \to \mathcal{C}_{Z/X} \to i^*\Omega_{X/Y} \to 0$$

of Lemma 36.5.11 is exact and locally split.

Proof. Denote $Z \to Z'$ the universal first order thickening of Z over X. Denote $Z \to Z''$ the universal first order thickening of Z over Y. By Lemma 36.5.10 here is a canonical morphism $Z' \to Z''$ so that we have a commutative diagram

$$\begin{array}{ccccc} Z & \xrightarrow{i'} & Z' & \xrightarrow{a} & X \\ & \searrow^{j'} & \downarrow k & & \downarrow f \\ & & Z'' & \xrightarrow{b} & Y \end{array}$$

In the proof of Lemma 36.5.11 we identified the sequence above with the sequence

$$\mathcal{C}_{Z/Z''} \to \mathcal{C}_{Z/Z'} \to (i')^*\Omega_{Z'/Z''} \to 0$$

Let $U'' \subset Z''$ be an affine open. Denote $U \subset Z$ and $U' \subset Z'$ the corresponding affine open subschemes. As f is formally smooth there exists a morphism $h : U'' \to X$ which agrees with i on U and such that $f \circ h$ equals $b|_{U''}$. Since Z' is the universal first order thickening we obtain a unique morphism $g : U'' \to Z'$ such that $g = a \circ h$. The universal property of Z'' implies that $k \circ g$ is the inclusion map $U'' \to Z''$. Hence g is a left inverse to k. Picture

$$\begin{array}{ccc} U & \longrightarrow & Z' \\ \downarrow & \nearrow_{g} & \downarrow k \\ U'' & \longrightarrow & Z'' \end{array}$$

Thus g induces a map $\mathcal{C}_{Z/Z'}|_U \to \mathcal{C}_{Z/Z''}|_U$ which is a left inverse to the map $\mathcal{C}_{Z/Z''} \to \mathcal{C}_{Z/Z'}$ over U. □

36.10. Smoothness over a Noetherian base

It turns out that if the base is Noetherian then we can get away with less in the formulation of formal smoothness. In some sense the following lemmas are the beginning of deformation theory.

Lemma 36.10.1. *Let $f : X \to S$ be a morphism of schemes. Let $x \in X$. Assume that S is locally Noetherian and f locally of finite type. The following are equivalent:*

(1) *f is smooth at x,*

(2) *for every solid commutative diagram*

$$\begin{array}{ccc} X & \xleftarrow{} & \mathrm{Spec}(B) \\ {\scriptstyle f}\downarrow & \overset{\alpha}{\nwarrow} & \downarrow{\scriptstyle i} \\ S & \xleftarrow{\beta} & \mathrm{Spec}(B') \end{array}$$

where $B' \to B$ is a surjection of local rings with $Ker(B' \to B)$ of square zero, and α mapping the closed point of $\mathrm{Spec}(B)$ to x there exists a dotted arrow making the diagram commute,

(3) *same as in (2) but with $B' \to B$ ranging over small extensions (see Algebra, Definition 10.134.1), and*

(4) *same as in (2) but with $B' \to B$ ranging over small extensions such that α induces an isomorphism $\kappa(x) \to \kappa(\mathfrak{m})$ where $\mathfrak{m} \subset B$ is the maximal ideal.*

Proof. Choose an affine neighbourhood $V \subset S$ of $f(x)$ and choose an affine neighbourhood $U \subset X$ of x such that $f(U) \subset V$. For any "test" diagram as in (2) the morphism α will map $\mathrm{Spec}(B)$ into U and the morphism β will map $\mathrm{Spec}(B')$ into V (see Schemes, Section 25.13). Hence the lemma reduces to the morphism $f|_U : U \to V$ of affines. (Indeed, V is Noetherian and $f|_U$ is of finite type, see Properties, Lemma 27.5.2 and Morphisms, Lemma 28.16.2.) In this affine case the lemma is identical to Algebra, Lemma 10.134.2. □

Sometimes it is useful to know that one only needs to check the lifting criterion for small extensions "centered" at points of finite type (see Morphisms, Section 28.17).

Lemma 36.10.2. *Let $f : X \to S$ be a morphism of schemes. Let $x \in X$. Assume that S is locally Noetherian and f locally of finite type. The following are equivalent:*

(1) *f is smooth,*

(2) *for every solid commutative diagram*

$$\begin{array}{ccc} X & \xleftarrow{} & \mathrm{Spec}(B) \\ {\scriptstyle f}\downarrow & \overset{\alpha}{\nwarrow} & \downarrow{\scriptstyle i} \\ S & \xleftarrow{\beta} & \mathrm{Spec}(B') \end{array}$$

where $B' \to B$ is a small extension of Artinian local rings and β of finite type (!) there exists a dotted arrow making the diagram commute.

Proof. If f is smooth, then the infinitesimal lifting criterion (Lemma 36.9.7) says f is formally smooth and (2) holds.

Assume (2). The set of points $x \in X$ where f is not smooth forms a closed subset T of X. By the discussion in Morphisms, Section 28.17, if $T \neq \emptyset$ there exists a point $x \in T \subset X$ such that the morphism

$$\mathrm{Spec}(\kappa(x)) \to X \to S$$

is of finite type (namely, pick any point x of T which is closed in an affine open of X). By Morphisms, Lemma 28.17.2 given any local Artinian ring B' with residue field $\kappa(x)$ then any morphism $\beta : \mathrm{Spec}(B') \to S$ is of finite type. Thus we see that all the diagrams used in Lemma 36.10.1 (4) correspond to diagrams as in the current lemma (2). Whence $X \to S$ is smooth a x a contradiction. □

36.11. Pushouts in the category of schemes

In this section we collect some results on pushouts in the category of schemes. See Categories, Section 36.11 for a general discussion of pushouts in any category.

Lemma 36.11.1. *Let $X \to X'$ be a thickening of schemes and let $X \to Y$ be an affine morphism of schemes. Then there exists a pushout*

$$\begin{array}{ccc} X & \longrightarrow & X' \\ {\scriptstyle f}\downarrow & & \downarrow{\scriptstyle f'} \\ Y & \longrightarrow & Y \amalg_X X' \end{array}$$

in the category of schemes. Moreover $Y' = Y \amalg_X X'$ is a thickening of Y and

$$\mathcal{O}_{Y'} = \mathcal{O}_Y \times_{f_*\mathcal{O}_X} f'_*\mathcal{O}_{X'}$$

as sheaves on $|Y| = |Y'|$.

Proof. We first construct Y' as a ringed space. Namely, as topological space we take $Y' = Y$. Denote $f' : X' \to Y'$ the map of topological spaces which equals f. As structure sheaf $\mathcal{O}_{Y'}$ we take the right hand side of the equation of the lemma. To see that Y' is a scheme, we have to show that any point has an affine neighbourhood. Since the formation of the fibre product of sheaves commutes with restricting to opens, we may assume Y is affine. Then X is affine (as f is affine) and X' is affine as well (see Lemma 36.2.3). Say $Y \leftarrow X \to X'$ corresponds to $B \to A \leftarrow A'$. Set $B' = B \times_A A'$; this is the global sections of $\mathcal{O}_{Y'}$. As $A' \to A$ is surjective with locally nilpotent kernel we see that $B' \to B$ is surjective with locally nilpotent kernel. Hence $\mathrm{Spec}(B') = \mathrm{Spec}(B)$ (as topological spaces). We claim that $Y' = \mathrm{Spec}(B')$. To see this we will show for $g' \in B'$ with image $g \in B$ that $\mathcal{O}_{Y'}(D(g)) = B'_{g'}$. Namely, by More on Algebra, Lemma 15.4.3 we see that

$$(B')_{g'} = B_g \times_{A_h} A'_{h'}$$

where $h \in A$, $h' \in A'$ are the images of g'. Since B_g, resp. A_h, resp. $A'_{h'}$ is equal to $\mathcal{O}_Y(D(g))$, resp. $f_*\mathcal{O}_X(D(g))$, resp. $f'_*\mathcal{O}_{X'}(D(g))$ the claim follows.

Finally, we prove the universal property of the pushout holds for Y' and the morphisms $Y \to Y'$ and $X' \to Y'$. Namely, let S be a scheme and let $b : Y \to S$ and $a' : X' \to S$ be morphisms such that

$$\begin{array}{ccc} X & \longrightarrow & X' \\ \downarrow & & \downarrow{\scriptstyle a'} \\ Y & \xrightarrow{b} & S \end{array}$$

commutes. Note that $a' = b \circ f'$ on underlying topological spaces. Denote also $(a')^\sharp : b^{-1}\mathcal{O}_S \to f'_*\mathcal{O}_{X'}$ the map which is adjoint to $(a')^\sharp : (a')^{-1}\mathcal{O}_S = (f')^{-1}b^{-1}\mathcal{O}_S \to \mathcal{O}_{X'}$. Then we get a map

$$b^{-1}\mathcal{O}_S \xrightarrow{(b^\sharp, (a')^\sharp)} \mathcal{O}_Y \times_{f_*\mathcal{O}_X} f'_*\mathcal{O}_{X'} = \mathcal{O}_{Y'}$$

which defines a morphism of ringed spaces $b' : Y' \to S$ compatible with a' and b. Since $Y \subset Y'$ is a thickening it follows that b' is a morphism of locally ringed spaces, i.e., a morphism of schemes. This finishes the proof. $\square$

In the following lemma we use the fibre product of categories as defined in Categories, Example 4.29.3.

Lemma 36.11.2. *Let $X \to X'$ be a thickening of schemes and let $X \to Y$ be an affine morphism of schemes. Let $Y' = Y \amalg_X X'$ be the pushout (see Lemma 36.11.1). Base change gives a functor*

$$F : (Sch/Y') \longrightarrow (Sch/Y) \times_{(Sch/Y')} (Sch/X')$$

given by $V' \longmapsto (V' \times_{Y'} Y, V' \times_{Y'} X', 1)$ which has a left adjoint

$$G : (Sch/Y) \times_{(Sch/Y')} (Sch/X') \longrightarrow (Sch/Y')$$

which sends the triple (V, U', φ) to the pushout $V \amalg_{(V \times_Y X)} U'$. Finally, $F \circ G$ is isomorphic to the identity functor.

Proof. Let (V, U', φ) be an object of the fibre product category. Set $U = U' \times_{X'} X$. Note that $U \to U'$ is a thickening. Since $\varphi : V \times_Y X \to U' \times_{X'} X = U$ is an isomorphism we have a morphism $U \to V$ over $X \to Y$ which identifies U with the fibre product $X \times_Y V$. In particular $U \to V$ is affine, see Morphisms, Lemma 28.13.8. Hence we can apply Lemma 36.11.1 to get a pushout $V' = V \amalg_U U'$. Denote $V' \to Y'$ the morphism we obtain in virtue of the fact that V' is a pushout and because we are given morphisms $V \to Y$ and $U' \to X'$ agreeing on U as morphisms into Y'. Setting $G(V, U', \varphi) = V'$ gives the functor G.

Let us prove that G is a left adjoint to F. Let Z be a scheme over Y'. We have to show that

$$\mathrm{Mor}(V', Z) = \mathrm{Mor}((V, U', \varphi), F(Z))$$

where the morphism sets are taking in their respective categories. Let $g' : V' \to Z$ be a morphism. Denote $\tilde{g}$, resp. $\tilde{f}'$ the composition of g' with the morphism $V \to V'$, resp. $U' \to V'$. Base change $\tilde{g}$, resp. $\tilde{f}'$ by $Y \to Y'$, resp. $X' \to Y'$ to get a morphism $g : V \to Z \times_{Y'} Y$, resp. $f' : U' \to Z \times_{Y'} X'$. Then (g, f') is an element of the right hand side of the equation above (details omitted). Conversely, suppose that $(g, f') : (V, U', \varphi) \to F(Z)$ is an element of the right hand side. We may consider the composition $\tilde{g} : V \to Z$, resp. $\tilde{f}' : U' \to Z$ of g, resp. f by $Z \times_{Y'} X' \to Z$, resp. $Z \times_{Y'} Y \to Z$. Then $\tilde{g}$ and $\tilde{f}'$ agree as morphism from U to Z. By the universal property of pushout, we obtain a morphism $g' : V' \to Z$, i.e., an element of the left hand side. We omit the verification that these constructions are mutually inverse.

To prove that $F \circ G$ is isomorphic to the identity we have to show that the adjunction mapping $(V, U', \varphi) \to F(G(V, U', \varphi))$ is an isomorphism. To do this we may work affine locally. Say $X = \mathrm{Spec}(A)$, $X' = \mathrm{Spec}(A')$, and $Y = \mathrm{Spec}(B)$. Then $A' \to A$ and $B \to A$ are ring maps as in More on Algebra, Lemma 15.4.5 and $Y' = \mathrm{Spec}(B')$ with $B' = B \times_A A'$. Next, suppose that $V = \mathrm{Spec}(D)$, $U' = \mathrm{Spec}(C')$ and φ is given by an A-algebra isomorphism $D \otimes_B A \to C' \otimes_{A'} A = C'/IC'$. Set $D' = D \times_{C'/IC'} C'$. In this case the statement we have to prove is that $D' \otimes_{B'} B \cong D$ and $D' \otimes_{B'} A' \cong C'$. This is a special case of More on Algebra, Lemma 15.4.5. $\square$

Lemma 36.11.3. *Let $X \to X'$ be a thickening of schemes and let $X \to Y$ be an affine morphism of schemes. Let $Y' = Y \amalg_X X'$ be the pushout (see Lemma 36.11.1). Let $V' \to Y'$ be a morphism of schemes. Set $V = Y \times_{Y'} V'$, $U' = X' \times_{Y'} V'$, and $U = X \times_{Y'} V'$. There is an equivalence of categories between*

(1) *quasi-coherent $\mathcal{O}_{V'}$-modules flat over Y', and*

(2) *the category of triples* $(\mathcal{G}, \mathcal{F}', \varphi)$ *where*
   (a) $\mathcal{G}$ *is a quasi-coherent* $\mathcal{O}_V$*-module flat over* Y*,*
   (b) $\mathcal{F}'$ *is a quasi-coherent* $\mathcal{O}_{U'}$*-module flat over* X*, and*
   (c) $\varphi : (U \to V)^*\mathcal{G} \to (U \to U')^*\mathcal{F}'$ *is an isomorphism of* $\mathcal{O}_U$*-modules.*

The equivalence maps $\mathcal{G}'$ *to* $((V \to V')^*\mathcal{G}', (U' \to V')^*\mathcal{G}', can)$*. Suppose* $\mathcal{G}'$ *corresponds to the triple* $(\mathcal{G}, \mathcal{F}', \varphi)$*. Then*

(a) $\mathcal{G}'$ *is a finite type* $\mathcal{O}_{V'}$*-module if and only if* $\mathcal{G}$ *and* $\mathcal{F}'$ *are finite type* $\mathcal{O}_Y$ *and* $\mathcal{O}_{U'}$*-modules.*
(b) *if* $V' \to Y'$ *is locally of finite presentation, then* $\mathcal{G}'$ *is an* $\mathcal{O}_{V'}$*-module of finite presentation if and only if* $\mathcal{G}$ *and* $\mathcal{F}'$ *are* $\mathcal{O}_Y$ *and* $\mathcal{O}_{U'}$*-modules of finite presentation.*

Proof. A quasi-inverse functor assigns to the triple $(\mathcal{G}, \mathcal{F}', \varphi)$ the fibre product

$$(V \to V')_*\mathcal{G} \times_{(U \to V')_*\mathcal{F}} (U' \to V')_*\mathcal{F}'$$

where $\mathcal{F} = (U \to U')^*\mathcal{F}'$. This works, because on affines we recover the equivalence of More on Algebra, Lemma 15.4.12. Some details omitted.

Parts (a) and (b) follow from More on Algebra, Lemmas 15.4.11 and 15.4.13. □

Lemma 36.11.4. *In the situation of Lemma 36.11.2. If* $V' = G(V, U', \varphi)$ *for some triple* (V, U', φ)*, then*

(1) $V' \to Y'$ *is locally of finite type if and only if* $V \to Y$ *and* $U' \to X'$ *are locally of finite type,*
(2) $V' \to Y'$ *is flat if and only if* $V \to Y$ *and* $U' \to X'$ *are flat,*
(3) $V' \to Y'$ *is flat and locally of finite presentation if and only if* $V \to Y$ *and* $U' \to X'$ *are flat and locally of finite presentation,*
(4) $V' \to Y'$ *is smooth if and only if* $V \to Y$ *and* $U' \to X'$ *are smooth,*
(5) $V' \to Y'$ *is étale if and only if* $V \to Y$ *and* $U' \to X'$ *are étale, and*
(6) *add more here as needed.*

If W' *is flat over* Y'*, then the adjunction mapping* $G(F(W')) \to W'$ *is an isomorphism. Hence* F *and* G *define mutually quasi-inverse functors between the category of schemes flat over* Y' *and the category of triples* (V, U', φ) *with* $V \to Y$ *and* $U' \to X'$ *flat.*

Proof. Looking over affine pieces the assertions of this lemma are equivalent to the corresponding assertions of More on Algebra, Lemma 15.4.14. □

36.12. Openness of the flat locus

This result takes some work to prove, and (perhaps) deserves its own section. Here it is.

Theorem 36.12.1. *Let* S *be a scheme. Let* $f : X \to S$ *be a morphism which is locally of finite presentation. Let* $\mathcal{F}$ *be a quasi-coherent* $\mathcal{O}_X$*-module which is locally of finite presentation. Then*

$$U = \{x \in X \mid \mathcal{F} \text{ is flat over } S \text{ at } x\}$$

is open in X*.*

Proof. We may test for openness locally on X hence we may assume that f is a morphism of affine schemes. In this case the theorem is exactly Algebra, Theorem 10.123.4. □

Lemma 36.12.2. *Let S be a scheme. Let*

$$\begin{array}{ccc} X' & \xrightarrow{g'} & X \\ {\scriptstyle f'}\downarrow & & \downarrow{\scriptstyle f} \\ S' & \xrightarrow{g} & S \end{array}$$

be a cartesian diagram of schemes. Let $\mathcal{F}$ be a quasi-coherent $\mathcal{O}_X$-module. Let $x' \in X'$ with images $x = g'(x')$ and $s' = g'(x')$.

1. *If $\mathcal{F}$ is flat over S at x, then $(g')^*\mathcal{F}$ is flat over S' at x'.*
2. *If g is flat at s' and $(g')^*\mathcal{F}$ is flat over S' at x', then $\mathcal{F}$ is flat over S at x.*

In particular, if g is flat, f is locally of finite presentation, and $\mathcal{F}$ is locally of finite presentation, then formation of the open subset of Theorem 36.12.1 commutes with base change.

Proof. Consider the commutative diagram of local rings

$$\begin{array}{ccc} \mathcal{O}_{X',x'} & \longleftarrow & \mathcal{O}_{X,x} \\ \uparrow & & \uparrow \\ \mathcal{O}_{S',s'} & \longleftarrow & \mathcal{O}_{S,s} \end{array}$$

Note that $\mathcal{O}_{X',x'}$ is a localization of $\mathcal{O}_{X,x} \otimes_{\mathcal{O}_{S,s}} \mathcal{O}_{S',s'}$, and that $((g')^*\mathcal{F})_{x'}$ is equal to $\mathcal{F}_x \otimes_{\mathcal{O}_{X,x}} \mathcal{O}_{X',x'}$. Hence the lemma follows from Algebra, Lemma 10.94.1. □

36.13. Critère de platitude par fibres

Consider a commutative diagram of schemes (left hand diagram)

$$\begin{array}{ccc} X & \xrightarrow{f} & Y \\ & \searrow \quad \swarrow & \\ & S & \end{array} \qquad \begin{array}{ccc} X_s & \xrightarrow{f_s} & Y_s \\ & \searrow \quad \swarrow & \\ & \mathrm{Spec}(\kappa(s)) & \end{array}$$

and a quasi-coherent $\mathcal{O}_X$-module $\mathcal{F}$. Given a point $x \in X$ lying over $s \in S$ with image $y = f(x)$ we consider the question: Is $\mathcal{F}$ flat over Y at x? If $\mathcal{F}$ is flat over S at x, then the theorem states this question is intimately related to the question of whether the restriction of $\mathcal{F}$ to the fibre

$$\mathcal{F}_s = (X_s \to X)^*\mathcal{F}$$

is flat over Y_s at x. Below you will find a "Noetherian" version, a "finitely presented" version, and earlier we treated a "nilpotent" version, see Lemma 36.8.2.

Theorem 36.13.1. *Let S be a scheme. Let $f : X \to Y$ be a morphism of schemes over S. Let $\mathcal{F}$ be a quasi-coherent $\mathcal{O}_X$-module. Let $x \in X$. Set $y = f(x)$ and $s \in S$ the image of x in S. Assume S, X, Y locally Noetherian, $\mathcal{F}$ coherent, and $\mathcal{F}_x \neq 0$. Then the following are equivalent:*

1. *$\mathcal{F}$ is flat over S at x, and $\mathcal{F}_s$ is flat over Y_s at x, and*
2. *Y is flat over S at y and $\mathcal{F}$ is flat over Y at x.*

Proof. Consider the ring maps

$$\mathcal{O}_{S,s} \longrightarrow \mathcal{O}_{Y,y} \longrightarrow \mathcal{O}_{X,x}$$

and the module $\mathcal{F}_x$. The stalk of $\mathcal{F}_s$ at x is the module $\mathcal{F}_x/\mathfrak{m}_s\mathcal{F}_x$ and the local ring of Y_s at y is $\mathcal{O}_{Y,y}/\mathfrak{m}_s\mathcal{O}_{Y,y}$. Thus the implication (1) $\Rightarrow$ (2) is Algebra, Lemma 10.93.14. If (2) holds, then the first ring map is faithfully flat and $\mathcal{F}_x$ is flat over $\mathcal{O}_{Y,y}$ so by Algebra, Lemma 10.36.3 we see that $\mathcal{F}_x$ is flat over $\mathcal{O}_{S,s}$. Moreover, $\mathcal{F}_x/\mathfrak{m}_s\mathcal{F}_x$ is the base change of the flat module $\mathcal{F}_x$ by $\mathcal{O}_{Y,y} \to \mathcal{O}_{Y,y}/\mathfrak{m}_s\mathcal{O}_{Y,y}$, hence flat by Algebra, Lemma 10.36.6. □

Here is the non-Noetherian version.

Theorem 36.13.2. *Let S be a scheme. Let $f : X \to Y$ be a morphism of schemes over S. Let $\mathcal{F}$ be a quasi-coherent $\mathcal{O}_X$-module. Assume*

(1) *X is locally of finite presentation over S,*
(2) *$\mathcal{F}$ an $\mathcal{O}_X$-module of finite presentation, and*
(3) *Y is locally of finite type over S.*

Let $x \in X$. Set $y = f(x)$ and let $s \in S$ be the image of x in S. If $\mathcal{F}_x \neq 0$, then the following are equivalent:

(1) *$\mathcal{F}$ is flat over S at x, and $\mathcal{F}_s$ is flat over Y_s at x, and*
(2) *Y is flat over S at y and $\mathcal{F}$ is flat over Y at x.*

Moreover, the set of points x where (1) and (2) hold is open in Supp($\mathcal{F}$).

Proof. Consider the ring maps

$$\mathcal{O}_{S,s} \longrightarrow \mathcal{O}_{Y,y} \longrightarrow \mathcal{O}_{X,x}$$

and the module $\mathcal{F}_x$. The stalk of $\mathcal{F}_s$ at x is the module $\mathcal{F}_x/\mathfrak{m}_s\mathcal{F}_x$ and the local ring of Y_s at y is $\mathcal{O}_{Y,y}/\mathfrak{m}_s\mathcal{O}_{Y,y}$. Thus the implication (1) $\Rightarrow$ (2) is Algebra, Lemma 10.122.9. If (2) holds, then the first ring map is faithfully flat and $\mathcal{F}_x$ is flat over $\mathcal{O}_{Y,y}$ so by Algebra, Lemma 10.36.3 we see that $\mathcal{F}_x$ is flat over $\mathcal{O}_{S,s}$. Moreover, $\mathcal{F}_x/\mathfrak{m}_s\mathcal{F}_x$ is the base change of the flat module $\mathcal{F}_x$ by $\mathcal{O}_{Y,y} \to \mathcal{O}_{Y,y}/\mathfrak{m}_s\mathcal{O}_{Y,y}$, hence flat by Algebra, Lemma 10.36.6.

By Morphisms, Lemma 28.22.11 the morphism f is locally of finite presentation. Consider the set

$$U = \{x \in X \mid \mathcal{F} \text{ flat at } x \text{ over both } Y \text{ and } S\}. \tag{36.13.2.1}$$

This set is open in X by Theorem 36.12.1. Note that if $x \in U$, then $\mathcal{F}_s$ is flat at x over Y_s as a base change of a flat module under the morphism $Y_s \to Y$, see Morphisms, Lemma 28.26.6. Hence at every point of $U \cap \text{Supp}(\mathcal{F})$ condition (1) is satisfied. On the other hand, it is clear that if $x \in \text{Supp}(\mathcal{F})$ satisfies (1) and (2), then $x \in U$. Thus the open set we are looking for is $U \cap \text{Supp}(\mathcal{F})$. □

These theorems are often used in the following simplified forms. We give only the global statements – of course there are also pointwise versions.

Lemma 36.13.3. *Let S be a scheme. Let $f : X \to Y$ be a morphism of schemes over S. Assume*

(1) *S, X, Y are locally Noetherian,*
(2) *X is flat over S,*
(3) *for every $s \in S$ the morphism $f_s : X_s \to Y_s$ is flat.*

Then f is flat. If f is also surjective, then Y is flat over S.

Proof. This is a special case of Theorem 36.13.1. □

Lemma 36.13.4. *Let S be a scheme. Let $f : X \to Y$ be a morphism of schemes over S. Assume*

(1) *X is locally of finite presentation over S,*
(2) *X is flat over S,*
(3) *for every $s \in S$ the morphism $f_s : X_s \to Y_s$ is flat, and*
(4) *Y is locally of finite type over S.*

Then f is flat. If f is also surjective, then Y is flat over S.

Proof. This is a special case of Theorem 36.13.2. □

Lemma 36.13.5. *Let S be a scheme. Let $f : X \to Y$ be a morphism of schemes over S. Let $\mathcal{F}$ be a quasi-coherent $\mathcal{O}_X$-module. Assume*

(1) *X is locally of finite presentation over S,*
(2) *$\mathcal{F}$ an $\mathcal{O}_X$-module of finite presentation,*
(3) *$\mathcal{F}$ is flat over S, and*
(4) *Y is locally of finite type over S.*

Then the set

$$U = \{x \in X \mid \mathcal{F} \text{ flat at } x \text{ over } Y\}.$$

is open in X and its formation commutes with arbitrary base change: If $S' \to S$ is a morphism of schemes, and U' is the set of points of $X' = X \times_S S'$ where $\mathcal{F}' = \mathcal{F} \times_S S'$ is flat over $Y' = Y \times_S S'$, then $U' = U \times_S S'$.

Proof. By Morphisms, Lemma 28.22.11 the morphism f is locally of finite presentation. Hence U is open by Theorem 36.12.1. Because we have assumed that $\mathcal{F}$ is flat over S we see that Theorem 36.13.2 implies

$$U = \{x \in X \mid \mathcal{F}_s \text{ flat at } x \text{ over } Y_s\}.$$

where s always denotes the image of x in S. (This description also works trivially when $\mathcal{F}_x = 0$.) Moreover, the assumptions of the lemma remain in force for the morphism $f' : X' \to Y'$ and the sheaf $\mathcal{F}'$. Hence U' has a similar description. In other words, it suffices to prove that given $s' \in S'$ mapping to $s \in S$ we have

$$\{x' \in X'_{s'} \mid \mathcal{F}'_{s'} \text{ flat at } x' \text{ over } Y'_{s'}\}$$

is the inverse image of the corresponding locus in X_s. This is true by Lemma 36.12.2 because in the cartesian diagram

$$\begin{array}{ccc} X'_{s'} & \longrightarrow & X_s \\ \downarrow & & \downarrow \\ Y'_{s'} & \longrightarrow & Y_s \end{array}$$

the horizontal morphisms are flat as they are base changes by the flat morphism $\mathrm{Spec}(\kappa(s')) \to \mathrm{Spec}(\kappa(s))$. □

Lemma 36.13.6. *Let S be a scheme. Let $f : X \to Y$ be a morphism of schemes over S. Assume*

(1) *X is locally of finite presentation over S,*
(2) *X is flat over S, and*

(3) Y *is locally of finite type over* S.

Then the set

$$U = \{x \in X \mid X \text{ flat at } x \text{ over } Y\}.$$

is open in X *and its formation commutes with arbitrary base change.*

Proof. This is a special case of Lemma 36.13.5. □

The following lemma is a variant of Algebra, Lemma 10.93.4. Note that the hypothesis that $(\mathcal{F}_s)_x$ is a flat $\mathcal{O}_{X_s,x}$-module means that $(\mathcal{F}_s)_x$ is a free $\mathcal{O}_{X_s,x}$-module which is always the case if $x \in X_s$ is a generic point of an irreducible component of X_s and X_s is reduced (namely, in this case $\mathcal{O}_{X_s,x}$ is a field, see Algebra, Lemma 10.24.4).

Lemma 36.13.7. *Let* $f : X \to S$ *be a morphism of schemes of finite presentation. Let* $\mathcal{F}$ *be a finitely presented* $\mathcal{O}_X$*-module. Let* $x \in X$ *with image* $s \in S$*. If* $\mathcal{F}$ *is flat at* x *over* S *and* $(\mathcal{F}_s)_x$ *is a flat* $\mathcal{O}_{X_s,x}$*-module, then* $\mathcal{F}$ *is finite free in a neighbourhood of* x*.*

Proof. If $\mathcal{F}_x \otimes \kappa(x)$ is zero, then $\mathcal{F}_x = 0$ by Nakayama's lemma (Algebra, Lemma 10.18.1) and hence $\mathcal{F}$ is zero in a neighbourhood of x (Modules, Lemma 17.9.5) and the lemma holds. Thus we may assume $\mathcal{F}_x \otimes \kappa(x)$ is not zero and we see that Theorem 36.13.2 applies with $f = \mathrm{id} : X \to X$. We conclude that $\mathcal{F}_x$ is flat over $\mathcal{O}_{X,x}$. Hence $\mathcal{F}_x$ is free, see Algebra, Lemma 10.73.4 for example. Choose an open neighbourhood $x \in U \subset X$ and sections $s_1, \ldots, s_r \in \mathcal{F}(U)$ which map to a basis in $\mathcal{F}_x$. The corresponding map $\psi : \mathcal{O}_U^{\oplus r} \to \mathcal{F}|_U$ is surjective after shrinking U (Modules, Lemma 17.9.5). Then $\mathrm{Ker}(\psi)$ is of finite type (see Modules, Lemma 17.11.3) and $\mathrm{Ker}(\psi)_x = 0$. Whence after shrinking U once more ψ is an isomorphism. □

36.14. Normalization revisited

Normalization commutes with smooth base change.

Lemma 36.14.1. *Let* $f : Y \to X$ *be a smooth morphism of schemes. Let* $\mathcal{A}$ *be a quasi-coherent sheaf of* $\mathcal{O}_X$*-algebras. The integral closure of* $\mathcal{O}_Y$ *in* $f^*\mathcal{A}$ *is equal to* $f^*\mathcal{A}'$ *where* $\mathcal{A}' \subset \mathcal{A}$ *is the integral closure of* $\mathcal{O}_X$ *in* $\mathcal{A}$*.*

Proof. This is a translation of Algebra, Lemma 10.138.4 into the language of schemes. Details omitted. □

Lemma 36.14.2 (Normalization commutes with smooth base change)**.** *Let*

$$\begin{array}{ccc} Y_2 & \longrightarrow & Y_1 \\ {\scriptstyle f_2}\downarrow & & \downarrow{\scriptstyle f_1} \\ X_2 & \xrightarrow{\varphi} & X_1 \end{array}$$

be a fibre square in the category of schemes. Assume f_1 *is quasi-compact and quasi-separated, and* φ *is smooth. Let* $Y_i \to X_i' \to X_i$ *be the normalization of* X_i *in* Y_i*. Then* $X_2' \cong X_2 \times_{X_1} X_1'$*.*

Proof. The base change of the factorization $Y_1 \to X_1' \to X_1$ to X_2 is a factorization $Y_2 \to X_2 \times_{X_1} X_1' \to X_1$ and $X_2 \times_{X_1} X_1' \to X_1$ is integral (Morphisms, Lemma 28.44.6). Hence we get a morphism $h : X_2' \to X_2 \times_{X_1} X_1'$ by the universal property of Morphisms, Lemma 28.48.4. Observe that X_2' is the relative spectrum of the integral closure of $\mathcal{O}_{X_2}$ in $f_{2,*}\mathcal{O}_{Y_2}$. If $\mathcal{A}' \subset f_{1,*}\mathcal{O}_{Y_1}$ denotes the integral closure of $\mathcal{O}_{X_2}$, then $X_2 \times_{X_1} X_1'$ is the relative spectrum of $\varphi^*\mathcal{A}'$, see Constructions, Lemma 26.4.6. By Cohomology of Schemes, Lemma 29.5.2 we know that $f_{2,*}\mathcal{O}_{Y_2} = \varphi^* f_{1,*}\mathcal{O}_{Y_1}$. Hence the result follows from Lemma 36.14.1. □

Lemma 36.14.3 (Normalization and smooth morphisms)**.** *Let $X \to Y$ be a smooth morphism of schemes. Assume every quasi-compact open of Y has finitely many irreducible components. Then the same is true for X and there is a canonical isomorphism $X^\nu = X \times_Y Y^\nu$.*

Proof. As $X \to Y$ is flat we see that generic points of irreducible components of X map to generic points of irreducible components of Y, see Morphisms, Lemma 28.26.8. On the other hand, the fibres of $X \to Y$ are locally Noetherian because they are locally of finite type over a field. Thus every quasi-compact open of X has finitely many irreducible components. Note that $X_{red} = X \times_Y Y_{red}$ as a scheme smooth over a reduced scheme is reduced, see Descent, Lemma 34.14.1. Hence we may assume that X and Y are reduced (as the normalization of a scheme is equal to the normalization of its reduction by definition). Next, note that $X' = X \times_Y Y^\nu$ is a normal scheme by Descent, Lemma 34.14.2. The morphism $X' \to Y^\nu$ is smooth (hence flat) thus the generic points of irreducible components of X' lie over generic points of irreducible components of Y^ν. Since $Y^\nu \to Y$ is birational we conclude that $X' \to X$ is birational too (because $X' \to Y^\nu$ induces an isomorphism on fibres over generic points of Y). We conclude that there exists a factorization $X^\nu \to X' \to X$, see Morphisms, Lemma 28.48.15 which is an isomorphism as X' is normal and integral over X. □

36.15. Normal morphisms

In the article [**DM69**] of Deligne and Mumford the notion of a normal morphism is mentioned. This is just one in a series of types[1] of morphisms that can all be defined similarly. Over time we will add these in their own sections as needed.

Definition 36.15.1. Let $f : X \to Y$ be a morphism of schemes. Assume that all the fibres X_y are locally Noetherian schemes.

(1) Let $x \in X$, and $y = f(x)$. We say that f is *normal at x* if f is flat at x, and the scheme X_y is geometrically normal at x over $\kappa(y)$ (see Varieties, Definition 32.8.1).
(2) We say f is a *normal morphism* if f is normal at every point of X.

So the condition that the morphism $X \to Y$ is normal is stronger than just requiring all the fibres to be normal locally Noetherian schemes.

Lemma 36.15.2. *Let $f : X \to Y$ be a morphism of schemes. Assume all fibres of f are locally Noetherian. The following are equivalent*

(1) *f is normal, and*

[1] The other types are coprof $\leq k$, Cohen-Macaulay, (S_k), regular, (R_k), and reduced. See [**DG67**, IV Definition 6.8.1.].

(2) *f is flat and its fibres are geometrically normal schemes.*

Proof. This follows directly from the definitions. □

Lemma 36.15.3. *A smooth morphism is normal.*

Proof. Let $f : X \to Y$ be a smooth morphism. As f is locally of finite presentation, see Morphisms, Lemma 28.35.8 the fibres X_y are locally of finite type over a field, hence locally Noetherian. Moreover, f is flat, see Morphisms, Lemma 28.35.9. Finally, the fibres X_y are smooth over a field (by Morphisms, Lemma 28.35.5) and hence geometrically normal by Varieties, Lemma 32.15.4. Thus f is normal by Lemma 36.15.2. □

We want to show that this notion is local on the source and target for the smooth topology. First we deal with the property of having locally Noetherian fibres.

Lemma 36.15.4. *The property $\mathcal{P}(f) =$ "the fibres of f are locally Noetherian" is local in the fppf topology on the source and the target.*

Proof. Let $f : X \to Y$ be a morphism of schemes. Let $\{\varphi_i : Y_i \to Y\}_{i \in I}$ be an fppf covering of Y. Denote $f_i : X_i \to Y_i$ the base change of f by φ_i. Let $i \in I$ and let $y_i \in Y_i$ be a point. Set $y = \varphi_i(y_i)$. Note that

$$X_{i,y_i} = \mathrm{Spec}(\kappa(y_i)) \times_{\mathrm{Spec}(\kappa(y))} X_y.$$

Moreover, as φ_i is of finite presentation the field extension $\kappa(y) \subset \kappa(y_i)$ is finitely generated. Hence in this situation we have that X_y is locally Noetherian if and only if X_{i,y_i} is locally Noetherian, see Varieties, Lemma 32.9.1. This fact implies locality on the target.

Let $\{X_i \to X\}$ be an fppf covering of X. Let $y \in Y$. In this case $\{X_{i,y} \to X_y\}$ is an fppf covering of the fibre. Hence the locality on the source follows from Descent, Lemma 34.12.1. □

Lemma 36.15.5. *The property $\mathcal{P}(f) =$ "the fibres of f are locally Noetherian and f is normal" is local in the fppf topology on the target and local in the smooth topology on the source.*

Proof. We have $\mathcal{P}(f) = \mathcal{P}_1(f) \wedge \mathcal{P}_2(f) \wedge \mathcal{P}_3(f)$ where $\mathcal{P}_1(f) =$"the fibres of f are locally Noetherian", $\mathcal{P}_2(f) =$"f is flat", and $\mathcal{P}_3(f) =$"the fibres of f are geometrically normal". We have already seen that $\mathcal{P}_1$ and $\mathcal{P}_2$ are local in the fppf topology on the source and the target, see Lemma 36.15.4, and Descent, Lemmas 34.19.13 and 34.23.1. Thus we have to deal with $\mathcal{P}_3$.

Let $f : X \to Y$ be a morphism of schemes. Let $\{\varphi_i : Y_i \to Y\}_{i \in I}$ be an fpqc covering of Y. Denote $f_i : X_i \to Y_i$ the base change of f by φ_i. Let $i \in I$ and let $y_i \in Y_i$ be a point. Set $y = \varphi_i(y_i)$. Note that

$$X_{i,y_i} = \mathrm{Spec}(\kappa(y_i)) \times_{\mathrm{Spec}(\kappa(y))} X_y.$$

Hence in this situation we have that X_y is geometrically normal if and only if X_{i,y_i} is geometrically normal, see Varieties, Lemma 32.8.4. This fact implies $\mathcal{P}_3$ is fpqc local on the target.

Let $\{X_i \to X\}$ be a smooth covering of X. Let $y \in Y$. In this case $\{X_{i,y} \to X_y\}$ is a smooth covering of the fibre. Hence the locality of $\mathcal{P}_3$ for the smooth topology on the source follows from Descent, Lemma 34.14.2. Combining the above the lemma follows. □

36.16. Regular morphisms

Compare with Section 36.15. The algebraic version of this notion is discussed in More on Algebra, Section 15.29.

Definition 36.16.1. Let $f : X \to Y$ be a morphism of schemes. Assume that all the fibres X_y are locally Noetherian schemes.

(1) Let $x \in X$, and $y = f(x)$. We say that f is *regular at* x if f is flat at x, and the scheme X_y is geometrically regular at x over $\kappa(y)$ (see Varieties, Definition 32.10.1).
(2) We say f is a *regular morphism* if f is regular at every point of X.

The condition that the morphism $X \to Y$ is regular is stronger than just requiring all the fibres to be regular locally Noetherian schemes.

Lemma 36.16.2. *Let $f : X \to Y$ be a morphism of schemes. Assume all fibres of f are locally Noetherian. The following are equivalent*

(1) *f is regular,*
(2) *f is flat and its fibres are geometrically regular schemes,*
(3) *for every pair of affine opens $U \subset X$, $V \subset Y$ with $f(U) \subset V$ the ring map $\mathcal{O}(V) \to \mathcal{O}(U)$ is regular,*
(4) *there exists an open covering $Y = \bigcup_{j \in J} V_j$ and open coverings $f^{-1}(V_j) = \bigcup_{i \in I_j} U_i$ such that each of the morphisms $U_i \to V_j$ is regular, and*
(5) *there exists an affine open covering $Y = \bigcup_{j \in J} V_j$ and affine open coverings $f^{-1}(V_j) = \bigcup_{i \in I_j} U_i$ such that the ring maps $\mathcal{O}(V_j) \to \mathcal{O}(U_i)$ are regular.*

Proof. The equivalence of (1) and (2) is immediate from the definitions. Let $x \in X$ with $y = f(x)$. By definition f is flat at x if and only if $\mathcal{O}_{Y,y} \to \mathcal{O}_{X,x}$ is a flat ring map, and X_y is geometrically regular at x over $\kappa(y)$ if and only if $\mathcal{O}_{X_y,x} = \mathcal{O}_{X,x}/\mathfrak{m}_y\mathcal{O}_{X,x}$ is a geometrically regular algebra over $\kappa(y)$. Hence Whether or not f is regular at x depends only on the local homomorphism of local rings $\mathcal{O}_{Y,y} \to \mathcal{O}_{X,x}$. Thus the equivalence of (1) and (4) is clear.

Recall (More on Algebra, Definition 15.29.1) that a ring map $A \to B$ is regular if and only if it is flat and the fibre rings $B \otimes_A \kappa(\mathfrak{p})$ are Noetherian and geometrically regular for all primes $\mathfrak{p} \subset A$. By Varieties, Lemma 32.10.3 this is equivalent to $\mathrm{Spec}(B \otimes_A \kappa(\mathfrak{p}))$ being a geometrically regular scheme over $\kappa(\mathfrak{p})$. Thus we see that (2) implies (3). It is clear that (3) implies (5). Finally, assume (5). This implies that f is flat (see Morphisms, Lemma 28.26.3). Moreover, if $y \in Y$, then $y \in V_j$ for some j and we see that $X_y = \bigcup_{i \in I_j} U_{i,y}$ with each $U_{i,y}$ geometrically regular over $\kappa(y)$ by Varieties, Lemma 32.10.3. Another application of Varieties, Lemma 32.10.3 shows that X_y is geometrically regular. Hence (2) holds and the proof of the lemma is finished. □

Lemma 36.16.3. *A smooth morphism is regular.*

Proof. Let $f : X \to Y$ be a smooth morphism. As f is locally of finite presentation, see Morphisms, Lemma 28.35.8 the fibres X_y are locally of finite type over a field, hence locally Noetherian. Moreover, f is flat, see Morphisms, Lemma 28.35.9. Finally, the fibres X_y are smooth over a field (by Morphisms, Lemma 28.35.5) and hence geometrically regular by Varieties, Lemma 32.15.4. Thus f is regular by Lemma 36.16.2. □

Lemma 36.16.4. *The property $\mathcal{P}(f) =$ "the fibres of f are locally Noetherian and f is regular" is local in the fppf topology on the target and local in the smooth topology on the source.*

Proof. We have $\mathcal{P}(f) = \mathcal{P}_1(f) \wedge \mathcal{P}_2(f) \wedge \mathcal{P}_3(f)$ where $\mathcal{P}_1(f) =$"the fibres of f are locally Noetherian", $\mathcal{P}_2(f) =$"f is flat", and $\mathcal{P}_3(f) =$"the fibres of f are geometrically regular". We have already seen that $\mathcal{P}_1$ and $\mathcal{P}_2$ are local in the fppf topology on the source and the target, see Lemma 36.15.4, and Descent, Lemmas 34.19.13 and 34.23.1. Thus we have to deal with $\mathcal{P}_3$.

Let $f : X \to Y$ be a morphism of schemes. Let $\{\varphi_i : Y_i \to Y\}_{i \in I}$ be an fpqc covering of Y. Denote $f_i : X_i \to Y_i$ the base change of f by φ_i. Let $i \in I$ and let $y_i \in Y_i$ be a point. Set $y = \varphi_i(y_i)$. Note that

$$X_{i,y_i} = \operatorname{Spec}(\kappa(y_i)) \times_{\operatorname{Spec}(\kappa(y))} X_y.$$

Hence in this situation we have that X_y is geometrically regular if and only if X_{i,y_i} is geometrically regular, see Varieties, Lemma 32.10.4. This fact implies $\mathcal{P}_3$ is fpqc local on the target.

Let $\{X_i \to X\}$ be a smooth covering of X. Let $y \in Y$. In this case $\{X_{i,y} \to X_y\}$ is a smooth covering of the fibre. Hence the locality of $\mathcal{P}_3$ for the smooth topology on the source follows from Descent, Lemma 34.14.4. Combining the above the lemma follows. □

36.17. Cohen-Macaulay morphisms

Compare with Section 36.15. Note that, as pointed out in Algebra, Section 10.153 and Varieties, Section 32.11 "geometrically Cohen-Macaulay" is the same as plain Cohen-Macaulay.

Definition 36.17.1. Let $f : X \to Y$ be a morphism of schemes. Assume that all the fibres X_y are locally Noetherian schemes.

(1) Let $x \in X$, and $y = f(x)$. We say that f is *Cohen-Macaulay at x* if f is flat at x, and the local ring of the scheme X_y at x is Cohen-Macaulay.
(2) We say f is a *Cohen-Macaulay morphism* if f is Cohen-Macaulay at every point of X.

Here is a translation.

Lemma 36.17.2. *Let $f : X \to Y$ be a morphism of schemes. Assume all fibres of f are locally Noetherian. The following are equivalent*

(1) *f is Cohen-Macaulay, and*
(2) *f is flat and its fibres are Cohen-Macaulay schemes.*

Proof. This follows directly from the definitions. □

Lemma 36.17.3. *Let $f : X \to Y$ be a morphism of schemes. Assume that all the fibres X_y are locally Noetherian schemes. Let $Y' \to Y$ be locally of finite type. Let $f' : X' = X_{Y'} \to Y$ be the base change of f. Let $x' \in X'$ be a point with image $x \in X$.*

(1) *If f is Cohen-Macaulay at x, then the base change $f' : X' \to Y'$ is Cohen-Macaulay at x'.*
(2) *If $Y' \to Y$ is flat at $f'(x')$ and f' is Cohen-Macaulay at x', then f is Cohen-Macaulay at x.*

Proof. Note that the assumption on $Y' \to Y$ means that for $y' \in Y'$ mapping to $y \in Y$ the field extension $\kappa(y) \subset \kappa(y')$ is finitely generated. Hence also all the fibres $X'_{y'} = (X_y)_{\kappa(y')}$ are locally Noetherian, see Varieties, Lemma 32.9.1. Thus the lemma makes sense. Set $y' = f'(x')$ and $y = f(x)$. Hence we get the following commutative diagram of local rings

$$\begin{array}{ccc} \mathcal{O}_{X',x'} & \longleftarrow & \mathcal{O}_{X,x} \\ \uparrow & & \uparrow \\ \mathcal{O}_{Y',y'} & \longleftarrow & \mathcal{O}_{Y,y} \end{array}$$

where the upper left corner is a localization of the tensor product of the upper right and lower left corners over the lower right corner.

Assume f is Cohen-Macaulay at x. The flatness of $\mathcal{O}_{Y,y} \to \mathcal{O}_{X,x}$ implies the flatness of $\mathcal{O}_{Y',y'} \to \mathcal{O}_{X',x'}$, see Algebra, Lemma 10.94.1. The fact that $\mathcal{O}_{X,x}/\mathfrak{m}_y\mathcal{O}_{X,x}$ is Cohen-Macaulay implies that $\mathcal{O}_{X',x'}/\mathfrak{m}_{y'}\mathcal{O}_{X',x'}$, see Varieties, Lemma 32.11.1. Hence we see that f' is Cohen-Macaulay at x'.

Assume $Y' \to Y$ is flat at y' and f' is Cohen-Macaulay at x'. The flatness of $\mathcal{O}_{Y',y'} \to \mathcal{O}_{X',x'}$ and $\mathcal{O}_{Y,y} \to \mathcal{O}_{Y',y'}$ implies the flatness of $\mathcal{O}_{Y,y} \to \mathcal{O}_{X,x}$, see Algebra, Lemma 10.94.1. The fact that $\mathcal{O}_{X',x'}/\mathfrak{m}_{y'}\mathcal{O}_{X',x'}$ is Cohen-Macaulay implies that $\mathcal{O}_{X,x}/\mathfrak{m}_y\mathcal{O}_{X,x}$, see Varieties, Lemma 32.11.1. Hence we see that f is Cohen-Macaulay at x. □

Lemma 36.17.4. *Let $f : X \to S$ be a morphism of schemes which is flat and locally of finite presentation. Let*

$$W = \{x \in X \mid f \text{ is Cohen-Macaulay at } x\}$$

Then

1. $W = \{x \in X \mid \mathcal{O}_{X_{f(x)},x}$ *is Cohen-Macaulay*$\}$,
2. W *is open in* X,
3. W *dense in every fibre of* $X \to S$,
4. *the formation of W commutes with arbitrary base change of f: For any morphism $g : S' \to S$, consider the base change $f' : X' \to S'$ of f and the projection $g' : X' \to X$. Then the corresponding set W' for the morphism f' is equal to $W' = (g')^{-1}(W)$.*

Proof. As f is flat with locally Noetherian fibres the equality in (1) holds by definition. Parts (2) and (3) follow from Algebra, Lemma 10.124.5. Part (4) follows either from Algebra, Lemma 10.124.7 or Varieties, Lemma 32.11.1. □

Lemma 36.17.5. *Let $f : X \to S$ be a morphism of schemes which is flat and locally of finite presentation. For $d \geq 0$ there exist opens $U_d \subset X$ with the following properties*

1. $W = \bigcup_{d \geq 0} U_d$ *is dense in every fibre of f, and*
2. $U_d \to S$ *is of relative dimension d (see Morphisms, Definition 28.30.1).*

Proof. This follows by combining Lemma 36.17.4 with Morphisms, Lemma 28.30.4. □

Lemma 36.17.6. *Let $f : X \to S$ be a morphism of schemes which is flat and locally of finite presentation. Suppose $x' \rightsquigarrow x$ is a specialization of points of X with image $s' \rightsquigarrow s$ in S. If x is a generic point of an irreducible component of X_s then $\dim_{x'}(X_{s'}) = \dim_x(X_s)$.*

Proof. The point x is contained in U_d for some d, where U_d as in Lemma 36.17.5. □

Lemma 36.17.7. *Let $f : X \to S$ be a morphism of schemes which is flat and locally of finite presentation. Let $n \geq 0$ and let $U \subset S$ be a dense open such that $\dim(X_u) \leq n$ for all $u \in U$. Then $\dim(X_s) \leq n$ for all $s \in S$.*

Proof. Let $W \subset X$ be the open subscheme constructed and studied in Lemmas 36.17.4 and 36.17.5. Note that every generic point of every fibre is contained in W, hence it suffices to prove the result for W. Since $W = \bigcup_{d \geq 0} U_d$, it suffices to prove that $U_d = \emptyset$ for $d > n$. Since f is flat and locally of finite presentation it is open hence $f(U_d)$ is open (Morphisms, Lemma 28.26.9). Thus if U_d is not empty, then $f(U_d) \cap U \neq \emptyset$. The lemma follows. □

Lemma 36.17.8. *The property $\mathcal{P}(f) =$ "the fibres of f are locally Noetherian and f is Cohen-Macaulay" is local in the fppf topology on the target and local in the syntomic topology on the source.*

Proof. We have $\mathcal{P}(f) = \mathcal{P}_1(f) \wedge \mathcal{P}_2(f)$ where $\mathcal{P}_1(f) =$"f is flat", and $\mathcal{P}_2(f) =$"the fibres of f are locally Noetherian and Cohen-Macaulay". We know that $\mathcal{P}_1$ is local in the fppf topology on the source and the target, see Descent, Lemmas 34.19.13 and 34.23.1. Thus we have to deal with $\mathcal{P}_2$.

Let $f : X \to Y$ be a morphism of schemes. Let $\{\varphi_i : Y_i \to Y\}_{i \in I}$ be an fppf covering of Y. Denote $f_i : X_i \to Y_i$ the base change of f by φ_i. Let $i \in I$ and let $y_i \in Y_i$ be a point. Set $y = \varphi_i(y_i)$. Note that

$$X_{i,y_i} = \mathrm{Spec}(\kappa(y_i)) \times_{\mathrm{Spec}(\kappa(y))} X_y.$$

and that $\kappa(y) \subset \kappa(y_i)$ is a finitely generated field extension. Hence if X_y is locally Noetherian, then X_{i,y_i} is locally Noetherian, see Varieties, Lemma 32.9.1. And if in addition X_y is Cohen-Macaulay, then X_{i,y_i} is Cohen-Macaulay, see Varieties, Lemma 32.11.1. Thus $\mathcal{P}_2$ is fppf local on the target.

Let $\{X_i \to X\}$ be a syntomic covering of X. Let $y \in Y$. In this case $\{X_{i,y} \to X_y\}$ is a syntomic covering of the fibre. Hence the locality of $\mathcal{P}_2$ for the syntomic topology on the source follows from Descent, Lemma 34.13.2. Combining the above the lemma follows. □

36.18. Slicing Cohen-Macaulay morphisms

The results in this section eventually lead to the assertion that the fppf topology is the same as the "finitely presented, flat, quasi-finite" topology. The following lemma is very closely related to Divisors, Lemma 30.10.7.

Lemma 36.18.1. *Let $f : X \to S$ be a morphism of schemes. Let $x \in X$ be a point with image $s \in S$. Let $h \in \mathfrak{m}_x \subset \mathcal{O}_{X,x}$. Assume*

1. *f is locally of finite presentation,*
2. *f is flat at x, and*
3. *the image $\overline{h}$ of h in $\mathcal{O}_{X_s,x} = \mathcal{O}_{X,x}/\mathfrak{m}_s\mathcal{O}_{X,x}$ is a nonzerodivisor.*

Then there exists an affine open neighbourhood $U \subset X$ of x such that h comes from $h \in \Gamma(U, \mathcal{O}_U)$ and such that $D = V(h)$ is an effective Cartier divisor in U with $x \in D$ and $D \to S$ flat and locally of finite presentation.

Proof. We are going to prove this by reducing to the Noetherian case. By openness of flatness (see Theorem 36.12.1) we may assume, after replacing X by an open neighbourhood of x, that $X \to S$ is flat. We may also assume that X and S are affine. After possible shrinking X a bit we may assume that there exists an $h \in \Gamma(X, \mathcal{O}_X)$ which maps to our given h.

We may write $S = \mathrm{Spec}(A)$ and we may write $A = \mathrm{colim}_i A_i$ as a directed colimit of finite type $\mathbf{Z}$ algebras. Then by Algebra, Lemma 10.154.1 or Limits, Lemmas 31.9.1, 31.7.2, and 31.9.1 we can find a cartesian diagram

$$\begin{array}{ccc} X & \longrightarrow & X_0 \\ {\scriptstyle f}\downarrow & & \downarrow{\scriptstyle f_0} \\ S & \longrightarrow & S_0 \end{array}$$

with f_0 flat and of finite presentation, X_0 affine, and S_0 affine and Noetherian. Let $x_0 \in X_0$, resp. $s_0 \in S_0$ be the image of x, resp. s. We may also assume there exists an element $h_0 \in \Gamma(X_0, \mathcal{O}_{X_0})$ which restricts to h on X. (If you used the algebra reference above then this is clear; if you used the references to the chapter on limits then this follows from Limits, Lemma 31.9.1 by thinking of h as a morphism $X \to \mathbf{A}^1_S$.) Note that $\mathcal{O}_{X_s,x}$ is a localization of $\mathcal{O}_{(X_0)_{s_0},x_0} \otimes_{\kappa(s_0)} \kappa(s)$, so that $\mathcal{O}_{(X_0)_{s_0},x_0} \to \mathcal{O}_{X_s,x}$ is a flat local ring map, in particular faithfully flat. Hence the image $\overline{h}_0 \in \mathcal{O}_{(X_0)_{s_0},x_0}$ is contained in $\mathfrak{m}_{(X_0)_{s_0},x_0}$ and is a nonzerodivisor. We claim that after replacing X_0 by a principal open neighbourhood of x_0 the element h_0 is a nonzerodivisor in $B_0 = \Gamma(X_0, \mathcal{O}_{X_0})$ such that B_0/h_0B_0 is flat over $A_0 = \Gamma(S_0, \mathcal{O}_{S_0})$. If so then

$$0 \to B_0 \xrightarrow{h_0} B_0 \to B_0/h_0B_0 \to 0$$

is a short exact sequence of flat A_0-modules. Hence this remains exact on tensoring with A (by Algebra, Lemma 10.36.11) and the lemma follows.

It remains to prove the claim above. The corresponding algebra statement is the following (we drop the subscript $_0$ here): Let $A \to B$ be a flat, finite type ring map of Noetherian rings. Let $\mathfrak{q} \subset B$ be a prime lying over $\mathfrak{p} \subset A$. Assume $h \in \mathfrak{q}$ maps to a nonzerodivisor in $B_\mathfrak{q}/\mathfrak{p}B_\mathfrak{q}$. Goal: show that after possible replacing B by B_g for some $g \in B$, $g \notin \mathfrak{q}$ the element h becomes a nonzerodivisor and B/hB becomes flat over A. By Algebra, Lemma 10.93.2 we see that h is a nonzerodivisor in $B_\mathfrak{q}$ and that $B_\mathfrak{q}/hB_\mathfrak{q}$ is flat over A. By openness of flatness, see Algebra, Theorem 10.123.4 or Theorem 36.12.1 we see that B/hB is flat over A after replacing B by B_g for some $g \in B$, $g \notin \mathfrak{q}$. Finally, let $I = \{b \in B \mid hb = 0\}$ be the annihilator of h. Then $IB_\mathfrak{q} = 0$ as h is a nonzerodivisor in $B_\mathfrak{q}$. Also I is finitely generated as B is Noetherian. Hence there exists a $g \in B$, $g \notin \mathfrak{q}$ such that $IB_g = 0$. After replacing B by B_g we see that h is a nonzerodivisor. □

Lemma 36.18.2. *Let $f : X \to S$ be a morphism of schemes. Let $x \in X$ be a point with image $s \in S$. Let $h_1, \ldots, h_r \in \mathcal{O}_{X,x}$. Assume*

(1) *f is locally of finite presentation,*

(2) *f is flat at x, and*
(3) *the images of $h_1, \ldots, h_r$ in $\mathcal{O}_{X_s,x} = \mathcal{O}_{X,x}/\mathfrak{m}_s\mathcal{O}_{X,x}$ form a regular sequence.*

Then there exists an affine open neighbourhood $U \subset X$ of x such that $h_1, \ldots, h_r$ come from $h_1, \ldots, h_r \in \Gamma(U, \mathcal{O}_U)$ and such that $Z = V(h_1, \ldots, h_r) \to U$ is a regular immersion with $x \in Z$ and $Z \to S$ flat and locally of finite presentation. Moreover, the base change $Z_{S'} \to U_{S'}$ is a regular immersion for any scheme S' over S.

Proof. (Our conventions on regular sequences imply that $h_i \in \mathfrak{m}_x$ for each i.) The case $r = 1$ follows from Lemma 36.18.1 combined with Divisors, Lemma 30.10.1 to see that $V(h_1)$ remains an effective Cartier divisor after base change. The case $r > 1$ follows from a straightforward induction on r (applying the result for $r = 1$ exactly r times; details omitted).

Another way to prove the lemma is using the material from Divisors, Section 30.14. Namely, first by openness of flatness (see Theorem 36.12.1) we may assume, after replacing X by an open neighbourhood of x, that $X \to S$ is flat. We may also assume that X and S are affine. After possible shrinking X a bit we may assume that we have $h_1, \ldots, h_r \in \Gamma(X, \mathcal{O}_X)$. Set $Z = V(h_1, \ldots, h_r)$. Note that X_s is a Noetherian scheme (because it is an algebraic $\kappa(s)$-scheme, see Varieties, Section 32.13) and that the topology on X_s is induced from the topology on X (see Schemes, Lemma 25.18.5). Hence after shrinking X a bit more we may assume that $Z_s \subset X_s$ is a regular immersion cut out by the r elements $h_i|_{X_s}$, see Divisors, Lemma 30.12.8 and its proof. It is also clear that $r = \dim_x(X_s) - \dim_x(Z_s)$ because

$$\begin{aligned} \dim_x(X_s) &= \dim(\mathcal{O}_{X_s,x}) + \operatorname{trdeg}_{\kappa(s)}(\kappa(x)), \\ \dim_x(Z_s) &= \dim(\mathcal{O}_{Z_s,x}) + \operatorname{trdeg}_{\kappa(s)}(\kappa(x)), \\ \dim(\mathcal{O}_{X_s,x}) &= \dim(\mathcal{O}_{Z_s,x}) + r \end{aligned}$$

the first two equalities by Algebra, Lemma 10.110.3 and the second by r times applying Algebra, Lemma 10.57.11. Hence Divisors, Lemma 30.14.6 part (3) applies to show that (after Zariski shrinking X) the morphism $Z \to X$ is a regular immersion to which Divisors, Lemma 30.14.4 applies (which gives the flatness and the statement on base change). □

Lemma 36.18.3. *Let $f : X \to S$ be a morphism of schemes. Let $x \in X$ be a point with image $s \in S$. Assume*

(1) *f is locally of finite presentation,*
(2) *f is flat at x, and*
(3) *$\mathcal{O}_{X_s,x}$ has depth ≥ 1.*

Then there exists an affine open neighbourhood $U \subset X$ of x and an effective Cartier divisor $D \subset U$ containing x such that $D \to S$ is flat and of finite presentation.

Proof. Pick any $h \in \mathfrak{m}_x \subset \mathcal{O}_{X,x}$ which maps to a nonzerodivisor in $\mathcal{O}_{X_s,x}$ and apply Lemma 36.18.1. □

Lemma 36.18.4. *Let $f : X \to S$ be a morphism of schemes. Let $x \in X$ be a point with image $s \in S$. Assume*

(1) *f is locally of finite presentation,*
(2) *f is Cohen-Macaulay at x, and*
(3) *x is a closed point of X_s.*

Then there exists a regular immersion $Z \to X$ containing x such that

(a) *$Z \to S$ is flat and locally of finite presentation,*
(b) *$Z \to S$ is locally quasi-finite, and*
(c) *$Z_s = \{x\}$ set theoretically.*

Proof. We may and do replace S by an affine open neighbourhood of s. We will prove the lemma for affine S by induction on $d = \dim_x(X_s)$.

The case $d = 0$. In this case we show that we may take Z to be an open neighbourhood of x. (Note that an open immersion is a regular immersion.) Namely, if $d = 0$, then $X \to S$ is quasi-finite at x, see Morphisms, Lemma 28.30.5. Hence there exists an affine open neighbourhood $U \subset X$ such that $U \to S$ is quasi-finite, see Morphisms, Lemma 28.49.2. Thus after replacing X by U we see that the fibre X_s is a finite discrete set. Hence after replacing X by a further affine open neighbourhood of X we see that that $f^{-1}(\{s\}) = \{x\}$ (because the topology on X_s is induced from the topology on X, see Schemes, Lemma 25.18.5). This proves the lemma in this case.

Next, assume $d > 0$. Note that because x is a closed point of its fibre the extension $\kappa(s) \subset \kappa(x)$ is finite (by the Hilbert Nullstellensatz, see Morphisms, Lemma 28.21.3). Thus we see

$$\text{depth}(\mathcal{O}_{X_s,x}) = \dim(\mathcal{O}_{X_s,x}) = d > 0$$

the first equality as $\mathcal{O}_{X_s,x}$ is Cohen-Macaulay and the second by Morphisms, Lemma 28.29.1. Thus we may apply Lemma 36.18.3 to find a diagram

$$\begin{array}{ccccc} D & \longrightarrow & U & \longrightarrow & X \\ & \searrow & & \swarrow & \downarrow \\ & & S & & \end{array}$$

with $x \in D$. Note that $\mathcal{O}_{D_s,x} = \mathcal{O}_{X_s,x}/(\overline{h})$ for some nonzerodivisor $\overline{h}$, see Divisors, Lemma 30.10.1. Hence $\mathcal{O}_{D_s,x}$ is Cohen-Macaulay of dimension one less than the dimension of $\mathcal{O}_{X_s,x}$, see Algebra, Lemma 10.99.2 for example. Thus the morphism $D \to S$ is flat, locally of finite presentation, and Cohen-Macaulay at x with $\dim_x(D_s) = \dim_x(X_s) - 1 = d - 1$. By induction hypothesis we can find a regular immersion $Z \to D$ having properties (a), (b), (c). As $Z \to D \to U$ are both regular immersions, we see that also $Z \to U$ is a regular immersion by Divisors, Lemma 30.13.7. This finishes the proof. □

Lemma 36.18.5. *Let $f : X \to S$ be a flat morphism of schemes which is locally of finite presentation Let $s \in S$ be a point in the image of f. Then there exists a commutative diagram*

$$\begin{array}{ccc} S' & \longrightarrow & X \\ & {\scriptstyle g}\searrow \quad \swarrow{\scriptstyle f} & \\ & S & \end{array}$$

where $g : S' \to S$ is flat, locally of finite presentation, locally quasi-finite, and $s \in g(S')$.

Proof. The fibre X_s is not empty by assumption. Hence there exists a closed point $x \in X_s$ where f is Cohen-Macaulay, see Lemma 36.17.4. Apply Lemma 36.18.4 and set $S' = S$. □

The following lemma shows that sheaves for the fppf topology are the same thing as sheaves for the "quasi-finite, flat, finite presentation" topology.

Lemma 36.18.6. *Let S be a scheme. Let $\mathcal{U} = \{S_i \to S\}_{i \in I}$ be an fppf covering of S, see Topologies, Definition 33.7.1. Then there exists an fppf covering $\mathcal{V} = \{T_j \to S\}_{j \in J}$ which refines (see Sites, Definition 7.8.1) $\mathcal{U}$ such that each $T_j \to S$ is locally quasi-finite.*

Proof. For every $s \in S$ there exists an $i \in I$ such that s is in the image of $S_i \to S$. By Lemma 36.18.5 we can find a morphism $g_s : T_s \to S$ such that $s \in g_s(T_s)$ which is flat, locally of finite presentation and locally quasi-finite and such that g_s factors through $S_i \to S$. Hence $\{T_s \to S\}$ is the desired covering of S that refines $\mathcal{U}$. □

36.19. Generic fibres

Some results on the relationship between generic fibres and nearby fibres.

Lemma 36.19.1. *Let $f : X \to Y$ be a finite type morphism of schemes. Assume Y irreducible with generic point η. If $X_\eta = \emptyset$ then there exists a nonempty open $V \subset Y$ such that $X_V = V \times_Y X = \emptyset$.*

Proof. Follows immediately from the more general Morphisms, Lemma 28.8.4. □

Lemma 36.19.2. *Let $f : X \to Y$ be a finite type morphism of schemes. Assume Y irreducible with generic point η. If $X_\eta \neq \emptyset$ then there exists a nonempty open $V \subset Y$ such that $X_V = V \times_Y X \to V$ is surjective.*

Proof. This follows, upon taking affine opens, from Algebra, Lemma 10.28.2. (Of course it also follows from generic flatness.) □

Lemma 36.19.3. *Let $f : X \to Y$ be a finite type morphism of schemes. Assume Y irreducible with generic point η. If $Z \subset X$ is a closed subset with Z_η nowhere dense in X_η, then there exists a nonempty open $V \subset Y$ such that Z_y is nowhere dense in X_y for all $y \in V$.*

Proof. Let $Y' \subset Y$ be the reduction of Y. Set $X' = Y' \times_Y X$ and $Z' = Y' \times_Y Z$. As $Y' \to Y$ is a universal homeomorphism by Morphisms, Lemma 28.45.4 we see that it suffices to prove the lemma for $Z' \subset X' \to Y'$. Thus we may assume that Y is integral, see Properties, Lemma 27.3.4. By Morphisms, Proposition 28.28.1 there exists a nonempty affine open $V \subset Y$ such that $X_V \to V$ and $Z_V \to Z$ are flat and of finite presentation. We claim that V works. Pick $y \in V$. If Z_y has a nonempty interior, then Z_y contains a generic point ξ of an irreducible component of X_y. Note that $\eta \rightsquigarrow f(\xi)$. Since $Z_V \to V$ is flat we can choose a specialization $\xi' \rightsquigarrow \xi$, $\xi' \in Z$ with $f(\xi') = \eta$, see Morphisms, Lemma 28.26.8. By Lemma 36.17.6 we see that

$$\dim_{\xi'}(Z_\eta) = \dim_\xi(Z_y) = \dim_\xi(X_y) = \dim_{\xi'}(X_\eta).$$

Hence some irreducible component of Z_η passing through ξ' has dimension $\dim_{\xi'}(X_\eta)$ which contradicts the assumption that Z_η is nowhere dense in X_η and we win. □

Lemma 36.19.4. *Let $f : X \to Y$ be a finite type morphism of schemes. Assume Y irreducible with generic point η. Let $U \subset X$ be an open subscheme such that U_η is scheme theoretically dense in X_η. Then there exists a nonempty open $V \subset Y$ such that U_y is scheme theoretically dense in X_y for all $y \in V$.*

Proof. Let $Y' \subset Y$ be the reduction of Y. Let $X' = Y' \times_Y X$ and $U' = Y' \times_Y U$. As $Y' \to Y$ induces a bijection on points, and as $U' \to U$ and $X' \to X$ induce isomorphisms of scheme theoretic fibres, we may replace Y by Y' and X by X'. Thus we may assume that Y is integral, see Properties, Lemma 27.3.4. We may also replace Y by a nonempty affine open. In other words we may assume that $Y = \mathrm{Spec}(A)$ where A is a domain with fraction field K.

As f is of finite type we see that X is quasi-compact. Write $X = X_1 \cup \ldots \cup X_n$ for some affine opens X_i. By Morphisms, Definition 28.7.1 we see that $U_i = X_i \cap U$ is an open subscheme of X_i such that $U_{i,\eta}$ is scheme theoretically dense in $X_{i,\eta}$. Thus it suffices to prove the result for the pairs (X_i, U_i), in other words we may assume that X is affine.

Write $X = \mathrm{Spec}(B)$. Note that B_K is Noetherian as it is a finite type K-algebra. Hence U_η is quasi-compact. Thus we can find finitely many $g_1, \ldots, g_m \in B$ such that $D(g_j) \subset U$ and such that $U_\eta = D(g_1)_\eta \cup \ldots \cup D(g_m)_\eta$. The fact that U_η is scheme theoretically dense in X_η means that $B_K \to \bigoplus_j (B_K)_{g_j}$ is injective, see Morphisms, Example 28.7.4. By Algebra, Lemma 10.21.4 this is equivalent to the injectivity of $B_K \to \bigoplus_{j=1,\ldots,m} B_K$, $b \mapsto (g_1 b, \ldots, g_m b)$. Let M be the cokernel of this map over A, i.e., such that we have an exact sequence

$$0 \to I \to B \xrightarrow{(g_1,\ldots,g_m)} \bigoplus\nolimits_{j=1,\ldots,m} B \to M \to 0$$

After replacing A by A_h for some nonzero h we may assume that B is a flat, finitely presented A-algebra, and that M is flat over A, see Algebra, Lemma 10.112.3. The flatness of B over A implies that B is torsion free as an A-module, see More on Algebra, Lemma 15.15.3. Hence $B \subset B_K$. By assumption $I_K = 0$ which implies that $I = 0$ (as $I \subset B \subset B_K$ is a subset of I_K). Hence now we have a short exact sequence

$$0 \to B \xrightarrow{(g_1,\ldots,g_m)} \bigoplus\nolimits_{j=1,\ldots,m} B \to M \to 0$$

with M flat over A. Hence for every homomorphism $A \to \kappa$ where κ is a field, we obtain a short exact sequence

$$0 \to B \otimes_A \kappa \xrightarrow{(g_1\otimes 1,\ldots,g_m\otimes 1)} \bigoplus\nolimits_{j=1,\ldots,m} B \otimes_A \kappa \to M \otimes_A \kappa \to 0$$

see Algebra, Lemma 10.36.11. Reversing the arguments above this means that $\bigcup D(g_j \otimes 1)$ is scheme theoretically dense in $\mathrm{Spec}(B \otimes_A \kappa)$. As $\bigcup D(g_j \otimes 1) = \bigcup D(g_j)_\kappa \subset U_\kappa$ we obtain that U_κ is scheme theoretically dense in X_κ which is what we wanted to prove. □

Suppose given a morphism of schemes $f : X \to Y$ and a point $y \in Y$. Recall that the fibre X_y is homeomorphic to the subset $f^{-1}(\{y\})$ of X with induced topology, see Schemes, Lemma 25.18.5. Suppose given a closed subset $T(y) \subset X_y$. Let T be the closure of $T(y)$ in X. Endow T with the induced reduced scheme structure. Then T is a closed subscheme of X with the property that $T_y = T(y)$ set-theoretically. In fact T is the smallest closed subscheme of X with this property. Thus it is

"harmless" to denote a closed subset of X_y by T_y if we so desire. In the following lemma we apply this to the generic fibre of f.

Lemma 36.19.5. *Let $f : X \to Y$ be a finite type morphism of schemes. Assume Y irreducible with generic point η. Let $X_\eta = Z_{1,\eta} \cup \ldots \cup Z_{n,\eta}$ be a covering of the generic fibre by closed subsets of X_η. Let Z_i be the closure of $Z_{i,\eta}$ in X (see discussion above). Then there exists a nonempty open $V \subset Y$ such that $X_y = Z_{1,y} \cup \ldots \cup Z_{n,y}$ for all $y \in V$.*

Proof. If Y is Noetherian then $U = X \setminus (Z_1 \cup \ldots \cup Z_n)$ is of finite type over Y and we can directly apply Lemma 36.19.1 to get that $U_V = \emptyset$ for a nonempty open $V \subset Y$. In general we argue as follows. As the question is topological we may replace Y by its reduction. Thus Y is integral, see Properties, Lemma 27.3.4. After shrinking Y we may assume that $X \to Y$ is flat, see Morphisms, Proposition 28.28.1. In this case every point x in X_y is a specialization of a point $x' \in X_\eta$ by Morphisms, Lemma 28.26.8. As the Z_i are closed in X and cover the generic fibre this implies that $X_y = \bigcup Z_{i,y}$ for $y \in Y$ as desired. □

The following lemma says that generic fibres of morphisms whose source is reduced are reduced.

Lemma 36.19.6. *Let $f : X \to Y$ be a morphism of schemes. Let $\eta \in Y$ be a generic point of an irreducible component of Y. Then $(X_\eta)_{red} = (X_{red})_\eta$.*

Proof. Choose an affine neighbourhood $\mathrm{Spec}(A) \subset Y$ of η. Choose an affine open $\mathrm{Spec}(B) \subset X$ mapping into $\mathrm{Spec}(A)$ via the morphism f. Let $\mathfrak{p} \subset A$ be the minimal prime corresponding to η. Let B_{red} be the quotient of B by $\sqrt{(0)}$. The algebraic content of the lemma is that $B_{red} \otimes_A \kappa(\mathfrak{p})$ is reduced. To prove this, suppose that $x \in B_{red} \otimes_A \kappa(\mathfrak{p})$ is nilpotent. Say $x^n = 0$ for some $n > 0$. Pick an $f \in A$, $f \notin \mathfrak{p}$ such that fx is the image of $y \in B_{red}$. Then $gy^n \in \mathfrak{p}B_{red}$ for some $g \in A$, $g \notin \mathfrak{p}$. By Algebra, Lemma 10.24.4 we see that $\mathfrak{p}A_\mathfrak{p}$ is locally nilpotent. By Algebra, Lemma 10.14.1 we see that $\mathfrak{p}(B_{red})_\mathfrak{p}$ is locally nilpotent. Hence we conclude that gy^n is nilpotent in $(B_{red})_\mathfrak{p}$. Thus there exists a $h \in A$, $h \notin \mathfrak{p}$ and an $m > 0$ such that $h(gy^n)^m = 0$ in B_{red}. This implies that hgy is nilpotent in B_{red}, i.e., that $hgy = 0$. Of course this means that $x = 0$ as desired. □

Lemma 36.19.7. *Let $f : X \to Y$ be a morphism of schemes. Assume that Y is irreducible and f is of finite type. There exists a diagram*

$$\begin{array}{ccccc} X' & \xrightarrow{g'} & X_V & \longrightarrow & X \\ \downarrow{\scriptstyle f'} & & \downarrow & & \downarrow{\scriptstyle f} \\ Y' & \xrightarrow{g} & V & \longrightarrow & Y \end{array}$$

where

(1) *V is a nonempty open of Y,*
(2) *$X_V = V \times_Y X$,*
(3) *$g : Y' \to V$ is a finite universal homeomorphism,*
(4) *$X' = (Y' \times_Y X)_{red} = (Y' \times_V X_V)_{red}$,*
(5) *g' is a finite universal homeomorphism,*
(6) *Y' is an integral affine scheme,*
(7) *f' is flat and of finite presentation, and*

(8) *the generic fibre of f' is geometrically reduced.*

Proof. Let $V = \mathrm{Spec}(A)$ be a nonempty affine open of Y. By assumption the radical of A is a prime ideal $\mathfrak{p}$. Let $K = f.f(A/\mathfrak{p})$ be the fraction field. Let p be the characteristic of K if positive and 1 if the characteristic is zero. By Varieties, Lemma 32.4.11 there exists a finite purely inseparable field extension $K \subset K'$ such that $X_{K'}$ is geometrically reduced over K'. Choose elements $x_1, \ldots, x_n \in K'$ which generate K' over K and such that some p-power of x_i is in $A/\mathfrak{p}$. Let $A' \subset K'$ be the finite A-subalgebra of K' generated by $x_1, \ldots, x_n$. Note that A' is a domain with fraction field K'. By Algebra, Lemma 10.43.2 we see that $A \to A'$ is a universal homeomorphism. Set $Y' = \mathrm{Spec}(A')$. Set $X' = (Y' \times_Y X)_{red}$. The generic fibre of $X' \to Y'$ is $(X_K)_{red}$ by Lemma 36.19.6 which is geometrically reduced by construction. Note that $X' \to X_V$ is a finite universal homeomorphism as the composition of the reduction morphism $X' \to Y' \times_Y X$ (see Morphisms, Lemma 28.45.4) and the base change of g. At this point all of the properties of the lemma hold except for possibly (7). This can be achieved by shrinking Y' and hence V, see Morphisms, Proposition 28.28.1. □

Lemma 36.19.8. *Let $f : X \to Y$ be a morphism of schemes. Assume that Y is irreducible and f is of finite type. There exists a diagram*

$$\begin{array}{ccccc} X' & \xrightarrow{g'} & X_V & \longrightarrow & X \\ {\scriptstyle f'}\downarrow & & \downarrow & & \downarrow{\scriptstyle f} \\ Y' & \xrightarrow{g} & V & \longrightarrow & Y \end{array}$$

where

(1) *V is a nonempty open of Y,*
(2) *$X_V = V \times_Y X$,*
(3) *$g : Y' \to V$ is surjective finite étale,*
(4) *$X' = Y' \times_Y X = Y' \times_V X_V$,*
(5) *g' is surjective finite étale,*
(6) *Y' is an irreducible affine scheme, and*
(7) *all irreducible components of the generic fibre of f' are geometrically irreducible.*

Proof. Let $V = \mathrm{Spec}(A)$ be a nonempty affine open of Y. By assumption the radical of A is a prime ideal $\mathfrak{p}$. Let $K = f.f(A/\mathfrak{p})$ be the fraction field. By Varieties, Lemma 32.6.14 there exists a finite separable field extension $K \subset K'$ such that all irreducible components of $X_{K'}$ are geometrically irreducible over K'. Choose an element $\alpha \in K'$ which generates K' over K, see Fields, Lemma 9.18.1. Let $P(T) \in K[T]$ be the minimal polynomial for α over K. After replacing α by $f\alpha$ for some $f \in A$, $f \notin \mathfrak{p}$ we may assume that there exists a monic polynomial $T^d + a_1T^{d-1} + \ldots + a_d \in A[T]$ which maps to $P(T) \in K[T]$ under the map $A[T] \to K[T]$. Set $A' = A[T]/(P)$. Then $A \to A'$ is a finite free ring map such that there exists a unique prime $\mathfrak{q}$ lying over $\mathfrak{p}$, such that $K = \kappa(\mathfrak{p}) \subset \kappa(\mathfrak{q}) = K'$ is finite separable, and such that $\mathfrak{p}A'_{\mathfrak{q}}$ is the maximal ideal of $A'_{\mathfrak{q}}$. Hence $g : Y' = \mathrm{Spec}(A') \to V = \mathrm{Spec}(A)$ is étale at $\mathfrak{q}$, see Algebra, Lemma 10.136.7. This means that there exists an open $W \subset \mathrm{Spec}(A')$ such that $g|_W : W \to \mathrm{Spec}(A)$ is étale. Since g is finite and since $\mathfrak{q}$ is the only point lying over $\mathfrak{p}$ we see that $Z = g(Y' \setminus W)$ is a closed subset of V not

containing $\mathfrak{p}$. Hence after replacing V by a principal affine open of V which does not meet Z we obtain that g is finite étale. □

Lemma 36.19.9. *Let S be an integral scheme with generic point η. Let $f : X \to S$ and $g : Y \to S$ be morphisms of schemes such that*

(1) *f, g are locally of finite type,*
(2) *X_η, Y_η are integral with generic points x, y, and*
(3) *$\kappa(x) \cong \kappa(y)$ as $\kappa(\eta)$-extensions.*

Then there exist open subschemes $x \in U \subset X$, $y \in V \subset Y$ and an S-isomorphism $U \to V$ which induces the given isomorphism of residue fields.

Proof. The question is local around the points η, x, y. Hence we may replace S, X, Y by affine neighbourhoods of η, x, y and hence reduce to the case that S, X, Y are affine. Say $S = \mathrm{Spec}(R)$ and $X = \mathrm{Spec}(A)$, $Y = \mathrm{Spec}(B)$. By Algebra, Lemma 10.112.3 we may also assume that A and B are flat and of finite presentation over R. Denote $K = f.f.(R)$. The rings A, B are torsion free as R-modules because A, B are flat over R, see More on Algebra, Lemma 15.15.3. Since $A \otimes_R K$ and $B \otimes_R K$ are domains by assumption it follows that A and B are domains. Set $L = f.f.(A)$ and $M = f.f.(B)$. Let $\varphi : L \to M$ be the given isomorphism of K-extensions.

Choose elements $x_1, \ldots, x_n \in A$ which generate A as an R-algebra, and choose elements $y_1, \ldots, y_m \in B$ which generate B as an R-algebra. Write $\varphi(x_i) = b_i/b$ for some $b, b_i \in B$. In other words, b is a common denominator for the elements $\varphi(x_i) \in M = f.f.(B)$. Similarly, write $\varphi^{-1}(y_j) = a_j/a$ for some $a, a_j \in A$. Note that $\varphi(a) \in B_b$ because a can be written as a polynomial in the x_i. Similarly we have $\varphi^{-1}(b) \in A_a$. Thus φ gives an isomorphism

$$A_a \longrightarrow B_b$$

of R-algebras and the lemma is proven. □

36.20. Relative assassins

Lemma 36.20.1. *Let $f : X \to S$ be a morphism of schemes. Let $\mathcal{F}$ be a quasi-coherent $\mathcal{O}_X$-module. Let $\xi \in Ass_{X/S}(\mathcal{F})$ and set $Z = \overline{\{\xi\}} \subset X$. If f is locally of finite type and $\mathcal{F}$ is a finite type $\mathcal{O}_X$-module, then there exists a nonempty open $V \subset Z$ such that for every $s \in f(V)$ the generic points of V_s are elements of $Ass_{X/S}(\mathcal{F})$.*

Proof. We may replace S by an affine open neighbourhood of $f(\xi)$ and X by an affine open neighbourhood of ξ. Hence we may assume $S = \mathrm{Spec}(A)$, $X = \mathrm{Spec}(B)$ and that f is given by the finite type ring map $A \to B$, see Morphisms, Lemma 28.16.2. Moreover, we may write $\mathcal{F} = \widetilde{M}$ for some finite B-module M, see Properties, Lemma 27.16.1. Let $\mathfrak{q} \subset B$ be the prime corresponding to ξ and let $\mathfrak{p} \subset A$ be the corresponding prime of A. By assumption $\mathfrak{q} \in \mathrm{Ass}_B(M \otimes_A \kappa(\mathfrak{p}))$, see Algebra, Remark 10.62.6 and Divisors, Lemma 30.2.2. With this notation $Z = V(\mathfrak{q}) \subset \mathrm{Spec}(B)$. In particular $f(Z) \subset V(\mathfrak{p})$. Hence clearly it suffices to prove the lemma after replacing A, B, and M by $A/\mathfrak{p}A$, $B/\mathfrak{p}B$, and $M/\mathfrak{p}M$. In other words we may assume that A is a domain with fraction field K and $\mathfrak{q} \subset B$ is an associated prime of $M \otimes_A K$.

At this point we can use generic flatness. Namely, by Algebra, Lemma 10.112.3 there exists a nonzero $g \in A$ such that M_g is flat as an A_g-module. After replacing A by A_g we may assume that M is flat as an A-module.

In this case, by Algebra, Lemma 10.62.4 we see that $\mathfrak{q}$ is also an associated prime of M. Hence we obtain an injective B-module map $B/\mathfrak{q} \to M$. Let Q be the cokernel so that we obtain a short exact sequence

$$0 \to B/\mathfrak{q} \to M \to Q \to 0$$

of finite B-modules. After applying generic flatness Algebra, Lemma 10.112.3 once more, this time to the B-module Q, we may assume that Q is a flat A-module. In particular we may assume the short exact sequence above is universally injective, see Algebra, Lemma 10.36.11. In this situation $(B/\mathfrak{q}) \otimes_A \kappa(\mathfrak{p}') \subset M \otimes_A \kappa(\mathfrak{p}')$ for any prime $\mathfrak{p}'$ of A. The lemma follows as a minimal prime $\mathfrak{q}'$ of the support of $(B/\mathfrak{q}) \otimes_A \kappa(\mathfrak{p}')$ is an associated prime of $(B/\mathfrak{q}) \otimes_A \kappa(\mathfrak{p}')$ by Divisors, Lemma 30.2.8. □

Lemma 36.20.2. *Let $f : X \to Y$ be a morphism of schemes. Let $\mathcal{F}$ be a quasi-coherent $\mathcal{O}_X$-module. Let $U \subset X$ be an open subscheme. Assume*

(1) *f is of finite type,*
(2) *$\mathcal{F}$ is of finite type,*
(3) *Y is irreducible with generic point η, and*
(4) *$Ass_{X_\eta}(\mathcal{F}_\eta)$ is not contained in U_η.*

Then there exists a nonempty open subscheme $V \subset Y$ such that for all $y \in V$ the set $Ass_{X_y}(\mathcal{F}_y)$ is not contained in U_y.

Proof. Let $\xi \in \mathrm{Ass}_{X_\eta}(\mathcal{F}_\eta)$ be a point which is not contained in U_η. Set $Z = \overline{\{\xi\}}$. By assumption $U \cap Z$ is not dense in the irreducible scheme Z_η. Hence by Lemma 36.19.3 after replacing Y by a nonempty open we may assume that $U_y \cap Z_y$ is nowhere dense in Z_y. On the other hand, by Lemma 36.20.1 there exists a nonempty open $V \subset Z$ such that every generic point of V_y is an associated point of $\mathcal{F}_y$. By Lemma 36.19.2 the set $f(V)$ contains a nonempty open subset of Y and we win. □

Lemma 36.20.3. *Let $f : X \to Y$ be a morphism of schemes. Let $\mathcal{F}$ be a quasi-coherent $\mathcal{O}_X$-module. Let $U \subset X$ be an open subscheme. Assume*

(1) *f is of finite type,*
(2) *$\mathcal{F}$ is of finite type,*
(3) *Y is irreducible with generic point η, and*
(4) *$Ass_{X_\eta}(\mathcal{F}_\eta) \subset U_\eta$.*

Then there exists a nonempty open subscheme $V \subset Y$ such that for all $y \in V$ we have $Ass_{X_y}(\mathcal{F}_y) \subset U_y$.

Proof. (This proof is the same as the proof of Lemma 36.19.4. We urge the reader to read that proof first.) Since the statement is about fibres it is clear that we may replace Y by its reduction. Hence we may assume that Y is integral, see Properties, Lemma 27.3.4. We may also assume that $Y = \mathrm{Spec}(A)$ is affine. Then A is a domain with fraction field K.

As f is of finite type we see that X is quasi-compact. Write $X = X_1 \cup \ldots \cup X_n$ for some affine opens X_i and set $\mathcal{F}_i = \mathcal{F}|_{X_i}$. By assumption the generic fibre of

$U_i = X_i \cap U$ contains $\mathrm{Ass}_{X_{i,\eta}}(\mathcal{F}_{i,\eta})$. Thus it suffices to prove the result for the triples $(X_i, \mathcal{F}_i, U_i)$, in other words we may assume that X is affine.

Write $X = \mathrm{Spec}(B)$. Let N be a finite B-module such that $\mathcal{F} = \widetilde{N}$. Note that B_K is Noetherian as it is a finite type K-algebra. Hence U_η is quasi-compact. Thus we can find finitely many $g_1, \ldots, g_m \in B$ such that $D(g_j) \subset U$ and such that $U_\eta = D(g_1)_\eta \cup \ldots \cup D(g_m)_\eta$. Since $\mathrm{Ass}_{X_\eta}(\mathcal{F}_\eta) \subset U_\eta$ we see that $N_K \to \bigoplus_j (N_K)_{g_j}$ is injective. By Algebra, Lemma 10.21.4 this is equivalent to the injectivity of $N_K \to \bigoplus_{j=1,\ldots,m} N_K$, $n \mapsto (g_1 n, \ldots, g_m n)$. Let I and M be the kernel and cokernel of this map over A, i.e., such that we have an exact sequence

$$0 \to I \to N \xrightarrow{(g_1,\ldots,g_m)} \bigoplus\nolimits_{j=1,\ldots,m} N \to M \to 0$$

After replacing A by A_h for some nonzero h we may assume that B is a flat, finitely presented A-algebra and that both M and N are flat over A, see Algebra, Lemma 10.112.3. The flatness of N over A implies that N is torsion free as an A-module, see More on Algebra, Lemma 15.15.3. Hence $N \subset N_K$. By construction $I_K = 0$ which implies that $I = 0$ (as $I \subset N \subset N_K$ is a subset of I_K). Hence now we have a short exact sequence

$$0 \to N \xrightarrow{(g_1,\ldots,g_m)} \bigoplus\nolimits_{j=1,\ldots,m} N \to M \to 0$$

with M flat over A. Hence for every homomorphism $A \to \kappa$ where κ is a field, we obtain a short exact sequence

$$0 \to N \otimes_A \kappa \xrightarrow{(g_1\otimes 1,\ldots,g_m\otimes 1)} \bigoplus\nolimits_{j=1,\ldots,m} N \otimes_A \kappa \to M \otimes_A \kappa \to 0$$

see Algebra, Lemma 10.36.11. Reversing the arguments above this means that $\bigcup D(g_j \otimes 1)$ contains $\mathrm{Ass}_{B\otimes_A \kappa}(N \otimes_A \kappa)$. As $\bigcup D(g_j \otimes 1) = \bigcup D(g_j)_\kappa \subset U_\kappa$ we obtain that U_κ contains $\mathrm{Ass}_{X\otimes\kappa}(\mathcal{F} \otimes \kappa)$ which is what we wanted to prove. □

Lemma 36.20.4. *Let $f : X \to S$ be a morphism which is locally of finite type. Let $\mathcal{F}$ be a quasi-coherent $\mathcal{O}_X$-module of finite type. Let $U \subset X$ be an open subscheme. Let $g : S' \to S$ be a morphism of schemes, let $f' : X' = X_{S'} \to S'$ be the base change of f, let $g' : X' \to X$ be the projection, set $\mathcal{F}' = (g')^*\mathcal{F}$, and set $U' = (g')^{-1}(U)$. Finally, let $s' \in S'$ with image $s = g(s')$. In this case*

$$Ass_{X_s}(\mathcal{F}_s) \subset U_s \Leftrightarrow Ass_{X'_{s'}}(\mathcal{F}'_{s'}) \subset U'_{s'}.$$

Proof. This follows immediately from Divisors, Lemma 30.7.2. See also Divisors, Remark 30.7.3. □

Lemma 36.20.5. *Let $f : X \to Y$ be a morphism of finite presentation. Let $\mathcal{F}$ be a quasi-coherent $\mathcal{O}_X$-module of finite presentation. Let $U \subset X$ be an open subscheme such that $U \to Y$ is quasi-compact. Then the set*

$$E = \{y \in Y \mid Ass_{X_y}(\mathcal{F}_y) \subset U_y\}$$

is locally constructible in Y.

Proof. Let $y \in Y$. We have to show that there exists an open neighbourhood V of y in Y such that $E \cap V$ is constructible in V. Thus we may assume that Y is affine. Write $Y = \mathrm{Spec}(A)$ and $A = \mathrm{colim}\, A_i$ as a directed limit of finite type $\mathbf{Z}$-algebras. By Limits, Lemma 31.9.1 we can find an i and a morphism $f_i : X_i \to \mathrm{Spec}(A_i)$ of finite presentation whose base change to Y recovers f. After possibly increasing i

we may assume there exists a quasi-coherent $\mathcal{O}_{X_i}$-module $\mathcal{F}_i$ of finite presentation whose pullback to X is isomorphic to $\mathcal{F}$, see Limits, Lemma 31.9.2. After possibly increasing i one more time we may assume there exists an open subscheme $U_i \subset X_i$ whose inverse image in X is U, see Limits, Lemma 31.3.8. By Lemma 36.20.4 it suffices to prove the lemma for f_i. Thus we reduce to the case where Y is the spectrum of a Noetherian ring.

We will use the criterion of Topology, Lemma 5.15.3 to prove that E is constructible in case Y is a Noetherian scheme. To see this let $Z \subset Y$ be an irreducible closed subscheme. We have to show that $E \cap Z$ either contains a nonempty open subset or is not dense in Z. This follows from Lemmas 36.20.2 and 36.20.3 applied to the base change $(X, \mathcal{F}, U) \times_Y Z$ over Z. □

36.21. Reduced fibres

Lemma 36.21.1. *Let $f : X \to Y$ be a morphism of schemes. Assume Y irreducible with generic point η and f of finite type. If X_η is nonreduced, then there exists a nonempty open $V \subset Y$ such that for all $y \in V$ the fibre X_y is nonreduced.*

Proof. Let $Y' \subset Y$ be the reduction of Y. Let $X' \to Y'$ be the base change of f. Note that $Y' \to Y$ induces a bijection on points and that $X' \to X$ identifies fibres. Hence we may assume that Y' is reduced, i.e., integral, see Properties, Lemma 27.3.4. We may also replace Y by an affine open. Hence we may assume that $Y = \mathrm{Spec}(A)$ with A a domain. Denote $K = f.f.(A)$ the fraction field of A. Pick an affine open $\mathrm{Spec}(B) = U \subset X$ and a section $h_\eta \in \Gamma(U_\eta, \mathcal{O}_{U_\eta}) = B_K$ which is nonzero and nilpotent. After shrinking Y we may assume that h comes from $h \in \Gamma(U, \mathcal{O}_U) = B$. After shrinking Y a bit more we may assume that h is nilpotent. Let $I = \{b \in B \mid hb = 0\}$ be the annihilator of h. Then $C = B/I$ is a finite type A-algebra whose generic fiber $(B/I)_K$ is nonzero (as $h_\eta \neq 0$). We apply generic flatness to $A \to C$ and $A \to B/hB$, see Algebra, Lemma 10.112.3, and we obtain a $g \in A$, $g \neq 0$ such that C_g is free as an A_g-module and $(B/hB)_g$ is flat as an A_g-module. Replace Y by $D(g) \subset Y$. Now we have the short exact sequence

$$0 \to C \to B \to B/hB \to 0.$$

with B/hB flat over A and with C nonzero free as an A-module. It follows that for any homomorphism $A \to \kappa$ to a field the ring $C \otimes_A \kappa$ is nonzero and the sequence

$$0 \to C \otimes_A \kappa \to B \otimes_A \kappa \to B/hB \otimes_A \kappa \to 0$$

is exact, see Algebra, Lemma 10.36.11. Note that $B/hB \otimes_A \kappa = (B \otimes_A \kappa)/h(B \otimes_A \kappa)$ by right exactness of tensor product. Thus we conclude that multiplication by h is not zero on $B \otimes_A \kappa$. This clearly means that for any point $y \in Y$ the element h restricts to a nonzero element of U_y, whence X_y is nonreduced. □

Lemma 36.21.2. *Let $f : X \to Y$ be a morphism of schemes. Let $g : Y' \to Y$ be any morphism, and denote $f' : X' \to Y'$ the base change of f. Then*

$$\{y' \in Y' \mid X'_{y'} \text{ is geometrically reduced}\}$$
$$= g^{-1}(\{y \in Y \mid X_y \text{ is geometrically reduced}\}).$$

Proof. This comes down to the statement that for $y' \in Y'$ with image $y \in Y$ the fibre $X'_{y'} = X_y \times_y y'$ is geometrically reduced over $\kappa(y')$ if and only if X_y is geometrically reduced over $\kappa(y)$. This follows from Varieties, Lemma 32.4.6. □

Lemma 36.21.3. *Let $f : X \to Y$ be a morphism of schemes. Assume Y irreducible with generic point η and f of finite type. If X_η is not geometrically reduced, then there exists a nonempty open $V \subset Y$ such that for all $y \in V$ the fibre X_y is not geometrically reduced.*

Proof. Apply Lemma 36.19.7 to get

$$\begin{array}{ccccc} X' & \xrightarrow{g'} & X_V & \longrightarrow & X \\ \downarrow{\scriptstyle f'} & & \downarrow & & \downarrow{\scriptstyle f} \\ Y' & \xrightarrow{g} & V & \longrightarrow & Y \end{array}$$

with all the properties mentioned in that lemma. Let η' be the generic point of Y'. Consider the morphism $X' \to X_{Y'}$ (which is the reduction morphism) and the resulting morphism of generic fibres $X'_{\eta'} \to X_{\eta'}$. Since $X'_{\eta'}$ is geometrically reduced, and X_η is not this cannot be an isomorphism, see Varieties, Lemma 32.4.6. Hence $X_{\eta'}$ is nonreduced. Hence by Lemma 36.21.1 the fibres of $X_{Y'} \to Y'$ are nonreduced at all points $y' \in V'$ of a nonempty open $V' \subset Y'$. Since $g : Y' \to V$ is a homeomorphism Lemma 36.21.2 proves that $g(V')$ is the open we are looking for. □

Lemma 36.21.4. *Let $f : X \to Y$ be a morphism of schemes. Assume*

(1) *Y is irreducible with generic point η,*
(2) *X_η is geometrically reduced, and*
(3) *f is of finite type.*

Then there exists a nonempty open subscheme $V \subset Y$ such that $X_V \to V$ has geometrically reduced fibres.

Proof. Let $Y' \subset Y$ be the reduction of Y. Let $X' \to Y'$ be the base change of f. Note that $Y' \to Y$ induces a bijection on points and that $X' \to X$ identifies fibres. Hence we may assume that Y' is reduced, i.e., integral, see Properties, Lemma 27.3.4. We may also replace Y by an affine open. Hence we may assume that $Y = \mathrm{Spec}(A)$ with A a domain. Denote $K = f.f.(A)$ the fraction field of A. After shrinking Y a bit we may also assume that $X \to Y$ is flat and of finite presentation, see Morphisms, Proposition 28.28.1.

As X_η is geometrically reduced there exists an open dense subset $V \subset X_\eta$ such that $V \to \mathrm{Spec}(K)$ is smooth, see Varieties, Lemma 32.15.7. Let $U \subset X$ be the set of points where f is smooth. By Morphisms, Lemma 28.35.15 we see that $V \subset U_\eta$. Thus the generic fibre of U is dense in the generic fibre of X. Since X_η is reduced, it follows that U_η is scheme theoretically dense in X_η, see Morphisms, Lemma 28.7.8. We note that as $U \to Y$ is smooth all the fibres of $U \to Y$ are geometrically reduced. Thus it suffices to show that, after shrinking Y, for all $y \in Y$ the scheme U_y is scheme theoretically dense in X_y, see Morphisms, Lemma 28.7.9. This follows from Lemma 36.19.4. □

Lemma 36.21.5. *Let $f : X \to Y$ be a morphism of finite presentation. Then the set*

$$E = \{y \in Y \mid X_y \text{ is geometrically reduced}\}$$

is locally constructible in Y.

Proof. Let $y \in Y$. We have to show that there exists an open neighbourhood V of y in Y such that $E \cap V$ is constructible in V. Thus we may assume that Y is affine. Write $Y = \mathrm{Spec}(A)$ and $A = \mathrm{colim}\, A_i$ as a directed limit of finite type $\mathbf{Z}$-algebras. By Limits, Lemma 31.9.1 we can find an i and a morphism $f_i : X_i \to \mathrm{Spec}(A_i)$ of finite presentation whose base change to Y recovers f. By Lemma 36.21.2 it suffices to prove the lemma for f_i. Thus we reduce to the case where Y is the spectrum of a Noetherian ring.

We will use the criterion of Topology, Lemma 5.15.3 to prove that E is constructible in case Y is a Noetherian scheme. To see this let $Z \subset Y$ be an irreducible closed subscheme. We have to show that $E \cap Z$ either contains a nonempty open subset or is not dense in Z. If X_ξ is geometrically reduced, then Lemma 36.21.4 (applied to the morphism $X_Z \to Z$) implies that all fibres X_y are geometrically reduced for a nonempty open $V \subset Z$. If X_ξ is not geometrically reduced, then Lemma 36.21.3 (applied to the morphism $X_Z \to Z$) implies that all fibres X_y are geometrically reduced for a nonempty open $V \subset Z$. Thus we win. □

36.22. Irreducible components of fibres

Lemma 36.22.1. *Let $f : X \to Y$ be a morphism of schemes. Assume Y irreducible with generic point η and f of finite type. If X_η has n irreducible components, then there exists a nonempty open $V \subset Y$ such that for all $y \in V$ the fibre X_y has at least n irreducible components.*

Proof. As the question is purely topological we may replace X and Y by their reductions. In particular this implies that Y is integral, see Properties, Lemma 27.3.4. Let $X_\eta = X_{1,\eta} \cup \ldots \cup X_{n,\eta}$ be the decomposition of X_η into irreducible components. Let $X_i \subset X$ be the reduced closed subscheme whose generic fibre is $X_{i,\eta}$. Note that $Z_{i,j} = X_i \cap X_j$ is a closed subset of X_i whose generic fibre $Z_{i,j,\eta}$ is nowhere dense in $X_{i,\eta}$. Hence after shrinking Y we may assume that $Z_{i,j,y}$ is nowhere dense in $X_{i,y}$ for every $y \in Y$, see Lemma 36.19.3. After shrinking Y some more we may assume that $X_y = \bigcup X_{i,y}$ for $y \in Y$, see Lemma 36.19.5. Moreover, after shrinking Y we may assume that each $X_i \to Y$ is flat and of finite presentation, see Morphisms, Proposition 28.28.1. The morphisms $X_i \to Y$ are open, see Morphisms, Lemma 28.26.9. Thus there exists an open neighbourhood V of η which is contained in $f(X_i)$ for each i. For each $y \in V$ the schemes $X_{i,y}$ are nonempty closed subsets of X_y, we have $X_y = \bigcup X_{i,y}$ and the intersections $Z_{i,j,y} = X_{i,y} \cap X_{j,y}$ are not dense in $X_{i,y}$. Clearly this implies that X_y has at least n irreducible components. □

Lemma 36.22.2. *Let $f : X \to Y$ be a morphism of schemes. Let $g : Y' \to Y$ be any morphism, and denote $f' : X' \to Y'$ the base change of f. Then*

$$\{y' \in Y' \mid X'_{y'} \text{ is geometrically irreducible}\}$$
$$= g^{-1}(\{y \in Y \mid X_y \text{ is geometrically irreducible}\}).$$

Proof. This comes down to the statement that for $y' \in Y'$ with image $y \in Y$ the fibre $X'_{y'} = X_y \times_y y'$ is geometrically irreducible over $\kappa(y')$ if and only if X_y is geometrically irreducible over $\kappa(y)$. This follows from Varieties, Lemma 32.6.2. □

Lemma 36.22.3. *Let $f : X \to Y$ be a morphism of schemes. Let*

$$n_{X/Y} : Y \to \{0, 1, 2, 3, \ldots, \infty\}$$

be the function which associates to $y \in Y$ the number of irreducible components of $(X_y)_K$ where K is a separably closed extension of $\kappa(y)$. This is well defined and if $g : Y' \to Y$ is a morphism then

$$n_{X'/Y'} = n_{X/Y} \circ g$$

where $X' \to Y'$ is the base change of f.

Proof. Suppose that $y' \in Y'$ has image $y \in Y$. Suppose $K \supset \kappa(y)$ and $K' \supset \kappa(y')$ are separably closed extensions. Then we may choose a commutative diagram

$$\begin{array}{ccccc} K & \longrightarrow & K'' & \longleftarrow & K' \\ \uparrow & & & & \uparrow \\ \kappa(y) & & \longrightarrow & & \kappa(y') \end{array}$$

of fields. The result follows as the morphisms of schemes

$$(X'_{y'})_{K'} \longleftarrow (X'_{y'})_{K''} = (X_y)_{K''} \longrightarrow (X_y)_K$$

induce bijections between irreducible components, see Varieties, Lemma 32.6.7. □

Lemma 36.22.4. *Let A be a domain with fraction field K. Let $P \in A[x_1, \ldots, x_n]$. Denote $\overline{K}$ the algebraic closure of K. Assume P is irreducible in $\overline{K}[x_1, \ldots, x_n]$. Then there exists a $f \in A$ such that $P^\varphi \in \kappa[x_1, \ldots, x_n]$ is irreducible for all homomorphisms $\varphi : A_f \to \kappa$ into fields.*

Proof. There exists an automorphism Ψ of $A[x_1, \ldots, x_n]$ over A such that $\Psi(P) = ax_n^d +$ lower order terms in x_n with $a \neq 0$, see Algebra, Lemma 10.109.2. We may replace P by $\Psi(P)$ and we may replace A by A_a. Thus we may assume that P is monic in x_n of degree $d > 0$. For $i = 1, \ldots, n-1$ let d_i be the degree of P in x_i. Note that this implies that P^φ is monic of degree d in x_n and has degree $\leq d_i$ in x_i for every homomorphism $\varphi : A \to \kappa$ where κ is a field. Thus if P^φ is reducible, then we can write

$$P^\varphi = Q_1 Q_2$$

with Q_1, Q_2 monic of degree $e_1, e_2 \geq 0$ in x_n with $e_1 + e_2 = d$ and having degree $\leq d_i$ in x_i for $i = 1, \ldots, n-1$. In other words we can write

$$Q_j = x_n^{e_j} + \sum\nolimits_{0 \leq l < e_j} \left(\sum\nolimits_{L \in \mathcal{L}} a_{j,l,L} x^L \right) x_n^l \tag{36.22.4.1}$$

where the sum is over the set $\mathcal{L}$ of multi-indices L of the form $L = (l_1, \ldots, l_{n-1})$ with $0 \leq l_i \leq d_i$. For any $e_1, e_2 \geq 0$ with $e_1 + e_2 = d$ we consider the A-algebra

$$B_{e_1,e_2} = A[\{a_{1,l,L}\}_{0 \leq l < e_1, L \in \mathcal{L}}, \{a_{2,l,L}\}_{0 \leq l < e_2, L \in \mathcal{L}}]/(\text{relations})$$

where the (relations) is the ideal generated by the coefficients of the polynomial

$$P - Q_1 Q_2 \in A[\{a_{1,l,L}\}_{0 \leq l < e_1, L \in \mathcal{L}}, \{a_{2,l,L}\}_{0 \leq l < e_2, L \in \mathcal{L}}][x_1, \ldots, x_n]$$

with Q_1 and Q_2 defined as in (36.22.4.1). OK, and the assumption that P is irreducible over $\overline{K}$ implies that there does not exist any A-algebra homomorphism $B_{e_1,e_2} \to \overline{K}$. By the Hilbert Nullstellensatz, see Algebra, Theorem 10.31.1 this means that $B_{e_1,e_2} \otimes_A K = 0$. As B_{e_1,e_2} is a finitely generated A-algebra this signifies that we can find an $f_{e_1,e_2} \in A$ such that $(B_{e_1,e_2})_{f_{e_1,e_2}} = 0$. By construction this means that if $\varphi : A_{f_{e_1,e_2}} \to \kappa$ is a homomorphism to a field, then P^φ does not have

a factorization $P^\varphi = Q_1Q_2$ with Q_1 of degree e_1 in x_n and Q_2 of degree e_2 in x_n. Thus taking $f = \prod_{e1,e_2\geq 0, e_1+e_2=d} f_{e_1,e_2}$ we win. □

Lemma 36.22.5. *Let $f : X \to Y$ be a morphism of schemes. Assume*

(1) *Y is irreducible with generic point η,*
(2) *X_η is geometrically irreducible, and*
(3) *f is of finite type.*

Then there exists a nonempty open subscheme $V \subset Y$ such that $X_V \to V$ has geometrically irreducible fibres.

First proof of Lemma 36.22.5. We give two proofs of the lemma. These are essentially equivalent; the second is more self contained but a bit longer. Choose a diagram

$$\begin{array}{ccccc} X' & \xrightarrow{g'} & X_V & \longrightarrow & X \\ \downarrow{\scriptstyle f'} & & \downarrow & & \downarrow{\scriptstyle f} \\ Y' & \xrightarrow{g} & V & \longrightarrow & Y \end{array}$$

as in Lemma 36.19.7. Note that the generic fibre of f' is the reduction of the geometric fibre of f (see Lemma 36.19.6) and hence is geometrically irreducible. Suppose that the lemma holds for the morphism f'. Then after shrinking V all the fibres of f' are geometrically irreducible. As $X' = (Y' \times_V X_V)_{red}$ this implies that all the fibres of $Y' \times_V X_V$ are geometrically irreducible. Hence by Lemma 36.22.2 all the fibres of $X_V \to V$ are geometrically irreducible and we win. In this way we see that we may assume that the generic fibre is geometrically reduced as well as geometrically irreducible and we may assume $Y = \mathrm{Spec}(A)$ with A a domain.

Let $x \in X_\eta$ be the generic point. As X_η is geometrically irreducible and reduced we see that $L = \kappa(x)$ is a finitely generated extension of $K = \kappa(\eta) = f.f.(A)$ which is geometrically reduced and geometrically irreducible, see Varieties, Lemmas 32.4.2 and 32.6.6. In particular the field extension $K \subset L$ is separable, see Algebra, Lemma 10.41.1. Hence we can find $x_1, \ldots, x_{r+1} \in L$ which generate L over K and such that $x_1, \ldots, x_r$ is a transcendence basis for L over K, see Algebra, Lemma 10.39.3. Let $P \in K(x_1, \ldots, x_r)[T]$ be the minimal polynomial for x_{r+1}. Clearing denominators we may assume that P has coefficients in $A[x_1, \ldots, x_r]$. Note that as L is geometrically reduced and geometrically irreducible over K, the polynomial P is irreducible in $\overline{K}[x_1, \ldots, x_r, T]$ where $\overline{K}$ is the algebraic closure of K. Denote

$$B' = A[x_1, \ldots, x_{r+1}]/(P(x_{r+1}))$$

and set $X' = \mathrm{Spec}(B')$. By construction the fraction field of B' is isomorphic to $L = \kappa(x)$ as K-extensions. Hence there exists an open $U \subset X$, and open $U' \subset X'$ and a Y-isomorphism $U \to U'$, see Lemma 36.19.9. Here is a diagram:

$$\begin{array}{ccccccc} X & \longleftarrow & U & = & U' & \longrightarrow & X' = \mathrm{Spec}(B') \\ & \searrow & \downarrow & & \downarrow & \swarrow & \\ & & Y & = & Y & & \end{array}$$

Note that $U_\eta \subset X_\eta$ and $U'_\eta \subset X'_\eta$ are dense opens. Thus after shrinking Y by applying Lemma 36.19.3 we obtain that U_y is dense in X_y and U'_y is dense in X'_y for all $y \in Y$. Thus it suffices to prove the lemma for $X' \to Y$ which is the content of Lemma 36.22.4. □

Second proof of Lemma 36.22.5. Let $Y' \subset Y$ be the reduction of Y. Let $X' \to X$ be the reduction of X. Note that $X' \to X \to Y$ factors through Y', see Schemes, Lemma 25.12.6. As $Y' \to Y$ and $X' \to X$ are universal homeomorphisms by Morphisms, Lemma 28.45.4 we see that it suffices to prove the lemma for $X' \to Y'$. Thus we may assume that X and Y are reduced. In particular Y is integral, see Properties, Lemma 27.3.4. Thus by Morphisms, Proposition 28.28.1 there exists a nonempty affine open $V \subset Y$ such that $X_V \to V$ is flat and of finite presentation. After replacing Y by V we may assume, in addition to (1), (2), (3) that Y is integral affine, X is reduced, and f is flat and of finite presentation. In particular f is universally open, see Morphisms, Lemma 28.26.9.

Pick a nonempty affine open $U \subset X$. Then $U \to Y$ is flat and of finite presentation with geometrically irreducible generic fibre. The complement $X_\eta \setminus U_\eta$ is nowhere dense. Thus after shrinking Y we may assume $U_y \subset X_y$ is open dense for all $y \in Y$, see Lemma 36.19.3. Thus we may replace X by U and we reduce to the case where Y is integral affine and X is reduced affine, flat and of finite presentation over Y with geometrically irreducible generic fibre X_η.

Write $X = \mathrm{Spec}(B)$ and $Y = \mathrm{Spec}(A)$. Then A is a domain, B is reduced, $A \to B$ is flat of finite presentation, and B_K is geometrically irreducible over $K = f.f.(A)$. In particular we see that B_K is a domain. Let $L = f.f.(B_K)$ be its fraction field. Note that L is a finitely generated field extension of K as B is an A-algebra of finite presentation. Let $K \subset K'$ be a finite purely inseparable extension such that $(L \otimes_K K')_{red}$ is a separably generated field extension, see Algebra, Lemma 10.42.3. Choose $x_1, \ldots, x_n \in K'$ which generate the field extension K' over K, and such that $x_i^{q_i} \in A$ for some prime power q_i (proof existence x_i omitted). Let A' be the A-subalgebra of K' generated by $x_1, \ldots, x_n$. Then A' is a finite A-subalgebra $A' \subset K'$ whose fraction field is K'. Note that $\mathrm{Spec}(A') \to \mathrm{Spec}(A)$ is a universal homeomorphism, see Algebra, Lemma 10.43.2. Hence it suffices to prove the result after base changing to $\mathrm{Spec}(A')$. We are going to replace A by A' and B by $(B \otimes_A A')_{red}$ to arrive at the situation where L is a separably generated field extension of K. Of course it may happen that $(B \otimes_A A')_{red}$ is no longer flat, or of finite presentation over A', but this can be remedied by replacing A' by A'_f for a suitable $f \in A'$, see Algebra, Lemma 10.112.3.

At this point we know that A is a domain, B is reduced, $A \to B$ is flat and of finite presentation, B_K is a domain, and $L = f.f.(B_K)$ is a separably generated field extension of $K = f.f.(A)$. By Algebra, Lemma 10.39.3 we may write $L = K(x_1, \ldots, x_{r+1})$ where $x_1, \ldots, x_r$ are algebraically independent over K, and x_{r+1} is separable over $K(x_1, \ldots, x_r)$. After clearing denominators we may assume that the minimal polynomial $P \in K(x_1, \ldots, x_r)[T]$ of x_{r+1} over $K(x_1, \ldots, x_r)$ has coefficients in $A[x_1, \ldots, x_r]$. Note that since L/K is separable and since L is geometrically irreducible over K, the polynomial P is irreducible over the algebraic closure $\overline{K}$ of K. Denote

$$B' = A[x_1, \ldots, x_{r+1}]/(P(x_{r+1})).$$

By construction the fraction fields of B and B' are isomorphic as K-extensions. Hence there exists an isomorphism of A-algebras $B_h \cong B'_{h'}$ for suitable $h \in B$ and $h' \in B'$, see Lemma 36.19.9. In other words X and $X' = \mathrm{Spec}(B')$ have a common

affine open U. Here is a diagram:

$$\begin{array}{ccccc} X = \mathrm{Spec}(B) & \longleftarrow & U & \longrightarrow & \mathrm{Spec}(B') = X' \\ & \searrow & \downarrow & \swarrow & \\ & & Y = \mathrm{Spec}(A) & & \end{array}$$

After shrinking Y once more (by applying Lemma 36.19.3 to $Z = X \setminus U$ in X and $Z' = X' \setminus U$ in X') we see that U_y is dense in X_y and U_y is dense in X'_y for all $y \in Y$. Thus it suffices to prove the lemma for $X' \to Y$ which is the content of Lemma 36.22.4. □

Lemma 36.22.6. *Let $f : X \to Y$ be a morphism of schemes. Let $n_{X/Y}$ be the function on Y counting the numbers of geometrically irreducible components of fibres of f introduced in Lemma 36.22.3. Assume f of finite type. Let $y \in Y$ be a point. Then there exists a nonempty open $V \subset \overline{\{y\}}$ such that $n_{X/Y}|_V$ is constant.*

Proof. Let Z be the reduced induced scheme structure on $\overline{\{y\}}$. Let $f_Z : X_Z \to Z$ be the base change of f. Clearly it suffices to prove the lemma for f_Z and the generic point of Z. Hence we may assume that Y is an integral scheme, see Properties, Lemma 27.3.4. Our goal in this case is to produce a nonempty open $V \subset Y$ such that $n_{X/Y}|_V$ is constant.

We apply Lemma 36.19.8 to $f : X \to Y$ and we get $g : Y' \to V \subset Y$. As $g : Y' \to V$ is surjective finite étale, in particular open (see Morphisms, Lemma 28.37.13), it suffices to prove that there exists an open $V' \subset Y'$ such that $n_{X'/Y'}|_{V'}$ is constant, see Lemma 36.22.3. Thus we see that we may assume that all irreducible components of the generic fibre X_η are geometrically irreducible over $\kappa(\eta)$.

At this point suppose that $X_\eta = X_{1,\eta} \bigcup \ldots \bigcup X_{n,\eta}$ is the decomposition of the generic fibre into (geometrically) irreducible components. In particular $n_{X/Y}(\eta) = n$. Let X_i be the closure of $X_{i,\eta}$ in X. After shrinking Y we may assume that $X = \bigcup X_i$, see Lemma 36.19.5. After shrinking Y some more we see that each fibre of f has at least n irreducible components, see Lemma 36.22.1. Hence $n_{X/Y}(y) \geq n$ for all $y \in Y$. After shrinking Y some more we obtain that $X_{i,y}$ is geometrically irreducible for each i and all $y \in Y$, see Lemma 36.22.5. Since $X_y = \bigcup X_{i,y}$ this shows that $n_{X/Y}(y) \leq n$ and finishes the proof. □

Lemma 36.22.7. *Let $f : X \to Y$ be a morphism of schemes. Let $n_{X/Y}$ be the function on Y counting the numbers of geometrically irreducible components of fibres of f introduced in Lemma 36.22.3. Assume f of finite presentation. Then the level sets*

$$E_n = \{y \in Y \mid n_{X/Y}(y) = n\}$$

of $n_{X/Y}$ are locally constructible in Y.

Proof. Fix n. Let $y \in Y$. We have to show that there exists an open neighbourhood V of y in Y such that $E_n \cap V$ is constructible in V. Thus we may assume that Y is affine. Write $Y = \mathrm{Spec}(A)$ and $A = \mathrm{colim}\, A_i$ as a directed limit of finite type $\mathbf{Z}$-algebras. By Limits, Lemma 31.9.1 we can find an i and a morphism $f_i : X_i \to \mathrm{Spec}(A_i)$ of finite presentation whose base change to Y recovers f. By Lemma 36.22.3 it suffices to prove the lemma for f_i. Thus we reduce to the case where Y is the spectrum of a Noetherian ring.

We will use the criterion of Topology, Lemma 5.15.3 to prove that E_n is constructible in case Y is a Noetherian scheme. To see this let $Z \subset Y$ be an irreducible closed subscheme. We have to show that $E_n \cap Z$ either contains a nonempty open subset or is not dense in Z. Let $\xi \in Z$ be the generic point. Then Lemma 36.22.6 shows that $n_{X/Y}$ is constant in a neighbourhood of ξ in Z. This clearly implies what we want. □

36.23. Connected components of fibres

Lemma 36.23.1. *Let $f : X \to Y$ be a morphism of schemes. Assume Y irreducible with generic point η and f of finite type. If X_η has n connected components, then there exists a nonempty open $V \subset Y$ such that for all $y \in V$ the fibre X_y has at least n connected components.*

Proof. As the question is purely topological we may replace X and Y by their reductions. In particular this implies that Y is integral, see Properties, Lemma 27.3.4. Let $X_\eta = X_{1,\eta} \cup \ldots \cup X_{n,\eta}$ be the decomposition of X_η into connected components. Let $X_i \subset X$ be the reduced closed subscheme whose generic fibre is $X_{i,\eta}$. Note that $Z_{i,j} = X_i \cap X_j$ is a closed subset of X whose generic fibre $Z_{i,j,\eta}$ is empty. Hence after shrinking Y we may assume that $Z_{i,j} = \emptyset$, see Lemma 36.19.1. After shrinking Y some more we may assume that $X_y = \bigcup X_{i,y}$ for $y \in Y$, see Lemma 36.19.5. Moreover, after shrinking Y we may assume that each $X_i \to Y$ is flat and of finite presentation, see Morphisms, Proposition 28.28.1. The morphisms $X_i \to Y$ are open, see Morphisms, Lemma 28.26.9. Thus there exists an open neighbourhood V of η which is contained in $f(X_i)$ for each i. For each $y \in V$ the schemes $X_{i,y}$ are nonempty closed subsets of X_y, we have $X_y = \bigcup X_{i,y}$ and the intersections $Z_{i,j,y} = X_{i,y} \cap X_{j,y}$ are empty! Clearly this implies that X_y has at least n connected components. □

Lemma 36.23.2. *Let $f : X \to Y$ be a morphism of schemes. Let $g : Y' \to Y$ be any morphism, and denote $f' : X' \to Y'$ the base change of f. Then*

$$\begin{aligned} &\{y' \in Y' \mid X'_{y'} \text{ is geometrically connected}\} \\ &= g^{-1}(\{y \in Y \mid X_y \text{ is geometrically connected}\}). \end{aligned}$$

Proof. This comes down to the statement that for $y' \in Y'$ with image $y \in Y$ the fibre $X'_{y'} = X_y \times_y y'$ is geometrically connected over $\kappa(y')$ if and only if X_y is geometrically connected over $\kappa(y)$. This follows from Varieties, Lemma 32.5.3. □

Lemma 36.23.3. *Let $f : X \to Y$ be a morphism of schemes. Let*

$$n_{X/Y} : Y \to \{0, 1, 2, 3, \ldots, \infty\}$$

be the function which associates to $y \in Y$ the number of connected components of $(X_y)_K$ where K is a separably closed extension of $\kappa(y)$. This is well defined and if $g : Y' \to Y$ is a morphism then

$$n_{X'/Y'} = n_{X/Y} \circ g$$

where $X' \to Y'$ is the base change of f.

Proof. Suppose that $y' \in Y'$ has image $y \in Y$. Suppose $K \supset \kappa(y)$ and $K' \supset \kappa(y')$ are separably closed extensions. Then we may choose a commutative diagram

$$\begin{array}{ccccc} K & \longrightarrow & K'' & \longleftarrow & K' \\ \uparrow & & & & \uparrow \\ \kappa(y) & & \longrightarrow & & \kappa(y') \end{array}$$

of fields. The result follows as the morphisms of schemes

$$(X'_{y'})_{K'} \longleftarrow (X'_{y'})_{K''} = (X_y)_{K''} \longrightarrow (X_y)_K$$

induce bijections between connected components, see Varieties, Lemma 32.5.6. □

Lemma 36.23.4. *Let $f : X \to Y$ be a morphism of schemes. Assume*

(1) *Y is irreducible with generic point η,*
(2) *X_η is geometrically connected, and*
(3) *f is of finite type.*

Then there exists a nonempty open subscheme $V \subset Y$ such that $X_V \to V$ has geometrically connected fibres.

Proof. Choose a diagram

$$\begin{array}{ccccc} X' & \xrightarrow{g'} & X_V & \longrightarrow & X \\ {\scriptstyle f'}\downarrow & & \downarrow & & \downarrow{\scriptstyle f} \\ Y' & \xrightarrow{g} & V & \longrightarrow & Y \end{array}$$

as in Lemma 36.19.8. Note that the generic fibre of f' is geometrically connected (for example by Lemma 36.23.3). Suppose that the lemma holds for the morphism f'. This means that there exists a nonempty open $W \subset Y'$ such that every fibre of $X' \to Y'$ over W is geometrically connected. Then, as g is an open morphism by Morphisms, Lemma 28.37.13 all the fibres of f at points of the nonempty open $V = g(W)$ are geometrically connected, see Lemma 36.23.3. In this way we see that we may assume that the irreducible components of the generic fibre X_η are geometrically irreducible.

Let Y' be the reduction of Y, and set $X' = Y' \times_Y X$. Then it suffices to prove the lemma for the morphism $X' \to Y'$ (for example by Lemma 36.23.3 once again). Since the generic fibre of $X' \to Y'$ is the same as the generic fibre of $X \to Y$ we see that we may assume that Y is irreducible and reduced (i.e., integral, see Properties, Lemma 27.3.4) and that the irreducible components of the generic fibre X_η are geometrically irreducible.

At this point suppose that $X_\eta = X_{1,\eta} \bigcup \ldots \bigcup X_{n,\eta}$ is the decomposition of the generic fibre into (geometrically) irreducible components. Let X_i be the closure of $X_{i,\eta}$ in X. After shrinking Y we may assume that $X = \bigcup X_i$, see Lemma 36.19.5. Let $Z_{i,j} = X_i \cap X_j$. Let

$$\{1, \ldots, n\} \times \{1, \ldots, n\} = I \coprod J$$

where $(i, j) \in I$ if $Z_{i,j,\eta} = \emptyset$ and $(i, j) \in J$ if $Z_{i,j,\eta} \neq \emptyset$. After shrinking Y we may assume that $Z_{i,j} = \emptyset$ for all $(i, j) \in I$, see Lemma 36.19.1. After shrinking Y we obtain that $X_{i,y}$ is geometrically irreducible for each i and all $y \in Y$, see

Lemma 36.22.5. After shrinking Y some more we achieve the situation where each $Z_{i,j} \to Y$ is flat and of finite presentation for all $(i,j) \in J$, see Morphisms, Proposition 28.28.1. This means that $f(Z_{i,j}) \subset Y$ is open, see Morphisms, Lemma 28.26.9. We claim that

$$V = \bigcap\nolimits_{(i,j)\in J} f(Z_{i,j})$$

works, i.e., that X_y is geometrically connected for each $y \in V$. Namely, the fact that X_η is connected implies that the equivalence relation generated by the pairs in J has only one equivalence class. Now if $y \in V$ and $K \supset \kappa(y)$ is a separably closed extension, then the irreducible components of $(X_y)_K$ are the fibres $(X_{i,y})_K$. Moreover, we see by construction and $y \in V$ that $(X_{i,y})_K$ meets $(X_{j,y})_K$ if and only $(i,j) \in J$. Hence the remark on equivalence classes shows that $(X_y)_K$ is connected and we win. □

Lemma 36.23.5. *Let $f : X \to Y$ be a morphism of schemes. Let $n_{X/Y}$ be the function on Y counting the numbers of geometrically connected components of fibres of f introduced in Lemma 36.23.3. Assume f of finite type. Let $y \in Y$ be a point. Then there exists a nonempty open $V \subset \overline{\{y\}}$ such that $n_{X/Y}|_V$ is constant.*

Proof. Let Z be the reduced induced scheme structure on $\overline{\{y\}}$. Let $f_Z : X_Z \to Z$ be the base change of f. Clearly it suffices to prove the lemma for f_Z and the generic point of Z. Hence we may assume that Y is an integral scheme, see Properties, Lemma 27.3.4. Our goal in this case is to produce a nonempty open $V \subset Y$ such that $n_{X/Y}|_V$ is constant.

We apply Lemma 36.19.8 to $f : X \to Y$ and we get $g : Y' \to V \subset Y$. As $g : Y' \to V$ is surjective finite étale, in particular open (see Morphisms, Lemma 28.37.13), it suffices to prove that there exists an open $V' \subset Y'$ such that $n_{X'/Y'}|_{V'}$ is constant, see Lemma 36.22.3. Thus we see that we may assume that all irreducible components of the generic fibre X_η are geometrically irreducible over $\kappa(\eta)$. By Varieties, Lemma 32.6.15 this implies that also the connected components of X_η are geometrically connected.

At this point suppose that $X_\eta = X_{1,\eta} \bigcup \ldots \bigcup X_{n,\eta}$ is the decomposition of the generic fibre into (geometrically) connected components. In particular $n_{X/Y}(\eta) = n$. Let X_i be the closure of $X_{i,\eta}$ in X. After shrinking Y we may assume that $X = \bigcup X_i$, see Lemma 36.19.5. After shrinking Y some more we see that each fibre of f has at least n connected components, see Lemma 36.23.1. Hence $n_{X/Y}(y) \geq n$ for all $y \in Y$. After shrinking Y some more we obtain that $X_{i,y}$ is geometrically connected for each i and all $y \in Y$, see Lemma 36.23.4. Since $X_y = \bigcup X_{i,y}$ this shows that $n_{X/Y}(y) \leq n$ and finishes the proof. □

Lemma 36.23.6. *Let $f : X \to Y$ be a morphism of schemes. Let $n_{X/Y}$ be the function on Y counting the numbers of geometrically connected components of fibres of f introduced in Lemma 36.23.3. Assume f of finite presentation. Then the level sets*

$$E_n = \{y \in Y \mid n_{X/Y}(y) = n\}$$

of $n_{X/Y}$ are locally constructible in Y.

Proof. Fix n. Let $y \in Y$. We have to show that there exists an open neighbourhood V of y in Y such that $E_n \cap V$ is constructible in V. Thus we may assume

that Y is affine. Write $Y = \mathrm{Spec}(A)$ and $A = \mathrm{colim}\, A_i$ as a directed limit of finite type $\mathbf{Z}$-algebras. By Limits, Lemma 31.9.1 we can find an i and a morphism $f_i : X_i \to \mathrm{Spec}(A_i)$ of finite presentation whose base change to Y recovers f. By Lemma 36.23.3 it suffices to prove the lemma for f_i. Thus we reduce to the case where Y is the spectrum of a Noetherian ring.

We will use the criterion of Topology, Lemma 5.15.3 to prove that E_n is constructible in case Y is a Noetherian scheme. To see this let $Z \subset Y$ be an irreducible closed subscheme. We have to show that $E_n \cap Z$ either contains a nonempty open subset or is not dense in Z. Let $\xi \in Z$ be the generic point. Then Lemma 36.23.5 shows that $n_{X/Y}$ is constant in a neighbourhood of ξ in Z. This clearly implies what we want. □

Lemma 36.23.7. *Let $f : X \to S$ be a morphism of schemes. Assume that*

(1) *S is the spectrum of a discrete valuation ring,*
(2) *f is flat,*
(3) *X is connected,*
(4) *the closed fibre X_s is reduced.*

Then the generic fibre X_η is connected.

Proof. Write $Y = \mathrm{Spec}(R)$ and let $\pi \in R$ be a uniformizer. To get a contradiction assume that X_η is disconnected. This means there exists a nontrivial idempotent $e \in \Gamma(X_\eta, \mathcal{O}_{X_\eta})$. Let $U = \mathrm{Spec}(A)$ be any affine open in X. Note that π is a nonzerodivisor on A as A is flat over R, see More on Algebra, Lemma 15.15.3 for example. Then $e|_{U_\eta}$ corresponds to an element $e \in A[1/\pi]$. Let $z \in A$ be an element such that $e = z/\pi^n$ with $n \geq 0$ minimal. Note that $z^2 = \pi^n z$. This means that $z \bmod \pi A$ is nilpotent if $n > 0$. By assumption $A/\pi A$ is reduced, and hence minimality of n implies $n = 0$. Thus we conclude that $e \in A$! In other words $e \in \Gamma(X, \mathcal{O}_X)$. As X is connected it follows that e is a trivial idempotent which is a contradiction. □

36.24. Connected components meeting a section

The results in this section are in particular applicable to a group scheme $G \to S$ and its neutral section $e : S \to G$.

Situation 36.24.1. Here $f : X \to Y$ be a morphism of schemes, and $s : Y \to X$ is a section of f. For every $y \in Y$ we denote X_y^0 the connected component of X_y containing $s(y)$. Finally, we set $X^0 = \bigcup_{y \in Y} X_y^0$.

Lemma 36.24.2. *Let $f : X \to Y$, $s : Y \to X$ be as in Situation 36.24.1. If $g : Y' \to Y$ is any morphism, consider the base change diagram*

$$\begin{array}{ccc} X' & \xrightarrow{g'} & X \\ {\scriptstyle s'}\uparrow \downarrow{\scriptstyle f'} & & {\scriptstyle f}\downarrow \uparrow{\scriptstyle s} \\ Y' & \xrightarrow{g} & Y \end{array}$$

so that we obtain $(X')^0 \subset X'$. Then $(X')^0 = (g')^{-1}(X^0)$.

Proof. Let $y' \in Y'$ with image $y \in Y$. We may think of X_y^0 as a closed subscheme of X_y, see for example Morphisms, Definition 28.27.3. As $s(y) \in X_y^0$ we conclude from Varieties, Lemma 32.5.14 that X_y^0 is a geometrically connected scheme over

$\kappa(y)$. Hence $X^0_y \times_y y' \to X'_{y'}$ is a connected closed subscheme which contains $s'(y')$. Thus $X^0_y \times_y y' \subset (X'_{y'})^0$. The other inclusion $X^0_y \times_y y' \supset (X'_{y'})^0$ is clear as the image of $(X'_{y'})^0$ in X_y is a connected subset of X_y which contains $s(y)$. □

Lemma 36.24.3. *Let $f : X \to Y$, $s : Y \to X$ be as in Situation 36.24.1. Assume f of finite type. Let $y \in Y$ be a point. Then there exists a nonempty open $V \subset \overline{\{y\}}$ such that the inverse image of X^0 in the base change X_V is open and closed in X_V.*

Proof. Let $Z \subset Y$ be the induced reduced closed subscheme structure on $\overline{\{y\}}$. Let $f_Z : X_Z \to Z$ and $s_Z : Z \to X_Z$ be the base changes of f and s. By Lemma 36.24.2 we have $(X_Z)^0 = (X^0)_Z$. Hence it suffices to prove the lemma for the morphism $X_Z \to Z$ and the point $x \in X_Z$ which maps to the generic point of Z. In other words we have reduced the problem to the case where Y is an integral scheme (see Properties, Lemma 27.3.4) with generic point η. Our goal is to show that after shrinking Y the subset X^0 becomes an open and closed subset of X.

Note that the scheme X_η is of finite type over a field, hence Noetherian. Thus its connected components are open as well as closed. Hence we may write $X_\eta = X^0_\eta \coprod T_\eta$ for some open and closed subset T_η of X_η. Next, let $T \subset X$ be the closure of T_η and let $X^{00} \subset X$ be the closure of X^0_η. Note that T_η, resp. X^0_η is the generic fibre of T, resp. X^{00}, see discussion preceding Lemma 36.19.5. Moreover, that lemma implies that after shrinking Y we may assume that $X = X^{00} \cup T$ (set theoretically). Note that $(T \cap X^{00})_\eta = T_\eta \cap X^0_\eta = \emptyset$. Hence after shrinking Y we may assume that $T \cap X^{00} = \emptyset$, see Lemma 36.19.1. In particular X^{00} is open in X. Note that X^0_η is connected and has a rational point, namely $s(\eta)$, hence it is geometrically connected, see Varieties, Lemma 32.5.14. Thus after shrinking Y we may assume that all fibres of $X^{00} \to Y$ are geometrically connected, see Lemma 36.23.4. At this point it follows that the fibres X^{00}_y are open, closed, and connected subsets of X_y containing $\sigma(y)$. It follows that $X^0 = X^{00}$ and we win. □

Lemma 36.24.4. *Let $f : X \to Y$, $s : Y \to X$ be as in Situation 36.24.1. If f is of finite presentation then X^0 is locally constructible in X.*

Proof. Let $x \in X$. We have to show that there exists an open neighbourhood U of x such that $X^0 \cap U$ is constructible in U. This reduces us to the case where Y is affine. Write $Y = \mathrm{Spec}(A)$ and $A = \mathrm{colim}\, A_i$ as a directed limit of finite type $\mathbf{Z}$-algebras. By Limits, Lemma 31.9.1 we can find an i and a morphism $f_i : X_i \to \mathrm{Spec}(A_i)$ of finite presentation, endowed with a section $s_i : \mathrm{Spec}(A_i) \to X_i$ whose base change to Y recovers f and the section s. By Lemma 36.24.2 it suffices to prove the lemma for f_i, s_i. Thus we reduce to the case where Y is the spectrum of a Noetherian ring.

Assume Y is a Noetherian affine scheme. Since f is of finite presentation, i.e., of finite type, we see that X is a Noetherian scheme too, see Morphisms, Lemma 28.16.6. In order to prove the lemma in this case it suffices to show that for every irreducible closed subset $Z \subset X$ the intersection $Z \cap X^0$ either contains a nonempty open of Z or is not dense in Z, see Topology, Lemma 5.15.3. Let $x \in Z$ be the generic point, and let $y = f(x)$. By Lemma 36.24.3 there exists a nonempty open subset $V \subset \overline{\{y\}}$ such that $X^0 \cap X_V$ is open and closed in X_V. Since $f(Z) \subset \overline{\{y\}}$ and $f(x) = y \in V$ we see that $W = f^{-1}(V) \cap Z$ is a nonempty open subset of Z.

It follows that $X^0 \cap W$ is open and closed in W. Since W is irreducible we see that $X^0 \cap W$ is either empty or equal to W. This proves the lemma. □

Lemma 36.24.5. *Let $f : X \to Y$, $s : Y \to X$ be as in Situation 36.24.1. Let $y \in Y$ be a point. Assume*

(1) *f is of finite presentation and flat, and*
(2) *the fibre X_y is geometrically reduced.*

Then X^0 is a neighbourhood of X^0_y in X.

Proof. We may replace Y with an affine open neighbourhood of y. Write $Y = \mathrm{Spec}(A)$ and $A = \mathrm{colim}\, A_i$ as a directed limit of finite type $\mathbf{Z}$-algebras. By Limits, Lemma 31.9.1 we can find an i and a morphism $f_i : X_i \to \mathrm{Spec}(A_i)$ of finite presentation, endowed with a section $s_i : \mathrm{Spec}(A_i) \to X_i$ whose base change to Y recovers f and the section s. After possibly increasing i we may also assume that f_i is flat, see Limits, Lemma 31.7.6. Let y_i be the image of y in Y_i. Note that $X_y = (X_{i,y_i}) \times_{y_i} y$. Hence X_{i,y_i} is geometrically reduced, see Varieties, Lemma 32.4.6. By Lemma 36.24.2 it suffices to prove the lemma for the system $f_i, s_i, y_i \in Y_i$. Thus we reduce to the case where Y is the spectrum of a Noetherian ring.

Assume Y is the spectrum of a Noetherian ring. Since f is of finite presentation, i.e., of finite type, we see that X is a Noetherian scheme too, see Morphisms, Lemma 28.16.6. Let $x \in X^0$ be a point lying over y. By Topology, Lemma 5.15.4 it suffices to prove that for any irreducible closed $Z \subset X$ passing through x the intersection $X^0 \cap Z$ is dense in Z. In particular it suffices to prove that the generic point $x' \in Z$ is in X^0. By Properties, Lemma 27.5.9 we can find a discrete valuation ring R and a morphism $\mathrm{Spec}(R) \to X$ which maps the special point to x and the generic point to x'. We are going to think of $\mathrm{Spec}(R)$ as a scheme over Y via the composition $\mathrm{Spec}(R) \to X \to Y$. By Lemma 36.24.2 we have that $(X_R)^0$ is the inverse image of X^0. By construction we have a second section $t : \mathrm{Spec}(R) \to X_R$ (besides the base change s_R of s) of the structure morphism $X_R \to \mathrm{Spec}(R)$ such that $t(\eta_R)$ is a point of X_R which maps to x' and $t(0_R)$ is a point of X_R which maps to x. Note that $t(0_R)$ is in $(X_R)^0$ and that $t(\eta_R) \rightsquigarrow t(0_R)$. Thus it suffices to prove that this implies that $t(\eta_R) \in (X_R)^0$. Hence it suffices to prove the lemma in the case where Y is the spectrum of a discrete valuation ring and y its closed point.

Assume Y is the spectrum of a discrete valuation ring and y is its closed point. Our goal is to prove that X^0 is a neighbourhood of X^0_y. Note that X^0_y is open and closed in X_y as X_y has finitely many irreducible components. Hence the complement $C = X_y \setminus X^0_y$ is closed in X. Thus $U = X \setminus C$ is an open neighbourhood of X^0_y and $U^0 = X^0$. Hence it suffices to prove the result for the morphism $U \to Y$. In other words, we may assume that X_y is connected. Suppose that X is disconnected, say $X = X_1 \amalg \ldots \amalg X_n$ is a decomposition into connected components. Then $s(Y)$ is completely contained in one of the X_i. Say $s(Y) \subset X_1$. Then $X^0 \subset X_1$. Hence we may replace X by X_1 and assume that X is connected. At this point Lemma 36.23.7 implies that X_η is connected, i.e., $X^0 = X$ and we win. □

Lemma 36.24.6. *Let $f : X \to Y$, $s : Y \to X$ be as in Situation 36.24.1. Assume*

(1) *f is of finite presentation and flat, and*
(2) *all fibres of f are geometrically reduced.*

Then X^0 is open in X.

Proof. This is an immediate consequence of Lemma 36.24.5. □

36.25. Dimension of fibres

Lemma 36.25.1. *Let $f : X \to Y$ be a morphism of schemes. Assume Y irreducible with generic point η and f of finite type. If X_η has dimension n, then there exists a nonempty open $V \subset Y$ such that for all $y \in V$ the fibre X_y has dimension n.*

Proof. Let $Z = \{x \in X \mid \dim_x(X_{f(x)}) > n\}$. By Morphisms, Lemma 28.29.4 this is a closed subset of X. By assumption $Z_\eta = \emptyset$. Hence by Lemma 36.19.1 we may shrink Y and assume that $Z = \emptyset$. Let $Z' = \{x \in X \mid \dim_x(X_{f(x)}) > n-1\} = \{x \in X \mid \dim_x(X_{f(x)}) = n\}$. As before this is a closed subset of X. By assumption we have $Z'_\eta \neq \emptyset$. Hence after shrinking Y we may assume that $Z' \to Y$ is surjective, see Lemma 36.19.2. Hence we win. □

Lemma 36.25.2. *Let $f : X \to Y$ be a morphism of finite type. Let*

$$n_{X/Y} : Y \to \{0, 1, 2, 3, \ldots, \infty\}$$

be the function which associates to $y \in Y$ the dimension of X_y. If $g : Y' \to Y$ is a morphism then

$$n_{X'/Y'} = n_{X/Y} \circ g$$

where $X' \to Y'$ is the base change of f.

Proof. This follows from Morphisms, Lemma 28.29.3. □

Lemma 36.25.3. *Let $f : X \to Y$ be a morphism of schemes. Let $n_{X/Y}$ be the function on Y giving the dimension of fibres of f introduced in Lemma 36.25.2. Assume f of finite presentation. Then the level sets*

$$E_n = \{y \in Y \mid n_{X/Y}(y) = n\}$$

of $n_{X/Y}$ are locally constructible in Y.

Proof. Fix n. Let $y \in Y$. We have to show that there exists an open neighbourhood V of y in Y such that $E_n \cap V$ is constructible in V. Thus we may assume that Y is affine. Write $Y = \operatorname{Spec}(A)$ and $A = \operatorname{colim} A_i$ as a directed limit of finite type $\mathbf{Z}$-algebras. By Limits, Lemma 31.9.1 we can find an i and a morphism $f_i : X_i \to \operatorname{Spec}(A_i)$ of finite presentation whose base change to Y recovers f. By Lemma 36.25.2 it suffices to prove the lemma for f_i. Thus we reduce to the case where Y is the spectrum of a Noetherian ring.

We will use the criterion of Topology, Lemma 5.15.3 to prove that E_n is constructible in case Y is a Noetherian scheme. To see this let $Z \subset Y$ be an irreducible closed subscheme. We have to show that $E_n \cap Z$ either contains a nonempty open subset or is not dense in Z. Let $\xi \in Z$ be the generic point. Then Lemma 36.25.1 shows that $n_{X/Y}$ is constant in a neighbourhood of ξ in Z. This implies what we want. □

36.26. Limit arguments

Some lemmas involving limits of schemes, and Noetherian approximation. We stick mostly to the affine case. Some of these lemmas are special cases of lemmas in the chapter on limits.

Lemma 36.26.1. *Let $f : X \to S$ be a morphism of affine schemes, which is of finite presentation. Then there exists a cartesian diagram*

$$\begin{array}{ccc} X_0 & \xleftarrow{g} & X \\ {\scriptstyle f_0}\downarrow & & \downarrow{\scriptstyle f} \\ S_0 & \longleftarrow & S \end{array}$$

such that

(1) *X_0, S_0 are affine schemes,*
(2) *S_0 of finite type over $\mathbf{Z}$,*
(3) *f_0 is finite of finite type.*

Proof. Write $S = \mathrm{Spec}(A)$ and $X = \mathrm{Spec}(B)$. As f is of finite presentation we see that B is of finite presentation as an A-algebra, see Morphisms, Lemma 28.22.2. Thus the lemma follows from Algebra, Lemma 10.121.15. □

Lemma 36.26.2. *Let $f : X \to S$ be a morphism of affine schemes, which is of finite presentation. Let $\mathcal{F}$ be a quasi-coherent $\mathcal{O}_X$-module of finite presentation. Then there exists a diagram as in Lemma 36.26.1 such that there exists a coherent $\mathcal{O}_{X_0}$-module $\mathcal{F}_0$ with $g^*\mathcal{F}_0 = \mathcal{F}$.*

Proof. Write $S = \mathrm{Spec}(A)$, $X = \mathrm{Spec}(B)$, and $\mathcal{F} = \widetilde{M}$. As f is of finite presentation we see that B is of finite presentation as an A-algebra, see Morphisms, Lemma 28.22.2. As $\mathcal{F}$ is of finite presentation over $\mathcal{O}_X$ we see that M is of finite presentation as a B-module, see Properties, Lemma 27.16.2. Thus the lemma follows from Algebra, Lemma 10.121.15. □

Lemma 36.26.3. *Let $f : X \to S$ be a morphism of affine schemes, which is of finite presentation. Let $\mathcal{F}$ be a quasi-coherent $\mathcal{O}_X$-module of finite presentation and flat over S. Then we may choose a diagram as in Lemma 36.26.2 and sheaf $\mathcal{F}_0$ such that in addition $\mathcal{F}_0$ is flat over S_0.*

Proof. Write $S = \mathrm{Spec}(A)$, $X = \mathrm{Spec}(B)$, and $\mathcal{F} = \widetilde{M}$. As f is of finite presentation we see that B is of finite presentation as an A-algebra, see Morphisms, Lemma 28.22.2. As $\mathcal{F}$ is of finite presentation over $\mathcal{O}_X$ we see that M is of finite presentation as a B-module, see Properties, Lemma 27.16.2. As $\mathcal{F}$ is flat over S we see that M is flat over A, see Morphisms, Lemma 28.26.2. Thus the lemma follows from Algebra, Lemma 10.154.1. □

Lemma 36.26.4. *Let $f : X \to S$ be a morphism of affine schemes, which is of finite presentation and flat. Then there exists a diagram as in Lemma 36.26.1 such that in addition f_0 is flat.*

Proof. This is a special case of Lemma 36.26.3. □

Lemma 36.26.5. *Let $f : X \to S$ be a morphism of affine schemes, which is smooth. Then there exists a diagram as in Lemma 36.26.1 such that in addition f_0 is smooth.*

Proof. Write $S = \mathrm{Spec}(A)$, $X = \mathrm{Spec}(B)$, and as f is smooth we see that B is smooth as an A-algebra, see Morphisms, Lemma 28.35.2. Hence the lemma follows from Algebra, Lemma 10.131.14. □

Lemma 36.26.6. *Let $f : X \to S$ be a morphism of affine schemes, which is of finite presentation with geometrically reduced fibres. Then there exists a diagram as in Lemma 36.26.1 such that in addition f_0 has geometrically reduced fibres.*

Proof. Apply Lemma 36.26.1 to get a cartesian diagram

$$\begin{array}{ccc} X_0 & \xleftarrow{g} & X \\ {\scriptstyle f_0}\downarrow & & \downarrow{\scriptstyle f} \\ S_0 & \xleftarrow{h} & S \end{array}$$

of affine schemes with $X_0 \to S_0$ a finite type morphism of schemes of finite type over $\mathbf{Z}$. By Lemma 36.21.5 the set $E \subset S_0$ of points where the fibre of f_0 is geometrically reduced is a constructible subset. By Lemma 36.21.2 we have $h(S) \subset E$. Write $S_0 = \mathrm{Spec}(A_0)$ and $S = \mathrm{Spec}(A)$. Write $A = \mathrm{colim}_i A_i$ as a direct colimit of finite type A_0-algebras. By Limits, Lemma 31.3.7 we see that $\mathrm{Spec}(A_i) \to S_0$ has image contained in E for some i. After replacing S_0 by $\mathrm{Spec}(A_i)$ and X_0 by $X_0 \times_{S_0} \mathrm{Spec}(A_i)$ we see that all fibres of f_0 are geometrically reduced. □

Lemma 36.26.7. *Let $f : X \to S$ be a morphism of affine schemes, which is of finite presentation with geometrically irreducible fibres. Then there exists a diagram as in Lemma 36.26.1 such that in addition f_0 has geometrically irreducible fibres.*

Proof. Apply Lemma 36.26.1 to get a cartesian diagram

$$\begin{array}{ccc} X_0 & \xleftarrow{g} & X \\ {\scriptstyle f_0}\downarrow & & \downarrow{\scriptstyle f} \\ S_0 & \xleftarrow{h} & S \end{array}$$

of affine schemes with $X_0 \to S_0$ a finite type morphism of schemes of finite type over $\mathbf{Z}$. By Lemma 36.22.7 the set $E \subset S_0$ of points where the fibre of f_0 is geometrically irreducible is a constructible subset. By Lemma 36.22.2 we have $h(S) \subset E$. Write $S_0 = \mathrm{Spec}(A_0)$ and $S = \mathrm{Spec}(A)$. Write $A = \mathrm{colim}_i A_i$ as a direct colimit of finite type A_0-algebras. By Limits, Lemma 31.3.7 we see that $\mathrm{Spec}(A_i) \to S_0$ has image contained in E for some i. After replacing S_0 by $\mathrm{Spec}(A_i)$ and X_0 by $X_0 \times_{S_0} \mathrm{Spec}(A_i)$ we see that all fibres of f_0 are geometrically irreducible. □

Lemma 36.26.8. *Let $f : X \to S$ be a morphism of affine schemes, which is of finite presentation with geometrically connected fibres. Then there exists a diagram as in Lemma 36.26.1 such that in addition f_0 has geometrically connected fibres.*

Proof. Apply Lemma 36.26.1 to get a cartesian diagram

$$\begin{array}{ccc} X_0 & \xleftarrow{g} & X \\ {\scriptstyle f_0}\downarrow & & \downarrow{\scriptstyle f} \\ S_0 & \xleftarrow{h} & S \end{array}$$

of affine schemes with $X_0 \to S_0$ a finite type morphism of schemes of finite type over $\mathbf{Z}$. By Lemma 36.23.6 the set $E \subset S_0$ of points where the fibre of f_0 is geometrically connected is a constructible subset. By Lemma 36.23.2 we have $h(S) \subset E$. Write $S_0 = \mathrm{Spec}(A_0)$ and $S = \mathrm{Spec}(A)$. Write $A = \mathrm{colim}_i A_i$ as a direct colimit of finite type A_0-algebras. By Limits, Lemma 31.3.7 we see that $\mathrm{Spec}(A_i) \to S_0$

has image contained in E for some i. After replacing S_0 by $\mathrm{Spec}(A_i)$ and X_0 by $X_0 \times_{S_0} \mathrm{Spec}(A_i)$ we see that all fibres of f_0 are geometrically connected. □

Lemma 36.26.9. *Let $d \geq 0$ be an integer. Let $f : X \to S$ be a morphism of affine schemes, which is of finite presentation all of whose fibres have dimension d. Then there exists a diagram as in Lemma 36.26.1 such that in addition all fibres of f_0 have dimension d.*

Proof. Apply Lemma 36.26.1 to get a cartesian diagram

$$\begin{array}{ccc} X_0 & \xleftarrow{g} & X \\ {\scriptstyle f_0}\downarrow & & \downarrow{\scriptstyle f} \\ S_0 & \xleftarrow{h} & S \end{array}$$

of affine schemes with $X_0 \to S_0$ a finite type morphism of schemes of finite type over $\mathbf{Z}$. By Lemma 36.25.3 the set $E \subset S_0$ of points where the fibre of f_0 has dimension d is a constructible subset. By Lemma 36.25.2 we have $h(S) \subset E$. Write $S_0 = \mathrm{Spec}(A_0)$ and $S = \mathrm{Spec}(A)$. Write $A = \mathrm{colim}_i A_i$ as a direct colimit of finite type A_0-algebras. By Limits, Lemma 31.3.7 we see that $\mathrm{Spec}(A_i) \to S_0$ has image contained in E for some i. After replacing S_0 by $\mathrm{Spec}(A_i)$ and X_0 by $X_0 \times_{S_0} \mathrm{Spec}(A_i)$ we see that all fibres of f_0 have dimension d. □

Lemma 36.26.10. *Let $f : X \to S$ be a morphism of affine schemes, which is standard syntomic (see Morphisms, Definition 28.32.1). Then there exists a diagram as in Lemma 36.26.1 such that in addition f_0 is standard syntomic.*

Proof. This lemma is a copy of Algebra, Lemma 10.129.12. □

Lemma 36.26.11. *(Noetherian approximation and combining properties.) Let P, Q be properties of morphisms of schemes which are stable under base change. Let $f : X \to S$ be a morphism of finite presentation of affine schemes. Assume we can find cartesian diagrams*

$$\begin{array}{ccc} X_1 & \longleftarrow & X \\ {\scriptstyle f_1}\downarrow & & \downarrow{\scriptstyle f} \\ S_1 & \longleftarrow & S \end{array} \quad \text{and} \quad \begin{array}{ccc} X_2 & \longleftarrow & X \\ {\scriptstyle f_2}\downarrow & & \downarrow{\scriptstyle f} \\ S_2 & \longleftarrow & S \end{array}$$

of affine schemes, with S_1, S_2 of finite type over $\mathbf{Z}$ and f_1, f_2 of finite type such that f_1 has property P and f_2 has property Q. Then we can find a cartesian diagram

$$\begin{array}{ccc} X_0 & \longleftarrow & X \\ {\scriptstyle f_0}\downarrow & & \downarrow{\scriptstyle f} \\ S_0 & \longleftarrow & S \end{array}$$

of affine schemes with S_0 of finite type over $\mathbf{Z}$ and f_0 of finite type such that f_0 has both property P and property Q.

Proof. The given pair of diagrams correspond to cocartesian diagrams of rings

$$\begin{array}{ccc} B_1 & \longrightarrow & B \\ \uparrow & & \uparrow \\ A_1 & \longrightarrow & A \end{array} \quad \text{and} \quad \begin{array}{ccc} B_2 & \longrightarrow & B \\ \uparrow & & \uparrow \\ A_2 & \longrightarrow & A \end{array}$$

Let $A_0 \subset A$ be a finite type $\mathbf{Z}$-subalgebra of A containing the image of both $A_1 \to A$ and $A_2 \to A$. Such a subalgebra exists because by assumption both A_1 and A_2 are of finite type over $\mathbf{Z}$. Note that the rings $B_{0,1} = B_1 \otimes_{A_1} A_0$ and $B_{0,2} = B_2 \otimes_{A_2} A_0$ are finite type A_0-algebras with the property that $B_{0,1} \otimes_{A_0} A \cong B \cong B_{0,2} \otimes_{A_0} A$ as A-algebras. As A is the directed colimit of its finite type A_0-subalgebras, by Limits, Lemma 31.9.1 we may assume after enlarging A_0 that there exists an isomorphism $B_{0,1} \cong B_{0,2}$ as A_0-algebras. Since properties P and Q are assumed stable under base change we conclude that setting $S_0 = \operatorname{Spec}(A_0)$ and

$$X_0 = X_1 \times_{S_1} S_0 = \operatorname{Spec}(B_{0,1}) \cong \operatorname{Spec}(B_{0,2}) = X_2 \times_{S_2} S_0$$

works. □

36.27. Étale neighbourhoods

It turns out that some properties of morphisms are easier to study after doing an étale base change. It is convenient to introduce the following terminology.

Definition 36.27.1. Let S be a scheme. Let $s \in S$ be a point.

(1) An *étale neighbourhood of* (S, s) is a pair (U, u) together with an étale morphism of schemes $\varphi : U \to S$ such that $\varphi(u) = s$.
(2) A *morphism of étale neighbourhoods* $f : (V, v) \to (U, u)$ of (S, s) is simply a morphism of S-schemes $f : V \to U$ such that $f(v) = u$.
(3) An *elementary étale neighbourhood* is an étale neighbourhood $\varphi : (U, u) \to (S, s)$ such that $\kappa(s) = \kappa(u)$.

If $f : (V, v) \to (U, u)$ is a morphism of étale neighbourhoods, then f is automatically étale, see Morphisms, Lemma 28.37.18. Hence it turns (V, v) into an étale neighbourhood of (U, u). Of course, since the composition of étale morphisms is étale (Morphisms, Lemma 28.37.3) we see that conversely any étale neighbourhood (V, v) of (U, u) is an étale neighbourhood of (S, s) as well. We also remark that if $U \subset S$ is an open neighbourhood of s, then $(U, s) \to (S, s)$ is an étale neighbourhood. This follows from the fact that an open immersion is étale (Morphisms, Lemma 28.37.9). We will use these remarks without further mention throughout this section.

Note that $\kappa(s) \subset \kappa(u)$ is a finite separable extension if $(U, u) \to (S, s)$ is an étale neighbourhood, see Morphisms, Lemma 28.37.15.

Lemma 36.27.2. *Let S be a scheme. Let $s \in S$. Let $\kappa(s) \subset k$ be a finite separable field extension. Then there exists an étale neighbourhood $(U, u) \to (S, s)$ such that the field extension $\kappa(s) \subset \kappa(u)$ is isomorphic to $\kappa(s) \subset k$.*

Proof. We may assume S is affine. In this case the lemma follows from Algebra, Lemma 10.136.16. □

Lemma 36.27.3. *Let S be a scheme, and let s be a point of S. The category of étale neighborhoods has the following properties:*

(1) *Let $(U_i, u_i)_{i=1,2}$ be two étale neighborhoods of s in S. Then there exists a third étale neighborhood (U, u) and morphisms $(U, u) \to (U_i, u_i)$, $i = 1, 2$.*
(2) *Let $h_1, h_2 : (U, u) \to (U', u')$ be two morphisms between étale neighborhoods of s. Assume h_1, h_2 induce the same map $\kappa(u') \to \kappa(u)$ of residue fields. Then there exist an étale neighborhood (U'', u'') and a morphism $h : (U'', u'') \to (U, u)$ which equalizes h_1 and h_2, i.e., such that $h_1 \circ h = h_2 \circ h$.*

Proof. For part (1), consider the fibre product $U = U_1 \times_S U_2$. It is étale over both U_1 and U_2 because étale morphisms are preserved under base change, see Morphisms, Lemma 28.37.4. There is a point of U mapping to both u_1 and u_2 for example by the description of points of a fibre product in Schemes, Lemma 25.17.5. For part (2), define U'' as the fibre product

$$\begin{array}{ccc} U'' & \longrightarrow & U \\ \downarrow & & \downarrow{\scriptstyle (h_1,h_2)} \\ U' & \xrightarrow{\Delta} & U' \times_S U'. \end{array}$$

Since h_1 and h_2 induce the same map of residue fields $\kappa(u') \to \kappa(u)$ there exists a point $u'' \in U''$ lying over u' with $\kappa(u'') = \kappa(u')$. In particular $U'' \neq \emptyset$. Moreover, since U' is étale over S, so is the fibre product $U' \times_S U'$ (see Morphisms, Lemmas 28.37.4 and 28.37.3). Hence the vertical arrow (h_1, h_2) is étale by Morphisms, Lemma 28.37.18. Therefore U'' is étale over U' by base change, and hence also étale over S (because compositions of étale morphisms are étale). Thus (U'', u'') is a solution to the problem. □

Lemma 36.27.4. *Let S be a scheme, and let s be a point of S. The category of elementary étale neighborhoods of (S, s) is cofiltered (see Categories, Definition 4.20.1).*

Proof. This is immediate from the definitions and Lemma 36.27.3. □

Lemma 36.27.5. *Let S be a scheme. Let $s \in S$. Then we have*

$$\mathcal{O}^h_{S,s} = \operatorname{colim}_{(U,u)} \mathcal{O}(U)$$

where the colimit is over the filtered category which is opposite to the category of elementary étale neighbourhoods (U, u) of (S, s).

Proof. Let $\operatorname{Spec}(A) \subset S$ be an affine neighbourhood of s. Let $\mathfrak{p} \subset A$ be the prime ideal corresponding to s. With these choices we have canonical isomorphisms $\mathcal{O}_{S,s} = A_{\mathfrak{p}}$ and $\kappa(s) = \kappa(\mathfrak{p})$. A cofinal system of elementary étale neighbourhoods is given by those elementary étale neighbourhoods (U, u) such that U is affine and $U \to S$ factors through $\operatorname{Spec}(A)$. In other words, we see that the right hand side is equal to $\operatorname{colim}_{(B,\mathfrak{q})} B$ where the colimit is over étale A-algebras B endowed with a prime $\mathfrak{q}$ lying over $\mathfrak{p}$ with $\kappa(\mathfrak{p}) = \kappa(\mathfrak{q})$. Thus the lemma follows from Algebra, Lemma 10.143.21. □

36.28. Slicing smooth morphisms

In this section we explain a result that roughly states that smooth coverings of a scheme S can be refined by étale coverings. The technique to prove this relies on a slicing argument.

Lemma 36.28.1. *Let $f : X \to S$ be a morphism of schemes. Let $x \in X$ be a point with image $s \in S$. Let $h \in \mathfrak{m}_x \subset \mathcal{O}_{X,x}$. Assume*

(1) *f is smooth at x, and*
(2) *the image $d\overline{h}$ of dh in*

$$\Omega_{X_s/s,x} \otimes_{\mathcal{O}_{X_s,x}} \kappa(x) = \Omega_{X/S,x} \otimes_{\mathcal{O}_{X,x}} \kappa(x)$$

is nonzero.

Then there exists an affine open neighbourhood $U \subset X$ of x such that h comes from $h \in \Gamma(U, \mathcal{O}_U)$ and such that $D = V(h)$ is an effective Cartier divisor in U with $x \in D$ and $D \to S$ smooth.

Proof. As f is smooth at x we may assume, after replacing X by an open neighbourhood of x that f is smooth. In particular we see that f is flat and locally of finite presentation. By Lemma 36.18.1 we already know there exists an open neighbourhood $U \subset X$ of x such that h comes from $h \in \Gamma(U, \mathcal{O}_U)$ and such that $D = V(h)$ is an effective Cartier divisor in U with $x \in D$ and $D \to S$ flat and of finite presentation. By Morphisms, Lemma 28.34.15 we have a short exact sequence

$$\mathcal{C}_{D/U} \to i^*\Omega_{U/S} \to \Omega_{D/S} \to 0$$

where $i : D \to U$ is the closed immersion and $\mathcal{C}_{D/U}$ is the conormal sheaf of D in U. As D is an effective Cartier divisor cut out by $h \in \Gamma(U, \mathcal{O}_U)$ we see that $\mathcal{C}_{D/U} = h \cdot \mathcal{O}_S$. Since $U \to S$ is smooth the sheaf $\Omega_{U/S}$ is finite locally free, hence its pullback $i^*\Omega_{U/S}$ is finite locally free also. The first arrow of the sequence maps the free generator h to the section $\mathrm{d}h|_D$ of $i^*\Omega_{U/S}$ which has nonzero value in the fibre $\Omega_{U/S,x} \otimes \kappa(x)$ by assumption. By right exactness of $\otimes\kappa(x)$ we conclude that

$$\dim_{\kappa(x)} \big(\Omega_{D/S,x} \otimes \kappa(x)\big) = \dim_{\kappa(x)} \big(\Omega_{U/S,x} \otimes \kappa(x)\big) - 1.$$

By Morphisms, Lemma 28.35.14 we see that $\Omega_{U/S,x} \otimes \kappa(x)$ can be generated by at most $\dim_x(U_s)$ elements. By the displayed formula we see that $\Omega_{D/S,x} \otimes \kappa(x)$ can be generated by at most $\dim_x(U_s) - 1$ elements. Note that $\dim_x(D_s) = \dim_x(U_s) - 1$ for example because $\dim(\mathcal{O}_{D_s,x}) = \dim(\mathcal{O}_{U_s,x}) - 1$ by Algebra, Lemma 10.57.11 (also $D_s \subset U_s$ is effective Cartier, see Divisors, Lemma 30.10.1) and then using Morphisms, Lemma 28.29.1. Thus we conclude that $\Omega_{D/S,x} \otimes \kappa(x)$ can be generated by at most $\dim_x(D_s)$ elements and we conclude that $D \to S$ is smooth at x by Morphisms, Lemma 28.35.14 again. After shrinking U we get that $D \to S$ is smooth and we win. □

Lemma 36.28.2. *Let $f : X \to S$ be a morphism of schemes. Let $x \in X$ be a point with image $s \in S$. Assume*

(1) *f is smooth at x, and*
(2) *the map*

$$\Omega_{X_s/s,x} \otimes_{\mathcal{O}_{X_s,x}} \kappa(x) \longrightarrow \Omega_{\kappa(x)/\kappa(s)}$$

has a nonzero kernel.

Then there exists an affine open neighbourhood $U \subset X$ of x and an effective Cartier divisor $D \subset U$ containing x such that $D \to S$ is smooth.

Proof. Write $k = \kappa(s)$ and $R = \mathcal{O}_{X_s,x}$. Denote $\mathfrak{m}$ the maximal ideal of R and $\kappa = R/\mathfrak{m}$ so that $\kappa = \kappa(x)$. As formation of modules of differentials commutes with

localization (see Algebra, Lemma 10.125.8) we have $\Omega_{X_s/s,x} = \Omega_{R/k}$. By Algebra, Lemma 10.125.9 there is an exact sequence

$$\mathfrak{m}/\mathfrak{m}^2 \xrightarrow{\mathrm{d}} \Omega_{R/k} \otimes_R \kappa \to \Omega_{\kappa/k} \to 0.$$

Hence if (2) holds, there exists an element $\overline{h} \in \mathfrak{m}$ such that $\mathrm{d}\overline{h}$ is nonzero. Choose a lift $h \in \mathcal{O}_{X,x}$ of $\overline{h}$ and apply Lemma 36.28.1. □

Remark 36.28.3. The second condition in Lemma 36.28.2 is necessary even if x is a closed point of a positive dimensional fibre. An example is the following: Let k be a field of characteristic $p > 0$ which is imperfect. Let $a \in k$ be an element which is not a pth power. Let $\mathfrak{m} = (x, y^p - a) \subset k[x, y]$. This corresponds to a closed point w of $X = \mathbf{A}^2_k$. Set $S = \mathbf{A}^1_k$ and let $f : X \to S$ be the morphism corresponding to $k[x] \to k[x, y]$. Then there does not exist any commutative diagram

$$\begin{array}{ccc} S' & \xrightarrow{\quad h \quad} & X \\ & {\scriptstyle g}\searrow \quad \swarrow{\scriptstyle f} & \\ & S & \end{array}$$

with g étale and w in the image of h. This is clear as the residue field extension $\kappa(f(w)) \subset \kappa(w)$ is purely inseparable, but for any $s' \in S'$ with $g(s') = f(w)$ the extension $\kappa(f(w)) \subset \kappa(s')$ would be separable.

If you assume the residue field extension is separable then the phenomenon of Remark 36.28.3 does not happen. Here is the precise result.

Lemma 36.28.4. *Let $f : X \to S$ be a morphism of schemes. Let $x \in X$ be a point with image $s \in S$. Assume*

(1) *f is smooth at x,*
(2) *the residue field extension $\kappa(s) \subset \kappa(x)$ is separable, and*
(3) *x is not a generic point of X_s.*

Then there exists an affine open neighbourhood $U \subset X$ of x and an effective Cartier divisor $D \subset U$ containing x such that $D \to S$ is smooth.

Proof. Write $k = \kappa(s)$ and $R = \mathcal{O}_{X_s,x}$. Denote $\mathfrak{m}$ the maximal ideal of R and $\kappa = R/\mathfrak{m}$ so that $\kappa = \kappa(x)$. As formation of modules of differentials commutes with localization (see Algebra, Lemma 10.125.8) we have $\Omega_{X_s/s,x} = \Omega_{R/k}$. By assumption (2) and Algebra, Lemma 10.133.4 the map

$$\mathrm{d} : \mathfrak{m}/\mathfrak{m}^2 \longrightarrow \Omega_{R/k} \otimes_R \kappa(\mathfrak{m})$$

is injective. Assumption (3) implies that $\mathfrak{m}/\mathfrak{m}^2 \neq 0$. Thus there exists an element $\overline{h} \in \mathfrak{m}$ such that $\mathrm{d}\overline{h}$ is nonzero. Choose a lift $h \in \mathcal{O}_{X,x}$ of $\overline{h}$ and apply Lemma 36.28.1. □

The subscheme Z constructed in the following lemma is really a complete intersection in an affine open neighbourhood of x. If we ever need this we will explicitly formulate a separate lemma stating this fact.

Lemma 36.28.5. *Let $f : X \to S$ be a morphism of schemes. Let $x \in X$ be a point with image $s \in S$. Assume*

(1) *f is smooth at x, and*
(2) *x is a closed point of X_s and $\kappa(s) \subset \kappa(x)$ is separable.*

Then there exists an immersion $Z \to X$ containing x such that

(1) *$Z \to S$ is étale, and*
(2) *$Z_s = \{x\}$ set theoretically.*

Proof. We may and do replace S by an affine open neighbourhood of s. We may and do replace X by an affine open neighbourhood of x such that $X \to S$ is smooth. We will prove the lemma for smooth morphisms of affines by induction on $d = \dim_x(X_s)$.

The case $d = 0$. In this case we show that we may take Z to be an open neighbourhood of x. Namely, if $d = 0$, then $X \to S$ is quasi-finite at x, see Morphisms, Lemma 28.30.5. Hence there exists an affine open neighbourhood $U \subset X$ such that $U \to S$ is quasi-finite, see Morphisms, Lemma 28.49.2. Thus after replacing X by U we see that X is quasi-finite and smooth over S, hence smooth of relative dimension 0 over S, hence étale over S. Moreover, the fibre X_s is a finite discrete set. Hence after replacing X by a further affine open neighbourhood of X we see that that $f^{-1}(\{s\}) = \{x\}$ (because the topology on X_s is induced from the topology on X, see Schemes, Lemma 25.18.5). This proves the lemma in this case.

Next, assume $d > 0$. Note that because x is a closed point of its fibre the extension $\kappa(s) \subset \kappa(x)$ is finite (by the Hilbert Nullstellensatz, see Morphisms, Lemma 28.21.3). Thus we see $\Omega_{\kappa(x)/\kappa(s)} = 0$ as this holds for algebraic separable field extensions. Thus we may apply Lemma 36.28.2 to find a diagram

$$\begin{array}{ccccc} D & \longrightarrow & U & \longrightarrow & X \\ & \searrow & & \swarrow & \downarrow \\ & & S & & \end{array}$$

with $x \in D$. Note that $\dim_x(D_s) = \dim_x(X_s) - 1$ for example because $\dim(\mathcal{O}_{D_s,x}) = \dim(\mathcal{O}_{X_s,x}) - 1$ by Algebra, Lemma 10.57.11 (also $D_s \subset X_s$ is effective Cartier, see Divisors, Lemma 30.10.1) and then using Morphisms, Lemma 28.29.1. Thus the morphism $D \to S$ is smooth with $\dim_x(D_s) = \dim_x(X_s) - 1 = d - 1$. By induction hypothesis we can find an immersion $Z \to D$ as desired, which finishes the proof. □

Lemma 36.28.6. *Let $f : X \to S$ be a smooth morphism of schemes. Let $s \in S$ be a point in the image of f. Then there exists an étale neighbourhood $(S', s') \to (S, s)$ and a S-morphism $S' \to X$.*

First proof of Lemma 36.28.6. By assumption $X_s \neq \emptyset$. By Varieties, Lemma 32.15.6 there exists a closed point $x \in X_s$ such that $\kappa(x)$ is a finite separable field extension of $\kappa(s)$. Hence by Lemma 36.28.5 there exists an immersion $Z \to X$ such that $Z \to S$ is étale and such that $x \in Z$. Take $(S', s') = (Z, x)$. □

Second proof of Lemma 36.28.6. Pick a point $x \in X$ with $f(x) = s$. Choose a diagram

$$\begin{array}{ccccc} X & \longleftarrow & U & \xrightarrow{\pi} & \mathbf{A}^d_V \\ \downarrow & & \downarrow & \swarrow & \\ Y & \longleftarrow & V & & \end{array}$$

with π étale, $x \in U$ and $V = \mathrm{Spec}(R)$ affine, see Morphisms, Lemma 28.37.20. In particular $s \in V$. The morphism $\pi : U \to \mathbf{A}^d_V$ is open, see Morphisms, Lemma

28.37.13. Thus $W = \pi(V) \cap \mathbf{A}^d_s$ is a nonempty open subset of $\mathbf{A}^d_s$. Let $w \in W$ be a point with $\kappa(s) \subset \kappa(w)$ finite separable, see Varieties, Lemma 32.15.5. By Algebra, Lemma 10.108.1 there exist d elements $\overline{f}_1, \ldots, \overline{f}_d \in \kappa(s)[x_1, \ldots, x_d]$ which generate the maximal ideal corresponding to w in $\kappa(s)[x_1, \ldots, x_n]$. After replacing R by a principal localization we may assume there are $f_1, \ldots, f_d \in R[x_1, \ldots, x_d]$ which map to $\overline{f}_1, \ldots, \overline{f}_d \in \kappa(s)[x_1, \ldots, x_d]$. Consider the R-algebra

$$R' = R[x_1, \ldots, x_d]/(f_1, \ldots, f_d)$$

and set $S' = \mathrm{Spec}(R')$. By construction we have a closed immersion $j : S' \to \mathbf{A}^d_V$ over V. By construction the fibre of $S' \to V$ over s is a single point s' whose residue field is finite separable over $\kappa(s)$. Let $\mathfrak{q}' \subset R'$ be the corresponding prime. By Algebra, Lemma 10.129.11 we see that $(R')_g$ is a relative global complete intersection over R for some $g \in R'$, $g \notin \mathfrak{q}$. Thus $S' \to V$ is flat and of finite presentation in a neighbourhood of s', see Algebra, Lemma 10.129.14. By construction the scheme theoretic fibre of $S' \to V$ over s is $\mathrm{Spec}(\kappa(s'))$. Hence it follows from Morphisms, Lemma 28.37.15 that $S' \to S$ is étale at s'. Set

$$S'' = U \times_{\pi, \mathbf{A}^d_V, j} S'.$$

By construction there exists a point $s'' \in S''$ which maps to s' via the projection $p : S'' \to S'$. Note that p is étale as the base change of the étale morphism π, see Morphisms, Lemma 28.37.4. Choose a small affine neighbourhood $S''' \subset S''$ of s'' which maps into the nonempty open neighbourhood of $s' \in S'$ where the morphism $S' \to S$ is étale. Then the étale neighbourhood $(S''', s'') \to (S, s)$ is a solution to the problem posed by the lemma. □

The following lemma shows that sheaves for the smooth topology are the same thing as sheaves for the étale topology.

Lemma 36.28.7. *Let S be a scheme. Let $\mathcal{U} = \{S_i \to S\}_{i \in I}$ be a smooth covering of S, see Topologies, Definition 33.5.1. Then there exists an étale covering $\mathcal{V} = \{T_j \to S\}_{j \in J}$ (see Topologies, Definition 33.4.1) which refines (see Sites, Definition 7.8.1) $\mathcal{U}$.*

Proof. For every $s \in S$ there exists an $i \in I$ such that s is in the image of $S_i \to S$. By Lemma 36.28.6 we can find an étale morphism $g_s : T_s \to S$ such that $s \in g_s(T)_s$ and such that g_s factors through $S_i \to S$. Hence $\{T_s \to S\}$ is an étale covering of S that refines $\mathcal{U}$. □

36.29. Finite free locally dominates étale

In this section we explain a result that roughly states that étale coverings of a scheme S can be refined by Zariski coverings of finite locally free covers of S.

Lemma 36.29.1. *Let S be a scheme. Let $s \in S$. Let $f : (U, u) \to (S, s)$ be an étale neighbourhood. There exists an affine open neighbourhood $s \in V \subset S$ and a surjective, finite locally free morphism $\pi : T \to V$ such that for every $t \in \pi^{-1}(s)$ there exists an open neighbourhood $t \in W_t \subset T$ and a commutative diagram*

$$\begin{array}{ccccc} T & \longleftarrow & W_t & \xrightarrow{h_t} & U \\ \downarrow{\scriptstyle \pi} & & & \searrow & \downarrow \\ V & & \longrightarrow & & S \end{array}$$

with $h_t(t) = u$.

Proof. The problem is local on S hence we may replace S by any open neighbourhood of s. We may also replace U by an open neighbourhood of u. Hence, by Morphisms, Lemma 28.37.14 we may assume that $U \to S$ is a standard étale morphism of affine schemes. In this case the lemma (with $V = S$) follows from Algebra, Lemma 10.136.18. □

Lemma 36.29.2. *Let $f : U \to S$ be a surjective étale morphism of affine schemes. There exists a surjective, finite locally free morphism $\pi : T \to S$ and a finite open covering $T = T_1 \cup \ldots \cup T_n$ such that each $T_i \to S$ factors through $U \to S$. Diagram:*

$$\begin{array}{ccccc} & & \coprod T_i & & \\ & \swarrow & & \searrow & \\ T & & & & U \\ & \searrow^{\pi} & & \swarrow_{f} & \\ & & S & & \end{array}$$

where the south-west arrow is a Zariski-covering.

Proof. This is a restatement of Algebra, Lemma 10.136.19. □

Remark 36.29.3. In terms of topologies the lemmas above mean the following. Let S be any scheme. Let $\{f_i : U_i \to S\}$ be an étale covering of S. There exists a Zariski open covering $S = \bigcup V_j$, for each j a finite locally free, surjective morphism $W_j \to V_j$, and for each j a Zariski open covering $\{W_{j,k} \to W_j\}$ such that the family $\{W_{j,k} \to S\}$ refines the given étale covering $\{f_i : U_i \to S\}$. What does this mean in practice? Well, for example, suppose we have a descent problem which we know how to solve for Zariski coverings and for fppf coverings of the form $\{\pi : T \to S\}$ with π finite locally free and surjective. Then this descent problem has an affirmative answer for étale coverings as well. This trick was used by Gabber in his proof that $\mathrm{Br}(X) = \mathrm{Br}'(X)$ for an affine scheme X, see [**Hoo82**].

36.30. Étale localization of quasi-finite morphisms

Now we come to a series of lemmas around the theme "quasi-finite morphisms become finite after étale localization". The general idea is the following. Suppose given a morphism of schemes $f : X \to S$ and a point $s \in S$. Let $\varphi : (U, u) \to (S, s)$ be an étale neighbourhood of s in S. Consider the fibre product $X_U = U \times_S X$ and the basic diagram

$$\begin{array}{ccccc} V & \longrightarrow & X_U & \longrightarrow & X \\ & \searrow & \downarrow & & \downarrow f \\ & & U & \xrightarrow{\varphi} & S \end{array} \tag{36.30.0.1}$$

where $V \subset X_U$ is open. Is there some standard model for the morphism $f_U : X_U \to U$, or for the morphism $V \to U$ for suitable opens V? Of course the answer is no in general. But for quasi-finite morphisms we can say something.

Lemma 36.30.1. *Let $f : X \to S$ be a morphism of schemes. Let $x \in X$. Set $s = f(x)$. Assume that*

(1) *f is locally of finite type, and*
(2) *$x \in X_s$ is isolated*[2]*.*

Then there exist

(a) *an elementary étale neighbourhood $(U, u) \to (S, s)$,*
(b) *an open subscheme $V \subset X_U$ (see 36.30.0.1)*

such that

(i) *$V \to U$ is a finite morphism,*
(ii) *there is a unique point v of V mapping to u in U, and*
(iii) *the point v maps to x under the morphism $X_U \to X$, inducing $\kappa(x) = \kappa(v)$.*

Moreover, for any elementary étale neighbourhood $(U', u') \to (U, u)$ setting $V' = U' \times_U V \subset X_{U'}$ the triple (U', u', V') satisfies the properties (i), (ii), and (iii) as well.

Proof. Let $Y \subset X$, $W \subset S$ be affine opens such that $f(Y) \subset W$ and such that $x \in Y$. Note that x is also an isolated point of the fibre of the morphism $f|_Y : Y \to W$. If we can prove the theorem for $f|_Y : Y \to W$, then the theorem follows for f. Hence we reduce to the case where f is a morphism of affine schemes. This case is Algebra, Lemma 10.136.22. □

In the preceding and following lemma we do not assume that the morphism f is separated. This means that the opens V, V_i created in them are not necessarily closed in X_U. Moreover, if we choose the neighbourhood U to be affine, then each V_i is affine, but the intersections $V_i \cap V_j$ need not be affine (in the nonseparated case).

Lemma 36.30.2. *Let $f : X \to S$ be a morphism of schemes. Let $x_1, \ldots, x_n \in X$ be points having the same image s in S. Assume that*

(1) *f is locally of finite type, and*
(2) *$x_i \in X_s$ is isolated for $i = 1, \ldots, n$.*

Then there exist

(a) *an elementary étale neighbourhood $(U, u) \to (S, s)$,*
(b) *for each i an open subscheme $V_i \subset X_U$,*

such that for each i we have

(i) *$V_i \to U$ is a finite morphism,*
(ii) *there is a unique point v_i of V_i mapping to u in U, and*
(iii) *the point v_i maps to x_i in X and $\kappa(x_i) = \kappa(v_i)$.*

Proof. We will use induction on n. Namely, suppose $(U, u) \to (S, s)$ and $V_i \subset X_U$, $i = 1, \ldots, n-1$ work for $x_1, \ldots, x_{n-1}$. Since $\kappa(s) = \kappa(u)$ the fibre $(X_U)_u = X_s$. Hence there exists a unique point $x'_n \in X_u \subset X_U$ corresponding to $x_n \in X_s$. Also x'_n is isolated in X_u. Hence by Lemma 36.30.1 there exists an elementary étale neighbourhood $(U', u') \to (U, u)$ and an open $V_n \subset X_{U'}$ which works for x'_n and hence for x_n. By the final assertion of Lemma 36.30.1 the open subschemes $V'_i = U' \times_U V_i$ for $i = 1, \ldots, n-1$ still work with respect to $x_1, \ldots, x_{n-1}$. Hence we win. □

[2]In the presence of (1) this means that f is quasi-finite at x, see Morphisms, Lemma 28.21.6.

If we allow a nontrivial field extension $\kappa(s) \subset \kappa(u)$, i.e., general étale neighbourhoods, then we can split the points as follows.

Lemma 36.30.3. *Let $f : X \to S$ be a morphism of schemes. Let $x_1, \ldots, x_n \in X$ be points having the same image s in S. Assume that*

(1) *f is locally of finite type, and*
(2) *$x_i \in X_s$ is isolated for $i = 1, \ldots, n$.*

Then there exist

(a) *an étale neighbourhood $(U, u) \to (S, s)$,*
(b) *for each i an integer m_i and open subschemes $V_{i,j} \subset X_U$, $j = 1, \ldots, m_i$*

such that we have

(i) *each $V_{i,j} \to U$ is a finite morphism,*
(ii) *there is a unique point $v_{i,j}$ of $V_{i,j}$ mapping to u in U with $\kappa(u) \subset \kappa(v_{i,j})$ finite purely inseparable,*
(iv) *if $v_{i,j} = v_{i',j'}$, then $i = i'$ and $j = j'$, and*
(iii) *the points $v_{i,j}$ map to x_i in X and no other points of $(X_U)_u$ map to x_i.*

Proof. This proof is a variant of the proof of Algebra, Lemma 10.136.24 in the language of schemes. By Morphisms, Lemma 28.21.6 the morphism f is quasi-finite at each of the points x_i. Hence $\kappa(s) \subset \kappa(x_i)$ is finite for each i (Morphisms, Lemma 28.21.5). For each i, let $\kappa(s) \subset L_i \subset \kappa(x_i)$ be the subfield such that $L_i/\kappa(s)$ is separable, and $\kappa(x_i)/L_i$ is purely inseparable. Choose a finite Galois extension $\kappa(s) \subset L$ such that there exist $\kappa(s)$-embeddings $L_i \to L$ for $i = 1, \ldots, n$. Choose an étale neighbourhood $(U, u) \to (S, s)$ such that $L \cong \kappa(u)$ as $\kappa(s)$-extensions (Lemma 36.27.2).

Let $y_{i,j}$, $j = 1, \ldots, m_i$ be the points of X_U lying over $x_i \in X$ and $u \in U$. By Schemes, Lemma 25.17.5 these points $y_{i,j}$ correspond exactly to the primes in the rings $\kappa(u) \otimes_{\kappa(s)} \kappa(x_i)$. This also explains why there are finitely many; in fact $m_i = [L_i : \kappa(s)]$ but we do not need this. By our choice of L (and elementary field theory) we see that $\kappa(u) \subset \kappa(y_{i,j})$ is finite purely inseparable for each pair i, j. Also, by Morphisms, Lemma 28.21.13 for example, the morphism $X_U \to U$ is quasi-finite at the points $y_{i,j}$ for all i, j.

Apply Lemma 36.30.2 to the morphism $X_U \to U$, the point $u \in U$ and the points $y_{i,j} \in (X_U)_u$. This gives an étale neighbourhood $(U', u') \to (U, u)$ with $\kappa(u) = \kappa(u')$ and opens $V_{i,j} \subset X_{U'}$ with the properties (i), (ii), and (iii) of that lemma. We claim that the étale neighbourhood $(U', u') \to (S, s)$ and the opens $V_{i,j} \subset X_{U'}$ are a solution to the problem posed by the lemma. We omit the verifications. □

Lemma 36.30.4. *Let $f : X \to S$ be a morphism of schemes. Let $s \in S$. Let $x_1, \ldots, x_n \in X_s$. Assume that*

(1) *f is locally of finite type,*
(2) *f is separated, and*
(3) *$x_1, \ldots, x_n$ are pairwise distinct isolated points of X_s.*

Then there exists an elementary étale neighbourhood $(U, u) \to (S, s)$ and a decomposition

$$U \times_S X = W \coprod V_1 \coprod \ldots \coprod V_n$$

into open and closed subschemes such that the morphisms $V_i \to U$ are finite, the fibres of $V_i \to U$ over u are singletons $\{v_i\}$, each v_i maps to x_i with $\kappa(x_i) = \kappa(v_i)$, and the fibre of $W \to U$ over u contains no points mapping to any of the x_i.

Proof. Choose $(U, u) \to (S, s)$ and $V_i \subset X_U$ as in Lemma 36.30.2. Since $X_U \to U$ is separated (Schemes, Lemma 25.21.13) and $V_i \to U$ is finite hence proper (Morphisms, Lemma 28.44.10) we see that $V_i \subset X_U$ is closed by Morphisms, Lemma 28.42.7. Hence $V_i \cap V_j$ is a closed subset of V_i which does not contain v_i. Hence the image of $V_i \cap V_j$ in U is a closed set (because $V_i \to U$ proper) not containing u. After shrinking U we may therefore assume that $V_i \cap V_j = \emptyset$ for all i, j. This gives the decomposition as in the lemma. □

Here is the variant where we reduce to purely inseparable field extensions.

Lemma 36.30.5. *Let $f : X \to S$ be a morphism of schemes. Let $s \in S$. Let $x_1, \ldots, x_n \in X_s$. Assume that*

(1) *f is locally of finite type,*
(2) *f is separated, and*
(3) *$x_1, \ldots, x_n$ are pairwise distinct isolated points of X_s.*

Then there exists an étale neighbourhood $(U, u) \to (S, s)$ and a decomposition

$$U \times_S X = W \coprod \coprod_{i=1,\ldots,n} \coprod_{j=1,\ldots,m_i} V_{i,j}$$

into open and closed subschemes such that the morphisms $V_{i,j} \to U$ are finite, the fibres of $V_{i,j} \to U$ over u are singletons $\{v_{i,j}\}$, each $v_{i,j}$ maps to x_i, $\kappa(u) \subset \kappa(v_{i,j})$ is purely inseparable, and the fibre of $W \to U$ over u contains no points mapping to any of the x_i.

Proof. This is proved in exactly the same way as the proof of Lemma 36.30.4 except that it uses Lemma 36.30.3 instead of Lemma 36.30.2. □

The following version may be a little easier to parse.

Lemma 36.30.6. *Let $f : X \to S$ be a morphism of schemes. Let $s \in S$. Assume that*

(1) *f is locally of finite type,*
(2) *f is separated, and*
(3) *X_s has at most finitely many isolated points.*

Then there exists an elementary étale neighbourhood $(U, u) \to (S, s)$ and a decomposition

$$U \times_S X = W \coprod V$$

into open and closed subschemes such that the morphism $V \to U$ is finite, and the fibre W_u of the morphism $W \to U$ contains no isolated points. In particular, if $f^{-1}(s)$ is a finite set, then $W_u = \emptyset$.

Proof. This is clear from Lemma 36.30.4 by choosing $x_1, \ldots, x_n$ the complete set of isolated points of X_s and setting $V = \bigcup V_i$. □

36.31. Zariski's Main Theorem

We can use the results Section 36.30 to prove the scheme theoretic version of Zariski's main theorem.

Lemma 36.31.1. *Let $f : X \to S$ be a morphism of schemes. Assume f is of finite type and separated. Let S' be the normalization of S in X, see Morphisms, Definition 28.48.3. Picture:*

$$\begin{array}{ccc} X & \xrightarrow{f'} & S' \\ & {\scriptstyle f}\searrow \quad \swarrow{\scriptstyle \nu} & \\ & S & \end{array}$$

Then there exists an open subscheme $U' \subset S'$ such that

(1) *$(f')^{-1}(U') \to U'$ is an isomorphism, and*
(2) *$(f')^{-1}(U') \subset X$ is the set of points at which f is quasi-finite.*

Proof. By Morphisms, Lemma 28.49.2 the subset $U \subset X$ of points where f is quasi-finite is open. The lemma is equivalent to

(a) $U' = f'(U) \subset S'$ is open,
(b) $U = f^{-1}(U')$, and
(c) $U \to U'$ is an isomorphism.

Let $x \in U$ be arbitrary. We claim there exists an open neighbourhood $f'(x) \in V \subset S'$ such that $(f')^{-1}V \to V$ is an isomorphism. We first prove the claim implies the lemma. Namely, then $(f')^{-1}V \cong V$ is both locally of finite type over S (as an open subscheme of X) and for $v \in V$ the residue field extension $\kappa(v) \supset \kappa(\nu(v))$ is algebraic (as $V \subset S'$ and S' is integral over S). Hence the fibres of $V \to S$ are discrete (Morphisms, Lemma 28.21.2) and $(f')^{-1}V \to S$ is locally quasi-finite (Morphisms, Lemma 28.21.8). This implies $(f')^{-1}V \subset U$ and $V \subset U'$. Since x was arbitrary we see that (a), (b), and (c) are true.

Let $s = f(x)$. Let $(T, t) \to (S, s)$ be an elementary étale neighbourhood. Denote by a subscript $_T$ the base change to T. Let $y = (x, t) \in X_T$ be the unique point in the fibre X_t lying over x. Note that $U_T \subset X_T$ is the set of points where f_T is quasi-finite, see Morphisms, Lemma 28.21.13. Note that

$$X_T \xrightarrow{f'_T} S'_T \xrightarrow{\nu_T} T$$

is the normalization of T in X_T, see Lemma 36.14.2. Suppose that the claim holds for $y \in U_T \subset X_T \to S'_T \to T$, i.e., suppose that we can find an open neighbourhood $f'_T(y) \in V' \subset S'_T$ such that $(f'_T)^{-1}V' \to V'$ is an isomorphism. The morphism $S'_T \to S'$ is étale hence the image $V \subset S'$ of V' is open. Observe that $f'(x) \in V$ as $f'_T(y) \in V'$. Observe that

$$\begin{array}{ccc} (f'_T)^{-1}V' & \longrightarrow & (f')^{-1}(V) \\ \downarrow & & \downarrow \\ V' & \longrightarrow & V \end{array}$$

is a fibre square (as $S'_T \times_{S'} X = X_T$). Since the left vertical arrow is an isomorphism and $\{V' \to V\}$ is a étale covering, we conclude that the right vertical arrow is an

isomorphism by Descent, Lemma 34.19.15. In other words, the claim holds for $x \in U \subset X \to S' \to S$.

By the result of the previous paragraph we may replace S by an elementary étale neighbourhood of $s = f(x)$ in order to prove the claim. Thus we may assume there is a decomposition

$$X = V \amalg W$$

into open and closed subschemes where $V \to S$ is finite and $x \in V$, see Lemma 36.30.4. Since X is a disjoint union of V and W over S and since $V \to S$ is finite we see that the normalization of S in X is the morphism

$$X = V \amalg W \longrightarrow V \amalg W' \longrightarrow S$$

where W' is the normalization of S in W, see Morphisms, Lemmas 28.48.7, 28.44.4, and 28.48.8. The claim follows and we win. □

Lemma 36.31.2. *Let $f : X \to S$ be a morphism of schemes. Assume f is quasi-finite and separated. Let S' be the normalization of S in X, see Morphisms, Definition 28.48.3. Picture:*

$$\begin{array}{ccc} X & \xrightarrow{f'} & S' \\ & {\scriptstyle f}\searrow \quad \swarrow{\scriptstyle \nu} & \\ & S & \end{array}$$

Then f' is a quasi-compact open immersion and ν is integral. In particular f is quasi-affine.

Proof. This follows from Lemma 36.31.1. Namely, by that lemma there exists an open subscheme $U' \subset S'$ such that $(f')^{-1}(U') = X$ (!) and $X \to U'$ is an isomorphism! In other words, f' is an open immersion. Note that f' is quasi-compact as f is quasi-compact and $\nu : S' \to S$ is separated (Schemes, Lemma 25.21.15). It follows that f is quasi-affine by Morphisms, Lemma 28.14.3. □

Lemma 36.31.3. *Let $f : X \to S$ be a morphism of schemes. Assume f is quasi-finite and separated and assume that S is quasi-compact and quasi-separated. Then there exists a factorization*

$$\begin{array}{ccc} X & \xrightarrow{j} & T \\ & {\scriptstyle f}\searrow \quad \swarrow{\scriptstyle \pi} & \\ & S & \end{array}$$

where j is a quasi-compact open immersion and π is finite.

Proof. Let $X \to S' \to S$ be as in the conclusion of Lemma 36.31.2. By Properties, Lemma 27.20.13 we can write $\nu_*\mathcal{O}_{S'} = \operatorname{colim}_{i \in I} \mathcal{A}_i$ as a directed colimit of finite quasi-coherent $\mathcal{O}_X$-algebras $\mathcal{A}_i \subset \nu_*\mathcal{O}_{S'}$. Then $\pi_i : T_i = \underline{\operatorname{Spec}}_S(\mathcal{A}_i) \to S$ is a finite morphism for each i. Note that the transition morphisms $T_{i'} \to T_i$ are affine and that $S' = \lim T_i$.

By Limits, Lemma 31.3.8 there exists an i and a quasi-compact open $U_i \subset T_i$ whose inverse image in S' equals $f'(X)$. For $i' \geq i$ let $U_{i'}$ be the inverse image of U_i in $T_{i'}$. Then $X \cong f'(X) = \lim_{i' \geq i} U_{i'}$, see Limits, Lemma 31.2.2. By Limits, Lemma 31.3.13 we see that $X \to U_{i'}$ is a closed immersion for some $i' \geq i$. (In fact $X \cong U_{i'}$ for sufficiently large i' but we don't need this.) Hence $X \to T_{i'}$ is an immersion.

By Morphisms, Lemma 28.3.2 we can factor this as $X \to T \to T_{i'}$ where the first arrow is an open immersion and the second a closed immersion. Thus we win. □

Lemma 36.31.4. *Let $f : X \to S$ be a morphism of schemes and $U \subset S$ an open. If*

(1) *f is separated, locally of finite presentation*[3]*, and flat,*
(2) *$f^{-1}(U) \to U$ is an isomorphism, and*
(3) *$U \subset S$ is retrocompact and scheme theoretically dense,*

then f is an open immersion.

Proof. The image $f(X) \subset S$ is open (Morphisms, Lemma 28.26.9) hence we may replace S by $f(X)$. Thus we have to prove that f is an isomorphism. We may assume S is affine. We can reduce to the case that X is quasi-compact because it suffices to show that any quasi-compact open $X' \subset X$ whose image is S maps isomorphically to S. Thus we may assume f is quasi-compact. All the fibers of f have dimension 0, see Lemma 36.17.7. Hence f is quasi-finite, see Morphisms, Lemma 28.30.5. Let $s \in S$. Choose an elementary étale neighbourhood $g : (T, t) \to (S, s)$ such that $X \times_S T = V \amalg W$ with $V \to T$ finite and $W_t = \emptyset$, see Lemma 36.30.6. Denote $\pi : V \amalg W \to T$ the given morphism. Since π is flat and locally of finite presentation, we see that $\pi(V)$ is open in T (Morphisms, Lemma 28.26.9). After shrinking T we may assume that $T = \pi(V)$. Since f is an isomorphism over U we see that π is an isomorphism over $g^{-1}U$. Since $\pi(V) = T$ this implies that $\pi^{-1}g^{-1}U$ is contained in V. By Morphisms, Lemma 28.26.13 we see that $\pi^{-1}g^{-1}U \subset V \amalg W$ is scheme theoretically dense. Hence we deduce that $W = \emptyset$. Thus $X \times_S T = V$ is finite over T. Shrinking T once more we may assume T is affine. Then V is affine too and we see that

$$\Gamma(T, \mathcal{O}_T) = \Gamma(g^{-1}U, \mathcal{O}_T) = \Gamma(\pi^{-1}g^{-1}U, \mathcal{O}_V) = \Gamma(V, \mathcal{O}_V)$$

because the inverse image of U is schematically dense in both T and V (see above). Thus $X \times_S T \to T$ is an isomorphism. This implies that f is an isomorphism, for example by Descent, Lemma 34.19.15. □

Lemma 36.31.5. *Let $f : X \to S$ be a morphism of schemes. The following are equivalent:*

(1) *f is finite,*
(2) *f is proper with finite fibres.*
(3) *f is universally closed, separated, locally of finite type and has finite fibres.*

Proof. We have (1) implies (2) by Morphisms, Lemmas 28.44.10, 28.21.10, and 28.44.9. By definition (2) implies (3).

Assume (3). Pick $s \in S$. By Morphisms, Lemma 28.21.7 we see that all the finitely many points of X_s are isolated in X_s. Choose an elementary étale neighbourhood $(U, u) \to (S, s)$ and decomposition $X_U = V \coprod W$ as in Lemma 36.30.6. Note that $W_u = \emptyset$ because all points of X_s are isolated. Since f is universally closed we see that the image of W in U is a closed set not containing u. After shrinking U we may assume that $W = \emptyset$. In other words we see that $X_U = V$ is finite over U. Since $s \in S$ was arbitrary this means there exists a family $\{U_i \to S\}$ of étale morphisms whose images cover S such that the base changes $X_{U_i} \to U_i$ are finite. Note that

[3]You can replace "finite presentation" with "finite type" by More on Flatness, Lemma 37.10.12.

$\{U_i \to S\}$ is an étale covering, see Topologies, Definition 33.4.1. Hence it is an fpqc covering, see Topologies, Lemma 33.8.6. Hence we conclude f is finite by Descent, Lemma 34.19.21. □

As a consequence we have the following useful results.

Lemma 36.31.6. *Let $f : X \to S$ be a morphism of schemes. Let $s \in S$. Assume that f is proper and $f^{-1}(\{s\})$ is a finite set. Then there exists an open neighbourhood $V \subset S$ of s such that $f|_{f^{-1}(V)} : f^{-1}(V) \to V$ is finite.*

Proof. The morphism f is quasi-finite at all the points of $f^{-1}(\{s\})$ by Morphisms, Lemma 28.21.7. By Morphisms, Lemma 28.49.2 the set of points at which f is quasi-finite is an open $U \subset X$. Let $Z = X \setminus U$. Then $s \notin f(Z)$. Since f is proper the set $f(Z) \subset S$ is closed. Choose any open neighbourhood $V \subset S$ of s with $Z \cap V = \emptyset$. Then $f^{-1}(V) \to V$ is locally quasi-finite and proper. Hence it is quasi-finite (Morphisms, Lemma 28.21.9), hence has finite fibres (Morphisms, Lemma 28.21.10), hence is finite by Lemma 36.31.5. □

Lemma 36.31.7. *Let $f : Y \to X$ be a quasi-finite morphism. There exists a dense open $U \subset X$ such that $f|_{f^{-1}(U)} : f^{-1}(U) \to U$ is finite.*

Proof. If $U_i \subset X$, $i \in I$ is a collection of opens such that the restrictions $f|_{f^{-1}(U_i)} : f^{-1}(U_i) \to U_i$ are finite, then with $U = \bigcup U_i$ the restriction $f|_{f^{-1}(U)} : f^{-1}(U) \to U$ is finite, see Morphisms, Lemma 28.44.3. Thus the problem is local on X and we may assume that X is affine.

Assume X is affine. Write $Y = \bigcup_{j=1,\ldots,m} V_j$ with V_j affine. This is possible since f is quasi-finite and hence in particular quasi-compact. Each $V_j \to X$ is quasi-finite and separated. Let $\eta \in X$ be a generic point of an irreducible component of X. We see from Morphisms, Lemmas 28.21.10 and 28.47.1 that there exists an open neighbourhood $\eta \in U_\eta$ such that $f^{-1}(U_\eta) \cap V_j \to U_\eta$ is finite. We may choose U_η such that it works for each $j = 1, \ldots, m$. Note that the collection of generic points of X is dense in X. Thus we see there exists a dense open $W = \bigcup_\eta U_\eta$ such that each $f^{-1}(W) \cap V_j \to W$ is finite. It suffices to show that there exists a dense open $U \subset W$ such that $f|_{f^{-1}(U)} : f^{-1}(U) \to U$ is finite. Thus we may replace X by an affine open subscheme of W and assume that each $V_j \to X$ is finite.

Assume X is affine, $Y = \bigcup_{j=1,\ldots,m} V_j$ with V_j affine, and the restrictions $f|_{V_j} : V_j \to X$ are finite. Set

$$\Delta_{ij} = \left(\overline{V_i \cap V_j} \setminus V_i \cap V_j\right) \cap V_j.$$

This is a nowhere dense closed subset of V_j because it is the boundary of the open subset $V_i \cap V_j$ in V_j. By Morphisms, Lemma 28.46.7 the image $f(\Delta_{ij})$ is a nowhere dense closed subset of X. By Topology, Lemma 5.20.2 the union $T = \bigcup f(\Delta_{ij})$ is a nowhere dense closed subset of X. Thus $U = X \setminus T$ is a dense open subset of X. We claim that $f|_{f^{-1}(U)} : f^{-1}(U) \to U$ is finite. To see this let $U' \subset U$ be an affine open. Set $Y' = f^{-1}(U') = U' \times_X Y$, $V'_j = Y' \cap V_j = U' \times_X V_j$. Consider the restriction

$$f' = f|_{Y'} : Y' \longrightarrow U'$$

of f. This morphism now has the property that $Y' = \bigcup_{j=1,\ldots,m} V'_j$ is an affine open covering, each $V'_j \to U'$ is finite, and $V'_i \cap V'_j$ is (open and) closed both in V'_i and

V'_j. Hence $V'_i \cap V'_j$ is affine, and the map

$$\mathcal{O}(V'_i) \otimes_{\mathbf{Z}} \mathcal{O}(V'_j) \longrightarrow \mathcal{O}(V'_i \cap V'_j)$$

is surjective. This implies that Y' is separated, see Schemes, Lemma 25.21.8. Finally, consider the commutative diagram

$$\begin{array}{ccc} \coprod_{j=1,\ldots,m} V'_j & \longrightarrow & Y' \\ & \searrow \quad \swarrow & \\ & U' & \end{array}$$

The south-east arrow is finite, hence proper, the horizontal arrow is surjective, and the south-west arrow is separated. Hence by Morphisms, Lemma 28.42.8 we conclude that $Y' \to U'$ is proper. Since it is also quasi-finite, we see that it is finite by Lemma 36.31.5, and we win. □

Lemma 36.31.8. *Let $f : X \to S$ be flat, locally of finite presentation, separated, locally quasi-finite with universally bounded fibres. Then there exist closed subsets*

$$\emptyset = Z_{-1} \subset Z_0 \subset Z_1 \subset Z_2 \subset \ldots \subset Z_n = S$$

such that with $S_r = Z_r \setminus Z_{r-1}$ the stratification $S = \coprod_{r=0,\ldots,n} S_r$ is characterized by the following universal property: Given $g : T \to S$ the projection $X \times_S T \to T$ is finite locally free of degree r if and only if $g(T) \subset S_r$ (set theoretically).

Proof. Let n be an integer bounding the degree of the fibres of $X \to S$. By Morphisms, Lemma 28.50.4 we see that any base change has degrees of fibres bounded by n also. In particular, all the integers r that occur in the statement of the lemma will be $\leq n$. We will prove the lemma by induction on n. The base case is $n = 0$ which is obvious.

We claim the set of points $s \in S$ with $\deg_{\kappa(s)}(X_s) = n$ is an open subset $S_n \subset S$ and that $X \times_S S_n \to S_n$ is finite locally free of degree n. Namely, suppose that $s \in S$ is such a point. Choose an elementary étale morphism $(U, u) \to (S, s)$ and a decomposition $U \times_S X = W \amalg V$ as in Lemma 36.30.6. Since $V \to U$ is finite, flat, and locally of finite presentation, we see that $V \to U$ is finite locally free, see Morphisms, Lemma 28.46.2. After shrinking U to a smaller neighbourhood of u we may assume $V \to U$ is finite locally free of some degree d, see Morphisms, Lemma 28.46.5. As $u \mapsto s$ and $W_u = \emptyset$ we see that $d = n$. Since n is the maximum degree of a fibre we see that $W = \emptyset$! Thus $U \times_S X \to U$ is finite locally free of degree n. By Descent, Lemma 34.19.28 we conclude that $X \to S$ is finite locally free of degree n over $\mathrm{Im}(U \to S)$ which is an open neighbourhood of s (Morphisms, Lemma 28.37.13). This proves the claim.

Let $S' = S \setminus S_n$ endowed with the reduced induced scheme structure and set $X' = X \times_S S'$. Note that the degrees of fibres of $X' \to S'$ are universally bounded by $n - 1$. By induction we find a stratification $S' = S_0 \amalg \ldots \amalg S_{n-1}$ adapted to the morphism $X' \to S'$. We claim that $S = \coprod_{r=0,\ldots,n} S_r$ works for the morphism $X \to S$. Let $g : T \to S$ be a morphism of schemes and assume that $X \times_S T \to T$ is finite locally free of degree r. As remarked above this implies that $r \leq n$. If $r = n$, then it is clear that $T \to S$ factors through S_n. If $r < n$, then $g(T) \subset S' = S \setminus S_d$ (set theoretically) hence $T_{red} \to S$ factors through S', see Schemes, Lemma 25.12.6. Note that $X \times_S T_{red} \to T_{red}$ is also finite locally free of degree r as a base change.

By the universal property of the stratification $S' = \coprod_{r=0,\ldots,n-1} S_r$ we see that $g(T) = g(T_{red})$ is contained in S_r. Conversely, suppose that we have $g : T \to S$ such that $g(T) \subset S_r$ (set theoretically). If $r = n$, then g factors through S_n and it is clear that $X \times_S T \to T$ is finite locally free of degree n as a base change. If $r < n$, then $X \times_S T \to T$ is a morphism which is separated, flat, and locally of finite presentation, such that the restriction to T_{red} is finite locally free of degree r. Since $T_{red} \to T$ is a universal homeomorphism, we conclude that $X \times_S T_{red} \to X \times_S T$ is a universal homeomorphism too and hence $X \times_S T \to T$ is universally closed (as this is true for the finite morphism $X \times_S T_{red} \to T_{red}$). It follows that $X \times_S T \to T$ is finite, for example by Lemma 36.31.5. Then we can use Morphisms, Lemma 28.46.2 to see that $X \times_S T \to T$ is finite locally free. Finally, the degree is r as all the fibres have degree r. □

Lemma 36.31.9. *Let $f : X \to S$ be a morphism of schemes which is flat, locally of finite presentation, separated, and quasi-finite. Then there exist closed subsets*

$$\emptyset = Z_{-1} \subset Z_0 \subset Z_1 \subset Z_2 \subset \ldots \subset S$$

such that with $S_r = Z_r \setminus Z_{r-1}$ the stratification $S = \coprod S_r$ is characterized by the following universal property: Given a morphism $g : T \to S$ the projection $X \times_S T \to T$ is finite locally free of degree r if and only if $g(T) \subset S_r$ (set theoretically). Moreover, the inclusion maps $S_r \to S$ are quasi-compact.

Proof. The question is local on S, hence we may assume that S is affine. By Morphisms, Lemma 28.50.8 the fibres of f are universally bounded in this case. Hence the existence of the stratification follows from Lemma 36.31.8.

We will show that $U_r = S \setminus Z_r \to S$ is quasi-compact for each $r \geq 0$. This will prove the final statement by elementary topology. Since a composition of quasi-compact maps is quasi-compact it suffices to prove that $U_r \to U_{r-1}$ is quasi-compact. Choose an affine open $W \subset U_{r-1}$. Write $W = \mathrm{Spec}(A)$. Then $Z_r \cap W = V(I)$ for some ideal $I \subset A$ and $X \times_S \mathrm{Spec}(A/I) \to \mathrm{Spec}(A/I)$ is finite locally free of degree r. Note that $A/I = \mathrm{colim}\, A/I_i$ where $I_i \subset I$ runs through the finitely generated ideals. By Limits, Lemma 31.7.7 we see that $X \times_S \mathrm{Spec}(A/I_i) \to \mathrm{Spec}(A/I_i)$ is finite locally free of degree r for some i. (This uses that $X \to S$ is of finite presentation, as it is locally of finite presentation, separated, and quasi-compact.) Hence $\mathrm{Spec}(A/I_i) \to \mathrm{Spec}(A) = W$ factors (set theoretically) through $Z_r \cap W$. It follows that $Z_r \cap W = V(I_i)$ is the zero set of a finite subset of elements of A. This means that $W \setminus Z_r$ is a finite union of standard opens, hence quasi-compact, as desired. □

Lemma 36.31.10. *Let $f : X \to S$ be a flat, locally of finite presentation, separated, and locally quasi-finite morphism of schemes. Then there exist open subschemes*

$$S = U_0 \supset U_1 \supset U_2 \supset \ldots$$

such that a morphism $\mathrm{Spec}(k) \to S$ factors through U_d if and only if $X \times_S \mathrm{Spec}(k)$ has degree $\geq d$ over k.

Proof. The statement simply means that the collection of points where the degree of the fibre is $\geq d$ is open. Thus we can work locally on S and assume S is affine. In this case, for every $W \subset X$ quasi-compact open, the set of points $U_d(W)$ where the fibres of $W \to S$ have degree $\geq d$ is open by Lemma 36.31.9. Since $U_d = \bigcup_W U_d(W)$ the result follows. □

Lemma 36.31.11. *Let $f : X \to S$ be a morphism of schemes which is flat, locally of finite presentation, and locally quasi-finite. Let $g \in \Gamma(X, \mathcal{O}_X)$ nonzero. Then there exist an open $V \subset X$ such that $g|_V \neq 0$, an open $U \subset S$ fitting into a commutative diagram*

$$\begin{array}{ccc} V & \longrightarrow & X \\ {\scriptstyle \pi}\downarrow & & \downarrow{\scriptstyle f} \\ U & \longrightarrow & S, \end{array}$$

a quasi-coherent subsheaf $\mathcal{F} \subset \mathcal{O}_U$, an integer $r > 0$, and an injective $\mathcal{O}_U$-module map $\mathcal{F}^{\oplus r} \to \pi_*\mathcal{O}_V$ whose image contains $g|_V$.

Proof. We may assume X and S affine. We obtain a filtration $\emptyset = Z_{-1} \subset Z_0 \subset Z_1 \subset Z_2 \subset \ldots \subset Z_n = S$ as in Lemmas 36.31.8 and 36.31.9. Let $T \subset X$ be the scheme theoretic support of the finite $\mathcal{O}_X$-module $\mathrm{Im}(g : \mathcal{O}_X \to \mathcal{O}_X)$. Note that T is the support of g as a section of $\mathcal{O}_X$ (Modules, Definition 17.5.1) and for any open $V \subset X$ we have $g|_V \neq 0$ if and only if $V \cap T \neq \emptyset$. Let r be the largest integer such that $f(T) \subset Z_r$ set theoretically. Let $\xi \in T$ be a generic point of an irreducible component of T such that $f(\xi) \notin Z_{r-1}$ (and hence $f(\xi) \in Z_r$). We may replace S by an affine neighbourhood of $f(\xi)$ contained in $S \setminus Z_{r-1}$. Write $S = \mathrm{Spec}(A)$ and let $I = (a_1, \ldots, a_m) \subset A$ be a finitely generated ideal such that $V(I) = Z_r$ (set theoretically, see Algebra, Lemma 10.27.1). Since the support of g is contained in $f^{-1}V(I)$ by our choice of r we see that there exists an integer N such that $a_j^N g = 0$ for $j = 1, \ldots, m$. Replacing a_j by a_j^r we may assume that $Ig = 0$. For any A-module M write $M[I]$ for the I-torsion of M, i.e., $M[I] = \{m \in M \mid Im = 0\}$. Write $X = \mathrm{Spec}(B)$, so $g \in B[I]$. Since $A \to B$ is flat we see that

$$B[I] = A[I] \otimes_A B \cong A[I] \otimes_{A/I} B/IB$$

By our choice of Z_r, the A/I-module B/IB is finite locally free of rank r. Hence after replacing S by a smaller affine open neighbourhood of $f(\xi)$ we may assume that $B/IB \cong (A/IA)^{\oplus r}$ as A/I-modules. Choose a map $\psi : A^{\oplus r} \to B$ which reduces modulo I to the isomorphism of the previous sentence. Then we see that the induced map

$$A[I]^{\oplus r} \longrightarrow B[I]$$

is an isomorphism. The lemma follows by taking $\mathcal{F}$ the quasi-coherent sheaf associated to the A-module $A[I]$ and the map $\mathcal{F}^{\oplus r} \to \pi_*\mathcal{O}_V$ the one corresponding to $A[I]^{\oplus r} \subset A^{\oplus r} \to B$. □

Lemma 36.31.12. *Let $f : X \to Y$ be a separated, locally quasi-finite morphism with Y affine. Then every finite set of points of X is contained in an open affine of X.*

Proof. Let $x_1, \ldots, x_n \in X$. Choose a quasi-compact open $U \subset X$ with $x_i \in U$. Then $U \to Y$ is quasi-affine by Lemma 36.31.2. Hence there exists an affine open $V \subset U$ containing $x_1, \ldots, x_n$ by Properties, Lemma 27.27.5. □

Lemma 36.31.13. *Let $U \to X$ be a surjective étale morphism of schemes. Assume X is quasi-compact and quasi-separated. Then there exists a surjective integral morphism $Y \to X$ where Y is a scheme, such that for every $y \in Y$ there is an open neighbourhood $V \subset Y$ such that $V \to X$ factors through U. In fact, we may assume $Y \to X$ is finite and of finite presentation.*

Proof. Since X is quasi-compact, there exist finitely many affine opens $U_i \subset U$ such that $U' = \coprod U_i \to X$ is surjective. After replacing U by U', we see that we may assume U is affine. Then there exists an integer d bounding the degree of the geometric fibres of $U \to X$ (see Morphisms, Lemma 28.50.8). We will prove the lemma by induction on d for all quasi-compact and separated schemes U mapping surjective and étale onto X. If $d = 1$, then $U = X$ and the result holds with $Y = U$. Assume $d > 1$.

We apply Lemma 36.31.2 and we obtain a factorization

$$\begin{array}{ccc} U & \xrightarrow{\quad j \quad} & Y \\ & \searrow \quad \swarrow_{\pi} & \\ & X & \end{array}$$

with π integral and j a quasi-compact open immersion. We may and do assume that $j(U)$ is scheme theoretically dense in Y. Note that

$$U \times_X Y = U \amalg W$$

where the first summand is the image of $U \to U \times_X Y$ (which is closed by Schemes, Lemma 25.21.11 and open because it is étale as a morphism between schemes étale over Y) and the second summand is the (open and closed) complement. The image $V \subset Y$ of W is an open subscheme containing $Y \setminus U$.

The étale morphism $W \to Y$ has geometric fibres of cardinality $< d$. Namely, this is clear for geometric points of $U \subset Y$ by inspection. Since $U \subset Y$ is dense, it holds for all geometric points of Y for example by Lemma 36.31.8 (the degree of the fibres of a quasi-compact étale morphism does not go up under specialization). Thus we may apply the induction hypothesis to $W \to V$ and find a surjective integral morphism $Z \to V$ with Z a scheme, which Zariski locally factors through W. Choose a factorization $Z \to Z' \to Y$ with $Z' \to Y$ integral and $Z \to Z'$ open immersion (Lemma 36.31.2). After replacing Z' by the scheme theoretic closure of Z in Z' we may assume that Z is scheme theoretically dense in Z'. After doing this we have $Z' \times_Y V = Z$. Finally, let $T \subset Y$ be the induced reduced closed subscheme structure on $Y \setminus V$. Consider the morphism

$$Z' \amalg T \longrightarrow X$$

This is a surjective integral morphism by construction. Since $T \subset U$ it is clear that the morphism $T \to X$ factors through U. On the other hand, let $z \in Z'$ be a point. If $z \notin Z$, then z maps to a point of $Y \setminus V \subset U$ and we find a neighbourhood of z on which the morphism factors through U. If $z \in Z$, then we have a neighbourhood $V \subset Z$ which factors through $W \subset U \times_X Y$ and hence through U. This proves existence.

Assume we have found $Y \to X$ integral and surjective which Zariski locally factors through U. Choose a finite affine open covering $Y = \bigcup V_j$ such that $V_j \to X$ factors through U. We can write $Y = \lim Y_i$ with $Y_i \to X$ finite and of finite presentation, see Limits, Lemma 31.6.2. For large enough i we can find affine opens $V_{i,j} \subset Y_i$ whose inverse image in Y recovers V_j, see Limits, Lemma 31.3.8. For even larger i the morphisms $V_j \to U$ over X come from morphisms $V_{i,j} \to U$ over X, see Limits, Proposition 31.5.1. This finishes the proof. $\square$

36.32. Application to morphisms with connected fibres

In this section we prove some lemmas that produce morphisms all of whose fibres are geometrically connected or geometrically integral. This will be useful in our study of the local structure of morphisms of finite type later.

Lemma 36.32.1. *Consider a diagram of morphisms of schemes*

$$\begin{array}{ccc} Z & \xrightarrow{\sigma} & X \\ & \searrow & \downarrow \\ & & Y \end{array}$$

an a point $y \in Y$. Assume

(1) *$X \to Y$ is of finite presentation and flat,*
(2) *$Z \to Y$ is finite locally free,*
(3) *$Z_y \neq \emptyset$,*
(4) *all fibres of $X \to Y$ are geometrically reduced, and*
(5) *X_y is geometrically connected over $\kappa(y)$.*

Then there exists an open $X^0 \subset X$ such that $X^0_y = X_y$ and such that all nonempty fibres of $X^0 \to Y$ are geometrically connected.

Proof. In this proof we will use that flat, finite presentation, finite locally free are properties that are preserved under base change and composition. We will also use that a finite locally free morphism is both open and closed. You can find these facts as Morphisms, Lemmas 28.26.7, 28.22.4, 28.46.4, 28.26.5, 28.22.3, 28.46.3, 28.26.9, and 28.44.10.

Note that $X_Z \to Z$ is flat morphism of finite presentation which has a section s coming from σ. Let X^0_Z denote the subset of X_Z defined in Situation 36.24.1. By Lemma 36.24.6 it is an open subset of X_Z.

The pullback $X_{Z \times_Y Z}$ of X to $Z \times_Y Z$ comes equipped with two sections s_0, s_1, namely the base changes of s by $\mathrm{pr}_0, \mathrm{pr}_1 : Z \times_Y Z \to Z$. The construction of Situation 36.24.1 gives two subsets $(X_{Z \times_Y Z})^0_{s_0}$ and $(X_{Z \times_Y Z})^0_{s_1}$. By Lemma 36.24.2 these are the inverse images of X^0_Z under the morphisms $1_X \times \mathrm{pr}_0, 1_X \times \mathrm{pr}_1 : X_{Z \times_Y Z} \to X_Z$. In particular these subsets are open.

Let $(Z \times_Y Z)_y = \{z_1, \ldots, z_n\}$. As X_y is geometrically connected, we see that the fibres of $(X_{Z \times_Y Z})^0_{s_0}$ and $(X_{Z \times_Y Z})^0_{s_1}$ over each z_i agree (being equal to the whole fibre). Another way to say this is that

$$s_0(z_i) \in (X_{Z \times_Y Z})^0_{s_1} \quad \text{and} \quad s_1(z_i) \in (X_{Z \times_Y Z})^0_{s_0}.$$

Since the sets $(X_{Z \times_Y Z})^0_{s_0}$ and $(X_{Z \times_Y Z})^0_{s_1}$ are open in $X_{Z \times_Y Z}$ there exists an open neighbourhood $W \subset Z \times_Y Z$ of $(Z \times_Y Z)_y$ such that

$$s_0(W) \subset (X_{Z \times_Y Z})^0_{s_1} \quad \text{and} \quad s_1(W) \subset (X_{Z \times_Y Z})^0_{s_0}.$$

Then it follows directly from the construction in Situation 36.24.1 that

$$p^{-1}(W) \cap (X_{Z \times_Y Z})^0_{s_0} = p^{-1}(W) \cap (X_{Z \times_Y Z})^0_{s_1}$$

where $p : X_{Z \times_Y Z} \to Z \times_W Z$ is the projection. Because $Z \times_Y Z \to Y$ is finite locally free, hence open and closed, there exists an open neighbourhood $V \subset Y$ of y such that $q^{-1}(V) \subset W$, where $q : Z \times_Y Z \to Y$ is the structure morphism. To

prove the lemma we may replace Y by V. After we do this we see that $X_Z^0 \subset Y_Z$ is an open such that

$$(1_X \times \mathrm{pr}_0)^{-1}(X_Z^0) = (1_X \times \mathrm{pr}_1)^{-1}(X_Z^0).$$

This means that the image $X^0 \subset X$ of X_Z^0 is an open such that $(X_Z \to X)^{-1}(X^0) = X_Z^0$, see Descent, Lemma 34.9.2. At this point it is clear that X^0 is the desired open subscheme. □

Lemma 36.32.2. *Let $h : Y \to S$ be a morphism of schemes. Let $s \in S$ be a point. Let $T \subset Y_s$ be an open subscheme. Assume*

(1) *h is flat and of finite presentation,*
(2) *all fibres of h are geometrically reduced, and*
(3) *T is geometrically connected over $\kappa(s)$.*

Then we can find an elementary étale neighbourhood $(S', s') \to (S, s)$ and an open $V \subset Y_{S'}$ such that

(a) *all fibres of $V \to S'$ are geometrically connected,*
(b) *$V_{s'} = T \times_s s'$.*

Proof. The problem is clearly local on S, hence we may replace S by an affine open neighbourhood of s. The topology on Y_s is induced from the topology on X, see Schemes, Lemma 25.18.5. Hence we can find a quasi-compact open $V \subset Y$ such that $V_s = T$. The restriction of h to V is quasi-compact (as S affine and V quasi-compact), quasi-separated, locally of finite presentation, and flat hence flat of finite presentation. Thus after replacing Y by V we may assume, in addition to (1) and (2) that $Y_s = T$ and S affine.

Pick a point $y \in Y_s$ such that h is Cohen-Macaulay at y, see Lemma 36.17.4. By Lemma 36.18.4 there exists a diagram

$$\begin{array}{ccc} Z & \longrightarrow & Y \\ & \searrow & \downarrow \\ & & S \end{array}$$

such that $Z \to S$ is flat, locally of finite presentation, locally quasi-finite with $Z_s = \{z\}$. Apply Lemma 36.30.1 to find an elementary neighbourhood $(S', s') \to (S, s)$ and an open $Z' \subset Z_{S'} = S' \times_S Z$ with $Z' \to S'$ finite with a unique point $z' \in Z'$ lying over s. Note that $Z' \to S'$ is also locally of finite presentation and flat (as an open of the base change of $Z \to S$), hence $Z' \to S'$ is finite locally free, see Morphisms, Lemma 28.46.2. Note that $Y_{S'} \to S'$ is flat and of finite presentation with geometrically reduced fibres as a base change of h. Also $Y_{s'} = Y_s$ is geometrically connected. Apply Lemma 36.32.1 to $Z' \to Y_{S'}$ over S' to get $V \subset Y_{S'}$ satisfying (2) whose fibres over S' are either empty or geometrically connected. As $V \to S'$ is open (Morphisms, Lemma 28.26.9), after shrinking S' we may assume $V \to S'$ is surjective, whence (1) holds. □

Lemma 36.32.3. *Let $h : Y \to S$ be a morphism of schemes. Let $s \in S$ be a point. Let $T \subset Y_s$ be an open subscheme. Assume*

(1) *h is of finite presentation,*
(2) *h is normal, and*
(3) *T is geometrically irreducible over $\kappa(s)$.*

Then we can find an elementary étale neighbourhood $(S', s') \to (S, s)$ and an open $V \subset Y_{S'}$ such that

(a) *all fibres of $V \to S'$ are geometrically integral,*
(b) *$V_{s'} = T \times_s s'$.*

Proof. Apply Lemma 36.32.2 to find an elementary étale neighbourhood $(S', s') \to (S, s)$ and an open $V \subset Y_{S'}$ such that all fibres of $V \to S'$ are geometrically integral and $V_{s'} = T \times_s s'$. Note that $V \to S'$ is open, see Morphisms, Lemma 28.26.9 Hence after replacing S' by the image of $V \to S'$ we see that all fibres of $V \to S'$ are nonempty. As V is an open of the base change of h all fibres of $V \to S'$ are geometrically normal, see Lemma 36.15.2. In particular, they are geometrically reduced. To finish the proof we have to show they are geometrically irreducible. But, if $t \in S'$ then V_t is of finite type over $\kappa(t)$ and hence $V_t \times_{\kappa(t)} \overline{\kappa(t)}$ is of finite type over $\overline{\kappa(t)}$ hence Noetherian. By choice of $S' \to S$ the scheme $V_t \times_{\kappa(t)} \overline{\kappa(t)}$ is connected. Hence $V_t \times_{\kappa(t)} \overline{\kappa(t)}$ is irreducible by Properties, Lemma 27.7.6 and we win. □

36.33. Application to the structure of finite type morphisms

The result in this section can be found in [**GR71**]. Loosely stated it says that a finite type morphism is étale locally on the source and target the composition of a finite morphism by a smooth morphism with geometrically connected fibres of relative dimension equal to the fibre dimension of the original morphism.

Lemma 36.33.1. *Let $f : X \to S$ be a morphism. Let $x \in X$ and set $s = f(x)$. Assume that f is locally of finite type and that $n = \dim_x(X_s)$. Then there exists a commutative diagram*

$$\begin{array}{ccc} X & \xleftarrow{g} & X' \\ \downarrow & & \downarrow{\scriptstyle \pi} \\ & & Y \\ & & \downarrow{\scriptstyle h} \\ S & = & S \end{array} \qquad \begin{array}{ccc} x & \longleftarrow & x' \\ \downarrow & & \downarrow \\ & & y \\ & & \downarrow \\ s & = & s \end{array}$$

and a point $x' \in X'$ with $g(x') = x$ such that with $y = \pi(x')$ we have

(1) *$h : Y \to S$ is smooth of relative dimension n,*
(2) *$g : (X', x') \to (X, x)$ is an elementary étale neighbourhood,*
(3) *π is finite, and $\pi^{-1}(\{y\}) = \{x'\}$, and*
(4) *$\kappa(y)$ is a purely transcendental extension of $\kappa(s)$.*

Moreover, if f is locally of finite presentation then π is of finite presentation.

Proof. The problem is local on X and S, hence we may assume that X and S are affine. By Algebra, Lemma 10.119.3 after replacing X by a standard open neighbourhood of x in X we may assume there is a factorization

$$X \xrightarrow{\pi} \mathbf{A}^n_S \longrightarrow S$$

such that π is quasi-finite and such that $\kappa(\pi(x))$ is purely transcendental over $\kappa(s)$. By Lemma 36.30.1 there exists an elementary étale neighbourhood

$$(Y, y) \to (\mathbf{A}^n_S, \pi(x))$$

and an open $X' \subset X \times_{\mathbf{A}^n_S} Y$ which contains a unique point x' lying over y such that $X' \to Y$ is finite. This proves (1) – (4) hold. For the final assertion, use Morphisms, Lemma 28.22.11. □

Lemma 36.33.2. *Let $f : X \to S$ be a morphism. Let $x \in X$ and set $s = f(x)$. Assume that f is locally of finite type and that $n = \dim_x(X_s)$. Then there exists a commutative diagram*

$$\begin{array}{ccc} X & \xleftarrow{g} & X' \\ \downarrow & & \downarrow \pi \\ & & Y' \\ & & \downarrow h \\ S & \xleftarrow{e} & S' \end{array} \qquad \begin{array}{ccc} x & \longleftarrow & x' \\ \downarrow & & \downarrow \\ & & y' \\ & & \downarrow \\ s & \longleftarrow & s' \end{array}$$

and a point $x' \in X'$ with $g(x') = x$ such that with $y' = \pi(x')$, $s' = h(y')$ we have

(1) *$h : Y' \to S'$ is smooth of relative dimension n,*
(2) *all fibres of $Y' \to S'$ are geometrically integral,*
(3) *$g : (X', x') \to (X, x)$ is an elementary étale neighbourhood,*
(4) *π is finite, and $\pi^{-1}(\{y'\}) = \{x'\}$,*
(5) *$\kappa(y')$ is a purely transcendental extension of $\kappa(s')$, and*
(6) *$e : (S', s') \to (S, s)$ is an elementary étale neighbourhood.*

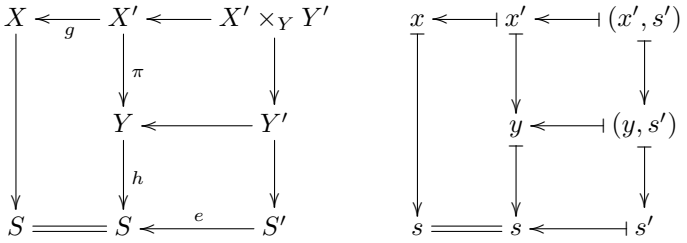
Moreover, if f is locally of finite presentation, then π is of finite presentation.

Proof. The question is local on S, hence we may replace S by an affine open neighbourhood of s. Next, we apply Lemma 36.33.1 to get a commutative diagram

$$\begin{array}{ccc} X & \xleftarrow{g} & X' \\ \downarrow & & \downarrow \pi \\ & & Y \\ & & \downarrow h \\ S & = & S \end{array} \qquad \begin{array}{ccc} x & \longleftarrow & x' \\ \downarrow & & \downarrow \\ & & y \\ & & \downarrow \\ s & = & s \end{array}$$

where h is smooth of relative dimension n and $\kappa(y)$ is a purely transcendental extension of $\kappa(s)$. Since the question is local on X also, we may replace Y by an affine neighbourhood of y (and X' by the inverse image of this under π). As S is affine this guarantees that $Y \to S$ is quasi-compact, separated and smooth, in particular of finite presentation. Let T be the connected component of Y_s containing y. As Y_s is Noetherian we see that T is open. We also see that T is geometrically connected over $\kappa(s)$ by Varieties, Lemma 32.5.14. Since T is also smooth over $\kappa(s)$ it is geometrically normal, see Varieties, Lemma 32.15.4. We conclude that T is geometrically irreducible over $\kappa(s)$ (as a connected Noetherian normal scheme is irreducible, see Properties, Lemma 27.7.6). Finally, note that the smooth morphism h is normal by Lemma 36.15.3. At this point we have verified all assumption of Lemma 36.32.3 hold for the morphism $h : Y \to S$ and open $T \subset Y_s$. As a result of applying Lemma 36.32.3 we obtain $e : S' \to S$, $s' \in S'$, Y' as in the commutative

diagram



where $e : (S', s') \to (S, s)$ is an elementary étale neighbourhood, and where $Y' \subset Y_{S'}$ is an open neighbourhood all of whose fibres over S' are geometrically irreducible, such that $Y'_{s'} = T$ via the identification $Y_s = Y_{S',s'}$. Let $(y, s') \in Y'$ be the point corresponding to $y \in T$; this is also the unique point of $Y \times_S S'$ lying over y with residue field equal to $\kappa(y)$ which maps to s' in S'. Similarly, let $(x', s') \in X' \times_Y Y' \subset X' \times_S S'$ be the unique point over x' with residue field equal to $\kappa(x')$ lying over s'. Then the outer part of this diagram is a solution to the problem posed in the lemma. Some minor details omitted. □

Lemma 36.33.3. *Assumption and notation as in Lemma 36.33.2. In addition to properties (1) – (6) we may also arrange it so that*

(7) *S', Y', X' are affine.*

Proof. Note that if Y' is affine, then X' is affine as π is finite. Choose an affine open neighbourhood $U' \subset S'$ of s'. Choose an affine open neighbourhood $V' \subset h^{-1}(U')$ of y'. Let $W' = h(V')$. This is an open neighbourhood of s' in S', see Morphisms, Lemma 28.35.10, contained in U'. Choose an affine open neighbourhood $U'' \subset W'$ of s'. Then $h^{-1}(U'') \cap V'$ is affine because it is equal to $U'' \times_{U'} V'$. By construction $h^{-1}(U'') \cap V' \to U''$ is a surjective smooth morphism whose fibres are (nonempty) open subschemes of geometrically integral fibres of $Y' \to S'$, and hence geometrically integral. Thus we may replace S' by U'' and Y' by $h^{-1}(U'') \cap V'$. □

The significance of the property $\pi^{-1}(\{y'\}) = \{x'\}$ is partially explained by the following lemma.

Lemma 36.33.4. *Let $\pi : X \to Y$ be a finite morphism. Let $x \in X$ with $y = \pi(x)$ such that $\pi^{-1}(\{y\}) = \{x\}$. Then*

(1) *For every neighbourhood $U \subset X$ of x in X, there exists a neighbourhood $V \subset Y$ of y such that $\pi^{-1}(V) \subset U$.*
(2) *The ring map $\mathcal{O}_{Y,y} \to \mathcal{O}_{X,x}$ is finite.*
(3) *If π is of finite presentation, then $\mathcal{O}_{Y,y} \to \mathcal{O}_{X,x}$ is of finite presentation.*
(4) *For any quasi-coherent $\mathcal{O}_X$-module $\mathcal{F}$ we have $\mathcal{F}_x = \pi_*\mathcal{F}_y$ as $\mathcal{O}_{Y,y}$-modules.*

Proof. The first assertion is purely topological; use that π is a continuous and closed map such that $\pi^{-1}(\{y\}) = \{x\}$. To prove the second and third parts we may assume $X = \mathrm{Spec}(B)$ and $Y = \mathrm{Spec}(A)$. Then $A \to B$ is a finite ring map and y corresponds to a prime $\mathfrak{p}$ of A such that there exists a unique prime $\mathfrak{q}$ of B lying over $\mathfrak{p}$. Then $B_\mathfrak{q} = B_\mathfrak{p}$, see Algebra, Lemma 10.38.11. In other words, the map $A_\mathfrak{p} \to B_\mathfrak{q}$ is equal to the map $A_\mathfrak{p} \to B_\mathfrak{p}$ you get from localizing $A \to B$ at $\mathfrak{p}$. Thus (2) and (3) follow from simple properties of localization (some details omitted). For the final statement, suppose that $\mathcal{F} = \widetilde{M}$ for some B-module M. Then $\mathcal{F} = M_\mathfrak{q}$

and $\pi_*\mathcal{F}_y = M_{\mathfrak{p}}$. By the above these localizations agree. Alternatively you can use part (1) and the definition of stalks to see that $\mathcal{F}_x = \pi_*\mathcal{F}_y$ directly. □

36.34. Application to the fppf topology

We can use the above étale localization techniques to prove the following result describing the fppf topology as being equal to the topology "generated by" Zariski coverings and by coverings of the form $\{f : T \to S\}$ where f is surjective finite locally free.

Lemma 36.34.1. *Let S be a scheme. Let $\{S_i \to S\}_{i\in I}$ be an fppf covering. Then there exist*

1. *a Zariski open covering $S = \bigcup U_j$,*
2. *surjective finite locally free morphisms $W_j \to U_j$,*
3. *Zariski open coverings $W_j = \bigcup_k W_{j,k}$,*
4. *surjective finite locally free morphisms $T_{j,k} \to W_{j,k}$*

such that the fppf covering $\{T_{j,k} \to S\}$ refines the given covering $\{S_i \to S\}$.

Proof. We may assume that each $S_i \to S$ is locally quasi-finite, see Lemma 36.18.6.

Fix a point $s \in S$. Pick an $i \in I$ and a point $s_i \in S_i$ mapping to s. Choose an elementary étale neighbourhood $(S', s) \to (S, s)$ such that there exists an open

$$S_i \times_S S' \supset V$$

which contains a unique point $v \in V$ mapping to $s \in S'$ and such that $V \to S'$ is finite, see Lemma 36.30.1. Then $V \to S'$ is finite locally free, because it is finite and because $S_i \times_S S' \to S'$ is flat and locally of finite presentation as a base change of the morphism $S_i \to S$, see Morphisms, Lemmas 28.22.4, 28.26.7, and 28.46.2. Hence $V \to S'$ is open, and after shrinking S' we may assume that $V \to S'$ is surjective finite locally free. Since we can do this for every point of S we conclude that $\{S_i \to S\}$ can be refined by a covering of the form $\{V_a \to S\}_{a\in A}$ where each $V_a \to S$ factors as $V_a \to S'_a \to S$ with $S'_a \to S$ étale and $V_a \to S'_a$ surjective finite locally free.

By Remark 36.29.3 there exists a Zariski open covering $S = \bigcup U_j$, for each j a finite locally free, surjective morphism $W_j \to U_j$, and for each j a Zariski open covering $\{W_{j,k} \to W_j\}$ such that the family $\{W_{j,k} \to S\}$ refines the étale covering $\{S'_a \to S\}$, i.e., for each pair j, k there exists an $a(j, k)$ and a factorization $W_{j,K} \to S'_a \to S$ of the morphism $W_{j,K} \to S$. Set $T_{j,k} = W_{j,k} \times_{S'_a} V_a$ and everything is clear. □

36.35. Closed points in fibres

Some of the material in this section is taken from the preprint [**OP10**].

Lemma 36.35.1. *Let $f : X \to S$ be a morphism of schemes. Let $Z \subset X$ be a closed subscheme. Let $s \in S$. Assume*

1. *S is irreducible with generic point η,*
2. *X is irreducible,*
3. *f is dominant,*
4. *f is locally of finite type,*
5. *$\dim(X_s) \leq \dim(X_\eta)$,*
6. *Z is locally principal in X, and*
7. *$Z_\eta = \emptyset$.*

Then the fibre Z_s is (set theoretically) a union of irreducible components of X_s.

Proof. Let X_{red} denote the reduction of X. Then $Z \cap X_{red}$ is a locally principal closed subscheme of X_{red}, see Divisors, Lemma 30.9.10. Hence we may assume that X is reduced. In other words X is integral, see Properties, Lemma 27.3.4. In this case the morphism $X \to S$ factors through S_{red}, see Schemes, Lemma 25.12.6. Thus we may replace S by S_{red} and assume that S is integral too.

The assertion that f is dominant signifies that the generic point of X is mapped to η, see Morphisms, Lemma 28.8.5. Moreover, the scheme X_η is an integral scheme which is locally of finite type over the field $\kappa(\eta)$. Hence $d = \dim(X_\eta) \geq 0$ is equal to $\dim_\xi(X_\eta)$ for every point ξ of X_η, see Algebra, Lemmas 10.108.4 and 10.108.5. In view of Morphisms, Lemma 28.29.4 and condition (5) we conclude that $\dim_x(X_s) = d$ for every $x \in X_s$.

In the Noetherian case the assertion can be proved as follows. If the lemma does not holds there exists $x \in Z_s$ which is a generic point of an irreducible component of Z_s but not a generic point of any irreducible component of X_s. Then we see that $\dim_x(Z_s) \leq d - 1$, because $\dim_x(X_s) = d$ and in a neighbourhood of x in X_s the closed subscheme Z_s does not contain any of the irreducible components of X_s. Hence after replacing X by an open neighbourhood of x we may assume that $\dim_z(Z_{f(z)}) \leq d - 1$ for all $z \in Z$, see Morphisms, Lemma 28.29.4. Let $\xi' \in Z$ be a generic point of an irreducible component of Z and set $s' = f(\xi)$. As $Z \neq X$ is locally principal we see that $\dim(\mathcal{O}_{X,\xi}) = 1$, see Algebra, Lemma 10.57.10 (this is where we use X is Noetherian). Let $\xi \in X$ be the generic point of X and let ξ_1 be a generic point of any irreducible component of $X_{s'}$ which contains ξ'. Then we see that we have the specializations

$$\xi \rightsquigarrow \xi_1 \rightsquigarrow \xi'.$$

As $\dim(\mathcal{O}_{X,\xi}) = 1$ one of the two specializations has to be an equality. By assumption $s' \neq \eta$, hence the first specialization is not an equality. Hence $\xi' = \xi_1$ is a generic point of an irreducible component of $X_{s'}$. Applying Morphisms, Lemma 28.29.4 one more time this implies $\dim_{\xi'}(Z_{s'}) = \dim_{\xi'}(X_{s'}) \geq \dim(X_\eta) = d$ which gives the desired contradiction.

In the general case we reduce to the Noetherian case as follows. If the lemma is false then there exists a point $x \in X$ lying over s such that x is a generic point of an irreducible component of Z_s, but not a generic point of any of the irreducible components of X_s. Let $U \subset S$ be an affine neighbourhood of s and let $V \subset X$ be an affine neighbourhood of x with $f(V) \subset U$. Write $U = \mathrm{Spec}(A)$ and $V = \mathrm{Spec}(B)$ so that $f|_V$ is given by a ring map $A \to B$. Let $\mathfrak{q} \subset B$, resp. $\mathfrak{p} \subset A$ be the prime corresponding to x, resp. s. After possibly shrinking V we may assume $Z \cap V$ is cut out by some element $g \in B$. Denote $K = f.f.(A)$. What we know at this point is the following:

(1) $A \subset B$ is a finitely generated extension of domains,
(2) the element $g \otimes 1$ is invertible in $B \otimes_A K$,
(3) $d = \dim(B \otimes_A K) = \dim(B \otimes_A \kappa(\mathfrak{p}))$,
(4) $g \otimes 1$ is not a unit of $B \otimes_A \kappa(\mathfrak{p})$, and
(5) $g \otimes 1$ is not in any of the minimal primes of $B \otimes_A \kappa(\mathfrak{p})$.

We are seeking a contradiction.

Pick elements $x_1, \ldots, x_n \in B$ which generate B over A. For a finitely generated $\mathbf{Z}$-algebra $A_0 \subset A$ let $B_0 \subset B$ be the A_0-subalgebra generated by $x_1, \ldots, x_n$, denote $K_0 = f.f.(A_0)$, and set $\mathfrak{p}_0 = A_0 \cap \mathfrak{p}$. We claim that when A_0 is large enough then (1) – (5) also hold for the system $(A_0 \subset B_0, g, \mathfrak{p}_0)$.

We prove each of the conditions in turn. Part (1) holds by construction. For part (2) write $(g \otimes 1)h = 1$ for some $h \otimes 1/a \in B \otimes_A K$. Write $g = \sum a_I x^I$, $h = \sum a'_I x^I$ (multi-index notation) for some coefficients $a_I, a'_I \in A$. As soon as A_0 contains a and the a_I, a'_I then (2) holds because $B_0 \otimes_{A_0} K_0 \subset B \otimes_A K$ (as localizations of the injective map $B_0 \to B$). To achieve (3) consider the exact sequence

$$0 \to I \to A[X_1, \ldots, X_n] \to B \to 0$$

which defines I where the second map sends X_i to x_i. Since $\otimes$ is right exact we see that $I \otimes_A K$, respectively $I \otimes_A \kappa(\mathfrak{p})$ is the kernel of the surjection $K[X_1, \ldots, X_n] \to B \otimes_A K$, respectively $\kappa(\mathfrak{p})[X_1, \ldots, X_n] \to B \otimes_A \kappa(\mathfrak{p})$. As a polynomial ring over a field is Noetherian there exist finitely many elements $h_j \in I$, $j = 1, \ldots, m$ which generate $I \otimes_A K$ and $I \otimes_A \kappa(\mathfrak{p})$. Write $h_j = \sum a_{j,I} X^I$. As soon as A_0 contains all $a_{j,I}$ we get to the situation where

$$B_0 \otimes_{A_0} K_0 \otimes_{K_0} K = B \otimes_A K \quad \text{and} \quad B_0 \otimes_{A_0} \kappa(\mathfrak{p}_0) \otimes_{\kappa(\mathfrak{p}_0)} \kappa(\mathfrak{p}) = B \otimes_A \kappa(\mathfrak{p}).$$

By either Morphisms, Lemma 28.29.3 or Algebra, Lemma 10.110.5 we see that the dimension equalities of (3) are satisfied. Part (4) is immediate. As $B_0 \otimes_{A_0} \kappa(\mathfrak{p}_0) \subset B \otimes_A \kappa(\mathfrak{p})$ each minimal prime of $B_0 \otimes_{A_0} \kappa(\mathfrak{p}_0)$ lies under a minimal prime of $B \otimes_A \kappa(\mathfrak{p})$ by Algebra, Lemma 10.28.6. This implies that (5) holds. In this way we reduce the problem to the Noetherian case which we have dealt with above. □

Here is an algebraic application of the lemma above. The fourth assumption of the lemma holds if $A \to B$ is flat, see Lemma 36.35.3.

Lemma 36.35.2. *Let $A \to B$ be a local homomorphism of local rings, and $g \in \mathfrak{m}_B$. Assume*

(1) *A and B are domains and $A \subset B$,*
(2) *B is essentially of finite type over A,*
(3) *g is not contained in any minimal prime over $\mathfrak{m}_A B$, and*
(4) *$\dim(B/\mathfrak{m}_A B) + trdeg_{\kappa(\mathfrak{m}_A)}(\kappa(\mathfrak{m}_B)) = trdeg_A(B)$.*

Then $A \subset B/gB$, i.e., the generic point of $\mathrm{Spec}(A)$ *is in the image of the morphism* $\mathrm{Spec}(B/gB) \to \mathrm{Spec}(A)$.

Proof. Note that the two assertions are equivalent by Algebra, Lemma 10.28.6. To start the proof let C be an A-algebra of finite type and $\mathfrak{q}$ a prime of C such that $B = C_{\mathfrak{q}}$. Of course we may assume that C is a domain and that $g \in C$. After replacing C by a localization we see that $\dim(C/\mathfrak{m}_A C) = \dim(B/\mathfrak{m}_A B) + \mathrm{trdeg}_{\kappa(\mathfrak{m}_A)}(\kappa(\mathfrak{m}_B))$, see Morphisms, Lemma 28.29.1. Setting $K = f.f.(A)$ we see by the same reference that $\dim(C \otimes_A K) = \mathrm{trdeg}_A(B)$. Hence assumption (4) means that the generic and closed fibres of the morphism $\mathrm{Spec}(C) \to \mathrm{Spec}(A)$ have the same dimension.

Suppose that the lemma is false. Then $(B/gB) \otimes_A K = 0$. This means that $g \otimes 1$ is invertible in $B \otimes_A K = C_{\mathfrak{q}} \otimes_A K$. As $C_{\mathfrak{q}}$ is a limit of principal localizations we conclude that $g \otimes 1$ is invertible in $C_h \otimes_A K$ for some $h \in C$, $h \notin \mathfrak{q}$. Thus after replacing C by C_h we may assume that $(C/gC) \otimes_A K = 0$. We do one more replacement of C to make sure that the minimal primes of $C/\mathfrak{m}_A C$ correspond

one-to-one with the minimal primes of $B/\mathfrak{m}_A B$. At this point we apply Lemma 36.35.1 to $X = \mathrm{Spec}(C) \to \mathrm{Spec}(A) = S$ and the locally closed subscheme $Z = \mathrm{Spec}(C/gC)$. Since $Z_K = \emptyset$ we see that $Z \otimes \kappa(\mathfrak{m}_A)$ has to contain an irreducible component of $X \otimes \kappa(\mathfrak{m}_A) = \mathrm{Spec}(C/\mathfrak{m}_A C)$. But this contradicts the assumption that g is not contained in any prime minimal over $\mathfrak{m}_A B$. The lemma follows. □

Lemma 36.35.3. *Let $A \to B$ be a local homomorphism of local rings. Assume*

(1) *A and B are domains and $A \subset B$,*
(2) *B is essentially of finite type over A, and*
(3) *B is flat over A.*

Then we have

$$\dim(B/\mathfrak{m}_A B) + \mathit{trdeg}_{\kappa(\mathfrak{m}_A)}(\kappa(\mathfrak{m}_B)) = \mathit{trdeg}_A(B).$$

Proof. Let C be an A-algebra of finite type and $\mathfrak{q}$ a prime of C such that $B = C_{\mathfrak{q}}$. We may assume C is a domain. We have $\dim_{\mathfrak{q}}(C/\mathfrak{m}_A C) = \dim(B/\mathfrak{m}_A B) + \mathrm{trdeg}_{\kappa(\mathfrak{m}_A)}(\kappa(\mathfrak{m}_B))$, see Morphisms, Lemma 28.29.1. Setting $K = f.f.(A)$ we see by the same reference that $\dim(C \otimes_A K) = \mathrm{trdeg}_A(B)$. Thus we are really trying to prove that $\dim_{\mathfrak{q}}(C/\mathfrak{m}_A C) = \dim(C \otimes_A K)$. Choose a valuation ring A' in K dominating A, see Algebra, Lemma 10.46.2. Set $C' = C \otimes_A A'$. Choose a prime $\mathfrak{q}'$ of C' lying over $\mathfrak{q}$; such a prime exists because

$$C'/\mathfrak{m}_{A'} C' = C/\mathfrak{m}_A C \otimes_{\kappa(\mathfrak{m}_A)} \kappa(\mathfrak{m}_{A'})$$

which proves that $C/\mathfrak{m}_A C \to C'/\mathfrak{m}_{A'} C'$ is faithfully flat. This also proves that $\dim_{\mathfrak{q}}(C/\mathfrak{m}_A C) = \dim_{\mathfrak{q}'}(C'/\mathfrak{m}_{A'} C')$, see Algebra, Lemma 10.110.6. Note that $B' = C'_{\mathfrak{q}'}$ is a localization of $B \otimes_A A'$. Hence B' is flat over A'. The generic fibre $B' \otimes_{A'} K$ is a localization of $B \otimes_A K$. Hence B' is a domain. If we prove the lemma for $A' \subset B'$, then we get the equality $\dim_{\mathfrak{q}'}(C'/\mathfrak{m}_{A'} C') = \dim(C' \otimes_{A'} K)$ which implies the desired equality $\dim_{\mathfrak{q}}(C/\mathfrak{m}_A C) = \dim(C \otimes_A K)$ by what was said above. This reduces the lemma to the case where A is a valuation ring.

Let $A \subset B$ be as in the lemma with A a valuation ring. As before write $B = C_{\mathfrak{q}}$ for some domain C of finite type over A. By Algebra, Lemma 10.119.9 we obtain $\dim(C/\mathfrak{m}_A C) = \dim(C \otimes_A K)$ and we win. □

Lemma 36.35.4. *Let $f : X \to S$ be a morphism of schemes. Let $x \rightsquigarrow x'$ be a specialization of points in X. Set $s = f(x)$ and $s' = f(x')$. Assume*

(1) *x' is a closed point of $X_{s'}$, and*
(2) *f is locally of finite type.*

Then the set

$$\{x_1 \in X \text{ such that } f(x_1) = s \text{ and } x_1 \text{ is closed in } X_s \text{ and } x \rightsquigarrow x_1 \rightsquigarrow x'\}$$

is dense in the closure of x in X_s.

Proof. We apply Schemes, Lemma 25.20.4 to the specialization $x \rightsquigarrow x'$. This produces a morphism $\varphi : \mathrm{Spec}(B) \to X$ where B is a valuation ring such that φ maps the generic point to x and the closed point to x'. We may also assume that $\kappa(x) = f.f.(B)$. Let $A = B \cap \kappa(s)$. Note that this is a valuation ring (see

Algebra, Lemma 10.46.5) which dominates the image of $\mathcal{O}_{S,s'} \to \kappa(s)$. Consider the commutative diagram

$$\begin{array}{ccccc} \mathrm{Spec}(B) & \longrightarrow & X_A & \longrightarrow & X \\ & \searrow & \downarrow & & \downarrow \\ & & \mathrm{Spec}(A) & \longrightarrow & S \end{array}$$

The generic (resp. closed) point of B maps to a point x_A (resp. x'_A) of X_A lying over the generic (resp. closed) point of $\mathrm{Spec}(A)$. Note that x'_A is a closed point of the special fibre of X_A by Morphisms, Lemma 28.21.4. Note that the generic fibre of $X_A \to \mathrm{Spec}(A)$ is isomorphic to X_s. Thus we have reduced the lemma to the case where S is the spectrum of a valuation ring, $s = \eta \in S$ is the generic point, and $s' \in S$ is the closed point.

We will prove the lemma by induction on $\dim_x(X_\eta)$. If $\dim_x(X_\eta) = 0$, then there are no other points of X_η specializing to x and x is closed in its fibre, see Morphisms, Lemma 28.21.6, and the result holds. Assume $\dim_x(X_\eta) > 0$.

Let $X' \subset X$ be the reduced induced scheme structure on the irreducible closed subscheme $\overline{\{x\}}$ of X, see Schemes, Definition 25.12.5. To prove the lemma we may replace X by X' as this only decreases $\dim_x(X_\eta)$. Hence we may also assume that X is an integral scheme and that x is its generic point. In addition, we may replace X by an affine neighbourhood of x'. Thus we have $X = \mathrm{Spec}(B)$ where $A \subset B$ is a finite type extension of domains. Note that in this case $\dim_x(X_\eta) = \dim(X_\eta) = \dim(X_{s'})$, and that in fact $X_{s'}$ is equidimensional, see Algebra, Lemma 10.119.9.

Let $W \subset X_\eta$ be a proper closed subset (this is the subset we want to "avoid"). As X_s is of finite type over a field we see that W has finitely many irreducible components $W = W_1 \cup \ldots \cup W_n$. Let $\mathfrak{q}_j \subset B$, $j = 1, \ldots, r$ be the corresponding prime ideals. Let $\mathfrak{q} \subset B$ be the maximal ideal corresponding to the point x'. Let $\mathfrak{p}_1, \ldots, \mathfrak{p}_s \subset B$ be the minimal primes lying over $\mathfrak{m}_A B$. There are finitely many as these correspond to the irreducible components of the Noetherian scheme $X_{s'}$. Moreover, each of these irreducible components has dimension > 0 (see above) hence we see that $\mathfrak{p}_i \neq \mathfrak{q}$ for all i. Now, pick an element $g \in \mathfrak{q}$ such that $g \notin \mathfrak{q}_j$ for all j and $g \notin \mathfrak{p}_i$ for all i, see Algebra, Lemma 10.14.3. Denote $Z \subset X$ the locally principal closed subscheme defined by h. Let $Z_\eta = Z_{1,\eta} \cup \ldots \cup Z_{n,\eta}$, $n \geq 0$ be the decomposition of the generic fibre of Z into irreducible components (finitely many as the generic fibre is Noetherian). Denote $Z_i \subset X$ the closure of $Z_{i,\eta}$. After replacing X by a smaller affine neighbourhood we may assume that $x \in Z_i$ for each $i = 1, \ldots, n$. By construction $Z \cap X_{s'}$ does not contain any irreducible component of $X_{s'}$. Hence by Lemma 36.35.1 we conclude that $Z_\eta \neq \emptyset$! In other words $n \geq 1$. Letting $x_1 \in Z_1$ be the generic point we see that $x_1 \rightsquigarrow x'$ and $f(x_1) = \eta$. Also, by construction $Z_{1,\eta} \cap W_j \subset W_j$ is a proper closed subset. Hence every irreducible component of $Z_{1,\eta} \cap W_j$ has codimension ≥ 2 in X_η whereas $\mathrm{codim}(Z_{1,\eta}, X_\eta) = 1$ by Algebra, Lemma 10.57.10. Thus $W \cap Z_{1,\eta}$ is a proper closed subset. At this point we see that the induction hypothesis applies to $Z_1 \to S$ and the specialization $x_1 \rightsquigarrow x'$. This produces a closed point x_2 of $Z_{1,\eta}$ not contained in W which specializes to x'. Thus we obtain $x \rightsquigarrow x_2 \rightsquigarrow x'$, the point x_2 is closed in X_η, and $x_2 \notin W$ as desired. $\square$

Remark 36.35.5. The proof of Lemma 36.35.4 actually shows that there exists a sequence of specializations

$$x \rightsquigarrow x_1 \rightsquigarrow x_2 \rightsquigarrow \ldots \rightsquigarrow x_d \rightsquigarrow x'$$

where all x_i are in the fibre X_s, each specialization is immediate, and x_d is a closed point of X_s. The integer $d = \mathrm{trdeg}_{\kappa(s)}(\kappa(x)) = \dim(\overline{\{x\}})$ where the closure is taken in X_s. Moreover, the points x_i can be chosen to avoid any closed subset of X_s which does not contain the point x.

Examples, Section 78.25 shows that the following lemma is false if A is not assumed Noetherian.

Lemma 36.35.6. *Let $\varphi : A \to B$ be a local ring map of local rings. Let $V \subset \mathrm{Spec}(B)$ be an open subscheme which contains at least one prime not lying over $\mathfrak{m}_A$. Assume A is Noetherian, φ essentially of finite type, and $A/\mathfrak{m}_A \subset B/\mathfrak{m}_B$ is finite. Then there exists a $\mathfrak{q} \in V$, $\mathfrak{m}_A \neq \mathfrak{q} \cap A$ such that $A \to B/\mathfrak{q}$ is the localization of a quasi-finite ring map.*

Proof. Since A is Noetherian and $A \to B$ is essentially of finite type, we know that B is Noetherian too. By Properties, Lemma 27.6.4 the topological space $\mathrm{Spec}(B)\backslash\{\mathfrak{m}_B\}$ is Jacobson. Hence we can choose a closed point $\mathfrak{q}$ which is contained in the nonempty open

$$V \setminus \{\mathfrak{q} \subset B \mid \mathfrak{m}_A = \mathfrak{q} \cap A\}.$$

(Nonempty by assumption, open because $\{\mathfrak{m}_A\}$ is a closed subset of $\mathrm{Spec}(A)$.) Then $\mathrm{Spec}(B/\mathfrak{q})$ has two points, namely $\mathfrak{m}_B$ and $\mathfrak{q}$ and $\mathfrak{q}$ does not lie over $\mathfrak{m}_A$. Write $B/\mathfrak{q} = C_{\mathfrak{m}}$ for some finite type A-algebra C and prime ideal $\mathfrak{m}$. Then $A \to C$ is quasi-finite at $\mathfrak{m}$ by Algebra, Lemma 10.116.2 (2). Hence by Algebra, Lemma 10.117.14 we see that after replacing C by a principal localization the ring map $A \to C$ is quasi-finite. □

Lemma 36.35.7. *Let $f : X \to S$ be a morphism of schemes. Let $x \in X$ with image $s \in S$. Let $U \subset X$ be an open subscheme. Assume f locally of finite type, S locally Noetherian, x a closed point of X_s, and assume there exists a point $x' \in U$ with $x' \rightsquigarrow x$ and $f(x') \neq s$. Then there exists a closed subscheme $Z \subset X$ such that (a) $x \in Z$, (b) $f|_Z : Z \to S$ is quasi-finite at x, and (c) there exists a $z \in Z$, $z \in U$, $z \rightsquigarrow x$ and $f(z) \neq s$.*

Proof. This is a reformulation of Lemma 36.35.6. Namely, set $A = \mathcal{O}_{S,s}$ and $B = \mathcal{O}_{X,x}$. Denote $V \subset \mathrm{Spec}(B)$ the inverse image of U. The ring map $f^\sharp : A \to B$ is essentially of finite type. By assumption there exists at least one point of V which does not map to the closed point of $\mathrm{Spec}(A)$. Hence all the assumptions of Lemma 36.35.6 hold and we obtain a prime $\mathfrak{q} \subset B$ which does not lie over $\mathfrak{m}_A$ and such that $A \to B/\mathfrak{q}$ is the localization of a quasi-finite ring map. Let $z \in X$ be the image of the point $\mathfrak{q}$ under the canonical morphism $\mathrm{Spec}(B) \to X$. Set $Z = \overline{\{z\}}$ with the induced reduced scheme structure. As $z \rightsquigarrow x$ we see that $x \in Z$ and $\mathcal{O}_{Z,x} = B/\mathfrak{q}$. By construction $Z \to S$ is quasi-finite at x. □

Remark 36.35.8. We can use Lemma 36.35.6 or its variant Lemma 36.35.7 to give an alternative proof of Lemma 36.35.4 in case S is locally Noetherian. Here is a rough sketch. Namely, first replace S by the spectrum of the local ring at s'. Then we may use induction on $\dim(S)$. The case $\dim(S) = 0$ is trivial because

then $s' = s$. Replace X by the reduced induced scheme structure on $\overline{\{x\}}$. Apply Lemma 36.35.7 to $X \to S$ and $x' \mapsto s'$ and any nonempty open $U \subset X$ containing x. This gives us a closed subscheme $x' \in Z \subset X$ a point $z \in Z$ such that $Z \to S$ is quasi-finite at x' and such that $f(z) \neq s'$. Then z is a closed point of $X_{f(z)}$, and $z \rightsquigarrow x'$. As $f(z) \neq s'$ we see $\dim(\mathcal{O}_{S,f(z)}) < \dim(S)$. Since x is the generic point of X we see $x \rightsquigarrow z$, hence $s = f(x) \rightsquigarrow f(z)$. Apply the induction hypothesis to $s \rightsquigarrow f(z)$ and $z \mapsto f(z)$ to win.

Lemma 36.35.9. *Suppose that $f : X \to S$ is locally of finite type, S locally Noetherian, $x \in X$ a closed point of its fibre X_s, and $U \subset X$ an open subscheme such that $U \cap X_s = \emptyset$ and $x \in \overline{U}$, then the conclusions of Lemma 36.35.7 hold.*

Proof. Namely, we can reduce this to the cited lemma as follows: First we replace X and S by affine neighbourhoods of x and s. Then X is Noetherian, in particular U is quasi-compact (see Morphisms, Lemma 28.16.6 and Topology, Lemmas 5.8.2 and 5.11.13). Hence there exists a specialization $x' \rightsquigarrow x$ with $x' \in U$ (see Morphisms, Lemma 28.6.5). Note that $f(x') \neq s$. Thus we see all hypotheses of the lemma are satisfied and we win. □

36.36. Stein factorization

Stein factorization is the statement that a proper morphism $f : X \to S$ with $f_*\mathcal{O}_X = \mathcal{O}_S$ has connected fibres.

Lemma 36.36.1. *Let S be a scheme. Let $f : X \to S$ be a universally closed, quasi-compact and quasi-separated morphism. There exists a factorization*

$$\begin{array}{ccc} X & \xrightarrow{\quad} & S' \\ & {\scriptstyle f'} \searrow \quad \swarrow {\scriptstyle \pi} & \\ & {\scriptstyle f} \searrow \quad \swarrow & \\ & S & \end{array}$$

with the following properties:

1. *the morphism f' is universally closed, quasi-compact, quasi-separated and surjective,*
2. *the morphism $\pi : S' \to S$ is integral,*
3. *we have $f'_*\mathcal{O}_X = \mathcal{O}_{S'}$,*
4. *we have $S' = \underline{\mathrm{Spec}}_S(f_*\mathcal{O}_X)$, and*
5. *S' is the normalization of S in X, see Morphisms, Definition 28.48.3.*

Proof. We just define S' as the normalization of S in X, so (5) and (2) hold automatically. By Morphisms, Lemma 28.48.9 we see that (4) holds. The morphism f' is universally closed by Morphisms, Lemma 28.42.7. It is quasi-compact by Schemes, Lemma 25.21.15 and quasi-separated by Schemes, Lemma 25.21.14.

To show the remaining statements we may assume the base scheme S is affine, say $S = \mathrm{Spec}(R)$. Then $S' = \mathrm{Spec}(A)$ with $A = \Gamma(X, \mathcal{O}_X)$ an integral R-algebra. Thus it is clear that $f'_*\mathcal{O}_X$ is $\mathcal{O}_{S'}$ (because $f'_*\mathcal{O}_X$ is quasi-coherent, by Schemes, Lemma 25.24.1, and hence equal to $\widetilde{A}$). This proves (3).

Let us show that f' is surjective. As f' is universally closed (see above) the image of f' is a closed subset $V(I) \subset S' = \mathrm{Spec}(A)$. Pick $h \in I$. Then $h|_X = f^\sharp(h)$ is a global section of the structure sheaf of X which vanishes at every point. As X is

quasi-compact this means that $h|_X$ is a nilpotent section, i.e., $h^n|X = 0$ for some $n > 0$. But $A = \Gamma(X, \mathcal{O}_X)$, hence $h^n = 0$. In other words I is contained in the radical ideal of A and we conclude that $V(I) = S'$ as desired. □

Lemma 36.36.2. *Let $f : X \to S$ be a morphism of schemes. Let $s \in S$. Then X_s is geometrically connected, if and only if for every étale neighbourhood $(U, u) \to (S, s)$ the base change $X_U \to U$ has connected fibre X_u.*

Proof. If X_s is geometrically connected, then any base change of it is connected. On the other hand, suppose that X_s is not geometrically connected. Then by Varieties, Lemma 32.5.11 we see that $X_s \times_{\mathrm{Spec}(\kappa(s))} \mathrm{Spec}(k)$ is disconnected for some finite separable field extension $\kappa(s) \subset k$. By Lemma 36.27.2 there exists an affine étale neighbourhood $(U, u) \to (S, s)$ such that $\kappa(s) \subset \kappa(u)$ is identified with $\kappa(s) \subset k$. In this case X_u is disconnected. □

Theorem 36.36.3 (Stein factorization; Noetherian case)**.** *Let S be a locally Noetherian scheme. Let $f : X \to S$ be a proper morphism. There exists a factorization*

$$\begin{array}{ccc} X & \xrightarrow{\quad} & S' \\ & {\scriptstyle f'} \searrow \quad \swarrow {\scriptstyle \pi} & \\ & f \searrow \quad \swarrow & \\ & S & \end{array}$$

with the following properties:

1. *the morphism f' is proper with geometrically connected fibres,*
2. *the morphism $\pi : S' \to S$ is finite,*
3. *we have $f'_*\mathcal{O}_X = \mathcal{O}_{S'}$,*
4. *we have $S' = \underline{\mathrm{Spec}}_S(f_*\mathcal{O}_X)$, and*
5. *S' is the normalization of S in X, see Morphisms, Definition 28.48.3.*

Proof. Let $f = \pi \circ f'$ be the factorization of Lemma 36.36.1. Note that besides the conclusions of Lemma 36.36.1 we also have that f' is separated (Schemes, Lemma 25.21.14) and finite type (Morphisms, Lemma 28.16.8). Hence f' is proper. By Cohomology of Schemes, Lemma 29.17.2 we see that $f_*\mathcal{O}_X$ is a coherent $\mathcal{O}_S$-module. Hence we see that π is finite, i.e., (2) holds.

This proves all but the most interesting assertion, namely that all the fibres of f' are geometrically connected. It is clear from the discussion above that we may replace S by S', and we may therefore assume that S is Noetherian, affine, $f : X \to S$ is proper, and $f_*\mathcal{O}_X = \mathcal{O}_S$. Let $s \in S$ be a point of S. We have to show that X_s is geometrically connected. By Lemma 36.36.2 we see that it suffices to show X_u is connected for every étale neighbourhood $(U, u) \to (S, s)$. We may assume U is affine. Thus U is Noetherian (Morphisms, Lemma 28.16.6), the base change $f_U : X_U \to U$ is proper (Morphisms, Lemma 28.42.5), and that also $(f_U)_*\mathcal{O}_{X_U} = \mathcal{O}_U$ (Cohomology of Schemes, Lemma 29.5.2). Hence after replacing $(f : X \to S, s)$ by the base change $(f_U : X_U \to U, u)$ it suffices to prove that the fibre X_s is connected.

At this point we apply the theorem on formal functions, more precisely Cohomology of Schemes, Lemma 29.18.7. It tells us that

$$\mathcal{O}^\wedge_{S,s} = \lim_n H^0(X_n, \mathcal{O}_{X_n})$$

where X_n is the nth infinitesimal neighbourhood of X_s. Since the underlying topological space of X_n is equal to that of X_s we see that if $X_s = T_1 \coprod T_2$ is a disjoint

union of nonempty open and closed subschemes, then similarly $X_n = T_{1,n} \coprod T_{2,n}$ for all n. And this in turn means $H^0(X_n, \mathcal{O}_{X_n})$ contains a nontrivial idempotent $e_{1,n}$, namely the function which is identically 1 on $T_{1,n}$ and identically 0 on $T_{2,n}$. It is clear that $e_{1,n+1}$ restricts to $e_{1,n}$ on X_n. Hence $e_1 = \lim e_{1,n}$ is a nontrivial idempotent of the limit. This contradicts the fact that $\mathcal{O}_{S,s}^\wedge$ is a local ring. Thus the assumption was wrong, i.e., X_s is connected, and we win. □

Theorem 36.36.4 (Stein factorization; general case)**.** *Let S be a scheme. Let $f : X \to S$ be a proper morphism. There exists a factorization*

$$\begin{array}{ccc} X & \xrightarrow{f'} & S' \\ & {\scriptstyle f}\searrow \quad \swarrow{\scriptstyle \pi} & \\ & S & \end{array}$$

with the following properties:

1. *the morphism f' is proper with geometrically connected fibres,*
2. *the morphism $\pi : S' \to S$ is integral,*
3. *we have $f'_*\mathcal{O}_X = \mathcal{O}_{S'}$,*
4. *we have $S' = \underline{\mathrm{Spec}}_S(f_*\mathcal{O}_X)$, and*
5. *S' is the normalization of S in X, see Morphisms, Definition 28.48.3.*

Proof. We may apply Lemma 36.36.1 to get the morphism $f' : X \to S'$. Note that besides the conclusions of Lemma 36.36.1 we also have that f' is separated (Schemes, Lemma 25.21.14) and finite type (Morphisms, Lemma 28.16.8). Hence f' is proper. At this point we have proved all of the statements except for the statement that f' has geometrically connected fibres.

We may assume that $S = \mathrm{Spec}(R)$ is affine. Set $R' = \Gamma(X, \mathcal{O}_X)$. Then $S' = \mathrm{Spec}(R')$. Thus we may replace S by S' and assume that $S = \mathrm{Spec}(R)$ is affine $R = \Gamma(X, \mathcal{O}_X)$. Next, let $s \in S$ be a point. Let $U \to S$ be an étale morphism of affine schemes and let $u \in U$ be a point mapping to s. Let $X_U \to U$ be the base change of X. By Lemma 36.36.2 it suffices to show that the fibre of $X_U \to U$ over u is connected. By Cohomology of Schemes, Lemma 29.5.2 we see that $\Gamma(X_U, \mathcal{O}_{X_U}) = \Gamma(U, \mathcal{O}_U)$. Hence we have to show: Given $S = \mathrm{Spec}(R)$ affine, $X \to S$ proper with $\Gamma(X, \mathcal{O}_X) = R$ and $s \in S$ is a point, the fibre X_s is connected.

By Limits, Lemma 31.12.5 we can write $X = \lim X_i$ with $X_i \to S$ proper and of finite presentation. Then $X_s = \lim X_{i,s}$. Assume $X_s = U \amalg V$ with U and V open. We have to show U or V is empty. We can find an i such that $X_{i,s} = U_i \amalg V_i$ for opens U_i, V_i whose inverse image in X_s are U, V (Limits, Lemma 31.3.8). Set $R_i = \Gamma(X_i, \mathcal{O}_{X_i})$ and $S_i = \mathrm{Spec}(R_i)$. Observe that we have ring maps $R \to R_i \to R$ whose composition is id_R. Namely, we have the first because X_i is a scheme over S and the second because we have $X \to X_i$ and $R = \Gamma(X, \mathcal{O}_X)$. Then

$$\begin{array}{ccccc} X & \longrightarrow & X_i & & \\ \downarrow & & \downarrow & & \\ S & \longrightarrow & S_i & \longrightarrow & S \end{array}$$

is commutative. Let $s_i \in S_i$ be the image of s. Observe that $\kappa(s_i) = \kappa(s)$. We obtain a factorization $X_s \to X_{i,s_i} \to X_{i,s}$. Thus U, V are the inverse images of

the inverse images of U_i, V_i in X_{i,s_i}. Hence it suffices to prove the result for the morphism $X_i \to S$, i.e., we may assume $X \to S$ is proper and of finite presentation.

Assume $X \to S$ is proper of finite presentation, $S = \mathrm{Spec}(R)$ is affine, $R = \Gamma(X, \mathcal{O}_X)$ and $s \in S$ is a point. We have to show X_s is connected. Write $R = \mathrm{colim}\, R_i$ as a direct colimit with R_i of finite type over $\mathbf{Z}$. Set $S_i = \mathrm{Spec}(R_i)$. Then there exists $0 \in I$ and $X_0 \to S_0$ such that $X = X_0 \times_{S_0} S$, see Limits, Lemma 31.9.1. Then $X_i = X_0 \times_{S_0} S_i \to S_i$ is proper for i large enough, see Limits, Lemma 31.12.4. Let $s_i \in S_i$ be the image of s. Then $s = \lim s_i$ (equality of schemes) and $X_s = \lim X_{i,s_i}$ because limits commute with fibre products. Assume $X_s = U \amalg V$ with U and V open. We have to show U or V is empty. We can find an i such that $X_{i,s_i} = U_i \amalg V_i$ for opens U_i, V_i whose inverse image in X_s are U, V (Limits, Lemma 31.3.8). Set $R_i' = \Gamma(X_i, \mathcal{O}_{X_i})$ and $S_i' = \mathrm{Spec}(R_i')$. Here is a diagram

$$\begin{array}{ccccc} X & \longrightarrow & & & X_i \\ \downarrow & & & \swarrow & \downarrow \\ S & \longrightarrow & S_i' & \longrightarrow & S_i \end{array}$$

By Theorem 36.36.3 the fibres of $X_i \to S_i'$ are connected. Let $s_i' \in S_i'$ be the image of s. Consider the factorization $X_s \to X_{i,s_i'} \to X_{i,s_i}$. Since the fibre $X_{i,s_i'}$ is connected, either the inverse image of U_i or the inverse image of V_i in $X_{i,s_i'}$ is empty, hence either U or V is empty. This finishes the proof. □

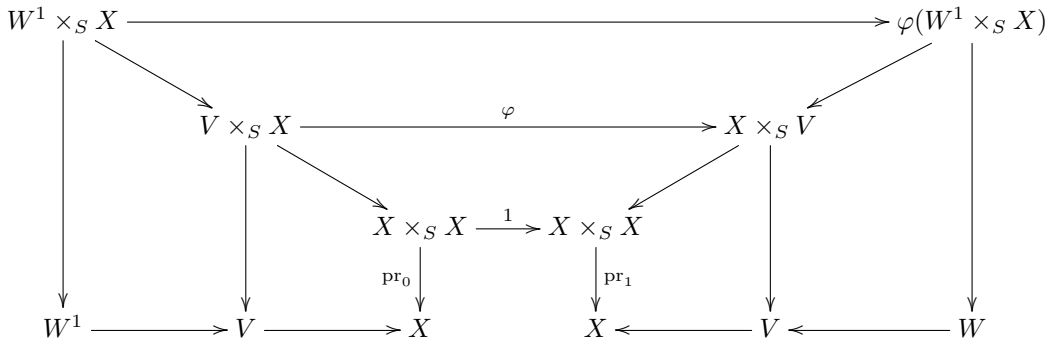
36.37. Descending separated locally quasi-finite morphisms

In this section we show that "separated locally quasi-finite morphisms satisfy descent for fppf-coverings". See Descent, Definition 34.32.1 for terminology. This is in the marvellous (for many reasons) paper by Raynaud and Gruson hidden in the proof of [**GR71**, Lemma 5.7.1]. It can also be found in [**Mur95**], and [**ABD+66**, Exposé X, Lemma 5.4] under the additional hypothesis that the morphism is locally of finite presentation. Here is the formal statement.

Lemma 36.37.1. *Let S be a scheme. Let $\{X_i \to S\}_{i \in I}$ be an fppf covering, see Topologies, Definition 33.7.1. Let $(V_i/X_i, \varphi_{ij})$ be a descent datum relative to $\{X_i \to S\}$. If each morphism $V_i \to X_i$ is separated and locally quasi-finite, then the descent datum is effective.*

Proof. Being separated and being locally quasi-finite are properties of morphisms of schemes which are preserved under any base change, see Schemes, Lemma 25.21.13 and Morphisms, Lemma 28.21.13. Hence Descent, Lemma 34.32.2 applies and it suffices to prove the statement of the lemma in case the fppf-covering is given by a single $\{X \to S\}$ flat surjective morphism of finite presentation of affines. Say $X = \mathrm{Spec}(A)$ and $S = \mathrm{Spec}(R)$ so that $R \to A$ is a faithfully flat ring map. Let (V, φ) be a descent datum relative to X over S and assume that $\pi : V \to X$ is separated and locally quasi-finite.

Let $W^1 \subset V$ be any affine open. Consider $W = \mathrm{pr}_1(\varphi(W^1 \times_S X)) \subset V$. Here is a picture



Ok, and now since $X \to S$ is flat and of finite presentation it is universally open (Morphisms, Lemma 28.26.9). Hence we conclude that W is open. Moreover, it is also clearly the case that W is quasi-compact, and $W^1 \subset W$. Moreover, we note that $\varphi(W \times_S X) = X \times_S W$ by the cocycle condition for φ. Hence we obtain a new descent datum (W, φ') by restricting φ to $W \times_S X$. Note that the morphism $W \to X$ is quasi-compact, separated and locally quasi-finite. This implies that it is separated and quasi-finite by definition. Hence it is quasi-affine by Lemma 36.31.2. Thus by Descent, Lemma 34.34.1 we see that the descent datum (W, φ') is effective.

In other words, we find that there exists an open covering $V = \bigcup W_i$ by quasi-compact opens W_i which are stable for the descent morphism φ. Moreover, for each such quasi-compact open $W \subset V$ the corresponding descent data (W, φ') is effective. It is an exercise to show this means the original descent datum is effective by glueing the schemes obtained from descending the opens W_i (details omitted). □

36.38. Relative finite presentation

Let $R \to A$ be a finite type ring map. Let M be an A-module. In More on Algebra, Section 15.52 we defined what it means for M to be finitely presented relative to R. We also proved this notion has good localization properties and glues. Hence we can define the corresponding global notion as follows.

Definition 36.38.1. Let $f : X \to S$ be a morphism of schemes which is locally of finite type. Let $\mathcal{F}$ be a quasi-coherent $\mathcal{O}_X$-module. We say $\mathcal{F}$ is *finitely presented relative to S* or *of finite presentation relative to S* if there exists an affine open covering $S = \bigcup V_i$ and for every i an affine open covering $f^{-1}(V_i) = \bigcup_j U_{ij}$ such that $\mathcal{F}(U_{ij})$ is a $\mathcal{O}_X(U_{ij})$-module of finite presentation relative to $\mathcal{O}_S(V_i)$.

Note that this implies that $\mathcal{F}$ is a finite type $\mathcal{O}_X$-module. If $X \to S$ is just locally of finite type, then $\mathcal{F}$ may be of finite presentation relative to S, without $X \to S$ being locally of finite presentation. We will see that $X \to S$ is locally of finite presentation if and only if $\mathcal{O}_X$ is of finite presentation relative to S.

Lemma 36.38.2. *Let $f : X \to S$ be a morphism of schemes which is locally of finite type. Let $\mathcal{F}$ be a quasi-coherent $\mathcal{O}_X$-module. The following are equivalent*

(1) *$\mathcal{F}$ is of finite presentation relative to S,*

(2) *for every affine opens $U \subset X$, $V \subset S$ with $f(U) \subset V$ the $\mathcal{O}_X(U)$-module $\mathcal{F}(U)$ is finitely presented relative to $\mathcal{O}_S(V)$.*

Moreover, if this is true, then for every open subschemes $U \subset X$ and $V \subset S$ with $f(U) \subset V$ the restriction $\mathcal{F}|_U$ is of finite presentation relative to V.

Proof. The final statement is clear from the equivalence of (1) and (2). It is also clear that (2) implies (1). Assume (1) holds. Let $S = \bigcup V_i$ and $f^{-1}(V_i) = \bigcup U_{ij}$ be affine open coverings as in Definition 36.38.1. Let $U \subset X$ and $V \subset S$ be as in (2). By More on Algebra, Lemma 15.52.8 it suffices to find a standard open covering $U = \bigcup U_k$ of U such that $\mathcal{F}(U_k)$ is finitely presented relative to $\mathcal{O}_S(V)$. In other words, for every $u \in U$ it suffices to find a standard affine open $u \in U' \subset U$ such that $\mathcal{F}(U')$ is finitely presented relative to $\mathcal{O}_S(V)$. Pick i such that $f(u) \in V_i$ and then pick j such that $u \in U_{ij}$. By Schemes, Lemma 25.11.5 we can find $v \in V' \subset V \cap V_i$ which is standard affine open in V' and V_i. Then $f^{-1}V' \cap U$, resp. $f^{-1}V' \cap U_{ij}$ are standard affine opens of U, resp. U_{ij}. Applying the lemma again we can find $u \in U' \subset f^{-1}V' \cap U \cap U_{ij}$ which is standard affine open in both $f^{-1}V' \cap U$ and $f^{-1}V' \cap U_{ij}$. Thus U' is also a standard affine open of U and U_{ij}. By More on Algebra, Lemma 15.52.4 the assumption that $\mathcal{F}(U_{ij})$ is finitely presented relative to $\mathcal{O}_S(V_i)$ implies that $\mathcal{F}(U')$ is finitely presented relative to $\mathcal{O}_S(V_i)$. Since $\mathcal{O}_X(U') = \mathcal{O}_X(U') \otimes_{\mathcal{O}_S(V_i)} \mathcal{O}_S(V')$ we see from More on Algebra, Lemma 15.52.5 that $\mathcal{F}(U')$ is finitely presented relative to $\mathcal{O}_S(V')$. Applying More on Algebra, Lemma 15.52.4 again we conclude that $\mathcal{F}(U')$ is finitely presented relative to $\mathcal{O}_S(V)$. This finishes the proof. □

Lemma 36.38.3. *Let $f : X \to S$ be a morphism of schemes which is locally of finite type. Let $\mathcal{F}$ be a quasi-coherent $\mathcal{O}_X$-module.*

(1) *If f is locally of finite presentation, then $\mathcal{F}$ is of finite presentation relative to S if and only if $\mathcal{F}$ is of finite presentation.*
(2) *The morphism f is locally of finite presentation if and only if $\mathcal{O}_X$ is of finite presentation relative to S.*

Proof. Follows immediately from the definitions, see discussion following More on Algebra, Definition 15.52.2. □

Lemma 36.38.4. *Let $\pi : X \to Y$ be a finite morphism of schemes locally of finite type over a base scheme S. Let $\mathcal{F}$ be a quasi-coherent $\mathcal{O}_X$-module. Then $\mathcal{F}$ is of finite presentation relative to S if and only if $\pi_*\mathcal{F}$ is of finite presentation relative to S.*

Proof. Translation of the result of More on Algebra, Lemma 15.52.3 into the language of schemes. □

Lemma 36.38.5. *Let $f : X \to S$ be a morphism of schemes which is locally of finite type. Let $\mathcal{F}$ be a quasi-coherent $\mathcal{O}_X$-module. Let $S' \to S$ be a morphism of schemes, set $X' = X \times_S S'$ and denote $\mathcal{F}'$ the pullback of $\mathcal{F}$ to X'. If $\mathcal{F}$ is of finite presentation relative to S, then $\mathcal{F}'$ is of finite presentation relative to S'.*

Proof. Translation of the result of More on Algebra, Lemma 15.52.5 into the language of schemes. □

Lemma 36.38.6. *Let $X \to Y \to S$ be morphisms of schemes which are locally of finite type. Let $\mathcal{G}$ be a quasi-coherent $\mathcal{O}_Y$-module. If $f : X \to Y$ is locally of*

finite presentation and $\mathcal{G}$ of finite presentation relative to S, then $f^*\mathcal{G}$ is of finite presentation relative to S.

Proof. Translation of the result of More on Algebra, Lemma 15.52.6 into the language of schemes. □

Lemma 36.38.7. *Let $X \to Y \to S$ be morphisms of schemes which are locally of finite type. Let $\mathcal{F}$ be a quasi-coherent $\mathcal{O}_X$-module. If $Y \to S$ is locally of finite presentation and $\mathcal{F}$ is of finite presentation relative to Y, then $\mathcal{F}$ is of finite presentation relative to S.*

Proof. Translation of the result of More on Algebra, Lemma 15.52.7 into the language of schemes. □

Lemma 36.38.8. *Let $X \to S$ be a morphism of schemes which is locally of finite type. Let $0 \to \mathcal{F}' \to \mathcal{F} \to \mathcal{F}'' \to 0$ be a short exact sequence of quasi-coherent $\mathcal{O}_X$-modules.*

1. *If $\mathcal{F}', \mathcal{F}''$ are finitely presented relative to S, then so is $\mathcal{F}$.*
2. *If $\mathcal{F}'$ is a finite type $\mathcal{O}_X$-module and $\mathcal{F}$ is finitely presented relative to S, then $\mathcal{F}''$ is finitely presented relative to S.*

Proof. Translation of the result of More on Algebra, Lemma 15.52.9 into the language of schemes. □

Lemma 36.38.9. *Let $X \to S$ be a morphism of schemes which is locally of finite type. Let $\mathcal{F}, \mathcal{F}'$ be quasi-coherent $\mathcal{O}_X$-modules. If $\mathcal{F} \oplus \mathcal{F}'$ is finitely presented relative to S, then so are $\mathcal{F}$ and $\mathcal{F}'$.*

Proof. Translation of the result of More on Algebra, Lemma 15.52.10 into the language of schemes. □

36.39. Relative pseudo-coherence

This section is the analogue of More on Algebra, Section 15.53 for schemes. We strongly urge the reader to take a look at that section first.

Lemma 36.39.1. *Let $X \to S$ be a finite type morphism of affine schemes. Let E be an object of $D(\mathcal{O}_X)$. Let $m \in \mathbf{Z}$. The following are equivalent*

1. *for some closed immersion $i : X \to \mathbf{A}^n_S$ the object Ri_*E of $D(\mathcal{O}_{\mathbf{A}^n_S})$ is m-pseudo-coherent, and*
2. *for all closed immersions $i : X \to \mathbf{A}^n_S$ the object Ri_*E of $D(\mathcal{O}_{\mathbf{A}^n_S})$ is m-pseudo-coherent.*

Proof. Say $S = \mathrm{Spec}(R)$ and $X = \mathrm{Spec}(A)$. Let i correspond to the surjection $\alpha : R[x_1, \ldots, x_n] \to A$ and let $X \to \mathbf{A}^m_S$ correspond to $\beta : R[y_1, \ldots, y_m] \to A$. Choose $f_j \in R[x_1, \ldots, x_n]$ with $\alpha(f_j) = \beta(y_j)$ and $g_i \in R[y_1, \ldots, y_m]$ with $\beta(g_i) = \alpha(x_i)$. Then we get a commutative diagram

$$\begin{array}{ccc} R[x_1, \ldots, x_n, y_1, \ldots, y_m] & \xrightarrow[y_j \mapsto f_j]{} & R[x_1, \ldots, x_n] \\ \downarrow{\scriptstyle x_i \mapsto g_i} & & \downarrow \\ R[y_1, \ldots, y_m] & \longrightarrow & A \end{array}$$

corresponding to the commutative diagram of closed immersions

$$\begin{array}{ccc} \mathbf{A}^{n+m}_S & \longleftarrow & \mathbf{A}^n_S \\ \uparrow & & \uparrow \\ \mathbf{A}^m_S & \longleftarrow & X \end{array}$$

Thus it suffices to show that under a closed immersion

$$f : \mathbf{A}^m_S \to \mathbf{A}^{n+m}_S$$

an object E of $D(\mathcal{O}_{\mathbf{A}^m_S})$ is m-pseudo-coherent if and only if Rf_*E is m-pseudo-coherent. This follows from Derived Categories of Schemes, Lemma 35.10.5 and the fact that $f_*\mathcal{O}_{\mathbf{A}^m_S}$ is pseudo-coherent on $\mathbf{A}^{n+m}_S$. This is straightforward to prove directly, but it also follows from Derived Categories of Schemes, Lemma 35.9.3 and More on Algebra, Lemma 15.53.3. □

Recall that if $f : X \to S$ is a morphism of scheme which is locally of finite type, then for every pair of affine opens $U \subset X$ and $V \subset S$ such that $f(U) \subset V$, the ring map $\mathcal{O}_S(V) \to \mathcal{O}_X(U)$ is of finite type (Morphisms, Lemma 28.16.2). Hence there always exist closed immersions $U \to \mathbf{A}^n_V$ and the following definition makes sense.

Definition 36.39.2. Let $f : X \to S$ be a morphism of schemes which is locally of finite type. Let E be an object of $D(\mathcal{O}_X)$. Let $\mathcal{F}$ be an $\mathcal{O}_X$-module. Fix $m \in \mathbf{Z}$.

(1) We say E is *m-pseudo-coherent relative to S* if there exists an affine open covering $S = \bigcup V_i$ and for each i an affine open covering $f^{-1}(V_i) = \bigcup U_{ij}$ such that the equivalent conditions of Lemma 36.39.1 are satisfied for each of the pairs $(U_{ij} \to V_i, E|_{U_{ij}})$.
(2) We say E is *pseudo-coherent relative to S* if E is m-pseudo-coherent relative to S for all $m \in \mathbf{Z}$.
(3) We say $\mathcal{F}$ is *m-pseudo-coherent relative to S* if $\mathcal{F}$ viewed as an object of $D(\mathcal{O}_X)$ is m-pseudo-coherent relative to S.
(4) We say $\mathcal{F}$ is *pseudo-coherent relative to S* if $\mathcal{F}$ viewed as an object of $D(\mathcal{O}_X)$ is pseudo-coherent relative to S.

If X is quasi-compact and E is m-pseudo-coherent relative to S for some m, then E is bounded above. We first prove the condition of relative pseudo-coherence localizes well.

Lemma 36.39.3. *Let S be an affine scheme. Let $V \subset S$ be a standard open. Let $X \to V$ be a finite type morphism of affine schemes. Let $U \subset X$ be an affine open. Let E be an object of $D(\mathcal{O}_X)$. If the equivalent conditions of Lemma 36.39.1 are satisfied for the pair $(X \to V, E)$, then the equivalent conditions of Lemma 36.39.1 are satisfied for the pair $(U \to S, E|_U)$.*

Proof. Write $S = \mathrm{Spec}(R)$, $V = D(f)$, $X = \mathrm{Spec}(A)$, and $U = D(g)$. Assume the equivalent conditions of Lemma 36.39.1 are satisfied for the pair $(X \to V, E)$.

Choose $R_f[x_1, \ldots, x_n] \to A$ surjective. Write $R_f = R[x_0]/(fx_0 - 1)$. Then $R[x_0, x_1, \ldots, x_n] \to A$ is surjective, and $R_f[x_1, \ldots, x_n]$ is pseudo-coherent as an $R[x_0, \ldots, x_n]$-module. Thus we have

$$X \to \mathbf{A}^n_V \to \mathbf{A}^{n+1}_S$$

and we can apply Derived Categories of Schemes, Lemma 35.10.5 to conclude that the pushfoward E' of E to $\mathbf{A}^{n+1}_S$ is m-pseudo-coherent.

Choose an element $g' \in R[x_0, x_1, \ldots, x_n]$ which maps to $g \in A$. Consider the surjection $R[x_0, \ldots, x_{n+1}] \to R[x_0, \ldots, x_n, 1/g']$. We obtain

$$\begin{array}{ccccc} X & \longleftarrow & U & & \\ \downarrow & & \downarrow & \searrow & \\ \mathbf{A}^{n+1}_S & \longleftarrow & D(g') & \longrightarrow & \mathbf{A}^{n+2}_S \end{array}$$

where the lower left arrow is an open immersion and the lower right arrow is a closed immersion. We conclude as before that the pushforward of $E'|_{D(g')}$ to $\mathbf{A}^{n+2}_S$ is m-pseudo-coherent. Since this is also the pushforward of $E|_U$ to $\mathbf{A}^{n+2}_S$ we conclude the lemma is true. □

Lemma 36.39.4. *Let $X \to S$ be a finite type morphism of affine schemes. Let E be an object of $D(\mathcal{O}_X)$. Let $m \in \mathbf{Z}$. Let $X = \bigcup U_i$ be a standard affine open covering. The following are equivalent*

1. *the equivalent conditions of Lemma 36.39.1 hold for the pairs $(U_i \to S, E|_{U_i})$,*
2. *the equivalent conditions of Lemma 36.39.1 hold for the pair $(X \to S, E)$.*

Proof. The implication (2) $\Rightarrow$ (1) is Lemma 36.39.3. Assume (1). Say $S = \mathrm{Spec}(R)$ and $X = \mathrm{Spec}(A)$ and $U_i = D(f_i)$. Write $1 = \sum f_i g_i$ in A. Consider the surjections

$$R[x_i, y_i, z_i] \to R[x_i, y_i, z_i]/(\sum y_i z_i - 1) \to A.$$

which sends y_i to f_i and z_i to g_i. Note that $R[x_i, y_i, z_i]/(\sum y_i z_i - 1)$ is pseudo-coherent as an $R[x_i, y_i, z_i]$-module. Thus it suffices to prove that the pushforward of E to $T = \mathrm{Spec}(R[x_i, y_i, z_i]/(\sum y_i z_i - 1))$ is m-pseudo-coherent, see Derived Categories of Schemes, Lemma 35.10.5. For each i_0 it suffices to prove the restriction of this pushforward to $W_{i_0} = \mathrm{Spec}(R[x_i, y_i, z_i, 1/y_{i_0}]/(\sum y_i z_i - 1))$ is m-pseudo-coherent. Note that there is a commutative diagram

$$\begin{array}{ccc} X & \longleftarrow & U_{i_0} \\ \downarrow & & \downarrow \\ T & \longleftarrow & W_{i_0} \end{array}$$

which implies that the pushforward of E to T restricted to W_{i_0} is the pushforward of $E|_{U_{i_0}}$ to W_{i_0}. Since $R[x_i, y_i, z_i, 1/y_{i_0}]/(\sum y_i z_i - 1)$ is isomorphic to a polynomial ring over R this proves what we want. □

Lemma 36.39.5. *Let $f : X \to S$ be a morphism of schemes which is locally of finite type. Let E be an object of $D(\mathcal{O}_X)$. Fix $m \in \mathbf{Z}$. The following are equivalent*

1. *E is m-pseudo-coherent relative to S,*
2. *for every affine opens $U \subset X$ and $V \subset S$ with $f(U) \subset V$ the equivalent conditions of Lemma 36.39.1 are satisfied for the pair $(U \to V, E|_U)$.*

Moreover, if this is true, then for every open subschemes $U \subset X$ and $V \subset S$ with $f(U) \subset V$ the restriction $E|_U$ is m-pseudo-coherent relative to V.

Proof. The final statement is clear from the equivalence of (1) and (2). It is also clear that (2) implies (1). Assume (1) holds. Let $S = \bigcup V_i$ and $f^{-1}(V_i) = \bigcup U_{ij}$ be affine open coverings as in Definition 36.39.2. Let $U \subset X$ and $V \subset S$ be as in (2). By Lemma 36.39.4 it suffices to find a standard open covering $U = \bigcup U_k$ of U such that the equivalent conditions of Lemma 36.39.1 are satisfied for the pairs $(U_k \to V, E|_{U_k})$. In other words, for every $u \in U$ it suffices to find a standard affine open $u \in U' \subset U$ such that the equivalent conditions of Lemma 36.39.1 are satisfied for the pair $(U' \to V, E|_{U'})$. Pick i such that $f(u) \in V_i$ and then pick j such that $u \in U_{ij}$. By Schemes, Lemma 25.11.5 we can find $v \in V' \subset V \cap V_i$ which is standard affine open in V' and V_i. Then $f^{-1}V' \cap U$, resp. $f^{-1}V' \cap U_{ij}$ are standard affine opens of U, resp. U_{ij}. Applying the lemma again we can find $u \in U' \subset f^{-1}V' \cap U \cap U_{ij}$ which is standard affine open in both $f^{-1}V' \cap U$ and $f^{-1}V' \cap U_{ij}$. Thus U' is also a standard affine open of U and U_{ij}. By Lemma 36.39.3 the assumption that the equivalent conditions of Lemma 36.39.1 are satisfied for the pair $(U_{ij} \to V_i, E|_{U_{ij}})$ implies that the equivalent conditions of Lemma 36.39.1 are satisfied for the pair $(U' \to V, E|_{U'})$. □

For objects of the derived category whose cohomology sheaves are quasi-coherent, we can relate relative m-pseudo-coherence to the notion defined in More on Algebra, Definition 15.53.4. We will use the fact that for an affine scheme $U = \mathrm{Spec}(A)$ the functor $R\Gamma(U, -)$ induces an equivalence between $D_{QCoh}(\mathcal{O}_U)$ and $D(A)$, see Derived Categories of Schemes, Lemma 35.3.4. This functor is compatible with pullbacks: if E is an object of $D_{QCoh}(\mathcal{O}_U)$ and $A \to B$ is a ring map corresponding to a morphism of affine schemes $g : V = \mathrm{Spec}(B) \to \mathrm{Spec}(A) = U$, then $R\Gamma(V, Lg^*E) = R\Gamma(U, E) \otimes^{\mathbf{L}}_A B$. See Derived Categories of Schemes, Lemma 35.3.6.

Lemma 36.39.6. *Let $f : X \to S$ be a morphism of schemes which is locally of finite type. Let E be an object of $D_{QCoh}(\mathcal{O}_X)$. Fix $m \in \mathbf{Z}$. The following are equivalent*

1. *E is m-pseudo-coherent relative to S,*
2. *there exists an affine open covering $S = \bigcup V_i$ and for each i an affine open covering $f^{-1}(V_i) = \bigcup U_{ij}$ such that the complex of $\mathcal{O}_X(U_{ij})$-modules $R\Gamma(U_{ij}, E)$ is m-pseudo-coherent relative to $\mathcal{O}_S(V_i)$, and*
3. *for every affine opens $U \subset X$ and $V \subset S$ with $f(U) \subset V$ the complex of $\mathcal{O}_X(U)$-modules $R\Gamma(U, E)$ is m-pseudo-coherent relative to $\mathcal{O}_S(V)$.*

Proof. Let U and V be as in (2) and choose a closed immersion $i : U \to \mathbf{A}^n_V$. A formal argument, using Lemma 36.39.5, shows it suffices to prove that $Ri_*(E|_U)$ is m-pseudo-coherent if and only if $R\Gamma(U, E)$ is m-pseudo-coherent relative to $\mathcal{O}_S(V)$. Say $U = \mathrm{Spec}(A)$, $V = \mathrm{Spec}(R)$, and $\mathbf{A}^n_V = \mathrm{Spec}(R[x_1, \ldots, x_n]$. By the remarks preceding the lemma, $E|_U$ is quasi-isomorphic to the complex of quasi-coherent sheaves on U associated to the object $R\Gamma(U, E)$ of $D(A)$. Note that $R\Gamma(U, E) = R\Gamma(\mathbf{A}^n_V, Ri_*(E|_U))$ as i is a closed immersion (and hence i_* is exact). Thus Ri_*E is associated to $R\Gamma(U, E)$ viewed as an object of $D(R[x_1, \ldots, x_n])$. We conclude as m-pseudo-coherence of $Ri_*(E|_U)$ is equivalent to m-pseudo-coherence of $R\Gamma(E, U)$ in $D(R[x_1, \ldots, x_n])$ by Derived Categories of Schemes, Lemma 35.9.3 which is equivalent to $R\Gamma(U, E)$ is m-pseudo-coherent relative to $R = \mathcal{O}_S(V)$ by definition. □

Lemma 36.39.7. *Let $i : X \to Y$ morphism of schemes locally of finite type over a base scheme S. Assume that i induces a homeomorphism of X with a closed subset*

of Y. Let E be an object of $D(\mathcal{O}_X)$. Then E is m-pseudo-coherent relative to S if and only if Ri_*E is m-pseudo-coherent relative to S.

Proof. By Morphisms, Lemma 28.45.2 the morphism i is affine. Thus we may assume S, Y, and X are affine. Say $S = \mathrm{Spec}(R)$, $Y = \mathrm{Spec}(A)$, and $X = \mathrm{Spec}(B)$. The condition means that $A/\mathrm{rad}(A) \to B/\mathrm{rad}(B)$ is surjective. As B is of finite type over A, we can find $b_1, \ldots, b_m \in \mathrm{rad}(B)$ which generate B as an A-algebra. Say $b_j^N = 0$ for all j. Consider the diagram of rings

$$\begin{array}{ccccc} B & \longleftarrow & R[x_i, y_j]/(y_j^N) & \longleftarrow & R[x_i, y_j] \\ \uparrow & & \uparrow & \nearrow & \\ A & \longleftarrow & R[x_i] & & \end{array}$$

which translates into a diagram

$$\begin{array}{ccccc} X & \longrightarrow & T & \longrightarrow & \mathbf{A}_S^{n+m} \\ \downarrow & & \downarrow & \swarrow & \\ Y & \longrightarrow & \mathbf{A}_S^n & & \end{array}$$

of affine schemes. By Lemma 36.39.5 we see that E is m-pseudo-coherent relative to S if and only if its pushforward to $\mathbf{A}_S^{n+m}$ is m-pseudo-coherent. By Derived Categories of Schemes, Lemma 35.10.5 we see that this is true if and only if its pushforward to T is m-pseudo-coherent. The same lemma shows that this holds if and only if the pushforward to $\mathbf{A}_S^n$ is m-pseudo-coherent. Again by Lemma 36.39.5 this holds if and only if Ri_*E is m-pseudo-coherent relative to S. □

Lemma 36.39.8. *Let $\pi : X \to Y$ be a finite morphism of schemes locally of finite type over a base scheme S. Let E be an object of $D_{QCoh}(\mathcal{O}_X)$. Then E is m-pseudo-coherent relative to S if and only if $R\pi_*E$ is m-pseudo-coherent relative to S.*

Proof. Translation of the result of More on Algebra, Lemma 15.53.5 into the language of schemes. Observe that $R\pi_*$ indeed maps $D_{QCoh}(\mathcal{O}_X)$ into $D_{QCoh}(\mathcal{O}_Y)$ by Derived Categories of Schemes, Lemma 35.4.1. To do the translation use Lemma 36.39.5. □

Lemma 36.39.9. *Let $f : X \to S$ be a morphism of schemes which is locally of finite type. Let (E, E', E'') be a distinguished triangle of $D(\mathcal{O}_X)$. Let $m \in \mathbf{Z}$.*

1. *If E is $(m+1)$-pseudo-coherent relative to S and E' is m-pseudo-coherent relative to S then E'' is m-pseudo-coherent relative to S.*
2. *If E, E'' are m-pseudo-coherent relative to S, then E' is m-pseudo-coherent relative to S.*
3. *If E' is $(m+1)$-pseudo-coherent relative to S and E'' is m-pseudo-coherent relative to S, then E is $(m+1)$-pseudo-coherent relative to S.*

Moreover, if two out of three of E, E', E'' are pseudo-coherent relative to S, the so is the third.

Proof. Immediate from Lemma 36.39.5 and Cohomology, Lemma 20.33.4. □

Lemma 36.39.10. *Let $X \to S$ be a morphism of schemes which is locally of finite type. Let $\mathcal{F}$ be an $\mathcal{O}_X$-module. Then*

(1) *$\mathcal{F}$ is m-pseudo-coherent relative to S for all $m > 0$,*
(2) *$\mathcal{F}$ is 0-pseudo-coherent relative to S if and only if $\mathcal{F}$ is a finite type $\mathcal{O}_X$-module,*
(3) *$\mathcal{F}$ is (-1)-pseudo-coherent relative to S if and only if $\mathcal{F}$ is quasi-coherent and finitely presented relative to S.*

Proof. Part (1) is immediate from the definition. To see part (3) we may work locally on X (both properties are local). Thus we may assume X and S are affine. Choose a closed immersion $i : X \to \mathbf{A}^n_S$. Then we see that $\mathcal{F}$ is (-1)-pseudo-coherent relative to S if and only if $i_*\mathcal{F}$ is (-1)-pseudo-coherent, which is true if and only if $i_*\mathcal{F}$ is an $\mathcal{O}_{\mathbf{A}^n_S}$-module of finite presentation, see Cohomology, Lemma 20.33.9. A module of finite presentation is quasi-coherent, see Modules, Lemma 17.11.2. By Morphisms, Lemma 28.4.1 we see that $\mathcal{F}$ is quasi-coherent if and only if $i_*\mathcal{F}$ is quasi-coherent. Having said this part (3) follows. The proof of (2) is similar but less involved. □

Lemma 36.39.11. *Let $X \to S$ be a morphism of schemes which is locally of finite type. Let $m \in \mathbf{Z}$. Let E, K be objects of $D(\mathcal{O}_X)$. If $E \oplus K$ is m-pseudo-coherent relative to S so are E and K.*

Proof. Follows from Cohomology, Lemma 20.33.6 and the definitions. □

Lemma 36.39.12. *Let $X \to S$ be a morphism of schemes which is locally of finite type. Let $m \in \mathbf{Z}$. Let $\mathcal{F}^\bullet$ be a (locally) bounded above complex of $\mathcal{O}_X$-modules such that $\mathcal{F}^i$ is $(m - i)$-pseudo-coherent relative to S for all i. Then $\mathcal{F}^\bullet$ is m-pseudo-coherent relative to S.*

Proof. Follows from Cohomology, Lemma 20.33.7 and the definitions. □

Lemma 36.39.13. *Let $X \to S$ be a morphism of schemes which is locally of finite type. Let $m \in \mathbf{Z}$. Let E be an object of $D(\mathcal{O}_X)$. If E is (locally) bounded above and $H^i(E)$ is $(m-i)$-pseudo-coherent relative to S for all i, then E is m-pseudo-coherent relative to S.*

Proof. Follows from Cohomology, Lemma 20.33.8 and the definitions. □

Lemma 36.39.14. *Let $X \to S$ be a morphism of schemes which is locally of finite type. Let $m \in \mathbf{Z}$. Let E be an object of $D(\mathcal{O}_X)$ which is m-pseudo-coherent relative to S. Let $S' \to S$ be a morphism of schemes. Set $X' = X \times_S S'$ and denote E' the derived pullback of E to X'. If S' and X are Tor independent over S, then E' is is m-pseudo-coherent relative to S'.*

Proof. The problem is local on X and X' hence we may assume X, S, S', and X' are affine. Choose a closed immersion $i : X \to \mathbf{A}^n_S$ and denote $i' : X' \to \mathbf{A}^n_{S'}$ the base change to S'. Denote $g : X' \to X$ and $g' : \mathbf{A}^n_{S'} \to \mathbf{A}^n_S$ the projections, so $E' = Lg^*E$. Since X and S' are tor-independent over S, the base change map (Cohomology, Remark 20.27.2) induces an isomorphism

$$Ri'_*(Lg^*E) = L(g')^*Ri_*E$$

Namely, for a point $x' \in X'$ lying over $x \in X$ the base change map on stalks at x' is the map

$$E_x \otimes^{\mathbf{L}}_{\mathcal{O}_{\mathbf{A}^n_S,x}} \mathcal{O}_{\mathbf{A}^n_{S'},x'} \longrightarrow E_x \otimes^{\mathbf{L}}_{\mathcal{O}_{X,x}} \mathcal{O}_{X',x'}$$

coming from the closed immersions i and i'. Note that the source is quasi-isomorphic to a localization of $E_x \otimes^{\mathbf{L}}_{\mathcal{O}_{S,s}} \mathcal{O}_{S',s'}$ which is isomorphic to the target as $\mathcal{O}_{X',x'}$ is isomorphic to (the same) localization of $\mathcal{O}_{X,x} \otimes^{\mathbf{L}}_{\mathcal{O}_{S,s}} \mathcal{O}_{S',s'}$ by assumption. We conclude the lemma holds by an application of Cohomology, Lemma 20.33.3. □

Lemma 36.39.15. *Let $f : X \to Y$ be a morphism of schemes locally of finite type over a base S. Let $m \in \mathbf{Z}$. Let E be an object of $D(\mathcal{O}_Y)$. Assume*

(1) *$\mathcal{O}_X$ is pseudo-coherent relative to Y[4], and*
(2) *E is m-pseudo-coherent relative to S.*

Then Lf^*E is m-pseudo-coherent relative to S.

Proof. The problem is local on X. Thus we may assume X, Y, and S are affine. Arguing as in the proof of More on Algebra, Lemma 15.53.13 we can find a commutative diagram

$$\begin{array}{ccccc} X & \xrightarrow{i} & \mathbf{A}^m_Y & \xrightarrow{j} & \mathbf{A}^{n+m}_S \\ {\scriptstyle f}\downarrow & \swarrow{\scriptstyle p} & & \swarrow & \\ Y & \longrightarrow & \mathbf{A}^n_S & & \end{array}$$

Observe that

$$Ri_*Lf^*E = Ri_*Li^*Lp^*E = Lp^*E \otimes^{\mathbf{L}}_{\mathcal{O}_{\mathbf{A}^n_Y}} Ri_*\mathcal{O}_X$$

by Derived Categories of Schemes, Lemma 35.16.1. By assumption and the fact that Y is affine, we can represent $Ri_*\mathcal{O}_X = i_*\mathcal{O}_X$ by a complexes of finite free $\mathcal{O}_{\mathbf{A}^n_Y}$-modules $\mathcal{F}^\bullet$, with $\mathcal{F}^i = 0$ for $i > 0$ (details omitted; use Derived Categories of Schemes, Lemma 35.9.3 and More on Algebra, Lemma 15.53.7). By assumption E is bounded above, say $H^i(E) = 0$ for $i > a$. Represent E by a complex $\mathcal{E}^\bullet$ of $\mathcal{O}_Y$-modules with $\mathcal{E}^i = 0$ for $i > a$. Then the derived tensor product above is represented by $\mathrm{Tot}(p^*\mathcal{E}^\bullet \otimes_{\mathcal{O}_{\mathbf{A}^n_Y}} \mathcal{F}^\bullet)$.

Thus we have to show that $j_*\mathrm{Tot}(p^*\mathcal{E}^\bullet \otimes_{\mathcal{O}_{\mathbf{A}^n_Y}} \mathcal{F}^\bullet)$ is m-pseudo-coherent as a complex of $\mathcal{O}_{\mathbf{A}^{n+m}_S}$-modules. Note that $\mathrm{Tot}(p^*\mathcal{E}^\bullet \otimes_{\mathcal{O}_{\mathbf{A}^n_Y}} \mathcal{F}^\bullet)$ has a filtration by subcomplexes with successive quotients the complexes $p^*\mathcal{E}^\bullet \otimes_{\mathcal{O}_{\mathbf{A}^n_Y}} \mathcal{F}^i[-i]$. Note that for $i \ll 0$ the complexes $p^*\mathcal{E}^\bullet \otimes_{\mathcal{O}_{\mathbf{A}^n_Y}} \mathcal{F}^i[-i]$ have zero cohomology in degrees $\leq m$ and hence are m-pseudo-coherent. Hence, applying Lemma 36.39.9 and induction, it suffices to show that $p^*\mathcal{E}^\bullet \otimes_{\mathcal{O}_{\mathbf{A}^n_Y}} \mathcal{F}^i[-i]$ is pseudo-coherent relative to S for all i. Note that $\mathcal{F}^i = 0$ for $i > 0$. Since also $\mathcal{F}^i$ is finite free this reduces to proving that $p^*\mathcal{E}^\bullet$ is m-pseudo-coherent relative to R which follows from Lemma 36.39.14 for instance. □

Lemma 36.39.16. *Let $f : X \to Y$ be a morphism of schemes locally of finite type over a base S. Let $m \in \mathbf{Z}$. Let E be an object of $D(\mathcal{O}_X)$. Assume $\mathcal{O}_Y$ is pseudo-coherent relative to S[5]. Then the following are equivalent*

(1) *E is m-pseudo-coherent relative to Y, and*
(2) *E is m-pseudo-coherent relative to S.*

[4]This means f is pseudo-coherent, see Definition 36.40.2.
[5]This means $Y \to S$ is pseudo-coherent, see Definition 36.40.2.

Proof. The question is local on X, hence we may assume X, Y, and S are affine. Arguing as in the proof of More on Algebra, Lemma 15.53.13 we can find a commutative diagram

$$\begin{array}{ccccc} X & \xrightarrow[i]{} & \mathbf{A}^m_Y & \xrightarrow[j]{} & \mathbf{A}^{n+m}_S \\ {\scriptstyle f}\downarrow & \swarrow{\scriptstyle p} & & \swarrow & \\ Y & \longrightarrow & \mathbf{A}^n_S & & \end{array}$$

The assumption that $\mathcal{O}_Y$ is pseudo-coherent relative to S implies that $\mathcal{O}_{\mathbf{A}^m_Y}$ is pseudo-coherent relative to $\mathbf{A}^m_S$ (by flat base change; this can be seen by using for example Lemma 36.39.14). This in turn implies that $j_*\mathcal{O}_{\mathbf{A}^n_Y}$ is pseudo-coherent as an $\mathcal{O}_{\mathbf{A}^{n+m}_S}$-module. Then the equivalence of the lemma follows from Derived Categories of Schemes, Lemma 35.10.5. □

Lemma 36.39.17. *Let*

$$\begin{array}{ccc} X & \xrightarrow[i]{} & P \\ & \searrow \quad \swarrow & \\ & S & \end{array}$$

be a commutative diagram of schemes. Assume i is a closed immersion and $P \to S$ flat and locally of finite presentation. Let E be an object of $D(\mathcal{O}_X)$. Then the following are equivalent

(1) *E is m-pseudo-coherent relative to S,*
(2) *Ri_*E is m-pseudo-coherent relative to S, and*
(3) *Ri_*E is m-pseudo-coherent on P.*

Proof. The equivalence of (1) and (2) is Lemma 36.39.8. The equivalence of (2) and (3) follows from Lemma 36.39.16 applied to $\mathrm{id} : P \to P$ provided we can show that $\mathcal{O}_P$ is pseudo-coherent relative to S. This follows from More on Algebra, Lemma 15.54.4 and the definitions. □

36.40. Pseudo-coherent morphisms

Avoid reading this section at all cost. If you need some of this material, first take a look at the corresponding algebra sections, see More on Algebra, Sections 15.48, 15.53, and 15.54. For now the only thing you need to know is that a ring map $A \to B$ is pseudo-coherent if and only if $B = A[x_1, \ldots, x_n]/I$ and B as an $A[x_1, \ldots, x_n]$-module has a resolution by finite free $A[x_1, \ldots, x_n]$-modules.

Lemma 36.40.1. *Let $f : X \to S$ be a morphism of schemes. The following are equivalent*

(1) *there exist an affine open covering $S = \bigcup V_j$ and for each j an affine open covering $f^{-1}(V_j) = \bigcup U_{ji}$ such that $\mathcal{O}_S(V_j) \to \mathcal{O}_X(U_{ij})$ is a pseudo-coherent ring map,*
(2) *for every pair of affine opens $U \subset X$, $V \subset S$ such that $f(U) \subset V$ the ring map $\mathcal{O}_S(V) \to \mathcal{O}_X(U)$ is pseudo-coherent, and*
(3) *f is locally of finite type and $\mathcal{O}_X$ is pseudo-coherent relative to S.*

Proof. To see the equivalence of (1) and (2) it suffices to check conditions (1)(a), (b), (c) of Morphisms, Definition 28.15.1 for the property of being a pseudo-coherent

ring map. These properties follow (using localization is flat) from More on Algebra, Lemmas 15.53.12, 15.53.11, and 15.53.16.

If (1) holds, then f is locally of finite type as a pseudo-coherent ring map is of finite type by definition. Moreover, (1) implies via Lemma 36.39.6 and the definitions that $\mathcal{O}_X$ is pseudo-coherent relative to S. Conversely, if (3) holds, then we see that for every U and V as in (2) the ring $\mathcal{O}_X(U)$ is of finite type over $\mathcal{O}_S(V)$ and $\mathcal{O}_X(U)$ is as a module pseudo-coherent relative to $\mathcal{O}_S(V)$, see Lemmas 36.39.5 and 36.39.6. This is the definition of a pseudo-coherent ring map, hence (2) and (1) hold. □

Definition 36.40.2. A morphism of schemes $f : X \to S$ is called *pseudo-coherent* if the equivalent conditions of Lemma 36.40.1 are satisfied. In this case we also say that X is pseudo-coherent over S.

Beware that a base change of a pseudo-coherent morphism is not pseudo-coherent in general.

Lemma 36.40.3. *A flat base change of a pseudo-coherent morphism is pseudo-coherent.*

Proof. This translates into the following algebra result: Let $A \to B$ be a pseudo-coherent ring map. Let $A \to A'$ be flat. Then $A' \to B \otimes_A A'$ is pseudo-coherent. This follows from the more general More on Algebra, Lemma 15.53.12. □

Lemma 36.40.4. *A composition of pseudo-coherent morphisms of schemes is pseudo-coherent.*

Proof. This translates into the following algebra result: If $A \to B \to C$ are composable pseudo-coherent ring maps then $A \to C$ is pseudo-coherent. This follows from either More on Algebra, Lemma 15.53.13 or More on Algebra, Lemma 15.53.15. □

Lemma 36.40.5. *A pseudo-coherent morphism is locally of finite presentation.*

Proof. Immediate from the definitions. □

Lemma 36.40.6. *A flat morphism which is locally of finite presentation is pseudo-coherent.*

Proof. This follows from the fact that a flat ring map of finite presentation is pseudo-coherent (and even perfect), see More on Algebra, Lemma 15.54.4. □

Lemma 36.40.7. *Let $f : X \to Y$ be a morphism of schemes pseudo-coherent over a base scheme S. Then f is pseudo-coherent.*

Proof. This translates into the following algebra result: If $R \to A \to B$ are composable ring maps and $R \to A$, $R \to B$ pseudo-coherent, then $R \to B$ is pseudo-coherent. This follows from More on Algebra, Lemma 15.53.15. □

Lemma 36.40.8. *Let $f : X \to S$ be a morphism of schemes. If S is locally Noetherian, then f is pseudo-coherent if and only if f is locally of finite type.*

Proof. This translates into the following algebra result: If $R \to A$ is a finite type ring map with R Noetherian, then $R \to A$ is pseudo-coherent if and only if $R \to A$ is of finite type. To see this, note that a pseudo-coherent ring map is of finite type by definition. Conversely, if $R \to A$ is of finite type, then we can write

$A = R[x_1, \ldots, x_n]/I$ and it follows from More on Algebra, Lemma 15.48.16 that A is pseudo-coherent as an $R[x_1, \ldots, x_n]$-module, i.e., $R \to A$ is a pseudo-coherent ring map. □

Lemma 36.40.9. *The property $\mathcal{P}(f) =$ "f is pseudo-coherent" is fpqc local on the base.*

Proof. We will use the criterion of Descent, Lemma 34.18.4 to prove this. By Definition 36.40.2 being pseudo-coherent is Zariski local on the base. By Lemma 36.40.3 being pseudo-coherent is preserved under flat base change. The final hypothesis (3) of Descent, Lemma 34.18.4 translates into the following algebra statement: Let $A \to B$ be a faithfully flat ring map. Let $C = A[x_1, \ldots, x_n]/I$ be an A-algebra. If $C \otimes_A B$ is pseudo-coherent as an $B[x_1, \ldots, x_n]$-module, then C is pseudo-coherent as a $A[x_1, \ldots, x_n]$-module. This is More on Algebra, Lemma 15.48.15. □

Lemma 36.40.10. *Let $A \to B$ be a flat ring map of finite presentation. Let $I \subset B$ be an ideal. Then $A \to B/I$ is pseudo-coherent if and only if I is pseudo-coherent as a B-module.*

Proof. Choose a presentation $B = A[x_1, \ldots, x_n]/J$. Note that B is pseudo-coherent as an $A[x_1, \ldots, x_n]$-module because $A \to B$ is a pseudo-coherent ring map by Lemma 36.40.6. Note that $A \to B/I$ is pseudo-coherent if and only if B/I is pseudo-coherent as an $A[x_1, \ldots, x_n]$-module. By More on Algebra, Lemma 15.48.11 we see this is equivalent to the condition that B/I is pseudo-coherent as an B-module. This proves the lemma as the short exact sequence $0 \to I \to B \to B/I \to 0$ shows that I is pseudo-coherent if and only if B/I is (see More on Algebra, Lemma 15.48.6). □

The following lemma will be obsoleted by the stronger Lemma 36.40.12.

Lemma 36.40.11. *The property $\mathcal{P}(f) =$ "f is pseudo-coherent" is syntomic local on the source.*

Proof. We will use the criterion of Descent, Lemma 34.22.3 to prove this. It follows from Lemmas 36.40.6 and 36.40.4 that being pseudo-coherent is preserved under precomposing with flat morphisms locally of finite presentation, in particular under precomposing with syntomic morphisms (see Morphisms, Lemmas 28.32.7 and 28.32.6). It is clear from Definition 36.40.2 that being pseudo-coherent is Zariski local on the source and target. Hence, according to the aforementioned Descent, Lemma 34.22.3 it suffices to prove the following: Suppose $X' \to X \to Y$ are morphisms of affine schemes with $X' \to X$ syntomic and $X' \to Y$ pseudo-coherent. Then $X \to Y$ is pseudo-coherent. To see this, note that in any case $X \to Y$ is of finite presentation by Descent, Lemma 34.10.1. Choose a closed immersion $X \to \mathbf{A}^n_Y$. By Algebra, Lemma 10.129.18 we can find an affine open covering $X' = \bigcup_{i=1,\ldots,n} X'_i$ and syntomic morphisms $W_i \to \mathbf{A}^n_Y$ lifting the morphisms $X'_i \to X$, i.e., such that there are fibre product diagrams

$$\begin{array}{ccc} X'_i & \longrightarrow & W_i \\ \downarrow & & \downarrow \\ X & \longrightarrow & \mathbf{A}^n_Y \end{array}$$

After replacing X' by $\coprod X'_i$ and setting $W = \coprod W_i$ we obtain a fibre product diagram

$$\begin{array}{ccc} X' & \longrightarrow & W \\ \downarrow & & \downarrow h \\ X & \longrightarrow & \mathbf{A}^n_Y \end{array}$$

with $W \to \mathbf{A}^n_Y$ flat and of finite presentation and $X' \to Y$ still pseudo-coherent. Since $W \to \mathbf{A}^n_Y$ is open (see Morphisms, Lemma 28.26.9) and $X' \to X$ is surjective we can find $f \in \Gamma(\mathbf{A}^n_Y, \mathcal{O})$ such that $X \subset D(f) \subset \mathrm{Im}(h)$. Write $Y = \mathrm{Spec}(R)$, $X = \mathrm{Spec}(A)$, $X' = \mathrm{Spec}(A')$ and $W = \mathrm{Spec}(B)$, $A = R[x_1, \ldots, x_n]/I$ and $A' = B/IB$. Then $R \to A'$ is pseudo-coherent. Picture

$$\begin{array}{ccc} A' = B/IB & \longleftarrow & B \\ \uparrow & & \uparrow \\ A = R[x_1, \ldots, x_n]/I & \longleftarrow & R[x_1, \ldots, x_n] \end{array}$$

By Lemma 36.40.10 we see that IB is pseudo-coherent as a B-module. The ring map $R[x_1, \ldots, x_n]_f \to B_f$ is faithfully flat by our choice of f above. This implies that $I_f \subset R[x_1, \ldots, x_n]_f$ is pseudo-coherent, see More on Algebra, Lemma 15.48.15. Applying Lemma 36.40.10 one more time we see that $R \to A$ is pseudo-coherent. □

Lemma 36.40.12. *The property $\mathcal{P}(f) =$ "f is pseudo-coherent" is fppf local on the source.*

Proof. Let $f : X \to S$ be a morphism of schemes. Let $\{g_i : X_i \to X\}$ be an fppf covering such that each composition $f \circ g_i$ is pseudo-coherent. According to Lemma 36.34.1 there exist

(1) a Zariski open covering $X = \bigcup U_j$,
(2) surjective finite locally free morphisms $W_j \to U_j$,
(3) Zariski open coverings $W_j = \bigcup_k W_{j,k}$,
(4) surjective finite locally free morphisms $T_{j,k} \to W_{j,k}$

such that the fppf covering $\{h_{j,k} : T_{j,k} \to X\}$ refines the given covering $\{X_i \to X\}$. Denote $\psi_{j,k} : T_{j,k} \to X_{\alpha(j,k)}$ the morphisms that witness the fact that $\{T_{j,k} \to X\}$ refines the given covering $\{X_i \to X\}$. Note that $T_{j,k} \to X$ is a flat, locally finitely presented morphism, so both X_i and $T_{j,k}$ are pseudo-coherent over X by Lemma 36.40.6. Hence $\psi_{j,k} : T_{j,k} \to X_i$ is pseudo-coherent, see Lemma 36.40.7. Hence $T_{j,k} \to S$ is pseudo coherent as the composition of $\psi_{j,k}$ and $f \circ g_{\alpha(j,k)}$, see Lemma 36.40.4. Thus we see we have reduced the lemma to the case of a Zariski open covering (which is OK) and the case of a covering given by a single surjective finite locally free morphism which we deal with in the following paragraph.

Assume that $X' \to X \to S$ is a sequence of morphisms of schemes with $X' \to X$ surjective finite locally free and $X' \to Y$ pseudo-coherent. Our goal is to show that $X \to S$ is pseudo-coherent. Note that by Descent, Lemma 34.10.3 the morphism $X \to S$ is locally of finite presentation. It is clear that the problem reduces to the case that X', X and S are affine and $X' \to X$ is free of some rank $r > 0$. The corresponding algebra problem is the following: Suppose $R \to A \to A'$ are ring maps such that $R \to A'$ is pseudo-coherent, $R \to A$ is of finite presentation, and $A' \cong A^{\oplus r}$ as an A-module. Goal: Show $R \to A$ is pseudo-coherent. The assumption

that $R \to A'$ is pseudo-coherent means that A' as an A'-module is pseudo-coherent relative to R. By More on Algebra, Lemma 15.53.5 this implies that A' as an A-module is pseudo-coherent relative to R. Since $A' \cong A^{\oplus r}$ as an A-module we see that A as an A-module is pseudo-coherent relative to R, see More on Algebra, Lemma 15.53.8. This by definition means that $R \to A$ is pseudo-coherent and we win. □

36.41. Perfect morphisms

In order to understand the material in this section you have to understand the material of the section on pseudo-coherent morphisms just a little bit. For now the only thing you need to know is that a ring map $A \to B$ is perfect if and only if it is pseudo-coherent and B has finite tor dimension as an A-module.

Lemma 36.41.1. *Let $f : X \to S$ be a morphism of schemes which is locally of finite type. The following are equivalent*

(1) *there exist an affine open covering $S = \bigcup V_j$ and for each j an affine open covering $f^{-1}(V_j) = \bigcup U_{ji}$ such that $\mathcal{O}_S(V_j) \to \mathcal{O}_X(U_{ij})$ is a perfect ring map, and*
(2) *for every pair of affine opens $U \subset X$, $V \subset S$ such that $f(U) \subset V$ the ring map $\mathcal{O}_S(V) \to \mathcal{O}_X(U)$ is perfect.*

Proof. Assume (1) and let U, V be as in (2). It follows from Lemma 36.40.1 that $\mathcal{O}_S(V) \to \mathcal{O}_X(U)$ is pseudo-coherent. Hence it suffices to prove that the property of a ring map being "of finite tor dimension" satisfies conditions (1)(a), (b), (c) of Morphisms, Definition 28.15.1. These properties follow from More on Algebra, Lemmas 15.49.9, 15.49.12, and 15.49.13. Some details omitted. □

Definition 36.41.2. A morphism of schemes $f : X \to S$ is called *perfect* if the equivalent conditions of Lemma 36.40.1 are satisfied. In this case we also say that X is perfect over S.

Note that a perfect morphism is in particular pseudo-coherent, hence locally of finite presentation. Beware that a base change of a perfect morphism is not perfect in general.

Lemma 36.41.3. *A flat base change of a perfect morphism is perfect.*

Proof. This translates into the following algebra result: Let $A \to B$ be a perfect ring map. Let $A \to A'$ be flat. Then $A' \to B \otimes_A A'$ is perfect. This result for pseudo-coherent ring maps we have seen in Lemma 36.40.3. The corresponding fact for finite tor dimension follows from More on Algebra, Lemma 15.49.12. □

Lemma 36.41.4. *A composition of perfect morphisms of schemes is perfect.*

Proof. This translates into the following algebra result: If $A \to B \to C$ are composable perfect ring maps then $A \to C$ is perfect. We have seen this is the case for pseudo-coherent in Lemma 36.40.4 and its proof. By assumption there exist integers n, m such that B has tor dimension $\leq n$ over A and C has tor dimension $\leq m$ over B. Then for any A-module M we have

$$M \otimes^{\mathbf{L}}_A C = (M \otimes^{\mathbf{L}}_A B) \otimes^{\mathbf{L}}_B C$$

and the spectral sequence of More on Algebra, Example 15.46.4 shows that $\mathrm{Tor}^A_p(M, C) = 0$ for $p > n + m$ as desired. □

Lemma 36.41.5. *Let $f : X \to S$ be a morphism of schemes. The following are equivalent*

(1) *f is flat and perfect, and*
(2) *f is flat and locally of finite presentation.*

Proof. The implication (2) $\Rightarrow$ (1) is More on Algebra, Lemma 15.54.4. The converse follows from the fact that a pseudo-coherent morphism is locally of finite presentation, see Lemma 36.40.5. □

Lemma 36.41.6. *Let $f : X \to S$ be a morphism of schemes. Assume S is regular and f is locally of finite type. Then f is perfect.*

Proof. See More on Algebra, Lemma 15.54.5. □

Lemma 36.41.7. *A regular immersion of schemes is perfect. A Koszul-regular immersion of schemes is perfect.*

Proof. Since a regular immersion is a Koszul-regular immersion, see Divisors, Lemma 30.13.2, it suffices to prove the second statement. This translates into the following algebraic statement: Suppose that $I \subset A$ is an ideal generated by a Koszul-regular sequence $f_1, \ldots, f_r$ of A. Then $A \to A/I$ is a perfect ring map. Since $A \to A/I$ is surjective this is a presentation of A/I by a polynomial algebra over A. Hence it suffices to see that A/I is pseudo-coherent as an A-module and has finite tor dimension. By definition of a Koszul sequence the Koszul complex $K(A, f_1, \ldots, f_r)$ is a finite free resolution of A/I. Hence A/I is a perfect complex of A-modules and we win. □

Lemma 36.41.8. *Let*

$$\begin{array}{ccc} X & \xrightarrow{f} & Y \\ & \searrow \quad \swarrow & \\ & S & \end{array}$$

be a commutative diagram of morphisms of schemes. Assume $Y \to S$ smooth and $X \to S$ perfect. Then $f : X \to Y$ is perfect.

Proof. We can factor f as the composition

$$X \longrightarrow X \times_S Y \longrightarrow Y$$

where the first morphism is the map $i = (1, f)$ and the second morphism is the projection. Since $Y \to S$ is flat, see Morphisms, Lemma 28.35.9, we see that $X \times_S Y \to Y$ is perfect by Lemma 36.41.3. As $Y \to S$ is smooth, also $X \times_S Y \to X$ is smooth, see Morphisms, Lemma 28.35.5. Hence i is a section of a smooth morphism, therefore i is a regular immersion, see Divisors, Lemma 30.14.7. This implies that i is perfect, see Lemma 36.41.7. We conclude that f is perfect because the composition of perfect morphisms is perfect, see Lemma 36.41.4. □

Remark 36.41.9. It is not true that a morphism between schemes X, Y perfect over a base S is perfect. An example is $S = \mathrm{Spec}(k)$, $X = \mathrm{Spec}(k)$, $Y = \mathrm{Spec}(k[x]/(x^2)$ and $X \to Y$ the unique S-morphism.

Lemma 36.41.10. *The property $\mathcal{P}(f) =$ "f is perfect" is fpqc local on the base.*

Proof. We will use the criterion of Descent, Lemma 34.18.4 to prove this. By Definition 36.41.2 being perfect is Zariski local on the base. By Lemma 36.41.3 being perfect is preserved under flat base change. The final hypothesis (3) of Descent, Lemma 34.18.4 translates into the following algebra statement: Let $A \to B$ be a faithfully flat ring map. Let $C = A[x_1, \ldots, x_n]/I$ be an A-algebra. If $C \otimes_A B$ is perfect as an $B[x_1, \ldots, x_n]$-module, then C is perfect as a $A[x_1, \ldots, x_n]$-module. This is More on Algebra, Lemma 15.50.12. □

Lemma 36.41.11. *Let $f : X \to S$ be a pseudo-coherent morphism of schemes. Then f is perfect if and only if for every $x \in X$ the ring $\mathcal{O}_{X,x}$ has finite tor dimension as an $\mathcal{O}_{S,f(x)}$-module.*

Proof. This translates into the following algebra problem. Suppose that $A \to B$ is a pseudo-coherent ring map. Write $B = A[x_1, \ldots, x_n]/I$. Then the following are equivalent

(1) $B_{\mathfrak{q}}$ has finite tor dimension over $A_{\mathfrak{p}}$ for all $\mathfrak{q}$ (with $\mathfrak{p} = A \cap \mathfrak{q}$), and
(2) B is perfect as an $A[x_1, \ldots, x_n]$-module.

The implication (2) $\Rightarrow$ (1) is clear. For the converse, consider a prime $\mathfrak{q}$ of B lying over $\mathfrak{p}$ as in (1). Let $\mathfrak{q}'$ be the prime of $A[x_1, \ldots, x_n]$ corresponding to $\mathfrak{q}$. By More on Algebra, Lemma 15.50.17 applied to $A_{\mathfrak{p}} \to A[x_1, \ldots, x_n]_{\mathfrak{q}'}$ we see that $B_{\mathfrak{q}}$ is a perfect $A[x_1, \ldots, x_n]_{\mathfrak{q}'}$-module. Hence B is a perfect $A[x_1, \ldots, x_n]$-module by More on Algebra, Lemma 15.50.16. Some details omitted. □

Lemma 36.41.12. *The property $\mathcal{P}(f) =$ "f is perfect" is fppf local on the source.*

Proof. Let $\{g_i : X_i \to X\}_{i \in I}$ be an fppf covering of schemes and let $f : X \to S$ be a morphism such that each $f \circ g_i$ is perfect. By Lemma 36.40.12 we conclude that f is pseudo-coherent. Hence by Lemma 36.41.11 it suffices to check that $\mathcal{O}_{X,x}$ is an $\mathcal{O}_{S,f(x)}$-module of finite tor dimension for all $x \in X$. Pick $i \in I$ and $x_i \in X_i$ mapping to x. Then we see that $\mathcal{O}_{X_i,x_i}$ has finite tor dimension over $\mathcal{O}_{S,f(x)}$ and that $\mathcal{O}_{X,x} \to \mathcal{O}_{X_i,x_i}$ is faithfully flat. The desired conclusion follows from More on Algebra, Lemma 15.49.14. □

Lemma 36.41.13. *Let $i : Z \to Y$ and $j : Y \to X$ be immersions of schemes. Assume*

(1) *X is locally Noetherian,*
(2) *$j \circ i$ is a regular immersion, and*
(3) *i is perfect.*

Then i and j are regular immersions.

Proof. Since X (and hence Y) is locally Noetherian all 4 types of regular immersions agree, and moreover we may check whether a morphism is a regular immersion on the level of local rings, see Divisors, Lemma 30.12.8. Thus the result follows from Divided Power Algebra, Lemma 23.7.5. □

36.42. Local complete intersection morphisms

In Divisors, Section 30.13 we have defined 4 different types of regular immersions: regular, Koszul-regular, H_1-regular, and quasi-regular. In this section we consider

morphisms $f : X \to S$ which locally on X factors as

$$\begin{array}{ccc} X & \xrightarrow{i} & \mathbf{A}^n_S \\ & \searrow \quad \swarrow & \\ & S & \end{array}$$

where i is a $*$-regular immersion for $* \in \{\emptyset, Koszul, H_1, quasi\}$. However, we don't know how to prove that this condition is independent of the factorization if $* = \emptyset$, i.e., when we require i to be a regular immersion. On the other hand, we want a local complete intersection morphism to be perfect, which is only going to be true if $* = Koszul$ or $* = \emptyset$. Hence we will define a *local complete intersection morphism* or *Koszul morphism* to be a morphism of schemes $f : X \to S$ that locally on X has a factorization as above with i a Koszul-regular immersion. To see that this works we first prove this is independent of the chosen factorizations.

Lemma 36.42.1. *Let S be a scheme. Let U, P, P' be schemes over S. Let $u \in U$. Let $i : U \to P$, $i' : U \to P'$ be immersions over S. Assume P and P' smooth over S. Then the following are equivalent*

(1) *i is a Koszul-regular immersion in a neighbourhood of x, and*
(2) *i' is a Koszul-regular immersion in a neighbourhood of x.*

Proof. Assume i is a Koszul-regular immersion in a neighbourhood of x. Consider the morphism $j = (i, i') : U \to P \times_S P' = P''$. Since $P'' = P \times_S P' \to P$ is smooth, it follows from Divisors, Lemma 30.14.8 that j is a Koszul-regular immersion, whereupon it follows from Divisors, Lemma 30.14.11 that i' is a Koszul-regular immersion. □

Before we state the definition, let us make the following simple remark. Let $f : X \to S$ be a morphism of schemes which is locally of finite type. Let $x \in X$. Then there exist an open neighbourhood $U \subset X$ and a factorization of $f|_U$ as the composition of an immersion $i : U \to \mathbf{A}^n_S$ followed by the projection $\mathbf{A}^n_S \to S$ which is smooth. Picture

$$\begin{array}{ccccc} X & \longleftarrow & U & \xrightarrow{i} & \mathbf{A}^n_S = P \\ & \searrow & \downarrow & \swarrow_{\pi} & \\ & & S & & \end{array}$$

In fact you can do this with any affine open neighbourhood U of x in X, see Morphisms, Lemma 28.40.2.

Definition 36.42.2. Let $f : X \to S$ be a morphism of schemes.

(1) Let $x \in X$. We say that f is *Koszul at x* if f is of finite type at x and there exists an open neighbourhood and a factorization of $f|_U$ as $\pi \circ i$ where $i : U \to P$ is a Koszul-regular immersion and $\pi : P \to S$ is smooth.
(2) We say f is a *Koszul morphism*, or that f is a *local complete intersection morphism* if f is Koszul at every point.

We have seen above that the choice of the factorization $f|_U = \pi \circ i$ is irrelevant, i.e., given a factorization of $f|_U$ as an immersion i followed by a smooth morphism π, whether or not i is Koszul regular in a neighbourhood of x is an intrinsic property of f at x. Let us record this here explicitly as a lemma so that we can refer to it

Lemma 36.42.3. *Let $f : X \to S$ be a local complete intersection morphism. Let P be a scheme smooth over S. Let $U \subset X$ be an open subscheme and $i : U \to P$ an immersion of schemes over S. Then i is a Koszul-regular immersion.*

Proof. This is the defining property of a local complete intersection morphism. See discussion above. □

It seems like a good idea to collect here some properties in common with all Koszul morphisms.

Lemma 36.42.4. *Let $f : X \to S$ be a local complete intersection morphism. Then*

(1) *f is locally of finite presentation,*
(2) *f is pseudo-coherent, and*
(3) *f is perfect.*

Proof. Since a perfect morphism is pseudo-coherent (because a perfect ring map is pseudo-coherent) and a pseudo-coherent morphism is locally of finite presentation (because a pseudo-coherent ring map is of finite presentation) it suffices to prove the last statement. Being perfect is a local property, hence we may assume that f factors as $\pi \circ i$ where π is smooth and i is a Koszul-regular immersion. A Koszul-regular immersion is perfect, see Lemma 36.41.7. A smooth morphism is perfect as it is flat and locally of finite presentation, see Lemma 36.41.5. Finally a composition of perfect morphisms is perfect, see Lemma 36.41.4. □

Lemma 36.42.5. *Let $f : X = \mathrm{Spec}(B) \to S = \mathrm{Spec}(A)$ be a morphism of affine schemes. Then f is a local complete intersection morphism if and only if $A \to B$ is a local complete intersection homomorphism, see More on Algebra, Definition 15.22.2.*

Proof. Follows immediately from the definitions. □

Beware that a base change of a Koszul morphism is not Koszul in general.

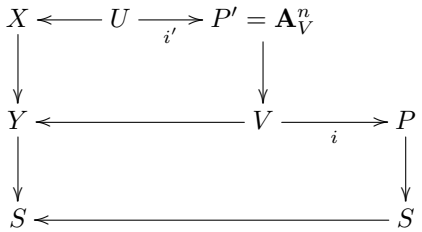
Lemma 36.42.6. *A flat base change of a local complete intersection morphism is a local complete intersection morphism.*

Proof. Omitted. Hint: This is true because a base change of a smooth morphism is smooth and a flat base change of a Koszul-regular immersion is a Koszul-regular immersion, see Divisors, Lemma 30.13.3. □

Lemma 36.42.7. *A composition of local complete intersection morphisms is a local complete intersection morphism.*

Proof. Let $g : Y \to S$ and $f : X \to Y$ be local complete intersection morphisms. Let $x \in X$ and set $y = f(x)$. Choose an open neighbourhood $V \subset Y$ of y and a factorization $g|_V = \pi \circ i$ for some Koszul-regular immersion $i : V \to P$ and smooth morphism $\pi : P \to S$. Next choose an open neighbourhood U of $x \in X$ and a factorization $f|_U = \pi' \circ i'$ for some Koszul-regular immersion $i' : U \to P'$ and smooth morphism $\pi' : P' \to Y$. In fact, we may assume that $P' = \mathbf{A}^n_V$, see

discussion preceding and following Definition 36.42.2. Picture:



Set $P'' = \mathbf{A}^n_P$. Then $U \to P' \to P''$ is a Koszul-regular immersion as a composition of Koszul-regular immersions, namely i' and the flat base change of i via $P'' \to P$, see Divisors, Lemma 30.13.3 and Divisors, Lemma 30.13.7. Also $P'' \to P \to S$ is smooth as a composition of smooth morphisms, see Morphisms, Lemma 28.35.4. Hence we conclude that $X \to S$ is Koszul at x as desired. □

Lemma 36.42.8. *Let $f : X \to S$ be a morphism of schemes. The following are equivalent*

(1) *f is flat and a local complete intersection morphism, and*
(2) *f is syntomic.*

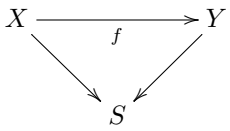
Proof. Assume (2). By Morphisms, Lemma 28.32.10 for every point x of X there exist affine open neighbourhoods U of x and V of $f(x)$ such that $f|_U : U \to V$ is standard syntomic. This means that $U = \mathrm{Spec}(R[x_1, \ldots, x_n]/(f_1, \ldots, f_c)) \to V = \mathrm{Spec}(R)$ where $R[x_1, \ldots, x_n]/(f_1, \ldots, f_c)$ is a relative global complete intersection over R. By Algebra, Lemma 10.129.13 the sequence $f_1, \ldots, f_c$ is a regular sequence in each local ring $R[x_1, \ldots, x_n]_{\mathfrak{q}}$ for every prime $\mathfrak{q} \supset (f_1, \ldots, f_c)$. Consider the Koszul complex $K_\bullet = K_\bullet(R[x_1, \ldots, x_n], f_1, \ldots, f_c)$ with homology groups $H_i = H_i(K_\bullet)$. By More on Algebra, Lemma 15.20.2 we see that $(H_i)_{\mathfrak{q}} = 0$, $i > 0$ for every $\mathfrak{q}$ as above. On the other hand, by More on Algebra, Lemma 15.19.6 we see that H_i is annihilated by $(f_1, \ldots, f_c)$. Hence we see that $H_i = 0, i > 0$ and $f_1, \ldots, f_c$ is a Koszul-regular sequence. This proves that $U \to V$ factors as a Koszul-regular immersion $U \to \mathbf{A}^n_V$ followed by a smooth morphism as desired.

Assume (1). Then f is a flat and locally of finite presentation (Lemma 36.42.4). Hence, according to Morphisms, Lemma 28.32.10 it suffices to show that the local rings $\mathcal{O}_{X_s,x}$ are local complete intersection rings. Choose, locally on X, a factorization $f = \pi \circ i$ for some Koszul-regular immersion $i : X \to P$ and smooth morphism $\pi : P \to S$. Note that $X \to P$ is a relative quasi-regular immersion over S, see Divisors, Definition 30.14.2. Hence according to Divisors, Lemma 30.14.4 we see that $X \to P$ is a regular immersion and the same remains true after any base change. Thus each fibre is a regular immersion, whence all the local rings of all the fibres of X are local complete intersections. □

Lemma 36.42.9. *A regular immersion of schemes is a local complete intersection morphism. A Koszul-regular immersion of schemes is a local complete intersection morphism.*

Proof. Since a regular immersion is a Koszul-regular immersion, see Divisors, Lemma 30.13.2, it suffices to prove the second statement. The second statement follows immediately from the definition. □

Lemma 36.42.10. *Let*

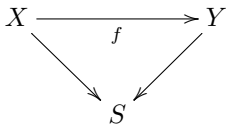


be a commutative diagram of morphisms of schemes. Assume $Y \to S$ smooth and $X \to S$ is a local complete intersection morphism. Then $f : X \to Y$ is a local complete intersection morphism.

Proof. Immediate from the definitions. □

The following lemma is of a different nature.

Lemma 36.42.11. *Let*



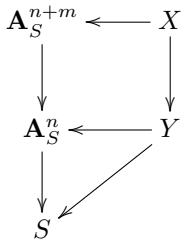
be a commutative diagram of morphisms of schemes. Assume

(1) *S is locally Noetherian,*
(2) *$Y \to S$ is locally of finite type,*
(3) *$f : X \to Y$ is perfect,*
(4) *$X \to S$ is a local complete intersection morphism.*

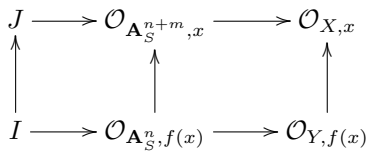
Then $X \to Y$ is a local complete intersection morphism and $Y \to S$ is Koszul at $f(x)$ for all $x \in X$.

Proof. In the course of this proof all schemes will be locally Noetherian and all rings will be Noetherian. We will use without further mention that regular sequences and Koszul regular sequences agree in this setting, see More on Algebra, Lemma 15.20.6. Moreover, whether an ideal (resp. ideal sheaf) is regular may be checked on local rings (resp. stalks), see Algebra, Lemma 10.65.8 (resp. Divisors, Lemma 30.12.8)

The question is local. Hence we may assume S, X, Y are affine. In this situation we may choose a commutative diagram



whose horizontal arrows are closed immersions. Let $x \in X$ be a point and consider the corresponding commutative diagram of local rings



where J and I are the kernels of the horizontal arrows. Since $X \to S$ is a local complete intersection morphism, the ideal J is generated by a regular sequence. Since $X \to Y$ is perfect the ring $\mathcal{O}_{X,x}$ has finite tor dimension over $\mathcal{O}_{Y,f(x)}$. Hence we may apply Divided Power Algebra, Lemma 23.7.6 to conclude that I and J/I are generated by regular sequences. By our initial remarks, this finishes the proof. □

Lemma 36.42.12. *The property $\mathcal{P}(f) =$ "f is a local complete intersection morphism" is fpqc local on the base.*

Proof. Let $f : X \to S$ be a morphism of schemes. Let $\{S_i \to S\}$ be an fpqc covering of S. Assume that each base change $f_i : X_i \to S_i$ of f is a local complete intersection morphism. Note that this implies in particular that f is locally of finite type, see Lemma 36.42.4 and Descent, Lemma 34.19.8. Let $x \in X$. Choose an open neighbourhood U of x and an immersion $j : U \to \mathbf{A}^n_S$ over S (see discussion preceding Definition 36.42.2). We have to show that j is a Koszul-regular immersion. Since f_i is a local complete intersection morphism, we see that the base change $j_i : U \times_S S_i \to \mathbf{A}^n_{S_i}$ is a Koszul-regular immersion, see Lemma 36.42.3. Because $\{\mathbf{A}^n_{S_i} \to \mathbf{A}^n_S\}$ is a fpqc covering we see from Descent, Lemma 34.19.30 that j is a Koszul-regular immersion as desired. □

Lemma 36.42.13. *The property $\mathcal{P}(f) =$ "f is a local complete intersection morphism" is syntomic local on the source.*

Proof. We will use the criterion of Descent, Lemma 34.22.3 to prove this. It follows from Lemmas 36.42.8 and 36.42.7 that being a local complete intersection morphism is preserved under precomposing with syntomic morphisms. It is clear from Definition 36.42.2 that being a local complete intersection morphism is Zariski local on the source and target. Hence, according to the aforementioned Descent, Lemma 34.22.3 it suffices to prove the following: Suppose $X' \to X \to Y$ are morphisms of affine schemes with $X' \to X$ syntomic and $X' \to Y$ a local complete intersection morphism. Then $X \to Y$ is a local complete intersection morphism. To see this, note that in any case $X \to Y$ is of finite presentation by Descent, Lemma 34.10.1. Choose a closed immersion $X \to \mathbf{A}^n_Y$. By Algebra, Lemma 10.129.18 we can find an affine open covering $X' = \bigcup_{i=1,\ldots,n} X'_i$ and syntomic morphisms $W_i \to \mathbf{A}^n_Y$ lifting the morphisms $X'_i \to X$, i.e., such that there are fibre product diagrams

$$\begin{array}{ccc} X'_i & \longrightarrow & W_i \\ \downarrow & & \downarrow \\ X & \longrightarrow & \mathbf{A}^n_Y \end{array}$$

After replacing X' by $\coprod X'_i$ and setting $W = \coprod W_i$ we obtain a fibre product diagram of affine schemes

$$\begin{array}{ccc} X' & \longrightarrow & W \\ \downarrow & & \downarrow h \\ X & \longrightarrow & \mathbf{A}^n_Y \end{array}$$

with $h : W \to \mathbf{A}^n_Y$ syntomic and $X' \to Y$ still a local complete intersection morphism. Since $W \to \mathbf{A}^n_Y$ is open (see Morphisms, Lemma 28.26.9) and $X' \to X$ is

surjective we see that X is contained in the image of $W \to \mathbf{A}^n_Y$. Choose a closed immersion $W \to \mathbf{A}^{n+m}_Y$ over $\mathbf{A}^n_Y$. Now the diagram looks like

$$\begin{array}{ccccc} X' & \longrightarrow & W & \longrightarrow & \mathbf{A}^{n+m}_Y \\ \downarrow & & \downarrow{\scriptstyle h} & \swarrow & \\ X & \longrightarrow & \mathbf{A}^n_Y & & \end{array}$$

Because h is syntomic and hence a local complete intersection morphism (see above) the morphism $W \to \mathbf{A}^{n+m}_Y$ is a Koszul-regular immersion. Because $X' \to Y$ is a local complete intersection morphism the morphism $X' \to \mathbf{A}^{n+m}_Y$ is a Koszul-regular immersion. We conclude from Divisors, Lemma 30.13.8 that $X' \to W$ is a Koszul-regular immersion. Hence, since being a Koszul-regular immersion is fpqc local on the target (see Descent, Lemma 34.19.30) we conclude that $X \to \mathbf{A}^n_Y$ is a Koszul-regular immersion which is what we had to show. □

Lemma 36.42.14. *Let S be a scheme. Let $f : X \to Y$ be a morphism of schemes over S. Assume both X and Y are flat and locally of finite presentation over S. Then the set*

$$\{x \in X \mid f \text{ Koszul at } x\}.$$

is open in X and its formation commutes with arbitrary base change $S' \to S$.

Proof. The set is open by definition (see Definition 36.42.2). Let $S' \to S$ be a morphism of schemes. Set $X' = S' \times_S X$, $Y' = S' \times_S Y$, and denote $f' : X' \to Y'$ the base change of f. Let $x' \in X'$ be a point such that f' is Koszul at x'. Denote $s' \in S'$, $x \in X$, $y' \in Y'$, $y \in Y$, $s \in S$ the image of x'. Note that f is locally of finite presentation, see Morphisms, Lemma 28.22.11. Hence we may choose an affine neighbourhood $U \subset X$ of x and an immersion $i : U \to \mathbf{A}^n_Y$. Denote $U' = S' \times_S U$ and $i' : U' \to \mathbf{A}^n_{Y'}$ the base change of i. The assumption that f' is Koszul at x' implies that i' is a Koszul-regular immersion in a neighbourhood of x', see Lemma 36.42.3. The scheme X' is flat and locally of finite presentation over S' as a base change of X (see Morphisms, Lemmas 28.26.7 and 28.22.4). Hence i' is a relative H_1-regular immersion over S' in a neighbourhood of x' (see Divisors, Definition 30.14.2). Thus the base change $i'_{s'} : U'_{s'} \to \mathbf{A}^n_{Y'_{s'}}$ is a H_1-regular immersion in an open neighbourhood of x', see Divisors, Lemma 30.14.1 and the discussion following Divisors, Definition 30.14.2. Since $s' = \mathrm{Spec}(\kappa(s')) \to \mathrm{Spec}(\kappa(s)) = s$ is a surjective flat universally open morphism (see Morphisms, Lemma 28.24.4) we conclude that the base change $i_s : U_s \to \mathbf{A}^n_{Y_s}$ is an H_1-regular immersion in a neighbourhood of x, see Descent, Lemma 34.19.30. Finally, note that $\mathbf{A}^n_Y$ is flat and locally of finite presentation over S, hence Divisors, Lemma 30.14.6 implies that i is a (Koszul-)regular immersion in a neighbourhood of x as desired. □

Lemma 36.42.15. *Let $f : X \to Y$ be a local complete intersection morphism of schemes. Then f is unramified if and only if f is formally unramified and in this case the conormal sheaf $\mathcal{C}_{X/Y}$ is finite locally free on X.*

Proof. The first assertion follows immediately from Lemma 36.4.8 and the fact that a local complete intersection morphism is locally of finite type. To compute the conormal sheaf of f we choose, locally on X, a factorization of f as $f = p \circ i$ where $i : X \to V$ is a Koszul-regular immersion and $V \to Y$ is smooth. By Lemma

36.9.11 we see that $\mathcal{C}_{X/Y}$ is a locally direct summand of $\mathcal{C}_{X/V}$ which is finite locally free as i is a Koszul-regular (hence quasi-regular) immersion, see Divisors, Lemma 30.13.5. □

Lemma 36.42.16. *Let $Z \to Y \to X$ be formally unramified morphisms of schemes. Assume that $Z \to Y$ is a local complete intersection morphism. The exact sequence*

$$0 \to i^*\mathcal{C}_{Y/X} \to \mathcal{C}_{Z/X} \to \mathcal{C}_{Z/Y} \to 0$$

of Lemma 36.5.12 is short exact.

Proof. The question is local on Z hence we may assume there exists a factorization $Z \to \mathbf{A}^n_Y \to Y$ of the morphism $Z \to Y$. Then we get a commutative diagram

$$\begin{array}{ccccc} Z & \xrightarrow{i'} & \mathbf{A}^n_Y & \longrightarrow & \mathbf{A}^n_X \\ \| & & \downarrow & & \downarrow \\ Z & \xrightarrow{i} & Y & \longrightarrow & X \end{array}$$

As $Z \to Y$ is a local complete intersection morphism, we see that $Z \to \mathbf{A}^n_Y$ is a Koszul-regular immersion. Hence by Divisors, Lemma 30.13.6 the sequence

$$0 \to (i')^*\mathcal{C}_{\mathbf{A}^n_Y/\mathbf{A}^n_X} \to \mathcal{C}_{Z/\mathbf{A}^n_X} \to \mathcal{C}_{Z/\mathbf{A}^n_Y} \to 0$$

is exact and locally split. Note that $i^*\mathcal{C}_{Y/X} = (i')^*\mathcal{C}_{\mathbf{A}^n_Y/\mathbf{A}^n_X}$ by Lemma 36.5.7 and note that the diagram

$$\begin{array}{ccc} (i')^*\mathcal{C}_{\mathbf{A}^n_Y/\mathbf{A}^n_X} & \longrightarrow & \mathcal{C}_{Z/\mathbf{A}^n_X} \\ \cong\uparrow & & \uparrow \\ i^*\mathcal{C}_{Y/X} & \longrightarrow & \mathcal{C}_{Z/X} \end{array}$$

is commutative. Hence the lower horizontal arrow is a locally split injection. This proves the lemma. □

36.43. Exact sequences of differentials and conormal sheaves

In this section we collect some results on exact sequences of conormal sheaves and sheaves of differentials. In some sense these are all realizations of the triangle of cotangent complexes associated to a pair of composable morphisms of schemes.

In the sequences below each of the maps are as constructed in either Morphisms, Lemma 28.34.8 or Lemma 36.5.5. Let $g : Z \to Y$ and $f : Y \to X$ be morphisms of schemes.

(1) There is a canonical exact sequence

$$g^*\Omega_{Y/X} \to \Omega_{Z/X} \to \Omega_{Z/Y} \to 0,$$

see Morphisms, Lemma 28.34.9. If $g : Z \to Y$ is formally smooth, then this sequence is a short exact sequence, see Lemma 36.9.9.

(2) If g is formally unramified, then there is a canonical exact sequence

$$\mathcal{C}_{Z/Y} \to g^*\Omega_{Y/X} \to \Omega_{Z/X} \to 0,$$

see Lemma 36.5.10. If $f \circ g : Z \to X$ is formally smooth, then this sequence is a short exact sequence, see Lemma 36.9.10.

(3) If g and $f \circ g$ are formally unramified, then there is a canonical exact sequence
$$\mathcal{C}_{Z/X} \to \mathcal{C}_{Z/Y} \to g^*\Omega_{Y/X} \to 0,$$
see Lemma 36.5.11. If $f : Y \to X$ is formally smooth, then this sequence is a short exact sequence, see Lemma 36.9.11.

(4) If g and f are formally unramified, then there is a canonical exact sequence
$$g^*\mathcal{C}_{Y/X} \to \mathcal{C}_{Z/X} \to \mathcal{C}_{Z/Y} \to 0.$$
see Lemma 36.5.12. If $g : Z \to Y$ is a local complete intersection morphism, then this sequence is a short exact sequence, see Lemma 36.42.16.

36.44. Weakly étale morphisms

A ring homomorphism $A \to B$ is weakly étale if both $A \to B$ and $B \otimes_A B \to B$ are flat, see More on Algebra, Definition 15.60.1. The analogous notion for morphisms of schemes is the following.

Definition 36.44.1. A morphism of schemes $X \to Y$ is *weakly étale* or *absolutely flat* if both $X \to Y$ and the diagonal morphism $X \to X \times_Y X$ are flat.

An étale morphism is weakly étale and conversely it turns out that a weakly étale morphism is indeed somewhat like an étale morphism. For example, if $X \to Y$ is weakly étale, then $L_{X/Y} = 0$, as follows from Cotangent, Lemma 66.8.4. We will prove a very precise result relating weakly étale morphisms to étale morphisms later (see Pro-étale Cohomology, Section 46.9). In this section we stick with the basics.

Lemma 36.44.2. *Let $f : X \to Y$ be a morphism of schemes. The following are equivalent*

(1) *$X \to Y$ is weakly étale, and*
(2) *for every $x \in X$ the ring map $\mathcal{O}_{Y,f(x)} \to \mathcal{O}_{X,x}$ is weakly étale.*

Proof. Observe that under both assumptions (1) and (2) the morphism f is flat. Thus we may assume f is flat. Let $x \in X$ with image $y = f(x)$ in Y. There are canonical maps of rings
$$\mathcal{O}_{X,x} \otimes_{\mathcal{O}_{Y,y}} \mathcal{O}_{X,x} \longrightarrow \mathcal{O}_{X \times_Y X, \Delta_{X/Y}(x)} \longrightarrow \mathcal{O}_{X,x}$$
where the first map is a localization (hence flat) and the second map is a surjection (hence an epimorphism of rings). Condition (1) means that for all x the second arrow is flat. Condition (2) is that for all x the composition is flat. These conditions are equivalent by Algebra, Lemma 10.36.3 and More on Algebra, Lemma 15.60.2. □

Lemma 36.44.3. *Let $X \to Y$ be a morphism of schemes such that $X \to X \times_Y X$ is flat. Let $\mathcal{F}$ be an $\mathcal{O}_X$-module. If $\mathcal{F}$ is flat over Y, then $\mathcal{F}$ is flat over X.*

Proof. Let $x \in X$ with image $y = f(x)$ in Y. Since $X \to X \times_Y X$ is flat, we see that $\mathcal{O}_{X,x} \otimes_{\mathcal{O}_{Y,y}} \mathcal{O}_{X,x} \to \mathcal{O}_{X,x}$ is flat. Hence the result follows from More on Algebra, Lemma 15.60.2 and the definitions. □

Lemma 36.44.4. *Let $f : X \to S$ be a morphism of schemes. The following are equivalent*

(1) *The morphism f is weakly étale.*

(2) *For every affine opens $U \subset X$, $V \subset S$ with $f(U) \subset V$ the ring map $\mathcal{O}_S(V) \to \mathcal{O}_X(U)$ is weakly étale.*
(3) *There exists an open covering $S = \bigcup_{j \in J} V_j$ and open coverings $f^{-1}(V_j) = \bigcup_{i \in I_j} U_i$ such that each of the morphisms $U_i \to V_j$, $j \in J, i \in I_j$ is weakly étale.*
(4) *There exists an affine open covering $S = \bigcup_{j \in J} V_j$ and affine open coverings $f^{-1}(V_j) = \bigcup_{i \in I_j} U_i$ such that the ring map $\mathcal{O}_S(V_j) \to \mathcal{O}_X(U_i)$ is of weakly étale, for all $j \in J, i \in I_j$.*

Moreover, if f is weakly étale then for any open subschemes $U \subset X$, $V \subset S$ with $f(U) \subset V$ the restriction $f|_U : U \to V$ is weakly-étale.

Proof. Suppose given open subschemes $U \subset X$, $V \subset S$ with $f(U) \subset V$. Then $U \times_V U \subset X \times_Y X$ is open (Schemes, Lemma 25.17.3) and the diagonal $\Delta_{U/V}$ of $f|_U : U \to V$ is the restriction $\Delta_{X/Y}|_U : U \to U \times_V U$. Since flatness is a local property of morphisms of schemes (Morphisms, Lemma 28.26.3) the final statement of the lemma is follows as well as the equivalence of (1) and (3). If X and Y are affine, then $X \to Y$ is weakly étale if and only if $\mathcal{O}_Y(Y) \to \mathcal{O}_X(X)$ is weakly étale (use again Morphisms, Lemma 28.26.3). Thus (1) and (3) are also equivalent to (2) and (4). □

Lemma 36.44.5. *Let $X \to Y \to Z$ be morphisms of schemes.*

(1) *If $X \to X \times_Y X$ and $Y \to Y \times_Z Y$ are flat, then $X \to X \times_Z X$ is flat.*
(2) *If $X \to Y$ and $Y \to Z$ are weakly étale, then $X \to Z$ is weakly étale.*

Proof. Part (1) follows from the factorization

$$X \to X \times_Y X \to X \times_Z X$$

of the diagonal of X over Z, the fact that

$$X \times_Y X = (X \times_Z X) \times_{(Y \times_Z Y)} Y,$$

the fact that a base change of a flat morphism is flat, and the fact that the composition of flat morphisms is flat (Morphisms, Lemmas 28.26.7 and 28.26.5). Part (2) follows from part (1) and the fact (just used) that the composition of flat morphisms is flat. □

Lemma 36.44.6. *Let $X \to Y$ and $Y' \to Y$ be morphisms of schemes and let $X' = Y' \times_Y X$ be the base change of X.*

(1) *If $X \to X \times_Y X$ is flat, then $X' \to X' \times_{Y'} X'$ is flat.*
(2) *If $X \to Y$ is weakly étale, then $X' \to Y'$ is weakly étale.*

Proof. Assume $X \to X \times_Y X$ is flat. The morphism $X' \to X' \times_{Y'} X'$ is the base change of $X \to X \times_Y X$ by $Y' \to Y$. Hence it is flat by Morphisms, Lemmas 28.26.7. This proves (1). Part (2) follows from (1) and the fact (just used) that the base change of a flat morphism is flat. □

Lemma 36.44.7. *Let $X \to Y \to Z$ be morphisms of schemes. Assume that $X \to Y$ is flat and surjective and that $X \to X \times_Z X$ is flat. Then $Y \to Y \times_Z Y$ is flat.*

Proof. Consider the commutative diagram

$$\begin{array}{ccc} X & \longrightarrow & X \times_Z X \\ \downarrow & & \downarrow \\ Y & \longrightarrow & Y \times_Z Y \end{array}$$

The top horizontal arrow is flat and the vertical arrows are flat. Hence X is flat over $Y \times_Z Y$. By Morphisms, Lemma 28.26.11 we see that Y is flat over $Y \times_Z Y$. □

Lemma 36.44.8. *Let $f : X \to Y$ be a weakly étale morphism of schemes. Then f is formally unramified, i.e., $\Omega_{X/Y} = 0$.*

Proof. Recall that f is formally unramified if and only if $\Omega_{X/Y} = 0$ by Lemma 36.4.7. Via Lemma 36.44.4 and Morphisms, Lemma 28.34.5 this follows from the case of rings which is More on Algebra, Lemma 15.60.12. □

Lemma 36.44.9. *Let $f : X \to Y$ be a morphism of schemes. Then $X \to Y$ is weakly étale in each of the following cases*

1. *$X \to Y$ is a flat monomorphism,*
2. *$X \to Y$ is an open immersion,*
3. *$X \to Y$ is flat and unramified,*
4. *$X \to Y$ is étale.*

Proof. If (1) holds, then $\Delta_{X/Y}$ is an isomorphism, hence certainly f is weakly étale. Case (2) is a special case of (1). The diagonal of an unramified morphism is an open immersion (Morphisms, Lemma 28.36.13), hence flat. Thus a flat unramified morphism is weakly étale. An étale morphism is flat and unramified (Morphisms, Lemma 28.37.5), hence (4) follows from (3). □

Lemma 36.44.10. *Let $f : X \to Y$ be a morphism of schemes. If Y is reduced and f weakly étale, then X is reduced.*

Proof. Via Lemma 36.44.4 this follows from the case of rings which is More on Algebra, Lemma 15.60.8. □

The following lemma uses a nontrivial result about weakly étale ring maps.

Lemma 36.44.11. *Let $f : X \to Y$ be a morphism of schemes. The following are equivalent*

1. *f is weakly étale, and*
2. *for $x \in X$ the local ring map $\mathcal{O}_{Y,f(x)} \to \mathcal{O}_{X,x}$ induces an isomorphism on strict henselizations.*

Proof. Let $x \in X$ be a point with image $y = f(x)$ in Y. Choose a separable algebraic closure κ^{sep} of $\kappa(x)$. Let $\mathcal{O}^{sh}_{X,x}$ be the strict henselization corresponding to κ^{sep} and $\mathcal{O}^{sh}_{Y,y}$ the strict henselization relative to the separable algebraic closure of $\kappa(y)$ in κ^{sep}. Consider the commutative diagram

$$\begin{array}{ccc} \mathcal{O}_{X,x} & \longrightarrow & \mathcal{O}^{sh}_{X,x} \\ \uparrow & & \uparrow \\ \mathcal{O}_{Y,y} & \longrightarrow & \mathcal{O}^{sh}_{Y,y} \end{array}$$

local homomorphisms of local rings, see Algebra, Lemma 10.143.26. Since the strict henselization is a filtered colimit of étale ring maps, More on Algebra, Lemma 15.60.13 shows the horizontal maps are weakly étale. Moreover, the horizontal maps are faithfully flat by More on Algebra, Lemma 15.32.1.

Assume f weakly étale. By Lemma 36.44.2 the left vertical arrow is weakly étale. By More on Algebra, Lemmas 15.60.9 and 15.60.11 the right vertical arrow is weakly étale. By More on Algebra, Theorem 15.60.24 we conclude the right vertical map is an isomorphism.

Assume $\mathcal{O}_{Y,y}^{sh} \to \mathcal{O}_{X,x}^{sh}$ is an isomorphism. Then $\mathcal{O}_{Y,y} \to \mathcal{O}_{X,x}^{sh}$ is weakly étale. Since $\mathcal{O}_{X,x} \to \mathcal{O}_{X,x}^{sh}$ is faithfully flat we conclude that $\mathcal{O}_{Y,y} \to \mathcal{O}_{X,x}$ is weakly étale by More on Algebra, Lemma 15.60.10. Thus (2) implies (1) by Lemma 36.44.2. □

Lemma 36.44.12. *Let $f : X \to Y$ be a morphism of schemes. If Y is a normal scheme and f weakly étale, then X is a normal scheme.*

Proof. By More on Algebra, Lemma 15.32.5 a scheme S is normal if and only if for all $s \in S$ the strict henselization of $\mathcal{O}_{S,s}$ is a normal domain. Hence the lemma follows from Lemma 36.44.11. □

Lemma 36.44.13. *Let S be a scheme. Let $f : X \to Y$ be a morphism of schemes over S. If X, Y are weakly étale over S, then f is weakly étale.*

Proof. We will use Morphisms, Lemmas 28.26.7 and 28.26.5 without further mention. Write $X \to Y$ as the composition $X \to X \times_S Y \to Y$. The second morphism is flat as the base change of the flat morphism $X \to S$. The first is the base change of the flat morphism $Y \to Y \times_S Y$ by the morphism $X \times_S Y \to Y \times_S Y$, hence flat. Thus $X \to Y$ is flat. The morphism $X \times_Y X \to X \times_S X$ is an immersion. Thus Lemma 36.44.3 implies, that since X is flat over $X \times_S X$ it follows that X is flat over $X \times_Y X$. □

36.45. Reduced fibre theorem

In this section we discuss the simplest kind of theorem of the kind advertised by the title. Although the proof of the result is kind of laborious, in essence it follows in a straightforward manner from Epp's result on eliminating ramification, see More on Algebra, Theorem 15.62.21.

Let A be a Dedekind domain with fraction field K. Let X be a scheme flat and of finite type over A. Let L be a finite extension of K. Let B be the integral closure of A in L. Then B is a Dedekind domain (Algebra, Lemma 10.114.11). Let $X_B = X \times_{\mathrm{Spec}(A)} \mathrm{Spec}(B)$ be the base change. Then $X_B \to \mathrm{Spec}(B)$ is of finite type (Morphisms, Lemma 28.16.4). Hence X_B is Noetherian (Morphisms, Lemma 28.16.6). Thus the normalization $\nu : Y \to X_B$ exists (see Morphisms, Definition 28.48.12 and the discussion following). Picture

(36.45.0.1)
$$\begin{array}{ccccc} Y & \xrightarrow{\nu} & X_B & \longrightarrow & X \\ & \searrow & \downarrow & & \downarrow \\ & & \mathrm{Spec}(B) & \longrightarrow & \mathrm{Spec}(A) \end{array}$$

We sometimes call Y the *normalized base change* of X. In general the morphism ν may not be finite. But if A is a Nagata ring (a condition that is virtually always

satisfied in practice) then ν is of finite and Y is of finite type over B, see Morphisms, Lemmas 28.48.17 and 28.19.1.

Taking the normalized base change commutes with composition. More precisely, if $K \subset L \subset M$ are finite extensions of fields with integral closures $A \subset B \subset C$ then the normalized base change Z of $Y \to \mathrm{Spec}(B)$ relative to $L \subset M$ is equal to the normalized base change of $X \to \mathrm{Spec}(A)$ relative to $K \subset M$.

Theorem 36.45.1. *Let A be a Dedekind ring with fraction field K. Let X be a scheme flat and of finite type over A. Assume A is a Nagata ring. There exists a finite extension $K \subset L$ such that the normalized base change Y is smooth over* $\mathrm{Spec}(B)$ *at all generic points of all fibres.*

Proof. During the proof we will repeatedly use that formation of the set of points where a (flat, finitely presented) morphism like $X \to \mathrm{Spec}(A)$ is smooth commutes with base change, see Morphisms, Lemma 28.35.15.

We first choose a finite extension $K \subset L$ such that $(X_L)_{red}$ is geometrically reduced over L, see Varieties, Lemma 32.4.11. Since $Y \to (X_B)_{red}$ is birational we see applying Varieties, Lemma 32.4.8 that Y_L is geometrically reduced over L as well. Hence $Y_L \to \mathrm{Spec}(L)$ is smooth on a dense open $V \subset Y_L$ by Varieties, Lemma 32.15.7. Thus the smooth locus $U \subset Y$ of the morphism $Y \to \mathrm{Spec}(B)$ is open (by Morphisms, Definition 28.35.1) and is dense in the generic fibre. Replacing A by B and X by Y we reduce to the case treated in the next paragraph.

Assume X is normal and the smooth locus $U \subset X$ of $X \to \mathrm{Spec}(A)$ is dense in the generic fibre. This implies that U is dense in all but finitely many fibres, see Lemma 36.19.3. Let $x_1, \ldots, x_r \in X \setminus U$ be the finitely many generic points of irreducible components of $X \setminus U$ which are moreover generic points of irreducible components of fibres of $X \to \mathrm{Spec}(A)$. Set $\mathcal{O}_i = \mathcal{O}_{X,x_i}$. Let A_i be the localization of A at the maximal ideal corresponding to the image of x_i in $\mathrm{Spec}(A)$. By More on Algebra, Proposition 15.62.23 there exist finite extensions $K \subset K_i$ which are solutions for the extension of discrete valuation rings $A_i \to \mathcal{O}_i$. Let $K \subset L$ be a finite extension dominating all of the extensions $K \subset K_i$. Then $K \subset L$ is still a solution for $A_i \to \mathcal{O}_i$ by More on Algebra, Lemma 15.62.4.

Consider the diagram (36.45.0.1) with the extension L/K we just produced. Note that $U_B \subset X_B$ is smooth over B, hence normal (for example use Algebra, Lemma 10.149.7). Thus $Y \to X_B$ is an isomorphism over U_B. Let $y \in Y$ be a generic point of an irreducible component of a fibre of $Y \to \mathrm{Spec}(B)$ lying over the maximal ideal $\mathfrak{m} \subset B$. Assume that $y \notin U_B$. Then y maps to one of the points x_i. It follows that $\mathcal{O}_{Y,y}$ is a local ring of the integral closure of $\mathcal{O}_i$ in $R(X) \otimes_K L$ (details omitted). Hence because $K \subset L$ is a solution for $A_i \to \mathcal{O}_i$ we see that $B_{\mathfrak{m}} \to \mathcal{O}_{Y,y}$ is formally smooth (this is the definition of being a "solution"). In other words, $\mathfrak{m}\mathcal{O}_{Y,y} = \mathfrak{m}_y$ and the residue field extension is separable. Hence the local ring of the fibre at y is $\kappa(y)$. This implies the fibre is smooth over $\kappa(\mathfrak{m})$ at y for example by Algebra, Lemma 10.133.5. This finishes the proof. □

36.46. Other chapters

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CHAPTER 37

More on Flatness

37.1. Introduction

In this chapter, we discuss some advanced results on flat modules and flat morphisms of schemes. Most of these results can be found in the paper [**GR71**] by Raynaud and Gruson.

Before reading this chapter we advise the reader to take a look at the following results (this list also serves as a pointer to previous results):

(1) General discussion on flat modules in Algebra, Section 10.36.
(2) The relationship between Tor-groups and flatness, see Algebra, Section 10.70.
(3) Criteria for flatness, see Algebra, Section 10.93 (Noetherian case), Algebra, Section 10.95 (Artinian case), Algebra, Section 10.122 (non-Noetherian case), and finally More on Morphisms, Section 36.13.
(4) Generic flatness, see Algebra, Section 10.112 and Morphisms, Section 28.28.
(5) Openness of the flat locus, see Algebra, Section 10.123 and More on Morphisms, Section 36.12.
(6) Flattening, see More on Algebra, Sections 15.9, 15.10, 15.11, 15.12, and 15.13.
(7) Additional results in More on Algebra, Sections 15.14, 15.15, 15.16, and 15.17.

37.2. Lemmas on étale localization

In this section we list some lemmas on étale localization which will be useful later in this chapter. Please skip this section on a first reading.

Lemma 37.2.1. *Let $i : Z \to X$ be a closed immersion of affine schemes. Let $Z' \to Z$ be an étale morphism with Z' affine. Then there exists an étale morphism $X' \to X$ with X' affine such that $Z' \cong Z \times_X X'$ as schemes over Z.*

Proof. See Algebra, Lemma 10.136.11. □

Lemma 37.2.2. *Let*

$$\begin{array}{ccc} X & \longleftarrow & X' \\ \downarrow & & \downarrow \\ S & \longleftarrow & S' \end{array}$$

be a commutative diagram of schemes with $X' \to X$ and $S' \to S$ étale. Let $s' \in S'$ be a point. Then

$$X' \times_{S'} \operatorname{Spec}(\mathcal{O}_{S',s'}) \longrightarrow X \times_S \operatorname{Spec}(\mathcal{O}_{S',s'})$$

is étale.

Proof. This is true because $X' \to X_{S'}$ is étale as a morphism of schemes étale over X, see Morphisms, Lemma 28.37.18 and the base change of an étale morphism is étale, see Morphisms, Lemma 28.37.4. □

Lemma 37.2.3. *Let $X \to T \to S$ be morphisms of schemes with $T \to S$ étale. Let $\mathcal{F}$ be a quasi-coherent $\mathcal{O}_X$-module. Let $x \in X$ be a point. Then*

$$\mathcal{F} \text{ flat over } S \text{ at } x \Leftrightarrow \mathcal{F} \text{ flat over } T \text{ at } x$$

In particular $\mathcal{F}$ is flat over S if and only if $\mathcal{F}$ is flat over T.

Proof. As an étale morphism is a flat morphism (see Morphisms, Lemma 28.37.12) the implication "$\Leftarrow$" follows from Algebra, Lemma 10.36.3. For the converse assume that $\mathcal{F}$ is flat at x over S. Denote $\tilde{x} \in X \times_S T$ the point lying over x in X and over the image of x in T in T. Then $(X \times_S T \to X)^*\mathcal{F}$ is flat at $\tilde{x}$ over T via $\mathrm{pr}_2 : X \times_S T \to T$, see Morphisms, Lemma 28.26.6. The diagonal $\Delta_{T/S} : T \to T \times_S T$ is an open immersion; combine Morphisms, Lemmas 28.36.13 and 28.37.5. So X is identified with open subscheme of $X \times_S T$, the restriction of pr_2 to this open is the given morphism $X \to T$, the point $\tilde{x}$ corresponds to the point x in this open, and $(X \times_S T \to X)^*\mathcal{F}$ restricted to this open is $\mathcal{F}$. Whence we see that $\mathcal{F}$ is flat at x over T. □

Lemma 37.2.4. *Let $T \to S$ be an étale morphism. Let $t \in T$ with image $s \in S$. Let M be a $\mathcal{O}_{T,t}$-module. Then*

$$M \text{ flat over } \mathcal{O}_{S,s} \Leftrightarrow M \text{ flat over } \mathcal{O}_{T,t}.$$

Proof. We may replace S by an affine neighbourhood of s and after that T by an affine neighbourhood of t. Set $\mathcal{F} = (\mathrm{Spec}(\mathcal{O}_{T,t}) \to T)_*\widetilde{M}$. This is a quasi-coherent sheaf (see Schemes, Lemma 25.24.1 or argue directly) on T whose stalk at t is M (details omitted). Apply Lemma 37.2.3. □

Lemma 37.2.5. *Let S be a scheme and $s \in S$ a point. Denote $\mathcal{O}^h_{S,s}$ (resp. $\mathcal{O}^{sh}_{S,s}$) the henselization (resp. strict henselization), see Algebra, Definition 10.143.18. Let M^{sh} be a $\mathcal{O}^{sh}_{S,s}$-module. The following are equivalent*

1. *M^{sh} is flat over $\mathcal{O}_{S,s}$,*
2. *M^{sh} is flat over $\mathcal{O}^h_{S,s}$, and*
3. *M^{sh} is flat over $\mathcal{O}^{sh}_{S,s}$.*

If $M^{sh} = M^h \otimes_{\mathcal{O}^h_{S,s}} \mathcal{O}^{sh}_{S,s}$ this is also equivalent to

4. *M^h is flat over $\mathcal{O}_{S,s}$, and*
5. *M^h is flat over $\mathcal{O}^h_{S,s}$.*

If $M^h = M \otimes_{\mathcal{O}_{S,s}} \mathcal{O}^h_{S,s}$ this is also equivalent to

6. *M is flat over $\mathcal{O}_{S,s}$.*

Proof. We may assume that S is an affine scheme. It is shown in Algebra, Lemmas 10.143.21 and 10.143.27 that $\mathcal{O}^h_{S,s}$ and $\mathcal{O}^{sh}_{S,s}$ are filtered colimits of the rings $\mathcal{O}_{T,t}$ where $T \to S$ is étale and affine. Hence the local ring maps $\mathcal{O}_{S,s} \to \mathcal{O}^h_{S,s} \to \mathcal{O}^{sh}_{S,s}$ are flat as directed colimits of étale ring maps, see Algebra, Lemma 10.36.2. Hence (3) $\Rightarrow$ (2) $\Rightarrow$ (1) and (5) $\Rightarrow$ (4) follow from Algebra, Lemma 10.36.3. Of course these maps are faithfully flat, see Algebra, Lemma 10.36.16. Hence the equivalences

(6) $\Leftrightarrow$ (5) and (5) $\Leftrightarrow$ (3) follow from Algebra, Lemma 10.36.7. Thus it suffices to show that (1) $\Rightarrow$ (2) $\Rightarrow$ (3) and (4) $\Rightarrow$ (5).

Assume (1). By Lemma 37.2.4 we see that M^{sh} is flat over $\mathcal{O}_{T,t}$ for any étale neighbourhood $(T,t) \to (S,s)$. Since $\mathcal{O}_{S,s}^h$ and $\mathcal{O}_{S,s}^{sh}$ are directed colimits of local rings of the form $\mathcal{O}_{T,t}$ (see above) we conclude that M^{sh} is flat over $\mathcal{O}_{S,s}^h$ and $\mathcal{O}_{S,s}^{sh}$ by Algebra, Lemma 10.36.5. Thus (1) implies (2) and (3). Of course this implies also (2) $\Rightarrow$ (3) by replacing $\mathcal{O}_{S,s}$ by $\mathcal{O}_{S,s}^h$. The same argument applies to prove (4) $\Rightarrow$ (5). □

Lemma 37.2.6. *Let $g : T \to S$ be a finite flat morphism of schemes. Let $\mathcal{G}$ be a quasi-coherent $\mathcal{O}_S$-module. Let $t \in T$ be a point with image $s \in S$. Then*

$$t \in \mathit{WeakAss}(g^*\mathcal{G}) \Leftrightarrow s \in \mathit{WeakAss}(\mathcal{G})$$

Proof. The implication "$\Leftarrow$" follows immediately from Divisors, Lemma 30.6.4. Assume $t \in \text{WeakAss}(g^*\mathcal{G})$. Let $\text{Spec}(A) \subset S$ be an affine open neighbourhood of s. Let $\mathcal{G}$ be the quasi-coherent sheaf associated to the A-module M. Let $\mathfrak{p} \subset A$ be the prime ideal corresponding to s. As g is finite flat we have $g^{-1}(\text{Spec}(A)) = \text{Spec}(B)$ for some finite flat A-algebra B. Note that $g^*\mathcal{G}$ is the quasi-coherent $\mathcal{O}_{\text{Spec}(B)}$-module associated to the B-module $M \otimes_A B$ and $g_*g^*\mathcal{G}$ is the quasi-coherent $\mathcal{O}_{\text{Spec}(A)}$-module associated to the A-module $M \otimes_A B$. By Algebra, Lemma 10.73.4 we have $B_\mathfrak{p} \cong A_\mathfrak{p}^{\oplus n}$ for some integer $n \geq 0$. Note that $n \geq 1$ as we assumed there exists at least one point of T lying over s. Hence we see by looking at stalks that

$$s \in \text{WeakAss}(\mathcal{G}) \Leftrightarrow s \in \text{WeakAss}(g_*g^*\mathcal{G})$$

Now the assumption that $t \in \text{WeakAss}(g^*\mathcal{G})$ implies that $s \in \text{WeakAss}(g_*g^*\mathcal{G})$ by Divisors, Lemma 30.6.3 and hence by the above $s \in \text{WeakAss}(\mathcal{G})$. □

Lemma 37.2.7. *Let $h : U \to S$ be an étale morphism of schemes. Let $\mathcal{G}$ be a quasi-coherent $\mathcal{O}_S$-module. Let $u \in U$ be a point with image $s \in S$. Then*

$$u \in \mathit{WeakAss}(h^*\mathcal{G}) \Leftrightarrow s \in \mathit{WeakAss}(\mathcal{G})$$

Proof. After replacing S and U by affine neighbourhoods of s and u we may assume that g is a standard étale morphism of affines, see Morphisms, Lemma 28.37.14. Thus we may assume $S = \text{Spec}(A)$ and $X = \text{Spec}(A[x, 1/g]/(f))$, where f is monic and f' is invertible in $A[x, 1/g]$. Note that $A[x, 1/g]/(f) = (A[x]/(f))_g$ is also the localization of the finite free A-algebra $A[x]/(f)$. Hence we may think of U as an open subscheme of the scheme $T = \text{Spec}(A[x]/(f))$ which is finite locally free over S. This reduces us to Lemma 37.2.6 above. □

37.3. The local structure of a finite type module

The key technical lemma that makes a lot of the arguments in this chapter work is the geometric Lemma 37.3.2.

Lemma 37.3.1. *Let $f : X \to S$ be a finite type morphism of affine schemes. Let $\mathcal{F}$ be a finite type quasi-coherent $\mathcal{O}_X$-module. Let $x \in X$ with image $s = f(x)$ in S. Set $\mathcal{F}_s = \mathcal{F}|_{X_s}$. Then there exist a closed immersion $i : Z \to X$ of finite presentation, and a quasi-coherent finite type $\mathcal{O}_Z$-module $\mathcal{G}$ such that $i_*\mathcal{G} = \mathcal{F}$ and $Z_s = \mathit{Supp}(\mathcal{F}_s)$.*

Proof. Say the morphism $f : X \to S$ is given by the ring map $A \to B$ and that $\mathcal{F}$ is the quasi-coherent sheaf associated to the B-module M. By Morphisms, Lemma 28.16.2 we know that $A \to B$ is a finite type ring map, and by Properties, Lemma 27.16.1 we know that M is a finite B-module. In particular the support of $\mathcal{F}$ is the closed subscheme of $\mathrm{Spec}(B)$ cut out by the annihilator $I = \{x \in B \mid xm = 0 \ \forall m \in M\}$ of M, see Algebra, Lemma 10.37.5. Let $\mathfrak{q} \subset B$ be the prime ideal corresponding to x and let $\mathfrak{p} \subset A$ be the prime ideal corresponding to s. Note that $X_s = \mathrm{Spec}(B \otimes_A \kappa(\mathfrak{p}))$ and that $\mathcal{F}_s$ is the quasi-coherent sheaf associated to the $B \otimes_A \kappa(\mathfrak{p})$ module $M \otimes_A \kappa(\mathfrak{p})$. By Morphisms, Lemma 28.5.3 the support of $\mathcal{F}_s$ is equal to $V(I(B \otimes_A \kappa(\mathfrak{p})))$. Since $B \otimes_A \kappa(\mathfrak{p})$ is of finite type over $\kappa(\mathfrak{p})$ there exist finitely many elements $f_1, \ldots, f_m \in I$ such that

$$I(B \otimes_A \kappa(\mathfrak{p})) = (f_1, \ldots, f_n)(B \otimes_A \kappa(\mathfrak{p})).$$

Denote $i : Z \to X$ the closed subscheme cut out by $(f_1, \ldots, f_m)$, in a formula $Z = \mathrm{Spec}(B/(f_1, \ldots, f_m))$. Since M is annihilated by I we can think of M as an $B/(f_1, \ldots, f_m)$-module. In other words, $\mathcal{F}$ is the pushforward of a finite type module on Z. As $Z_s = \mathrm{Supp}(\mathcal{F}_s)$ by construction, this proves the lemma. □

Lemma 37.3.2. *Let $f : X \to S$ be morphism of schemes which is locally of finite type. Let $\mathcal{F}$ be a finite type quasi-coherent $\mathcal{O}_X$-module. Let $x \in X$ with image $s = f(x)$ in S. Set $\mathcal{F}_s = \mathcal{F}|_{X_s}$ and $n = \dim_x(\mathit{Supp}(\mathcal{F}_s))$. Then we can construct*

1. *elementary étale neighbourhoods $g : (X', x') \to (X, x)$, $e : (S', s') \to (S, s)$,*
2. *a commutative diagram*

$$\begin{array}{ccccc} X & \xleftarrow{g} & X' & \xleftarrow{i} & Z' \\ {\scriptstyle f}\downarrow & & \downarrow & & \downarrow{\scriptstyle \pi} \\ & & & & Y' \\ & & & & \downarrow{\scriptstyle h} \\ S & \xleftarrow{e} & S' & = & S' \end{array}$$

3. *a point $z' \in Z'$ with $i(z') = x'$, $y' = \pi(z')$, $h(y') = s'$,*
4. *a finite type quasi-coherent $\mathcal{O}_{Z'}$-module $\mathcal{G}$,*

such that the following properties hold

1. *X', Z', Y', S' are affine schemes,*
2. *i is a closed immersion of finite presentation,*
3. *$i_*(\mathcal{G}) \cong g^*\mathcal{F}$,*
4. *π is finite and $\pi^{-1}(\{y'\}) = \{z'\}$,*
5. *the extension $\kappa(s') \subset \kappa(y')$ is purely transcendental,*
6. *h is smooth of relative dimension n with geometrically integral fibres.*

Proof. Let $V \subset S$ be an affine neighbourhood of s. Let $U \subset f^{-1}(V)$ be an affine neighbourhood of x. Then it suffices to prove the lemma for $f|_U : U \to V$ and $\mathcal{F}|_U$. Hence in the rest of the proof we assume that X and S are affine.

First, suppose that $X_s = \mathrm{Supp}(\mathcal{F}_s)$, in particular $n = \dim_x(X_s)$. Apply More on Morphisms, Lemmas 36.33.2 and 36.33.3. This gives us a commutative diagram

$$\begin{array}{ccccc} X & \xleftarrow{g} & X' & & \\ \downarrow & & & \searrow^{\pi} & \\ & & & & Y' \\ & & & & \downarrow{\scriptstyle h} \\ S & \xleftarrow{e} & S' & & \end{array}$$

and point $x' \in X'$. We set $Z' = X'$, $i = \mathrm{id}$, and $\mathcal{G} = g^*\mathcal{F}$ to obtain a solution in this case.

In general choose a closed immersion $Z \to X$ and a sheaf $\mathcal{G}$ on Z as in Lemma 37.3.1. Applying the result of the previous paragraph to $Z \to S$ and $\mathcal{G}$ we obtain a diagram

$$\begin{array}{ccccc} X & \xleftarrow{} & Z & \xleftarrow{g} & Z' \\ \downarrow{\scriptstyle f} & & \downarrow{\scriptstyle f|_Z} & & \downarrow{\scriptstyle \pi} \\ & & & & Y' \\ & & & & \downarrow{\scriptstyle h} \\ S & = & S & \xleftarrow{e} & S' \end{array}$$

and point $z' \in Z'$ satisfying all the required properties. We will use Lemma 37.2.1 to embed Z' into a scheme étale over X. We cannot apply the lemma directly as we want X' to be a scheme over S'. Instead we consider the morphisms

$$Z' \longrightarrow Z \times_S S' \longrightarrow X \times_S S'$$

The first morphism is étale by Morphisms, Lemma 28.37.18. The second is a closed immersion as a base change of a closed immersion. Finally, as X, S, S', Z, Z' are all affine we may apply Lemma 37.2.1 to get an étale morphism of affine schemes $X' \to X \times_S S'$ such that

$$Z' = (Z \times_S S') \times_{(X \times_S S')} X' = Z \times_X X'.$$

As $Z \to X$ is a closed immersion of finite presentation, so is $Z' \to X'$. Let $x' \in X'$ be the point corresponding to $z' \in Z'$. Then the completed diagram

$$\begin{array}{ccccc} X & \xleftarrow{} & X' & \xleftarrow{i} & Z' \\ \downarrow & & \downarrow & & \downarrow{\scriptstyle \pi} \\ & & & & Y' \\ & & & & \downarrow{\scriptstyle h} \\ S & \xleftarrow{e} & S' & = & S' \end{array}$$

is a solution of the original problem. □

Lemma 37.3.3. *Assumptions and notation as in Lemma 37.3.2. If f is locally of finite presentation then π is of finite presentation. In this case the following are equivalent*

(1) *$\mathcal{F}$ is an $\mathcal{O}_X$-module of finite presentation in a neighbourhood of x,*
(2) *$\mathcal{G}$ is an $\mathcal{O}_{Z'}$-module of finite presentation in a neighbourhood of z', and*
(3) *$\pi_*\mathcal{G}$ is an $\mathcal{O}_{Y'}$-module of finite presentation in a neighbourhood of y'.*

Still assuming f locally of finite presentation the following are equivalent to each other

(a) *$\mathcal{F}_x$ is an $\mathcal{O}_{X,x}$-module of finite presentation,*
(b) *$\mathcal{G}_{z'}$ is an $\mathcal{O}_{Z',z'}$-module of finite presentation, and*
(c) *$(\pi_*\mathcal{G})_{y'}$ is an $\mathcal{O}_{Y',y'}$-module of finite presentation.*

Proof. Assume f locally of finite presentation. Then $Z' \to S$ is locally of finite presentation as a composition of such, see Morphisms, Lemma 28.22.3. Note that $Y' \to S$ is also locally of finite presentation as a composition of a smooth and an étale morphism. Hence Morphisms, Lemma 28.22.11 implies π is locally of finite presentation. Since π is finite we conclude that it is also separated and quasi-compact, hence π is actually of finite presentation.

To prove the equivalence of (1), (2), and (3) we also consider: (4) $g^*\mathcal{F}$ is a $\mathcal{O}_{X'}$-module of finite presentation in a neighbourhood of x'. The pullback of a module of finite presentation is of finite presentation, see Modules, Lemma 17.11.4. Hence (1) $\Rightarrow$ (4). The étale morphism g is open, see Morphisms, Lemma 28.37.13. Hence for any open neighbourhood $U' \subset X'$ of x', the image $g(U')$ is an open neighbourhood of x and the map $\{U' \to g(U')\}$ is an étale covering. Thus (4) $\Rightarrow$ (1) by Descent, Lemma 34.6.3. Using Descent, Lemma 34.6.10 and some easy topological arguments (see More on Morphisms, Lemma 36.33.4) we see that (4) $\Leftrightarrow$ (2) $\Leftrightarrow$ (3).

To prove the equivalence of (a), (b), (c) consider the ring maps

$$\mathcal{O}_{X,x} \to \mathcal{O}_{X',x'} \to \mathcal{O}_{Z',z'} \leftarrow \mathcal{O}_{Y',y'}$$

The first ring map is faithfully flat. Hence $\mathcal{F}_x$ is of finite presentation over $\mathcal{O}_{X,x}$ if and only if $g^*\mathcal{F}_{x'}$ is of finite presentation over $\mathcal{O}_{X',x'}$, see Algebra, Lemma 10.78.2. The second ring map is surjective (hence finite) and finitely presented by assumption, hence $g^*\mathcal{F}_{x'}$ is of finite presentation over $\mathcal{O}_{X',x'}$ if and only if $\mathcal{G}_{z'}$ is of finite presentation over $\mathcal{O}_{Z',z'}$, see Algebra, Lemma 10.7.4. Because π is finite, of finite presentation, and $\pi^{-1}(\{y'\}) = \{x'\}$ the ring homomorphism $\mathcal{O}_{Y',y'} \leftarrow \mathcal{O}_{Z',z'}$ is finite and of finite presentation, see More on Morphisms, Lemma 36.33.4. Hence $\mathcal{G}_{z'}$ is of finite presentation over $\mathcal{O}_{Z',z'}$ if and only if $\pi_*\mathcal{G}_{y'}$ is of finite presentation over $\mathcal{O}_{Y',y'}$, see Algebra, Lemma 10.7.4. □

Lemma 37.3.4. *Assumptions and notation as in Lemma 37.3.2. The following are equivalent*

(1) *$\mathcal{F}$ is flat over S in a neighbourhood of x,*
(2) *$\mathcal{G}$ is flat over S' in a neighbourhood of z', and*
(3) *$\pi_*\mathcal{G}$ is flat over S' in a neighbourhood of y'.*

The following are equivalent also

(a) *$\mathcal{F}_x$ is flat over $\mathcal{O}_{S,s}$,*
(b) *$\mathcal{G}_{z'}$ is flat over $\mathcal{O}_{S',s'}$, and*
(c) *$(\pi_*\mathcal{G})_{y'}$ is flat over $\mathcal{O}_{S',s'}$.*

Proof. To prove the equivalence of (1), (2), and (3) we also consider: (4) $g^*\mathcal{F}$ is flat over S in a neighbourhood of x'. We will use Lemma 37.2.3 to equate flatness over S and S' without further mention. The étale morphism g is flat and open,

see Morphisms, Lemma 28.37.13. Hence for any open neighbourhood $U' \subset X'$ of x', the image $g(U')$ is an open neighbourhood of x and the map $U' \to g(U')$ is surjective and flat. Thus (4) $\Leftrightarrow$ (1) by Morphisms, Lemma 28.26.11. Note that

$$\Gamma(X', g^*\mathcal{F}) = \Gamma(Z', \mathcal{G}) = \Gamma(Y', \pi_*\mathcal{G})$$

Hence the flatness of $g^*\mathcal{F}$, $\mathcal{G}$ and $\pi_*\mathcal{G}$ over S' are all equivalent (this uses that X', Z', Y', and S' are all affine). Some omitted topological arguments (compare More on Morphisms, Lemma 36.33.4) regarding affine neighbourhoods now show that (4) $\Leftrightarrow$ (2) $\Leftrightarrow$ (3).

To prove the equivalence of (a), (b), (c) consider the commutative diagram of local ring maps

$$\begin{array}{ccccccc} \mathcal{O}_{X',x'} & \xrightarrow{\iota} & \mathcal{O}_{Z',z'} & \xleftarrow{\alpha} & \mathcal{O}_{Y',y'} & \xleftarrow{\beta} & \mathcal{O}_{S',s'} \\ \uparrow{\scriptstyle \gamma} & & & & & & \uparrow{\scriptstyle \epsilon} \\ \mathcal{O}_{X,x} & & & \xleftarrow{\varphi} & & & \mathcal{O}_{S,s} \end{array}$$

We will use Lemma 37.2.4 to equate flatness over $\mathcal{O}_{S,s}$ and $\mathcal{O}_{S',s'}$ without further mention. The map γ is faithfully flat. Hence $\mathcal{F}_x$ is flat over $\mathcal{O}_{S,s}$ if and only if $g^*\mathcal{F}_{x'}$ is flat over $\mathcal{O}_{S',s'}$, see Algebra, Lemma 10.36.8. As $\mathcal{O}_{S',s'}$-modules the modules $g^*\mathcal{F}_{x'}$, $\mathcal{G}_{z'}$, and $\pi_*\mathcal{G}_{y'}$ are all isomorphic, see More on Morphisms, Lemma 36.33.4. This finishes the proof. □

37.4. One step dévissage

In this section we explain what is a one step dévissage of a module. A one step dévissage exist étale locally on base and target. We discuss base change, Zariski shrinking and étale localization of a one step dévissage.

Definition 37.4.1. Let S be a scheme. Let X be locally of finite type over S. Let $\mathcal{F}$ be a quasi-coherent $\mathcal{O}_X$-module of finite type. Let $s \in S$ be a point. A *one step dévissage of $\mathcal{F}/X/S$ over s* is given by morphisms of schemes over S

$$X \xleftarrow{i} Z \xrightarrow{\pi} Y$$

and a quasi-coherent $\mathcal{O}_Z$-module $\mathcal{G}$ of finite type such that

(1) X, S, Z and Y are affine,
(2) i is a closed immersion of finite presentation,
(3) $\mathcal{F} \cong i_*\mathcal{G}$,
(4) π is finite, and
(5) the structure morphism $Y \to S$ is smooth with geometrically irreducible fibres of dimension $\dim(\mathrm{Supp}(\mathcal{F}_s))$.

In this case we say $(Z, Y, i, \pi, \mathcal{G})$ is a one step dévissage of $\mathcal{F}/X/S$ over s.

Note that such a one step dévissage can only exist if X and S are affine. In the definition above we only require X to be (locally) of finite type over S and we continue working in this setting below. In [**GR71**] the authors use consistently the setup where $X \to S$ is locally of finite presentation and $\mathcal{F}$ quasi-coherent $\mathcal{O}_X$-module of finite type. The advantage of this choice is that it "makes sense" to ask for $\mathcal{F}$ to be of finite presentation as an $\mathcal{O}_X$-module, whereas in our setting it "does not make sense". Please see More on Morphisms, Section 36.38 for a discussion; the observations made there show that in our setup we may consider the condition of $\mathcal{F}$

being "locally of finite presentation relative to S", and we could work consistently with this notion. Instead however, we will rely on the results of Lemma 37.3.3 and the observations in Remark 37.6.3 to deal with this issue in an ad hoc fashion whenever it comes up.

Definition 37.4.2. Let S be a scheme. Let X be locally of finite type over S. Let $\mathcal{F}$ be a quasi-coherent $\mathcal{O}_X$-module of finite type. Let $x \in X$ be a point with image s in S. A *one step dévissage of $\mathcal{F}/X/S$ at x* is a system $(Z, Y, i, \pi, \mathcal{G}, z, y)$, where $(Z, Y, i, \pi, \mathcal{G})$ is a one step dévissage of $\mathcal{F}/X/S$ over s and

(1) $\dim_x(\mathrm{Supp}(\mathcal{F}_s)) = \dim(\mathrm{Supp}(\mathcal{F}_s))$,
(2) $z \in Z$ is a point with $i(z) = x$ and $\pi(z) = y$,
(3) we have $\pi^{-1}(\{y\}) = \{z\}$,
(4) the extension $\kappa(s) \subset \kappa(y)$ is purely transcendental.

A one step dévissage of $\mathcal{F}/X/S$ at x can only exist if X and S are affine. Condition (1) assures us that $Y \to S$ has relative dimension equal to $\dim_x(\mathrm{Supp}(\mathcal{F}_s))$ via condition (5) of Definition 37.4.1.

Lemma 37.4.3. *Let $f : X \to S$ be morphism of schemes which is locally of finite type. Let $\mathcal{F}$ be a finite type quasi-coherent $\mathcal{O}_X$-module. Let $x \in X$ with image $s = f(x)$ in S. Then there exists a commutative diagram of pointed schemes*

$$\begin{array}{ccc} (X, x) & \xleftarrow{g} & (X', x') \\ {\scriptstyle f}\downarrow & & \downarrow \\ (S, s) & \longleftarrow & (S', s') \end{array}$$

such that $(S', s') \to (S, s)$ and $(X', x') \to (X, x)$ are elementary étale neighbourhoods, and such that $g^*\mathcal{F}/X'/S'$ has a one step dévissage at x'.

Proof. This is immediate from Definition 37.4.2 and Lemma 37.3.2. □

Lemma 37.4.4. *Let S, X, $\mathcal{F}$, s be as in Definition 37.4.1. Let $(Z, Y, i, \pi, \mathcal{G})$ be a one step dévissage of $\mathcal{F}/X/S$ over s. Let $(S', s') \to (S, s)$ be any morphism of pointed schemes. Given this data let X', Z', Y', i', π' be the base changes of X, Z, Y, i, π via $S' \to S$. Let $\mathcal{F}'$ be the pullback of $\mathcal{F}$ to X' and let $\mathcal{G}'$ be the pullback of $\mathcal{G}$ to Z'. If S' is affine, then $(Z', Y', i', \pi', \mathcal{G}')$ is a one step dévissage of $\mathcal{F}'/X'/S'$ over s'.*

Proof. Fibre products of affines are affine, see Schemes, Lemma 25.17.2. Base change preserves closed immersions, morphisms of finite presentation, finite morphisms, smooth morphisms, morphisms with geometrically irreducible fibres, and morphisms of relative dimension n, see Morphisms, Lemmas 28.2.4, 28.22.4, 28.44.6, 28.35.5, 28.30.2, and More on Morphisms, Lemma 36.22.2. We have $i'_*\mathcal{G}' \cong \mathcal{F}'$ because pushforward along the finite morphism i commutes with base change, see Cohomology of Schemes, Lemma 29.5.1. We have $\dim(\mathrm{Supp}(\mathcal{F}_s)) = \dim(\mathrm{Supp}(\mathcal{F}'_{s'}))$ by Morphisms, Lemma 28.29.3 because

$$\mathrm{Supp}(\mathcal{F}_s) \times_s s' = \mathrm{Supp}(\mathcal{F}'_{s'}).$$

This proves the lemma. □

Lemma 37.4.5. *Let S, X, $\mathcal{F}$, x, s be as in Definition 37.4.2. Let $(Z, Y, i, \pi, \mathcal{G}, z, y)$ be a one step dévissage of $\mathcal{F}/X/S$ at x. Let $(S', s') \to (S, s)$ be a morphism of pointed schemes which induces an isomorphism $\kappa(s) = \kappa(s')$. Let $(Z', Y', i', \pi', \mathcal{G}')$ be as constructed in Lemma 37.4.4 and let $x' \in X'$ (resp. $z' \in Z'$, $y' \in Y'$) be the unique point mapping to both $x \in X$ (resp. $z \in Z$, $y \in Y$) and $s' \in S'$. If S' is affine, then $(Z', Y', i', \pi', \mathcal{G}', z', y')$ is a one step dévissage of $\mathcal{F}'/X'/S'$ at x'.*

Proof. By Lemma 37.4.4 $(Z', Y', i', \pi', \mathcal{G}')$ is a one step dévissage of $\mathcal{F}'/X'/S'$ over s'. Properties (1) – (4) of Definition 37.4.2 hold for $(Z', Y', i', \pi', \mathcal{G}', z', y')$ as the assumption that $\kappa(s) = \kappa(s')$ insures that the fibres $X'_{s'}$, $Z'_{s'}$, and $Y'_{s'}$ are isomorphic to X_s, Z_s, and Y_s. □

Definition 37.4.6. Let S, X, $\mathcal{F}$, x, s be as in Definition 37.4.2. Let $(Z, Y, i, \pi, \mathcal{G}, z, y)$ be a one step dévissage of $\mathcal{F}/X/S$ at x. Let us define a *standard shrinking* of this situation to be given by standard opens $S' \subset S$, $X' \subset X$, $Z' \subset Z$, and $Y' \subset Y$ such that $s \in S'$, $x \in X'$, $z \in Z'$, and $y \in Y'$ and such that

$$(Z', Y', i|_{Z'}, \pi|_{Z'}, \mathcal{G}|_{Z'}, z, y)$$

is a one step dévissage of $\mathcal{F}|_{X'}/X'/S'$ at x.

Lemma 37.4.7. *With assumption and notation as in Definition 37.4.6 we have:*

1. *If $S' \subset S$ is a standard open neighbourhood of s, then setting $X' = X_{S'}$, $Z' = Z_{S'}$ and $Y' = Y_{S'}$ we obtain a standard shrinking.*
2. *Let $W \subset Y$ be a standard open neighbourhood of y. Then there exists a standard shrinking with $Y' = W \times_S S'$.*
3. *Let $U \subset X$ be an open neighbourhood of x. Then there exists a standard shrinking with $X' \subset U$.*

Proof. Part (1) is immediate from Lemma 37.4.5 and the fact that the inverse image of a standard open under a morphism of affine schemes is a standard open, see Algebra, Lemma 10.16.4.

Let $W \subset Y$ as in (2). Because $Y \to S$ is smooth it is open, see Morphisms, Lemma 28.35.10. Hence we can find a standard open neighbourhood S' of s contained in the image of W. Then the fibres of $W_{S'} \to S'$ are nonempty open subschemes of the fibres of $Y \to S$ over S' and hence geometrically irreducible too. Setting $Y' = W_{S'}$ and $Z' = \pi^{-1}(Y')$ we see that $Z' \subset Z$ is a standard open neighbourhood of z. Let $\overline{h} \in \Gamma(Z, \mathcal{O}_Z)$ be a function such that $Z' = D(\overline{h})$. As $i : Z \to X$ is a closed immersion, we can find a function $h \in \Gamma(X, \mathcal{O}_X)$ such that $i^\sharp(h) = \overline{h}$. Take $X' = D(h) \subset X$. In this way we obtain a standard shrinking as in (2).

Let $U \subset X$ be as in (3). We may after shrinking U assume that U is a standard open. By More on Morphisms, Lemma 36.33.4 there exists a standard open $W \subset Y$ neighbourhood of y such that $\pi^{-1}(W) \subset i^{-1}(U)$. Apply (2) to get a standard shrinking X', S', Z', Y' with $Y' = W_{S'}$. Since $Z' \subset \pi^{-1}(W) \subset i^{-1}(U)$ we may replace X' by $X' \cap U$ (still a standard open as U is also standard open) without violating any of the conditions defining a standard shrinking. Hence we win. □

Lemma 37.4.8. *Let S, X, $\mathcal{F}$, x, s be as in Definition 37.4.2. Let $(Z, Y, i, \pi, \mathcal{G}, z, y)$ be a one step dévissage of $\mathcal{F}/X/S$ at x. Let*

$$\begin{array}{ccc} (Y, y) & \longleftarrow & (Y', y') \\ \downarrow & & \downarrow \\ (S, s) & \longleftarrow & (S', s') \end{array}$$

be a commutative diagram of pointed schemes such that the horizontal arrows are elementary étale neighbourhoods. Then there exists a commutative diagram

$$\begin{array}{ccccc} & & (X'', x'') & \longleftarrow & (Z'', z'') \\ & \swarrow & \downarrow & \swarrow & \downarrow \\ (X, x) & \longleftarrow & (Z, z) & (S'', s'') & \longleftarrow & (Y'', y'') \\ \downarrow & & \downarrow & \swarrow & \swarrow \\ (S, s) & \longleftarrow & (Y, y) & & \end{array}$$

of pointed schemes with the following properties:

1. *$(S'', s'') \to (S', s')$ is an elementary étale neighbourhood and the morphism $S'' \to S$ is the composition $S'' \to S' \to S$,*
2. *Y'' is an open subscheme of $Y' \times_{S'} S''$,*
3. *$Z'' = Z \times_Y Y''$,*
4. *$(X'', x'') \to (X, x)$ is an elementary étale neighbourhood, and*
5. *$(Z'', Y'', i'', \pi'', \mathcal{G}'', z'', y'')$ is a one step dévissage at x'' of the sheaf $\mathcal{F}''$.*

Here $\mathcal{F}''$ (resp. $\mathcal{G}''$) is the pullback of $\mathcal{F}$ (resp. $\mathcal{G}$) via the morphism $X'' \to X$ (resp. $Z'' \to Z$) and $i'' : Z'' \to X''$ and $\pi'' : Z'' \to Y''$ are as in the diagram.

Proof. Let $(S'', s'') \to (S', s')$ be any elementary étale neighbourhood with S'' affine. Let $Y'' \subset Y' \times_{S'} S''$ be any affine open neighbourhood containing the point $y'' = (y', s'')$. Then we obtain an affine (Z'', z'') by (3). Moreover $Z_{S''} \to X_{S''}$ is a closed immersion and $Z'' \to Z_{S''}$ is an étale morphism. Hence Lemma 37.2.1 applies and we can find an étale morphism $X'' \to X_{S'}$ of affines such that $Z'' \cong X'' \times_{X_{S'}} Z_{S'}$. Denote $i'' : Z'' \to X''$ the corresponding closed immersion. Setting $x'' = i''(z'')$ we obtain a commutative diagram as in the lemma. Properties (1), (2), (3), and (4) hold by construction. Thus it suffices to show that (5) holds for a suitable choice of $(S'', s'') \to (S', s')$ and Y''.

We first list those properties which hold for any choice of $(S'', s'') \to (S', s')$ and Y'' as in the first paragraph. As we have $Z'' = X'' \times_X Z$ by construction we see that $i''_*\mathcal{G}'' = \mathcal{F}''$ (with notation as in the statement of the lemma), see Cohomology of Schemes, Lemma 29.5.1. Set $n = \dim(\mathrm{Supp}(\mathcal{F}_s)) = \dim_x(\mathrm{Supp}(\mathcal{F}_s))$. The morphism $Y'' \to S''$ is smooth of relative dimension n (because $Y' \to S'$ is smooth of relative dimension n as the composition $Y' \to Y_{S'} \to S'$ of an étale and smooth morphism of relative dimension n and because base change preserves smooth morphisms of relative dimension n). We have $\kappa(y'') = \kappa(y)$ and $\kappa(s) = \kappa(s'')$ hence $\kappa(y'')$ is a purely transcendental extension of $\kappa(s'')$. The morphism of fibres $X''_{s''} \to X_s$ is an étale morphism of affine schemes over $\kappa(s) = \kappa(s'')$ mapping the point x'' to the

point x and pulling back $\mathcal{F}_s$ to $\mathcal{F}''_{s''}$. Hence

$$\dim(\text{Supp}(\mathcal{F}''_{s''})) = \dim(\text{Supp}(\mathcal{F}_s)) = n = \dim_x(\text{Supp}(\mathcal{F}_s)) = \dim_{x''}(\text{Supp}(\mathcal{F}''_{s''}))$$

because dimension is invariant under étale localization, see Descent, Lemma 34.17.2. As $\pi'' : Z'' \to Y''$ is the base change of π we see that π'' is finite and as $\kappa(y) = \kappa(y'')$ we see that $\pi^{-1}(\{y''\}) = \{z''\}$.

At this point we have verified all the conditions of Definition 37.4.1 except we have not verified that $Y'' \to S''$ has geometrically irreducible fibres. Of course in general this is not going to be true, and it is at this point that we will use that $\kappa(s) \subset \kappa(y)$ is purely transcendental. Namely, let $T \subset Y'_{s'}$ be the irreducible component of $Y'_{s'}$ containing $y' = (y, s')$. Note that T is an open subscheme of $Y'_{s'}$ as this is a smooth scheme over $\kappa(s')$. By Varieties, Lemma 32.5.14 we see that T is geometrically connected because $\kappa(s') = \kappa(s)$ is algebraically closed in $\kappa(y') = \kappa(y)$. As T is smooth we see that T is geometrically irreducible. Hence More on Morphisms, Lemma 36.32.3 applies and we can find an elementary étale morphism $(S'', s'') \to (S', s')$ and an affine open $Y'' \subset Y'_{S''}$ such that all fibres of $Y'' \to S''$ are geometrically irreducible and such that $T = Y''_{s''}$. After shrinking (first Y'' and then S'') we may assume that both Y'' and S'' are affine. This finishes the proof of the lemma. □

Lemma 37.4.9. *Let S, X, $\mathcal{F}$, s be as in Definition 37.4.1. Let $(Z, Y, i, \pi, \mathcal{G})$ be a one step dévissage of $\mathcal{F}/X/S$ over s. Let $\xi \in Y_s$ be the (unique) generic point. Then there exists an integer $r > 0$ and an $\mathcal{O}_Y$-module map*

$$\alpha : \mathcal{O}_Y^{\oplus r} \longrightarrow \pi_*\mathcal{G}$$

such that

$$\alpha : \kappa(\xi)^{\oplus r} \longrightarrow (\pi_*\mathcal{G})_\xi \otimes_{\mathcal{O}_{Y,\xi}} \kappa(\xi)$$

is an isomorphism. Moreover, in this case we have

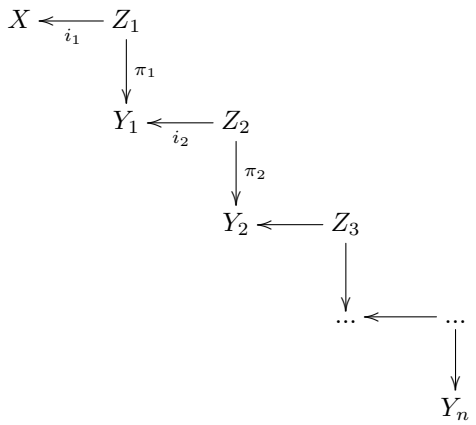
$$\dim(\mathit{Supp}(\mathit{Coker}(\alpha)_s)) < \dim(\mathit{Supp}(\mathcal{F}_s)).$$

Proof. By assumption the schemes S and Y are affine. Write $S = \text{Spec}(A)$ and $Y = \text{Spec}(B)$. As π is finite the $\mathcal{O}_Y$-module $\pi_*\mathcal{G}$ is a finite type quasi-coherent $\mathcal{O}_Y$-module. Hence $\pi_*\mathcal{G} = \widetilde{N}$ for some finite B-module N. Let $\mathfrak{p} \subset B$ be the prime ideal corresponding to ξ. To obtain α set $r = \dim_{\kappa(\mathfrak{p})} N \otimes_B \kappa(\mathfrak{p})$ and pick $x_1, \ldots, x_r \in N$ which form a basis of $N \otimes_B \kappa(\mathfrak{p})$. Take $\alpha : B^{\oplus r} \to N$ to be the map given by the formula $\alpha(b_1, \ldots, b_r) = \sum b_i x_i$. It is clear that $\alpha : \kappa(\mathfrak{p})^{\oplus r} \to N \otimes_B \kappa(\mathfrak{p})$ is an isomorphism as desired. Finally, suppose α is any map with this property. Then $N' = \text{Coker}(\alpha)$ is a finite B-module such that $N' \otimes \kappa(\mathfrak{p}) = 0$. By Nakayama's lemma (Algebra, Lemma 10.18.1) we see that $N'_\mathfrak{p} = 0$. Since the fibre Y_s is geometrically irreducible of dimension n with generic point ξ and since we have just seen that ξ is not in the support of $\text{Coker}(\alpha)$ the last assertion of the lemma holds. □

37.5. Complete dévissage

In this section we explain what is a complete dévissage of a module and prove that such exist. The material in this section is mainly bookkeeping.

Definition 37.5.1. Let S be a scheme. Let X be locally of finite type over S. Let $\mathcal{F}$ be a quasi-coherent $\mathcal{O}_X$-module of finite type. Let $s \in S$ be a point. A *complete dévissage of $\mathcal{F}/X/S$ over s* is given by a diagram



of schemes over S, finite type quasi-coherent $\mathcal{O}_{Z_k}$-modules $\mathcal{G}_k$, and $\mathcal{O}_{Y_k}$-module maps

$$\alpha_k : \mathcal{O}_{Y_k}^{\oplus r_k} \longrightarrow \pi_{k,*}\mathcal{G}_k, \quad k = 1, \ldots, n$$

satisfying the following properties:

(1) $(Z_1, Y_1, i_1, \pi_1, \mathcal{G}_1)$ is a one step dévissage of $\mathcal{F}/X/S$ over s,
(2) the map α_k induces an isomorphism

$$\kappa(\xi_k)^{\oplus r_k} \longrightarrow (\pi_{k,*}\mathcal{G}_k)_{\xi_k} \otimes_{\mathcal{O}_{Y_k,\xi_k}} \kappa(\xi_k)$$

where $\xi_k \in (Y_k)_s$ is the unique generic point,
(3) for $k = 2, \ldots, n$ the system $(Z_k, Y_k, i_k, \pi_k, \mathcal{G}_k)$ is a one step dévissage of $\mathrm{Coker}(\alpha_{k-1})/Y_{k-1}/S$ over s,
(4) $\mathrm{Coker}(\alpha_n) = 0$.

In this case we say that $(Z_k, Y_k, i_k, \pi_k, \mathcal{G}_k, \alpha_k)_{k=1,\ldots,n}$ is a complete dévissage of $\mathcal{F}/X/S$ over s.

Definition 37.5.2. Let S be a scheme. Let X be locally of finite type over S. Let $\mathcal{F}$ be a quasi-coherent $\mathcal{O}_X$-module of finite type. Let $x \in X$ be a point with image $s \in S$. A *complete dévissage of $\mathcal{F}/X/S$ at x* is given by a system

$$(Z_k, Y_k, i_k, \pi_k, \mathcal{G}_k, \alpha_k, z_k, y_k)_{k=1,\ldots,n}$$

such that $(Z_k, Y_k, i_k, \pi_k, \mathcal{G}_k, \alpha_k)$ is a complete dévissage of $\mathcal{F}/X/S$ over s, and such that

(1) $(Z_1, Y_1, i_1, \pi_1, \mathcal{G}_1, z_1, y_1)$ is a one step dévissage of $\mathcal{F}/X/S$ at x,
(2) for $k = 2, \ldots, n$ the system $(Z_k, Y_k, i_k, \pi_k, \mathcal{G}_k, z_k, y_k)$ is a one step dévissage of $\mathrm{Coker}(\alpha_{k-1})/Y_{k-1}/S$ at y_{k-1}.

Again we remark that a complete dévissage can only exist if X and S are affine.

Lemma 37.5.3. *Let S, X, $\mathcal{F}$, s be as in Definition 37.5.1. Let $(S', s') \to (S, s)$ be any morphism of pointed schemes. Let $(Z_k, Y_k, i_k, \pi_k, \mathcal{G}_k, \alpha_k)_{k=1,\ldots,n}$ be a complete dévissage of $\mathcal{F}/X/S$ over s. Given this data let $X', Z'_k, Y'_k, i'_k, \pi'_k$ be the base changes of X, Z_k, Y_k, i_k, π_k via $S' \to S$. Let $\mathcal{F}'$ be the pullback of $\mathcal{F}$ to X' and let $\mathcal{G}'_k$ be*

the pullback of $\mathcal{G}_k$ to Z'_k. Let α'_k be the pullback of α_k to Y'_k. If S' is affine, then $(Z'_k, Y'_k, i'_k, \pi'_k, \mathcal{G}'_k, \alpha'_k)_{k=1,\ldots,n}$ is a complete dévissage of $\mathcal{F}'/X'/S'$ over s'.

Proof. By Lemma 37.4.4 we know that the base change of a one step dévissage is a one step dévissage. Hence it suffices to prove that formation of $\mathrm{Coker}(\alpha_k)$ commutes with base change and that condition (2) of Definition 37.5.1 is preserved by base change. The first is true as $\pi'_{k,*}\mathcal{G}'_k$ is the pullback of $\pi_{k,*}\mathcal{G}_k$ (by Cohomology of Schemes, Lemma 29.5.1) and because $\otimes$ is right exact. The second because by the same token we have

$$(\pi_{k,*}\mathcal{G}_k)_{\xi_k} \otimes_{\mathcal{O}_{Y_k,\xi_k}} \kappa(\xi_k) \otimes_{\kappa(\xi_k)} \kappa(\xi'_k) \cong (\pi'_{k,*}\mathcal{G}'_k)_{\xi'_k} \otimes_{\mathcal{O}_{Y'_k,\xi'_k}} \kappa(\xi'_k)$$

with obvious notation. □

Lemma 37.5.4. *Let S, X, $\mathcal{F}$, x, s be as in Definition 37.5.2. Let $(S', s') \to (S, s)$ be a morphism of pointed schemes which induces an isomorphism $\kappa(s) = \kappa(s')$. Let $(Z_k, Y_k, i_k, \pi_k, \mathcal{G}_k, \alpha_k, z_k, y_k)_{k=1,\ldots,n}$ be a complete dévissage of $\mathcal{F}/X/S$ at x. Let $(Z'_k, Y'_k, i'_k, \pi'_k, \mathcal{G}'_k, \alpha'_k)_{k=1,\ldots,n}$ be as constructed in Lemma 37.5.3 and let $x' \in X'$ (resp. $z'_k \in Z'$, $y'_k \in Y'$) be the unique point mapping to both $x \in X$ (resp. $z_k \in Z_k$, $y_k \in Y_k$) and $s' \in S'$. If S' is affine, then $(Z'_k, Y'_k, i'_k, \pi'_k, \mathcal{G}'_k, \alpha'_k, z'_k, y'_k)_{k=1,\ldots,n}$ is a complete dévissage of $\mathcal{F}'/X'/S'$ at x'.*

Proof. Combine Lemma 37.5.3 and Lemma 37.4.5. □

Definition 37.5.5. Let S, X, $\mathcal{F}$, x, s be as in Definition 37.5.2. Consider a complete dévissage $(Z_k, Y_k, i_k, \pi_k, \mathcal{G}_k, \alpha_k, z_k, y_k)_{k=1,\ldots,n}$ of $\mathcal{F}/X/S$ at x. Let us define a *standard shrinking* of this situation to be given by standard opens $S' \subset S$, $X' \subset X$, $Z'_k \subset Z_k$, and $Y'_k \subset Y_k$ such that $s_k \in S'$, $x_k \in X'$, $z_k \in Z'$, and $y_k \in Y'$ and such that

$$(Z'_k, Y'_k, i'_k, \pi'_k, \mathcal{G}'_k, \alpha'_k, z_k, y_k)_{k=1,\ldots,n}$$

is a one step dévissage of $\mathcal{F}'/X'/S'$ at x where $\mathcal{G}'_k = \mathcal{G}_k|_{Z'_k}$ and $\mathcal{F}' = \mathcal{F}|_{X'}$.

Lemma 37.5.6. *With assumption and notation as in Definition 37.5.5 we have:*

1. *If $S' \subset S$ is a standard open neighbourhood of s, then setting $X' = X_{S'}$, $Z'_k = Z_{S'}$ and $Y'_k = Y_{S'}$ we obtain a standard shrinking.*
2. *Let $W \subset Y_n$ be a standard open neighbourhood of y. Then there exists a standard shrinking with $Y'_n = W \times_S S'$.*
3. *Let $U \subset X$ be an open neighbourhood of x. Then there exists a standard shrinking with $X' \subset U$.*

Proof. Part (1) is immediate from Lemmas 37.5.4 and 37.4.7.

Proof of (2). For convenience denote $X = Y_0$. We apply Lemma 37.4.7 (2) to find a standard shrinking S', Y'_{n-1}, Z'_n, Y'_n of the one step dévissage of $\mathrm{Coker}(\alpha_{n-1})/Y_{n-1}/S$ at y_{n-1} with $Y'_n = W \times_S S'$. We may repeat this procedure and find a standard shrinking $S'', Y''_{n-2}, Z''_{n-1}, Y''_{n-1}$ of the one step dévissage of $\mathrm{Coker}(\alpha_{n-2})/Y_{n-2}/S$ at y_{n-2} with $Y''_{n-1} = Y'_{n-1} \times_S S''$. We may continue in this manner until we obtain $S^{(n)}, Y_0^{(n)}, Z_1^{(n)}, Y_1^{(n)}$. At this point it is clear that we obtain our desired standard shrinking by taking $S^{(n)}$, $X^{(n)}$, $Z_k^{(n-k)} \times_S S^{(n)}$, and $Y_k^{(n-k)} \times_S S^{(n)}$ with the desired property.

Proof of (3). We use induction on the length of the complete dévissage. First we apply Lemma 37.4.7 (3) to find a standard shrinking S', X', Z'_1, Y'_1 of the one

step dévissage of $\mathcal{F}/X/S$ at x with $X' \subset U$. If $n = 1$, then we are done. If $n > 1$, then by induction we can find a standard shrinking S'', Y_1'', Z_k'', and Y_k'' of the complete dévissage $(Z_k, Y_k, i_k, \pi_k, \mathcal{G}_k, \alpha_k, z_k, y_k)_{k=2,\ldots,n}$ of $\mathrm{Coker}(\alpha_1)/Y_1/S$ at x such that $Y_1'' \subset Y_1'$. Using Lemma 37.4.7 (2) we can find $S''' \subset S'$, $X''' \subset X'$, Z_1''' and $Y_1''' = Y_1'' \times_S S'''$ which is a standard shrinking. The solution to our problem is to take

$$S''', X''', Z_1''', Y_1''', Z_2'' \times_S S''', Y_2'' \times_S S''', \ldots, Z_n'' \times_S S''', Y_n'' \times_S S'''$$

This ends the proof of the lemma. □

Proposition 37.5.7. *Let S be a scheme. Let X be locally of finite type over S. Let $x \in X$ be a point with image $s \in S$. There exists a commutative diagram*

$$\begin{array}{ccc} (X, x) & \xleftarrow{g} & (X', x') \\ \downarrow & & \downarrow \\ (S, s) & \longleftarrow & (S', s') \end{array}$$

of pointed schemes such that the horizontal arrows are elementary étale neighbourhoods and such that $g^*\mathcal{F}/X'/S'$ has a complete dévissage at x.

Proof. We prove this by induction on the integer $d = \dim_x(\mathrm{Supp}(\mathcal{F}_s))$. By Lemma 37.4.3 there exists a diagram

$$\begin{array}{ccc} (X, x) & \xleftarrow{g} & (X', x') \\ \downarrow & & \downarrow \\ (S, s) & \longleftarrow & (S', s') \end{array}$$

of pointed schemes such that the horizontal arrows are elementary étale neighbourhoods and such that $g^*\mathcal{F}/X'/S'$ has a one step dévissage at x'. The local nature of the problem implies that we may replace $(X, x) \to (S, s)$ by $(X', x') \to (S', s')$. Thus after doing so we may assume that there exists a one step dévissage $(Z_1, Y_1, i_1, \pi_1, \mathcal{G}_1)$ of $\mathcal{F}/X/S$ at x.

We apply Lemma 37.4.9 to find a map

$$\alpha_1 : \mathcal{O}_{Y_1}^{\oplus r_1} \longrightarrow \pi_{1,*}\mathcal{G}_1$$

which induces an isomorphism of vector spaces over $\kappa(\xi_1)$ where $\xi_1 \in Y_1$ is the unique generic point of the fibre of Y_1 over s. Moreover $\dim_{y_1}(\mathrm{Supp}(\mathrm{Coker}(\alpha_1)_s)) < d$. It may happen that the stalk of $\mathrm{Coker}(\alpha_1)_s$ at y_1 is zero. In this case we may shrink Y_1 by Lemma 37.4.7 (2) and assume that $\mathrm{Coker}(\alpha_1) = 0$ so we obtain a complete dévissage of length zero.

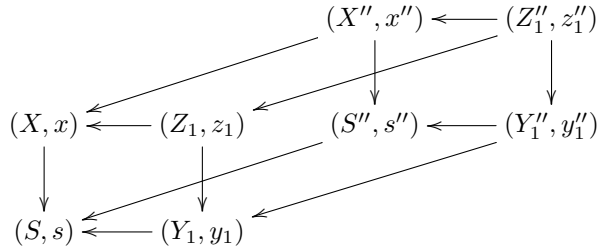
Assume now that the stalk of $\mathrm{Coker}(\alpha_1)_s$ at y_1 is not zero. In this case, by induction, there exists a commutative diagram

$$\begin{array}{ccc} (Y_1, y_1) & \xleftarrow{h} & (Y_1', y_1') \\ \downarrow & & \downarrow \\ (S, s) & \longleftarrow & (S', s') \end{array} \tag{37.5.7.1}$$

of pointed schemes such that the horizontal arrows are elementary étale neighbourhoods and such that $h^*\mathrm{Coker}(\alpha_1)/Y_1'/S'$ has a complete dévissage

$$(Z_k, Y_k, i_k, \pi_k, \mathcal{G}_k, \alpha_k, z_k, y_k)_{k=2,\ldots,n}$$

at y_1'. (In particular $i_2 : Z_2 \to Y_1'$ is a closed immersion into Y_2'.) At this point we apply Lemma 37.4.8 to $S, X, \mathcal{F}, x, s$, the system $(Z_1, Y_1, i_1, \pi_1, \mathcal{G}_1)$ and diagram (37.5.7.1). We obtain a diagram



with all the properties as listed in the referenced lemma. In particular $Y_1'' \subset Y_1' \times_{S'} S''$. Set $X_1 = Y_1' \times_{S'} S''$ and let $\mathcal{F}_1$ denote the pullback of $\mathrm{Coker}(\alpha_1)$. By Lemma 37.5.4 the system

$$(Z_k \times_{S'} S'', Y_k \times_{S'} S'', i_k'', \pi_k'', \mathcal{G}_k'', \alpha_k'', z_k'', y_k'')_{k=2,\ldots,n} \tag{37.5.7.2}$$

is a complete dévissage of $\mathcal{F}_1$ to X_1. Again, the nature of the problem allows us to replace $(X, x) \to (S, s)$ by $(X'', x'') \to (S'', s'')$. In this we see that we may assume:

(a) There exists a one step dévissage $(Z_1, Y_1, i_1, \pi_1, \mathcal{G}_1)$ of $\mathcal{F}/X/S$ at x,
(b) there exists an $\alpha_1 : \mathcal{O}_{Y_1}^{\oplus r_1} \to \pi_{1,*}\mathcal{G}_1$ such that $\alpha \otimes \kappa(\xi_1)$ is an isomorphism,
(c) $Y_1 \subset X_1$ is open, $y_1 = x_1$, and $\mathcal{F}_1|_{Y_1} \cong \mathrm{Coker}(\alpha_1)$, and
(d) there exists a complete dévissage $(Z_k, Y_k, i_k, \pi_k, \mathcal{G}_k, \alpha_k, z_k, y_k)_{k=2,\ldots,n}$ of $\mathcal{F}_1/X_1/S$ at x_1.

To finish the proof all we have to do is shrink the one step dévissage and the complete dévissage such that they fit together to a complete dévissage. (We suggest the reader do this on their own using Lemmas 37.4.7 and 37.5.6 instead of reading the proof that follows.) Since $Y_1 \subset X_1$ is an open neighbourhood of x_1 we may apply Lemma 37.5.6 (3) to find a standard shrinking $S', X_1', Z_2', Y_2', \ldots, Y_n'$ of the datum (d) so that $X_1' \subset Y_1$. Note that X_1' is also a standard open of the affine scheme Y_1. Next, we shrink the datum (a) as follows: first we shrink the base S to S', see Lemma 37.4.7 (1) and then we shrink the result to S'', X'', Z_1'', Y_1'' using Lemma 37.4.7 (2) such that eventually $Y_1'' = X_1' \times_S S''$ and $S'' \subset S'$. Then we see that

$$Z_1'', Y_1'', Z_2' \times_{S'} S'', Y_2' \times_{S'} S'', \ldots, Y_n' \times_{S'} S''$$

gives the complete dévissage we were looking for. □

Some more bookkeeping gives the following consequence.

Lemma 37.5.8. *Let $X \to S$ be a finite type morphism of schemes. Let $\mathcal{F}$ be a finite type quasi-coherent $\mathcal{O}_X$-module. Let $s \in S$ be a point. There exists an elementary étale neighbourhood $(S', s') \to (S, s)$ and étale morphisms $h_i : Y_i \to X_{S'}$, $i = 1, \ldots, n$ such that for each i there exists a complete dévissage of $\mathcal{F}_i/Y_i/S'$ over s', where $\mathcal{F}_i$ is the pullback of $\mathcal{F}$ to Y_i and such that $X_s = (X_{S'})_{s'} \subset \bigcup h_i(Y_i)$.*

Proof. For every point $x \in X_s$ we can find a diagram

$$\begin{array}{ccc} (X, x) & \xleftarrow[g]{} & (X', x') \\ \downarrow & & \downarrow \\ (S, s) & \longleftarrow & (S', s') \end{array}$$

of pointed schemes such that the horizontal arrows are elementary étale neighbourhoods and such that $g^*\mathcal{F}/X'/S'$ has a complete dévissage at x'. As $X \to S$ is of finite type the fibre X_s is quasi-compact, and since each $g : X' \to X$ as above is open we can cover X_s by a finite union of $g(X'_{s'})$. Thus we can find a finite family of such diagrams

$$\begin{array}{ccc} (X, x) & \xleftarrow[g_i]{} & (X'_i, x'_i) \\ \downarrow & & \downarrow \\ (S, s) & \longleftarrow & (S'_i, s'_i) \end{array} \qquad i = 1, \ldots, n$$

such that $X_s = \bigcup g_i(X'_i)$. Set $S' = S'_1 \times_S \ldots \times_S S'_n$ and let $Y_i = X_i \times_{S'_i} S'$ be the base change of X'_i to S'. By Lemma 37.5.3 we see that the pullback of $\mathcal{F}$ to Y_i has a complete dévissage over s and we win. □

37.6. Translation into algebra

It may be useful to spell out algebraically what it means to have a complete dévissage. We introduce the following notion (which is not that useful so we give it an impossibly long name).

Definition 37.6.1. Let $R \to S$ be a ring map. Let $\mathfrak{q}$ be a prime of S lying over the prime $\mathfrak{p}$ of R. A *elementary étale localization of the ring map $R \to S$ at $\mathfrak{q}$* is given by a commutative diagram of rings and accompanying primes

$$\begin{array}{ccc} S & \longrightarrow & S' \\ \uparrow & & \uparrow \\ R & \longrightarrow & R' \end{array} \qquad \begin{array}{ccc} \mathfrak{q} & \longrightarrow & \mathfrak{q}' \\ \downarrow & & \downarrow \\ \mathfrak{p} & \longrightarrow & \mathfrak{p}' \end{array}$$

such that $R \to R'$ and $S \to S'$ are étale ring maps and $\kappa(\mathfrak{p}) = \kappa(\mathfrak{p}')$ and $\kappa(\mathfrak{q}) = \kappa(\mathfrak{q}')$.

Definition 37.6.2. Let $R \to S$ be a finite type ring map. Let $\mathfrak{r}$ be a prime of R. Let N be a finite S-module. A *complete dévissage of $N/S/R$ over $\mathfrak{r}$* is given by R-algebra maps

$$\begin{array}{ccccccccccc} & & A_1 & & A_2 & & \ldots & & A_n & & \\ & \nearrow & & \nwarrow & & \nearrow & & \nwarrow & & \nearrow & \\ S & & & & B_1 & & \ldots & & \ldots & & B_n \end{array}$$

finite A_i-modules M_i and B_i-module maps $\alpha_i : B_i^{\oplus r_i} \to M_i$ such that

(1) $S \to A_1$ is surjective and of finite presentation,
(2) $B_i \to A_{i+1}$ is surjective and of finite presentation,
(3) $B_i \to A_i$ is finite,
(4) $R \to B_i$ is smooth with geometrically irreducible fibres,

(5) $N \cong M_1$ as S-modules,
(6) $\mathrm{Coker}(\alpha_i) \cong M_{i+1}$ as B_i-modules,
(7) $\alpha_i : \kappa(\mathfrak{p}_i)^{\oplus r_i} \to M_i \otimes_{B_i} \kappa(\mathfrak{p}_i)$ is an isomorphism where $\mathfrak{p}_i = \mathfrak{r}B_i$, and
(8) $\mathrm{Coker}(\alpha_n) = 0$.

In this situation we say that $(A_i, B_i, M_i, \alpha_i)_{i=1,\ldots,n}$ is a complete dévissage of $N/S/R$ over $\mathfrak{r}$.

Remark 37.6.3. Note that the R-algebras B_i for all i and A_i for $i \geq 2$ are of finite presentation over R. If S is of finite presentation over R, then it is also the case that A_1 is of finite presentation over R. In this case all the ring maps in the complete dévissage are of finite presentation. See Algebra, Lemma 10.6.2. Still assuming S of finite presentation over R the following are equivalent

(1) M is of finite presentation over S,
(2) M_1 is of finite presentation over A_1,
(3) M_1 is of finite presentation over B_1,
(4) each M_i is of finite presentation both as an A_i-module and as a B_i-module.

The equivalences (1) $\Leftrightarrow$ (2) and (2) $\Leftrightarrow$ (3) follow from Algebra, Lemma 10.7.4. If M_1 is finitely presented, so is $\mathrm{Coker}(\alpha_1)$ (see Algebra, Lemma 10.5.3) and hence M_2, etc.

Definition 37.6.4. Let $R \to S$ be a finite type ring map. Let $\mathfrak{q}$ be a prime of S lying over the prime $\mathfrak{r}$ of R. Let N be a finite S-module. A *complete dévissage of $N/S/R$ at $\mathfrak{q}$* is given by a complete dévissage $(A_i, B_i, M_i, \alpha_i)_{i=1,\ldots,n}$ of $N/S/R$ over $\mathfrak{r}$ and prime ideals $\mathfrak{q}_i \subset B_i$ lying over $\mathfrak{r}$ such that

(1) $\kappa(\mathfrak{r}) \subset \kappa(\mathfrak{q}_i)$ is purely transcendental,
(2) there is a unique prime $\mathfrak{q}'_i \subset A_i$ lying over $\mathfrak{q}_i \subset B_i$,
(3) $\mathfrak{q} = \mathfrak{q}'_1 \cap S$ and $\mathfrak{q}_i = \mathfrak{q}'_{i+1} \cap A_i$,
(4) $R \to B_i$ has relative dimension $\dim_{\mathfrak{q}_i}(\mathrm{Supp}(M_i \otimes_R \kappa(\mathfrak{r})))$.

Remark 37.6.5. Let $A \to B$ be a finite type ring map and let N be a finite B-module. Let $\mathfrak{q}$ be a prime of B lying over the prime $\mathfrak{r}$ of A. Set $X = \mathrm{Spec}(B)$, $S = \mathrm{Spec}(A)$ and $\mathcal{F} = \widetilde{N}$ on X. Let x be the point corresponding to $\mathfrak{q}$ and let $s \in S$ be the point corresponding to $\mathfrak{p}$. Then

(1) if there exists a complete dévissage of $\mathcal{F}/X/S$ over s then there exists a complete dévissage of $N/B/A$ over $\mathfrak{p}$, and
(2) there exists a complete dévissage of $\mathcal{F}/X/S$ at x if and only if there exists a complete dévissage of $N/B/A$ at $\mathfrak{q}$.

There is just a small twist in that we omitted the condition on the relative dimension in the formulation of "a complete dévissage of $N/B/A$ over $\mathfrak{p}$" which is why the implication in (1) only goes in one direction. The notion of a complete dévissage at $\mathfrak{q}$ does have this condition built in. In any case we will only use that existence for $\mathcal{F}/X/S$ implies the existence for $N/B/A$.

Lemma 37.6.6. *Let $R \to S$ be a finite type ring map. Let M be a finite S-module. Let $\mathfrak{q}$ be a prime ideal of S. There exists an elementary étale localization $R' \to S', \mathfrak{q}', \mathfrak{p}'$ of the ring map $R \to S$ at $\mathfrak{q}$ such that there exists a complete dévissage of $(M \otimes_S S')/S'/R'$ at $\mathfrak{q}'$.*

Proof. This is a reformulation of Proposition 37.5.7 via Remark 37.6.5 $\square$

37.7. Localization and universally injective maps

Lemma 37.7.1. *Let $R \to S$ be a ring map. Let N be a S-module. Assume*

(1) *R is a local ring with maximal ideal $\mathfrak{m}$,*
(2) *$\overline{S} = S/\mathfrak{m}S$ is Noetherian, and*
(3) *$\overline{N} = N/\mathfrak{m}_R N$ is a finite $\overline{S}$-module.*

Let $\Sigma \subset S$ be the multiplicative subset of elements which are not a zerodivisor on $\overline{N}$. Then $\Sigma^{-1}S$ is a semi-local ring whose spectrum consists of primes $\mathfrak{q} \subset S$ contained in an element of $Ass_S(\overline{N})$. Moreover, any maximal ideal of $\Sigma^{-1}S$ corresponds to an associated prime of $\overline{N}$ over $\overline{S}$.

Proof. Note that $\mathrm{Ass}_S(\overline{N}) = \mathrm{Ass}_{\overline{S}}(\overline{N})$, see Algebra, Lemma 10.60.13. This is a finite set by Algebra, Lemma 10.60.5. Say $\{\mathfrak{q}_1, \ldots, \mathfrak{q}_r\} = \mathrm{Ass}_S(\overline{N})$. We have $\Sigma = S \setminus (\bigcup \mathfrak{q}_i)$ by Algebra, Lemma 10.60.9. By the description of $\mathrm{Spec}(\Sigma^{-1}S)$ in Algebra, Lemma 10.16.5 and by Algebra, Lemma 10.14.3 we see that the primes of $\Sigma^{-1}S$ correspond to the primes of S contained in one of the $\mathfrak{q}_i$. Hence the maximal ideals of $\Sigma^{-1}S$ correspond one-to-one with the maximal (w.r.t. inclusion) elements of the set $\{\mathfrak{q}_1, \ldots, \mathfrak{q}_r\}$. This proves the lemma. □

Lemma 37.7.2. *Assumption and notation as in Lemma 37.7.1. Assume moreover that*

(1) *S is local and $R \to S$ is a local homomorphism,*
(2) *S is essentially of finite presentation over R,*
(3) *N is finitely presented over S, and*
(4) *N is flat over R.*

Then each $s \in \Sigma$ defines a universally injective R-module map $s : N \to N$, and the map $N \to \Sigma^{-1}N$ is R-universally injective.

Proof. By Algebra, Lemma 10.122.4 the sequence $0 \to N \to N \to N/sN \to 0$ is exact and N/sN is flat over R. This implies that $s : N \to N$ is universally injective, see Algebra, Lemma 10.36.11. The map $N \to \Sigma^{-1}N$ is universally injective as the directed colimit of the maps $s : N \to N$. □

Lemma 37.7.3. *Let $R \to S$ be a ring map. Let N be an S-module. Let $S \to S'$ be a ring map. Assume*

(1) *$R \to S$ is a local homomorphism of local rings*
(2) *S is essentially of finite presentation over R,*
(3) *N is of finite presentation over S,*
(4) *N is flat over R,*
(5) *$S \to S'$ is flat, and*
(6) *the image of $\mathrm{Spec}(S') \to \mathrm{Spec}(S)$ contains all primes $\mathfrak{q}$ of S lying over $\mathfrak{m}_R$ such that $\mathfrak{q}$ is an associated prime of $N/\mathfrak{m}_R N$.*

Then $N \to N \otimes_S S'$ is R-universally injective.

Proof. Set $N' = N \otimes_R S'$. Consider the commutative diagram

$$\begin{array}{ccc} N & \longrightarrow & N' \\ \downarrow & & \downarrow \\ \Sigma^{-1}N & \longrightarrow & \Sigma^{-1}N' \end{array}$$

where $\Sigma \subset S$ is the set of elements which are not a zerodivisor on $N/\mathfrak{m}_R N$. If we can show that the map $N \to \Sigma^{-1}N'$ is universally injective, then $N \to N'$ is too (see Algebra, Lemma 10.77.10).

By Lemma 37.7.1 the ring $\Sigma^{-1}S$ is a semi-local ring whose maximal ideals correspond to associated primes of $N/\mathfrak{m}_R N$. Hence the image of $\mathrm{Spec}(\Sigma^{-1}S') \to \mathrm{Spec}(\Sigma^{-1}S)$ contains all these maximal ideals by assumption. By Algebra, Lemma 10.36.15 the ring map $\Sigma^{-1}S \to \Sigma^{-1}S'$ is faithfully flat. Hence $\Sigma^{-1}N \to \Sigma^{-1}N'$, which is the map

$$N \otimes_S \Sigma^{-1}S \longrightarrow N \otimes_S \Sigma^{-1}S'$$

is universally injective, see Algebra, Lemmas 10.77.11 and 10.77.8. Finally, we apply Lemma 37.7.2 to see that $N \to \Sigma^{-1}N$ is universally injective. As the composition of universally injective module maps is universally injective (see Algebra, Lemma 10.77.9) we conclude that $N \to \Sigma^{-1}N'$ is universally injective and we win. □

Lemma 37.7.4. *Let $R \to S$ be a ring map. Let N be an S-module. Let $S \to S'$ be a ring map. Assume*

1. *$R \to S$ is of finite presentation and N is of finite presentation over S,*
2. *N is flat over R,*
3. *$S \to S'$ is flat, and*
4. *the image of $\mathrm{Spec}(S') \to \mathrm{Spec}(S)$ contains all primes $\mathfrak{q}$ such that $\mathfrak{q}$ is an associated prime of $N \otimes_R \kappa(\mathfrak{p})$ where $\mathfrak{p}$ is the inverse image of $\mathfrak{q}$ in R.*

Then $N \to N \otimes_S S'$ is R-universally injective.

Proof. By Algebra, Lemma 10.77.12 it suffices to show that $N_\mathfrak{q} \to (N \otimes_R S')_\mathfrak{q}$ is a $R_\mathfrak{p}$-universally injective for any prime $\mathfrak{q}$ of S lying over $\mathfrak{p}$ in R. Thus we may apply Lemma 37.7.3 to the ring maps $R_\mathfrak{p} \to S_\mathfrak{q} \to S'_\mathfrak{q}$ and the module $N_\mathfrak{q}$. □

The reader may want to compare the following lemma to Algebra, Lemmas 10.93.1 and 10.122.4. In each case the conclusion is that the map $u : M \to N$ is universally injective with flat cokernel.

Lemma 37.7.5. *Let $(R, \mathfrak{m})$ be a local ring. Let $u : M \to N$ be an R-module map. If M is a projective R-module, N is a flat R-module, and $\overline{u} : M/\mathfrak{m}M \to N/\mathfrak{m}N$ is injective then u is universally injective.*

Proof. By Algebra, Theorem 10.80.4 the module M is free. If we show the result holds for every finitely generated direct summand of M, then the lemma follows. Hence we may assume that M is finite free. Write $N = \mathrm{colim}_i N_i$ as a directed colimit of finite free modules, see Algebra, Theorem 10.76.4. Note that $u : M \to N$ factors through N_i for some i (as M is finite free). Denote $u_i : M \to N_i$ the corresponding R-module map. As $\overline{u}$ is injective we see that $\overline{u_i} : M/\mathfrak{m}M \to N_i/\mathfrak{m}N_i$ is injective and remains injective on composing with the maps $N_i/\mathfrak{m}N_i \to N_{i'}/\mathfrak{m}N_{i'}$ for all $i' \geq i$. As M and $N_{i'}$ are finite free over the local ring R this implies that $M \to N_{i'}$ is a split injection for all $i' \geq i$. Hence for any R-module Q we see that $M \otimes_R Q \to N_{i'} \otimes_R Q$ is injective for all $i' \geq i$. As $- \otimes_R Q$ commutes with colimits we conclude that $M \otimes_R Q \to N_{i'} \otimes_R Q$ is injective as desired. □

Lemma 37.7.6. *Assumption and notation as in Lemma 37.7.1. Assume moreover that N is projective as an R-module. Then each $s \in \Sigma$ defines a universally injective R-module map $s : N \to N$, and the map $N \to \Sigma^{-1}N$ is R-universally injective.*

Proof. Pick $s \in \Sigma$. By Lemma 37.7.5 the map $s : N \to N$ is universally injective. The map $N \to \Sigma^{-1}N$ is universally injective as the directed colimit of the maps $s : N \to N$. □

37.8. Completion and Mittag-Leffler modules

Lemma 37.8.1. *Let R be a ring. Let $I \subset R$ be an ideal. Let A be a set. Assume R is Noetherian and complete with respect to I. The completion $(\bigoplus_{\alpha\in A} R)^\wedge$ is flat and Mittag-Leffler.*

Proof. By More on Algebra, Lemma 15.18.1 the map $(\bigoplus_{\alpha\in A} R)^\wedge \to \prod_{\alpha\in A} R$ is universally injective. Thus, by Algebra, Lemmas 10.77.7 and 10.84.6 it suffices to show that $\prod_{\alpha\in A} R$ is flat and Mittag-Leffler. By Algebra, Proposition 10.85.5 (and Algebra, Lemma 10.85.4) we see that $\prod_{\alpha\in A} R$ is flat. Thus we conclude because a product of copies of R is Mittag-Leffler, see Algebra, Lemma 10.86.3. □

Lemma 37.8.2. *Let R be a ring. Let $I \subset R$ be an ideal. Let M be an R-module. Assume*

1. *R is Noetherian and I-adically complete,*
2. *M is flat over R, and*
3. *M/IM is a projective R/I-module.*

Then the I-adic completion $M^\wedge$ is a flat Mittag-Leffler R-module.

Proof. Choose a surjection $F \to M$ where F is a free R-module. By Algebra, Lemma 10.91.20 the module $M^\wedge$ is a direct summand of the module $F^\wedge$. Hence it suffices to prove the lemma for F. In this case the lemma follows from Lemma 37.8.1. □

In Lemmas 37.8.3 and 37.8.4 the assumption that S be Noetherian holds if $R \to S$ is of finite type, see Algebra, Lemma 10.29.1.

Lemma 37.8.3. *Let R be a ring. Let $I \subset R$ be an ideal. Let $R \to S$ be a ring map, and N an S-module. Assume*

1. *R is a Noetherian ring,*
2. *S is a Noetherian ring,*
3. *N is a finite S-module, and*
4. *for any finite R-module Q, any $\mathfrak{q} \in Ass_S(Q \otimes_R N)$ satisfies $IS + \mathfrak{q} \neq S$.*

Then the map $N \to N^\wedge$ of N into the I-adic completion of N is universally injective as a map of R-modules.

Proof. We have to show that for any finite R-module Q the map $Q \otimes_R N \to Q \otimes_R N^\wedge$ is injective, see Algebra, Theorem 10.77.3. As there is a canonical map $Q \otimes_R N^\wedge \to (Q \otimes_R N)^\wedge$ it suffices to prove that the canonical map $Q \otimes_R N \to (Q \otimes_R N)^\wedge$ is injective. Hence we may replace N by $Q \otimes_R N$ and it suffices to prove the injectivity for the map $N \to N^\wedge$.

Let $K = \mathrm{Ker}(N \to N^\wedge)$. It suffices to show that $K_\mathfrak{q} = 0$ for $\mathfrak{q} \in \mathrm{Ass}(N)$ as N is a submodule of $\prod_{\mathfrak{q}\in\mathrm{Ass}(N)} N_\mathfrak{q}$, see Algebra, Lemma 10.60.18. Pick $\mathfrak{q} \in \mathrm{Ass}(N)$. By the last assumption we see that there exists a prime $\mathfrak{q}' \supset IS + \mathfrak{q}$. Since $K_\mathfrak{q}$ is a localization of $K_{\mathfrak{q}'}$ it suffices to prove the vanishing of $K_{\mathfrak{q}'}$. Note that $K = \bigcap I^n N$, hence $K_{\mathfrak{q}'} \subset \bigcap I^n N_{\mathfrak{q}'}$. Hence $K_{\mathfrak{q}'} = 0$ by Algebra, Lemma 10.47.6. □

Lemma 37.8.4. *Let R be a ring. Let $I \subset R$ be an ideal. Let $R \to S$ be a ring map, and N an S-module. Assume*

(1) *R is a Noetherian ring,*
(2) *S is a Noetherian ring,*
(3) *N is a finite S-module,*
(4) *N is flat over R, and*
(5) *for any prime $\mathfrak{q} \subset S$ which is an associated prime of $N \otimes_R \kappa(\mathfrak{p})$ where $\mathfrak{p} = R \cap \mathfrak{q}$ we have $IS + \mathfrak{q} \neq S$.*

Then the map $N \to N^\wedge$ of N into the I-adic completion of N is universally injective as a map of R-modules.

Proof. This follows from Lemma 37.8.3 because Algebra, Lemma 10.62.5 and Remark 10.62.6 guarantee that the set of associated primes of tensor products $N \otimes_R Q$ are contained in the set of associated primes of the modules $N \otimes_R \kappa(\mathfrak{p})$. □

37.9. Projective modules

The following lemma can be used to prove projectivity by Noetherian induction on the base, see Lemma 37.9.2.

Lemma 37.9.1. *Let R be a ring. Let $I \subset R$ be an ideal. Let $R \to S$ be a ring map, and N an S-module. Assume*

(1) *R is Noetherian and I-adically complete,*
(2) *$R \to S$ is of finite type,*
(3) *N is a finite S-module,*
(4) *N is flat over R,*
(5) *N/IN is projective as a R/I-module, and*
(6) *for any prime $\mathfrak{q} \subset S$ which is an associated prime of $N \otimes_R \kappa(\mathfrak{p})$ where $\mathfrak{p} = R \cap \mathfrak{q}$ we have $IS + \mathfrak{q} \neq S$.*

Then N is projective as an R-module.

Proof. By Lemma 37.8.4 the map $N \to N^\wedge$ is universally injective. By Lemma 37.8.2 the module $N^\wedge$ is Mittag-Leffler. By Algebra, Lemma 10.84.6 we conclude that N is Mittag-Leffler. Hence N is countably generated, flat and Mittag-Leffler as an R-module, whence projective by Algebra, Lemma 10.88.1. □

Lemma 37.9.2. *Let R be a ring. Let $R \to S$ be a ring map. Assume*

(1) *R is Noetherian,*
(2) *$R \to S$ is of finite type and flat, and*
(3) *every fibre ring $S \otimes_R \kappa(\mathfrak{p})$ is geometrically integral over $\kappa(\mathfrak{p})$.*

Then S is projective as an R-module.

Proof. Consider the set

$$\{I \subset R \mid S/IS \text{ not projective as } R/I\text{-module}\}$$

We have to show this set is empty. To get a contradiction assume it is nonempty. Then it contains a maximal element I. Let $J = \sqrt{I}$ be its radical. If $I \neq J$, then S/JS is projective as a R/J-module, and S/IS is flat over R/I and J/I is a nilpotent ideal in R/I. Applying Algebra, Lemma 10.72.5 we see that S/IS is a projective R/I-module, which is a contradiction. Hence we may assume that I is

a radical ideal. In other words we are reduced to proving the lemma in case R is a reduced ring and S/IS is a projective R/I-module for every nonzero ideal I of R.

Assume R is a reduced ring and S/IS is a projective R/I-module for every nonzero ideal I of R. By generic flatness, Algebra, Lemma 10.112.1 (applied to a localization R_g which is a domain) or the more general Algebra, Lemma 10.112.7 there exists a nonzero $f \in R$ such that S_f is free as an R_f-module. Denote $R^\wedge = \lim R/(f^n)$ the (f)-adic completion of R. Note that the ring map

$$R \longrightarrow R_f \times R^\wedge$$

is a faithfully flat ring map, see Algebra, Lemma 10.91.3. Hence by faithfully flat descent of projectivity, see Algebra, Theorem 10.90.5 it suffices to prove that $S \otimes_R R^\wedge$ is a projective $R^\wedge$-module. To see this we will use the criterion of Lemma 37.9.1. First of all, note that $S/fS = (S \otimes_R R^\wedge)/f(S \otimes_R R^\wedge)$ is a projective $R/(f)$-module and that $S \otimes_R R^\wedge$ is flat and of finite type over $R^\wedge$ as a base change of such. Next, suppose that $\mathfrak{p}^\wedge$ is a prime ideal of $R^\wedge$. Let $\mathfrak{p} \subset R$ be the corresponding prime of R. As $R \to S$ has geometrically integral fibre rings, the same is true for the fibre rings of any base change. Hence $\mathfrak{q}^\wedge = \mathfrak{p}^\wedge(S \otimes_R R^\wedge)$, is a prime ideals lying over $\mathfrak{p}^\wedge$ and it is the unique associated prime of $S \otimes_R \kappa(\mathfrak{p}^\wedge)$. Thus we win if $f(S \otimes_R R^\wedge) + \mathfrak{q}^\wedge \neq S \otimes_R R^\wedge$. This is true because $\mathfrak{p}^\wedge + fR^\wedge \neq R^\wedge$ as f lies in the radical of the f-adically complete ring $R^\wedge$ and because $R^\wedge \to S \otimes_R R^\wedge$ is surjective on spectra as its fibres are nonempty (irreducible spaces are nonempty). □

Lemma 37.9.3. *Let R be a ring. Let $R \to S$ be a ring map. Assume*

(1) *$R \to S$ is of finite presentation and flat, and*
(2) *every fibre ring $S \otimes_R \kappa(\mathfrak{p})$ is geometrically integral over $\kappa(\mathfrak{p})$.*

Then S is projective as an R-module.

Proof. We can find a cocartesian diagram of rings

$$\begin{array}{ccc} S_0 & \longrightarrow & S \\ \uparrow & & \uparrow \\ R_0 & \longrightarrow & R \end{array}$$

such that R_0 is of finite type over $\mathbf{Z}$, the map $R_0 \to S_0$ is of finite type and flat with geometrically integral fibres, see More on Morphisms, Lemmas 36.26.4, 36.26.6, 36.26.7, and 36.26.11. By Lemma 37.9.2 we see that S_0 is a projective R_0-module. Hence $S = S_0 \otimes_{R_0} R$ is a projective R-module, see Algebra, Lemma 10.89.1. □

Remark 37.9.4. Lemma 37.9.3 is a key step in the development of results in this chapter. The analogue of this lemma in [**GR71**] is [**GR71**, I Proposition 3.3.1]: If $R \to S$ is smooth with geometrically integral fibres, then S is projective as an R-module. This is a special case of Lemma 37.9.3, but as we will later improve on this lemma anyway, we do not gain much from having a stronger result at this point. We briefly sketch the proof of this as it is given in [**GR71**].

(1) First reduce to the case where R is Noetherian as above.
(2) Since projectivity descends through faithfully flat ring maps, see Algebra, Theorem 10.90.5 we may work locally in the fppf topology on R, hence

we may assume that $R \to S$ has a section $\sigma : S \to R$. (Just by the usual trick of base changing to S.) Set $I = \mathrm{Ker}(S \to R)$.

(3) Localizing a bit more on R we may assume that I/I^2 is a free R-module and that the completion $S^\wedge$ of S with respect to I is isomorphic to $R[[t_1, \ldots, t_n]]$, see Morphisms, Lemma 28.35.20. Here we are using that $R \to S$ is smooth.

(4) To prove that S is projective as an R-module, it suffices to prove that S is flat, countably generated and Mittag-Leffler as an R-module, see Algebra, Lemma 10.88.1. The first two properties are evident. Thus it suffices to prove that S is Mittag-Leffler as an R-module. By Algebra, Lemma 10.86.4 the module $R[[t_1, \ldots, t_n]]$ is Mittag-Leffler over R. Hence Algebra, Lemma 10.84.6 shows that it suffices to show that the $S \to S^\wedge$ is universally injective as a map of R-modules.

(5) Apply Lemma 37.7.4 to see that $S \to S^\wedge$ is R-universally injective. Namely, as $R \to S$ has geometrically integral fibres, any associated point of any fibre ring is just the generic point of the fibre ring which is in the image of $\mathrm{Spec}(S^\wedge) \to \mathrm{Spec}(S)$.

There is an analogy between the proof as sketched just now, and the development of the arguments leading to the proof of Lemma 37.9.3. In both a completion plays an essential role, and both times the assumption of having geometrically integral fibres assures one that the map from S to the completion of S is R-universally injective.

37.10. Flat finite type modules, Part I

In some cases given a ring map $R \to S$ of finite presentation and a finite S-module N the flatness of N over R implies that N is of finite presentation. In this section we prove this is true "pointwise". We remark that the first proof of Proposition 37.10.3 uses the geometric results of Section 37.3 but not the existence of a complete dévissage.

Lemma 37.10.1. *Let $(R, \mathfrak{m})$ be a local ring. Let $R \to S$ be a finitely presented flat ring map with geometrically integral fibres. Write $\mathfrak{p} = \mathfrak{m}S$. Let $\mathfrak{q} \subset S$ be a prime ideal lying over $\mathfrak{m}$. Let N be a finite S-module. There exist $r \geq 0$ and an S-module map*

$$\alpha : S^{\oplus r} \longrightarrow N$$

such that $\alpha : \kappa(\mathfrak{p})^{\oplus r} \to N \otimes_S \kappa(\mathfrak{p})$ is an isomorphism. For any such α the following are equivalent:

(1) *$N_\mathfrak{q}$ is R-flat,*
(2) *α is R-universally injective and $\mathrm{Coker}(\alpha)_\mathfrak{q}$ is R-flat,*
(3) *α is injective and $\mathrm{Coker}(\alpha)_\mathfrak{q}$ is R-flat,*
(4) *$\alpha_\mathfrak{p}$ is an isomorphism and $\mathrm{Coker}(\alpha)_\mathfrak{q}$ is R-flat, and*
(5) *$\alpha_\mathfrak{q}$ is injective and $\mathrm{Coker}(\alpha)_\mathfrak{q}$ is R-flat.*

Proof. To obtain α set $r = \dim_{\kappa(\mathfrak{p})} N \otimes_S \kappa(\mathfrak{p})$ and pick $x_1, \ldots, x_r \in N$ which form a basis of $N \otimes_S \kappa(\mathfrak{p})$. Define $\alpha(s_1, \ldots, s_r) = \sum s_i x_i$. This proves the existence.

Fix an α. The most interesting implication is (1) $\Rightarrow$ (2) which we prove first. Assume (1). Because $S/\mathfrak{m}S$ is a domain with fraction field $\kappa(\mathfrak{p})$ we see that $(S/\mathfrak{m}S)^{\oplus r} \to N_\mathfrak{p}/\mathfrak{m}N_\mathfrak{p} = N \otimes_S \kappa(\mathfrak{p})$ is injective. Hence by Lemmas 37.7.5 and

37.9.3. the map $S^{\oplus r} \to N_{\mathfrak{p}}$ is R-universally injective. It follows that $S^{\oplus r} \to N$ is R-universally injective, see Algebra, Lemma 10.77.10. Then also the localization $\alpha_{\mathfrak{q}}$ is R-universally injective, see Algebra, Lemma 10.77.13. We conclude that $\mathrm{Coker}(\alpha)_{\mathfrak{q}}$ is R-flat by Algebra, Lemma 10.77.7.

The implication (2) $\Rightarrow$ (3) is immediate. If (3) holds, then $\alpha_{\mathfrak{p}}$ is injective as a localization of an injective module map. By Nakayama's lemma (Algebra, Lemma 10.18.1) $\alpha_{\mathfrak{p}}$ is surjective too. Hence (3) $\Rightarrow$ (4). If (4) holds, then $\alpha_{\mathfrak{p}}$ is an isomorphism, so α is injective as $S_{\mathfrak{q}} \to S_{\mathfrak{p}}$ is injective. Namely, elements of $S \setminus \mathfrak{p}$ are nonzerodivisors on S by a combination of Lemmas 37.7.6 and 37.9.3. Hence (4) $\Rightarrow$ (5). Finally, if (5) holds, then $N_{\mathfrak{q}}$ is R-flat as an extension of flat modules, see Algebra, Lemma 10.36.12. Hence (5) $\Rightarrow$ (1) and the proof is finished. □

Lemma 37.10.2. *Let $(R, \mathfrak{m})$ be a local ring. Let $R \to S$ be a ring map of finite presentation. Let N be a finite S-module. Let $\mathfrak{q}$ be a prime of S lying over $\mathfrak{m}$. Assume that $N_{\mathfrak{q}}$ is flat over R, and assume there exists a complete dévissage of $N/S/R$ at $\mathfrak{q}$. Then N is a finitely presented S-module, free as an R-module, and there exists an isomorphism*

$$N \cong B_1^{\oplus r_1} \oplus \ldots \oplus B_n^{\oplus r_n}$$

as R-modules where each B_i is a smooth R-algebra with geometrically irreducible fibres.

Proof. Let $(A_i, B_i, M_i, \alpha_i, \mathfrak{q}_i)_{i=1,\ldots,n}$ be the given complete dévissage. We prove the lemma by induction on n. Note that N is finitely presented as an S-module if and only if M_1 is finitely presented as an B_1-module, see Remark 37.6.3. Note that $N_{\mathfrak{q}} \cong (M_1)_{\mathfrak{q}_1}$ as R-modules because (a) $N_{\mathfrak{q}} \cong (M_1)_{\mathfrak{q}_1'}$ where $\mathfrak{q}_1'$ is the unique prime in A_1 lying over $\mathfrak{q}_1$ and (b) $(A_1)_{\mathfrak{q}_1'} = (A_1)_{\mathfrak{q}_1}$ by Algebra, Lemma 10.38.11, so (c) $(M_1)_{\mathfrak{q}_1'} \cong (M_1)_{\mathfrak{q}_1}$. Hence $(M_1)_{\mathfrak{q}_1}$ is a flat R-module. Thus we may replace (S, N) by (B_1, M_1) in order to prove the lemma. By Lemma 37.10.1 the map $\alpha_1 : B_1^{\oplus r_1} \to M_1$ is R-universally injective and $\mathrm{Coker}(\alpha_1)_{\mathfrak{q}}$ is R-flat. Note that $(A_i, B_i, M_i, \alpha_i, \mathfrak{q}_i)_{i=2,\ldots,n}$ is a complete dévissage of $\mathrm{Coker}(\alpha_1)/B_1/R$ at $\mathfrak{q}_1$. Hence the induction hypothesis implies that $\mathrm{Coker}(\alpha_1)$ is finitely presented as a B_1-module, free as an R-module, and has a decomposition as in the lemma. This implies that M_1 is finitely presented as a B_1-module, see Algebra, Lemma 10.5.3. It further implies that $M_1 \cong B_1^{\oplus r_1} \oplus \mathrm{Coker}(\alpha_1)$ as R-modules, hence a decomposition as in the lemma. Finally, B_1 is projective as an R-module by Lemma 37.9.3 hence free as an R-module by Algebra, Theorem 10.80.4. This finishes the proof. □

Proposition 37.10.3. *Let $f : X \to S$ be a morphism of schemes. Let $\mathcal{F}$ be a quasi-coherent sheaf on X. Let $x \in X$ with image $s \in S$. Assume that*

1. *f is locally of finite presentation,*
2. *$\mathcal{F}$ is of finite type, and*
3. *$\mathcal{F}$ is flat at x over S.*

Then there exists an elementary étale neighbourhood $(S', s') \to (S, s)$ and an open subscheme

$$V \subset X \times_S \mathrm{Spec}(\mathcal{O}_{S',s'})$$

which contains the unique point of $X \times_S \mathrm{Spec}(\mathcal{O}_{S',s'})$ mapping to x such that the pullback of $\mathcal{F}$ to V is an $\mathcal{O}_V$-module of finite presentation and flat over $\mathcal{O}_{S',s'}$.

First proof. This proof is longer but does not use the existence of a complete dévissage. The problem is local around x and s, hence we may assume that X and S are affine. During the proof we will finitely many times replace S by an elementary étale neighbourhood of (S, s). The goal is then to find (after such a replacement) an open $V \subset X \times_S \mathrm{Spec}(\mathcal{O}_{S,s})$ containing x such that $\mathcal{F}|_V$ is flat over S and finitely presented. Of course we may also replace S by $\mathrm{Spec}(\mathcal{O}_{S,s})$ at any point of the proof, i.e., we may assume S is a local scheme. We will prove the lemma by induction on the integer $n = \dim_x(\mathrm{Supp}(\mathcal{F}_s))$.

We can choose

(1) elementary étale neighbourhoods $g : (X', x') \to (X, x)$, $e : (S', s') \to (S, s)$,
(2) a commutative diagram

$$\begin{array}{ccccc} X & \xleftarrow{g} & X' & \xleftarrow{i} & Z' \\ & & & & \downarrow{\scriptstyle \pi} \\ {\scriptstyle f}\downarrow & & & & Y' \\ & & & & \downarrow{\scriptstyle h} \\ S & \xleftarrow{e} & S' & = & S' \end{array}$$

(3) a point $z' \in Z'$ with $i(z') = x'$, $y' = \pi(z')$, $h(y') = s'$,
(4) a finite type quasi-coherent $\mathcal{O}_{Z'}$-module $\mathcal{G}$,

as in Lemma 37.3.2. We are going to replace S by $\mathrm{Spec}(\mathcal{O}_{S',s'})$, see remarks in first paragraph of the proof. Consider the diagram

$$\begin{array}{ccccc} X_{\mathcal{O}_{S',s'}} & \xleftarrow{g} & X'_{\mathcal{O}_{S',s'}} & \xleftarrow{i} & Z'_{\mathcal{O}_{S',s'}} \\ & \searrow^{f} & \downarrow & & \downarrow{\scriptstyle \pi} \\ & & & & Y'_{\mathcal{O}_{S',s'}} \\ & & & \swarrow_{h} & \\ & & \mathrm{Spec}(\mathcal{O}_{S',s'}) & & \end{array}$$

Here we have base changed the schemes X', Z', Y' over S' via $\mathrm{Spec}(\mathcal{O}_{S',s'}) \to S'$ and the scheme X over S via $\mathrm{Spec}(\mathcal{O}_{S',s'}) \to S$. It is still the case that g is étale, see Lemma 37.2.2. After replacing X by $X_{\mathcal{O}_{S',s'}}$, X' by $X'_{\mathcal{O}_{S',s'}}$, Z' by $Z'_{\mathcal{O}_{S',s'}}$, and Y' by $Y'_{\mathcal{O}_{S',s'}}$ we may assume we have a diagram as Lemma 37.3.2 where in addition $S = S'$ is a local scheme with closed point s. By Lemmas 37.3.3 and 37.3.4 the result for $Y' \to S$, the sheaf $\pi_*\mathcal{G}$, and the point y' implies the result for $X \to S$, $\mathcal{F}$ and x. Hence we may assume that S is local and $X \to S$ is a smooth morphism of affines with geometrically irreducible fibres of dimension n.

The base case of the induction: $n = 0$. As $X \to S$ is smooth with geometrically irreducible fibres of dimension 0 we see that $X \to S$ is an open immersion, see Descent, Lemma 34.21.2. As S is local and the closed point is in the image of $X \to S$ we conclude that $X = S$. Thus we see that $\mathcal{F}$ corresponds to a finite flat $\mathcal{O}_{S,s}$ module. In this case the result follows from Algebra, Lemma 10.73.4 which tells us that $\mathcal{F}$ is in fact finite free.

The induction step. Assume the result holds whenever the dimension of the support in the closed fibre is $< n$. Write $S = \mathrm{Spec}(A)$, $X = \mathrm{Spec}(B)$ and $\mathcal{F} = \widetilde{N}$ for some B-module N. Note that A is a local ring; denote its maximal ideal $\mathfrak{m}$. Then $\mathfrak{p} = \mathfrak{m}B$ is the unique minimal prime lying over $\mathfrak{m}$ as $X \to S$ has geometrically irreducible fibres. Finally, let $\mathfrak{q} \subset B$ be the prime corresponding to x. By Lemma 37.10.1 we can choose a map

$$\alpha : B^{\oplus r} \to N$$

such that $\kappa(\mathfrak{p})^{\oplus r} \to N \otimes_B \kappa(\mathfrak{p})$ is an isomorphism. Moreover, as $N_\mathfrak{q}$ is A-flat the lemma also shows that α is injective and that $\mathrm{Coker}(\alpha)_\mathfrak{q}$ is A-flat. Set $Q = \mathrm{Coker}(\alpha)$. Note that the support of $Q/\mathfrak{m}Q$ does not contain $\mathfrak{p}$. Hence it is certainly the case that $\dim_\mathfrak{q}(\mathrm{Supp}(Q/\mathfrak{m}Q)) < n$. Combining everything we know about Q we see that the induction hypothesis applies to Q. It follows that there exists an elementary étale morphism $(S', s) \to (S, s)$ such that the conclusion holds for $Q \otimes_A A'$ over $B \otimes_A A'$ where $A' = \mathcal{O}_{S',s'}$. After replacing A by A' we have an exact sequence

$$0 \to B^{\oplus r} \to N \to Q \to 0$$

(here we use that α is injective as mentioned above) of finite B-modules and we also get an element $g \in B$, $g \notin \mathfrak{q}$ such that Q_g is finitely presented over B_g and flat over A. Since localization is exact we see that

$$0 \to B_g^{\oplus r} \to N_g \to Q_g \to 0$$

is still exact. As B_g and Q_g are flat over A we conclude that N_g is flat over A, see Algebra, Lemma 10.36.12, and as B_g and Q_g are finitely presented over B_g the same holds for N_g, see Algebra, Lemma 10.5.3. □

Second proof. We apply Proposition 37.5.7 to find a commutative diagram

$$\begin{array}{ccc} (X, x) & \xleftarrow{g} & (X', x') \\ \downarrow & & \downarrow \\ (S, s) & \longleftarrow & (S', s') \end{array}$$

of pointed schemes such that the horizontal arrows are elementary étale neighbourhoods and such that $g^*\mathcal{F}/X'/S'$ has a complete dévissage at x. (In particular S' and X' are affine.) By Morphisms, Lemma 28.26.11 we see that $g^*\mathcal{F}$ is flat at x' over S and by Lemma 37.2.3 we see that it is flat at x' over S'. Via Remark 37.6.5 we deduce that

$$\Gamma(X', g^*\mathcal{F})/\Gamma(X', \mathcal{O}_{X'})/\Gamma(S', \mathcal{O}_{S'})$$

has a complete dévissage at the prime of $\Gamma(X', \mathcal{O}_{X'})$ corresponding to x'. We may base change this complete dévissage to the local ring $\mathcal{O}_{S',s'}$ of $\Gamma(S', \mathcal{O}_{S'})$ at the prime corresponding to s'. Thus Lemma 37.10.2 implies that

$$\Gamma(X', \mathcal{F}') \otimes_{\Gamma(S', \mathcal{O}_{S'})} \mathcal{O}_{S',s'}$$

is flat over $\mathcal{O}_{S',s'}$ and of finite presentation over $\Gamma(X', \mathcal{O}_{X'}) \otimes_{\Gamma(S', \mathcal{O}_{S'})} \mathcal{O}_{S',s'}$. In other words, the restriction of $\mathcal{F}$ to $X' \times_{S'} \mathrm{Spec}(\mathcal{O}_{S',s'})$ is of finite presentation and flat over $\mathcal{O}_{S',s'}$. Since the morphism $X' \times_{S'} \mathrm{Spec}(\mathcal{O}_{S',s'}) \to X \times_S \mathrm{Spec}(\mathcal{O}_{S',s'})$ is étale (Lemma 37.2.2) its image $V \subset X \times_S \mathrm{Spec}(\mathcal{O}_{S',s'})$ is an open subscheme, and by étale descent the restriction of $\mathcal{F}$ to V is of finite presentation and flat over $\mathcal{O}_{S',s'}$. (Results used: Morphisms, Lemma 28.37.13, Descent, Lemma 34.6.3, and Morphisms, Lemma 28.26.11.) □

Lemma 37.10.4. *Let $f : X \to S$ be a morphism of schemes which is locally of finite type. Let $\mathcal{F}$ be a quasi-coherent $\mathcal{O}_X$-module of finite type. Let $s \in S$. Then the set*

$$\{x \in X_s \mid \mathcal{F} \text{ flat over } S \text{ at } x\}$$

is open in the fibre X_s.

Proof. Suppose $x \in U$. Choose an elementary étale neighbourhood $(S', s') \to (S, s)$ and open $V \subset X \times_S \operatorname{Spec}(\mathcal{O}_{S',s'})$ as in Proposition 37.10.3. Note that $X_{s'} = X_s$ as $\kappa(s) = \kappa(s')$. If $x' \in V \cap X_{s'}$, then the pullback of $\mathcal{F}$ to $X \times_S S'$ is flat over S' at x'. Hence $\mathcal{F}$ is flat at x' over S, see Morphisms, Lemma 28.26.11. In other words $X_s \cap V \subset U$ is an open neighbourhood of x in U. □

Lemma 37.10.5. *Let $f : X \to S$ be a morphism of schemes. Let $\mathcal{F}$ be a quasi-coherent sheaf on X. Let $x \in X$ with image $s \in S$. Assume that*

(1) *f is locally of finite type,*
(2) *$\mathcal{F}$ is of finite type, and*
(3) *$\mathcal{F}$ is flat at x over S.*

Then there exists an elementary étale neighbourhood $(S', s') \to (S, s)$ and an open subscheme

$$V \subset X \times_S \operatorname{Spec}(\mathcal{O}_{S',s'})$$

which contains the unique point of $X \times_S \operatorname{Spec}(\mathcal{O}_{S',s'})$ mapping to x such that the pullback of $\mathcal{F}$ to V is flat over $\mathcal{O}_{S',s'}$.

Proof. (The only difference between this and Proposition 37.10.3 is that we do not assume f is of finite presentation.) The question is local on X and S, hence we may assume X and S are affine. Write $X = \operatorname{Spec}(B)$, $S = \operatorname{Spec}(A)$ and write $B = A[x_1, \ldots, x_n]/I$. In other words we obtain a closed immersion $i : X \to \mathbf{A}^n_S$. Denote $t = i(x) \in \mathbf{A}^n_S$. We may apply Proposition 37.10.3 to $\mathbf{A}^n_S \to S$, the sheaf $i_*\mathcal{F}$ and the point t. We obtain an elementary étale neighbourhood $(S', s') \to (S, s)$ and an open subscheme

$$W \subset \mathbf{A}^n_{\mathcal{O}_{S',s'}}$$

such that the pullback of $i_*\mathcal{F}$ to W is flat over $\mathcal{O}_{S',s'}$. This means that $V := W \cap \big(X \times_S \operatorname{Spec}(\mathcal{O}_{S',s'})\big)$ is the desired open subscheme. □

Lemma 37.10.6. *Let $f : X \to S$ be a morphism of schemes. Let $\mathcal{F}$ be a quasi-coherent sheaf on X. Let $s \in S$. Assume that*

(1) *f is of finite presentation,*
(2) *$\mathcal{F}$ is of finite type, and*
(3) *$\mathcal{F}$ is flat over S at every point of the fibre X_s.*

Then there exists an elementary étale neighbourhood $(S', s') \to (S, s)$ and an open subscheme

$$V \subset X \times_S \operatorname{Spec}(\mathcal{O}_{S',s'})$$

which contains the fibre $X_s = X \times_S s'$ such that the pullback of $\mathcal{F}$ to V is an $\mathcal{O}_V$-module of finite presentation and flat over $\mathcal{O}_{S',s'}$.

Proof. For every point $x \in X_s$ we can use Proposition 37.10.3 to find an elementary étale neighbourhood $(S_x, s_x) \to (S, s)$ and an open $V_x \subset X \times_S \operatorname{Spec}(\mathcal{O}_{S_x,s_x})$ such that $x \in X_s = X \times_S s_x$ is contained in V_x and such that the pullback of $\mathcal{F}$ to V_x is an $\mathcal{O}_{V_x}$-module of finite presentation and flat over $\mathcal{O}_{S_x,s_x}$. In particular we may

view the fibre $(V_x)_{s_x}$ as an open neighbourhood of x in X_s. Because X_s is quasi-compact we can find a finite number of points $x_1, \ldots, x_n \in X_s$ such that X_s is the union of the $(V_{x_i})_{s_{x_i}}$. Choose an elementary étale neighbourhood $(S', s') \to (S, s)$ which dominates each of the neighbourhoods (S_{x_i}, s_{x_i}), see More on Morphisms, Lemma 36.27.4. Set $V = \bigcup V_i$ where V_i is the inverse images of the open V_{x_i} via the morphism

$$X \times_S \operatorname{Spec}(\mathcal{O}_{S',s'}) \longrightarrow X \times_S \operatorname{Spec}(\mathcal{O}_{S_{x_i},s_{x_i}})$$

By construction V contains X_s and by construction the pullback of $\mathcal{F}$ to V is an $\mathcal{O}_V$-module of finite presentation and flat over $\mathcal{O}_{S',s'}$. □

Lemma 37.10.7. *Let $f : X \to S$ be a morphism of schemes. Let $\mathcal{F}$ be a quasi-coherent sheaf on X. Let $s \in S$. Assume that*

(1) *f is of finite type,*
(2) *$\mathcal{F}$ is of finite type, and*
(3) *$\mathcal{F}$ is flat over S at every point of the fibre X_s.*

Then there exists an elementary étale neighbourhood $(S', s') \to (S, s)$ and an open subscheme

$$V \subset X \times_S \operatorname{Spec}(\mathcal{O}_{S',s'})$$

which contains the fibre $X_s = X \times_S s'$ such that the pullback of $\mathcal{F}$ to V is flat over $\mathcal{O}_{S',s'}$.

Proof. (The only difference between this and Lemma 37.10.6 is that we do not assume f is of finite presentation.) For every point $x \in X_s$ we can use Lemma 37.10.5 to find an elementary étale neighbourhood $(S_x, s_x) \to (S, s)$ and an open $V_x \subset X \times_S \operatorname{Spec}(\mathcal{O}_{S_x,s_x})$ such that $x \in X_s = X \times_S s_x$ is contained in V_x and such that the pullback of $\mathcal{F}$ to V_x is flat over $\mathcal{O}_{S_x,s_x}$. In particular we may view the fibre $(V_x)_{s_x}$ as an open neighbourhood of x in X_s. Because X_s is quasi-compact we can find a finite number of points $x_1, \ldots, x_n \in X_s$ such that X_s is the union of the $(V_{x_i})_{s_{x_i}}$. Choose an elementary étale neighbourhood $(S', s') \to (S, s)$ which dominates each of the neighbourhoods (S_{x_i}, s_{x_i}), see More on Morphisms, Lemma 36.27.4. Set $V = \bigcup V_i$ where V_i is the inverse images of the open V_{x_i} via the morphism

$$X \times_S \operatorname{Spec}(\mathcal{O}_{S',s'}) \longrightarrow X \times_S \operatorname{Spec}(\mathcal{O}_{S_{x_i},s_{x_i}})$$

By construction V contains X_s and by construction the pullback of $\mathcal{F}$ to V is flat over $\mathcal{O}_{S',s'}$. □

Lemma 37.10.8. *Let S be a scheme. Let X be locally of finite type over S. Let $x \in X$ with image $s \in S$. If X is flat at x over S, then there exists an elementary étale neighbourhood $(S', s') \to (S, s)$ and an open subscheme*

$$V \subset X \times_S \operatorname{Spec}(\mathcal{O}_{S',s'})$$

which contains the unique point of $X \times_S \operatorname{Spec}(\mathcal{O}_{S',s'})$ mapping to x such that $V \to \operatorname{Spec}(\mathcal{O}_{S',s'})$ is flat and of finite presentation.

Proof. The question is local on X and S, hence we may assume X and S are affine. Write $X = \operatorname{Spec}(B)$, $S = \operatorname{Spec}(A)$ and write $B = A[x_1, \ldots, x_n]/I$. In other words we obtain a closed immersion $i : X \to \mathbf{A}^n_S$. Denote $t = i(x) \in \mathbf{A}^n_S$. We may apply

Proposition 37.10.3 to $\mathbf{A}^n_S \to S$, the sheaf $\mathcal{F} = i_*\mathcal{O}_X$ and the point t. We obtain an elementary étale neighbourhood $(S', s') \to (S, s)$ and an open subscheme

$$W \subset \mathbf{A}^n_{\mathcal{O}_{S',s'}}$$

such that the pullback of $i_*\mathcal{O}_X$ is flat and of finite presentation. This means that $V := W \cap \big(X \times_S \mathrm{Spec}(\mathcal{O}_{S',s'})\big)$ is the desired open subscheme. □

Lemma 37.10.9. *Let $f : X \to S$ be a morphism which is locally of finite presentation. Let $\mathcal{F}$ be a quasi-coherent $\mathcal{O}_X$-module of finite type. If $x \in X$ and $\mathcal{F}$ is flat at x over S, then $\mathcal{F}_x$ is an $\mathcal{O}_{X,x}$-module of finite presentation.*

Proof. Let $s = f(x)$. By Proposition 37.10.3 there exists an elementary étale neighbourhood $(S', s') \to (S, s)$ such that the pullback of $\mathcal{F}$ to $X \times_S \mathrm{Spec}(\mathcal{O}_{S',s'})$ is of finite presentation in a neighbourhood of the point $x' \in X_{s'} = X_s$ corresponding to x. The ring map

$$\mathcal{O}_{X,x} \longrightarrow \mathcal{O}_{X \times_S \mathrm{Spec}(\mathcal{O}_{S',s'}),x'} = \mathcal{O}_{X \times_S S',x'}$$

is flat and local as a localization of an étale ring map. Hence $\mathcal{F}_x$ is of finite presentation over $\mathcal{O}_{X,x}$ by descent, see Algebra, Lemma 10.78.2 (and also that a flat local ring map is faithfully flat, see Algebra, Lemma 10.36.16). □

Lemma 37.10.10. *Let $f : X \to S$ be a morphism which is locally of finite type. Let $x \in X$ with image $s \in S$. If f is flat at x over S, then $\mathcal{O}_{X,x}$ is essentially of finite presentation over $\mathcal{O}_{S,s}$.*

Proof. We may assume X and S affine. Write $X = \mathrm{Spec}(B)$, $S = \mathrm{Spec}(A)$ and write $B = A[x_1, \ldots, x_n]/I$. In other words we obtain a closed immersion $i : X \to \mathbf{A}^n_S$. Denote $t = i(x) \in \mathbf{A}^n_S$. We may apply Lemma 37.10.9 to $\mathbf{A}^n_S \to S$, the sheaf $\mathcal{F} = i_*\mathcal{O}_X$ and the point t. We conclude that $\mathcal{O}_{X,x}$ is of finite presentation over $\mathcal{O}_{\mathbf{A}^n_S,t}$ which implies what we want. □

Lemma 37.10.11. *Let $f : X \to S$ be a morphism of schemes. Let $\mathcal{F}$ be a quasi-coherent $\mathcal{O}_X$-module. Let $U \subset S$ be open. Assume*

1. *f is locally of finite presentation,*
2. *$\mathcal{F}$ is of finite type and flat over S,*
3. *$U \subset S$ is retrocompact and scheme theoretically dense,*
4. *$\mathcal{F}|_{f^{-1}U}$ is of finite presentation.*

Then $\mathcal{F}$ is of finite presentation.

Proof. The problem is local on X and S, hence we may assume X and S affine. Write $S = \mathrm{Spec}(A)$ and $X = \mathrm{Spec}(B)$. Let N be a finite B-module such that $\mathcal{F}$ is the quasi-coherent sheaf associated to N. We have $U = D(f_1) \cup \ldots \cup D(f_n)$ for some $f_i \in A$, see Algebra, Lemma 10.27.1. As U is schematically dense the map $A \to A_{f_1} \times \ldots \times A_{f_n}$ is injective. Pick a prime $\mathfrak{q} \subset B$ lying over $\mathfrak{p} \subset A$ corresponding to $x \in X$ mapping to $s \in S$. By Lemma 37.10.9 the module $N_\mathfrak{q}$ is of finite presentation over $B_\mathfrak{q}$. Choose a surjection $\varphi : B^{\oplus m} \to N$ of B-modules. Choose $k_1, \ldots, k_t \in \mathrm{Ker}(\varphi)$ and set $N' = B^{\oplus m}/\sum Bk_j$. There is a canonical surjection $N' \to N$ and N is the filtered colimit of the B-modules N' constructed in this manner. Thus we see that we can choose $k_1, \ldots, k_t$ such that (a) $N'_{f_i} \cong N_{f_i}$, $i = 1, \ldots, n$ and (b) $N'_\mathfrak{q} \cong N_\mathfrak{q}$. This in particular implies that $N'_\mathfrak{q}$ is flat over A. By

openness of flatness, see Algebra, Theorem 10.123.4 we conclude that there exists a $g \in B$, $g \notin \mathfrak{q}$ such that N'_g is flat over A. Consider the commutative diagram

$$\begin{array}{ccc} N'_g & \longrightarrow & N_g \\ \downarrow & & \downarrow \\ \prod N'_{gf_i} & \longrightarrow & \prod N_{gf_i} \end{array}$$

The bottom arrow is an isomorphism by choice of $k_1, \ldots, k_t$. The left vertical arrow is an injective map as $A \to \prod A_{f_i}$ is injective and N'_g is flat over A. Hence the top horizontal arrow is injective, hence an isomorphism. This proves that N_g is of finite presentation over B_g. We conclude by applying Algebra, Lemma 10.22.2. □

Lemma 37.10.12. *Let $f : X \to S$ be a morphism of schemes. Let $U \subset S$ be open. Assume*

(1) *f is locally of finite type and flat,*
(2) *$U \subset S$ is retrocompact and scheme theoretically dense,*
(3) *$f|_{f^{-1}U} : f^{-1}U \to U$ is locally of finite presentation.*

Then f is of locally of finite presentation.

Proof. The question is local on X and S, hence we may assume X and S affine. Choose a closed immersion $i : X \to \mathbf{A}^n_S$ and apply Lemma 37.10.11 to $i_*\mathcal{O}_X$. Some details omitted. □

37.11. Flat finitely presented modules

In some cases given a ring map $R \to S$ of finite presentation and a finitely presented S-module N the flatness of N over R implies that N is projective as an R-module, at least after replacing S by an étale extension. In this section we collect a some results of this nature.

Lemma 37.11.1. *Let R be a ring. Let $R \to S$ be a finitely presented flat ring map with geometrically integral fibres. Let $\mathfrak{q} \subset S$ be a prime ideal lying over the prime $\mathfrak{r} \subset R$. Set $\mathfrak{p} = \mathfrak{r}S$. Let N be a finitely presented S-module. There exists $r \geq 0$ and an S-module map*

$$\alpha : S^{\oplus r} \longrightarrow N$$

such that $\alpha : \kappa(\mathfrak{p})^{\oplus r} \to N \otimes_S \kappa(\mathfrak{p})$ is an isomorphism. For any such α the following are equivalent:

(1) *$N_\mathfrak{q}$ is R-flat,*
(2) *there exists an $f \in R$, $f \notin \mathfrak{r}$ such that $\alpha_f : S_f^{\oplus r} \to N_f$ is R_f-universally injective and a $g \in S$, $g \notin \mathfrak{q}$ such that $\mathit{Coker}(\alpha)_g$ is R-flat,*
(3) *$\alpha_\mathfrak{r}$ is $R_\mathfrak{r}$-universally injective and $\mathit{Coker}(\alpha)_\mathfrak{q}$ is R-flat*
(4) *$\alpha_\mathfrak{r}$ is injective and $\mathit{Coker}(\alpha)_\mathfrak{q}$ is R-flat,*
(5) *$\alpha_\mathfrak{p}$ is an isomorphism and $\mathit{Coker}(\alpha)_\mathfrak{q}$ is R-flat, and*
(6) *$\alpha_\mathfrak{q}$ is injective and $\mathit{Coker}(\alpha)_\mathfrak{q}$ is R-flat.*

Proof. To obtain α set $r = \dim_{\kappa(\mathfrak{p})} N \otimes_S \kappa(\mathfrak{p})$ and pick $x_1, \ldots, x_r \in N$ which form a basis of $N \otimes_S \kappa(\mathfrak{p})$. Define $\alpha(s_1, \ldots, s_r) = \sum s_i x_i$. This proves the existence.

Fix a choice of α. We may apply Lemma 37.10.1 to the map $\alpha_\mathfrak{r} : S_\mathfrak{r}^{\oplus r} \to N_\mathfrak{r}$. Hence we see that (1), (3), (4), (5), and (6) are all equivalent. Since it is also clear that (2) implies (3) we see that all we have to do is show that (1) implies (2).

Assume (1). By openness of flatness, see Algebra, Theorem 10.123.4, the set

$$U_1 = \{\mathfrak{q}' \subset S \mid N_{\mathfrak{q}'} \text{ is flat over } R\}$$

is open in $\mathrm{Spec}(S)$. It contains $\mathfrak{q}$ by assumption and hence $\mathfrak{p}$. Because $S^{\oplus r}$ and N are finitely presented S-modules the set

$$U_2 = \{\mathfrak{q}' \subset S \mid \alpha_{\mathfrak{q}'} \text{ is an isomorphism}\}$$

is open in $\mathrm{Spec}(S)$, see Algebra, Lemma 10.74.2. It contains $\mathfrak{p}$ by (5). As $R \to S$ is finitely presented and flat the map $\Phi : \mathrm{Spec}(S) \to \mathrm{Spec}(R)$ is open, see Algebra, Proposition 10.38.8. For any prime $\mathfrak{r}' \in \Phi(U_1 \cap U_2)$ we see that there exists a prime $\mathfrak{q}'$ lying over $\mathfrak{r}'$ such that $N_{\mathfrak{q}'}$ is flat and such that $\alpha_{\mathfrak{q}'}$ is an isomorphism, which implies that $\alpha \otimes \kappa(\mathfrak{p}')$ is an isomorphism where $\mathfrak{p}' = \mathfrak{r}'S$. Thus $\alpha_{\mathfrak{r}'}$ is $R_{\mathfrak{r}'}$-universally injective by the implication (1) $\Rightarrow$ (3). Hence if we pick $f \in R$, $f \notin \mathfrak{r}$ such that $D(f) \subset \Phi(U_1 \cap U_2)$ then we conclude that α_f is R_f-universally injective, see Algebra, Lemma 10.77.12. The same reasoning also shows that for any $\mathfrak{q}' \in U_1 \cap \Phi^{-1}(\Phi(U_1 \cap U_2))$ the module $\mathrm{Coker}(\alpha)_{\mathfrak{q}'}$ is R-flat. Note that $\mathfrak{q} \in U_1 \cap \Phi^{-1}(\Phi(U_1 \cap U_2))$. Hence we can find a $g \in S$, $g \notin \mathfrak{q}$ such that $D(g) \subset U_1 \cap \Phi^{-1}(\Phi(U_1 \cap U_2))$ and we win. □

Lemma 37.11.2. *Let $R \to S$ be a ring map of finite presentation. Let N be a finitely presented S-module flat over R. Let $\mathfrak{r} \subset R$ be a prime ideal. Assume there exists a complete dévissage of $N/S/R$ over $\mathfrak{r}$. Then there exists an $f \in R$, $f \notin \mathfrak{r}$ such that*

$$N_f \cong B_1^{\oplus r_1} \oplus \ldots \oplus B_n^{\oplus r_n}$$

as R-modules where each B_i is a smooth R_f-algebra with geometrically irreducible fibres. Moreover, N_f is projective as an R_f-module.

Proof. Let $(A_i, B_i, M_i, \alpha_i)_{i=1,\ldots,n}$ be the given complete dévissage. We prove the lemma by induction on n. Note that the assertions of the lemma are entirely about the structure of N as an R-module. Hence we may replace N by M_1, and we may think of M_1 as a B_1-module. See Remark 37.6.3 in order to see why M_1 is of finite presentation as a B_1-module. By Lemma 37.11.1 we may, after replacing R by R_f for some $f \in R$, $f \notin \mathfrak{r}$, assume the map $\alpha_1 : B_1^{\oplus r_1} \to M_1$ is R-universally injective. Since M_1 and $B_1^{\oplus r_1}$ are R-flat and finitely presented as B_1-modules we see that $\mathrm{Coker}(\alpha_1)$ is R-flat (Algebra, Lemma 10.77.7) and finitely presented as a B_1-module. Note that $(A_i, B_i, M_i, \alpha_i)_{i=2,\ldots,n}$ is a complete dévissage of $\mathrm{Coker}(\alpha_1)$. Hence the induction hypothesis implies that, after replacing R by R_f for some $f \in R$, $f \notin \mathfrak{r}$, we may assume that $\mathrm{Coker}(\alpha_1)$ has a decomposition as in the lemma and is projective. In particular $M_1 = B_1^{\oplus r_1} \oplus \mathrm{Coker}(\alpha_1)$. This proves the statement regarding the decomposition. The statement on projectivity follows as B_1 is projective as an R-module by Lemma 37.9.3. □

Remark 37.11.3. There is a variant of Lemma 37.11.2 where we weaken the flatness condition by assuming only that N is flat at some given prime $\mathfrak{q}$ lying over $\mathfrak{r}$ but where we strengthen the dévissage condition by assuming the existence of a complete dévissage *at* $\mathfrak{q}$. Compare with Lemma 37.10.2.

The following is the main result of this section.

Proposition 37.11.4. *Let $f : X \to S$ be a morphism of schemes. Let $\mathcal{F}$ be a quasi-coherent sheaf on X. Let $x \in X$ with image $s \in S$. Assume that*

(1) *f is locally of finite presentation,*

(2) $\mathcal{F}$ *is of finite presentation, and*
(3) $\mathcal{F}$ *is flat at* x *over* S.

Then there exists a commutative diagram of pointed schemes

$$\begin{array}{ccc} (X, x) & \xleftarrow{g} & (X', x') \\ \downarrow & & \downarrow \\ (S, s) & \longleftarrow & (S', s') \end{array}$$

whose horizontal arrows are elementary étale neighbourhoods such that X', S' *are affine and such that* $\Gamma(X', g^*\mathcal{F})$ *is a projective* $\Gamma(S', \mathcal{O}_{S'})$*-module.*

Proof. By openness of flatness, see More on Morphisms, Theorem 36.12.1 we may replace X by an open neighbourhood of x and assume that $\mathcal{F}$ is flat over S. Next, we apply Proposition 37.5.7 to find a diagram as in the statement of the proposition such that $g^*\mathcal{F}/X'/S'$ has a complete dévissage over s'. (In particular S' and X' are affine.) By Morphisms, Lemma 28.26.11 we see that $g^*\mathcal{F}$ is flat over S and by Lemma 37.2.3 we see that it is flat over S'. Via Remark 37.6.5 we deduce that

$$\Gamma(X', g^*\mathcal{F})/\Gamma(X', \mathcal{O}_{X'})/\Gamma(S', \mathcal{O}_{S'})$$

has a complete dévissage over the prime of $\Gamma(S', \mathcal{O}_{S'})$ corresponding to s'. Thus Lemma 37.11.2 implies that the result of the proposition holds after replacing S' by a standard open neighbourhood of s'. □

In the rest of this section we prove a number of variants on this result. The first is a "global" version.

Lemma 37.11.5. *Let* $f : X \to S$ *be a morphism of schemes. Let* $\mathcal{F}$ *be a quasi-coherent sheaf on* X*. Let* $s \in S$*. Assume that*

(1) f *is of finite presentation,*
(2) $\mathcal{F}$ *is of finite presentation, and*
(3) $\mathcal{F}$ *is flat over* S *at every point of the fibre* X_s.

Then there exists an elementary étale neighbourhood $(S', s') \to (S, s)$ *and a commutative diagram of schemes*

$$\begin{array}{ccc} X & \xleftarrow{g} & X' \\ \downarrow & & \downarrow \\ S & \longleftarrow & S' \end{array}$$

such that g *is étale,* $X_s \subset g(X')$*, the schemes* X', S' *are affine, and such that* $\Gamma(X', g^*\mathcal{F})$ *is a projective* $\Gamma(S', \mathcal{O}_{S'})$*-module.*

Proof. For every point $x \in X_s$ we can use Proposition 37.11.4 to find a commutative diagram

$$\begin{array}{ccc} (X, x) & \xleftarrow{g_x} & (Y_x, y_x) \\ \downarrow & & \downarrow \\ (S, s) & \longleftarrow & (S_x, s_x) \end{array}$$

whose horizontal arrows are elementary étale neighbourhoods such that Y_x, S_x are affine and such that $\Gamma(Y_x, g_x^*\mathcal{F})$ is a projective $\Gamma(S_x, \mathcal{O}_{S_x})$-module. In particular $g_x(Y_x) \cap X_s$ is an open neighbourhood of x in X_s. Because X_s is quasi-compact we

can find a finite number of points $x_1, \ldots, x_n \in X_s$ such that X_s is the union of the $g_{x_i}(Y_{x_i}) \cap X_s$. Choose an elementary étale neighbourhood $(S', s') \to (S, s)$ which dominates each of the neighbourhoods (S_{x_i}, s_{x_i}), see More on Morphisms, Lemma 36.27.4. We may also assume that S' is affine. Set $X' = \coprod Y_{x_i} \times_{S_{x_i}} S'$ and endow it with the obvious morphism $g : X' \to X$. By construction $g(X')$ contains X_s and

$$\Gamma(X', g^*\mathcal{F}) = \bigoplus \Gamma(Y_{x_i}, g_{x_i}^*\mathcal{F}) \otimes_{\Gamma(S_{x_i}, \mathcal{O}_{S_{x_i}})} \Gamma(S', \mathcal{O}_{S'}).$$

This is a projective $\Gamma(S', \mathcal{O}_{S'})$-module, see Algebra, Lemma 10.89.1. □

The following two lemmas are reformulations of the results above in case $\mathcal{F} = \mathcal{O}_X$.

Lemma 37.11.6. *Let $f : X \to S$ be locally of finite presentation. Let $x \in X$ with image $s \in S$. If f is flat at x over S, then there exists a commutative diagram of pointed schemes*

$$\begin{array}{ccc} (X, x) & \xleftarrow{g} & (X', x') \\ \downarrow & & \downarrow \\ (S, s) & \longleftarrow & (S', s') \end{array}$$

whose horizontal arrows are elementary étale neighbourhoods such that X', S' are affine and such that $\Gamma(X', \mathcal{O}_{X'})$ is a projective $\Gamma(S', \mathcal{O}_{S'})$-module.

Proof. This is a special case of Proposition 37.11.4. □

Lemma 37.11.7. *Let $f : X \to S$ be of finite presentation. Let $s \in S$. If X is flat over S at all points of X_s, then there exists an elementary étale neighbourhood $(S', s') \to (S, s)$ and a commutative diagram of schemes*

$$\begin{array}{ccc} X & \xleftarrow{g} & X' \\ \downarrow & & \downarrow \\ S & \longleftarrow & S' \end{array}$$

with g étale, $X_s \subset g(X')$, such that X', S' are affine, and such that $\Gamma(X', \mathcal{O}_{X'})$ is a projective $\Gamma(S', \mathcal{O}_{S'})$-module.

Proof. This is a special case of Lemma 37.11.5. □

The following lemmas explain consequences of Proposition 37.11.4 in case we only assume the morphism and the sheaf are of finite type (and not necessarily of finite presentation).

Lemma 37.11.8. *Let $f : X \to S$ be a morphism of schemes. Let $\mathcal{F}$ be a quasi-coherent sheaf on X. Let $x \in X$ with image $s \in S$. Assume that*

(1) *f is locally of finite presentation,*
(2) *$\mathcal{F}$ is of finite type, and*
(3) *$\mathcal{F}$ is flat at x over S.*

Then there exists an elementary étale neighbourhood $(S', s') \to (S, s)$ and a commutative diagram of pointed schemes

$$\begin{array}{ccc} (X, x) & \xleftarrow[g]{} & (X', x') \\ \downarrow & & \downarrow \\ (S, s) & \longleftarrow & (\mathrm{Spec}(\mathcal{O}_{S',s'}), s') \end{array}$$

such that $X' \to X \times_S \mathrm{Spec}(\mathcal{O}_{S',s'})$ is étale, $\kappa(x) = \kappa(x')$, the scheme X' is affine of finite presentation over $\mathcal{O}_{S',s'}$, the sheaf $g^*\mathcal{F}$ is of finite presentation over $\mathcal{O}_{X'}$, and such that $\Gamma(X', g^*\mathcal{F})$ is a free $\mathcal{O}_{S',s'}$-module.

Proof. To prove the lemma we may replace (S, s) by any elementary étale neighbourhood, and we may also replace S by $\mathrm{Spec}(\mathcal{O}_{S,s})$. Hence by Proposition 37.10.3 we may assume that $\mathcal{F}$ is finitely presented and flat over S in a neighbourhood of x. In this case the result follows from Proposition 37.11.4 because Algebra, Theorem 10.80.4 assures us that projective = free over a local ring. □

Lemma 37.11.9. *Let $f : X \to S$ be a morphism of schemes. Let $\mathcal{F}$ be a quasi-coherent sheaf on X. Let $x \in X$ with image $s \in S$. Assume that*

1. *f is locally of finite type,*
2. *$\mathcal{F}$ is of finite type, and*
3. *$\mathcal{F}$ is flat at x over S.*

Then there exists an elementary étale neighbourhood $(S', s') \to (S, s)$ and a commutative diagram of pointed schemes

$$\begin{array}{ccc} (X, x) & \xleftarrow[g]{} & (X', x') \\ \downarrow & & \downarrow \\ (S, s) & \longleftarrow & (\mathrm{Spec}(\mathcal{O}_{S',s'}), s') \end{array}$$

such that $X' \to X \times_S \mathrm{Spec}(\mathcal{O}_{S',s'})$ is étale, $\kappa(x) = \kappa(x')$, the scheme X' is affine, and such that $\Gamma(X', g^*\mathcal{F})$ is a free $\mathcal{O}_{S',s'}$-module.

Proof. (The only difference with Lemma 37.11.8 is that we do not assume f is of finite presentation.) The problem is local on X and S. Hence we may assume X and S are affine, say $X = \mathrm{Spec}(B)$ and $S = \mathrm{Spec}(A)$. Since B is a finite type A-algebra we can find a surjection $A[x_1, \ldots, x_n] \to B$. In other words, we can choose a closed immersion $i : X \to \mathbf{A}^n_S$. Set $t = i(x)$ and $\mathcal{G} = i_*\mathcal{F}$. Note that $\mathcal{G}_t \cong \mathcal{F}_x$ are $\mathcal{O}_{S,s}$-modules. Hence $\mathcal{G}$ is flat over S at t. We apply Lemma 37.11.8 to the morphism $\mathbf{A}^n_S \to S$, the point t, and the sheaf $\mathcal{G}$. Thus we can find an elementary étale neighbourhood $(S', s') \to (S, s)$ and a commutative diagram of pointed schemes

$$\begin{array}{ccc} (\mathbf{A}^n_S, t) & \xleftarrow[h]{} & (Y, y) \\ \downarrow & & \downarrow \\ (S, s) & \longleftarrow & (\mathrm{Spec}(\mathcal{O}_{S',s'}), s') \end{array}$$

such that $Y \to \mathbf{A}^n_{\mathcal{O}_{S',s'}}$ is étale, $\kappa(t) = \kappa(y)$, the scheme Y is affine, and such that $\Gamma(Y, h^*\mathcal{G})$ is a projective $\mathcal{O}_{S',s'}$-module. Then a solution to the original problem is given by the closed subscheme $X' = Y \times_{\mathbf{A}^n_S} X$ of Y. □

Lemma 37.11.10. *Let $f : X \to S$ be a morphism of schemes. Let $\mathcal{F}$ be a quasi-coherent sheaf on X. Let $s \in S$. Assume that*

(1) *f is of finite presentation,*
(2) *$\mathcal{F}$ is of finite type, and*
(3) *$\mathcal{F}$ is flat over S at all points of X_s.*

Then there exists an elementary étale neighbourhood $(S', s') \to (S, s)$ and a commutative diagram of schemes

$$\begin{array}{ccc} X & \xleftarrow{g} & X' \\ \downarrow & & \downarrow \\ S & \longleftarrow & \mathrm{Spec}(\mathcal{O}_{S',s'}) \end{array}$$

such that $X' \to X \times_S \mathrm{Spec}(\mathcal{O}_{S',s'})$ is étale, $X_s = g((X')_{s'})$, the scheme X' is affine of finite presentation over $\mathcal{O}_{S',s'}$, the sheaf $g^*\mathcal{F}$ is of finite presentation over $\mathcal{O}_{X'}$, and such that $\Gamma(X', g^*\mathcal{F})$ is a free $\mathcal{O}_{S',s'}$-module.

Proof. For every point $x \in X_s$ we can use Lemma 37.11.8 to find an elementary étale neighbourhood $(S_x, s_x) \to (S, s)$ and a commutative diagram

$$\begin{array}{ccc} (X, x) & \xleftarrow{g_x} & (Y_x, y_x) \\ \downarrow & & \downarrow \\ (S, s) & \longleftarrow & (\mathrm{Spec}(\mathcal{O}_{S_x,s_x}), s_x) \end{array}$$

such that $Y_x \to X \times_S \mathrm{Spec}(\mathcal{O}_{S_x,s_x})$ is étale, $\kappa(x) = \kappa(y_x)$, the scheme Y_x is affine of finite presentation over $\mathcal{O}_{S_x,s_x}$, the sheaf $g_x^*\mathcal{F}$ is of finite presentation over $\mathcal{O}_{Y_x}$, and such that $\Gamma(Y_x, g_x^*\mathcal{F})$ is a free $\mathcal{O}_{S_x,s_x}$-module. In particular $g_x((Y_x)_{s_x})$ is an open neighbourhood of x in X_s. Because X_s is quasi-compact we can find a finite number of points $x_1, \ldots, x_n \in X_s$ such that X_s is the union of the $g_{x_i}((Y_{x_i})_{s_{x_i}})$. Choose an elementary étale neighbourhood $(S', s') \to (S, s)$ which dominates each of the neighbourhoods (S_{x_i}, s_{x_i}), see More on Morphisms, Lemma 36.27.4. Set

$$X' = \coprod Y_{x_i} \times_{\mathrm{Spec}(\mathcal{O}_{S_{x_i},s_{x_i}})} \mathrm{Spec}(\mathcal{O}_{S',s'})$$

and endow it with the obvious morphism $g : X' \to X$. By construction $X_s = g(X'_{s'})$ and

$$\Gamma(X', g^*\mathcal{F}) = \bigoplus \Gamma(Y_{x_i}, g_{x_i}^*\mathcal{F}) \otimes_{\mathcal{O}_{S_{x_i},s_{x_i}}} \mathcal{O}_{S',s'}.$$

This is a free $\mathcal{O}_{S',s'}$-module as a direct sum of base changes of free modules. Some minor details omitted. □

Lemma 37.11.11. *Let $f : X \to S$ be a morphism of schemes. Let $\mathcal{F}$ be a quasi-coherent sheaf on X. Let $s \in S$. Assume that*

(1) *f is of finite type,*
(2) *$\mathcal{F}$ is of finite type, and*
(3) *$\mathcal{F}$ is flat over S at all points of X_s.*

Then there exists an elementary étale neighbourhood $(S', s') \to (S, s)$ and a commutative diagram of schemes

$$\begin{array}{ccc} X & \xleftarrow{g} & X' \\ \downarrow & & \downarrow \\ S & \longleftarrow & \mathrm{Spec}(\mathcal{O}_{S',s'}) \end{array}$$

such that $X' \to X \times_S \mathrm{Spec}(\mathcal{O}_{S',s'})$ is étale, $X_s = g((X')_{s'})$, the scheme X' is affine, and such that $\Gamma(X', g^*\mathcal{F})$ is a free $\mathcal{O}_{S',s'}$-module.

Proof. (The only difference with Lemma 37.11.10 is that we do not assume f is of finite presentation.) For every point $x \in X_s$ we can use Lemma 37.11.9 to find an elementary étale neighbourhood $(S_x, s_x) \to (S, s)$ and a commutative diagram

$$\begin{array}{ccc} (X, x) & \xleftarrow{g_x} & (Y_x, y_x) \\ \downarrow & & \downarrow \\ (S, s) & \longleftarrow & (\mathrm{Spec}(\mathcal{O}_{S_x,s_x}), s_x) \end{array}$$

such that $Y_x \to X \times_S \mathrm{Spec}(\mathcal{O}_{S_x,s_x})$ is étale, $\kappa(x) = \kappa(y_x)$, the scheme Y_x is affine, and such that $\Gamma(Y_x, g_x^*\mathcal{F})$ is a free $\mathcal{O}_{S_x,s_x}$-module. In particular $g_x((Y_x)_{s_x})$ is an open neighbourhood of x in X_s. Because X_s is quasi-compact we can find a finite number of points $x_1, \ldots, x_n \in X_s$ such that X_s is the union of the $g_{x_i}((Y_{x_i})_{s_{x_i}})$. Choose an elementary étale neighbourhood $(S', s') \to (S, s)$ which dominates each of the neighbourhoods (S_{x_i}, s_{x_i}), see More on Morphisms, Lemma 36.27.4. Set

$$X' = \coprod Y_{x_i} \times_{\mathrm{Spec}(\mathcal{O}_{S_{x_i},s_{x_i}})} \mathrm{Spec}(\mathcal{O}_{S',s'})$$

and endow it with the obvious morphism $g : X' \to X$. By construction $X_s = g(X'_{s'})$ and

$$\Gamma(X', g^*\mathcal{F}) = \bigoplus \Gamma(Y_{x_i}, g_{x_i}^*\mathcal{F}) \otimes_{\mathcal{O}_{S_{x_i},s_{x_i}}} \mathcal{O}_{S',s'}.$$

This is a free $\mathcal{O}_{S',s'}$-module as a direct sum of base changes of free modules. □

37.12. Flat finite type modules, Part II

The following lemma will be superseded by the stronger Lemma 37.12.3 below.

Lemma 37.12.1. *Let $(R, \mathfrak{m})$ be a local ring. Let $R \to S$ be of finite presentation. Let N be a finitely presented S-module which is free as an R-module. Let M be an R-module. Let $\mathfrak{q}$ be a prime of S lying over $\mathfrak{m}$. Then*

1. *if $\mathfrak{q} \in WeakAss_S(M \otimes_R N)$ then $\mathfrak{m} \in WeakAss_R(M)$ and $\overline{\mathfrak{q}} \in Ass_{\overline{S}}(\overline{N})$,*
2. *if $\mathfrak{m} \in WeakAss_R(M)$ and $\overline{\mathfrak{q}} \in Ass_{\overline{S}}(\overline{N})$ is a maximal element then $\mathfrak{q} \in WeakAss_S(M \otimes_R N)$.*

Here $\overline{S} = S/\mathfrak{m}S$, $\overline{\mathfrak{q}} = \mathfrak{q}\overline{S}$, and $\overline{N} = N/\mathfrak{m}N$.

Proof. Suppose that $\overline{\mathfrak{q}} \notin \mathrm{Ass}_{\overline{S}}(\overline{N})$. By Algebra, Lemmas 10.60.9, 10.60.5, and 10.14.3 there exists an element $\overline{g} \in \overline{\mathfrak{q}}$ which is not a zerodivisor on $\overline{N}$. Let $g \in \mathfrak{q}$ be an element which maps to $\overline{g}$ in $\overline{\mathfrak{q}}$. By Lemma 37.7.6 the map $g : N \to N$ is R-universally injective. In particular we see that $g : M \otimes_R N \to M \otimes_R N$ is injective. Clearly this implies that $\mathfrak{q} \notin \mathrm{WeakAss}_S(M \otimes_R N)$. We conclude that $\mathfrak{q} \in \mathrm{WeakAss}_S(M \otimes_R N)$ implies $\overline{\mathfrak{q}} \in \mathrm{Ass}_{\overline{S}}(\overline{N})$.

Assume $\mathfrak{q} \in \text{WeakAss}_S(M \otimes_R N)$. Let $z \in M \otimes_R N$ be an element whose annihilator in S has radical $\mathfrak{q}$. As N is a free R-module, we can find a finite free direct summand $F \subset N$ such that $z \in M \otimes_R F$. The radical of the annihilator of $z \in M \otimes_R F$ in R is $\mathfrak{m}$ (by our assumption on z and because $\mathfrak{q}$ lies over $\mathfrak{m}$). Hence we see that $\mathfrak{m} \in \text{WeakAss}(M \otimes_R F)$ which implies that $\mathfrak{m} \in \text{WeakAss}(M)$ by Algebra, Lemma 10.63.3. This finishes the proof of (1).

Assume that $\mathfrak{m} \in \text{WeakAss}_R(M)$ and $\overline{\mathfrak{q}} \in \text{Ass}_{\overline{S}}(\overline{N})$ is a maximal element. Let $y \in M$ be an element whose annihilator $I = \text{Ann}_R(y)$ has radical $\mathfrak{m}$. Then $R/I \subset M$ and by flatness of N over R we get $N/IN = R/I \otimes_R N \subset M \otimes_R N$. Hence it is enough to show that $\mathfrak{q} \in \text{WeakAss}(N/IN)$. Write $\overline{\mathfrak{q}} = (\overline{g}_1, \ldots, \overline{g}_n)$ for some $\overline{g}_i \in \overline{S}$. Choose lifts $g_i \in \mathfrak{q}$. Consider the map

$$\Psi : N/IN \longrightarrow N/IN^{\oplus n}, \quad z \longmapsto (g_1 z, \ldots, g_n z).$$

We may think of this as a map of free R/I-modules. As the ring R/I is auto-associated (since $\mathfrak{m}/I$ is locally nilpotent) and since $\Psi \otimes R/\mathfrak{m}$ isn't injective (since $\overline{\mathfrak{q}} \in \text{Ass}(\overline{N})$) we see by More on Algebra, Lemma 15.8.4 that Ψ isn't injective. Pick $z \in N/IN$ nonzero in the kernel of Ψ. The annihilator of z contains I and g_i, whence its radical $J = \sqrt{\text{Ann}_S(z)}$ contains $\mathfrak{q}$. Let $\mathfrak{q}' \supset J$ be a minimal prime over J. Then $\mathfrak{q}' \in \text{WeakAss}(M \otimes_R N)$ (by definition) and by (1) we see that $\overline{\mathfrak{q}}' \in \text{Ass}(\overline{N})$. Then since $\mathfrak{q} \subset \mathfrak{q}'$ by construction the maximality of $\overline{\mathfrak{q}}$ implies $\mathfrak{q} = \mathfrak{q}'$ whence $\mathfrak{q} \in \text{WeakAss}(M \otimes_R N)$. This proves part (2) of the lemma. □

Lemma 37.12.2. *Let S be a scheme. Let $f : X \to S$ be locally of finite type. Let $x \in X$ with image $s \in S$. Let $\mathcal{F}$ be a finite type quasi-coherent sheaf on X. Let $\mathcal{G}$ be a quasi-coherent sheaf on S. If $\mathcal{F}$ is flat at x over S, then*

$$x \in \textit{WeakAss}_X(\mathcal{F} \otimes_{\mathcal{O}_X} f^*\mathcal{G}) \Leftrightarrow s \in \textit{WeakAss}_S(\mathcal{G}) \textit{ and } x \in \textit{Ass}_{X_s}(\mathcal{F}_s).$$

Proof. The question is local on X and S, hence we may assume X and S are affine. Write $X = \text{Spec}(B)$, $S = \text{Spec}(A)$ and write $B = A[x_1, \ldots, x_n]/I$. In other words we obtain a closed immersion $i : X \to \mathbf{A}^n_S$ over S. Denote $t = i(x) \in \mathbf{A}^n_S$. Note that $i_*\mathcal{F}$ is a finite type quasi-coherent sheaf on $\mathbf{A}^n_S$ which is flat at t over S and note that

$$i_*(\mathcal{F} \otimes_{\mathcal{O}_X} f^*\mathcal{G}) = i_*\mathcal{F} \otimes_{\mathcal{O}_{\mathbf{A}^n_S}} p^*\mathcal{G}$$

where $p : \mathbf{A}^n_S \to S$ is the projection. Note that t is a weakly associated point of $i_*(\mathcal{F} \otimes_{\mathcal{O}_X} f^*\mathcal{G})$ if and only if x is a weakly associated point of $\mathcal{F} \otimes_{\mathcal{O}_X} f^*\mathcal{G}$, see Divisors, Lemma 30.6.3. Similarly $x \in \text{Ass}_{X_s}(\mathcal{F}_s)$ if and only if $t \in \text{Ass}_{\mathbf{A}^n_s}((i_*\mathcal{F})_s)$ (see Algebra, Lemma 10.60.13). Hence it suffices to prove the lemma in case $X = \mathbf{A}^n_S$. In particular we may assume that $X \to S$ is of finite presentation.

Recall that $\text{Ass}_{X_s}(\mathcal{F}_s)$ is a locally finite subset of the locally Noetherian scheme X_s, see Divisors, Lemma 30.2.5. After replacing X by a suitable affine neighbourhood of x we may assume that

$$(*) \text{ if } x' \in \text{Ass}_{X_s}(\mathcal{F}_s) \text{ and } x \rightsquigarrow x' \text{ then } x = x'.$$

(Proof omitted. Hint: using Algebra, Lemma 10.14.3 invert a function which does not vanish at x but does vanish in all the finitely many points of $\text{Ass}_{X_s}(\mathcal{F}_s)$ which are specializations of x but not equal to x.) In words, no point of $\text{Ass}_{X_s}(\mathcal{F}_s)$ is a proper specialization of x.

Suppose given a commutative diagram

$$\begin{array}{ccc} (X, x) & \xleftarrow{g} & (X', x') \\ \downarrow & & \downarrow \\ (S, s) & \xleftarrow{e} & (S', s') \end{array}$$

of pointed schemes whose horizontal arrows are elementary étale neighbourhoods. Then it suffices to prove the statement for x', s', $g^*\mathcal{F}$ and $e^*\mathcal{G}$, see Lemma 37.2.7. Note that property $(*)$ is preserved by such an étale localization by the same lemma (if there is a proper specialization $x' \rightsquigarrow x''$ on $X'_{s'}$ then this maps to a proper specialization on X_s because the fibres of an étale morphism are discrete). We may also replace S by the spectrum of its local ring as the condition of being an associated point of a quasi-coherent sheaf depends only on the stalk of the sheaf. Again property $(*)$ is preserved by this as well. Thus we may first apply Proposition 37.10.3 to reduce to the case where $\mathcal{F}$ is of finite presentation and flat over S, whereupon we may use Proposition 37.11.4 to reduce to the case that $X \to S$ is a morphism of affines and $\Gamma(X, \mathcal{F})$ is a finitely presented $\Gamma(X, \mathcal{O}_X)$-module which is projective as a $\Gamma(S, \mathcal{O}_S)$-module. Localizing S once more we may assume that $\Gamma(S, \mathcal{O}_S)$ is a local ring such that s corresponds to the maximal ideal. In this case Algebra, Theorem 10.80.4 guarantees that $\Gamma(X, \mathcal{F})$ is free as an $\Gamma(S, \mathcal{O}_S)$-module. The implication $x \in \text{WeakAss}_X(\mathcal{F} \otimes_{\mathcal{O}_X} f^*\mathcal{G}) \Rightarrow s \in \text{WeakAss}_S(\mathcal{G})$ and $x \in \text{Ass}_{X_s}(\mathcal{F}_s)$ follows from part (1) of Lemma 37.12.1. The converse implication follows from part (2) of Lemma 37.12.1 as property $(*)$ insures that the prime corresponding to x gives rise to a maximal element of $\text{Ass}_{\overline{S}}(\overline{N})$ exactly as in the statement of part (2) of Lemma 37.12.1. □

Lemma 37.12.3. *Let $R \to S$ be a ring map which is essentially of finite type. Let N be a localization of a finite S-module flat over R. Let M be an R-module. Then*

$$\mathit{WeakAss}_S(M \otimes_R N) = \bigcup\nolimits_{\mathfrak{p} \in \mathit{WeakAss}_R(M)} \mathit{Ass}_{S \otimes_R \kappa(\mathfrak{p})}(N \otimes_R \kappa(\mathfrak{p}))$$

Proof. This lemma is a translation of Lemma 37.12.2 into algebra. Details of translation omitted. □

Lemma 37.12.4. *Let $f : X \to S$ be a morphism which is locally of finite type. Let $\mathcal{F}$ be a finite type quasi-coherent sheaf on X which is flat over S. Let $\mathcal{G}$ be a quasi-coherent sheaf on S. Then we have*

$$\mathit{WeakAss}_X(\mathcal{F} \otimes_{\mathcal{O}_X} f^*\mathcal{G}) = \bigcup\nolimits_{s \in \mathit{WeakAss}_S(\mathcal{G})} \mathit{Ass}_{X_s}(\mathcal{F}_s)$$

Proof. Immediate consequence of Lemma 37.12.2. □

Theorem 37.12.5. *Let $f : X \to S$ be a morphism of schemes. Let $\mathcal{F}$ be a quasi-coherent $\mathcal{O}_X$-module. Assume*

1. *$X \to S$ is locally of finite presentation,*
2. *$\mathcal{F}$ is an $\mathcal{O}_X$-module of finite type, and*
3. *the set of weakly associated points of S is locally finite in S.*

Then $U = \{x \in X \mid \mathcal{F} \text{ flat at } x \text{ over } S\}$ is open in X and $\mathcal{F}|_U$ is an $\mathcal{O}_U$-module of finite presentation and flat over S.

Proof. Let $x \in X$ be such that $\mathcal{F}$ is flat at x over S. We have to find an open neighbourhood of x such that $\mathcal{F}$ restricts to a S-flat finitely presented module on this neighbourhood. The problem is local on X and S, hence we may assume that X and S are affine. As $\mathcal{F}_x$ is a finitely presented $\mathcal{O}_{X,x}$-module by Lemma 37.10.9 we conclude from Algebra, Lemma 10.120.5 there exists a finitely presented $\mathcal{O}_X$-module $\mathcal{F}'$ and a map $\varphi : \mathcal{F}' \to \mathcal{F}$ which induces an isomorphism $\varphi_x : \mathcal{F}'_x \to \mathcal{F}_x$. In particular we see that $\mathcal{F}'$ is flat over S at x, hence by openness of flatness More on Morphisms, Theorem 36.12.1 we see that after shrinking X we may assume that $\mathcal{F}'$ is flat over S. As $\mathcal{F}$ is of finite type after shrinking X we may assume that φ is surjective, see Modules, Lemma 17.9.4 or alternatively use Nakayama's lemma (Algebra, Lemma 10.18.1). By Lemma 37.12.4 we have

$$\mathrm{WeakAss}_X(\mathcal{F}') \subset \bigcup\nolimits_{s \in \mathrm{WeakAss}(S)} \mathrm{Ass}_{X_s}(\mathcal{F}'_s)$$

As $\mathrm{WeakAss}(S)$ is finite by assumption and since $\mathrm{Ass}_{X_s}(\mathcal{F}'_s)$ is finite by Divisors, Lemma 30.2.5 we conclude that $\mathrm{WeakAss}_X(\mathcal{F}')$ is finite. Using Algebra, Lemma 10.14.3 we may, after shrinking X once more, assume that $\mathrm{WeakAss}_X(\mathcal{F}')$ is contained in the generalization of x. Now consider $\mathcal{K} = \mathrm{Ker}(\varphi)$. We have $\mathrm{WeakAss}_X(\mathcal{K}) \subset \mathrm{WeakAss}_X(\mathcal{F}')$ (by Divisors, Lemma 30.5.4) but on the other hand, φ_x is an isomorphism, also $\varphi_{x'}$ is an isomorphism for all $x' \rightsquigarrow x$. We conclude that $\mathrm{WeakAss}_X(\mathcal{K}) = \emptyset$ whence $\mathcal{K} = 0$ by Divisors, Lemma 30.5.5. □

Lemma 37.12.6. *Let $R \to S$ be a ring map of finite presentation. Let M be a finite S-module. Assume $\mathrm{WeakAss}_S(S)$ is finite. Then*

$$U = \{\mathfrak{q} \subset S \mid M_{\mathfrak{q}} \text{ flat over } R\}$$

is open in $\mathrm{Spec}(S)$ *and for every $g \in S$ such that $D(g) \subset U$ the localization M_g is a finitely presented S_g-module flat over R.*

Proof. Follows immediately from Theorem 37.12.5. □

Lemma 37.12.7. *Let $f : X \to S$ be a morphism of schemes which is locally of finite type. Assume the set of weakly associated points of S is locally finite in S. Then the set of points $x \in X$ where f is flat is an open subscheme $U \subset X$ and $U \to S$ is flat and locally of finite presentation.*

Proof. The problem is local on X and S, hence we may assume that X and S are affine. Then $X \to S$ corresponds to a finite type ring map $A \to B$. Choose a surjection $A[x_1, \ldots, x_n] \to B$ and consider B as an $A[x_1, \ldots, x_n]$-module. An application of Lemma 37.12.6 finishes the proof. □

Lemma 37.12.8. *Let $f : X \to S$ be a morphism of schemes which is locally of finite type and flat. If S is integral, then f is locally of finite presentation.*

Proof. Special case of Lemma 37.12.7. □

Lemma 37.12.9. *Let $A \to B$ be a finite type, flat ring map with A an integral domain. Then B is a finitely presented A-algebra.*

Proof. Special case of Lemma 37.12.8. It is also a consequence of More on Algebra, Proposition 15.16.8. □

Remark 37.12.10 (Finite type version of Theorem 37.12.5)**.** Let $f : X \to S$ be a morphism of schemes. Let $\mathcal{F}$ be a quasi-coherent $\mathcal{O}_X$-module. Assume

(1) $X \to S$ is locally of finite type,
(2) $\mathcal{F}$ is an $\mathcal{O}_X$-module of finite type, and
(3) the set of weakly associated points of S is locally finite in S.

Then $U = \{x \in X \mid \mathcal{F} \text{ flat at } x \text{ over } S\}$ is open in X and $\mathcal{F}|_U$ is flat over S and locally finitely presented relative to S (see More on Morphisms, Definition 36.38.1). If we ever need this result in the stacks project we will convert this remark into a lemma with a proof.

Remark 37.12.11 (Algebra version of Remark 37.12.10)**.** Let $R \to S$ be a ring map of finite type. Let M be a finite S-module. Assume $\text{WeakAss}_S(S)$ is finite. Then

$$U = \{\mathfrak{q} \subset S \mid M_\mathfrak{q} \text{ flat over } R\}$$

is open in $\text{Spec}(S)$ and for every $g \in S$ such that $D(g) \subset U$ the localization M_g is flat over R and an S_g-module finitely presented relative to R (see More on Algebra, Definition 15.52.2). If we ever need this result in the stacks project we will convert this remark into a lemma with a proof.

37.13. Examples of relatively pure modules

In the short section we discuss some examples of results that will serve as motivation for the notion of a *relatively pure module* and the concept of an *impurity* which we will introduce later. Each of the examples is stated as a lemma. Note the similarity with the condition on associated primes to the conditions appearing in Lemmas 37.7.4, 37.8.3, 37.8.4, and 37.9.1. See also Algebra, Lemma 10.62.1 for a discussion.

Lemma 37.13.1. *Let R be a local ring with maximal ideal $\mathfrak{m}$. Let $R \to S$ be a ring map. Let N be an S-module. Assume*

(1) *N is projective as an R-module, and*
(2) *$S/\mathfrak{m}S$ is Noetherian and $N/\mathfrak{m}N$ is a finite $S/\mathfrak{m}S$-module.*

Then for any prime $\mathfrak{q} \subset S$ which is an associated prime of $N \otimes_R \kappa(\mathfrak{p})$ where $\mathfrak{p} = R \cap \mathfrak{q}$ we have $\mathfrak{q} + \mathfrak{m}S \neq S$.

Proof. Note that the hypotheses of Lemmas 37.7.1 and 37.7.6 are satisfied. We will use the conclusions of these lemmas without further mention. Let $\Sigma \subset S$ be the multiplicative set of elements which are not zerodivisors on $N/\mathfrak{m}N$. The map $N \to \Sigma^{-1}N$ is R-universally injective. Hence we see that any $\mathfrak{q} \subset S$ which is an associated prime of $N \otimes_R \kappa(\mathfrak{p})$ is also an associated prime of $\Sigma^{-1}N \otimes_R \kappa(\mathfrak{p})$. Clearly this implies that $\mathfrak{q}$ corresponds to a prime of $\Sigma^{-1}S$. Thus $\mathfrak{q} \subset \mathfrak{q}'$ where $\mathfrak{q}'$ corresponds to an associated prime of $N/\mathfrak{m}N$ and we win. □

The following lemma gives another (slightly silly) example of this phenomenon.

Lemma 37.13.2. *Let R be a ring. Let $I \subset R$ be an ideal. Let $R \to S$ be a ring map. Let N be an S-module. If N is I-adically complete, then for any R-module M and for any prime $\mathfrak{q} \subset S$ which is an associated prime of $N \otimes_R M$ we have $\mathfrak{q} + IS \neq S$.*

Proof. Let $S^\wedge$ denote the I-adic completion of S. Note that N is an $S^\wedge$-module, hence also $N \otimes_R M$ is an $S^\wedge$-module. Let $z \in N \otimes_R M$ be an element such that $\mathfrak{q} = \text{Ann}_S(z)$. Since $z \neq 0$ we see that $\text{Ann}_{S^\wedge}(z) \neq S^\wedge$. Hence $\mathfrak{q}S^\wedge \neq S^\wedge$. Hence there exists a maximal ideal $\mathfrak{m} \subset S^\wedge$ with $\mathfrak{q}S^\wedge \subset \mathfrak{m}$. Since $IS^\wedge \subset \mathfrak{m}$ by Algebra, Lemma 10.91.11 we win. □

Note that the following lemma gives an alternative proof of Lemma 37.13.1 as a projective module over a local ring is free, see Algebra, Theorem 10.80.4.

Lemma 37.13.3. *Let R be a local ring with maximal ideal $\mathfrak{m}$. Let $R \to S$ be a ring map. Let N be an S-module. Assume N is isomorphic as an R-module to a direct sum of finite R-modules. Then for any R-module M and for any prime $\mathfrak{q} \subset S$ which is an associated prime of $N \otimes_R M$ we have $\mathfrak{q} + \mathfrak{m}S \neq S$.*

Proof. Write $N = \bigoplus_{i \in I} M_i$ with each M_i a finite R-module. Let M be an R-module and let $\mathfrak{q} \subset S$ be an associated prime of $N \otimes_R M$ such that $\mathfrak{q} + \mathfrak{m}S = S$. Let $z \in N \otimes_R M$ be an element with $\mathfrak{q} = \text{Ann}_S(z)$. After modifying the direct sum decomposition a little bit we may assume that $z \in M_1 \otimes_R M$ for some element $1 \in I$. Write $1 = f + \sum x_j g_j$ for some $f \in \mathfrak{q}$, $x_j \in \mathfrak{m}$, and $g_j \in S$. For any $g \in S$ denote g' the R-linear map

$$M_1 \to N \xrightarrow{g} N \to M_1$$

where the first arrow is the inclusion map, the second arrow is multiplication by g and the third arrow is the projection map. Because each $x_j \in R$ we obtain the equality

$$f' + \sum x_j g'_j = \text{id}_{M_1} \in \text{End}_R(M_1)$$

By Nakayama's lemma (Algebra, Lemma 10.18.1) we see that f' is surjective, hence by Algebra, Lemma 10.15.4 we see that f' is an isomorphism. In particular the map

$$M_1 \otimes_R M \to N \otimes_R M \xrightarrow{f} N \otimes_R M \to M_1 \otimes_R M$$

is an isomorphism. This contradicts the assumption that $fz = 0$. □

Lemma 37.13.4. *Let R be a henselian local ring with maximal ideal $\mathfrak{m}$. Let $R \to S$ be a ring map. Let N be an S-module. Assume N is countably generated and Mittag-Leffler as an R-module. Then for any R-module M and for any prime $\mathfrak{q} \subset S$ which is an associated prime of $N \otimes_R M$ we have $\mathfrak{q} + \mathfrak{m}S \neq S$.*

Proof. This lemma reduces to Lemma 37.13.3 by Algebra, Lemma 10.143.32. □

Suppose $f : X \to S$ is a morphism of schemes and $\mathcal{F}$ is a quasi-coherent module on X. Let $\xi \in \text{Ass}_{X/S}(\mathcal{F})$ and let $Z = \overline{\{\xi\}}$. Picture

$$\begin{array}{ccc} \xi & & Z \longrightarrow X \\ \downarrow & & \searrow \quad \downarrow f \\ f(\xi) & & S \end{array}$$

Note that $f(Z) \subset \overline{\{f(\xi)\}}$ and that $f(Z)$ is closed if and only if equality holds, i.e., $f(Z) = \overline{\{f(\xi)\}}$. It follows from Lemma 37.13.1 that if S, X are affine, the fibres X_s are Noetherian, $\mathcal{F}$ is of finite type, and $\Gamma(X, \mathcal{F})$ is a projective $\Gamma(S, \mathcal{O}_S)$-module, then $f(Z) = \overline{\{f(\xi)\}}$ is a closed subset. Slightly different analogous statements holds for the cases described in Lemmas 37.13.2, 37.13.3, and 37.13.4.

37.14. Impurities

We want to formalize the phenomenon of which we gave examples in Section 37.13 in terms of specializations of points of $\mathrm{Ass}_{X/S}(\mathcal{F})$. We also want to work locally around a point $s \in S$. In order to do so we make the following definitions.

Situation 37.14.1. Here S, X are schemes and $f : X \to S$ is a finite type morphism. Also, $\mathcal{F}$ is a finite type quasi-coherent $\mathcal{O}_X$-module. Finally s is a point of S.

In this situation consider a morphism $g : T \to S$, a point $t \in T$ with $g(t) = s$, a specialization $t' \rightsquigarrow t$, and a point $\xi \in X_T$ in the base change of X lying over t'. Picture

$$\begin{array}{ccc} \xi & & X_T \longrightarrow X \\ \downarrow & & \downarrow \quad\quad \downarrow \\ t' \rightsquigarrow t & & T \xrightarrow{g} S \\ \downarrow & & \downarrow g \\ s & & S \end{array} \tag{37.14.1.1}$$

Moreover, denote $\mathcal{F}_T$ the pullback of $\mathcal{F}$ to X_T.

Definition 37.14.2. In Situation 37.14.1 we say a diagram (37.14.1.1) defines an *impurity of $\mathcal{F}$ above s* if $\xi \in \mathrm{Ass}_{X_T/T}(\mathcal{F}_T)$ and $\overline{\{\xi\}} \cap X_t = \emptyset$. We will indicate this by saying "let $(g : T \to S, t' \rightsquigarrow t, \xi)$ be an impurity of $\mathcal{F}$ above s".

Lemma 37.14.3. *In Situation 37.14.1. If there exists an impurity of $\mathcal{F}$ above s, then there exists an impurity $(g : T \to S, t' \rightsquigarrow t, \xi)$ of $\mathcal{F}$ above s such that g is locally of finite presentation and t a closed point of the fibre of g above s.*

Proof. Let $(g : T \to S, t' \rightsquigarrow t, \xi)$ be any impurity of $\mathcal{F}$ above s. We apply Limits, Lemma 31.13.1 to $t \in T$ and $Z = \overline{\{\xi\}}$ to obtain an open neighbourhood $V \subset T$ of t, a commutative diagram

$$\begin{array}{ccc} V & \xrightarrow{a} & T' \\ \downarrow & & \downarrow b \\ T & \xrightarrow{g} & S, \end{array}$$

and a closed subscheme $Z' \subset X_{T'}$ such that

(1) the morphism $b : T' \to S$ is locally of finite presentation,
(2) we have $Z' \cap X_{a(t)} = \emptyset$, and
(3) $Z \cap X_V$ maps into Z' via the morphism $X_V \to X_{T'}$.

As t' specializes to t we may replace T by the open neighbourhood V of t. Thus we have a commutative diagram

$$\begin{array}{ccccc} X_T & \longrightarrow & X_{T'} & \longrightarrow & X \\ \downarrow & & \downarrow & & \downarrow \\ T & \xrightarrow{a} & T' & \xrightarrow{b} & S \end{array}$$

where $b \circ a = g$. Let $\xi' \in X_{T'}$ denote the image of ξ. By Divisors, Lemma 30.7.2 we see that $\xi' \in \mathrm{Ass}_{X_{T'}/T'}(\mathcal{F}_{T'})$. Moreover, by construction the closure of $\overline{\{\xi'\}}$ is

contained in the closed subset Z' which avoids the fibre $X_{a(t)}$. In this way we see that $(T' \to S, a(t') \rightsquigarrow a(t), \xi')$ is an impurity of $\mathcal{F}$ above s.

Thus we may assume that $g : T \to S$ is locally of finite presentation. Let $Z = \overline{\{\xi\}}$. By assumption $Z_t = \emptyset$. By More on Morphisms, Lemma 36.19.1 this means that $Z_{t''} = \emptyset$ for t'' in an open subset of $\overline{\{t\}}$. Since the fibre of $T \to S$ over s is a Jacobson scheme, see Morphisms, Lemma 28.17.10 we find that there exist a closed point $t'' \in \overline{\{t\}}$ such that $Z_{t''} = \emptyset$. Then $(g : T \to S, t' \rightsquigarrow t'', \xi)$ is the desired impurity. □

Lemma 37.14.4. *In Situation 37.14.1. Let $(g : T \to S, t' \rightsquigarrow t, \xi)$ be an impurity of $\mathcal{F}$ above s. Assume S is affine and that T is written $T = \lim_{i \in I} T_i$ as a directed colimit of affine schemes over S. Then for some i the triple $(T_i \to S, t'_i \rightsquigarrow t_i, \xi_i)$ is an impurity of $\mathcal{F}$ above s.*

Proof. The notation in the statement means this: Let $f_i : T \to T_i$ be the projection morphisms, let $t_i = f_i(t)$ and $t'_i = f_i(t')$. Finally $\xi_i \in X_{T_i}$ is the image of ξ. By Divisors, Lemma 30.7.2 it is true that ξ_i is a point of the relative assassin of $\mathcal{F}_{T_i}$ over T_i. Thus the only point is to show that $\overline{\{\xi_i\}} \cap X_{t_i} = \emptyset$ for some i. Set $Z = \overline{\{\xi\}}$. Apply Limits, Lemma 31.13.1 to this situation to obtain an open neighbourhood $V \subset T$ of t, a commutative diagram

$$\begin{array}{ccc} V & \xrightarrow{a} & T' \\ \downarrow & & \downarrow b \\ T & \xrightarrow{g} & S, \end{array}$$

and a closed subscheme $Z' \subset X_{T'}$ such that

(1) the morphism $b : T' \to S$ is locally of finite presentation,
(2) we have $Z' \cap X_{a(t)} = \emptyset$, and
(3) $Z \cap X_V$ maps into Z' via the morphism $X_V \to X_{T'}$.

We may assume V is an affine open of T, hence by Limits, Lemmas 31.3.8 and 31.3.10 we can find an i and an affine open $V_i \subset T_i$ with $V = f_i^{-1}(V_i)$. By Limits, Proposition 31.5.1 after possibly increasing i a bit we can find a morphism $a_i : V_i \to T'$ such that $a = a_i \circ f_i|_V$. The induced morphism $X_{T_i} \to X_{T'}$ maps ξ_i into Z'. As $Z' \cap X_{a(t)} = \emptyset$ we conclude that $(T_i \to S, t'_i \rightsquigarrow t_i, \xi_i)$ is an impurity of $\mathcal{F}$ above s. □

Lemma 37.14.5. *In Situation 37.14.1. If there exists an impurity $(g : T \to S, t' \rightsquigarrow t, \xi)$ of $\mathcal{F}$ above s with g quasi-finite at t, then there exists an impurity $(g : T \to S, t' \rightsquigarrow t, \xi)$ such that $(T, t) \to (S, s)$ is an elementary étale neighbourhood.*

Proof. Let $(g : T \to S, t' \rightsquigarrow t, \xi)$ be an impurity of $\mathcal{F}$ above s such that g is quasi-finite at t. After shrinking T we may assume that g is locally of finite type. Apply More on Morphisms, Lemma 36.30.1 to $T \to S$ and $t \mapsto s$. This gives us a diagram

$$\begin{array}{ccccc} T & \longleftarrow & T \times_S U & \longleftarrow & V \\ \downarrow & & \downarrow & \swarrow & \\ S & \longleftarrow & U & & \end{array}$$

where $(U, u) \to (S, s)$ is an elementary étale neighbourhood and $V \subset T \times_S U$ is an open neighbourhood of $v = (t, u)$ such that $V \to U$ is finite and such that v is the unique point of V lying over u. Since the morphism $V \to T$ is étale hence flat we see that there exists a specialization $v' \rightsquigarrow v$ such that $v' \mapsto t'$. Note that $\kappa(t') \subset \kappa(v')$ is finite separable. Pick any point $\zeta \in X_{v'}$ mapping to $\xi \in X_{t'}$. By Divisors, Lemma 30.7.2 we see that $\zeta \in \mathrm{Ass}_{X_V/V}(\mathcal{F}_V)$. Moreover, the closure $\overline{\{\zeta\}}$ does not meet the fibre X_v as by assumption the closure $\overline{\{\xi\}}$ does not meet X_t. In other words $(V \to S, v' \rightsquigarrow v, \zeta)$ is an impurity of $\mathcal{F}$ above S.

Next, let $u' \in U'$ be the image of v' and let $\theta \in X_U$ be the image of ζ. Then $\theta \mapsto u'$ and $u' \rightsquigarrow u$. By Divisors, Lemma 30.7.2 we see that $\theta \in \mathrm{Ass}_{X_U/U}(\mathcal{F})$. Moreover, as $\pi : X_V \to X_U$ is finite we see that $\pi\big(\overline{\{\zeta\}}\big) = \overline{\{\pi(\zeta)\}}$. Since v is the unique point of V lying over u we see that $X_u \cap \overline{\{\pi(\zeta)\}} = \emptyset$ because $X_v \cap \overline{\{\zeta\}} = \emptyset$. In this way we conclude that $(U \to S, u' \rightsquigarrow u, \theta)$ is an impurity of $\mathcal{F}$ above s and we win. □

Lemma 37.14.6. *In Situation 37.14.1. Assume that S is locally Noetherian. If there exists an impurity of $\mathcal{F}$ above s, then there exists an impurity $(g : T \to S, t' \rightsquigarrow t, \xi)$ of $\mathcal{F}$ above s such that g is quasi-finite at t.*

Proof. We may replace S by an affine neighbourhood of s. By Lemma 37.14.3 we may assume that we have an impurity $(g : T \to S, t' \rightsquigarrow t, \xi)$ of such that g is locally of finite type and t a closed point of the fibre of g above s. We may replace T by the reduced induced scheme structure on $\overline{\{t'\}}$. Let $Z = \overline{\{\xi\}} \subset X_T$. By assumption $Z_t = \emptyset$ and the image of $Z \to T$ contains t'. By More on Morphisms, Lemma 36.20.1 there exists a nonempty open $V \subset Z$ such that for any $w \in f(V)$ any generic point ξ' of V_w is in $\mathrm{Ass}_{X_T/T}(\mathcal{F}_T)$. By More on Morphisms, Lemma 36.19.2 there exists a nonempty open $W \subset T$ with $W \subset f(V)$. By More on Morphisms, Lemma 36.35.7 there exists a closed subscheme $T' \subset T$ such that $t \in T'$, $T' \to S$ is quasi-finite at t, and there exists a point $z \in T' \cap W$, $z \rightsquigarrow t$ which does not map to s. Choose any generic point ξ' of the nonempty scheme V_z. Then $(T' \to S, z \rightsquigarrow t, \xi')$ is the desired impurity. □

In the following we will use the henselization $S^h = \mathrm{Spec}(\mathcal{O}^h_{S,s})$ of S at s, see Étale Cohomology, Definition 44.33.2. Since $S^h \to S$ maps to closed point of S^h to s and induces an isomorphism of residue fields, we will indicate $s \in S^h$ this closed point also. Thus $(S^h, s) \to (S, s)$ is a morphism of pointed schemes.

Lemma 37.14.7. *In Situation 37.14.1. If there exists an impurity $(S^h \to S, s' \rightsquigarrow s, \xi)$ of $\mathcal{F}$ above s then there exists an impurity $(T \to S, t' \rightsquigarrow t, \xi)$ of $\mathcal{F}$ above s where $(T, t) \to (S, s)$ is an elementary étale neighbourhood.*

Proof. We may replace S by an affine neighbourhood of s. Say $S = \mathrm{Spec}(A)$ and s corresponds to the prime $\mathfrak{p} \subset A$. Then $\mathcal{O}^h_{S,s} = \mathrm{colim}_{(T,t)} \Gamma(T, \mathcal{O}_T)$ where the limit is over the opposite of the cofiltered category of affine elementary étale neighbourhoods (T, t) of (S, s), see More on Morphisms, Lemma 36.27.5 and its proof. Hence $S^h = \lim_i T_i$ and we win by Lemma 37.14.4. □

Lemma 37.14.8. *In Situation 37.14.1 the following are equivalent*

(1) *there exists an impurity $(S^h \to S, s' \rightsquigarrow s, \xi)$ of $\mathcal{F}$ above s where S^h is the henselization of S at s,*

(2) *there exists an impurity* $(T \to S, t' \rightsquigarrow t, \xi)$ *of* $\mathcal{F}$ *above* s *such that* $(T, t) \to (S, s)$ *is an elementary étale neighbourhood, and*
(3) *there exists an impurity* $(T \to S, t' \rightsquigarrow t, \xi)$ *of* $\mathcal{F}$ *above* s *such that* $T \to S$ *is quasi-finite at* t.

Proof. As an étale morphism is locally quasi-finite it is clear that (2) implies (3). We have seen that (3) implies (2) in Lemma 37.14.5. We have seen that (1) implies (2) in Lemma 37.14.7. Finally, if $(T \to S, t' \rightsquigarrow t, \xi)$ is an impurity of $\mathcal{F}$ above s such that $(T, t) \to (S, s)$ is an elementary étale neighbourhood, then we can choose a factorization $S^h \to T \to S$ of the structure morphism $S^h \to S$. Choose any point $s' \in S^h$ mapping to t' and choose any $\xi' \in X_{s'}$ mapping to $\xi \in X_{t'}$. Then $(S^h \to S, s' \rightsquigarrow s, \xi')$ is an impurity of $\mathcal{F}$ above s. We omit the details. □

37.15. Relatively pure modules

The notion of a module pure relative to a base was introduced in [**GR71**].

Definition 37.15.1. Let $f : X \to S$ be a morphism of schemes which is of finite type. Let $\mathcal{F}$ be a finite type quasi-coherent $\mathcal{O}_X$-module.

(1) Let $s \in S$. We say $\mathcal{F}$ is *pure along* X_s if there is no impurity $(g : T \to S, t' \rightsquigarrow t, \xi)$ of $\mathcal{F}$ above s with $(T, t) \to (S, s)$ an elementary étale neighbourhood.
(2) We say $\mathcal{F}$ is *universally pure along* X_s if there does not exist any impurity of $\mathcal{F}$ above s.
(3) We say that X is *pure along* X_s if $\mathcal{O}_X$ is pure along X_s.
(4) We say $\mathcal{F}$ is *universally S-pure*, or *universally pure relative to* S if $\mathcal{F}$ is universally pure along X_s for every $s \in S$.
(5) We say $\mathcal{F}$ is *S-pure*, or *pure relative to* S if $\mathcal{F}$ is pure along X_s for every $s \in S$.
(6) We say that X is *S-pure* or *pure relative to* S if $\mathcal{O}_X$ is pure relative to S.

We intentionally restrict ourselves here to morphisms which are of finite type and not just morphisms which are locally of finite type, see Remark 37.15.2 for a discussion. In the situation of the definition Lemma 37.14.8 tells us that the following are equivalent

(1) $\mathcal{F}$ is pure along X_s,
(2) there is no impurity $(g : T \to S, t' \rightsquigarrow t, \xi)$ with g quasi-finite at t,
(3) there does not exist any impurity of the form $(S^h \to S, s' \rightsquigarrow s, \xi)$, where S^h is the henselization of S at s.

If we denote $X^h = X \times_S S^h$ and $\mathcal{F}^h$ the pullback of $\mathcal{F}$ to X^h, then we can formulate the last condition in the following more positive way:

(4) All points of $\mathrm{Ass}_{X^h/S^h}(\mathcal{F}^h)$ specialize to points of X_s.

In particular, it is clear that $\mathcal{F}$ is pure along X_s if and only if the pullback of $\mathcal{F}$ to $X \times_S \mathrm{Spec}(\mathcal{O}_{S,s})$ is pure along X_s.

Remark 37.15.2. Let $f : X \to S$ be a morphism which is locally of finite type and $\mathcal{F}$ a quasi-coherent finite type $\mathcal{O}_X$-module. In this case it is still true that (1) and (2) above are equivalent because the proof of Lemma 37.14.5 does not use that f is quasi-compact. It is also clear that (3) and (4) are equivalent. However, we don't know if (1) and (3) are equivalent. In this case it may sometimes be more

convenient to define purity using the equivalent conditions (3) and (4) as is done in [**GR71**]. On the other hand, for many applications it seems that the correct notion is really that of being universally pure.

A natural question to ask is if the property of being pure relative to the base is preserved by base change, i.e., if being pure is the same thing as being universally pure. It turns out that this is true over Noetherian base schemes (see Lemma 37.15.5), or if the sheaf is flat (see Lemmas 37.17.3 and 37.17.4). It is not true in general, even if the morphism and the sheaf are of finite presentation, see Examples, Section 78.26 for a counter example. First we match our usage of "universally" to the usual notion.

Lemma 37.15.3. *Let $f : X \to S$ be a morphism of schemes which is of finite type. Let $\mathcal{F}$ be a finite type quasi-coherent $\mathcal{O}_X$-module. Let $s \in S$. The following are equivalent*

1. *$\mathcal{F}$ is universally pure along X_s, and*
2. *for every morphism of pointed schemes $(S', s') \to (S, s)$ the pullback $\mathcal{F}_{S'}$ is pure along $X_{s'}$.*

In particular, $\mathcal{F}$ is universally pure relative to S if and only if every base change $\mathcal{F}_{S'}$ of $\mathcal{F}$ is pure relative to S'.

Proof. This is formal. □

Lemma 37.15.4. *Let $f : X \to S$ be a morphism of schemes which is of finite type. Let $\mathcal{F}$ be a finite type quasi-coherent $\mathcal{O}_X$-module. Let $s \in S$. Let $(S', s') \to (S, s)$ be a morphism of pointed schemes. If $S' \to S$ is quasi-finite at s' and $\mathcal{F}$ is pure along X_s, then $\mathcal{F}_{S'}$ is pure along $X_{s'}$.*

Proof. It $(T \to S', t' \rightsquigarrow t, \xi)$ is an impurity of $\mathcal{F}_{S'}$ above s' with $T \to S'$ quasi-finite at t, then $(T \to S, t' \to t, \xi)$ is an impurity of $\mathcal{F}$ above s with $T \to S$ quasi-finite at t, see Morphisms, Lemma 28.21.12. Hence the lemma follows immediately from the characterization (2) of purity given following Definition 37.15.1. □

Lemma 37.15.5. *Let $f : X \to S$ be a morphism of schemes which is of finite type. Let $\mathcal{F}$ be a finite type quasi-coherent $\mathcal{O}_X$-module. Let $s \in S$. If $\mathcal{O}_{S,s}$ is Noetherian then $\mathcal{F}$ is pure along X_s if and only if $\mathcal{F}$ is universally pure along X_s.*

Proof. First we may replace S by $\mathrm{Spec}(\mathcal{O}_{S,s})$, i.e., we may assume that S is Noetherian. Next, use Lemma 37.14.6 and characterization (2) of purity given in discussion following Definition 37.15.1 to conclude. □

Purity satisfies flat descent.

Lemma 37.15.6. *Let $f : X \to S$ be a morphism of schemes which is of finite type. Let $\mathcal{F}$ be a finite type quasi-coherent $\mathcal{O}_X$-module. Let $s \in S$. Let $(S', s') \to (S, s)$ be a morphism of pointed schemes. Assume $S' \to S$ is flat at s'.*

1. *If $\mathcal{F}_{S'}$ is pure along $X_{s'}$, then $\mathcal{F}$ is pure along X_s.*
2. *If $\mathcal{F}_{S'}$ is universally pure along $X_{s'}$, then $\mathcal{F}$ is universally pure along X_s.*

Proof. Let $(T \to S, t' \rightsquigarrow t, \xi)$ be an impurity of $\mathcal{F}$ above s. Set $T_1 = T \times_S S'$, and let t_1 be the unique point of T_1 mapping to t and s'. Since $T_1 \to T$ is flat at t_1, see Morphisms, Lemma 28.26.7, there exists a specialization $t_1' \rightsquigarrow t_1$ lying over $t' \rightsquigarrow t$, see Algebra, Section 10.38. Choose a point $\xi_1 \in X_{t_1'}$ which corresponds to a

generic point of $\mathrm{Spec}(\kappa(t_1') \otimes_{\kappa(t')} \kappa(\xi))$, see Schemes, Lemma 25.17.5. By Divisors, Lemma 30.7.2 we see that $\xi_1 \in \mathrm{Ass}_{X_{T_1}/T_1}(\mathcal{F}_{T_1})$. As the Zariski closure of $\{\xi_1\}$ in X_{T_1} maps into the Zariski closure of $\{\xi\}$ in X_T we conclude that this closure is disjoint from X_{t_1}. Hence $(T_1 \to S', t_1' \rightsquigarrow t_1, \xi_1)$ is an impurity of $\mathcal{F}_{S'}$ above s'. In other words we have proved the contrapositive to part (2) of the lemma. Finally, if $(T, t) \to (S, s)$ is an elementary étale neighbourhood, then $(T_1, t_1) \to (S', s')$ is an elementary étale neighbourhood too, and in this way we see that (1) holds. □

Lemma 37.15.7. *Let $i : Z \to X$ be a closed immersion of schemes of finite type over a scheme S. Let $s \in S$. Let $\mathcal{F}$ be a finite type, quasi-coherent sheaf on Z. Then $\mathcal{F}$ is (universally) pure along Z_s if and only if $i_*\mathcal{F}$ is (universally) pure along X_s.*

Proof. Omitted. □

37.16. Examples of relatively pure sheaves

Here are some example cases where it is possible to see what purity means.

Lemma 37.16.1. *Let $f : X \to S$ be a proper morphism of schemes. Then every finite type, quasi-coherent $\mathcal{O}_X$-module $\mathcal{F}$ is universally pure relative to S. In particular X is universally pure relative to S.*

Proof. Let $(g : T \to S, t' \rightsquigarrow t, \xi)$ be an impurity of $\mathcal{F}$ above $s \in S$. Since f is proper, it is universally closed. Hence $f_T : X_T \to T$ is closed. Since $f_T(\xi) = t'$ this implies that $t \in f(\overline{\{\xi\}})$ which is a contradiction. □

Lemma 37.16.2. *Let $f : X \to S$ be a separated, finite type morphism of schemes. Let $\mathcal{F}$ be a finite type, quasi-coherent $\mathcal{O}_X$-module. Assume that $\mathrm{Supp}(\mathcal{F}_s)$ is finite for every $s \in S$. Then the following are equivalent*

1. *$\mathcal{F}$ is pure relative to S,*
2. *the scheme theoretic support of $\mathcal{F}$ is finite over S, and*
3. *$\mathcal{F}$ is universally pure relative to S.*

In particular, given a quasi-finite separated morphism $X \to S$ we see that X is pure relative to S if and only if $X \to S$ is finite.

Proof. Let $Z \subset X$ be the scheme theoretic support of $\mathcal{F}$, see Morphisms, Definition 28.5.5. Then $Z \to S$ is a separated, finite type morphism of schemes with finite fibres. Hence it is separated and quasi-finite, see Morphisms, Lemma 28.21.10. By Lemma 37.15.7 it suffices to prove the lemma for $Z \to S$ and the sheaf $\mathcal{F}$ viewed as a finite type quasi-coherent module on Z. Hence we may assume that $X \to S$ is separated and quasi-finite and that $\mathrm{Supp}(\mathcal{F}) = X$.

It follows from Lemma 37.16.1 and Morphisms, Lemma 28.44.10 that (2) implies (3). Trivially (3) implies (1). Assume (1) holds. We will prove that (2) holds. It is clear that we may assume S is affine. By More on Morphisms, Lemma 36.31.3 we can find a diagram

$$\begin{array}{ccc} X & \xrightarrow{\quad j \quad} & T \\ & {\scriptstyle f}\searrow \quad \swarrow{\scriptstyle \pi} & \\ & S & \end{array}$$

with π finite and j a quasi-compact open immersion. If we show that j is closed, then j is a closed immersion and we conclude that $f = \pi \circ j$ is finite. To show that j is closed it suffices to show that specializations lift along j, see Schemes, Lemma 25.19.8. Let $x \in X$, set $t' = j(x)$ and let $t' \rightsquigarrow t$ be a specialization. We have to show $t \in j(X)$. Set $s' = f(x)$ and $s = \pi(t)$ so $s' \rightsquigarrow s$. By More on Morphisms, Lemma 36.30.4 we can find an elementary étale neighbourhood $(U, u) \to (S, s)$ and a decomposition

$$T_U = T \times_S U = V \amalg W$$

into open and closed subschemes, such that $V \to U$ is finite and there exists a unique point v of V mapping to u, and such that v maps to t in T. As $V \to T$ is étale, we can lift generalizations, see Morphisms, Lemmas 28.26.8 and 28.37.12. Hence there exists a specialization $v' \rightsquigarrow v$ such that v' maps to $t' \in T$. In particular we see that $v' \in X_U \subset T_U$. Denote $u' \in U$ the image of t'. Note that $v' \in \mathrm{Ass}_{X_U/U}(\mathcal{F})$ because $X_{u'}$ is a finite discrete set and $X_{u'} = \mathrm{Supp}(\mathcal{F}_{u'})$. As $\mathcal{F}$ is pure relative to S we see that v' must specialize to a point in X_u. Since v is the only point of V lying over u (and since no point of W can be a specialization of v') we see that $v \in X_u$. Hence $t \in X$. □

Lemma 37.16.3. *Let $f : X \to S$ be a finite type, flat morphism of schemes with geometrically integral fibres. Then X is universally pure over S.*

Proof. Let $\xi \in X$ with $s' = f(\xi)$ and $s' \rightsquigarrow s$ a specialization of S. If ξ is an associated point of $X_{s'}$, then ξ is the unique generic point because $X_{s'}$ is an integral scheme. Let ξ_0 be the unique generic point of X_s. As $X \to S$ is flat we can lift $s' \rightsquigarrow s$ to a specialization $\xi' \rightsquigarrow \xi_0$ in X, see Morphisms, Lemma 28.26.8. The $\xi \rightsquigarrow \xi'$ because ξ is the generic point of $X_{s'}$ hence $\xi \rightsquigarrow \xi_0$. This means that $(\mathrm{id}_S, s' \to s, \xi)$ is not an impurity of $\mathcal{O}_X$ above s. Since the assumption that f is finite type, flat with geometrically integral fibres is preserved under base change, we see that there doesn't exist an impurity after any base change. In this way we see that X is universally S-pure. □

Lemma 37.16.4. *Let $f : X \to S$ be a finite type, affine morphism of schemes. Let $\mathcal{F}$ be a finite type quasi-coherent $\mathcal{O}_X$-module such that $f_*\mathcal{F}$ is locally projective on S, see Properties, Definition 27.19.1. Then $\mathcal{F}$ is universally pure over S.*

Proof. After reducing to the case where S is the spectrum of a henselian local ring this follows from Lemma 37.13.1. □

37.17. A criterion for purity

We first prove that given a flat family of finite type quasi-coherent sheaves the points in the relative assassin specialize to points in the relative assassins of nearby fibres (if they specialize at all).

Lemma 37.17.1. *Let $f : X \to S$ be a morphism of schemes of finite type. Let $\mathcal{F}$ be a quasi-coherent $\mathcal{O}_X$-module of finite type. Let $s \in S$. Assume that $\mathcal{F}$ is flat over S at all points of X_s. Let $x' \in Ass_{X/S}(\mathcal{F})$ with $f(x') = s'$ such that $s' \rightsquigarrow s$ is a specialization in S. If x' specializes to a point of X_s, then $x' \rightsquigarrow x$ with $x \in Ass_{X_s}(\mathcal{F}_s)$.*

Proof. Let $x' \rightsquigarrow t$ be a specialization with $t \in X_s$. We may replace X by an affine neighbourhood of t and S by an affine neighbourhood of s. Choose a closed

immersion $i : X \to \mathbf{A}^n_S$. Then it suffices to prove the lemma for the module $i_*\mathcal{F}$ on $\mathbf{A}^n_S$ and the point $i(x')$. Hence we may assume $X \to S$ is of finite presentation.

Let $x' \rightsquigarrow t$ be a specialization with $t \in X_s$. Set $A = \mathcal{O}_{S,s}$, $B = \mathcal{O}_{X,t}$, and $N = \mathcal{F}_t$. Note that B is essentially of finite presentation over A and that N is a finite B-module flat over A. Also N is a finitely presented B-module by Lemma 37.10.9. Let $\mathfrak{q}' \subset B$ be the prime ideal corresponding to x' and let $\mathfrak{p}' \subset A$ be the prime ideal corresponding to s'. The assumption $x' \in \mathrm{Ass}_{X/S}(\mathcal{F})$ means that $\mathfrak{q}'$ is an associated prime of $N \otimes_A \kappa(\mathfrak{p}')$. Let $\Sigma \subset B$ be the multiplicative subset of elements which are not zerodivisors on $N/\mathfrak{m}_A N$. By Lemma 37.7.2 the map $N \to \Sigma^{-1}N$ is universally injective. In particular, we see that $N \otimes_A \kappa(\mathfrak{p}') \to \Sigma^{-1}N \otimes_A \kappa(\mathfrak{p}')$ is injective which implies that $\mathfrak{q}'$ is an associated prime of $\Sigma^{-1}N \otimes_A \kappa(\mathfrak{p}')$ and hence $\mathfrak{q}'$ is in the image of $\mathrm{Spec}(\Sigma^{-1}B) \to \mathrm{Spec}(B)$. Thus Lemma 37.7.1 implies that $\mathfrak{q}' \subset \mathfrak{q}$ for some prime $\mathfrak{q} \in \mathrm{Ass}_B(N/\mathfrak{m}_A N)$ (which in particular implies that $\mathfrak{m}_A = A \cap \mathfrak{q}$). If $x \in X_s$ denotes the point corresponding to $\mathfrak{q}$, then $x \in \mathrm{Ass}_{X_s}(\mathcal{F}_s)$ and $x' \rightsquigarrow x$ as desired. □

Lemma 37.17.2. *Let $f : X \to S$ be a morphism of schemes of finite type. Let $\mathcal{F}$ be a quasi-coherent $\mathcal{O}_X$-module of finite type. Let $s \in S$. Let $(S', s') \to (S, s)$ be an elementary étale neighbourhood and let*

$$\begin{array}{ccc} X & \xleftarrow{g} & X' \\ \downarrow & & \downarrow \\ S & \longleftarrow & S' \end{array}$$

be a commutative diagram of morphisms of schemes. Assume

1. *$\mathcal{F}$ is flat over S at all points of X_s,*
2. *$X' \to S'$ is of finite type,*
3. *$g^*\mathcal{F}$ is pure along $X'_{s'}$,*
4. *$g : X' \to X$ is étale, and*
5. *$g(X')$ contains $Ass_{X_s}(\mathcal{F}_s)$.*

In this situation $\mathcal{F}$ is pure along X_s if and only if the image of $X' \to X \times_S S'$ contains the points of $Ass_{X \times_S S'/S'}(\mathcal{F} \times_S S')$ lying over points in S' which specialize to s'.

Proof. Since the morphism $S' \to S$ is étale, we see that if $\mathcal{F}$ is pure along X_s, then $\mathcal{F} \times_S S'$ is pure along X_s, see Lemma 37.15.4. Since purity satisfies flat descent, see Lemma 37.15.6, we see that if $\mathcal{F} \times_S S'$ is pure along $X_{s'}$, then $\mathcal{F}$ is pure along X_s. Hence we may replace S by S' and assume that $S = S'$ so that $g : X' \to X$ is an étale morphism between schemes of finite type over S. Moreover, we may replace S by $\mathrm{Spec}(\mathcal{O}_{S,s})$ and assume that S is local.

First, assume that $\mathcal{F}$ is pure along X_s. In this case every point of $\mathrm{Ass}_{X/S}(\mathcal{F})$ specializes to a point of X_s by purity. Hence by Lemma 37.17.1 we see that every point of $\mathrm{Ass}_{X/S}(\mathcal{F})$ specializes to a point of $\mathrm{Ass}_{X_s}(\mathcal{F}_s)$. Thus every point of $\mathrm{Ass}_{X/S}(\mathcal{F})$ is in the image of g (as the image is open and contains $\mathrm{Ass}_{X_s}(\mathcal{F}_s)$).

Conversely, assume that $g(X')$ contains $\mathrm{Ass}_{X/S}(\mathcal{F})$. Let $S^h = \mathrm{Spec}(\mathcal{O}^h_{S,s})$ be the henselization of S at s. Denote $g^h : (X')^h \to X^h$ the base change of g by $S^h \to S$, and denote $\mathcal{F}^h$ the pullback of $\mathcal{F}$ to X^h. By Divisors, Lemma 30.7.2 and Remark 30.7.3 the relative assassin $\mathrm{Ass}_{X^h/S^h}(\mathcal{F}^h)$ is the inverse image of $\mathrm{Ass}_{X/S}(\mathcal{F})$ via

the projection $X^h \to X$. As we have assumed that $g(X')$ contains $\mathrm{Ass}_{X/S}(\mathcal{F})$ we conclude that the base change $g^h((X')^h) = g(X') \times_S S^h$ contains $\mathrm{Ass}_{X^h/S^h}(\mathcal{F}^h)$. In this way we reduce to the case where S is the spectrum of a henselian local ring. Let $x \in \mathrm{Ass}_{X/S}(\mathcal{F})$. To finish the proof of the lemma we have to show that x specializes to a point of X_s, see criterion (4) for purity in discussion following Definition 37.15.1. By assumption there exists a $x' \in X'$ such that $g(x') = x$. As $g : X' \to X$ is étale, we see that $x' \in \mathrm{Ass}_{X'/S}(g^*\mathcal{F})$, see Lemma 37.2.7 (applied to the morphism of fibres $X'_w \to X_w$ where $w \in S$ is the image of x'). Since $g^*\mathcal{F}$ is pure along X'_s we see that $x' \rightsquigarrow y$ for some $y \in X'_s$. Hence $x = g(x') \rightsquigarrow g(y)$ and $g(y) \in X_s$ as desired. □

Lemma 37.17.3. *Let $f : X \to S$ be a morphism of schemes. Let $\mathcal{F}$ be a quasi-coherent $\mathcal{O}_X$-module. Let $s \in S$. Assume*

(1) *f is of finite type,*
(2) *$\mathcal{F}$ is of finite type,*
(3) *$\mathcal{F}$ is flat over S at all points of X_s, and*
(4) *$\mathcal{F}$ is pure along X_s.*

Then $\mathcal{F}$ is universally pure along X_s.

Proof. We first make a preliminary remark. Suppose that $(S', s') \to (S, s)$ is an elementary étale neighbourhood. Denote $\mathcal{F}'$ the pullback of $\mathcal{F}$ to $X' = X \times_S S'$. By the discussion following Definition 37.15.1 we see that $\mathcal{F}'$ is pure along $X'_{s'}$. Moreover, $\mathcal{F}'$ is flat over S' along $X'_{s'}$. Then it suffices to prove that $\mathcal{F}'$ is universally pure along $X'_{s'}$. Namely, given any morphism $(T, t) \to (S, s)$ of pointed schemes the fibre product $(T', t') = (T \times_S S', (t, s'))$ is flat over (T, t) and hence if $\mathcal{F}_{T'}$ is pure along $X_{t'}$ then $\mathcal{F}_T$ is pure along X_t by Lemma 37.15.6. Thus during the proof we may always replace (s, S) by an elementary étale neighbourhood. We may also replace S by $\mathrm{Spec}(\mathcal{O}_{S,s})$ due to the local nature of the problem.

Choose an elementary étale neighbourhood $(S', s') \to (S, s)$ and a commutative diagram

$$\begin{array}{ccc} X & \xleftarrow{g} & X' \\ \downarrow & & \downarrow \\ S & \longleftarrow & \mathrm{Spec}(\mathcal{O}_{S',s'}) \end{array}$$

such that $X' \to X \times_S \mathrm{Spec}(\mathcal{O}_{S',s'})$ is étale, $X_s = g((X')_{s'})$, the scheme X' is affine, and such that $\Gamma(X', g^*\mathcal{F})$ is a free $\mathcal{O}_{S',s'}$-module, see Lemma 37.11.11. Note that $X' \to \mathrm{Spec}(\mathcal{O}_{S',s'})$ is of finite type (as a quasi-compact morphism which is the composition of an étale morphism and the base change of a finite type morphism). By our preliminary remarks in the first paragraph of the proof we may replace S by $\mathrm{Spec}(\mathcal{O}_{S',s'})$. Hence we may assume there exists a commutative diagram

$$\begin{array}{ccc} X & \xleftarrow{g} & X' \\ & \searrow \quad \swarrow & \\ & S & \end{array}$$

of schemes of finite type over S, where g is étale, $X_s \subset g(X')$, with S local with closed point s, with X' affine, and with $\Gamma(X', g^*\mathcal{F})$ a free $\Gamma(S, \mathcal{O}_S)$-module. Note that in this case $g^*\mathcal{F}$ is universally pure over S, see Lemma 37.16.4.

In this situation we apply Lemma 37.17.2 to deduce that $\mathrm{Ass}_{X/S}(\mathcal{F}) \subset g(X')$ from our assumption that $\mathcal{F}$ is pure along X_s and flat over S along X_s. By Divisors, Lemma 30.7.2 and Remark 30.7.3 we see that for any morphism of pointed schemes $(T, t) \to (S, s)$ we have

$$\mathrm{Ass}_{X_T/T}(\mathcal{F}_T) \subset (X_T \to X)^{-1}(\mathrm{Ass}_{X/S}(\mathcal{F})) \subset g(X') \times_S T = g_T(X'_T).$$

Hence by Lemma 37.17.2 applied to the base change of our displayed diagram to (T, t) we conclude that $\mathcal{F}_T$ is pure along X_t as desired. □

Lemma 37.17.4. *Let $f : X \to S$ be a finite type morphism of schemes. Let $\mathcal{F}$ be a finite type quasi-coherent $\mathcal{O}_X$-module. Assume $\mathcal{F}$ is flat over S. In this case $\mathcal{F}$ is pure relative to S if and only if $\mathcal{F}$ is universally pure relative to S.*

Proof. Immediate consequence of Lemma 37.17.3 and the definitions. □

Lemma 37.17.5. *Let I be a directed partially ordered set. Let $(S_i, g_{ii'})$ be an inverse system of affine schemes over I. Set $S = \lim_i S_i$ and $s \in S$. Denote $g_i : S \to S_i$ the projections and set $s_i = g_i(s)$. Suppose that $f : X \to S$ is a morphism of finite presentation, $\mathcal{F}$ a quasi-coherent $\mathcal{O}_X$-module of finite presentation which is pure along X_s and flat over S at all points of X_s. Then there exists an $i \in I$, a morphism of finite presentation $X_i \to S_i$, a quasi-coherent $\mathcal{O}_{X_i}$-module $\mathcal{F}_i$ of finite presentation which is pure along $(X_i)_{s_i}$ and flat over S_i at all points of $(X_i)_{s_i}$ such that $X \cong X_i \times_{S_i} S$ and such that the pullback of $\mathcal{F}_i$ to X is isomorphic to $\mathcal{F}$.*

Proof. Let $U \subset X$ be the set of points where $\mathcal{F}$ is flat over S. By More on Morphisms, Theorem 36.12.1 this is an open subscheme of X. By assumption $X_s \subset U$. As X_s is quasi-compact, we can find a quasi-compact open $U' \subset U$ with $X_s \subset U'$. By Limits, Lemma 31.9.1 we can find an $i \in I$ and a morphism of finite presentation $f_i : X_i \to S_i$ whose base change to S is isomorphic to f_i. Fix such a choice and set $X_{i'} = X_i \times_{S_i} S_{i'}$. Then $X = \lim_{i'} X_{i'}$ with affine transition morphisms. By Limits, Lemma 31.9.2 we can, after possible increasing i assume there exists a quasi-coherent $\mathcal{O}_{X_i}$-module $\mathcal{F}_i$ of finite presentation whose base change to S is isomorphic to $\mathcal{F}$. By Limits, Lemma 31.3.8 after possibly increasing i we may assume there exists an open $U'_i \subset X_i$ whose inverse image in X is U'. Note that in particular $(X_i)_{s_i} \subset U'_i$. By Limits, Lemma 31.9.3 (after increasing i once more) we may assume that $\mathcal{F}_i$ is flat on U'_i. In particular we see that $\mathcal{F}_i$ is flat along $(X_i)_{s_i}$.

Next, we use Lemma 37.11.5 to choose an elementary étale neighbourhood $(S'_i, s'_i) \to (S_i, s_i)$ and a commutative diagram of schemes

$$\begin{array}{ccc} X_i & \xleftarrow{g_i} & X'_i \\ \downarrow & & \downarrow \\ S_i & \longleftarrow & S'_i \end{array}$$

such that g_i is étale, $(X_i)_{s_i} \subset g_i(X'_i)$, the schemes X'_i, S'_i are affine, and such that $\Gamma(X'_i, g_i^*\mathcal{F}_i)$ is a projective $\Gamma(S'_i, \mathcal{O}_{S'_i})$-module. Note that $g_i^*\mathcal{F}_i$ is universally pure over S'_i, see Lemma 37.16.4. We may base change the diagram above to a diagram with morphisms $(S'_{i'}, s'_{i'}) \to (S_{i'}, s_{i'})$ and $g_{i'} : X'_{i'} \to X_{i'}$ over $S_{i'}$ for any $i' \geq i$ and we may base change the diagram to a diagram with morphisms $(S', s') \to (S, s)$ and $g : X' \to X$ over S.

At this point we can use our criterion for purity. Set $W'_i \subset X_i \times_{S_i} S'_i$ equal to the image of the étale morphism $X'_i \to X_i \times_{S_i} S'_i$. For every $i' \geq i$ we have similarly the image $W'_{i'} \subset X_{i'} \times_{S_{i'}} S'_{i'}$ and we have the image $W' \subset X \times_S S'$. Taking images commutes with base change, hence $W'_{i'} = W'_i \times_{S'_i} S'_{i'}$ and $W' = W_i \times_{S'_i} S'$. Because $\mathcal{F}$ is pure along X_s the Lemma 37.17.2 implies that

$$f^{-1}(\mathrm{Spec}(\mathcal{O}_{S',s'})) \cap \mathrm{Ass}_{X \times_S S'/S'}(\mathcal{F} \times_S S') \subset W' \tag{37.17.5.1}$$

By More on Morphisms, Lemma 36.20.5 we see that

$$E = \{t \in S' \mid \mathrm{Ass}_{X_t}(\mathcal{F}_t) \subset W'\} \quad \text{and} \quad E_{i'} = \{t \in S'_{i'} \mid \mathrm{Ass}_{X_t}(\mathcal{F}_{i',t}) \subset W'_{i'}\}$$

are locally constructible subsets of S' and $S'_{i'}$. By More on Morphisms, Lemma 36.20.4 we see that $E_{i'}$ is the inverse image of E_i under the morphism $S'_{i'} \to S'_i$ and that E is the inverse image of E_i under the morphism $S' \to S'_i$. Thus Equation (37.17.5.1) is equivalent to the assertion that $\mathrm{Spec}(\mathcal{O}_{S',s'})$ maps into E_i. As $\mathcal{O}_{S',s'} = \mathrm{colim}_{i' \geq i} \mathcal{O}_{S'_{i'},s'_{i'}}$ we see that $\mathrm{Spec}(\mathcal{O}_{S'_{i'},s'_{i'}})$ maps into E_i for some $i' \geq i$, see Limits, Lemma 31.3.7. Then, applying Lemma 37.17.2 to the situation over $S_{i'}$, we conclude that $\mathcal{F}_{i'}$ is pure along $(X_{i'})_{s_{i'}}$. □

Lemma 37.17.6. *Let $f : X \to S$ be a morphism of finite presentation. Let $\mathcal{F}$ be a quasi-coherent $\mathcal{O}_X$-module of finite presentation flat over S. Then the set*

$$U = \{s \in S \mid \mathcal{F} \text{ is pure along } X_s\}$$

is open in S.

Proof. Let $s \in U$. Using Lemma 37.11.5 we can find an elementary étale neighbourhood $(S', s') \to (S, s)$ and a commutative diagram

$$\begin{array}{ccc} X & \xleftarrow{g} & X' \\ \downarrow & & \downarrow \\ S & \longleftarrow & S' \end{array}$$

such that g is étale, $X_s \subset g(X')$, the schemes X', S' are affine, and such that $\Gamma(X', g^*\mathcal{F})$ is a projective $\Gamma(S', \mathcal{O}_{S'})$-module. Note that $g^*\mathcal{F}$ is universally pure over S', see Lemma 37.16.4. Set $W' \subset X \times_S S'$ equal to the image of the étale morphism $X' \to X \times_S S'$. Note that W is open and quasi-compact over S'. Set

$$E = \{t \in S' \mid \mathrm{Ass}_{X_t}(\mathcal{F}_t) \subset W'\}.$$

By More on Morphisms, Lemma 36.20.5 E is a constructible subset of S'. By Lemma 37.17.2 we see that $\mathrm{Spec}(\mathcal{O}_{S',s'}) \subset E$. By Morphisms, Lemma 28.23.4 we see that E contains an open neighbourhood V' of s'. Applying Lemma 37.17.2 once more we see that for any point s_1 in the image of V' in S the sheaf $\mathcal{F}$ is pure along X_{s_1}. Since $S' \to S$ is étale the image of V' in S is open and we win. □

37.18. How purity is used

Here are some examples of how purity can be used. The first lemma actually uses a slightly weaker form of purity.

Lemma 37.18.1. *Let $f : X \to S$ be a morphism of finite type. Let $\mathcal{F}$ be a quasi-coherent sheaf of finite type on X. Assume S is local with closed point s. Assume $\mathcal{F}$ is pure along X_s and that $\mathcal{F}$ is flat over S. Let $\varphi : \mathcal{F} \to \mathcal{G}$ of quasi-coherent $\mathcal{O}_X$-modules. Then the following are equivalent*

(1) *the map on stalks φ_x is injective for all $x \in Ass_{X_s}(\mathcal{F}_s)$, and*
(2) *φ is injective.*

Proof. Let $\mathcal{K} = \mathrm{Ker}(\varphi)$. Our goal is to prove that $\mathcal{K} = 0$. In order to do this it suffices to prove that $\mathrm{WeakAss}_X(\mathcal{K}) = \emptyset$, see Divisors, Lemma 30.5.5. We have $\mathrm{WeakAss}_X(\mathcal{K}) \subset \mathrm{WeakAss}_X(\mathcal{F})$, see Divisors, Lemma 30.5.4. As $\mathcal{F}$ is flat we see from Lemma 37.12.4 that $\mathrm{WeakAss}_X(\mathcal{F}) \subset \mathrm{Ass}_{X/S}(\mathcal{F})$. By purity any point x' of $\mathrm{Ass}_{X/S}(\mathcal{F})$ is a generalization of a point of X_s, and hence is the specialization of a point $x \in \mathrm{Ass}_{X_s}(\mathcal{F}_s)$, by Lemma 37.17.1. Hence the injectivity of φ_x implies the injectivity of $\varphi_{x'}$, whence $\mathcal{K}_{x'} = 0$. □

Proposition 37.18.2. *Let $f : X \to S$ be an affine, finitely presented morphism of schemes. Let $\mathcal{F}$ be a quasi-coherent $\mathcal{O}_X$-module of finite presentation, flat over S. Then the following are equivalent*

(1) *$f_*\mathcal{F}$ is locally projective on S, and*
(2) *$\mathcal{F}$ is pure relative to S.*

In particular, given a ring map $A \to B$ of finite presentation and a finitely presented B-module N flat over A we have: N is projective as an A-module if and only if $\widetilde{N}$ on $\mathrm{Spec}(B)$ is pure relative to $\mathrm{Spec}(A)$.

Proof. The implication (1) $\Rightarrow$ (2) is Lemma 37.16.4. Assume $\mathcal{F}$ is pure relative to S. Note that by Lemma 37.17.3 this implies $\mathcal{F}$ remains pure after any base change. By Descent, Lemma 34.6.7 it suffices to prove $f_*\mathcal{F}$ is fpqc locally projective on S. Pick $s \in S$. We will prove that the restriction of $f_*\mathcal{F}$ to an étale neighbourhood of s is locally projective. Namely, by Lemma 37.11.5, after replacing S by an affine elementary étale neighbourhood of s, we may assume there exists a diagram

$$\begin{array}{ccc} X & \xleftarrow{\quad g \quad} & X' \\ & \searrow \quad \swarrow & \\ & S & \end{array}$$

of schemes affine and of finite presentation over S, where g is étale, $X_s \subset g(X')$, and with $\Gamma(X', g^*\mathcal{F})$ a projective $\Gamma(S, \mathcal{O}_S)$-module. Note that in this case $g^*\mathcal{F}$ is universally pure over S, see Lemma 37.16.4. Hence by Lemma 37.17.2 we see that the open $g(X')$ contains the points of $\mathrm{Ass}_{X/S}(\mathcal{F})$ lying over $\mathrm{Spec}(\mathcal{O}_{S,s})$. Set

$$E = \{t \in S \mid \mathrm{Ass}_{X_t}(\mathcal{F}_t) \subset g(X')\}.$$

By More on Morphisms, Lemma 36.20.5 E is a constructible subset of S. We have seen that $\mathrm{Spec}(\mathcal{O}_{S,s}) \subset E$. By Morphisms, Lemma 28.23.4 we see that E contains an open neighbourhood of s. Hence after replacing S by an affine neighbourhood of s we may assume that $\mathrm{Ass}_{X/S}(\mathcal{F}) \subset g(X')$. By Lemma 37.7.4 this means that

$$\Gamma(X, \mathcal{F}) \longrightarrow \Gamma(X', g^*\mathcal{F})$$

is $\Gamma(S, \mathcal{O}_S)$-universally injective. By Algebra, Lemma 10.84.6 we conclude that $\Gamma(X, \mathcal{F})$ is Mittag-Leffler as an $\Gamma(S, \mathcal{O}_S)$-module. Since $\Gamma(X, \mathcal{F})$ is countably generated and flat as a $\Gamma(S, \mathcal{O}_S)$-module, we conclude it is projective by Algebra, Lemma 10.88.1. □

We can use the proposition to improve some of our earlier results. The following lemma is an improvement of Proposition 37.11.4.

Lemma 37.18.3. *Let $f : X \to S$ be a morphism which is locally of finite presentation. Let $\mathcal{F}$ be a quasi-coherent $\mathcal{O}_X$-module which is of finite presentation. Let $x \in X$ with $s = f(x) \in S$. If $\mathcal{F}$ is flat at x over S there exists an affine elementary étale neighbourhood $(S', s') \to (S, s)$ and an affine open $U' \subset X \times_S S'$ which contains $x' = (x, s')$ such that $\Gamma(U', \mathcal{F}|_{U'})$ is a projective $\Gamma(S', \mathcal{O}_{S'})$-module.*

Proof. During the proof we may replace X by an open neighbourhood of x and we may replace S by an elementary étale neighbourhood of s. Hence, by openness of flatness (see More on Morphisms, Theorem 36.12.1) we may assume that $\mathcal{F}$ is flat over S. We may assume S and X are affine. After shrinking X some more we may assume that any point of $\mathrm{Ass}_{X_s}(\mathcal{F}_s)$ is a generalization of x. This property is preserved on replacing (S, s) by an elementary étale neighbourhood. Hence we may apply Lemma 37.11.5 to arrive at the situation where there exists a diagram

$$\begin{array}{ccc} X & \xleftarrow{\quad g \quad} & X' \\ & \searrow \quad \swarrow & \\ & S & \end{array}$$

of schemes affine and of finite presentation over S, where g is étale, $X_s \subset g(X')$, and with $\Gamma(X', g^*\mathcal{F})$ a projective $\Gamma(S, \mathcal{O}_S)$-module. Note that in this case $g^*\mathcal{F}$ is universally pure over S, see Lemma 37.16.4.

Let $U \subset g(X')$ be an affine open neighbourhood of x. We claim that $\mathcal{F}|_U$ is pure along U_s. If we prove this, then the lemma follows because $\mathcal{F}|_U$ will be pure relative to S after shrinking S, see Lemma 37.17.6, whereupon the projectivity follows from Proposition 37.18.2. To prove the claim we have to show, after replacing (S, s) by an arbitrary elementary étale neighbourhood, that any point ξ of $\mathrm{Ass}_{U/S}(\mathcal{F}|_U)$ lying over some $s' \in S$, $s' \rightsquigarrow s$ specializes to a point of U_s. Since $U \subset g(X')$ we can find a $\xi' \in X'$ with $g(\xi') = \xi$. Because $g^*\mathcal{F}$ is pure over S, using Lemma 37.17.1, we see there exists a specialization $\xi' \rightsquigarrow x'$ with $x' \in \mathrm{Ass}_{X'_s}(g^*\mathcal{F}_s)$. Then $g(x') \in \mathrm{Ass}_{X_s}(\mathcal{F}_s)$ (see for example Lemma 37.2.7 applied to the étale morphism $X'_s \to X_s$ of Noetherian schemes) and hence $g(x') \rightsquigarrow x$ by our choice of X above! Since $x \in U$ we conclude that $g(x') \in U$. Thus $\xi = g(\xi') \rightsquigarrow g(x') \in U_s$ as desired. □

The following lemma is an improvement of Lemma 37.11.9.

Lemma 37.18.4. *Let $f : X \to S$ be a morphism which is locally of finite type. Let $\mathcal{F}$ be a quasi-coherent $\mathcal{O}_X$-module which is of finite type. Let $x \in X$ with $s = f(x) \in S$. If $\mathcal{F}$ is flat at x over S there exists an affine elementary étale neighbourhood $(S', s') \to (S, s)$ and an affine open $U' \subset X \times_S \mathrm{Spec}(\mathcal{O}_{S',s'})$ which contains $x' = (x, s')$ such that $\Gamma(U', \mathcal{F}|_{U'})$ is a free $\mathcal{O}_{S',s'}$-module.*

Proof. The question is Zariski local on X and S. Hence we may assume that X and S are affine. Then we can find a closed immersion $i : X \to \mathbf{A}^n_S$ over S. It is clear that it suffices to prove the lemma for the sheaf $i_*\mathcal{F}$ on $\mathbf{A}^n_S$ and the point $i(x)$. In this way we reduce to the case where $X \to S$ is of finite presentation. After replacing S by $\mathrm{Spec}(\mathcal{O}_{S',s'})$ and X by an open of $X \times_S \mathrm{Spec}(\mathcal{O}_{S',s'})$ we may assume that $\mathcal{F}$ is of finite presentation, see Proposition 37.10.3. In this case we may appeal to Lemma 37.18.3 and Algebra, Theorem 10.80.4 to conclude. □

Lemma 37.18.5. *Let $A \to B$ be a local ring map of local rings which is essentially of finite type. Let N be a finite B-module which is flat as an A-module. If A is henselian, then N is a filtered colimit*

$$N = \operatorname{colim}_i F_i$$

of free A-modules F_i such that all transition maps $u_i : F_i \to F_{i'}$ of the system induce injective maps $\overline{u}_i : F_i/\mathfrak{m}_A F_i \to F_{i'}/\mathfrak{m}_A F_{i'}$.

Proof. We can find a morphism of finite type $X \to S = \operatorname{Spec}(A)$ and a point $x \in X$ lying over the closed point s of S and a finite type quasi-coherent $\mathcal{O}_X$-module $\mathcal{F}$ such that $\mathcal{F}_x \cong N$ as an A-module. After shrinking X we may assume that each point of $\operatorname{Ass}_{X_s}(\mathcal{F}_s)$ specializes to x. By Lemma 37.18.4 we see that there exists a fundamental system of affine open neighbourhoods $U_i \subset X$ of x such that $\Gamma(U_i, \mathcal{F})$ is a free A-module F_i. Note that if $U_{i'} \subset U_i$, then

$$F_i/\mathfrak{m}_A F_i = \Gamma(U_{i,s}, \mathcal{F}_s) \longrightarrow \Gamma(U_{i',s}, \mathcal{F}_s) = F_{i'}/\mathfrak{m}_A F_{i'}$$

is injective because a section of the kernel would be supported at a closed subset of X_s not meeting x which is a contradiction to our choice of X above. □

37.19. Flattening functors

Let S be a scheme. Recall that a functor $F : (Sch/S)^{opp} \to Sets$ is called limit preserving if for every directed inverse system $\{T_i\}_{i \in I}$ of affine schemes with limit T we have $F(T) = \operatorname{colim}_i F(T_i)$.

Situation 37.19.1. Let $f : X \to S$ be a morphism of schemes. Let $u : \mathcal{F} \to \mathcal{G}$ be a homomorphism of quasi-coherent $\mathcal{O}_X$-modules. For any scheme T over S we will denote $u_T : \mathcal{F}_T \to \mathcal{G}_T$ the base change of u to T, in other words, u_T is the pullback of u via the projection morphism $X_T = X \times_S T \to X$. In this situation we can consider the functor

(37.19.1.1)

$$F_{iso} : (Sch/S)^{opp} \longrightarrow Sets, \quad T \longrightarrow \begin{cases} \{*\} & \text{if} \quad u_T \text{ is an isomorphism,} \\ \emptyset & \text{else.} \end{cases}$$

There are variants F_{inj}, F_{surj}, F_{zero} where we ask that u_T is injective, surjective, or zero.

Lemma 37.19.2. *In Situation 37.19.1.*

1. *Each of the functors F_{iso}, F_{inj}, F_{surj}, F_{zero} satisfies the sheaf property for the fpqc topology.*
2. *If f is quasi-compact and $\mathcal{G}$ is of finite type, then F_{surj} is limit preserving.*
3. *If f is quasi-compact and $\mathcal{F}$ of finite type, then F_{zero} is limit preserving.*
4. *If f is quasi-compact, $\mathcal{F}$ is of finite type, and $\mathcal{G}$ is of finite presentation, then F_{iso} is limit preserving.*

Proof. Let $\{T_i \to T\}_{i \in I}$ be an fpqc covering of schemes over S. Set $X_i = X_{T_i} = X \times_S T_i$ and $u_i = u_{T_i}$. Note that $\{X_i \to X_T\}_{i \in I}$ is an fpqc covering of X_T, see Topologies, Lemma 33.8.7. In particular, for every $x \in X_T$ there exists an $i \in I$ and an $x_i \in X_i$ mapping to x. Since $\mathcal{O}_{X_T,x} \to \mathcal{O}_{X_i,x_i}$ is flat, hence faithfully flat (see Algebra, Lemma 10.36.16) we conclude that $(u_i)_{x_i}$ is injective, surjective, bijective, or zero if and only if $(u_T)_x$ is injective, surjective, bijective, or zero. Whence part (1) of the lemma.

Proof of (2). Assume f quasi-compact and $\mathcal{G}$ of finite type. Let $T = \lim_{i\in I} T_i$ be a directed limit of affine S-schemes and assume that u_T is surjective. Set $X_i = X_{T_i} = X \times_S T_i$ and $u_i = u_{T_i} : \mathcal{F}_i = \mathcal{F}_{T_i} \to \mathcal{G}_i = \mathcal{G}_{T_i}$. To prove part (2) we have to show that u_i is surjective for some i. Pick $i_0 \in I$ and replace I by $\{i \mid i \geq i_0\}$. Since f is quasi-compact the scheme X_{i_0} is quasi-compact. Hence we may choose affine opens $W_1, \ldots, W_m \subset X$ and an affine open covering $X_{i_0} = U_{1,i_0} \cup \ldots \cup U_{m,i_0}$ such that U_{j,i_0} maps into W_j under the projection morphism $X_{i_0} \to X$. For any $i \in I$ let $U_{j,i}$ be the inverse image of U_{j,i_0}. Setting $U_j = \lim_i U_{j,i}$ we see that $X_T = U_1 \cup \ldots \cup U_m$ is an affine open covering of X_T. Now it suffices to show, for a given $j \in \{1, \ldots, m\}$ that $u_i|_{U_{j,i}}$ is surjective for some $i = i(j) \in I$. Using Properties, Lemma 27.16.1 this translates into the following algebra problem: Let A be a ring and let $u : M \to N$ be an A-module map. Suppose that $R = \operatorname{colim}_{i\in I} R_i$ is a directed colimit of A-algebras. If N is a finite A-module and if $u \otimes 1 : M \otimes_A R \to N \otimes_A R$ is surjective, then for some i the map $u \otimes 1 : M \otimes_A R_i \to N \otimes_A R_i$ is surjective. This is Algebra, Lemma 10.121.3 part (2).

Proof of (3). Exactly the same arguments as given in the proof of (2) reduces this to the following algebra problem: Let A be a ring and let $u : M \to N$ be an A-module map. Suppose that $R = \operatorname{colim}_{i\in I} R_i$ is a directed colimit of A-algebras. If M is a finite A-module and if $u \otimes 1 : M \otimes_A R \to N \otimes_A R$ is zero, then for some i the map $u \otimes 1 : M \otimes_A R_i \to N \otimes_A R_i$ is zero. This is Algebra, Lemma 10.121.3 part (1).

Proof of (4). Assume f quasi-compact and $\mathcal{F}, \mathcal{G}$ of finite presentation. Arguing in exactly the same manner as in the previous paragraph (using in addition also Properties, Lemma 27.16.2) part (3) translates into the following algebra statement: Let A be a ring and let $u : M \to N$ be an A-module map. Suppose that $R = \operatorname{colim}_{i\in I} R_i$ is a directed colimit of A-algebras. Assume M is a finite A-module, N is a finitely presented A-module, and $u \otimes 1 : M \otimes_A R \to N \otimes_A R$ is an isomorphism. Then for some i the map $u \otimes 1 : M \otimes_A R_i \to N \otimes_A R_i$ is an isomorphism. This is Algebra, Lemma 10.121.3 part (3). □

Situation 37.19.3. Let $(A, \mathfrak{m}_A)$ be a local ring. Denote $\mathcal{C}$ the category whose objects are A-algebras A' which are local rings such that the algebra structure $A \to A'$ is a local homomorphism of local rings. A morphism between objects A', A'' of $\mathcal{C}$ is a local homomorphism $A' \to A''$ of A-algebras. Let $A \to B$ be a local ring map of local rings and let M be a B-module. If A' is an object of $\mathcal{C}$ we set $B' = B \otimes_A A'$ and we set $M' = M \otimes_A A'$ as a B'-module. Given $A' \in \mathrm{Ob}(\mathcal{C})$, consider the condition

$$\forall \mathfrak{q} \in V(\mathfrak{m}_{A'} B' + \mathfrak{m}_B B') \subset \operatorname{Spec}(B') : M'_{\mathfrak{q}} \text{ is flat over } A'. \tag{37.19.3.1}$$

Note the similarity with More on Algebra, Equation (15.12.1.1). In particular, if $A' \to A''$ is a morphism of $\mathcal{C}$ and (37.19.3.1) holds for A', then it holds for A'', see More on Algebra, Lemma 15.12.2. Hence we obtain a functor

$$F_{lf} : \mathcal{C} \longrightarrow \mathit{Sets}, \quad A' \longrightarrow \begin{cases} \{*\} & \text{if (37.19.3.1) holds,} \\ \emptyset & \text{else.} \end{cases} \tag{37.19.3.2}$$

Lemma 37.19.4. *In Situation 37.19.3.*

(1) *If $A' \to A''$ is a flat morphism in $\mathcal{C}$ then $F_{fl}(A') = F_{lf}(A'')$.*
(2) *If $A \to B$ is essentially of finite presentation and M is a B-module of finite presentation, then F_{fl} is limit preserving: If $\{A_i\}_{i\in I}$ is a directed system of objects of $\mathcal{C}$, then $F_{fl}(\operatorname{colim}_i A_i) = \operatorname{colim}_i F_{fl}(A_i)$.*

Proof. Part (1) is a special case of More on Algebra, Lemma 15.12.3. Part (2) is a special case of More on Algebra, Lemma 15.12.4. □

Lemma 37.19.5. *In Situation 37.19.3 suppose that $B \to C$ is a local map of local A-algebras and that $M \cong N$ as B-modules. Denote $F'_{lf} : \mathcal{C} \to Sets$ the functor associated to the pair (C, N). If $B \to C$ is finite, then $F_{lf} = F'_{lf}$.*

Proof. Let A' be an object of $\mathcal{C}$. Set $C' = C \otimes_A A'$ and $N' = N \otimes_A A'$ similarly to the definitions of B', M' in Situation 37.19.3. Note that $M' \cong N'$ as B'-modules. The assumption that $B \to C$ is finite has two consequences: (a) $\mathfrak{m}_C = \sqrt{\mathfrak{m}_B C}$ and (b) $B' \to C'$ is finite. Consequence (a) implies that

$$V(\mathfrak{m}_{A'} C' + \mathfrak{m}_C C') = (\mathrm{Spec}(C') \to \mathrm{Spec}(B'))^{-1} V(\mathfrak{m}_{A'} B' + \mathfrak{m}_B B').$$

Suppose $\mathfrak{q} \subset V(\mathfrak{m}_{A'} B' + \mathfrak{m}_B B')$. Then $M'_\mathfrak{q}$ is flat over A' if and only if the $C'_\mathfrak{q}$-module $N'_\mathfrak{q}$ is flat over A' (because these are isomorphic as A'-modules) if and only if for every maximal ideal $\mathfrak{r}$ of $C'_\mathfrak{q}$ the module $N'_\mathfrak{r}$ is flat over A' (see Algebra, Lemma 10.36.19). As $B'_\mathfrak{q} \to C'_\mathfrak{q}$ is finite by (b), the maximal ideals of $C'_\mathfrak{q}$ correspond exactly to the primes of C' lying over $\mathfrak{q}$ (see Algebra, Lemma 10.33.20) and these primes are all contained in $V(\mathfrak{m}_{A'} C' + \mathfrak{m}_C C')$ by the displayed equation above. Thus the result of the lemma holds. □

Lemma 37.19.6. *In Situation 37.19.3 suppose that $B \to C$ is a flat local homomorphism of local rings. Set $N = M \otimes_B C$. Denote $F'_{lf} : \mathcal{C} \to Sets$ the functor associated to the pair (C, N). Then $F_{lf} = F'_{lf}$.*

Proof. Let A' be an object of $\mathcal{C}$. Set $C' = C \otimes_A A'$ and $N' = N \otimes_A A' = M' \otimes_{B'} C'$ similarly to the definitions of B', M' in Situation 37.19.3. Note that

$$V(\mathfrak{m}_{A'} B' + \mathfrak{m}_B B') = \mathrm{Spec}(\kappa(\mathfrak{m}_B) \otimes_A \kappa(\mathfrak{m}_{A'}))$$

and similarly for $V(\mathfrak{m}_{A'} C' + \mathfrak{m}_C C')$. The ring map

$$\kappa(\mathfrak{m}_B) \otimes_A \kappa(\mathfrak{m}_{A'}) \longrightarrow \kappa(\mathfrak{m}_C) \otimes_A \kappa(\mathfrak{m}_{A'})$$

is faithfully flat, hence $V(\mathfrak{m}_{A'} C' + \mathfrak{m}_C C') \to V(\mathfrak{m}_{A'} B' + \mathfrak{m}_B B')$ is surjective. Finally, if $\mathfrak{r} \in V(\mathfrak{m}_{A'} C' + \mathfrak{m}_C C')$ maps to $\mathfrak{q} \in V(\mathfrak{m}_{A'} B' + \mathfrak{m}_B B')$, then $M'_\mathfrak{q}$ is flat over A' if and only if $N'_\mathfrak{r}$ is flat over A' because $B' \to C'$ is flat, see Algebra, Lemma 10.36.8. The lemma follows formally from these remarks. □

Situation 37.19.7. Let $f : X \to S$ be a smooth morphism with geometrically irreducible fibres. Let $\mathcal{F}$ be a quasi-coherent $\mathcal{O}_X$-module of finite type. For any scheme T over S we will denote $\mathcal{F}_T$ the base change of $\mathcal{F}$ to T, in other words, $\mathcal{F}_T$ is the pullback of $\mathcal{F}$ via the projection morphism $X_T = X \times_S T \to X$. Note that $X_T \to T$ is smooth with geometrically irreducible fibres, see Morphisms, Lemma 28.35.5 and More on Morphisms, Lemma 36.22.2. Let $p \geq 0$ be an integer. Given a point $t \in T$ consider the condition

$$\mathcal{F}_T \text{ is free of rank } p \text{ in a neighbourhood of } \xi_t \tag{37.19.7.1}$$

where ξ_t is the generic point of the fibre X_t. This condition for all $t \in T$ is stable under base change, and hence we obtain a functor

(37.19.7.2)

$$H_p : (Sch/S)^{opp} \longrightarrow Sets, \quad T \longrightarrow \begin{cases} \{*\} & \text{if } \mathcal{F}_T \text{ satisfies (37.19.7.1) } \forall t \in T, \\ \emptyset & \text{else.} \end{cases}$$

Lemma 37.19.8. *In Situation 37.19.7.*

(1) *The functor H_p satisfies the sheaf property for the fpqc topology.*
(2) *If $\mathcal{F}$ is of finite presentation, then functor H_p is limit preserving.*

Proof. Let $\{T_i \to T\}_{i\in I}$ be an fpqc[1] covering of schemes over S. Set $X_i = X_{T_i} = X \times_S T_i$ and denote $\mathcal{F}_i$ the pullback of $\mathcal{F}$ to X_i. Assume that $\mathcal{F}_i$ satisfies (37.19.7.1) for all i. Pick $t \in T$ and let $\xi_t \in X_T$ denote the generic point of X_t. We have to show that $\mathcal{F}$ is free in a neighbourhood of ξ_t. For some $i \in I$ we can find a $t_i \in T_i$ mapping to t. Let $\xi_i \in X_i$ denote the generic point of X_{t_i}, so that ξ_i maps to ξ_t. The fact that $\mathcal{F}_i$ is free of rank p in a neighbourhood of ξ_i implies that $(\mathcal{F}_i)_{x_i} \cong \mathcal{O}_{X_i,x_i}^{\oplus p}$ which implies that $\mathcal{F}_{T,\xi_t} \cong \mathcal{O}_{X_T,\xi_t}^{\oplus p}$ as $\mathcal{O}_{X_T,\xi_t} \to \mathcal{O}_{X_i,x_i}$ is flat, see for example Algebra, Lemma 10.73.5. Thus there exists an affine neighbourhood U of ξ_t in X_T and a surjection $\mathcal{O}_U^{\oplus p} \to \mathcal{F}_U = \mathcal{F}_T|_U$, see Modules, Lemma 17.9.4. After shrinking T we may assume that $U \to T$ is surjective. Hence $U \to T$ is a smooth morphism of affines with geometrically irreducible fibres. Moreover, for every $t' \in T$ we see that the induced map

$$\alpha : \mathcal{O}_{U,\xi_{t'}}^{\oplus p} \longrightarrow \mathcal{F}_{U,\xi_{t'}}$$

is an isomorphism (since by the same argument as before the module on the right is free of rank p). It follows from Lemma 37.10.1 that

$$\Gamma(U, \mathcal{O}_U^{\oplus p}) \otimes_{\Gamma(T,\mathcal{O}_T)} \mathcal{O}_{T,t'} \longrightarrow \Gamma(U, \mathcal{F}_U) \otimes_{\Gamma(T,\mathcal{O}_T)} \mathcal{O}_{T,t'}$$

is injective for every $t' \in T$. Hence we see the surjection α is an isomorphism. This finishes the proof of (1).

Assume that $\mathcal{F}$ is of finite presentation. Let $T = \lim_{i\in I} T_i$ be a directed limit of affine S-schemes and assume that $\mathcal{F}_T$ satisfies (37.19.7.1). Set $X_i = X_{T_i} = X \times_S T_i$ and denote $\mathcal{F}_i$ the pullback of $\mathcal{F}$ to X_i. Let $U \subset X_T$ denote the open subscheme of points where $\mathcal{F}_T$ is flat over T, see More on Morphisms, Theorem 36.12.1. By assumption every generic point of every fibre is a point of U, i.e., $U \to T$ is a smooth surjective morphism with geometrically irreducible fibres. We may shrink U a bit and assume that U is quasi-compact. Using Limits, Lemma 31.3.8 we can find an $i \in I$ and a quasi-compact open $U_i \subset X_i$ whose inverse image in X_T is U. After increasing i we may assume that $\mathcal{F}_i|_{U_i}$ is flat over T_i, see Limits, Lemma 31.9.3. In particular, $\mathcal{F}_i|_{U_i}$ is finite locally free hence defines a locally constant rank function $\rho : U_i \to \{0, 1, 2, \ldots\}$. Let $(U_i)_p \subset U_i$ denote the open and closed subset where ρ has value p. Let $V_i \subset T_i$ be the image of $(U_i)_p$; note that V_i is open and quasi-compact. By assumption the image of $T \to T_i$ is contained in V_i. Hence there exists an $i' \geq i$ such that $T_{i'} \to T_i$ factors through V_i by Limits, Lemma 31.3.8. Then $\mathcal{F}_{i'}$ satisfies (37.19.7.1) as desired. Some details omitted. □

Situation 37.19.9. Let $f : X \to S$ be a morphism of schemes which is of finite type. Let $\mathcal{F}$ be a quasi-coherent $\mathcal{O}_X$-module of finite type. For any scheme T over S we will denote $\mathcal{F}_T$ the base change of $\mathcal{F}$ to T, in other words, $\mathcal{F}_T$ is the pullback of $\mathcal{F}$ via the projection morphism $X_T = X \times_S T \to X$. Note that $X_T \to T$ is of finite type and that $\mathcal{F}_T$ is an $\mathcal{O}_{X_T}$-module of finite type, see Morphisms, Lemma 28.16.4 and Modules, Lemma 17.9.2. Let $n \geq 0$. We say that $\mathcal{F}_T$ *is flat over T in*

[1] It is quite easy to show that H_p is a sheaf for the fppf topology using that flat morphisms of finite presentation are open. This is all we really need later on. But it is kind of fun to prove directly that it also satisfies the sheaf condition for the fpqc topology.

dimensions $\geq n$ if for every $t \in T$ the closed subset $Z \subset X_t$ of points where $\mathcal{F}_T$ is not flat over T (see Lemma 37.10.4) satisfies $\dim(Z) < n$ for all $t \in T$. Note that if this is the case, and if $T' \to T$ is a morphism, then $\mathcal{F}_{T'}$ is also flat in dimensions $\geq n$ over T', see Morphisms, Lemmas 28.26.6 and 28.29.3. Hence we obtain a functor (37.19.9.1)

$$F_n : (Sch/S)^{opp} \longrightarrow Sets, \quad T \longrightarrow \begin{cases} \{*\} & \text{if } \mathcal{F}_T \text{ is flat over } T \text{ in } \dim \geq n, \\ \emptyset & \text{else.} \end{cases}$$

Lemma 37.19.10. *In Situation 37.19.9.*

(1) *The functor F_n satisfies the sheaf property for the fpqc topology.*
(2) *If f is quasi-compact and locally of finite presentation and $\mathcal{F}$ is of finite presentation, then the functor F_n is limit preserving.*

Proof. Let $\{T_i \to T\}_{i \in I}$ be an fpqc covering of schemes over S. Set $X_i = X_{T_i} = X \times_S T_i$ and denote $\mathcal{F}_i$ the pullback of $\mathcal{F}$ to X_i. Assume that $\mathcal{F}_i$ is flat over T_i in dimensions $\geq n$ for all i. Let $t \in T$. Choose an index i and a point $t_i \in T_i$ mapping to t. Consider the cartesian diagram

$$\begin{array}{ccc} X_{\mathrm{Spec}(\mathcal{O}_{T,t})} & \longleftarrow & X_{\mathrm{Spec}(\mathcal{O}_{T_i,t_i})} \\ \downarrow & & \downarrow \\ \mathrm{Spec}(\mathcal{O}_{T,t}) & \longleftarrow & \mathrm{Spec}(\mathcal{O}_{T_i,t_i}) \end{array}$$

As the lower horizontal morphism is flat we see from More on Morphisms, Lemma 36.12.2 that the set $Z_i \subset X_{t_i}$ where $\mathcal{F}_i$ is not flat over T_i and the set $Z \subset X_t$ where $\mathcal{F}_T$ is not flat over T are related by the rule $Z_i = Z_{\kappa(t_i)}$. Hence we see that $\mathcal{F}_T$ is flat over T in dimensions $\geq n$ by Morphisms, Lemma 28.29.3.

Assume that f is quasi-compact and locally of finite presentation and that $\mathcal{F}$ is of finite presentation. In this paragraph we first reduce the proof of (2) to the case where f is of finite presentation. Let $T = \lim_{i \in I} T_i$ be a directed limit of affine S-schemes and assume that $\mathcal{F}_T$ is flat in dimensions $\geq n$. Set $X_i = X_{T_i} = X \times_S T_i$ and denote $\mathcal{F}_i$ the pullback of $\mathcal{F}$ to X_i. We have to show that $\mathcal{F}_i$ is flat in dimensions $\geq n$ for some i. Pick $i_0 \in I$ and replace I by $\{i \mid i \geq i_0\}$. Since T_{i_0} is affine (hence quasi-compact) there exist finitely many affine opens $W_j \subset S$, $j = 1, \ldots, m$ and an affine open overing $T_{i_0} = \bigcup_{j=1,\ldots,m} V_{j,i_0}$ such that $T_{i_0} \to S$ maps V_{j,i_0} into W_j. For $i \geq i_0$ denote $V_{j,i}$ the inverse image of V_{j,i_0} in T_i. If we can show, for each j, that there exists an i such that $\mathcal{F}_{V_{j,i_0}}$ is flat in dimensions $\geq n$, then we win. In this way we reduce to the case that S is affine. In this case X is quasi-compact and we can choose a finite affine open covering $X = W_1 \cup \ldots \cup W_m$. In this case the result for $(X \to S, \mathcal{F})$ is equivalent to the result for $(\coprod W_j, \coprod \mathcal{F}|_{W_j})$. Hence we may assume that f is of finite presentation.

Assume f is of finite presentation and $\mathcal{F}$ is of finite presentation. Let $U \subset X_T$ denote the open subscheme of points where $\mathcal{F}_T$ is flat over T, see More on Morphisms, Theorem 36.12.1. By assumption the dimension of every fibre of $Z = X_T \setminus U$ over T has dimension $\leq n$. By Limits, Lemma 31.14.3 we can find a closed subscheme $Z \subset Z' \subset X_T$ such that $\dim(Z'_t) < n$ for all $t \in T$ and such that $Z' \to X_T$ is of finite presentation. By Limits, Lemmas 31.9.1 and 31.7.4 there exists an $i \in I$ and a closed subscheme $Z'_i \subset X_i$ of finite presentation whose base change to T is Z'. By Limits, Lemma 31.14.1 we may assume all fibres of $Z'_i \to T_i$ have dimension

$< n$. By Limits, Lemma 31.9.3 we may assume that $\mathcal{F}_i|_{X_i \setminus T_i'}$ is flat over T_i. This implies that $\mathcal{F}_i$ is flat in dimensions $\geq n$; here we use that $Z' \to X_T$ is of finite presentation, and hence the complement $X_T \setminus Z'$ is quasi-compact! Thus part (2) is proved and the proof of the lemma is complete. □

Situation 37.19.11. Let $f : X \to S$ be a morphism of schemes. Let $\mathcal{F}$ be a quasi-coherent $\mathcal{O}_X$-module. For any scheme T over S we will denote $\mathcal{F}_T$ the base change of $\mathcal{F}$ to T, in other words, $\mathcal{F}_T$ is the pullback of $\mathcal{F}$ via the projection morphism $X_T = X \times_S T \to X$. Since the base change of a flat module is flat we obtain a functor

$$F_{flat} : (Sch/S)^{opp} \longrightarrow Sets, \quad T \longrightarrow \begin{cases} \{*\} & \text{if } \mathcal{F}_T \text{ is flat over } T, \\ \emptyset & \text{else.} \end{cases} \tag{37.19.11.1}$$

Lemma 37.19.12. *In Situation 37.19.11.*

(1) *The functor F_{flat} satisfies the sheaf property for the fpqc topology.*
(2) *If f is quasi-compact and locally of finite presentation and $\mathcal{F}$ is of finite presentation, then the functor F_{flat} is limit preserving.*

Proof. Part (1) follows from the following statement: If $T' \to T$ is a surjective flat morphism of schemes over S, then $\mathcal{F}_{T'}$ is flat over T' if and only if $\mathcal{F}_T$ is flat over T, see More on Morphisms, Lemma 36.12.2. Part (2) follows from Limits, Lemma 31.9.3 after reducing to the case where X and S are affine (compare with the proof of Lemma 37.19.10). □

37.20. Flattening stratifications

Just the definitions and an important baby case.

Definition 37.20.1. Let $X \to S$ be a morphism of schemes. Let $\mathcal{F}$ be a quasi-coherent $\mathcal{O}_X$-module. We say that the *universal flattening of $\mathcal{F}$ exists* if the functor F_{flat} defined in Situation 37.19.11 is representable by a scheme S' over S. We say that the *universal flattening of X exists* if the universal flattening of $\mathcal{O}_X$ exists.

Note that if the universal flattening S'[2] of $\mathcal{F}$ exists, then the morphism $S' \to S$ is a monomorphism of schemes such that $\mathcal{F}_{S'}$ is flat over S' and such that a morphism $T \to S$ factors through S' if and only if $\mathcal{F}_T$ is flat over T.

We define (compare with Topology, Remark 5.26.4) a (locally finite, scheme theoretic) *stratification* of a scheme S to be given by closed subschemes $Z_i \subset S$ such that $S = \bigcup Z_i$ (set theoretically), such that every point of S has a neighbourhood meeting only a finite number of Z_i, and such that I comes with a partial ordering such that

$$Z_i \cap Z_j = \bigcup\nolimits_{k \leq i,j} Z_k.$$

Setting $S_i = Z_i \setminus \bigcup_{j<i} Z_j$ the actual stratification is the decomposition $S = \coprod S_i$ into locally closed subschemes. We often only indicate the strata S_i and leave the construction of the closed subschemes Z_i to the reader. Given a stratification we obtain a monomorphism

$$S' = \coprod\nolimits_{i \in I} S_i \longrightarrow S.$$

[2]The scheme S' is sometimes called the *universal flatificator*. In [**GR71**] it is called the *platificateur universel*. Existence of the universal flattening should not be confused with the type of results discussed in More on Algebra, Section 15.17.

We will call this the *monomorphism associated to the stratification*. With this terminology we can define what it means to have a flattening stratification.

Definition 37.20.2. Let $X \to S$ be a morphism of schemes. Let $\mathcal{F}$ be a quasi-coherent $\mathcal{O}_X$-module. We say that $\mathcal{F}$ has a *flattening stratification* if the functor F_{flat} defined in Situation 37.19.11 is representable by a monomorphism $S' \to S$ associated to a stratification of S by locally closed subschemes. We say that X has a *flattening stratification* if $\mathcal{O}_X$ has a flattening stratification.

When a flattening stratification exists, it is often important to understand the index set labeling the strata and its partial ordering. This often has to do with ranks of modules, as in the baby case below.

Lemma 37.20.3. *Let S be a scheme. Let $\mathcal{F}$ be a finite type, quasi-coherent $\mathcal{O}_S$-module. Let $r \geq 0$. The functor*

$$F_r : (Sch/S)^{opp} \to Sets, \quad T \mapsto \begin{cases} \{*\} & \text{if} \quad \mathcal{F}_T \text{ locally free rank } r \\ \emptyset & \text{else.} \end{cases}$$

is representable by a monomorphism $S_r \to S$ of schemes. If $\mathcal{F}$ is of finite presentation, then $S_r \to S$ is of finite presentation.

Proof. We refer to the chapter on exercises for more information on *fitting ideals*. Let

$$\mathcal{I}_0 \subset \mathcal{I}_1 \subset \mathcal{I}_2 \subset \ldots \subset \mathcal{O}_S$$

be the fitting ideals of $\mathcal{F}$ as an $\mathcal{O}_S$-module. If $U \subset X$ is open, and

$$\bigoplus\nolimits_{i \in I} \mathcal{O}_U \to \mathcal{O}_U^{\oplus n} \to \mathcal{F}|_U \to 0$$

is a presentation of $\mathcal{F}$ over U, then $\mathcal{I}_k|_U$ is generated by the $(n-k) \times (n-k)$-minors of the matrix defining the first arrow of the presentation. In particular, $\mathcal{I}_k$ is locally generated by sections, whence quasi-coherent. For any morphism $g : T \to S$ we see that $\mathcal{F}_T$ is locally free of rank r if and only if $\mathcal{I}_r \cdot \mathcal{O}_T = \mathcal{O}_T$ and $\mathcal{I}_{r-1} \cdot \mathcal{O}_T = 0$. Hence, letting $Z_k \subset S$ denote the closed subscheme defined by $\mathcal{I}_k$ we see that $S_r = Z_r \setminus Z_{r-1}$ works. If $\mathcal{F}$ is of finite presentation, then each of the morphisms $Z_k \to X$ is of finite presentation as $\mathcal{I}_k$ is locally generated by finitely many minors. This implies that $Z_k \setminus Z_{r-1}$ is a retrocompact open in Z_k and hence the morphism $S_r \to Z_r$ is of finite presentation as well. □

Lemma 37.20.4. *Let S be a scheme. Let $\mathcal{F}$ be a quasi-coherent $\mathcal{O}_S$-module of finite presentation. There exists a flattening stratification $S' = \coprod_{r \geq 0} S_r$ for $\mathcal{F}$ (relative to $id_S : S \to S$) such that $\mathcal{F}|_{S_r}$ is locally free of rank r. Moreover, each $S_r \to S$ is of finite presentation.*

Proof. Suppose that $g : T \to S$ is a morphism of schemes such that the pullback $\mathcal{F}_T = g^*\mathcal{F}$ is flat. Then $\mathcal{F}_T$ is a flat $\mathcal{O}_T$-module of finite presentation. Hence $\mathcal{F}_T$ is finite locally free, see Properties, Lemma 27.18.2. Thus $T = \coprod_{r \geq 0} T_r$, where $\mathcal{F}_T|_{T_r}$ is locally free of rank r. This implies that

$$F_{flat} = \coprod\nolimits_{r \geq 0} F_r$$

in the category of Zariski sheaves on Sch/S. Hence it follows that F_{flat} is represented by $\coprod_{r \geq 0} S_r$ where S_r is as in Lemma 37.20.3. □

37.21. Flattening stratification over an Artinian ring

A flatting stratification exists when the base scheme is the spectrum of an Artinian ring.

Lemma 37.21.1. *Let S be the spectrum of an Artinian ring. For any scheme X over S, and any quasi-coherent $\mathcal{O}_X$-module there exists a universal flattening. In fact the universal flattening is given by a closed immersion $S' \to S$, and hence is a flattening stratification for $\mathcal{F}$ as well.*

Proof. Choose an affine open covering $X = \bigcup U_i$. Then F_{flat} is the product of the functors associated to each of the pairs $(U_i, \mathcal{F}|_{U_i})$. Hence it suffices to prove the result for each $(U_i, \mathcal{F}|_{U_i})$. In the affine case the lemma follows immediately from More on Algebra, Lemma 15.10.2. □

37.22. Flattening a map

Theorem 37.22.3 is the key to further flattening statements.

Lemma 37.22.1. *Let S be a scheme. Let $g : X' \to X$ be a flat morphism of schemes over S with X locally of finite type over S. Let $\mathcal{F}$ be a finite type $\mathcal{O}_X$-module which is flat over S. If $Ass_{X/S}(\mathcal{F}) \subset g(X')$ then the canonical map*

$$\mathcal{F} \longrightarrow g_* g^* \mathcal{F}$$

is injective, and remains injective after any base change.

Proof. The final assertion means that $\mathcal{F}_T \to (g_T)_* g_T^* \mathcal{F}_T$ is injective for any morphism $T \to S$. The assumption $\mathrm{Ass}_{X/S}(\mathcal{F}) \subset g(X')$ is preserved by base change, see Divisors, Lemma 30.7.2 and Remark 30.7.3. The same holds for the assumption of flatness and finite type. Hence it suffices to prove the injectivity of the displayed arrow. Let $\mathcal{K} = \mathrm{Ker}(\mathcal{F} \to g_* g^* \mathcal{F})$. Our goal is to prove that $\mathcal{K} = 0$. In order to do this it suffices to prove that $\mathrm{WeakAss}_X(\mathcal{K}) = \emptyset$, see Divisors, Lemma 30.5.5. We have $\mathrm{WeakAss}_X(\mathcal{K}) \subset \mathrm{WeakAss}_X(\mathcal{F})$, see Divisors, Lemma 30.5.4. As $\mathcal{F}$ is flat we see from Lemma 37.12.4 that $\mathrm{WeakAss}_X(\mathcal{F}) \subset \mathrm{Ass}_{X/S}(\mathcal{F})$. By assumption any point x of $\mathrm{Ass}_{X/S}(\mathcal{F})$ is the image of some $x' \in X'$. Since g is flat the local ring map $\mathcal{O}_{X,x} \to \mathcal{O}_{X',x'}$ is faithfully flat, hence the map

$$\mathcal{F}_x \longrightarrow g^* \mathcal{F}_{x'} = \mathcal{F}_x \otimes_{\mathcal{O}_{X,x}} \mathcal{O}_{X',x'}$$

is injective (see Algebra, Lemma 10.77.11). This implies that $\mathcal{K}_x = 0$ as desired. □

Lemma 37.22.2. *Let A be a ring. Let $u : M \to N$ be a surjective map of A-modules. If M is projective as an A-module, then there exists an ideal $I \subset A$ such that for any ring map $\varphi : A \to B$ the following are equivalent*

1. *$u \otimes 1 : M \otimes_A B \to N \otimes_A B$ is an isomorphism, and*
2. *$\varphi(I) = 0$.*

Proof. As M is projective we can find a projective A-module C such that $F = M \oplus C$ is a free R-module. By replacing u by $u \oplus 1 : F = M \oplus C \to N \oplus C$ we see that we may assume M is free. In this case let I be the ideal of A generated by coefficients of all the elements of $\mathrm{Ker}(u)$ with respect to some (fixed) basis of M. The reason this works is that, since u is surjective and $\otimes_A B$ is right exact, $\mathrm{Ker}(u \otimes 1)$ is the image of $\mathrm{Ker}(u) \otimes_A B$ in $M \otimes_A B$. □

Theorem 37.22.3. *In Situation 37.19.1 assume*

(1) f *is of finite presentation,*
(2) $\mathcal{F}$ *is of finite presentation, flat over* S*, and pure relative to* S*, and*
(3) u *is surjective.*

Then F_{iso} *is representable by a closed immersion* $Z \to S$*. Moreover* $Z \to S$ *is of finite presentation if* $\mathcal{G}$ *is of finite presentation.*

Proof. We will use without further mention that $\mathcal{F}$ is universally pure over S, see Lemma 37.17.3. By Lemma 37.19.2 and Descent, Lemmas 34.33.2 and 34.35.1 the question is local for the étale topology on S. Hence it suffices to prove, given $s \in S$, that there exists an étale neighbourhood of (S, s) so that the theorem holds.

Using Lemma 37.11.5 and after replacing S by an elementary étale neighbourhood of s we may assume there exists a commutative diagram

$$\begin{array}{ccc} X & \xleftarrow{g} & X' \\ & \searrow \quad \swarrow & \\ & S & \end{array}$$

of schemes of finite presentation over S, where g is étale, $X_s \subset g(X')$, the schemes X' and S are affine, $\Gamma(X', g^*\mathcal{F})$ a projective $\Gamma(S, \mathcal{O}_S)$-module. Note that $g^*\mathcal{F}$ is universally pure over S, see Lemma 37.16.4. Hence by Lemma 37.17.2 we see that the open $g(X')$ contains the points of $\mathrm{Ass}_{X/S}(\mathcal{F})$ lying over $\mathrm{Spec}(\mathcal{O}_{S,s})$. Set

$$E = \{t \in S \mid \mathrm{Ass}_{X_t}(\mathcal{F}_t) \subset g(X')\}.$$

By More on Morphisms, Lemma 36.20.5 E is a constructible subset of S. We have seen that $\mathrm{Spec}(\mathcal{O}_{S,s}) \subset E$. By Morphisms, Lemma 28.23.4 we see that E contains an open neighbourhood of s. Hence after replacing S by a smaller affine neighbourhood of s we may assume that $\mathrm{Ass}_{X/S}(\mathcal{F}) \subset g(X')$.

Since we have assumed that u is surjective we have $F_{iso} = F_{inj}$. From Lemma 37.22.1 it follows that $u : \mathcal{F} \to \mathcal{G}$ is injective if and only if $g^*u : g^*\mathcal{F} \to g^*\mathcal{G}$ is injective, and the same remains true after any base change. Hence we have reduced to the case where, in addition to the assumptions in the theorem, $X \to S$ is a morphism of affine schemes and $\Gamma(X, \mathcal{F})$ is a projective $\Gamma(S, \mathcal{O}_S)$-module. This case follows immediately from Lemma 37.22.2.

To see that Z is of finite presentation if $\mathcal{G}$ is of finite presentation, combine Lemma 37.19.2 part (4) with Limits, Remark 31.5.2. □

Lemma 37.22.4. *Let* $f : X \to S$ *be a morphism of schemes which is of finite presentation, flat, and pure. Let* Y *be a closed subscheme of* X*. Let* $F = f_*Y$ *be the Weil restriction functor of* Y *along* f*, defined by*

$$F : (Sch/S)^{opp} \to Sets, \quad T \mapsto \begin{cases} \{*\} & \text{if} \quad Y_T \to X_T \text{ is an isomorphism,} \\ \emptyset & \text{else.} \end{cases}$$

Then F *is representable by a closed immersion* $Z \to S$*. Moreover* $Z \to S$ *is of finite presentation if* $Y \to S$ *is.*

Proof. Let $\mathcal{I}$ be the ideal sheaf defining Y in X and let $u : \mathcal{O}_X \to \mathcal{O}_X/\mathcal{I}$ be the surjection. Then for an S-scheme T, the closed immersion $Y_T \to X_T$ is an isomorphism if and only if u_T is an isomorphism. Hence the result follows from Theorem 37.22.3. □

37.23. Flattening in the local case

In this section we start applying the earlier material to obtain a shadow of the flattening stratification.

Theorem 37.23.1. *In Situation 37.19.3 assume A is henselian, B is essentially of finite type over A, and M is a finite B-module. Then there exists an ideal $I \subset A$ such that A/I corepresents the functor F_{lf} on the category $\mathcal{C}$. In other words given a local homomorphism of local rings $\varphi : A \to A'$ with $B' = B \otimes_A A'$ and $M' = M \otimes_A A'$ the following are equivalent:*

(1) $\forall \mathfrak{q} \in V(\mathfrak{m}_{A'}B' + \mathfrak{m}_B B') \subset \mathrm{Spec}(B') : M'_{\mathfrak{q}}$ *is flat over A', and*
(2) $\varphi(I) = 0$.

If B is essentially of finite presentation over A and M of finite presentation over B, then I is a finitely generated ideal.

Proof. Choose a finite type ring map $A \to C$ and a finite C-module N and a prime $\mathfrak{q}$ of C such that $B = C_{\mathfrak{q}}$ and $M = N_{\mathfrak{q}}$. In the following, when we say "the theorem holds for $(N/C/A, \mathfrak{q})$ we mean that it holds for $(A \to B, M)$ where $B = C_{\mathfrak{q}}$ and $M = N_{\mathfrak{q}}$. By Lemma 37.19.6 the functor F_{lf} is unchanged if we replace B by a local ring flat over B. Hence, since A is henselian, we may apply Lemma 37.6.6 and assume that there exists a complete dévissage of $N/C/A$ at $\mathfrak{q}$.

Let $(A_i, B_i, M_i, \alpha_i, \mathfrak{q}_i)_{i=1,\ldots,n}$ be such a complete dévissage of $N/C/A$ at $\mathfrak{q}$. Let $\mathfrak{q}'_i \subset A_i$ be the unique prime lying over $\mathfrak{q}_i \subset B_i$ as in Definition 37.6.4. Since $C \to A_1$ is surjective and $N \cong M_1$ as C-modules, we see by Lemma 37.19.5 it suffices to prove the theorem holds for $(M_1/A_1/A, \mathfrak{q}'_1)$. Since $B_1 \to A_1$ is finite and $\mathfrak{q}_1$ is the only prime of B_1 over $\mathfrak{q}'_1$ we see that $(A_1)_{\mathfrak{q}'_1} \to (B_1)_{\mathfrak{q}_1}$ is finite (see Algebra, Lemma 10.38.11 or More on Morphisms, Lemma 36.33.4). Hence by Lemma 37.19.5 it suffices to prove the theorem holds for $(M_1/B_1/A, \mathfrak{q}_1)$.

At this point we may assume, by induction on the length n of the dévissage, that the theorem holds for $(M_2/B_2/A, \mathfrak{q}_2)$. (If $n = 1$, then $M_2 = 0$ which is flat over A.) Reversing the last couple of steps of the previous paragraph, using that $M_2 \cong \mathrm{Coker}(\alpha_2)$ as B_1-modules, we see that the theorem holds for $(\mathrm{Coker}(\alpha_1)/B_1/A, \mathfrak{q}_1)$.

Let A' be an object of $\mathcal{C}$. At this point we use Lemma 37.10.1 to see that if $(M_1 \otimes_A A')_{\mathfrak{q}'}$ is flat over A' for a prime $\mathfrak{q}'$ of $B_1 \otimes_A A'$ lying over $\mathfrak{m}_{A'}$, then $(\mathrm{Coker}(\alpha_1) \otimes_A A')_{\mathfrak{q}'}$ is flat over A'. Hence we conclude that F_{lf} is a subfunctor of the functor F'_{lf} associated to the module $\mathrm{Coker}(\alpha_1)_{\mathfrak{q}_1}$ over $(B_1)_{\mathfrak{q}_1}$. By the previous paragraph we know F'_{lf} is corepresented by A/J for some ideal $J \subset A$. Hence we may replace A by A/J and assume that $\mathrm{Coker}(\alpha_1)_{\mathfrak{q}_1}$ is flat over A.

Since $\mathrm{Coker}(\alpha_1)$ is a B_1-module for which there exist a complete dévissage of $N_1/B_1/A$ at $\mathfrak{q}_1$ and since $\mathrm{Coker}(\alpha_1)_{\mathfrak{q}_1}$ is flat over A by Lemma 37.10.2 we see that $\mathrm{Coker}(\alpha_1)$ is free as an A-module, in particular flat as an A-module. Hence Lemma 37.10.1 implies $F_{lf}(A')$ is nonempty if and only if $\alpha \otimes 1_{A'}$ is injective. Let $N_1 = \mathrm{Im}(\alpha_1) \subset M_1$ so that we have exact sequences

$$0 \to N_1 \to M_1 \to \mathrm{Coker}(\alpha_1) \to 0 \quad \text{and} \quad B_1^{\oplus r_1} \to N_1 \to 0$$

The flatness of $\mathrm{Coker}(\alpha_1)$ implies the first sequence is universally exact (see Algebra, Lemma 10.77.5). Hence $\alpha \otimes 1_{A'}$ is injective if and only if $B_1^{\oplus r_1} \otimes_A A' \to N_1 \otimes_A A'$ is an isomorphism. Finally, Theorem 37.22.3 applies to show this functor is

corepresentable by A/I for some ideal I and we conclude F_{lf} is corepresentable by A/I also.

To prove the final statement, suppose that $A \to B$ is essentially of finite presentation and M of finite presentation over B. Let $I \subset A$ be the ideal such that F_{lf} is corepresented by A/I. Write $I = \bigcup I_\lambda$ where I_λ ranges over the finitely generated ideals contained in I. Then, since $F_{lf}(A/I) = \{*\}$ we see that $F_{lf}(A/I_\lambda) = \{*\}$ for some λ, see Lemma 37.19.4 part (2). Clearly this implies that $I = I_\lambda$. □

Remark 37.23.2. Here is a scheme theoretic reformulation of Theorem 37.23.1. Let $(X, x) \to (S, s)$ be a morphism of pointed schemes which is locally of finite type. Let $\mathcal{F}$ be a finite type quasi-coherent $\mathcal{O}_X$-module. Assume S henselian local with closed point s. There exists a closed subscheme $Z \subset S$ with the following property: for any morphism of pointed schemes $(T, t) \to (S, s)$ the following are equivalent

(1) $\mathcal{F}_T$ is flat over T at all points of the fibre X_t which map to $x \in X_s$, and
(2) $\mathrm{Spec}(\mathcal{O}_{T,t}) \to S$ factors through Z.

Moreover, if $X \to S$ is of finite presentation at x and $\mathcal{F}_x$ of finite presentation over $\mathcal{O}_{X,x}$, then $Z \to S$ is of finite presentation.

At this point we can obtain some very general results completely for free from the result above. Note that perhaps the most interesting case is when $E = X_s$!

Lemma 37.23.3. *Let S be the spectrum of a henselian local ring with closed point s. Let $X \to S$ be a morphism of schemes which is locally of finite type. Let $\mathcal{F}$ be a finite type quasi-coherent $\mathcal{O}_X$-module. Let $E \subset X_s$ be a subset. There exists a closed subscheme $Z \subset S$ with the following property: for any morphism of pointed schemes $(T, t) \to (S, s)$ the following are equivalent*

(1) *$\mathcal{F}_T$ is flat over T at all points of the fibre X_t which map to a point of $E \subset X_s$, and*
(2) *$\mathrm{Spec}(\mathcal{O}_{T,t}) \to S$ factors through Z.*

Moreover, if $X \to S$ is locally of finite presentation, $\mathcal{F}$ is of finite presentation, and $E \subset X_s$ is closed and quasi-compact, then $Z \to S$ is of finite presentation.

Proof. For $x \in X_s$ denote $Z_x \subset S$ the closed subscheme we found in Remark 37.23.2. Then it is clear that $Z = \bigcap_{x \in E} Z_x$ works!

To prove the final statement assume X locally of finite presentation, $\mathcal{F}$ of finite presentation and Z closed and quasi-compact. First, choose finitely many affine opens $W_j \subset X$ such that $E \subset \bigcup W_j$. It clearly suffices to prove the result for each morphism $W_j \to S$ with sheaf $\mathcal{F}|_{X_j}$ and closed subset $E \cap W_j$. Hence we may assume X is affine. In this case, More on Algebra, Lemma 15.12.4 shows that the functor defined by (1) is "limit preserving". Hence we can show that $Z \to S$ is of finite presentation exactly as in the last part of the proof of Theorem 37.23.1. □

Remark 37.23.4. Tracing the proof of Lemma 37.23.3 to its origins we find a long and winding road. But if we assume that

(1) f is of finite type,
(2) $\mathcal{F}$ is a finite type $\mathcal{O}_X$-module,
(3) $E = X_s$, and
(4) S is the spectrum of a Noetherian complete local ring.

then there is a proof relying completely on more elementary algebra as follows: first we reduce to the case where X is affine by taking a finite affine open cover. In this case Z exists by More on Algebra, Lemma 15.13.3. The key step in this proof is constructing the closed subscheme Z step by step inside the truncations $\mathrm{Spec}(\mathcal{O}_{S,s}/\mathfrak{m}_s^n)$. This relies on the fact that flattening stratifications always exist when the base is Artinian, and the fact that $\mathcal{O}_{S,s} = \lim \mathcal{O}_{S,s}/\mathfrak{m}_s^n$.

37.24. Flat finite type modules, Part III

The following lemma improves Algebra, Lemma 10.122.4.

Lemma 37.24.1. *Let $\varphi : A \to B$ be a local ring homomorphism of local rings which is essentially of finite type. Let M be a flat A-module, N a finite B-module and $u : N \to M$ an A-module map such that $\overline{u} : N/\mathfrak{m}_A N \to M/\mathfrak{m}_A M$ is injective. Then u is A-universally injective, N is of finite presentation over B, and N is flat over A.*

Proof. Let $A \to A^h$ be the henselization of A. Let B' be the localization of $B \otimes_A A^h$ at the maximal ideal $\mathfrak{m}_B \otimes A^h + B \otimes \mathfrak{m}_{A^h}$. Since $B \to B'$ is flat (hence faithfully flat, see Algebra, Lemma 10.36.16), we may replace $A \to B$ with $A^h \to B'$, M by $M \otimes_A A^h$, and N by $N \otimes_B B'$, see Algebra, Lemmas 10.78.2 and 10.36.8. Thus we may and do assume that A is a henselian local ring.

Write $B = C/I$ where C is the localization of a polynomial algebra over A at a prime. If we can show that N is finitely presented as a C-module, then a fortiori this shows that N is finitely presented as a B-module (see discussion in More on Morphisms, Section 36.38; more precisely, see Algebra, Lemma 10.6.4). Hence we may assume that B is essentially of finite presentation over A (even the localization of a polynomial algebra). Next, write $N = B^{\oplus n}/K$ for some submodule $K \subset B^{\oplus n}$. Since $B/\mathfrak{m}_A B$ is Noetherian (as it is essentially of finite type over a field), there exist finitely many elements $k_1, \ldots, k_s \in K$ such that for $K' = \sum Bk_i$ and $N' = B^{\oplus n}/K'$ the canonical surjection $N' \to N$ induces an isomorphism $N'/\mathfrak{m}_A N' \cong N/\mathfrak{m}_A N$. Thus, if we can prove the lemma for the composition $u' : N' \to M$, then u' is injective, hence $N' = N$ and N is of finite presentation. In this way we reduce to the case where N is of finite presentation over B!

Assume A is a henselian local ring, B is essentially of finite presentation over A, N of finite presentation over B and let us temporarily make the additional assumption that N is flat over A. Then N is a filtered colimit $N = \mathrm{colim}_i F_i$ of free A-modules F_i such that the transition maps $u_{ii'} : F_i \to F_{i'}$ are injective modulo $\mathfrak{m}_A$, see Lemma 37.18.5. Each of the compositions $u_i : F_i \to M$ is A-universally injective by Lemma 37.7.5 wherefore $u = \mathrm{colim}\, u_i$ is A-universally injective as desired.

Assume A is a henselian local ring, B is essentially of finite presentation over A, N of finite presentation over B. By Theorem 37.23.1 there exists a finitely generated ideal $I \subset A$ such that N/IN is flat over A/I and such that N/I^2N is not flat over A/I^2 unless $I = 0$. The result of the previous paragraph shows that the lemma

holds for $u \bmod I : N/IN \to M/IM$ over A/I. Consider the commutative diagram

$$\begin{array}{ccccccc} 0 & \longrightarrow & M \otimes_A I/I^2 & \longrightarrow & M/I^2M & \longrightarrow & M/IM \longrightarrow 0 \\ & & \uparrow{\scriptstyle u} & & \uparrow{\scriptstyle u} & & \uparrow{\scriptstyle u} \\ & & N \otimes_A I/I^2 & \longrightarrow & N/I^2N & \longrightarrow & N/IN \longrightarrow 0 \end{array}$$

whose rows are exact by right exactness of $\otimes$ and the fact that M is flat over A. Note that the left vertical arrow is the map $N/IN \otimes_{A/I} I/I^2 \to M/IM \otimes_{A/I} I/I^2$, hence is injective. A diagram chase shows that the lower left arrow is injective, i.e., $\mathrm{Tor}^1_{A/I^2}(I/I^2, M/I^2) = 0$ see Algebra, Remark 10.70.8. Hence N/I^2N is flat over A/I^2 by Algebra, Lemma 10.93.8 a contradiction unless $I = 0$. □

Theorem 37.24.2. *Let $f : X \to S$ be locally of finite type. Let $\mathcal{F}$ be an $\mathcal{O}_X$-module of finite type. Let $x \in X$ with image $s \in S$. The following are equivalent*

1. *$\mathcal{F}$ is flat at x over S, and*
2. *for every $x' \in Ass_{X_s}(\mathcal{F}_s)$ which specializes to x we have that $\mathcal{F}$ is flat at x' over S.*

Proof. It is clear that (1) implies (2) as $\mathcal{F}_{x'}$ is a localization of $\mathcal{F}_x$ for every point which specializes to x. Set $A = \mathcal{O}_{S,s}$, $B = \mathcal{O}_{X,x}$ and $N = \mathcal{F}_x$. Let $\Sigma \subset B$ be the multiplicative subset of B of elements which act as nonzerodivisors on $N/\mathfrak{m}_A N$. Assumption (2) implies that $\Sigma^{-1}N$ is A-flat by the description of $\mathrm{Spec}(\Sigma^{-1}N)$ in Lemma 37.7.1. On the other hand, the map $N \to \Sigma^{-1}N$ is injective modulo $\mathfrak{m}_A$ by construction. Hence applying Lemma 37.24.1 we win. □

Now we apply this directly to obtain the following useful results.

Lemma 37.24.3. *Let S be a local scheme with closed point s. Let $f : X \to S$ be locally of finite type. Let $\mathcal{F}$ be a finite type $\mathcal{O}_X$-module. Assume that*

1. *every point of $Ass_{X/S}(\mathcal{F})$ specializes to a point of the closed fibre X_s[3],*
2. *$\mathcal{F}$ is flat over S at every point of X_s.*

Then $\mathcal{F}$ is flat over S.

Proof. This is immediate from the fact that it suffices to check for flatness at points of the relative assassin of $\mathcal{F}$ over S by Theorem 37.24.2. □

37.25. Universal flattening

If $f : X \to S$ is a proper, finitely presented morphism of schemes then one can find a universal flattening of f. In this section we discuss this and some of its variants.

Lemma 37.25.1. *In Situation 37.19.7. For each $p \geq 0$ the functor H_p (37.19.7.2) is representable by a locally closed immersion $S_p \to S$. If $\mathcal{F}$ is of finite presentation, then $S_p \to S$ is of finite presentation.*

Proof. For each S we will prove the statement for all $p \geq 0$ concurrently. The functor H_p is a sheaf for the fppf topology by Lemma 37.19.8. Hence combining Descent, Lemma 34.35.1, More on Morphisms, Lemma 36.37.1 , and Descent, Lemma 34.20.1 we see that the question is local for the étale topology on S. In particular, the question is Zariski local on S.

[3]For example this holds if f is finite type and $\mathcal{F}$ is pure along X_s, or if f is proper.

For $s \in S$ denote ξ_s the unique generic point of the fibre X_s. Note that for every $s \in S$ the restriction $\mathcal{F}_s$ of $\mathcal{F}$ is locally free of some rank $p(s) \geq 0$ in some neighbourhood of ξ_s. (As X_s is irreducible and smooth this follows from generic flatness for $\mathcal{F}_s$ over X_s, see Algebra, Lemma 10.112.1 although this is overkill.) For future reference we note that

$$p(s) = \dim_{\kappa(\xi_s)}(\mathcal{F}_{\xi_s} \otimes_{\mathcal{O}_{X,\xi_s}} \kappa(\xi_s)).$$

In particular $H_{p(s)}(s)$ is nonempty and $H_q(s)$ is empty if $q \neq p(s)$.

Let $U \subset X$ be an open subscheme. As $f : X \to S$ is smooth, it is open. It is immediate from (37.19.7.2) that the functor H_p for the pair $(f|_U : U \to f(U), \mathcal{F}|_U)$ and the functor H_p for the pair $(f|_{f^{-1}(f(U))}, \mathcal{F}|_{f^{-1}(f(U))})$ are the same. Hence to prove the existence of S_p over $f(U)$ we may always replace X by U.

Pick $s \in S$. There exists an affine open neighbourhood U of ξ_s such that $\mathcal{F}|_U$ can be generated by at most $p(s)$ elements. By the arguments above we see that in order to prove the statement for $H_{p(s)}$ in an neighbourhood of s we may assume that $\mathcal{F}$ is generated by $p(s)$ elements, i.e., that there exists a surjection

$$u : \mathcal{O}_X^{\oplus p(s)} \longrightarrow \mathcal{F}$$

In this case it is clear that $H_{p(s)}$ is equal to F_{iso} (37.19.1.1) for the map u (this follows immediately from Lemma 37.18.1 but also from Lemma 37.11.1 after shrinking a bit more so that both S and X are affine.) Thus we may apply Theorem 37.22.3 to see that $H_{p(s)}$ is representable by a closed immersion in a neighbourhood of s.

The result follows formally from the above. Namely, the arguments above show that locally on S the function $s \mapsto p(s)$ is bounded. Hence we may use induction on $p = \max_{s \in S} p(s)$. The functor H_p is representable by a closed immersion $S_p \to S$ by the above. Replace S by $S \setminus S_p$ which drops the maximum by at least one and we win by induction hypothesis.

To see that $S_p \to S$ is of finite presentation if $\mathcal{F}$ is of finite presentation combine Lemma 37.19.8 part (2) with Limits, Remark 31.5.2. □

Lemma 37.25.2. *In Situation 37.19.9. Let $h : X' \to X$ be an étale morphism. Set $\mathcal{F}' = h^*\mathcal{F}$ and $f' = f \circ h$. Let F'_n be (37.19.9.1) associated to $(f' : X' \to S, \mathcal{F}')$. Then F_n is a subfunctor of F'_n and if $h(X') \supset Ass_{X/S}(\mathcal{F})$, then $F_n = F'_n$.*

Proof. Let $T \to S$ be any morphism. Then $h_T : X'_T \to X_T$ is étale as a base change of the étale morphism g. For $t \in T$ denote $Z \subset X_t$ the set of points where $\mathcal{F}_T$ is not flat over T, and similarly denote $Z' \subset X'_t$ the set of points where $\mathcal{F}'_T$ is not flat over T. As $\mathcal{F}'_T = h_T^*\mathcal{F}_T$ we see that $Z' = h_t^{-1}(Z)$, see Morphisms, Lemma 28.26.11. Hence $Z' \to Z$ is an étale morphism, so $\dim(Z') \leq \dim(Z)$ (for example by Descent, Lemma 34.17.2 or just because an étale morphism is smooth of relative dimension 0). This implies that $F_n \subset F'_n$.

Finally, suppose that $h(X') \supset \mathrm{Ass}_{X/S}(\mathcal{F})$ and that $T \to S$ is a morphism such that $F'_n(T)$ is nonempty, i.e., such that $\mathcal{F}'_T$ is flat in dimensions $\geq n$ over T. Pick a point $t \in T$ and let $Z \subset X_t$ and $Z' \subset X'_t$ be as above. To get a contradiction assume that $\dim(Z) \geq n$. Pick a generic point $\xi \in Z$ corresponding to a component of dimension $\geq n$. Let $x \in \mathrm{Ass}_{X_t}(\mathcal{F}_t)$ be a generalization of ξ. Then x maps to a point of $\mathrm{Ass}_{X/S}(\mathcal{F})$ by Divisors, Lemma 30.7.2 and Remark 30.7.3. Thus we see that x is in the image of h_T, say $x = h_T(x')$ for some $x' \in X'_T$. But $x' \notin Z'$ as

$x \rightsquigarrow \xi$ and $\dim(Z') < n$. Hence $\mathcal{F}'_T$ is flat over T at x' which implies that $\mathcal{F}_T$ is flat at x over T (by Morphisms, Lemma 28.26.11). Since this holds for every such x we conclude that $\mathcal{F}_T$ is flat over T at ξ by Theorem 37.24.2 which is the desired contradiction. □

Lemma 37.25.3. *Assume that $X \to S$ is a smooth morphism of affine schemes with geometrically irreducible fibres of dimension d and that $\mathcal{F}$ is a quasi-coherent $\mathcal{O}_X$-module of finite presentation. Then $F_d = \coprod_{p=0,\ldots,c} H_p$ for some $c \geq 0$ with F_d as in (37.19.9.1) and H_p as in (37.19.7.2).*

Proof. As X is affine and $\mathcal{F}$ is quasi-coherent of finite presentation we know that $\mathcal{F}$ can be generated by $c \geq 0$ elements. Then $\dim_{\kappa(x)}(\mathcal{F}_x \otimes \kappa(x))$ in any point $x \in X$ never exceeds c. In particular $H_p = \emptyset$ for $p > c$. Moreover, note that there certainly is an inclusion $\coprod H_p \to F_d$. Having said this the content of the lemma is that, if a base change $\mathcal{F}_T$ is flat in dimensions $\geq d$ over T and if $t \in T$, then $\mathcal{F}_T$ is free of some rank r in an open neighbourhood $U \subset X_T$ of the unique generic point ξ of X_t. Namely, then H_r contains the image of U which is an open neighbourhood of t. The existence of U follows from More on Morphisms, Lemma 36.13.7. □

Lemma 37.25.4. *In Situation 37.19.9. Let $s \in S$ let $d \geq 0$. Assume*

1. *there exists a complete dévissage of $\mathcal{F}/X/S$ over some point $s \in S$,*
2. *X is of finite presentation over S,*
3. *$\mathcal{F}$ is an $\mathcal{O}_X$-module of finite presentation, and*
4. *$\mathcal{F}$ is flat in dimensions $\geq d+1$ over S.*

Then after possibly replacing S by an open neighbourhood of s the functor F_d (37.19.9.1) is representable by a monomorphism $Z_d \to S$ of finite presentation.

Proof. A preliminary remark is that X, S are affine schemes and that it suffices to prove F_d is representable by a closed subscheme on the category of affine schemes over S. Hence throughout the proof of the lemma we work in the category of affine schemes over S.

Let $(Z_k, Y_k, i_k, \pi_k, \mathcal{G}_k, \alpha_k)_{k=1,\ldots,n}$ be a complete dévissage of $\mathcal{F}/X/S$ over s, see Definition 37.5.1. We will use induction on the length n of the dévissage. Recall that $Y_k \to S$ is smooth with geometrically irreducible fibres, see Definition 37.4.1. Let d_k be the relative dimension of Y_k over S. Recall that $i_{k,*}\mathcal{G}_k = \mathrm{Coker}(\alpha_k)$ and that i_k is a closed immersion. By the definitions referenced above we have $d_1 = \dim(\mathrm{Supp}(\mathcal{F}_s))$ and

$$d_k = \dim(\mathrm{Supp}(\mathrm{Coker}(\alpha_{k-1})_s)) = \dim(\mathrm{Supp}(\mathcal{G}_{k,s}))$$

for $k = 2, \ldots, n$. It follows that $d_1 > d_2 > \ldots > d_n \geq 0$ because α_k is an isomorphism in the generic point of $(Y_k)_s$.

Note that i_1 is a closed immersion and $\mathcal{F} = i_{1,*}\mathcal{G}_1$. Hence for any morphism of schemes $T \to S$ with T affine, we have $\mathcal{F}_T = i_{1,T,*}\mathcal{G}_{1,T}$ and $i_{1,T}$ is still a closed immersion of schemes over T. Thus $\mathcal{F}_T$ is flat in dimensions $\geq d$ over T if and only if $\mathcal{G}_{1,T}$ is flat in dimensions $\geq d$ over T. Because $\pi_1 : Z_1 \to Y_1$ is finite we see in the same manner that $\mathcal{G}_{1,T}$ is flat in dimensions $\geq d$ over T if and only if $\pi_{1,T,*}\mathcal{G}_{1,T}$ is flat in dimensions $\geq d$ over T. The same arguments work for "flat in dimensions $\geq d+1$" and we conclude in particular that $\pi_{1,*}\mathcal{G}_1$ is flat over S in dimensions $\geq d+1$ by our assumption on $\mathcal{F}$.

Suppose that $d_1 > d$. It follows from the discussion above that in particular $\pi_{1,*}\mathcal{G}_1$ is flat over S at the generic point of $(Y_1)_s$. By Lemma 37.11.1 we may replace S by an affine neighbourhood of s and assume that α_1 is S-universally injective. Because α_1 is S-universally injective, for any morphism $T \to S$ with T affine, we have a short exact sequence

$$0 \to \mathcal{O}_{Y_{1,T}}^{\oplus r_1} \to \pi_{1,T,*}\mathcal{G}_{1,T} \to \mathrm{Coker}(\alpha_1)_T \to 0$$

and still the first arrow is T-universally injective. Hence the set of points of $(Y_1)_T$ where $\pi_{1,T,*}\mathcal{G}_{1,T}$ is flat over T is the same as the set of points of $(Y_1)_T$ where $\mathrm{Coker}(\alpha_1)_T$ is flat over S. In this way the question reduces to the sheaf $\mathrm{Coker}(\alpha_1)$ which has a complete dévissage of length $n-1$ and we win by induction.

If $d_1 < d$ then F_d is represented by S and we win.

The last case is the case $d_1 = d$. This case follows from a combination of Lemma 37.25.3 and Lemma 37.25.1. □

Theorem 37.25.5. *In Situation 37.19.9. Assume moreover that f is of finite presentation, that $\mathcal{F}$ is an $\mathcal{O}_X$-module of finite presentation, and that $\mathcal{F}$ is pure relative to S. Then F_n is representable by a monomorphism $Z_n \to S$ of finite presentation.*

Proof. The functor F_n is a sheaf for the fppf topology by Lemma 37.19.10. Hence combining Descent, Lemma 34.35.1, More on Morphisms, Lemma 36.37.1 , and Descent, Lemmas 34.19.29 and 34.19.11 we see that the question is local for the étale topology on S.

In particular the situation is local for the Zariski topology on S and we may assume that S is affine. In this case the dimension of the fibres of f is bounded above, hence we see that F_n is representable for n large enough. Thus we may use descending induction on n. Suppose that we know F_{n+1} is representable by a monomorphism $Z_{n+1} \to S$ of finite presentation. Consider the base change $X_{n+1} = Z_{n+1} \times_S X$ and the pullback $\mathcal{F}_{n+1}$ of $\mathcal{F}$ to X_{n+1}. The morphism $Z_{n+1} \to S$ is quasi-finite as it is a monomorphism of finite presentation, hence Lemma 37.15.4 implies that $\mathcal{F}_{n+1}$ is pure relative to Z_{n+1}. Since F_n is a subfunctor of F_{n+1} we conclude that in order to prove the result for F_n it suffices to prove the result for the corresponding functor for the situation $\mathcal{F}_{n+1}/X_{n+1}/Z_{n+1}$. In this way we reduce to proving the result for F_n in case $S_{n+1} = S$, i.e., we may assume that $\mathcal{F}$ is flat in dimensions $\geq n+1$ over S.

Fix n and assume $\mathcal{F}$ is flat in dimensions $\geq n+1$ over S. To finish the proof we have to show that F_n is representable by a monomorphism $Z_n \to S$ of finite presentation. Since the question is local in the étale topology on S it suffices to show that for every $s \in S$ there exists an elementary étale neighbourhood $(S', s') \to (S, s)$ such that the result holds after base change to S'. Thus by Lemma 37.5.8 we may assume there exist étale morphisms $h_j : Y_j \to X$, $j = 1, \ldots, m$ such that for each i there exists a complete dévissage of $\mathcal{F}_j/Y_j/S$ over s, where $\mathcal{F}_j$ is the pullback of $\mathcal{F}$ to Y_j and such that $X_s \subset \bigcup h_j(Y_j)$. Note that by Lemma 37.25.2 the sheaves $\mathcal{F}_j$ are still flat over in dimensions $\geq n+1$ over S. Set $W = \bigcup h_j(Y_j)$, which is a quasi-compact open of X. As $\mathcal{F}$ is pure along X_s we see that

$$E = \{t \in S \mid \mathrm{Ass}_{X_t}(\mathcal{F}_t) \subset W\}.$$

contains all generalizations of s. By More on Morphisms, Lemma 36.20.5 E is a constructible subset of S. We have seen that $\mathrm{Spec}(\mathcal{O}_{S,s}) \subset E$. By Morphisms, Lemma 28.23.4 we see that E contains an open neighbourhood of s. Hence after shrinking S we may assume that $E = S$. It follows from Lemma 37.25.2 that it suffices to prove the lemma for the functor F_n associated to $X = \coprod Y_j$ and $\mathcal{F} = \coprod \mathcal{F}_j$. If $F_{j,n}$ denotes the functor for $Y_j \to S$ and the sheaf $\mathcal{F}_i$ we see that $F_n = \prod F_{j,n}$. Hence it suffices to prove each $F_{j,n}$ is representable by some monomorphism $Z_{j,n} \to S$ of finite presentation, since then

$$Z_n = Z_{1,n} \times_S \ldots \times_S Z_{m,n}$$

Thus we have reduced the theorem to the special case handled in Lemma 37.25.4. □

We make explicit what the theorem means in terms of universal flattenings in the following lemma.

Lemma 37.25.6. *Let $f : X \to S$ be a morphism of schemes. Let $\mathcal{F}$ be a quasi-coherent $\mathcal{O}_X$-module.*

(1) *If f is of finite presentation, $\mathcal{F}$ is an $\mathcal{O}_X$-module of finite presentation, and $\mathcal{F}$ is pure relative to S, then there exists a universal flattening $S' \to S$ of $\mathcal{F}$. Moreover $S' \to S$ is a monomorphism of finite presentation.*
(2) *If f is of finite presentation and X is pure relative to S, then there exists a universal flattening $S' \to S$ of X. Moreover $S' \to S$ is a monomorphism of finite presentation.*
(3) *If f is proper and of finite presentation and $\mathcal{F}$ is an $\mathcal{O}_X$-module of finite presentation, then there exists a universal flattening $S' \to S$ of $\mathcal{F}$. Moreover $S' \to S$ is a monomorphism of finite presentation.*
(4) *If f is proper and of finite presentation then there exists a universal flattening $S' \to S$ of X.*

Proof. These statements follow immediately from Theorem 37.25.5 applied to $F_0 = F_{flat}$ and the fact that if f is proper then $\mathcal{F}$ is automatically pure over the base, see Lemma 37.16.1. □

37.26. Blowing up and flatness

In this section we begin our discussion of results of the form: "After a blowup the strict transform becomes flat". We will use the following (more or less standard) notation in this section. If $X \to S$ is a morphism of schemes, $\mathcal{F}$ is a quasi-coherent module on X, and $T \to S$ is a morphism of schemes, then we denote $\mathcal{F}_T$ the pullback of $\mathcal{F}$ to the base change $X_T = X \times_S T$.

Remark 37.26.1. Let S be a quasi-compact and quasi-separated scheme. Let $f : X \to S$ be a morphism of schemes. Let $\mathcal{F}$ be a quasi-coherent module on X. Let $U \subset S$ be a quasi-compact open subscheme. Given a U-admissible blowup $S' \to S$ we denote X' the strict transform of X and $\mathcal{F}'$ the strict transform of $\mathcal{F}$ which we think of as a quasi-coherent module on X' (via Divisors, Lemma 30.19.2). Let P be a property of $\mathcal{F}/X/S$ which is stable under strict transform (as above) for U-admissible blowups. The general problem in this section is: Show (under auxiliary conditions on $\mathcal{F}/X/S$) there exists a U-admissible blowup $S' \to S$ such that the strict transform $\mathcal{F}'/X'/S'$ has P.

The general strategy will be to use that a composition of U-admissible blowups is a U-admissible blowup, see Divisors, Lemma 30.20.2. In fact, we will make use of the more precise Divisors, Lemma 30.18.12 and combine it with Divisors, Lemma 30.19.6. The result is that it suffices to find a sequence of U-admissible blowups

$$S = S_0 \leftarrow S_1 \leftarrow \ldots \leftarrow S_n$$

such that, setting $\mathcal{F}_0 = \mathcal{F}$ and $X_0 = X$ and setting $\mathcal{F}_i/X_i$ equal to the strict transform of $\mathcal{F}_{i-1}/X_{i-1}$, we arrive at $\mathcal{F}_n/X_n/S_n$ with property P.

In particular, choose a finite type quasi-coherent sheaf of ideals $\mathcal{I} \subset \mathcal{O}_S$ such that $V(\mathcal{I}) = S \setminus U$, see Properties, Lemma 27.22.1. Let $S' \to S$ be the blowup in $\mathcal{I}$ and let $E \subset S'$ be the exceptional divisor (Divisors, Lemma 30.18.4). Then we see that we've reduced the problem to the case where there exists an effective Cartier divisor $D \subset S$ whose support is $X \setminus U$. In particular we may assume U is scheme theoretically dense in S (Divisors, Lemma 30.9.4).

Suppose that P is local on S: If $S = \bigcup S_i$ is a finite open covering by quasi-compact opens and P holds for $\mathcal{F}_{S_i}/X_{S_i}/S_i$ then P holds for $\mathcal{F}/X/S$. In this case the general problem above is local on S as well, i.e., if given $s \in S$ we can find a quasi-compact open neighbourhood W of s such that the problem for $\mathcal{F}_W/X_W/W$ is solvable, then the problem is solvable for $\mathcal{F}/X/S$. This follows from Divisors, Lemmas 30.20.3 and 30.20.4.

Lemma 37.26.2. *Let R be a local ring. Let M be a finite R-module. Let $k \geq 0$. Assume that $Fit_k(M) = (f)$ for some $f \in R$. Let M' be the quotient of M by $\{x \in M \mid fx = 0\}$. Then M' can be generated by k elements.*

Proof. Choose generators $x_1, \ldots, x_n \in M$ corresponding to the surjection $R^{\oplus n} \to M$. Since R is local if a set of elements $E \subset (f)$ generates (f), then some $e \in E$ generates (f), see Algebra, Lemma 10.18.1. Hence we may pick $z_1, \ldots, z_{n-k}$ in the kernel of $R^{\oplus n} \to M$ such that some $(n-k) \times (n-k)$ minor of the $n \times (n-k)$ matrix $A = (z_{ij})$ is (f). After renumbering the x_i we may assume the first minor $\det(z_{ij})_{1 \leq i,j \leq n-k}$ generates (f), i.e., $\det(z_{ij})_{1 \leq i,j \leq n-k} = uf$ for some unit $u \in R$. Every other minor is a multiple of f. By Algebra, Lemma 10.14.6 there exists a $n - k \times n - k$ matrix B such that

$$AB = f \begin{pmatrix} u1_{n-k \times n-k} \\ C \end{pmatrix}$$

for some matrix C with coefficients in R. This implies that for every $i \leq n - k$ the element $y_i = ux_i + \sum_j c_{ji} x_j$ is annihilated by f. Since $M/\sum Ry_i$ is generated by the images of $x_{n-k+1}, \ldots, x_n$ we win. $\square$

Lemma 37.26.3. *Let R be a ring and let $f \in R$. Let $r, d \geq 0$ be integers. Let $R \to S$ be a ring map and let M be an S-module. Assume*

1. *$R \to S$ is of finite presentation and flat,*
2. *every fibre ring $S \otimes_R \kappa(\mathfrak{p})$ is geometrically integral over R,*
3. *M is a finite S-module,*
4. *M_f is a finitely presented S_f-module,*
5. *for all $\mathfrak{p} \in R$, $f \notin \mathfrak{p}$ with $\mathfrak{q} = \mathfrak{p}S$ the module $M_\mathfrak{q}$ is free of rank r over $S_\mathfrak{q}$.*

Then there exists a finitely generated ideal $I \subset R$ with $V(f) = V(I)$ such that for all $a \in I$ with $R' = R[\frac{I}{a}]$ the quotient

$$M' = (M \otimes_R R')/a\text{-power torsion}$$

over $S' = S \otimes_R R'$ satisfies the following: for every prime $\mathfrak{p}' \subset R'$ there exists a $g \in S'$, $g \notin \mathfrak{p}'S'$ such that M'_g is a free S'_g-module of rank r.

Proof. Choose a surjection $S^{\oplus n} \to M$, which is possible by (1). Choose a finite submodule $K \subset \mathrm{Ker}(S^{\oplus n} \to M)$ such that $S^{\oplus n}/K \to M$ becomes an isomorphism after inverting f. This is possible by (2). Set $M_1 = S^{\oplus n}/K$ and suppose we can prove the lemma for M_1. Say $I \subset R$ is the corresponding ideal. Then for $a \in I$ the map

$$M'_1 = (M_1 \otimes_R R')/a\text{-power torsion} \longrightarrow M' = (M \otimes_R R')/a\text{-power torsion}$$

is surjective. It is also an isomorphism after inverting a in R' as $R'_a = R_f$, see Algebra, Lemma 10.54.3. But a is a nonzerodivisor on M'_1, whence the displayed map is an isomorphism. Thus it suffices to prove the lemma in case M is a finitely presented S-module.

Assume M is a finitely presented S-module satisfying (3). Then $J = \mathrm{Fit}_r(M) \subset S$ is a finitely generated ideal. By Lemma 37.9.3 we can write S as a direct summand of a free R-module: $\bigoplus_{\alpha \in A} R = S \oplus C$. For any element $h \in S$ writing $h = \sum a_\alpha$ in the decomposition above, we say that the a_α are the coefficents of h. Let $I' \subset R$ be the ideal generated by the coefficients of the elements of J. Multiplication by an element of S defines an R-linear map $S \to S$, hence I' is generated by the coefficients of the generators of J, i.e., I' is a finitely generated ideal. We claim that $I = fI'$ works.

We first check that $V(f) = V(I)$. The inclusion $V(f) \subset V(I)$ is clear. Conversely, if $f \notin \mathfrak{p}$, then $\mathfrak{q} = \mathfrak{p}S$ is not an element of $V(J)$ by property (3) and the fact that formation of fitting ideals commute with base change (More on Algebra, Lemma 15.5.4). Hence there is an element of J which does not map to zero in $S \otimes_R \kappa(\mathfrak{p})$. Thus there exists an element of I' which is not contained in $\mathfrak{p}$, so $\mathfrak{p} \notin V(fI') = V(I)$.

Let $a \in I$ and let $\mathfrak{p}' \subset R' = R[\frac{I}{a}]$ be a prime ideal. Set $S' = S \otimes_S R'$ and $\mathfrak{q}' = \mathfrak{p}'S'$. Every element g of $JS' = \mathrm{Fit}_r(M \otimes_S S')$ can be written as $g = \sum_\alpha c_\alpha$ for some $c_\alpha \in IS'$. Since $IR' = aR'$ we can write $c_\alpha = ac'_\alpha$ for some $c'_\alpha \in R'$ and $g = (\sum c'_\alpha)a = g'a$ in S'. Moreover, we can find some $g_0 \in JS'$ such that $a = c_\alpha$ for some α. For this element $g_0 = g'_0 a$ where g'_0 is a unit in $S'_{\mathfrak{q}'}$. Thus we see that $JS'_{\mathfrak{q}'}$ is the principal ideal generated by the nonzerodivisor a. It follows from Lemma 37.26.2 that $M'_{\mathfrak{q}'}$ can be generated by r elements. Since M' is finite, there exist $m_1, \ldots, m_r \in M'$ and $g \in S'$, $g \notin \mathfrak{q}'$ such that the corresponding map $(S')^{\oplus r} \to M'$ becomes surjective after inverting g.

Finally, consider the finitely generated ideal $J' = \mathrm{Fit}_{k-1}(M')$. Note that $J'S'_g$ is generated by the coefficients of relations between $m_1, \ldots, m_r$ (compatibility of fitting ideal with base change). Thus it suffices to show that $J' = 0$, see More on Algebra, Lemma 15.5.7. Since $R'_a = R_f$ (see above) and $M'_a = M_f$ we see from (3) that J'_a maps to zero in $S_{\mathfrak{q}''}$ for any prime $\mathfrak{q}'' \subset S'$ of the form $\mathfrak{q}'' = \mathfrak{p}''S'$ where $\mathfrak{p}'' \subset R'_a$. Since $S'_a \subset \prod_{\mathfrak{q}'' \text{ as above}} S'_{\mathfrak{q}''}$ (as $(S'_a)_{\mathfrak{p}''} \subset S'_{\mathfrak{q}''}$ by Lemma 37.7.4) we see that $J'R'_a = 0$. Since a is a nonzerodivisor in R' we conclude that $J' = 0$ and we win. $\square$

Lemma 37.26.4. *Let S be a quasi-compact and quasi-separated scheme. Let $X \to S$ be a morphism of schemes. Let $\mathcal{F}$ be a quasi-coherent module on X. Let $U \subset S$ be a quasi-compact open. Assume*

(1) *$X \to S$ is affine, of finite presentation, flat, geometrically integral fibres,*
(2) *$\mathcal{F}$ is a module of finite type,*
(3) *$\mathcal{F}_U$ is of finite presentation,*
(4) *$\mathcal{F}$ is flat over S at all generic points of fibres lying over points of U.*

Then there exists a U-admissible blowup $S' \to S$ and an open subscheme $V \subset X_{S'}$ such that (a) the strict transform $\mathcal{F}'$ of $\mathcal{F}$ restricts to a finitely locally free $\mathcal{O}_V$-module and (b) $V \to S'$ is surjective.

Proof. Given $\mathcal{F}/X/S$ and $U \subset S$ with hypotheses as in the lemma, denote P the property "$\mathcal{F}$ is flat over S at all generic points of fibres". It is clear that P is preserved under strict transform, see Divisors, Lemma 30.19.3 and Morphisms, Lemma 28.26.6. It is also clear that P is local on S. Hence any and all observations of Remark 37.26.1 apply to the problem posed by the lemma.

Consider the function $r : U \to \mathbf{Z}_{\geq 0}$ which assigns to $u \in U$ the integer

$$r(u) = \dim_{\kappa(\xi_u)}(\mathcal{F}_{\xi_u} \otimes \kappa(\xi_u))$$

where ξ_u is the generic point of the fibre X_u. By More on Morphisms, Lemma 36.13.7 and the fact that the image of an open in X_S in S is open, we see that $r(u)$ is locally constant. Accordingly $U = U_0 \amalg U_1 \amalg \ldots \amalg U_c$ is a finite disjoint union of open and closed subschemes where r is constant with value i on U_i. By Divisors, Lemma 30.20.5 we can find a U-admissible blowup to decompose S into the disjoint union of two schemes, the first containing U_0 and the second $U_1 \cup \ldots \cup U_c$. Repeating this $c-1$ more times we may assume that S is a disjoint union $S = S_0 \amalg S_1 \amalg \ldots \amalg S_c$ with $U_i \subset S_i$. Thus we may assume the function r defined above is constant, say with value r.

By Remark 37.26.1 we see that we may assume that we have an effective Cartier divisor $D \subset S$ whose support is $S \setminus U$. Another application of Remark 37.26.1 combined with Divisors, Lemma 30.9.2 tells us we may assume that $S = \mathrm{Spec}(R)$ and $D = \mathrm{Spec}(R/(f))$ for some nonzerodivisor $f \in R$. This case is handled by Lemma 37.26.3. □

Lemma 37.26.5. *Let $A \to C$ be a finite locally free ring map of rank d. Let $h \in C$ be an element such that C_h is étale over A. Let $J \subset C$ be an ideal. Set $I = Fit_0(C/J)$ where we think of C/J as a finite A-module. Then $IC_h = JJ'$ for some ideal $J' \subset C_h$. If J is finitely generated so are I and J'.*

Proof. We will use basic properties of fitting ideals, see More on Algebra, Lemma 15.5.4. Then IC is the fitting ideal of $C/J \otimes_A C$. Note that $C \to C \otimes_A C$, $c \mapsto 1 \otimes c$ has a section (the multiplication map). By assumption $C \to C \otimes_A C$ is étale at every prime in the image of $\mathrm{Spec}(C_h)$ under this section. Hence the multiplication map $C \otimes_A C_h \to C_h$ is étale in particular flat, see Algebra, Lemma 10.136.9. Hence there exists a C_h-algebra such that $C \otimes_A C_h \cong C_h \oplus C'$ as C_h-algebras, see Algebra, Lemma 10.136.10. Thus $(C/J) \otimes_A C_h \cong (C_h/J_h) \oplus C'/I'$ as C_h-modules for some ideal $I' \subset C'$. Hence $IC_h = JJ'$ with $J' = \mathrm{Fit}_0(C'/I')$ where we view C'/J' as a C_h-module. □

Lemma 37.26.6. *Let $A \to B$ be an étale ring map. Let $a \in A$ be a nonzerodivisor. Let $J \subset B$ be a finite type ideal with $V(J) \subset V(aB)$. For every $\mathfrak{q} \subset B$ there exists a finite type ideal $I \subset A$ with $V(I) \subset V(a)$ and $g \in B$, $g \notin \mathfrak{q}$ such that $IB_g = JJ'$ for some finite type ideal $J' \subset B_g$.*

Proof. We may replace B by a principal localization at an element $g \in B$, $g \notin \mathfrak{q}$. Thus we may assume that B is standard étale, see Algebra, Proposition 10.136.17. Thus we may assume B is a localization of $C = A[x]/(f)$ for some monic $f \in A[x]$ of some degree d. Say $B = C_h$ for some $h \in C$. Choose elements $h_1, \ldots, h_n \in C$ which generate J over B. The condition $V(J) \subset V(aB)$ signifies that $a^m = \sum b_i h_i$ in B for some large m. Set $h_{n+1} = a^m$. As in Lemma 37.26.5 we take $I = \mathrm{Fit}_0(C/(h_1, \ldots, h_{r+1}))$. Since the module $C/(h_1, \ldots, h_{r+1})$ is annihilated by a^m we see that $a^{dm} \in I$ which implies that $V(I) \subset V(a)$. □

Lemma 37.26.7. *Let S be a quasi-compact and quasi-separated scheme. Let $X \to S$ be a morphism of schemes. Let $\mathcal{F}$ be a quasi-coherent module on X. Let $U \subset S$ be a quasi-compact open. Assume there exist finitely many commutative diagrams*

$$\begin{array}{ccccc} & & X_i & \xrightarrow{j_i} & X \\ & & \downarrow & & \downarrow \\ S_i^* & \longrightarrow & S_i & \xrightarrow{e_i} & S \end{array}$$

where

(1) *$e_i : S_i \to S$ are quasi-compact étale morphisms and $S = \bigcup e_i(S_i)$,*
(2) *$j_i : X_i \to X$ are étale morphisms and $X = \bigcup j_i(X_i)$,*
(3) *$S_i^* \to S_i$ is an $e_i^{-1}(U)$-admissible blowup such that the strict transform $\mathcal{F}_i^*$ of $j_i^*\mathcal{F}$ is flat over S_i^*.*

Then there exists a U-admissible blowup $S' \to S$ such that the strict transform of $\mathcal{F}$ is flat over S'.

Proof. We claim that the hypotheses of the lemma are preserved under U-admissible blowups. Namely, suppose $b : S' \to S$ is a U-admissible blowup in the quasi-coherent sheaf of ideals $\mathcal{I}$. Moreover, let $S_i' \to S_i$ be the blowup in the quasi-coherent sheaf of ideals $\mathcal{J}_i$. Then the collection of morphisms $e_i' : S_i' = S_i \times_S S' \to S'$ and $j_i' : X_i' = X_i \times_S S' \to X \times_S S'$ satisfy conditions (1), (2), (3) for the strict transform $\mathcal{F}'$ of $\mathcal{F}$ relative to the blowup $S' \to S$. First, observe that S_i' is the blowup of S_i in the pullback of $\mathcal{I}$, see Divisors, Lemma 30.18.3. Second, consider the blowup $S_i'^* \to S_i'$ of S_i' in the pullback of the ideal $\mathcal{J}_i$. By Divisors, Lemma 30.18.10 we get a commutative diagram

$$\begin{array}{ccc} S_i'^* & \longrightarrow & S_i' \\ \downarrow & \searrow & \downarrow \\ S_i^* & \longrightarrow & S_i \end{array}$$

and all the morphisms in the diagram above are blowups. Hence by Divisors, Lemmas 30.19.3 and 30.19.6 we see

$$\begin{aligned} & \text{the strict transform of } (j_i')^*\mathcal{F}' \text{ under } S_i'^* \to S_i' \\ = {} & \text{the strict transform of } j_i^*\mathcal{F} \text{ under } S_i'^* \to S_i \\ = {} & \text{the strict transform of } \mathcal{F}_i' \text{ under } S_i'^* \to S_i' \\ = {} & \text{the pullback of } \mathcal{F}_i^* \text{ via } X_i \times_{S_i} S_i'^* \to X_i \end{aligned}$$

which is therefore flat over $S_i'^*$ (Morphisms, Lemma 28.26.6). Having said this, we see that all observations of Remark 37.26.1 apply to the problem of finding a U-admissible blowup such that the strict transform of $\mathcal{F}$ becomes flat over the base under assumptions as in the lemma. In particular, we may assume that $S \setminus U$ is the support of an effective Cartier divisor $D \subset S$. Another application of Remark 37.26.1 combined with Divisors, Lemma 30.9.2 shows we may assume that $S = \mathrm{Spec}(A)$ and $D = \mathrm{Spec}(A/(a))$ for some nonzerodivisor $a \in A$.

Pick an i and $s \in S_i$. Lemma 37.26.6 implies we can find an open neighbourhood $s \in W_i \subset S_i$ and a finite type quasi-coherent ideal $\mathcal{I} \subset \mathcal{O}_S$ such that $\mathcal{I} \cdot \mathcal{O}_{W_i} = \mathcal{J}_i\mathcal{J}_i'$ for some finite type quasi-coherent ideal $\mathcal{J}_i' \subset \mathcal{O}_{W_i}$ and such that $V(\mathcal{I}) \subset V(a) = S \setminus U$. Since S_i is quasi-compact we can replace S_i by a finite collection $W_1, \ldots, W_n$ of these opens and assume that for each i there exists a quasi-coherent sheaf of ideals $\mathcal{I}_i \subset \mathcal{O}_S$ such that $\mathcal{I}_i \cdot \mathcal{O}_{S_i} = \mathcal{J}_i\mathcal{J}_i'$ for some finite type quasi-coherent ideal $\mathcal{J}_i' \subset \mathcal{O}_{S_i}$. As in the discussion of the first paragraph of the proof, consider the blowup S' of S in the product $\mathcal{I}_1 \ldots \mathcal{I}_n$ (this blowup is U-admissible by construction). The base change of $S' \to S$ to S_i is the blowup in

$$\mathcal{J}_i \cdot \mathcal{J}_i'\mathcal{I}_1 \ldots \hat{\mathcal{I}}_i \ldots \mathcal{I}_n$$

which factors through the given blowup $S_i^* \to S_i$ (Divisors, Lemma 30.18.10). In the notation of the diagram above this means that $S_i'^* = S_i'$. Hence after replacing S by S' we arrive in the situation that $j_i^*\mathcal{F}$ is flat over S_i. Hence $j_i^*\mathcal{F}$ is flat over S, see Lemma 37.2.3. By Morphisms, Lemma 28.26.11 we see that $\mathcal{F}$ is flat over S. □

Theorem 37.26.8. *Let S be a quasi-compact and quasi-separated scheme. Let X be a scheme over S. Let $\mathcal{F}$ be a quasi-coherent module on X. Let $U \subset S$ be a quasi-compact open. Assume*

1. *X is quasi-compact,*
2. *X is locally of finite presentation over S,*
3. *$\mathcal{F}$ is a module of finite type,*
4. *$\mathcal{F}_U$ is of finite presentation, and*
5. *$\mathcal{F}_U$ is flat over U.*

Then there exists a U-admissible blowup $S' \to S$ such that the strict transform $\mathcal{F}'$ of $\mathcal{F}$ is an $\mathcal{O}_{X \times_S S'}$-module of finite presentation and flat over S'.

Proof. We first prove that we can find a U-admissible blowup such that the strict transform is flat. The question is étale local on the source and the target, see Lemma 37.26.7 for a precise statement. In particular, we may assume that $S = \mathrm{Spec}(R)$ and $X = \mathrm{Spec}(A)$ are affine. For $s \in S$ write $\mathcal{F}_s = \mathcal{F}|_{X_s}$ (pullback of $\mathcal{F}$ to the fibre). As $X \to S$ is of finite type $d = \max_{s \in S} \dim(\mathrm{Supp}(\mathcal{F}_s))$ is an integer. We will do induction on d.

Let $x \in X$ be a point of X lying over $s \in S$ with $\dim_x(\mathrm{Supp}(\mathcal{F}_s)) = d$. Apply Lemma 37.3.2 to get $g : X' \to X$, $e : S' \to S$, $i : Z' \to X'$, and $\pi : Z' \to Y'$. Observe that $Y' \to S'$ is a smooth morphism of affines with geometrically irreducible fibres of dimension d. Because the problem is étale local it suffices to prove the theorem for $g^*\mathcal{F}/X'/S'$. Because $i : Z' \to X'$ is a closed immersion of finite presentation (and since strict transform commutes with affine pushforward, see Divisors, Lemma 30.19.4) it suffices to prove the flattening result for $\mathcal{G}$. Since π is finite (hence also affine) it suffices to prove the flattening result for $\pi_*\mathcal{G}/Y'/S'$. Thus we may assume that $X \to S$ is a smooth morphism of affines with geometrically irreducible fibres of dimension d.

Next, we apply a blow up as in Lemma 37.26.4. Doing so we reach the situation where there exists an open $V \subset X$ surjecting onto S such that $\mathcal{F}|_V$ is finite locally free. Let $\xi \in X$ be the generic point of X_s. Let $r = \dim_{\kappa(\xi)} \mathcal{F}_\xi \otimes \kappa(\xi)$. Choose a map $\alpha : \mathcal{O}_X^{\oplus r} \to \mathcal{F}$ which induces an isomorphism $\kappa(\xi)^{\oplus r} \to \mathcal{F}_\xi \otimes \kappa(\xi)$. Because $\mathcal{F}$ is locally free over V we find an open neighbourhood W of ξ where α is an isomorphism. Shrink S to an affine open neighbourhood of s such that $W \to S$ is surjective. Say $\mathcal{F}$ is the quasi-coherent module associated to the A-module N. Since $\mathcal{F}$ is flat over S at all generic points of fibres (in fact at all points of W), we see that

$$\alpha_{\mathfrak{p}} : A_{\mathfrak{p}}^{\oplus r} \to N_{\mathfrak{p}}$$

is universally injective for all primes $\mathfrak{p}$ of R, see Lemma 37.10.1. Hence α is universally injective, see Algebra, Lemma 10.77.12. Set $\mathcal{H} = \mathrm{Coker}(\alpha)$. By Divisors, Lemma 30.19.7 we see that, given a U-admissible blowup $S' \to S$ the strict transforms of $\mathcal{F}'$ and $\mathcal{H}'$ fit into an exact sequence

$$0 \to \mathcal{O}_{X \times_S S'}^{\oplus r} \to \mathcal{F}' \to \mathcal{H}' \to 0$$

Hence Lemma 37.10.1 also shows that $\mathcal{F}'$ is flat at a point x' if and only if $\mathcal{H}'$ is flat at that point. In particular $\mathcal{H}_U$ is flat over U and $\mathcal{H}_U$ is a module of finite presentation. We may apply the induction hypothesis to $\mathcal{H}$ to see that there exists a U-admissible blowup such that the strict transform $\mathcal{H}'$ is flat as desired.

To finish the proof of the theorem we still have to show that $\mathcal{F}'$ is a module of finite presentation (after possibly another U-admissible blowup). This follows from Lemma 37.10.11 as we can assume $U \subset S$ is scheme theoretically dense (see third paragraph of Remark 37.26.1). This finishes the proof of the theorem. □

37.27. Applications

In this section we apply some of the results above.

Lemma 37.27.1. *Let S be a quasi-compact and quasi-separated scheme. Let X be a scheme over S. Let $U \subset S$ be a quasi-compact open. Assume*

(1) *$X \to S$ is of finite type and quasi-separated, and*
(2) *$X_U \to U$ is flat and locally of finite presentation.*

Then there exists a U-admissible blowup $S' \to S$ such that the strict transform of X is flat and of finite presentation over S'.

Proof. Since $X \to S$ is quasi-compact and quasi-separated by assumption, the strict transform of X with respect to a blowing up $S' \to S$ is also quasi-compact and quasi-separated. Hence to prove the lemma it suffices to find a U-admissible

blowup such that the strict transform is flat and locally of finite presentation. Let $X = W_1 \cup \ldots \cup W_n$ be a finite affine open covering. If we can find a U-admissible blowup $S_i \to S$ such that the strict transform of W_i is flat and locally of finite presentation, then there exists a U-admissble blowing up $S' \to S$ dominating all $S_i \to S$ which does the job (see Divisors, Lemma 30.20.4; see also Remark 37.26.1). Hence we may assume X is affine.

Assume X is affine. By Morphisms, Lemma 28.40.2 we can choose an immersion $j : X \to \mathbf{A}^n_S$ over S. Let $V \subset \mathbf{A}^n_S$ be a quasi-compact open subscheme such that j induces a closed immersion $i : X \to V$ over S. Apply Theorem 37.26.8 to $V \to S$ and the quasi-coherent module $i_*\mathcal{O}_X$ to obtain a U-admissible blowup $S' \to S$ such that the strict transform of $i_*\mathcal{O}_X$ is flat over S' and of finite presentation over $\mathcal{O}_{V \times_S S'}$. Let X' be the strict transform of X with respect to $S' \to S$. Let $i' : X' \to V \times_S S'$ be the induced morphism. Since taking strict transform commutes with pushforward along affine morphisms (Divisors, Lemma 30.19.4), we see that $i'_*\mathcal{O}_{X'}$ is flat over S and of finite presentation as a $\mathcal{O}_{V \times_S S'}$-module. This implies the lemma. □

Lemma 37.27.2. *Let $\varphi : X \to S$ be a separated morphism of finite type with S quasi-compact and quasi-separated. Let $U \subset S$ be a quasi-compact open such that $\varphi^{-1}U \to U$ is an isomorphism. Then there exists a U-admissible blowup $S' \to S$ such that the strict transform X' of X is isomorphic to an open subscheme of S'.*

Proof. The discussion in Remark 37.26.1 applies. Thus we may do a first U-admissible blowup and assume the complement $S \setminus U$ is the support of an effective Cartier divisor D. In particular U is scheme theoretically dense in S. Next, we do another U-admissible blowup to get to the situation where $X \to S$ is flat and of finite presentation, see Lemma 37.27.1. In this case the result follows from More on Morphisms, Lemma 36.31.4. □

The following lemma says that a proper modification can be dominated by a blowup.

Lemma 37.27.3. *Let $\varphi : X \to S$ be a proper morphism with S quasi-compact and quasi-separated. Let $U \subset S$ be a quasi-compact open such that $\varphi^{-1}U \to U$ is an isomorphism. Then there exists a U-admissible blowup $S' \to S$ which dominates X, i.e., such that there exists a factorization $S' \to X \to S$ of the blowup morphism.*

Proof. The discussion in Remark 37.26.1 applies. Thus we may do a first U-admissible blowup and assume the complement $S \setminus U$ is the support of an effective Cartier divisor D. In particular U is scheme theoretically dense in S. Choose another U-admissible blowup $S' \to S$ such that the strict transform X' of X is an open subscheme of S', see Lemma 37.27.2. Since $X' \to S'$ is proper, and $U \subset S'$ is dense, we see that $X' = S'$. Some details omitted. □

37.28. Other chapters

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CHAPTER 38

Groupoid Schemes

38.1. Introduction

This chapter is devoted to generalities concerning groupoid schemes. See for example the beautiful paper [**KM97**] by Keel and Mori.

38.2. Notation

Let S be a scheme. If U, T are schemes over S we denote $U(T)$ for the set of T-valued points of U *over* S. In a formula: $U(T) = \mathrm{Mor}_S(T, U)$. We try to reserve the letter T to denote a "test scheme" over S, as in the discussion that follows. Suppose we are given schemes X, Y over S and a morphism of schemes $f : X \to Y$ over S. For any scheme T over S we get an induced map of sets

$$f : X(T) \longrightarrow Y(T)$$

which as indicated we denote by f also. In fact this construction is functorial in the scheme T/S. Yoneda's Lemma, see Categories, Lemma 4.3.5, says that f determines and is determined by this transformation of functors $f : h_X \to h_Y$. More generally, we use the same notation for maps between fibre products. For example, if X, Y, Z are schemes over S, and if $m : X \times_S Y \to Z \times_S Z$ is a morphism of schemes over S, then we think of m as corresponding to a collection of maps between T-valued points

$$X(T) \times Y(T) \longrightarrow Z(T) \times Z(T).$$

And so on and so forth.

We continue our convention to label projection maps starting with index 0, so we have $\mathrm{pr}_0 : X \times_S Y \to X$ and $\mathrm{pr}_1 : X \times_S Y \to Y$.

38.3. Equivalence relations

Recall that a *relation* R on a set A is just a subset of $R \subset A \times A$. We usually write aRb to indicate $(a, b) \in R$. We say the relation is *transitive* if $aRb, bRc \Rightarrow aRc$. We say the relation is *reflexive* if aRa for all $a \in A$. We say the relation is *symmetric* if $aRb \Rightarrow bRa$. A relation is called an *equivalence relation* if it is transitive, reflexive and symmetric.

In the setting of schemes we are going to relax the notion of a relation a little bit and just require $R \to A \times A$ to be a map. Here is the definition.

Definition 38.3.1. Let S be a scheme. Let U be a scheme over S.

(1) A *pre-relation* on U over S is any morphism $j : R \to U \times_S U$. In this case we set $t = \mathrm{pr}_0 \circ j$ and $s = \mathrm{pr}_1 \circ j$, so that $j = (t, s)$.
(2) A *relation* on U over S is a monomorphism $j : R \to U \times_S U$.

(3) A *pre-equivalence relation* is a pre-relation $j : R \to U \times_S U$ such that the image of $j : R(T) \to U(T) \times U(T)$ is an equivalence relation for all T/S.
(4) We say a morphism $R \to U \times_S U$ is an *equivalence relation on U over S* if and only if for every T/S the T-valued points of R define an equivalence relation on the set of T-valued points of U.

In other words, an equivalence relation is a pre-equivalence relation such that j is a relation.

Lemma 38.3.2. *Let S be a scheme. Let U be a scheme over S. Let $j : R \to U \times_S U$ be a pre-relation. Let $g : U' \to U$ be a morphism of schemes. Finally, set*

$$R' = (U' \times_S U') \times_{U \times_S U} R \xrightarrow{j'} U' \times_S U'$$

Then j' is a pre-relation on U' over S. If j is a relation, then j' is a relation. If j is a pre-equivalence relation, then j' is a pre-equivalence relation. If j is an equivalence relation, then j' is an equivalence relation.

Proof. Omitted. □

Definition 38.3.3. Let S be a scheme. Let U be a scheme over S. Let $j : R \to U \times_S U$ be a pre-relation. Let $g : U' \to U$ be a morphism of schemes. The pre-relation $j' : R' \to U' \times_S U'$ is called the *restriction*, or *pullback* of the pre-relation j to U'. In this situation we sometimes write $R' = R|_{U'}$.

Lemma 38.3.4. *Let $j : R \to U \times_S U$ be a pre-relation. Consider the relation on points of the scheme U defined by the rule*

$$x \sim y \Leftrightarrow \exists\ r \in R : t(r) = x, s(r) = y.$$

If j is a pre-equivalence relation then this is an equivalence relation.

Proof. Suppose that $x \sim y$ and $y \sim z$. Pick $r \in R$ with $t(r) = x$, $s(r) = y$ and pick $r' \in R$ with $t(r') = y$, $s(r') = z$. Pick a field K fitting into the following commutative diagram

$$\begin{array}{ccc} \kappa(r) & \longrightarrow & K \\ \uparrow & & \uparrow \\ \kappa(y) & \longrightarrow & \kappa(r') \end{array}$$

Denote $x_K, y_K, z_K : \mathrm{Spec}(K) \to U$ the morphisms

$$\begin{array}{c} \mathrm{Spec}(K) \to \mathrm{Spec}(\kappa(r)) \to \mathrm{Spec}(\kappa(x)) \to U \\ \mathrm{Spec}(K) \to \mathrm{Spec}(\kappa(r)) \to \mathrm{Spec}(\kappa(y)) \to U \\ \mathrm{Spec}(K) \to \mathrm{Spec}(\kappa(r')) \to \mathrm{Spec}(\kappa(z)) \to U \end{array}$$

By construction $(x_K, y_K) \in j(R(K))$ and $(y_K, z_K) \in j(R(K))$. Since j is a pre-equivalence relation we see that also $(x_K, z_K) \in j(R(K))$. This clearly implies that $x \sim z$.

The proof that $\sim$ is reflexive and symmetric is omitted. □

38.4. Group schemes

Let us recall that a *group* is a pair (G, m) where G is a set, and $m : G \times G \to G$ is a map of sets with the following properties:

(1) (associativity) $m(g, m(g', g'')) = m(m(g, g'), g'')$ for all $g, g', g'' \in G$,
(2) (identity) there exists a unique element $e \in G$ (called the *identity*, *unit*, or 1 of G) such that $m(g, e) = m(e, g) = g$ for all $g \in G$, and
(3) (inverse) for all $g \in G$ there exists a $i(g) \in G$ such that $m(g, i(g)) = m(i(g), g) = e$, where e is the identity.

Thus we obtain a map $e : \{*\} \to G$ and a map $i : G \to G$ so that the quadruple (G, m, e, i) satisfies the axioms listed above.

A *homomorphism of groups* $\psi : (G, m) \to (G', m')$ is a map of sets $\psi : G \to G'$ such that $m'(\psi(g), \psi(g')) = \psi(m(g, g'))$. This automatically insures that $\psi(e) = e'$ and $i'(\psi(g)) = \psi(i(g))$. (Obvious notation.) We will use this below.

Definition 38.4.1. Let S be a scheme.

(1) A *group scheme over* S is a pair (G, m), where G is a scheme over S and $m : G \times_S G \to G$ is a morphism of schemes over S with the following property: For every scheme T over S the pair $(G(T), m)$ is a group.
(2) A *morphism* $\psi : (G, m) \to (G', m')$ *of group schemes over* S is a morphism $\psi : G \to G'$ of schemes over S such that for every T/S the induced map $\psi : G(T) \to G'(T)$ is a homomorphism of groups.

Let (G, m) be a group scheme over the scheme S. By the discussion above (and the discussion in Section 38.2) we obtain morphisms of schemes over S: (identity) $e : S \to G$ and (inverse) $i : G \to G$ such that for every T the quadruple $(G(T), m, e, i)$ satisfies the axioms of a group listed above.

Let (G, m), (G', m') be group schemes over S. Let $f : G \to G'$ be a morphism of schemes over S. It follows from the definition that f is a morphism of group schemes over S if and only if the following diagram is commutative:

$$\begin{array}{ccc} G \times_S G & \xrightarrow{f \times f} & G' \times_S G' \\ \downarrow{\scriptstyle m} & & \downarrow{\scriptstyle m} \\ G & \xrightarrow{f} & G' \end{array}$$

Lemma 38.4.2. *Let (G, m) be a group scheme over S. Let $S' \to S$ be a morphism of schemes. The pullback $(G_{S'}, m_{S'})$ is a group scheme over S'.*

Proof. Omitted. □

Definition 38.4.3. Let S be a scheme. Let (G, m) be a group scheme over S.

(1) A *closed subgroup scheme* of G is a closed subscheme $H \subset G$ such that $m|_{H \times_S H}$ factors through H and induces a group scheme structure on H over S.
(2) An *open subgroup scheme* of G is an open subscheme $G' \subset G$ such that $m|_{G' \times_S G'}$ factors through G' and induces a group scheme structure on G' over S.

Alternatively, we could say that H is a closed subgroup scheme of G if it is a group scheme over S endowed with a morphism of group schemes $i : H \to G$ over S which identifies H with a closed subscheme of G.

Definition 38.4.4. Let S be a scheme. Let (G, m) be a group scheme over S.

(1) We say G is a *smooth group scheme* if the structure morphism $G \to S$ is smooth.
(2) We say G is a *flat group scheme* if the structure morphism $G \to S$ is flat.
(3) We say G is a *separated group scheme* if the structure morphism $G \to S$ is separated.

Add more as needed.

38.5. Examples of group schemes

Example 38.5.1 (Multiplicative group scheme)**.** Consider the functor which associates to any scheme T the group $\Gamma(T, \mathcal{O}_T^*)$ of units in the global sections of the structure sheaf. This is representable by the scheme

$$\mathbf{G}_m = \mathrm{Spec}(\mathbf{Z}[x, x^{-1}])$$

The morphism giving the group structure is the morphism

$$\begin{aligned} \mathbf{G}_m \times \mathbf{G}_m &\to \mathbf{G}_m \\ \mathrm{Spec}(\mathbf{Z}[x, x^{-1}] \otimes_{\mathbf{Z}} \mathbf{Z}[x, x^{-1}]) &\to \mathrm{Spec}(\mathbf{Z}[x, x^{-1}]) \\ \mathbf{Z}[x, x^{-1}] \otimes_{\mathbf{Z}} \mathbf{Z}[x, x^{-1}] &\leftarrow \mathbf{Z}[x, x^{-1}] \\ x \otimes x &\leftarrow x \end{aligned}$$

Hence we see that $\mathbf{G}_m$ is a group scheme over $\mathbf{Z}$. For any scheme S the base change $\mathbf{G}_{m,S}$ is a group scheme over S whose functor of points is

$$T/S \longmapsto \mathbf{G}_{m,S}(T) = \mathbf{G}_m(T) = \Gamma(T, \mathcal{O}_T^*)$$

as before.

Example 38.5.2 (Roots of unity)**.** Let $n \in \mathbf{N}$. Consider the functor which associates to any scheme T the subgroup of $\Gamma(T, \mathcal{O}_T^*)$ consisting of nth roots of unity. This is representable by the scheme

$$\mu_n = \mathrm{Spec}(\mathbf{Z}[x]/(x^n - 1)).$$

The morphism giving the group structure is the morphism

$$\begin{aligned} \mu_n \times \mu_n &\to \mu_n \\ \mathrm{Spec}(\mathbf{Z}[x]/(x^n - 1) \otimes_{\mathbf{Z}} \mathbf{Z}[x]/(x^n - 1)) &\to \mathrm{Spec}(\mathbf{Z}[x]/(x^n - 1)) \\ \mathbf{Z}[x]/(x^n - 1) \otimes_{\mathbf{Z}} \mathbf{Z}[x]/(x^n - 1) &\leftarrow \mathbf{Z}[x]/(x^n - 1) \\ x \otimes x &\leftarrow x \end{aligned}$$

Hence we see that μ_n is a group scheme over $\mathbf{Z}$. For any scheme S the base change $\mu_{n,S}$ is a group scheme over S whose functor of points is

$$T/S \longmapsto \mu_{n,S}(T) = \mu_n(T) = \{f \in \Gamma(T, \mathcal{O}_T^*) \mid f^n = 1\}$$

as before.

Example 38.5.3 (Additive group scheme)**.** Consider the functor which associates to any scheme T the group $\Gamma(T, \mathcal{O}_T)$ of global sections of the structure sheaf. This is representable by the scheme

$$\mathbf{G}_a = \mathrm{Spec}(\mathbf{Z}[x])$$

The morphism giving the group structure is the morphism

$$\begin{aligned} \mathbf{G}_a \times \mathbf{G}_a &\to \mathbf{G}_a \\ \text{Spec}(\mathbf{Z}[x] \otimes_{\mathbf{Z}} \mathbf{Z}[x]) &\to \text{Spec}(\mathbf{Z}[x]) \\ \mathbf{Z}[x] \otimes_{\mathbf{Z}} \mathbf{Z}[x] &\leftarrow \mathbf{Z}[x] \\ x \otimes 1 + 1 \otimes x &\leftarrow x \end{aligned}$$

Hence we see that $\mathbf{G}_a$ is a group scheme over $\mathbf{Z}$. For any scheme S the base change $\mathbf{G}_{a,S}$ is a group scheme over S whose functor of points is

$$T/S \longmapsto \mathbf{G}_{a,S}(T) = \mathbf{G}_a(T) = \Gamma(T, \mathcal{O}_T)$$

as before.

Example 38.5.4 (General linear group scheme)**.** Let $n \geq 1$. Consider the functor which associates to any scheme T the group

$$\text{GL}_n(\Gamma(T, \mathcal{O}_T))$$

of invertible $n \times n$ matrices over the global sections of the structure sheaf. This is representable by the scheme

$$\text{GL}_n = \text{Spec}(\mathbf{Z}[\{x_{ij}\}_{1 \leq i,j \leq n}][1/d])$$

where $d = \det((x_{ij}))$ with (x_{ij}) the $n \times n$ matrix with entry x_{ij} in the (i, j)-spot. The morphism giving the group structure is the morphism

$$\begin{aligned} \text{GL}_n \times \text{GL}_n &\to \text{GL}_n \\ \text{Spec}(\mathbf{Z}[x_{ij}, 1/d] \otimes_{\mathbf{Z}} \mathbf{Z}[x_{ij}, 1/d]) &\to \text{Spec}(\mathbf{Z}[x_{ij}, 1/d]) \\ \mathbf{Z}[x_{ij}, 1/d] \otimes_{\mathbf{Z}} \mathbf{Z}[x_{ij}, 1/d] &\leftarrow \mathbf{Z}[x_{ij}, 1/d] \\ \sum x_{ik} \otimes x_{kj} &\leftarrow x_{ij} \end{aligned}$$

Hence we see that GL_n is a group scheme over $\mathbf{Z}$. For any scheme S the base change $\text{GL}_{n,S}$ is a group scheme over S whose functor of points is

$$T/S \longmapsto \text{GL}_{n,S}(T) = \text{GL}_n(T) = \text{GL}_n(\Gamma(T, \mathcal{O}_T))$$

as before.

Example 38.5.5. The determinant defines a morphisms of group schemes

$$\det : \text{GL}_n \longrightarrow \mathbf{G}_m$$

over $\mathbf{Z}$. By base change it gives a morphism of group schemes $\text{GL}_{n,S} \to \mathbf{G}_{m,S}$ over any base scheme S.

Example 38.5.6 (Constant group)**.** Let G be an abstract group. Consider the functor which associates to any scheme T the group of locally constant maps $T \to G$ (where T has the Zariski topology and G the discrete topology). This is representable by the scheme

$$G_{\text{Spec}(\mathbf{Z})} = \coprod\nolimits_{g \in G} \text{Spec}(\mathbf{Z}).$$

The morphism giving the group structure is the morphism

$$G_{\text{Spec}(\mathbf{Z})} \times_{\text{Spec}(\mathbf{Z})} G_{\text{Spec}(\mathbf{Z})} \longrightarrow G_{\text{Spec}(\mathbf{Z})}$$

which maps the component corresponding to the pair (g, g') to the component corresponding to gg'. For any scheme S the base change G_S is a group scheme over S whose functor of points is

$$T/S \longmapsto G_S(T) = \{f : T \to G \text{ locally constant}\}$$

as before.

38.6. Properties of group schemes

In this section we collect some simple properties of group schemes which hold over any base.

Lemma 38.6.1. *Let S be a scheme. Let G be a group scheme over S. Then $G \to S$ is separated (resp. quasi-separated) if and only if the identity morphism $e : S \to G$ is a closed immersion (resp. quasi-compact).*

Proof. We recall that by Schemes, Lemma 25.21.12 we have that e is an immersion which is a closed immersion (resp. quasi-compact) if $G \to S$ is separated (resp. quasi-separated). For the converse, consider the diagram

$$\begin{array}{ccc} G & \xrightarrow[\Delta_{G/S}]{} & G \times_S G \\ \downarrow & & \downarrow{\scriptstyle (g,g') \mapsto m(i(g),g')} \\ S & \xrightarrow{e} & G \end{array}$$

It is an exercise in the functorial point of view in algebraic geometry to show that this diagram is cartesian. In other words, we see that $\Delta_{G/S}$ is a base change of e. Hence if e is a closed immersion (resp. quasi-compact) so is $\Delta_{G/S}$, see Schemes, Lemma 25.18.2 (resp. Schemes, Lemma 25.19.3). □

Lemma 38.6.2. *Let S be a scheme. Let G be a group scheme over S. Let T be a scheme over S and let $\psi : T \to G$ be a morphism over S. If T is flat over S, then the morphism*

$$T \times_S G \longrightarrow G, \quad (t, g) \longmapsto m(\psi(t), g)$$

is flat. In particular, if G is flat over S, then $m : G \times_S G \to G$ is flat.

Proof. Consider the diagram

$$\begin{array}{ccccc} T \times_S G & \xrightarrow[(t,g) \mapsto (t, m(\psi(t),g))]{} & T \times_S G & \xrightarrow{\text{pr}} & G \\ & & \downarrow & & \downarrow \\ & & T & \longrightarrow & S \end{array}$$

The left top horizontal arrow is an isomorphism and the square is cartesian. Hence the lemma follows from Morphisms, Lemma 28.26.7. □

Lemma 38.6.3. *Let (G, m, e, i) be a group scheme over the scheme S. Denote $f : G \to S$ the structure morphism. Assume f is flat. Then there exist canonical isomorphisms*

$$\Omega_{G/S} \cong f^*\mathcal{C}_{S/G} \cong f^*e^*\Omega_{G/S}$$

where $\mathcal{C}_{S/G}$ denotes the conormal sheaf of the immersion e. In particular, if S is the spectrum of a field, then $\Omega_{G/S}$ is a free $\mathcal{O}_G$-module.

Proof. In Morphisms, Lemma 28.34.5 we identified $\Omega_{G/S}$ with the conormal sheaf of the diagonal morphism $\Delta_{G/S}$. In the proof of Lemma 38.6.1 we showed that $\Delta_{G/S}$ is a base change of the immersion e by the morphism $(g, g') \mapsto m(i(g), g')$. This morphism is isomorphic to the morphism $(g, g') \mapsto m(g, g')$ hence is flat by Lemma 38.6.2. Hence we get the first isomorphism by Morphisms, Lemma 28.33.4. By Morphisms, Lemma 28.34.16 we have $\mathcal{C}_{S/G} \cong e^*\Omega_{G/S}$.

If S is the spectrum of a field, then $G \to S$ is flat, and any $\mathcal{O}_S$-module on S is free. □

38.7. Properties of group schemes over a field

In this section we collect some simple properties of group schemes over a field.

Lemma 38.7.1. *If (G, m) is a group scheme over a field k, then the multiplication map $m : G \times_k G \to G$ is open.*

Proof. The multiplication map is isomorphic to the projection map $\mathrm{pr}_0 : G \times_k G \to G$ because the diagram

$$\begin{array}{ccc} G \times_k G & \xrightarrow[(g,g') \mapsto (m(g,g'),g')]{} & G \times_k G \\ \downarrow{\scriptstyle m} & & \downarrow{\scriptstyle (g,g') \mapsto g} \\ G & \xrightarrow{\mathrm{id}} & G \end{array}$$

is commutative with isomorphisms as horizontal arrows. The projection is open by Morphisms, Lemma 28.24.4. □

Lemma 38.7.2. *Let G be a group scheme over a field. Then G is a separated scheme.*

Proof. Say $S = \mathrm{Spec}(k)$ with k a field, and let G be a group scheme over S. By Lemma 38.6.1 we have to show that $e : S \to G$ is a closed immersion. By Morphisms, Lemma 28.21.2 the image of $e : S \to G$ is a closed point of G. It is clear that $\mathcal{O}_G \to e_*\mathcal{O}_S$ is surjective, since $e_*\mathcal{O}_S$ is a skyscraper sheaf supported at the neutral element of G with value k. We conclude that e is a closed immersion by Schemes, Lemma 25.24.2. □

Lemma 38.7.3. *Let G be a group scheme over a field k. Then*

1. *every local ring $\mathcal{O}_{G,g}$ of G has a unique minimal prime ideal,*
2. *there is exactly one irreducible component Z of G passing through e, and*
3. *Z is geometrically irreducible over k.*

Proof. For any point $g \in G$ there exists a field extension $k \subset K$ and a K-valued point $g' \in G(K)$ mapping to g. If we think of g' as a K-rational point of the group scheme G_K, then we see that $\mathcal{O}_{G,g} \to \mathcal{O}_{G_K,g'}$ is a faithfully flat local ring map (as $G_K \to G$ is flat, and a local flat ring map is faithfully flat, see Algebra, Lemma 10.36.16). The result for $\mathcal{O}_{G_K,g'}$ implies the result for $\mathcal{O}_{G,g}$, see Algebra, Lemma 10.28.5. Hence in order to prove (1) it suffices to prove it for k-rational points g of G. In this case translation by g defines an automorphism $G \to G$ which maps e to g. Hence $\mathcal{O}_{G,g} \cong \mathcal{O}_{G,e}$. In this way we see that (2) implies (1), since irreducible components passing through e correspond one to one with minimal prime ideals of $\mathcal{O}_{G,e}$.

In order to prove (2) and (3) it suffices to prove (2) when k is algebraically closed. In this case, let Z_1, Z_2 be two irreducible components of G passing through e. Since k is algebraically closed the closed subscheme $Z_1 \times_k Z_2 \subset G \times_k G$ is irreducible too, see Varieties, Lemma 32.6.4. Hence $m(Z_1 \times_k Z_2)$ is contained in an irreducible component of G. On the other hand it contains Z_1 and Z_2 since $m|_{e \times G} = \mathrm{id}_G$ and $m|_{G \times e} = \mathrm{id}_G$. We conclude $Z_1 = Z_2$ as desired. □

Remark 38.7.4. Warning: The result of Lemma 38.7.3 does not mean that every irreducible component of G/k is geometrically irreducible. For example the group scheme $\mu_{3,\mathbf{Q}} = \mathrm{Spec}(\mathbf{Q}[x]/(x^3 - 1))$ over $\mathbf{Q}$ has two irreducible components corresponding to the factorization $x^3 - 1 = (x - 1)(x^2 + x + 1)$. The first factor corresponds to the irreducible component passing through the identity, and the second irreducible component is not geometrically irreducible over $\mathrm{Spec}(\mathbf{Q})$.

Lemma 38.7.5. *Let G be a group scheme which is locally of finite type over a field k. Then G is equidimensional and $\dim(G) = \dim_g(G)$ for all $g \in G$. For any closed point $g \in G$ we have $\dim(G) = \dim(\mathcal{O}_{G,g})$.*

Proof. Let us first prove that $\dim_g(G) = \dim_{g'}(G)$ for any pair of points $g, g' \in G$. By Morphisms, Lemma 28.29.3 we may extend the ground field at will. Hence we may assume that both g and g' are defined over k. Hence there exists an automorphism of G mapping g to g', whence the equality. By Morphisms, Lemma 28.29.1 we have $\dim_g(G) = \dim(\mathcal{O}_{G,g}) + \mathrm{trdeg}_k(\kappa(g))$. On the other hand, the dimension of G (or any open subset of G) is the supremum of the dimensions of the local rings of of G, see Properties, Lemma 27.11.4. Clearly this is maximal for closed points g in which case $\mathrm{trdeg}_k(\kappa(g)) = 0$ (by the Hilbert Nullstellensatz, see Morphisms, Section 28.17). Hence the lemma follows. □

The following result is sometimes referred to as Cartier's theorem.

Lemma 38.7.6. *Let G be a group scheme which is locally of finite type over a field k of characteristic zero. Then the structure morphism $G \to \mathrm{Spec}(k)$ is smooth, i.e., G is a smooth group scheme.*

Proof. By Lemma 38.6.3 the module of differentials of G over k is free. Hence smoothness follows from Varieties, Lemma 32.15.1. □

Remark 38.7.7. Any group scheme over a field of characteristic 0 is reduced, see [**Per75**, I, Theorem 1.1 and I, Corollary 3.9, and II, Theorem 2.4] and also [**Per76**, Proposition 4.2.8]. This was a question raised in [**Oor66**, page 80]. We have seen in Lemma 38.7.6 that this holds when the group scheme is locally of finite type.

Lemma 38.7.8. *Let G be a group scheme which is locally of finite type over a perfect field k of characteristic $p > 0$ (see Lemma 38.7.6 for the characteristic zero case). If G is reduced then the structure morphism $G \to \mathrm{Spec}(k)$ is smooth, i.e., G is a smooth group scheme.*

Proof. By Lemma 38.6.3 the sheaf $\Omega_{G/k}$ is free. Hence the lemma follows from Varieties, Lemma 32.15.2. □

Remark 38.7.9. Let k be a field of characteristic $p > 0$. Let $\alpha \in k$ be an element which is not a pth power. The closed subgroup scheme

$$G = V(x^p + \alpha y^p) \subset \mathbf{G}^2_{a,k}$$

is reduced and irreducible but not smooth (not even normal).

Lemma 38.7.10. *Let G be a group scheme over a perfect field k. Then the reduction G_{red} of G is a closed subgroup scheme of G.*

Proof. Omitted. Hint: Use that $G_{red} \times_k G_{red}$ is reduced by Varieties, Lemmas 32.4.3 and 32.4.7. □

The next lemma will be generalized slightly in More on Groupoids, Lemma 39.10.2. Namely, if $G' \to G$ is a morphism of group schemes over a field whose image is open, then its image is closed.

Lemma 38.7.11. *Let G be group scheme over a field k. Let $G' \subset G$ be an open subgroup scheme. Then G' is open and closed in G.*

Proof. Suppose that $k \subset K$ is a field extension such that $G'_K \subset G_K$ is closed. Then it follows from Morphisms, Lemma 28.26.10 that G' is closed (as $G_K \to G$ is flat, quasi-compact and surjective). Hence it suffices to prove the lemma after replacing k by some extension. Choose K to be an algebraically closed field extension of very large cardinality. Then by Varieties, Lemma 32.12.2, we see that G_K is a Jacobson scheme all of whose closed points have residue field equal to K. In other words we may assume G is a Jacobson scheme all of whose closed points have residue field k.

Let $Z = G \setminus G'$. We have to show that Z is open. Because G is Jacobson and Z is closed the closed points of Z are dense in Z. Moreover any closed point $z \in Z$ is a k-rational point and hence we translation by z defines an automorphism $L_z : G \to G$, $g \mapsto m(z, g)$ with $e \mapsto z$. As G' is a subgroup scheme we conclude that $L_z(G') \subset Z$. Altogether we see that

$$Z = \bigcup\nolimits_{z \in Z(k)} L_z(G')$$

is a union of open subsets, and hence open as desired. □

Lemma 38.7.12. *Let $i : G' \to G$ be an immersion of group schemes over a field k. Then i is a closed immersion, i.e., $i(G')$ is a closed subgroup scheme of G.*

Proof. To show that i is a closed immersion it suffices to show that $i(G')$ is a closed subset of G. Let $k \subset k'$ be a perfect extension of k. If $i(G'_{k'}) \subset G_{k'}$ is closed, then $i(G') \subset G$ is closed by Morphisms, Lemma 28.26.10 (as $G_{k'} \to G$ is flat, quasi-compact and surjective). Hence we may and do assume k is perfect. We will use without further mention that products of reduced schemes over k are reduced. We may replace G' and G by their reductions, see Lemma 38.7.10. Let $\overline{G'} \subset G$ be the closure of $i(G')$ viewed as a reduced closed subscheme. By Varieties, Lemma 32.14.1 we conclude that $\overline{G'} \times_k \overline{G'}$ is the closure of the image of $G' \times_k G' \to G \times_k G$. Hence

$$m\Big(\overline{G'} \times_k \overline{G'}\Big) \subset \overline{G'}$$

as m is continuous. It follows that $\overline{G'} \subset G$ is a (reduced) closed subgroup scheme. By Lemma 38.7.11 we see that $i(G') \subset \overline{G'}$ is also closed which implies that $i(G') = \overline{G'}$ as desired. □

Lemma 38.7.13. *Let G be a group scheme over a field. There exists an open and closed subscheme $G' \subset G$ which is a countable union of affines.*

Proof. Let $e \in U(k)$ be a quasi-compact open neighbourhood of the identity element. By replacing U by $U \cap i(U)$ we may assume that U is invariant under the inverse map. As G is separated this is still a quasi-compact set. Set

$$G' = \bigcup\nolimits_{n \geq 1} m_n(U \times_k \ldots \times_k U)$$

where $m_n : G \times_k \ldots \times_k G \to G$ is the n-slot multiplication map $(g_1, \ldots, g_n) \mapsto m(m(\ldots(m(g_1, g_2), g_3), \ldots), g_n)$. Each of these maps are open (see Lemma 38.7.1) hence G' is an open subgroup scheme. By Lemma 38.7.11 it is also a closed subgroup scheme. □

Remark 38.7.14. If G is a group scheme over a field, is there always a quasi-compact open and closed subgroup scheme? Or is there a counter example?

38.8. Actions of group schemes

Let (G, m) be a group and let V be a set. Recall that a *(left) action* of G on V is given by a map $a : G \times V \to V$ such that

(1) (associativity) $a(m(g, g'), v) = a(g, a(g', v))$ for all $g, g' \in G$ and $v \in V$, and
(2) (identity) $a(e, v) = v$ for all $v \in V$.

We also say that V is a *G-set* (this usually means we drop the a from the notation – which is abuse of notation). A *map of G-sets* $\psi : V \to V'$ is any set map such that $\psi(a(g, v)) = a(g, \psi(v))$ for all $v \in V$.

Definition 38.8.1. Let S be a scheme. Let (G, m) be a group scheme over S.

(1) An *action of G on the scheme X/S* is a morphism $a : G \times_S X \to X$ over S such that for every T/S the map $a : G(T) \times X(T) \to X(T)$ defines the structure of a $G(T)$-set on $X(T)$.
(2) Suppose that X, Y are schemes over S each endowed with an action of G. An *equivariant* or more precisely a *G-equivariant* morphism $\psi : X \to Y$ is a morphism of schemes over S such that for every T/S the map $\psi : X(T) \to Y(T)$ is a morphism of $G(T)$-sets.

In situation (1) this means that the diagrams

(38.8.1.1)
$$\begin{array}{ccc} G \times_S G \times_S X & \xrightarrow{1_G \times a} & G \times_S X \\ {\scriptstyle m \times 1_X}\downarrow & & \downarrow{\scriptstyle a} \\ G \times_S X & \xrightarrow{a} & X \end{array} \qquad \begin{array}{ccc} G \times_S X & \xrightarrow{a} & X \\ {\scriptstyle e \times 1_X}\uparrow & \nearrow_{1_X} & \\ X & & \end{array}$$

are commutative. In situation (2) this just means that the diagram

$$\begin{array}{ccc} G \times_S X & \xrightarrow{\mathrm{id} \times f} & G \times_S Y \\ {\scriptstyle a}\downarrow & & \downarrow{\scriptstyle a} \\ X & \xrightarrow{f} & Y \end{array}$$

commutes.

Definition 38.8.2. Let S, $G \to S$, and $X \to S$ as in Definition 38.8.1. Let $a : G \times_S X \to X$ be an action of G on X/S. We say the action is *free* if for every scheme T over S the action $a : G(T) \times X(T) \to X(T)$ is a free action of the group $G(T)$ on the set $X(T)$.

Lemma 38.8.3. *Situation as in Definition 38.8.2, The action a is free if and only if*

$$G \times_S X \to X \times_S X, \quad (g, x) \mapsto (a(g, x), x)$$

is a monomorphism.

Proof. Immediate from the definitions. □

38.9. Principal homogeneous spaces

In Cohomology on Sites, Definition 21.5.1 we have defined a torsor for a sheaf of groups on a site. Suppose $\tau \in \{Zariski, étale, smooth, syntomic, fppf\}$ is a topology and (G, m) is a group scheme over S. Since τ is stronger than the canonical topology (see Descent, Lemma 34.9.3) we see that $\underline{G}$ (see Sites, Definition 7.13.3) is a sheaf of groups on $(Sch/S)_\tau$. Hence we already know what it means to have a torsor for $\underline{G}$ on $(Sch/S)_\tau$. A special situation arises if this sheaf is representable. In the following definitions we define directly what it means for the representing scheme to be a G-torsor.

Definition 38.9.1. Let S be a scheme. Let (G, m) be a group scheme over S. Let X be a scheme over S, and let $a : G \times_S X \to X$ be an action of G on X.

(1) We say X is a *pseudo G-torsor* or that X is *formally principally homogeneous under G* if the induced morphism of schemes $G \times_S X \to X \times_S X$, $(g, x) \mapsto (a(g, x), x)$ is an isomorphism of schemes over S.
(2) A pseudo G-torsor X is called *trivial* if there exists an G-equivariant isomorphism $G \to X$ over S where G acts on G by left multiplication.

It is clear that if $S' \to S$ is a morphism of schemes then the pullback $X_{S'}$ of a pseudo G-torsor over S is a pseudo $G_{S'}$-torsor over S'.

Lemma 38.9.2. *In the situation of Definition 38.9.1.*

(1) *The scheme X is a pseudo G-torsor if and only if for every scheme T over S the set $X(T)$ is either empty or the action of the group $G(T)$ on $X(T)$ is simply transitive.*
(2) *A pseudo G-torsor X is trivial if and only if the morphism $X \to S$ has a section.*

Proof. Omitted. □

Definition 38.9.3. Let S be a scheme. Let (G, m) be a group scheme over S. Let X be a pseudo G-torsor over S.

(1) We say X is a *principal homogeneous space* or a *G-torsor* if there exists a fpqc covering[1] $\{S_i \to S\}_{i \in I}$ such that each $X_{S_i} \to S_i$ has a section (i.e., is a trivial pseudo G_{S_i}-torsor).
(2) Let $\tau \in \{Zariski, étale, smooth, syntomic, fppf\}$. We say X is a *G-torsor in the τ topology*, or a *τ G-torsor*, or simply a *τ torsor* if there exists a τ covering $\{S_i \to S\}_{i \in I}$ such that each $X_{S_i} \to S_i$ has a section.
(3) If X is a G-torsor, then we say that it is *quasi-isotrivial* if it is a torsor for the étale topology.

[1] This means that the default type of torsor is a pseudo torsor which is trivial on an fpqc covering. This is the definition in [**ABD+66**, Exposé IV, 6.5]. It is a little bit inconvenient for us as we most often work in the fppf topology.

(4) If X is a G-torsor, then we say that it is *locally trivial* if it is a torsor for the Zariski topology.

We sometimes say "let X be a G-torsor over S" to indicate that X is a scheme over S equipped with an action of G which turns it into a principal homogeneous space over S. Next we show that this agrees with the notation introduced earlier when both apply.

Lemma 38.9.4. *Let S be a scheme. Let (G, m) be a group scheme over S. Let X be a scheme over S, and let $a : G \times_S X \to X$ be an action of G on X. Let $\tau \in \{Zariski, étale, smooth, syntomic, fppf\}$. Then X is a G-torsor in the τ-topology if and only if $\underline{X}$ is a $\underline{G}$-torsor on $(Sch/S)_\tau$.*

Proof. Omitted. □

Remark 38.9.5. Let (G, m) be a group scheme over the scheme S. In this situation we have the following natural types of questions:

(1) If $X \to S$ is a pseudo G-torsor and $X \to S$ is surjective, then is X necessarily a G-torsor?
(2) Is every $\underline{G}$-torsor on $(Sch/S)_{fppf}$ representable? In other words, does every $\underline{G}$-torsor come from a fppf G-torsor?
(3) Is every G-torsor an fppf (resp. smooth, resp. étale, resp. Zariski) torsor?

In general the answers to these questions is no. To get a positive answer we need to impose additional conditions on $G \to S$. For example: If S is the spectrum of a field, then the answer to (1) is yes because then $\{X \to S\}$ is a fpqc covering trivializing X. If $G \to S$ is affine, then the answer to (2) is yes (insert future reference here). If $G = \mathrm{GL}_{n,S}$ then the answer to (3) is yes and in fact any $\mathrm{GL}_{n,S}$-torsor is locally trivial (insert future reference here).

38.10. Equivariant quasi-coherent sheaves

We think of "functions" as dual to "space". Thus for a morphism of spaces the map on functions goes the other way. Moreover, we think of the sections of a sheaf of modules as "functions". This leads us naturally to the direction of the arrows chosen in the following definition.

Definition 38.10.1. Let S be a scheme, let (G, m) be a group scheme over S, and let $a : G \times_S X \to X$ be an action of the group scheme G on X/S. An *G-equivariant quasi-coherent $\mathcal{O}_X$-module*, or simply a *equivariant quasi-coherent $\mathcal{O}_X$-module*, is a pair $(\mathcal{F}, \alpha)$, where $\mathcal{F}$ is a quasi-coherent $\mathcal{O}_X$-module, and α is a $\mathcal{O}_{G \times_S X}$-module map

$$\alpha : a^*\mathcal{F} \longrightarrow \mathrm{pr}_1^*\mathcal{F}$$

where $\mathrm{pr}_1 : G \times_S X \to X$ is the projection such that

(1) the diagram

$$\begin{array}{ccc} (1_G \times a)^*\mathrm{pr}_2^*\mathcal{F} & \xrightarrow{\mathrm{pr}_{12}^*\alpha} & \mathrm{pr}_2^*\mathcal{F} \\ {\scriptstyle (1_G \times a)^*\alpha}\uparrow & & \uparrow{\scriptstyle (m \times 1_X)^*\alpha} \\ (1_G \times a)^*a^*\mathcal{F} & = & (m \times 1_X)^*a^*\mathcal{F} \end{array}$$

is a commutative in the category of $\mathcal{O}_{G \times_S G \times_S X}$-modules, and

(2) the pullback
$$(e \times 1_X)^*\alpha : \mathcal{F} \longrightarrow \mathcal{F}$$
is the identity map.

For explanation compare with the relevant diagrams of Equation (38.8.1.1).

Note that the commutativity of the first diagram guarantees that $(e \times 1_X)^*\alpha$ is an idempotent operator on $\mathcal{F}$, and hence condition (2) is just the condition that it is an isomorphism.

Lemma 38.10.2. *Let S be a scheme. Let G be a group scheme over S. Let $f : X \to Y$ be a G-equivariant morphism between S-schemes endowed with G-actions. Then pullback f^* given by $(\mathcal{F}, \alpha) \mapsto (f^*\mathcal{F}, (1_G \times f)^*\alpha)$ defines a functor from the category of G-equivariant sheaves on X to the category of quasi-coherent G-equivariant sheaves on Y.*

Proof. Omitted. □

38.11. Groupoids

Recall that a groupoid is a category in which every morphism is an isomorphism, see Categories, Definition 4.2.5. Hence a groupoid has a set of objects Ob, a set of arrows Arrows, a *source* and *target* map $s, t : \text{Arrows} \to \text{Ob}$, and a *composition law* $c : \text{Arrows} \times_{s,\text{Ob},t} \text{Arrows} \to \text{Arrows}$. These maps satisfy exactly the following axioms

(1) (associativity) $c \circ (1, c) = c \circ (c, 1)$ as maps $\text{Arrows} \times_{s,\text{Ob},t} \text{Arrows} \times_{s,\text{Ob},t} \text{Arrows} \to \text{Arrows}$,
(2) (identity) there exists a map $e : \text{Ob} \to \text{Arrows}$ such that
   (a) $s \circ e = t \circ e = \text{id}$ as maps $\text{Ob} \to \text{Ob}$,
   (b) $c \circ (1, e \circ s) = c \circ (e \circ t, 1) = 1$ as maps $\text{Arrows} \to \text{Arrows}$,
(3) (inverse) there exists a map $i : \text{Arrows} \to \text{Arrows}$ such that
   (a) $s \circ i = t$, $t \circ i = s$ as maps $\text{Arrows} \to \text{Ob}$, and
   (b) $c \circ (1, i) = e \circ t$ and $c \circ (i, 1) = e \circ s$ as maps $\text{Arrows} \to \text{Arrows}$.

If this is the case the maps e and i are uniquely determined and i is a bijection. Note that if $(\text{Ob}', \text{Arrows}', s', t', c')$ is a second groupoid category, then a functor $f : (\text{Ob}, \text{Arrows}, s, t, c) \to (\text{Ob}', \text{Arrows}', s', t', c')$ is given by a pair of set maps $f : \text{Ob} \to \text{Ob}'$ and $f : \text{Arrows} \to \text{Arrows}'$ such that $s' \circ f = f \circ s$, $t' \circ f = f \circ t$, and $c' \circ (f, f) = f \circ c$. The compatibility with identity and inverse is automatic. We will use this below. (Warning: The compatibility with identity has to be imposed in the case of general categories.)

Definition 38.11.1. Let S be a scheme.

(1) A *groupoid scheme over S*, or simply a *groupoid over S* is a quintuple (U, R, s, t, c) where U and R are schemes over S, and $s, t : R \to U$ and $c : R \times_{s,U,t} R \to R$ are morphisms of schemes over S with the following property: For any scheme T over S the quintuple
$$(U(T), R(T), s, t, c)$$
is a groupoid category in the sense described above.

(2) A *morphism* $f : (U, R, s, t, c) \to (U', R', s', t', c')$ *of groupoid schemes over* S is given by morphisms of schemes $f : U \to U'$ and $f : R \to R'$ with the following property: For any scheme T over S the maps f define a functor from the groupoid category $(U(T), R(T), s, t, c)$ to the groupoid category $(U'(T), R'(T), s', t', c')$.

Let (U, R, s, t, c) be a groupoid over S. Note that, by the remarks preceding the definition and the Yoneda lemma, there are unique morphisms of schemes $e : U \to R$ and $i : R \to R$ over S such that for every scheme T over S the induced map $e : U(T) \to R(T)$ is the identity, and $i : R(T) \to R(T)$ is the inverse of the groupoid category. The septuple (U, R, s, t, c, e, i) satisfies commutative diagrams corresponding to each of the axioms (1), (2)(a), (2)(b), (3)(a) and (3)(b) above, and conversely given a septuple with this property the quintuple (U, R, s, t, c) is a groupoid scheme. Note that i is an isomorphism, and e is a section of both s and t. Moreover, given a groupoid scheme over S we denote

$$j = (t, s) : R \longrightarrow U \times_S U$$

which is compatible with our conventions in Section 38.3 above. We sometimes say "let (U, R, s, t, c, e, i) be a groupoid over S" to stress the existence of identity and inverse.

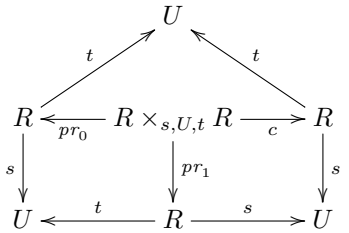
Lemma 38.11.2. *Given a groupoid scheme (U, R, s, t, c) over S the morphism $j : R \to U \times_S U$ is a pre-equivalence relation.*

Proof. Omitted. This is a nice exercise in the definitions. □

Lemma 38.11.3. *Given an equivalence relation $j : R \to U$ over S there is a unique way to extend it to a groupoid (U, R, s, t, c) over S.*

Proof. Omitted. This is a nice exercise in the definitions. □

Lemma 38.11.4. *Let S be a scheme. Let (U, R, s, t, c) be a groupoid over S. In the commutative diagram*



the two lower squares are fibre product squares. Moreover, the triangle on top (which is really a square) is also cartesian.

Proof. Omitted. Exercise in the definitions and the functorial point of view in algebraic geometry. □

Lemma 38.11.5. *Let S be a scheme. Let (U, R, s, t, c, e, i) be a groupoid over S. The diagram*

(38.11.5.1)
$$\begin{array}{ccccc} R \times_{t,U,t} R & \xrightarrow[pr_0]{pr_1} & R & \xrightarrow{t} & U \\ \downarrow{\scriptstyle pr_0 \times c \circ (i,1)} & & \downarrow{\scriptstyle id_R} & & \downarrow{\scriptstyle id_U} \\ R \times_{s,U,t} R & \xrightarrow[pr_0]{c} & R & \xrightarrow{t} & U \\ \downarrow{\scriptstyle pr_1} & & \downarrow{\scriptstyle s} & & \\ R & \xrightarrow[t]{s} & U & & \end{array}$$

is commutative. The two top rows are isomorphic via the vertical maps given. The two lower left squares are cartesian.

Proof. The commutativity of the diagram follows from the axioms of a groupoid. Note that, in terms of groupoids, the top left vertical arrow assigns to a pair of morphisms (α, β) with the same target, the pair of morphisms $(\alpha, \alpha^{-1} \circ \beta)$. In any groupoid this defines a bijection between Arrows $\times_{t,\mathrm{Ob},t}$ Arrows and Arrows $\times_{s,\mathrm{Ob},t}$ Arrows. Hence the second assertion of the lemma. The last assertion follows from Lemma 38.11.4. □

38.12. Quasi-coherent sheaves on groupoids

See the introduction of Section 38.10 for our choices in direction of arrows.

Definition 38.12.1. Let S be a scheme, let (U, R, s, t, c) be a groupoid scheme over S. A *quasi-coherent module on* (U, R, s, t, c) is a pair $(\mathcal{F}, \alpha)$, where $\mathcal{F}$ is a quasi-coherent $\mathcal{O}_U$-module, and α is a $\mathcal{O}_R$-module map

$$\alpha : t^*\mathcal{F} \longrightarrow s^*\mathcal{F}$$

such that

(1) the diagram

$$\begin{array}{ccccc} & & \mathrm{pr}_1^*t^*\mathcal{F} & \xrightarrow[\mathrm{pr}_1^*\alpha]{} & \mathrm{pr}_1^*s^*\mathcal{F} & & \\ & \nearrow & & & & \searrow & \\ \mathrm{pr}_0^*s^*\mathcal{F} & & & & & & c^*s^*\mathcal{F} \\ & \nwarrow{\scriptstyle \mathrm{pr}_0^*\alpha} & & & & {\scriptstyle c^*\alpha}\nearrow & \\ & & \mathrm{pr}_0^*t^*\mathcal{F} & = & c^*t^*\mathcal{F} & & \end{array}$$

is a commutative in the category of $\mathcal{O}_{R \times_{s,U,t} R}$-modules, and

(2) the pullback

$$e^*\alpha : \mathcal{F} \longrightarrow \mathcal{F}$$

is the identity map.

Compare with the commutative diagrams of Lemma 38.11.4.

The commutativity of the first diagram forces the operator $e^*\alpha$ to be idempotent. Hence the second condition can be reformulated as saying that $e^*\alpha$ is an isomorphism. In fact, the condition implies that α is an isomorphism.

Lemma 38.12.2. *Let S be a scheme, let (U, R, s, t, c) be a groupoid scheme over S. If $(\mathcal{F}, \alpha)$ is a quasi-coherent module on (U, R, s, t, c) then α is an isomorphism.*

Proof. Pull back the commutative diagram of Definition 38.12.1 by the morphism $(i, 1) : R \to R \times_{s,U,t} R$. Then we see that $i^*\alpha \circ \alpha = s^*e^*\alpha$. Pulling back by the morphism $(1, i)$ we obtain the relation $\alpha \circ i^*\alpha = t^*e^*\alpha$. By the second assumption these morphisms are the identity. Hence $i^*\alpha$ is an inverse of α. □

Lemma 38.12.3. *Let S be a scheme. Consider a morphism $f : (U, R, s, t, c) \to (U', R', s', t', c')$ of groupoid schemes over S. Then pullback f^* given by*

$$(\mathcal{F}, \alpha) \mapsto (f^*\mathcal{F}, f^*\alpha)$$

defines a functor from the category of quasi-coherent sheaves on (U', R', s', t', c') to the category of quasi-coherent sheaves on (U, R, s, t, c).

Proof. Omitted. □

Lemma 38.12.4. *Let S be a scheme. Consider a morphism $f : (U, R, s, t, c) \to (U', R', s', t', c')$ of groupoid schemes over S. Assume that*

(1) *$f : U \to U'$ is quasi-compact and quasi-separated,*
(2) *the square*

$$\begin{array}{ccc} R & \xrightarrow{f} & R' \\ {\scriptstyle t}\downarrow & & \downarrow{\scriptstyle t'} \\ U & \xrightarrow{f} & U' \end{array}$$

is cartesian, and
(3) *s' and t' are flat.*

Then pushforward f_* given by

$$(\mathcal{F}, \alpha) \mapsto (f_*\mathcal{F}, f_*\alpha)$$

defines a functor from the category of quasi-coherent sheaves on (U, R, s, t, c) to the category of quasi-coherent sheaves on (U, R, s, t, c) which is right adjoint to pullback as defined in Lemma 38.12.3.

Proof. Since $U \to U'$ is quasi-compact and quasi-separated we see that f_* transforms quasi-coherent sheaves into quasi-coherent sheaves (Schemes, Lemma 25.24.1). Moreover, since the squares

$$\begin{array}{ccc} R & \xrightarrow{f} & R' \\ {\scriptstyle t}\downarrow & & \downarrow{\scriptstyle t'} \\ U & \xrightarrow{f} & U' \end{array} \quad \text{and} \quad \begin{array}{ccc} R & \xrightarrow{f} & R' \\ {\scriptstyle s}\downarrow & & \downarrow{\scriptstyle s'} \\ U & \xrightarrow{f} & U' \end{array}$$

are cartesian we find that $(t')^*f_*\mathcal{F} = f_*t^*\mathcal{F}$ and $(s')^*f_*\mathcal{F} = f_*s^*\mathcal{F}$, see Cohomology of Schemes, Lemma 29.5.2. Thus it makes sense to think of $f_*\alpha$ as a map $(t')^*f_*\mathcal{F} \to (s')^*f_*\mathcal{F}$. A similar argument shows that $f_*\alpha$ satisfies the cocycle condition. The functor is adjoint to the pullback functor since pullback and pushforward on modules on ringed spaces are adjoint. Some details omitted. □

Lemma 38.12.5. *Let S be a scheme. Let (U, R, s, t, c) be a groupoid scheme over S. The category of quasi-coherent modules on (U, R, s, t, c) has colimits.*

Proof. Let $i \mapsto (\mathcal{F}_i, \alpha_i)$ be a diagram over the index category $\mathcal{I}$. We can form the colimit $\mathcal{F} = \operatorname{colim} \mathcal{F}_i$ which is a quasi-coherent sheaf on U, see Schemes, Section 25.24. Since colimits commute with pullback we see that $s^*\mathcal{F} = \operatorname{colim} s^*\mathcal{F}_i$ and similarly $t^*\mathcal{F} = \operatorname{colim} t^*\mathcal{F}_i$. Hence we can set $\alpha = \operatorname{colim} \alpha_i$. We omit the proof that $(\mathcal{F}, \alpha)$ is the colimit of the diagram in the category of quasi-coherent modules on (U, R, s, t, c). □

Lemma 38.12.6. *Let S be a scheme. Let (U, R, s, t, c) be a groupoid scheme over S. If s, t are flat, then the category of quasi-coherent modules on (U, R, s, t, c) is abelian.*

Proof. Let $\varphi : (\mathcal{F}, \alpha) \to (\mathcal{G}, \beta)$ be a homomorphism of quasi-coherent modules on (U, R, s, t, c). Since s is flat we see that

$$0 \to s^*\mathrm{Ker}(\varphi) \to s^*\mathcal{F} \to s^*\mathcal{G} \to s^*\mathrm{Coker}(\varphi) \to 0$$

is exact and similarly for pullback by t. Hence α and β induce isomorphisms $\kappa : t^*\mathrm{Ker}(\varphi) \to s^*\mathrm{Ker}(\varphi)$ and $\lambda : t^*\mathrm{Coker}(\varphi) \to s^*\mathrm{Coker}(\varphi)$ which satisfy the cocycle condition. Then it is straightforward to verify that $(\mathrm{Ker}(\varphi), \kappa)$ and $(\mathrm{Coker}(\varphi), \lambda)$ are a kernel and cokernel in the category of quasi-coherent modules on (U, R, s, t, c). Moreover, the condition $\mathrm{Coim}(\varphi) = \mathrm{Im}(\varphi)$ follows because it holds over U. □

38.13. Colimits of quasi-coherent modules

In this section we prove some technical results saying that under suitable assumptions every quasi-coherent module on a groupoid is a filtered colimit of "small" quasi-coherent modules.

Lemma 38.13.1. *Let (U, R, s, t, c) be a groupoid scheme over S. Assume s, t are flat, quasi-compact, and quasi-separated. For any quasi-coherent module $\mathcal{G}$ on U, there exists a canonical isomorphism $\alpha : t^*t_*s^*\mathcal{G} \to s^*t_*s^*\mathcal{G}$ which turns $(t_*s^*\mathcal{G}, \alpha)$ into a quasi-coherent module on (U, R, s, t, c). This construction defines a functor*

$$QCoh(\mathcal{O}_U) \longrightarrow QCoh(U, R, s, t, c)$$

which is a right adjoint to the forgetful functor $(\mathcal{F}, \beta) \mapsto \mathcal{F}$.

Proof. The pushforward of a quasi-coherent module along a quasi-compact and quasi-separated morphism is quasi-coherent, see Schemes, Lemma 25.24.1. Hence $t_*s^*\mathcal{G}$ is quasi-coherent. With notation as in Lemma 38.11.4 we have

$$t^*t_*s^*\mathcal{G} = \mathrm{pr}_{0,*}c^*s^*\mathcal{G} = \mathrm{pr}_{0,*}\mathrm{pr}_1^*s^*\mathcal{G} = s^*t_*s^*\mathcal{G}$$

The middle equality because $s \circ c = s \circ \mathrm{pr}_1$ as morphisms $R \times_{s,U,t} R \to U$, and the first and the last equality because we know that base change and pushforward commute in these steps by Cohomology of Schemes, Lemma 29.5.2.

To verify the cocycle condition of Definition 38.12.1 for α and the adjointness property we describe the construction $\mathcal{G} \mapsto (\mathcal{G}, \alpha)$ in another way. Consider the groupoid scheme $(R, R \times_{s,U,s} R, \mathrm{pr}_0, \mathrm{pr}_1, \mathrm{pr}_{02})$ associated to the equivalence relation $R \times_{s,U,s} R$ on R, see Lemma 38.11.3. There is a morphism

$$f : (R, R \times_{s,U,s} R, \mathrm{pr}_1, \mathrm{pr}_0, \mathrm{pr}_{02}) \longrightarrow (U, R, s, t, c)$$

of groupoid schemes given by $t : R \to U$ and $R \times_{t,U,t} R \to R$ given by $(r_0, r_1) \mapsto r_0 \circ r_1^{-1}$ (we omit the verification of the commutativity of the required diagrams).

Since $t, s : R \to U$ are quasi-compact, quasi-separated, and flat, and since we have a cartesian square

$$\begin{array}{ccc} R \times_{s,U,s} R & \xrightarrow[(r_0,r_1)\mapsto r_0 \circ r_1^{-1}]{} & R \\ {\scriptstyle \mathrm{pr}_0}\downarrow & & \downarrow{\scriptstyle t} \\ R & \xrightarrow{t} & U \end{array}$$

by Lemma 38.11.5 it follows that Lemma 38.12.4 applies to f. Note that

$$QCoh(R, R \times_{s,U,s} R, \mathrm{pr}_0, \mathrm{pr}_1, \mathrm{pr}_{02}) = QCoh(\mathcal{O}_U)$$

by the theory of descent of quasi-coherent sheaves as $\{t : R \to U\}$ is an fpqc covering, see Descent, Proposition 34.5.2. Observe that pullback along f agrees with the forgetful functor and that pushforward agrees with the construction that assigns to $\mathcal{G}$ the pair $(\mathcal{G}, \alpha)$. We omit the precise verifications. Thus the lemma follows from Lemma 38.12.4. □

Lemma 38.13.2. *Let $f : Y \to X$ be a morphism of schemes. Let $\mathcal{F}$ be a quasi-coherent $\mathcal{O}_X$-module, let $\mathcal{G}$ be a quasi-coherent $\mathcal{O}_Y$-module, and let $\varphi : \mathcal{G} \to f^*\mathcal{F}$ be a module map. Assume*

(1) *φ is injective,*
(2) *f is quasi-compact, quasi-separated, flat, and surjective,*
(3) *X, Y are locally Noetherian, and*
(4) *$\mathcal{G}$ is a coherent $\mathcal{O}_Y$-module.*

Then $\mathcal{F} \cap f_*\mathcal{G}$ defined as the pullback

$$\begin{array}{ccc} \mathcal{F} & \longrightarrow & f_*f^*\mathcal{F} \\ \uparrow & & \uparrow \\ \mathcal{F} \cap f_*\mathcal{G} & \longrightarrow & f_*\mathcal{G} \end{array}$$

is a coherent $\mathcal{O}_X$-module.

Proof. We will freely use the characterization of coherent modules of Cohomology of Schemes, Lemma 29.9.1 as well as the fact that coherent modules form a Serre subcategory of $QCoh(\mathcal{O}_X)$, see Cohomology of Schemes, Lemma 29.9.3. If f has a section σ, then we see that $\mathcal{F} \cap f_*\mathcal{G}$ is contained in the image of $\sigma^*\mathcal{G} \to \sigma^*f^*\mathcal{F} = \mathcal{F}$, hence coherent. In general, to show that $\mathcal{F} \cap f_*\mathcal{G}$ is coherent, it suffices the show that $f^*(\mathcal{F} \cap f_*\mathcal{G})$ is coherent (see Descent, Lemma 34.6.1). Since f is flat this is equal to $f^*\mathcal{F} \cap f^*f_*\mathcal{G}$. Since f is flat, quasi-compact, and quasi-separated we see $f^*f_*\mathcal{G} = p_*q^*\mathcal{G}$ where $p, q : Y \times_X Y \to Y$ are the projections, see Cohomology of Schemes, Lemma 29.5.2. Since p has a section we win. □

Let S be a scheme. Let (U, R, s, t, c) be a groupoid in schemes over S. Assume that U is locally Noetherian. In the lemma below we say that a quasi-coherent sheaf $(\mathcal{F}, \alpha)$ on (U, R, s, t, c) is *coherent* if $\mathcal{F}$ is a coherent $\mathcal{O}_U$-module.

Lemma 38.13.3. *Let (U, R, s, t, c) be a groupoid scheme over S. Assume that*

(1) *U, R are Noetherian,*
(2) *s, t are flat, quasi-compact, and quasi-separated.*

Then every quasi-coherent module $(\mathcal{F}, \alpha)$ on (U, R, s, t, c) is a filtered colimit of coherent modules.

Proof. We will use the characterization of Cohomology of Schemes, Lemma 29.9.1 of coherent modules on locally Noetherian scheme without further mention. Write $\mathcal{F} = \operatorname{colim} \mathcal{H}_i$ with $\mathcal{H}_i$ coherent, see Properties, Lemma 27.20.6. Given a quasi-coherent sheaf $\mathcal{H}$ on U we denote $t_*s^*\mathcal{H}$ the quasi-coherent sheaf on (U, R, s, t, c) of Lemma 38.13.1. There is an adjunction map $\mathcal{F} \to t_*s^*\mathcal{F}$ in $QCoh(U, R, s, t, c)$. Consider the pullback diagram

$$\begin{array}{ccc} \mathcal{F} & \longrightarrow & t_*s^*\mathcal{F} \\ \uparrow & & \uparrow \\ \mathcal{F}_i & \longrightarrow & t_*s^*\mathcal{H}_i \end{array}$$

in other words $\mathcal{F}_i = \mathcal{F} \cap t_*s^*\mathcal{H}_i$. Then $\mathcal{F}_i$ is coherent by Lemma 38.13.2. On the other hand, the diagram above is a pullback diagram in $QCoh(U, R, s, t, c)$ also as restriction to U is an exact functor by (the proof of) Lemma 38.12.6. Finally, because t is quasi-compact and quasi-separated we see that t_* commutes with colimits (see Cohomology of Schemes, Lemma 29.6.1). Hence $t_*s^*\mathcal{F} = \operatorname{colim} t_*\mathcal{H}_i$ and hence $\mathcal{F} = \operatorname{colim} \mathcal{F}_i$ as desired. □

Here is a curious lemma that is useful when working with groupoids on fields. In fact, this is the standard argument to prove that any representation of an algebraic group is a colimit of finite dimensional representations.

Lemma 38.13.4. *Let (U, R, s, t, c) be a groupoid scheme over S. Assume that*

(1) *U, R are affine,*
(2) *there exist $e_i \in \mathcal{O}_R(R)$ such that every element $g \in \mathcal{O}_R(R)$ can be uniquely written as $\sum s^*(f_i)e_i$ for some $f_i \in \mathcal{O}_U(U)$.*

Then every quasi-coherent module $(\mathcal{F}, \alpha)$ on (U, R, s, t, c) is a filtered colimit of finite type quasi-coherent modules.

Proof. The assumption means that $\mathcal{O}_R(R)$ is a free $\mathcal{O}_U(U)$-module via s with basis e_i. Hence for any quasi-coherent $\mathcal{O}_U$-module $\mathcal{G}$ we see that $s^*\mathcal{G}(R) = \bigoplus_i \mathcal{G}(U)e_i$. We will write $s(-)$ to indicate pullback of sections by s and similarly for other morphisms. Let $(\mathcal{F}, \alpha)$ be a quasi-coherent module on (U, R, s, t, c). Let $\sigma \in \mathcal{F}(U)$. By the above we can write

$$\alpha(t(\sigma)) = \sum s(\sigma_i)e_i$$

for some unique $\sigma_i \in \mathcal{F}(U)$ (all but finitely many are zero of course). We can also write

$$c(e_i) = \sum \mathrm{pr}_1(f_{ij})\mathrm{pr}_0(e_j)$$

as functions on $R \times_{s,U,t} R$. Then the commutativity of the diagram in Definition 38.12.1 means that

$$\sum \mathrm{pr}_1(\alpha(t(\sigma_i)))\mathrm{pr}_0(e_i) = \sum \mathrm{pr}_1(s(\sigma_i)f_{ij})\mathrm{pr}_0(e_j)$$

(calculation omitted). Picking off the coefficients of $\mathrm{pr}_0(e_l)$ we see that $\alpha(t(\sigma_l)) = \sum s(\sigma_i)f_{il}$. Hence the submodule $\mathcal{G} \subset \mathcal{F}$ generated by the elements σ_i defines a finite type quasi-coherent module preserved by α. Hence it is a subobject of $\mathcal{F}$ in $QCoh(U, R, s, t, c)$. This submodule contains σ (as one sees by pulling back the first relation by e). Hence we win. □

We suggest the reader skip the rest of this section. Let S be a scheme. Let (U, R, s, t, c) be a groupoid in schemes over S. Let κ be a cardinal. In the following we will say that a quasi-coherent sheaf $(\mathcal{F}, \alpha)$ on (U, R, s, t, c) is κ-generated if $\mathcal{F}$ is a κ-generated $\mathcal{O}_U$-module, see Properties, Definition 27.21.1.

Lemma 38.13.5. *Let (U, R, s, t, c) be a groupoid scheme over S. Let κ be a cardinal. There exists a set T and a family $(\mathcal{F}_t, \alpha_t)_{t \in T}$ of κ-generated quasi-coherent modules on (U, R, s, t, c) such that every κ-generated quasi-coherent module on (U, R, s, t, c) is isomorphic to one of the $(\mathcal{F}_t, \alpha_t)$.*

Proof. For each quasi-coherent module $\mathcal{F}$ on U there is a (possibly empty) set of maps $\alpha : t^*\mathcal{F} \to s^*\mathcal{F}$ such that $(\mathcal{F}, \alpha)$ is a quasi-coherent modules on (U, R, s, t, c). By Properties, Lemma 27.21.2 there exists a set of isomorphism classes of κ-generated quasi-coherent $\mathcal{O}_U$-modules. □

Lemma 38.13.6. *Let (U, R, s, t, c) be a groupoid scheme over S. Assume that s, t are flat. There exists a cardinal κ such that every quasi-coherent module $(\mathcal{F}, \alpha)$ on (U, R, s, t, c) is the directed colimit of its κ-generated quasi-coherent submodules.*

Proof. In the statement of the lemma and in this proof a *submodule* of a quasi-coherent module $(\mathcal{F}, \alpha)$ is a quasi-coherent submodule $\mathcal{G} \subset \mathcal{F}$ such that $\alpha(t^*\mathcal{G}) = s^*\mathcal{G}$ as subsheaves of $s^*\mathcal{F}$. This makes sense because since s, t are flat the pullbacks s^* and t^* are exact, i.e., preserve subsheaves. The proof will be a repeat of the proof of Properties, Lemma 27.21.3. We urge the reader to read that proof first.

Choose an affine open covering $U = \bigcup_{i \in I} U_i$. For each pair i, j choose affine open coverings

$$U_i \cap U_j = \bigcup\nolimits_{k \in I_{ij}} U_{ijk} \quad \text{and} \quad s^{-1}(U_i) \cap t^{-1}(U_j) = \bigcup\nolimits_{k \in J_{ij}} W_{ijk}.$$

Write $U_i = \operatorname{Spec}(A_i)$, $U_{ijk} = \operatorname{Spec}(A_{ijk})$, $W_{ijk} = \operatorname{Spec}(B_{ijk})$. Let κ be any infinite cardinal $\geq$ than the cardinality of any of the sets I, I_{ij}, J_{ij}.

Let $(\mathcal{F}, \alpha)$ be a quasi-coherent module on (U, R, s, t, c). Set $M_i = \mathcal{F}(U_i)$, $M_{ijk} = \mathcal{F}(U_{ijk})$. Note that

$$M_i \otimes_{A_i} A_{ijk} = M_{ijk} = M_j \otimes_{A_j} A_{ijk}$$

and that α gives isomorphisms

$$\alpha|_{W_{ijk}} : M_i \otimes_{A_i, t} B_{ijk} \longrightarrow M_j \otimes_{A_j, s} B_{ijk}$$

see Schemes, Lemma 25.7.3. Using the axiom of choice we choose a map

$$(i, j, k, m) \mapsto S(i, j, k, m)$$

which associates to every $i, j \in I$, $k \in I_{ij}$ or $k \in J_{ij}$ and $m \in M_i$ a finite subset $S(i, j, k, m) \subset M_j$ such that we have

$$m \otimes 1 = \sum\nolimits_{m' \in S(i,j,k,m)} m' \otimes a_{m'} \quad \text{or} \quad \alpha(m \otimes 1) = \sum\nolimits_{m' \in S(i,j,k,m)} m' \otimes b_{m'}$$

in M_{ijk} for some $a_{m'} \in A_{ijk}$ or $b_{m'} \in B_{ijk}$. Moreover, let's agree that $S(i, i, k, m) = \{m\}$ for all $i, j = i, k, m$ when $k \in I_{ij}$. Fix such a collection $S(i, j, k, m)$

Given a family $\mathcal{S} = (S_i)_{i \in I}$ of subsets $S_i \subset M_i$ of cardinality at most κ we set $\mathcal{S}' = (S'_i)$ where

$$S'_j = \bigcup\nolimits_{(i,j,k,m) \text{ such that } m \in S_i} S(i, j, k, m)$$

Note that $S_i \subset S_i'$. Note that S_i' has cardinality at most κ because it is a union over a set of cardinality at most κ of finite sets. Set $\mathcal{S}^{(0)} = \mathcal{S}$, $\mathcal{S}^{(1)} = \mathcal{S}'$ and by induction $\mathcal{S}^{(n+1)} = (\mathcal{S}^{(n)})'$. Then set $\mathcal{S}^{(\infty)} = \bigcup_{n \geq 0} \mathcal{S}^{(n)}$. Writing $\mathcal{S}^{(\infty)} = (S_i^{(\infty)})$ we see that for any element $m \in S_i^{(\infty)}$ the image of m in M_{ijk} can be written as a finite sum $\sum m' \otimes a_{m'}$ with $m' \in S_j^{(\infty)}$. In this way we see that setting

$$N_i = A_i\text{-submodule of } M_i \text{ generated by } S_i^{(\infty)}$$

we have

$$N_i \otimes_{A_i} A_{ijk} = N_j \otimes_{A_j} A_{ijk} \quad \text{and} \quad \alpha(N_i \otimes_{A_i,t} B_{ijk}) = N_j \otimes_{A_j,s} B_{ijk}$$

as submodules of M_{ijk} or $M_j \otimes_{A_j,s} B_{ijk}$. Thus there exists a quasi-coherent submodule $\mathcal{G} \subset \mathcal{F}$ with $\mathcal{G}(U_i) = N_i$ such that $\alpha(t^*\mathcal{G}) = s^*\mathcal{G}$ as submodules of $s^*\mathcal{F}$. In other words, $(\mathcal{G}, \alpha|_{t^*\mathcal{G}})$ is a submodule of $(\mathcal{F}, \alpha)$. Moreover, by construction $\mathcal{G}$ is κ-generated.

Let $\{(\mathcal{G}_t, \alpha_t)\}_{t \in T}$ be the set of κ-generated quasi-coherent submodules of $(\mathcal{F}, \alpha)$. If $t, t' \in T$ then $\mathcal{G}_t + \mathcal{G}_{t'}$ is also a κ-generated quasi-coherent submodule as it is the image of the map $\mathcal{G}_t \oplus \mathcal{G}_{t'} \to \mathcal{F}$. Hence the system (ordered by inclusion) is directed. The arguments above show that every section of $\mathcal{F}$ over U_i is in one of the $\mathcal{G}_t$ (because we can start with $\mathcal{S}$ such that the given section is an element of S_i). Hence $\operatorname{colim}_t \mathcal{G}_t \to \mathcal{F}$ is both injective and surjective as desired. □

38.14. Groupoids and group schemes

There are many ways to construct a groupoid out of an action a of a group G on a set V. We choose the one where we think of an element $g \in G$ as an arrow with source v and target $a(g, v)$. This leads to the following construction for group actions of schemes.

Lemma 38.14.1. *Let S be a scheme. Let Y be a scheme over S. Let (G, m) be a group scheme over Y with identity e_G and inverse i_G. Let X/Y be a scheme over Y and let $a : G \times_Y X \to X$ be an action of G on X/Y. Then we get a groupoid scheme (U, R, s, t, c, e, i) over S in the following manner:*

1. *We set $U = X$, and $R = G \times_Y X$.*
2. *We set $s : R \to U$ equal to $(g, x) \mapsto x$.*
3. *We set $t : R \to U$ equal to $(g, x) \mapsto a(g, x)$.*
4. *We set $c : R \times_{s,U,t} R \to R$ equal to $((g, x), (g', x')) \mapsto (m(g, g'), x')$.*
5. *We set $e : U \to R$ equal to $x \mapsto (e_G(x), x)$.*
6. *We set $i : R \to R$ equal to $(g, x) \mapsto (i_G(g), a(g, x))$.*

Proof. Omitted. Hint: It is enough to show that this works on the set level. For this use the description above the lemma describing g as an arrow from v to $a(g, v)$. □

Lemma 38.14.2. *Let S be a scheme. Let Y be a scheme over S. Let (G, m) be a group scheme over Y. Let X be a scheme over Y and let $a : G \times_Y X \to X$ be an action of G on X over Y. Let (U, R, s, t, c) be the groupoid scheme constructed in Lemma 38.14.1. The rule $(\mathcal{F}, \alpha) \mapsto (\mathcal{F}, \alpha)$ defines an equivalence of categories between G-equivariant $\mathcal{O}_X$-modules and the category of quasi-coherent modules on (U, R, s, t, c).*

Proof. The assertion makes sense because $t = a$ and $s = \mathrm{pr}_1$ as morphisms $R = G \times_Y X \to X$, see Definitions 38.10.1 and 38.12.1. Using the translation in Lemma 38.14.1 the commutativity requirements of the two definitions match up exactly. □

38.15. The stabilizer group scheme

Given a groupoid scheme we get a group scheme as follows.

Lemma 38.15.1. *Let S be a scheme. Let (U, R, s, t, c) be a groupoid over S. The scheme G defined by the cartesian square*

$$\begin{array}{ccc} G & \longrightarrow & R \\ \downarrow & & \downarrow{\scriptstyle j=(t,s)} \\ U & \xrightarrow{\Delta} & U \times_S U \end{array}$$

is a group scheme over U with composition law m induced by the composition law c.

Proof. This is true because in a groupoid category the set of self maps of any object forms a group. □

Since Δ is an immersion we see that $G = j^{-1}(\Delta_{U/S})$ is a locally closed subscheme of R. Thinking of it in this way, the structure morphism $j^{-1}(\Delta_{U/S}) \to U$ is induced by either s or t (it is the same), and m is induced by c.

Definition 38.15.2. Let S be a scheme. Let (U, R, s, t, c) be a groupoid over S. The group scheme $j^{-1}(\Delta_{U/S}) \to U$ is called the *stabilizer of the groupoid scheme* (U, R, s, t, c).

In the literature the stabilizer group scheme is often denoted S (because the word stabilizer starts with an "s" presumably); we cannot do this since we have already used S for the base scheme.

Lemma 38.15.3. *Let S be a scheme. Let (U, R, s, t, c) be a groupoid over S, and let G/U be its stabilizer. Denote R_t/U the scheme R seen as a scheme over U via the morphism $t : R \to U$. There is a canonical left action*

$$a : G \times_U R_t \longrightarrow R_t$$

induced by the composition law c.

Proof. In terms of points over T/S we define $a(g, r) = c(g, r)$. □

Lemma 38.15.4. *Let S be a scheme. Let (U, R, s, t, c) be a groupoid scheme over S. Let G be the stabilizer group scheme of R. Let*

$$G_0 = G \times_{U, pr_0} (U \times_S U) = G \times_S U$$

as a group scheme over $U \times_S U$. The action of G on R of Lemma 38.15.3 induces an action of G_0 on R over $U \times_S U$ which turns R into a pseudo G_0-torsor over $U \times_S U$.

Proof. This is true because in a groupoid category $\mathcal{C}$ the set $\mathrm{Mor}_{\mathcal{C}}(x, y)$ is a principal homogeneous set under the group $\mathrm{Mor}_{\mathcal{C}}(y, y)$. □

Lemma 38.15.5. *Let S be a scheme. Let (U, R, s, t, c) be a groupoid scheme over S. Let $p \in U \times_S U$ be a point. Denote R_p the scheme theoretic fibre of $j = (t, s) : R \to U \times_S U$. If $R_p \neq \emptyset$, then the action*

$$G_{0,\kappa(p)} \times_{\kappa(p)} R_p \longrightarrow R_p$$

(see Lemma 38.15.4) which turns R_p into a $G_{\kappa(p)}$-torsor over $\kappa(p)$.

Proof. The action is a pseudo-torsor by the lemma cited in the statement. And if R_p is not the empty scheme, then $\{R_p \to p\}$ is an fpqc covering which trivializes the pseudo-torsor. □

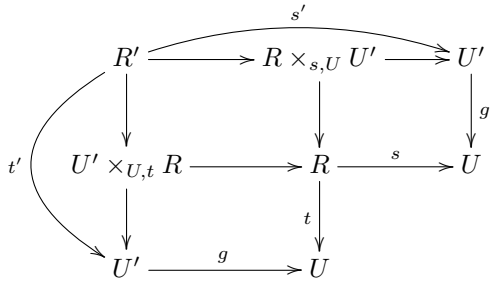
38.16. Restricting groupoids

Consider a (usual) groupoid $\mathcal{C} = (\mathrm{Ob}, \mathrm{Arrows}, s, t, c)$. Suppose we have a map of sets $g : \mathrm{Ob}' \to \mathrm{Ob}$. Then we can construct a groupoid $\mathcal{C}' = (\mathrm{Ob}', \mathrm{Arrows}', s', t', c')$ by thinking of a morphism between elements x', y' of Ob' as a morphisms in $\mathcal{C}$ between $g(x'), g(y')$. In other words we set

$$\mathrm{Arrows}' = \mathrm{Ob}' \times_{g,\mathrm{Ob},t} \mathrm{Arrows} \times_{s,\mathrm{Ob},g} \mathrm{Ob}'.$$

with obvious choices for s', t', and c'. There is a canonical functor $\mathcal{C}' \to \mathcal{C}$ which is fully faithful, but not necessarily essentially surjective. This groupoid $\mathcal{C}'$ endowed with the functor $\mathcal{C}' \to \mathcal{C}$ is called the *restriction* of the groupoid $\mathcal{C}$ to Ob'.

Lemma 38.16.1. *Let S be a scheme. Let (U, R, s, t, c) be a groupoid scheme over S. Let $g : U' \to U$ be a morphism of schemes. Consider the following diagram*



where all the squares are fibre product squares. Then there is a canonical composition law $c' : R' \times_{s',U',t'} R' \to R'$ such that (U', R', s', t', c') is a groupoid scheme over S and such that $U' \to U$, $R' \to R$ defines a morphism $(U', R', s', t', c') \to (U, R, s, t, c)$ of groupoid schemes over S. Moreover, for any scheme T over S the functor of groupoids

$$(U'(T), R'(T), s', t', c') \to (U(T), R(T), s, t, c)$$

is the restriction (see above) of $(U(T), R(T), s, t, c)$ via the map $U'(T) \to U(T)$.

Proof. Omitted. □

Definition 38.16.2. Let S be a scheme. Let (U, R, s, t, c) be a groupoid scheme over S. Let $g : U' \to U$ be a morphism of schemes. The morphism of groupoids $(U', R', s', t', c') \to (U, R, s, t, c)$ constructed in Lemma 38.16.1 is called the *restriction of (U, R, s, t, c) to U'*. We sometime use the notation $R' = R|_{U'}$ in this case.

Lemma 38.16.3. *The notions of restricting groupoids and (pre-)equivalence relations defined in Definitions 38.16.2 and 38.3.3 agree via the constructions of Lemmas 38.11.2 and 38.11.3.*

Proof. What we are saying here is that R' of Lemma 38.16.1 is also equal to

$$R' = (U' \times_S U') \times_{U \times_S U} R \longrightarrow U' \times_S U'$$

In fact this might have been a clearer way to state that lemma. □

Lemma 38.16.4. *Let S be a scheme. Let (U, R, s, t, c) be a groupoid scheme over S. Let $g : U' \to U$ be a morphism of schemes. Let (U', R', s', t', c') be the restriction of (U, R, s, t, c) via g. Let G be the stabilizer of (U, R, s, t, c) and let G' be the stabilizer of (U', R', s', t', c'). Then G' is the base change of G by g, i.e., there is a canonical identification $G' = U' \times_{g,U} G$.*

Proof. Omitted. □

38.17. Invariant subschemes

In this section we discuss briefly the notion of an invariant subscheme.

Definition 38.17.1. Let (U, R, s, t, c) be a groupoid scheme over the base scheme S.

(1) We say an open $W \subset U$ is *R-invariant* if $t(s^{-1}(W)) \subset W$.
(2) A closed subscheme $Z \subset U$ is called *R-invariant* if $t^{-1}(Z) = s^{-1}(Z)$. Here we use the scheme theoretic inverse image, see Schemes, Definition 25.17.7.
(3) A monomorphism of schemes $T \to U$ is *R-invariant* if $T \times_{U,t} R = R \times_{s,U} T$ as schemes over R.

For an open subscheme $W \subset U$ the R-invariance is also equivalent to requiring that $s^{-1}(W) = t^{-1}(W)$. If $W \subset U$ is R-equivariant then the restriction of R to W is just $R_W = s^{-1}(W) = t^{-1}(W)$. Similarly, if $Z \subset U$ is an R-invariant closed subscheme, then the restriction of R to Z is just $R_Z = s^{-1}(Z) = t^{-1}(Z)$.

Lemma 38.17.2. *Let S be a scheme. Let (U, R, s, t, c) be a groupoid scheme over S.*

(1) *If s and t are open, then for every open $W \subset U$ the open $s(t^{-1}(W))$ is R-invariant.*
(2) *If s and t are open and quasi-compact, then U has an open covering consisting of R-invariant quasi-compact open subschemes.*

Proof. Assume s and t open and $W \subset U$ open. Since s is open the set $W' = s(t^{-1}(W))$ is an open subset of U. Now it is quite easy to using the functorial point of view that this is an R-invariant open subset of U, but we are going to argue this directly by some diagrams, since we think it is instructive. Note that $t^{-1}(W')$ is the image of the morphism

$$A := t^{-1}(W) \times_{s|_{t^{-1}(W)}, U, t} R \xrightarrow{\mathrm{pr}_1} R$$

and that $s^{-1}(W')$ is the image of the morphism

$$B := R \times_{s, U, s|_{t^{-1}(W)}} t^{-1}(W) \xrightarrow{\mathrm{pr}_0} R.$$

The schemes A, B on the left of the arrows above are open subschemes of $R \times_{s,U,t} R$ and $R \times_{s,U,s} R$ respectively. By Lemma 38.11.4 the diagram

$$\begin{array}{ccc} R \times_{s,U,t} R & \xrightarrow[(\mathrm{pr}_1, c)]{} & R \times_{s,U,s} R \\ & {\scriptstyle \mathrm{pr}_1} \searrow \quad \swarrow {\scriptstyle \mathrm{pr}_0} & \\ & R & \end{array}$$

is commutative, and the horizontal arrow is an isomorphism. Moreover, it is clear that $(\mathrm{pr}_1, c)(A) = B$. Hence we conclude $s^{-1}(W') = t^{-1}(W')$, and W' is R-invariant. This proves (1).

Assume now that s, t are both open and quasi-compact. Then, if $W \subset U$ is a quasi-compact open, then also $W' = s(t^{-1}(W))$ is a quasi-compact open, and invariant by the discussion above. Letting W range over all affine opens of U we see (2). $\square$

38.18. Quotient sheaves

Let $\tau \in \{Zariski, étale, fppf, smooth, syntomic\}$. Let S be a scheme. Let $j : R \to U \times_S U$ be a pre-relation over S. Say U, R, S are objects of a τ-site Sch_τ (see Topologies, Section 33.2). Then we can consider the functors

$$h_U, h_R : (Sch/S)_\tau^{opp} \longrightarrow Sets.$$

These are sheaves, see Descent, Lemma 34.9.3. The morphism j induces a map $j : h_R \to h_U \times h_U$. For each object $T \in \mathrm{Ob}((Sch/S)_\tau)$ we can take the equivalence relation $\sim_T$ generated by $j(T) : R(T) \to U(T) \times U(T)$ and consider the quotient. Hence we get a presheaf

$$(Sch/S)_\tau^{opp} \longrightarrow Sets, \quad T \longmapsto U(T)/\sim_T \tag{38.18.0.1}$$

Definition 38.18.1. Let τ, S, and the pre-relation $j : R \to U \times_S U$ be as above. In this setting the *quotient sheaf* U/R associated to j is the sheafification of the presheaf (38.18.0.1) in the τ-topology. If $j : R \to U \times_S U$ comes from the action of a group scheme G/S on U as in Lemma 38.14.1 then we sometimes denote the quotient sheaf U/G.

This means exactly that the diagram

$$h_R \rightrightarrows h_U \longrightarrow U/R$$

is a coequalizer diagram in the category of sheaves of sets on $(Sch/S)_\tau$. Using the Yoneda embedding we may view $(Sch/S)_\tau$ as a full subcategory of sheaves on $(Sch/S)_\tau$ and hence identify schemes with representable functors. Using this abuse of notation we will often depict the diagram above simply

$$R \overset{s}{\underset{t}{\rightrightarrows}} U \longrightarrow U/R$$

We will mostly work with the fppf topology when considering quotient sheaves of groupoids/equivalence relations.

Definition 38.18.2. In the situation of Definition 38.18.1. We say that the pre-relation j has a *representable quotient* if the sheaf U/R is representable. We will say a groupoid (U, R, s, t, c) has a *representable quotient* if the quotient U/R with $j = (t, s)$ is representable.

The following lemma characterizes schemes M representing the quotient. It applies for example if $\tau = fppf$, $U \to M$ is flat, of finite presentation and surjective, and $R \cong U \times_M U$.

Lemma 38.18.3. *In the situation of Definition 38.18.1. Assume there is a scheme M, and a morphism $U \to M$ such that*

1. *the morphism $U \to M$ equalizes s, t,*
2. *the morphism $U \to M$ induces a surjection of sheaves $h_U \to h_M$ in the τ-topology, and*
3. *the induced map $(t, s) : R \to U \times_M U$ induces a surjection of sheaves $h_R \to h_{U \times_M U}$ in the τ-topology.*

In this case M represents the quotient sheaf U/R.

Proof. Condition (1) says that $h_U \to h_M$ factors through U/R. Condition (2) says that $U/R \to h_M$ is surjective as a map of sheaves. Condition (3) says that $U/R \to h_M$ is injective as a map of sheaves. Hence the lemma follows. □

The following lemma is wrong if we do not require j to be a pre-equivalence relation (but just a pre-relation say).

Lemma 38.18.4. *Let $\tau \in \{Zariski, étale, fppf, smooth, syntomic\}$. Let S be a scheme. Let $j : R \to U \times_S U$ be a pre-equivalence relation over S. Assume U, R, S are objects of a τ-site Sch_τ. For $T \in \mathrm{Ob}((Sch/S)_\tau)$ and $a, b \in U(T)$ the following are equivalent:*

1. *a and b map to the same element of $(U/R)(T)$, and*
2. *there exists a τ-covering $\{f_i : T_i \to T\}$ of T and morphisms $r_i : T_i \to R$ such that $a \circ f_i = s \circ r_i$ and $b \circ f_i = t \circ r_i$.*

In other words, in this case the map of τ-sheaves

$$h_R \longrightarrow h_U \times_{U/R} h_U$$

is surjective.

Proof. Omitted. Hint: The reason this works is that the presheaf (38.18.0.1) in this case is really given by $T \mapsto U(T)/j(R(T))$ as $j(R(T)) \subset U(T) \times U(T)$ is an equivalence relation, see Definition 38.3.1. □

Lemma 38.18.5. *Let $\tau \in \{Zariski, étale, fppf, smooth, syntomic\}$. Let S be a scheme. Let $j : R \to U \times_S U$ be a pre-equivalence relation over S and $g : U' \to U$ a morphism of schemes over S. Let $j' : R' \to U' \times_S U'$ be the restriction of j to U'. Assume U, U', R, S are objects of a τ-site Sch_τ. The map of quotient sheaves*

$$U'/R' \longrightarrow U/R$$

is injective. If g defines a surjection $h_{U'} \to h_U$ of sheaves in the τ-topology (for example if $\{g : U' \to U\}$ is a τ-covering), then $U'/R' \to U/R$ is an isomorphism.

Proof. Suppose $\xi, \xi' \in (U'/R')(T)$ are sections which map to the same section of U/R. Then we can find a τ-covering $\mathcal{T} = \{T_i \to T\}$ of T such that $\xi|_{T_i}, \xi'|_{T_i}$ are given by $a_i, a_i' \in U'(T_i)$. By Lemma 38.18.4 and the axioms of a site we may after refining $\mathcal{T}$ assume there exist morphisms $r_i : T_i \to R$ such that $g \circ a_i = s \circ r_i$, $g \circ a_i' = t \circ r_i$. Since by construction $R' = R \times_{U \times_S U} (U' \times_S U')$ we see that $(r_i, (a_i, a_i')) \in R'(T_i)$ and this shows that a_i and a_i' define the same section of U'/R' over T_i. By the sheaf condition this implies $\xi = \xi'$.

If $h_{U'} \to h_U$ is a surjection of sheaves, then of course $U'/R' \to U/R$ is surjective also. If $\{g : U' \to U\}$ is a τ-covering, then the map of sheaves $h_{U'} \to h_U$ is surjective, see Sites, Lemma 7.13.4. Hence $U'/R' \to U/R$ is surjective also in this case. □

Lemma 38.18.6. *Let $\tau \in \{Zariski, étale, fppf, smooth, syntomic\}$. Let S be a scheme. Let (U, R, s, t, c) be a groupoid scheme over S. Let $g : U' \to U$ a morphism of schemes over S. Let (U', R', s', t', c') be the restriction of (U, R, s, t, c) to U'. Assume U, U', R, S are objects of a τ-site Sch_τ. The map of quotient sheaves*

$$U'/R' \longrightarrow U/R$$

is injective. If the composition

$$U' \times_{g,U,t} R \xrightarrow[pr_1]{} R \xrightarrow[s]{} U, \quad \text{with } h : R \to U$$

defines a surjection of sheaves in the τ-topology then the map is bijective. This holds for example if $\{h : U' \times_{g,U,t} R \to U\}$ is a τ-covering, or if $U' \to U$ defines a surjection of sheaves in the τ-topology, or if $\{g : U' \to U\}$ is a covering in the τ-topology.

Proof. Injectivity follows on combining Lemmas 38.11.2 and 38.18.5. To see surjectivity (see Sites, Section 7.12 for a characterization of surjective maps of sheaves) we argue as follows. Suppose that T is a scheme and $\sigma \in U/R(T)$. There exists a covering $\{T_i \to T\}$ such that $\sigma|_{T_i}$ is the image of some element $f_i \in U(T_i)$. Hence we may assume that σ if the image of $f \in U(T)$. By the assumption that h is a surjection of sheaves, we can find a τ-covering $\{\varphi_i : T_i \to T\}$ and morphisms $f_i : T_i \to U' \times_{g,U,t} R$ such that $f \circ \varphi_i = h \circ f_i$. Denote $f_i' = \mathrm{pr}_0 \circ f_i : T_i \to U'$. Then we see that $f_i' \in U'(T_i)$ maps to $g \circ f_i' \in U(T_i)$ and that $g \circ f_i' \sim_{T_i} h \circ f_i = f \circ \varphi_i$ notation as in (38.18.0.1). Namely, the element of $R(T_i)$ giving the relation is $\mathrm{pr}_1 \circ f_i$. This means that the restriction of σ to T_i is in the image of $U'/R'(T_i) \to U/R(T_i)$ as desired.

If $\{h\}$ is a τ-covering, then it induces a surjection of sheaves, see Sites, Lemma 7.13.4. If $U' \to U$ is surjective, then also h is surjective as s has a section (namely the neutral element e of the groupoid scheme). □

Lemma 38.18.7. *Let S be a scheme. Let $f : (U, R, j) \to (U', R', j')$ be a morphism between equivalence relations over S. Assume that*

$$\begin{array}{ccc} R & \xrightarrow[f]{} & R' \\ {\scriptstyle s}\downarrow & & \downarrow{\scriptstyle s'} \\ U & \xrightarrow{f} & U' \end{array}$$

is cartesian. For any $\tau \in \{Zariski, étale, fppf, smooth, syntomic\}$ the diagram

$$\begin{array}{ccc} U & \longrightarrow & U/R \\ \downarrow & & \downarrow{\scriptstyle f} \\ U' & \longrightarrow & U'/R' \end{array}$$

is a fibre product square of τ-sheaves.

Proof. By Lemma 38.18.4 the quotient sheaves have a simple description which we will use below without further mention. We first show that

$$U \longrightarrow U' \times_{U'/R'} U/R$$

is injective. Namely, assume $a, b \in U(T)$ map to the same element on the right hand side. Then $f(a) = f(b)$. After replacing T by the members of a τ-covering we may assume that there exists an $r \in R(T)$ such that $a = s(r)$ and $b = t(r)$. Then $r' = f(r)$ is a T-valued point of R' with $s'(r') = t'(r')$. Hence $r' = e'(f(a))$ (where e' is the identity of the groupoid scheme associated to j', see Lemma 38.11.3). Because the first diagram of the lemma is cartesian this implies that r has to equal $e(a)$. Thus $a = b$.

Finally, we show that the displayed arrow is surjective. Let T be a scheme over S and let $(a', \bar{b})$ be a section of the sheaf $U' \times_{U'/R'} U/R$ over T. After replacing T by the members of a τ-covering we may assume that $\bar{b}$ is the class of an element $b \in U(T)$. After replacing T by the members of a τ-covering we may assume that there exists an $r' \in R'(T)$ such that $a' = t(r')$ and $s'(r') = f(b)$. Because the first diagram of the lemma is cartesian we can find $r \in R(T)$ such that $s(r) = b$ and $f(r) = r'$. Then it is clear that $a = t(r) \in U(T)$ is a section which maps to $(a', \bar{b})$. □

38.19. Separation conditions

This really means conditions on the morphism $j : R \to U \times_S U$ when given a groupoid (U, R, s, t, c) over S. As in the previous section we first formulate the corresponding diagram.

Lemma 38.19.1. *Let S be a scheme. Let (U, R, s, t, c) be a groupoid over S. Let $G \to U$ be the stabilizer group scheme. The commutative diagram*

$$\begin{array}{ccccc} R & \xrightarrow{f \mapsto (f, s(f))} & R \times_{s,U} U & \longrightarrow & U \\ \downarrow{\scriptstyle \Delta_{R/U \times_S U}} & & \downarrow & & \downarrow \\ R \times_{(U \times_S U)} R & \xrightarrow{(f,g) \mapsto (f, f^{-1} \circ g)} & R \times_{s,U} G & \longrightarrow & G \end{array}$$

the two left horizontal arrows are isomorphisms and the right square is a fibre product square.

Proof. Omitted. Exercise in the definitions and the functorial point of view in algebraic geometry. □

Lemma 38.19.2. *Let S be a scheme. Let (U, R, s, t, c) be a groupoid over S. Let $G \to U$ be the stabilizer group scheme.*

(1) *The following are equivalent*
   (a) *$j : R \to U \times_S U$ is separated,*
   (b) *$G \to U$ is separated, and*
   (c) *$e : U \to G$ is a closed immersion.*
(2) *The following are equivalent*
   (a) *$j : R \to U \times_S U$ is quasi-separated,*
   (b) *$G \to U$ is quasi-separated, and*
   (c) *$e : U \to G$ is quasi-compact.*

Proof. The group scheme $G \to U$ is the base change of $R \to U \times_S U$ by the diagonal morphism $U \to U \times_S U$, see Lemma 38.15.1. Hence if j is separated (resp. quasi-separated), then $G \to U$ is separated (resp. quasi-separated). (See Schemes, Lemma 25.21.13). Thus (a) $\Rightarrow$ (b) in both (1) and (2).

If $G \to U$ is separated (resp. quasi-separated), then the morphism $U \to G$, as a section of the structure morphism $G \to U$ is a closed immersion (resp. quasi-compact), see Schemes, Lemma 25.21.12. Thus (b) $\Rightarrow$ (a) in both (1) and (2).

By the result of Lemma 38.19.1 (and Schemes, Lemmas 25.18.2 and 25.19.3) we see that if e is a closed immersion (resp. quasi-compact) $\Delta_{R/U \times_S U}$ is a closed immersion (resp. quasi-compact). Thus (c) $\Rightarrow$ (a) in both (1) and (2). □

38.20. Finite flat groupoids, affine case

Let S be a scheme. Let (U, R, s, t, c) be a groupoid scheme over S. Assume $U = \mathrm{Spec}(A)$, and $R = \mathrm{Spec}(B)$ are affine. In this case we get two ring maps $s^\sharp, t^\sharp : A \longrightarrow B$. Let C be the equalizer of $s^\sharp$ and $t^\sharp$. In a formula

$$C = \{a \in A \mid t^\sharp(a) = s^\sharp(a)\}. \tag{38.20.0.1}$$

We will sometimes call this the *ring of R-invariant functions on U*. What properties does $M = \mathrm{Spec}(C)$ have? The first observation is that the diagram

$$\begin{array}{ccc} R & \xrightarrow[s]{} & U \\ {\scriptstyle t}\downarrow & & \downarrow \\ U & \longrightarrow & M \end{array}$$

is commutative, i.e., the morphism $U \to M$ equalizes s, t. Moreover, if T is any affine scheme, and if $U \to T$ is a morphism which equalizes s, t, then $U \to T$ factors through $U \to M$. In other words, $U \to M$ is a coequalizer in the category of affine schemes.

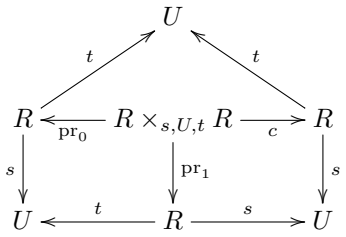
We would like to find conditions that guarantee the morphism $U \to M$ is really a "quotient" in the category of schemes. We will discuss this at length elsewhere (insert future reference here); here we just discuss some special cases. Namely, we will focus on the case where s, t are finite locally free.

Example 38.20.1. Let k be a field. Let $U = \mathrm{GL}_{2,k}$. Let $B \subset \mathrm{GL}_2$ be the closed subgroup scheme of upper triangular matrices. Then the quotient sheaf $\mathrm{GL}_{2,k}/B$ (in the Zariski, étale or fppf topology, see Definition 38.18.1) is representable by the projective line: $\mathbf{P}^1 = \mathrm{GL}_{2,k}/B$. (Details omitted.) On the other hand, the ring of invariant functions in this case is just k. Note that in this case the morphisms $s, t : R = \mathrm{GL}_{2,k} \times_k B \to \mathrm{GL}_{2,k} = U$ are smooth of relative dimension 3.

Recall that in Exercises, Exercises 79.15.6 and 79.15.7 we have defined the determinant and the norm for finitely locally free modules and finite locally free ring extensions. If $\varphi : A \to B$ is a finite locally free ring map, then we will denote $\mathrm{Norm}_\varphi(b) \in A$ the norm of $b \in B$.

Lemma 38.20.2. *Let S be a scheme. Let (U, R, s, t, c) be a groupoid scheme over S. Assume $U = \mathrm{Spec}(A)$, and $R = \mathrm{Spec}(B)$ are affine, and $s, t : R \to U$ finite locally free. Let C be as in (38.20.0.1). Let $f \in A$. Then $\mathrm{Norm}_{s^\sharp}(t^\sharp(f)) \in C$.*

Proof. Consider the commutative diagram



of Lemma 38.11.4. Think of $f \in \Gamma(U, \mathcal{O}_U)$. The commutativity of the top part of the diagram shows that $pr_0^\sharp(t^\sharp(f)) = c^\sharp(t^\sharp(f))$ as elements of $\Gamma(R \times_{S,U,t} R, \mathcal{O})$. Looking at the right lower cartesian square the compatibility of the norm construction with base change shows that $s^\sharp(\mathrm{Norm}_{s^\sharp}(t^\sharp(f))) = \mathrm{Norm}_{\mathrm{pr}_1}(c^\sharp(t^\sharp(f)))$. Similarly we get $t^\sharp(\mathrm{Norm}_{s^\sharp}(t^\sharp(f))) = \mathrm{Norm}_{\mathrm{pr}_1}(\mathrm{pr}_0^\sharp(t^\sharp(f)))$. Hence by the first equality of this proof we see that $s^\sharp(\mathrm{Norm}_{s^\sharp}(t^\sharp(f))) = t^\sharp(\mathrm{Norm}_{s^\sharp}(t^\sharp(f)))$ as desired. □

Lemma 38.20.3. *Let S be a scheme. Let (U, R, s, t, c) be a groupoid scheme over S. Assume $s, t : R \to U$ finite locally free. Then*

$$U = \coprod\nolimits_{r \geq 1} U_r$$

is a disjoint union of R-invariant opens such that the restriction R_r of R to U_r has the property that $s, t : R_r \to U_r$ are finite locally free of rank 1.

Proof. By Morphisms, Lemma 28.46.5 there exists a decomposition $U = \coprod_{r \geq 0} U_r$ such that $s : s^{-1}(U_r) \to U_r$ is finite locally free of rank r. As s is surjective we see that $U_0 = \emptyset$. Note that $u \in U_r \Leftrightarrow$ the scheme theoretic fibre $s^{-1}(u)$ has degree r over $\kappa(u)$. Now, if $z \in R$ with $s(z) = u$ and $t(z) = u'$ then $\mathrm{pr}_1^{-1}(z)$ see diagram of Lemma 38.11.4 is a scheme over $\kappa(z)$ which is the base change of both $s^{-1}(u)$ and $s^{-1}(u')$ via $\kappa(u) \to \kappa(z)$ and $\kappa(u') \to \kappa(z)$ by the properties of that diagram. Hence we see that the open subsets U_r are R-invariant. In particular the restriction of R to U_r is just $s^{-1}(U_r)$ and $s : R_r \to U_r$ is finite locally free of rank r. As $t : R_r \to U_r$ is isomorphic to s by the inverse of R_r we see that it has also rank r. □

Lemma 38.20.4. *Let S be a scheme. Let (U, R, s, t, c) be a groupoid scheme over S. Assume $U = \mathrm{Spec}(A)$, and $R = \mathrm{Spec}(B)$ are affine, and $s, t : R \to U$ finite locally free. Let $C \subset A$ be as in (38.20.0.1). Then A is integral over C.*

Proof. First, by Lemma 38.20.3 we know that (U, R, s, t, c) is a disjoint union of groupoid schemes (U_r, R_r, s, t, c) such that each $s, t : R_r \to U_r$ has constant rank r. As U is quasi-compact, we have $U_r = \emptyset$ for almost all r. It suffices to prove the lemma for each (U_r, R_r, s, t, c) and hence we may assume that s, t are finite locally free of rank r.

Assume that s, t are finite locally free of rank r. Let $f \in A$. Consider the element $x - f \in A[x]$, where we think of x as the coordinate on $\mathbf{A}^1$. Since

$$(U \times \mathbf{A}^1, R \times \mathbf{A}^1, s \times \mathrm{id}_{\mathbf{A}^1}, t \times \mathrm{id}_{\mathbf{A}^1}, c \times \mathrm{id}_{\mathbf{A}^1})$$

is also a groupoid scheme with finite source and target, we may apply Lemma 38.20.2 to it and we see that $P(x) = \mathrm{Norm}_{s^\sharp}(t^\sharp(x - f))$ is an element of $C[x]$.

Because $s^\sharp : A \to B$ is finite locally free of rank r we see that P is monic of degree r. Moreover $P(f) = 0$ by Cayley-Hamilton (Algebra, Lemma 10.15.1). $\square$

Lemma 38.20.5. *Let S be a scheme. Let (U, R, s, t, c) be a groupoid scheme over S. Assume $U = \mathrm{Spec}(A)$, and $R = \mathrm{Spec}(B)$ are affine, and $s, t : R \to U$ finite locally free. Let $C \subset A$ be as in (38.20.0.1). Let $C \to C'$ be a ring map, and set $U' = \mathrm{Spec}(A \otimes_C C')$, $R' = \mathrm{Spec}(B \otimes_C C')$. Then*

1. *the maps s, t, c induce maps s', t', c' such that (U', R', s', t', c') is a groupoid scheme, and*
2. *there is a canonical map $\varphi : C' \to C^1$ of C' into the R'-invariant functions C^1 on U' with the properties*
   (a) *for every $f \in C^1$ there exists an $n > 0$ such that f^n is in the image of φ, and*
   (b) *for every $f \in Ker(\varphi)$ there exists an $n > 0$ such that $f^n = 0$.*
3. *if $C \to C'$ is flat then φ is an isomorphism.*

Proof. The proof of part (1) is omitted. Let us denote $A' = A \otimes_C C'$ and $B' = B \otimes_C C'$. Then we have

$$C^1 = \{x \in A' \mid (t')^\sharp(x) = (s')^\sharp(x)\} = \{a \in A \otimes_C C' \mid t^\sharp \otimes 1(x) = s^\sharp \otimes 1(x)\}.$$

In other words, C^1 is the kernel of the difference map $(t^\sharp - s^\sharp) \otimes 1$ which is just the base change of the C-linear map $t^\sharp - s^\sharp : A \to B$ by $C \to C'$. Hence (3) follows.

Proof of part (2)(b). Since $C \to A$ is integral (Lemma 38.20.4) and injective we see that $\mathrm{Spec}(A) \to \mathrm{Spec}(C)$ is surjective, see Algebra, Lemma 10.33.15. Thus also $\mathrm{Spec}(A') \to \mathrm{Spec}(C')$ is surjective as a base change of a surjective morphism (Morphisms, Lemma 28.11.4). Hence $\mathrm{Spec}(C^1) \to \mathrm{Spec}(C')$ is surjective also. This implies that the kernel of φ is contained in the radical of the ring C', i.e., (2)(b) holds.

Proof of part (2)(a). By Lemma 38.20.3 we know that A is a finite product of rings A_r and B is a finite product of rings B_r such that the groupoid scheme decomposes accordingly (see the proof of Lemma 38.20.4). Then also C is a product of rings C_r and correspondingly C' decomposes as a product. Hence we may and do assume that the ring maps $s^\sharp, t^\sharp : A \to B$ are finite locally free of a fixed rank r. Let $f \in C^1 \subset A' = A \otimes_C C'$. We may replace C' by a finitely generated C-subalgebra of C' and hence we may assume that $C' = C[X_1, \ldots, X_n]/I$ for some ideal I. Choose a lift $\tilde{f} \in A \otimes_C C[X_i] = A[X_i]$ of the element f. Note that $f^r = \mathrm{Norm}_{(s')^\sharp}((t')^\sharp(f))$ in A as $t^\sharp(f) = s^\sharp(f)$. Hence we see that

$$h = \mathrm{Norm}_{s^\sharp \otimes 1}(t^\sharp \otimes 1(f)) \in A[X_i]$$

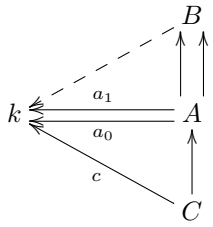
is invariant according to Lemma 38.20.2 and maps to f^r in A'. Since $C \to C[X_i]$ is flat we see from (3) that $h \in C[X_i]$. Hence it follows that f^r is in the image of φ. $\square$

Lemma 38.20.6. *Let S be a scheme. Let (U, R, s, t, c) be a groupoid scheme over S. Assume $U = \mathrm{Spec}(A)$, and $R = \mathrm{Spec}(B)$ are affine, and $s, t : R \to U$ finite locally free. Let $C \subset A$ be as in (38.20.0.1). Then $U \to M = \mathrm{Spec}(C)$ has the following properties:*

(1) *the map on points* $|U| \to |M|$ *is surjective and* $u_0, u_1 \in |U|$ *map to the same point if and only if there exists a* $r \in |R|$ *with* $t(r) = u_0$ *and* $s(r) = u_1$*, in a formula*
$$|M| = |U|/|R|$$
(2) *for any algebraically closed field* k *we have*
$$M(k) = U(k)/R(k)$$

Proof. Let k be an algebraically closed field. Since $C \to A$ is integral (Lemma 38.20.4) and injective we see that $\mathrm{Spec}(A) \to \mathrm{Spec}(C)$ is surjective, see Algebra, Lemma 10.33.15. Thus $|M| \to |U|$ is surjective. Let $C \to k$ be a ring map. Since surjective morphisms are preserved under base change (Morphisms, Lemma 28.11.4) we see that $A \otimes_C k$ is not zero. Now $k \subset A \otimes_C k$ is a nonzero integral extension. Hence any residue field of $A \otimes_C k$ is an algebraic extension of k, hence equal to k. Thus we see that $U(k) \to M(k)$ is surjective.

Let $a_0, a_1 : A \to k$ be ring maps. If there exists a ring map $b : B \to k$ such that $a_0 = b \circ t^\sharp$ and $a_1 = b \circ s^\sharp$ then we see that $a_0|_C = a_1|_C$ by definition. Conversely, suppose that $a_0|_C = a_1|_C$. Let us name this algebra map $c : C \to k$. Consider the diagram



We are trying to construct the dotted arrow, and if we do then part (2) follows, which in turn implies part (1). Since $A \to B$ is finite and faithfully flat there exist finitely many ring maps $b_1, \ldots, b_n : B \to k$ such that $b_i \circ s^\sharp = a_1$. If the dotted arrow does not exist, then we see that none of the $a_i' = b_i \circ t^\sharp$, $i = 1, \ldots, n$ is equal to a_0. Hence the maximal ideals
$$\mathfrak{m}_i' = \mathrm{Ker}(a_i' \otimes 1 : A \otimes_C k \to k)$$
of $A \otimes_C k$ are distinct from $\mathfrak{m} = \mathrm{Ker}(a_0 \otimes 1 : A \otimes_C k \to k)$. By Algebra, Lemma 10.14.3 we would get an element $f \in A \otimes_C k$ with $f \in \mathfrak{m}$, but $f \notin \mathfrak{m}_i'$ for $i = 1, \ldots, n$. Consider the norm
$$g = \mathrm{Norm}_{s^\sharp \otimes 1}(t^\sharp \otimes 1(f)) \in A \otimes_C k$$
By Lemma 38.20.2 this lies in the invariants $C^1 \subset A \otimes_C k$ of the base change groupoid (base change via the map $c : C \to k$). On the one hand, $a_1(g) \in k^*$ since the value of $t^\sharp(f)$ at all the points (which correspond to $b_1, \ldots, b_n$) lying over a_1 is invertible (insert future reference on property determinant here). On the other hand, since $f \in \mathfrak{m}$, we see that f is not a unit, hence $t^\sharp(f)$ is not a unit (as $t^\sharp \otimes 1$ is faithfully flat), hence its norm is not a unit (insert future reference on property determinant here). We conclude that C^1 contains an element which is not nilpotent and not a unit. We will now show that this leads to a contradiction. Namely, apply Lemma 38.20.5 to the map $c : C \to C' = k$, then we see that the map of k into the invariants C^1 is injective and moreover, that for any element $x \in C^1$ there exists an integer $n > 0$ such that $x^n \in k$. Hence every element of C^1 is either a unit or nilpotent. $\square$

Lemma 38.20.7. *Let S be a scheme. Let (U, R, s, t, c) be a groupoid scheme over S. Assume*

(1) *$U = \mathrm{Spec}(A)$, and $R = \mathrm{Spec}(B)$ are affine, and*
(2) *there exist elements $x_i \in A$, $i \in I$ such that $B = \bigoplus_{i \in I} s^\sharp(A)t^\sharp(x_i)$.*

Then $A = \bigoplus_{i \in I} Cx_i$, and $B \cong A \otimes_C A$ where $C \subset A$ is the R-invariant functions on U as in (38.20.0.1).

Proof. During this proof we will write $s, t : A \to B$ instead of $s^\sharp, t^\sharp$, and similarly $c : B \to B \otimes_{s,A,t} B$. We write $p_0 : B \to B \otimes_{s,A,t} B$, $b \mapsto b \otimes 1$ and $p_1 : B \to B \otimes_{s,A,t} B$, $b \mapsto 1 \otimes b$. By Lemma 38.11.5 and the definition of C we have the following commutative diagram

$$\begin{array}{ccccc}
B \otimes_{s,A,t} B & \overset{c}{\underset{p_0}{\leftleftarrows}} & B & \xleftarrow{t} & A \\
\uparrow{\scriptstyle p_1} & & \uparrow{\scriptstyle s} & & \uparrow \\
B & \overset{s}{\underset{t}{\leftleftarrows}} & A & \longleftarrow & C
\end{array}$$

Moreover the tow left squares are cocartesian in the category of rings, and the top row is isomorphic to the diagram

$$B \otimes_{t,A,t} B \overset{p_1}{\underset{p_0}{\leftleftarrows}} B \xleftarrow{t} A$$

which is an equalizer diagram according to Descent, Lemma 34.3.6 because condition (2) implies in particular that s (and hence also then isomorphic arrow t) is faithfully flat. The lower row is an equalizer diagram by definition of C. We can use the x_i and get a commutative diagram

$$\begin{array}{ccccc}
B \otimes_{s,A,t} B & \overset{c}{\underset{p_0}{\leftleftarrows}} & B & \xleftarrow{t} & A \\
\uparrow{\scriptstyle p_1} & & \uparrow{\scriptstyle s} & & \uparrow \\
\bigoplus_{i \in I} Bx_i & \overset{s}{\underset{t}{\leftleftarrows}} & \bigoplus_{i \in I} Ax_i & \longleftarrow & \bigoplus_{i \in I} Cx_i
\end{array}$$

where in the right vertical arrow we map x_i to x_i, in the middle vertical arrow we map x_i to $t(x_i)$ and in the left vertical arrow we map x_i to $c(t(x_i)) = t(x_i) \otimes 1 = p_0(t(x_i))$ (equality by the commutativity of the top part of the diagram in Lemma 38.11.4). Then the diagram commutes. Moreover the middle vertical arrow is an isomorphism by assumption. Since the left two squares are cocartesian we conclude that also the left vertical arrow is an isomorphism. On the other hand, the horizontal rows are exact (i.e., they are equalizers). Hence we conclude that also the right vertical arrow is an isomorphism. □

Proposition 38.20.8. *Let S be a scheme. Let (U, R, s, t, c) be a groupoid scheme over S. Assume*

(1) *$U = \mathrm{Spec}(A)$, and $R = \mathrm{Spec}(B)$ are affine,*
(2) *$s, t : R \to U$ finite locally free, and*
(3) *$j = (t, s)$ is an equivalence.*

In this case, let $C \subset A$ be as in (38.20.0.1). Then $U \to M = \mathrm{Spec}(C)$ is finite locally free and $R = U \times_M U$. Moreover, M represents the quotient sheaf U/R in the fppf topology (see Definition 38.18.1).

Proof. During this proof we use the notation $s, t : A \to B$ instead of the notation $s^\sharp, t^\sharp$. By Lemma 38.18.3 it suffices to show that $C \to A$ is finite locally free and that the map

$$t \otimes s : A \otimes_C A \longrightarrow B$$

is an isomorphism. First, note that j is a monomorphism, and also finite (since already s and t are finite). Hence we see that j is a closed immersion by Morphisms, Lemma 28.44.13. Hence $A \otimes_C A \to B$ is surjective.

We will perform base change by flat ring maps $C \to C'$ as in Lemma 38.20.5, and we will use that formation of invariants commutes with flat base change, see part (3) of the lemma cited. We will show below that for every prime $\mathfrak{p} \subset C$, there exists a local flat ring map $C_\mathfrak{p} \to C'_\mathfrak{p}$ such that the result holds after a base change to $C'_\mathfrak{p}$. This implies immediately that $A \otimes_C A \to B$ is injective (use Algebra, Lemma 10.22.1). It also implies that $C \to A$ is flat, by combining Algebra, Lemmas 10.36.16, 10.36.19, and 10.36.7. Then since $U \to \mathrm{Spec}(C)$ is surjective also (Lemma 38.20.6) we conclude that $C \to A$ is faithfully flat. Then the isomorphism $B \cong A \otimes_C A$ implies that A is a finitely presented C-module, see Algebra, Lemma 10.78.2. Hence A is finite locally free over C, see Algebra, Lemma 10.73.2.

By Lemma 38.20.3 we know that A is a finite product of rings A_r and B is a finite product of rings B_r such that the groupoid scheme decomposes accordingly (see the proof of Lemma 38.20.4). Then also C is a product of rings C_r and correspondingly C' decomposes as a product. Hence we may and do assume that the ring maps $s, t : A \to B$ are finite locally free of a fixed rank r.

The local ring maps $C_\mathfrak{p} \to C'_\mathfrak{p}$ we are going to use are any local flat ring maps such that the residue field of $C'_\mathfrak{p}$ is infinite. By Algebra, Lemma 10.146.1 such local ring maps exist.

Assume C is a local ring with maximal ideal $\mathfrak{m}$ and infinite residue field, and assume that $s, t : A \to B$ is finite locally free of constant rank $r > 0$. Since $C \subset A$ is integral (Lemma 38.20.4) all primes lying over $\mathfrak{m}$ are maximal, and all maximal ideals of A lie over $\mathfrak{m}$. Similarly for $C \subset B$. Pick a maximal ideal $\mathfrak{m}'$ of A lying over $\mathfrak{m}$ (exists by Lemma 38.20.6). Since $t : A \to B$ is finite locally free there exist at most finitely many maximal ideals of B lying over $\mathfrak{m}'$. Hence we conclude (by Lemma 38.20.6 again) that A has finitely many maximal ideals, i.e., A is semi-local. This in turn implies that B is semi-local as well. OK, and now, because $t \otimes s : A \otimes_C A \to B$ is surjective, we can apply Algebra, Lemma 10.73.7 to the ring map $C \to A$, the A-module $M = B$ (seen as an A-module via t) and the C-submodule $s(A) \subset B$. This lemma implies that there exist $x_1, \ldots, x_r \in A$ such that M is free over A on the basis $s(x_1), \ldots, s(x_r)$. Hence we conclude that $C \to A$ is finite free and $B \cong A \otimes_C A$ by applying Lemma 38.20.7. □

38.21. Finite flat groupoids

In this section we prove a lemma that will help to show that the quotient of a scheme by a finite flat equivalence relation is a scheme, provided that each equivalence class is contained in an affine. See Properties of Spaces, Proposition 48.11.1.

Lemma 38.21.1. *Let S be a scheme. Let (U, R, s, t, c) be a groupoid scheme over S. Assume s, t are finite locally free. Let $u \in U$ be a point such that $t(s^{-1}(\{u\}))$*

is contained in an affine open of U. Then there exists an R-invariant affine open neighbourhood of u in U.

Proof. Since s is finite locally free it has finite fibres. Hence $t(s^{-1}(\{u\})) = \{u_1, \ldots, u_n\}$ is a finite set. Note that $u \in \{u_1, \ldots, u_n\}$. Let $W \subset U$ be an affine open containing $\{u_1, \ldots, u_n\}$, in particular $u \in W$. Consider $Z = R \setminus s^{-1}(W) \cap t^{-1}(W)$. This is a closed subset of R. The image $t(Z)$ is a closed subset of U which can be loosely described as the set of points of U which are R-equivalent to a point of $U \setminus W$. Hence $W' = U \setminus t(Z)$ is an R-invariant, open subscheme of U contained in W, and $\{u_1, \ldots, u_n\} \subset W'$. Picture

$$\{u_1, \ldots, u_n\} \subset W' \subset W \subset U.$$

Let $f \in \Gamma(W, \mathcal{O}_W)$ be an element such that $\{u_1, \ldots, u_n\} \subset D(f) \subset W'$. Such an f exists by Algebra, Lemma 10.14.3. By our choice of W' we have $s^{-1}(W') \subset t^{-1}(W)$, and hence we get a diagram

$$\begin{array}{ccc} s^{-1}(W') & \xrightarrow{t} & W \\ {\scriptstyle s}\downarrow & & \\ W' & & \end{array}$$

The vertical arrow is finite locally free by assumption. Set

$$g = \mathrm{Norm}_s(t^\sharp f) \in \Gamma(W', \mathcal{O}_{W'})$$

By construction g is a function on W' which is nonzero in u, as $t^\sharp(f)$ is nonzero in each of the points of R lying over u, since f is nonzero in $u_1, \ldots, u_n$. Similarly, $D(g) \subset W'$ is equal to the set of points w such that f is not zero in any of the points equivalent to w. This means that $D(g)$ is an R-invariant affine open of W'. The final picture is

$$\{u_1, \ldots, u_n\} \subset D(g) \subset D(f) \subset W' \subset W \subset U$$

and hence we win. □

38.22. Other chapters

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CHAPTER 39

More on Groupoid Schemes

39.1. Introduction

This chapter is devoted to advanced topics on groupoid schemes. Even though the results are stated in terms of groupoid schemes, the reader should keep in mind the 2-cartesian diagram

$$\begin{array}{ccc} R & \longrightarrow & U \\ \downarrow & & \downarrow \\ U & \longrightarrow & [U/R] \end{array} \tag{39.1.0.1}$$

where $[U/R]$ is the quotient stack, see Groupoids in Spaces, Remark 59.19.4. Many of the results are motivated by thinking about this diagram. See for example the beautiful paper [**KM97**] by Keel and Mori.

39.2. Notation

We continue to abide by the conventions and notation introduced in Groupoids, Section 38.2.

39.3. Useful diagrams

We briefly restate the results of Groupoids, Lemmas 38.11.4 and 38.11.5 for easy reference in this chapter. Let S be a scheme. Let (U, R, s, t, c) be a groupoid scheme over S. In the commutative diagram

$$\begin{array}{ccccc} & & U & & \\ & \overset{t}{\nearrow} & & \overset{t}{\nwarrow} & \\ R & \xleftarrow[\mathrm{pr}_0]{} & R \times_{s,U,t} R & \xrightarrow[c]{} & R \\ \downarrow{\scriptstyle s} & & \downarrow{\scriptstyle \mathrm{pr}_1} & & \downarrow{\scriptstyle s} \\ U & \xleftarrow{t} & R & \xrightarrow{s} & U \end{array} \tag{39.3.0.2}$$

the two lower squares are fibre product squares. Moreover, the triangle on top (which is really a square) is also cartesian.

The diagram

$$
\begin{array}{ccccc}
 & & R \times_{t,U,t} R & \xrightarrow[\mathrm{pr}_0]{\mathrm{pr}_1} & R & \xrightarrow{t} & U \\
 & & \downarrow{\scriptstyle \mathrm{pr}_0 \times c \circ (i,1)} & & \downarrow{\scriptstyle \mathrm{id}_R} & & \downarrow{\scriptstyle \mathrm{id}_U} \\
 & & R \times_{s,U,t} R & \xrightarrow[\mathrm{pr}_0]{c} & R & \xrightarrow{t} & U \\
 & & \downarrow{\scriptstyle \mathrm{pr}_1} & & \downarrow{\scriptstyle s} & & \\
 & & R & \xrightarrow[t]{s} & U & &
\end{array}
\tag{39.3.0.3}
$$

is commutative. The two top rows are isomorphic via the vertical maps given. The two lower left squares are cartesian.

39.4. Sheaf of differentials

The following lemma is the analogue of Groupoids, Lemma 38.6.3.

Lemma 39.4.1. *Let S be a scheme. Let (U, R, s, t, c) be a groupoid scheme over S. The sheaf of differentials of R seen as a scheme over U via t is a quotient of the pullback via t of the conormal sheaf of the immersion $e : U \to R$. In a formula: there is a canonical surjection $t^*\mathcal{C}_{U/R} \to \Omega_{R/U}$. If s is flat, then this map is an isomorphism.*

Proof. Note that $e : U \to R$ is an immersion as it is a section of the morphism s, see Schemes, Lemma 25.21.12. Consider the following diagram

$$
\begin{array}{ccccc}
R & \xrightarrow[(1,i)]{} & R \times_{s,U,t} R & \xrightarrow[(\mathrm{pr}_0, i \circ \mathrm{pr}_1)]{} & R \times_{t,U,t} R \\
\downarrow{\scriptstyle t} & & \downarrow{\scriptstyle c} & & \\
U & \xrightarrow{e} & R & &
\end{array}
$$

The square on the left is cartesian, because if $a \circ b = e$, then $b = i(a)$. The composition of the horizontal maps is the diagonal morphism of $t : R \to U$. The right top horizontal arrow is an isomorphism. Hence since $\Omega_{R/U}$ is the conormal sheaf of the composition it is isomorphic to the conormal sheaf of $(1, i)$. By Morphisms, Lemma 28.33.4 we get the surjection $t^*\mathcal{C}_{U/R} \to \Omega_{R/U}$ and if c is flat, then this is an isomorphism. Since c is a base change of s by the properties of Diagram (39.3.0.3) we conclude that if s is flat, then c is flat, see Morphisms, Lemma 28.26.7. □

39.5. Properties of groupoids

Let (U, R, s, t, c) be a groupoid scheme. The idea behind the results in this section is that $s : R \to U$ is a base changes of the morphism $U \to [U/R]$ (see Diagram (39.1.0.1). Hence the local properties of $s : R \to U$ should reflect local properties of the morphism $U \to [U/R]$. This doesn't work, because $[U/R]$ is not always an algebraic stack, and hence we cannot speak of geometric or algebraic properties of $U \to [U/R]$. But it turns out that we can make some of it work without even referring to the quotient stack at all.

Here is a first example of such a result. The open $W \subset U'$ found in the lemma is roughly speaking the locus where the morphism $U' \to [U/R]$ has property $\mathcal{P}$.

Lemma 39.5.1. *Let S be a scheme. Let (U, R, s, t, c, e, i) be a groupoid over S. Let $g : U' \to U$ be a morphism of schemes. Denote h the composition*

$$h : U' \times_{g,U,t} R \xrightarrow[pr_1]{} R \xrightarrow[s]{} U.$$

Let $\mathcal{P}, \mathcal{Q}, \mathcal{R}$ be properties of morphisms of schemes. Assume

1. *$\mathcal{R} \Rightarrow \mathcal{Q}$,*
2. *$\mathcal{Q}$ is preserved under base change and composition,*
3. *for any morphism $f : X \to Y$ which has $\mathcal{Q}$ there exists a largest open $W(\mathcal{P}, f) \subset X$ such that $f|_{W(\mathcal{P},f)}$ has $\mathcal{P}$, and*
4. *for any morphism $f : X \to Y$ which has $\mathcal{Q}$, and any morphism $Y' \to Y$ which has $\mathcal{R}$ we have $Y' \times_Y W(\mathcal{P}, f) = W(\mathcal{P}, f')$, where $f' : X_{Y'} \to Y'$ is the base change of f.*

If s, t have $\mathcal{R}$ and g has $\mathcal{Q}$, then there exists an open subscheme $W \subset U'$ such that $W \times_{g,U,t} R = W(\mathcal{P}, h)$.

Proof. Note that the following diagram is commutative

$$\begin{array}{ccc} U' \times_{g,U,t} R \times_{t,U,t} R & \xrightarrow[\mathrm{pr}_{12}]{} & R \times_{t,U,t} R \\ {\scriptstyle \mathrm{pr}_{01}}\downarrow\downarrow{\scriptstyle \mathrm{pr}_{02}} & & {\scriptstyle \mathrm{pr}_0}\downarrow\downarrow{\scriptstyle \mathrm{pr}_1} \\ U' \times_{g,U,t} R & \xrightarrow{\mathrm{pr}_1} & R \end{array}$$

with both squares cartesian (this uses that the two maps $t \circ \mathrm{pr}_i : R \times_{t,U,t} R \to U$ are equal). Combining this with the properties of diagram (39.3.0.3) we get a commutative diagram

$$\begin{array}{ccc} U' \times_{g,U,t} R \times_{t,U,t} R & \xrightarrow[c\circ(i,1)]{} & R \\ {\scriptstyle \mathrm{pr}_{01}}\downarrow\downarrow{\scriptstyle \mathrm{pr}_{02}} & & {\scriptstyle t}\downarrow\downarrow{\scriptstyle s} \\ U' \times_{g,U,t} R & \xrightarrow{h} & U \end{array}$$

where both squares are cartesian.

Assume s, t have $\mathcal{R}$ and g has $\mathcal{Q}$. Then h has $\mathcal{Q}$ as a composition of s (which has $\mathcal{R}$ hence $\mathcal{Q}$) and a base change of g (which has $\mathcal{Q}$). Thus $W(\mathcal{P}, h) \subset U' \times_{g,U,t} R$ exists. By our assumptions we have $\mathrm{pr}_{01}^{-1}(W(\mathcal{P}, h)) = \mathrm{pr}_{02}^{-1}(W(\mathcal{P}, h))$ since both are the largest open on which $c \circ (i, 1)$ has $\mathcal{P}$. Note that the projection $U' \times_{g,U,t} R \to U'$ has a section, namely $\sigma : U' \to U' \times_{g,U,t} R$, $u' \mapsto (u', e(g(u')))$. Also via the isomorphism

$$(U' \times_{g,U,t} R) \times_{U'} (U' \times_{g,U,t} R) = U' \times_{g,U,t} R \times_{t,U,t} R$$

the two projections of the left hand side to $U' \times_{g,U,t} R$ agree with the morphisms pr_{01} and pr_{02} on the right hand side. Since $\mathrm{pr}_{01}^{-1}(W(\mathcal{P}, h)) = \mathrm{pr}_{02}^{-1}(W(\mathcal{P}, h))$ we conclude that $W(\mathcal{P}, h)$ is the inverse image of a subset of U, which is necessarily the open set $W = \sigma^{-1}(W(\mathcal{P}, h))$. □

Remark 39.5.2. Warning: Lemma 39.5.1 should be used with care. For example, it applies to $\mathcal{P}$ ="flat", $\mathcal{Q}$ ="empty", and $\mathcal{R}$ ="flat and locally of finite presentation". But given a morphism of schemes $f : X \to Y$ the largest open $W \subset X$ such that $f|_W$ is flat is *not* the set of points where f is flat!

Remark 39.5.3. Notwithstanding the warning in Remark 39.5.2 there are some cases where Lemma 39.5.1 can be used without causing too much ambiguity. We give a list. In each case we omit the verification of assumptions (1) and (2) and we give references which imply (3) and (4). Here is the list:

(1) $\mathcal{Q} = \mathcal{R} =$"locally of finite type", and $\mathcal{P} =$"relative dimension $\leq d$". See Morphisms, Definition 28.30.1 and Morphisms, Lemmas 28.29.4 and 28.29.3.
(2) $\mathcal{Q} = \mathcal{R} =$"locally of finite type", and $\mathcal{P} =$"locally quasi-finite". This is the case $d = 0$ of the previous item, see Morphisms, Lemma 28.30.5.
(3) $\mathcal{Q} = \mathcal{R} =$"locally of finite type", and $\mathcal{P} =$"unramified". See Morphisms, Lemmas 28.36.3 and 28.36.15.

What is interesting about the cases listed above is that we do not need to assume that s, t are flat to get a conclusion about the locus where the morphism h has property $\mathcal{P}$. We continue the list:

(4) $\mathcal{Q} =$"locally of finite presentation", $\mathcal{R} =$"flat and locally of finite presentation", and $\mathcal{P} =$"flat". See More on Morphisms, Theorem 36.12.1 and Lemma 36.12.2.
(5) $\mathcal{Q} =$"locally of finite presentation", $\mathcal{R} =$"flat and locally of finite presentation", and $\mathcal{P} =$"Cohen-Macaulay". See More on Morphisms, Definition 36.17.1 and More on Morphisms, Lemmas 36.17.3 and 36.17.4.
(6) $\mathcal{Q} =$"locally of finite presentation", $\mathcal{R} =$"flat and locally of finite presentation", and $\mathcal{P} =$"syntomic" use Morphisms, Lemma 28.32.12 (the locus is automatically open).
(7) $\mathcal{Q} =$"locally of finite presentation", $\mathcal{R} =$"flat and locally of finite presentation", and $\mathcal{P} =$"smooth". See Morphisms, Lemma 28.35.15 (the locus is automatically open).
(8) $\mathcal{Q} =$"locally of finite presentation", $\mathcal{R} =$"flat and locally of finite presentation", and $\mathcal{P} =$"étale". See Morphisms, Lemma 28.37.17 (the locus is automatically open).

Here is the second result. The R-invariant open $W \subset U$ should be thought of as the inverse image of the largest open of $[U/R]$ over which the morphism $U \to [U/R]$ has property $\mathcal{P}$.

Lemma 39.5.4. *Let S be a scheme. Let (U, R, s, t, c) be a groupoid over S. Let $\tau \in \{Zariski, fppf, étale, smooth, syntomic\}$[1]. Let $\mathcal{P}$ be a property of morphisms of schemes which is τ-local on the target (Descent, Definition 34.18.1). Assume $\{s : R \to U\}$ and $\{t : R \to U\}$ are coverings for the τ-topology. Let $W \subset U$ be the maximal open subscheme such that $s|_{s^{-1}(W)} : s^{-1}(W) \to W$ has property $\mathcal{P}$. Then W is R-invariant, see Groupoids, Definition 38.17.1.*

Proof. The existence and properties of the open $W \subset U$ are described in Descent, Lemma 34.18.3. In Diagram (39.3.0.2) let $W_1 \subset R$ be the maximal open subscheme over which the morphism $\mathrm{pr}_1 : R \times_{s,U,t} R \to R$ has property $\mathcal{P}$. It follows from the aforementioned Descent, Lemma 34.18.3 and the assumption that $\{s : R \to U\}$ and $\{t : R \to U\}$ are coverings for the τ-topology that $t^{-1}(W) = W_1 = s^{-1}(W)$ as desired. □

[1]The fact that *fpqc* is missing is not a typo.

Lemma 39.5.5. *Let S be a scheme. Let (U, R, s, t, c) be a groupoid over S. Let $G \to U$ be its stabilizer group scheme. Let $\tau \in \{fppf, étale, smooth, syntomic\}$. Let $\mathcal{P}$ be a property of morphisms which is τ-local on the target. Assume $\{s : R \to U\}$ and $\{t : R \to U\}$ are coverings for the τ-topology. Let $W \subset U$ be the maximal open subscheme such that $G_W \to W$ has property $\mathcal{P}$. Then W is R-invariant (see Groupoids, Definition 38.17.1).*

Proof. The existence and properties of the open $W \subset U$ are described in Descent, Lemma 34.18.3. The morphism

$$G \times_{U,t} R \longrightarrow R \times_{s,U} G, \quad (g, r) \longmapsto (r, r^{-1} \circ g \circ r)$$

is an isomorphism over R (where $\circ$ denotes composition in the groupoid). Hence $s^{-1}(W) = t^{-1}(W)$ by the properties of W proved in the aforementioned Descent, Lemma 34.18.3. □

39.6. Comparing fibres

Let (U, R, s, t, c, e, i) be a groupoid scheme over S. Diagram (39.3.0.2) gives us a way to compare the fibres of the map $s : R \to U$ in a groupoid. For a point $u \in U$ we will denote $F_u = s^{-1}(u)$ the scheme theoretic fibre of $s : R \to U$ over u. For example the diagram implies that if $u, u' \in U$ are points such that $s(r) = u$ and $t(r) = u'$, then $(F_u)_{\kappa(r)} \cong (F_{u'})_{\kappa(r)}$. This is a special case of the more general and more precise Lemma 39.6.1 below. To see this take $r' = i(r)$.

A pair (X, x) consisting of a scheme X and a point $x \in X$ is sometimes called the *germ of X at x*. A *morphism of germs* $f : (X, x) \to (S, s)$ is a morphism $f : U \to S$ defined on an open neighbourhood of x with $f(x) = s$. Two such f, f' are said to give the same morphism of germs if and only if f and f' agree in some open neighbourhood of x. Let $\tau \in \{Zariski, étale, smooth, syntomic, fppf\}$. We temporarily introduce the following concept: We say that two morphisms of germs $f : (X, x) \to (S, s)$ and $f' : (X', x') \to (S', s')$ are *isomorphic locally on the base in the τ-topology*, if there exists a pointed scheme (S'', s'') and morphisms of germs $g : (S'', s'') \to (S, s)$, and $g' : (S'', s'') \to (S', s')$ such that

(1) g and g' are an open immersion (resp. étale, smooth, syntomic, flat and locally of finite presentation) at s'',
(2) there exists an isomorphism

$$(S'' \times_{g,S,f} X, \tilde{x}) \cong (S'' \times_{g',S',f'} X', \tilde{x}')$$

of germs over the germ (S'', s'') for some choice of points $\tilde{x}$ and $\tilde{x}'$ lying over (s'', x) and (s'', x').

Finally, we simply say that the maps of germs $f : (X, x) \to (S, s)$ and $f' : (X', x') \to (S', s')$ are *flat locally on the base isomorphic* if there exist S'', s'', g, g' as above but with (1) replaced by the condition that g and g' are flat at s'' (this is much weaker than any of the τ conditions above as a flat morphism need not be open).

Lemma 39.6.1. *Let S be a scheme. Let (U, R, s, t, c) be a groupoid over S. Let $r, r' \in R$ with $t(r) = t(r')$ in U. Set $u = s(r)$, $u' = s(r')$. Denote $F_u = s^{-1}(u)$ and $F_{u'} = s^{-1}(u')$ the scheme theoretic fibres.*

(1) *There exists a common field extension $\kappa(u) \subset k$, $\kappa(u') \subset k$ and an isomorphism $(F_u)_k \cong (F_{u'})_k$.*

(2) *We may choose the isomorphism of (1) such that a point lying over r maps to a point lying over r'.*
(3) *If the morphisms s, t are flat then the morphisms of germs $s : (R, r) \to (U, u)$ and $s : (R, r') \to (U, u')$ are flat locally on the base isomorphic.*
(4) *If the morphisms s, t are étale (resp. smooth, syntomic, or flat and locally of finite presentation) then the morphisms of germs $s : (R, r) \to (U, u)$ and $s : (R, r') \to (U, u')$ are locally on the base isomorphic in the étale (resp. smooth, syntomic, or fppf) topology.*

Proof. We repeatedly use the properties and the existence of diagram (39.3.0.2). By the properties of the diagram (and Schemes, Lemma 25.17.5) there exists a point ξ of $R \times_{s,U,t} R$ with $\mathrm{pr}_0(\xi) = r$ and $c(\xi) = r'$. Let $\tilde{r} = \mathrm{pr}_1(\xi) \in R$.

Proof of (1). Set $k = \kappa(\tilde{r})$. Since $t(\tilde{r}) = u$ and $s(\tilde{r}) = u'$ we see that k is a common extension of both $\kappa(u)$ and $\kappa(u')$ and in fact that both $(F_u)_k$ and $(F_{u'})_k$ are isomorphic to the fibre of $\mathrm{pr}_1 : R \times_{s,U,t} R \to R$ over $\tilde{r}$. Hence (1) is proved.

Part (2) follows since the point ξ maps to r, resp. r'.

Part (3) is clear from the above (using the point ξ for $\tilde{u}$ and $\tilde{u}'$) and the definitions.

If s and t are flat and of finite presentation, then they are open morphisms (Morphisms, Lemma 28.26.9). Hence the image of some affine open neighbourhood V'' of $\tilde{r}$ will cover an open neighbourhood V of u, resp. V' of u'. These can be used to show that properties (1) and (2) of the definition of "locally on the base isomorphic in the τ-topology". □

39.7. Cohen-Macaulay presentations

Given any groupoid (U, R, s, t, c) with s, t flat and locally of finite presentation there exists an "equivalent" groupoid (U', R', s', t', c') such that s' and t' are Cohen-Macaulay morphisms (and locally of finite presentation). See More on Morphisms, Section 36.17 for more information on Cohen-Macaulay morphisms. Here "equivalent" can be taken to mean that the quotient stacks $[U/R]$ and $[U'/R']$ are equivalent stacks, see Groupoids in Spaces, Section 59.19 and Section 59.24.

Lemma 39.7.1. *Let S be a scheme. Let (U, R, s, t, c) be a groupoid over S. Assume s and t are flat and locally of finite presentation. Then there exists an open $U' \subset U$ such that*

(1) *$t^{-1}(U') \subset R$ is the largest open subscheme of R on which the morphism s is Cohen-Macaulay,*
(2) *$s^{-1}(U') \subset R$ is the largest open subscheme of R on which the morphism t is Cohen-Macaulay,*
(3) *the morphism $t|_{s^{-1}(U')} : s^{-1}(U') \to U$ is surjective,*
(4) *the morphism $s|_{t^{-1}(U')} : t^{-1}(U') \to U$ is surjective, and*
(5) *the restriction $R' = s^{-1}(U') \cap t^{-1}(U')$ of R to U' defines a groupoid (U', R', s', t', c') which has the property that the morphisms s' and t' are Cohen-Macaulay and locally of finite presentation.*

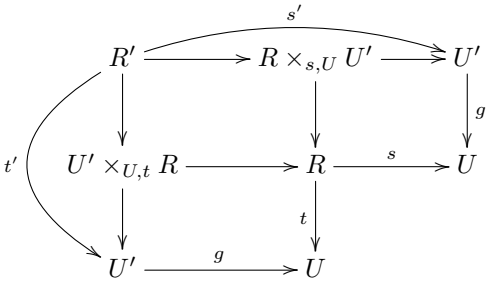
Proof. Apply Lemma 39.5.1 with $g = \mathrm{id}$ and $\mathcal{Q}$ ="locally of finite presentation", $\mathcal{R}$ ="flat and locally of finite presentation", and $\mathcal{P}$ ="Cohen-Macaulay", see Remark 39.5.3. This gives us an open $U' \subset U$ such that Let $t^{-1}(U') \subset R$ is the largest open subscheme of R on which the morphism s is Cohen-Macaulay. This proves

(1). Let $i : R \to R$ be the inverse of the groupoid. Since i is an isomorphism, and $s \circ i = t$ and $t \circ i = s$ we see that $s^{-1}(U')$ is also the largest open of R on which t is Cohen-Macaulay. This proves (2). By More on Morphisms, Lemma 36.17.4 the open subset $t^{-1}(U')$ is dense in every fibre of $s : R \to U$. This proves (3). Same argument for (4). Part (5) is a formal consequence of (1) and (2) and the discussion of restrictions in Groupoids, Section 38.16. □

39.8. Restricting groupoids

In this section we collect a bunch of lemmas on properties of groupoids which are inherited by restrictions. Most of these lemmas can be proved by contemplating the defining diagram

(39.8.0.1)



of a restriction. See Groupoids, Lemma 38.16.1.

Lemma 39.8.1. *Let S be a scheme. Let (U, R, s, t, c) be a groupoid scheme over S. Let $g : U' \to U$ be a morphism of schemes. Let (U', R', s', t', c') be the restriction of (U, R, s, t, c) via g.*

1. *If s, t are locally of finite type and g is locally of finite type, then s', t' are locally of finite type.*
2. *If s, t are locally of finite presentation and g is locally of finite presentation, then s', t' are locally of finite presentation.*
3. *If s, t are flat and g is flat, then s', t' are flat.*
4. *Add more here.*

Proof. The property of being locally of finite type is stable under composition and arbitrary base change, see Morphisms, Lemmas 28.16.3 and 28.16.4. Hence (1) is clear from Diagram (39.8.0.1). For the other cases, see Morphisms, Lemmas 28.22.3, 28.22.4, 28.26.5, and 28.26.7. □

The following lemma could have been used to prove the results of the preceding lemma in a more uniform way.

Lemma 39.8.2. *Let S be a scheme. Let (U, R, s, t, c) be a groupoid scheme over S. Let $g : U' \to U$ be a morphism of schemes. Let (U', R', s', t', c') be the restriction of (U, R, s, t, c) via g, and let $h = s \circ pr_1 : U' \times_{g,U,t} R \to U$. If $\mathcal{P}$ is a property of morphisms of schemes such that*

1. *h has property $\mathcal{P}$, and*
2. *$\mathcal{P}$ is preserved under base change,*

then s', t' have property $\mathcal{P}$.

Proof. This is clear as s' is the base change of h by Diagram (39.8.0.1) and t' is isomorphic to s' as a morphism of schemes. □

Lemma 39.8.3. *Let S be a scheme. Let (U, R, s, t, c) be a groupoid scheme over S. Let $g : U' \to U$ and $g' : U'' \to U'$ be morphisms of schemes. Set $g'' = g \circ g'$. Let (U', R', s', t', c') be the restriction of R to U'. Let $h = s \circ pr_1 : U' \times_{g,U,t} R \to U$, let $h' = s' \circ pr_1 : U'' \times_{g',U',t} R \to U'$, and let $h'' = s \circ pr_1 : U'' \times_{g'',U,t} R \to U$. The following diagram is commutative*

$$\begin{array}{ccccc} U'' \times_{g',U',t} R' & \longleftarrow & (U' \times_{g,U,t} R) \times_U (U'' \times_{g'',U,t} R) & \longrightarrow & U'' \times_{g'',U,t} R \\ \downarrow{\scriptstyle h'} & & \downarrow & & \downarrow{\scriptstyle h''} \\ U' & \xleftarrow{pr_0} & U' \times_{g,U,t} R & \xrightarrow{h} & U \end{array}$$

with both squares cartesian where the left upper horizontal arrow is given by the rule

$$\begin{array}{ccc} (U' \times_{g,U,t} R) \times_U (U'' \times_{g'',U,t} R) & \longrightarrow & U'' \times_{g',U',t} R' \\ ((u', r_0), (u'', r_1)) & \longmapsto & (u'', (c(r_1, i(r_0)), (g'(u''), u'))) \end{array}$$

with notation as explained in the proof.

Proof. We work this out by exploiting the functorial point of view and reducing the lemma to a statement on arrows in restrictions of a groupoid category. In the last formula of the lemma the notation $((u', r_0), (u'', r_1))$ indicates a T-valued point of $(U' \times_{g,U,t} R) \times_U (U'' \times_{g'',U,t} R)$. This means that u', u'', r_0, r_1 are T-valued points of U', U'', R, R and that $g(u') = t(r_0)$, $g(g'(u'')) = g''(u'') = t(r_1)$, and $s(r_0) = s(r_1)$. It would be more correct here to write $g \circ u' = t \circ r_0$ and so on but this makes the notation even more unreadable. If we think of r_1 and r_0 as arrows in a groupoid category then we can represent this by the picture

$$t(r_0) = g(u') \xleftarrow{r_0} s(r_0) = s(r_1) \xrightarrow{r_1} t(r_1) = g(g'(u''))$$

This diagram in particular demonstrates that the composition $c(r_1, i(r_0))$ makes sense. Recall that

$$R' = R \times_{(t,s),U \times_S U, g \times g} U' \times_S U'$$

hence a T-valued point of R' looks like $(r, (u'_0, u'_1))$ with $t(r) = g(u'_0)$ and $s(r) = g(u'_1)$. In particular given $((u', r_0), (u'', r_1))$ as above we get the T-valued point $(c(r_1, i(r_0)), (g'(u''), u'))$ of R' because we have $t(c(r_1, i(r_0))) = t(r_1) = g(g'(u''))$ and $s(c(r_1, i(r_0))) = s(i(r_0)) = t(r_0) = g(u')$. We leave it to the reader to show that the left square commutes with this definition.

To show that the left square is cartesian, suppose we are given (v'', p') and (v', p) which are T-valued points of $U'' \times_{g',U',t} R'$ and $U' \times_{g,U,t} R$ with $v' = s'(p')$. This also means that $g'(v'') = t'(p')$ and $g(v') = t(p)$. By the discussion above we know that we can write $p' = (r, (u'_0, u'_1))$ with $t(r) = g(u'_0)$ and $s(r) = g(u'_1)$. Using this notation we see that $v' = s'(p') = u'_1$ and $g'(v'') = t'(p') = u'_0$. Here is a picture

$$s(p) \xrightarrow{p} g(v') = g(u'_1) \xrightarrow{r} g(u'_0) = g(g'(v''))$$

What we have to show is that there exists a unique T-valued point $((u', r_0), (u'', r_1))$ as above such that $v' = u'$, $p = r_0$, $v'' = u''$ and $p' = (c(r_1, i(r_0)), (g'(u''), u'))$. Comparing the two diagrams above it is clear that we have no choice but to take

$$((u', r_0), (u'', r_1)) = ((v', p), (v'', c(r, p))$$

Some details omitted. □

Lemma 39.8.4. *Let S be a scheme. Let (U, R, s, t, c) be a groupoid scheme over S. Let $g : U' \to U$ and $g' : U'' \to U'$ be morphisms of schemes. Set $g'' = g \circ g'$. Let (U', R', s', t', c') be the restriction of R to U'. Let $h = s \circ pr_1 : U' \times_{g,U,t} R \to U$, let $h' = s' \circ pr_1 : U'' \times_{g',U',t} R \to U'$, and let $h'' = s \circ pr_1 : U'' \times_{g'',U,t} R \to U$. Let $\tau \in \{Zariski, étale, smooth, syntomic, fppf, fpqc\}$. Let $\mathcal{P}$ be a property of morphisms of schemes which is preserved under base change, and which is local on the target for the τ-topology. If*

(1) *$h(U' \times_U R)$ is open in U,*
(2) *$\{h : U' \times_U R \to h(U' \times_U R)\}$ is a τ-covering,*
(3) *h' has property $\mathcal{P}$,*

then h'' has property $\mathcal{P}$. Conversely, if

(a) *$\{t : R \to U\}$ is a τ-covering,*
(d) *h'' has property $\mathcal{P}$,*

then h' has property $\mathcal{P}$.

Proof. This follows formally from the properties of the diagram of Lemma 39.8.3. In the first case, note that the image of the morphism h'' is contained in the image of h, as $g'' = g \circ g'$. Hence we may replace the U in the lower right corner of the diagram by $h(U' \times_U R)$. This explains the significance of conditions (1) and (2) in the lemma. In the second case, note that $\{\mathrm{pr}_0 : U' \times_{g,U,t} R \to U'\}$ is a τ-covering as a base change of τ and condition (a). □

39.9. Properties of groupoids on fields

A "groupoid on a field" indicates a groupoid scheme (U, R, s, t, c) where U is the spectrum of a field. It does **not** mean that (U, R, s, t, c) is defined over a field, more precisely, it does **not** mean that the morphisms $s, t : R \to U$ are equal. Given any field k, an abstract group G and a group homomorphism $\varphi : G \to \mathrm{Aut}(k)$ we obtain a groupoid scheme (U, R, s, t, c) over $\mathbf{Z}$ by setting

$$\begin{aligned} U &= \mathrm{Spec}(k) \\ R &= \coprod_{g \in G} \mathrm{Spec}(k) \\ s &= \coprod_{g \in G} \mathrm{Spec}(\mathrm{id}_k) \\ t &= \coprod_{g \in G} \mathrm{Spec}(\varphi(g)) \\ c &= \text{composition in } G \end{aligned}$$

This example still is a groupoid scheme over $\mathrm{Spec}(k^G)$. Hence, if G is finite, then $U = \mathrm{Spec}(k)$ is finite over $\mathrm{Spec}(k^G)$. In some sense our goal in this section is to show that suitable finiteness conditions on s, t force any groupoid on a field to be defined over a finite index subfield $k' \subset k$.

If k is a field and (G, m) is a group scheme over k with structure morphism $p : G \to \mathrm{Spec}(k)$, then $(\mathrm{Spec}(k), G, p, p, m)$ is an example of a groupoid on a field (and in this case of course the whole structure is defined over a field). Hence this section can be viewed as the analogue of Groupoids, Section 38.7.

Lemma 39.9.1. *Let S be a scheme. Let (U, R, s, t, c) be a groupoid scheme over S. If U is the spectrum of a field, then the composition morphism $c : R \times_{s,U,t} R \to R$ is open.*

Proof. The composition is isomorphic to the projection map $\mathrm{pr}_1 : R \times_{t,U,t} R \to R$ by Diagram (39.3.0.3). The projection is open by Morphisms, Lemma 28.24.4. □

Lemma 39.9.2. *Let S be a scheme. Let (U, R, s, t, c) be a groupoid scheme over S. If U is the spectrum of a field, then R is a separated scheme.*

Proof. By Groupoids, Lemma 38.7.2 the stabilizer group scheme $G \to U$ is separated. By Groupoids, Lemma 38.19.2 the morphism $j = (t, s) : R \to U \times_S U$ is separated. As U is the spectrum of a field the scheme $U \times_S U$ is affine (by the construction of fibre products in Schemes, Section 25.17). Hence R is a separated scheme, see Schemes, Lemma 25.21.13. □

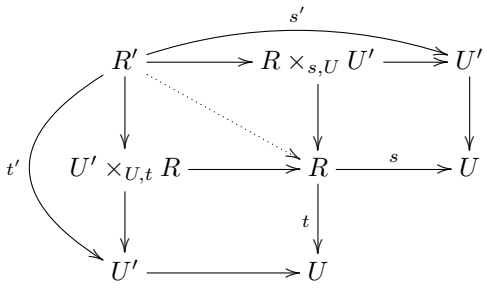
Lemma 39.9.3. *Let S be a scheme. Let (U, R, s, t, c) be a groupoid scheme over S. Assume $U = \mathrm{Spec}(k)$ with k a field. For any points $r, r' \in R$ there exists a field extension $k \subset k'$ and points $r_1, r_2 \in R \times_{s,\mathrm{Spec}(k)} \mathrm{Spec}(k')$ and a diagram*

$$R \xleftarrow{pr_0} R \times_{s,\mathrm{Spec}(k)} \mathrm{Spec}(k') \xrightarrow{\varphi} R \times_{s,\mathrm{Spec}(k)} \mathrm{Spec}(k') \xrightarrow{pr_0} R$$

such that φ is an isomorphism of schemes over $\mathrm{Spec}(k')$, we have $\varphi(r_1) = r_2$, $pr_0(r_1) = r$, and $pr_0(r_2) = r'$.

Proof. This is a special case of Lemma 39.6.1 parts (1) and (2). □

Lemma 39.9.4. *Let S be a scheme. Let (U, R, s, t, c) be a groupoid scheme over S. Assume $U = \mathrm{Spec}(k)$ with k a field. Let $k \subset k'$ be a field extension, $U' = \mathrm{Spec}(k')$ and let (U', R', s', t', c') be the restriction of (U, R, s, t, c) via $U' \to U$. In the defining diagram*



all the morphisms are surjective, flat, and universally open. The dotted arrow $R' \to R$ is in addition affine.

Proof. The morphism $U' \to U$ equals $\mathrm{Spec}(k') \to \mathrm{Spec}(k)$, hence is affine, surjective and flat. The morphisms $s, t : R \to U$ and the morphism $U' \to U$ are universally open by Morphisms, Lemma 28.24.4. Since R is not empty and U is the spectrum of a field the morphisms $s, t : R \to U$ are surjective and flat. Then you conclude by using Morphisms, Lemmas 28.11.4, 28.11.2, 28.24.3, 28.13.8, 28.13.7, 28.26.7, and 28.26.5. □

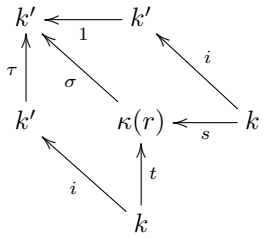
Lemma 39.9.5. *Let S be a scheme. Let (U, R, s, t, c) be a groupoid scheme over S. Assume $U = \mathrm{Spec}(k)$ with k a field. For any point $r \in R$ there exist*

1. *a field extension $k \subset k'$ with k' algebraically closed,*
2. *a point $r' \in R'$ where (U', R', s', t', c') is the restriction of (U, R, s, t, c) via $\mathrm{Spec}(k') \to \mathrm{Spec}(k)$*

such that

(1) *the point r' maps to r under the morphism $R' \to R$, and*
(2) *the maps $s', t' : R' \to \mathrm{Spec}(k')$ induce isomorphisms $k' \to \kappa(r')$.*

Proof. Translating the geometric statement into a statement on fields, this means that we can find a diagram



where $i : k \to k'$ is the embedding of k into k', the maps $s, t : k \to \kappa(r)$ are induced by $s, t : R \to U$, and the map $\tau : k' \to k'$ is an automorphism. To produce such a diagram we may proceed in the following way:

(1) Pick $i : k \to k'$ a field map with k' algebraically closed of very large transcendence degree over k.
(2) Pick an embedding $\sigma : \kappa(r) \to k'$ such that $\sigma \circ s = i$. Such a σ exists because we can just choose a transcendence basis $\{x_\alpha\}_{\alpha \in A}$ of $\kappa(r)$ over k and find $y_\alpha \in k'$, $\alpha \in A$ which are algebraically independent over $i(k)$, and map $s(k)(\{x_\alpha\})$ into k' by the rules $s(\lambda) \mapsto i(\lambda)$ for $\lambda \in k$ and $x_\alpha \mapsto y_\alpha$ for $\alpha \in A$. Then extend to $\tau : \kappa(\alpha) \to k'$ using that k' is algebraically closed.
(3) Pick an automorphism $\tau : k' \to k'$ such that $\tau \circ i = \sigma \circ t$. To do this pick a transcendence basis $\{x_\alpha\}_{\alpha \in A}$ of k over its prime field. On the one hand, extend $\{i(x_\alpha)\}$ to a transcendence basis of k' by adding $\{y_\beta\}_{\beta \in B}$ and extend $\{\sigma(t(x_\alpha))\}$ to a transcendence basis of k' by adding $\{z_\gamma\}_{\gamma \in C}$. As k' is algebraically closed we can extend the isomorphism $\sigma \circ t \circ i^{-1} : i(k) \to \sigma(t(k))$ to an isomorphism $\tau' : \overline{i(k)} \to \overline{\sigma(t(k))}$ of their algebraic closures in k'. As k' has large transcendence degree we see that the sets B and C have the same cardinality. Thus we can use a bijection $B \to C$ to extend τ' to an isomorphism
$$\overline{i(k)}(\{y_\beta\}) \longrightarrow \overline{\sigma(t(k))}(\{z_\gamma\})$$
and then since k' is the algebraic closure of both sides we see that this extends to an automorphism $\tau : k' \to k'$ as desired.

This proves the lemma. □

Lemma 39.9.6. *Let S be a scheme. Let (U, R, s, t, c) be a groupoid scheme over S. Assume $U = \mathrm{Spec}(k)$ with k a field. If $r \in R$ is a point such that s, t induce isomorphisms $k \to \kappa(r)$, then the map*
$$R \longrightarrow R, \quad x \longmapsto c(r, x)$$
(see proof for precise notation) is an automorphism $R \to R$ which maps e to r.

Proof. This is completely obvious if you think about groupoids in a functorial way. But we will also spell it out completely. Denote $a : U \to R$ the morphism with image r such that $s \circ a = \mathrm{id}_U$ which exists by the hypothesis that $s : k \to \kappa(r)$ is an

isomorphism. Similarly, denote $b : U \to R$ the morphism with image r such that $t \circ b = \mathrm{id}_U$. Note that $b = a \circ (t \circ a)^{-1}$, in particular $a \circ s \circ b = b$.

Consider the morphism $\Psi : R \to R$ given on T-valued points by

$$(f : T \to R) \longmapsto (c(a \circ t \circ f, f) : T \to R)$$

To see this is defined we have to check that $s \circ a \circ t \circ f = t \circ f$ which is obvious as $s \circ a = 1$. Note that $\Phi(e) = a$, so that in order to prove the lemma it suffices to show that Φ is an automorphism of R. Let $\Phi : R \to R$ be the morphism given on T-valued points by

$$(g : T \to R) \longmapsto (c(i \circ b \circ t \circ g, g) : T \to R).$$

This is defined because $s \circ i \circ b \circ t \circ g = t \circ b \circ t \circ g = t \circ g$. We claim that Φ and Ψ are inverse to each other. To see this we compute

$$\begin{aligned}
& c(a \circ t \circ c(i \circ b \circ t \circ g, g), c(i \circ b \circ t \circ g, g)) \\
&= c(a \circ t \circ i \circ b \circ t \circ g, c(i \circ b \circ t \circ g, g)) \\
&= c(a \circ s \circ b \circ t \circ g, c(i \circ b \circ t \circ g, g)) \\
&= c(b \circ t \circ g, c(i \circ b \circ t \circ g, g)) \\
&= c(c(b \circ t \circ g, i \circ b \circ t \circ g), g)) \\
&= c(e, g) \\
&= g
\end{aligned}$$

where we have used the relation $a \circ s \circ b = b$ shown above. In the other direction we have

$$\begin{aligned}
& c(i \circ b \circ t \circ c(a \circ t \circ f, f), c(a \circ t \circ f, f)) \\
&= c(i \circ b \circ t \circ a \circ t \circ f, c(a \circ t \circ f, f)) \\
&= c(i \circ a \circ (t \circ a)^{-1} \circ t \circ a \circ t \circ f, c(a \circ t \circ f, f)) \\
&= c(i \circ a \circ t \circ f, c(a \circ t \circ f, f)) \\
&= c(c(i \circ a \circ t \circ f, a \circ t \circ f), f) \\
&= c(e, f) \\
&= f
\end{aligned}$$

The lemma is proved. □

Lemma 39.9.7. *Let S be a scheme. Let (U, R, s, t, c) be a groupoid scheme over S. Assume $U = \mathrm{Spec}(k)$ with k a field. By abuse of notation denote $e \in R$ the image of the identity morphism $e : U \to R$. Then*

1. *every local ring $\mathcal{O}_{R,r}$ of R has a unique minimal prime ideal,*
2. *there is exactly one irreducible component Z of R passing through e, and*
3. *Z is geometrically irreducible over k via either s or t.*

Proof. Let $r \in R$ be a point. In this proof we will use the correspondence between irreducible components of R passing through a point r and minimal primes of the local ring $\mathcal{O}_{R,r}$ without further mention. Choose $k \subset k'$ and $r' \in R'$ as in Lemma 39.9.5. Note that $\mathcal{O}_{R,r} \to \mathcal{O}_{R',r'}$ is faithfully flat and local, see Lemma 39.9.4. Hence the result for $r' \in R'$ implies the result for $r \in R$. In other words we may assume that $s, t : k \to \kappa(r)$ are isomorphisms. By Lemma 39.9.6 there exists an

automorphism moving e to r. Hence we may assume $r = e$, i.e., part (1) follows from part (2).

We first prove (2) in case k is separably algebraically closed. Namely, let $X, Y \subset R$ be irreducible components passing through e. Then by Varieties, Lemma 32.6.4 and 32.6.3 the scheme $X \times_{s,U,t} Y$ is irreducible as well. Hence $c(X \times_{s,U,t} Y) \subset R$ is an irreducible subset. We claim it contains both X and Y (as subsets of R). Namely, let T be the spectrum of a field. If $x : T \to X$ is a T-valued point of X, then $c(x, e \circ s \circ x) = x$ and $e \circ s \circ x$ factors through Y as $e \in Y$. Similarly for points of Y. This clearly implies that $X = Y$, i.e., there is a unique irreducible component of R passing through e.

Proof of (2) and (3) in general. Let $k \subset k'$ be a separable algebraic closure, and let (U', R', s', t', c') be the restriction of (U, R, s, t, c) via $\mathrm{Spec}(k') \to \mathrm{Spec}(k)$. By the previous paragraph there is exactly one irreducible component Z' of R' passing through e'. Denote $e'' \in R \times_{s,U} U'$ the base change of e. As $R' \to R \times_{s,U} U'$ is faithfully flat, see Lemma 39.9.4, and $e' \mapsto e''$ we see that there is exactly one irreducible component Z'' of $R \times_{s,k} k'$ passing through e''. This implies, as $R \times_k k' \to R$ is faithfully flat, that there is exactly one irreducible component Z of R passing through e. This proves (2).

To prove (3) let $Z''' \subset R \times_k k'$ be an arbitrary irreducible component of $Z \times_k k'$. By Varieties, Lemma 32.6.12 we see that $Z''' = \sigma(Z'')$ for some $\sigma \in \mathrm{Gal}(k'/k)$. Since $\sigma(e'') = e''$ we see that $e'' \in Z'''$ and hence $Z''' = Z''$. This means that Z is geometrically irreducible over $\mathrm{Spec}(k)$ via the morphism s. The same argument implies that Z is geometrically irreducible over $\mathrm{Spec}(k)$ via the morphism t. □

Lemma 39.9.8. *Let S be a scheme. Let (U, R, s, t, c) be a groupoid scheme over S. Assume $U = \mathrm{Spec}(k)$ with k a field. Assume s, t are locally of finite type. Then*

(1) *R is equidimensional,*
(2) *$\dim(R) = \dim_r(R)$ for all $r \in R$,*
(3) *for any $r \in R$ we have $\mathrm{trdeg}_{s(k)}(\kappa(r)) = \mathrm{trdeg}_{t(k)}(\kappa(r))$, and*
(4) *for any closed point $r \in R$ we have $\dim(R) = \dim(\mathcal{O}_{R,r})$.*

Proof. Let $r, r' \in R$. Then $\dim_r(R) = \dim_{r'}(R)$ by Lemma 39.9.3 and Morphisms, Lemma 28.29.3. By Morphisms, Lemma 28.29.1 we have

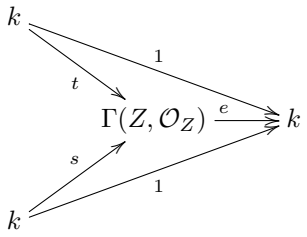
$$\dim_r(R) = \dim(\mathcal{O}_{R,r}) + \mathrm{trdeg}_{s(k)}(\kappa(r)) = \dim(\mathcal{O}_{R,r}) + \mathrm{trdeg}_{t(k)}(\kappa(r)).$$

On the other hand, the dimension of R (or any open subset of R) is the supremum of the dimensions of the local rings of of R, see Properties, Lemma 27.11.4. Clearly this is maximal for closed points r in which case $\mathrm{trdeg}_k(\kappa(r)) = 0$ (by the Hilbert Nullstellensatz, see Morphisms, Section 28.17). Hence the lemma follows. □

Lemma 39.9.9. *Let S be a scheme. Let (U, R, s, t, c) be a groupoid scheme over S. Assume $U = \mathrm{Spec}(k)$ with k a field. Assume s, t are locally of finite type. Then $\dim(R) = \dim(G)$ where G is the stabilizer group scheme of R.*

Proof. Let $Z \subset R$ be the irreducible component passing through e (see Lemma 39.9.7) thought of as an integral closed subscheme of R. Let k'_s, resp. k'_t be the integral closure of $s(k)$, resp. $t(k)$ in $\Gamma(Z, \mathcal{O}_Z)$. Recall that k'_s and k'_t are fields, see

Varieties, Lemma 32.17.4. By Varieties, Proposition 32.18.1 we have $k'_s = k'_t$ as subrings of $\Gamma(Z, \mathcal{O}_Z)$. As e factors through Z we obtain a commutative diagram



This on the one hand shows that $k'_s = s(k)$, $k'_t = t(k)$, so $s(k) = t(k)$, which combined with the diagram above implies that $s = t$! In other words, we conclude that Z is a closed subscheme of $G = R \times_{(t,s),U \times_S U,\Delta} U$. The lemma follows as both G and R are equidimensional, see Lemma 39.9.8 and Groupoids, Lemma 38.7.5. □

Remark 39.9.10. Warning: Lemma 39.9.9 is wrong without the condition that s and t are locally of finite type. An easy example is to start with the action

$$\mathbf{G}_{m,\mathbf{Q}} \times_{\mathbf{Q}} \mathbf{A}^1_{\mathbf{Q}} \to \mathbf{A}^1_{\mathbf{Q}}$$

and restrict the corresponding groupoid scheme to the generic point of $\mathbf{A}^1_{\mathbf{Q}}$. In other words restrict via the morphism $\mathrm{Spec}(\mathbf{Q}(x)) \to \mathrm{Spec}(\mathbf{Q}[x]) = \mathbf{A}^1_{\mathbf{Q}}$. Then you get a groupoid scheme (U, R, s, t, c) with $U = \mathrm{Spec}(\mathbf{Q}(x))$ and

$$R = \mathrm{Spec}\left(\mathbf{Q}(x)[y]\left[\frac{1}{P(xy)}, P \in \mathbf{Q}[T], P \neq 0\right]\right)$$

In this case $\dim(R) = 1$ and $\dim(G) = 0$.

Lemma 39.9.11. *Let S be a scheme. Let (U, R, s, t, c) be a groupoid scheme over S. Assume*

(1) *$U = \mathrm{Spec}(k)$ with k a field,*
(2) *s, t are locally of finite type, and*
(3) *the characteristic of k is zero.*

Then $s, t : R \to U$ are smooth.

Proof. By Lemma 39.4.1 the sheaf of differentials of $R \to U$ is free. Hence smoothness follows from Varieties, Lemma 32.15.1. □

Lemma 39.9.12. *Let S be a scheme. Let (U, R, s, t, c) be a groupoid scheme over S. Assume*

(1) *$U = \mathrm{Spec}(k)$ with k a field,*
(2) *s, t are locally of finite type,*
(3) *R is reduced, and*
(4) *k is perfect.*

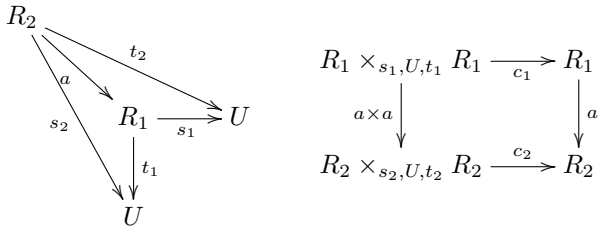
Then $s, t : R \to U$ are smooth.

Proof. By Lemma 39.4.1 the sheaf $\Omega_{R/U}$ is free. Hence the lemma follows from Varieties, Lemma 32.15.2. □

39.10. Morphisms of groupoids on fields

This section studies morphisms between groupoids on fields. This is slightly more general, but very akin to, studying morphisms of groupschemes over a field.

Situation 39.10.1. Let S be a scheme. Let $U = \mathrm{Spec}(k)$ be a scheme over S with k a field. Let (U, R_1, s_1, t_1, c_1), (U, R_2, s_2, t_2, c_2) be groupoid schemes over S with identical first component. Let $a : R_1 \to R_2$ be a morphism such that (id_U, a) defines a morphism of groupoid schemes over S, see Groupoids, Definition 38.11.1. In particular, the following diagrams commute



The following lemma is a generalization of Groupoids, Lemma 38.7.11.

Lemma 39.10.2. *Notation and assumptions as in Situation 39.10.1. If $a(R_1)$ is open in R_2, then $a(R_1)$ is closed in R_2.*

Proof. Let $r_2 \in R_2$ be a point in the closure of $a(R_1)$. We want to show $r_2 \in a(R_1)$. Pick $k \subset k'$ and $r_2' \in R_2'$ adapted to (U, R_2, s_2, t_2, c_2) and r_2 as in Lemma 39.9.5. Let R_i' be the restriction of R_i via the morphism $U' = \mathrm{Spec}(k') \to U = \mathrm{Spec}(k)$. Let $a' : R_1' \to R_2'$ be the base change of a. The diagram

$$\begin{array}{ccc} R_1' & \xrightarrow[a']{} & R_2' \\ {\scriptstyle p_1}\downarrow & & \downarrow{\scriptstyle p_2} \\ R_1 & \xrightarrow{a} & R_2 \end{array}$$

is a fibre square. Hence the image of a' is the inverse image of the image of a via the morphism $p_2 : R_2' \to R_2$. By Lemma 39.9.4 the map p_2 is surjective and open. Hence by Topology, Lemma 5.5.4 we see that r_2' is in the closure of $a'(R_1')$. This means that we may assume that $r_2 \in R_2$ has the property that the maps $k \to \kappa(r_2)$ induced by s_2 and t_2 are isomorphisms.

In this case we can use Lemma 39.9.6. This lemma implies $c(r_2, a(R_1))$ is an open neighbourhood of r_2. Hence $a(R_1) \cap c(r_2, a(R_1)) \neq \emptyset$ as we assumed that r_2 was a point of the closure of $a(R_1)$. Using the inverse of R_2 and R_1 we see this means $c_2(a(R_1), a(R_1))$ contains r_2. As $c_2(a(R_1), a(R_1)) \subset a(c_1(R_1, R_1)) = a(R_1)$ we conclude $r_2 \in a(R_1)$ as desired. □

Lemma 39.10.3. *Notation and assumptions as in Situation 39.10.1. Let $Z \subset R_2$ be the reduced closed subscheme (see Schemes, Definition 25.12.5) whose underlying topological space is the closure of the image of $a : R_1 \to R_2$. Then $c_2(Z \times_{s_2,U,t_2} Z) \subset Z$ set theoretically.*

Proof. Consider the commutative diagram

$$\begin{array}{ccc} R_1 \times_{s_1,U,t_1} R_1 & \longrightarrow & R_1 \\ \downarrow & & \downarrow \\ R_2 \times_{s_2,U,t_2} R_2 & \longrightarrow & R_2 \end{array}$$

By Varieties, Lemma 32.14.2 the closure of the image of the left vertical arrow is (set theoretically) $Z \times_{s_2,U,t_2} Z$. Hence the result follows. □

Lemma 39.10.4. *Notation and assumptions as in Situation 39.10.1. Assume that k is perfect. Let $Z \subset R_2$ be the reduced closed subscheme (see Schemes, Definition 25.12.5) whose underlying topological space is the closure of the image of $a : R_1 \to R_2$. Then*

$$(U, Z, s_2|_Z, t_2|_Z, c_2|_Z)$$

is a groupoid scheme over S.

Proof. We first explain why the statement makes sense. Since U is the spectrum of a perfect field k, the scheme Z is geometrically reduced over k (via either projection), see Varieties, Lemma 32.4.3. Hence the scheme $Z \times_{s_2,U,t_2} Z \subset Z$ is reduced, see Varieties, Lemma 32.4.7. Hence by Lemma 39.10.3 we see that c induces a morphism $Z \times_{s_2,U,t_2} Z \to Z$. Finally, it is clear that e_2 factors through Z and that the map $i_2 : R_2 \to R_2$ preserves Z. Since the morphisms of the septuple $(U, R_2, s_2, t_2, c_2, e_2, i_2)$ satisfies the axioms of a groupoid, it follows that after restricting to Z they satisfy the axioms. □

Lemma 39.10.5. *Notation and assumptions as in Situation 39.10.1. If the image $a(R_1)$ is a locally closed subset of R_2 then it is a closed subset.*

Proof. Let $k \subset k'$ be a perfect closure of the field k. Let R'_i be the restriction of R_i via the morphism $U' = \mathrm{Spec}(k') \to \mathrm{Spec}(k)$. Note that the morphisms $R'_i \to R_i$ are universal homeomorphisms as compositions of base changes of the universal homeomorphism $U' \to U$ (see diagram in statement of Lemma 39.9.4). Hence it suffices to prove that $a'(R'_1)$ is closed in R'_2. In other words, we may assume that k is perfect.

If k is perfect, then the closure of the image is a groupoid scheme $Z \subset R_2$, by Lemma 39.10.4. By the same lemma applied to $\mathrm{id}_{R_1} : R_1 \to R_1$ we see that $(R_2)_{red}$ is a groupoid scheme. Thus we may apply Lemma 39.10.2 to the morphism $a|_{(R_2)_{red}} : (R_2)_{red} \to Z$ to conclude that Z equals the image of a. □

Lemma 39.10.6. *Notation and assumptions as in Situation 39.10.1. Assume that $a : R_1 \to R_2$ is a quasi-compact morphism. Let $Z \subset R_2$ be the scheme theoretic image (see Morphisms, Definition 28.6.2) of $a : R_1 \to R_2$. Then*

$$(U, Z, s_2|_Z, t_2|_Z, c_2|_Z)$$

is a groupoid scheme over S.

Proof. The main difficulty is to show that $c_2|_{Z \times_{s_2,U,t_2} Z}$ maps into Z. Consider the commutative diagram

$$\begin{array}{ccc} R_1 \times_{s_1,U,t_1} R_1 & \longrightarrow & R_1 \\ \downarrow{\scriptstyle a \times a} & & \downarrow \\ R_2 \times_{s_2,U,t_2} R_2 & \longrightarrow & R_2 \end{array}$$

By Varieties, Lemma 32.14.3 we see that the scheme theoretic image of $a \times a$ is $Z \times_{s_2,U,t_2} Z$. By the commutativity of the diagram we conclude that $Z \times_{s_2,U,t_2} Z$ maps into Z by the bottom horizontal arrow. As in the proof of Lemma 39.10.4 it is also true that $i_2(Z) \subset Z$ and that e_2 factors through Z. Hence we conclude as in the proof of that lemma. □

Lemma 39.10.7. *Let S be a scheme. Let (U, R, s, t, c) be a groupoid scheme over S. Assume U is the spectrum of a field. Let $Z \subset U \times_S U$ be the reduced closed subscheme (see Schemes, Definition 25.12.5) whose underlying topological space is the closure of the image of $j = (t, s) : R \to U \times_S U$. Then $pr_{02}(Z \times_{pr_1,U,pr_0} Z) \subset Z$ set theoretically.*

Proof. As $(U, U \times_S U, \mathrm{pr}_1, \mathrm{pr}_0, \mathrm{pr}_{02})$ is a groupoid scheme over S this is a special case of Lemma 39.10.3. But we can also prove it directly as follows.

Write $U = \mathrm{Spec}(k)$. Denote R_s (resp. Z_s, resp. U_s^2) the scheme R (resp. Z, resp. $U \times_S U$) viewed as a scheme over k via s (resp. $\mathrm{pr}_1|_Z$, resp. pr_1). Similarly, denote ${}_tR$ (resp. ${}_tZ$, resp. ${}_tU^2$) the scheme R (resp. Z, resp. $U \times_S U$) viewed as a scheme over k via t (resp. $\mathrm{pr}_0|_Z$, resp. pr_0). The morphism j induces morphisms of schemes $j_s : R_s \to U_s^2$ and ${}_tj : {}_tR \to {}_tU^2$ over k. Consider the commutative diagram

$$\begin{array}{ccc} R_s \times_{k\ {}_tR} & \xrightarrow{c} & R \\ {\scriptstyle j_s \times {}_tj}\downarrow & & \downarrow{\scriptstyle j} \\ U_s^2 \times_{k\ {}_tU^2} & \longrightarrow & U \times_S U \end{array}$$

By Varieties, Lemma 32.14.2 we see that the closure of the image of $j_s \times {}_tj$ is $Z_s \times_{k\ {}_tZ} Z$. By the commutativity of the diagram we conclude that $Z_s \times_{k\ {}_tZ} Z$ maps into Z by the bottom horizontal arrow. □

Lemma 39.10.8. *Let S be a scheme. Let (U, R, s, t, c) be a groupoid scheme over S. Assume U is the spectrum of a perfect field. Let $Z \subset U \times_S U$ be the reduced closed subscheme (see Schemes, Definition 25.12.5) whose underlying topological space is the closure of the image of $j = (t, s) : R \to U \times_S U$. Then*

$$(U, Z, pr_0|_Z, pr_1|_Z, pr_{02}|_{Z \times_{pr_1,U,pr_0} Z})$$

is a groupoid scheme over S.

Proof. As $(U, U \times_S U, \mathrm{pr}_1, \mathrm{pr}_0, \mathrm{pr}_{02})$ is a groupoid scheme over S this is a special case of Lemma 39.10.4. But we can also prove it directly as follows.

We first explain why the statement makes sense. Since U is the spectrum of a perfect field k, the scheme Z is geometrically reduced over k (via either projection), see Varieties, Lemma 32.4.3. Hence the scheme $Z \times_{\mathrm{pr}_1,U,\mathrm{pr}_0} Z \subset Z$ is reduced, see Varieties, Lemma 32.4.7. Hence by Lemma 39.10.7 we see that pr_{02} induces a morphism $Z \times_{\mathrm{pr}_1,U,\mathrm{pr}_0} Z \to Z$. Finally, it is clear that $\Delta_{U/S}$ factors through Z

and that the map $\sigma : U \times_S U \to U \times_S U$, $(x, y) \mapsto (y, x)$ preserves Z. Since $(U, U \times_S U, \mathrm{pr}_0, \mathrm{pr}_1, \mathrm{pr}_{02}, \Delta_{U/S}, \sigma)$ satisfies the axioms of a groupoid, it follows that after restricting to Z they satisfy the axioms. □

Lemma 39.10.9. *Let S be a scheme. Let (U, R, s, t, c) be a groupoid scheme over S. Assume U is the spectrum of a field and assume R is quasi-compact (equivalently s, t are quasi-compact). Let $Z \subset U \times_S U$ be the scheme theoretic image (see Morphisms, Definition 28.6.2) of $j = (t, s) : R \to U \times_S U$. Then*

$$(U, Z, pr_0|_Z, pr_1|_Z, pr_{02}|_{Z \times_{pr_1, U, pr_0} Z})$$

is a groupoid scheme over S.

Proof. As $(U, U \times_S U, \mathrm{pr}_1, \mathrm{pr}_0, \mathrm{pr}_{02})$ is a groupoid scheme over S this is a special case of Lemma 39.10.6. But we can also prove it directly as follows.

The main difficulty is to show that $\mathrm{pr}_{02}|_{Z \times_{\mathrm{pr}_1, U, \mathrm{pr}_0} Z}$ maps into Z. Write $U = \mathrm{Spec}(k)$. Denote R_s (resp. Z_s, resp. U_s^2) the scheme R (resp. Z, resp. $U \times_S U$) viewed as a scheme over k via s (resp. $\mathrm{pr}_1|_Z$, resp. pr_1). Similarly, denote ${}_tR$ (resp. ${}_tZ$, resp. ${}_tU^2$) the scheme R (resp. Z, resp. $U \times_S U$) viewed as a scheme over k via t (resp. $\mathrm{pr}_0|_Z$, resp. pr_0). The morphism j induces morphisms of schemes $j_s : R_s \to U_s^2$ and ${}_tj : {}_tR \to {}_tU^2$ over k. Consider the commutative diagram

$$\begin{array}{ccc} R_s \times_{k\ {}_tR} & \xrightarrow{c} & R \\ {\scriptstyle j_s \times_t j}\downarrow & & \downarrow{\scriptstyle j} \\ U_s^2 \times_{k\ {}_tU^2} & \longrightarrow & U \times_S U \end{array}$$

By Varieties, Lemma 32.14.3 we see that the scheme theoretic image of $j_s \times {}_tj$ is $Z_s \times_{k\ {}_tZ} Z$. By the commutativity of the diagram we conclude that $Z_s \times_{k\ {}_tZ} Z$ maps into Z by the bottom horizontal arrow. As in the proof of Lemma 39.10.8 it is also true that $\sigma(Z) \subset Z$ and that $\Delta_{U/S}$ factors through Z. Hence we conclude as in the proof of that lemma. □

39.11. Slicing groupoids

The following lemma shows that we may slice a Cohen-Macaulay groupoid scheme in order to reduce the dimension of the fibres, provided that the dimension of the stabilizer is small. This is an essential step in the process of improving a given presentation of a quotient stack.

Situation 39.11.1. Let S be a scheme. Let (U, R, s, t, c) be a groupoid scheme over S. Let $g : U' \to U$ be a morphism of schemes. Let $u \in U$ be a point, and let $u' \in U'$ be a point such that $g(u') = u$. Given these data, denote (U', R', s', t', c') the restriction of (U, R, s, t, c) via the morphism g. Denote $G \to U$ the stabilizer group scheme of R, which is a locally closed subscheme of R. Denote h the composition

$$h = s \circ \mathrm{pr}_1 : U' \times_{g,U,t} R \longrightarrow U.$$

Denote $F_u = s^{-1}(u)$ (scheme theoretic fibre), and G_u the scheme theoretic fibre of G over u. Similarly for R' we denote $F'_{u'} = (s')^{-1}(u')$. Because $g(u') = u$ we have

$$F'_{u'} = h^{-1}(u) \times_{\mathrm{Spec}(\kappa(u))} \mathrm{Spec}(\kappa(u')).$$

The point $e(u) \in R$ may be viewed as a point on G_u and F_u also, and $e'(u')$ is a point of R' (resp. $G'_{u'}$, resp. $F'_{u'}$) which maps to $e(u)$ in R (resp. G_u, resp. F_u).

Lemma 39.11.2. *Let S be a scheme. Let (U, R, s, t, c, e, i) be a groupoid scheme over S. Let $G \to U$ be the stabilizer group scheme. Assume s and t are Cohen-Macaulay and locally of finite presentation. Let $u \in U$ be a finite type point of the scheme U, see Morphisms, Definition 28.17.3. With notation as in Situation 39.11.1, set*

$$d_1 = \dim(G_u), \quad d_2 = \dim_{e(u)}(F_u).$$

If $d_2 > d_1$, then there exist an affine scheme U' and a morphism $g : U' \to U$ such that (with notation as in Situation 39.11.1)

1. *g is an immersion*
2. *$u \in U'$,*
3. *g is locally of finite presentation,*
4. *the morphism $h : U' \times_{g,U,t} R \longrightarrow U$ is Cohen-Macaulay at $(u, e(u))$, and*
5. *we have $\dim_{e'(u)}(F'_u) = d_2 - 1$.*

Proof. Let $\mathrm{Spec}(A) \subset U$ be an affine neighbourhood of u such that u corresponds to a closed point of U, see Morphisms, Lemma 28.17.4. Let $\mathrm{Spec}(B) \subset R$ be an affine neighbourhood of $e(u)$ which maps via j into the open $\mathrm{Spec}(A) \times_S \mathrm{Spec}(A) \subset U \times_S U$. Let $\mathfrak{m} \subset A$ be the maximal ideal corresponding to u. Let $\mathfrak{q} \subset B$ be the prime ideal corresponding to $e(u)$. Pictures:

$$\begin{array}{ccc} B & \xleftarrow{s} & A \\ {\scriptstyle t}\uparrow & & \\ A & & \end{array} \quad \text{and} \quad \begin{array}{ccc} B_{\mathfrak{q}} & \xleftarrow{s} & A_{\mathfrak{m}} \\ {\scriptstyle t}\uparrow & & \\ A_{\mathfrak{m}} & & \end{array}$$

Note that the two induced maps $s, t : \kappa(\mathfrak{m}) \to \kappa(\mathfrak{q})$ are equal and isomorphisms as $s \circ e = t \circ e = \mathrm{id}_U$. In particular we see that $\mathfrak{q}$ is a maximal ideal as well. The ring maps $s, t : A \to B$ are of finite presentation and flat. By assumption the ring

$$\mathcal{O}_{F_u, e(u)} = B_{\mathfrak{q}}/s(\mathfrak{m})B_{\mathfrak{q}}$$

is Cohen-Macaulay of dimension d_2. The equality of dimension holds by Morphisms, Lemma 28.29.1.

Let R'' be the restriction of R to $u = \mathrm{Spec}(\kappa(u))$ via the morphism $\mathrm{Spec}(\kappa(u)) \to U$. As $u \to U$ is locally of finite type, we see that $(\mathrm{Spec}(\kappa(u)), R'', s'', t'', c'')$ is a groupoid scheme with s'', t'' locally of finite type, see Lemma 39.8.1. By Lemma 39.9.9 this implies that $\dim(G'') = \dim(R'')$. We also have $\dim(R'') = \dim_{e''}(R'') = \dim(\mathcal{O}_{R'', e''})$, see Lemma 39.9.8. By Groupoids, Lemma 38.16.4 we have $G'' = G_u$. Hence we conclude that $\dim(\mathcal{O}_{R'', e''}) = d_1$.

As a scheme R'' is

$$R'' = R \times_{(U \times_S U)} \Big(\mathrm{Spec}(\kappa(\mathfrak{m})) \times_S \mathrm{Spec}(\kappa(\mathfrak{m})) \Big)$$

Hence an affine open neighbourhood of e'' is the spectrum of the ring

$$B \otimes_{(A \otimes A)} (\kappa(\mathfrak{m}) \otimes \kappa(\mathfrak{m})) = B/s(\mathfrak{m})B + t(\mathfrak{m})B$$

We conclude that

$$\mathcal{O}_{R'', e''} = B_{\mathfrak{q}}/s(\mathfrak{m})B_{\mathfrak{q}} + t(\mathfrak{m})B_{\mathfrak{q}}$$

and so now we know that this ring has dimension d_1.

We claim this implies we can find an element $f \in \mathfrak{m}$ such that

$$\dim(B_\mathfrak{q}/(s(\mathfrak{m})B_\mathfrak{q} + fB_\mathfrak{q}) < d_2$$

Namely, suppose $\mathfrak{n}_j \supset s(\mathfrak{m})B_\mathfrak{q}$, $j = 1, \ldots, m$ correspond to the minimal primes of the local ring $B_\mathfrak{q}/s(\mathfrak{m})B_\mathfrak{q}$. There are finitely many as this ring is Noetherian (since it is essentially of finite type over a field – but also because a Cohen-Macaulay ring is Noetherian). By the Cohen-Macaulay condition we have $\dim(B_\mathfrak{q}/\mathfrak{n}_j) = d_2$, for example by Algebra, Lemma 10.99.4. Note that $\dim(B_\mathfrak{q}/(\mathfrak{n}_j + t(\mathfrak{m})B_\mathfrak{q})) \leq d_1$ as it is a quotient of the ring $\mathcal{O}_{R'',e''} = B_\mathfrak{q}/s(\mathfrak{m})B_\mathfrak{q} + t(\mathfrak{m})B_\mathfrak{q}$ which has dimension d_1. As $d_1 < d_2$ this implies that $\mathfrak{m} \not\subset t^{-1}(\mathfrak{n}_i)$. By prime avoidance, see Algebra, Lemma 10.14.3, we can find $f \in \mathfrak{m}$ with $t(f) \notin \mathfrak{n}_j$ for $j = 1, \ldots, m$. For this choice of f we have the displayed inequality above, see Algebra, Lemma 10.57.11.

Set $A' = A/fA$ and $U' = \mathrm{Spec}(A')$. Then it is clear that $U' \to U$ is an immersion, locally of finite presentation and that $u \in U'$. Thus (1), (2) and (3) of the lemma hold. The morphism

$$U' \times_{g,U,t} R \longrightarrow U$$

factors through $\mathrm{Spec}(A)$ and corresponds to the ring map

$$B/t(f)B = A/(f) \otimes_{A,t} B \xleftarrow{s} A$$

Now, we see $t(f)$ is not a zerodivisor on $B_\mathfrak{q}/s(\mathfrak{m})B_\mathfrak{q}$ as this is a Cohen-Macaulay ring of positive dimension and f is not contained in any minimal prime, see for example Algebra, Lemma 10.99.2. Hence by Algebra, Lemma 10.122.5 we conclude that $s : A_\mathfrak{m} \to B_\mathfrak{q}/t(f)B_\mathfrak{q}$ is flat with fibre ring $B_\mathfrak{q}/(s(\mathfrak{m})B_\mathfrak{q} + t(f)B_\mathfrak{q})$ which is Cohen-Macaulay by Algebra, Lemma 10.99.2 again. This implies part (4) of the lemma. To see part (5) note that by Diagram (39.8.0.1) the fibre F'_u is equal to the fibre of h over u. Hence $\dim_{e'(u)}(F'_u) = \dim(B_\mathfrak{q}/(s(\mathfrak{m})B_\mathfrak{q} + t(f)B_\mathfrak{q}))$ by Morphisms, Lemma 28.29.1 and the dimension of this ring is $d_2 - 1$ by Algebra, Lemma 10.99.2 once more. This proves the final assertion of the lemma and we win. □

Now that we know how to slice we can combine it with the preceding material to get the following "optimal" result. It is optimal in the sense that since G_u is a locally closed subscheme of F_u one always has the inequality $\dim(G_u) = \dim_{e(u)}(G_u) \leq \dim_{e(u)}(F_u)$ so it is not possible to slice more than in the lemma.

Lemma 39.11.3. *Let S be a scheme. Let (U, R, s, t, c, e, i) be a groupoid scheme over S. Let $G \to U$ be the stabilizer group scheme. Assume s and t are Cohen-Macaulay and locally of finite presentation. Let $u \in U$ be a finite type point of the scheme U, see Morphisms, Definition 28.17.3. With notation as in Situation 39.11.1 there exist an affine scheme U' and a morphism $g : U' \to U$ such that*

(1) *g is an immersion,*
(2) *$u \in U'$,*
(3) *g is locally of finite presentation,*
(4) *the morphism $h : U' \times_{g,U,t} R \longrightarrow U$ is Cohen-Macaulay and locally of finite presentation,*
(5) *the morphisms $s', t' : R' \to U'$ are Cohen-Macaulay and locally of finite presentation, and*
(6) *$\dim_{e(u)}(F'_u) = \dim(G'_u)$.*

Proof. As s is locally of finite presentation the scheme F_u is locally of finite type over $\kappa(u)$. Hence $\dim_{e(u)}(F_u) < \infty$ and we may argue by induction on $\dim_{e(u)}(F_u)$.

If $\dim_{e(u)}(F_u) = \dim(G_u)$ there is nothing to prove. Assume $\dim_{e(u)}(F_u) > \dim(G_u)$. This means that Lemma 39.11.2 applies and we find a morphism $g : U' \to U$ which has properties (1), (2), (3), instead of (6) we have $\dim_{e(u)}(F'_u) < \dim_{e(u)}(F_u)$, and instead of (4) and (5) we have that the composition

$$h = s \circ \mathrm{pr}_1 : U' \times_{g,U,t} R \longrightarrow U$$

is Cohen-Macaulay at the point $(u, e(u))$. We apply Remark 39.5.3 and we obtain an open subscheme $U'' \subset U'$ such that $U'' \times_{g,U,t} R \subset U' \times_{g,U,t} R$ is the largest open subscheme on which h is Cohen-Macaulay. Since $(u, e(u)) \in U'' \times_{g,U,t} R$ we see that $u \in U''$. Hence we may replace U' by U'' and assume that in fact h is Cohen-Macaulay everywhere! By Lemma 39.8.2 we conclude that s', t' are locally of finite presentation and Cohen-Macaulay (use Morphisms, Lemma 28.22.4 and More on Morphisms, Lemma 36.17.3).

By construction $\dim_{e'(u)}(F'_u) < \dim_{e(u)}(F_u)$, so we may apply the induction hypothesis to (U', R', s', t', c') and the point $u \in U'$. Note that u is also a finite type point of U' (for example you can see this using the characterization of finite type points from Morphisms, Lemma 28.17.4). Let $g' : U'' \to U'$ and $(U'', R'', s'', t'', c'')$ be the solution of the corresponding problem starting with (U', R', s', t', c') and the point $u \in U'$. We claim that the composition

$$g'' = g \circ g' : U'' \longrightarrow U$$

is a solution for the original problem. Properties (1), (2), (3), (5), and (6) are immediate. To see (4) note that the morphism

$$h'' = s \circ \mathrm{pr}_1 : U'' \times_{g'',U,t} R \longrightarrow U$$

is locally of finite presentation and Cohen-Macaulay by an application of Lemma 39.8.4 (use More on Morphisms, Lemma 36.17.8 to see that Cohen-Macaulay morphisms are fppf local on the target). □

In case the stabilizer group scheme has fibres of dimension 0 this leads to the following slicing lemma.

Lemma 39.11.4. *Let S be a scheme. Let (U, R, s, t, c, e, i) be a groupoid scheme over S. Let $G \to U$ be the stabilizer group scheme. Assume s and t are Cohen-Macaulay and locally of finite presentation. Let $u \in U$ be a finite type point of the scheme U, see Morphisms, Definition 28.17.3. Assume that $G \to U$ is locally quasi-finite. With notation as in Situation 39.11.1 there exist an affine scheme U' and a morphism $g : U' \to U$ such that*

1. *g is an immersion,*
2. *$u \in U'$,*
3. *g is locally of finite presentation,*
4. *the morphism $h : U' \times_{g,U,t} R \longrightarrow U$ is flat, locally of finite presentation, and locally quasi-finite, and*
5. *the morphisms $s', t' : R' \to U'$ are flat, locally of finite presentation, and locally quasi-finite.*

Proof. Take $g : U' \to U$ as in Lemma 39.11.3. Since $h^{-1}(u) = F'_u$ we see that h has relative dimension ≤ 0 at $(u, e(u))$. Hence, by Remark 39.5.3, we obtain an open subscheme $U'' \subset U'$ such that $u \in U''$ and $U'' \times_{g,U,t} R$ is the maximal open subscheme of $U' \times_{g,U,t} R$ on which h has relative dimension ≤ 0. After replacing U' by U'' we see that h has relative dimension ≤ 0. This implies that h is locally quasi-finite by Morphisms, Lemma 28.30.5. Since it is still locally of finite presentation and Cohen-Macaulay we see that it is flat, locally of finite presentation and locally quasi-finite, i.e., (4) above holds. This implies that s' is flat, locally of finite presentation and locally quasi-finite as a base change of h, see Lemma 39.8.2. □

39.12. Étale localization of groupoids

In this section we begin applying the étale localization techniques of More on Morphisms, Section 36.30 to groupoid schemes. More advanced material of this kind can be found in More on Groupoids in Spaces, Section 60.11. Lemma 39.12.2 will be used to prove results on algebraic spaces separated and quasi-finite over a scheme, namely Morphisms of Spaces, Proposition 49.43.2 and its corollary Morphisms of Spaces, Lemma 49.44.1.

Lemma 39.12.1. *Let S be a scheme. Let (U, R, s, t, c) be a groupoid scheme over S. Let $p \in S$ be a point, and let $u \in U$ be a point lying over p. Assume that*

(1) *$U \to S$ is locally of finite type,*
(2) *$U \to S$ is quasi-finite at u,*
(3) *$U \to S$ is separated,*
(4) *$R \to S$ is separated,*
(5) *s, t are flat and locally of finite presentation, and*
(6) *$s^{-1}(\{u\})$ is finite.*

Then there exists an étale neighbourhood $(S', p') \to (S, p)$ with $\kappa(p) = \kappa(p')$ and a base change diagram

$$\begin{array}{ccccc} R' \coprod W' & = & S' \times_S R & \longrightarrow & R \\ & & \downarrow{\scriptstyle t'} \quad \downarrow{\scriptstyle s'} & & \downarrow{\scriptstyle t} \quad \downarrow{\scriptstyle s} \\ U' \coprod W & = & S' \times_S U & \longrightarrow & U \\ & & \downarrow & & \downarrow \\ & & S' & \longrightarrow & S \end{array}$$

where the equal signs are decompositions into open and closed subschemes such that

(a) *there exists a point u' of U' mapping to u in U,*
(b) *the fibre $(U')_{p'}$ equals $t'\big((s')^{-1}(\{u'\})\big)$ set theoretically,*
(c) *the fibre $(R')_{p'}$ equals $(s')^{-1}\big((U')_{p'}\big)$ set theoretically,*
(d) *the schemes U' and R' are finite over S',*
(e) *we have $s'(R') \subset U'$ and $t'(R') \subset U'$,*
(f) *we have $c'(R' \times_{s',U',t'} R') \subset R'$ where c' is the base change of c, and*
(g) *the morphisms s', t', c' determine a groupoid structure by taking the system $(U', R', s'|_{R'}, t'|_{R'}, c'|_{R' \times_{s',U',t'} R'})$.*

Proof. Let us denote $f : U \to S$ the structure morphism of U. By assumption (6) we can write $s^{-1}(\{u\}) = \{r_1, \ldots, r_n\}$. Since this set is finite, we see that s is quasi-finite at each of these finitely many inverse images, see Morphisms, Lemma 28.21.7. Hence we see that $f \circ s : R \to S$ is quasi-finite at each r_i (Morphisms, Lemma 28.21.12). Hence r_i is isolated in the fibre R_p, see Morphisms, Lemma 28.21.6. Write $t(\{r_1, \ldots, r_n\}) = \{u_1, \ldots, u_m\}$. Note that it may happen that $m < n$ and note that $u \in \{u_1, \ldots, u_m\}$. Since t is flat and locally of finite presentation, the morphism of fibres $t_p : R_p \to U_p$ is flat and locally of finite presentation (Morphisms, Lemmas 28.26.7 and 28.22.4), hence open (Morphisms, Lemma 28.26.9). The fact that each r_i is isolated in R_p implies that each $u_j = t(r_i)$ is isolated in U_p. Using Morphisms, Lemma 28.21.6 again, we see that f is quasi-finite at $u_1, \ldots, u_m$.

Denote $F_u = s^{-1}(u)$ and $F_{u_j} = s^{-1}(u_j)$ the scheme theoretic fibres. Note that F_u is finite over $\kappa(u)$ as it is locally of finite type over $\kappa(u)$ with finitely many points (for example it follows from the much more general Morphisms, Lemma 28.50.8). By Lemma 39.6.1 we see that F_u and F_{u_j} become isomorphic over a common field extension of $\kappa(u)$ and $\kappa(u_j)$. Hence we see that F_{u_j} is finite over $\kappa(u_j)$. In particular we see $s^{-1}(\{u_j\})$ is a finite set for each $j = 1, \ldots, m$. Thus we see that assumptions (2) and (6) hold for each u_j also (above we saw that $U \to S$ is quasi-finite at u_j). Hence the argument of the first paragraph applies to each u_j and we see that $R \to U$ is quasi-finite at each of the points of

$$\{r_1, \ldots, r_N\} = s^{-1}(\{u_1, \ldots, u_m\})$$

Note that $t(\{r_1, \ldots, r_N\}) = \{u_1, \ldots, u_m\}$ and $t^{-1}(\{u_1, \ldots, u_m\}) = \{r_1, \ldots, r_N\}$ since R is a groupoid[2]. Moreover, we have $\mathrm{pr}_0(c^{-1}(\{r_1, \ldots, r_N\})) = \{r_1, \ldots, r_N\}$ and $\mathrm{pr}_1(c^{-1}(\{r_1, \ldots, r_N\})) = \{r_1, \ldots, r_N\}$. Similarly we get $e(\{u_1, \ldots, u_m\}) \subset \{r_1, \ldots, r_N\}$ and $i(\{r_1, \ldots, r_N\}) = \{r_1, \ldots, r_N\}$.

We may apply More on Morphisms, Lemma 36.30.4 to the pairs $(U \to S, \{u_1, \ldots, u_m\})$ and $(R \to S, \{r_1, \ldots, r_N\})$ to get an étale neighbourhood $(S', p') \to (S, p)$ which induces an identification $\kappa(p) = \kappa(p')$ such that $S' \times_S U$ and $S' \times_S R$ decompose as

$$S' \times_S U = U' \coprod W, \quad S' \times_S R = R' \coprod W'$$

with $U' \to S'$ finite and $(U')_{p'}$ mapping bijectively to $\{u_1, \ldots, u_m\}$, and $R' \to S'$ finite and $(R')_{p'}$ mapping bijectively to $\{r_1, \ldots, r_N\}$. Moreover, no point of $W_{p'}$ (resp. $(W')_{p'}$) maps to any of the points u_j (resp. r_i). At this point (a), (b), (c), and (d) of the lemma are satisfied. Moreover, the inclusions of (e) and (f) hold on fibres over p', i.e., $s'((R')_{p'}) \subset (U')_{p'}$, $t'((R')_{p'}) \subset (U')_{p'}$, and $c'((R' \times_{s',U',t'} R')_{p'}) \subset (R')_{p'}$.

We claim that we can replace S' by a Zariski open neighbourhood of p' so that the inclusions of (e) and (f) hold. For example, consider the set $E = (s'|_{R'})^{-1}(W)$. This is open and closed in R' and does not contain any points of R' lying over p'. Since $R' \to S'$ is closed, after replacing S' by $S' \setminus (R' \to S')(E)$ we reach a situation where E is empty. In other words s' maps R' into U'. Note that this property is preserved under further shrinking S'. Similarly, we can arrange it so that t' maps R' into U'. At this point (e) holds. In the same manner, consider the

[2]Explanation in groupoid language: The original set $\{r_1, \ldots, r_n\}$ was the set of arrows with source u. The set $\{u_1, \ldots, u_m\}$ was the set of objects isomorphic to u. And $\{r_1, \ldots, r_N\}$ is the set of all arrows between all the objects equivalent to u.

set $E = (c'|_{R' \times_{s',U',t'} R'})^{-1}(W')$. It is open and closed in the scheme $R' \times_{s',U',t'} R'$ which is finite over S', and does not contain any points lying over p'. Hence after replacing S' by $S' \setminus (R' \times_{s',U',t'} R' \to S')(E)$ we reach a situation where E is empty. In other words we obtain the inclusion in (f). We may repeat the argument also with the identity $e' : S' \times_S U \to S' \times_S R$ and the inverse $i' : S' \times_S R \to S' \times_S R$ so that we may assume (after shrinking S' some more) that $(e'|_{U'})^{-1}(W') = \emptyset$ and $(i'|_{R'})^{-1}(W') = \emptyset$.

At this point we see that we may consider the structure

$$(U', R', s'|_{R'}, t'|_{R'}, c'|_{R' \times_{t',U',s'} R'}, e'|_{U'}, i'|_{R'}).$$

The axioms of a groupoid scheme over S' hold because they hold for the groupoid scheme $(S' \times_S U, S' \times_S R, s', t', c', e', i')$. □

Lemma 39.12.2. *Let S be a scheme. Let (U, R, s, t, c) be a groupoid scheme over S. Let $p \in S$ be a point, and let $u \in U$ be a point lying over p. Assume assumptions (1) – (6) of Lemma 39.12.1 hold as well as*

(7) *$j : R \to U \times_S U$ is universally closed*[3].

Then we can choose $(S', p') \to (S, p)$ and decompositions $S' \times_S U = U' \amalg W$ and $S' \times_S R = R' \amalg W'$ and $u' \in U'$ such that (a) – (g) of Lemma 39.12.1 hold as well as

(h) *R' is the restriction of $S' \times_S R$ to U'.*

Proof. We apply Lemma 39.12.1 for the groupoid (U, R, s, t, c) over the scheme S with points p and u. Hence we get an étale neighbourhood $(S', p') \to (S, p)$ and disjoint union decompositions

$$S' \times_S U = U' \amalg W, \quad S' \times_S R = R' \amalg W'$$

and $u' \in U'$ satisfying conclusions (a), (b), (c), (d), (e), (f), and (g). We may shrink S' to a smaller neighbourhood of p' without affecting the conclusions (a) – (g). We will show that for a suitable shrinking conclusion (h) holds as well. Let us denote j' the base change of j to S'. By conclusion (e) it is clear that

$$j'^{-1}(U' \times_{S'} U') = R' \amalg Rest$$

for some open and closed $Rest$ piece. Since $U' \to S'$ is finite by conclusion (d) we see that $U' \times_{S'} U'$ is finite over S'. Since j is universally closed, also j' is universally closed, and hence $j'|_{Rest}$ is universally closed too. By conclusions (b) and (c) we see that the fibre of

$$(U' \times_{S'} U' \to S') \circ j'|_{Rest} : Rest \longrightarrow S'$$

over p' is empty. Hence, since $Rest \to S'$ is closed as a composition of closed morphisms, after replacing S' by $S' \setminus \mathrm{Im}(Rest \to S')$, we may assume that $Rest = \emptyset$. And this is exactly the condition that R' is the restriction of $S' \times_S R$ to the open subscheme $U' \subset S' \times_S U$, see Groupoids, Lemma 38.16.3 and its proof. □

39.13. Other chapters

Preliminaries

(1) Introduction
(2) Conventions
(3) Set Theory
(4) Categories

[3] In view of the other conditions this is equivalent to requiring j to be proper.

(5) Topology
(6) Sheaves on Spaces
(7) Sites and Sheaves
(8) Stacks
(9) Fields
(10) Commutative Algebra
(11) Brauer Groups
(12) Homological Algebra
(13) Derived Categories
(14) Simplicial Methods
(15) More on Algebra
(16) Smoothing Ring Maps
(17) Sheaves of Modules
(18) Modules on Sites
(19) Injectives
(20) Cohomology of Sheaves
(21) Cohomology on Sites
(22) Differential Graded Algebra
(23) Divided Power Algebra
(24) Hypercoverings

Schemes

(25) Schemes
(26) Constructions of Schemes
(27) Properties of Schemes
(28) Morphisms of Schemes
(29) Cohomology of Schemes
(30) Divisors
(31) Limits of Schemes
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(33) Topologies on Schemes
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(39) More on Groupoid Schemes
(40) Étale Morphisms of Schemes

Topics in Scheme Theory

(41) Chow Homology
(42) Adequate Modules
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Algebraic Spaces

(47) Algebraic Spaces
(48) Properties of Algebraic Spaces
(49) Morphisms of Algebraic Spaces
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Deformation Theory

(64) Formal Deformation Theory
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Algebraic Stacks

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(68) Examples of Stacks
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CHAPTER 40

Étale Morphisms of Schemes

40.1. Introduction

In this Chapter, we discuss étale morphisms of schemes. We illustrate some of the more important concepts by working with the Noetherian case. Our principal goal is to collect for the reader with enough commutative algebra results to start reading a treatise on étale cohomology. An auxiliary goal is to provide enough evidence to ensure that the reader stops calling the phrase "the étale topology of schemes" an exercise in general nonsense, if (s)he does indulge in such blasphemy.

We will refer to the other chapters of the stacks project for standard results in algebraic geometry (on schemes and commutative algebra). We will provide detailed proofs of the new results that we state here.

40.2. Conventions

In this chapter, frequently schemes will be assumed locally Noetherian and frequently rings will be assumed Noetherian. But in all the statements we will reiterate this when necessary, and make sure we list all the hypotheses! On the other hand, here are some general facts that we will use often and are useful to keep in mind:

(1) A ring homomorphism $A \to B$ of finite type with A Noetherian is of finite presentation. See Algebra, Lemma 10.29.4.
(2) A morphism (locally) of finite type between locally Noetherian schemes is automatically (locally) of finite presentation. See Morphisms, Lemma 28.22.9.
(3) Add more like this here.

40.3. Unramified morphisms

We first define the notion of unramified morphisms for local rings, and then globalize it to get one for arbitrary schemes.

Definition 40.3.1. Let A, B be Noetherian local rings. A local homomorphism $A \to B$ is said to be *unramified homomorphism of local rings* if

(1) $\mathfrak{m}_A B = \mathfrak{m}_B$,
(2) $\kappa(\mathfrak{m}_B)$ is a finite separable extension of $\kappa(\mathfrak{m}_A)$, and
(3) B is essentially of finite type over A (this means that B is the localization of a finite type A-algebra at a prime).

This is the local version of the definition in Algebra, Section 10.142. In that section a ring map $R \to S$ is defined to be unramified if and only if it is of finite type, and

$\Omega_{S/R} = 0$. It is shown in Algebra, Lemmas 10.142.5 and 10.142.7 that given a ring map $R \to S$ of finite type, and a prime $\mathfrak{q}$ of S lying over $\mathfrak{p} \subset R$, then we have

$$R \to S \text{ is unramified at } \mathfrak{q} \Leftrightarrow \mathfrak{p}S_{\mathfrak{q}} = \mathfrak{q}S_{\mathfrak{q}} \text{ and } \kappa(\mathfrak{p}) \subset \kappa(\mathfrak{q}) \text{ finite separable}$$

Thus we see that for a local homomorphism of local rings the properties of our definition above are closely related to the question of being unramified. In fact, we have proved the following lemma.

Lemma 40.3.2. *Let $A \to B$ be of finite type with A a Noetherian ring. Let $\mathfrak{q}$ be a prime of B lying over $\mathfrak{p} \subset A$. Then $A \to B$ is unramified at $\mathfrak{q}$ if and only if $A_{\mathfrak{p}} \to B_{\mathfrak{q}}$ is an unramified homomorphism of local rings.*

Proof. See discussion above. □

We will characterize the property of being unramified in terms of completions. For a Noetherian local ring A we denote $A^\wedge$ the completion of A with respect to the maximal ideal. It is also a Noetherian local ring, see Algebra, Lemma 10.91.10.

Lemma 40.3.3. *Let A, B be Noetherian local rings. Let $A \to B$ be a local homomorphism.*

1. *if $A \to B$ is an unramified homomorphism of local rings, then $B^\wedge$ is a finite $A^\wedge$ module,*
2. *if $A \to B$ is an unramified homomorphism of local rings and $\kappa(\mathfrak{m}_A) = \kappa(\mathfrak{m}_B)$, then $A^\wedge \to B^\wedge$ is surjective,*
3. *if $A \to B$ is an unramified homomorphism of local rings and $\kappa(\mathfrak{m}_A)$ is separably closed, then $A^\wedge \to B^\wedge$ is surjective,*
4. *if A and B are complete discrete valuation rings, then $A \to B$ is an unramified homomorphism of local rings if and only the uniformizer for A maps to a uniformizer for B, and the residue field extension is finite separable (and B is essentially of finite type over A).*

Proof. Part (1) is a special case of Algebra, Lemma 10.91.18. For part (2), note that the $\kappa(\mathfrak{m}_A)$-vector space $B^\wedge/\mathfrak{m}_{A^\wedge}B^\wedge$ is generated by 1. Hence by Nakayama's lemma (Algebra, Lemma 10.18.1) the map $A^\wedge \to B^\wedge$ is surjective. Part (3) is a special case of part (2). Part (4) is immediate from the definitions. □

Lemma 40.3.4. *Let A, B be Noetherian local rings. Let $A \to B$ be a local homomorphism such that B is essentially of finite type over A. The following are equivalent*

1. *$A \to B$ is an unramified homomorphism of local rings*
2. *$A^\wedge \to B^\wedge$ is an unramified homomorphism of local rings, and*
3. *$A^\wedge \to B^\wedge$ is unramified.*

Proof. The equivalence of (1) and (2) follows from the fact that $\mathfrak{m}_A A^\wedge$ is the maximal ideal of $A^\wedge$ (and similarly for B) and faithful flatness of $B \to B^\wedge$. For example if $A^\wedge \to B^\wedge$ is unramified, then $\mathfrak{m}_A B^\wedge = (\mathfrak{m}_A B)B^\wedge = \mathfrak{m}_B B^\wedge$ and hence $\mathfrak{m}_A B = \mathfrak{m}_B$.

Assume the equivalent conditions (1) and (2). By Lemma 40.3.3 we see that $A^\wedge \to B^\wedge$ is finite. Hence $A^\wedge \to B^\wedge$ is of finite presentation, and by Algebra, Lemma 10.142.7 we conclude that $A^\wedge \to B^\wedge$ is unramified at $\mathfrak{m}_{B^\wedge}$. Since $B^\wedge$ is local we conclude that $A^\wedge \to B^\wedge$ is unramified.

Assume (3). By Algebra, Lemma 10.142.5 we conclude that $A^\wedge \to B^\wedge$ is an unramified homomorphism of local rings, i.e., (2) holds. □

Definition 40.3.5. (See Morphisms, Definition 28.36.1 for the definition in the general case.) Let Y be a locally Noetherian scheme. Let $f : X \to Y$ be locally of finite type. Let $x \in X$.

(1) We say f is *unramified at* x if $\mathcal{O}_{Y,f(x)} \to \mathcal{O}_{X,x}$ is an unramified homomorphism of local rings.
(2) The morphism $f : X \to Y$ is said to be *unramified* if it is unramified at all points of X.

Let us prove that this definition agrees with the definition in the chapter on morphisms of schemes. This in particular guarantees that the set of points where a morphism is unramified is open.

Lemma 40.3.6. *Let Y be a locally Noetherian scheme. Let $f : X \to Y$ be locally of finite type. Let $x \in X$. The morphism f is unramified at x in the sense of Definition 40.3.5 if and only if it is unramified in the sense of Morphisms, Definition 28.36.1.*

Proof. This follows from Lemma 40.3.2 and the definitions. □

Here are some results on unramified morphisms. The formulations as given in this list apply only to morphisms locally of finite type between locally Noetherian schemes. In each case we give a reference to the general result as proved earlier in the project, but in some cases one can prove the result more easily in the Noetherian case. Here is the list:

(1) Unramifiedness is local on the source and the target in the Zariski topology.
(2) Unramified morphisms are stable under base change and composition. See Morphisms, Lemmas 28.36.5 and 28.36.4.
(3) Unramified morphisms of schemes are locally quasi-finite and quasi-compact unramified morphisms are quasi-finite. See Morphisms, Lemma 28.36.10
(4) Unramified morphisms have relative dimension 0. See Morphisms, Definition 28.30.1 and Morphisms, Lemma 28.30.5.
(5) A morphism is unramified if and only if all its fibres are unramified. That is, unramifiedness can be checked on the scheme theoretic fibres. See Morphisms, Lemma 28.36.12.
(6) Let X and Y be unramified over a base scheme S. Any S-morphism from X to Y is unramified. See Morphisms, Lemma 28.36.16.

40.4. Three other characterizations of unramified morphisms

The following theorem gives three equivalent notions of being unramified at a point. See Morphisms, Lemma 28.36.14 for (part of) the statement for general schemes.

Theorem 40.4.1. *Let Y be a locally Noetherian scheme. Let $f : X \to Y$ be a morphism of schemes which is locally of finite type. Let x be a point of X. The following are equivalent*

(1) *f is unramified at x,*
(2) *the stalk $\Omega_{X/Y,x}$ of the module of relative differentials at x is trivial,*

(3) *there exist open neighbourhoods U of x and V of $f(x)$, and a commutative diagram*

$$\begin{array}{ccc} U & \xrightarrow{i} & \mathbf{A}^n_V \\ & \searrow \quad \swarrow & \\ & V & \end{array}$$

where i is a closed immersion defined by a quasi-coherent sheaf of ideals $\mathcal{I}$ such that the differentials dg for $g \in \mathcal{I}_{i(x)}$ generate $\Omega_{\mathbf{A}^n_V/V,i(x)}$, and

(4) *the diagonal $\Delta_{X/Y} : X \to X \times_Y X$ is a local isomorphism at x.*

Proof. The equivalence of (1) and (2) is proved in Morphisms, Lemma 28.36.14.

If f is unramified at x, then f is unramified in an open neighbourhood of x; this does not follow immediately from Definition 40.3.5 of this chapter but it does follow from Morphisms, Definition 28.36.1 which we proved to be equivalent in Lemma 40.3.6. Choose affine opens $V \subset Y$, $U \subset X$ with $f(U) \subset V$ and $x \in U$, such that f is unramified on U, i.e., $f|_U : U \to V$ is unramified. By Morphisms, Lemma 28.36.13 the morphism $U \to U \times_V U$ is an open immersion. This proves that (1) implies (4).

If $\Delta_{X/Y}$ is a local isomorphism at x, then $\Omega_{X/Y,x} = 0$ by Morphisms, Lemma 28.34.7. Hence we see that (4) implies (2). At this point we know that (1), (2) and (4) are all equivalent.

Assume (3). The assumption on the diagram combined with Morphisms, Lemma 28.34.15 show that $\Omega_{U/V,x} = 0$. Since $\Omega_{U/V,x} = \Omega_{X/Y,x}$ we conclude (2) holds.

Finally, assume that (2) holds. To prove (3) we may localize on X and Y and assume that X and Y are affine. Say $X = \mathrm{Spec}(B)$ and $Y = \mathrm{Spec}(A)$. The point $x \in X$ corresponds to a prime $\mathfrak{q} \subset B$. Our assumption is that $\Omega_{B/A,\mathfrak{q}} = 0$ (see Morphisms, Lemma 28.34.5 for the relationship between differentials on schemes and modules of differentials in commutative algebra). Since Y is locally Noetherian and f locally of finite type we see that A is Noetherian and $B \cong A[x_1, \ldots, x_n]/(f_1, \ldots, f_m)$, see Properties, Lemma 27.5.2 and Morphisms, Lemma 28.16.2. In particular, $\Omega_{B/A}$ is a finite B-module. Hence we can find a single $g \in B$, $g \notin \mathfrak{q}$ such that the principal localization $(\Omega_{B/A})_g$ is zero. Hence after replacing B by B_g we see that $\Omega_{B/A} = 0$ (formation of modules of differentials commutes with localization, see Algebra, Lemma 10.125.8). This means that $\mathrm{d}(f_j)$ generate the kernel of the canonical map $\Omega_{A[x_1,\ldots,x_n]/A} \otimes_A B \to \Omega_{B/A}$. Thus the surjection $A[x_1, \ldots, x_n] \to B$ of A-algebras gives the commutative diagram of (3), and the theorem is proved. □

How can we use this theorem? Well, here are a few remarks:

(1) Suppose that $f : X \to Y$ and $g : Y \to Z$ are two morphisms locally of finite type between locally Noetherian schemes. There is a canonical short exact sequence

$$f^*(\Omega_{Y/Z}) \to \Omega_{X/Z} \to \Omega_{X/Y} \to 0$$

see Morphisms, Lemma 28.34.9. The theorem therefore implies that if $g \circ f$ is unramified, then so is f. This is Morphisms, Lemma 28.36.16.

(2) Since $\Omega_{X/Y}$ is isomorphic to the conormal sheaf of the diagonal morphism (Morphisms, Lemma 28.34.7) we see that if $X \to Y$ is a monomorphism of

locally Noetherian schemes and locally of finite type, then $X \to Y$ is unramified. In particular, open and closed immersions of locally Noetherian schemes are unramified. See Morphisms, Lemmas 28.36.7 and 28.36.8.

(3) The theorem also implies that the set of points where a morphism $f : X \to Y$ (locally of finite type of locally Noetherian schemes) is not unramified is the support of the coherent sheaf $\Omega_{X/Y}$. This allows one to give a scheme theoretic definition to the "ramification locus".

40.5. The functorial characterization of unramified morphisms

In basic algebraic geometry we learn that some classes of morphisms can be characterized functorially, and that such descriptions are quite useful. Unramified morphisms too have such a characterization.

Theorem 40.5.1. *Let $f : X \to S$ be a morphism of schemes. Assume S is a locally Noetherian scheme, and f is locally of finite type. Then the following are equivalent:*

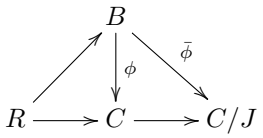
(1) *f is unramified,*
(2) *the morphism f is formally unramified: for any affine S-scheme T and subscheme T_0 of T defined by a square-zero ideal, the natural map*

$$\mathrm{Hom}_S(T, X) \longrightarrow \mathrm{Hom}_S(T_0, X)$$

is injective.

Proof. See More on Morphisms, Lemma 36.4.8 for a more general statement and proof. What follows is a sketch of the proof in the current case.

Firstly, one checks both properties are local on the source and the target. This we may assume that S and X are affine. Say $X = \mathrm{Spec}(B)$ and $S = \mathrm{Spec}(R)$. Say $T = \mathrm{Spec}(C)$. Let J be the square-zero ideal of C with $T_0 = \mathrm{Spec}(C/J)$. Assume that we are given the diagram



Secondly, one checks that the association $\phi' \mapsto \phi' - \phi$ gives a bijection between the set of liftings of $\bar{\phi}$ and the module $\mathrm{Der}_R(B, J)$. Thus, we obtain the implication (1) $\Rightarrow$ (2) via the description of unramified morphisms having trivial module of differentials, see Theorem 40.4.1.

To obtain the reverse implication, consider the surjection $q : C = (B \otimes_R B)/I^2 \to B = C/J$ defined by the square zero ideal $J = I/I^2$ where I is the kernel of the multiplication map $B \otimes_R B \to B$. We already have a lifting $B \to C$ defined by, say, $b \mapsto b \otimes 1$. Thus, by the same reasoning as above, we obtain a bijective correspondence between liftings of $\mathrm{id} : B \to C/J$ and $\mathrm{Der}_R(B, J)$. The hypothesis therefore implies that the latter module is trivial. But we know that $J \cong \Omega_{B/R}$. Thus, B/R is unramified. $\square$

40.6. Topological properties of unramified morphisms

The first topological result that will be of utility to us is one which says that unramified and separated morphisms have "nice" sections. The material in this section does not require any Noetherian hypotheses.

Proposition 40.6.1. *Sections of unramified morphisms.*

(1) *Any section of an unramified morphism is an open immersion.*
(2) *Any section of a separated morphism is a closed immersion.*
(3) *Any section of an unramified separated morphism is open and closed.*

Proof. Fix a base scheme S. If $f : X' \to X$ is any S-morphism, then the graph $\Gamma_f : X' \to X' \times_S X$ is obtained as the base change of the diagonal $\Delta_{X/S} : X \to X \times_S X$ via the projection $X' \times_S X \to X \times_S X$. If $g : X \to S$ is separated (resp. unramified) then the diagonal is a closed immersion (resp. open immersion) by Schemes, Definition 25.21.3 (resp. Morphisms, Lemma 28.36.13). Hence so is the graph as a base change (by Schemes, Lemma 25.18.2). In the special case $X' = S$, we obtain (1), resp. (2). Part (3) follows on combining (1) and (2). □

We can now explicitly describe the sections of unramified morphisms.

Theorem 40.6.2. *Let Y be a connected scheme. Let $f : X \to Y$ be unramified and separated. Every section of f is an isomorphism onto a connected component. There exists a bijective correspondence*

$$\text{sections of } f \leftrightarrow \left\{ \begin{array}{c} \text{connected components } X' \text{ of } X \text{ such that} \\ \text{the induced map } X' \to Y \text{ is an isomorphism} \end{array} \right\}$$

In particular, given $x \in X$ there is at most one section passing through x.

Proof. Direct from Proposition 40.6.1 part (3). □

The preceding theorem gives us some idea of the "rigidity" of unramified morphisms. Further indication is provided by the following proposition which, besides being intrinsically interesting, is also useful in the theory of the algebraic fundamental group (see [**Gro71**, Exposé V]). See also the more general Morphisms, Lemma 28.36.17.

Proposition 40.6.3. *Let S is be a scheme. Let $\pi : X \to S$ be unramified and separated. Let Y be an S-scheme and $y \in Y$ a point. Let $f, g : Y \to X$ be two S-morphisms. Assume*

(1) *Y is connected*
(2) *$x = f(y) = g(y)$, and*
(3) *the induced maps $f^\sharp, g^\sharp : \kappa(x) \to \kappa(y)$ on residue fields are equal.*

Then $f = g$.

Proof. The maps $f, g : Y \to X$ define maps $f', g' : Y \to X_Y = Y \times_S X$ which are sections of the structure map $X_Y \to Y$. Note that $f = g$ if and only if $f' = g'$. The structure map $X_Y \to Y$ is the base change of π and hence unramified and separated also (see Morphisms, Lemmas 28.36.5 and Schemes, Lemma 25.21.13). Thus according to Theorem 40.6.2 it suffices to prove that f' and g' pass through the same point of X_Y. And this is exactly what the hypotheses (2) and (3) guarantee, namely $f'(y) = g'(y) \in X_Y$. □

40.7. Universally injective, unramified morphisms

Recall that a morphism of schemes $f : X \to Y$ is universally injective if any base change of f is injective (on underlying topological spaces), see Morphisms, Definition 28.12.1. Universally injective and unramified morphisms can be characterized as follows.

Lemma 40.7.1. *Let $f : X \to S$ be a morphism of schemes. The following are equivalent:*

(1) *f is unramified and a monomorphism,*
(2) *f is unramified and universally injective,*
(3) *f is locally of finite type and a monomorphism,*
(4) *f is universally injective, locally of finite type, and formally unramified,*
(5) *f is locally of finite type and X_y is either empty or $X_y \to y$ is an isomorphism for all $y \in Y$.*

Proof. We have seen in More on Morphisms, Lemma 36.4.8 that being formally unramified and locally of finite type is the same thing as being unramified. Hence (4) is equivalent to (2). A monomorphism is certainly universally injective and formally unramified hence (3) implies (4). It is clear that (1) implies (3). Finally, if (2) holds, then $\Delta : X \to X \times_S X$ is both an open immersion (Morphisms, Lemma 28.36.13) and surjective (Morphisms, Lemma 28.12.2) hence an isomorphism, i.e., f is a monomorphism. In this way we see that (2) implies (1).

Condition (3) implies (5) because monomorphisms are preserved under base change (Schemes, Lemma 25.23.5) and because of the description of monomorphisms towards the spectra of fields in Schemes, Lemma 25.23.10. Condition (5) implies (4) by Morphisms, Lemmas 28.12.2 and 28.36.12. □

This leads to the following useful characterization of closed immersions.

Lemma 40.7.2. *Let $f : X \to S$ be a morphism of schemes. The following are equivalent:*

(1) *f is a closed immersion,*
(2) *f is a proper monomorphism,*
(3) *f is proper, unramified, and universally injective,*
(4) *f is universally closed, unramified, and a monomorphism,*
(5) *f is universally closed, unramified, and universally injective,*
(6) *f is universally closed, locally of finite type, and a monomorphism,*
(7) *f is universally closed, universally injective, locally of finite type, and formally unramified.*

Proof. The equivalence of (4) – (7) follows immediately from Lemma 40.7.1.

Let $f : X \to S$ satisfy (6). Then f is separated, see Schemes, Lemma 25.23.3 and has finite fibres. Hence More on Morphisms, Lemma 36.31.5 shows f is finite. Then Morphisms, Lemma 28.44.13 implies f is a closed immersion, i.e., (1) holds.

Note that (1) $\Rightarrow$ (2) because a closed immersion is proper and a monomorphism (Morphisms, Lemma 28.42.6 and Schemes, Lemma 25.23.7). By Lemma 40.7.1 we see that (2) implies (3). It is clear that (3) implies (5). □

Here is another result of a similar flavor.

Lemma 40.7.3. *Let $\pi : X \to S$ be a morphism of schemes. Let $s \in S$. Assume that*

(1) *π is finite,*
(2) *π is unramified,*
(3) *$\pi^{-1}(\{s\}) = \{x\}$, and*
(4) *$\kappa(s) \subset \kappa(x)$ is purely inseparable*[1].

Then there exists an open neighbourhood U of s such that $\pi|_{\pi^{-1}(U)} : \pi^{-1}(U) \to U$ is a closed immersion.

Proof. The question is local on S. Hence we may assume that $S = \mathrm{Spec}(A)$. By definition of a finite morphism this implies $X = \mathrm{Spec}(B)$. Note that the ring map $\varphi : A \to B$ defining π is a finite unramified ring map. Let $\mathfrak{p} \subset A$ be the prime corresponding to s. Let $\mathfrak{q} \subset B$ be the prime corresponding to x. By Conditions (2), (3) and (4) imply that $B_\mathfrak{q}/\mathfrak{p}B_\mathfrak{q} = \kappa(\mathfrak{p})$. Algebra, Lemma 10.38.11 we have $B_\mathfrak{q} = B_\mathfrak{p}$ (note that a finite ring map satisfies going up, see Algebra, Section 10.38.) Hence we see that $B_\mathfrak{p}/\mathfrak{p}B_\mathfrak{p} = \kappa(\mathfrak{p})$. As B is a finite A-module we see from Nakayama's lemma (see Algebra, Lemma 10.18.1) that $B_\mathfrak{p} = \varphi(A_\mathfrak{p})$. Hence (using the finiteness of B as an A-module again) there exists a $f \in A$, $f \notin \mathfrak{p}$ such that $B_f = \varphi(A_f)$ as desired. □

The topological results presented above will be used to give a functorial characterization of étale morphisms similar to Theorem 40.5.1.

40.8. Examples of unramified morphisms

Here are a few examples.

Example 40.8.1. Let k be a field. Unramified quasi-compact morphisms $X \to \mathrm{Spec}(k)$ are affine. This is true because X has dimension 0 and is Noetherian, hence is a finite discrete set, and each point gives an affine open, so X is a finite disjoint union of affines hence affine. Noether normalization forces X to be the spectrum of a finite k-algebra A. This algebra is a product of finite separable field extensions of k. Thus, an unramified quasi-compact morphism to $\mathrm{Spec}(k)$ corresponds to a finite number of finite separable field extensions of k. In particular, an unramified morphism with a connected source and a one point target is forced to be a finite separable field extension. As we will see later, $X \to \mathrm{Spec}(k)$ is étale if and only if it is unramified. Thus, in this case at least, we obtain a very easy description of the étale topology of a scheme. Of course, the cohomology of this topology is another story.

Example 40.8.2. Property (3) in Theorem 40.4.1 gives us a canonical source of examples for unramified morphisms. Fix a ring R and an integer n. Let $I = (g_1, \cdots, g_m)$ be an ideal in $R[x_1, \cdots, x_n]$. Let $\mathfrak{q} \subset R[x_1, \ldots, x_n]$ be a prime. Assume $I \subset \mathfrak{q}$ and that the matrix

$$\left(\frac{\partial g_i}{\partial x_j}\right) \bmod \mathfrak{q} \quad \in \quad \mathrm{Mat}(n \times m, \kappa(\mathfrak{q}))$$

has rank n. Then the morphism $f : Z = \mathrm{Spec}(R[x_1, \cdots, x_n]/I) \to \mathrm{Spec}(R)$ is unramified at the point $x \in Z \subset \mathbf{A}^n_R$ corresponding to $\mathfrak{q}$. Clearly we must have $m \geq n$. In the extreme case $m = n$, i.e., the differential of the map $\mathbf{A}^n_R \to \mathbf{A}^n_R$

[1] In view of condition (2) this is equivalent to $\kappa(s) = \kappa(x)$.

defined by the g_i's is an isomorphism of the tangent spaces, then f is also flat x and, hence, is an étale map (see Algebra, Definition 10.130.6, Lemma 10.130.7 and Example 10.130.8).

Example 40.8.3. Fix an extension of number fields L/K with rings of integers $\mathcal{O}_L$ and $\mathcal{O}_K$. The injection $K \to L$ defines a morphism $f : \mathrm{Spec}(\mathcal{O}_L) \to \mathrm{Spec}(\mathcal{O}_K)$. As discussed above, the points where f is unramified in our sense correspond to the set of points where f is unramified in the conventional sense. In the conventional sense, the locus of ramification in $\mathrm{Spec}(\mathcal{O}_L)$ can be defined by vanishing set of the different; this is an ideal in $\mathcal{O}_L$. In fact, the different is nothing but the annihilator of the module $\Omega_{\mathcal{O}_L/\mathcal{O}_K}$. Similarly, the discriminant is an ideal in $\mathcal{O}_K$, namely it is the norm of the different. The vanishing set of the discriminant is precisely the set of points of K which ramify in L. Thus, denoting by X the complement of the closed subset defined by the different in $\mathrm{Spec}(\mathcal{O}_L)$, we obtain a morphism $X \to \mathrm{Spec}(\mathcal{O}_L)$ which is unramified. Furthermore, this morphism is also flat, as any local homomorphism of discrete valuation rings is flat, and hence this morphism is actually étale. If L/K is Galois, then denoting by Y the complement of the closed subset defined by the discriminant in $\mathrm{Spec}(\mathcal{O}_K)$, we see that we get even a finite étale morphism $X \to Y$. Thus, this is an example of a finite étale covering.

40.9. Flat morphisms

This section simply exists to summarize the properties of flatness that will be useful to us. Thus, we will be content with stating the theorems precisely and giving references for the proofs.

After briefly recalling the necessary facts about flat modules over Noetherian rings, we state a theorem of Grothendieck which gives sufficient conditions for "hyperplane sections" of certain modules to be flat.

Definition 40.9.1. Flatness of modules and rings.

(1) A module N over a ring A is said to be *flat* if the functor $M \mapsto M \otimes_A N$ is exact.
(2) If this functor is also faithful, we say that N is *faithfully flat* over A.
(3) A morphism of rings $f : A \to B$ is said to be *flat (resp. faithfully flat)* if the functor $M \mapsto M \otimes_A B$ is exact (resp. faithful and exact).

Here is a list of facts with references to the algebra chapter.

(1) Free and projective modules are flat. This is clear for free modules and follows for projective modules as they are direct summands of free modules and $\otimes$ commutes with direct sums.
(2) Flatness is a local property, that is, M is flat over A if and only if $M_{\mathfrak{p}}$ is flat over $A_{\mathfrak{p}}$ for all $\mathfrak{p} \in \mathrm{Spec}(A)$. See Algebra, Lemma 10.36.19.
(3) If M is a flat A-module and $A \to B$ is a ring map, then $M \otimes_A B$ is a flat B-module. See Algebra, Lemma 10.36.6.
(4) Finite flat modules over local rings are free. See Algebra, Lemma 10.73.4.
(5) If $f : A \to B$ is a morphism of arbitrary rings, f is flat if and only if the induced maps $A_{f^{-1}(\mathfrak{q})} \to B_{\mathfrak{q}}$ are flat for all $\mathfrak{q} \in \mathrm{Spec}(B)$. See Algebra, Lemma 10.36.19
(6) If $f : A \to B$ is a local homomorphism of local rings, f is flat if and only if it is faithfully flat. See Algebra, Lemma 10.36.16.

(7) A map $A \to B$ of rings is faithfully flat if and only if it is flat and the induced map on spectra is surjective. See Algebra, Lemma 10.36.15.
(8) If A is a noetherian local ring, the completion $A^\wedge$ is faithfully flat over A. See Algebra, Lemma 10.91.4.
(9) Let A be a Noetherian local ring and M an A-module. Then M is flat over A if and only if $M \otimes_A A^\wedge$ is flat over $A^\wedge$. (Combine the previous statement with Algebra, Lemma 10.36.7.)

Before we move on to the geometric category, we present Grothendieck's theorem, which provides a convenient recipe for producing flat modules.

Theorem 40.9.2. *Let A, B be Noetherian local rings. Let $f : A \to B$ be a local homomorphism. If M is a finite B-module that is flat as an A-module, and $t \in \mathfrak{m}_B$ is an element such that multiplication by t is injective on $M/\mathfrak{m}_A M$, then M/tM is also A-flat.*

Proof. See Algebra, Lemma 10.93.1. See also [**Mat70**, Section 20]. □

Definition 40.9.3. (See Morphisms, Definition 28.26.1). Let $f : X \to Y$ be a morphism of schemes. Let $\mathcal{F}$ be a quasi-coherent $\mathcal{O}_X$-module.

(1) Let $x \in X$. We say $\mathcal{F}$ is *flat over Y at $x \in X$* if $\mathcal{F}_x$ is a flat $\mathcal{O}_{Y,f(x)}$-module. This uses the map $\mathcal{O}_{Y,f(x)} \to \mathcal{O}_{X,x}$ to think of $\mathcal{F}_x$ as a $\mathcal{O}_{Y,f(x)}$-module.
(2) Let $x \in X$. We say f is *flat at $x \in X$* if $\mathcal{O}_{Y,f(x)} \to \mathcal{O}_{X,x}$ is flat.
(3) We say f is *flat* if it is flat at all points of X.
(4) A morphism $f : X \to Y$ that is flat and surjective is sometimes said to be *faithfully flat*.

Once again, here is a list of results:

(1) The property (of a morphism) of being flat is, by fiat, local in the Zariski topology on the source and the target.
(2) Open immersions are flat. (This is clear because it induces isomorphisms on local rings.)
(3) Flat morphisms are stable under base change and composition. Morphisms, Lemmas 28.26.7 and 28.26.5.
(4) If $f : X \to Y$ is flat, then the pullback functor $QCoh(\mathcal{O}_Y) \to QCoh(\mathcal{O}_X)$ is exact. This is immediate by looking at stalks.
(5) Let $f : X \to Y$ be a morphism of schemes, and assume Y is quasi-compact and quasi-separated. In this case if the functor f^* is exact then f is flat. (Proof omitted. Hint: Use Properties, Lemma 27.20.1 to see that Y has "enough" ideal sheaves and use the characterization of flatness in Algebra, Lemma 10.36.4.)

40.10. Topological properties of flat morphisms

We "recall" below some openness properties that flat morphisms enjoy.

Theorem 40.10.1. *Let Y be a locally Noetherian scheme. Let $f : X \to Y$ be a morphism which is locally of finite type. Let $\mathcal{F}$ be a coherent $\mathcal{O}_X$-module. The set of points in X where $\mathcal{F}$ is flat over S is an open set. In particular the set of points where f is flat is open in X.*

Proof. See More on Morphisms, Theorem 36.12.1. □

Theorem 40.10.2. *Let Y be a locally Noetherian scheme. Let $f : X \to Y$ be a morphism which is flat and locally of finite type. Then f is (universally) open.*

Proof. See Morphisms, Lemma 28.26.9. □

Theorem 40.10.3. *A faithfully flat quasi-compact morphism is a quotient map for the Zariski topology.*

Proof. See Morphisms, Lemma 28.26.10. □

An important reason to study flat morphisms is that they provide the adequate framework for capturing the notion of a family of schemes parametrized by the points of another scheme. Naively one may think that any morphism $f : X \to S$ should be thought of as a family parametrized by the points of S. However, without a flatness restriction on f, really bizarre things can happen in this so-called family. For instance, we aren't guaranteed that relative dimension (dimension of the fibres) is constant in a family. Other numerical invariants, such as the Hilbert polynomial, too may change from fibre to fibre. Flatness prevents such things from happening and, therefore, provides some "continuity" to the fibres.

40.11. Étale morphisms

In this section, we will define étale morphisms and prove a number of important properties about them. The most important one, no doubt, is the functorial characterization presented in Theorem 40.16.1. Following this, we will also discuss a few properties of rings which are insensitive to an étale extension (properties which hold for a ring if and only if they hold for all its étale extensions) to motivate the basic tenet of étale cohomology – étale morphisms are the algebraic analogue of local isomorphisms.

As the title suggests, we will define the class of étale morphisms – the class of morphisms (whose surjective families) we shall deem to be coverings in the category of schemes over a base scheme S in order to define the étale site $S_{étale}$. Intuitively, an étale morphism is supposed to capture the idea of a covering space and, therefore, should be close to a local isomorphism. If we're working with varieties over algebraically closed fields, this last statement can be made into a definition provided we replace "local isomorphism" with "formal local isomorphism" (isomorphism after completion). One can then give a definition over any base field by asking that the base change to the algebraic closure be étale (in the aforementioned sense). But, rather than proceeding via such aesthetically displeasing constructions, we will adopt a cleaner, albeit slightly more abstract, algebraic approach.

Definition 40.11.1. Let A, B be Noetherian local rings. A local homomorphism $f : A \to B$ is said to be a *étale homomorphism of local rings* if it is flat and unramified homomorphism of local rings (please see Definition 40.3.1).

This is the local version of the definition of an étale ring map in Algebra, Section 10.136. The exact definition given in that section is that it is a smooth ring map of relative dimension 0. It is shown (in Algebra, Lemma 10.136.2) that an étale R-algebra S always has a presentation

$$S = R[x_1, \ldots, x_n]/(f_1, \ldots, f_n)$$

such that

$$g = \det \begin{pmatrix} \partial f_1/\partial x_1 & \partial f_2/\partial x_1 & \ldots & \partial f_n/\partial x_1 \\ \partial f_1/\partial x_2 & \partial f_2/\partial x_2 & \ldots & \partial f_n/\partial x_2 \\ \ldots & \ldots & \ldots & \ldots \\ \partial f_1/\partial x_n & \partial f_2/\partial x_n & \ldots & \partial f_n/\partial x_n \end{pmatrix}$$

maps to an invertible element in S. The following two lemmas link the two notions.

Lemma 40.11.2. *Let $A \to B$ be of finite type with A a Noetherian ring. Let $\mathfrak{q}$ be a prime of B lying over $\mathfrak{p} \subset A$. Then $A \to B$ is étale at $\mathfrak{q}$ if and only if $A_\mathfrak{p} \to B_\mathfrak{q}$ is an étale homomorphism of local rings.*

Proof. See Algebra, Lemmas 10.136.3 (flatness of étale maps), 10.136.5 (étale maps are unramified) and 10.136.7 (flat and unramified maps are étale). □

Lemma 40.11.3. *Let A, B be Noetherian local rings. Let $A \to B$ be a local homomorphism such that B is essentially of finite type over A. The following are equivalent*

1. *$A \to B$ is an étale homomorphism of local rings*
2. *$A^\wedge \to B^\wedge$ is an étale homomorphism of local rings, and*
3. *$A^\wedge \to B^\wedge$ is étale.*

Moreover, in this case $B^\wedge \cong (A^\wedge)^{\oplus n}$ as $A^\wedge$-modules for some $n \geq 1$.

Proof. To see the equivalences of (1), (2) and (3), as we have the corresponding results for unramified ring maps (Lemma 40.3.4) it suffices to prove that $A \to B$ is flat if and only if $A^\wedge \to B^\wedge$ is flat. This is clear from our lists of properties of flat maps since the ring maps $A \to A^\wedge$ and $B \to B^\wedge$ are faithfully flat. For the final statement, by Lemma 40.3.3 we see that $B^\wedge$ is a finite flat $A^\wedge$ module. Hence it is finite free by our list of properties on flat modules in Section 40.9. □

The integer n which occurs in the lemma above is nothing other than the degree $[\kappa(\mathfrak{m}_B) : \kappa(\mathfrak{m}_A)]$ of the residue field extension. In particular, if $\kappa(\mathfrak{m}_A)$ is separably closed, we see that $A^\wedge \to B^\wedge$ is an isomorphism, which vindicates our earlier claims.

Definition 40.11.4. (See Morphisms, Definition 28.37.1.) Let Y be a locally Noetherian scheme. Let $f : X \to Y$ be a morphism of schemes which is locally of finite type.

1. Let $x \in X$. We say f is *étale at $x \in X$* if $\mathcal{O}_{Y,f(x)} \to \mathcal{O}_{X,x}$ is an étale homomorphism of local rings.
2. The morphism is said to be *étale* if it is étale at all its points.

Let us prove that this definition agrees with the definition in the chapter on morphisms of schemes. This in particular guarantees that the set of points where a morphism is étale is open.

Lemma 40.11.5. *Let Y be a locally Noetherian scheme. Let $f : X \to Y$ be locally of finite type. Let $x \in X$. The morphism f is étale at x in the sense of Definition 40.11.4 if and only if it is unramified at x in the sense of Morphisms, Definition 28.37.1.*

Proof. This follows from Lemma 40.11.2 and the definitions. □

Here are some results on étale morphisms. The formulations as given in this list apply only to morphisms locally of finite type between locally Noetherian schemes. In each case we give a reference to the general result as proved earlier in the project, but in some cases one can prove the result more easily in the Noetherian case. Here is the list:

(1) An étale morphism is unramified. (Clear from our definitions.)
(2) Étaleness is local on the source and the target in the Zariski topology.
(3) Étale morphisms are stable under base change and composition. See Morphisms, Lemmas 28.37.4 and 28.37.3.
(4) Étale morphisms of schemes are locally quasi-finite and quasi-compact étale morphisms are quasi-finite. (This is true because it holds for unramified morphisms as seen earlier.)
(5) Étale morphisms have relative dimension 0. See Morphisms, Definition 28.30.1 and Morphisms, Lemma 28.30.5.
(6) A morphism is étale if and only if it is flat and all its fibres are étale. See Morphisms, Lemma 28.37.8.
(7) Étale morphisms are open. This is true because an étale morphism is flat, and Theorem 40.10.2.
(8) Let X and Y be étale over a base scheme S. Any S-morphism from X to Y is étale. See Morphisms, Lemma 28.37.18.

40.12. The structure theorem

We present a theorem which describes the local structure of étale and unramified morphisms. Besides its obvious independent importance, this theorem also allows us to make the transition to another definition of étale morphisms that captures the geometric intuition better than the one we've used so far.

To state it we need the notion of a *standard étale ring map*, see Algebra, Definition 10.136.14. Namely, suppose that R is a ring and $f, g \in R[t]$ are polynomials such that

(a) f is a monic polynomial, and
(b) $f' = \mathrm{d}f/\mathrm{d}t$ is invertible in the localization $R[t]_g/(f)$.

Then the map

$$R \longrightarrow R[t]_g/(f) = R[t, 1/g]/(f)$$

is a standard étale algebra, and any standard étale algebra is isomorphic to one of these. It is a pleasant exercise to prove that such a ring map is flat, and unramified and hence étale (as expected of course). A special case of a standard étale ring map is any ring map

$$R \longrightarrow R[t]_{f'}/(f) = R[t, 1/f']/(f)$$

with f a monic polynomial, and any standard étale algebra is (isomorphic to) a principal localization of one of these.

Theorem 40.12.1. *Let $f : A \to B$ be an étale homomorphism of local rings. Then there exist $f, g \in A[t]$ such that*

(1) *$B' = A[t]_g/(f)$ is standard étale – see (a) and (b) above, and*
(2) *B is isomorphic to a localization of B' at a prime.*

Proof. Write $B = B'_{\mathfrak{q}}$ for some finite type A-algebra B' (we can do this because B is essentially of finite type over A). By Lemma 40.11.2 we see that $A \to B'$ is étale at $\mathfrak{q}$. Hence we may apply Algebra, Proposition 10.136.17 to see that a principal localization of B' is standard étale. □

Here is the version for unramified homomorphisms of local rings.

Theorem 40.12.2. *Let $f : A \to B$ be an unramified morphism of local rings. Then there exist $f, g \in A[t]$ such that*

(1) *$B' = A[t]_g/(f)$ is standard étale – see (a) and (b) above, and*
(2) *B is isomorphic to a quotient of a localization of B' at a prime.*

Proof. Write $B = B'_{\mathfrak{q}}$ for some finite type A-algebra B' (we can do this because B is essentially of finite type over A). By Lemma 40.3.2 we see that $A \to B'$ is unramified at $\mathfrak{q}$. Hence we may apply Algebra, Proposition 10.142.8 to see that a principal localization of B' is a quotient of a standard étale A-algebra. □

Via standard lifting arguments, one then obtains the following geometric statement which will be of essential use to us.

Theorem 40.12.3. *Let $\varphi : X \to Y$ be a morphism of schemes. Let $x \in X$. If φ is étale at x, then there exist exist affine opens $V \subset Y$ and $U \subset X$ with $x \in U$ and $\varphi(U) \subset V$ such that we have the following diagram*

$$\begin{array}{ccccc} X & \longleftarrow & U & \xrightarrow[j]{} & \operatorname{Spec}(R[t]_{f'}/(f)) \\ \downarrow & & \downarrow & & \downarrow \\ Y & \longleftarrow & V & = & \operatorname{Spec}(R) \end{array}$$

where j is an open immersion, and $f \in R[t]$ is monic.

Proof. This is equivalent to Morphisms, Lemma 28.37.14 although the statements differ slightly. □

40.13. Étale and smooth morphisms

An étale morphism is smooth of relative dimension zero. The projection $\mathbf{A}^n_S \to S$ is a standard example of a smooth morphism of relative dimension n. It turns out that any smooth morphism is étale locally of this form. Here is the precise statement.

Theorem 40.13.1. *Let $\varphi : X \to Y$ be a morphism of schemes. Let $x \in X$. If φ is smooth at x, then there exist exist and integer $n \geq 0$ and affine opens $V \subset Y$ and $U \subset X$ with $x \in U$ and $\varphi(U) \subset V$ such that there exists a commutative diagram*

$$\begin{array}{ccccc} X & \longleftarrow & U & \xrightarrow[\pi]{} & \mathbf{A}^n_R & = & \operatorname{Spec}(R[x_1, \ldots, x_n]) \\ \downarrow & & \downarrow & & \downarrow & \swarrow & \\ Y & \longleftarrow & V & = & \operatorname{Spec}(R) & & \end{array}$$

where π is étale.

Proof. See Morphisms, Lemma 28.37.20. □

40.14. Topological properties of étale morphisms

We present a few of the topological properties of étale and unramified morphisms. First, we give what Grothendieck calls the *fundamental property of étale morphisms*, see [**Gro71**, Exposé I.5].

Theorem 40.14.1. *Let $f : X \to Y$ be a morphism of schemes. The following are equivalent:*

(1) *f is an open immersion,*
(2) *f is universally injective and étale, and*
(3) *f is a flat monomorphism, locally of finite presentation.*

Proof. An open immersion is universally injective since any base change of an open immersion is an open immersion. Moreover, it is étale by Morphisms, Lemma 28.37.9. Hence (1) implies (2).

Assume f is universally injective and étale. Since f is étale it is flat and locally of finite presentation, see Morphisms, Lemmas 28.37.12 and 28.37.11. By Lemma 40.7.1 we see that f is a monomorphism. Hence (2) implies (3).

Assume f is flat, locally of finite presentation, and a monomorphism. Then f is open, see Morphisms, Lemma 28.26.9. Thus we may replace Y by $f(X)$ and we may assume f is surjective. Then f is open and bijective hence a homeomorphism. Hence f is quasi-compact. Hence Descent, Lemma 34.21.1 shows that f is an isomorphism and we win. □

Here is another result of a similar flavor.

Lemma 40.14.2. *Let $\pi : X \to S$ be a morphism of schemes. Let $s \in S$. Assume that*

(1) *π is finite,*
(2) *π is étale,*
(3) *$\pi^{-1}(\{s\}) = \{x\}$, and*
(4) *$\kappa(s) \subset \kappa(x)$ is purely inseparable*[2].

Then there exists an open neighbourhood U of s such that $\pi|_{\pi^{-1}(U)} : \pi^{-1}(U) \to U$ is an isomorphism.

Proof. By Lemma 40.7.3 there exists an open neighbourhood U of s such that $\pi|_{\pi^{-1}(U)} : \pi^{-1}(U) \to U$ is a closed immersion. But a morphism which is étale and a closed immersion is an open immersion (for example by Theorem 40.14.1). Hence after shrinking U we obtain an isomorphism. □

40.15. Topological invariance of the étale topology

Next, we present an extremely crucial theorem which, roughly speaking, says that étaleness is a topological property.

Theorem 40.15.1. *Let X and Y be two schemes over a base scheme S. Let S_0 be a closed subscheme of S whose ideal sheaf has square zero. Denote X_0 (resp. Y_0) the base change $S_0 \times_S X$ (resp. $S_0 \times_S Y$). If X is étale over S, then the map*

$$\mathrm{Mor}_S(Y, X) \longrightarrow \mathrm{Mor}_{S_0}(Y_0, X_0)$$

is bijective.

[2] In view of condition (2) this is equivalent to $\kappa(s) = \kappa(x)$.

Proof. After base changing via $Y \to S$, we may assume that $Y = S$. In this case the theorem states that any S-morphism $\sigma_0 : S_0 \to X$ actually factors uniquely through a section $S \to X$ of the étale structure morphism $X \to S$.

Existence. Since we have equality of underlying topological spaces $|S_0| = |S|$ and $|X_0| = |X|$, by Theorem 40.6.2, the section σ_0 is uniquely determined by a connected component X' of X such that the base change $X'_0 = S_0 \times_S X'$ maps isomorphically to S_0. In particular, $X' \to S$ is a universal homeomorphism and therefore universally injective. Since $X' \to S$ is étale, it follows from Theorem 40.14.1 that $X' \to S$ is an isomorphism and, therefore, it has an inverse σ which is the required section.

Uniqueness. This follows from Theorem 40.5.1, or directly from Theorem 40.6.2, or, if one carefuly observes, from our proof itself. □

From the proof of preceeding theorem, we also obtain one direction of the promised functorial characterization of étale morphisms. The following theorem will be strengthened in Étale Cohomology, Theorem 44.46.1.

Theorem 40.15.2 (Une equivalence remarquable de catégories)**.** *Let S be a scheme. Let $S_0 \subset S$ be a closed subscheme defined by an ideal with square zero. The functor*

$$X \longmapsto X_0 = S_0 \times_S X$$

defines an equivalence of categories

$$\{\textit{schemes } X \textit{ étale over } S\} \leftrightarrow \{\textit{schemes } X_0 \textit{ étale over } S_0\}$$

Proof. By Theorem 40.15.1 we see that this functor is fully faithful. It remains to show that the functor is essentially surjective. Let $Y \to S_0$ be an étale morphism of schemes.

Suppose that the result holds if S and Y are affine. In that case, we choose an affine open covering $Y = \bigcup V_j$ such that each V_j maps into an affine open of S. By assumption (affine case) we can find étale morphisms $W_j \to S$ such that $W_{j,0} \cong V_j$ (as schemes over S_0). Let $W_{j,j'} \subset W_j$ be the open subscheme whose underlying topological space corresponds to $V_j \cap V_{j'}$. Because we have isomorphisms

$$W_{j,j',0} \cong V_j \cap V_{j'} \cong W_{j',j,0}$$

as schemes over S_0 we see by fully faithfulness that we obtain isomorphisms $\theta_{j,j'} : W_{j,j'} \to W_{j',j}$ of schemes over S. We omit the verification that these isomorphisms satisfy the cocycle condition of Schemes, Section 25.14. Applying Schemes, Lemma 25.14.2 we obtain a scheme $X \to S$ by glueing the schemes W_j along the identifications $\theta_{j,j'}$. It is clear that $X \to S$ is étale and $X_0 \cong Y$ by construction.

Thus it suffices to show the lemma in case S and Y are affine. Say $S = \mathrm{Spec}(R)$ and $S_0 = \mathrm{Spec}(R/I)$ with $I^2 = 0$. By Algebra, Lemma 10.136.2 we know that Y is the spectrum of a ring $\overline{A}$ with

$$\overline{A} = (R/I)[x_1, \ldots, x_n]/(\overline{f}_1, \ldots, \overline{f}_n)$$

such that

$$\overline{g} = \det \begin{pmatrix} \partial\overline{f}_1/\partial x_1 & \partial\overline{f}_2/\partial x_1 & \ldots & \partial\overline{f}_n/\partial x_1 \\ \partial\overline{f}_1/\partial x_2 & \partial\overline{f}_2/\partial x_2 & \ldots & \partial\overline{f}_n/\partial x_2 \\ \ldots & \ldots & \ldots & \ldots \\ \partial\overline{f}_1/\partial x_n & \partial\overline{f}_2/\partial x_n & \ldots & \partial\overline{f}_n/\partial x_n \end{pmatrix}$$

maps to an invertible element in A. Choose any lifts $f_i \in R[x_1, \ldots, x_n]$. Since I is nilpotent it follows that the determinant of the matrix of partials of the f_i is invertible in the algebra A defined by

$$A = R[x_1, \ldots, x_n]/(f_1, \ldots, f_n)$$

Hence $R \to A$ is étale and $(R/I) \otimes_R A \cong \overline{A}$. To prove the general case one argues with glueing affine pieces. □

40.16. The functorial characterization

We finally present the promised functorial characterization. Thus there are four ways to think about étale morphisms of schemes:

(1) as a smooth morphism of relative dimension 0,
(2) as locally finitely presented, flat, and unramified morphisms,
(3) using the structure theorem, and
(4) using the functorial characterization.

Theorem 40.16.1. *Let $f : X \to S$ be a morphism that is locally of finite presentation. The following are equivalent*

(1) *f is étale,*
(2) *for all affine S-schemes Y, and closed subschemes $Y_0 \subset Y$ defined by square-zero ideals, the natural map*

$$\mathrm{Mor}_S(Y, X) \longrightarrow \mathrm{Mor}_S(Y_0, X)$$

is bijective.

Proof. This is More on Morphisms, Lemma 36.6.9. □

This characterization says that solutions to the equations defining X can be lifted uniquely through nilpotent thickenings.

40.17. Étale local structure of unramified morphisms

In the chapter More on Morphisms, Section 36.30 the reader can find some results on the étale local structure of quasi-finite morphisms. In this section we want to combine this with the topological properties of unramified morphisms we have seen in this chapter. The basic overall picture to keep in mind is

$$\begin{array}{ccccc} V & \longrightarrow & X_U & \longrightarrow & X \\ & \searrow & \downarrow & & \downarrow f \\ & & U & \longrightarrow & S \end{array}$$

see More on Morphisms, Equation (36.30.0.1). We start with a very general case.

Lemma 40.17.1. *Let $f : X \to S$ be a morphism of schemes. Let $x_1, \ldots, x_n \in X$ be points having the same image s in S. Assume f is unramified at each x_i. Then there exists an étale neighbourhood $(U, u) \to (S, s)$ and opens $V_{i,j} \subset X_U$, $i = 1, \ldots, n$, $j = 1, \ldots, m_i$ such that*

(1) *$V_{i,j} \to U$ is a closed immersion passing through u,*
(2) *u is not in the image of $V_{i,j} \cap V_{i',j'}$ unless $i = i'$ and $j = j'$, and*
(3) *any point of $(X_U)_u$ mapping to x_i is in some $V_{i,j}$.*

Proof. By Morphisms, Definition 28.36.1 there exists an open neighbourhood of each x_i which is locally of finite type over S. Replacing X by an open neighbourhood of $\{x_1, \ldots, x_n\}$ we may assume f is locally of finite type. Apply More on Morphisms, Lemma 36.30.3 to get the étale neighbourhood (U, u) and the opens $V_{i,j}$ finite over U. By Lemma 40.7.3 after possibly shrinking U we get that $V_{i,j} \to U$ is a closed immersion. □

Lemma 40.17.2. *Let $f : X \to S$ be a morphism of schemes. Let $x_1, \ldots, x_n \in X$ be points having the same image s in S. Assume f is separated and f is unramified at each x_i. Then there exists an étale neighbourhood $(U, u) \to (S, s)$ and a disjoint union decomposition*

$$X_U = W \amalg \coprod\nolimits_{i,j} V_{i,j}$$

such that

(1) *$V_{i,j} \to U$ is a closed immersion passing through u,*
(2) *the fibre W_u contains no point mapping to any x_i.*

In particular, if $f^{-1}(\{s\}) = \{x_1, \ldots, x_n\}$, then the fibre W_u is empty.

Proof. Apply Lemma 40.17.1. We may assume U is affine, so X_U is separated. Then $V_{i,j} \to X_U$ is a closed map, see Morphisms, Lemma 28.42.7. Suppose $(i, j) \neq (i', j')$. Then $V_{i,j} \cap V_{i',j'}$ is closed in $V_{i,j}$ and its image in U does not contain u. Hence after shrinking U we may assume that $V_{i,j} \cap V_{i',j'} = \emptyset$. Moreover, $\bigcup V_{i,j}$ is a closed and open subscheme of X_U and hence has an open and closed complement W. This finishes the proof. □

The following lemma is in some sense much weaker than the preceding one but it may be useful to state it explicitly here. It says that a finite unramified morphism is étale locally on the base a closed immersion.

Lemma 40.17.3. *Let $f : X \to S$ be a finite unramified morphism of schemes. Let $s \in S$. There exists an étale neighbourhood $(U, u) \to (S, s)$ and a disjoint union decomposition*

$$X_U = \coprod\nolimits_j V_j$$

such that each $V_j \to U$ is a closed immersion.

Proof. Since $X \to S$ is finite the fibre over S is a finite set $\{x_1, \ldots, x_n\}$ of points of X. Apply Lemma 40.17.2 to this set (a finite morphism is separated, see Morphisms, Section 28.44). The image of W in U is a closed subset (as $X_U \to U$ is finite, hence proper) which does not contain u. After removing this from U we see that $W = \emptyset$ as desired. □

40.18. Étale local structure of étale morphisms

This is a bit silly, but perhaps helps form intuition about étale morphisms. We simply copy over the results of Section 40.17 and change "closed immersion" into "isomorphism".

Lemma 40.18.1. *Let $f : X \to S$ be a morphism of schemes. Let $x_1, \ldots, x_n \in X$ be points having the same image s in S. Assume f is étale at each x_i. Then there exists an étale neighbourhood $(U, u) \to (S, s)$ and opens $V_{i,j} \subset X_U$, $i = 1, \ldots, n$, $j = 1, \ldots, m_i$ such that*

(1) *$V_{i,j} \to U$ is an isomorphism,*

(2) *u is not in the image of $V_{i,j} \cap V_{i',j'}$ unless $i = i'$ and $j = j'$, and*
(3) *any point of $(X_U)_u$ mapping to x_i is in some $V_{i,j}$.*

Proof. An étale morphism is unramified, hence we may apply Lemma 40.17.1. Now $V_{i,j} \to U$ is a closed immersion and étale. Hence it is an open immersion, for example by Theorem 40.14.1. Replace U by the intersection of the images of $V_{i,j} \to U$ to get the lemma. □

Lemma 40.18.2. *Let $f : X \to S$ be a morphism of schemes. Let $x_1, \ldots, x_n \in X$ be points having the same image s in S. Assume f is separated and f is étale at each x_i. Then there exists an étale neighbourhood $(U, u) \to (S, s)$ and a disjoint union decomposition*

$$X_U = W \amalg \coprod\nolimits_{i,j} V_{i,j}$$

such that

(1) *$V_{i,j} \to U$ is an isomorphism,*
(2) *the fibre W_u contains no point mapping to any x_i.*

In particular, if $f^{-1}(\{s\}) = \{x_1, \ldots, x_n\}$, then the fibre W_u is empty.

Proof. An étale morphism is unramified, hence we may apply Lemma 40.17.2. As in the proof of Lemma 40.18.1 the morphisms $V_{i,j} \to U$ are open immersions and we win after replacing U by the intersection of their images. □

The following lemma is in some sense much weaker than the preceding one but it may be useful to state it explicitly here. It says that a finite étale morphism is étale locally on the base a "topological covering space", i.e., a finite product of copies of the base.

Lemma 40.18.3. *Let $f : X \to S$ be a finite étale morphism of schemes. Let $s \in S$. There exists an étale neighbourhood $(U, u) \to (S, s)$ and a disjoint union decomposition*

$$X_U = \coprod\nolimits_j V_j$$

such that each $V_j \to U$ is an isomorphism.

Proof. An étale morphism is unramified, hence we may apply Lemma 40.17.3. As in the proof of Lemma 40.18.1 we see that $V_{i,j} \to U$ is an open immersion and we win after replacing U by the intersection of their images. □

40.19. Permanence properties

In what follows, we present a few "permanence" properties of étale homomorphisms of Noetherian local rings (as defined in Definition 40.11.1). See More on Algebra, Sections 15.31 and 15.32 for the analogue of this material for the completion and henselization of a Noetherian local ring.

Lemma 40.19.1. *Let A, B be Noetherian local rings. Let $A \to B$ be a étale homomorphism of local rings. Then* $\dim(A) = \dim(B)$.

Proof. See for example Algebra, Lemma 10.106.7. □

Proposition 40.19.2. *Let A, B be Noetherian local rings. Let $f : A \to B$ be an étale homomorphism of local rings. Then $\mathrm{depth}(A) = \mathrm{depth}(B)$*

Proof. See Algebra, Lemma 10.149.1. □

Proposition 40.19.3. *Let A, B be Noetherian local rings. Let $f : A \to B$ be an étale homomorphism of local rings. Then A is Cohen-Macaulay if and only if B is so.*

Proof. A local ring A is Cohen-Macaulay if and only $\dim(A) = \text{depth}(A)$. As both of these invariants is preserved under an étale extension, the claim follows. □

Proposition 40.19.4. *Let A, B be Noetherian local rings. Let $f : A \to B$ be an étale homomorphism of local rings. Then A is regular if and only if B is so.*

Proof. If B is regular, then A is regular by Algebra, Lemma 10.105.8. Assume A is regular. Let $\mathfrak{m}$ be the maximal ideal of A. Then $\dim_{\kappa(\mathfrak{m})} \mathfrak{m}/\mathfrak{m}^2 = \dim(A) = \dim(B)$ (see Lemma 40.19.1). On the other hand, $\mathfrak{m}B$ is the maximal ideal of B and hence $\mathfrak{m}_B/\mathfrak{m}_B = \mathfrak{m}B/\mathfrak{m}^2B$ is generated by at most $\dim(B)$ elements. Thus B is regular. (You can also use the slightly more general Algebra, Lemma 10.106.8.) □

Proposition 40.19.5. *Let A, B be Noetherian local rings. Let $f : A \to B$ be an étale homomorphism of local rings. Then A is reduced if and only if B is so.*

Proof. It is clear from the faithful flatness of $A \to B$ that if B is reduced, so is A. See also Algebra, Lemma 10.150.2. Conversely, assume A is reduced. By assumption B is a localization of a finite type A-algebra B' at some prime $\mathfrak{q}$. After replacing B' by a localization we may assume that B' is étale over A, see Lemma 40.11.2. Then we see that Algebra, Lemma 10.149.6 applies to $A \to B'$ and B' is reduced. Hence B is reduced. □

Remark 40.19.6. The result on "reducedness" does not hold with a weaker definition of étale local ring maps $A \to B$ where one drops the assumption that B is essentially of finite type over A. Namely, it can happen that a Noetherian local domain A has nonreduced completion $A^\wedge$, see Examples, Section 78.10. But the ring map $A \to A^\wedge$ is flat, and $\mathfrak{m}_A A^\wedge$ is the maximal ideal of $A^\wedge$ and of course A and $A^\wedge$ have the same residue fields. This is why it is important to consider this notion only for ring extensions which are essentially of finite type (or essentially of finite presentation if A is not Noetherian).

Proposition 40.19.7. *Let A, B be Noetherian local rings. Let $f : A \to B$ be an étale homomorphism of local rings. Then A is a normal domain if and only if B is so.*

Proof. See Algebra, Lemma 10.150.3 for descending normality. Conversely, assume A is normal. By assumption B is a localization of a finite type A-algebra B' at some prime $\mathfrak{q}$. After replacing B' by a localization we may assume that B' is étale over A, see Lemma 40.11.2. Then we see that Algebra, Lemma 10.149.7 applies to $A \to B'$ and we conclude that B' is normal. Hence B is a normal domain. □

The preceeding propositions give some indication as to why we'd like to think of étale maps as "local isomorphisms". Another property that gives an excellent indication that we have the "right" definition is the fact that for $\mathbf{C}$-schemes of finite type, a morphism is étale if and only if the associated morphism on analytic spaces (the $\mathbf{C}$-valued points given the complex topology) is a local isomorphism in the analytic sense (open embedding locally on the source). This fact can be proven with the aid of the structure theorem and the fact that the analytification commutes with the formation of the completed local rings – the details are left to the reader.

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CHAPTER 41

Chow Homology and Chern Classes

41.1. Introduction

In this chapter we discuss Chow homology groups and the construction of chern classes of vector bundles as elements of operational Chow cohomology groups (everything with **Z**-coefficients). We follow the first few chapters of [**Ful98**], except that we have been less precise about the supports of the cycles involved. More classical discussions of Chow groups can be found in [**Sam56**], [**Che58a**], and [**Che58b**]. Of course there are many others.

To make the material a little bit more challenging we decided to treat a somewhat more general case than is usually done. Namely we assume our schemes X are locally of finite type over a fixed locally Noetherian base scheme which is universally catenary and has a given dimension function. This seems to be all that is needed to be able to define the Chow homology groups $A_*(X)$ and the action of capping with chern classes on them. This is an indication that we should be able to define these also for algebraic stacks locally of finite type over such a base.

In another chapter we will define the intersection products on $A_*(X)$ using Serre's Tor-formula in case X is nonsingular (see [**Ser00**], or [**Ser65**]) and we have a good moving lemma. See (insert future reference here).

41.2. Determinants of finite length modules

The material in this section is related to the material in the paper [**KM76**] and to the material in the thesis [**Ros09**]. If you have a good reference then please email stacks.project@gmail.com.

Given any field κ and any finite dimensional κ-vector space V we set $\det_\kappa(V) = \wedge^n(V)$ where $n = \dim_\kappa(V)$. We want to generalize this slightly.

Definition 41.2.1. Let R be a local ring with maximal ideal $\mathfrak{m}$ and residue field κ. Let M be a finite length R-module. Say $l = \text{length}_R(M)$.

1. Given elements $x_1, \ldots, x_r \in M$ we denote $\langle x_1, \ldots, x_r \rangle = Rx_1 + \ldots + Rx_r$ the R-submodule of R generated by $x_1, \ldots, x_r$.
2. We will say an l-tuple of elements $(e_1, \ldots, e_l)$ of M is *admissible* if $\mathfrak{m}e_i \in \langle e_1, \ldots, e_{i-1} \rangle$ for $i = 1, \ldots, l$.
3. A *symbol* $[e_1, \ldots, e_l]$ will mean $(e_1, \ldots, e_l)$ is an admissible l-tuple.
4. An *admissible relation* between symbols is one of the following:
   (a) if $(e_1, \ldots, e_l)$ is an admissible sequence and for some $1 \leq a \leq l$ we have $e_a \in \langle e_1, \ldots, e_{a-1} \rangle$, then $[e_1, \ldots, e_l] = 0$,

(b) if $(e_1, \ldots, e_l)$ is an admissible sequence and for some $1 \leq a \leq l$ we have $e_a = \lambda e'_a + x$ with $\lambda \in R^*$, and $x \in \langle e_1, \ldots, e_{a-1} \rangle$, then

$$[e_1, \ldots, e_l] = \overline{\lambda}[e_1, \ldots, e_{a-1}, e'_a, e_{a+1}, \ldots, e_l]$$

where $\overline{\lambda} \in \kappa^*$ is the image of λ in the residue field, and

(c) if $(e_1, \ldots, e_l)$ is an admissible sequence and $\mathfrak{m} e_a \subset \langle e_1, \ldots, e_{a-2} \rangle$ then

$$[e_1, \ldots, e_l] = -[e_1, \ldots, e_{a-2}, e_a, e_{a-1}, e_{a+1}, \ldots, e_l].$$

(5) We define the *determinant of the finite length R-module M* to be

$$\det_\kappa(M) = \left\{ \frac{\kappa\text{-vector space generated by symbols}}{\kappa\text{-linear combinations of admissible relations}} \right\}$$

We stress that always $l = \text{length}_R(M)$. We also stress that it does not follow that the symbol $[e_1, \ldots, e_l]$ is additive in the entries (this will typically not be the case). Before we can show that the determinant $\det_\kappa(M)$ actually has dimension 1 we have to show that it has dimension at most 1.

Lemma 41.2.2. *With notations as above we have* $\dim_\kappa(\det_\kappa(M)) \leq 1$.

Proof. Fix an admissible sequence $(f_1, \ldots, f_l)$ of M such that

$$\text{length}_R(\langle f_1, \ldots, f_i \rangle) = i$$

for $i = 1, \ldots, l$. Such an admissible sequence exists exactly because M has length l. We will show that any element of $\det_\kappa(M)$ is a κ-multiple of the symbol $[f_1, \ldots, f_l]$. This will prove the lemma.

Let $(e_1, \ldots, e_l)$ be an admissible sequence of M. It suffices to show that $[e_1, \ldots, e_l]$ is a multiple of $[f_1, \ldots, f_l]$. First assume that $\langle e_1, \ldots, e_l \rangle \neq M$. Then there exists an $i \in [1, \ldots, l]$ such that $e_i \in \langle e_1, \ldots, e_{i-1} \rangle$. It immediately follows from the first admissible relation that $[e_1, \ldots, e_n] = 0$ in $\det_\kappa(M)$. Hence we may assume that $\langle e_1, \ldots, e_l \rangle = M$. In particular there exists a smallest index $i \in \{1, \ldots, l\}$ such that $f_1 \in \langle e_1, \ldots, e_i \rangle$. This means that $e_i = \lambda f_1 + x$ with $x \in \langle e_1, \ldots, e_{i-1} \rangle$ and $\lambda \in R^*$. By the second admissible relation this means that $[e_1, \ldots, e_l] = \overline{\lambda}[e_1, \ldots, e_{i-1}, f_1, e_{i+1}, \ldots, e_l]$. Note that $\mathfrak{m} f_1 = 0$. Hence by applying the third admissible relation $i - 1$ times we see that

$$[e_1, \ldots, e_l] = (-1)^{i-1}\overline{\lambda}[f_1, e_1, \ldots, e_{i-1}, e_{i+1}, \ldots, e_l].$$

Note that it is also the case that $\langle f_1, e_1, \ldots, e_{i-1}, e_{i+1}, \ldots, e_l \rangle = M$. By induction suppose we have proven that our original symbol is equal to a scalar times

$$[f_1, \ldots, f_j, e_{j+1}, \ldots, e_l]$$

for some admissible sequence $(f_1, \ldots, f_j, e_{j+1}, \ldots, e_l)$ whose elements generate M, i.e., with $\langle f_1, \ldots, f_j, e_{j+1}, \ldots, e_l \rangle = M$. Then we find the smallest i such that $f_{j+1} \in \langle f_1, \ldots, f_j, e_{j+1}, \ldots, e_i \rangle$ and we go through the same process as above to see that

$$[f_1, \ldots, f_j, e_{j+1}, \ldots, e_l] = (\text{scalar})[f_1, \ldots, f_j, f_{j+1}, e_{j+1}, \ldots, \hat{e_i}, \ldots, e_l]$$

Continuing in this vein we obtain the desired result. □

Before we show that $\det_\kappa(M)$ always has dimension 1, let us show that it agrees with the usual top exterior power in the case the module is a vector space over κ.

Lemma 41.2.3. *Let R be a local ring with maximal ideal $\mathfrak{m}$ and residue field κ. Let M be a finite length R-module which is annihilated by $\mathfrak{m}$. Let $l = n = \dim_\kappa(M)$. Then the map*

$$\det_\kappa(M) \longrightarrow \wedge^l_\kappa(M), \quad [e_1, \ldots, e_l] \longmapsto e_1 \wedge \ldots \wedge e_l$$

is an isomorphism.

Proof. It is clear that the rule described in the lemma gives a κ-linear map since all of the admissible relations are satisfied by the usual symbols $e_1 \wedge \ldots \wedge e_l$. It is also clearly a surjective map. Since by Lemma 41.2.2 the left hand side has dimension at most one we see that the map is an isomorphism. □

Lemma 41.2.4. *Let R be a local ring with maximal ideal $\mathfrak{m}$ and residue field κ. Let M be a finite length R-module. The determinant $\det_\kappa(M)$ defined above is a κ-vector space of dimension 1. It is generated by the symbol $[f_1, \ldots, f_l]$ for any admissible sequence such that $\langle f_1, \ldots f_l \rangle = M$.*

Proof. We know $\det_\kappa(M)$ has dimension at most 1, and in fact that it is generated by $[f_1, \ldots, f_l]$, by Lemma 41.2.2 and its proof. We will show by induction on $l = \text{length}(M)$ that it is nonzero. For $l = 1$ it follows from Lemma 41.2.3. Choose a nonzero element $f \in M$ with $\mathfrak{m}f = 0$. Set $\overline{M} = M/\langle f \rangle$, and denote the quotient map $x \mapsto \overline{x}$. We will define a surjective map

$$\psi : \det_k(M) \to \det_\kappa(\overline{M})$$

which will prove the lemma since by induction the determinant of $\overline{M}$ is nonzero.

We define ψ on symbols as follows. Let $(e_1, \ldots, e_l)$ be an admissible sequence. If $f \notin \langle e_1, \ldots, e_l \rangle$ then we simply set $\psi([e_1, \ldots, e_l]) = 0$. If $f \in \langle e_1, \ldots, e_l \rangle$ then we choose an i minimal such that $f \in \langle e_1, \ldots, e_i \rangle$ and write $e_i = \lambda f + x$ for some $\lambda \in R$ and $x \in \langle e_1, \ldots, e_{i-1} \rangle$. In this case we set

$$\psi([e_1, \ldots, e_l]) = \overline{\lambda}[\overline{e}_1, \ldots, \overline{e}_{i-1}, \overline{e}_{i+1}, \ldots, \overline{e}_l].$$

Note that it is indeed the case that $(\overline{e}_1, \ldots, \overline{e}_{i-1}, \overline{e}_{i+1}, \ldots, \overline{e}_l)$ is an admissible sequence in $\overline{M}$, so this makes sense. Let us show that extending this rule κ-linearly to linear combinations of symbols does indeed lead to a map on determinants. To do this we have to show that the admissible relations are mapped to zero.

Type (a) relations. Suppose we have $(e_1, \ldots, e_l)$ an admissible sequence and for some $1 \leq a \leq l$ we have $e_a \in \langle e_1, \ldots, e_{a-1} \rangle$. Suppose that $f \in \langle e_1, \ldots, e_i \rangle$ with i minimal. Then it is immediate that $i \neq a$. Since it is also the case that $\overline{e}_a \in \langle \overline{e}_1, \ldots, \hat{\overline{e}}_i, \ldots, \overline{e}_{a-1} \rangle$ we see immediately that the same admissible relation for $\det_\kappa(\overline{M})$ forces the symbol $[\overline{e}_1, \ldots, \overline{e}_{i-1}, \overline{e}_{i+1}, \ldots, \overline{e}_l]$ to be zero as desired.

Type (b) relations. Suppose we have $(e_1, \ldots, e_l)$ an admissible sequence and for some $1 \leq a \leq l$ we have $e_a = \lambda e'_a + x$ with $\lambda \in R^*$, and $x \in \langle e_1, \ldots, e_{a-1} \rangle$. Suppose that $f \in \langle e_1, \ldots, e_i \rangle$ with i minimal. Say $e_i = \mu f + y$ with $y \in \langle e_1, \ldots, e_{i-1} \rangle$. If $i < a$ then the desired equality is

$$\overline{\lambda}[\overline{e}_1, \ldots, \overline{e}_{i-1}, \overline{e}_{i+1}, \ldots, \overline{e}_l] = \overline{\lambda}[\overline{e}_1, \ldots, \overline{e}_{i-1}, \overline{e}_{i+1}, \ldots, \overline{e}_{a-1}, \overline{e}'_a, \overline{e}_{a+1}, \ldots, \overline{e}_l]$$

which follows from $\overline{e}_a = \lambda \overline{e}'_a + \overline{x}$ and the corresponding admissible relation for $\det_\kappa(\overline{M})$. If $i > a$ then the desired equality is

$$\overline{\lambda}[\overline{e}_1, \ldots, \overline{e}_{i-1}, \overline{e}_{i+1}, \ldots, \overline{e}_l] = \overline{\lambda}[\overline{e}_1, \ldots, \overline{e}_{a-1}, \overline{e}'_a, \overline{e}_{a+1}, \ldots, \overline{e}_{i-1}, \overline{e}_{i+1}, \ldots, \overline{e}_l]$$

which follows from $\overline{e}_a = \lambda\overline{e}'_a + \overline{x}$ and the corresponding admissible relation for $\det_\kappa(\overline{M})$. The interesting case is when $i = a$. In this case we have $e_a = \lambda e'_a + x = \mu f + y$. Hence also $e'_a = \lambda^{-1}(\mu f + y - x)$. Thus we see that

$$\psi([e_1, \ldots, e_l]) = \overline{\mu}[\overline{e}_1, \ldots, \overline{e}_{i-1}, \overline{e}_{i+1}, \ldots, \overline{e}_l] = \psi(\overline{\lambda}[e_1, \ldots, e_{a-1}, e'_a, e_{a+1}, \ldots, e_l])$$

as desired.

Type (c) relations. Suppose that $(e_1, \ldots, e_l)$ is an admissible sequence and $\mathfrak{m}e_a \subset \langle e_1, \ldots, e_{a-2}\rangle$. Suppose that $f \in \langle e_1, \ldots, e_i\rangle$ with i minimal. Say $e_i = \lambda f + x$ with $x \in \langle e_1, \ldots, e_{i-1}\rangle$. If $i < a - 1$, then the desired equality is

$$\overline{\lambda}[\overline{e}_1, \ldots, \overline{e}_{i-1}, \overline{e}_{i+1}, \ldots, \overline{e}_l] = \overline{\lambda}[\overline{e}_1, \ldots, \overline{e}_{i-1}, \overline{e}_{i+1}, \ldots, \overline{e}_{a-2}, \overline{e}_a, \overline{e}_{a-1}, \overline{e}_{a+1}, \ldots, \overline{e}_l]$$

which follows from the type (c) admissible relation for $\det_\kappa(\overline{M})$. Similarly, if $i > a$, then the desired equality is

$$\overline{\lambda}[\overline{e}_1, \ldots, \overline{e}_{i-1}, \overline{e}_{i+1}, \ldots, \overline{e}_l] = \overline{\lambda}[\overline{e}_1, \ldots, \overline{e}_{a-2}, \overline{e}_a, \overline{e}_{a-1}, \overline{e}_{a+1}, \ldots, \overline{e}_{i-1}, \overline{e}_{i+1}, \ldots, \overline{e}_l]$$

which follows from the type (c) admissible relation for $\det_\kappa(\overline{M})$. If $i = a$, then the desired equality is

$$\overline{\lambda}[\overline{e}_1, \ldots, \overline{e}_{a-1}, \overline{e}_{a+1}, \ldots, \overline{e}_l] = \overline{\lambda}[\overline{e}_1, \ldots, \overline{e}_{a-2}, \overline{e}_{a-1}, \overline{e}_{a+1}, \ldots, \overline{e}_l]$$

which is tautological. Finally, the interesting case is $i = a-1$. This case itself splits into two cases as to whether $f \in \langle e_1, \ldots, e_{a-2}, e_a\rangle$ or not. If not, then we see that the desired equality is

$$\overline{\lambda}[\overline{e}_1, \ldots, \overline{e}_{a-2}, \overline{e}_a, \ldots, \overline{e}_l] = \overline{\lambda}[\overline{e}_1, \ldots, \overline{e}_{a-2}, \overline{e}_a, \overline{e}_{a+1}, \ldots, \overline{e}_l]$$

which is tautological since after switching e_{a-1} and e_a the smallest index such that f is in the becomes equal to $i' = a$ and it is again e_a which is removed. On the other hand, suppose that $f \in \langle e_1, \ldots, e_{a-2}, e_a\rangle$. In this case we see that we can, besides the equality $e_{a-1} = \lambda f + x$ of above, also write $e_a = \mu f + y$ with $y \in \langle e_1, \ldots, e_{a-2}\rangle$. Clearly this means that both $e_a \in \langle e_1, \ldots, e_{a-1}\rangle$ and $e_{a-1} \in \langle e_1, \ldots, e_{a-2}, e_a\rangle$. Thus we can use relations of type (a) and the compatibility of ψ with these shown above to see that both

$$\psi([e_1, \ldots, e_l]) \quad \text{and} \quad \psi([e_1, \ldots, e_{a-2}, e_a, e_{a-1}, e_{a+1}, \ldots, e_l])$$

are zero, as desired.

At this point we have shown that ψ is well defined, and all that remains is to show that it is surjective. To see this let $(\overline{f}_2, \ldots, \overline{f}_l)$ be an admissible sequence in $\overline{M}$. We can choose lifts $f_2, \ldots, f_l \in M$, and then $(f, f_2, \ldots, f_l)$ is an admissible sequence in M. Since $\psi([f, f_2, \ldots, f_l]) = [f_2, \ldots, f_l]$ we win. □

Let R be a local ring with maximal ideal $\mathfrak{m}$ and residue field κ. Note that if $\varphi : M \to N$ is an isomorphism of finite length R-modules, then we get an isomorphism

$$\det_\kappa(\varphi) : \det_\kappa(M) \to \det_\kappa(N)$$

simply by the rule

$$\det_\kappa(\varphi)([e_1, \ldots, e_l]) = [\varphi(e_1), \ldots, \varphi(e_l)]$$

for any symbol $[e_1, \ldots, e_l]$ for M. Hence we see that $\det_\kappa$ is a functor

$$\left\{\begin{array}{c}\text{finite length } R\text{-modules}\\ \text{with isomorphisms}\end{array}\right\} \longrightarrow \left\{\begin{array}{c}\text{1-dimensional } \kappa\text{-vector spaces}\\ \text{with isomorphisms}\end{array}\right\} \tag{41.2.4.1}$$

This is typical for a "determinant functor" (see [**Knu02**]), as is the following additivity property.

Lemma 41.2.5. *Let $(R, \mathfrak{m}, \kappa)$ be a local ring. For every short exact sequence*

$$0 \to K \to L \to M \to 0$$

of finite length R-modules there exists a canonical isomorphism

$$\gamma_{K \to L \to M} : \det_\kappa(K) \otimes_\kappa \det_\kappa(M) \longrightarrow \det_\kappa(L)$$

defined by the rule on nonzero symbols

$$[e_1, \ldots, e_k] \otimes [\overline{f}_1, \ldots, \overline{f}_m] \longrightarrow [e_1, \ldots, e_k, f_1, \ldots, f_m]$$

with the following properties:

(1) *For every isomorphism of short exact sequences, i.e., for every commutative diagram*

$$\begin{array}{ccccccccc} 0 & \longrightarrow & K & \longrightarrow & L & \longrightarrow & M & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle u} & & \downarrow{\scriptstyle v} & & \downarrow{\scriptstyle w} & & \\ 0 & \longrightarrow & K' & \longrightarrow & L' & \longrightarrow & M' & \longrightarrow & 0 \end{array}$$

with short exact rows and isomorphisms u, v, w we have

$$\gamma_{K' \to L' \to M'} \circ (\det_\kappa(u) \otimes \det_\kappa(w)) = \det_\kappa(v) \circ \gamma_{K \to L \to M},$$

(2) *for every commutative square of finite length R-modules with exact rows and columns*

$$\begin{array}{ccccccccc} & & 0 & & 0 & & 0 & & \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & A & \longrightarrow & B & \longrightarrow & C & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & D & \longrightarrow & E & \longrightarrow & F & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & G & \longrightarrow & H & \longrightarrow & I & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ & & 0 & & 0 & & 0 & & \end{array}$$

the following diagram is commutative

$$\begin{array}{ccc} \det_\kappa(A) \otimes \det_\kappa(C) \otimes \det_\kappa(G) \otimes \det_\kappa(I) & \xrightarrow{\gamma_{A \to B \to C} \otimes \gamma_{G \to H \to I}} & \det_\kappa(B) \otimes \det_\kappa(H) \\ \downarrow{\scriptstyle \epsilon} & & \downarrow{\scriptstyle \gamma_{B \to E \to H}} \\ & & \det_\kappa(E) \\ & & \uparrow{\scriptstyle \gamma_{D \to E \to F}} \\ \det_\kappa(A) \otimes \det_\kappa(G) \otimes \det_\kappa(C) \otimes \det_\kappa(I) & \xrightarrow{\gamma_{A \to D \to G} \otimes \gamma_{C \to F \to I}} & \det_\kappa(D) \otimes \det_\kappa(F) \end{array}$$

where ϵ is the switch of the factors in the tensor product times $(-1)^{cg}$ with $c = length_R(C)$ and $g = length_R(G)$, and

(3) *the map $\gamma_{K\to L\to M}$ agrees with the usual isomorphism if $0 \to K \to L \to M \to 0$ is actually a short exact sequence of κ-vector spaces.*

Proof. The significance of taking nonzero symbols in the explicit description of the map $\gamma_{K\to L\to M}$ is simply that if $(e_1, \ldots, e_l)$ is an admissible sequence in K, and $(\overline{f}_1, \ldots, \overline{f}_m)$ is an admissible sequence in M, then it is not guaranteed that $(e_1, \ldots, e_l, f_1, \ldots, f_m)$ is an admissible sequence in L (where of course $f_i \in L$ signifies a lift of $\overline{f}_i$). However, if the symbol $[e_1, \ldots, e_l]$ is nonzero in $\det_\kappa(K)$, then necessarily $K = \langle e_1, \ldots, e_k\rangle$ (see proof of Lemma 41.2.2), and in this case it is true that $(e_1, \ldots, e_k, f_1, \ldots, f_m)$ is an admissible sequence. Moreover, by the admissible relations of type (b) for $\det_\kappa(L)$ we see that the value of $[e_1, \ldots, e_k, f_1, \ldots, f_m]$ in $\det_\kappa(L)$ is independent of the choice of the lifts f_i in this case also. Given this remark, it is clear that an admissible relation for $e_1, \ldots, e_k$ in K translates into an admissible relation among $e_1, \ldots, e_k, f_1, \ldots, f_m$ in L, and similarly for an admissible relation among the $\overline{f}_1, \ldots, \overline{f}_m$. Thus γ defines a linear map of vector spaces as claimed in the lemma.

By Lemma 41.2.4 we know $\det_\kappa(L)$ is generated by any single symbol $[x_1, \ldots, x_{k+m}]$ such that $(x_1, \ldots, x_{k+m})$ is an admissible sequence with $L = \langle x_1, \ldots, x_{k+m}\rangle$. Hence it is clear that the map $\gamma_{K\to L\to M}$ is surjective and hence an isomorphism.

Property (1) holds because

$$\begin{aligned} & \det_\kappa(v)([e_1, \ldots, e_k, f_1, \ldots, f_m]) \\ = \ & [v(e_1), \ldots, v(e_k), v(f_1), \ldots, v(f_m)] \\ = \ & \gamma_{K'\to L'\to M'}([u(e_1), \ldots, u(e_k)] \otimes [w(f_1), \ldots, w(f_m)]). \end{aligned}$$

Property (2) means that given a symbol $[\alpha_1, \ldots, \alpha_a]$ generating $\det_\kappa(A)$, a symbol $[\gamma_1, \ldots, \gamma_c]$ generating $\det_\kappa(C)$, a symbol $[\zeta_1, \ldots, \zeta_g]$ generating $\det_\kappa(G)$, and a symbol $[\iota_1, \ldots, \iota_i]$ generating $\det_\kappa(I)$ we have

$$\begin{aligned} & [\alpha_1, \ldots, \alpha_a, \tilde{\gamma}_1, \ldots, \tilde{\gamma}_c, \tilde{\zeta}_1, \ldots, \tilde{\zeta}_g, \tilde{\iota}_1, \ldots, \tilde{\iota}_i] \\ = \ & (-1)^{cg}[\alpha_1, \ldots, \alpha_a, \tilde{\zeta}_1, \ldots, \tilde{\zeta}_g, \tilde{\gamma}_1, \ldots, \tilde{\gamma}_c, \tilde{\iota}_1, \ldots, \tilde{\iota}_i] \end{aligned}$$

(for suitable lifts $\tilde{x}$ in E) in $\det_\kappa(E)$. This holds because we may use the admissible relations of type (c) cg times in the following order: move the $\tilde{\zeta}_1$ past the elements $\tilde{\gamma}_c, \ldots, \tilde{\gamma}_1$ (allowed since $\mathfrak{m}\tilde{\zeta}_1 \subset A$), then move $\tilde{\zeta}_2$ past the elements $\tilde{\gamma}_c, \ldots, \tilde{\gamma}_1$ (allowed since $\mathfrak{m}\tilde{\zeta}_2 \subset A + R\tilde{\zeta}_1$), and so on.

Part (3) of the lemma is obvious. This finishes the proof. □

We can use the maps γ of the lemma to define more general maps γ as follows. Suppose that $(R, \mathfrak{m}, \kappa)$ is a local ring. Let M be a finite length R-module and suppose we are given a finite filtration (see Homology, Definition 12.16.1)

$$M = F^n \supset F^{n+1} \supset \ldots \supset F^{m-1} \supset F^m = 0.$$

Then there is a canonical isomorphism

$$\gamma_{(M,F)} : \bigotimes\nolimits_i \det_\kappa(F^i/F^{i+1}) \longrightarrow \det_\kappa(M)$$

well defined up to sign(!). One can make the sign explicit either by giving a well defined order of the terms in the tensor product (starting with higher indices unfortunately), and by thinking of the target category for the functor $\det_\kappa$ as the category of 1-dimensional super vector spaces. See [**KM76**, Section 1].

Here is another typical result for determinant functors. It is not hard to show. The tricky part is usually to show the existence of a determinant functor.

Lemma 41.2.6. *Let $(R, \mathfrak{m}, \kappa)$ be any local ring. The functor*

$$\det_\kappa : \left\{\begin{matrix}\text{finite length } R\text{-modules} \\ \text{with isomorphisms}\end{matrix}\right\} \longrightarrow \left\{\begin{matrix}1\text{-dimensional } \kappa\text{-vector spaces} \\ \text{with isomorphisms}\end{matrix}\right\}$$

endowed with the maps $\gamma_{K\to L\to M}$ is characterized by the following properties

(1) *its restriction to the subcategory of modules annihilated by $\mathfrak{m}$ is isomorphic to the usual determinant functor (see Lemma 41.2.3), and*
(2) *(1), (2) and (3) of Lemma 41.2.5 hold.*

Proof. Omitted. □

Lemma 41.2.7. *Let $(R, \mathfrak{m}, \kappa)$ be a local ring. Let $I \subset \mathfrak{m}$ be an ideal, and set $R' = R/I$. Let $\det_{R,\kappa}$ denote the determinant functor on the category Mod_R^f finite length R-modules and denote $\det_{R',\kappa}$ the determinant on the category $Mod_{R'}^f$ of finite length R'-modules. Then $Mod_{R'}^f \subset Mod_R^f$ is a full subcategory and there exists an isomorphism of functors*

$$\det_{R,\kappa}|_{Mod_{R'}^f} = \det_{R',\kappa}$$

compatible with the isomorphisms $\gamma_{K\to L\to M}$ for either of these functors.

Proof. This can be shown by using the characterization of the pair $(\det_{R',\kappa}, \gamma)$ in Lemma 41.2.6. But really the isomorphism is obtained by mapping a symbol $[x_1, \ldots, x_l] \in \det_{R,\kappa}(M)$ to the corresponding symbol $[x_1, \ldots, x_l] \in \det_{R',\kappa}(M)$ which "obviously" works. □

Here is a case where we can compute the determinant of a linear map. In fact there is nothing mysterious about this in any case, see Example 41.2.9 for a random example.

Lemma 41.2.8. *Let R be a local ring with residue field κ. Let $u \in R^*$ be a unit. Let M be a module of finite length over R. Denote $u_M : M \to M$ the map multiplication by u. Then*

$$\det_\kappa(u_M) : \det_\kappa(M) \longrightarrow \det_\kappa(M)$$

is multiplication by $\overline{u}^l$ where $l = length_R(M)$ and $\overline{u} \in \kappa^*$ is the image of u.

Proof. Denote $f_M \in \kappa^*$ the element such that $\det_\kappa(u_M) = f_M \mathrm{id}_{\det_\kappa(M)}$. Suppose that $0 \to K \to L \to M \to 0$ is a short exact sequence of finite R-modules. Then we see that u_k, u_L, u_M give an isomorphism of short exact sequences. Hence by Lemma 41.2.5 (1) we conclude that $f_K f_M = f_L$. This means that by induction on length it suffices to prove the lemma in the case of length 1 where it is trivial. □

Example 41.2.9. Consider the local ring $R = \mathbf{Z}_p$. Set $M = \mathbf{Z}_p/(p^2) \oplus \mathbf{Z}_p/(p^3)$. Let $u : M \to M$ be the map given by the matrix

$$u = \begin{pmatrix} a & b \\ pc & d \end{pmatrix}$$

where $a, b, c, d \in \mathbf{Z}_p$, and $a, d \in \mathbf{Z}_p^*$. In this case $\det_\kappa(u)$ equals multiplication by $a^2d^3 \bmod p \in \mathbf{F}_p^*$. This can easily be seen by consider the effect of u on the symbol $[p^2e, pe, pf, e, f]$ where $e = (0, 1) \in M$ and $f = (1, 0) \in M$.

41.3. Periodic complexes

Of course there is a very general notion of periodic complexes. We can require periodicity of the maps, or periodicity of the objects. We will add these here as needed. For the moment we only need the following cases.

Definition 41.3.1. Let R be a ring.

(1) A 2-*periodic complex* over R is given by a quadruple (M, N, φ, ψ) consisting of R-modules M, N and R-module maps $\varphi : M \to N$, $\psi : N \to M$ such that

$$\ldots \longrightarrow M \xrightarrow{\varphi} N \xrightarrow{\psi} M \xrightarrow{\varphi} N \longrightarrow \ldots$$

is a complex. In this setting we define the *cohomology modules* of the complex to be the R-modules

$$H^0(M, N, \varphi, \psi) = \mathrm{Ker}(\varphi)/\mathrm{Im}(\psi), \quad \text{and} \quad H^1(M, N, \varphi, \psi) = \mathrm{Ker}(\psi)/\mathrm{Im}(\varphi).$$

We say the 2-periodic complex is *exact* if the cohomology groups are zero.

(2) A $(2, 1)$-*periodic complex* over R is given by a triple (M, φ, ψ) consisting of an R-module M and R-module maps $\varphi : M \to M$, $\psi : M \to M$ such that

$$\ldots \longrightarrow M \xrightarrow{\varphi} M \xrightarrow{\psi} M \xrightarrow{\varphi} M \longrightarrow \ldots$$

is a complex. Since this is a special case of a 2-periodic complex we have its *cohomology modules* $H^0(M, \varphi, \psi)$, $H^1(M, \varphi, \psi)$ and a notion of exactness.

In the following we will use any result proved for 2-periodic complexes without further mention for $(2, 1)$-periodic complexes. It is clear that the collection of 2-periodic complexes (resp. $(2, 1)$-periodic complexes) forms a category with morphisms $(f, g) : (M, N, \varphi, \psi) \to (M', N', \varphi', \psi')$ pairs of morphisms $f : M \to M'$ and $g : N \to N'$ such that $\varphi' \circ f = f \circ \varphi$ and $\psi' \circ g = g \circ \psi$. In fact it is an abelian category, with kernels and cokernels as in Homology, Lemma 12.12.3. Also, note that a special case are the $(2, 1)$-periodic complexes of the form $(M, 0, \psi)$. In this special case we have

$$H^0(M, 0, \psi) = \mathrm{Coker}(\psi), \quad \text{and} \quad H^1(M, 0, \psi) = \mathrm{Ker}(\psi).$$

Definition 41.3.2. Let R be a local ring. Let (M, N, φ, ψ) be a 2-periodic complex over R whose cohomology groups have finite length over R. In this case we define the *multiplicity* of (M, N, φ, ψ) to be the integer

$$e_R(M, N, \varphi, \psi) = \mathrm{length}_R(H^0(M, N, \varphi, \psi)) - \mathrm{length}_R(H^1(M, N, \varphi, \psi))$$

We will sometimes (especially in the case of a $(2, 1)$-periodic complex with $\varphi = 0$) call this the *Herbrand quotient*[1].

Lemma 41.3.3. *Let R be a local ring.*

(1) *If (M, N, φ, ψ) is a 2-periodic complex such that M, N have finite length. Then $e_R(M, N, \varphi, \psi) = \mathrm{length}_R(M) - \mathrm{length}_R(N)$.*

(2) *If (M, φ, ψ) is a $(2, 1)$-periodic complex such that M has finite length. Then $e_R(M, \varphi, \psi) = 0$.*

[1] If the residue field of R is finite with q elements it is customary to call the Herbrand quotient $h(M, N, \varphi, \psi) = q^{e_R(M,N,\varphi,\psi)}$ which is equal to the number of elements of H^0 divided by the number of elements of H^1.

(3) *Suppose that we have a short exact sequence of* $(2,1)$*-periodic complexes*

$$0 \to (M_1, N_1, \varphi_1, \psi_1) \to (M_2, N_2, \varphi_2, \psi_2) \to (M_3, N_3, \varphi_3, \psi_3) \to 0$$

If two out of three have cohomology modules of finite length so does the third and we have

$$e_R(M_2, N_2, \varphi_2, \psi_2) = e_R(M_1, N_1, \varphi_1, \psi_1) + e_R(M_3, N_3, \varphi_3, \psi_3).$$

Proof. Proof of (3). Abbreviate $A = (M_1, N_1, \varphi_1, \psi_1)$, $B = (M_2, N_2, \varphi_2, \psi_2)$ and $C = (M_3, N_3, \varphi_3, \psi_3)$. We have a long exact cohomology sequence

$$\ldots \to H^1(C) \to H^0(A) \to H^0(B) \to H^0(C) \to H^1(A) \to H^1(B) \to H^1(C) \to \ldots$$

This gives a finite exact sequence

$$0 \to I \to H^0(A) \to H^0(B) \to H^0(C) \to H^1(A) \to H^1(B) \to K \to 0$$

with $0 \to K \to H^1(C) \to I \to 0$ a filtration. By additivity of the length function (Algebra, Lemma 10.48.3) we see the result. The proofs of (1) and (2) are omitted. □

Let R be a local ring with residue field κ. Let (M, φ, ψ) be a $(2,1)$-periodic complex over R. Assume that M has finite length and that (M, φ, ψ) is exact. We are going to use the determinant construction to define an invariant of this situation. See Section 41.2. Let us abbreviate $K_\varphi = \mathrm{Ker}(\varphi)$, $I_\varphi = \mathrm{Im}(\varphi)$, $K_\psi = \mathrm{Ker}(\psi)$, and $I_\psi = \mathrm{Im}(\psi)$. The short exact sequences

$$0 \to K_\varphi \to M \to I_\varphi \to 0, \quad 0 \to K_\psi \to M \to I_\psi \to 0$$

give isomorphisms

$$\gamma_\varphi : \det_\kappa(K_\varphi) \otimes \det_\kappa(I_\varphi) \longrightarrow \det_\kappa(M), \quad \gamma_\psi : \det_\kappa(K_\psi) \otimes \det_\kappa(I_\psi) \longrightarrow \det_\kappa(M),$$

see Lemma 41.2.5. On the other hand the exactness of the complex gives equalities $K_\varphi = I_\psi$, and $K_\psi = I_\varphi$ and hence an isomorphism

$$\sigma : \det_\kappa(K_\varphi) \otimes \det_\kappa(I_\varphi) \longrightarrow \det_\kappa(K_\psi) \otimes \det_\kappa(I_\psi)$$

by switching the factors. Using this notation we can define our invariant.

Definition 41.3.4. Let R be a local ring with residue field κ. Let (M, φ, ψ) be a $(2,1)$-periodic complex over R. Assume that M has finite length and that (M, φ, ψ) is exact. The *determinant of* (M, φ, ψ) is the element

$$\det_\kappa(M, \varphi, \psi) \in \kappa^*$$

such that the composition

$$\det_\kappa(M) \xrightarrow{\gamma_\psi \circ \sigma \circ \gamma_\varphi^{-1}} \det_\kappa(M)$$

is multiplication by $(-1)^{\mathrm{length}_R(I_\varphi)\mathrm{length}_R(I_\psi)} \det_\kappa(M, \varphi, \psi)$.

Remark 41.3.5. Here is a more down to earth description of the determinant introduced above. Let R be a local ring with residue field κ. Let (M, φ, ψ) be a $(2,1)$-periodic complex over R. Assume that M has finite length and that (M, φ, ψ) is exact. Let us abbreviate $I_\varphi = \mathrm{Im}(\varphi)$, $I_\psi = \mathrm{Im}(\psi)$ as above. Assume that $\mathrm{length}_R(I_\varphi) = a$ and $\mathrm{length}_R(I_\psi) = b$, so that $a + b = \mathrm{length}_R(M)$ by exactness. Choose admissible sequences $x_1, \ldots, x_a \in I_\varphi$ and $y_1, \ldots, y_b \in I_\psi$ such that the symbol $[x_1, \ldots, x_a]$ generates $\det_\kappa(I_\varphi)$ and the symbol $[x_1, \ldots, x_b]$ generates $\det_\kappa(I_\psi)$.

Choose $\tilde{x}_i \in M$ such that $\varphi(\tilde{x}_i) = x_i$. Choose $\tilde{y}_j \in M$ such that $\psi(\tilde{y}_j) = y_j$. Then $\det_\kappa(M, \varphi, \psi)$ is characterized by the equality

$$[x_1, \ldots, x_a, \tilde{y}_1, \ldots, \tilde{y}_b] = (-1)^{ab} \det_\kappa(M, \varphi, \psi)[y_1, \ldots, y_b, \tilde{x}_1, \ldots, \tilde{x}_a]$$

in $\det_\kappa(M)$. This also explains the sign.

Lemma 41.3.6. *Let R be a local ring with residue field κ. Let (M, φ, ψ) be a $(2,1)$-periodic complex over R. Assume that M has finite length and that (M, φ, ψ) is exact. Then*

$$\det_\kappa(M, \varphi, \psi) \det_\kappa(M, \psi, \varphi) = 1.$$

Proof. Omitted. □

Lemma 41.3.7. *Let R be a local ring with residue field κ. Let (M, φ, φ) be a $(2,1)$-periodic complex over R. Assume that M has finite length and that (M, φ, φ) is exact. Then $length_R(M) = 2length_R(Im(\varphi))$ and*

$$\det_\kappa(M, \varphi, \psi) = (-1)^{length_R(Im(\varphi))} = (-1)^{\frac{1}{2} length_R(M)}$$

Proof. Follows directly from the sign rule in the definitions. □

Lemma 41.3.8. *Let R be a local ring with residue field κ. Let M be a finite length R-module.*

(1) *if $\varphi : M \to M$ is an isomorphism then $\det_\kappa(M, \varphi, 0) = \det_\kappa(\varphi)$.*
(2) *if $\psi : M \to M$ is an isomorphism then $\det_\kappa(M, 0, \psi) = \det_\kappa(\psi)^{-1}$.*

Proof. Let us prove (1). Set $\psi = 0$. Then we may, with notation as above Definition 41.3.4, identify $K_\varphi = I_\psi = 0$, $I_\varphi = K_\psi = M$. With these identifications, the map

$$\gamma_\varphi : \kappa \otimes \det_\kappa(M) = \det_\kappa(K_\varphi) \otimes \det_\kappa(I_\varphi) \longrightarrow \det_\kappa(M)$$

is identified with $\det_\kappa(\varphi^{-1})$. On the other hand the map γ_ψ is identified with the identity map. Hence $\gamma_\psi \circ \sigma \circ \gamma_\varphi^{-1}$ is equal to $\det_\kappa(\varphi)$ in this case. Whence the result. We omit the proof of (2). □

Lemma 41.3.9. *Let R be a local ring with residue field κ. Suppose that we have a short exact sequence of $(2,1)$-periodic complexes*

$$0 \to (M_1, \varphi_1, \psi_1) \to (M_2, \varphi_2, \psi_2) \to (M_3, \varphi_3, \psi_3) \to 0$$

with all M_i of finite length, and each (M_1, φ_1, ψ_1) exact. Then

$$\det_\kappa(M_2, \varphi_2, \psi_2) = \det_\kappa(M_1, \varphi_1, \psi_1) \det_\kappa(M_3, \varphi_3, \psi_3).$$

in κ^*.

Proof. Let us abbreviate $I_{\varphi,i} = \mathrm{Im}(\varphi_i)$, $K_{\varphi,i} = \mathrm{Ker}(\varphi_i)$, $I_{\psi,i} = \mathrm{Im}(\psi_i)$, and $K_{\psi,i} = \mathrm{Ker}(\psi_i)$. Observe that we have a commutative square

$$\begin{array}{ccccccccc}
 & & 0 & & 0 & & 0 & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & K_{\varphi,1} & \longrightarrow & K_{\varphi,2} & \longrightarrow & K_{\varphi,3} & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & M_1 & \longrightarrow & M_2 & \longrightarrow & M_3 & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & I_{\varphi,1} & \longrightarrow & I_{\varphi,2} & \longrightarrow & I_{\varphi,3} & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
 & & 0 & & 0 & & 0 & &
\end{array}$$

of finite length R-modules with exact rows and columns. The top row is exact since it can be identified with the sequence $I_{\psi,1} \to I_{\psi,2} \to I_{\psi,3} \to 0$ of images, and similarly for the bottom row. There is a similar diagram involving the modules $I_{\psi,i}$ and $K_{\psi,i}$. By definition $\det_\kappa(M_2, \varphi_2, \psi_2)$ corresponds, up to a sign, to the composition of the left vertical maps in the following diagram

$$\begin{array}{ccc}
\det_\kappa(M_1) \otimes \det_\kappa(M_3) & \xrightarrow{\gamma} & \det_\kappa(M_2) \\
\downarrow{\scriptstyle \gamma^{-1}\otimes\gamma^{-1}} & & \downarrow{\scriptstyle \gamma^{-1}} \\
\det_\kappa(K_{\varphi,1}) \otimes \det_\kappa(I_{\varphi,1}) \otimes \det_\kappa(K_{\varphi,3}) \otimes \det_\kappa(I_{\varphi,3}) & \xrightarrow{\gamma\otimes\gamma} & \det_\kappa(K_{\varphi,2}) \otimes \det_\kappa(I_{\varphi,2}) \\
\downarrow{\scriptstyle \sigma\otimes\sigma} & & \downarrow{\scriptstyle \sigma} \\
\det_\kappa(K_{\psi,1}) \otimes \det_\kappa(I_{\psi,1}) \otimes \det_\kappa(K_{\psi,3}) \otimes \det_\kappa(I_{\psi,3}) & \xrightarrow{\gamma\otimes\gamma} & \det_\kappa(K_{\psi,2}) \otimes \det_\kappa(I_{\psi,2}) \\
\downarrow{\scriptstyle \gamma\otimes\gamma} & & \downarrow{\scriptstyle \gamma} \\
\det_\kappa(M_1) \otimes \det_\kappa(M_3) & \xrightarrow{\gamma} & \det_\kappa(M_2)
\end{array}$$

The top and bottom squares are commutative up to sign by applying Lemma 41.2.5 (2). The middle square is trivially commutative (we are just switching factors). Hence we see that $\det_\kappa(M_2, \varphi_2, \psi_2) = \epsilon \det_\kappa(M_1, \varphi_1, \psi_1) \det_\kappa(M_3, \varphi_3, \psi_3)$ for some sign ϵ. And the sign can be worked out, namely the outer rectangle in the diagram above commutes up to

$$\begin{aligned}
\epsilon &= (-1)^{\mathrm{length}(I_{\varphi,1})\mathrm{length}(K_{\varphi,3})+\mathrm{length}(I_{\psi,1})\mathrm{length}(K_{\psi,3})} \\
&= (-1)^{\mathrm{length}(I_{\varphi,1})\mathrm{length}(I_{\psi,3})+\mathrm{length}(I_{\psi,1})\mathrm{length}(I_{\varphi,3})}
\end{aligned}$$

(proof omitted). It follows easily from this that the signs work out as well. □

Example 41.3.10. Let k be a field. Consider the ring $R = k[T]/(T^2)$ of dual numbers over k. Denote t the class of T in R. Let $M = R$ and $\varphi = ut$, $\psi = vt$ with $u, v \in k^*$. In this case $\det_k(M)$ has generator $e = [t, 1]$. We identify $I_\varphi = K_\varphi = I_\psi = K_\psi = (t)$. Then $\gamma_\varphi(t \otimes t) = u^{-1}[t, 1]$ (since $u^{-1} \in M$ is a lift of $t \in I_\varphi$) and $\gamma_\psi(t \otimes t) = v^{-1}[t, 1]$ (same reason). Hence we see that $\det_k(M, \varphi, \psi) = -u/v \in k^*$.

Example 41.3.11. Let $R = \mathbf{Z}_p$ and let $M = \mathbf{Z}_p/(p^l)$. Let $\varphi = p^b u$ and $\varphi = p^a v$ with $a, b \geq 0$, $a + b = l$ and $u, v \in \mathbf{Z}_p^*$. Then a computation as in Example 41.3.10 shows that

$$\begin{aligned} \det_{\mathbf{F}_p}(\mathbf{Z}_p/(p^l), p^b u, p^a v) &= (-1)^{ab} u^a / v^b \bmod p \\ &= (-1)^{\mathrm{ord}_p(\alpha)\mathrm{ord}_p(\beta)} \frac{\alpha^{\mathrm{ord}_p(\beta)}}{\beta^{\mathrm{ord}_p(\alpha)}} \bmod p \end{aligned}$$

with $\alpha = p^b u, \beta = p^a v \in \mathbf{Z}_p$. See Lemma 41.4.10 for a more general case (and a proof).

Example 41.3.12. Let $R = k$ be a field. Let $M = k^{\oplus a} \oplus k^{\oplus b}$ be $l = a + b$ dimensional. Let φ and ψ be the following diagonal matrices

$$\varphi = \mathrm{diag}(u_1, \ldots, u_a, 0, \ldots, 0), \quad \psi = \mathrm{diag}(0, \ldots, 0, v_1, \ldots, v_b)$$

with $u_i, v_j \in k^*$. In this case we have

$$\det_k(M, \varphi, \psi) = \frac{u_1 \ldots u_a}{v_1 \ldots v_b}.$$

This can be seen by a direct computation or by computing in case $l = 1$ and using the additivity of Lemma 41.3.9.

Example 41.3.13. Let $R = k$ be a field. Let $M = k^{\oplus a} \oplus k^{\oplus a}$ be $l = 2a$ dimensional. Let φ and ψ be the following block matrices

$$\varphi = \begin{pmatrix} 0 & U \\ 0 & 0 \end{pmatrix}, \quad \psi = \begin{pmatrix} 0 & V \\ 0 & 0 \end{pmatrix},$$

with $U, V \in \mathrm{Mat}(a \times a, k)$ invertible. In this case we have

$$\det_k(M, \varphi, \psi) = (-1)^a \frac{\det(U)}{\det(V)}.$$

This can be seen by a direct computation. The case $a = 1$ is similar to the computation in Example 41.3.10.

Example 41.3.14. Let $R = k$ be a field. Let $M = k^{\oplus 4}$. Let

$$\varphi = \begin{pmatrix} 0 & 0 & 0 & 0 \\ u_1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & u_2 & 0 \end{pmatrix} \quad \varphi = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & v_2 & 0 \\ 0 & 0 & 0 & 0 \\ v_1 & 0 & 0 & 0 \end{pmatrix}$$

with $u_1, u_2, v_1, v_2 \in k^*$. Then we have

$$\det_k(M, \varphi, \psi) = -\frac{u_1 u_2}{v_1 v_2}.$$

Next we come to the analogue of the fact that the determinant of a composition of linear endomorphisms is the product of the determinants. To avoid very long formulae we write $I_\varphi = \mathrm{Im}(\varphi)$, and $K_\varphi = \mathrm{Ker}(\varphi)$ for any R-module map $\varphi : M \to M$. We also denote $\varphi\psi = \varphi \circ \psi$ for a pair of morphisms $\varphi, \psi : M \to M$.

Lemma 41.3.15. *Let R be a local ring with residue field κ. Let M be a finite length R-module. Let α, β, γ be endomorphisms of M. Assume that*

(1) *$I_\alpha = K_{\beta\gamma}$, and similarly for any permutation of α, β, γ,*
(2) *$K_\alpha = I_{\beta\gamma}$, and similarly for any permutation of α, β, γ.*

Then

(1) *The triple* $(M, \alpha, \beta\gamma)$ *is an exact* $(2,1)$*-periodic complex.*
(2) *The triple* $(I_\gamma, \alpha, \beta)$ *is an exact* $(2,1)$*-periodic complex.*
(3) *The triple* $(M/K_\beta, \alpha, \gamma)$ *is an exact* $(2,1)$*-periodic complex.*
(4) *We have*

$$\det_\kappa(M, \alpha, \beta\gamma) = \det_\kappa(I_\gamma, \alpha, \beta) \det_\kappa(M/K_\beta, \alpha, \gamma).$$

Proof. It is clear that the assumptions imply part (1) of the lemma.

To see part (1) note that the assumptions imply that $I_{\gamma\alpha} = I_{\alpha\gamma}$, and similarly for kernels and any other pair of morphisms. Moreover, we see that $I_{\gamma\beta} = I_{\beta\gamma} = K_\alpha \subset I_\gamma$ and similarly for any other pair. In particular we get a short exact sequence

$$0 \to I_{\beta\gamma} \to I_\gamma \xrightarrow{\alpha} I_{\alpha\gamma} \to 0$$

and similarly we get a short exact sequence

$$0 \to I_{\alpha\gamma} \to I_\gamma \xrightarrow{\beta} I_{\beta\gamma} \to 0.$$

This proves $(I_\gamma, \alpha, \beta)$ is an exact $(2,1)$-periodic complex. Hence part (2) of the lemma holds.

To see that α, γ give well defined endomorphisms of M/K_β we have to check that $\alpha(K_\beta) \subset K_\beta$ and $\gamma(K_\beta) \subset K_\beta$. This is true because $\alpha(K_\beta) = \alpha(I_{\gamma\alpha}) = I_{\alpha\gamma\alpha} \subset I_{\alpha\gamma} = K_\beta$, and similarly in the other case. The kernel of the map $\alpha : M/K_\beta \to M/K_\beta$ is $K_{\beta\alpha}/K_\beta = I_\gamma/K_\beta$. Similarly, the kernel of $\gamma : M/K_\beta \to M/K_\beta$ is equal to I_α/K_β. Hence we conclude that (3) holds.

We introduce $r = \text{length}_R(K_\alpha)$, $s = \text{length}_R(K_\beta)$ and $t = \text{length}_R(K_\gamma)$. By the exact sequences above and our hypotheses we have $\text{length}_R(I_\alpha) = s + t$, $\text{length}_R(I_\beta) = r + t$, $\text{length}_R(I_\gamma) = r + s$, and $\text{length}(M) = r + s + t$. Choose

(1) an admissible sequence $x_1, \ldots, x_r \in K_\alpha$ generating K_α
(2) an admissible sequence $y_1, \ldots, y_s \in K_\beta$ generating K_β,
(3) an admissible sequence $z_1, \ldots, z_t \in K_\gamma$ generating K_γ,
(4) elements $\tilde{x}_i \in M$ such that $\beta\gamma\tilde{x}_i = x_i$,
(5) elements $\tilde{y}_i \in M$ such that $\alpha\gamma\tilde{y}_i = y_i$,
(6) elements $\tilde{z}_i \in M$ such that $\beta\alpha\tilde{z}_i = z_i$.

With these choices the sequence $y_1, \ldots, y_s, \alpha\tilde{z}_1, \ldots, \alpha\tilde{z}_t$ is an admissible sequence in I_α generating it. Hence, by Remark 41.3.5 the determinant $D = \det_\kappa(M, \alpha, \beta\gamma)$ is the unique element of κ^* such that

$$\begin{aligned}&[y_1, \ldots, y_s, \alpha\tilde{z}_1, \ldots, \alpha\tilde{z}_s, \tilde{x}_1, \ldots, \tilde{x}_r] \\ &= (-1)^{r(s+t)} D[x_1, \ldots, x_r, \gamma\tilde{y}_1, \ldots, \gamma\tilde{y}_s, \tilde{z}_1, \ldots, \tilde{z}_t]\end{aligned}$$

By the same remark, we see that $D_1 = \det_\kappa(M/K_\beta, \alpha, \gamma)$ is characterized by

$$[y_1, \ldots, y_s, \alpha\tilde{z}_1, \ldots, \alpha\tilde{z}_t, \tilde{x}_1, \ldots, \tilde{x}_r] = (-1)^{rt} D_1[y_1, \ldots, y_s, \gamma\tilde{x}_1, \ldots, \gamma\tilde{x}_r, \tilde{z}_1, \ldots, \tilde{z}_t]$$

By the same remark, we see that $D_2 = \det_\kappa(I_\gamma, \alpha, \beta)$ is characterized by

$$[y_1, \ldots, y_s, \gamma\tilde{x}_1, \ldots, \gamma\tilde{x}_r, \tilde{z}_1, \ldots, \tilde{z}_t] = (-1)^{rs} D_2[x_1, \ldots, x_r, \gamma\tilde{y}_1, \ldots, \gamma\tilde{y}_s, \tilde{z}_1, \ldots, \tilde{z}_t]$$

Combining the formulas above we see that $D = D_1 D_2$ as desired. □

Lemma 41.3.16. *Let* R *be a local ring with residue field* κ*. Let* $\alpha : (M, \varphi, \psi) \to (M', \varphi', \psi')$ *be a morphism of* $(2,1)$*-periodic complexes over* R*. Assume*

(1) M*,* M' *have finite length,*

(2) (M, φ, ψ), (M', φ', ψ') *are exact,*
(3) *the maps* φ, ψ *induce the zero map on* $K = Ker(\alpha)$, *and*
(4) *the maps* φ, ψ *induce the zero map on* $Q = Coker(\alpha)$.

Denote $N = \alpha(M) \subset M'$. *We obtain two short exact sequences of* $(2,1)$*-periodic complexes*

$$0 \to (N, \varphi', \psi') \to (M', \varphi', \psi') \to (Q, 0, 0) \to 0$$
$$0 \to (K, 0, 0) \to (M, \varphi, \psi) \to (N, \varphi', \psi') \to 0$$

which induce two isomorphisms $\alpha_i : Q \to K$, $i = 0, 1$. *Then*

$$\det_\kappa(M, \varphi, \psi) = \det_\kappa(\alpha_0^{-1} \circ \alpha_1) \det_\kappa(M', \varphi', \psi')$$

In particular, if $\alpha_0 = \alpha_1$, *then* $\det_\kappa(M, \varphi, \psi) = \det_\kappa(M', \varphi', \psi')$.

Proof. There are (at least) two ways to prove this lemma. One is to produce an enormous commutative diagram using the properties of the determinants. The other is to use the characterization of the determinants in terms of admissible sequences of elements. It is the second approach that we will use.

First let us explain precisely what the maps α_i are. Namely, α_0 is the composition

$$\alpha_0 : Q = H^0(Q, 0, 0) \to H^1(N, \varphi', \psi') \to H^2(K, 0, 0) = K$$

and α_1 is the composition

$$\alpha_1 : Q = H^1(Q, 0, 0) \to H^2(N, \varphi', \psi') \to H^3(K, 0, 0) = K$$

coming from the boundary maps of the short exact sequences of complexes displayed in the lemma. The fact that the complexes (M, φ, ψ), (M', φ', ψ') are exact implies these maps are isomorphisms.

We will use the notation $I_\varphi = \mathrm{Im}(\varphi)$, $K_\varphi = \mathrm{Ker}(\varphi)$ and similarly for the other maps. Exactness for M and M' means that $K_\varphi = I_\psi$ and three similar equalities. We introduce $k = \mathrm{length}_R(K)$, $a = \mathrm{length}_R(I_\varphi)$, $b = \mathrm{length}_R(I_\psi)$. Then we see that $\mathrm{length}_R(M) = a + b$, and $\mathrm{length}_R(N) = a + b - k$, $\mathrm{length}_R(Q) = k$ and $\mathrm{length}_R(M') = a+b$. The exact sequences below will show that also $\mathrm{length}_R(I_{\varphi'}) = a$ and $\mathrm{length}_R(I_{\psi'}) = b$.

The assumption that $K \subset K_\varphi = I_\psi$ means that φ factors through N to give an exact sequence

$$0 \to \alpha(I_\psi) \to N \xrightarrow{\varphi\alpha^{-1}} I_\psi \to 0.$$

Here $\varphi\alpha^{-1}(x') = y$ means $x' = \alpha(x)$ and $y = \varphi(x)$. Similarly, we have

$$0 \to \alpha(I_\varphi) \to N \xrightarrow{\psi\alpha^{-1}} I_\varphi \to 0.$$

The assumption that ψ' induces the zero map on Q means that $I_{\psi'} = K_{\varphi'} \subset N$. This means the quotient $\varphi'(N) \subset I_{\varphi'}$ is identified with Q. Note that $\varphi'(N) = \alpha(I_\varphi)$. Hence we conclude there is an isomorphism

$$\varphi' : Q \to I_{\varphi'}/\alpha(I_\varphi)$$

simply described by $\varphi'(x' \bmod N) = \varphi'(x') \bmod \alpha(I_\varphi)$. In exactly the same way we get

$$\psi' : Q \to I_{\psi'}/\alpha(I_\psi)$$

Finally, note that α_0 is the composition

$$Q \xrightarrow{\varphi'} I_{\varphi'}/\alpha(I_\varphi) \xrightarrow{\psi\alpha^{-1}|_{I_{\varphi'}/\alpha(I_\varphi)}} K$$

and similarly $\alpha_1 = \varphi\alpha^{-1}|_{I_{\psi'}/\alpha(I_\psi)} \circ \psi'$.

To shorten the formulas below we are going to write αx instead of $\alpha(x)$ in the following. No confusion should result since all maps are indicated by greek letters and elements by roman letters. We are going to choose

(1) an admissible sequence $z_1, \ldots, z_k \in K$ generating K,
(2) elements $z_i' \in M$ such that $\varphi z_i' = z_i$,
(3) elements $z_i'' \in M$ such that $\psi z_i'' = z_i$,
(4) elements $x_{k+1}, \ldots, x_a \in I_\varphi$ such that $z_1, \ldots, z_k, x_{k+1}, \ldots, x_a$ is an admissible sequence generating I_φ,
(5) elements $\tilde{x}_i \in M$ such that $\varphi\tilde{x}_i = x_i$,
(6) elements $y_{k+1}, \ldots, y_b \in I_\psi$ such that $z_1, \ldots, z_k, y_{k+1}, \ldots, y_b$ is an admissible sequence generating I_ψ,
(7) elements $\tilde{y}_i \in M$ such that $\psi\tilde{y}_i = y_i$, and
(8) elements $w_1, \ldots, w_k \in M'$ such that $w_1 \bmod N, \ldots, w_k \bmod N$ are an admissible sequence in Q generating Q.

By Remark 41.3.5 the element $D = \det_\kappa(M, \varphi, \psi) \in \kappa^*$ is characterized by

$$\begin{aligned} & [z_1, \ldots, z_k, x_{k+1}, \ldots, x_a, z_1'', \ldots, z_k'', \tilde{y}_{k+1}, \ldots, \tilde{y}_b] \\ = \quad & (-1)^{ab} D[z_1, \ldots, z_k, y_{k+1}, \ldots, y_b, z_1', \ldots, z_k', \tilde{x}_{k+1}, \ldots, \tilde{x}_a] \end{aligned}$$

Note that by the discussion above $\alpha x_{k+1}, \ldots, \alpha x_a, \varphi w_1, \ldots, \varphi w_k$ is an admissible sequence generating $I_{\varphi'}$ and $\alpha y_{k+1}, \ldots, \alpha y_b, \psi w_1, \ldots, \psi w_k$ is an admissible sequence generating $I_{\psi'}$. Hence by Remark 41.3.5 the element $D' = \det_\kappa(M', \varphi', \psi') \in \kappa^*$ is characterized by

$$\begin{aligned} & [\alpha x_{k+1}, \ldots, \alpha x_a, \varphi' w_1, \ldots, \varphi' w_k, \alpha\tilde{y}_{k+1}, \ldots, \alpha\tilde{y}_b, w_1, \ldots, w_k] \\ = \quad & (-1)^{ab} D'[\alpha y_{k+1}, \ldots, \alpha y_b, \psi' w_1, \ldots, \psi' w_k, \alpha\tilde{x}_{k+1}, \ldots, \alpha\tilde{x}_a, w_1, \ldots, w_k] \end{aligned}$$

Note how in the first, resp. second displayed formula the the first, resp. last k entries of the symbols on both sides are the same. Hence these formulas are really equivalent to the equalities

$$\begin{aligned} & [\alpha x_{k+1}, \ldots, \alpha x_a, \alpha z_1'', \ldots, \alpha z_k'', \alpha\tilde{y}_{k+1}, \ldots, \alpha\tilde{y}_b] \\ = \quad & (-1)^{ab} D[\alpha y_{k+1}, \ldots, \alpha y_b, \alpha z_1', \ldots, \alpha z_k', \alpha\tilde{x}_{k+1}, \ldots, \alpha\tilde{x}_a] \end{aligned}$$

and

$$\begin{aligned} & [\alpha x_{k+1}, \ldots, \alpha x_a, \varphi' w_1, \ldots, \varphi' w_k, \alpha\tilde{y}_{k+1}, \ldots, \alpha\tilde{y}_b] \\ = \quad & (-1)^{ab} D'[\alpha y_{k+1}, \ldots, \alpha y_b, \psi' w_1, \ldots, \psi' w_k, \alpha\tilde{x}_{k+1}, \ldots, \alpha\tilde{x}_a] \end{aligned}$$

in $\det_\kappa(N)$. Note that $\varphi' w_1, \ldots, \varphi' w_k$ and $\alpha z_1'', \ldots, z_k''$ are admissible sequences generating the module $I_{\varphi'}/\alpha(I_\varphi)$. Write

$$[\varphi' w_1, \ldots, \varphi' w_k] = \lambda_0 [\alpha z_1'', \ldots, \alpha z_k'']$$

in $\det_\kappa(I_{\varphi'}/\alpha(I_\varphi))$ for some $\lambda_0 \in \kappa^*$. Similarly, write

$$[\psi' w_1, \ldots, \psi' w_k] = \lambda_1 [\alpha z_1', \ldots, \alpha z_k']$$

in $\det_\kappa(I_{\psi'}/\alpha(I_\psi))$ for some $\lambda_1 \in \kappa^*$. On the one hand it is clear that

$$\alpha_i([w_1, \ldots, w_k]) = \lambda_i [z_1, \ldots, z_k]$$

for $i = 0, 1$ by our description of α_i above, which means that

$$\det_\kappa(\alpha_0^{-1} \circ \alpha_1) = \lambda_1/\lambda_0$$

and on the other hand it is clear that

$$\begin{aligned} & \lambda_0[\alpha x_{k+1}, \ldots, \alpha x_a, \alpha z_1'', \ldots, \alpha z_k'', \alpha \tilde{y}_{k+1}, \ldots, \alpha \tilde{y}_b] \\ = \quad & [\alpha x_{k+1}, \ldots, \alpha x_a, \varphi' w_1, \ldots, \varphi' w_k, \alpha \tilde{y}_{k+1}, \ldots, \alpha \tilde{y}_b] \end{aligned}$$

and

$$\begin{aligned} & \lambda_1[\alpha y_{k+1}, \ldots, \alpha y_b, \alpha z_1', \ldots, \alpha z_k', \alpha \tilde{x}_{k+1}, \ldots, \alpha \tilde{x}_a] \\ = \quad & [\alpha y_{k+1}, \ldots, \alpha y_b, \psi' w_1, \ldots, \psi' w_k, \alpha \tilde{x}_{k+1}, \ldots, \alpha \tilde{x}_a] \end{aligned}$$

which imply $\lambda_0 D = \lambda_1 D'$. The lemma follows. □

41.4. Symbols

The correct generality for this construction is perhaps the situation of the following lemma.

Lemma 41.4.1. *Let A be a Noetherian local ring. Let M be a finite A-module of dimension 1. Assume $\varphi, \psi : M \to M$ are two injective A-module maps, and assume $\varphi(\psi(M)) = \psi(\varphi(M))$, for example if φ and ψ commute. Then $\text{length}_R(M/\varphi\psi M) < \infty$ and $(M/\varphi\psi M, \varphi, \psi)$ is an exact $(2,1)$-periodic complex.*

Proof. Let $\mathfrak{q}$ be a minimal prime of the support of M. Then $M_\mathfrak{q}$ is a finite length $A_\mathfrak{q}$-module, see Algebra, Lemma 10.59.3. Hence both φ and ψ induce isomorphisms $M_\mathfrak{q} \to M_\mathfrak{q}$. Thus the support of $M/\varphi\psi M$ is $\{\mathfrak{m}_A\}$ and hence it has finite length (see lemma cited above). Finally, the kernel of φ on $M/\varphi\psi M$ is clearly $\psi M/\varphi\psi M$, and hence the kernel of φ is the image of ψ on $M/\varphi\psi M$. Similarly the other way since $M/\varphi\psi M = M/\psi\varphi M$ by assumption. □

Lemma 41.4.2. *Let A be a Noetherian local ring. Let $a, b \in A$.*

1. *If M is a finite A-module of dimension 1 such that a, b are nonzerodivisors on M, then $\text{length}_A(M/abM) < \infty$ and $(M/abM, a, b)$ is a $(2,1)$-periodic exact complex.*
2. *If a, b are nonzerodivisors and $\dim(A) = 1$ then $\text{length}_A(A/(ab)) < \infty$ and $(A/(ab), a, b)$ is a $(2,1)$-periodic exact complex.*

In particular, in these cases $\det_\kappa(M/abM, a, b) \in \kappa^*$, resp. $\det_\kappa(A/(ab), a, b) \in \kappa^*$ are defined.

Proof. Follows from Lemma 41.4.1. □

Definition 41.4.3. Let A be a Noetherian local ring with residue field κ. Let $a, b \in A$. Let M be a finite A-module of dimension 1 such that a, b are nonzerodivisors on M. We define the *symbol associated to M, a, b* to be the element

$$d_M(a, b) = \det_\kappa(M/abM, a, b) \in \kappa^*$$

Lemma 41.4.4. *Let A be a Noetherian local ring. Let $a, b, c \in A$. Let M be a finite A-module with $\dim(M) = 1$. Assume a, b, c are nonzerodivisors on M. Then*

$$d_M(a, bc) = d_M(a, b) d_M(a, c)$$

and $d_M(a, b) d_M(b, a) = 1$.

Proof. The first statement is immediate from Lemma 41.3.15 above. The second comes from Lemma 41.3.6. □

Definition 41.4.5. Let A be a Noetherian local domain of dimension 1 with residue field κ. Let K be the fraction field of A. We define the *tame symbol* of A to be the map

$$K^* \times K^* \longrightarrow \kappa^*, \quad (x, y) \longmapsto d_A(x, y)$$

where $d_A(x, y)$ is extended to $K^* \times K^*$ by the multiplicativity of Lemma 41.4.4.

It is clear that we may extend more generally $d_M(-, -)$ to certain rings of fractions of A (even if A is not a domain).

Lemma 41.4.6. *Let A be a Noetherian local ring. Let M be a finite A-module of dimension 1. Let $b \in A$ be a nonzerodivisor on M, and let $u \in A^*$. Then*

$$d_M(u, b) = u^{length_M(M/bM)} \bmod \mathfrak{m}_A.$$

In particular, if $M = A$, then $d_A(u, b) = u^{ord_A(b)} \bmod \mathfrak{m}_A$.

Proof. Note that in this case $M/ubM = M/bM$ on which multiplication by b is zero. Hence $d_M(u, b) = \det_\kappa(u|_{M/bM})$ by Lemma 41.3.8. The lemma then follows from Lemma 41.2.8. □

Lemma 41.4.7. *Let A be a Noetherian local ring. Let $a, b \in A$. Let*

$$0 \to M \to M' \to M'' \to 0$$

be a short exact sequence of A-modules of dimension 1 such that a, b are nonzerodivisors on all three A-modules. Then

$$d_{M'}(a, b) = d_M(a, b) d_{M''}(a, b)$$

in κ^*.

Proof. It is easy to see that this leads to a short exact sequence of exact $(2, 1)$-periodic complexes

$$0 \to (M/abM, a, b) \to (M'/abM', a, b) \to (M''/abM'', a, b) \to 0$$

Hence the lemma follows from Lemma 41.3.9. □

Lemma 41.4.8. *Let A be a Noetherian local ring. Let $\alpha : M \to M'$ be a homomorphism of finite A-modules of dimension 1. Let $a, b \in A$. Assume*

(1) *a, b are nonzerodivisors on both M and M', and*
(2) *$\dim(Ker(\alpha)), \dim(Coker(\alpha)) \leq 0$.*

Then $d_M(a, b) = d_{M'}(a, b)$.

Proof. If $a \in A^*$, then the equality follows from the equality $\text{length}(M/bM) = \text{length}(M'/bM')$ and Lemma 41.4.6. Similarly if b is a unit the lemma holds as well (by the symmetry of Lemma 41.4.4). Hence we may assume that $a, b \in \mathfrak{m}_A$. This in particular implies that $\mathfrak{m}$ is not an associated prime of M, and hence $\alpha : M \to M'$ is injective. This permits us to think of M as a submodule of M'. By assumption M'/M is a finite A-module with support $\{\mathfrak{m}_A\}$ and hence has finite length. Note that for any third module M'' with $M \subset M'' \subset M'$ the maps $M \to M''$ and $M'' \to M'$ satisfy the assumptions of the lemma as well. This reduces us, by induction on the length of M'/M, to the case where $\text{length}_A(M'/M) = 1$. Finally, in this case consider the map

$$\overline{\alpha} : M/abM \longrightarrow M'/abM'.$$

By construction the cokernel Q of $\overline{\alpha}$ has length 1. Since $a, b \in \mathfrak{m}_A$, they act trivially on Q. It also follows that the kernel K of $\overline{\alpha}$ has length 1 and hence also a, b act trivially on K. Hence we may apply Lemma 41.3.16. Thus it suffices to see that the two maps $\alpha_i : Q \to K$ are the same. In fact, both maps are equal to the map $q = x' \bmod \mathrm{Im}(\overline{\alpha}) \mapsto abx' \in K$. We omit the verification. □

Lemma 41.4.9. *Let A be a Noetherian local ring. Let M be a finite A-module with $\dim(M) = 1$. Let $a, b \in A$ nonzerodivisors on M. Let $\mathfrak{q}_1, \ldots, \mathfrak{q}_t$ be the minimal primes in the support of M. Then*

$$d_M(a,b) = \prod\nolimits_{i=1,\ldots,t} d_{A/\mathfrak{q}_i}(a,b)^{length_{A_{\mathfrak{q}_i}}(M_{\mathfrak{q}_i})}$$

as elements of κ^*.

Proof. Choose a filtration by A-submodules

$$0 = M_0 \subset M_1 \subset \ldots \subset M_n = M$$

such that each quotient M_j/M_{j-1} is isomorphic to $A/\mathfrak{p}_j$ for some prime ideal $\mathfrak{p}_j$ of A. See Algebra, Lemma 10.59.1. For each j we have either $\mathfrak{p}_j = \mathfrak{q}_i$ for some i, or $\mathfrak{p}_j = \mathfrak{m}_A$. Moreover, for a fixed i, the number of j such that $\mathfrak{p}_j = \mathfrak{q}_i$ is equal to $\mathrm{length}_{A_{\mathfrak{q}_i}}(M_{\mathfrak{q}_i})$ by Algebra, Lemma 10.59.5. Hence $d_{M_j}(a,b)$ is defined for each j and

$$d_{M_j}(a,b) = \begin{cases} d_{M_{j-1}}(a,b) d_{A/\mathfrak{q}_i}(a,b) & \text{if} \quad \mathfrak{p}_j = \mathfrak{q}_i \\ d_{M_{j-1}}(a,b) & \text{if} \quad \mathfrak{p}_j = \mathfrak{m}_A \end{cases}$$

by Lemma 41.4.7 in the first instance and Lemma 41.4.8 in the second. Hence the lemma. □

Lemma 41.4.10. *Let A be a discrete valuation ring with fraction field K. For nonzero $x, y \in K$ we have*

$$d_A(x,y) = (-1)^{ord_A(x) ord_A(y)} \frac{x^{ord_A(y)}}{y^{ord_A(x)}} \bmod \mathfrak{m}_A,$$

in other words the symbol is equal to the usual tame symbol.

Proof. By multiplicativity it suffices to prove this when $x, y \in A$. Let $t \in A$ be a uniformizer. Write $x = t^b u$ and $y = t^b v$ for some $a, b \geq 0$ and $u, v \in A^*$. Set $l = a + b$. Then $t^{l-1}, \ldots, t^b$ is an admissible sequence in $(x)/(xy)$ and $t^{l-1}, \ldots, t^a$ is an admissible sequence in $(y)/(xy)$. Hence by Remark 41.3.5 we see that $d_A(x,y)$ is characterized by the equation

$$[t^{l-1}, \ldots, t^b, v^{-1}t^{b-1}, \ldots, v^{-1}] = (-1)^{ab} d_A(x,y) [t^{l-1}, \ldots, t^a, u^{-1}t^{a-1}, \ldots, u^{-1}].$$

Hence by the admissible relations for the symbols $[x_1, \ldots, x_l]$ we see that

$$d_A(x,y) = (-1)^{ab} u^a / v^b \bmod \mathfrak{m}_A$$

as desired. □

We add the following lemma here. It is very similar to Algebra, Lemma 10.113.2.

Lemma 41.4.11. *Let R be a local Noetherian domain of dimension 1 with maximal ideal $\mathfrak{m}$. Let $a, b \in \mathfrak{m}$ be nonzero. There exists a finite ring extension $R \subset R'$ with same field of fractions, and $t, a', b' \in R'$ such that $a = ta'$ and $b = tb'$ and $R' = a'R' + b'R'$.*

Proof. Set $I = (a, b)$. The idea is to blow up R in I as in the proof of Algebra, Lemma 10.113.2. Instead of doing the algebraic argument we work geometrically. Let $X = \text{Proj}(\bigoplus I^d/I^{d+1})$. By Divisors, Lemma 30.18.7 this is an integral scheme. The morphism $X \to \text{Spec}(R)$ is projective by Divisors, Lemma 30.18.11. By Algebra, Lemma 10.107.2 and the fact that X is quasi-compact we see that the fibre of $X \to \text{Spec}(R)$ over $\mathfrak{m}$ is finite. By Properties, Lemma 27.27.5 there exists an affine open $U \subset X$ containing this fibre. Hence $X = U$ because $X \to \text{Spec}(R)$ is closed. In other words X is affine, say $X = \text{Spec}(R')$. By Morphisms, Lemma 28.16.2 we see that $R \to R'$ is of finite type. Since $X \to \text{Spec}(R)$ is proper and affine it is integral (see Morphisms, Lemma 28.44.7). Hence $R \to R'$ is of finite type and integral, hence finite (Algebra, Lemma 10.33.5). By Divisors, Lemma 30.18.4 we see that IR' is a locally principal ideal. Since R' is semi-local we see that IR' is principal, see Algebra, Lemma 10.73.6, say $IR' = (t)$. Then we have $a = a't$ and $b = b't$ and everything is clear. □

Lemma 41.4.12. *Let A be a Noetherian local ring. Let $a, b \in A$. Let M be a finite A-module of dimension 1 on which each of a, b, $b - a$ are nonzerodivisors. Then*

$$d_M(a, b - a)d_M(b, b) = d_M(b, b - a)d_M(a, b)$$

in κ^*.

Proof. By Lemma 41.4.9 it suffices to show the relation when $M = A/\mathfrak{q}$ for some prime $\mathfrak{q} \subset A$ with $\dim(A/\mathfrak{q}) = 1$.

In case $M = A/\mathfrak{q}$ we may replace A by $A/\mathfrak{q}$ and a, b by their images in $A/\mathfrak{q}$. Hence we may assume $A = M$ and A a local Noetherian domain of dimension 1. The reason is that the residue field κ of A and $A/\mathfrak{q}$ are the same and that for any $A/\mathfrak{q}$-module M the determinant taken over A or over $A/\mathfrak{q}$ are canonically identified. See Lemma 41.2.7.

It suffices to show the relation when both a, b are in the maximal ideal. Namely, the case where one or both are units follows from Lemma 41.4.6.

Choose an extension $A \subset A'$ and factorizations $a = ta'$, $b = tb'$ as in Lemma 41.4.11. Note that also $b - a = t(b' - a')$ and that $A' = (a', b') = (a', b' - a') = (b' - a', b')$. Here and in the following we think of A' as an A-module and a, b, a', b', t as A-module endomorphisms of A'. We will use the notation $d^A_{A'}(a', b')$ and so on to indicate

$$d^A_{A'}(a', b') = \det_\kappa(A'/a'b'A', a', b')$$

which is defined by Lemma 41.4.1. The upper index ^{A} is used to distinguish this from the already defined symbol $d_{A'}(a', b')$ which is different (for example because it has values in the residue field of A' which may be different from κ). By Lemma 41.4.8 we see that $d_A(a, b) = d^A_{A'}(a, b)$, and similarly for the other combinations. Using this and multiplicativity we see that it suffices to prove

$$d^A_{A'}(a', b' - a')d^A_{A'}(b', b') = d^A_{A'}(b', b' - a')d^A_{A'}(a', b')$$

Now, since $(a', b') = A'$ and so on we have

$$\begin{array}{ccc} A'/(a'(b' - a')) & \cong & A'/(a') \oplus A'/(b' - a') \\ A'/(b'(b' - a')) & \cong & A'/(b') \oplus A'/(b' - a') \\ A'/(a'b') & \cong & A'/(a') \oplus A'/(b') \end{array}$$

Moreover, note that multiplication by $b' - a'$ on $A/(a')$ is equal to multiplication by b', and that multiplication by $b' - a'$ on $A/(b')$ is equal to multiplication by $-a'$. Using Lemmas 41.3.8 and 41.3.9 we conclude

$$\begin{aligned} d^A_{A'}(a', b' - a') &= \det_\kappa(b'|_{A'/(a')})^{-1} \det_\kappa(a'|_{A'/(b'-a')}) \\ d^A_{A'}(b', b' - a') &= \det_\kappa(-a'|_{A'/(b')})^{-1} \det_\kappa(b'|_{A'/(b'-a')}) \\ d^A_{A'}(a', b') &= \det_\kappa(b'|_{A'/(a')})^{-1} \det_\kappa(a'|_{A'/(b')}) \end{aligned}$$

Hence we conclude that

$$(-1)^{\text{length}_A(A'/(b'))} d^A_{A'}(a', b' - a') = d^A_{A'}(b', b' - a') d^A_{A'}(a', b')$$

the sign coming from the $-a'$ in the second equality above. On the other hand, by Lemma 41.3.7 we have $d^A_{A'}(b', b') = (-1)^{\text{length}_A(A'/(b'))}$, and the lemma is proved. □

The tame symbol is a Steinberg symbol.

Lemma 41.4.13. *Let A be a Noetherian local domain of dimension 1. Let $K = f.f.(A)$. For $x \in K \setminus \{0, 1\}$ we have*

$$d_A(x, 1 - x) = 1$$

Proof. Write $x = a/b$ with $a, b \in A$. The hypothesis implies, since $1 - x = (b - a)/b$, that also $b - a \neq 0$. Hence we compute

$$d_A(x, 1 - x) = d_A(a, b - a) d_A(a, b)^{-1} d_A(b, b - a)^{-1} d_A(b, b)$$

Thus we have to show that $d_A(a, b - a) d_A(b, b) = d_A(b, b - a) d_A(a, b)$. This is Lemma 41.4.12. □

41.5. Lengths and determinants

In this section we use the determinant to compare lattices. The key lemma is the following.

Lemma 41.5.1. *Let R be a noetherian local ring. Let $\mathfrak{q} \subset R$ be a prime with $\dim(R/\mathfrak{q}) = 1$. Let $\varphi : M \to N$ be a homomorphism of finite R-modules. Assume there exist $x_1, \ldots, x_l \in M$ and $y_1, \ldots, y_l \in M$ with the following properties*

1. *$M = \langle x_1, \ldots, x_l \rangle$,*
2. *$\langle x_1, \ldots, x_i \rangle / \langle x_1, \ldots, x_{i-1} \rangle \cong R/\mathfrak{q}$ for $i = 1, \ldots, l$,*
3. *$N = \langle y_1, \ldots, y_l \rangle$, and*
4. *$\langle y_1, \ldots, y_i \rangle / \langle y_1, \ldots, y_{i-1} \rangle \cong R/\mathfrak{q}$ for $i = 1, \ldots, l$.*

Then φ is injective if and only if $\varphi_\mathfrak{q}$ is an isomorphism, and in this case we have

$$length_R(Coker(\varphi)) = ord_{R/\mathfrak{q}}(f)$$

where $f \in \kappa(\mathfrak{q})$ is the element such that

$$[\varphi(x_1), \ldots, \varphi(x_l)] = f[y_1, \ldots, y_l]$$

in $\det_{\kappa(\mathfrak{q})}(N_\mathfrak{q})$.

Proof. First, note that the lemma holds in case $l = 1$. Namely, in this case x_1 is a basis of M over $R/\mathfrak{q}$ and y_1 is a basis of N over $R/\mathfrak{q}$ and we have $\varphi(x_1) = f y_1$ for some $f \in R$. Thus φ is injective if and only if $f \notin \mathfrak{q}$. Moreover, $\text{Coker}(\varphi) = R/(f, \mathfrak{q})$ and hence the lemma holds by definition of $\text{ord}_{R/q}(f)$ (see Algebra, Definition 10.115.2).

In fact, suppose more generally that $\varphi(x_i) = f_i y_i$ for some $f_i \in R$, $f_i \notin \mathfrak{q}$. Then the induced maps

$$\langle x_1, \ldots, x_i\rangle/\langle x_1, \ldots, x_{i-1}\rangle \longrightarrow \langle y_1, \ldots, y_i\rangle/\langle y_1, \ldots, y_{i-1}\rangle$$

are all injective and have cokernels isomorphic to $R/(f_i, \mathfrak{q})$. Hence we see that

$$\text{length}_R(\text{Coker}(\varphi)) = \sum \text{ord}_{R/\mathfrak{q}}(f_i).$$

On the other hand it is clear that

$$[\varphi(x_1), \ldots, \varphi(x_l)] = f_1 \ldots f_l [y_1, \ldots, y_l]$$

in this case from the admissible relation (b) for symbols. Hence we see the result holds in this case also.

We prove the general case by induction on l. Assume $l > 1$. Let $i \in \{1, \ldots, l\}$ be minimal such that $\varphi(x_1) \in \langle y_1, \ldots, y_i\rangle$. We will argue by induction on i. If $i = 1$, then we get a commutative diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \langle x_1\rangle & \longrightarrow & \langle x_1, \ldots, x_l\rangle & \longrightarrow & \langle x_1, \ldots, x_l\rangle/\langle x_1\rangle & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & \langle y_1\rangle & \longrightarrow & \langle y_1, \ldots, y_l\rangle & \longrightarrow & \langle y_1, \ldots, y_l\rangle/\langle y_1\rangle & \longrightarrow & 0 \end{array}$$

and the lemma follows from the snake lemma and induction on l. Assume now that $i > 1$. Write $\varphi(x_1) = a_1 y_1 + \ldots + a_{i-1} y_{i-1} + a y_i$ with $a_j, a \in R$ and $a \notin \mathfrak{q}$ (since otherwise i was not minimal). Set

$$x'_j = \begin{cases} x_j & \text{if} \quad j = 1 \\ a x_j & \text{if} \quad j \geq 2 \end{cases} \quad \text{and} \quad y'_j = \begin{cases} y_j & \text{if} \quad j < i \\ a y_j & \text{if} \quad j \geq i \end{cases}$$

Let $M' = \langle x'_1, \ldots, x'_l\rangle$ and $N' = \langle y'_1, \ldots, y'_l\rangle$. Since $\varphi(x'_1) = a_1 y'_1 + \ldots + a_{i-1} y'_{i-1} + y'_i$ by construction and since for $j > 1$ we have $\varphi(x'_j) = a\varphi(x_i) \in \langle y'_1, \ldots, y'_l\rangle$ we get a commutative diagram of R-modules and maps

$$\begin{array}{ccc} M' & \xrightarrow{\varphi'} & N' \\ \downarrow & & \downarrow \\ M & \xrightarrow{\varphi} & N \end{array}$$

By the result of the second paragraph of the proof we know that $\text{length}_R(M/M') = (l-1)\text{ord}_{R/\mathfrak{q}}(a)$ and similarly $\text{length}_R(M/M') = (l-i+1)\text{ord}_{R/\mathfrak{q}}(a)$. By a diagram chase this implies that

$$\text{length}_R(\text{Coker}(\varphi')) = \text{length}_R(\text{Coker}(\varphi)) + i\ \text{ord}_{R/\mathfrak{q}}(a).$$

On the other hand, it is clear that writing

$$[\varphi(x_1), \ldots, \varphi(x_l)] = f[y_1, \ldots, y_l], \quad [\varphi'(x'_1), \ldots, \varphi(x'_l)] = f'[y'_1, \ldots, y'_l]$$

we have $f' = a^i f$. Hence it suffices to prove the lemma for the case that $\varphi(x_1) = a_1 y_1 + \ldots a_{i-1} y_{i-1} + y_i$, i.e., in the case that $a = 1$. Next, recall that

$$[y_1, \ldots, y_l] = [y_1, \ldots, y_{i-1}, a_1 y_1 + \ldots a_{i-1} y_{i-1} + y_i, y_{i+1}, \ldots, y_l]$$

by the admissible relations for symbols. The sequence $y_1, \ldots, y_{i-1}, a_1 y_1 + \ldots + a_{i-1} y_{i-1} + y_i, y_{i+1}, \ldots, y_l$ satisfies the conditions (3), (4) of the lemma also. Hence,

we may actually assume that $\varphi(x_1) = y_i$. In this case, note that we have $\mathfrak{q}x_1 = 0$ which implies also $\mathfrak{q}y_i = 0$. We have

$$[y_1, \ldots, y_l] = -[y_1, \ldots, y_{i-2}, y_i, y_{i-1}, y_{i+1}, \ldots, y_l]$$

by the third of the admissible relations defining $\det_{\kappa(\mathfrak{q})}(N_\mathfrak{q})$. Hence we may replace $y_1, \ldots, y_l$ by the sequence $y'_1, \ldots, y'_l = y_1, \ldots, y_{i-2}, y_i, y_{i-1}, y_{i+1}, \ldots, y_l$ (which also satisfies conditions (3) and (4) of the lemma). Clearly this decreases the invariant i by 1 and we win by induction on i. □

To use the previous lemma we show that often sequences of elements with the required properties exist.

Lemma 41.5.2. *Let R be a local Noetherian ring. Let $\mathfrak{q} \subset R$ be a prime ideal. Let M be a finite R-module such that $\mathfrak{q}$ is one of the minimal primes of the support of M. Then there exist $x_1, \ldots, x_l \in M$ such that*

(1) *the support of $M/\langle x_1, \ldots, x_l\rangle$ does not contain $\mathfrak{q}$, and*
(2) *$\langle x_1, \ldots, x_i\rangle/\langle x_1, \ldots, x_{i-1}\rangle \cong R/\mathfrak{q}$ for $i = 1, \ldots, l$.*

Moreover, in this case $l = length_{R_\mathfrak{q}}(M_\mathfrak{q})$.

Proof. The condition that $\mathfrak{q}$ is a minimal prime in the support of M implies that $l = \text{length}_{R_\mathfrak{q}}(M_\mathfrak{q})$ is finite (see Algebra, Lemma 10.59.3). Hence we can find $y_1, \ldots, y_l \in M_\mathfrak{q}$ such that $\langle y_1, \ldots, y_i\rangle/\langle y_1, \ldots, y_{i-1}\rangle \cong \kappa(\mathfrak{q})$ for $i = 1, \ldots, l$. We can find $f_i \in R$, $f_i \notin \mathfrak{q}$ such that $f_i y_i$ is the image of some element $z_i \in M$. Moreover, as R is Noetherian we can write $\mathfrak{q} = (g_1, \ldots, g_t)$ for some $g_j \in R$. By assumption $g_j y_i \in \langle y_1, \ldots, y_{i-1}\rangle$ inside the module $M_\mathfrak{q}$. By our choice of z_i we can find some further elements $f_{ji} \in R$, $f_{ij} \notin \mathfrak{q}$ such that $f_{ij} g_j z_i \in \langle z_1, \ldots, z_{i-1}\rangle$ (equality in the module M). The lemma follows by taking

$$x_1 = f_{11} f_{12} \ldots f_{1t} z_1, \quad x_2 = f_{11} f_{12} \ldots f_{1t} f_{21} f_{22} \ldots f_{2t} z_2,$$

and so on. Namely, since all the elements f_i, f_{ij} are invertible in $R_\mathfrak{q}$ we still have that $R_\mathfrak{q}x_1 + \ldots + R_\mathfrak{q}x_i / R_\mathfrak{q}x_1 + \ldots + R_\mathfrak{q}x_{i-1} \cong \kappa(\mathfrak{q})$ for $i = 1, \ldots, l$. By construction, $\mathfrak{q}x_i \in \langle x_1, \ldots, x_{i-1}\rangle$. Thus $\langle x_1, \ldots, x_i\rangle/\langle x_1, \ldots, x_{i-1}\rangle$ is an R-module generated by one element, annihilated $\mathfrak{q}$ such that localizing at $\mathfrak{q}$ gives a q-dimensional vector space over $\kappa(\mathfrak{q})$. Hence it is isomorphic to $R/\mathfrak{q}$. □

Here is the main result of this section. We will see below the various different consequences of this proposition. The reader is encouraged to first prove the easier Lemma 41.5.4 his/herself.

Proposition 41.5.3. *Let R be a local Noetherian ring with residue field κ. Suppose that (M, φ, ψ) is a $(2,1)$-periodic complex over R. Assume*

(1) *M is a finite R-module,*
(2) *the cohomology modules of (M, φ, ψ) are of finite length, and*
(3) *$\dim(Supp(M)) = 1$.*

Let $\mathfrak{q}_i$, $i = 1, \ldots, t$ be the minimal primes of the support of M. Then we have[2]

$$-e_R(M, \varphi, \psi) = \sum\nolimits_{i=1,\ldots,t} ord_{R/\mathfrak{q}_i}\left(\det_{\kappa(\mathfrak{q}_i)}(M_{\mathfrak{q}_i}, \varphi_{\mathfrak{q}_i}, \psi_{\mathfrak{q}_i})\right)$$

[2] Obviously we could get rid of the minus sign by redefining $\det_\kappa(M, \varphi, \psi)$ as the inverse of its current value, see Definition 41.3.4.

Proof. We first reduce to the case $t = 1$ in the following way. Note that $\text{Supp}(M) = \{\mathfrak{m}, \mathfrak{q}_1, \ldots, \mathfrak{q}_t\}$, where $\mathfrak{m} \subset R$ is the maximal ideal. Let M_i denote the image of $M \to M_{\mathfrak{q}_i}$, so $\text{Supp}(M_i) = \{\mathfrak{m}, \mathfrak{q}_i\}$. The map φ (resp. ψ) induces an R-module map $\varphi_i : M_i \to M_i$ (resp. $\psi_i : M_i \to M_i$). Thus we get a morphism of $(2,1)$-periodic complexes

$$(M, \varphi, \psi) \longrightarrow \bigoplus\nolimits_{i=1,\ldots,t} (M_i, \varphi_i, \psi_i).$$

The kernel and cokernel of this map have support equal to $\{\mathfrak{m}\}$ (or are zero). Hence by Lemma 41.3.3 these $(2,1)$-periodic complexes have multiplicity 0. In other words we have

$$e_R(M, \varphi, \psi) = \sum\nolimits_{i=1,\ldots,t} e_R(M_i, \varphi_i, \psi_i)$$

On the other hand we clearly have $M_{\mathfrak{q}_i} = M_{i,\mathfrak{q}_i}$, and hence the terms of the right hand side of the formula of the lemma are equal to the expressions

$$\text{ord}_{R/\mathfrak{q}_i} \left(\det_{\kappa(\mathfrak{q}_i)}(M_{i,\mathfrak{q}_i}, \varphi_{i,\mathfrak{q}_i}, \psi_{i,\mathfrak{q}_i}) \right)$$

In other words, if we can prove the lemma for each of the modules M_i, then the lemma holds. This reduces us to the case $t = 1$.

Assume we have a $(2,1)$-periodic complex (M, φ, ψ) over a Noetherian local ring with M a finite R-module, $\text{Supp}(M) = \{\mathfrak{m}, \mathfrak{q}\}$, and finite length cohomology modules. The proof in this case follows from Lemma 41.5.1 and careful bookkeeping. Denote $K_\varphi = \text{Ker}(\varphi)$, $I_\varphi = \text{Im}(\varphi)$, $K_\psi = \text{Ker}(\psi)$, and $I_\psi = \text{Im}(\psi)$. Since R is Noetherian these are all finite R-modules. Set

$$a = \text{length}_{R_\mathfrak{q}}(I_{\varphi,\mathfrak{q}}) = \text{length}_{R_\mathfrak{q}}(K_{\psi,\mathfrak{q}}), \quad b = \text{length}_{R_\mathfrak{q}}(I_{\psi,\mathfrak{q}}) = \text{length}_{R_\mathfrak{q}}(K_{\varphi,\mathfrak{q}}).$$

Equalities because the complex becomes exact after localizing at $\mathfrak{q}$. Note that $l = \text{length}_{R_\mathfrak{q}}(M_\mathfrak{q})$ is equal to $l = a + b$.

We are going to use Lemma 41.5.2 to choose sequences of elements in finite R-modules N with support contained in $\{\mathfrak{m}, \mathfrak{q}\}$. In this case $N_\mathfrak{q}$ has finite length, say $n \in \mathbf{N}$. Let us call a sequence $w_1, \ldots, w_n \in N$ with properties (1) and (2) of Lemma 41.5.2 a "good sequence". Note that the quotient $N/\langle w_1, \ldots, w_n \rangle$ of N by the submodule generated by a good sequence has support (contained in) $\{\mathfrak{m}\}$ and hence has finite length (Algebra, Lemma 10.59.3). Moreover, the symbol $[w_1, \ldots, w_n] \in \det_{\kappa(\mathfrak{q})}(N_\mathfrak{q})$ is a generator, see Lemma 41.2.4.

Having said this we choose good sequences

$$\begin{array}{llllll} x_1, \ldots, x_b & \text{in} & K_\varphi, & t_1, \ldots, t_a & \text{in} & K_\psi, \\ y_1, \ldots, y_a & \text{in} & I_\varphi \cap \langle t_1, \ldots t_a \rangle, & s_1, \ldots, s_b & \text{in} & I_\psi \cap \langle x_1, \ldots, x_b \rangle. \end{array}$$

We will adjust our choices a little bit as follows. Choose lifts $\tilde{y}_i \in M$ of $y_i \in I_\varphi$ and $\tilde{s}_i \in M$ of $s_i \in I_\psi$. It may not be the case that $\mathfrak{q}\tilde{y}_1 \subset \langle x_1, \ldots, x_b \rangle$ and it may not be the case that $\mathfrak{q}\tilde{s}_1 \subset \langle t_1, \ldots, t_a \rangle$. However, using that $\mathfrak{q}$ is finitely generated (as in the proof of Lemma 41.5.2) we can find a $d \in R$, $d \notin \mathfrak{q}$ such that $\mathfrak{q}d\tilde{y}_1 \subset \langle x_1, \ldots, x_b \rangle$ and $\mathfrak{q}d\tilde{s}_1 \subset \langle t_1, \ldots, t_a \rangle$. Thus after replacing y_i by dy_i, $\tilde{y}_i$ by $d\tilde{y}_i$, s_i by ds_i and $\tilde{s}_i$ by $d\tilde{s}_i$ we see that we may assume also that $x_1, \ldots, x_b, \tilde{y}_1, \ldots, \tilde{y}_b$ and $t_1, \ldots, t_a, \tilde{s}_1, \ldots, \tilde{s}_b$ are good sequences in M.

Finally, we choose a good sequence $z_1, \ldots, z_l$ in the finite R-module

$$\langle x_1, \ldots, x_b, \tilde{y}_1, \ldots, \tilde{y}_a \rangle \cap \langle t_1, \ldots, t_a, \tilde{s}_1, \ldots, \tilde{s}_b \rangle.$$

Note that this is also a good sequence in M.

Since $I_{\varphi,\mathfrak{q}} = K_{\psi,\mathfrak{q}}$ there is a unique element $h \in \kappa(\mathfrak{q})$ such that $[y_1, \ldots, y_a] = h[t_1, \ldots, t_a]$ inside $\det_{\kappa(\mathfrak{q})}(K_{\psi,\mathfrak{q}})$. Similarly, as $I_{\psi,\mathfrak{q}} = K_{\varphi,\mathfrak{q}}$ there is a unique element $h \in \kappa(\mathfrak{q})$ such that $[s_1, \ldots, s_b] = g[x_1, \ldots, x_b]$ inside $\det_{\kappa(\mathfrak{q})}(K_{\varphi,\mathfrak{q}})$. We can also do this with the three good sequences we have in M. All in all we get the following identities

$$\begin{aligned}
[y_1, \ldots, y_a] &= h[t_1, \ldots, t_a] \\
[s_1, \ldots, s_b] &= g[x_1, \ldots, x_b] \\
[z_1, \ldots, z_l] &= f_\varphi[x_1, \ldots, x_b, \tilde{y}_1, \ldots, \tilde{y}_a] \\
[z_1, \ldots, z_l] &= f_\psi[t_1, \ldots, t_a, \tilde{s}_1, \ldots, \tilde{s}_b]
\end{aligned}$$

for some $g, h, f_\varphi, f_\psi \in \kappa(\mathfrak{q})$.

Having set up all this notation let us compute $\det_{\kappa(\mathfrak{q})}(M, \varphi, \psi)$. Namely, consider the element $[z_1, \ldots, z_l]$. Under the map $\gamma_\psi \circ \sigma \circ \gamma_\varphi^{-1}$ of Definition 41.3.4 we have

$$\begin{aligned}
[z_1, \ldots, z_l] &= f_\varphi[x_1, \ldots, x_b, \tilde{y}_1, \ldots, \tilde{y}_a] \\
&\mapsto f_\varphi[x_1, \ldots, x_b] \otimes [y_1, \ldots, y_a] \\
&\mapsto f_\varphi h/g[t_1, \ldots, t_a] \otimes [s_1, \ldots, s_b] \\
&\mapsto f_\varphi h/g[t_1, \ldots, t_a, \tilde{s}_1, \ldots, \tilde{s}_b] \\
&= f_\varphi h/f_\psi g[z_1, \ldots, z_l]
\end{aligned}$$

This means that $\det_{\kappa(\mathfrak{q})}(M_\mathfrak{q}, \varphi_\mathfrak{q}, \psi_\mathfrak{q})$ is equal to $f_\varphi h/f_\psi g$ up to a sign.

We abbreviate the following quantities

$$\begin{aligned}
k_\varphi &= \text{length}_R(K_\varphi/\langle x_1, \ldots, x_b\rangle) \\
k_\psi &= \text{length}_R(K_\psi/\langle t_1, \ldots, t_a\rangle) \\
i_\varphi &= \text{length}_R(I_\varphi/\langle y_1, \ldots, y_a\rangle) \\
i_\psi &= \text{length}_R(I_\psi/\langle s_1, \ldots, s_a\rangle) \\
m_\varphi &= \text{length}_R(M/\langle x_1, \ldots, x_b, \tilde{y}_1, \ldots, \tilde{y}_a\rangle) \\
m_\psi &= \text{length}_R(M/\langle t_1, \ldots, t_a, \tilde{s}_1, \ldots, \tilde{s}_b\rangle) \\
\delta_\varphi &= \text{length}_R(\langle x_1, \ldots, x_b, \tilde{y}_1, \ldots, \tilde{y}_a\rangle\langle z_1, \ldots, z_l\rangle) \\
\delta_\psi &= \text{length}_R(\langle t_1, \ldots, t_a, \tilde{s}_1, \ldots, \tilde{s}_b\rangle\langle z_1, \ldots, z_l\rangle)
\end{aligned}$$

Using the exact sequences $0 \to K_\varphi \to M \to I_\varphi \to 0$ we get $m_\varphi = k_\varphi + i_\varphi$. Similarly we have $m_\psi = k_\psi + i_\psi$. We have $\delta_\varphi + m_\varphi = \delta_\psi + m_\psi$ since this is equal to the colength of $\langle z_1, \ldots, z_l\rangle$ in M. Finally, we have

$$\delta_\varphi = \text{ord}_{R/\mathfrak{q}}(f_\varphi), \quad \delta_\psi = \text{ord}_{R/\mathfrak{q}}(f_\psi)$$

by our first application of the key Lemma 41.5.1.

Next, let us compute the multiplicity of the periodic complex

$$\begin{aligned} e_R(M,\varphi,\psi) &= \text{length}_R(K_\varphi/I_\psi) - \text{length}_R(K_\psi/I_\varphi) \\ &= \text{length}_R(\langle x_1,\ldots,x_b\rangle/\langle s_1,\ldots,s_b\rangle) + k_\varphi - i_\psi \\ &\quad -\text{length}_R(\langle t_1,\ldots,t_a\rangle/\langle y_1,\ldots,y_a\rangle) - k_\psi + i_\varphi \\ &= \text{ord}_{R/\mathfrak{q}}(g/h) + k_\varphi - i_\psi - k_\psi + i_\varphi \\ &= \text{ord}_{R/\mathfrak{q}}(g/h) + m_\varphi - m_\psi \\ &= \text{ord}_{R/\mathfrak{q}}(g/h) + \delta_\psi - \delta_\varphi \\ &= \text{ord}_{R/\mathfrak{q}}(f_\psi g/f_\varphi h) \end{aligned}$$

where we used the key Lemma 41.5.1 twice in the third equality. By our computation of $\det_{\kappa(\mathfrak{q})}(M_\mathfrak{q},\varphi_\mathfrak{q},\psi_\mathfrak{q})$ this proves the proposition. □

In most applications the following lemma suffices.

Lemma 41.5.4. *Let R be a Noetherian local ring with maximal ideal $\mathfrak{m}$. Let M be a finite R-module, and let $\psi : M \to M$ be an R-module map. Assume that*

(1) *$Ker(\psi)$ and $Coker(\psi)$ have finite length, and*
(2) *$\dim(Supp(M)) \leq 1$.*

Write $Supp(M) = \{\mathfrak{m},\mathfrak{q}_1,\ldots,\mathfrak{q}_t\}$ and denote $f_i \in \kappa(\mathfrak{q}_i)^*$ the element such that $\det_{\kappa(\mathfrak{q}_i)}(\psi_{\mathfrak{q}_i}) : \det_{\kappa(\mathfrak{q}_i)}(M_{\mathfrak{q}_i}) \to \det_{\kappa(\mathfrak{q}_i)}(M_{\mathfrak{q}_i})$ is multiplication by f_i. Then we have

$$length_R(Coker(\psi)) - length_R(Ker(\psi)) = \sum\nolimits_{i=1,\ldots,t} ord_{R/\mathfrak{q}_i}(f_i).$$

Proof. Recall that $H^0(M,0,\psi) = \text{Coker}(\psi)$ and $H^1(M,0,\psi) = \text{Ker}(\psi)$, see remarks above Definition 41.3.2. The lemma follows by combining Proposition 41.5.3 with Lemma 41.3.8.

Alternative proof. Reduce to the case $\text{Supp}(M) = \{\mathfrak{m},\mathfrak{q}\}$ as in the proof of Proposition 41.5.3. Then directly combine Lemmas 41.5.1 and 41.5.2 to prove this specific case of Proposition 41.5.3. There is much less bookkeeping in this case, and the reader is encouraged to work this out. Details omitted. □

Lemma 41.5.5. *Let R be a Noetherian local ring with maximal ideal $\mathfrak{m}$. Let M be a finite R-module. Let $x \in R$. Assume that*

(1) *$\dim(Supp(M)) \leq 1$, and*
(2) *$\dim(M/xM) \leq 0$.*

Write $Supp(M) = \{\mathfrak{m},\mathfrak{q}_1,\ldots,\mathfrak{q}_t\}$. Then

$$length_R(M_x) - length_R({}_xM) = \sum\nolimits_{i=1,\ldots,t} ord_{R/\mathfrak{q}_i}(x) length_{R_{\mathfrak{q}_i}}(M_{\mathfrak{q}_i}).$$

where $M_x = M/xM$ and ${}_xM = Ker(x : M \to M)$.

Proof. This is a special case of Lemma 41.5.4. To see that $f_i = x^{\text{length}_{R_{\mathfrak{q}_i}}(M_{\mathfrak{q}_i})}$ see Lemma 41.2.8. □

Lemma 41.5.6. *Let R be a Noetherian local ring with maximal ideal $\mathfrak{m}$. Let $I \subset R$ be an ideal and let $x \in R$. Assume x is a nonzerodivisor on R/I and that $\dim(R/I) = 1$. Then*

$$length_R(R/(x,I)) = \sum\nolimits_i length_R(R/(x,\mathfrak{q}_i)) length_{R_{\mathfrak{q}_i}}((R/I)_{\mathfrak{q}_i})$$

where $\mathfrak{q}_1, \ldots, \mathfrak{q}_n$ are the minimal primes over I. More generally if M is any finite Cohen-Macaulay module of dimension 1 over R and $\dim(M/xM) = 0$, then

$$length_R(M/xM) = \sum\nolimits_i length_R(R/(x, \mathfrak{q}_i)) length_{R_{\mathfrak{q}_i}}(M_{\mathfrak{q}_i}).$$

where $\mathfrak{q}_1, \ldots, \mathfrak{q}_t$ are the minimal primes of the support of M.

Proof. These are special cases of Lemma 41.5.5. □

Lemma 41.5.7. *Let A be a Noetherian local ring. Let M be a finite A-module. Let $a, b \in A$. Assume*

1. $\dim(A) = 1$,
2. *both a and b are nonzerodivisors in A,*
3. *A has no embedded primes,*
4. *M has no embedded associated primes,*
5. *$Supp(M) = \mathrm{Spec}(A)$.*

Let $I = \{x \in A \mid x(a/b) \in A\}$. Let $\mathfrak{q}_1, \ldots, \mathfrak{q}_t$ be the minimal primes of A. Then $(a/b)IM \subset M$ and

$$length_A(M/(a/b)IM) - length_A(M/IM) = \sum\nolimits_i length_{A_{\mathfrak{q}_i}}(M_{\mathfrak{q}_i}) ord_{A/\mathfrak{q}_i}(a/b)$$

Proof. Since M has no embedded associated primes, and since the support of M is $\mathrm{Spec}(A)$ we see that $\mathrm{Ass}(M) = \{\mathfrak{q}_1, \ldots, \mathfrak{q}_t\}$. Hence a, b are nonzerodivisors on M. Note that

$$\begin{aligned}
&\mathrm{length}_A(M/(a/b)IM) \\
&= \mathrm{length}_A(bM/aIM) \\
&= \mathrm{length}_A(M/aIM) - \mathrm{length}_A(M/bM) \\
&= \mathrm{length}_A(M/aM) + \mathrm{length}_A(aM/aIM) - \mathrm{length}_A(M/bM) \\
&= \mathrm{length}_A(M/aM) + \mathrm{length}_A(M/IM) - \mathrm{length}_A(M/bM)
\end{aligned}$$

as the injective map $b : M \to bM$ maps $(a/b)IM$ to aIM and the injective map $a : M \to aM$ maps IM to aIM. Hence the left hand side of the equation of the lemma is equal to

$$\mathrm{length}_A(M/aM) - \mathrm{length}_A(M/bM).$$

Applying the second formula of Lemma 41.5.6 with $x = a, b$ respectively and using Algebra, Definition 10.115.2 of the ord-functions we get the result. □

41.6. Application to tame symbol

In this section we apply the results above to show the following lemma.

Lemma 41.6.1. *Let A be a 2-dimensional Noetherian local domain. Let $K = f.f.(A)$. Let $f, g \in K^*$. Let $\mathfrak{q}_1, \ldots, \mathfrak{q}_t$ be the height 1 primes $\mathfrak{q}$ of A such that either f or g is not an element of $A^*_{\mathfrak{q}}$. Then we have*

$$\sum\nolimits_{i=1,\ldots,t} ord_{A/\mathfrak{q}_i}(d_{A_{\mathfrak{q}_i}}(f, g)) = 0$$

We can also write this as

$$\sum\nolimits_{height(\mathfrak{q})=1} ord_{A/\mathfrak{q}}(d_{A_{\mathfrak{q}}}(f, g)) = 0$$

since at any height one prime $\mathfrak{q}$ of A where $f, g \in A^*_{\mathfrak{q}}$ we have $d_{A_{\mathfrak{q}}}(f, g) = 1$ by Lemma 41.4.6.

Proof. Since the tame symbols $d_{A_\mathfrak{q}}(f, g)$ are additive (Lemma 41.4.4) and the order functions $\text{ord}_{A/\mathfrak{q}}$ are additive (Algebra, Lemma 10.115.1) it suffices to prove the formula when $f = a \in A$ and $g = b \in A$. In this case we see that we have to show

$$\sum\nolimits_{\text{height}(\mathfrak{q})=1} \text{ord}_{A/\mathfrak{q}}(\det_\kappa(A_\mathfrak{q}/(ab), a, b)) = 0$$

By Proposition 41.5.3 this is equivalent to showing that

$$e_A(A/(ab), a, b) = 0.$$

Since the complex $A/(ab) \xrightarrow{a} A/(ab) \xrightarrow{b} A/(ab) \xrightarrow{a} A/(ab)$ is exact we win. □

41.7. Setup

We will throughout work over a locally Noetherian universally catenary base S endowed with a dimension function δ. Although it is likely possible to generalize (parts of) the discussion in the chapter, it seems that this is a good first approximation. We usually do not assume our schemes are separated or quasi-compact. Many interesting algebraic stacks are non-separated and/or non-quasi-compact and this is a good case study to see how to develop a reasonable theory for those as well. In order to reference these hypotheses we give it a number.

Situation 41.7.1. Here S is a locally Noetherian, and universally catenary scheme. Moreover, we assume S is endowed with a dimension function $\delta : S \longrightarrow \mathbf{Z}$.

See Morphisms, Definition 28.18.1 for the notion of a universally catenary scheme, and see Topology, Definition 5.19.1 for the notion of a dimension function. Recall that any locally Noetherian catenary scheme locally has a dimension function, see Properties, Lemma 27.11.3. Moreover, there are lots of schemes which are universally catenary, see Morphisms, Lemma 28.18.4.

Let (S, δ) be as in Situation 41.7.1. Any scheme X locally of finite type over S is locally Noetherian and catenary. In fact, X has a canonical dimension function

$$\delta = \delta_{X/S} : X \longrightarrow \mathbf{Z}$$

associated to $(f : X \to S, \delta)$ given by the rule $\delta_{X/S}(x) = \delta(f(x)) + \text{trdeg}_{\kappa(f(x))}\kappa(x)$. See Morphisms, Lemma 28.31.2. Moreover, if $h : X \to Y$ is a morphism of schemes locally of finite type over S, and $x \in X$, $y = h(x)$, then obviously $\delta_{X/S}(x) = \delta_{Y/S}(y) + \text{trdeg}_{\kappa(y)}\kappa(x)$. We will freely use this function and its properties in the following.

Here are the basic examples of setups as above. In fact, the main interest lies in the case where the base is the spectrum of a field, or the case where the base is the spectrum of a Dedekind ring (e.g. $\mathbf{Z}$, or a discrete valuation ring).

Example 41.7.2. Here $S = \text{Spec}(k)$ and k is a field. We set $\delta(pt) = 0$ where pt indicates the unique point of S. The pair (S, δ) is an example of a situation as in Situation 41.7.1 by Morphisms, Lemma 28.18.4.

Example 41.7.3. Here $S = \text{Spec}(A)$, where A is a Noetherian domain of dimension 1. For example we could consider $A = \mathbf{Z}$. We set $\delta(\mathfrak{p}) = 0$ if $\mathfrak{p}$ is a maximal ideal and $\delta(\mathfrak{p}) = 1$ if $\mathfrak{p} = (0)$ corresponds to the generic point. This is an example of Situation 41.7.1 by Morphisms, Lemma 28.18.4.

In good cases δ corresponds to the dimension function.

Lemma 41.7.4. *Let (S, δ) be as in Situation 41.7.1. Assume in addition S is a Jacobson scheme, and $\delta(s) = 0$ for every closed point s of S. Let X be locally of finite type over S. Let $Z \subset X$ be an integral closed subscheme and let $\xi \in Z$ be its generic point. The following integers are the same:*

(1) $\delta_{X/S}(\xi)$,
(2) $\dim(Z)$, *and*
(3) $\dim(\mathcal{O}_{Z,z})$ *where z is a closed point of Z.*

Proof. Let $X \to S$, $\xi \in Z \subset X$ be as in the lemma. Since X is locally of finite type over S we see that X is Jacobson, see Morphisms, Lemma 28.17.9. Hence closed points of X are dense in every closed subset of Z and map to closed points of S. Hence given any chain of irreducible closed subsets of Z we can end it with a closed point of Z. It follows that $\dim(Z) = \sup_z(\dim(\mathcal{O}_{Z,z})$ (see Properties, Lemma 27.11.4) where $z \in Z$ runs over the closed points of Z. Note that $\dim(\mathcal{O}_{Z,z}) = \delta(\xi) - \delta(z))$ by the properties of a dimension function. For each closed $z \in Z$ the field extension $\kappa(z) \supset \kappa(f(z))$ is finite, see Morphisms, Lemma 28.17.8. Hence $\delta_{X/S}(z) = \delta(f(z)) = 0$ for $z \in Z$ closed. It follows that all three integers are equal. □

In the situation of the lemma above the value of δ at the generic point of a closed irreducible subset is the dimension of the irreducible closed subset. However, in general we cannot expect the equality to hold. For example if $S = \mathrm{Spec}(\mathbf{C}[[t]])$ and $X = \mathrm{Spec}(\mathbf{C}((t)))$ then we would get $\delta(x) = 1$ for the unique point of X, but $\dim(X) = 0$. Still we want to think of $\delta_{X/S}$ as giving the dimension of the irreducible closed subschemes. Thus we introduce the following terminology.

Definition 41.7.5. Let (S, δ) as in Situation 41.7.1. For any scheme X locally of finite type over S and any irreducible closed subset $Z \subset X$ we define

$$\dim_\delta(Z) = \delta(\xi)$$

where $\xi \in Z$ is the generic point of Z. We will call this the *δ-dimension of Z*. If Z is a closed subscheme of X, then we define $\dim_\delta(Z)$ as the supremum of the δ-dimensions of its irreducible components.

41.8. Cycles

Since we are not assuming our schemes are quasi-compact we have to be a little careful when defining cycles. We have to allow infinite sums because a rational function may have infinitely many poles for example. In any case, if X is quasi-compact then a cycle is a finite sum as usual.

Definition 41.8.1. Let (S, δ) be as in Situation 41.7.1. Let X be locally of finite type over S. Let $k \in \mathbf{Z}$.

(1) A collection of closed subschemes $\{Z_i\}_{i \in I}$ of X is said to be *locally finite* if for every quasi-compact open $U \subset X$ the set

$$\#\{i \in I \mid Z_i \cap U \neq \emptyset\}$$

is finite.

(2) A *cycle on X* is a formal sum

$$\alpha = \sum n_Z [Z]$$

where the sum is over integral closed subschemes $Z \subset X$, each $n_Z \in \mathbf{Z}$, and the collection $\{Z; n_Z \neq 0\}$ is locally finite.

(3) A *k-cycle*, on X is a cycle
$$\alpha = \sum n_Z [Z]$$
where $n_Z \neq 0 \Rightarrow \dim_\delta(Z) = k$.

(4) The abelian group of all k-cycles on X is denoted $Z_k(X)$.

In other words, a k-cycle on X is a locally finite formal $\mathbf{Z}$-linear combination of integral closed subschemes of δ-dimension k. Addition of k-cycles $\alpha = \sum n_Z[Z]$ and $\beta = \sum m_Z[Z]$ is given by
$$\alpha + \beta = \sum (n_Z + m_Z)[Z],$$
i.e., by adding the coefficients.

41.9. Cycle associated to a closed subscheme

Lemma 41.9.1. *Let (S, δ) be as in Situation 41.7.1. Let X be locally of finite type over S. Let $Z \subset X$ be a closed subscheme.*

(1) *The collection of irreducible components of Z is locally finite.*

(2) *Let $Z' \subset Z$ be an irreducible component and let $\xi \in Z'$ be its generic point. Then*
$$\mathit{length}_{\mathcal{O}_{X,\xi}} \mathcal{O}_{Z,\xi} < \infty$$

(3) *If $\dim_\delta(Z) \leq k$ and $\xi \in Z$ with $\delta(\xi) = k$, then ξ is a generic point of an irreducible component of Z.*

Proof. Let $U \subset X$ be a quasi-compact open subscheme. Then U is a Noetherian scheme, and hence has a Noetherian underlying topological space (Properties, Lemma 27.5.5). Hence every subspace is Noetherian and has finitely many irreducible components (see Topology, Lemma 5.8.2). This proves (1).

Let $Z' \subset Z$, $\xi \in Z'$ be as in (2). Then $\dim(\mathcal{O}_{Z,\xi}) = 0$ (for example by Properties, Lemma 27.11.4). Hence $\mathcal{O}_{Z,\xi}$ is Noetherian local ring of dimension zero, and hence has finite length over itself (see Algebra, Proposition 10.57.6). Hence, it also has finite length over $\mathcal{O}_{X,\xi}$, see Algebra, Lemma 10.48.12.

Assume $\xi \in Z$ and $\delta(\xi) = k$. Consider the closure $Z' = \overline{\{\xi\}}$. It is an irreducible closed subscheme with $\dim_\delta(Z') = k$ by definition. Since $\dim_\delta(Z) = k$ it must be an irreducible component of Z. Hence we see (3) holds. □

Definition 41.9.2. Let (S, δ) be as in Situation 41.7.1. Let X be locally of finite type over S. Let $Z \subset X$ be a closed subscheme.

(1) For any irreducible component $Z' \subset Z$ with generic point ξ the integer $m_{Z',Z} = \text{length}_{\mathcal{O}_{X,\xi}} \mathcal{O}_{Z,\xi}$ (Lemma 41.9.1) is called the *multiplicity of Z' in Z*.

(2) Assume $\dim_\delta(Z) \leq k$. The *k-cycle associated to Z* is
$$[Z]_k = \sum m_{Z',Z} [Z']$$
where the sum is over the irreducible components of Z of δ-dimension k. (This is a k-cycle by Lemma 41.9.1.)

It is important to note that we only define $[Z]_k$ if the δ-dimension of Z does not exceed k. In other words, by convention, if we write $[Z]_k$ then this implies that $\dim_\delta(Z) \leq k$.

41.10. Cycle associated to a coherent sheaf

Lemma 41.10.1. *Let (S, δ) be as in Situation 41.7.1. Let X be locally of finite type over S. Let $\mathcal{F}$ be a coherent $\mathcal{O}_X$-module.*

1. *The collection of irreducible components of the support of $\mathcal{F}$ is locally finite.*
2. *Let $Z' \subset Supp(\mathcal{F})$ be an irreducible component and let $\xi \in Z'$ be its generic point. Then*
$$length_{\mathcal{O}_{X,\xi}} \mathcal{F}_\xi < \infty$$
3. *If $\dim_\delta(Supp(\mathcal{F})) \leq k$ and $\xi \in Z$ with $\delta(\xi) = k$, then ξ is a generic point of an irreducible component of $Supp(\mathcal{F})$.*

Proof. By Cohomology of Schemes, Lemma 29.9.7 the support Z of $\mathcal{F}$ is a closed subset of X. We may think of Z as a reduced closed subscheme of X (Schemes, Lemma 25.12.4). Hence (1) and (3) follow immediately by applying Lemma 41.9.1 to $Z \subset X$.

Let $\xi \in Z'$ be as in (2). In this case for any specialization $\xi' \rightsquigarrow \xi$ in X we have $\mathcal{F}_{\xi'} = 0$. Recall that the non-maximal primes of $\mathcal{O}_{X,\xi}$ correspond to the points of X specializing to ξ (Schemes, Lemma 25.13.2). Hence $\mathcal{F}_\xi$ is a finite $\mathcal{O}_{X,\xi}$-module whose support is $\{\mathfrak{m}_\xi\}$. Hence it has finite length by Algebra, Lemma 10.59.3. □

Definition 41.10.2. Let (S, δ) be as in Situation 41.7.1. Let X be locally of finite type over S. Let $\mathcal{F}$ be a coherent $\mathcal{O}_X$-module.

1. For any irreducible component $Z' \subset \mathrm{Supp}(\mathcal{F})$ with generic point ξ the integer $m_{Z',\mathcal{F}} = \mathrm{length}_{\mathcal{O}_{X,\xi}} \mathcal{F}_\xi$ (Lemma 41.10.1) is called the *multiplicity of Z' in $\mathcal{F}$*.
2. Assume $\dim_\delta(\mathrm{Supp}(\mathcal{F})) \leq k$. The *$k$-cycle associated to $\mathcal{F}$* is
$$[\mathcal{F}]_k = \sum m_{Z',\mathcal{F}}[Z']$$
where the sum is over the irreducible components of $\mathrm{Supp}(\mathcal{F})$ of δ-dimension k. (This is a k-cycle by Lemma 41.10.1.)

It is important to note that we only define $[\mathcal{F}]_k$ if $\mathcal{F}$ is coherent and the δ-dimension of $\mathrm{Supp}(\mathcal{F})$ does not exceed k. In other words, by convention, if we write $[\mathcal{F}]_k$ then this implies that $\mathcal{F}$ is coherent on X and $\dim_\delta(\mathrm{Supp}(\mathcal{F})) \leq k$.

Lemma 41.10.3. *Let (S, δ) be as in Situation 41.7.1. Let X be locally of finite type over S. Let $Z \subset X$ be a closed subscheme. If $\dim_\delta(Z) \leq k$, then $[Z]_k = [\mathcal{O}_Z]_k$.*

Proof. This is because in this case the multiplicities $m_{Z',Z}$ and $m_{Z',\mathcal{O}_Z}$ agree by definition. □

Lemma 41.10.4. *Let (S, δ) be as in Situation 41.7.1. Let X be locally of finite type over S. Let $0 \to \mathcal{F} \to \mathcal{G} \to \mathcal{H} \to 0$ be a short exact sequence of coherent sheaves on X. Assume that the δ-dimension of the supports of $\mathcal{F}$, $\mathcal{G}$, and $\mathcal{H}$ is $\leq k$. Then $[\mathcal{G}]_k = [\mathcal{F}]_k + [\mathcal{H}]_k$.*

Proof. Follows immediately from additivity of lengths, see Algebra, Lemma 10.48.3. □

41.11. Preparation for proper pushforward

Lemma 41.11.1. *Let (S, δ) be as in Situation 41.7.1. Let X, Y be locally of finite type over S. Let $f : X \to Y$ be a morphism. Assume X, Y integral and $\dim_\delta(X) = \dim_\delta(Y)$. Then either $f(X)$ is contained in a proper closed subscheme of Y, or f is dominant and the extension of function fields $R(Y) \subset R(X)$ is finite.*

Proof. The closure $\overline{f(X)} \subset Y$ is irreducible as X is irreducible (Topology, Lemmas 5.7.2 and 5.7.3). If $\overline{f(X)} \neq Y$, then we are done. If $\overline{f(X)} = Y$, then f is dominant and by Morphisms, Lemma 28.8.5 we see that the generic point η_Y of Y is in the image of f. Of course this implies that $f(\eta_X) = \eta_Y$, where $\eta_X \in X$ is the generic point of X. Since $\delta(\eta_X) = \delta(\eta_Y)$ we see that $R(Y) = \kappa(\eta_Y) \subset \kappa(\eta_X) = R(X)$ is an extension of transcendence degree 0. Hence $R(Y) \subset R(X)$ is a finite extension by Morphisms, Lemma 28.47.4 (which applies by Morphisms, Lemma 28.16.8). □

Lemma 41.11.2. *Let (S, δ) be as in Situation 41.7.1. Let X, Y be locally of finite type over S. Let $f : X \to Y$ be a morphism. Assume f is quasi-compact, and $\{Z_i\}_{i \in I}$ is a locally finite collection of closed subsets of X. Then $\{\overline{f(Z_i)}\}_{i \in I}$ is a locally finite collection of closed subsets of Y.*

Proof. Let $V \subset Y$ be a quasi-compact open subset. Since f is quasi-compact the open $f^{-1}(V)$ is quasi-compact. Hence the set $\{i \in I \mid Z_i \cap f^{-1}(V) \neq \emptyset\}$ is finite by assumption (Definition 41.8.1). Since this is the same as the set

$$\{i \in I \mid f(Z_i) \cap V \neq \emptyset\} = \{i \in I \mid \overline{f(Z_i)} \cap V \neq \emptyset\}$$

the lemma is proved. □

41.12. Proper pushforward

Definition 41.12.1. Let (S, δ) be as in Situation 41.7.1. Let X, Y be locally of finite type over S. Let $f : X \to Y$ be a morphism. Assume f is proper.

(1) Let $Z \subset X$ be an integral closed subscheme with $\dim_\delta(Z) = k$. We define

$$f_*[Z] = \begin{cases} 0 & \text{if} \quad \dim_\delta(f(Z)) < k, \\ \deg(Z/f(Z))[f(Z)] & \text{if} \quad \dim_\delta(f(Z)) = k. \end{cases}$$

Here we think of $f(Z) \subset Y$ as an integral closed subscheme. The degree of Z over $f(Z)$ is finite if $\dim_\delta(f(Z)) = \dim_\delta(Z)$ by Lemma 41.11.1.

(2) Let $\alpha = \sum n_Z [Z]$ be a k-cycle on X. The *pushforward* of α as the sum

$$f_*\alpha = \sum n_Z f_*[Z]$$

where each $f_*[Z]$ is defined as above. The sum is locally finite by Lemma 41.11.2 above.

By definition the proper pushforward of cycles

$$f_* : Z_k(X) \longrightarrow Z_k(Y)$$

is a homomorphism of abelian groups. It turns $X \mapsto Z_k(X)$ into a covariant functor on the category of schemes locally of finite type over S with morphisms equal to proper morphisms.

Lemma 41.12.2. *Let (S, δ) be as in Situation 41.7.1. Let X, Y, and Z be locally of finite type over S. Let $f : X \to Y$ and $g : Y \to Z$ be proper morphisms. Then $g_* \circ f_* = (g \circ f)_*$ as maps $Z_k(X) \to Z_k(Z)$.*

Proof. Let $W \subset X$ be an integral closed subscheme of dimension k. Consider $W' = f(Z) \subset Y$ and $W'' = g(f(Z)) \subset Z$. Since f, g are proper we see that W' (resp. W'') is an integral closed subscheme of Y (resp. Z). We have to show that $g_*(f_*[W]) = (f \circ g)_*[W]$. If $\dim_\delta(W'') < k$, then both sides are zero. If $\dim_\delta(W'') = k$, then we see the induced morphisms

$$W \longrightarrow W' \longrightarrow W''$$

both satisfy the hypotheses of Lemma 41.11.1. Hence

$$g_*(f_*[W]) = \deg(W/W')\deg(W'/W'')[W''], \quad (f \circ g)_*[W] = \deg(W/W'')[W''].$$

Then we can apply Morphisms, Lemma 28.47.6 to conclude. □

Lemma 41.12.3. *Let (S, δ) be as in Situation 41.7.1. Let X, Y be locally of finite type over S. Let $f : X \to Y$ be a morphism. Assume f is proper.*

(1) *Let $Z \subset X$ be a closed subscheme with $\dim_\delta(Z) \leq k$. Then*

$$f_*[Z]_k = [f_*\mathcal{O}_Z]_k.$$

(2) *Let $\mathcal{F}$ be a coherent sheaf on X such that $\dim_\delta(Supp(\mathcal{F})) \leq k$. Then*

$$f_*[\mathcal{F}]_k = [f_*\mathcal{F}]_k.$$

Note that the statement makes sense since $f_*\mathcal{F}$ and $f_*\mathcal{O}_Z$ are coherent $\mathcal{O}_Y$-modules by Cohomology of Schemes, Lemma 29.17.2.

Proof. Part (1) follows from (2) and Lemma 41.10.3. Let $\mathcal{F}$ be a coherent sheaf on X. Assume that $\dim_\delta(\mathrm{Supp}(\mathcal{F})) \leq k$. By Cohomology of Schemes, Lemma 29.9.7 there exists a closed subscheme $i : Z \to X$ and a coherent $\mathcal{O}_Z$-module $\mathcal{G}$ such that $i_*\mathcal{G} \cong \mathcal{F}$ and such that the support of $\mathcal{F}$ is Z. Let $Z' \subset Y$ be the scheme theoretic image of $f|_Z : Z \to Y$.Consider the commutative diagram of schemes

$$\begin{array}{ccc} Z & \xrightarrow{i} & X \\ {\scriptstyle f|_Z}\downarrow & & \downarrow{\scriptstyle f} \\ Z' & \xrightarrow{i'} & Y \end{array}$$

We have $f_*\mathcal{F} = f_*i_*\mathcal{G} = i'_*(f|_Z)_*\mathcal{G}$ by going around the diagram in two ways. Suppose we know the result holds for closed immersions and for $f|_Z$. Then we see that

$$f_*[\mathcal{F}]_k = f_*i_*[\mathcal{G}]_k = (i')_*(f|_Z)_*[\mathcal{G}]_k = (i')_*[(f|_Z)_*\mathcal{G}]_k = [(i')_*(f|_Z)_*\mathcal{G}]_k = [f_*\mathcal{F}]_k$$

as desired. The case of a closed immersion is straightforward (omitted). Note that $f|_Z : Z \to Z'$ is a dominant morphism (see Morphisms, Lemma 28.6.3). Thus we have reduced to the case where $\dim_\delta(X) \leq k$ and $f : X \to Y$ is proper and dominant.

Assume $\dim_\delta(X) \leq k$ and $f : X \to Y$ is proper and dominant. Since f is dominant, for every irreducible component $Z \subset Y$ with generic point η there exists a point $\xi \in X$ such that $f(\xi) = \eta$. Hence $\delta(\eta) \leq \delta(\xi) \leq k$. Thus we see that in the expressions

$$f_*[\mathcal{F}]_k = \sum n_Z[Z], \quad \text{and} \quad [f_*\mathcal{F}]_k = \sum m_Z[Z].$$

whenever $n_Z \neq 0$, or $m_Z \neq 0$ the integral closed subscheme Z is actually an irreducible component of Y of δ-dimension k. Pick such an integral closed subscheme $Z \subset Y$ and denote η its generic point. Note that for any $\xi \in X$ with $f(\xi) = \eta$ we

have $\delta(\xi) \geq k$ and hence ξ is a generic point of an irreducible component of X of δ-dimension k as well (see Lemma 41.9.1). Since f is quasi-compact and X is locally Noetherian, there can be only finitely many of these and hence $f^{-1}(\{\eta\})$ is finite. By Morphisms, Lemma 28.47.1 there exists an open neighbourhood $\eta \in V \subset Y$ such that $f^{-1}(V) \to V$ is finite. Replacing Y by V and X by $f^{-1}(V)$ we reduce to the case where Y is affine, and f is finite.

Write $Y = \mathrm{Spec}(R)$ and $X = \mathrm{Spec}(A)$ (possible as a finite morphism is affine). Then R and A are Noetherian rings and A is finite over R. Moreover $\mathcal{F} = \widetilde{M}$ for some finite A-module M. Note that $f_*\mathcal{F}$ corresponds to M viewed as an R-module. Let $\mathfrak{p} \subset R$ be the minimal prime corresponding to $\eta \in Y$. The coefficient of Z in $[f_*\mathcal{F}]_k$ is clearly $\mathrm{length}_{R_\mathfrak{p}}(M_\mathfrak{p})$. Let $\mathfrak{q}_i$, $i = 1, \ldots, t$ be the primes of A lying over $\mathfrak{p}$. Then $A_\mathfrak{p} = \prod A_{\mathfrak{q}_i}$ since $A_\mathfrak{p}$ is an Artinian ring being finite over the dimension zero local Noetherian ring $R_\mathfrak{p}$. Clearly the coefficient of Z in $f_*[\mathcal{F}]_k$ is

$$\sum\nolimits_{i=1,\ldots,t} [\kappa(\mathfrak{q}_i) : \kappa(\mathfrak{p})] \mathrm{length}_{A_{\mathfrak{q}_i}}(M_{\mathfrak{q}_i})$$

Hence the desired equality follows from Algebra, Lemma 10.48.12. □

41.13. Preparation for flat pullback

Recall that a morphism $f : X \to Y$ which is locally of finite type is said to have relative dimension r if every nonempty fibre is equidimensional of dimension r. See Morphisms, Definition 28.30.1.

Lemma 41.13.1. *Let (S, δ) be as in Situation 41.7.1. Let X, Y be locally of finite type over S. Let $f : X \to Y$ be a morphism. Assume f is flat of relative dimension r. For any closed subset $Z \subset Y$ we have*

$$\dim_\delta(f^{-1}(Z)) = \dim_\delta(Z) + r.$$

If Z is irreducible and $Z' \subset f^{-1}(Z)$ is an irreducible component, then Z' dominates Z and $\dim_\delta(Z') = \dim_\delta(Z) + r$.

Proof. It suffices to prove the final statement. We may replace Y by the integral closed subscheme Z and X by the scheme theoretic inverse image $f^{-1}(Z) = Z \times_Y X$. Hence we may assume $Z = Y$ is integral and f is a flat morphism of relative dimension r. Since Y is locally Noetherian the morphism f which is locally of finite type, is actually locally of finite presentation. Hence Morphisms, Lemma 28.26.9 applies and we see that f is open. Let $\xi \in X$ be a generic point of an irreducible component of X. By the openness of f we see that $f(\xi)$ is the generic point η of $Z = Y$. Note that $\dim_\xi(X_\eta) = r$ by assumption that f has relative dimension r. On the other hand, since ξ is a generic point of X we see that $\mathcal{O}_{X,\xi} = \mathcal{O}_{X_\eta,\xi}$ has only one prime ideal and hence has dimension 0. Thus by Morphisms, Lemma 28.29.1 we conclude that the transcendence degree of $\kappa(\xi)$ over $\kappa(\eta)$ is r. In other words, $\delta(\xi) = \delta(\eta) + r$ as desired. □

Here is the lemma that we will use to prove that the flat pullback of a locally finite collection of closed subschemes is locally finite.

Lemma 41.13.2. *Let (S, δ) be as in Situation 41.7.1. Let X, Y be locally of finite type over S. Let $f : X \to Y$ be a morphism. Assume $\{Z_i\}_{i \in I}$ is a locally finite collection of closed subsets of Y. Then $\{f^{-1}(Z_i)\}_{i \in I}$ is a locally finite collection of closed subsets of Y.*

Proof. Let $U \subset X$ be a quasi-compact open subset. Since the image $f(U) \subset Y$ is a quasi-compact subset there exists a quasi-compact open $V \subset Y$ such that $f(U) \subset V$. Note that

$$\{i \in I \mid f^{-1}(Z_i) \cap U \neq \emptyset\} \subset \{i \in I \mid Z_i \cap V \neq \emptyset\}.$$

Since the right hand side is finite by assumption we win. □

41.14. Flat pullback

In the following we use $f^{-1}(Z)$ to denote the *scheme theoretic inverse image* of a closed subscheme $Z \subset Y$ for a morphism of schemes $f : X \to Y$. We recall that the scheme theoretic inverse image is the fibre product

$$\begin{array}{ccc} f^{-1}(Z) & \longrightarrow & X \\ \downarrow & & \downarrow \\ Z & \longrightarrow & Y \end{array}$$

and it is also the closed subscheme of X cut out by the quasi-coherent sheaf of ideals $f^{-1}(\mathcal{I})\mathcal{O}_X$, if $\mathcal{I} \subset \mathcal{O}_Y$ is the quasi-coherent sheaf of ideals corresponding to Z in Y. (This is discussed in Schemes, Section 25.4 and Lemma 25.17.6 and Definition 25.17.7.)

Definition 41.14.1. Let (S, δ) be as in Situation 41.7.1. Let X, Y be locally of finite type over S. Let $f : X \to Y$ be a morphism. Assume f is flat of relative dimension r.

(1) Let $Z \subset Y$ be an integral closed subscheme of δ-dimension k. We define $f^*[Z]$ to be the $(k + r)$-cycle on X to the scheme theoretic inverse image

$$f^*[Z] = [f^{-1}(Z)]_{k+r}.$$

This makes sense since $\dim_\delta(f^{-1}(Z)) = k + r$ by Lemma 41.13.1.

(2) Let $\alpha = \sum n_i[Z_i]$ be a k-cycle on Y. The *flat pullback of α by f* is the sum

$$f^*\alpha = \sum n_i f^*[Z_i]$$

where each $f^*[Z_i]$ is defined as above. The sum is locally finite by Lemma 41.13.2.

(3) We denote $f^* : Z_k(Y) \to Z_{k+r}(X)$ the map of abelian groups so obtained.

An open immersion is flat. This is an important though trivial special case of a flat morphism. If $U \subset X$ is open then sometimes the pullback by $j : U \to X$ of a cycle is called the *restriction* of the cycle to U. Note that in this case the maps

$$j^* : Z_k(X) \longrightarrow Z_k(U)$$

are all *surjective*. The reason is that given any integral closed subscheme $Z' \subset U$, we can take the closure of Z of Z' in X and think of it as a reduced closed subscheme of X (see Schemes, Lemma 25.12.4). And clearly $Z \cap U = Z'$, in other words $j^*[Z] = [Z']$ whence the surjectivity. In fact a little bit more is true.

Lemma 41.14.2. *Let (S, δ) be as in Situation 41.7.1. Let X be locally of finite type over S. Let $U \subset X$ be an open subscheme, and denote $i : Y = X \setminus U \to X$ as a reduced closed subscheme of X. For every $k \in \mathbf{Z}$ the sequence*

$$Z_k(Y) \xrightarrow{i_*} Z_k(X) \xrightarrow{j^*} Z_k(U) \longrightarrow 0$$

is an exact complex of abelian groups.

Proof. By the description above the basis elements $[Z]$ of the free abelian group $Z_k(X)$ map either to (distinct) basis elements $[Z \cap U]$ or to zero if $Z \subset Y$. Hence the lemma is clear. □

Lemma 41.14.3. *Let (S, δ) be as in Situation 41.7.1. Let X, Y, Z be locally of finite type over S. Let $f : X \to Y$ and $g : Y \to Z$ be flat morphisms of relative dimensions r and s. Then $g \circ f$ is flat of relative dimension $r + s$ and*

$$f^* \circ g^* = (g \circ f)^*$$

as maps $Z_k(Z) \to Z_{k+r+s}(X)$.

Proof. The composition is flat of relative dimension $r + s$ by Morphisms, Lemma 28.30.3. Suppose that

(1) $W \subset Z$ is a closed integral subscheme of δ-dimension k,
(2) $W' \subset Y$ is a closed integral subscheme of δ-dimension $k + s$ with $W' \subset g^{-1}(W)$, and
(3) $W'' \subset Y$ is a closed integral subscheme of δ-dimension $k + s + r$ with $W'' \subset f^{-1}(W')$.

We have to show that the coefficient n of $[W'']$ in $(g \circ f)^*[W]$ agrees with the coefficient m of $[W'']$ in $f^*(g^*[W])$. That it suffices to check the lemma in these cases follows from Lemma 41.13.1. Let $\xi'' \in W''$, $\xi' \in W'$ and $\xi \in W$ be the generic points. Consider the local rings $A = \mathcal{O}_{Z,\xi}$, $B = \mathcal{O}_{Y,\xi'}$ and $C = \mathcal{O}_{X,\xi''}$. Then we have local flat ring maps $A \to B$, $B \to C$ and moreover

$$n = \text{length}_C(C/\mathfrak{m}_A C), \quad \text{and} \quad m = \text{length}_C(C/\mathfrak{m}_B C)\text{length}_B(B/\mathfrak{m}_A B)$$

Hence the equality follows from Algebra, Lemma 10.48.14. □

Lemma 41.14.4. *Let (S, δ) be as in Situation 41.7.1. Let X, Y be locally of finite type over S. Let $f : X \to Y$ be a flat morphism of relative dimension r.*

(1) *Let $Z \subset Y$ be a closed subscheme with $\dim_\delta(Z) \leq k$. Then we have $\dim_\delta(f^{-1}(Z)) \leq k + r$ and $[f^{-1}(Z)]_{k+r} = f^*[Z]_k$ in $Z_{k+r}(X)$.*
(2) *Let $\mathcal{F}$ be a coherent sheaf on Y with $\dim_\delta(Supp(\mathcal{F})) \leq k$. Then we have $\dim_\delta(Supp(f^*\mathcal{F})) \leq k + r$ and*

$$f^*[\mathcal{F}]_k = [f^*\mathcal{F}]_{k+r}$$

in $Z_{k+r}(X)$.

Proof. Part (1) follows from part (2) by Lemma 41.10.3 and the fact that $f^*\mathcal{O}_Z = \mathcal{O}_{f^{-1}(Z)}$.

Proof of (2). As X, Y are locally Noetherian we may apply Cohomology of Schemes, Lemma 29.9.1 to see that $\mathcal{F}$ is of finite type, hence $f^*\mathcal{F}$ is of finite type (Modules, Lemma 17.9.2), hence $f^*\mathcal{F}$ is coherent (Cohomology of Schemes, Lemma 29.9.1 again). Thus the lemma makes sense. Let $W \subset Y$ be an integral closed subscheme of δ-dimension k, and let $W' \subset X$ be an integral closed subscheme of dimension

$k + r$ mapping into W under f. We have to show that the coefficient n of $[W]$ in $f^*[\mathcal{F}]_k$ agrees with the coefficient m of $[W]$ in $[f^*\mathcal{F}]_{k+r}$. Let $\xi \in W$ and $\xi' \in W'$ be the generic points. Let $A = \mathcal{O}_{Y,\xi}$, $B = \mathcal{O}_{X,\xi'}$ and set $M = \mathcal{F}_\xi$ as an A-module. (Note that M has finite length by our dimension assumptions, but we actually do not need to verify this. See Lemma 41.10.1.) We have $f^*\mathcal{F}_{\xi'} = B \otimes_A M$. Thus we see that

$$n = \text{length}_B(B \otimes_A M) \quad \text{and} \quad m = \text{length}_A(M)\text{length}_B(B/\mathfrak{m}_A B)$$

Thus the equality follows from Algebra, Lemma 10.48.13. □

41.15. Push and pull

In this section we verify that proper pushforward and flat pullback are compatible when this makes sense. By the work we did above this is a consequence of cohomology and base change.

Lemma 41.15.1. *Let (S, δ) be as in Situation 41.7.1. Let*

$$\begin{array}{ccc} X' & \xrightarrow{g'} & X \\ {\scriptstyle f'}\downarrow & & \downarrow{\scriptstyle f} \\ Y' & \xrightarrow{g} & Y \end{array}$$

be a fibre product diagram of schemes locally of finite type over S. Assume $f : X \to Y$ proper and $g : Y' \to Y$ flat of relative dimension r. Then also f' is proper and g' is flat of relative dimension r. For any k-cycle α on X we have

$$g^* f_* \alpha = f'_*(g')^* \alpha$$

in $Z_{k+r}(Y')$.

Proof. The assertion that f' is proper follows from Morphisms, Lemma 28.42.5. The assertion that g' is flat of relative dimension r follows from Morphisms, Lemmas 28.30.2 and 28.26.7. It suffices to prove the equality of cycles when $\alpha = [W]$ for some integral closed subscheme $W \subset X$ of δ-dimension k. Note that in this case we have $\alpha = [\mathcal{O}_W]_k$, see Lemma 41.10.3. By Lemmas 41.12.3 and 41.14.4 it therefore suffices to show that $f'_*(g')^*\mathcal{O}_W$ is isomorphic to $g^* f_* \mathcal{O}_W$. This follows from cohomology and base change, see Cohomology of Schemes, Lemma 29.5.2. □

Lemma 41.15.2. *Let (S, δ) be as in Situation 41.7.1. Let X, Y be locally of finite type over S. Let $f : X \to Y$ be a finite locally free morphism of degree d (see Morphisms, Definition 28.46.1). Then f is both proper and flat of relative dimension 0, and*

$$f_* f^* \alpha = d\alpha$$

for every $\alpha \in Z_k(Y)$.

Proof. A finite locally free morphism is flat and finite by Morphisms, Lemma 28.46.2, and a finite morphism is proper by Morphisms, Lemma 28.44.10. We omit showing that a finite morphism has relative dimension 0. Thus the formula makes sense. To prove it, let $Z \subset Y$ be an integral closed subscheme of δ-dimension k. It suffices to prove the formula for $\alpha = [Z]$. Since the base change of a finite locally free morphism is finite locally free (Morphisms, Lemma 28.46.4) we see that $f_* f^* \mathcal{O}_Z$ is a finite locally free sheaf of rank d on Z. Hence

$$f_* f^*[Z] = f_* f^*[\mathcal{O}_Z]_k = [f_* f^* \mathcal{O}_Z]_k = d[Z]$$

where we have used Lemmas 41.14.4 and 41.12.3. □

41.16. Preparation for principal divisors

Recall that if Z is an irreducible closed subset of a scheme X, then the codimension of Z in X is equal to the dimension of the local ring $\mathcal{O}_{X,\xi}$, where $\xi \in Z$ is the generic point. See Properties, Lemma 27.11.4.

Definition 41.16.1. Let X be a locally Noetherian scheme. Assume X is integral. Let $f \in R(X)^*$. For every integral closed subscheme $Z \subset X$ of codimension 1 we define the *order of vanishing of f along Z* as the integer

$$\mathrm{ord}_Z(f) = \mathrm{ord}_{\mathcal{O}_{X,\xi}}(f)$$

where the right hand side is the notion of Algebra, Definition 10.115.2 and ξ is the generic point of Z.

Of course it can happen that $\mathrm{ord}_Z(f) < 0$. In this case we say that f has a *pole* along Z and that $-\mathrm{ord}_Z(f) > 0$ is the *order of pole of f along Z*. Note that for $f, g \in R(X)^*$ we have

$$\mathrm{ord}_Z(fg) = \mathrm{ord}_Z(f) + \mathrm{ord}_Z(g).$$

Lemma 41.16.2. *Let (S, δ) be as in Situation 41.7.1. Let X be locally of finite type over S. Assume X is integral. If $Z \subset X$ is an integral closed subscheme of codimension 1, then $\dim_\delta(Z) = \dim_\delta(X) - 1$.*

Proof. This is more or less the defining property of a dimension function. □

Lemma 41.16.3. *Let (S, δ) be as in Situation 41.7.1. Let X be locally of finite type over S. Assume X is integral. Let $f \in R(X)^*$. Then the set*

$$\{Z \subset X \mid Z \text{ is integral, closed of codimension } 1 \text{ and } ord_Z(f) \neq 0\}$$

is locally finite in X.

Proof. This is true simply because there exists a nonempty open subscheme $U \subset X$ such that f corresponds to a section of $\Gamma(U, \mathcal{O}_X^*)$, and hence the codimension 1 irreducibles which can occur in the set of the lemma are all irreducible components of $X \setminus U$. Hence Lemma 41.9.1 gives the desired result. □

Lemma 41.16.4. *Let $f : X \to Y$ be a morphism of schemes. Let $\xi \in Y$ be a point. Assume that*

1. *X, Y are integral,*
2. *X is locally Noetherian*
3. *f is proper, dominant and $R(X) \subset R(Y)$ is finite, and*
4. *$\dim(\mathcal{O}_{Y,\xi}) = 1$.*

Then there exists an open neighbourhood $V \subset Y$ of ξ such that $f|_{f^{-1}(V)} : f^{-1}(V) \to V$ is finite.

Proof. By Cohomology of Schemes, Lemma 29.19.2 it suffices to prove that $f^{-1}(\{\xi\})$ is finite. We replace Y by an affine open, say $Y = \mathrm{Spec}(R)$. Note that R is Noetherian, as X is assumed locally Noetherian. Since f is proper it is quasi-compact. Hence we can find a finite affine open covering $X = U_1 \cup \ldots \cup U_n$ with each $U_i = \mathrm{Spec}(A_i)$. Note that $R \to A_i$ is a finite type injective homomorphism of domains with $f.f.(R) \subset f.f.(A_i)$ finite. Thus the lemma follows from Algebra, Lemma 10.107.2. □

41.17. Principal divisors

The following definition is the key to everything that follows.

Definition 41.17.1. Let (S, δ) be as in Situation 41.7.1. Let X be locally of finite type over S. Assume X is integral with $\dim_\delta(X) = n$. Let $f \in R(X)^*$. The *principal divisor associated to* f is the $(n-1)$-cycle

$$\mathrm{div}(f) = \mathrm{div}_X(f) = \sum \mathrm{ord}_Z(f)[Z]$$

where the sum is over integral closed subschemes of codimension 1 and $\mathrm{ord}_Z(f)$ is as in Definition 41.16.1. This makes sense by Lemmas 41.16.2 and 41.16.3.

Lemma 41.17.2. *Let (S, δ) be as in Situation 41.7.1. Let X be locally of finite type over S. Assume X is integral with $\dim_\delta(X) = n$. Let $f, g \in R(X)^*$. Then*

$$\mathit{div}(fg) = \mathit{div}(f) + \mathit{div}(g)$$

in $Z_{n-1}(X)$.

Proof. This is clear from the additivity of the ord functions. □

An important role in the discussion of principal divisors is played by the "universal" principal divisor $[0] - [\infty]$ on $\mathbf{P}^1_S$. To make this more precise, let us denote

$$D_0, D_\infty \subset \mathbf{P}^1_S = \underline{\mathrm{Proj}}_S(\mathcal{O}_S[X_0, X_1])$$

the closed subscheme cut out by the section X_1, resp. X_0 of $\mathcal{O}(1)$. These are effective Cartier divisors, see Divisors, Definition 30.9.1 and Lemma 30.9.20. The following lemma says that loosely speaking we have "$\mathrm{div}(X_1/X_0) = [D_0] - [D_1]$" and that this is the universal principal divisor.

Lemma 41.17.3. *Let (S, δ) be as in Situation 41.7.1. Let X be locally of finite type over S. Assume X is integral and $n = \dim_\delta(X)$. Let $f \in R(X)^*$. Let $U \subset X$ be a nonempty open such that f corresponds to a section $f \in \Gamma(U, \mathcal{O}_X^*)$. Let $Y \subset X \times_S \mathbf{P}^1_S$ be the closure of the graph of $f : U \to \mathbf{P}^1_S$. Then*

1. *the projection morphism $p : Y \to X$ is proper,*
2. *$p|_{p^{-1}(U)} : p^{-1}(U) \to U$ is an isomorphism,*
3. *the pullbacks $q^{-1}D_0$ and $q^{-1}D_\infty$ via the morphism $q : Y \to \mathbf{P}^1_S$ are effective Cartier divisors on Y,*
4. *we have*
$$\mathit{div}_Y(f) = [q^{-1}D_0]_{n-1} - [q^{-1}D_\infty]_{n-1}$$
5. *we have*
$$\mathit{div}_X(f) = p_* \mathit{div}_Y(f)$$
6. *if we view $Y_0 = q^{-1}D_0$, and $Y_\infty = q^{-1}D_\infty$ as closed subschemes of X via the morphism p then we have*
$$\mathit{div}_X(f) = [Y_0]_{n-1} - [Y_\infty]_{n-1}$$

Proof. Since X is integral, we see that U is integral. Hence Y is integral, and $(1, f)(U) \subset Y$ is an open dense subscheme. Also, note that the closed subscheme $Y \subset X \times_S \mathbf{P}^1_S$ does not depend on the choice of the open U, since after all it is the closure of the one point set $\{\eta'\} = \{(1, f)(\eta)\}$ where $\eta \in X$ is the generic point. Having said this let us prove the assertions of the lemma.

For (1) note that p is the composition of the closed immersion $Y \to X \times_S \mathbf{P}^1_S = \mathbf{P}^1_X$ with the proper morphism $\mathbf{P}^1_X \to X$. As a composition of proper morphisms is proper (Morphisms, Lemma 28.42.4) we conclude.

It is clear that $Y \cap U \times_S \mathbf{P}^1_S = (1, f)(U)$. Thus (2) follows. It also follows that $\dim_\delta(Y) = n$.

Note that $q(\eta') = f(\eta)$ is not contained in D_0 or D_∞ since $f \in R(X)^*$. Hence $q^{-1}D_0$ and $q^{-1}D_\infty$ are effective Cartier divisors on Y by Divisors, Lemma 30.9.12. Thus we see (3). It also follows that $\dim_\delta(q^{-1}D_0) = n-1$ and $\dim_\delta(q^{-1}D_\infty) = n-1$.

Consider the effective Cartier divisor $q^{-1}D_0$. At every point $\xi \in q^{-1}D_0$ we have $f \in \mathcal{O}_{Y,\xi}$ and the local equation for $q^{-1}D_0$ is given by f. In particular, if $\delta(\xi) = n-1$ so ξ is the generic point of a integral closed subscheme Z of δ-dimension $n-1$, then we see that the coefficient of $[Z]$ in $\mathrm{div}_Y(f)$ is

$$\mathrm{ord}_Z(f) = \mathrm{length}_{\mathcal{O}_{Y,\xi}}(\mathcal{O}_{Y,\xi}/f\mathcal{O}_{Y,\xi}) = \mathrm{length}_{\mathcal{O}_{Y,\xi}}(\mathcal{O}_{q^{-1}D_0,\xi})$$

which is the coefficient of $[Z]$ in $[q^{-1}D_0]_{n-1}$. A similar argument using the rational function $1/f$ shows that $-[q^{-1}D_\infty]$ agrees with the terms with negative coefficients in the expression for $\mathrm{div}_Y(f)$. Hence (4) follows.

Note that $D_0 \to S$ is an isomorphism. Hence we see that $X \times_S D_0 \to X$ is an isomorphism as well. Clearly we have $q^{-1}D_0 = Y \cap X \times_S D_0$ (scheme theoretic intersection) inside $X \times_S \mathbf{P}^1_S$. Hence it is really the case that $Y_0 \to X$ is a closed immersion. By the same token we see that

$$p_*\mathcal{O}_{q^{-1}D_0} = \mathcal{O}_{Y_0}$$

and hence by Lemma 41.12.3 we have $p_*[q^{-1}D_0]_{n-1} = [Y_0]_{n-1}$. Of course the same is true for D_∞ and Y_∞. Hence to finish the proof of the lemma it suffices to prove the last assertion.

Let $Z \subset X$ be an integral closed subscheme of δ-dimension $n-1$. We want to show that the coefficient of $[Z]$ in $\mathrm{div}(f)$ is the same as the coefficient of $[Z]$ in $[Y_0]_{n-1} - [Y_\infty]_{n-1}$. We may apply Lemma 41.16.4 to the morphism $p : Y \to X$ and the generic point $\xi \in Z$. Hence we may replace X by an affine open neighbourhood of ξ and assume that $p : Y \to X$ is finite. Write $X = \mathrm{Spec}(R)$ and $Y = \mathrm{Spec}(A)$ with p induced by a finite homomorphism $R \to A$ of Noetherian domains which induces an isomorphism $f.f.(R) \cong f.f.(A)$ of fraction fields. Now we have $f \in f.f.(R)$ and a prime $\mathfrak{p} \subset R$ with $\dim(R_\mathfrak{p}) = 1$. The coefficient of $[Z]$ in $\mathrm{div}_X(f)$ is $\mathrm{ord}_{R_\mathfrak{p}}(f)$. The coefficient of $[Z]$ in $p_*\mathrm{div}_Y(f)$ is

$$\sum\nolimits_{\mathfrak{q} \text{ lying over } \mathfrak{p}} [\kappa(\mathfrak{q}) : \kappa(\mathfrak{p})]\mathrm{ord}_{A_\mathfrak{q}}(f)$$

The desired equality therefore follows from Algebra, Lemma 10.115.8. □

This lemma will be superseded by the more general Lemma 41.20.1.

Lemma 41.17.4. *Let (S, δ) be as in Situation 41.7.1. Let X, Y be locally of finite type over S. Assume X, Y are integral and $n = \dim_\delta(Y)$. Let $f : X \to Y$ be a flat morphism of relative dimension r. Let $g \in R(Y)^*$. Then*

$$f^*(div_Y(g)) = div_X(g)$$

in $Z_{n+r-1}(X)$.

Proof. Note that since f is flat it is dominant so that f induces an embedding $R(Y) \subset R(X)$, and hence we may think of g as an element of $R(X)^*$. Let $Z \subset X$ be an integral closed subscheme of δ-dimension $n + r - 1$. Let $\xi \in Z$ be its generic point. If $\dim_\delta(f(Z)) > n - 1$, then we see that the coefficient of $[Z]$ in the left and right hand side of the equation is zero. Hence we may assume that $Z' = \overline{f(Z)}$ is an integral closed subscheme of Y of δ-dimension $n - 1$. Let $\xi' = f(\xi)$. It is the generic point of Z'. Set $A = \mathcal{O}_{Y,\xi'}$, $B = \mathcal{O}_{X,\xi}$. The ring map $A \to B$ is a flat local homomorphism of Noetherian local domains of dimension 1. We have $g \in f.f.(A)$. What we have to show is that

$$\text{ord}_A(g)\text{length}_B(B/\mathfrak{m}_A B) = \text{ord}_B(g).$$

This follows from Algebra, Lemma 10.48.13 (details omitted). □

41.18. Two fun results on principal divisors

The first lemma implies that the pushforward of a principal divisor along a generically finite morphism is a principal divisor.

Lemma 41.18.1. *Let (S, δ) be as in Situation 41.7.1. Let X, Y be locally of finite type over S. Assume X, Y are integral and $n = \dim_\delta(X) = \dim_\delta(Y)$. Let $p : X \to Y$ be a dominant proper morphism. Let $f \in R(X)^*$. Set*

$$g = Nm_{R(X)/R(Y)}(f).$$

Then we have $p_* div(f) = div(g)$.

Proof. Let $Z \subset Y$ be an integral closed subscheme of δ-dimension $n - 1$. We want to show that the coefficient of $[Z]$ in $p_*\text{div}(f)$ and $\text{div}(g)$ are equal. We may apply Lemma 41.16.4 to the morphism $p : X \to X$ and the generic point $\xi \in Z$. Hence we may replace X by an affine open neighbourhood of ξ and assume that $p : Y \to X$ is finite. Write $X = \text{Spec}(R)$ and $Y = \text{Spec}(A)$ with p induced by a finite homomorphism $R \to A$ of Noetherian domains which induces an finite field extension $f.f.(R) \subset f.f.(A)$ of fraction fields. Now we have $f \in f.f.(A)$, $g = \text{Nm}(f) \in f.f.(R)$, and a prime $\mathfrak{p} \subset R$ with $\dim(R_\mathfrak{p}) = 1$. The coefficient of $[Z]$ in $\text{div}_Y(g)$ is $\text{ord}_{R_\mathfrak{p}}(g)$. The coefficient of $[Z]$ in $p_*\text{div}_X(f)$ is

$$\sum\nolimits_{\mathfrak{q} \text{ lying over } \mathfrak{p}} [\kappa(\mathfrak{q}) : \kappa(\mathfrak{p})]\text{ord}_{A_\mathfrak{q}}(f)$$

The desired equality therefore follows from Algebra, Lemma 10.115.8. □

The following lemma says that the degree of a principal divisor on a proper curve is zero.

Lemma 41.18.2. *Let K be any field. Let X be a 1-dimensional integral scheme endowed with a proper morphism $c : X \to \text{Spec}(K)$. Let $f \in K(X)^*$ be an invertible rational function. Then*

$$\sum\nolimits_{x \in X \text{ closed}} [\kappa(x) : K] ord_{\mathcal{O}_{X,x}}(f) = 0$$

where ord is as in Algebra, Definition 10.115.2. In other words, $c_* div(f) = 0$.

Proof. Consider the diagram

$$\begin{array}{ccc} Y & \xrightarrow{p} & X \\ {\scriptstyle q}\downarrow & & \downarrow{\scriptstyle c} \\ \mathbf{P}^1_K & \xrightarrow{c'} & \mathrm{Spec}(K) \end{array}$$

that we constructed in Lemma 41.17.3 starting with X and the rational function f over $S = \mathrm{Spec}(K)$. We will use all the results of this lemma without further mention. We have to show that $c_*\mathrm{div}_X(f) = c_*p_*\mathrm{div}_Y(f) = 0$. This is the same as proving that $c'_*q_*\mathrm{div}_Y(f) = 0$. If $q(Y)$ is a closed point of $\mathbf{P}^1_K$ then we see that $\mathrm{div}_X(f) = 0$ and the lemma holds. Thus we may assume that q is dominant. Since $\mathrm{div}_Y(f) = [q^{-1}D_0]_0 - [q^{-1}D_\infty]_0$ we see (by definition of flat pullback) that $\mathrm{div}_Y(f) = q^*([D_0]_0 - [D_\infty]_0)$. Suppose we can show that $q : Y \to \mathbf{P}^1_K$ is finite locally free of degree d (see Morphisms, Definition 28.46.1). Then byy Lemma 41.15.2 we get $q_*\mathrm{div}_Y(f) = d([D_0]_0 - [D_\infty]_0)$. Since clearly $c'_*[D_0]_0 = c'_*[D_\infty]_0$ we win.

It remains to show that q is finite locally free. (It will automatically have some given degree as $\mathbf{P}^1_K$ is connected.) Since $\dim(\mathbf{P}^1_K) = 1$ we see that q is finite for example by Lemma 41.16.4. All local rings of $\mathbf{P}^1_K$ at closed points are regular local rings of dimension 1 (in other words discrete valuation rings), since they are localizations of $K[T]$ (see Algebra, Lemma 10.108.1). Hence for $y \in Y$ closed the local ring $\mathcal{O}_{Y,y}$ will be flat over $\mathcal{O}_{\mathbf{P}^1_K,q(y)}$ as soon as it is torsion free. This is obviously the case as $\mathcal{O}_{Y,y}$ is a domain and q is dominant. Thus q is flat. Hence q is finite locally free by Morphisms, Lemma 28.46.2. □

41.19. Rational equivalence

In this section we define *rational equivalence* on k-cycles. We will allow locally finite sums of images of principal divisors (under closed immersions). This leads to some pretty strange phenomena, see Example 41.19.3. However, if we do not allow these then we do not know how to prove that capping with chern classes of line bundles factors through rational equivalence.

Definition 41.19.1. Let (S, δ) be as in Situation 41.7.1. Let X be a scheme locally of finite type over S. Let $k \in \mathbf{Z}$.

(1) Given any locally finite collection $\{W_j \subset X\}$ of integral closed subschemes with $\dim_\delta(W_j) = k + 1$, and any $f_j \in R(W_j)^*$ we may consider
$$\sum (i_j)_*\mathrm{div}(f_j) \in Z_k(X)$$
where $i_j : W_j \to X$ is the inclusion morphism. This makes sense as the morphism $\coprod i_j : \coprod W_j \to X$ is proper.

(2) We say that $\alpha \in Z_k(X)$ is *rationally equivalent to zero* if α is a cycle of the form displayed above.

(3) We say $\alpha, \beta \in Z_k(X)$ are *rationally equivalent* and we write $\alpha \sim_{rat} \beta$ if $\alpha - \beta$ is rationally equivalent to zero.

(4) We define
$$A_k(X) = Z_k(X)/\sim_{rat}$$
to be the *Chow group of k-cycles on X*. This is sometimes called the *Chow group of k-cycles module rational equivalence on X*.

There are many other interesting (adequate) equivalence relations. Rational equivalence is the coarsest one of them all. A very simple but important lemma is the following.

Lemma 41.19.2. *Let (S, δ) be as in Situation 41.7.1. Let X be a scheme locally of finite type over S. Let $U \subset X$ be an open subscheme, and denote $i : Y = X \setminus U \to X$ as a reduced closed subscheme of X. Let $k \in \mathbf{Z}$. Suppose $\alpha, \beta \in Z_k(X)$. If $\alpha|_U \sim_{rat} \beta|_U$ then there exist a cycle $\gamma \in Z_k(Y)$ such that*

$$\alpha \sim_{rat} \beta + i_*\gamma.$$

In other words, the sequence

$$A_k(Y) \xrightarrow{i_*} A_k(X) \xrightarrow{j_*} A_k(U) \longrightarrow 0$$

is an exact complex of abelian groups.

Proof. Let $\{W_j\}_{j \in J}$ be a locally finite collection of integral closed subschemes of δ-dimension $k+1$, and let $f_j \in R(W_j)^*$ be elements such that $(\alpha - \beta)|_U = \sum (i_j)_* \mathrm{div}(f_j)$ as in the definition. Set $W_j' \subset X$ equal to the closure of W_j. Suppose that $V \subset X$ is a quasi-compact open. Then also $V \cap U$ is quasi-compact open in U as V is Noetherian. Hence the set $\{j \in J \mid W_j \cap V \neq \emptyset\} = \{j \in J \mid W_j' \cap V \neq \emptyset\}$ is finite since $\{W_j\}$ is locally finite. In other words we see that $\{W_j'\}$ is also locally finite. Since $R(W_j) = R(W_j')$ we see that

$$\alpha - \beta - \sum (i_j')_* \mathrm{div}(f_j)$$

is a cycle supported on Y and the lemma follows (see Lemma 41.14.2). □

Example 41.19.3. Here is a "strange" example. Suppose that S is the spectrum of a field k with δ as in Example 41.7.2. Suppose that $X = C_1 \cup C_2 \cup \ldots$ is an infinite union of curves $C_j \cong \mathbf{P}^1_k$ glued together in the following way: The point $\infty \in C_j$ is glued transversally to the point $0 \in C_{j+1}$ for $j = 1, 2, 3, \ldots$. Take the point $0 \in C_1$. This gives a zero cycle $[0] \in Z_0(X)$. The "strangeness" in this situation is that actually $[0] \sim_{rat} 0$! Namely we can choose the rational function $f_j \in R(C_j)$ to be the function which has a simple zero at 0 and a simple pole at ∞ and no other zeros or poles. Then we see that the sum $\sum (i_j)_* \mathrm{div}(f_j)$ is exactly the 0-cycle $[0]$. In fact it turns out that $A_0(X) = 0$ in this example. If you find this too bizarre, then you can just make sure your spaces are always quasi-compact (so X does not even exist for you).

Remark 41.19.4. Let (S, δ) be as in Situation 41.7.1. Let X be a scheme locally of finite type over S. Suppose we have infinite collections $\alpha_i, \beta_i \in Z_k(X)$, $i \in I$ of k-cycles on X. Suppose that the supports of α_i and β_i form locally finite collections of closed subsets of X so that $\sum \alpha_i$ and $\sum \beta_i$ are defined as cycles. Moreover, assume that $\alpha_i \sim_{rat} \beta_i$ for each i. Then it is not clear that $\sum \alpha_i \sim_{rat} \sum \beta_i$. Namely, the problem is that the rational equivalences may be given by locally finite families $\{W_{i,j}, f_{i,j} \in R(W_{i,j})^*\}_{j \in J_i}$ but the union $\{W_{i,j}\}_{i \in I, j \in J_i}$ may not be locally finite.

In many cases in practice, one has a locally finite family of closed subsets $\{T_i\}_{i \in I}$ such that α_i, β_i are supported on T_i and such that $\alpha_i = \beta_i$ in $A_k(T_i)$, in other words, the families $\{W_{i,j}, f_{i,j} \in R(W_{i,j})^*\}_{j \in J_i}$ consist of subschemes $W_{i,j} \subset T_i$. In this case it is true that $\sum \alpha_i \sim_{rat} \sum \beta_i$ on X, simply because the family $\{W_{i,j}\}_{i \in I, j \in J_i}$ is automatically locally finite in this case.

41.20. Properties of rational equivalence

Lemma 41.20.1. *Let (S, δ) be as in Situation 41.7.1. Let X, Y be schemes locally of finite type over S. Let $f : X \to Y$ be a flat morphism of relative dimension r. Let $\alpha \sim_{rat} \beta$ be rationally equivalent k-cycles on Y. Then $f^*\alpha \sim_{rat} f^*\beta$ as $(k+r)$-cycles on X.*

Proof. What do we have to show? Well, suppose we are given a collection

$$i_j : W_j \longrightarrow Y$$

of closed immersions, with each W_j integral of δ-dimension $k+1$ and rational functions $f_j \in R(W_j)^*$. Moreover, assume that the collection $\{i_j(W_j)\}_{j \in J}$ is locally finite on Y. Then we have to show that

$$f^*(\sum i_{j,*}\mathrm{div}(f_j))$$

is rationally equivalent to zero on X.

Consider the fibre products

$$i'_j : W'_j = W_j \times_Y X \longrightarrow X.$$

For each j, consider the collection $\{W'_{j,l}\}_{l \in L_j}$ of irreducible components $W'_{j,l} \subset W'_j$ having δ-dimension $k+1$. We may write

$$[W'_j]_{k+1} = \sum\nolimits_{l \in L_j} n_{j,l}[W'_{j,l}]_{k+1}$$

for some $n_{j,l} > 0$. By Lemma 41.13.1 we see that $W'_{j,l} \to W_j$ is dominant and hence we can let $f_{j,l} \in R(W'_{j,l})^*$ denote the image of f_j under the map of fields $R(W_j) \to R(W'_{j,l})$. We claim that

(1) the collection $\{W'_{j,l}\}_{j \in J, l \in L_j}$ is locally finite on X, and
(2) with obvious notation $f^*(\sum i_{j,*}\mathrm{div}(f_j)) = \sum i'_{j,l,*}\mathrm{div}(f_{j,l}^{n_{j,l}})$.

Clearly this claim implies the lemma.

To show (1), note that $\{W'_j\}$ is a locally finite collection of closed subschemes of X by Lemma 41.13.2. Hence if $U \subset X$ is quasi-compact, then U meets only finitely many W'_j. By Lemma 41.9.1 the collection of irreducible components of each W_j is locally finite as well. Hence we see only finitely many $W'_{j,l}$ meet U as desired.

Let $Z \subset X$ be an integral closed subscheme of δ-dimension $k+r$. We have to show that the coefficient n of $[Z]$ in $f^*(\sum i_{j,*}\mathrm{div}(f_j))$ is equal to the coefficient m of $[Z]$ in $\sum i'_{j,l,*}\mathrm{div}(f_{j,l}^{n_{j,l}})$. Let Z' be the closure of $f(Z)$ which is an integral closed subscheme of Y. By Lemma 41.13.1 we have $\dim_\delta(Z') \geq k$. If $\dim_\delta(Z') > k$, then the coefficients n and m are both zero, since the generic point of Z will not be contained in any W'_j or $W'_{j,l}$. Hence we may assume that $\dim_\delta(Z') = k$.

We are going to translate the equality of n and m into algebra. Namely, let $\xi' \in Z'$ and $\xi \in Z$ be the generic points. Set $A = \mathcal{O}_{Y,\xi'}$ and $B = \mathcal{O}_{X,\xi}$. Note that A, B are Noetherian, $A \to B$ is flat, local, and that $\mathfrak{m}_A B$ is an ideal of definition of the local ring B. There are finitely many j such that W_j passes through ξ', and these correspond to prime ideals

$$\mathfrak{p}_1, \ldots, \mathfrak{p}_T \subset A$$

with the property that $\dim(A/\mathfrak{p}_t) = 1$ for each $t = 1, \ldots, T$. The rational functions f_j correspond to elements $f_t \in \kappa(\mathfrak{p}_t)^*$. Say $\mathfrak{p}_t$ corresponds to W_j. By construction, the closed subschemes $W'_{j,l}$ which meet ξ correspond $1-1$ with minimal primes

$$\mathfrak{p}_t B \subset \mathfrak{q}_{t,1}, \ldots, \mathfrak{q}_{t,S_t} \subset B$$

over $\mathfrak{p}_t B$. The integers $n_{j,l}$ correspond to the integers

$$n_{t,s} = \text{length}_{B_{\mathfrak{q}_{t,s}}}((B/\mathfrak{p}_t B)_{B_{\mathfrak{q}_{t,s}}})$$

The rational functions $f_{j,l}$ correspond to the images $f_{t,s} \in \kappa(\mathfrak{q}_{t,s})^*$ of the elements $f_t \in \kappa(\mathfrak{p}_t)^*$. Putting everything together we see that

$$n = \sum \text{ord}_{A/\mathfrak{p}_t}(f_t)\text{length}_B(B/\mathfrak{m}_A B)$$

and that

$$m = \sum \text{ord}_{B/\mathfrak{q}_{t,s}}(f_{t,s})\text{length}_{B_{\mathfrak{q}_{t,s}}}((B/\mathfrak{p}_t B)_{B_{\mathfrak{q}_{t,s}}})$$

Note that it suffices to prove the equality for each $t \in \{1, \ldots, T\}$ separately. Writing $f_t = x/y$ for some nonzero $\overline{x}, \overline{y} \in A/\mathfrak{p}_t$ coming from $x, y \in A$ we see that it suffices to prove

$$\text{length}_{A/\mathfrak{p}_t}(A/(\mathfrak{p}_t, x))\text{length}_B(B/\mathfrak{m}_A B) = \text{length}_B(B/(x, \mathfrak{p}_t)B)$$

(equality uses Algebra, Lemma 10.48.13) equals

$$\sum\nolimits_{s=1,\ldots,S_t} \text{ord}_{B/\mathfrak{q}_{t,s}}(B/(x, \mathfrak{q}_{t,s}))\text{length}_{B_{\mathfrak{q}_{t,s}}}((B/\mathfrak{p}_t B)_{B_{\mathfrak{q}_{t,s}}})$$

and similarly for y. Note that as $x \notin \mathfrak{p}_t$ we see that x is a nonzerodivisor on $A/\mathfrak{p}_t$. As $A \to B$ is flat it follows that x is a nonzerodivisor on the module $M = B/\mathfrak{p}_t B$. Hence the equality above follows from Lemma 41.5.6. □

Lemma 41.20.2. *Let (S, δ) be as in Situation 41.7.1. Let X, Y be schemes locally of finite type over S. Let $p : X \to Y$ be a proper morphism. Suppose $\alpha, \beta \in Z_k(X)$ are rationally equivalent. Then $p_*\alpha$ is rationally equivalent to $p_*\beta$.*

Proof. What do we have to show? Well, suppose we are given a collection

$$i_j : W_j \longrightarrow X$$

of closed immersions, with each W_j integral of δ-dimension $k+1$ and rational functions $f_j \in R(W_j)^*$. Moreover, assume that the collection $\{i_j(W_j)\}_{j \in J}$ is locally finite on X. Then we have to show that

$$p_*\left(\sum i_{j,*}\text{div}(f_j)\right)$$

is rationally equivalent to zero on X.

Note that the sum is equal to

$$\sum p_* i_{j,*}\text{div}(f_j).$$

Let $W'_j \subset Y$ be the integral closed subscheme which is the image of $p \circ i_j$. The collection $\{W'_j\}$ is locally finite in Y by Lemma 41.11.2. Hence it suffices to show, for a given j, that either $p_* i_{j,*}\text{div}(f_j) = 0$ or that it is equal to $i'_{j,*}\text{div}(g_j)$ for some $g_j \in R(W'_j)^*$.

The arguments above therefore reduce us to the case of a since integral closed subscheme $W \subset X$ of δ-dimension $k+1$. Let $f \in R(W)^*$. Let $W' = p(W)$ as above. We get a commutative diagram of morphisms

$$\begin{array}{ccc} W & \xrightarrow{i} & X \\ {\scriptstyle p'}\downarrow & & \downarrow{\scriptstyle p} \\ W' & \xrightarrow{i'} & Y \end{array}$$

Note that $p_* i_* \mathrm{div}(f) = i'_* (p')_* \mathrm{div}(f)$ by Lemma 41.12.2. As explained above we have to show that $(p')_* \mathrm{div}(f)$ is the divisor of a rational function on W' or zero. There are three cases to distinguish.

The case $\dim_\delta(W') < k$. In this case automatically $(p')_* \mathrm{div}(f) = 0$ and there is nothing to prove.

The case $\dim_\delta(W') = k$. Let us show that $(p')_* \mathrm{div}(f) = 0$ in this case. Let $\eta \in W'$ be the generic point. Note that $c : W_\eta \to \mathrm{Spec}(K)$ is a proper integral curve over $K = \kappa(\eta)$ whose function field $K(W_\eta)$ is identified with $R(W)$. Here is a diagram

$$\begin{array}{ccc} W_\eta & \longrightarrow & W \\ {\scriptstyle c}\downarrow & & \downarrow{\scriptstyle p'} \\ \mathrm{Spec}(K) & \longrightarrow & W' \end{array}$$

Let us denote $f_\eta \in K(W_\eta)^*$ the rational function corresponding to $f \in R(W)^*$. Moreover, the closed points ξ of W_η correspond $1-1$ to the closed integral subschemes $Z = Z_\xi \subset W$ of δ-dimension k with $p'(Z) = W'$. Note that the multiplicity of Z_ξ in $\mathrm{div}(f)$ is equal to $\mathrm{ord}_{\mathcal{O}_{W_\eta,\xi}}(f_\eta)$ simply because the local rings $\mathcal{O}_{W_\eta,\xi}$ and $\mathcal{O}_{W,\xi}$ are identified (as subrings of their fraction fields). Hence we see that the multiplicity of $[W']$ in $(p')_* \mathrm{div}(f)$ is equal to the multiplicity of $[\mathrm{Spec}(K)]$ in $c_* \mathrm{div}(f_\eta)$. By Lemma 41.18.2 this is zero.

The case $\dim_\delta(W') = k+1$. In this case Lemma 41.18.1 applies, and we see that indeed $p'_* \mathrm{div}(f) = \mathrm{div}(g)$ for some $g \in R(W')^*$ as desired. □

41.21. Different characterizations of rational equivalence

Let (S, δ) be as in Situation 41.7.1. Let X be a scheme locally of finite type over S. Given any closed subscheme $Z \subset X \times_S \mathbf{P}^1_S = X \times \mathbf{P}^1$ we let Z_0, resp. Z_∞ be the scheme theoretic closed subscheme $Z_0 = \mathrm{pr}_2^{-1}(D_0)$, resp. $Z_\infty = \mathrm{pr}_2^{-1}(D_\infty)$. Here D_0, D_∞ are as defined just above Lemma 41.17.3.

Lemma 41.21.1. *Let (S, δ) be as in Situation 41.7.1. Let X be a scheme locally of finite type over S. Let $W \subset X \times_S \mathbf{P}^1_S$ be an integral closed subscheme of δ-dimension $k+1$. Assume $W \neq W_0$, and $W \neq W_\infty$. Then*

1. *W_0, W_∞ are effective Cartier divisors of W,*
2. *W_0, W_∞ can be viewed as closed subschemes of X and*
$$[W_0]_k \sim_{rat} [W_\infty]_k,$$
3. *for any locally finite family of integral closed subschemes $W_i \subset X \times_S \mathbf{P}^1_S$ of δ-dimension $k+1$ with $W_i \neq (W_i)_0$ and $W_i \neq (W_i)_\infty$ we have $\sum([(W_i)_0]_k - [(W_i)_\infty]_k) \sim_{rat} 0$ on X, and*

(4) *for any* $\alpha \in Z_k(X)$ *with* $\alpha \sim_{rat} 0$ *there exists a locally finite family of integral closed subschemes* $W_i \subset X \times_S \mathbf{P}^1_S$ *as above such that* $\alpha = \sum([(W_i)_0]_k - [(W_i)_\infty]_k)$.

Proof. Part (1) follows from Divisors, Lemma 30.9.12 since the generic point of W is not mapped into D_0 or D_∞ under the projection $X \times_S \mathbf{P}^1_S \to \mathbf{P}^1_S$ by assumption.

Since $X \times_S D_0 \to X$ is an isomorphism we see that W_0 is isomorphic to a closed subscheme of X. Similarly for W_∞. Consider the morphism $p : W \to X$. It is proper and on W we have $[W_0]_k \sim_{rat} [W_\infty]_k$. Hence part (2) follows from Lemma 41.20.2 as clearly $p_*[W_0]_k = [W_0]_k$ and similarly for W_∞.

The only content of statement (3) is, given parts (1) and (2), that the collection $\{(W_i)_0, (W_i)_\infty\}$ is a locally finite collection of closed subschemes of X. This is clear.

Suppose that $\alpha \sim_{rat} 0$. By definition this means there exist integral closed subschemes $V_i \subset X$ of δ-dimension $k+1$ and rational functions $f_i \in R(V_i)^*$ such that the family $\{V_i\}_{i \in I}$ is locally finite in X and such that $\alpha = \sum (V_i \to X)_* \mathrm{div}(f_i)$. Let

$$W_i \subset V_i \times_S \mathbf{P}^1_S \subset X \times_S \mathbf{P}^1_S$$

be the closure of the graph of the rational map f_i as in Lemma 41.17.3. Then we have that $(V_i \to X)_* \mathrm{div}(f_i)$ is equal to $[(W_i)_0]_k - [(W_i)_\infty]_k$ by that same lemma. Hence the result is clear. □

Lemma 41.21.2. *Let* (S, δ) *be as in Situation 41.7.1. Let* X *be a scheme locally of finite type over* S. *Let* Z *be a closed subscheme of* $X \times \mathbf{P}^1$. *Assume* $\dim_\delta(Z) \leq k+1$, $\dim_\delta(Z_0) \leq k$, $\dim_\delta(Z_\infty) \leq k$ *and assume any embedded point* ξ *(Divisors, Definition 30.4.1) of* Z *has* $\delta(\xi) < k$. *Then*

$$[Z_0]_k \sim_{rat} [Z_\infty]_k$$

as k*-cycles on* X.

Proof. Let $\{W_i\}_{i \in I}$ be the collection of irreducible components of Z which have δ-dimension $k+1$. Write

$$[Z]_{k+1} = \sum n_i [W_i]$$

with $n_i > 0$ as per definition. Note that $\{W_i\}$ is a locally finite collection of closed subsets of $X \times_S \mathbf{P}^1_S$ by Lemma 41.9.1. We claim that

$$[Z_0]_k = \sum n_i [(W_i)_0]_k$$

and similarly for $[Z_\infty]_k$. If we prove this then the lemma follows from Lemma 41.21.1.

Let $Z' \subset X$ be an integral closed subscheme of δ-dimension k. To prove the equality above it suffices to show that the coefficient n of $[Z']$ in $[Z_0]_k$ is the same as the coefficient m of $[Z']$ in $\sum n_i [(W_i)_0]_k$. Let $\xi' \in Z'$ be the generic point. Set $\xi = (\xi', 0) \in X \times_S \mathbf{P}^1_S$. Consider the local ring $A = \mathcal{O}_{X \times_S \mathbf{P}^1_S, \xi}$. Let $I \subset A$ be the ideal cutting out Z, in other words so that $A/I = \mathcal{O}_{Z,\xi}$. Let $t \in A$ be the element cutting out $X \times_S D_0$ (i.e., the coordinate of $\mathbf{P}^1$ at zero pulled back). By our choice of $\xi' \in Z'$ we have $\delta(\xi) = k$ and hence $\dim(A/I) = 1$. Since ξ is not an embedded point by definition we see that A/I is Cohen-Macaulay. Since $\dim_\delta(Z_0) = k$ we see that $\dim(A/(t, I)) = 0$ which implies that t is a nonzerodivisor on A/I. Finally, the irreducible closed subschemes W_i passing through ξ correspond

to the minimal primes $I \subset \mathfrak{q}_i$ over I. The multiplicities n_i correspond to the lengths $\text{length}_{A_{\mathfrak{q}_i}}(A/I)_{\mathfrak{q}_i}$. Hence we see that

$$n = \text{length}_A(A/(t, I))$$

and

$$m = \sum \text{length}_A(A/(t, \mathfrak{q}_i))\text{length}_{A_{\mathfrak{q}_i}}(A/I)_{\mathfrak{q}_i}$$

Thus the result follows from Lemma 41.5.6. □

Lemma 41.21.3. *Let (S, δ) be as in Situation 41.7.1. Let X be a scheme locally of finite type over S. Let $\mathcal{F}$ be a coherent sheaf on $X \times \mathbf{P}^1$. Let $i_0, i_\infty : X \to X \times \mathbf{P}^1$ be the closed immersion such that $i_t(x) = (x, t)$. Denote $\mathcal{F}_0 = i_0^*\mathcal{F}$ and $\mathcal{F}_\infty = i_\infty^*\mathcal{F}$. Assume*

1. *$\dim_\delta(Supp(\mathcal{F})) \leq k + 1$,*
2. *$\dim_\delta(Supp(\mathcal{F}_0)) \leq k$, $\dim_\delta(Supp(\mathcal{F}_\infty)) \leq k$, and*
3. *any nonmaximal associated point (insert future reference here) $\xi \in Supp(\mathcal{F})$ of $\mathcal{F}$ has $\delta(\xi) < k$.*

Then

$$[\mathcal{F}_0]_k \sim_{rat} [\mathcal{F}_\infty]_k$$

as k-cycles on X.

Proof. Let $\{W_i\}_{i \in I}$ be the collection of irreducible components of $\text{Supp}(\mathcal{F})$ which have δ-dimension $k + 1$. Write

$$[\mathcal{F}]_{k+1} = \sum n_i[W_i]$$

with $n_i > 0$ as per definition. Note that $\{W_i\}$ is a locally finite collection of closed subsets of $X \times_S \mathbf{P}^1_S$ by Lemma 41.10.1. We claim that

$$[\mathcal{F}_0]_k = \sum n_i[(W_i)_0]_k$$

and similarly for $[\mathcal{F}_\infty]_k$. If we prove this then the lemma follows from Lemma 41.21.1.

Let $Z' \subset X$ be an integral closed subscheme of δ-dimension k. To prove the equality above it suffices to show that the coefficient n of $[Z']$ in $[\mathcal{F}_0]_k$ is the same as the coefficient m of $[Z']$ in $\sum n_i[(W_i)_0]_k$. Let $\xi' \in Z'$ be the generic point. Set $\xi = (\xi', 0) \in X \times_S \mathbf{P}^1_S$. Consider the local ring $A = \mathcal{O}_{X \times_S \mathbf{P}^1_S, \xi}$. Let $M = \mathcal{F}_\xi$ as an A-module. Let $t \in A$ be the element cutting out $X \times_S D_0$ (i.e., the coordinate of $\mathbf{P}^1$ at zero pulled back). By our choice of $\xi' \in Z'$ we have $\delta(\xi) = k$ and hence $\dim(M) = 1$. Since ξ is not an associated point of $\mathcal{F}$ by definition we see that M is Cohen-Macaulay module. Since $\dim_\delta(\text{Supp}(\mathcal{F}_0)) = k$ we see that $\dim(M/tM) = 0$ which implies that t is a nonzerodivisor on M. Finally, the irreducible closed subschemes W_i passing through ξ correspond to the minimal primes $\mathfrak{q}_i$ of $\text{Ass}(M)$. The multiplicities n_i correspond to the lengths $\text{length}_{A_{\mathfrak{q}_i}} M_{\mathfrak{q}_i}$. Hence we see that

$$n = \text{length}_A(M/tM)$$

and

$$m = \sum \text{length}_A(A/(t, \mathfrak{q}_i)A)\text{length}_{A_{\mathfrak{q}_i}} M_{\mathfrak{q}_i}$$

Thus the result follows from Lemma 41.5.6. □

41.22. Rational equivalence and K-groups

In this section we compare the cycle groups $Z_k(X)$ and the Chow groups $A_k(X)$ with certain K_0-groups of abelian categories of coherent sheaves on X. We avoid having to talk about $K_1(\mathcal{A})$ for an abelian category $\mathcal{A}$ by dint of Homology, Lemma 12.10.3. In particular, the motivation for the precise form of Lemma 41.22.4 is that lemma.

Let us introduce the following notation. Let (S, δ) be as in Situation 41.7.1. Let X be a scheme locally of finite type over S. We denote $\mathrm{Coh}(X) = \mathrm{Coh}(\mathcal{O}_X)$ the category of coherent sheaves on X. It is an abelian category, see Cohomology of Schemes, Lemma 29.9.2. For any $k \in \mathbf{Z}$ we let $\mathrm{Coh}_{\leq k}(X)$ be the full subcategory of $\mathrm{Coh}(X)$ consisting of those coherent sheaves $\mathcal{F}$ having $\dim_\delta(\mathrm{Supp}(\mathcal{F})) \leq k$.

Lemma 41.22.1. *Let us introduce the following notation. Let (S, δ) be as in Situation 41.7.1. Let X be a scheme locally of finite type over S. The categories $Coh_{\leq k}(X)$ are Serre subcategories of the abelian category $Coh(X)$.*

Proof. Omitted. The definition of a Serre subcategory is Homology, Definition 12.9.1. □

Lemma 41.22.2. *Let (S, δ) be as in Situation 41.7.1. Let X be a scheme locally of finite type over S. There are maps*

$$Z_k(X) \longrightarrow K_0(Coh_{\leq k}(X)/Coh_{\leq k-1}(X)) \longrightarrow Z_k(X)$$

whose composition is the identity. The first is the map

$$\sum n_Z[Z] \mapsto \Big[\bigoplus\nolimits_{n_Z>0} \mathcal{O}_Z^{\oplus n_Z}\Big] - \Big[\bigoplus\nolimits_{n_Z<0} \mathcal{O}_Z^{\oplus -n_Z}\Big]$$

and the second comes from the map $\mathcal{F} \mapsto [\mathcal{F}]_k$. If X is quasi-compact, then both maps are isomorphisms.

Proof. Note that the direct sum $\bigoplus_{n_Z>0} \mathcal{O}_Z^{\oplus n_Z}$ is indeed a coherent sheaf on X since the family $\{Z \mid n_Z > 0\}$ is locally finite on X. The map $\mathcal{F} \to [\mathcal{F}]_k$ is additive on $\mathrm{Coh}_{\leq k}(X)$, see Lemma 41.10.4. And $[\mathcal{F}]_k = 0$ if $\mathcal{F} \in \mathrm{Coh}_{\leq k-1}(X)$. This implies we have the left map as shown in the lemma. It is clear that their composition is the identity.

In case X is quasi-compact we will show that the right arrow is injective. Suppose that $q \in K_0(\mathrm{Coh}_{\leq k}(X)/\mathrm{Coh}_{\leq k+1}(X))$ maps to zero in $Z_k(X)$. By Homology, Lemma 12.10.3 we can find a $\tilde{q} \in K_0(\mathrm{Coh}_{\leq k}(X))$ mapping to q. Write $\tilde{q} = [\mathcal{F}] - [\mathcal{G}]$ for some $\mathcal{F}, \mathcal{G} \in K_0(\mathrm{Coh}_{\leq k}(X))$. Since X is quasi-compact we may apply Cohomology of Schemes, Lemma 29.12.3. This shows that there exist integral closed subschemes $Z_j, T_i \subset X$ and (nonzero) ideal sheaves $\mathcal{I}_j \subset \mathcal{O}_{Z_j}$, $\mathcal{I}_i \subset \mathcal{O}_{T_i}$ such that $\mathcal{F}$, resp. $\mathcal{G}$ have filtrations whose successive quotients are the sheaves $\mathcal{I}_j$, resp. $\mathcal{I}_i$. In particular we see that $\dim_\delta(Z_j), \dim_\delta(T_i) \leq k$. In other words we have

$$[\mathcal{F}] = \sum\nolimits_j [\mathcal{I}_j], \quad [\mathcal{G}] = \sum\nolimits_i [\mathcal{I}_i],$$

in $K_0(\mathrm{Coh}_{\leq k}(X))$. Our assumption is that $\sum_j [\mathcal{I}_j]_k - \sum_i [\mathcal{I}_i]_k = 0$. It is clear that we may throw out the indices j, resp. i such that $\dim_\delta(Z_j) < k$, resp. $\dim_\delta(T_i) < k$, since the corresponding sheaves are in $\mathrm{Coh}_{k-1}(X)$ and also do not contribute to the cycle. Moreover, the exact sequences $0 \to \mathcal{I}_j \to \mathcal{O}_{Z_j} \to \mathcal{O}_{Z_j}/\mathcal{I}_j \to 0$ and

$0 \to \mathcal{I}_i \to \mathcal{O}_{T_i} \to \mathcal{O}_{Z_i}/\mathcal{I}_i \to 0$ show similarly that we may replace $\mathcal{I}_j$, resp. $\mathcal{I}_i$ by $\mathcal{O}_{Z_j}$, resp. $\mathcal{O}_{T_i}$. OK, and finally, at this point it is clear that our assumption

$$\sum\nolimits_j [\mathcal{O}_{Z_j}]_k - \sum\nolimits_i [\mathcal{O}_{T_i}]_k = 0$$

implies that in $K_0(\mathrm{Coh}_k(X))$ we have also $\sum_j [\mathcal{O}_{Z_j}] - \sum_i [\mathcal{O}_{T_i}] = 0$ as desired. □

Remark 41.22.3. It seems likely that the arrows of Lemma 41.22.2 are not isomorphisms if X is not quasi-compact. For example, suppose X is an infinite disjoint union $X = \coprod_{n \in \mathbf{N}} \mathbf{P}^1_k$ over a field k. Let $\mathcal{F}$, resp. $\mathcal{G}$ be the coherent sheaf on X whose restriction to the nth summand is equal to the skyscraper sheaf at 0 associated to $\mathcal{O}_{\mathbf{P}^1_k,0}/\mathfrak{m}_0^n$, resp. $\kappa(0)^{\oplus n}$. The cycle associated to $\mathcal{F}$ is equal to the cycle associated to $\mathcal{G}$, namely both are equal to $\sum n[0_n]$ where $0_n \in X$ denotes 0 on the nth component of X. But there seems to be no way to show that $[\mathcal{F}] = [\mathcal{G}]$ in $K_0(\mathrm{Coh}(X))$ since any proof we can envision uses infinitely many relations.

Lemma 41.22.4. *Let (S, δ) be as in Situation 41.7.1. Let X be a scheme locally of finite type over S. Let $\mathcal{F}$ be a coherent sheaf on X. Let*

$$\ldots \longrightarrow \mathcal{F} \xrightarrow{\varphi} \mathcal{F} \xrightarrow{\psi} \mathcal{F} \xrightarrow{\varphi} \mathcal{F} \longrightarrow \ldots$$

be a complex as in Homology, Equation (12.10.2.1). Assume that

(1) $\dim_\delta(\mathit{Supp}(\mathcal{F})) \leq k + 1$.
(2) $\dim_\delta(\mathit{Supp}(H^i(\mathcal{F}, \varphi, \psi))) \leq k$ *for* $i = 0, 1$.

Then we have

$$[H^0(\mathcal{F}, \varphi, \psi)]_k \sim_{rat} [H^1(\mathcal{F}, \varphi, \psi)]_k$$

as k-cycles on X.

Proof. Let $\{W_j\}_{j \in J}$ be the collection of irreducible components of $\mathrm{Supp}(\mathcal{F})$ which have δ-dimension $k+1$. Note that $\{W_j\}$ is a locally finite collection of closed subsets of X by Lemma 41.10.1. For every j, let $\xi_j \in W_j$ be the generic point. Set

$$f_j = \det_{\kappa(\xi_j)}(\mathcal{F}_{\xi_j}, \varphi_{\xi_j}, \psi_{\xi_j}) \in R(W_j)^*.$$

See Definition 41.3.4 for notation. We claim that

$$-[H^0(\mathcal{F}, \varphi, \psi)]_k + [H^1(\mathcal{F}, \varphi, \psi)]_k = \sum (W_j \to X)_* \mathrm{div}(f_j)$$

If we prove this then the lemma follows.

Let $Z \subset X$ be an integral closed subscheme of δ-dimension k. To prove the equality above it suffices to show that the coefficient n of $[Z]$ in $[H^0(\mathcal{F}, \varphi, \psi)]_k - [H^1(\mathcal{F}, \varphi, \psi)]_k$ is the same as the coefficient m of $[Z]$ in $\sum (W_j \to X)_* \mathrm{div}(f_j)$. Let $\xi \in Z$ be the generic point. Consider the local ring $A = \mathcal{O}_{X,\xi}$. Let $M = \mathcal{F}_\xi$ as an A-module. Denote $\varphi, \psi : M \to M$ the action of φ, ψ on the stalk. By our choice of $\xi \in Z$ we have $\delta(\xi) = k$ and hence $\dim(M) = 1$. Finally, the integral closed subschemes W_j passing through ξ correspond to the minimal primes $\mathfrak{q}_i$ of $\mathrm{Supp}(M)$. In each case the element $f_j \in R(W_j)^*$ corresponds to the element $\det_{\kappa(\mathfrak{q}_i)}(M_{\mathfrak{q}_i}, \varphi, \psi)$ in $\kappa(\mathfrak{q}_i)^*$. Hence we see that

$$n = -e_A(M, \varphi, \psi)$$

and

$$m = \sum \mathrm{ord}_{A/\mathfrak{q}_i}(\det_{\kappa(\mathfrak{q}_i)}(M_{\mathfrak{q}_i}, \varphi, \psi))$$

Thus the result follows from Proposition 41.5.3. □

Lemma 41.22.5. *Let (S, δ) be as in Situation 41.7.1. Let X be a scheme locally of finite type over S. Denote $B_k(X)$ the image of the map*

$$K_0(Coh_{\leq k}(X)/Coh_{\leq k-1}(X)) \longrightarrow K_0(Coh_{\leq k+1}(X)/Coh_{\leq k-1}(X)).$$

There is a commutative diagram

$$\begin{array}{ccccc} K_0\left(\frac{Coh_{\leq k}(X)}{Coh_{\leq k-1}(X)}\right) & \longrightarrow & B_k(X) & \hookrightarrow & K_0\left(\frac{Coh_{\leq k+1}(X)}{Coh_{\leq k-1}(X)}\right) \\ \downarrow & & \downarrow & & \\ Z_k(X) & \longrightarrow & A_k(X) & & \end{array}$$

where the left vertical arrow is the one from Lemma 41.22.2. If X is quasi-compact then both vertical arrows are isomorphisms.

Proof. Suppose we have an element $[A] - [B]$ of $K_0(\mathrm{Coh}_{\leq k}(X)/\mathrm{Coh}_{\leq k-1}(X))$ which maps to zero in $B_k(X)$, i.e., in $K_0(\mathrm{Coh}_{\leq k+1}(X)/\mathrm{Coh}_{\leq k-1}(X))$. Suppose $[A] = [\mathcal{A}]$ and $[B] = [\mathcal{B}]$ for some coherent sheaves $\mathcal{A}, \mathcal{B}$ on X supported in δ-dimension $\leq k$. The assumption that $[A] - [B]$ maps to zero in the group $K_0(\mathrm{Coh}_{\leq k+1}(X)/\mathrm{Coh}_{\leq k-1}(X))$ means that there exists coherent sheaves $\mathcal{A}', \mathcal{B}'$ on X supported in δ-dimension $\leq k-1$ such that $[\mathcal{A} \oplus \mathcal{A}'] - [\mathcal{B} \oplus \mathcal{B}']$ is zero in $K_0(\mathrm{Coh}_{k+1}(X))$ (use part (1) of Homology, Lemma 12.10.3). By part (2) of Homology, Lemma 12.10.3 this means there exists a $(2,1)$-periodic complex $(\mathcal{F}, \varphi, \psi)$ in the category $\mathrm{Coh}_{\leq k+1}(X)$ such that $\mathcal{A} \oplus \mathcal{A}' = H^0(\mathcal{F}, \varphi, \psi)$ and $\mathcal{B} \oplus \mathcal{B}' = H^1(\mathcal{F}, \varphi, \psi)$. By Lemma 41.22.4 this implies that

$$[\mathcal{A} \oplus \mathcal{A}']_k \sim_{rat} [\mathcal{B} \oplus \mathcal{B}']_k$$

This proves that $[A] - [B]$ maps to zero via the composition

$$K_0(\mathrm{Coh}_{\leq k}(X)/\mathrm{Coh}_{\leq k-1}(X)) \longrightarrow Z_k(X) \longrightarrow A_k(X).$$

In other words this proves the commutative diagram exists.

Next, assume that X is quasi-compact. By Lemma 41.22.2 the left vertical arrow is bijective. Hence it suffices to show any $\alpha \in Z_k(X)$ which is rationally equivalent to zero maps to zero in $B_k(X)$ via the inverse of the left vertical arrow composed with the horizontal arrow. By Lemma 41.21.1 we see that $\alpha = \sum([(W_i)_0]_k - [(W_i)_\infty]_k)$ for some closed integral subschemes $W_i \subset X \times_S \mathbf{P}^1_S$ of δ-dimension $k+1$. Moreover the family $\{W_i\}$ is finite because X is quasi-compact. Note that the ideal sheaves $\mathcal{I}_i, \mathcal{J}_i \subset \mathcal{O}_{W_i}$ of the effective Cartier divisors $(W_i)_0, (W_i)_\infty$ are isomorphic (as $\mathcal{O}_{W_i}$-modules). This is true because the ideal sheaves of D_0 and D_∞ on $\mathbf{P}^1$ are isomorphic and $\mathcal{I}_i, \mathcal{J}_i$ are the pullbacks of these. (Some details omitted.) Hence we have short exact sequences

$$0 \to \mathcal{I}_i \to \mathcal{O}_{W_i} \to \mathcal{O}_{(W_i)_0} \to 0, \quad 0 \to \mathcal{J}_i \to \mathcal{O}_{W_i} \to \mathcal{O}_{(W_i)_\infty} \to 0$$

of coherent $\mathcal{O}_{W_i}$-modules. Also, since $[(W_i)_0]_k = [p_*\mathcal{O}_{(W_i)_0}]_k$ in $Z_k(X)$ we see that the inverse of the left vertical arrow maps $[(W_i)_0]_k$ to the element $[p_*\mathcal{O}_{(W_i)_0}]$ in $K_0(\mathrm{Coh}_{\leq k}(X)/\mathrm{Coh}_{\leq k-1}(X))$. Thus we have

$$\begin{aligned} \alpha &= \sum \left([(W_i)_0]_k - [(W_i)_\infty]_k\right) \\ &\mapsto \sum \left([p_*\mathcal{O}_{(W_i)_0}] - [p_*\mathcal{O}_{(W_i)_\infty}]\right) \\ &= \sum \left([p_*\mathcal{O}_{W_i}] - [p_*\mathcal{I}_i] - [p_*\mathcal{O}_{W_i}] + [p_*\mathcal{J}_i]\right) \end{aligned}$$

in $K_0(\mathrm{Coh}_{\leq k+1}(X)/\mathrm{Coh}_{\leq k-1}(X))$. By what was said above this is zero, and we win. □

Remark 41.22.6. Let (S, δ) be as in Situation 41.7.1. Let X be a scheme locally of finite type over S. Assume X is quasi-compact. The result of Lemma 41.22.5 in particular gives a map

$$A_k(X) \longrightarrow K_0(\mathrm{Coh}(X)/\mathrm{Coh}_{\leq k-1}(X)).$$

We have not been able to find a statement or conjecture in the literature as to whether this map is should be injective or not. If X is connected nonsingular, then, using the isomorphism $K_0(X) = K^0(X)$ (see insert future reference here) and chern classes (see below), one can show that the map is an isomorphism up to $(p-1)!$-torsion where $p = \dim_\delta(X) - k$.

41.23. Preparation for the divisor associated to an invertible sheaf

For the following remarks, see Divisors, Section 30.15. Let X be a scheme. Let $\mathcal{L}$ be an invertible $\mathcal{O}_X$-module. Let $\xi \in X$ be a point. If $s_\xi, s'_\xi \in \mathcal{L}_\xi$ generate $\mathcal{L}_\xi$ as $\mathcal{O}_{X,\xi}$-module, then there exists a unit $u \in \mathcal{O}^*_{X,\xi}$ such that $s_\xi = us'_\xi$. The stalk of the sheaf of meromorphic sections $\mathcal{K}_X(\mathcal{L})$ of $\mathcal{L}$ at x is equal to $\mathcal{K}_{X,x} \otimes_{\mathcal{O}_{X,x}} \mathcal{L}_x$. Thus the image of any meromorphic section s of $\mathcal{L}$ in the stalk at x can be written as $s = fs_\xi$ with $f \in \mathcal{K}_{X,x}$. Below we will abbreviate this by saying $f = s/s_\xi$. Also, if X is integral we have $\mathcal{K}_{X,x} = R(X)$ is equal to the function field of X, so $s/s_\xi \in R(X)$. If s is a *regular* meromorphic section (see Divisors, Definition 30.15.11), then actually $f \in R(X)^*$. (On an integral scheme a regular meromorphic section is the same thing as a nonzero meromorphic section.) Hence the following definition makes sense.

Definition 41.23.1. Let X be a locally Noetherian scheme. Assume X is integral. Let $\mathcal{L}$ be an invertible $\mathcal{O}_X$-module. Let $s \in \Gamma(X, \mathcal{K}_X(\mathcal{L}))$ be a regular meromorphic section of $\mathcal{L}$. For every integral closed subscheme $Z \subset X$ of codimension 1 we define the *order of vanishing of s along Z* as the integer

$$\mathrm{ord}_{Z,\mathcal{L}}(s) = \mathrm{ord}_{\mathcal{O}_{X,\xi}}(s/s_\xi)$$

where the right hand side is the notion of Algebra, Definition 10.115.2, $\xi \in Z$ is the generic point, and $s_\xi \in \mathcal{L}_\xi$ is a generator.

Lemma 41.23.2. *Let (S, δ) be as in Situation 41.7.1. Let X be locally of finite type over S. Assume X is integral. Let $\mathcal{L}$ be an invertible $\mathcal{O}_X$-module. Let $s \in \mathcal{K}_X(\mathcal{L})$ be a regular (i.e., nonzero) meromorphic section of $\mathcal{L}$. Then the set*

$$\{Z \subset X \mid Z \text{ is irreducible, closed of codimension } 1 \text{ and } ord_{Z,\mathcal{L}}(s) \neq 0\}$$

is locally finite in X.

Proof. This is true simply because there exists a nonempty open subscheme $U \subset X$ such that s corresponds to a section of $\Gamma(U, \mathcal{L})$ which generates $\mathcal{L}$ over U. Hence the codimension 1 irreducibles which can occur in the set of the lemma are all irreducible components of $X \setminus U$. Hence Lemma 41.9.1 gives the desired result. □

Lemma 41.23.3. *Let (S, δ) be as in Situation 41.7.1. Let X be locally of finite type over S. Assume X is integral and $n = \dim_\delta(X)$. Let $\mathcal{L}$ be an invertible $\mathcal{O}_X$-module. Let $s, s' \in \mathcal{K}_X(\mathcal{L})$ be nonzero meromorphic sections of $\mathcal{L}$. Then $f = s/s'$ is an element of $R(X)^*$ and we have*

$$\sum ord_{Z,\mathcal{L}}(s)[Z] = \sum ord_{Z,\mathcal{L}}(s')[Z] + div(f)$$

(where the sums are over integral closed subschemes $Z \subset X$ of δ-dimension $n-1$) as elements of $Z_{n-1}(X)$.

Proof. This is clear from the definitions. Note that Lemma 41.23.2 guarantees that the sums are indeed elements of $Z_{n-1}(X)$. □

41.24. The divisor associated to an invertible sheaf

The material above allows us to define the divisor associated to an invertible sheaf.

Definition 41.24.1. Let (S, δ) be as in Situation 41.7.1. Let X be locally of finite type over S. Assume X is integral and $n = \dim_\delta(X)$. Let $\mathcal{L}$ be an invertible $\mathcal{O}_X$-module.

(1) For any nonzero meromorphic section s of $\mathcal{L}$ we define the *Weil divisor associated to s* as

$$\mathrm{div}_{\mathcal{L}}(s) := \sum \mathrm{ord}_{Z,\mathcal{L}}(s)[Z] \in Z_{n-1}(X)$$

where the sum is over integral closed subschemes $Z \subset X$ of δ-dimension $n-1$.

(2) We define *Weil divisor associated to $\mathcal{L}$*

$$c_1(\mathcal{L}) \cap [X] = \text{class of } \mathrm{div}_{\mathcal{L}}(s) \in A_{n-1}(X)$$

where s is any nonzero meromorphic section of $\mathcal{L}$ over X. This is well defined by Lemma 41.23.3.

There are some cases where it is easy to compute the Weil divisor associated to an invertible sheaf.

Lemma 41.24.2. *Let (S, δ) be as in Situation 41.7.1. Let X be locally of finite type over S. Assume X is integral and $n = \dim_\delta(X)$. Let $\mathcal{L}$ be an invertible $\mathcal{O}_X$-module. Let $s \in \Gamma(X, \mathcal{L})$ be a nonzero global section. Then*

$$div_{\mathcal{L}}(s) = [Z(s)]_{n-1}$$

in $Z_{n-1}(X)$ and

$$c_1(\mathcal{L}) \cap [X] = [Z(s)]_{n-1}$$

in $A_{n-1}(X)$.

Proof. Let $Z \subset X$ be an integral closed subscheme of δ-dimension $n-1$. Let $\xi \in Z$ be its generic point. Choose a generator $s_\xi \in \mathcal{L}_\xi$. Write $s = f s_\xi$ for some $f \in \mathcal{O}_{X,\xi}$. By definition of $Z(s)$, see Divisors, Definition 30.9.18 we see that $Z(s)$ is cut out by a quasi-coherent sheaf of ideals $\mathcal{I} \subset \mathcal{O}_X$ such that $\mathcal{I}_\xi = (f)$. Hence $\mathrm{length}_{\mathcal{O}_{X,x}}(\mathcal{O}_{Z(s),\xi}) = \mathrm{length}_{\mathcal{O}_{X,x}}(\mathcal{O}_{X,\xi}/(f)) = \mathrm{ord}_{\mathcal{O}_{X,x}}(f)$ as desired. □

Lemma 41.24.3. *Let (S, δ) be as in Situation 41.7.1. Let X be locally of finite type over S. Assume X is integral and $n = \dim_\delta(X)$. Let $\mathcal{L}$, $\mathcal{N}$ be invertible $\mathcal{O}_X$-modules. Then*

(1) *Let s, resp. t be a nonzero meromorphic section of $\mathcal{L}$, resp. $\mathcal{N}$. Then st is a nonzero meromorphic section of $\mathcal{L} \otimes \mathcal{N}$, and*
$$div_{\mathcal{L}\otimes\mathcal{N}}(st) = div_{\mathcal{L}}(s) + div_{\mathcal{N}}(t)$$
in $Z_{n-1}(X)$.

(2) *We have*
$$c_1(\mathcal{L}) \cap [X] + c_1(\mathcal{N}) \cap [X] = c_1(\mathcal{L} \otimes_{\mathcal{O}_X} \mathcal{N}) \cap [X]$$
in $A_{n-1}(X)$.

Proof. Let s, resp. t be a nonzero meromorphic section of $\mathcal{L}$, resp. $\mathcal{N}$. Then st is a nonzero meromorphic section of $\mathcal{L}\otimes\mathcal{N}$. Let $Z \subset X$ be an integral closed subscheme of δ-dimension $n-1$. Let $\xi \in Z$ be its generic point. Choose generators $s_\xi \in \mathcal{L}_\xi$, and $t_\xi \in \mathcal{N}_\xi$. Then $s_\xi t_\xi$ is a generator for $(\mathcal{L} \otimes \mathcal{N})_\xi$. So $st/(s_\xi t_\xi) = (s/s_\xi)(t/t_\xi)$. Hence we see that
$$\mathrm{div}_{\mathcal{L}\otimes\mathcal{N},Z}(st) = \mathrm{div}_{\mathcal{L},Z}(s) + \mathrm{div}_{\mathcal{N},Z}(t)$$
by the additivity of the ord_Z function. □

The following lemma will be superseded by the more general Lemma 41.25.4.

Lemma 41.24.4. *Let (S, δ) be as in Situation 41.7.1. Let X, Y be locally of finite type over S. Assume X, Y are integral and $n = \dim_\delta(Y)$. Let $\mathcal{L}$ be an invertible $\mathcal{O}_Y$-module. Let $f : X \to Y$ be a flat morphism of relative dimension r. Let $\mathcal{L}$ be an invertible sheaf on Y. Then*
$$f^*(c_1(\mathcal{L}) \cap [Y]) = c_1(f^*\mathcal{L}) \cap [X]$$
in $A_{n+r-1}(X)$.

Proof. Let s be a nonzero meromorphic section of $\mathcal{L}$. We will show that actually $f^*\mathrm{div}_{\mathcal{L}}(s) = \mathrm{div}_{f^*\mathcal{L}}(f^*s)$ and hence the lemma holds. To see this let $\xi \in Y$ be a point and let $s_\xi \in \mathcal{L}_\xi$ be a generator. Write $s = g s_\xi$ with $g \in R(X)^*$. Then there is an open neighbourhood $V \subset Y$ of ξ such that $s_\xi \in \mathcal{L}(V)$ and such that s_ξ generates $\mathcal{L}|_V$. Hence we see that
$$\mathrm{div}_{\mathcal{L}}(s)|_V = \mathrm{div}(g)|_V.$$
In exactly the same way, since f^*s_ξ generates $\mathcal{L}$ over $f^{-1}(V)$ and since $f^*s = g f^*s_\xi$ we also have
$$\mathrm{div}_{\mathcal{L}}(f^*s)|_{f^{-1}(V)} = \mathrm{div}(g)|_{f^{-1}(V)}.$$
Thus the desired equality of cycles over $f^{-1}(V)$ follows from the corresponding result for pullbacks of principal divisors, see Lemma 41.17.4. □

41.25. Intersecting with Cartier divisors

Definition 41.25.1. Let (S, δ) be as in Situation 41.7.1. Let X be locally of finite type over S. Let $\mathcal{L}$ be an invertible $\mathcal{O}_X$-module. We define, for every integer k, an operation
$$c_1(\mathcal{L}) \cap - : Z_{k+1}(X) \to A_k(X)$$
called *intersection with the first chern class of $\mathcal{L}$.*

(1) Given an integral closed subscheme $i : W \to X$ with $\dim_\delta(W) = k+1$ we define
$$c_1(\mathcal{L}) \cap [W] = i_*(c_1(i^*\mathcal{L}) \cap [W])$$
where the right hand side is defined in Definition 41.24.1.

(2) For a general $(k+1)$-cycle $\alpha = \sum n_i[W_i]$ we set
$$c_1(\mathcal{L}) \cap \alpha = \sum n_i c_1(\mathcal{L}) \cap [W_i]$$

Write each $c_1(\mathcal{L}) \cap W_i = \sum_j n_{i,j}[Z_{i,j}]$ with $\{Z_{i,j}\}_j$ a locally finite sum of integral closed subschemes of W_i. Since $\{W_i\}$ is a locally finite collection of integral closed subschemes on X, it follows easily that $\{Z_{i,j}\}_{i,j}$ is a locally finite collection of closed subschemes of X. Hence $c_1(\mathcal{L}) \cap \alpha = \sum n_i n_{i,j}[Z_{i,j}]$ is a cycle. Another, more convenient, way to think about this is to observe that the morphism $\coprod W_i \to X$ is proper. Hence $c_1(\mathcal{L}) \cap \alpha$ can be viewed as the pushforward of a class in $A_k(\coprod W_i) = \prod A_k(W_i)$. This also explains why the result is well defined up to rational equivalence on X.

The main goal for the next few sections is to show that intersecting with $c_1(\mathcal{L})$ factors through rational equivalence, and is commutative. This is not a triviality.

Lemma 41.25.2. *Let (S, δ) be as in Situation 41.7.1. Let X be locally of finite type over S. Let $\mathcal{L}$, $\mathcal{N}$ be an invertible sheaves on X. Then*
$$c_1(\mathcal{L}) \cap \alpha + c_1(\mathcal{N}) \cap \alpha = c_1(\mathcal{L} \otimes_{\mathcal{O}_X} \mathcal{N}) \cap \alpha$$
in $A_k(X)$ for every $\alpha \in Z_{k-1}(X)$. Moreover, $c_1(\mathcal{O}_X) \cap \alpha = 0$ for all α.

Proof. The additivity follows directly from Lemma 41.24.3 and the definitions. To see that $c_1(\mathcal{O}_X) \cap \alpha = 0$ consider the section $1 \in \Gamma(X, \mathcal{O}_X)$. This restricts to an everywhere nonzero section on any integral closed subscheme $W \subset X$. Hence $c_1(\mathcal{O}_X) \cap [W] = 0$ as desired. □

The following lemma is a useful result in order to compute the intersection product of the c_1 of an invertible sheaf and the cycle associated to a closed subscheme. Recall that $Z(s) \subset X$ denotes the zero scheme of a global section s of an invertible sheaf on a scheme X, see Divisors, Definition 30.9.18.

Lemma 41.25.3. *Let (S, δ) be as in Situation 41.7.1. Let X be locally of finite type over S. Let $\mathcal{L}$ be an invertible $\mathcal{O}_X$-module. Let $Z \subset X$ be a closed subscheme. Assume $\dim_\delta(Z) \leq k+1$. Let $s \in \Gamma(Z, \mathcal{L}|_Z)$. Assume*

(1) *$\dim_\delta(Z(s)) \leq k$, and*
(2) *for every generic point ξ of an irreducible component of $Z(s)$ of dimension k the multiplication by s induces an injection $\mathcal{O}_{Z,\xi} \to (\mathcal{L}|_Z)_\xi$.*

This holds for example if s is a regular section of $\mathcal{L}|_Z$. Then
$$[Z(s)]_k = c_1(\mathcal{L}) \cap [Z]_{k+1}$$
in $A_k(X)$.

Proof. Write
$$[Z]_{k+1} = \sum n_i [W_i]$$
where $W_i \subset Z$ are the irreducible components of Z of δ-dimension $k+1$ and $n_i > 0$. By assumption the restriction $s_i = s|_{W_i} \in \Gamma(W_i, \mathcal{L}|_{W_i})$ is not zero, and hence is a regular section. By Lemma 41.24.2 we see that $[Z(s_i)]_k$ represents $c_1(\mathcal{L}|_{W_i})$. Hence by definition
$$c_1(\mathcal{L}) \cap [Z]_{k+1} = \sum n_i [Z(s_i)]_k$$
In fact, the proof below will show that we have
$$[Z(s)]_k = \sum n_i [Z(s_i)]_k \tag{41.25.3.1}$$

as k-cycles on X.

Let $Z' \subset X$ be an integral closed subscheme of δ-dimension k. Let $\xi' \in Z'$ be its generic point. We want to compare the coefficient n of $[Z']$ in the expression $\sum n_i[Z(s_i)]_k$ with the coefficient m of $[Z']$ in the expression $[Z(s)]_k$. Choose a generator $s_{\xi'} \in \mathcal{L}_\xi$. Let $\mathcal{I} \subset \mathcal{O}_X$ be the ideal sheaf of Z. Write $A = \mathcal{O}_{X,\xi'}$, $L = \mathcal{L}_{\xi'}$ and $I = \mathcal{I}_{\xi'}$. Then $L = As_{\xi'}$ and $L/IL = (A/I)s_{\xi'} = (\mathcal{L}|_Z)_{\xi'}$. Write $s = fs_{\xi'}$ for some (unique) $f \in A/I$. Hypothesis (2) means that $f : A/I \to A/I$ is injective. Since $\dim_\delta(Z) \leq k+1$ and $\dim_\delta(Z') = k$ we have $\dim(A/I) = 0$ or 1. We have

$$m = \text{length}_A(A/(f, I))$$

which is finite in either case.

If $\dim(A/I) = 0$, then $f : A/I \to A/I$ being injective implies that $f \in (A/I)^*$. Hence in this case m is zero. Moreover, the condition $\dim(A/I) = 0$ means that ξ' does not lie on any irreducible component of δ-dimension $k+1$, i.e., $n = 0$ as well.

Now, let $\dim(A/I) = 1$. Since A is a Noetherian local ring there are finitely many minimal primes $\mathfrak{q}_1, \ldots, \mathfrak{q}_t \supset I$ over I. These correspond 1-1 with W_i passing through ξ'. Moreover $n_i = \text{length}_{A_{\mathfrak{q}_i}}((A/I)_{\mathfrak{q}_i})$. Also, the multiplicity of $[Z']$ in $[Z(s_i)]_k$ is $\text{length}_A(A/(f, \mathfrak{q}_i))$. Hence the equation to prove in this case is

$$\text{length}_A(A/(f, I)) = \sum \text{length}_{A_{\mathfrak{q}_i}}((A/I)_{\mathfrak{q}_i})\text{length}_A(A/(f, \mathfrak{q}_i))$$

which follows from Lemma 41.5.6. □

Lemma 41.25.4. *Let (S, δ) be as in Situation 41.7.1. Let X, Y be locally of finite type over S. Let $f : X \to Y$ be a flat morphism of relative dimension r. Let $\mathcal{L}$ be an invertible sheaf on Y. Let α be a k-cycle on Y. Then*

$$f^*(c_1(\mathcal{L}) \cap \alpha) = c_1(f^*\mathcal{L}) \cap f^*\alpha$$

in $A_{k+r-1}(X)$.

Proof. Write $\alpha = \sum n_i[W_i]$. We claim it suffices to show that $f^*(c_1(\mathcal{L}) \cap [W_i]) = c_1(f^*\mathcal{L}) \cap f^*[W_i]$ for each i. Proof of this claim is omitted. (Remarks: it is clear in the quasi-compact case. Something similar happened in the proof of Lemma 41.20.1, and one can copy the method used there here. Another possibility is to check the cycles and rational equivalences used for all W_i combined at each step form a locally finite collection).

Let $W \subset Y$ be an integral closed subscheme of δ-dimension k. We have to show that $f^*(c_1(\mathcal{L}) \cap [W]) = c_1(f^*\mathcal{L}) \cap f^*[W]$. Consider the following fibre product diagram

$$\begin{array}{ccc} W' = W \times_Y X & \longrightarrow & X \\ \downarrow & & \downarrow \\ W & \longrightarrow & Y \end{array}$$

and let $W'_i \subset W'$ be the irreducible components of δ-dimension $k+r$. Write $[W']_{k+r} = \sum n_i[W'_i]$ with $n_i > 0$ as per definition. So $f^*[W] = \sum n_i[W'_i]$. Choose a nonzero meromorphic section s of $\mathcal{L}|_W$. Since each $W'_i \to W$ is dominant we see that $s_i = s|_{W'_i}$ is a nonzero meromorphic section for each i. We claim that we have the following equality of cycles

$$\sum n_i \text{div}_{\mathcal{L}|_{W_i}}(s_i) = f^*\text{div}_{\mathcal{L}|_W}(s)$$

in $Z_{k+r-1}(X)$.

Having formulated the problem as an equality of cycles we may work locally on Y. Hence we may assume Y and also W affine, and $s = p/q$ for some nonzero sections $p \in \Gamma(W, \mathcal{L})$ and $q \in \Gamma(W, \mathcal{O})$. If we can show both

$$\sum n_i \mathrm{div}_{\mathcal{L}|_{W_i}}(p_i) = f^* \mathrm{div}_{\mathcal{L}|_W}(p), \quad \text{and} \quad \sum n_i \mathrm{div}_{\mathcal{O}|_{W_i}}(q_i) = f^* \mathrm{div}_{\mathcal{O}|_W}(q)$$

(with obvious notations) then we win by the additivity, see Lemma 41.24.3. Thus we may assume that $s \in \Gamma(W, \mathcal{L}|_W)$. In this case we may apply the equality (41.25.3.1) obtained in the proof of Lemma 41.25.3 to see that

$$\sum n_i \mathrm{div}_{\mathcal{L}|_{W_i}}(s_i) = [Z(s')]_{k+r-1}$$

where $s' \in f^*\mathcal{L}|_{W'}$ denotes the pullback of s to W'. On the other hand we have

$$f^* \mathrm{div}_{\mathcal{L}|_W}(s) = f^*[Z(s)]_{k-1} = [f^{-1}(Z(s))]_{k+r-1},$$

by Lemmas 41.24.2 and 41.14.4. Since $Z(s') = f^{-1}(Z(s))$ we win. □

Lemma 41.25.5. *Let (S, δ) be as in Situation 41.7.1. Let X, Y be locally of finite type over S. Let $f : X \to Y$ be a proper morphism. Let $\mathcal{L}$ be an invertible sheaf on Y. Let s be a nonzero meromorphic section s of $\mathcal{L}$ on Y. Assume X, Y integral, f dominant, and $\dim_\delta(X) = \dim_\delta(Y)$. Then*

$$f_* \left(div_{f^*\mathcal{L}}(f^* s) \right) = [R(X) : R(Y)] div_{\mathcal{L}}(s).$$

In particular

$$f_*(c_1(f^*\mathcal{L}) \cap [X]) = c_1(\mathcal{L}) \cap f_*[Y].$$

Proof. The last equation follows from the first since $f_*[X] = [R(X) : R(Y)][Y]$ by definition. It turns out that we can re-use Lemma 41.18.1 to prove this. Namely, since we are trying to prove an equality of cycles, we may work locally on Y. Hence we may assume that $\mathcal{L} = \mathcal{O}_Y$. In this case s corresponds to a rational function $g \in R(Y)$, and we are simply trying to prove

$$f_* \left(\mathrm{div}_X(g) \right) = [R(X) : R(Y)] \mathrm{div}_Y(g).$$

Comparing with the result of the aforementioned Lemma 41.18.1 we see this true since $\mathrm{Nm}_{R(X)/R(Y)}(g) = g^{[R(X):R(Y)]}$ as $g \in R(Y)^*$. □

Lemma 41.25.6. *Let (S, δ) be as in Situation 41.7.1. Let X, Y be locally of finite type over S. Let $p : X \to Y$ be a proper morphism. Let $\alpha \in Z_{k+1}(X)$. Let $\mathcal{L}$ be an invertible sheaf on Y. Then*

$$p_*(c_1(p^*\mathcal{L}) \cap \alpha) = c_1(\mathcal{L}) \cap p_*\alpha$$

in $A_k(Y)$.

Proof. Suppose that p has the property that for every integral closed subscheme $W \subset X$ the map $p|_W : W \to Y$ is a closed immersion. Then, by definition of capping with $c_1(\mathcal{L})$ the lemma holds.

We will use this remark to reduce to a special case. Namely, write $\alpha = \sum n_i [W_i]$ with $n_i \neq 0$ and W_i pairwise distinct. Let $W_i' \subset Y$ be the image of W_i (as an

integral closed subscheme). Consider the diagram

$$\begin{array}{ccc} X' = \coprod W_i & \xrightarrow{q} & X \\ {\scriptstyle p'}\downarrow & & \downarrow{\scriptstyle p} \\ Y' = \coprod W_i' & \xrightarrow{q'} & Y. \end{array}$$

Since $\{W_i\}$ is locally finite on X, and p is proper we see that $\{W_i'\}$ is locally finite on Y and that q, q', p' are also proper morphisms. We may think of $\sum n_i[W_i]$ also as a k-cycle $\alpha' \in Z_k(X')$. Clearly $q_*\alpha' = \alpha$. We have $q_*(c_1(q^*p^*\mathcal{L}) \cap \alpha') = c_1(p^*\mathcal{L}) \cap q_*\alpha'$ and $(q')_*(c_1((q')^*\mathcal{L}) \cap p'_*\alpha') = c_1(\mathcal{L}) \cap q'_*p'_*\alpha'$ by the initial remark of the proof. Hence it suffices to prove the lemma for the morphism p' and the cycle $\sum n_i[W_i]$. Clearly, this means we may assume X, Y integral, $f : X \to Y$ dominant and $\alpha = [X]$. In this case the result follows from Lemma 41.25.5. □

41.26. Cartier divisors and K-groups

In this section we describe how the intersection with the first chern class of an invertible sheaf $\mathcal{L}$ corresponds to tensoring with $\mathcal{L} - \mathcal{O}$ in K-groups.

Lemma 41.26.1. *Let (S, δ) be as in Situation 41.7.1. Let X be locally of finite type over S. Let $\mathcal{L}$ be an invertible $\mathcal{O}_X$-module. Let $\mathcal{F}$ be a coherent $\mathcal{O}_X$-module. Let $s \in \Gamma(X, \mathcal{K}_X(\mathcal{L}))$ be a meromorphic section of $\mathcal{L}$. Assume*

(1) $\dim_\delta(X) \leq k + 1$,
(2) *X has no embedded points,*
(3) *$\mathcal{F}$ has no embedded associated points,*
(4) *the support of $\mathcal{F}$ is X, and*
(5) *the section s is regular meromorphic.*

In this situation let $\mathcal{I} \subset \mathcal{O}_X$ be the ideal of denominators of s, see Divisors, Definition 30.15.15. Then we have the following:

(1) *there are short exact sequences*

$$\begin{array}{ccccccccc} 0 & \to & \mathcal{I}\mathcal{F} & \xrightarrow{1} & \mathcal{F} & \to & \mathcal{Q}_1 & \to & 0 \\ 0 & \to & \mathcal{I}\mathcal{F} & \xrightarrow{s} & \mathcal{F} \otimes_{\mathcal{O}_X} \mathcal{L} & \to & \mathcal{Q}_2 & \to & 0 \end{array}$$

(2) *the coherent sheaves $\mathcal{Q}_1$, $\mathcal{Q}_2$ are supported in δ-dimension $\leq k$,*
(3) *the section s restricts to a regular meromorphic section s_i on every irreducible component X_i of X of δ-dimension $k + 1$, and*
(4) *writing $[\mathcal{F}]_{k+1} = \sum m_i[X_i]$ we have*

$$[\mathcal{Q}_2]_k - [\mathcal{Q}_1]_k = \sum m_i (X_i \to X)_* div_{\mathcal{L}|_{X_i}}(s_i)$$

in $Z_k(X)$, in particular

$$[\mathcal{Q}_2]_k - [\mathcal{Q}_1]_k = c_1(\mathcal{L}) \cap [\mathcal{F}]_{k+1}$$

in $A_k(X)$.

Proof. Recall from Divisors, Lemma 30.15.16 the existence of injective maps $1 : \mathcal{I}\mathcal{F} \to \mathcal{F}$ and $s : \mathcal{I}\mathcal{F} \to \mathcal{F} \otimes_{\mathcal{O}_X} \mathcal{L}$ whose cokernels are supported on a closed nowhere dense subsets T. Denote $\mathcal{Q}_i$ there cokernels as in the lemma. We conclude that $\dim_\delta(\mathrm{Supp}(\mathcal{Q}_i)) \leq k$. By Divisors, Lemmas 30.15.4 and 30.15.12 the pullbacks s_i are defined and are regular meromorphic sections for $\mathcal{L}|_{X_i}$. The equality of cycles

in (4) implies the equality of cycle classes in (4). Hence the only remaining thing to show is that

$$[\mathcal{Q}_2]_k - [\mathcal{Q}_1]_k = \sum m_i (X_i \to X)_* \mathrm{div}_{\mathcal{L}|_{X_i}}(s_i)$$

holds in $Z_k(X)$. To see this, let $Z \subset X$ be an integral closed subscheme of δ-dimension k. Let $\xi \in Z$ be the generic point. Let $A = \mathcal{O}_{X,\xi}$ and $M = \mathcal{F}_\xi$. Moreover, choose a generator $s_\xi \in \mathcal{L}_\xi$. Then we can write $s = (a/b)s_\xi$ where $a, b \in A$ are nonzerodivisors. In this case $I = \mathcal{I}_\xi = \{x \in A \mid x(a/b) \in A\}$. In this case the coefficient of $[Z]$ in the left hand side is

$$\mathrm{length}_A(M/(a/b)IM) - \mathrm{length}_A(M/IM)$$

and the coefficient of $[Z]$ in the right hand side is

$$\sum \mathrm{length}_{A_{\mathfrak{q}_i}}(M_{\mathfrak{q}_i}) \mathrm{ord}_{A/\mathfrak{q}_i}(a/b)$$

where $\mathfrak{q}_1, \ldots, \mathfrak{q}_t$ are the minimal primes of the 1-dimensional local ring A. Hence the result follows from Lemma 41.5.7. □

Lemma 41.26.2. *Let (S, δ) be as in Situation 41.7.1. Let X be locally of finite type over S. Let $\mathcal{L}$ be an invertible $\mathcal{O}_X$-module. Let $\mathcal{F}$ be a coherent $\mathcal{O}_X$-module. Assume $\dim_\delta(\mathit{Support}(\mathcal{F})) \leq k+1$. Then the element*

$$[\mathcal{F} \otimes_{\mathcal{O}_X} \mathcal{L}] - [\mathcal{F}] \in K_0(\mathit{Coh}_{\leq k+1}(X)/\mathit{Coh}_{\leq k-1}(X))$$

lies in the subgroup $B_k(X)$ of Lemma 41.22.5 and maps to the element $c_1(\mathcal{L}) \cap [\mathcal{F}]_{k+1}$ via the map $B_k(X) \to A_k(X)$.

Proof. Let

$$0 \to \mathcal{K} \to \mathcal{F} \to \mathcal{F}' \to 0$$

be the short exact sequence constructed in Divisors, Lemma 30.4.5. This in particular means that $\mathcal{F}'$ has no embedded associated points. Since the support of $\mathcal{K}$ is nowhere dense in the support of $\mathcal{F}$ we see that $\dim_\delta(\mathrm{Supp}(\mathcal{K})) \leq k$. We may re-apply Divisors, Lemma 30.4.5 starting with $\mathcal{K}$ to get a short exact sequence

$$0 \to \mathcal{K}'' \to \mathcal{K} \to \mathcal{K}' \to 0$$

where now $\dim_\delta(\mathrm{Supp}(\mathcal{K}'')) < k$ and $\mathcal{K}'$ has no embedded associated points. Suppose we can prove the lemma for the coherent sheaves $\mathcal{F}'$ and $\mathcal{K}'$. Then we see from the equations

$$[\mathcal{F}]_{k+1} = [\mathcal{F}']_{k+1} + [\mathcal{K}']_{k+1} + [\mathcal{K}'']_{k+1}$$

(use Lemma 41.10.4),

$$[\mathcal{F} \otimes_{\mathcal{O}_X} \mathcal{L}] - [\mathcal{F}] = [\mathcal{F}' \otimes_{\mathcal{O}_X} \mathcal{L}] - [\mathcal{F}'] + [\mathcal{K}' \otimes_{\mathcal{O}_X} \mathcal{L}] - [\mathcal{K}'] + [\mathcal{K}'' \otimes_{\mathcal{O}_X} \mathcal{L}] - [\mathcal{K}'']$$

(use the $\otimes \mathcal{L}$ is exact) and the trivial vanishing of $[\mathcal{K}'']_{k+1}$ and $[\mathcal{K}'' \otimes_{\mathcal{O}_X} \mathcal{L}] - [\mathcal{K}'']$ in $K_0(\mathrm{Coh}_{\leq k+1}(X)/\mathrm{Coh}_{\leq k-1}(X))$ that the result holds for $\mathcal{F}$. What this means is that we may assume that the sheaf $\mathcal{F}$ has no embedded associated points.

Assume X, $\mathcal{F}$ as in the lemma, and assume in addition that $\mathcal{F}$ has no embedded associated points. Consider the sheaf of ideals $\mathcal{I} \subset \mathcal{O}_X$, the corresponding closed subscheme $i : Z \to X$ and the coherent $\mathcal{O}_Z$-module $\mathcal{G}$ constructed in Divisors, Lemma 30.4.6. Recall that Z is a locally Noetherian scheme without embedded points, $\mathcal{G}$ is a coherent sheaf without embedded associated points, with $\mathrm{Supp}(\mathcal{G}) = Z$ and such that $i_*\mathcal{G} = \mathcal{F}$. Moreover, set $\mathcal{N} = \mathcal{L}|_Z$.

By Divisors, Lemma 30.15.13 the invertible sheaf $\mathcal{N}$ has a regular meromorphic section s over Z. Let us denote $\mathcal{J} \subset \mathcal{O}_Z$ the sheaf of denominators of s. By Lemma 41.26.1 there exist short exact sequences

$$\begin{array}{ccccccccc} 0 & \to & \mathcal{J}\mathcal{G} & \xrightarrow{1} & \mathcal{G} & \to & \mathcal{Q}_1 & \to & 0 \\ 0 & \to & \mathcal{J}\mathcal{G} & \xrightarrow{s} & \mathcal{G} \otimes_{\mathcal{O}_Z} \mathcal{N} & \to & \mathcal{Q}_2 & \to & 0 \end{array}$$

such that $\dim_\delta(\mathrm{Supp}(\mathcal{Q}_i)) \leq k$ and such that the cycle $[\mathcal{Q}_2]_k - [\mathcal{Q}_1]_k$ is a representative of $c_1(\mathcal{N}) \cap [\mathcal{G}]_{k+1}$. We see (using the fact that $i_*(\mathcal{G} \otimes \mathcal{N}) = \mathcal{F} \otimes \mathcal{L}$ by the projection formula, see Cohomology, Lemma 20.8.2) that

$$[\mathcal{F} \otimes_{\mathcal{O}_X} \mathcal{L}] - [\mathcal{F}] = [i_*\mathcal{Q}_2] - [i_*\mathcal{Q}_1]$$

in $K_0(\mathrm{Coh}_{\leq k+1}(X)/\mathrm{Coh}_{\leq k-1}(X))$. This already shows that $[\mathcal{F} \otimes_{\mathcal{O}_X} \mathcal{L}] - [\mathcal{F}]$ is an element of $B_k(X)$. Moreover we have

$$\begin{aligned} [i_*\mathcal{Q}_2]_k - [i_*\mathcal{Q}_1]_k &= i_*\left([\mathcal{Q}_2]_k - [\mathcal{Q}_1]_k\right) \\ &= i_*\left(c_1(\mathcal{N}) \cap [\mathcal{G}]_{k+1}\right) \\ &= c_1(\mathcal{L}) \cap i_*[\mathcal{G}]_{k+1} \\ &= c_1(\mathcal{L}) \cap [\mathcal{F}]_{k+1} \end{aligned}$$

by the above and Lemmas 41.25.6 and 41.12.3. And this agree with the image of the element under $B_k(X) \to A_k(X)$ by definition. Hence the lemma is proved. □

41.27. Blowing up lemmas

In this section we prove some lemmas on representing Cartier divisors by suitable effective Cartier divisors on blow-ups. These lemmas can be found in [**Ful98**, Section 2.4]. We have adapted the formulation so they also work in the non-finite type setting. It may happen that the morphism b of Lemma 41.27.7 is a composition of infinitely many blow ups, but over any given quasi-compact open $W \subset X$ one needs only finitely many blow-ups (and this is the result of loc. cit.).

Lemma 41.27.1. *Let (S, δ) be as in Situation 41.7.1. Let X, Y be locally of finite type over S. Let $f : X \to Y$ be a proper morphism. Let $D \subset Y$ be an effective Cartier divisor. Assume X, Y integral, $n = \dim_\delta(X) = \dim_\delta(Y)$ and f dominant. Then*

$$f_*[f^{-1}(D)]_{n-1} = [R(X) : R(Y)][D]_{n-1}.$$

In particular if f is birational then $f_*[f^{-1}(D)]_{n-1} = [D]_{n-1}$.

Proof. Immediate from Lemma 41.25.5 and the fact that D is the zero scheme of the canonical section 1_D of $\mathcal{O}_X(D)$. □

Lemma 41.27.2. *Let (S, δ) be as in Situation 41.7.1. Let X be locally of finite type over S. Assume X integral with $\dim_\delta(X) = n$. Let $\mathcal{L}$ be an invertible $\mathcal{O}_X$-module. Let s be a nonzero meromorphic section of $\mathcal{L}$. Let $U \subset X$ be the maximal open subscheme such that s corresponds to a section of $\mathcal{L}$ over U. There exists a projective morphism*

$$\pi : X' \longrightarrow X$$

such that

(1) *X' is integral,*
(2) *$\pi|_{\pi^{-1}(U)} : \pi^{-1}(U) \to U$ is an isomorphism,*

(3) *there exist effective Cartier divisors $D, E \subset X'$ such that*
$$\pi^*\mathcal{L} = \mathcal{O}_{X'}(D - E),$$
(4) *the meromorphic section s corresponds, via the isomorphism above, to the meromorphic section $1_D \otimes (1_E)^{-1}$ (see Divisors, Definition 30.9.14),*
(5) *we have*
$$\pi_*([D]_{n-1} - [E]_{n-1}) = div_{\mathcal{L}}(s)$$
in $Z_{n-1}(X)$.

Proof. Let $\mathcal{I} \subset \mathcal{O}_X$ be the quasi-coherent ideal sheaf of denominators of s. Namely, we declare a local section f of $\mathcal{O}_X$ to be a local section of $\mathcal{I}$ if and only if fs is a local section of $\mathcal{L}$. On any affine open $U = \mathrm{Spec}(A)$ of X write $\mathcal{L}|_U = \widetilde{L}$ for some invertible A-module L. Then A is a Noetherian domain with fraction field $K = R(X)$ and we may think of $s|_U$ as an element of $L \otimes_A K$ (see Divisors, Lemma 30.15.7). Let $I = \{x \in A \mid xs \in L\}$. Then we see that $\mathcal{I}|_U = \widetilde{I}$ (details omitted) and hence $\mathcal{I}$ is quasi-coherent.

Consider the closed subscheme $Z \subset X$ defined by $\mathcal{I}$. It is clear that $U = X \setminus Z$. This suggests we should blow up Z. Let
$$\pi : X' = \underline{\mathrm{Proj}}_X\left(\bigoplus\nolimits_{n\geq 0} \mathcal{I}^n\right) \longrightarrow X$$
be the blowing up of X along Z. The quasi-coherent sheaf of $\mathcal{O}_X$-algebras $\bigoplus_{n\geq 0}\mathcal{I}^n$ is generated in degree 1 over $\mathcal{O}_X$. Moreover, the degree 1 part is a coherent $\mathcal{O}_X$-module, in particular of finite type. Hence we see that π is projective and $\mathcal{O}_{X'}(1)$ is relatively very ample.

By Divisors, Lemma 30.18.7 we have X' is integral. By Divisors, Lemma 30.18.4 there exists an effective Cartier divisor $E \subset X'$ such that $\pi^{-1}\mathcal{I} \cdot \mathcal{O}_{X'} = \mathcal{I}_E$. Also, by the same lemma we see that $\pi^{-1}(U) \cong U$.

Denote s' the pullback of the meromorphic section s to a meromorphic section of $\mathcal{L}' = \pi^*\mathcal{L}$ over X'. It follows from the fact that $\mathcal{I}s \subset \mathcal{L}$ that $\mathcal{I}_E s' \subset \mathcal{L}'$. In other words, s' gives rise to an $\mathcal{O}_{X'}$-linear map $\mathcal{I}_E \to \mathcal{L}'$, or in other words a section $t \in \mathcal{L}' \otimes \mathcal{O}_{X'}(E)$. By Divisors, Lemma 30.9.20 we obtain a unique effective Cartier divisor $D \subset X'$ such that $\mathcal{L}' \otimes \mathcal{O}_{X'}(E) \cong \mathcal{O}_{X'}(D)$ with t corresponding to 1_D. Reversing this procedure we conclude that $\mathcal{L}' = \mathcal{O}_{X'}(-E) \cong \mathcal{O}_{X'}(D)$ with s' corresponding to $1_D \otimes 1_E^{-1}$ as in (4).

We still have to prove (5). By Lemma 41.25.5 we have
$$\pi_*(\mathrm{div}_{\mathcal{L}'}(s')) = \mathrm{div}_{\mathcal{L}}(s).$$
Hence it suffices to show that $\mathrm{div}_{\mathcal{L}'}(s') = [D]_{n-1} - [E]_{n-1}$. This follows from the equality $s' = 1_D \otimes 1_E^{-1}$ and additivity, see Lemma 41.24.3. □

Definition 41.27.3. Let (S, δ) be as in Situation 41.7.1. Let X be locally of finite type over S. Assume X integral and $\dim_\delta(X) = n$. Let D_1, D_2 be two effective Cartier divisors in X. Let $Z \subset X$ be an integral closed subscheme with $\dim_\delta(Z) = n - 1$. The *ϵ-invariant* of this situation is
$$\epsilon_Z(D_1, D_2) = n_Z \cdot m_Z$$
where n_Z, resp. m_Z is the coefficient of Z in the $(n-1)$-cycle $[D_1]_{n-1}$, resp. $[D_2]_{n-1}$.

Lemma 41.27.4. *Let (S, δ) be as in Situation 41.7.1. Let X be locally of finite type over S. Assume X integral and $\dim_\delta(X) = n$. Let D_1, D_2 be two effective Cartier divisors in X. Let Z be an open and closed subscheme of the scheme $D_1 \cap D_2$. Assume $\dim_\delta(D_1 \cap D_2 \setminus Z) \leq n-2$. Then there exists a morphism $b : X' \to X$, and Cartier divisors D'_1, D'_2, E on X' with the following properties*

(1) *X' is integral,*
(2) *b is projective,*
(3) *b is the blow up of X in the closed subscheme Z,*
(4) *$E = b^{-1}(Z)$,*
(5) *$b^{-1}(D_1) = D'_1 + E$, and $b^{-1}D_2 = D'_2 + E$,*
(6) *$\dim_\delta(D'_1 \cap D'_2) \leq n-2$, and if $Z = D_1 \cap D_2$ then $D'_1 \cap D'_2 = \emptyset$,*
(7) *for every integral closed subscheme W' with $\dim_\delta(W') = n-1$ we have*
   (a) *if $\epsilon_{W'}(D'_1, E) > 0$, then setting $W = b(W')$ we have $\dim_\delta(W) = n-1$ and*
   $$\epsilon_{W'}(D'_1, E) < \epsilon_W(D_1, D_2),$$
   (b) *if $\epsilon_{W'}(D'_2, E) > 0$, then setting $W = b(W')$ we have $\dim_\delta(W) = n-1$ and*
   $$\epsilon_{W'}(D'_2, E) < \epsilon_W(D_1, D_2),$$

Proof. Note that the quasi-coherent ideal sheaf $\mathcal{I} = \mathcal{I}_{D_1} + \mathcal{I}_{D_2}$ defines the scheme theoretic intersection $D_1 \cap D_2 \subset X$. Since Z is a union of connected components of $D_1 \cap D_2$ we see that for every $z \in Z$ the kernel of $\mathcal{O}_{X,z} \to \mathcal{O}_{Z,z}$ is equal to $\mathcal{I}_z$. Let $b : X' \to X$ be the blow up of X in Z. (So Zariski locally around Z it is the blow up of X in $\mathcal{I}$.) Denote $E = b^{-1}(Z)$ the corresponding effective Cartier divisor, see Divisors, Lemma 30.18.4. Since $Z \subset D_1$ we have $E \subset f^{-1}(D_1)$ and hence $D_1 = D'_1 + E$ for some effective Cartier divisor $D'_1 \subset X'$, see Divisors, Lemma 30.9.8. Similarly $D_2 = D'_2 + E$. This takes care of assertions (1) – (5).

Note that if W' is as in (7) (a) or (7) (b), then the image W of W' is contained in $D_1 \cap D_2$. If W is not contained in Z, then b is an isomorphism at the generic point of W and we see that $\dim_\delta(W) = \dim_\delta(W') = n-1$ which contradicts the assumption that $\dim_\delta(D_1 \cap D_2 \setminus Z) \leq n-2$. Hence $W \subset Z$. This means that to prove (6) and (7) we may work locally around Z on X.

Thus we may assume that $X = \mathrm{Spec}(A)$ with A a Noetherian domain, and $D_1 = \mathrm{Spec}(A/a)$, $D_2 = \mathrm{Spec}(A/b)$ and $Z = D_1 \cap D_2$. Set $I = (a, b)$. Since A is a domain and $a, b \neq 0$ we can cover the blow up by two patches, namely $U = \mathrm{Spec}(A[s]/(as - b))$ and $V = \mathrm{Spec}(A[t]/(bt - a))$. These patches are glued using the isomorphism $A[s, s^{-1}]/(as - b) \cong A[t, t^{-1}]/(bt - a)$ which maps s to t^{-1}. The effective Cartier divisor E is described by $\mathrm{Spec}(A[s]/(as - b, a)) \subset U$ and $\mathrm{Spec}(A[t]/(bt - a, b)) \subset V$. The closed subscheme D'_1 corresponds to $\mathrm{Spec}(A[t]/(bt - a, t)) \subset U$. The closed subscheme D'_2 corresponds to $\mathrm{Spec}(A[s]/(as - b, s)) \subset V$. Since "$ts = 1$" we see that $D'_1 \cap D'_2 = \emptyset$.

Suppose we have a prime $\mathfrak{q} \subset A[s]/(as - b)$ of height one with $s, a \in \mathfrak{q}$. Let $\mathfrak{p} \subset A$ be the corresponding prime of A. Observe that $a, b \in \mathfrak{p}$. By the dimension formula we see that $\dim(A_\mathfrak{p}) = 1$ as well. The final assertion to be shown is that

$$\mathrm{ord}_{A_\mathfrak{p}}(a)\, \mathrm{ord}_{A_\mathfrak{p}}(b) > \mathrm{ord}_{B_\mathfrak{q}}(a)\, \mathrm{ord}_{B_\mathfrak{q}}(s)$$

where $B = A[s]/(as - b)$. By Algebra, Lemma 10.118.1 we have $\text{ord}_{A_\mathfrak{p}}(x) \geq \text{ord}_{B_\mathfrak{q}}(x)$ for $x = a, b$. Since $\text{ord}_{B_\mathfrak{q}}(s) > 0$ we win by additivity of the ord function and the fact that $as = b$. □

Definition 41.27.5. Let X be a scheme. Let $\{D_i\}_{i \in I}$ be a locally finite collection of effective Cartier divisors on X. Suppose given a function $I \to \mathbf{Z}_{\geq 0}$, $i \mapsto n_i$. The *sum of the effective Cartier divisors* $D = \sum n_i D_i$, is the unique effective Cartier divisor $D \subset X$ such that on any quasi-compact open $U \subset X$ we have $D|_U = \sum_{D_i \cap U \neq \emptyset} n_i D_i|_U$ is the sum as in Divisors, Definition 30.9.6.

Lemma 41.27.6. *Let (S, δ) be as in Situation 41.7.1. Let X be locally of finite type over S. Assume X integral and $\dim_\delta(X) = n$. Let $\{D_i\}_{i \in I}$ be a locally finite collection of effective Cartier divisors on X. Suppose given $n_i \geq 0$ for $i \in I$. Then*

$$[D]_{n-1} = \sum\nolimits_i n_i [D_i]_{n-1}$$

in $Z_{n-1}(X)$.

Proof. Since we are proving an equality of cycles we may work locally on X. Hence this reduces to a finite sum, and by induction to a sum of two effective Cartier divisors $D = D_1 + D_2$. By Lemma 41.24.2 we see that $D_1 = \text{div}_{\mathcal{O}_X(D_1)}(1_{D_1})$ where 1_{D_1} denotes the canonical section of $\mathcal{O}_X(D_1)$. Of course we have the same statement for D_2 and D. Since $1_D = 1_{D_1} \otimes 1_{D_2}$ via the identification $\mathcal{O}_X(D) = \mathcal{O}_X(D_1) \otimes \mathcal{O}_X(D_2)$ we win by Lemma 41.24.3. □

Lemma 41.27.7. *Let (S, δ) be as in Situation 41.7.1. Let X be locally of finite type over S. Assume X integral and $\dim_\delta(X) = d$. Let $\{D_i\}_{i \in I}$ be a locally finite collection of effective Cartier divisors on X. Assume that for all $\{i, j, k\} \subset I$, $\#\{i, j, k\} = 3$ we have $D_i \cap D_j \cap D_k = \emptyset$. Then there exist*

1. *an open subscheme $U \subset X$ with $\dim_\delta(X \setminus U) \leq d - 3$,*
2. *a morphism $b : U' \to U$, and*
3. *effective Cartier divisors $\{D'_j\}_{j \in J}$ on U'*

with the following properties:

1. *b is proper morphism $b : U' \to U$,*
2. *U' is integral,*
3. *b is an isomorphism over the complement of the union of the pairwise intersections of the $D_i|_U$,*
4. *$\{D'_j\}_{j \in J}$ is a locally finite collection of effective Cartier divisors on U',*
5. *$\dim_\delta(D'_j \cap D'_{j'}) \leq d - 2$ if $j \neq j'$, and*
6. *$b^{-1}(D_i|_U) = \sum n_{ij} D'_j$ for certain $n_{ij} \geq 0$.*

Moreover, if X is quasi-compact, then we may assume $U = X$ in the above.

Proof. Let us first prove this in the quasi-compact case, since it is perhaps the most interesting case. In this case we produce inductively a sequence of blowups

$$X = X_0 \xleftarrow{b_0} X_1 \xleftarrow{b_1} X_2 \leftarrow \ldots$$

and finite sets of effective Cartier divisors $\{D_{n,i}\}_{i \in I_n}$. At each stage these will have the property that any triple intersection $D_{n,i} \cap D_{n,j} \cap D_{n,k}$ is empty. Moreover, for each $n \geq 0$ we will have $I_{n+1} = I_n \coprod P(I_n)$ where $P(I_n)$ denotes the set of pairs of elements of I_n. Finally, we will have

$$b_n^{-1}(D_{n,i}) = D_{n+1,i} + \sum\nolimits_{i' \in I_n, i' \neq i} D_{n+1,\{i,i'\}}$$

We conclude that for each $n \geq 0$ we have $(b_0 \circ \ldots \circ b_n)^{-1}(D_i)$ is a nonnegative integer combination of the divisors $D_{n+1,j}$, $j \in I_{n+1}$.

To start the induction we set $X_0 = X$ and $I_0 = I$ and $D_{0,i} = D_i$.

Given $(X_n, \{D_{n,i}\}_{i \in I_n})$ let X_{n+1} be the blow up of X_n in the closed subscheme $Z_n = \bigcup_{\{i,i'\} \in P(I_n)} D_{n,i} \cap D_{n,i'}$. Note that the closed subschemes $D_{n,i} \cap D_{n,i'}$ are pairwise disjoint by our assumption on triple intersections. In other words we may write $Z_n = \coprod_{\{i,i'\} \in P(I_n)} D_{n,i} \cap D_{n,i'}$. Moreover, in a Zariski neighbourhood of $D_{n,i} \cap D_{n,i'}$ the morphism b_n is equal to the blow up of the scheme X_n in the closed subscheme $D_{n,i} \cap D_{n,i'}$, and the results of Lemma 41.27.4 apply. Hence setting $D_{n+1,\{i,i'\}} = b_n^{-1}(D_i \cap D_{i'})$ we get an effective Cartier divisor. The Cartier divisors $D_{n+1,\{i,i'\}}$ are pairwise disjoint. Clearly we have $b_n^{-1}(D_{n,i}) \supset D_{n+1,\{i,i'\}}$ for every $i' \in I_n$, $i' \neq i$. Hence, applying Divisors, Lemma 30.9.8 we see that indeed $b^{-1}(D_{n,i}) = D_{n+1,i} + \sum_{i' \in I_n, i' \neq i} D_{n+1,\{i,i'\}}$ for some effective Cartier divisor $D_{n+1,i}$ on X_{n+1}. In a neighbourhood of $D_{n+1,\{i,i'\}}$ these divisors $D_{n+1,i}$ play the role of the primed divisors of Lemma 41.27.4. In particular we conclude that $D_{n+1,i} \cap D_{n+1,i'} = \emptyset$ if $i \neq i'$, $i, i' \in I_n$ by part (6) of Lemma 41.27.4. This already implies that triple intersections of the divisors $D_{n+1,i}$ are zero.

OK, and at this point we can use the quasi-compactness of X to conclude that the invariant

(41.27.7.1)
$$\epsilon(X, \{D_i\}_{i \in I}) = \max\{\epsilon_Z(D_i, D_{i'}) \mid Z \subset X, \dim_\delta(Z) = d - 1, \{i, i'\} \in P(I)\}$$

is finite, since after all each D_i has at most finitely many irreducible components. We claim that for some n the invariant $\epsilon(X_n, \{D_{n,i}\}_{i \in I_n})$ is zero. Namely, if not then by Lemma 41.27.4 we have a strictly decreasing sequence

$$\epsilon(X, \{D_i\}_{i \in I}) = \epsilon(X_0, \{D_{0,i}\}_{i \in I_0}) > \epsilon(X_1, \{D_{1,i}\}_{i \in I_1}) > \ldots$$

of positive integers which is a contradiction. Take n with invariant $\epsilon(X_n, \{D_{n,i}\}_{i \in I_n})$ equal to zero. This means that there is no integral closed subscheme $Z \subset X_n$ and no pair of indices $i, i' \in I_n$ such that $\epsilon_Z(D_{n,i}, D_{n,i'}) > 0$. In other words, $\dim_\delta(D_{n,i}, D_{n,i'}) \leq d - 2$ for all pairs $\{i, i'\} \in P(I_n)$ as desired.

Next, we come to the general case where we no longer assume that the scheme X is quasi-compact. The problem with the idea from the first part of the proof is that we may get and infinite sequence of blow ups with centers dominating a fixed point of X. In order to avoid this we cut out suitable closed subsets of codimension ≥ 3 at each stage. Namely, we will construct by induction a sequence of morphisms having the following shape

$$\begin{array}{ccccccc}
X = X_0 & & & & & & \\
\uparrow{\scriptstyle j_0} & & & & & & \\
U_0 & \xleftarrow{b_0} & X_1 & & & & \\
 & & \uparrow{\scriptstyle j_1} & & & & \\
 & & U_1 & \xleftarrow{b_1} & X_2 & & \\
 & & & & \uparrow{\scriptstyle j_2} & & \\
 & & & & U_2 & \xleftarrow{b_2} & X_3
\end{array}$$

Each of the morphisms $j_n : U_n \to X_n$ will be an open immersion. Each of the morphisms $b_n : X_{n+1} \to U_n$ will be a proper birational morphism of integral schemes. As in the quasi-compact case we will have effective Cartier divisors $\{D_{n,i}\}_{i \in I_n}$ on X_n. At each stage these will have the property that any triple intersection $D_{n,i} \cap D_{n,j} \cap D_{n,k}$ is empty. Moreover, for each $n \geq 0$ we will have $I_{n+1} = I_n \coprod P(I_n)$ where $P(I_n)$ denotes the set of pairs of elements of I_n. Finally, we will arrange it so that

$$b_n^{-1}(D_{n,i}|_{U_n}) = D_{n+1,i} + \sum\nolimits_{i' \in I_n, i' \neq i} D_{n+1,\{i,i'\}}$$

We start the induction by setting $X_0 = X$, $I_0 = I$ and $D_{0,i} = D_i$.

Given $(X_n, \{D_{n,i}\})$ we construct the open subscheme U_n as follows. For each pair $\{i, i'\} \in P(I_n)$ consider the closed subscheme $D_{n,i} \cap D_{n,i'}$. This has "good" irreducible components which have δ-dimension $d-2$ and "bad" irreducible components which have δ-dimension $d-1$. Let us set

$$\text{Bad}(i,i') = \bigcup\nolimits_{W \subset D_{n,i} \cap D_{n,i'} \text{ irred. comp. with } \dim_\delta(W) = d-1} W$$

and similarly

$$\text{Good}(i,i') = \bigcup\nolimits_{W \subset D_{n,i} \cap D_{n,i'} \text{ irred. comp. with } \dim_\delta(W) = d-2} W.$$

Then $D_{n,i} \cap D_{n,i'} = \text{Bad}(i,i') \cup \text{Good}(i,i')$ and moreover we have $\dim_\delta(\text{Bad}(i,i') \cap \text{Good}(i,i')) \leq d-3$. Here is our choice of U_n:

$$U_n = X_n \setminus \bigcup\nolimits_{\{i,i'\} \in P(I_n)} \text{Bad}(i,i') \cap \text{Good}(i,i').$$

By our condition on triple intersections of the divisors $D_{n,i}$ we see that the union is actually a disjoint union. Moreover, we see that (as a scheme)

$$D_{n,i}|_{U_n} \cap D_{n,i'}|_{U_n} = Z_{n,i,i'} \coprod G_{n,i,i'}$$

where $Z_{n,i,i'}$ is δ-equidimension of dimension $d-1$ and $G_{n,i,i'}$ is δ-equidimensional of dimension $d-2$. (So topologically $Z_{n,i,i'}$ is the union of the bad components but throw out intersections with good components.) Finally we set

$$Z_n = \bigcup\nolimits_{\{i,i'\} \in P(I_n)} Z_{n,i,i'} = \coprod\nolimits_{\{i,i'\} \in P(I_n)} Z_{n,i,i'},$$

and we let $b_n : X_{n+1} \to X_n$ be the blow up in Z_n. Note that Lemma 41.27.4 applies to the morphism $b_n : X_{n+1} \to X_n$ locally around each of the loci $D_{n,i}|_{U_n} \cap D_{n,i'}|_{U_n}$. Hence, exactly as in the first part of the proof we obtain effective Cartier divisors $D_{n+1,\{i,i'\}}$ for $\{i,i'\} \in P(I_n)$ and effective Cartier divisors $D_{n+1,i}$ for $i \in I_n$ such that $b_n^{-1}(D_{n,i}|_{U_n}) = D_{n+1,i} + \sum_{i' \in I_n, i' \neq i} D_{n+1,\{i,i'\}}$. For each n denote $\pi_n : X_n \to X$ the morphism obtained as the composition $j_0 \circ \ldots \circ j_{n-1} \circ b_{n-1}$.

Claim: given any quasi-compact open $V \subset X$ for all sufficiently large n the maps

$$\pi_n^{-1}(V) \leftarrow \pi_{n+1}^{-1}(V) \leftarrow \ldots$$

are all isomorphisms. Namely, if the map $\pi_n^{-1}(V) \leftarrow \pi_{n+1}^{-1}(V)$ is not an isomorphism, then $Z_{n,i,i'} \cap \pi_n^{-1}(V) \neq \emptyset$ for some $\{i,i'\} \in P(I_n)$. Hence there exists an irreducible component $W \subset D_{n,i} \cap D_{n,i'}$ with $\dim_\delta(W) = d-1$. In particular we see that $\epsilon_W(D_{n,i}, D_{n,i'}) > 0$. Applying Lemma 41.27.4 repeatedly we see that

$$\epsilon_W(D_{n,i}, D_{n,i'}) < \epsilon(V, \{D_i|_V\}) - n$$

with $\epsilon(V, \{D_i|_V\})$ as in (41.27.7.1). Since V is quasi-compact, we have $\epsilon(V, \{D_i|_V\}) < \infty$ and taking $n > \epsilon(V, \{D_i|_V\})$ we see the result.

Note that by construction the difference $X_n \setminus U_n$ has $\dim_\delta(X_n \setminus U_n) \leq d-3$. Let $T_n = \pi_n(X_n \setminus U_n)$ be its image in X. Traversing in the diagram of maps above using each b_n is closed it follows that $T_0 \cup \ldots \cup T_n$ is a closed subset of X for each n. Any $t \in T_n$ satisfies $\delta(t) \leq d-3$ by construction. Hence $\overline{T_n} \subset X$ is a closed subset with $\dim_\delta(T_n) \leq d-3$. By the claim above we see that for any quasi-compact open $V \subset X$ we have $T_n \cap V \neq \emptyset$ for at most finitely many n. Hence $\{\overline{T_n}\}_{n\geq 0}$ is a locally finite collection of closed subsets, and we may set $U = X \setminus \bigcup \overline{T_n}$. This will be U as in the lemma.

Note that $U_n \cap \pi_n^{-1}(U) = \pi_n^{-1}(U)$ by construction of U. Hence all the morphisms

$$b_n : \pi_{n+1}^{-1}(U) \longrightarrow \pi_n^{-1}(U)$$

are proper. Moreover, by the claim they eventually become isomorphisms over each quasi-compact open of X. Hence we can define

$$U' = \lim_n \pi_n^{-1}(U).$$

The induced morphism $b : U' \to U$ is proper since this is local on U, and over each compact open the limit stabilizes. Similarly we set $J = \bigcup_{n\geq 0} I_n$ using the inclusions $I_n \to I_{n+1}$ from the construction. For $j \in J$ choose an n_0 such that j corresponds to $i \in I_{n_0}$ and define $D'_j = \lim_{n\geq n_0} D_{n,i}$. Again this makes sense as locally over X the morphisms stabilize. The other claims of the lemma are verified as in the case of a quasi-compact X. □

41.28. Intersecting with effective Cartier divisors

To be able to prove the commutativity of intersection products we need a little more precision in terms of supports of the cycles. Here is the relevant notion.

Definition 41.28.1. Let (S, δ) be as in Situation 41.7.1. Let X be locally of finite type over S. Let D be an effective Cartier divisor on X, and denote $i : D \to X$ the closed immersion. We define, for every integer k, a *Gysin homomorphism*

$$i^* : Z_{k+1}(X) \to A_k(D).$$

(1) Given a integral closed subscheme $W \subset X$ with $\dim_\delta(W) = k+1$ we define
   (a) if $W \not\subset D$, then $i^*[W] = [D \cap W]_k$ as a k-cycle on D, and
   (b) if $W \subset D$, then $i^*[W] = i'_*(c_1(\mathcal{O}_X(D)|_W) \cap [W])$, where $i' : W \to D$ is the induced closed immersion.

(2) For a general $(k+1)$-cycle $\alpha = \sum n_j[W_j]$ we set

$$i^*\alpha = \sum n_j i^*[W_j]$$

(3) We denote $D \cdot \alpha = i_* i^* \alpha$ the pushforward of the class to a class on X.

In fact, as we will see later, this Gysin homomorphism i^* can be viewed as an example of a non-flat pullback. Thus we will sometimes informally call the class $i^*\alpha$ the *pullback* of the class α.

Lemma 41.28.2. *Let (S, δ) be as in Situation 41.7.1. Let X be locally of finite type over S. Let D be an effective Cartier divisor on X. Let α be a $(k+1)$-cycle on X. Then $D \cdot \alpha = c_1(\mathcal{O}_X(D)) \cap \alpha$ in $A_k(X)$.*

Proof. Write $\alpha = \sum n_j[W_j]$ where $i_j : W_j \to X$ are integral closed subschemes with $\dim_\delta(W_j) = k$. Since D is the zero scheme of the canonical section 1_D of $\mathcal{O}_X(D)$ we see that $D \cap W_j$ is the zero scheme of the restriction $1_D|_{W_j}$. Hence for each j such that $W_j \not\subset D$ we have $c_1(\mathcal{O}_X(D)) \cap [W_j] = [D \cap W_j]_k$ by Lemma 41.25.3. So we have

$$c_1(\mathcal{O}_X(D)) \cap \alpha = \sum\nolimits_{W_j \not\subset D} n_j[D \cap W_j]_k + \sum\nolimits_{W_j \subset D} n_j i_{j,*}(c_1(\mathcal{O}_X(D)|_{W_j}) \cap [W_j])$$

in $A_k(X)$ by Definition 41.25.1. The right hand side matches (termwise) the push-forward of the class $i^*\alpha$ on D from Definition 41.28.1. Hence we win. □

The following lemma will be superseded later.

Lemma 41.28.3. *Let (S, δ) be as in Situation 41.7.1. Let X be locally of finite type over S. Let D be an effective Cartier divisor on X. Let $W \subset X$ be a closed subscheme such that $D' = W \cap D$ is an effective Cartier divisor on W.*

$$\begin{array}{ccc} D' & \xrightarrow{i'} & W \\ {\scriptstyle i''}\downarrow & & \downarrow \\ D & \xrightarrow{i} & X \end{array}$$

For any $(k+1)$-cycle on W we have $i^*\alpha = (i'')_*(i')^*\alpha$ in $A_k(D)$.

Proof. Suppose $\alpha = [Z]$ for some integral closed subscheme $Z \subset W$. In case $Z \not\subset D$ we have $Z \cap D' = Z \cap D$ scheme theoretically. Hence the equality holds as cycles. In case $Z \subset D$ we also have $Z \subset D'$ and the equality holds since $\mathcal{O}_X(D)|_Z \cong \mathcal{O}_W(D')|_Z$ and the definition of i^* and $(i')^*$ in these cases. □

Lemma 41.28.4. *Let (S, δ) be as in Situation 41.7.1. Let X be locally of finite type over S. Let $i : D \to X$ be an effective Cartier divisor on X.*

1. *Let $Z \subset X$ be a closed subscheme such that $\dim_\delta(Z) \leq k+1$ and such that $D \cap Z$ is an effective Cartier divisor on Z. Then $i^*[Z]_{k+1} = [D \cap Z]_k$.*
2. *Let $\mathcal{F}$ be a coherent sheaf on X such that $\dim_\delta(\mathit{Support}(\mathcal{F})) \leq k+1$ and $1_D : \mathcal{F} \to \mathcal{F} \otimes_{\mathcal{O}_X} \mathcal{O}_X(D)$ is injective. Then*
$$i^*[\mathcal{F}]_{k+1} = [i^*\mathcal{F}]_k$$
in $A_k(D)$.

Proof. Assume $Z \subset X$ as in (1). Then set $\mathcal{F} = \mathcal{O}_Z$. The assumption that $D \cap Z$ is an effective Cartier divisor is equivalent to the assumption that $1_D : \mathcal{F} \to \mathcal{F} \otimes_{\mathcal{O}_X} \mathcal{O}_X(D)$ is injective. Moreover $[Z]_{k+1} = [\mathcal{F}]_{k+1}]$ and $[D \cap Z]_k = [\mathcal{O}_{D\cap Z}]_k = [i^*\mathcal{F}]_k$. See Lemma 41.10.3. Hence part (1) follows from part (2).

Write $[\mathcal{F}]_{k+1} = \sum m_j[W_j]$ with $m_j > 0$ and pairwise distinct integral closed subschemes $W_j \subset X$ of δ-dimension $k+1$. The assumption that $1_D : \mathcal{F} \to \mathcal{F} \otimes_{\mathcal{O}_X} \mathcal{O}_X(D)$ is injective implies that $W_j \not\subset D$ for all j. By definition we see that

$$i^*[\mathcal{F}]_{k+1} = \sum [D \cap W_j]_k.$$

We claim that

$$\sum [D \cap W_j]_k = [i^*\mathcal{F}]_k$$

as cycles. Let $Z \subset D$ be an integral closed subscheme of δ-dimension k. Let $\xi \in Z$ be its generic point. Let $A = \mathcal{O}_{X,\xi}$. Let $M = \mathcal{F}_\xi$. Let $f \in A$ be an

element generating the ideal of D, i.e., such that $\mathcal{O}_{D,\xi} = A/fA$. By assumption $\dim(M) = 1$, $f : M \to M$ is injective, and $\text{length}_A(M/fM) < \infty$. Moreover, $\text{length}_A(M/fM)$ is the coefficient of $[Z]$ in $[i^*\mathcal{F}]_k$. On the other hand, let $\mathfrak{q}_1, \ldots, \mathfrak{q}_t$ be the minimal primes in the support of M. Then

$$\sum \text{length}_{A_{\mathfrak{q}_i}}(M_{\mathfrak{q}_i})\text{ord}_{A/\mathfrak{q}_i}(f)$$

is the coefficient of $[Z]$ in $\sum [D \cap W_j]_k$. Hence we see the equality by Lemma 41.5.6. □

Lemma 41.28.5. *Let (S, δ) be as in Situation 41.7.1. Let X be locally of finite type over S. Let $\{i_j : D_j \to X\}_{j\in J}$ be a locally finite collection of effective Cartier divisors on X. Let $n_j > 0$, $j \in J$. Set $D = \sum_{j\in J} n_j D_j$, and denote $i : D \to X$ the inclusion morphism. Let $\alpha \in Z_{k+1}(X)$. Then*

$$p : \coprod\nolimits_{j\in J} D_j \longrightarrow D$$

is proper and

$$i^*\alpha = p_*\left(\sum n_j i_j^*\alpha\right)$$

in $A_k(D)$.

Proof. The proof of this lemma is made a bit longer than expected by a subtlety concerning infinite sums of rational equivalences. In the quasi-compact case the family D_j is finite and the result is altogether easy and a straightforward consequence of Lemmas 41.24.2 and 41.24.3 and the definitions.

The morphism p is proper since the family $\{D_j\}_{j\in J}$ is locally finite. Write $\alpha = \sum_{a\in A} m_a[W_a]$ with $W_a \subset X$ an integral closed subscheme of δ-dimension $k + 1$. Denote $i_a : W_a \to X$ the closed immersion. We assume that $m_a \neq 0$ for all $a \in A$ such that $\{W_a\}_{a\in A}$ is locally finite on X.

Observe that by Definition 41.28.1 the class $i^*\alpha$ is the class of a cycle $\sum m_a\beta_a$ for certain $\beta_a \in Z_k(W_a \cap D)$. Namely, if $W_a \not\subset D$ then $\beta_a = [D \cap W_a]_k$ and if $W_a \subset D$, then β_a is a cycle representing $c_1(\mathcal{O}_X(D)) \cap [W_a]$.

For each $a \in A$ write $J = J_{a,1} \coprod J_{a,2} \coprod J_{a,3}$ where

(1) $j \in J_{a,1}$ if and only if $W_a \cap D_j = \emptyset$,
(2) $j \in J_{a,2}$ if and only if $W_a \neq W_a \cap D_1 \neq \emptyset$, and
(3) $j \in J_{a,3}$ if and only if $W_a \subset D_j$.

Since the family $\{D_j\}$ is locally finite we see that $J_{a,3}$ is a finite set. For every $a \in A$ and $j \in J$ we choose a cycle $\beta_{a,j} \in Z_k(W_a \cap D_j)$ as follows

(1) if $j \in J_{a,1}$ we set $\beta_{a,j} = 0$,
(2) if $j \in J_{a,2}$ we set $\beta_{a,j} = [D_j \cap W_a]_k$, and
(3) if $j \in J_{a,3}$ we choose $\beta_{a,j} \in Z_k(W_a)$ representing $c_1(i_a^*\mathcal{O}_X(D_j)) \cap [W_j]$.

We claim that

$$\beta_a \sim_{rat} \sum\nolimits_{j\in J} n_j\beta_{a,j}$$

in $A_k(W_a \cap D)$.

Case I: $W_a \not\subset D$. In this case $J_{a,3} = \emptyset$. Thus it suffices to show that $[D \cap W_a]_k = \sum n_j[D_j \cap W_a]_k$ as cycles. This is Lemma 41.27.6.

Case II: $W_a \subset D$. In this case β_a is a cycle representing $c_1(i_a^*\mathcal{O}_X(D)) \cap [W_a]$. Write $D = D_{a,1} + D_{a,2} + D_{a,3}$ with $D_{a,s} = \sum_{j \in J_{a,s}} n_j D_j$. By Lemma 41.24.3 we have

$$\begin{aligned} c_1(i_a^*\mathcal{O}_X(D)) \cap [W_a] &= c_1(i_a^*\mathcal{O}_X(D_{a,1})) \cap [W_a] + c_1(i_a^*\mathcal{O}_X(D_{a,2})) \cap [W_a] \\ &\quad + c_1(i_a^*\mathcal{O}_X(D_{a,3})) \cap [W_a]. \end{aligned}$$

It is clear that the first term of the sum is zero. Since $J_{a,3}$ is finite we see that the last term agrees with $\sum_{j \in J_{a,3}} n_j c_1(i_a^*\mathcal{L}_j) \cap [W_a]$, see Lemma 41.24.3. This is represented by $\sum_{j \in J_{a,3}} n_j \beta_{a,j}$. Finally, by Case I we see that the middle term is represented by the cycle $\sum_{j \in J_{a,2}} n_j [D_j \cap W_a]_k = \sum_{j \in J_{a,2}} n_j \beta_{a,j}$. Whence the claim in this case.

At this point we are ready to finish the proof of the lemma. Namely, we have $i^*D \sim_{rat} \sum m_a \beta_a$ by our choice of β_a. For each a we have $\beta_a \sim_{rat} \sum_j \beta_{a,j}$ with the rational equivalence taking place on $D \cap W_a$. Since the collection of closed subschemes $D \cap W_a$ is locally finite on D, we see that also $\sum m_a \beta_a \sim_{rat} \sum_{a,j} m_a \beta_{a,j}$ on D! (See Remark 41.19.4.) Ok, and now it is clear that $\sum_a m_a \beta_{a,j}$ (viewed as a cycle on D_j) represents $i_j^*\alpha$ and hence $\sum_{a,j} m_a \beta_{a,j}$ represents $p_* \sum_j i_j^*\alpha$ and we win. □

Lemma 41.28.6. *Let (S, δ) be as in Situation 41.7.1. Let X be locally of finite type over S. Assume X integral and $\dim_\delta(X) = n$. Let D, D' be effective Cartier divisors on X. Assume $\dim_\delta(D \cap D') = n - 2$. Let $i : D \to X$, resp. $i' : D' \to X$ be the corresponding closed immersions. Then*

1. *there exists a cycle $\alpha \in Z_{n-2}(D \cap D')$ whose pushforward to D represents $i^*[D']_{n-1} \in A_{n-2}(D)$ and whose pushforward to D' represents $(i')^*[D]_{n-1} \in A_{n-2}(D')$, and*
2. *we have*
$$D \cdot [D']_{n-1} = D' \cdot [D]_{n-1}$$
in $A_{n-2}(X)$.

Proof. Part (3) is a trivial consequence of parts (1) and (2). Because of symmetry we only need to prove (1). Let us write $[D]_{n-1} = \sum n_a [Z_a]$ and $[D']_{n-1} = \sum m_b [Z_b]$ with Z_a the irreducible components of D and $[Z_b]$ the irreducible components of D'. According to Definition 41.28.1, we have $i^*D' = \sum m_b i^*[Z_b]$ and $(i')^*D = \sum n_a (i')^*[Z_a]$. By assumption, none of the irreducible components Z_b is contained in D, and hence $i^*[Z_b] = [Z_b \cap D]_{n-2}$ by definition. Similarly $(i')^*[Z_a] = [Z_a \cap D']_{n-2}$. Hence we are trying to prove the equality of cycles

$$\sum n_a [Z_a \cap D']_{n-2} = \sum m_b [Z_b \cap D]_{n-2}$$

which are indeed supported on $D \cap D'$. Let $W \subset X$ be an integral closed subscheme with $\dim_\delta(W) = n - 2$. Let $\xi \in W$ be its generic point. Set $R = \mathcal{O}_{X,\xi}$. It is a Noetherian local domain. Note that $\dim(R) = 2$. Let $f \in R$, resp. $f' \in R$ be an element defining the ideal of D, resp. D'. By assumption $\dim(R/(f, f')) = 0$. Let $\mathfrak{q}'_1, \ldots, \mathfrak{q}'_t \subset R$ be the minimal primes over (f'), let $\mathfrak{q}_1, \ldots, \mathfrak{q}_s \subset R$ be the minimal primes over (f). The equality above comes down to the equality

$$\sum_{i=1,\ldots,s} \text{length}_{R_{\mathfrak{q}_i}}(R_{\mathfrak{q}_i}/(f)) \text{ord}_{R/\mathfrak{q}_i}(f') = \sum_{j=1,\ldots,t} \text{length}_{R_{\mathfrak{q}'_j}}(R_{\mathfrak{q}'_j}/(f')) \text{ord}_{R/\mathfrak{q}_j}(f).$$

By Lemma 41.5.5 applied with $M = R/(f)$ the left hand side of this equation is equal to

$$\text{length}_R(R/(f, f')) - \text{length}_R(\text{Ker}(f' : R/(f) \to R/(f)))$$

OK, and now we note that $\text{Ker}(f' : R/(f) \to R/(f))$ is canonically isomorphic to $((f) \cap (f'))/(ff')$ via the map $x \bmod (f) \mapsto f'x \bmod (ff')$. Hence the left hand side is

$$\text{length}_R(R/(f, f')) - \text{length}_R((f) \cap (f')/(ff'))$$

Since this is symmetric in f and f' we win. □

Lemma 41.28.7. *Let (S, δ) be as in Situation 41.7.1. Let X be locally of finite type over S. Assume X integral and $\dim_\delta(X) = n$. Let $\{D_j\}_{j\in J}$ be a locally finite collection of effective Cartier divisors on X. Let $n_j, m_j \geq 0$ be collections of nonnegative integers. Set $D = \sum n_j D_j$ and $D' = \sum m_j D_j$. Assume that $\dim_\delta(D_j \cap D_{j'}) = n - 2$ for every $j \neq j'$. Then $D \cdot [D']_{n-1} = D' \cdot [D]_{n-1}$ in $A_{n-2}(X)$.*

Proof. This lemma is a trivial consequence of Lemmas 41.27.6 and 41.28.6 in case the sums are finite, e.g., if X is quasi-compact. Hence we suggest the reader skip the proof.

Here is the proof in the general case. Let $i_j : D_j \to X$ be the closed immersions Let $p : \coprod D_j \to X$ denote coproduct of the morphisms i_j. Let $\{Z_a\}_{a\in A}$ be the collection of irreducible components of $\bigcup D_j$. For each j we write

$$[D_j]_{n-1} = \sum d_{j,a}[Z_a].$$

By Lemma 41.27.6 we have

$$[D]_{n-1} = \sum n_j d_{j,a}[Z_a], \quad [D']_{n-1} = \sum m_j d_{j,a}[Z_a].$$

By Lemma 41.28.5 we have

$$D \cdot [D']_{n-1} = p_* \left(\sum n_j i_j^*[D']_{n-1} \right), \quad D' \cdot [D]_{n-1} = p_* \left(\sum m_{j'} i_{j'}^*[D]_{n-1} \right).$$

As in the definition of the Gysin homomorphisms (see Definition 41.28.1) we choose cycles $\beta_{a,j}$ on $D_j \cap Z_a$ representing $i_j^*[Z_a]$. (Note that in fact $\beta_{a,j} = [D_j \cap Z_a]_{n-2}$ if Z_a is not contained in D_j, i.e., there is no choice in that case.) Now since p is a closed immersion when restricted to each of the D_j we can (and we will) view $\beta_{a,j}$ as a cycle on X. Plugging in the formulas for $[D]_{n-1}$ and $[D']_{n-1}$ obtained above we see that

$$D \cdot [D']_{n-1} = \sum\nolimits_{j,j',a} n_j m_{j'} d_{j',a} \beta_{a,j}, \quad D' \cdot [D]_{n-1} = \sum\nolimits_{j,j',a} m_{j'} n_j d_{j,a} \beta_{a,j'}.$$

Moreover, with the same conventions we also have

$$D_j \cdot [D_{j'}]_{n-1} = \sum d_{j',a} \beta_{a,j}.$$

In these terms Lemma 41.28.6 (see also its proof) says that for $j \neq j'$ the cycles $\sum d_{j',a}\beta_{a,j}$ and $\sum d_{j,a}\beta_{a,j'}$ are equal as cycles! Hence we see that

$$\begin{aligned} D \cdot [D']_{n-1} &= \sum\nolimits_{j,j',a} n_j m_{j'} d_{j',a}\beta_{a,j} \\ &= \sum\nolimits_{j \neq j'} n_j m_{j'} \left(\sum\nolimits_a d_{j',a}\beta_{a,j}\right) + \sum\nolimits_{j,a} n_j m_j d_{j,a}\beta_{a,j} \\ &= \sum\nolimits_{j \neq j'} n_j m_{j'} \left(\sum\nolimits_a d_{j,a}\beta_{a,j'}\right) + \sum\nolimits_{j,a} n_j m_j d_{j,a}\beta_{a,j} \\ &= \sum\nolimits_{j,j',a} m_{j'} n_j d_{j,a}\beta_{a,j'} \\ &= D' \cdot [D]_{n-1} \end{aligned}$$

and we win. □

Here is the key lemma of this chapter. A stronger version of this lemma asserts that $D \cdot [D']_{n-1} = D' \cdot [D]_{n-1}$ holds in $A_{n-2}(D \cap D')$ for suitable representatives of the dot products involved. The first proof of the lemma together with Lemmas 41.28.5, 41.28.6, and 41.28.7 can be modified to show this (see [**Ful98**]). It is not so clear how to modify the second proof to prove the refined version. An application of the refined version is a proof that the Gysin homomorphism factors through rational equivalence. We will show this by another method later.

Lemma 41.28.8. *Let (S, δ) be as in Situation 41.7.1. Let X be locally of finite type over S. Assume X integral and $\dim_\delta(X) = n$. Let D, D' be effective Cartier divisors on X. Then*

$$D \cdot [D']_{n-1} = D' \cdot [D]_{n-1}$$

in $A_{n-2}(X)$.

First proof of Lemma 41.28.8. First, let us prove this in case X is quasi-compact. In this case, apply Lemma 41.27.7 to X and the two element set $\{D, D'\}$ of effective Cartier divisors. Thus we get a proper morphism $b : X' \to X$, a finite collection of effective Cartier divisors $D'_j \subset X'$ intersecting pairwise in codimension ≥ 2, with $b^{-1}(D) = \sum n_j D'_j$, and $b^{-1}(D') = \sum m_j D'_j$. Note that $b_*[b^{-1}(D)]_{n-1} = [D]_{n-1}$ in $Z_{n-1}(X)$ and similarly for D', see Lemma 41.27.1. Hence, by Lemma 41.25.6 we have

$$D \cdot [D']_{n-1} = b_*\left(b^{-1}(D) \cdot [b^{-1}(D')]_{n-1}\right)$$

in $A_{n-2}(X)$ and similarly for the other term. Hence the lemma follows from the equality $b^{-1}(D) \cdot [b^{-1}(D')]_{n-1} = b^{-1}(D') \cdot [b^{-1}(D)]_{n-1}$ in $A_{n-2}(X')$ of Lemma 41.28.7.

Note that in the proof above, each referenced lemma works also in the general case (when X is not assumed quasi-compact). The only minor change in the general case is that the morphism $b : U' \to U$ we get from applying Lemma 41.27.7 has as its target an open $U \subset X$ whose complement has codimension ≥ 3. Hence by Lemma 41.19.2 we see that $A_{n-2}(U) = A_{n-2}(X)$ and after replacing X by U the rest of the proof goes through unchanged. □

Second proof of Lemma 41.28.8. Let $\mathcal{I} = \mathcal{O}_X(-D)$ and $\mathcal{I}' = \mathcal{O}_X(-D')$ be the invertible ideal sheaves of D and D'. We denote $\mathcal{I}_{D'} = \mathcal{I} \otimes_{\mathcal{O}_X} \mathcal{O}_{D'}$ and $\mathcal{I}'_D = \mathcal{I}' \otimes_{\mathcal{O}_X} \mathcal{O}_D$. We can restrict the inclusion map $\mathcal{I} \to \mathcal{O}_X$ to D' to get a map

$$\varphi : \mathcal{I}_{D'} \longrightarrow \mathcal{O}_{D'}$$

and similarly

$$\psi : \mathcal{I}'_D \longrightarrow \mathcal{O}_D$$

It is clear that

$$\mathrm{Coker}(\varphi) \cong \mathcal{O}_{D \cap D'} \cong \mathrm{Coker}(\psi)$$

and

$$\mathrm{Ker}(\varphi) \cong \frac{\mathcal{I} \cap \mathcal{I}'}{\mathcal{I}\mathcal{I}'} \cong \mathrm{Ker}(\psi).$$

Hence we see that

$$\gamma = [\mathcal{I}_{D'}] - [\mathcal{O}_{D'}] = [\mathcal{I}'_D] - [\mathcal{O}_D]$$

in $K_0(\mathrm{Coh}_{\leq n-1}(X))$. On the other hand it is clear that

$$[\mathcal{I}'_D]_{n-1} = [D]_{n-1}, \quad [\mathcal{I}_{D'}]_{n-1} = [D']_{n-1}.$$

and that

$$\mathcal{O}_X(D') \otimes \mathcal{I}'_D = \mathcal{O}_D, \quad \mathcal{O}_X(D) \otimes \mathcal{I}_{D'} = \mathcal{O}_{D'}.$$

By Lemma 41.26.2 (applied two times) this means that the element γ is an element of $B_{n-2}(X)$, and maps to both $c_1(\mathcal{O}_X(D')) \cap [D]_{n-1}$ and to $c_1(\mathcal{O}_X(D)) \cap [D']_{n-1}$ and we win (since the map $B_{n-2}(X) \to A_{n-2}(X)$ is well defined – which is the key to this proof). □

41.29. Commutativity

At this point we can start using the material above and start proving more interesting results.

Lemma 41.29.1. *Let (S, δ) be as in Situation 41.7.1. Let X be locally of finite type over S. Assume X integral and $\dim_\delta(X) = n$. Let $\mathcal{L}$, $\mathcal{N}$ be invertible on X. Choose a nonzero meromorphic section s of $\mathcal{L}$ and a nonzero meromorphic section t of $\mathcal{N}$. Set $\alpha = div_{\mathcal{L}}(s)$ and $\beta = div_{\mathcal{N}}(t)$. Then*

$$c_1(\mathcal{N}) \cap \alpha = c_1(\mathcal{L}) \cap \beta$$

in $A_{n-2}(X)$.

Proof. By Lemma 41.27.2 (applied twice) there exists a proper morphism $\pi : X' \to X$ and effective Cartier divisors D_1, E_1, D_2, E_2 on X' such that

$$b^*\mathcal{L} = \mathcal{O}_{X'}(D_1 - E_1), \quad b^*\mathcal{N} = \mathcal{O}_{X'}(D_2 - E_2),$$

and such that

$$\alpha = \pi_*([D_1]_{n-1} - [E_1]_{n-1}), \quad \beta = \pi_*([D_2]_{n-1} - [E_2]_{n-1}).$$

By the projection formula of Lemma 41.25.6 and the additivity of Lemma 41.25.2 it is enough to show the equality

$$c_1(\mathcal{O}_{X'}(D_1)) \cap [D_2]_{n-1} = c_1(\mathcal{O}_{X'}(D_2)) \cap [D_1]_{n-1}$$

and three other similar equalities involving D_i and E_j. By Lemma 41.28.2 this is the same as showing that $D_1 \cdot [D_2]_{n-1} = D_2 \cdot [D_1]_{n-1}$ and so on. Thus the result follows from Lemma 41.28.8. □

Lemma 41.29.2. *Let (S, δ) be as in Situation 41.7.1. Let X be locally of finite type over S. Let $\mathcal{L}$ be invertible on X. The operation $\alpha \mapsto c_1(\mathcal{L}) \cap \alpha$ factors through rational equivalence to give an operation*

$$c_1(\mathcal{L}) \cap - : A_{k+1}(X) \to A_k(X)$$

Proof. Let $\alpha \in Z_{k+1}(X)$, and $\alpha \sim_{rat} 0$. We have to show that $c_1(\mathcal{L}) \cap \alpha$ as defined in Definition 41.25.1 is zero. By Definition 41.19.1 there exists a locally finite family $\{W_j\}$ of integral closed subschemes with $\dim_\delta(W_j) = k+2$ and rational functions $f_j \in R(W_j)^*$ such that

$$\alpha = \sum (i_j)_* \mathrm{div}_{W_j}(f_j)$$

Note that $p : \coprod W_j \to X$ is a proper morphism, and hence $\alpha = p_*\alpha'$ where $\alpha' \in Z_{k+1}(\coprod W_j)$ is the sum of the principal divisors $\mathrm{div}_{W_j}(f_j)$. By the projection formula (Lemma 41.25.6) we have $c_1(\mathcal{L}) \cap \alpha = p_*(c_1(p^*\mathcal{L}) \cap \alpha')$. Hence it suffices to show that each $c_1(\mathcal{L}|_{W_j}) \cap \mathrm{div}_{W_j}(f_j)$ is zero. In other words we may assume that X is integral and $\alpha = \mathrm{div}_X(f)$ for some $f \in R(X)^*$.

Assume X is integral and $\alpha = \mathrm{div}_X(f)$ for some $f \in R(X)^*$. We can think of f as a regular meromorphic section of the invertible sheaf $\mathcal{N} = \mathcal{O}_X$. Choose a meromorphic section s of $\mathcal{L}$ and denote $\beta = \mathrm{div}_{\mathcal{L}}(s)$. By Lemma 41.29.1 we conclude that

$$c_1(\mathcal{L}) \cap \alpha = c_1(\mathcal{O}_X) \cap \beta.$$

However, by Lemma 41.25.2 we see that the right hand side is zero in $A_k(X)$ as desired. □

For any integer $s \geq 0$ we will denote

$$c_1(\mathcal{L})^s \cap - : A_{k+s}(X) \to A_k(X)$$

the s-fold iterate of the operation $c_1(\mathcal{L}) \cap -$. This makes sense by the lemma above.

Lemma 41.29.3. *Let (S, δ) be as in Situation 41.7.1. Let X be locally of finite type over S. Let $\mathcal{L}$, $\mathcal{N}$ be invertible on X. For any $\alpha \in A_{k+2}(X)$ we have*

$$c_1(\mathcal{L}) \cap c_1(\mathcal{N}) \cap \alpha = c_1(\mathcal{N}) \cap c_1(\mathcal{L}) \cap \alpha$$

as elements of $A_k(X)$.

Proof. Write $\alpha = \sum m_j[Z_j]$ for some locally finite collection of integral closed subschemes $Z_j \subset X$ with $\dim_\delta(Z_j) = k+2$. Consider the proper morphism $p : \coprod Z_j \to X$. Set $\alpha' = \sum m_j[Z_j]$ as a $(k+2)$-cycle on $\coprod Z_j$. By several applications of Lemma 41.25.6 we see that $c_1(\mathcal{L}) \cap c_1(\mathcal{N}) \cap \alpha = p_*(c_1(p^*\mathcal{L}) \cap c_1(p^*\mathcal{N}) \cap \alpha')$ and $c_1(\mathcal{N}) \cap c_1(\mathcal{L}) \cap \alpha = p_*(c_1(p^*\mathcal{N}) \cap c_1(p^*\mathcal{L}) \cap \alpha')$. Hence it suffices to prove the formula in case X is integral and $\alpha = [X]$. In this case the result follows from Lemma 41.29.1 and the definitions. □

41.30. Gysin homomorphisms

We want to show the Gysin homomorphisms factor through rational equivalence. One method (see [**Ful98**]) is to prove a more precise version of the key Lemma 41.28.8 keeping track of supports. Having obtained this one can find analogues of the lemmas of Section 41.29 for the Gysin homomorphism and get the result. We will use another method.

Lemma 41.30.1. *Let (S, δ) be as in Situation 41.7.1. Let X be locally of finite type over S. Let X be integral and $n = \dim_\delta(X)$. Let $a \in \Gamma(X, \mathcal{O}_X)$ be a nonzero function. Let $i : D = Z(a) \to X$ be the closed immersion of the zero scheme of a. Let $f \in R(X)^*$. In this case $i^*div_X(f) = 0$ in $A_{n-2}(D)$.*

Proof. Write $\mathrm{div}_X(f) = \sum n_j[Z_j]$ for some integral closed subschemes $Z_j \subset X$ of δ-dimension $n-1$. We may assume that the family $\{Z_j\}_{j\in J}$ is locally finite and that $f \in \Gamma(U, \mathcal{O}_U^*)$ where $U = X \setminus \bigcup Z_j$ (see Lemma 41.16.3 and its proof).

Write $J = J_1 \coprod J_2$ where $J_1 = \{j \in J \mid Z_j \subset D\}$. Note that $\mathcal{O}_X(D) \cong \mathcal{O}_X$ because a^{-1} is a trivializing global section. Hence by Definition 41.28.1 of i^* we see that $i^*\mathrm{div}_X(f)$ is represented by

$$\sum\nolimits_{j\in J_2} n_j[D \cap Z_j]_{n-2}.$$

Namely, the terms involving $c_1(\mathcal{O}_X(D)|_{Z_j}) \cap Z_j$ may be dropped since $c_1(\mathcal{O}) \cap -$ is the zero operation anyway (see Lemma 41.25.2).

For each j let $\xi_j \in Z_j$ be its generic point. Let $B_j = \mathcal{O}_{X,\xi_j}$, which has residue field $\kappa_j = \kappa(\xi_j) = R(Z_j)$. For $j \in J_1$, let

$$f_j = d_{B_j}(f, a)$$

be the tame symbol, see Definition 41.4.5. We claim that we have the following equality of cycles

$$\sum\nolimits_{j\in J_2} n_j[D \cap Z_j]_{n-2} = \sum\nolimits_{j\in J_1} (Z_j \to D)_*\mathrm{div}_{Z_j}(f_j)$$

on D. Indeed, note that $[D\cap Z_j]_{n-2} = \mathrm{div}_{Z_j}(a)$. Hence $n_j[D\cap Z_j]_{n-2} = \mathrm{div}_{Z_j}(a^{n_j})$. Since $n_j = \mathrm{ord}_{B_j}(f)$ we see that in fact also $n_j[D \cap Z_j]_{n-2} = \mathrm{div}_{Z_j}(d_{B_j}(a, f))$, as a is a unit in B_j see Lemma 41.4.6. Note that $d_{B_j}(f, a) = d_{B_j}(a, f)^{-1}$, see Lemma 41.4.4. Hence altogether we are trying to show that

$$\sum\nolimits_{j\in J} (Z_j \to D)_*\mathrm{div}_{Z_j}(d_{B_j}(a, f)) = 0$$

as an $(n-2)$-cycle. Consider any codimension 2 integral closed subscheme $W \subset X$ with generic point $\zeta \in X$. Set $A = \mathcal{O}_{X,\zeta}$. Applying Lemma 41.6.1 to (A, a, f) we see that the coefficient of $[W]$ in the expression above is zero as desired. □

Lemma 41.30.2. *Let (S, δ) be as in Situation 41.7.1. Let X be locally of finite type over S. Let X be integral and $n = \dim_\delta(X)$. Let $i : D \to X$ be an effective Cartier divisor. Let $f \in R(X)^*$. In this case $i^*div_X(f) = 0$ in $A_{n-2}(D)$.*

Proof. This proof is a repeat of the proof of Lemma 41.30.1. So make sure you've read that one first.

Write $\mathrm{div}_X(f) = \sum n_j[Z_j]$ for some integral closed subschemes $Z_j \subset X$ of δ-dimension $n-1$. We may assume that the family $\{Z_j\}_{j\in J}$ is locally finite and that $f \in \Gamma(U, \mathcal{O}_U^*)$ where $U = X \setminus \bigcup Z_j$ (see Lemma 41.16.3 and its proof).

Write $J = J_1 \coprod J_2$ where $J_1 = \{j \in J \mid Z_j \subset D\}$. For each j let $\xi_j \in Z_j$ be its generic point. Let us write $\mathcal{L} = \mathcal{O}_X(D)$. Choose $\tilde{s}_j \in \mathcal{L}_{\xi_j}$ a generator. Denote $s_j \in \mathcal{L}_{\xi_j} \otimes \kappa(\xi_j)$ the corresponding nonzero meromorphic section of $\mathcal{L}|_{Z_j}$. Then by Definition 41.28.1 of i^* we see that $i^*\mathrm{div}_X(f)$ is represented by the cycle

$$\sum\nolimits_{j\in J_2} n_j[D \cap Z_j]_{n-2} + \sum\nolimits_{j\in J_2} n_j\mathrm{div}_{\mathcal{L}|_{Z_j}}(s_j)$$

on D. Our goal is to show that this is rationally equivalent to zero on D.

Let $B_j = \mathcal{O}_{X,\xi_j}$, which has residue field $\kappa_j = \kappa(\xi_j) = R(Z_j)$. Write $s = a_j\tilde{s}_j$ for some $a_j \in B_j$. For $j \in J_1$ let

$$f_j = d_{B_j}(f, a_j) \in \kappa_j^* = R(Z_j)^*$$

be the tame symbol, see Definition 41.4.5. We claim that we have the following equality of cycles

$$\sum\nolimits_{j\in J_2} n_j[D\cap Z_j]_{n-2} + \sum\nolimits_{j\in J_2} n_j \mathrm{div}_{\mathcal{L}|_{Z_j}}(s_j) = \sum\nolimits_{j\in J_1} (Z_j \to D)_* \mathrm{div}_{Z_j}(f_j)$$

on D. This will clearly prove the lemma.

Note that for $j \in J_2$ we have $[D\cap Z_j]_{n-2} = \mathrm{div}_{\mathcal{L}|_{Z_j}}(s|_{Z_j})$. Since $s|_{Z_j} = a_j|_{Z_j} s_j$ we see that $[D\cap Z_j]_{n-2} = \mathrm{div}_{\mathcal{L}|_{Z_j}}(s_j) + \mathrm{div}_{Z_j}(a_j|_{Z_j})$. Hence, still for $j \in J_2$, we have

$$n_j[D\cap Z_j]_{n-2} = n_j \mathrm{div}_{\mathcal{L}|_{Z_j}}(s_j) + \mathrm{div}_{Z_j}((a_j|_{Z_j})^{n_j})$$

Since $n_j = \mathrm{ord}_{B_j}(f)$ we see that $\mathrm{div}_{Z_j}((a_j|_{Z_j})^{n_j}) = \mathrm{div}_{Z_j}(d_{B_j}(a_j, f))$, as a_j is a unit in B_j (since $j \in J_2$), see Lemma 41.4.6. Note that $d_{B_j}(f, a_j) = d_{B_j}(a_j, f)^{-1}$, see Lemma 41.4.4. Hence altogether we are trying to show that

$$\sum\nolimits_{j\in J} n_j \mathrm{div}_{\mathcal{L}|_{Z_j}}(s_j) = \sum\nolimits_{j\in J} (Z_j \to D)_* \mathrm{div}_{Z_j}(d_{B_j}(a_j, f)) \tag{41.30.2.1}$$

as an $(n-2)$-cycle.

Consider any codimension 2 integral closed subscheme $W \subset X$ with generic point $\zeta \in X$. Set $A = \mathcal{O}_{X,\zeta}$. Choose a generator $s_\zeta \in \mathcal{L}_\zeta$. For those j such that $\zeta \in Z_j$ we may write $\tilde{s}_j = b_j s_\zeta$ with $b_j \in B_j^*$. We may also write $s = a_\zeta s_\zeta$ for some $a_\zeta \in A$. Then we see that $a_j = b_j a_\zeta$. The coefficient of $[W]$ on the right hand side of Equation (41.30.2.1) is

$$\sum\nolimits_{\zeta\in Z_j} n_j \mathrm{ord}_{A/\mathfrak{q}_j}(\overline{b_j}).$$

where $\mathfrak{q}_j \subset A$ is the height one prime corresponding to Z_j. Note that $B_j = A_{\mathfrak{q}_j}$ in this case. The coefficient of $[W]$ on the left hand side of Equation (41.30.2.1) is

$$\sum\nolimits_{\zeta\in Z_j} \mathrm{ord}_{A/\mathfrak{q}_j}(d_{A_{\mathfrak{q}_j}}(b_j a_\zeta, f)).$$

Since b_j is a unit, and $n_j = \mathrm{ord}_{A_{\mathfrak{q}_j}}(f)$ we see that $d_{A_{\mathfrak{q}_j}}(b_j a_\zeta, f) = \overline{b_j}^{n_j} d_{A_{\mathfrak{q}_j}}(a_\zeta, f)$ by Lemmas 41.4.4 and 41.4.6. By additivity of ord we see that it suffices to prove

$$0 = \sum\nolimits_{\zeta\in Z_j} \mathrm{ord}_{A/\mathfrak{q}_j}(d_{A_{\mathfrak{q}_j}}(a_\zeta, f))$$

which is Lemma 41.6.1. □

Lemma 41.30.3. *Let (S, δ) be as in Situation 41.7.1. Let X be locally of finite type over S. Let $i : D \to X$ be an effective Cartier divisor on X. The Gysin homomorphism factors through rational equivalence to give a map $i^* : A_{k+1}(X) \to A_k(D)$.*

Proof. Let $\alpha \in Z_{k+1}(X)$ and assume that $\alpha \sim_{rat} 0$. This means there exists a locally finite collection of integral closed subschemes $W_j \subset X$ of δ-dimension $k+2$ and $f_j \in R(W_j)^*$ such that $\alpha = \sum i_{j,*}\mathrm{div}_{W_j}(f_j)$. By construction of the map i^* we see that $i^*\alpha = \sum i^* i_{j,*}\mathrm{div}_{W_j}(f_j)$ where each cycle $i^* i_{j,*}\mathrm{div}_{W_j}(f_j)$ is supported on $D \cap W_j$. If we can show that each $i^* i_{j,*}\mathrm{div}_{W_j}(f_j)$ is rationally equivalent on $W_j \cap D$, then we see that $i^*\alpha \sim_{rat} 0$ (this is clear if the sum is finite, in general see Remark 41.19.4).

Pick an index j. If $W_j \subset D$, then we see that $i^* i_{j,*}\mathrm{div}_{W_j}(f_j)$ is simply equal to

$$i'_{j,*} c_1(\mathcal{O}_X(D)|_{W_j}) \cap \mathrm{div}_{W_j}(f_j)$$

where $i'_j : W_j \to D$ is the inclusion map. This is rationally equivalent to zero by Lemma 41.29.2. If $W_j \not\subset D$, then we see that $i^*i_{j,*}\mathrm{div}_{W_j}(f_j)$ is simply equal to

$$(i')^*\mathrm{div}_{W_j}(f_j)$$

where $i' : D \cap W_j \to W_j$ is the corresponding closed immersion (see Lemma 41.28.3). Hence in this case Lemma 41.30.2 applies, and we win. □

41.31. Relative effective Cartier divisors

Lemma 41.31.1. *Let $A \to B$ be a ring map. Let $f \in B$. Assume that*

(1) *$A \to B$ is flat,*
(2) *f is a nonzerodivisor, and*
(3) *$A \to B/fB$ is flat.*

Then for every ideal $I \subset A$ the map $f : B/IB \to B/IB$ is injective.

Proof. Note that $IB = I \otimes_A B$ and $I(B/fB) = I \otimes_A B/fB$ by the flatness of B and B/fB over A. In particular $IB/fIB \cong I \otimes_A B/fB$ maps injectively into B/fB. Hence the result follows from the snake lemma applied to the diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & I \otimes_A B & \longrightarrow & B & \longrightarrow & B/IB & \longrightarrow & 0 \\ & & \downarrow f & & \downarrow f & & \downarrow f & & \\ 0 & \longrightarrow & I \otimes_A B & \longrightarrow & B & \longrightarrow & B/IB & \longrightarrow & 0 \end{array}$$

with exact rows. □

Lemma 41.31.2. *Let (S, δ) be as in Situation 41.7.1. Let X, Y be locally of finite type over S. Let $p : X \to Y$ be a flat morphism of relative dimension r. Let $i : D \to X$ be an effective Cartier divisor with the property that $p|_D : D \to Y$ is flat of relative dimension $r - 1$. Let $\mathcal{L} = \mathcal{O}_X(D)$. For any $\alpha \in A_{k+1}(Y)$ we have*

$$i^*p^*\alpha = (p|_D)^*\alpha$$

in $A_{k+r}(D)$ and

$$c_1(\mathcal{L}) \cap p^*\alpha = i_*((p|_D)^*\alpha)$$

in $A_{k+r}(X)$.

Proof. Let $W \subset Y$ be an integral closed subvariety of δ-dimension $k + 1$. By Lemma 41.31.1 we see that $D \cap p^{-1}W$ is an effective Cartier divisor on $p^{-1}W$. By Lemma 41.28.4 we see that $i^*[p^{-1}W]_{k+r+1} = [D \cap W]_{k+r} = [(p|_D)^{-1}(W)]_{k+r}$. Since by definition $p^*[W] = [p^{-1}W]_{k+r+1}$ and $(p|_D)^*[W] = [(p|_D)^{-1}(W)]_{k+r}$ we see we have equality of cycles. Hence if $\alpha = \sum m_j[W_j]$, then we get $i^*\alpha = \sum m_j i^*[W_j] = \sum m_j (p|_D)^*[W_j]$ as cycles. This proves then first equality. To deduce the second from the first apply Lemma 41.28.2. □

41.32. Affine bundles

Lemma 41.32.1. *Let (S, δ) be as in Situation 41.7.1. Let X, Y be locally of finite type over S. Let $f : X \to Y$ be a flat morphism of relative dimension r. Assume that for every $y \in Y$, there exists an open neighbourhood $U \subset Y$ such that $f|_{f^{-1}(U)} : f^{-1}(U) \to U$ is identified with the morphism $U \times \mathbf{A}^r \to U$. Then $f^* : A_k(Y) \to A_{k+r}(X)$ is surjective for all $k \in \mathbf{Z}$.*

Proof. Let $\alpha \in A_{k+r}(X)$. Write $\alpha = \sum m_j[W_j]$ with $m_j \neq 0$ and W_j pairwise distinct integral closed subschemes of δ-dimension $k+r$. Then the family $\{W_j\}$ is locally finite in X. For any quasi-compact open $V \subset Y$ we see that $f^{-1}(V) \cap W_j$ is nonempty only for finitely many j. Hence the collection $Z_j = \overline{f(W_j)}$ of closures of images is a locally finite collection of integral closed subschemes of Y.

Consider the fibre product diagrams

$$\begin{array}{ccc} f^{-1}(Z_j) & \longrightarrow & X \\ {\scriptstyle f_j}\downarrow & & \downarrow{\scriptstyle f} \\ Z_j & \longrightarrow & Y \end{array}$$

Suppose that $[W_j] \in Z_{k+r}(f^{-1}(Z_j))$ is rationally equivalent to $f_j^*\beta_j$ for some k-cycle $\beta_j \in A_k(Z_j)$. Then $\beta = \sum m_j\beta_j$ will be a k-cycle on Y and $f^*\beta = \sum m_j f_j^*\beta_j$ will be rationally equivalent to α (see Remark 41.19.4). This reduces us to the case Y integral, and $\alpha = [W]$ for some integral closed subscheme of X dominating Y. In particular we may assume that $d = \dim_\delta(Y) < \infty$.

Hence we can use induction on $d = \dim_\delta(Y)$. If $d < k$, then $A_{k+r}(X) = 0$ and the lemma holds. By assumption there exists a dense open $V \subset Y$ such that $f^{-1}(V) \cong V \times \mathbf{A}^r$ as schemes over V. Suppose that we can show that $\alpha|_{f^{-1}(V)} = f^*\beta$ for some $\beta \in Z_k(V)$. By Lemma 41.14.2 we see that $\beta = \beta'|_V$ for some $\beta' \in Z_k(Y)$. By the exact sequence $A_k(f^{-1}(Y \setminus V)) \to A_k(X) \to A_k(f^{-1}(V))$ of Lemma 41.19.2 we see that $\alpha - f^*\beta'$ comes from a cycle $\alpha' \in A_{k+r}(f^{-1}(Y \setminus V))$. Since $\dim_\delta(Y \setminus V) < d$ we win by induction on d.

Thus we may assume that $X = Y \times \mathbf{A}^r$. In this case we can factor f as

$$X = Y \times \mathbf{A}^r \to Y \times \mathbf{A}^{r-1} \to \ldots \to Y \times \mathbf{A}^1 \to Y.$$

Hence it suffices to do the case $r = 1$. By the argument in the second paragraph of the proof we are reduced to the case $\alpha = [W]$, Y integral, and $W \to Y$ dominant. Again we can do induction on $d = \dim_\delta(Y)$. If $W = Y \times \mathbf{A}^1$, then $[W] = f^*[Y]$. Lastly, $W \subset Y \times \mathbf{A}^1$ is a proper inclusion, then $W \to Y$ induces a finite field extension $R(Y) \subset R(W)$. Let $P(T) \in R(Y)[T]$ be the monic irreducible polynomial such that the generic fibre of $W \to Y$ is cut out by P in $\mathbf{A}^1_{R(Y)}$. Let $V \subset Y$ be a nonempty open such that $P \in \Gamma(V, \mathcal{O}_Y)[T]$, and such that $W \cap f^{-1}(V)$ is still cut out by P. Then we see that $\alpha|_{f^{-1}(V)} \sim_{rat} 0$ and hence $\alpha \sim_{rat} \alpha'$ for some cycle α' on $(Y \setminus V) \times \mathbf{A}^1$. By induction on the dimension we win. □

Remark 41.32.2. We will see later (Lemma 41.33.3) that if X is a vectorbundle over Y then the pullback map $A_k(Y) \to A_{k+r}(X)$ is an isomorphism. Is this true in general?

41.33. Projective space bundle formula

Let (S, δ) be as in Situation 41.7.1. Let X be locally of finite type over S. Consider a finite locally free $\mathcal{O}_X$-module $\mathcal{E}$ of rank r. Our convention is that the *projective bundle associated to* $\mathcal{E}$ is the morphism

$$\mathbf{P}(\mathcal{E}) = \underline{\mathrm{Proj}}_X(\mathrm{Sym}^*(\mathcal{E})) \xrightarrow{\pi} X$$

over X with $\mathcal{O}_{\mathbf{P}(\mathcal{E})}(1)$ normalized so that $\pi_*(\mathcal{O}_{\mathbf{P}(\mathcal{E})}(1)) = \mathcal{E}$. In particular there is a surjection $\pi^*\mathcal{E} \to \mathcal{O}_{\mathbf{P}(\mathcal{E})}(1)$. We will say informally "let $(\pi : P \to X, \mathcal{O}_P(1))$ be the projective bundle associated to $\mathcal{E}$" to denote the situation where $P = \mathbf{P}(\mathcal{E})$ and $\mathcal{O}_P(1) = \mathcal{O}_{\mathbf{P}(\mathcal{E})}(1)$.

Lemma 41.33.1. *Let (S, δ) be as in Situation 41.7.1. Let X be locally of finite type over S. Let $\mathcal{E}$ be a finite locally free $\mathcal{O}_X$-module $\mathcal{E}$ of rank r. Let $(\pi : P \to X, \mathcal{O}_P(1))$ be the projective bundle associated to $\mathcal{E}$. For any $\alpha \in A_k(X)$ the element*

$$\pi_* \left(c_1(\mathcal{O}_P(1))^s \cap \pi^*\alpha \right) \in A_{k+r-1-s}(X)$$

is 0 if $s < r-1$ and is equal to α when $s = r-1$.

Proof. Let $Z \subset X$ be an integral closed subscheme of δ-dimension k. Note that $\pi^*[Z] = [\pi^{-1}(Z)]$ as $\pi^{-1}(Z)$ is integral of δ-dimension $r-1$. If $s < r-1$, then by construction $c_1(\mathcal{O}_P(1))^s \cap \pi^*[Z]$ is represented by a $(k+r-1-s)$-cycle supported on $\pi^{-1}(Z)$. Hence the pushforward of this cycle is zero for dimension reasons.

Let $s = r-1$. By the argument given above we see that $\pi_*(c_1(\mathcal{O}_P(1))^s \cap \pi^*\alpha) = n[Z]$ for some $n \in \mathbf{Z}$. We want to show that $n = 1$. For the same dimension reasons as above it suffices to prove this result after replacing X by $X \setminus T$ where $T \subset Z$ is a proper closed subset. Let ξ be the generic point of Z. We can choose elements $e_1, \ldots, e_{r-1} \in \mathcal{E}_\xi$ which form part of a basis of $\mathcal{E}_\xi$. These give rational sections $s_1, \ldots, s_{r-1}$ of $\mathcal{O}_P(1)|_{\pi^{-1}(Z)}$ whose common zero set is the closure of the image a rational section of $\mathbf{P}(\mathcal{E}|_Z) \to Z$ union a closed subset whose support maps to a proper closed subset T of Z. After removing T from X (and correspondingly $\pi^{-1}(T)$ from P), we see that $s_1, \ldots, s_n$ form a sequence of global sections $s_i \in \Gamma(\pi^{-1}(Z), \mathcal{O}_{\pi^{-1}(Z)}(1))$ whose common zero set is the image of a section $Z \to \pi^{-1}(Z)$. Hence we see successively that

$$\begin{aligned}
\pi^*[Z] &= [\pi^{-1}(Z)] \\
c_1(\mathcal{O}_P(1)) \cap \pi^*[Z] &= [Z(s_1)] \\
c_1(\mathcal{O}_P(1))^2 \cap \pi^*[Z] &= [Z(s_1) \cap Z(s_2)] \\
\ldots &= \ldots \\
c_1(\mathcal{O}_P(1))^{r-1} \cap \pi^*[Z] &= [Z(s_1) \cap \ldots \cap Z(s_{r-1})]
\end{aligned}$$

by repeated applications of Lemma 41.25.3. Since the pushforward by π of the image of a section of π over Z is clearly $[Z]$ we see the result when $\alpha = [Z]$. We omit the verification that these arguments imply the result for a general cycle $\alpha = \sum n_j[Z_j]$. □

Lemma 41.33.2 (Projective space bundle formula)**.** *Let (S, δ) be as in Situation 41.7.1. Let X be locally of finite type over S. Let $\mathcal{E}$ be a finite locally free $\mathcal{O}_X$-module $\mathcal{E}$ of rank r. Let $(\pi : P \to X, \mathcal{O}_P(1))$ be the projective bundle associated to $\mathcal{E}$. The map*

$$\bigoplus\nolimits_{i=0}^{r-1} A_{k+i}(X) \longrightarrow A_{k+r-1}(P),$$

$$(\alpha_0, \ldots, \alpha_{r-1}) \longmapsto \pi^*\alpha_0 + c_1(\mathcal{O}_P(1)) \cap \pi^*\alpha_1 + \ldots + c_1(\mathcal{O}_P(1))^{r-1} \cap \pi^*\alpha_{r-1}$$

is an isomorphism.

Proof. Fix $k \in \mathbf{Z}$. We first show the map is injective. Suppose that $(\alpha_0, \ldots, \alpha_{r-1})$ is an element of the left hand side that maps to zero. By Lemma 41.33.1 we see that

$$0 = \pi_*(\pi^*\alpha_0 + c_1(\mathcal{O}_P(1)) \cap \pi^*\alpha_1 + \ldots + c_1(\mathcal{O}_P(1))^{r-1} \cap \pi^*\alpha_{r-1}) = \alpha_{r-1}$$

Next, we see that

$$0 = \pi_*(c_1(\mathcal{O}_P(1)) \cap (\pi^*\alpha_0 + c_1(\mathcal{O}_P(1)) \cap \pi^*\alpha_1 + \ldots + c_1(\mathcal{O}_P(1))^{r-2} \cap \pi^*\alpha_{r-2})) = \alpha_{r-2}$$

and so on. Hence the map is injective.

It remains to show the map is surjective. Let X_i, $i \in I$ be the irreducible components of X. Then $P_i = \mathbf{P}(\mathcal{E}|_{X_i})$, $i \in I$ are the irreducible components of P. If the map is surjective for each of the morphisms $P_i \to X_i$, then the map is surjective for $\pi : P \to X$. Details omitted. Hence we may assume X is irreducible. Thus $\dim_\delta(X) < \infty$ and in particular we may use induction on $\dim_\delta(X)$.

The result is clear if $\dim_\delta(X) < k$. Let $\alpha \in A_{k+r-1}(P)$. For any locally closed subscheme $T \subset X$ denote $\gamma_T : \bigoplus A_{k+i}(T) \to A_{k+r-1}(\pi^{-1}(T))$ the map

$$\gamma_T(\alpha_0, \ldots, \alpha_{r-1}) = \pi^*\alpha_0 + \ldots + c_1(\mathcal{O}_{\pi^{-1}(T)}(1))^{r-1} \cap \pi^*\alpha_{r-1}.$$

Suppose for some nonempty open $U \subset X$ we have $\alpha|_{\pi^{-1}(U)} = \gamma_U(\alpha_0, \ldots, \alpha_{r-1})$. Then we may choose lifts $\alpha_i' \in A_{k+i}(X)$ and we see that $\alpha - \gamma_X(\alpha_0', \ldots, \alpha_{r-1}')$ is by Lemma 41.19.2 rationally equivalent to a k-cycle on $P_Y = \mathbf{P}(\mathcal{E}|_Y)$ where $Y = X \setminus U$ as a reduced closed subscheme. Note that $\dim_\delta(Y) < \dim_\delta(X)$. By induction the result holds for $P_Y \to Y$ and hence the result holds for α. Hence we may replace X by any nonempty open of X.

In particular we may assume that $\mathcal{E} \cong \mathcal{O}_X^{\oplus r}$. In this case $\mathbf{P}(\mathcal{E}) = X \times \mathbf{P}^{r-1}$. Let us use the stratification

$$\mathbf{P}^{r-1} = \mathbf{A}^{r-1} \coprod \mathbf{A}^{r-2} \coprod \ldots \coprod \mathbf{A}^0$$

The closure of each stratum is a $\mathbf{P}^{r-1-i}$ which is a representative of $c_1(\mathcal{O}(1))^i \cap [\mathbf{P}^{r-1}]$. Hence P has a similar stratification

$$P = U^{r-1} \coprod U^{r-2} \coprod \ldots \coprod U^0$$

Let P^i be the closure of U^i. Let $\pi^i : P^i \to X$ be the restriction of π to P^i. Let $\alpha \in A_{k+r-1}(P)$. By Lemma 41.32.1 we can write $\alpha|_{U^{r-1}} = \pi^*\alpha_0|_{U^{r-1}}$ for some $\alpha_0 \in A_k(X)$. Hence the difference $\alpha - \pi^*\alpha_0$ is the image of some $\alpha' \in A_{k+r-1}(P^{r-2})$. By Lemma 41.32.1 again we can write $\alpha'|_{U^{r-2}} = (\pi^{r-2})^*\alpha_1|_{U^{r-2}}$ for some $\alpha_1 \in A_{k+1}(X)$. By Lemma 41.31.2 we see that the image of $(\pi^{r-2})^*\alpha_1$ represents $c_1(\mathcal{O}_P(1)) \cap \pi^*\alpha_1$. We also see that $\alpha - \pi^*\alpha_0 - c_1(\mathcal{O}_P(1)) \cap \pi^*\alpha_1$ is the image of some $\alpha'' \in A_{k+r-1}(P^{r-3})$. And so on. □

Lemma 41.33.3. *Let (S, δ) be as in Situation 41.7.1. Let X be locally of finite type over S. Let $\mathcal{E}$ be a finite locally free sheaf of rank r on X. Let*

$$p : E = \underline{\mathrm{Spec}}(Sym^*(\mathcal{E})) \longrightarrow X$$

be the associated vector bundle over X. Then $p^* : A_k(X) \to A_{k+r}(E)$ is an isomorphism for all k.

Proof. For surjectivity see Lemma 41.32.1. Let $(\pi : P \to X, \mathcal{O}_P(1))$ be the projective space bundle associated to the finite locally free sheaf $\mathcal{E} \oplus \mathcal{O}_X$. Let $s \in \Gamma(P, \mathcal{O}_P(1))$ correspond to the global section $(0, 1) \in \Gamma(X, \mathcal{E} \oplus \mathcal{O}_X)$. Let $D = Z(s) \subset P$. Note that $(\pi|_D : D \to X, \mathcal{O}_P(1)|_D)$ is the projective space bundle associated to $\mathcal{E}$. We denote $\pi_D = \pi|_D$ and $\mathcal{O}_D(1) = \mathcal{O}_P(1)|_D$. Moreover, D is an effective Cartier divisor on P. Hence $\mathcal{O}_P(D) = \mathcal{O}_P(1)$ (see Divisors, Lemma 30.9.20). Also there is an isomorphism $E \cong P \setminus D$. Denote $j : E \to P$ the corresponding open immersion. For injectivity we use that the kernel of

$$j^* : A_{k+r}(P) \longrightarrow A_{k+r}(E)$$

are the cycles supported in the effective Cartier divisor D, see Lemma 41.19.2. So if $p^*\alpha = 0$, then $\pi^*\alpha = i_*\beta$ for some $\beta \in A_{k+r}(D)$. By Lemma 41.33.2 we may write

$$\beta = \pi_D^*\beta_0 + \ldots + c_1(\mathcal{O}_D(1))^{r-1} \cap \pi_D^*\beta_{r-1}.$$

for some $\beta_i \in A_{k+i}(X)$. By Lemmas 41.31.2 and 41.25.6 this implies

$$\pi^*\alpha = i_*\beta = c_1(\mathcal{O}_P(1)) \cap \pi^*\beta_0 + \ldots + c_1(\mathcal{O}_D(1))^r \cap \pi^*\beta_{r-1}.$$

Since the rank of $\mathcal{E} \oplus \mathcal{O}_X$ is $r + 1$ this contradicts Lemma 41.25.6 unless all α and all β_i are zero. □

41.34. The Chern classes of a vector bundle

We can use the projective space bundle formula to define the chern classes of a rank r vector bundle in terms of the expansion of $c_1(\mathcal{O}(1))^r$ in terms of the lower powers, see formula (41.34.1.1). The reason for the signs will be explained later.

Definition 41.34.1. Let (S, δ) be as in Situation 41.7.1. Let X be locally of finite type over S. Assume X is integral and $n = \dim_\delta(X)$. Let $\mathcal{E}$ be a finite locally free sheaf of rank r on X. Let $(\pi : P \to X, \mathcal{O}_P(1))$ be the projective space bundle associated to $\mathcal{E}$.

(1) By Lemma 41.33.2 there are elements $c_i \in A_{n-i}(X)$, $i = 0, \ldots, r$ such that $c_0 = [X]$, and

$$\sum\nolimits_{i=0}^{r} (-1)^i c_1(\mathcal{O}_P(1))^i \cap \pi^* c_{r-i} = 0. \tag{41.34.1.1}$$

(2) With notation as above we set $c_i(\mathcal{E}) \cap [X] = c_i$ as an element of $A_{n-i}(X)$. We call these the *chern classes of $\mathcal{E}$ on X*.

(3) The *total chern class of $\mathcal{E}$ on X* is the combination

$$c(\mathcal{E}) \cap [X] = c_0(\mathcal{E}) \cap [X] + c_1(\mathcal{E}) \cap [X] + \ldots + c_r(\mathcal{E}) \cap [X]$$

which is an element of $A_*(X) = \bigoplus_{k \in \mathbf{Z}} A_k(X)$.

Let us check that this does not give a new notion in case the vector bundle has rank 1.

Lemma 41.34.2. *Let (S, δ) be as in Situation 41.7.1. Let X be locally of finite type over S. Assume X is integral and $n = \dim_\delta(X)$. Let $\mathcal{L}$ be an invertible $\mathcal{O}_X$-module. The first chern class of $\mathcal{L}$ on X of Definition 41.34.1 is equal to the Weil divisor associated to $\mathcal{L}$ by Definition 41.24.1.*

Proof. In this proof we use $c_1(\mathcal{L}) \cap [X]$ to denote the construction of Definition 41.24.1. Since $\mathcal{L}$ has rank 1 we have $\mathbf{P}(\mathcal{L}) = X$ and $\mathcal{O}_{\mathbf{P}(\mathcal{L})}(1) = \mathcal{L}$ by our normalizations. Hence (41.34.1.1) reads

$$(-1)^1 c_1(\mathcal{L}) \cap c_0 + (-1)^0 c_1 = 0$$

Since $c_0 = [X]$, we conclude $c_1 = c_1(\mathcal{L}) \cap [X]$ as desired. □

Remark 41.34.3. We could also rewrite equation 41.34.1.1 as

$$\sum\nolimits_{i=0}^{r} c_1(\mathcal{O}_P(-1))^i \cap \pi^* c_{r-i} = 0. \tag{41.34.3.1}$$

but we find it easier to work with the tautological quotient sheaf $\mathcal{O}_P(1)$ instead of its dual.

41.35. Intersecting with chern classes

Definition 41.35.1. Let (S, δ) be as in Situation 41.7.1. Let X be locally of finite type over S. Let $\mathcal{E}$ be a finite locally free sheaf of rank r on X. We define, for every integer k and any $0 \leq j \leq r$, an operation

$$c_j(\mathcal{E}) \cap - : Z_k(X) \to A_{k-j}(X)$$

called *intersection with the jth chern class of $\mathcal{E}$.*

(1) Given an integral closed subscheme $i : W \to X$ of δ-dimension k we define

$$c_j(\mathcal{E}) \cap [W] = i_*(c_j(i^*\mathcal{E}) \cap [W]) \in A_{k-j}(X)$$

where $c_j(i^*\mathcal{E}) \cap [W]$ is as defined in Definition 41.34.1.

(2) For a general k-cycle $\alpha = \sum n_i [W_i]$ we set

$$c_j(\mathcal{E}) \cap \alpha = \sum n_i c_j(\mathcal{E}) \cap [W_i]$$

Again, if $\mathcal{E}$ has rank 1 then this agrees with our previous definition.

Lemma 41.35.2. *Let (S, δ) be as in Situation 41.7.1. Let X be locally of finite type over S. Let $\mathcal{E}$ be a finite locally free sheaf of rank r on X. Let $(\pi : P \to X, \mathcal{O}_P(1))$ be the projective bundle associated to $\mathcal{E}$. For $\alpha \in Z_k(X)$ the elements $c_j(\mathcal{E}) \cap \alpha$ are the unique elements α_j of $A_{k-j}(X)$ such that $\alpha_0 = \alpha$ and*

$$\sum\nolimits_{i=0}^{r} (-1)^i c_1(\mathcal{O}_P(1))^i \cap \pi^*(\alpha_{r-i}) = 0$$

holds in the Chow group of P.

Proof. The uniqueness of $\alpha_0, \ldots, \alpha_r$ such that $\alpha_0 = \alpha$ and such that the displayed equation holds follows from the projective space bundle formula Lemma 41.33.2. The identity holds by definition for $\alpha = [X]$. For a general k-cycle α on X write $\alpha = \sum n_a [W_a]$ with $n_a \neq 0$, and $i_a : W_a \to X$ pairwise distinct integral closed subschemes. Then the family $\{W_a\}$ is locally finite on X. Set $P_a = \pi^{-1}(W_a) = \mathbf{P}(\mathcal{E}|_{W_a})$. Denote $i'_a : P_a \to P$ the corresponding closed immersions. Consider the fibre product diagram

$$\begin{array}{ccccc} P' & = & \coprod P_a & \xrightarrow{i'_a} & P \\ \downarrow{\scriptstyle \pi'} & & \downarrow{\scriptstyle \pi_a} & & \downarrow{\scriptstyle \pi} \\ X' & = & \coprod W_a & \xrightarrow{i_a} & X \end{array}$$

The morphism $p : X' \to X$ is proper. Moreover $\pi' : P' \to X'$ together with the invertible sheaf $\mathcal{O}_{P'}(1) = \coprod \mathcal{O}_{P_a}(1)$ which is also the pullback of $\mathcal{O}_P(1)$ is the projective bundle associated to $\mathcal{E}' = p^*\mathcal{E}$. By definition

$$c_j(\mathcal{E}) \cap [\alpha] = \sum i_{a,*}(c_j(\mathcal{E}|_{W_a}) \cap [W_a]).$$

Write $\beta_{a,j} = c_j(\mathcal{E}|_{W_a}) \cap [W_a]$ which is an element of $A_{k-j}(W_a)$. We have

$$\sum\nolimits_{i=0}^{r} (-1)^i c_1(\mathcal{O}_{P_a}(1))^i \cap \pi_a^*(\beta_{a,r-i}) = 0$$

for each a by definition. Thus clearly we have

$$\sum\nolimits_{i=0}^{r} (-1)^i c_1(\mathcal{O}_{P'}(1))^i \cap (\pi')^*(\beta_{r-i}) = 0$$

with $\beta_j = \sum n_a \beta_{a,j} \in A_{k-j}(X')$. Denote $p' : P' \to P$ the morphism $\coprod i'_a$. We have $\pi^* p_* \beta_j = p'_*(\pi')^* \beta_j$ by Lemma 41.15.1. By the projection formula of Lemma 41.25.6 we conclude that

$$\sum\nolimits_{i=0}^{r} (-1)^i c_1(\mathcal{O}_P(1))^i \cap \pi^*(p_* \beta_j) = 0$$

Since $p_*\beta_j$ is a representative of $c_j(\mathcal{E}) \cap \alpha$ we win. □

This characterization of chern classes allows us to prove many more properties.

Lemma 41.35.3. *Let (S, δ) be as in Situation 41.7.1. Let X be locally of finite type over S. Let $\mathcal{E}$ be a finite locally free sheaf of rank r on X. If $\alpha \sim_{rat} \beta$ are rationally equivalent k-cycles on X then $c_j(\mathcal{E}) \cap \alpha = c_j(\mathcal{E}) \cap \beta$ in $A_{k-j}(X)$.*

Proof. By Lemma 41.35.2 the elements $\alpha_j = c_j(\mathcal{E}) \cap \alpha$, $j \geq 1$ and $\beta_j = c_j(\mathcal{E}) \cap \beta$, $j \geq 1$ are uniquely determined by the *same* equation in the chow group of the projective bundle associated to $\mathcal{E}$. (This of course relies on the fact that flat pullback is compatible with rational equivalence, see Lemma 41.20.1.) Hence they are equal. □

In other words capping with chern classes of finite locally free sheaves factors through rational equivalence to give maps

$$c_j(\mathcal{E}) \cap - : A_k(X) \to A_{k-j}(X).$$

Lemma 41.35.4. *Let (S, δ) be as in Situation 41.7.1. Let X, Y be locally of finite type over S. Let $\mathcal{E}$ be a finite locally free sheaf of rank r on Y. Let $f : X \to Y$ be a flat morphism of relative dimension r. Let α be a k-cycle on Y. Then*

$$f^*(c_j(\mathcal{E}) \cap \alpha) = c_j(f^*\mathcal{E}) \cap f^*\alpha$$

Proof. Write $\alpha_j = c_j(\mathcal{E}) \cap \alpha$, so $\alpha_0 = \alpha$. By Lemma 41.35.2 we have

$$\sum\nolimits_{i=0}^{r} (-1)^i c_1(\mathcal{O}_P(1))^i \cap \pi^*(\alpha_{r-i}) = 0$$

in the chow group of the projective bundle $(\pi : P \to Y, \mathcal{O}_P(1))$ associated to $\mathcal{E}$. Consider the fibre product diagram

$$\begin{array}{ccc} P_X = \mathbf{P}(f^*\mathcal{E}) & \xrightarrow{f_P} & P \\ \downarrow{\scriptstyle \pi_X} & & \downarrow{\scriptstyle \pi} \\ X & \xrightarrow{f} & Y \end{array}$$

Note that $\mathcal{O}_{P_X}(1) = f_P^*\mathcal{O}_P(1)$. By Lemmas 41.25.4 and 41.14.3 we see that

$$\sum\nolimits_{i=0}^{r}(-1)^i c_1(\mathcal{O}_{P_X}(1))^i \cap \pi_X^*(f^*\alpha_{r-i}) = 0$$

holds in the chow group of P_X. Since $f^*\alpha_0 = f^*\alpha$ the lemma follows from the uniqueness in Lemma 41.35.2. □

Lemma 41.35.5. *Let (S, δ) be as in Situation 41.7.1. Let X, Y be locally of finite type over S. Let $\mathcal{E}$ be a finite locally free sheaf of rank r on X. Let $p : X \to Y$ be a proper morphism. Let α be a k-cycle on X. Let $\mathcal{E}$ be a finite locally free sheaf on Y. Then*

$$p_*(c_j(p^*\mathcal{E}) \cap \alpha) = c_j(\mathcal{E}) \cap p_*\alpha$$

Proof. Write $\alpha_j = c_j(p^*\mathcal{E}) \cap \alpha$, so $\alpha_0 = \alpha$. By Lemma 41.35.2 we have

$$\sum\nolimits_{i=0}^{r}(-1)^i c_1(\mathcal{O}_P(1))^i \cap \pi_X^*(\alpha_{r-i}) = 0$$

in the chow group of the projective bundle $(\pi_X : P_X \to X, \mathcal{O}_{P_X}(1))$ associated to $p^*\mathcal{E}$. Let $(\pi : P \to Y, \mathcal{O}_P(1))$ be the projective bundle associated to $\mathcal{E}$. Consider the fibre product diagram

$$\begin{array}{ccc} P_X = \mathbf{P}(p^*\mathcal{E}) & \xrightarrow[p_P]{} & P \\ {\scriptstyle \pi_X}\downarrow & & \downarrow{\scriptstyle \pi} \\ X & \xrightarrow{p} & Y \end{array}$$

Note that $\mathcal{O}_{P_X}(1) = p_P^*\mathcal{O}_P(1)$. Pushing the displayed equality above to P and using Lemmas 41.15.1, 41.25.6 and 41.14.3 we see that

$$\sum\nolimits_{i=0}^{r}(-1)^i c_1(\mathcal{O}_P(1))^i \cap \pi^*(p_*\alpha_{r-i}) = 0$$

holds in the chow group of P. Since $p_*\alpha_0 = p_*\alpha$ the lemma follows from the uniqueness in Lemma 41.35.2. □

Lemma 41.35.6. *Let (S, δ) be as in Situation 41.7.1. Let X be locally of finite type over S. Let $\mathcal{E}$, $\mathcal{F}$ be finite locally free sheaves on X of ranks r and s. For any $\alpha \in A_k(X)$ we have*

$$c_i(\mathcal{E}) \cap c_j(\mathcal{F}) \cap \alpha = c_j(\mathcal{F}) \cap c_i(\mathcal{E}) \cap \alpha$$

as elements of $A_{k-i-j}(X)$.

Proof. Consider

$$\pi : \mathbf{P}(\mathcal{E}) \times_X \mathbf{P}(\mathcal{F}) \longrightarrow X$$

with invertible sheaves $\mathcal{L} = \mathrm{pr}_1^*\mathcal{O}_{\mathbf{P}(\mathcal{E})}(1)$ and $\mathcal{N} = \mathrm{pr}_2^*\mathcal{O}_{\mathbf{P}(\mathcal{F})}(1)$. Write $\alpha_{i,j}$ for the left hand side and $\beta_{i,j}$ for the right hand side. Also write $\alpha_j = c_j(\mathcal{F}) \cap \alpha$ and $\beta_i = c_i(\mathcal{E}) \cap \alpha$. In particular this means that $\alpha_0 = \alpha = \beta_0$, and $\alpha_{0,j} = \alpha_j = \beta_{0,j}$, $\alpha_{i,0} = \beta_i = \beta_{i,0}$. From Lemma 41.35.2 (pulled back to the space above using

Lemma 41.25.4 for the first two) we see that

$$\begin{aligned} 0 &= \sum\nolimits_{j=0,\ldots,s} (-1)^j c_1(\mathcal{N})^j \cap \pi^*\alpha_{s-j} \\ 0 &= \sum\nolimits_{i=0,\ldots,r} (-1)^i c_1(\mathcal{L})^i \cap \pi^*\beta_{r-i} \\ 0 &= \sum\nolimits_{i=0,\ldots,r} (-1)^i c_1(\mathcal{L})^i \cap \pi^*\alpha_{r-i,s-j} \\ 0 &= \sum\nolimits_{j=0,\ldots,s} (-1)^j c_1(\mathcal{N})^j \cap \pi^*\beta_{r-i,s-j} \end{aligned}$$

We can combine the first and the third of these to get

$$\begin{aligned} &(-1)^{r+s} c_1(\mathcal{L})^r \cap c_1(\mathcal{N})^s \cap \pi^*\alpha \\ &= \sum\nolimits_{j=1,\ldots,s} (-1)^{r+j-1} c_1(\mathcal{L})^r \cap c_1(\mathcal{N})^j \cap \pi^*\alpha_{s-j} \\ &= \sum\nolimits_{j=1,\ldots,s} (-1)^{j-1+r} c_1(\mathcal{N})^j \cap c_1(\mathcal{L})^r \cap \pi^*\alpha_{0,s-j} \\ &= \sum\nolimits_{j=1}^{s} \sum\nolimits_{i=1}^{r} (-1)^{i+j} c_1(\mathcal{N})^j \cap c_1(\mathcal{L})^i \cap \pi^*\alpha_{r-i,s-j} \end{aligned}$$

using that capping with $c_1(\mathcal{L})$ commutes with capping with $c_1(\mathcal{N})$. In exactly the same way one shows that

$$(-1)^{r+s} c_1(\mathcal{L})^r \cap c_1(\mathcal{N})^s \cap \pi^*\alpha = \sum\nolimits_{j=1}^{s} \sum\nolimits_{i=1}^{r} (-1)^{i+j} c_1(\mathcal{N})^j \cap c_1(\mathcal{L})^i \cap \pi^*\beta_{r-i,s-j}$$

By the projective space bundle formula Lemma 41.33.2 applied twice these representations are unique. Whence the result. □

41.36. Polynomial relations among chern classes

Definition 41.36.1. Let $P(x_{i,j}) \in \mathbf{Z}[x_{i,j}]$ be a polynomial. We write P as a finite sum

$$\sum\nolimits_s \sum\nolimits_{I=((i_1,j_1),(i_2,j_2),\ldots,(i_s,j_s))} a_I x_{i_1,j_1} \ldots x_{i_s,j_s}.$$

Let (S, δ) be as in Situation 41.7.1. Let X be locally of finite type over S. Let $\mathcal{E}_i$ be a finite collection of finite locally free sheaves on X. We say that P is a *polynomial relation among the chern classes* and we write $P(c_j(\mathcal{E}_i)) = 0$ if for any morphism $f : Y \to X$ of an integral scheme locally of finite type over S the cycle

$$\sum\nolimits_s \sum\nolimits_{I=((i_1,j_1),(i_2,j_2),\ldots,(i_s,j_s))} a_I \; c_{j_1}(f^*\mathcal{E}_{i_1}) \cap \ldots \cap c_{j_s}(f^*\mathcal{E}_{i_s}) \cap [Y]$$

is zero in $A_*(Y)$.

This is not an elegant definition but it will do for now. It makes sense because we showed in Lemma 41.35.6 that capping with chern classes of vector bundles is commutative. By our definitions and results above this is equivalent with requiring all the operations

$$\sum\nolimits_s \sum\nolimits_I a_I \; c_{j_1}(f^*\mathcal{E}_{i_1}) \cap \ldots \cap c_{j_s}(f^*\mathcal{E}_{i_s}) \cap - : A_*(Y) \to A_*(Y)$$

to be zero for all morphisms $f : Y \to X$ which are locally of finite type.

An example of such a relation is the relation

$$c_1(\mathcal{L} \otimes_{\mathcal{O}_X} \mathcal{N}) = c_1(\mathcal{L}) + c_1(\mathcal{N})$$

proved in Lemma 41.25.2. More generally, here is what happens when we tensor an arbitrary locally free sheaf by an invertible sheaf.

Lemma 41.36.2. *Let (S, δ) be as in Situation 41.7.1. Let X be locally of finite type over S. Let $\mathcal{E}$ be a finite locally free sheaf of rank r on X. Let $\mathcal{L}$ be an invertible sheaf on X. Then*

$$c_i(\mathcal{E} \otimes \mathcal{L}) = \sum\nolimits_{j=0}^{i} \binom{r-i+j}{j} c_{i-j}(\mathcal{E}) c_1(\mathcal{L})^j \tag{41.36.2.1}$$

is a valid polynomial relation in the sense described above.

Proof. This should hold for any triple $(X, \mathcal{E}, \mathcal{L})$. In particular it should hold when X is integral, and in fact by definition of a polynomial relation it is enough to prove it holds when capping with $[X]$ for such X. Thus assume that X is integral. Let $(\pi : P \to X, \mathcal{O}_P(1))$, resp. $(\pi' : P' \to X, \mathcal{O}_{P'}(1))$ be the projective space bundle associated to $\mathcal{E}$, resp. $\mathcal{E} \otimes \mathcal{L}$. Consider the canonical morphism

$$\begin{array}{ccc} P & \xrightarrow{g} & P' \\ & {\scriptstyle\pi}\searrow \quad \swarrow{\scriptstyle\pi'} & \\ & X & \end{array}$$

see Constructions, Lemma 26.20.1. It has the property that $g^*\mathcal{O}_{P'}(1) = \mathcal{O}_P(1) \otimes \pi^*\mathcal{L}$. This means that we have

$$\sum\nolimits_{i=0}^{r} (-1)^i (\xi + x)^i \cap \pi^*(c_{r-i}(\mathcal{E} \otimes \mathcal{L}) \cap [X]) = 0$$

in $A_*(P)$, where ξ represents $c_1(\mathcal{O}_P(1))$ and x represents $c_1(\pi^*\mathcal{L})$. By simple algebra this is equivalent to

$$\sum\nolimits_{i=0}^{r} (-1)^i \xi^i \left(\sum\nolimits_{j=i}^{r} (-1)^{j-i} \binom{j}{i} x^{j-i} \cap \pi^*(c_{r-j}(\mathcal{E} \otimes \mathcal{L}) \cap [X]) \right) = 0$$

Comparing with Equation (41.34.1.1) it follows from this that

$$c_{r-i}(\mathcal{E}) \cap [X] = \sum\nolimits_{j=i}^{r} \binom{j}{i} (-c_1(\mathcal{L}))^{j-i} \cap c_{r-j}(\mathcal{E} \otimes \mathcal{L}) \cap [X]$$

Reworking this (getting rid of minus signs, and renumbering) we get the desired relation. □

Some example cases of (41.36.2.1) are

$$c_1(\mathcal{E} \otimes \mathcal{L}) = c_1(\mathcal{E}) + r c_1(\mathcal{L})$$

$$c_2(\mathcal{E} \otimes \mathcal{L}) = c_2(\mathcal{E}) + (r-1) c_1(\mathcal{E}) c_1(\mathcal{L}) + \binom{r}{2} c_1(\mathcal{L})^2$$

$$c_3(\mathcal{E} \otimes \mathcal{L}) = c_3(\mathcal{E}) + (r-2) c_2(\mathcal{E}) c_1(\mathcal{L}) + \binom{r-1}{2} c_1(\mathcal{E}) c_1(\mathcal{L})^2 + \binom{r}{3} c_1(\mathcal{L})^3$$

41.37. Additivity of chern classes

All of the preliminary lemmas follow trivially from the final result.

Lemma 41.37.1. *Let (S, δ) be as in Situation 41.7.1. Let X be locally of finite type over S. Let $\mathcal{E}$, $\mathcal{F}$ be finite locally free sheaves on X of ranks r, $r-1$ which fit into a short exact sequence*

$$0 \to \mathcal{O}_X \to \mathcal{E} \to \mathcal{F} \to 0$$

Then

$$c_r(\mathcal{E}) = 0, \quad c_j(\mathcal{E}) = c_j(\mathcal{F}), \quad j = 0, \ldots, r-1$$

are valid polynomial relations among chern classes.

Proof. By Definition 41.36.1 it suffices to show that if X is integral then $c_j(\mathcal{E}) \cap [X] = c_j(\mathcal{F}) \cap [X]$. Let $(\pi : P \to X, \mathcal{O}_P(1))$, resp. $(\pi' : P' \to X, \mathcal{O}_{P'}(1))$ denote the projective space bundle associated to $\mathcal{E}$, resp. $\mathcal{F}$. The surjection $\mathcal{E} \to \mathcal{F}$ gives rise to a closed immersion

$$i : P' \longrightarrow P$$

over X. Moreover, the element $1 \in \Gamma(X, \mathcal{O}_X) \subset \Gamma(X, \mathcal{E})$ gives rise to a global section $s \in \Gamma(P, \mathcal{O}_P(1))$ whose zero set is exactly P'. Hence P' is an effective Cartier divisor on P such that $\mathcal{O}_P(P') \cong \mathcal{O}_P(1)$. Hence we see that

$$c_1(\mathcal{O}_P(1)) \cap \pi^*\alpha = i_*((\pi')^*\alpha)$$

for any cycle class α on X by Lemma 41.31.2. By Lemma 41.35.2 we see that $\alpha_j = c_j(\mathcal{F}) \cap [X]$, $j = 0, \ldots, r-1$ satisfy

$$\sum\nolimits_{j=0}^{r-1} (-1)^j c_1(\mathcal{O}_{P'}(1))^j \cap (\pi')^*\alpha_j = 0$$

Pushing this to P and using the remark above as well as Lemma 41.25.6 we get

$$\sum\nolimits_{j=0}^{r-1} (-1)^j c_1(\mathcal{O}_P(1))^{j+1} \cap \pi^*\alpha_j = 0$$

By the uniqueness of Lemma 41.35.2 we conclude that $c_r(\mathcal{E}) \cap [X] = 0$ and $c_j(\mathcal{E}) \cap [X] = \alpha_j = c_j(\mathcal{F}) \cap [X]$ for $j = 0, \ldots, r-1$. Hence the lemma holds. □

Lemma 41.37.2. *Let (S, δ) be as in Situation 41.7.1. Let X be locally of finite type over S. Let $\mathcal{E}$, $\mathcal{F}$ be finite locally free sheaves on X of ranks r, $r-1$ which fit into a short exact sequence*

$$0 \to \mathcal{L} \to \mathcal{E} \to \mathcal{F} \to 0$$

where $\mathcal{L}$ is an invertible sheaf Then

$$c(\mathcal{E}) = c(\mathcal{L})c(\mathcal{F})$$

is a valid polynomial relation among chern classes.

Proof. This relation really just says that $c_i(\mathcal{E}) = c_i(\mathcal{F}) + c_1(\mathcal{L})c_{i-1}(\mathcal{F})$. By Lemma 41.37.1 we have $c_j(\mathcal{E} \otimes \mathcal{L}^{\otimes -1}) = c_j(\mathcal{E} \otimes \mathcal{L}^{\otimes -1})$ for $j = 0, \ldots, r$ (were we set $c_r(\mathcal{F}) = 0$ by convention). Applying Lemma 41.36.2 we deduce

$$\sum_{j=0}^{i} \binom{r-i+j}{j} (-1)^j c_{i-j}(\mathcal{E}) c_1(\mathcal{L})^j = \sum_{j=0}^{i} \binom{r-1-i+j}{j} (-1)^j c_{i-j}(\mathcal{F}) c_1(\mathcal{L})^j$$

Setting $c_i(\mathcal{E}) = c_i(\mathcal{F}) + c_1(\mathcal{L})c_{i-1}(\mathcal{F})$ gives a "solution" of this equation. The lemma follows if we show that this is the only possible solution. We omit the verification. □

Lemma 41.37.3. *Let (S, δ) be as in Situation 41.7.1. Let X be a scheme locally of finite type over S. Suppose that $\mathcal{E}$ sits in an exact sequence*

$$0 \to \mathcal{E}_1 \to \mathcal{E} \to \mathcal{E}_2 \to 0$$

of finite locally free sheaves $\mathcal{E}_i$ of rank r_i. Then

$$c(\mathcal{E}) = c(\mathcal{E}_1)c(\mathcal{E}_2)$$

is a polynomial relation among chern classes.

Proof. We may assume that X is integral and we have to show the identity when capping against $[X]$. By induction on r_1. The case $r_1 = 1$ is Lemma 41.37.2. Assume $r_1 > 1$. Let $(\pi : P \to X, \mathcal{O}_P(1))$ denote the projective space bundle associated to $\mathcal{E}_1$. Note that

(1) $\pi^* : A_*(X) \to A_*(P)$ is injective, and
(2) $\pi^*\mathcal{E}_1$ sits in a short exact sequence $0 \to \mathcal{F} \to \pi^*\mathcal{E}_1 \to \mathcal{L} \to 0$ where $\mathcal{L}$ is invertible.

The first assertion follows from the projective space bundle formula and the second follows from the definition of a projective space bundle. (In fact $\mathcal{L} = \mathcal{O}_P(1)$.) Let $Q = \pi^*\mathcal{E}/\mathcal{F}$, which sits in an exact sequence $0 \to \mathcal{L} \to Q \to \pi^*\mathcal{E}_2 \to 0$. By induction we have

$$\begin{aligned} c(\pi^*\mathcal{E}) \cap [P] &= c(\mathcal{F}) \cap c(\pi^*\mathcal{E}/\mathcal{F}) \cap [P] \\ &= c(\mathcal{F}) \cap c(\mathcal{L}) \cap c(\pi^*\mathcal{E}_2) \cap [P] \\ &= c(\pi^*\mathcal{E}_1) \cap c(\pi^*\mathcal{E}_2) \cap [P] \end{aligned}$$

Since $[P] = \pi^*[X]$ we win by Lemma 41.35.4. □

Lemma 41.37.4. *Let (S, δ) be as in Situation 41.7.1. Let X be locally of finite type over S. Let $\mathcal{L}_i$, $i = 1, \ldots, r$ be invertible $\mathcal{O}_X$-modules on X. Let $\mathcal{E}$ be a finite locally free rank r $\mathcal{O}_X$-module endowed with a filtration*

$$0 = \mathcal{E}_0 \subset \mathcal{E}_1 \subset \mathcal{E}_2 \subset \ldots \subset \mathcal{E}_r = \mathcal{E}$$

such that $\mathcal{E}_i/\mathcal{E}_{i-1} \cong \mathcal{L}_i$. Set $c_1(\mathcal{L}_i) = x_i$. Then

$$c(\mathcal{E}) = \prod\nolimits_{i=1}^{r}(1 + x_i)$$

is a valid polynomial relation among chern classes in the sense of Definition 41.36.1.

Proof. Apply Lemma 41.37.2 and induction. □

41.38. The splitting principle

In our setting it is not so easy to say what the splitting principle exactly says/is. Here is a possible formulation.

Lemma 41.38.1. *Let (S, δ) be as in Situation 41.7.1. Let X be locally of finite type over S. Let $\mathcal{E}$ be a finite locally free sheaf $\mathcal{E}$ on X of rank r. There exists a projective flat morphism of relative dimension d $\pi : P \to X$ such that*

(1) *for any morphism $f : Y \to X$ the map $\pi_Y^* : A_*(Y) \to A_{*+r}(Y \times_X P)$ is injective, and*
(2) *$\pi^*\mathcal{E}$ has a filtration with successive quotients $\mathcal{L}_1, \ldots, \mathcal{L}_r$ for some invertible $\mathcal{O}_P$-modules $\mathcal{L}_i$.*

Proof. Omitted. Hint: Use a composition of projective space bundles, i.e., a flag variety over X. □

The splitting principle refers to the practice of symbolically writing

$$c(\mathcal{E}) = \prod(1 + x_i)$$

with $x_i = c_1(\mathcal{L}_i)$. The expressions x_i are then called the *Chern roots* of $\mathcal{E}$. In order to prove polynomial relations among chern classes of vector bundles it is permissible to do calculations using the chern roots.

For example, let us calculate the chern classes of the dual vector bundle $\mathcal{E}^\wedge$. Note that if $\mathcal{E}$ has a filtration with subquotients invertible sheaves $\mathcal{L}_i$ then $\mathcal{E}^\wedge$ has a filtration with subquotients the invertible sheaves $\mathcal{L}_i^{-1}$. Hence if x_i are the chern roots of $\mathcal{E}$, then the $-x_i$ are the chern roots of $\mathcal{E}^\wedge$. It follows that

$$c_j(\mathcal{E}^\wedge) = (-1)^j c_j(\mathcal{E})$$

is a valid polynomial relation among chern classes.

In the same vain, let us compute the chern classes of a tensor product of vector bundles. Namely, suppose that $\mathcal{E}$, $\mathcal{F}$ are finite locally free of ranks r, s. Write

$$c(\mathcal{E}) = \prod\nolimits_{i=1}^{r}(1 + x_i), \quad c(\mathcal{E}) = \prod\nolimits_{j=1}^{s}(1 + y_j)$$

where x_i, y_j are the chern roots of $\mathcal{E}$, $\mathcal{F}$. Then we see that

$$c(\mathcal{E} \otimes_{\mathcal{O}_X} \mathcal{F}) = \prod\nolimits_{i,j}(1 + x_i + y_j)$$

Here are some examples of what this means in terms of chern classes

$$c_1(\mathcal{E} \otimes \mathcal{F}) = rc_1(\mathcal{F}) + sc_1(\mathcal{E})$$
$$c_2(\mathcal{E} \otimes \mathcal{F}) = r^2c_2(\mathcal{F}) + rsc_1(\mathcal{F})c_1(\mathcal{E}) + s^2c_2(\mathcal{E})$$

41.39. Chern classes and tensor product

We define the *Chern character* of a finite locally free sheaf of rank r to be the formal expression

$$ch(\mathcal{E}) := \sum\nolimits_{i=1}^{r} e^{x_i}$$

if the x_i are the chern roots of $\mathcal{E}$. Writing this in terms of chern classes $c_i = c_i(\mathcal{E})$ we see that

$$ch(\mathcal{E}) = r + c_1 + \frac{1}{2}(c_1^2 - 2c_2) + \frac{1}{6}(c_1^3 - 3c_1c_2 + 3c_3) + \frac{1}{24}(c_1^4 - 4c_1^2c_2 + 4c_1c_3 + 2c_2^2 - 4c_4) + \ldots$$

What does it mean that the coefficients are rational numbers? Well this simply means that we think of these as operations

$$ch_j(\mathcal{E}) \cap - : A_k(X) \longrightarrow A_{k-j}(X) \otimes_{\mathbf{Z}} \mathbf{Q}$$

and we think of polynomial relations among them as relations between these operations with values in the groups $A_{k-j}(Y) \otimes_{\mathbf{Z}} \mathbf{Q}$ for varying Y. By the above we have in case of an exact sequence

$$0 \to \mathcal{E}_1 \to \mathcal{E} \to \mathcal{E}_2 \to 0$$

that

$$ch(\mathcal{E}) = ch(\mathcal{E}_1) + ch(\mathcal{E}_2)$$

Using the Chern character we can express the compatibility of the chern classes and tensor product as follows:

$$ch(\mathcal{E}_1 \otimes_{\mathcal{O}_X} \mathcal{E}_2) = ch(\mathcal{E}_1)ch(\mathcal{E}_2)$$

This follows directly from the discussion of the chern roots of the tensor product in the previous section.

41.40. Todd classes

A final class associated to a vector bundle $\mathcal{E}$ of rank r is its *Todd class* $Todd(\mathcal{E})$. In terms of the chern roots $x_1, \ldots, x_r$ it is defined as

$$Todd(\mathcal{E}) = \prod_{i=1}^{r} \frac{x_i}{1 - e^{-x_i}}$$

In terms of the chern classes $c_i = c_i(\mathcal{E})$ we have

$$Todd(\mathcal{E}) = 1 + \frac{1}{2}c_1 + \frac{1}{12}(c_1^2 + c_2) + \frac{1}{24}c_1c_2 + \frac{1}{720}(-c_1^4 + 4c_1^2c_2 + 3c_2^2 + c_1c_3 - c_4) + \ldots$$

We have made the appropriate remarks about denominators in the previous section. It is the case that given an exact sequence

$$0 \to \mathcal{E}_1 \to \mathcal{E} \to \mathcal{E}_2 \to 0$$

we have

$$Todd(\mathcal{E}) = Todd(\mathcal{E}_1)Todd(\mathcal{E}_2).$$

41.41. Grothendieck-Riemann-Roch

Let (S, δ) be as in Situation 41.7.1. Let X, Y be locally of finite type over S. Let $\mathcal{E}$ be a finite locally free sheaf $\mathcal{E}$ on X of rank r. Let $f : X \to Y$ be a proper smooth morphism. Assume that $R^i f_* \mathcal{E}$ are locally free sheaves on Y of finite rank (for example if Y is a point). The Grothendieck-Riemann-Roch theorem implies that in this case we have

$$f_*(Todd(T_{X/Y})ch(\mathcal{E})) = \sum (-1)^i ch(R^i f_* \mathcal{E})$$

Here

$$T_{X/Y} = \mathcal{H}om_{\mathcal{O}_X}(\Omega_{X/Y}, \mathcal{O}_X)$$

is the relative tangent bundle of X over Y. The theorem is more general and becomes easier to prove when formulated in correct generality. We will return to this elsewhere (insert future reference here).

41.42. Other chapters

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CHAPTER 42

Adequate Modules

42.1. Introduction

For any scheme X the category $QCoh(\mathcal{O}_X)$ of quasi-coherent modules is abelian and a weak Serre subcategory of the abelian category of all $\mathcal{O}_X$-modules. The same thing works for the category of quasi-coherent modules on an algebraic space X viewed as a subcategory of the category of all $\mathcal{O}_X$-modules on the small étale site of X. Moreover, for a quasi-compact and quasi-separated morphism $f : X \to Y$ the pushforward f_* and higher direct images preserve quasi-coherency.

Next, let X be a scheme and let $\mathcal{O}$ be the structure sheaf on one of the big sites of X, say, the big fppf site. The category of quasi-coherent $\mathcal{O}$-modules is abelian (in fact it is equivalent to the category of usual quasi-coherent $\mathcal{O}_X$-modules on the scheme X we mentioned above) but its imbedding into $Mod(\mathcal{O})$ is not exact. An example is the map of quasi-coherent modules

$$\mathcal{O}_{\mathbf{A}^1_k} \longrightarrow \mathcal{O}_{\mathbf{A}^1_k}$$

on $\mathbf{A}^1_k = \mathrm{Spec}(k[x])$ given by multiplication by x. In the abelian category of quasi-coherent sheaves this map is injective, whereas in the abelian category of all $\mathcal{O}$-modules on the big site of $\mathbf{A}^1_k$ this map has a nontrivial kernel as we see by evaluating on sections over $\mathrm{Spec}(k[x]/(x)) = \mathrm{Spec}(k)$. Moreover, for a quasi-compact and quasi-separated morphism $f : X \to Y$ the functor $f_{big,*}$ does not preserve quasi-coherency.

In this chapter we introduce a larger category of modules, closely related to quasi-coherent modules, which "fixes" the two problems mentioned above.

42.2. Conventions

In this chapter we fix $\tau \in \{Zar, étale, smooth, syntomic, fppf\}$ and we fix a big τ-site Sch_τ as in Topologies, Section 33.2. All schemes will be objects of Sch_τ. In particular, given a scheme S we obtain sites $(Aff/S)_\tau \subset (Sch/S)_\tau$. The structure sheaf $\mathcal{O}$ on these sites is defined by the rule $\mathcal{O}(T) = \Gamma(T, \mathcal{O}_T)$.

All rings A will be such that $\mathrm{Spec}(A)$ is (isomorphic to) an object of Sch_τ. Given a ring A we denote Alg_A the category of A-algebras whose objects are the A-algebras B of the form $B = \Gamma(U, \mathcal{O}_U)$ where S is an affine object of Sch_τ. Thus given an affine scheme $S = \mathrm{Spec}(A)$ the functor

$$(Aff/S)_\tau \longrightarrow Alg_A, \quad U \longmapsto \mathcal{O}(U)$$

is an equivalence.

42.3. Adequate functors

In this section we discuss a topic closely related to direct images of quasi-coherent sheaves. Most of this material was taken from the paper [**Jaf97**].

Definition 42.3.1. Let A be a ring. A *module-valued functor* is a functor $F : Alg_A \to Ab$ such that

(1) for every object B of Alg_A the group $F(B)$ is endowed with the structure of a B-module, and
(2) for any morphism $B \to B'$ of Alg_A the map $F(B) \to F(B')$ is B-linear.

A *morphism of module-valued functors* is a transformation of functors $\varphi : F \to G$ such that $F(B) \to G(B)$ is B-linear for all $B \in \mathrm{Ob}(Alg_A)$.

Let $S = \mathrm{Spec}(A)$ be an affine scheme. The category of module-valued functors on Alg_A is equivalent to the category $PMod((Aff/S)_\tau, \mathcal{O})$ of presheaves of $\mathcal{O}$-modules. The equivalence is given by the rule which assigns to the module-valued functor F the presheaf $\mathcal{F}$ defined by the rule $\mathcal{F}(U) = F(\mathcal{O}(U))$. This is clear from the equivalence $(Aff/S)_\tau \to Alg_A$, $U \mapsto \mathcal{O}(U)$ given in Section 42.2. The quasi-inverse sets $F(B) = \mathcal{F}(\mathrm{Spec}(B))$.

An important special case of a module-valued functor comes about as follows. Let M be an A-module. Then we will denote $\underline{M}$ the module-valued functor $B \mapsto M \otimes_A B$ (with obvious B-module structure). Note that if $M \to N$ is a map of A-modules then there is an associated morphism $\underline{M} \to \underline{N}$ of module-valued functors. Conversely, any morphism of module-valued functors $\underline{M} \to \underline{N}$ comes from an A-module map $M \to N$ as the reader can see by evaluating on $B = A$. In other words Mod_A is a full subcategory of the category of module-valued functors on Alg_A.

Given and A-module map $\varphi : M \to N$ then $\mathrm{Coker}(\underline{M} \to \underline{N}) = \underline{Q}$ where $Q = \mathrm{Coker}(M \to N)$ because $\otimes$ is right exact. But this isn't the case for the kernel in general: for example an injective map of A-modules need not be injective after base change. Thus the following definition makes sense.

Definition 42.3.2. Let A be a ring. A module-valued functor F on Alg_A is called

(1) *adequate* if there exists a map of A-modules $M \to N$ such that F is isomorphic to $\mathrm{Ker}(\underline{M} \to \underline{N})$.
(2) *linearly adequate* if F is isomorphic to the kernel of a map $\underline{A^{\oplus n}} \to \underline{A^{\oplus m}}$.

Note that F is adequate if and only if there exists an exact sequence $0 \to F \to \underline{M} \to \underline{N}$ and F is linearly adequate if and only if there exists an exact sequence $0 \to F \to \underline{A^{\oplus n}} \to \underline{A^{\oplus m}}$.

Let A be a ring. In this section we will show the category of adequate functors on Alg_A is abelian (Lemmas 42.3.10 and 42.3.11) and has a set of generators (Lemma 42.3.6). We will also see that it is a weak Serre subcategory of the category of all module-valued functors on Alg_A (Lemma 42.3.16) and that it has arbitrary colimits (Lemma 42.3.12).

Lemma 42.3.3. *Let A be a ring. Let F be an adequate functor on Alg_A. If $B = \mathrm{colim}\, B_i$ is a filtered colimit of A-algebras, then $F(B) = \mathrm{colim}\, F(B_i)$.*

Proof. This holds because for any A-module M we have $M \otimes_A B = \mathrm{colim}\, M \otimes_A B_i$ (see Algebra, Lemma 10.11.9) and because filtered colimits commute with exact sequences, see Algebra, Lemma 10.8.9. □

Remark 42.3.4. Consider the category $Alg_{fp,A}$ whose objects are A-algebras B of the form $B = A[x_1, \ldots, x_n]/(f_1, \ldots, f_m)$ and whose morphisms are A-algebra maps. Every A-algebra B is a filtered colimit of finitely presented A-algebra, i.e., a filtered colimit of objects of $Alg_{fp,A}$. By Lemma 42.3.3 we conclude every adequate functor F is determined by its restriction to $Alg_{fp,A}$. For some questions we can therefore restrict to functors on $Alg_{fp,A}$. For example, the category of adequate functors does not depend on the choice of the big τ-site chosen in Section 42.2.

Lemma 42.3.5. *Let A be a ring. Let F be an adequate functor on Alg_A. If $B \to B'$ is flat, then $F(B) \otimes_B B' \to F(B')$ is an isomorphism.*

Proof. Choose an exact sequence $0 \to F \to \underline{M} \to \underline{N}$. This gives the diagram

$$\begin{array}{ccccccc} 0 & \longrightarrow & F(B) \otimes_B B' & \longrightarrow & (M \otimes_A B) \otimes_B B' & \longrightarrow & (N \otimes_A B) \otimes_B B' \\ & & \downarrow & & \downarrow & & \downarrow \\ 0 & \longrightarrow & F(B') & \longrightarrow & M \otimes_A B' & \longrightarrow & N \otimes_A B' \end{array}$$

where the rows are exact (the top one because $B \to B'$ is flat). Since the right two vertical arrows are isomorphisms, so is the left one. □

Lemma 42.3.6. *Let A be a ring. Let F be an adequate functor on Alg_A. Then there exists a surjection $L \to F$ with L a direct sum of linearly adequate functors.*

Proof. Choose an exact sequence $0 \to F \to \underline{M} \to \underline{N}$ where $\underline{M} \to \underline{N}$ is given by $\varphi : M \to N$. By Lemma 42.3.3 it suffices to construct $L \to F$ such that $L(B) \to F(B)$ is surjective for every finitely presented A-algebra B. Hence it suffices to construct, given a finitely presented A-algebra B and an element $\xi \in F(B)$ a map $L \to F$ with L linearly adequate such that ξ is in the image of $L(B) \to F(B)$. (Because there is a set worth of such pairs (B, ξ) up to isomorphism.)

To do this write $\sum_{i=1,\ldots,n} m_i \otimes b_i$ the image of ξ in $\underline{M}(B) = M \otimes_A B$. We know that $\sum \varphi(m_i) \otimes b_i = 0$ in $N \otimes_A B$. As N is a filtered colimit of finitely presented A-modules, we can find a finitely presented A-module N', a commutative diagram of A-modules

$$\begin{array}{ccc} A^{\oplus n} & \longrightarrow & N' \\ {\scriptstyle m_1,\ldots,m_n}\downarrow & & \downarrow \\ M & \longrightarrow & N \end{array}$$

such that $(b_1, \ldots, b_n)$ maps to zero in $N' \otimes_A B$. Choose a presentation $A^{\oplus l} \to A^{\oplus k} \to N' \to 0$. Choose a lift $A^{\oplus n} \to A^{\oplus k}$ of the map $A^{\oplus n} \to N'$ of the diagram. Then we see that there exist $(c_1, \ldots, c_l) \in B^{\oplus l}$ such that $(b_1, \ldots, b_n, c_1, \ldots, c_l)$ maps to zero in $B^{\oplus k}$ under the map $B^{\oplus n} \oplus B^{\oplus l} \to B^{\oplus k}$. Consider the commutative diagram

$$\begin{array}{ccc} A^{\oplus n} \oplus A^{\oplus l} & \longrightarrow & A^{\oplus k} \\ \downarrow & & \downarrow \\ M & \longrightarrow & N \end{array}$$

where the left vertical arrow is zero on the summand $A^{\oplus l}$. Then we see that L equal to the kernel of $\underline{A^{\oplus n+l}} \to \underline{A^{\oplus k}}$ works because the element $(b_1, \ldots, b_n, c_1, \ldots, c_l) \in L(B)$ maps to ξ. □

Consider a graded A-algebra $B = \bigoplus_{d \geq 0} B_d$. Then there are two A-algebra maps $p, a : B \to B[t, t^{-1}]$, namely $p : b \mapsto b$ and $a : b \mapsto t^{\deg(b)} b$ where b is homogeneous. If F is a module-valued functor on Alg_A, then we define

$$F(B)^{(k)} = \{\xi \in F(B) \mid t^k F(p)(\xi) = F(a)(\xi)\}. \tag{42.3.6.1}$$

For functors which behave well with respect to flat ring extensions this gives a direct sum decomposition. This amounts to the fact that representations of $\mathbf{G}_m$ are completely reducible.

Lemma 42.3.7. *Let A be a ring. Let F be a module-valued functor on Alg_A. Assume that for $B \to B'$ flat the map $F(B) \otimes_B B' \to F(B')$ is an isomorphism. Let B be a graded A-algebra. Then*

1. *$F(B) = \bigoplus_{k \in \mathbf{Z}} F(B)^{(k)}$, and*
2. *the map $B \to B_0 \to B$ induces map $F(B) \to F(B)$ whose image is contained in $F(B)^{(0)}$.*

Proof. Let $x \in F(B)$. The map $p : B \to B[t, t^{-1}]$ is free hence we know that

$$F(B[t, t^{-1}]) = \bigoplus\nolimits_{k \in \mathbf{Z}} F(p)(F(B)) \cdot t^k = \bigoplus\nolimits_{k \in \mathbf{Z}} F(B) \cdot t^k$$

as indicated we drop the $F(p)$ in the rest of the proof. Write $F(a)(x) = \sum t^k x_k$ for some $x_k \in F(B)$. Denote $\epsilon : B[t, t^{-1}] \to B$ the B-algebra map $t \mapsto 1$. Note that the compositions $\epsilon \circ p, \epsilon \circ a : B \to B[t, t^{-1}] \to B$ are the identity. Hence we see that

$$x = F(\epsilon)(F(a)(x)) = F(\epsilon)(\sum t^k x_k) = \sum x_k.$$

On the other hand, we claim that $x_k \in F(B)^{(k)}$. Namely, consider the commutative diagram

$$\begin{array}{ccc} B & \xrightarrow{a} & B[t, t^{-1}] \\ {\scriptstyle a'}\downarrow & & \downarrow{\scriptstyle f} \\ B[s, s^{-1}] & \xrightarrow{g} & B[t, s, t^{-1}, s^{-1}] \end{array}$$

where $a'(b) = s^{\deg(b)} b$, $f(b) = b$, $f(t) = st$ and $g(b) = t^{\deg(b)} b$ and $g(s) = s$. Then

$$F(g)(F(a'))(x) = F(g)(\sum s^k x_k) = \sum s^k F(a)(x_k)$$

and going the other way we see

$$F(f)(F(a))(x) = F(f)(\sum t^k x_k) = \sum (st)^k x_k.$$

Since $B \to B[s, t, s^{-1}, t^{-1}]$ is free we see that $F(B[t, s, t^{-1}, s^{-1}]) = \bigoplus_{k,l \in \mathbf{Z}} F(B) \cdot t^k s^l$ and comparing coefficients in the expressions above we find $F(a)(x_k) = t^k x_k$ as desired.

Finally, the image of $F(B_0) \to F(B)$ is contained in $F(B)^{(0)}$ because $B_0 \to B \xrightarrow{a} B[t, t^{-1}]$ is equal to $B_0 \to B \xrightarrow{p} B[t, t^{-1}]$. □

As a particular case of Lemma 42.3.7 note that

$$\underline{M}(B)^{(k)} = M \otimes_A B_k$$

where B_k is the degree k part of the graded A-algebra B.

Lemma 42.3.8. *Let A be a ring. Given a solid diagram*

$$\begin{array}{ccccccc} 0 & \longrightarrow & L & \longrightarrow & \underline{A^{\oplus n}} & \longrightarrow & \underline{A^{\oplus m}} \\ & & \downarrow{\scriptstyle \varphi} & \swarrow & & & \\ & & \underline{M} & & & & \end{array}$$

of module-valued functors on Alg_A with exact row there exists a dotted arrow making the diagram commute.

Proof. Suppose that the map $A^{\oplus n} \to A^{\oplus m}$ is given by the $m \times n$-matrix (a_{ij}). Consider the ring $B = A[x_1, \ldots, x_n]/(\sum a_{ij} x_j)$. The element $(x_1, \ldots, x_n) \in \underline{A^{\oplus n}}(B)$ maps to zero in $\underline{A^{\oplus m}}(B)$ hence is the image of a unique element $\xi \in L(B)$. Note that ξ has the following universal property: for any A-algebra C and any $\xi' \in L(C)$ there exists an A-algebra map $B \to C$ such that ξ maps to ξ' via the map $L(B) \to L(C)$.

Note that B is a graded A-algebra, hence we can use Lemmas 42.3.7 and 42.3.5 to decompose the values of our functors on B into graded pieces. Note that $\xi \in L(B)^{(1)}$ as $(x_1, \ldots, x_n)$ is an element of degree one in $\underline{A^{\oplus n}}(B)$. Hence we see that $\varphi(\xi) \in \underline{M}(B)^{(1)} = M \otimes_A B_1$. Since B_1 is generated by $x_1, \ldots, x_n$ as an A-module we can write $\varphi(\xi) = \sum m_i \otimes x_i$. Consider the map $A^{\oplus n} \to M$ which maps the ith basis vector to m_i. By construction the associated map $\underline{A^{\oplus n}} \to \underline{M}$ maps the element ξ to $\varphi(\xi)$. It follows from the universal property mentioned above that the diagram commutes. □

Lemma 42.3.9. *Let A be a ring. Let $\varphi : F \to \underline{M}$ be a map of module-valued functors on Alg_A with F adequate. Then Coker(φ) is adequate.*

Proof. By Lemma 42.3.6 we may assume that $F = \bigoplus L_i$ is a direct sum of linearly adequate functors. Choose exact sequences $0 \to L_i \to \underline{A^{\oplus n_i}} \to \underline{A^{\oplus m_i}}$. For each i choose a map $A^{\oplus n_i} \to M$ as in Lemma 42.3.8. Consider the diagram

$$\begin{array}{ccccccc} 0 & \longrightarrow & \bigoplus L_i & \longrightarrow & \bigoplus \underline{A^{\oplus n_i}} & \longrightarrow & \bigoplus \underline{A^{\oplus m_i}} \\ & & \downarrow & \swarrow & & & \\ & & \underline{M} & & & & \end{array}$$

Consider the A-modules

$$Q = \mathrm{Coker}(\bigoplus A^{\oplus n_i} \to M \oplus \bigoplus A^{\oplus m_i}) \quad \text{and} \quad P = \mathrm{Coker}(\bigoplus A^{\oplus n_i} \to \bigoplus A^{\oplus m_i}).$$

Then we see that Coker(φ) is isomorphic to the kernel of $\underline{Q} \to \underline{P}$. □

Lemma 42.3.10. *Let A be a ring. Let $\varphi : F \to G$ be a map of adequate functors on Alg_A. Then Coker(φ) is adequate.*

Proof. Choose an injection $G \to \underline{M}$. Then we have an injection $G/F \to \underline{M}/F$. By Lemma 42.3.9 we see that $\underline{M}/F$ is adequate, hence we can find an injection $\underline{M}/F \to \underline{N}$. Composing we obtain an injection $G/F \to \underline{N}$. By Lemma 42.3.9 the cokernel of the induced map $G \to \underline{N}$ is adequate hence we can find an injection $\underline{N}/G \to \underline{K}$. Then $0 \to G/F \to \underline{N} \to \underline{K}$ is exact and we win. □

Lemma 42.3.11. *Let A be a ring. Let $\varphi : F \to G$ be a map of adequate functors on Alg_A. Then Ker(φ) is adequate.*

Proof. Choose an injection $F \to \underline{M}$ and an injection $G \to \underline{N}$. Denote $F \to \underline{M \oplus N}$ the diagonal map so that

$$\begin{array}{ccc} F & \longrightarrow & G \\ \downarrow & & \downarrow \\ \underline{M \oplus N} & \longrightarrow & \underline{N} \end{array}$$

commutes. By Lemma 42.3.10 we can find a module map $M \oplus N \to K$ such that F is the kernel of $\underline{M \oplus N} \to \underline{K}$. Then $\mathrm{Ker}(\varphi)$ is the kernel of $\underline{M \oplus N} \to \underline{K \oplus N}$. □

Lemma 42.3.12. *Let A be a ring. An arbitrary direct sum of adequate functors on Alg_A is adequate. A colimit of adequate functors is adequate.*

Proof. The statement on direct sums is immediate. A general colimit can be written as a kernel of a map between direct sums, see Categories, Lemma 4.14.11. Hence this follows from Lemma 42.3.11. □

Lemma 42.3.13. *Let A be a ring. Let F, G be module-valued functors on Alg_A. Let $\varphi : F \to G$ be a transformation of functors. Assume*

1. *φ is additive,*
2. *for every A-algebra B and $\xi \in F(B)$ and unit $u \in B^*$ we have $\varphi(u\xi) = u\varphi(\xi)$ in $G(B)$, and*
3. *for any flat ring map $B \to B'$ we have $G(B) \otimes_B B' = G(B')$.*

Then φ is a morphism of module-valued functors.

Proof. Let B be an A-algebra, $\xi \in F(B)$, and $b \in B$. We have to show that $\varphi(b\xi) = b\varphi(\xi)$. Consider the ring map

$$B \to B' = B[x, y, x^{-1}, y^{-1}]/(x + y - b).$$

This ring map is faithfully flat, hence $G(B) \subset G(B')$. On the other hand

$$\varphi(b\xi) = \varphi((x+y)\xi) = \varphi(x\xi) + \varphi(y\xi) = x\varphi(\xi) + y\varphi(\xi) = (x+y)\varphi(\xi) = b\varphi(\xi)$$

because x, y are units in B'. Hence we win. □

Lemma 42.3.14. *Let A be a ring. Let $0 \to \underline{M} \to G \to L \to 0$ be a short exact sequence of module-valued functors on Alg_A with L linearly adequate. Then G is adequate.*

Proof. We first point out that for any flat A-algebra map $B \to B'$ the map $G(B) \otimes_B B' \to G(B')$ is an isomorphism. Namely, this holds for $\underline{M}$ and L, see Lemma 42.3.5 and hence follows for G by the five lemma. In particular, by Lemma 42.3.7 we see that $G(B) = \bigoplus_{k \in \mathbf{Z}} G(B)^{(k)}$ for any graded A-algebra B.

Choose an exact sequence $0 \to L \to \underline{A^{\oplus n}} \to \underline{A^{\oplus m}}$. Suppose that the map $A^{\oplus n} \to A^{\oplus m}$ is given by the $m \times n$-matrix (a_{ij}). Consider the graded A-algebra $B = A[x_1, \ldots, x_n]/(\sum a_{ij} x_j)$. The element $(x_1, \ldots, x_n) \in \underline{A^{\oplus n}}(B)$ maps to zero in $\underline{A^{\oplus m}}(B)$ hence is the image of a unique element $\xi \in L(B)$. Observe that $\xi \in L(B)^{(1)}$. The map

$$\mathrm{Hom}_A(B, C) \longrightarrow L(C), \quad f \longmapsto L(f)(\xi)$$

defines an isomorphism of functors. The reason is that f is determined by the images $c_i = f(x_i) \in C$ which have to satisfy the relations $\sum a_{ij} c_j = 0$. And $L(C)$ is the set of n-tuples $(c_1, \ldots, c_n)$ satisfying the relations $\sum a_{ij} c_j = 0$.

Since the value of each of the functors $\underline{M}$, G, L on B is a direct sum of its weight spaces (by the lemma mentioned above) exactness of $0 \to \underline{M} \to G \to L \to 0$ implies the sequence $0 \to \underline{M}(B)^{(1)} \to G(B)^{(1)} \to L(B)^{(1)} \to 0$ is exact. Thus we may choose an element $\theta \in G(B)^{(1)}$ mapping to ξ.

Consider the graded A-algebra

$$C = A[x_1, \ldots, x_n, y_1, \ldots, y_n]/(\sum a_{ij}x_j, \sum a_{ij}y_j)$$

There are three graded A-algebra homomorphisms $p_1, p_2, m : B \to C$ defined by the rules

$$p_1(x_i) = x_i, \quad p_1(x_i) = y_i, \quad m(x_i) = x_i + y_i.$$

We will show that the element

$$\tau = G(m)(\theta) - G(p_1)(\theta) - G(p_2)(\theta) \in G(C)$$

is zero. First, τ maps to zero in $L(C)$ by a direct calculation. Hence τ is an element of $\underline{M}(C)$. Moreover, since m, p_1, p_2 are graded algebra maps we see that $\tau \in G(C)^{(1)}$ and since $\underline{M} \subset G$ we conclude

$$\tau \in \underline{M}(C)^{(1)} = M \otimes_A C_1.$$

We may write uniquely $\tau = \underline{M}(p_1)(\tau_1) + \underline{M}(p_2)(\tau_2)$ with $\tau_i \in M \otimes_A B_1 = \underline{M}(B)^{(1)}$ because $C_1 = p_1(B_1) \oplus p_2(B_1)$. Consider the ring map $q_1 : C \to B$ defined by $x_i \mapsto x_i$ and $y_i \mapsto 0$. Then $\underline{M}(q_1)(\tau) = \underline{M}(q_1)(\underline{M}(p_1)(\tau_1) + \underline{M}(p_2)(\tau_2)) = \tau_1$. On the other hand, because $q_1 \circ m = q_1 \circ p_1$ we see that $G(q_1)(\tau) = -G(q_1 \circ p_2)(\tau)$. Since $q_1 \circ p_2$ factors as $B \to A \to B$ we see that $G(q_1 \circ p_2)(\tau)$ is in $G(B)^{(0)}$, see Lemma 42.3.7. Hence $\tau_1 = 0$ because it is in $G(B)^{(0)} \cap \underline{M}(B)^{(1)} \subset G(B)^{(0)} \cap G(B)^{(1)} = 0$. Similarly $\tau_2 = 0$, whence $\tau = 0$.

Since $\theta \in G(B)$ we obtain a transformation of functors

$$\psi : L(-) = \mathrm{Hom}_A(B, -) \longrightarrow G(-)$$

by mapping $f : B \to C$ to $G(f)(\theta)$. Since θ is a lift of ξ the map ψ is a right inverse of $G \to L$. In terms of ψ the statements proved above have the following meaning: $\tau = 0$ means that ψ is additive and $\theta \in G(B)^{(1)}$ implies that for any A-algebra D we have $\psi(ul) = u\psi(l)$ in $G(D)$ for $l \in L(D)$ and $u \in D^*$ a unit. This implies that ψ is a morphism of module-valued functors, see Lemma 42.3.13. Clearly this implies that $G \cong \underline{M} \oplus L$ and we win. □

Remark 42.3.15. Let A be a ring. The proof of Lemma 42.3.14 shows that any extension $0 \to \underline{M} \to E \to L \to 0$ of module-valued functors on Alg_A with L linearly adequate splits. It uses only the following properties of the module-valued functor $F = \underline{M}$:

(1) $F(B) \otimes_B B' \to F(B')$ is an isomorphism for a flat ring map $B \to B'$, and
(2) $F(C)^{(1)} = F(p_1)(F(B)^{(1)}) \oplus F(p_2)(F(B)^{(1)})$ where $B = A[x_1, \ldots, x_n]/(\sum a_{ij}x_j)$ and $C = A[x_1, \ldots, x_n, y_1, \ldots, y_n]/(\sum a_{ij}x_j, \sum a_{ij}y_j)$.

These two properties hold for any adequate functor F; details omitted. Hence we see that L is a projective object of the abelian category of adequate functors.

Lemma 42.3.16. *Let A be a ring. Let $0 \to F \to G \to H \to 0$ be a short exact sequence of module-valued functors on Alg_A. If F and H are adequate, so is G.*

Proof. Choose an exact sequence $0 \to F \to \underline{M} \to \underline{N}$. If we can show that $(\underline{M} \oplus G)/F$ is adequate, then G is the kernel of the map of adequate functors $(\underline{M} \oplus G)/F \to \underline{N}$, hence adequate by Lemma 42.3.11. Thus we may assume $F = \underline{M}$.

We can choose a surjection $L \to H$ where L is a direct sum of linearly adequate functors, see Lemma 42.3.6. If we can show that the pullback $G \times_H L$ is adequate, then G is the cokernel of the map $\mathrm{Ker}(L \to H) \to G \times_H L$ hence adequate by Lemma 42.3.10. Thus we may assume that $H = \bigoplus L_i$ is a direct sum of linearly adequate functors. By Lemma 42.3.14 each of the pullbacks $G \times_H L_i$ is adequate. By Lemma 42.3.12 we see that $\bigoplus G \times_H L_i$ is adequate. Then G is the cokernel of

$$\bigoplus\nolimits_{i \neq i'} F \longrightarrow \bigoplus G \times_H L_i$$

where ξ in the summand (i, i') maps to $(0, \ldots, 0, \xi, 0, \ldots, 0, -\xi, 0, \ldots, 0)$ with nonzero entries in the summands i and i'. Thus G is adequate by Lemma 42.3.10. □

Lemma 42.3.17. *Let $A \to A'$ be a ring map. If F is an adequate functor on Alg_A, then its restriction F' to $Alg_{A'}$ is adequate too.*

Proof. Choose an exact sequence $0 \to F \to \underline{M} \to \underline{N}$. Then $F'(B') = F(B') = \mathrm{Ker}(M \otimes_A B' \to N \otimes_A B')$. Since $M \otimes_A B' = M \otimes_A A' \otimes_{A'} B'$ and similarly for N we see that F' is the kernel of $\underline{M \otimes_A A'} \to \underline{N \otimes_A A'}$. □

Lemma 42.3.18. *Let $A \to A'$ be a ring map. If F' is an adequate functor on $Alg_{A'}$, then the module-valued functor $F : B \mapsto F'(A' \otimes_A B)$ on Alg_A is adequate too.*

Proof. Choose an exact sequence $0 \to F' \to \underline{M'} \to \underline{N'}$. Then

$$\begin{aligned} F(B) &= F'(A' \otimes_A B) \\ &= \mathrm{Ker}(M' \otimes_{A'} (A' \otimes_A B) \to N' \otimes_{A'} (A' \otimes_A B)) \\ &= \mathrm{Ker}(M' \otimes_A B \to N' \otimes_A B) \end{aligned}$$

Thus F is the kernel of $\underline{M} \to \underline{N}$ where $M = M'$ and $N = N'$ viewed as A-modules. □

Lemma 42.3.19. *Let $A = A_1 \times \ldots \times A_n$ be a product of rings. An adequate functor over A is the same thing as a sequence $F_1, \ldots, F_n$ of adequate functors F_i over A_i.*

Proof. This is true because an A-algebra B is canonically a product $B_1 \times \ldots \times B_n$ and the same thing holds for A-modules. Setting $F(B) = \coprod F_i(B_i)$ gives the correspondence. Details omitted. □

Lemma 42.3.20. *Let $A \to A'$ be a ring map and let F be a module-valued functor on Alg_A such that*

1. *the restriction F' of F to the category of A'-algebras is adequate, and*
2. *for any A-algebra B the sequence*

$$0 \to F(B) \to F(B \otimes_A A') \to F(B \otimes_A A' \otimes_A A')$$

is exact.

Then F is adequate.

Proof. The functors $B \to F(B \otimes_A A')$ and $B \mapsto F(B \otimes_A A' \otimes_A A')$ are adequate, see Lemmas 42.3.18 and 42.3.17. Hence F as a kernel of a map of adequate functors is adequate, see Lemma 42.3.11. □

42.4. Higher exts of adequate functors

Let A be a ring. In Lemma 42.3.16 we have seen that any extension of adequate functors in the category of module-valued functors on Alg_A is adequate. In this section we show that the same remains true for higher ext groups.

Lemma 42.4.1. *Let A be a ring. For every module-valued functor F on Alg_A there exists a morphism $Q(F) \to F$ of module-valued functors on Alg_A such that (1) $Q(F)$ is adequate and (2) for every adequate functor G the map $\mathrm{Hom}(G, Q(F)) \to \mathrm{Hom}(G, F)$ is a bijection.*

Proof. Choose a set $\{L_i\}_{i \in I}$ of linearly adequate functors such that every linearly adequate functor is isomorphic to one of the L_i. This is possible. Suppose that we can find $Q(F) \to F$ with (1) and (2)' or every $i \in I$ the map $\mathrm{Hom}(L_i, Q(F)) \to \mathrm{Hom}(L_i, F)$ is a bijection. Then (2) holds. Namely, combining Lemmas 42.3.6 and 42.3.11 we see that every adequate functor G sits in an exact sequence

$$K \to L \to G \to 0$$

with K and L direct sums of linearly adequate functors. Hence (2)' implies that $\mathrm{Hom}(L, Q(F)) \to \mathrm{Hom}(L, F)$ and $\mathrm{Hom}(K, Q(F)) \to \mathrm{Hom}(K, F)$ are bijections, whence the same thing for G.

Consider the category $\mathcal{I}$ whose objects are pairs (i, φ) where $i \in I$ and $\varphi : L_i \to F$ is a morphism. A morphism $(i, \varphi) \to (i', \varphi')$ is a map $\psi : L_i \to L_{i'}$ such that $\varphi' \circ \psi = \varphi$. Set

$$Q(F) = \mathrm{colim}_{(i,\varphi) \in \mathrm{Ob}(\mathcal{I})} L_i$$

There is a natural map $Q(F) \to F$, by Lemma 42.3.12 it is adequate, and by construction it has property (2)'. □

Lemma 42.4.2. *Let A be a ring. Denote $\mathcal{P}$ the category of module-valued functors on Alg_A and $\mathcal{A}$ the category of adequate functors on Alg_A. Denote $i : \mathcal{A} \to \mathcal{P}$ the inclusion functor. Denote $Q : \mathcal{P} \to \mathcal{A}$ the construction of Lemma 42.4.1. Then*

1. *i is fully faithful, exact, and its image is a weak Serre subcategory,*
2. *$\mathcal{P}$ has enough injectives,*
3. *the functor Q is a right adjoint to i hence left exact,*
4. *Q transforms injectives into injectives,*
5. *$\mathcal{A}$ has enough injectives.*

Proof. This lemma just collects some facts we have already seen so far. Part (1) is clear from the definitions, the characterization of weak Serre subcategories (see Homology, Lemma 12.9.3), and Lemmas 42.3.10, 42.3.11, and 42.3.16. Recall that $\mathcal{P}$ is equivalent to the category $PMod((Aff/\operatorname{Spec}(A))_\tau, \mathcal{O})$. Hence (2) by Injectives, Proposition 19.8.5. Part (3) follows from Lemma 42.4.1 and Categories, Lemma 4.24.3. Parts (4) and (5) follow from Homology, Lemmas 12.25.1 and 12.25.3. □

Let A be a ring. As in Formal Deformation Theory, Section 64.10 given an A-algebra B and an B-module N we set $B[N]$ equal to the R-algebra with underlying B-module $B \oplus N$ with multiplication given by $(b, m)(b', m') = (bb', bm' + b'm)$. Note that this construction is functorial in the pair (B, N) where morphism $(B, N) \to (B', N')$ is given by an A-algebra map $B \to B'$ and an B-module map $N \to N'$. In some sense the functor TF of pairs defined in the following lemma is the tangent

space of F. Below we will only consider pairs (B, N) such that $B[N]$ is an object of Alg_A.

Lemma 42.4.3. *Let A be a ring. Let F be a module valued functor. For every $B \in \mathrm{Ob}(Alg_A)$ and B-module N there is a canonical decomposition*

$$F(B[N]) = F(B) \oplus TF(B, N)$$

characterized by the following properties

1. *$TF(B, N) = Ker(F(B[N]) \to F(B))$,*
2. *there is a B-module structure $TF(B, N)$ compatible with $B[N]$-module structure on $F(B[N])$,*
3. *TF is a functor from the category of pairs (B, N),*
4. *there are canonical maps $N \otimes_B F(B) \to TF(B, N)$ inducing a transformation between functors defined on the category of pairs (B, N),*
5. *$TF(B, 0) = 0$ and the map $TF(B, N) \to TF(B, N')$ is zero when $N \to N'$ is the zero map.*

Proof. Since $B \to B[N] \to B$ is the identity we see that $F(B) \to F(B[N])$ is a direct summand whose complement is $TF(N, B)$ as defined in (1). This construction is functorial in the pair (B, N) simply because given a morphism of pairs $(B, N) \to (B', N')$ we obtain a commutative diagram

$$\begin{array}{ccccc} B' & \longrightarrow & B'[N'] & \longrightarrow & B' \\ \uparrow & & \uparrow & & \uparrow \\ B & \longrightarrow & B[N] & \longrightarrow & B \end{array}$$

in Alg_A. The B-module structure comes from the $B[N]$-module structure and the ring map $B \to B[N]$. The map in (4) is the composition

$$N \otimes_B F(B) \longrightarrow B[N] \otimes_{B[N]} F(B[N]) \longrightarrow F(B[N])$$

whose image is contained in $TF(B, N)$. (The first arrow uses the inclusions $N \to B[N]$ and $F(B) \to F(B[N])$ and the second arrow is the multiplication map.) If $N = 0$, then $B = B[N]$ hence $TF(B, 0) = 0$. If $N \to N'$ is zero then it factors as $N \to 0 \to N'$ hence the induced map is zero since $TF(B, 0) = 0$. □

Let A be a ring. Let M be an A-module. Then the module-valued functor $\underline{M}$ has tangent space $T\underline{M}$ given by the rule $T\underline{M}(B, N) = N \otimes_A M$. In particular, for B given, the functor $N \mapsto T\underline{M}(B, N)$ is additive and right exact. It turns out this also holds for injective module-valued functors.

Lemma 42.4.4. *Let A be a ring. Let I be an injective object of the category of module-valued functors. Then for any $B \in \mathrm{Ob}(Alg_A)$ and short exact sequence $0 \to N_1 \to N \to N_2 \to 0$ of B-modules the sequence*

$$TI(B, N_1) \to TI(B, N) \to TI(B, N_2) \to 0$$

is exact.

Proof. We will use the results of Lemma 42.4.3 without further mention. Denote $h : Alg_A \to Sets$ the functor given by $h(C) = \mathrm{Mor}_A(B[N], C)$. Similarly for h_1 and h_2. The map $B[N] \to B[N_2]$ corresponding to the surjection $N \to N_2$ is surjective.

It corresponds to a map $h_2 \to h$ such that $h_2(C) \to h(C)$ is injective for all A-algebras C. On the other hand, there are two maps $p, q : h \to h_1$, corresponding to the zero map $N_1 \to N$ and the injection $N_1 \to N$. Note that

$$h_2 \longrightarrow h \rightrightarrows h_1$$

is an equalizer diagram. Denote $\mathcal{O}_h$ the module-valued functor $C \mapsto \bigoplus_{h(C)} C$. Similarly for $\mathcal{O}_{h_1}$ and $\mathcal{O}_{h_2}$. Note that

$$\mathrm{Hom}_{\mathcal{P}}(\mathcal{O}_h, F) = F(B[N])$$

where $\mathcal{P}$ is the category of of module-valued functors on Alg_A. We claim there is an equalizer diagram

$$\mathcal{O}_{h_2} \longrightarrow \mathcal{O}_h \rightrightarrows \mathcal{O}_{h_1}$$

in $\mathcal{P}$. Namely, suppose that $C \in \mathrm{Ob}(Alg_A)$ and $\xi = \sum_{i=1,\ldots,n} c_i \cdot f_i$ where $c_i \in C$ and $f_i : B[N] \to C$ is an element of $\mathcal{O}_h(C)$. If $p(\xi) = q(\xi)$, then we see that

$$\sum c_i \cdot f_i \circ z = \sum c_i \cdot f_i \circ y$$

where $z, y : B[N_1] \to B[N]$ are the maps $z : (b, m_1) \mapsto (b, 0)$ and $y : (b, m_1) \mapsto (b, m_1)$. This means that for every i there exists a j such that $f_j \circ z = f_i \circ y$. Clearly, this implies that $f_i(N_1) = 0$, i.e., f_i factors through a unique map $\overline{f}_i : B[N_2] \to C$. Hence ξ is the image of $\overline{\xi} = \sum c_i \cdot \overline{f}_i$. Since I is injective, it transforms this equalizer diagram into a coequalizer diagram

$$I(B[N_1]) \rightrightarrows I(B[N]) \longrightarrow I(B[N_2])$$

This diagram is compatible with the direct sum decompositions $I(B[N]) = I(B) \oplus TI(B, N)$ and $I(B[N_i]) = I(B) \oplus TI(B, N_i)$. The zero map $N \to N_1$ induces the zero map $TI(B, N) \to TI(B, N_1)$. Thus we see that the coequalizer property above means we have an exact sequence $TI(B, N_1) \to TI(B, N) \to TI(B, N_2) \to 0$ as desired. □

Lemma 42.4.5. *Let A be a ring. Let F be a module-valued functor such that for any $B \in \mathrm{Ob}(Alg_A)$ the functor $TF(B, -)$ on B-modules transforms a short exact sequence of B-modules into a right exact sequence. Then*

1. *$TF(B, N_1 \oplus N_2) = TF(B, N_1) \oplus TF(B, N_2)$,*
2. *there is a second functorial B-module structure on $TF(B, N)$ defined by setting $x \cdot b = TF(B, b \cdot 1_N)(x)$ for $x \in TF(B, N)$ and $b \in B$,*
3. *the canonical map $N \otimes_B F(B) \to TF(B, N)$ of Lemma 42.4.3 is B-linear also with respect to the second B-module structure,*
4. *given a finitely presented B-module N there is a canonical isomorphism $TF(B, B) \otimes_B N \to TF(B, N)$ where the tensor product uses the second B-module structure on $TF(B, B)$.*

Proof. We will use the results of Lemma 42.4.3 without further mention. The maps $N_1 \to N_1 \oplus N_2$ and $N_2 \to N_1 \oplus N_2$ give a map $TF(B, N_1) \oplus TF(B, N_2) \to TF(B, N_1 \oplus N_2)$ which is injective since the maps $N_1 \oplus N_2 \to N_1$ and $N_1 \oplus N_2 \to N_2$ induce an inverse. Since TF is right exact we see that $TF(B, N_1) \to TF(B, N_1 \oplus N_2) \to TF(B, N_2) \to 0$ is exact. Hence $TF(B, N_1) \oplus TF(B, N_2) \to TF(B, N_1 \oplus N_2)$ is an isomorphism. This proves (1).

To see (2) the only thing we need to show is that $x \cdot (b_1 + b_2) = x \cdot b_1 + x \cdot b_2$. (Associativity and additivity are clear.) To see this consider

$$N \xrightarrow{(b_1, b_2)} N \oplus N \xrightarrow{+} N$$

and apply $TF(B, -)$.

Part (3) follows immediately from the fact that $N \otimes_B F(B) \to TF(B, N)$ is functorial in the pair (B, N).

Suppose N is a finitely presented B-module. Choose a presentation $B^{\oplus m} \to B^{\oplus n} \to N \to 0$. This gives an exact sequence

$$TF(B, B^{\oplus m}) \to TF(B, B^{\oplus n}) \to TF(B, N) \to 0$$

by right exactness of $TF(B, -)$. By part (1) we can write $TF(B, B^{\oplus m}) = TF(B, B)^{\oplus m}$ and $TF(B, B^{\oplus n}) = TF(B, B)^{\oplus n}$. Next, suppose that $B^{\oplus m} \to B^{\oplus n}$ is given by the matrix $T = (b_{ij})$. Then the induced map $TF(B, B)^{\oplus m} \to TF(B, B)^{\oplus n}$ is given by the matrix with entries $TF(B, b_{ij} \cdot 1_B)$. This combined with right exactness of $\otimes$ proves (4). □

Example 42.4.6. Let F be a module-valued functor as in Lemma 42.4.5. It is not always the case that the two module structures on $TF(B, N)$ agree. Here is an example. Suppose $A = \mathbf{F}_p$ where p is a prime. Set $F(B) = B$ but with B-module structure given by $b \cdot x = b^p x$. Then $TF(B, N) = N$ with B-module structure given by $b \cdot x = b^p x$ for $x \in N$. However, the second B-module structure is given by $x \cdot b = bx$. Note that in this case the canonical map $N \otimes_B F(B) \to TF(B, N)$ is zero as raising an element $n \in B[N]$ to the pth power is zero.

In the following lemma we will frequently use the observation that if $0 \to F \to G \to H \to 0$ is an exact sequence of module-valued functors on Alg_A, then for any pair (B, N) the sequence $0 \to TF(B, N) \to TG(B, N) \to TH(B, N) \to 0$ is exact. This follows from the fact that $0 \to F(B[N]) \to G(B[N]) \to H(B[N]) \to 0$ is exact.

Lemma 42.4.7. *Let A be a ring. For F a module-valued functor on Alg_A say $(*)$ holds if for all $B \in \mathrm{Ob}(Alg_A)$ the functor $TF(B, -)$ on B-modules transforms a short exact sequence of B-modules into a right exact sequence. Let $0 \to F \to G \to H \to 0$ be a short exact sequence of module-valued functors on Alg_A.*

1. *If $(*)$ holds for F, G then $(*)$ holds for H.*
2. *If $(*)$ holds for F, H then $(*)$ holds for G.*
3. *If $H' \to H$ is morphism of module-valued functors on Alg_A and $(*)$ holds for F, G, H, and H', then $(*)$ holds for $G \times_H H'$.*

Proof. Let B be given. Let $0 \to N_1 \to N_2 \to N_3 \to 0$ be a short exact sequence of B-modules. Part (1) follows from a diagram chase in the diagram

$$\begin{array}{ccccccccc}
0 & \longrightarrow & TF(B, N_1) & \longrightarrow & TG(B, N_1) & \longrightarrow & TH(B, N_1) & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & TF(B, N_2) & \longrightarrow & TG(B, N_2) & \longrightarrow & TH(B, N_2) & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & TF(B, N_3) & \longrightarrow & TG(B, N_3) & \longrightarrow & TH(B, N_3) & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & & & \\
 & & 0 & & 0 & & & &
\end{array}$$

with exact horizontal rows and exact columns involving TF and TG. To prove part (2) we do a diagram chase in the diagram

$$\begin{array}{ccccccccc}
0 & \longrightarrow & TF(B, N_1) & \longrightarrow & TG(B, N_1) & \longrightarrow & TH(B, N_1) & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & TF(B, N_2) & \longrightarrow & TG(B, N_2) & \longrightarrow & TH(B, N_2) & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & TF(B, N_3) & \longrightarrow & TG(B, N_3) & \longrightarrow & TH(B, N_3) & \longrightarrow & 0 \\
 & & \downarrow & & & & \downarrow & & \\
 & & 0 & & & & 0 & &
\end{array}$$

with exact horizontal rows and exact columns involving TF and TH. Part (3) follows from part (2) as $G \times_H H'$ sits in the exact sequence $0 \to F \to G \times_H H' \to H' \to 0$. □

Most of the work in this section was done in order to prove the following key vanishing result.

Lemma 42.4.8. *Let A be a ring. Let M, P be A-modules with P of finite presentation. Then $Ext^i_{\mathcal{P}}(\underline{P}, \underline{M}) = 0$ for $i > 0$ where $\mathcal{P}$ is the category of module-valued functors on Alg_A.*

Proof. Choose an injective resolution $\underline{M} \to I^\bullet$ in $\mathcal{P}$, see Lemma 42.4.2. By Derived Categories, Lemma 13.26.2 any element of $\mathrm{Ext}^i_{\mathcal{P}}(\underline{P}, \underline{M})$ comes from a morphism $\varphi : \underline{P} \to I^i$ with $d^i \circ \varphi = 0$. We will prove that the Yoneda extension

$$E : 0 \to \underline{M} \to I^0 \to \ldots \to I^{i-1} \times_{\mathrm{Ker}(d^i)} \underline{P} \to \underline{P} \to 0$$

of $\underline{P}$ by $\underline{M}$ associated to φ is trivial, which will prove the lemma by Derived Categories, Lemma 13.26.5.

For F a module-valued functor on Alg_A say $(*)$ holds if for all $B \in \mathrm{Ob}(Alg_A)$ the functor $TF(B, -)$ on B-modules transforms a short exact sequence of B-modules into a right exact sequence. Recall that the module-valued functors $\underline{M}, I^n, \underline{P}$ each have property $(*)$, see Lemma 42.4.4 and the remarks preceding it. By splitting $0 \to \underline{M} \to I^\bullet$ into short exact sequences we find that each of the functors $\mathrm{Im}(d^{n-1}) =$

$\mathrm{Ker}(d^n) \subset I^n$ has property $(*)$ by Lemma 42.4.7 and also that $I^{i-1} \times_{\mathrm{Ker}(d^i)} \underline{P}$ has property $(*)$.

Thus we may assume the Yoneda extension is given as

$$E : 0 \to \underline{M} \to F_{i-1} \to \ldots \to F_0 \to \underline{P} \to 0$$

where each of the module-valued functors F_j has property $(*)$. Set $G_j(B) = TF_j(B, B)$ viewed as a B-module via the *second* B-module structure defined in Lemma 42.4.5. Since TF_j is a functor on pairs we see that G_j is a module-valued functor on Alg_A. Moreover, since E is an exact sequence the sequence $G_{j+1} \to G_j \to G_{j-1}$ is exact (see remark preceding Lemma 42.4.7). Observe that $T\underline{M}(B, B) = M \otimes_A B = \underline{M}(B)$ and that the two B-module structures agree on this. Thus we obtain a Yoneda extension

$$E' : 0 \to \underline{M} \to G_{i-1} \to \ldots \to G_0 \to \underline{P} \to 0$$

Moreover, the canonical maps

$$F_j(B) = B \otimes_B F_j(B) \longrightarrow TF_j(B, B) = G_j(B)$$

of Lemma 42.4.3 (4) are B-linear by Lemma 42.4.5 (3) and functorial in B. Hence a map

$$\begin{array}{ccccccccccc}
0 & \longrightarrow & \underline{M} & \longrightarrow & F_{i-1} & \longrightarrow & \ldots & \longrightarrow & F_0 & \longrightarrow & \underline{P} & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle 1} & & \downarrow & & & & \downarrow & & \downarrow{\scriptstyle 1} & & \\
0 & \longrightarrow & \underline{M} & \longrightarrow & G_{i-1} & \longrightarrow & \ldots & \longrightarrow & G_0 & \longrightarrow & \underline{P} & \longrightarrow & 0
\end{array}$$

of Yoneda extensions. In particular we see that E and E' have the same class in $\mathrm{Ext}^i_{\mathcal{P}}(\underline{P}, \underline{M})$ by the lemma on Yoneda Exts mentioned above. Finally, let N be a A-module of finite presentation. Then we see that

$$0 \to T\underline{M}(A, N) \to TF_{i-1}(A, N) \to \ldots \to TF_0(A, N) \to T\underline{P}(A, N) \to 0$$

is exact. By Lemma 42.4.5 (4) with $B = A$ this translates into the exactness of the sequence of A-modules

$$0 \to M \otimes_A N \to G_{i-1}(A) \otimes_A N \to \ldots \to G_0(A) \otimes_A N \to P \otimes_A N \to 0$$

Hence the sequence of A-modules $0 \to M \to G_{i-1}(A) \to \ldots \to G_0(A) \to P \to 0$ is universally exact, in the sense that it remains exact on tensoring with any finitely presented A-module N. Let $K = \mathrm{Ker}(G_0(A) \to P)$ so that we have exact sequences

$$0 \to K \to G_0(A) \to P \to 0 \quad \text{and} \quad G_2(A) \to G_1(A) \to K \to 0$$

Tensoring the second sequence with N we obtain that $K \otimes_A N = \mathrm{Coker}(G_2(A) \otimes_A N \to G_1(A) \otimes_A N)$. Exactness of $G_2(A) \otimes_A N \to G_1(A) \otimes_A N \to G_0(A) \otimes_A N$ then implies that $K \otimes_A N \to G_0(A) \otimes_A N$ is injective. By Algebra, Theorem 10.77.3 this means that the A-module extension $0 \to K \to G_0(A) \to P \to 0$ is exact, and because P is assumed of finite presentation this means the sequence is split, see Algebra, Lemma 10.77.4. Any splitting $P \to G_0(A)$ defines a map $\underline{P} \to G_0$ which splits the surjection $G_0 \to \underline{P}$. Thus the Yoneda extension E' is equivalent to the trivial Yoneda extension and we win. □

Lemma 42.4.9. *Let A be a ring. Let M be an A-module. Let L be a linearly adequate functor on Alg_A. Then $Ext^i_{\mathcal{P}}(L, \underline{M}) = 0$ for $i > 0$ where $\mathcal{P}$ is the category of module-valued functors on Alg_A.*

Proof. Since L is linearly adequate there exists an exact sequence

$$0 \to L \to \underline{A^{\oplus m}} \to \underline{A^{\oplus n}} \to \underline{P} \to 0$$

Here $P = \mathrm{Coker}(A^{\oplus m} \to A^{\oplus n})$ is the cokernel of the map of finite free A-modules which is given by the definition of linearly adequate functors. By Lemma 42.4.8 we have the vanishing of $\mathrm{Ext}^i_{\mathcal{P}}(\underline{P}, \underline{M})$ and $\mathrm{Ext}^i_{\mathcal{P}}(\underline{A}, \underline{M})$ for $i > 0$. Let $K = \mathrm{Ker}(\underline{A^{\oplus n}} \to \underline{P})$. By the long exact sequence of Ext groups associated to the exact sequence $0 \to K \to \underline{A^{\oplus n}} \to \underline{P} \to 0$ we conclude that $\mathrm{Ext}^i_{\mathcal{P}}(K, \underline{M}) = 0$ for $i > 0$. Repeating with the sequence $0 \to L \to \underline{A^{\oplus m}} \to K \to 0$ we win. □

Lemma 42.4.10. *With notation as in Lemma 42.4.2 we have $R^pQ(F) = 0$ for all $p > 0$ and any adequate functor F.*

Proof. Choose an exact sequence $0 \to F \to \underline{M^0} \to \underline{M^1}$. Set $M^2 = \mathrm{Coker}(M^0 \to M^1)$ so that $0 \to F \to \underline{M^0} \to \underline{M^1} \to \underline{M^2} \to 0$ is a resolution. By Derived Categories, Lemma 13.20.3 we obtain a spectral sequence

$$R^pQ(\underline{M^q}) \Rightarrow R^{p+q}Q(F)$$

Since $Q(\underline{M^q}) = \underline{M^q}$ it suffices to prove $R^pQ(\underline{M}) = 0$, $p > 0$ for any A-module M.

Choose an injective resolution $\underline{M} \to I^\bullet$ in the category $\mathcal{P}$. Suppose that $R^iQ(\underline{M})$ is nonzero. Then $\mathrm{Ker}(Q(I^i) \to Q(I^{i+1}))$ is strictly bigger than the image of $Q(I^{i-1}) \to Q(I^i)$. Hence by Lemma 42.3.6 there exists a linearly adequate functor L and a map $\varphi : L \to Q(I^i)$ mapping into the kernel of $Q(I^i) \to Q(I^{i+1})$ which does not factor through the image of $Q(I^{i-1}) \to Q(I^i)$. Because Q is a left adjoint to the inclusion functor the map φ corresponds to a map $\varphi' : L \to I^i$ with the same properties. Thus φ' gives a nonzero element of $\mathrm{Ext}^i_{\mathcal{P}}(L, \underline{M})$ contradicting Lemma 42.4.9. □

42.5. Adequate modules

In Descent, Section 34.7 we have seen that quasi-coherent modules on a scheme S are the same as quasi-coherent modules on any of the big sites $(Sch/S)_\tau$ associated to S. We have seen that there are two issues with this identification:

(1) $QCoh(\mathcal{O}_S) \to Mod((Sch/S)_\tau, \mathcal{O})$, $\mathcal{F} \mapsto \mathcal{F}^a$ is not exact in general, and
(2) given a quasi-compact and quasi-separated morphism $f : X \to S$ the functor f_* does not preserve quasi-coherent sheaves on the big sites in general.

Part (1) means that we cannot define a triangulated subcategory of $D(\mathcal{O})$ consisting of complexes whose cohomology sheaves are quasi-coherent. Part (2) means that $Rf_*\mathcal{F}$ isn't a complex with quasi-coherent cohomology sheaves even when $\mathcal{F}$ is quasi-coherent and f is quasi-compact and quasi-separated. Moreover, the examples given in the proofs of Descent, Lemma 34.7.13 and Descent, Proposition 34.7.14 are not of a pathological nature.

In this section we discuss a slightly larger category of $\mathcal{O}$-modules on $(Sch/S)_\tau$ with contains the quasi-coherent modules, is abelian, and is preserved under f_* when f is quasi-compact and quasi-separated. To do this, suppose that S is a scheme. Let $\mathcal{F}$ be a presheaf of $\mathcal{O}$-modules on $(Sch/S)_\tau$. For any affine object $U = \mathrm{Spec}(A)$ of

$(Sch/S)_\tau$ we can restrict $\mathcal{F}$ to $(Aff/U)_\tau$ to get a presheaf of $\mathcal{O}$-modules on this site. The corresponding module-valued functor, see Section 42.3, will be denoted

$$F = F_{\mathcal{F},A} : Alg_A \longrightarrow Ab, \quad B \longmapsto \mathcal{F}(\mathrm{Spec}(B))$$

The assignment $\mathcal{F} \mapsto F_{\mathcal{F},A}$ is an exact functor of abelian categories.

Definition 42.5.1. A sheaf of $\mathcal{O}$-modules $\mathcal{F}$ on $(Sch/S)_\tau$ is *adequate* if there exists a τ-covering $\{\mathrm{Spec}(A_i) \to S\}_{i\in I}$ such that $F_{\mathcal{F},A_i}$ is adequate for all $i \in I$.

We will see below that the category of adequate $\mathcal{O}$-modules is independent of the chosen topology τ.

Lemma 42.5.2. *Let S be a scheme. Let $\mathcal{F}$ be an adequate $\mathcal{O}$-module on $(Sch/S)_\tau$. For any affine scheme $\mathrm{Spec}(A)$ over S the functor $F_{\mathcal{F},A}$ is adequate.*

Proof. Let $\{\mathrm{Spec}(A_i) \to S\}_{i\in I}$ be a τ-covering such that $F_{\mathcal{F},A_i}$ is adequate for all $i \in I$. We can find a standard affine τ-covering $\{\mathrm{Spec}(A'_j) \to \mathrm{Spec}(A)\}_{j=1,\ldots,m}$ such that $\mathrm{Spec}(A'_j) \to \mathrm{Spec}(A) \to S$ factors through $\mathrm{Spec}(A_{i(j)})$ for some $i(j) \in I$. Then we see that $F_{\mathcal{F},A'_j}$ is the restriction of $F_{\mathcal{F},A_{i(j)}}$ to the category of A'_j-algebras. Hence $F_{\mathcal{F},A'_j}$ is adequate by Lemma 42.3.17. By Lemma 42.3.19 the sequence $F_{\mathcal{F},A'_j}$ corresponds to an adequate "product" functor F' over $A' = A'_1 \times \ldots \times A'_m$. As $\mathcal{F}$ is a sheaf (for the Zariski topology) this product functor F' is equal to $F_{\mathcal{F},A'}$, i.e., is the restriction of F to A'-algebras. Finally, $\{\mathrm{Spec}(A') \to \mathrm{Spec}(A)\}$ is a τ-covering. It follows from Lemma 42.3.20 that $F_{\mathcal{F},A}$ is adequate. □

Lemma 42.5.3. *Let $S = \mathrm{Spec}(A)$ be an affine scheme. The category of adequate $\mathcal{O}$-modules on $(Sch/S)_\tau$ is equivalent to the category of adequate module-valued functors on Alg_A.*

Proof. Given an adequate module $\mathcal{F}$ the functor $F_{\mathcal{F},A}$ is adequate by Lemma 42.5.2. Given an adequate functor F we choose an exact sequence $0 \to F \to \underline{M} \to \underline{N}$ and we consider the $\mathcal{O}$-module $\mathcal{F} = \mathrm{Ker}(M^a \to N^a)$ where M^a denotes the quasi-coherent $\mathcal{O}$-module on $(Sch/S)_\tau$ associated to the quasi-coherent sheaf $\widetilde{M}$ on S. Note that $F = F_{\mathcal{F},A}$, in particular the module $\mathcal{F}$ is adequate by definition. We omit the proof that the constructions define mutually inverse equivalences of categories. □

Lemma 42.5.4. *Let $f : T \to S$ be a morphism of schemes. The pullback $f^*\mathcal{F}$ of an adequate $\mathcal{O}$-module $\mathcal{F}$ on $(Sch/S)_\tau$ is an adequate $\mathcal{O}$-module on $(Sch/T)_\tau$.*

Proof. The pullback map $f^* : Mod((Sch/S)_\tau, \mathcal{O}) \to Mod((Sch/T)_\tau, \mathcal{O})$ is given by restriction, i.e., $f^*\mathcal{F}(V) = \mathcal{F}(V)$ for any scheme V over T. Hence this lemma follows immediately from Lemma 42.5.2 and the definition. □

Here is a characterization of the category of adequate $\mathcal{O}$-modules. To understand the significance, consider a map $\mathcal{G} \to \mathcal{H}$ of quasi-coherent $\mathcal{O}_S$-modules on a scheme S. The cokernel of the associated map $\mathcal{G}^a \to \mathcal{H}^a$ of $\mathcal{O}$-modules is quasi-coherent because it is equal to $(\mathcal{H}/\mathcal{G})^a$. But the kernel of $\mathcal{G}^a \to \mathcal{H}^a$ in general isn't quasi-coherent. However, it is adequate.

Lemma 42.5.5. *Let S be a scheme. Let $\mathcal{F}$ be an $\mathcal{O}$-module on $(Sch/S)_\tau$. The following are equivalent*

(1) *$\mathcal{F}$ is adequate,*

(2) *there exists an affine open covering* $S = \bigcup S_i$ *and maps of quasi-coherent* $\mathcal{O}_{S_i}$*-modules* $\mathcal{G}_i \to \mathcal{H}_i$ *such that* $\mathcal{F}|_{(Sch/S_i)_\tau}$ *is the kernel of* $\mathcal{G}_i^a \to \mathcal{H}_i^a$
(3) *there exists a* τ*-covering* $\{S_i \to S\}_{i\in I}$ *and maps of* $\mathcal{O}_{S_i}$*-quasi-coherent modules* $\mathcal{G}_i \to \mathcal{H}_i$ *such that* $\mathcal{F}|_{(Sch/S_i)_\tau}$ *is the kernel of* $\mathcal{G}_i^a \to \mathcal{H}_i^a$*,*
(4) *there exists a* τ*-covering* $\{f_i : S_i \to S\}_{i\in I}$ *such that each* $f_i^*\mathcal{F}$ *is adequate,*
(5) *for any affine scheme* U *over* S *the restriction* $\mathcal{F}|_{(Sch/U)_\tau}$ *is the kernel of a map* $\mathcal{G}^a \to \mathcal{H}^a$ *of quasi-coherent* $\mathcal{O}_U$*-modules.*

Proof. Let $U = \mathrm{Spec}(A)$ be an affine scheme over S. Set $F = F_{\mathcal{F},A}$. By definition, the functor F is adequate if and only if there exists a map of A-modules $M \to N$ such that $F = \mathrm{Ker}(\underline{M} \to \underline{N})$. Combining with Lemmas 42.5.2 and 42.5.3 we see that (1) and (5) are equivalent.

It is clear that (5) implies (2) and (2) implies (3). If (3) holds then we can refine the covering $\{S_i \to S\}$ such that each $S_i = \mathrm{Spec}(A_i)$ is affine. Then we see, by the prelimiary remarks of the proof, that $F_{\mathcal{F},A_i}$ is adequate. Thus $\mathcal{F}$ is adequate by definition. Hence (3) implies (1).

Finally, (4) is equivalent to (1) using Lemma 42.5.4 for one direction and that a composition of τ-coverings is a τ-covering for the other. □

Just like is true for quasi-coherent sheaves the category of adequate modules is independent of the topology.

Lemma 42.5.6. *Let* $\mathcal{F}$ *be an adequate* $\mathcal{O}$*-module on* $(Sch/S)_\tau$*. For any surjective flat morphism* $\mathrm{Spec}(B) \to \mathrm{Spec}(A)$ *of affines over* S *the extended Čech complex*

$$0 \to \mathcal{F}(\mathrm{Spec}(A)) \to \mathcal{F}(\mathrm{Spec}(B)) \to \mathcal{F}(\mathrm{Spec}(B \otimes_A B)) \to \ldots$$

is exact. In particular $\mathcal{F}$ *satisfies the sheaf condition for fpqc coverings, and is a sheaf of* $\mathcal{O}$*-modules on* $(Sch/S)_{fppf}$*.*

Proof. With $A \to B$ as in the lemma let $F = F_{\mathcal{F},A}$. This functor is adequate by Lemma 42.5.2. By Lemma 42.3.5 since $A \to B$, $A \to B \otimes_A B$, etc are flat we see that $F(B) = F(A) \otimes_A B$, $F(B \otimes_A B) = F(A) \otimes_A B \otimes_A B$, etc. Exactness follows from Descent, Lemma 34.3.6.

Thus $\mathcal{F}$ satisfies the sheaf condition for τ-coverings (in particular Zariski coverings) and any faithfully flat covering of an affine by an affine. Arguing as in the proofs of Descent, Lemma 34.5.1 and Descent, Proposition 34.5.2 we conclude that $\mathcal{F}$ satisfies the sheaf condition for all fpqc coverings (made out of objects of $(Sch/S)_\tau$). Details omitted. □

Lemma 42.5.6 shows in particular that for any pair of topologies τ, τ' the collection of adequate modules for the τ-topology and the τ'-topology are identical (as presheaves of modules on the underlying category Sch/S).

Definition 42.5.7. Let S be a scheme. The category of adequate $\mathcal{O}$-modules on $(Sch/S)_\tau$ is denoted $Adeq(\mathcal{O})$ or $Adeq((Sch/S)_\tau, \mathcal{O})$. If we want to think just about the abelian category of adequate modules without choosing a topology we simply write $Adeq(S)$.

Lemma 42.5.8. *Let* S *be a scheme. Let* $\mathcal{F}$ *be an adequate* $\mathcal{O}$*-module on* $(Sch/S)_\tau$*.*

(1) *The restriction* $\mathcal{F}|_{S_{Zar}}$ *is a quasi-coherent* $\mathcal{O}_S$*-module on the scheme* S*.*
(2) *The restriction* $\mathcal{F}|_{S_{étale}}$ *is the quasi-coherent module associated to* $\mathcal{F}|_{S_{Zar}}$*.*

(3) *For any affine scheme U over S we have $H^q(U, \mathcal{F}) = 0$ for all $q > 0$.*
(4) *There is a canonical isomorphism*

$$H^q(S, \mathcal{F}|_{S_{Zar}}) = H^q((Sch/S)_\tau, \mathcal{F}).$$

Proof. By Lemma 42.3.5 and Lemma 42.5.2 we see that for any flat morphism of affines $U \to V$ over S we have $\mathcal{F}(U) = \mathcal{F}(V) \otimes_{\mathcal{O}(V)} \mathcal{O}(U)$. This works in particular if $U \subset V \subset S$ are affine opens of S, hence $\mathcal{F}|_{S_{Zar}}$ is quasi-coherent. Thus (1) holds.

Let $S' \to S$ be an étale morphism of schemes. Then for $U \subset S'$ affine open mapping into an affine open $V \subset S$ we see that $\mathcal{F}(U) = \mathcal{F}(V) \otimes_{\mathcal{O}(V)} \mathcal{O}(U)$ because $U \to V$ is étale, hence flat. Therefore $\mathcal{F}|_{S'_{Zar}}$ is the pullback of $\mathcal{F}|_{S_{Zar}}$. This proves (2).

We are going to apply Cohomology on Sites, Lemma 21.11.8 to the site $(Sch/S)_\tau$ with $\mathcal{B}$ the set of affine schemes over S and Cov the set of standard affine τ-coverings. Assumption (3) of the lemma is satisfied by Descent, Lemma 34.7.8 and Lemma 42.5.6 for the case of a covering by a single affine. Hence we conclude that $H^p(U, \mathcal{F}) = 0$ for every affine scheme U over S. This proves (3). In exactly the same way as in the proof of Descent, Proposition 34.7.10 this implies the equality of cohomologies (4). □

Remark 42.5.9. Let S be a scheme. We have functors $u : QCoh(\mathcal{O}_S) \to Adeq(\mathcal{O})$ and $v : Adeq(\mathcal{O}) \to QCoh(\mathcal{O}_S)$. Namely, the functor $u : \mathcal{F} \mapsto \mathcal{F}^a$ comes from taking the associated $\mathcal{O}$-module which is adequate by Lemma 42.5.5. Conversely, the functor v comes from restriction $v : \mathcal{G} \mapsto \mathcal{G}|_{S_{Zar}}$, see Lemma 42.5.8. Since $\mathcal{F}^a$ can be described as the pullback of $\mathcal{F}$ under a morphism of ringed topoi $((Sch/S)_\tau, \mathcal{O}) \to (S_{Zar}, \mathcal{O}_S)$, see Descent, Remark 34.7.6 and since restriction is the pushforward we see that u and v are adjoint as follows

$$\mathcal{H}om_{\mathcal{O}_S}(\mathcal{F}, v\mathcal{G}) = \mathcal{H}om_{\mathcal{O}}(u\mathcal{F}, \mathcal{G})$$

where $\mathcal{O}$ denotes the structure sheaf on the big site. It is immediate from the description that the adjunction mapping $\mathcal{F} \to vu\mathcal{F}$ is an isomorphism for all quasi-coherent sheaves.

Lemma 42.5.10. *Let S be a scheme. Let $\mathcal{F}$ be a presheaf of $\mathcal{O}$-modules on $(Sch/S)_\tau$. If for every affine scheme $\mathrm{Spec}(A)$ over S the functor $F_{\mathcal{F},A}$ is adequate, then the sheafification of $\mathcal{F}$ is an adequate $\mathcal{O}$-module.*

Proof. Let $U = \mathrm{Spec}(A)$ be an affine scheme over S. Set $F = F_{\mathcal{F},A}$. The sheafification $\mathcal{F}^\# = (\mathcal{F}^+)^+$, see Sites, Section 7.10. By construction

$$(\mathcal{F})^+(U) = \mathrm{colim}_{\mathcal{U}} \check{H}^0(\mathcal{U}, \mathcal{F})$$

where the colimit is over coverings in the site $(Sch/S)_\tau$. Since U is affine it suffices to take the limit over standard affine τ-coverings $\mathcal{U} = \{U_i \to U\}_{i \in I} = \{\mathrm{Spec}(A_i) \to \mathrm{Spec}(A)\}_{i \in I}$ of U. Since each $A \to A_i$ and $A \to A_i \otimes_A A_j$ is flat we see that

$$\check{H}^0(\mathcal{U}, \mathcal{F}) = \mathrm{Ker}(\prod F(A) \otimes_A A_i \to \prod F(A) \otimes_A A_i \otimes_A A_j)$$

by Lemma 42.3.5. Since $A \to \prod A_i$ is faithfully flat we see that this always is canonically isomorphic to $F(A)$ by Descent, Lemma 34.3.6. Thus the presheaf $(\mathcal{F})^+$ has the same value as $\mathcal{F}$ on all affine schemes over S. Repeating the argument once more we deduce the same thing for $\mathcal{F}^\# = ((\mathcal{F})^+)^+$. Thus $F_{\mathcal{F},A} = F_{\mathcal{F}^\#,A}$ and we conclude that $\mathcal{F}^\#$ is adequate. □

Lemma 42.5.11. *Let S be a scheme.*

(1) *The category $Adeq(\mathcal{O})$ is abelian.*
(2) *The functor $Adeq(\mathcal{O}) \to Mod((Sch/S)_\tau, \mathcal{O})$ is exact.*
(3) *If $0 \to \mathcal{F}_1 \to \mathcal{F}_2 \to \mathcal{F}_3 \to 0$ is a short exact sequence of $\mathcal{O}$-modules and $\mathcal{F}_1$ and $\mathcal{F}_3$ are adequate, then $\mathcal{F}_2$ is adequate.*
(4) *The category $Adeq(\mathcal{O})$ has colimits and $Adeq(\mathcal{O}) \to Mod((Sch/S)_\tau, \mathcal{O})$ commutes with them.*

Proof. Let $\varphi : \mathcal{F} \to \mathcal{G}$ be a map of adequate $\mathcal{O}$-modules. To prove (1) and (2) it suffices to show that $\mathcal{K} = \mathrm{Ker}(\varphi)$ and $\mathcal{Q} = \mathrm{Coker}(\varphi)$ computed in $Mod((Sch/S)_\tau, \mathcal{O})$ are adequate. Let $U = \mathrm{Spec}(A)$ be an affine scheme over S. Let $F = F_{\mathcal{F},A}$ and $G = F_{\mathcal{G},A}$. By Lemmas 42.3.11 and 42.3.10 the kernel K and cokernel Q of the induced map $F \to G$ are adequate functors. Because the kernel is computed on the level of presheaves, we see that $K = F_{\mathcal{K},A}$ and we conclude $\mathcal{K}$ is adequate. To prove the result for the cokernel, denote $\mathcal{Q}'$ the presheaf cokernel of φ. Then $Q = F_{\mathcal{Q}',A}$ and $\mathcal{Q} = (\mathcal{Q}')^\#$. Hence $\mathcal{Q}$ is adequate by Lemma 42.5.10.

Let $0 \to \mathcal{F}_1 \to \mathcal{F}_2 \to \mathcal{F}_3 \to 0$ is a short exact sequence of $\mathcal{O}$-modules and $\mathcal{F}_1$ and $\mathcal{F}_3$ are adequate. Let $U = \mathrm{Spec}(A)$ be an affine scheme over S. Let $F_i = F_{\mathcal{F}_i,A}$. The sequence of functors

$$0 \to F_1 \to F_2 \to F_3 \to 0$$

is exact, because for $V = \mathrm{Spec}(B)$ affine over U we have $H^1(V, \mathcal{F}_1) = 0$ by Lemma 42.5.8. Since F_1 and F_3 are adequate functors by Lemma 42.5.2 we see that F_2 is adequate by Lemma 42.3.16. Thus $\mathcal{F}_2$ is adequate.

Let $\mathcal{I} \to Adeq(\mathcal{O})$, $i \mapsto \mathcal{F}_i$ be a diagram. Denote $\mathcal{F} = \mathrm{colim}_i \mathcal{F}_i$ the colimit computed in $Mod((Sch/S)_\tau, \mathcal{O})$. To prove (4) it suffices to show that $\mathcal{F}$ is adequate. Let $\mathcal{F}' = \mathrm{colim}_i \mathcal{F}_i$ be the colimit computed in presheaves of $\mathcal{O}$-modules. Then $\mathcal{F} = (\mathcal{F}')^\#$. Let $U = \mathrm{Spec}(A)$ be an affine scheme over S. Let $F_i = F_{\mathcal{F}_i,A}$. By Lemma 42.3.12 the functor $\mathrm{colim}_i F_i = F_{\mathcal{F}',A}$ is adequate. Lemma 42.5.10 shows that $\mathcal{F}$ is adequate. □

The following lemma tells us that the total direct image $Rf_*\mathcal{F}$ of an adequate module under a quasi-compact and quasi-separated morphism is a complex whose cohomology sheaves are adequate.

Lemma 42.5.12. *Let $f : T \to S$ be a quasi-compact and quasi-separated morphism of schemes. For any adequate $\mathcal{O}_T$-module on $(Sch/T)_\tau$ the pushforward $f_*\mathcal{F}$ and the higher direct images $R^i f_*\mathcal{F}$ are adequate $\mathcal{O}_S$-modules on $(Sch/S)_\tau$.*

Proof. First we explain how to compute the higher direct images. Choose an injective resolution $\mathcal{F} \to \mathcal{I}^\bullet$. Then $R^i f_*\mathcal{F}$ is the ith cohomology sheaf of the complex $f_*\mathcal{I}^\bullet$. Hence $R^i f_*\mathcal{F}$ is the sheaf associated to the presheaf which associates to an object U/S of $(Sch/S)_\tau$ the module

$$\begin{aligned}
\frac{\mathrm{Ker}(f_*\mathcal{I}^i(U) \to f_*\mathcal{I}^{i+1}(U))}{\mathrm{Im}(f_*\mathcal{I}^{i-1}(U) \to f_*\mathcal{I}^i(U))} &= \frac{\mathrm{Ker}(\mathcal{I}^i(U \times_S T) \to \mathcal{I}^{i+1}(U \times_S T))}{\mathrm{Im}(\mathcal{I}^{i-1}(U \times_S T) \to \mathcal{I}^i(U \times_S T))} \\
&= H^i(U \times_S T, \mathcal{F}) \\
&= H^i((Sch/U \times_S T)_\tau, \mathcal{F}|_{(Sch/U \times_S T)_\tau}) \\
&= H^i(U \times_S T, \mathcal{F}|_{(U \times_S T)_{Zar}})
\end{aligned}$$

The first equality by Topologies, Lemma 33.7.12 (and its analogues for other topologies), the second equality by definition of cohomology of $\mathcal{F}$ over an object of $(Sch/T)_\tau$, the third equality by Cohomology on Sites, Lemma 21.8.1, and the last equality by Lemma 42.5.8. Thus by Lemma 42.5.10 it suffices to prove the claim stated in the following paragraph.

Let A be a ring. Let T be a scheme quasi-compact and quasi-separated over A. Let $\mathcal{F}$ be an adequate $\mathcal{O}_T$-module on $(Sch/T)_\tau$. For an A-algebra B set $T_B = T \times_{\mathrm{Spec}(A)} \mathrm{Spec}(B)$ and denote $\mathcal{F}_B = \mathcal{F}|_{(T_B)_{Zar}}$ the restriction of $\mathcal{F}$ to the small Zariski site of T_B. (Recall that this is a "usual" quasi-coherent sheaf on the scheme T_B, see Lemma 42.5.8.) Claim: The functor

$$B \longmapsto H^q(T_B, \mathcal{F}_B)$$

is adequate. We will prove the lemma by the usual procedure of cutting T into pieces.

Case I: T is affine. In this case the schemes T_B are all affine and $H^q(T_B, \mathcal{F}_B) = 0$ for all $q \geq 1$. The functor $B \mapsto H^0(T_B, \mathcal{F}_B)$ is adequate by Lemma 42.3.18.

Case II: T is separated. Let n be the minimal number of affines needed to cover T. We argue by induction on n. The base case is Case I. Choose an affine open covering $T = V_1 \cup \ldots \cup V_n$. Set $V = V_1 \cup \ldots \cup V_{n-1}$ and $U = V_n$. Observe that

$$U \cap V = (V_1 \cap V_n) \cup \ldots \cup (V_{n-1} \cap V_n)$$

is also a union of $n-1$ affine opens as T is separated, see Schemes, Lemma 25.21.8. Note that for each B the base changes U_B, V_B and $(U \cap V)_B = U_B \cap V_B$ behave in the same way. Hence we see that for each B we have a long exact sequence

$$0 \to H^0(T_B, \mathcal{F}_B) \to H^0(U_B, \mathcal{F}_B) \oplus H^0(V_B, \mathcal{F}_B) \to H^0((U \cap V)_B, \mathcal{F}_B) \to H^1(T_B, \mathcal{F}_B) \to \ldots$$

functorial in B, see Cohomology, Lemma 20.9.2. By induction hypothesis the functors $B \mapsto H^q(U_B, \mathcal{F}_B)$, $B \mapsto H^q(V_B, \mathcal{F}_B)$, and $B \mapsto H^q((U \cap V)_B, \mathcal{F}_B)$ are adequate. Using Lemmas 42.3.11 and 42.3.10 we see that our functor $B \mapsto H^q(T_B, \mathcal{F}_B)$ sits in the middle of a short exact sequence whose outer terms are adequate. Thus the claim follows from Lemma 42.3.16.

Case III: General quasi-compact and quasi-separated case. The proof is again by induction on the number n of affines needed to cover T. The base case $n = 1$ is Case I. Choose an affine open covering $T = V_1 \cup \ldots \cup V_n$. Set $V = V_1 \cup \ldots \cup V_{n-1}$ and $U = V_n$. Note that since T is quasi-separated $U \cap V$ is a quasi-compact open of an affine scheme, hence Case II applies to it. The rest of the argument proceeds in exactly the same manner as in the paragraph above and is omitted. □

42.6. Parasitic adequate modules

In this section we start comparing adequate modules and quasi-coherent modules on a scheme S. Recall that there are functors $u : QCoh(\mathcal{O}_S) \to Adeq(\mathcal{O})$ and $v : Adeq(\mathcal{O}) \to QCoh(\mathcal{O}_S)$ satisfying the adjunction

$$\mathcal{H}om_{QCoh(\mathcal{O}_S)}(\mathcal{F}, v\mathcal{G}) = \mathcal{H}om_{Adeq(\mathcal{O})}(u\mathcal{F}, \mathcal{G})$$

and such that $\mathcal{F} \to vu\mathcal{F}$ is an isomorphism for every quasi-coherent sheaf $\mathcal{F}$, see Remark 42.5.9. Hence u is a fully faithful embedding and we can identify $QCoh(\mathcal{O}_S)$ with a full subcategory of $Adeq(\mathcal{O})$. The functor v is exact but u is not left exact in general. The kernel of v is the subcategory of parasitic adequate modules.

In Descent, Definition 34.8.1 we give the definition of a parasitic module. For adequate modules the notion does not depend on the chosen topology.

Lemma 42.6.1. *Let S be a scheme. Let $\mathcal{F}$ be an adequate $\mathcal{O}$-module on $(Sch/S)_\tau$. The following are equivalent:*

1. *$v\mathcal{F} = 0$,*
2. *$\mathcal{F}$ is parasitic,*
3. *$\mathcal{F}$ is parasitic for the τ-topology,*
4. *$\mathcal{F}(U) = 0$ for all $U \subset S$ open, and*
5. *there exists an affine open covering $S = \bigcup U_i$ such that $\mathcal{F}(U_i) = 0$ for all i.*

Proof. The implications (2) $\Rightarrow$ (3) $\Rightarrow$ (4) $\Rightarrow$ (5) are immediate from the definitions. Assume (5). Suppose that $S = \bigcup U_i$ is an affine open covering such that $\mathcal{F}(U_i) = 0$ for all i. Let $V \to S$ be a flat morphism. There exists an affine open covering $V = \bigcup V_j$ such that each V_j maps into some U_i. As the morphism $V_j \to S$ is flat, also $V_j \to U_i$ is flat. Hence the corresponding ring map $A_i = \mathcal{O}(U_i) \to \mathcal{O}(V_j) = B_j$ is flat. Thus by Lemma 42.5.2 and Lemma 42.3.5 we see that $\mathcal{F}(U_i) \otimes_{A_i} B_j \to \mathcal{F}(V_j)$ is an isomorphism. Hence $\mathcal{F}(V_j) = 0$. Since $\mathcal{F}$ is a sheaf for the Zariski topology we conclude that $\mathcal{F}(V) = 0$. In this way we see that (5) implies (2).

This proves the equivalence of (2), (3), (4), and (5). As (1) is equivalent to (3) (see Remark 42.5.9) we conclude that all five conditions are equivalent. □

Let S be a scheme. The subcategory of parasitic adequate modules is a Serre subcategory of $Adeq(\mathcal{O})$. The quotient is the category of quasi-coherent modules.

Lemma 42.6.2. *Let S be a scheme. The subcategory $\mathcal{C} \subset Adeq(\mathcal{O})$ of parasitic adequate modules is a Serre subcategory. Moreover, the functor v induces an equivalence of categories*

$$Adeq(\mathcal{O})/\mathcal{C} = QCoh(\mathcal{O}_S).$$

Proof. The category $\mathcal{C}$ is the kernel of the exact functor $v : Adeq(\mathcal{O}) \to QCoh(\mathcal{O}_S)$, see Lemma 42.6.1. Hence it is a Serre subcategory by Homology, Lemma 12.9.4. By Homology, Lemma 12.9.6 we obtain an induced exact functor $\overline{v} : Adeq(\mathcal{O})/\mathcal{C} \to QCoh(\mathcal{O}_S)$. Because u is a right inverse to v we see right away that $\overline{v}$ is essentially surjective. We see that $\overline{v}$ is faithful by Homology, Lemma 12.9.7. Because u is a right inverse to v we finally conclude that $\overline{v}$ is fully faithful. □

Lemma 42.6.3. *Let $f : T \to S$ be a quasi-compact and quasi-separated morphism of schemes. For any parasitic adequate $\mathcal{O}_T$-module on $(Sch/T)_\tau$ the pushforward $f_*\mathcal{F}$ and the higher direct images $R^i f_*\mathcal{F}$ are parasitic adequate $\mathcal{O}_S$-modules on $(Sch/S)_\tau$.*

Proof. We have already seen in Lemma 42.5.12 that these higher direct images are adequate. Hence it suffices to show that $(R^i f_*\mathcal{F})(U_i) = 0$ for any τ-covering $\{U_i \to S\}$ open. And $R^i f_*\mathcal{F}$ is parasitic by Descent, Lemma 34.8.3. □

42.7. Derived categories of adequate modules, I

Let S be a scheme. We continue the discussion started in Section 42.6. The exact functor v induces a functor

$$D(Adeq(\mathcal{O})) \longrightarrow D(QCoh(\mathcal{O}_S))$$

and similarly for bounded versions.

Lemma 42.7.1. *Let S be a scheme. Let $\mathcal{C} \subset Adeq(\mathcal{O})$ denote the full subcategory consisting of parasitic adequate modules. Then*

$$D(Adeq(\mathcal{O}))/D_{\mathcal{C}}(Adeq(\mathcal{O})) = D(QCoh(\mathcal{O}_S))$$

and similarly for the bounded versions.

Proof. Follows immediately from Derived Categories, Lemma 13.12.3. □

Next, we look for a description the other way around by looking at the functors

$$K^+(QCoh(\mathcal{O}_S)) \longrightarrow K^+(Adeq(\mathcal{O})) \longrightarrow D^+(Adeq(\mathcal{O})) \longrightarrow D^+(QCoh(\mathcal{O}_S)).$$

In some cases the derived category of adequate modules is a localization of the homotopy category of complexes of quasi-coherent modules at universal quasi-isomorphisms. Let S be a scheme. A map of complexes $\varphi : \mathcal{F}^\bullet \to \mathcal{G}^\bullet$ of quasi-coherent $\mathcal{O}_S$-modules is said to be a *universal quasi-isomorphism* if for every morphism of schemes $f : T \to S$ the pullback $f^*\varphi$ is a quasi-isomorphism.

Lemma 42.7.2. *Let $U = \mathrm{Spec}(A)$ be an affine scheme. The bounded below derived category $D^+(Adeq(\mathcal{O}))$ is the localization of $K^+(QCoh(\mathcal{O}_U))$ at the multiplicative subset of universal quasi-isomorphisms.*

Proof. If $\varphi : \mathcal{F}^\bullet \to \mathcal{G}^\bullet$ is a morphism of complexes of quasi-coherent $\mathcal{O}_U$-modules, then $u\varphi : u\mathcal{F}^\bullet \to u\mathcal{G}^\bullet$ is a quasi-isomorphism if and only if φ is a universal quasi-isomorphism. Hence the collection S of universal quasi-isomorphisms is a saturated multiplicative system compatible with the triangulated structure by Derived Categories, Lemma 13.5.3. Hence $S^{-1}K^+(QCoh(\mathcal{O}_U))$ exists and is a triangulated category, see Derived Categories, Proposition 13.5.5. We obtain a canonical functor $can : S^{-1}K^+(QCoh(\mathcal{O}_U)) \to D^+(Adeq(\mathcal{O}))$ by Derived Categories, Lemma 13.5.6.

Note that, almost by definition, every adequate module on U has an embedding into a quasi-coherent sheaf, see Lemma 42.5.5. Hence by Derived Categories, Lemma 13.15.4 given $\mathcal{F}^\bullet \in \mathrm{Ob}(K^+(Adeq(\mathcal{O})))$ there exists a quasi-isomorphism $\mathcal{F}^\bullet \to u\mathcal{G}^\bullet$ where $\mathcal{G}^\bullet \in \mathrm{Ob}(K^+(QCoh(\mathcal{O}_U)))$. This proves that can is essentially surjective.

Similarly, suppose that $\mathcal{F}^\bullet$ and $\mathcal{G}^\bullet$ are bounded below complexes of quasi-coherent $\mathcal{O}_U$-modules. A morphism in $D^+(Adeq(\mathcal{O}))$ between these consists of a pair $f : u\mathcal{F}^\bullet \to \mathcal{H}^\bullet$ and $s : u\mathcal{G}^\bullet \to \mathcal{H}^\bullet$ where s is a quasi-isomorphism. Pick a quasi-isomorphism $s' : \mathcal{H}^\bullet \to u\mathcal{E}^\bullet$. Then we see that $s' \circ f : \mathcal{F} \to \mathcal{E}^\bullet$ and the universal quasi-isomorphism $s' \circ s : \mathcal{G}^\bullet \to \mathcal{E}^\bullet$ give a morphism in $S^{-1}K^+(QCoh(\mathcal{O}_U))$ mapping to the given morphism. This proves the "fully" part of full faithfulness. Faithfulness is proved similarly. □

Lemma 42.7.3. *Let $U = \mathrm{Spec}(A)$ be an affine scheme. The inclusion functor*

$$Adeq(\mathcal{O}) \to Mod((Sch/U)_\tau, \mathcal{O})$$

has a right adjoint A^1. Moreover, the adjunction mapping $A(\mathcal{F}) \to \mathcal{F}$ is an isomorphism for every adequate module $\mathcal{F}$.

[1]This is the "adequator".

Proof. By Topologies, Lemma 33.7.11 (and similarly for the other topologies) we may work with $\mathcal{O}$-modules on $(\mathit{Aff}/U)_\tau$. Denote $\mathcal{P}$ the category of module-valued functors on Alg_A and $\mathcal{A}$ the category of adequate functors on Alg_A. Denote $i : \mathcal{A} \to \mathcal{P}$ the inclusion functor. Denote $Q : \mathcal{P} \to \mathcal{A}$ the construction of Lemma 42.4.1. We have the commutative diagram

$$\begin{array}{ccccc} \mathit{Adeq}(\mathcal{O}) & \xrightarrow[k]{} & \mathit{Mod}((\mathit{Aff}/U)_\tau, \mathcal{O}) & \xrightarrow[j]{} & \mathit{PMod}((\mathit{Aff}/U)_\tau, \mathcal{O}) \\ \| & & & & \| \\ \mathcal{A} & & \xrightarrow{\qquad\qquad i \qquad\qquad} & & \mathcal{P} \end{array} \tag{42.7.3.1}$$

The left vertical equality is Lemma 42.5.3 and the right vertical equality was explained in Section 42.3. Define $A(\mathcal{F}) = Q(j(\mathcal{F}))$. Since j is fully faithful it follows immediately that A is a right adjoint of the inclusion functor k. Also, since k is fully faithful too, the final assertion follows formally. □

The functor A is a right adjoint hence left exact. Since the inclusion functor is exact, see Lemma 42.5.11 we conclude that A transforms injectives into injectives, and that the category $\mathit{Adeq}(\mathcal{O})$ has enough injectives, see Homology, Lemma 12.25.3 and Injectives, Theorem 19.8.4. This also follows from the equivalence in (42.7.3.1) and Lemma 42.4.2.

Lemma 42.7.4. *Let $U = \mathrm{Spec}(A)$ be an affine scheme. For any object $\mathcal{F}$ of $\mathit{Adeq}(\mathcal{O})$ we have $R^pA(\mathcal{F}) = 0$ for all $p > 0$ where A is as in Lemma 42.7.3.*

Proof. With notation as in the proof of Lemma 42.7.3 choose an injective resolution $k(\mathcal{F}) \to \mathcal{I}^\bullet$ in the category of $\mathcal{O}$-modules on $(\mathit{Aff}/U)_\tau$. By Cohomology on Sites, Lemmas 21.12.2 and Lemma 42.5.8 the complex $j(\mathcal{I}^\bullet)$ is exact. On the other hand, each $j(\mathcal{I}^n)$ is an injective object of the category of presheaves of modules by Cohomology on Sites, Lemma 21.12.1. It follows that $R^pA(\mathcal{F}) = R^pQ(j(k(\mathcal{F})))$. Hence the result now follows from Lemma 42.4.10. □

Let S be a scheme. By the discussion in Section 42.5 the embedding $\mathit{Adeq}(\mathcal{O}) \subset \mathit{Mod}((\mathit{Sch}/S)_\tau, \mathcal{O})$ exhibits $\mathit{Adeq}(\mathcal{O})$ as a weak Serre subcategory of the category of all $\mathcal{O}$-modules. Denote

$$D_{\mathit{Adeq}}(\mathcal{O}) \subset D(\mathcal{O}) = D(\mathit{Mod}((\mathit{Sch}/S)_\tau, \mathcal{O}))$$

the triangulated subcategory of complexes whose cohomology sheaves are adequate, see Derived Categories, Section 13.12. We obtain a canonical functor

$$D(\mathit{Adeq}(\mathcal{O})) \longrightarrow D_{\mathit{Adeq}}(\mathcal{O})$$

see Derived Categories, Equation (13.12.1.1).

Lemma 42.7.5. *If $U = \mathrm{Spec}(A)$ is an affine scheme, then the bounded below version*

$$D^+(\mathit{Adeq}(\mathcal{O})) \longrightarrow D^+_{\mathit{Adeq}}(\mathcal{O}) \tag{42.7.5.1}$$

of the functor above is an equivalence.

Proof. Let $A : \mathit{Mod}(\mathcal{O}) \to \mathit{Adeq}(\mathcal{O})$ be the right adjoint to the inclusion functor constructed in Lemma 42.7.3. Since A is left exact and since $\mathit{Mod}(\mathcal{O})$ has enough injectives, A has a right derived functor $RA : D^+_{\mathit{Adeq}}(\mathcal{O}) \to D^+(\mathit{Adeq}(\mathcal{O}))$. We claim that RA is a quasi-inverse to (42.7.5.1). To see this the key fact is that if $\mathcal{F}$ is an

adequate module, then the adjunction map $\mathcal{F} \to RA(\mathcal{F})$ is a quasi-isomorphism by Lemma 42.7.4.

Namely, to prove the lemma in full it suffices to show:

(1) Given $\mathcal{F}^\bullet \in K^+(Adeq(\mathcal{O}))$ the canonical map $\mathcal{F}^\bullet \to RA(\mathcal{F}^\bullet)$ is a quasi-isomorphism, and
(2) given $\mathcal{G}^\bullet \in K^+(Mod(\mathcal{O}))$ the canonical map $RA(\mathcal{G}^\bullet) \to \mathcal{G}^\bullet$ is a quasi-isomorphism.

Both (1) and (2) follow from the key fact via a spectral sequence argument using one of the spectral sequences of Derived Categories, Lemma 13.20.3. Some details omitted. □

Lemma 42.7.6. *Let $U = \mathrm{Spec}(A)$ be an affine scheme. Let $\mathcal{F}$ and $\mathcal{G}$ be adequate $\mathcal{O}$-modules. For any $i \geq 0$ the natural map*

$$Ext^i_{Adeq(\mathcal{O})}(\mathcal{F}, \mathcal{G}) \longrightarrow Ext^i_{Mod(\mathcal{O})}(\mathcal{F}, \mathcal{G})$$

is an isomorphism.

Proof. By definition these ext groups are computed as hom sets in the derived category. Hence this follows immediately from Lemma 42.7.5. □

42.8. Pure extensions

We want to characterize extensions of quasi-coherent sheaves on the big site of an affine schemes in terms of algebra. To do this we introduce the following notion.

Definition 42.8.1. Let A be a ring.

(1) An A-module P is said to be *pure projective* if for every universally exact sequence $0 \to K \to M \to N \to 0$ of A-module the sequence $0 \to \mathrm{Hom}_A(P, K) \to \mathrm{Hom}_A(P, M) \to \mathrm{Hom}_A(P, N) \to 0$ is exact.
(2) An A-module I is said to be *pure injective* if for every universally exact sequence $0 \to K \to M \to N \to 0$ of A-module the sequence $0 \to \mathrm{Hom}_A(N, I) \to \mathrm{Hom}_A(M, I) \to \mathrm{Hom}_A(K, I) \to 0$ is exact.

Let's characterize pure projectives.

Lemma 42.8.2. *Let A be a ring.*

(1) *A module is pure projective if and only if it is a direct summand of a direct sum of finitely presented A-modules.*
(2) *For any module M there exists a universally exact sequence $0 \to N \to P \to M \to 0$ with P pure projective.*

Proof. First note that a finitely presented A-module is pure projective by Algebra, Theorem 10.77.3. Hence a direct summand of a direct sum of finitely presented A-modules is indeed pure projective. Let M be any A-module. Write $M = \mathrm{colim}_{i \in I} P_i$ as a filtered colimit of finitely presented A-modules. Consider the sequence

$$0 \to N \to \bigoplus P_i \to M \to 0.$$

For any finitely presented A-module P the map $\mathrm{Hom}_A(P, \bigoplus P_i) \to \mathrm{Hom}_A(P, M)$ is surjective, as any map $P \to M$ factors through some P_i. Hence by Algebra, Theorem 10.77.3 this sequence is universally exact. This proves (2). If now M is pure projective, then the sequence is split and we see that M is a direct summand of $\bigoplus P_i$. □

Let's characterize pure injectives.

Lemma 42.8.3. *Let A be a ring. For any A-module M set $M^\wedge = \mathrm{Hom}_{\mathbf{Z}}(M, \mathbf{Q}/\mathbf{Z})$.*

1. *For any A-module M the A-module $M^\wedge$ is pure injective.*
2. *An A-module I is pure injective if and only if the map $I \to (I^\wedge)^\wedge$ splits.*
3. *For any module M there exists a universally exact sequence $0 \to M \to I \to N \to 0$ with I pure injective.*

Proof. We will use the properties of the functor $M \mapsto M^\wedge$ found in More on Algebra, Section 15.40 without further mention. Part (1) holds because $\mathrm{Hom}_A(N, M^\wedge) = \mathrm{Hom}_{\mathbf{Z}}(N \otimes_A M, \mathbf{Q}/\mathbf{Z})$ and because $\mathbf{Q}/\mathbf{Z}$ is injective in the category of abelian groups. Hence if $I \to (I^\wedge)^\wedge$ is split, then I is pure injective. We claim that for any A-module M the evaluation map $ev : M \to (M^\wedge)^\wedge$ is universally injective. To see this note that $ev^\wedge : ((M^\wedge)^\wedge)^\wedge \to M^\wedge$ has a right inverse, namely $ev' : M^\wedge \to ((M^\wedge)^\wedge)^\wedge$. Then for any A-module N applying the exact faithful functor $^\wedge$ to the map $N \otimes_A M \to N \otimes_A (M^\wedge)^\wedge$ gives

$$\mathrm{Hom}_A(N, ((M^\wedge)^\wedge)^\wedge) = \Big(N \otimes_A (M^\wedge)^\wedge\Big)^\wedge \to \Big(N \otimes_A M\Big)^\wedge = \mathrm{Hom}_A(N, M^\wedge)$$

which is surjective by the existence of the right inverse. The claim follows. The claim implies (3) and the necessity of the condition in (2). □

Before we continue we make the following observation which we will use frequently in the rest of this section.

Lemma 42.8.4. *Let A be a ring.*

1. *Let $L \to M \to N$ be a universally exact sequence of A-modules. Let $K = Im(M \to N)$. Then $K \to N$ is universally injective.*
2. *Any universally exact complex can be split into universally exact short exact sequences.*

Proof. Proof of (1). For any A-module T the sequence $L \otimes_A T \to M \otimes_A T \to K \otimes_A T \to 0$ is exact by right exactness of $\otimes$. By assumption the sequence $L \otimes_A T \to M \otimes_A T \to N \otimes_A T$ is exact. Combined this shows that $K \otimes_A T \to N \otimes_A T$ is injective.

Part (2) means the following: Suppose that $M^\bullet$ is a universally exact complex of A-modules. Set $K^i = \mathrm{Ker}(d^i) \subset M^i$. Then the short exact sequences $0 \to K^i \to M^i \to K^{i+1} \to 0$ are universally exact. This follows immediately from part (1). □

Definition 42.8.5. Let A be a ring. Let M be an A-module.

1. A *pure projective resolution* $P_\bullet \to M$ is a universally exact sequence
$$\ldots \to P_1 \to P_0 \to M \to 0$$
with each P_i pure projective.
2. A *pure injective resolution* $M \to I^\bullet$ is a universally exact sequence
$$0 \to M \to I^0 \to I^1 \to \ldots$$
with each I^i pure injective.

These resolutions satisfy the usual uniqueness properties among the class of all universally exact left or right resolutions.

Lemma 42.8.6. *Let A be a ring.*

(1) *Any A-module has a pure projective resolution.*

Let $M \to N$ be a map of A-modules. Let $P_\bullet \to M$ be a pure projective resolution and let $N_\bullet \to N$ be a universally exact resolution.

(2) *There exists a map of complexes $P_\bullet \to N_\bullet$ inducing the given map*

$$M = Coker(P_1 \to P_0) \to Coker(N_1 \to N_0) = N$$

(3) *two maps $\alpha, \beta : P_\bullet \to N_\bullet$ inducing the same map $M \to N$ are homotopic.*

Proof. Part (1) follows immediately from Lemma 42.8.2. Before we prove (2) and (3) note that by Lemma 42.8.4 we can split the universally exact complex $N_\bullet \to N \to 0$ into universally exact short exact sequences $0 \to K_0 \to N_0 \to N \to 0$ and $0 \to K_i \to N_i \to K_{i-1} \to 0$.

Proof of (2). Because P_0 is pure projective we can find a map $P_0 \to N_0$ lifting the map $P_0 \to M \to N$. We obtain an induced map $P_1 \to F_0 \to N_0$ wich ends up in K_0. Since P_1 is pure projective we may lift this to a map $P_1 \to N_1$. This in turn induces a map $P_2 \to P_1 \to N_1$ which maps to zero into N_0, i.e., into K_1. Hence we may lift to get a map $P_2 \to N_2$. Repeat.

Proof of (3). To show that α, β are homotopic it suffices to show the difference $\gamma = \alpha - \beta$ is homotopic to zero. Note that the image of $\gamma_0 : P_0 \to N_0$ is contained in K_0. Hence we may lift γ_0 to a map $h_0 : P_0 \to N_1$. Consider the map $\gamma_1' = \gamma_1 - h_0 \circ d_{P,1} : P_1 \to N_1$. By our choice of h_0 we see that the image of γ_1' is contained in K_1. Since P_1 is pure projective may lift γ_1' to a map $h_1 : P_1 \to N_2$. At this point we have $\gamma_1 = h_0 \circ d_{F,1} + d_{N,2} \circ h_1$. Repeat. □

Lemma 42.8.7. *Let A be a ring.*

(1) *Any A-module has a pure injective resolution.*

Let $M \to N$ be a map of A-modules. Let $M \to M^\bullet$ be a universally exact resolution and let $N \to I^\bullet$ be a pure injective resolution.

(2) *There exists a map of complexes $M^\bullet \to I^\bullet$ inducing the given map*

$$M = Ker(M^0 \to M^1) \to Ker(I^0 \to I^1) = N$$

(3) *two maps $\alpha, \beta : M^\bullet \to I^\bullet$ inducing the same map $M \to N$ are homotopic.*

Proof. This lemma is dual to Lemma 42.8.6. The proof is identical, except one has to reverse all the arrows. □

Using the material above we can define pure extension groups as follows. Let A be a ring and let M, N be A-modules. Choose a pure injective resolution $N \to I^\bullet$. By Lemma 42.8.7 the complex

$$\mathrm{Hom}_A(M, I^\bullet)$$

is well defined up to homotopy. Hence its ith cohomology module is a well defined invariant of M and N.

Definition 42.8.8. Let A be a ring and let M, N be A-modules. The ith *pure extension module* $\mathrm{Pext}^i_A(M, N)$ is the ith cohomology module of the complex $\mathrm{Hom}_A(M, I^\bullet)$ where $I^\bullet$ is a pure injective resolution of N.

Warning: It is not true that an exact sequence of A-modules gives rise to a long exact sequence of pure extensions groups. (You need a universally exact sequence for this.) We collect some facts which are obvious from the material above.

Lemma 42.8.9. *Let A be a ring.*

(1) *$Pext^i_A(M, N) = 0$ for $i > 0$ whenever N is pure injective,*
(2) *$Pext^i_A(M, N) = 0$ for $i > 0$ whenever M is pure projective, in particular if M is an A-module of finite presentation,*
(3) *$Pext^i_A(M, N)$ is also the ith cohomology module of the complex $\mathrm{Hom}_A(P_\bullet, N)$ where $P_\bullet$ is a pure projective resolution of M.*

Proof. To see (3) consider the double complex

$$A^{\bullet,\bullet} = \mathrm{Hom}_A(P_\bullet, I^\bullet)$$

Each of its rows is exact except in degree 0 where its cohomology is $\mathrm{Hom}_A(M, I^q)$. Each of its columns is exact except in degree 0 where its cohomology is $\mathrm{Hom}_A(P_p, N)$. Hence the two spectral sequences associated to this complex in Homology, Section 12.22 degenerate, giving the equality. □

42.9. Higher exts of quasi-coherent sheaves on the big site

It turns out that the module-valued functor $\underline{I}$ associated to a pure injective module I gives rise to an injective object in the category of adequate functors on Alg_A. Warning: It is not true that a pure projective module gives rise to a projective object in the category of adequate functors. We do have plenty of projective objects, namely, the linearly adequate functors.

Lemma 42.9.1. *Let A be a ring. Let $\mathcal{A}$ be the category of adequate functors on Alg_A. The injective objects of $\mathcal{A}$ are exactly the functors $\underline{I}$ where I is a pure injective A-module.*

Proof. Let I be an injective object of $\mathcal{A}$. Choose an embedding $I \to \underline{M}$ for some A-module M. As I is injective we see that $\underline{M} = I \oplus F$ for some module-valued functor F. Then $M = I(A) \oplus F(A)$ and it follows that $I = \underline{I(A)}$. Thus we see that any injective object is of the form $\underline{I}$ for some A-module I. It is clear that the module I has to be pure injective since any universally exact sequence $0 \to M \to N \to L \to 0$ gives rise to an exact sequence $0 \to \underline{M} \to \underline{N} \to \underline{L} \to 0$ of $\mathcal{A}$.

Finally, suppose that I is a pure injective A-module. Choose an embedding $\underline{I} \to J$ into an injective object of $\mathcal{A}$ (see Lemma 42.4.2). We have seen above that $J = \underline{I'}$ for some A-module I' which is pure injective. As $\underline{I} \to \underline{I'}$ is injective the map $I \to I'$ is universally injective. By assumption on I it splits. Hence $\underline{I}$ is a summand of $J = \underline{I'}$ whence an injective object of the category $\mathcal{A}$. □

Let $U = \mathrm{Spec}(A)$ be an affine scheme. Let M be an A-module. We will use the notation M^a to denote the quasi-coherent sheaf of $\mathcal{O}$-modules on $(Sch/U)_\tau$ associated to the quasi-coherent sheaf $\widetilde{M}$ on U. Now we have all the notation in place to formulate the following lemma.

Lemma 42.9.2. *Let $U = \mathrm{Spec}(A)$ be an affine scheme. Let M, N be A-modules. For all i we have a canonical isomorphism*

$$Ext^i_{Mod(\mathcal{O})}(M^a, N^a) = Pext^i_A(M, N)$$

functorial in M and N.

Proof. Let us construct a canonical arrow from right to left. Namely, if $N \to I^\bullet$ is a pure injective resolution, then $M^a \to (I^\bullet)^a$ is an exact complex of (adequate) $\mathcal{O}$-modules. Hence any element of $\mathrm{Pext}^i_A(M,N)$ gives rise to a map $N^a \to M^a[i]$ in $D(\mathcal{O})$, i.e., an element of the group on the left.

To prove this map is an isomorphism, note that we may replace $\mathrm{Ext}^i_{Mod(\mathcal{O})}(M^a,N^a)$ by $\mathrm{Ext}^i_{Adeq(\mathcal{O})}(M^a,N^a)$, see Lemma 42.7.6. Let $\mathcal{A}$ be the category of adequate functors on Alg_A. We have seen that $\mathcal{A}$ is equivalent to $Adeq(\mathcal{O})$, see Lemma 42.5.3; see also the proof of Lemma 42.7.3. Hence now it suffices to prove that

$$\mathrm{Ext}^i_{\mathcal{A}}(\underline{M},\underline{N}) = \mathrm{Pext}^i_A(M,N)$$

However, this is clear from Lemma 42.9.1 as a pure injective resolution $N \to I^\bullet$ exactly corresponds to an injective resolution of $\underline{N}$ in $\mathcal{A}$. □

42.10. Derived categories of adequate modules, II

Let S be a scheme. Denote $\mathcal{O}_S$ the structure sheaf of S and $\mathcal{O}$ the structure sheaf of the big site $(Sch/S)_\tau$. In Descent, Remark 34.7.4 we constructed a morphism of ringed sites

$$f : ((Sch/S)_\tau, \mathcal{O}) \longrightarrow (S_{Zar}, \mathcal{O}_S). \tag{42.10.0.1}$$

In the previous sections have seen that the functor $f_* : Mod(\mathcal{O}) \to Mod(\mathcal{O}_S)$ transforms adequate sheaves into quasi-coherent sheaves, and induces an exact functor $v : Adeq(\mathcal{O}) \to QCoh(\mathcal{O}_S)$, and in fact that $f_* = v$ induces an equivalence $Adeq(\mathcal{O})/\mathcal{C} \to QCoh(\mathcal{O}_S)$ where $\mathcal{C}$ is the subcategory of parasitic adequate modules. Moreover, the functor f^* transforms quasi-coherent modules into adequate modules, and induces a functor $u : QCoh(\mathcal{O}_S) \to Adeq(\mathcal{O})$ which is a left adjoint to v.

There is a very similar relationship between $D_{Adeq}(\mathcal{O})$ and $D_{QCoh}(S)$. First we explain why the category $D_{Adeq}(\mathcal{O})$ is independent of the chosen topology.

Remark 42.10.1. Let S be a scheme. Let $\tau, \tau' \in \{Zar, étale, smooth, syntomic, fppf\}$. Denote $\mathcal{O}_\tau$, resp. $\mathcal{O}_{\tau'}$ the structure sheaf $\mathcal{O}$ viewed as a sheaf on $(Sch/S)_\tau$, resp. $(Sch/S)_{\tau'}$. Then $D_{Adeq}(\mathcal{O}_\tau)$ and $D_{Adeq}(\mathcal{O}_{\tau'})$ are canonically isomorphic. This follows from Cohomology on Sites, Lemma 21.22.4. Namely, assume τ is stronger than the topology τ', let $\mathcal{C} = (Sch/S)_{fppf}$, and let $\mathcal{B}$ the collection of affine schemes over S. Assumptions (1) and (2) we've seen above. Assumption (3) is clear and assumption (4) follows from Lemma 42.5.8.

Remark 42.10.2. Let S be a scheme. The morphism f see (42.10.0.1) induces adjoint functors $Rf_* : D_{Adeq}(\mathcal{O}) \to D_{QCoh}(S)$ and $Lf^* : D_{QCoh}(S) \to D_{Adeq}(\mathcal{O})$. Moreover $Rf_*Lf^* \cong \mathrm{id}_{D_{QCoh}(S)}$.

We sketch the proof. By Remark 42.10.1 we may assume the topology τ is the Zariski topology. We will use the existence of the unbounded total derived functors Lf^* and Rf_* on $\mathcal{O}$-modules and their adjointness, see Cohomology on Sites, Lemma 21.19.1. In this case f_* is just the restriction to the subcategory S_{Zar} of $(Sch/S)_{Zar}$. Hence it is clear that $Rf_* = f_*$ induces $Rf_* : D_{Adeq}(\mathcal{O}) \to D_{QCoh}(S)$. Suppose that $\mathcal{G}^\bullet$ is an object of $D_{QCoh}(S)$. We may choose a system $\mathcal{K}_1^\bullet \to \mathcal{K}_2^\bullet \to \ldots$ of

bounded above complexes of flat $\mathcal{O}_S$-modules whose transition maps are termwise split injectives and a diagram

$$\begin{array}{ccccc} \mathcal{K}_1^\bullet & \longrightarrow & \mathcal{K}_2^\bullet & \longrightarrow & \ldots \\ \downarrow & & \downarrow & & \\ \tau_{\leq 1}\mathcal{G}^\bullet & \longrightarrow & \tau_{\leq 2}\mathcal{G}^\bullet & \longrightarrow & \ldots \end{array}$$

with the properties (1), (2), (3) listed in Derived Categories, Lemma 13.27.1 where $\mathcal{P}$ is the collection of flat $\mathcal{O}_S$-modules. Then $Lf^*\mathcal{G}^\bullet$ is computed by $\operatorname{colim} f^*\mathcal{K}_n^\bullet$, see Cohomology on Sites, Lemmas 21.18.1 and 21.18.3 (note that our sites have enough points by Étale Cohomology, Lemma 44.30.1). We have to see that $H^i(Lf^*\mathcal{G}^\bullet) = \operatorname{colim} H^i(f^*\mathcal{K}_n^\bullet)$ is adequate for each i. By Lemma 42.5.11 we conclude that it suffices to show that each $H^i(f^*\mathcal{K}_n^\bullet)$ is adequate.

The adequacy of $H^i(f^*\mathcal{K}_n^\bullet)$ is local on S, hence we may assume that $S = \operatorname{Spec}(A)$ is affine. Because S is affine $D_{QCoh}(S) = D(QCoh(\mathcal{O}_S))$, see the discussion in Derived Categories of Schemes, Section 35.3. Hence there exists a quasi-isomorphism $\mathcal{F}^\bullet \to \mathcal{K}_n^\bullet$ where $\mathcal{F}^\bullet$ is a bounded above complex of flat quasi-coherent modules. Then $f^*\mathcal{F}^\bullet \to f^*\mathcal{K}_n^\bullet$ is a quasi-isomorphism, and the cohomology sheaves of $f^*\mathcal{F}^\bullet$ are adequate.

The final assertion $Rf_*Lf^* \cong \mathrm{id}_{D_{QCoh}(S)}$ follows from the explicit description of the functors above. (In plain English: if $\mathcal{F}$ is quasi-coherent and $p > 0$, then $L_pf^*\mathcal{F}$ is a parasitic adequate module.)

Remark 42.10.3. Remark 42.10.2 above implies we have an equivalence of derived categories

$$D_{Adeq}(\mathcal{O})/D_{\mathcal{C}}(\mathcal{O}) \longrightarrow D_{QCoh}(S)$$

where $\mathcal{C}$ is the category of parasitic adequate modules. Namely, it is clear that $D_{\mathcal{C}}(\mathcal{O})$ is the kernel of Rf_*, hence a functor as indicated. For any object X of $D_{Adeq}(\mathcal{O})$ the map $Lf^*Rf_*X \to X$ maps to a quasi-isomorphism in $D_{QCoh}(S)$, hence $Lf^*Rf_*X \to X$ is an isomorphism in $D_{Adeq}(\mathcal{O})/D_{\mathcal{C}}(\mathcal{O})$. Finally, for X, Y objects of $D_{Adeq}(\mathcal{O})$ the map

$$Rf_* : \mathrm{Hom}_{D_{Adeq}(\mathcal{O})/D_{\mathcal{C}}(\mathcal{O})}(X, Y) \to \mathrm{Hom}_{D_{QCoh}(S)}(Rf_*X, Rf_*Y)$$

is bijective as Lf^* gives an inverse (by the remarks above).

42.11. Other chapters

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CHAPTER 43

Dualizing Complexes

43.1. Introduction

A reference is the book [**Har66**].

The goals of this chapter are the following:

(1) Define what it means to have a dualizing complex $\omega_A^\bullet$ over a Noetherian ring A, namely
   (a) we have $\omega_A^\bullet \in D^+(A)$,
   (b) the cohomology modules $H^i(\omega_A^\bullet)$ are all finite A-modules,
   (c) $\omega_A^\bullet$ has finite injective dimension, and
   (d) we have $A \to R\operatorname{Hom}_A(\omega_A^\bullet, \omega_A^\bullet)$ is a quasi-isomorphism.
(2) List elementary properties of dualizing complexes.
(3) Show a dualizing complex gives rise to a dimension function.
(4) Show a dualizing complex gives rise to a good notion of a reflexive hull.
(5) Prove the finiteness theorem when a dualizing complex exists.

43.2. Essential surjections and injections

We will mostly work in categories of modules, but we may as well make the definition in general.

Definition 43.2.1. Let $\mathcal{A}$ be an abelian category.

(1) An injection $A \subset B$ of $\mathcal{A}$ is *essential*, or we say that B is an *essential extension of* A, if every nonzero subobject $B' \subset B$ has nonzero intersection with A.
(2) A surjection $f : A \to B$ of $\mathcal{A}$ is *essential* if for every proper subobject $A' \subset A$ we have $f(A') \neq B$.

Some lemmas about this notion.

Lemma 43.2.2. *Let $\mathcal{A}$ be an abelian category.*

(1) *If $A \subset B$ and $B \subset C$ are essential extensions, then $A \subset C$ is an essential extension.*
(2) *If $A \subset B$ is an essential extension and $C \subset B$ is a subobject, then $A \cap C \subset C$ is an essential extension.*
(3) *If $A \to B$ and $B \to C$ are essential surjections, then $A \to C$ is an essential surjection.*
(4) *Given an essential surjection $f : A \to B$ and a surjection $A \to C$ with kernel K, the morphism $C \to B/f(K)$ is an essential surjection.*

Proof. Omitted. □

Lemma 43.2.3. *Let R be a ring. Let M be an R-module. Let $E = \operatorname{colim} E_i$ be a filtered colimit of R-modules. Suppose given a compatible system of essential injections $M \to E_i$ of R-modules. Then $M \to E$ is an essential extension of M.*

Proof. Immediate from the definitions and the fact that filtered colimits are exact (Algebra, Lemma 10.8.9). □

Lemma 43.2.4. *Let R be a ring. Let $M \subset N$ be R-modules. The following are equivalent*

(1) *$M \subset N$ is an essential extension,*
(2) *for all $x \in N$ there exists an $f \in R$ such that $fx \in M$ and $fx \neq 0$.*

Proof. Assume (1) and let $x \in N$ be a nonzero element. By (1) we have $Rx \cap M \neq 0$. This implies (2).

Assume (2). Let $N' \subset N$ be a nonzero submodule. Pick $x \in N'$ nonzero. By (2) we can find $f \in$ with $fx \in N$ and $fx \neq 0$. Thus $N' \cap M \neq 0$. □

43.3. Injective modules

Some results about injective modules over rings.

Lemma 43.3.1. *Let R be a ring. Any product of injective R-modules is injective.*

Proof. Special case of Homology, Lemma 12.23.3. □

Lemma 43.3.2. *Let $R \to S$ be a flat ring map. If E is an injective S-module, then E is injective as an R-module.*

Proof. This is true because $\operatorname{Hom}_R(M, E) = \operatorname{Hom}_S(M \otimes_R S, E)$ by Algebra, Lemma 10.13.3 and the fact that tensoring with S is exact. □

Lemma 43.3.3. *Let $R \to S$ be an epimorphism of rings. Let E be an S-module. If E is injective as an R-module, then E is an injective S-module.*

Proof. This is true because $\operatorname{Hom}_R(N, E) = \operatorname{Hom}_S(N, E)$ for any S-module N, see Algebra, Lemma 10.102.14. □

Lemma 43.3.4. *Let $R \to S$ be a ring map. If E is an injective R-module, then $\operatorname{Hom}_R(S, E)$ is an injective S-module.*

Proof. This is true because $\operatorname{Hom}_S(N, \operatorname{Hom}_R(S, E)) = \operatorname{Hom}_R(N, E)$ by Algebra, Lemma 10.13.4. □

Lemma 43.3.5. *Let R be a ring. Let I be an injective R-module. Let $E \subset I$ be a submodule. The following are equivalent*

(1) *E is injective, and*
(2) *for all $E \subset E' \subset I$ with $E \subset E'$ essential we have $E = E'$.*

In particular, an R-module is injective if and only if every essential extension is trivial.

Proof. The final assertion follows from the first and the fact that the category of R-modules has enough injectives (More on Algebra, Section 15.40).

Assume (1). Let $E \subset E' \subset I$ as in (2). Then the map $\mathrm{id}_E : E \to E$ can be extended to a map $\alpha : E' \to E$. The kernel of α has to be zero because it intersects E trivially and E' is an essential extension. Hence $E = E'$.

Assume (2). Let $M \subset N$ be R-modules and let $\varphi : M \to E$ be an R-module map. In order to prove (1) we have to show that φ extends to a morphism $N \to E$. Consider the set $\mathcal{S}$ of pairs (M', φ') where $M \subset M' \subset N$ and $\varphi' : M' \to E$ is an R-module map agreeing with φ on M. We define an ordering on $\mathcal{S}$ by the rule $(M', \varphi') \leq (M'', \varphi'')$ if and only if $M' \subset M''$ and $\varphi''|_{M'} = \varphi'$. It is clear that we can take the maximum of a totally ordered subset of $\mathcal{S}$. Hence by Zorn's lemma we may assume (M, φ) is a maximal element.

Choose an extension $\psi : N \to I$ of φ composed with the inclusion $E \to I$. This is possible as I is injective. If $\psi(N) \subset E$, then ψ is the desired extension. If $\psi(N)$ is not contained in E, then by (2) the inclusion $E \subset E + \psi(N)$ is not essential. hence we can find a nonzero submodule $K \subset E + \psi(N)$ meeting E in 0. This means that $M' = \psi^{-1}(E + K)$ strictly contains M. Thus we can extend φ to M' using

$$M' \xrightarrow{\psi|_{M'}} E + K \to (E + K)/K = E$$

This contradicts the maximality of (M, φ). □

Example 43.3.6. Let R be a reduced ring. Let $\mathfrak{p} \subset R$ be a minimal prime so that $K = R_{\mathfrak{p}}$ is a field (Algebra, Lemma 10.24.4). Then K is an injective R-module. Namely, we have $\mathrm{Hom}_R(M, K) = \mathrm{Hom}_K(M_{\mathfrak{p}}, K)$ for any R-module M. Since localization is an exact functor and taking duals is an exact functor on K-vector spaces we conclude $\mathrm{Hom}_R(-, K)$ is an exact functor, i.e., K is an injective R-module.

Lemma 43.3.7. *Let R be a ring. Let E be an R-module. The following are equivalent*

(1) *E is an injective R-module, and*
(2) *given an ideal $I \subset R$ and a module map $\varphi : I \to E$ there exists an extension of φ to and R-module map $R \to E$.*

Proof. The implication (1) $\Rightarrow$ (2) follows from the definitions. Thus we assume (2) holds and we prove (1). First proof: Since R is a generator for the category of R-modules, this follows from Injectives, Lemma 19.11.5.

Second proof: We have to show that every essential extension $E \subset E'$ is trivial. Pick $x \in E'$ and set $I = \{f \in R \mid fx \in E\}$. The map $I \to E$, $f \mapsto fx$ extends to $\psi : R \to E$ by (2). Then $x' = x - \psi(1)$ is an element of E' whose annihilator in E'/E is I and which is annihilated by I as an element of E'. Thus $Rx' = (R/I)x'$ does not intersect E. Since $E \subset E'$ is an essential extension it follows that $x' \in E$ as desired. □

Lemma 43.3.8. *Let R be a Noetherian ring. A direct sum of injective modules is injective.*

Proof. Let E_i be a family of injective modules parametrized by a set I. Set $E = \bigcup E_i$. To show that E is injective we use Lemma 43.3.7. Thus let $\varphi : I \to E$ be a module map from an ideal of R into E. As I is a finite R-module (because R is Noetherian) we can find finitely many elements $i_1, \ldots, i_r \in I$ such that φ maps into $\bigcup_{j=1,\ldots,r} E_{i_j}$. Then we can extend φ into $\bigcup_{j=1,\ldots,r} E_{i_j}$ using the injectivity of the modules E_{i_j}. □

Lemma 43.3.9. *Let R be a Noetherian ring. Let I be an injective R-module.*

(1) *Let $f \in R$. Then $E = \{x \in I \mid \exists\ n,\ f^n x = 0\}$ is an injective submodule of I.*
(2) *Let $J \subset R$ be an ideal. Then the J-power torsion submodule $I[J^\infty]$ is an injective submodule of I.*

Proof. We will use Lemma 43.3.5 to prove (1). Suppose that $E \subset E' \subset I$ and that E' is an essential extension of E. We will show that $E' = E$. If not, then we can find $x \in E'$ and $x \notin E$. Let $J = \{a \in R \mid ax \in E'\}$. Since R is Noetherian we can choose x with J maximal. Since R is Noetherian we can write $J = (g_1, \ldots, g_t)$ for some $g_i \in R$. Say f^{n_i} annihilates $g_i x$. Set $n = \max\{n_i\}$. Then $x' = f^n x$ is an element of E' not in E and is annihilated by J. By maximality of J we see that $Rx' = (R/J)x' \cap E = (0)$. Hence E' is not an essential extension of E a contradiction.

To prove (2) write $J = (f_1, \ldots, f_t)$. Then $I[J^\infty]$ is equal to

$$(\ldots((I[f_1^\infty])[f_2^\infty])\ldots)[f_t^\infty]$$

and the result follows from (1) and induction. □

43.4. Projective covers

In this section we briefly discuss projective covers.

Definition 43.4.1. Let R be a ring. A surjection $P \to M$ of R-modules is said to be a *projective cover*, or sometimes a *projective envelope*, if P is a projective R-module and $P \to M$ is an essential surjection.

Projective covers do not always exist. For example, if k is a field and $R = k[x]$ is the polynomial ring over k, then the module $M = R/(x)$ does not have a projective cover. Namely, for any surjection $f : P \to M$ with P projective over R, the proper submodule $(x-1)P$ surjects onto M. Hence f is not essential.

Lemma 43.4.2. *Let R be a ring and let M be an R-module. If a projective cover of M exists, then it is unique up to isomorphism.*

Proof. Let $P \to M$ and $P' \to M$ be projective covers. Because P is a projective R-module and $P' \to M$ is surjective, we can find an R-module map $\alpha : P \to P'$ compatible with the maps to M. Since $P' \to M$ is essential, we see that α is surjective. As P' is a projective R-module we can choose a direct sum decomposition $P = \mathrm{Ker}(\alpha) \oplus P'$. Since $P' \to M$ is surjective and since $P \to M$ is essential we conclude that $\mathrm{Ker}(\alpha)$ is zero as desired. □

Here is an example where projective covers exist.

Lemma 43.4.3. *Let $(R, \mathfrak{m}, \kappa)$ be a local ring. Any finite R-module has a projective cover.*

Proof. Let M be a finite R-module. Let $r = \dim_\kappa(M/\mathfrak{m}M)$. Choose $x_1, \ldots, x_r \in M$ mapping to a basis of $M/\mathfrak{m}M$. Consider the map $f : R^{\oplus r} \to M$. By Nakayama's lemma this is a surjection (Algebra, Lemma 10.18.1). If $N \subset R^{\oplus R}$ is a proper submodule, then $N/\mathfrak{m}N \to \kappa^{\oplus r}$ is not surjective (by Nakayama's lemma again) hence $N/\mathfrak{m}N \to M/\mathfrak{m}M$ is not surjective. Thus f is an essential surjection. □

43.5. Injective hulls

In this section we briefly discuss injective hulls.

Definition 43.5.1. Let R be a ring. A injection $M \to I$ of R-modules is said to be an *injective hull* if I is a injective R-module and $M \to I$ is an essential injection.

Injective hulls always exist.

Lemma 43.5.2. *Let R be a ring. Any R-module has an injective hull.*

Proof. Let M be an R-module. By More on Algebra, Section 15.40 the category of R-modules has enough injectives. Choose an injection $M \to I$ with I an injective R-module. Consider the set $\mathcal{S}$ of submodules $M \subset E \subset I$ such that E is an essential extension of M. We order $\mathcal{S}$ by inclusion. If $\{E_\alpha\}$ is a totally ordered subset of $\mathcal{S}$, then $\bigcup E_\alpha$ is an essential extension of M too (Lemma 43.2.3). Thus we can apply Zorn's lemma and find a maximal element $E \in \mathcal{S}$. We claim $M \subset E$ is an injective hull, i.e., E is an injective R-module. This follows from Lemma 43.3.5. □

Lemma 43.5.3. *Let R be a ring. Let M, N be R-modules and let $M \to E$ and $N \to E'$ be injective hulls. Then*

(1) *for any R-module map $\varphi : M \to N$ there exists an R-module map $\psi : E \to E'$ such that*

$$\begin{array}{ccc} M & \longrightarrow & E \\ {\scriptstyle\varphi}\downarrow & & \downarrow{\scriptstyle\psi} \\ N & \longrightarrow & E' \end{array}$$

commutes,

(2) *if φ is injective, then ψ is injective,*

(3) *if φ is an essential injection, then ψ is an isomorphism,*

(4) *if φ is an isomorphism, then ψ is an isomorphism,*

(5) *if $M \to I$ is an embedding of M into an injective R-module, then there is an isomorphism $I \cong E \oplus I'$ compatible with the embeddings of M,*

In particular, the injective hull E of M is unique up to isomorphism.

Proof. Part (1) follows from the fact that E' is an injective R-module. Part (2) follows as $\mathrm{Ker}(\psi) \cap M = 0$ and E is an essential extension of M. Assume φ is an essential injection. Then $E \cong \psi(E) \subset E'$ by (2) which implies $E' = \psi(E) \oplus E''$ because E is injective. Since E' is an essential extension of M (Lemma 43.2.2) we get $E'' = 0$. Part (4) is a special case of (3). Assume $M \to I$ as in (5). Choose a map $\alpha : E \to I$ extending the map $M \to I$. Arguing as before we see that α is injective. Thus as before $\alpha(E)$ splits off from I. This proves (5). □

Example 43.5.4. Let R be a domain with fraction field K. Then $R \subset K$ is an injective hull of R. Namely, by Example 43.3.6 we see that K is an injective R-module and by Lemma 43.2.4 we see that $R \subset K$ is an essential extension.

Definition 43.5.5. An object X of an additive category is called *indecomposable* if it is nonzero and if $X = Y \oplus Z$, then either $Y = 0$ or $Z = 0$.

Lemma 43.5.6. *Let R be a ring. Let E be an indecomposable injective R-module. Then*

(1) *E is the injective hull of any nonzero submodule of E,*

(2) *the intersection of any two nonzero submodules of E is nonzero,*
(3) *$End_R(E, E)$ is a noncommutative local ring with maximal ideal those $\varphi : E \to E$ whose kernel is nonzero, and*
(4) *the set of zerodivisors on E is a prime ideal $\mathfrak{p}$ of R and E is an injective $R_\mathfrak{p}$-module.*

Proof. Part (1) follows from Lemma 43.5.3. Part (2) follows from part (1) and the definition of injective hulls.

Proof of (3). Set $A = \mathrm{End}_R(E, E)$ and $I = \{\varphi \in A \mid \mathrm{Ker}(f) \neq 0\}$. The statement means that I is a two sided ideal and that any $\varphi \in A$, $\varphi \notin I$ is invertible. Suppose φ and ψ are not injective. Then $\mathrm{Ker}(\varphi) \cap \mathrm{Ker}(\psi)$ is nonzero by (2). Hence $\varphi + \psi \in I$. It follows that I is a two sided ideal. If $\varphi \in A$, $\varphi \notin I$, then $E \cong \varphi(E) \subset E$ is an injective submodule, hence $E = \varphi(E)$ because E is indecomposable.

Proof of (4). Consider the ring map $R \to A$ and let $\mathfrak{p} \subset R$ be the inverse image of the maximal ideal I. Then it is clear that $\mathfrak{p}$ is a prime ideal and that $R \to A$ extends to $R_\mathfrak{p} \to A$. Thus E is an $R_\mathfrak{p}$-module. It follows from Lemma 43.3.3 that E is injective as an $R_\mathfrak{p}$-module. □

Lemma 43.5.7. *Let $\mathfrak{p} \subset R$ be a prime of a ring R. Let E be the injective hull of $R/\mathfrak{p}$. Then*

(1) *E is indecomposable,*
(2) *E is the injective hull of $\kappa(\mathfrak{p})$,*
(3) *E is the injective hull of $\kappa(\mathfrak{p})$ over the ring $R_\mathfrak{p}$.*

Proof. As $R/\mathfrak{p} \subset \kappa(\mathfrak{p})$ we can extend the embedding to a map $\kappa(\mathfrak{p}) \to E$. Hence (2) holds. For $f \in R$, $f \notin \mathfrak{p}$ the map $f : \kappa(\mathfrak{p}) \to \kappa(\mathfrak{p})$ is an isomorphism hence the map $f : E \to E$ is an isomorphism, see Lemma 43.5.3. Thus E is an $R_\mathfrak{p}$-module. It is injective as an $R_\mathfrak{p}$-module by Lemma 43.3.3. Finally, let $E' \subset E$ be a nonzero injective R-submodule. Then $J = (R/\mathfrak{p}) \cap E'$ is nonzero. After shrinking E' we may assume that E' is the injective hull of J (see Lemma 43.5.3 for example). Observe that $R/\mathfrak{p}$ is an essential extension of J for example by Lemma 43.2.4. Hence $E' \to E$ is an isomorphism by Lemma 43.5.3 part (3). Hence E is indecomposable. □

Lemma 43.5.8. *Let R be a Noetherian ring. Let E be an indecomposable injective R-module. Then there exists a prime ideal $\mathfrak{p}$ of R such that E is the injective hull of $\kappa(\mathfrak{p})$.*

Proof. Let $\mathfrak{p}$ be the prime ideal found in Lemma 43.5.6. Say $\mathfrak{p} = (f_1, \ldots, f_r)$. Pick a nonzero element $x \in \bigcap \mathrm{Ker}(f_i : E \to E)$, see Lemma 43.5.6. Then $(R_\mathfrak{p})x$ is a module isomorphic to $\kappa(\mathfrak{p})$ inside E. We conclude by Lemma 43.5.6. □

Proposition 43.5.9 (Structure injective modules over Noetherian rings)**.** *Let R be a Noetherian ring. Every injective module is a direct sum of indecomposable injective modules. Every indecomposable injective module is the injective hull of the residue field at a prime.*

Proof. The second statement is Lemma 43.5.8. For the second statement, let I be an injective R-module. We will use transfinite induction to construct $I_\alpha \subset I$ for ordinals α which are direct sums of indecomposable injective R-modules $E_{\beta+1}$ for $\beta < \alpha$. For $\alpha = 0$ we let $I_0 = 0$. Suppose given an ordinal α such that I_α has been constructed. Then I_α is an injective R-module by Lemma 43.3.8. Hence

$I \cong I_\alpha \oplus I'$. If $I' = 0$ we are done. If not, then I' has an associated prime by Algebra, Lemma 10.60.7. Thus I' contains a copy of $R/\mathfrak{p}$ for some prime $\mathfrak{p}$. Hence I' contains an indecomposable submodule E by Lemmas 43.5.3 and 43.5.7. Set $I_{\alpha+1} = I_\alpha \oplus E_\alpha$. If α is a limit ordinal and I_β has been constructed for $\beta < \alpha$, then we set $I_\alpha = \bigcup_{\beta<\alpha} I_\beta$. Observe that $I_\alpha = \bigoplus_{\beta<\alpha} E_{\beta+1}$. This concludes the proof. □

43.6. Duality over Artinian local rings

Let $(R, \mathfrak{m}, \kappa)$ be an artinian local ring. Recall that this implies R is Noetherian and that R has finite length as an R-module. Moreover an R-module is finite if and only if it has finite length. We will use these facts without further mention in this section. Please see Algebra, Sections 10.48 and 10.49 and Algebra, Proposition 10.57.6 for more details.

Lemma 43.6.1. *Let $(R, \mathfrak{m}, \kappa)$ be an artinian local ring. Let E be an injective hull of κ. For every finite R-module M we have*

$$\mathit{length}_R(M) = \mathit{length}_R(\mathrm{Hom}_R(M, E))$$

In particular, the injective hull E of κ is a finite R-module.

Proof. Because E is an essential extension of κ we have $\kappa = E[\mathfrak{m}]$ where $E[\mathfrak{m}]$ is the $\mathfrak{m}$-torsion in E (notation as in More on Algebra, Section 15.56). Hence $\mathrm{Hom}_R(\kappa, E) \cong \kappa$ and the equality of lengths holds for $M = \kappa$. We prove the displayed equality of the lemma by induction on the length of M. If M is nonzero there exists a surjection $M \to \kappa$ with kernel M'. Since the functor $M \mapsto \mathrm{Hom}_R(M, E)$ is exact we obtain a short exact sequence

$$0 \to \mathrm{Hom}_R(\kappa, E) \to \mathrm{Hom}_R(M, E) \to \mathrm{Hom}_R(M', E) \to 0.$$

Additivity of length for this sequence and the sequence $0 \to M' \to M \to \kappa \to 0$ and the equality for M' (induction hypothesis) and κ implies the equality for M. The final statement of the lemma follows as $E = \mathrm{Hom}_R(R, E)$. □

Lemma 43.6.2. *Let $(R, \mathfrak{m}, \kappa)$ be an artinian local ring. Let E be an injective hull of κ. For any finite R-module M the evaluation map*

$$M \longrightarrow \mathrm{Hom}_R(\mathrm{Hom}_R(M, E), E)$$

is an isomorphism. In particular $R = \mathrm{Hom}_R(E, E)$.

Proof. Observe that the displayed arrow is injective. Namely, if $x \in M$ is a nonzero element, then there is a nonzero map $Rx \to \kappa$ which we can extend to a map $\varphi : M \to E$ that doesn't vanish on x. Since the source and target of the arrow have the same length by Lemma 43.6.1 we conclude it is an isomorphism. The final statement follows on taking $M = R$. □

To state the next lemma, denote Mod_R^{fg} the category of finite R-modules over a ring R.

Lemma 43.6.3. *Let $(R, \mathfrak{m}, \kappa)$ be an artinian local ring. Let E be an injective hull of κ. The functor $D(-) = \mathrm{Hom}_R(-, E)$ induces an exact anti-equivalence $Mod_R^{fg} \to Mod_R^{fg}$ and $D \circ D \cong id$.*

Proof. We have seen that $D \circ D = \mathrm{id}$ on Mod_R^{fg} in Lemma 43.6.2. It follows immediately that D is an anti-equivalence. □

Lemma 43.6.4. *Assumptions and notation as in Lemma 43.6.3. Let $I \subset R$ be an ideal and M a finite R-module. Then*

$$D(M[I]) = D(M)/ID(M) \quad \text{and} \quad D(M/IM) = D(M)[I]$$

Proof. Say $I = (f_1, \ldots, f_t)$. Consider the map

$$M^{\oplus t} \xrightarrow{f_1, \ldots, f_t} M$$

with cokernel M/IM. Applying the exact functor D we conclude that $D(M/IM)$ is $D(M)[I]$. The other case is proved in the same way. □

43.7. Injective hull of the residue field

Most of our results will be for Noetherian local rings in this section.

Lemma 43.7.1. *Let $R \to S$ be a surjective map of local rings with kernel I. Let E be the injective hull of the residue field of R over R. Then $E[I]$ is the injective hull of the residue field of S over S.*

Proof. Observe that $E[I] = \mathrm{Hom}_R(S, E)$ as $S = R/I$. Hence $E[I]$ is an injective S-module by Lemma 43.3.4. Since E is an essential extension of $\kappa = R/\mathfrak{m}_R$ it follows that $E[I]$ is an essential extension of κ as well. The result follows. □

Lemma 43.7.2. *Let $(R, \mathfrak{m}, \kappa)$ be a local ring. Let E be the injective hull of κ. Let M be a $\mathfrak{m}$-power torsion R-module with $n = \dim_\kappa(M[\mathfrak{m}]) < \infty$. Then M is isomorphic to a submodule of $E^{\oplus n}$.*

Proof. Observe that $E^{\oplus n}$ is the injective hull of $\kappa^{\oplus n} = M[\mathfrak{m}]$. Thus there is an R-module map $M \to E^{\oplus n}$ which is injective on $M[\mathfrak{m}]$. Since M is $\mathfrak{m}$-power torsion the inclusion $M[\mathfrak{m}] \subset M$ is an essential extension (for example by Lemma 43.2.4) we conclude that the kernel of $M \to E^{\oplus n}$ is zero. □

Lemma 43.7.3. *Let $(R, \mathfrak{m}, \kappa)$ be a Noetherian local ring. Let E be an injective hull of κ over R. Let E_n be an injective hull of κ over $R/\mathfrak{m}^n$. Then $E = \bigcup E_n$ and $E_n = E[\mathfrak{m}^n]$.*

Proof. We have $E_n = E[\mathfrak{m}^n]$ by Lemma 43.7.1. We have $E = \bigcup E_n$ because $\bigcup E_n = E[\mathfrak{m}^\infty]$ is an injective R-submodule which contains κ, see Lemma 43.3.9. □

The following lemma tells us the injective hull of the residue field of a Noetherian local ring only depends on the completion.

Lemma 43.7.4. *Let $R \to S$ be a flat local homomorphism of local Noetherian rings such that $R/\mathfrak{m}_R \cong S/\mathfrak{m}_R S$. Then the injective hull of the residue field of R is the injective hull of the residue field of S.*

Proof. Set $\kappa = R/\mathfrak{m}_R = S/\mathfrak{m}_S$. Let E_R be the injective hull of κ over R. Let E_S be the injective hull of κ over S. Observe that E_S is an injective R-module by Lemma 43.3.2. Choose an extension $E_R \to E_S$ of the identification of residue fields. This map is an isomorphism by Lemma 43.7.3 because $R \to S$ induces an isomorphism $R/\mathfrak{m}_R^n \to S/\mathfrak{m}_S^n$ for all n. □

Lemma 43.7.5. *Let $(R, \mathfrak{m}, \kappa)$ be a Noetherian local ring. Let E be an injective hull of κ over R. Then $\mathrm{Hom}_R(E, E)$ is canonically isomorphic to the completion of R.*

Proof. Write $E = \bigcup E_n$ with $E_n = E[\mathfrak{m}^n]$ as in Lemma 43.7.3. Any endomorphism of E preserves this filtration. Hence

$$\mathrm{Hom}_R(E, E) = \lim \mathrm{Hom}_R(E_n, E_n)$$

The lemma follows as $\mathrm{Hom}_R(E_n, E_n) = \mathrm{Hom}_{R/\mathfrak{m}^n}(E_n, E_n) = R/\mathfrak{m}^n$ by Lemma 43.6.2. □

Lemma 43.7.6. *Let $(R, \mathfrak{m}, \kappa)$ be a Noetherian local ring. Let E be an injective hull of κ over R. Then E satisfies the descending chain condition.*

Proof. If $E \subset M_1 \subset M_2 \ldots$ is a sequence of submodules, then

$$\mathrm{Hom}_R(E, E) \to \mathrm{Hom}_R(M_1, E) \to \mathrm{Hom}_R(M_2, E) \to \ldots$$

is sequence of surjections. By Lemma 43.7.5 each of these is a module over the completion $R^\wedge = \mathrm{Hom}_R(E, E)$. Since $R^\wedge$ is Noetherian (Algebra, Lemma 10.91.10) the sequence stabilizes: $\mathrm{Hom}_R(M_n, E) = \mathrm{Hom}_R(M_{n+1}, E) = \ldots$. Since E is injective, this can only happen if $\mathrm{Hom}_R(M_n/M_{n+1}, E)$ is zero. However, if M_n/M_{n+1} is nonzero, then it contains a nonzero element annihilated by $\mathfrak{m}$, because E is $\mathfrak{m}$-power torsion by Lemma 43.7.3. In this case M_n/M_{n+1} has a nonzero map into E, contradicting the assumed vanishing. This finishes the proof. □

Lemma 43.7.7. *Let $(R, \mathfrak{m}, \kappa)$ be a Noetherian local ring. Let E be an injective hull of κ.*

1. *For an R-module M the following are equivalent:*
   (a) *M satisfies the ascending chain condition,*
   (b) *M is a finite R-module, and*
   (c) *there exist n, m and an exact sequence $R^{\oplus m} \to R^{\oplus n} \to M \to 0$.*
2. *For an R-module M the following are equivalent:*
   (a) *M satisfies the descending chain condition,*
   (b) *M is $\mathfrak{m}$-power torsion and $\dim_\kappa(M[\mathfrak{m}]) < \infty$, and*
   (c) *there exist n, m and an exact sequence $0 \to M \to E^{\oplus n} \to E^{\oplus m}$.*

Proof. We omit the proof of (1).

Let M be an R-module with the descending chain condition. Let $x \in M$. Then $\mathfrak{m}^n x$ is a descending chain of submodules, hence stabilizes. Thus $\mathfrak{m}^n x = \mathfrak{m}^{n+1} x$ for some n. By Nakayama's lemma (Algebra, Lemma 10.18.1) this implies $\mathfrak{m}^n x = 0$, i.e., x is $\mathfrak{m}$-power torsion. Since $M[\mathfrak{m}]$ is a vector space over κ it has to be finite dimensional in order to have the descending chain condition.

Assume that M is $\mathfrak{m}$-power torsion and has a finite dimensional $\mathfrak{m}$-torsion submodule $M[\mathfrak{m}]$. By Lemma 43.7.2 we see that M is a submodule of $E^{\oplus n}$ for some n. Consider the quotient $N = E^{\oplus n}/M$. By Lemma 43.7.6 the module E has the descending chain condition hence so do $E^{\oplus n}$ and N. Therefore N satisfies (2)(a) which implies N satisfies (2)(b) by the second paragraph of the proof. Thus by Lemma 43.7.2 again we see that N is a submodule of $E^{\oplus m}$ for some m. Thus we have a short exact sequence $0 \to M \to E^{\oplus n} \to E^{\oplus m}$.

Assume we have a short exact sequence $0 \to M \to E^{\oplus n} \to E^{\oplus m}$. Since E satisfies the descending chain condition by Lemma 43.7.6 so does M. □

Proposition 43.7.8 (Matlis duality)**.** *Let $(R, \mathfrak{m}, \kappa)$ be a complete local Noetherian ring. Let E be an injective hull of κ over R. The functor $D(-) = \mathrm{Hom}_R(-, E)$ induces an anti-equivalence*

$$\left\{\begin{array}{c} R\text{-modules with the} \\ \text{descending chain condition} \end{array}\right\} \longleftrightarrow \left\{\begin{array}{c} R\text{-modules with the} \\ \text{ascending chain condition} \end{array}\right\}$$

and we have $D \circ D = id$ on either side of the equivalence.

Proof. By Lemma 43.7.5 we have $R = \mathrm{Hom}_R(E, E) = D(E)$. Of course we have $E = \mathrm{Hom}_R(R, E) = D(R)$. Since E is injective the functor D is exact. The result now follows immediately from the description of the categories in Lemma 43.7.7. □

43.8. Local cohomology

Let A be a ring and let $I \subset A$ be a finitely generated ideal (if I is not finitely generated perhaps a different definition should be used). Consider the functor

$$\Gamma_Z : \mathrm{Mod}_A \longrightarrow \mathrm{Mod}_A, \quad M \longmapsto \bigcup M[I^n]$$

which sends M to the submodule of I-power torsion. This functor is left exact and has a derived extension $R\Gamma_Z : D(A) \to D(A)$. Note that for every $K \in D(A)$ there is a canonical map $R\Gamma_Z(K) \to K$.

Lemma 43.8.1. *With notation as above. The functors Γ_Z and $R\Gamma_Z$ depend only on the closed subset $Z = V(I) \subset \mathrm{Spec}(A)$.*

Proof. Let M be an A-module. Let $x \in M$. If $x \in \Gamma_Z(M)$, then x maps to zero in M_f for all $f \in I$. Hence x maps to zero in $M_\mathfrak{p}$ for all $\mathfrak{p} \not\supset I$. Conversely, if x maps to zero in $M_\mathfrak{p}$ for all $\mathfrak{p} \not\supset I$, then x maps to zero in M_f for all $f \in I$. Hence if $I = (f_1, \ldots, f_r)$, then $f_i^{n_i} x = 0$ for some $n_i \geq 1$. It follows that $x \in M[I^{\sum n_i}]$. Thus $\Gamma_Z(M)$ is the kernel of $M \to \prod_{\mathfrak{p} \notin Z} M_\mathfrak{p}$. □

Lemma 43.8.2. *With notation as above. For any A-module M we have*

$$R^q\Gamma_Z(M) = \mathrm{colim}_n \, Ext^q_A(A/I^n, M)$$

Proof. This follows immediately from the fact that $M[I^n] = \mathrm{Hom}_A(A/I^n, M)$. □

Lemma 43.8.3. *With notation as above. If A is Noetherian and the A-module M is I-power torsion, then the canonical map $R\Gamma_Z(M) \to M$ is an isomorphism in $D(A)$.*

Proof. By Lemma 43.3.9 we see that M has an injective resolution $M \to J^0 \to J^1 \to \ldots$ with $J^p = J^p[I^\infty]$ completely I-power torsion. The result follows. □

Lemma 43.8.4. *If A is a Noetherian ring and $I = (f_1, \ldots, f_r)$ an ideal. There is a canonical isomorphism*

$$(A \to \prod\nolimits_{i_0} A_{f_{i_0}} \to \prod\nolimits_{i_0 < i_1} A_{f_{i_0} f_{i_1}} \to \ldots \to A_{f_1 \ldots f_r}) \longrightarrow R\Gamma_Z(A)$$

in $D(A)$.

Proof. There is a canonical map of complexes

$$(A \to \prod\nolimits_{i_0} A_{f_{i_0}} \to \prod\nolimits_{i_0 < i_1} A_{f_{i_0} f_{i_1}} \to \ldots \to A_{f_1 \ldots f_r}) \longrightarrow A.$$

Applying $R\Gamma_Z$ we get a map

$$R\Gamma_Z(A \to \prod\nolimits_{i_0} A_{f_{i_0}} \to \prod\nolimits_{i_0<i_1} A_{f_{i_0}f_{i_1}} \to \ldots \to A_{f_1\ldots f_r}) \longrightarrow R\Gamma_Z(A).$$

This map is an isomorphism as clearly $R\Gamma_Z(A_f) = 0$ for all $f \in I$. On the other hand, the homology modules of the complex $A \to \prod_{i_0} A_{f_{i_0}} \to \prod_{i_0<i_1} A_{f_{i_0}f_{i_1}} \to \ldots \to A_{f_1\ldots f_r}$ are I-power torsion by More on Algebra, Lemmas 15.19.13 and 15.19.6. Thus the canonical map

$$\begin{array}{c} R\Gamma_Z(A \to \prod_{i_0} A_{f_{i_0}} \to \prod_{i_0<i_1} A_{f_{i_0}f_{i_1}} \to \ldots \to A_{f_1\ldots f_r}) \\ \downarrow \\ (A \to \prod_{i_0} A_{f_{i_0}} \to \prod_{i_0<i_1} A_{f_{i_0}f_{i_1}} \to \ldots \to A_{f_1\ldots f_r}) \end{array}$$

is an isomorphism in $D(A)$ by Lemma 43.8.3 and the spectral sequence of Derived Categories, Lemma 13.20.3. □

Lemma 43.8.5. *If $A \to B$ is a homomorphism of Noetherian rings and $I \subset A$ is an ideal, then*

$$R\Gamma_Z(A) \otimes^{\mathbf{L}}_A B = R\Gamma_Y(B)$$

where $Y = V(IB) \subset \mathrm{Spec}(B)$.

Proof. This is clear from Lemma 43.8.4 and the fact that formation of the extended Čech complex commutes with base change. □

43.9. Other chapters

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CHAPTER 44

Étale Cohomology

44.1. Introduction

These are the notes of a course on étale cohomology taught by Johan de Jong at Columbia University in the Fall of 2009. The original note takers were Thibaut Pugin, Zachary Maddock and Min Lee. Over time we will add references to background material in the rest of the stacks project and provide rigorous proofs of all the statements.

44.2. Which sections to skip on a first reading?

We want to use the material in this chapter for the development of theory related to algebraic spaces, Deligne-Mumford stacks, algebraic stacks, etc. Thus we have added some pretty technical material to the original exposition of étale cohomology for schemes. The reader can recognize this material by the frequency of the word "topos", or by discussions related to set theory, or by proofs dealing with very general properties of morphisms of schemes. Some of these discussions can be skipped on a first reading.

In particular, we suggest that the reader skip the following sections:

(1) Comparing big and small topoi, Section 44.39.
(2) Recovering morphisms, Section 44.41.
(3) Push and pull, Section 44.42.
(4) Property (A), Section 44.43.
(5) Property (B), Section 44.44.
(6) Property (C), Section 44.45.
(7) Topological invariance of the small étale site, Section 44.46.
(8) Integral universally injective morphisms, Section 44.48.
(9) Big sites and pushforward, Section 44.49.
(10) Exactness of big lower shriek, Section 44.50.

Besides these sections there are some sporadic results that may be skipped that the reader can recognize by the keywords given above.

44.3. Prologue

These lectures are about another cohomology theory. The first thing to remark is that the Zariski topology is not entirely satisfactory. One of the main reasons that it fails to give the results that we would want is that if X is a complex variety and $\mathcal{F}$ is a constant sheaf then

$$H^i(X, \mathcal{F}) = 0, \quad \text{for all } i > 0.$$

The reason for that is the following. In an irreducible scheme (a variety in particular), any two nonempty open subsets meet, and so the restriction mappings of

a constant sheaf are surjective. We say that the sheaf is *flasque*. In this case, all higher Čech cohomology groups vanish, and so do all higher Zariski cohomology groups. In other words, there are "not enough" open sets in the Zariski topology to detect this higher cohomology.

On the other hand, if X is a smooth projective complex variety, then

$$H^{2\dim X}_{Betti}(X(\mathbf{C}), \Lambda) = \Lambda \quad \text{for } \Lambda = \mathbf{Z},\ \mathbf{Z}/n\mathbf{Z},$$

where $X(\mathbf{C})$ means the set of complex points of X. This is a feature that would be nice to replicate in algebraic geometry. In positive characteristic in particular.

44.4. The étale topology

It is very hard to simply "add" extra open sets to refine the Zariski topology. One efficient way to define a topology is to consider not only open sets, but also some schemes that lie over them. To define the étale topology, one considers all morphisms $\varphi : U \to X$ which are étale. If X is a smooth projective variety over $\mathbf{C}$, then this means

(1) U is a disjoint union of smooth varieties ; and
(2) φ is (analytically) locally an isomorphism.

The word "analytically" refers to the usual (transcendental) topology over $\mathbf{C}$. So the second condition means that the derivative of φ has full rank everywhere (and in particular all the components of U have the same dimension as X).

A double cover – loosely defined as a finite degree 2 map between varieties – for example

$$\mathrm{Spec}(\mathbf{C}[t]) \longrightarrow \mathrm{Spec}(\mathbf{C}[t]), \quad t \longmapsto t^2$$

will not be an étale morphism if it has a fibre consisting of a single point. In the example this happens when $t = 0$. For a finite map between varieties over $\mathbf{C}$ to be étale all the fibers should have the same number of points. Removing the point $t = 0$ from the source of the map in the example will make the morphism étale. But we can remove other points from the source of the morphism also, and the morphism will still be étale. To consider the étale topology, we have to look at all such morphisms. Unlike the Zariski topology, these need not be merely be open subsets of X, even though their images always are.

Definition 44.4.1. A family of morphisms $\{\varphi_i : U_i \to X\}_{i\in I}$ is called an *étale covering* if each φ_i is an étale morphism and their images cover X, i.e., $X = \bigcup_{i\in I} \varphi_i(U_i)$.

This "defines" the étale topology. In other words, we can now say what the sheaves are. An *étale sheaf* $\mathcal{F}$ of sets (resp. abelian groups, vector spaces, etc) on X is the data:

(1) for each étale morphism $\varphi : U \to X$ a set (resp. abelian group, vector space, etc) $\mathcal{F}(U)$,
(2) for each pair U, U' of étale schemes over X, and each morphism $U \to U'$ over X (which is automatically étale) a restriction map $\rho^U_{U'} : \mathcal{F}(U) \to \mathcal{F}(U')$

These data have to satisfy the following *sheaf axiom*:

(*) for every étale covering $\{\varphi_i : U_i \to X\}_{i\in I}$, the diagram

$$\emptyset \longrightarrow \mathcal{F}(U) \longrightarrow \Pi_{i\in I}\mathcal{F}(U_i) \rightrightarrows \Pi_{i,j\in I}\mathcal{F}(U_i \times_U U_j)$$

is exact in the category of sets (resp. abelian groups, vector spaces, etc).

Remark 44.4.2. In the last statement, it is essential not to forget the case where $i = j$ which is in general a highly nontrivial condition (unlike in the Zariski topology). In fact, frequently important coverings have only one element.

Since the identity is an étale morphism, we can compute the global sections of an étale sheaf, and cohomology will simply be the corresponding right-derived functors. In other words, once more theory has been developed and statements have been made precise, there will be no obstacle to defining cohomology.

44.5. Feats of the étale topology

For a natural number $n \in \mathbf{N} = \{1, 2, 3, 4, \dots\}$ it is true that

$$H^2_{étale}(\mathbf{P}^1_{\mathbf{C}}, \mathbf{Z}/n\mathbf{Z}) = \mathbf{Z}/n\mathbf{Z}.$$

More generally, if X is a complex variety, then its étale Betti numbers with coefficients in a finite field agree with the usual Betti numbers of $X(\mathbf{C})$, i.e.,

$$\dim_{\mathbf{F}_q} H^{2i}_{étale}(X, \mathbf{F}_q) = \dim_{\mathbf{F}_q} H^{2i}_{Betti}(X(\mathbf{C}), \mathbf{F}_q).$$

This is extremely satisfactory. However, these equalities only hold for torsion coefficients, not in general. For integer coefficients, one has

$$H^2_{étale}(\mathbf{P}^1_{\mathbf{C}}, \mathbf{Z}) = 0.$$

There are ways to get back to nontorsion coefficients from torsion ones by a limit procedure which we will come to shortly.

44.6. A computation

How do we compute the cohomology of $\mathbf{P}^1_{\mathbf{C}}$ with coefficients $\Lambda = \mathbf{Z}/n\mathbf{Z}$? We use Čech cohomology. A covering of $\mathbf{P}^1_{\mathbf{C}}$ is given by the two standard opens U_0, U_1, which are both isomorphic to $\mathbf{A}^1_{\mathbf{C}}$, and which intersection is isomorphic to $\mathbf{A}^1_{\mathbf{C}} \setminus \{0\} = \mathbf{G}_{m,\mathbf{C}}$. It turns out that the Mayer-Vietoris sequence holds in étale cohomology. This gives an exact sequence

$$H^{i-1}_{étale}(U_0 \cap U_1, \Lambda) \to H^i_{étale}(\mathbf{P}^1_C, \Lambda) \to H^i_{étale}(U_0, \Lambda) \oplus H^i_{étale}(U_1, \Lambda) \to H^i_{étale}(U_0 \cap U_1, \Lambda).$$

To get the answer we expect, we would need to show that the direct sum in the third term vanishes. In fact, it is true that, as for the usual topology,

$$H^q_{étale}(\mathbf{A}^1_{\mathbf{C}}, \Lambda) = 0 \quad \text{for } q \geq 1,$$

and

$$H^q_{étale}(\mathbf{A}^1_{\mathbf{C}} \setminus \{0\}, \Lambda) = \begin{cases} \Lambda & \text{if } q = 1, \text{ and} \\ 0 & \text{for } q \geq 2. \end{cases}$$

These results are already quite hard (what is an elementary proof?). Let us explain how we would compute this once the machinery of étale cohomology is at our disposal.

Higher cohomology. This is taken care of by the following general fact: if X is an affine curve over $\mathbf{C}$, then

$$H^q_{étale}(X, \mathbf{Z}/n\mathbf{Z}) = 0 \quad \text{for } q \geq 2.$$

This is proved by considering the generic point of the curve and doing some Galois cohomology. So we only have to worry about the cohomology in degree 1.

Cohomology in degree 1. We use the following identifications:

$$H^1_{étale}(X, \mathbf{Z}/n\mathbf{Z}) = \left\{ \begin{array}{l} \text{sheaves of sets } \mathcal{F} \text{ on the étale site } X_{\text{ét}} \text{ endowed with an} \\ \text{action } \mathbf{Z}/n\mathbf{Z} \times \mathcal{F} \to \mathcal{F} \text{ such that } \mathcal{F} \text{ is a } \mathbf{Z}/n\mathbf{Z}\text{-torsor.} \end{array} \right\} \Big/ \cong$$

$$= \left\{ \begin{array}{l} \text{morphisms } Y \to X \text{ which are finite étale together} \\ \text{with a free } \mathbf{Z}/n\mathbf{Z} \text{ action such that } X = Y/(\mathbf{Z}/n\mathbf{Z}). \end{array} \right\} \Big/ \cong .$$

The first identification is very general (it is true for any cohomology theory on a site) and has nothing to do with the étale topology. The second identification is a consequence of descent theory. The last set describes a collection of geometric objects on which we can get our hands.

The curve $\mathbf{A}^1_{\mathbf{C}}$ has no nontrivial finite étale covering and hence $H^1_{étale}(\mathbf{A}^1_{\mathbf{C}}, \mathbf{Z}/n\mathbf{Z}) = 0$. This can be seen either topologically or by using the argument in the next paragraph.

Let us describe the finite étale coverings $\varphi : Y \to \mathbf{A}^1_{\mathbf{C}} \setminus \{0\}$. It suffices to consider the case where Y is connected, which we assume. We are going to find out what Y can be by applying the Riemann-Hurwitz formula (of course this is a bit silly, and you can go ahead and skip the next section if you like). Say that this morphism is n to 1, and consider a projective compactification

$$\begin{array}{ccc} Y & \hookrightarrow & \bar{Y} \\ \downarrow{\scriptstyle \varphi} & & \downarrow{\scriptstyle \bar{\varphi}} \\ \mathbf{A}^1_{\mathbf{C}} \setminus \{0\} & \hookrightarrow & \mathbf{P}^1_{\mathbf{C}} \end{array}$$

Even though φ is étale and does not ramify, $\bar{\varphi}$ may ramify at 0 and ∞. Say that the preimages of 0 are the points $y_1, \ldots, y_r$ with indices of ramification $e_1, \ldots e_r$, and that the preimages of ∞ are the points $y'_1, \ldots, y'_s$ with indices of ramification $d_1, \ldots d_s$. In particular, $\sum e_i = n = \sum d_j$. Applying the Riemann-Hurwitz formula, we get

$$2g_Y - 2 = -2n + \sum (e_i - 1) + \sum (d_j - 1)$$

and therefore $g_Y = 0$, $r = s = 1$ and $e_1 = d_1 = n$. Hence $Y \cong \mathbf{A}^1_{\mathbf{C}} \setminus \{0\}$, and it is easy to see that $\varphi(z) = \lambda z^n$ for some $\lambda \in \mathbf{C}^*$. After reparametrizing Y we may assume $\lambda = 1$. Thus our covering is given by taking the nth root of the coordinate on $\mathbf{A}^1_{\mathbf{C}} \setminus \{0\}$.

Remember that we need to classify the coverings of $\mathbf{A}^1_{\mathbf{C}} \setminus \{0\}$ together with free $\mathbf{Z}/n\mathbf{Z}$-actions on them. In our case any such action corresponds to an automorphism of Y sending z to $\zeta_n z$, where ζ_n is a primitive nth root of unity. There are $\phi(n)$ such actions (here $\phi(n)$ means the Euler function). Thus there are exactly $\phi(n)$ connected finite étale coverings with a given free $\mathbf{Z}/n\mathbf{Z}$-action, each corresponding to a primitive nth root of unity. We leave it to the reader to see that the disconnected finite étale degree n coverings of $\mathbf{A}^1_{\mathbf{C}} \setminus \{0\}$ with a given free $\mathbf{Z}/n\mathbf{Z}$-action correspond one-to-one with nth roots of 1 which are not primitive. In other words, this computation shows that

$$H^1_{étale}(\mathbf{A}^1_{\mathbf{C}} \setminus \{0\}, \mathbf{Z}/n\mathbf{Z}) = \mu_n(\mathbf{C}) \cong \mathbf{Z}/n\mathbf{Z}.$$

The first identification is canonical, the second isn't. We remark that since the proof of Riemann-Hurwitz does not use this fact, the above actually constitutes a proof (provided we fill in the details on vanishing, etc).

44.7. Nontorsion coefficients

To study nontorsion coefficients, one makes the following definition:

$$H^i_{étale}(X, \mathbf{Q}_\ell) := \left(\lim_n H^i_{étale}(X, \mathbf{Z}/\ell^n\mathbf{Z})\right) \otimes_{\mathbf{Z}_\ell} \mathbf{Q}_\ell.$$

The symbol $\lim_n$ denote the *limit* of the system of cohomology groups $H^i_{étale}(X, \mathbf{Z}/\ell^n\mathbf{Z})$ indexed by n, see Categories, Section 4.21. Thus we will need to study systems of sheaves satisfying some compatibility conditions.

44.8. Sheaf theory

At this point we start talking about sites and sheaves in earnest. There is an amazing amount of useful abstract material that could fit in the next few sections. Some of this material is worked out in earlier chapters, such as the chapter on sites, modules on sites, and cohomology on sites. We try to refrain from adding to much material here, just enough so the material later in this chapter makes sense.

44.9. Presheaves

A reference for this section is Sites, Section 7.2.

Definition 44.9.1. Let $\mathcal{C}$ be a category. A *presheaf of sets* (respectively, an *abelian presheaf*) on $\mathcal{C}$ is a functor $\mathcal{C}^{opp} \to \mathit{Sets}$ (resp. Ab).

Terminology. If $U \in \mathrm{Ob}(\mathcal{C})$, then elements of $\mathcal{F}(U)$ are called *sections* of $\mathcal{F}$ on U; for $\varphi : V \to U$ in $\mathcal{C}$, the map $\mathcal{F}(\varphi) : \mathcal{F}(V) \to \mathcal{F}(U)$ is denoted $s \mapsto \mathcal{F}(\varphi)(s) = \varphi^*(s) = s|_V$ and called *restriction mapping*. This last notation is ambiguous since the restriction map depends on φ, but it is a standard abuse of notation. We also use the notation $\Gamma(U, \mathcal{F}) = \mathcal{F}(U)$.

Saying that $\mathcal{F}$ is a functor means that if $W \to V \to U$ are morphisms in $\mathcal{C}$ and $s \in \Gamma(U, \mathcal{F})$ then $(s|_V)|_W = s|_W$, with the abuse of notation just seen. Moreover, the restriction mappings corresponding to the identity morphisms $\mathrm{id}_U : U \to U$ are the identity.

The category of presheaves of sets (respectively of abelian presheaves) on $\mathcal{C}$ is denoted $\mathit{PSh}(\mathcal{C})$ (resp. $\mathit{PAb}(\mathcal{C})$). It is the category of functors from $\mathcal{C}^{opp}$ to Sets (resp. Ab), which is to say that the morphisms of presheaves are natural transformations of functors. We only consider the categories $\mathit{PSh}(\mathcal{C})$ and $\mathit{PAb}(\mathcal{C})$ when the category $\mathcal{C}$ is small. (Our convention is that a category is small unless otherwise mentioned, and if it isn't small it should be listed in Categories, Remark 4.2.2.)

Example 44.9.2. Given an object $X \in \mathrm{Ob}(\mathcal{C})$, we consider the functor

$$\begin{array}{rccc} h_X : & \mathcal{C}^{opp} & \longrightarrow & \mathit{Sets} \\ & U & \longmapsto & h_X(U) = \mathrm{Mor}_{\mathcal{C}}(U, X) \\ & V \xrightarrow{\varphi} U & \longmapsto & \varphi \circ - : h_X(U) \to h_X(V). \end{array}$$

It is a presheaf, called the *representable presheaf associated to* X. It is not true that representable presheaves are sheaves in every topology on every site.

Lemma 44.9.3 (Yoneda)**.** *Let $\mathcal{C}$ be a category, and $X, Y \in \mathrm{Ob}(\mathcal{C})$. There is a natural bijection*

$$\begin{array}{ccc} \mathrm{Mor}_{\mathcal{C}}(X,Y) & \longrightarrow & \mathrm{Mor}_{PSh(\mathcal{C})}(h_X, h_Y) \\ \psi & \longmapsto & h_\psi = \psi \circ - : h_X \to h_Y. \end{array}$$

Proof. See Categories, Lemma 4.3.5. □

44.10. Sites

Definition 44.10.1. Let $\mathcal{C}$ be a category. A *family of morphisms with fixed target* $\mathcal{U} = \{\varphi_i : U_i \to U\}_{i \in I}$ is the data of

(1) an object $U \in \mathcal{C}$;
(2) a set I (possibly empty) ; and
(3) for all $i \in I$, a morphism $\varphi_i : U_i \to U$ of $\mathcal{C}$ with target U.

There is a notion of a *morphism of families of morphisms with fixed target*. A special case of that is the notion of a *refinement*. A reference for this material is Sites, Section 7.8.

Definition 44.10.2. A *site*[1] consists of a category $\mathcal{C}$ and a set $\mathrm{Cov}(\mathcal{C})$ consisting of families of morphisms with fixed target called *coverings*, such that

(1) (isomorphism) if $\varphi : V \to U$ is an isomorphism in $\mathcal{C}$, then $\{\varphi : V \to U\}$ is a covering,
(2) (locality) if $\{\varphi_i : U_i \to U\}_{i \in I}$ is a covering and for all $i \in I$ we are given a covering $\{\psi_{ij} : U_{ij} \to U_i\}_{j \in I_i}$, then
$$\{\varphi_i \circ \psi_{ij} : U_{ij} \to U\}_{(i,j) \in \prod_{i \in I} \{i\} \times I_i}$$
is also a covering, and
(3) (base change) if $\{U_i \to U\}_{i \in I}$ is a covering and $V \to U$ is a morphism in $\mathcal{C}$, then
   (a) for all $i \in I$ the fibre product $U_i \times_U V$ exists in $\mathcal{C}$, and
   (b) $\{U_i \times_U V \to V\}_{i \in I}$ is a covering.

For us the category underlying a site is always "small", i.e., its collection of objects form a set, and the collection of coverings of a site is a set as well (as in the definition above). We will mostly, in this chapter, leave out the arguments that cut down the collection of objects and coverings to a set. For further discussion, see Sites, Remark 7.6.3.

Example 44.10.3. If X is a topological space, then it has an associated site X_{Zar} defined as follows: the objects of X_{Zar} are the open subsets of X, the morphisms between these are the inclusion mappings, and the coverings are the usual topological (surjective) coverings. Observe that if $U, V \subset W \subset X$ are open subsets then $U \times_W V = U \cap V$ exists: this category has fiber products. All the verifications are trivial and everything works as expected.

[1] What we call a site is a called a category endowed with a pretopology in [**AGV71**, Exposé II, Définition 1.3]. In [**Art62**] it is called a category with a Grothendieck topology.

44.11. Sheaves

Definition 44.11.1. A presheaf $\mathcal{F}$ of sets (resp. abelian presheaf) on a site $\mathcal{C}$ is said to be a *separated presheaf* if for all coverings $\{\varphi_i : U_i \to U\}_{i\in I} \in \mathrm{Cov}(\mathcal{C})$ the map

$$\mathcal{F}(U) \longrightarrow \prod\nolimits_{i\in I} \mathcal{F}(U_i)$$

is injective. Here the map is $s \mapsto (s|_{U_i})_{i\in I}$. The presheaf $\mathcal{F}$ is a *sheaf* if for all coverings $\{\varphi_i : U_i \to U\}_{i\in I} \in \mathrm{Cov}(\mathcal{C})$, the diagram

$$\mathcal{F}(U) \longrightarrow \prod\nolimits_{i\in I} \mathcal{F}(U_i) \rightrightarrows \prod\nolimits_{i,j\in I} \mathcal{F}(U_i \times_U U_j), \tag{44.11.1.1}$$

where the first map is $s \mapsto (s|_{U_i})_{i\in I}$ and the two maps on the right are $(s_i)_{i\in I} \mapsto (s_i|_{U_i\times_U U_j})$ and $(s_i)_{i\in I} \mapsto (s_j|_{U_i\times_U U_j})$, is an equalizer diagram in the category of sets (resp. abelian groups).

Remark 44.11.2. For the empty covering (where $I = \emptyset$), this implies that $\mathcal{F}(\emptyset)$ is an empty product, which is a final object in the corresponding category (a singleton, for both *Sets* and *Ab*).

Example 44.11.3. Working this out for the site X_{Zar} associated to a topological space, see Example 44.10.3, gives the usual notion of sheaves.

Definition 44.11.4. We denote $Sh(\mathcal{C})$ (resp. $Ab(\mathcal{C})$) the full subcategory of $PSh(\mathcal{C})$ (resp. $PAb(\mathcal{C})$) whose objects are sheaves. This is the *category of sheaves of sets* (resp. *abelian sheaves*) on $\mathcal{C}$.

44.12. The example of G-sets

Let G be a group and define a site $\mathcal{T}_G$ as follows: the underlying category is the category of G-sets, i.e., its objects are sets endowed with a left G-action and the morphisms are equivariant maps; and the coverings of $\mathcal{T}_G$ are the families $\{\varphi_i : U_i \to U\}_{i\in I}$ satisfying $U = \bigcup_{i\in I} \varphi_i(U_i)$.

There is a special object in the site $\mathcal{T}_G$, namely the G-set G endowed with its natural action by left translations. We denote it ${}_GG$. Observe that there is a natural group isomorphism

$$\begin{array}{rccc} \rho: & G^{opp} & \longrightarrow & \mathrm{Aut}_{G\text{-}Sets}({}_GG) \\ & g & \longmapsto & (h \mapsto hg). \end{array}$$

In particular, for any presheaf $\mathcal{F}$, the set $\mathcal{F}({}_GG)$ inherits a G-action via ρ. (Note that by contravariance of $\mathcal{F}$, the set $\mathcal{F}({}_GG)$ is again a left G-set.) In fact, the functor

$$\begin{array}{ccc} Sh(\mathcal{T}_G) & \longrightarrow & G\text{-}Sets \\ \mathcal{F} & \longmapsto & \mathcal{F}({}_GG) \end{array}$$

is an equivalence of categories. Its quasi-inverse is the functor $X \mapsto h_X$. Without giving the complete proof (which can be found in Sites, Section 7.9) let us try to explain why this is true.

(1) If S is a G-set, we can decompose it into orbits $S = \coprod_{i\in I} O_i$. The sheaf axiom for the covering $\{O_i \to S\}_{i\in I}$ says that

$$\mathcal{F}(S) \longrightarrow \prod\nolimits_{i\in I} \mathcal{F}(O_i) \rightrightarrows \prod\nolimits_{i,j\in I} \mathcal{F}(O_i \times_S O_j)$$

is an equalizer. Observing that fibered products in G-*Sets* are induced from fibered products in *Sets*, and using the fact that $\mathcal{F}(\emptyset)$ is a G-singleton, we get that

$$\prod_{i,j\in I} \mathcal{F}(O_i \times_S O_j) = \prod_{i\in I} \mathcal{F}(O_i)$$

and the two maps above are in fact the same. Therefore the sheaf axiom merely says that $\mathcal{F}(S) = \prod_{i\in I} \mathcal{F}(O_i)$.

(2) If S is the G-set $S = G/H$ and $\mathcal{F}$ is a sheaf on $\mathcal{T}_G$, then we claim that

$$\mathcal{F}(G/H) = \mathcal{F}({}_GG)^H$$

and in particular $\mathcal{F}(\{*\}) = \mathcal{F}({}_GG)^G$. To see this, let's use the sheaf axiom for the covering $\{{}_GG \to G/H\}$ of S. We have

$$\begin{array}{rcl} {}_GG \times_{G/H} {}_GG & \cong & G \times H \\ (g_1, g_2) & \longmapsto & (g_1, g_1 g_2^{-1}) \end{array}$$

is a disjoint union of copies of ${}_GG$ (as a G-set). Hence the sheaf axiom reads

$$\mathcal{F}(G/H) \longrightarrow \mathcal{F}({}_GG) \rightrightarrows \prod_{h\in H} \mathcal{F}({}_GG)$$

where the two maps on the right are $s \mapsto (s)_{h\in H}$ and $s \mapsto (hs)_{h\in H}$. Therefore $\mathcal{F}(G/H) = \mathcal{F}({}_GG)^H$ as claimed.

This doesn't quite prove the claimed equivalence of categories, but it shows at least that a sheaf $\mathcal{F}$ is entirely determined by its sections over ${}_GG$. Details (and set theoretical remarks) can be found in Sites, Section 7.9.

44.13. Sheafification

Definition 44.13.1. Let $\mathcal{F}$ be a presheaf on the site $\mathcal{C}$ and $\mathcal{U} = \{U_i \to U\} \in \mathrm{Cov}(\mathcal{C})$. We define the *zeroth Čech cohomology group* of $\mathcal{F}$ with respect to $\mathcal{U}$ by

$$\check{H}^0(\mathcal{U}, \mathcal{F}) = \left\{ (s_i)_{i\in I} \in \prod\nolimits_{i\in I} \mathcal{F}(U_i) \text{ such that } s_i|_{U_i\times_U U_j} = s_j|_{U_i\times_U U_j} \right\}.$$

There is a canonical map $\mathcal{F}(U) \to \check{H}^0(\mathcal{U}, \mathcal{F})$, $s \mapsto (s|_{U_i})_{i\in I}$. We say that a *morphism of coverings* from a covering $\mathcal{V} = \{V_j \to V\}_{j\in J}$ to $\mathcal{U}$ is a triple (χ, α, χ_j), where $\chi : V \to U$ is a morphism, $\alpha : J \to I$ is a map of sets, and for all $j \in J$ the morphism χ_j fits into a commutative diagram

$$\begin{array}{ccc} V_j & \xrightarrow[\chi_j]{} & U_{\alpha(j)} \\ \downarrow & & \downarrow \\ V & \xrightarrow{\chi} & U. \end{array}$$

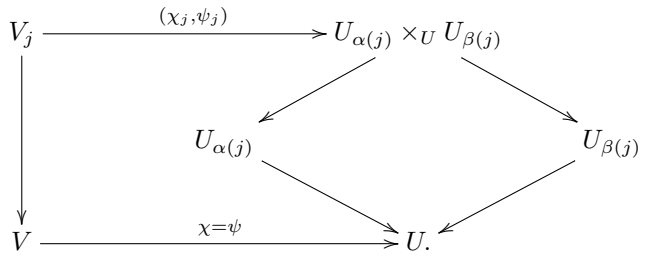
Given the data $\chi, \alpha, \{\chi_j\}_{i\in J}$ we define

$$\begin{array}{rcl} \check{H}^0(\mathcal{U}, \mathcal{F}) & \longrightarrow & \check{H}^0(\mathcal{V}, \mathcal{F}) \\ (s_i)_{i\in I} & \longmapsto & \left(\chi_j^* \left(s_{\alpha(j)}\right)\right)_{j\in J}. \end{array}$$

We then claim that

(1) the map is well-defined, and
(2) depends only on χ and is independent of the choice of $\alpha, \{\chi_j\}_{i\in J}$.

We omit the proof of the first fact. To see part (2), consider another triple (ψ, β, ψ_j) with $\chi = \psi$. Then we have the commutative diagram



Given a section $s \in \mathcal{F}(\mathcal{U})$, its image in $\mathcal{F}(V_j)$ under the map given by $(\chi, \alpha, \{\chi_j\}_{i \in J})$ is $\chi_j^* s_{\alpha(j)}$, and its image under the map given by $(\psi, \beta, \{\psi_j\}_{i \in J})$ is $\psi_j^* s_{\beta(j)}$. These two are equal since by assumption $s \in \check{H}(\mathcal{U}, \mathcal{F})$ and hence both are equal to the pullback of the common value

$$s_{\alpha(j)}|_{U_{\alpha(j)} \times_U U_{\beta(j)}} = s_{\beta(j)}|_{U_{\alpha(j)} \times_U U_{\beta(j)}}$$

pulled back by the map (χ_j, ψ_j) in the diagram.

Theorem 44.13.2. *Let $\mathcal{C}$ be a site and $\mathcal{F}$ a presheaf on $\mathcal{C}$.*

(1) *The rule*
$$U \mapsto \mathcal{F}^+(U) := \operatorname{colim}_{\mathcal{U} \text{ covering of } U} \check{H}^0(\mathcal{U}, \mathcal{F})$$
is a presheaf. And the colimit is a directed one.
(2) *There is a canonical map of presheaves $\mathcal{F} \to \mathcal{F}^+$.*
(3) *If $\mathcal{F}$ is a separated presheaf then $\mathcal{F}^+$ is a sheaf and the map in (2) is injective.*
(4) *$\mathcal{F}^+$ is a separated presheaf.*
(5) *$\mathcal{F}^\# = (\mathcal{F}^+)^+$ is a sheaf, and the canonical map induces a functorial isomorphism*
$$\operatorname{Hom}_{PSh(\mathcal{C})}(\mathcal{F}, \mathcal{G}) = \operatorname{Hom}_{Sh(\mathcal{C})}(\mathcal{F}^\#, \mathcal{G})$$
for any $\mathcal{G} \in Sh(\mathcal{C})$.

Proof. See Sites, Theorem 7.10.10. □

In other words, this means that the natural map $\mathcal{F} \to \mathcal{F}^\#$ is a left adjoint to the forgetful functor $Sh(\mathcal{C}) \to PSh(\mathcal{C})$.

44.14. Cohomology

The following is the basic result that makes it possible to define cohomology for abelian sheaves on sites.

Theorem 44.14.1. *The category of abelian sheaves on a site is an abelian category which has enough injectives.*

Proof. See Modules on Sites, Lemma 18.3.1 and Injectives, Theorem 19.7.4. □

So we can define cohomology as the right-derived functors of the sections functor: if $U \in \mathrm{Ob}(\mathcal{C})$ and $\mathcal{F} \in Ab(\mathcal{C})$,

$$H^p(U, \mathcal{F}) := R^p\Gamma(U, \mathcal{F}) = H^p(\Gamma(U, \mathcal{I}^\bullet))$$

where $\mathcal{F} \to \mathcal{I}^\bullet$ is an injective resolution. To do this, we should check that the functor $\Gamma(U, -)$ is left exact. This is true and is part of why the category $Ab(\mathcal{C})$ is abelian, see Modules on Sites, Lemma 18.3.1. For more general discussion of cohomology on sites (including the global sections functor and its right derived functors), see Cohomology on Sites, Section 21.3.

44.15. The fpqc topology

Before doing étale cohomology we study a bit the fpqc topology, since it works well for quasi-coherent sheaves.

Definition 44.15.1. Let T be a scheme. An *fpqc covering* of T is a family $\{\varphi_i : T_i \to T\}_{i \in I}$ such that

(1) each φ_i is a flat morphism and $\bigcup_{i \in I} \varphi_i(T_i) = T$, and
(2) for each affine open $U \subset T$ there exists a finite set K, a map $\mathbf{i} : K \to I$ and affine opens $U_{\mathbf{i}(k)} \subset T_{\mathbf{i}(k)}$ such that $U = \bigcup_{k \in K} \varphi_{\mathbf{i}(k)}(U_{\mathbf{i}(k)})$.

Remark 44.15.2. The first condition corresponds to fp, which stands for *fidèlement plat*, faithfully flat in french, and the second to qc, *quasi-compact*. The second part of the first condition is unnecessary when the second condition holds.

Example 44.15.3. Examples of fpqc coverings.

(1) Any Zariski open covering of T is an fpqc covering.
(2) A family $\{\mathrm{Spec}(B) \to \mathrm{Spec}(A)\}$ is an fpqc covering if and only if $A \to B$ is a faithfully flat ring map.
(3) If $f : X \to Y$ is flat, surjective and quasi-compact, then $\{f : X \to Y\}$ is an fpqc covering.
(4) The morphism $\varphi : \coprod_{x \in \mathbf{A}^1_k} \mathrm{Spec}(\mathcal{O}_{\mathbf{A}^1_k, x}) \to \mathbf{A}^1_k$, where k is a field, is flat and surjective. It is not quasi-compact, and in fact the family $\{\varphi\}$ is not an fpqc covering.
(5) Write $\mathbf{A}^2_k = \mathrm{Spec}(k[x, y])$. Denote $i_x : D(x) \to \mathbf{A}^2_k$ and $i_y : D(y) \hookrightarrow \mathbf{A}^2_k$ the standard opens. Then the families $\{i_x, i_y, \mathrm{Spec}(k[[x, y]]) \to \mathbf{A}^2_k\}$ and $\{i_x, i_y, \mathrm{Spec}(\mathcal{O}_{\mathbf{A}^2_k, 0}) \to \mathbf{A}^2_k\}$ are fpqc coverings.

Lemma 44.15.4. *The collection of fpqc coverings on the category of schemes satisfies the axioms of site.*

Proof. See Topologies, Lemma 33.8.7. □

It seems that this lemma allows us to define the fpqc site of the category of schemes. However, there is a set theoretical problem that comes up when considering the fpqc topology, see Topologies, Section 33.8. It comes from our requirement that sites are "small", but that no small category of schemes can contain a cofinal system of fpqc coverings of a given nonempty scheme. Although this does not strictly speaking prevent us from defining "partial" fpqc sites, it does not seem prudent to do so. The work-around is to allow the notion of a sheaf for the fpqc topology (see below) but to prohibit considering the category of all fpqc sheaves.

Definition 44.15.5. Let S be a scheme. The category of schemes over S is denoted Sch/S. Consider a functor $\mathcal{F} : (Sch/S)^{opp} \to Sets$, in other words a presheaf of sets. We say $\mathcal{F}$ *satisfies the sheaf property for the fpqc topology* if for every fpqc covering $\{U_i \to U\}_{i \in I}$ of schemes over S the diagram (44.11.1.1) is an equalizer diagram.

We similarly say that $\mathcal{F}$ *satisfies the sheaf property for the Zariski topology* if for every open covering $U = \bigcup_{i \in I} U_i$ the diagram (44.11.1.1) is an equalizer diagram. See Schemes, Definition 25.15.3. Clearly, this is equivalent to saying that for every scheme T over S the restriction of $\mathcal{F}$ to the opens of T is a (usual) sheaf.

Lemma 44.15.6. *Let $\mathcal{F}$ be a presheaf on Sch/S. Then $\mathcal{F}$ satisfies the sheaf property for the fpqc topology if and only if*

(1) *$\mathcal{F}$ satisfies the sheaf property with respect to the Zariski topology, and*
(2) *for every faithfully flat morphism $\mathrm{Spec}(B) \to \mathrm{Spec}(A)$ of affine schemes over S, the sheaf axiom holds for the covering $\{\mathrm{Spec}(B) \to \mathrm{Spec}(A)\}$. Namely, this means that*

$$\mathcal{F}(\mathrm{Spec}(A)) \longrightarrow \mathcal{F}(\mathrm{Spec}(B)) \rightrightarrows \mathcal{F}(\mathrm{Spec}(B \otimes_A B))$$

is an equalizer diagram.

Proof. See Topologies, Lemma 33.8.13. □

An alternative way to think of a presheaf $\mathcal{F}$ on Sch/S which satisfies the sheaf condition for the fpqc topology is as the following data:

(1) for each T/S, a usual (i.e., Zariski) sheaf $\mathcal{F}_T$ on T_{Zar},
(2) for every map $f : T' \to T$ over S, a restriction mapping $f^{-1}\mathcal{F}_T \to \mathcal{F}_{T'}$

such that

(a) the restriction mappings are functorial,
(b) if $f : T' \to T$ is an open immersion then the restriction mapping $f^{-1}\mathcal{F}_T \to \mathcal{F}_{T'}$ is an isomorphism, and
(c) for every faithfully flat morphism $\mathrm{Spec}(B) \to \mathrm{Spec}(A)$ over S, the diagram

$$\mathcal{F}_{\mathrm{Spec}(A)}(\mathrm{Spec}(A)) \longrightarrow \mathcal{F}_{\mathrm{Spec}(B)}(\mathrm{Spec}(B)) \rightrightarrows \mathcal{F}_{\mathrm{Spec}(B \otimes_A B)}(\mathrm{Spec}(B \otimes_A B))$$

is an equalizer.

Data (1) and (2) and conditions (a), (b) give the data of a presheaf on Sch/S satisfying the sheaf condition for the Zariski topology. By Lemma 44.15.6 condition (c) then suffices to get the sheaf condition for the fpqc topology.

Example 44.15.7. Consider the presheaf

$$\begin{array}{rccc} \mathcal{F} : & (Sch/S)^{opp} & \longrightarrow & Ab \\ & T/S & \longmapsto & \Gamma(T, \Omega_{T/S}). \end{array}$$

The compatibility of differentials with localization implies that $\mathcal{F}$ is a sheaf on the Zariski site. However, it does not satisfy the sheaf condition for the fpqc topology. Namely, consider the case $S = \mathrm{Spec}(\mathbf{F}_p)$ and the morphism

$$\varphi : V = \mathrm{Spec}(\mathbf{F}_p[v]) \to U = \mathrm{Spec}(\mathbf{F}_p[u])$$

given by mapping u to v^p. The family $\{\varphi\}$ is an fpqc covering, yet the restriction mapping $\mathcal{F}(U) \to \mathcal{F}(V)$ sends the generator $\mathrm{d}u$ to $\mathrm{d}(v^p) = 0$, so it is the zero map, and the diagram

$$\mathcal{F}(U) \xrightarrow{0} \mathcal{F}(V) \rightrightarrows \mathcal{F}(V \times_U V)$$

is not an equalizer. We will see later that $\mathcal{F}$ does in fact give rise to a sheaf on the étale and smooth sites.

Lemma 44.15.8. *Any representable presheaf on Sch/S satisfies the sheaf condition for the fpqc topology.*

Proof. See Descent, Lemma 34.9.3. □

We will return to this later, since the proof of this fact uses descent for quasi-coherent sheaves, which we will discuss in the next section. A fancy way of expressing the lemma is to say that *the fpqc topology is weaker than the canonical topology*, or that the fpqc topology is *subcanonical*. In the setting of sites this is discussed in Sites, Section 7.13.

Remark 44.15.9. The fpqc is the finest topology that we will see. Hence any presheaf satisfying the sheaf condition for the fpqc topology will be a sheaf in the subsequent sites (étale, smooth, etc). In particular representable presheaves will be sheaves on the étale site of a scheme for example.

Example 44.15.10. Let S be a scheme. Consider the additive group scheme $\mathbf{G}_{a,S} = \mathbf{A}^1_S$ over S, see Groupoids, Example 38.5.3. The associated representable presheaf is given by

$$h_{\mathbf{G}_{a,S}}(T) = \mathrm{Mor}_S(T, \mathbf{G}_{a,S}) = \Gamma(T, \mathcal{O}_T).$$

By the above we now know that this is a presheaf of sets which satisfies the sheaf condition for the fpqc topology. On the other hand, it is clearly a presheaf of rings as well. Hence we can think of this as a functor

$$\begin{array}{rccc} \mathcal{O}: & (Sch/S)^{opp} & \longrightarrow & Rings \\ & T/S & \longmapsto & \Gamma(T, \mathcal{O}_T) \end{array}$$

which satisfies the sheaf condition for the fpqc topology. Correspondingly there is a notion of $\mathcal{O}$-module, and so on and so forth.

44.16. Faithfully flat descent

Definition 44.16.1. Let $\mathcal{U} = \{t_i : T_i \to T\}_{i \in I}$ be a family of morphisms of schemes with fixed target. A *descent datum* for quasi-coherent sheaves with respect to $\mathcal{U}$ is a family $(\mathcal{F}_i, \varphi_{ij})_{i,j \in I}$ where

(1) for all i, $\mathcal{F}_i$ is a quasi-coherent sheaf on T_i ; and
(2) for all $i, j \in I$ the map $\varphi_{ij} : \mathrm{pr}_0^*\mathcal{F}_i \cong \mathrm{pr}_1^*\mathcal{F}_j$ is an isomorphism on $T_i \times_T T_j$ such that the diagrams

$$\begin{array}{ccc} \mathrm{pr}_0^*\mathcal{F}_i & \xrightarrow{\mathrm{pr}_{01}^*\varphi_{ij}} & \mathrm{pr}_1^*\mathcal{F}_j \\ & {\scriptstyle \mathrm{pr}_{02}^*\varphi_{ik}} \searrow \quad \swarrow {\scriptstyle \mathrm{pr}_{12}^*\varphi_{jk}} & \\ & \mathrm{pr}_2^*\mathcal{F}_k & \end{array}$$

commute on $T_i \times_T T_j \times_T T_k$.

This descent datum is called *effective* if there exist a quasi-coherent sheaf $\mathcal{F}$ over T and $\mathcal{O}_{T_i}$-module isomorphisms $\varphi_i : t_i^*\mathcal{F} \cong \mathcal{F}_i$ satisfying the cocycle condition, namely

$$\varphi_{ij} = \mathrm{pr}_1^*(\varphi_j) \circ \mathrm{pr}_0^*(\varphi_i)^{-1}.$$

In this and the next section we discuss some ingredients of the proof of the following theorem, as well as some related material.

Theorem 44.16.2. *If $\mathcal{V} = \{T_i \to T\}_{i\in I}$ is an fpqc covering, then all descent data for quasi-coherent sheaves with respect to $\mathcal{V}$ are effective.*

Proof. See Descent, Proposition 34.5.2. □

In other words, the fibered category of quasi-coherent sheaves is a stack on the fpqc site. The proof of the theorem is in two steps. The first one is to realize that for Zariski coverings this is easy (or well-known) using standard glueing of sheaves (see Sheaves, Section 6.33) and the locality of quasi-coherence. The second step is the case of an fpqc covering of the form $\{\mathrm{Spec}(B) \to \mathrm{Spec}(A)\}$ where $A \to B$ is a faithfully flat ring map. This is a lemma in algebra, which we now present.

Descent of modules. If $A \to B$ is a ring map, we consider the complex

$$(B/A)_\bullet : \qquad B \to B \otimes_A B \to B \otimes_A B \otimes_A B \to \cdots$$

where B is in degree 0, $B \otimes_A B$ in degree 1, etc, and the maps are given by

$$\begin{aligned} b &\mapsto 1 \otimes b - b \otimes 1, \\ b_0 \otimes b_1 &\mapsto 1 \otimes b_0 \otimes b_1 - b_0 \otimes 1 \otimes b_1 + b_0 \otimes b_1 \otimes 1, \\ &\text{etc.} \end{aligned}$$

Lemma 44.16.3. *If $A \to B$ is faithfully flat, then the complex $(B/A)_\bullet$ is exact in positive degrees, and $H^0((B/A)_\bullet) = A$.*

Proof. See Descent, Lemma 34.3.6. □

Grothendieck proves this in three steps. Firstly, he assumes that the map $A \to B$ has a section, and constructs an explicit homotopy to the complex where A is the only nonzero term, in degree 0. Secondly, he observes that to prove the result, it suffices to do so after a faithfully flat base change $A \to A'$, replacing B with $B' = B \otimes_A A'$. Thirdly, he applies the faithfully flat base change $A \to A' = B$ and remarks that the map $A' = B \to B' = B \otimes_A B$ has a natural section.

The same strategy proves the following lemma.

Lemma 44.16.4. *If $A \to B$ is faithfully flat and M is an A-module, then the complex $(B/A)_\bullet \otimes_A M$ is exact in positive degrees, and $H^0((B/A)_\bullet \otimes_A M) = M$.*

Proof. See Descent, Lemma 34.3.6. □

Definition 44.16.5. Let $A \to B$ be a ring map and N a B-module. A *descent datum* for N with respect to $A \to B$ is an isomorphism $\varphi : N \otimes_A B \cong B \otimes_A N$ of $B \otimes_A B$-modules such that the diagram of $B \otimes_A B \otimes_A B$-modules

$$\begin{array}{ccc} N \otimes_A B \otimes_A B & \xrightarrow{\varphi_{02}} & B \otimes_A N \otimes_A B \\ & {\scriptstyle \varphi_{01}} \searrow \quad \swarrow {\scriptstyle \varphi_{12}} & \\ & B \otimes_A B \otimes_A N & \end{array}$$

commutes.

If $N' = B \otimes_A M$ for some A-module M, then it has a canonical descent datum given by the map

$$\begin{aligned} \varphi_{\mathrm{can}} : \quad N' \otimes_A B &\to B \otimes_A N' \\ b_0 \otimes m \otimes b_1 &\mapsto b_0 \otimes b_1 \otimes m. \end{aligned}$$

Definition 44.16.6. A descent datum (N, φ) is called *effective* if there exists an A-module M such that $(N, \varphi) \cong (B \otimes_A M, \varphi_{\text{can}})$, with the obvious notion of isomorphism of descent data.

Theorem 44.16.2 is a consequence the following result.

Theorem 44.16.7. *If $A \to B$ is faithfully flat then all descent data with respect to $A \to B$ is effective.*

Proof. See Descent, Proposition 34.3.9. See also Descent, Remark 34.3.11 for an alternative view of the proof. □

Remarks 44.16.8. The results on descent of modules have several applications:

(1) The exactness of the Čech complex in positive degrees for the covering $\{\text{Spec}(B) \to \text{Spec}(A)\}$ where $A \to B$ is faithfully flat. This will give some vanishing of cohomology.
(2) If (N, φ) is a descent datum with respect to a faithfully flat map $A \to B$, then the corresponding A-module is given by
$$M = \ker \begin{pmatrix} N & \longrightarrow & B \otimes_A N \\ n & \longmapsto & 1 \otimes n - \varphi(n \otimes 1) \end{pmatrix}.$$
See Descent, Proposition 34.3.9.

44.17. Quasi-coherent sheaves

We can apply the descent of modules to study quasi-coherent sheaves.

Proposition 44.17.1. *For any quasi-coherent sheaf $\mathcal{F}$ on S the presheaf*
$$\begin{array}{rccc} \mathcal{F}^a : & Sch/S & \to & Ab \\ & (f : T \to S) & \mapsto & \Gamma(T, f^*\mathcal{F}) \end{array}$$
is an $\mathcal{O}$-module which satisfies the sheaf condition for the fpqc topology.

Proof. This is proved in Descent, Lemma 34.7.1. We indicate the proof here. As established in Lemma 44.15.6, it is enough to check the sheaf property on Zariski coverings and faithfully flat morphisms of affine schemes. The sheaf property for Zariski coverings is standard scheme theory, since $\Gamma(U, i^*\mathcal{F}) = \mathcal{F}(U)$ when $i : U \hookrightarrow S$ is an open immersion.

For $\{\text{Spec}(B) \to \text{Spec}(A)\}$ with $A \to B$ faithfully flat and $\mathcal{F}|_{\text{Spec}(A)} = \widetilde{M}$ this corresponds to the fact that $M = H^0((B/A)_\bullet \otimes_A M)$, i.e., that
$$0 \to M \to B \otimes_A M \to B \otimes_A B \otimes_A M$$
is exact by Lemma 44.16.4. □

There is an abstract notion of a quasi-coherent sheaf on a ringed site. We briefly introduce this here. For more information please consult Modules on Sites, Section 18.23. Let $\mathcal{C}$ be a category, and let U be an object of $\mathcal{C}$. Then $\mathcal{C}/U$ indicates the category of objects over U, see Categories, Example 4.2.13. If $\mathcal{C}$ is a site, then $\mathcal{C}/U$ is a site as well, namely the coverings of V/U are families $\{V_i/U \to V/U\}$ of morphisms of $\mathcal{C}/U$ with fixed target such that $\{V_i \to V\}$ is a covering of $\mathcal{C}$. Moreover, given any sheaf $\mathcal{F}$ on $\mathcal{C}$ the *restriction* $\mathcal{F}|_{\mathcal{C}/U}$ (defined in the obvious manner) is a sheaf as well. See Sites, Section 7.24 for details.

Definition 44.17.2. Let $\mathcal{C}$ be a *ringed site*, i.e., a site endowed with a sheaf of rings $\mathcal{O}$. A sheaf of $\mathcal{O}$-modules $\mathcal{F}$ on $\mathcal{C}$ is called *quasi-coherent* if for all $U \in \mathrm{Ob}(\mathcal{C})$ there exists a covering $\{U_i \to U\}_{i \in I}$ of $\mathcal{C}$ such that the restriction $\mathcal{F}|_{\mathcal{C}/U_i}$ is isomorphic to the cokernel of an $\mathcal{O}$-linear map of free $\mathcal{O}$-modules

$$\bigoplus\nolimits_{k \in K} \mathcal{O}|_{\mathcal{C}/U_i} \longrightarrow \bigoplus\nolimits_{l \in L} \mathcal{O}|_{\mathcal{C}/U_i}.$$

The direct sum over K is the sheaf associated to the presheaf $V \mapsto \bigoplus_{k \in K} \mathcal{O}(V)$ and similarly for the other.

Although it is useful to be able to give a general definition as above this notion is not well behaved in general.

Remark 44.17.3. In the case where $\mathcal{C}$ has a final object, e.g. S, it suffices to check the condition of the definition for $U = S$ in the above statement. See Modules on Sites, Lemma 18.23.3.

Theorem 44.17.4 (Meta theorem on quasi-coherent sheaves)**.** *Let S be a scheme. Let $\mathcal{C}$ be a site. Assume that*

1. *the underlying category $\mathcal{C}$ is a full subcategory of Sch/S,*
2. *any Zariski covering of $T \in \mathrm{Ob}(\mathcal{C})$ can be refined by a covering of $\mathcal{C}$,*
3. *S/S is an object of $\mathcal{C}$,*
4. *every covering of $\mathcal{C}$ is an fpqc covering of schemes.*

Then the presheaf $\mathcal{O}$ is a sheaf on $\mathcal{C}$ and any quasi-coherent $\mathcal{O}$-module on $(\mathcal{C}, \mathcal{O})$ is of the form $\mathcal{F}^a$ for some quasi-coherent sheaf $\mathcal{F}$ on S.

Proof. After some formal arguments this is exactly Theorem 44.16.2. Details omitted. In Descent, Proposition 34.7.11 we prove a more precise version of the theorem for the big Zariski, fppf, étale, smooth, and syntomic sites of S, as well as the small Zariski and étale sites of S. □

In other words, there is no difference between quasi-coherent modules on the scheme S and quasi-coherent $\mathcal{O}$-modules on sites $\mathcal{C}$ as in the theorem. More precise statements for the big and small sites $(Sch/S)_{fppf}$, $S_{étale}$, etc can be found in Descent, Section 34.7. In this chapter we will sometimes refer to a "site as in Theorem 44.17.4" in order to conveniently state results which hold in any of those situations.

44.18. Cech cohomology

Our next goal is to use descent theory to show that $H^i(\mathcal{C}, \mathcal{F}^a) = H^i_{Zar}(S, \mathcal{F})$ for all quasi-coherent sheaves $\mathcal{F}$ on S, and any site $\mathcal{C}$ as in Theorem 44.17.4. To this end, we introduce Čech cohomology on sites. See [**Art62**] and Cohomology on Sites, Sections 21.9, 21.10 and 21.11 for more details.

Definition 44.18.1. Let $\mathcal{C}$ be a category, $\mathcal{U} = \{U_i \to U\}_{i \in I}$ a family of morphisms of $\mathcal{C}$ with fixed target, and $\mathcal{F} \in PAb(\mathcal{C})$ an abelian presheaf. We define the *Čech complex* $\check{\mathcal{C}}^\bullet(\mathcal{U}, \mathcal{F})$ by

$$\prod_{i_0 \in I} \mathcal{F}(U_{i_0}) \to \prod_{i_0, i_1 \in I} \mathcal{F}(U_{i_0} \times_U U_{i_1}) \to \prod_{i_0, i_1, i_2 \in I} \mathcal{F}(U_{i_0} \times_U U_{i_1} \times_U U_{i_2}) \to \cdots$$

where the first term is in degree 0, and the maps are the usual ones. Again, it is essential to allow the case $i_0 = i_1$ etc. The *Čech cohomology groups* are defined by

$$\check{H}^p(\mathcal{U}, \mathcal{F}) = H^p(\check{\mathcal{C}}^\bullet(\mathcal{U}, \mathcal{F})).$$

Lemma 44.18.2. *The functor $\check{\mathcal{C}}^\bullet(\mathcal{U}, -)$ is exact on the category $PAb(\mathcal{C})$.*

In other words, if $0 \to \mathcal{F}_1 \to \mathcal{F}_2 \to \mathcal{F}_3 \to 0$ is a short exact sequence of presheaves of abelian groups, then

$$0 \to \check{\mathcal{C}}^\bullet(\mathcal{U}, \mathcal{F}_1) \to \check{\mathcal{C}}^\bullet(\mathcal{U}, \mathcal{F}_2) \to \check{\mathcal{C}}^\bullet(\mathcal{U}, \mathcal{F}_3) \to 0$$

is a short exact sequence of complexes.

Proof. This follows at once from the definition of a short exact sequence of presheaves. Namely, as the category of abelian presheaves is the category of functors on some category with values in Ab, it is automatically an abelian category: a sequence $\mathcal{F}_1 \to \mathcal{F}_2 \to \mathcal{F}_3$ is exact in PAb if and only if for all $U \in \mathrm{Ob}(\mathcal{C})$, the sequence $\mathcal{F}_1(U) \to \mathcal{F}_2(U) \to \mathcal{F}_3(U)$ is exact in Ab. So the complex above is merely a product of short exact sequences in each degree. See also Cohomology on Sites, Lemma 21.10.1. □

This shows that $\check{H}^\bullet(\mathcal{U}, -)$ is a δ-functor. We now proceed to show that it is a universal δ-functor. We thus need to show that it is an *effaceable* functor. We start by recalling the Yoneda lemma.

Lemma 44.18.3 (Yoneda Lemma)**.** *For any presheaf $\mathcal{F}$ on a category $\mathcal{C}$ there is a functorial isomorphism*

$$\mathrm{Hom}_{PSh(\mathcal{C})}(h_U, \mathcal{F}) = \mathcal{F}(U).$$

Proof. See Categories, Lemma 4.3.5. □

Given a set E we denote (in this section) $\mathbf{Z}[E]$ the free abelian group on E. In a formula $\mathbf{Z}[E] = \bigoplus_{e \in E} \mathbf{Z}$, i.e., $\mathbf{Z}[E]$ is a free $\mathbf{Z}$-module having a basis consisting of the elements of E. Using this notation we introduce the free abelian presheaf on a presheaf of sets.

Definition 44.18.4. Let $\mathcal{C}$ be a category. Given a presheaf of sets $\mathcal{G}$, we define the *free abelian presheaf on* $\mathcal{G}$, denoted $\mathbf{Z}_\mathcal{G}$, by the rule

$$\mathbf{Z}_\mathcal{G}(U) = \mathbf{Z}[\mathcal{G}(U)]$$

for $U \in \mathrm{Ob}(\mathcal{C})$ with restriction maps induced by the restriction maps of $\mathcal{G}$. In the special case $\mathcal{G} = h_U$ we write simply $\mathbf{Z}_U = \mathbf{Z}_{h_U}$.

The functor $\mathcal{G} \mapsto \mathbf{Z}_\mathcal{G}$ is left adjoint to the forgetful functor $PAb(\mathcal{C}) \to PSh(\mathcal{C})$. Thus, for any presheaf $\mathcal{F}$, there is a canonical isomorphism

$$\mathrm{Hom}_{PAb(\mathcal{C})}(\mathbf{Z}_U, \mathcal{F}) = \mathrm{Hom}_{PSh(\mathcal{C})}(h_U, \mathcal{F}) = \mathcal{F}(U)$$

the last equality by the Yoneda lemma. In particular, we have the following result.

Lemma 44.18.5. *The Čech complex $\check{\mathcal{C}}^\bullet(\mathcal{U}, \mathcal{F})$ can be described explicitly as follows*

$$\begin{aligned}
\check{\mathcal{C}}^\bullet(\mathcal{U}, \mathcal{F}) &= \left(\prod_{i_0 \in I} \mathrm{Hom}_{PAb(\mathcal{C})}(\mathbf{Z}_{U_{i_0}}, \mathcal{F}) \to \prod_{i_0, i_1 \in I} \mathrm{Hom}_{PAb(\mathcal{C})}(\mathbf{Z}_{U_{i_0} \times_U U_{i_1}}, \mathcal{F}) \to \ldots \right) \\
&= \mathrm{Hom}_{PAb(\mathcal{C})} \left(\left(\bigoplus_{i_0 \in I} \mathbf{Z}_{U_{i_0}} \leftarrow \bigoplus_{i_0, i_1 \in I} \mathbf{Z}_{U_{i_0} \times_U U_{i_1}} \leftarrow \ldots \right), \mathcal{F} \right)
\end{aligned}$$

Proof. This follows from the formula above. See Cohomology on Sites, Lemma 21.10.3. □

This reduces us to studying only the complex in the first argument of the last Hom.

Lemma 44.18.6. *The complex of abelian presheaves*

$$\mathbf{Z}^\bullet_{\mathcal{U}} \quad : \quad \bigoplus_{i_0 \in I} \mathbf{Z}_{U_{i_0}} \leftarrow \bigoplus_{i_0, i_1 \in I} \mathbf{Z}_{U_{i_0} \times_U U_{i_1}} \leftarrow \bigoplus_{i_0, i_1, i_2 \in I} \mathbf{Z}_{U_{i_0} \times_U U_{i_1} \times_U U_{i_2}} \leftarrow \ldots$$

is exact in all degrees except 0 *in* $PAb(\mathcal{C})$.

Proof. For any $V \in \mathrm{Ob}(\mathcal{C})$ the complex of abelian groups $\mathbf{Z}^\bullet_{\mathcal{U}}(V)$ is

$$\mathbf{Z}\left[\coprod_{i_0 \in I} \mathrm{Mor}_{\mathcal{C}}(V, U_{i_0})\right] \leftarrow \mathbf{Z}\left[\coprod_{i_0, i_1 \in I} \mathrm{Mor}_{\mathcal{C}}(V, U_{i_0} \times_U U_{i_1})\right] \leftarrow \ldots =$$

$$\bigoplus_{\varphi : V \to U} \left(\mathbf{Z}\left[\coprod_{i_0 \in I} \mathrm{Mor}_{\varphi}(V, U_{i_0})\right] \leftarrow \mathbf{Z}\left[\coprod_{i_0, i_1 \in I} \mathrm{Mor}_{\varphi}(V, U_{i_0}) \times \mathrm{Mor}_{\varphi}(V, U_{i_1})\right] \leftarrow \ldots \right)$$

where

$$\mathrm{Mor}_{\varphi}(V, U_i) = \{V \to U_i \text{ such that } V \to U_i \to U \text{ equals } \varphi\}.$$

Set $S_\varphi = \coprod_{i \in I} \mathrm{Mor}_{\varphi}(V, U_i)$, so that

$$\mathbf{Z}^\bullet_{\mathcal{U}}(V) = \bigoplus_{\varphi : V \to U} \left(\mathbf{Z}[S_\varphi] \leftarrow \mathbf{Z}[S_\varphi \times S_\varphi] \leftarrow \mathbf{Z}[S_\varphi \times S_\varphi \times S_\varphi] \leftarrow \ldots\right).$$

Thus it suffices to show that for each $S = S_\varphi$, the complex

$$\mathbf{Z}[S] \leftarrow \mathbf{Z}[S \times S] \leftarrow \mathbf{Z}[S \times S \times S] \leftarrow \ldots$$

is exact in negative degrees. To see this, we can give an explicit homotopy. Fix $s \in S$ and define $K : n_{(s_0, \ldots, s_p)} \mapsto n_{(s, s_0, \ldots, s_p)}$. One easily checks that K is a nullhomotopy for the operator

$$\delta : \eta_{(s_0, \ldots, s_p)} \mapsto \sum\nolimits_{i=0}^{p} (-1)^p \eta_{(s_0, \ldots, \hat{s}_i, \ldots, s_p)}.$$

See Cohomology on Sites, Lemma 21.10.4 for more details. □

Lemma 44.18.7. *Let* $\mathcal{C}$ *be a category. If* $\mathcal{I}$ *is an injective object of* $PAb(\mathcal{C})$ *and* $\mathcal{U}$ *is a family of morphisms with fixed target in* $\mathcal{C}$*, then* $\check{H}^p(\mathcal{U}, \mathcal{I}) = 0$ *for all* $p > 0$.

Proof. The Čech complex is the result of applying the functor $\mathrm{Hom}_{PAb(\mathcal{C})}(-, \mathcal{I})$ to the complex $\mathbf{Z}^\bullet_{\mathcal{U}}$, i.e.,

$$\check{H}^p(\mathcal{U}; \mathcal{I}) = H^p(\mathrm{Hom}_{PAb(\mathcal{C})}(\mathbf{Z}^\bullet_{\mathcal{U}}, \mathcal{I})).$$

But we have just seen that $\mathbf{Z}^\bullet_{\mathcal{U}}$ is exact in negative degrees, and the functor $\mathrm{Hom}_{PAb(\mathcal{C})}(-, \mathcal{I})$ is exact, hence $\mathrm{Hom}_{PAb(\mathcal{C})}(\mathbf{Z}^\bullet_{\mathcal{U}}, \mathcal{I})$ is exact in positive degrees. □

Theorem 44.18.8. *On* $PAb(\mathcal{C})$ *the functors* $\check{H}^p(\mathcal{U}, -)$ *are the right derived functors of* $\check{H}^0(\mathcal{U}, -)$.

Proof. By the Lemma 44.18.7, the functors $\check{H}^p(\mathcal{U}, -)$ are universal δ-functors since they are effaceable. So are the right derived functors of $\check{H}^0(\mathcal{U}, -)$. Since they agree in degree 0, they agree by the universal property of universal δ-functors. For more details see Cohomology on Sites, Lemma 21.10.6. □

Remark 44.18.9. Observe that all of the preceding statements are about presheaves so we haven't made use of the topology yet.

44.19. The Cech-to-cohomology spectral sequence

This spectral sequence is fundamental in proving foundational results on cohomology of sheaves.

Lemma 44.19.1. *The forgetful functor $Ab(\mathcal{C}) \to PAb(\mathcal{C})$ transforms injectives into injectives.*

Proof. This is formal using the fact that the forgetful functor has a left adjoint, namely sheafification, which is an exact functor. For more details see Cohomology on Sites, Lemma 21.11.1. □

Theorem 44.19.2. *Let $\mathcal{C}$ be a site. For any covering $\mathcal{U} = \{U_i \to U\}_{i \in I}$ of $U \in \mathrm{Ob}(\mathcal{C})$ and any abelian sheaf $\mathcal{F}$ on $\mathcal{C}$ there is a spectral sequence*

$$E_2^{p,q} = \check{H}^p(\mathcal{U}, \underline{H}^q(\mathcal{F})) \Rightarrow H^{p+q}(U, \mathcal{F}),$$

where $\underline{H}^q(\mathcal{F})$ is the abelian presheaf $V \mapsto H^q(V, \mathcal{F})$.

Proof. Choose an injective resolution $\mathcal{F} \to \mathcal{I}^\bullet$ in $Ab(\mathcal{C})$, and consider the double complex $\check{\mathcal{C}}^\bullet(\mathcal{U}, \mathcal{I}^\bullet)$ and the maps

$$\begin{array}{ccc} \Gamma(U, I^\bullet) & \longrightarrow & \check{\mathcal{C}}^\bullet(\mathcal{U}, \mathcal{I}^\bullet) \\ & & \uparrow \\ & & \check{\mathcal{C}}^\bullet(\mathcal{U}, \mathcal{F}) \end{array}$$

Here the horizontal map is the natural map $\Gamma(U, I^\bullet) \to \check{\mathcal{C}}^0(\mathcal{U}, \mathcal{I}^\bullet)$ to the left column, and the vertical map is induced by $\mathcal{F} \to \mathcal{I}^0$ and lands in the bottom row. By assumption, $\mathcal{I}^\bullet$ is a complex of injectives in $Ab(\mathcal{C})$, hence by Lemma 44.19.1, it is a complex of injectives in $PAb(\mathcal{C})$. Thus, the rows of the double complex are exact in positive degrees, and the kernel of the horizontal map is equal to $\Gamma(U, \mathcal{I}^\bullet)$, since $\mathcal{I}^\bullet$ is a complex of sheaves. In particular, the cohomology of the total complex is the standard cohomology of the global sections functor $H^0(U, \mathcal{F})$.

For the vertical direction, the qth cohomology group of the pth column is

$$\prod_{i_0, \ldots, i_p} H^q(U_{i_0} \times_U \ldots \times_U U_{i_p}, \mathcal{F}) = \prod_{i_0, \ldots, i_p} \underline{H}^q(\mathcal{F})(U_{i_0} \times_U \ldots \times_U U_{i_p})$$

in the entry $E_1^{p,q}$. So this is a standard double complex spectral sequence, and the E_2-page is as prescribed. For more details see Cohomology on Sites, Lemma 21.11.5. □

Remark 44.19.3. This is a Grothendieck spectral sequence for the composition of functors

$$Ab(\mathcal{C}) \longrightarrow PAb(\mathcal{C}) \xrightarrow{\check{H}^0} Ab.$$

44.20. Big and small sites of schemes

Let S be a scheme. Let τ be one of the topologies we will be discussing. Thus $\tau \in \{fppf, syntomic, smooth, étale, Zariski\}$. Of course if you are only interested in the étale topology, then you can simply assume $\tau = étale$ throughout. Moreover, we will discuss étale morphisms, étale coverings, and étale sites in more detail starting in Section 44.25. In order to proceed with the discussion of cohomology

of quasi-coherent sheaves it is convenient to introduce the big τ-site and in case $\tau \in \{\acute{e}tale, Zariski\}$, the small τ-site of S. In order to do this we first introduce the notion of a τ-covering.

Definition 44.20.1. (See Topologies, Definitions 33.7.1, 33.6.1, 33.5.1, 33.4.1, and 33.3.1.) Let $\tau \in \{fppf, syntomic, smooth, \acute{e}tale, Zariski\}$. A family of morphisms of schemes $\{f_i : T_i \to T\}_{i \in I}$ with fixed target is called a *τ-covering* if and only if each f_i is flat of finite presentation, syntomic, smooth, étale, resp. an open immersion, and we have $\bigcup f_i(T_i) = T$.

It turns out that the class of all τ-coverings satisfies the axioms (1), (2) and (3) of Definition 44.10.2 (our definition of a site), see Topologies, Lemmas 33.7.3, 33.6.3, 33.5.3, 33.4.3, and 33.3.2. In order to be able to compare any of these new topologies to the fpqc topology and to use the preceding results on descent on modules we single out a special class of τ-coverings of affine schemes called standard coverings.

Definition 44.20.2. (See Topologies, Definitions 33.7.5, 33.6.5, 33.5.5, 33.4.5, and 33.3.4.) Let $\tau \in \{fppf, syntomic, smooth, \acute{e}tale, Zariski\}$. Let T be an affine scheme. A *standard τ-covering* of T is a family $\{f_j : U_j \to T\}_{j=1,\ldots,m}$ with each U_j is affine, and each f_j flat and of finite presentation, standard syntomic, standard smooth, étale, resp. the immersion of a standard principal open in T and $T = \bigcup f_j(U_j)$.

Lemma 44.20.3. *Let $\tau \in \{fppf, syntomic, smooth, \acute{e}tale, Zariski\}$. Any τ-covering of an affine scheme can be refined by a standard τ-covering.*

Proof. See Topologies, Lemmas 33.7.4, 33.6.4, 33.5.4, 33.4.4, and 33.3.3. □

Finally, we come to our definition of the sites we will be working with. This is actually somewhat involved since, contrary to what happens in [**AGV71**], we do not want to choose a universe. Instead we pick a "partial universe" (which just means a suitably large set), and consider all schemes contained in this set. Of course we make sure that our favorite base scheme S is contained in the partial universe. Having picked the underlying category we pick a suitably large set of τ-coverings which turns this into a site. The details are in the chapter on topologies on schemes; there is a lot of freedom in the choices made, but in the end the actual choices made will not affect the étale (or other) cohomology of S (just as in [**AGV71**] the actual choice of universe doesn't matter at the end). Moreover, the way the material is written the reader who is happy using strongly inaccessible cardinals (i.e., universes) can do so as a substitute.

Definition 44.20.4. Let S be a scheme. Let $\tau \in \{fppf, syntomic, smooth, \acute{e}tale, Zariski\}$.

(1) A *big τ-site of S* is any of the sites $(Sch/S)_\tau$ constructed as explained above and in more detail in Topologies, Definitions 33.7.8, 33.6.8, 33.5.8, 33.4.8, and 33.3.7.
(2) If $\tau \in \{\acute{e}tale, Zariski\}$, then the *small τ-site of S* is the full subcategory S_τ of $(Sch/S)_\tau$ whose objects are schemes T over S whose structure morphism $T \to S$ is étale, resp. an open immersion. A covering in S_τ is a covering $\{U_i \to U\}$ in $(Sch/S)_\tau$ such that U is an object of S_τ.

The underlying category of the site $(Sch/S)_\tau$ has reasonable "closure" properties, i.e., given a scheme T in it any locally closed subscheme of T is isomorphic to an

object of $(Sch/S)_\tau$. Other such closure properties are: closed under fibre products of schemes, taking countable disjoint unions, taking finite type schemes over a given scheme, given an affine scheme $\mathrm{Spec}(R)$ one can complete, localize, or take the quotient of R by an ideal while staying inside the category, etc. On the other hand, for example arbitrary disjoint unions of schemes in $(Sch/S)_\tau$ will take you outside of it. Also note that, given an object T of $(Sch/S)_\tau$ there will exist τ-coverings $\{T_i \to T\}_{i\in I}$ (as in Definition 44.20.1) which are not coverings in $(Sch/S)_\tau$ for example because the schemes T_i are not objects of the category $(Sch/S)_\tau$. But our choice of the sites $(Sch/S)_\tau$ is such that there always does exist a covering $\{U_j \to T\}_{j\in J}$ of $(Sch/S)_\tau$ which refines the covering $\{T_i \to T\}_{i\in I}$, see Topologies, Lemmas 33.7.7, 33.6.7, 33.5.7, 33.4.7, and 33.3.6. We will mostly ignore these issues in this chapter.

If $\mathcal{F}$ is a sheaf on $(Sch/S)_\tau$ or S_τ, then we denote

$$H^p_\tau(U, \mathcal{F}), \text{ in particular } H^p_\tau(S, \mathcal{F})$$

the cohomology groups of $\mathcal{F}$ over the object U of the site, see Section 44.14. Thus we have $H^p_{fppf}(S, \mathcal{F})$, $H^p_{syntomic}(S, \mathcal{F})$, $H^p_{smooth}(S, \mathcal{F})$, $H^p_{étale}(S, \mathcal{F})$, and $H^p_{Zar}(S, \mathcal{F})$. The last two are potentially ambiguous since they might refer to either the big or small étale or Zariski site. However, this ambiguity is harmless by the following lemma.

Lemma 44.20.5. *Let $\tau \in \{étale, Zariski\}$. If $\mathcal{F}$ is an abelian sheaf defined on $(Sch/S)_\tau$, then the cohomology groups of $\mathcal{F}$ over S agree with the cohomology groups of $\mathcal{F}|_{S_\tau}$ over S.*

Proof. By Topologies, Lemmas 33.3.13 and 33.4.13 the functors $S_\tau \to (Sch/S)_\tau$ satisfy the hypotheses of Sites, Lemma 7.20.8. Hence our lemma follows from Cohomology on Sites, Lemma 21.8.2. □

For completeness we state and prove the invariance under choice of partial universe of the cohomology groups we are considering. We will prove invariance of the small étale topos in Lemma 44.21.3 below. For notation and terminology used in this lemma we refer to Topologies, Section 33.10.

Lemma 44.20.6. *Let $\tau \in \{fppf, syntomic, smooth, étale, Zariski\}$. Let S be a scheme. Let $(Sch/S)_\tau$ and $(Sch'/S)_\tau$ be two big τ-sites of S, and assume that the first is contained in the second. In this case*

(1) *for any abelian sheaf $\mathcal{F}'$ defined on $(Sch'/S)_\tau$ and any object U of $(Sch/S)_\tau$ we have*
$$H^p_\tau(U, \mathcal{F}'|_{(Sch/S)_\tau}) = H^p_\tau(U, \mathcal{F}')$$
In words: the cohomology of $\mathcal{F}'$ over U computed in the bigger site agrees with the cohomology of $\mathcal{F}'$ restricted to the smaller site over U.

(2) *for any abelian sheaf $\mathcal{F}$ on $(Sch/S)_\tau$ there is an abelian sheaf $\mathcal{F}'$ on $(Sch/S)'_\tau$ whose restriction to $(Sch/S)_\tau$ is isomorphic to $\mathcal{F}$.*

Proof. By Topologies, Lemma 33.10.2 the inclusion functor $(Sch/S)_\tau \to (Sch'/S)_\tau$ satisfies the assumptions of Sites, Lemma 7.20.8. This implies (2) and (1) follows from Cohomology on Sites, Lemma 21.8.2. □

44.21. The étale topos

A *topos* is the category of sheaves of sets on a site, see Sites, Definition 7.16.1. Hence it is customary to refer to the use the phrase "étale topos of a scheme" to refer to the category of sheaves on the small étale site of a scheme. Here is the formal definition.

Definition 44.21.1. Let S be a scheme.

(1) The *étale topos*, or the *small étale topos* of S is the category $Sh(S_{étale})$ of sheaves of sets on the small étale site of S.
(2) The *Zariski topos*, or the *small Zariski topos* of S is the category $Sh(S_{Zar})$ of sheaves of sets on the small Zariski site of S.
(3) For $\tau \in \{fppf, syntomic, smooth, étale, Zariski\}$ a *big τ-topos* is the category of sheaves of set on a big τ-topos of S.

Note that the small Zariski topos of S is simply the category of sheaves of sets on the underlying topological space of S, see Topologies, Lemma 33.3.11. Whereas the small étale topos does not depend on the choices made in the construction of the small étale site, in general the big topoi do depend on those choices.

Here is a lemma, which is one of many possible lemmas expressing the fact that it doesn't matter too much which site we choose to define the small étale topos of a scheme.

Lemma 44.21.2. *Let S be a scheme. Let $S_{affine,étale}$ denote the full subcategory of $S_{étale}$ whose objects are those $U/S \in \mathrm{Ob}(S_{étale})$ with U affine. A covering of $S_{affine,étale}$ will be a standard étale covering, see Topologies, Definition 33.4.5. Then restriction*

$$\mathcal{F} \longmapsto \mathcal{F}|_{S_{affine,étale}}$$

defines an equivalence of topoi $Sh(S_{étale}) \cong Sh(S_{affine,étale})$.

Proof. This you can show directly from the definitions, and is a good exercise. But it also follows immediately from Sites, Lemma 7.28.1 by checking that the inclusion functor $S_{affine,étale} \to S_{étale}$ is a special cocontinuous functor (see Sites, Definition 7.28.2). □

Lemma 44.21.3. *Let S be a scheme. The étale topos of S is independent (up to canonical equivalence) of the construction of the small étale site in Definition 44.20.4.*

Proof. We have to show, given two big étale sites $Sch_{étale}$ and $Sch'_{étale}$ containing S, then $Sh(S_{étale}) \cong Sh(S'_{étale})$ with obvious notation. By Topologies, Lemma 33.10.1 we may assume $Sch_{étale} \subset Sch'_{étale}$. By Sets, Lemma 3.9.9 any affine scheme étale over S is isomorphic to an object of both $Sch_{étale}$ and $Sch'_{étale}$. Thus the induced functor $S_{affine,étale} \to S'_{affine,étale}$ is an equivalence. Moreover, it is clear that both this functor and a quasi-inverse map transform standard étale coverings into standard étale coverings. Hence the result follows from Lemma 44.21.2. □

44.22. Cohomology of quasi-coherent sheaves

We start with a simple lemma (which holds in greater generality than stated). It says that the Čech complex of a standard covering is equal to the Čech complex of an fpqc covering of the form $\{\mathrm{Spec}(B) \to \mathrm{Spec}(A)\}$ with $A \to B$ faithfully flat.

Lemma 44.22.1. *Let $\tau \in \{fppf, syntomic, smooth, étale, Zariski\}$. Let S be a scheme. Let $\mathcal{F}$ be an abelian sheaf on $(Sch/S)_\tau$, or on S_τ in case $\tau = étale$, and let $\mathcal{U} = \{U_i \to U\}_{i\in I}$ be a standard τ-covering of this site. Let $V = \coprod_{i\in I} U_i$. Then*

(1) *V is an affine scheme,*
(2) *$\mathcal{V} = \{V \to U\}$ is an fpqc covering.*
(3) *the Čech complexes $\check{\mathcal{C}}^\bullet(\mathcal{U}, \mathcal{F})$ and $\check{\mathcal{C}}^\bullet(\mathcal{V}, \mathcal{F})$ agree.*

Proof. As the covering is a standard τ-covering each of the schemes U_i is affine and I is a finite set. Hence V is an affine scheme. It is clear that $V \to U$ is flat and surjective, hence $\mathcal{V}$ is an fpqc covering, see Example 44.15.3. Note that $\mathcal{U}$ is a refinement of $\mathcal{V}$ and hence there is a map of Čech complexes $\check{\mathcal{C}}^\bullet(\mathcal{V}, \mathcal{F}) \to \check{\mathcal{C}}^\bullet(\mathcal{U}, \mathcal{F})$, see Cohomology on Sites, Equation (21.9.2.1). Next, we observe that if $T = \coprod_{j\in J} T_j$ is a disjoint union of schemes in the site on which $\mathcal{F}$ is defined then the family of morphisms with fixed target $\{T_j \to T\}_{j\in J}$ is a Zariski covering, and so

$$\mathcal{F}(T) = \mathcal{F}(\coprod\nolimits_{j\in J} T_j) = \prod\nolimits_{j\in J} \mathcal{F}(T_j) \tag{44.22.1.1}$$

by the sheaf condition of $\mathcal{F}$. This implies the map of Čech complexes above is an isomorphism in each degree because

$$V \times_U \ldots \times_U V = \prod\nolimits_{i_0,\ldots i_p} U_{i_0} \times_U \ldots \times_U U_{i_p}$$

as schemes. □

Note that Equality (44.22.1.1) is false for a general presheaf. Even for sheaves it does not hold on any site, since coproducts may not lead to coverings, and may not be disjoint. But it does for all the usual ones (at least all the ones we will study).

Remark 44.22.2. In the statement of Lemma 44.22.1, $\mathcal{U}$ is a refinement of $\mathcal{V}$, so this does not mean that it suffices to look at coverings with a single morphism to compute Čech cohomology $\check{H}^n(U, \mathcal{F})$ (which is defined as the colimit over all coverings $\mathcal{U}$ of U of the Cech cohomology groups of $\mathcal{F}$ with respect to $\mathcal{U}$).

Lemma 44.22.3 (Locality of cohomology)**.** *Let $\mathcal{C}$ be a site, $\mathcal{F}$ an abelian sheaf on $\mathcal{C}$, U an object of $\mathcal{C}$, $p > 0$ an integer and $\xi \in H^p(U, \mathcal{F})$. Then there exists a covering $\mathcal{U} = \{U_i \to U\}_{i\in I}$ of U in $\mathcal{C}$ such that $\xi|_{U_i} = 0$ for all $i \in I$.*

Proof. Choose an injective resolution $\mathcal{F} \to \mathcal{I}^\bullet$. Then ξ is represented by a cocycle $\tilde{\xi} \in \mathcal{I}^p(U)$ with $d^p(\tilde{\xi}) = 0$. By assumption, the sequence $\mathcal{I}^{p-1} \to \mathcal{I}^p \to \mathcal{I}^{p+1}$ in exact in $Ab(\mathcal{C})$, which means that there exists a covering $\mathcal{U} = \{U_i \to U\}_{i\in I}$ such that $\tilde{\xi}|_{U_i} = d^{p-1}(\xi_i)$ for some $\xi_i \in \mathcal{I}^{p-1}(U_i)$. Since the cohomology class $\xi|_{U_i}$ is represented by the cocycle $\tilde{\xi}|_{U_i}$ which is a coboundary, it vanishes. For more details see Cohomology on Sites, Lemma 21.8.3. □

Theorem 44.22.4. *Let S be a scheme and $\mathcal{F}$ a quasi-coherent $\mathcal{O}_S$-module. Let $\mathcal{C}$ be either $(Sch/S)_\tau$ for $\tau \in \{fppf, syntomic, smooth, étale, Zariski\}$ or $S_{étale}$. Then*

$$H^p(S, \mathcal{F}) = H^p_\tau(S, \mathcal{F}^a)$$

for all $p \geq 0$ where

(1) *the left hand side indicates the usual cohomology of the sheaf $\mathcal{F}$ on the underlying topological space of the scheme S, and*

(2) *the right hand side indicates cohomology of the abelian sheaf $\mathcal{F}^a$ (see Proposition 44.17.1) on the site $\mathcal{C}$.*

Proof. We are going to show that $H^p(U, f^*\mathcal{F}) = H^p_\tau(U, \mathcal{F}^a)$ for any object $f : U \to S$ of the site $\mathcal{C}$. The result is true for $p = 0$ by the sheaf property.

Assume that U is affine. Then we want to prove that $H^p_\tau(U, \mathcal{F}^a) = 0$ for all $p > 0$. We use induction on p.

$p = 1$ Pick $\xi \in H^1_\tau(U, \mathcal{F}^a)$. By Lemma 44.22.3, there exists an fpqc covering $\mathcal{U} = \{U_i \to U\}_{i\in I}$ such that $\xi|_{U_i} = 0$ for all $i \in I$. Up to refining $\mathcal{U}$, we may assume that $\mathcal{U}$ is a standard τ-covering. Applying the spectral sequence of Theorem 44.19.2, we see that ξ comes from a cohomology class $\check{\xi} \in \check{H}^1(\mathcal{U}, \mathcal{F}^a)$. Consider the covering $\mathcal{V} = \{\coprod_{i\in I} U_i \to U\}$. By Lemma 44.22.1, $\check{H}^\bullet(\mathcal{U}, \mathcal{F}^a) = \check{H}^\bullet(\mathcal{V}, \mathcal{F}^a)$. On the other hand, since $\mathcal{V}$ is a covering of the form $\{\mathrm{Spec}(B) \to \mathrm{Spec}(A)\}$ and $f^*\mathcal{F} = \widetilde{M}$ for some A-module M, we see the Čech complex $\check{\mathcal{C}}^\bullet(\mathcal{V}, \mathcal{F})$ is none other than the complex $(B/A)_\bullet \otimes_A M$. Now by Lemma 44.16.4, $H^p((B/A)_\bullet \otimes_A M) = 0$ for $p > 0$, hence $\check{\xi} = 0$ and so $\xi = 0$.

$p > 1$ Pick $\xi \in H^p_\tau(U, \mathcal{F}^a)$. By Lemma 44.22.3, there exists an fpqc covering $\mathcal{U} = \{U_i \to U\}_{i\in I}$ such that $\xi|_{U_i} = 0$ for all $i \in I$. Up to refining $\mathcal{U}$, we may assume that $\mathcal{U}$ is a standard τ-covering. We apply the spectral sequence of Theorem 44.19.2. Observe that the intersections $U_{i_0} \times_U \ldots \times_U U_{i_p}$ are affine, so that by induction hypothesis the cohomology groups

$$E_2^{p,q} = \check{H}^p(\mathcal{U}, \underline{H}^q(\mathcal{F}^a))$$

vanish for all $0 < q < p$. We see that ξ must come from a $\check{\xi} \in \check{H}^p(\mathcal{U}, \mathcal{F}^a)$. Replacing $\mathcal{U}$ with the covering $\mathcal{V}$ containing only one morphism and using Lemma 44.16.4 again, we see that the Čech cohomology class $\check{\xi}$ must be zero, hence $\xi = 0$.

Next, assume that U is separated. Choose an affine open covering $U = \bigcup_{i\in I} U_i$ of U. The family $\mathcal{U} = \{U_i \to U\}_{i\in I}$ is then an fpqc covering, and all the intersections $U_{i_0} \times_S \cdots \times_S U_{i_p}$ are affine since U is separated. So all rows of the spectral sequence of Theorem 44.19.2 are zero, except the zeroth row. Therefore

$$H^p_\tau(S, \mathcal{F}^a) = \check{H}^p(\mathcal{U}, \mathcal{F}^a) = \check{H}^p(\mathcal{U}, \mathcal{F}) = H^p(S, \mathcal{F})$$

where the last equality results from standard scheme theory, see Cohomology of Schemes, Lemma 29.2.5.

The general case is technical and (to extend the proof as given here) requires a discussion about maps of spectral sequences, so we won't treat it. It follows from Descent, Proposition 34.7.10 (whose proof takes a slightly different approach) combined with Cohomology on Sites, Lemma 21.8.1. □

Remark 44.22.5. Comment on Theorem 44.22.4. Since S is a final object in the category $\mathcal{C}$, the cohomology groups on the right-hand side are merely the right derived functors of the global sections functor. In fact the proof shows that $H^p(U, f^*\mathcal{F}) = H^p_\tau(U, \mathcal{F}^a)$ for any object $f : U \to S$ of the site $\mathcal{C}$.

44.23. Examples of sheaves

Let S and τ be as in Section 44.20. We have already seen that any representable presheaf is a sheaf on $(Sch/S)_\tau$ or S_τ, see Lemma 44.15.8 and Remark 44.15.9. Here are some special cases.

Definition 44.23.1. On any of the sites $(Sch/S)_\tau$ or S_τ of Section 44.20.

(1) The sheaf $T \mapsto \Gamma(T, \mathcal{O}_T)$ is denoted $\mathcal{O}_S$, or $\mathbf{G}_a$, or $\mathbf{G}_{a,S}$ if we want to indicate the base scheme.
(2) Similarly, the sheaf $T \mapsto \Gamma(T, \mathcal{O}_T^*)$ is denoted $\mathcal{O}_S^*$, or $\mathbf{G}_m$, or $\mathbf{G}_{m,S}$ if we want to indicate the base scheme.
(3) The *constant sheaf* $\underline{\mathbf{Z}/n\mathbf{Z}}$ on any site is the sheafification of the constant presheaf $U \mapsto \mathbf{Z}/n\mathbf{Z}$.

The first is a sheaf by Theorem 44.17.4 for example. The second is a sub presheaf of the first, which is easily seen to be a sheaf itself. The third is a sheaf by definition. Note that each of these sheaves is representable. The first and second by the schemes $\mathbf{G}_{a,S}$ and $\mathbf{G}_{m,S}$, see Groupoids, Section 38.4. The third by the finite étale group scheme $\mathbf{Z}/n\mathbf{Z}_S$ sometimes denoted $(\mathbf{Z}/n\mathbf{Z})_S$ which is just n copies of S endowed with the obvious group scheme structure over S, see Groupoids, Example 38.5.6 and the following remark.

Remark 44.23.2. Let G be an abstract group. On any of the sites $(Sch/S)_\tau$ or S_τ of Section 44.20 the sheafification $\underline{G}$ of the constant presheaf associated to G in the *Zariski topology* of the site already gives

$$\Gamma(U, \underline{G}) = \{\text{Zariski locally constant maps } U \to G\}$$

This Zariski sheaf is representable by the group scheme G_S according to Groupoids, Example 38.5.6. By Lemma 44.15.8 any representable presheaf satisfies the sheaf condition for the τ-topology as well, and hence we conclude that the Zariski sheafification $\underline{G}$ above is also the τ-sheafification.

Definition 44.23.3. Let S be a scheme. The *structure sheaf* of S is the sheaf of rings $\mathcal{O}_S$ on any of the sites S_{Zar}, $S_{étale}$, or $(Sch/S)_\tau$ discussed above.

If there is some possible confusion as to which site we are working on then we will indicate this by using indices. For example we may use $\mathcal{O}_{S_{étale}}$ to stress the fact that we are working on the small étale site of S.

Remark 44.23.4. In the terminology introduced above a special case of Theorem 44.22.4 is

$$H^p_{fppf}(X, \mathbf{G}_a) = H^p_{étale}(X, \mathbf{G}_a) = H^p_{Zar}(X, \mathbf{G}_a) = H^p(X, \mathcal{O}_X)$$

for all $p \geq 0$. Moreover, we could use the notation $H^p_{fppf}(X, \mathcal{O}_X)$ to indicate the cohomology of the structure sheaf on the big fppf site of X.

44.24. Picard groups

The following theorem is sometimes called "Hilbert 90".

Theorem 44.24.1. *For any scheme X we have canonical identifications*

$$\begin{aligned} H^1_{fppf}(X, \mathbf{G}_m) &= H^1_{syntomic}(X, \mathbf{G}_m) \\ &= H^1_{smooth}(X, \mathbf{G}_m) \\ &= H^1_{étale}(X, \mathbf{G}_m) \\ &= H^1_{Zar}(X, \mathbf{G}_m) \\ &= Pic(X) \\ &= H^1(X, \mathcal{O}_X^*) \end{aligned}$$

Proof. Let τ be one of the topologies considered in Section 44.20. By Cohomology on Sites, Lemma 21.7.1 we see that $H^1_\tau(X, \mathbf{G}_m) = H^1_\tau(X, \mathcal{O}_\tau^*) = \mathrm{Pic}(\mathcal{O}_\tau)$ where $\mathcal{O}_\tau$ is the structure sheaf of the site $(Sch/X)_\tau$. Now an invertible $\mathcal{O}_\tau$-module is a quasi-coherent $\mathcal{O}_\tau$-module. By Theorem 44.17.4 or the more precise Descent, Proposition 34.7.11 we see that $\mathrm{Pic}(\mathcal{O}_\tau) = \mathrm{Pic}(X)$. The last equality is proved in the same way. □

44.25. The étale site

At this point we start exploring the étale site of a scheme in more detail. As a first step we discuss a little the notion of an étale morphism.

44.26. Étale morphisms

For more details, see Morphisms, Section 28.37 for the formal definition and Étale Morphisms, Sections 40.11, 40.12, 40.13, 40.14, 40.16, and 40.19 for a survey of interesting properties of étale morphisms.

Recall that an algebra A over an algebraically closed field k is *smooth* if it is of finite type and the module of differentials $\Omega_{A/k}$ is finite locally free of rank equal to the dimension. A scheme X over k is *smooth* over k if it is locally of finite type and each affine open is the spectrum of a smooth k-algebra. If k is not algebraically closed then an A-algebra is said to be a smooth k-algebra if $A \otimes_k \overline{k}$ is a smooth $\overline{k}$-algebra. A ring map $A \to B$ is smooth if it is flat, finitely presented, and for all primes $\mathfrak{p} \subset A$ the fibre ring $\kappa(\mathfrak{p}) \otimes_A B$ is smooth over the residue field $\kappa(\mathfrak{p})$. More generally, a morphism of schemes is *smooth* if it is flat, locally of finite presentation, and the geometric fibers are smooth.

For these facts please see Morphisms, Section 28.35. Using this we may define an étale morphism as follows.

Definition 44.26.1. A morphism of schemes is *étale* if it is smooth of relative dimension 0.

In particular, a morphism of schemes $X \to S$ is étale if it is smooth and $\Omega_{X/S} = 0$.

Proposition 44.26.2. *Facts on étale morphisms.*

1. *Let k be a field. A morphism of schemes $U \to \mathrm{Spec}(k)$ is étale if and only if $U \cong \coprod_{i \in I} \mathrm{Spec}(k_i)$ such that for each $i \in I$ the ring k_i is a field which is a finite separable extension of k.*
2. *Let $\varphi : U \to S$ be a morphism of schemes. The following conditions are equivalent:*
   (a) *φ is étale,*

(b) *φ is locally finitely presented, flat, and all its fibres are étale,*
(c) *φ is flat, unramified and locally of finite presentation.*

(3) *A ring map $A \to B$ is étale if and only if $B \cong A[x_1, \ldots, x_n]/(f_1, \ldots, f_n)$ such that $\Delta = \det\left(\frac{\partial f_i}{\partial x_j}\right)$ is invertible in B.*
(4) *The base change of an étale morphism is étale.*
(5) *Compositions of étale morphisms are étale.*
(6) *Fibre products and products of étale morphisms are étale.*
(7) *An étale morphism has relative dimension 0.*
(8) *Let $Y \to X$ be an étale morphism. If X is reduced (respectively regular) then so is Y.*
(9) *Étale morphisms are open.*
(10) *If $X \to S$ and $Y \to S$ are étale, then any S-morphism $X \to Y$ is also étale.*

Proof. We have proved these facts (and more) in the preceding chapters. Here is a list of references: (1) Morphisms, Lemma 28.37.7. (2) Morphisms, Lemmas 28.37.8 and 28.37.16. (3) Algebra, Lemma 10.136.2. (4) Morphisms, Lemma 28.37.4. (5) Morphisms, Lemma 28.37.3. (6) Follows formally from (4) and (5). (7) Morphisms, Lemmas 28.37.6 and 28.30.5. (8) See Algebra, Lemmas 10.149.6 and 10.149.5, see also more results of this kind in Étale Morphisms, Section 40.19. (9) See Morphisms, Lemma 28.26.9 and 28.37.12. (10) See Morphisms, Lemma 28.37.18. □

Definition 44.26.3. A ring map $A \to B$ is called *standard étale* if $B \cong (A[t]/(f))_g$ with $f, g \in A[t]$, with f monic, and $\mathrm{d}f/\mathrm{d}t$ invertible in B.

It is true that a standard étale ring map is étale. Namely, suppose that $B = (A[t]/(f))_g$ with $f, g \in A[t]$, with f monic, and $\mathrm{d}f/\mathrm{d}t$ invertible in B. Then $A[t]/(f)$ is a finite free A-module of rank equal to the degree of the monic polynomial f. Hence B, as a localization of this free algebra is finitely presented and flat over A. To finish the proof that B is étale it suffices to show that the fibre rings

$$\kappa(\mathfrak{p}) \otimes_A B \cong \kappa(\mathfrak{p}) \otimes_A (A[t]/(f))_g \cong \kappa(\mathfrak{p})[t, 1/\overline{g}]/(\overline{f})$$

are finite products of finite separable field extensions. Here $\overline{f}, \overline{g} \in \kappa(\mathfrak{p})[t]$ are the images of f and g. Let

$$\overline{f} = \overline{f}_1 \ldots \overline{f}_a \overline{f}_{a+1}^{e_1} \ldots \overline{f}_{a+b}^{e_b}$$

be the factorization of $\overline{f}$ into powers of pairwise distinct irreducible monic factors $\overline{f}_i$ with $e_1, \ldots, e_b > 0$. By assumption $\mathrm{d}\overline{f}/\mathrm{d}t$ is invertible in $\kappa(\mathfrak{p})[t, 1/\overline{g}]$. Hence we see that at least all the $\overline{f}_i$, $i > a$ are invertible. We conclude that

$$\kappa(\mathfrak{p})[t, 1/\overline{g}]/(\overline{f}) \cong \prod\nolimits_{i \in I} \kappa(\mathfrak{p})[t]/(\overline{f}_i)$$

where $I \subset \{1, \ldots, a\}$ is the subset of indices i such that $\overline{f}_i$ does not divide $\overline{g}$. Moreover, the image of $\mathrm{d}\overline{f}/\mathrm{d}t$ in the factor $\kappa(\mathfrak{p})[t]/(\overline{f}_i)$ is clearly equal to a unit times $\mathrm{d}\overline{f}_i/\mathrm{d}t$. Hence we conclude that $\kappa_i = \kappa(\mathfrak{p})[t]/(\overline{f}_i)$ is a finite field extension of $\kappa(\mathfrak{p})$ generated by one element whose minimal polynomial is separable, i.e., the field extension $\kappa(\mathfrak{p}) \subset \kappa_i$ is finite separable as desired.

It turns out that any étale ring map is locally standard étale. To formulate this we introduce the following notation. A ring map $A \to B$ is *étale at a prime* $\mathfrak{q}$ of B if there exists $h \in B$, $h \notin \mathfrak{q}$ such that $A \to B_h$ is étale. Here is the result.

Theorem 44.26.4. *A ring map $A \to B$ is étale at a prime $\mathfrak{q}$ if and only if there exists $g \in B$, $g \notin \mathfrak{q}$ such that B_g is standard étale over A.*

Proof. See Algebra, Proposition 10.136.17. □

44.27. Étale coverings

We recall the definition.

Definition 44.27.1. An *étale covering* of a scheme U is a family of morphisms of schemes $\{\varphi_i : U_i \to U\}_{i \in I}$ such that

(1) each φ_i is an étale morphism ;
(2) the U_i cover U, i.e., $U = \bigcup_{i \in I} \varphi_i(U_i)$.

Lemma 44.27.2. *Any étale covering is an fpqc covering.*

Proof. (See also Topologies, Lemma 33.8.6.) Let $\{\varphi_i : U_i \to U\}_{i \in I}$ be an étale covering. Since an étale morphism is flat, and the elements of the covering should cover its target, the property fp (faithfully flat) is satisfied. To check the property qc (quasi-compact), let $V \subset U$ be an affine open, and write $\varphi_i^{-1} = \bigcup_{j \in J_i} V_{ij}$ for some affine opens $V_{ij} \subset U_i$. Since φ_i is open (as étale morphisms are open), we see that $V = \bigcup_{i \in I} \bigcup_{j \in J_i} \varphi_i(V_{ij})$ is an open covering of U. Further, since V is quasi-compact, this covering has a finite refinement. □

So any statement which is true for fpqc coverings remains true *a fortiori* for étale coverings. For instance, the étale site is subcanonical.

Definition 44.27.3. (For more details see Section 44.20, or Topologies, Section 33.4.) Let S be a scheme. The *big étale site over S* is the site $(Sch/S)_{étale}$, see Definition 44.20.4. The *small étale site over S* is the site $S_{étale}$, see Definition 44.20.4. We define similarly the *big* and *small Zariski sites* on S, denoted $(Sch/S)_{Zar}$ and S_{Zar}.

Loosely speaking the big étale site of S is made up out of schemes over S and coverings the étale coverings. The small étale site of S is made up out of schemes étale over S with coverings the étale coverings. Actually any morphism between objects of $S_{étale}$ is étale, in virtue of Proposition 44.26.2, hence to check that $\{U_i \to U\}_{i \in I}$ in $S_{étale}$ is a covering it suffices to check that $\coprod U_i \to U$ is surjective.

The small étale site has fewer objects than the big étale site, it contains only the "opens" of the étale topology on S. It is a full subcategory of the big étale site, and its topology is induced from the topology on the big site. Hence it is true that the restriction functor from the big étale site to the small one is exact and maps injectives to injectives. This has the following consequence.

Proposition 44.27.4. *Let S be a scheme and $\mathcal{F}$ an abelian sheaf on $(Sch/S)_{étale}$. Then $\mathcal{F}|_{S_{étale}}$ is a sheaf on $S_{étale}$ and*

$$H^p_{étale}(S, \mathcal{F}|_{S_{étale}}) = H^p_{étale}(S, \mathcal{F})$$

for all $p \geq 0$.

Proof. This is a special case of Lemma 44.20.5. □

In accordance with the general notation introduced in Section 44.20 we write $H^p_{étale}(S, \mathcal{F})$ for the above cohomology group.

44.28. Kummer theory

Let $n \in \mathbf{N}$ and consider the functor μ_n defined by

$$\begin{array}{ccc} Sch^{opp} & \longrightarrow & Ab \\ S & \longmapsto & \mu_n(S) = \{t \in \Gamma(S, \mathcal{O}_S^*) \mid t^n = 1\}. \end{array}$$

By Groupoids, Example 38.5.2 this is a representable functor, and the scheme representing it is denoted μ_n also. By Lemma 44.15.8 this functor satisfies the sheaf condition for the fpqc topology (in particular, it is also satisfies the sheaf condition for the étale, Zariski, etc topology).

Lemma 44.28.1. *If $n \in \mathcal{O}_S^*$ then*

$$0 \to \mu_{n,S} \to \mathbf{G}_{m,S} \xrightarrow{(\cdot)^n} \mathbf{G}_{m,S} \to 0$$

is a short exact sequence of sheaves on both the small and big étale site of S.

Proof. By definition the sheaf $\mu_{n,S}$ is the kernel of the map $(\cdot)^n$. Hence it suffices to show that the last map is surjective. Let U be a scheme over S. Let $f \in \mathbf{G}_m(U) = \Gamma(U, \mathcal{O}_U^*)$. We need to show that we can find an étale cover of U over the members of which the restriction of f is an nth power. Set

$$U' = \underline{\mathrm{Spec}}_U(\mathcal{O}_U[T]/(T^n - f)) \xrightarrow{\pi} U.$$

(See Constructions, Section 26.3 or 26.4 for a discussion of the relative spectrum.) Let $\mathrm{Spec}(A) \subset U$ be an affine open, and say $f|_{\mathrm{Spec}(A)}$ corresponds to the unit $a \in A^*$. Then $\pi^{-1}(\mathrm{Spec}(A)) = \mathrm{Spec}(B)$ with $B = A[T]/(T^n - a)$. The ring map $A \to B$ is finite free of rank n, hence it is faithfully flat, and hence we conclude that $\mathrm{Spec}(B) \to \mathrm{Spec}(A)$ is surjective. Since this holds for every affine open in U we conclude that π is surjective. In addition, n and T^{n-1} are invertible in B, so $nT^{n-1} \in B^*$ and the ring map $A \to B$ is standard étale, in particular étale. Since this holds for every affine open of U we conclude that π is étale. Hence $\mathcal{U} = \{\pi : U' \to U\}$ is an étale covering. Moreover, $f|_{U'} = (f')^n$ where f' is the class of T in $\Gamma(U', \mathcal{O}_{U'}^*)$, so $\mathcal{U}$ has the desired property. □

Remark 44.28.2. Lemma 44.28.1 is false when "étale" is replaced with "Zariski". Since the étale topology is coarser than the smooth topology, see Topologies, Lemma 33.5.2 it follows that the sequence is also exact in the smooth topology.

By Theorem 44.24.1 and Lemma 44.28.1 and general properties of cohomology we obtain the long exact cohomology sequence

$$\begin{array}{ccccccc} 0 & \longrightarrow & H^0_{étale}(S, \mu_{n,S}) & \longrightarrow & \Gamma(S, \mathcal{O}_S^*) & \xrightarrow{(\cdot)^n} & \Gamma(S, \mathcal{O}_S^*) \\ & & \searrow & & & & \\ & & H^1_{étale}(S, \mu_{n,S}) & \longrightarrow & \mathrm{Pic}(S) & \xrightarrow{(\cdot)^n} & \mathrm{Pic}(S) \\ & & & & \searrow & & \\ & & & & H^2_{étale}(S, \mu_{n,S}) & \longrightarrow & \cdots \end{array}$$

at least if n is invertible on S. When n is not invertible on S we can apply the following lemma.

Lemma 44.28.3. *For any $n \in \mathbf{N}$ the sequence*

$$0 \to \mu_{n,S} \to \mathbf{G}_{m,S} \xrightarrow{(\cdot)^n} \mathbf{G}_{m,S} \to 0$$

is a short exact sequence of sheaves on the site $(Sch/S)_{fppf}$ and $(Sch/S)_{syntomic}$.

Proof. By definition the sheaf $\mu_{n,S}$ is the kernel of the map $(\cdot)^n$. Hence it suffices to show that the last map is surjective. Since the syntomic topology is stronger than the fppf topology, see Topologies, Lemma 33.7.2, it suffices to prove this for the syntomic topology. Let U be a scheme over S. Let $f \in \mathbf{G}_m(U) = \Gamma(U, \mathcal{O}_U^*)$. We need to show that we can find a syntomic cover of U over the members of which the restriction of f is an nth power. Set

$$U' = \underline{\mathrm{Spec}}_U(\mathcal{O}_U[T]/(T^n - f)) \xrightarrow{\pi} U.$$

(See Constructions, Section 26.3 or 26.4 for a discussion of the relative spectrum.) Let $\mathrm{Spec}(A) \subset U$ be an affine open, and say $f|_{\mathrm{Spec}(A)}$ corresponds to the unit $a \in A^*$. Then $\pi^{-1}(\mathrm{Spec}(A)) = \mathrm{Spec}(B)$ with $B = A[T]/(T^n - a)$. The ring map $A \to B$ is finite free of rank n, hence it is faithfully flat, and hence we conclude that $\mathrm{Spec}(B) \to \mathrm{Spec}(A)$ is surjective. Since this holds for every affine open in U we conclude that π is surjective. In addition, B is a global relative complete intersection over A, so the ring map $A \to B$ is standard syntomic, in particular syntomic. Since this holds for every affine open of U we conclude that π is syntomic. Hence $\mathcal{U} = \{\pi : U' \to U\}$ is a syntomic covering. Moreover, $f|_{U'} = (f')^n$ where f' is the class of T in $\Gamma(U', \mathcal{O}_{U'}^*)$, so $\mathcal{U}$ has the desired property. □

Remark 44.28.4. Lemma 44.28.3 is false for the smooth, étale, or Zariski topology.

By Theorem 44.24.1 and Lemma 44.28.3 and general properties of cohomology we obtain the long exact cohomology sequence

$$\begin{array}{ccccccc}
0 & \longrightarrow & H^0_{fppf}(S, \mu_{n,S}) & \longrightarrow & \Gamma(S, \mathcal{O}_S^*) & \xrightarrow{(\cdot)^n} & \Gamma(S, \mathcal{O}_S^*) \\
& & \searrow & & & & \\
& & H^1_{fppf}(S, \mu_{n,S}) & \longrightarrow & \mathrm{Pic}(S) & \xrightarrow{(\cdot)^n} & \mathrm{Pic}(S) \\
& & & & \searrow & & \\
& & H^2_{fppf}(S, \mu_{n,S}) & \longrightarrow & \cdots & &
\end{array}$$

for any scheme S and any integer n. Of course there is a similar sequence with syntomic cohomology.

Let $n \in \mathbf{N}$ and let S be any scheme. There is another more direct way to describe the first cohomology group with values in μ_n. Consider pairs $(\mathcal{L}, \alpha)$ where $\mathcal{L}$ is an invertible sheaf on S and $\alpha : \mathcal{L}^{\otimes n} \to \mathcal{O}_S$ is a trivialization of the nth tensor power of $\mathcal{L}$. Let $(\mathcal{L}', \alpha')$ be a second such pair. An isomorphism $\varphi : (\mathcal{L}, \alpha) \to (\mathcal{L}', \alpha')$ is an isomorphism $\varphi : \mathcal{L} \to \mathcal{L}'$ of invertible sheaves such that the diagram

$$\begin{array}{ccc}
\mathcal{L}^{\otimes n} & \xrightarrow[\alpha]{} & \mathcal{O}_S \\
{\scriptstyle \varphi^{\otimes n}}\downarrow & & \downarrow{\scriptstyle 1} \\
(\mathcal{L}')^{\otimes n} & \xrightarrow{\alpha'} & \mathcal{O}_S
\end{array}$$

commutes. Thus we have
(44.28.4.1)

$$Isom_S((\mathcal{L},\alpha),(\mathcal{L}',\alpha')) = \begin{cases} \emptyset & \text{if they are not isomorphic} \\ H^0(S,\mu_{n,S})\cdot\varphi & \text{if } \varphi \text{ isomorphism of pairs} \end{cases}$$

Moreover, given two pairs $(\mathcal{L},\alpha)$, $(\mathcal{L}',\alpha')$ the tensor product

$$(\mathcal{L},\alpha)\otimes(\mathcal{L}',\alpha') = (\mathcal{L}\otimes\mathcal{L}',\alpha\otimes\alpha')$$

is another pair. The pair $(\mathcal{O}_S,1)$ is an identity for this tensor product operation, and an inverse is given by

$$(\mathcal{L},\alpha)^{-1} = (\mathcal{L}^{\otimes -1},\alpha^{\otimes -1}).$$

Hence the collection of isomorphism classes of pairs forms an abelian group. Note that

$$(\mathcal{L},\alpha)^{\otimes n} = (\mathcal{L}^{\otimes n},\alpha^{\otimes n}) \xrightarrow{\alpha} (\mathcal{O}_S,1)$$

hence every element of this group has order dividing n. We warn the reader that this group is in general **not** the n-torsion in $\mathrm{Pic}(S)$.

Lemma 44.28.5. *Let S be a scheme. There is a canonical identification*

$$H^1_{étale}(S,\mu_n) = \textit{group of pairs } (\mathcal{L},\alpha) \textit{ up to isomorphism as above}$$

if n is invertible on S. In general we have

$$H^1_{fppf}(S,\mu_n) = \textit{group of pairs } (\mathcal{L},\alpha) \textit{ up to isomorphism as above.}$$

The same result holds with fppf replaced by syntomic.

Proof. We first prove the second isomorphism. Let $(\mathcal{L},\alpha)$ be a pair as above. Choose an affine open covering $S = \bigcup U_i$ such that $\mathcal{L}|_{U_i} \cong \mathcal{O}_{U_i}$. Say $s_i \in \mathcal{L}(U_i)$ is a generator. Then $\alpha(s_i^{\otimes n}) = f_i \in \mathcal{O}_S^*(U_i)$. Writing $U_i = \mathrm{Spec}(A_i)$ we see there exists a global relative complete intersection $A_i \to B_i = A_i[T]/(T^n - f_i)$ such that f_i maps to an nth power in B_i. In other words, setting $V_i = \mathrm{Spec}(B_i)$ we obtain a syntomic covering $\mathcal{V} = \{V_i \to S\}_{i\in I}$ and trivializations $\varphi_i : (\mathcal{L},\alpha)|_{V_i} \to (\mathcal{O}_{V_i},1)$.

We will use this result (the existence of the covering $\mathcal{V}$) to associate to this pair a cohomology class in $H^1_{syntomic}(S,\mu_{n,S})$. We give two (equivalent) constructions.

First construction: using Čech cohomology. Over the double overlaps $V_i \times_S V_j$ we have the isomorphism

$$(\mathcal{O}_{V_i\times_S V_j},1) \xrightarrow{\mathrm{pr}_0^*\varphi_i^{-1}} (\mathcal{L}|_{V_i\times_S V_j},\alpha|_{V_i\times_S V_j}) \xrightarrow{\mathrm{pr}_1^*\varphi_j} (\mathcal{O}_{V_i\times_S V_j},1)$$

of pairs. By (44.28.4.1) this is given by an element $\zeta_{ij} \in \mu_n(V_i\times_S V_j)$. We omit the verification that these ζ_{ij}'s give a 1-cocycle, i.e., give an element $(\zeta_{i_0i_1}) \in \check{C}(\mathcal{V},\mu_n)$ with $d(\zeta_{i_0i_1}) = 0$. Thus its class is an element in $\check{H}^1(\mathcal{V},\mu_n)$ and by Theorem 44.19.2 it maps to a cohomology class in $H^1_{syntomic}(S,\mu_{n,S})$.

Second construction: Using torsors. Consider the presheaf

$$\mu_n(\mathcal{L},\alpha) : U \longmapsto Isom_U((\mathcal{O}_U,1),(\mathcal{L},\alpha)|_U)$$

on $(Sch/S)_{syntomic}$. We may view this as a subpresheaf of $\mathcal{H}om_{\mathcal{O}}(\mathcal{O},\mathcal{L})$ (internal hom sheaf, see Modules on Sites, Section 18.27). Since the conditions defining this subpresheaf are local, we see that it is a sheaf. By (44.28.4.1) this sheaf has a free action of the sheaf $\mu_{n,S}$. Hence the only thing we have to check is that it locally has sections. This is true because of the existence of the trivializing cover $\mathcal{V}$. Hence

$\mu_n(\mathcal{L}, \alpha)$ is a $\mu_{n,S}$-torsor and by Cohomology on Sites, Lemma 21.5.3 we obtain a corresponding element of $H^1_{syntomic}(S, \mu_{n,S})$.

Ok, now we have to still show the following

(1) The two constructions give the same cohomology class.
(2) Isomorphic pairs give rise to the same cohomology class.
(3) The cohomology class of $(\mathcal{L}, \alpha) \otimes (\mathcal{L}', \alpha')$ is the sum of the cohomology classes of $\mathcal{L}, \alpha)$ and $(\mathcal{L}', \alpha')$.
(4) If the cohomology class is trivial, then the pair is trivial.
(5) Any element of $H^1_{syntomic}(S, \mu_{n,S})$ is the cohomology class of a pair.

We omit the proof of (1). Part (2) is clear from the second construction, since isomorphic torsors give the same cohomology classes. Part (3) is clear from the first construction, since the resulting Cech classes add up. Part (4) is clear from the second construction since a torsor is trivial if and only if it has a global section, see Cohomology on Sites, Lemma 21.5.2.

Part (5) can be seen as follows (although a direct proof would be preferable). Suppose $\xi \in H^1_{syntomic}(S, \mu_{n,S})$. Then ξ maps to an element $\overline{\xi} \in H^1_{syntomic}(S, \mathbf{G}_{m,S})$ with $n\overline{\xi} = 0$. By Theorem 44.24.1 we see that $\overline{\xi}$ corresponds to an invertible sheaf $\mathcal{L}$ whose nth tensor power is isomorphic to $\mathcal{O}_S$. Hence there exists a pair $(\mathcal{L}, \alpha')$ whose cohomology class ξ' has the same image $\overline{\xi'}$ in $H^1_{syntomic}(S, \mathbf{G}_{m,S})$. Thus it suffices to show that $\xi - \xi'$ is the class of a pair. By construction, and the long exact cohomology sequence above, we see that $\xi - \xi' = \partial(f)$ for some $f \in H^0(S, \mathcal{O}_S^*)$. Consider the pair $(\mathcal{O}_S, f)$. We omit the verification that the cohomology class of this pair is $\partial(f)$, which finishes the proof of the first identification (with fppf replaced with syntomic).

To see the first, note that if n is invertible on S, then the covering $\mathcal{V}$ constructed in the first part of the proof is actually an étale covering (compare with the proof of Lemma 44.28.1). The rest of the proof is independent of the topology, apart from the very last argument which uses that the Kummer sequence is exact, i.e., uses Lemma 44.28.1. □

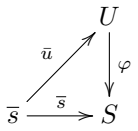
44.29. Neighborhoods, stalks and points

We can associate to any geometric point of S a stalk functor which is exact. A map of sheaves on $S_{étale}$ is an isomorphism if and only if it is an isomorphism on all these stalks. A complex of abelian sheaves is exact if and only if the complex of stalks is exact at all geometric points. Altogether this means that the small étale site of a scheme S has enough points. It also turns out that any point of the small étale topos of S (an abstract notion) is given by a geometric point. Thus in some sense the small étale topos of S can be understood in terms of geometric points and neighbourhoods.

Definition 44.29.1. Let S be a scheme.

(1) A *geometric point* of S is a morphism $\mathrm{Spec}(k) \to S$ where k is algebraically closed. Such a point is usually denoted $\overline{s}$, i.e., by an overlined small case letter. We often use $\overline{s}$ to denote the scheme $\mathrm{Spec}(k)$ as well as the morphism, and we use $\kappa(\overline{s})$ to denote k.
(2) We say $\overline{s}$ *lies over* s to indicate that $s \in S$ is the image of $\overline{s}$.

(3) An *étale neighborhood* of a geometric point $\overline{s}$ of S is a commutative diagram



where φ is an étale morphism of schemes. We write $(U, \overline{u}) \to (S, \overline{s})$.

(4) A *morphism of étale neighborhoods* $(U, \overline{u}) \to (U', \overline{u}')$ is an S-morphism $h : U \to U'$ such that $\overline{u}' = h \circ \overline{u}$.

Remark 44.29.2. Since U and U' are étale over S, any S-morphism between them is also étale, see Proposition 44.26.2. In particular all morphisms of étale neighborhoods are étale.

Remark 44.29.3. Let S be a scheme and $s \in S$ a point. In More on Morphisms, Definition 36.27.1 we defined the notion of an étale neighbourhood $(U, u) \to (S, s)$ of (S, s). If $\overline{s}$ is a geometric point of S lying over s, then any étale neighbourhood $(U, \overline{u}) \to (S, \overline{s})$ gives rise to an étale neighbourhood (U, u) of (S, s) by taking $u \in U$ to be the unique point of U such that $\overline{u}$ lies over u. Conversely, given an étale neighbourhood (U, u) of (S, s) the residue field extension $\kappa(s) \subset \kappa(u)$ is finite separable (see Proposition 44.26.2) and hence we can find an embedding $\kappa(u) \subset \kappa(\overline{s})$ over $\kappa(s)$. In other words, we can find a geometric point $\overline{u}$ of U lying over u such that $(U, \overline{u})$ is an étale neighbourhood of $(S, \overline{s})$. We will use these observations to go between the two types of étale neighbourhoods.

Lemma 44.29.4. *Let S be a scheme, and let $\overline{s}$ be a geometric point of S. The category of étale neighborhoods is cofiltered. More precisely:*

(1) *Let $(U_i, \overline{u}_i)_{i=1,2}$ be two étale neighborhoods of $\overline{s}$ in S. Then there exists a third étale neighborhood $(U, \overline{u})$ and morphisms $(U, \overline{u}) \to (U_i, \overline{u}_i)$, $i = 1, 2$.*

(2) *Let $h_1, h_2 : (U, \overline{u}) \to (U', \overline{u}')$ be two morphisms between étale neighborhoods of $\overline{s}$. Then there exist an étale neighborhood $(U'', \overline{u}'')$ and a morphism $h : (U'', \overline{u}'') \to (U, \overline{u})$ which equalizes h_1 and h_2, i.e., such that $h_1 \circ h = h_2 \circ h$.*

Proof. For part (1), consider the fibre product $U = U_1 \times_S U_2$. It is étale over both U_1 and U_2 because étale morphisms are preserved under base change, see Proposition 44.26.2. The map $\overline{s} \to U$ defined by $(\overline{u}_1, \overline{u}_2)$ gives it the structure of an étale neighborhood mapping to both U_1 and U_2. For part (2), define U'' as the fibre product

$$\begin{array}{ccc} U'' & \longrightarrow & U \\ \downarrow & & \downarrow{\scriptstyle (h_1, h_2)} \\ U' & \xrightarrow{\Delta} & U' \times_S U'. \end{array}$$

Since $\overline{u}$ and $\overline{u}'$ agree over S with $\overline{s}$, we see that $\overline{u}'' = (\overline{u}, \overline{u}')$ is a geometric point of U''. In particular $U'' \neq \emptyset$. Moreover, since U' is étale over S, so is the fibre product $U' \times_S U'$ (see Proposition 44.26.2). Hence the vertical arrow (h_1, h_2) is étale by Remark 44.29.2 above. Therefore U'' is étale over U' by base change, and hence also étale over S (because compositions of étale morphisms are étale). Thus $(U'', \overline{u}'')$ is a solution to the problem. □

Lemma 44.29.5. *Let S be a scheme. Let $\overline{s}$ be a geometric point of S. Let $(U, \overline{u})$ an étale neighborhood of $\overline{s}$. Let $\mathcal{U} = \{\varphi_i : U_i \to U\}_{i \in I}$ be an étale covering. Then there exist $i \in I$ and $\overline{u}_i : \overline{s} \to U_i$ such that $\varphi_i : (U_i, \overline{u}_i) \to (U, \overline{u})$ is a morphism of étale neighborhoods.*

Proof. As $U = \bigcup_{i \in I} \varphi_i(U_i)$, the fibre product $\overline{s} \times_{\overline{u}, U, \varphi_i} U_i$ is not empty for some i. Then look at the cartesian diagram

$$\begin{array}{ccc} \overline{s} \times_{\overline{u}, U, \varphi_i} U_i & \xrightarrow[\mathrm{pr}_2]{} & U_i \\ {\scriptstyle \sigma}\uparrow \big\downarrow{\scriptstyle \mathrm{pr}_1} & & \big\downarrow{\scriptstyle \varphi_i} \\ \mathrm{Spec}(k) = \overline{s} & \xrightarrow{\overline{u}} & U \end{array}$$

The projection pr_1 is the base change of an étale morphisms so it is étale, see Proposition 44.26.2. Therefore, $\overline{s} \times_{\overline{u}, U, \varphi_i} U_i$ is a disjoint union of finite separable extensions of k, by Proposition 44.26.2. Here $\overline{s} = \mathrm{Spec}(k)$. But k is algebraically closed, so all these extensions are trivial, and there exists a section σ of pr_1. The composition $\mathrm{pr}_2 \circ \sigma$ gives a map compatible with $\overline{u}$. □

Definition 44.29.6. Let S be a scheme. Let $\mathcal{F}$ be a presheaf on $S_{étale}$. Let $\overline{s}$ be a geometric point of S. The *stalk* of $\mathcal{F}$ at $\overline{s}$ is

$$\mathcal{F}_{\overline{s}} = \mathrm{colim}_{(U, \overline{u})} \mathcal{F}(U)$$

where $(U, \overline{u})$ runs over all étale neighborhoods of $\overline{s}$ in S.

By Lemma 44.29.4, this colimit is over a filtered index category, namely the opposite of the category of étale neighbourhoods. In other words, an element of $\mathcal{F}_{\overline{s}}$ can be thought of as a triple $(U, \overline{u}, \sigma)$ where $\sigma \in \mathcal{F}(U)$. Two triples $(U, \overline{u}, \sigma)$, $(U', \overline{u}', \sigma')$ define the same element of the stalk if there exists a third étale neighbourhood $(U'', \overline{u}'')$ and morphisms of étale neighbourhoods $h : (U'', \overline{u}'') \to (U, \overline{u})$, $h' : (U'', \overline{u}'') \to (U', \overline{u}')$ such that $h^*\sigma = (h')^*\sigma'$ in $\mathcal{F}(U'')$. See Categories, Section 4.19.

Lemma 44.29.7. *Let S be a scheme. Let $\overline{s}$ be a geometric point of S. Consider the functor*

$$\begin{aligned} u : S_{étale} &\longrightarrow Sets, \\ U &\longmapsto |U_{\overline{s}}| = \{\overline{u} \text{ such that } (U, \overline{u}) \text{ is an étale neighbourhood of } \overline{s}\}. \end{aligned}$$

Here $|U_{\overline{s}}|$ denotes the underlying set of the geometric fibre. Then u defines a point p of the site $S_{étale}$ (Sites, Definition 7.31.2) and its associated stalk functor $\mathcal{F} \mapsto \mathcal{F}_p$ (Sites, Equation 7.31.1.1) is the functor $\mathcal{F} \mapsto \mathcal{F}_{\overline{s}}$ defined above.

Proof. In the proof of Lemma 44.29.5 we have seen that the scheme $U_{\overline{s}}$ is a disjoint union of schemes isomorphic to $\overline{s}$. Thus we can also think of $|U_{\overline{s}}|$ as the set of geometric points of U lying over $\overline{s}$, i.e., as the collection of morphisms $\overline{u} : \overline{s} \to U$ fitting into the diagram of Definition 44.29.1. From this it follows that $u(S)$ is a singleton, and that $u(U \times_V W) = u(U) \times_{u(V)} u(W)$ whenever $U \to V$ and $W \to V$ are morphisms in $S_{étale}$. And, given a covering $\{U_i \to U\}_{i \in I}$ in $S_{étale}$ we see that $\coprod u(U_i) \to u(U)$ is surjective by Lemma 44.29.5. Hence Sites, Proposition 7.32.2 applies, so p is a point of the site $S_{étale}$. Finally, the our functor $\mathcal{F} \mapsto \mathcal{F}_{\overline{s}}$ is given by exactly the same colimit as the functor $\mathcal{F} \mapsto \mathcal{F}_p$ associated to p in Sites, Equation 7.31.1.1 which proves the final assertion. □

Remark 44.29.8. Let S be a scheme and let $\overline{s} : \mathrm{Spec}(k) \to S$ and $\overline{s}' : \mathrm{Spec}(k') \to S$ be two geometric points of S. A *morphism* $a : \overline{s} \to \overline{s}'$ *of geometric points* is simply a morphism $a : \mathrm{Spec}(k) \to \mathrm{Spec}(k')$ such that $a \circ \overline{s}' = \overline{s}$. Given such a morphism we obtain a functor from the category of étale neighbourhoods of $\overline{s}'$ to the category of étale neighbourhoods of $\overline{s}$ by the rule $(U, \overline{u}') \mapsto (U, \overline{u}' \circ a)$. Hence we obtain a canonical map

$$\mathcal{F}_{\overline{s}'} = \mathrm{colim}_{(U,\overline{u}')} \mathcal{F}(U) \longrightarrow \mathrm{colim}_{(U,\overline{u})} \mathcal{F}(U) = \mathcal{F}_{\overline{s}}$$

from Categories, Lemma 4.14.7. Using the description of elements of stalks as triples this maps the element of $\mathcal{F}_{\overline{s}'}$ represented by the triple $(U, \overline{u}', \sigma)$ to the element of $\mathcal{F}_{\overline{s}}$ represented by the triple $(U, \overline{u}' \circ a, \sigma)$. Since the functor above is clearly an equivalence we conclude that this canonical map is an isomorphism of stalk functors.

Let us make sure we have the map of stalks corresponding to a pointing in the correct direction. Note that the above means, according to Sites, Definition 7.36.2, that a defines a morphism $a : p \to p'$ between the points p, p' of the site $S_{étale}$ associated to $\overline{s}, \overline{s}'$ by Lemma 44.29.7. There are more general morphisms of points (corresponding to specializations of points of S) which we will describe later, and which will not be isomorphisms (insert future reference here).

Lemma 44.29.9. *Let S be a scheme. Let $\overline{s}$ be a geometric point of S.*

1. *The stalk functor $PAb(S_{étale}) \to Ab$, $\mathcal{F} \mapsto \mathcal{F}_{\overline{s}}$ is exact.*
2. *We have $(\mathcal{F}^\#)_{\overline{s}} = \mathcal{F}_{\overline{s}}$ for any presheaf of sets $\mathcal{F}$ on $S_{étale}$.*
3. *The functor $Ab(S_{étale}) \to Ab$, $\mathcal{F} \mapsto \mathcal{F}_{\overline{s}}$ is exact.*
4. *Similarly the functors $PSh(S_{étale}) \to Sets$ and $Sh(S_{étale}) \to Sets$ given by the stalk functor $\mathcal{F} \mapsto \mathcal{F}_{\overline{x}}$ are exact (see Categories, Definition 4.23.1) and commute with arbitrary colimits.*

Proof. Before we indicate how to prove this by direct arguments we note that the result follows from the general material in Modules on Sites, Section 18.35. This is true because $\mathcal{F} \mapsto \mathcal{F}_{\overline{s}}$ comes from a point of the small étale site of S, see Lemma 44.29.7. We will only give a direct proof of (1), (2) and (3), and omit a direct proof of (4).

Exactness as a functor on $PAb(S_{étale})$ is formal from the fact that directed colimits commute with all colimits and with finite limits. The identification of the stalks in (2) is via the map

$$\kappa : \mathcal{F}_{\overline{s}} \longrightarrow (\mathcal{F}^\#)_{\overline{s}}$$

induced by the natural morphism $\mathcal{F} \to \mathcal{F}^\#$, see Theorem 44.13.2. We claim that this map is an isomorphism of abelian groups. We will show injectivity and omit the proof of surjectivity.

Let $\sigma \in \mathcal{F}_{\overline{s}}$. There exists an étale neighborhood $(U, \overline{u}) \to (S, \overline{s})$ such that σ is the image of some section $s \in \mathcal{F}(U)$. If $\kappa(\sigma) = 0$ in $(\mathcal{F}^\#)_{\overline{s}}$ then there exists a morphism of étale neighborhoods $(U', \overline{u}') \to (U, \overline{u})$ such that $s|_{U'}$ is zero in $\mathcal{F}^\#(U')$. It follows there exists an étale covering $\{U'_i \to U'\}_{i \in I}$ such that $s|_{U'_i} = 0$ in $\mathcal{F}(U'_i)$ for all i. By Lemma 44.29.5 there exist $i \in I$ and a morphism $\overline{u}'_i : \overline{s} \to U'_i$ such that $(U'_i, \overline{u}'_i) \to (U', \overline{u}') \to (U, \overline{u})$ are morphisms of étale neighborhoods. Hence $\sigma = 0$ since $(U'_i, \overline{u}'_i) \to (U, \overline{u})$ is a morphism of étale neighbourhoods such that we have $s|_{U'_i} = 0$. This proves κ is injective.

To show that the functor $Ab(S_{étale}) \to Ab$ is exact, consider any short exact sequence in $Ab(S_{étale})$: $0 \to \mathcal{F} \to \mathcal{G} \to \mathcal{H} \to 0$. This gives us the exact sequence of presheaves

$$0 \to \mathcal{F} \to \mathcal{G} \to \mathcal{H} \to \mathcal{H}/^p\mathcal{G} \to 0,$$

where $/^p$ denotes the quotient in $PAb(S_{étale})$. Taking stalks at $\overline{s}$, we see that $(\mathcal{H}/^p\mathcal{G})_{\overline{s}} = (\mathcal{H}/\mathcal{G})_{\overline{s}} = 0$, since the sheafification of $\mathcal{H}/^p\mathcal{G}$ is 0. Therefore,

$$0 \to \mathcal{F}_{\overline{s}} \to \mathcal{G}_{\overline{s}} \to \mathcal{H}_{\overline{s}} \to 0 = (\mathcal{H}/^p\mathcal{G})_{\overline{s}}$$

is exact, since taking stalks is exact as a functor from presheaves. □

Theorem 44.29.10. *Let S be a scheme. A map $a : \mathcal{F} \to \mathcal{G}$ of sheaves of sets is injective (resp. surjective) if and only if the map on stalks $a_{\overline{s}} : \mathcal{F}_{\overline{s}} \to \mathcal{G}_{\overline{s}}$ is injective (resp. surjective) for all geometric points of S. A sequence of abelian sheaves on $S_{étale}$ is exact if and only if it is exact on all stalks at geometric points of S.*

Proof. The necessity of exactness on stalks follows from Lemma 44.29.9. For the converse, it suffices to show that a map of sheaves is surjective (respectively injective) if and only if it is surjective (respectively injective) on all stalks. We prove this in the case of surjectivity, and omit the proof in the case of injectivity.

Let $\alpha : \mathcal{F} \to \mathcal{G}$ be a map of abelian sheaves such that $\mathcal{F}_{\overline{s}} \to \mathcal{G}_{\overline{s}}$ is surjective for all geometric points. Fix $U \in \mathrm{Ob}(S_{étale})$ and $s \in \mathcal{G}(U)$. For every $u \in U$ choose some $\overline{u} \to U$ lying over u and an étale neighborhood $(V_u, \overline{v}_u) \to (U, \overline{u})$ such that $s|_{V_u} = \alpha(s_{V_u})$ for some $s_{V_u} \in \mathcal{F}(V_u)$. This is possible since α is surjective on stalks. Then $\{V_u \to U\}_{u \in U}$ is an étale covering on which the restrictions of s are in the image of the map α. Thus, α is surjective, see Sites, Section 7.12. □

Remarks 44.29.11. On points of the geometric sites.

(1) Theorem 44.29.10 says that the family of points of $S_{étale}$ given by the geometric points of S (Lemma 44.29.7) is conservative, see Sites, Definition 7.37.1. In particular $S_{étale}$ has enough points.
(2) Suppose $\mathcal{F}$ is a sheaf on the big étale site of S. Let $T \to S$ be an object of the big étale site of S, and let $\overline{t}$ be a geometric point of T. Then we define $\mathcal{F}_{\overline{t}}$ as the stalk of the restriction $\mathcal{F}|_{T_{étale}}$ of $\mathcal{F}$ to the small étale site of T. In other words, we can define the stalk of $\mathcal{F}$ at any geometric point of any scheme $T/S \in \mathrm{Ob}((Sch/S)_{étale})$.
(3) The big étale site of S also has enough points, by considering all geometric points of all objects of this site, see (2).

The following lemma should be skipped on a first reading.

Lemma 44.29.12. *Let S be a scheme.*

(1) *Let p be a point of the small étale site $S_{étale}$ of S given by a functor $u : S_{étale} \to Sets$. Then there exists a geometric point $\overline{s}$ of S such that p is isomorphic to the point of $S_{étale}$ associated to $\overline{s}$ in Lemma 44.29.7.*
(2) *Let $p : Sh(pt) \to Sh(S_{étale})$ be a point of the small étale topos of S. Then p comes from a geometric point of S, i.e., the stalk functor $\mathcal{F} \mapsto \mathcal{F}_p$ is isomorphic to a stalk functor as defined in Definition 44.29.6.*

Proof. By Sites, Lemma 7.31.7 there is a one to one correspondence between points of the site and points of the associated topos, hence it suffices to prove (1). By Sites, Proposition 7.32.2 the functor u has the following properties: (a) $u(S) = \{*\}$,

(b) $u(U \times_V W) = u(U) \times_{u(V)} u(W)$, and (c) if $\{U_i \to U\}$ is an étale covering, then $\coprod u(U_i) \to u(U)$ is surjective. In particular, if $U' \subset U$ is an open subscheme, then $u(U') \subset u(U)$. Moreover, by Sites, Lemma 7.31.7 we can write $u(U) = p^{-1}(h_U^\#)$, in other words $u(U)$ is the stalk of the representable sheaf h_U. If $U = V \amalg W$, then we see that $h_U = (h_V \amalg h_W)^\#$ and we get $u(U) = u(V) \amalg u(W)$ since p^{-1} is exact.

Consider the restriction of u to S_{Zar}. By Sites, Examples 7.32.4 and 7.32.5 there exists a unique point $s \in S$ such that for $S' \subset S$ open we have $u(S') = \{*\}$ if $s \in S'$ and $u(S') = \emptyset$ if $s \notin S'$. Note that if $\varphi : U \to S$ is an object of $S_{étale}$ then $\varphi(U) \subset S$ is open (see Proposition 44.26.2) and $\{U \to \varphi(U)\}$ is an étale covering. Hence we conclude that $u(U) = \emptyset \Leftrightarrow s \in \varphi(U)$.

Pick a geometric point $\overline{s} : \overline{s} \to S$ lying over s, see Definition 44.29.1 for customary abuse of notation. Suppose that $\varphi : U \to S$ is an object of $S_{étale}$ with U affine. Note that φ is separated, and that the fibre U_s of φ over s is an affine scheme over $\mathrm{Spec}(\kappa(s))$ which is the spectrum of a finite product of finite separable extensions k_i of $\kappa(s)$. Hence we may apply Étale Morphisms, Lemma 40.18.2 to get an étale neighbourhood $(V, \overline{v})$ of $(S, \overline{s})$ such that

$$U \times_S V = U_1 \amalg \ldots \amalg U_n \amalg W$$

with $U_i \to V$ an isomorphism and W having no point lying over $\overline{v}$. Thus we conclude that

$$u(U) \times u(V) = u(U \times_S V) = u(U_1) \amalg \ldots \amalg u(U_n) \amalg u(W)$$

and of course also $u(U_i) = u(V)$. After shrinking V a bit we can assume that V has exactly one point lying over s, and hence W has no point lying over s. By the above this then gives $u(W) = \emptyset$. Hence we obtain

$$u(U) \times u(V) = u(U_1) \amalg \ldots \amalg u(U_n) = \coprod\nolimits_{i=1,\ldots,n} u(V)$$

Note that $u(V) \neq \emptyset$ as s is in the image of $V \to S$. In particular, we see that in this situation $u(U)$ is a finite set with n elements.

Consider the limit

$$\lim\nolimits_{(V,\overline{v})} u(V)$$

over the category of étale neighbourhoods $(V, \overline{v})$ of $\overline{s}$. It is clear that we get the same value when taking the limit over the subcategory of $(V, \overline{v})$ with V affine. By the previous paragraph (applied with the roles of V and U switched) we see that in this case $u(V)$ is always a finite nonempty set. Moreover, the limit is cofiltered, see Lemma 44.29.4. Hence by Categories, Section 4.20 the limit is nonempty. Pick an element x from this limit. This means we obtain a $x_{V,\overline{v}} \in u(V)$ for every étale neighbourhood $(V, \overline{v})$ of $(S, \overline{s})$ such that for every morphism of étale neighbourhoods $\varphi : (V', \overline{v}') \to (V, \overline{v})$ we have $u(\varphi)(x_{V',\overline{v}'}) = x_{V,\overline{v}}$.

We will use the choice of x to construct a functorial bijective map

$$c : |U_{\overline{s}}| \longrightarrow u(U)$$

for $U \in \mathrm{Ob}(S_{étale})$ which will conclude the proof. See Lemma 44.29.7 and its proof for a description of $|U_{\overline{s}}|$. First we claim that it suffices to construct the map for U affine. We omit the proof of this claim. Assume $U \to S$ in $S_{étale}$ with U affine, and let $\overline{u} : \overline{s} \to U$ be an element of $|U_{\overline{s}}|$. Choose a $(V, \overline{v})$ such that $U \times_S V$ decomposes as in the third paragraph of the proof. Then the pair $(\overline{u}, \overline{v})$ gives a geometric

point of $U \times_S V$ lying over $\overline{v}$ and determines one of the components U_i of $U \times_S V$. More precisely, there exists a section $\sigma : V \to U \times_S V$ of the projection pr_U such that $(\overline{u}, \overline{v}) = \sigma \circ \overline{v}$. Set $c(\overline{u}) = u(\mathrm{pr}_U)(u(\sigma)(x_{V,\overline{v}})) \in u(U)$. We have to check this is independent of the choice of $(V, \overline{v})$. By Lemma 44.29.4 the category of étale neighbourhoods is cofiltered. Hence it suffice to show that given a morphism of étale neighbourhood $\varphi : (V', \overline{v}') \to (V, \overline{v})$ and a choice of a section $\sigma' : V' \to U \times_S V'$ of the projection such that $(\overline{u}, \overline{v'}) = \sigma' \circ \overline{v}'$ we have $u(\sigma')(x_{V',\overline{v}'}) = u(\sigma)(x_{V,\overline{v}})$. Consider the diagram

$$\begin{array}{ccc} V' & \xrightarrow{\varphi} & V \\ \downarrow{\scriptstyle \sigma'} & & \downarrow{\scriptstyle \sigma} \\ U \times_S V' & \xrightarrow{1\times\varphi} & U \times_S V \end{array}$$

Now, it may not be the case that this diagram commutes. The reason is that the schemes V' and V may not be connected, and hence the decompositions used to construct σ' and σ above may not be unique. But we do know that $\sigma \circ \varphi \circ \overline{v}' = (1 \times \varphi) \circ \sigma' \circ \overline{v}'$ by construction. Hence, since $U \times_S V$ is étale over S, there exists an open neighbourhood $V'' \subset V'$ of $\overline{v'}$ such that the diagram does commute when restricted to V'', see Morphisms, Lemma 28.36.17. This means we may extend the diagram above to

$$\begin{array}{ccccc} V'' & \longrightarrow & V' & \xrightarrow{\varphi} & V \\ \downarrow{\scriptstyle \sigma'|_{V''}} & & \downarrow{\scriptstyle \sigma'} & & \downarrow{\scriptstyle \sigma} \\ U \times_S V'' & \longrightarrow & U \times_S V' & \xrightarrow{1\times\varphi} & U \times_S V \end{array}$$

such that the left square and the outer rectangle commute. Since u is a functor this implies that $x_{V'',\overline{v}'}$ maps to the same element in $u(U \times_S V)$ no matter which route we take through the diagram. On the other hand, it maps to the elements $x_{V',\overline{v}'}$ and $x_{V,\overline{v}}$ in $u(V')$ and $u(V)$. This implies the desired equality $u(\sigma')(x_{V',\overline{v}'}) = u(\sigma)(x_{V,\overline{v}})$.

In a similar manner one proves that the construction $c : |U_{\overline{s}}| \to u(U)$ is functorial in U; details omitted. And finally, by the results of the third paragraph it is clear that the map c is bijective which ends the proof of the lemma. □

44.30. Points in other topologies

In this section we briefly discuss the existence of points for some sites other than the étale site of a scheme. We refer to Sites, Section 7.37 and Topologies, Section 33.2 ff for the terminology used in this section. All of the geometric sites have enough points.

Lemma 44.30.1. *Let S be a scheme. All of the following sites have enough points S_{Zar}, $S_{étale}$, $(Sch/S)_{Zar}$, $(Aff/S)_{Zar}$, $(Sch/S)_{étale}$, $(Aff/S)_{étale}$, $(Sch/S)_{smooth}$, $(Aff/S)_{smooth}$, $(Sch/S)_{syntomic}$, $(Aff/S)_{syntomic}$, $(Sch/S)_{fppf}$, and $(Aff/S)_{fppf}$.*

Proof. For each of the big sites the associated topos is equivalent to the topos defined by the site $(Aff/S)_\tau$, see Topologies, Lemmas 33.3.10, 33.4.11, 33.5.9, 33.6.9, and 33.7.11. The result for the sites $(Aff/S)_\tau$ follows immediately from Deligne's result Sites, Proposition 7.38.3.

The result for S_{Zar} is clear. The result for $S_{étale}$ either follows from (the proof of) Theorem 44.29.10 or from Lemma 44.21.2 and Deligne's result applied to $S_{affine,étale}$. □

The lemma above guarantees the existence of points, but it doesn't tell us what these points look like. We can explicitly construct *some* points as follows. Suppose $\overline{s} : \mathrm{Spec}(k) \to S$ is a geometric point with k algebraically closed. Consider the functor

$$u : (Sch/S)_{fppf} \longrightarrow Sets, \quad u(U) = U(k) = \mathrm{Mor}_S(\mathrm{Spec}(k), U).$$

Note that $U \mapsto U(k)$ commutes with direct limits as $S(k) = \{\overline{s}\}$ and $(U_1 \times_U U_2)(k) = U_1(k) \times_{U(k)} U_2(k)$. Moreover, if $\{U_i \to U\}$ is an fppf covering, then $\coprod U_i(k) \to U(k)$ is surjective. By Sites, Proposition 7.32.2 we see that u defines a point p of $(Sch/S)_{fppf}$ with stalks

$$\mathcal{F}_p = \mathrm{colim}_{(U,x)} \mathcal{F}(U)$$

where the colimit is over pairs $U \to S$, $x \in U(k)$ as usual. But... this category has an initial object, namely $(\mathrm{Spec}(k), \mathrm{id})$, hence we see that

$$\mathcal{F}_p = \mathcal{F}(\mathrm{Spec}(k))$$

which isn't terribly interesting! In fact, in general these points won't form a conservative family of points. A more interesting type of point is described in the following remark.

Remark 44.30.2. Let $S = \mathrm{Spec}(A)$ be an affine scheme. Let (p, u) be a point of the site $(Aff/S)_{fppf}$. Let $B = \mathcal{O}_p$ be the stalk of the structure sheaf at the point p. Since $A = \Gamma(S, \mathcal{O})$ we see that B is an A-algebra. Ignoring set theoretical difficulties, we see that $\mathrm{Spec}(B)$ is an object of $(Aff/S)_{fppf}$. Recall that

$$B = \mathrm{colim}_{(U,x)} \mathcal{O}(U) = \mathrm{colim}_{(\mathrm{Spec}(C),x)} C$$

where $x \in u(\mathrm{Spec}(C))$. Hence there are canonical maps $C \to B$ and we see that the system has an initial object $(\mathrm{Spec}(B), x_B)$ for a suitable $x_B \in u(\mathrm{Spec}(B))$. It follows that $\mathcal{F}_p = \mathcal{F}(\mathrm{Spec}(B))$ for any sheaf $\mathcal{F}$ on $(Aff/S)_{fppf}$. In other words, every stalk functor is representable. It is straightforward to see that if $\mathcal{F} \mapsto \mathcal{F}(B)$ comes from a point, then B has to be a local A-algebra such that for every faithfully flat, finitely presented ring map $B \to B'$ there is a section $B' \to B$. Conversely, any such A-algebra B gives rise to a point. Details omitted.

44.31. Supports of abelian sheaves

First we talk about supports of local sections.

Lemma 44.31.1. *Let S be a scheme. Let $\mathcal{F}$ be a subsheaf of the final object of the étale topos of S (see Sites, Example 7.10.2). Then there exists a unique open $W \subset S$ such that $\mathcal{F} = h_W$.*

Proof. The condition means that $\mathcal{F}(U)$ is a singleton or empty for all $\varphi : U \to S$ in $\mathrm{Ob}(S_{étale})$. In particular local sections always glue. If $\mathcal{F}(U) \neq \emptyset$, then $\mathcal{F}(\varphi(U)) \neq \emptyset$ because $\{\varphi : U \to \varphi(U)\}$ is a covering. Hence we can take $W = \bigcup_{\varphi : U \to S, \mathcal{F}(U) \neq \emptyset} \varphi(U)$. □

Lemma 44.31.2. *Let S be a scheme. Let $\mathcal{F}$ be an abelian sheaf on $S_{étale}$. Let $\sigma \in \mathcal{F}(U)$ be a local section. There exists an open subset $W \subset U$ such that*

(1) *$W \subset U$ is the largest Zariski open subset of U such that $\sigma|_W = 0$,*
(2) *for every $\varphi : V \to U$ in $S_{étale}$ we have*
$$\sigma|_V = 0 \Leftrightarrow \varphi(V) \subset W,$$
(3) *for every geometric point $\overline{u}$ of U we have*
$$(U, \overline{u}, \sigma) = 0 \text{ in } \mathcal{F}_{\overline{s}} \Leftrightarrow \overline{u} \in W$$
where $\overline{s} = (U \to S) \circ \overline{u}$.

Proof. Since $\mathcal{F}$ is a sheaf in the étale topology the restriction of $\mathcal{F}$ to U_{Zar} is a sheaf on U in the Zariski topology. Hence there exists a Zariski open W having property (1), see Modules, Lemma 17.5.2. Let $\varphi : V \to U$ be an arrow of $S_{étale}$. Note that $\varphi(V) \subset U$ is an open subset and that $\{V \to \varphi(V)\}$ is an étale covering. Hence if $\sigma|_V = 0$, then by the sheaf condition for $\mathcal{F}$ we see that $\sigma|_{\varphi(V)} = 0$. This proves (2). To prove (3) we have to show that if $(U, \overline{u}, \sigma)$ defines the zero element of $\mathcal{F}_{\overline{s}}$, then $\overline{u} \in W$. This is true because the assumption means there exists a morphism of étale neighbourhoods $(V, \overline{v}) \to (U, \overline{u})$ such that $\sigma|_V = 0$. Hence by (2) we see that $V \to U$ maps into W, and hence $\overline{u} \in W$. □

Let S be a scheme. Let $s \in S$. Let $\mathcal{F}$ be a sheaf on $S_{étale}$. By Remark 44.29.8 the isomorphism class of the stalk of the sheaf $\mathcal{F}$ at a geometric points lying over s is well defined.

Definition 44.31.3. Let S be a scheme. Let $\mathcal{F}$ be an abelian sheaf on $S_{étale}$.

(1) The *support of $\mathcal{F}$* is the set of points $s \in S$ such that $\mathcal{F}_{\overline{s}} \neq 0$ for any (some) geometric point $\overline{s}$ lying over s.
(2) Let $\sigma \in \mathcal{F}(U)$ be a section. The *support of σ* is the closed subset $U \setminus W$, where $W \subset U$ is the largest open subset of U on which σ restricts to zero (see Lemma 44.31.2).

In general the support of an abelian sheaf is not closed. For example, suppose that $S = \mathrm{Spec}(\mathbf{A}^1_{\mathbf{C}})$. Let $i_t : \mathrm{Spec}(\mathbf{C}) \to S$ be the inclusion of the point $t \in \mathbf{C}$. We will see later that $\mathbf{F}_t = i_{t,*}(\mathbf{Z}/2\mathbf{Z})$ is an abelian sheaf whose support is exactly $\{t\}$, see Section 44.47. Then
$$\bigoplus\nolimits_{n \in \mathbf{N}} \mathcal{F}_n$$
is an abelian sheaf with support $\{1, 2, 3, \ldots\} \subset S$. This is true because taking stalks commutes with colimits, see Lemma 44.29.9. Thus an example of an abelian sheaf whose support is not closed. Here are some basic facts on supports of sheaves and sections.

Lemma 44.31.4. *Let S be a scheme. Let $\mathcal{F}$ be an abelian sheaf on $S_{étale}$. Let $U \in \mathrm{Ob}(S_{étale})$ and $\sigma \in \mathcal{F}(U)$.*

(1) *The support of σ is closed in U.*
(2) *The support of $\sigma + \sigma'$ is contained in the union of the supports of $\sigma, \sigma' \in \mathcal{F}(U)$.*
(3) *If $\varphi : \mathcal{F} \to \mathcal{G}$ is a map of abelian sheaves on $S_{étale}$, then the support of $\varphi(\sigma)$ is contained in the support of $\sigma \in \mathcal{F}(U)$.*
(4) *The support of $\mathcal{F}$ is the union of the images of the supports of all local sections of $\mathcal{F}$.*
(5) *If $\mathcal{F} \to \mathcal{G}$ is surjective then the support of $\mathcal{G}$ is a subset of the support of $\mathcal{F}$.*

(6) *If $\mathcal{F} \to \mathcal{G}$ is injective then the support of $\mathcal{F}$ is a subset of the support of $\mathcal{G}$.*

Proof. Part (1) holds by definition. Parts (2) and (3) hold because they holds for the restriction of $\mathcal{F}$ and $\mathcal{G}$ to U_{Zar}, see Modules, Lemma 17.5.2. Part (4) is a direct consequence of Lemma 44.31.2 part (3). Parts (5) and (6) follow from the other parts. □

Lemma 44.31.5. *The support of a sheaf of rings on $S_{étale}$ is closed.*

Proof. This is true because (according to our conventions) a ring is 0 if and only if $1 = 0$, and hence the support of a sheaf of rings is the support of the unit section. □

44.32. Henselian rings

We begin by stating a theorem which has already been used many times in the stacks project. There are many versions of this result; here we just state the algebraic version.

Theorem 44.32.1. *Let $A \to B$ be finite type ring map and $\mathfrak{p} \subset A$ a prime ideal. Then there exist an étale ring map $A \to A'$ and a prime $\mathfrak{p}' \subset A'$ lying over $\mathfrak{p}$ such that*

(1) *$\kappa(\mathfrak{p}) = \kappa(\mathfrak{p}')$,*
(2) *$B \otimes_A A' = B_1 \times \cdots \times B_r \times C$,*
(3) *$A' \to B_i$ is finite and there exists a unique prime $q_i \subset B_i$ lying over $\mathfrak{p}'$, and*
(4) *all irreducible components of the fibre $\mathrm{Spec}(C \otimes_{A'} \kappa(\mathfrak{p}'))$ of C over $\mathfrak{p}'$ have dimension at least 1.*

Proof. See Algebra, Lemma 10.136.23, or see [**GD67**, Théorème 18.12.1]. For a slew of versions in terms of morphisms of schemes, see More on Morphisms, Section 36.30. □

Recall Hensel's lemma. There are many versions of this lemma. Here are two:

(f) if $f \in \mathbf{Z}_p[T]$ monic and $f \bmod p = g_0 h_0$ with $gcd(g_0, h_0) = 1$ then f factors as $f = gh$ with $\bar{g} = g_0$ and $\bar{h} = h_0$,
(r) if $f \in \mathbf{Z}_p[T]$, monic $a_0 \in \mathbf{F}_p$, $\bar{f}(a_0) = 0$ but $\bar{f}'(a_0) \neq 0$ then there exists $a \in \mathbf{Z}_p$ with $f(a) = 0$ and $\bar{a} = a_0$.

Both versions are true (we will see this later). The first version asks for lifts of factorizations into coprime parts, and the second version asks for lifts of simple roots modulo the maximal ideal. It turns out that requiring these conditions for a general local ring are equivalent, and are equivalent to many other conditions. We use the root lifting property as the definition of a henselian local ring as it is often the easiest one to check.

Definition 44.32.2. (See Algebra, Definition 10.143.1.) A local ring $(R, \mathfrak{m}, \kappa)$ is called *henselian* if for all $f \in R[T]$ monic, for all $a_0 \in \kappa$ such that $\bar{f}(a_0) = 0$ and $\bar{f}'(a_0) \neq 0$, there exists an $a \in R$ such that $f(a) = 0$ and $a \bmod \mathfrak{m} = a_0$.

A good example of henselian local rings to keep in mind is complete local rings. Recall (Algebra, Definition 10.147.1) that a complete local ring is a local ring $(R, \mathfrak{m})$ such that $R \cong \lim_n R/\mathfrak{m}^n$, i.e., it is complete and separated for the $\mathfrak{m}$-adic topology.

Theorem 44.32.3. *Complete local rings are henselian.*

Proof. Newton's method. See Algebra, Lemma 10.143.10. □

Theorem 44.32.4. *Let $(R, \mathfrak{m}, \kappa)$ be a local ring. The following are equivalent:*

1. *R is henselian,*
2. *for any $f \in R[T]$ and any factorization $\bar{f} = g_0 h_0$ in $\kappa[T]$ with $\gcd(g_0, h_0) = 1$, there exists a factorization $f = gh$ in $R[T]$ with $\bar{g} = g_0$ and $\bar{h} = h_0$,*
3. *any finite R-algebra S is isomorphic to a finite product of finite local rings,*
4. *any finite type R-algebra A is isomorphic to a product $A \cong A' \times C$ where $A' \cong A_1 \times \ldots \times A_r$ is a product of finite local R-algebras and all the irreducible components of $C \otimes_R \kappa$ have dimension at least 1,*
5. *if A is an étale R-algebra and $\mathfrak{n}$ is a maximal ideal of A lying over $\mathfrak{m}$ such that $\kappa \cong A/\mathfrak{n}$, then there exists an isomorphism $\varphi : A \cong R \times A'$ such that $\varphi(\mathfrak{n}) = \mathfrak{m} \times A' \subset R \times A'$.*

Proof. This is just a subset of the results from Algebra, Lemma 10.143.3. Note that part (5) above corresponds to part (8) of Algebra, Lemma 10.143.3 but is formulated slightly differently. □

Lemma 44.32.5. *If R is henselian and A is a finite R-algebra, then A is a finite product of henselian local rings.*

Proof. See Algebra, Lemma 10.143.4. □

Definition 44.32.6. A local ring R is called *strictly henselian* if it is henselian and its residue field is separably closed.

Example 44.32.7. In the case $R = \mathbf{C}[[t]]$, the étale R-algebras are finite products of the trivial extension $R \to R$ and the extensions $R \to R[X, X^{-1}]/(X^n - t)$. The latter ones factor through the open $D(t) \subset \mathrm{Spec}(R)$, so any étale covering can be refined by the covering $\{\mathrm{id} : \mathrm{Spec}(R) \to \mathrm{Spec}(R)\}$. We will see below that this is a somewhat general fact on étale coverings of spectra of henselian rings. This will show that higher étale cohomology of the spectrum of a strictly henselian ring is zero.

Theorem 44.32.8. *Let $(R, \mathfrak{m}, \kappa)$ be a local ring and $\kappa \subset \kappa^{sep}$ a separable algebraic closure. There exist canonical flat local ring maps $R \to R^h \to R^{sh}$ where*

1. *R^h, R^{sh} are filtered colimits of étale R-algebras,*
2. *R^h is henselian, R^{sh} is strictly henselian,*
3. *$\mathfrak{m}R^h$ (resp. $\mathfrak{m}R^{sh}$) is the maximal ideal of R^h (resp. R^{sh}), and*
4. *$\kappa = R^h/\mathfrak{m}R^h$, and $\kappa^{sep} = R^{sh}/\mathfrak{m}R^{sh}$ as extensions of κ.*

Proof. The structure of R^h and R^{sh} is described in Algebra, Lemmas 10.143.16 and 10.143.17. □

The rings constructed in Theorem 44.32.8 are called respectively the *henselization* and the *strict henselization* of the local ring R, see Algebra, Definition 10.143.18. Many of the properties of R are reflected in its (strict) henselization, see More on Algebra, Section 15.32.

44.33. Stalks of the structure sheaf

In this section we identify the stalk of the structure sheaf at a geometric point with the strict henselization of the local ring at the corresponding "usual" point.

Lemma 44.33.1. *Let S be a scheme. Let $\overline{s}$ be a geometric point of S lying over $s \in S$. Let $\kappa = \kappa(s)$ and let $\kappa \subset \kappa^{sep} \subset \kappa(\overline{s})$ denote the separable algebraic closure of κ in $\kappa(\overline{s})$. Then there is a canonical identification*

$$(\mathcal{O}_{S,s})^{sh} \cong \mathcal{O}_{S,\overline{s}}$$

where the left hand side is the strict henselization of the local ring $\mathcal{O}_{S,s}$ as described in Theorem 44.32.8 and right hand side is the stalk of the structure sheaf $\mathcal{O}_S$ on $S_{étale}$ at the geometric point $\overline{s}$.

Proof. Let $\mathrm{Spec}(A) \subset S$ be an affine neighbourhood of s. Let $\mathfrak{p} \subset A$ be the prime ideal corresponding to s. With these choices we have canonical isomorphisms $\mathcal{O}_{S,s} = A_{\mathfrak{p}}$ and $\kappa(s) = \kappa(\mathfrak{p})$. Thus we have $\kappa(\mathfrak{p}) \subset \kappa^{sep} \subset \kappa(\overline{s})$. Recall that

$$\mathcal{O}_{S,\overline{s}} = \mathrm{colim}_{(U,\overline{u})} \mathcal{O}(U)$$

where the limit is over the étale neighbourhoods of $(S, \overline{s})$. A cofinal system is given by those étale neighbourhoods $(U, \overline{u})$ such that U is affine and $U \to S$ factors through $\mathrm{Spec}(A)$. In other words, we see that

$$\mathcal{O}_{S,\overline{s}} = \mathrm{colim}_{(B,\mathfrak{q},\phi)} B$$

where the colimit is over étale A-algebras B endowed with a prime $\mathfrak{q}$ lying over $\mathfrak{p}$ and a $\kappa(\mathfrak{p})$-algebra map $\phi : \kappa(\mathfrak{q}) \to \kappa(\overline{s})$. Note that since $\kappa(\mathfrak{q})$ is finite separable over $\kappa(\mathfrak{p})$ the image of ϕ is contained in κ^{sep}. Via these translations the result of the lemma is equivalent to the result of Algebra, Lemma 10.143.27. □

Definition 44.33.2. Let S be a scheme. Let $\overline{s}$ be a geometric point of S lying over the point $s \in S$.

(1) The *étale local ring of S at $\overline{s}$* is the stalk of the structure sheaf $\mathcal{O}_S$ on $S_{étale}$ at $\overline{s}$. We sometimes call this the *strict henselization of $\mathcal{O}_{S,s}$* relative to the geometric point $\overline{s}$. Notation used: $\mathcal{O}_{S,\overline{s}} = \mathcal{O}^{sh}_{S,s}$.
(2) The *henselization of $\mathcal{O}_{S,s}$* is the henselization of the local ring of S at s. See Algebra, Definition 10.143.18, and Theorem 44.32.8. Notation: $\mathcal{O}^h_{S,s}$.
(3) The *strict henselization of S at $\overline{s}$* is the scheme $\mathrm{Spec}(\mathcal{O}^{sh}_{S,s})$.
(4) The *henselization of S at s* is the scheme $\mathrm{Spec}(\mathcal{O}^h_{S,s})$.

Lemma 44.33.3. *Let S be a scheme. Let $s \in S$. Then we have*

$$\mathcal{O}^h_{S,s} = \mathrm{colim}_{(U,u)} \mathcal{O}(U)$$

where the colimit is over the filtered category of étale neighbourhoods (U, u) of (S, s) such that $\kappa(s) = \kappa(u)$.

Proof. This lemma is a copy of More on Morphisms, Lemma 36.27.5. □

Remark 44.33.4. Let S be a scheme. Let $s \in S$. If S is locally noetherian then $\mathcal{O}^h_{S,s}$ is also noetherian and it has the same completion:

$$\widehat{\mathcal{O}_{S,s}} \cong \widehat{\mathcal{O}^h_{S,s}}.$$

In particular, $\mathcal{O}_{S,s} \subset \mathcal{O}_{S,s}^h \subset \widehat{\mathcal{O}_{S,s}}$. The henselization of $\mathcal{O}_{S,s}$ is in general much smaller than its completion and inherits many of its properties. For example, if $\mathcal{O}_{S,s}$ is reduced, then so is $\mathcal{O}_{S,s}^h$, but this is not true for the completion in general. Insert future references here.

Lemma 44.33.5. *Let S be a scheme. The small étale site $S_{étale}$ endowed with its structure sheaf $\mathcal{O}_S$ is a locally ringed site, see Modules on Sites, Definition 18.39.4.*

Proof. This follows because the stalks $\mathcal{O}_{S,s}^{sh} = \mathcal{O}_{S,\overline{s}}$ are local, and because $S_{étale}$ has enough points, see Lemma 44.33.1, Theorem 44.29.10, and Remarks 44.29.11. See Modules on Sites, Lemmas 18.39.2 and 18.39.3 for the fact that this implies the small étale site is locally ringed. □

44.34. Functoriality of small étale topos

So far we haven't yet discussed the functoriality of the étale site, in other words what happens when given a morphism of schemes. A precise formal discussion can be found in Topologies, Section 33.4. In this and the next sections we discuss this material briefly specifically in the setting of small étale sites.

Let $f : X \to Y$ be a morphism of schemes. We obtain a functor

$$u : Y_{étale} \longrightarrow X_{étale}, \quad V/Y \longmapsto X \times_Y V/X. \tag{44.34.0.1}$$

This functor has the following important properties

(1) u(final object) = final object,
(2) u preserves fibre products,
(3) if $\{V_j \to V\}$ is a covering in $Y_{étale}$, then $\{u(V_j) \to u(V)\}$ is a covering in $X_{étale}$.

Each of these is easy to check (omitted). As a consequence we obtain what is called a *morphism of sites*

$$f_{small} : X_{étale} \longrightarrow Y_{étale},$$

see Sites, Definition 7.15.1 and Sites, Proposition 7.15.6. It is not necessary to know about the abstract notion in detail in order to work with étale sheaves and étale cohomology. It usually suffices to know that there are functors $f_{small,*}$ (pushforward) and f_{small}^{-1} (pullback) on étale sheaves, and to know some of their simple properties. We will discuss these properties in the next sections, but we will sometimes refer to the more abstract material for proofs since that is often the natural setting to prove them.

44.35. Direct images

Let us define the pushforward of a presheaf.

Definition 44.35.1. Let $f : X \to Y$ be a morphism of schemes. Let $\mathcal{F}$ a presheaf of sets on $X_{étale}$. The *direct image*, or *pushforward* of $\mathcal{F}$ (under f) is

$$f_*\mathcal{F} : Y_{étale}^{opp} \longrightarrow Sets, \quad (V/Y) \longmapsto \mathcal{F}(X \times_Y V/X).$$

We sometimes write $f_* = f_{small,*}$ to distinguish from other direct image functors (such as usual Zariski pushforward or $f_{big,*}$).

This is a well-defined étale presheaf since the base change of an étale morphism is again étale. A more categorical way of saying this is that $f_*\mathcal{F}$ is the composition of functors $\mathcal{F} \circ u$ where u is as in Equation (44.34.0.1). This makes it clear that the construction is functorial in the presheaf $\mathcal{F}$ and hence we obtain a functor

$$f_* = f_{small,*} : PSh(X_{étale}) \longrightarrow PSh(Y_{étale})$$

Note that if $\mathcal{F}$ is a presheaf of abelian groups, then $f_*\mathcal{F}$ is also a presheaf of abelian groups and we obtain

$$f_* = f_{small,*} : PAb(X_{étale}) \longrightarrow PAb(Y_{étale})$$

as before (i.e., defined by exactly the same rule).

Remark 44.35.2. We claim that the direct image of a sheaf is a sheaf. Namely, if $\{V_j \to V\}$ is an étale covering in $Y_{étale}$ then $\{X \times_Y V_j \to X \times_Y V\}$ is an étale covering in $X_{étale}$. Hence the sheaf condition for $\mathcal{F}$ with respect to $\{X \times_Y V_i \to X \times_Y V\}$ is equivalent to the sheaf condition for $f_*\mathcal{F}$ with respect to $\{V_i \to V\}$. Thus if $\mathcal{F}$ is a sheaf, so is $f_*\mathcal{F}$.

Definition 44.35.3. Let $f : X \to Y$ be a morphism of schemes. Let $\mathcal{F}$ a sheaf of sets on $X_{étale}$. The *direct image*, or *pushforward* of $\mathcal{F}$ (under f) is

$$f_*\mathcal{F} : Y_{étale}^{opp} \longrightarrow Sets, \quad (V/Y) \longmapsto \mathcal{F}(X \times_Y V/X)$$

which is a sheaf by Remark 44.35.2. We sometimes write $f_* = f_{small,*}$ to distinguish from other direct image functors (such as usual Zariski pushforward or $f_{big,*}$).

The exact same discussion as above applies and we obtain functors

$$f_* = f_{small,*} : Sh(X_{étale}) \longrightarrow Sh(Y_{étale})$$

and

$$f_* = f_{small,*} : Ab(X_{étale}) \longrightarrow Ab(Y_{étale})$$

called *direct image* again.

The functor f_* on abelian sheaves is left exact. (See Homology, Section 12.7 for what it means for a functor between abelian categories to be left exact.) Namely, if $0 \to \mathcal{F}_1 \to \mathcal{F}_2 \to \mathcal{F}_3$ is exact on $X_{étale}$, then for every $U/X \in \mathrm{Ob}(X_{étale})$ the sequence of abelian groups $0 \to \mathcal{F}_1(U) \to \mathcal{F}_2(U) \to \mathcal{F}_3(U)$ is exact. Hence for every $V/Y \in \mathrm{Ob}(Y_{étale})$ the sequence of abelian groups $0 \to f_*\mathcal{F}_1(V) \to f_*\mathcal{F}_2(V) \to f_*\mathcal{F}_3(V)$ is exact, because this is the previous sequence with $U = X \times_Y V$.

Definition 44.35.4. Let $f : X \to Y$ be a morphism of schemes. The right derived functors $\{R^p f_*\}_{p \geq 1}$ of $f_* : Ab(X_{étale}) \to Ab(Y_{étale})$ are called *higher direct images*.

The higher direct images and their derived category variants are discussed in more detail in (insert future reference here).

44.36. Inverse image

In this section we briefly discuss pullback of sheaves on the small étale sites. The precise construction of this is in Topologies, Section 33.4.

Definition 44.36.1. Let $f : X \to Y$ be a morphism of schemes. The *inverse image*, or *pullback*[2] functors are the functors

$$f^{-1} = f^{-1}_{small} : Sh(Y_{étale}) \longrightarrow Sh(X_{étale})$$

and

$$f^{-1} = f^{-1}_{small} : Ab(Y_{étale}) \longrightarrow Ab(X_{étale})$$

which are left adjoint to $f_* = f_{small,*}$. Thus f^{-1} thus characterized by the fact that

$$\mathrm{Hom}_{Sh(X_{étale})}(f^{-1}\mathcal{G}, \mathcal{F}) = \mathrm{Hom}_{Sh(Y_{étale})}(\mathcal{G}, f_*\mathcal{F})$$

functorially, for any $\mathcal{F} \in Sh(X_{étale})$ and $\mathcal{G} \in Sh(Y_{étale})$. We similarly have

$$\mathrm{Hom}_{Ab(X_{étale})}(f^{-1}\mathcal{G}, \mathcal{F}) = \mathrm{Hom}_{Ab(Y_{étale})}(\mathcal{G}, f_*\mathcal{F})$$

for $\mathcal{F} \in Ab(X_{étale})$ and $\mathcal{G} \in Ab(Y_{étale})$.

It is not trivial that such an adjoint exists. On the other hand, it exists in a fairly general setting, see Remark 44.36.3 below. The general machinery shows that $f^{-1}\mathcal{G}$ is the sheaf associated to the presheaf

$$U/X \longmapsto \mathrm{colim}_{U \to X \times_Y V} \mathcal{G}(V/Y) \tag{44.36.1.1}$$

where the colimit is over the category of pairs $(V/Y, \varphi : U/X \to X \times_Y V/X)$. To see this apply Sites, Proposition 7.15.6 to the functor u of Equation (44.34.0.1) and use the description of $u_s = (u_p\ )^\#$ in Sites, Sections 7.14 and 7.5. We will occasionally use this formula for the pullback in order to prove some of its basic properties.

Lemma 44.36.2. *Let $f : X \to Y$ be a morphism of schemes.*

1. *The functor $f^{-1} : Ab(Y_{étale}) \to Ab(X_{étale})$ is exact.*
2. *The functor $f^{-1} : Sh(Y_{étale}) \to Sh(X_{étale})$ is exact, i.e., it commutes with finite limits and colimits, see Categories, Definition 4.23.1.*
3. *Let $\overline{x} \to X$ be a geometric point. Let $\mathcal{G}$ be a sheaf on $Y_{étale}$. Then there is a canonical identification*
$$(f^{-1}\mathcal{G})_{\overline{x}} = \mathcal{G}_{\overline{y}}.$$
where $\overline{y} = f \circ \overline{x}$.
4. *For any $V \to Y$ étale we have $f^{-1}h_V = h_{X \times_Y V}$.*

Proof. The exactness of f^{-1} on sheaves of sets is a consequence of Sites, Proposition 7.15.6 applied to our functor u of Equation (44.34.0.1). In fact the exactness of pullback is part of the definition of of a morphism of topoi (or sites if you like). Thus we see (2) holds. It implies part (1) since given an abelian sheaf $\mathcal{G}$ on $Y_{étale}$ the underlying sheaf of sets of $f^{-1}\mathcal{F}$ is the same as f^{-1} of the underlying sheaf of sets of $\mathcal{F}$, see Sites, Section 7.43. See also Modules on Sites, Lemma 18.30.2. In the literature (1) and (2) are sometimes deduced from (3) via Theorem 44.29.10.

Part (3) is a general fact about stalks of pullbacks, see Sites, Lemma 7.33.1. We will also prove (3) directly as follows. Note that by Lemma 44.29.9 taking stalks commutes with sheafification. Now recall that $f^{-1}\mathcal{G}$ is the sheaf associated to the presheaf

$$U \longrightarrow \mathrm{colim}_{U \to X \times_Y V} \mathcal{G}(V),$$

[2]We use the notation f^{-1} for pullbacks of sheaves of sets or sheaves of abelian groups, and we reserve f^* for pullbacks of sheaves of modules via a morphism of ringed sites/topoi.

see Equation (44.36.1.1). Thus we have

$$\begin{aligned}(f^{-1}\mathcal{G})_{\overline{x}} &= \operatorname{colim}_{(U,\overline{u})} f^{-1}\mathcal{G}(U) \\ &= \operatorname{colim}_{(U,\overline{u})} \operatorname{colim}_{a:U\to X\times_Y V} \mathcal{G}(V) \\ &= \operatorname{colim}_{(V,\overline{v})} \mathcal{G}(V) \\ &= \mathcal{G}_{\overline{y}}\end{aligned}$$

in the third equality the pair $(U, \overline{u})$ and the map $a : U \to X \times_Y V$ corresponds to the pair $(V, a \circ \overline{u})$.

Part (4) can be proved in a similar manner by identifying the colimits which define $f^{-1}h_V$. Or you can use Yoneda's lemma (Categories, Lemma 4.3.5) and the functorial equalities

$$\operatorname{Mor}_{Sh(X_{étale})}(f^{-1}h_V, \mathcal{F}) = \operatorname{Mor}_{Sh(Y_{étale})}(h_V, f_*\mathcal{F}) = f_*\mathcal{F}(V) = \mathcal{F}(X \times_Y V)$$

combined with the fact that representable presheaves are sheaves. See also Sites, Lemma 7.14.5 for a completely general result. □

The pair of functors (f_*, f^{-1}) define a morphism of small étale topoi

$$f_{small} : Sh(X_{étale}) \longrightarrow Sh(Y_{étale})$$

Many generalities on cohomology of sheaves hold for topoi and morphisms of topoi. We will try to point out when results are general and when they are specific to the étale topos.

Remark 44.36.3. More generally, let $\mathcal{C}_1, \mathcal{C}_2$ be sites, and assume they have final objects and fibre products. Let $u : \mathcal{C}_2 \to \mathcal{C}_1$ be a functor satisfying:

(1) if $\{V_i \to V\}$ is a covering of $\mathcal{C}_2$, then $\{u(V_i) \to V_i\}$ is a covering of $\mathcal{C}_1$ (we say that u is *continuous*), and
(2) u commutes with finite limits (i.e., u is left exact, i.e., u preserves fibre products and final objects).

Then one can define $f_* : Sh(\mathcal{C}_1) \to Sh(\mathcal{C}_2)$ by $f_*\mathcal{F}(V) = \mathcal{F}(u(V))$. Moreover, there exists an exact functor f^{-1} which is left adjoint to f_*, see Sites, Definition 7.15.1 and Proposition 7.15.6. Warning: It is not enough to require simply that u is continuous and commutes with fibre products in order to get a morphism of topoi.

44.37. Functoriality of big topoi

Given a morphism of schemes $f : X \to Y$ there are a whole host of morphisms of topoi associated to f, see Topologies, Section 33.9 for a list. Perhaps the most used ones are the morphisms of topoi

$$f_{big} = f_{big,\tau} : Sh((Sch/X)_\tau) \longrightarrow Sh((Sch/Y)_\tau)$$

where $\tau \in \{Zariski, étale, smooth, syntomic, fppf\}$. These each correspond to a continuous functor

$$(Sch/Y)_\tau \longrightarrow (Sch/X)_\tau, \quad V/Y \longmapsto X \times_Y V/X$$

which preserves final objects, fibre products and covering, and hence defines a morphism of sites

$$f_{big} : (Sch/X)_\tau \longrightarrow (Sch/Y)_\tau.$$

See Topologies, Sections 33.3, 33.4, 33.5, 33.6, and 33.7. In particular, pushforward along f_{big} is given by the rule

$$(f_{big,*}\mathcal{F})(V/Y) = \mathcal{F}(X \times_Y V/X)$$

It turns out that these morphisms of topoi have an inverse image functor f_{big}^{-1} which is very easy to describe. Namely, we have

$$(f_{big}^{-1}\mathcal{G})(U/X) = \mathcal{G}(U/Y)$$

where the structure morphism of U/Y is the composition of the structure morphism $U \to X$ with f, see Topologies, Lemmas 33.3.15, 33.4.15, 33.5.10, 33.6.10, and 33.7.12.

44.38. Functoriality and sheaves of modules

In this section we are going to reformulate some of the material explained in Descent, Section 34.7 in the setting of étale topologies. Let $f : X \to Y$ be a morphism of schemes. We have seen above, see Sections 44.34, 44.35, and 44.36 that this induces a morphism f_{small} of small étale sites. In Descent, Remark 34.7.4 we have seen that f also induces a natural map

$$f_{small}^\sharp : \mathcal{O}_{Y_{étale}} \longrightarrow f_{small,*}\mathcal{O}_{X_{étale}}$$

of sheaves of rings on $Y_{étale}$ such that $(f_{small}, f_{small}^\sharp)$ is a morphism of ringed sites. See Modules on Sites, Definition 18.6.1 for the definition of a morphism of ringed sites. Let us just recall here that $f_{small}^\sharp$ is defined by the compatible system of maps

$$\mathrm{pr}_V^\sharp : \mathcal{O}(V) \longrightarrow \mathcal{O}(X \times_Y V)$$

for V varying over the objects of $Y_{étale}$.

It is clear that this construction is compatible with compositions of morphisms of schemes. More precisely, if $f : X \to Y$ and $g : Y \to Z$ are morphisms of schemes, then we have

$$(g_{small}, g_{small}^\sharp) \circ (f_{small}, f_{small}^\sharp) = ((g \circ f)_{small}, (g \circ f)_{small}^\sharp)$$

as morphisms of ringed topoi. Moreover, by Modules on Sites, Definition 18.13.1 we see that given a morphism $f : X \to Y$ of schemes we get well defined pullback and direct image functors

$$f_{small}^* : \mathit{Mod}(\mathcal{O}_{Y_{étale}}) \longrightarrow \mathit{Mod}(\mathcal{O}_{X_{étale}}),$$
$$f_{small,*} : \mathit{Mod}(\mathcal{O}_{X_{étale}}) \longrightarrow \mathit{Mod}(\mathcal{O}_{Y_{étale}})$$

which are adjoint in the usual way. If $g : Y \to Z$ is another morphism of schemes, then we have $(g \circ f)_{small}^* = f_{small}^* \circ g_{small}^*$ and $(g \circ f)_{small,*} = g_{small,*} \circ f_{small,*}$ because of what we said about compositions.

There is quite a bit of difference between the category of all $\mathcal{O}_X$ modules on X and the category between all $\mathcal{O}_{X_{étale}}$-modules on $X_{étale}$. But the results of Descent, Section 34.7 tell us that there is not much difference between considering quasi-coherent modules on S and quasi-coherent modules on $S_{étale}$. (We have already seen this in Theorem 44.17.4 for example.) In particular, if $f : X \to Y$ is any morphism of schemes, then the pullback functors f_{small}^* and f^* match for quasi-coherent sheaves, see Descent, Proposition 34.7.14. Moreover, the same is true for

pushforward provided f is quasi-compact and quasi-separated, see Descent, Lemma 34.7.15.

A few words about functoriality of the structure sheaf on big sites. Let $f : X \to Y$ be a morphism of schemes. Choose any of the topologies $\tau \in \{Zariski, étale, smooth, syntomic, fppf\}$. Then the morphism $f_{big} : (Sch/X)_\tau \to (Sch/Y)_\tau$ becomes a morphism of ringed sites by a map

$$f_{big}^\sharp : \mathcal{O}_Y \longrightarrow f_{big,*}\mathcal{O}_X$$

see Descent, Remark 34.7.4. In fact it is given by the same construction as in the case of small sites explained above.

44.39. Comparing big and small topoi

Let X be a scheme. In Topologies, Lemma 33.4.13 we have introduced comparison morphisms $\pi_X : (Sch/X)_{étale} \to X_{étale}$ and $i_X : Sh(X_{étale}) \to Sh((Sch/X)_{étale})$ with $\pi_X \circ i_X = \text{id}$ and $\pi_{X,*} = i_X^{-1}$. In Descent, Remark 34.7.4 we have extended these to a morphism of ringed sites

$$\pi_X : ((Sch/X)_{étale}, \mathcal{O}) \to (X_{étale}, \mathcal{O}_X)$$

and a morphism of ringed topoi

$$i_X : (Sh(X_{étale}), \mathcal{O}_X) \to (Sh((Sch/X)_{étale}), \mathcal{O})$$

Note that the restriction $i_X^{-1} = \pi_{X,*}$ (see Topologies, Definition 33.4.14) transforms $\mathcal{O}$ into $\mathcal{O}_X$. Hence $i_X^*\mathcal{F} = i_X^{-1}\mathcal{F}$ for any $\mathcal{O}$-module $\mathcal{F}$ on $(Sch/X)_{étale}$. In particular i_X^* is exact. This functor is often denoted $\mathcal{F} \mapsto \mathcal{F}|_{X_{étale}}$.

Lemma 44.39.1. *Let X be a scheme.*

(1) *$\mathcal{I}|_{X_{étale}}$ is injective in $Ab(X_{étale})$ for $\mathcal{I}$ injective in $Ab((Sch/X)_{étale})$, and*
(2) *$\mathcal{I}|_{X_{étale}}$ is injective in $Mod(X_{étale}, \mathcal{O}_X)$ for $\mathcal{I}$ injective in $Mod((Sch/X)_{étale}, \mathcal{O})$.*

Proof. This follows formally from the fact that the restriction functor $\pi_{X,*} = i_X^{-1}$ is an exact left adjoint of $i_{X,*}$, see Homology, Lemma 12.25.1. □

Let $f : X \to Y$ be a morphism of schemes. The commutative diagram of Topologies, Lemma 33.4.16 (3) leads to a commutative diagram of ringed sites

$$\begin{array}{ccc} (T_{étale}, \mathcal{O}_T) & \xleftarrow{\pi_T} & ((Sch/T)_{étale}, \mathcal{O}) \\ \downarrow{\scriptstyle f_{small}} & & \downarrow{\scriptstyle f_{big}} \\ (S_{étale}, \mathcal{O}_S) & \xleftarrow{\pi_S} & ((Sch/S)_{étale}, \mathcal{O}) \end{array}$$

as one easily sees by writing out the definitions of $f_{small}^\sharp$, $f_{big}^\sharp$, $\pi_S^\sharp$, and $\pi_T^\sharp$. In particular this means that

$$(f_{big,*}\mathcal{F})|_{Y_{étale}} = f_{small,*}(\mathcal{F}|_{X_{étale}}) \tag{44.39.1.1}$$

for any sheaf $\mathcal{F}$ on $(Sch/X)_{étale}$ and if $\mathcal{F}$ is a sheaf of $\mathcal{O}$-modules, then (44.39.1.1) is an isomorphism of $\mathcal{O}_Y$-modules on $Y_{étale}$.

Lemma 44.39.2. *Let $f : X \to Y$ be a morphism of schemes.*

(1) *For any* $\mathcal{F} \in Ab((Sch/X)_{étale})$ *we have*
$$(Rf_{big,*}\mathcal{F})|_{Y_{étale}} = Rf_{small,*}(\mathcal{F}|_{X_{étale}}).$$
in $D(Y_{étale})$.

(2) *For any object* $\mathcal{F}$ *of* $Mod((Sch/X)_{étale}, \mathcal{O})$ *we have*
$$(Rf_{big,*}\mathcal{F})|_{Y_{étale}} = Rf_{small,*}(\mathcal{F}|_{X_{étale}}).$$
in $D(Mod(Y_{étale}, \mathcal{O}_Y))$.

Proof. Follows immediately from Lemma 44.39.1 and (44.39.1.1) on choosing an injective resolution of $\mathcal{F}$. □

44.40. Comparing topologies

In this section we start studying what happens when you compare sheaves with respect to different topologies.

Lemma 44.40.1. *Let* S *be a scheme. Let* $\mathcal{F}$ *be a sheaf of sets on* $S_{étale}$. *Let* $s, t \in \mathcal{F}(S)$. *Then there exists an open* $W \subset S$ *characterized by the following property: A morphism* $f : T \to S$ *factors through* W *if and only if* $s|_T = t|_T$ *(restriction is pullback by* f_{small}*).*

Proof. Consider the presheaf which assigns to $U \in \mathrm{Ob}(S_{étale})$ the emptyset if $s|_U \neq t|_U$ and a singleton else. It is clear that this is a subsheaf of the final object of $Sh(S_{étale})$. By Lemma 44.31.1 we find an open $W \subset S$ representing this presheaf. For a geometric point $\overline{x}$ of S we see that $\overline{x} \in W$ if and only if the stalks of s and t at $\overline{x}$ agree. By the description of stalks of pullbacks in Lemma 44.36.2 we see that W has the desired property. □

Lemma 44.40.2. *Let* S *be a scheme. Let* $\tau \in \{Zariski, étale\}$. *Consider the morphism*
$$\pi : (Sch/S)_\tau \longrightarrow S_\tau$$
of Topologies, Lemma 33.3.13 or 33.4.13. Let $\mathcal{F}$ *be a sheaf on* S_τ. *Then* $\pi^{-1}\mathcal{F}$ *is given by the rule*
$$\pi^{-1}\mathcal{F}(T) = \Gamma(T_\tau, f_{small}^{-1}\mathcal{F})$$
where $f : T \to S$. *Moreover,* $\pi^{-1}\mathcal{F}$ *satisfies the sheaf condition with respect to fpqc coverings.*

Proof. Observe that we have a morphism $i_f : Sh(T_\tau) \to Sh(Sch/S)_\tau)$ such that $\pi \circ i_f = f_{small}$ as morphisms $T_\tau \to S_\tau$, see Topologies, Lemmas 33.3.12, 33.3.16, 33.4.12, and 33.4.16. Since pullback is transitive we see that $i_f^{-1}\pi^{-1}\mathcal{F} = f_{small}^{-1}\mathcal{F}$ as desired.

Let $\{g_i : T_i \to T\}_{i \in I}$ be an fpqc covering. The final statement means the following: Given a sheaf $\mathcal{G}$ on T_τ and given sections $s_i \in \Gamma(T_i, g_{i,small}^{-1}\mathcal{G})$ whose pullbacks to $T_i \times_T T_j$ agree, there is a unique section s of $\mathcal{G}$ over T whose pullback to T_i agrees with s_i.

Let $V \to T$ be an object of T_τ and let $t \in \mathcal{G}(V)$. For every i there is a largest open $W_i \subset T_i \times_T V$ such that the pullbacks of s_i and t agree as sections of the pullback of $\mathcal{G}$ to $W_i \subset T_i \times_T V$, see Lemma 44.40.1. Because s_i and s_j agree over $T_i \times_T T_j$ we find that W_i and W_j pullback to the same open over $T_i \times_T T_j \times_T V$. By Descent,

Lemma 34.9.2 we find an open $W \subset V$ whose inverse image to $T_i \times_T V$ recovers W_i.

By construction of $g_{i,small}^{-1}\mathcal{G}$ there exists a τ-covering $\{T_{ij} \to T_i\}_{j \in J_i}$, for each j an open immersion or étale morphism $V_{ij} \to T$, a section $t_{ij} \in \mathcal{G}(V_{ij})$, and commutative diagrams

$$\begin{array}{ccc} T_{ij} & \longrightarrow & V_{ij} \\ \downarrow & & \downarrow \\ T_i & \longrightarrow & T \end{array}$$

such that $s_i|_{T_{ij}}$ is the pullback of t_{ij}. In other words, after replacing the covering $\{T_i \to T\}$ by $\{T_{ij} \to T\}$ we may assume there are factorizations $T_i \to V_i \to T$ with $V_i \in \mathrm{Ob}(T_\tau)$ and sections $t_i \in \mathcal{G}(V_i)$ pulling back to s_i over T_i. By the result of the previous paragraph we find opens $W_i \subset V_i$ such that $t_i|_{W_i}$ "agrees with" every s_j over $T_j \times_T W_i$. Note that $T_i \to V_i$ factors through W_i. Hence $\{W_i \to T\}$ is a τ-covering and the lemma is proven. □

44.41. Recovering morphisms

In this section we prove that the rule which associates to a scheme its locally ringed small étale topos is fully faithful in a suitable sense, see Theorem 44.41.5.

Lemma 44.41.1. *Let $f : X \to Y$ be a morphism of schemes. The morphism of ringed sites $(f_{small}, f_{small}^\sharp)$ associated to f is a morphism of locally ringed sites, see Modules on Sites, Definition 18.39.8.*

Proof. Note that the assertion makes sense since we have seen that $(X_{étale}, \mathcal{O}_{X_{étale}})$ and $(Y_{étale}, \mathcal{O}_{Y_{étale}})$ are locally ringed sites, see Lemma 44.33.5. Moreover, we know that $X_{étale}$ has enough points, see Theorem 44.29.10 and Remarks 44.29.11. Hence it suffices to prove that $(f_{small}, f_{small}^\sharp)$ satisfies condition (3) of Modules on Sites, Lemma 18.39.7. To see this take a point p of $X_{étale}$. By Lemma 44.29.12 p corresponds to a geometric point $\overline{x}$ of X. By Lemma 44.36.2 the point $q = f_{small} \circ p$ corresponds to the geometric point $\overline{y} = f \circ \overline{x}$ of Y. Hence the assertion we have to prove is that the induced map of stalks

$$\mathcal{O}_{Y,\overline{y}} \longrightarrow \mathcal{O}_{X,\overline{x}}$$

is a local ring map. Suppose that $a \in \mathcal{O}_{Y,\overline{y}}$ is an element of the left hand side which maps to an element of the maximal ideal of the right hand side. Suppose that a is the equivalence class of a triple $(V, \overline{v}, a)$ with $V \to Y$ étale, $\overline{v} : \overline{x} \to V$ over Y, and $a \in \mathcal{O}(V)$. It maps to the equivalence class of $(X \times_Y V, \overline{x} \times \overline{v}, \mathrm{pr}_V^\sharp(a))$ in the local ring $\mathcal{O}_{X,\overline{x}}$. But it is clear that being in the maximal ideal means that pulling back $\mathrm{pr}_V^\sharp(a)$ to an element of $\kappa(\overline{x})$ gives zero. Hence also pulling back a to $\kappa(\overline{x})$ is zero. Which means that a lies in the maximal ideal of $\mathcal{O}_{Y,\overline{y}}$. □

Lemma 44.41.2. *Let X, Y be schemes. Let $f : X \to Y$ be a morphism of schemes. Let t be a 2-morphism from $(f_{small}, f_{small}^\sharp)$ to itself, see Modules on Sites, Definition 18.8.1. Then $t = id$.*

Proof. This means that $t : f_{small}^{-1} \to f_{small}^{-1}$ is a transformation of functors such that the diagram

$$\begin{array}{ccc} f_{small}^{-1}\mathcal{O}_Y & \xleftarrow{\quad t \quad} & f_{small}^{-1}\mathcal{O}_Y \\ & {\scriptstyle f_{small}^\sharp} \searrow \quad \swarrow {\scriptstyle f_{small}^\sharp} & \\ & \mathcal{O}_X & \end{array}$$

is commutative. Suppose $V \to Y$ is étale with V affine. By Morphisms, Lemma 28.40.2 we may choose an immersion $i : V \to \mathbf{A}_Y^n$ over Y. In terms of sheaves this means that i induces an injection $h_i : h_V \to \prod_{j=1,\ldots,n} \mathcal{O}_Y$ of sheaves. The base change i' of i to X is an immersion (Schemes, Lemma 25.18.2). Hence $i' : X \times_Y V \to \mathbf{A}_X^n$ is an immersion, which in turn means that $h_{i'} : h_{X\times_Y V} \to \prod_{j=1,\ldots,n} \mathcal{O}_X$ is an injection of sheaves. Via the identification $f_{small}^{-1} h_V = h_{X\times_Y V}$ of Lemma 44.36.2 the map $h_{i'}$ is equal to

$$f_{small}^{-1} h_V \xrightarrow{f^{-1}h_i} \prod_{j=1,\ldots,n} f_{small}^{-1}\mathcal{O}_Y \xrightarrow{\prod f^\sharp} \prod_{j=1,\ldots,n} \mathcal{O}_X$$

(verification omitted). This means that the map $t : f_{small}^{-1} h_V \to f_{small}^{-1} h_V$ fits into the commutative diagram

$$\begin{array}{ccccc} f_{small}^{-1} h_V & \xrightarrow{f^{-1}h_i} & \prod_{j=1,\ldots,n} f_{small}^{-1}\mathcal{O}_Y & \xrightarrow{\prod f^\sharp} & \prod_{j=1,\ldots,n} \mathcal{O}_X \\ \downarrow {\scriptstyle t} & & \downarrow {\scriptstyle \prod t} & & \downarrow {\scriptstyle \mathrm{id}} \\ f_{small}^{-1} h_V & \xrightarrow{f^{-1}h_i} & \prod_{j=1,\ldots,n} f_{small}^{-1}\mathcal{O}_Y & \xrightarrow{\prod f^\sharp} & \prod_{j=1,\ldots,n} \mathcal{O}_X \end{array}$$

The commutativity of the right square holds by our assumption on t explained above. Since the composition of the horizontal arrows is injective by the discussion above we conclude that the left vertical arrow is the identity map as well. Any sheaf of sets on $Y_{étale}$ admits a surjection from a (huge) coproduct of sheaves of the form h_V with V affine (combine Lemma 44.21.2 with Sites, Lemma 7.13.5). Thus we conclude that $t : f_{small}^{-1} \to f_{small}^{-1}$ is the identity transformation as desired. □

Lemma 44.41.3. *Let X, Y be schemes. Any two morphisms $a, b : X \to Y$ of schemes for which there exists a 2-isomorphism $(a_{small}, a_{small}^\sharp) \cong (b_{small}, b_{small}^\sharp)$ in the 2-category of ringed topoi are equal.*

Proof. Let us argue this carefuly since it is a bit confusing. Let $t : a_{small}^{-1} \to b_{small}^{-1}$ be the 2-isomorphism. Consider any open $V \subset Y$. Note that h_V is a subsheaf of the final sheaf $*$. Thus both $a_{small}^{-1} h_V = h_{a^{-1}(V)}$ and $b_{small}^{-1} h_V = h_{b^{-1}(V)}$ are subsheaves of the final sheaf. Thus the isomorphism

$$t : a_{small}^{-1} h_V = h_{a^{-1}(V)} \to b_{small}^{-1} h_V = h_{b^{-1}(V)}$$

has to be the identity, and $a^{-1}(V) = b^{-1}(V)$. It follows that a and b are equal on underlying topological spaces. Next, take a section $f \in \mathcal{O}_Y(V)$. This determines and is determined by a map of sheaves of sets $f : h_V \to \mathcal{O}_Y$. Pull this back and

apply t to get a commutative diagram

$$\begin{array}{ccccccc}
h_{b^{-1}(V)} & = & b_{small}^{-1}h_V & \xleftarrow[t]{} & a_{small}^{-1}h_V & = & h_{a^{-1}(V)} \\
 & & \downarrow{\scriptstyle b_{small}^{-1}(f)} & & \downarrow{\scriptstyle a_{small}^{-1}(f)} & & \\
 & & b_{small}^{-1}\mathcal{O}_Y & \xleftarrow[t]{} & a_{small}^{-1}\mathcal{O}_Y & & \\
 & & & {\scriptstyle b^\sharp}\searrow \quad \swarrow{\scriptstyle a^\sharp} & & & \\
 & & & \mathcal{O}_X & & &
\end{array}$$

where the triangle is commutative by definition of a 2-isomorphism in Modules on Sites, Section 18.8. Above we have seen that the composition of the top horizontal arrows comes from the identity $a^{-1}(V) = b^{-1}(V)$. Thus the commutativity of the diagram tells us that $a^\sharp_{small}(f) = b^\sharp_{small}(f)$ in $\mathcal{O}_X(a^{-1}(V)) = \mathcal{O}_X(b^{-1}(V))$. Since this holds for every open V and every $f \in \mathcal{O}_Y(V)$ we conclude that $a = b$ as morphisms of schemes. □

Lemma 44.41.4. *Let X, Y be affine schemes. Let*

$$(g, g^\#) : (Sh(X_{étale}), \mathcal{O}_X) \longrightarrow (Sh(Y_{étale}), \mathcal{O}_Y)$$

be a morphism of locally ringed topoi. Then there exists a unique morphism of schemes $f : X \to Y$ such that $(g, g^\#)$ is 2-isomorphic to $(f_{small}, f^\sharp_{small})$, see Modules on Sites, Definition 18.8.1.

Proof. In this proof we write $\mathcal{O}_X$ for the structure sheaf of the small étale site $X_{étale}$, and similarly for $\mathcal{O}_Y$. Say $Y = \mathrm{Spec}(B)$ and $X = \mathrm{Spec}(A)$. Since $B = \Gamma(Y_{étale}, \mathcal{O}_Y)$, $A = \Gamma(X_{étale}, \mathcal{O}_X)$ we see that $g^\sharp$ induces a ring map $\varphi : B \to A$. Let $f = \mathrm{Spec}(\varphi) : X \to Y$ be the corresponding morphism of affine schemes. We will show this f does the job.

Let $V \to Y$ be an affine scheme étale over Y. Thus we may write $V = \mathrm{Spec}(C)$ with C an étale B-algebra. We can write

$$C = B[x_1, \ldots, x_n]/(P_1, \ldots, P_n)$$

with P_i polynomials such that $\Delta = \det(\partial P_i/\partial x_j)$ is invertible in C, see for example Algebra, Lemma 10.136.2. If T is a scheme over Y, then a T-valued point of V is given by n sections of $\Gamma(T, \mathcal{O}_T)$ which satisfy the polynomial equations $P_1 = 0, \ldots, P_n = 0$. In other words, the sheaf h_V on $Y_{étale}$ is the equalizer of the two maps

$$\prod_{i=1,\ldots,n} \mathcal{O}_Y \underset{b}{\overset{a}{\rightrightarrows}} \prod_{j=1,\ldots,n} \mathcal{O}_Y$$

where $b(h_1, \ldots, h_n) = 0$ and $a(h_1, \ldots, h_n) = (P_1(h_1, \ldots, h_n), \ldots, P_n(h_1, \ldots, h_n))$. Since g^{-1} is exact we conclude that the top row of the following solid commutative diagram is an equalizer diagram as well:

$$\begin{array}{ccccc}
g^{-1}h_V & \longrightarrow & \prod_{i=1,\ldots,n} g^{-1}\mathcal{O}_Y & \underset{g^{-1}b}{\overset{g^{-1}a}{\rightrightarrows}} & \prod_{j=1,\ldots,n} g^{-1}\mathcal{O}_Y \\
\vdots & & \downarrow{\scriptstyle \prod g^\sharp} & & \downarrow{\scriptstyle \prod g^\sharp} \\
h_{X\times_Y V} & \longrightarrow & \prod_{i=1,\ldots,n} \mathcal{O}_X & \underset{b'}{\overset{a'}{\rightrightarrows}} & \prod_{j=1,\ldots,n} \mathcal{O}_X
\end{array}$$

Here b' is the zero map and a' is the map defined by the images $P_i' = \varphi(P_i) \in A[x_1, \ldots, x_n]$ via the same rule $a'(h_1, \ldots, h_n) = (P_1'(h_1, \ldots, h_n), \ldots, P_n'(h_1, \ldots, h_n))$. that a was defined by. The commutativity of the diagram follows from the fact that $\varphi = g^\sharp$ on global sections. The lower row is an equalizer diagram also, by exactly the same arguments as before since $X \times_Y V$ is the affine scheme $\mathrm{Spec}(A \otimes_B C)$ and $A \otimes_B C = A[x_1, \ldots, x_n]/(P_1', \ldots, P_n')$. Thus we obtain a unique dotted arrow $g^{-1}h_V \to h_{X \times_Y V}$ fitting into the diagram

We claim that the map of sheaves $g^{-1}h_V \to h_{X \times_Y V}$ is an isomorphism. Since the small étale site of X has enough points (Theorem 44.29.10) it suffices to prove this on stalks. Hence let $\overline{x}$ be a geometric point of X, and denote p the associate point of the small étale topos of X. Set $q = g \circ p$. This is a point of the small étale topos of Y. By Lemma 44.29.12 we see that q corresponds to a geometric point $\overline{y}$ of Y. Consider the map of stalks

$$(g^\sharp)_p : \mathcal{O}_{Y,\overline{y}} = \mathcal{O}_{Y,q} = (g^{-1}\mathcal{O}_Y)_p \longrightarrow \mathcal{O}_{X,p} = \mathcal{O}_{X,\overline{x}}$$

Since $(g, g^\sharp)$ is a morphism of *locally* ringed topoi $(g^\sharp)_p$ is a local ring homomorphism of strictly henselian local rings. Applying localization to the big commutative diagram above and Algebra, Lemma 10.143.31 we conclude that $(g^{-1}h_V)_p \to (h_{X \times_Y V})_p$ is an isomorphism as desired.

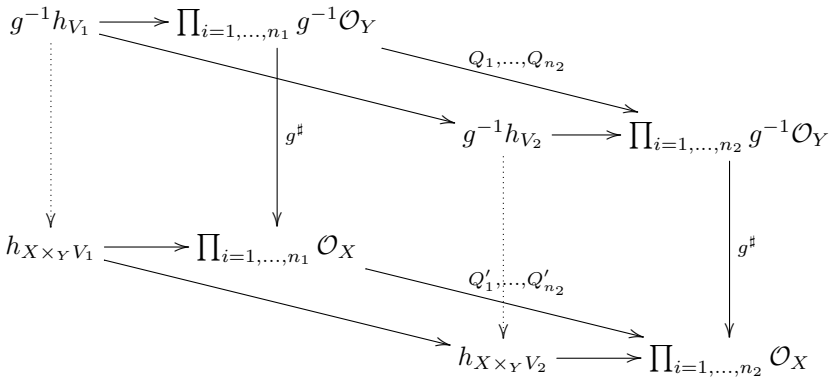
We claim that the isomorphisms $g^{-1}h_V \to h_{X \times_Y V}$ are functorial. Namely, suppose that $V_1 \to V_2$ is a morphism of affine schemes étale over Y. Write $V_i = \mathrm{Spec}(C_i)$ with

$$C_i = B[x_{i,1}, \ldots, x_{i,n_i}]/(P_{i,1}, \ldots, P_{i,n_i})$$

The morphism $V_1 \to V_2$ is given by a B-algebra map $C_2 \to C_1$ which in turn is given by some polynomials $Q_j \in B[x_{1,1}, \ldots, x_{1,n_1}]$ for $j = 1, \ldots, n_2$. Then it is an easy matter to show that the diagram of sheaves

$$\begin{array}{ccc} h_{V_1} & \longrightarrow & \prod_{i=1,\ldots,n_1} \mathcal{O}_Y \\ \downarrow & & \downarrow{\scriptstyle Q_1,\ldots,Q_{n_2}} \\ h_{V_2} & \longrightarrow & \prod_{i=1,\ldots,n_2} \mathcal{O}_Y \end{array}$$

is commutative, and pulling back to $X_{étale}$ we obtain the solid commutative diagram



where $Q_j' \in A[x_{1,1}, \ldots, x_{1,n_1}]$ is the image of Q_j via φ. Since the dotted arrows exist, make the two squares commute, and the horizontal arrows are injective we

see that the whole diagram commutes. This proves functoriality (and also that the construction of $g^{-1}h_V \to h_{X\times_Y V}$ is independent of the choice of the presentation, although we strictly speaking do not need to show this).

At this point we are able to show that $f_{small,*} \cong g_*$. Namely, let $\mathcal{F}$ be a sheaf on $X_{étale}$. For every $V \in \mathrm{Ob}(X_{étale})$ affine we have

$$\begin{aligned}(g_*\mathcal{F})(V) &= \mathrm{Mor}_{Sh(Y_{étale})}(h_V, g_*\mathcal{F}) \\ &= \mathrm{Mor}_{Sh(X_{étale})}(g^{-1}h_V, \mathcal{F}) \\ &= \mathrm{Mor}_{Sh(X_{étale})}(h_{X\times_Y V}, \mathcal{F}) \\ &= \mathcal{F}(X \times_Y V) \\ &= f_{small,*}\mathcal{F}(V)\end{aligned}$$

where in the third equality we use the isomorphism $g^{-1}h_V \cong h_{X\times_Y V}$ constructed above. These isomorphisms are clearly functorial in $\mathcal{F}$ and functorial in V as the isomorphisms $g^{-1}h_V \cong h_{X\times_Y V}$ are functorial. Now any sheaf on $Y_{étale}$ is determined by the restriction to the subcategory of affine schemes (Lemma 44.21.2), and hence we obtain an isomorphism of functors $f_{small,*} \cong g_*$ as desired.

Finally, we have to check that, via the isomorphism $f_{small,*} \cong g_*$ above, the maps $f^\sharp_{small}$ and $g^\sharp$ agree. By construction this is already the case for the global sections of $\mathcal{O}_Y$, i.e., for the elements of B. We only need to check the result on sections over an affine V étale over Y (by Lemma 44.21.2 again). Writing $V = \mathrm{Spec}(C)$, $C = B[x_i]/(P_j)$ as before it suffices to check that the coordinate functions x_i are mapped to the same sections of $\mathcal{O}_X$ over $X \times_Y V$. And this is exactly what it means that the diagram

$$\begin{array}{ccc} g^{-1}h_V & \longrightarrow & \prod_{i=1,\ldots,n} g^{-1}\mathcal{O}_Y \\ \downarrow & & \downarrow \prod g^\sharp \\ h_{X\times_Y V} & \longrightarrow & \prod_{i=1,\ldots,n} \mathcal{O}_X \end{array}$$

commutes. Thus the lemma is proved. □

Here is a version for general schemes.

Theorem 44.41.5. *Let X, Y be schemes. Let*

$$(g, g^\#) : (Sh(X_{étale}), \mathcal{O}_X) \longrightarrow (Sh(Y_{étale}), \mathcal{O}_Y)$$

be a morphism of locally ringed topoi. Then there exists a unique morphism of schemes $f : X \to Y$ such that $(g, g^\#)$ is isomorphic to $(f_{small}, f^\sharp_{small})$. In other words, the construction

$$Sch \longrightarrow \text{Locally ringed topoi}, \quad X \longrightarrow (X_{étale}, \mathcal{O}_X)$$

is fully faithful (morphisms up to 2-isomorphisms on the right hand side).

Proof. You can prove this theorem by carefuly adjusting the arguments of the proof of Lemma 44.41.4 to the global setting. However, we want to indicate how we can glue the result of that lemma to get a global morphism due to the rigidity provided by the result of Lemma 44.41.2. Unfortunately, this is a bit messy.

Let us prove existence when Y is affine. In this case choose an affine open covering $X = \bigcup U_i$. For each i the inclusion morphism $j_i : U_i \to X$ induces a morphism

of locally ringed topoi $(j_{i,small}, j_{i,small}^\sharp) : (Sh(U_{i,étale}), \mathcal{O}_{U_i}) \to (Sh(X_{étale}), \mathcal{O}_X)$ by Lemma 44.41.1. We can compose this with $(g, g^\sharp)$ to obtain a morphism of locally ringed topoi

$$(g, g^\sharp) \circ (j_{i,small}, j_{i,small}^\sharp) : (Sh(U_{i,étale}), \mathcal{O}_{U_i}) \to (Sh(X_{étale}), \mathcal{O}_X)$$

see Modules on Sites, Lemma 18.39.9. By Lemma 44.41.4 there exists a unique morphism of schemes $f_i : U_i \to Y$ and a 2-isomorphism

$$t_i : (f_{i,small}, f_{i,small}^\sharp) \longrightarrow (g, g^\sharp) \circ (j_{i,small}, j_{i,small}^\sharp).$$

Set $U_{i,i'} = U_i \cap U_{i'}$, and denote $j_{i,i'} : U_{i,i'} \to U_i$ the inclusion morphism. Since we have $j_i \circ j_{i,i'} = j_{i'} \circ j_{i',i}$ we see that

$$\begin{aligned}(g, g^\sharp) \circ (j_{i,small}, j_{i,small}^\sharp) \circ (j_{i,i',small}, j_{i,i',small}^\sharp) = \\ (g, g^\sharp) \circ (j_{i',small}, j_{i',small}^\sharp) \circ (j_{i',i,small}, j_{i',i,small}^\sharp)\end{aligned}$$

Hence by uniqueness (see Lemma 44.41.3) we conclude that $f_i \circ j_{i,i'} = f_{i'} \circ j_{i',i}$, in other words the morphisms of schemes $f_i = f \circ j_i$ are the restrictions of a global morphism of schemes $f : X \to Y$. Consider the diagram of 2-isomorphisms (where we drop the components $^\sharp$ to ease the notation)

$$\begin{array}{ccc} g \circ j_{i,small} \circ j_{i,i',small} & \xrightarrow{t_i \star \mathrm{id}_{j_{i,i',small}}} & f_{small} \circ j_{i,small} \circ j_{i,i',small} \\ \| & & \| \\ g \circ j_{i',small} \circ j_{i',i,small} & \xrightarrow{t_{i'} \star \mathrm{id}_{j_{i',i,small}}} & f_{small} \circ j_{i',small} \circ j_{i',i,small} \end{array}$$

The notation $\star$ indicates horizontal composition, see Categories, Definition 4.27.1 in general and Sites, Section 7.35 for our particular case. By the result of Lemma 44.41.2 this diagram commutes. Hence for any sheaf $\mathcal{G}$ on $Y_{étale}$ the isomorphisms $t_i : f_{small}^{-1}\mathcal{G}|_{U_i} \to g^{-1}\mathcal{G}|_{U_i}$ agree over $U_{i,i'}$ and we obtain a global isomorphism $t : f_{small}^{-1}\mathcal{G} \to g^{-1}\mathcal{G}$. It is clear that this isomorphism is functorial in $\mathcal{G}$ and is compatible with the maps $f_{small}^\sharp$ and $g^\sharp$ (because it is compatible with these maps locally). This proves the theorem in case Y is affine.

In the general case, let $V \subset Y$ be an affine open. Then h_V is a subsheaf of the final sheaf $*$ on $Y_{étale}$. As g is exact we see that $g^{-1}h_V$ is a subsheaf of the final sheaf on $X_{étale}$. Hence by Lemma 44.31.1 there exists an open subscheme $W \subset X$ such that $g^{-1}h_V = h_W$. By Modules on Sites, Lemma 18.39.11 there exists a commutative diagram of morphisms of locally ringed topoi

$$\begin{array}{ccc} (Sh(W_{étale}), \mathcal{O}_W) & \longrightarrow & (Sh(X_{étale}), \mathcal{O}_X) \\ \downarrow{\scriptstyle g'} & & \downarrow{\scriptstyle g} \\ (Sh(V_{étale}), \mathcal{O}_V) & \longrightarrow & (Sh(Y_{étale}), \mathcal{O}_Y) \end{array}$$

where the horizontal arrows are the localization morphisms (induced by the inclusion morphisms $V \to Y$ and $W \to X$) and where g' is induced from g. By the result of the preceding paragraph we obtain a morphism of schemes $f' : W \to V$ and a 2-isomorphism $t : (f'_{small}, (f'_{small})^\sharp) \to (g', (g')^\sharp)$. Exactly as before these morphisms f' (for varying affine opens $V \subset Y$) agree on overlaps by uniqueness,

so we get a morphism $f : X \to Y$. Moreover, the 2-isomorphisms t are compatible on overlaps by Lemma 44.41.2 again and we obtain a global 2-isomorphism $(f_{small}, (f_{small})^\sharp) \to (g, (g)^\sharp)$. as desired. Some details omitted. □

44.42. Push and pull

Let $f : X \to Y$ be a morphism of schemes. Here is a list of conditions we will consider in the following:

(A) For every étale morphism $U \to X$ and $u \in U$ there exist an étale morphism $V \to Y$ and a disjoint union decomposition $X \times_Y V = W \amalg W'$ and a morphism $h : W \to U$ over X with u in the image of h.
(B) For every $V \to Y$ étale, and every étale covering $\{U_i \to X \times_Y V\}$ there exists an étale covering $\{V_j \to V\}$ such that for each j we have $X \times_Y V_j = \coprod W_{ji}$ where $W_{ij} \to X \times_Y V$ factors through $U_i \to X \times_Y V$ for some i.
(C) For every $U \to X$ étale, there exists a $V \to Y$ étale and a surjective morphism $X \times_Y V \to U$ over X.

It turns out that each of these properties has meaning in terms of the behaviour of the functor $f_{small,*}$. We will work this out in the next few sections.

44.43. Property (A)

Please see Section 44.42 for the definition of propery (A).

Lemma 44.43.1. *Let $f : X \to Y$ be a morphism of schemes. Assume (A).*

(1) *$f_{small,*} : Ab(X_{étale}) \to Ab(Y_{étale})$ reflects injections and surjections,*
(2) *$f_{small}^{-1} f_{small,*}\mathcal{F} \to \mathcal{F}$ is surjective for any abelian sheaf $\mathcal{F}$ on $X_{étale}$,*
(3) *$f_{small,*} : Ab(X_{étale}) \to Ab(Y_{étale})$ is faithful.*

Proof. Let $\mathcal{F}$ be an abelian sheaf on $X_{étale}$. Let U be an object of $X_{étale}$. By assumption we can find a covering $\{W_i \to U\}$ in $X_{étale}$ such that each W_i is an open and closed subscheme of $X \times_Y V_i$ for some object V_i of $Y_{étale}$. The sheaf condition shows that

$$\mathcal{F}(U) \subset \prod \mathcal{F}(W_i)$$

and that $\mathcal{F}(W_i)$ is a direct summand of $\mathcal{F}(X \times_Y V_i) = f_{small,*}\mathcal{F}(V_i)$. Hence it is clear that $f_{small,*}$ reflects injections.

Next, suppose that $a : \mathcal{G} \to \mathcal{F}$ is a map of abelian sheaves such that $f_{small,*}a$ is surjective. Let $s \in \mathcal{F}(U)$ with U as above. With W_i, V_i as above we see that it suffices to show that $s|_{W_i}$ is étale locally the image of a section of $\mathcal{G}$ under a. Since $\mathcal{F}(W_i)$ is a direct summand of $\mathcal{F}(X \times_Y V_i)$ it suffices to show that for any $V \in \mathrm{Ob}(Y_{étale})$ any element $s \in \mathcal{F}(X \times_Y V)$ is étale locally on $X \times_Y V$ the image of a section of $\mathcal{G}$ under a. Since $\mathcal{F}(X \times_Y V) = f_{small,*}\mathcal{F}(V)$ we see by assumption that there exists a covering $\{V_j \to V\}$ such that s is the image of $s_j \in f_{small,*}\mathcal{G}(V_j) = \mathcal{G}(X \times_Y V_j)$. This proves $f_{small,*}$ reflects surjections.

Parts (2), (3) follow formally from part (1), see Modules on Sites, Lemma 18.15.1. □

Lemma 44.43.2. *Let $f : X \to Y$ be a separated locally quasi-finite morphism of schemes. Then property (A) above holds.*

Proof. Let $U \to X$ be an étale morphism and $u \in U$. The geometric statement (A) reduces directly to the case where U and Y are affine schemes. Denote $x \in X$ and $y \in Y$ the images of u. Since $X \to Y$ is locally quasi-finite, and $U \to X$ is locally quasi-finite (see Morphisms, Lemma 28.37.6) we see that $U \to Y$ is locally quasi-finite (see Morphisms, Lemma 28.21.12). Moreover both $X \to Y$ and $U \to Y$ are separated. Thus More on Morphisms, Lemma 36.30.5 applies to both morphisms. This means we may pick an étale neighbourhood $(V, v) \to (Y, y)$ such that

$$X \times_Y V = W \amalg R, \quad U \times_Y V = W' \amalg R'$$

and points $w \in W$, $w' \in W'$ such that

(1) W, R are open and closed in $X \times_Y V$,
(2) W', R' are open and closed in $U \times_Y V$,
(3) $W \to V$ and $W' \to V$ are finite,
(4) w, w' map to v,
(5) $\kappa(v) \subset \kappa(w)$ and $\kappa(v) \subset \kappa(w')$ are purely inseparable, and
(6) no other point of W or W' maps to v.

Here is a commutative diagram

$$\begin{array}{ccccc} U & \longleftarrow & U \times_Y V & \longleftarrow & W' \amalg R' \\ \downarrow & & \downarrow & & \downarrow \\ X & \longleftarrow & X \times_Y V & \longleftarrow & W \amalg R \\ \downarrow & & \downarrow & & \\ Y & \longleftarrow & V & & \end{array}$$

After shrinking V we may assume that W' maps into W: just remove the image the inverse image of R in W'; this is a closed set (as $W' \to V$ is finite) not containing v. Then $W' \to W$ is finite because both $W \to V$ and $W' \to V$ are finite. Hence $W' \to W$ is finite étale, and there is exactly one point in the fibre over w with $\kappa(w) = \kappa(w')$. Hence $W' \to W$ is an isomorphism in an open neighbourhood W° of w, see Étale Morphisms, Lemma 40.14.2. Since $W \to V$ is finite the image of $W \setminus W^\circ$ is a closed subset T of V not containing v. Thus after replacing V by $V \setminus T$ we may assume that $W' \to W$ is an isomorphism. Now the decomposition $X \times_Y V = W \amalg R$ and the morphism $W \to U$ are as desired and we win. □

Lemma 44.43.3. *Let $f : X \to Y$ be an integral morphism of schemes. Then property (A) holds.*

Proof. Let $U \to X$ be étale, and let $u \in U$ be a point. We have to find $V \to Y$ étale, a disjoint union decomposition $X \times_Y V = W \amalg W'$ and an X-morphism $W \to U$ with u in the image. We may shrink U and Y and assume U and Y are affine. In this case also X is affine, since an integral morphism is affine by definition. Write $Y = \mathrm{Spec}(A)$, $X = \mathrm{Spec}(B)$ and $U = \mathrm{Spec}(C)$. Then $A \to B$ is an integral ring map, and $B \to C$ is an étale ring map. By Algebra, Lemma 10.136.3 we can find a finite A-subalgebra $B' \subset B$ and an étale ring map $B' \to C'$ such that $C = B \otimes_{B'} C'$. Thus the question reduces to the étale morphism $U' = \mathrm{Spec}(C') \to X' = \mathrm{Spec}(B')$ over the finite morphism $X' \to Y$. In this case the result follows from Lemma 44.43.2. □

Lemma 44.43.4. *Let $f : X \to Y$ be a morphism of schemes. Denote $f_{small} : Sh(X_{étale}) \to Sh(Y_{étale})$ the associated morphism of small étale topoi. Assume at least one of the following*

(1) *f is integral, or*
(2) *f is separated and locally quasi-finite.*

Then the functor $f_{small,*} : Ab(X_{étale}) \to Ab(Y_{étale})$ has the following properties

(1) *the map $f_{small}^{-1} f_{small,*}\mathcal{F} \to \mathcal{F}$ is always surjective,*
(2) *$f_{small,*}$ is faithful, and*
(3) *$f_{small,*}$ reflects injections and surjections.*

Proof. Combine Lemmas 44.43.2, 44.43.3, and 44.43.1. □

44.44. Property (B)

Please see Section 44.42 for the definition of propery (B).

Lemma 44.44.1. *Let $f : X \to Y$ be a morphism of schemes. Assume (B) holds. Then the functor $f_{small,*} : Sh(X_{étale}) \to Sh(Y_{étale})$ transforms surjections into surjections.*

Proof. This follows from Sites, Lemma 7.40.2. □

Lemma 44.44.2. *Let $f : X \to Y$ be a morphism of schemes. Suppose*

(1) *$V \to Y$ is an étale morphism of schemes,*
(2) *$\{U_i \to X \times_Y V\}$ is an étale covering, and*
(3) *$v \in V$ is a point.*

Assume that for any such data there exists an étale neighbourhood $(V', v') \to (V, v)$, a disjoint union decomposition $X \times_Y V' = \coprod W_i'$, and morphisms $W_i' \to U_i$ over $X \times_Y V$. Then property (B) holds.

Proof. Omitted. □

Lemma 44.44.3. *Let $f : X \to Y$ be a finite morphism of schemes. Then property (B) holds.*

Proof. Consider $V \to Y$ étale, $\{U_i \to X \times_Y V\}$ an étale covering, and $v \in V$. We have to find a $V' \to V$ and decomposition and maps as in Lemma 44.44.2. We may shrink V and Y, hence we may assume that V and Y are affine. Since X is finite over Y, this also implies that X is affine. During the proof we may (finitely often) replace (V, v) by an étale neighbourhood (V', v') and correspondingly the covering $\{U_i \to X \times_Y V\}$ by $\{V' \times_V U_i \to X \times_Y V'\}$.

Since $X \times_Y V \to V$ is finite there exist finitely many (pairwise distinct) points $x_1, \ldots, x_n \in X \times_Y V$ mapping to v. We may apply More on Morphisms, Lemma 36.30.5 to $X \times_Y V \to V$ and the points $x_1, \ldots, x_n$ lying over v and find an étale neighbourhood $(V', v') \to (V, v)$ such that

$$X \times_Y V' = R \amalg \coprod T_a$$

with $T_a \to V'$ finite with exactly one point p_a lying over v' and moreover $\kappa(v') \subset \kappa(p_a)$ purely inseparable, and such that $R \to V'$ has empty fibre over v'. Because $X \to Y$ is finite, also $R \to V'$ is finite. Hence after shrinking V' we may assume that $R = \emptyset$. Thus we may assume that $X \times_Y V = X_1 \amalg \ldots \amalg X_n$ with exactly one

point $x_l \in X_l$ lying over v with moreover $\kappa(v) \subset \kappa(x_l)$ purely inseparable. Note that this property is preserved under refinement of the étale neighbourhood (V, v).

For each l choose an i_l and a point $u_l \in U_{i_l}$ mapping to x_l. Now we apply property (A) for the finite morphism $X \times_Y V \to V$ and the étale morphisms $U_{i_l} \to X \times_Y V$ and the points u_l. This is permissible by Lemma 44.43.3 This gives produces an étale neighbourhood $(V', v') \to (V, v)$ and decompositions

$$X \times_Y V' = W_l \amalg R_l$$

and X-morphisms $a_l : W_l \to U_{i_l}$ whose image contains u_{i_l}. Here is a picture:

$$\begin{array}{ccccccccc}
 & & & & & & U_{i_l} & & \\
 & \nearrow & & & & & \downarrow & & \\
W_l & \longrightarrow & W_l \amalg R_l & = & X \times_Y V' & \longrightarrow & X \times_Y V & \longrightarrow & X \\
 & & & & \downarrow & & \downarrow & & \downarrow \\
 & & & & V' & \longrightarrow & V & \longrightarrow & Y
\end{array}$$

After replacing (V, v) by (V', v') we conclude that each x_l is contained in an open and closed neighbourhood W_l such that the inclusion morphism $W_l \to X \times_Y V$ factors through $U_i \to X \times_Y V$ for some i. Replacing W_l by $W_l \cap X_l$ we see that these open and closed sets are disjoint and moreover that $\{x_1, \ldots, x_n\} \subset W_1 \cup \ldots \cup W_n$. Since $X \times_Y V \to V$ is finite we may shrink V and assume that $X \times_Y V = W_1 \amalg \ldots \amalg W_n$ as desired. □

Lemma 44.44.4. *Let $f : X \to Y$ be an integral morphism of schemes. Then property (B) holds.*

Proof. Consider $V \to Y$ étale, $\{U_i \to X \times_Y V\}$ an étale covering, and $v \in V$. We have to find a $V' \to V$ and decomposition and maps as in Lemma 44.44.2. We may shrink V and Y, hence we may assume that V and Y are affine. Since X is integral over Y, this also implies that X and $X \times_Y V$ are affine. We may refine the covering $\{U_i \to X \times_Y V\}$, and hence we may assume that $\{U_i \to X \times_Y V\}_{i=1,\ldots,n}$ is a standard étale covering. Write $Y = \mathrm{Spec}(A)$, $X = \mathrm{Spec}(B)$, $V = \mathrm{Spec}(C)$, and $U_i = \mathrm{Spec}(B_i)$. Then $A \to B$ is an integral ring map, and $B \otimes_A C \to B_i$ are étale ring maps. By Algebra, Lemma 10.136.3 we can find a finite A-subalgebra $B' \subset B$ and an étale ring map $B' \otimes_A C \to B'_i$ for $i = 1, \ldots, n$ such that $B_i = B \otimes_{B'} B'_i$. Thus the question reduces to the étale covering $\{\mathrm{Spec}(B'_i) \to X' \times_Y V\}_{i=1,\ldots,n}$ with $X' = \mathrm{Spec}(B')$ finite over Y. In this case the result follows from Lemma 44.44.3. □

Lemma 44.44.5. *Let $f : X \to Y$ be a morphism of schemes. Assume f is integral (for example finite). Then*

1. *$f_{small,*}$ transforms surjections into surjections (on sheaves of sets and on abelian sheaves),*
2. *$f_{small}^{-1} f_{small,*} \mathcal{F} \to \mathcal{F}$ is surjective for any abelian sheaf $\mathcal{F}$ on $X_{étale}$,*
3. *$f_{small,*} : Ab(X_{étale}) \to Ab(Y_{étale})$ is faithful and reflects injections and surjections, and*
4. *$f_{small,*} : Ab(X_{étale}) \to Ab(Y_{étale})$ is exact.*

Proof. Parts (2), (3) we have seen in Lemma 44.43.4. Part (1) follows from Lemmas 44.44.4 and 44.44.1. Part (4) is a consequence of part (1), see Modules on Sites, Lemma 18.15.2. □

44.45. Property (C)

Please see Section 44.42 for the definition of propery (C).

Lemma 44.45.1. *Let $f : X \to Y$ be a morphism of schemes. Assume (C) holds. Then the functor $f_{small,*} : Sh(X_{étale}) \to Sh(Y_{étale})$ reflects injections and surjections.*

Proof. Follows from Sites, Lemma 7.40.4. We omit the verification that property (C) implies that the functor $Y_{étale} \to X_{étale}$, $V \mapsto X \times_Y V$ satisfies the assumption of Sites, Lemma 7.40.4. □

Remark 44.45.2. Property (C) holds if $f : X \to Y$ is an open immersion. Namely, if $U \in \mathrm{Ob}(X_{étale})$, then we can view U also as an object of $Y_{étale}$ and $U \times_Y X = U$. Hence property (C) does not imply that $f_{small,*}$ is exact as this is not the case for open immersions (in general).

Lemma 44.45.3. *Let $f : X \to Y$ be a morphism of schemes. Assume that for any $V \to Y$ étale we have that*

(1) *$X \times_Y V \to V$ has property (C), and*
(2) *$X \times_Y V \to V$ is closed.*

Then the functor $Y_{étale} \to X_{étale}$, $V \mapsto X \times_Y V$ is almost cocontinuous, see Sites, Definition 7.41.3.

Proof. Let $V \to Y$ be an object of $Y_{étale}$ and let $\{U_i \to X \times_Y V\}_{i \in I}$ be a covering of $X_{étale}$. By assumption (1) for each i we can find an étale morphism $h_i : V_i \to V$ and a surjective morphism $X \times_Y V_i \to U_i$ over $X \times_Y V$. Note that $\bigcup h_i(V_i) \subset V$ is an open set containing the closed set $Z = \mathrm{Im}(X \times_Y V \to V)$. Let $h_0 : V_0 = V \setminus Z \to V$ be the open immersion. It is clear that $\{V_i \to V\}_{i \in I \cup \{0\}}$ is an étale covering such that for each $i \in I \cup \{0\}$ we have either $V_i \times_Y X = \emptyset$ (namely if $i = 0$), or $V_i \times_Y X \to V \times_Y X$ factors through $U_i \to X \times_Y V$ (if $i \neq 0$). Hence the functor $Y_{étale} \to X_{étale}$ is almost cocontinuous. □

Lemma 44.45.4. *Let $f : X \to Y$ be an integral morphism of schemes which defines a homeomorphism of X with a closed subset of Y. Then property (C) holds.*

Proof. Let $g : U \to X$ be an étale morphism. We need to find an object $V \to Y$ of $Y_{étale}$ and a surjective morphism $X \times_Y V \to U$ over X. Suppose that for every $u \in U$ we can find an object $V_u \to Y$ of $Y_{étale}$ and a morphism $h_u : X \times_Y V_u \to U$ over X with $u \in \mathrm{Im}(h_u)$. Then we can take $V = \coprod V_u$ and $h = \coprod h_u$ and we win. Hence given a point $u \in U$ we find a pair (V_u, h_u) as above. To do this we may shrink U and assume that U is affine. In this case $g : U \to X$ is locally quasi-finite. Let $g^{-1}(g(\{u\})) = \{u, u_2, \ldots, u_n\}$. Since there are no specializations $u_i \rightsquigarrow u$ we may replace U by an affine neighbourhood so that $g^{-1}(g(\{u\})) = \{u\}$.

The image $g(U) \subset X$ is open, hence $f(g(U))$ is locally closed in Y. Choose an open $V \subset Y$ such that $f(g(U)) = f(X) \cap V$. It follows that g factors through $X \times_Y V$ and that the resulting $\{U \to X \times_Y V\}$ is an étale covering. Since f has property (B) , see Lemma 44.44.4, we see that there exists an étale covering $\{V_j \to V\}$ such

that $X \times_Y V_j \to X \times_Y V$ factor through U. This implies that $V' = \coprod V_j$ is étale over Y and that there is a morphism $h : X \times_Y V' \to U$ whose image surjects onto $g(U)$. Since u is the only point in its fibre it must be in the image of h and we win. □

We urge the reader to think of the following lemma as a way station[3] on the journey towards the ultimate truth regarding $f_{small,*}$ for integral universally injective morphisms.

Lemma 44.45.5. *Let $f : X \to Y$ be a morphism of schemes. Assume that f is universally injective and integral (for example a closed immersion). Then*

(1) *$f_{small,*} : Sh(X_{étale}) \to Sh(Y_{étale})$ reflects injections and surjections,*
(2) *$f_{small,*} : Sh(X_{étale}) \to Sh(Y_{étale})$ commutes with pushouts and coequalizers (and more generally finite connected colimits),*
(3) *$f_{small,*}$ transforms surjections into surjections (on sheaves of sets and on abelian sheaves),*
(4) *the map $f_{small}^{-1} f_{small,*} \mathcal{F} \to \mathcal{F}$ is surjective for any sheaf (of sets or of abelian groups) $\mathcal{F}$ on $X_{étale}$,*
(5) *the functor $f_{small,*}$ is faithful (on sheaves of sets and on abelian sheaves),*
(6) *$f_{small,*} : Ab(X_{étale}) \to Ab(Y_{étale})$ is exact, and*
(7) *the functor $Y_{étale} \to X_{étale}$, $V \mapsto X \times_Y V$ is almost cocontinuous.*

Proof. By Lemmas 44.43.3, 44.44.4 and 44.45.4 we know that the morphism f has properties (A), (B), and (C). Moreover, by Lemma 44.45.3 we know that the functor $Y_{étale} \to X_{étale}$ is almost cocontinuous. Now we have

(1) property (C) implies (1) by Lemma 44.45.1,
(2) almost continuous implies (2) by Sites, Lemma 7.41.6,
(3) property (B) implies (3) by Lemma 44.44.1.

Properties (4), (5), and (6) follow formally from the first three, see Sites, Lemma 7.40.1 and Modules on Sites, Lemma 18.15.2. Property (7) we saw above. □

44.46. Topological invariance of the small étale site

In the following theorem we show that the small étale site is a topological invariant in the following sense: If $f : X \to Y$ is a morphism of schemes which is a universal homeomorphism, then $X_{étale} \cong Y_{étale}$ as sites. This improves the result of Étale Morphisms, Theorem 40.15.2.

Theorem 44.46.1. *Let $f : X \to Y$ be a morphism of schemes. Assume f is integral, universally injective and surjective (i.e., f is a universal homeomorphism, see Morphisms, Lemma 28.45.3). The functor*

$$V \longmapsto V_X = X \times_Y V$$

defines an equivalence of categories

$$\{\text{schemes } V \text{ étale over } Y\} \leftrightarrow \{\text{schemes } U \text{ étale over } X\}$$

Proof. We claim that it suffices to prove that the functor defines an equivalence

$$\{\text{affine schemes } V \text{ étale over } Y\} \leftrightarrow \{\text{affine schemes } U \text{ étale over } X\} \tag{44.46.1.1}$$

when X and Y are affine. We omit the proof of this claim.

[3] A way station is a place where people stop to eat and rest when they are on a long journey.

Assume X and Y affine. Let us prove (44.46.1.1) is fully faithful. Suppose that V, V' are affine schemes étale over Y, and that $\varphi : V_X \to V'_X$ is a morphism over X. To prove that $\varphi = \psi_X$ for some $\psi : V \to V'$ over Y we may work locally on V. The graph

$$\Gamma_\varphi \subset (V \times_Y V')_X$$

of φ is an open and closed subscheme, see Étale Morphisms, Proposition 40.6.1. Since f is a universal homeomorphism we see that there exists an open and closed subscheme $\Gamma \subset V \times_Y V'$ with $\Gamma_X = \Gamma_\varphi$. We see that Γ is an affine scheme endowed with an étale, universally injective, and surjective morphism $\Gamma \to V$. This implies that $\Gamma \to V$ is an isomorphism (see Étale Morphisms, Theorem 40.14.1), and hence Γ is the graph of a morphism $\psi : V \to V'$ over Y as desired.

Let us prove (44.46.1.1) is essentially surjective. Let $U \to X$ be an affine scheme étale over X. We have to find $V \to Y$ étale (and affine) such that $X \times_Y V$ is isomorphic to U over X. Note that an étale morphism of affines has universally bounded fibres, see Morphisms, Lemmas 28.37.6 and 28.50.8. Hence we can do induction on the integer n bounding the degree of the fibres of $U \to X$. See Morphisms, Lemma 28.50.7 for a description of this integer in the case of an étale morphism. If $n = 1$, then $U \to X$ is an open immersion (see Étale Morphisms, Theorem 40.14.1), and the result is clear. Assume $n > 1$.

By Lemma 44.45.4 there exists an étale morphism of schemes $W \to Y$ and a surjective morphism $W_X \to U$ over X. As U is quasi-compact we may replace W by a disjoint union of finitely many affine opens of W, hence we may assume that W is affine as well. Here is a diagram

$$\begin{array}{ccccc} U & \longleftarrow & U \times_Y W & = & W_X \amalg R \\ \downarrow & & \downarrow & & \\ X & \longleftarrow & W_X & & \\ \downarrow & & \downarrow & & \\ Y & \longleftarrow & W & & \end{array}$$

The disjoint union decomposition arises because by construction the étale morphism of affine schemes $U \times_Y W \to W_X$ has a section. OK, and now we see that the morphism $R \to X \times_Y W$ is an étale morphism of affine schemes whose fibres have degree universally bounded by $n - 1$. Hence by induction assumption there exists a scheme $V' \to W$ étale such that $R \cong W_X \times_W V'$. Taking $V'' = W \amalg V'$ we find a scheme V'' étale over W whose base change to W_X is isomorphic to $U \times_Y W$ over $X \times_Y W$.

At this point we can use descent to find V over Y whose base change to X is isomorphic to U over X. Namely, by the fully faithfulness of the functor (44.46.1.1) corresponding to the universal homeomorphism $X \times_Y (W \times_Y W) \to (W \times_Y W)$ there exists a unique isomorphism $\varphi : V'' \times_Y W \to W \times_Y V''$ whose base change to $X \times_Y (W \times_Y W)$ is the canonical descent datum for $U \times_Y W$ over $X \times_Y W$. In particular φ satisfies the cocycle condition. Hence by Descent, Lemma 34.33.1 we see that φ is effective (recall that all schemes above are affine). Thus we obtain $V \to Y$ and an isomorphism $V'' \cong W \times_Y V$ such that the canonical descent datum on $W \times_Y V/W/Y$ agrees with φ. Note that $V \to Y$ is étale, by Descent, Lemma

34.19.27. Moreover, there is an isomorphism $V_X \cong U$ which comes from descending the isomorphism

$$V_X \times_X W_X = X \times_Y V \times_Y W = (X \times_Y W) \times_W (W \times_Y V) \cong W_X \times_W V'' \cong U \times_Y W$$

which we have by construction. Some details omitted. □

Remark 44.46.2. In the situation of Theorem 44.46.1 it is also true that $V \mapsto V_X$ induces an equivalence between those étale morphisms $V \to Y$ with V affine and those étale morphisms $U \to X$ with U affine. This follows for example from Limits, Proposition 31.10.2.

44.47. Closed immersions and pushforward

Before stating and proving Proposition 44.47.4 in its correct generality we briefly state and prove it for closed immersions. Namely, some of the preceding arguments are quite a bit easier to follow in the case of a closed immersion and so we repeat them here in their simplified form.

In the rest of this section $i : Z \to X$ is a closed immersion. The functor

$$Sch/X \longrightarrow Sch/Z, \quad U \longmapsto U_Z = Z \times_X U$$

will be denoted $U \mapsto U_Z$ as indicated. Since being a closed immersion is preserved under arbitrary base change the scheme U_Z is a closed subscheme of U.

Lemma 44.47.1. *Let $i : Z \to X$ be a closed immersion of schemes. Let U, U' be schemes étale over X. Let $h : U_Z \to U'_Z$ be a morphism over Z. Then there exists a diagram*

$$U \xleftarrow{a} W \xrightarrow{b} U'$$

such that $a_Z : W_Z \to U_Z$ is an isomorphism and $h = b_Z \circ (a_Z)^{-1}$.

Proof. Consider the scheme $M = U \times_Y U'$. The graph $\Gamma_h \subset M_Z$ of h is open. This is true for example as Γ_h is the image of a section of the étale morphism $\mathrm{pr}_{1,Z} : M_Z \to U_Z$, see Étale Morphisms, Proposition 40.6.1. Hence there exists an open subscheme $W \subset M$ whose intersection with the closed subset M_Z is Γ_h. Set $a = \mathrm{pr}_1|_W$ and $b = \mathrm{pr}_2|_W$. □

Lemma 44.47.2. *Let $i : Z \to X$ be a closed immersion of schemes. Let $V \to Z$ be an étale morphism of schemes. There exist étale morphisms $U_i \to X$ and morphisms $U_{i,Z} \to V$ such that $\{U_{i,Z} \to V\}$ is a Zariski covering of V.*

Proof. Since we only have to find a Zariski covering of V consisting of schemes of the form U_Z with U étale over X, we may Zariski localize on X and V. Hence we may assume X and V affine. In the affine case this is Algebra, Lemma 10.136.11. □

If $\overline{x} : \mathrm{Spec}(k) \to X$ is a geometric point of X, then either $\overline{x}$ factors (uniquely) through the closed subscheme Z, or $Z_{\overline{x}} = \emptyset$. If $\overline{x}$ factors through Z we say that $\overline{x}$ is a geometric point of Z (because it is) and we use the notation "$\overline{x} \in Z$" to indicate this.

Lemma 44.47.3. *Let $i : Z \to X$ be a closed immersion of schemes. Let $\mathcal{G}$ be a sheaf of sets on $Z_{étale}$. Let $\overline{x}$ be a geometric point of X. Then*

$$(i_{small,*}\mathcal{G})_{\overline{x}} = \begin{cases} * & \text{if} \quad \overline{x} \notin Z \\ \mathcal{F}_{\overline{x}} & \text{if} \quad \overline{x} \in Z \end{cases}$$

where $*$ denotes a singleton set.

Proof. Note that $i_{small,*}\mathcal{G}|_{U_{étale}} = *$ is the final object in the category of étale sheaves on U, i.e., the sheaf which associates a singleton set to each scheme étale over U. This explains the value of $(i_{small,*}\mathcal{G})_{\overline{x}}$ if $\overline{x} \notin Z$.

Next, suppose that $\overline{x} \in Z$. Note that

$$(i_{small,*}\mathcal{G})_{\overline{x}} = \operatorname{colim}_{(U,\overline{u})} \mathcal{G}(U_Z)$$

and on the other hand

$$\mathcal{G}_{\overline{x}} = \operatorname{colim}_{(V,\overline{v})} \mathcal{G}(V).$$

Let $\mathcal{C}_1 = \{(U,\overline{u})\}^{opp}$ be the opposite of the category of étale neighbourhoods of $\overline{x}$ in X, and let $\mathcal{C}_2 = \{(V,\overline{v})\}^{opp}$ be the opposite of the category of étale neighbourhoods of $\overline{x}$ in Z. The canonical map

$$\mathcal{G}_{\overline{x}} \longrightarrow (i_{small,*}\mathcal{G})_{\overline{x}}$$

corresponds to the functor $F : \mathcal{C}_1 \to \mathcal{C}_2$, $F(U,\overline{u}) = (U_Z, \overline{x})$. Now Lemmas 44.47.2 and 44.47.1 imply that $\mathcal{C}_1$ is cofinal in $\mathcal{C}_2$, see Categories, Definition 4.17.1. Hence it follows that the displayed arrow is an isomorphism, see Categories, Lemma 4.17.2. □

Proposition 44.47.4. *Let $i : Z \to X$ be a closed immersion of schemes.*

(1) *The functor*

$$i_{small,*} : Sh(Z_{étale}) \longrightarrow Sh(X_{étale})$$

is fully faithful and its essential image is those sheaves of sets $\mathcal{F}$ on $X_{étale}$ whose restriction to $X \setminus Z$ is isomorphic to $*$, and

(2) *the functor*

$$i_{small,*} : Ab(Z_{étale}) \longrightarrow Ab(X_{étale})$$

is fully faithful and its essential image is those abelian sheaves on $X_{étale}$ whose support is contained in Z.

In both cases i_{small}^{-1} is a left inverse to the functor $i_{small,*}$.

Proof. Let's discuss the case of sheaves of sets. For any sheaf $\mathcal{G}$ on Z the morphism $i_{small}^{-1} i_{small,*}\mathcal{G} \to \mathcal{G}$ is an isomorphism by Lemma 44.47.3 (and Theorem 44.29.10). This implies formally that $i_{small,*}$ is fully faithful, see Sites, Lemma 7.40.1. It is clear that $i_{small,*}\mathcal{G}|_{U_{étale}} \cong *$ where $U = X \setminus Z$. Conversely, suppose that $\mathcal{F}$ is a sheaf of sets on X such that $\mathcal{F}|_{U_{étale}} \cong *$. Consider the adjunction mapping

$$\mathcal{F} \longrightarrow i_{small,*} i_{small}^{-1} \mathcal{F}$$

Combining Lemmas 44.47.3 and 44.36.2 we see that it is an isomorphism. This finishes the proof of (1). The proof of (2) is identical. □

44.48. Integral universally injective morphisms

Here is the general version of Proposition 44.47.4.

Proposition 44.48.1. *Let $f : X \to Y$ be a morphism of schemes which is integral and universally injective.*

(1) *The functor*

$$f_{small,*} : Sh(X_{étale}) \longrightarrow Sh(Y_{étale})$$

is fully faithful and its essential image is those sheaves of sets $\mathcal{F}$ on $Y_{étale}$ whose restriction to $Y \setminus f(X)$ is isomorphic to $*$, and

(2) *the functor*

$$f_{small,*} : Ab(X_{étale}) \longrightarrow Ab(Y_{étale})$$

is fully faithful and its essential image is those abelian sheaves on $Y_{étale}$ whose support is contained in $f(X)$.

In both cases f_{small}^{-1} is a left inverse to the functor $f_{small,*}$.

Proof. We may factor f as

$$X \xrightarrow{h} Z \xrightarrow{i} Y$$

where h is integral, universally injective and surjective and $i : Z \to Y$ is a closed immersion. Apply Proposition 44.47.4 to i and apply Theorem 44.46.1 to h. □

44.49. Big sites and pushforward

In this section we prove some technical results on $f_{big,*}$ for certain types of morphisms of schemes.

Lemma 44.49.1. *Let $\tau \in \{Zariski, étale, smooth, syntomic, fppf\}$. Let $f : X \to Y$ be a monomorphism of schemes. Then the canonical map $f_{big}^{-1} f_{big,*} \mathcal{F} \to \mathcal{F}$ is an isomorphism for any sheaf $\mathcal{F}$ on $(Sch/X)_\tau$.*

Proof. In this case the functor $(Sch/X)_\tau \to (Sch/Y)_\tau$ is continuous, cocontinuous and fully faithful. Hence the result follows from Sites, Lemma 7.20.7. □

Remark 44.49.2. In the situation of Lemma 44.49.1 it is true that the canonical map $\mathcal{F} \to f_{big}^{-1} f_{big!} \mathcal{F}$ is an isomorphism for any sheaf of sets $\mathcal{F}$ on $(Sch/X)_\tau$. The proof is the same. This also holds for sheaves of abelian groups. However, note that the functor $f_{big!}$ for sheaves of abelian groups is defined in Modules on Sites, Section 18.16 and is in general different from $f_{big!}$ on sheaves of sets. The result for sheaves of abelian groups follows from Modules on Sites, Lemma 18.16.4.

Lemma 44.49.3. *Let $f : X \to Y$ be a closed immersion of schemes. Let $U \to X$ be a syntomic (resp. smooth, resp. étale) morphism. Then there exist syntomic (resp. smooth, resp. étale) morphisms $V_i \to Y$ and morphisms $V_i \times_Y X \to U$ such that $\{V_i \times_Y X \to U\}$ is a Zariski covering of U.*

Proof. Let us prove the lemma when $\tau = syntomic$. The question is local on U. Thus we may assume that U is an affine scheme mapping into an affine of Y. Hence we reduce to proving the following case: $Y = \mathrm{Spec}(A)$, $X = \mathrm{Spec}(A/I)$, and $U = \mathrm{Spec}(\overline{B})$, where $A/I \to \overline{B}$ be a syntomic ring map. By Algebra, Lemma 10.129.18 we can find elements $\overline{g}_i \in \overline{B}$ such that $\overline{B}_{\overline{g}_i} = A_i/IA_i$ for certain syntomic ring maps $A \to A_i$. This proves the lemma in the syntomic case. The proof of the smooth case is the same except it uses Algebra, Lemma 10.130.19. In the étale case use Algebra, Lemma 10.136.11. □

Lemma 44.49.4. *Let $f : X \to Y$ be a closed immersion of schemes. Let $\{U_i \to X\}$ be a syntomic (resp. smooth, resp. étale) covering. There exists a syntomic (resp. smooth, resp. étale) covering $\{V_j \to Y\}$ such that for each j, either $V_j \times_Y X = \emptyset$, or the morphism $V_j \times_Y X \to X$ factors through U_i for some i.*

Proof. For each i we can choose syntomic (resp. smooth, resp. étale) morphisms $g_{ij} : V_{ij} \to Y$ and morphisms $V_{ij} \times_Y X \to U_i$ over X, such that $\{V_{ij} \times_Y X \to U_i\}$ are Zariski coverings, see Lemma 44.49.3. This in particular implies that $\bigcup_{ij} g_{ij}(V_{ij})$ contains the closed subset $f(X)$. Hence the family of syntomic (resp. smooth, resp. étale) maps g_{ij} together with the open immersion $Y \setminus f(X) \to Y$ forms the desired syntomic (resp. smooth, resp. étale) covering of Y. □

Lemma 44.49.5. *Let $f : X \to Y$ be a closed immersion of schemes. Let $\tau \in \{syntomic, smooth, étale\}$. The functor $V \mapsto X \times_Y V$ defines an almost cocontinuous functor (see Sites, Definition 7.41.3) $(Sch/Y)_\tau \to (Sch/X)_\tau$ between big τ sites.*

Proof. We have to show the following: given a morphism $V \to Y$ and any syntomic (resp. smooth, resp. étale) covering $\{U_i \to X \times_Y V\}$, there exists a smooth (resp. smooth, resp. étale) covering $\{V_j \to V\}$ such that for each j, either $X \times_Y V_j$ is empty, or $X \times_Y V_j \to Z \times_Y V$ factors through one of the U_i. This follows on applying Lemma 44.49.4 above to the closed immersion $X \times_Y V \to V$. □

Lemma 44.49.6. *Let $f : X \to Y$ be a closed immersion of schemes. Let $\tau \in \{syntomic, smooth, étale\}$.*

1. *The pushforward $f_{big,*} : Sh((Sch/X)_\tau) \to Sh((Sch/Y)_\tau)$ commutes with coequalizers and pushouts.*
2. *The pushforward $f_{big,*} : Ab((Sch/X)_\tau) \to Ab((Sch/Y)_\tau)$ is exact.*

Proof. This follows from Sites, Lemma 7.41.6, Modules on Sites, Lemma 18.15.3, and Lemma 44.49.5 above. □

Remark 44.49.7. In Lemma 44.49.6 the case $\tau = fppf$ is missing. The reason is that given a ring A, an ideal I and a faithfully flat, finitely presented ring map $A/I \to \overline{B}$, there is no reason to think that one can find *any* flat finitely presented ring map $A \to B$ with $B/IB \neq 0$ such that $A/I \to B/IB$ factors through $\overline{B}$. Hence the proof of Lemma 44.49.5 does not work for the fppf topology. In fact it is likely false that $f_{big,*} : Ab((Sch/X)_{fppf}) \to Ab((Sch/Y)_{fppf})$ is exact when f is a closed immersion. If you know an example, please email stacks.project@gmail.com.

44.50. Exactness of big lower shriek

This is just the following technical result. Note that the functor $f_{big!}$ has nothing whatsoever to do with cohomology with compact support in general.

Lemma 44.50.1. *Let $\tau \in \{Zariski, étale, smooth, syntomic, fppf\}$. Let $f : X \to Y$ be a morphism of schemes. Let*

$$f_{big} : Sh((Sch/X)_\tau) \longrightarrow Sh((Sch/Y)_\tau)$$

be the corresponding morphism of topoi as in Topologies, Lemma 33.3.15, 33.4.15, 33.5.10, 33.6.10, or 33.7.12.

(1) *The functor* $f_{big}^{-1} : Ab((Sch/Y)_\tau) \to Ab((Sch/X)_\tau)$ *has a left adjoint*
$$f_{big!} : Ab((Sch/X)_\tau) \to Ab((Sch/Y)_\tau)$$
which is exact.

(2) *The functor* $f_{big}^* : Mod((Sch/Y)_\tau, \mathcal{O}) \to Mod((Sch/X)_\tau, \mathcal{O})$ *has a left adjoint*
$$f_{big!} : Mod((Sch/X)_\tau, \mathcal{O}) \to Mod((Sch/Y)_\tau, \mathcal{O})$$
which is exact.

Moreover, the two functors $f_{big!}$ *agree on underlying sheaves of abelian groups.*

Proof. Recall that f_{big} is the morphism of topoi associated to the continuous and cocontinuous functor $u : (Sch/X)_\tau \to (Sch/Y)_\tau$, $U/X \mapsto U/Y$. Moreover, we have $f_{big}^{-1}\mathcal{O} = \mathcal{O}$. Hence the existence of $f_{big!}$ follows from Modules on Sites, Lemma 18.16.2, respectively Modules on Sites, Lemma 18.40.1. Note that if U is an object of $(Sch/X)_\tau$ then the functor u induces an equivalence of categories
$$u' : (Sch/X)_\tau/U \longrightarrow (Sch/Y)_\tau/U$$
because both sides of the arrow are equal to $(Sch/U)_\tau$. Hence the agreement of $f_{big!}$ on underlying abelian sheaves follows from the discussion in Modules on Sites, Remark 18.40.2. The exactness of $f_{big!}$ follows from Modules on Sites, Lemma 18.16.3 as the functor u above which commutes with fibre products and equalizers. □

Next, we prove a technical lemma that will be useful later when comparing sheaves of modules on different sites associated to algebraic stacks.

Lemma 44.50.2. *Let* X *be a scheme. Let* $\tau \in \{Zariski, étale, smooth, syntomic, fppf\}$. *Let* $\mathcal{C}_1 \subset \mathcal{C}_2 \subset (Sch/X)_\tau$ *be full subcategories with the following properties:*

(1) *For an object* U/X *of* $\mathcal{C}_t$,
   (a) *if* $\{U_i \to U\}$ *is a covering of* $(Sch/X)_\tau$, *then* U_i/X *is an object of* $\mathcal{C}_t$,
   (b) $U \times \mathbf{A}^1/X$ *is an object of* $\mathcal{C}_t$.

(2) X/X *is an object of* $\mathcal{C}_t$.

We endow $\mathcal{C}_t$ *with the structure of a site whose coverings are exactly those coverings* $\{U_i \to U\}$ *of* $(Sch/X)_\tau$ *with* $U \in \mathrm{Ob}(\mathcal{C}_t)$. *Then*

(i) *The functor* $\mathcal{C}_1 \to \mathcal{C}_2$ *is fully faithful, continuous, and cocontinuous.*

Denote $g : Sh(\mathcal{C}_1) \to Sh(\mathcal{C}_2)$ *the corresponding morphism of topoi. Denote* $\mathcal{O}_t$ *the restriction of* $\mathcal{O}$ *to* $\mathcal{C}_t$. *Denote* $g_!$ *the functor of Modules on Sites, Definition 18.16.1.*

(ii) *The canonical map* $g_!\mathcal{O}_1 \to \mathcal{O}_2$ *is an isomorphism.*

Proof. Assertion (i) is immediate from the definitions. In this proof all schemes are schemes over X and all morphisms of schemes are morphisms of schemes over X. Note that g^{-1} is given by restriction, so that for an object U of $\mathcal{C}_1$ we have $\mathcal{O}_1(U) = \mathcal{O}_2(U) = \mathcal{O}(U)$. Recall that $g_!\mathcal{O}_1$ is the sheaf associated to the presheaf $g_{p!}\mathcal{O}_1$ which associates to V in $\mathcal{C}_2$ the group
$$\mathrm{colim}_{V \to U} \mathcal{O}(U)$$
where U runs over the objects of $\mathcal{C}_1$ and the colimit is taken in the category of abelian groups. Below we will use frequently that if
$$V \to U \to U'$$

are morphisms with $U, U' \in \mathrm{Ob}(\mathcal{C}_1)$ and if $f' \in \mathcal{O}(U')$ restricts to $f \in \mathcal{O}(U)$, then $(V \to U, f)$ and $(V \to U', f')$ define the same element of the colimit. Also, $g_!\mathcal{O}_1 \to \mathcal{O}_2$ maps the element $(V \to U, f)$ simply to the pullback of f to V.

Surjectivity. Let V be a scheme and let $h \in \mathcal{O}(V)$. Then we obtain a morphism $V \to X \times \mathbf{A}^1$ induced by h and the structure morphism $V \to X$. Writing $\mathbf{A}^1 = \mathrm{Spec}(\mathbf{Z}[x])$ we see the element $x \in \mathcal{O}(X \times \mathbf{A}^1)$ pulls back to h. Since $X \times \mathbf{A}^1$ is an object of $\mathcal{C}_1$ by assumptions (1)(b) and (2) we obtain the desired surjectivity.

Injectivity. Let V be a scheme. Let $s = \sum_{i=1,\ldots,n}(V \to U_i, f_i)$ be an element of the colimit displayed above. For any i we can use the morphism $f_i : U_i \to X \times \mathbf{A}^1$ to see that $(V \to U_i, f_i)$ defines the same element of the colimit as $(f_i : V \to X \times \mathbf{A}^1, x)$. Then we can consider

$$f_1 \times \ldots \times f_n : V \to X \times \mathbf{A}^n$$

and we see that s is equivalent in the colimit to

$$\sum\nolimits_{i=1,\ldots,n}(f_1 \times \ldots \times f_n : V \to X \times \mathbf{A}^n, x_i) = (f_1 \times \ldots \times f_n : V \to X \times \mathbf{A}^n, x_1 + \ldots + x_n)$$

Now, if $x_1 + \ldots + x_n$ restricts to zero on V, then we see that $f_1 \times \ldots \times f_n$ factors through $X \times \mathbf{A}^{n-1} = V(x_1 + \ldots + x_n)$. Hence we see that s is equivalent to zero in the colimit. □

44.51. Étale cohomology

In the following sections we prove some basic results on étale cohomology.

44.52. Colimits

We recall that if $(\mathcal{F}_i, \varphi_{ii'})$ is a diagram of sheaves on a site $\mathcal{C}$ its colimit (in the category of sheaves) is the sheafification of the presheaf $U \mapsto \mathrm{colim}_i \mathcal{F}_i(U)$. See Sites, Lemma 7.10.13. If the system is directed, U is a quasi-compact object of $\mathcal{C}$ which has a cofinal system of coverings by quasi-compact objects, then $\mathcal{F}(U) = \mathrm{colim}\, \mathcal{F}_i(U)$, see Sites, Lemma 7.11.2. See Cohomology on Sites, Lemma 21.16.1 for a result dealing with higher cohomology groups of colimits of abelian sheaves.

We first state and prove a very general result on colimits and cohomology and then we explain what it means in some special cases.

Theorem 44.52.1. *Let $X = \lim_{i \in I} X_i$ be a limit of a directed system of schemes with affine transition morphisms $f_{i'i} : X_{i'} \to X_i$. We assume that X_i is quasi-compact and quasi-separated for all $i \in I$. Assume given*

1. *an abelian sheaf $\mathcal{F}_i$ on $(X_i)_{étale}$ for all $i \in I$,*
2. *for $i' \geq i$ a map $\varphi_{i'i} : f_{i'i}^{-1}\mathcal{F}_i \to \mathcal{F}_{i'}$ of abelian sheaves on $(X_{i'})_{étale}$*

such that $\varphi_{i''i} = \varphi_{i''i'} \circ f_{i''i'}^{-1}\varphi_{i'i}$ whenever $i'' \geq i' \geq i$. Denote $f_i : X \to X_i$ the projection and set $\mathcal{F} = \mathrm{colim}\, f_i^{-1}\mathcal{F}_i$. Then

$$\mathrm{colim}_{i \in I} H^p_{étale}(X_i, \mathcal{F}_i) = H^p_{étale}(X, \mathcal{F}).$$

for all $p \geq 0$.

Proof. Let us use the affine étale sites of X and X_i as introduced in Lemma 44.21.2. We claim that

$$X_{affine,étale} = \mathrm{colim}(X_i)_{affine,étale}$$

as sites (see Sites, Lemma 7.11.4). If we prove this, then the theorem follows from Cohomology on Sites, Lemma 21.16.2. The category of schemes of finite presentation over X is the colimit of the categories of schemes of finite presentation over X_i, see Limits, Lemma 31.9.1. The same holds for the subcategories of affine objects étale over X by Limits, Lemmas 31.3.10 and 31.7.8. Finally, if $\{U^j \to U\}$ is a covering of $X_{affine,étale}$ and if $U_i^j \to U_i$ is morphism of affine schemes étale over X_i whose base change to X is $U^j \to U$, then we see that the base change of $\{U_i^j \to U_i\}$ to some $X_{i'}$ is a covering for i' large enough, see Limits, Lemma 31.7.11. □

The following two results are special cases of the theorem above.

Lemma 44.52.2. *Let X be a quasi-compact and quasi-separated scheme. Let $(\mathcal{F}_i, \varphi_{ij})$ be a system of abelian sheaves on $X_{étale}$ over the partially ordered set I. If I is directed then*

$$\operatorname{colim}_{i\in I} H^p_{étale}(X, \mathcal{F}_i) = H^p_{étale}(X, \operatorname{colim}_{i\in I} \mathcal{F}_i).$$

Proof. This is a special case of Theorem 44.52.1. We also sketch a direct proof. We prove it for all X at the same time, by induction on p.

(1) For any quasi-compact and quasi-separated scheme X and any étale covering $\mathcal{U}$ of X, show that there exists a refinement $\mathcal{V} = \{V_j \to X\}_{j\in J}$ with J finite and each V_j quasi-compact and quasi-separated such that all $V_{j_0} \times_X \cdots \times_X V_{j_p}$ are also quasi-compact and quasi-separated.
(2) Using the previous step and the definition of colimits in the category of sheaves, show that the theorem holds for $p = 0$ and all X.
(3) Using the locality of cohomology (Lemma 44.22.3), the Čech-to-cohomology spectral sequence (Theorem 44.19.2) and the fact that the induction hypothesis applies to all $V_{j_0} \times_X \cdots \times_X V_{j_p}$ in the above situation, prove the induction step $p \to p + 1$.

□

Lemma 44.52.3. *Let A be a ring, $(I, \leq)$ a directed poset and (B_i, φ_{ij}) a system of A-algebras. Set $B = \operatorname{colim}_{i\in I} B_i$. Let $X \to \operatorname{Spec}(A)$ be a quasi-compact and quasi-separated morphism of schemes. Let $\mathcal{F}$ an abelian sheaf on $X_{étale}$. Denote $Y_i = X \times_{\operatorname{Spec}(A)} \operatorname{Spec}(B_i)$, $Y = X \times_{\operatorname{Spec}(A)} \operatorname{Spec}(B)$, $\mathcal{G}_i = (Y_i \to X)^{-1}\mathcal{F}$ and $\mathcal{G} = (Y \to X)^{-1}\mathcal{F}$. Then*

$$H^p_{étale}(Y, \mathcal{G}) = \operatorname{colim}_{i\in I} H^p_{étale}(X_i, \mathcal{G}_i).$$

Proof. This is a special case of Theorem 44.52.1. We also outline a direct proof as follows.

(1) Given $V \to Y$ étale with V quasi-compact and quasi-separated, there exist $i \in I$ and $V_i \to Y_i$ such that $V = V_i \times_{Y_i} Y$. If all the schemes considered were affine, this would correspond to the following algebra statement: if $B = \operatorname{colim} B_i$ and $B \to C$ is étale, then there exist $i \in I$ and $B_i \to C_i$ étale such that $C \cong B \otimes_{B_i} C_i$. This is proved in Algebra, Lemma 10.136.3.
(2) In the situation of (1) show that $\mathcal{G}(V) = \operatorname{colim}_{i'\geq i} \mathcal{G}_{i'}(V_{i'})$ where $V_{i'}$ is the base change of V_i to $Y_{i'}$.
(3) By (1), we see that for every étale covering $\mathcal{V} = \{V_j \to Y\}_{j\in J}$ with J finite and the V_js quasi-compact and quasi-separated, there exists $i \in I$ and an étale covering $\mathcal{V}_i = \{V_{ij} \to Y_i\}_{j\in J}$ such that $\mathcal{V} \cong \mathcal{V}_i \times_{Y_i} Y$.

(4) Show that (2) and (3) imply

$$\check{H}^*(\mathcal{V}, \mathcal{G}) = \text{colim}_{i \in I} \check{H}^*(\mathcal{V}_i, \mathcal{G}_i).$$

(5) Cleverly use the Čech-to-cohomology spectral sequence (Theorem 44.19.2).

□

Lemma 44.52.4. *Let $f : X \to Y$ be a morphism of schemes and $\mathcal{F} \in Ab(X_{étale})$. Then $R^p f_* \mathcal{F}$ is the sheaf associated to the presheaf*

$$(V \to Y) \longmapsto H^p_{étale}(X \times_Y V, \mathcal{F}|_{X \times_Y V}).$$

Proof. This lemma is valid for topological spaces, and the proof in this case is the same. See Cohomology on Sites, Lemma 21.8.4 for details. □

Lemma 44.52.5. *Let S be a scheme. Let $X = \lim_{i \in I} X_i$ be a limit of a directed system of schemes over S with affine transition morphisms $f_{i'i} : X_{i'} \to X_i$. We assume the structure morphism $g_i : X_i \to S$ is quasi-compact and quasi-separated for all $i \in I$ and we set $g : X \to S$. Assume given*

(1) *an abelian sheaf $\mathcal{F}_i$ on $(X_i)_{étale}$ for all $i \in I$,*
(2) *for $i' \geq i$ a map $\varphi_{i'i} : f_{i'i}^{-1} \mathcal{F}_i \to \mathcal{F}_{i'}$ of abelian sheaves on $(X_{i'})_{étale}$*

such that $\varphi_{i''i} = \varphi_{i''i'} \circ f_{i''i'}^{-1} \varphi_{i'i}$ whenever $i'' \geq i' \geq i$. Denote $f_i : X \to X_i$ the projection and set $\mathcal{F} = \text{colim } f_i^{-1} \mathcal{F}_i$. Then

$$\text{colim}_{i \in I} R^p g_{i,*} \mathcal{F}_i = R^p g_* \mathcal{F}$$

for all $p \geq 0$.

Proof. Recall (Lemma 44.52.4) that $R^p g_{i,*} \mathcal{F}_i$ is the sheaf associated to the presheaf $U \mapsto H^p_{étale}(U \times_S X_i, \mathcal{F}_i)$ and similarly for $R^p g_* \mathcal{F}$. Moreover, the colimit of a system of sheaves is the sheafification of the colimit on the level of presheaves. Note that every object of $S_{étale}$ has a covering by quasi-compact and quasi-separated objects (e.g., affine schemes). Moreover, if U is a quasi-compact and quasi-separated object, then we have

$$\text{colim } H^p_{étale}(U \times_S X_i, \mathcal{F}_i) = H^p_{étale}(U \times_S X, \mathcal{F})$$

by Theorem 44.52.1. Thus the lemma follows. □

44.53. Stalks of higher direct images

Theorem 44.53.1. *Let $f : X \to S$ be a quasi-compact and quasi-separated morphism of schemes, $\mathcal{F}$ an abelian sheaf on $X_{étale}$, and $\overline{s}$ a geometric point of S. Then*

$$(R^p f_* \mathcal{F})_{\overline{s}} = H^p_{étale}(X \times_S \text{Spec}(\mathcal{O}^{sh}_{S,\overline{s}}), p^{-1} \mathcal{F})$$

where $p : X \times_S \text{Spec}(\mathcal{O}^{sh}_{S,\overline{s}}) \to X$ is the projection.

Proof. Let $\mathcal{I}$ be the category of étale neighborhoods of $\overline{s}$ on S. By Lemma 44.52.4 we have

$$(R^p f_* \mathcal{F})_{\overline{s}} = \text{colim}_{(V, \overline{v}) \in \mathcal{I}^{opp}} H^p(X \times_S V, \mathcal{F}|_{X \times_S V}).$$

We may replace $\mathcal{I}$ by the initial subcategory consisting of affine étale neighbourhoods of $\overline{s}$. Observe that

$$\text{Spec}(\mathcal{O}^{sh}_{S,\overline{s}}) = \lim_{(V, \overline{v}) \in \mathcal{I}} V$$

by Lemma 44.33.1 and Limits, Lemma 31.2.1. Since fibre products commute with limits we also obtain

$$X \times_S \mathrm{Spec}(\mathcal{O}^{sh}_{S,\overline{s}}) = \lim_{(V,\overline{v})\in\mathcal{I}} X \times_S V$$

We conclude by Lemma 44.52.3. □

44.54. The Leray spectral sequence

Lemma 44.54.1. *Let $f : X \to Y$ be a morphism and $\mathcal{I}$ an injective object of $Ab(X_{étale})$. Let $V \in \mathrm{Ob}(Y_{étale})$. Then*

1. *for any covering $\mathcal{V} = \{V_j \to V\}_{j\in J}$ we have $\check{H}^p(\mathcal{V}, f_*\mathcal{I}) = 0$ for all $p > 0$,*
2. *$f_*\mathcal{I}$ is acyclic for the functor $\Gamma(V, -)$, and*
3. *if $g : Y \to Z$, then $f_*\mathcal{I}$ is acyclic for g_*.*

Proof. Observe that $\check{\mathcal{C}}^\bullet(\mathcal{V}, f_*\mathcal{I}) = \check{\mathcal{C}}^\bullet(\mathcal{V} \times_Y X, \mathcal{I})$ which has vanishing higher cohomology groups by Lemma 44.18.7. This proves (1). The second statement follows as a sheaf which has vanishing higher Čech cohomology groups for any covering has vanishing higher cohomology groups. This a wonderful exercise in using the Čech-to-cohomology spectral sequence, but see Cohomology on Sites, Lemma 21.11.8 for details and a more precise and general statement. Part (3) is a consequence of (2) and the description of $R^p g_*$ in Lemma 44.52.4. □

Using the formalism of Grothendieck spectral sequences, this gives the following.

Proposition 44.54.2 (Leray spectral sequence)**.** *Let $f : X \to Y$ be a morphism of schemes and $\mathcal{F}$ an étale sheaf on X. Then there is a spectral sequence*

$$E_2^{p,q} = H^p_{étale}(Y, R^q f_*\mathcal{F}) \Rightarrow H^{p+q}_{étale}(X, \mathcal{F}).$$

Proof. See Lemma 44.54.1 and see Derived Categories, Section 13.21. □

44.55. Vanishing of finite higher direct images

The next goal is to prove that the higher direct images of a finite morphism of schemes vanish.

Lemma 44.55.1. *Let R be a strictly henselian local ring. Set $S = \mathrm{Spec}(R)$ and let $\overline{s}$ be its closed point. Then the global sections functor $\Gamma(S, -) : Ab(S_{étale}) \to Ab$ is exact. In fact we have $\Gamma(S, \mathcal{F}) = \mathcal{F}_{\overline{s}}$ for any sheaf of sets $\mathcal{F}$. In particular*

$$\forall p \geq 1, \quad H^p_{étale}(S, \mathcal{F}) = 0$$

for all $\mathcal{F} \in Ab(S_{étale})$.

Proof. If we show that $\Gamma(S, \mathcal{F}) = \mathcal{F}_{\overline{s}}$ the $\Gamma(S, -)$ is exact as the stalk functor is exact. Let $(U, \overline{u})$ be an étale neighbourhood of $\overline{s}$. Pick an affine open neighborhood $\mathrm{Spec}(A)$ of $\overline{u}$ in U. Then $R \to A$ is étale and $\kappa(\overline{s}) = \kappa(\overline{u})$. By Theorem 44.32.4 we see that $A \cong R \times A'$ as an R-algebra compatible with maps to $\kappa(\overline{s}) = \kappa(\overline{u})$. Hence we get a section

$$\begin{array}{ccc} \mathrm{Spec}(A) & \longrightarrow & U \\ & \nwarrow & \downarrow \\ & & S \end{array}$$

It follows that in the system of étale neighbourhoods of $\overline{s}$ the identity map $(S, \overline{s}) \to (S, \overline{s})$ is cofinal. Hence $\Gamma(S, \mathcal{F}) = \mathcal{F}_{\overline{s}}$. The final statement of the lemma follows

as the higher derived functors of an exact functor are zero, see Derived Categories, Lemma 13.16.8. □

Proposition 44.55.2. *Let $f : X \to Y$ be a finite morphism of schemes.*

(1) *For any geometric point $\overline{y} : \mathrm{Spec}(k) \to Y$ we have*

$$(f_*\mathcal{F})_{\overline{y}} = \prod\nolimits_{\overline{x}:\mathrm{Spec}(k)\to X,\ f(\overline{x})=\overline{y}} \mathcal{F}_{\overline{x}}.$$

for $\mathcal{F}$ in $Sh(X_{étale})$ and

$$(f_*\mathcal{F})_{\overline{y}} = \bigoplus\nolimits_{\overline{x}:\mathrm{Spec}(k)\to X,\ f(\overline{x})=\overline{y}} \mathcal{F}_{\overline{x}}.$$

for $\mathcal{F}$ in $Ab(X_{étale})$.

(2) *For any $q \geq 1$ we have $R^q f_*\mathcal{F} = 0$.*

Proof. Let $X^{sh}_{\overline{y}}$ denote the fiber product $X \times_Y \mathrm{Spec}(\mathcal{O}^{sh}_{Y,\overline{y}})$. By Theorem 44.53.1 the stalk of $R^q f_*\mathcal{F}$ at $\overline{y}$ is computed by $H^q_{étale}(X^{sh}_{\overline{y}}, \mathcal{F})$. Since f is finite, $X^{sh}_{\overline{y}}$ is finite over $\mathrm{Spec}(\mathcal{O}^{sh}_{Y,\overline{y}})$, thus $X^{sh}_{\overline{y}} = \mathrm{Spec}(A)$ for some ring A finite over $\mathcal{O}^{sh}_{Y,\overline{y}}$. Since the latter is strictly henselian, Lemma 44.32.5 implies that A is a finite product of henselian local rings $A = A_1 \times \cdots \times A_r$. Since the residue field of $\mathcal{O}^{sh}_{Y,\overline{y}}$ is separably closed the same is true for each A_i. Hence A_i is strictly henselian. This implies that $X^{sh}_{\overline{y}} = \coprod_{i=1}^r \mathrm{Spec}(A_i)$. The vanishing of Lemma 44.55.1 implies that $(R^q f_*\mathcal{F})_{\overline{y}} = 0$ for $q > 0$ which implies (2) by Theorem 44.29.10. Part (1) follows from the corresponding statement of Lemma 44.55.1. □

Lemma 44.55.3. *Consider a cartesian square*

$$\begin{array}{ccc} X' & \xrightarrow{g'} & X \\ {\scriptstyle f'}\downarrow & & \downarrow{\scriptstyle f} \\ Y' & \xrightarrow{g} & Y \end{array}$$

of schemes with f a finite morphism. For any sheaf of sets $\mathcal{F}$ on $X_{étale}$ we have $f'_*(g')^{-1}\mathcal{F} = g^{-1}f_*\mathcal{F}$.

Proof. In great generality there is a pullback map $g^{-1}f_*\mathcal{F} \to f'_*(g')^{-1}\mathcal{F}$, see Sites, Section 7.44. To check this map is an isomorphism it suffices to check on stalks (Theorem 44.29.10). This is clear from the description of stalks in Proposition 44.55.2 and Lemma 44.36.2. □

The following lemma is a case of cohomological descent dealing with étale sheaves and finite surjective morphisms. We will significantly generalize this result once we prove the proper base change theorem.

Lemma 44.55.4. *Let $f : X \to Y$ be a surjective finite morphism of schemes. Set $f_n : X_n \to Y$ equal to the $(n+1)$-fold fibre product of X over Y. For $\mathcal{F} \in Ab(Y_{étale})$ set $\mathcal{F}_n = f_{n,*}f_n^{-1}\mathcal{F}$. There is an exact sequence*

$$0 \to \mathcal{F} \to \mathcal{F}_0 \to \mathcal{F}_1 \to \mathcal{F}_2 \to \ldots$$

on $X_{étale}$. Moreover, there is a spectral sequence

$$E_1^{p,q} = H^q_{étale}(X_p, f_p^{-1}\mathcal{F})$$

converging to $H^{p+q}(Y_{étale}, \mathcal{F})$. This spectral sequence is functorial in $\mathcal{F}$.

Proof. If we prove the first statement of the lemma, then we obtain a spectral sequence with $E_1^{p,q} = H^q_{étale}(Y, \mathcal{F})$ convering to $H^{p+q}(Y_{étale}, \mathcal{F})$, see Derived Categories, Lemma 13.20.3. On the other hand, since $R^i f_{p,*} f_p^{-1}\mathcal{F} = 0$ for $i > 0$ (Proposition 44.55.2) we get

$$H^q_{étale}(X_p, f_p^{-1}\mathcal{F}) = H^q_{étale}(Y, f_{p,*}f_p^{-1}\mathcal{F}) = H^q_{étale}(Y, \mathcal{F}_p)$$

by Proposition 44.54.2 and we get the spectral sequence of the lemma.

To prove the first statement of the lemma, observe that X_n forms a simplicial scheme over Y, see Simplicial, Example 14.3.5. Observe moreover, that for each of the projections $d_j : X_{n+1} \to X_n$ there is a map $d_j^{-1} f_n^{-1}\mathcal{F} \to f_{n+1}^{-1}\mathcal{F}$. These maps induce maps

$$\delta_j : \mathcal{F}_n \to \mathcal{F}_{n+1}$$

for $j = 0, \ldots, n+1$. We use the alternating sum of these maps to define the differentials $\mathcal{F}_n \to \mathcal{F}_{n+1}$. Similarly, there is a canonical augmentation $\mathcal{F} \to \mathcal{F}_0$, namely this is just the canonical map $\mathcal{F} \to f_* f^{-1}\mathcal{F}$. To check that this sequence of sheaves is an exact complex it suffices to check on stalks at geometric points (Theorem 44.29.10). Thus we let $\overline{y} : \mathrm{Spec}(k) \to Y$ be a geometric point. Let $E = \{\overline{x} : \mathrm{Spec}(k) \to X \mid f(\overline{x}) = \overline{y}\}$. Then E is a finite nonempty set and we see that

$$(\mathcal{F}_n)_{\overline{y}} = \bigoplus\nolimits_{e \in E^{n+1}} \mathcal{F}_{\overline{y}}$$

by Proposition 44.55.2 and Lemma 44.36.2. Thus we have to see that given an abelian group M the sequence

$$0 \to M \to \bigoplus\nolimits_{e \in E} M \to \bigoplus\nolimits_{e \in E^2} M \to \ldots$$

is exact. Here the first map is the diagonal map and the map $\bigoplus_{e \in E^{n+1}} M \to \bigoplus_{e \in E^{n+2}} M$ is the alternating sum of the maps induced by the $(n+2)$ projections $E^{n+2} \to E^{n+1}$. This can be shown directly or deduced by applying Simplicial, Lemma 14.24.9 to the map $E \to \{*\}$. □

Remark 44.55.5. In the situation of Lemma 44.55.4 if $\mathcal{G}$ is a sheaf of sets on $Y_{étale}$, then we have

$$\Gamma(Y, \mathcal{G}) = \mathrm{Equalizer}(\ \Gamma(X_0, f_0^{-1}\mathcal{G}) \rightrightarrows \Gamma(X_1, f_1^{-1}\mathcal{G})\ )$$

This is proved in exactly the same way, by showing that the sheaf $\mathcal{G}$ is the equalizer of the two maps $f_{0,*}f_0^{-1}\mathcal{G} \to f_{1,*}f_1^{-1}\mathcal{G}$.

Here is a fun generalization of Lemma 44.55.1.

Lemma 44.55.6. *Let S be a scheme all of whose local rings are strictly henselian. Then for any abelian sheaf $\mathcal{F}$ on $S_{étale}$ we have $H^i(S_{étale}, \mathcal{F}) = H^i(S_{Zar}, \mathcal{F})$.*

Proof. Let $\epsilon : S_{étale} \to S_{Zar}$ be the morphism of sites given by the inclusion functor. The Zariski sheaf $R^p\epsilon_*\mathcal{F}$ is the sheaf associated to the presheaf $U \mapsto H^p_{étale}(U, \mathcal{F})$. Thus the stalk at $x \in X$ is $\mathrm{colim}\, H^p_{étale}(U, \mathcal{F}) = H^p_{étale}(\mathrm{Spec}(\mathcal{O}_{X,x}), \mathcal{G}_x)$ where $\mathcal{G}_x$ denotes the pullback of $\mathcal{F}$ to $\mathrm{Spec}(\mathcal{O}_{X,x})$, see Lemma 44.52.3. Thus the higher direct images of $R^p\epsilon_*\mathcal{F}$ are zero by Lemma 44.55.1 and we conclude by the Leray spectral sequence. □

Lemma 44.55.7. *Let S be an affine scheme such that (1) all points are closed, and (2) all residue fields are separably algebraically closed. Then for any abelian sheaf $\mathcal{F}$ on $S_{étale}$ we have $H^i(S_{étale}, \mathcal{F}) = 0$ for $i > 0$.*

Proof. Condition (1) implies that the underlying topological space of S is profinite, see Algebra, Lemma 10.24.6. Thus the higher cohomology groups of an abelian sheaf on the topological space S (i.e., Zariski cohomology) is trivial, see Topology, Lemma 5.21.3 and Cohomology on Sites, Lemma 21.37.1. The local rings are strictly henselian by Algebra, Lemma 10.143.11. Thus étale cohomology of S is computed by Zariski cohomology by Lemma 44.55.6 and the proof is done. □

44.56. Schemes étale over a point

In this section we describe schemes étale over the spectrum of a field. Before we state the result we introduce the category of G-sets for a topological group G.

Definition 44.56.1. Let G be a topological group. A *G-set*, sometime called a *discrete G-set*, is a set X endowed with a left action $a : G \times X \to X$ such that a is continuous when X is given the discrete topology and $G \times X$ the product topology. A *morphism of G-sets* $f : X \to Y$ is simply any G-equivariant map from X to Y. The category of G-sets is denoted *G-Sets*.

The condition that $a : G \times X \to X$ is continuous signifies simply that the stabilizer of any $x \in X$ is open in G. If G is an abstract group G (i.e., a group but not a topological group) then this agrees with our preceding definition (see for example Sites, Example 7.6.5) provided we endow G with the discrete topology.

Recall that if $K \subset L$ is an infinite Galois extension the Galois group $G = \mathrm{Gal}(L/K)$ comes endowed with a canonical topology. Namely the open subgroups are the subgroups of the form $\mathrm{Gal}(L/K') \subset G$ where K'/K is a finite subextension of L/K. The index of an open subgroup is always finite. We say that G is a profinite (topological) group.

Lemma 44.56.2. *Let K be a field. Let K^{sep} a separable closure of K. Consider the profinite group*

$$G = Aut_{\mathrm{Spec}(K)}(\mathrm{Spec}(K^{sep}))^{opp} = Gal(K^{sep}/K)$$

The functor

$$\begin{array}{ccc} \text{schemes étale over } K & \longrightarrow & G\text{-Sets} \\ X/K & \longmapsto & \mathrm{Mor}_{\mathrm{Spec}(K)}(\mathrm{Spec}(K^{sep}), X) \end{array}$$

is an equivalence of categories.

Proof. A scheme X over K is étale over K if and only if $X \cong \coprod_{i \in I} \mathrm{Spec}(K_i)$ with each K_i a finite separable extension of K. The functor of the lemma associates to X the G-set

$$\coprod\nolimits_i \mathrm{Hom}_K(K_i, K^{sep})$$

with its natural left G-action. Each element has an open stabilizer by definition of the topology on G. Conversely, any G-set S is a disjoint union of its orbits. Say $S = \coprod S_i$. Pick $s_i \in S_i$ and denote $G_i \subset G$ its open stabilizer. By Galois theory the fields $(K^{sep})^{G_i}$ are finite separable field extensions of K, and hence the scheme

$$\coprod\nolimits_i \mathrm{Spec}((K^{sep})^{G_i})$$

is étale over K. This gives an inverse to the functor of the lemma. Some details omitted. □

Remark 44.56.3. Under the correspondence of the lemma, the coverings in the small étale site $\mathrm{Spec}(K)_{étale}$ of K correspond to surjective families of maps in G-*Sets*.

44.57. Galois action on stalks

In this section we define an action of the absolute Galois group of a residue field of a point s of S on the stalk functor at any geometric point lying over s.

Galois action on stalks. Let S be a scheme. Let $\overline{s}$ be a geometric point of S. Let $\sigma \in \mathrm{Aut}(\kappa(\overline{s})/\kappa(s))$. Define an action of σ on the stalk $\mathcal{F}_{\overline{s}}$ of a sheaf $\mathcal{F}$ as follows

$$\begin{array}{ccc} \mathcal{F}_{\overline{s}} & \longrightarrow & \mathcal{F}_{\overline{s}} \\ (U, \overline{u}, t) & \longmapsto & (U, \overline{u} \circ \mathrm{Spec}(\sigma), t). \end{array} \tag{44.57.0.1}$$

where we use the description of elements of the stalk in terms of triples as in the discussion following Definition 44.29.6. This is a left action, since if $\sigma_i \in \mathrm{Aut}(\kappa(\overline{s})/\kappa(s))$ then

$$\begin{aligned} \sigma_1 \cdot (\sigma_2 \cdot (U, \overline{u}, t)) &= \sigma_1 \cdot (U, \overline{u} \circ \mathrm{Spec}(\sigma_2), t) \\ &= (U, \overline{u} \circ \mathrm{Spec}(\sigma_2) \circ \mathrm{Spec}(\sigma_1), t) \\ &= (U, \overline{u} \circ \mathrm{Spec}(\sigma_1 \circ \sigma_2), t) \\ &= (\sigma_1 \circ \sigma_2) \cdot (U, \overline{u}, t) \end{aligned}$$

It is clear that this action is functorial in the sheaf $\mathcal{F}$. We note that we could have defined this action by referring directly to Remark 44.29.8.

Definition 44.57.1. Let S be a scheme. Let $\overline{s}$ be a geometric point lying over the point s of S. Let $\kappa(s) \subset \kappa(s)^{sep} \subset \kappa(\overline{s})$ denote the separable algebraic closure of $\kappa(s)$ in the algebraically closed field $\kappa(\overline{s})$.

1. In this situation the *absolute Galois group* of $\kappa(s)$ is $\mathrm{Gal}(\kappa(s)^{sep}/\kappa(s))$. It is sometimes denoted $\mathrm{Gal}_{\kappa(s)}$.
2. The geometric point $\overline{s}$ is called *algebraic* if $\kappa(s) \subset \kappa(\overline{s})$ is an algebraic closure of $\kappa(s)$.

Example 44.57.2. The geometric point $\mathrm{Spec}(\mathbf{C}) \to \mathrm{Spec}(\mathbf{Q})$ is not algebraic.

Let $\kappa(s) \subset \kappa(s)^{sep} \subset \kappa(\overline{s})$ be as in the definition. Note that as $\kappa(\overline{s})$ is algebraically closed the map

$$\mathrm{Aut}(\kappa(\overline{s})/\kappa(s)) \longrightarrow \mathrm{Gal}(\kappa(s)^{sep}/\kappa(s)) = \mathrm{Gal}_{\kappa(s)}$$

is surjective. Suppose $(U, \overline{u})$ is an étale neighbourhood of $\overline{s}$, and say $\overline{u}$ lies over the point u of U. Since $U \to S$ is étale, the residue field extension $\kappa(s) \subset \kappa(u)$ is finite separable. This implies the following

1. If $\sigma \in \mathrm{Aut}(\kappa(\overline{s})/\kappa(s)^{sep})$ then σ acts trivially on $\mathcal{F}_{\overline{s}}$.
2. More precisely, the action of $\mathrm{Aut}(\kappa(\overline{s})/\kappa(s))$ determines and is determined by an action of the absolute Galois group $\mathrm{Gal}_{\kappa(s)}$ on $\mathcal{F}_{\overline{s}}$.
3. Given $(U, \overline{u}, t)$ representing an element ξ of $\mathcal{F}_{\overline{s}}$ any element of $\mathrm{Gal}(\kappa(s)^{sep}/K)$ acts trivially, where $\kappa(s) \subset K \subset \kappa(s)^{sep}$ is the image of $\overline{u}^\sharp : \kappa(u) \to \kappa(\overline{s})$.

Altogether we see that $\mathcal{F}_{\overline{s}}$ becomes a $\mathrm{Gal}_{\kappa(s)}$-set (see Definition 44.56.1). Hence we may think of the stalk functor as a functor

$$Sh(S_{étale}) \longrightarrow \mathrm{Gal}_{\kappa(s)}\text{-}Sets, \quad \mathcal{F} \longmapsto \mathcal{F}_{\overline{s}}$$

and from now on we usually do think about the stalk functor in this way.

Theorem 44.57.3. *Let $S = \mathrm{Spec}(K)$ with K a field. Let $\overline{s}$ be a geometric point of S. Let $G = Gal_{\kappa(s)}$ denote the absolute Galois group. Then the functor above induces an equivalence of categories*

$$Sh(S_{étale}) \longrightarrow G\text{-}Sets, \quad \mathcal{F} \longmapsto \mathcal{F}_{\overline{s}}.$$

Proof. Let us construct the inverse to this functor. In Lemma 44.56.2 we have seen that given a G-set M there exists an étale morphism $X \to \mathrm{Spec}(K)$ such that $\mathrm{Mor}_K(\mathrm{Spec}(K^{sep}), X)$ is isomorphic to M as a G-set. Consider the sheaf $\mathcal{F}$ on $\mathrm{Spec}(K)_{étale}$ defined by the rule $U \mapsto \mathrm{Mor}_K(U, X)$. This is a sheaf as the étale topology is subcanonical. Then we see that $\mathcal{F}_{\overline{s}} = \mathrm{Mor}_K(\mathrm{Spec}(K^{sep}), X) = M$ as G-sets (details omitted). This gives the inverse of the functor and we win. □

Remark 44.57.4. Another way to state the conclusions of Lemmas 44.56.2 and Theorem 44.57.3 is to say that every sheaf on $\mathrm{Spec}(K)_{étale}$ is representable by a scheme X étale over $\mathrm{Spec}(K)$. This does not mean that every sheaf is representable in the sense of Sites, Definition 7.13.3. The reason is that in our construction of $\mathrm{Spec}(K)_{étale}$ we chose a sufficiently large set of schemes étale over $\mathrm{Spec}(K)$, whereas sheaves on $\mathrm{Spec}(K)_{étale}$ form a proper class.

Lemma 44.57.5. *Assumptions and notations as in Theorem 44.57.3. There is a functorial bijection*

$$\Gamma(S, \mathcal{F}) = (\mathcal{F}_{\overline{s}})^G$$

Proof. We can prove this using formal arguments and the result of Theorem 44.57.3 as follows. Given a sheaf $\mathcal{F}$ corresponding to the G-set $M = \mathcal{F}_{\overline{s}}$ we have

$$\begin{aligned} \Gamma(S, \mathcal{F}) &= \mathrm{Mor}_{Sh(S_{étale})}(h_{\mathrm{Spec}(K)}, \mathcal{F}) \\ &= \mathrm{Mor}_{G\text{-}Sets)}(\{*\}, M) \\ &= M^G \end{aligned}$$

Here the first identification is explained in Sites, Sections 7.2 and 7.13, the second results from Theorem 44.57.3 and the third is clear. We will also give a direct proof[4].

Suppose that $t \in \Gamma(S, \mathcal{F})$ is a global section. Then the triple $(S, \overline{s}, t)$ defines an element of $\mathcal{F}_{\overline{s}}$ which is clearly invariant under the action of G. Conversely, suppose that $(U, \overline{u}, t)$ defines an element of $\mathcal{F}_{\overline{s}}$ which is invariant. Then we may shrink U and assume $U = \mathrm{Spec}(L)$ for some finite separable field extension of K, see Proposition 44.26.2. In this case the map $\mathcal{F}(U) \to \mathcal{F}_{\overline{s}}$ is injective, because for any morphism of étale neighbourhoods $(U', \overline{u}') \to (U, \overline{u})$ the restriction map $\mathcal{F}(U) \to \mathcal{F}(U')$ is injective since $U' \to U$ is a covering of $S_{étale}$. After enlarging L a bit we may assume $K \subset L$ is a finite Galois extension. At this point we use that

$$\mathrm{Spec}(L) \times_{\mathrm{Spec}(K)} \mathrm{Spec}(L) = \coprod\nolimits_{\sigma \in \mathrm{Gal}(L/K)} \mathrm{Spec}(L)$$

[4]For the doubting Thomases out there.

where the maps $\mathrm{Spec}(L) \to \mathrm{Spec}(L \otimes_K L)$ come from the ring maps $a \otimes b \mapsto a\sigma(b)$. Hence we see that the condition that $(U, \overline{u}, t)$ is invariant under all of G implies that $t \in \mathcal{F}(\mathrm{Spec}(L))$ maps to the same element of $\mathcal{F}(\mathrm{Spec}(L) \times_{\mathrm{Spec}(K)} \mathrm{Spec}(L))$ via restriction by either projection (this uses the injectivity mentioned above; details omitted). Hence the sheaf condition of $\mathcal{F}$ for the étale covering $\{\mathrm{Spec}(L) \to \mathrm{Spec}(K)\}$ kicks in and we conclude that t comes from a unique section of $\mathcal{F}$ over $\mathrm{Spec}(K)$. □

Remark 44.57.6. Let S be a scheme and let $\overline{s} : \mathrm{Spec}(k) \to S$ be a geometric point of S. By definition this means that k is algebraically closed. In particular the absolute Galois group of k is trivial. Hence by Theorem 44.57.3 the category of sheaves on $\mathrm{Spec}(k)_{étale}$ is equivalent to the category of sets. The equivalence is given by taking sections over $\mathrm{Spec}(k)$. This finally provides us with an alternative definition of the stalk functor. Namely, the functor

$$Sh(S_{étale}) \longrightarrow Sets, \quad \mathcal{F} \longmapsto \mathcal{F}_{\overline{s}}$$

is isomorphic to the functor

$$Sh(S_{étale}) \longrightarrow Sh(\mathrm{Spec}(k)_{étale}) = Sets, \quad \mathcal{F} \longmapsto \overline{s}^*\mathcal{F}$$

To prove this rigorously one can use Lemma 44.36.2 part (3) with $f = \overline{s}$. Moreover, having said this the general case of Lemma 44.36.2 part (3) follows from functoriality of pullbacks.

44.58. Cohomology of a point

As a consequence of the discussion in the preceding two sections we obtain the equivalence of étale cohomology of the spectrum of a field with Galois cohomology.

Definition 44.58.1. Let G be a topological group. A *G-module*, sometime called a *discrete G-module*, is an abelian group M endowed with a left action $a : G \times M \to M$ by group homomorphisms such that a is continuous when M is given the discrete topology and $G \times M$ the product topology. A *morphism of G-modules* $f : M \to N$ is simply any G-equivariant homomorphism from M to N. The category of G-modules is denoted Mod_G.

The condition that $a : G \times M \to M$ is continuous signifies simply that the stabilizer of any $x \in M$ is open in G. If G is an abstract group G (i.e., a group but not a topological group) then this corresponds to the notion of an abelian group endowed with a G-action provided we endow G with the discrete topology.

Lemma 44.58.2. *Let $S = \mathrm{Spec}(K)$ with K a field. Let $\overline{s}$ be a geometric point of S. Let $G = Gal_{\kappa(s)}$ denote the absolute Galois group. The stalk functor induces an equivalence of categories*

$$Ab(S_{étale}) \longrightarrow Mod_G, \quad \mathcal{F} \longmapsto \mathcal{F}_{\overline{s}}.$$

Proof. In Theorem 44.57.3 we have seen the equivalence between sheaves of sets and G-sets. The current lemma follows formally from this as an abelian sheaf is just a sheaf of sets endowed with a commutative group law, and a G-module is just a G-set endowed with a commutative group law. □

The category Mod_G has enough injectives, see Injectives, Lemma 19.3.1. Consider the left exact functor

$$Mod_G \longrightarrow Ab, \quad M \longmapsto M^G = \{x \in M \mid g \cdot x = x \ \forall g \in G\}$$

We sometimes denote $M^G = H^0(G, M)$ and sometimes we write $M^G = \Gamma_G(M)$. This functor has a total right derived functor $R\Gamma_G(M)$ and ith right derived functor $R^i\Gamma_G(M) = H^i(G, M)$ for any $i \geq 0$.

Definition 44.58.3. Let G be a topological group.

(1) The right derived functors $H^i(G, M)$ are called the *continuous group cohomology groups* of M.
(2) If G is an abstract group endowed with the discrete topology then the $H^i(G, M)$ are called the *group cohomology groups* of M.
(3) If G is a Galois group, then the groups $H^i(G, M)$ are called the *Galois cohomology groups* of M.
(4) If G is the absolute Galois group of a field K, then the groups $H^i(G, M)$ are sometimes called the *Galois cohomology groups of K with coefficients in M*.

Lemma 44.58.4. *Notation and assumptions as in Lemma 44.58.2. Let $\mathcal{F}$ be an abelian sheaf on $\mathrm{Spec}(K)_{étale}$ which corresponds to the G-module M. Then*

(1) *in $D(Ab)$ we have a canonical isomorphism $R\Gamma(S, \mathcal{F}) = R\Gamma_G(M)$,*
(2) *$H^0_{étale}(S, \mathcal{F}) = M^G$, and*
(3) *$H^q_{étale}(S, \mathcal{F}) = H^q(G, M)$.*

Proof. Combine Lemma 44.58.2 with Lemma 44.57.5. □

Example 44.58.5. Sheaves on $\mathrm{Spec}(K)_{étale}$. Let $G = \mathrm{Gal}(K^{sep}/K)$ be the absolute Galois group of K.

(1) The constant sheaf $\underline{\mathbf{Z}/n\mathbf{Z}}$ corresponds to the module $\mathbf{Z}/n\mathbf{Z}$ with trivial G-action,
(2) the sheaf $\mathbf{G}_m|_{\mathrm{Spec}(K)_{étale}}$ corresponds to $(K^{sep})^*$ with its G-action,
(3) the sheaf $\mathbf{G}_a|_{\mathrm{Spec}(K^{sep})}$ corresponds to $(K^{sep}, +)$ with its G-action, and
(4) the sheaf $\mu_n|_{\mathrm{Spec}(K^{sep})}$ corresponds to $\mu_n(K^{sep})$ with its G-action.

By Remark 44.23.4 and Theorem 44.24.1 we have the following identifications for cohomology groups:

$$H^0_{étale}(S_{étale}, \mathbf{G}_m) = \Gamma(S, \mathcal{O}_S^*)$$
$$H^1_{étale}(S_{étale}, \mathbf{G}_m) = H^1_{Zar}(S, \mathcal{O}_S^*) = \mathrm{Pic}(S)$$
$$H^i_{étale}(S_{étale}, \mathbf{G}_a) = H^i_{Zar}(S, \mathcal{O}_S)$$

Also, for any quasi-coherent sheaf $\mathcal{F}$ on $S_{étale}$ we have

$$H^i(S_{étale}, \mathcal{F}) = H^i_{Zar}(S, \mathcal{F}),$$

see Theorem 44.22.4. In particular, this gives the following sequence of equalities

$$0 = \mathrm{Pic}(\mathrm{Spec}(K)) = H^1_{étale}(\mathrm{Spec}(K)_{étale}, \mathbf{G}_m) = H^1(G, (K^{sep})^*)$$

which is none other than Hilbert's 90 theorem. Similarly, for $i \geq 1$,

$$0 = H^i(\mathrm{Spec}(K), \mathcal{O}) = H^i_{étale}(\mathrm{Spec}(K)_{étale}, \mathbf{G}_a) = H^i(G, K^{sep})$$

where the K^{sep} indicates K^{sep} as a Galois module with addition as group law. In this way we may consider the work we have done so far as a complicated way of computing Galois cohomology groups.

44.59. Cohomology of curves

The next task at hand is to compute the étale cohomology of a smooth curve with torsion coefficients, and in particular show that it vanishes in degree at least 3. To prove this, we will compute cohomology at the generic point, which amounts to some Galois cohomology. We now review without proofs. the relevant facts about Brauer groups. For references, see [**Ser62**], [**Ser97**] or [**Wei48**].

44.60. Brauer groups

Brauer groups of fields, defined using finite central simple algebras, are discussed in the chapter Brauer Groups, Section 11.1. Here we give a synopsis.

Theorem 44.60.1. *Let K be a field. For a unital, associative (not necessarily commutative) K-algebra A the following are equivalent*

1. *A is finite central simple K-algebra,*
2. *A is a finite dimensional K-vector space, K is the center of A, and A has no nontrivial two-sided ideal,*
3. *there exists $d \geq 1$ such that $A \otimes_K \bar{K} \cong Mat(d \times d, \bar{K})$,*
4. *there exists $d \geq 1$ such that $A \otimes_K K^{sep} \cong Mat(d \times d, K^{sep})$,*
5. *there exist $d \geq 1$ and a finite Galois extension $K \subset K'$ such that $A \otimes_{K'} K' \cong Mat(d \times d, K')$,*
6. *there exist $n \geq 1$ and a finite central skew field D over K such that $A \cong Mat(n \times n, D)$.*

The integer d is called the degree of A.

Proof. This is a copy of Brauer Groups, Lemma 11.8.6. □

Lemma 44.60.2. *Let A be a finite central simple algebra over K. Then*

$$\begin{array}{ccc} A \otimes_K A^{opp} & \longrightarrow & End_K(A) \\ a \otimes a' & \longmapsto & (x \mapsto axa') \end{array}$$

is an isomorphism of algebras over K.

Proof. See Brauer Groups, Lemma 11.4.10. □

Definition 44.60.3. Two finite central simple algebras A_1 and A_2 over K are called *similar*, or *equivalent* if there exist $m, n \geq 1$ such that $\mathrm{Mat}(n \times n, A_1) \cong \mathrm{Mat}(m \times m, A_2)$. We write $A_1 \sim A_2$.

Definition 44.60.4. Let K be a field. The *Brauer group* of K is the set $\mathrm{Br}(K)$ of similarity classes of finite central simple algebras over K, endowed with the group law induced by tensor product (over K). The class of A in $\mathrm{Br}(K)$ is denoted by $[A]$. The neutral element is $[K] = [\mathrm{Mat}(d \times d, K)]$ for any $d \geq 1$.

The previous lemma thus mean that inverses exist, and that $-[A] = [A^{opp}]$. The Brauer group is always torsion, but not finitely generated in general. We will see that $A^{\otimes \deg A} \sim K$ for any finite central simple algebra A (insert future reference here).

Lemma 44.60.5. *Let K be a field and $\mathcal{G} = Gal(K^{sep}|K))$. Then the set of isomorphism classes of central simple algebras of degree d over K is in bijection with the non-abelian cohomology $H^1_{cont}(\mathcal{G}, PGL_d(K^{sep}))$.*

Sketch of proof. The Skolem-Noether theorem (see Brauer Groups, Theorem 11.6.1) implies that for any field L the group $\mathrm{Aut}_{L\text{-Algebras}}(\mathrm{Mat}_d(L))$ equals $\mathrm{PGL}_d(L)$. By Theorem 44.60.1, we see that central simple algebras of degree d correspond to forms of the K-algebra $\mathrm{Mat}_d(K)$, which in turn correspond to $H^1_{cont}(\mathcal{G}, \mathrm{PGL}_d(K^{sep}))$. For more details on twisting, see for example [**Sil86**]. □

If A is a finite central simple algebra over K, we denote ξ_A the corresponding cohomology class in $H^1_{cont}(\mathcal{G}, \mathrm{PGL}_{\deg A}(K^{sep}))$. Consider now the short exact sequence

$$1 \to (K^{sep})^* \to \mathrm{GL}_d(K^{sep}) \to \mathrm{PGL}_d(K^{sep}) \to 1,$$

which gives rise to a long exact cohomology sequence (up to degree 2) with coboundary map

$$\delta_d : H^1_{cont}(\mathcal{G}, \mathrm{PGL}_d(K^{sep})) \to H^2(\mathcal{G}, (K^{sep})^*).$$

Explicitly, this is given as follows: if ξ is a cohomology class represented by the 1-cocycle (g_σ), then $\delta_d(\xi)$ is the class of the 2-cocycle $((g_\sigma^\tau)^{-1} g_{\sigma\tau} g_\tau^{-1})$.

Theorem 44.60.6. *The map*

$$\begin{array}{rccc} \delta : & Br(K) & \longrightarrow & H^2(\mathcal{G}, (K^{sep})^*) \\ & [A] & \longmapsto & \delta_{\deg A}(\xi_A) \end{array}$$

is a group isomorphism.

Proof. Omitted. Hints: In the abelian case ($d = 1$), one has the identification

$$H^1(\mathcal{G}, \mathrm{GL}_d(K^{sep})) = H^1_{étale}(\mathrm{Spec}(K), \mathrm{GL}_d(\mathcal{O}))$$

the latter of which is trivial by fpqc descent. If this were true in the non-abelian case, this would readily imply injectivity of δ. (See [**Del77**].) Rather, to prove this, one can reinterpret $\delta([A])$ as the obstruction to the existence of a K-vector space V with a left A-module structure and such that $\dim_K V = \deg A$. In the case where V exists, one has $A \cong \mathrm{End}_K(V)$. For surjectivity, pick a cohomology class $\xi \in H^2(\mathcal{G}, (K^{sep})^*)$, then there exists a finite Galois extension $K \subset K' \subset K^{sep}$ such that ξ is the image of some $\xi' \in H^2_{cont}(\mathrm{Gal}(K'|K), (K')^*)$. Then write down an explicit central simple algebra over K using the data K', ξ'. □

The Brauer group of a scheme. Let S be a scheme. An $\mathcal{O}_S$-algebra $\mathcal{A}$ is called *Azumaya* if it is étale locally a matrix algebra, i.e., if there exists an étale covering $\mathcal{U} = \{\varphi_i : \mathcal{U}_i \to S\}_{i \in I}$ such that $\varphi_i^* \mathcal{A} \cong \mathrm{Mat}_{d_i}(\mathcal{O}_{\mathcal{U}_i})$ for some $d_i \geq 1$. Two such $\mathcal{A}$ and $\mathcal{B}$ are called *equivalent* if there exist finite locally free $\mathcal{O}_S$-modules $\mathcal{F}$ and $\mathcal{G}$ which have positive rank at every $s \in S$ such that

$$\mathcal{A} \otimes_{\mathcal{O}_S} \mathrm{End}(\mathcal{F}) \cong \mathcal{B} \otimes_{\mathcal{O}_S} \mathrm{End}(\mathcal{G})$$

as $\mathcal{O}_S$-algebras. The *Brauer group* of S is the set $\mathrm{Br}(S)$ of equivalence classes of Azumaya $\mathcal{O}_S$-algebras with the operation induced by tensor product (over $\mathcal{O}_S$).

In this setting, the analogue of the isomorphism δ of Theorem 44.60.6 is a map

$$\delta_S : \mathrm{Br}(S) \to H^2_{étale}(S, \mathbf{G}_m).$$

It is true that δ_S is injective (the previous argument still works). If S is quasi-compact or connected, then $\mathrm{Br}(S)$ is a torsion group, so in this case the image of δ_S is contained in the *cohomological Brauer group* of S

$$\mathrm{Br}'(S) := H^2_{\acute{e}tale}(S, \mathbf{G}_m)_{\mathrm{torsion}}.$$

So if S is quasi-compact or connected, there is an inclusion $\mathrm{Br}(S) \subset \mathrm{Br}'(S)$. This is not always an equality: there exists a nonseparated singular surface S for which $\mathrm{Br}(S) \subset \mathrm{Br}'(S)$ is a strict inclusion. If S is quasi-projective, then $\mathrm{Br}(S) = \mathrm{Br}'(S)$. However, it is not known whether this holds for a smooth proper variety over $\mathbf{C}$, say.

Proposition 44.60.7. *Let K be a field, $\mathcal{G} = Gal(K^{sep}|K)$ and suppose that for any finite extension K' of K, $Br(K') = 0$. Then*

1. *for all $q \geq 1$, $H^q(\mathcal{G}, (K^{sep})^*) = 0$; and*
2. *for any torsion $\mathcal{G}$-module M and any $q \geq 2$, $H^q_{cont}(\mathcal{G}, M) = 0$.*

Proof. See [**Ser97**] for proofs. □

Definition 44.60.8. A field K is called C_r if for every $0 < d^r < n$ and every $f \in K[T_1, \ldots, T_n]$ homogeneous of degree d, there exist $\alpha = (\alpha_1, \ldots, \alpha_n)$, $\alpha_i \in K$ not all zero, such that $f(\alpha) = 0$. Such an α is called a *nontrivial solution* of f.

Example 44.60.9. An algebraically closed field is C_r.

In fact, we have the following simple lemma.

Lemma 44.60.10. *Let k be an algebraically closed field. Let $f_1, \ldots, f_s \in k[T_1, \ldots, T_n]$ be homogeneous polynomials of degree $d_1, \ldots, d_s$ with $d_i > 0$. If $s < n$, then $f_1 = \ldots = f_s = 0$ have a common nontrivial solution.*

Proof. Omitted. □

The following result computes the Brauer group of C_1 fields.

Theorem 44.60.11. *Let K be a C_1 field. Then $Br(K) = 0$.*

Proof. Let D be a finite dimensional division algebra over K with center K. We have seen that

$$D \otimes_K K^{sep} \cong \mathrm{Mat}_d(K^{sep})$$

uniquely up to inner isomorphism. Hence the determinant $\det : \mathrm{Mat}_d(K^{sep}) \to K^{sep}$ is Galois invariant and descends to a homogeneous degree d map

$$\det = N_{\mathrm{red}} : D \longrightarrow K$$

called the *reduced norm*. Since K is C_1, if $d > 1$, then there exists a nonzero $x \in D$ with $N_{\mathrm{red}}(x) = 0$. This clearly implies that x is not invertible, which is a contradiction. Hence $\mathrm{Br}(K) = 0$. □

Theorem 44.60.12 (Tsen's theorem)**.** *The function field of a variety of dimension r over an algebraically closed field k is C_r.*

Proof. For projective space one can show directly that the field $k(x_1, \ldots, x_r)$ is C_r (exercise).

General case. Without loss of generality, we may assume X to be projective. Let $f \in K[T_1, \ldots, T_n]_d$ with $0 < d^r < n$. Say the coefficients of f are in $\Gamma(X, \mathcal{O}_X(H))$ for some ample $H \subset X$. Let $\alpha = (\alpha_1, \ldots, \alpha_n)$ with $\alpha_i \in \Gamma(X, \mathcal{O}_X(eH))$. Then

$f(\alpha) \in \Gamma(X, \mathcal{O}_X((de+1)H))$. Consider the system of equations $f(\alpha) = 0$. Then by asymptotic Riemann-Roch,

- the number of variables is $n \dim_K \Gamma(X, \mathcal{O}_X(eH)) \sim n\frac{e^r}{r!}(H^r)$; and
- the number of equations is $\dim_K \Gamma(X, \mathcal{O}_X((de+1)H)) \sim \frac{(de+1)^r}{r!}(H^r)$.

Since $n > d^r$, there are more variables than equations, and since there is a trivial solution, there are also nontrivial solutions. □

Definition 44.60.13. We call *variety* a separated, geometrically irreducible and geometrically reduced scheme of finite type over a field, and *curve* a variety of dimension 1.

Lemma 44.60.14. *Let C be a curve over an algebraically closed field k. Then the Brauer group of the function field of C is zero: $Br(k(C)) = 0$.*

Proof. This is clear from Tsen's theorem, Theorem 44.60.12. □

Lemma 44.60.15. *Let k be an algebraically closed field and $k \subset K$ a field extension of transcendence degree 1. Then for all $q \geq 1$, $H^q_{étale}(\mathrm{Spec}(K), \mathbf{G}_m) = 0$.*

Proof. It suffices to show that if $K \subset K'$ is a finite field extension, then $\mathrm{Br}(K') = 0$. Now observe that $K' = \mathrm{colim}\, K''$, where K'' runs over the finitely generated subextensions of k contained in K' of transcendence degree 1. By some result in [**Har77**], each K'' is the function field of a curve, hence has trivial Brauer group by Lemma 44.60.14. It now suffices to observe that $\mathrm{Br}(K') = \mathrm{colim}\, \mathrm{Br}(K'')$. □

44.61. Higher vanishing for the multiplicative group

In this section, we fix an algebraically closed field k and a smooth curve X over k. We denote $i_x : x \hookrightarrow X$ the inclusion of a closed point of X and $j : \eta \hookrightarrow X$ the inclusion of the generic point. We also denote X^0 the set of closed points of X.

Theorem 44.61.1 (The Fundamental Exact Sequence)**.** *There is a short exact sequence of étale sheaves on X*

$$0 \longrightarrow \mathbf{G}_{m,X} \longrightarrow j_*\mathbf{G}_{m,\eta} \xrightarrow{\div} \bigoplus_{x \in X^0} i_{x*}\underline{\mathbf{Z}} \longrightarrow 0.$$

Proof. Let $\varphi : \mathcal{U} \to X$ be an étale morphism. Then by properties v and vi of étale morphisms (Proposition 44.26.2), $\mathcal{U} = \coprod_i \mathcal{U}_i$ where each $\mathcal{U}_i$ is a smooth curve mapping to X. The above sequence for X is a product of the corresponding sequences for each $\mathcal{U}_i$, so it suffices to treat the case where $\mathcal{U}$ is connected, hence irreducible. In this case, there is a well known exact sequence (see [**Har77**])

$$1 \longrightarrow \Gamma(\mathcal{U}, \mathcal{O}^*_{\mathcal{U}}) \longrightarrow k(\mathcal{U})^* \xrightarrow{\div} \bigoplus_{y \in \mathcal{U}^0} \mathbf{Z}_y.$$

This amounts to a sequence

$$\Gamma(\mathcal{U}, \mathcal{O}^*_{\mathcal{U}}) \longrightarrow \Gamma(\eta \times_X \mathcal{U}, \mathcal{O}^*_{\eta \times_X \mathcal{U}}) \xrightarrow{\div} \bigoplus_{x \in X^0} \Gamma(x \times_X \mathcal{U}, \underline{\mathbf{Z}})$$

which, unfolding definitions, is nothing but a sequence

$$\mathbf{G}_m(\mathcal{U}) \longrightarrow j_*\mathbf{G}_{m,\eta}(\mathcal{U}) \xrightarrow{\div} \bigoplus_{x \in X^0} i_{x*}\underline{\mathbf{Z}}(\mathcal{U}).$$

This defines the maps in the Fundamental Exact Sequence and shows it is exact except possibly at the last step. To see surjectivity, let us recall (from [**Har77**] again) that if C is a nonsingular curve and D is a divisor on C, then there exists a Zariski open covering $\{\mathcal{V}_j \to C\}$ of C such that $D|_{\mathcal{V}_j} = \div(f_j)$ for some $f_j \in k(C)^*$. □

Lemma 44.61.2. *For any $q \geq 1$, $R^q j_* \mathbf{G}_{m,\eta} = 0$.*

Proof. We need to show that $(R^q j_* \mathbf{G}_{m,\eta})_{\bar{x}} = 0$ for every geometric point $\bar{x}$ of X.

(1) Assume that $\bar{x}$ lies over a closed point x of X. Let $\mathrm{Spec}(A)$ be an open neighborhood of x in X, and K the fraction field of A, so that
$$\mathrm{Spec}(\mathcal{O}^{sh}_{X,\bar{x}}) \times_X \eta = \mathrm{Spec}(\mathcal{O}^{sh}_{X,\bar{x}} \otimes_A K).$$
The ring $\mathcal{O}^{sh}_{X,\bar{x}} \otimes_A K$ is a localization of the discrete valuation ring $\mathcal{O}^{sh}_{X,\bar{x}}$, so it is either $\mathcal{O}^{sh}_{X,\bar{x}}$ again, or its fraction field $K^{sh}_{\bar{x}}$. But since some local uniformizer gets inverted, it must be the latter. Hence
$$(R^q j_* \mathbf{G}_{m,\eta})_{(X,\bar{x})} = H^q_{étale}(\mathrm{Spec}\, K^{sh}_{\bar{x}}, \mathbf{G}_m).$$
Now recall that $\mathcal{O}^{sh}_{X,\bar{x}} = \mathrm{colim}_{(\mathcal{U},\bar{u})\to\bar{x}} \mathcal{O}(\mathcal{U}) = \mathrm{colim}_{A\subset B} B$ where $A \to B$ is étale, hence $K^{sh}_{\bar{x}}$ is an algebraic extension of $k(X)$, and we may apply Lemma 44.60.15 to get the vanishing.

(2) Assume that $\bar{x} = \bar{\eta}$ lies over the generic point η of X (in fact, this case is superfluous). Then $\mathcal{O}_{X,\bar{\eta}} = \kappa(\eta)^{sep}$ and thus
$$\begin{aligned}(R^q j_* \mathbf{G}_{m,\eta})_{\bar{\eta}} &= H^q_{étale}(\mathrm{Spec}(\kappa(\eta)^{sep}) \times_X \eta, \mathbf{G}_m) \\ &= H^q_{étale}(\mathrm{Spec}(\kappa(\eta)^{sep}), \mathbf{G}_m) \\ &= 0 \quad \text{for } q \geq 1\end{aligned}$$
since the corresponding Galois group is trivial.

□

Lemma 44.61.3. *For all $p \geq 1$, $H^p_{étale}(X, j_* \mathbf{G}_{m,\eta}) = 0$.*

Proof. The Leray spectral sequence reads
$$E_2^{p,q} = H^p_{étale}(X, R^q j_* \mathbf{G}_{m,\eta}) \Rightarrow H^{p+q}_{étale}(\eta, \mathbf{G}_{m,\eta}),$$
which vanishes for $p + q \geq 1$ by Lemma 44.60.15. Taking $q = 0$, we get the desired vanishing. □

Lemma 44.61.4. *For all $q \geq 1$, $H^q_{étale}(X, \bigoplus_{x\in X^0} i_{x*}\underline{\mathbf{Z}}) = 0$.*

Proof. For X quasi-compact and quasi-separated, cohomology commutes with colimits, so it suffices to show the vanishing of $H^q_{étale}(X, i_{x*}\underline{\mathbf{Z}})$. But then the inclusion i_x of a closed point is finite so $R^p i_{x*}\underline{\mathbf{Z}} = 0$ for all $p \geq 1$ by Proposition 44.55.2. Applying the Leray spectral sequence, we see that $H^q_{étale}(X, i_{x*}\underline{\mathbf{Z}}) = H^q_{étale}(x, \underline{\mathbf{Z}})$. Finally, since x is the spectrum of an algebraically closed field, all higher cohomology on x vanishes. □

Concluding this series of lemmata, we get the following result.

Theorem 44.61.5. *Let X be a smooth curve over an algebraically closed field. Then*
$$H^q_{étale}(X, \mathbf{G}_m) = 0 \quad \textit{for all } q \geq 2.$$

Proof. See discussion above. □

We also get the cohomology long exact sequence

$$0 \to H^0_{étale}(X, \mathbf{G}_m) \to H^0_{étale}(X, j_*\mathbf{G}_{m\eta}) \xrightarrow{\div} H^0_{étale}(X, \bigoplus i_{x*}\underline{\mathbf{Z}}) \to H^1_{étale}(X, \mathbf{G}_m) \to 0$$

although this is the familiar

$$0 \to H^0_{Zar}(X, \mathcal{O}_X^*) \to k(X)^* \xrightarrow{\div} \mathrm{Div}(X) \to \mathrm{Pic}(X) \to 0.$$

We would like to use the Kummer sequence to deduce some information about the cohomology group of a curve with finite coefficients. In order to get vanishing in the long exact sequence, we review some facts about Picard groups.

44.62. Picard groups of curves

Let X be a smooth projective curve over an algebraically closed field k. There exists a short exact sequence

$$0 \to \mathrm{Pic}^0(X) \to \mathrm{Pic}(X) \xrightarrow{\deg} \mathbf{Z} \to 0.$$

The abelian group $\mathrm{Pic}^0(X)$ can be identified with $\mathrm{Pic}^0(X) = \underline{\mathrm{Pic}}^0_{X/k}(k)$, i.e., the k-valued points of an abelian variety $\underline{\mathrm{Pic}}^0_{X/k}$ of dimension $g = g(X)$ over k.

Definition 44.62.1. An *abelian variety* over k is a proper smooth connected group scheme over k (i.e., a proper group variety over k).

Proposition 44.62.2. *Let A be an abelian variety over an algebraically closed field k. Then*

1. *A is projective over k;*
2. *A is a commutative group scheme;*
3. *the morphism $[n] : A \to A$ is surjective for all $n \geq 1$, in other words $A(k)$ is a divisible abelian group;*
4. *$A[n] = Ker(A \xrightarrow{[n]} A)$ is a finite flat group scheme of rank $n^{2\dim A}$ over k. It is reduced if and only if $n \in k^*$;*
5. *if $n \in k^*$ then $A(k)[n] = A[n](k) \cong (\mathbf{Z}/n\mathbf{Z})^{2\dim(A)}$.*

Proof. See Mumford's book on abelian varieties, [**Mum70**]. □

Consequently, if $n \in k^*$ then $\mathrm{Pic}^0(X)[n] \cong (\mathbf{Z}/n\mathbf{Z})^{2g}$ as abelian groups.

Lemma 44.62.3. *Let X be a smooth projective of genus g over an algebraically closed field k and $n \geq 1$, $n \in k^*$. Then there are canonical identifications*

$$H^q_{étale}(X, \mu_n) = \begin{cases} \mu_n(k) & \text{if } q = 0 \ ; \\ Pic^0(X)[n] & \text{if } q = 1 \ ; \\ \mathbf{Z}/n\mathbf{Z} & \text{if } q = 2 \ ; \\ 0 & \text{if } q \geq 3. \end{cases}$$

Since $\mu_n \cong \underline{\mathbf{Z}/n\mathbf{Z}}$, this gives (noncanonical) identifications

$$H^q_{étale}(X, \underline{\mathbf{Z}/n\mathbf{Z}}) \cong \begin{cases} \mathbf{Z}/n\mathbf{Z} & \text{if } q = 0 \ ; \\ (\mathbf{Z}/n\mathbf{Z})^{2g} & \text{if } q = 1 \ ; \\ \mathbf{Z}/n\mathbf{Z} & \text{if } q = 2 \ ; \\ 0 & \text{if } q \geq 3. \end{cases}$$

Proof. The Kummer sequence $0 \to \mu_{n,X} \to \mathbf{G}_{m,X} \xrightarrow{(\cdot)^n} \mathbf{G}_{m,X} \to 0$ give the long exact cohomology sequence

$$\begin{array}{ccccccc} 0 & \longrightarrow & \mu_n(k) & \longrightarrow & k^* & \xrightarrow{(\cdot)^n} & k^* \\ & & & & \searrow & & \\ & & H^1_{étale}(X,\mu_n) & \longrightarrow & \mathrm{Pic}(X) & \xrightarrow{(\cdot)^n} & \mathrm{Pic}(X) \\ & & & & \searrow & & \\ & & H^2_{étale}(X,\mu_n) & \longrightarrow & 0 & \longrightarrow & 0 \cdots \end{array}$$

The n power map $k^* \to k^*$ is surjective since k is algebraically closed. So we need to compute the kernel and cokernel of the map $\mathrm{Pic}(X) \xrightarrow{(\cdot)^n} \mathrm{Pic}(X)$. Consider the commutative diagram with exact rows

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \mathrm{Pic}^0(X) & \longrightarrow & \mathrm{Pic}(X) & \xrightarrow{\deg} & \mathbf{Z} & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle (\cdot)^n} & & \downarrow{\scriptstyle (\cdot)^n} & & \downarrow{\scriptstyle n} & & \\ 0 & \longrightarrow & \mathrm{Pic}^0(X) & \longrightarrow & \mathrm{Pic}(X) & \xrightarrow{\deg} & \mathbf{Z} & \longrightarrow & 0 \end{array}$$

where the left vertical map is surjective by Proposition 44.62.2 (3). Applying the snake lemma gives the desired identifications. □

Lemma 44.62.4. *Let X be an affine smooth curve over an algebraically closed field k and $n \in k^*$. Then*

1. *$H^0_{étale}(X,\mu_n) = \mu_n(k)$;*
2. *$H^1_{étale}(X,\mu_n) \cong (\mathbf{Z}/n\mathbf{Z})^{2g+r-1}$, where r is the number of points in $\bar{X} - X$ for some smooth projective compactification $\bar{X}$ of X ; and*
3. *for all $q \geq 2$, $H^q_{étale}(X,\mu_n) = 0$.*

Proof. Write $X = \bar{X} - \{x_1, \ldots, x_r\}$. Then $\mathrm{Pic}(X) = \mathrm{Pic}(\bar{X})/R$, where R is the subgroup generated by $\mathcal{O}_{\bar{X}}(x_i)$, $1 \leq i \leq r$. Since $r \geq 1$, we see that $\mathrm{Pic}^0(X) \to \mathrm{Pic}(X)$ is surjective, hence $\mathrm{Pic}(X)$ is divisible. Applying the Kummer sequence, we get *i* and *iii*. For *ii*, recall that

$$\begin{aligned} H^1_{étale}(X,\mu_n) &= \left\{(\mathcal{L},\alpha) | \mathcal{L} \in \mathrm{Pic}(X), \alpha : \mathcal{L}^{\otimes n} \cong \mathcal{O}_X\right\} \Big/ \cong \\ &= \left\{(\bar{\mathcal{L}},\ D,\ \bar{\alpha})\right\} / \tilde{R} \end{aligned}$$

where $\bar{\mathcal{L}} \in \mathrm{Pic}^0(\bar{X})$, D is a divisor on $\bar{X}$ supported on $\{x_1, \cdots, x_r\}$ and $\bar{\alpha} : \bar{\mathcal{L}}^{\otimes n} \cong \mathcal{O}_{\bar{X}}(D)$ is an isomorphism. Note that D must have degree 0. Further $\tilde{R}$ is the subgroup of triples of the form $(\mathcal{O}_{\bar{X}}(D'), nD', 1^{\otimes n})$ where D' is supported on $\{x_1, \cdots, x_r\}$ and has degree 0. Thus, we get an exact sequence

$$0 \longrightarrow H^1_{étale}(\bar{X},\mu_n) \longrightarrow H^1_{étale}(X,\mu_n) \longrightarrow \bigoplus_{i=1}^{r} \mathbf{Z}/n\mathbf{Z} \xrightarrow{\sum} \mathbf{Z}/n\mathbf{Z} \longrightarrow 0$$

where the middle map sends the class of a triple $(\bar{\mathcal{L}}, D, \bar{\alpha})$ with $D = \sum_{i=1}^r a_i(x_i)$ to the r-tuple $(a_i)_{i=1}^r$. It now suffices to use Lemma 44.62.3 to count ranks. □

Remark 44.62.5. The "natural" way to prove the previous corollary is to excise X from $\bar{X}$. This is possible, we just haven't developed that theory.

Our main goal is to prove the following result.

Theorem 44.62.6. *Let X be a separated, finite type, dimension 1 scheme over an algebraically closed field k and $\mathcal{F}$ a torsion sheaf on $X_{étale}$. Then*

$$H^q_{étale}(X, \mathcal{F}) = 0, \quad \forall q \geq 3.$$

If X affine then also $H^2_{étale}(X, \mathcal{F}) = 0$.

Proof. See discussion below. □

Recall that an abelian sheaf is called a *torsion sheaf* if all of its stalks are torsion groups. We have computed the cohomology of constant sheaves. We now generalize the latter notion to get all the way to torsion sheaves.

44.63. Extension by zero

The general material in Modules on Sites, Section 18.19 allows us to make the following definition.

Definition 44.63.1. Let $j : U \to X$ be an étale morphism of schemes.

(1) The restriction functor $j^{-1} : Sh(X_{étale}) \to Sh(U_{étale})$ has a left adjoint $j^{Sh}_! : Sh(X_{étale}) \to Sh(U_{étale})$.
(2) The restriction functor $j^{-1} : Ab(X_{étale}) \to Ab(U_{étale})$ has a left adjoint which is denoted $j_! : Ab(U_{étale}) \to Ab(X_{étale})$ and called *extension by zero*.
(3) Let Λ be a ring. The restriction functor $j^{-1} : Mod(X_{étale}, \Lambda) \to Mod(U_{étale}, \Lambda)$ has a left adjoint which is denoted $j_! : Mod(U_{étale}, \Lambda) \to Mod(X_{étale}, \Lambda)$ and called *extension by zero*.

If $\mathcal{F}$ is an abelian sheaf on $X_{étale}$, then $j_!\mathcal{F} \neq j^{Sh}_!\mathcal{F}$ in general. On the other hand $j_!$ for sheaves of Λ-modules agrees with $j_!$ on underlying abelian sheaves (Modules on Sites, Remark 18.19.5). The functor $j_!$ is characterized by the functorial isomorphism

$$\mathrm{Hom}_X(j_!\mathcal{F}, \mathcal{G}) = \mathrm{Hom}_U(\mathcal{F}, j^{-1}\mathcal{G})$$

for all $\mathcal{F} \in Ab(U_{étale})$ and $\mathcal{G} \in Ab(X_{étale})$. Similarly for sheaves of Λ-modules.

To describe it more explicitly, recall that j^{-1} is just the restriction via the functor $U_{étale} \to X_{étale}$. In other words, $j^{-1}\mathcal{G}(U') = \mathcal{G}(U')$ for U' étale over U. For $\mathcal{F} \in Ab(U_{étale})$ we consider the presheaf

$$j^{PSh}_!\mathcal{F} : X_{étale} \longrightarrow Ab, \quad V \longmapsto \bigoplus\nolimits_{V \to U} \mathcal{F}(V)$$

Then $j_!\mathcal{F}$ is the sheafification of $j^{PSh}_!\mathcal{F}$.

Exercise 44.63.2. Prove directly that $j_!$ is left adjoint to j^{-1} and that j_* is right adjoint to j^{-1}.

Proposition 44.63.3. *Let $j : U \to X$ be an étale morphism of schemes. Let $\mathcal{F}$ in $Ab(U_{étale})$. If $\overline{x} : \mathrm{Spec}(k) \to X$ is a geometric point of X, then*

$$(j_!\mathcal{F})_{\overline{x}} = \bigoplus\nolimits_{\overline{u}:\mathrm{Spec}(k)\to U,\ f(\overline{u})=\overline{x}} \mathcal{F}_{\overline{u}}.$$

In particular, $j_!$ is an exact functor.

Proof. Exactness of $j_!$ is very general, see Modules on Sites, Lemma 18.19.3. Of course it does also follow from the description of stalks. The formula for the stalk of $j_!\mathcal{F}$ can be deduced directly from the explicit description of $j_!$ given above. On the other hand, we can deduce it from the very general Modules on Sites, Lemma 18.37.1 and the description of points of the small étale site in terms of geometric points, see Lemma 44.29.12. □

Lemma 44.63.4 (Extension by zero commutes with base change)**.** *Let $f : Y \to X$ be a morphism of schemes. Let $j : V \to X$ be an étale morphism. Consider the fibre product*

$$\begin{array}{ccc} V' = Y \times_X V & \xrightarrow{j'} & Y \\ {\scriptstyle f'}\downarrow & & \downarrow{\scriptstyle f} \\ V & \xrightarrow{j} & X \end{array}$$

Then we have $j'_!f'^{-1} = f^{-1}j_!$ on abelian sheaves and on sheaves of modules.

Proof. This is true because $j'_!f'^{-1}$ is left adjoint to $f'_*(j')^{-1}$ and $f^{-1}j_!$ is left adjoint to $j^{-1}f_*$. Further $f'_*(j')^{-1} = j^{-1}f_*$ because f_* commutes with étale localization (by construction). In fact, the lemma holds very generally in the setting of a morphism of sites, see Modules on Sites, Lemma 18.20.1. □

Lemma 44.63.5. *Let $j : U \to X$ be finite and étale. Then $j_! = j_*$ on abelian sheaves and sheaves of Λ-modules.*

Proof. We prove this in the case of abelian sheaves. By Modules on Sites, Remark 18.19.7 there is a natural transformation $j_! \to j_*$. It suffices to check this is an isomorphism étale locally on X. Thus we may assume $U \to X$ is a finite disjoint union of isomorphisms, see Étale Morphisms, Lemma 40.18.3. We omit the proof in this case. □

Lemma 44.63.6. *Let X be a scheme. Let $Z \subset X$ be a closed subscheme and let $U \subset X$ be the complement. Denote $i : Z \to X$ and $j : U \to X$ the inclusion morphisms. For every abelian sheaf on $X_{étale}$ there is a canonical short exact sequence*

$$0 \to j_!j^{-1}\mathcal{F} \to \mathcal{F} \to i_*i^{-1}\mathcal{F} \to 0$$

on $X_{étale}$.

Proof. We obtain the maps by the adjointness properties of the functors involved. For a geometric point $\overline{x}$ in X we have either $\overline{x} \in U$ in which case the map on the left hand side is an isomorphism on stalks and the stalk of $i_*i^{-1}\mathcal{F}$ is zero or $\overline{x} \in Z$ in which case the map on the right hand side is an isomorphism on stalks and the stalk of $j_!j^{-1}\mathcal{F}$ is zero. Here we have used the description of stalks of Lemma 44.47.3 and Proposition 44.63.3. □

44.64. Locally constant sheaves

This section is the analogue of Modules on Sites, Section 18.42 for the étale site.

Definition 44.64.1. Let X be a scheme. Let $\mathcal{F}$ be a sheaf of sets on $X_{étale}$.

(1) Let E be a set. We say $\mathcal{F}$ is the *constant sheaf with value E* if $\mathcal{F}$ is the sheafification of the presheaf $U \mapsto E$. Notation: $\underline{E}_X$ or $\underline{E}$.
(2) We say $\mathcal{F}$ is a *constant sheaf* if it is isomorphic to a sheaf as in (1).

(3) We say $\mathcal{F}$ is *locally constant* if there exists a covering $\{U_i \to X\}$ such that $\mathcal{F}|_{U_i}$ is a constant sheaf.
(4) We say that $\mathcal{F}$ is *finite locally constant* if it is locally constant and the values are finite sets.

Let $\mathcal{F}$ be a sheaf of abelian groups on $X_{étale}$.

(1) Let A be an abelian group. We say $\mathcal{F}$ is the *constant sheaf with value* A if $\mathcal{F}$ is the sheafification of the presheaf $U \mapsto A$. Notation: $\underline{A}_X$ or $\underline{A}$.
(2) We say $\mathcal{F}$ is a *constant sheaf* if it is isomorphic as an abelian sheaf to a sheaf as in (1).
(3) We say $\mathcal{F}$ is *locally constant* if there exists a covering $\{U_i \to X\}$ such that $\mathcal{F}|_{U_i}$ is a constant sheaf.
(4) We say that $\mathcal{F}$ is *finite locally constant* if it is locally constant and the values are finite abelian groups.

Let Λ be a ring. Let $\mathcal{F}$ be a sheaf of Λ-modules on $X_{étale}$.

(1) Let M be a Λ-module. We say $\mathcal{F}$ is the *constant sheaf with value* M if $\mathcal{F}$ is the sheafification of the presheaf $U \mapsto M$. Notation: $\underline{M}_X$ or $\underline{M}$.
(2) We say $\mathcal{F}$ is a *constant sheaf* if it is isomorphic as a sheaf of Λ-modules to a sheaf as in (1).
(3) We say $\mathcal{F}$ is *locally constant* if there exists a covering $\{U_i \to X\}$ such that $\mathcal{F}|_{U_i}$ is a constant sheaf.

Lemma 44.64.2. *Let $f : X \to Y$ be a morphism of schemes. If $\mathcal{G}$ is a locally constant sheaf of sets, abelian groups, or Λ-modules on $Y_{étale}$, the same is true for $f^{-1}\mathcal{G}$ on $X_{étale}$.*

Proof. Holds for any morphism of topoi, see Modules on Sites, Lemma 18.42.2. □

Lemma 44.64.3. *Let $f : X \to Y$ be a finite étale morphism of schemes. If $\mathcal{F}$ is a (finite) locally constant sheaf of sets, (finite) locally constant sheaf of abelian groups, or (finite type) locally constant sheaf of Λ-modules on $X_{étale}$, the same is true for $f_*\mathcal{F}$ on $Y_{étale}$.*

Proof. The construction of f_* commutes with étale localization. A finite étale morphism is locally isomorphic to a disjoint union of isomorphisms, see Étale Morphisms, Lemma 40.18.3. Thus the lemma says that if $\mathcal{F}_i$, $i = 1, \ldots, n$ are (finite) locally constant sheaves of sets, then $\prod_{i=1,\ldots,n} \mathcal{F}_i$ is too. This is clear. Similarly for sheaves of abelian groups and modules. □

Lemma 44.64.4. *Let X be a scheme and $\mathcal{F}$ a sheaf of sets on $X_{étale}$. Then the following are equivalent*

(1) *$\mathcal{F}$ is finite locally constant, and*
(2) *$\mathcal{F} = h_U$ for some finite étale morphism $U \to X$.*

Proof. A finite étale morphism is locally isomorphic to a disjoint union of isomorphisms, see Étale Morphisms, Lemma 40.18.3. Thus (2) implies (1). Conversely, if $\mathcal{F}$ is finite locally constant, then there exists an étale covering $\{X_i \to X\}$ such that $\mathcal{F}|_{X_i}$ is representable by $U_i \to X_i$ finite étale. Arguing exactly as in the proof of Descent, Lemma 34.35.1 we obtain a descent datum for schemes (U_i, φ_{ij}) relative to $\{X_i \to X\}$ (details omitted). This descent datum is effective for example by Descent, Lemma 34.33.1 and the resulting morphism of schemes $U \to X$ is finite étale by Descent, Lemmas 34.19.21 and 34.19.27. □

Lemma 44.64.5. *Let X be a scheme.*

(1) *Let $\varphi : \mathcal{F} \to \mathcal{G}$ be a map of locally constant sheaves of sets on $X_{étale}$. If $\mathcal{F}$ is finite locally constant, there exists an étale covering $\{U_i \to X\}$ such that $\varphi|_{U_i}$ is the map of constant sheaves associated to a map of sets.*
(2) *Let $\varphi : \mathcal{F} \to \mathcal{G}$ be a map of locally constant sheaves of abelian groups on $X_{étale}$. If $\mathcal{F}$ is finite locally constant, there exists an étale covering $\{U_i \to X\}$ such that $\varphi|_{U_i}$ is the map of constant abelian sheaves associated to a map of abelian groups.*
(3) *Let Λ be a ring. Let $\varphi : \mathcal{F} \to \mathcal{G}$ be a map of locally constant sheaves of Λ-modules on $X_{étale}$. If $\mathcal{F}$ is of finite type, then there exists an étale covering $\{U_i \to X\}$ such that $\varphi|_{U_i}$ is the map of constant sheaves of Λ-modules associated to a map of Λ-modules.*

Proof. This holds on any site, see Modules on Sites, Lemma 18.42.3. □

Lemma 44.64.6. *Let X be a scheme.*

(1) *The category of finite locally constant sheaves of sets is closed under finite limits and colimits inside $Sh(X_{étale})$.*
(2) *The category of finite locally constant abelian sheaves is a weak Serre subcategory of $Ab(X_{étale})$.*
(3) *Let Λ be a Noetherian ring. The category of finite type, locally constant sheaves of Λ-modules on $X_{étale}$ is a weak Serre subcategory of $Mod(X_{étale}, \Lambda)$.*

Proof. This holds on any site, see Modules on Sites, Lemma 18.42.5. □

Lemma 44.64.7. *Let X be a scheme. Let Λ be a ring. The tensor product of two locally constant sheaves of Λ-modules on $X_{étale}$ is a locally constant sheaf of Λ-modules.*

Proof. This holds on any site, see Modules on Sites, Lemma 18.42.6. □

Lemma 44.64.8. *Let X be a connected scheme. Let Λ be a ring and let $\mathcal{F}$ be a locally constant sheaf of Λ-modules. Then there exists a Λ-module M and an étale covering $\{U_i \to X\}$ such that $\mathcal{F}|_{U_i} \cong \underline{M}|_{U_i}$.*

Proof. Choose an étale covering $\{U_i \to X\}$ such that $\mathcal{F}|_{U_i}$ is constant, say $\mathcal{F}|_{U_i} \cong \underline{M_i}_{U_i}$. Observe that $U_i \times_X U_j$ is empty if M_i is not isomorphic to M_j. For each Λ-module M let $I_M = \{i \in I \mid M_i \cong M\}$. As étale morphisms are open we see that $U_M = \bigcup_{i \in I_M} \mathrm{Im}(U_i \to X)$ is an open subset of X. Then $X = \coprod U_M$ is a disjoint open covering of X. As X is connected only one U_M is nonempty and the lemma follows. □

44.65. Constructible sheaves

Let X be a scheme. A *constructible locally closed subscheme* of X is a locally closed subscheme $T \subset X$ such that the underlying topological space of T is a constructible subset of X. If $T, T' \subset X$ are locally closed subschemes with the same underlying topological space, then $T_{étale} \cong T'_{étale}$ by the topological invariance of the étale site (Theorem 44.46.1). Thus in the following definition we may assume are locally closed subschemes are reduced.

Definition 44.65.1. Let X be a scheme.

(1) A sheaf of sets on $X_{étale}$ is *constructible* if for every affine open $U \subset X$ there exists a finite decomposition of U into constructible locally closed subschemes $U = \coprod_i U_i$ such that $\mathcal{F}|_{U_i}$ is finite locally constant for all i.
(2) A sheaf of abelian groups on $X_{étale}$ is *constructible* if for every affine open $U \subset X$ there exists a finite decomposition of U into constructible locally closed subschemes $U = \coprod_i U_i$ such that $\mathcal{F}|_{U_i}$ is finite locally constant for all i.
(3) Let Λ be a Noetherian ring. A sheaf of Λ-modules on $X_{étale}$ is *constructible* if for every affine open $U \subset X$ there exists a finite decomposition of U into constructible locally closed subschemes $U = \coprod_i U_i$ such that $\mathcal{F}|_{U_i}$ is of finite type and locally constant for all i.

It seems that this is the accepted definition. An alternative, which lends itself more readily to generalizations beyond the étale site of a scheme, would have been to define constructible sheaves by starting with h_U, $j_{U!}\mathbf{Z}/n\mathbf{Z}$, and $j_{U!}\underline{\Lambda}$ where U runs over all quasi-compact and quasi-separated objects of $X_{étale}$, and then take the smallest full subcategory of $Sh(X_{étale})$, $Ab(X_{étale})$, and $Mod(X_{étale}, \underline{\Lambda})$ containing these and closed under finite limits and colimits. It follows from Lemma 44.65.6 and Lemmas 44.67.5, 44.67.7, and 44.67.6 that this produces the same category if X is quasi-compact and quasi-separated. In general this does not produce the same category however.

A disjoint union decomposition $U = \coprod U_i$ of a scheme by locally closed subschemes will be called a *partition* of U (compare with Topology, Section 5.26).

Lemma 44.65.2. *Let X be a quasi-compact and quasi-separated scheme. Let $\mathcal{F}$ be a sheaf of sets on $X_{étale}$. The following are equivalent*

(1) *$\mathcal{F}$ is constructible,*
(2) *there exists an open covering $X = \bigcup U_i$ such that $\mathcal{F}|_{U_i}$ is constructible, and*
(3) *there exists a partition $X = \bigcup X_i$ by constructible locally closed subschemes such that $\mathcal{F}|_{X_i}$ is finite locally constant.*

A similar statement holds for abelian sheaves and sheaves of Λ-modules if Λ is Noetherian.

Proof. It is clear that (1) implies (2).

Assume (2). For every $x \in X$ we can find an i and an affine open neighbourhood $V_x \subset U_i$ of x. Hence we can find a finite affine open covering $X = \bigcup V_j$ such that for each j there exists a finite decomposition $V_j = \coprod V_{j,k}$ by locally closed constructible subsets such that $\mathcal{F}|_{V_{j,k}}$ is finite locally constant. By Topology, Lemma 5.14.5 each $V_{j,k}$ is constructible as a subset of X. By Topology, Lemma 5.26.6 we can find a finite stratification $X = \coprod X_l$ with constructible locally closed strata such that each $V_{j,k}$ is a union of X_l. Thus (3) holds.

Assume (3) holds. Let $U \subset X$ be an affine open. Then $U \cap X_i$ is a constructible locally closed subset of U (for example by Properties, Lemma 27.2.1) and $U = \coprod U \cap X_i$ is a partition of U as in Definition 44.65.1. Thus (1) holds. □

Lemma 44.65.3. *Let X be a quasi-compact and quasi-separated scheme. Let $\mathcal{F}$ be a sheaf of sets, abelian groups, Λ-modules (with Λ Noetherian) on $X_{étale}$. If there*

exist constructible locally closed subschemes $T_i \subset X$ such that (a) $X = \bigcup T_j$ and (b) $\mathcal{F}|_{T_j}$ is constructible, then $\mathcal{F}$ is constructible.

Proof. First, we can assume the covering is finite as X is quasi-compact in the spectral topology (Topology, Lemma 5.22.2 and Properties, Lemma 27.2.3). Observe that each T_i is a quasi-compact and quasi-separated scheme in its own right (because it is constructible in X; details omitted). Thus we can find a finite partition $T_i = \coprod T_{i,j}$ into locally closed constructible parts of T_i such that $\mathcal{F}|_{T_{i,j}}$ is finite locally constant (Lemma 44.65.2). By Topology, Lemma 5.14.12 we see that $T_{i,j}$ is a constructible locally closed subscheme of X. Then we can apply Topology, Lemma 5.26.6 to $X = \bigcup T_{i,j}$ to find the desired partition of X. □

Lemma 44.65.4. *Let X be a scheme. Checking constructibility of a sheaf of sets, abelian groups, Λ-modules (with Λ Noetherian) can be done Zariski locally on X.*

Proof. The statement means if $X = \bigcup U_i$ is an open covering such that $\mathcal{F}|_{U_i}$ is constructible, then $\mathcal{F}$ is constructible. If $U \subset X$ is affine open, then $U = \bigcup U \cap U_i$ and $\mathcal{F}|_{U \cap U_i}$ is constructible (it is trivial that the restriction of a constructible sheaf to an open is constructible). It follows from Lemma 44.65.2 that $\mathcal{F}|_U$ is constructible, i.e., a suitable partition of U exists. □

Lemma 44.65.5. *Let $f : X \to Y$ be a morphism of schemes. If $\mathcal{F}$ is a constructible sheaf of sets, abelian groups, or Λ-modules (with Λ Noetherian) on $Y_{étale}$, the same is true for $f^{-1}\mathcal{F}$ on $X_{étale}$.*

Proof. By Lemma 44.65.4 this reduces to the case where X and Y are affine. By Lemma 44.65.2 it suffices to find a finite partition of X by constructible locally closed subschemes such that $f^{-1}\mathcal{F}$ is finite locally constant on each of them. To find it we just pull back the partition of Y adapted to $\mathcal{F}$ and use Lemma 44.64.2. □

Lemma 44.65.6. *Let X be a scheme.*

(1) *The category of constructible sheaves of sets is closed under finite limits and colimits inside $Sh(X_{étale})$.*
(2) *The category of constructible abelian sheaves is a weak Serre subcategory of $Ab(X_{étale})$.*
(3) *Let Λ be a Noetherian ring. The category of constructible sheaves of Λ-modules on $X_{étale}$ is a weak Serre subcategory of $Mod(X_{étale}, \Lambda)$.*

Proof. We prove (3). We will use the criterion of Homology, Lemma 12.9.3. Suppose that $\varphi : \mathcal{F} \to \mathcal{G}$ is a map of constructible sheaves of Λ-modules. We have to show that $\mathcal{K} = \mathrm{Ker}(\varphi)$ and $\mathcal{Q} = \mathrm{Coker}(\varphi)$ are constructible. Similarly, suppose that $0 \to \mathcal{F} \to \mathcal{E} \to \mathcal{G} \to 0$ is a short exact sequence of sheaves of Λ-modules with $\mathcal{F}$, $\mathcal{G}$ constructible. We have to show that $\mathcal{E}$ is constructible. In both cases we can replace X with the members of an affine open covering. Hence we may assume X is affine. The we may further replace X by the members of a finite partition of X by constructible locally closed subschemes on which $\mathcal{F}$ and $\mathcal{G}$ are of finite type and locally constant. Thus we may apply Lemma 44.64.6 to conclude.

The proofs of (1) and (2) are very similar and are omitted. □

Lemma 44.65.7. *Let X be a scheme. Let Λ be a Noetherian ring. The tensor product of two constructible sheaves of Λ-modules on $X_{étale}$ is a constructible sheaf of Λ-modules.*

Proof. The question immediately reduces to the case where X is affine. Since any two partitions of X with constructible locally closed strata have a common refinement of the same type and since pullbacks commute with tensor product we reduce to Lemma 44.64.7. □

Lemma 44.65.8. *Let X be a quasi-compact and quasi-separated scheme.*

(1) *Let $\mathcal{F} \to \mathcal{G}$ be a map of constructible sheaves of sets on $X_{étale}$. Then the set of points $x \in X$ where $\mathcal{F}_{\overline{x}} \to \mathcal{F}_{\overline{x}}$ is surjective, resp. injective, resp. is isomorphic to a given map of sets, is constructible in X.*
(2) *Let $\mathcal{F}$ be a constructible abelian sheaf on $X_{étale}$. The support of $\mathcal{F}$ is constructible.*
(3) *Let Λ be a Noetherian ring. Let $\mathcal{F}$ be a constructible sheaf of Λ-modules on $X_{étale}$. The support of $\mathcal{F}$ is constructible.*

Proof. Proof of (1). Let $X = \coprod X_i$ be a partion of X by locally closed constructible subschemes such that both $\mathcal{F}$ and $\mathcal{G}$ are finite locally constant over the parts (use Lemma 44.65.2 for both $\mathcal{F}$ and $\mathcal{G}$ and choose a common refinement). Then apply Lemma 44.64.5 to the restriction of the map to each part.

The proof of (2) and (3) is omitted. □

The following lemma will turn out to be very useful later on. It roughly says that the category of constructible sheaves has a kind of weak "Noetherian" property.

Lemma 44.65.9. *Let X be a quasi-compact and quasi-separated scheme. Let $\mathcal{F} = \operatorname{colim}_{i \in I} \mathcal{F}_i$ be a filtered colimit of sheaves of sets, abelian sheaves, or sheaves of modules.*

(1) *If $\mathcal{F}$ and $\mathcal{F}_i$ are constructible sheaves of sets, then the ind-object $\mathcal{F}_i$ is essentially constant with value $\mathcal{F}$.*
(2) *If $\mathcal{F}$ and $\mathcal{F}_i$ are constructible sheaves of abelian groups, then the ind-object $\mathcal{F}_i$ is essentially constant with value $\mathcal{F}$.*
(3) *Let Λ be a Noetherian ring. If $\mathcal{F}$ and $\mathcal{F}_i$ are constructible sheaves of Λ-modules, then the ind-object $\mathcal{F}_i$ is essentially constant with value $\mathcal{F}$.*

Proof. Proof of (1). We will use without further mention that finite limits and colimits of constructible sheaves are constructible (Lemma 44.64.6). For each i let $T_i \subset X$ be the set of points $x \in X$ where $\mathcal{F}_{i,\overline{x}} \to \mathcal{F}_{\overline{x}}$ is not surjective. Because $\mathcal{F}_i$ and $\mathcal{F}$ are constructible T_i is a constructible subset of X (Lemma 44.65.8). Since the stalks of $\mathcal{F}$ are finite and since $\mathcal{F} = \operatorname{colim}_{i \in I} \mathcal{F}_i$ we see that for all $x \in X$ we have $x \notin T_i$ for i large enough. Since X is a spectral space by Properties, Lemma 27.2.3 the constructible topology on X is quasi-compact by Topology, Lemma 5.22.2. Thus $T_i = \emptyset$ for i large enough. Thus $\mathcal{F}_i \to \mathcal{F}$ is surjective for i large enough. Assume now that $\mathcal{F}_i \to \mathcal{F}$ is surjective for all i. Choose $i \in I$. For $i' \geq i$ denote $S_{i'} \subset X$ the set of points x such that the number of elements in $\operatorname{Im}(\mathcal{F}_{i,\overline{x}} \to \mathcal{F}_{\overline{x}})$ is equal to the number of elements in $\operatorname{Im}(\mathcal{F}_{i,\overline{x}} \to \mathcal{F}_{i',\overline{x}})$. Because $\mathcal{F}_i$, $\mathcal{F}_{i'}$ and $\mathcal{F}$ are constructible $S_{i'}$ is a constructible subset of X (details omitted; hint: use Lemma 44.65.8). Since the stalks of $\mathcal{F}_i$ and $\mathcal{F}$ are finite and since $\mathcal{F} = \operatorname{colim}_{i' \geq i} \mathcal{F}_{i'}$ we see that for all $x \in X$ we have $x \notin S_{i'}$ for i' large enough. By the same argument as above we can find a large i' such that $S_{i'} = \emptyset$. Thus $\mathcal{F}_i \to \mathcal{F}_{i'}$ factors through $\mathcal{F}$ as desired.

Proof of (2). Observe that a constructible abelian sheaf is a constructible sheaf of sets. Thus case (2) follows from (1).

Proof of (3). We will use without further mention that the category of constructible sheaves of Λ-modules is abelian (Lemma 44.64.6). For each i let $\mathcal{Q}_i$ be the cokernel of the map $\mathcal{F}_i \to \mathcal{F}$. The support T_i of $\mathcal{Q}_i$ is a constructible subset of X as $\mathcal{Q}_i$ is constructible (Lemma 44.65.8). Since the stalks of $\mathcal{F}$ are finite Λ-modules and since $\mathcal{F} = \operatorname{colim}_{i \in I} \mathcal{F}_i$ we see that for all $x \in X$ we have $x \notin T_i$ for i large enough. Since X is a spectral space by Properties, Lemma 27.2.3 the constructible topology on X is quasi-compact by Topology, Lemma 5.22.2. Thus $T_i = \emptyset$ for i large enough. This proves the first assertion. For the second, assume now that $\mathcal{F}_i \to \mathcal{F}$ is surjective for all i. Choose $i \in I$. For $i' \geq i$ denote $\mathcal{K}_{i'}$ the image of $\operatorname{Ker}(\mathcal{F}_i \to \mathcal{F})$ in $\mathcal{F}_{i'}$. The support $S_{i'}$ of $\mathcal{K}_{i'}$ is a constructible subset of X as $\mathcal{K}_{i'}$ is constructible. Since the stalks of $\operatorname{Ker}(\mathcal{F}_i \to \mathcal{F})$ are finite Λ-modules and since $\mathcal{F} = \operatorname{colim}_{i' \geq i} \mathcal{F}_{i'}$ we see that for all $x \in X$ we have $x \notin S_{i'}$ for i' large enough. By the same argument as above we can find a large i' such that $S_{i'} = \emptyset$. Thus $\mathcal{F}_i \to \mathcal{F}_{i'}$ factors through $\mathcal{F}$ as desired. □

44.66. Auxiliary lemmas on morphisms

Some lemmas that are useful for proving functioriality properties of constructible sheaves.

Lemma 44.66.1. *Let $U \to X$ be an étale morphism of quasi-compact and quasi-separated schemes (for example an étale morphism of Noetherian schemes). Then there exists a partition $X = \coprod_i X_i$ by constructible locally closed subschemes such that $X_i \times_X U \to X_i$ is finite étale for all i.*

Proof. If $U \to X$ is separated, then this is More on Morphisms, Lemma 36.31.9. In general, we may assume X is affine. Choose a finite affine open covering $U = \bigcup U_j$. Apply the previous case to all the morphisms $U_j \to X$ and $U_j \cap U_{j'} \to X$ and choose a common refinement $X = \coprod X_i$ of the resulting partitions. After refining the partition further we may assume X_i affine as well. Fix i and set $V = U \times_X X_i$. The morphisms $V_j = U_j \times_X X_i \to X_i$ and $V_{jj'} = (U_j \cap U_{j'}) \times_X X_i \to X_i$ are finite étale. Hence V_j and $V_{jj'}$ are affine schemes and $V_{jj'} \subset V_j$ is closed as well as open (since $V_{jj'} \to X_i$ is proper, so Morphisms, Lemma 28.42.7 applies). Then $V = \bigcup V_j$ is separated because $\mathcal{O}(V_j) \to \mathcal{O}(V_{jj'})$ is surjective, see Schemes, Lemma 25.21.8. Thus the previous case applies to $V \to X_i$ and we can further refine the partition if needed (it actually isn't but we don't need this). □

In the Noetherian case one can prove the preceding lemma by Noetherian induction and the following amusing lemma.

Lemma 44.66.2. *Let $f : X \to Y$ be a morphism of schemes which is quasi-compact, quasi-separated, and locally of finite type. If η is a generic point of on irreducible component of Y such that $f^{-1}(\eta)$ is finite, then there exists an open $V \subset Y$ containing η such that $f^{-1}(V) \to V$ is finite.*

Proof. This is Morphisms, Lemma 28.47.1. □

The statement of the following lemma can be strengthened a bit.

Lemma 44.66.3. *Let $f : Y \to X$ be a quasi-finite and finitely presented morphism of affine schemes.*

(1) *There exists a surjective morphism of affine schemes $X' \to X$ and a closed subscheme $Z' \subset Y' = X' \times_X Y$ such that*

(a) $Z' \subset Y'$ *is a thickening, and*
(b) $Z' \to X'$ *is a finite étale morphism.*

(2) *There exists a finite partition* $X = \coprod X_i$ *by locally closed, constructible, affine strata, and surjective finite locally free morphisms* $X_i' \to X_i$ *such that the reduction of* $Y_i' = X_i' \times_X Y \to X_i'$ *is isomorphic to* $\coprod_{j=1}^{n_i} (X_i')_{red} \to (X_i')_{red}$ *for some* n_i.

Proof. Setting $X' = \coprod X_i'$ we see that (2) implies (1). Write $X = \mathrm{Spec}(A)$ and $Y = \mathrm{Spec}(B)$. Write A as a filtered colimit of finite type $\mathbf{Z}$-algebras A_i. Since B is an A-algebra of finite presentation, we see that there exists $0 \in I$ and a finite type ring map $A_0 \to B_0$ such that $B = \mathrm{colim}\, B_i$ with $B_i = A_i \otimes_{A_0} B_0$, see Algebra, Lemma 10.121.6. For i sufficiently large we see that $A_i \to B_i$ is quasi-finite, see Limits, Lemma 31.14.2. Thus we reduce to the case of finite type algebras over $\mathbf{Z}$, in particular we reduce to the Noetherian case. (Details omitted.)

Assume X and Y Noetherian. In this case any locally closed subset of X is constructible. By Lemma 44.66.2 and Noetherian induction we see that there is a finite partition $X = \coprod X_i$ of X by locally closed strata such that $Y \times_X X_i \to X_i$ is finite. We can refine this partition to get affine strata. Thus after replacing X by $X' = \coprod X_i$ we may assume $Y \to X$ is finite.

Assume X and Y Noetherian and $Y \to X$ finite. Suppose that we can prove (2) after base change by a surjective, flat, quasi-finite morphism $U \to X$. Thus we have a partition $U = \coprod U_i$ and finite locally free morphisms $U_i' \to U_i$ such that $U_i' \times_X Y \to U_i'$ is isomorphic to $\coprod_{j=1}^{n_i} (U_i')_{red} \to (U_i')_{red}$ for some n_i. Then, by the argument in the previous paragraph, we can find a partition $X = \coprod X_j$ with locally closed affine strata such that $X_j \times_X U_i \to X_j$ is finite for all i, j. By Morphisms, Lemma 28.46.2 each $X_j \times_X U_i \to X_j$ is finite locally free. Hence $X_j \times_X U_i' \to X_j$ is finite locally free (Morphisms, Lemma 28.46.3). It follows that $X = \coprod X_j$ and $X_j' = \coprod_i X_j \times_X U_i'$ is a solution for $Y \to X$. Thus it suffices to prove the result (in the Noetherian case) after a surjective flat quasi-finite base change.

Applying Morphisms, Lemma 28.46.6 we see we may assume that Y is a closed subscheme of an affine scheme Z which is (set theoretically) a finite union $Z = \bigcup_{i \in I} Z_i$ of closed subschemes mapping isomorphically to X. In this case we will find a finite partition of $X = \coprod X_j$ with affine locally closed strata that works (in other words $X_j' = X_j$). Set $T_i = Y \cap Z_i$. This is a closed subscheme of X. As X is Noetherian we can find a finite partition of $X = \coprod X_j$ by affine locally closed subschemes, such that each $X_j \times_X T_i$ is (set theoretically) a union of strata $X_j \times_X Z_i$. Replacing X by X_j we see that we may assume $I = I_1 \amalg I_2$ with $Z_i \subset Y$ for $i \in I_1$ and $Z_i \cap Y = \emptyset$ for $i \in I_2$. Replacing Z by $\bigcup_{i \in I_1} Z_i$ we see that we may assume $Y = Z$. Finally, we can replace X again by the members of a partition as above such that for every $i, i' \subset I$ the intersection $Z_i \cap Z_{i'}$ is either empty or (set theoretically) equal to Z_i and $Z_{i'}$. This clearly means that Y is (set theoretically) equal to a disjoint union of the Z_i which is what we wanted to show. □

44.67. More on constructible sheaves

Let Λ be a Noetherian ring. Let X be a scheme. We often consider $X_{étale}$ as a ringed site with sheaf of rings $\underline{\Lambda}$. In case of abelian sheaves we often take $\Lambda = \mathbf{Z}/n\mathbf{Z}$ for a suitable integer n.

Lemma 44.67.1. *Let $j : U \to X$ be an étale morphism of quasi-compact and quasi-separated schemes.*

(1) *The sheaf h_U is a constructible sheaf of sets.*
(2) *The sheaf $j_!\underline{M}$ is a constructible abelian sheaf for a finite abelian group M.*
(3) *If Λ is a Noetherian ring and M is a finite Λ-module, then $j_!\underline{M}$ is a constructible sheaf of Λ-modules on $X_{étale}$.*

Proof. By Lemma 44.66.1 there is a partition $\coprod_i X_i$ such that $\pi_i : j^{-1}(X_i) \to X_i$ is finite étale. The restriction of h_U to X_i is $h_{j^{-1}(X_i)}$ which is finite locally constant by Lemma 44.64.4. For cases (2) and (3) we note that

$$j_!(\underline{M})|_{X_i} = \pi_{i!}(\underline{M}) = \pi_{i*}(\underline{M})$$

by Lemmas 44.63.4 and 44.63.5. Thus it suffices to show the lemma for $\pi : Y \to X$ finite étale. This is Lemma 44.64.3. □

Lemma 44.67.2. *Let X be a quasi-compact and quasi-separated scheme.*

(1) *Let $\mathcal{F}$ be a sheaf of sets on $X_{étale}$. Then $\mathcal{F}$ is a filtered colimit of constructible sheaves of sets.*
(2) *Let $\mathcal{F}$ be a torsion abelian sheaf on $X_{étale}$. Then $\mathcal{F}$ is a filtered colimit of constructible abelian sheaves.*
(3) *Let Λ be a Noetherian ring and $\mathcal{F}$ a sheaf of Λ-modules on $X_{étale}$. Then $\mathcal{F}$ is a filtered colimit of constructible sheaves of Λ-modules.*

Proof. Let $\mathcal{B}$ be the collection of quasi-compact and quasi-separated objects of $X_{étale}$. By Modules on Sites, Lemma 18.29.6 any sheaf of sets is a filtered colimit of sheaves of the form

$$\text{Coequalizer}\Big(\coprod\nolimits_{j=1,\ldots,m} h_{V_j} \longrightarrow\!\!\!\!\!\rightarrow \coprod\nolimits_{i=1,\ldots,n} j_{U_i}\Big)$$

with V_j and U_i quasi-compact and quasi-separated objects of $X_{étale}$. By Lemmas 44.67.1 and 44.65.6 these coequalizers are constructible. This proves (1).

Let Λ be a Noetherian ring. By Modules on Sites, Lemma 18.29.6 Λ-modules $\mathcal{F}$ is a filtered colimit of modules of the form

$$\text{Coker}\Big(\bigoplus\nolimits_{j=1,\ldots,m} j_{V_j!}\underline{\Lambda}_{V_j} \longrightarrow \bigoplus\nolimits_{i=1,\ldots,n} j_{U_i!}\underline{\Lambda}_{U_i}\Big)$$

with V_j and U_i quasi-compact and quasi-separated objects of $X_{étale}$. By Lemmas 44.67.1 and 44.65.6 these cokernels are constructible. This proves (3).

Proof of (2). First write $\mathcal{F} = \bigcup \mathcal{F}[n]$ where $\mathcal{F}[n]$ is the n-torsion subsheaf. Then we can view $\mathcal{F}[n]$ as a sheaf of $\mathbf{Z}/n\mathbf{Z}$-modules and apply (3). □

Lemma 44.67.3. *Let $f : X \to Y$ be a surjective morphism of quasi-compact and quasi-separated schemes.*

(1) *Let $\mathcal{F}$ be a sheaf of sets on $Y_{étale}$. Then $\mathcal{F}$ is constructible if and only if $f^{-1}\mathcal{F}$ is constructible.*
(2) *Let $\mathcal{F}$ be an abelian sheaf on $Y_{étale}$. Then $\mathcal{F}$ is constructible if and only if $f^{-1}\mathcal{F}$ is constructible.*
(3) *Let Λ be a Noetherian ring. Let $\mathcal{F}$ be sheaf of Λ-modules on $Y_{étale}$. Then $\mathcal{F}$ is constructible if and only if $f^{-1}\mathcal{F}$ is constructible.*

Proof. One implication follows from Lemma 44.65.5. For the converse, assume $f^{-1}\mathcal{F}$ is constructible. Write $\mathcal{F} = \operatorname{colim} \mathcal{F}_i$ as a filtered colimit of constructible sheaves (of sets, abelian groups, or modules) using Lemma 44.67.2. Since f^{-1} is a left adjoint it commutes with colimits (Categories, Lemma 4.24.3) and we see that $f^{-1}\mathcal{F} = \operatorname{colim} f^{-1}\mathcal{F}_i$. By Lemma 44.65.9 we see that $f^{-1}\mathcal{F}_i \to f^{-1}\mathcal{F}$ is surjective for all i large enough. Since f is surjective we conclude (by looking at stalks using Lemma 44.36.2 and Theorem 44.29.10) that $\mathcal{F}_i \to \mathcal{F}$ is surjective for all i large enough. Thus $\mathcal{F}$ is the quotient of a constructible sheaf $\mathcal{G}$. Applying the argument once more to $\mathcal{G} \times_{\mathcal{F}} \mathcal{G}$ or the kernel of $\mathcal{G} \to \mathcal{F}$ we conclude using that f^{-1} is exact and that the category of constructible sheaves (of sets, abelian groups, or modules) is preserved under finite (co)limits or (co)kernels inside $Sh(Y_{étale})$, $Sh(X_{étale})$, $Ab(Y_{étale})$, $Ab(X_{étale})$, $Mod(Y_{étale}, \Lambda)$, and $Mod(X_{étale}, \Lambda)$, see Lemma 44.65.6. □

Lemma 44.67.4. *Let $f : X \to Y$ be a finite étale morphism of schemes. Let Λ be a Noetherian ring. If $\mathcal{F}$ is a constructible sheaf of sets, constructible sheaf of abelian groups, or constructible sheaf of Λ-modules on $X_{étale}$, the same is true for $f_*\mathcal{F}$ on $Y_{étale}$.*

Proof. By Lemma 44.65.4 it suffices to check this Zariski locally on Y and by Lemma 44.67.3 we may replace Y by an étale cover (the construction of f_* commutes with étale localization). A finite étale morphism is étale locally isomorphic to a disjoint union of isomorphisms, see Étale Morphisms, Lemma 40.18.3. Thus, in the case of sheaves of sets, the lemma says that if $\mathcal{F}_i$, $i = 1, \ldots, n$ are constructible sheaves of sets, then $\prod_{i=1,\ldots,n} \mathcal{F}_i$ is too. This is clear. Similarly for sheaves of abelian groups and modules. □

Lemma 44.67.5. *Let X be a quasi-compact and quasi-separated scheme. The category of constructible sheaves of sets is the full subcategory of $Sh(X_{étale})$ consisting of sheaves $\mathcal{F}$ which are coequalizers*

$$\mathcal{F}_1 \rightrightarrows \mathcal{F}_0 \longrightarrow \mathcal{F}$$

such that $\mathcal{F}_i$, $i = 0, 1$ is a finite coproduct of sheaves of the form h_U with U a quasi-compact and quasi-separated object of $X_{étale}$.

Proof. In the proof of Lemma 44.67.2 we have seen that sheaves of this form are constructible. For the converse, suppose that for every constructible sheaf of sets $\mathcal{F}$ we can find a surjection $\mathcal{F}_0 \to \mathcal{F}$ with $\mathcal{F}_0$ as in the lemma. Then we find our surjection $\mathcal{F}_1 \to \mathcal{F}_0 \times_{\mathcal{F}} \mathcal{F}_0$ because the latter is constructible by Lemma 44.65.6.

By Topology, Lemma 5.26.6 we may choose a finite stratification $X = \coprod_{i \in I} X_i$ such that $\mathcal{F}$ is finite locally constant on each stratum. We will prove the result by induction on the cardinality of I. Let $i \in I$ be a minimal element in the partial ordering of I. Then $X_i \subset X$ is closed. By induction, there exist finitely many quasi-compact and quasi-separated objects U_α of $(X \setminus X_i)_{étale}$ and a surjective map $\coprod h_{U_\alpha} \to \mathcal{F}|_{X \setminus X_i}$. These determine a map

$$\coprod h_{U_\alpha} \to \mathcal{F}$$

which is surjective after restricting to $X \setminus X_i$. By Lemma 44.64.4 we see that $\mathcal{F}|_{X_i} = h_V$ for some scheme V finite étale over X_i. Let $\overline{v}$ be a geometric point of V lying over $\overline{x} \in X_i$. We may think of $\overline{v}$ as an element of the stalk $\mathcal{F}_{\overline{x}} = V_{\overline{x}}$.

Thus we can find an étale neighbourhood $(U, \overline{u})$ of $\overline{x}$ and a section $s \in \mathcal{F}(U)$ whose stalk at $\overline{x}$ gives $\overline{v}$. Thinking of s as a map $s : h_U \to \mathcal{F}$, restricting to X_i we obtain a morphism $s|_{X_i} : U \times_X X_i \to V$ over X_i which maps $\overline{u}$ to $\overline{v}$. Since V is quasi-compact (finite over the closed subscheme X_i of the quasi-compact scheme X) a finite number $s^{(1)}, \ldots, s^{(m)}$ of these sections of $\mathcal{F}$ over $U^{(1)}, \ldots, U^{(m)}$ will determine a jointly surjective map

$$\coprod s^{(j)}|_{X_i} : \coprod U^{(j)} \times_X X_i \longrightarrow V$$

Then we obtain the surjection

$$\coprod h_{U_\alpha} \amalg \coprod h_{U^{(j)}} \to \mathcal{F}$$

as desired. □

Lemma 44.67.6. *Let X be a quasi-compact and quasi-separated scheme. Let Λ be a Noetherian ring. The category of constructible sheaves of Λ-modules is exactly the category of modules of the form*

$$\mathit{Coker}\Big(\bigoplus\nolimits_{j=1,\ldots,m} j_{V_j!}\underline{\Lambda}_{V_j} \longrightarrow \bigoplus\nolimits_{i=1,\ldots,n} j_{U_i!}\underline{\Lambda}_{U_i}\Big)$$

with V_j and U_i quasi-compact and quasi-separated objects of $X_{étale}$. In fact, we can even assume U_i and V_j affine.

Proof. In the proof of Lemma 44.67.2 we have seen modules of this form are constructible. Since the category of constructible modules is abelian (Lemma 44.65.6) it suffices to prove that given a constructible module $\mathcal{F}$ there is a surjection

$$\bigoplus\nolimits_{i=1,\ldots,n} j_{U_i!}\underline{\Lambda}_{U_i} \longrightarrow \mathcal{F}$$

for some affine objects U_i in $X_{étale}$. By Modules on Sites, Lemma 18.29.6 there is a surjection

$$\Psi : \bigoplus\nolimits_{i\in I} j_{U_i!}\underline{\Lambda}_{U_i} \longrightarrow \mathcal{F}$$

with U_i affine and the direct sum over a possibly infinite index set I. For every finite subset $I' \subset I$ set

$$T_{I'} = \mathrm{Supp}(\mathrm{Coker}(\bigoplus\nolimits_{i\in I'} j_{U_i!}\underline{\Lambda}_{U_i} \longrightarrow \mathcal{F}))$$

By the very definition of constructible sheaves, the set $T_{I'}$ is a constructible subset of X. We want to show that $T_{I'} = \emptyset$ for some I'. Since every stalk $\mathcal{F}_{\overline{x}}$ is a finite type Λ-module and since Ψ is surjective, for every $x \in X$ there is an I' such that $x \notin T_{I'}$. In other words we have $\emptyset = \bigcap_{I'\subset I \text{ finite}} T_{I'}$. Since X is a spectral space by Properties, Lemma 27.2.3 the constructible topology on X is quasi-compact by Topology, Lemma 5.22.2. Thus $T_{I'} = \emptyset$ for some $I' \subset I$ finite as desired. □

Lemma 44.67.7. *Let X be a quasi-compact and quasi-separated scheme. The category of constructible abelian sheaves is exactly the category of abelian sheaves of the form*

$$\mathit{Coker}\Big(\bigoplus\nolimits_{j=1,\ldots,m} j_{V_j!}\underline{\mathbf{Z}/m_j\mathbf{Z}}_{V_j} \longrightarrow \bigoplus\nolimits_{i=1,\ldots,n} j_{U_i!}\underline{\mathbf{Z}/n_i\mathbf{Z}}_{U_i}\Big)$$

with V_j and U_i quasi-compact and quasi-separated objects of $X_{étale}$ and m_j, n_i positive integers. In fact, we can even assume U_i and V_j affine.

Proof. This follows from Lemma 44.67.6 applied with $\Lambda = \mathbf{Z}/n\mathbf{Z}$ and the fact that, since X is quasi-compact, every constructible abelian sheaf is annihilated by some positive integer n (details omitted). □

Lemma 44.67.8. *Let X be a quasi-compact and quasi-separated scheme. Let Λ be a Noetherian ring. Let $\mathcal{F}$ be a constructible sheaf of sets, abelian groups, or Λ-modules on $X_{étale}$. Let $\mathcal{G} = \operatorname{colim} \mathcal{G}_i$ be a filtered colimit of sheaves of sets, abelian groups, or Λ-modules. Then*

$$\operatorname{Mor}(\mathcal{F}, \mathcal{G}) = \operatorname{colim} \operatorname{Mor}(\mathcal{F}, \mathcal{G}_i)$$

in the category of sheaves of sets, abelian groups, or Λ-modules on $X_{étale}$.

Proof. The case of sheaves of sets. By Lemma 44.67.5 it suffices to prove the lemma for h_U where U is a quasi-compact and quasi-separated object of $X_{étale}$. Recall that $\operatorname{Mor}(h_U, \mathcal{G}) = \mathcal{G}(U)$. Hence the result follows from Sites, Lemma 7.11.2.

In the case of abelian sheaves or sheaves of modules, the result follows in the same way using Lemmas 44.67.7 and 44.67.6. For the case of abelian sheaves, we add that $\operatorname{Mor}(j_{U!}\underline{\mathbf{Z}/n\mathbf{Z}}, \mathcal{G})$ is equal to the n-torsion elements of $\mathcal{G}(U)$. □

Lemma 44.67.9. *Let $f : X \to Y$ be a finite and finitely presented morphism of schemes. Let Λ be a Noetherian ring. If $\mathcal{F}$ is a constructible sheaf of sets, abelian groups, or Λ-modules on $X_{étale}$, then $f_*\mathcal{F}$ is too.*

Proof. It suffices to prove this when X and Y are affine by Lemma 44.65.4. By Lemmas 44.55.3 and 44.67.3 we may base change to any affine scheme surjective over X. By Lemma 44.66.3 this reduces us to the case of a finite étale morphism (because a thickening leads to an equivalence of étale topoi and even small étale sites, see Theorem 44.46.1). The finite étale case is Lemma 44.67.4. □

Lemma 44.67.10. *Let $X = \lim_{i \in I} X_i$ be a limit of a directed system of schemes with affine transition morphisms. We assume that X_i is quasi-compact and quasi-separated for all $i \in I$.*

1. *The category of constructible sheaves of sets on $X_{étale}$ is the colimit of the categories of constructible sheaves of sets on $(X_i)_{étale}$.*
2. *The category of constructible abelian sheaves on $X_{étale}$ is the colimit of the categories of constructible abelian sheaves on $(X_i)_{étale}$.*
3. *Let Λ be a Noetherian ring. The category of constructible sheaves of Λ-modules on $X_{étale}$ is the colimit of the categories of constructible sheaves of Λ-modules on $(X_i)_{étale}$.*

Proof. Proof of (1). Denote $f_i : X \to X_i$ the projection maps. There are 3 parts to the proof corresponding to "faithful", "fully faithful", and "essentially surjective".

Faithful. Choose $0 \in I$ and let $\mathcal{F}_0, \mathcal{G}_0$ be constructible sheaves on X_0. Suppose that $a, b : \mathcal{F}_0 \to \mathcal{G}_0$ are maps such that $f_0^{-1}a = f_0^{-1}b$. Let $E \subset X_0$ be the set of points $x \in X_0$ such that $a_{\overline{x}} = b_{\overline{x}}$. By Lemma 44.65.8 the subset $E \subset X_0$ is constructible. By assumption $X \to X_0$ maps into E. By Limits, Lemma 31.3.7 we find an $i \geq 0$ such that $X_i \to X_0$ maps into E. Hence $f_{i0}^{-1}a = f_{i0}^{-1}b$.

Fully faithful. Choose $0 \in I$ and let $\mathcal{F}_0, \mathcal{G}_0$ be constructible sheaves on X_0. Suppose that $a : f_0^{-1}\mathcal{F}_0 \to f_0^{-1}\mathcal{G}_0$ is a map. We claim there is an i and a map $a_i : f_{i0}^{-1}\mathcal{F}_0 \to f_{i0}^{-1}\mathcal{G}_0$ which pulls back to a on X. By Lemma 44.67.5 we can replace $\mathcal{F}_0$ by a finite

coproduct of sheaves represented by quasi-compact and quasi-separated objects of $(X_0)_{étale}$. Thus we have to show: If $U_0 \to X_0$ is such an object of $(X_0)_{étale}$, then

$$f_0^{-1}\mathcal{G}(U) = \text{colim}_{i \geq 0} f_{i0}^{-1}\mathcal{G}(U_i)$$

where $U = X \times_{X_0} U_0$ and $U_i = X_i \times_{X_0} U_0$. This is a special case of Theorem 44.52.1.

Essentially surjective. We have to show every constructible $\mathcal{F}$ on X is isomorphic to $f_i^{-1}\mathcal{F}$ for some constructible $\mathcal{F}_i$ on X_i. Applying Lemma 44.67.5 and using the results of the previous two paragraphs, we see that it suffices to prove this for h_U for some quasi-compact and quasi-separated object U of $X_{étale}$. In this case we have to show that U is the base change of a quasi-compact and quasi-separated scheme étale over X_i for some i. This follows from Limits, Lemmas 31.9.1 and 31.7.8.

Proof of (3). The argument is very similar to the argument for sheaves of sets, but using Lemma 44.67.6 instead of Lemma 44.67.5. Details omitted. Part (2) follows from part (3) because every constructible abelian sheaf over a quasi-compact scheme is a constructible sheaf of $\mathbf{Z}/n\mathbf{Z}$-modules for some n. □

Lemma 44.67.11. *Let X be an irreducible scheme with generic point η.*

1. *Let $S' \subset S$ be an inclusion of sets. If we have $\underline{S'} \subset \mathcal{G} \subset \underline{S}$ in $Sh(X_{étale})$ and $S' = \mathcal{G}_{\overline{\eta}}$, then $\mathcal{G} = \underline{S'}$.*
2. *Let $A' \subset A$ be an inclusion of abelian groups. If we have $\underline{A'} \subset \mathcal{G} \subset \underline{A}$ in $Ab(X_{étale})$ and $A' = \mathcal{G}_{\overline{\eta}}$, then $\mathcal{G} = \underline{A'}$.*
3. *Let $M' \subset M$ be an inclsuion of modules over a ring Λ. If we have $\underline{M'} \subset \mathcal{G} \subset \underline{M}$ in $Mod(X_{étale}, \underline{\Lambda})$ and $M' = \mathcal{G}_{\overline{\eta}}$, then $\mathcal{G} = \underline{M'}$.*

Proof. This is true because for every étale morphism $U \to X$ with $U \neq \emptyset$ the point η is in the image. □

Lemma 44.67.12. *Let X be an integral normal scheme with function field K. Let E be a set.*

1. *Let $g : \text{Spec}(K) \to X$ be the inclusion of the generic point. Then $g_*\underline{E} = \underline{E}$.*
2. *Let $j : U \to X$ be the inclusion of a nonempty open. Then $j_*\underline{E} = \underline{E}$.*

Proof. Proof of (1). Let $x \in X$ be a point. Let $\mathcal{O}_{X,\overline{x}}$ be a strict henselization of $\mathcal{O}_{X,x}$. By More on Algebra, Lemma 15.32.5 we see that $\mathcal{O}_{X,\overline{x}}$ is a normal domain. Hence $\text{Spec}(K) \times_X \text{Spec}(\mathcal{O}_{X,\overline{x}})$ is irreducible. It follows that the stalk $(g_*\underline{E}_{\underline{x}}$ is equal to E, see Theorem 44.53.1.

Proof of (2). Since g factors through j there is a map $j_*\underline{E} \to g_*\underline{E}$. This map is injective because for every scheme V étale over X the set $\text{Spec}(K) \times_X V$ is dense in $U \times_X V$. On the other hand, we have a map $\underline{E} \to j_*\underline{E}$ and we conclude. □

44.68. Constructible sheaves on Noetherian schemes

If X is a Noetherian scheme then any locally closed subset is a constructible locally closed subset (Topology, Lemma 5.15.1). Hence an abelian sheaf $\mathcal{F}$ on $X_{étale}$ is constructible if and only if there exists a finite partition $X = \coprod X_i$ such that $\mathcal{F}|_{X_i}$ is finite locally constant. (By convention a partition of a topological space has locally closed parts, see Topology, Section 5.26.) In other words, we can omit the adjective "constructible" in Definition 44.65.1. Actually, the category of constructible sheaves

on Noetherian schemes has some additional properties which we will catalogue in this section.

Proposition 44.68.1. *Let X be a Noetherian scheme. Let Λ be a Noetherian ring.*

1. *Any sub or quotient sheaf of a constructible sheaf of sets is constructible.*
2. *The category of constructible abelian sheaves on $X_{étale}$ is a (strong) Serre subcategory of $Ab(X_{étale})$. In particular, every sub and quotient sheaf of a constructible abelian sheaf on $X_{étale}$ is constructible.*
3. *The category of constructible sheaves of Λ-modules on $X_{étale}$ is a (strong) Serre subcategory of $Mod(X_{étale}, \Lambda)$. In particular, every submodule and quotient module of a constructible sheaf of Λ-modules on $X_{étale}$ is constructible.*

Proof. Proof of (1). Let $\mathcal{G} \subset \mathcal{F}$ with $\mathcal{F}$ a constructible sheaf of sets on $X_{étale}$. Let $\eta \in X$ be a generic point of an irreducible component of X. By Noetherian induction it suffices to find an open neighbourhood U of η such that $\mathcal{G}|_U$ is locally constant. To do this we may replace X by an étale neighbourhood of η. Hence we may assume $\mathcal{F}$ is constant and X is irreducible.

Say $\mathcal{F} = \underline{S}$ for some finite set S. Then $S' = \mathcal{G}_{\overline{\eta}} \subset S$ say $S' = \{s_1, \ldots, s_t\}$. Pick an étale neighbourhood $(U, \overline{u})$ of $\overline{\eta}$ and sections $\sigma_1, \ldots, \sigma_t \in \mathcal{G}(U)$ which map to s_i in $\mathcal{G}_{\overline{\eta}} \subset S$. Since σ_i maps to an element $s_i \in S' \subset S = \Gamma(X, \mathcal{F})$ we see that the two pullbacks of σ_i to $U \times_X U$ are the same as sections of $\mathcal{G}$. By the sheaf condition for $\mathcal{G}$ we find that σ_i comes from a section of $\mathcal{G}$ over the open $\mathrm{Im}(U \to X)$ of X. Shrinking X we may assume $\underline{S'} \subset \mathcal{G} \subset \underline{S}$. Then we see that $\underline{S'} = \mathcal{G}$ by Lemma 44.67.11.

Let $\mathcal{F} \to \mathcal{Q}$ be a surjection with $\mathcal{F}$ a constructible sheaf of sets on $X_{étale}$. Then set $\mathcal{G} = \mathcal{F} \times_{\mathcal{Q}} \mathcal{F}$. By the first part of the proof we see that $\mathcal{G}$ is constructible as a subsheaf of $\mathcal{F} \times \mathcal{F}$. This in turn implies that $\mathcal{Q}$ is constructible, see Lemma 44.65.6.

Proof of (3). we already know that constructible sheaves of modules form a weak Serre subcategory, see Lemma 44.65.6. Thus it suffices to show the statement on submodules.

Let $\mathcal{G} \subset \mathcal{F}$ be a submodule of a constructible sheaf of Λ-modules on $X_{étale}$. Let $\eta \in X$ be a generic point of an irreducible component of X. By Noetherian induction it suffices to find an open neighbourhood U of η such that $\mathcal{G}|_U$ is locally constant. To do this we may replace X by an étale neighbourhood of η. Hence we may assume $\mathcal{F}$ is constant and X is irreducible.

Say $\mathcal{F} = \underline{M}$ for some finite Λ-module M. Then $M' = \mathcal{G}_{\overline{\eta}} \subset M$. Pick finitely many elements $s_1, \ldots, s_t$ generating M' as a Λ-module. (This is possible as Λ is Noetherian and M is finite.) Pick an étale neighbourhood $(U, \overline{u})$ of $\overline{\eta}$ and sections $\sigma_1, \ldots, \sigma_t \in \mathcal{G}(U)$ which map to s_i in $\mathcal{G}_{\overline{\eta}} \subset M$. Since σ_i maps to an element $s_i \in M' \subset M = \Gamma(X, \mathcal{F})$ we see that the two pullbacks of σ_i to $U \times_X U$ are the same as sections of $\mathcal{G}$. By the sheaf condition for $\mathcal{G}$ we find that σ_i comes from a section of $\mathcal{G}$ over the open $\mathrm{Im}(U \to X)$ of X. Shrinking X we may assume $\underline{M'} \subset \mathcal{G} \subset \underline{M}$. Then we see that $\underline{M'} = \mathcal{G}$ by Lemma 44.67.11.

Proof of (2). This follows in the usual manner from (3). Details omitted. □

The following lemma tells us that every object of the abelian category of constructible sheaves on X is "Noetherian", i.e., satisfies a.c.c. for subobjects.

Lemma 44.68.2. *Let X be a Noetherian scheme. Let Λ be a Noetherian ring. Consider inclusions*

$$\mathcal{F}_1 \subset \mathcal{F}_2 \subset \mathcal{F}_3 \subset \ldots \subset \mathcal{F}$$

in the category of sheaves of sets, abelian groups, or Λ-modules. If $\mathcal{F}$ is constructible, then for some n we have $\mathcal{F}_n = \mathcal{F}_{n+1} = \mathcal{F}_{n+2} = \ldots$.

Proof. By Proposition 44.68.1 we see that $\mathcal{F}_i$ and $\operatorname{colim} \mathcal{F}_i$ are constructible. Then the lemma follows from Lemma 44.65.9. □

Lemma 44.68.3. *Let X be a Noetherian scheme.*

(1) *Let $\mathcal{F}$ be a constructible sheaf of sets on $X_{étale}$. There exist an injective map of sheaves*
$$\mathcal{F} \longrightarrow \coprod\nolimits_{i=1,\ldots,n} f_{i,*}\underline{E_i}$$
where $f_i : Y_i \to X$ is a finite morphism and E_i is a finite set.

(2) *Let $\mathcal{F}$ be a constructible abelian sheaf on $X_{étale}$. There exist an injective map of abelian sheaves*
$$\mathcal{F} \longrightarrow \bigoplus\nolimits_{i=1,\ldots,n} f_{i,*}\underline{M_i}$$
where $f_i : Y_i \to X$ is a finite morphism and M_i is a finite abelian group.

(3) *Let Λ be a Noetherian ring. Let $\mathcal{F}$ be a constructible sheaf of Λ-modules on $X_{étale}$. There exist an injective map of sheaves of modules*
$$\mathcal{F} \longrightarrow \bigoplus\nolimits_{i=1,\ldots,n} f_{i,*}\underline{M_i}$$
where $f_i : Y_i \to X$ is a finite morphism and M_i is a finite Λ-module.

Moreover, we may assume each Y_i is irreducible, reduced, maps onto an irreducible and reduced closed subscheme $Z_i \subset X$ such that $Y_i \to Z_i$ is finite étale over a nonempty open of Z_i.

Proof. Proof of (1). Because we have the ascending chain condition for subsheaves of $\mathcal{F}$ (Lemma 44.68.2), it suffices to show that for every point $x \in X$ we can find a map $\varphi : \mathcal{F} \to f_*\underline{E}$ where $f : Y \to X$ is finite and E is a finite set such that $\varphi_{\overline{x}} : \mathcal{F}_{\overline{x}} \to (f_*S)_{\overline{x}}$ is injective. (This argument can be avoided by picking a partition of X as in Lemma 44.65.2 and constructing a $Y_i \to X$ for each irreducible component of each part.) Let $Z \subset X$ be the induced reduced scheme structure (Schemes, Definition 25.12.5) on $\overline{\{x\}}$. Since $\mathcal{F}$ is constructible, there is a finite separable extension $\kappa(x) \subset \operatorname{Spec}(K)$ such that $\mathcal{F}|_{\operatorname{Spec}(K)}$ is the constant sheaf with value E for some finite set E. Let $Y \to Z$ be the normalization of Z in $\operatorname{Spec}(K)$. By Morphisms, Lemma 28.48.10 we see that Y is a normal integral scheme. As $\kappa(x) \subset K$ is finite, it is clear that K is the function field of Y. Denote $g : \operatorname{Spec}(K) \to Y$ the inclusion. The map $\mathcal{F}|_{\operatorname{Spec}(K)} \to \underline{E}$ is adjoint to a map $\mathcal{F}|_Y \to g_*\underline{E} = \underline{E}$ (Lemma 44.67.12). This in turn is adjoint to a map $\varphi : \mathcal{F} \to f_*\underline{E}$. Observe that the stalk of φ at a geometric point $\overline{x}$ is injective: we may take a lift $\overline{y} \in Y$ of $\overline{x}$ and the commutative diagram

$$\begin{array}{ccc} \mathcal{F}_{\overline{x}} & = & (\mathcal{F}|_Y)_{\overline{y}} \\ \downarrow & & \| \\ (f_*\underline{E})_{\overline{x}} & \longrightarrow & \underline{E}_{\overline{y}} \end{array}$$

proves the injectivity. We are not yet done, however, as the morphism $f : Y \to Z$ is integral but in general not finite[5].

To fix the problem stated in the last sentence of the previous paragraph, we write $Y = \lim_{i \in I} Y_i$ with Y_i irreducible, integral, and finite over Z. Namely, apply Properties, Lemma 27.20.13 to $f_*\mathcal{O}_Y$ viewed as a sheaf of $\mathcal{O}_Z$-algebras and apply the functor $\underline{\mathrm{Spec}}_Z$. Then $f_*\underline{E} = \mathrm{colim}\, f_{i,*}\underline{E}$ by Lemma 44.52.5. By Lemma 44.67.8 the map $\mathcal{F} \to f_*\underline{E}$ factors through $f_{i,*}\underline{E}$ for some i. Since $Y_i \to Z$ is a finite morphism of integral schemes and since the function field extension induced by this morphism is finite separable, we see that the morphism is finite étale over a nonempty open of Z (use Algebra, Lemma 10.133.9; details omitted). This finishes the proof of (1).

The proofs of (2) and (3) are identical to the proof of (1). □

In the following lemma we use a standard trick to reduce a very general statement to the Noetherian case.

Lemma 44.68.4. *Let X be a quasi-compact and quasi-separated scheme.*

(1) *Let $\mathcal{F}$ be a constructible sheaf of sets on $X_{étale}$. There exist an injective map of sheaves*
$$\mathcal{F} \longrightarrow \coprod\nolimits_{i=1,\ldots,n} f_{i,*}\underline{E_i}$$
where $f_i : Y_i \to X$ is a finite and finitely presented morphism and E_i is a finite set.

(2) *Let $\mathcal{F}$ be a constructible abelian sheaf on $X_{étale}$. There exist an injective map of abelian sheaves*
$$\mathcal{F} \longrightarrow \bigoplus\nolimits_{i=1,\ldots,n} f_{i,*}\underline{M_i}$$
where $f_i : Y_i \to X$ is a finite and finitely presented morphism and M_i is a finite abelian group.

(3) *Let Λ be a Noetherian ring. Let $\mathcal{F}$ be a constructible sheaf of Λ-modules on $X_{étale}$. There exist an injective map of sheaves of modules*
$$\mathcal{F} \longrightarrow \bigoplus\nolimits_{i=1,\ldots,n} f_{i,*}\underline{M_i}$$
where $f_i : Y_i \to X$ is a finite and finitely presented morphism and M_i is a finite Λ-module.

Proof. We will reduce this lemma to the Noetherian case by absolute Noetherian approximation. Namely, by Limits, Proposition 31.4.4 we can write $X = \lim_{t \in T} X_t$ with each X_t of finite type over $\mathrm{Spec}(\mathbf{Z})$ and with affine transition morphisms. By Lemma 44.67.10 the category of constructible sheaves (of sets, abelian groups, or Λ-modules) on $X_{étale}$ is the colimit of the corresponding categories for X_t. Thus our constructible sheaf $\mathcal{F}$ is the pullback of a similar constructible sheaf $\mathcal{F}_t$ over X_t for some t. Then we apply the Noetherian case (Lemma 44.68.3) to find an injection
$$\mathcal{F}_t \longrightarrow \coprod\nolimits_{i=1,\ldots,n} f_{i,*}\underline{E_i} \quad \text{or} \quad \mathcal{F}_t \longrightarrow \bigoplus\nolimits_{i=1,\ldots,n} f_{i,*}\underline{M_i}$$
over X_t for some finite morphisms $f_i : Y_i \to X_t$. Since X_t is Noetherian the morphisms f_i are of finite presentation. Since pullback is exact and since formation of $f_{i,*}$ commutes with base change (Lemma 44.55.3), we conclude. □

[5] If X is a Nagata scheme, for example of finite type over a field, then $Y \to Z$ is finite.

44.69. Cohomology with support in a closed subscheme

Let X be a scheme and let $Z \subset X$ be a closed subscheme. Let $\mathcal{F}$ be an abelian sheaf on $X_{étale}$. We let

$$\Gamma_Z(X, \mathcal{F}) = \{s \in \mathcal{F}(X) \mid \text{Supp}(s) \subset Z\}$$

be the sections with support in Z (Definition 44.31.3). This is a left exact functor which is not exact in general. Hence we obtain a derived functor

$$R\Gamma_Z(X, -) : D(X_{étale}) \longrightarrow D(Ab)$$

and cohomology groups with support in Z defined by $H^q_Z(X, \mathcal{F}) = R^q\Gamma_Z(X, \mathcal{F})$.

Let $\mathcal{I}$ be an injective abelian sheaf on $X_{étale}$. Then the restriction map $\mathcal{I}(X) \to \mathcal{I}(U)$ is surjective (Cohomology on Sites, Lemma 21.12.6) with kernel $\Gamma_Z(X, \mathcal{I})$. It immediately follows that for $K \in D(X_{étale})$ there is a distinguished triangle

$$R\Gamma_Z(X, K) \to R\Gamma(X, K) \to R\Gamma(U, K) \to R\Gamma_Z(X, K)[1]$$

in $D(Ab)$. As a consequence we obtain a long exact cohomology sequence

$$\ldots \to H^i_Z(X, K) \to H^i(X, K) \to H^i(U, K) \to H^{i+1}_Z(X, K) \to \ldots$$

for any K in $D(X_{étale})$.

For an abelian sheaf $\mathcal{F}$ on $X_{étale}$ we can consider the *subsheaf of sections with support in* Z, denoted $\mathcal{H}_Z(\mathcal{F})$, defined by the rule

$$\mathcal{H}_Z(\mathcal{F})(U) = \{s \in \mathcal{F}(U) \mid \text{Supp}(s) \subset U \times_X Z\}$$

Here we use the support of a section from Definition 44.31.3. Using the equivalence of Proposition 44.47.4 we may view $\mathcal{H}_Z(\mathcal{F})$ as an abelian sheaf on $Z_{étale}$. Thus we obtain a functor

$$Ab(X_{étale}) \longrightarrow Ab(Z_{étale}), \quad \mathcal{F} \longmapsto \mathcal{H}_Z(\mathcal{F})$$

which is left exact, but in general not exact.

Lemma 44.69.1. *Let $i : Z \to X$ be a closed immersion of schemes. Let $\mathcal{I}$ be an injective abelian sheaf on $X_{étale}$. Then $\mathcal{H}_Z(\mathcal{I})$ is an injective abelian sheaf on $Z_{étale}$.*

Proof. Observe that for any abelian sheaf $\mathcal{G}$ on $Z_{étale}$ we have

$$\text{Hom}_Z(\mathcal{G}, \mathcal{H}_Z(\mathcal{F})) = \text{Hom}_X(i_*\mathcal{G}, \mathcal{F})$$

because after all any section of $i_*\mathcal{G}$ has support in Z. Since i_* is exact (Section 44.47) and as $\mathcal{I}$ is injective on $X_{étale}$ we conclude that $\mathcal{H}_Z(\mathcal{I})$ is injective on $Z_{étale}$. □

Denote

$$R\mathcal{H}_Z : D(X_{étale}) \longrightarrow D(Z_{étale})$$

the derived functor. We set $\mathcal{H}^q_Z(\mathcal{F}) = R^q\mathcal{H}_Z(\mathcal{F})$ so that $\mathcal{H}^0_Z(\mathcal{F}) = \mathcal{H}_Z(\mathcal{F})$. By the lemma above we have a Grothendieck spectral sequence

$$E_2^{p,q} = H^p(Z, \mathcal{H}^q_Z(\mathcal{F})) \Rightarrow H^{p+q}_Z(X, \mathcal{F})$$

Lemma 44.69.2. *Let $i : Z \to X$ be a closed immersion of schemes. Let $\mathcal{G}$ be an injective abelian sheaf on $Z_{étale}$. Then $\mathcal{H}^p_Z(i_*\mathcal{G}) = 0$ for $p > 0$.*

Proof. This is true because the functor i_* is exact and transforms injective abelian sheaves into injective abelian sheaves (Cohomology on Sites, Lemma 21.14.2). □

44.70. Affine analog of proper base change

In this section we discuss a result by Ofer Gabber, see [**Gab94**].

Lemma 44.70.1. *Let X be an integral normal scheme with separably closed function field.*

(1) *A separated étale morphism $U \to X$ is a disjoint union of open immersions.*
(2) *All local ring of X are strictly henselian.*

Proof. Let R be a normal domain whose fraction field is separably algebraically closed. Let $R \to A$ be an étale ring map. Then $A \otimes_R K$ is as a K-algebra a finite product $\prod_{i=1,\ldots,n} K$ of copies of K. Let e_i, $i = 1, \ldots, n$ be the corresponding idempotents of $A \otimes_R K$. Since A is normal (Algebra, Lemma 10.149.7) the idempotents e_i are in A (Algebra, Lemma 10.34.11). Hence $A = \prod Ae_i$ and we may assume $A \otimes_R K = K$. Since $A \subset A \otimes_R K = K$ (by flatness of $R \to A$ and since $R \subset K$) we conclude that A is a domain. By the same argument we conclude that $A \otimes_R A \subset (A \otimes_R A) \otimes_R K = K$. It follows that the map $A \otimes_R A \to A$ is injective as well as surjective. Thus $R \to A$ defines an open immersion by Morphisms, Lemma 28.12.2 and Étale Morphisms, Theorem 40.14.1.

Let $f : U \to X$ be a separated étale morphism. Let $\eta \in X$ be the generic point and let $f^{-1}(\{\eta\}) = \{\xi_i\}_{i \in I}$. The result of the previous paragraph shows the following: For any affine open $U' \subset U$ whose image in X is contained in an affine we have $U' = \coprod_{i \in I} U'_i$ where U'_i is the set of point of U' which are specializations of ξ_i. Moreover, the morphism $U'_i \to X$ is an open immersion. It follows that $U_i = \overline{\{\xi_i\}}$ is an open and closed subscheme of U and that $U_i \to X$ is locally on the source an isomorphism. By Morphisms, Lemma 28.10.7 the fact that $U_i \to X$ is separated, implies that $U_i \to X$ is injective and we conclude that $U_i \to X$ is an open immersion, i.e., (1) holds.

Part (2) follows from part (1) and the description of the strict henselization of $\mathcal{O}_{X,x}$ as the local ring at $\overline{x}$ on the étale site of X (Lemma 44.33.1). □

Lemma 44.70.2. *Let X be an affine integral normal scheme with separably closed function field. Let $Z \subset X$ be a closed subscheme. Let $V \to Z$ be an étale morphism with V affine. Then V is a finite disjoint union of open subschemes of Z. If $V \to Z$ is surjective and finite étale, then $V \to Z$ has a section.*

Proof. By Algebra, Lemma 10.136.11 we can lift V to an affine scheme U étale over X. Apply Lemma 44.70.1 to $U \to X$ to get the first statement.

The final statement is a consequence of the first. Let $V = \coprod_{i=1,\ldots,n} V_i$ be a finite decomposition into open and closed subschemes with $V_i \to Z$ an open immersion. As $V \to Z$ is finite we see that $V_i \to Z$ is also closed. Let $U_i \subset Z$ be the image. Then we have a decomposition into open and closed subshemes

$$Z = \coprod_{(A,B)} \bigcap_{i \in A} U_i \cap \bigcap_{i \in B} U_i^c$$

where the disjoint union is over $\{1, \ldots, n\} = A \amalg B$ where A has at least one element. Each of the strata is contained in a single U_i and we find our section. □

Lemma 44.70.3. *Let X be a normal integral affine scheme with with separably closed function field. Let $Z \subset X$ be a closed subscheme. For any finite abelian group M we have $H^1_{étale}(Z, \underline{M}) = 0$.*

Proof. By Cohomology on Sites, Lemma 21.5.3 an element of $H^1_{étale}(Z, \underline{M})$ corresponds to a $\underline{M}$-torsor $\mathcal{F}$ on $Z_{étale}$. Such a torsor is clearly a finite locally constant sheaf. Hence $\mathcal{F}$ is representable by a scheme V finite étale over Z, Lemma 44.64.4. Of course $V \to Z$ is surjective as a torsor is locally trivial. Since $V \to Z$ has a section by Lemma 44.70.2 we are done. □

Lemma 44.70.4. *Let X be a normal integral affine scheme with with separably closed function field. Let $Z \subset X$ be a closed subscheme. For any finite abelian group M we have $H^q_{étale}(Z, \underline{M}) = 0$ for $q \geq 1$.*

Proof. We have seen that the result is true for H^1 in Lemma 44.70.3. We will prove the result for $q \geq 2$ by induction on q. Let $\xi \in H^q_{étale}(Z, \underline{M})$.

Let $X = \mathrm{Spec}(R)$. Let $I \subset R$ be the set of elements $f \in R$ sch that $\xi|_{Z \cap D(f)} = 0$. All local rings of Z are strictly henselian by Lemma 44.70.1 and Algebra, Lemma 10.143.30. Hence étale cohomology on Z or open subschemes of Z is equal to Zariski cohomology, see Lemma 44.55.6. In particular ξ is Zariski locally trivial. It follows that for every prime $\mathfrak{p}$ of R there exists an $f \in I$ with $f \notin \mathfrak{p}$. Thus if we can show that I is an ideal, then $1 \in I$ and we're done. It is clear that $f \in I$, $r \in R$ implies $rf \in I$. Thus we now assume that $f, g \in I$ and we show that $f + g \in I$. Note that

$$D(f + g) \cap Z = D(f(f + g)) \cap Z \cup D(g(f + g)) \cap Z$$

By Mayer-Vietoris (Cohomology, Lemma 20.9.2 which applies as étale cohomology on open subschemes of Z equals Zariski cohomology) we have an exact sequence

$$\begin{array}{c} H^{q-1}_{étale}(D(fg(f + g)) \cap Z, \underline{M}) \\ \downarrow \\ H^q_{étale}(D(f + g) \cap Z, \underline{M}) \\ \downarrow \\ H^q_{étale}(D(f(f + g)) \cap Z, \underline{M}) \oplus H^q_{étale}(D(g(f + g)) \cap Z, \underline{M}) \end{array}$$

and the result follows as the first group is zero by induction. □

Lemma 44.70.5. *Let X be an affine scheme.*

1. *There exists an integral surjective morphism $X' \to X$ such that for every closed subscheme $Z' \subset X'$, every finite abelian group M, and every $q \geq 1$ we have $H^q_{étale}(Z', \underline{M}) = 0$.*
2. *For any closed subscheme $Z \subset X$, finite abelian group M, $q \geq 1$, and $\xi \in H^q_{étale}(Z, \underline{M})$ there exists a finite surjective morphism $X' \to X$ of finite presentation such that ξ pulls back to zero in $H^q_{étale}(X' \times_X Z, \underline{M})$.*

Proof. Write $X = \mathrm{Spec}(A)$. Write $A = \mathbf{Z}[x_i]/J$ for some ideal J. Let R be the integral closure of $\mathbf{Z}[x_i]$ in an algebraic closure of the fraction field of $\mathbf{Z}[x_i]$. Let $A' = R/JR$ and set $X' = \mathrm{Spec}(A')$. This gives an example as in (1) by Lemma 44.70.4.

Proof of (2). Let $X' \to X$ be the integral surjective morphism we found above. Certainly, ξ maps to zero in $H^q_{étale}(X' \times_X Z, \underline{M})$. We may write X' as a limit $X' = \lim X'_i$ of schemes finite and of finite presentation over X; this is easy to do in our current affine case, but it is a special case of the more general Limits, Lemma 31.6.2. By Lemma 44.52.3 we see that ξ maps to zero in $H^q_{étale}(X'_i \times_X Z, \underline{M})$ for some i large enough. □

Lemma 44.70.6. *Let X be an affine scheme. Let $\mathcal{F}$ be a torsion abelian sheaf on $X_{étale}$. Let $Z \subset X$ be a closed subscheme. Let $\xi \in H^q_{étale}(Z, \mathcal{F}|_Z)$ for some $q > 0$. Then there exists an injective map $\mathcal{F} \to \mathcal{F}'$ of torsion abelian sheaves on $X_{étale}$ such that the image of ξ in $H^q_{étale}(Z, \mathcal{F}'|_Z)$ is zero.*

Proof. By Lemmas 44.67.2 and 44.52.2 we can find a map $\mathcal{G} \to \mathcal{F}$ with $\mathcal{G}$ a constructible abelian sheaf and ξ coming from an element ζ of $H^q_{étale}(Z, \mathcal{G}|_Z)$. Suppose we can find an injective map $\mathcal{G} \to \mathcal{G}'$ of torsion abelian sheaves on $X_{étale}$ such that the image of ζ in $H^q_{étale}(Z, \mathcal{G}'|_Z)$ is zero. Then we can take $\mathcal{F}'$ to be the pushout

$$\mathcal{F}' = \mathcal{G}' \amalg_{\mathcal{G}} \mathcal{F}$$

and we conclude the result of the lemma holds. (Observe that restriction to Z is exact, so commutes with finite limits and colimits and moreover it commutes with arbitrary colimits as a left adjoint to pushforward.) Thus we may assume $\mathcal{F}$ is constructible.

Assume $\mathcal{F}$ is constructible. By Lemma 44.68.4 it suffices to prove the result when $\mathcal{F}$ is of the form $f_*\underline{M}$ where M is a finite abelian group and $f : Y \to X$ is a finite morphism of finite presentation (such sheaves are still constructible by Lemma 44.67.9 but we won't need this). Since formation of f_* commutes with any base change (Lemma 44.55.3) we see that the restriction of $f_*\underline{M}$ to Z is equal to the pushforward of $\underline{M}$ via $Y \times_X Z \to Z$. By the Leray spectral sequence (Proposition 44.54.2) and vanishing of higher direct images (Proposition 44.55.2), we find

$$H^q_{étale}(Z, f_*\underline{M}|_Z) = H^q(Y \times_X Z, \underline{M}).$$

By Lemma 44.70.5 we can find a finite surjective morphism $Y' \to Y$ of finite presentation such that ξ maps to zero in $H^q(Y' \times_X Z, \underline{M})$. Denoting $f' : Y' \to X$ the compostion $Y' \to Y \to X$ we claim the map

$$f_*\underline{M} \longrightarrow f'_*\underline{M}$$

is injective which finishes the proof by what was said above. To see the desired injectivity we can look at stalks. Namely, if $\overline{x} : \mathrm{Spec}(k) \to X$ is a geometric point, then

$$(f_*\underline{M})_{\overline{x}} = \bigoplus\nolimits_{f(\overline{y})=\overline{x}} M$$

by Proposition 44.55.2 and similarly for the other sheaf. Since $Y' \to Y$ is surjective and finite we see that the induced map on geometric points lifting $\overline{x}$ is surjective too and we conclude. □

The lemma above will take care of higher cohomology groups in Gabber's result. The following lemma will be used to deal with global sections.

Lemma 44.70.7. *Let X be a quasi-compact and quasi-separated scheme. Let $i : Z \to X$ be a closed immersion. Assume that*

(1) *for any sheaf $\mathcal{F}$ on X_{Zar} the map $\Gamma(X, \mathcal{F}) \to \Gamma(Z, i^{-1}\mathcal{F})$ is bijective, and*

(2) *for any finite morphism $X' \to X$ assumption (1) holds for $Z \times_X X' \to X'$.*

Then for any sheaf $\mathcal{F}$ on $X_{étale}$ we have $\Gamma(X, \mathcal{F}) = \Gamma(Z, i^{-1}_{small}\mathcal{F})$.

Proof. Let $\mathcal{F}$ be a sheaf on $X_{étale}$. There is a canonical (base change) map

$$i^{-1}(\mathcal{F}|_{X_{Zar}}) \longrightarrow (i^{-1}_{small}\mathcal{F})|_{Z_{Zar}}$$

of sheaves on Z_{Zar}. This map is injective as can be seen by looking on stalks. The stalk on the left hand side at $z \in Z$ is the stalk of $\mathcal{F}|_{X_{Zar}}$ at z. The stalk on the right hand side is the colimit over all elementary étale neighbourhoods $(U, u) \to (X, z)$ such that $U \times_X Z \to Z$ has a section over a neighbourhood of z. As étale morphisms are open, the image of $U \to X$ is an open neighbourhood of z in X and injectivity follows.

It follows from this and assumption (1) that the map $\Gamma(X, \mathcal{F}) \to \Gamma(Z, i^{-1}_{small}\mathcal{F})$ is injective. By (2) the same thing is true on all X' finite over X.

Let $s \in \Gamma(Z, i^{-1}_{small}\mathcal{F})$. By construction of $i^{-1}_{small}\mathcal{F}$ there exists an étale covering $\{V_j \to Z\}$, étale morphisms $U_j \to X$, sections $s_j \in \mathcal{F}(U_j)$ and morphisms $V_j \to U_j$ over X such that $s|_{V_j}$ is the pullback of s_j. Observe that every closed subscheme $T \subset X$ meets Z by assumption (1) applied to the sheaf $(T \to X)_*\underline{\mathbf{Z}}$ for example. Thus we see that $\coprod U_j \to X$ is surjective. By More on Morphisms, Lemma 36.31.13 we can find a finite surjective morphism $X' \to X$ such that $X' \to X$ Zariski locally factors through $\coprod U_j \to X$. It follows that $s|_{Z'}$ Zariski locally comes from a section of $\mathcal{F}|_{X'}$. In other words, $s|_{Z'}$ comes from $t' \in \Gamma(X', \mathcal{F}|_{X'})$ by assumption (2). By injectivity we conclude that the two pullbacks of t' to $X' \times_X X'$ are the same (after all this is true for the pullbacks of s to $Z' \times_Z Z'$). Hence we conclude t' comes from a section of $\mathcal{F}$ over X by Remark 44.55.5. □

Lemma 44.70.8. *Let X be a topological space and let $Z \subset X$ be a closed subset. Suppose that for every $x \in X$ the intersection $Z \cap \overline{\{x\}}$ is connected (in particular nonempty). Then for any sheaf $\mathcal{F}$ on X we have $\Gamma(X, \mathcal{F}) = \Gamma(Z, \mathcal{F}|_Z)$.*

Proof. Let's view a global section of $\mathcal{F}$ as an assignment $x \mapsto s_x \in \mathcal{F}_x$ satisfying the continuity property (*) introduced in Sheaves, Section 6.17. If $x \rightsquigarrow z$ is a specialization on X, then there is a corresponding map on stalks $\mathcal{F}_z \to \mathcal{F}_x$. Thus, given a global section $s = (s_z)_{z \in Z}$ of $\mathcal{F}|_Z$ we can assign to every $x \in X$ a value s_x by chooseing a $z \in Z \cap \overline{\{x\}}$ and taking the image of s_z. The fact that s_x is independent of the choice of z comes from the fact that we assumed $Z \cap \overline{\{x\}}$ is connected (details omitted). It is clear that this rule satisfies (*) and provides us with a section $\tilde{s}$ of $\mathcal{F}$ over X which restricts to s. □

Lemma 44.70.9. *Let (A, I) be a henselian pair. Set $X = \mathrm{Spec}(A)$ and $Z = \mathrm{Spec}(A/I)$. For any sheaf $\mathcal{F}$ on $X_{étale}$ we have $\Gamma(X, \mathcal{F}) = \Gamma(Z, \mathcal{F}|_Z)$.*

Proof. Combine Lemmas 44.70.7 and 44.70.8 and More on Algebra, Lemmas 15.7.7 and 15.7.8. □

Finally, we can state and prove Gabber's theorem.

Theorem 44.70.10 (Gabber)**.** *Let (A, I) be a henselian pair. Set $X = \mathrm{Spec}(A)$ and $Z = \mathrm{Spec}(A/I)$. For any torsion abelian sheaf $\mathcal{F}$ on $X_{étale}$ we have $H^q_{étale}(X, \mathcal{F}) = H^q_{étale}(Z, \mathcal{F}|_Z)$.*

Proof. The result holds for $q = 0$ by Lemma 44.70.9. Let $q \geq 1$. Suppose the result has been shown in all degrees $< q$. Let $\mathcal{F}$ be a torsion abelian sheaf. Let $\mathcal{F} \to \mathcal{F}'$ be an injective map of torsion abelian sheaves (to be chosen later) with cokernel $\mathcal{G}$ so that we have the short exact sequence

$$0 \to \mathcal{F} \to \mathcal{F}' \to \mathcal{Q} \to 0$$

of torsion abelian sheaves on $X_{étale}$. This gives a map of long exact cohomology sequences over X and Z part of which looks like

$$\begin{array}{ccccccc} H^{q-1}_{étale}(X, \mathcal{F}') & \longrightarrow & H^{q-1}_{étale}(X, \mathcal{Q}) & \longrightarrow & H^q_{étale}(X, \mathcal{F}) & \longrightarrow & H^q_{étale}(X, \mathcal{F}') \\ \downarrow & & \downarrow & & \downarrow & & \downarrow \\ H^{q-1}_{étale}(Z, \mathcal{F}'|_Z) & \longrightarrow & H^{q-1}_{étale}(Z, \mathcal{Q}|_Z) & \longrightarrow & H^q_{étale}(Z, \mathcal{F}|_Z) & \longrightarrow & H^q_{étale}(Z, \mathcal{F}'|_Z) \end{array}$$

Using this commutative diagram of abelian groups with exact rows we will finish the proof.

Injectivity for $\mathcal{F}$. Let ξ be a nonzero element of $H^q_{étale}(X, \mathcal{F})$. By Lemma 44.70.6 applied with $Z = X$ (!) we can find $\mathcal{F} \subset \mathcal{F}'$ such that ξ maps to zero to the right. Then ξ is the image of an element of $H^{q-1}_{étale}(X, \mathcal{Q})$ and bijectivity for $q - 1$ implies ξ does not map to zero in $H^q_{étale}(Z, \mathcal{F}|_Z)$.

Surjectivity for $\mathcal{F}$. Let ξ be an element of $H^q_{étale}(Z, \mathcal{F}|_Z)$. By Lemma 44.70.6 applied with $Z = Z$ we can find $\mathcal{F} \subset \mathcal{F}'$ such that ξ maps to zero to the right. Then ξ is the image of an element of $H^{q-1}_{étale}(Z, \mathcal{Q}|_Z)$ and bijectivity for $q-1$ implies ξ is in the image of the vertical map. □

Lemma 44.70.11. *Let (A, I) be a henselian pair. Set $X = \mathrm{Spec}(A)$ and $Z = \mathrm{Spec}(A/I)$. The functor*

$$U \longmapsto U \times_X Z$$

is an equivalence of categories between finite étale schemes over X and finite étale schemes over Z.

Proof. This is a translation of More on Algebra, Lemma 15.7.9. □

44.71. Higher vanishing for torsion sheaves

The goal of this section is to prove the result that follows now.

Theorem 44.71.1. *Let X be an affine curve over an algebraically closed field k and $\mathcal{F}$ a torsion sheaf on $X_{étale}$. Then $H^q_{étale}(X, \mathcal{F}) = 0$ for all $q \geq 2$.*

Proof. We begin by reducing the proof to a more simpler statement.

(1) *If suffices to prove the vanishing when $\mathcal{F}$ is a constructible sheaf.*

Using the compatibility of étale cohomology with colimits and Lemma 44.67.2, we have $\mathrm{colim}\, H^q_{étale}(X, \mathcal{F}) = H^q_{étale}(X, \mathrm{colim}\, \mathcal{F}_i)$ for some constructible sheaves $\mathcal{F}_i$, whence the result.

(2) *It suffices to assume that $\mathcal{F} = j_!\mathcal{G}$ where $\mathcal{U} \subset X$ is open, $\mathcal{G}$ is finite locally constant on $\mathcal{U}$ smooth.*

Choose a nonempty open $\mathcal{U} \subset X$ such that $\mathcal{F}|_{\mathcal{U}}$ is finite locally constant, and consider the exact sequence

$$0 \to j_!(\mathcal{F}|_{\mathcal{U}}) \to \mathcal{F} \to Q \to 0.$$

By looking at stalks we get $Q_{\bar{x}} = 0$ unless $\bar{x} \in X - U$. It follows that $Q = \bigoplus_{x \in X-U} i_{x*}(Q_x)$ which has no higher cohomology.

(3) *It suffices to assume that X is smooth and affine (over k), $\mathcal{G}$ is a finite locally constant sheaf on a open $\mathcal{U}$ of X and $\mathcal{F} = j_!\mathcal{G}$.*

Let $\mathcal{U}$, X and $\mathcal{G}$ be as in the step 2, and consider the commutative diagram

$$\begin{array}{ccc} & & X^\nu \\ & {\scriptstyle j^\nu}\nearrow & \downarrow{\scriptstyle \nu} \\ \mathcal{U} & \xrightarrow{\ j\ } & X \end{array}$$

where $\nu : X^\nu \to X$ is the normalization of X. Since ν is finite, $H^*_{étale}(X, j_!\mathcal{G}) = H^*_{étale}(X^\nu, j^\nu_!\mathcal{G})$, which implies that $\nu_*((j^\nu)_!\mathcal{G}) = j_!\mathcal{G}$ by looking at stalks. We are thus reduced to proving the following lemma. □

Lemma 44.71.2. *Let X be a smooth affine curve over an algebraically closed field k, $j : \mathcal{U} \hookrightarrow X$ an open immersion and $\mathcal{F}$ a finite locally constant sheaf on $\mathcal{U}_{étale}$. Then for all $q \geq 2$, $H^q_{étale}(X, j_!\mathcal{F}) = 0$.*

Proof. The proof of this follows the "méthode de la trace" as explained in [**AGV71**, Exposé IX, §5]. See discussion below. □

Definition 44.71.3. Let $f : Y \to X$ be a finite étale morphism. There are pairs of adjoint functors $(f_!, f^{-1})$ and (f^{-1}, f_*) on $Ab(X_{étale})$. The adjunction map $\mathrm{id} \to f_*f^{-1}$ is called *restriction*. Since f is finite, $f_! = f_*$ and the adjunction map $f_*f^{-1} = f_!f^{-1} \to \mathrm{id}$ is called the *trace*.

The trace map is characterized by the following two properties:

(1) it commutes with étale localization ; and
(2) if $f : Y = \coprod_{i=1}^d X \to X$ then the trace map is just the sum map $f_*f^{-1}\mathcal{F} = \mathcal{F}^{\oplus d} \to \mathcal{F}$.

It follows that if f has constant degree d, then the composition $\mathcal{F} \xrightarrow{res} f_*f^{-1}\mathcal{F} \xrightarrow{trace} \mathcal{F}$ is multiplication by d. The "méthode" then essentially consists in the following observation: if $\mathcal{F}$ is an abelian sheaf on $X_{étale}$ such that multiplication by d is an isomorphism $\mathcal{F} \cong \mathcal{F}$, and if furthermore $H^q_{étale}(Y, f^{-1}\mathcal{F}) = 0$ then $H^q_{étale}(X, \mathcal{F}) = 0$ as well. Indeed, multiplication by d induces an isomorphism on $H^q_{étale}(X, \mathcal{F})$ which factors through $H^q_{étale}(Y, f^{-1}\mathcal{F}) = 0$.

Using this method, we further reduce the proof of Lemma 44.71.2 to a yet simpler statement.

(4) *We may assume that $\mathcal{F}$ is killed by a prime ℓ.*

Writing $\mathcal{F} = \mathcal{F}_1 \oplus \cdots \oplus \mathcal{F}_r$ where $\mathcal{F}_i$ is ℓ_i-primary for some prime ℓ_i, we may assume that ℓ^n kills $\mathcal{F}$ for some prime ℓ. Now consider the exact sequence

$$0 \to \mathcal{F}[\ell] \to \mathcal{F} \to \mathcal{F}/\mathcal{F}[\ell] \to 0.$$

Applying the exact functor $j_!$ and looking at the long exact cohomology sequence, we see that it suffices to assume that $\mathcal{F}$ is ℓ-torsion, which we do.

(5) *There exists a finite étale morphism $f : \mathcal{V} \to \mathcal{U}$ of degree prime to ℓ such that $f^{-1}\mathcal{F}$ has a filtration*

$$0 \subset \mathcal{G}_1 \subset \mathcal{G}_2 \subset \cdots \subset \mathcal{G}_s = f^{-1}\mathcal{F}$$

with $\mathcal{G}_i/\mathcal{G}_{i-1} \cong \underline{\mathbf{Z}/\ell\mathbf{Z}}_{\mathcal{V}}$ for all $i \leq s$.

Since $\mathcal{F}$ is finite locally constant, there exists a finite étale Galois cover $h : \mathcal{U}' \to \mathcal{U}$ such that $h^{-1}\mathcal{F} \cong \underline{A}_{\mathcal{U}'}$ for some finite abelian group A. Note that $A \cong (\mathbf{Z}/\ell\mathbf{Z})^{\oplus m}$ for some m. Saying that the cover is *Galois* means that the finite group $G = \mathrm{Aut}(\mathcal{U}'|\mathcal{U})$ has (maximal) cardinality $\#G = \deg h$. Now let $H \subset G$ be the ℓ-Sylow, and set

$$\mathcal{U}' \xrightarrow{\pi} \mathcal{V} = \mathcal{U}'/H \xrightarrow{f} \mathcal{U}.$$

The quotient exists by taking invariants (schemes are affine). By construction, $\deg f = \#G/\#H$ is prime to ℓ. The sheaf $\mathcal{G} = f^{-1}\mathcal{F}$ is then a finite locally constant sheaf on $\mathcal{V}$ and

$$\pi^{-1}\mathcal{G} = h^{-1}\mathcal{F} \cong \underline{(\mathbf{Z}/\ell\mathbf{Z})}^{\oplus m}_{\mathcal{U}'}.$$

Moreover,

$$H^0_{étale}(\mathcal{V}, \mathcal{G}) = H^0_{étale}(\mathcal{U}', \pi^{-1}\mathcal{G})^H = \left((\mathbf{Z}/\ell\mathbf{Z})^{\oplus m}\right)^H \neq 0,$$

where the first equality follows from writing out the sheaf condition for $\mathcal{G}$ (again, schemes are affine), and the last inequality is an exercise in linear algebra over $\mathbf{F}_\ell$. Following, we have found a subsheaf $\underline{\mathbf{Z}/\ell\mathbf{Z}}_{\mathcal{V}} \hookrightarrow \mathcal{G}$. Repeating the argument for the quotient $\mathcal{G}/\underline{\mathbf{Z}/\ell\mathbf{Z}}_{\mathcal{V}}$ if necessary, we eventually get a subsheaf of $\mathcal{G}$ with quotient $\underline{\mathbf{Z}/\ell\mathbf{Z}}_{\mathcal{V}}$. This is the first step of the filtration.

Exercise 44.71.4. Let $f : X \to Y$ be a finite étale morphism with Y noetherian, and X, Y irreducible. Then there exists a finite étale Galois morphism $X' \to Y$ which dominates X over Y.

(6) *We consider the normalization Y of X in $\mathcal{V}$, that is, we have the commutative diagram*

$$\begin{array}{ccc} \mathcal{V} & \xrightarrow{j'} & Y \\ \downarrow{\scriptstyle f} & & \downarrow{\scriptstyle f'} \\ \mathcal{U} & \xrightarrow{j} & X. \end{array}$$

Then there is an injection $H^q_{étale}(X, j_!\mathcal{F}) \hookrightarrow H^q_{étale}(Y, j'_!f^{-1}\mathcal{F})$ for all q.

We have seen that the composition $\mathcal{F} \xrightarrow{res} f_*f^{-1}\mathcal{F} \xrightarrow{trace} \mathcal{F}$ is multiplication by the degree of f, which is prime to ℓ. On the other hand,

$$j_!f_*f^{-1}\mathcal{F} = j_!f_!f^{-1}\mathcal{F} = f'_*j'_!f^{-1}\mathcal{F}$$

since f and f' are both finite and the above diagram is commutative. Hence applying $j_!$ to the previous sequence gives a sequence

$$j_!\mathcal{F} \longrightarrow f'^*j'_!f^{-1}\mathcal{F} \longrightarrow j_!\mathcal{F}.$$

Taking cohomology, we see that $H^q_{étale}(X, j_!\mathcal{F})$ injects into $H^q_{étale}(X, f'^*j'_!f^{-1}\mathcal{F})$. But since f' is finite, this is merely $H^q_{étale}(Y, j'_!f^{-1}\mathcal{F})$, as desired.

(7) *It suffices to prove $H^q_{étale}(Y, j'_!\underline{\mathbf{Z}/\ell\mathbf{Z}}) = 0$.*

By Step 3, it suffices to show vanishing of $H^q_{étale}(Y, j'_! f^{-1}\mathcal{F})$. But then by Step 2, we may assume that $f^{-1}\mathcal{F}$ has a finite filtration with quotients isomorphic to $\underline{\mathbf{Z}/n\mathbf{Z}}$, whence the claim.

Finally, we are reduced to proving the following lemma.

Lemma 44.71.5. *Let X be a smooth affine curve over an algebraically closed field, $j : \mathcal{U} \hookrightarrow X$ an open immersion and ℓ a prime number. Then for all $q \geq 2$, $H^q_{étale}(X, j_!\underline{\mathbf{Z}/\ell\mathbf{Z}}) = 0$.*

Proof. Consider the short exact sequence

$$0 \longrightarrow j_!\underline{\mathbf{Z}/\ell\mathbf{Z}}_{\mathcal{U}} \longrightarrow \underline{\mathbf{Z}/\ell\mathbf{Z}}_X \longrightarrow \bigoplus_{x \in X - \mathcal{U}} i_{x*}(\underline{\mathbf{Z}/\ell\mathbf{Z}}) \longrightarrow 0.$$

We know that the cohomology of the middle sheaf vanishes in degree at least 2 by Lemma 44.62.4 and that of the skyscraper sheaf on the right vanishes in degree at least 1. Thus applying the long exact cohomology sequence, we get the vanishing of $j_!\underline{\mathbf{Z}/\ell\mathbf{Z}}_{\mathcal{U}}$ in degree at least 2. This finishes the proof of the lemma, hence of Lemma 44.71.2, hence of Theorem 44.71.1. □

Remarks 44.71.6. Here are some remarks about what happened above.

- This method is very general. For instance, it applies in Galois cohomology, and this is essentially how Proposition 44.60.7 is proved.
- In fact, we have overlooked the case where ℓ is the characteristic of the field k, since the Kummer sequence is not exact then and we cannot use Lemma 44.62.4 anymore. The result is still true, as shown by considering the *Artin-Schreier* exact sequence for a scheme S of characteristic $p > 0$, namely
$$0 \longrightarrow \underline{\mathbf{Z}/p\mathbf{Z}}_S \longrightarrow \mathbf{G}_{a,S} \xrightarrow{F-1} \mathbf{G}_{a,S} \longrightarrow 0$$
where $F - 1$ is the map $x \mapsto x^p - x$. Using this, it can be shown that is S is affine then $H^q_{étale}(S, \underline{\mathbf{Z}/p\mathbf{Z}}) = 0$ for all $q \geq 2$. It also follows that if S is a Noetherian scheme of characteristic p and dimension d, then $H^q_{étale}(S, \underline{\mathbf{Z}/p\mathbf{Z}}) = 0$ for all $q \geq 2 + d$ (this uses Cohomology, Lemma 20.21.5).
- If X is separated of finite type over an algebraically closed field k of characteristic p, then it can be shown that in fact $H^q_{étale}(X, \underline{\mathbf{Z}/p\mathbf{Z}}) = 0$ for $q >= dim(X) + 1$. Sketch of proof: Let $d = \dim(X)$. It suffices to show that $H^{d+1}_{étale}(X, \underline{\mathbf{Z}/p\mathbf{Z}}) = 0$. Using Chow's lemma one can show that $H^d(X, \mathcal{F})$ is a finite dimensional k-vector space for every coherent sheaf $\mathcal{F}$ on X. For any p-linear endomorphism F of a finite dimensional k-vector space, the map $1 - F$ is surjective (as k is assumed algebraically closed), so the Artin-Schreier sequence does the rest.
- If X is a projective curve over an algebraically closed field then $H^q_{étale}(X, \mathcal{F}) = 0$ for all $q \geq 3$ and all torsion sheaves $\mathcal{F}$ on $X_{étale}$. This can be shown using Serre's Mayer Vietoris argument, thereby proving Theorem 44.62.6.
- We can prove using the same methods vanishing of higher cohomology on 1-dimensional schemes of finite type over an algebraically closed field. However, it is easier to reduce to the case of a curve by using the topological invariance of étale cohomology as stated below.

Proposition 44.71.7 (Topological invariance of étale cohomology)**.** *Let X be a scheme and $X_0 \hookrightarrow X$ a closed immersion defined by a nilpotent sheaf of ideals. Then the étale sites $X_{étale}$ and $(X_0)_{étale}$ are isomorphic. In particular, for any sheaf $\mathcal{F}$ on $X_{étale}$, $H^q(X, \mathcal{F}) = H^q(X_0, \mathcal{F}|_{X_0})$ for all q.*

Proof. Follows from the more general Theorem 44.46.1 and formal arguments. □

44.72. The proper base change theorem

The proper base change theorem is stated and proved in this section. Our approach follows roughly the proof in [**AGV71**, XII, Theorem 5.1] using Gabber's ideas (from the affine case) to slightly simplify the arguments.

Lemma 44.72.1. *Let (A, I) be a henselian pair. Let $f : X \to \mathrm{Spec}(A)$ be a proper morphism of schemes. Let $Z = X \times_{\mathrm{Spec}(A)} \mathrm{Spec}(A/I)$. For any sheaf $\mathcal{F}$ on the topological space associated to X we have $\Gamma(X, \mathcal{F}) = \Gamma(Z, \mathcal{F}|_Z)$.*

Proof. We will use Lemma 44.70.8 to prove this. To do this let $Y \subset X$ be an irreducible closed subscheme. We have to show that $Y \cap Z = Y \times_{\mathrm{Spec}(A)} \mathrm{Spec}(A/I)$ is connected. Thus we may assume that X is irreducible and we have to show that Z is connected. Let $X \to \mathrm{Spec}(B) \to \mathrm{Spec}(A)$ be the Stein factorization of f (More on Morphisms, Theorem 36.36.4). Then $A \to B$ is integral and (B, IB) is a henselian pair (More on Algebra, Lemma 15.7.7). Thus we may assume the fibres of $X \to \mathrm{Spec}(A)$ are geometrically connected. On the other hand, the image $T \subset \mathrm{Spec}(A)$ of f is irreducible and closed as X is proper over A. Hence $T \cap V(I)$ is connected by More on Algebra, Lemma 15.7.8. Now $Y \times_{\mathrm{Spec}(A)} \mathrm{Spec}(A/I) \to T \cap V(I)$ is a surjective closed map with connected fibres. The result now follows from Topology, Lemma 5.6.4. □

Lemma 44.72.2. *Let (A, I) be a henselian pair. Let $f : X \to \mathrm{Spec}(A)$ be a proper morphism of schemes. Let $i : Z \to X$ be the closed immersion of $X \times_{\mathrm{Spec}(A)} \mathrm{Spec}(A/I)$ into X. For any sheaf $\mathcal{F}$ on $X_{étale}$ we have $\Gamma(X, \mathcal{F}) = \Gamma(Z, i_{small}^{-1}\mathcal{F})$.*

Proof. This follows from Lemma 44.70.7 and 44.72.1 and the fact that any scheme finite over X is proper over $\mathrm{Spec}(A)$. □

Theorem 44.72.3. *Let $f : X \to Y$ be a proper morphism of schemes. Let $g : Y' \to Y$ be a morphism of schemes giving rise to the base change diagram*

$$\begin{array}{ccc} X' & \xrightarrow{g'} & X \\ {\scriptstyle f'}\downarrow & & \downarrow{\scriptstyle f} \\ Y' & \xrightarrow{g} & Y \end{array}$$

Let $\mathcal{F}$ be an abelian torsion sheaf on $X_{étale}$. Then the base change map

$$g^{-1}Rf_*\mathcal{F} \longrightarrow Rf'_*(g')^{-1}\mathcal{F}$$

is an isomorphism.

Proof. The map is defined in Cohomology on Sites, Lemma 21.15.1. We omit the proof. □

As an application of the proper base change theorem we obtain the following.

Lemma 44.72.4. *Let $f : X \to Y$ be a proper morphism of schemes all of whose fibres have dimension $\leq n$. Then for any abelian torsion sheaf $\mathcal{F}$ on $X_{étale}$ we have $R^q f_* \mathcal{F} = 0$ for $q > 2n$.*

Proof. Omitted. Hints: By the proper base change theorem it suffices to prove that for a proper scheme X over an algebraically closed field, the étale cohomology of $\mathcal{F}$ vanishes above $2 \dim X$. By the proper base change theorem and dévissage (using Chow's lemma for example) one can reduce to the case where the dimension of X is 1. The case of curves is Theorem 44.62.6. See also Remarks 44.71.6. □

44.73. The trace formula

A typical course in étale cohomology would normally state and prove the proper and smooth base change theorems, purity and Poincaré duality. All of these can be found in [**Del77**, Arcata]. Instead, we are going to study the trace formula for the frobenius, following the account of Deligne in [**Del77**, Rapport]. We will only look at dimension 1, but using proper base change this is enough for the general case. Since all the cohomology groups considered will be étale, we drop the subscript $_{étale}$. Let us now describe the formula we are after. Let X be a finite type scheme of dimension 1 over a finite field k, ℓ a prime number and $\mathcal{F}$ a constructible, flat $\mathbf{Z}/\ell^n\mathbf{Z}$ sheaf. Then

$$\sum\nolimits_{x \in X(k)} \mathrm{Tr}(\mathrm{Frob}|\mathcal{F}_{\bar{x}}) = \sum\nolimits_{i=0}^{2} (-1)^i \mathrm{Tr}(\pi_X^* | H_c^i(X \otimes_k \bar{k}, \mathcal{F})) \tag{44.73.0.1}$$

as elements of $\mathbf{Z}/\ell^n\mathbf{Z}$. As we will see, this formulation is slightly wrong as stated. Let us nevertheless describe the symbols that occur therein.

44.74. Frobenii

In this section we will prove a "baffling" theorem. A topological analogue of the baffling theorem is the following.

Exercise 44.74.1. Let X be a topological space and $g : X \to X$ a continuous map such that $g^{-1}(U) = U$ for all opens U of X. Then g induces the identity on cohomology on X (for any coefficients).

We now turn to the statement for the étale site.

Lemma 44.74.2. *Let X be a scheme and $g : X \to X$ a morphism. Assume that for all $\varphi : U \to X$ étale, there is an isomorphism*

$$\begin{array}{ccc} U & \xrightarrow{\sim} & U \times_{\varphi, X, g} X \\ & {\scriptstyle \varphi} \searrow \quad \swarrow {\scriptstyle pr_2} & \\ & X & \end{array}$$

functorial in U. Then g induces the identity on cohomology (for any sheaf).

Proof. The proof is formal and without difficulty. □

Definition 44.74.3. Let X be a scheme in characteristic p. The *absolute frobenius* of X is the morphism $F_X : X \to X$ which is the identity on the induced topological space, and which takes a function to its pth power. Thus $F_X^\sharp : \mathcal{O}_X \to \mathcal{O}_X$ is given by $g \mapsto g^p$.

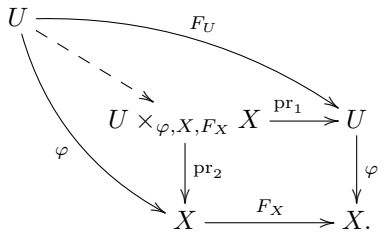
Theorem 44.74.4 (The Baffling Theorem)**.** *Let X be a scheme in characteristic $p > 0$. Then the absolute frobenius induces (by pullback) the trivial map on cohomology, i.e., for all integers $j \geq 0$,*

$$F_X^* : H^j(X, \underline{\mathbf{Z}/n\mathbf{Z}}) \longrightarrow H^j(X, \underline{\mathbf{Z}/n\mathbf{Z}})$$

is the identity.

This theorem is purely formal. It is a good idea, however, to review how to compute the pullback of a cohomology class. Let us simply say that in the case where cohomology agrees with Čech cohomology, it suffices to pull back (using the fiber products on a site) the Čech cocycles. The general case is quite technical, see Hypercoverings, Theorem 24.9.1. To prove the theorem, we merely verify that the assumption of Lemma 44.74.2 holds for the frobenius.

Proof of Theorem 44.74.4. We need to verify the existence of a functorial isomorphism as above. For an étale morphism $\varphi : U \to S$, consider the diagram



The dotted arrow is an étale morphism which induces an isomorphism on the underlying topological spaces, so it is an isomorphism. □

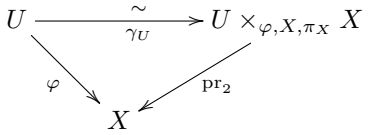
Definition 44.74.5. Let k be a finite field with $q = p^f$ elements. Let X be a scheme over k. The *geometric frobenius* of X is the morphism $\pi_X : X \to X$ over $\mathrm{Spec}(k)$ which equals F_X^f.

Since π_X is a morphism over k, we can base change it to any scheme over k. In particular we can base chage it to the algebraic closure $\bar{k}$ and get a morphism $\pi_X : X_{\bar{k}} \to X_{\bar{k}}$. Using π_X also for this base change should not be confusing as $X_{\bar{k}}$ does not have a geometric frobenius of its own.

Lemma 44.74.6. *Let $\mathcal{F}$ be a sheaf on $X_{étale}$. Then there are canonical isomorphisms $\pi_X^{-1}\mathcal{F} \cong \mathcal{F}$ and $\mathcal{F} \cong \pi_{X*}\mathcal{F}$.*

This is false for the fppf site.

Proof. Let $\varphi : U \to X$ be étale. Recall that $\pi_{X*}\mathcal{F}(U) = \mathcal{F}(U \times_{\varphi,X,\pi_X} X)$. Since $\pi_X = F_X^f$, it follows from the proof of Theorem 44.74.4 that there is a functorial isomorphism



where $\gamma_U = (\varphi, F_U^f)$. Now we define an isomorphism

$$\mathcal{F}(U) \longrightarrow \pi_{X*}\mathcal{F}(U) = \mathcal{F}(U \times_{\varphi,X,\pi_X} X)$$

by taking the restriction map of $\mathcal{F}$ along γ_U^{-1}. The other isomorphism is analogous. □

Remark 44.74.7. It may or may not be the case that F_U^f equals π_U.

We continue discussion cohomology of sheaves on our scheme X over the finite field k with $q = p^f$ elements. Fix an algebraic clsoure $\bar{k}$ of k and write $G_k = \mathrm{Gal}(\bar{k}/k)$ for the absolute Galois group of k. Let $\mathcal{F}$ be an abelian sheaf on $X_{étale}$. Consider the We will define a left G_k-module structure cohomology group $H^j(X_{\bar{k}}, \mathcal{F}|_{X_{\bar{k}}})$ as follows: if $\sigma \in G_k$, the diagram

$$\begin{array}{ccc} X_{\bar{k}} & \xrightarrow{\mathrm{Spec}(\sigma)\times \mathrm{id}_X} & X_{\bar{k}} \\ & \searrow \quad \swarrow & \\ & X & \end{array}$$

commutes. Thus we can set, for $\xi \in H^j(X_{\bar{k}}, \mathcal{F}|_{X_{\bar{k}}})$

$$\sigma \cdot \xi := (\mathrm{Spec}(\sigma) \times \mathrm{id}_X)^* \xi \in H^j(X_{\bar{k}}, (\mathrm{Spec}(\sigma) \times \mathrm{id}_X)^{-1}\mathcal{F}|X_{\bar{k}}) = H^j(X_{\bar{k}}, \mathcal{F}|_{X_{\bar{k}}}),$$

where the last equality follows from the commutativity of the previous diagram. This endows the latter group with the structure of a G_k-module.

Lemma 44.74.8. *In the situation above denote $\alpha : X \to \mathrm{Spec}(k)$ the structure morphism. Consider the stalk $(R^j\alpha_*\mathcal{F})_{\mathrm{Spec}(\bar{k})}$ endowed with its natural Galois action as in Section 44.57. Then the identification*

$$(R^j\alpha_*\mathcal{F})_{\mathrm{Spec}(\bar{k})} \cong H^j(X_{\bar{k}}, \mathcal{F}|_{X_{\bar{k}}})$$

from Theorem 44.53.1 is an isomorphism of G_k-modules.

A similar result holds comparing $(R^j\alpha_!\mathcal{F})_{\mathrm{Spec}(\bar{k})}$ with $H_c^j(X_{\bar{k}}, \mathcal{F}|_{X_{\bar{k}}})$.

Proof. Omitted. □

Definition 44.74.9. The *arithmetic frobenius* is the map $\mathrm{frob}_k : \bar{k} \to \bar{k}$, $x \mapsto x^q$ of G_k.

Theorem 44.74.10. *Let $\mathcal{F}$ be an abelian sheaf on $X_{étale}$. Then for all $j \geq 0$, $frob_k$ acts on the cohomology group $H^j(X_{\bar{k}}, \mathcal{F}|_{X_{\bar{k}}})$ as the inverse of the map π_X^*.*

The map π_X^* is defined by the composition

$$H^j(X_{\bar{k}}, \mathcal{F}|_{X_{\bar{k}}}) \xrightarrow{\pi_{X\,\bar{k}}^*} H^j(X_{\bar{k}}, (\pi_X^{-1}\mathcal{F})|_{X_{\bar{k}}}) \cong H^j(X_{\bar{k}}, \mathcal{F}|_{X_{\bar{k}}}).$$

where the last isomorphism comes from the canonical isomorphism $\pi_X^{-1}\mathcal{F} \cong \mathcal{F}$ of Lemma 44.74.6.

Proof. The composition $X_{\bar{k}} \xrightarrow{\mathrm{Spec}(\mathrm{frob}_k)} X_{\bar{k}} \xrightarrow{\pi_X} X_{\bar{k}}$ is equal to $F_{X_{\bar{k}}}^f$, hence the result follows from the baffling theorem suitably generalized to nontrivial coefficients. Note that the previous composition commutes in the sense that $F_{X_{\bar{k}}}^f = \pi_X \circ \mathrm{Spec}(\mathrm{frob}_k) = \mathrm{Spec}(\mathrm{frob}_k) \circ \pi_X$. □

Definition 44.74.11. If $x \in X(k)$ is a rational point and $\bar{x} : \mathrm{Spec}(\bar{k}) \to X$ the geometric point lying over x, we let $\pi_x : \mathcal{F}_{\bar{x}} \to \mathcal{F}_{\bar{x}}$ denote the action by frob_k^{-1} and call it the *geometric frobenius*[6]

We can now make a more precise statement (albeit a false one) of the trace formula (44.73.0.1). Let X be a finite type scheme of dimension 1 over a finite field k, ℓ a prime number and $\mathcal{F}$ a constructible, flat $\mathbf{Z}/\ell^n\mathbf{Z}$ sheaf. Then

$$\sum\nolimits_{x \in X(k)} \mathrm{Tr}(\pi_X | \mathcal{F}_{\bar{x}}) = \sum\nolimits_{i=0}^{2} (-1)^i \mathrm{Tr}(\pi_X^* | H_c^i(X_{\bar{k}}, \mathcal{F})) \tag{44.74.11.1}$$

as elements of $\mathbf{Z}/\ell^n\mathbf{Z}$. The reason this equation is wrong is that the trace in the right-hand side does not make sense for the kind of sheaves considered. Before addressing this issue, we try to motivate the appearance of the geometric frobenius (apart from the fact that it is a natural morphism!).

Let us consider the case where $X = \mathbf{P}^1_k$ and $\mathcal{F} = \underline{\mathbf{Z}/\ell\mathbf{Z}}$. For any point, the Galois module $\mathcal{F}_{\bar{x}}$ is trivial, hence for any morphism φ acting on $\mathcal{F}_{\bar{x}}$, the left-hand side is

$$\sum\nolimits_{x \in X(k)} \mathrm{Tr}(\varphi | \mathcal{F}_{\bar{x}}) = \#\mathbf{P}^1_k(k) = q + 1.$$

Now $\mathbf{P}^1_k$ is proper, so compactly supported cohomology equals standard cohomology, and so for a morphism $\pi : \mathbf{P}^1_k \to \mathbf{P}^1_k$, the right-hand side equals

$$\mathrm{Tr}(\pi^* | H^0(\mathbf{P}^1_{\bar{k}}, \underline{\mathbf{Z}/\ell\mathbf{Z}})) + \mathrm{Tr}(\pi^* | H^2(\mathbf{P}^1_{\bar{k}}, \underline{\mathbf{Z}/\ell\mathbf{Z}})).$$

The Galois module $H^0(\mathbf{P}^1_{\bar{k}}, \underline{\mathbf{Z}/\ell\mathbf{Z}}) = \mathbf{Z}/\ell\mathbf{Z}$ is trivial, since the pullback of the identity is the identity. Hence the first trace is 1, regardless of π. For the second trace, we need to compute the pullback of a map $\pi : \mathbf{P}^1_{\bar{k}} \to \mathbf{P}^1_{\bar{k}}$ on $H^2(\mathbf{P}^1_{\bar{k}}, \underline{\mathbf{Z}/\ell\mathbf{Z}})$. This is a good exercise and the answer is multiplication by the degree of π. In other words, this works as in the familiar situation of complex cohomology. In particular, if π is the geometric frobenius we get

$$\mathrm{Tr}(\pi_X^* | H^2(\mathbf{P}^1_{\bar{k}}, \underline{\mathbf{Z}/\ell\mathbf{Z}})) = q$$

and if π is the arithmetic frobenius then we get

$$\mathrm{Tr}(\mathrm{frob}_k^* | H^2(\mathbf{P}^1_{\bar{k}}, \underline{\mathbf{Z}/\ell\mathbf{Z}})) = q^{-1}.$$

The latter option is clearly wrong.

Remark 44.74.12. The computation of the degrees can be done by lifting (in some obvious sense) to characteristic 0 and considering the situation with complex coefficients. This method almost never works, since lifting is in general impossible for schemes which are not projective space.

The question remains as to why we have to consider compactly supported cohomology. In fact, in view of Poincaré duality, it is not strictly necessary for smooth varieties, but it involves adding in certain powers of q. For example, let us consider the case where $X = \mathbf{A}^1_k$ and $\mathcal{F} = \underline{\mathbf{Z}/\ell\mathbf{Z}}$. The action on stalks is again trivial, so we only need look at the action on cohomology. But then π_X^* acts as the identity on $H^0(\mathbf{A}^1_{\bar{k}}, \underline{\mathbf{Z}/\ell\mathbf{Z}})$ and as multiplication by q on $H^2_c(\mathbf{A}^1_{\bar{k}}, \underline{\mathbf{Z}/\ell\mathbf{Z}})$.

[6] This notation is not standard. This operator is denoted F_x in [**Del77**]. We will likely change this notation in the future.

44.75. Traces

We now explain how to take the trace of an endomorphism of a module over a noncommutative ring. Fix a finite ring Λ with cardinality prime to p. Typically, Λ is the group ring $(\mathbf{Z}/\ell^n\mathbf{Z})[G]$ for some finite group G. By convention, all the Λ-modules considered will be left Λ-modules.

We introduce the following notation: We set $\Lambda^\natural$ to be the quotient of Λ by its additive subgroup generated by the commutators (i.e., the elements of the form $ab - ba$, $a, b \in \Lambda$). Note that $\Lambda^\natural$ is not a ring.

For instance, the module $(\mathbf{Z}/\ell^n\mathbf{Z})[G]^\natural$ is the dual of the class functions, so

$$(\mathbf{Z}/\ell^n\mathbf{Z})[G]^\natural = \bigoplus\nolimits_{\text{conjugacy classes of } G} \mathbf{Z}/\ell^n\mathbf{Z}.$$

For a free Λ-module, we have $\mathrm{End}_\Lambda(\Lambda^{\oplus m}) = \mathrm{Mat}_n(\Lambda)$. Note that since the modules are left modules, representation of endomorphism by matrices is a right action: if $a \in \mathrm{End}(\Lambda^{\oplus m})$ has matrix A and $v \in \Lambda$, then $a(v) = vA$.

Definition 44.75.1. The *trace* of the endomorphism a is the sum of the diagonal entries of a matrix representing it. This defines an additive map $\mathrm{Tr} : \mathrm{End}_\Lambda(\Lambda^{\oplus m}) \to \Lambda^\natural$.

Exercise 44.75.2. Given maps

$$\Lambda^{\oplus n} \xrightarrow{a} \Lambda^{\oplus n} \xrightarrow{b} \Lambda^{\oplus m}$$

show that $\mathrm{Tr}(ab) = \mathrm{Tr}(ba)$.

We extend the definition of the trace to a finite projective Λ-module P and an endomorphism φ of P as follows. Write P as the summand of a free Λ-module, i.e., consider maps $P \xrightarrow{a} \Lambda^{\oplus n} \xrightarrow{b} P$ with

(1) $\Lambda^{\oplus n} = \mathrm{Im}(a) \oplus \ker(b)$; and
(2) $b \circ a = \mathrm{id}_P$.

Then we set $\mathrm{Tr}(\varphi) = \mathrm{Tr}(a\varphi b)$. It is easy to check that this is well-defined, using the previous exercise.

44.76. Why derived categories?

With this definition of the trace, let us now discuss another issue with the formula as stated. Let C be a smooth projective curve over k. Then there is a correspondence between finite locally constant sheaves $\mathcal{F}$ on $C_{étale}$ which stalks are isomorphic to $(\mathbf{Z}/\ell^n\mathbf{Z})^{\oplus m}$ on the one hand, and continuous representations $\rho : \pi_1(C, \bar{c}) \to \mathrm{GL}_m(\mathbf{Z}/\ell^n\mathbf{Z}))$ (for some fixed choice of $\bar{c}$) on the other hand. We denote $\mathcal{F}_\rho$ the sheaf corresponding to ρ. Then $H^2(C_{\bar{k}}, \mathcal{F}_\rho)$ is the group of coinvariants for the action of $\rho(\pi_1(C, \bar{c}))$ on $(\mathbf{Z}/\ell^n\mathbf{Z})^{\oplus m}$, and there is a short exact sequence

$$0 \longrightarrow \pi_1(C_{\bar{k}}, \bar{c}) \longrightarrow \pi_1(C, \bar{c}) \longrightarrow G_k \longrightarrow 0.$$

For instance, let $\mathbf{Z} = \mathbf{Z}\sigma$ act on $\mathbf{Z}/\ell^2\mathbf{Z}$ via $\sigma(x) = (1 + \ell)x$. The coinvariants are $(\mathbf{Z}/\ell^2\mathbf{Z})_\sigma = \mathbf{Z}/\ell\mathbf{Z}$, which is not a flat $\mathbf{Z}/\ell\mathbf{Z}$-module. Hence we cannot take the trace of some action on $H^2(C_{\bar{k}}, \mathcal{F}_\rho)$, at least not in the sense of the previous section.

In fact, our goal is to consider a trace formula for ℓ-adic coefficients. But $\mathbf{Q}_\ell = \mathbf{Z}_\ell[1/\ell]$ and $\mathbf{Z}_\ell = \lim \mathbf{Z}/\ell^n\mathbf{Z}$, and even for a flat $\mathbf{Z}/\ell^n\mathbf{Z}$ sheaf, the individual cohomology groups may not be flat, so we cannot compute traces. One possible remedy is consider the total derived complex $R\Gamma(C_{\bar{k}}, \mathcal{F}_\rho)$ in the derived category $D(\mathbf{Z}/\ell^n\mathbf{Z})$ and show that it is a perfect object, which means that it is quasi-isomorphic to a finite complex of finite free module. For such complexes, we can define the trace, but this will require an account of derived categories.

44.77. Derived categories

To set up notation, let $\mathcal{A}$ be an abelian category. Let $\mathrm{Comp}(\mathcal{A})$ be the abelian category of complexes in $\mathcal{A}$. Let $K(\mathcal{A})$ be the category of complexes up to homotopy, with objects equal to complexes in $\mathcal{A}$ and objects equal to homotopy classes of morphisms of complexes. This is not an abelian category. Loosely speaking, $D(A)$ is defined to be the category obtained by inverting all quasi-isomorphisms in $\mathrm{Comp}(\mathcal{A})$ or, equivalently, in $K(\mathcal{A})$. Moreover, we can define $\mathrm{Comp}^+(\mathcal{A}), K^+(\mathcal{A}), D^+(\mathcal{A})$ analogously using only bounded below complexes. Similarly, we can define $\mathrm{Comp}^-(\mathcal{A}), K^-(\mathcal{A}), D^-(\mathcal{A})$ using bounded above complexes, and we can define $\mathrm{Comp}^b(\mathcal{A}), K^b(\mathcal{A}), D^b(\mathcal{A})$ using bounded complexes.

Remark 44.77.1. Notes on derived categories.

(1) There are some set-theoretical problems when $\mathcal{A}$ is somewhat arbitrary, which we will happily disregard.
(2) The categories $K(A)$ and $D(A)$ may be endowed with the structure of triangulated category, but we will not need these structures in the following discussion.
(3) The categories $\mathrm{Comp}(\mathcal{A})$ and $K(\mathcal{A})$ can also be defined when $\mathcal{A}$ is an additive category.

The homology functor $H^i : \mathrm{Comp}(\mathcal{A}) \to \mathcal{A}$ taking a complex $K^\bullet \mapsto H^i(K^\bullet)$ extends to functors $H^i : K(\mathcal{A}) \to \mathcal{A}$ and $H^i : D(\mathcal{A}) \to \mathcal{A}$.

Lemma 44.77.2. *An object E of $D(\mathcal{A})$ is contained in $D^+(\mathcal{A})$ if and only if $H^i(E) = 0$ for all $i \ll 0$. Similar statements hold for D^- and D^+.*

Proof. Hint: use truncation functors. See Derived Categories, Lemma 13.10.6. □

Lemma 44.77.3. *Morphisms between objects in the derived category.*

(1) *Let $I^\bullet \in Comp^+(\mathcal{A})$ with I^n injective for all $n \in \mathbf{Z}$. Then*
$$\mathrm{Hom}_{D(\mathcal{A})}(K^\bullet, I^\bullet) = \mathrm{Hom}_{K(\mathcal{A})}(K^\bullet, I^\bullet).$$
(2) *Let $P^\bullet \in Comp^-(\mathcal{A})$ with P^n is projective for all $n \in \mathbf{Z}$. Then*
$$\mathrm{Hom}_{D(\mathcal{A})}(P^\bullet, K^\bullet) = \mathrm{Hom}_{K(\mathcal{A})}(P^\bullet, K^\bullet).$$
(3) *If $\mathcal{A}$ has enough injectives and $\mathcal{I} \subset \mathcal{A}$ is the additive subcategory of injectives, then $D^+(\mathcal{A}) \cong K^+(\mathcal{I})$ (as triangulated categories).*
(4) *If $\mathcal{A}$ has enough projectives and $\mathcal{P} \subset \mathcal{A}$ is the additive subcategory of projectives, then $D^-(\mathcal{A}) \cong K^-(\mathcal{P})$.*

Proof. Omitted. □

Definition 44.77.4. Let $F : \mathcal{A} \to \mathcal{B}$ be a left exact functor and assume that $\mathcal{A}$ has enough injectives. We define the *total right derived functor of* F as the functor $RF : D^+(\mathcal{A}) \to D^+(\mathcal{B})$ fitting into the diagram

$$\begin{array}{ccc} D^+(\mathcal{A}) & \xrightarrow{RF} & D^+(\mathcal{B}) \\ \uparrow & & \uparrow \\ K^+(\mathcal{I}) & \xrightarrow{F} & K^+(\mathcal{B}). \end{array}$$

This is possible since the left vertical arrow is invertible by the previous lemma. Similarly, let $G : \mathcal{A} \to \mathcal{B}$ be a right exact functor and assume that $\mathcal{A}$ has enough projectives. We define the *total right derived functor of* G as the functor $LG : D^-(\mathcal{A}) \to D^-(\mathcal{B})$ fitting into the diagram

$$\begin{array}{ccc} D^-(\mathcal{A}) & \xrightarrow{LG} & D^-(\mathcal{B}) \\ \uparrow & & \uparrow \\ K^-(\mathcal{P}) & \xrightarrow{G} & K^-(\mathcal{B}). \end{array}$$

This is possible since the left vertical arrow is invertible by the previous lemma.

Remark 44.77.5. In these cases, it is true that $R^iF(K^\bullet) = H^i(RF(K^\bullet))$, where the left hand side is defined to be ith homology of the complex $F(K^\bullet)$.

44.78. Filtered derived category

It turns out we have to do it all again and build the filtered derived category also.

Definition 44.78.1. Let $\mathcal{A}$ be an abelian category. Let $\mathrm{Fil}(\mathcal{A})$ be the category of filtered objects (A, F) of $\mathcal{A}$, where F is a filtration of the form

$$A \supseteq \cdots \supseteq F^nA \supseteq F^{n+1}A \supseteq \cdots \supseteq 0.$$

This is an additive category. We denote $\mathrm{Fil}^f(\mathcal{A})$ the full subcategory of $\mathrm{Fil}(\mathcal{A})$ whose objects (A, F) have finite filtration. This is also an additive category. An object $I \in \mathrm{Fil}^f(\mathcal{A})$ is called *filtered injective* (respectively *projective*) provided that $\mathrm{gr}^p(I) = \mathrm{gr}^p_F(I) = F^pI/F^{p+1}I$ is injective (resp. projective) in $\mathcal{A}$ for all p. The categories $\mathrm{Comp}(\mathrm{Fil}^f(\mathcal{A})) \supseteq \mathrm{Comp}^+(\mathrm{Fil}^f(\mathcal{A}))$ and $K(\mathrm{Fil}^f(\mathcal{A})) \supseteq K^+(\mathrm{Fil}^f(\mathcal{A}))$ are defined as before.
A morphism $\alpha : K^\bullet \to L^\bullet$ of complexes in $\mathrm{Comp}(\mathrm{Fil}^f(\mathcal{A}))$ is called a *filtered quasi-isomorphism* provided that

$$\mathrm{gr}^p(\alpha) : \mathrm{gr}^p(K^\bullet) \to \mathrm{gr}^p(L^\bullet)$$

is a quasi-isomorphism for all $p \in \mathbf{Z}$. Finally, we define $DF(\mathcal{A})$ (resp. $DF^+(\mathcal{A})$) by inverting the filtered quasi-isomorphisms in $K(\mathrm{Fil}^f(\mathcal{A}))$ (resp. $K^+(\mathrm{Fil}^f(\mathcal{A}))$).

Lemma 44.78.2. *If $\mathcal{A}$ has enough injectives, then $DF^+(\mathcal{A}) \cong K^+(\mathcal{I})$, where $\mathcal{I}$ is the full additive subcategory of $Fil^f(\mathcal{A})$ consisting of filtered injective objects. Similarly, if $\mathcal{A}$ has enough projectives, then $DF^-(\mathcal{A}) \cong K^+(\mathcal{P})$, where $\mathcal{P}$ is the full additive subcategory of $Fil^f(\mathcal{A})$ consisting of filtered projective objects.*

Proof. Omitted. □

44.79. Filtered derived functors

And then there are the filtered derived functors.

Definition 44.79.1. Let $T : \mathcal{A} \to \mathcal{B}$ be a left exact functor and assume that $\mathcal{A}$ has enough injectives. Define $RT : DF^+(\mathcal{A}) \to DF^+(\mathcal{B})$ to fit in the diagram

$$\begin{array}{ccc} DF^+(\mathcal{A}) & \xrightarrow{RT} & DF^+(\mathcal{B}) \\ \uparrow & & \uparrow \\ K^+(\mathcal{I}) & \xrightarrow{T} & K^+(\mathrm{Fil}^f(\mathcal{B})). \end{array}$$

This is well-defined by the previous lemma. Let $G : \mathcal{A} \to \mathcal{B}$ be a right exact functor and assume that $\mathcal{A}$ has enough projectives. Define $LG : DF^+(\mathcal{A}) \to DF^+(\mathcal{B})$ to fit in the diagram

$$\begin{array}{ccc} DF^-(\mathcal{A}) & \xrightarrow{LG} & DF^-(\mathcal{B}) \\ \uparrow & & \uparrow \\ K^-(\mathcal{P}) & \xrightarrow{G} & K^-(\mathrm{Fil}^f(\mathcal{B})). \end{array}$$

Again, this is well-defined by the previous lemma. The functors RT, resp. LG, are called the *filtered derived functor* of T, resp. G.

Proposition 44.79.2. *In the situation above, we have*

$$\mathrm{gr}^p \circ RT = RT \circ \mathrm{gr}^p$$

where the RT on the left is the filtered derived functor while the one on the right is the total derived functor. That is, there is a commuting diagram

$$\begin{array}{ccc} DF^+(\mathcal{A}) & \xrightarrow{RT} & DF^+(\mathcal{B}) \\ {\scriptstyle gr^p}\downarrow & & \downarrow{\scriptstyle gr^p} \\ D^+(\mathcal{A}) & \xrightarrow{RT} & D^+(\mathcal{B}). \end{array}$$

Proof. Omitted. □

Given $K^\bullet \in DF^+(\mathcal{B})$, we get a spectral sequence

$$E_1^{p,q} = H^{p+q}(\mathrm{gr}^p K^\bullet) \Rightarrow H^{p+q}(\text{forget filt}(K^\bullet)).$$

44.80. Application of filtered complexes

Let $\mathcal{A}$ be an abelian category with enough injectives, and $0 \to L \to M \to N \to 0$ a short exact sequence in $\mathcal{A}$. Consider $\widetilde{M} \in \mathrm{Fil}^f(\mathcal{A})$ to be M along with the filtration defined by

$$F^1 M = L,\ F^n M = M \text{ for } n \leq 0, \text{ and } F^n M = 0 \text{ for } n \geq 2.$$

By definition, we have

$$\text{forget filt}(\widetilde{M}) = M, \quad \mathrm{gr}^0(\widetilde{M}) = N, \quad \mathrm{gr}^1(\widetilde{M}) = L$$

and $\mathrm{gr}^n(\widetilde{M}) = 0$ for all other $n \neq 0, 1$. Let $T : \mathcal{A} \to \mathcal{B}$ be a left exact functor. Assume that $\mathcal{A}$ has enough injectives. Then $RT(\widetilde{M}) \in DF^+(\mathcal{B})$ is a filtered complex with

$$\mathrm{gr}^p(RT(\widetilde{M})) \overset{\text{qis}}{=} \begin{cases} 0 & \text{if} \quad p \neq 0, 1, \\ RT(N) & \text{if} \quad p = 0, \\ RT(L) & \text{if} \quad p = 1. \end{cases}$$

and forget $\mathrm{filt}(RT(\widetilde{M})) \overset{\text{qis}}{=} RT(M)$. The spectral sequence applied to $RT(\widetilde{M})$ gives

$$E_1^{p,q} = R^{p+q}T(\mathrm{gr}^p(\widetilde{M})) \Rightarrow R^{p+q}T(\text{forget } \mathrm{filt}(\widetilde{M})).$$

Unwinding the spectral sequence gives us the long exact sequence

$$\begin{array}{ccccccc} 0 & \longrightarrow & T(L) & \longrightarrow & T(M) & \longrightarrow & T(N) \\ & & & & & \searrow & \\ & & R^1T(L) & \longrightarrow & R^1T(M) & \longrightarrow & \cdots \end{array}$$

This will be used as follows. Let X/k be a scheme of finite type. Let $\mathcal{F}$ be a flat constructible $\mathbf{Z}/\ell^n\mathbf{Z}$-module. Then we want to show that the trace

$$\mathrm{Tr}(\pi_X^* | R\Gamma_c(X_{\bar{k}}, \mathcal{F})) \in \mathbf{Z}/\ell^n\mathbf{Z}$$

is additive on short exact sequences. To see this, it will not be enough to work with $R\Gamma_c(X_{\bar{k}}, -) \in D^+(\mathbf{Z}/\ell^n\mathbf{Z})$, but we will have to use the filtered derived category.

44.81. Perfectness

Let Λ be a (possibly noncommutative) ring, Mod_Λ the category of left Λ-modules, $K(\Lambda) = K(\mathrm{Mod}_\Lambda)$ its homotopy category, and $D(\Lambda) = D(\mathrm{Mod}_\Lambda)$ the derived category.

Definition 44.81.1. We denote by $K_{perf}(\Lambda)$ the category whose objects are bounded complexes of finite projective Λ-modules, and whose morphisms are morphisms of complexes up to homotopy. The functor $K_{perf}(\Lambda) \to D(\Lambda)$ is fully faithful (Derived Categories, Lemma 13.18.8). Denote $D_{perf}(\Lambda)$ its essential image. An object of $D(\Lambda)$ is called *perfect* if it is in $D_{perf}(\Lambda)$.

Proposition 44.81.2. *Let $K \in D_{perf}(\Lambda)$ and $f \in End_{D(\Lambda)}(K)$. Then the trace $Tr(f) \in \Lambda^\natural$ is well defined.*

Proof. We will use Derived Categories, Lemma 13.18.8 without further mention in this proof. Let $P^\bullet$ be a bounded complex of finite projective Λ-modules and let $\alpha : P^\bullet \to K$ be an isomorphism in $D(\Lambda)$. Then $\alpha^{-1} \circ f \circ \alpha$ corresponds to a morphism of complexes $f^\bullet : P^\bullet \to P^\bullet$ well defined up to homotopy. Set

$$\mathrm{Tr}(f) = \sum_i (-1)^i \mathrm{Tr}(f^i : P^i \to P^i) \in \Lambda^\natural.$$

Given $P^\bullet$ and α, this is independent of the choice of $f^\bullet$. Namely, any other choice is of the form $\tilde{f}^\bullet = f^\bullet + dh + hd$ for some $h^i : P^i \to P^{i-1} (i \in \mathbf{Z})$. But

$$\begin{aligned} \mathrm{Tr}(dh) &= \sum_i (-1)^i \mathrm{Tr}(P^i \xrightarrow{dh} P^i) \\ &= \sum_i (-1)^i \mathrm{Tr}(P^{i-1} \xrightarrow{hd} P^{i-1}) \\ &= -\sum_i (-1)^{i-1} \mathrm{Tr}(P^{i-1} \xrightarrow{hd} P^{i-1}) \\ &= -\mathrm{Tr}(hd) \end{aligned}$$

and so $\sum_i (-1)^i \mathrm{Tr}((dh+hd)|_{P^i}) = 0$. Furthermore, this is independent of the choice of $(P^\bullet, \alpha)$: suppose $(Q^\bullet, \beta)$ is another choice. The compositions

$$Q^\bullet \xrightarrow{\beta} K \xrightarrow{\alpha^{-1}} P^\bullet \quad \text{and} \quad P^\bullet \xrightarrow{\alpha} K \xrightarrow{\beta^{-1}} Q^\bullet$$

are representable by morphisms of complexes $\gamma_1^\bullet$ and $\gamma_2^\bullet$ respectively, such that $\gamma_1^\bullet \circ \gamma_2^\bullet$ is homotopic to the identity. Thus, the morphism of complexes $\gamma_2^\bullet \circ f^\bullet \circ \gamma_1^\bullet : Q^\bullet \to Q^\bullet$ represents the morphism $\beta^{-1} \circ f \circ \beta$ in $D(\Lambda)$. Now

$$\begin{aligned} \mathrm{Tr}(\gamma_2^\bullet \circ f^\bullet \circ \gamma_1^\bullet|_{Q^\bullet}) &= \mathrm{Tr}(\gamma_1^\bullet \circ \gamma_2^\bullet \circ f^\bullet|_{P^\bullet}) \\ &= \mathrm{Tr}(f^\bullet|_{P^\bullet}) \end{aligned}$$

by the fact that $\gamma_1^\bullet \circ \gamma_2^\bullet$ is homotopic to the identity and the independence of the choice of $f^\bullet$ we saw above. □

44.82. Filtrations and perfect complexes

We now present a filtered version of the category of perfect complexes. An object (M, F) of $\mathrm{Fil}^f(\mathrm{Mod}_\Lambda)$ is called *filtered finite projective* if for all p, $\mathrm{gr}_F^p(M)$ is finite and projective. We then consider the homotopy category $K\mathrm{F}_{\mathrm{perf}}(\Lambda)$ of bounded complexes of filtered finite projective objects of $\mathrm{Fil}^f(\mathrm{Mod}_\Lambda)$. We have a diagram of categories

$$\begin{array}{ccc} K\mathrm{F}(\Lambda) & \supseteq & K\mathrm{F}_{\mathrm{perf}}(\Lambda) \\ \downarrow & & \downarrow \\ D\mathrm{F}(\Lambda) & \supseteq & D\mathrm{F}_{\mathrm{perf}}(\Lambda) \end{array}$$

where the vertical functor on the right is fully faithful and the category $D\mathrm{F}_{\mathrm{perf}}(\Lambda)$ is its essential image, as before.

Lemma 44.82.1 (Additivity)**.** *Let $K \in D\mathrm{F}_{perf}(\Lambda)$ and $f \in End_{D\mathrm{F}}(K)$. Then*

$$Tr(f|_K) = \sum\nolimits_{p \in \mathbf{Z}} Tr(f|_{gr^p K}).$$

Proof. By Proposition 44.81.2, we may assume we have a bounded complex $P^\bullet$ of filtered finite projectives of $\mathrm{Fil}^f(\mathrm{Mod}_\Lambda)$ and a map $f^\bullet : P^\bullet \to P^\bullet$ in $\mathrm{Comp}(\mathrm{Fil}^f(\mathrm{Mod}_\Lambda))$. So the lemma follows from the following result, which proof is left to the reader. □

Lemma 44.82.2. *Let $P \in Fil^f(Mod_\Lambda)$ be filtered finite projective, and $f : P \to P$ an endomorphism in $Fil^f(Mod_\Lambda)$. Then*

$$Tr(f|_P) = \sum\nolimits_p Tr(f|_{gr^p(P)}).$$

Proof. Omitted. □

44.83. Characterizing perfect objects

For the commutative case see More on Algebra, Sections 15.48, 15.49, and 15.50.

Definition 44.83.1. Let Λ be a (possibly noncommutative) ring. An object $K \in D(\Lambda)$ has *finite Tor-dimension* if there exist $a, b \in \mathbf{Z}$ such that for any right Λ-module N, we have $H^i(N \otimes^{\mathbf{L}}_{\Lambda} K) = 0$ for all $i \notin [a, b]$.

This in particular means that $K \in D^b(\Lambda)$ as we see by taking $N = \Lambda$.

Lemma 44.83.2. *Let Λ be a left noetherian ring and $K \in D(\Lambda)$. Then K is perfect if and only if the two following conditions hold:*

(1) *K has finite Tor-dimension ; and*
(2) *for all $i \in \mathbf{Z}$, $H^i(K)$ is a finite Λ-module.*

Proof. See More on Algebra, Lemma 15.50.2 for the proof in the commutative case. □

The reader is strongly urged to try and prove this. The proof relies on the fact that a finite module on a finitely left-presented ring is flat if and only if it is projective.

Remark 44.83.3. A variant of this lemma is to consider a Noetherian scheme X and the category $D_{perf}(\mathcal{O}_X)$ of complexes which are locally quasi-isomorphic to a finite complex of finite locally free $\mathcal{O}_X$-modules. Objects K of $D_{perf}(\mathcal{O}_X)$ can be characterized by having coherent cohomology sheaves and bounded tor dimension.

44.84. Complexes with constructible cohomology

Let Λ be a ring. Let X a scheme. Let $K(X, \Lambda)$ the homotopy category of sheaves of Λ-modules on $X_{étale}$. Denote $D(X, \Lambda)$ the corresponding derived category. We denote by $D^b(X, \Lambda)$ (respectively D^+, D^-) the full subcategory of bounded (resp. above, below) complexes in $D(X, \Lambda)$.

Definition 44.84.1. Let X be a scheme. Let Λ be a Noetherian ring. We denote $D_c(X, \Lambda)$ the full subcategory of $D(X, \Lambda)$ of complexes whose cohomology sheaves are constructible sheaves of Λ-modules.

This definition makes sense by Lemma 44.65.6 and Derived Categories, Section 13.12. Thus we see that $D_c(X, \Lambda)$ is a strictly full, saturated triangulated subcategory of $D_c(X, \Lambda)$.

Lemma 44.84.2. *Let Λ be a Noetherian ring. If $j : U \to X$ is an étale morphism of schemes, then*

(1) *$K|_U \in D_c(U, \Lambda)$ if $K \in D_c(X, \Lambda)$, and*
(2) *$j_!M \in D_c(X, \Lambda)$ if $M \in D_c(U, \Lambda)$ and the morphism j is quasi-compact and quasi-separated.*

Proof. The first assertion is clear. The second follows from the fact that $j_!$ is exact and Lemma 44.67.1. □

Lemma 44.84.3. *Let Λ be a Noetherian ring. Let $f : X \to Y$ be a morphism of schemes. If $K \in D_c(Y, \Lambda)$ then $Lf^*K \in D_c(X, \Lambda)$.*

Proof. This follows as $f^{-1} = f^*$ is exact and Lemma 44.65.5. □

Lemma 44.84.4. *Let X be a quasi-compact and quasi-separated scheme. Let Λ be a Noetherian ring. Let $K \in D(X, \Lambda)$ and $b \in \mathbf{Z}$ such that $H^b(K)$ is constructible. Then there exist a sheaf $\mathcal{F}$ which is a finite direct sum of $j_{U!}\underline{\Lambda}$ with $U \in \mathrm{Ob}(X_{étale})$ affine and a map $\mathcal{F}[-b] \to K$ in $D(X, \Lambda)$ inducing a surjection $\mathcal{F} \to H^b(K)$.*

Proof. Represent K by a complex $\mathcal{K}^\bullet$ of sheaves of Λ-modules. Consider the surjection

$$\mathrm{Ker}(\mathcal{K}^b \to \mathcal{K}^{b+1}) \longrightarrow H^b(K)$$

By Modules on Sites, Lemma 18.29.5 we may choose a surjection $\bigoplus_{i \in I} j_{U_i!}\underline{\Lambda} \to \mathrm{Ker}(\mathcal{K}^b \to \mathcal{K}^{b+1})$ with U_i affine. For $I' \subset I$ finite, denote $\mathcal{H}_{I'} \subset H^b(K)$ the image of $\bigoplus_{i \in I'} j_{U_i!}\underline{\Lambda}$. By Lemma 44.65.9 we see that $\mathcal{H}_{I'} = H^b(K)$ for some $I' \subset I$ finite. The lemma follows taking $\mathcal{F} = \bigoplus_{i \in I'} j_{U_i!}\underline{\Lambda}$. □

Lemma 44.84.5. *Let X be a quasi-compact and quasi-separated scheme. Let Λ be a Noetherian ring. Let $K \in D^-(X, \Lambda)$. Then the following are equivalent*

1. *K is in $D_c(X, \Lambda)$,*
2. *K can be represented by a bounded above complex whose terms are finite direct sums of $j_{U!}\underline{\Lambda}$ with $U \in \mathrm{Ob}(X_{étale})$ affine,*
3. *K can be represented by a bounded above complex of flat constructible sheaves of Λ-modules.*

Proof. It is clear that (2) implies (3) and that (3) implies (1). Assume K is in $D_c^-(X, \Lambda)$. Say $H^i(K) = 0$ for $i > b$. By induction on a we will construct a complex $\mathcal{F}^a \to \ldots \to \mathcal{F}^b$ such that each $\mathcal{F}^i$ is a finite direct sum of of $j_{U!}\underline{\Lambda}$ with $U \in \mathrm{Ob}(X_{étale})$ affine and a map $\mathcal{F}^\bullet \to K$ which induces an isomorphism $H^i(\mathcal{F}^\bullet) \to H^i(K)$ for $i > a$ and a surjection $H^a(\mathcal{F}^\bullet) \to H^a(K)$. For $a = b$ this can be done by Lemma 44.84.4. Given such a datum choose a distinguished triangle

$$\mathcal{F}^\bullet \to K \to L \to \mathcal{F}^\bullet[1]$$

Then we see that $H^i(L) = 0$ for $i \geq a$. Choose $\mathcal{F}^{a-1}[-a+1] \to L$ as in Lemma 44.84.4. The composition $\mathcal{F}^{a-1}[-a+1] \to L \to \mathcal{F}^\bullet$ corresponds to a map $\mathcal{F}^{a-1} \to \mathcal{F}^a$ such that the composition with $\mathcal{F}^a \to \mathcal{F}^{a+1}$ is zero. By TR4 we obtain a map

$$(\mathcal{F}^{a-1} \to \ldots \to \mathcal{F}^b) \to K$$

in $D(X, \Lambda)$. This finishes the induction step and the proof of the lemma. □

Lemma 44.84.6. *Let X be a scheme. Let Λ be a Noetherian ring. Let $K, L \in D_c^-(X, \Lambda)$. Then $K \otimes^{\mathbf{L}}_{\Lambda} L$ is in $D_c^-(X, \Lambda)$.*

Proof. This follows from Lemmas 44.84.5 and 44.65.7. □

Definition 44.84.7. Let X be a scheme. Let Λ be a Noetherian ring. We denote $D_{ctf}(X, \Lambda)$ the full subcategory of $D_c(X, \Lambda)$ consisting of objects having locally finite tor dimension.

This is a strictly full, saturated triangulated subcategory of $D_c(X, \Lambda)$ and $D(X, \Lambda)$. By our conventions, see Cohomology on Sites, Definition 21.34.1, we see that

$$D_{ctf}(X, \Lambda) \subset D^b(X, \Lambda)$$

if X is quasi-compact. A good way to think about objects of $D_{ctf}(X, \Lambda)$ is given in Lemma 44.84.9.

Remark 44.84.8. The situation with objects of $D_{ctf}(X, \Lambda)$ is different from $D_{perf}(\mathcal{O}_X)$ in Remark 44.83.3. Namely, it can happen that a complex of $\mathcal{O}_X$-modules is locally quasi-isomorphic to a finite complex of finite locally free $\mathcal{O}_X$-modules, without being globally quasi-isomorphic to a bounded complex of locally free $\mathcal{O}_X$-modules. The following lemma shows this does not happen for D_{ctf} on a Noetherian scheme.

Lemma 44.84.9. *Let Λ be a Noetherian ring. Let X be a quasi-compact and quasi-separated scheme. Let $K \in D(X, \Lambda)$. The following are equivalent*

(1) *$K \in D_{ctf}(X, \Lambda)$, and*
(2) *K can be represented by a finite complex of constructible flat sheaves of Λ-modules.*

In fact, if K has tor amplitude in $[a, b]$ then we can represent K by a complex $\mathcal{F}^a \to \ldots \to \mathcal{F}^b$ with $\mathcal{F}^p$ a constructible flat sheaf of Λ-modules.

Proof. It is clear that a finite complex of constructible flat sheaves of Λ-modules has finite tor dimension. It is also clear that it is an object of $D_c(X, \Lambda)$. Thus we see that (2) implies (1).

Assume (1). Choose $a, b \in \mathbf{Z}$ such that $H^i(K \otimes^{\mathbf{L}}_{\Lambda} \mathcal{G}) = 0$ if $i \notin [a, b]$ for all sheaves of Λ-modules $\mathcal{G}$. We will prove the final assertion holds by induction on $b - a$. If $a = b$, then $K = H^a(K)[-a]$ is a flat constructible sheaf and the result holds. Next, assume $b > a$. Represent K by a complex $\mathcal{K}^\bullet$ of sheaves of Λ-modules. Consider the surjection

$$\mathrm{Ker}(\mathcal{K}^b \to \mathcal{K}^{b+1}) \longrightarrow H^b(K)$$

By Lemma 44.67.6 we can find finitely many affine schemes U_i étale over X and a surjection $\bigoplus j_{U_i!}\underline{\Lambda}_{U_i} \to H^b(K)$. After replacing U_i by standard étale coverings $\{U_{ij} \to U_i\}$ we may assume this surjection lifts to a map $\mathcal{F} = \bigoplus j_{U_i!}\underline{\Lambda}_{U_i} \to \mathrm{Ker}(\mathcal{K}^b \to \mathcal{K}^{b+1})$. This map determines a distinguished triangle

$$\mathcal{F}[-b] \to K \to L \to \mathcal{F}[-b+1]$$

in $D(X, \Lambda)$. Since $D_{ctf}(X, \Lambda)$ is a triangulated subcategory we see that L is in it too. In fact L has tor amplitude in $[a, b-1]$ as $\mathcal{F}$ surjects onto $H^b(K)$ (details omitted). By induction hypothesis we can find a finite complex $\mathcal{F}^a \to \ldots \to \mathcal{F}^{b-1}$ of flat constructible sheaves of Λ-modules representing L. The map $L \to \mathcal{F}[-b+1]$ corresponds to a map $\mathcal{F}^b \to \mathcal{F}$ annihilating the image of $\mathcal{F}^{b-1} \to \mathcal{F}^b$. Then it follows from axiom TR3 that K is represented by the complex

$$\mathcal{F}^a \to \ldots \to \mathcal{F}^{b-1} \to \mathcal{F}^b$$

which finishes the proof. □

Remark 44.84.10. Let Λ be a Noetherian ring. Let X be a scheme. For a bounded complex $K^\bullet$ of constructible flat Λ-modules on $X_{étale}$ each stalk $K^p_{\overline{x}}$ is a finite projective Λ-module. Hence the stalks of the complex are perfect complexes of Λ-modules.

Remark 44.84.11. Lemma 44.84.9 can be used to prove that if $f : X \to Y$ is a separated, finite type morphism of schemes and Y is noetherian, then $Rf_!$ induces a functor $D_{ctf}(X, \Lambda) \to D_{ctf}(Y, \Lambda)$. We only need this fact in the case where Y is the spectrum of a field and X is a curve.

Lemma 44.84.12. *Let Λ be a Noetherian ring. If $j : U \to X$ is an étale morphism of schemes, then*

(1) $K|_U \in D_{ctf}(U,\Lambda)$ *if* $K \in D_{ctf}(X,\Lambda)$*, and*
(2) $j_!M \in D_{ctf}(X,\Lambda)$ *if* $M \in D_{ctf}(U,\Lambda)$ *and the morphism* j *is quasi-compact and quasi-separated.*

Proof. Perhaps the easiest way to prove this lemma is to reduce to the case where X is affine and then apply Lemma 44.84.9 to translate it into a statement about finite complexes of flat constructible sheaves of Λ-modules where the result follows from Lemma 44.67.1. □

Lemma 44.84.13. *Let* Λ *be a Noetherian ring. Let* $f : X \to Y$ *be a morphism of schemes. If* $K \in D_{ctf}(Y,\Lambda)$ *then* $Lf^*K \in D_{ctf}(X,\Lambda)$*.*

Proof. Apply Lemma 44.84.9 to reduce this to a question about finite complexes of flat constructible sheaves of Λ-modules. Then the statement follows as $f^{-1} = f^*$ is exact and Lemma 44.65.5. □

Lemma 44.84.14. *Let* X *be a connected scheme. Let* Λ *be a Noetherian ring. Let* $K \in D_{ctf}(X,\Lambda)$ *have locally constant cohomology sheaves. Then there exists a finite complex of finite projective* Λ*-modules* $M^\bullet$ *and an étale covering* $\{U_i \to X\}$ *such that* $K|_{U_i} \cong \underline{M^\bullet}|_{U_i}$ *in* $D(U_i,\Lambda)$*.*

Proof. Choose an étale covering $\{U_i \to X\}$ such that $K|_{U_i}$ is constant, say $K|_{U_i} \cong \underline{M_i^\bullet}_{U_i}$ for some finite complex of finite Λ-modules $M_i^\bullet$. See Cohomology on Sites, Lemma 21.39.1. Observe that $U_i \times_X U_j$ is empty if $M_i^\bullet$ is not isomorphic to $M_j^\bullet$ in $D(\Lambda)$. For each complex of Λ-modules $M^\bullet$ let $I_{M^\bullet} = \{i \in I \mid M_i^\bullet \cong M^\bullet \text{ in } D(\Lambda)\}$. As étale morphisms are open we see that $U_{M^\bullet} = \bigcup_{i \in I_{M^\bullet}} \mathrm{Im}(U_i \to X)$ is an open subset of X. Then $X = \coprod U_{M^\bullet}$ is a disjoint open covering of X. As X is connected only one $U_{M^\bullet}$ is nonempty. As K is in $D_{ctf}(X,\Lambda)$ we see that $M^\bullet$ is a perfect complex of Λ-modules, see More on Algebra, Lemma 15.50.2. Hence we may assume $M^\bullet$ is a finite complex of finite projective Λ-modules. □

44.85. Cohomology of nice complexes

The following is a special case of a more general result about compactly supported cohomology of objects of $D_{ctf}(X,\Lambda)$.

Proposition 44.85.1. *Let* X *be a projective curve over a field* k*,* Λ *a finite ring and* $K \in D_{ctf}(X,\Lambda)$*. Then* $R\Gamma(X_{\bar{k}},K) \in D_{perf}(\Lambda)$*.*

Sketch of proof. The first step is to show:

(1) *The cohomology of* $R\Gamma(X_{\bar{k}},K)$ *is bounded.*

Consider the spectral sequence

$$H^i(X_{\bar{k}}, \underline{H}^j(K)) \Rightarrow H^{i+j}(R\Gamma(X_{\bar{k}},K)).$$

Since K is bounded and Λ is finite, the sheaves $\underline{H}^j(K)$ are torsion. Moreover, $X_{\bar{k}}$ has finite cohomological dimension, so the left-hand side is nonzero for finitely many i and j only. Therefore, so is the right-hand side.

(2) *The cohomology groups* $H^{i+j}(R\Gamma(X_{\bar{k}},K))$ *are finite.*

Since the sheaves $\underline{H}^j(K)$ are constructible, the groups $H^i(X_{\bar{k}}, \underline{H}^j(K))$ are finite, [7] so it follows by the spectral sequence again.

[7] In Section 44.71 where we proved vanishing of cohomology, we should have proved – using the exact same arguments – that étale cohomology with values in a torsion sheaf is finite. Maybe that section should be updated. It's flabbergasting that we even forgot to mention it.

(3) $R\Gamma(X_{\bar{k}}, K)$ *has finite Tor-dimension.*

Let N be a right Λ-module (in fact, since Λ is finite, it suffices to assume that N is finite). By the projection formula (change of module),

$$N \otimes^{\mathbf{L}}_{\Lambda} R\Gamma(X_{\bar{k}}, K) = R\Gamma(X_{\bar{k}}, N \otimes^{\mathbf{L}}_{\Lambda} K).$$

Therefore,

$$H^i(N \otimes^{\mathbf{L}}_{\Lambda} R\Gamma(X_{\bar{k}}, K)) = H^i(R\Gamma(X_{\bar{k}}, N \otimes^{\mathbf{L}}_{\Lambda} K)).$$

Now consider the spectral sequence

$$H^i(X_{\bar{k}}, \underline{H}^j(N \otimes^{\mathbf{L}}_{\Lambda} K)) \Rightarrow H^{i+j}(R\Gamma(X_{\bar{k}}, N \otimes^{\mathbf{L}}_{\Lambda} K)).$$

Since K has finite Tor-dimension, $\underline{H}^j(N \otimes^{\mathbf{L}}_{\Lambda} K)$ vanishes universally for j small enough, and the left-hand side vanishes whenever $i < 0$. Therefore $R\Gamma(X_{\bar{k}}, K)$ has finite Tor-dimension, as claimed. So it is a perfect complex by Lemma 44.83.2. □

44.86. Lefschetz numbers

The fact that the total cohomology of a constructible complex of finite tor dimension is a perfect complex is the key technical reason why cohomology behaves well, and allows us to define rigorously the traces occurring in the trace formula.

Definition 44.86.1. Let Λ be a finite ring, X a projective curve over a finite field k and $K \in D_{ctf}(X, \Lambda)$ (for instance $K = \underline{\Lambda}$). There is a canonical map $c_K : \pi_X^{-1} K \to K$, and its base change $c_K|_{X_{\bar{k}}}$ induces an action denoted π_X^* on the perfect complex $R\Gamma(X_{\bar{k}}, K|_{X_{\bar{k}}})$. The *global Lefschetz number* of K is the trace $\mathrm{Tr}(\pi_X^* \,|_{R\Gamma(X_{\bar{k}}, K)})$ of that action. It is an element of $\Lambda^\natural$.

Definition 44.86.2. With Λ, X, k, K as in Definition 44.86.1. Since $K \in D_{ctf}(X, \Lambda)$, for any geometric point $\bar{x}$ of X, the complex $K_{\bar{x}}$ is a perfect complex (in $D_{perf}(\Lambda)$). As we have seen in Section 44.74, the Frobenius π_X acts on $K_{\bar{x}}$. The *local Lefschetz number* of K is the sum

$$\sum_{x \in X(k)} \mathrm{Tr}(\pi_X \,|_{K_{\overline{x}}})$$

which is again an element of $\Lambda^\natural$.

At last, we can formulate precisely the trace formula.

Theorem 44.86.3 (Lefschetz Trace Formula)**.** *Let X be a projective curve over a finite field k, Λ a finite ring and $K \in D_{ctf}(X, \Lambda)$. Then the global and local Lefschetz numbers of K are equal, i.e.,*

$$Tr(\pi_X^* \,|_{R\Gamma(X_{\bar{k}}, K)}) = \sum\nolimits_{x \in X(k)} Tr(\pi_X \,|_{K_{\bar{x}}}) \tag{44.86.3.1}$$

in $\Lambda^\natural$.

Proof. See discussion below. □

We will use, rather than prove, the trace formula. Nevertheless, we will give quite a few details of the proof of the theorem as given in [**Del77**] (some of the things that are not adequately explained are listed in Section 44.93).

We only stated the formula for curves, and in some weak sense it is a consequence of the following result.

Theorem 44.86.4 (Weil)**.** *Let C be a nonsingular projective curve over an algebraically closed field k, and $\varphi : C \to C$ a k-endomorphism of C distinct from the identity. Let $V(\varphi) = \Delta_C \cdot \Gamma_\varphi$, where Δ_C is the diagonal, Γ_φ is the graph of φ, and the intersection number is taken on $C \times C$. Let $J = \underline{Pic}^0_{C/k}$ be the jacobian of C and denote $\varphi^* : J \to J$ the action induced by φ by taking pullbacks. Then*

$$V(\varphi) = 1 - Tr_J(\varphi^*) + \deg \varphi.$$

Proof. The number $V(\varphi)$ is the number of fixed points of φ, it is equal to

$$V(\varphi) = \sum_{c \in |C| : \varphi(c) = c} m_{\mathrm{Fix}(\varphi)}(c)$$

where $m_{\mathrm{Fix}(\varphi)}(c)$ is the multiplicity of c as a fixed point of φ, namely the order or vanishing of the image of a local uniformizer under $\varphi - \mathrm{id}_C$. Proofs of this theorem can be found in [**Lan02, Wei48**]. □

Example 44.86.5. Let $C = E$ be an elliptic curve and $\varphi = [n]$ be multiplication by n. Then $\varphi^* = \varphi^t$ is multiplication by n on the jacobian, so it has trace $2n$ and degree n^2. On the other hand, the fixed points of φ are the points $p \in E$ such that $np = p$, which is the $(n-1)$-torsion, which has cardinality $(n-1)^2$. So the theorem reads

$$(n-1)^2 = 1 - 2n + n^2.$$

Jacobians. We now discuss without proofs the correspondence between a curve and its jacobian which is used in Weil's proof. Let C be a nonsingular projective curve over an algebraically closed field k and choose a base point $c_0 \in C(k)$. Denote by $A^1(C \times C)$ (or $\mathrm{Pic}(C \times C)$, or $\mathrm{CaCl}(C \times C)$) the abelian group of codimension 1 divisors of $C \times C$. Then

$$A^1(C \times C) = \mathrm{pr}_1^*(A^1(C)) \oplus \mathrm{pr}_2^*(A^1(C)) \oplus R$$

where

$$R = \{Z \in A^1(C \times C) \mid Z|_{C \times \{c_0\}} \sim_{\mathrm{rat}} 0 \text{ and } Z|_{\{c_0\} \times C} \sim_{\mathrm{rat}} 0\}.$$

In other words, R is the subgroup of line bundles which pull back to the trivial one under either projection. Then there is a canonical isomorphism of abelian groups $R \cong \mathrm{End}(J)$ which maps a divisor Z in R to the endomorphism

$$\begin{array}{ccc} J & \to & J \\ [\mathcal{O}_C(D)] & \mapsto & (\mathrm{pr}_1|_Z)_*(\mathrm{pr}_2|_Z)^*(D). \end{array}$$

The aforementioned correspondence is the following. We denote by σ the automorphism of $C \times C$ that switches the factors.

$\mathrm{End}(J)$	R
composition of α, β	$\mathrm{pr}_{13*}(\mathrm{pr}_{12}{}^*(\alpha) \circ \mathrm{pr}_{23}{}^*(\beta))$
id_J	$\Delta_C - \{c_0\} \times C - C \times \{c_0\}$
φ^*	$\Gamma_\varphi - C \times \{\varphi(c_0)\} - \sum_{\varphi(c)=c_0} \{c\} \times C$
the trace form $\alpha, \beta \mapsto \mathrm{Tr}(\alpha\beta)$	$\alpha, \beta \mapsto -\int_{C\times C} \alpha.\sigma^*\beta$
the Rosati involution $\alpha \mapsto \alpha^\dagger$	$\alpha \mapsto \sigma^*\alpha$
positivity of Rosati $\mathrm{Tr}(\alpha\alpha^\dagger) > 0$	Hodge index theorem on $C \times C$ $-\int_{C\times C} \alpha\sigma^*\alpha > 0.$

In fact, in light of the Kunneth formula, the subgroup R corresponds to the $1,1$ hodge classes in $H^1(C) \otimes H^1(C)$.

Weil's proof. Using this correspondence, we can prove the trace formula. We have

$$\begin{aligned} V(\varphi) &= \int_{C\times C} \Gamma_\varphi.\Delta \\ &= \int_{C\times C} \Gamma_\varphi.\left(\Delta_C - \{c_0\} \times C - C \times \{c_0\}\right) + \int_{C\times C} \Gamma_\varphi.\left(\{c_0\} \times C + C \times \{c_0\}\right). \end{aligned}$$

Now, on the one hand

$$\int_{C\times C} \Gamma_\varphi.\left(\{c_0\} \times C + C \times \{c_0\}\right) = 1 + \deg\varphi$$

and on the other hand, since R is the orthogonal of the ample divisor $\{c_0\} \times C + C \times \{c_0\}$,

$$\begin{aligned} &\int_{C\times C} \Gamma_\varphi.\left(\Delta_C - \{c_0\} \times C - C \times \{c_0\}\right) \\ = &\int_{C\times C} \left(\Gamma_\varphi - C \times \{\varphi(c_0)\} - \sum_{\varphi(c)=c_0} \{c\} \times C\right).\left(\Delta_C - \{c_0\} \times C - C \times \{c_0\}\right) \\ = &-\mathrm{Tr}_J(\varphi^* \circ \mathrm{id}_J). \end{aligned}$$

Recapitulating, we have

$$V(\varphi) = 1 - \mathrm{Tr}_J(\varphi^*) + \deg\varphi$$

which is the trace formula.

Lemma 44.86.6. *Consider the situation of Theorem 44.86.4 and let ℓ be a prime number invertible in k. Then*

$$\sum\nolimits_{i=0}^{2}(-1)^i Tr(\varphi^*|_{H^i(C,\underline{\mathbf{Z}/\ell^n\mathbf{Z}})}) = V(\varphi) \mod \ell^n.$$

Sketch of proof. Observe first that the assumption makes sense because $H^i(C, \underline{\mathbf{Z}/\ell^n\mathbf{Z}})$ is a free $\mathbf{Z}/\ell^n\mathbf{Z}$-module for all i. The trace of φ^* on the 0th degree cohomology is 1. The choice of a primitive ℓ^nth root of unity in k gives an isomorphism

$$H^i(C, \underline{\mathbf{Z}/\ell^n\mathbf{Z}}) \cong H^i(C, \mu_{\ell^n})$$

compatibly with the action of the geometric Frobenius. On the other hand, $H^1(C, \mu_{\ell^n}) = J[\ell^n]$. Therefore,

$$\begin{aligned} \mathrm{Tr}(\varphi^*|_{H^1(C,\underline{\mathbf{Z}/\ell^n\mathbf{Z}})}) &= \mathrm{Tr}_J(\varphi^*) \mod \ell^n \\ &= \mathrm{Tr}_{\mathbf{Z}/\ell^n\mathbf{Z}}(\varphi^* : J[\ell^n] \to J[\ell^n]). \end{aligned}$$

Moreover, $H^2(C, \mu_{\ell^n}) = \mathrm{Pic}(C)/\ell^n\mathrm{Pic}(C) \cong \mathbf{Z}/\ell^n\mathbf{Z}$ where φ^* is multiplication by $\deg\varphi$. Hence

$$\mathrm{Tr}(\varphi^*|_{H^2(C,\underline{\mathbf{Z}/\ell^n\mathbf{Z}})}) = \deg\varphi.$$

Thus we have

$$\sum_{i=0}^{2}(-1)^i\mathrm{Tr}(\varphi^*|_{H^i(C,\underline{\mathbf{Z}/\ell^n\mathbf{Z}})}) = 1 - \mathrm{Tr}_J(\varphi^*) + \deg\varphi \mod \ell^n$$

and the corollary follows from Theorem 44.86.4. □

An alternative way to prove this corollary is to show that

$$X \mapsto H^*(X, \mathbf{Q}_\ell) = \mathbf{Q}_\ell \otimes \lim_n H^*(X, \mathbf{Z}/\ell^n\mathbf{Z})$$

defines a Weil cohomology theory on smooth projective varieties over k. Then the trace formula

$$V(\varphi) = \sum_{i=0}^{2}(-1)^i\mathrm{Tr}(\varphi^*|_{H^i(C,\mathbf{Q}_\ell)})$$

is a formal consequence of the axioms (it's an exercise in linear algebra, the proof is the same as in the topological case).

44.87. Preliminaries and sorites

Notation: We fix the notation for this section. We denote by A a commutative ring, Λ a (possibly noncommutative) ring with a ring map $A \to \Lambda$ which image lies in the center of Λ. We let G be a finite group, Γ a *monoid extension of G by* $\mathbf{N}$, meaning that there is an exact sequence

$$1 \to G \to \tilde{\Gamma} \to \mathbf{Z} \to 1$$

and Γ consists of those elements of $\tilde{\Gamma}$ which image is nonnegative. Finally, we let P be an $A[\Gamma]$-module which is finite and projective as an $A[G]$-module, and M a $\Lambda[\Gamma]$-module which is finite and projective as a Λ-module.

Our goal is to compute the trace of $1 \in \mathbf{N}$ acting over Λ on the coinvariants of G on $P \otimes_A M$, that is, the number

$$\mathrm{Tr}_\Lambda\left(1; (P \otimes_A M)_G\right) \in \Lambda^\natural.$$

The element $1 \in \mathbf{N}$ will correspond to the Frobenius.

Lemma 44.87.1. *Let $e \in G$ denote the neutral element. The map*

$$\begin{array}{ccc} \Lambda[G] & \longrightarrow & \Lambda^\natural \\ \sum \lambda_g \cdot g & \longmapsto & \lambda_e \end{array}$$

factors through $\Lambda[G]^\natural$. We denote $\varepsilon : \Lambda[G]^\natural \to \Lambda^\natural$ the induced map.

Proof. We have to show the map annihilates commutators. One has

$$\left(\sum \lambda_g g\right)\left(\sum \mu_g g\right) - \left(\sum \mu_g g\right)\left(\sum \lambda_g g\right) = \sum_g \left(\sum_{g_1 g_2 = g} \lambda_{g_1}\mu_{g_2} - \mu_{g_1}\lambda_{g_2}\right) g$$

The coefficient of e is

$$\sum_g \left(\lambda_g \mu_{g^{-1}} - \mu_g \lambda_{g^{-1}}\right) = \sum_g \left(\lambda_g \mu_{g^{-1}} - \mu_{g^{-1}} \lambda_g\right)$$

which is a sum of commutators, hence it it zero in $\Lambda^\natural$. □

Definition 44.87.2. Let $f : P \to P$ be an endomorphism of a finite projective $\Lambda[G]$-module P. We define

$$\mathrm{Tr}^G_\Lambda(f; P) := \varepsilon\left(\mathrm{Tr}_{\Lambda[G]}(f; P)\right)$$

to be the *G-trace of f on P*.

Lemma 44.87.3. *Let $f : P \to P$ be an endomorphism of the finite projective $\Lambda[G]$-module P. Then*

$$Tr_\Lambda(f; P) = \#G \cdot Tr^G_\Lambda(f; P).$$

Proof. By additivity, reduce to the case $P = \Lambda[G]$. In that case, f is given by right multiplication by some element $\sum \lambda_g \cdot g$ of $\Lambda[G]$. In the basis $(g)_{g\in G}$, the matrix of f has coefficient $\lambda_{g_2^{-1}g_1}$ in the (g_1, g_2) position. In particular, all diagonal coefficients are λ_e, and there are $\#G$ such coefficients. □

Lemma 44.87.4. *The map $A \to \Lambda$ defines an A-module structure on $\Lambda^\natural$.*

Proof. This is clear. □

Lemma 44.87.5. *Let P be a finite projective $A[G]$-module and M a $\Lambda[G]$-module, finite projective as a Λ-module. Then $P \otimes_A M$ is a finite projective $\Lambda[G]$-module, for the structure induced by the diagonal action of G.*

Note that $P \otimes_A M$ is naturally a Λ-module since M is. Explicitly, together with the diagonal action this reads

$$\left(\sum \lambda_g g\right)(p \otimes m) = \sum gp \otimes \lambda_g gm.$$

Proof. For any $\Lambda[G]$-module N one has

$$\mathrm{Hom}_{\Lambda[G]}\left(P \otimes_A M, N\right) = \mathrm{Hom}_{A[G]}\left(P, \mathrm{Hom}_\Lambda(M, N)\right)$$

where the G-action on $\mathrm{Hom}_\Lambda(M, N)$ is given by $(g \cdot \varphi)(m) = g\varphi(g^{-1}m)$. Now it suffices to observe that the right-hand side is a composition of exact functors, because of the projectivity of P and M. □

Lemma 44.87.6. *With assumptions as in Lemma 44.87.5, let $u \in End_{A[G]}(P)$ and $v \in End_{\Lambda[G]}(M)$. Then*

$$Tr^G_\Lambda\left(u \otimes v; P \otimes_A M\right) = Tr^G_A(u; P) \cdot Tr_\Lambda(v; M).$$

Sketch of proof. Reduce to the case $P = A[G]$. In that case, u is right multiplication by some element $a = \sum a_g g$ of $A[G]$, which we write $u = R_a$. There is an isomorphism of $\Lambda[G]$-modules

$$\begin{array}{rccc} \varphi: & A[G] \otimes_A M & \cong & (A[G] \otimes_A M)' \\ & g \otimes m & \longmapsto & g \otimes g^{-1} m \end{array}$$

where $(A[G] \otimes_A M)'$ has the module structure given by the left G-action, together with the Λ-linearity on M. This transport of structure changes $u \otimes v$ into $\sum_g a_g R_g \otimes g^{-1} v$. In other words,

$$\varphi \circ (u \otimes v) \circ \varphi^{-1} = \sum_g a_g R_g \otimes g^{-1} v.$$

Working out explicitly both sides of the equation, we have to show

$$\mathrm{Tr}_\Lambda^G \left(\sum_g a_g R_g \otimes g^{-1} v \right) = a_e \cdot \mathrm{Tr}_\Lambda(v; M).$$

This is done by showing that

$$\mathrm{Tr}_\Lambda^G \left(a_g R_g \otimes g^{-1} v \right) = \begin{cases} 0 & \text{if } g \neq e \\ a_e \mathrm{Tr}_\Lambda(v; M) & \text{if } g = e \end{cases}$$

by reducing to $M = \Lambda$. □

Notation: Consider the monoid extension $1 \to G \to \Gamma \to \mathbf{N} \to 1$ and let $\gamma \in \Gamma$. Then we write $Z_\gamma = \{g \in G \mid g\gamma = \gamma g\}$.

Lemma 44.87.7. *Let P be a $\Lambda[\Gamma]$-module, finite and projective as a $\Lambda[G]$-module, and $\gamma \in \Gamma$. Then*

$$Tr_\Lambda(\gamma; P) = \#Z_\gamma \cdot Tr_\Lambda^{Z_\gamma}(\gamma; P).$$

Proof. This follows readily from Lemma 44.87.3. □

Lemma 44.87.8. *Let P be an $A[\Gamma]$-module, finite projective as $A[G]$-module. Let M be a $\Lambda[\Gamma]$-module, finite projective as a Λ-module. Then*

$$Tr_\Lambda^{Z_\gamma}(\gamma; P \otimes_A M) = Tr_A^{Z_\gamma}(\gamma; P) \cdot Tr_\Lambda(\gamma; M).$$

Proof. This follows directly from Lemma 44.87.6. □

Lemma 44.87.9. *Let P be a $\Lambda[\Gamma]$-module, finite projective as $\Lambda[G]$-module. Then the coinvariants $P_G = \Lambda \otimes_{\Lambda[G]} P$ form a finite projective Λ-module, endowed with an action of $\Gamma/G = \mathbf{N}$. Moreover, we have*

$$Tr_\Lambda(1; P_G) = \sum\nolimits'_{\gamma \mapsto 1} Tr_\Lambda^{Z_\gamma}(\gamma; P)$$

where $\sum'_{\gamma \mapsto 1}$ means taking the sum over the G-conjugacy classes in Γ.

Sketch of proof. We first prove this after multiplying by $\#G$.

$$\#G \cdot \mathrm{Tr}_\Lambda(1; P_G) = \mathrm{Tr}_\Lambda\left(\sum_{\gamma \mapsto 1} \gamma; P_G \right) = \mathrm{Tr}_\Lambda\left(\sum_{\gamma \mapsto 1} \gamma; P \right)$$

where the second equality follows by considering the commutative triangle

$$P^G \xrightarrow{a} P \xrightarrow{b} P_G, \qquad P_G \xrightarrow{c} P$$

where a is the canonical inclusion, b the canonical surjection and $c = \sum_{\gamma\mapsto 1}\gamma$. Then we have

$$\left(\sum_{\gamma\mapsto 1}\gamma\right)\Big|_P = a\circ c\circ b \qquad \text{and} \qquad \left(\sum_{\gamma\mapsto 1}\gamma\right)\Big|_{P_G} = b\circ a\circ c$$

hence they have the same trace. We then have

$$\#G\cdot \mathrm{Tr}_\Lambda(1;P_G) = {\sum_{\gamma\mapsto 1}}' \frac{\#G}{\#Z_\gamma}\mathrm{Tr}_\Lambda(\gamma;P) = \#G{\sum_{\gamma\mapsto 1}}'\mathrm{Tr}_\Lambda^{Z_\gamma}(\gamma;P).$$

To finish the proof, reduce to case Λ torsion-free by some universality argument. See [**Del77**] for details. □

Remark 44.87.10. Let us try to illustrate the content of the formula of Lemma 44.87.8. Suppose that Λ, viewed as a trivial Γ-module, admits a finite resolution $0\to P_r\to\ldots\to P_1\to P_0\to\Lambda\to 0$ by some $\Lambda[\Gamma]$-modules P_i which are finite and projective as $\Lambda[G]$-modules. In that case

$$H_*\left((P_\bullet)_G\right) = \mathrm{Tor}_*^{\Lambda[G]}(\Lambda,\Lambda) = H_*(G,\Lambda);$$

and

$$\mathrm{Tr}_\Lambda^{Z_\gamma}(\gamma;P_\bullet) = \frac{1}{\#Z_\gamma}\mathrm{Tr}_\Lambda(\gamma;P_\bullet) = \frac{1}{\#Z_\gamma}\mathrm{Tr}(\gamma;\Lambda) = \frac{1}{\#Z_\gamma}.$$

Therefore, Lemma 44.87.8 says

$$\mathrm{Tr}_\Lambda(1;P_G) = \mathrm{Tr}\left(1\,\big|_{H_*(G,\Lambda)}\right) = {\sum_{\gamma\mapsto 1}}'\frac{1}{\#Z_\gamma}.$$

This can be interpreted as a point count on the stack BG. If $\Lambda = \mathbf{F}_\ell$ with ℓ prime to $\#G$, then $H_*(G,\Lambda)$ is $\mathbf{F}_\ell$ in degree 0 (and 0 in other degrees) and the formula reads

$$1 = \sum\nolimits_{\substack{\sigma\text{-conjugacy}\\ \text{classes}\langle\gamma\rangle}}\frac{1}{\#Z_\gamma} \mod \ell.$$

This is in some sense a "trivial" trace formula for G. Later we will see that (44.86.3.1) can in some cases be viewed as a highly nontrivial trace formula for a certain type of group, see Section 44.102.

44.88. Proof of the trace formula

Theorem 44.88.1. *Let k be a finite field and X a finite type, separated scheme of dimension at most 1 over k. Let Λ be a finite ring whose cardinality is prime to that of k, and $K\in D_{ctf}(X,\Lambda)$. Then*

$$Tr\left(\pi_X^*\big|_{R\Gamma_c(X_{\bar k},K)}\right) = \sum\nolimits_{x\in X(k)} Tr\left(\pi_x\big|_{K_{\bar x}}\right) \tag{44.88.1.1}$$

in $\Lambda^\natural$.

Please see Remark 44.88.2 for some remarks on the statement. Notation: For short, we write

$$T'(X,K) = \sum\nolimits_{x\in X(k)}\mathrm{Tr}\big(\pi_x\big|_{K_{\bar x}}\big)$$

for the right-hand side of (44.88.1.1) and

$$T''(X,K) = \mathrm{Tr}\big(\pi_x^*\big|_{R\Gamma_c(X_{\bar k},K)}\big)$$

for the left-hand side.

Proof of Theorem 44.88.1. The proof proceeds in a number of steps.

Step 1. *Let $j : \mathcal{U} \hookrightarrow X$ be an open immersion with complement $Y = X - \mathcal{U}$ and $i : Y \hookrightarrow X$. Then $T''(X, K) = T''(\mathcal{U}, j^{-1}K) + T''(Y, i^{-1}K)$ and $T'(X, K) = T'(\mathcal{U}, j^{-1}K) + T'(Y, i^{-1}K)$.*

This is clear for T'. For T'' use the exact sequence

$$0 \to j_!j^{-1}K \to K \to i_*i^{-1}K \to 0$$

to get a filtration on K. This gives rise to an object $\widetilde{K} \in D\mathrm{F}(X, \Lambda)$ whose graded pieces are $j_!j^{-1}K$ and $i_*i^{-1}K$, both of which lie in $D_{ctf}(X, \Lambda)$. Then, by filtered derived abstract nonsense (INSERT REFERENCE), $R\Gamma_c(X_{\bar{k}}, K) \in D\mathrm{F}_{perf}(\Lambda)$, and it comes equipped with π_x^* in $D\mathrm{F}_{perf}(\Lambda)$. By the discussion of traces on filtered complexes (INSERT REFERENCE) we get

$$\begin{aligned} \mathrm{Tr}\left(\pi_X^*\big|_{R\Gamma_c(X_{\bar{k}}, K)}\right) &= \mathrm{Tr}\left(\pi_X^*\big|_{R\Gamma_c(X_{\bar{k}}, j_!j^{-1}K)}\right) + \mathrm{Tr}\left(\pi_X^*\big|_{R\Gamma_c(X_{\bar{k}}, i_*i^{-1}K)}\right) \\ &= T''(U, i^{-1}K) + T''(Y, i^{-1}K). \end{aligned}$$

Step 2. *The theorem holds if* $\dim X \leq 0$.

Indeed, in that case

$$R\Gamma_c(X_{\bar{k}}, K) = R\Gamma(X_{\bar{k}}, K) = \Gamma(X_{\bar{k}}, K) = \bigoplus\nolimits_{\bar{x} \in X_{\bar{k}}} K_{\bar{x}} \leftarrow \pi_X *.$$

Since the fixed points of $\pi_X : X_{\bar{k}} \to X_{\bar{k}}$ are exactly the points $\bar{x} \in X_{\bar{k}}$ which lie over a k-rational point $x \in X(k)$ we get

$$\mathrm{Tr}\big(\pi_X^*|_{R\Gamma_c(X_{\bar{k}}, K)}\big) = \sum\nolimits_{x \in X(k)} \mathrm{Tr}(\pi_{\bar{x}}|_{K_{\bar{x}}}).$$

Step 3. *It suffices to prove the equality $T'(\mathcal{U}, \mathcal{F}) = T''(\mathcal{U}, \mathcal{F})$ in the case where*

- *$\mathcal{U}$ is a smooth irreducible affine curve over k ;*
- *$\mathcal{U}(k) = \emptyset$;*
- *$K = \mathcal{F}$ is a finite locally constant sheaf of Λ-modules on $\mathcal{U}$ whose stalk(s) are finite projective Λ-modules ; and*
- *Λ is killed by a power of a prime ℓ and $\ell \in k^*$.*

Indeed, because of Step 2, we can throw out any finite set of points. But we have only finitely many rational points, so we may assume there are none[8]. We may assume that $\mathcal{U}$ is smooth irreducible and affine by passing to irreducible components and throwing away the bad points if necessary. The assumptions of $\mathcal{F}$ come from unwinding the definition of $D_{ctf}(X, \Lambda)$ and those on Λ from considering its primary decomposition.

For the remainder of the proof, we consider the situation

$$\begin{array}{ccc} \mathcal{V} & \longrightarrow & Y \\ {\scriptstyle f}\downarrow & & \downarrow{\scriptstyle \bar{f}} \\ \mathcal{U} & \longrightarrow & X \end{array}$$

where $\mathcal{U}$ is as above, f is a finite étale Galois covering, $\mathcal{V}$ is connected and the horizontal arrows are projective completions. Denoting $G = \mathrm{Aut}(\mathcal{V}|\mathcal{U})$, we also assume (as we may) that $f^{-1}\mathcal{F} = \underline{M}$ is constant, where the module $M = \Gamma(\mathcal{V}, f^{-1}\mathcal{F})$ is a

[8]At this point, there should be an evil laugh in the background.

$\Lambda[G]$-module which is finite and projective over Λ. This corresponds to the trivial monoid extension

$$1 \to G \to \Gamma = G \times \mathbf{N} \to \mathbf{N} \to 1.$$

In that context, using the reductions above, we need to show that $T''(\mathcal{U}, \mathcal{F}) = 0$.

Step 4. *There is a natural action of G on $f_* f^{-1}\mathcal{F}$ and the trace map $f_* f^{-1}\mathcal{F} \to \mathcal{F}$ defines an isomorphism*

$$(f_* f^{-1}\mathcal{F}) \otimes_{\Lambda[G]} \Lambda = (f_* f^{-1}\mathcal{F})_G \cong \mathcal{F}.$$

To prove this, simply unwind everything at a geometric point.

Step 5. *Let $A = \mathbf{Z}/\ell^n\mathbf{Z}$ with $n \gg 0$. Then $f_* f^{-1}\mathcal{F} \cong (f_*\underline{A}) \otimes_{\underline{A}} \underline{M}$ with diagonal G-action.*

Step 6. *There is a canonical isomorphism $(f_*\underline{A} \otimes_{\underline{A}} \underline{M}) \otimes_{\Lambda[G]} \underline{\Lambda} \cong \mathcal{F}$.*

In fact, this is a derived tensor product, because of the projectivity assumption on $\mathcal{F}$.

Step 7. *There is a canonical isomorphism*

$$R\Gamma_c(\mathcal{U}_{\bar{k}}, \mathcal{F}) = (R\Gamma_c(\mathcal{U}_{\bar{k}}, f_*A) \otimes^{\mathbf{L}}_A M) \otimes^{\mathbf{L}}_{\Lambda[G]} \Lambda,$$

compatible with the action of $\pi^*_{\mathcal{U}}$.

This comes from the universal coefficient theorem, i.e., the fact that $R\Gamma_c$ commutes with $\otimes^{\mathbf{L}}$, and the flatness of $\mathcal{F}$ as a Λ-module.

We have

$$\begin{aligned} \mathrm{Tr}(\pi^*_{\mathcal{U}}\big|_{R\Gamma_c(\mathcal{U}_{\bar{k}},\mathcal{F})}) &= {\sum_{g\in G}}' \mathrm{Tr}^{Z_g}_{\Lambda}\left((g,\pi^*_{\mathcal{U}})\big|_{R\Gamma_c(\mathcal{U}_{\bar{k}},f_*A)\otimes^{\mathbf{L}}_A M}\right) \\ &= {\sum_{g\in G}}' \mathrm{Tr}^{Z_g}_{A}((g,\pi^*_{\mathcal{U}})\big|_{R\Gamma_c(\mathcal{U}_{\bar{k}},f_*A)}) \cdot \mathrm{Tr}_{\Lambda}(g|_M) \end{aligned}$$

where Γ acts on $R\Gamma_c(\mathcal{U}_{\bar{k}}, \mathcal{F})$ by G and $(e, 1)$ acts via $\pi^*_{\mathcal{U}}$. So the monoidal extension is given by $\Gamma = G \times \mathbf{N} \to \mathbf{N}$, $\gamma \mapsto 1$. The first equality follows from Lemma 44.87.9 and the second from Lemma 44.87.8.

Step 8. *It suffices to show that $Tr^{Z_g}_A\left((g,\pi^*_{\mathcal{U}})\big|_{R\Gamma_c(\mathcal{U}_{\bar{k}},f_*A)}\right) \in A$ maps to zero in Λ.*

Recall that

$$\begin{aligned} \#Z_g \cdot \mathrm{Tr}^{Z_g}_A\left((g,\pi^*_{\mathcal{U}})\big|_{R\Gamma_c(\mathcal{U}_{\bar{k}},f_*A)}\right) &= \mathrm{Tr}_A\left((g,\pi^*_{\mathcal{U}})\big|_{R\Gamma_c(\mathcal{U}_{\bar{k}},f_*A)}\right) \\ &= \mathrm{Tr}_A\left((g^{-1}\pi_{\mathcal{V}})^*\big|_{R\Gamma_c(\mathcal{V}_{\bar{k}},A)}\right). \end{aligned}$$

The first equality is Lemma 44.87.7, the second is the Leray spectral sequence, using the finiteness of f and the fact that we are only taking traces over A. Now since $A = \mathbf{Z}/\ell^n\mathbf{Z}$ with $n \gg 0$ and $\#Z_g = \ell^a$ for some (fixed) a, it suffices to show the following result.

Step 9. *$Tr_A\left((g^{-1}\pi_{\mathcal{V}})^*\big|_{R\Gamma_c(\mathcal{V},A)}\right) = 0$ in A.*

By additivity again, we have

$$\begin{aligned}\mathrm{Tr}_A\left((g^{-1}\pi_{\mathcal{V}})^*\big|_{R\Gamma_c(\mathcal{V}_{\bar{k}}A)}\right) + \mathrm{Tr}_A\left((g^{-1}\pi_{\mathcal{V}})^*\big|_{R\Gamma_c(Y-\mathcal{V})_{\bar{k}},A)}\right)\\ = \mathrm{Tr}_A\left((g^{-1}\pi_Y)^*\big|_{R\Gamma(Y_{\bar{k}},A)}\right)\end{aligned}$$

The latter trace is the number of fixed points of $g^{-1}\pi_Y$ on Y, by Weil's trace formula Theorem 44.86.4. Moreover, by the 0-dimensional case already proven in step 2,

$$\mathrm{Tr}_A\left((g^{-1}\pi_{\mathcal{V}})^*\big|_{R\Gamma_c(Y-\mathcal{V})_{\bar{k}},A)}\right)$$

is the number of fixed points of $g^{-1}\pi_Y$ on $(Y-\mathcal{V})_{\bar{k}}$. Therefore,

$$\mathrm{Tr}_A\left((g^{-1}\pi_{\mathcal{V}})^*\big|_{R\Gamma_c(\mathcal{V}_{\bar{k}},A)}\right)$$

is the number of fixed points of $g^{-1}\pi_Y$ on $\mathcal{V}_{\bar{k}}$. But there are no such points: if $\bar{y} \in Y_{\bar{k}}$ is fixed under $g^{-1}\pi_Y$, then $\bar{f}(\bar{y}) \in X_{\bar{k}}$ is fixed under π_X. But $\mathcal{U}$ has no k-rational point, so we must have $\bar{f}(\bar{y}) \in (X-\mathcal{U})_{\bar{k}}$ and so $\bar{y} \notin \mathcal{V}_{\bar{k}}$, a contradiction. This finishes the proof. □

Remark 44.88.2. Remarks on Theorem 44.88.1.

(1) This formula holds in any dimension. By a dévissage lemma (which uses proper base change etc.) it reduces to the current statement – in that generality.
(2) The complex $R\Gamma_c(X_{\bar{k}}, K)$ is defined by choosing an open immersion $j : X \hookrightarrow \bar{X}$ with $\bar{X}$ projective over k of dimension at most 1 and setting

$$R\Gamma_c(X_{\bar{k}}, K) := R\Gamma(\bar{X}_{\bar{k}}, j_!K).$$

This is independent of the choice of $\bar{X}$ follows from (insert reference here). We define $H^i_c(X_{\bar{k}}, K)$ to be the ith cohomology group of $R\Gamma_c(X_{\bar{k}}, K)$.

Remark 44.88.3. Even though all we did are reductions and mostly algebra, the trace formula Theorem 44.88.1 is much stronger than Weil's geometric trace formula (Theorem 44.86.4) because it applies to coefficient systems (sheaves), not merely constant coefficients.

44.89. Applications

OK, having indicated the proof of the trace formula, let's try to use it for something.

44.90. On l-adic sheaves

Definition 44.90.1. Let X be a noetherian scheme. A *$\mathbf{Z}_\ell$-sheaf* on X, or simply a ℓ-adic sheaf is an inverse system $\{\mathcal{F}_n\}_{n\geq 1}$ where

(1) $\mathcal{F}_n$ is a constructible $\mathbf{Z}/\ell^n\mathbf{Z}$-module on $X_{étale}$, and
(2) the transition maps $\mathcal{F}_{n+1} \to \mathcal{F}_n$ induce isomorphisms $\mathcal{F}_{n+1} \otimes_{\mathbf{Z}/\ell^{n+1}\mathbf{Z}} \mathbf{Z}/\ell^n\mathbf{Z} \cong \mathcal{F}_n$.

We say that $\mathcal{F}$ is *lisse* if each $\mathcal{F}_n$ is locally constant. A *morphism* of such is merely a morphism of inverse systems.

Lemma 44.90.2. *Let $\{\mathcal{G}_n\}_{n\geq 1}$ be an inverse system of constructible $\mathbf{Z}/\ell^n\mathbf{Z}$-modules. Suppose that for all $k \geq 1$, the maps*

$$\mathcal{G}_{n+1}/\ell^k\mathcal{G}_{n+1} \to \mathcal{G}_n/\ell^k\mathcal{G}_n$$

are isomorphisms for all $n \gg 0$ (where the bound possibly depends on k). In other words, assume that the system $\{\mathcal{G}_n/\ell^k\mathcal{G}_n\}_{n\geq 1}$ is eventually constant, and call $\mathcal{F}_k$ the corresponding sheaf. Then the system $\{\mathcal{F}_k\}_{k\geq 1}$ forms a $\mathbf{Z}_\ell$-sheaf on X.

Proof. The proof is obvious. □

Lemma 44.90.3. *The category of $\mathbf{Z}_\ell$-sheaves on X is abelian.*

Proof. Let $\Phi = \{\varphi_n\}_{n\geq 1} : \{\mathcal{F}_n\} \to \{\mathcal{G}_n\}$ be a morphism of $\mathbf{Z}_\ell$-sheaves. Set

$$\mathrm{Coker}(\Phi) = \left\{\mathrm{Coker}\left(\mathcal{F}_n \xrightarrow{\varphi_n} \mathcal{G}_n\right)\right\}_{n\geq 1}$$

and $\mathrm{Ker}(\Phi)$ is the result of Lemma 44.90.2 applied to the inverse system

$$\left\{\bigcap_{m\geq n} \mathrm{Im}\left(\mathrm{Ker}(\varphi_m) \to \mathrm{Ker}(\varphi_n)\right)\right\}_{n\geq 1}.$$

That this defines an abelian category is left to the reader. □

Example 44.90.4. Let $X = \mathrm{Spec}(\mathbf{C})$ and $\Phi : \mathbf{Z}_\ell \to \mathbf{Z}_\ell$ be multiplication by ℓ. More precisely,

$$\Phi = \left\{\mathbf{Z}/\ell^n\mathbf{Z} \xrightarrow{\ell} \mathbf{Z}/\ell^n\mathbf{Z}\right\}_{n\geq 1}.$$

To compute the kernel, we consider the inverse system

$$\cdots \to \mathbf{Z}/\ell\mathbf{Z} \xrightarrow{0} \mathbf{Z}/\ell\mathbf{Z} \xrightarrow{0} \mathbf{Z}/\ell\mathbf{Z}.$$

Since the images are always zero, $\mathrm{Ker}(\Phi)$ is zero as a system.

Remark 44.90.5. If $\mathcal{F} = \{\mathcal{F}_n\}_{n\geq 1}$ is a $\mathbf{Z}_\ell$-sheaf on X and $\bar{x}$ is a geometric point then $M_n = \{\mathcal{F}_{n,\bar{x}}\}$ is an inverse system of finite $\mathbf{Z}/\ell^n\mathbf{Z}$-modules such that $M_{n+1} \to M_n$ is surjective and $M_n = M_{n+1}/\ell^n M_{n+1}$. It follows that

$$M = \lim_n M_n = \lim \mathcal{F}_{n,\bar{x}}$$

is a finite $\mathbf{Z}_\ell$-module. Indeed, $M/\ell M = M_1$ is finite over $\mathbf{F}_\ell$, so by Nakayama M is finite over $\mathbf{Z}_\ell$. Therefore, $M \cong \mathbf{Z}_\ell^{\oplus r} \oplus \oplus_{i=1}^t \mathbf{Z}_\ell/\ell^{e_i}\mathbf{Z}_\ell$ for some $r, t \geq 0$, $e_i \geq 1$. The module $M = \mathcal{F}_{\bar{x}}$ is called the *stalk* of $\mathcal{F}$ at $\bar{x}$.

Definition 44.90.6. A $\mathbf{Z}_\ell$-sheaf $\mathcal{F}$ is *torsion* if $\ell^n : \mathcal{F} \to \mathcal{F}$ is the zero map for some n. The abelian category of $\mathbf{Q}_\ell$-sheaves on X is the quotient of the abelian category of $\mathbf{Z}_\ell$-sheaves by the Serre subcategory of torsion sheaves. In other words, its objects are $\mathbf{Z}_\ell$-sheaves on X, and if $\mathcal{F}, \mathcal{G}$ are two such, then

$$\mathrm{Hom}_{\mathbf{Q}_\ell}(\mathcal{F}, \mathcal{G}) = \mathrm{Hom}_{\mathbf{Z}_\ell}(\mathcal{F}, \mathcal{G}) \otimes_{\mathbf{Z}_\ell} \mathbf{Q}_\ell.$$

We denote by $\mathcal{F} \mapsto \mathcal{F} \otimes \mathbf{Q}_\ell$ the quotient functor (right adjoint to the inclusion). If $\mathcal{F} = \mathcal{F}' \otimes \mathbf{Q}_\ell$ where $\mathcal{F}'$ is a $\mathbf{Z}_\ell$-sheaf and $\bar{x}$ is a geometric point, then the *stalk* of $\mathcal{F}$ at $\bar{x}$ is $\mathcal{F}_{\bar{x}} = \mathcal{F}'_{\bar{x}} \otimes \mathbf{Q}_\ell$.

Remark 44.90.7. Since a $\mathbf{Z}_\ell$-sheaf is only defined on a noetherian scheme, it is torsion if and only if its stalks are torsion.

Definition 44.90.8. If X is a separated scheme of finite type over an algebraically closed field k and $\mathcal{F} = \{\mathcal{F}_n\}_{n\geq 1}$ is a $\mathbf{Z}_\ell$-sheaf on X, then we define

$$H^i(X,\mathcal{F}) := \lim_n H^i(X,\mathcal{F}_n) \qquad \text{and} \qquad H^i_c(X,\mathcal{F}) := \lim_n H^i_c(X,\mathcal{F}_n).$$

If $\mathcal{F} = \mathcal{F}' \otimes \mathbf{Q}_\ell$ for a $\mathbf{Z}_\ell$-sheaf $\mathcal{F}'$ then we set

$$H^i_c(X,\mathcal{F}) := H^i_c(X,\mathcal{F}') \otimes_{\mathbf{Z}_\ell} \mathbf{Q}_\ell.$$

We call these the *ℓ-adic cohomology* of X with coefficients $\mathcal{F}$.

44.91. L-functions

Definition 44.91.1. Let X be a scheme of finite type over a finite field k. Let Λ be a finite ring of order prime to the characteristic of k and $\mathcal{F}$ a constructible flat Λ-module on $X_{étale}$. Then we set

$$L(X,\mathcal{F}) := \prod_{x\in|X|} \det\left(1 - \pi_x^*\ T^{\deg x}\Big|_{\mathcal{F}_{\bar{x}}}\right)^{-1} \in \Lambda[[T]]$$

where $|X|$ is the set of closed points of X, $\deg x = [\kappa(x) : k]$ and $\bar{x}$ is a geometric point lying over x. This definition clearly generalizes to the case where $\mathcal{F}$ is replace by a $K \in D_{ctf}(X,\Lambda)$. We call this the *L-function of $\mathcal{F}$*.

Remark 44.91.2. Intuitively, T should be thought of as $T = t^f$ where $p^f = \#k$. The definitions are then independent of the size of the ground field.

Definition 44.91.3. Now assume that $\mathcal{F}$ is a $\mathbf{Q}_\ell$-sheaf on X. In this case we define

$$L(X,\mathcal{F}) := \prod_{x\in|X|} \det\left(1 - \pi_x^*\ T^{\deg x}\Big|_{\mathcal{F}_{\bar{x}}}\right)^{-1} \in \mathbf{Q}_\ell[[T]].$$

Note that this product converges since there are finitely many points of a given degree. We call this the *L-function of $\mathcal{F}$*.

44.92. Cohomological interpretation

This is how Grothendieck interpreted the L-function.

Theorem 44.92.1 (Finite Coefficients)**.** *Let X be a scheme of finite type over a finite field k. Let Λ be a finite ring of order prime to the characteristic of k and $\mathcal{F}$ a constructible flat Λ-module on $X_{étale}$. Then*

$$L(X,\mathcal{F}) = \det\left(1 - \pi_X^*\ T\Big|_{R\Gamma_c(X_{\bar{k}},\mathcal{F})}\right)^{-1} \in \Lambda[[T]].$$

Proof. Omitted. □

Thus far, we don't even know whether each cohomology group $H^i_c(X_{\bar{k}},\mathcal{F})$ is free.

Theorem 44.92.2 (Adic sheaves)**.** *Let X be a scheme of finite type over a finite field k, and $\mathcal{F}$ a $\mathbf{Q}_\ell$-sheaf on X. Then*

$$L(X,\mathcal{F}) = \prod_i \det\left(1 - \pi_X^*\ T\Big|_{H^i_c(X_{\bar{k}},\mathcal{F})}\right)^{(-1)^{i+1}} \in \mathbf{Q}_\ell[[T]].$$

Proof. This is sketched below. □

Remark 44.92.3. Since we have only developed some theory of traces and not of determinants, Theorem 44.92.1 is harder to prove than Theorem 44.92.2. We will only prove the latter, for the former see [**Del77**]. Observe also that there is no version of this theorem more general for $\mathbf{Z}_\ell$ coefficients since there is no ℓ-torsion.

We reduce the proof of Theorem 44.92.2 to a trace formula. Since $\mathbf{Q}_\ell$ has characteristic 0, it suffices to prove the equality after taking logarithmic derivatives. More precisely, we apply $T\frac{d}{dT}\log$ to both sides. We have on the one hand

$$\begin{aligned} T\frac{d}{dT}\log L(X,\mathcal{F}) &= T\frac{d}{dT}\log\prod_{x\in|X|}\det\left(1-\pi_x^*\ T^{\deg x}\Big|_{\mathcal{F}_{\bar{x}}}\right)^{-1} \\ &= \sum_{x\in|X|} T\frac{d}{dT}\log\left(\det\left(1-\pi_x^*\ T^{\deg x}\Big|_{\mathcal{F}_{\bar{x}}}\right)^{-1}\right) \\ &= \sum_{x\in|X|}\deg x\sum_{n\geq 1}\mathrm{Tr}\left((\pi_x^n)^*\big|_{\mathcal{F}_{\bar{x}}}\right)T^{n\deg x} \end{aligned}$$

where the last equality results from the formula

$$T\frac{d}{dT}\log\left(\det\left(1-fT|_M\right)^{-1}\right)=\sum_{n\geq 1}\mathrm{Tr}(f^n|_M)T^n$$

which holds for any commutative ring Λ and any endomorphism f of a finite projective Λ-module M. On the other hand, we have

$$\begin{aligned} T\frac{d}{dT}\log\left(\prod_i\det\left(1-\pi_X^*\ T\Big|_{H_c^i(X_{\bar{k}},\mathcal{F})}\right)^{(-1)^{i+1}}\right) \\ = \quad \sum_i(-1)^i\sum_{n\geq 1}\mathrm{Tr}\left((\pi_X^n)^*\big|_{H_c^i(X_{\bar{k}},\mathcal{F})}\right)T^n \end{aligned}$$

by the same formula again. Now, comparing powers of T and using the Mobius inversion formula, we see that Theorem 44.92.2 is a consequence of the following equality

$$\sum_{d|n}d\sum_{\substack{x\in|X|\\ \deg x=d}}\mathrm{Tr}\left(\left(\pi_X^{n/d}\right)^*\Big|_{\mathcal{F}_{\bar{x}}}\right)=\sum_i(-1)^i\mathrm{Tr}\left((\pi_X^n)^*\,|_{H_c^i(X_{\bar{k}},\mathcal{F})}\right).$$

Writing k_n for the degree n extension of k, $X_n=X\times_{\mathrm{Spec}\,k}\mathrm{Spec}(k_n)$ and ${}_n\mathcal{F}=\mathcal{F}|_{X_n}$, this boils down to

$$\sum_{x\in X_n(k_n)}\mathrm{Tr}\left(\pi_X^*\big|_{{}_n\mathcal{F}_{\bar{x}}}\right)=\sum_i(-1)^i\mathrm{Tr}\left((\pi_X^n)^*\,|_{H_c^i((X_n)_{\bar{k}},{}_n\mathcal{F})}\right)$$

which is a consequence of Theorem 44.92.5.

Theorem 44.92.4. *Let X/k be as above, let Λ be a finite ring with $\#\Lambda\in k^*$ and $K\in D_{ctf}(X,\Lambda)$. Then $R\Gamma_c(X_{\bar{k}},K)\in D_{perf}(\Lambda)$ and*

$$\sum_{x\in X(k)}Tr(\pi_x\,|_{K_{\bar{x}}})=Tr\left(\pi_X^*\,|_{R\Gamma_c(X_{\bar{k}},K)}\right).$$

Proof. Note that we have already proved this (REFERENCE) when $\dim X\leq 1$. The general case follows easily from that case together with the proper base change theorem. □

Theorem 44.92.5. *Let X be a separated scheme of finite type over a finite field k and $\mathcal{F}$ be a $\mathbf{Q}_\ell$-sheaf on X. Then $\dim_{\mathbf{Q}_\ell} H^i_c(X_{\bar{k}}, \mathcal{F})$ is finite for all i, and is nonzero for $0 \leq i \leq 2\dim X$ only. Furthermore, we have*

$$\sum_{x \in X(k)} Tr(\pi_x \,|_{\mathcal{F}_{\bar{x}}}) = \sum_i (-1)^i \, Tr\left(\pi_X^* \big|_{H^i_c(X_{\bar{k}}, \mathcal{F})}\right).$$

Proof. We explain how to deduce this from Theorem 44.92.4. We first use some étale cohomology arguments to reduce the proof to an algebraic statement which we subsequently prove.

Let $\mathcal{F}$ be as in the theorem. We can write $\mathcal{F}$ as $\mathcal{F}' \otimes \mathbf{Q}_\ell$ where $\mathcal{F}' = \{\mathcal{F}'_n\}$ is a $\mathbf{Z}_\ell$-sheaf without torsion, i.e., $\ell : \mathcal{F}' \to \mathcal{F}'$ has trivial kernel in the category of $\mathbf{Z}_\ell$-sheaves. Then each $\mathcal{F}'_n$ is a flat constructible $\mathbf{Z}/\ell^n\mathbf{Z}$-module on $X_{étale}$, so $\mathcal{F}'_n \in D_{ctf}(X, \mathbf{Z}/\ell^n\mathbf{Z})$ and $\mathcal{F}'_{n+1} \otimes^{\mathbf{L}}_{\mathbf{Z}/\ell^{n+1}\mathbf{Z}} \mathbf{Z}/\ell^n\mathbf{Z} = \mathcal{F}'_n$. Note that the last equality holds also for standard (non-derived) tensor product, since $\mathcal{F}'_n$ is flat (it is the same equality). Therefore,

(1) the complex $K_n = R\Gamma_c(X_{\bar{k}}, \mathcal{F}'_n)$ is perfect, and it is endowed with an endomorphism $\pi_n : K_n \to K_n$ in $D(\mathbf{Z}/\ell^n\mathbf{Z})$;
(2) there are identifications
$$K_{n+1} \otimes^{\mathbf{L}}_{\mathbf{Z}/\ell^{n+1}\mathbf{Z}} \mathbf{Z}/\ell^n\mathbf{Z} = K_n$$
in $D_{perf}(\mathbf{Z}/\ell^n\mathbf{Z})$, compatible with the endomorphisms π_{n+1} and π_n (see [**Del77**, Rapport 4.12]) ;
(3) the equality $\mathrm{Tr}(\pi_X^* \,|_{K_n}) = \sum_{x \in X(k)} \mathrm{Tr}\left(\pi_x \,\big|_{(\mathcal{F}'_n)_{\bar{x}}}\right)$ holds ; and
(4) for each $x \in X(k)$, the elements $\mathrm{Tr}\left(\pi_x\big|_{\mathcal{F}'_{n,\bar{x}}}\right) \in \mathbf{Z}/\ell^n\mathbf{Z}$ form an element of $\mathbf{Z}_\ell$ which is equal to $\mathrm{Tr}(\pi_x \,|_{\mathcal{F}_{\bar{x}}}) \in \mathbf{Q}_\ell$.

It thus suffices to prove the following algebra lemma. □

Lemma 44.92.6. *Suppose we have $K_n \in D_{perf}(\mathbf{Z}/\ell^n\mathbf{Z})$, $\pi_n : K_n \to K_n$ and isomorphisms $\varphi_n : K_{n+1} \otimes^{\mathbf{L}}_{\mathbf{Z}/\ell^{n+1}\mathbf{Z}} \mathbf{Z}/\ell^n\mathbf{Z} \cong K_n$ compatible with π_{n+1} and π_n. Then*

(1) *the elements $t_n = Tr(\pi_n \,|_{K_n}) \in \mathbf{Z}/\ell^n\mathbf{Z}$ form an element $t_\infty = \{t_n\}$ of $\mathbf{Z}_\ell$;*
(2) *the $\mathbf{Z}_\ell$-module $H^i_\infty = \lim_n H^i(k_n)$ is finite and is nonzero for finitely many i only ; and*
(3) *the operators $H^i(\pi_n) : H^i(K_n) \to H^i(K_n)$ are compatible and define $\pi^i_\infty : H^i_\infty \to H^i_\infty$ satisfying*
$$\sum (-1)^i \, Tr\left(\pi^i_\infty \big|_{H^i_\infty \otimes_{\mathbf{Z}_\ell} \mathbf{Q}_\ell}\right) = t_\infty.$$

Proof. Since $\mathbf{Z}/\ell^n\mathbf{Z}$ is a local ring and K_n is perfect, each K_n can be represented by a finite complex $K_n^\bullet$ of finite free $\mathbf{Z}/\ell^n\mathbf{Z}$-modules such that the map $K_n^p \to K_n^{p+1}$ has image contained in ℓK_n^{p+1}. It is a fact that such a complex is unique up to isomorphism. Moreover π_n can be represented by a morphism of complexes $\pi_n^\bullet : K_n^\bullet \to K_n^\bullet$ (which is unique up to homotopy). By the same token the isomorphism $\varphi_n : K_{n+1} \otimes^{\mathbf{L}}_{\mathbf{Z}/\ell^{n+1}\mathbf{Z}} \mathbf{Z}/\ell^n\mathbf{Z} \to K_n$ is represented by a map of complexes

$$\varphi_n^\bullet : K_{n+1}^\bullet \otimes_{\mathbf{Z}/\ell^{n+1}\mathbf{Z}} \mathbf{Z}/\ell^n\mathbf{Z} \to K_n^\bullet.$$

In fact, $\varphi_n^\bullet$ is an isomorphism of complexes, thus we see that

- there exist $a, b \in \mathbf{Z}$ independent of n such that $K_n^i = 0$ for all $i \notin [a, b]$; and
- the rank of K_n^i is independent of n.

Therefore, the module $K_\infty^i = \lim_n \{K_n^i, \varphi_n^i\}$ is a finite free $\mathbf{Z}_\ell$-module and $K_\infty^\bullet$ is a finite complex of finite free $\mathbf{Z}_\ell$-modules. By induction on the number of nonzero terms, one can prove that $H^i(K_\infty^\bullet) = \lim_n H^i(K_n^\bullet)$ (this is not true for unbounded complexes). We conclude that $H_\infty^i = H^i(K_\infty^\bullet)$ is a finite $\mathbf{Z}_\ell$-module. This proves *ii*. To prove the remainder of the lemma, we need to overcome the possible non-commutativity of the diagrams

$$\begin{array}{ccc} K_{n+1}^\bullet & \xrightarrow{\varphi_n^\bullet} & K_n^\bullet \\ {\scriptstyle \pi_{n+1}^\bullet}\downarrow & & \downarrow{\scriptstyle \pi_n^\bullet} \\ K_{n+1}^\bullet & \xrightarrow[\varphi_n^\bullet]{} & K_n^\bullet. \end{array}$$

However, this diagram does commute in the derived category, hence it commutes up to homotopy. We inductively replace $\pi_n^\bullet$ for $n \geq 2$ by homotopic maps of complexes making these diagrams commute. Namely, if $h^i : K_{n+1}^i \to K_n^{i-1}$ is a homotopy, i.e.,

$$\pi_n^\bullet \circ \varphi_n^\bullet - \varphi_n^\bullet \circ \pi_{n+1}^\bullet = dh + hd,$$

then we choose $\tilde{h}^i : K_{n+1}^i \to K_{n+1}^{i-1}$ lifting h^i. This is possible because K_{n+1}^i free and $K_{n+1}^{i-1} \to K_n^{i-1}$ is surjective. Then replace $\pi_n^\bullet$ by $\tilde{\pi}_n^\bullet$ defined by

$$\tilde{\pi}_{n+1}^\bullet = \pi_{n+1}^\bullet + d\tilde{h} + \tilde{h}d.$$

With this choice of $\{\pi_n^\bullet\}$, the above diagrams commute, and the maps fit together to define an endomorphism $\pi_\infty^\bullet = \lim_n \pi_n^\bullet$ of $K_\infty^\bullet$. Then part *i* is clear: the elements $t_n = \sum (-1)^i \mathrm{Tr}\left(\pi_n^i \big|_{K_n^i}\right)$ fit into an element t_∞ of $\mathbf{Z}_\ell$. Moreover

$$\begin{aligned} t_\infty &= \sum (-1)^i \mathrm{Tr}_{\mathbf{Z}_\ell}\left(\pi_\infty^i\big|_{K_\infty^i}\right) \\ &= \sum (-1)^i \mathrm{Tr}_{\mathbf{Q}_\ell}\left(\pi_\infty^i\big|_{K_\infty^i \otimes_{\mathbf{Z}_\ell} \mathbf{Q}_\ell}\right) \\ &= \sum (-1)^i \mathrm{Tr}\left(\pi_\infty\big|_{H^i(K_\infty^\bullet \otimes \mathbf{Q}_\ell)}\right) \end{aligned}$$

where the last equality follows from the fact that $\mathbf{Q}_\ell$ is a field, so the complex $K_\infty^\bullet \otimes \mathbf{Q}_\ell$ is quasi-isomorphic to its cohomology $H^i(K_\infty^\bullet \otimes \mathbf{Q}_\ell)$. The latter is also equal to $H^i(K_\infty^\bullet) \otimes_{\mathbf{Z}} \mathbf{Q}_\ell = H_\infty^i \otimes \mathbf{Q}_\ell$, which finishes the proof of the lemma, and also that of Theorem 44.92.5. □

44.93. List of things which we should add above

What did we skip the proof of in the lectures so far:

(1) curves and their Jacobians,
(2) proper base change theorem,
(3) inadequate discussion of $R\Gamma_c$,
(4) more generally, given $f : X \to S$ finite type, separated S quasi-projective, discussion of $Rf_!$ on étale sheaves.
(5) discussion of $\otimes^{\mathbf{L}}$
(6) discussion of why $R\Gamma_c$ commutes with $\otimes^{\mathbf{L}}$

44.94. Examples of L-functions

We use Theorem 44.92.2 for curves to give examples of L-functions

44.95. Constant sheaves

Let k be a finite field, X a smooth, geometrically irreducible curve over k and $\mathcal{F} = \mathbf{Q}_\ell$ the constant sheaf. If $\bar{x}$ is a geometric point of X, the Galois module $\mathcal{F}_{\bar{x}} = \overline{\mathbf{Q}_\ell}$ is trivial, so

$$\det\left(1 - \pi_x^* \, T^{\deg x}\Big|_{\mathcal{F}_{\bar{x}}}\right)^{-1} = \frac{1}{1 - T^{\deg x}}.$$

Applying Theorem 44.92.2, we get

$$\begin{aligned} L(X, \mathcal{F}) &= \prod_{i=0}^{2} \det\left(1 - \pi_X^* \, T\big|_{H_c^i(X_{\bar{k}}, \mathbf{Q}_\ell)}\right)^{(-1)^{i+1}} \\ &= \frac{\det\left(1 - \pi_X^* \, T\big|_{H_c^1(X_{\bar{k}}, \mathbf{Q}_\ell)}\right)}{\det\left(1 - \pi_X^* \, T\big|_{H_c^0(X_{\bar{k}}, \mathbf{Q}_\ell)}\right) \cdot \det\left(1 - \pi_X^* \, T\big|_{H_c^2(X_{\bar{k}}, \mathbf{Q}_\ell)}\right)}. \end{aligned}$$

To compute the latter, we distinguish two cases.

Projective case. Assume that X is projective, so $H_c^i(X_{\bar{k}}, \mathbf{Q}_\ell) = H^i(X_{\bar{k}}, \mathbf{Q}_\ell)$, and we have

$$H^i(X_{\bar{k}}, \mathbf{Q}_\ell) = \begin{cases} \mathbf{Q}_\ell & \text{if } i = 0, \text{ and } \pi_X^* \text{ acts as } 1 \ ; \\ \mathbf{Q}_\ell^{2g} & \text{if } i = 1 \ ; \\ \mathbf{Q}_\ell & \text{if } i = 2, \text{ and } \pi_X^* \text{ acts as multiplication by } q = \deg \pi_X. \end{cases}$$

We do not know much about the action of π_X^* on the degree 1 cohomology. Let us call $\alpha_1, \ldots, \alpha_{2g}$ its eigenvalues in $\bar{\mathbf{Q}}_\ell$. Putting everything together, Theorem 44.92.2 yields the equality

$$\prod_{x \in |X|} \frac{1}{1 - T^{\deg x}} = \frac{\det\left(1 - \pi_X^* \, T\big|_{H^1(X_{\bar{k}}, \mathbf{Q}_\ell)}\right)}{(1 - T)(1 - qT)}$$

from which we deduce the following result.

Lemma 44.95.1. *Let X be a smooth, projective, geometrically irreducible curve over a finite field k. Then*

1. *the L-function $L(X, \mathbf{Q}_\ell)$ is a rational function ;*
2. *the eigenvalues $\alpha_1, \ldots, \alpha_{2g}$ of π_X^* on $H^1(X_{\bar{k}}, \mathbf{Q}_\ell)$ are algebraic integers independent of ℓ,*
3. *the number of rational points of X on k_n, where $[k_n : k] = n$, is*
$$\#X(k_n) = 1 - \sum\nolimits_{i=1}^{2g} \alpha_i^n + q^n,$$
4. *for each i, $|\alpha_i| < q$.*

Proof. Part (3) is Theorem 44.92.5 applied to $\mathcal{F} = \underline{\mathbf{Q}_\ell}$ on $X \otimes k_n$. For part (4), use the following result. □

Exercise 44.95.2. Let $\alpha_1, \ldots, \alpha_n \in \mathbf{C}$. Then for any conic sector containing the positive real axis of the form $C_\varepsilon = \{z \in \mathbf{C} \mid |\arg z| < \varepsilon\}$ with $\varepsilon > 0$, there exists an integer $k \geq 1$ such that $\alpha_1^k, \ldots, \alpha_n^k \in C_\varepsilon$.

Then prove that $|\alpha_i| \leq q$ for all i. Then, use elementary considerations on complex numbers to prove (as in the proof of the prime number theorem) that $|\alpha_i| < q$. In fact, the Riemann hypothesis says that for all $|\alpha_i| = \sqrt{q}$ for all i. We will come back to this later.

Affine case. Assume now that X is affine, say $X = \bar{X} - \{x_1, \ldots, x_n\}$ where $j : X \hookrightarrow \bar{X}$ is a projective nonsingular completion. Then $H^0_c(X_{\bar{k}}, \mathbf{Q}_\ell) = 0$ and $H^2_c(X_{\bar{k}}, \mathbf{Q}_\ell) = H^2(\bar{X}_{\bar{k}}, \mathbf{Q}_\ell)$ so Theorem 44.92.2 reads

$$L(X, \mathbf{Q}_\ell) = \prod_{x \in |X|} \frac{1}{1 - T^{\deg x}} = \frac{\det\left(1 - \pi_X^* \; T\big|_{H^1_c(X_{\bar{k}}, \mathbf{Q}_\ell)}\right)}{1 - qT}.$$

On the other hand, the previous case gives

$$\begin{aligned} L(X, \mathbf{Q}_\ell) &= L(\bar{X}, \mathbf{Q}_\ell) \prod_{i=1}^{n} \left(1 - T^{\deg x_i}\right) \\ &= \frac{\prod_{i=1}^{n}(1 - T^{\deg x_i}) \prod_{j=1}^{2g}(1 - \alpha_j T)}{(1 - T)(1 - qT)}. \end{aligned}$$

Therefore, we see that $\dim H^1_c(X_{\bar{k}}, \mathbf{Q}_\ell) = 2g + \sum_{i=1}^n \deg(x_i) - 1$, and the eigenvalues $\alpha_1, \ldots, \alpha_{2g}$ of $\pi^*_{\bar{X}}$ acting on the degree 1 cohomology are roots of unity. More precisely, each x_i gives a complete set of $\deg(x_i)$th roots of unity, and one occurrence of 1 is omitted. To see this directly using coherent sheaves, consider the short exact sequence on $\bar{X}$

$$0 \to j_! \mathbf{Q}_\ell \to \mathbf{Q}_\ell \to \bigoplus_{i=1}^{n} \mathbf{Q}_{\ell, x_i} \to 0.$$

The long exact cohomology sequence reads

$$0 \to \mathbf{Q}_\ell \to \bigoplus_{i=1}^{n} \mathbf{Q}_\ell^{\oplus \deg x_i} \to H^1_c(X_{\bar{k}}, \mathbf{Q}_\ell) \to H^1_c(\bar{X}_{\bar{k}}, \mathbf{Q}_\ell) \to 0$$

where the action of Frobenius on $\bigoplus_{i=1}^n \mathbf{Q}_\ell^{\oplus \deg x_i}$ is by cyclic permutation of each term; and $H^2_c(X_{\bar{k}}, \mathbf{Q}_\ell) = H^2_c(\bar{X}_{\bar{k}}, \mathbf{Q}_\ell)$.

44.96. The Legendre family

Let k be a finite field of odd characteristic, $X = \mathrm{Spec}(k[\lambda, \frac{1}{\lambda(\lambda-1)}])$, and consider the family of elliptic curves $f : E \to X$ on $\mathbf{P}^2_X$ whose affine equation is $y^2 = x(x-1)(x-\lambda)$. We set $\mathcal{F} = Rf^1_* \mathbf{Q}_\ell = \left\{R^1 f_* \mathbf{Z}/\ell^n \mathbf{Z}\right\}_{n \geq 1} \otimes \mathbf{Q}_\ell$. In this situation, the following is true

- for each $n \geq 1$, the sheaf $R^1 f_*(\mathbf{Z}/\ell^n \mathbf{Z})$ is finite locally constant – in fact, it is free of rank 2 over $\mathbf{Z}/\ell^n \mathbf{Z}$;
- the system $\{R^1 f_* \mathbf{Z}/\ell^n \mathbf{Z}\}_{n \geq 1}$ is a lisse ℓ-adic sheaf ; and
- for all $x \in |X|$, $\det\left(1 - \pi_x \; T^{\deg x}\big|_{\mathcal{F}_{\bar{x}}}\right) = (1 - \alpha_x T^{\deg x})(1 - \beta_x T^{\deg x})$ where α_x, β_x are the eigenvalues of the geometric frobenius of E_x acting on $H^1(E_{\bar{x}}, \mathbf{Q}_\ell)$.

Note that E_x is only defined over $\kappa(x)$ and not over k. The proof of these facts uses the proper base change theorem and the local acyclicity of smooth morphisms. For details, see [**Del77**]. It follows that

$$L(E/X) := L(X, \mathcal{F}) = \prod_{x \in |X|} \frac{1}{(1 - \alpha_x T^{\deg x})(1 - \beta_x T^{\deg x})}.$$

Applying Theorem 44.92.2 we get

$$L(E/X) = \prod_{i=0}^{2} \det\left(1 - \pi_X^* \ T\big|_{H_c^i(X_{\bar{k}}, \mathcal{F})}\right)^{(-1)^{i+1}},$$

and we see in particular that this is a rational function. Furthermore, it is relatively easy to show that $H_c^0(X_{\bar{k}}, \mathcal{F}) = H_c^2(X_{\bar{k}}, \mathcal{F}) = 0$, so we merely have

$$L(E/X) = \det\left(1 - \pi_X^* T\big|_{H_c^1(X, \mathcal{F})}\right).$$

To compute this determinant explicitly, consider the Leray spectral sequence for the proper morphism $f : E \to X$ over $\mathbf{Q}_\ell$, namely

$$H_c^i(X_{\bar{k}}, R^j f_* \mathbf{Q}_\ell) \Rightarrow H_c^{i+j}(E_{\bar{k}}, \mathbf{Q}_\ell)$$

which degenerates. We have $f_* \mathbf{Q}_\ell = \mathbf{Q}_\ell$ and $R^1 f_* \mathbf{Q}_\ell = \mathcal{F}$. The sheaf $R^2 f_* \mathbf{Q}_\ell = \mathbf{Q}_\ell(-1)$ is the *Tate twist* of $\mathbf{Q}_\ell$, i.e., it is the sheaf $\mathbf{Q}_\ell$ where the Galois action is given by multiplication by $\#\kappa(x)$ on the stalk at $\bar{x}$. It follows that, for all $n \geq 1$,

$$\begin{aligned}
\#E(k_n) &= \sum (-1)^i \mathrm{Tr}\left(\pi_E^{n\,*}\big|_{H_c^i(E_{\bar{k}}, \mathbf{Q}_\ell)}\right) \\
&= \sum_{i,j} (-1)^{i+j} \mathrm{Tr}\left(\pi_X^{n\,*}\big|_{H_c^i(X_{\bar{k}}, R^j f_* \mathbf{Q}_\ell)}\right) \\
&= (q^n - 2) + \mathrm{Tr}\left(\pi_X^{n\,*}\big|_{H_c^1(X_{\bar{k}}, \mathcal{F})}\right) + q^n(q^n - 2) \\
&= q^{2n} - q^n - 2 + \mathrm{Tr}\left(\pi_X^{n\,*}\big|_{H_c^1(X_{\bar{k}}, \mathcal{F})}\right)
\end{aligned}$$

where the first equality follows from Theorem 44.92.5, the second one from the Leray spectral sequence and the third one by writing down the higher direct images of $\mathbf{Q}_\ell$ under f. Alternatively, we could write

$$\#E(k_n) = \sum_{x \in X(k_n)} \#E_x(k_n)$$

and use the trace formula for each curve. We can also find the number of k_n-rational points simply by counting. The zero section contributes $q^n - 2$ points (we omit the points where $\lambda = 0, 1$) hence

$$\#E(k_n) = q^n - 2 + \#\left\{y^2 = x(x-1)(x-\lambda), \lambda \neq 0, 1\right\}.$$

Now we have

$$\#\left\{y^2 = x(x-1)(x-\lambda),\ \lambda \neq 0,1\right\}$$

$$= \#\left\{y^2 = x(x-1)(x-\lambda) \text{ in } \mathbf{A}^3\right\} - \#\left\{y^2 = x^2(x-1)\right\} - \#\left\{y^2 = x(x-1)^2\right\}$$

$$= \#\left\{\lambda = \tfrac{-y^2}{x(x-1)} + x,\ x \neq 0,1\right\} + \#\left\{y^2 = x(x-1)(x-\lambda), x = 0,1\right\} - 2(q^n - \varepsilon_n)$$

$$= q^n(q^n - 2) + 2q^n - 2(q^n - \varepsilon_n)$$

$$= q^{2n} - 2q^n + 2\varepsilon_n$$

where $\varepsilon_n = 1$ if -1 is a square in k_n, 0 otherwise, i.e.,

$$\varepsilon_n = \frac{1}{2}\left(1 + \left(\frac{-1}{k_n}\right)\right) = \frac{1}{2}\left(1 + (-1)^{\frac{q^n-1}{2}}\right).$$

Thus $\#E(k_n) = q^{2n} - q^n - 2 + 2\varepsilon_n$. Comparing with the previous formula, we find

$$\mathrm{Tr}\left(\pi_X^{n\,*}\big|_{H_c^1(X_{\bar{k}},\mathcal{F})}\right) = 2\varepsilon_n = 1 + (-1)^{\frac{q^n-1}{2}},$$

which implies, by elementary algebra of complex numbers, that if -1 is a square in k_n^*, then $\dim H_c^1(X_{\bar{k}}, \mathcal{F}) = 2$ and the eigenvalues are 1 and 1. Therefore, in that case we have

$$L(E/X) = (1-T)^2.$$

44.97. Exponential sums

A standard problem in number theory is to evaluate sums of the form

$$S_{a,b}(p) = \sum_{x \in \mathbf{F}_p - \{0,1\}} e^{\frac{2\pi i x^a (x-1)^b}{p}}.$$

In our context, this can be interpreted as a cohomological sum as follows. Consider the base scheme $S = \mathrm{Spec}(\mathbf{F}_p[x, \frac{1}{x(x-1)}])$ and the affine curve $f : X \to \mathbf{P}^1 - \{0,1,\infty\}$ over S given by the equation $y^{p-1} = x^a(x-1)^b$. This is a finite étale Galois cover with group $\mathbf{F}_p^*$ and there is a splitting

$$f_*(\bar{\mathbf{Q}}_\ell^*) = \bigoplus_{\chi : \mathbf{F}_p^* \to \bar{\mathbf{Q}}_\ell^*} \mathcal{F}_\chi$$

where χ varies over the characters of $\mathbf{F}_p^*$ and $\mathcal{F}_\chi$ is a rank 1 lisse $\mathbf{Q}_\ell$-sheaf on which $\mathbf{F}_p^*$ acts via χ on stalks. We get a corresponding decomposition

$$H_c^1(X_{\bar{k}}, \mathbf{Q}_\ell) = \bigoplus_\chi H^1(\mathbf{P}^1_{\bar{k}} - \{0,1,\infty\}, \mathcal{F}_\chi)$$

and the cohomological interpretation of the exponential sum is given by the trace formula applied to $\mathcal{F}_\chi$ over $\mathbf{P}^1 - \{0,1,\infty\}$ for some suitable χ. It reads

$$S_{a,b}(p) = -\mathrm{Tr}\left(\pi_X^*\big|_{H^1(\mathbf{P}^1_{\bar{k}} - \{0,1,\infty\}, \mathcal{F}_\chi)}\right).$$

The general yoga of Weil suggests that there should be some cancellation in the sum. Applying (roughly) the Riemann-Hurwitz formula, we see that

$$2g_X - 2 \approx -2(p-1) + 3(p-2) \approx p$$

so $g_X \approx p/2$, which also suggests that the χ-pieces are small.

44.98. Trace formula in terms of fundamental groups

In the following sections we reformulate the trace formula completely in terms of the fundamental group of a curve, except if the curve happens to be $\mathbf{P}^1$.

44.99. Fundamental groups

X connected scheme $\overline{x} \to X$ geometric point consider the functor

$$\begin{array}{rccc} F_{\overline{x}}: & \begin{array}{c}\text{finite étale}\\ \text{schemes over } X\end{array} & \longrightarrow & \text{finite sets} \\ & Y/X & \longmapsto & F_{\overline{x}}(Y) = \left\{ \begin{array}{c}\text{geom points } \overline{y}\\ \text{of } Y \text{ lying over } \overline{x}\end{array} \right\} = Y_{\overline{x}} \end{array}$$

Set

$$\pi_1(X, \overline{x}) = Aut(F_{\overline{x}}) = \text{ set of automorphisms of the functor } F_{\overline{x}}$$

Note that for every finite étale $Y \to X$ there is an action

$$\pi_1(X, \overline{x}) \times F_{\overline{x}}(Y) \to F_{\overline{x}}(Y)$$

Definition 44.99.1. A subgroup of the form $\text{Stab}(\overline{y} \in F_{\overline{x}}(Y)) \subset \pi_1(X, \overline{x})$ is called *open*.

Theorem 44.99.2 (Grothendieck)**.** *Let X be a connected scheme.*

1. *There is a topology on $\pi_1(X, \overline{x})$ such that the open subgroups form a fundamental system of open nbhds of $e \in \pi_1(X, \overline{x})$.*
2. *With topology of (1) the group $\pi_1(X, \overline{x})$ is a profinite group.*
3. *The functor*

$$\begin{array}{ccc} \begin{array}{c}\textit{schemes finite}\\ \textit{étale over } X\end{array} & \to & \begin{array}{c}\textit{finite discrete continuous}\\ \pi_1(X,\overline{x})\textit{-sets}\end{array} \\ Y/X & \mapsto & F_{\overline{x}}(Y) \textit{ with its natural action} \end{array}$$

is an equivalence of categories.

Proof. See [**Gro71**]. □

Proposition 44.99.3. *Let X be an integral normal Noetherian scheme. Let $\overline{y} \to X$ be an algebraic geometric point lying over the generic point $\eta \in X$. Then*

$$\pi_x(X, \overline{\eta}) = Gal(M/\kappa(\eta))$$

($\kappa(\eta)$, function field of X) where

$$\kappa(\overline{\eta}) \supset M \supset \kappa(\eta) = k(X)$$

is the max sub-extension such that for every finite sub extension $M \supset L \supset \kappa(\eta)$ the normalization of X in L is finite étale over X.

Proof. Omitted. □

Change of base point. For any $\overline{x}_1$, $\overline{x}_2$ geom. points of X there exists an isom. of fibre functions

$$\mathcal{F}_{\overline{x}_1} \cong \mathcal{F}_{\overline{x}_2}$$

(This is a path from $\overline{x}_1$ to $\overline{x}_2$.) Conjugation by this path gives isom

$$\pi_1(X, \overline{x}_1) \cong \pi_1(X, \overline{x}_2)$$

well defined up to inner actions.

Functoriality. For any morphism $X_1 \to X_2$ of connected schemes any $\overline{x} \in X_1$ there is a canonical map

$$\pi_1(X_1, \overline{x}) \to \pi_1(X_2, \overline{x})$$

(Why? because the fibre functor ...)

Base field. Let X be a variety over a field k. Then we get

$$\pi_1(X, \overline{x}) \to \pi_1(Spec(k), \overline{x}) =^{\text{prop}} Gal(k^{\text{sep}}/k)$$

This map is surjective if and only if X is geometrically connected over k. So in the geometrically connected case we get s.e.s. of profinite groups

$$1 \to \pi_1(X_{\overline{k}}, \overline{x}) \to \pi_1(X, \overline{x}) \to Gal(k^{\text{sep}}/k) \to 1$$

($\pi_1(X_{\overline{k}}, \overline{x})$: geometric fundamental group of X, $\pi_1(X, \overline{x})$: arithmetic fundamental group of X)

Comparison. If X is a variety over $\mathbf{C}$ then

$$\pi_1(X, \overline{x}) = \text{ profinite completion of } \pi_1(X(\mathbf{C})(\text{ usual topology}), x)$$

(have $x \in X(\mathbf{C})$)

Frobenii. X variety over k, $\#k < \infty$. For any $x \in X$ closed point, let

$$F_x \in \pi_1(x, \overline{x}) = \mathrm{Gal}(\kappa(x)^{\text{sep}}/\kappa(x))$$

be the geometric frobenius. Let $\overline{\eta}$ be an alg. geom. gen. pt. Then

$$\pi_1(X, \overline{\eta}) \leftarrow^{\cong} \pi_1(X, \overline{x}) \overset{\text{functoriality}}{\underset{\leftarrow}{}} \pi_1(x, \overline{x})$$

Easy fact:

$$\begin{array}{ccccc} \pi_1(X, \overline{\eta}) & \to^{\deg} & \pi_1(\mathrm{Spec}(k), \overline{\eta}) * & = Gal(k^{sep}/k) \\ & & & \| \\ & & & \widehat{\mathbf{Z}} \cdot F_{\mathrm{Spec}(k)} \\ F_x & \mapsto & & \deg(x) \cdot F_{\mathrm{Spec}(k)} \end{array}$$

Recall: $\deg(x) = [\kappa(x) : k]$

Fundamental groups and lisse sheaves. Let X be a connected scheme, $\overline{x}$ geom. pt. There are equivalences of categories

$$\begin{array}{lccc} (\Lambda \text{ finite ring}) & \begin{array}{c}\text{fin. loc. const. sheaves of} \\ \Lambda\text{-modules of } X_{étale}\end{array} & \leftrightarrow & \begin{array}{c}\text{finite(discrete) } \Lambda\text{-modules} \\ \text{with continuous } \pi_1(X,\overline{x})\text{-action}\end{array} \\ (l \text{ a prime}) & \begin{array}{c}\text{lisse } l\text{-adic} \\ \text{sheaves}\end{array} & \leftrightarrow & \begin{array}{c}\text{finitely generated } \mathbf{Z}_l\text{-modules } M \text{ with continuous} \\ \pi_1(X,\overline{x}) \text{ action where we use } l\text{-adic topology on } M\end{array} \end{array}$$

In particular lisse $\mathbf{Q}_l$-sheaves correspond to continuous homomorphisms

$$\pi_1(X, \overline{x}) \to \mathrm{GL}_r(\mathbf{Q}_l), \quad r \geq 0$$

Notation: A module with action (M, ρ) corresponds to the sheaf $\mathcal{F}_\rho$.

Trace formulas. X variety over k, $\#k < \infty$.

(1) Λ finite ring $(\#\Lambda, \#k) = 1$

$$\rho : \pi_1(X, \overline{x}) \to \mathrm{GL}_r(\Lambda)$$

continuous. For every $n \geq 1$ we have

$$\sum_{d|n} d \left(\sum_{\substack{x \in |X|, \\ \deg(x) = d}} \mathrm{Tr}(\rho(F_x^{n/d})) \right) = \mathrm{Tr}\left((\pi_x^n)^* \,\middle|_{R\Gamma_c(X_{\overline{k}}, \mathcal{F}_\rho)} \right)$$

(2) $l \neq char(k)$ prime, $\rho : \pi_1(X, \overline{x}) \to \mathrm{GL}_r(\mathbf{Q}_l)$. For any $n \geq 1$

$$\sum_{d|n} d \left(\sum_{\substack{x \in |X| \\ \deg(x)=d}} \mathrm{Tr}\left(\rho(F_x^{n/d})\right) \right) = \sum_{i=0}^{2\dim X} (-1)^i \mathrm{Tr}\left(\pi_X^* \Big|_{H_c^i(X_{\overline{k}}, \mathcal{F}_\rho)}\right)$$

Weil conjectures. (Deligne-Weil I, 1974) X smooth proj. over k, $\#k = q$, then the eigenvalues of π_X^* on $H^i(X_{\overline{k}}, \mathbf{Q}_l)$ are algebraic integers α with $|\alpha| = q^{1/2}$.

Deligne's conjectures. (almost completely proved by Lafforgue + $\cdots$) Let X be a normal variety over k finite

$$\rho : \pi_1(X, \overline{x}) \longrightarrow \mathrm{GL}_r(\mathbf{Q}_l)$$

continuous. Assume: ρ irreducible $\det(\rho)$ of finite order. Then

(1) there exists a number field E such that for all $x \in |X|$(closed points) the char. poly of $\rho(F_x)$ has coefficients in E.
(2) for any $x \in |X|$ the eigenvalues $\alpha_{x,i}$, $i = 1, \ldots, r$ of $\rho(F_x)$ have complex absolute value 1. (these are algebraic numbers not necessary integers)
(3) for every finite place λ(not dividing p), of E (maybe after enlarging E a bit) there exists

$$\rho\lambda : \pi_1(X, \overline{x}) \to \mathrm{GL}_r(E_\lambda)$$

compatible with ρ. (some char. polys of F_x's)

Theorem 44.99.4 (Deligne, Weil II)**.** *For a sheaf $\mathcal{F}_\rho$ with ρ satisfying the conclusions of the conjecture above then the eigenvalues of π_X^* on $H_c^i(X_{\overline{k}}, \mathcal{F}_\rho)$ are algebraic numbers α with absolute values*

$$|\alpha| = q^{w/2}, \text{ for } w \in \mathbf{Z},\ w \leq i$$

Moreover, if X smooth and proj. then $w = i$.

Proof. See [**Del74a**]. □

44.100. Profinite groups, cohomology and homology

Let G be a profinite group.

Cohomology. Consider the category of discrete modules with continuous G-action. This category has enough injectives and we can define

$$H^i(G, M) = R^i H^0(G, M) = R^i(M \mapsto M^G)$$

Also there is a derived version $RH^0(G, -)$.

Homology. Consider the category of compact abelian groups with continuous G-action. This category has enough projectives and we can define

$$H_i(G, M) = L_i H_0(G, M) = L_i(M \mapsto M_G)$$

and there is also a derived version.

Trivial duality. The functor $M \mapsto M^\wedge = Hom_{cont}(M, S^1)$ exchanges the categories above and

$$H^i(G, M)^\wedge = H_i(G, M^\wedge)$$

Moreover, this functor maps torsion discrete G-modules to profinite continuous G-modules and vice versa, and if M is either a discrete or profinite continuous G-module, then $M^\wedge = \mathrm{Hom}(M, \mathbf{Q}/\mathbf{Z})$.

Notes on Homology.

(1) If we look at Λ-modules for a finite ring Λ then we can identify
$$H_i(G, M) = Tor_i^{\Lambda[[G]]}(M, \Lambda)$$
where $\Lambda[[G]]$ is the limit of the group algebras of the finite quotients of G.

(2) If $G \lhd \Gamma$, and Γ is also profinite then
- $H^0(G, -)$: discrete Γ-module$\to$ discrete Γ/G-modules
- $H_0(G, -)$: compact Γ-modules $\to$ compact Γ/G-modules

and hence the profinite group Γ/G acts on the cohomology groups of G with values in a Γ-module. In other words, there are derived functors
$$RH^0(G, -) : D^+(\text{discrete } \Gamma\text{-modules}) \longrightarrow D^+(\text{discrete } \Gamma/G\text{-modules})$$
and similarly for $LH_0(G, -)$.

44.101. Cohomology of curves, revisited

Let k be a field, X be geometrically connected, smooth curve over k. We have the fundamental short exact sequence
$$1 \to \pi_1(X_{\overline{k}}, \overline{\eta}) \to \pi_1(X, \overline{\eta}) \to \mathrm{Gal}(k^{sep}/k) \to 1$$
If Λ is a finite ring with $\#\Lambda \in k^*$ and M a finite Λ-module, and we are given
$$\rho : \pi_1(X, \overline{\eta}) \to \mathrm{Aut}_\Lambda(M)$$
continuous, then $\mathcal{F}_\rho$ denotes the associated sheaf on $X_{étale}$.

Lemma 44.101.1. *There is a canonical isomorphism*
$$H_c^2(X_{\overline{k}}, \mathcal{F}_\rho) = (M)_{\pi_1(X_{\overline{k}}, \overline{\eta})}(-1)$$
as Gal(k^{sep}/k)-modules.

Here the subscript $_{\pi_1(X_{\overline{k}}, \overline{\eta})}$ indicates co-invariants, and (-1) indicates the Tate twist i.e., $\sigma \in \mathrm{Gal}(k^{sep}/k)$ acts via
$$\chi_{cycl}(\sigma)^{-1}.\sigma \text{ on RHS}$$
where
$$\chi_{cycl} : \mathrm{Gal}(k^{sep}/k) \to \prod\nolimits_{l \neq char(k)} \mathbf{Z}_l^*$$
is the cyclotomic character.

Reformulation (Deligne, Weil II, page 338). For any finite locally constant sheaf $\mathcal{F}$ on X there is a maximal quotient $\mathcal{F} \to \mathcal{F}''$ with $\mathcal{F}''/X_{\overline{k}}$ a constant sheaf, hence
$$\mathcal{F}'' = (X \to \mathrm{Spec}(k))^{-1} F''$$
where F'' is a sheaf $\mathrm{Spec}(k)$, i.e., a $\mathrm{Gal}(k^{sep}/k)$-module. Then
$$H_c^2(X_{\overline{k}}, \mathcal{F}) \to H_c^2(X_{\overline{k}}, \mathcal{F}'') \to F''(-1)$$
is an isomorphism.

Proof of Lemma 44.101.1. Let $Y \to^{\varphi} X$ be the finite étale Galois covering corresponding to $\mathrm{Ker}(\rho) \subset \pi_1(X, \overline{\eta})$. So

$$\mathrm{Aut}(Y/X) = Ind(\rho)$$

is Galois group. Then $\varphi^*\mathcal{F}_\rho = \underline{M}_Y$ and

$$\varphi_*\varphi^*\mathcal{F}_\rho \to \mathcal{F}_\rho$$

which gives

$$\begin{aligned} & H^2_c(X_{\overline{k}}, \varphi_*\varphi^*\mathcal{F}_\rho) \to H^2_c(X_{\overline{k}}, \mathcal{F}_\rho) \\ & = H^2_c(Y_{\overline{k}}, \varphi^*\mathcal{F}_\rho) \\ & = H^2_c(Y_{\overline{k}}, \underline{M}) = \oplus_{\substack{\text{irred. comp. of} \\ Y_{\overline{k}}}} M \end{aligned}$$

$$\mathrm{Im}(\rho) \to H^2_c(Y_{\overline{k}}, \underline{M}) = \oplus_{\substack{\text{irred. comp. of} \\ Y_{\overline{k}}}} M \to_{\mathrm{Im}(\rho)\text{equivalent}} H^2_c(X_{\overline{k}}, \mathcal{F}_\rho) \to^{\substack{\text{trivial Im}(\rho) \\ \text{action}}}$$

irreducible curve $C/\overline{k}$, $H^2_c(C, \underline{M}) = M$.
Since

$$\begin{matrix} \text{set of irreducible} \\ \text{components of } Y_k \end{matrix} = \frac{Im(\rho)}{Im(\rho|_{\pi_1(X_{\overline{k}}, \overline{\eta})})}$$

We conclude that $H^2_c(X_{\overline{k}}, \mathcal{F}_\rho)$ is a quotient of $M_{\pi_1(X_{\overline{k}}, \overline{\eta})}$. On the other hand, there is a surjection

$$\mathcal{F}_\rho \to \mathcal{F}'' = \begin{matrix} \text{sheaf on } X \text{ associated to} \\ (M)_{\pi_1(X_{\overline{k}}, \overline{\eta})} \leftarrow \pi_1(X, \overline{\eta}) \end{matrix}$$

$$H^2_c(X_{\overline{k}}, \mathcal{F}_\rho) \to M_{\pi_1(X_{\overline{k}}, \overline{\eta})}$$

The twist in Galois action comes from the fact that $H^2_c(X_{\overline{k}}, \mu_n) =^{\mathrm{can}} \mathbf{Z}/n\mathbf{Z}$. □

Remark 44.101.2. Thus we conclude that if X is also projective then we have functorially in the representation ρ the identifications

$$H^0(X_{\overline{k}}, \mathcal{F}_\rho) = M^{\pi_1(X_{\overline{k}}, \overline{\eta})}$$

and

$$H^2_c(X_{\overline{k}}, \mathcal{F}_\rho) = M_{\pi_1(X_{\overline{k}}, \overline{\eta})}(-1)$$

Of course if X is not projective, then $H^0_c(X_{\overline{k}}, \mathcal{F}_\rho) = 0$.

Proposition 44.101.3. *Let X/k as before but $X_{\overline{k}} \neq \mathbf{P}^1_{\overline{k}}$ The functors $(M, \rho) \mapsto H^{2-i}_c(X_{\overline{k}}, \mathcal{F}_\rho)$ are the left derived functor of $(M, \rho) \mapsto H^2_c(X_{\overline{k}}, \mathcal{F}_\rho)$ so*

$$H^{2-i}_c(X_{\overline{k}}, \mathcal{F}_\rho) = H_i(\pi_1(X_{\overline{k}}, \overline{\eta}), M)(-1)$$

Moreover, there is a derived version, namely

$$R\Gamma_c(X_{\overline{k}}, \mathcal{F}_\rho) = LH_0(\pi_1(X_{\overline{k}}, \overline{\eta}), M(-1)) = M(-1) \otimes^{\mathbf{L}}_{\Lambda[[\pi_1(X_{\overline{k}}, \overline{\eta})]]} \Lambda$$

in $D(\Lambda[[\widehat{\mathbf{Z}}]])$. Similarly, the functors $(M, \rho) \mapsto H^i(X_{\overline{k}}, \mathcal{F}_\rho)$ are the right derived functor of $(M, \rho) \mapsto M^{\pi_1(X_{\overline{k}}, \overline{\eta})}$ so

$$H^i(X_{\overline{k}}, \mathcal{F}_\rho) = H^i(\pi_1(X_{\overline{k}}, \overline{\eta}), M)$$

Moreover, in this case there is a derived version too.

Proof. (Idea) Show both sides are universal δ-functors. □

Remark 44.101.4. By the proposition and Trivial duality then you get

$$H_c^{2-i}(X_{\overline{k}}, \mathcal{F}_\rho) \times H^i(X_{\overline{k}}, \mathcal{F}_\rho^\wedge(1)) \to \mathbf{Q}/\mathbf{Z}$$

a perfect pairing. If X is projective then this is Poincare duality.

44.102. Abstract trace formula

Suppose given an extension of profinite groups,

$$1 \to G \to \Gamma \xrightarrow{\deg} \widehat{\mathbf{Z}} \to 1$$

We say Γ *has an abstract trace formula* if and only if there exist

(1) an integer $q \geq 1$, and
(2) for every $d \geq 1$ a finite set S_d and for each $x \in S_d$ a conjugacy class $F_x \in \Gamma$ with $\deg(F_x) = d$

such that the following hold

(1) for all ℓ not dividing q have $\mathrm{cd}_\ell(G) < \infty$, and
(2) for all finite rings Λ with $q \in \Lambda^*$, for all finite projective Λ-modules M with continuous Γ-action, for all $n > 0$ we have

$$\sum\nolimits_{d|n} d \left(\sum\nolimits_{x \in S_d} \mathrm{Tr}(F_x^{n/d} \,|_M) \right) = q^n \mathrm{Tr}(F^n \,\Big|_{M \otimes^{\mathbf{L}}_{\Lambda[[G]]} \Lambda})$$

in $\Lambda^\natural$.

Here $M \otimes^{\mathbf{L}}_{\Lambda[[G]]} \Lambda = LH_0(G, M)$ denotes derived homology, and $F = 1$ in $\Gamma/G = \widehat{\mathbf{Z}}$.

Remark 44.102.1. Here are some observations concerning this notion.

(1) If modeling projective curves then we can use cohomology and we don't need factor q^n.
(2) The only examples I know are $\Gamma = \pi_1(X, \overline{\eta})$ where X is smooth, geometrically irreducible and $K(\pi, 1)$ over finite field. In this case $q = (\#k)^{\dim X}$. Modulo the proposition, we proved this for curves in this course.
(3) Given the integer q then the sets S_d are uniquely determined. (You can multiple q by an integer m and then replace S_d by m^d copies of S_d without changing the formula.)

Example 44.102.2. Fix an integer $q \geq 1$

$$\begin{array}{ccccccccc} 1 & \to & G = \widehat{\mathbf{Z}}^{(q)} & \to & \Gamma & \to & \widehat{\mathbf{Z}} & \to & 1 \\ & & = \prod_{l \nmid q} \mathbf{Z}_l & & F & \mapsto & 1 & & \end{array}$$

with $FxF^{-1} = ux$, $u \in (\widehat{\mathbf{Z}}^{(q)})^*$. Just using the trivial modules $\mathbf{Z}/m\mathbf{Z}$ we see

$$q^n - (qu)^n \equiv \sum\nolimits_{d|n} d \# S_d$$

in $\mathbf{Z}/m\mathbf{Z}$ for all $(m, q) = 1$ (up to $u \to u^{-1}$) this implies $qu = a \in \mathbf{Z}$ and $|a| < q$. The special case $a = 1$ does occur with

$$\Gamma = \pi_1^t(\mathbf{G}_{m, \mathbf{F}_p}, \overline{\eta}), \quad \#S_1 = q - 1, \quad \text{and} \quad \#S_2 = \frac{(q^2 - 1) - (q - 1)}{2}$$

44.103. Automorphic forms and sheaves

References: See especially the amazing papers [**Dri83**], [**Dri84**] and [**Dri80**] by Drinfeld.

Unramified cusp forms. Let k be a finite field of characteristic p. Let X geometrically irreducible projective smooth curve over k. Set $K = k(X)$ equal to the function field of X. Let v be a place of K which is the same thing as a closed point $x \in X$. Let K_v be the completion of K at v, which is the same thing as the fraction field of the completion of the local ring of X at x, i.e., $K_v = f.f.(\widehat{O_{X,x}})$. Denote $O_v \subset K_v$ the ring of integers. We further set

$$O = \prod\nolimits_v O_v \subset \mathbf{A} = \prod_v{}' K_v$$

and we let Λ be any ring with p invertible in Λ.

Definition 44.103.1. An *unramified cusp form on $GL_2(\mathbf{A})$ with values in Λ*[9] is a function

$$f : \mathrm{GL}_2(\mathbf{A}) \to \Lambda$$

such that

(1) $f(x\gamma) = f(x)$ for all $x \in \mathrm{GL}_2(\mathbf{A})$ and all $\gamma \in \mathrm{GL}_2(K)$
(2) $f(ux) = f(x)$ for all $x \in \mathrm{GL}_2(\mathbf{A})$ and all $u \in \mathrm{GL}_2(O)$
(3) for all $x \in \mathrm{GL}_2(\mathbf{A})$,

$$\int_{\mathbf{A} \mod K} f\left(x \begin{pmatrix} 1 & z \\ 0 & 1 \end{pmatrix}\right) dz = 0$$

see [**dJ01**, Section 4.1] for an explanation of how to make sense out of this for a general ring Λ in which p is invertible.

Hecke Operators. For v a place of K and f an unramified cusp form we set

$$T_v(f)(x) = \int_{g \in M_v} f(g^{-1}x) dg,$$

and

$$U_v(f)(x) = f\left(\begin{pmatrix} \pi_v^{-1} & 0 \\ 0 & \pi_v^{-1} \end{pmatrix} x\right)$$

Notations used: here $\pi_v \in O_v$ is a uniformizer

$$M_v = \{h \in Mat(2 \times 2,\ O_v) \,|\det h = \pi_v O_v^*\}$$

and $dg =$ is the Haar measure on $\mathrm{GL}_2(K_v)$ with $\int_{\mathrm{GL}_2(O_v)} dg = 1$. Explicitly we have

$$T_v(f)(x) = f\left(\begin{pmatrix} \pi_v^{-1} & 0 \\ 0 & 1 \end{pmatrix} x\right) + \sum_{i=1}^{q_v} f\left(\begin{pmatrix} 1 & 0 \\ -\pi_v^{-1}\lambda_i & \pi_v^{-1} \end{pmatrix} x\right)$$

with $\lambda_i \in O_v$ a set of representatives of $O_v/(\pi_v) = \kappa_v$, $q_v = \#\kappa_v$.

Eigenforms. An *eigenform* f is an unramified cusp form such that some value of f is a unit and $T_v f = t_v f$ and $U_v f = u_v f$ for some (uniquely determined) $t_v,\ u_v \in \Lambda$.

[9] This is likely nonstandard notation.

Theorem 44.103.2. *Given an eigenform f with values in $\overline{\mathbf{Q}}_l$ and eigenvalues $u_v \in \overline{\mathbf{Z}}_l^*$ then there exists*

$$\rho : \pi_1(X) \to GL_2(E)$$

continuous, absolutely irreducible where E is a finite extension of $\mathbf{Q}_\ell$ contained in $\overline{\mathbf{Q}}_l$ such that $t_v = Tr(\rho(F_v))$, and $u_v = q_v^{-1} \det(\rho(F_v))$ for all places v.

Proof. See [**Dri80**]. □

Theorem 44.103.3. *Suppose $\mathbf{Q}_l \subset E$ finite, and*

$$\rho : \pi_1(X) \to GL_2(E)$$

absolutely irreducible, continuous. Then there exists an eigenform f with values in $\overline{\mathbf{Q}}_l$ whose eigenvalues t_v, u_v satisfy the equalities $t_v = Tr(\rho(F_v))$ and $u_v = q_v^{-1} \det(\rho(F_v))$.

Proof. See [**Dri83**]. □

Remark 44.103.4. We now have, thanks to Lafforgue and many other mathematicians, complete theorems like this two above for GL_n and allowing ramification! In other words, the full global Langlands correspondence for GL_n is known for function fields of curves over finite fields. At the same time this does not mean there aren't a lot of interesting questions left to answer about the fundamental groups of curves over finite fields, as we shall see below.

Central character. If f is an eigenform then

$$\begin{array}{rccc} \chi_f : & O^* \backslash \mathbf{A}^* / K^* & \to & \Lambda^* \\ & (1, \ldots, \pi_v, 1, \ldots, 1) & \mapsto & u_v^{-1} \end{array}$$

is called the *central character*. If corresponds to the determinant of ρ via normalizations as above. Set

$$C(\Lambda) = \left\{ \begin{array}{c} \text{unr. cusp forms } f \text{ with coefficients in } \Lambda \\ \text{such that } U_v f = \varphi_v^{-1} f \forall v \end{array} \right\}$$

Proposition 44.103.5. *If Λ is Noetherian then $C(\Lambda)$ is a finitely generated Λ-module. Moreover, if Λ is a field with prime subfield $\mathbf{F} \subset \Lambda$ then*

$$C(\Lambda) = (C(\mathbf{F})) \otimes_{\mathbf{F}} \Lambda$$

compatibly with T_v acting.

Proof. See [**dJ01**, Proposition 4.7]. □

This proposition trivially implies the following lemma.

Lemma 44.103.6. *Algebraicity of eigenvalues. If Λ is a field then the eigenvalues t_v for $f \in C(\Lambda)$ are algebraic over the prime subfield $\mathbf{F} \subset \Lambda$.*

Proof. Follows from Proposition 44.103.5. □

Combining all of the above we can do the following very useful trick.

Lemma 44.103.7. *Switching l. Let E be a number field. Start with*

$$\rho : \pi_1(X) \to SL_2(E_\lambda)$$

absolutely irreducible continuous, where λ is a place of E not lying above p. Then for any second place λ' of E not lying above p there exists a finite extension $E'_{\lambda'}$ and a absolutely irreducible continuous representation

$$\rho' : \pi_1(X) \to SL_2(E'_{\lambda'})$$

which is compatible with ρ in the sense that the characteristic polynomials of all Frobenii are the same.

Note how this is an instance of Deligne's conjecture!

Proof. To prove the switching lemma use Theorem 44.103.3 to obtain $f \in C(\overline{\mathbf{Q}}_l)$ eigenform ass. to ρ. Next, use Proposition 44.103.5 to see that we may choose $f \in C(E')$ with $E \subset E'$ finite. Next we may complete E' to see that we get $f \in C(E'_{\lambda'})$ eigenform with $E'_{\lambda'}$ a finite extension of $E_{\lambda'}$. And finally we use Theorem 44.103.2 to obtain $\rho' : \pi_1(X) \to SL_2(E'_{\lambda'})$ abs. irred. and continuous after perhaps enlarging $E'_{\lambda'}$ a bit again. □

Speculation: If for a (topological) ring Λ we have

$$\begin{pmatrix} \rho : \pi_1(X) \to SL_2(\Lambda) \\ \text{abs irred} \end{pmatrix} \leftrightarrow \text{ eigen forms in } C(\Lambda)$$

then all eigenvalues of $\rho(F_v)$ algebraic (won't work in an easy way if Λ is a finite ring. Based on the speculation that the Langlands correspondence works more generally than just over fields one arrives at the following conjecture.

Conjecture. (See [**dJ01**]) For any continuous

$$\rho : \pi_1(X) \to \mathrm{GL}_n(\mathbf{F}_l[[t]])$$

we have $\#\rho(\pi_1(X_{\overline{k}})) < \infty$.

A rephrasing in the language of sheaves: "For any lisse sheaf of $\overline{\mathbf{F}_l((t))}$-modules the geom monodromy is finite."

Theorem 44.103.8. *The Conjecture holds if $n \leq 2$.*

Proof. See [**dJ01**]. □

Theorem 44.103.9. *Conjecture holds if $l > 2n$ modulo some unproven things.*

Proof. See [**Gai07**]. □

It turns out the conjecture is useful for something. See work of Drinfeld on Kashiwara's conjectures. But there is also the much more down to earth application as follows.

Theorem 44.103.10. *(See* [**dJ01**, Theorem 3.5]*) Suppose*

$$\rho_0 : \pi_1(X) \to GL_n(\mathbf{F}_l)$$

is a continuous, $l \neq p$. Assume

1. *Conj. holds for X,*
2. *$\rho_0\big|_{\pi_1(X_{\overline{k}})}$ abs. irred., and*
3. *l does not divide n.*

Then the universal determination ring R_{univ} of ρ_0 is finite flat over $\mathbf{Z}_l$.

Explanation: There is a representation $\rho_{\text{univ}} : \pi_1(X) \to \mathrm{GL}_n(R_{\text{univ}})$ (Univ. Defo ring) R_{univ} loc. complete, residue field $\mathbf{F}_l$ and $(R_{\text{univ}} \to \mathbf{F}_l) \circ \rho_{\text{univ}} \cong \rho_0$. And given any $R \to \mathbf{F}_l$, R local complete and $\rho : \pi_1(X) \to \mathrm{GL}_n(R)$ then there exists $\psi : R_{\text{univ}} \to R$ such that $\psi \circ \rho_{\text{univ}} \cong \rho$. The theorem says that the morphism

$$\mathrm{Spec}(R_{\text{univ}}) \longrightarrow \mathrm{Spec}(\mathbf{Z}_l)$$

is finite and flat. In particular, such a ρ_0 lifts to a $\rho : \pi_1(X) \to \mathrm{GL}_n(\overline{\mathbf{Q}}_l)$.

Notes:

(1) The theorem on deformations is easy.
(2) Any result towards the conjecture seems hard.
(3) It would be interesting to have more conjectures on $\pi_1(X)$!

44.104. Counting points

Let X be a smooth, geometrically irreducible, projective curve over k and $q = \#k$. Trace formula gives:
there exists algebraic integers $w_1, \ldots, w_{2g}$ such that

$$\#X(k_n) = q^n - \sum\nolimits_{i=1}^{2g_X} w_i^n + 1.$$

If $\sigma \in \mathrm{Aut}(X)$ then for all i, there exists j such that $\sigma(w_i) = w_j$.

Riemann-Hypothesis. For all i we have $|\omega_i| = \sqrt{q}$.

This was formulated by Emil Artin, in 1924, for hyperelliptic curves. Proved by Weil 1940. Weil gave two proofs

- using intersection theory on $X \times X$, using the Hodge index theorem, and
- using the Jacobian of X.

There is another proof whose initial idea is due to Stephanov, and which was given by Bombieri: it uses the function field $k(X)$ and its Frobenius operator (1969). The starting point is that given $f \in k(X)$ one observes that $f^q - f$ is a rational function which vanishes in all the $\mathbf{F}_q$-rational points of X, and that one can try to use this idea to give an upper bound for the number of points.

44.105. Precise form of Chebotarev

As a first application let us prove a precise form of Chebotarev for a finite étale Galois covering of curves. Let $\varphi : Y \to X$ be a finite étale Galois covering with group G. This corresponds to a homomorphism

$$\pi_1(X) \longrightarrow G = \mathrm{Aut}(Y/X)$$

Assume $Y_{\overline{k}} =$ irreducible. If $C \subset G$ is a conjugacy class then for all $n > 0$, we have

$$\left| \#\left\{x \in X(k_n) \mid F_x \in C\right\} - \frac{\#C}{\#G} \cdot \#X(k_n) \right| \leq (\#C)(2g-2)\sqrt{q^n}$$

(Warning: Please check the coefficient $\#C$ on the right hand side carefuly before using.)

Sketch. Write

$$\varphi_*(\overline{\mathbf{Q}}_l) = \oplus_{\pi \in \widehat{G}} \mathcal{F}_\pi$$

where $\widehat{G}$ is the set of isomorphism classes of irred representations of G over $\overline{\mathbf{Q}}_l$. For $\pi \in \widehat{G}$ let $\chi_\pi : G \to \overline{\mathbf{Q}}_l$ be the character of π. Then

$$H^*(Y_{\overline{k}}, \overline{\mathbf{Q}}_l) = \oplus_{\pi\in\widehat{G}} H^*(Y_{\overline{k}}, \overline{\mathbf{Q}}_l)_\pi =_{(\varphi \text{ finite }\,)} \oplus_{\pi\in\widehat{G}} H^*(X_{\overline{k}}, \mathcal{F}_\pi)$$

If $\pi \neq 1$ then we have

$$H^0(X_{\overline{k}}, \mathcal{F}_\pi) = H^2(X_{\overline{k}}, \mathcal{F}_\pi) = 0,\ \dim H^1(X_{\overline{k}}, \mathcal{F}_\pi) = (2g_X - 2)d_\pi^2$$

(can get this from trace formula for acting on ...) and we see that

$$\left| \sum_{x\in X(k_n)} \chi_\pi(\mathcal{F}_x) \right| \leq (2g_X - 2)d_\pi^2\sqrt{q^n}$$

Write $1_C = \sum_\pi a_\pi\chi_\pi$, then $a_\pi = \langle 1_C, \chi_\pi\rangle$, and $a_1 = \langle 1_C, \chi_1\rangle = \frac{\#C}{\#G}$ where

$$\langle f, h\rangle = \frac{1}{\#G}\sum_{g\in G} f(g)\overline{h(g)}$$

Thus we have the relation

$$\frac{\#C}{\#G} = ||1_C||^2 = \sum |a_\pi|^2$$

Final step:

$$\begin{aligned} \#\left\{x \in X(k_n) \mid F_x \in C\right\} &= \sum_{x\in X(k_n)} 1_C(x) \\ &= \sum_{x\in X(k_n)} \sum_\pi a_\pi\chi_\pi(F_x) \\ &= \underbrace{\frac{\#C}{\#G}\#X(k_n)}_{\text{term for } \pi=1} + \underbrace{\sum_{\pi\neq 1} a_\pi \sum_{x\in X(k_n)} \chi_\pi(F_x)}_{\text{error term (to be bounded by } E)} \end{aligned}$$

We can bound the error term by

$$\begin{aligned} |E| &\leq \sum_{\substack{\pi\in\widehat{G},\\ \pi\neq 1}} |a_\pi|(2g-2)d_\pi^2\sqrt{q^n} \\ &\leq \sum_{\pi\neq 1} \frac{\#C}{\#G}(2g_X - 2)d_\pi^3\sqrt{q^n} \end{aligned}$$

By Weil's conjecture, $\#X(k_n) \sim q^n$. □

44.106. How many primes decompose completely?

This section gives a second application of the Riemann Hypothesis for curves over a finite field. For number theorists it may be nice to look at the paper by Ihara, entitled "How many primes decompose completely in an infinite unramified Galois extension of a global field?", see [**Iha83**]. Consider the fundamental exact sequence

$$1 \to \pi_1(X_{\overline{k}}) \to \pi_1(X) \xrightarrow{\deg} \widehat{\mathbf{Z}} \to 1$$

Proposition 44.106.1. *There exists a finite set $x_1, \ldots, x_n$ of closed points of X such that that set of <u>all</u> frobenius elements corresponding to these points topologically generate $\pi_1(X)$.*

Another way to state this is: There exist $x_1, \ldots, x_n \in |X|$ such that the smallest normal closed subgroup Γ of $\pi_1(X)$ containing 1 frobenius element for each x_i is all of $\pi_1(X)$. i.e., $\Gamma = \pi_1(X)$.

Proof. Pick $N \gg 0$ and let

$$\{x_1, \ldots, x_n\} = \begin{array}{c}\text{set of all closed points of} \\ X \text{ of degree} \leq N \text{ over } k\end{array}$$

Let $\Gamma \subset \pi_1(X)$ be as in variant statement for these points. Assume $\Gamma \neq \pi_1(X)$. We can pick $\Gamma \lhd \pi_1(X)$ with $U \neq \pi_1(X)$. By R.H. for X this set I will have some x_{i_1} of degree N, some x_{i_2} of degree $N-1$. This shows $\Gamma \to^{\deg} \widehat{\mathbf{Z}}$ and z_0 and so also U. This exactly means if $Y \to X$ is the finite étale Galois covering as to U, then $Y_{\overline{k}}$ irreducible.

$$Y \to^G X,\ G = \pi_1(X)/U$$

By construction all points of X of degree $\leq N$, split completely in Y. So, in particular

$$\#Y(k_N) \geq (\#G)\#X(k_N)$$

Use R.H. on both sides. So you get

$$q^N + 1 + 2g_Y q^{N/2} \geq \#G\#X(k_N) \geq \#G(q^N + 1 - 2g_X q^{N/2})$$

Since $2g_Y - 2 = (\#G)(2g_X - 2)$,

$$q^N + 1 + (\#G)(2g_X - 1) + 1)q^{N/2} \geq \#G(q^N + 1 - 2g_X q^{N/2})$$

□

Weird Question. Set $W_X = \deg^{-1}(\mathbf{Z}) \subset \pi_1(X)$. Is it true that for some finite set of closed points $x_1, \ldots, x_n$ of X the set of all frobenii corresponding to these points *algebraically* generate W_X?

By a Baire category argument this translates into the same question for all Frobenii.

44.107. How many points are there really?

If the genus of the curve is large relative to q, then the main term in the formula $\#X(k) = q - \sum \omega_i + 1$ is not q but the second term $\sum \omega_i$ which can (a priori) have size about $2g_X\sqrt{q}$. In the paper [**VD83**] the authors Drinfeld and Vladut show that this maximum is (as predicted by Ihara earlier) actually at most about $g\sqrt{q}$.

Fix q and let k be a field with k elements. Set

$$A(q) = \limsup_{g_X \to \infty} \frac{\#X(k)}{g_X}$$

where X runs over geometrically irreducible smooth projective curves over k. With this definition we have the following results:

- RH $\Rightarrow$ $A(q) \leq 2\sqrt{q}$
- Ihara $\Rightarrow$ $A(q) \leq \sqrt{2q}$
- DV $A(q) \leq \sqrt{q} - 1$ (actually this is sharp if q is a square)

Proof. Given X let $w_1, \ldots, w_{2g}$ and $g = g_X$ be as before. Set $\alpha_i = \frac{w_i}{\sqrt{q}}$, so $|\alpha_i| = 1$. If α_i occurs then $\overline{\alpha}_i = \alpha_i^{-1}$ also occurs. Then

$$N = \#X(k) \leq X(k_r) = q^r + 1 - (\sum_i \alpha_i^r) q^{r/2}$$

Rewriting we see that for every $r \geq 1$

$$-\sum_i \alpha_i^r \geq Nq^{-r/2} - q^{r/2} - q^{-r/2}$$

Observe that

$$0 \leq |\alpha_i^n + \alpha_i^{n-1} + \cdots + \alpha_i + 1|^2 = (n+1) + \sum_{j=1}^{n} (n+1-j)(\alpha_i^j + \alpha_i^{-j})$$

So

$$\begin{aligned} 2g(n+1) &\geq -\sum_i \left(\sum_{j=1}^{n} (n+1-j)(\alpha_i^j + \alpha_i^{-j}) \right) \\ &= -\sum_{j=1}^{n} (n+1-j) \left(\sum_i \alpha_i^j + \sum_i \alpha_i^{-j} \right) \end{aligned}$$

Take half of this to get

$$\begin{aligned} g(n+1) &\geq -\sum_{j=1}^{n} (n+1-j)(\sum_i \alpha_i^j) \\ &\geq N \sum_{j=1}^{n} (n+1-j) q^{-j/2} - \sum_{j=1}^{n} (n+1-j)(q^{j/2} + q^{-j/2}) \end{aligned}$$

This gives

$$\frac{N}{g} \leq \left(\sum_{j=1}^{n} \frac{n+1-j}{n+1} q^{-j/2} \right)^{-1} \cdot \left(1 + \frac{1}{g} \sum_{j=1}^{n} \frac{n+1-j}{n+1} (q^{j/2} + q^{-j/2}) \right)$$

Fix n let $g \to \infty$

$$A(q) \leq \left(\sum_{j=1}^{n} \frac{n+1-j}{n+1} q^{-j/2} \right)^{-1}$$

So

$$A(q) \leq \lim_{n \to \infty} (\cdots) = \left(\sum_{j=1}^{\infty} q^{-j/2} \right)^{-1} = \sqrt{q} - 1$$

□

44.108. Other chapters

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CHAPTER 45

Crystalline Cohomology

45.1. Introduction

This chapter is based on a lecture series given by Johan de Jong held in 2012 at Columbia University. The goals of this chapter are to give a quick introduction to crystalline cohomology. A reference is the book [**Ber74**].

We have moved the more elementary purely algebraic discussion of divided power rings to a preliminary chapter as it is also useful in discussing Tate resolutions in commutative algebra. Please see Divided Power Algebra, Section 23.1.

45.2. Divided power envelope

The construction of the following lemma will be dubbed the divided power envelope. It will play an important role later.

Lemma 45.2.1. *Let (A, I, γ) be a divided power ring. Let $A \to B$ be a ring map. Let $J \subset B$ be an ideal with $IB \subset J$. There exists a homomorphism of divided power rings*

$$(A, I, \gamma) \longrightarrow (D, \bar{J}, \bar{\gamma})$$

such that

$$\mathrm{Hom}_{(A,I,\gamma)}((D, \bar{J}, \bar{\gamma}), (C, K, \delta)) = \mathrm{Hom}_{(A,I)}((B, J), (C, K))$$

functorially in the divided power algebra (C, K, δ) over (A, I, γ). Here the LHS is morphisms of divided power rings over (A, I, γ) and the RHS is morphisms of (ring, ideal) pairs over (A, I).

Proof. Denote $\mathcal{C}$ the category of divided power rings (C, K, δ). Consider the functor $F : \mathcal{C} \longrightarrow \mathit{Sets}$ defined by

$$F(C, K, \delta) = \left\{ (\varphi, \psi) \middle| \begin{array}{c} \varphi : (A, I, \gamma) \to (C, K, \delta) \text{ homomorphism of divided power rings} \\ \psi : (B, J) \to (C, K) \text{ an } A\text{-algebra homomorphism with } \psi(J) \subset K \end{array} \right\}$$

We will show that Divided Power Algebra, Lemma 23.3.3 applies to this functor which will prove the lemma. Suppose that $(\varphi, \psi) \in F(C, K, \delta)$. Let $C' \subset C$ be the subring generated by $\varphi(A)$, $\psi(B)$, and $\delta_n(\psi(f))$ for all $f \in J$. Let $K' \subset K \cap C'$ be the ideal of C' generated by $\varphi(I)$ and $\delta_n(\psi(f))$ for $f \in J$. Then $(C', K', \delta|_{K'})$ is a divided power ring and C' has cardinality bounded by the cardinal $\kappa = |A| \otimes |B|^{\aleph_0}$. Moreover, φ factors as $A \to C' \to C$ and ψ factors as $B \to B' \to B$. This proves assumption (1) of Divided Power Algebra, Lemma 23.3.3 holds. Assumption (2) is clear as limits in the category of divided power rings commute with the forgetful functor $(C, K, \delta) \mapsto (C, K)$, see Divided Power Algebra, Lemma 23.3.2 and its proof. □

Definition 45.2.2. Let (A, I, γ) be a divided power ring. Let $A \to B$ be a ring map. Let $J \subset B$ be an ideal with $IB \subset J$. The divided power algebra $(D, \bar{J}, \bar{\gamma})$ constructed in Lemma 45.2.1 is called the *divided power envelope of J in B relative to* (A, I, γ) and is denoted $D_B(J)$ or $D_{B,\gamma}(J)$.

Let $(A, I, \gamma) \to (C, K, \delta)$ be a homomorphism of divided power rings. The universal property of $D_{B,\gamma}(J) = (D, \bar{J}, \bar{\gamma})$ is

$$\begin{array}{c}\text{ring maps } B \to C \\ \text{which map } J \text{ into } K\end{array} \longleftrightarrow \begin{array}{c}\text{divided power homomorphisms} \\ (D, \bar{J}, \bar{\gamma}) \to (C, K, \delta)\end{array}$$

and the correspondence is given by precomposing with the map $B \to D$ which corresponds to id_D. Here are some properties of $(D, \bar{J}, \bar{\gamma})$ which follow directly from the universal property. There are A-algebra maps

$$B \longrightarrow D \longrightarrow B/J \tag{45.2.2.1}$$

The first arrow maps J into $\bar{J}$ and $\bar{J}$ is the kernel of the second arrow. The elements $\bar{\gamma}_n(x)$ where $n > 0$ and x is an element in the image of $J \to D$ generate $\bar{J}$ as an ideal in D and generate D as a B-algebra.

Lemma 45.2.3. *Let (A, I, γ) be a divided power ring. Let $\varphi : B' \to B$ be a surjection of A-algebras with kernel K. Let $IB \subset J \subset B$ be an ideal. Let $J' \subset B'$ be the inverse image of J. Write $D_{B',\gamma}(J') = (D', \bar{J}', \bar{\gamma})$. Then $D_{B,\gamma}(J) = (D'/K', \bar{J}'/K', \bar{\gamma})$ where K' is the ideal generated by the elements $\bar{\gamma}_n(k)$ for $n \geq 1$ and $k \in K$.*

Proof. Write $D_{B,\gamma}(J) = (D, \bar{J}, \bar{\gamma})$. The universal property of D' gives us a homomorphism $D' \to D$ of divided power algebras. As $B' \to B$ and $J' \to J$ are surjective, we see that $D' \to D$ is surjective (see remarks above). It is clear that $\bar{\gamma}_n(k)$ is in the kernel for $n \geq 1$ and $k \in K$, i.e., we obtain a homomorphism $D'/K' \to D$. Conversely, there exists a divided power structure on $\bar{J}'/K' \subset D'/K'$, see Divided Power Algebra, Lemma 23.4.3. Hence the universal property of D gives an inverse $D \to D'/K'$ and we win. □

In the situation of Definition 45.2.2 we can choose a surjection $P \to B$ where P is a polynomial algebra over A and let $J' \subset P$ be the inverse image of J. The previous lemma describes $D_{B,\gamma}(J)$ in terms of $D_{P,\gamma}(J')$. Note that γ extends to a divided power structure γ' on IP by Divided Power Algebra, Lemma 23.4.2. Hence $D_{P,\gamma}(J') = D_{P,\gamma'}(J')$ is an example of a special case of divided power envelopes we describe in the following lemma.

Lemma 45.2.4. *Let (B, I, γ) be a divided power algebra. Let $I \subset J \subset B$ be an ideal. Let $(D, \bar{J}, \bar{\gamma})$ be the divided power envelope of J relative to γ. Choose elements $f_t \in J$, $t \in T$ such that $J = I + (f_t)$. Then there exists a surjection*

$$\Psi : B\langle x_t \rangle \longrightarrow D$$

of divided power rings mapping x_t to the image of f_t in D. The kernel of Ψ is generated by the elements $x_t - f_t$ and all

$$\delta_n\left(\sum r_t x_t - r_0\right)$$

whenever $\sum r_t f_t = r_0$ in B for some $r_t \in B$, $r_0 \in I$.

Proof. In the statement of the lemma we think of $B\langle x_t\rangle$ as a divided power ring with ideal $J' = IB\langle x_t\rangle + B\langle x_t\rangle_+$, see Divided Power Algebra, Remark 23.5.2. The existence of Ψ follows from the universal property of divided power polynomial rings. Surjectivity of Ψ follows from the fact that its image is a divided power subring of D, hence equal to D by the universal property of D. It is clear that $x_t - f_t$ is in the kernel. Set

$$\mathcal{R} = \{(r_0, r_t) \in I \oplus \bigoplus\nolimits_{t\in T} B \mid \sum r_t f_t = r_0 \text{ in } B\}$$

If $(r_0, r_t) \in \mathcal{R}$ then it is clear that $\sum r_t x_t - r_0$ is in the kernel. As Ψ is a homomorphism of divided power rings and $\sum r_t x_t = r_0 \in J'$ it follows that $\delta_n(\sum r_t x_t - r_0)$ is in the kernel as well. Let $K \subset B\langle x_t\rangle$ be the ideal generated by $x_t - f_t$ and the elements $\delta_n(\sum r_t x_t - r_0)$ for $(r_0, r_t) \in \mathcal{R}$. To show that $K = \mathrm{Ker}(\Psi)$ it suffices to show that δ extends to $B\langle x_t\rangle/K$. Namely, if so the universal property of D gives a map $D \to B\langle x_t\rangle/K$ inverse to Ψ. Hence we have to show that $K \cap J'$ is preserved by δ_n, see Divided Power Algebra, Lemma 23.4.3. Let $K' \subset B\langle x_t\rangle$ be the ideal generated by the elements

(1) $\delta_m(\sum r_t x_t - r_0)$ where $m > 0$ and $(r_0, r_t) \in \mathcal{R}$,
(2) $x_{t'}^{[m]}(x_t - f_t)$ where $m > 0$ and $t', t \in I$.

We claim that $K' = K \cap J'$. The claim proves that $K \cap J'$ is preserved by δ_n, $n > 0$ by the criterion of Divided Power Algebra, Lemma 23.4.3 (2)(c) and a computation of δ_n of the elements listed which we leave to the reader. To prove the claim note that $K' \subset K \cap J'$. Conversely, if $h \in K \cap J'$ then, modulo K' we can write

$$h = \sum r_t(x_t - f_t)$$

for some $r_t \in B$. As $h \in K \cap J' \subset J'$ we see that $r_0 = \sum r_t f_t \in I$. Hence $(r_0, r_t) \in \mathcal{R}$ and we see that

$$h = \sum r_t x_t - r_0$$

is in K' as desired. □

Lemma 45.2.5. *Let (A, I, γ) be a divided power ring. Let B be an A-algebra and $IB \subset J \subset B$ an ideal. Let x_i be a set of variables. Then*

$$D_{B[x_i],\gamma}(JB[x_i] + (x_i)) = D_{B,\gamma}(J)\langle x_i\rangle$$

Proof. One possible proof is to deduce this from Lemma 45.2.4 as any relation between x_i in $B[x_i]$ is trivial. On the other hand, the lemma follows from the universal property of the divided power polynomial algebra and the universal property of divided power envelopes. □

Conditions (1) and (2) of the following lemma hold if $B \to B'$ is flat at all primes of $V(IB') \subset \mathrm{Spec}(B')$ and is very closely related to that condition, see Algebra, Lemma 10.93.8. It in particular says that taking the divided power envelope commutes with localization.

Lemma 45.2.6. *Let (A, I, γ) be a divided power ring. Let $B \to B'$ be a homomorphism of A-algebras. Assume that*

(1) *$B/IB \to B'/IB'$ is flat, and*
(2) *$Tor_1^B(B', B/IB) = 0$.*

Then for any ideal $IB \subset J \subset B$ the canonical map

$$D_B(J) \otimes_B B' \longrightarrow D_{B'}(JB')$$

is an isomorphism.

Proof. Set $D = D_B(J)$ and denote $\bar{J} \subset D$ its divided power ideal with divided power structure $\bar{\gamma}$. The universal property of D produces a B-algebra map $D \to D_{B'}(JB')$, whence a map as in the lemma. It suffices to show that the divided powers $\bar{\gamma}$ extend to $D \otimes_B B'$ since then the universal property of $D_{B'}(JB')$ will produce a map $D_{B'}(JB') \to D \otimes_B B'$ inverse to the one in the lemma.

Choose a surjection $P \to B'$ where P is a polynomial algebra over B. In particular $B \to P$ is flat, hence $D \to D \otimes_B P$ is flat by Algebra, Lemma 10.36.6. Then $\bar{\gamma}$ extends to $D \otimes_B P$ by Divided Power Algebra, Lemma 23.4.2; we will denote this extension $\bar{\gamma}$ also. Set $\mathfrak{a} = \mathrm{Ker}(P \to B')$ so that we have the short exact sequence

$$0 \to \mathfrak{a} \to P \to B' \to 0$$

Thus $\mathrm{Tor}_1^B(B', B/IB) = 0$ implies that $\mathfrak{a} \cap IP = I\mathfrak{a}$. Now we have the following commutative diagram

$$\begin{array}{ccccc}
B/J \otimes_B \mathfrak{a} & \xrightarrow[\beta]{} & B/J \otimes_B P & \longrightarrow & B/J \otimes_B B' \\
\uparrow & & \uparrow & & \uparrow \\
D \otimes_B \mathfrak{a} & \xrightarrow{\alpha} & D \otimes_B P & \longrightarrow & D \otimes_B B' \\
\uparrow & & \uparrow & & \uparrow \\
\bar{J} \otimes_B \mathfrak{a} & \longrightarrow & \bar{J} \otimes_B P & \longrightarrow & \bar{J} \otimes_B B'
\end{array}$$

This diagram is exact even with 0's added at the top and the right. We have to show the divided powers on the ideal $\bar{J} \otimes_B P$ preserve the ideal $\mathrm{Im}(\alpha) \cap \bar{J} \otimes_B P$, see Divided Power Algebra, Lemma 23.4.3. Consider the exact sequence

$$0 \to \mathfrak{a}/I\mathfrak{a} \to P/IP \to B'/IB' \to 0$$

(which uses that $\mathfrak{a} \cap IP = I\mathfrak{a}$ as seen above). As B'/IB' is flat over B/IB this sequence remains exact after applying $B/J \otimes_{B/IB} -$, see Algebra, Lemma 10.36.11. Hence

$$\mathrm{Ker}(B/J \otimes_{B/IB} \mathfrak{a}/I\mathfrak{a} \to B/J \otimes_{B/IB} P/IP) = \mathrm{Ker}(\mathfrak{a}/J\mathfrak{a} \to P/JP)$$

is zero. Thus β is injective. It follows that $\mathrm{Im}(\alpha) \cap \bar{J} \otimes_B P$ is the image of $\bar{J} \otimes \mathfrak{a}$. Now if $f \in \bar{J}$ and $a \in \mathfrak{a}$, then $\bar{\gamma}_n(f \otimes a) = \bar{\gamma}_n(f) \otimes a^n$ hence the result is clear. □

The following lemma is a special case of [**dJ95**, Proposition 2.1.7] which in turn is a generalization of [**Ber74**, Proposition 2.8.2].

Lemma 45.2.7. *Let $(B, I, \gamma) \to (B', I', \gamma')$ be a homomorphism of divided power rings. Let $I \subset J \subset B$ and $I' \subset J' \subset B'$ be ideals. Assume*

(1) *$B/I \to B'/I'$ is flat, and*
(2) *$J' = JB' + I'$.*

Then the canonical map

$$D_{B,\gamma}(J) \otimes_B B' \longrightarrow D_{B',\gamma'}(J')$$

is an isomorphism.

Proof. Set $D = D_B(J)$ and denote $\bar{J} \subset D$ its divided power ideal with divided power structure $\bar{\gamma}$. The universal property of D produces a homomorphism of divided power rings $D \to D_{B'}(J')$, whence a map as in the lemma. It suffices to show that there exist divided powers on the image of $D \otimes_B I' + \bar{J} \otimes_B B' \to D \otimes_B B'$ compatible with $\bar{\gamma}$ and γ' since then the universal property of $D_{B'}(J')$ will produce a map $D_{B'}(J') \to D \otimes_B B'$ inverse to the one in the lemma.

Choose elements $f_t \in J$ which generate J/I. Set $\mathcal{R} = \{(r_0, r_t) \in I \oplus \bigoplus_{t \in T} B \mid \sum r_t f_t = r_0 \text{ in } B\}$ as in the proof of Lemma 45.2.4. This lemma shows that

$$D = B\langle x_t \rangle / K$$

where K is generated by the elements $x_t - f_t$ and $\delta_n(\sum r_t x_t - r_0)$ for $(r_0, r_t) \in \mathcal{R}$. Thus we see that

$$D \otimes_B B' = B'\langle x_t \rangle / K' \tag{45.2.7.1}$$

where K' is generated by the images in $B'\langle x_t \rangle$ of the generators of K listed above. Let $f'_t \in B'$ be the image of f_t. By assumption (1) we see that the elements $f'_t \in J'$ generate J'/I' and we see that $x_t - f'_t \in K'$. Set

$$\mathcal{R}' = \{(r'_0, r'_t) \in I' \oplus \bigoplus\nolimits_{t \in T} B' \mid \sum r'_t f'_t = r'_0 \text{ in } B'\}$$

To finish the proof we have to show that $\delta'_n(\sum r'_t x_t - r'_0) \in K'$ for $(r'_0, r'_t) \in \mathcal{R}'$, because then the presentation (45.2.7.1) of $D \otimes_B B'$ is identical to the presentation of $D_{B',\gamma'}(J')$ obtain in Lemma 45.2.4 from the generators f'_t. Suppose that $(r'_0, r'_t) \in \mathcal{R}'$. Then $\sum r'_t f'_t = 0$ in B'/I'. As $B/I \to B'/I'$ is flat by assumption (1) we can apply the equational criterion of flatness (Algebra, Lemma 10.36.10) to see that there exist an $m > 0$ and $r_{jt} \in B$ and $c_j \in B'$, $j = 1, \ldots, m$ such that

$$r_{j0} = \sum r_{jt} f_t \in I \text{ for } j = 1, \ldots, m, \quad \text{and} \quad r'_t = \sum c_j r_{jt}.$$

Note that this also implies that $r'_0 = \sum c_j r_{j0}$. Then we have

$$\begin{aligned} \delta'_n(\sum r'_t x_t - r'_0) &= \delta'_n(\sum c_j (\sum r_{jt} x_t - r_{j0})) \\ &= \sum c_1^{n_1} \ldots c_m^{n_m} \delta_{n_1}(\sum r_{1t} x_t - r_{10}) \ldots \delta_{n_m}(\sum r_{mt} x_t - r_{m0}) \end{aligned}$$

where the sum is over $n_1 + \ldots + n_m = n$. This proves what we want. □

45.3. Some explicit divided power thickenings

The constructions in this section will help us to define the connection on a crystal in modules on the crystalline site.

Lemma 45.3.1. *Let (A, I, γ) be a divided power ring. Let M be an A-module. Let $B = A \oplus M$ as an A-algebra where M is an ideal of square zero and set $J = I \oplus M$. Set*

$$\delta_n(x + z) = \gamma_n(x) + \gamma_{n-1}(x) z$$

for $x \in I$ and $z \in M$. Then δ is a divided power structure and $A \to B$ is a homomorphism of divided power rings from (A, I, γ) to (B, J, δ).

Proof. We have to check conditions (1) – (5) of Divided Power Algebra, Definition 23.2.1. We will prove this directly for this case, but please see the proof of the next

lemma for a method which avoids calculations. Conditions (1) and (3) are clear. Condition (2) follows from

$$\begin{aligned}\delta_n(x+z)\delta_m(x+z) &= (\gamma_n(x)+\gamma_{n-1}(x)z)(\gamma_m(x)+\gamma_{m-1}(x)z)\\ &= \gamma_n(x)\gamma_m(x)+\gamma_n(x)\gamma_{m-1}(x)z+\gamma_{n-1}(x)\gamma_m(x)z\\ &= \frac{(n+m)!}{n!m!}\gamma_{n+m}(x)+\left(\frac{(n+m-1)!}{n!(m-1)!}+\frac{(n+m-1)!}{(n-1)!m!}\right)\gamma_{n+m-1}(x)z\\ &= \frac{(n+m)!}{n!m!}\delta_{n+m}(x+z)\end{aligned}$$

Condition (5) follows from

$$\begin{aligned}\delta_n(\delta_m(x+z)) &= \delta_n(\gamma_m(x)+\gamma_{m-1}(x)z)\\ &= \gamma_n(\gamma_m(x))+\gamma_{n-1}(\gamma_m(x))\gamma_{m-1}(x)z\\ &= \frac{(nm)!}{n!(m!)^n}\gamma_{nm}(x)+\frac{((n-1)m)!}{(n-1)!(m!)^{n-1}}\gamma_{(n-1)m}(x)\gamma_{m-1}(x)z\\ &= \frac{(nm)!}{n!(m!)^n}(\gamma_{nm}(x)+\gamma_{nm-1}(x)z)\end{aligned}$$

by elementary number theory. To prove (4) we have to see that

$$\delta_n(x+x'+z+z') = \gamma_n(x+x')+\gamma_{n-1}(x+x')(z+z')$$

is equal to

$$\sum\nolimits_{i=0}^{n}(\gamma_i(x)+\gamma_{i-1}(x)z)(\gamma_{n-i}(x')+\gamma_{n-i-1}(x')z')$$

This follows easily on collecting the coefficients of 1, z, and z' and using condition (4) for γ. □

Lemma 45.3.2. *Let (A, I, γ) be a divided power ring. Let M, N be A-modules. Let $q : M \times M \to N$ be an A-bilinear map. Let $B = A \oplus M \oplus N$ as an A-algebra with multiplication*

$$(x, z, w) \cdot (x', z', w') = (xx', xz' + x'z, xw' + x'w + q(z, z') + q(z', z))$$

and set $J = I \oplus M \oplus N$. Set

$$\delta_n(x, z, w) = (\gamma_n(x), \gamma_{n-1}(x)z, \gamma_{n-1}(z)w + \gamma_{n-2}(x)q(z, z))$$

for $(a, m, n) \in J$. Then δ is a divided power structure and $A \to B$ is a homomorphism of divided power rings from (A, I, γ) to (B, J, δ).

Proof. Suppose we want to prove that property (4) of Divided Power Algebra, Definition 23.2.1 is satisfied. Pick (x, z, w) and (x', z', w') in J. Pick a map

$$A_0 = \mathbf{Z}\langle s, s'\rangle \longrightarrow A, \quad s \longmapsto x, s' \longmapsto x'$$

which is possible by the universal property of divided power polynomial rings. Set $M_0 = A_0 \oplus A_0$ and $N_0 = A_0 \oplus A_0 \oplus M_0 \otimes_{A_0} M_0$. Let $q_0 : M_0 \times M_0 \to N_0$ be the obvious map. Define $M_0 \to M$ as the A_0-linear map which sends the basis vectors of M_0 to z and z'. Define $N_0 \to N$ as the A_0 linear map which sends the first two basis vectors of N_0 to w and w' and uses $M_0 \otimes_{A_0} M_0 \to M \otimes_A M \xrightarrow{q} N$ on the last summand. Then we see that it suffices to prove the identity (4) for the situation (A_0, M_0, N_0, q_0). Similarly for the other identities. This reduces us to the case of a

Z-torsion free ring A and A-torsion free modules. In this case all we have to do is show that

$$n!\delta_n(x, z, w) = (x, z, w)^n$$

in the ring A, see Divided Power Algebra, Lemma 23.2.2. To see this note that

$$(x, z, w)^2 = (x^2, 2xz, 2xw + 2q(z, z))$$

and by induction

$$(x, z, w)^n = (x^n, nx^{n-1}z, nx^{n-1}w + n(n-1)x^{n-2}q(z, z))$$

On the other hand,

$$n!\delta_n(x, z, w) = (n!\gamma_n(x), n!\gamma_{n-1}(x)z, n!\gamma_{n-1}(x)w + n!\gamma_{n-2}(x)q(z, z))$$

which matches. This finishes the proof. □

45.4. Compatibility

This section isn't required reading; it explains how our discussion fits with that of [**Ber74**]. Consider the following technical notion.

Definition 45.4.1. Let (A, I, γ) and (B, J, δ) be divided power rings. Let $A \to B$ be a ring map. We say δ *is compatible with* γ if there exists a divided power structure $\bar{\gamma}$ on $J + IB$ such that

$$(A, I, \gamma) \to (B, J + IB, \bar{\gamma}) \quad \text{and} \quad (B, J, \delta) \to (B, J + IB, \bar{\gamma})$$

are homomorphisms of divided power rings.

Let p be a prime number. Let (A, I, γ) be a divided power ring. Let $A \to C$ be a ring map with p nilpotent in C. Assume that γ extends to IC (see Divided Power Algebra, Lemma 23.4.2). In this situation, the (big affine) crystalline site of $\mathrm{Spec}(C)$ over $\mathrm{Spec}(A)$ as defined in [**Ber74**] is the opposite of the category of systems

$$(B, J, \delta, A \to B, C \to B/J)$$

where

(1) (B, J, δ) is a divided power ring with p nilpotent in B,
(2) δ is compatible with γ, and
(3) the diagram

$$\begin{array}{ccc} B & \longrightarrow & B/J \\ \uparrow & & \uparrow \\ A & \longrightarrow & C \end{array}$$

is commutative.

The conditions "γ extends to C and δ compatible with γ" are used in [**Ber74**] to insure that the crystalline cohomology of $\mathrm{Spec}(C)$ is the same as the crystalline cohomology of $\mathrm{Spec}(C/IC)$. We will avoid this issue by working exclusively with C such that $IC = 0$[1]. In this case, for a system $(B, J, \delta, A \to B, C \to B/J)$ as above, the commutativity of the displayed diagram above implies $IB \subset J$ and compatibility is equivalent to the condition that $(A, I, \gamma) \to (B, J, \delta)$ is a homomorphism of divided power rings.

[1] Of course there will be a price to pay.

45.5. Affine crystalline site

In this section we discuss the algebraic variant of the crystalline site. Our basic situation in which we discuss this material will be as follows.

Situation 45.5.1. Here p is a prime number, (A, I, γ) is a divided power ring such that A is a $\mathbf{Z}_{(p)}$-algebra, and $A \to C$ is a ring map such that $IC = 0$ and such that p is nilpotent in C.

Usually the prime number p will be contained in the divided power ideal I.

Definition 45.5.2. In Situation 45.5.1.

(1) A *divided power thickening* of C over (A, I, γ) is a homomorphism of divided power algebras $(A, I, \gamma) \to (B, J, \delta)$ such that p is nilpotent in B and a ring map $C \to B/J$ such that

$$\begin{array}{ccc} B & \longrightarrow & B/J \\ \uparrow & & \uparrow \\ & & C \\ & & \uparrow \\ A & \longrightarrow & A/I \end{array}$$

is commutative.

(2) A *homomorphism of divided power thickenings*

$$(B, J, \delta, C \to B/J) \longrightarrow (B', J', \delta', C \to B'/J')$$

is a homomorphism $\varphi : B \to B'$ of divided power A-algebras such that $C \to B/J \to B'/J'$ is the given map $C \to B'/J'$.

(3) We denote $\mathrm{CRIS}(C/A, I, \gamma)$ or simply $\mathrm{CRIS}(C/A)$ the category of divided power thickenings of C over (A, I, γ).

(4) We denote $\mathrm{Cris}(C/A, I, \gamma)$ or simply $\mathrm{Cris}(C/A)$ the full subcategory consisting of $(B, J, \delta, C \to B/J)$ such that $C \to B/J$ is an isomorphism. We often denote such an object $(B \to C, \delta)$ with $J = \mathrm{Ker}(B \to C)$ being understood.

Note that for a divided power thickening (B, J, δ) as above the ideal J is locally nilpotent, see Divided Power Algebra, Lemma 23.2.6. There is a canonical functor

$$\mathrm{CRIS}(C/A) \longrightarrow C\text{-algebras}, \quad (B, J, \delta) \longmapsto B/J \tag{45.5.2.1}$$

This category does not have equalizers or fibre products in general. It also doesn't have an initial object (= empty colimit) in general.

Lemma 45.5.3. *In Situation 45.5.1.*

(1) *CRIS(C/A) has products,*

(2) *CRIS(C/A) has all finite nonempty colimits and (45.5.2.1) commutes with these, and*

(3) *Cris(C/A) has all finite nonempty colimits and Cris$(C/A) \to$ CRIS(C/A) commutes with them.*

Proof. The empty product is $(C, 0, \emptyset)$. If (B_t, J_t, δ_t) is a family of objects of $\mathrm{CRIS}(C/A)$ then we can form the product $(\prod B_t, \prod J_t, \prod \delta_t)$ as in Divided Power Algebra, Lemma 23.3.4. The map $C \to \prod B_t / \prod J_t = \prod B_t/J_t$ is clear.

Given two objects (B, J, γ) and (B', J', γ') of $\mathrm{CRIS}(C/A)$ we can form a cocartesian diagram

$$\begin{array}{ccc} (B, J, \delta) & \longrightarrow & (B'', J'', \delta'') \\ \uparrow & & \uparrow \\ (A, I, \gamma) & \longrightarrow & (B', J', \delta') \end{array}$$

in the category of divided power rings. Then we see that we have

$$B''/J'' = B/J \otimes_{A/I} B'/J' \longleftarrow C \otimes_{A/I} C$$

see Divided Power Algebra, Remark 23.3.5. Denote $J'' \subset K \subset B''$ the ideal such that

$$\begin{array}{ccc} B''/J'' & \longrightarrow & B''/K \\ \uparrow & & \uparrow \\ C \otimes_{A/I} C & \longrightarrow & C \end{array}$$

is a pushout, i.e., $B''/K \cong B/J \otimes_C B'/J'$. Let $D_{B''}(K) = (D, \bar{K}, \bar{\delta})$ be the divided power envelope of K in B'' relative to (B'', J'', δ''). Then it is easily verified that $(D, \bar{K}, \bar{\delta})$ is a coproduct of (B, J, δ) and (B', J', δ') in $\mathrm{CRIS}(C/A)$.

Next, we come to coequalizers. Let $\alpha, \beta : (B, J, \delta) \to (B', J', \delta')$ be morphisms of $\mathrm{CRIS}(C/A)$. Consider $B'' = B'/(\alpha(b) - \beta(b))$. Let $J'' \subset B''$ be the image of J'. Let $D_{B''}(J'') = (D, \bar{J}, \bar{\delta})$ be the divided power envelope of J'' in B'' relative to (B', J', δ'). Then it is easily verified that $(D, \bar{J}, \bar{\delta})$ is the coequalizer of (B, J, δ) and (B', J', δ') in $\mathrm{CRIS}(C/A)$.

By Categories, Lemma 4.18.6 we have all finite nonempty colimits in $\mathrm{CRIS}(C/A)$. The constructions above shows that (45.5.2.1) commutes with them. This formally implies part (3) as $\mathrm{Cris}(C/A)$ is the fibre category of (45.5.2.1) over C. □

Remark 45.5.4. In Situation 45.5.1 we denote $\mathrm{Cris}^\wedge(C/A)$ the category whose objects are pairs $(B \to C, \delta)$ such that

(1) B is a p-adically complete A-algebra,
(2) $B \to C$ is a surjection of A-algebras,
(3) δ is a divided power structure on $\mathrm{Ker}(B \to C)$,
(4) $A \to B$ is a homomorphism of divided power rings.

Morphisms are defined as in Definition 45.5.2. Then $\mathrm{Cris}(C/A) \subset \mathrm{Cris}^\wedge(C/A)$ is the full subcategory consisting of those B such that p is nilpotent in B. Conversely, any object $(B \to C, \delta)$ of $\mathrm{Cris}^\wedge(C/A)$ is equal to the limit

$$(B \to C, \delta) = \lim_e (B/p^e B \to C, \delta)$$

where for $e \gg 0$ the object $(B/p^e B \to C, \delta)$ lies in $\mathrm{Cris}(C/A)$, see Divided Power Algebra, Lemma 23.4.5. In particular, we see that $\mathrm{Cris}^\wedge(C/A)$ is a full subcategory of the category of pro-objects of $\mathrm{Cris}(C/A)$, see Categories, Remark 4.22.4.

Lemma 45.5.5. *In Situation 45.5.1. Let $P \to C$ be a surjection of A-algebras with kernel J. Write $D_{P,\gamma}(J) = (D, \bar{J}, \bar{\gamma})$. Let $(D^\wedge, J^\wedge, \bar{\gamma}^\wedge)$ be the p-adic completion of D, see Divided Power Algebra, Lemma 23.4.5. For every $e \geq 1$ set $P_e = P/p^eP$ and $J_e \subset P_e$ the image of J and write $D_{P_e,\gamma}(J_e) = (D_e, \bar{J}_e, \bar{\gamma})$. Then for all e large enough we have*

(1) *$p^eD \subset \bar{J}$ and $p^eD^\wedge \subset \bar{J}^\wedge$ are preserved by divided powers,*
(2) *$D^\wedge/p^eD^\wedge = D/p^eD = D_e$ as divided power rings,*
(3) *$(D_e, \bar{J}_e, \bar{\gamma})$ is an object of $Cris(C/A)$,*
(4) *$(D^\wedge, \bar{J}^\wedge, \bar{\gamma}^\wedge)$ is equal to $\lim_e (D_e, \bar{J}_e, \bar{\gamma})$, and*
(5) *$(D^\wedge, \bar{J}^\wedge, \bar{\gamma}^\wedge)$ is an object of $Cris^\wedge(C/A)$.*

Proof. Part (1) follows from Divided Power Algebra, Lemma 23.4.5. It is a general property of p-adic completion that $D/p^eD = D^\wedge/p^eD^\wedge$. Since D/p^eD is a divided power ring and since $P \to D/p^eD$ factors through P_e, the universal property of D_e produces a map $D_e \to D/p^eD$. Conversely, the universal property of D produces a map $D \to D_e$ which factors through D/p^eD. We omit the verification that these maps are mutually inverse. This proves (2). If e is large enough, then $p^eC = 0$, hence we see (3) holds. Part (4) follows from Divided Power Algebra, Lemma 23.4.5. Part (5) is clear from the definitions. □

Lemma 45.5.6. *In Situation 45.5.1. Let P be a polynomial algebra over A and let $P \to C$ be a surjection of A-algebras with kernel J. With $(D_e, \bar{J}_e, \bar{\gamma})$ as in Lemma 45.5.5: for every object (B, J_B, δ) of $CRIS(C/A)$ there exists an e and a morphism $D_e \to B$ of $CRIS(C/A)$.*

Proof. We can find an A-algebra homomorphism $P \to B$ lifting the map $C \to B/J_B$. By our definition of $\mathrm{CRIS}(C/A)$ we see that $p^eB = 0$ for some e hence $P \to B$ factors as $P \to P_e \to B$. By the universal property of the divided power envelope we conclude that $P_e \to B$ factors through D_e. □

Lemma 45.5.7. *In Situation 45.5.1. Let P be a polynomial algebra over A and let $P \to C$ be a surjection of A-algebras with kernel J. Let $(D, \bar{J}, \bar{\gamma})$ be the p-adic completion of $D_{P,\gamma}(J)$. For every object $(B \to C, \delta)$ of $Cris^\wedge(C/A)$ there exists a morphism $D \to B$ of $Cris^\wedge(C/A)$.*

Proof. We can find an A-algebra homomorphism $P \to B$ compatible with maps to C. By our definition of $\mathrm{Cris}(C/A)$ we see that $P \to B$ factors as $P \to D_{P,\gamma}(J) \to B$. As B is p-adically complete we can factor this map through D. □

45.6. Module of differentials

In this section we develop a theory of modules of differentials for divided power rings.

Definition 45.6.1. Let A be a ring. Let (B, J, δ) be a divided power ring. Let $A \to B$ be a ring map. Let M be an B-module. A *divided power A-derivation* into M is a map $\theta : B \to M$ which is additive, annihilates the elements of A, satisfies the Leibniz rule $\theta(bb') = b\theta(b') + b'\theta(b)$ and satisfies

$$\theta(\gamma_n(x)) = \gamma_{n-1}(x)\theta(x)$$

for all $n \geq 1$ and all $x \in J$.

In the situation of the definition, just as in the case of usual derivations, there exists a *universal divided power A-derivation*

$$\mathrm{d}_{B/A,\delta} : B \to \Omega_{B/A,\delta}$$

such that any divided power A-derivation $\theta : B \to M$ is equal to $\theta = \xi \circ d_{B/A,\delta}$ for some B-linear map $\Omega_{B/A,\delta} \to M$. If $(A, I, \gamma) \to (B, J, \delta)$ is a homomorphism of divided power rings, then we can forget the divided powers on A and consider the divided power derivations of B over A. Here are some basic properties of the divided power module of differentials.

Lemma 45.6.2. *Let A be a ring. Let (B, J, δ) be a divided power ring and $A \to B$ a ring map.*

(1) *Consider $B[x]$ with divided power ideal $(JB[x], \delta')$ where δ' is the extension of δ to $B[x]$. Then*
$$\Omega_{B[x]/A,\delta'} = \Omega_{B/A,\delta} \otimes_B B[x] \oplus B[x]dx.$$
(2) *Consider $B\langle x\rangle$ with divided power ideal $(JB\langle x\rangle + B\langle x\rangle_+, \delta')$. Then*
$$\Omega_{B\langle x\rangle/A,\delta'} = \Omega_{B/A,\delta} \otimes_B B\langle x\rangle \oplus B\langle x\rangle dx.$$
(3) *Let $K \subset J$ be an ideal preserved by δ_n for all $n > 0$. Set $B' = B/K$ and denote δ' the induced divided power on J/K. Then $\Omega_{B'/A,\delta'}$ is the quotient of $\Omega_{B/A,\delta} \otimes_B B'$ by the B'-submodule generated by dk for $k \in K$.*

Proof. These are proved directly from the construction of $\Omega_{B/A,\delta}$ as the free B-module on the elements $\mathrm{d}b$ modulo the relations

(1) $\mathrm{d}(b + b') = \mathrm{d}b + \mathrm{d}b'$, $b, b' \in B$,
(2) $\mathrm{d}a = 0$, $a \in A$,
(3) $\mathrm{d}(bb') = b\mathrm{d}b' + b'\mathrm{d}b$, $b, b' \in B$,
(4) $\mathrm{d}\delta_n(f) = \delta_{n-1}(f)\mathrm{d}f$, $f \in J$, $n > 1$.

Note that the last relation explains why we get "the same" answer for the divided power polynomial algebra and the usual polynomial algebra: in the first case x is an element of the divided power ideal and hence $\mathrm{d}x^{[n]} = x^{[n-1]}\mathrm{d}x$. □

Let (A, I, γ) be a divided power ring. In this setting the correct version of the powers of I is given by the divided powers

$$I^{[n]} = \text{ideal generate by } \gamma_{e_1}(x_1) \ldots \gamma_{e_t}(x_t) \text{ with } \sum e_j \geq n \text{ and } x_j \in I.$$

Of course we have $I^n \subset I^{[n]}$. Note that $I^{[1]} = I$. Sometimes we also set $I^{[0]} = A$.

Lemma 45.6.3. *Let $(A, I, \gamma) \to (B, J, \delta)$ be a homomorphism of divided power rings. Let $(B(1), J(1), \delta(1))$ be the coproduct of (B, J, δ) with itself over (A, I, γ), i.e., such that*

$$\begin{array}{ccc} (B, J, \delta) & \longrightarrow & (B(1), J(1), \delta(1)) \\ \uparrow & & \uparrow \\ (A, I, \gamma) & \longrightarrow & (B, J, \delta) \end{array}$$

is cocartesian. Denote $K = Ker(B(1) \to B)$. Then $K \cap J(1) \subset J(1)$ is preserved by the divided power structure and

$$\Omega_{B/A,\delta} = K/\left(K^2 + (K \cap J(1))^{[2]}\right)$$

canonically.

Proof. The fact that $K \cap J(1) \subset J(1)$ is preserved by the divided power structure follows from the fact that $B(1) \to B$ is a homomorphism of divided power rings.

Recall that K/K^2 has a canonical B-module structure. Denote $s_0, s_1 : B \to B(1)$ the two coprojections and consider the map $\mathrm{d} : B \to K/K^2 + (K \cap J(1))^{[2]}$ given by $b \mapsto s_1(b) - s_0(b)$. It is clear that d is additive, annihilates A, and satisfies the Leibniz rule. We claim that d is an A-derivation. Let $x \in J$. Set $y = s_1(x)$ and $z = s_0(x)$. Denote δ the divided power structure on $J(1)$. We have to show that $\delta_n(y) - \delta_n(z) = \delta_{n-1}(y)(y - z)$ modulo $K^2 + (K \cap J(1))^{[2]}$ for $n \geq 1$. We will show this by induction on n. It is true for $n = 1$. Let $n > 1$ and that it holds for all smaller values. Note that

$$\delta_n(z - y) = \sum\nolimits_{i=0}^{n} (-1)^{n-i} \delta_i(z) \delta_{n-i}(y)$$

is an element of $K^2 + (K \cap J(1))^{[2]}$. From this and induction we see that working modulo $K^2 + (K \cap J(1))^{[2]}$ we have

$$\begin{aligned} &\delta_n(y) - \delta_n(z) \\ &= \delta_n(y) + \sum\nolimits_{i=0}^{n-1} (-1)^{n-i} \delta_i(z) \delta_{n-i}(y) \\ &= \delta_n(y) + (-1)^n \delta_n(y) + \sum\nolimits_{i=1}^{n-1} (-1)^{n-i} (\delta_i(y) - \delta_{i-1}(y)(y - z)) \delta_{n-i}(y) \end{aligned}$$

Using that $\delta_i(y)\delta_{n-i}(y) = \binom{n}{i}\delta_n(y)$ and that $\delta_{i-1}(y)\delta_{n-i}(y) = \binom{n-1}{i}\delta_{n-1}(y)$ the reader easily verifies that this expression comes out to give $\delta_{n-1}(y)(y-z)$ as desired.

Let M be a B-module. Let $\theta : B \to M$ be a divided power A-derivation. Set $D = B \oplus M$ where M is an ideal of square zero. Define a divided power structure on $J \oplus M \subset D$ by setting $\delta_n(x + m) = \delta_n(x) + \delta_{n-1}(x)m$ for $n > 1$, see Lemma 45.3.1. There are two divided power algebra homomorphisms $B \to D$: the first is given by the inclusion and the second by the map $b \mapsto b + \theta(b)$. Hence we get a canonical homomorphism $B(1) \to D$ of divided power algebras over (A, I, γ). This induces a map $K \to M$ which annihilates K^2 (as M is an ideal of square zero) and $(K \cap J(1))^{[2]}$ as $M^{[2]} = 0$. The composition $B \to K/K^2 + (K \cap J(1))^{[2]} \to M$ equals θ by construction. It follows that d is a universal divided power A-derivation and we win. □

Remark 45.6.4. Let $A \to B$ be a ring map and let (J, δ) be a divided power structure on B. The universal module $\Omega_{B/A,\delta}$ comes with a little bit of extra structure, namely the B-submodule N of $\Omega_{B/A,\delta}$ generated by $\mathrm{d}_{B/A,\delta}(J)$. In terms of the isomorphism given in Lemma 45.6.3 this corresponds to the image of $K \cap J(1)$ in $\Omega_{B/A,\delta}$. Consider the A-algebra $D = B \oplus \Omega^1_{B/A,\delta}$ with ideal $\bar{J} = J \oplus N$ and divided powers $\bar{\delta}$ as in the proof of the lemma. Then $(D, \bar{J}, \bar{\delta})$ is a divided power ring and the two maps $B \to D$ given by $b \mapsto b$ and $b \mapsto b + \mathrm{d}_{B/A,\delta}(b)$ are homomorphisms of divided power rings over A. Moreover, N is the smallest submodule of $\Omega_{B/A,\delta}$ such that this is true.

Lemma 45.6.5. *In Situation 45.5.1. Let (B, J, δ) be an object of CRIS(C/A). Let $(B(1), J(1), \delta(1))$ be the coproduct of (B, J, δ) with itself in CRIS(C/A). Denote $K = Ker(B(1) \to B)$. Then $K \cap J(1) \subset J(1)$ is preserved by the divided power*

structure and

$$\Omega_{B/A,\delta} = K/\left(K^2 + (K \cap J(1))^{[2]}\right)$$

canonically.

Proof. Word for word the same as the proof of Lemma 45.6.3. The only point that has to be checked is that the divided power ring $D = B \oplus M$ is an object of $\mathrm{CRIS}(C/A)$ and that the two maps $B \to C$ are morphisms of $\mathrm{CRIS}(C/A)$. Since $D/(J \oplus M) = B/J$ we can use $C \to B/J$ to view D as an object of $\mathrm{CRIS}(C/A)$ and the statement on morphisms is clear from the construction. □

Lemma 45.6.6. *Let (A, I, γ) be a divided power ring. Let $A \to B$ be a ring map and let $IB \subset J \subset B$ be an ideal. Let $D_{B,\gamma}(J) = (D, \bar{J}, \bar{\gamma})$ be the divided power envelope. Then we have*

$$\Omega_{D/A,\bar{\gamma}} = \Omega_{B/A} \otimes_B D$$

Proof. We will prove this first when B is flat over A. In this case γ extends to a divided power structure γ' on IB, see Divided Power Algebra, Lemma 23.4.2. Hence $D = D_{B',\gamma'}(J)$ is equal to a quotient of the divided power ring (D', J', δ) where $D' = B\langle x_t\rangle$ and $J' = IB\langle x_t\rangle + B\langle x_t\rangle_+$ by the elements $x_t - f_t$ and $\delta_n(\sum r_t x_t - r_0)$, see Lemma 45.2.4 for notation and explanation. Write $\mathrm{d} : D' \to \Omega_{D'/A,\delta}$ for the universal derivation. Note that

$$\Omega_{D'/A,\delta} = \Omega_{B/A} \otimes_B D' \oplus \bigoplus D'\mathrm{d}x_t,$$

see Lemma 45.6.2. We conclude that $\Omega_{D/A,\bar{\gamma}}$ is the quotient of $\Omega_{D'/A,\delta} \otimes_{D'} D$ by the submodule generated by d applied to the generators of the kernel of $D' \to D$ listed above, see Lemma 45.6.2. Since $\mathrm{d}(x_t - f_t) = -\mathrm{d}f_t + \mathrm{d}x_t$ we see that we have $\mathrm{d}x_t = \mathrm{d}f_t$ in the quotient. In particular we see that $\Omega_{B/A} \otimes_B D \to \Omega_{D/A,\gamma}$ is surjective with kernel given by the images of d applied to the elements $\delta_n(\sum r_t x_t - r_0)$. However, given a relation $\sum r_t f_t - r_0 = 0$ in B with $r_t \in B$ and $r_0 \in IB$ we see that

$$\begin{aligned}\mathrm{d}\delta_n(\sum r_t x_t - r_0) &= \delta_{n-1}(\sum r_t x_t - r_0)\mathrm{d}(\sum r_t x_t - r_0) \\ &= \delta_{n-1}(\sum r_t x_t - r_0)\left(\sum r_t \mathrm{d}(x_t - f_t) + \sum (x_t - f_t)\mathrm{d}r_t\right)\end{aligned}$$

because $\sum r_t f_t - r_0 = 0$ in B. Hence this is already zero in $\Omega_{B/A} \otimes_A D$ and we win in the case that B is flat over A.

In the general case we write B as a quotient of a polynomial ring $P \to B$ and let $J' \subset P$ be the inverse image of J. Then $D = D'/K'$ with notation as in Lemma 45.2.3. By the case handled in the first paragraph of the proof we have $\Omega_{D'/A,\bar{\gamma}'} = \Omega_{P/A} \otimes_P D'$. Then $\Omega_{D/A,\bar{\gamma}}$ is the quotient of $\Omega_{P/A} \otimes_P D$ by the submodule generated by $\mathrm{d}\bar{\gamma}'_n(k)$ where k is an element of the kernel of $P \to B$, see Lemma 45.6.2 and the description of K' from Lemma 45.2.3. Since $\mathrm{d}\bar{\gamma}'_n(k) = \bar{\gamma}'_{n-1}(k)\mathrm{d}k$ we see again that it suffices to divided by the submodule generated by $\mathrm{d}k$ with $k \in \mathrm{Ker}(P \to B)$ and since $\Omega_{B/A}$ is the quotient of $\Omega_{P/A} \otimes_A B$ by these elements (Algebra, Lemma 10.125.9) we win. □

Remark 45.6.7. Let B be a ring. Write $\Omega_B = \Omega_{B/\mathbf{Z}}$ for the absolute[2] module of differentials of B. Let $\mathrm{d} : B \to \Omega_B$ denote the universal derivation. Set $\Omega^i_B = \wedge^i_B(\Omega_B)$ as in Algebra, Section 10.12. The absolute *de Rham complex*

$$\Omega^0_B \to \Omega^1_B \to \Omega^2_B \to \ldots$$

Here $\mathrm{d} : \Omega^p_B \to \Omega^{p+1}_B$ is defined by the rule

$$\mathrm{d}\left(b_0 \mathrm{d}b_1 \wedge \ldots \wedge \mathrm{d}b_p\right) = \mathrm{d}b_0 \wedge \mathrm{d}b_1 \wedge \ldots \wedge \mathrm{d}b_p$$

which we will show is well defined; note that $\mathrm{d} \circ \mathrm{d} = 0$ so we get a complex. Recall that Ω_B is the B-module generated by elements $\mathrm{d}b$ subject to the relations $\mathrm{d}(a+b) = \mathrm{d}a + \mathrm{d}b$ and $\mathrm{d}(ab) = b\mathrm{d}a + a\mathrm{d}b$ for $a, b \in B$. To prove that our map is well defined for $p = 1$ we have to show that the elements

$$a\mathrm{d}(b+c) - a\mathrm{d}b - a\mathrm{d}c \quad \text{and} \quad a\mathrm{d}(bc) - ac\mathrm{d}b - ab\mathrm{d}c, \quad a, b, c \in B$$

are mapped to zero by our rule. This is clear by direct computation (using the Leibniz rule). Thus we get a map

$$\Omega_B \otimes_{\mathbf{Z}} \ldots \otimes_{\mathbf{Z}} \Omega_B \longrightarrow \Omega^{p+1}_B$$

defined by the formula

$$\omega_1 \otimes \ldots \otimes \omega_p \longmapsto \sum (-1)^{i+1} \omega_1 \wedge \ldots \wedge \mathrm{d}(\omega_i) \wedge \ldots \wedge \omega_p$$

which matches our rule above on elements of the form $b_0 \mathrm{d}b_1 \otimes \mathrm{d}b_2 \otimes \ldots \otimes \mathrm{d}b_p$. It is clear that this map is alternating. To finish we have to show that

$$\omega_1 \otimes \ldots \otimes f\omega_i \otimes \ldots \otimes \omega_p \quad \text{and} \quad \omega_1 \otimes \ldots \otimes f\omega_j \otimes \ldots \otimes \omega_p$$

are mapped to the same element. By $\mathbf{Z}$-linearity and the alternating property, it is enough to show this for $p = 2$, $i = 1$, $j = 2$, $\omega_1 = a_1 \mathrm{d}b_1$ and $\omega_2 = a_2 \mathrm{d}b_2$. Thus we need to show that

$$\begin{aligned} &\mathrm{d}fa_1 \wedge \mathrm{d}b_1 \wedge a_2 \mathrm{d}b_2 - fa_1 \mathrm{d}b_1 \wedge \mathrm{d}a_2 \wedge \mathrm{d}b_2 \\ &= \mathrm{d}a_1 \wedge \mathrm{d}b_1 \wedge fa_2 \mathrm{d}b_2 - a_1 \mathrm{d}b_1 \wedge \mathrm{d}fa_2 \wedge \mathrm{d}b_2 \end{aligned}$$

in other words that

$$(a_2 \mathrm{d}fa_1 + fa_1 \mathrm{d}a_2 - fa_2 \mathrm{d}a_1 - a_1 \mathrm{d}fa_2) \wedge \mathrm{d}b_1 \wedge \mathrm{d}b_2 = 0.$$

This follows from the Leibniz rule.

Lemma 45.6.8. *Let B be a ring. Let $\pi : \Omega_B \to \Omega$ be a surjective B-module map. Denote $d : B \to \Omega$ the composition of π with $d_B : B \to \Omega_B$. Set $\Omega^i = \wedge^i_B(\Omega)$. Assume that the kernel of π is generated, as a B-module, by elements $\omega \in \Omega_B$ such that $d_B(\omega) \in \Omega^2_B$ maps to zero in Ω^2. Then there is a de Rham complex*

$$\Omega^0 \to \Omega^1 \to \Omega^2 \to \ldots$$

whose differential is defined by the rule

$$d : \Omega^p \to \Omega^{p+1}, \quad d\left(f_0 df_1 \wedge \ldots \wedge df_p\right) = df_0 \wedge df_1 \wedge \ldots \wedge df_p$$

[2]This actually makes sense: if Ω_B is the module of differentials where we only assume the Leibniz rule and not the vanishing of $\mathrm{d}1$, then the Leibniz rule gives $\mathrm{d}1 = \mathrm{d}(1 \cdot 1) = 1\mathrm{d}1 + 1\mathrm{d}1 = 2\mathrm{d}1$ and hence $\mathrm{d}1 = 0$ in Ω_B.

Proof. We will show that there exists a commutative diagram

$$\begin{array}{ccccccc} \Omega_B^0 & \xrightarrow{\mathrm{d}_B} & \Omega_B^1 & \xrightarrow{\mathrm{d}_B} & \Omega_B^2 & \xrightarrow{\mathrm{d}_B} & \ldots \\ \downarrow & & \downarrow{\scriptstyle \pi} & & \downarrow{\scriptstyle \wedge^2\pi} & & \\ \Omega^0 & \xrightarrow{\mathrm{d}} & \Omega^1 & \xrightarrow{\mathrm{d}} & \Omega^2 & \xrightarrow{\mathrm{d}} & \ldots \end{array}$$

the description of the map d will follow from the construction of d_B in Remark 45.6.7. Since the left most vertical arrow is an isomorphism we have the first square. Because π is surjective, to get the second square it suffices to show that d_B maps the kernel of π into the kernel of $\wedge^2\pi$. We are given that any element of the kernel of π is of the form $\sum b_i\omega_i$ with $\pi(\omega_i) = 0$ and $\wedge^2\pi(\mathrm{d}_B(\omega_i)) = 0$. By the Leibniz rule for d_B we have $\mathrm{d}_B(\sum b_i\omega_i) = \sum b_i\mathrm{d}_B(\omega_i) + \sum \mathrm{d}_B(b_i) \wedge \omega_i$. Hence this maps to zero under $\wedge^2\pi$.

For $i > 1$ we note that $\wedge^i\pi$ is surjective with kernel the image of $\mathrm{Ker}(\pi) \wedge \Omega_B^{i-1} \to \Omega_B^i$. For $\omega_1 \in \mathrm{Ker}(\pi)$ and $\omega_2 \in \Omega_B^{i-1}$ we have

$$\mathrm{d}_B(\omega_1 \wedge \omega_2) = \mathrm{d}_B(\omega_1) \wedge \omega_2 - \omega_1 \wedge \mathrm{d}_B(\omega_2)$$

which is in the kernel of $\wedge^{i+1}\pi$ by what we just proved above. Hence we get the $(i+1)$st square in the diagram above. This concludes the proof. □

Remark 45.6.9. Let $A \to B$ be a ring map and let (J, δ) be a divided power structure on B. Set $\Omega^i_{B/A,\delta} = \wedge^i_B \Omega_{B/A,\delta}$ where $\Omega_{B/A,\delta}$ is the target of the universal divided power A-derivation $\mathrm{d} = \mathrm{d}_{B/A} : B \to \Omega_{B/A,\delta}$. Note that $\Omega_{B/A,\delta}$ is the quotient of Ω_B by the B-submodule generated by the elements $\mathrm{d}a = 0$ for $a \in A$ and $\mathrm{d}\delta_n(x) - \delta_{n-1}(x)\mathrm{d}x$ for $x \in J$. We claim Lemma 45.6.8 applies. To see this it suffices to verify the elements $\mathrm{d}a$ and $\mathrm{d}\delta_n(x) - \delta_{n-1}(x)\mathrm{d}x$ of Ω_B are mapped to zero in $\Omega^2_{B/A,\delta}$. This is clear for the first, and for the last we observe that

$$\mathrm{d}(\delta_{n-1}(x)) \wedge \mathrm{d}x = \delta_{n-2}(x)\mathrm{d}x \wedge \mathrm{d}x = 0$$

in $\Omega^2_{B/A,\delta}$ as desired. Hence we obtain a *divided power de Rham complex*

$$\Omega^0_{B/A,\delta} \to \Omega^1_{B/A,\delta} \to \Omega^2_{B/A,\delta} \to \ldots$$

which will play an important role in the sequel.

Remark 45.6.10. Let B be a ring. Let $\Omega_B \to \Omega$ be a quotient satisfying the assumptions of Lemma 45.6.8. Let M be a B-module. A *connection* is an additive map

$$\nabla : M \longrightarrow M \otimes_B \Omega$$

such that $\nabla(bm) = b\nabla(m) + m \otimes \mathrm{d}b$ for $b \in B$ and $m \in M$. In this situation we can define maps

$$\nabla : M \otimes_B \Omega^i \longrightarrow M \otimes_B \Omega^{i+1}$$

by the rule $\nabla(m \otimes \omega) = \nabla(m) \wedge \omega + m \otimes \mathrm{d}\omega$. This works because if $b \in B$, then

$$\begin{aligned} \nabla(bm \otimes \omega) - \nabla(m \otimes b\omega) &= \nabla(bm) \otimes \omega + bm \otimes \mathrm{d}\omega - \nabla(m) \otimes b\omega - m \otimes \mathrm{d}(b\omega) \\ &= b\nabla(m) \otimes \omega + m \otimes \mathrm{d}b \wedge \omega + bm \otimes \mathrm{d}\omega \\ &\quad - b\nabla(m) \otimes \omega - bm \otimes \mathrm{d}(\omega) - m \otimes \mathrm{d}b \wedge \omega = 0 \end{aligned}$$

As is customary we say the connection is *integrable* if and only if the composition

$$M \xrightarrow{\nabla} M \otimes_B \Omega^1 \xrightarrow{\nabla} M \otimes_B \Omega^2$$

is zero. In this case we obtain a complex

$$M \xrightarrow{\nabla} M \otimes_B \Omega^1 \xrightarrow{\nabla} M \otimes_B \Omega^2 \xrightarrow{\nabla} M \otimes_B \Omega^3 \xrightarrow{\nabla} M \otimes_B \Omega^4 \to \ldots$$

which is called the de Rham complex of the connection.

Remark 45.6.11. Let $\varphi : B \to B'$ be a ring map. Let $\Omega_B \to \Omega$ and $\Omega_{B'} \to \Omega'$ be quotients satisfying the assumptions of Lemma 45.6.8. Assume that the map $\Omega_B \to \Omega_{B'}$, $b_1 \mathrm{d}b_2 \mapsto \varphi(b_1)\mathrm{d}\varphi(b_2)$ fits into a commutative diagram

$$\begin{array}{ccccc} B & \longrightarrow & \Omega_B & \longrightarrow & \Omega \\ \downarrow & & \downarrow & & \downarrow \varphi \\ B' & \longrightarrow & \Omega_{B'} & \longrightarrow & \Omega' \end{array}$$

In this situation, given any pair (M, ∇) where M is a B-module and $\nabla : M \to M \otimes_B \Omega$ is a connection we obtain a *base change* $(M \otimes_B B', \nabla')$ where

$$\nabla' : M \otimes_B B' \longrightarrow (M \otimes_B B') \otimes_{B'} \Omega' = M \otimes_B \Omega'$$

is defined by the rule

$$\nabla'(m \otimes b') = \sum m_i \otimes b' \mathrm{d}\varphi(b_i) + m \otimes \mathrm{d}b'$$

if $\nabla(m) = \sum m_i \otimes \mathrm{d}b_i$. If ∇ is integrable, then so is ∇', and in this case there is a canonical map of de Rham complexes

$$M \otimes_B \Omega^\bullet \longrightarrow (M \otimes_B B') \otimes_{B'} (\Omega')^\bullet = M \otimes_B (\Omega')^\bullet \tag{45.6.11.1}$$

which maps $m \otimes \eta$ to $m \otimes \varphi(\eta)$.

Lemma 45.6.12. *Let $A \to B$ be a ring map and let (J, δ) be a divided power structure on B. Let p be a prime number. Assume that A is a $\mathbf{Z}_{(p)}$-algebra and that p is nilpotent in B/J. Then we have*

$$\lim_e \Omega_{B_e/A,\bar{\delta}} = \lim_e \Omega_{B/A,\delta}/p^e\Omega_{B/A,\delta} = \lim_e \Omega_{B^\wedge/A,\delta^\wedge}/p^e\Omega_{B^\wedge/A,\delta^\wedge}$$

see proof for notation and explanation.

Proof. By Divided Power Algebra, Lemma 23.4.5 we see that δ extends to $B_e = B/p^eB$ for all sufficiently large e. Hence the first limit make sense. The lemma also produces a divided power structure $\delta^\wedge$ on the completion $B^\wedge = \lim_e B_e$, hence the last limit makes sense. By Lemma 45.6.2 and the fact that $\mathrm{d}p^e = 0$ (always) we see that the surjection $\Omega_{B/A,\delta} \to \Omega_{B_e/A,\bar{\delta}}$ has kernel $p^e\Omega_{B/A,\delta}$. Similarly for the kernel of $\Omega_{B^\wedge/A,\delta^\wedge} \to \Omega_{B_e/A,\bar{\delta}}$. Hence the lemma is clear. □

45.7. Divided power schemes

Some remarks on how to globalize the previous notions.

Definition 45.7.1. Let $\mathcal{C}$ be a site. Let $\mathcal{O}$ be a sheaf of rings on $\mathcal{C}$. Let $\mathcal{I} \subset \mathcal{O}$ be a sheaf of ideals. A *divided power structure* γ on $\mathcal{I}$ is a sequence of maps $\gamma_n : \mathcal{I} \to \mathcal{I}$, $n \geq 1$ such that for any object U of $\mathcal{C}$ the triple

$$(\mathcal{O}(U), \mathcal{I}(U), \gamma)$$

is a divided power ring.

To be sure this applies in particular to sheaves of rings on topological spaces. But it's good to be a little bit more general as the structure sheaf of the crystalline site lives on a... site! A triple $(\mathcal{C}, \mathcal{I}, \gamma)$ as in the definition above is sometimes called a *divided power topos* in this chapter. Given a second $(\mathcal{C}', \mathcal{I}', \gamma')$ and given a morphism of ringed topoi $(f, f^\sharp) : (Sh(\mathcal{C}), \mathcal{O}) \to (Sh(\mathcal{C}'), \mathcal{O}')$ we say that $(f, f^\sharp)$ induces a *morphism of divided power topoi* if $f^\sharp(f^{-1}\mathcal{I}') \subset \mathcal{I}$ and the diagrams

$$\begin{array}{ccc} f^{-1}\mathcal{I}' & \xrightarrow{f^\sharp} & \mathcal{I} \\ {\scriptstyle f^{-1}\gamma'_n}\downarrow & & \downarrow{\scriptstyle \gamma_n} \\ f^{-1}\mathcal{I}' & \xrightarrow{f^\sharp} & \mathcal{I} \end{array}$$

are commutative for all $n \geq 1$. If f comes from a morphism of sites induced by a functor $u : \mathcal{C}' \to \mathcal{C}$ then this just means that

$$(\mathcal{O}'(U'), \mathcal{I}'(U'), \gamma') \longrightarrow (\mathcal{O}(u(U')), \mathcal{I}(u(U')), \gamma)$$

is a homomorphism of divided power rings for all $U' \in \mathrm{Ob}(\mathcal{C}')$.

In the case of schemes we require the divided power ideal to be **quasi-coherent**. But apart from this the definition is exactly the same as in the case of topoi. Here it is.

Definition 45.7.2. A *divided power scheme* is a triple $(S, \mathcal{I}, \gamma)$ where S is a scheme, $\mathcal{I}$ is a quasi-coherent sheaf of ideals, and γ is a divided power structure on $\mathcal{I}$. A *morphism of divided power schemes* $(S, \mathcal{I}, \gamma) \to (S', \mathcal{I}', \gamma')$ is a morphism of schemes $f : S \to S'$ such that $f^{-1}\mathcal{I}'\mathcal{O}_S \subset \mathcal{I}$ and such that

$$(\mathcal{O}_S(U'), \mathcal{I}(U'), \gamma) \longrightarrow (\mathcal{O}_{S'}(f^{-1}U'), \mathcal{I}(f^{-1}U'), \gamma)$$

is a homomorphism of divided power rings for all $U' \subset S'$ open.

Recall that there is a 1-to-1 correspondence between quasi-coherent sheaves of ideals and closed immersions, see Morphisms, Section 28.2. Thus given a divided power scheme $(T, \mathcal{J}, \gamma)$ we get a canonical closed immersion $U \to T$ defined by $\mathcal{J}$. Conversely, given a closed immersion $U \to T$ and a divided power structure γ on the sheaf of ideals $\mathcal{J}$ associated to $U \to T$ we obtain a divided power scheme $(T, \mathcal{J}, \gamma)$. In many situations we only want to consider such triples (U, T, γ) when the morphism $U \to T$ is a thickening, see More on Morphisms, Definition 36.2.1.

Definition 45.7.3. A triple (U, T, γ) as above is called a *divided power thickening* if $U \to T$ is a thickening.

Fibre products of divided power schemes exist when one of the three is a divided power thickening. Here is a formal statement.

Lemma 45.7.4. *Let $(U', T', \delta') \to (S'_0, S', \gamma')$ and $(S_0, S, \gamma) \to (S'_0, S', \gamma')$ be morphisms of divided power schemes. If (U', T', δ') is a divided power thickening, then there exists a divided power scheme (T_0, T, δ) and*

$$\begin{array}{ccc} T & \longrightarrow & T' \\ \downarrow & & \downarrow \\ S & \longrightarrow & S' \end{array}$$

which is a cartesian diagram in the category of divided power schemes.

Proof. Omitted. Hints: If T exists, then $T_0 = S_0 \times_{S'_0} U'$ (argue as in Divided Power Algebra, Remark 23.3.5). Since T' is a divided power thickening, we see that T (if it exists) will be a divided power thickening too. Hence we can define T as the scheme with underlying topological space the underlying topological space of $T_0 = S_0 \times_{S'_0} U'$ and as structure sheaf on affine pieces the ring given by Lemma 45.5.3. □

We make the following observation. Suppose that (U, T, γ) is triple as above. Assume that T is a scheme over $\mathbf{Z}_{(p)}$ and that p is locally nilpotent on U. Then

(1) p locally nilpotent on $T \Leftrightarrow U \to T$ is a thickening (see Divided Power Algebra, Lemma 23.2.6), and
(2) $p^e\mathcal{O}_T$ is locally on T preserved by γ for $e \gg 0$ (see Divided Power Algebra, Lemma 23.4.5).

This suggest that good results on divided power thickenings will be available under the following hypotheses.

Situation 45.7.5. Here p is a prime number and $(S, \mathcal{I}, \gamma)$ is a divided power scheme over $\mathbf{Z}_{(p)}$. We set $S_0 = V(\mathcal{I}) \subset S$. Finally, $X \to S_0$ is a morphism of schemes such that p is locally nilpotent on X.

It is in this situation that we will define the big and small crystalline sites.

45.8. The big crystalline site

We first define the big site. Given a divided power scheme $(S, \mathcal{I}, \gamma)$ we say $(T, \mathcal{J}, \delta)$ is a divided power scheme over $(S, \mathcal{I}, \gamma)$ if T comes endowed with a morphism $T \to S$ of divided power schemes. Similarly, we say a divided power thickening (U, T, δ) is a divided power thickening over $(S, \mathcal{I}, \gamma)$ if T comes endowed with a morphism $T \to S$ of divided power schemes.

Definition 45.8.1. In Situation 45.7.5.

(1) A *divided power thickening of X relative to* $(S, \mathcal{I}, \gamma)$ is given by a divided power thickening (U, T, δ) over $(S, \mathcal{I}, \gamma)$ and an S-morphism $U \to X$.
(2) A *morphism of divided power thickenings of X relative to* $(S, \mathcal{I}, \gamma)$ is defined in the obvious manner.

The category of divided power thickenings of X relative to $(S, \mathcal{I}, \gamma)$ is denoted $\mathrm{CRIS}(X/S, \mathcal{I}, \gamma)$ or simply $\mathrm{CRIS}(X/S)$.

For any (U, T, δ) in $\mathrm{CRIS}(X/S)$ we have that p is locally nilpotent on T, see discussion after Definition 45.7.3. A good way to visualize all the data associated to (U, T, δ) is the commutative diagram

$$\begin{array}{ccc} T & \longleftarrow & U \\ \downarrow & & \downarrow \\ & & X \\ & & \downarrow \\ S & \longleftarrow & S_0 \end{array}$$

where $S_0 = V(\mathcal{I}) \subset S$. Morphisms of $\mathrm{CRIS}(X/S)$ can be similarly visualized as huge commutative diagrams. In particular, there is a canonical forgetful functor

$$\mathrm{CRIS}(X/S) \longrightarrow Sch/X, \quad (U, T, \delta) \longmapsto U \tag{45.8.1.1}$$

as well as its one sided inverse (and left adjoint)

$$Sch/X \longrightarrow \mathrm{CRIS}(X/S), \quad U \longmapsto (U, U, \emptyset) \tag{45.8.1.2}$$

which is sometimes useful.

Lemma 45.8.2. *In Situation 45.7.5. The category CRIS(X/S) has all finite nonempty limits, in particular products of pairs and fibre products. The functor (45.8.1.1) commutes with limits.*

Proof. Omitted. Hint: See Lemma 45.5.3 for the affine case. See also Divided Power Algebra, Remark 23.3.5. □

Lemma 45.8.3. *In Situation 45.7.5. Let*

$$\begin{array}{ccc} (U_3, T_3, \delta_3) & \longrightarrow & (U_2, T_2, \delta_2) \\ \downarrow & & \downarrow \\ (U_1, T_1, \delta_1) & \longrightarrow & (U, T, \delta) \end{array}$$

be a fibre square in the category of divided power thickenings of X relative to $(S, \mathcal{I}, \gamma)$. If $T_2 \to T$ is flat, then $T_3 = T_1 \times_T T_2$ (as schemes).

Proof. This is true because a divided power structure extends uniquely along a flat ring map. See Divided Power Algebra, Lemma 23.4.2. □

The lemma above means that the base change of a flat morphism of divided power thickenings is another flat morphism, and in fact is the "usual" base change of the morphism. This implies that the following definition makes sense.

Definition 45.8.4. In Situation 45.7.5.

1. A family of morphisms $\{(U_i, T_i, \delta_i) \to (U, T, \delta)\}$ of divided power thickenings of X/S is a *Zariski, étale, smooth, syntomic, or fppf covering* if and only if the family of morphisms of schemes $\{T_i \to T\}$ is one.
2. The *big crystalline site* of X over $(S, \mathcal{I}, \gamma)$, is the category $\mathrm{CRIS}(X/S)$ endowed with the Zariski topology.
3. The topos of sheaves on $\mathrm{CRIS}(X/S)$ is denoted $(X/S)_{\mathrm{CRIS}}$ or sometimes $(X/S, \mathcal{I}, \gamma)_{\mathrm{CRIS}}$[3].

There are some obvious functorialities concerning these topoi.

Remark 45.8.5 (Functoriality)**.** Let p be a prime number. Let $(S, \mathcal{I}, \gamma) \to (S', \mathcal{I}', \gamma')$ be a morphism of divided power schemes over $\mathbf{Z}_{(p)}$. Set $S_0 = V(\mathcal{I})$ and $S_0' = V(\mathcal{I}')$. Let

$$\begin{array}{ccc} X & \xrightarrow[f]{} & Y \\ \downarrow & & \downarrow \\ S_0 & \longrightarrow & S_0' \end{array}$$

[3]This clashes with our convention to denote the topos associated to a site $\mathcal{C}$ by $Sh(\mathcal{C})$.

be a commutative diagram of morphisms of schemes and assume p is locally nilpotent on X and Y. Then we get a continuous and cocontinuous functor

$$\mathrm{CRIS}(X/S) \longrightarrow \mathrm{CRIS}(Y/S')$$

by letting (U, T, δ) correspond to (U, T, δ) with $U \to X \to Y$ as the S'-morphism from U to Y. Hence we get a morphism of topoi

$$f_{\mathrm{CRIS}} : (X/S)_{\mathrm{CRIS}} \longrightarrow (Y/S')_{\mathrm{CRIS}}$$

see Sites, Section 7.20.

Remark 45.8.6 (Comparison with Zariski site)**.** In Situation 45.7.5. The functor (45.8.1.1) is continuous, cocontinuous, and commutes with products and fibred products. Hence we obtain a morphism of topoi

$$U_{X/S} : (X/S)_{\mathrm{CRIS}} \longrightarrow Sh((Sch/X)_{Zar})$$

from the big crystalline topos of X/S to the big Zariski topos of X. See Sites, Section 7.20.

Remark 45.8.7 (Structure morphism)**.** In Situation 45.7.5. Consider the closed subscheme $S_0 = V(\mathcal{I}) \subset S$. If we assume that p is locally nilpotent on S_0 (which is always the case in practice) then we obtain a situation as in Definition 45.8.1 with S_0 instead of X. Hence we get a site $\mathrm{CRIS}(S_0/S)$. If $f : X \to S_0$ is the structure morphism of X over S, then we get a commutative diagram of morphisms of ringed topoi

$$\begin{array}{ccc} (X/S)_{\mathrm{CRIS}} & \xrightarrow{f_{\mathrm{CRIS}}} & (S_0/S)_{\mathrm{CRIS}} \\ \downarrow{\scriptstyle U_{X/S}} & & \downarrow{\scriptstyle U_{S_0/S}} \\ Sh((Sch/X)_{Zar}) & \xrightarrow{f_{big}} & Sh((Sch/S_0)_{Zar}) \\ & & \searrow \\ & & Sh((Sch/S)_{Zar}) \end{array}$$

by Remark 45.8.5. We think of the composition $(X/S)_{\mathrm{CRIS}} \to Sh((Sch/S)_{Zar})$ as the structure morphism of the big crystalline site. Even if p is not locally nilpotent on S_0 the structure morphism

$$(X/S)_{\mathrm{CRIS}} \longrightarrow Sh((Sch/S)_{Zar})$$

is defined as we can take the lower route through the diagram above. Thus it is the morphism of topoi corresponding to the cocontinuous functor $\mathrm{CRIS}(X/S) \to (Sch/S)_{Zar}$ given by the rule $(U, T, \delta)/S \mapsto T/S$, see Sites, Section 7.20.

Remark 45.8.8 (Compatibilities)**.** The morphisms defined above satisfy numerous compatibilities. For example, in the situation of Remark 45.8.5 we obtain a commutative diagram of ringed topoi

$$\begin{array}{ccc} (X/S)_{\mathrm{CRIS}} & \longrightarrow & (Y/S')_{\mathrm{CRIS}} \\ \downarrow & & \downarrow \\ Sh((Sch/S)_{Zar}) & \longrightarrow & Sh((Sch/S')_{Zar}) \end{array}$$

where the vertical arrows are the structure morphisms.

45.9. The crystalline site

Since (45.8.1.1) commutes with products and fibre products, we see that looking at those (U, T, δ) such that $U \to X$ is an open immersion defines a full subcategory preserved under fibre products (and more generally finite nonempty limits). Hence the following definition makes sense.

Definition 45.9.1. In Situation 45.7.5.

(1) The (small) *crystalline site* of X over $(S, \mathcal{I}, \gamma)$, denoted $\mathrm{Cris}(X/S, \mathcal{I}, \gamma)$ or simply $\mathrm{Cris}(X/S)$ is the full subcategory of $\mathrm{CRIS}(X/S)$ consisting of those (U, T, δ) in $\mathrm{CRIS}(X/S)$ such that $U \to X$ is an open immersion. It comes endowed with the Zariski topology.
(2) The topos of sheaves on $\mathrm{Cris}(X/S)$ is denoted $(X/S)_{\mathrm{cris}}$ or sometimes $(X/S, \mathcal{I}, \gamma)_{\mathrm{cris}}$[4].

For any (U, T, δ) in $\mathrm{Cris}(X/S)$ the morphism $U \to X$ defines an object of the small Zariski site X_{Zar} of X. Hence a canonical forgetful functor

$$\mathrm{Cris}(X/S) \longrightarrow X_{Zar}, \quad (U, T, \delta) \longmapsto U \tag{45.9.1.1}$$

and a left adjoint

$$X_{Zar} \longrightarrow \mathrm{Cris}(X/S), \quad U \longmapsto (U, U, \emptyset) \tag{45.9.1.2}$$

which is sometimes useful.

We can compare the small and big crystalline sites, just like we can compare the small and big Zariski sites of a scheme, see Topologies, Lemma 33.3.13.

Lemma 45.9.2. *Assumptions as in Definition 45.8.1. The inclusion functor*

$$\mathit{Cris}(X/S) \to \mathit{CRIS}(X/S)$$

commutes with finite nonempty limits, is fully faithful, continuous, and cocontinuous. There are morphisms of topoi

$$(X/S)_{cris} \xrightarrow{i} (X/S)_{CRIS} \xrightarrow{\pi} (X/S)_{cris}$$

whose composition is the identity and of which the first is induced by the inclusion functor. Moreover, $\pi_* = i^{-1}$.

Proof. For the first assertion see Lemma 45.8.2. This gives us a morphism of topoi $i : (X/S)_{\mathrm{cris}} \to (X/S)_{\mathrm{CRIS}}$ and a left adjoint $i_!$ such that $i^{-1}i_! = i^{-1}i_* = \mathrm{id}$, see Sites, Lemmas 7.20.5, 7.20.6, and 7.20.7. We claim that $i_!$ is exact. If this is true, then we can define π by the rules $\pi^{-1} = i_!$ and $\pi_* = i^{-1}$ and everything is clear. To prove the claim, note that we already know that $i_!$ is right exact and preserves fibre products (see references given). Hence it suffices to show that $i_!* = *$ where $*$ indicates the final object in the category of sheaves of sets. To see this it suffices to produce a set of objects (U_i, T_i, δ_i), $i \in I$ of $\mathrm{Cris}(X/S)$ such that

$$\coprod\nolimits_{i \in I} h_{(U_i, T_i, \delta_i)} \to *$$

is surjective in $(X/S)_{\mathrm{CRIS}}$ (details omitted; hint: use that $\mathrm{Cris}(X/S)$ has products and that the functor $\mathrm{Cris}(X/S) \to \mathrm{CRIS}(X/S)$ commutes with them). In the affine case this follows from Lemma 45.5.6. We omit the proof in general. □

[4]This clashes with our convention to denote the topos associated to a site $\mathcal{C}$ by $Sh(\mathcal{C})$.

Remark 45.9.3 (Functoriality)**.** Let p be a prime number. Let $(S, \mathcal{I}, \gamma) \to (S', \mathcal{I}', \gamma')$ be a morphism of divided power schemes over $\mathbf{Z}_{(p)}$. Let

$$\begin{array}{ccc} X & \xrightarrow{f} & Y \\ \downarrow & & \downarrow \\ S_0 & \longrightarrow & S'_0 \end{array}$$

be a commutative diagram of morphisms of schemes and assume p is locally nilpotent on X and Y. By analogy with Topologies, Lemma 33.3.16 we define

$$f_{\text{cris}} : (X/S)_{\text{cris}} \longrightarrow (Y/S')_{\text{cris}}$$

by the formula $f_{\text{cris}} = \pi_Y \circ f_{\text{CRIS}} \circ i_X$ where i_X and π_Y are as in Lemma 45.9.2 for X and Y and where f_{CRIS} is as in Remark 45.8.5.

Remark 45.9.4 (Comparison with Zariski site)**.** In Situation 45.7.5. The functor (45.9.1.1) is continuous, cocontinuous, and commutes with products and fibred products. Hence we obtain a morphism of topoi

$$u_{X/S} : (X/S)_{\text{cris}} \longrightarrow Sh(X_{Zar})$$

relating the small crystalline topos of X/S with the small Zariski topos of X. See Sites, Section 7.20.

Lemma 45.9.5. *In Situation 45.7.5. Let $X' \subset X$ and $S' \subset S$ be open subschemes such that X' maps into S'. Then there is a fully faithful functor $Cris(X'/S') \to Cris(X/S)$ which gives rise to a morphism of topoi fitting into the commutative diagram*

$$\begin{array}{ccc} (X'/S')_{cris} & \longrightarrow & (X/S)_{cris} \\ {\scriptstyle u_{X'/S'}}\downarrow & & \downarrow{\scriptstyle u_{X/S}} \\ Sh(X'_{Zar}) & \longrightarrow & Sh(X_{Zar}) \end{array}$$

Moreover, this diagram is an example of localization of morphisms of topoi as in Sites, Lemma 7.30.1.

Proof. The fully faithful functor comes from thinking of objects of $\text{Cris}(X'/S')$ as divided power thickenings (U, T, δ) of X where $U \to X$ factors through $X' \subset X$ (since then automatically $T \to S$ will factor through S'). This functor is clearly cocontinuous hence we obtain a morphism of topoi as indicated. Let $h_{X'} \in Sh(X_{Zar})$ be the representable sheaf associated to X' viewed as an object of X_{Zar}. It is clear that $Sh(X'_{Zar})$ is the localization $Sh(X_{Zar})/h_{X'}$. On the other hand, the category $\text{Cris}(X/S)/u_{X/S}^{-1}h_{X'}$ (see Sites, Lemma 7.29.3) is canonically identified with $\text{Cris}(X'/S')$ by the functor above. This finishes the proof. □

Remark 45.9.6 (Structure morphism)**.** In Situation 45.7.5. Consider the closed subscheme $S_0 = V(\mathcal{I}) \subset S$. If we assume that p is locally nilpotent on S_0 (which is always the case in practice) then we obtain a situation as in Definition 45.8.1 with S_0 instead of X. Hence we get a site $\text{Cris}(S_0/S)$. If $f : X \to S_0$ is the structure

morphism of X over S, then we get a commutative diagram of ringed topoi

$$\begin{array}{ccccc} (X/S)_{\text{cris}} & \xrightarrow[f_{\text{cris}}]{} & (S_0/S)_{\text{cris}} & & \\ \downarrow{\scriptstyle u_{X/S}} & & \downarrow{\scriptstyle u_{S_0/S}} & & \\ Sh(X_{Zar}) & \xrightarrow{f_{small}} & Sh(S_{0,Zar}) & & \\ & & & \searrow & \\ & & & & Sh(S_{Zar}) \end{array}$$

see Remark 45.9.3. We think of the composition $(X/S)_{\text{cris}} \to Sh(S_{Zar})$ as the structure morphism of the crystalline site. Even if p is not locally nilpotent on S_0 the structure morphism

$$\tau_{X/S} : (X/S)_{\text{cris}} \longrightarrow Sh(S_{Zar})$$

is defined as we can take the lower route through the diagram above.

Remark 45.9.7 (Compatibilities)**.** The morphisms defined above satisfy numerous compatibilities. For example, in the situation of Remark 45.9.3 we obtain a commutative diagram of ringed topoi

$$\begin{array}{ccc} (X/S)_{\text{cris}} & \longrightarrow & (Y/S')_{\text{cris}} \\ \downarrow & & \downarrow \\ Sh((Sch/S)_{Zar}) & \longrightarrow & Sh((Sch/S')_{Zar}) \end{array}$$

where the vertical arrows are the structure morphisms.

45.10. Sheaves on the crystalline site

Notation and assumptions as in Situation 45.7.5. In order to discuss the small and big crystalline sites of X/S simultaneously in this section we let

$$\mathcal{C} = \text{CRIS}(X/S) \quad \text{or} \quad \mathcal{C} = \text{Cris}(X/S).$$

A sheaf $\mathcal{F}$ on $\mathcal{C}$ gives rise to a *restriction* $\mathcal{F}_T$ for every object (U, T, δ) of $\mathcal{C}$. Namely, $\mathcal{F}_T$ is the Zariski sheaf on the scheme T defined by the rule

$$\mathcal{F}_T(W) = \mathcal{F}(U \cap W, W, \delta|_W)$$

for $W \subset T$ is open. Moreover, if $f : T \to T'$ is a morphism between objects (U, T, δ) and (U', T', δ') of $\mathcal{C}$, then there is a canonical *comparison* map

$$c_f : f^{-1}\mathcal{F}_{T'} \longrightarrow \mathcal{F}_T. \tag{45.10.0.1}$$

Namely, if $W' \subset T'$ is open then f induces a morphism

$$f|_{f^{-1}W'} : (U \cap f^{-1}(W'), f^{-1}W', \delta|_{f^{-1}W'}) \longrightarrow (U' \cap W', W', \delta|_{W'})$$

of $\mathcal{C}$, hence we can use the restriction mapping $(f|_{f^{-1}W'})^*$ of $\mathcal{F}$ to define a map $\mathcal{F}_{T'}(W') \to \mathcal{F}_T(f^{-1}W')$. These maps are clearly compatible with further restriction, hence define an f-map from $\mathcal{F}_{T'}$ to $\mathcal{F}_T$ (see Sheaves, Section 6.21 and especially Sheaves, Definition 6.21.7). Thus a map c_f as in (45.10.0.1). Note that if f is an open immersion, then c_f is an isomorphism, because in that case $\mathcal{F}_T$ is just the restriction of $\mathcal{F}_{T'}$ to T.

Conversely, given Zariski sheaves $\mathcal{F}_T$ for every object (U, T, δ) of $\mathcal{C}$ and comparison maps c_f as above which (a) are isomorphisms for open immersions, and (b) satisfy a suitable cocycle condition, we obtain a sheaf on $\mathcal{C}$. This is proved exactly as in Topologies, Lemma 33.3.18.

The *structure sheaf* on $\mathcal{C}$ is the sheaf $\mathcal{O}_{X/S}$ defined by the rule

$$\mathcal{O}_{X/S} : (U, T, \delta) \longmapsto \Gamma(T, \mathcal{O}_T)$$

This is a sheaf by the definition of coverings in $\mathcal{C}$. Suppose that $\mathcal{F}$ is a sheaf of $\mathcal{O}_{X/S}$-modules. In this case the comparison mappings (45.10.0.1) define a comparison map

$$c_f : f^*\mathcal{F}_T \longrightarrow \mathcal{F}_{T'} \tag{45.10.0.2}$$

of $\mathcal{O}_T$-modules.

Another type of example comes by starting with a sheaf $\mathcal{G}$ on $(Sch/X)_{Zar}$ or X_{Zar} (depending on whether $\mathcal{C} = \mathrm{CRIS}(X/S)$ or $\mathcal{C} = \mathrm{Cris}(X/S)$). Then $\underline{\mathcal{G}}$ defined by the rule

$$\underline{\mathcal{G}} : (U, T, \delta) \longmapsto \mathcal{G}(U)$$

is a sheaf on $\mathcal{C}$. In particular, if we take $\mathcal{G} = \mathbf{G}_a = \mathcal{O}_X$, then we obtain

$$\underline{\mathbf{G}_a} : (U, T, \delta) \longmapsto \Gamma(U, \mathcal{O}_U)$$

There is a surjective map of sheaves $\mathcal{O}_{X/S} \to \underline{\mathbf{G}_a}$ defined by the canonical maps $\Gamma(T, \mathcal{O}_T) \to \Gamma(U, \mathcal{O}_U)$ for objects (U, T, δ). The kernel of this map is denoted $\mathcal{J}_{X/S}$, hence a short exact sequence

$$0 \to \mathcal{J}_{X/S} \to \mathcal{O}_{X/S} \to \underline{\mathbf{G}_a} \to 0$$

Note that $\mathcal{J}_{X/S}$ comes equipped with a canonical divided power structure. After all, for each object (U, T, δ) the third component δ *is* a divided power structure on the kernel of $\mathcal{O}_T \to \mathcal{O}_U$. Hence the (big) crystalline topos is a divided power topos.

45.11. Crystals in modules

It turns out that a crystal is a very general gadget. However, the definition may be a bit hard to parse, so we first give the definition in the case of modules on the crystalline sites.

Definition 45.11.1. In Situation 45.7.5. Let $\mathcal{C} = \mathrm{CRIS}(X/S)$ or $\mathcal{C} = \mathrm{Cris}(X/S)$. Let $\mathcal{F}$ be a sheaf of $\mathcal{O}_{X/S}$-modules on $\mathcal{C}$.

1. We say $\mathcal{F}$ is *locally quasi-coherent* if for every object (U, T, δ) of $\mathcal{C}$ the restriction $\mathcal{F}_T$ is a quasi-coherent $\mathcal{O}_T$-module.
2. We say $\mathcal{F}$ is *quasi-coherent* if it is quasi-coherent in the sense of Modules on Sites, Definition 18.23.1.
3. We say $\mathcal{F}$ is a *crystal in $\mathcal{O}_{X/S}$-modules* if all the comparison maps (45.10.0.2) are isomorphisms.

It turns out that we can relate these notions as follows.

Lemma 45.11.2. *With notation $X/S, \mathcal{I}, \gamma, \mathcal{C}, \mathcal{F}$ as in Definition 45.11.1. The following are equivalent*

1. *$\mathcal{F}$ is quasi-coherent, and*
2. *$\mathcal{F}$ is locally quasi-coherent and a crystal in $\mathcal{O}_{X/S}$-modules.*

Proof. Assume (1). Let $f : (U', T', \delta') \to (U, T, \delta)$ be an object of $\mathcal{C}$. We have to prove (a) $\mathcal{F}_T$ is a quasi-coherent $\mathcal{O}_T$-module and (b) $c_f : f^*\mathcal{F}_T \to \mathcal{F}_{T'}$ is an isomorphism. The assumption means that we can find a covering $\{(T_i, U_i, \delta_i) \to (T, U, \delta)\}$ and for each i the restriction of $\mathcal{F}$ to $\mathcal{C}/(T_i, U_i, \delta_i)$ has a global presentation. Since it suffices to prove (a) and (b) Zariski locally, we may replace $f : (T', U', \delta') \to (T, U, \delta)$ by the base change to (T_i, U_i, δ_i) and assume that $\mathcal{F}$ restricted to $\mathcal{C}/(T, U, \delta)$ has a global presentation

$$\bigoplus\nolimits_{j\in J} \mathcal{O}_{X/S}|_{\mathcal{C}/(U,T,\delta)} \longrightarrow \bigoplus\nolimits_{i\in I} \mathcal{O}_{X/S}|_{\mathcal{C}/(U,T,\delta)} \longrightarrow \mathcal{F}|_{\mathcal{C}/(U,T,\delta)} \longrightarrow 0$$

It is clear that this gives a presentation

$$\bigoplus\nolimits_{j\in J} \mathcal{O}_T \longrightarrow \bigoplus\nolimits_{i\in I} \mathcal{O}_T \longrightarrow \mathcal{F}_T \longrightarrow 0$$

and hence (a) holds. Moreover, the presentation restricts to T' to give a similar presentation of $\mathcal{F}_{T'}$, whence (b) holds.

Assume (2). Let (U, T, δ) be an object of $\mathcal{C}$. We have to find a covering of (U, T, δ) such that $\mathcal{F}$ has a global presentation when we restrict to the localization of $\mathcal{C}$ at the members of the covering. Thus we may assume that T is affine. In this case we can choose a presentation

$$\bigoplus\nolimits_{j\in J} \mathcal{O}_T \longrightarrow \bigoplus\nolimits_{i\in I} \mathcal{O}_T \longrightarrow \mathcal{F}_T \longrightarrow 0$$

as $\mathcal{F}_T$ is assumed to be a quasi-coherent $\mathcal{O}_T$-module. Then by the crystal property of $\mathcal{F}$ we see that this pulls back to a presentation of $\mathcal{F}_{T'}$ for any morphism $f : (U', T', \delta') \to (U, T, \delta)$ of $\mathcal{C}$. Thus the desired presentation of $\mathcal{F}|_{\mathcal{C}/(U,T,\delta)}$. □

Definition 45.11.3. If $\mathcal{F}$ satisfies the equivalent conditions of Lemma 45.11.2, then we say that $\mathcal{F}$ is a *crystal in quasi-coherent modules*. We say that $\mathcal{F}$ is a *crystal in finite locally free modules* if, in addition, $\mathcal{F}$ is finite locally free.

Of course, as Lemma 45.11.2 shows, this notation is somewhat heavy since a quasi-coherent module is always a crystal. But it is standard terminology in the literature.

Remark 45.11.4. To formulate the general notion of a crystal we use the language of stacks and strongly cartesian morphisms, see Stacks, Definition 8.4.1 and Categories, Definition 4.31.1. In Situation 45.7.5 let $p : \mathcal{C} \to \mathrm{Cris}(X/S)$ be a stack. A *crystal in objects of $\mathcal{C}$ on X relative to S* is a *cartesian section* $\sigma : \mathrm{Cris}(X/S) \to \mathcal{C}$, i.e., a functor σ such that $p \circ \sigma = \mathrm{id}$ and such that $\sigma(f)$ is strongly cartesian for all morphisms f of $\mathrm{Cris}(X/S)$. Similarly for the big crystalline site.

45.12. Sheaf of differentials

In this section we will stick with the (small) crystalline site as it seems more natural. We globalize Definition 45.6.1 as follows.

Definition 45.12.1. In Situation 45.7.5 let $\mathcal{F}$ be a sheaf of $\mathcal{O}_{X/S}$-modules on $\mathrm{Cris}(X/S)$. An *S-derivation* $D : \mathcal{O}_{X/S} \to \mathcal{F}$ is a map of sheaves such that for every object (U, T, δ) of $\mathrm{Cris}(X/S)$ the map

$$D : \Gamma(T, \mathcal{O}_T) \longrightarrow \Gamma(T, \mathcal{F})$$

is a divided power $\Gamma(V, \mathcal{O}_V)$-derivation where $V \subset S$ is any open such that $T \to S$ factors through V.

This means that D is additive, satisfies the Leibniz rule, annihilates functions coming from S, and satisfies $D(f^{[n]}) = f^{[n-1]}D(f)$ for a local section f of the divided power ideal $\mathcal{J}_{X/S}$. This is a special case of a very general notion which we now describe.

Please compare the following discussion with Modules on Sites, Section 18.32. Let $\mathcal{C}$ be a site, let $\mathcal{A} \to \mathcal{B}$ be a map of sheaves of rings on $\mathcal{C}$, let $\mathcal{J} \subset \mathcal{B}$ be a sheaf of ideals, let δ be a divided power structure on $\mathcal{J}$, and let $\mathcal{F}$ be a sheaf of $\mathcal{B}$-modules. Then there is a notion of a *divided power $\mathcal{A}$-derivation* $D : \mathcal{B} \to \mathcal{F}$. This means that D is $\mathcal{A}$-linear, satisfies the Leibnize rule, and satisfies $D(\delta_n(x)) = \delta_{n-1}(x)D(x)$ for local sections x of $\mathcal{J}$. In this situation there exists a *universal divided power $\mathcal{A}$-derivation*

$$\mathrm{d}_{\mathcal{B}/\mathcal{A},\delta} : \mathcal{B} \longrightarrow \Omega_{\mathcal{B}/\mathcal{A},\delta}$$

Moreover, $\mathrm{d}_{\mathcal{B}/\mathcal{A},\delta}$ is the composition

$$\mathcal{B} \longrightarrow \Omega_{\mathcal{B}/\mathcal{A}} \longrightarrow \Omega_{\mathcal{B}/\mathcal{A},\delta}$$

where the first map is the universal derivation constructed in the proof of Modules on Sites, Lemma 18.32.2 and the second arrow is the quotient by the submodule generated by the local sections $\mathrm{d}_{\mathcal{B}/\mathcal{A}}(\delta_n(x)) - \delta_{n-1}(x)\mathrm{d}_{\mathcal{B}/\mathcal{A}}(x)$.

We translate this into a relative notion as follows. Suppose $(f, f^\sharp) : (Sh(\mathcal{C}), \mathcal{O}) \to (Sh(\mathcal{C}'), \mathcal{O}')$ is a morphism of ringed topoi, $\mathcal{J} \subset \mathcal{O}$ a sheaf of ideals, δ a divided power structure on $\mathcal{J}$, and $\mathcal{F}$ a sheaf of $\mathcal{O}$-modules. In this situation we say $D : \mathcal{O} \to \mathcal{F}$ is a divided power $\mathcal{O}'$-derivation if D is a divided power $f^{-1}\mathcal{O}'$-derivation as defined above. Moreover, we write

$$\Omega_{\mathcal{O}/\mathcal{O}',\delta} = \Omega_{\mathcal{O}/f^{-1}\mathcal{O}',\delta}$$

which is the receptacle of the universal divided power $\mathcal{O}'$-derivation.

Applying this to the structure morphism

$$(X/S)_{\mathrm{Cris}} \longrightarrow Sh(S_{Zar})$$

(see Remark 45.9.6) we recover the notion of Definition 45.12.1 above. In particular, there is a universal divided power derivation

$$d_{X/S} : \mathcal{O}_{X/S} \to \Omega_{X/S}$$

Note that we omit from the notation the decoration indicating the module of differentials is compatible with divided powers (it seems unlikely anybody would ever consider the usual module of differentials of the structure sheaf on the crystalline site).

Lemma 45.12.2. *Let $(T, \mathcal{J}, \delta)$ be a divided power scheme. Let $T \to S$ be a morphism of schemes. The quotient $\Omega_{T/S} \to \Omega_{T/S,\delta}$ described above is a quasi-coherent $\mathcal{O}_T$-module. For $W \subset T$ affine open mapping into $V \subset S$ affine open we have*

$$\Gamma(W, \Omega_{T/S,\delta}) = \Omega_{\Gamma(W,\mathcal{O})/\Gamma(V,\mathcal{O}_V),\delta}$$

where the right hand side is as constructed in Section 45.6.

Proof. Omitted. □

Lemma 45.12.3. *In Situation 45.7.5. For (U, T, δ) in $Cris(X/S)$ the restriction $(\Omega_{X/S})_T$ to T is $\Omega_{T/S,\delta}$ and the restriction $d_{X/S}|_T$ is equal to $d_{T/S,\delta}$.*

Proof. Omitted. □

Lemma 45.12.4. *In Situation 45.7.5. For any affine object (U, T, δ) of Cris(X/S) mapping into an affine open $V \subset S$ we have*

$$\Gamma((U, T, \delta), \Omega_{X/S}) = \Omega_{\Gamma(T,\mathcal{O})/\Gamma(V,\mathcal{O}_V),\delta}$$

where the right hand side is as constructed in Section 45.6.

Proof. Combine Lemmas 45.12.2 and 45.12.3. □

Lemma 45.12.5. *In Situation 45.7.5. Let (U, T, δ) be an object of Cris(X/S). Let*

$$(U(1), T(1), \delta(1)) = (U, T, \delta) \times (U, T, \delta)$$

in Cris(X/S). Let $\mathcal{K} \subset \mathcal{O}_{T(1)}$ be the quasi-coherent sheaf of ideals corresponding to the closed immersion $\Delta : T \to T(1)$. Then $\mathcal{K} \subset \mathcal{J}_{T(1)}$ is preserved by the divided structure on $\mathcal{J}_{T(1)}$ and we have

$$(\Omega_{X/S})_T = \mathcal{K}/\mathcal{K}^{[2]}$$

Proof. Note that $U = U(1)$ as $U \to X$ is an open immersion and as (45.9.1.1) commutes with products. Hence we see that $\mathcal{K} \subset \mathcal{J}_{T(1)}$. Given this fact the lemma follows by working affine locally on T and using Lemmas 45.12.4 and 45.6.5. □

It turns out that $\Omega_{X/S}$ is not a crystal in quasi-coherent $\mathcal{O}_{X/S}$-modules. But it does satisfy two closely related properties (compare with Lemma 45.11.2).

Lemma 45.12.6. *In Situation 45.7.5. The sheaf of differentials $\Omega_{X/S}$ has the following two properties:*

(1) *$\Omega_{X/S}$ is locally quasi-coherent, and*
(2) *for any morphism $(U, T, \delta) \to (U', T', \delta')$ of Cris(X/S) where $f : T \to T'$ is a closed immersion the map $c_f : f^*(\Omega_{X/S})_{T'} \to (\Omega_{X/S})_T$ is surjective.*

Proof. Part (1) follows from a combination of Lemmas 45.12.2 and 45.12.3. Part (2) follows from the fact that $(\Omega_{X/S})_T = \Omega_{T/S,\delta}$ is a quotient of $\Omega_{T/S}$ and that $f^*\Omega_{T'/S} \to \Omega_{T/S}$ is surjective. □

45.13. Two universal thickenings

The constructions in this section will help us define a connection on a crystal in modules on the crystalline site. In some sense the constructions here are the "sheafified, universal" versions of the constructions in Section 45.3.

Remark 45.13.1. In Situation 45.7.5. Let (U, T, δ) be an object of Cris(X/S). Write $\Omega_{T/S,\delta} = (\Omega_{X/S})_T$, see Lemma 45.12.3. We explicitly describe a first order thickening T' of T. Namely, set

$$\mathcal{O}_{T'} = \mathcal{O}_T \oplus \Omega_{T/S,\delta}$$

with algebra structure such that $\Omega_{T/S,\delta}$ is an ideal of square zero. Let $\mathcal{J} \subset \mathcal{O}_T$ be the ideal sheaf of the closed immersion $U \to T$. Set $\mathcal{J}' = \mathcal{J} \oplus \Omega_{T/S,\delta}$. Define a divided power structure on $\mathcal{J}'$ by setting

$$\delta'_n(f, \omega) = (\delta_n(f), \delta_{n-1}(f)\omega),$$

see Lemma 45.3.1. There are two ring maps

$$p_0, p_1 : \mathcal{O}_T \to \mathcal{O}_{T'}$$

The first is given by $f \mapsto (f, 0)$ and the second by $f \mapsto (f, \mathrm{d}_{T/S,\delta} f)$. Note that both are compatible with the divided power structures on $\mathcal{J}$ and $\mathcal{J}'$ and so is the

quotient map $\mathcal{O}_{T'} \to \mathcal{O}_T$. Thus we get an object (U, T', δ') of $\mathrm{Cris}(X/S)$ and a commutative diagram

$$\begin{array}{ccccc} & & T & & \\ & \swarrow^{\mathrm{id}} & \downarrow{\scriptstyle i} & \searrow^{\mathrm{id}} & \\ T & \xleftarrow{p_0} & T' & \xrightarrow{p_1} & T \end{array}$$

of $\mathrm{Cris}(X/S)$ such that i is a first order thickening whose ideal sheaf is identified with $\Omega_{T/S,\delta}$ and such that $p_1^* - p_0^* : \mathcal{O}_T \to \mathcal{O}_{T'}$ is identified with the universal derivation $\mathrm{d}_{T/S,\delta}$ composed with the inclusion $\Omega_{T/S,\delta} \to \mathcal{O}_{T'}$.

Remark 45.13.2. In Situation 45.7.5. Let (U, T, δ) be an object of $\mathrm{Cris}(X/S)$. Write $\Omega_{T/S,\delta} = (\Omega_{X/S})_T$, see Lemma 45.12.3. We also write $\Omega^2_{T/S,\delta}$ for its second exterior power. We explicitly describe a second order thickening T'' of T. Namely, set

$$\mathcal{O}_{T''} = \mathcal{O}_T \oplus \Omega_{T/S,\delta} \oplus \Omega_{T/S,\delta} \oplus \Omega^2_{T/S,\delta}$$

with algebra structure defined in the following way

$$(f, \omega_1, \omega_2, \eta) \cdot (f', \omega_1', \omega_2', \eta') = (ff', f\omega_1' + f'\omega_1, f\omega_2' + f'\omega_2', f\eta' + f'\eta + \omega_1 \wedge \omega_2' + \omega_1' \wedge \omega_2).$$

Let $\mathcal{J} \subset \mathcal{O}_T$ be the ideal sheaf of the closed immersion $U \to T$. Let $\mathcal{J}''$ be the inverse image of $\mathcal{J}$ under the projection $\mathcal{O}_{T''} \to \mathcal{O}_T$. Define a divided power structure on $\mathcal{J}''$ by setting

$$\delta''_n(f, \omega_1, \omega_2, \eta) = (\delta_n(f), \delta_{n-1}(f)\omega_1, \delta_{n-1}(f)\omega_2, \delta_{n-1}(f)\eta + \delta_{n-2}(f)\omega_1 \wedge \omega_2)$$

see Lemma 45.3.2. There are three ring maps $q_0, q_1, q_2 : \mathcal{O}_T \to \mathcal{O}_{T''}$ given by

$$\begin{aligned} q_0(f) &= (f, 0, 0, 0), \\ q_1(f) &= (f, \mathrm{d}f, 0, 0), \\ q_2(f) &= (f, \mathrm{d}f, \mathrm{d}f, 0) \end{aligned}$$

where $\mathrm{d} = \mathrm{d}_{T/S,\delta}$. Note that all three are compatible with the divided power structures on $\mathcal{J}$ and $\mathcal{J}''$. There are three ring maps $q_{01}, q_{12}, q_{02} : \mathcal{O}_{T'} \to \mathcal{O}_{T''}$ where $\mathcal{O}_{T'}$ is as in Remark 45.13.1. Namely, set

$$\begin{aligned} q_{01}(f, \omega) &= (f, \omega, 0, 0), \\ q_{12}(f, \omega) &= (f, \mathrm{d}f, \omega, \mathrm{d}\omega), \\ q_{02}(f, \omega) &= (f, \omega, \omega, 0) \end{aligned}$$

These are also compatible with the given divided power structures. Let's do the verifications for q_{12}: Note that q_{12} is a ring homomorphism as

$$\begin{aligned} q_{12}(f, \omega) q_{12}(g, \eta) &= (f, \mathrm{d}f, \omega, \mathrm{d}\omega)(g, \mathrm{d}g, \eta, \mathrm{d}\eta) \\ &= (fg, f\mathrm{d}g + g\mathrm{d}f, f\eta + g\omega, f\mathrm{d}\eta + g\mathrm{d}\omega + \mathrm{d}f \wedge \eta + \mathrm{d}g \wedge \omega) \\ &= q_{12}(fg, f\eta + g\omega) = q_{12}((f, \omega)(g, \eta)) \end{aligned}$$

Note that q_{12} is compatible with divided powers because

$$\begin{aligned} \delta''_n(q_{12}(f, \omega)) &= \delta''_n((f, \mathrm{d}f, \omega, \mathrm{d}\omega)) \\ &= (\delta_n(f), \delta_{n-1}(f)\mathrm{d}f, \delta_{n-1}(f)\omega, \delta_{n-1}(f)\mathrm{d}\omega + \delta_{n-2}(f)\mathrm{d}(f) \wedge \omega) \\ &= q_{12}((\delta_n(f), \delta_{n-1}(f)\omega)) = q_{12}(\delta'_n(f, \omega)) \end{aligned}$$

The verifications for q_{01} and q_{02} are easier. Note that $q_0 = q_{01} \circ p_0$, $q_1 = q_{01} \circ p_1$, $q_1 = q_{12} \circ p_0$, $q_2 = q_{12} \circ p_1$, $q_0 = q_{02} \circ p_0$, and $q_2 = q_{02} \circ p_1$. Thus (U, T'', δ'') is an object of $\mathrm{Cris}(X/S)$ and we get morphisms

$$T'' \begin{array}{c} \longrightarrow \\ \longrightarrow \\ \longrightarrow \end{array} T' \begin{array}{c} \longrightarrow \\ \longrightarrow \end{array} T$$

of $\mathrm{Cris}(X/S)$ satisfying the relations described above. In applications we will use $q_i : T'' \to T$ and $q_{ij} : T'' \to T'$ to denote the morphisms associated to the ring maps described above.

45.14. The de Rham complex

In Situation 45.7.5. Working on the (small) crystalline site, we define $\Omega^i_{X/S} = \wedge^i_{\mathcal{O}_{X/S}} \Omega_{X/S}$ for $i \geq 0$. The universal S-derivation $\mathrm{d}_{X/S}$ gives rise to the *de Rham complex*

$$\mathcal{O}_{X/S} \to \Omega^1_{X/S} \to \Omega^2_{X/S} \to \ldots$$

on $\mathrm{Cris}(X/S)$, see Lemma 45.12.4 and Remark 45.6.9.

45.15. Connections

In Situation 45.7.5. Given an $\mathcal{O}_{X/S}$-module $\mathcal{F}$ on $\mathrm{Cris}(X/S)$ a *connection* is a map of abelian sheaves

$$\nabla : \mathcal{F} \longrightarrow \mathcal{F} \otimes_{\mathcal{O}_{X/S}} \Omega_{X/S}$$

such that $\nabla(fs) = f\nabla(s) + s \otimes \mathrm{d}f$ for local sections s, f of $\mathcal{F}$ and $\mathcal{O}_{X/S}$. Given a connection there are canonical maps $\nabla : \mathcal{F} \otimes_{\mathcal{O}_{X/S}} \Omega^i_{X/S} \longrightarrow \mathcal{F} \otimes_{\mathcal{O}_{X/S}} \Omega^{i+1}_{X/S}$ defined by the rule $\nabla(s \otimes \omega) = \nabla(s) \wedge \omega + s \otimes \mathrm{d}\omega$ as in Remark 45.6.10. We say the connection is *integrable* if $\nabla \circ \nabla = 0$. If ∇ is integrable we obtain the *de Rham complex*

$$\mathcal{F} \to \mathcal{F} \otimes_{\mathcal{O}_{X/S}} \Omega^1_{X/S} \to \mathcal{F} \otimes_{\mathcal{O}_{X/S}} \Omega^2_{X/S} \to \ldots$$

on $\mathrm{Cris}(X/S)$. It turns out that any crystal in $\mathcal{O}_{X/S}$-modules comes equipped with a canonical integrable connection.

Lemma 45.15.1. *In Situation 45.7.5. Let $\mathcal{F}$ be a crystal in $\mathcal{O}_{X/S}$-modules on $\mathrm{Cris}(X/S)$. Then $\mathcal{F}$ comes equipped with a canonical integrable connection.*

Proof. Say (U, T, δ) is an object of $\mathrm{Cris}(X/S)$. Let (U, T', δ') be the infinitesimal thickening of T by $(\Omega_{X/S})_T = \Omega_{T/S,\delta}$ constructed in Remark 45.13.1. It comes with projections $p_0, p_1 : T' \to T$ and a diagonal $i : T \to T(1)$. By assumption we get isomorphisms

$$p_0^*\mathcal{F}_T \xrightarrow{c_0} \mathcal{F}_{T'} \xleftarrow{c_1} p_1^*\mathcal{F}_T$$

of $\mathcal{O}_{T'}$-modules. Pulling $c = c_1^{-1} \circ c_0$ back to T by i we obtain the identity map of $\mathcal{F}_T$. Hence if $s \in \Gamma(T, \mathcal{F}_T)$ then $\nabla(s) = p_1^*s - c(p_0^*s)$ is a section of $p_1^*\mathcal{F}_T$ which vanishes on pulling back by Δ. Hence $\nabla(s)$ is a section of

$$\mathcal{F}_T \otimes_{\mathcal{O}_T} \Omega_{T/S,\delta}$$

because this is the kernel of $p_1^*\mathcal{F}_T \to \mathcal{F}_T$ as $\Omega_{T/S,\delta}$ is the kernel of $\mathcal{O}_{T'} \to \mathcal{O}_T$ by construction.

The collection of maps

$$\nabla : \Gamma(T, \mathcal{F}_T) \to \Gamma(T, \mathcal{F}_T \otimes_{\mathcal{O}_T} \Omega_{T/S,\delta})$$

so obtained is functorial in T because the construction of T' is functorial in T. Hence we obtain a connection.

To show that the connection is integrable we consider the object (U, T'', δ'') constructed in Remark 45.13.2. Because $\mathcal{F}$ is a sheaf we see that

$$\begin{array}{ccc} q_0^*\mathcal{F}_T & \xrightarrow[q_{01}^*c]{} & q_1^*\mathcal{F}_T \\ & {\scriptstyle q_{02}^*c} \searrow \quad \swarrow {\scriptstyle q_{12}^*c} & \\ & q_2^*\mathcal{F}_T & \end{array}$$

is a commutative map of $\mathcal{O}_{T''}$-modules. For $s \in \Gamma(T, \mathcal{F}_T)$ we have $c(p_0^*s) = p_1^*s - \nabla(s)$. Write $\nabla(s) = \sum p_1^*s_i \cdot \omega_i$ where s_i is a local section of $\mathcal{F}_T$ and ω_i is a local section of $\Omega_{T/S,\delta}$. We think of ω_i as a local section of the structure sheaf of $\mathcal{O}_{T'}$ and hence we write product instead of tensor product. On the one hand

$$\begin{aligned} q_{12}^*c \circ q_{01}^*c(q_0^*s) &= q_{12}^*c(q_1^*s - \sum q_1^*s_i \cdot q_{01}^*\omega_i) \\ &= q_2^*s - \sum q_2^*s_i \cdot q_{12}^*\omega_i - \sum q_2^*s_i \cdot q_{01}^*\omega_i + \sum q_{12}^*\nabla(s_i) \cdot q_{01}^*\omega_i \end{aligned}$$

and on the other hand

$$q_{02}^*c(q_0^*s) = q_2^*s - \sum q_2^*s_i \cdot q_{02}^*\omega_i.$$

From the formulae of Remark 45.13.2 we see that $q_{01}^*\omega_i + q_{12}^*\omega_i - q_{02}^*\omega_i = \mathrm{d}\omega_i$. Hence the difference of the two expressions above is

$$\sum q_2^*s_i \cdot \mathrm{d}\omega_i - \sum q_{12}^*\nabla(s_i) \cdot q_{01}^*\omega_i$$

Note that $q_{12}^*\omega \cdot q_{01}^*\omega' = \omega' \wedge \omega = -\omega \wedge \omega'$ by the definition of the multiplication on $\mathcal{O}_{T''}$. Thus the expression above is $\nabla^2(s)$ viewed as a section of the subsheaf $\mathcal{F}_T \otimes \Omega^2_{T/S,\delta}$ of $q_2^*\mathcal{F}$. Hence we get the integrability condition. □

45.16. Cosimplicial algebra

This section should be moved somewhere else. A *cosimplicial ring* is a cosimplicial object in the category of rings. Given a ring R, a *cosimplicial R-algebra* is a cosimplicial object in the category of R-algebras. A *cosimplicial ideal* in a cosimplicial ring A_* is given by an ideal $I_n \subset A_n$ for all n such that $A(f)(I_n) \subset I_m$ for all $f : [n] \to [m]$ in Δ.

Let A_* be a cosimplicial ring. Let $\mathcal{C}$ be the category of pairs (A, M) where A is a ring and M is a module over A. A morphism $(A, M) \to (A', M')$ consists of a ring map $A \to A'$ and an A-module map $M \to M'$ where M' is viewed as an A-module via $A \to A'$ and the A'-module structure on M'. Having said this we can define a *cosimplicial module M_* over A_** as a cosimplicial object (A_*, M_*) of $\mathcal{C}$ whose first entry is equal to A_*. A *homomorphism $\varphi_* : M_* \to N_*$ of cosimplicial modules over A_** is a morphism $(A_*, M_*) \to (A_*, N_*)$ of cosimplicial objects in $\mathcal{C}$ whose first component is 1_{A_*}.

A *homotopy* between homomorphisms $\varphi_*, \psi_* : M_* \to N_*$ of cosimplicial modules over A_* is a homotopy between the associated maps $(A_*, M_*) \to (A_*, N_*)$ whose first component is the trivial homotopy (dual to Simplicial, Example 14.24.3). We spell out what this means. Such a homotopy is a homotopy

$$h : M_* \longrightarrow \mathrm{Hom}(\Delta[1], N_*)$$

between φ_* and ψ_* as homomorphisms of cosimplicial abelian groups such that for each n the map $h_n : M_n \to \prod_{\alpha \in \Delta[1]_n} N_n$ is A_n-linear. The following lemma is a version of Simplicial, Lemma 14.26.3 for cosimplicial modules.

Lemma 45.16.1. *Let A_* be a cosimplicial ring. Let $\varphi_*, \psi_* : K_* \to M_*$ be homomorphisms of cosimplicial A_*-modules.*

(1) *If φ_* and ψ_* are homotopic, then*
$$\varphi_* \otimes 1, \psi_* \otimes 1 : K_* \otimes_{A_*} L_* \longrightarrow M_* \otimes_{A_*} L_*$$
are homotopic for any cosimplicial A_*-module L_*.

(2) *If φ_* and ψ_* are homotopic, then*
$$\wedge^i(\varphi_*), \wedge^i(\psi_*) : \wedge^i(K_*) \longrightarrow \wedge^i(M_*)$$
are homotopic.

(3) *If φ_* and ψ_* are homotopic, and $A_* \to B_*$ is a homomorphism of cosimplicial rings, then*
$$\varphi_* \otimes 1, \psi_* \otimes 1 : K_* \otimes_{A_*} B_* \longrightarrow M_* \otimes_{A_*} B_*$$
are homotopic as homomorphisms of cosimplicial B_*-modules.

(4) *If $I_* \subset A_*$ is a cosimplicial ideal, then the induced maps*
$$\varphi_*^\wedge, \psi_*^\wedge : K_*^\wedge \longrightarrow M_*^\wedge$$
between completions are homotopic.

(5) *Add more here as needed, for example symmetric powers.*

Proof. Let $h : M_* \longrightarrow \mathrm{Hom}(\Delta[1], N_*)$ be the given homotopy. In degree n we have
$$h_n = (h_{n,\alpha}) : K_n \longrightarrow \prod\nolimits_{\alpha \in \Delta[1]_n} K_n$$
see Simplicial, Section 14.26. In order for a collection of $h_{n,\alpha}$ to form a homotopy, it is necessary and sufficient if for every $f : [n] \to [m]$ we have
$$h_{m,\alpha} \circ M_*(f) = N_*(f) \circ h_{n,\alpha \circ f}$$
see Simplicial, Equation (14.26.1.1). We also should have that $\psi_n = h_{n,0:[n]\to[1]}$ and $\varphi_n = h_{n,1:[n]\to[1]}$.

In each of the cases of the lemma we can produce the corresponding maps. Case (1). We can use the homotopy $h \otimes 1$ defined in degree n by setting
$$(h \otimes 1)_{n,\alpha} = h_{n,\alpha} \otimes 1_{L_n} : K_n \otimes_{A_n} L_n \longrightarrow M_n \otimes_{A_n} L_n.$$
Case (2). We can use the homotopy $\wedge^i h$ defined in degree n by setting
$$\wedge^i(h)_{n,\alpha} = \wedge^i(h_{n,\alpha}) : \wedge_{A_n}(K_n) \longrightarrow \wedge^i_{A_n}(M_n).$$
Case (3). We can use the homotopy $h \otimes 1$ defined in degree n by setting
$$(h \otimes 1)_{n,\alpha} = h_{n,\alpha} \otimes 1 : K_n \otimes_{A_n} B_n \longrightarrow M_n \otimes_{A_n} B_n.$$
Case (4). We can use the homotopy $h^\wedge$ defined in degree n by setting
$$(h^\wedge)_{n,\alpha} = h^\wedge_{n,\alpha} : K_n^\wedge \longrightarrow M_n^\wedge.$$
This works because each $h_{n,\alpha}$ is A_n-linear. $\square$

45.17. Crystals in quasi-coherent modules

In Situation 45.5.1. Set $X = \mathrm{Spec}(C)$ and $S = \mathrm{Spec}(A)$. We are going to classify crystals in quasi-coherent modules on $\mathrm{Cris}(X/S)$. Before we do so we fix some notation.

Choose a polynomial ring $P = A[x_i]$ over A and a surjection $P \to C$ of A-algebras with kernel $J = \mathrm{Ker}(P \to C)$. Set

$$D = \lim_e D_{P,\gamma}(J)/p^e D_{P,\gamma}(J) \tag{45.17.0.1}$$

for the p-adically completed divided power envelope. This ring comes with a divided power ideal $\bar{J}$ and divided power structure $\bar{\gamma}$, see Lemma 45.5.5. Set $D_e = D/p^e D$ and denote $\bar{J}_e$ the image of $\bar{J}$ in D_e. We will use the short hand

$$\Omega_D = \lim_e \Omega_{D_e/A,\bar{\gamma}} = \lim_e \Omega_{D/A,\bar{\gamma}}/p^e\Omega_{D/A,\bar{\gamma}} \tag{45.17.0.2}$$

for the p-adic completion of the module of divided power differentials, see Lemma 45.6.12. It is also the p-adic completion of $\Omega_{D_{P,\gamma}(J)/A,\bar{\gamma}}$ which is free on $\mathrm{d}x_i$, see Lemma 45.6.6. Hence any element of Ω_D can be written uniquely as a sum $\sum f_i \mathrm{d}x_i$ with for all e only finitely many f_i not in $p^e D$. Moreover, the maps $\mathrm{d}_{D_e/A,\bar{\gamma}} : D_e \to \Omega_{D_e/A,\bar{\gamma}}$ fit together to define a divided power A-derivation

$$\mathrm{d} : D \longrightarrow \Omega_D \tag{45.17.0.3}$$

on p-adic completions.

We will also need the "products $\mathrm{Spec}(D(n))$ of $\mathrm{Spec}(D)$", see Proposition 45.21.1 and its proof for an explanation. Formally these are defined as follows. For $n \geq 0$ let $J(n) = \mathrm{Ker}(P \otimes_A \ldots \otimes_A P \to C)$ where the tensor product has $n+1$ factors. We set

$$D(n) = \lim_e D_{P\otimes_A\ldots\otimes_A P,\gamma}(J(n))/p^e D_{P\otimes_A\ldots\otimes_A P,\gamma}(J(n)) \tag{45.17.0.4}$$

equal to the p-adic completion of the divided power envelope. We denote $\bar{J}(n)$ its divided power ideal and $\bar{\gamma}(n)$ its divided powers. We also introduce $D(n)_e = D(n)/p^e D(n)$ as well as the p-adically completed module of differentials

$$\Omega_{D(n)} = \lim_e \Omega_{D(n)_e/A,\bar{\gamma}} = \lim_e \Omega_{D(n)/A,\bar{\gamma}}/p^e\Omega_{D(n)/A,\bar{\gamma}} \tag{45.17.0.5}$$

and derivation

$$\mathrm{d} : D(n) \longrightarrow \Omega_{D(n)} \tag{45.17.0.6}$$

Of course we have $D = D(0)$. Note that the rings $D(0), D(1), D(2), \ldots$ form a cosimplicial object in the category of divided power rings.

Lemma 45.17.1. *Let D and $D(n)$ be as in (45.17.0.1) and (45.17.0.4). The coprojection $P \to P \otimes_A \ldots \otimes_A P$, $f \mapsto f \otimes 1 \otimes \ldots \otimes 1$ induces an isomorphism*

$$D(n) = \lim_e D\langle \xi_i(j) \rangle / p^e D\langle \xi_i(j) \rangle \tag{45.17.1.1}$$

of algebras over D with

$$\xi_i(j) = x_i \otimes 1 \otimes \ldots \otimes 1 - 1 \otimes \ldots \otimes 1 \otimes x_i \otimes 1 \otimes \ldots \otimes 1$$

for $j = 1, \ldots, n$.

Proof. We have

$$P \otimes_A \ldots \otimes_A P = P[\xi_i(j)]$$

and $J(n)$ is generated by J and the elements $\xi_i(j)$. Hence the lemma follows from Lemma 45.2.5. □

Lemma 45.17.2. *Let D and $D(n)$ be as in (45.17.0.1) and (45.17.0.4). Then $(D, \bar{J}, \bar{\gamma})$ and $(D(n), \bar{J}(n), \bar{\gamma}(n))$ are objects of $Cris^\wedge(C/A)$, see Remark 45.5.4, and*

$$D(n) = \coprod\nolimits_{j=0,\ldots,n} D$$

in $Cris^\wedge(C/A)$.

Proof. The first assertion is clear. For the second, if $(B \to C, \delta)$ is an object of $\mathrm{Cris}^\wedge(C/A)$, then we have

$$\mathrm{Mor}_{\mathrm{Cris}^\wedge(C/A)}(D, B) = \mathrm{Hom}_A((P, J), (B, \mathrm{Ker}(B \to C)))$$

and similarly for $D(n)$ replacing (P, J) by $(P \otimes_A \ldots \otimes_A P, J(n))$. The property on coproducts follows as $P \otimes_A \ldots \otimes_A P$ is a coproduct. □

In the lemma below we will consider pairs (M, ∇) satisfying the following conditions

(1) M is a p-adically complete D-module,
(2) $\nabla : M \to M \otimes^\wedge_D \Omega_D$ is a connection, i.e., $\nabla(fm) = m \otimes \mathrm{d}f + f\nabla(m)$,
(3) ∇ is integrable (see Remark 45.6.10), and
(4) ∇ is *topologically quasi-nilpotent*: If we write $\nabla(m) = \sum \theta_i(m)\mathrm{d}x_i$ for some operators $\theta_i : M \to M$, then for any $m \in M$ there are only finitely many pairs (i, k) such that $\theta_i^k(m) \notin pM$.

The operators θ_i are sometimes denoted $\nabla_{\partial/\partial x_i}$ in the literature. In the following lemma we construct a functor from crystals in quasi-coherent modules on $\mathrm{Cris}(X/S)$ to the category of such pairs. We will show this functor is an equivalent in Proposition 45.17.4.

Lemma 45.17.3. *In the situation above there is a functor*

$$\begin{array}{c}\text{crystals in quasi-coherent} \\ \mathcal{O}_{X/S}\text{-modules on } Cris(X/S)\end{array} \longrightarrow \begin{array}{c}\text{pairs } (M, \nabla) \text{ satisfying} \\ \text{(1), (2), (3), and (4)}\end{array}$$

Proof. Let $\mathcal{F}$ be a crystal in quasi-coherent modules on X/S. Set $T_e = \mathrm{Spec}(D_e)$ so that $(X, T_e, \bar{\gamma})$ is an object of $\mathrm{Cris}(X/S)$ for $e \gg 0$. We have morphisms

$$(X, T_e, \bar{\gamma}) \to (X, T_{e+1}, \bar{\gamma}) \to \ldots$$

which are closed immersions. We set

$$M = \lim_e \Gamma((X, T_e, \bar{\gamma}), \mathcal{F}) = \lim_e \Gamma(T_e, \mathcal{F}_{T_e}) = \lim_e M_e$$

Note that since $\mathcal{F}$ is locally quasi-coherent we have $\mathcal{F}_{T_e} = \widetilde{M_e}$. Since $\mathcal{F}$ is a crystal we have $M_e = M_{e+1}/p^e M_{e+1}$. Hence we see that $M_e = M/p^e M$ and that M is p-adically complete.

By Lemma 45.15.1 we know that $\mathcal{F}$ comes endowed with a canonical integrable connection $\nabla : \mathcal{F} \to \mathcal{F} \otimes \Omega_{X/S}$. If we evaluate this connection on the objects T_e constructed above we obtain a canonical integrable connection

$$\nabla : M \longrightarrow M \otimes^\wedge_D \Omega_D$$

To see that this is topologically nilpotent we work out what this means.

Now we can do the same procedure for the rings $D(n)$. This produces a p-adically complete $D(n)$-module $M(n)$. Again using the crystal property of $\mathcal{F}$ we obtain isomorphisms

$$M \otimes^\wedge_{D,p_0} D(1) \to M(1) \leftarrow M \otimes^\wedge_{D,p_1} D(1)$$

compare with the proof of Lemma 45.15.1. Denote c the composition from left to right. Pick $m \in M$. Write $\xi_i = x_i \otimes 1 - 1 \otimes x_i$. Using (45.17.1.1) we can write uniquely

$$c(m \otimes 1) = \sum\nolimits_K \theta_K(m) \otimes \prod \xi_i^{[k_i]}$$

for some $\theta_K(m) \in M$ where the sum is over multi-indices $K = (k_i)$ with $k_i \geq 0$ and $\sum k_i < \infty$. Set $\theta_i = \theta_K$ where K has a 1 in the ith spot and zeros elsewhere. We have

$$\nabla(m) = \sum \theta_i(m) \mathrm{d}x_i.$$

as can be seen by comparing with the definition of ∇. Namely, the defining equation is $p_1^* m = \nabla(m) - c(p_0^* m)$ in Lemma 45.15.1 but the sign works out because in the stacks project we consistently use $\mathrm{d}f = p_1(f) - p_0(f)$ modulo the ideal of the diagonal squared, and hence $\xi_i = x_i \otimes 1 - 1 \otimes x_i$ maps to $-\mathrm{d}x_i$ modulo the ideal of the diagonal squared.

Denote $q_i : D \to D(2)$ and $q_{ij} : D(1) \to D(2)$ the coprojections corresponding to the indices i, j. As in the last paragraph of the proof of Lemma 45.15.1 we see that

$$q_{02}^* c = q_{12}^* c \circ q_{01}^* c.$$

This means that

$$\sum\nolimits_{K''} \theta_{K''}(m) \otimes \prod \zeta_i''^{[k_i'']} = \sum\nolimits_{K', K} \theta_{K'}(\theta_K(m)) \otimes \prod \zeta_i'^{[k_i']} \prod \zeta_i^{[k_i]}$$

in $M \otimes^\wedge_{D,q_2} D(2)$ where

$$\begin{aligned} \zeta_i &= x_i \otimes 1 \otimes 1 - 1 \otimes x_i \otimes 1, \\ \zeta_i' &= 1 \otimes x_i \otimes 1 - 1 \otimes 1 \otimes x_i, \\ \zeta_i'' &= x_i \otimes 1 \otimes 1 - 1 \otimes 1 \otimes x_i. \end{aligned}$$

In particular $\zeta_i'' = \zeta_i + \zeta_i'$ and we have that $D(2)$ is the p-adic completion of the divided power polynomial ring in ζ_i, ζ_i' over $q_2(D)$, see Lemma 45.17.1. Comparing coefficients in the expression above it follows immediately that $\theta_i \circ \theta_j = \theta_j \circ \theta_i$ (this provides an alternative proof of the integrability of ∇) and that

$$\theta_K(m) = (\prod \theta_i^{k_i})(m).$$

In particular, as the sum expressing $c(m \otimes 1)$ above has to converge p-adically we conclude that for each i and each $m \in M$ only a finite number of $\theta_i^k(m)$ are allowed to be nonzero modulo p. □

Proposition 45.17.4. *The functor*

$$\begin{array}{c} \text{crystals in quasi-coherent} \\ \mathcal{O}_{X/S}\text{-modules on } \mathit{Cris}(X/S) \end{array} \longrightarrow \begin{array}{c} \text{pairs } (M, \nabla) \text{ satisfying} \\ \text{(1), (2), (3), and (4)} \end{array}$$

of Lemma 45.17.3 is an equivalence of categories.

Proof. Let (M, ∇) be given. We are going to construct a crystal in quasi-coherent modules $\mathcal{F}$. Write $\nabla(m) = \sum \theta_i(m) \mathrm{d}x_i$. Then $\theta_i \circ \theta_j = \theta_j \circ \theta_i$ and we can set $\theta_K(m) = (\prod \theta_i^{k_i})(m)$ for any multi-index $K = (k_i)$ with $k_i \geq 0$ and $\sum k_i < \infty$.

Let (U, T, δ) be any object of $\mathrm{Cris}(X/S)$ with T affine. Say $T = \mathrm{Spec}(B)$ and the ideal of $U \to T$ is $J_B \subset B$. By Lemma 45.5.6 there exists an integer e and a morphism

$$f : (U, T, \delta) \longrightarrow (X, T_e, \bar{\gamma})$$

where $T_e = \mathrm{Spec}(D_e)$ as in the proof of Lemma 45.17.3. Choose such an e and f; denote $f : D \to B$ also the corresponding divided power A-algebra map. We will set $\mathcal{F}_T$ equal to the quasi-coherent sheaf of $\mathcal{O}_T$-modules associated to the B-module

$$M \otimes_{D,f} B.$$

However, we have to show that this is independent of the choice of f. Suppose that $g : D \to B$ is a second such morphism. Since f and g are morphisms in $\mathrm{Cris}(X/S)$ we see that the image of $f - g : D \to B$ is contained in the divided power ideal J_B. Write $\xi_i = f(x_i) - g(x_i) \in J_B$. By analogy with the proof of Lemma 45.17.3 we define an isomorphism

$$c_{f,g} : M \otimes_{D,f} B \longrightarrow M \otimes_{D,g} B$$

by the formula

$$m \otimes 1 \longmapsto \sum\nolimits_K \theta_K(m) \otimes \prod \xi_i^{[k_i]}$$

which makes sense by our remarks above and the fact that ∇ is topologically quasi-nilpotent (so the sum is finite!). A computation shows that

$$c_{g,h} \circ c_{f,g} = c_{f,h}$$

if given a third morphism $h : (U, T, \delta) \longrightarrow (X, T_e, \bar{\gamma})$. It is also true that $c_{f,f} = 1$. Hence these maps are all isomorphisms and we see that the module $\mathcal{F}_T$ is independent of the choice of f.

If $a : (U', T', \delta') \to (U, T, \delta)$ is a morphism of affine objects of $\mathrm{Cris}(X/S)$, then choosing $f' = f \circ a$ it is clear that there exists a canonical isomorphism $a^*\mathcal{F}_T \to \mathcal{F}_{T'}$. We omit the verification that this map is independent of the choice of f. Using these maps as the restriction maps it is clear that we obtain a crystal in quasi-coherent modules on the full subcategory of $\mathrm{Cris}(X/S)$ consisting of affine objects. We omit the proof that this extends to a crystal on all of $\mathrm{Cris}(X/S)$. We also omit the proof that this procedure is a functor and that it is quasi-inverse to the functor constructed in Lemma 45.17.3. □

Lemma 45.17.5. *In Situation 45.5.1. Let $A \to P' \to C$ be ring maps with $A \to P'$ smooth and $P' \to C$ surjective with kernel J'. Let D' be the p-adic completion of $D_{P',\gamma}(J')$. There are homomorphisms of divided power A-algebras*

$$a : D \longrightarrow D', \quad b : D' \longrightarrow D$$

compatible with the maps $D \to C$ and $D' \to C$ such that $a \circ b = id_{D'}$. These maps induce an equivalence of categories of pairs (M, ∇) satisfying (1), (2), (3), and (4) over D and pairs (M', ∇') satisfying (1), (2), (3), and (4) over D'. In particular, the equivalence of categories of Proposition 45.17.4 also holds for the corresponding functor towards pairs over D'.

Proof. We can pick the map $P = A[x_i] \to C$ such that it factors through a surjection of A-algebras $P \to P'$ (we may have to increase the number of variables in P to do this). Hence we obtain a surjective map $a : D \to D'$ by functoriality of divided power envelopes and completion. Pick e large enough so that D_e is a divided power thickening of C over A. Then $D_e \to C$ is a surjection whose kernel is locally nilpotent, see Divided Power Algebra, Lemma 23.2.6. Setting $D'_e = D'/p^e D'$ we see that the kernel of $D_e \to D'_e$ is locally nilpotent. Hence by Algebra, Lemma 10.131.16 we can find a lift $\beta_e : P' \to D_e$ of the map $P' \to D'_e$. Note that $D_{e+i+1} \to D_{e+i} \times_{D'_{e+i}} D'_{e+i+1}$ is surjective with square zero kernel for any

$i \geq 0$ because $p^{e+i}D \to p^{e+i}D'$ is surjective. Applying the usual lifting property (Algebra, Proposition 10.131.13) successively to the diagrams

$$\begin{array}{ccc} P' & \longrightarrow & D_{e+i} \times_{D'_{e+i}} D'_{e+i+1} \\ \uparrow & & \uparrow \\ A & \longrightarrow & D_{e+i+1} \end{array}$$

we see that we can find an A-algebra map $\beta : P' \to D$ whose composition with a is the given map $P' \to D'$. By the universal property of the divided power envelope we obtain a map $D_{P',\gamma}(J') \to D$. As D is p-adically complete we obtain $b : D' \to D$ such that $a \circ b = \mathrm{id}_{D'}$.

Consider the base change functor

$$(M, \nabla) \longmapsto (M \otimes^{\wedge}_{D} D', \nabla')$$

from pairs for D to pairs for D', see Remark 45.6.11. Similarly, we have the base change functor corresponding to the divided power homomorphism $D' \to D$. To finish the proof of the lemma we have to show that the base change for the compositions $b \circ a : D \to D$ and $a \circ b : D' \to D'$ are isomorphic to the identity functor. This is clear for the second as $a \circ b = \mathrm{id}_{D'}$. To prove it for the first, we use the functorial isomorphism

$$c_{\mathrm{id}_D, boa} : M \otimes_{D,\mathrm{id}_D} D \longrightarrow M \otimes_{D,boa} D$$

of the proof of Proposition 45.17.4. The only thing to prove is that these maps are horizontal, which we omit.

The last statement of the proof now follows. □

Remark 45.17.6. The equivalence of Proposition 45.17.4 holds if we start with a surjection $P \to C$ where P/A satisfies the strong lifting property of Algebra, Lemma 10.131.16. To prove this we can argue as in the proof of Lemma 45.17.5. (Details will be added here if we ever need this.) Presumably there is also a direct proof of this result, but the advantage of using polynomial rings is that the rings $D(n)$ are p-adic completions of divided power polynomial rings and the algebra is simplified.

45.18. General remarks on cohomology

In this section we do a bit of work to translate the cohomology of modules on the cristalline site of an affine scheme into an algebraic question.

Lemma 45.18.1. *In Situation 45.7.5. Let $\mathcal{F}$ be a locally quasi-coherent $\mathcal{O}_{X/S}$-module on Cris(X/S). Then we have*

$$H^p((U, T, \delta), \mathcal{F}) = 0$$

for all $p > 0$ and all (U, T, δ) with T or U affine.

Proof. As $U \to T$ is a thickening we see that U is affine if and only if T is affine, see Limits, Lemma 31.10.1. Having said this, let us apply Cohomology on Sites, Lemma 21.11.8 to the collection $\mathcal{B}$ of affine objects (U, T, δ) and the collection Cov of affine open coverings $\mathcal{U} = \{(U_i, T_i, \delta_i) \to (U, T, \delta)\}$. The Čech complex $\check{C}^*(\mathcal{U}, \mathcal{F})$ for such a covering is simply the Cech complex of the quasi-coherent $\mathcal{O}_T$-module $\mathcal{F}_T$ (here

we are using the assumption that $\mathcal{F}$ is locally quasi-coherent) with respect to the affine open covering $\{T_i \to T\}$ of the affine scheme T. Hence the Čech cohomology is zero by Cohomology of Schemes, Lemma 29.2.5 and 29.2.2. Thus the hypothesis of Cohomology on Sites, Lemma 21.11.8 are satisfied and we win. □

Lemma 45.18.2. *In Situation 45.7.5. Assume moreover X and S are affine schemes. Consider the full subcategory $\mathcal{C} \subset Cris(X/S)$ consisting of divided power thickenings (X, T, δ) endowed with the chaotic topology (see Sites, Example 7.6.6). For any locally quasi-coherent $\mathcal{O}_{X/S}$-module $\mathcal{F}$ we have*

$$R\Gamma(\mathcal{C}, \mathcal{F}|_{\mathcal{C}}) = R\Gamma(Cris(X/S), \mathcal{F})$$

Proof. We will use without further mention that $\mathcal{C}$ and $\mathrm{Cris}(X/S)$ have products and fibre products, see Lemma 45.8.2. Note that the inclusion functor $u : \mathcal{C} \to \mathrm{Cris}(X/S)$ is fully faithful, continuous and commutes with products and fibre products. We claim it defines a morphism of ringed sites

$$f : (\mathrm{Cris}(X/S), \mathcal{O}_{X/S}) \longrightarrow (Sh(\mathcal{C}), \mathcal{O}_{X/S}|_{\mathcal{C}})$$

To see this we will use Sites, Lemma 7.15.5. Note that $\mathcal{C}$ has fibre products and u commutes with them so the categories $\mathcal{I}^u_{(U,T,\delta)}$ are disjoint unions of directed categories (by Sites, Lemma 7.5.1 and Categories, Lemma 4.19.7). Hence it suffices to show that $\mathcal{I}^u_{(U,T,\delta)}$ is connected. Nonempty follows from Lemma 45.5.6 and connectedness follows from the fact that $\mathcal{C}$ has products and that u commutes with them (compare with the proof of Sites, Lemma 7.5.2).

Note that $f_*\mathcal{F} = \mathcal{F}|_{\mathcal{C}}$. Hence the lemma follows if $R^p f_*\mathcal{F} = 0$ for $p > 0$, see Cohomology on Sites, Lemma 21.14.6. By Cohomology on Sites, Lemma 21.8.4 it suffices to show that $H^p((X, T, \delta), \mathcal{F}) = 0$ for all (X, T, δ). This follows from Lemma 45.18.1. □

Lemma 45.18.3. *In Situation 45.5.1. Set $\mathcal{C} = (Cris(C/A))^{opp}$ and $\mathcal{C}^\wedge = (Cris^\wedge(C/A))^{opp}$ endowed with the chaotic topology, see Remark 45.5.4 for notation. There is a morphism of topoi*

$$g : Sh(\mathcal{C}) \longrightarrow Sh(\mathcal{C}^\wedge)$$

such that if $\mathcal{F}$ is a sheaf of abelian groups on $\mathcal{C}$, then

$$R^p g_*\mathcal{F}(B \to C, \delta) = \begin{cases} \lim_e \mathcal{F}(B_e \to C, \delta) & \text{if } p = 0 \\ R^1 \lim_e \mathcal{F}(B_e \to C, \delta) & \text{if } p = 1 \\ 0 & \text{else} \end{cases}$$

where $B_e = B/p^e B$ for $e \gg 0$.

Proof. Any functor between categories defines a morphism between chaotic topoi in the same direction, for example because such a functor can be considered as a cocontinuous functor between sites, see Sites, Section 7.20. Proof of the description of $g_*\mathcal{F}$ is omitted. Note that in the statement we take $(B_e \to C, \delta)$ is an object of $\mathrm{Cris}(C/A)$ only for e large enough. Let $\mathcal{I}$ be an injective abelian sheaf on $\mathcal{C}$. Then the transition maps

$$\mathcal{I}(B_e \to C, \delta) \leftarrow \mathcal{I}(B_{e+1} \to C, \delta)$$

are surjective as the morphisms

$$(B_e \to C, \delta) \longrightarrow (B_{e+1} \to C, \delta)$$

are monomorphisms in the category $\mathcal{C}$. Hence for an injective abelian sheaf both sides of the displayed formula of the lemma agree. Taking an injective resolution of $\mathcal{F}$ one easily obtains the result (sheaves are presheaves, so exactness is measured on the level of groups of sections over objects). □

Lemma 45.18.4. *Let $\mathcal{C}$ be a category endowed with the chaotic topology. Let X be an object of $\mathcal{C}$ such that every object of $\mathcal{C}$ has a morphism towards X. Assume that $\mathcal{C}$ has products. Then for every abelian sheaf $\mathcal{F}$ on $\mathcal{C}$ the total cohomology $R\Gamma(\mathcal{C}, \mathcal{F})$ is represented by the complex*

$$\mathcal{F}(X) \to \mathcal{F}(X \times X) \to \mathcal{F}(X \times X \times X) \to \ldots$$

associated to the cosimplicial abelian group $[n] \mapsto \mathcal{F}(X^n)$.

Proof. Note that $H^q(X^p, \mathcal{F}) = 0$ for all $q > 0$ as sheaves are presheaves on $\mathcal{C}$. The assumption on X is that $h_X \to *$ is surjective. Using that $H^q(X, \mathcal{F}) = H^p(h_X, \mathcal{F})$ and $H^p(\mathcal{C}, \mathcal{F}) = H^p(*, \mathcal{F})$ we see that our statement is a special case of Cohomology on Sites, Lemma 21.13.2. □

45.19. Cosimplicial preparations

In this section we compare crystalline cohomology with de Rham cohomology. We follow [**BdJ11**].

Example 45.19.1. Suppose that A_* is any cosimplicial ring. Consider the cosimplicial module M_* defined by the rule

$$M_n = \bigoplus\nolimits_{i=0,\ldots,n} A_n e_i$$

For a map $f : [n] \to [m]$ define $M_*(f) : M_n \to M_m$ to be the unique $A_*(f)$-linear map which maps e_i to $e_{f(i)}$. We claim the identity on M_* is homotopic to 0. Namely, a homotopy is given by a map of cosimplicial modules

$$h : M_* \longrightarrow \mathrm{Hom}(\Delta[1], M_*)$$

see Section 45.16. For $j \in \{0, \ldots, n+1\}$ we let $\alpha_j^n : [n] \to [1]$ be the map defined by $\alpha_j^n(i) = 0 \Leftrightarrow i < j$. Then $\Delta[1]_n = \{\alpha_0^n, \ldots, \alpha_{n+1}^n\}$ and correspondingly $\mathrm{Hom}(\Delta[1], M_*)_n = \prod_{j=0,\ldots,n+1} M_n$, see Simplicial, Sections 14.24 and 14.26. Instead of using this product representation, we think of an element in $\mathrm{Hom}(\Delta[1], M_*)_n$ as a function $\Delta[1]_n \to M_n$. Using this notation, we define h in degree n by the rule

$$h_n(e_i)(\alpha_j^n) = \begin{cases} e_i & \text{if} \quad i < j \\ 0 & \text{else} \end{cases}$$

We first check h is a morphism of cosimplicial modules. Namely, for $f : [n] \to [m]$ we will show that

$$h_m \circ M_*(f) = \mathrm{Hom}(\Delta[1], M_*)(f) \circ h_n \tag{45.19.1.1}$$

The left hand side of (45.19.1.1) evaluated at e_i and then in turn evaluated at α_j^m is

$$h_m(e_{f(i)})(\alpha_j^m) = \begin{cases} e_{f(i)} & \text{if} \quad f(i) < j \\ 0 & \text{else} \end{cases}$$

Note that $\alpha^m_j \circ f = \alpha^n_{j'}$ where $0 \leq j' \leq n+1$ is the unique index such that $f(i) < j$ if and only if $i < j'$. Thus the right hand side of (45.19.1.1) evaluated at e_i and then in turn evaluated at α^m_j is

$$M_*(f)(h_n(e_i)(\alpha^m_j \circ f) = M_*(f)(h_n(e_i)(\alpha^n_{j'})) = \begin{cases} e_{f(i)} & \text{if} \quad i < j' \\ 0 & \text{else} \end{cases}$$

It follows from our description of j' that the two answers are equal. Hence h is a map of cosimplicial modules. Let $0 : \Delta[0] \to \Delta[1]$ and $1 : \Delta[0] \to \Delta[1]$ be the obvious maps, and denote $ev_0, ev_1 : \mathrm{Hom}(\Delta[1], M_*) \to M_*$ the corresponding evaluation maps. The reader verifies readily that the the compositions

$$ev_0 \circ h, ev_1 \circ h : M_* \longrightarrow M_*$$

are 0 and 1 respectively, whence h is the desired homotopy between 0 and 1.

Lemma 45.19.2. *With notation as in (45.17.0.5) the complex*

$$\Omega_{D(0)} \to \Omega_{D(1)} \to \Omega_{D(2)} \to \cdots$$

is homotopic to zero as a $D(*)$-cosimplicial module.

Proof. We are going to use the principle of Simplicial, Lemma 14.26.3 and more specifically Lemma 45.16.1 which tells us that homotopic maps between (co)simplicial objects are transformed by any functor into homotopic maps. The complex of the lemma is equal to the p-adic completion of the base change of the cosimplicial module

$$M_* = \big(\Omega_{P/A} \to \Omega_{P \otimes_A P/A} \to \Omega_{P \otimes_A P \otimes_A P/A} \to \cdots\big)$$

via the cosimplicial ring map $P \otimes_A \ldots \otimes_A P \to D(n)$. This follows from Lemma 45.6.6, see comments following (45.17.0.2). Hence it suffices to show that the cosimplicial module M_* is homotopic to zero (uses base change and p-adic completion). We can even assume $A = \mathbf{Z}$ and $P = \mathbf{Z}[\{x_i\}_{i \in I}]$ as we can use base change with $\mathbf{Z} \to A$. In this case $P^{\otimes n+1}$ is the polynomial algebra on the elements

$$x_i(e) = 1 \otimes \cdots \otimes x_i \otimes \cdots \otimes 1$$

with x_i in the eth slot. The modules of the complex are free on the generators $\mathrm{d}x_i(e)$. Note that if $f : [n] \to [m]$ is a map then we see that

$$M_*(f)(\mathrm{d}x_i(e)) = \mathrm{d}x_i(f(e))$$

Hence we see that M_* is a direct sum over I of copies of the module studied in Example 45.19.1 and we win. □

Lemma 45.19.3. *With notation as in (45.17.0.4) and (45.17.0.5), given any cosimplicial module M_* over $D(*)$ and $i > 0$ the cosimplicial module*

$$M_0 \otimes^\wedge_{D(0)} \Omega^i_{D(0)} \to M_1 \otimes^\wedge_{D(1)} \Omega^i_{D(1)} \to M_2 \otimes^\wedge_{D(2)} \Omega^i_{D(2)} \to \ldots$$

is homotopic to zero, where $\Omega^i_{D(n)}$ is the p-adic completion of the ith exterior power of $\Omega_{D(n)}$.

Proof. By Lemma 45.19.2 the endomorphisms 0 and 1 of $\Omega_{D(*)}$ are homotopic. If we apply the functor $\wedge^i$ we see that the same is true for the cosimplicial module $\wedge^i \Omega_{D(*)}$, see Lemma 45.16.1. Another application of the same lemma shows the p-adic completion $\Omega^i_{D(*)}$ is homotopy equivalent to zero. Tensoring with M_* we see that $M_* \otimes_{D(*)} \Omega^i_{D(*)}$ is homotopic to zero, see Lemma 45.16.1 again. A final application of the p-adic completion functor finishes the proof. □

45.20. Divided power Poincaré lemma

Just the simplest possible version.

Lemma 45.20.1. *Let A be a ring. Let $P = A\langle x_i \rangle$ be a divided power polynomial ring over A. For any A-module M the complex*

$$0 \to M \to M \otimes_A P \to M \otimes_A \Omega^1_{P/A,\delta} \to M \otimes_A \Omega^2_{P/A,\delta} \to \ldots$$

is exact. Let D be the p-adic completion of P. Let Ω^i_D be the p-adic completion of the ith exterior power of $\Omega_{D/A,\delta}$. For any p-adically complete A-module M the complex

$$0 \to M \to M \otimes^\wedge_A D \to M \otimes^\wedge_A \Omega^1_D \to M \otimes^\wedge_A \Omega^2_D \to \ldots$$

is exact.

Proof. It suffices to show that the complex

$$E : (0 \to A \to P \to \Omega^1_{P/A,\delta} \to \Omega^2_{P/A,\delta} \to \ldots)$$

is homotopy equivalent to zero as a complex of A-modules. For every multi-index $K = (k_i)$ we can consider the subcomplex $E(K)$ which in degree j consists of

$$\bigoplus\nolimits_{I=\{i_1,\ldots,i_j\} \subset \mathrm{Supp}(K)} A \prod\nolimits_{i \notin I} x_i^{[k_i]} \prod\nolimits_{i \in I} x_i^{[k_i-1]} \mathrm{d}x_{i_1} \wedge \ldots \wedge \mathrm{d}x_{i_j}$$

Since $E = \bigoplus E(K)$ we see that it suffices to prove each of the complexes $E(K)$ is homotopic to zero. If $K = 0$, then $E(K) : (A \to A)$ is homotopic to zero. If K has nonempty (finite) support S, then the complex $E(K)$ is isomorphic to the complex

$$0 \to A \to \bigoplus\nolimits_{s \in S} A \to \wedge^2(\bigoplus\nolimits_{s \in S} A) \to \ldots \to \wedge^{\#S}(\bigoplus\nolimits_{s \in S} A) \to 0$$

which is homotopic to zero, for example by More on Algebra, Lemma 15.19.5. □

An alternative (more direct) approach to the following lemma is explained in Example 45.25.2.

Lemma 45.20.2. *Let A be a ring. Let (B, J, δ) be a divided power ring. Let $P = B\langle x_i \rangle$ be a divided power polynomial ring over B with divided power ideal $J = IP + B\langle x_i \rangle_+$ as usual. Let M be a B-module endowed with an integrable connection $\nabla : M \to M \otimes_B \Omega^1_{B/A,\delta}$. Then the map of de Rham complexes*

$$M \otimes_B \Omega^*_{B/A,\delta} \longrightarrow M \otimes_P \Omega^*_{P/A,\delta}$$

is a quasi-isomorphism. Let D, resp. D' be the p-adic completion of B, resp. P and let Ω^i_D, resp. $\Omega^i_{D'}$ be the p-adic completion of $\Omega^i_{B/A,\delta}$, resp. $\Omega^i_{P/A,\delta}$. Let M be a p-adically complete D-module endowed with an integral connection $\nabla : M \to M \otimes^\wedge_D \Omega^1_D$. Then the map of de Rham complexes

$$M \otimes^\wedge_D \Omega^*_D \longrightarrow M \otimes^\wedge_D \Omega^*_{D'}$$

is a quasi-isomorphism.

Proof. Consider the decreasing filtration F^* on $\Omega^*_{B/A,\delta}$ given by the subcomplexes $F^i(\Omega^*_{B/A,\delta}) = \sigma_{\geq i}\Omega^*_{B/A,\delta}$. See Homology, Section 12.13. This induces a decreasing filtration F^* on $\Omega^*_{P/A,\delta}$ by setting

$$F^i(\Omega^*_{P/A,\delta}) = F^i(\Omega^*_{B/A,\delta}) \wedge \Omega^*_{P/A,\delta}.$$

We have a split short exact sequence

$$0 \to \Omega^1_{B/A,\delta} \otimes_B P \to \Omega^1_{P/A,\delta} \to \Omega^1_{P/B,\delta} \to 0$$

and the last module is free on $\mathrm{d}x_i$. It follows from this that $F^i(\Omega^*_{P/A,\delta}) \to \Omega^*_{P/A,\delta}$ is a termwise split injection and that

$$\mathrm{gr}^i_F(\Omega^*_{B/A,\delta}) = \Omega^i_{B/A,\delta} \otimes_B \Omega^*_{P/B,\delta}$$

as complexes. Thus we can define a filtration F^* on $M \otimes_B \Omega^*_{B/A,\delta}$ by setting

$$F^i(M \otimes_B \Omega^*_{P/A,\delta}) = M \otimes_B F^i(\Omega^*_{P/A,\delta})$$

and we have

$$\mathrm{gr}^i_F(M \otimes_B \Omega^*_{P/A,\delta}) = M \otimes_B \Omega^i_{B/A,\delta} \otimes_B \Omega^*_{P/B,\delta}$$

as complexes. By Lemma 45.20.1 each of these complexes is quasi-isomorphic to $M \otimes_B \Omega^i_{B/A,\delta}$ placed in degree 0. Hence we see that the first displayed map of the lemma is a morphism of filtered complexes which induces a quasi-isomorphism on graded pieces. This implies that it is a quasi-isomorphism, for example by the spectral sequence associated to a filtered complex, see Homology, Section 12.21.

The proof of the second quasi-isomorphism is exactly the same. □

45.21. Cohomology in the affine case

Let's go back to the situation studied in Section 45.17. We start with (A, I, γ) and $A/I \to C$ and set $X = \mathrm{Spec}(C)$ and $S = \mathrm{Spec}(A)$. Then we choose a polynomial ring P over A and a surjection $P \to C$ with kernel J. We obtain D and $D(n)$ see (45.17.0.1) and (45.17.0.4). Set $T(n)_e = \mathrm{Spec}(D(n)/p^e D(n))$ so that $(X, T(n)_e, \delta(n))$ is an object of $\mathrm{Cris}(X/S)$. Let $\mathcal{F}$ be a sheaf of $\mathcal{O}_{X/S}$-modules and set

$$M(n) = \lim_e \Gamma((X, T(n)_e, \delta(n)), \mathcal{F})$$

for $n = 0, 1, 2, 3, \ldots$. This forms a cosimplicial module over the cosimplicial ring $D(0), D(1), D(2), \ldots$.

Proposition 45.21.1. *With notations as above assume that*

(1) *$\mathcal{F}$ is locally quasi-coherent, and*
(2) *for any morphism $(U, T, \delta) \to (U', T', \delta')$ of Cris(X/S) where $f : T \to T'$ is a closed immersion the map $c_f : f^*\mathcal{F}_{T'} \to \mathcal{F}_T$ is surjective.*

Then the complex

$$M(0) \to M(1) \to M(2) \to \ldots$$

computes $R\Gamma(Cris(X/S), \mathcal{F})$.

Proof. Using assumption (1) and Lemma 45.18.2 we see that $R\Gamma(\mathrm{Cris}(X/S), \mathcal{F})$ is isomorphic to $R\Gamma(\mathcal{C}, \mathcal{F})$. Note that the categories $\mathcal{C}$ used in Lemmas 45.18.2 and 45.18.3 agree. Let $f : T \to T'$ be a closed immersion as in (2). Surjectivity of $c_f : f^*\mathcal{F}_{T'} \to \mathcal{F}_T$ is equivalent to surjectivity of $\mathcal{F}_{T'} \to f_*\mathcal{F}_T$. Hence, if $\mathcal{F}$ satisfies (1) and (2), then we obtain a short exact sequence

$$0 \to \mathcal{K} \to \mathcal{F}_{T'} \to f_*\mathcal{F}_T \to 0$$

of quasi-coherent $\mathcal{O}_{T'}$-modules on T', see Schemes, Section 25.24 and in particular Lemma 25.24.1. Thus, if T' is affine, then we conclude that the restriction map $\mathcal{F}(U', T', \delta') \to \mathcal{F}(U, T, \delta)$ is surjective by the vanishing of $H^1(T', \mathcal{K})$, see Cohomology of Schemes, Lemma 29.2.2. Hence the transition maps of the inverse systems in

Lemma 45.18.3 are surjective. We conclude that that $R^p g_*(\mathcal{F}|_{\mathcal{C}}) = 0$ for all $p \geq 1$ where g is as in Lemma 45.18.3. The object D of the category $\mathcal{C}^\wedge$ satisfies the assumption of Lemma 45.18.4 by Lemma 45.5.7 with

$$D \times \ldots \times D = D(n)$$

in $\mathcal{C}$ because $D(n)$ is the $n+1$-fold coproduct of D in $\mathrm{Cris}^\wedge(C/A)$, see Lemma 45.17.2. Thus we win. □

Lemma 45.21.2. *Assumptions and notation as in Proposition 45.21.1. Then*

$$H^j(Cris(X/S), \mathcal{F} \otimes_{\mathcal{O}_{X/S}} \Omega^i_{X/S}) = 0$$

for all $i > 0$ and all $j \geq 0$.

Proof. Using Lemma 45.12.6 it follows that $\mathcal{H} = \mathcal{F} \otimes_{\mathcal{O}_{X/S}} \Omega^i_{X/S}$ also satisfies assumptions (1) and (2) of Proposition 45.21.1. Write $M(n)_e = \Gamma((X, T(n)_e, \delta(n)), \mathcal{F})$ so that $M(n) = \lim_e M(n)_e$. Then

$$\begin{aligned} \lim_e \Gamma((X, T(n)_e, \delta(n)), \mathcal{H}) &= \lim_e M(n)_e \otimes_{D(n)_e} \Omega_{D(n)}/p^e\Omega_{D(n)} \\ &= \lim_e M(n)_e \otimes_{D(n)} \Omega_{D(n)} \end{aligned}$$

By Lemma 45.19.3 the cosimplicial modules

$$M(0)_e \otimes_{D(0)} \Omega^i_{D(0)} \to M(1)_e \otimes_{D(1)} \Omega^i_{D(1)} \to M(2)_e \otimes_{D(2)} \Omega^i_{D(2)} \to \ldots$$

are homotopic to zero. Because the transition maps $M(n)_{e+1} \to M(n)_e$ are surjective, we see that the inverse limit of the associated complexes are acyclic[5]. Hence the vanishing of cohomology of $\mathcal{H}$ by Proposition 45.21.1. □

Proposition 45.21.3. *Assumptions as in Proposition 45.21.1 but now assume that $\mathcal{F}$ is a crystal in quasi-coherent modules. Let (M, ∇) be the corresponding module with connection over D, see Proposition 45.17.4. Then the complex*

$$M \otimes^\wedge_D \Omega^*_D$$

computes $R\Gamma(Cris(X/S), \mathcal{F})$.

Proof. We will prove this using the two spectral sequences associated to the double complex $K^{*,*}$ with terms

$$K^{a,b} = M \otimes^\wedge_D \Omega^a_{D(b)}$$

What do we know so far? Well, Lemma 45.19.3 tells us that each column $K^{a,*}$, $a > 0$ is acyclic. Proposition 45.21.1 tells us that the first column $K^{0,*}$ is quasi-isomorphic to $R\Gamma(\mathrm{Cris}(X/S), \mathcal{F})$. Hence the first spectral sequence associated to the double complex shows that there is a canonical quasi-isomorphism of $R\Gamma(\mathrm{Cris}(X/S), \mathcal{F})$ with $\mathrm{Tot}(K^{*,*})$.

Next, let's consider the rows $K^{*,b}$. By Lemma 45.17.1 each of the $b+1$ maps $D \to D(b)$ presents $D(b)$ as the p-adic completion of a divided power polynomial algebra over D. Hence Lemma 45.20.2 shows that the map

$$M \otimes^\wedge_D \Omega^*_D \longrightarrow M \otimes^\wedge_{D(b)} \Omega^*_{D(b)} = K^{*,b}$$

[5]Actually, they are even homotopic to zero as the homotopies fit together, but we don't need this. The reason for this roundabout argument is that the limit $\lim_e M(n)_e \otimes_{D(n)} \Omega^i_{D(n)}$ isn't the p-adic completion of $M(n) \otimes_{D(n)} \Omega^i_{D(n)}$ as with the assumptions of the lemma we don't know that $M(n)_e = M(n)_{e+1}/p^e M(n)_{e+1}$. If $\mathcal{F}$ is a crystal then this does hold.

is a quasi-isomorphism. Note that each of these maps defines the *same* map on cohomology (and even the same map in the derived category) as the inverse is given by the co-diagonal map $D(b) \to D$ (corresponding to the multiplication map $P \otimes_A \ldots \otimes_A P \to P$). Hence if we look at the E_1 page of the second spectral sequence we obtain

$$E_1^{a,b} = H^a(M \otimes_D^\wedge \Omega_D^*)$$

with differentials

$$E_1^{a,0} \xrightarrow{0} E_1^{a,1} \xrightarrow{1} E_1^{a,2} \xrightarrow{0} E_1^{a,3} \xrightarrow{1} \ldots$$

as each of these is the alternation sum of the given identifications $H^a(M \otimes_D^\wedge \Omega_D^*) = E_1^{a,0} = E_1^{a,1} = \ldots$. Thus we see that the E_2 page is equal $H^a(M \otimes_D^\wedge \Omega_D^*)$ on the first row and zero elsewhere. It follows that the identification of $M \otimes_D^\wedge \Omega_D^*$ with the first row induces a quasi-isomorphism of $M \otimes_D^\wedge \Omega_D^*$ with $\mathrm{Tot}(K^{*,*})$. □

Lemma 45.21.4. *Assumptions as in Proposition 45.21.3. Let $A \to P' \to C$ be ring maps with $A \to P'$ smooth and $P' \to C$ surjective with kernel J'. Let D' be the p-adic completion of $D_{P',\gamma}(J')$. Let (M', ∇') be the pair over D' corresponding to $\mathcal{F}$, see Lemma 45.17.5. Then the complex*

$$M' \otimes_{D'}^\wedge \Omega_{D'}^*$$

computes $R\Gamma(Cris(X/S), \mathcal{F})$.

Proof. Choose $a : D \to D'$ and $b : D' \to D$ as in Lemma 45.17.5. Note that the base change $M = M' \otimes_{D',b} D$ with its connection ∇ corresponds to $\mathcal{F}$. Hence we know that $M \otimes_D^\wedge \Omega_D^*$ computes the crystalline cohomology of $\mathcal{F}$, see Proposition 45.21.3. Hence it suffices to show that the base change maps (induced by a and b)

$$M' \otimes_{D'}^\wedge \Omega_{D'}^* \longrightarrow M \otimes_D^\wedge \Omega_D^* \quad \text{and} \quad M \otimes_D^\wedge \Omega_D^* \longrightarrow M' \otimes_{D'}^\wedge \Omega_{D'}^*$$

are quasi-isomorphisms. Since $a \circ b = \mathrm{id}_{D'}$ we see that the composition one way around is the identity on the complex $M' \otimes_{D'}^\wedge \Omega_{D'}^*$. Hence it suffices to show that the map

$$M \otimes_D^\wedge \Omega_D^* \longrightarrow M \otimes_D^\wedge \Omega_D^*$$

induced by $b \circ a : D \to D$ is a quasi-isomorphism. (Note that we have the same complex on both sides as $M = M' \otimes_{D',b}^\wedge D$, hence $M \otimes_{D,b\circ a}^\wedge D = M' \otimes_{D',b\circ a\circ b}^\wedge D = M' \otimes_{D',b}^\wedge D = M$.) In fact, we claim that for any divided power A-algebra homomorphism $\rho : D \to D$ compatible with the augmentation to C the induced map $M \otimes_D^\wedge \Omega_D^* \to M \otimes_{D,\rho}^\wedge \Omega_D^*$ is a quasi-isomorphism.

Write $\rho(x_i) = x_i + z_i$. The elements z_i are in the divided power ideal of D because ρ is compatible with the augmentation to C. Hence we can factor the map ρ as a composition

$$D \xrightarrow{\sigma} D\langle \xi_i \rangle^\wedge \xrightarrow{\tau} D$$

where the first map is given by $x_i \mapsto x_i + \xi_i$ and the second map is the divided power D-algebra map which maps ξ_i to z_i. (This uses the universal properties of polynomial algebra, divided power polynomial algebras, divided power envelopes, and p-adic completion.) Note that there exists an *automorphism* α of $D\langle \xi_i \rangle^\wedge$ with $\alpha(x_i) = x_i - \xi_i$ and $\alpha(\xi_i) = \xi_i$. Applying Lemma 45.20.2 to $\alpha \circ \sigma$ (which maps x_i to x_i) and using that α is an isomorphism we conclude that σ induces a quasi-isomorphism of $M \otimes_D^\wedge \Omega_D^*$ with $M \otimes_{D,\sigma}^\wedge \Omega_{D\langle x_i \rangle^\wedge}^*$. On the other hand the map τ has as a left inverse the map $D \to D\langle x_i \rangle^\wedge$, $x_i \mapsto x_i$ and we conclude (using Lemma

45.20.2 once more) that τ induces a quasi-isomorphism of $M \otimes^\wedge_{D,\sigma} \Omega^*_{D\langle x_i\rangle^\wedge}$ with $M \otimes^\wedge_{D,\tau\circ\sigma} \Omega^*_D$. Composing these two quasi-isomorphisms we obtain that ρ induces a quasi-isomorphism $M \otimes^\wedge_D \Omega^*_D \to M \otimes^\wedge_{D,\rho} \Omega^*_D$ as desired. □

45.22. Two counter examples

Before we turn to some of the successes of crystalline cohomology, let us give two examples which explain why crystalline cohomology does not work very well if the schemes in question are either not proper over the base, or singular. The first example can be found in [**BO83**].

Example 45.22.1. Let $A = \mathbf{Z}_p$ with divided power ideal (p) endowed with its unique divided powers γ. Let $C = \mathbf{F}_p[x,y]/(x^2, xy, y^2)$. We choose the presentation

$$C = P/J = \mathbf{Z}_p[x,y]/(x^2, xy, y^2, p)$$

Let $D = D_{P,\gamma}(J)^\wedge$ with divided power ideal $(\bar{J}, \bar{\gamma})$ as in Section 45.17. We will denote x, y also the images of x and y in D. Consider the element

$$\tau = \bar{\gamma}_p(x^2)\bar{\gamma}_p(y^2) - \bar{\gamma}_p(xy)^2 \in D$$

We note that $p\tau = 0$ as

$$p!\bar{\gamma}_p(x^2)\bar{\gamma}_p(y^2) = x^{2p}\bar{\gamma}_p(y^2) = \bar{\gamma}_p(x^2y^2) = x^py^p\bar{\gamma}_p(xy) = p!\bar{\gamma}_p(xy)^2$$

in D. We also note that $\mathrm{d}\tau = 0$ in Ω_D as

$$\begin{aligned}
\mathrm{d}(\bar{\gamma}_p(x^2)\bar{\gamma}_p(y^2)) &= \bar{\gamma}_{p-1}(x^2)\bar{\gamma}_p(y^2)\mathrm{d}x^2 + \bar{\gamma}_p(x^2)\bar{\gamma}_{p-1}(y^2)\mathrm{d}y^2 \\
&= 2x\bar{\gamma}_{p-1}(x^2)\bar{\gamma}_p(y^2)\mathrm{d}x + 2y\bar{\gamma}_p(x^2)\bar{\gamma}_{p-1}(y^2)\mathrm{d}y \\
&= 2/(p-1)!(x^{2p-1}\bar{\gamma}_p(y^2)\mathrm{d}x + y^{2p-1}\bar{\gamma}_p(x^2)\mathrm{d}y) \\
&= 2/(p-1)!(x^{p-1}\bar{\gamma}_p(xy^2)\mathrm{d}x + y^{p-1}\bar{\gamma}_p(x^2y)\mathrm{d}y) \\
&= 2/(p-1)!(x^{p-1}y^p\bar{\gamma}_p(xy)\mathrm{d}x + x^py^{p-1}\bar{\gamma}_p(xy)\mathrm{d}y) \\
&= 2\bar{\gamma}_{p-1}(xy)\bar{\gamma}_p(xy)(y\mathrm{d}x + x\mathrm{d}y) \\
&= \mathrm{d}(\bar{\gamma}_p(xy)^2)
\end{aligned}$$

Finally, we claim that $\tau \neq 0$ in D. To see this it suffices to produce an object $(B \to \mathbf{F}_p[x,y]/(x^2,xy,y^2), \delta)$ of $\mathrm{Cris}(C/S)$ such that τ does not map to zero in B. To do this take

$$B = \mathbf{F}_p[x,y,u,v]/(x^3, x^2y, xy^2, y^3, xu, yu, xv, yv, u^2, v^2)$$

with the obvious surjection to C. Let $K = \mathrm{Ker}(B \to C)$ and consider the map

$$\delta_p : K \longrightarrow K, \quad ay^2 + bxy + cy^2 + du + ev + fuv \longmapsto a^pu + c^pv$$

One checks this satisfies the assumptions (1), (2), (3) of Divided Power Algebra, Lemma 23.5.3 and hence defines a divided power structure. Moreover, we see that τ maps to uv which is not zero in B. Set $X = \mathrm{Spec}(C)$ and $S = \mathrm{Spec}(A)$. We draw the following conclusions

(1) $H^0(\mathrm{Cris}(X/S), \mathcal{O}_{X/S})$ has p-torsion, and
(2) pulling back by frobenius $F^* : H^0(\mathrm{Cris}(X/S), \mathcal{O}_{X/S}) \to H^0(\mathrm{Cris}(X/S), \mathcal{O}_{X/S})$ is not injective.

Namely, τ defines a nonzero torsion element of $H^0(\mathrm{Cris}(X/S), \mathcal{O}_{X/S})$ by Proposition 45.21.3. Similarly, $F^*(\tau) = \sigma(\tau)$ where $\sigma : D \to D$ is the map induced by any lift of Frobenius on P. If we choose $\sigma(x) = x^p$ and $\sigma(y) = y^p$, then an easy computation shows that $F^*(\tau) = 0$.

The next example shows that even for affine n-space crystalline cohomology does not give the correct thing.

Example 45.22.2. Let $A = \mathbf{Z}_p$ with divided power ideal (p) endowed with its unique divided powers γ. Let $C = \mathbf{F}_p[x_1, \ldots, x_r]$. We choose the presentation

$$C = P/J = P/pP \quad \text{with} \quad P = \mathbf{Z}_p[x_1, \ldots, x_r]$$

Note that pP has divided powers by Divided Power Algebra, Lemma 23.4.2. Hence setting $D = P^\wedge$ with divided power ideal (p) we obtain a situation as in Section 45.17. We conclude that $R\Gamma(\mathrm{Cris}(X/S), \mathcal{O}_{X/S})$ is represented by the complex

$$D \to \Omega^1_D \to \Omega^2_D \to \ldots \to \Omega^r_D$$

see Proposition 45.21.3. Assuming $r > 0$ we conclude the following

(1) The cristalline cohomology of the cristalline structure sheaf of $X = \mathbf{A}^r_{\mathbf{F}_p}$ over $S = \mathrm{Spec}(\mathbf{Z}_p)$ is zero except in degrees $0, \ldots, r$.
(2) We have $H^0(\mathrm{Cris}(X/S), \mathcal{O}_{X/S}) = \mathbf{Z}_p$.
(3) The cohomology group $H^r(\mathrm{Cris}(X/S), \mathcal{O}_{X/S})$ is infinite and is not a torsion abelian group.
(4) The cohomology group $H^r(\mathrm{Cris}(X/S), \mathcal{O}_{X/S})$ is not separated for the p-adic topology.

While the first two statements are reasonable, parts (3) and (4) are disconcerting! The truth of these statements follows immediately from working out what the complex displayed above looks like. Let's just do this in case $r = 1$. Then we are just looking at the two term complex of p-adically complete modules

$$\mathrm{d} : D = \Big(\bigoplus\nolimits_{n \geq 0} \mathbf{Z}_p x^n\Big)^\wedge \longrightarrow \Omega^1_D = \Big(\bigoplus\nolimits_{n \geq 1} \mathbf{Z}_p x^{n-1} \mathrm{d}x\Big)^\wedge$$

The map is given by $\mathrm{diag}(0, 1, 2, 3, 4, \ldots)$ except that the first summand is missing on the right hand side. Now it is clear that $\bigoplus_{n>0} \mathbf{Z}_p/n\mathbf{Z}_p$ is a subgroup of the cokernel, hence the cokernel is infinite. In fact, the element

$$\omega = \sum\nolimits_{e>0} p^e x^{p^{2e}-1} \mathrm{d}x$$

is clearly not a torsion element of the cokernel. But it gets worse. Namely, consider the element

$$\eta = \sum\nolimits_{e>0} p^e x^{p^e-1} \mathrm{d}x$$

For every $t > 0$ the element η is congruent to $\sum_{e>t} p^e x^{p^e-1} \mathrm{d}x$ modulo the image of d which is divisible by p^t. But η is not in the image of d because it would have to be the image of $a + \sum_{e>0} x^{p^e}$ for some $a \in \mathbf{Z}_p$ which is not an element of the left hand side. In fact, $p^N \eta$ is similarly not in the image of d for any integer N. This implies that η "generates" a copy of $\mathbf{Q}_p$ inside of $H^1_{\mathrm{cris}}(\mathbf{A}^1_{\mathbf{F}_p}/\mathrm{Spec}(\mathbf{Z}_p))$.

45.23. Applications

In this section we collect some applications of the material in the previous sections.

Proposition 45.23.1. *In Situation 45.7.5. Let $\mathcal{F}$ be a crystal in quasi-coherent modules on Cris(X/S). The truncation map of complexes*

$$(\mathcal{F} \to \mathcal{F} \otimes_{\mathcal{O}_{X/S}} \Omega^1_{X/S} \to \mathcal{F} \otimes_{\mathcal{O}_{X/S}} \Omega^2_{X/S} \to \cdots) \longrightarrow \mathcal{F}[0],$$

while not a quasi-isomorphism, becomes a quasi-isomorphism after applying $Ru_{X/S,*}$. In fact, for any $i > 0$, we have

$$Ru_{X/S,*}(\mathcal{F} \otimes_{\mathcal{O}_{X/S}} \Omega^i_{X/S}) = 0.$$

Proof. By Lemma 45.15.1 we get a de Rham complex as indicated in the lemma. We abbreviate $\mathcal{H} = \mathcal{F} \otimes \Omega^i_{X/S}$. Let $X' \subset X$ be an affine open subscheme which maps into an affine open subscheme $S' \subset S$. Then

$$(Ru_{X/S,*}\mathcal{H})|_{X'_{Zar}} = Ru_{X'/S',*}(\mathcal{H}|_{\mathrm{Cris}(X'/S')}),$$

see Lemma 45.9.5. Thus Lemma 45.21.2 shows that $Ru_{X/S,*}\mathcal{H}$ is a complex of sheaves on X_{Zar} whose cohomology on any affine open is trivial. As X has a basis for its topology consisting of affine opens this implies that $Ru_{X/S,*}\mathcal{H}$ is quasi-isomorphic to zero. □

Remark 45.23.2. The proof of Proposition 45.23.1 shows that the conclusion

$$Ru_{X/S,*}(\mathcal{F} \otimes_{\mathcal{O}_{X/S}} \Omega^i_{X/S}) = 0$$

for $i > 0$ is true for any $\mathcal{O}_{X/S}$-module $\mathcal{F}$ which satisfies conditions (1) and (2) of Proposition 45.21.1. This applies to the following non-crystals: $\Omega^i_{X/S}$ for all i, and any sheaf of the form $\underline{\mathcal{F}}$, where $\mathcal{F}$ is a quasi-coherent $\mathcal{O}_X$-module. In particular, it applies to the sheaf $\underline{\mathcal{O}_X} = \underline{\mathbf{G}_a}$. But note that we need something like Lemma 45.15.1 to produce a de Rham complex which requires $\mathcal{F}$ to be a crystal. Hence (currently) the collection of sheaves of modules for which the full statement of Proposition 45.23.1 holds is exactly the category of crystals in quasi-coherent modules.

In Situation 45.7.5. Let $\mathcal{F}$ be a crystal in quasi-coherent modules on Cris(X/S). Let (U, T, δ) be an object of Cris(X/S). Proposition 45.23.1 allows us to construct a canonical map

$$R\Gamma(\mathrm{Cris}(X/S), \mathcal{F}) \longrightarrow R\Gamma(T, \mathcal{F}_T \otimes_{\mathcal{O}_T} \Omega^*_{T/S,\delta}) \tag{45.23.2.1}$$

Namely, we have $R\Gamma(\mathrm{Cris}(X/S), \mathcal{F}) = R\Gamma(\mathrm{Cris}(X/S), \mathcal{F} \otimes \Omega^*_{X/S})$, we can restrict global cohomology classes to T, and $\Omega_{X/S}$ restricts to $\Omega_{T/S,\delta}$ by Lemma 45.12.3.

45.24. Some further results

In this section we mention some results whose proof is missing. We will formulate these as a series of remarks and we will convert them into actual lemmas and propositions only when we add detailed proofs.

Remark 45.24.1 (Higher direct images)**.** Let p be a prime number. Let $(S, \mathcal{I}, \gamma) \to (S', \mathcal{I}', \gamma')$ be a morphism of divided power schemes over $\mathbf{Z}_{(p)}$. Let

$$\begin{array}{ccc} X & \xrightarrow{f} & X' \\ \downarrow & & \downarrow \\ S_0 & \longrightarrow & S'_0 \end{array}$$

be a commutative diagram of morphisms of schemes and assume p is locally nilpotent on X and X'. Let $\mathcal{F}$ be an $\mathcal{O}_{X/S}$-module on $\mathrm{Cris}(X/S)$. Then $Rf_{\mathrm{cris},*}\mathcal{F}$ can be computed as follows.

Given an object (U', T', δ') of $\mathrm{Cris}(X'/S')$ set $U = X \times_{X'} U' = f^{-1}(U')$ (an open subscheme of X). Denote (T_0, T, δ) the divided power scheme over S such that

$$\begin{array}{ccc} T & \longrightarrow & T' \\ \downarrow & & \downarrow \\ S & \longrightarrow & S' \end{array}$$

is cartesian in the category of divided power schemes, see Lemma 45.7.4. There is an induced morphism $U \to T_0$ and we obtain a morphism $(U/T)_{\mathrm{cris}} \to (X/S)_{\mathrm{cris}}$, see Remark 45.9.3. Let $\mathcal{F}_U$ be the pullback of $\mathcal{F}$. Let $\tau_{U/T} : (U/T)_{\mathrm{cris}} \to T_{Zar}$ be the structure morphism. Then we have

$$(Rf_{\mathrm{cris},*}\mathcal{F})_{T'} = R(T \to T')_* \left(R\tau_{U/T,*}\mathcal{F}_U\right) \tag{45.24.1.1}$$

where the left hand side is the restriction (see Section 45.10).

Hints: First, show that $\mathrm{Cris}(U/T)$ is the localization (in the sense of Sites, Lemma 7.29.3) of $\mathrm{Cris}(X/S)$ at the sheaf of sets $f^{-1}_{\mathrm{cris}} h_{(U',T',\delta')}$. Next, reduce the statement to the case where $\mathcal{F}$ is an injective module and pushforward of modules using that the pullback of an injective $\mathcal{O}_{X/S}$-module is an injective $\mathcal{O}_{U/T}$-module on $\mathrm{Cris}(U/T)$. Finally, check the result holds for plain pushforward.

Remark 45.24.2 (Mayer-Vietoris)**.** In the situation of Remark 45.24.1 suppose we have an open covering $X = X' \cup X''$. Denote $X''' = X' \cap X''$. Let f', f'', and f''' be the restriction of f to X', X'', and X'''. Moreover, Let $\mathcal{F}'$, $\mathcal{F}''$, and $\mathcal{F}'''$ be the restriction of $\mathcal{F}$ to the crystalline sites of X', X'', and X'''. Then there exists a distinguished triangle

$$Rf_{\mathrm{cris},*}\mathcal{F} \longrightarrow Rf'_{\mathrm{cris},*}\mathcal{F}' \oplus Rf''_{\mathrm{cris},*}\mathcal{F}'' \longrightarrow Rf'''_{\mathrm{cris},*}\mathcal{F}''' \longrightarrow Rf_{\mathrm{cris},*}\mathcal{F}[1]$$

in $D(\mathcal{O}_{X'/S'})$.

Hints: This is a formal consequence of the fact that the subcategories $\mathrm{Cris}(X'/S)$, $\mathrm{Cris}(X''/S)$, $\mathrm{Cris}(X'''/S)$ correspond to open subobjects of the final sheaf on $\mathrm{Cris}(X/S)$ and that the last is the intersection of the first two.

Remark 45.24.3 (Čech complex)**.** Let p be a prime number. Let (A, I, γ) be a divided power ring with A a $\mathbf{Z}_{(p)}$-algebra. Set $S = \mathrm{Spec}(A)$ and $S_0 = \mathrm{Spec}(A/I)$. Let X be a separated[6] scheme over S_0 such that p is locally nilpotent on X. Let $\mathcal{F}$ be a crystal in quasi-coherent $\mathcal{O}_{X/S}$-modules.

[6]This assumption is not strictly necessary, as using hypercoverings the construction of the remark can be extended to the general case.

Choose an affine open covering $X = \bigcup_{\lambda\in\Lambda} U_\lambda$ of X. Write $U_\lambda = \mathrm{Spec}(C_\lambda)$. Choose a polynomial algebra P_λ over A and a surjection $P_\lambda \to C_\lambda$. Having fixed these choices we can construct a Čech complex which computes $R\Gamma(\mathrm{Cris}(X/S), \mathcal{F})$.

Given $n \geq 0$ and $\lambda_0, \ldots, \lambda_n \in \Lambda$ write $U_{\lambda_0\ldots\lambda_n} = U_{\lambda_0} \cap \ldots \cap U_{\lambda_n}$. This is an affine scheme by assumption. Write $U_{\lambda_0\ldots\lambda_n} = \mathrm{Spec}(C_{\lambda_0\ldots\lambda_n})$. Set

$$P_{\lambda_0\ldots\lambda_n} = P_{\lambda_0} \otimes_A \ldots \otimes_A P_{\lambda_n}$$

which comes with a canonical surjection onto $C_{\lambda_0\ldots\lambda_n}$. Denote the kernel $J_{\lambda_0\ldots\lambda_n}$ and set $D_{\lambda_0\ldots\lambda_n}$ the p-adically completed divided power envelope of $J_{\lambda_0\ldots\lambda_n}$ in $P_{\lambda_0\ldots\lambda_n}$ relative to γ. Let $M_{\lambda_0\ldots\lambda_n}$ be the $P_{\lambda_0\ldots\lambda_n}$-module corresponding to the restriction of $\mathcal{F}$ to $\mathrm{Cris}(U_{\lambda_0\ldots\lambda_n}/S)$ via Proposition 45.17.4. By construction we obtain a cosimplicial divided power ring $D(*)$ having in degree n the ring

$$D(n) = \prod\nolimits_{\lambda_0\ldots\lambda_n} D_{\lambda_0\ldots\lambda_n}$$

(use that divided power envelopes are functorial and the trivial cosimplicial structure on the ring $P(*)$ defined similarly). Since $M_{\lambda_0\ldots\lambda_n}$ is the "value" of $\mathcal{F}$ on the objects $\mathrm{Spec}(D_{\lambda_0\ldots\lambda_n})$ we see that $M(*)$ defined by the rule

$$M(n) = \prod\nolimits_{\lambda_0\ldots\lambda_n} M_{\lambda_0\ldots\lambda_n}$$

forms a cosimplicial $D(*)$-module. Now we claim that we have

$$R\Gamma(\mathrm{Cris}(X/S), \mathcal{F}) = s(M(*))$$

Here $s(-)$ denotes the cochain complex associated to a cosimplicial module (see Simplicial, Section 14.23).

Hints: The proof of this is similar to the proof of Proposition 45.21.1 (in particular the result holds for any module satisfying the assumptions of that proposition).

Remark 45.24.4 (Alternating Čech complex)**.** Let p be a prime number. Let (A, I, γ) be a divided power ring with A a $\mathbf{Z}_{(p)}$-algebra. Set $S = \mathrm{Spec}(A)$ and $S_0 = \mathrm{Spec}(A/I)$. Let X be a separated quasi-compact scheme over S_0 such that p is locally nilpotent on X. Let $\mathcal{F}$ be a crystal in quasi-coherent $\mathcal{O}_{X/S}$-modules.

Choose a finite affine open covering $X = \bigcup_{\lambda\in\Lambda} U_\lambda$ of X and a total ordering on Λ. Write $U_\lambda = \mathrm{Spec}(C_\lambda)$. Choose a polynomial algebra P_λ over A and a surjection $P_\lambda \to C_\lambda$. Having fixed these choices we can construct an alternating Čech complex which computes $R\Gamma(\mathrm{Cris}(X/S), \mathcal{F})$.

We are going to use the notation introduced in Remark 45.24.3. Denote $\Omega_{\lambda_0\ldots\lambda_n}$ the p-adically completed module of differentials of $D_{\lambda_0\ldots\lambda_n}$ over A compatible with the divided power structure. Let ∇ be the integrable connection on $M_{\lambda_0\ldots\lambda_n}$ coming from Proposition 45.17.4. Consider the double complex $M^{\bullet,\bullet}$ with terms

$$M^{n,m} = \bigoplus\nolimits_{\lambda_0<\ldots<\lambda_n} M_{\lambda_0\ldots\lambda_n} \otimes^\wedge_{D_{\lambda_0\ldots\lambda_n}} \Omega^m_{D_{\lambda_0\ldots\lambda_n}}.$$

For the differential d_1 (increasing n) we use the usual Čech differential and for the differential d_2 we use the connection, i.e., the differential of the de Rham complex. We claim that

$$R\Gamma(\mathrm{Cris}(X/S), \mathcal{F}) = \mathrm{Tot}(M^{\bullet,\bullet})$$

Here $\mathrm{Tot}(-)$ denotes the total complex associated to a double complex, see Homology, Definition 12.22.2.

Hints: We have

$$R\Gamma(\mathrm{Cris}(X/S), \mathcal{F}) = R\Gamma(\mathrm{Cris}(X/S), \mathcal{F} \otimes_{\mathcal{O}_{X/S}} \Omega^\bullet_{X/S})$$

by Proposition 45.23.1. The right hand side of the formula is simply the alternating Čech complex for the covering $X = \bigcup_{\lambda\in\Lambda} U_\lambda$ (which induces an open covering of the final sheaf of $\mathrm{Cris}(X/S)$) and the complex $\mathcal{F} \otimes_{\mathcal{O}_{X/S}} \Omega^\bullet_{X/S}$, see Proposition 45.21.3. Now the result follows from a general result in cohomology on sites, namely that the alternating Čech complex computes the cohomology provided it gives the correct answer on all the pieces (insert future reference here).

Remark 45.24.5 (Quasi-coherence)**.** In the situation of Remark 45.24.1 assume that $S \to S'$ is quasi-compact and quasi-separated and that $X \to S_0$ is quasi-compact and quasi-separated. Then for a crystal in quasi-coherent $\mathcal{O}_{X/S}$-modules $\mathcal{F}$ the sheaves $R^i f_{\mathrm{cris},*}\mathcal{F}$ are locally quasi-coherent.

Hints: We have to show that the restrictions to T' are quasi-coherent $\mathcal{O}_{T'}$-modules, where (U', T', δ') is any object of $\mathrm{Cris}(X'/S')$. It suffices to do this when T' is affine. We use the formula (45.24.1.1), the fact that $T \to T'$ is quasi-compact and quasi-separated (as T is affine over the base change of T' by $S \to S'$), and Cohomology of Schemes, Lemma 29.4.4 to see that it suffices to show that the sheaves $R^i\tau_{U/T,*}\mathcal{F}_U$ are quasi-coherent. Note that $U \to T_0$ is also quasi-compact and quasi-separated, see Schemes, Lemmas 25.21.15 and 25.21.15.

This reduces us to proving that $R^i\tau_{X/S,*}\mathcal{F}$ is quasi-coherent on S in the case that p locally nilpotent on S. Here $\tau_{X/S}$ is the structure morphism, see Remark 45.9.6. We may work locally on S, hence we may assume S affine (see Lemma 45.9.5). Induction on the number of affines covering X and Mayer-Vietoris (Remark 45.24.2) reduces the question to the case where X is also affine (as in the proof of Cohomology of Schemes, Lemma 29.4.4). Say $X = \mathrm{Spec}(C)$ and $S = \mathrm{Spec}(A)$ so that (A, I, γ) and $A \to C$ are as in Situation 45.5.1. Choose a polynomial algebra P over A and a surjection $P \to C$ as in Section 45.17. Let (M, ∇) be the module corresponding to $\mathcal{F}$, see Proposition 45.17.4. Applying Proposition 45.21.3 we see that $R\Gamma(\mathrm{Cris}(X/S), \mathcal{F})$ is represented by $M \otimes_D \Omega^*_D$. Note that completion isn't necessary as p is nilpotent in A! We have to show that this is compatible with taking principal opens in $S = \mathrm{Spec}(A)$. Suppose that $g \in A$. Then we conclude that similarly $R\Gamma(\mathrm{Cris}(X_g/S_g), \mathcal{F})$ is computed by $M_g \otimes_{D_g} \Omega^*_{D_g}$ (again this uses that p-adic completion isn't necessary). Hence we conclude because localization is an exact functor on A-modules.

Remark 45.24.6 (Boundedness)**.** In the situation of Remark 45.24.1 assume that $S \to S'$ is quasi-compact and quasi-separated and that $X \to S_0$ is of finite type and quasi-separated. Then there exists an integer i_0 such that for any crystal in quasi-coherent $\mathcal{O}_{X/S}$-modules $\mathcal{F}$ we have $R^i f_{\mathrm{cris},*}\mathcal{F} = 0$ for all $i > i_0$.

Hints: Arguing as in Remark 45.24.5 (using Cohomology of Schemes, Lemma 29.4.4) we reduce to proving that $H^i(\mathrm{Cris}(X/S), \mathcal{F}) = 0$ for $i \gg 0$ in the situation of Proposition 45.21.3 when C is a finite type algebra over A. This is clear as we can choose a finite polynomial algebra and we see that $\Omega^i_D = 0$ for $i \gg 0$.

Remark 45.24.7 (Specific boundedness)**.** In Situation 45.7.5 let $\mathcal{F}$ be a crystal in quasi-coherent $\mathcal{O}_{X/S}$-modules. Assume that S_0 has a unique point and that $X \to S_0$ is of finite presentation.

(1) If $\dim X = d$ and X/S_0 has embedding dimension e, then $H^i(\mathrm{Cris}(X/S), \mathcal{F}) = 0$ for $i > d + e$.
(2) If X is separated and can be covered by q affines, and X/S_0 has embedding dimension e, then $H^i(\mathrm{Cris}(X/S), \mathcal{F}) = 0$ for $i > q + e$.

Hints: In case (1) we can use that

$$H^i(\mathrm{Cris}(X/S), \mathcal{F}) = H^i(X_{Zar}, Ru_{X/S,*}\mathcal{F})$$

and that $Ru_{X/S,*}\mathcal{F}$ is locally calculated by a de Rham complex constructed using an embedding of X into a smooth scheme of dimension e over S (see Lemma 45.21.4). These de Rham complexes are zero in all degrees $> e$. Hence (1) follows from Cohomology, Lemma 20.21.5. In case (2) we use the alternating Čech complex (see Remark 45.24.4) to reduce to the case X affine. In the affine case we prove the result using the de Rham complex associated to an embedding of X into a smooth scheme of dimension e over S (it takes some work to construct such a thing).

Remark 45.24.8 (Base change map)**.** In the situation of Remark 45.24.1 assume $S = \mathrm{Spec}(A)$ and $S' = \mathrm{Spec}(A')$ are affine. Let $\mathcal{F}'$ be an $\mathcal{O}_{X'/S'}$-module. Let $\mathcal{F}$ be the pullback of $\mathcal{F}'$. Then there is a canonical base change map

$$L(S' \to S)^* R\tau_{X'/S',*}\mathcal{F}' \longrightarrow R\tau_{X/S,*}\mathcal{F}$$

where $\tau_{X/S}$ and $\tau_{X'/S'}$ are the structure morphisms, see Remark 45.9.6. On global sections this gives a base change map

$$R\Gamma(\mathrm{Cris}(X'/S'), \mathcal{F}') \otimes^{\mathbf{L}}_{A'} A \longrightarrow R\Gamma(\mathrm{Cris}(X/S), \mathcal{F}) \tag{45.24.8.1}$$

in $D(A)$.

Hint: Compose the very general base change map of Cohomology on Sites, Remark 21.19.2 with the canonical map $Lf^*_{\mathrm{cris}}\mathcal{F}' \to f^*_{\mathrm{cris}}\mathcal{F}' = \mathcal{F}$.

Remark 45.24.9 (Base change isomorphism)**.** The map (45.24.8.1) is an isomorphism provided all of the following conditions are satisfied:

(1) p is nilpotent in A',
(2) $\mathcal{F}'$ is a crystal in quasi-coherent $\mathcal{O}_{X'/S'}$-modules,
(3) $X' \to S'_0$ is a quasi-compact, quasi-separated morphism,
(4) $X = X' \times_{S'_0} S_0$,
(5) $\mathcal{F}'$ is a flat $\mathcal{O}_{X'/S'}$-module,
(6) $X' \to S'_0$ is a local complete intersection morphism (see More on Morphisms, Definition 36.42.2; this holds for example if $X' \to S'_0$ is syntomic or smooth),
(7) X' and S_0 are Tor independent over S'_0 (see More on Algebra, Definition 15.45.1; this holds for example if either $S_0 \to S'_0$ or $X' \to S'_0$ is flat).

Hints: Condition (1) means that in the arguments below p-adic completion does nothing and can be ignored. Using condition (3) and Mayer Vietoris (see Remark 45.24.2) this reduces to the case where X' is affine. In fact by condition (6), after shrinking further, we can assume that $X' = \mathrm{Spec}(C')$ and we are given a presentation $C' = A'/I'[x_1, \ldots, x_n]/(\bar{f}'_1, \ldots, \bar{f}'_c)$ where $\bar{f}'_1, \ldots, \bar{f}'_c$ is a Koszul-regular sequence in A'/I'. (This means that smooth locally $\bar{f}'_1, \ldots, \bar{f}'_c$ forms a regular sequence, see More on Algebra, Lemma 15.20.17.) We choose a lift of $\bar{f}'_i$ to an element $f'_i \in A'[x_1, \ldots, x_n]$. By (4) we see that $X = \mathrm{Spec}(C)$ with $C = A/I[x_1, \ldots, x_n]/(\bar{f}_1, \ldots, \bar{f}_c)$ where $f_i \in A[x_1, \ldots, x_n]$ is the image of f'_i. By

property (7) we see that $\bar{f}_1, \ldots, \bar{f}_c$ is a Koszul-regular sequence in $A/I[x_1, \ldots, x_n]$. The divided power envelope of $I'A'[x_1, \ldots, x_n] + (f'_1, \ldots, f'_c)$ in $A'[x_1, \ldots, x_n]$ relative to γ' is

$$D' = A'[x_1, \ldots, x_n]\langle \xi_1, \ldots, \xi_c \rangle / (\xi_i - f'_i)$$

see Lemma 45.2.4. Then you check that $\xi_1 - f'_1, \ldots, \xi_n - f'_n$ is a Koszul-regular sequence in the ring $A'[x_1, \ldots, x_n]\langle \xi_1, \ldots, \xi_c \rangle$. Similarly the divided power envelope of $IA[x_1, \ldots, x_n] + (f_1, \ldots, f_c)$ in $A[x_1, \ldots, x_n]$ relative to γ is

$$D = A[x_1, \ldots, x_n]\langle \xi_1, \ldots, \xi_c \rangle / (\xi_i - f_i)$$

and $\xi_1 - f_1, \ldots, \xi_n - f_n$ is a Koszul-regular sequence in the ring $A[x_1, \ldots, x_n]\langle \xi_1, \ldots, \xi_c \rangle$. It follows that $D' \otimes^{\mathbf{L}}_{A'} A = D$. Condition (2) implies $\mathcal{F}'$ corresponds to a pair (M', ∇) consisting of a D'-module with connection, see Proposition 45.17.4. Then $M = M' \otimes_{D'} D$ corresponds to the pullback $\mathcal{F}$. By assumption (5) we see that M' is a flat D'-module, hence

$$M = M' \otimes_{D'} D = M' \otimes_{D'} D' \otimes^{\mathbf{L}}_{A'} A = M' \otimes^{\mathbf{L}}_{A'} A$$

Since the modules of differentials $\Omega_{D'}$ and Ω_D (as defined in Section 45.17) are free D'-modules on the same generators we see that

$$M \otimes_D \Omega^\bullet_D = M' \otimes_{D'} \Omega^\bullet_{D'} \otimes_{D'} D = M' \otimes_{D'} \Omega^\bullet_{D'} \otimes^{\mathbf{L}}_{A'} A$$

which proves what we want by Proposition 45.21.3.

Remark 45.24.10 (Rlim)**.** Let p be a prime number. Let (A, I, γ) be a divided power ring with A an algebra over $\mathbf{Z}_{(p)}$ with p nilpotent in A/I. Set $S = \mathrm{Spec}(A)$ and $S_0 = \mathrm{Spec}(A/I)$. Let X be a scheme over S_0 with p locally nilpotent on X. Let $\mathcal{F}$ be any $\mathcal{O}_{X/S}$-module. For $e \gg 0$ we have $(p^e) \subset I$ is preserved by γ, see Divided Power Algebra, Lemma 23.4.5. Set $S_e = \mathrm{Spec}(A/p^eA)$ for $e \gg 0$. Then $\mathrm{Cris}(X/S_e)$ is a full subcategory of $\mathrm{Cris}(X/S)$ and we denote $\mathcal{F}_e$ the restriction of $\mathcal{F}$ to $\mathrm{Cris}(X/S_e)$. Then

$$R\Gamma(\mathrm{Cris}(X/S), \mathcal{F}) = R\lim_e R\Gamma(\mathrm{Cris}(X/S_e), \mathcal{F}_e)$$

Hints: Suffices to prove this for $\mathcal{F}$ injective. In this case the sheaves $\mathcal{F}_e$ are injective modules too, the transition maps $\Gamma(\mathcal{F}_{e+1}) \to \Gamma(\mathcal{F}_e)$ are surjective, and we have $\Gamma(\mathcal{F}) = \lim_e \Gamma(\mathcal{F}_e)$ because any object of $\mathrm{Cris}(X/S)$ is locally an object of one of the categories $\mathrm{Cris}(X/S_e)$ by definition of $\mathrm{Cris}(X/S)$.

Remark 45.24.11 (Comparison)**.** Let p be a prime number. Let (A, I, γ) be a divided power ring with p nilpotent in A. Set $S = \mathrm{Spec}(A)$ and $S_0 = \mathrm{Spec}(A/I)$. Let Y be a smooth scheme over S and set $X = Y \times_S S_0$. Let $\mathcal{F}$ be a crystal in quasi-coherent $\mathcal{O}_{X/S}$-modules. Then

(1) γ extends to a divided power structure on the ideal of X in Y so that (X, Y, γ) is an object of $\mathrm{Cris}(X/S)$,
(2) the restriction $\mathcal{F}_Y$ (see Section 45.10) comes endowed with a canonical integrable connection $\nabla : \mathcal{F}_Y \to \mathcal{F}_Y \otimes_{\mathcal{O}_Y} \Omega_{Y/S}$, and
(3) we have

$$R\Gamma(\mathrm{Cris}(X/S), \mathcal{F}) = R\Gamma(Y, \mathcal{F}_Y \otimes_{\mathcal{O}_Y} \Omega^\bullet_{Y/S})$$

in $D(A)$.

Hints: See Divided Power Algebra, Lemma 23.4.2 for (1). See Lemma 45.15.1 for (2). For Part (3) note that there is a map, see (45.23.2.1). This map is an isomorphism when X is affine, see Lemma 45.21.4. This shows that $Ru_{X/S,*}\mathcal{F}$ and $\mathcal{F}_Y \otimes \Omega^\bullet_{Y/S}$ are quasi-isomorphic as complexes on $Y_{Zar} = X_{Zar}$. Since $R\Gamma(\mathrm{Cris}(X/S), \mathcal{F}) = R\Gamma(X_{Zar}, Ru_{X/S,*}\mathcal{F})$ the result follows.

Remark 45.24.12 (Perfectness)**.** Let p be a prime number. Let (A, I, γ) be a divided power ring with p nilpotent in A. Set $S = \mathrm{Spec}(A)$ and $S_0 = \mathrm{Spec}(A/I)$. Let X be a proper smooth scheme over S_0. Let $\mathcal{F}$ be a crystal in finite locally free quasi-coherent $\mathcal{O}_{X/S}$-modules. Then $R\Gamma(\mathrm{Cris}(X/S), \mathcal{F})$ is a perfect object of $D(A)$.

Hints: By Remark 45.24.9 we have

$$R\Gamma(\mathrm{Cris}(X/S), \mathcal{F}) \otimes^{\mathbf{L}}_A A/I \cong R\Gamma(\mathrm{Cris}(X/S_0), \mathcal{F}|_{\mathrm{Cris}(X/S_0)})$$

By Remark 45.24.11 we have

$$R\Gamma(\mathrm{Cris}(X/S_0), \mathcal{F}|_{\mathrm{Cris}(X/S_0)}) = R\Gamma(X, \mathcal{F}_X \otimes \Omega^\bullet_{X/S_0})$$

Using the stupid filtration on the de Rham complex we see that the last displayed complex is perfect in $D(A/I)$ as soon as the complexes

$$R\Gamma(X, \mathcal{F}_X \otimes \Omega^q_{X/S_0})$$

are perfect complexes in $D(A/I)$, see More on Algebra, Lemma 15.50.4. This is true by standard arguments in coherent cohomology using that $\mathcal{F}_X \otimes \Omega^q_{X/S_0}$ is a finite locally free sheaf and $X \to S_0$ is proper and flat (insert future reference here). Applying More on Algebra, Lemma 15.51.4 we see that

$$R\Gamma(\mathrm{Cris}(X/S), \mathcal{F}) \otimes^{\mathbf{L}}_A A/I^n$$

is a perfect object of $D(A/I^n)$ for all n. This isn't quite enough unless A is Noetherian. Namely, even though I is locally nilpotent by our assumption that p is nilpotent, see Divided Power Algebra, Lemma 23.2.6, we cannot conclude that $I^n = 0$ for some n. A counter example is $\mathbf{F}_p\langle x\rangle$. To prove it in general when $\mathcal{F} = \mathcal{O}_{X/S}$ the argument of `http://math.columbia.edu/~dejong/wordpress/?p=2227` works. When the coefficients $\mathcal{F}$ are non-trivial the argument of [**Fal99**] seems to be as follows. Reduce to the case $pA = 0$ by More on Algebra, Lemma 15.51.4. In this case the Frobenius map $A \to A$, $a \mapsto a^p$ factors as $A \to A/I \xrightarrow{\varphi} A$ (as $x^p = 0$ for $x \in I$). Set $X^{(1)} = X \otimes_{A/I,\varphi} A$. The absolute Frobenius morphism of X factors through a morphism $F_X : X \to X^{(1)}$ (a kind of relative Frobenius). Affine locally if $X = \mathrm{Spec}(C)$ then $X^{(1)} = \mathrm{Spec}(C \otimes_{A/I,\varphi} A)$ and F_X corresponds to $C \otimes_{A/I,\varphi} A \to C$, $c \otimes a \mapsto c^p a$. This defines morphisms of ringed topoi

$$(X/S)_{\mathrm{cris}} \xrightarrow{(F_X)_{\mathrm{cris}}} (X^{(1)}/S)_{\mathrm{cris}} \xrightarrow{u_{X^{(1)}/S}} Sh(X^{(1)}_{Zar})$$

whose composition is denoted Frob_X. One then shows that $R\mathrm{Frob}_{X,*}\mathcal{F}$ is representable by a perfect complex of $\mathcal{O}_{X^{(1)}}$-modules(!) by a local calculation.

Remark 45.24.13 (Complete perfectness)**.** Let p be a prime number. Let (A, I, γ) be a divided power ring with A a p-adically complete ring and p nilpotent in A/I. Set $S = \mathrm{Spec}(A)$ and $S_0 = \mathrm{Spec}(A/I)$. Let X be a proper smooth scheme over S_0. Let $\mathcal{F}$ be a crystal in finite locally free quasi-coherent $\mathcal{O}_{X/S}$-modules. Then $R\Gamma(\mathrm{Cris}(X/S), \mathcal{F})$ is a perfect object of $D(A)$.

Hints: We know that $K = R\Gamma(\mathrm{Cris}(X/S), \mathcal{F})$ is the derived limit $K = R\lim K_e$ of the cohomologies over A/p^eA, see Remark 45.24.10. Each K_e is a perfect complex of $D(A/p^eA)$ by Remark 45.24.12. Since A is p-adically complete the result follows from More on Algebra, Lemma 15.58.3.

Remark 45.24.14 (Complete comparison)**.** Let p be a prime number. Let (A, I, γ) be a divided power ring with A a Noetherian p-adically complete ring and p nilpotent in A/I. Set $S = \mathrm{Spec}(A)$ and $S_0 = \mathrm{Spec}(A/I)$. Let Y be a proper smooth scheme over S and set $X = Y \times_S S_0$. Let $\mathcal{F}$ be a finite type crystal in quasi-coherent $\mathcal{O}_{X/S}$-modules. Then

(1) there exists a coherent $\mathcal{O}_Y$-module $\mathcal{F}_Y$ endowed with integrable connection
$$\nabla : \mathcal{F}_Y \longrightarrow \mathcal{F}_Y \otimes_{\mathcal{O}_Y} \Omega_{Y/S}$$
such that $\mathcal{F}_Y/p^e\mathcal{F}_Y$ is the module with connection over A/p^eA found in Remark 45.24.11, and

(2) we have
$$R\Gamma(\mathrm{Cris}(X/S), \mathcal{F}) = R\Gamma(Y, \mathcal{F}_Y \otimes_{\mathcal{O}_Y} \Omega^\bullet_{Y/S})$$
in $D(A)$.

Hints: The existence of $\mathcal{F}_Y$ is Grothendieck's existence theorem (insert future reference here). The isomorphism of cohomologies follows as both sides are computed as $R\lim$ of the versions modulo p^e (see Remark 45.24.10 for the left hand side; use the theorem on formal functions, see Cohomology of Schemes, Theorem 29.18.5 for the right hand side). Each of the versions modulo p^e are isomorphic by Remark 45.24.11.

45.25. Pulling back along purely inseparable maps

By an α_p-cover we mean a morphism of the form
$$X' = \mathrm{Spec}(C[z]/(z^p - c)) \longrightarrow \mathrm{Spec}(C) = X$$
where C is an $\mathbf{F}_p$-algebra and $c \in C$. Equivalently, X' is an α_p-torsor over X. An *iterated α_p-cover*[7] is a morphism of schemes in characteristic p which is locally on the target a composition of finitely many α_p-covers. In this section we prove that pullback along such a morphism induces a quasi-isomorphism on crystalline cohomology after inverting the prime p. In fact, we prove a precise version of this result. We beging with a preliminary lemma whose formulation needs some notation.

Assume we have a ring map $B \to B'$ and quotients $\Omega_B \to \Omega$ and $\Omega_{B'} \to \Omega'$ satisfying the assumptions of Remark 45.6.11. Thus (45.6.11.1) provides a canonical map of complexes
$$c^\bullet_M : M \otimes_B \Omega^\bullet \longrightarrow M \otimes_B (\Omega')^\bullet$$
for all B-modules M endowed with integrable connection $\nabla : M \to M \otimes_B \Omega_B$.

Suppose we have $a \in B$, $z \in B'$, and a map $\theta : B' \to B'$ satisfying the following assumptions

(1) $\mathrm{d}(a) = 0$,

[7] This is nonstandard notation.

(2) $\Omega' = B' \otimes_B \Omega \oplus B' \mathrm{d}z$; we write $\mathrm{d}(f) = \mathrm{d}_1(f) + \partial_z(f)\mathrm{d}z$ with $\mathrm{d}_1(f) \in B' \otimes \Omega$ and $\partial_z(f) \in N'$ for all $f \in B'$,
(3) $\theta : B' \to B'$ is B-linear,
(4) $\partial_z \circ \theta = a$,
(5) $B \to B'$ is universally injective (and hence $\Omega \to \Omega'$ is injective),
(6) $af - \theta(\partial_z(f)) \in B$ for all $f \in B'$,
(7) $(\theta \otimes 1)(\mathrm{d}_1(f)) - \mathrm{d}_1(\theta(f)) \in \Omega$ for all $f \in B'$ where $\theta \otimes 1 : B' \otimes \Omega \to B' \otimes \Omega$

These conditions are not logically independent. For example, assumption (4) implies that $\partial_z(af - \theta(\partial_z(f))) = 0$. Hence if the image of $B \to B'$ is the collection of elements annihilated by ∂_z, then (6) follows. A similar argument can be made for condition (7).

Lemma 45.25.1. *In the situation above there exists a map of complexes*

$$e_M^\bullet : M \otimes_B (\Omega')^\bullet \longrightarrow M \otimes_B \Omega^\bullet$$

such that $c_M^\bullet \circ e_M^\bullet$ and $e_M^\bullet \circ c_M^\bullet$ are homotopic to multiplication by a.

Proof. In this proof all tensor products are over B. Assumption (2) implies that

$$M \otimes (\Omega')^i = (B' \otimes M \otimes \Omega^i) \oplus (B'\mathrm{d}z \otimes M \otimes \Omega^{i-1})$$

for all $i \geq 0$. A collection of additive generators for $M \otimes (\Omega')^i$ is formed by elements of the form $f\omega$ and elements of the form $f\mathrm{d}z \wedge \eta$ where $f \in B'$, $\omega \in M \otimes \Omega^i$, and $\eta \in M \otimes \Omega^{i-1}$.

For $f \in B'$ we write

$$\epsilon(f) = af - \theta(\partial_z(f)) \quad \text{and} \quad \epsilon'(f) = (\theta \otimes 1)(\mathrm{d}_1(f)) - \mathrm{d}_1(\theta(f))$$

so that $\epsilon(f) \in B$ and $\epsilon'(f) \in \Omega$ by assumptions (6) and (7). We define $e_M^\bullet$ by the rules $e_M^i(f\omega) = \epsilon(f)\omega$ and $e_M^i(f\mathrm{d}z \wedge \eta) = \epsilon'(f) \wedge \eta$. We will see below that the collection of maps e_M^i is a map of complexes.

We define

$$h^i : M \otimes_B (\Omega')^i \longrightarrow M \otimes_B (\Omega')^{i-1}$$

by the rules $h^i(f\omega) = 0$ and $h^i(f\mathrm{d}z \wedge \eta) = \theta(f)\eta$ for elements as above. We claim that

$$\mathrm{d} \circ h + h \circ \mathrm{d} = a - c_M^\bullet \circ e_M^\bullet$$

Note that multiplication by a is a map of complexes by (1). Hence, since $c_M^\bullet$ is an injective map of complexes by assumption (5), we conclude that $e_M^\bullet$ is a map of complexes. To prove the claim we compute

$$\begin{aligned}
(\mathrm{d} \circ h + h \circ \mathrm{d})(f\omega) &= h\left(\mathrm{d}(f) \wedge \omega + f\nabla(\omega)\right) \\
&= \theta(\partial_z(f))\omega \\
&= af\omega - \epsilon(f)\omega \\
&= af\omega - c_M^i(e_M^i(f\omega))
\end{aligned}$$

The second equality because $\mathrm{d}z$ does not occur in $\nabla(\omega)$ and the third equality by assumption (6). Similarly, we have

$$\begin{aligned}(\mathrm{d} \circ h + h \circ \mathrm{d})(f\mathrm{d}z \wedge \eta) &= \mathrm{d}(\theta(f)\eta) + h\left(\mathrm{d}(f) \wedge \mathrm{d}z \wedge \eta - f\mathrm{d}z \wedge \nabla(\eta)\right) \\ &= \mathrm{d}(\theta(f)) \wedge \eta + \theta(f)\nabla(\eta) - (\theta \otimes 1)(\mathrm{d}_1(f)) \wedge \eta - \theta(f)\nabla(\eta) \\ &= \mathrm{d}_1(\theta(f)) \wedge \eta + \partial_z(\theta(f))\mathrm{d}z \wedge \eta - (\theta \otimes 1)(\mathrm{d}_1(f)) \wedge \eta \\ &= af\mathrm{d}z \wedge \eta - \epsilon'(f) \wedge \eta \\ &= af\mathrm{d}z \wedge \eta - c^i_M(e^i_M(f\mathrm{d}z \wedge \eta))\end{aligned}$$

The second equality because $\mathrm{d}(f) \wedge \mathrm{d}z \wedge \eta = -\mathrm{d}z \wedge \mathrm{d}_1(f) \wedge \eta$. The fourth equality by assumption (4). On the other hand it is immediate from the definitions that $e^i_M(c^i_M(\omega)) = \epsilon(1)\omega = a\omega$. This proves the lemma. □

Example 45.25.2. A standard example of the situation above occurs when $B' = B\langle z\rangle$ is the divided power polynomial ring over a divided power ring (B, J, δ) with divided powers δ' on $J' = B'_+ + JB' \subset B'$. Namely, we take $\Omega = \Omega_{B,\delta}$ and $\Omega' = \Omega_{B',\delta'}$. In this case we can take $a = 1$ and

$$\theta(\sum b_m z^{[m]}) = \sum b_m z^{[m+1]}$$

Note that

$$f - \theta(\partial_z(f)) = f(0)$$

equals the constant term. It follows that in this case Lemma 45.25.1 recovers the crystalline Poincaré lemma (Lemma 45.20.2).

Lemma 45.25.3. *In Situation 45.5.1. Assume D and Ω_D are as in (45.17.0.1) and (45.17.0.2). Let $\lambda \in D$. Let D' be the p-adic completion of*

$$D[z]\langle\xi\rangle/(\xi - (z^p - \lambda))$$

and let $\Omega_{D'}$ be the p-adic completion of the module of divided power differentials of D' over A. For any pair (M, ∇) over D satisfying (1), (2), (3), and (4) the canonical map of complexes (45.6.11.1)

$$c^\bullet_M : M \otimes^\wedge_D \Omega^\bullet_D \longrightarrow M \otimes^\wedge_D \Omega^\bullet_{D'}$$

has the following property: There exists a map $e^\bullet_M$ in the opposite direction such that both $c^\bullet_M \circ e^\bullet_M$ and $e^\bullet_M \circ c^\bullet_M$ are homotopic to multiplication by p.

Proof. We will prove this using Lemma 45.25.1 with $a = p$. Thus we have to find $\theta : D' \to D'$ and prove (1), (2), (3), (4), (5), (6), (7). We first collect some information about the rings D and D' and the modules Ω_D and $\Omega_{D'}$.

Writing

$$D[z]\langle\xi\rangle/(\xi - (z^p - \lambda)) = D\langle\xi\rangle[z]/(z^p - \xi - \lambda)$$

we see that D' is the p-adic completion of the free D-module

$$\bigoplus\nolimits_{i=0,\ldots,p-1} \bigoplus\nolimits_{n\geq 0} z^i \xi^{[n]} D$$

where $\xi^{[0]} = 1$. It follows that $D \to D'$ has a continuous D-linear section, in particular $D \to D'$ is universally injective, i.e., (5) holds. We think of D' as a divided power algebra over A with divided power ideal $\overline{J}' = \overline{J}D' + (\xi)$. Then D' is

also the p-adic completion of the divided power envelope of the ideal generated by $z^p - \lambda$ in D, see Lemma 45.2.4. Hence

$$\Omega_{D'} = \Omega_D \otimes^\wedge_D D' \oplus D'\mathrm{d}z$$

by Lemma 45.6.6. This proves (2). Note that (1) is obvious.

At this point we construct θ. (We wrote a PARI/gp script theta.gp verifying some of the formulas in this proof which can be found in the scripts subdirectory of the stacks project.) Before we do so we compute the derivative of the elements $z^i\xi^{[n]}$. We have $\mathrm{d}z^i = iz^{i-1}\mathrm{d}z$. For $n \geq 1$ we have

$$\mathrm{d}\xi^{[n]} = \xi^{[n-1]}\mathrm{d}\xi = -\xi^{[n-1]}\mathrm{d}\lambda + pz^{p-1}\xi^{[n-1]}\mathrm{d}z$$

because $\xi = z^p - \lambda$. For $0 < i < p$ and $n \geq 1$ we have

$$\begin{aligned}
\mathrm{d}(z^i\xi^{[n]}) &= iz^{i-1}\xi^{[n]}\mathrm{d}z + z^i\xi^{[n-1]}\mathrm{d}\xi \\
&= iz^{i-1}\xi^{[n]}\mathrm{d}z + z^i\xi^{[n-1]}\mathrm{d}(z^p - \lambda) \\
&= -z^i\xi^{[n-1]}\mathrm{d}\lambda + (iz^{i-1}\xi^{[n]} + pz^{i+p-1}\xi^{[n-1]})\mathrm{d}z \\
&= -z^i\xi^{[n-1]}\mathrm{d}\lambda + (iz^{i-1}\xi^{[n]} + pz^{i-1}(\xi + \lambda)\xi^{[n-1]})\mathrm{d}z \\
&= -z^i\xi^{[n-1]}\mathrm{d}\lambda + ((i + pn)z^{i-1}\xi^{[n]} + p\lambda z^{i-1}\xi^{[n-1]})\mathrm{d}z
\end{aligned}$$

the last equality because $\xi\xi^{[n-1]} = n\xi^{[n]}$. Thus we see that

$$\begin{aligned}
\partial_z(z^i) &= iz^{i-1} \\
\partial_z(\xi^{[n]}) &= pz^{p-1}\xi^{[n-1]} \\
\partial_z(z^i\xi^{[n]}) &= (i + pn)z^{i-1}\xi^{[n]} + p\lambda z^{i-1}\xi^{[n-1]}
\end{aligned}$$

Motivated by these formulas we define θ by the rules

$$\begin{array}{rcll}
\theta(z^j) & = & p\frac{z^{j+1}}{j+1} & j = 0, \ldots p-1, \\
\theta(z^{p-1}\xi^{[m]}) & = & \xi^{[m+1]} & m \geq 1, \\
\theta(z^j\xi^{[m]}) & = & \frac{pz^{j+1}\xi^{[m]} - \theta(p\lambda z^j\xi^{[m-1]})}{(j+1+pm)} & 0 \leq j < p-1, m \geq 1
\end{array}$$

where in the last line we use induction on m to define our choice of θ. Working this out we get (for $0 \leq j < p-1$ and $1 \leq m$)

$$\theta(z^j\xi^{[m]}) = \tfrac{pz^{j+1}\xi^{[m]}}{(j+1+pm)} - \tfrac{p^2\lambda z^{j+1}\xi^{[m-1]}}{(j+1+pm)(j+1+p(m-1))} + \ldots + \tfrac{(-1)^m p^{m+1}\lambda^m z^{j+1}}{(j+1+pm)\ldots(j+1)}$$

although we will not use this expression below. It is clear that θ extends uniquely to a p-adically continuous D-linear map on D'. By construction we have (3) and (4). It remains to prove (6) and (7).

Proof of (6) and (7). As θ is D-linear and continuous it suffices to prove that $p - \theta \circ \partial_z$, resp. $(\theta \otimes 1) \circ \mathrm{d}_1 - \mathrm{d}_1 \circ \theta$ gives an element of D, resp. Ω_D when evaluated on the elements $z^i\xi^{[n]}$[8]. Set $D_0 = \mathbf{Z}_{(p)}[\lambda]$ and $D'_0 = \mathbf{Z}_{(p)}[z, \lambda]\langle\xi\rangle/(\xi - z^p + \lambda)$. Observe that each of the expressions above is an element of D'_0 or $\Omega_{D'_0}$. Hence it suffices to prove the result in the case of $D_0 \to D'_0$. Note that D_0 and D'_0 are torsion free rings and that $D_0 \otimes \mathbf{Q} = \mathbf{Q}[\lambda]$ and $D'_0 \otimes \mathbf{Q} = \mathbf{Q}[z, \lambda]$. Hence $D_0 \subset D'_0$ is the

[8]This can be done by direct computation: It turns out that $p - \theta \circ \partial_z$ evaluated on $z^i\xi^{[n]}$ gives zero except for 1 which is mapped to p and ξ which is mapped to $-p\lambda$. It turns out that $(\theta \otimes 1) \circ \mathrm{d}_1 - \mathrm{d}_1 \circ \theta$ evaluated on $z^i\xi^{[n]}$ gives zero except for $z^{p-1}\xi$ which is mapped to $-\lambda$.

subring of elements annihilated by ∂_z and (6) follows from (4), see the discussion directly preceding Lemma 45.25.1. Similarly, we have $\mathrm{d}_1(f) = \partial_\lambda(f)\mathrm{d}\lambda$ hence

$$((\theta \otimes 1) \circ \mathrm{d}_1 - \mathrm{d}_1 \circ \theta)(f) = (\theta(\partial_\lambda(f)) - \partial_\lambda(\theta(f)))\,\mathrm{d}\lambda$$

Applying ∂_z to the coefficient we obtain

$$\begin{aligned}\partial_z(\theta(\partial_\lambda(f)) - \partial_\lambda(\theta(f))) &= p\partial_\lambda(f) - \partial_z(\partial_\lambda(\theta(f))) \\ &= p\partial_\lambda(f) - \partial_\lambda(\partial_z(\theta(f))) \\ &= p\partial_\lambda(f) - \partial_\lambda(pf) = 0\end{aligned}$$

whence the coefficient does not depend on z as desired. This finishes the proof of the lemma. □

Note that an iterated α_p-cover $X' \to X$ (as defined in the introduction to this section) is finite locally free. Hence if X is connected the degree of $X' \to X$ is constant and is a power of p.

Lemma 45.25.4. *Let p be a prime number. Let $(S, \mathcal{I}, \gamma)$ be a divided power scheme over $\mathbf{Z}_{(p)}$ with $p \in \mathcal{I}$. We set $S_0 = V(\mathcal{I}) \subset S$. Let $f : X' \to X$ be an iterated α_p-cover of schemes over S_0 with constant degree q. Let $\mathcal{F}$ be any crystal in quasi-coherent sheaves on X and set $\mathcal{F}' = f^*_{cris}\mathcal{F}$. In the distinguished triangle*

$$Ru_{X/S,*}\mathcal{F} \longrightarrow f_*Ru_{X'/S,*}\mathcal{F}' \longrightarrow E \longrightarrow Ru_{X/S,*}\mathcal{F}[1]$$

the object E has cohomology sheaves annihilated by q.

Proof. Note that $X' \to X$ is a homeomorphism hence we can identify the underlying topological spaces of X and X'. The question is clearly local on X, hence we may assume X, X', and S affine and $X' \to X$ given as a composition

$$X' = X_n \to X_{n-1} \to X_{n-2} \to \ldots \to X_0 = X$$

where each morphism $X_{i+1} \to X_i$ is an α_p-cover. Denote $\mathcal{F}_i$ the pullback of $\mathcal{F}$ to X_i. It suffices to prove that each of the maps

$$R\Gamma(\mathrm{Cris}(X_i/S), \mathcal{F}_i) \longrightarrow R\Gamma(\mathrm{Cris}(X_{i+1}/S), \mathcal{F}_{i+1})$$

fits into a triangle whose third member has cohomology groups annihilated by p. (This uses axiom TR4 for the triangulated category $D(X)$. Details omitted.)

Hence we may assume that $S = \mathrm{Spec}(A)$, $X = \mathrm{Spec}(C)$, $X' = \mathrm{Spec}(C')$ and $C' = C[z]/(z^p - c)$ for some $c \in C$. Choose a polynomial algebra P over A and a surjection $P \to C$. Let D be the p-adically completed divided power envelop of $\mathrm{Ker}(P \to C)$ in P as in (45.17.0.1). Set $P' = P[z]$ with surjection $P' \to C'$ mapping z to the class of z in C'. Choose a lift $\lambda \in D$ of $c \in C$. Then we see that the p-adically completed divided power envelope D' of $\mathrm{Ker}(P' \to C')$ in P' is isomorphic to the p-adic completion of $D[z]\langle\xi\rangle/(\xi - (z^p - \lambda))$, see Lemma 45.25.3 and its proof. Thus we see that the result follows from this lemma by the computation of cohomology of crystals in quasi-coherent modules in Proposition 45.21.3. □

The bound in the following lemma is probably not optimal.

Lemma 45.25.5. *With notations and assumptions as in Lemma 45.25.4 the map*

$$f^* : H^i(\mathit{Cris}(X/S), \mathcal{F}) \longrightarrow H^i(\mathit{Cris}(X'/S), \mathcal{F}')$$

has kernel and cokernel annihilated by q^{i+1}.

Proof. This follows from the fact that E has nonzero cohomology sheaves in degrees -1 and up, so that the spectral sequence $H^a(\mathcal{H}^b(E)) \Rightarrow H^{a+b}(E)$ converges. This combined with the long exact cohomology sequence associated to a distinguished triangle gives the bound. □

In Situation 45.7.5 assume that $p \in \mathcal{I}$. Set

$$X^{(1)} = X \times_{S_0, F_{S_0}} S_0.$$

Denote $F_{X/S_0} : X \to X^{(1)}$ the relative Frobenius morphism.

Lemma 45.25.6. *In the situation above, assume that $X \to S_0$ is smooth of relative dimension d. Then F_{X/S_0} is an iterated α_p-cover of degree p^d. Hence Lemmas 45.25.4 and 45.25.5 apply to this situation. In particular, for any crystal in quasi-coherent modules $\mathcal{G}$ on $Cris(X^{(1)}/S)$ the map*

$$F^*_{X/S_0} : H^i(Cris(X^{(1)}/S), \mathcal{G}) \longrightarrow H^i(Cris(X/S), F^*_{X/S_0, cris}\mathcal{G})$$

has kernel and cokernel annihilated by $p^{d(i+1)}$.

Proof. It suffices to prove the first statement. To see this we may assume that X is étale over $\mathbf{A}^d_{S_0}$, see Morphisms, Lemma 28.37.20. Denote $\varphi : X \to \mathbf{A}^d_{S_0}$ this étale morphism. In this case the relative Frobenius of X/S_0 fits into a diagram

$$\begin{array}{ccc} X & \longrightarrow & X^{(1)} \\ \downarrow & & \downarrow \\ \mathbf{A}^d_{S_0} & \longrightarrow & \mathbf{A}^d_{S_0} \end{array}$$

where the lower horizontal arrow is the relative frobenius morphism of $\mathbf{A}^d_{S_0}$ over S_0. This is the morphism which raises all the coordinates to the pth power, hence it is an iterated α_p-cover. The proof is finished by observing that the diagram is a fibre square, see the proof of Étale Cohomology, Theorem 44.74.4. □

45.26. Frobenius action on crystalline cohomology

In this section we prove that Frobenius pullback induces a quasi-isomorphism on crystalline cohomology after inverting the prime p. But in order to even formulate this we need to work in a special situation.

Situation 45.26.1. In Situation 45.7.5 assume the following

(1) $S = \mathrm{Spec}(A)$ for some divided power ring (A, I, γ) with $p \in I$,
(2) there is given a homomorphism of divided power rings $\sigma : A \to A$ such that $\sigma(x) = x^p \bmod pA$ for all $x \in A$.

In Situation 45.26.1 the morphism $\mathrm{Spec}(\sigma) : S \to S$ is a lift of the absolute Frobenius $F_{S_0} : S_0 \to S_0$ and since the diagram

$$\begin{array}{ccc} X & \xrightarrow[F_X]{} & X \\ \downarrow & & \downarrow \\ S_0 & \xrightarrow{F_{S_0}} & S_0 \end{array}$$

is commutative where $F_X : X \to X$ is the absolute Frobenius morphism of X. Thus we obtain a morphism of crystalline topoi

$$(F_X)_{\text{cris}} : (X/S)_{\text{cris}} \longrightarrow (X/S)_{\text{cris}}$$

see Remark 45.9.3. Here is the terminology concerning F-crystals following the notation of Saavedra, see [**SR72**].

Definition 45.26.2. In Situation 45.26.1 an *F-crystal on X/S (relative to σ)* is a pair $(\mathcal{E}, F_\mathcal{E})$ given by a crystal in finite locally free $\mathcal{O}_{X/S}$-modules $\mathcal{E}$ together with a map

$$F_\mathcal{E} : (F_X)^*_{\text{cris}}\mathcal{E} \longrightarrow \mathcal{E}$$

An F-crystal is called *nondegenerate* if there exists an integer $i \geq 0$ a map $V : \mathcal{E} \to (F_X)^*_{\text{cris}}\mathcal{E}$ such that $V \circ F_\mathcal{E} = p^i\text{id}$.

Remark 45.26.3. Let $(\mathcal{E}, F)$ be an F-crystal as in Definition 45.26.2. In the literature the nondegeneracy condition is often part of the definition of an F-crystal. Moreover, often it is also assumed that $F \circ V = p^n\text{id}$. What is needed for the result below is that there exists an integer $j \geq 0$ such that $\text{Ker}(F)$ and $\text{Coker}(F)$ are killed by p^j. If the rank of $\mathcal{E}$ is bounded (for example if X is quasi-compact), then both of these conditions follow from the nondegeneracy condition as formulated in the definition. Namely, suppose R is a ring, $r \geq 1$ is an integer and $K, L \in \text{Mat}(r \times r, R)$ are matrices with $KL = p^i 1_{r\times r}$. Then $\det(K)\det(L) = p^{ri}$. Let L' be the adjugate matrix of L, i.e., $L'L = LL' = \det(L)$. Set $K' = p^{ri}K$ and $j = ri + i$. Then we have $K'L = p^j 1_{r\times r}$ as $KL = p^i$ and

$$LK' = LK\det(L)\det(M) = LKLL'\det(M) = Lp^iL'\det(M) = p^j 1_{r\times r}$$

It follows that if V is as in Definition 45.26.2 then setting $V' = p^N V$ where $N > i \cdot \text{rank}(\mathcal{E})$ we get $V' \circ F = p^{N+i}$ and $F \circ V' = p^{N+i}$.

Theorem 45.26.4. *In Situation 45.26.1 let $(\mathcal{E}, F_\mathcal{E})$ be a nondegenerate F-crystal. Assume A is a p-adically complete Noetherian ring and that $X \to S_0$ is proper smooth. Then the canonical map*

$$F_\mathcal{E} \circ (F_X)^*_{cris} : R\Gamma(Cris(X/S), \mathcal{E}) \otimes^{\mathbf{L}}_{A,\sigma} A \longrightarrow R\Gamma(Cris(X/S), \mathcal{E})$$

becomes an isomorphism after inverting p.

Proof. We first write the arrow as a composition of three arrows. Namely, set

$$X^{(1)} = X \times_{S_0, F_{S_0}} S_0$$

and denote $F_{X/S_0} : X \to X^{(1)}$ the relative Frobenius morphism. Denote $\mathcal{E}^{(1)}$ the base change of $\mathcal{E}$ by $\text{Spec}(\sigma)$, in other words the pullback of $\mathcal{E}$ to $\text{Cris}(X^{(1)}/S)$ by the morphism of crystalline topoi associated to the commutative diagram

$$\begin{array}{ccc} X^{(1)} & \longrightarrow & X \\ \downarrow & & \downarrow \\ S & \xrightarrow{\text{Spec}(\sigma)} & S \end{array}$$

Then we have the base change map

$$R\Gamma(\text{Cris}(X/S), \mathcal{E}) \otimes^{\mathbf{L}}_{A,\sigma} A \longrightarrow R\Gamma(\text{Cris}(X^{(1)}/S), \mathcal{E}^{(1)}) \tag{45.26.4.1}$$

see Remark 45.24.8. Note that the composition of $F_{X/S_0} : X \to X^{(1)}$ with the projection $X^{(1)} \to X$ is the absolute Frobenius morphism F_X. Hence we see that $F_{X/S_0}^* \mathcal{E}^{(1)} = (F_X)_{\text{cris}}^* \mathcal{E}$. Thus pullback by F_{X/S_0} is a map

$$F_{X/S_0}^* : R\Gamma(\text{Cris}(X^{(1)}/S), \mathcal{E}^{(1)}) \longrightarrow R\Gamma(\text{Cris}(X/S), (F_X)_{\text{cris}}^* \mathcal{E}) \tag{45.26.4.2}$$

Finally we can use $F_{\mathcal{E}}$ to get a map

$$R\Gamma(\text{Cris}(X/S), (F_X)_{\text{cris}}^* \mathcal{E}) \longrightarrow R\Gamma(\text{Cris}(X/S), \mathcal{E}) \tag{45.26.4.3}$$

The map of the theorem is the composition of the three maps (45.26.4.1), (45.26.4.2), and (45.26.4.3) above. The first is a quasi-isomorphism modulo all powers of p by Remark 45.24.9. Hence it is a quasi-isomorphism since the complexes involved are perfect in $D(A)$ see Remark 45.24.13. The third map is a quasi-isomorphism after inverting p simply because $F_{\mathcal{E}}$ has an inverse up to a power of p, see Remark 45.26.3. Finally, the second is an isomorphism after inverting p by Lemma 45.25.6. □

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CHAPTER 46

Pro-étale Cohomology

46.1. Introduction

The material in this chapter and more can be found in the preprint [**BS13**].

The goal of this chapter is to introduce the pro-étale topology and show how it simplifies the introduction of ℓ-adic cohomology in algebraic geometry.

A brief overview of the history of this material as we have understood it. In [**Gro77**, Exposés V and VI] Grothendieck et al developed a theory for dealing with ℓ-adic sheaves as inverse systems of sheaves of $\mathbf{Z}/\ell^n\mathbf{Z}$-modules. In his second paper on the Weil conjectures ([**Del74a**]) Deligne introduced a derived category of ℓ-adic sheaves as a certain 2-limit of categories of complexes of sheaves of $\mathbf{Z}/\ell^n\mathbf{Z}$-modules on the étale site of a scheme X. This approach is used in the paper by Beilinson, Bernstein, and Deligne ([**BBD82**]) as the basis for their beautiful theory of perverse sheaves. In a paper entitled "Continuous Étale Cohomology" ([**Jan88**]) Uwe Jannsen discusses an important variant of the cohomology of a ℓ-adic sheaf on a variety over a field. His paper is followed up by a paper of Torsten Ekedahl ([**Eke90**]) who discusses the adic formalism needed to work comfortably with derived categories defined as limits.

The goal of this chapter is to show that, if we work with the pro-étale site of a scheme, then one can avoid some of the technicalities these authors encountered. This comes at the expense of having to work with non-Noetherian schemes, even when one is only interested in working with ℓ-adic sheaves and cohomology of such on varieties over an algebraically closed field.

46.2. Some topology

Some preliminaries. We have defined *spectral spaces* and *spectral maps* of spectral spaces in Topology, Section 5.22. The spectrum of a ring is a spectral space, see Algebra, Lemma 10.24.2.

Lemma 46.2.1. *Let X be a spectral space. Let $X_0 \subset X$ be the set of closed points. The following are equivalent*

1. *Every open covering of X can be refined by a finite disjoint union decomposition $X = \coprod U_i$ with U_i open and closed in X.*
2. *The composition $X_0 \to X \to \pi_0(X)$ is bijective.*

Moreover, if X_0 is closed in X and every point of X specializes to a unique point of X_0, then these conditions are satisfied.

Proof. We will use without further mention that X_0 is quasi-compact (Topology, Lemma 5.11.9) and $\pi_0(X)$ is profinite (Topology, Lemma 5.22.7). Picture

$$\begin{array}{ccc} X_0 & \longrightarrow & X \\ & \searrow^{f} & \downarrow{\scriptstyle \pi} \\ & & \pi_0(X) \end{array}$$

If (2) holds, the continuous bijective map $f : X_0 \to \pi_0(X)$ is a homeomorphism by Topology, Lemma 5.16.8. Given an open covering $X = \bigcup U_i$, we get an open covering $\pi_0(X) = \bigcup f(X_0 \cap U_i)$. By Topology, Lemma 5.21.3 we can find a finite open covering of the form $\pi_0(X) = \coprod V_j$ which refines this covering. Since $X_0 \to \pi_0(X)$ is bijective each connected component of X has a unique closed point, whence is equal to the set of points specializing to this closed point. Hence $\pi^{-1}(V_j)$ is the set of points specializing to the points of $f^{-1}(V_j)$. Now, if $f^{-1}(V_j) \subset X_0 \cap U_i \subset U_i$, then it follows that $\pi^{-1}(V_j) \subset U_i$ (because the open set U_i is closed under generalizations). In this way we see that the open covering $X = \coprod \pi^{-1}(V_j)$ refines the covering we started out with. In this way we see that (2) implies (1).

Assume (1). Let $x, y \in X$ be closed points. Then we have the open covering $X = (X \setminus \{x\}) \cup (X \setminus \{y\})$. It follows from (1) that there exists a disjoint union decomposition $X = U \amalg V$ with U and V open (and closed) and $x \in U$ and $y \in V$. In particular we see that every connected component of X has at most one closed point. By Topology, Lemma 5.11.8 every connected component (being closed) also does have a closed point. Thus $X_0 \to \pi_0(X)$ is bijective. In this way we see that (1) implies (2).

Assume X_0 is closed in X and every point specializes to a unique point of X_0. Then X_0 is a spectral space (Topology, Lemma 5.22.3) consisting of closed points, hence profinite (Topology, Lemma 5.22.6). Let $x, y \in X_0$ be distinct. By Topology, Lemma 5.21.3 we can find a disjoint union decomposition $X_0 = U_0 \amalg V_0$ with U_0 and V_0 open and closed. Let $\{U_i\}$ be the set of quasi-compact open subsets of X such that $U_0 = X_0 \cap U_i$. Similarly, let $\{V_j\}$ be the set of quasi-compact open subsets of X such that and $V_0 = X_0 \cap V_j$. If $U_i \cap V_j$ is nonempty for all i, j, then there exists a point ξ contained in all of them (use the $U_i \cap V_j$ is constructible, hence closed in the constructible topology, and use Topology, Lemmas 5.22.2 and 5.11.6). However, since X is sober and V_0 is closed in X, the intersection $\bigcap U_i$ is the set of points specializing to U_0. Similarly, $\bigcap V_j$ is the set of points specializing to V_0. Since $U_0 \cap V_0$ is empty this is a contradiction. Thus we find disjoint quasi-compact opens $U, V \subset X$ such that $U_0 = X_0 \cap U$ and $V_0 = X_0 \cap V$. Observe that $X = U \cup V = U \amalg V$ as $X_0 \subset U \cup V$ (use Topology, Lemma 5.11.8). This proves that x, y are not in the same connected component of X. In other words, $X_0 \to \pi_0(X)$ is injective. The map is also surjective by Topology, Lemma 5.11.8 and the fact that connected components are closed. In this way we see that the final condition implies (1). □

Example 46.2.2. Let T be a profinite space. Let $t \in T$ be a point and assume that $T \setminus \{t\}$ is not quasi-compact. Let $X = T \times \{0, 1\}$. Consider the topology on X with a subbase given by the sets $U \times \{0, 1\}$ for $U \subset T$ open, $X \setminus \{(t, 0)\}$, and $U \times \{1\}$ for $U \subset T$ open with $t \notin U$. The set of closed points of X is $X_0 = T \times \{0\}$ and $(t, 1)$ is in the closure of X_0. Moreover, $X_0 \to \pi_0(X)$ is a bijection. This example

shows that conditions (1) and (2) of Lemma 46.2.1 do no imply the set of closed points is closed.

It turns out it is more convenient to work with spectral spaces which have the slightly stronger property mentioned in the final statement of Lemma 46.2.1. We give this property a name.

Definition 46.2.3. A spectral space X is *w-local* if the set of closed points X_0 is closed and every point of X specializes to a unique closed point. A continuous map $f : X \to Y$ of w-local spaces is *w-local* if it is spectral and maps any closed point of X to a closed point of Y.

We have seen in the proof of Lemma 46.2.1 that in this case $X_0 \to \pi_0(X)$ is a homeomorphism and that $X_0 \cong \pi_0(X)$ is a profinite space. Moreover, a connected component of X is exactly the set of points specializing to a given $x \in X_0$.

Lemma 46.2.4. *Let X be a w-local spectral space. If $Y \subset X$ is closed, then Y is w-local.*

Proof. The subset $Y_0 \subset Y$ of closed points is closed because $Y_0 = X_0 \cap Y$. Since X is w-local, every $y \in Y$ specializes to a unique point of X_0. This specialization is in Y, and hence also in Y_0, because $\overline{\{y\}} \subset Y$. In conclusion, Y is w-local. □

Lemma 46.2.5. *Let X be a spectral space. Let*

$$\begin{array}{ccc} Y & \longrightarrow & T \\ \downarrow & & \downarrow \\ X & \longrightarrow & \pi_0(X) \end{array}$$

be a cartesian diagram in the category of topological spaces with T profinite. Then Y is spectral and $T = \pi_0(Y)$. If moreover X is w-local, then Y is w-local, $Y \to X$ is w-local, and the set of closed points of Y is the inverse image of the set of closed points of X.

Proof. Note that Y is a closed subspace of $X \times T$ as $\pi_0(X)$ is a profinite space hence Hausdorff (use Topology, Lemmas 5.22.7 and 5.3.4). Since $X \times T$ is spectral (Topology, Lemma 5.22.8) it follows that Y is spectral (Topology, Lemma 5.22.3). Let $Y \to \pi_0(Y) \to T$ be the canonical factorization (Topology, Lemma 5.6.8). It is clear that $\pi_0(Y) \to T$ is surjective. The fibres of $Y \to T$ are homeomorphic to the fibres of $X \to \pi_0(X)$. Hence these fibres are connected. It follows that $\pi_0(Y) \to T$ is injective. We conclude that $\pi_0(Y) \to T$ is a homeomorphism by Topology, Lemma 5.16.8.

Next, assume that X is w-local and let $X_0 \subset X$ be the set of closed points. The inverse image $Y_0 \subset Y$ of X_0 in Y maps bijectively onto T as $X_0 \to \pi_0(X)$ is a bijection by Lemma 46.2.1. Moreover, Y_0 is quasi-compact as a closed subset of the spectral space Y. Hence $Y_0 \to \pi_0(Y) = T$ is a homeomorphism by Topology, Lemma 5.16.8. It follows that all points of Y_0 are closed in Y. Conversely, if $y \in Y$ is a closed point, then it is closed in the fibre of $Y \to \pi_0(Y) = T$ and hence its image x in X is closed in the (homeomorphic) fibre of $X \to \pi_0(X)$. This implies $x \in X_0$ and hence $y \in Y_0$. Thus Y_0 is the collection of closed points of Y and for each $y \in Y_0$ the set of generalizations of y is the fibre of $Y \to \pi_0(Y)$. The lemma follows. □

46.3. Local isomorphisms

We start with a definition.

Definition 46.3.1. Let $\varphi : A \to B$ be a ring map.

(1) We say $A \to B$ is a *local isomorphism* if for every prime $\mathfrak{q} \subset B$ there exists a $g \in B$, $g \notin \mathfrak{q}$ such that $A \to B_g$ induces an open immersion $\mathrm{Spec}(B_g) \to \mathrm{Spec}(A)$.
(2) We say $A \to B$ *identifies local rings* if for every prime $\mathfrak{q} \subset B$ the canonical map $A_{\varphi^{-1}(\mathfrak{q})} \to B_{\mathfrak{q}}$ is an isomorphism.

We list some elementary properties.

Lemma 46.3.2. *Let $A \to B$ and $A \to A'$ be ring maps. Let $B' = B \otimes_A A'$ be the base change of B.*

(1) *If $A \to B$ is a local isomorphism, then $A' \to B'$ is a local isomorphism.*
(2) *If $A \to B$ identifies local rings, then $A' \to B'$ identifies local rings.*

Proof. Omitted. □

Lemma 46.3.3. *Let $A \to B$ and $B \to C$ be ring maps.*

(1) *If $A \to B$ and $B \to C$ are local isomorphisms, then $A \to C$ is a local isomorphism.*
(2) *If $A \to B$ and $B \to C$ identify local rings, then $A \to C$ identifies local rings.*

Proof. Omitted. □

Lemma 46.3.4. *Let A be a ring. Let $B \to C$ be an A-algebra homomorphism.*

(1) *If $A \to B$ and $A \to C$ are local isomorphisms, then $B \to C$ is a local isomorphism.*
(2) *If $A \to B$ and $A \to C$ identify local rings, then $B \to C$ identifies local rings.*

Proof. Omitted. □

Lemma 46.3.5. *Let $A \to B$ be a local isomorphism. Then*

(1) *$A \to B$ is étale,*
(2) *$A \to B$ identifies local rings,*
(3) *$A \to B$ is quasi-finite.*

Proof. Omitted. □

Lemma 46.3.6. *Let $A \to B$ be a local isomorphism. Then there exist $n \geq 0$, $g_1, \ldots, g_n \in B$, $f_1, \ldots, f_n \in A$ such that $(g_1, \ldots, g_n) = B$ and $A_{f_i} \cong B_{g_i}$.*

Proof. Omitted. □

Lemma 46.3.7. *Let $p : (Y, \mathcal{O}_Y) \to (X, \mathcal{O}_X)$ and $q : (Z, \mathcal{O}_Z) \to (X, \mathcal{O}_X)$ be morphisms of locally ringed spaces. If $\mathcal{O}_Y = p^{-1}\mathcal{O}_X$, then*

$$\mathrm{Mor}_{LRS/(X,\mathcal{O}_X)}((Z, \mathcal{O}_Z), (Y, \mathcal{O}_Y)) \longrightarrow \mathrm{Mor}_{Top/X}(Z, Y), \quad (f, f^\sharp) \longmapsto f$$

is bijective. Here $LRS/(X, \mathcal{O}_X)$ is the category of locally ringed spaces over X and Top/X is the category of topological space over X.

Proof. This is immediate from the definitions. □

Lemma 46.3.8. *Let A be a ring. Set $X = \mathrm{Spec}(A)$. The functor*

$$B \longmapsto \mathrm{Spec}(B)$$

from the category of A-algebras B such that $A \to B$ identifies local rings to the category of topological spaces over X is fully faithful.

Proof. This follows from Lemma 46.3.7 and the fact that if $A \to B$ identifies local rings, then the pullback of the structure sheaf of $\mathrm{Spec}(A)$ via $p : \mathrm{Spec}(B) \to \mathrm{Spec}(A)$ is equal to the structure sheaf of $\mathrm{Spec}(B)$. □

46.4. Ind-Zariski algebra

We start with a definition.

Definition 46.4.1. A ring map $A \to B$ is said to be *ind-Zariski* if B can be written as a filtered colimit $B = \mathrm{colim}\, B_i$ with each $A \to B_i$ a local isomorphism.

An example of an Ind-Zariski map is a localization $A \to S^{-1}A$, see Algebra, Lemma 10.9.9. The category of ind-Zariski algebras is closed under several natural operations.

Lemma 46.4.2. *Let $A \to B$ and $A \to A'$ be ring maps. Let $B' = B \otimes_A A'$ be the base change of B. If $A \to B$ is ind-Zariski, then $A' \to B'$ is ind-Zariski.*

Proof. Omitted. □

Lemma 46.4.3. *Let $A \to B$ and $B \to C$ be ring maps. If $A \to B$ and $B \to C$ are ind-Zariski, then $A \to C$ is ind-Zariski.*

Proof. Omitted. □

Lemma 46.4.4. *Let A be a ring. Let $B \to C$ be an A-algebra homomorphism. If $A \to B$ and $A \to C$ are ind-Zariski, then $B \to C$ is ind-Zariski.*

Proof. Omitted. □

Lemma 46.4.5. *A filtered colimit of ind-Zariski A-algebras is ind-Zariski over A.*

Proof. Omitted. □

Lemma 46.4.6. *Let $A \to B$ be ind-Zariski. Then $A \to B$ identifies local rings,*

Proof. Omitted. □

46.5. Constructing w-local affine schemes

An affine scheme X is called *w-local* if its underlying topological space is w-local (Definition 46.2.3). It turns out given any ring A there is a canonical faithfully flat ind-Zariski ring map $A \to A_w$ such that $\mathrm{Spec}(A_w)$ is w-local. The key to constructing A_w is the following simple lemma.

Lemma 46.5.1. *Let A be a ring. Set $X = \mathrm{Spec}(A)$. Let $Z \subset X$ be a locally closed subscheme which of the form $D(f) \cap V(I)$ for some $f \in A$ and ideal $I \subset A$. Then*

1. *there exists a multiplicative subset $S \subset A$ such that $\mathrm{Spec}(S^{-1}A)$ maps by a homeomorphism to the set of points of X specializing to Z,*
2. *the A-algebra $A^\sim_Z = S^{-1}A$ depends only on the underlying locally closed subset $Z \subset X$,*

(3) *Z is a closed subscheme of* $\mathrm{Spec}(A^\sim_Z)$,

If $A \to A'$ is a ring map and $Z' \subset X' = \mathrm{Spec}(A')$ is a locally closed subscheme of the same form which maps into Z, then there is a unique A-algebra map $A^\sim_Z \to (A')^\sim_{Z'}$.

Proof. Let $S \subset A$ be the multiplicative set of elements which map to invertible elements of $\Gamma(Z, \mathcal{O}_Z) = (A/I)_f$. If $\mathfrak{p}$ is a prime of A which does not specialize to Z, then $\mathfrak{p}$ generates the unit ideal in $(A/I)_f$. Hence we can write $f^n = g + h$ for some $n \geq 0$, $g \in \mathfrak{p}$, $h \in I$. Then $g \in S$ and we see that $\mathfrak{p}$ is not in the spectrum of $S^{-1}A$. Conversely, if $\mathfrak{p}$ does specialize to Z, say $\mathfrak{p} \subset \mathfrak{q} \supset I$ with $f \notin \mathfrak{q}$, then we see that $S^{-1}A$ maps to $A_\mathfrak{q}$ and hence $\mathfrak{p}$ is in the spectrum of $S^{-1}A$. This proves (1).

The isomorphism class of the localization $S^{-1}A$ depends only on the corresponding subset $\mathrm{Spec}(S^{-1}A) \subset \mathrm{Spec}(A)$, whence (2) holds. By construction $S^{-1}A$ maps surjectively onto $(A/I)_f$, hence (3). The final statement follows as the multiplicative subset $S' \subset A'$ corresponding to Z' contains the image of the multiplicative subset S. □

Let A be a ring. Let $E \subset A$ be a finite subset. We get a stratification of $X = \mathrm{Spec}(A)$ into locally closed subschemes by looking at the vanishing behaviour of the elements of E. More precisely, given a disjoint union decomposition $E = E' \amalg E''$ we set

(46.5.1.1)
$$Z(E', E'') = \bigcap\nolimits_{f \in E'} D(f) \cap \bigcap\nolimits_{f \in E''} V(f) = D(\prod\nolimits_{f \in E'} f) \cap V(\sum\nolimits_{f \in E''} fA)$$

The points of $Z(E', E'')$ are exactly those $x \in X$ such that $f \in E'$ maps to a nonzero element in $\kappa(x)$ and $f \in E''$ maps to zero in $\kappa(x)$. Thus it is clear that

$$X = \coprod\nolimits_{E = E' \amalg E''} Z(E', E'') \tag{46.5.1.2}$$

set theoretically. Observe that each stratum is constructible.

Lemma 46.5.2. *Let $X = \mathrm{Spec}(A)$ as above. Given any finite stratification $X = \coprod T_i$ by constructible subsets, there exists a finite subset $E \subset A$ such that the stratification (46.5.1.2) refines $X = \coprod T_i$.*

Proof. We may write $T_i = \bigcup_j U_{i,j} \cap V_{i,j}^c$ as a finite union for some $U_{i,j}$ and $V_{i,j}$ quasi-compact open in X. Then we may write $U_{i,j} = \bigcup D(f_{i,j,k})$ and $V_{i,j} = \bigcup D(g_{i,j,l})$. Then we set $E = \{f_{i,j,k}\} \cup \{g_{i,j,l}\}$. This does the job, because the stratification (46.5.1.2) is the one whose strata are labeled by the vanishing pattern of the elements of E which clearly refines the given stratification. □

We continue the discussion. Given a finite subset $E \subset A$ we set

$$A_E = \prod\nolimits_{E = E' \amalg E''} A^\sim_{Z(E', E'')} \tag{46.5.2.1}$$

with notation as in Lemma 46.5.1. This makes sense because (46.5.1.1) shows that each $Z(E', E'')$ has the correct shape. We take the spectrum of this ring and denote it

$$X_E = \mathrm{Spec}(A_E) = \coprod\nolimits_{E = E' \amalg E''} X_{E', E''} \tag{46.5.2.2}$$

with $X_{E', E''} = \mathrm{Spec}(A^\sim_{Z(E', E'')})$. Note that

$$Z_E = \coprod\nolimits_{E = E' \amalg E''} Z(E', E'') \longrightarrow X_E \tag{46.5.2.3}$$

is a closed subscheme. By construction the closed subscheme Z_E contains all the closed points of the affine scheme X_E as every point of $X_{E',E''}$ specializes to a point of $Z(E', E'')$.

Let $I(A)$ be the partially ordered set of all finite subsets of A. This is a directed partially ordered set. For $E_1 \subset E_2$ there is a canonical transition map $A_{E_1} \to A_{E_2}$ of A-algebras. Namely, given a decomposition $E_2 = E_2' \amalg E_2''$ we set $E_1' = E_1 \cap E_2'$ and $E_1'' = E_1 \cap E_2''$. Then observe that $Z(E_1', E_1'') \subset Z(E_2', E_2'')$ hence a unique A-algebra map $A^\sim_{Z(E_1', E_1'')} \to A^\sim_{Z(E_2', E_2'')}$ by Lemma 46.5.1. Using these maps collectively we obtain the desired ring map $A_{E_1} \to A_{E_2}$. Observe that the corresponding map of affine schemes

$$X_{E_2} \longrightarrow X_{E_1} \tag{46.5.2.4}$$

maps Z_{E_2} into Z_{E_1}. By uniqueness we obtain a system of A-algebras over $I(A)$ and we set

$$A_w = \operatorname{colim}_{E \in I(A)} A_E \tag{46.5.2.5}$$

This A-algebra is ind-Zariski and faithfully flat over A. Finally, we set $X_w = \operatorname{Spec}(A_w)$ and endow it with the closed subscheme $Z = \lim_{E \in I(A)} Z_E$. In a formula

$$X_w = \lim_{E \in I(A)} X_E \supset Z = \lim_{E \in I(A)} Z_E \tag{46.5.2.6}$$

Lemma 46.5.3. *Let $X = \operatorname{Spec}(A)$ be an affine scheme. With $A \to A_w$, $X_w = \operatorname{Spec}(A_w)$, and $Z \subset X_w$ as above.*

1. *$A \to A_w$ is ind-Zariski and faithfully flat,*
2. *$X_w \to X$ induces a bijection $Z \to X$,*
3. *Z is the set of closed points of X_w,*
4. *Z is a reduced scheme, and*
5. *every point of X_w specializes to a unique point of Z.*

In particular, X_w is w-local (Definition 46.2.3).

Proof. The map $A \to A_w$ is ind-Zariski by construction. For every E the morphism $Z_E \to X$ is a bijection, hence (2). As $Z \subset X_w$ we conclude $X_w \to X$ is surjective and $A \to A_w$ is faithfully flat by Algebra, Lemma 10.36.15. This proves (1).

Suppose that $y \in X_w$, $y \notin Z$. Then there exists an E such that the image of y in X_E is not contained in Z_E. Then for all $E \subset E'$ also y maps to an element of $X_{E'}$ not contained in $Z_{E'}$. Let $T_{E'} \subset X_{E'}$ be the reduced closed subscheme which is the closure of the image of y. It is clear that $T = \lim_{E \subset E'} T_{E'}$ is the closure of y in X_w. For every $E \subset E'$ the scheme $T_{E'} \cap Z_{E'}$ is nonempty by construction of $X_{E'}$. Hence $\lim T_{E'} \cap Z_{E'}$ is nonempty and we conclude that $T \cap Z$ is nonempty. Thus y is not a closed point. It follows that every closed point of X_w is in Z.

Suppose that $y \in X_w$ specializes to $z, z' \in Z$. We will show that $z = z'$ which will finish the proof of (3) and will imply (5). Let $x, x' \in X$ be the images of z and z'. Since $Z \to X$ is bijective it suffices to show that $x = x'$. If $x \neq x'$, then there exists an $f \in A$ such that $x \in D(f)$ and $x' \in V(f)$ (or vice versa). Set $E = \{f\}$ so that

$$X_E = \operatorname{Spec}(A_f) \amalg \operatorname{Spec}(A^\sim_{V(f)})$$

Then we see that z and z' map x_E and x'_E which are in different parts of the given decomposition of X_E above. But then it impossible for x_E and x'_E to be specializations of a common point. This is the desired contradiction.

Recall that given a finite subset $E \subset A$ we have Z_E is a disjoint union of the locally closed subschemes $Z(E', E'')$ each isomorphic to the spectrum of $(A/I)_f$ where I is the ideal generated by E'' and f the product of the elements of E'. Any nilpotent element b of $(A/I)_f$ is the class of g/f^n for some $g \in A$. Then setting $E' = E \cup \{g\}$ the reader verifies that b is pulls back to zero under the transition map $Z_{E'} \to Z_E$ of the system. This proves (4). □

Remark 46.5.4. Let A be a ring. Let κ be an infinite cardinal bigger or equal than the cardinality of A. Then the cardinality of A_w (Lemma 46.5.3) is at most κ. Namely, each A_E has cardinality at most κ and the set of finite subsets of A has cardinality at most κ as well. Thus the result follows as $\kappa \otimes \kappa = \kappa$, see Sets, Section 3.6.

Lemma 46.5.5 (Universal property of the construction)**.** *Let A be a ring. Let $A \to A_w$ be the ring map constructed in Lemma 46.5.3. For any ring map $A \to B$ such that* $\mathrm{Spec}(B)$ *is w-local, there is a unique factorization $A \to A_w \to B$ such that* $\mathrm{Spec}(B) \to \mathrm{Spec}(A_w)$ *is w-local.*

Proof. Denote $Y = \mathrm{Spec}(B)$ and $Y_0 \subset Y$ the set of closed points. Denote $f : Y \to X$ the given morphism. Recall that Y_0 is profinite, in particular every constructible subset of Y_0 is open and closed. Let $E \subset A$ be a finite subset. Recall that $A_w = \operatorname{colim} A_E$ and that the set of closed points of $\mathrm{Spec}(A_w)$ is the limit of the closed subsets $Z_E \subset X_E = \mathrm{Spec}(A_E)$. Thus it suffices to show there is a unique factorization $A \to A_E \to B$ such that $Y \to X_E$ maps Y_0 into Z_E. Since $Z_E \to X = \mathrm{Spec}(A)$ is bijective, and since the strata $Z(E', E'')$ are constructible we see that

$$Y_0 = \coprod f^{-1}(Z(E', E'')) \cap Y_0$$

is a disjoint union decomposition into open and closed subsets. As $Y_0 = \pi_0(Y)$ we obtain a corresponding decomposition of Y into open and closed pieces. Thus it suffices to construct the factorization in case $f(Y_0) \subset Z(E', E'')$ for some decomposition $E = E' \amalg E''$. In this case $f(Y)$ is contained in the set of points of X specializing to $Z(E', E'')$ which is homeomorphic to $X_{E',E''}$. Thus we obtain a unique continuous map $Y \to X_{E',E''}$ over X. By Lemma 46.3.7 this corresponds to a unique morphism of schemes $Y \to X_{E',E''}$ over X. This finishes the proof. □

Recall that the spectrum of a ring is profinite if and only if every point is closed. There are in fact a whole slew of equivalent conditions that imply this. See Algebra, Lemma 10.24.6 or Topology, Lemma 5.22.6.

Lemma 46.5.6. *Let A be a ring such that* $\mathrm{Spec}(A)$ *is profinite. Let $A \to B$ be a ring map. Then* $\mathrm{Spec}(B)$ *is profinite in each of the following cases:*

(1) *if $\mathfrak{q}, \mathfrak{q}' \subset B$ lie over the same prime of A, then neither $\mathfrak{q} \subset \mathfrak{q}'$, nor $\mathfrak{q}' \subset \mathfrak{q}$,*
(2) *$A \to B$ induces algebraic extensions of residue fields,*
(3) *$A \to B$ is a local isomorphism,*
(4) *$A \to B$ identifies local rings,*
(5) *$A \to B$ is weakly étale,*
(6) *$A \to B$ is quasi-finite,*
(7) *$A \to B$ is unramified,*
(8) *$A \to B$ is étale,*
(9) *B is a filtered colimit of A-algebras as in (1) – (8),*
(10) *etc.*

Proof. By the references mentioned above (Algebra, Lemma 10.24.6 or Topology, Lemma 5.22.6) there are no specializations between distinct points of $\mathrm{Spec}(A)$ and $\mathrm{Spec}(B)$ is profinite if and only if there are no specializations between distinct points of $\mathrm{Spec}(B)$. These specializations can only happen in the fibres of $\mathrm{Spec}(B) \to \mathrm{Spec}(A)$. In this way we see that (1) is true.

The assumption in (2) implies all primes of B are maximal by Algebra, Lemma 10.32.9. Thus (2) holds. If $A \to B$ is a local isomorphism or identifies local rings, then the residue field extensions are trivial, so (3) and (4) follow from (2). If $A \to B$ is weakly étale, then More on Algebra, Lemma 15.60.16 tells us it induces separable algebraic residue field extensions, so (5) follows from (2). If $A \to B$ is quasi-finite, then the fibres are finite discrete topological spaces. Hence (6) follows from (1). Hence (3) follows from (1). Cases (7) and (8) follow from this as unramified and étale ring map are quasi-finite (Algebra, Lemmas 10.142.6 and 10.136.6). If $B = \operatorname{colim} B_i$ is a filtered colimit of A-algebras, then $\mathrm{Spec}(B) = \operatorname{colim} \mathrm{Spec}(B_i)$, hence if each $\mathrm{Spec}(B_i)$ is profinite, so is $\mathrm{Spec}(B)$. This proves (9). □

Lemma 46.5.7. *Let A be a ring. Let $V(I) \subset \mathrm{Spec}(A)$ be a closed subset which is a profinite topological space. Then there exists an ind-Zariski ring map $A \to B$ such that $\mathrm{Spec}(B)$ is w-local, the set of closed points is $V(IB)$, and $A/I \cong B/IB$.*

Proof. Let $A \to A_w$ and $Z \subset Y = \mathrm{Spec}(A_w)$ as in Lemma 46.5.3. Let $T \subset Z$ be the inverse image of $V(I)$. Then $T \to V(I)$ is a homeomorphism by Topology, Lemma 5.16.8. Let $B = (A_w)^\sim_T$, see Lemma 46.5.1. It is clear that B is w-local with closed points $V(IB)$. The ring map $A/I \to B/IB$ is ind-Zariski and induces a homeomorphism on underlying topological spaces. Hence it is an isomorphism by Lemma 46.3.8. □

Lemma 46.5.8. *Let A be a ring such that $X = \mathrm{Spec}(A)$ is w-local. Let $I \subset A$ be the radical ideal cutting out the set X_0 of closed points in X. Let $A \to B$ be a ring map inducing algebraic extensions on residue fields at primes. Then*

1. *every point of $Z = V(IB)$ is a closed point of $\mathrm{Spec}(B)$,*
2. *there exists an ind-Zariski ring map $B \to C$ such that*
   (a) *$B/IB \to C/IC$ is an isomorphism,*
   (b) *the space $Y = \mathrm{Spec}(C)$ is w-local,*
   (c) *the induced map $p : Y \to X$ is w-local, and*
   (d) *$p^{-1}(X_0)$ is the set of closed points of Y.*

Proof. By Lemma 46.5.6 applied to $A/I \to B/IB$ all points of $Z = V(IB) = \mathrm{Spec}(B/IB)$ are closed, in fact $\mathrm{Spec}(B/IB)$ is a profinite space. To finish the proof we apply Lemma 46.5.7 to $IB \subset B$. □

46.6. Identifying local rings versus ind-Zariski

An ind-Zariski ring map $A \to B$ identifies local rings (Lemma 46.4.6). The converse does not hold (Examples, Section 78.31). However, it turns out that there is a kind of structure theorem for ring maps which identify local rings in terms of ind-Zariski ring maps, see Proposition 46.6.6.

Let A be a ring. Let $X = \mathrm{Spec}(A)$. The space of connected components $\pi_0(X)$ is a profinite space by Topology, Lemma 5.22.7 (and Algebra, Lemma 10.24.2).

Lemma 46.6.1. *Let A be a ring. Let $X = \mathrm{Spec}(A)$. Let $T \subset \pi_0(X)$ be a closed subset. There exists a surjective ind-Zariski ring map $A \to B$ such that $\mathrm{Spec}(B) \to \mathrm{Spec}(A)$ induces a homeomorphism of $\mathrm{Spec}(B)$ with the inverse image of T in X.*

Proof. Let $Z \subset X$ be the inverse image of T. Then Z is the intersection $Z = \bigcap Z_\alpha$ of the open and closed subsets of X containing Z, see Topology, Lemma 5.11.12. For each α we have $Z_\alpha = \mathrm{Spec}(A_\alpha)$ where $A \to A_\alpha$ is a local isomorphism (a localization at an idempotent). Setting $B = \mathrm{colim}\, A_\alpha$ proves the lemma. □

Lemma 46.6.2. *Let A be a ring and let $X = \mathrm{Spec}(A)$. Let T be a profinite space and let $T \to \pi_0(X)$ be a continuous map. There exists an ind-Zariski ring map $A \to B$ such that with $Y = \mathrm{Spec}(B)$ the diagram*

$$\begin{array}{ccc} Y & \longrightarrow & \pi_0(Y) \\ \downarrow & & \downarrow \\ X & \longrightarrow & \pi_0(X) \end{array}$$

is cartesian in the category of topological spaces and such that $\pi_0(Y) = T$ as spaces over $\pi_0(X)$.

Proof. Namely, write $T = \lim T_i$ as the limit of an inverse system finite discrete spaces over a directed partially ordered set (see Topology, Lemma 5.21.2). For each i let $Z_i = \mathrm{Im}(T \to \pi_0(X) \times T_i)$. This is a closed subset. Observe that $X \times T_i$ is the spectrum of $A_i = \prod_{t \in T_i} A$ and that $A \to A_i$ is a local isomorphism. By Lemma 46.6.1 we see that $Z_i \subset \pi_0(X \times T_i) = \pi_0(X) \times T_i$ corresponds to a surjection $A_i \to B_i$ which is ind-Zariski such that $\mathrm{Spec}(B_i) = X \times_{\pi_0(X)} Z_i$ as subsets of $X \times T_i$. The transition maps $T_i \to T_{i'}$ induce maps $Z_i \to Z_{i'}$ and $X \times_{\pi_0(X)} Z_i \to X \times_{\pi_0(X)} Z_{i'}$. Hence ring maps $B_{i'} \to B_i$ (Lemmas 46.3.8 and 46.4.6). Set $B = \mathrm{colim}\, B_i$. Because $T = \lim Z_i$ we have $X \times_{\pi_0(X)} T = \lim X \times_{\pi_0(X)} Z_i$ and hence $Y = \mathrm{Spec}(B) = \lim \mathrm{Spec}(B_i)$ fits into the cartesian diagram

$$\begin{array}{ccc} Y & \longrightarrow & T \\ \downarrow & & \downarrow \\ X & \longrightarrow & \pi_0(X) \end{array}$$

of topological spaces. By Lemma 46.2.5 we conclude that $T = \pi_0(Y)$. □

Example 46.6.3. Let k be a field. Let T be a profinite topological space. There exists an ind-Zariski ring map $k \to A$ such that $\mathrm{Spec}(A)$ is homeomorphic to T. Namely, just apply Lemma 46.6.2 to $T \to \pi_0(\mathrm{Spec}(k)) = \{*\}$. In fact, in this case we have

$$A = \mathrm{colim}\, \mathrm{Map}(T_i, k)$$

whenever we write $T = \lim T_i$ as a filtered limit with each T_i finite.

Lemma 46.6.4. *Let $A \to B$ be ring map such that*

1. *$A \to B$ identifies local rings,*
2. *the topological spaces $\mathrm{Spec}(B)$, $\mathrm{Spec}(A)$ are w-local,*
3. *$\mathrm{Spec}(B) \to \mathrm{Spec}(A)$ is w-local, and*
4. *$\pi_0(\mathrm{Spec}(B)) \to \pi_0(\mathrm{Spec}(A))$ is bijective.*

Then $A \to B$ is an isomorphism

Proof. Let $X_0 \subset X = \mathrm{Spec}(A)$ and $Y_0 \subset Y = \mathrm{Spec}(B)$ be the sets of closed points. By assumption Y_0 maps into X_0 and the induced map $Y_0 \to X_0$ is a bijection. As a space $\mathrm{Spec}(A)$ is the disjoint union of the spectra of the local rings of A at closed points. Similarly for B. Hence $X \to Y$ is a bijection. Since $A \to B$ is flat we have going down (Algebra, Lemma 10.36.17). Thus Algebra, Lemma 10.38.11 shows for any prime $\mathfrak{q} \subset B$ lying over $\mathfrak{p} \subset A$ we have $B_\mathfrak{q} = B_\mathfrak{p}$. Since $B_\mathfrak{q} = A_\mathfrak{p}$ by assumption, we see that $A_\mathfrak{p} = B_\mathfrak{p}$ for all primes $\mathfrak{p}$ of A. Thus $A = B$ by Algebra, Lemma 10.22.1. □

Lemma 46.6.5. *Let $A \to B$ be ring map such that*

(1) *$A \to B$ identifies local rings,*
(2) *the topological spaces $\mathrm{Spec}(B)$, $\mathrm{Spec}(A)$ are w-local, and*
(3) *$\mathrm{Spec}(B) \to \mathrm{Spec}(A)$ is w-local.*

Then $A \to B$ is ind-Zariski.

Proof. Set $X = \mathrm{Spec}(A)$ and $Y = \mathrm{Spec}(B)$. Let $X_0 \subset X$ and $Y_0 \subset Y$ be the set of closed points. Let $A \to A'$ be the ind-Zariski morphism of affine schemes such that with $X' = \mathrm{Spec}(A')$ the diagram

$$\begin{array}{ccc} X' & \longrightarrow & \pi_0(X') \\ \downarrow & & \downarrow \\ X & \longrightarrow & \pi_0(X) \end{array}$$

is cartesian in the category of topological spaces and such that $\pi_0(X') = \pi_0(Y)$ as spaces over $\pi_0(X)$, see Lemma 46.6.2. By Lemma 46.2.5 we see that X' is w-local and the set of closed points $X'_0 \subset X'$ is the inverse image of X_0.

We obtain a continuous map $Y \to X'$ of underlying topological spaces over X identifying $\pi_0(Y)$ with $\pi_0(X')$. By Lemma 46.3.8 (and Lemma 46.4.6) this is corresponds to a morphism of affine schemes $Y \to X'$ over X. Since $Y \to X$ maps Y_0 into X_0 we see that $Y \to X'$ maps Y_0 into X'_0, i.e., $Y \to X'$ is w-local. By Lemma 46.6.4 we see that $Y \cong X'$ and we win. □

The following proposition is a warm up for the type of result we will prove later.

Proposition 46.6.6. *Let $A \to B$ be a ring map which identifies local rings. Then there exists a faithfully flat, ind-Zariski ring map $B \to B'$ such that $A \to B'$ is ind-Zariski.*

Proof. Let $A \to A_w$, resp. $B \to B_w$ be the faithfully flat, ind-Zariski ring map constructed in Lemma 46.5.3 for A, resp. B. Since $\mathrm{Spec}(B_w)$ is w-local, there exists a unique factorization $A \to A_w \to B_w$ such that $\mathrm{Spec}(B_w) \to \mathrm{Spec}(A_w)$ is w-local by Lemma 46.5.5. Note that $A_w \to B_w$ identifies local rings, see Lemma 46.3.4. By Lemma 46.6.5 this means $A_w \to B_w$ is ind-Zariski. Since $B \to B_w$ is faithfully flat, ind-Zariski (Lemma 46.5.3) and the composition $A \to B \to B_w$ is ind-Zariski (Lemma 46.4.3) the proposition is proved. □

The proposition above allows us to characterize the affine, weakly contractible objects in the pro-Zariski site of an affine scheme.

Lemma 46.6.7. *Let A be a ring. The following are equivalent*

(1) *every faithfully flat ring map $A \to B$ identifying local rings has a section,*

(2) *every faithfully flat ind-Zariski ring map $A \to B$ has a section, and*
(3) *A satisfies*
   (a) $\mathrm{Spec}(A)$ *is w-local, and*
   (b) $\pi_0(\mathrm{Spec}(A))$ *is extremally disconnected.*

Proof. The equivalence of (1) and (2) follows immediately from Proposition 46.6.6.

Assume (3)(a) and (3)(b). Let $A \to B$ be faithfully flat and ind-Zariski. We will use without further mention the fact that a flat map $A \to B$ is faithfully flat if and only if every closed point of $\mathrm{Spec}(A)$ is in the image of $\mathrm{Spec}(B) \to \mathrm{Spec}(A)$ We will show that $A \to B$ has a section.

Let $I \subset A$ be an ideal such that $V(I) \subset \mathrm{Spec}(A)$ is the set of closed points of $\mathrm{Spec}(A)$. We may replace B by the ring C constructed in Lemma 46.5.8 for $A \to B$ and $I \subset A$. Thus we may assume $\mathrm{Spec}(B)$ is w-local such that the set of closed points of $\mathrm{Spec}(B)$ is $V(IB)$.

Assume $\mathrm{Spec}(B)$ is w-local and the set of closed points of $\mathrm{Spec}(B)$ is $V(IB)$. Choose a continuous section to the surjective continuous map $V(IB) \to V(I)$. This is possible as $V(I) \cong \pi_0(\mathrm{Spec}(A))$ is extremally disconnected, see Topology, Proposition 5.24.6. The image is a closed subspace $T \subset \pi_0(\mathrm{Spec}(B)) \cong V(JB)$ mapping homeomorphically onto $\pi_0(A)$. Replacing B by the ind-Zariski quotient ring constructed in Lemma 46.6.1 we see that we may assume $\pi_0(\mathrm{Spec}(B)) \to \pi_0(\mathrm{Spec}(A))$ is bijective. At this point $A \to B$ is an isomorphism by Lemma 46.6.4.

Assume (1) or equivalently (2). Let $A \to A_w$ be the ring map constructed in Lemma 46.5.3. By (1) there is a section $A_w \to A$. Thus $\mathrm{Spec}(A)$ is homeomorphic to a closed subset of $\mathrm{Spec}(A_w)$. By Lemma 46.2.4 we see (3)(a) holds. Finally, let $T \to \pi_0(A)$ be a surjective map with T an extremally disconnected, quasi-compact, Hausdorff topological space (Topology, Lemma 5.24.9). Choose $A \to B$ as in Lemma 46.6.2 adapted to $T \to \pi_0(\mathrm{Spec}(A))$. By (1) there is a section $B \to A$. Thus we see that $T = \pi_0(\mathrm{Spec}(B)) \to \pi_0(\mathrm{Spec}(A))$ has a section. A formal categorical argument, using Topology, Proposition 5.24.6, implies that $\pi_0(\mathrm{Spec}(A))$ is extremally disconnected. □

Lemma 46.6.8. *Let A be a ring. There exists a faithfully flat, ind-Zariski ring map $A \to B$ such that B satisfies the equivalent conditions of Lemma 46.6.7.*

Proof. We first apply Lemma 46.5.3 to see that we may assume that $\mathrm{Spec}(A)$ is w-local. Choose an extremally disconnected space T and a surjective continuous map $T \to \pi_0(\mathrm{Spec}(A))$, see Topology, Lemma 5.24.9. Note that T is profinite. Apply Lemma 46.6.2 to find an ind-Zariski ring map $A \to B$ such that $\pi_0(\mathrm{Spec}(B)) \to \pi_0(\mathrm{Spec}(A))$ realizes $T \to \pi_0(\mathrm{Spec}(A))$ and such that

$$\begin{array}{ccc} \mathrm{Spec}(B) & \longrightarrow & \pi_0(\mathrm{Spec}(B)) \\ \downarrow & & \downarrow \\ \mathrm{Spec}(A) & \longrightarrow & \pi_0(\mathrm{Spec}(A)) \end{array}$$

is cartesian in the category of topological spaces. Note that $\mathrm{Spec}(B)$ is w-local, that $\mathrm{Spec}(B) \to \mathrm{Spec}(A)$ is w-local, and that the set of closed points of $\mathrm{Spec}(B)$ is the inverse image of the set of closed points of $\mathrm{Spec}(A)$, see Lemma 46.2.5. Thus condition (3) of Lemma 46.6.7 holds for B. □

46.7. Ind-étale algebra

We start with a definition.

Definition 46.7.1. A ring map $A \to B$ is said to be *ind-étale* if B can be written as a filtered colimit of étale A-algebras.

The category of ind-étale algebras is closed under a number of natural operations.

Lemma 46.7.2. *Let $A \to B$ and $A \to A'$ be ring maps. Let $B' = B \otimes_A A'$ be the base change of B. If $A \to B$ is ind-étale, then $A' \to B'$ is ind-étale.*

Proof. Omitted. □

Lemma 46.7.3. *Let $A \to B$ and $B \to C$ be ring maps. If $A \to B$ and $B \to C$ are ind-étale, then $A \to C$ is ind-étale.*

Proof. Omitted. □

Lemma 46.7.4. *A filtered colimit of ind-étale A-algebras is ind-étale over A.*

Proof. Omitted. □

Lemma 46.7.5. *Let A be a ring. Let $B \to C$ be an A-algebra map of ind-étale A-algebras. Then C is an ind-étale B-algebra.*

Proof. Write $B = \operatorname{colim} B_i$ and $C = \operatorname{colim} C_j$ as filtered colimits of étale A-algebras. Then

$$C = B \otimes_B C = \operatorname{colim}_{(i,j)} B \otimes_{B_i} C_j$$

where the colimit is over the partially ordered set of pairs (i, j) such that $B_i \to B \to C$ factors through $C_j \to C$. Note that the factorization $B_i \to C_j$ is étale by Algebra, Lemma 10.136.9. Some details omitted. □

Lemma 46.7.6. *Let $A \to B$ be ind-étale. Then $A \to B$ is weakly étale (More on Algebra, Definition 15.60.1).*

Proof. This follows from More on Algebra, Lemma 15.60.13. □

Lemma 46.7.7. *Let A be a ring and let $I \subset A$ be an ideal. The base change functor*

$$\textit{ind-étale } A\textit{-algebras} \longrightarrow \textit{ind-étale } A/I\textit{-algebras}, \quad C \longmapsto C/IC$$

has a fully faithful right adjoint v. In particular, given an ind-étale A/I-algebra $\overline{C}$ there exists an ind-étale A-algebra $C = v(\overline{C})$ such that $\overline{C} = C/IC$.

Proof. Let $\overline{C}$ be an ind-étale A/I-algebra. Consider the category $\mathcal{C}$ of factorizations $A \to B \to \overline{C}$ where $A \to B$ is étale. (We ignore some set theoretical issues in this proof.) We will show that this category is directed and that $C = \operatorname{colim}_{\mathcal{C}} B$ is an ind-étale A-algebra such that $\overline{C} = C/IC$.

Suppose that $A \to B \to \overline{C}$ and $A \to B' \to \overline{C}$ are two objects of $\mathcal{C}$. Then $A \to B \otimes_A B' \to \overline{C}$ is another. Suppose that $f, g : B \to B'$ are two maps between objects $A \to B \to \overline{C}$ and $A \to B' \to \overline{C}$ of $\mathcal{C}$. Then a coequalizer is $A \to B' \otimes_{f,B,g} B' \to \overline{C}$. Thus the category $\mathcal{C}$ is directed.

Write $\overline{C} = \operatorname{colim} \overline{B_i}$ as a filtered colimit with $\overline{B_i}$ étale over A/I. For every i there exists $A \to B_i$ étale with $\overline{B_i} = B_i/IB_i$, see Algebra, Lemma 10.136.11. Thus

$C \to \overline{C}$ is surjective. Since $C/IC \to \overline{C}$ is ind-étale (Lemma 46.7.5) we see that it is flat. Hence $\overline{C}$ is a localization of C/IC at some multiplicative subset $S \subset C/IC$ (Algebra, Lemma 10.103.2). Take an $f \in C$ mapping to an element of $S \subset C/IC$. Choose $A \to B \to \overline{C}$ in $\mathcal{C}$ and $g \in B$ mapping to f in the colimit. Then we see that $A \to B_g \to \overline{C}$ is an object of $\mathcal{C}$ as well. Thus f is an invertible element of C. It follows that $C/IC = \overline{C}$.

Next, we claim that for an ind-étale algebra D over A we have

$$\mathrm{Mor}_A(D, C) = \mathrm{Mor}_{A/I}(D/ID, \overline{C})$$

Namely, let $D/ID \to \overline{C}$ be an A/I-algebra map. Write $D = \mathrm{colim}_{i \in I} D_i$ as a filtered colimit over a partially ordered set I with D_i étale over A. By choice of $\mathcal{C}$ we obtain a transformation $I \to \mathcal{C}$ and hence a map $D \to C$ compatible with maps to $\overline{C}$. Whence the claim.

It follows that the functor v defined by the rule

$$\overline{C} \longmapsto v(\overline{C}) = \mathrm{colim}_{A \to B \to \overline{C}} B$$

is a right adjoint to the base change functor u as required by the lemma. The functor v is fully faithful because $u \circ v = \mathrm{id}$ by construction, see Categories, Lemma 4.24.2. □

46.8. Constructing ind-étale algebras

Let A be a ring. Recall that any étale ring map $A \to B$ is isomorphic to a standard smooth ring map of relative dimension 0. Such a ring map is of the form

$$A \longrightarrow A[x_1, \ldots, x_n]/(f_1, \ldots, f_n)$$

where the determinant of the $n \times n$-matrix with entries $\partial f_i/\partial x_j$ is invertible in the quotient ring. See Algebra, Lemma 10.136.2.

Let $S(A)$ be the set of all *faithfully flat*[1] standard smooth A-algebras of relative dimension 0. Let $I(A)$ be the partially ordered (by inclusion) set of finite subsets E of $S(A)$. Note that $I(A)$ is a directed partially ordered set. For $E = \{A \to B_1, \ldots, A \to B_n\}$ set

$$B_E = B_1 \otimes_A \ldots \otimes_A B_n$$

Observe that B_E is a faithfully flat étale A-algebra. For $E \subset E'$, there is a canonical transition map $B_E \to B_{E'}$ of étale A-algebras. Namely, say $E = \{A \to B_1, \ldots, A \to B_n\}$ and $E' = \{A \to B_1, \ldots, A \to B_{n+m}\}$ then $B_E \to B_{E'}$ sends $b_1 \otimes \ldots \otimes b_n$ to the element $b_1 \otimes \ldots \otimes b_n \otimes 1 \otimes \ldots \otimes 1$ of $B_{E'}$. This construction defines a system of faithfully flat étale A-algebras over $I(A)$ and we set

$$T(A) = \mathrm{colim}_{E \in I(A)} B_E$$

Observe that $T(A)$ is a faithfully flat ind-étale A-algebra (Algebra, Lemma 10.36.20). By construction given any faithfully flat étale A-algebra B there is a (non-unique) A-algebra map $B \to T(A)$. Namely, pick some $(A \to B_0) \in S(A)$ and an isomorphism $B \cong B_0$. Then the canonical coprojection

$$B \to B_0 \to T(A) = \mathrm{colim}_{E \in I(A)} B_E$$

is the desired map.

[1] In the presence of flatness, e.g., for smooth or étale ring maps, this just means that the induced map on spectra is surjective. See Algebra, Lemma 10.36.15.

Lemma 46.8.1. *Given a ring A there exists a faithfully flat ind-étale A-algebra C such that every faithfully flat étale ring map $C \to B$ has a section.*

Proof. Set $T^1(A) = T(A)$ and $T^{n+1}(A) = T(T^n(A))$. Let

$$C = \operatorname{colim} T^n(A)$$

This algebra is faithfully flat over each $T^n(A)$ and in particular over A, see Algebra, Lemma 10.36.20. Moreover, C is ind-étale over A by Lemma 46.7.4. If $C \to B$ is étale, then there exists an n and an étale ring map $T^n(A) \to B'$ such that $B = C \otimes_{T^n(A)} B'$, see Algebra, Lemma 10.136.3. If $C \to B$ is faithfully flat, then $\operatorname{Spec}(B) \to \operatorname{Spec}(C) \to \operatorname{Spec}(T^n(A))$ is surjective, hence $\operatorname{Spec}(B') \to \operatorname{Spec}(T^n(A))$ is surjective. In other words, $T^n(A) \to B'$ is faithfully flat. By our construction, there is a $T^n(A)$-algebra map $B' \to T^{n+1}(A)$. This induces a C-algebra map $B \to C$ which finishes the proof. □

Remark 46.8.2. Let A be a ring. Let κ be an infinite cardinal bigger or equal than the cardinality of A. Then the cardinality of $T(A)$ is at most κ. Namely, each B_E has cardinality at most κ and the index set $I(A)$ has cardinality at most κ as well. Thus the result follows as $\kappa \otimes \kappa = \kappa$, see Sets, Section 3.6. It follows that the ring constructed in the proof of Lemma 46.8.1 has cardinality at most κ as well.

Remark 46.8.3. The construction $A \mapsto T(A)$ is functorial in the following sense: If $A \to A'$ is a ring map, then we can construct a commutative diagram

$$\begin{array}{ccc} A & \longrightarrow & T(A) \\ \downarrow & & \downarrow \\ A' & \longrightarrow & T(A') \end{array}$$

Namely, given $(A \to A[x_1, \ldots, x_n]/(f_1, \ldots, f_n))$ in $S(A)$ we can use the ring map $\varphi : A \to A'$ to obtain a corresponding element $(A' \to A'[x_1, \ldots, x_n]/(f_1^\varphi, \ldots, f_n^\varphi))$ of $S(A')$ where f^φ means the polynomial obtained by applying φ to the coefficients of the polynomial f. Moreover, there is a commutative diagram

$$\begin{array}{ccc} A & \longrightarrow & A[x_1, \ldots, x_n]/(f_1, \ldots, f_n) \\ \downarrow & & \downarrow \\ A' & \longrightarrow & A'[x_1, \ldots, x_n]/(f_1^\varphi, \ldots, f_n^\varphi) \end{array}$$

which is a in the category of rings. For $E \subset S(A)$ finite, set $E' = \varphi(E)$ and define $B_E \to B_{E'}$ in the obvious manner. Taking the colimit gives the desired map $T(A) \to T(A')$, see Categories, Lemma 4.14.7.

Lemma 46.8.4. *Let A be a ring such that every faithfully flat étale ring map $A \to B$ has a section. Then the same is true for every quotient ring A/I.*

Proof. Omitted. □

Lemma 46.8.5. *Let A be a ring such that every faithfully flat étale ring map $A \to B$ has a section. Then every local ring of A at a maximal ideal is strictly henselian.*

Proof. Let $\mathfrak{m}$ be a maximal ideal of A. Let $A \to B$ be an étale ring map and let $\mathfrak{q} \subset B$ be a prime lying over $\mathfrak{m}$. By the description of the strict henselization $A^{sh}_{\mathfrak{m}}$ in Algebra, Lemma 10.143.27 it suffices to show that $A_{\mathfrak{m}} = B_{\mathfrak{q}}$. Note that there are finitely many primes $\mathfrak{q} = \mathfrak{q}_1, \mathfrak{q}_2, \ldots, \mathfrak{q}_n$ lying over $\mathfrak{m}$ and there are no specializations between them as an étale ring map is quasi-finite, see Algebra, Lemma 10.136.6. Thus $\mathfrak{q}_i$ is a maximal ideal and we can find $g \in \mathfrak{q}_2 \cap \ldots \cap \mathfrak{q}_n$, $g \notin \mathfrak{q}$ (Algebra, Lemma 10.14.3). After replacing B by B_g we see that $\mathfrak{q}$ is the only prime of B lying over $\mathfrak{m}$. The image $U \subset \mathrm{Spec}(A)$ of $\mathrm{Spec}(B) \to \mathrm{Spec}(A)$ is open (Algebra, Proposition 10.38.8). Thus the complement $\mathrm{Spec}(A) \setminus U$ is closed and we can find $f \in A$, $f \notin \mathfrak{p}$ such that $\mathrm{Spec}(A) = U \cup D(f)$. The ring map $A \to B \times A_f$ is faithfully flat and étale, hence has a section $\sigma : B \times A_f \to A$ by assumption on A. Observe that σ is étale, hence flat as a map between étale A-algebras (Algebra, Lemma 10.136.9). Since $\mathfrak{q}$ is the only prime of $B \times A_f$ lying over A we find that $A_{\mathfrak{p}} \to B_{\mathfrak{q}}$ has a section which is also flat. Thus $A_{\mathfrak{p}} \to B_{\mathfrak{q}} \to A_{\mathfrak{p}}$ are flat local ring maps whose composition is the identity. Since a flat local homomorphism of local rings is injective we conclude these maps are isomorphisms as desired. □

Lemma 46.8.6. *Let A be a ring such that every faithfully flat étale ring map $A \to B$ has a section. Let $Z \subset \mathrm{Spec}(A)$ be a closed subscheme of the form $D(f) \cap V(I)$ and let $A \to A^\sim_Z$ be as constructed in Lemma 46.5.1. Then every faithfully flat étale ring map $A^\sim_Z \to C$ has a section.*

Proof. There exists an étale ring map $A \to B'$ such that $C = B' \otimes_A A^\sim_Z$ as $A^\sim_Z$-algebras. The image $U' \subset \mathrm{Spec}(A)$ of $\mathrm{Spec}(B') \to \mathrm{Spec}(A)$ is open and contains $V(I)$, hence we can find $f \in I$ such that $\mathrm{Spec}(A) = U' \cup D(f)$. Then $A \to B' \times A_f$ is étale and faithfully flat. By assumption there is a section $B' \times A_f \to A$. Localizing we obtain the desired section $C \to A^\sim_Z$. □

Lemma 46.8.7. *Let $A \to B$ be a ring map inducing algebraic extensions on residue fields. There exists a commutative diagram*

$$\begin{array}{ccc} B & \longrightarrow & D \\ \uparrow & & \uparrow \\ A & \longrightarrow & C \end{array}$$

with the following properties:

1. *$A \to C$ is faithfully flat and ind-étale,*
2. *$B \to D$ is faithfully flat and ind-étale,*
3. *$\mathrm{Spec}(C)$ is w-local,*
4. *$\mathrm{Spec}(D)$ is w-local,*
5. *$\mathrm{Spec}(D) \to \mathrm{Spec}(C)$ is w-local,*
6. *the set of closed points of $\mathrm{Spec}(D)$ is the inverse image of the set of closed points of $\mathrm{Spec}(C)$,*
7. *the set of closed points of $\mathrm{Spec}(C)$ surjects onto $\mathrm{Spec}(A)$,*
8. *the set of closed points of $\mathrm{Spec}(D)$ surjects onto $\mathrm{Spec}(B)$,*
9. *for $\mathfrak{m} \subset C$ maximal the local ring $C_{\mathfrak{m}}$ is strictly henselian.*

Proof. There is a faithfully flat, ind-Zariski ring map $A \to A'$ such that $\mathrm{Spec}(A')$ is w-local and such that the set of closed points of $\mathrm{Spec}(A')$ maps onto $\mathrm{Spec}(A)$, see Lemma 46.5.3. Let $I \subset A'$ be the ideal such that $V(I)$ is the set of closed points of $\mathrm{Spec}(A')$. Choose $A' \to C'$ as in Lemma 46.8.1. Note that the local rings $C'_{\mathfrak{m}'}$ at

maximal ideals $\mathfrak{m}' \subset C'$ are strictly henselian by Lemma 46.8.5. We apply Lemma 46.5.8 to $A' \to C'$ and $I \subset A'$ to get $C' \to C$ with $C'/IC' \cong C/IC$. Note that since $A' \to C'$ is faithfully flat, $\mathrm{Spec}(C'/IC')$ surjects onto the set of closed points of A' and in particular onto $\mathrm{Spec}(A)$. Moreover, as $V(IC) \subset \mathrm{Spec}(C)$ is the set of closed points of C and $C' \to C$ is ind-Zariski (and identifies local rings) we obtain properties (1), (3), (7), and (9).

Denote $J \subset C$ the ideal such that $V(J)$ is the set of closed points of $\mathrm{Spec}(C)$. Set $D' = B \otimes_A C$. The ring map $C \to D'$ induces algebraic residue field extensions. Keep in mind that since $V(J) \to \mathrm{Spec}(A)$ is surjective the map $T = V(JD) \to \mathrm{Spec}(B)$ is surjective too. Apply Lemma 46.5.8 to $C \to D'$ and $J \subset C$ to get $D' \to D$ with $D'/JD' \cong D/JD$. All of the remaining properties given in the lemma are immediate from the results of Lemma 46.5.8. □

46.9. Weakly étale versus pro-étale

Recall that a ring homomorphism $A \to B$ is *weakly étale* if $A \to B$ is flat and $B \otimes_A B \to B$ is flat. We have proved some properties of such ring maps in More on Algebra, Section 15.60. In particular, if $A \to B$ is a local homomorphism, and A is a strictly henselian local rings, then $A = B$, see More on Algebra, Theorem 15.60.24. Using this theorem and the work we've done above we obtain the following structure theorem for weakly étale ring maps.

Proposition 46.9.1. *Let $A \to B$ be a weakly étale ring map. Then there exists a faithfully flat, ind-étale ring map $B \to B'$ such that $A \to B'$ is ind-étale.*

Proof. The ring map $A \to B$ induces (separable) algebraic extensions of residue fields, see More on Algebra, Lemma 15.60.16. Thus we may apply Lemma 46.8.7 and choose a diagram

$$\begin{array}{ccc} B & \longrightarrow & D \\ \uparrow & & \uparrow \\ A & \longrightarrow & C \end{array}$$

with the properties as listed in the lemma. Note that $C \to D$ is weakly étale by More on Algebra, Lemma 15.60.11. Pick a maximal ideal $\mathfrak{m} \subset D$. By construction this lies over a maximal ideal $\mathfrak{m}' \subset C$. By More on Algebra, Theorem 15.60.24 the ring map $C_{\mathfrak{m}'} \to D_{\mathfrak{m}}$ is an isomorphism. As every point of $\mathrm{Spec}(C)$ specializes to a closed point we conclude that $C \to D$ identifies local rings. Thus Proposition 46.6.6 applies to the ring map $C \to D$. Pick $D \to D'$ faithfully flat and ind-Zariski such that $C \to D'$ is ind-Zariski. Then $B \to D'$ is a solution to the problem posed in the proposition. □

46.10. Constructing w-contractible covers

In this section we construct w-contractible covers of affine schemes.

Definition 46.10.1. Let A be a ring. We say A is *w-contractible* if every faithfully flat weakly-etale ring map $A \to B$ has a section.

We remark that by Proposition 46.9.1 an equivalent definition would be to ask that every faithfully flat, ind-étale ring map $A \to B$ has a section. Here is a key observation that will allow us to construct w-contractible rings.

Lemma 46.10.2. *Let A be a ring. The following are equivalent*

(1) *A is w-contractible,*
(2) *every faithfully flat, ind-étale ring map $A \to B$ has a section, and*
(3) *A satisfies*
   (a) $\mathrm{Spec}(A)$ *is w-local,*
   (b) $\pi_0(\mathrm{Spec}(A))$ *is extremally disconnected, and*
   (c) *for every maximal ideal $\mathfrak{m} \subset A$ the local ring $A_\mathfrak{m}$ is strictly henselian.*

Proof. The equivalence of (1) and (2) follows immediately from Proposition 46.9.1.

Assume (3)(a), (3)(b), and (3)(c). Let $A \to B$ be faithfully flat and ind-étale. We will use without further mention the fact that a flat map $A \to B$ is faithfully flat if and only if every closed point of $\mathrm{Spec}(A)$ is in the image of $\mathrm{Spec}(B) \to \mathrm{Spec}(A)$ We will show that $A \to B$ has a section.

Let $I \subset A$ be an ideal such that $V(I) \subset \mathrm{Spec}(A)$ is the set of closed points of $\mathrm{Spec}(A)$. We may replace B by the ring C constructed in Lemma 46.5.8 for $A \to B$ and $I \subset A$. Thus we may assume $\mathrm{Spec}(B)$ is w-local such that the set of closed points of $\mathrm{Spec}(B)$ is $V(IB)$. In this case $A \to B$ identifies local rings by condition (3)(c) as it suffices to check this at maximal ideals of B which lie over maximal ideals of A. Thus $A \to B$ has a section by Lemma 46.6.7.

Assume (1) or equivalently (2). We have (3)(c) by Lemma 46.8.5. Properties (3)(a) and (3)(b) follow from Lemma 46.6.7. □

Proposition 46.10.3. *For every ring A there exists a faithfully flat, ind-étale ring map $A \to D$ such that D is w-contractible.*

Proof. Applying Lemma 46.8.7 to $\mathrm{id}_A : A \to A$ we find a faithfully flat, ind-étale ring map $A \to C$ such that C is w-local and such that every local ring at a maximal ideal of C is strictly henselian. Choose an extremally disconnected space T and a surjective continuous map $T \to \pi_0(\mathrm{Spec}(C))$, see Topology, Lemma 5.24.9. Note that T is profinite. Apply Lemma 46.6.2 to find an ind-Zariski ring map $C \to D$ such that $\pi_0(\mathrm{Spec}(D)) \to \pi_0(\mathrm{Spec}(C))$ realizes $T \to \pi_0(\mathrm{Spec}(C))$ and such that

$$\begin{array}{ccc} \mathrm{Spec}(D) & \longrightarrow & \pi_0(\mathrm{Spec}(D)) \\ \downarrow & & \downarrow \\ \mathrm{Spec}(C) & \longrightarrow & \pi_0(\mathrm{Spec}(C)) \end{array}$$

is cartesian in the category of topological spaces. Note that $\mathrm{Spec}(D)$ is w-local, that $\mathrm{Spec}(D) \to \mathrm{Spec}(C)$ is w-local, and that the set of closed points of $\mathrm{Spec}(D)$ is the inverse image of the set of closed points of $\mathrm{Spec}(C)$, see Lemma 46.2.5. Thus it is still true that the local rings of D at its maximal ideals are strictly henselian (as they are isomorphic to the local rings at the corresponding maximal ideals of C). It follows from Lemma 46.10.2 that D is w-contractible. □

Remark 46.10.4. Let A be a ring. Let κ be an infinite cardinal bigger or equal than the cardinality of A. Then the cardinality of the ring D constructed in Proposition 46.10.3 is at most

$$\kappa^{2^{2^{2^\kappa}}}.$$

Namely, the ring map $A \to D$ is constructed as a composition

$$A \to A_w = A' \to C' \to C \to D.$$

Here the first three steps of the construction are carried out in the first paragraph of the proof of Lemma 46.8.7. For the first step we have $|A_w| \leq \kappa$ by Remark 46.5.4. We have $|C'| \leq \kappa$ by Remark 46.8.2. Then $|C| \leq \kappa$ because C is a localization of $(C')_w$ (it is constructed from C' by an application of Lemma 46.5.7 in the proof of Lemma 46.5.8). Thus C has at most 2^κ maximal ideals. Finally, the ring map $C \to D$ identifies local rings and the cardinality of the set of maximal ideals of D is at most $2^{2^{2^\kappa}}$ by Topology, Remark 5.24.10. Since $D \subset \prod_{\mathfrak{m} \subset D} D_{\mathfrak{m}}$ we see that D has at most the size displayed above.

Lemma 46.10.5. *Let $A \to B$ be a quasi-finite and finitely presented ring map. If the residue fields of A are separably algebraically closed and $\mathrm{Spec}(A)$ is extremally disconnected, then $\mathrm{Spec}(B)$ is extremally disconnected.*

Proof. Set $X = \mathrm{Spec}(A)$ and $Y = \mathrm{Spec}(B)$. Choose a finite partition $X = \coprod X_i$ and $X_i' \to X_i$ as in Étale Cohomology, Lemma 44.66.3. Because X is extremally disconnected, every constructible locally closed subset is open and closed, hence we see that X is topologically the disjoint union of the strata X_i. Thus we may replace X by the X_i and assume there exists a surjective finite locally free morphism $X' \to X$ such that $(X' \times_X Y)_{red}$ is isomorphic to a finite disjoint union of copies of X'_{red}. Picture

$$\begin{array}{ccc} \coprod_{i=1,\ldots,r} X' & \longrightarrow & Y \\ \downarrow & & \downarrow \\ X' & \longrightarrow & X \end{array}$$

The assumption on the residue fields of A implies that this diagram is a fibre product diagram on underlying sets of points (details omitted). Since X is extremally disconnected and X' is Hausdorff (Lemma 46.5.6), the continuous map $X' \to X$ has a continuous section σ. Then $\coprod_{i=1,\ldots,r} \sigma(X) \to Y$ is a bijective continuous map. By Topology, Lemma 5.16.8 we see that it is a homeomorphism and the proof is done. □

Lemma 46.10.6. *Let $A \to B$ be a finite and finitely presented ring map. If A is w-contractible, so is B.*

Proof. We will use the criterion of Lemma 46.10.2. Set $X = \mathrm{Spec}(A)$ and $Y = \mathrm{Spec}(B)$. As $Y \to X$ is a finite morphism, we see that the set of closed points Y_0 of Y is the inverse image of the set of closed points X_0 of X. Moreover, every point of Y specializes to a unique point of Y_0 as (a) this is true for X and (b) the map $X \to Y$ is separated. For every $y \in Y_0$ with image $x \in X_0$ we see that $\mathcal{O}_{Y,y}$ is strictly henselian by Algebra, Lemma 10.143.4 applied to $\mathcal{O}_{X,x} \to B \otimes_A \mathcal{O}_{X,x}$. It remains to show that Y_0 is extremally disconnected. To do this we look at $X_0 \times_X Y \to X_0$ where $X_0 \subset X$ is the reduced induced scheme structure. Note that the underlying topological space of $X_0 \times_X Y$ agrees with Y_0. Now the desired result follows from Lemma 46.10.5. □

Lemma 46.10.7. *Let A be a ring. Let $Z \subset \mathrm{Spec}(A)$ be a closed subset of the form $Z = V(f_1, \ldots, f_r)$. Set $B = A^\sim_Z$, see Lemma 46.5.1. If A is w-contractible, so is B.*

Proof. Let $A_Z^\sim \to B$ be a weakly étale faithfully flat ring map. Consider the ring map

$$A \longrightarrow A_{f_1} \times \ldots \times A_{f_r} \times B$$

this is faithful flat and weakly étale. If A is w-contractible, then there is a section σ. Consider the morphism

$$\mathrm{Spec}(A_Z^\sim) \to \mathrm{Spec}(A) \xrightarrow{\mathrm{Spec}(\sigma)} \coprod \mathrm{Spec}(A_{f_i}) \amalg \mathrm{Spec}(B)$$

Every point of $Z \subset \mathrm{Spec}(A_Z^\sim)$ maps into the component $\mathrm{Spec}(B)$. Since every point of $\mathrm{Spec}(A_Z^\sim)$ specializes to a point of Z we find a morphism $\mathrm{Spec}(A_Z^\sim) \to \mathrm{Spec}(B)$ as desired. □

46.11. The pro-étale site

The (small) pro-étale site of a scheme has some remarkable properties. In particular, it has enough w-contractible objects which implies a number of useful consequences for the derived category of abelian sheaves and for inverse systems of sheaves. Thus it is well adapted to deal with some of the intricacies of working with ℓ-adic sheaves.

On the other hand, the pro-étale topology is a bit like the fpqc topology (see Topologies, Section 33.8) in that the topos of sheaves on the small pro-étale site of a scheme depends on the choice of the underlying category of schemes. Thus we cannot speak of *the* pro-étale topos of a scheme. However, it will be true that the cohomology groups of a sheaf are unchanged if we enlarge our underlying category of schemes.

Another curiosity is that we define pro-étale coverings using weakly étale morphisms of schemes, see More on Morphisms, Section 36.44. The reason is that, on the one hand, it is somewhat awkward to define the notion of a pro-étale morphism of schemes, and on the other, Proposition 46.9.1 assures us that we obtain the same sheaves with the definition that follows.

Definition 46.11.1. Let T be a scheme. A *pro-étale covering of* T is a family of morphisms $\{f_i : T_i \to T\}_{i \in I}$ of schemes such that each f_i is weakly-étale and such that for every affine open $U \subset T$ there exists $n \geq 0$, a map $a : \{1, \ldots, n\} \to I$ and affine opens $V_j \subset T_{a(j)}$, $j = 1, \ldots, n$ with $\bigcup_{j=1}^n f_{a(j)}(V_j) = U$.

To be sure this condition implies that $T = \bigcup f_i(T_i)$. Here is a lemma that will allow us to recognize pro-étale coverings. It will also allow us to reduce many lemmas about pro-étale coverings to the corresponding results for fpqc coverings.

Lemma 46.11.2. *Let T be a scheme. Let $\{f_i : T_i \to T\}_{i \in I}$ be a family of morphisms of schemes with target T. The following are equivalent*

1. *$\{f_i : T_i \to T\}_{i \in I}$ is a pro-étale covering,*
2. *each f_i is weakly étale and $\{f_i : T_i \to T\}_{i \in I}$ is an fpqc covering,*
3. *each f_i is weakly étale and for every affine open $U \subset T$ there exist quasi-compact opens $U_i \subset T_i$ which are almost all empty, such that $U = \bigcup f_i(U_i)$,*
4. *each f_i is weakly étale and there exists an affine open covering $T = \bigcup_{\alpha \in A} U_\alpha$ and for each $\alpha \in A$ there exist $i_{\alpha,1}, \ldots, i_{\alpha,n(\alpha)} \in I$ and quasi-compact opens $U_{\alpha,j} \subset T_{i_{\alpha,j}}$ such that $U_\alpha = \bigcup_{j=1,\ldots,n(\alpha)} f_{i_{\alpha,j}}(U_{\alpha,j})$.*

If T is quasi-separated, these are also equivalent to

(5) *each f_i is weakly étale, and for every $t \in T$ there exist $i_1, \ldots, i_n \in I$ and quasi-compact opens $U_j \subset T_{i_j}$ such that $\bigcup_{j=1,\ldots,n} f_{i_j}(U_j)$ is a (not necessarily open) neighbourhood of t in T.*

Proof. The equivalence of (1) and (2) is immediate from the definitions. Hence the lemma follows from Topologies, Lemma 33.8.2. □

Lemma 46.11.3. *Any étale covering and any Zariski covering is a pro-étale covering.*

Proof. This follows from the corresponding result for fpqc coverings (Topologies, Lemma 33.8.6), Lemma 46.11.2, and the fact that an étale morphism is a weakly étale morphism, see More on Morphisms, Lemma 36.44.9. □

Lemma 46.11.4. *Let T be a scheme.*

(1) *If $T' \to T$ is an isomorphism then $\{T' \to T\}$ is a pro-étale covering of T.*
(2) *If $\{T_i \to T\}_{i \in I}$ is a pro-étale covering and for each i we have a pro-étale covering $\{T_{ij} \to T_i\}_{j \in J_i}$, then $\{T_{ij} \to T\}_{i \in I, j \in J_i}$ is a pro-étale covering.*
(3) *If $\{T_i \to T\}_{i \in I}$ is a pro-étale covering and $T' \to T$ is a morphism of schemes then $\{T' \times_T T_i \to T'\}_{i \in I}$ is a pro-étale covering.*

Proof. This follows from the fact that composition and base changes of weakly étale morphisms are weakly étale (More on Morphisms, Lemmas 36.44.5 and 36.44.6), Lemma 46.11.2, and the corresponding results for fpqc coverings, see Topologies, Lemma 33.8.7. □

Lemma 46.11.5. *Let T be an affine scheme. Let $\{T_i \to T\}_{i \in I}$ be a pro-étale covering of T. Then there exists a pro-étale covering $\{U_j \to T\}_{j=1,\ldots,n}$ which is a refinement of $\{T_i \to T\}_{i \in I}$ such that each U_j is an affine scheme. Moreover, we may choose each U_j to be open affine in one of the T_i.*

Proof. This follows directly from the definition. □

Thus we define the corresponding standard coverings of affines as follows.

Definition 46.11.6. Let T be an affine scheme. A *standard pro-étale covering* of T is a family $\{f_i : T_i \to T\}_{i=1,\ldots,n}$ with each T_j is affine, each f_i is weakly étale, and $T = \bigcup f_i(T_i)$.

We interrupt the discussion for any explanation of the notion of w-contractible rings in terms of pro-étale coverings.

Lemma 46.11.7. *Let $T = \mathrm{Spec}(A)$ be an affine scheme. The following are equivalent*

(1) *A is w-contractible, and*
(2) *every pro-étale covering of T can be refined by a Zariski covering of the form $T = \coprod_{i=1,\ldots,n} U_i$.*

Proof. Assume A is w-contractible. By Lemma 46.11.5 it suffices to prove we can refine every standard pro-étale covering $\{f_i : T_i \to T\}_{i=1,\ldots,n}$ by a Zariski covering of T. The morphism $\coprod T_i \to T$ is a surjective weakly étale morphism of affine schemes. Hence by Definition 46.10.1 there exists a morphism $\sigma : T \to \coprod T_i$ over T. Then the Zariski covering $T = \coprod \sigma^{-1}(T_i)$ refines $\{f_i : T_i \to T\}$.

Conversely, assume (2). If $A \to B$ is faithfully flat and weakly étale, then $\{\mathrm{Spec}(B) \to T\}$ is a pro-étale covering. Hence there exists a Zariski covering $T = \coprod U_i$ and morphisms $U_i \to \mathrm{Spec}(B)$ over T. Since $T = \coprod U_i$ we obtain $T \to \mathrm{Spec}(B)$, i.e., an A-algebra map $B \to A$. This means A is w-contractible. □

We follow the general outline given in Topologies, Section 33.2 for constructing the big pro-étale site we will be working with. However, because we need a bit larger rings to accommodate for the size of certain constructions we modify the constructions slightly.

Definition 46.11.8. A *big pro-étale site* is any site $Sch_{pro\text{-}étale}$ as in Sites, Definition 7.6.2 constructed as follows:

(1) Choose any set of schemes S_0, and any set of pro-étale coverings Cov_0 among these schemes.
(2) Change the function $Bound$ of Sets, Equation (3.9.1.1) into
$$Bound(\kappa) = \max\{\kappa^{2^{2^{2^\kappa}}}, \kappa^{\aleph_0}, \kappa^+\}.$$
(3) As underlying category take any category Sch_α constructed as in Sets, Lemma 3.9.2 starting with the set S_0 and the function $Bound$.
(4) Choose any set of coverings as in Sets, Lemma 3.11.1 starting with the category Sch_α and the class of pro-étale coverings, and the set Cov_0 chosen above.

See the remarks following Topologies, Definition 33.3.5 for motivation and explanation regarding the definition of big sites.

Before we continue with the introduction of the big and small pro-étale sites of a scheme, let us point out that (1) our category contains many weakly contractible objects, and (2) the topology on a big pro-étale site $Sch_{pro\text{-}étale}$ is in some sense induced from the pro-étale topology on the category of all schemes.

Lemma 46.11.9. *Let $Sch_{pro\text{-}étale}$ be a big pro-étale site as in Definition 46.11.8. Let $T = \mathrm{Spec}(A)$ be an affine object of $Sch_{pro\text{-}étale}$. If A is w-contractible, then T is a weakly contractible (Sites, Definition 7.39.2) object of $Sch_{pro\text{-}étale}$.*

Proof. Let $\mathcal{F} \to \mathcal{G}$ be a surjection of sheaves on $Sch_{pro\text{-}étale}$. Let $s \in \mathcal{G}(T)$. We have to show that s is in the image of $\mathcal{F}(T) \to \mathcal{G}(T)$. We can find a covering $\{T_i \to T\}$ of $Sch_{pro\text{-}étale}$ such that s lifts to a section of $\mathcal{F}$ over T_i (Sites, Definition 7.12.1). By Lemma 46.11.7 we can refine $\{T_i \to T\}$ by a Zariski covering of the form $T = \coprod_{j=1,\ldots,m} V_j$. Hence we get $t_j \in \mathcal{F}(U_j)$ mapping to $s|_{U_j}$. Since Zariski coverings are coverings in $Sch_{pro\text{-}étale}$ (Lemma 46.11.3) we conclude that $\mathcal{F}(T) = \prod \mathcal{F}(U_j)$. Thus, taking $t = (t_1, \ldots, t_m) \in \mathcal{F}(T)$ is a section mapping to s. □

Lemma 46.11.10. *Let $Sch_{pro\text{-}étale}$ be a big pro-étale site as in Definition 46.11.8. For every object T of $Sch_{pro\text{-}étale}$ there exists a covering $\{T_i \to T\}$ in $Sch_{pro\text{-}étale}$ with each T_i affine and the spectrum of a w-contractible ring. In particular, T_i is weakly contractible in $Sch_{pro\text{-}étale}$.*

Proof. For those readers who do not care about set-theoretical issues this lemma is a trivial consequence of Lemma 46.11.9 and Proposition 46.10.3. Here are the details. Choose an affine open covering $T = \bigcup U_i$. Write $U_i = \mathrm{Spec}(A_i)$. Choose faithfully flat, ind-étale ring maps $A_i \to D_i$ such that D_i is w-contractible as in

Proposition 46.10.3. The family of morphisms $\{\mathrm{Spec}(D_i) \to T\}$ is a pro-étale covering. If we can show that $\mathrm{Spec}(D_i)$ is isomorphic to an object, say T_i, of $Sch_{pro\text{-}étale}$, then $\{T_i \to T\}$ will be combinatorially equivalent to a covering of $Sch_{pro\text{-}étale}$ by the construction of $Sch_{pro\text{-}étale}$ in Definition 46.11.8 and more precisely the application of Sets, Lemma 3.11.1 in the last step. To prove $\mathrm{Spec}(D_i)$ is isomorphic to an object of $Sch_{pro\text{-}étale}$, it suffices to prove that $|D_i| \leq Bound(\mathrm{Size}(T))$ by the construction of $Sch_{pro\text{-}étale}$ in Definition 46.11.8 and more precisely the application of Sets, Lemma 3.9.2 in step (3). Since $|A_i| \leq \mathrm{size}(U_i) \leq \mathrm{size}(T)$ by Sets, Lemmas 3.9.4 and 3.9.7 we get $|D_i| \leq \kappa^{2^{2^{2^\kappa}}}$ where $\kappa = \mathrm{size}(T)$ by Remark 46.10.4. Thus by our choice of the function $Bound$ in Definition 46.11.8 we win. □

Lemma 46.11.11. *Let $Sch_{pro\text{-}étale}$ be a big pro-étale site as in Definition 46.11.8. Let $T \in \mathrm{Ob}(Sch_{pro\text{-}étale})$. Let $\{T_i \to T\}_{i\in I}$ be an arbitrary pro-étale covering of T. There exists a covering $\{U_j \to T\}_{j\in J}$ of T in the site $Sch_{pro\text{-}étale}$ which refines $\{T_i \to T\}_{i\in I}$.*

Proof. Namely, we first let $\{V_k \to T\}$ be a covering as in Lemma 46.11.10. Then the pro-étale coverings $\{T_i \times_T V_k \to V_k\}$ can be refined by a finite disjoint open covering $V_k = V_{k,1} \amalg \ldots \amalg V_{k,n_k}$, see Lemma 46.11.7. Then $\{V_{k,i} \to T\}$ is a covering of $Sch_{pro\text{-}étale}$ which refines $\{T_i \to T\}_{i\in I}$. □

Definition 46.11.12. Let S be a scheme. Let $Sch_{pro\text{-}étale}$ be a big pro-étale site containing S.

(1) The *big pro-étale site of S*, denoted $(Sch/S)_{pro\text{-}étale}$, is the site $Sch_{pro\text{-}étale}/S$ introduced in Sites, Section 7.24.
(2) The *small pro-étale site of S*, which we denote $S_{pro\text{-}étale}$, is the full subcategory of $(Sch/S)_{pro\text{-}étale}$ whose objects are those U/S such that $U \to S$ is weakly étale. A covering of $S_{pro\text{-}étale}$ is any covering $\{U_i \to U\}$ of $(Sch/S)_{pro\text{-}étale}$ with $U \in \mathrm{Ob}(S_{pro\text{-}étale})$.
(3) The *big affine pro-étale site of S*, denoted $(Aff/S)_{pro\text{-}étale}$, is the full subcategory of $(Sch/S)_{pro\text{-}étale}$ whose objects are affine U/S. A covering of $(Aff/S)_{pro\text{-}étale}$ is any covering $\{U_i \to U\}$ of $(Sch/S)_{pro\text{-}étale}$ which is a standard pro-étale covering.

It is not completely clear that the small pro-étale site and the big affine pro-étale site are sites. We check this now.

Lemma 46.11.13. *Let S be a scheme. Let $Sch_{pro\text{-}étale}$ be a big pro-étale site containing S. Both $S_{pro\text{-}étale}$ and $(Aff/S)_{pro\text{-}étale}$ are sites.*

Proof. Let us show that $S_{pro\text{-}étale}$ is a site. It is a category with a given set of families of morphisms with fixed target. Thus we have to show properties (1), (2) and (3) of Sites, Definition 7.6.2. Since $(Sch/S)_{pro\text{-}étale}$ is a site, it suffices to prove that given any covering $\{U_i \to U\}$ of $(Sch/S)_{pro\text{-}étale}$ with $U \in \mathrm{Ob}(S_{pro\text{-}étale})$ we also have $U_i \in \mathrm{Ob}(S_{pro\text{-}étale})$. This follows from the definitions as the composition of weakly étale morphisms is weakly étale.

To show that $(Aff/S)_{pro\text{-}étale}$ is a site, reasoning as above, it suffices to show that the collection of standard pro-étale coverings of affines satisfies properties (1), (2) and (3) of Sites, Definition 7.6.2. This follows from Lemma 46.11.2 and the corresponding result for standard fpqc coverings (Topologies, Lemma 33.8.10). □

Lemma 46.11.14. *Let S be a scheme. Let $Sch_{pro\text{-}étale}$ be a big pro-étale site containing S. Let Sch be the category of all schemes.*

1. *The categories $Sch_{pro\text{-}étale}$, $(Sch/S)_{pro\text{-}étale}$, $S_{pro\text{-}étale}$, and $(Aff/S)_{pro\text{-}étale}$ have fibre products agreeing with fibre products in Sch.*
2. *The categories $Sch_{pro\text{-}étale}$, $(Sch/S)_{pro\text{-}étale}$, $S_{pro\text{-}étale}$ have equalizers agreeing with equalizers in Sch.*
3. *The categories $(Sch/S)_{pro\text{-}étale}$, and $S_{pro\text{-}étale}$ both have a final object, namely S/S.*
4. *The category $Sch_{pro\text{-}étale}$ has a final object agreeing with the final object of Sch, namely $\mathrm{Spec}(\mathbf{Z})$.*

Proof. The category $Sch_{pro\text{-}étale}$ contains $\mathrm{Spec}(\mathbf{Z})$ and is closed under products and fibre products by construction, see Sets, Lemma 3.9.9. Suppose we have $U \to S$, $V \to U$, $W \to U$ morphisms of schemes with $U, V, W \in \mathrm{Ob}(Sch_{pro\text{-}étale})$. The fibre product $V \times_U W$ in $Sch_{pro\text{-}étale}$ is a fibre product in Sch and is the fibre product of V/S with W/S over U/S in the category of all schemes over S, and hence also a fibre product in $(Sch/S)_{pro\text{-}étale}$. This proves the result for $(Sch/S)_{pro\text{-}étale}$. If $U \to S$, $V \to U$ and $W \to U$ are weakly étale then so is $V \times_U W \to S$ (see More on Morphisms, Section 36.44) and hence we get fibre products for $S_{pro\text{-}étale}$. If U, V, W are affine, so is $V \times_U W$ and hence we get fibre products for $(Aff/S)_{pro\text{-}étale}$.

Let $a, b : U \to V$ be two morphisms in $Sch_{pro\text{-}étale}$. In this case the equalizer of a and b (in the category of schemes) is

$$V \times_{\Delta_{V/\mathrm{Spec}(\mathbf{Z})}, V \times_{\mathrm{Spec}(\mathbf{Z})} V, (a,b)} (U \times_{\mathrm{Spec}(\mathbf{Z})} U)$$

which is an object of $Sch_{pro\text{-}étale}$ by what we saw above. Thus $Sch_{pro\text{-}étale}$ has equalizers. If a and b are morphisms over S, then the equalizer (in the category of schemes) is also given by

$$V \times_{\Delta_{V/S}, V \times_S V, (a,b)} (U \times_S U)$$

hence we see that $(Sch/S)_{pro\text{-}étale}$ has equalizers. Moreover, if U and V are weakly-étale over S, then so is the equalizer above as a fibre product of schemes weakly étale over S. Thus $S_{pro\text{-}étale}$ has equalizers. The statements on final objects is clear. □

Next, we check that the big affine pro-étale site defines the same topos as the big pro-étale site.

Lemma 46.11.15. *Let S be a scheme. Let $Sch_{pro\text{-}étale}$ be a big pro-étale site containing S. The functor $(Aff/S)_{pro\text{-}étale} \to (Sch/S)_{pro\text{-}étale}$ is a special cocontinuous functor. Hence it induces an equivalence of topoi from $Sh((Aff/S)_{pro\text{-}étale})$ to $Sh((Sch/S)_{pro\text{-}étale})$.*

Proof. The notion of a special cocontinuous functor is introduced in Sites, Definition 7.28.2. Thus we have to verify assumptions (1) – (5) of Sites, Lemma 7.28.1. Denote the inclusion functor $u : (Aff/S)_{pro\text{-}étale} \to (Sch/S)_{pro\text{-}étale}$. Being cocontinuous just means that any pro-étale covering of T/S, T affine, can be refined by a standard pro-étale covering of T. This is the content of Lemma 46.11.5. Hence (1) holds. We see u is continuous simply because a standard pro-étale covering is a pro-étale covering. Hence (2) holds. Parts (3) and (4) follow immediately from the fact that u is fully faithful. And finally condition (5) follows from the fact that every scheme has an affine open covering. □

Lemma 46.11.16. *Let $Sch_{pro\text{-}étale}$ be a big pro-étale site. Let $f : T \to S$ be a morphism in $Sch_{pro\text{-}étale}$. The functor $T_{pro\text{-}étale} \to (Sch/S)_{pro\text{-}étale}$ is cocontinuous and induces a morphism of topoi*

$$i_f : Sh(T_{pro\text{-}étale}) \longrightarrow Sh((Sch/S)_{pro\text{-}étale})$$

For a sheaf $\mathcal{G}$ on $(Sch/S)_{pro\text{-}étale}$ we have the formula $(i_f^{-1}\mathcal{G})(U/T) = \mathcal{G}(U/S)$. The functor i_f^{-1} also has a left adjoint $i_{f,!}$ which commutes with fibre products and equalizers.

Proof. Denote the functor $u : T_{pro\text{-}étale} \to (Sch/S)_{pro\text{-}étale}$. In other words, given a weakly étale morphism $j : U \to T$ corresponding to an object of $T_{pro\text{-}étale}$ we set $u(U \to T) = (f \circ j : U \to S)$. This functor commutes with fibre products, see Lemma 46.11.14. Moreover, $T_{pro\text{-}étale}$ has equalizers and u commutes with them by Lemma 46.11.14. It is clearly cocontinuous. It is also continuous as u transforms coverings to coverings and commutes with fibre products. Hence the lemma follows from Sites, Lemmas 7.20.5 and 7.20.6. □

Lemma 46.11.17. *Let S be a scheme. Let $Sch_{pro\text{-}étale}$ be a big pro-étale site containing S. The inclusion functor $S_{pro\text{-}étale} \to (Sch/S)_{pro\text{-}étale}$ satisfies the hypotheses of Sites, Lemma 7.20.8 and hence induces a morphism of sites*

$$\pi_S : (Sch/S)_{pro\text{-}étale} \longrightarrow S_{pro\text{-}étale}$$

and a morphism of topoi

$$i_S : Sh(S_{pro\text{-}étale}) \longrightarrow Sh((Sch/S)_{pro\text{-}étale})$$

such that $\pi_S \circ i_S = id$. Moreover, $i_S = i_{id_S}$ with i_{id_S} as in Lemma 46.11.16. In particular the functor $i_S^{-1} = \pi_{S,*}$ is described by the rule $i_S^{-1}(\mathcal{G})(U/S) = \mathcal{G}(U/S)$.

Proof. In this case the functor $u : S_{pro\text{-}étale} \to (Sch/S)_{pro\text{-}étale}$, in addition to the properties seen in the proof of Lemma 46.11.16 above, also is fully faithful and transforms the final object into the final object. The lemma follows from Sites, Lemma 7.20.8. □

Definition 46.11.18. In the situation of Lemma 46.11.17 the functor $i_S^{-1} = \pi_{S,*}$ is often called the *restriction to the small pro-étale site*, and for a sheaf $\mathcal{F}$ on the big pro-étale site we denote $\mathcal{F}|_{S_{pro\text{-}étale}}$ this restriction.

With this notation in place we have for a sheaf $\mathcal{F}$ on the big site and a sheaf $\mathcal{G}$ on the big site that

$$\begin{aligned}\mathrm{Mor}_{Sh(S_{pro\text{-}étale})}(\mathcal{F}|_{S_{pro\text{-}étale}}, \mathcal{G}) &= \mathrm{Mor}_{Sh((Sch/S)_{pro\text{-}étale})}(\mathcal{F}, i_{S,*}\mathcal{G}) \\ \mathrm{Mor}_{Sh(S_{pro\text{-}étale})}(\mathcal{G}, \mathcal{F}|_{S_{pro\text{-}étale}}) &= \mathrm{Mor}_{Sh((Sch/S)_{pro\text{-}étale})}(\pi_S^{-1}\mathcal{G}, \mathcal{F})\end{aligned}$$

Moreover, we have $(i_{S,*}\mathcal{G})|_{S_{pro\text{-}étale}} = \mathcal{G}$ and we have $(\pi_S^{-1}\mathcal{G})|_{S_{pro\text{-}étale}} = \mathcal{G}$.

Lemma 46.11.19. *Let $Sch_{pro\text{-}étale}$ be a big pro-étale site. Let $f : T \to S$ be a morphism in $Sch_{pro\text{-}étale}$. The functor*

$$u : (Sch/T)_{pro\text{-}étale} \longrightarrow (Sch/S)_{pro\text{-}étale}, \quad V/T \longmapsto V/S$$

is cocontinuous, and has a continuous right adjoint

$$v : (Sch/S)_{pro\text{-}étale} \longrightarrow (Sch/T)_{pro\text{-}étale}, \quad (U \to S) \longmapsto (U \times_S T \to T).$$

They induce the same morphism of topoi

$$f_{big} : Sh((Sch/T)_{pro\text{-}étale}) \longrightarrow Sh((Sch/S)_{pro\text{-}étale})$$

We have $f_{big}^{-1}(\mathcal{G})(U/T) = \mathcal{G}(U/S)$. *We have* $f_{big,*}(\mathcal{F})(U/S) = \mathcal{F}(U \times_S T/T)$. *Also,* f_{big}^{-1} *has a left adjoint* $f_{big!}$ *which commutes with fibre products and equalizers.*

Proof. The functor u is cocontinuous, continuous, and commutes with fibre products and equalizers (details omitted; compare with proof of Lemma 46.11.16). Hence Sites, Lemmas 7.20.5 and 7.20.6 apply and we deduce the formula for f_{big}^{-1} and the existence of $f_{big!}$. Moreover, the functor v is a right adjoint because given U/T and V/S we have $\mathrm{Mor}_S(u(U), V) = \mathrm{Mor}_T(U, V \times_S T)$ as desired. Thus we may apply Sites, Lemmas 7.21.1 and 7.21.2 to get the formula for $f_{big,*}$. □

Lemma 46.11.20. *Let* $Sch_{pro\text{-}étale}$ *be a big pro-étale site. Let* $f : T \to S$ *be a morphism in* $Sch_{pro\text{-}étale}$.

(1) *We have* $i_f = f_{big} \circ i_T$ *with* i_f *as in Lemma 46.11.16 and* i_T *as in Lemma 46.11.17.*
(2) *The functor* $S_{pro\text{-}étale} \to T_{pro\text{-}étale}$, $(U \to S) \mapsto (U \times_S T \to T)$ *is continuous and induces a morphism of topoi*

$$f_{small} : Sh(T_{pro\text{-}étale}) \longrightarrow Sh(S_{pro\text{-}étale}).$$

We have $f_{small,*}(\mathcal{F})(U/S) = \mathcal{F}(U \times_S T/T)$.
(3) *We have a commutative diagram of morphisms of sites*

$$\begin{array}{ccc} T_{pro\text{-}étale} & \xleftarrow{\pi_T} & (Sch/T)_{pro\text{-}étale} \\ {\scriptstyle f_{small}}\downarrow & & \downarrow{\scriptstyle f_{big}} \\ S_{pro\text{-}étale} & \xleftarrow{\pi_S} & (Sch/S)_{pro\text{-}étale} \end{array}$$

so that $f_{small} \circ \pi_T = \pi_S \circ f_{big}$ *as morphisms of topoi.*
(4) *We have* $f_{small} = \pi_S \circ f_{big} \circ i_T = \pi_S \circ i_f$.

Proof. The equality $i_f = f_{big} \circ i_T$ follows from the equality $i_f^{-1} = i_T^{-1} \circ f_{big}^{-1}$ which is clear from the descriptions of these functors above. Thus we see (1).

The functor $u : S_{pro\text{-}étale} \to T_{pro\text{-}étale}$, $u(U \to S) = (U \times_S T \to T)$ transforms coverings into coverings and commutes with fibre products, see Lemmas 46.11.4 and 46.11.14. Moreover, both $S_{pro\text{-}étale}$, $T_{pro\text{-}étale}$ have final objects, namely S/S and T/T and $u(S/S) = T/T$. Hence by Sites, Proposition 7.15.6 the functor u corresponds to a morphism of sites $T_{pro\text{-}étale} \to S_{pro\text{-}étale}$. This in turn gives rise to the morphism of topoi, see Sites, Lemma 7.16.2. The description of the pushforward is clear from these references.

Part (3) follows because π_S and π_T are given by the inclusion functors and f_{small} and f_{big} by the base change functors $U \mapsto U \times_S T$.

Statement (4) follows from (3) by precomposing with i_T. □

In the situation of the lemma, using the terminology of Definition 46.11.18 we have: for $\mathcal{F}$ a sheaf on the big pro-étale site of T

$$(f_{big,*}\mathcal{F})|_{S_{pro\text{-}étale}} = f_{small,*}(\mathcal{F}|_{T_{pro\text{-}étale}}),$$

This equality is clear from the commutativity of the diagram of sites of the lemma, since restriction to the small pro-étale site of T, resp. S is given by $\pi_{T,*}$, resp. $\pi_{S,*}$. A similar formula involving pullbacks and restrictions is false.

Lemma 46.11.21. *Given schemes X, Y, Y in $Sch_{pro\text{-}étale}$ and morphisms $f : X \to Y$, $g : Y \to Z$ we have $g_{big} \circ f_{big} = (g \circ f)_{big}$ and $g_{small} \circ f_{small} = (g \circ f)_{small}$.*

Proof. This follows from the simple description of pushforward and pullback for the functors on the big sites from Lemma 46.11.19. For the functors on the small sites this follows from the description of the pushforward functors in Lemma 46.11.20. □

We can think about a sheaf on the big pro-étale site of S as a collection of sheaves on the small pro-étale site on schemes over S.

Lemma 46.11.22. *Let S be a scheme contained in a big pro-étale site $Sch_{pro\text{-}étale}$. A sheaf $\mathcal{F}$ on the big pro-étale site $(Sch/S)_{pro\text{-}étale}$ is given by the following data:*

(1) *for every $T/S \in \mathrm{Ob}((Sch/S)_{pro\text{-}étale})$ a sheaf $\mathcal{F}_T$ on $T_{pro\text{-}étale}$,*
(2) *for every $f : T' \to T$ in $(Sch/S)_{pro\text{-}étale}$ a map $c_f : f_{small}^{-1}\mathcal{F}_T \to \mathcal{F}_{T'}$.*

These data are subject to the following conditions:

(i) *given any $f : T' \to T$ and $g : T'' \to T'$ in $(Sch/S)_{pro\text{-}étale}$ the composition $g_{small}^{-1}c_f \circ c_g$ is equal to $c_{f\circ g}$, and*
(ii) *if $f : T' \to T$ in $(Sch/S)_{pro\text{-}étale}$ is weakly étale then c_f is an isomorphism.*

Proof. Identical to the proof of Topologies, Lemma 33.4.18. □

Lemma 46.11.23. *Let S be a scheme. Let $S_{affine,pro\text{-}étale}$ denote the full subcategory of $S_{pro\text{-}étale}$ consisting of affine objects. A covering of $S_{affine,pro\text{-}étale}$ will be a standard étale covering, see Definition 46.11.6. Then restriction*

$$\mathcal{F} \longmapsto \mathcal{F}|_{S_{affine,étale}}$$

defines an equivalence of topoi $Sh(S_{pro\text{-}étale}) \cong Sh(S_{affine,pro\text{-}étale})$.

Proof. This you can show directly from the definitions, and is a good exercise. But it also follows immediately from Sites, Lemma 7.28.1 by checking that the inclusion functor $S_{affine,pro\text{-}étale} \to S_{pro\text{-}étale}$ is a special cocontinuous functor (see Sites, Definition 7.28.2). □

Lemma 46.11.24. *Let S be an affine scheme. Let S_{app} denote the full subcategory of $S_{pro\text{-}étale}$ consisting of affine objects U such that $\mathcal{O}(S) \to \mathcal{O}(U)$ is ind-étale. A covering of S_{app} will be a standard pro-étale covering, see Definition 46.11.6. Then restriction*

$$\mathcal{F} \longmapsto \mathcal{F}|_{S_{app}}$$

defines an equivalence of topoi $Sh(S_{pro\text{-}étale}) \cong Sh(S_{app})$.

Proof. By Lemma 46.11.23 we may replace $S_{pro\text{-}étale}$ by $S_{affine,pro\text{-}étale}$. The lemma follows from Sites, Lemma 7.28.1 by checking that the inclusion functor $S_{app} \to S_{affine,pro\text{-}étale}$ is a special cocontinuous functor, see Sites, Definition 7.28.2. The conditions of Sites, Lemma 7.28.1 follow immediately from the definition and the facts (a) any object U of $S_{affine,pro\text{-}étale}$ has a covering $\{V \to U\}$ with V ind-étale over X (Proposition 46.9.1) and (b) the functor u is fully faithful. □

Next we show that cohomology of sheaves is independent of the choice of a partial universe. Namely, the functor g_* of the lemma below is an embedding of pro-étale topoi which does not change cohomology.

Lemma 46.11.25. *Let S be a scheme. Let $S_{pro\text{-}étale} \subset S'_{pro\text{-}étale}$ be two small pro-étale sites of S as constructed in Definition 46.11.12. Then the inclusion functor satisfies the assumptions of Sites, Lemma 7.20.8. Hence there exist morphisms of topoi*

$$Sh(S_{pro\text{-}étale}) \xrightarrow{g} Sh(S'_{pro\text{-}étale}) \xrightarrow{f} Sh(S_{pro\text{-}étale})$$

whose composition is isomorphic to the identity and with $f_* = g^{-1}$. Moreover,

(1) *for $\mathcal{F}' \in Ab(S'_{pro\text{-}étale})$ we have $H^p(S'_{pro\text{-}étale}, \mathcal{F}') = H^p(S_{pro\text{-}étale}, g^{-1}\mathcal{F}')$,*
(2) *for $\mathcal{F} \in Ab(S_{pro\text{-}étale})$ we have*

$$H^p(S_{pro\text{-}étale}, \mathcal{F}) = H^p(S'_{pro\text{-}étale}, g_*\mathcal{F}) = H^p(S'_{pro\text{-}étale}, f^{-1}\mathcal{F}).$$

Proof. The inclusion functor is fully faithful and continuous. We have seen that $S_{pro\text{-}étale}$ and $S'_{pro\text{-}étale}$ have fibre products and final objects and that our functor commutes with these (Lemma 46.11.14). It follows from Lemma 46.11.11 that the inclusion functor is cocontinuous. Hence the existence of f and g follows from Sites, Lemma 7.20.8. The equality in (1) is Cohomology on Sites, Lemma 21.8.2. Part (2) follows from (1) as $\mathcal{F} = g^{-1}g_*\mathcal{F} = g^{-1}f^{-1}\mathcal{F}$. □

Lemma 46.11.26. *Let S be a scheme. The topology on each of the pro-étale sites $S_{pro\text{-}étale}$, $(Sch/S)_{pro\text{-}étale}$, $S_{affine,pro\text{-}étale}$, and $(Aff/S)_{pro\text{-}étale}$ is subcanonical.*

Proof. Combine Lemma 46.11.2 and Descent, Lemma 34.9.3. □

Lemma 46.11.27. *Let S be a scheme. The pro-étale sites $S_{pro\text{-}étale}$, $(Sch/S)_{pro\text{-}étale}$, $S_{affine,pro\text{-}étale}$, and $(Aff/S)_{pro\text{-}étale}$ and if S is affine S_{app} have enough quasi-compact, weakly contractible objects, see Sites, Definition 7.39.2.*

Proof. Follows immediately from Lemma 46.11.10. □

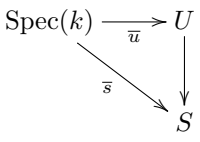
46.12. Points of the pro-étale site

We first apply Deligne's criterion to show that there are enough points.

Lemma 46.12.1. *Let S be a scheme. The pro-étale sites $S_{pro\text{-}étale}$, $(Sch/S)_{pro\text{-}étale}$, $S_{affine,pro\text{-}étale}$, and $(Aff/S)_{pro\text{-}étale}$ have enough points.*

Proof. The big topos is equivalent to the topos defined by $(Aff/S)_{pro\text{-}étale}$, see Lemma 46.11.15. The topos of sheaves on $S_{pro\text{-}étale}$ is equivalent to the topos associated to $S_{affine,pro\text{-}étale}$, see Lemma 46.11.23. The result for the sites $(Aff/S)_{pro\text{-}étale}$ and $S_{affine,pro\text{-}étale}$ follows immediately from Deligne's result Sites, Proposition 7.38.3. □

Let S be a scheme. Let $\overline{s} : \mathrm{Spec}(k) \to S$ be a geometric point. We define a *pro-étale neighbourhood* of $\overline{s}$ to be a commutative diagram



with $U \to S$ weakly étale. In exactly the same manner as in the chapter on étale cohomology one shows that the category of pro-étale neighbourhoods of $\overline{s}$ is cofiltered. Moreover, if $(U, \overline{u})$ is a pro-étale neighbourhood, and if $\{U_i \to U\}$ is a pro-étale covering, then there exists an i and a lift of $\overline{u}$ to a geometric point $\overline{u}_i$ of U_i. For $\mathcal{F}$ in $Sh(S_{pro\text{-}étale})$ define the *stalk of* $\mathcal{F}$ *at* $\overline{s}$ by the formula

$$\mathcal{F}_{\overline{s}} = \operatorname{colim}_{(U,\overline{u})} \mathcal{F}(U)$$

where the colimit is over all pro-étale neighbourhoods $(U, \overline{u})$ of $\overline{s}$ with $U \in \mathrm{Ob}(S_{pro\text{-}étale})$. A formal argument using the facts above shows the functor $\mathcal{F} \mapsto \mathcal{F}_{\overline{s}}$ defines a point of the topos $Sh(S_{pro\text{-}étale})$: it is an exact functor which commutes with arbitrary colimits. In fact, this functor has another description.

Lemma 46.12.2. *In the situation above the scheme* $\mathrm{Spec}(\mathcal{O}^{sh}_{S,\overline{s}})$ *is an object of* $X_{pro\text{-}étale}$ *and there is a canonical isomorphism*

$$\mathcal{F}(\mathrm{Spec}(\mathcal{O}^{sh}_{S,\overline{s}})) = \mathcal{F}_{\overline{s}}$$

functorial in $\mathcal{F}$.

Proof. The first statement is clear from the construction of the strict henselization as a filtered colimit of étale algebras over S, or by the characterization of weakly étale morphisms of More on Morphisms, Lemma 36.44.11. The second statement follows as by Olivier's theorem (More on Algebra, Theorem 15.60.24) the scheme $\mathrm{Spec}(\mathcal{O}^{sh}_{S,\overline{s}})$ is an initial object of the category of pro-étale neighbourhoods of $\overline{s}$. □

Contrary to the situation with the étale topos of S it is not true that every point of $Sh(S_{pro\text{-}étale})$ is of this form, and it is not true that the collection of points associated to geometric point is conservative. Namely, suppose that $S = \mathrm{Spec}(k)$ where k is an algebraically closed field. Let A be an abelian group. Consider the sheaf $\mathcal{F}$ on $S_{pro\text{-}étale}$ defined by the rule

$$\mathcal{F}(U) = \frac{\{\text{functions } U \to A\}}{\{\text{locally constant functions}\}}$$

Then $\mathcal{F}(U) = 0$ if $U = S = \mathrm{Spec}(k)$ but in general $\mathcal{F}$ is not zero. Namely, $S_{pro\text{-}étale}$ contains objects with infinitely many points. For example, let $E = \lim E_n$ be an inverse limit of finite sets with surjective transition maps, e.g., $E = \lim \mathbf{Z}/n\mathbf{Z}$. The scheme $\mathrm{Spec}(\operatorname{colim} \mathrm{Map}(E_n, k))$ is an object of $S_{pro\text{-}étale}$ because $\operatorname{colim} \mathrm{Map}(E_n, k)$ is weakly étale (even ind-Zariski) over k. Thus $\mathcal{F}$ is a nonzero abelian sheaf whose stalk at the unique geometric point of S is zero.

The solution is to use the existence of quasi-compact, weakly contractible objects. First, there are enough quasi-compact, weakly contractible objects by Lemma 46.11.27. Second, if $W \in \mathrm{Ob}(S_{pro\text{-}étale})$ is quasi-compact, weakly contractible, then the functor

$$Sh(S_{pro\text{-}étale}) \longrightarrow Sets, \quad \mathcal{F} \longmapsto \mathcal{F}(W)$$

is an exact functor $Sh(S_{pro\text{-}étale}) \to Sets$ which commutes with all limits. The functor

$$Ab(S_{pro\text{-}étale}) \longrightarrow Ab, \quad \mathcal{F} \longmapsto \mathcal{F}(W)$$

is exact and commutes with direct sums (as W is quasi-compact, see Sites, Lemma 7.11.2), hence commutes with all limits and colimits. Moreover, we can check exactness of a complex of abelian sheaves by evaluation at the quasi-compact, weakly contractible objects of $S_{pro\text{-}étale}$, see Cohomology on Sites, Proposition 21.37.2.

46.13. Compact generation

Let S be a scheme. The site $S_{pro\text{-}étale}$ has enough quasi-compact, weakly contractible objects U. For any sheaf of rings $\mathcal{A}$ on $S_{pro\text{-}étale}$ the corresponding objects $j_{U!}\mathcal{A}_U$ are compact objects of the derived category $D(\mathcal{A})$, see Cohomology on Sites, Lemma 21.38.7. Since every complex of $\mathcal{A}$-modules is quasi-isomorphic to a complex whose terms are direct sums of the modules $j_{U!}\mathcal{A}_U$ (details omitted). Thus we see that $D(\mathcal{A})$ is generated by its compact objects.

The same argument works for the big pro-étale site of S.

46.14. Generalities on derived completion

We urge the reader to skip this section on a first reading.

The algebra version of this material can be found in More on Algebra, Section 15.57. Let $\mathcal{O}$ be a sheaf of rings on a site $\mathcal{C}$. Let f be a global section of $\mathcal{O}$. We denote $\mathcal{O}_f$ the sheaf associated to the presheaf of localizations $U \mapsto \mathcal{O}(U)_f$.

Lemma 46.14.1. *Let $(\mathcal{C}, \mathcal{O})$ be a ringed site. Let f be a global section of $\mathcal{O}$.*

(1) *For $L, N \in D(\mathcal{O}_f)$ we have $R\mathcal{H}om_{\mathcal{O}}(L, N) = R\mathcal{H}om_{\mathcal{O}_f}(L, N)$. In particular the two $\mathcal{O}_f$-structures on $R\mathcal{H}om_{\mathcal{O}}(L, N)$ agree.*
(2) *For $K \in D(\mathcal{O})$ and $L \in D(\mathcal{O}_f)$ we have*

$$R\mathcal{H}om_{\mathcal{O}}(L, K) = R\mathcal{H}om_{\mathcal{O}_f}(L, R\mathcal{H}om_{\mathcal{O}}(\mathcal{O}_f, K))$$

In particular $R\mathcal{H}om_{\mathcal{O}}(\mathcal{O}_f, R\mathcal{H}om_{\mathcal{O}}(\mathcal{O}_f, K)) = R\mathcal{H}om_{\mathcal{O}}(\mathcal{O}_f, K)$.
(3) *If g is a second global section of $\mathcal{O}$, then*

$$R\mathcal{H}om_{\mathcal{O}}(\mathcal{O}_f, R\mathcal{H}om_{\mathcal{O}}(\mathcal{O}_g, K)) = R\mathcal{H}om_{\mathcal{O}}(\mathcal{O}_{gf}, K).$$

Proof. Proof of (1). Let $\mathcal{J}^\bullet$ be a K-injective complex of $\mathcal{O}_f$-modules representing N. By Cohomology on Sites, Lemma 21.20.3 it follows that $\mathcal{J}^\bullet$ is a K-injective complex of $\mathcal{O}$-modules as well. Let $\mathcal{F}^\bullet$ be a complex of $\mathcal{O}_f$-modules representing L. Then

$$R\mathcal{H}om_{\mathcal{O}}(L, N) = R\mathcal{H}om_{\mathcal{O}}(\mathcal{F}^\bullet, \mathcal{J}^\bullet) = R\mathcal{H}om_{\mathcal{O}_f}(\mathcal{F}^\bullet, \mathcal{J}^\bullet)$$

by Modules on Sites, Lemma 18.11.4 because $\mathcal{J}^\bullet$ is a K-injective complex of $\mathcal{O}$ and of $\mathcal{O}_f$-modules.

Proof of (2). Let $\mathcal{I}^\bullet$ be a K-injective complex of $\mathcal{O}$-modules representing K. Then $R\mathcal{H}om_{\mathcal{O}}(\mathcal{O}_f, K)$ is represented by $\mathcal{H}om_{\mathcal{O}}(\mathcal{O}_f, \mathcal{I}^\bullet)$ which is a K-injective complex of $\mathcal{O}_f$-modules and of $\mathcal{O}$-modules by Cohomology on Sites, Lemmas 21.20.4 and 21.20.3. Let $\mathcal{F}^\bullet$ be a complex of $\mathcal{O}_f$-modules representing L. Then

$$R\mathcal{H}om_{\mathcal{O}}(L, K) = R\mathcal{H}om_{\mathcal{O}}(\mathcal{F}^\bullet, \mathcal{I}^\bullet) = R\mathcal{H}om_{\mathcal{O}_f}(\mathcal{F}^\bullet, \mathcal{H}om_{\mathcal{O}}(\mathcal{O}_f, \mathcal{I}^\bullet))$$

by Modules on Sites, Lemma 18.27.5 and because $\mathcal{H}om_{\mathcal{O}}(\mathcal{O}_f, \mathcal{I}^\bullet)$ is a K-injective complex of $\mathcal{O}_f$-modules.

Proof of (3). This follows from the fact that $R\mathcal{H}om_{\mathcal{O}}(\mathcal{O}_g, \mathcal{I}^\bullet)$ is K-injective as a complex of $\mathcal{O}$-modules and the fact that $\mathcal{H}om_{\mathcal{O}}(\mathcal{O}_f, \mathcal{H}om_{\mathcal{O}}(\mathcal{O}_g, \mathcal{H})) = \mathcal{H}om_{\mathcal{O}}(\mathcal{O}_{gf}, \mathcal{H})$ for all sheaves of $\mathcal{O}$-modules $\mathcal{H}$. □

Let $K \in D(\mathcal{O})$. We denote $T(K, f)$ a derived limit (Derived Categories, Definition 13.30.2) of the system

$$\ldots \to K \xrightarrow{f} K \xrightarrow{f} K$$

in $D(\mathcal{O})$.

Lemma 46.14.2. *Let $(\mathcal{C}, \mathcal{O})$ be a ringed site. Let f be a global section of $\mathcal{O}$. Let $K \in D(\mathcal{O})$. The following are equivalent*

(1) $R\mathcal{H}om_{\mathcal{O}}(\mathcal{O}_f, K) = 0$,
(2) $R\mathcal{H}om_{\mathcal{O}}(L, K) = 0$ *for all* L *in* $D(\mathcal{O}_f)$,
(3) $T(K, f) = 0$.

Proof. It is clear that (2) implies (1). The implication (1) $\Rightarrow$ (2) follows from Lemma 46.14.1. A free resolution of the $\mathcal{O}$-module $\mathcal{O}_f$ is given by

$$0 \to \bigoplus\nolimits_{n \in \mathbf{N}} \mathcal{O} \to \bigoplus\nolimits_{n \in \mathbf{N}} \mathcal{O} \to \mathcal{O}_f \to 0$$

where the first map sends a local section $(x_0, x_1, \ldots)$ to $(fx_0 - x_1, fx_1 - x_2, \ldots)$ and the second map sends $(x_0, x_1, \ldots)$ to $x_0 + x_1/f + x_2/f^2 + \ldots$. Applying $\mathcal{H}om_{\mathcal{O}}(-, \mathcal{I}^\bullet)$ where $\mathcal{I}^\bullet$ is a K-injective complex of $\mathcal{O}$-modules representing K we get a short exact sequence of complexes

$$0 \to \mathcal{H}om_{\mathcal{O}}(\mathcal{O}_f, \mathcal{I}^\bullet) \to \prod \mathcal{I}^\bullet \to \prod \mathcal{I}^\bullet \to 0$$

because $\mathcal{I}^n$ is an injective $\mathcal{O}$-module. The products are products in $D(\mathcal{O})$, see Injectives, Lemma 19.13.4. This means that the object $T(K, f)$ is a representative of $R\mathcal{H}om_{\mathcal{O}}(\mathcal{O}_f, K)$ in $D(\mathcal{O})$. Thus the equivalence of (1) and (3). □

Lemma 46.14.3. *Let $(\mathcal{C}, \mathcal{O})$ be a ringed site. Let $K \in D(\mathcal{O})$. The rule which associates to U the set $\mathcal{I}(U)$ of sections $f \in \mathcal{O}(U)$ such that $T(K|_U, f) = 0$ is a sheaf of ideals in $\mathcal{O}$.*

Proof. We will use the results of Lemma 46.14.2 without further mention. If $f \in \mathcal{I}(U)$, and $g \in \mathcal{O}(U)$, then $\mathcal{O}_{U,gf}$ is an $\mathcal{O}_{U,f}$-module hence $R\mathcal{H}om_{\mathcal{O}}(\mathcal{O}_{U,gf}, K|_U) = 0$, hence $gf \in \mathcal{I}(U)$. Suppose $f, g \in \mathcal{O}(U)$. Then there is a short exact sequence

$$0 \to \mathcal{O}_{U,f+g} \to \mathcal{O}_{U,f(f+g)} \oplus \mathcal{O}_{U,g(f+g)} \to \mathcal{O}_{U,gf(f+g)} \to 0$$

because f, g generate the unit ideal in $\mathcal{O}(U)_{f+g}$. This follows from Algebra, Lemma 10.21.1 and the easy fact that the last arrow is surjective. Because $R\mathcal{H}om_{\mathcal{O}}(-, K|_U)$ is an exact functor of triangulated categories the vanishing of $R\mathcal{H}om_{\mathcal{O}_U}(\mathcal{O}_{U,f(f+g)}, K|_U)$, $R\mathcal{H}om_{\mathcal{O}_U}(\mathcal{O}_{U,g(f+g)}, K|_U)$, and $R\mathcal{H}om_{\mathcal{O}_U}(\mathcal{O}_{U,gf(f+g)}, K|_U)$, implies the vanishing of $R\mathcal{H}om_{\mathcal{O}_U}(\mathcal{O}_{U,f+g}, K|_U)$. We omit the verification of the sheaf condition. □

We can make the following definition for any ringed site.

Definition 46.14.4. Let $(\mathcal{C}, \mathcal{O})$ be a ringed site. Let $\mathcal{I} \subset \mathcal{O}$ be a sheaf of ideals. Let $K \in D(\mathcal{O})$. We say that K is *derived complete with respect to* $\mathcal{I}$ if for every object U of $\mathcal{C}$ and $f \in \mathcal{I}(U)$ the object $T(K|_U, f)$ of $D(\mathcal{O}_U)$ is zero.

It is clear that the full subcategory $D_{comp}(\mathcal{O}) = D_{comp}(\mathcal{O}, \mathcal{I}) \subset D(\mathcal{O})$ consisting of derived complete objects is a saturated triangulated subcategory, see Derived Categories, Definitions 13.3.4 and 13.6.1. This subcategory is preserved under products and homotopy limits in $D(\mathcal{O})$. But it is not preserved under countable direct sums in general.

Lemma 46.14.5. *Let $(\mathcal{C}, \mathcal{O})$ be a ringed site. Let $\mathcal{I} \subset \mathcal{O}$ be a sheaf of rings. If $K \in D(\mathcal{O})$ and $L \in D_{comp}(\mathcal{O})$, then $R\mathcal{H}om_{\mathcal{O}}(K, L) \in D_{comp}(\mathcal{O})$.*

Proof. Let U be an object of $\mathcal{C}$ and let $f \in \mathcal{I}(U)$. Recall that

$$\mathrm{Hom}_{D(\mathcal{O}_U)}(\mathcal{O}_{U,f}, R\mathcal{H}om_{\mathcal{O}}(K, L)|_U) = \mathrm{Hom}_{D(\mathcal{O}_U)}(K|_U \otimes^{\mathbf{L}}_{\mathcal{O}_U} \mathcal{O}_{U,f}, L|_U)$$

by Cohomology on Sites, Lemma 21.25.1. The right hand side is zero by Lemma 46.14.2 and the relationship between internal hom and actual hom, see Cohomology on Sites, Lemma 21.25.2. The same vanishing holds for all U'/U. Thus the object $R\mathcal{H}om_{\mathcal{O}_U}(\mathcal{O}_{U,f}, R\mathcal{H}om_{\mathcal{O}}(K, L)|_U)$ of $D(\mathcal{O}_U)$ has vanishing 0th cohomology sheaf (by locus citatus). Similarly for the other cohomology sheaves, i.e., $R\mathcal{H}om_{\mathcal{O}_U}(\mathcal{O}_{U,f}, R\mathcal{H}om_{\mathcal{O}}(K, L)|_U)$ is zero in $D(\mathcal{O}_U)$. By Lemma 46.14.2 we conclude. □

Lemma 46.14.6. *Let $(\mathcal{C}, \mathcal{O})$ be a ringed on a site. Let $f_1, \ldots, f_r$ be global sections of $\mathcal{O}$. Let $\mathcal{I} \subset \mathcal{O}$ be the ideal sheaf generated by $f_1, \ldots, f_r$. Then the inclusion functor $D_{comp}(\mathcal{O}) \to D(\mathcal{O})$ has a left adjoint, i.e., given any object K of $D(\mathcal{O})$ there exists a map $K \to K^\wedge$ with $K^\wedge$ in $D_{comp}(\mathcal{O})$ such that the map*

$$\mathrm{Hom}_{D(\mathcal{O})}(K^\wedge, E) \longrightarrow \mathrm{Hom}_{D(\mathcal{O})}(K, E)$$

is bijective whenever E is in $D_{comp}(\mathcal{O})$. In fact we have

$$K^\wedge = R\mathcal{H}om_{\mathcal{O}}(\mathcal{O} \to \prod\nolimits_{i_0} \mathcal{O}_{f_{i_0}} \to \prod\nolimits_{i_0<i_1} \mathcal{O}_{f_{i_0}f_{i_1}} \to \ldots \to \mathcal{O}_{f_1\ldots f_r}, K)$$

functorially in K.

Proof. Define $K^\wedge$ by the last displayed formula of the lemma. There is a map of complexes

$$(\mathcal{O} \to \prod\nolimits_{i_0} \mathcal{O}_{f_{i_0}} \to \prod\nolimits_{i_0<i_1} \mathcal{O}_{f_{i_0}f_{i_1}} \to \ldots \to \mathcal{O}_{f_1\ldots f_r}) \longrightarrow \mathcal{O}$$

which induces a map $K \to K^\wedge$. It suffices to prove that $K^\wedge$ is derived complete and that $K \to K^\wedge$ is an isomorphism if K is derived complete.

Let f be a global section of $\mathcal{O}$. By Lemma 46.14.1 the object $R\mathcal{H}om_{\mathcal{O}}(\mathcal{O}_f, K^\wedge)$ is equal to

$$R\mathcal{H}om_{\mathcal{O}}((\mathcal{O}_f \to \prod\nolimits_{i_0} \mathcal{O}_{ff_{i_0}} \to \prod\nolimits_{i_0<i_1} \mathcal{O}_{ff_{i_0}f_{i_1}} \to \ldots \to \mathcal{O}_{ff_1\ldots f_r}), K)$$

If $f = f_i$ for some i, then $f_1, \ldots, f_r$ generate the unit ideal in $\mathcal{O}_f$, hence the extended alternating Čech complex

$$\mathcal{O}_f \to \prod\nolimits_{i_0} \mathcal{O}_{ff_{i_0}} \to \prod\nolimits_{i_0<i_1} \mathcal{O}_{ff_{i_0}f_{i_1}} \to \ldots \to \mathcal{O}_{ff_1\ldots f_r}$$

is zero (even homotopic to zero). In this way we see that $K^\wedge$ is derived complete.

If K is derived complete, then $R\mathcal{H}om_{\mathcal{O}}(\mathcal{O}_f, K)$ is zero for all $f = f_{i_0} \ldots f_{i_p}$, $p \geq 0$. Thus $K \to K^\wedge$ is an isomorphism in $D(\mathcal{O})$. □

Lemma 46.14.7. *Let $\mathcal{C}$ be a site. Let $\mathcal{O} \to \mathcal{O}'$ be a homomorphism of sheaves of rings. Let $\mathcal{I} \subset \mathcal{O}$ be a sheaf of ideals. The inverse image of $D_{comp}(\mathcal{O}', \mathcal{I}\mathcal{O}')$ under the restriction functor $D(\mathcal{O}') \to D(\mathcal{O})$ is $D_{comp}(\mathcal{O}, \mathcal{I})$.*

Proof. Using Lemma 46.14.3 we see that $K' \in D(\mathcal{O}')$ is in $D_{comp}(\mathcal{O}', \mathcal{I}\mathcal{O}')$ if and only if $T(K'|_U, f)$ is zero for every local section $f \in \mathcal{I}(U)$. Observe that the cohomology sheaves of $T(K'|_U, f)$ are computed in the category of abelian sheaves, so it doesn't matter whether we think of f as a section of $\mathcal{O}$ or take the image of f as a section of $\mathcal{O}'$. The lemma follows immediately from this and the definition of derived complete objects. □

Lemma 46.14.8. *There exist a way to construct*

(1) *for every pair (A, I) consisting of a ring A and a finitely generated ideal $I \subset A$ a complex $K(A, I)$ of A-modules,*
(2) *a map $K(A, I) \to A$ of complexes of A-modules,*
(3) *for every ring map $A \to B$ and finitely generated ideal $I \subset A$ a map of complexes $K(A, I) \to K(B, IB)$,*

such that

(a) *for $A \to B$ and $I \subset A$ finitely generated the diagram*

$$\begin{array}{ccc} K(A, I) & \longrightarrow & A \\ \downarrow & & \downarrow \\ K(B, IB) & \longrightarrow & B \end{array}$$

commutes,

(b) *for $A \to B \to C$ and $I \subset A$ finitely generated the composition of the maps $K(A, I) \to K(B, IB) \to K(C, IC)$ is the map $K(A, I) \to K(C, IC)$.*
(c) *for $A \to B$ and a finitely generated ideal $I \subset A$ the induced map $K(A, I) \otimes^{\mathbf{L}}_A B \to K(B, IB)$ is an isomorphism in $D(B)$, and*
(d) *if $I = (f_1, \ldots, f_r) \subset A$ then there is a commutative diagram*

$$\begin{array}{ccc} (A \to \prod_{i_0} A_{f_{i_0}} \to \prod_{i_0 < i_1} A_{f_{i_0} f_{i_1}} \to \ldots \to A_{f_1 \ldots f_r}) & \longrightarrow & K(A, I) \\ \downarrow & & \downarrow \\ A & \xrightarrow{1} & A \end{array}$$

in $D(A)$ whose horizontal arrows are isomorphisms.

Proof. Let S be the set of rings A_0 of the form $A_0 = \mathbf{Z}[x_1, \ldots, x_n]/J$. Every finite type $\mathbf{Z}$-algebra is isomorphic to an element of S. Let $\mathcal{A}_0$ be the category whose objects are pairs (A_0, I_0) where $A_0 \in S$ and $I_0 \subset A_0$ is an ideal and whose morphisms $(A_0, I_0) \to (B_0, J_0)$ are ring maps $\varphi : A_0 \to B_0$ such that $J_0 = \varphi(I_0)B_0$.

Suppose we can construct $K(A_0, I_0) \to A_0$ functorially for objects of $\mathcal{A}_0$ having properties (a), (b), (c), and (d). Then we take

$$K(A, I) = \operatorname{colim}_{\varphi:(A_0,I_0)\to(A,I)} K(A_0, I_0)$$

where the colimit is over ring maps $\varphi : A_0 \to A$ such that $\varphi(I_0)A = I$ with (A_0, I_0) in $\mathcal{A}_0$. A morphism between $(A_0, I_0) \to (A, I)$ and $(A'_0, I'_0) \to (A, I)$ are given by maps $(A_0, I_0) \to (A'_0, I'_0)$ in $\mathcal{A}_0$ commuting with maps to A. The category of these $(A_0, I_0) \to (A, I)$ is filtered (details omitted). Moreover, $\operatorname{colim}_{\varphi:(A_0,I_0)\to(A,I)} A_0 = A$ so that $K(A, I)$ is a complex of A-modules. Finally, given $\varphi : A \to B$ and $I \subset A$ for every $(A_0, I_0) \to (A, I)$ in the colimit, the composition $(A_0, I_0) \to (B, IB)$ lives in the colimit for (B, IB). In this way we get a map on colimits. Properties (a),

(b), (c), and (d) follow readily from this and the corresponding properties of the complexes $K(A_0, I_0)$.

Endow $\mathcal{C}_0 = \mathcal{A}_0^{opp}$ with the chaotic topology. We equip $\mathcal{C}_0$ with the sheaf of rings $\mathcal{O} : (A, I) \mapsto A$. The ideals I fit together to give a sheaf of ideals $\mathcal{I} \subset \mathcal{O}$. Choose an injective resolution $\mathcal{O} \to \mathcal{J}^\bullet$. Consider the object

$$\mathcal{F}^\bullet = \bigcup\nolimits_n \mathcal{J}^\bullet[\mathcal{I}^n]$$

Let $U = (A, I) \in \mathrm{Ob}(\mathcal{C}_0)$. Since the topology in $\mathcal{C}_0$ is chaotic, the value $\mathcal{J}^\bullet(U)$ is a resolution of A by injective A-modules. Hence the value $\mathcal{F}^\bullet(U)$ is an object of $D(A)$ representing $R\Gamma_Z(A)$ where $Z = V(I)$, see Dualizing Complexes, Section 43.8. Choose a complex of $\mathcal{O}$-modules $\mathcal{K}^\bullet$ and a commutative diagram

$$\begin{array}{ccc} \mathcal{O} & \longrightarrow & \mathcal{J}^\bullet \\ \uparrow & & \uparrow \\ \mathcal{K}^\bullet & \longrightarrow & \mathcal{F}^\bullet \end{array}$$

where the horizontal arrows are quasi-isomorphisms. This is possible by the construction of the derived category $D(\mathcal{O})$. Set $K(A, I) = \mathcal{K}^\bullet(U)$ where $U = (A, I)$. Properties (a) and (b) are clear and properties (c) and (d) follow from Dualizing Complexes, Lemmas 43.8.4 and 43.8.5. □

Lemma 46.14.9. *Let $(\mathcal{C}, \mathcal{O})$ be a ringed site. Let $\mathcal{I} \subset \mathcal{O}$ be a finite type sheaf of ideals. There exists a map $K \to \mathcal{O}$ in $D(\mathcal{O})$ such that for every $U \in \mathrm{Ob}(\mathcal{C})$ such that $\mathcal{I}|_U$ is generated by $f_1, \ldots, f_r \in \mathcal{I}(U)$ there is an isomorphism*

$$(\mathcal{O}_U \to \prod\nolimits_{i_0} \mathcal{O}_{U,f_{i_0}} \to \prod\nolimits_{i_0<i_1} \mathcal{O}_{U,f_{i_0}f_{i_1}} \to \ldots \to \mathcal{O}_{U,f_1\ldots f_r}) \longrightarrow K|_U$$

compatible with maps to $\mathcal{O}_U$.

Proof. Let $\mathcal{C}' \subset \mathcal{C}$ be the full subcategory of objects U such that $\mathcal{I}|_U$ is generated by finitely many sections. Then $\mathcal{C}' \to \mathcal{C}$ is a special cocontinuous functor (Sites, Definition 7.28.2). Hence it suffices to work with $\mathcal{C}'$, see Sites, Lemma 7.28.1. in other words we may assume that for every object U of $\mathcal{C}$ there exists a finitely generated ideal $I \subset \mathcal{I}(U)$ such that $\mathcal{I}|_U = \mathrm{Im}(I \otimes \mathcal{O}_U \to \mathcal{O}_U)$. We will say that I generates $\mathcal{I}|_U$. Warning: We do not know that $\mathcal{I}(U)$ is a finitely generated ideal in $\mathcal{O}(U)$.

Let U be an object and $I \subset \mathcal{O}(U)$ a finitely generated ideal which generates $\mathcal{I}|_U$. On the category $\mathcal{C}/U$ consider the complex of presheaves

$$K^\bullet_{U,I} : U'/U \longmapsto K(\mathcal{O}(U'), I\mathcal{O}(U'))$$

with $K(-,-)$ as in Lemma 46.14.8. We claim that the sheafification of this is independent of the choice of I. Indeed, if $I' \subset \mathcal{O}(U)$ is a finitely generated ideal which also generates $\mathcal{I}|_U$, then there exists a covering $\{U_j \to U\}$ such that $I\mathcal{O}(U_j) = I'\mathcal{O}(U_j)$. (Hint: this works because both I and I' are finitely generated and generate $\mathcal{I}|_U$.) Hence $K^\bullet_{U,I}$ and $K^\bullet_{U,I'}$ are the *same* for any object lying over one of the U_j. The statement on sheafifications follows. Denote $K^\bullet_U$ the common value.

The independence of choice of I also shows that $K^\bullet_U|_{\mathcal{C}/U'} = K^\bullet_{U'}$ whenever we are given a morphism $U' \to U$ and hence a localization morphism $\mathcal{C}/U' \to \mathcal{C}/U$. Thus the complexes $K^\bullet_U$ glue to give a single well defined complex $K^\bullet$ of $\mathcal{O}$-modules. The existence of the map $K^\bullet \to \mathcal{O}$ and the quasi-isomorphism of the lemma follow

immediately from the corresponding properties of the complexes $K(-,-)$ in Lemma 46.14.8. □

Proposition 46.14.10. *Let $(\mathcal{C}, \mathcal{O})$ be a ringed site. Let $\mathcal{I} \subset \mathcal{O}$ be a finite type sheaf of ideals. There exists a left adjoint to the inclusion functor $D_{comp}(\mathcal{O}) \to D(\mathcal{O})$.*

Proof. Let $K \to \mathcal{O}$ in $D(\mathcal{O})$ be as constructed in Lemma 46.14.9. Let $E \in D(\mathcal{O})$. Then $E^\wedge = R\mathcal{H}om(K, E)$ together with the map $E \to E^\wedge$ will do the job. Namely, locally on the site $\mathcal{C}$ we recover the adjoint of Lemma 46.14.6. This shows that $E^\wedge$ is always derived complete and that $E \to E^\wedge$ is an isomorphism if E is derived complete. □

Remark 46.14.11 (Completed tensor product)**.** Let $(\mathcal{C}, \mathcal{O})$ be a ringed site. Let $\mathcal{I} \subset \mathcal{O}$ be a finite type sheaf of ideals. Denote $K \mapsto K^\wedge$ the adjoint of Proposition 46.14.10. Then we set

$$K \otimes^\wedge_{\mathcal{O}} L = (K \otimes^{\mathbf{L}}_{\mathcal{O}} L)^\wedge$$

This *completed tensor product* defines a functor $D_{comp}(\mathcal{O}) \times D_{comp}(\mathcal{O}) \to D_{comp}(\mathcal{O})$ such that we have

$$\mathrm{Hom}_{D_{comp}(\mathcal{O})}(K, R\mathcal{H}om_{\mathcal{O}}(L, M)) = \mathrm{Hom}_{D_{comp}(\mathcal{O})}(K \otimes^\wedge_{\mathcal{O}} L, M)$$

for $K, L, M \in D_{comp}(\mathcal{O})$. Note that $R\mathcal{H}om_{\mathcal{O}}(L, M) \in D_{comp}(\mathcal{O})$ by Lemma 46.14.5.

Lemma 46.14.12. *Let $\mathcal{C}$ be a site. Assume $\varphi : \mathcal{O} \to \mathcal{O}'$ is a flat homomorphism of sheaves of rings. Let $f_1, \ldots, f_r$ be global sections of $\mathcal{O}$ such that $\mathcal{O}/(f_1, \ldots, f_r) \cong \mathcal{O}'/(f_1, \ldots, f_r)$. Then the map of extended alternating Čech complexes*

$$\begin{array}{ccccccc} \mathcal{O} & \to & \prod_{i_0} \mathcal{O}_{f_{i_0}} & \to & \prod_{i_0<i_1} \mathcal{O}_{f_{i_0}f_{i_1}} & \to \ldots \to & \mathcal{O}_{f_1\ldots f_r} \\ & & \downarrow & & \downarrow & & \\ \mathcal{O}' & \to & \prod_{i_0} \mathcal{O}'_{f_{i_0}} & \to & \prod_{i_0<i_1} \mathcal{O}'_{f_{i_0}f_{i_1}} & \to \ldots \to & \mathcal{O}'_{f_1\ldots f_r} \end{array}$$

is a quasi-isomorphism.

Proof. Observe that the second complex is the tensor product of the first complex with $\mathcal{O}'$. We can write the first extended alternating Čech complex as a colimit of the Koszul complexes $K_n = K(\mathcal{O}, f_1^n, \ldots, f_r^n)$, see More on Algebra, Lemma 15.19.13. Hence it suffices to prove $K_n \to K_n \otimes_{\mathcal{O}} \mathcal{O}'$ is a quasi-isomorphism. Since $\mathcal{O} \to \mathcal{O}'$ is flat it suffices to show that $H^i \to H^i \otimes_{\mathcal{O}} \mathcal{O}'$ is an isomorphism where H^i is the ith cohomology sheaf $H^i = H^i(K_n)$. These sheaves are annihilated by $f_1^n, \ldots, f_r^n$, see More on Algebra, Lemma 15.19.6. Thus it suffices to show that $\mathcal{O}/(f_1^n, \ldots, f_r^n) \to \mathcal{O}'/(f_1^n, \ldots, f_r^n)$ is an isomorphism. Equivalently, we will show that $\mathcal{O}/(f_1, \ldots, f_r)^n \to \mathcal{O}'/(f_1, \ldots, f_r)^n$ is an isomorphism for all n. This holds for $n = 1$ by assumption. It follows for all n by induction using Modules on Sites, Lemma 18.28.13 applied to the ring map $\mathcal{O}/(f_1, \ldots, f_r)^{n+1} \to \mathcal{O}/(f_1, \ldots, f_r)^n$ and the module $\mathcal{O}'/(f_1, \ldots, f_r)^{n+1}$. □

Lemma 46.14.13. *Let $\mathcal{C}$ be a site with enough points. Let $\mathcal{O} \to \mathcal{O}'$ be a homomorphism of sheaves of rings. Let $\mathcal{I} \subset \mathcal{O}$ be a finite type sheaf of ideals. If $\mathcal{O} \to \mathcal{O}'$ is flat and $\mathcal{O}/\mathcal{I} \cong \mathcal{O}'/\mathcal{I}\mathcal{O}'$, then the restriction functor $D(\mathcal{O}') \to D(\mathcal{O})$ induces an equivalence $D_{comp}(\mathcal{O}', \mathcal{I}\mathcal{O}') \to D_{comp}(\mathcal{O}, \mathcal{I})$.*

Proof. Let $K \to \mathcal{O}$ be the morphism of $D(\mathcal{O})$ constructed in Lemma 46.14.9. Set $K' = K \otimes_{\mathcal{O}} \mathcal{O}'$. Then $K' \to \mathcal{O}'$ is a map in $D(\mathcal{O}')$ satisfying the same condition with respect to $\mathcal{I}' = \mathcal{I}\mathcal{O}'$. The map $K \to K'$ is a quasi-isomorphism by Lemma 46.14.12. Now, let $E \in D_{comp}(\mathcal{O}, \mathcal{I})$. By the proof of Proposition 46.14.10 we have the first equality in

$$E = R\mathcal{H}om_{\mathcal{O}}(K, E) = R\mathcal{H}om_{\mathcal{O}}(K', E)$$

Since K' is a complex of $\mathcal{O}'$ modules, this shows that E is the image of some $E' \in D(\mathcal{O}')$. By Lemma 46.14.7 we have $E' \in D_{comp}(\mathcal{O}')$. Thus the functor is essentially surjective. In fact the functor $E \mapsto R\mathcal{H}om_{\mathcal{O}}(K', E)$ is a quasi-inverse to the restriction functor. The formula above shows this in one direction. The other direction hinges on the fact that for $E' \in D_{comp}(\mathcal{O}')$ the map

$$R\mathcal{H}om_{\mathcal{O}}(K', E') \to R\mathcal{H}om_{\mathcal{O}}(\mathcal{O}', E')$$

is an isomorphism in $D(\mathcal{O}')$ for both $\mathcal{O}'$-module structures and the existence of a map $E' \to R\mathcal{H}om_{\mathcal{O}}(\mathcal{O}', E')$ which is $\mathcal{O}'$-linear for both $\mathcal{O}'$-module structures on the target. Details omitted. □

Lemma 46.14.14. *Let $f : (Sh(\mathcal{D}), \mathcal{O}') \to (Sh(\mathcal{C}), \mathcal{O})$ be a morphism of ringed topoi. Let $\mathcal{I} \subset \mathcal{O}$ and $\mathcal{I}' \subset \mathcal{O}'$ be sheaves of ideals such that $f^\sharp$ sends $f^{-1}\mathcal{I}$ into $\mathcal{I}'$. Then Rf_* sends $D_{comp}(\mathcal{O}', \mathcal{I}')$ into $D_{comp}(\mathcal{O}, \mathcal{I})$.*

Proof. We may assume f is given by a morphism of ringed sites corresponding to a continuous functor $\mathcal{C} \to \mathcal{D}$ (Modules on Sites, Lemma 18.7.2). Let U be an object of $\mathcal{C}$ and let g be a section of $\mathcal{I}$ over U. We have to show that $\mathrm{Hom}_{D(\mathcal{O}_U)}(\mathcal{O}_g, Rf_*K) = 0$ whenever K is derived complete with respect to $\mathcal{I}'$. (Because by Cohomology on Sites, Lemma 21.25.2 this, applied to all objects over U and all shifts of K, will imply that $R\mathcal{H}om_{\mathcal{O}}(\mathcal{O}_g, Rf_*K)$ is zero.) Let V in $\mathcal{D}$ be the image of U. Then

$$\mathrm{Hom}_{D(\mathcal{O}_U)}(\mathcal{O}_g, Rf_*K) = \mathrm{Hom}_{D(\mathcal{O}'_V)}(\mathcal{O}'_{g'}, K) = 0$$

where $g' = f^\sharp(g) \in \mathcal{I}'(V)$. This holds because $Lf^*\mathcal{O}_g = \mathcal{O}'_{g'}$ and Cohomology on Sites, Lemma 21.19.1. □

Lemma 46.14.15. *Let $f : (Sh(\mathcal{D}), \mathcal{O}') \to (Sh(\mathcal{C}), \mathcal{O})$ be a morphism of ringed topoi. Let $\mathcal{I} \subset \mathcal{O}$ and $\mathcal{I}' \subset \mathcal{O}'$ be finite type sheaves of ideals such that $f^\sharp$ sends $f^{-1}\mathcal{I}$ into $\mathcal{I}'$. Then Rf_* sends $D_{comp}(\mathcal{O}', \mathcal{I}')$ into $D_{comp}(\mathcal{O}, \mathcal{I})$ and has a left adjoint Lf^*_{comp} which is Lf^* followed by derived completion.*

Proof. The first statement we have seen in Lemma 46.14.14. Note that the second statement makes sense as we have a derived completion functor $D(\mathcal{O}') \to D_{comp}(\mathcal{O}', \mathcal{I}')$ by Proposition 46.14.10. OK, so now let $K \in D_{comp}(\mathcal{O}, \mathcal{I})$ and $M \in D_{comp}(\mathcal{O}', \mathcal{I}')$. Then we have

$$R\mathrm{Hom}(K, Rf_*M) = R\mathrm{Hom}(Lf^*K, M) = R\mathrm{Hom}(Lf^*_{comp}K, M)$$

by the universal property of derived completion. □

46.15. Derived completion in the constant Noetherian case

Let $\mathcal{C}$ be a site. Let Λ be a Noetherian ring and let $I \subset \Lambda$ be an ideal. Recall from Modules on Sites, Lemma 18.41.4 that

$$\underline{\Lambda}^\wedge = \lim \underline{\Lambda/I^n}$$

is a flat $\underline{\Lambda}$-algebra and that the map $\underline{\Lambda} \to \underline{\Lambda}^\wedge$ identifies quotients by I. Hence Lemma 46.14.13 tells us that

$$D_{comp}(\mathcal{C}, \Lambda) = D_{comp}(\mathcal{C}, \underline{\Lambda}^\wedge)$$

In particular the cohomology sheaves $H^i(K)$ of an object K of $D_{comp}(\mathcal{C}, \Lambda)$ are sheaves of $\underline{\Lambda}^\wedge$-modules. For notational convenience we often work with $D_{comp}(\mathcal{C}, \Lambda)$.

Lemma 46.15.1. *Let $\mathcal{C}$ be a site. Let Λ be a Noetherian ring and let $I \subset \Lambda$ be an ideal. The left adjoint to the inclusion functor $D_{comp}(\mathcal{C}, \Lambda) \to D(\mathcal{C}, \Lambda)$ of Proposition 46.14.10 sends K to*

$$K^\wedge = R\lim(K \otimes^{\mathbf{L}}_\Lambda \underline{\Lambda/I^n})$$

In particular, K is derived complete if and only if $K = R\lim(K \otimes^{\mathbf{L}}_\Lambda \underline{\Lambda/I^n})$.

Proof. Choose generators $f_1, \ldots, f_r$ of I. In More on Algebra, Lemma 15.19.13 we have seen that the extended alternating Čech complex

$$\Lambda \to \prod\nolimits_{i_0} \Lambda_{f_{i_0}} \to \prod\nolimits_{i_0 < i_1} \Lambda_{f_{i_0} f_{i_1}} \to \ldots \to \Lambda_{f_1 \ldots f_r}$$

is a colimit of the Koszul complexes $K^n = K(\Lambda, f_1^n, \ldots, f_r^n)$. Note that K^n is a finite chain complex of finite free Λ-modules with dual $\mathrm{Hom}_\Lambda(K^n, \Lambda) = K_n$ where K_n is the Koszul cochain complex sitting in degrees $-r, \ldots, 0$. By Lemma 46.14.6 the functor $K \mapsto K^\wedge$ is gotten by taking $R\mathcal{H}om$ from the extended alternating Čech complex into K:

$$K^\wedge = R\mathcal{H}om(\mathrm{colim}\, \underline{K^n}, K) = R\lim(K \otimes^{\mathbf{L}}_\Lambda \underline{K_n})$$

Use the fact that each K_n is a perfect complex to show that $R\mathcal{H}om(\mathrm{colim}\, \underline{K^n}, K) = R\lim R\mathcal{H}om(\underline{K^n}, K)$ and use Cohomology on Sites, Lemma 21.35.9 to convert to a derived tensor product — some arguments omitted. In More on Algebra, Lemma 15.57.13 we have seen that the pro-systems $\{K_n\}$ and $\{\Lambda/I^n\}$ of $D(\Lambda)$ are isomorphic. Thus the lemma follows. □

Lemma 46.15.2. *Let Λ be a Noetherian ring. Let $I \subset \Lambda$ be an ideal. Let $f : Sh(\mathcal{D}) \to Sh(\mathcal{C})$ be a morphism of topoi. Then Rf_* sends $D_{comp}(\mathcal{D}, \Lambda)$ into $D_{comp}(\mathcal{C}, \Lambda)$ and has a left adjoint Lf^*_{comp} which is Lf^* followed by derived completion. Moreover,*

1. *for K in $D_{comp}(\mathcal{D}, \Lambda)$ we have $Rf_*K = R\lim Rf_*(K \otimes^{\mathbf{L}}_\Lambda \underline{\Lambda/I^n})$.*
2. *for M in $D_{comp}(\mathcal{C}, \Lambda)$ we have $Lf^*_{comp}M = R\lim Lf^*(M \otimes^{\mathbf{L}}_\Lambda \underline{\Lambda/I^n})$.*

Proof. The first statement we have seen in Lemma 46.14.15. Thus we know that

$$Rf_*K = R\lim(Rf_*(K) \otimes^{\mathbf{L}}_\Lambda \underline{\Lambda/I^n})$$

by Lemma 46.15.1. Let $f_1, \ldots, f_r$ be generators of I. Let K_n be the Koszul complex on $f_1^n, \ldots, f_r^n$ over Λ sitting in cohomological degrees $-r, \ldots, 0$. Then $\{\Lambda/I^n\}$ and $\{K_n\}$ define the same pro-object of $D(\Lambda)$ by More on Algebra, Lemma 15.57.13. Since K_n is perfect tensoring with it commutes with Rf_* by Cohomology on Sites, Lemma 21.36.1. Thus we get (1).

By Lemma 46.15.1 we have

$$Lf^*_{comp}M = R\lim(Lf^*M \otimes^{\mathbf{L}}_\Lambda \underline{\Lambda/I^n})$$

Since Lf^* commutes with derived tensor product by Cohomology on Sites, Lemma 21.18.4 and since $Lf^*\underline{\Lambda/I^n} = \underline{\Lambda/I^n}$ we get (2). □

46.16. Derived completion on the pro-étale site

Let $\mathcal{C}$ be a site. Let Λ be a Noetherian ring. Let $I \subset \Lambda$ be an ideal. Although the general theory (see Sections 46.14 and 46.15) concerning $D_{comp}(\mathcal{C}, \Lambda)$ is quite satisfactory it is somewhat useless as it is hard to explicitly give examples of derived complete complexes. We know that

(1) every object M of $D(\mathcal{C}, \Lambda/I^n)$ restricts to a derived complete object of $D(\mathcal{C}, \Lambda)$, and
(2) for every $K \in D(\mathcal{C}, \Lambda)$ the derived completion $K^\wedge = R\lim(K \otimes^{\mathbf{L}}_\Lambda \underline{\Lambda/I^n})$ is derived complete.

The first type of objects are trivially complete and perhaps not interesting. The problem with (2) is that derived completion in general is somewhat mysterious, even in case $K = \underline{\Lambda}$. Namely, by definition of homotopy limits there is a distinguished triangle

$$R\lim(\underline{\Lambda/I^n}) \to \prod \underline{\Lambda/I^n} \to \prod \underline{\Lambda/I^n} \to R\lim(\underline{\Lambda/I^n})[1]$$

in $D(\mathcal{C}, \Lambda)$ where the products are in $D(\mathcal{C}, \Lambda)$. These are computed by taking products of injective resolutions (Injectives, Lemma 19.13.4), so we see that the sheaf $H^p(\prod \underline{\Lambda/I^n})$ is the sheafification of the presheaf

$$U \longmapsto \prod H^p(U, \Lambda/I^n).$$

As an explicit example, if $X = \mathrm{Spec}(\mathbf{C}[t, t^{-1}])$, $\mathcal{C} = X_{étale}$, $\Lambda = \mathbf{Z}$, $I = (2)$, and $p = 1$, then we get the sheafification of the presheaf

$$U \mapsto \prod H^1(U_{étale}, \mathbf{Z}/2^n\mathbf{Z})$$

for U étale over X. Note that $H^1(X_{étale}, \mathbf{Z}/m\mathbf{Z})$ is cyclic of order m with generator α_m given by the finite étale $\mathbf{Z}/m\mathbf{Z}$-covering given by the equation $t = s^m$ (see Étale Cohomology, Section 44.6). Then the section

$$\alpha = (\alpha_{2^n}) \in \prod H^1(X_{étale}, \mathbf{Z}/2^n\mathbf{Z})$$

of the presheaf above does not restrict to zero on any nonempty étale scheme over X, whence the sheaf associated to the presheaf is not zero.

However, on the pro-étale site this phenomenon does not occur. The reason is that we have enough (quasi-compact) weakly contractible objects. In the following proposition we collect some results about derived completion in the Noetherian constant case for sites having enough weakly contractible objects (see Sites, Definition 7.39.2).

Proposition 46.16.1. *Let $\mathcal{C}$ be a site. Assume $\mathcal{C}$ has enough weakly contractible objects. Let Λ be a Noetherian ring. Let $I \subset \Lambda$ be an ideal.*

(1) *The category of derived complete sheaves Λ-modules is a weak Serre subcategory of Mod$(\mathcal{C}, \Lambda)$.*
(2) *A sheaf $\mathcal{F}$ of Λ-modules satisfies $\mathcal{F} = \lim \mathcal{F}/I^n\mathcal{F}$ if and only if $\mathcal{F}$ is derived complete and $\bigcap I^n\mathcal{F} = 0$.*
(3) *The sheaf $\underline{\Lambda}^\wedge$ is derived complete.*
(4) *If $\ldots \to \mathcal{F}_3 \to \mathcal{F}_2 \to \mathcal{F}_1$ is an inverse system of derived complete sheaves of Λ-modules, then $\lim \mathcal{F}_n$ is derived complete.*
(5) *An object $K \in D(\mathcal{C}, \Lambda)$ is derived complete if and only if each cohomology sheaf $H^p(K)$ is derived complete.*

(6) *An object $K \in D_{comp}(\mathcal{C}, \Lambda)$ is bounded above if and only if $K \otimes^{\mathbf{L}}_{\Lambda} \underline{\Lambda/I}$ is bounded above.*
(7) *An object $K \in D_{comp}(\mathcal{C}, \Lambda)$ is bounded if $K \otimes^{\mathbf{L}}_{\Lambda} \underline{\Lambda/I}$ has finite tor dimension.*

Proof. Let $\mathcal{B} \subset \mathrm{Ob}(\mathcal{C})$ be a subset such that every $U \in \mathcal{B}$ is weakly contractible and every object of $\mathcal{C}$ has a covering by elements of $\mathcal{B}$. We will use the results of Cohomology on Sites, Lemma 21.37.1 and Proposition 21.37.2 without further mention.

Recall that $R\lim$ commutes with $R\Gamma(U, -)$, see Injectives, Lemma 19.13.6. Let $f \in I$. Recall that $T(K, f)$ is the homotopy limit of the system

$$\ldots K \xrightarrow{f} K \xrightarrow{f} K$$

in $D(\mathcal{C}, \Lambda)$. Thus

$$R\Gamma(U, T(K, f)) = T(R\Gamma(U, K), f).$$

Since we can test isomorphisms of maps between objects of $D(\mathcal{C}, \Lambda)$ by evaluating at $U \in \mathcal{B}$ we conclude an object K of $D(\mathcal{C}, \Lambda)$ is derived complete if and only if for every $U \in \mathcal{B}$ the object $R\Gamma(U, K)$ is derived complete as an object of $D(\Lambda)$.

The remark above implies that items (1), (5) follow from the corresponding results for modules over rings, see More on Algebra, Lemmas 15.57.1 and 15.57.6. In the same way (2) can be deduced from More on Algebra, Proposition 15.57.5 as $(I^n\mathcal{F})(U) = I^n \cdot \mathcal{F}(U)$ for $U \in \mathcal{B}$ (by exactness of evaluating at U).

Proof of (4). The homotopy limit $R\lim \mathcal{F}_n$ is in $D_{comp}(X, \Lambda)$ (see discussion following Definition 46.14.4). By part (5) just proved we conclude that $\lim \mathcal{F}_n = H^0(R\lim \mathcal{F}_n)$ is derived complete. Part (3) is a special case of (4).

Proof of (6) and (7). Follows from Lemma 46.15.1 and Cohomology on Sites, Lemma 21.34.8 and the computation of homotopy limits in Cohomology on Sites, Proposition 21.37.2. □

46.17. Comparison with the étale site

Let X be a scheme. With suitable choices of sites (as in Topologies, Remark 33.9.1) the functor $u : X_{étale} \to X_{pro\text{-}étale}$ sending U/X to U/X defines a morphism of sites

$$\epsilon : X_{pro\text{-}étale} \longrightarrow X_{étale}$$

This follows from Sites, Proposition 7.15.6. A fundamental fact about this comparison morphism is the following.

Lemma 46.17.1. *Let X be a scheme. Let $Y = \lim Y_i$ be an inverse limit of quasi-compact and quasi-separated schemes étale over X with affine transition morphisms. For any sheaf $\mathcal{F}$ on $X_{étale}$ we have $\epsilon^{-1}\mathcal{F}(Y) = \mathrm{colim}\, \mathcal{F}(Y_i)$.*

Proof. Let $\mathcal{F} = h_U$ be a representable sheaf on $X_{étale}$ with U an object of $X_{étale}$. In this case $\epsilon^{-1}h_U = h_{u(U)}$ where $u(U)$ is U viewed as an object of $X_{pro\text{-}étale}$ (Sites, Lemma 7.14.5). Then

$$h_{u(U)}(Y) = \mathrm{Mor}_X(Y, U) = \mathrm{colim}\, \mathrm{Mor}_X(Y_i, U) = \mathrm{colim}\, h_U(Y_i)$$

by Limits, Proposition 31.5.1. Hence the lemma holds for every representable sheaf. Since every sheaf is a coequalizer of a map of coproducts of representable sheaves (Sites, Lemma 7.13.5) we obtain the result in general. □

Lemma 46.17.2. *Let X be a scheme. For every sheaf $\mathcal{F}$ on $X_{étale}$ the adjunction map $\mathcal{F} \to \epsilon_*\epsilon^{-1}\mathcal{F}$ is an isomorphism.*

Proof. Suppose that U is a quasi-compact and quasi-separated scheme étale over X. Then

$$\epsilon_*\epsilon^{-1}\mathcal{F}(U) = \epsilon^{-1}\mathcal{F}(U) = \mathcal{F}(U)$$

the second equality by (a special case of) Lemma 46.17.1. Since every object of $X_{étale}$ has a covering by quasi-compact and quasi-separated objects we conclude. □

Lemma 46.17.3. *Let X be an affine scheme. For injective abelian sheaf $\mathcal{I}$ on $X_{étale}$ we have $H^p(X_{pro\text{-}étale}, \epsilon^{-1}\mathcal{I}) = 0$ for $p > 0$.*

Proof. We are going to use Cohomology on Sites, Lemma 21.11.8 to prove this. The idea is simple: We show that every standard pro-étale covering of X is a limit of coverings in $X_{étale}$. If this holds then Lemma 46.17.1 will kick in to show the Čech cohomology groups of $\epsilon^{-1}\mathcal{I}$ are colimits of those of $\mathcal{I}$ which are zero in positive degree.

Here are the details. Let $\mathcal{B} \subset \mathrm{Ob}(X_{pro\text{-}étale})$ be the set of affine schemes U over X such that $\mathcal{O}(X) \to \mathcal{O}(U)$ is ind-étale. Let Cov be the set of pro-étale coverings $\{U_i \to U\}_{i=1,\ldots,n}$ with $U, U_i \in \mathcal{B}$ such that $\mathcal{O}(U) \to \mathcal{O}(U_i)$ is ind-étale for $i = 1, \ldots, n$. Properties (1) and (2) of Cohomology on Sites, Lemma 21.11.8 hold for $\mathcal{B}$ and Cov by Proposition 46.9.1 (it also follows from Lemma 46.11.10).

To check condition (3) suppose that $\{U_i \to U\}_{i=1,\ldots,n}$ is an element of Cov. Then we can write $U_i = \lim_{a \in A_i} U_{i,a}$ with $U_{i,a} \to U$ étale and $U_{i,a}$ affine. Next we write $U = \lim_{b \in B} U_b$ with U_b affine and $U_b \to U$ étale. By Limits, Lemma 31.9.1 for each i and $a \in A_i$ we can choose a $b(i,a) \in B$ and for all $b \geq b(i,a)$ an affine scheme $U_{i,a,b}$ étale over U_b such that $U_{i,a} = \lim_{b \geq b(i,a)} U_{i,a,b}$[2]. Moreover, any transition map $U_{i,a} \to U_{i,a'}$ comes from an essentially unique morphism $U_{i,a,b} \to U_{i,a',b}$ for b large enough (by the same reference). Finally, given $a_1 \in A_1, \ldots, a_n \in A_n$ the morphism $U_{1,a_1} \amalg \ldots \amalg U_{n,a_n} \to U$ is surjective, hence for b large enough the map $U_{1,a_1,b} \amalg \ldots \amalg U_{n,a_n,b} \to U_b$ is surjective by Limits, Lemma 31.7.11. Let $\mathcal{D}$ be the category of coverings $\{U_{i,a_i,b} \to U_b\}_{i=1,\ldots,n}$ so obtained. This category is cofiltered. We claim that, given $i_0, \ldots, i_p \in \{1, \ldots, n\}$ we have

$$U_{i_0} \times_U U_{i_1} \times_U \ldots \times_U U_{i_p} = \lim_{\mathcal{D}} U_{i_0,a_{i_0},b} \times_{U_b} U_{i_1,a_{i_1},b} \times_{U_b} \ldots \times_{U_b} U_{i_p,a_{i_p},b}$$

This is clear from the fact that it holds for $p = -1$ (i.e., $U = \lim_{\mathcal{D}} U_b$) and for $p = 0$ (i.e., $U_i = \lim_{\mathcal{D}} U_{i,a_i,b}$) and the fact that fibre products commute with limits. Then finally it follows from Lemma 46.17.1 that

$$\check{\mathcal{C}}^\bullet(\{U_i \to U\}, \epsilon^{-1}\mathcal{I}) = \mathrm{colim}_{\mathcal{D}^{opp}} \check{\mathcal{C}}^\bullet(\{U_{i,a_i,b} \to U_b\}, \mathcal{I})$$

Since each of the Čech complexes on the right hand side is acyclic in positive degrees (Cohomology on Sites, Lemma 21.11.2) it follows that the one on the left is too. This prove condition (3) of Cohomology on Sites, Lemma 21.11.8. Since $X \in \mathcal{B}$ the lemma follows. □

Lemma 46.17.4. *Let X be a scheme. For an abelian sheaf $\mathcal{F}$ on $X_{étale}$ we have $R\epsilon_*(\epsilon^{-1}\mathcal{F}) = \mathcal{F}$.*

[2]To be sure, we pick $U_{i,a,b} = U_b \times_{U_{b(i,a)}} U_{i,a,b(i,a)}$ although this isn't necessary for what follows.

Proof. Let $\mathcal{I}$ be an injective abelian sheaf on $X_{étale}$. Recall that $R^q\epsilon_*(\epsilon^{-1}\mathcal{I})$ is the sheaf associated to $U \mapsto H^q(U_{pro\text{-}étale}, \epsilon^{-1}\mathcal{I})$, see Cohomology on Sites, Lemma 21.8.4. By Lemma 46.17.3 we see that this is zero for $q > 0$ and U affine and étale over X. Since every object of $X_{étale}$ has a covering by affine objects, it follows that $R^q\epsilon_*(\epsilon^{-1}\mathcal{I}) = 0$ for $q > 0$. Combined with Lemma 46.17.2 we conclude that $R\epsilon_*\epsilon^{-1}\mathcal{I} = \mathcal{I}$ for every injective abelian sheaf. Since every abelian sheaf has a resolution by injective sheaves, the result follows. (Hint: use Leray acyclicity theorem – Derived Categories, Lemma 13.16.7.) □

Lemma 46.17.5. *Let X be a scheme. For an abelian sheaf $\mathcal{F}$ on $X_{étale}$ we have*

$$H^i(X_{étale}, \mathcal{F}) = H^i(X_{pro\text{-}étale}, \epsilon^{-1}\mathcal{F})$$

for all i.

Proof. Immediate consequence of Lemma 46.17.4 and the Leray spectral sequence (Cohomology on Sites, Lemma 21.14.6). □

Lemma 46.17.6. *Let X be a scheme. Let $\mathcal{G}$ be a sheaf of (possibly noncommutative) groups on $X_{étale}$. We have*

$$H^1(X_{étale}, \mathcal{G}) = H^1(X_{pro\text{-}étale}, \epsilon^{-1}\mathcal{G})$$

where H^1 is defined as the set of isomorphism classes of torsors (see Cohomology on Sites, Section 21.5).

Proof. Since the functor ϵ^{-1} is fully faithful by Lemma 46.17.2 it is clear that the map $H^1(X_{étale}, \mathcal{G}) \to H^1(X_{pro\text{-}étale}, \epsilon^{-1}\mathcal{G})$ is injective. To show surjectivity it suffices to show that any $\epsilon^{-1}\mathcal{G}$-torsor $\mathcal{F}$ is étale locally trivial. To do this we may assume that X is affine. Thus we reduce to proving surjectivity for X affine.

Choose a covering $\{U \to X\}$ with (a) U affine, (b) $\mathcal{O}(X) \to \mathcal{O}(U)$ ind-étale, and (c) $\mathcal{F}(U)$ nonempty. We can do this by Proposition 46.9.1 and the fact that standard pro-étale coverings of X are cofinal among all pro-étale coverings of X (Lemma 46.11.5). Write $U = \lim U_i$ as a limit of affine schemes étale over X. Pick $s \in \mathcal{F}(U)$. Let $g \in \epsilon^{-1}\mathcal{G}(U \times_X U)$ be the unique section such that $g \cdot \mathrm{pr}_1^* s = \mathrm{pr}_2^* s$ in $\mathcal{F}(U \times_X U)$. Then g satisfies the cocycle condition

$$\mathrm{pr}_{12}^* g \cdot \mathrm{pr}_{23}^* g = \mathrm{pr}_{13}^* g$$

in $\epsilon^{-1}\mathcal{G}(U \times_X U \times_X U)$. By Lemma 46.17.1 we have

$$\epsilon^{-1}\mathcal{G}(U \times_X U) = \operatorname{colim} \mathcal{G}(U_i \times_X U_i)$$

and

$$\epsilon^{-1}\mathcal{G}(U \times_X U \times_X U) = \operatorname{colim} \mathcal{G}(U_i \times_X U_i \times_X U_i)$$

hence we can find an i and an element $g_i \in \mathcal{G}(U_i)$ mapping to g satisfying the cocycle condition. The cocycle g_i then defines a torsor for $\mathcal{G}$ on $X_{étale}$ whose pullback is isomorphic to $\mathcal{F}$ by construction. Some details omitted (namely, the relationship between torsors and 1-cocycles which should be added to the chapter on cohomology on sites). □

Lemma 46.17.7. *Let X be a scheme. Let Λ be a ring.*

(1) *The essential image of $\epsilon^{-1} : Mod(X_{étale}, \Lambda) \to Mod(X_{pro\text{-}étale}, \Lambda)$ is a weak Serre subcategory $\mathcal{C}$.*

(2) *The functor ϵ^{-1} defines an equivalence of categories of $D^+(X_{étale}, \Lambda)$ with $D^+_{\mathcal{C}}(X_{pro\text{-}étale}, \Lambda)$.*

Proof. To prove (1) we will prove conditions (1) – (4) of Homology, Lemma 12.9.3. Since ϵ^{-1} is fully faithful (Lemma 46.17.2) and exact, everything is clear except for condition (4). However, if

$$0 \to \epsilon^{-1}\mathcal{F}_1 \to \mathcal{G} \to \epsilon^{-1}\mathcal{F}_2 \to 0$$

is a short exact sequence of sheaves of Λ-modules on $X_{pro\text{-}étale}$, then we get

$$0 \to \epsilon_*\epsilon^{-1}\mathcal{F}_1 \to \epsilon_*\mathcal{G} \to \epsilon_*\epsilon^{-1}\mathcal{F}_2 \to R^1\epsilon_*\epsilon^{-1}\mathcal{F}_1$$

which by Lemma 46.17.4 is the same as a short exact sequence

$$0 \to \mathcal{F}_1 \to \epsilon_*\mathcal{G} \to \mathcal{F}_2 \to 0$$

Pulling pack we find that $\mathcal{G} = \epsilon^{-1}\epsilon_*\mathcal{G}$. This proves (1).

By (1) and the discussion in Derived Categories, Section 13.12 we obtain a strictly full, saturated, triangulated subcategory $D_{\mathcal{C}}(X_{pro\text{-}étale}, \Lambda)$. It is clear that ϵ^{-1} maps $D(X_{étale}, \Lambda)$ into $D_{\mathcal{C}}(X_{pro\text{-}étale}, \Lambda)$. If M is in $D^+(X_{étale}, \Lambda)$, then Lemma 46.17.4 shows that $M \to R\epsilon_*\epsilon^{-1}M$ is an isomorphism. If K is in $D^+_{\mathcal{C}}(X_{pro\text{-}étale}, \Lambda)$, then the spectral sequence

$$R^q\epsilon_*H^p(K) \Rightarrow H^{p+q}(R\epsilon_*K)$$

and the vanishing in Lemma 46.17.4 shows that $H^p(R\epsilon_*K) = R\epsilon_*H^p(K)$. Since ϵ is a flat morphism of ringed sites (ringed by the constant sheaf $\underline{\Lambda}$) we see that $\epsilon^{-1}R\epsilon_*K$ has cohomology sheaves $\epsilon^{-1}R\epsilon_*H^p(K)$. Since we've assumed $H^p(K)$ is in $\mathcal{C}$ we conclude by Lemma 46.17.4 once more that $\epsilon^{-1}R\epsilon_*K \to K$ is an isomorphism. In this way we see that ϵ^{-1} and $R\epsilon_*$ are quasi-inverse functors proving (2). □

Let Λ be a ring. In Modules on Sites, Section 18.42 we have defined the notion of a locally constant sheaf of Λ-modules on a site. If M is a Λ-module, then $\underline{M}$ is of finite presentation as a sheaf of $\underline{\Lambda}$-modules if and only if M is a finitely presented Λ-module, see Modules on Sites, Lemma 18.41.5.

Lemma 46.17.8. *Let X be a scheme. Let Λ be a ring. The functor ϵ^{-1} defines an equivalence of categories*

$$\left\{\begin{array}{c}\text{locally constant sheaves}\\ \text{of } \Lambda\text{-modules on } X_{étale}\\ \text{of finite presentation}\end{array}\right\} \longleftrightarrow \left\{\begin{array}{c}\text{locally constant sheaves}\\ \text{of } \Lambda\text{-modules on } X_{pro\text{-}étale}\\ \text{of finite presentation}\end{array}\right\}$$

Proof. Let $\mathcal{F}$ be a locally constant sheaf of Λ-modules on $X_{pro\text{-}étale}$ of finite presentation. Choose a pro-étale covering $\{U_i \to X\}$ such that $\mathcal{F}|_{U_i}$ is constant, say $\mathcal{F}|_{U_i} \cong \underline{M_i}_{U_i}$. Observe that $U_i \times_X U_j$ is empty if M_i is not isomorphic to M_j. For each Λ-module M let $I_M = \{i \in I \mid M_i \cong M\}$. As pro-étale coverings are fpqc coverings and by Descent, Lemma 34.9.2 we see that $U_M = \bigcup_{i \in I_M} \mathrm{Im}(U_i \to X)$ is an open subset of X. Then $X = \coprod U_M$ is a disjoint open covering of X. We may replace X by U_M for some M and assume that $M_i = M$ for all i.

Consider the sheaf $\mathcal{I} = \mathit{Isom}(\underline{M}, \mathcal{F})$. This sheaf is a torsor for $\mathcal{G} = \mathit{Isom}(\underline{M}, \underline{M})$. By Modules on Sites, Lemma 18.42.4 we have $\mathcal{G} = \underline{G}$ where $G = \mathit{Isom}_\Lambda(M, M)$. Since torsors for the étale topology and the pro-étale topology agree by Lemma 46.17.6 it follows that $\mathcal{I}$ has sections étale locally on X. Thus $\mathcal{F}$ is étale locally a constant sheaf which is what we had to show. □

Lemma 46.17.9. *Let X be a scheme. Let Λ be a Noetherian ring. Let $D_{flc}(X_{étale}, \Lambda)$, resp. $D_{flc}(X_{pro\text{-}étale}, \Lambda)$ be the full subcategory of $D(X_{étale}, \Lambda)$, resp. $D(X_{pro\text{-}étale}, \Lambda)$ consisting of those complexes whose cohomology sheaves are locally constant sheaves of Λ-modules of finite type. Then*

$$\epsilon^{-1} : D^+_{flc}(X_{étale}, \Lambda) \longrightarrow D^+_{flc}(X_{pro\text{-}étale}, \Lambda)$$

is an equivalence of categories.

Proof. The categories $D_{flc}(X_{étale}, \Lambda)$ and $D_{flc}(X_{pro\text{-}étale}, \Lambda)$ are strictly full, saturated, triangulated subcategories of $D(X_{étale}, \Lambda)$ and $D(X_{pro\text{-}étale}, \Lambda)$ by Modules on Sites, Lemma 18.42.5 and Derived Categories, Section 13.12 The statement of the lemma follows by combining Lemmas 46.17.7 and 46.17.8. □

Lemma 46.17.10. *Let X be a scheme. Let Λ be a Noetherian ring. Let K be an object of $D(X_{pro\text{-}étale}, \Lambda)$. Set $K_n = K \otimes^{\mathbf{L}}_{\Lambda} \underline{\Lambda/I^n}$. If K_1 is*

1. *in the essential image of $\epsilon^{-1} : D(X_{étale}, \Lambda/I) \to D(X_{pro\text{-}étale}, \Lambda/I)$, and*
2. *has tor amplitude in $[a, \infty)$ for some $a \in \mathbf{Z}$,*

then (1) and (2) hold for K_n as an object of $D(X_{pro\text{-}étale}, \Lambda/I^n)$.

Proof. For assertion (2) this follows from the more general Cohomology on Sites, Lemma 21.34.8. The second assertion follows from the fact that the essential image of ϵ^{-1} is a triangulated subcategory of $D^+(X_{pro\text{-}étale}, \Lambda/I^n)$ (Lemma 46.17.7), the distinguished triangles

$$K \otimes^{\mathbf{L}}_{\Lambda} \underline{I^n/I^{n+1}} \to K_{n+1} \to K_n \to K \otimes^{\mathbf{L}}_{\Lambda} \underline{I^n/I^{n+1}}[1]$$

and the isomorphism

$$K \otimes^{\mathbf{L}}_{\Lambda} \underline{I^n/I^{n+1}} = K_1 \otimes^{\mathbf{L}}_{\Lambda/I} \underline{I^n/I^{n+1}}$$

□

46.18. Cohomology of a point

Let Λ be a Noetherian ring complete with respect to an ideal $I \subset \Lambda$. Let k be a field. In this section we "compute"

$$H^i(\mathrm{Spec}(k)_{pro\text{-}étale}, \underline{\Lambda}^\wedge)$$

where $\underline{\Lambda}^\wedge = \lim \underline{\Lambda/I^n}$ as before. Let k^{sep} be a separable algebraic closure of k. Then

$$\mathcal{U} = \{\mathrm{Spec}(k^{sep}) \to \mathrm{Spec}(k)\}$$

is a pro-étale covering of $\mathrm{Spec}(k)$. We will use the Čech to cohomology spectral sequence with respect to this covering. Set $U_0 = \mathrm{Spec}(k^{sep})$ and

$$\begin{aligned} U_n &= \mathrm{Spec}(k^{sep}) \times_{\mathrm{Spec}(k)} \mathrm{Spec}(k^{sep}) \times_{\mathrm{Spec}(k)} \ldots \times_{\mathrm{Spec}(k)} \mathrm{Spec}(k^{sep}) \\ &= \mathrm{Spec}(k^{sep} \otimes_k k^{sep} \otimes_k \ldots \otimes_k k^{sep}) \end{aligned}$$

($n+1$ factors). Note that the underlying topological space $|U_0|$ of U_0 is a singleton and for $n \geq 1$ we have

$$|U_n| = G \times \ldots \times G \quad (n \text{ factors})$$

as profinite spaces where $G = \mathrm{Gal}(k^{sep}/k)$. Namely, every point of U_n has residue field k^{sep} and we identify $(\sigma_1, \ldots, \sigma_n)$ with the point corresponding to the surjection

$$k^{sep} \otimes_k k^{sep} \otimes_k \ldots \otimes_k k^{sep} \longrightarrow k^{sep}, \quad \lambda_0 \otimes \lambda_1 \otimes \ldots \lambda_n \longmapsto \lambda_0 \sigma_1(\lambda_1) \ldots \sigma_n(\lambda_n)$$

Then we compute

$$\begin{aligned} R\Gamma((U_n)_{pro\text{-}étale}, \underline{\Lambda}^\wedge) &= R\lim R\Gamma((U_n)_{pro\text{-}étale}, \underline{\Lambda/I^n}) \\ &= R\lim R\Gamma((U_n)_{étale}, \underline{\Lambda/I^n}) \\ &= \lim H^0(U_n, \underline{\Lambda/I^n}) \\ &= \mathrm{Maps}_{cont}(G \times \ldots \times G, \Lambda) \end{aligned}$$

The first equality because $R\Gamma$ commutes with derived limits and as $\Lambda^\wedge$ is the derived limit of the sheaves $\underline{\Lambda/I^n}$ by Proposition 46.16.1. The second equality by Lemma 46.17.5. The third equality by Étale Cohomology, Lemma 44.55.7. The fourth equality uses Étale Cohomology, Remark 44.23.2 to identify sections of the constant sheaf $\underline{\Lambda/I^n}$. Then it uses the fact that Λ is complete with respect to I and hence equal to $\lim \Lambda/I^n$ as a topological space, to see that $\lim \mathrm{Map}_{cont}(G, \Lambda/I^n) = \mathrm{Map}_{cont}(G, \Lambda)$ and similarly for higher powers of G. At this point Cohomology on Sites, Lemmas 21.11.3 and 21.11.6 tell us that

$$\Lambda \to \mathrm{Maps}_{cont}(G, \Lambda) \to \mathrm{Maps}_{cont}(G \times G, \Lambda) \to \ldots$$

computes the pro-étale cohomology. In other words, we see that

$$H^i(\mathrm{Spec}(k)_{pro\text{-}étale}, \underline{\Lambda}^\wedge) = H^i_{cont}(G, \Lambda)$$

where the right hand side is continuous cohomology as defined by Tate in [**Tat76**]. Of course, this is as it should be.

Lemma 46.18.1. *Let k be a field. Let $G = Gal(k^{sep}/k)$ be its absolute Galois group. Further,*

1. *let M be a profinite abelian group with a continuous G-action, or*
2. *let Λ be a Noetherian ring and $I \subset \Lambda$ an ideal an let M be an I-adically complete Λ-module with continuous G-action.*

Then there is a canonical sheaf $\underline{M}^\wedge$ on $\mathrm{Spec}(k)_{pro\text{-}étale}$ associated to M such that

$$H^i(\mathrm{Spec}(k), \underline{M}^\wedge) = H^i_{cont}(G, M)$$

as abelian groups or Λ-modules.

Proof. Proof in case (2). Set $M_n = M/I^nM$. Then $M = \lim M_n$ as M is assumed I-adically complete. Since the action of G is continuous we get continuous actions of G on M_n. By Étale Cohomology, Theorem 44.57.3 this action corresponds to a (locally constant) sheaf $\underline{M_n}$ of Λ/I^n-modules on $\mathrm{Spec}(k)_{étale}$. Pull back to $\mathrm{Spec}(k)_{pro\text{-}étale}$ by the comparison morphism ϵ and take the limit

$$\underline{M}^\wedge = \lim \epsilon^{-1} \underline{M_n}$$

to get the sheaf promised in the lemma. Exactly the same argument as given in the introduction of this section gives the comparison with Tate's continuous Galois cohomology. □

46.19. Weakly contractible hypercoverings

Let X be a scheme. For every object $U \in \mathrm{Ob}(X_{pro\text{-}étale})$ there exists a covering $\{V \to U\}$ of $X_{pro\text{-}étale}$ with V weakly contractible. This follows from Lemma 46.11.10 and the elementary fact that a disjoint union of weakly contractible objects in $X_{pro\text{-}étale}$ is weakly contractible (discussion of set theoretic issues omitted).

This observation leads to the existence of hypercoverings made up out weakly contractible objects.

Lemma 46.19.1. *Let X be a scheme.*

(1) *For every object U of $X_{pro\text{-}étale}$ there exists a hypercovering K of U in $X_{pro\text{-}étale}$ such that each term K_n consists of a single weakly contractible object of $X_{pro\text{-}étale}$ covering U.*
(2) *For every quasi-compact and quasi-separated object U of $X_{pro\text{-}étale}$ there exists a hypercovering K of U in $X_{pro\text{-}étale}$ such that each term K_n consists of a single affine and weakly contractible object of $X_{pro\text{-}étale}$ covering U.*

Proof. Let $\mathcal{B} \subset \mathrm{Ob}(X_{pro\text{-}étale})$ be the set of weakly contractible objects of $X_{pro\text{-}étale}$. We have seen above that every object of $X_{pro\text{-}étale}$ has a covering by an element of $\mathcal{B}$. Apply Hypercoverings, Lemma 24.11.1 to get (1).

Let $X_{qcqs,pro\text{-}étale} \subset X_{pro\text{-}étale}$ be the full subcategory consisting of quasi-compact and quasi-separated objects. Note that $X_{qcqs,pro\text{-}étale}$ is preserved under fibre products. A covering of $X_{qcqs,pro\text{-}étale}$ will be a finite pro-étale covering. Then $X_{qcqs,pro\text{-}étale} \to X_{pro\text{-}étale}$ is a special cocontinuous functor hence $X_{qcqs,pro\text{-}étale}$ defines the same topos as $X_{pro\text{-}étale}$. Details omitted; see Sites, Definition 7.28.2 and Lemma 7.28.1. In particular, if K is a hypercovering of an object U in $X_{qcqs,pro\text{-}étale}$ then K is a hypercovering of $X_{pro\text{-}étale}$. Let $\mathcal{B} \subset \mathrm{Ob}(X_{qcqs,pro\text{-}étale})$ be the set of affine and weakly contractible objects. By Lemma 46.11.10 and the fact that finite unions of affines are affine, for every object U of $X_{qcqs,pro\text{-}étale}$ there exists a covering $\{V \to U\}$ of $X_{qcqs,pro\text{-}étale}$ with $V \in \mathcal{B}$. Apply Hypercoverings, Lemma 24.11.1 to get (2). □

In the following lemma we use the Čech complex $\mathcal{F}(K)$ associated to a hypercovering K in a site. See Hypercoverings, Section 24.4. If K is a hypercovering of U and $K_n = \{U_n \to U\}$, then the Čech complex looks like this:

$$\mathcal{F}(K) = (\mathcal{F}(U_0) \to \mathcal{F}(U_1) \to \mathcal{F}(U_2) \to \ldots)$$

Lemma 46.19.2. *Let X be a scheme. Let $E \in D^+(X_{pro\text{-}étale})$ be represented by a bounded below complex $\mathcal{E}^\bullet$ of abelian sheaves. Let K be a hypercovering of $U \in \mathrm{Ob}(X_{pro\text{-}étale})$ with $K_n = \{U_n \to U\}$ where U_n is a weakly contractible object of $X_{pro\text{-}étale}$. Then*

$$R\Gamma(U, E) = \mathit{Tot}(\mathcal{E}^\bullet(K))$$

in $D(Ab)$.

Proof. If $E = \mathcal{E}[n]$ is the object associated to a single abelian sheaf on $X_{pro\text{-}étale}$, then the spectral sequence of Hypercoverings, Lemma 24.4.3 implies that

$$R\Gamma(X_{pro\text{-}étale}, \mathcal{E}) = \mathcal{E}(K)$$

because the higher cohomology groups of any sheaf over U_n vanish, see Cohomology on Sites, Lemma 21.37.1.

If $\mathcal{E}^\bullet$ is bounded below, then we can choose an injective resolution $\mathcal{E}^\bullet \to \mathcal{I}^\bullet$ and consider the map of complexes

$$\mathrm{Tot}(\mathcal{E}^\bullet(K)) \longrightarrow \mathrm{Tot}(\mathcal{I}^\bullet(K))$$

For every n the map $\mathcal{E}^\bullet(U_n) \to \mathcal{I}^\bullet(U_n)$ is a quasi-isomorphism because taking sections over U_n is exact. Hence the displayed map is a quasi-isomorphism by one

of the spectral sequences of Homology, Lemma 12.22.5. Using the result of the first paragraph we see that for every p the complex $\mathcal{I}^p(K)$ is acyclic in degrees $n > 0$ and computes $\mathcal{I}^p(U)$ in degree 0. Thus the other spectral sequence of Homology, Lemma 12.22.5 shows $\mathrm{Tot}(\mathcal{I}^\bullet(K))$ computes $R\Gamma(U, E) = \mathcal{I}^\bullet(U)$. □

Lemma 46.19.3. *Let X be a quasi-compact and quasi-separated scheme. The functor $R\Gamma(X, -) : D^+(X_{pro\text{-}étale}) \to D(Ab)$ commutes with direct sums and homotopy colimits.*

Proof. The statement means the following: Suppose we have a family of objects E_i of $D^+(X_{pro\text{-}étale})$ such that $\bigoplus E_i$ is an object of $D^+(X_{pro\text{-}étale})$. Then $R\Gamma(X, \bigoplus E_i) = \bigoplus R\Gamma(X, E_i)$. To see this choose a hypercovering K of X with $K_n = \{U_n \to X\}$ where U_n is an affine and weakly contractible scheme, see Lemma 46.19.1. Let N be an integer such that $H^p(E_i) = 0$ for $p < N$. Choose a complex of abelian sheaves $\mathcal{E}_i^\bullet$ representing E_i with $\mathcal{E}_i^p = 0$ for $p < N$. The termwise direct sum $\bigoplus \mathcal{E}_i^\bullet$ represents $\bigoplus E_i$ in $D(X_{pro\text{-}étale})$, see Injectives, Lemma 19.13.4. By Lemma 46.19.2 we have

$$R\Gamma(X, \bigoplus E_i) = \mathrm{Tot}((\bigoplus \mathcal{E}_i^\bullet)(K))$$

and

$$R\Gamma(X, E_i) = \mathrm{Tot}(\mathcal{E}_i^\bullet(K))$$

Since each U_n is quasi-compact we see that

$$\mathrm{Tot}((\bigoplus \mathcal{E}_i^\bullet)(K)) = \bigoplus \mathrm{Tot}(\mathcal{E}_i^\bullet(K))$$

by Modules on Sites, Lemma 18.29.2. The statement on homotopy colimits is a formal consequence of the fact that $R\Gamma$ is an exact functor of triangulated categories and the fact (just proved) that it commutes with direct sums. □

Remark 46.19.4. Let X be a scheme. Because $X_{pro\text{-}étale}$ has enough weakly contractible objects for all K in $D(X_{pro\text{-}étale})$ we have $K = R\lim \tau_{\geq -n} K$ by Cohomology on Sites, Proposition 21.37.2. Since $R\Gamma$ commutes with $R\lim$ by Injectives, Lemma 19.13.6 we see that

$$R\Gamma(X, K) = R\lim R\Gamma(X, \tau_{\geq -n} K)$$

in $D(Ab)$. This will allows us to extend some results from bounded below complexes to all complexes.

46.20. Functoriality of the pro-étale site

Let $f : X \to Y$ be a morphism of schemes. The functor $Y_{pro\text{-}étale} \to X_{pro\text{-}étale}$, $V \mapsto X \times_Y V$ induces a morphism of sites $f_{pro\text{-}étale} : X_{pro\text{-}étale} \to Y_{pro\text{-}étale}$, see Sites, Proposition 7.15.6. In fact, we obtain a commutative diagram of morphisms of sites

$$\begin{array}{ccc} X_{pro\text{-}étale} & \xrightarrow{\epsilon} & X_{étale} \\ \downarrow{\scriptstyle f_{pro\text{-}étale}} & & \downarrow{\scriptstyle f_{étale}} \\ Y_{pro\text{-}étale} & \xrightarrow{\epsilon} & Y_{étale} \end{array}$$

where ϵ is as in Section 46.17. In particular we have $\epsilon^{-1} f_{étale}^{-1} = f_{pro\text{-}étale}^{-1} \epsilon^{-1}$. Here is the corresponding result for pushforward.

Lemma 46.20.1. *Let $f : X \to Y$ be a morphism of schemes.*

(1) *Let $\mathcal{F}$ be a sheaf of sets on $X_{étale}$. Then we have $f_{pro\text{-}étale,*}\epsilon^{-1}\mathcal{F} = \epsilon^{-1}f_{étale,*}\mathcal{F}$.*
(2) *Let $\mathcal{F}$ be an abelian sheaf on $X_{étale}$. Then we have $Rf_{pro\text{-}étale,*}\epsilon^{-1}\mathcal{F} = \epsilon^{-1}Rf_{étale,*}\mathcal{F}$.*

Proof. Proof of (1). Let $\mathcal{F}$ be a sheaf of sets on $X_{étale}$. There is a canonical map $\epsilon^{-1}f_{étale,*}\mathcal{F} \to f_{pro\text{-}étale,*}\epsilon^{-1}\mathcal{F}$, see Sites, Section 7.44. To show it is an isomorphism we may work (Zariski) locally on Y, hence we may assume Y is affine. In this case every object of $Y_{pro\text{-}étale}$ has a covering by objects $V = \lim V_i$ which are limits of affine schemes V_i étale over Y (by Proposition 46.9.1 for example). Evaluating the map $\epsilon^{-1}f_{étale,*}\mathcal{F} \to f_{pro\text{-}étale,*}\epsilon^{-1}\mathcal{F}$ on V we obtain a map

$$\operatorname{colim}\Gamma(X \times_Y V_i, \mathcal{F}) \longrightarrow \Gamma(X \times_Y V, \epsilon^*\mathcal{F}).$$

see Lemma 46.17.1 for the left hand side. By Lemma 46.17.1 we have

$$\Gamma(X \times_Y V, \epsilon^*\mathcal{F}) = \Gamma(X \times_Y V, \mathcal{F})$$

Hence the result holds by Étale Cohomology, Lemma 44.52.3.

Proof of (2). Arguing in exactly the same manner as above we see that it suffices to show that

$$\operatorname{colim} H^i_{étale}(X \times_Y V_i, \mathcal{F}) \longrightarrow H^i_{étale}(X \times_Y V, \mathcal{F})$$

which follows once more from Étale Cohomology, Lemma 44.52.3. □

46.21. Finite morphisms and pro-étale sites

It is not clear that a finite morphism of schemes determines an exact pushforward on abelian pro-étale sheaves.

Lemma 46.21.1. *Let $f : Z \to X$ be a finite morphism of schemes which is locally of finite presentation. Then $f_{pro\text{-}étale,*} : Ab(Z_{pro\text{-}étale}) \to Ab(X_{pro\text{-}étale})$ is exact.*

Proof. The prove this we may work (Zariski) locally on X and assume that X is affine, say $X = \operatorname{Spec}(A)$. Then $Z = \operatorname{Spec}(B)$ for some finite A-algebra B of finite presentation. The construction in the proof of Proposition 46.10.3 produces a faithfully flat, ind-étale ring map $A \to D$ with D w-contractible. We may check exactness of a sequence of sheaves by evaluating on $U = \operatorname{Spec}(D)$ be such an object. Then $f_{pro\text{-}étale,*}\mathcal{F}$ evaluated at U is equal to $\mathcal{F}$ evaluated at $V = \operatorname{Spec}(D \otimes_A B)$. Since $D \otimes_A B$ is w-contractible by Lemma 46.10.6 evaluation at V is exact. □

46.22. Closed immersions and pro-étale sites

It is not clear (and likely false) that a closed immersion of schemes determines an exact pushforward on abelian pro-étale sheaves.

Lemma 46.22.1. *Let $i : Z \to X$ be a closed immersion morphism of affine schemes. Denote X_{app} and Z_{app} the sites introduced in Lemma 46.11.24. The base change functor*

$$u : X_{app} \to Z_{app}, \quad U \longmapsto u(U) = U \times_X Z$$

is continuous and has a fully faithful left adjoint v. For V in Z_{app} the morphism $V \to v(V)$ is a closed immersion identifying V with $u(v(V)) = v(V) \times_X Z$ and every point of $v(V)$ specializes to a point of V. The functor v is cocontinuous and sends coverings to coverings.

Proof. The existence of the adjoint follows immediately from Lemma 46.7.7 and the definitions. It is clear that u is continuous from the definition of coverings in X_{app}.

Write $X = \mathrm{Spec}(A)$ and $Z = \mathrm{Spec}(A/I)$. Let $V = \mathrm{Spec}(\overline{C})$ be an object of Z_{app} and let $v(V) = \mathrm{Spec}(C)$. We have seen in the statement of Lemma 46.7.7 that V equals $v(V) \times_X Z = \mathrm{Spec}(C/IC)$. Any $g \in C$ which maps to an invertible element of $C/IC = \overline{C}$ is invertible in C. Namely, we have the A-algebra maps $C \to C_g \to C/IC$ and by adjointness we obtain an C-algebra map $C_g \to C$. Thus every point of $v(V)$ specializes to a point of V.

Suppose that $\{V_i \to V\}$ is a covering in Z_{app}. Then $\{v(V_i) \to v(V)\}$ is a finite family of morphisms of Z_{app} such that every point of $V \subset v(V)$ is in the image of one of the maps $v(V_i) \to v(V)$. As the morphisms $v(V_i) \to v(V)$ are flat (since they are weakly étale) we conclude that $\{v(V_i) \to v(V)\}$ is jointly surjective. This proves that v sends coverings to coverings.

Let V be an object of Z_{app} and let $\{U_i \to v(V)\}$ be a covering in X_{app}. Then we see that $\{u(U_i) \to u(v(V)) = V\}$ is a covering of Z_{app}. By adjointness we obtain morphisms $v(u(U_i)) \to U_i$. Thus the family $\{v(u(U_i)) \to v(V)\}$ refines the given covering and we conclude that v is cocontinuous. □

Lemma 46.22.2. *Let $Z \to X$ be a closed immersion morphism of affine schemes. The corresponding morphism of topoi $i = i_{pro\text{-}étale}$ is equal to the morphism of topoi associated to the fully faithful cocontinuous functor $v : Z_{app} \to X_{app}$ of Lemma 46.22.1. It follows that*

1. *$i^{-1}\mathcal{F}$ is the sheaf associated to the presheaf $V \mapsto \mathcal{F}(v(V))$,*
2. *for a weakly contractible object V of Z_{app} we have $i^{-1}\mathcal{F}(V) = \mathcal{F}(v(V))$,*
3. *$i^{-1} : Sh(X_{pro\text{-}étale}) \to Sh(Z_{pro\text{-}étale})$ has a left adjoint $i_!^{Sh}$,*
4. *$i^{-1} : Ab(X_{pro\text{-}étale}) \to Ab(Z_{pro\text{-}étale})$ has a left adjoint $i_!$,*
5. *$id \to i^{-1}i_!^{Sh}$, $id \to i^{-1}i_!$, and $i^{-1}i_* \to id$ are isomorphisms, and*
6. *i_*, $i_!^{Sh}$ and $i_!$ are fully faithful.*

Proof. By Lemma 46.11.24 we may describe $i_{pro\text{-}étale}$ in terms of the morphism of sites $u : X_{app} \to Z_{app}$, $V \mapsto V \times_X Z$. The first statement of the lemma follows from Sites, Lemma 7.21.2 (but with the roles of u and v reversed).

Proof of (1). By the description of i as the morphism of topoi associated to v this holds by the construction, see Sites, Lemma 7.20.1.

Proof of (2). Since the functor v sends coverings to coverings by Lemma 46.22.1 we see that the presheaf $\mathcal{G} : V \mapsto \mathcal{F}(v(V))$ is a separated presheaf (Sites, Definition 7.10.9). Hence the sheafification of $\mathcal{G}$ is $\mathcal{G}^+$, see Sites, Theorem 7.10.10. Next, let V be a weakly contractible object of Z_{app}. Let $\mathcal{V} = \{V_i \to V\}_{i=1,\ldots,n}$ be any covering in Z_{app}. Set $\mathcal{V}' = \{\coprod V_i \to V\}$. Since v commutes with finite disjoint unions (as a left adjoint or by the construction) and since $\mathcal{F}$ sends finite disjoint unions into products, we see that

$$H^0(\mathcal{V}, \mathcal{G}) = H^0(\mathcal{V}', \mathcal{G})$$

(notation as in Sites, Section 7.10; compare with Étale Cohomology, Lemma 44.22.1). Thus we may assume the covering is given by a single morphism, like so $\{V' \to V\}$. Since V is weakly contractible, this covering can be refined by the trivial covering

$\{V \to V\}$. It therefore follows that the value of $\mathcal{G}^+ = i^{-1}\mathcal{F}$ on V is simply $\mathcal{F}(v(V))$ and (2) is proved.

Proof of (3). Every object of Z_{app} has a covering by weakly contractible objects (Lemma 46.11.27). By the above we see that we would have $i_!^{Sh} h_V = h_{v(V)}$ for V weakly contractible if $i_!^{Sh}$ existed. The existence of $i_!^{Sh}$ then follows from Sites, Lemma 7.23.1.

Proof of (4). Existence of $i_!$ follows in the same way by setting $i_!\mathbf{Z}_V = \mathbf{Z}_{v(V)}$ for V weakly contractible in Z_{app}, using similar for direct sums, and applying Homology, Lemma 12.25.6. Details omitted.

Proof of (5). Let V be a contractible object of Z_{app}. Then $i^{-1}i_!^{Sh}h_V = i^{-1}h_{v(V)} = h_{u(v(V))} = h_V$. (It is a general fact that $i^{-1}h_U = h_{u(U)}$.) Since the sheaves h_V for V contractible generate $Sh(Z_{app})$ (Sites, Lemma 7.13.5) we conclude $\mathrm{id} \to i^{-1}i_!^{Sh}$ is an isomorphism. Similarly for the map $\mathrm{id} \to i^{-1}i_!$. Then $(i^{-1}i_*\mathcal{H})(V) = i_*\mathcal{H}(v(V)) = \mathcal{H}(u(v(V))) = \mathcal{H}(V)$ and we find that $i^{-1}i_* \to \mathrm{id}$ is an isomorphism.

The fully faithfulness statements of (6) now follow from Categories, Lemma 4.24.2. □

Lemma 46.22.3. *Let $i : Z \to X$ be a closed immersion of schemes. Then*

(1) *$i^{-1}_{pro\text{-}étale}$ commutes with limits,*
(2) *$i_{pro\text{-}étale,*}$ is fully faithful, and*
(3) *$i^{-1}_{pro\text{-}étale} i_{pro\text{-}étale,*} \cong id_{Sh(Z_{pro\text{-}étale})}$.*

Proof. Assertions (2) and (3) are equivalent by Sites, Lemma 7.40.1. Parts (1) and (3) are (Zariski) local on X, hence we may assume that X is affine. In this case the result follows from Lemma 46.22.2. □

Lemma 46.22.4. *Let $i : Z \to X$ be an integral universally injective and surjective morphism of schemes. Then $i_{pro\text{-}étale,*}$ and $i^{-1}_{pro\text{-}étale}$ are quasi-inverse equivalences of categories of pro-étale topoi.*

Proof. There is an immediate reduction to the case that X is affine. Then Z is affine too. Set $A = \mathcal{O}(X)$ and $B = \mathcal{O}(Z)$. Then the categories of étale algebras over A and B are equivalent, see Étale Cohomology, Theorem 44.46.1 and Remark 44.46.2. Thus the categories of ind-étale algebras over A and B are equivalent. In other words the categories X_{app} and Z_{app} of Lemma 46.11.24 are equivalent. We omit the verification that this equivalence sends coverings to coverings and vice versa. Thus the result as Lemma 46.11.24 tells us the pro-étale topos is the topos of sheaves on X_{app}. □

Lemma 46.22.5. *Let $i : Z \to X$ be a closed immersion of schemes. Let $U \to X$ be an object of $X_{pro\text{-}étale}$ such that*

(1) *U is affine and weakly contractible, and*
(2) *every point of U specializes to a point of $U \times_X Z$.*

Then $i^{-1}_{pro\text{-}étale}\mathcal{F}(U \times_X Z) = \mathcal{F}(U)$ for all abelian sheaves on $X_{pro\text{-}étale}$.

Proof. Since pullback commutes with restriction, we may replace X by U. Thus we may assume that X is affine and weakly contractible and that every point of X specializes to a point of Z. By Lemma 46.22.2 part (1) it suffices to show that $v(Z) = X$ in this case. Thus we have to show: If A is a w-contractible ring, $I \subset A$

an ideal contained in the radical of A and $A \to B \to A/I$ is a factorization with $A \to B$ ind-étale, then there is a unique section $B \to A$ compatible with maps to A/I. Observe that $B/IB = A/I \times R$ as A/I-algebras. After replacing B by a localization we may assume $B/IB = A/I$. Note that $\mathrm{Spec}(B) \to \mathrm{Spec}(A)$ is surjective as the image contains $V(I)$ and hence all closed points and is closed under specialization. Since A is w-contractible there is a section $B \to A$. Since $B/IB = A/I$ this section is compatible with the map to A/I. We omit the proof of uniqueness (hint: use that A and B have isomorphic local rings at maximal ideals of A). □

Lemma 46.22.6. *Let $i : Z \to X$ be a closed immersion of schemes. If $X \setminus i(Z)$ is a retrocompact open of X, then $i_{pro\text{-}étale,*}$ is exact.*

Proof. The question is local on X hence we may assume X is affine. Say $X = \mathrm{Spec}(A)$ and $Z = \mathrm{Spec}(A/I)$. There exist $f_1, \ldots, f_r \in I$ such that $Z = V(f_1, \ldots, f_r)$ set theoretically, see Algebra, Lemma 10.27.1. By Lemma 46.22.4 we may assume that $Z = \mathrm{Spec}(A/(f_1, \ldots, f_r))$. In this case the functor $i_{pro\text{-}étale,*}$ is exact by Lemma 46.21.1. □

46.23. Extension by zero

The general material in Modules on Sites, Section 18.19 allows us to make the following definition.

Definition 46.23.1. Let $j : U \to X$ be a weakly étale morphism of schemes.

(1) The restriction functor $j^{-1} : \mathit{Sh}(X_{pro\text{-}étale}) \to \mathit{Sh}(U_{pro\text{-}étale})$ has a left adjoint $j_!^{Sh} : \mathit{Sh}(X_{pro\text{-}étale}) \to \mathit{Sh}(U_{pro\text{-}étale})$.
(2) The restriction functor $j^{-1} : \mathit{Ab}(X_{pro\text{-}étale}) \to \mathit{Ab}(U_{pro\text{-}étale})$ has a left adjoint which is denoted $j_! : \mathit{Ab}(U_{pro\text{-}étale}) \to \mathit{Ab}(X_{pro\text{-}étale})$ and called *extension by zero*.
(3) Let Λ be a ring. The functor $j^{-1} : \mathit{Mod}(X_{pro\text{-}étale}, \Lambda) \to \mathit{Mod}(U_{pro\text{-}étale}, \Lambda)$ has a left adjoint $j_! : \mathit{Mod}(U_{pro\text{-}étale}, \Lambda) \to \mathit{Mod}(X_{pro\text{-}étale}, \Lambda)$ and called *extension by zero*.

As usual we compare this to what happens in the étale case.

Lemma 46.23.2. *Let $j : U \to X$ be an étale morphism of schemes. Let $\mathcal{G}$ be an abelian sheaf on $U_{étale}$. Then $\epsilon^{-1} j_! \mathcal{G} = j_! \epsilon^{-1} \mathcal{G}$ as sheaves on $X_{pro\text{-}étale}$.*

Proof. This is true because both are left adjoints to $j_{pro\text{-}étale,*}\epsilon^{-1} = \epsilon^{-1} j_{étale,*}$, see Lemma 46.20.1. □

Lemma 46.23.3. *Let $j : U \to X$ be a weakly étale morphism of schemes. Let $i : Z \to X$ be a closed immersion such that $U \times_X Z = \emptyset$. Let $V \to X$ be an affine object of $X_{pro\text{-}étale}$ such that every point of V specializes to a point of $V_Z = Z \times_X V$. Then $j_! \mathcal{F}(V) = 0$ for all abelian sheaves on $U_{pro\text{-}étale}$.*

Proof. Let $\{V_i \to V\}$ be a pro-étale covering. The lemma follows if we can refine this covering to a covering where the members have no morphisms into U over X (see construction of $j_!$ in Modules on Sites, Section 18.19). First refine the covering to get a finite covering with V_i affine. For each i let $V_i = \mathrm{Spec}(A_i)$ and let $Z_i \subset V_i$ be the inverse image of Z. Set $W_i = \mathrm{Spec}(A_{i,Z_i}^\sim)$ with notation as in Lemma 46.5.1. Then $\coprod W_i \to V$ is weakly étale and the image contains all points of V_Z. Hence

the image contains all points of V by our assumption on specializations. Thus $\{W_i \to V\}$ is a pro-étale covering refining the given one. But each point in W_i specializes to a point lying over Z, hence there are no morphisms $W_i \to U$ over X. □

Lemma 46.23.4. *Let $j : U \to X$ be an open immersion of schemes. Then $id \cong j^{-1}j_!$ and $j^{-1}j_* \cong id$ and the functors $j_!$ and j_* are fully faithful.*

Proof. See Sites, Lemma 7.26.4 and Categories, Lemma 4.24.2. □

Here is the relationship between extension by zero and restriction to the complementary closed subscheme.

Lemma 46.23.5. *Let X be a scheme. Let $Z \subset X$ be a closed subscheme and let $U \subset X$ be the complement. Denote $i : Z \to X$ and $j : U \to X$ the inclusion morphisms. Assume that j is a quasi-compact morphism. For every abelian sheaf on $X_{pro\text{-}étale}$ there is a canonical short exact sequence*

$$0 \to j_!j^{-1}\mathcal{F} \to \mathcal{F} \to i_*i^{-1}\mathcal{F} \to 0$$

on $X_{pro\text{-}étale}$ where all the functors are for the pro-étale topology.

Proof. We obtain the maps by the adjointness properties of the functors involved. It suffices to show that $X_{pro\text{-}étale}$ has enough objects (Sites, Definition 7.39.2) on which the sequence evaluates to a short exact sequence. Let $V = \mathrm{Spec}(A)$ be an affine object of $X_{pro\text{-}étale}$ such that A is w-contractible (there are enough objects of this type). Then $V \times_X Z$ is cut out by an ideal $I \subset A$. The assumption that j is quasi-compact implies there exist $f_1, \ldots, f_r \in I$ such that $V(I) = V(f_1, \ldots, f_r)$. We obtain a faithfully flat, ind-Zariski ring map

$$A \longrightarrow A_{f_1} \times \ldots \times A_{f_r} \times A^\sim_{V(I)}$$

with $A^\sim_{V(I)}$ as in Lemma 46.5.1. Since $V_i = \mathrm{Spec}(A_{f_i}) \to X$ factors through U we have

$$j_!j^{-1}\mathcal{F}(V_i) = \mathcal{F}(V_i) \quad \text{and} \quad i_*i^{-1}\mathcal{F}(V_i) = 0$$

On the other hand, for the scheme $V^\sim = \mathrm{Spec}(A^\sim_{V(I)})$ we have

$$j_!j^{-1}\mathcal{F}(V^\sim) = 0 \quad \text{and} \quad \mathcal{F}(V^\sim) = i_*i^{-1}\mathcal{F}(V^\sim)$$

the first equality by Lemma 46.23.3 and the second by Lemmas 46.22.5 and 46.10.7. Thus the sequence evaluates to an exact sequence on $\mathrm{Spec}(A_{f_1} \times \ldots \times A_{f_r} \times A^\sim_{V(I)})$ and the lemma is proved. □

Lemma 46.23.6. *Let $j : U \to X$ be a quasi-compact open immersion morphism of schemes. The functor $j_! : Ab(U_{pro\text{-}étale}) \to Ab(X_{pro\text{-}étale})$ commutes with limits.*

Proof. Since $j_!$ is exact it suffices to show that $j_!$ commutes with products. The question is local on X, hence we may assume X affine. Let $\mathcal{G}$ be an abelian sheaf on $U_{pro\text{-}étale}$. Note that there always is a canonical map

$$j_!\mathcal{G} \to j_*\mathcal{G}$$

see Modules on Sites, Remark 18.19.7. In our particular case this map can be obtained from the fact that $j^{-1}j_*\mathcal{G} = \mathcal{G}$. Hence applying the exact sequence of Lemma 46.23.5 we get

$$0 \to j_!\mathcal{G} \to j_*\mathcal{G} \to i_*i^{-1}j_*\mathcal{G} \to 0$$

where $i : Z \to X$ is the inclusion of the reduced induced scheme structure on the complement $Z = X \setminus U$. The functors j_* and i_* commute with products as right adjoints. The functor i^{-1} commutes with products by Lemma 46.22.3. Hence $j_!$ does because on the pro-étale site products are exact (Cohomology on Sites, Proposition 21.37.2). □

46.24. Constructible sheaves on the pro-étale site

We stick to constructible sheaves of Λ-modules for a Noetherian ring. In the future we intend to discuss constructible sheaves of sets, groups, etc.

Definition 46.24.1. Let X be a scheme. Let Λ be a Noetherian ring. A sheaf of Λ-modules on $X_{pro\text{-}étale}$ is *constructible* if for every affine open $U \subset X$ there exists a finite decomposition of U into constructible locally closed subschemes $U = \coprod_i U_i$ such that $\mathcal{F}|_{U_i}$ is of finite type and locally constant for all i.

Again this does not give anything "new".

Lemma 46.24.2. *Let X be a scheme. Let Λ be a Noetherian ring. The functor ϵ^{-1} defines an equivalence of categories*

$$\left\{\begin{array}{c}\text{constructible sheaves of} \\ \Lambda\text{-modules on } X_{étale}\end{array}\right\} \longleftrightarrow \left\{\begin{array}{c}\text{constructible sheaves of} \\ \Lambda\text{-modules on } X_{pro\text{-}étale}\end{array}\right\}$$

between constructible sheaves of Λ-modules on $X_{étale}$ and constructible sheaves of Λ-modules on $X_{pro\text{-}étale}$.

Proof. By Lemma 46.17.2 the functor ϵ^{-1} is fully faithful and commutes with pullback (restriction) to the strata. Hence ϵ^{-1} of a constructible étale sheaf is a constructible pro-étale sheaf. To finish the proof let $\mathcal{F}$ be a constructible sheaf of Λ-modules on $X_{pro\text{-}étale}$ as in Definition 46.24.1. There is a canonical map

$$\epsilon^{-1}\epsilon_*\mathcal{F} \longrightarrow \mathcal{F}$$

We will show this map is an isomorphism. This will prove that $\mathcal{F}$ is in the essential image of ϵ^{-1} and finish the proof (details omitted).

To prove this we may assume that X is affine. In this case we have a finite partition $X = \coprod_i X_i$ by constructible locally closed strata such that $\mathcal{F}|_{X_i}$ is locally constant of finite type. Let $U \subset X$ be one of the open strata in the partition and let $Z \subset X$ be the reduced induced structure on the complement. By Lemma 46.23.5 we have a short exact sequence

$$0 \to j_!j^{-1}\mathcal{F} \to \mathcal{F} \to i_*i^{-1}\mathcal{F} \to 0$$

on $X_{pro\text{-}étale}$. Functoriality gives a commutative diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \epsilon^{-1}\epsilon_*j_!j^{-1}\mathcal{F} & \longrightarrow & \epsilon^{-1}\epsilon_*\mathcal{F} & \longrightarrow & \epsilon^{-1}\epsilon_*i_*i^{-1}\mathcal{F} & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & j_!j^{-1}\mathcal{F} & \longrightarrow & \mathcal{F} & \longrightarrow & i_*i^{-1}\mathcal{F} & \longrightarrow & 0 \end{array}$$

By induction on the length of the partition we know that on the one hand $\epsilon^{-1}\epsilon_*i^{-1}\mathcal{F} \to i^{-1}\mathcal{F}$ and $\epsilon^{-1}\epsilon_*j^{-1}\mathcal{F} \to j^{-1}\mathcal{F}$ are isomorphisms and on the other that $i^{-1}\mathcal{F} = \epsilon^{-1}\mathcal{A}$

and $j^{-1}\mathcal{F} = \epsilon^{-1}\mathcal{B}$ for some constructible sheaves of Λ-modules $\mathcal{A}$ on $Z_{étale}$ and $\mathcal{B}$ on $U_{étale}$. Then

$$\epsilon^{-1}\epsilon_* j_! j^{-1}\mathcal{F} = \epsilon^{-1}\epsilon_* j_! \epsilon^{-1}\mathcal{B} = \epsilon^{-1}\epsilon_*\epsilon^{-1} j_!\mathcal{B} = \epsilon^{-1} j_!\mathcal{B} = j_!\epsilon^{-1}\mathcal{B} = j_! j^{-1}\mathcal{F}$$

the second equality by Lemma 46.23.2, the third equality by Lemma 46.17.2, and the fourth equality by Lemma 46.23.2 again. Similarly, we have

$$\epsilon^{-1}\epsilon_* i_* i^{-1}\mathcal{F} = \epsilon^{-1}\epsilon_* i_*\epsilon^{-1}\mathcal{A} = \epsilon^{-1}\epsilon_*\epsilon^{-1} i_*\mathcal{A} = \epsilon^{-1} i_*\mathcal{A} = i_*\epsilon^{-1}\mathcal{A} = i_* i^{-1}\mathcal{F}$$

this time using Lemma 46.20.1. By the five lemma we conclude the vertical map in the middle of the big diagram is an isomorphism. □

Lemma 46.24.3. *Let X be a scheme. Let Λ be a Noetherian ring. The category of constructible sheaves of Λ-modules on $X_{pro\text{-}étale}$ is a weak Serre subcategory of $Mod(X_{pro\text{-}étale}, \Lambda)$.*

Proof. This is a formal consequence of Lemmas 46.24.2 and 46.17.7 and the result for the étale site (Étale Cohomology, Lemma 44.65.6). □

Lemma 46.24.4. *Let X be a scheme. Let Λ be a Noetherian ring. Let $D_c(X_{étale}, \Lambda)$, resp. $D_c(X_{pro\text{-}étale}, \Lambda)$ be the full subcategory of $D(X_{étale}, \Lambda)$, resp. $D(X_{pro\text{-}étale}, \Lambda)$ consisting of those complexes whose cohomology sheaves are constructible sheaves of Λ-modules. Then*

$$\epsilon^{-1} : D_c^+(X_{étale}, \Lambda) \longrightarrow D_c^+(X_{pro\text{-}étale}, \Lambda)$$

is an equivalence of categories.

Proof. The categories $D_c(X_{étale}, \Lambda)$ and $D_c(X_{pro\text{-}étale}, \Lambda)$ are strictly full, saturated, triangulated subcategories of $D(X_{étale}, \Lambda)$ and $D(X_{pro\text{-}étale}, \Lambda)$ by Étale Cohomology, Lemma 44.65.6 and Lemma 46.24.3 and Derived Categories, Section 13.12. The statement of the lemma follows by combining Lemmas 46.17.7 and 46.24.2. □

Lemma 46.24.5. *Let X be a scheme. Let Λ be a Noetherian ring. Let $K, L \in D_c^-(X_{pro\text{-}étale}, \Lambda)$. Then $K \otimes^{\mathbf{L}}_{\Lambda} L$ is in $D_c^-(X_{pro\text{-}étale}, \Lambda)$.*

Proof. Note that $H^i(K \otimes^{\mathbf{L}}_{\Lambda} L)$ is the same as $H^i(\tau_{\geq i-1}K \otimes^{\mathbf{L}}_{\Lambda} \tau_{\geq i-1}L)$. Thus we may assume K and L are bounded. In this case we can apply Lemma 46.24.4 to reduce to the case of the étale site, see Étale Cohomology, Lemma 44.84.6. □

Lemma 46.24.6. *Let X be a scheme. Let Λ be a Noetherian ring. Let K be an object of $D(X_{pro\text{-}étale}, \Lambda)$. Set $K_n = K \otimes^{\mathbf{L}}_{\Lambda} \underline{\Lambda/I^n}$. If K_1 is in $D_c^-(X_{pro\text{-}étale}, \Lambda/I)$, then K_n is in $D_c^-(X_{pro\text{-}étale}, \Lambda/I^n)$ for all n.*

Proof. Consider the distinguished triangles

$$K \otimes^{\mathbf{L}}_{\Lambda} \underline{I^n/I^{n+1}} \to K_{n+1} \to K_n \to K \otimes^{\mathbf{L}}_{\Lambda} \underline{I^n/I^{n+1}}[1]$$

and the isomorphisms

$$K \otimes^{\mathbf{L}}_{\Lambda} \underline{I^n/I^{n+1}} = K_1 \otimes^{\mathbf{L}}_{\Lambda/I} \underline{I^n/I^{n+1}}$$

By Lemma 46.24.5 we see that this tensor product has constructible cohomology sheaves (and vanishing when K_1 has vanishing cohomology). Hence by induction on n using Lemma 46.24.3 we see that each K_n has constructible cohomology sheaves. □

46.25. Constructible adic sheaves

In this section we define the notion of a constructible Λ-sheaf as well as some variants.

Definition 46.25.1. Let Λ be a Noetherian ring and let $I \subset \Lambda$ be an ideal. Let X be a scheme. Let $\mathcal{F}$ be a sheaf of Λ-modules on $X_{pro\text{-}étale}$.

(1) We say $\mathcal{F}$ is a *constructible Λ-sheaf* if $\mathcal{F} = \lim \mathcal{F}/I^n\mathcal{F}$ and each $\mathcal{F}/I^n\mathcal{F}$ is a constructible sheaf of Λ/I^n-modules.
(2) If $\mathcal{F}$ is a constructible Λ-sheaf, then we say $\mathcal{F}$ is *lisse* if each $\mathcal{F}/I^n\mathcal{F}$ is locally constant.
(3) We say $\mathcal{F}$ is *adic lisse*[3] if there exists a I-adically complete Λ-module M with M/IM finite such that $\mathcal{F}$ is locally isomorphic to
$$\underline{M}^\wedge = \lim \underline{M/I^nM}.$$
(4) We say $\mathcal{F}$ is *adic constructible*[4] if for every affine open $U \subset X$ there exists a decomposition $U = \coprod U_i$ into constructible locally closed subschemes such that $\mathcal{F}|_{U_i}$ is adic lisse.

The definition of a constructible Λ-sheaf is equivalent to the one in [**Gro77**, Exposé VI, Definition 1.1.1] when $\Lambda = \mathbf{Z}_\ell$ and $I = (\ell)$. It is clear that we have the implications

$$\begin{array}{ccc} \text{lisse adic} & \Longrightarrow & \text{adic constructible} \\ \Downarrow & & \Downarrow \\ \text{lisse constructible } \Lambda\text{-sheaf} & \Longrightarrow & \text{constructible } \Lambda\text{-sheaf} \end{array}$$

The vertical arrows can be inverted in some cases (see Lemmas 46.25.2 and 46.25.5). In general neither the category of adic constructible sheaves nor the category of adic constructible sheaves is closed under kernels and cokernels.

Namely, let X be an affine scheme whose underlying topological space $|X|$ is homeomorphic to $\Lambda = \mathbf{Z}_\ell$, see Example 46.6.3. Denote $f : |X| \to \mathbf{Z}_\ell = \Lambda$ a homeomorphism. We can think of f as a section of $\underline{\Lambda}^\wedge$ over X and multiplication by f then defines a two term complex
$$\underline{\Lambda}^\wedge \xrightarrow{f} \underline{\Lambda}^\wedge$$
on $X_{pro\text{-}étale}$. The sheaf $\underline{\Lambda}^\wedge$ is adic lisse. However, the cokernel of the map above, is not adic constructible, as the isomorphism type of the stalks of this cokernel attains infinitely many values: $\mathbf{Z}/\ell^n\mathbf{Z}$ and $\mathbf{Z}_\ell$. The cokernel is a constructible $\mathbf{Z}_\ell$-sheaf. However, the kernel is not even a constructible $\mathbf{Z}_\ell$-sheaf as it is zero a non-quasi-compact open but not zero.

Lemma 46.25.2. *Let X be a Noetherian scheme. Let Λ be a Noetherian ring and let $I \subset \Lambda$ be an ideal. Let $\mathcal{F}$ be a constructible Λ-sheaf on $X_{pro\text{-}étale}$. Then there exists a finite partition $X = \coprod X_i$ by locally closed subschemes such that the restriction $\mathcal{F}|_{X_i}$ is lisse.*

[3] This may be nonstandard notation.
[4] This may be nonstandard notation.

Proof. Let $R = \bigoplus I^n/I^{n+1}$. Observe that R is a Noetherian ring. Since each of the sheaves $\mathcal{F}/I^n\mathcal{F}$ is a constructible sheaf of $\Lambda/I^n\Lambda$-modules also $I^n\mathcal{F}/I^{n+1}\mathcal{F}$ is a constructible sheaf of Λ/I-modules and hence the pullback of a constructible sheaf $\mathcal{G}_n$ on $X_{étale}$ by Lemma 46.24.2. Set $\mathcal{G} = \bigoplus \mathcal{G}_n$. This is a sheaf of R-modules on $X_{étale}$ and the map

$$\mathcal{G}_0 \otimes_{\Lambda/I} \underline{R} \longrightarrow \mathcal{G}$$

is surjective because the maps

$$\mathcal{F}/I\mathcal{F} \otimes \underline{I^n/I^{n+1}} \to I^n\mathcal{F}/I^{n+1}\mathcal{F}$$

are surjective. Hence $\mathcal{G}$ is a constructible sheaf of R-modules by Étale Cohomology, Proposition 44.68.1. Choose a partition $X = \coprod X_i$ such that $\mathcal{G}|_{X_i}$ is a locally constant sheaf of R-modules of finite type (Étale Cohomology, Lemma 44.65.2). We claim this is a partition as in the lemma. Namely, replacing X by X_i we may assume $\mathcal{G}$ is locally constant. It follows that each of the sheaves $I^n\mathcal{F}/I^{n+1}\mathcal{F}$ is locally constant. Using the short exact sequences

$$0 \to I^n\mathcal{F}/I^{n+1}\mathcal{F} \to \mathcal{F}/I^{n+1}\mathcal{F} \to \mathcal{F}/I^n\mathcal{F} \to 0$$

induction and Modules on Sites, Lemma 18.42.5 the lemma follows. □

Lemma 46.25.3. *Let X be a weakly contractible affine scheme. Let Λ be a Noetherian ring and $I \subset \Lambda$ be an ideal. Let $\mathcal{F}$ be a sheaf of Λ-modules on $X_{pro\text{-}étale}$ such that*

(1) $\mathcal{F} = \lim \mathcal{F}/I^n\mathcal{F}$,
(2) $\mathcal{F}/I^n\mathcal{F}$ *is a constant sheaf of Λ/I^n-modules,*
(3) $\mathcal{F}/I\mathcal{F}$ *is of finite type.*

Then $\mathcal{F} \cong \underline{M}^\wedge$ where M is a finite $\Lambda^\wedge$-module.

Proof. Pick a Λ/I^n-module M_n such that $\mathcal{F}/I^n\mathcal{F} \cong \underline{M_n}$. Since we have the surjections $\mathcal{F}/I^{n+1}\mathcal{F} \to \mathcal{F}/I^n\mathcal{F}$ we conclude that there exist surjections $M_{n+1} \to M_n$ inducing isomorphisms $M_{n+1}/I^nM_{n+1} \to M_n$. Fix a choice of such surjections and set $M = \lim M_n$. Then M is an I-adically complete Λ-module with $M/I^nM = M_n$, see Algebra, Lemma 10.92.1. Since M_1 is a finite type Λ-module (Modules on Sites, Lemma 18.41.5) we see that M is a finite $\Lambda^\wedge$-module. Consider the sheaves

$$\mathcal{I}_n = Isom(\underline{M_n}, \mathcal{F}/I^n\mathcal{F})$$

on $X_{pro\text{-}étale}$. Modding out by I^n defines a transition map

$$\mathcal{I}_{n+1} \longrightarrow \mathcal{I}_n$$

By our choice of M_n the sheaf $\mathcal{I}_n$ is a torsor under

$$Isom(\underline{M_n}, \underline{M_n}) = \underline{\mathrm{Isom}_\Lambda(M_n, M_n)}$$

(Modules on Sites, Lemma 18.42.4) since $\mathcal{F}/I^n\mathcal{F}$ is (étale) locally isomorphic to $\underline{M_n}$. It follows from More on Algebra, Lemma 15.59.1 that the system of sheaves $(\mathcal{I}_n)$ is Mittag-Leffler. For each n let $\mathcal{I}'_n \subset \mathcal{I}_n$ be the image of $\mathcal{I}_N \to \mathcal{I}_n$ for all $N \gg n$. Then

$$\ldots \to \mathcal{I}'_3 \to \mathcal{I}'_2 \to \mathcal{I}'_1 \to *$$

is a sequence of sheaves of sets on $X_{pro\text{-}étale}$ with surjective transition maps. Since $*(X)$ is a singleton (not empty) and since evaluating at X transforms surjective maps of sheaves of sets into surjections of sets, we can pick $s \in \lim \mathcal{I}'_n(X)$. The sections define isomorphisms $\underline{M}^\wedge \to \lim \mathcal{F}/I^n\mathcal{F} = \mathcal{F}$ and the proof is done. □

Lemma 46.25.4. *Let X be a connected scheme. Let Λ be a Noetherian ring and let $I \subset \Lambda$ be an ideal. If $\mathcal{F}$ is a lisse constructible Λ-sheaf on $X_{pro\text{-}étale}$, then $\mathcal{F}$ is adic lisse.*

Proof. By Lemma 46.17.8 we have $\mathcal{F}/I^n\mathcal{F} = \epsilon^{-1}\mathcal{G}_n$ for some locally constant sheaf $\mathcal{G}_n$ of Λ/I^n-modules. By Étale Cohomology, Lemma 44.64.8 there exists a finite Λ/I^n-module M_n such that $\mathcal{G}_n$ is locally isomorphic to $\underline{M_n}$. Choose a covering $\{W_t \to X\}_{t\in T}$ with each W_t affine and weakly contractible. Then $\mathcal{F}|_{W_t}$ satisfies the assumptions of Lemma 46.25.3 and hence $\mathcal{F}|_{W_t} \cong \underline{N_t}^\wedge$ for some finite $\Lambda^\wedge$-module N_t. Note that $N_t/I^nN_t \cong M_n$ for all t and n. Hence $N_t \cong N_{t'}$ for all $t, t' \in T$, see More on Algebra, Lemma 15.59.2. This proves that $\mathcal{F}$ is adic lisse. □

Lemma 46.25.5. *Let X be a Noetherian scheme. Let Λ be a Noetherian ring and let $I \subset \Lambda$ be an ideal. Let $\mathcal{F}$ be a constructible Λ-sheaf on $X_{pro\text{-}étale}$. Then $\mathcal{F}$ is adic constructible.*

Proof. This is a consequence of Lemmas 46.25.2 and 46.25.4, the fact that a Noetherian scheme is locally connected (Topology, Lemma 5.8.6), and the definitions. □

It will be useful to identify the constructible Λ-sheaves inside the category of derived complete sheaves of Λ-modules. It turns out that the naive analogue of More on Algebra, Lemma 15.57.16 is wrong in this setting. However, here is the analogue of More on Algebra, Lemma 15.57.15.

Lemma 46.25.6. *Let X be a scheme. Let Λ be a ring and let $I \subset \Lambda$ be a finitely generated ideal. Let $\mathcal{F}$ be a sheaf of Λ-modules on $X_{pro\text{-}étale}$. If $\mathcal{F}$ is derived complete and $\mathcal{F}/I\mathcal{F} = 0$, then $\mathcal{F} = 0$.*

Proof. Assume that $\mathcal{F}/I\mathcal{F}$ is zero. Let $I = (f_1, \ldots, f_r)$. Let $i < r$ be the largest integer such that $\mathcal{G} = \mathcal{F}/(f_1, \ldots, f_i)\mathcal{F}$ is nonzero. If i does not exist, then $\mathcal{F} = 0$ which is what we want to show. Then $\mathcal{G}$ is derived complete as a cokernel of a map between derived complete modules, see Proposition 46.16.1. By our choice of i we have that $f_{i+1} : \mathcal{G} \to \mathcal{G}$ is surjective. Hence

$$\lim(\ldots \to \mathcal{G} \xrightarrow{f_{i+1}} \mathcal{G} \xrightarrow{f_{i+1}} \mathcal{G})$$

is nonzero, contradicting the derived completeness of $\mathcal{G}$. □

Lemma 46.25.7. *Let X be a weakly contractible affine scheme. Let Λ be a Noetherian ring and let $I \subset \Lambda$ be an ideal. Let $\mathcal{F}$ be a derived complete sheaf of Λ-modules on $X_{pro\text{-}étale}$ with $\mathcal{F}/I\mathcal{F}$ a locally constant sheaf of Λ/I-modules of finite type. Then there exists an integer t and a surjective map*

$$(\underline{\Lambda}^\wedge)^{\oplus t} \to \mathcal{F}$$

Proof. Since X is weakly contractible, there exists a finite disjoint open covering $X = \coprod U_i$ such that $\mathcal{F}/I\mathcal{F}|_{U_i}$ is isomorphic to the constant sheaf associated to a finite Λ/I-module M_i. Choose finitely many generators m_{ij} of M_i. We can find sections $s_{ij} \in \mathcal{F}(X)$ restricting to m_{ij} viewed as a section of $\mathcal{F}/I\mathcal{F}$ over U_i. Let t be the total number of s_{ij}. Then we obtain a map

$$\alpha : \underline{\Lambda}^{\oplus t} \longrightarrow \mathcal{F}$$

which is surjective modulo I by construction. By Lemma 46.15.1 the derived completion of $\underline{\Lambda}^{\oplus t}$ is the sheaf $(\underline{\Lambda}^\wedge)^{\oplus t}$. Since $\mathcal{F}$ is derived complete we see that α factors through a map

$$\alpha^\wedge : (\underline{\Lambda}^\wedge)^{\oplus t} \longrightarrow \mathcal{F}$$

Then $\mathcal{Q} = \mathrm{Coker}(\alpha^\wedge)$ is a derived complete sheaf of Λ-modules by Proposition 46.16.1. By construction $\mathcal{Q}/I\mathcal{Q} = 0$. It follows from Lemma 46.25.6 that $\mathcal{Q} = 0$ which is what we wanted to show. □

46.26. A suitable derived category

Let X be a scheme. It will turn out that for many schemes X a suitable derived category of ℓ-adic sheaves can be gotten by considering the derived complete objects K of $D(X_{pro\text{-}étale}, \Lambda)$ with the property that $K \otimes^{\mathbf{L}}_{\Lambda} \mathbf{F}_\ell$ is bounded with constructible cohomology sheaves. Here is the general definition.

Definition 46.26.1. Let Λ be a Noetherian ring and let $I \subset \Lambda$ be an ideal. Let X be a scheme. An object K of $D(X_{pro\text{-}étale}, \Lambda)$ is called *constructible* if

(1) K is derived complete with respect to I,
(2) $K \otimes^{\mathbf{L}}_{\Lambda} \underline{\Lambda/I}$ has constructible cohomology sheaves and locally has finite tor dimension.

We denote $D_{cons}(X, \Lambda)$ the full subcategory of constructible K in $D(X_{pro\text{-}étale}, \Lambda)$.

Recall that with our conventions a complex of finite tor dimension is bounded (Cohomology on Sites, Definition 21.34.1). In fact, let's collect everything proved so far in a lemma.

Lemma 46.26.2. *In the situation above suppose K is in $D_{cons}(X, \Lambda)$ and X is quasi-compact. Set $K_n = K \otimes^{\mathbf{L}}_{\Lambda} \underline{\Lambda/I^n}$. There exist a, b such that*

(1) *$K = R\lim K_n$ and $H^i(K) = 0$ for $i \notin [a, b]$,*
(2) *each K_n has tor amplitude in $[a, b]$,*
(3) *each K_n has constructible cohomology sheaves,*
(4) *each $K_n = \epsilon^{-1} L_n$ for some $L_n \in D_{ctf}(X_{étale}, \Lambda/I^n)$ (Étale Cohomology, Definition 44.84.7).*

Proof. By definition of local having finite tor dimension, we can find a, b such that K_1 has tor amplitude in $[a, b]$. Part (2) follows from Cohomology on Sites, Lemma 21.34.8. Then (1) follows as K is derived complete by the description of limits in Cohomology on Sites, Proposition 21.37.2 and the fact that $H^b(K_{n+1}) \to H^b(K_n)$ is surjective as $K_n = K_{n+1} \otimes^{\mathbf{L}}_{\Lambda} \underline{\Lambda/I^n}$. Part (3) follows from Lemma 46.24.6, Part (4) follows from Lemma 46.24.4 and the fact that L_n has finite tor dimension because K_n does (small argument omitted). □

Lemma 46.26.3. *Let X be a weakly contractible affine scheme. Let Λ be a Noetherian ring and let $I \subset \Lambda$ be an ideal. Let K be an object of $D_{cons}(X, \Lambda)$ such that the cohomology sheaves of $K \otimes^{\mathbf{L}}_{\Lambda} \underline{\Lambda/I}$ are locally constant. Then there exists a finite disjoint open covering $X = \coprod \overline{U_i}$ and for each i a finite collection of finite projective $\Lambda^\wedge$-modules $M_a, \ldots, M_b$ such that $K|_{U_i}$ is represented by a complex*

$$(\underline{M^a})^\wedge \to \ldots \to (\underline{M^b})^\wedge$$

in $D(U_{i,pro\text{-}étale}, \Lambda)$ for some maps of sheaves of Λ-modules $(\underline{M^i})^\wedge \to (\underline{M^{i+1}})^\wedge$.

Proof. We freely use the results of Lemma 46.26.2. Choose a, b as in that lemma. We will prove the lemma by induction on $b - a$. Let $\mathcal{F} = H^b(K)$. Note that $\mathcal{F}$ is a derived complete sheaf of Λ-modules by Proposition 46.16.1. Moreover $\mathcal{F}/I\mathcal{F}$ is a locally constant sheaf of Λ/I-modules of finite type. Apply Lemma 46.25.7 to get a surjection $\rho : (\underline{\Lambda}^\wedge)^{\oplus t} \to \mathcal{F}$.

If $a = b$, then $K = \mathcal{F}[-b]$. In this case we see that

$$\mathcal{F} \otimes^{\mathbf{L}}_{\Lambda} \underline{\Lambda/I} = \mathcal{F}/I\mathcal{F}$$

As X is weakly contractible and $\mathcal{F}/I\mathcal{F}$ locally constant, we can find a finite disjoint union decomposition $X = \coprod U_i$ by affine opens U_i and Λ/I-modules $\overline{M}_i$ such that $\mathcal{F}/I\mathcal{F}$ restricts to $\underline{\overline{M}_i}$ on U_i. After refining the covering we may assume the map

$$\rho|_{U_i} \bmod I : \underline{\Lambda/I}^{\oplus t} \longrightarrow \underline{\overline{M}_i}$$

is equal to $\underline{\alpha_i}$ for some surjective module map $\alpha_i : \Lambda/I^{\oplus t} \to \overline{M}_i$, see Modules on Sites, Lemma 18.42.3. Note that each $\overline{M}_i$ is a finite Λ/I-module. Since $\mathcal{F}/I\mathcal{F}$ has tor amplitude in $[0, 0]$ we conclude that $\overline{M}_i$ is a flat Λ/I-module. Hence $\overline{M}_i$ is finite projective (Algebra, Lemma 10.73.2). Hence we can find a projector $\overline{p}_i : (\Lambda/I)^{\oplus t} \to (\Lambda/I)^{\oplus t}$ whose image maps isomorphically to $\overline{M}_i$ under the map α_i. We can lift $\overline{p}_i$ to a projector $p_i : (\Lambda^\wedge)^{\oplus t} \to (\Lambda^\wedge)^{\oplus t}$[5]. Then $M_i = \mathrm{Im}(p_i)$ is a finite I-adically complete $\Lambda^\wedge$-module with $M_i/IM_i = \overline{M}_i$. Over U_i consider the maps

$$\underline{M_i}^\wedge \to (\underline{\Lambda}^\wedge)^{\oplus t} \to \mathcal{F}|_{U_i}$$

By construction the composition induces an isomorphism modulo I. The source and target are derived complete, hence so are the cokernel $\mathcal{Q}$ and the kernel $\mathcal{K}$. We have $\mathcal{Q}/I\mathcal{Q} = 0$ by construction hence $\mathcal{Q}$ is zero by Lemma 46.25.6. Then

$$0 \to \mathcal{K}/I\mathcal{K} \to \underline{\overline{M}_i} \to \mathcal{F}/I\mathcal{F} \to 0$$

is exact by the vanishing of Tor_1 see at the start of this paragraph; also use that $\underline{\Lambda}^\wedge/I\underline{\Lambda}^\wedge$ by Modules on Sites, Lemma 18.41.4 to see that $\underline{M_i}^\wedge/I\underline{M_i}^\wedge = \underline{\overline{M}_i}$. Hence $\mathcal{K}/I\mathcal{K} = 0$ by construction and we conclude that $\mathcal{K} = 0$ as before. This proves the result in case $a = b$.

If $b > a$, then we lift the map ρ to a map

$$\tilde{\rho} : (\underline{\Lambda}^\wedge)^{\oplus t}[-b] \longrightarrow K$$

in $D(X_{pro\text{-}étale}, \Lambda)$. This is possible as we can think of K as a complex of $\underline{\Lambda}^\wedge$-modules by discussion in the introduction to Section 46.15 and because $X_{pro\text{-}étale}$ is weakly contractible hence there is no obstruction to lifting the elements $\rho(e_s) \in H^0(X, \mathcal{F})$ to elements of $H^b(X, K)$. Fitting $\tilde{\rho}$ into a distinguished triangle

$$(\underline{\Lambda}^\wedge)^{\oplus t}[-b] \to K \to L \to (\underline{\Lambda}^\wedge)^{\oplus t}[-b+1]$$

we see that L is an object of $D_{cons}(X, \Lambda)$ such that $L \otimes^{\mathbf{L}}_{\Lambda} \underline{\Lambda/I}$ has tor amplitude contained in $[a, b-1]$ (details omitted). By induction we can describe L locally as stated in the lemma, say L is isomorphic to

$$(\underline{M^a})^\wedge \to \ldots \to (\underline{M^{b-1}})^\wedge$$

[5]Proof: by Algebra, Lemma 10.49.6 we can lift $\overline{p}_i$ to a compatible system of projectors $p_{i,n} : (\Lambda/I^n)^{\oplus t} \to (\Lambda/I^n)^{\oplus t}$ and then we set $p_i = \lim p_{i,n}$ which works because $\Lambda^\wedge = \lim \Lambda/I^n$.

The map $L \to (\underline{\Lambda}^\wedge)^{\oplus t}[-b+1]$ corresponds to a map $(\underline{M^{b-1}})^\wedge \to (\underline{\Lambda}^\wedge)^{\oplus t}$ which allows us to extend the complex by one. The corresponding complex is isomorphic to K in the derived category by the properties of triangulated categories. This finishes the proof. □

Motivated by what happens for constructible Λ-sheaves we introduce the following notion.

Definition 46.26.4. Let X be a scheme. Let Λ be a Noetherian ring and let $I \subset \Lambda$ be an ideal. Let $K \in D(X_{pro\text{-}étale}, \Lambda)$.

(1) We say K is *adic lisse*[6] if there exists a finite complex of finite projective $\Lambda^\wedge$-modules $M^\bullet$ such that K is locally isomorphic to
$$\underline{M^a}^\wedge \to \ldots \to \underline{M^b}^\wedge$$
(2) We say K is *adic constructible*[7] if for every affine open $U \subset X$ there exists a decomposition $U = \coprod U_i$ into constructible locally closed subschemes such that $K|_{U_i}$ is adic lisse.

The difference between the local structure obtained in Lemma 46.26.3 and the structure of an adic lisse complex is that the maps $\underline{M^i}^\wedge \to \underline{M^{i+1}}^\wedge$ in Lemma 46.26.3 need not be constant, whereas in the definition above they are required to be constant.

Lemma 46.26.5. *Let X be a weakly contractible affine scheme. Let Λ be a Noetherian ring and let $I \subset \Lambda$ be an ideal. Let K be an object of $D_{cons}(X, \Lambda)$ such that $K \otimes^{\mathbf{L}}_\Lambda \underline{\Lambda/I^n}$ is isomorphic in $D(X_{pro\text{-}étale}, \Lambda/I^n)$ to a complex of constant sheaves of Λ/I^n-modules. Then*
$$H^0(X, K \otimes^{\mathbf{L}}_\Lambda \Lambda/I^n)$$
has the Mittag-Leffler condition.

Proof. Say $K \otimes^{\mathbf{L}}_\Lambda \underline{\Lambda/I^n}$ is isomorphic to $\underline{E_n}$ for some object E_n of $D(\Lambda/I^n)$. Since $K \otimes^{\mathbf{L}}_\Lambda \underline{\Lambda/I}$ has finite tor dimension and has finite type cohomology sheaves we see that E_1 is perfect (see More on Algebra, Lemma 15.50.2). The transition maps
$$K \otimes^{\mathbf{L}}_\Lambda \underline{\Lambda/I^{n+1}} \to K \otimes^{\mathbf{L}}_\Lambda \underline{\Lambda/I^n}$$
locally come from (possibly many distinct) maps of complexes $E_{n+1} \to E_n$ in $D(\Lambda/I^{n+1})$ see Cohomology on Sites, Lemma 21.39.3. For each n choose one such map and observe that it induces an isomorphism $E_{n+1} \otimes^{\mathbf{L}}_{\Lambda/I^{n+1}} \Lambda/I^n \to E_n$ in $D(\Lambda/I^n)$. By More on Algebra, Lemma 15.58.3 we can find a finite complex $M^\bullet$ of finite projective $\Lambda^\wedge$-modules and isomorphisms $M^\bullet/I^n M^\bullet \to E_n$ in $D(\Lambda/I^n)$ compatible with the transition maps.

Now observe that for each finite collection of indices $n > m > k$ the triple of maps
$$H^0(X, K \otimes^{\mathbf{L}}_\Lambda \Lambda/I^n) \to H^0(X, K \otimes^{\mathbf{L}}_\Lambda \Lambda/I^m) \to H^0(X, K \otimes^{\mathbf{L}}_\Lambda \Lambda/I^k)$$
is isomorphic to
$$H^0(X, \underline{M^\bullet/I^n M^\bullet}) \to H^0(X, \underline{M^\bullet/I^m M^\bullet}) \to H^0(X, \underline{M^\bullet/I^k M^\bullet})$$

[6] This may be nonstandard notation

[7] This may be nonstandard notation.

Namely, choose any isomorphism

$$\underline{M^\bullet/I^nM^\bullet} \to K \otimes^{\mathbf{L}}_\Lambda \Lambda/I^n$$

induces similar isomorphisms module I^m and I^k and we see that the assertion is true. Thus to prove the lemma it suffices to show that the system $H^0(X, \underline{M^\bullet/I^nM^\bullet})$ has Mittag-Leffler. Since taking sections over X is exact, it suffices to prove that the system of Λ-modules

$$H^0(M^\bullet/I^nM^\bullet)$$

has Mittag-Leffler. Set $A = \Lambda^\wedge$ and consider the spectral sequence

$$\mathrm{Tor}^A_{-p}(H^q(M^\bullet), A/I^nA) \Rightarrow H^{p+q}(M^\bullet/I^nM^\bullet)$$

By More on Algebra, Lemma 15.18.3 the pro-systems $\{\mathrm{Tor}^A_{-p}(H^q(M^\bullet), A/I^nA)\}$ are zero for $p > 0$. Thus the pro-system $\{H^0(M^\bullet/I^nM^\bullet)\}$ is equal to the pro-system $\{H^0(M^\bullet)/I^nH^0(M^\bullet)\}$ and the lemma is proved. □

Lemma 46.26.6. *Let X be a connected scheme. Let Λ be a Noetherian ring and let $I \subset \Lambda$ be an ideal. If K is in $D_{cons}(X, \Lambda)$ such that $K \otimes_\Lambda \underline{\Lambda/I}$ has locally constant cohomology sheaves, then K is adic lisse (Definition 46.26.4).*

Proof. Write $K_n = K \otimes^{\mathbf{L}}_\Lambda \underline{\Lambda/I^n}$. We will use the results of Lemma 46.26.2 without further mention. By Cohomology on Sites, Lemma 21.39.5 we see that K_n has locally constant cohomology sheaves for all n. We have $K_n = \epsilon^{-1}L_n$ some L_n in $D_{ctf}(X_{étale}, \Lambda/I^n)$ with locally constant cohomology sheaves. By Étale Cohomology, Lemma 44.84.14 there exist perfect $M_n \in D(\Lambda/I^n)$ such that L_n is étale locally isomorphic to $\underline{M_n}$. The maps $L_{n+1} \to L_n$ corresponding to $K_{n+1} \to K_n$ induces isomorphisms $L_{n+1} \otimes^{\mathbf{L}}_{\Lambda/I^{n+1}} \underline{\Lambda/I^n} \to L_n$. Looking locally on X we conclude that there exist maps $M_{n+1} \to M_n$ in $D(\Lambda/I^{n+1})$ inducing isomorphisms $M_{n+1} \otimes_{\Lambda/I^{n+1}} \Lambda/I^n \to M_n$, see Cohomology on Sites, Lemma 21.39.3. Fix a choice of such maps. By More on Algebra, Lemma 15.58.3 we can find a finite complex $M^\bullet$ of finite projective $\Lambda^\wedge$-modules and isomorphisms $M^\bullet/I^nM^\bullet \to M_n$ in $D(\Lambda/I^n)$ compatible with the transition maps. To finish the proof we will show that K is locally isomorphic to

$$\underline{M^\bullet}^\wedge = \lim \underline{M^\bullet/I^nM^\bullet} = R\lim \underline{M^\bullet/I^nM^\bullet}$$

Let $E^\bullet$ be the dual complex to $M^\bullet$, see More on Algebra, Lemma 15.50.19 and its proof. Consider the objects

$$H_n = R\mathcal{H}om_{\Lambda/I^n}(\underline{M^\bullet/I^nM^\bullet}, K_n) = \underline{E^\bullet/I^nE^\bullet} \otimes^{\mathbf{L}}_{\Lambda/I^n} K_n$$

of $D(X_{pro\text{-}étale}, \Lambda/I^n)$. Modding out by I^n defines a transition map $H_{n+1} \to H_n$. Set $H = R\lim H_n$. Then H is an object of $D_{cons}(X, \Lambda)$ (details omitted) with $H \otimes^{\mathbf{L}}_\Lambda \underline{\Lambda/I^n} = H_n$. Choose a covering $\{W_t \to X\}_{t \in T}$ with each W_t affine and weakly contractible. By our choice of $M^\bullet$ we see that

$$\begin{aligned} H_n|_{W_t} &\cong R\mathcal{H}om_{\Lambda/I^n}(\underline{M^\bullet/I^nM^\bullet}, \underline{M^\bullet/I^nM^\bullet}) \\ &= \underline{\mathrm{Tot}(E^\bullet/I^nE^\bullet \otimes_{\Lambda/I^n} M^\bullet/I^nM^\bullet)} \end{aligned}$$

Thus we may apply Lemma 46.26.5 to $H = R\lim H_n$. We conclude the system $H^0(W_t, H_n)$ satisfies Mittag-Leffler. Since for all $n \gg 1$ there is an element of $H^0(W_t, H_n)$ which maps to an isomorphism in

$$H^0(W_t, H_1) = \mathrm{Hom}(\underline{M^\bullet/IM^\bullet}, K_1)$$

we find an element $(\varphi_{t,n})$ in the inverse limit which produces an isomorphism mod I. Then

$$R\lim \varphi_{t,n} : \underline{M^\bullet}^\wedge|_{W_t} = R\lim \underline{M^\bullet/I^nM^\bullet}|_{W_t} \longrightarrow R\lim K_n|_{W_t} = K|_{W_t}$$

is an isomorphism. This finishes the proof. □

Proposition 46.26.7. *Let X be a Noetherian scheme. Let Λ be a Noetherian ring and let $I \subset \Lambda$ be an ideal. Let K be an object of $D_{cons}(X, \Lambda)$. Then K is adic constructible (Definition 46.26.4).*

Proof. This is a consequence of Lemma 46.26.6 and the fact that a Noetherian scheme is locally connected (Topology, Lemma 5.8.6), and the definitions. □

46.27. Proper base change

In this section we explain how to prove the proper base change theorem for derived complete objects on the pro-étale site using the proper base change theorem for étale cohomology following the general theme that we use the pro-étale topology only to deal with "limit issues" and we use results proved for the étale topology to handle everything else.

Theorem 46.27.1. *Let $f : X \to Y$ be a proper morphism of schemes. Let $g : Y' \to Y$ be a morphism of schemes giving rise to the base change diagram*

$$\begin{array}{ccc} X' & \xrightarrow{g'} & X \\ {\scriptstyle f'}\downarrow & & \downarrow{\scriptstyle f} \\ Y' & \xrightarrow{g} & Y \end{array}$$

Let Λ be a Noetherian ring and let $I \subset \Lambda$ be an ideal such that Λ/I is torsion. Let K be an object of $D(X_{pro\text{-}étale})$ such that

1. *K is derived complete, and*
2. *$K \otimes^{\mathbf{L}}_{\Lambda} \underline{\Lambda/I^n}$ is bounded below with cohomology sheaves coming from $X_{étale}$,*
3. *Λ/I^n is a perfect Λ-module*[8].

Then the base change map

$$Lg^*_{comp} Rf_* K \longrightarrow Rf'_* L(g')^*_{comp} K$$

is an isomorphism.

Proof. We omit the construction of the base change map (this uses only formal properties of derived pushforward and completed derived pullback, compare with Cohomology on Sites, Remark 21.19.2). Write $K_n = K \otimes^{\mathbf{L}}_{\Lambda} \underline{\Lambda/I^n}$ so that $K = R\lim K_n$ because K is derived complete. By Lemma 46.15.2 we can unwind the left hand side

$$Lg^*_{comp} Rf_* K = R\lim Lg^*(Rf_* K) \otimes^{\mathbf{L}}_{\Lambda} \underline{\Lambda/I^n} = R\lim Lg^* Rf_* K_n$$

the last equality because Λ/I^n is a perfect module and the projection formula (Cohomology on Sites, Lemma 21.36.1). Using Lemma 46.15.2 we can unwind the right hand side

$$Rf'_* L(g')^*_{comp} K = Rf'_* R\lim L(g')^* K_n = R\lim Rf'_* L(g')^* K_n$$

[8] This assumption can be removed if K is a constructible complex, see [**BS13**].

the last equality because Rf'_* commutes with homotopy limits as it is a right adjoint. Thus it suffices to show the maps

$$Lg^*Rf_*K_n \longrightarrow Rf'_*L(g')^*K_n$$

are isomorphisms. By Lemma 46.17.7 and our second condition we can write $K_n = \epsilon^{-1}L_n$ for some $L_n \in D^+(X_{étale}, \Lambda/I^n)$. By Lemma 46.20.1 and the fact that ϵ^{-1} commutes with pullbacks we obtain

$$Lg^*Rf_*K_n = Lg^*Rf_*\epsilon^*L_n = Lg^*\epsilon^{-1}Rf_*L_n = \epsilon^{-1}Lg^*Rf_*L_n$$

and

$$Rf'_*L(g')^*K_n = Rf'_*L(g')^*\epsilon^{-1}L_n = Rf'_*\epsilon^{-1}L(g')^*L_n = \epsilon^{-1}Rf'_*L(g')^*L_n$$

(this also uses that L_n is bounded below). Finally, by the proper base change theorem for étale cohomology (Étale Cohomology, Theorem 44.72.3) we have

$$Lg^*Rf_*L_n = Rf'_*L(g')^*L_n$$

(again using that L_n is bounded below) and the theorem is proved. □

46.28. Other chapters

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CHAPTER 47

Algebraic Spaces

47.1. Introduction

Algebraic spaces were first introduced by Michael Artin, see [**Art69b**], [**Art70**], [**Art73**], [**Art71b**], [**Art71a**], [**Art69a**], [**Art69c**], and [**Art74**]. Some of the foundational material was developed jointly with Knutson, who produced the book [**Knu71**]. Artin defined (see [**Art69c**, Definition 1.3]) an algebraic space as a sheaf for the étale topology which is locally in the étale topology representable. In most of Artin's work the categories of schemes considered are schemes locally of finite type over a fixed excellent Noetherian base.

Our definition is slightly different. First of all we consider sheaves for the fppf topology. This is just a technical point and scarcely makes any difference. Second, we include the condition that the diagonal is representable.

After defining algebraic spaces we make some foundational observations. The main result in this chapter is that with our definitions an algebraic space is the same thing as an étale equivalence relation, see the discussion in Section 47.9 and Theorem 47.10.5. The analogue of this theorem in Artin's setting is [**Art69c**, Theorem 1.5], or [**Knu71**, Proposition II.1.7]. In other words, the sheaf defined by an étale equivalence relation has a representable diagonal. It follows that our definition agrees with Artin's original definition in a broad sense. It also means that one can give examples of algebraic spaces by simply writing down an étale equivalence relation.

In Section 47.13 we introduce various separation axioms on algebraic spaces that we have found in the literature. Finally in Section 47.14 we give some weird and not so weird examples of algebraic spaces.

47.2. General remarks

We work in a suitable big fppf site Sch_{fppf} as in Topologies, Definition 33.7.6. So, if not explicitly stated otherwise all schemes will be objects of Sch_{fppf}. We will record elsewhere what changes if you change the big fppf site (insert future reference here).

We will always work relative to a base S contained in Sch_{fppf}. And we will then work with the big fppf site $(Sch/S)_{fppf}$, see Topologies, Definition 33.7.8. The absolute case can be recovered by taking $S = \mathrm{Spec}(\mathbf{Z})$.

If U, T are schemes over S, then we denote $U(T)$ for the set of T-valued points *over* S. In a formula: $U(T) = \mathrm{Mor}_S(T, U)$.

Note that any fpqc covering is a universal effective epimorphism, see Descent, Lemma 34.9.3. Hence the topology on Sch_{fppf} is weaker than the canonical topology and all representable presheaves are sheaves.

47.3. Representable morphisms of presheaves

Let S be a scheme contained in Sch_{fppf}. Let $F, G : (Sch/S)^{opp}_{fppf} \to Sets$. Let $a : F \to G$ be a representable transformation of functors, see Categories, Definition 4.8.2. This means that for every $U \in \mathrm{Ob}((Sch/S)_{fppf})$ and any $\xi \in G(U)$ the fiber product $h_U \times_{\xi,G} F$ is representable. Choose a representing object V_ξ and an isomorphism $h_{V_\xi} \to h_U \times_G F$. By the Yoneda lemma, see Categories, Lemma 4.3.5, the projection $h_{V_\xi} \to h_U \times_G F \to h_U$ comes from a unique morphism of schemes $a_\xi : V_\xi \to U$. Suggestively we could represent this by the diagram

$$\begin{array}{ccccc} V_\xi & \rightsquigarrow & h_{V_\xi} & \longrightarrow & F \\ {\scriptstyle a_\xi}\downarrow & & \downarrow & & \downarrow{\scriptstyle a} \\ U & \rightsquigarrow & h_U & \xrightarrow{\xi} & G \end{array}$$

where the squiggly arrows represent the Yoneda embedding. Here are some lemmas about this notion that work in great generality.

Lemma 47.3.1. *Let S, X, Y be objects of Sch_{fppf}. Let $f : X \to Y$ be a morphism of schemes. Then*

$$h_f : h_X \longrightarrow h_Y$$

is a representable transformation of functors.

Proof. This is formal and relies only on the fact that the category $(Sch/S)_{fppf}$ has fibre products. □

Lemma 47.3.2. *Let S be a scheme contained in Sch_{fppf}. Let $F, G, H : (Sch/S)^{opp}_{fppf} \to Sets$. Let $a : F \to G$, $b : G \to H$ be representable transformations of functors. Then*

$$b \circ a : F \longrightarrow H$$

is a representable transformation of functors.

Proof. This is entirely formal and works in any category. □

Lemma 47.3.3. *Let S be a scheme contained in Sch_{fppf}. Let $F, G, H : (Sch/S)^{opp}_{fppf} \to Sets$. Let $a : F \to G$ be a representable transformations of functors. Let $b : H \to G$ be any transformation of functors. Consider the fibre product diagram*

$$\begin{array}{ccc} H \times_{b,G,a} F & \xrightarrow[b']{} & F \\ {\scriptstyle a'}\downarrow & & \downarrow{\scriptstyle a} \\ H & \xrightarrow{b} & G \end{array}$$

Then the base change a' is a representable transformation of functors.

Proof. This is entirely formal and works in any category. □

Lemma 47.3.4. *Let S be a scheme contained in Sch_{fppf}. Let $F_i, G_i : (Sch/S)^{opp}_{fppf} \to Sets$, $i = 1, 2$. Let $a_i : F_i \to G_i$, $i = 1, 2$ be representable transformations of functors. Then*

$$a_1 \times a_2 : F_1 \times F_2 \longrightarrow G_1 \times G_2$$

is a representable transformation of functors.

Proof. Write $a_1 \times a_2$ as the composition $F_1 \times F_2 \to G_1 \times F_2 \to G_1 \times G_2$. The first arrow is the base change of a_1 by the map $G_1 \times F_2 \to G_1$, and the second arrow is the base change of a_2 by the map $G_1 \times G_2 \to G_2$. Hence this lemma is a formal consequence of Lemmas 47.3.2 and 47.3.3. □

Lemma 47.3.5. *Let S be a scheme contained in Sch_{fppf}. Let $F, G : (Sch/S)^{opp}_{fppf} \to Sets$. Let $a : F \to G$ be a representable transformation of functors. If G is a sheaf, then so is F.*

Proof. Let $\{\varphi_i : T_i \to T\}$ be a covering of the site $(Sch/S)_{fppf}$. Let $s_i \in F(T_i)$ which satisfy the sheaf condition. Then $\sigma_i = a(s_i) \in G(T_i)$ satisfy the sheaf condition also. Hence there exists a unique $\sigma \in G(T)$ such that $\sigma_i = \sigma|_{T_i}$. By assumption $F' = h_T \times_{\sigma,G,a} F$ is a representable presheaf and hence (see remarks in Section 47.2) a sheaf. Note that $(\varphi_i, s_i) \in F'(T_i)$ satisfy the sheaf condition also, and hence come from some unique $(\mathrm{id}_T, s) \in F'(T)$. Clearly s is the section of F we are looking for. □

Lemma 47.3.6. *Let S be a scheme contained in Sch_{fppf}. Let $F, G : (Sch/S)^{opp}_{fppf} \to Sets$. Let $a : F \to G$ be a representable transformation of functors. Then $\Delta_{F/G} : F \to F \times_G F$ is representable.*

Proof. Let $U \in \mathrm{Ob}((Sch/S)_{fppf})$. Let $\xi = (\xi_1, \xi_2) \in (F \times_G F)(U)$. Set $\xi' = a(\xi_1) = a(\xi_2) \in G(U)$. By assumption there exist a scheme V and a morphism $V \to U$ representing the fibre product $h_U \times_{\xi',G} F$. In particular, the elements ξ_1, ξ_2 give morphisms $f_1, f_2 : U \to V$ over U. Because V represents the fibre product $h_U \times_{\xi',G} F$ and because $\xi' = a \circ \xi_1 = a \circ \xi_2$ we see that if $g : U' \to U$ is a morphism then

$$g^*\xi_1 = g^*\xi_2 \Leftrightarrow f_1 \circ g = f_2 \circ g.$$

In other words, we see that $h_U \times_{\xi, F \times_G F} F$ is represented by $V \times_{\Delta, V \times V, (f_1, f_2)} U$ which is a scheme. □

47.4. Lists of useful properties of morphisms of schemes

For ease of reference we list in the following remarks the properties of morphisms which possess some of the properties required of them in later results.

Remark 47.4.1. Here is a list of properties/types of morphisms which are *stable under arbitrary base change*:

(1) closed, open, and locally closed immersions, see Schemes, Lemma 25.18.2,
(2) quasi-compact, see Schemes, Lemma 25.19.3,
(3) universally closed, see Schemes, Definition 25.20.1,
(4) (quasi-)separated, see Schemes, Lemma 25.21.13,
(5) monomorphism, see Schemes, Lemma 25.23.5
(6) surjective, see Morphisms, Lemma 28.11.4,
(7) universally injective, see Morphisms, Lemma 28.12.2,
(8) affine, see Morphisms, Lemma 28.13.8,

(9) quasi-affine, see Morphisms, Lemma 28.14.5,
(10) (locally) of finite type, see Morphisms, Lemma 28.16.4,
(11) (locally) quasi-finite, see Morphisms, Lemma 28.21.13,
(12) (locally) of finite presentation, see Morphisms, Lemma 28.22.4,
(13) locally of finite type of relative dimension d, see Morphisms, Lemma 28.30.2,
(14) universally open, see Morphisms, Definition 28.24.1,
(15) flat, see Morphisms, Lemma 28.26.7,
(16) syntomic, see Morphisms, Lemma 28.32.4,
(17) smooth, see Morphisms, Lemma 28.35.5,
(18) unramified (resp. G-unramified), see Morphisms, Lemma 28.36.5,
(19) étale, see Morphisms, Lemma 28.37.4,
(20) proper, see Morphisms, Lemma 28.42.5,
(21) H-projective, see Morphisms, Lemma 28.43.8,
(22) (locally) projective, see Morphisms, Lemma 28.43.9,
(23) finite or integral, see Morphisms, Lemma 28.44.6,
(24) finite locally free, see Morphisms, Lemma 28.46.4.

Add more as needed.

Remark 47.4.2. Of the properties of morphisms which are stable under base change (as listed in Remark 47.4.1) the following are also *stable under compositions*:

(1) closed, open and locally closed immersions, see Schemes, Lemma 25.24.3,
(2) quasi-compact, see Schemes, Lemma 25.19.4,
(3) universally closed, see Morphisms, Lemma 28.42.4,
(4) (quasi-)separated, see Schemes, Lemma 25.21.13,
(5) monomorphism, see Schemes, Lemma 25.23.4,
(6) surjective, see Morphisms, Lemma 28.11.2,
(7) universally injective, see Morphisms, Lemma 28.12.5,
(8) affine, see Morphisms, Lemma 28.13.7,
(9) quasi-affine, see Morphisms, Lemma 28.14.4,
(10) (locally) of finite type, see Morphisms, Lemma 28.16.3,
(11) (locally) quasi-finite, see Morphisms, Lemma 28.21.12,
(12) (locally) of finite presentation, see Morphisms, Lemma 28.22.3,
(13) universally open, see Morphisms, Lemma 28.24.3,
(14) flat, see Morphisms, Lemma 28.26.5,
(15) syntomic, see Morphisms, Lemma 28.32.3,
(16) smooth, see Morphisms, Lemma 28.35.4,
(17) unramified (resp. G-unramified), see Morphisms, Lemma 28.36.4,
(18) étale, see Morphisms, Lemma 28.37.3,
(19) proper, see Morphisms, Lemma 28.42.4,
(20) H-projective, see Morphisms, Lemma 28.43.7,
(21) finite or integral, see Morphisms, Lemma 28.44.5,
(22) finite locally free, see Morphisms, Lemma 28.46.3.

Add more as needed.

Remark 47.4.3. Of the properties mentioned which are stable under base change (as listed in Remark 47.4.1) the following are also *fpqc local on the base* (and a fortiori fppf local on the base):

(1) for immersions we have this for

   (a) closed immersions, see Descent, Lemma 34.19.17,
   (b) open immersions, see Descent, Lemma 34.19.14, and
   (c) quasi-compact immersions, see Descent, Lemma 34.19.19,
(2) quasi-compact, see Descent, Lemma 34.19.1,
(3) universally closed, see Descent, Lemma 34.19.3,
(4) (quasi-)separated, see Descent, Lemmas 34.19.2, and 34.19.5,
(5) monomorphism, see Descent, Lemma 34.19.29,
(6) surjective, see Descent, Lemma 34.19.6,
(7) universally injective, see Descent, Lemma 34.19.7,
(8) affine, see Descent, Lemma 34.19.16,
(9) quasi-affine, see Descent, Lemma 34.19.18,
(10) (locally) of finite type, see Descent, Lemmas 34.19.8, and 34.19.10,
(11) (locally) quasi-finite, see Descent, Lemma 34.19.22,
(12) (locally) of finite presentation, see Descent, Lemmas 34.19.9, and 34.19.11,
(13) locally of finite type of relative dimension d, see Descent, Lemma 34.19.23,
(14) universally open, see Descent, Lemma 34.19.4,
(15) flat, see Descent, Lemma 34.19.13,
(16) syntomic, see Descent, Lemma 34.19.24,
(17) smooth, see Descent, Lemma 34.19.25,
(18) unramified (resp. G-unramified), see Descent, Lemma 34.19.26,
(19) étale, see Descent, Lemma 34.19.27,
(20) proper, see Descent, Lemma 34.19.12,
(21) finite or integral, see Descent, Lemma 34.19.21,
(22) finite locally free, see Descent, Lemma 34.19.28.

Note that the property of being an "immersion" may not be fpqc local on the base, but in Descent, Lemma 34.20.1 we proved that it is fppf local on the base.

47.5. Properties of representable morphisms of presheaves

Here is the definition that makes this work.

Definition 47.5.1. With S, and $a : F \to G$ representable as above. Let $\mathcal{P}$ be a property of morphisms of schemes which

(1) is preserved under any base change, see Schemes, Definition 25.18.3, and
(2) is fppf local on the base, see Descent, Definition 34.18.1.

In this case we say that a has *property* $\mathcal{P}$ if for every $U \in \mathrm{Ob}((Sch/S)_{fppf})$ and any $\xi \in G(U)$ the resulting morphism of schemes $V_\xi \to U$ has property $\mathcal{P}$.

It is important to note that we will only use this definition for properties of morphisms that are stable under base change, and local in the fppf topology on the base. This is not because the definition doesn't make sense otherwise; rather it is because we may want to give a different definition which is better suited to the property we have in mind.

Remark 47.5.2. Consider the property $\mathcal{P}$ ="surjective". In this case there could be some ambiguity if we say "let $F \to G$ be a surjective map". Namely, we could mean the notion defined in Definition 47.5.1 above, or we could mean a surjective map of presheaves, see Sites, Definition 7.3.1, or, if both F and G are sheaves, we could mean a surjective map of sheaves, see Sites, Definition 7.12.1, If not mentioned otherwise when discussing morphisms of algebraic spaces we will always mean the

first. See Lemma 47.5.9 for a case where surjectivity implies surjectivity as a map of sheaves.

Here is a sanity check.

Lemma 47.5.3. *Let S, X, Y be objects of Sch_{fppf}. Let $f : X \to Y$ be a morphism of schemes. Let $\mathcal{P}$ be as in Definition 47.5.1. Then $h_X \longrightarrow h_Y$ has property $\mathcal{P}$ if and only if f has property $\mathcal{P}$.*

Proof. Note that the lemma makes sense by Lemma 47.3.1. Proof omitted. □

Lemma 47.5.4. *Let S be a scheme contained in Sch_{fppf}. Let $F, G, H : (Sch/S)^{opp}_{fppf} \to Sets$. Let $\mathcal{P}$ be a property as in Definition 47.5.1 which is stable under composition. Let $a : F \to G$, $b : G \to H$ be representable transformations of functors. If a and b have property $\mathcal{P}$ so does $b \circ a : F \longrightarrow H$.*

Proof. Note that the lemma makes sense by Lemma 47.3.2. Proof omitted. □

Lemma 47.5.5. *Let S be a scheme contained in Sch_{fppf}. Let $F, G, H : (Sch/S)^{opp}_{fppf} \to Sets$. Let $\mathcal{P}$ be a property as in Definition 47.5.1. Let $a : F \to G$ be a representable transformations of functors. Let $b : H \to G$ be any transformation of functors. Consider the fibre product diagram*

$$\begin{array}{ccc} H \times_{b,G,a} F & \xrightarrow{b'} & F \\ {\scriptstyle a'}\downarrow & & \downarrow{\scriptstyle a} \\ H & \xrightarrow{b} & G \end{array}$$

If a has property $\mathcal{P}$ then also the base change a' has property $\mathcal{P}$.

Proof. Note that the lemma makes sense by Lemma 47.3.3. Proof omitted. □

Lemma 47.5.6. *Let S be a scheme contained in Sch_{fppf}. Let $F, G, H : (Sch/S)^{opp}_{fppf} \to Sets$. Let $\mathcal{P}$ be a property as in Definition 47.5.1. Let $a : F \to G$ be a representable transformations of functors. Let $b : H \to G$ be any transformation of functors. Consider the fibre product diagram*

$$\begin{array}{ccc} H \times_{b,G,a} F & \xrightarrow{b'} & F \\ {\scriptstyle a'}\downarrow & & \downarrow{\scriptstyle a} \\ H & \xrightarrow{b} & G \end{array}$$

Assume that b induces a surjective map of fppf sheaves $H^\# \to G^\#$. In this case, if a' has property $\mathcal{P}$, then also a has property $\mathcal{P}$.

Proof. First we remark that by Lemma 47.3.3 the transformation a' is representable. Let $U \in \mathrm{Ob}((Sch/S)_{fppf})$, and let $\xi \in G(U)$. By assumption there exists an fppf covering $\{U_i \to U\}_{i \in I}$ and elements $\xi_i \in H(U_i)$ mapping to $\xi|_U$ via b. From general category theory it follows that for each i we have a fibre product diagram

$$\begin{array}{ccc} U_i \times_{\xi_i,H,a'} (H \times_{b,G,a} F) & \longrightarrow & U \times_{\xi,G,a} F \\ \downarrow & & \downarrow \\ U_i & \longrightarrow & U \end{array}$$

By assumption the left vertical arrow is a morphism of schemes which has property $\mathcal{P}$. Since $\mathcal{P}$ is local in the fppf topology this implies that also the right vertical arrow has property $\mathcal{P}$ as desired. □

Lemma 47.5.7. *Let S be a scheme contained in Sch_{fppf}. Let $F_i, G_i : (Sch/S)^{opp}_{fppf} \to Sets$, $i = 1, 2$. Let $a_i : F_i \to G_i$, $i = 1, 2$ be representable transformations of functors. Let $\mathcal{P}$ be a property as in Definition 47.5.1 which is stable under composition. If a_1 and a_2 have property $\mathcal{P}$ so does $a_1 \times a_2 : F_1 \times F_2 \longrightarrow G_1 \times G_2$.*

Proof. Note that the lemma makes sense by Lemma 47.3.4. Proof omitted. □

Lemma 47.5.8. *Let S be a scheme contained in Sch_{fppf}. Let $F, G : (Sch/S)^{opp}_{fppf} \to Sets$. Let $a : F \to G$ be a representable transformation of functors. Let $\mathcal{P}$, $\mathcal{P}'$ be properties as in Definition 47.5.1. Suppose that for any morphism of schemes $f : X \to Y$ we have $\mathcal{P}(f) \Rightarrow \mathcal{P}'(f)$. If a has property $\mathcal{P}$ then a has property $\mathcal{P}'$.*

Proof. Formal. □

Lemma 47.5.9. *Let S be a scheme. Let $F, G : (Sch/S)^{opp}_{fppf} \to Sets$ be sheaves. Let $a : F \to G$ be representable, flat, locally of finite presentation, and surjective. Then $a : F \to G$ is surjective as a map of sheaves.*

Proof. Let T be a scheme over S and let $g : T \to G$ be a T-valued point of G. By assumption $T' = F \times_G T$ is (representable by) a scheme and the morphism $T' \to T$ is a flat, locally of finite presentation, and surjective. Hence $\{T' \to T\}$ is an fppf covering such that $g|_{T'} \in G(T')$ comes from an element of $F(T')$, namely the map $T' \to F$. This proves the map is surjective as a map of sheaves, see Sites, Definition 7.12.1. □

Here is a characterization of those functors for which the diagonal is representable.

Lemma 47.5.10. *Let S be a scheme contained in Sch_{fppf}. Let F be a presheaf of sets on $(Sch/S)_{fppf}$. The following are equivalent:*

1. *The diagonal $F \to F \times F$ is representable.*
2. *For every scheme U over S, $U/S \in \mathrm{Ob}((Sch/S)_{fppf})$ and any $\xi \in F(U)$ the map $\xi : h_U \to F$ is representable.*

Proof. This is completely formal, see Categories, Lemma 4.8.4. It depends only on the fact that the category $(Sch/S)_{fppf}$ has products of pairs of objects and fibre products, see Topologies, Lemma 33.7.10. □

In the situation of the lemma, for any morphism $\xi : h_U \to F$ as in the lemma, it makes sense to say that ξ has property $\mathcal{P}$, for any property as in Definition 47.5.1. In particular this holds for $\mathcal{P}$ = "surjective" and $\mathcal{P}$ = "étale", see Remark 47.4.3 above. We will use these in the definition of algebraic spaces below.

47.6. Algebraic spaces

Here is the definition.

Definition 47.6.1. Let S be a scheme contained in Sch_{fppf}. An *algebraic space over S* is a presheaf

$$F : (Sch/S)^{opp}_{fppf} \longrightarrow Sets$$

with the following properties

(1) The presheaf F is a sheaf.
(2) The diagonal morphism $F \to F \times F$ is representable.
(3) There exists a scheme $U \in \mathrm{Ob}(Sch_{fppf})$ and a map $h_U \to F$ which is surjective, and étale.

There are two differences with the "usual" definition, for example the definition in Knutson's book [**Knu71**].

The first is that we require F to be a sheaf in the fppf topology. One reason for doing this is that many natural examples of algebraic spaces satisfy the sheaf condition for the fppf coverings (and even for fpqc coverings). Also, one of the reasons that algebraic spaces have been so useful is via Michael Artin's results on algebraic spaces. Built into his method is a condition which guarantees the result is locally of finite presentation over S. Combined it somehow seems to us that the fppf topology is the natural topology to work with. In the end the category of algebraic spaces ends up being the same. See Bootstrap, Section 61.12.

The second is that we only require the diagonal map for F to be representable, whereas in [**Knu71**] it is required that it also be quasi-compact. If $F = h_U$ for some scheme U over S this corresponds to the condition that S be quasi-separated. Our point of view is to try to prove a certain number of the results that follow only assuming that the diagonal of F be representable, and simply add an addition hypothesis wherever this is necessary. In any case it has the pleasing consequence that the following lemma is true.

Lemma 47.6.2. *A scheme is an algebraic space. More precisely, given a scheme $T \in \mathrm{Ob}((Sch/S)_{fppf})$ the representable functor h_T is an algebraic space.*

Proof. The functor h_T is a sheaf by our remarks in Section 47.2. The diagonal $h_T \to h_T \times h_T = h_{T \times T}$ is representable because $(Sch/S)_{fppf}$ has fibre products. The identity map $h_T \to h_T$ is surjective étale. □

Definition 47.6.3. Let F, F' be algebraic spaces over S. A *morphism $f : F \to F'$ of algebraic spaces over S* is a transformation of functors from F to F'.

The category of algebraic spaces over S contains the category $(Sch/S)_{fppf}$ as a full subcategory via the Yoneda embedding $T/S \mapsto h_T$. From now on we no longer distinguish between a scheme T/S and the algebraic space it represents. Thus when we say "Let $f : T \to F$ be a morphism from the scheme T to the algebraic space F", we mean that $T \in \mathrm{Ob}((Sch/S)_{fppf})$, that F is an algebraic space over S, and that $f : h_T \to F$ is a morphism of algebraic spaces over S.

47.7. Fibre products of algebraic spaces

The category of algebraic spaces over S has both products and fibre products.

Lemma 47.7.1. *Let S be a scheme contained in Sch_{fppf}. Let F, G be algebraic spaces over S. Then $F \times G$ is an algebraic space, and is a product in the category of algebraic spaces over S.*

Proof. It is clear that $H = F \times G$ is a sheaf. The diagonal of H is simply the product of the diagonals of F and G. Hence it is representable by Lemma 47.3.4. Finally, if $U \to F$ and $V \to G$ are surjective étale morphisms, with $U, V \in \mathrm{Ob}((Sch/S)_{fppf})$, then $U \times V \to F \times G$ is surjective étale by Lemma 47.5.7. □

Lemma 47.7.2. *Let S be a scheme contained in Sch_{fppf}. Let H be a sheaf on $(Sch/S)_{fppf}$ whose diagonal is representable. Let F, G be algebraic spaces over S. Let $F \to H$, $G \to H$ be maps of sheaves. Then $F \times_H G$ is an algebraic space.*

Proof. We check the 3 conditions of Definition 47.6.1. A fibre product of sheaves is a sheaf, hence $F \times_H G$ is a sheaf. The diagonal of $F \times_H G$ is the left vertical arrow in

$$\begin{array}{ccc} F \times_H G & \longrightarrow & F \times G \\ {\scriptstyle\Delta}\downarrow & & \downarrow{\scriptstyle\Delta_F \times \Delta_G} \\ (F \times F) \times_{(H \times H)} (G \times G) & \longrightarrow & (F \times F) \times (G \times G) \end{array}$$

which is cartesian. Hence Δ is representable as the base change of the morphism on the right which is representable, see Lemmas 47.3.4 and 47.3.3. Finally, let $U, V \in \mathrm{Ob}((Sch/S)_{fppf})$ and $a : U \to F$, $b : V \to G$ be surjective and étale. As Δ_H is representable, we see that $U \times_H V$ is a scheme. The morphism

$$U \times_H V \longrightarrow F \times_H G$$

is surjective and étale as a composition of the base changes $U \times_H V \to U \times_H G$ and $U \times_H G \to F \times_H G$ of the étale surjective morphisms $U \to F$ and $V \to G$, see Lemmas 47.3.2 and 47.3.3. This proves the last condition of Definition 47.6.1 holds and we conclude that $F \times_H G$ is an algebraic space. □

Lemma 47.7.3. *Let S be a scheme contained in Sch_{fppf}. Let $F \to H$, $G \to H$ be morphisms of algebraic spaces over S. Then $F \times_H G$ is an algebraic space, and is a fibre product in the category of algebraic spaces over S.*

Proof. It follows from the stronger Lemma 47.7.2 that $F \times_H G$ is an algebraic space. It is clear that $F \times_H G$ is a fibre product in the category of algebraic spaces over S since that is a full subcategory of the category of (pre)sheaves of sets on $(Sch/S)_{fppf}$. □

47.8. Glueing algebraic spaces

In this section we really start abusing notation and not distinguish between schemes and the spaces they represent.

Lemma 47.8.1. *Let $S \in \mathrm{Ob}(Sch_{fppf})$. Let $U \in \mathrm{Ob}((Sch/S)_{fppf})$. Given a set I and sheaves F_i on $\mathrm{Ob}((Sch/S)_{fppf})$, if $U \cong \coprod_{i \in I} F_i$ as sheaves, then each F_i is representable by an open and closed subscheme U_i and $U \cong \coprod U_i$ as schemes.*

Proof. By assumption this means there exists an fppf covering $\{U_j \to U\}_{j \in J}$ such that each $U_j \to U$ factors through $F_{i(j)}$ for some $i(j) \in I$. Denote $V_j = \mathrm{Im}(U_j \to U)$. This is an open of U by Morphisms, Lemma 28.26.9, and $\{U_j \to V_j\}$ is an fppf covering. Hence it follows that $V_j \to U$ factors through $F_{i(j)}$ since $F_{i(j)}$ is a subsheaf. It follows from $F_i \cap F_{i'} = \emptyset$, $i \neq i'$ that $V_j \cap V_{j'} = \emptyset$ unless $i(j) = i(j')$. Hence we can take $U_i = \bigcup_{j,\ i(j)=i} V_j$ and everything is clear. □

Lemma 47.8.2. *Let $S \in \mathrm{Ob}(Sch_{fppf})$. Let F be an algebraic space over S. Given a set I and sheaves F_i on $\mathrm{Ob}((Sch/S)_{fppf})$, if $F \cong \coprod_{i \in I} F_i$ as sheaves, then each F_i is an algebraic space over S.*

Proof. It follows directly from the representability of $F \to F \times F$ that each diagonal morphism $F_i \to F_i \times F_i$ is representable. Choose a scheme U in $(Sch/S)_{fppf}$ and a surjective étale morphism $U \to \coprod F_i$ (this exist by hypothesis). By considering the inverse image of F_i we get a decomposition of U (as a sheaf) into a coproduct of sheaves. By Lemma 47.8.1 we get correspondingly $U \cong \coprod U_i$. Then it follows easily that $U_i \to F_i$ is surjective and étale (from the corresponding property of $U \to F$). □

The condition on the size of I and the F_i in the following lemma may be ignored by those not worried about set theoretic questions.

Lemma 47.8.3. *Let $S \in \mathrm{Ob}(Sch_{fppf})$. Suppose given a set I and algebraic spaces F_i, $i \in I$. Then $F = \coprod_{i \in I} F_i$ is an algebraic space provided I, and the F_i are not too "large": for example if we can choose surjective étale morphisms $U_i \to F_i$ such that $\coprod_{i \in I} U_i$ is isomorphic to an object of $(Sch/S)_{fppf}$, then F is an algebraic space.*

Proof. By construction F is a sheaf. We omit the verification that the diagonal morphism of F is representable. Finally, if U is an object of $(Sch/S)_{fppf}$ isomorphic to $\coprod_{i \in I} U_i$ then it is straightforward to verify that the resulting map $U \to \coprod F_i$ is surjective and étale. □

Here is the analogue of Schemes, Lemma 25.15.4.

Lemma 47.8.4. *Let $S \in \mathrm{Ob}(Sch_{fppf})$. Let F be a presheaf of sets on $(Sch/S)_{fppf}$. Assume*

1. *F is a sheaf,*
2. *there exists an index set I and subfunctors $F_i \subset F$ such that*
   (a) *each F_i is an algebraic space,*
   (b) *each $F_i \to F$ is a representable,*
   (c) *each $F_i \to F$ is an open immersion (see Definition 47.5.1),*
   (d) *the map of sheaves $\coprod F_i \to F$ is surjective, and*
   (e) *$\coprod F_i$ is an algebraic space (set theoretic condition, see Lemma 47.8.3).*

Then F is an algebraic space.

Proof. Let T, T' be objects of $(Sch/S)_{fppf}$. Let $T \to F$, $T' \to F$ morphisms. The assumptions imply that there exists an open covering $T = \bigcup V_i$ such that $V_i = T \times_F F_i$. Note that this in particular implies that $\coprod F_i \to F$ is surjective in the Zariski topology! Also write similarly $T' = \bigcup V_i'$ with $V_i' = T' \times_F F_i$.

To show that the diagonal $F \to F \times F$ is representable we have to show that $G = T \times_F T'$ is representable. Consider the subfunctors $G_i = G \times_F F_i$. Note that $G_i = V_i \times_{F_i} V_i'$, and hence is representable as F_i is an algebraic space. By the above the G_i form a Zariski covering of F. Hence by Schemes, Lemma 25.15.4 we see G is representable.

Choose a scheme $U \in \mathrm{Ob}((Sch/S)_{fppf})$ and a surjective étale morphism $U \to \coprod F_i$ (this exist by hypothesis). We may write $U = \coprod U_i$ with U_i the inverse image of F_i, see Lemma 47.8.1. We claim that $U \to F$ is surjective and étale. Surjectivity follows as $\coprod F_i \to F$ is surjective. Consider the fibre product $U \times_F T$ where $T \to F$ is as above. We have to show that $U \times_F T \to T$ is étale. Since $U \times_F T = \coprod U_i \times_F T$ it suffices to show each $U_i \times_F T \to T$ is étale. Since $U_i \times_F T = U_i \times_{F_i} V_i$ this

follows from the fact that $U_i \to F_i$ is étale and $V_i \to T$ is an open immersion (and Morphisms, Lemmas 28.37.9 and 28.37.3). □

47.9. Presentations of algebraic spaces

Given an algebraic space we can find a "presentation" of it.

Lemma 47.9.1. *Let F be an algebraic space over S. Let $f : U \to F$ be a surjective étale morphism from a scheme to F. Set $R = U \times_F U$. Then*

1. *$j : R \to U \times_S U$ defines an equivalence relation on U over S (see Groupoids, Definition 38.3.1).*
2. *the morphisms $s, t : R \to U$ are étale, and*
3. *the diagram*
$$R \rightrightarrows U \longrightarrow F$$
is a coequalizer diagram in $Sh((Sch/S)_{fppf})$.

Proof. Let T/S be an object of $(Sch/S)_{fppf}$. Then $R(T) = \{(a, b) \in U(T) \times U(T) \mid f \circ a = f \circ b\}$ which is clearly defines an equivalence relation on $U(T)$. The morphisms $s, t : R \to U$ are étale because the morphism $U \to F$ is étale.

To prove (3) we first show that $U \to F$ is a surjection of sheaves, see Sites, Definition 7.12.1. Let $\xi \in F(T)$ with T as above. Let $V = T \times_{\xi, F, f} U$. By assumption V is a scheme and $V \to T$ is surjective étale. Hence $\{V \to T\}$ is a covering for the fppf topology. Since $\xi|_V$ factors through U by construction we conclude $U \to F$ is surjective. Surjectivity implies that F is the coequalizer of the diagram by Sites, Lemma 7.12.3. □

This lemma suggests the following definitions.

Definition 47.9.2. Let S be a scheme. Let U be a scheme over S. An *étale equivalence relation* on U over S is an equivalence relation $j : R \to U \times_S U$ such that $s, t : R \to U$ are étale morphisms of schemes.

Definition 47.9.3. Let F be an algebraic space over S. A *presentation* of F is given by a scheme U over S and an étale equivalence relation R on U over S, and a surjective étale morphism $U \to F$ such that $R = U \times_F U$.

Equivalently we could ask for the existence of an isomorphism

$$U/R \cong F$$

where the quotient U/R is as defined in Groupoids, Section 38.18. To construct algebraic spaces we will study the converse question, namely, for which equivalence relations the quotient sheaf U/R is an algebraic space. It will finally turn out this is always the case if R is an étale equivalence relation on U over S, see Theorem 47.10.5.

47.10. Algebraic spaces and equivalence relations

Suppose given a scheme U over S and an étale equivalence relation R on U over S. We would like to show this defines an algebraic space. We will produce a series of lemmas that prove the quotient sheaf U/R (see Groupoids, Definition 38.18.1) has all the properties required of it in Definition 47.6.1.

Lemma 47.10.1. *Let S be a scheme. Let U be a scheme over S. Let $j = (s, t) : R \to U \times_S U$ be an étale equivalence relation on U over S. Let $U' \to U$ be an étale morphism. Let R' be the restriction of R to U', see Groupoids, Definition 38.3.3. Then $j' : R' \to U' \times_S U'$ is an étale equivalence relation also.*

Proof. It is clear from the description of s', t' in Groupoids, Lemma 38.16.1 that $s', t' : R' \to U'$ are étale as compositions of base changes of étale morphisms (see Morphisms, Lemma 28.37.4 and 28.37.3). □

We will often use the following lemma to find open subspaces of algebraic spaces. A slight improvement (with more general hypotheses) of this lemma is Bootstrap, Lemma 61.7.1.

Lemma 47.10.2. *Let S be a scheme. Let U be a scheme over S. Let $j = (s, t) : R \to U \times_S U$ be a pre-relation. Let $g : U' \to U$ be a morphism. Assume*

(1) *j is an equivalence relation,*
(2) *$s, t : R \to U$ are surjective, flat and locally of finite presentation,*
(3) *g is flat and locally of finite presentation.*

Let $R' = R|_{U'}$ be the restriction of R to U. Then $R'/U' \to R/U$ is representable, and is an open immersion.

Proof. By Groupoids, Lemma 38.3.2 the morphism $j' = (t', s') : R' \to U' \times_S U'$ defines an equivalence relation. Since g is flat and locally of finite presentation we see that g is universally open as well (Morphisms, Lemma 28.26.9). For the same reason s, t are universally open as well. Let $W^1 = g(U') \subset U$, and let $W = t(s^{-1}(W^1))$. Then W^1 and W are open in U. Moreover, as j is an equivalence relation we have $t(s^{-1}(W)) = W$ (see Groupoids, Lemma 38.17.2 for example).

By Groupoids, Lemma 38.18.5 the map of sheaves $F' = U'/R' \to F = U/R$ is injective. Let $a : T \to F$ be a morphism from a scheme into U/R. We have to show that $T \times_F F'$ is representable by an open subscheme of T.

The morphism a is given by the following data: an fppf covering $\{\varphi_j : T_j \to T\}_{j \in J}$ of T and morphisms $a_j : T_j \to U$ such that the maps

$$a_j \times a_{j'} : T_j \times_T T_{j'} \longrightarrow U \times_S U$$

factor through $j : R \to U \times_S U$ via some (unique) maps $r_{jj'} : T_j \times_T T_{j'} \to R$. The system (a_j) corresponds to a in the sense that the diagrams

$$\begin{array}{ccc} T_j & \xrightarrow{a_j} & U \\ \downarrow & & \downarrow \\ T & \xrightarrow{a} & F \end{array}$$

commute.

Consider the open subsets $W_j = a_j^{-1}(W) \subset T_j$. Since $t(s^{-1}(W)) = W$ we see that

$$W_j \times_T T_{j'} = r_{jj'}^{-1}(t^{-1}(W)) = r_{jj'}^{-1}(s^{-1}(W)) = T_j \times_T W_{j'}.$$

By Descent, Lemma 34.9.2 this means there exists an open $W_T \subset T$ such that $\varphi_j^{-1}(W_T) = W_j$ for all $j \in J$. We claim that $W_T \to T$ represents $T \times_F F' \to T$.

First, let us show that $W_T \to T \to F$ is an element of $F'(W_T)$. Since $\{W_j \to W_T\}_{j \in J}$ is an fppf covering of W_T, it is enough to show that each $W_j \to U \to F$

is an element of $F'(W_j)$ (as F' is a sheaf for the fppf topology). Consider the commutative diagram

$$\begin{array}{ccccc} W'_j & \longrightarrow & & & U' \\ \downarrow & \searrow & & & \downarrow{\scriptstyle g} \\ & & s^{-1}(W^1) & \xrightarrow[s]{} & W^1 \\ & & \downarrow{\scriptstyle t} & & \downarrow \\ W_j & \xrightarrow{a_j|_{W_j}} & W & \longrightarrow & F \end{array}$$

where $W'_j = W_j \times_W s^{-1}(W^1) \times_{W^1} U'$. Since t and g are surjective, flat and locally of finite presentation, so is $W'_j \to W_j$. Hence the restriction of the element $W_j \to U \to F$ to W'_j is an element of F' as desired.

Suppose that $f : T' \to T$ is a morphism of schemes such that $a|_{T'} \in F'(T')$. We have to show that f factors through the open W_T. Since $\{T' \times_T T_j \to T\}$ is an fppf covering of T' it is enough to show each $T' \times_T T_j \to T$ factors through W_T. Hence we may assume f factors as $\varphi_j \circ f_j : T' \to T_j \to T$ for some j. In this case the condition $a|_{T'} \in F'(T')$ means that there exists some fppf covering $\{\psi_i : T'_i \to T'\}_{i \in I}$ and some morphisms $b_i : T'_i \to U'$ such that

$$\begin{array}{ccccc} T'_i & \xrightarrow[b_i]{} & U' & \xrightarrow[g]{} & U \\ {\scriptstyle f_j \circ \psi_i}\downarrow & & & & \downarrow \\ T_j & \xrightarrow{a_j} & U & \longrightarrow & F \end{array}$$

is commutative. This commutativity means that there exists a morphism $r'_i : T'_i \to R$ such that $t \circ r'_i = a_j \circ f_j \circ \psi_i$, and $s \circ r'_i = g \circ b_i$. This implies that $\mathrm{Im}(f_j \circ \psi_i) \subset W_j$ and we win. □

The following lemma is not completely trivial although it looks like it should be trivial.

Lemma 47.10.3. *Let S be a scheme. Let U be a scheme over S. Let $j = (s, t) : R \to U \times_S U$ be an étale equivalence relation on U over S. If the quotient U/R is an algebraic space, then $U \to U/R$ is étale and surjective. Hence $(U, R, U \to U/R)$ is a presentation of the algebraic space U/R.*

Proof. Denote $c : U \to U/R$ the morphism in question. Let T be a scheme and let $a : T \to U/R$ be a morphism. We have to show that the morphism (of schemes) $\pi : T \times_{a,R/U,c} U \to T$ is étale and surjective. The morphism a corresponds to an fppf covering $\{\varphi_i : T_i \to T\}$ and morphisms $a_i : T_i \to U$ such that $a_i \times a_{i'} : T_i \times_T T_{i'} \to U \times_S U$ factors through R, and such that $c \circ a_i = \varphi_i \circ a$. Hence

$$T_i \times_{\varphi_i,T} T \times_{a,R/U,c} U = T_i \times_{c \circ a_i,R/U,c} U = T_i \times_{a_i,U} U \times_{c,R/U,c} U = T_i \times_{a_i,U,t} R.$$

Since t is étale and surjective we conclude that the base change of π to T_i is surjective and étale. Since the property of being surjective and étale is local on the base in the fpqc topology (see Remark 47.4.3) we win. □

Lemma 47.10.4. *Let S be a scheme. Let U be a scheme over S. Let $j = (s, t) : R \to U \times_S U$ be an étale equivalence relation on U over S. Assume that U is affine. Then the quotient $F = U/R$ is an algebraic space, and $U \to F$ is étale and surjective.*

Proof. Since $j : R \to U \times_S U$ is a monomorphism we see that j is separated (see Schemes, Lemma 25.23.3). Since U is affine we see that $U \times_S U$ (which comes equipped with a monomorphism into the affine scheme $U \times U$) is separated. Hence we see that R is separated. In particular the morphisms s, t are separated as well as étale.

Since the composition $R \to U \times_S U \to U$ is locally of finite type we conclude that j is locally of finite type (see Morphisms, Lemma 28.16.8). As j is also a monomorphism it has finite fibres and we see that j is locally quasi-finite by Morphisms, Lemma 28.21.7. Altogether we see that j is separated and locally quasi-finite.

Our first step is to show that the quotient map $c : U \to F$ is representable. Consider a scheme T and a morphism $a : T \to F$. We have to show that the sheaf $G = T \times_{a,F,c} U$ is representable. As seen in the proofs of Lemmas 47.10.2 and 47.10.3 there exists an fppf covering $\{\varphi_i : T_i \to T\}_{i \in I}$ and morphisms $a_i : T_i \to U$ such that $a_i \times a_{i'} : T_i \times_T T_{i'} \to U \times_S U$ factors through R, and such that $c \circ a_i = \varphi_i \circ a$. As in the proof of Lemma 47.10.3 we see that

$$\begin{aligned} T_i \times_{\varphi_i, T} G &= T_i \times_{\varphi_i, T} T \times_{a, R/U, c} U \\ &= T_i \times_{c \circ a_i, R/U, c} U \\ &= T_i \times_{a_i, U} U \times_{c, R/U, c} U \\ &= T_i \times_{a_i, U, t} R \end{aligned}$$

Since t is separated and étale, and in particular separated and locally quasi-finite (by Morphisms, Lemmas 28.36.10 and 28.37.16) we see that the restriction of G to each T_i is representable by a morphism of schemes $X_i \to T_i$ which is separated and locally quasi-finite. By Descent, Lemma 34.35.1 we obtain a descent datum $(X_i, \varphi_{ii'})$ relative to the fppf-covering $\{T_i \to T\}$. Since each $X_i \to T_i$ is separated and locally quasi-finite we see by More on Morphisms, Lemma 36.37.1 that this descent datum is effective. Hence by Descent, Lemma 34.35.1 (2) we conclude that G is representable as desired.

The second step of the proof is to show that $U \to F$ is surjective and étale. This is clear from the above since in the first step above we saw that $G = T \times_{a,F,c} U$ is a scheme over T which base changes to schemes $X_i \to T_i$ which are surjective and étale. Thus $G \to T$ is surjective and étale (see Remark 47.4.3). Alternatively one can reread the proof of Lemma 47.10.3 in the current situation.

The third and final step is to show that the diagonal map $F \to F \times F$ is representable. We first observe that the diagram

$$\begin{array}{ccc} R & \longrightarrow & F \\ {\scriptstyle j}\downarrow & & \downarrow{\scriptstyle \Delta} \\ U \times_S U & \longrightarrow & F \times F \end{array}$$

is a fibre product square. By Lemma 47.3.4 the morphism $U \times_S U \to F \times F$ is representable (note that $h_U \times h_U = h_{U \times_S U}$). Moreover, by Lemma 47.5.7 the

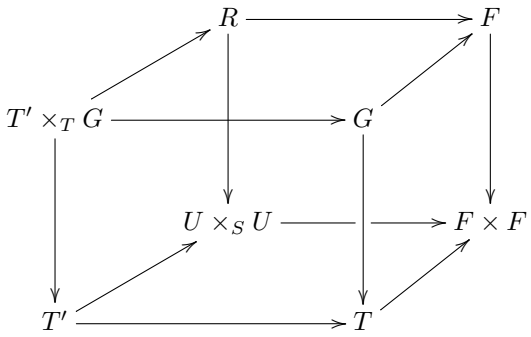
morphism $U \times_S U \to F \times F$ is surjective and étale (note also that étale and surjective occur in the lists of Remarks 47.4.3 and 47.4.2). It follows either from Lemma 47.3.3 and the diagram above, or by writing $R \to F$ as $R \to U \to F$ and Lemmas 47.3.1 and 47.3.2 that $R \to F$ is representable as well. Let T be a scheme and let $a : T \to F \times F$ be a morphism. We have to show that $G = T \times_{a,F\times F,\Delta} F$ is representable. By what was said above the morphism (of schemes)

$$T' = (U \times_S U) \times_{F\times F,a} T \longrightarrow T$$

is surjective and étale. Hence $\{T' \to T\}$ is an étale covering of T. Note also that

$$T' \times_T G = T' \times_{U\times_S U,j} R$$

as can be seen contemplating the following cube



Hence we see that the restriction of G to T' is representable by a scheme X, and moreover that the morphism $X \to T'$ is a base change of the morphism j. Hence $X \to T'$ is separated and locally quasi-finite (see second paragraph of the proof). By Descent, Lemma 34.35.1 we obtain a descent datum (X, φ) relative to the fppf-covering $\{T' \to T\}$. Since $X \to T$ is separated and locally quasi-finite we see by More on Morphisms, Lemma 36.37.1 that this descent datum is effective. Hence by Descent, Lemma 34.35.1 (2) we conclude that G is representable as desired. □

Theorem 47.10.5. *Let S be a scheme. Let U be a scheme over S. Let $j = (s, t) : R \to U \times_S U$ be an étale equivalence relation on U over S. Then the quotient U/R is an algebraic space, and $U \to U/R$ is étale and surjective, in other words $(U, R, U \to U/R)$ is a presentation of U/R.*

Proof. By Lemma 47.10.3 it suffices to prove that U/R is an algebraic space. Let $U' \to U$ be a surjective, étale morphism. Then $\{U' \to U\}$ is in particular an fppf covering. Let R' be the restriction of R to U', see Groupoids, Definition 38.3.3. According to Groupoids, Lemma 38.18.6 we see that $U/R \cong U'/R'$. By Lemma 47.10.1 R' is an étale equivalence relation on U'. Thus we may replace U by U'.

We apply the previous remark to $U' = \coprod U_i$, where $U = \bigcup U_i$ is an affine open covering of S. Hence we may and do assume that $U = \coprod U_i$ where each U_i is an affine scheme.

Consider the restriction R_i of R to U_i. By Lemma 47.10.1 this is an étale equivalence relation. Set $F_i = U_i/R_i$ and $F = U/R$. It is clear that $\coprod F_i \to F$ is surjective. By Lemma 47.10.2 each $F_i \to F$ is representable, and an open immersion. By Lemma 47.10.4 applied to (U_i, R_i) we see that F_i is an algebraic space. Then by Lemma 47.10.3 we see that $U_i \to F_i$ is étale and surjective. From Lemma 47.8.3 it follows

that $\coprod F_i$ is an algebraic space. Finally, we have verified all hypotheses of Lemma 47.8.4 and it follows that $F = U/R$ is an algebraic space. □

47.11. Algebraic spaces, retrofitted

We start building our arsenal of lemmas dealing with algebraic spaces.

Lemma 47.11.1. *Let S be a scheme contained in Sch_{fppf}. Let F be an algebraic space over S. Let $G \to F$ be a representable transformation of functors. Then G is an algebraic space.*

Proof. By Lemma 47.3.5 we see that G is a sheaf. The diagram

$$\begin{array}{ccc} G \times_F G & \longrightarrow & F \\ \downarrow & & \downarrow{\scriptstyle \Delta_F} \\ G \times G & \longrightarrow & F \times F \end{array}$$

is cartesian. Hence we see that $G \times_F G \to G \times G$ is representable by Lemma 47.3.3. By Lemma 47.3.6 we see that $G \to G \times_F G$ is representable. Hence $\Delta_G : G \to G \times G$ is representable as a composition of representable transformations, see Lemma 47.3.2. Finally, let U be an object of $(Sch/S)_{fppf}$ and let $U \to F$ be surjective and étale. By assumption $U \times_F G$ is representable by a scheme U'. By Lemma 47.5.5 the morphism $U' \to G$ is surjective and étale. This verifies the final condition of Definition 47.6.1 and we win. □

Lemma 47.11.2. *Let S be a scheme contained in Sch_{fppf}. Let F, G be algebraic spaces over S. Let $G \to F$ be a representable morphism. Let $U \in \mathrm{Ob}((Sch/S)_{fppf})$, and $q : U \to F$ surjective and étale. Set $V = G \times_F U$. Finally, let $\mathcal{P}$ be a property of morphisms of schemes as in Definition 47.5.1. Then $G \to F$ has property $\mathcal{P}$ if and only if $V \to U$ has property $\mathcal{P}$.*

Proof. (This lemma follows from Lemmas 47.5.5 and 47.5.6, but we give a direct proof here also.) It is clear from the definitions that if $G \to F$ has property $\mathcal{P}$, then $V \to U$ has property $\mathcal{P}$. Conversely, assume $V \to U$ has property $\mathcal{P}$. Let $T \to F$ be a morphism from a scheme to F. Let $T' = T \times_F G$ which is a scheme since $G \to F$ is representable. We have to show that $T' \to T$ has property $\mathcal{P}$. Consider the commutative diagram of schemes

$$\begin{array}{ccccc} V & \longleftarrow & T \times_F V & \longrightarrow & T \times_F G & = & T' \\ \downarrow & & \downarrow & & \downarrow & & \\ U & \longleftarrow & T \times_F U & \longrightarrow & T & & \end{array}$$

where both squares are fibre product squares. Hence we conclude the middle arrow has property $\mathcal{P}$ as a base change of $V \to U$. Finally, $\{T \times_F U \to T\}$ is a fppf covering as it is surjective étale, and hence we conclude that $T' \to T$ has property $\mathcal{P}$ as it is local on the base in the fppf topology. □

Lemma 47.11.3. *Let S be a scheme contained in Sch_{fppf}. Let $G \to F$ be a transformation of presheaves on $(Sch/S)_{fppf}$. Let $\mathcal{P}$ be a property of morphisms of schemes. Assume*

(1) $\mathcal{P}$ *is preserved under any base change, fppf local on the base, and morphisms of type* $\mathcal{P}$ *satisfy descent for fppf coverings, see Descent, Definition 34.32.1,*
(2) G *is a sheaf,*
(3) F *is an algebraic space,*
(4) *there exists a* $U \in \mathrm{Ob}((Sch/S)_{fppf})$ *and a surjective étale morphism* $U \to F$ *such that* $V = G \times_F U$ *is representable, and*
(5) $V \to U$ *has* $\mathcal{P}$.

Then G *is an algebraic space,* $G \to F$ *is representable and has property* $\mathcal{P}$.

Proof. Let $R = U \times_F U$, and denote $t, s : R \to U$ the projection morphisms as usual. Let T be a scheme and let $T \to F$ be a morphism. Then $U \times_F T \to T$ is surjective étale, hence $\{U \times_F T \to T\}$ is a covering for the étale topology. Consider

$$W = G \times_F (U \times_F T) = V \times_F T = V \times_U (U \times_F T).$$

It is a scheme since F is an algebraic space. The morphism $W \to U \times_F T$ has property $\mathcal{P}$ since it is a base change of $V \to U$. There is an isomorphism

$$\begin{aligned} W \times_T (U \times_F T) &= (G \times_F (U \times_F T)) \times_T (U \times_F T) \\ &= (U \times_F T) \times_T (G \times_F (U \times_F T)) \\ &= (U \times_F T) \times_T W \end{aligned}$$

over $(U \times_F T) \times_T (U \times_F T)$. The middle equality maps $((g, (u_1, t)), (u_2, t))$ to $((u_1, t), (g, (u_2, t)))$. This defines a descent datum for $W/U \times_F T/T$, see Descent, Definition 34.30.1. This follows from Descent, Lemma 34.35.1. Namely we have a sheaf $G \times_F T$, whose base change to $U \times_F T$ is represented by W and the isomorphism above is the one from the proof of Descent, Lemma 34.35.1. By assumption on $\mathcal{P}$ the descent datum above is representable. Hence by the last statement of Descent, Lemma 34.35.1 we see that $G \times_F T$ is representable. This proves that $G \to F$ is a representable transformation of functors.

As $G \to F$ is representable, we see that G is an algebraic space by Lemma 47.11.1. The fact that $G \to F$ has property $\mathcal{P}$ now follows from Lemma 47.11.2. □

Lemma 47.11.4. *Let* S *be a scheme contained in* Sch_{fppf}. *Let* F, G *be algebraic spaces over* S. *Let* $a : F \to G$ *be a morphism. Given any* $V \in \mathrm{Ob}((Sch/S)_{fppf})$ *and a surjective étale morphism* $q : V \to G$ *there exists a* $U \in \mathrm{Ob}((Sch/S)_{fppf})$ *and a commutative diagram*

$$\begin{array}{ccc} U & \xrightarrow{\alpha} & V \\ {\scriptstyle p}\downarrow & & \downarrow{\scriptstyle q} \\ F & \xrightarrow{a} & G \end{array}$$

with p *surjective and étale.*

Proof. First choose $W \in \mathrm{Ob}((Sch/S)_{fppf})$ with surjective étale morphism $W \to F$. Next, put $U = W \times_G V$. Since G is an algebraic space we see that U is isomorphic to an object of $(Sch/S)_{fppf}$. As q is surjective étale, we see that $U \to W$ is surjective étale (see Lemma 47.5.5). Thus $U \to F$ is surjective étale as a composition of surjective étale morphisms (see Lemma 47.5.4). □

47.12. Immersions and Zariski coverings of algebraic spaces

At this point an interesting phenomenon occurs. We have already defined the notion of an open immersion of algebraic spaces (through Definition 47.5.1) but we have yet to define the notion of a *point*[1]. Thus the *Zariski topology* of an algebraic space has already been defined, but there is no space yet!

Perhaps superfluously we formally introduce immersions as follows.

Definition 47.12.1. Let $S \in \mathrm{Ob}(Sch_{fppf})$ be a scheme. Let F be an algebraic space over S.

(1) A morphism of algebraic spaces over S is called an *open immersion* if it is representable, and an open immersion in the sense of Definition 47.5.1.
(2) An *open subspace* of F is a subfunctor $F' \subset F$ such that F' is an algebraic space and $F' \to F$ is an open immersion.
(3) A morphism of algebraic spaces over S is called a *closed immersion* if it is representable, and a closed immersion in the sense of Definition 47.5.1.
(4) A *closed subspace* of F is a subfunctor $F' \subset F$ such that F' is an algebraic space and $F' \to F$ is a closed immersion.
(5) A morphism of algebraic spaces over S is called an *immersion* if it is representable, and an immersion in the sense of Definition 47.5.1.
(6) A *locally closed subspace* of F is a subfunctor $F' \subset F$ such that F' is an algebraic space and $F' \to F$ is an immersion.

We note that these definitions make sense since an immersion is in particular a monomorphism (see Schemes, Lemma 25.23.7 and Lemma 47.5.8), and hence the image of an immersion $G \to F$ of algebraic spaces is a subfunctor $F' \subset F$ which is (canonically) isomorphic to G. Thus some of the discussion of Schemes, Section 25.10 carries over to the setting of algebraic spaces.

Lemma 47.12.2. *Let $S \in \mathrm{Ob}(Sch_{fppf})$ be a scheme. A composition of (closed, resp. open) immersions of algebraic spaces over S is a (closed, resp. open) immersion of algebraic spaces over S.*

Proof. See Lemma 47.5.4 and Remarks 47.4.3 (see very last line of that remark) and 47.4.2. □

Lemma 47.12.3. *Let $S \in \mathrm{Ob}(Sch_{fppf})$ be a scheme. A base change of a (closed, resp. open) immersion of algebraic spaces over S is a (closed, resp. open) immersion of algebraic spaces over S.*

Proof. See Lemma 47.5.5 and Remark 47.4.3 (see very last line of that remark). □

Lemma 47.12.4. *Let $S \in \mathrm{Ob}(Sch_{fppf})$ be a scheme. Let F be an algebraic space over S. Let F_1, F_2 be locally closed subspaces of F. If $F_1 \subset F_2$ as subfunctors of F, then F_1 is a locally closed subspace of F_2. Similarly for closed and open subspaces.*

Proof. Let $T \to F_2$ be a morphism with T a scheme. Since $F_2 \to F$ is a monomorphism, we see that $T \times_{F_2} F_1 = T \times_F F_1$. The lemma follows formally from this. □

[1] We will associate a topological space to an algebraic space in Properties of Spaces, Section 48.4, and its opens will correspond exactly to the open subspaces defined below.

Let us formally define the notion of a Zariski open covering of algebraic spaces. Note that in Lemma 47.8.4 we have already encountered such open coverings as a method for constructing algebraic spaces.

Definition 47.12.5. Let $S \in \mathrm{Ob}(Sch_{fppf})$ be a scheme. Let F be an algebraic space over S. A *Zariski covering* $\{F_i \subset F\}_{i \in I}$ of F is given by a set I, a collection of open subspaces $F_i \subset F$ such that $\coprod F_i \to F$ is a surjective map of sheaves.

Note that if T is a schemes, and $a : T \to F$ is a morphism, then each of the fibre products $T \times_F F_i$ is identified with an open subscheme $T_i \subset T$. The final condition of the definition signifies exactly that $T = \bigcup_{i \in I} T_i$.

It is clear that the collection F_{Zar} of open subspaces of F is a set (as $(Sch/S)_{fppf}$ is a site, hence a set). Moreover, we can turn F_{Zar} into a category by letting the morphisms be inclusions of subfunctors (which are automatically open immersions by Lemma 47.12.4). Finally, Definition 47.12.5 provides the notion of a Zariski covering $\{F_i \to F'\}_{i \in I}$ in the category F_{Zar}. Hence, just as in the case of a topological space (see Sites, Example 7.6.4) by suitably choosing a set of coverings we may obtain a Zariski site of the algebraic space F.

Definition 47.12.6. Let $S \in \mathrm{Ob}(Sch_{fppf})$ be a scheme. Let F be an algebraic space over S. A *small Zariski site* F_{Zar} of an algebraic space F is one of the sites described above.

Hence this gives a notion of what it means for something to be true Zariski locally on an algebraic space, which is how we will use this notion. In general the Zariski topology is not fine enough for our purposes. For example we can consider the category of Zariski sheaves on an algebraic space. It will turn out that this is not the correct thing to consider, even for quasi-coherent sheaves. One only gets the desired result when using the étale or fppf site of F to define quasi-coherent sheaves.

47.13. Separation conditions on algebraic spaces

A separation condition on an algebraic space F is a condition on the diagonal morphism $F \to F \times F$. Let us first list the properties the diagonal has automatically. Since the diagonal is representable by definition the following lemma makes sense (through Definition 47.5.1).

Lemma 47.13.1. *Let S be a scheme contained in Sch_{fppf}. Let F be an algebraic space over S. Let $\Delta : F \to F \times F$ be the diagonal morphism. Then*

(1) *Δ is locally of finite type,*
(2) *Δ is a monomorphism,*
(3) *Δ is separated, and*
(4) *Δ is locally quasi-finite.*

Proof. Let $F = U/R$ be a presentation of F. As in the proof of Lemma 47.10.4 the diagram

$$\begin{array}{ccc} R & \longrightarrow & F \\ {\scriptstyle j}\downarrow & & \downarrow{\scriptstyle \Delta} \\ U \times_S U & \longrightarrow & F \times F \end{array}$$

is cartesian. Hence according to Lemma 47.11.2 it suffices to show that j has the properties listed in the lemma. (Note that each of the properties (1) – (4) occur

in the lists of Remarks 47.4.1 and 47.4.3.) Since j is an equivalence relation it is a monomorphism. Hence it is separated by Schemes, Lemma 25.23.3. As R is an étale equivalence relation we see that $s, t : R \to U$ are étale. Hence s, t are locally of finite type. Then it follows from Morphisms, Lemma 28.16.8 that j is locally of finite type. Finally, as it is a monomorphism its fibres are finite. Thus we conclude that it is locally quasi-finite by Morphisms, Lemma 28.21.7. □

Here are some common types of separation conditions, relative to the base scheme S. There is also an absolute notion of these conditions which we will discuss in Properties of Spaces, Section 48.3. Moreover, we will discuss separation conditions for a morphism of algebraic spaces in Morphisms of Spaces, Section 49.5.

Definition 47.13.2. Let S be a scheme contained in Sch_{fppf}. Let F be an algebraic space over S. Let $\Delta : F \to F \times F$ be the diagonal morphism.

(1) We say F is *separated over* S if Δ is a closed immersion.
(2) We say F is *locally separated over* S[2] if Δ is an immersion.
(3) We say F is *quasi-separated over* S if Δ is quasi-compact.
(4) We say F is *Zariski locally quasi-separated over* S[3] if there exists a Zariski covering $F = \bigcup_{i \in I} F_i$ such that each F_i is quasi-separated.

Note that if the diagonal is quasi-compact (when F is separated or quasi-separated) then the diagonal is actually quasi-finite and separated, hence quasi-affine (by More on Morphisms, Lemma 36.31.2).

47.14. Examples of algebraic spaces

In this section we construct some examples of algebraic spaces. Some of these were suggested by B. Conrad. Since we do not yet have a lot of theory at our disposal the discussion is a bit awkward in some places.

Example 47.14.1. Let k be a field of characteristic $\neq 2$. Let $U = \mathbf{A}^1_k$. Set

$$j : R = \Delta \coprod \Gamma \longrightarrow U \times_k U$$

where $\Delta = \{(x, x) \mid x \in \mathbf{A}^1_k\}$ and $\Gamma = \{(x, -x) \mid x \in \mathbf{A}^1_k, x \neq 0\}$. It is clear that $s, t : R \to U$ are étale, and hence j is an étale equivalence relation. The quotient $X = U/R$ is an algebraic space by Theorem 47.10.5. Since R is quasi-compact we see that X is quasi-separated. On the other hand, X is not locally separated because the morphism j is not an immersion.

Example 47.14.2. Let k be a field. Let $k \subset k'$ be a degree 2 Galois extension with $\mathrm{Gal}(k'/k) = \{1, \sigma\}$. Let $S = \mathrm{Spec}(k[x])$ and $U = \mathrm{Spec}(k'[x])$. Note that

$$U \times_S U = \mathrm{Spec}((k' \otimes_k k')[x]) = \Delta(U) \coprod \Delta'(U)$$

where $\Delta' = (1, \sigma) : U \to U \times_S U$. Take

$$R = \Delta(U) \coprod \Delta'(U \setminus \{0_U\})$$

where $0_U \in U$ denotes the k'-rational point whose x-coordinate is zero. It is easy to see that R is an étale equivalence relation on U over S and hence $X = U/R$ is an algebraic space by Theorem 47.10.5. Here are some properties of X (some of which will not make sense until later):

[2] In the literature this often refers to quasi-separated and locally separated algebraic spaces.
[3] This definition was suggested by B. Conrad.

(1) $X \to S$ is an isomorphism over $S \setminus \{0_S\}$,
(2) the morphism $X \to S$ is étale (see Properties of Spaces, Definition 48.13.2)
(3) the fibre 0_X of $X \to S$ over 0_S is isomorphic to $\mathrm{Spec}(k') = 0_U$,
(4) X is not a scheme because if it where, then $\mathcal{O}_{X,0_X}$ would be a local domain $(\mathcal{O}, \mathfrak{m}, \kappa)$ with fraction field $k(x)$, with $x \in \mathfrak{m}$ and residue field $\kappa = k'$ which is impossible,
(5) X is not separated, but it is locally separated and quasi-separated,
(6) there exists a surjective, finite, étale morphism $S' \to S$ such that the base change $X' = S' \times_S X$ is a scheme (namely, if we base change to $S' = \mathrm{Spec}(k'[x])$ then U splits into two copies of S' and X' becomes isomorphic to the affine line with 0 doubled, see Schemes, Example 25.14.3), and
(7) if we think of X as a finite type algebraic space over $\mathrm{Spec}(k)$, then similarly the base change $X_{k'}$ is a scheme but X is not a scheme.

In particular, this gives an example of a descent datum for schemes relative to the covering $\{\mathrm{Spec}(k') \to \mathrm{Spec}(k)\}$ which is not effective.

See also Examples, Lemma 78.50.1, which shows that descent data need not be effective even for a projective morphism of schemes. That example gives a smooth separated algebraic space of dimension 3 over $\mathbf{C}$ which is not a scheme.

We will use the following lemma as a convenient way to construct algebraic spaces as quotients of schemes by free group actions.

Lemma 47.14.3. *Let $U \to S$ be a morphism of Sch_{fppf}. Let G be an abstract group. Let $G \to Aut_S(U)$ be a group homomorphism. Assume*

(∗) *if $u \in U$ is a point, and $g(u) = u$ for some non-identity element $g \in G$, then g induces a nontrivial automorphism of $\kappa(u)$.*

Then

$$j : R = \coprod\nolimits_{g \in G} U \longrightarrow U \times_S U, \quad (g, x) \longmapsto (g(x), x)$$

is an étale equivalence relation and hence

$$F = U/R$$

is an algebraic space by Theorem 47.10.5.

Proof. In the statement of the lemma the symbol $\mathrm{Aut}_S(U)$ denotes the group of automorphisms of U over S. Assume (∗) holds. Let us show that

$$j : R = \coprod\nolimits_{g \in G} U \longrightarrow U \times_S U, \quad (g, x) \longmapsto (g(x), x)$$

is a monomorphism. This signifies that if T is a nonempty scheme, and $h : T \to U$ is a T-valued point such that $g \circ h = g' \circ h$ then $g = g'$. Suppose $T \neq \emptyset$, $h : T \to U$ and $g \circ h = g' \circ h$. Let $t \in T$. Consider the composition $\mathrm{Spec}(\kappa(t)) \to \mathrm{Spec}(\kappa(h(t))) \to U$. Then we conclude that $g' \circ g^{-1}$ fixes $u = h(t)$ and acts as the identity on its residue field. Hence $g = g'$ by (∗).

Thus if (∗) holds we see that j is a relation (see Groupoids, Definition 38.3.1). Moreover, it is an equivalence relation since on T-valued points for a connected scheme T we see that $R(T) = G \times U(T) \to U(T) \times U(T)$ (recall that we always work over S). Moreover, the morphisms $s, t : R \to U$ are étale since R is a disjoint product of copies of U. This proves that $j : R \to U \times_S U$ is an étale equivalence relation. □

Given a scheme U and an action of a group G on U we say the action of G on U is *free* if condition $(*)$ of Lemma 47.14.3 holds. This is equivalent to the notion of a free action of the constant group scheme G_S on U as defined in Groupoids, Definition 38.8.2. The lemma can be interpreted as saying that quotients of schemes by free actions of groups exist in the category of algebraic spaces.

Definition 47.14.4. Notation $U \to S$, G, R as in Lemma 47.14.3. If the action of G on U satisfies $(*)$ we say G *acts freely* on the scheme U. In this case the algebraic space U/R is denoted U/G and is called the *quotient of U by G*.

This notation is consistent with the notation U/G introduced in Groupoids, Definition 38.18.1. We will later make sense of the quotient as an algebraic stack without any assumptions on the action whatsoever; when we do this we will use the notation $[U/G]$. Before we discuss the examples we prove some more lemmas to facilitate the discussion. Here is a lemma discussing the various separation conditions for this quotient when G is finite.

Lemma 47.14.5. *Notation and assumptions as in Lemma 47.14.3. Assume G is finite. Then*

(1) *if $U \to S$ is quasi-separated, then U/G is quasi-separated, and*
(2) *if $U \to S$ is separated, then U/G is separated.*

Proof. In the proof of Lemma 47.13.1 we saw that it suffices to prove the corresponding properties for the morphism $j : R \to U \times_S U$. If $U \to S$ is quasi-separated, then for every affine open $V \subset U$ the opens $g(V) \cap V$ are quasi-compact. It follows that j is quasi-compact. If $U \to S$ is separated, the diagonal $\Delta_{U/S}$ is a closed immersion. Hence $j : R \to U \times_S U$ is a finite coproduct of closed immersions with disjoint images. Hence j is a closed immersion. □

Lemma 47.14.6. *Notation and assumptions as in Lemma 47.14.3. If $\mathrm{Spec}(k) \to U/G$ is a morphism, then there exist*

(1) *a finite Galois extension $k \subset k'$,*
(2) *a finite subgroup $H \subset G$,*
(3) *an isomorphism $H \to Gal(k'/k)$, and*
(4) *an H-equivariant morphism $\mathrm{Spec}(k') \to U$.*

Conversely, such data determine a morphism $\mathrm{Spec}(k) \to U/G$.

Proof. Consider the fibre product $V = \mathrm{Spec}(k) \times_{U/G} U$. Here is a diagram

$$\begin{array}{ccc} V & \longrightarrow & U \\ \downarrow & & \downarrow \\ \mathrm{Spec}(k) & \longrightarrow & U/G \end{array}$$

This is a nonempty scheme étale over $\mathrm{Spec}(k)$ and hence is a disjoint union of spectra of fields finite separable over k (Morphisms, Lemma 28.37.7). So write $V = \coprod_{i \in I} \mathrm{Spec}(k_i)$. The action of G on U induces an action of G on $V = \coprod \mathrm{Spec}(k_i)$. Pick an i, and let $H \subset G$ be the stabilizer of i. Since

$$V \times_{\mathrm{Spec}(k)} V = \mathrm{Spec}(k) \times_{U/G} U \times_{U/G} U = \mathrm{Spec}(k) \times_{U/G} U \times G = V \times G$$

we see that (a) the orbit of $\mathrm{Spec}(k_i)$ is V and (b) $\mathrm{Spec}(k_i \otimes_k k_i) = \mathrm{Spec}(k_i) \times H$. Thus H is finite and is the Galois group of k_i/k. We omit the converse construction. □

It follows from this lemma for example that if k'/k is a finite Galois extension, then $\mathrm{Spec}(k')/\mathrm{Gal}(k'/k) \cong \mathrm{Spec}(k)$. What happens if the extension is infinite? Here is an example.

Example 47.14.7. Let $S = \mathrm{Spec}(\mathbf{Q})$. Let $U = \mathrm{Spec}(\overline{\mathbf{Q}})$. Let $G = \mathrm{Gal}(\overline{\mathbf{Q}}/\mathbf{Q})$ with obvious action on U. Then by construction property $(*)$ of Lemma 47.14.3 holds and we obtain an algebraic space

$$X = \mathrm{Spec}(\overline{\mathbf{Q}})/G \longrightarrow S = \mathrm{Spec}(\mathbf{Q}).$$

Of course this is totally ridiculous as an approximation of S! Namely, by the Artin-Schreier theorem, see [**Jac64**, Theorem 17, page 316], the only finite subgroups of $\mathrm{Gal}(\overline{\mathbf{Q}}/\mathbf{Q})$ are $\{1\}$ and the conjugates of the order two group $\mathrm{Gal}(\overline{\mathbf{Q}}/\overline{\mathbf{Q}} \cap \mathbf{R})$. Hence, if $\mathrm{Spec}(k) \to X$ is a morphism with k algebraic over $\mathbf{Q}$, then it follows from Lemma 47.14.6 and the theorem just mentioned that either k is $\overline{\mathbf{Q}}$ or isomorphic to $\overline{\mathbf{Q}} \cap \mathbf{R}$.

What is wrong with the example above is that the Galois group comes equipped with a topology, and this should somehow be part of any construction of a quotient of $\mathrm{Spec}(\overline{\mathbf{Q}})$. The following example is much more reasonable in my opinion and may actually occur in "nature".

Example 47.14.8. Let k be a field of characteristic zero. Let $U = \mathbf{A}^1_k$ and let $G = \mathbf{Z}$. As action we take $n(x) = x + n$, i.e., the action of $\mathbf{Z}$ on the affine line by translation. The only fixed point is the generic point and it is clearly the case that $\mathbf{Z}$ injects into the automorphism group of the field $k(x)$. (This is where we use the characteristic zero assumption.) Consider the morphism

$$\gamma : \mathrm{Spec}(k(x)) \longrightarrow X = \mathbf{A}^1_k/\mathbf{Z}$$

of the generic point of the affine line into the quotient. We claim that this morphism does not factor through any monomorphism $\mathrm{Spec}(L) \to X$ of the spectrum of a field to X. (Contrary to what happens for schemes, see Schemes, Section 25.13.) In fact, since $\mathbf{Z}$ does not have any finite subgroups we see from Lemma 47.14.6 that for any such factorization $k(x) = L$. Finally, γ is not a monomorphism since

$$\mathrm{Spec}(k(x)) \times_{\gamma, X, \gamma} \mathrm{Spec}(k(x)) \cong \mathrm{Spec}(k(x)) \times \mathbf{Z}.$$

This example suggests that in order to define points of an algebraic space X we should consider equivalence classes of morphisms from spectra of fields into X and not the set of monomorphisms from spectra of fields.

We finish with a truly awful example.

Example 47.14.9. Let k be a field. Let $A = \prod_{n \in \mathbf{N}} k$ be the infinite product. Set $U = \mathrm{Spec}(A)$ seen as a scheme over $S = \mathrm{Spec}(k)$. Note that the projection maps $\mathrm{pr}_n : A \to k$ define open and closed immersions $f_n : S \to U$. Set

$$R = U \coprod \coprod_{(n,m) \in \mathbf{N}^2,\ n \neq m} S$$

with morphism j equal to $\Delta_{U/S}$ on the component U and $j = (f_n, f_m)$ on the component S corresponding to (n, m). It is clear from the remark above that s, t are étale. It is also clear that j is an equivalence relation. Hence we obtain an algebraic space

$$X = U/R.$$

To see what this means we specialize to the case where the field k is finite with q elements. Let us first discuss the topological space $|U|$ associated to the scheme U a little bit. All elements of A satisfy $x^q = x$. Hence every residue field of A is isomorphic to k, and all points of U are closed. But the topology on U isn't the discrete topology. Let $u_n \in |U|$ be the point corresponding to f_n. As mentioned above the points u_n are the open points (and hence isolated). This implies there have to be other points since we know U is quasi-compact, see Algebra, Lemma 10.16.10 (hence not equal to an infinite discrete set). Another way to see this is because the (proper) ideal

$$I = \{x = (x_n) \in A \mid \text{all but a finite number of } x_n \text{ are zero}\}$$

is contained in a maximal ideal. Note also that every element x of A is of the form $x = ue$ where u is a unit and e is an idempotent. Hence a basis for the topology of A consists of open and closed subsets (see Algebra, Lemma 10.19.1.) So the topology on $|U|$ is totally disconnected, but nontrivial. Finally, note that $\{u_n\}$ is dense in $|U|$.

We will later define a topological space $|X|$ associated to X, see Properties of Spaces, Section 48.4. What can we say about $|X|$? It turns out that the map $|U| \to |X|$ is surjective and continuous. All the points u_n map to the same point x_0 of $|X|$, and none of the other points get identified. Since $\{u_n\}$ is dense in $|U|$ we conclude that the closure of x_0 in $|X|$ is $|X|$. In other words $|X|$ is irreducible and x_0 is a generic point of $|X|$. This seems bizarre since also x_0 is the image of a section $S \to X$ of the structure morphism $X \to S$ (and in the case of schemes this would imply it was a closed point, see Morphisms, Lemma 28.21.2).

Whatever you think is actually going on in this example, it certainly shows that some care has to be exercised when defining irreducible components, connectedness, etc of algebraic spaces.

47.15. Change of big site

In this section we briefly discuss what happens when we change big sites. The upshot is that we can always enlarge the big site at will, hence we may assume any set of schemes we want to consider is contained in the big fppf site over which we consider our algebraic space. Here is a precise statement of the result.

Lemma 47.15.1. *Suppose given big sites Sch_{fppf} and Sch'_{fppf}. Assume that Sch_{fppf} is contained in Sch'_{fppf}, see Topologies, Section 33.10. Let S be an object of Sch_{fppf}. Let*

$$g : Sh((Sch/S)_{fppf}) \longrightarrow Sh((Sch'/S)_{fppf}),$$
$$f : Sh((Sch'/S)_{fppf}) \longrightarrow Sh((Sch/S)_{fppf})$$

be the morphisms of topoi of Topologies, Lemma 33.10.2. Let F be a sheaf of sets on $(Sch/S)_{fppf}$. Then

1. *if F is representable by a scheme $X \in \mathrm{Ob}((Sch/S)_{fppf})$ over S, then $f^{-1}F$ is representable too, in fact it is representable by the same scheme X, now viewed as an object of $(Sch'/S)_{fppf}$, and*
2. *if F is an algebraic space over S, then $f^{-1}F$ is an algebraic space over S also.*

Proof. Let $X \in \mathrm{Ob}((Sch/S)_{fppf})$. Let us write h_X for the representable sheaf on $(Sch/S)_{fppf}$ associated to X, and h'_X for the representable sheaf on $(Sch'/S)_{fppf}$ associated to X. By the description of f^{-1} in Topologies, Section 33.10 we see that $f^{-1}h_X = h'_X$. This proves (1).

Next, suppose that F is an algebraic space over S. By Lemma 47.9.1 this means that $F = h_U/h_R$ for some étale equivalence relation $R \to U \times_S U$ in $(Sch/S)_{fppf}$. Since f^{-1} is an exact functor we conclude that $f^{-1}F = h'_U/h'_R$. Hence $f^{-1}F$ is an algebraic space over S by Theorem 47.10.5. □

Note that this lemma is purely set theoretical and has virtually no content. Moreover, it is not true (in general) that the restriction of an algebraic space over the bigger site is an algebraic space over the smaller site (simply by reasons of cardinality). Hence we can only ever use a simple lemma of this kind to enlarge the base category and never to shrink it.

Lemma 47.15.2. *Suppose Sch_{fppf} is contained in Sch'_{fppf}. Let S be an object of Sch_{fppf}. Denote $Spaces/S$ the category of algebraic spaces over S defined using Sch_{fppf}. Similarly, denote $Spaces'/S$ the category of algebraic spaces over S defined using Sch'_{fppf}. The construction of Lemma 47.15.1 defines a fully faithful functor*

$$Spaces/S \longrightarrow Spaces'/S$$

whose essential image consists of those $X' \in \mathrm{Ob}(Spaces'/S)$ such that there exists an object $U \in \mathrm{Ob}((Sch/S)_{fppf})$ and a surjective étale morphism $U \to X'$ in the category $Sh((Sch'/S)_{fppf})$.

Proof. In Sites, Lemma 7.20.8 we have seen that the functor $f^{-1} : Sh((Sch/S)_{fppf}) \to Sh((Sch'/S)_{fppf})$ is fully faithful. Hence we see that the displayed functor of the lemma is fully faithful. Suppose that $X' \in \mathrm{Ob}(Spaces'/S)$ such that there exists an object $U \in \mathrm{Ob}((Sch/S)_{fppf})$ and a surjective étale morphism $h'_U \to X$ (with notation as in the proof of Lemma 47.15.1). Let R be an object of $(Sch'/S)_{fppf}$ representing $h'_U \times_X h'_U$. Note that $R \to U \times_S U$ is a monomorphism of schemes which is locally of finite type (Lemma 47.13.1). Hence by Sets, Lemma 3.9.9 the scheme R is isomorphic to an object of $(Sch/S)_{fppf}$ and hence we may (after replacing R by an isomorphic scheme) assume $R \in \mathrm{Ob}((Sch/S)_{fppf})$. Now we use Lemma 47.9.1 and Theorem 47.10.5 to see that $X = h_U/h_R$ is an object of $Spaces/S$ such that $f^{-1}X \cong X'$ as desired. □

47.16. Change of base scheme

In this section we briefly discuss what happens when we change base schemes. The upshot is that given a morphism $S \to S'$ of base schemes, any algebraic space over S can be viewed as an algebraic space over S'. And, given an algebraic space F' over S' there is a base change F'_S which is an algebraic space over S. We explain only what happens in case $S \to S'$ is a morphism of the big fppf site under consideration, if only S or S' is contained in the big site, then one first enlarges the big site as in Section 47.15.

Lemma 47.16.1. *Suppose given a big site Sch_{fppf}. Let $g : S \to S'$ be morphism of Sch_{fppf}. Let $j : (Sch/S)_{fppf} \to (Sch/S')_{fppf}$ be the corresponding localization functor. Let F be a sheaf of sets on $(Sch/S)_{fppf}$. Then*

(1) *for a scheme T' over S' we have $j_!F(T'/S') = \coprod_{\varphi : T' \to S} F(T' \xrightarrow{\varphi} S)$,*

(2) *if F is representable by a scheme $X \in \mathrm{Ob}((Sch/S)_{fppf})$, then $j_!F$ is representable by $j(X)$ which is X viewed as a scheme over S', and*
(3) *if F is an algebraic space over S, then $j_!F$ is an algebraic space over S', and if $F = U/R$ is a presentation, then $j_!F = j(U)/j(R)$ is a presentation.*

Let F' be a sheaf of sets on $(Sch/S')_{fppf}$. Then

(4) *for a scheme T over S we have $j^{-1}F'(T/S) = F'(T/S')$,*
(5) *if F' is representable by a scheme $X' \in \mathrm{Ob}((Sch/S')_{fppf})$, then $j^{-1}F'$ is representable, namely by $X'_S = S \times_{S'} X'$, and*
(6) *if F' is an algebraic space, then $j^{-1}F'$ is an algebraic space, and if $F' = U'/R'$ is a presentation, then $j^{-1}F' = U'_S/R'_S$ is a presentation.*

Proof. The functors $j_!$, j_* and j^{-1} are defined in Sites, Lemma 7.24.7 where it is also shown that $j = j_{S/S'}$ is the localization of $(Sch/S')_{fppf}$ at the object S/S'. Hence all of the material on localization functors is available for j. The formula in (1) is Sites, Lemma 7.26.1. By definition $j_!$ is the left adjoint to restriction j^{-1}, hence $j_!$ is right exact. By Sites, Lemma 7.24.5 it also commutes with fibre products and equalizers. By Sites, Lemma 7.24.3 we see that $j_!h_X = h_{j(X)}$ hence (2) holds. If F is an algebraic space over S, then we can write $F = U/R$ (Lemma 47.9.1) and we get

$$j_!F = j(U)/j(R)$$

because $j_!$ being right exact commutes with coequalizers, and moreover $j(R) = j(U) \times_{j_!F} j(U)$ as $j_!$ commutes with fibre products. Since the morphisms $j(s), j(t) : j(R) \to j(U)$ are simply the morphisms $s, t : R \to U$ (but viewed as morphisms of schemes over S'), they are still étale. Thus $(j(U), j(R), s, t)$ is an étale equivalence relation. Hence by Theorem 47.10.5 we conclude that $j_!F$ is an algebraic space.

Proof of (4), (5), and (6). The description of j^{-1} is in Sites, Section 7.24. The restriction of the representable sheaf associated to X'/S' is the representable sheaf associated to $X'_S = S \times_{S'} Y'$ by Sites, Lemma 7.26.2. The restriction functor j^{-1} is exact, hence $j^{-1}F' = U'_S/R'_S$. Again by exactness the sheaf R'_S is still an equivalence relation on U'_S. Finally the two maps $R'_S \to U'_S$ are étale as base changes of the étale morphisms $R' \to U'$. Hence $j^{-1}F' = U'_S/R'_S$ is an algebraic space by Theorem 47.10.5 and we win. □

Note how the presentation $j_!F = j(U)/j(R)$ is just the presentation of F but viewed as a presentation by schemes over S'. Hence the following definition makes sense.

Definition 47.16.2. Let Sch_{fppf} be a big fppf site. Let $S \to S'$ be a morphism of this site.

(1) If F' is an algebraic space over S', then the *base change of F' to S* is the algebraic space $j^{-1}F'$ described in Lemma 47.16.1. We denote it F'_S.
(2) If F is an algebraic space over S, then F *viewed as an algebraic space over S'* is the algebraic space $j_!F$ over S' described in Lemma 47.16.1. We often simply denote this F; if not then we will write $j_!F$.

The algebraic space $j_!F$ comes equipped with a canonical morphism $j_!F \to S$ of algebraic spaces over S'. This is true simply because the sheaf $j_!F$ maps to h_S (see for example the explicit description in Lemma 47.16.1). In fact, in Sites, Lemma 7.24.4 we have seen that the category of sheaves on $(Sch/S)_{fppf}$ is equivalent to the category of pairs $(\mathcal{F}', \mathcal{F}' \to h_S)$ consisting of a sheaf on $(Sch/S')_{fppf}$ and

a map of sheaves $\mathcal{F}' \to h_S$. The equivalence assigns to the sheaf $\mathcal{F}$ the pair $(j_!\mathcal{F}, j_!\mathcal{F} \to h_S)$. This, combined with the above, leads to the following result for categories of algebraic spaces.

Lemma 47.16.3. *Let Sch_{fppf} be a big fppf site. Let $S \to S'$ be a morphism of this site. The construction above give an equivalence of categories*

$$\left\{\begin{array}{c}\text{category of algebraic}\\ \text{spaces over } S\end{array}\right\} \leftrightarrow \left\{\begin{array}{c}\text{category of pairs } (F', F' \to S) \text{ consisting}\\ \text{of an algebraic space } F' \text{ over } S' \text{ and a}\\ \text{morphism } F' \to S \text{ of algebraic spaces over } S'\end{array}\right\}$$

Proof. Let F be an algebraic space over S. The functor from left to right assigns the pair $(j_!F, j_!F \to S)$ ot F which is an object of the right hand side by Lemma 47.16.1. Since this defines an equivalence of categories of sheaves by Sites, Lemma 7.24.4 to finish the proof it suffices to show: if F is a sheaf and $j_!F$ is an algebraic space, then F is an algebraic space. To do this, write $j_!F = U'/R'$ as in Lemma 47.9.1 with $U', R' \in \mathrm{Ob}((Sch/S')_{fppf})$. Then the compositions $U' \to j_!F \to S$ and $R' \to j_!F \to S$ are morphisms of schemes over S'. Denote U, R the corresponding objects of $(Sch/S)_{fppf}$. The two morphisms $R' \to U'$ are morphisms over S and hence correspond to morphisms $R \to U$. Since these are simply the same morphisms (but viewed over S) we see that we get an étale equivalence relation over S. As $j_!$ defines an equivalence of categories of sheaves (see reference above) we see that $F = U/R$ and by Theorem 47.10.5 we see that F is an algebraic space. □

The following lemma is a slight rephrasing of the above.

Lemma 47.16.4. *Let Sch_{fppf} be a big fppf site. Let $S \to S'$ be a morphism of this site. Let F' be a sheaf on $(Sch/S')_{fppf}$. The following are equivalent:*

1. *The restriction $F'|_{(Sch/S)_{fppf}}$ is an algebraic space over S, and*
2. *the sheaf $h_S \times F'$ is an algebraic space over S'.*

Proof. The restriction and the product match under the equivalence of categories of Sites, Lemma 7.24.4 so that Lemma 47.16.3 above gives the result. □

We finish this section with a lemma on a compatibility.

Lemma 47.16.5. *Let Sch_{fppf} be a big fppf site. Let $S \to S'$ be a morphism of this site. Let F be an algebraic space over S. Let T be a scheme over S and let $f : T \to F$ be a morphism over S. Let $f' : T' \to F'$ be the morphism over S' we get from f by applying the equivalence of categories described in Lemma 47.16.3. For any property $\mathcal{P}$ as in Definition 47.5.1 we have $\mathcal{P}(f') \Leftrightarrow \mathcal{P}(f)$.*

Proof. Suppose that U is a scheme over S, and $U \to F$ is a surjective étale morphism. Denote U' the scheme U viewed as a scheme over S'. In Lemma 47.16.1 we have seen that $U' \to F'$ is surjective étale. Since

$$j(T \times_{f,\, F} U) = T' \times_{f',\, F'} U'$$

the morphism of schemes $T \times_{f,F} U \to U$ is identified with the morphism of schemes $T' \times_{f',F'} U' \to U'$. It is the same morphism, just viewed over different base schemes. Hence the lemma follows from Lemma 47.11.2. □

47.17. Other chapters

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CHAPTER 48

Properties of Algebraic Spaces

48.1. Introduction

Please see Spaces, Section 47.1 for a brief introduction to algebraic spaces, and please read some of that chapter for our basic definitions and conventions concerning algebraic spaces. In this chapter we start introducing some basic notions and properties of algebraic spaces. A fundamental reference for the case of quasi-separated algebraic spaces is [**Knu71**].

The discussion is somewhat awkward at times since we made the design decision to first talk about properties of algebraic spaces by themselves, and only later about properties of morphisms of algebraic spaces. We make an exception for this rule regarding *étale morphisms* of algebraic spaces, which we introduce in Section 48.13. But until that section whenever we say a morphism has a certain property, it automatically means the source of the morphism is a scheme (or perhaps the morphism is representable).

Some of the material in the chapter (especially regarding points) will be improved upon in the chapter on decent algebraic spaces.

48.2. Conventions

The standing assumption is that all schemes are contained in a big fppf site Sch_{fppf}. And all rings A considered have the property that $\mathrm{Spec}(A)$ is (isomorphic) to an object of this big site.

Let S be a scheme and let X be an algebraic space over S. In this chapter and the following we will write $X \times_S X$ for the product of X with itself (in the category of algebraic spaces over S), instead of $X \times X$. The reason is that we want to avoid confusion when changing base schemes, as in Spaces, Section 47.16.

48.3. Separation axioms

In this section we collect all the "absolute" separation conditions of algebraic spaces. Since in our language any algebraic space is an algebraic space over some definite base scheme, any absolute property of X over S corresponds to a conditions imposed on X viewed as an algebraic space over $\mathrm{Spec}(\mathbf{Z})$. Here is the precise formulation.

Definition 48.3.1. (Compare Spaces, Definition 47.13.2.) Consider a big fppf site $Sch_{fppf} = (Sch/\mathrm{Spec}(\mathbf{Z}))_{fppf}$. Let X be an algebraic space over $\mathrm{Spec}(\mathbf{Z})$. Let $\Delta : X \to X \times X$ be the diagonal morphism.

(1) We say X is *separated* if Δ is a closed immersion.
(2) We say X is *locally separated*[1] if Δ is an immersion.

[1] In the literature this often refers to quasi-separated and locally separated algebraic spaces.

(3) We say X is *quasi-separated* if Δ is quasi-compact.
(4) We say X is *Zariski locally quasi-separated*[2] if there exists a Zariski covering $X = \bigcup_{i \in I} X_i$ (see Spaces, Definition 47.12.5) such that each X_i is quasi-separated.

Let S is a scheme contained in Sch_{fppf}, and let X be an algebraic space over S. Then we say X is *separated*, *locally separated*, *quasi-separated*, or *Zariski locally quasi-separated* if X viewed as an algebraic space over $\mathrm{Spec}(\mathbf{Z})$ (see Spaces, Definition 47.16.2) has the corresponding property.

It is true that an algebraic space X over S which is separated (in the absolute sense above) is separated over S (and similarly for the other absolute separation properties above). This will be discussed in great detail in Morphisms of Spaces, Section 49.5. We will see in Lemma 48.6.6 that being Zariski locally separated is independent of the base scheme (hence equivalent to the absolute notion).

Lemma 48.3.2. *Let S be a scheme. Let X be an algebraic space over S. We have the following implications among the separation axioms of Definition 48.3.1:*

(1) *separated implies all the others,*
(2) *quasi-separated implies Zariski locally quasi-separated.*

Proof. Omitted. □

48.4. Points of algebraic spaces

As is clear from Spaces, Example 47.14.8 a point of an algebraic space should not be defined as a monomorphism from the spectrum of a field. Instead we define them as equivalence classes of morphisms of spectra of fields as equivalence classes of morphisms from spectra of fields. exactly as explained in Schemes, Section 25.13.

Let S be a scheme. Let F be a presheaf on $(Sch/S)_{fppf}$. Let K is a field. Consider a morphism

$$\mathrm{Spec}(K) \longrightarrow F.$$

By the Yoneda Lemma this is given by an element $p \in F(\mathrm{Spec}(K))$. We say that two such pairs $(\mathrm{Spec}(K), p)$ and $(\mathrm{Spec}(L), q)$ are *equivalent* if there exists a third field Ω and a commutative diagram

$$\begin{array}{ccc} \mathrm{Spec}(\Omega) & \longrightarrow & \mathrm{Spec}(L) \\ \downarrow & & \downarrow q \\ \mathrm{Spec}(K) & \xrightarrow{p} & F. \end{array}$$

In other words, there are field extensions $K \to \Omega$ and $L \to \Omega$ such that p and q map to the same element of $F(\mathrm{Spec}(\Omega))$. We omit the verification that this defines an equivalence relation.

Definition 48.4.1. Let S be a scheme. Let X be an algebraic space over S. A *point* of X is an equivalence class of morphisms from spectra of fields into X. The set of points of X is denoted $|X|$.

Note that if $f : X \to Y$ is a morphism of algebraic spaces over S, then there is an induced map $|f| : |X| \to |Y|$ which maps a representative $x : \mathrm{Spec}(K) \to X$ to the representative $f \circ x : \mathrm{Spec}(K) \to Y$.

[2] This notion was suggested by B. Conrad.

Lemma 48.4.2. *Let S be a scheme. Let X be a scheme over S. The points of X as a scheme are in canonical 1-1 correspondence with the points of X as an algebraic space.*

Proof. This is Schemes, Lemma 25.13.3. □

Lemma 48.4.3. *Let S be a scheme. Let*

$$\begin{array}{ccc} Z \times_Y X & \longrightarrow & X \\ \downarrow & & \downarrow \\ Z & \longrightarrow & Y \end{array}$$

be a cartesian diagram of algebraic spaces. Then the map of sets of points

$$|Z \times_Y X| \longrightarrow |Z| \times_{|Y|} |X|$$

is surjective.

Proof. Namely, suppose given fields K, L and morphisms $\mathrm{Spec}(K) \to X$, $\mathrm{Spec}(L) \to Z$, then the assumption that they agree as elements of $|Y|$ means that there is a common extension $K \subset M$ and $L \subset M$ such that $\mathrm{Spec}(M) \to \mathrm{Spec}(K) \to X \to Y$ and $\mathrm{Spec}(M) \to \mathrm{Spec}(L) \to Z \to Y$ agree. And this is exactly the condition which says you get a morphism $\mathrm{Spec}(M) \to Z \times_Y X$. □

Lemma 48.4.4. *Let S be a scheme. Let X be an algebraic space over S. Let $f : T \to X$ be a morphism from a scheme to X. The following are equivalent*

(1) *$f : T \to X$ is surjective (according to Spaces, Definition 47.5.1), and*
(2) *$|f| : |T| \to |X|$ is surjective.*

Proof. Assume (1). Let $x : \mathrm{Spec}(K) \to X$ be a morphism from the spectrum of a field into X. By assumption the morphism of schemes $\mathrm{Spec}(K) \times_X T \to \mathrm{Spec}(K)$ is surjective. Hence there exists a field extension $K \subset K'$ and a morphism $\mathrm{Spec}(K') \to \mathrm{Spec}(K) \times_X T$ such that the left square in the diagram

$$\begin{array}{ccccc} \mathrm{Spec}(K') & \longrightarrow & \mathrm{Spec}(K) \times_X T & \longrightarrow & T \\ \downarrow & & \downarrow & & \downarrow \\ \mathrm{Spec}(K) & = & \mathrm{Spec}(K) & \xrightarrow{x} & X \end{array}$$

is commutative. This shows that $|f| : |T| \to |X|$ is surjective.

Assume (2). Let $Z \to X$ be a morphism where Z is a scheme. We have to show that the morphism of schemes $Z \times_X T \to T$ is surjective, i.e., that $|Z \times_X T| \to |Z|$ is surjective. This follows from (2) and Lemma 48.4.3. □

Lemma 48.4.5. *Let S be a scheme. Let X be an algebraic space over S. Let $X = U/R$ be a presentation of X, see Spaces, Definition 47.9.3. Then the image of $|R| \to |U| \times |U|$ is an equivalence relation and $|X|$ is the quotient of $|U|$ by this equivalence relation.*

Proof. The assumption means that U is a scheme, $p : U \to X$ is a surjective, étale morphism, $R = U \times_X U$ is a scheme and defines an étale equivalence relation on U such that $X = U/R$ as sheaves. By Lemma 48.4.4 we see that $|U| \to |X|$ is surjective. By Lemma 48.4.3 the map

$$|R| \longrightarrow |U| \times_{|X|} |U|$$

is surjective. Hence the image of $|R| \to |U| \times |U|$ is exactly the set of pairs $(u_1, u_2) \in |U| \times |U|$ such that u_1 and u_2 have the same image in $|X|$. Combining these two statements we get the result of the lemma. □

Lemma 48.4.6. *Let S be a scheme. There exists a unique topology on the set of points of algebraic spaces over S with the following properties:*

(1) *for every morphism of algebraic spaces $X \to Y$ over S the map $|X| \to |Y|$ is continuous, and*
(2) *for every étale morphism $U \to X$ with U a scheme the map of topological spaces $|U| \to |X|$ is continuous and open.*

Proof. Let X be an algebraic space over S. Let $p : U \to X$ be a surjective étale morphism where U is a scheme over S. We define $W \subset |X|$ is open if and only if $|p|^{-1}(W)$ is an open subset of $|U|$. This is a topology on $|X|$.

Let us prove that the topology is independent of the choice of the presentation. To do this it suffices to show that if U' is a scheme, and $U' \to X$ is an étale morphism, then the map $|U'| \to |X|$ (with topology on $|X|$ defined using $U \to X$ as above) is open and continuous; which in addition will prove that (2) holds. Set $U'' = U \times_X U'$, so that we have the commutative diagram

$$\begin{array}{ccc} U'' & \longrightarrow & U' \\ \downarrow & & \downarrow \\ U & \longrightarrow & X \end{array}$$

As $U \to X$ and $U' \to X$ are étale we see that both $U'' \to U$ and $U'' \to U'$ are étale morphisms of schemes. Moreover, $U'' \to U'$ is surjective. Hence we get a commutative diagram of maps of sets

$$\begin{array}{ccc} |U''| & \longrightarrow & |U'| \\ \downarrow & & \downarrow \\ |U| & \longrightarrow & |X| \end{array}$$

The lower horizontal arrow is surjective (see Lemma 48.4.4 or Lemma 48.4.5) and continuous by definition of the topology on $|X|$. The top horizontal arrow is surjective, continuous, and open by Morphisms, Lemma 28.37.13. The left vertical arrow is continuous and open (by Morphisms, Lemma 28.37.13 again.) Hence it follows formally that the right vertical arrow is continuous and open.

To finish the proof we prove (1). Let $a : X \to Y$ be a morphism of algebraic spaces. According to Spaces, Lemma 47.11.4 we can find a diagram

$$\begin{array}{ccc} U & \xrightarrow[\alpha]{} & V \\ {\scriptstyle p}\downarrow & & \downarrow{\scriptstyle q} \\ X & \xrightarrow{a} & Y \end{array}$$

where U and V are schemes, and p and q are surjective and étale. This gives rise to the diagram

$$\begin{array}{ccc} |U| & \xrightarrow[\alpha]{} & |V| \\ {\scriptstyle p}\downarrow & & \downarrow{\scriptstyle q} \\ |X| & \xrightarrow{a} & |Y| \end{array}$$

where all but the lower horizontal arrows are known to be continuous and the two vertical arrows are surjective and open. It follows that the lower horizontal arrow is continuous as desired. □

Definition 48.4.7. Let S be a scheme. Let X be an algebraic space over S. The underlying *topological space* of X is the set of points $|X|$ endowed with the topology constructed in Lemma 48.4.6.

It turns out that this topological space carries the same information as the small Zariski site X_{Zar} of Spaces, Definition 47.12.6.

Lemma 48.4.8. *Let S be a scheme. Let X be an algebraic space over S.*

(1) *The rule $X' \mapsto |X'|$ defines an inclusion preserving bijection between open subspaces X' (see Spaces, Definition 47.12.1) of X, and opens of the topological space $|X|$.*
(2) *A family $\{X_i \subset X\}_{i \in I}$ of open subspaces of X is a Zariski covering (see Spaces, Definition 47.12.5) if and only if $|X| = \bigcup |X_i|$.*

In other words, the small Zariski site X_{Zar} of X is canonically identified with a site associated to the topological space $|X|$ (see Sites, Example 7.6.4).

Proof. In order to prove (1) let us construct the inverse of the rule. Namely, suppose that $W \subset |X|$ is open. Choose a presentation $X = U/R$ corresponding to the surjective étale map $p : U \to X$ and étale maps $s, t : R \to U$. By construction we see that $|p|^{-1}(W)$ is an open of U. Denote $W' \subset U$ the corresponding open subscheme. It is clear that $R' = s^{-1}(W') = t^{-1}(W')$ is a Zariski open of R which defines an étale equivalence relation on W'. By Spaces, Lemma 47.10.2 the morphism $X' = W'/R' \to X$ is an open immersion. Hence X' is an algebraic space by Spaces, Lemma 47.11.1. By construction $|X'| = W$, i.e., X' is a subspace of X corresponding to W. Thus (1) is proved.

To prove (2), note that if $\{X_i \subset X\}_{i \in I}$ is a collection of open subspaces, then it is a Zariski covering if and only if the $U = \bigcup U \times_X X_i$ is an open covering. This follows from the definition of a Zariski covering and the fact that the morphism $U \to X$ is surjective as a map of presheaves on $(Sch/S)_{fppf}$. On the other hand, we see that $|X| = \bigcup |X_i|$ if and only if $U = \bigcup U \times_X X_i$ by Lemma 48.4.5 (and the fact that the projections $U \times_X X_i \to X_i$ are surjective and étale). Thus the equivalence of (2) follows. □

Lemma 48.4.9. *Let S be a scheme. Let X, Y be algebraic spaces over S. Let $X' \subset X$ be an open subspace. Let $f : Y \to X$ be a morphism of algebraic spaces over S. Then f factors through X' if and only if $|f| : |Y| \to |X|$ factors through $|X'| \subset |X|$.*

Proof. By Spaces, Lemma 47.12.3 we see that $Y' = Y \times_X X' \to Y$ is an open immersion. If $|f|(|Y|) \subset |X'|$, then clearly $|Y'| = |Y|$. Hence $Y' = Y$ by Lemma 48.4.8. □

Lemma 48.4.10. *Let S be a scheme. Let X be an algebraic spaces over S. Let U be a scheme and let $f : U \to X$ be an étale morphism. Let $X' \subset X$ be the open subspace corresponding to the open $|f|(|U|) \subset |X|$ via Lemma 48.4.8. Then f factors through a surjective étale morphism $f' : U \to X'$. Moreover, if $R = U \times_X U$, then $R = U \times_{X'} U$ and X' has the presentation $X' = U/R$.*

Proof. The existence of the factorization follows from Lemma 48.4.9. The morphism f' is surjective according to Lemma 48.4.4. To see f' is étale, suppose that $T \to X'$ is a morphism where T is a scheme. Then $T \times_X U = T \times_{X'} U$ as $X" \to X$ is a monomorphism of sheaves. Thus the projection $T \times_{X'} U \to T$ is étale as we assumed f étale. We have $U \times_X U = U \times_{X'} U$ as $X' \to X$ is a monomorphism. Then $X' = U/R$ follows from Spaces, Lemma 47.9.1. □

Lemma 48.4.11. *Let S be a scheme. Let X be an algebraic space over S. Consider the map*

$$\{\mathrm{Spec}(k) \to X \text{ monomorphism}\} \longrightarrow |X|$$

This map is injective.

Proof. Suppose that $\varphi_i : \mathrm{Spec}(k_i) \to X$ are monomorphisms for $i = 1, 2$. If φ_1 and φ_2 define the same point of $|X|$, then we see that the scheme

$$Y = \mathrm{Spec}(k_1) \times_{\varphi_1, X, \varphi_2} \mathrm{Spec}(k_2)$$

is nonempty. Since the base change of a monomorphism is a monomorphism this means that the projection morphisms $Y \to \mathrm{Spec}(k_i)$ are monomorphisms. Hence $\mathrm{Spec}(k_1) = Y = \mathrm{Spec}(k_2)$ as schemes over X, see Schemes, Lemma 25.23.10. We conclude that $\varphi_1 = \varphi_2$, which proves the lemma. □

We will see in Decent Spaces, Lemma 50.9.1 that this map is a bijection when X is decent.

48.5. Quasi-compact spaces

Definition 48.5.1. Let S be a scheme. Let X be an algebraic space over S. We say X is *quasi-compact* if there exists a surjective étale morphism $U \to X$ with U quasi-compact.

Lemma 48.5.2. *Let S be a scheme. Let X be an algebraic space over S. Then X is quasi-compact if and only if $|X|$ is quasi-compact.*

Proof. Choose a scheme U and an étale surjective morphism $U \to X$. We will use Lemma 48.4.4. If U is quasi-compact, then since $|U| \to |X|$ is surjective we conclude that $|X|$ is quasi-compact. If $|X|$ is quasi-compact, then since $|U| \to |X|$ is open we see that there exists a quasi-compact open $U' \subset U$ such that $|U'| \to |X|$ is surjective (and still étale). Hence we win. □

Lemma 48.5.3. *A finite disjoint union of quasi-compact algebraic spaces is a quasi-compact algebraic space.*

Proof. This is clear from Lemma 48.5.2 and the corresponding topological fact. □

Example 48.5.4. The space $\mathbf{A}^1_{\mathbf{Q}}/\mathbf{Z}$ is a quasi-compact algebraic space.

Lemma 48.5.5. *Let S be a scheme. Let X be an algebraic space over S. Every point of $|X|$ has a fundamental system of open quasi-compact neighbourhoods. In particular $|X|$ is locally quasi-compact in the sense of Topology, Definition 5.12.1.*

Proof. This follows formally from the fact that there exists a scheme U and a surjective, open, continuous map $U \to |X|$ of topological spaces. To be a bit more precise, if $u \in U$ maps to $x \in |X|$, then the images of the affine neighbourhoods of u will give a fundamental system of quasi-compact open neighbourhoods of x. □

48.6. Special coverings

In this section we collect some straightforward lemmas on the existence of étale surjective coverings of algebraic spaces.

Lemma 48.6.1. *Let S be a scheme. Let X be an algebraic space over S. There exists a surjective étale morphism $U \to X$ where U is a disjoint union of affine schemes. We may in addition assume each of these affines maps into an affine open of S.*

Proof. Let $V \to X$ be a surjective étale morphism. Let $V = \bigcup_{i \in I} V_i$ be a Zariski open covering such that each V_i maps into an affine open of S. Then set $U = \coprod_{i \in I} V_i$ with induced morphism $U \to V \to X$. This is étale and surjective as a composition of étale and surjective representable transformations of functors (via the general principle Spaces, Lemma 47.5.4 and Morphisms, Lemmas 28.11.2 and 28.37.3). □

Lemma 48.6.2. *Let S be a scheme. Let X be an algebraic space over S. There exists a Zariski covering $X = \bigcup X_i$ such that each algebraic space X_i has a surjective étale covering by an affine scheme. We may in addition assume each X_i maps into an affine open of S.*

Proof. By Lemma 48.6.1 we can find a surjective étale morphism $U = \coprod U_i \to X$, with U_i affine and mapping into an affine open of S. Let $X_i \subset X$ be the open subspace of X such that $U_i \to X$ factors through an étale surjective morphism $U_i \to X_i$, see Lemma 48.4.10. Since $U = \bigcup U_i$ we see that $X = \bigcup X_i$. As $U_i \to X_i$ is surjective it follows that $X_i \to S$ maps into an affine open of S. □

Lemma 48.6.3. *Let S be a scheme. Let X be an algebraic space over S. Then X is quasi-compact if and only if there exists an étale surjective morphism $U \to X$ with U an affine scheme.*

Proof. If there exists an étale surjective morphism $U \to X$ with U affine then X is quasi-compact by Definition 48.5.1. Conversely, if X is quasi-compact, then $|X|$ is quasi-compact. Let $U = \coprod_{i \in I} U_i$ be a disjoint union of affine schemes with an étale and surjective map $\varphi : U \to X$ (Lemma 48.6.1). Then $|X| = \bigcup \varphi(|U_i|)$ and by quasi-compactness there is a finite subset $i_1, \ldots, i_n$ such that $|X| = \bigcup \varphi(|U_{i_j}|)$. Hence $U_{i_1} \cup \ldots \cup U_{i_n}$ is an affine scheme with a finite surjective morphism towards X. □

The following lemma will be obsoleted by the discussion of separated morphisms in the chapter on morphisms of algebraic spaces.

Lemma 48.6.4. *Let S be a scheme. Let X be an algebraic space over S. Let U be a separated scheme and $U \to X$ étale. Then $U \to X$ is separated, and $R = U \times_X U$ is a separated scheme.*

Proof. Let $X' \subset X$ be the open subscheme such that $U \to X$ factors through an étale surjection $U \to X'$, see Lemma 48.4.10. If $U \to X'$ is separated, then so is $U \to X$, see Spaces, Lemma 47.5.4 (as the open immersion $X' \to X$ is

separated by Spaces, Lemma 47.5.8 and Schemes, Lemma 25.23.7). Moreover, since $U \times_{X'} U = U \times_X U$ it suffices to prove the result after replacing X by X', i.e., we may assume $U \to X$ surjective. Consider the commutative diagram

$$\begin{array}{ccc} R = U \times_X U & \longrightarrow & U \\ \downarrow & & \downarrow \\ U & \longrightarrow & X \end{array}$$

In the proof of Spaces, Lemma 47.13.1 we have seen that $j : R \to U \times_S U$ is separated. The morphism of schemes $U \to S$ is separated as U is a separated scheme, see Schemes, Lemma 25.21.14. Hence $U \times_S U \to U$ is separated as a base change, see Schemes, Lemma 25.21.13. Hence the scheme $U \times_S U$ is separated (by the same lemma). Since j is separated we see in the same way that R is separated. Hence $R \to U$ is a separated morphism (by Schemes, Lemma 25.21.14 again). Thus by Spaces, Lemma 47.11.2 and the diagram above we conclude that $U \to X$ is separated. □

Lemma 48.6.5. *Let S be a scheme. Let X be an algebraic space over S. If there exists a quasi-separated scheme U and a surjective étale morphism $U \to X$ such that either of the projections $U \times_X U \to U$ is quasi-compact, then X is quasi-separated.*

Proof. We may think of X as an algebraic space over $\mathbf{Z}$. Consider the cartesian diagram

$$\begin{array}{ccc} U \times_X U & \longrightarrow & X \\ {\scriptstyle j}\downarrow & & \downarrow{\scriptstyle \Delta} \\ U \times U & \longrightarrow & X \times X \end{array}$$

Since U is quasi-separated the projection $U \times U \to U$ is quasi-separated (as a base change of a quasi-separated morphism of schemes, see Schemes, Lemma 25.21.13). Hence the assumption in the lemma implies j is quasi-compact by Schemes, Lemma 25.21.15. By Spaces, Lemma 47.11.2 we see that Δ is quasi-compact as desired. □

Lemma 48.6.6. *Let S be a scheme. Let X be an algebraic space over S. The following are equivalent*

1. *X is Zariski locally quasi-separated over S,*
2. *X is Zariski locally quasi-separated,*
3. *there exists a Zariski open covering $X = \bigcup X_i$ such that for each i there exists an affine scheme U_i and a quasi-compact surjective étale morphism $U_i \to X_i$, and*
4. *there exists a Zariski open covering $X = \bigcup X_i$ such that for each i there exists an affine scheme U_i which maps into an affine open of S and a quasi-compact surjective étale morphism $U_i \to X_i$.*

Proof. Assume $U_i \to X_i \subset X$ are as in (3). To prove (4) choose for each i a finite affine open covering $U_i = U_{i1} \cup \ldots \cup U_{in_i}$ such that each U_{ij} maps into an affine open of S. The compositions $U_{ij} \to U_i \to X_i$ are étale and quasi-compact (see Spaces, Lemma 47.5.4). Let $X_{ij} \subset X_i$ be the open subspace corresponding to the image of $|U_{ij}| \to |X_i|$, see Lemma 48.4.10. Note that $U_{ij} \to X_{ij}$ is quasi-compact as $X_{ij} \subset X_i$ is a monomorphism and as $U_{ij} \to X$ is quasi-compact. Then $X = \bigcup X_{ij}$ is a covering as in (4). The implication (4) $\Rightarrow$ (3) is immediate.

Assume (4). To show that X is Zariski locally quasi-separated over S it suffices to show that X_i is quasi-separated over S. Hence we may assume there exists an affine scheme U mapping into an affine open of S and a quasi-compact surjective étale morphism $U \to X$. Consider the fibre product square

$$\begin{array}{ccc} U \times_X U & \longrightarrow & U \times_S U \\ \downarrow & & \downarrow \\ X & \xrightarrow{\Delta_{X/S}} & X \times_S X \end{array}$$

The right vertical arrow is surjective étale (see Spaces, Lemma 47.5.7) and $U \times_S U$ is affine (as U maps into an affine open of S, see Schemes, Section 25.17), and $U \times_X U$ is quasi-compact because the projection $U \times_X U \to U$ is quasi-compact as a base change of $U \to X$. It follows from Spaces, Lemma 47.11.2 that $\Delta_{X/S}$ is quasi-compact as desired.

Assume (1). To prove (3) there is an immediate reduction to the case where X is quasi-separated over S. By Lemma 48.6.2 we can find a Zariski open covering $X = \bigcup X_i$ such that each X_i maps into an affine open of S, and such that there exist affine schemes U_i and surjective étale morphisms $U_i \to X_i$. Since $U_i \to S$ maps into an affine open of S we see that $U_i \times_S U_i$ is affine, see Schemes, Section 25.17. As X is quasi-separated over S, the morphisms

$$R_i = U_i \times_{X_i} U_i = U_i \times_X U_i \longrightarrow U_i \times_S U_i$$

as base changes of $\Delta_{X/S}$ are quasi-compact. Hence we conclude that R_i is a quasi-compact scheme. This in turn implies that each projection $R_i \to U_i$ is quasi-compact. Hence, applying Spaces, Lemma 47.11.2 to the covering $U_i \to X_i$ and the morphism $U_i \to X_i$ we conclude that the morphisms $U_i \to X_i$ are quasi-compact as desired.

At this point we see that (1), (3), and (4) are equivalent. Since (3) does not refer to the base scheme we conclude that these are also equivalent with (2). □

48.7. Properties of Spaces defined by properties of schemes

Any étale local property of schemes gives rise to a corresponding property of algebraic spaces via the following lemma.

Lemma 48.7.1. *Let S be a scheme. Let X be an algebraic space over S. Let $\mathcal{P}$ be a property of schemes which is local in the étale topology, see Descent, Definition 34.11.1. The following are equivalent*

1. *for some scheme U and surjective étale morphism $U \to X$ the scheme U has property $\mathcal{P}$, and*
2. *for every scheme U and every étale morphism $U \to X$ the scheme U has property $\mathcal{P}$.*

If X is representable this is equivalent to $\mathcal{P}(X)$.

Proof. Omitted. □

Definition 48.7.2. Let $\mathcal{P}$ be a property of schemes which is local in the étale topology. Let S be a scheme. Let X be an algebraic space over S. We say X *has property* $\mathcal{P}$ if any of the equivalent conditions of Lemma 48.7.1 hold.

Remark 48.7.3. Here is a list of properties which are local for the étale topology (keep in mind that the fpqc, fppf, syntomic, and smooth topologies are stronger than the étale topology):

(1) locally Noetherian, see Descent, Lemma 34.12.1,
(2) Jacobson, see Descent, Lemma 34.12.2,
(3) locally Noetherian and (S_k), see Descent, Lemma 34.13.1,
(4) Cohen-Macaulay, see Descent, Lemma 34.13.2,
(5) reduced, see Descent, Lemma 34.14.1,
(6) normal, see Descent, Lemma 34.14.2,
(7) locally Noetherian and (R_k), see Descent, Lemma 34.14.3,
(8) regular, see Descent, Lemma 34.14.4,
(9) Nagata, see Descent, Lemma 34.14.5.

Any étale local property of germs of schemes gives rise to a corresponding property of algebraic spaces. Here is the obligatory lemma.

Lemma 48.7.4. *Let $\mathcal{P}$ be a property of germs of schemes which is étale local, see Descent, Definition 34.17.1. Let S be a scheme. Let X be an algebraic space over S. Let $x \in |X|$ be a point of X. Consider étale morphisms $a : U \to X$ where U is a scheme. The following are equivalent*

(1) *for any $U \to X$ as above and $u \in U$ with $a(u) = x$ we have $\mathcal{P}(U, u)$, and*
(2) *for some $U \to X$ as above and $u \in U$ with $a(u) = x$ we have $\mathcal{P}(U, u)$.*

If X is representable, then this is equivalent to $\mathcal{P}(X, x)$.

Proof. Omitted. □

Definition 48.7.5. Let S be a scheme. Let X be an algebraic space over S. Let $x \in |X|$. Let $\mathcal{P}$ be a property of germs of schemes which is étale local. We say X *has property $\mathcal{P}$ at x* if any of the equivalent conditions of Lemma 48.7.4 hold.

48.8. Dimension at a point

We can use Descent, Lemma 34.17.2 to define the dimension of an algebraic space X at a point x. This will give us a different notion than the topological one (i.e., the dimension of $|X|$ at x).

Definition 48.8.1. Let S be a scheme. Let X be an algebraic space over S. Let $x \in |X|$ be a point of X. We define the *dimension of X at x* to be the element $\dim_x(X) \in \{0, 1, 2, \ldots, \infty\}$ such that $\dim_x(X) = \dim_u(U)$ for any (equivalently some) pair $(a : U \to X, u)$ consisting of an étale morphism $a : U \to X$ from a scheme to X and a point $u \in U$ with $a(u) = x$. See Definition 48.7.5, Lemma 48.7.4, and Descent, Lemma 34.17.2.

Warning: It is **not** the case that $\dim_x(X) = \dim_x(|X|)$ in general. A counter example is the algebraic space X of Spaces, Example 47.14.9. Namely, in this example we have $\dim_x(X) = 0$ and $\dim_x(|X|) = 1$ (this holds for any $x \in |X|$). In particular, it also means that the dimension of X (as defined below) is different from the dimension of $|X|$.

Definition 48.8.2. Let S be a scheme. Let X be an algebraic space over S. The *dimension* $\dim(X)$ of X is defined by the rule

$$\dim(X) = \sup_{x \in |X|} \dim_x(X)$$

By Properties, Lemma 27.10.2 we see that this is the usual notion if X is a scheme. There is another integer that measures the dimension of a scheme at a point, namely the dimension of the local ring. This invariant is compatible with étale morphisms also, see Section 48.20.

48.9. Reduced spaces

We have already defined reduced algebraic spaces in Section 48.7. Here we just prove some simple lemmas regarding reduced algebraic spaces. But first a lemma that will help us construct (locally) closed subspaces.

Lemma 48.9.1. *Let S be a scheme. Let $j : R \to U \times_S U$ be an étale equivalence relation. Let $X = U/R$ be the associated algebraic space (Spaces, Theorem 47.10.5). There is a canonical bijection*

R-invariant locally closed subschemes Z' of $U \leftrightarrow$ locally closed subspaces Z of X

Moreover, if $Z \to X$ is closed (resp. open) if and only if $Z' \to U$ is closed (resp. open).

Proof. Denote $\varphi : U \to X$ the canonical map. The bijection sends $Z \to X$ to $Z' = Z \times_X U \to U$. It is immediate from the definition that $Z' \to U$ is an immersion, resp. closed immersion, resp. open immersion if $Z \to X$ is so. It is also clear that Z' is R-invariant (see Groupoids, Definition 38.17.1).

Conversely, assume that $Z' \to U$ is an immersion which is R-invariant. Let R' be the restriction of R to Z', see Groupoids, Definition 38.16.2. Since $R' = R \times_{s,U} Z' = Z' \times_{U,t} R$ in this case we see that R' is an étale equivalence relation on Z'. By Spaces, Theorem 47.10.5 we see $Z = Z'/R'$ is an algebraic space. By construction we have $U \times_X Z = Z'$, so $U \times_X Z \to Z$ is an immersion. Note that the property "immersion" is preserved under base change and fppf local on the base (see Spaces, Section 47.4). Moreover, immersions are separated and locally quasi-finite (see Schemes, Lemma 25.23.7 and Morphisms, Lemma 28.21.14). Hence by More on Morphisms, Lemma 36.37.1 immersions satisfy descent for fppf covering. This means all the hypotheses of Spaces, Lemma 47.11.3 are satisfied for $Z \to X$, $\mathcal{P}$ ="immersion", and the étale surjective morphism $U \to X$. We conclude that $Z \to X$ is representable and an immersion, which is the definition of a subspace (see Spaces, Definition 47.12.1).

It is clear that these constructions are inverse to each other and we win. □

Lemma 48.9.2. *Let S be a scheme. Let X be an algebraic space over S. Let $T \subset |X|$ be a closed subset. There exists a unique closed subspace $Z \subset X$ with the following properties: (a) we have $|Z| = T$, and (b) Z is reduced.*

Proof. Let $U \to X$ be a surjective étale morphism, where U is a scheme. Set $R = U \times_X U$, so that $X = U/R$, see Spaces, Lemma 47.9.1. As usual we denote $s, t : R \to U$ the two projection morphisms. By Lemma 48.4.5 we see that T corresponds to a closed subset $T' \subset |U|$ such that $s^{-1}(T') = t^{-1}(T')$. Let $Z' \subset U$ be the reduced induced scheme structure on T'. In this case the fibre products $Z' \times_{U,t} R$ and $Z' \times_{U,s} R$ are closed subschemes of R (Schemes, Lemma 25.18.2) which are étale over Z' (Morphisms, Lemma 28.37.4), and hence reduced (because being reduced is local in the étale topology, see Remark 48.7.3). Since they have the same underlying topological space (see above) we conclude that $Z' \times_{U,t} R = Z' \times_{U,s} R$. Thus we can apply Lemma 48.9.1 to obtain a closed subspace $Z \subset X$ whose pullback

to U is Z'. By construction $|Z| = T$ and Z is reduced. This proves existence. We omit the proof of uniqueness. □

Lemma 48.9.3. *Let S be a scheme. Let X, Y be algebraic spaces over S. Let $Z \subset X$ be a closed subspace. Assume Y is reduced. A morphism $f : Y \to X$ factors through Z if and only if $f(|Y|) \subset |Z|$.*

Proof. Assume $f(|Y|) \subset |Z|$. Choose a diagram

$$\begin{array}{ccc} V & \xrightarrow[h]{} & U \\ {\scriptstyle b}\downarrow & & \downarrow{\scriptstyle a} \\ Y & \xrightarrow{f} & X \end{array}$$

where U, V are schemes, and the vertical arrows are surjective and étale. The scheme V is reduced, see Lemma 48.7.1. Hence h factors through $a^{-1}(Z)$ by Schemes, Lemma 25.12.6. So $a \circ h$ factors through Z. As $Z \subset X$ is a subsheaf, and $V \to Y$ is a surjection of sheaves on $(Sch/S)_{fppf}$ we conclude that $X \to Y$ factors through Z. □

Definition 48.9.4. Let S be a scheme, and let X be an algebraic space over S. Let $Z \subset |X|$ be a closed subset. An *algebraic space structure on Z* is given by a closed subspace Z' of X with $|Z'|$ equal to Z. The *reduced induced algebraic space structure* on Z is the one constructed in Lemma 48.9.2. The *reduction X_{red} of X* is the reduced induced algebraic space structure on $|X|$.

48.10. The schematic locus

Every algebraic space has a largest open subspace which is a scheme; this is more or less clear but we also write out the proof below. Of course this subspace may be empty, for example if $X = \mathbf{A}^1_{\mathbf{Q}}/\mathbf{Z}$ (the universal counter example). On the other hand, if X is for example quasi-separated, then this largest open subscheme is actually dense in X!

Lemma 48.10.1. *Let S be a scheme. Let X be an algebraic space over S. There exists a largest open subspace $X' \subset X$ which is a scheme.*

Proof. Let $U \to X$ be an étale surjective morphism, where U is a scheme. Let $R = U \times_X U$. The open subspaces of X correspond $1-1$ with open subschemes of U which are R-invariant. Hence there is a set of them. Let X_i, $i \in I$ be the set of open subspaces of X which are schemes, i.e., are representable. Consider the open subspace $X' \subset X$ whose underlying set of points is the open $\bigcup |X_i|$ of $|X|$. By Lemma 48.4.4 we see that

$$\coprod X_i \longrightarrow X'$$

is a surjective map of sheaves on $(Sch/S)_{fppf}$. But since each $X_i \to X'$ is representable by open immersions we see that in fact the map is surjective in the Zariski topology. (Because if $T \to X'$ is a morphism from a scheme into X', then $X_i \times'_X T$ is an open subscheme of T.) Hence we can apply Schemes, Lemma 25.15.4 to see that X' is a scheme. □

In the rest of this section we say that an open subspace X' of an algebraic space X is *dense* if the corresponding open subset $|X'| \subset |X|$ is dense.

Lemma 48.10.2. *Let S be a scheme. Let X be an algebraic space over S. If there exists a finite, étale, surjective morphism $U \to X$ where U is a scheme, then there exists a dense open subspace of X which is a scheme.*

Proof. Assume X is an algebraic space, U a scheme, and $U \to X$ is a finite étale surjective morphism. Write $R = U \times_X U$ and denote $s, t : R \to U$ the projections as usual. Note that s, t are surjective, finite and étale. Claim: The union of the R-invariant affine opens of U is topologically dense in U.

Proof of the claim[3]. Let $W \subset U$ be an affine open. Set $W' = t(s^{-1}(W)) \subset U$. Since $s^{-1}(W)$ is affine (hence quasi-compact) we see that $W' \subset U$ is a quasi-compact open. By Properties, Lemma 27.27.3 there exists a dense open $W'' \subset W'$ which is a separated scheme. Set $\Delta' = W' \setminus W''$. This is a nowhere dense closed subset of W''. Since $t|_{s^{-1}(W)} : s^{-1}(W) \to W'$ is open (because it is étale) we see that the inverse image $(t|_{s^{-1}(W)})^{-1}(\Delta') \subset s^{-1}(W)$ is a nowhere dense closed subset (see Topology, Lemma 5.20.6). Hence, by Morphisms, Lemma 28.46.7 we see that

$$\Delta = s\left((t|_{s^{-1}(W)})^{-1}(\Delta')\right)$$

is a nowhere dense closed subset of W. Pick any point $\eta \in W$, $\eta \notin \Delta$ which is a generic point of an irreducible component of W (and hence of U). By our choices above the finite set $t(s^{-1}(\{\eta\})) = \{\eta_1, \ldots, \eta_n\}$ is contained in the separated scheme W''. Note that the fibres of s is are finite discrete spaces, and that generalizations lift along the étale morphism t, see Morphisms, Lemmas 28.37.12 and 28.26.8. In this way we see that each η_i is a generic point of an irreducible component of W''. Thus, by Properties, Lemma 27.27.1 we can find an affine open $V \subset W''$ such that $\{\eta_1, \ldots, \eta_n\} \subset V$. By Groupoids, Lemma 38.21.1 this implies that η is contained in an R-invariant affine open subscheme of U. The claim follows as W was chosen as an arbitrary affine open of U and because the set of generic points of irreducible components of $W \setminus \Delta$ is dense in W.

Using the claim we can finish the proof. Namely, if $W \subset U$ is an R-invariant affine open, then the restriction R_W of R to W equals $R_W = s^{-1}(W) = t^{-1}(W)$ (see Groupoids, Definition 38.17.1 and discussion following it). In particular the maps $R_W \to W$ are finite étale also. It follows in particular that R_W is affine. Thus we see that W/R_W is a scheme, by Groupoids, Proposition 38.20.8. On the other hand, W/R_W is an open subspace of X by Spaces, Lemma 47.10.2. Hence having a dense collection of points contained in R-invariant affine open of U certainly implies that the schematic locus of X (see Lemma 48.10.1) is open dense in X. □

We will improve the following proposition to the case of decent algebraic spaces in Decent Spaces, Theorem 50.8.4.

Proposition 48.10.3. *Let S be a scheme. Let X be an algebraic space over S. If X is Zariski locally quasi-separated (for example if X is quasi-separated), then there exists a dense open subspace of X which is a scheme.*

Proof. By Lemma 48.10.1 and Lemma 48.6.6 we may assume that there exists an affine scheme U and a surjective, quasi-compact, étale morphism $U \to X$. Set $R = U \times_X U$, and denote $s, t : R \to U$ the projections as usual. Note that s, t are surjective, quasi-compact and étale, hence also quasi-finite (see Étale Morphisms,

[3]The claim is easier to prove if U is assumed quasi-separated, since in that case Properties, Lemma 27.27.1 may be applied immediately to the R-equivalence class of any generic point of U.

Section 40.11). By More on Morphisms, Lemma 36.31.7 there exists a dense open subscheme $W \subset U$ such that $s^{-1}(W) \to W$ is finite. By Descent, Lemma 34.19.21 being finite is fpqc (and in particular étale) local on the target. Hence we may apply More on Groupoids, Lemma 39.5.4 which says that the largest open $W \subset U$ over which s is finite is R-invariant. It is still dense of course. The restriction R_W of R to W equals $R_W = s^{-1}(W) = t^{-1}(W)$ (see Groupoids, Definition 38.17.1 and discussion following it). By construction $s_W, t_W : R_W \to W$ are finite étale. If we can show the open subspace $W/R_W \subset X$ (see Spaces, Lemma 47.10.2) contains a dense open subspace which is a scheme, then the proposition follows for X. This reduces us to Lemma 48.10.2. □

48.11. Obtaining a scheme

We have used in the previous section that the quotient U/R of an affine scheme U by an equivalence relation R is a scheme if the morphisms $s, t : R \to U$ are finite étale. This is a special case of the following result.

Proposition 48.11.1. *Let S be a scheme. Let (U, R, s, t, c) be a groupoid scheme over S. Assume*

1. *$s, t : R \to U$ finite locally free,*
2. *$j = (t, s)$ is an equivalence, and*
3. *for a dense set of points $u \in U$ the R-equivalence class $t(s^{-1}(\{u\}))$ is contained in an affine open of U.*

Then there exists a finite locally free morphism $U \to M$ of schemes over S such that $R = U \times_M U$ and such that M represents the quotient sheaf U/R in the fppf topology.

Proof. By assumption (3) and Groupoids, Lemma 38.21.1 we can find an open covering $U = \bigcup U_i$ such that each U_i is an R-invariant affine open of U. Set $R_i = R|_{U_i}$. Consider the fppf sheaves $F = U/R$ and $F_i = U_i/R_i$. By Spaces, Lemma 47.10.2 the morphisms $F_i \to F$ are representable and open immersions. By Groupoids, Proposition 38.20.8 the sheaves F_i are representable by affine schemes. Hence we conclude that F is representable by a scheme, see Schemes, Lemma 25.15.4. □

For example, if U is isomorphic to a locally closed subscheme of an affine scheme or isomorphic to a locally closed subscheme of $\mathrm{Proj}(A)$ for some graded ring A, then the third assumption holds by Properties, Lemma 27.27.5. In particular we can apply this to free actions of finite groups and finite group schemes on quasi-affine or quasi-projective schemes. For example, the quotient X/G of a quasi-projective variety X by a free action of a finite group G is a scheme. Here is a detailed statement.

Lemma 48.11.2. *Let S be a scheme. Let $G \to S$ be a group scheme. Let $X \to S$ be a morphism of schemes. Let $a : G \times_S X \to X$ be an action. Assume that*

1. *$G \to S$ is finite locally free,*
2. *the action a is free,*
3. *$X \to S$ is affine, or quasi-affine, or projective, or quasi-projective, or X is isomorphic to an open subscheme of an affine scheme or isomorphic to an open subscheme of $\mathrm{Proj}(A)$ for some graded ring A.*

Then the fppf quotient sheaf X/G is a scheme.

Proof. Since the action is free the morphism $j = (a, \mathrm{pr}) : G \times_S X \to X \times_S X$ is a monomorphism and hence an equivalence relation, see Groupoids, Lemma 38.8.3. The maps $s, t : G \times_S X \to X$ are finite locally free as we've assumed that $G \to S$ is finite locally free. To conclude it now suffices to prove the last assumption of Proposition 48.11.1 holds. Since the action of G is over S it suffices to prove that any finite set of points in a fibre of $X \to S$ is contained in an affine open of X. If X is isomorphic to an open subscheme of an affine scheme or isomorphic to an open subscheme of $\mathrm{Proj}(A)$ for some graded ring A this follows from Properties, Lemma 27.27.5. In the remaining cases, we may replace S by an affine open and we get back to the case we just dealt with. Some details omitted. □

48.12. Points on quasi-separated spaces

Points can behave very badly on algebraic spaces in the generality introduced in the stacks project. However, for quasi-separated spaces their behaviour is mostly like the behaviour of points on schemes. We prove a few results on this in this section.

The following lemma is a key lemma which we will use to prove that certain algebraic spaces are isomorphic to the spectrum of a field.

Lemma 48.12.1. *Let S be a scheme. Let k be a field. Let X be an algebraic space over S and assume that there exists a surjective étale morphism $\mathrm{Spec}(k) \to X$. If X is quasi-separated, then $X \cong \mathrm{Spec}(k')$ where $k' \subset k$ is a finite separable extension.*

Proof. Set $R = \mathrm{Spec}(k) \times_X \mathrm{Spec}(k)$, so that we have a fibre product diagram

$$\begin{array}{ccc} R & \xrightarrow[s]{} & \mathrm{Spec}(k) \\ {\scriptstyle t}\downarrow & & \downarrow \\ \mathrm{Spec}(k) & \longrightarrow & X \end{array}$$

By Spaces, Lemma 47.9.1 we know $X = \mathrm{Spec}(k)/R$ is the quotient sheaf. Because $\mathrm{Spec}(k) \to X$ is étale, the morphisms s and t are étale. Hence $R = \coprod_{i \in I} \mathrm{Spec}(k_i)$ is a disjoint union of spectra of fields, and both s and t induce finite separable field extensions $s, t : k \subset k_i$, see Morphisms, Lemma 28.37.7. Because

$$R = \mathrm{Spec}(k) \times_X \mathrm{Spec}(k) = (\mathrm{Spec}(k) \times_S \mathrm{Spec}(k)) \times_{X \times_S X, \Delta} X$$

and since Δ is quasi-compact by assumption we conclude that $R \to \mathrm{Spec}(k) \times_S \mathrm{Spec}(k)$ is quasi-compact. Hence R is quasi-compact as $\mathrm{Spec}(k) \times_S \mathrm{Spec}(k)$ is affine. We conclude that I is finite. This implies that s and t are finite locally free morphisms. Hence by Groupoids, Proposition 38.20.8 we conclude that $\mathrm{Spec}(k)/R$ is represented by $\mathrm{Spec}(k')$, with $k' \subset k$ finite locally free where

$$k' = \{x \in k \mid s_i(x) = t_i(x) \text{ for all } i \in I\}$$

It is easy to see that k' is a field. □

Remark 48.12.2. The lemma above holds for decent algebraic spaces, see Decent Spaces, Lemma 50.9.2. In fact a decent algebraic space with one point is a scheme, see Decent Spaces, Lemma 50.11.1. This also holds when X is locally separated, because a locally separated algebraic space is decent, see Decent Spaces, Lemma 50.12.2.

Lemma 48.12.3. *Let S be a scheme. Let X be an algebraic space over S. Let U be a scheme. Let $\varphi : U \to X$ be an étale morphism such that the projections $R = U \times_X U \to U$ are quasi-compact; for example if φ is quasi-compact. Then the fibres of*

$$|U| \to |X| \quad \text{and} \quad |R| \to |X|$$

are finite.

Proof. Denote $R = U \times_X U$, and $s, t : R \to U$ the projections. Let $u \in U$ be a point, and let $x \in |X|$ be its image. The fibre of $|U| \to |X|$ over x is equal to $s(t^{-1}(\{u\}))$ by Lemma 48.4.3, and the fibre of $|R| \to |X|$ over x is $t^{-1}(s(t^{-1}(\{u\})))$. Since $t : R \to U$ is étale and quasi-compact, it has finite fibres (as its fibres are disjoint unions of spectra of fields by Morphisms, Lemma 28.37.7 and quasi-compact). Hence we win. □

Lemma 48.12.4. *Let S be a scheme. Let X be a Zariski locally quasi-separated algebraic space over S. Then the topological space $|X|$ is sober (see Topology, Definition 5.7.4).*

Proof. Combining Topology, Lemma 5.7.5 and Lemma 48.6.6 we see that we may assume that there exists an affine scheme U and a surjective, quasi-compact, étale morphism $U \to X$. Set $R = U \times_X U$ with projection maps $s, t : R \to U$. Applying Lemma 48.12.3 we see that the fibres of s, t are finite. It follows all the assumptions of Topology, Lemma 5.18.7 are met, and we conclude that $|X|$ is Kolmogorov[4].

It remains to show that every irreducible closed subset $T \subset |X|$ has a generic point. By Lemma 48.9.2 there exists a closed subspace $Z \subset X$ with $|Z| = |T|$. Note that $U \times_X Z \to Z$ is a quasi-compact, surjective, étale morphism from an affine scheme to Z, hence Z is Zariski locally quasi-separated by Lemma 48.6.6. By Proposition 48.10.3 we see that there exists an open dense subspace $Z' \subset Z$ which is a scheme. This means that $|Z'| \subset T$ is open dense. Hence the topological space $|Z'|$ is irreducible, which means that Z' is an irreducible scheme. By Schemes, Lemma 25.11.1 we conclude that $|Z'|$ is the closure of a single point $\eta \in |Z'| \subset T$ and hence also $T = \overline{\{\eta\}}$, and we win. □

48.13. Étale morphisms of algebraic spaces

This section really belongs in the chapter on morphisms of algebraic spaces, but we need the notion of an algebraic space étale over another in order to define the small étale site of an algebraic space. Thus we need to do some preliminary work on étale morphisms from schemes to algebraic spaces, and étale morphisms between algebraic spaces. For more about étale morphisms of algebraic spaces, see Morphisms of Spaces, Section 49.35.

Lemma 48.13.1. *Let S be a scheme. Let X be an algebraic space over S. Let U, U' be schemes over S.*

(1) *If $U \to U'$ is an étale morphism of schemes, and if $U' \to X$ is an étale morphism from U' to X, then the composition $U \to X$ is an étale morphism from U to X.*

[4] Actually we use here also Schemes, Lemma 25.11.1 (soberness schemes), Morphisms, Lemmas 28.37.12 and 28.26.8 (generalizations lift along étale morphisms), Lemma 48.4.5 (points on an algebraic space in terms of a presentation), and Lemma 48.4.6 (openness quotient map).

(2) *If $\varphi : U \to X$ and $\varphi' : U' \to X$ are étale morphisms towards X, and if $\chi : U \to U'$ is a morphism of schemes such that $\varphi = \varphi' \circ \chi$, then χ is an étale morphism of schemes.*

Proof. Recall that our definition of an étale morphism from a scheme into an algebraic space comes from Spaces, Definition 47.5.1 via the fact that any morphism from a scheme into an algebraic space is representable. Part (1) of the lemma follows from this, the fact that étale morphisms are preserved under composition (Morphisms, Lemma 28.37.3) and Spaces, Lemmas 47.5.4 and 47.5.3 (which are formal). To prove part (2) choose a scheme W over S and a surjective étale morphism $W \to X$. Consider the base change $\chi_W : W \times_X U \to W \times_X U'$ of χ. As $W \times_X U$ and $W \times_X U'$ are étale over W, we conclude that χ_W is étale, by Morphisms, Lemma 28.37.19. On the other hand, in the commutative diagram

$$\begin{array}{ccc} W \times_X U & \longrightarrow & W \times_X U' \\ \downarrow & & \downarrow \\ U & \longrightarrow & U' \end{array}$$

the two vertical arrows are étale and surjective. Hence by Descent, Lemma 34.10.4 we conclude that $U \to U'$ is étale. □

Definition 48.13.2. Let S be a scheme. A morphism $f : X \to Y$ between algebraic spaces over S is called *étale* if and only if for every étale morphism $\varphi : U \to X$ where U is a scheme, the composition $\varphi \circ f$ is étale also.

If X and Y are schemes, then this agree with the usual notion of an étale morphism of schemes. In fact, whenever $X \to Y$ is a representable morphism of algebraic spaces, then this agrees with the notion defined via Spaces, Definition 47.5.1. This follows by combining Lemma 48.13.3 below and Spaces, Lemma 47.11.2.

Lemma 48.13.3. *Let S be a scheme. Let $f : X \to Y$ be a morphism of algebraic spaces over S. The following are equivalent:*

(1) *f is étale,*
(2) *there exists a surjective étale morphism $\varphi : U \to X$, where U is a scheme, such that the composition $f \circ \varphi$ is étale (as a morphism of algebraic spaces),*
(3) *there exists a surjective étale morphism $\psi : V \to Y$, where V is a scheme, such that the base change $V \times_X Y \to V$ is étale (as a morphism of algebraic spaces),*
(4) *there exists a commutative diagram*

$$\begin{array}{ccc} U & \longrightarrow & V \\ \downarrow & & \downarrow \\ X & \longrightarrow & Y \end{array}$$

where U, V are schemes, the vertical arrows are étale, and the left vertical arrow is surjective such that the horizontal arrow is étale.

Proof. Let us prove that (4) implies (1). Assume a diagram as in (4) given. Let $W \to X$ be an étale morphism with W a scheme. Then we see that $W \times_X U \to U$ is étale. Hence $W \times_X U \to V$ is étale, and also $W \times_X U \to Y$ is étale by Lemma

48.13.1 (1). Since also the projection $W \times_X U \to W$ is surjective and étale, we conclude from Lemma 48.13.1 (2) that $W \to Y$ is étale.

Let us prove that (1) implies (4). Assume (1). Choose a commutative diagram

$$\begin{array}{ccc} U & \longrightarrow & V \\ \downarrow & & \downarrow \\ X & \longrightarrow & Y \end{array}$$

where $U \to X$ and $V \to Y$ are surjective and étale, see Spaces, Lemma 47.11.4. By assumption the morphism $U \to Y$ is étale, and hence $U \to V$ is étale by Lemma 48.13.1 (2).

We omit the proof that (2) and (3) are also equivalent to (1). □

Lemma 48.13.4. *The composition of two étale morphisms of algebraic spaces is étale.*

Proof. This is immediate from the definition. □

Lemma 48.13.5. *The base change of an étale morphism of algebraic spaces by any morphism of algebraic spaces is étale.*

Proof. Let $X \to Y$ be an étale morphism of algebraic spaces over S. Let $Z \to Y$ be a morphism of algebraic spaces. Choose a scheme U and a surjective étale morphism $U \to X$. Choose a scheme W and a surjective étale morphism $W \to Z$. Then $U \to Y$ is étale, hence in the diagram

$$\begin{array}{ccc} W \times_Y U & \longrightarrow & W \\ \downarrow & & \downarrow \\ Z \times_Y X & \longrightarrow & Z \end{array}$$

the top horizontal arrow is étale. Moreover, the left vertical arrow is surjective and étale (verification omitted). Hence we conclude that the lower horizontal arrow is étale by Lemma 48.13.3. □

Lemma 48.13.6. *Let S be a scheme. Let X, Y, Z be algebraic spaces. Let $g : X \to Z$, $h : Y \to Z$ be étale morphisms and let $f : X \to Y$ be a morphism such that $h \circ f = g$. Then f is étale.*

Proof. Choose a commutative diagram

$$\begin{array}{ccc} U & \xrightarrow[\chi]{} & V \\ \downarrow & & \downarrow \\ X & \longrightarrow & Y \end{array}$$

where $U \to X$ and $V \to Y$ are surjective and étale, see Spaces, Lemma 47.11.4. By assumption the morphisms $\varphi : U \to X \to Z$ and $\psi : V \to Y \to Z$ are étale. Moreover, $\psi \circ \chi = \varphi$ by our assumption on f, g, h. Hence $U \to V$ is étale by Lemma 48.13.1 part (2). □

Lemma 48.13.7. *Let S be a scheme. If $X \to Y$ is an étale morphism of algebraic spaces over S, then the associated map $|X| \to |Y|$ of topological spaces is open.*

Proof. This is clear from the diagram in Lemma 48.13.3 and Lemma 48.4.6. □

Finally, here is a fun lemma. It is not true that an algebraic space with an étale morphism towards a scheme is a scheme, see Spaces, Example 47.14.2. But it is true if the target is the spectrum of a field.

Lemma 48.13.8. *Let S be a scheme. Let $X \to \mathrm{Spec}(k)$ be étale morphism over S, where k is a field. Then X is a scheme.*

Proof. Let U be an affine scheme, and let $U \to X$ be an étale morphism. By Definition 48.13.2 we see that $U \to \mathrm{Spec}(k)$ is an étale morphism. Hence $U = \coprod_{i=1,\ldots,n} \mathrm{Spec}(k_i)$ is a finite disjoint union of spectra of finite separable extensions k_i of k, see Morphisms, Lemma 28.37.7. The $R = U \times_X U \to U \times_{\mathrm{Spec}(k)} U$ is a monomorphism and $U \times_{\mathrm{Spec}(k)} U$ is also a finite disjoint union of spectra of finite separable extensions of k. Hence by Schemes, Lemma 25.23.10 we see that R is similarly a finite disjoint union of spectra of finite separable extensions of k. This U and R are affine and both projections $R \to U$ are finite locally free. Hence U/R is a scheme by Groupoids, Proposition 38.20.8. By Spaces, Lemma 47.10.2 it is also an open subspace of X. By Lemma 48.10.1 we conclude that X is a scheme. □

48.14. Spaces and fpqc coverings

Let S be a scheme. An algebraic space over S is defined as a sheaf in the fppf topology with additional properties. Hence it is not clear that it satisfies the sheaf property for the fpqc topology (see Topologies, Definition 33.8.12). In this section we discuss this question. However, when we say that the algebraic space X satisfies the sheaf property for the fpqc topology we really only consider fpqc coverings $\{f_i : T_i \to T\}_{i \in I}$ such that T, T_i are objects of the big site $(Sch/S)_{fppf}$ (as per our conventions, see Section 48.2). We first address the question as to whether an algebraic space is separated as a presheaf for the fpqc topology.

Lemma 48.14.1. *Let S be a scheme. Let X be an algebraic space over S. Let $\{f_i : T_i \to T\}_{i \in I}$ be a fpqc covering of schemes over S. Then the map*

$$\mathrm{Mor}_S(T, X) \longrightarrow \prod\nolimits_{i \in I} \mathrm{Mor}_S(T_i, X)$$

is injective.

Proof. Let $a, b : T \to X$ be two morphisms such that $a \circ f_i = b \circ f_i$ for all i. Consider the fibre product

$$T' = X \times_{\Delta_{X/S}, X \times_S X, (a,b)} T.$$

By Spaces, Lemma 47.13.1 the morphism $\Delta_{X/S}$ is a representable monomorphism which is locally of finite type. Hence $T' \to T$ is a monomorphism of schemes which is locally of finite type. Our assumption that $a \circ f_i = b \circ f_i$ implies that each $T_i \to T$ factors (uniquely) through T'. Consider the commutative diagram

$$\begin{array}{ccc} T_i \times_T T' & \longrightarrow & T' \\ \uparrow & \nearrow & \downarrow \\ T_i & \longrightarrow & T \end{array}$$

Since the projection $T_i \times_T T' \to T_i$ is a monomorphism with a section we conclude it is an isomorphism. Hence we conclude that $T' \to T$ is an isomorphism by Descent, Lemma 34.19.15. This means $a = b$ as desired. □

Lemma 48.14.2. *Let S be a scheme. Let X be an algebraic space over S. Let $X = \bigcup_{j \in J} X_j$ be a Zariski covering, see Spaces, Definition 47.12.5. If each X_j satisfies the sheaf property for the fpqc topology then X satisfies the sheaf property for the fpqc topology.*

Proof. Assume each X_j satisfies the sheaf property for the fpqc topology. Let $\{f_i : T_i \to T\}_{i \in I}$ be a fpqc covering of schemes over S. Let $a_i \in \mathrm{Mor}_S(T_i, X)$ be such that $a_i \circ \mathrm{pr}_0 = a_{i'} \circ \mathrm{pr}_1$ as morphisms $T_i \times_T T_{i'} \to X$. We have to prove that these morphisms come from a morphism $a : T \to X$. Consider the open subsets $T_{i,j} = a_i^{-1}(X_j)$. Then it is clear that $\mathrm{pr}_0^{-1}(T_{i,j}) = \mathrm{pr}_1^{-1}(T_{i',j})$ as open subsets of $T_i \times_T T_{i'}$. Hence there exist open subsets $U_j \subset T$ such that $T_{i,j} = f_i^{-1}(U_j)$, see Descent, Lemma 34.9.2. In particular, $\{T_{i,j} \to U_j\}_{i \in I}$ is a fpqc covering of U_j, and the morphisms $a_{i,j} = a_i|_{T_{i,j}}$ are morphisms into X_j. By assumption there exist morphisms $\alpha_j : U_j \to X_j$ such that $T_{i,j} \to U_j \to X_j$ agrees with $a_{i,j}$. By Lemma 48.14.1 we conclude that $\alpha_j|_{U_j \cap U_{j'}}$ agrees with $\alpha_{j'}|_{U_j \cap U_{j'}}$. Hence, since X is a sheaf for the Zariski topology we conclude that the α_j glue to a morphism $a : T \to X$. By construction we have $a \circ f_i|_{T_{i,j}} = a_{i,j} = a_i|_{T_{i,j}}$. Using the sheaf condition for the Zariski topology one more time we conclude that $a \circ f_i = a_i$ as desired. □

Lemma 48.14.3. *Let S be a scheme. Let X be an algebraic space over S. If X is Zariski locally quasi-separated over S, then X satisfies the sheaf condition for the fpqc topology.*

Proof. By Lemmas 48.6.6 and 48.14.2 we may assume there exists an affine scheme U and a surjective étale quasi-compact morphism $\varphi : U \to X$. By Lemma 48.6.4 the morphism $U \to X$ is also separated.

Let $\{f_i : T_i \to T\}_{i \in I}$ be a fpqc covering of schemes over S. Let $a_i \in \mathrm{Mor}_S(T_i, X)$ be such that $a_i \circ \mathrm{pr}_0 = a_j \circ \mathrm{pr}_1$ as morphisms $T_i \times_T T_j \to X$. We have to prove that these morphisms come from a morphism $a : T \to X$. Consider the schemes

$$W_i = T_i \times_{a_i, X, \varphi} U.$$

The structure morphisms $W_i \to T_i$ are surjective, separated, quasi-compact and étale, in particular also quasi-finite (see Morphisms, Lemma 28.37.6). Hence each $W_i \to T_i$ is quasi-affine, see More on Morphisms, Lemma 36.31.2. For each pair of indices $i, j \in I$ we have canonical isomorphisms

$$\begin{aligned} W_i \times_T T_j &= (T_i \times_{a_i, X, \varphi} U) \times_T T_j \\ &= (T_i \times_T T_j) \times_{a_i \circ \mathrm{pr}_0, X, \varphi} U \\ &= (T_i \times_T T_j) \times_{a_j \circ \mathrm{pr}_1, X, \varphi} U \\ &= T_i \times_T (T_j \times_{a_j, X, \varphi} U) \\ &= T_i \times_T W_j. \end{aligned}$$

These isomorphisms satisfy the cocycle condition of Descent, Definition 34.30.3 as a_i, a_j and a_k agree over $T_i \times_T T_j \times_T T_k$ (some details omitted). By Descent, Lemma 34.34.1 this descent datum is effective and we conclude that there exists a scheme $W \to T$ and isomorphisms $T_i \times_T W = W_i$ compatible with the canonical descent

datum and the one given above. In particular we see that $\{W_i \to W\}_{i\in I}$ is the base change of a fpqc covering, and hence a fpqc covering. Note that by construction the morphisms $b_i : W_i = T_i \times_X U \to U$ have the property $b_i \circ \mathrm{pr}_0 = b_j \circ \mathrm{pr}_1$ as morphisms $W_i \times_W W_j \to U$. Hence by Descent, Lemma 34.9.3 we see that there exists a morphism of schemes $b : W \to U$ which restricts to b_i on W_i for all i.

By Descent, Lemmas 34.19.6 and 34.19.27 the morphism $W \to T$ is surjective and étale. Hence, in order to see that b gives rise to the morphism $a : T \to X$ we are looking for, it suffices to show that $b\circ\mathrm{pr}_0 = b\circ\mathrm{pr}_1$ as morphisms $W \times_T W \to X$. For this we note that we do know that $b_i \circ \mathrm{pr}_0 = b_j \circ \mathrm{pr}_1$ as morphisms $W_i \times_T W_j \to X$, because

$$W_i \times_T W_j = (T_i \times_T T_j) \times_{(a_i,a_j),X\times_S X} U \times_S U$$

and $(a_i, a_j) : T_i \times_T T_j \to X \times_S X$ factors through $\Delta_{X/S}$ by assumption. In other words we conclude that over the members of the fpqc covering $\{W_i \times_T W_j \to W \times_T W\}$ the morphisms $b \circ \mathrm{pr}_0$ and $b \circ \mathrm{pr}_1$ agree, and hence by Lemma 48.14.1 they agree. As X is a sheaf for the fppf topology we obtain a unique morphism $a : T \to X$ whose composition with $W \to T$ agrees with b. We omit the final verification that $a|_{T_i} = a_i$. □

Remark 48.14.4. The proof of Lemma 48.14.3 works for any algebraic space which has a Zariski covering $X = \bigcup X_i$ such that for each i there exists a surjective étale separated quasi-compact morphism $U_i \to X_i$ where U_i is a scheme. This condition is slightly stronger than the condition of being very reasonable, and the current proof does not work even for very reasonable spaces. There are results in the literature, see David Rydh's paper [**Ryd07b**] and its references, to remedy this. On the other hand, it seems that the question for general algebraic spaces as defined in the stacks project is still open. If this is no longer the case, please email stacks.project@gmail.com so we can update this remark.

48.15. The étale site of an algebraic space

In this section we define the small étale site of an algebraic space. This is the analogue of the small étale site $S_{étale}$ of a scheme. Lemma 48.13.1 implies that in the definition below any morphism between objects of the étale site of X is étale, and that any scheme étale over an object of $X_{étale}$ is also an object of $X_{étale}$.

Definition 48.15.1. Let S be a scheme. Let Sch_{fppf} be a big fppf site containing S, and let $Sch_{étale}$ be the corresponding big étale site (i.e., having the same underlying category). Let X be an algebraic space over S. The *small étale site* $X_{étale}$ of X is defined as follows:

(1) An object of $X_{étale}$ is a morphism $\varphi : U \to X$ where $U \in \mathrm{Ob}((Sch/S)_{étale})$ is a scheme and φ is an étale morphism,
(2) a morphism $(\varphi : U \to X) \to (\varphi' : U' \to X)$ is given by a morphism of schemes $\chi : U \to U'$ such that $\varphi = \varphi' \circ \chi$, and
(3) a family of morphisms $\{(U_i \to X) \to (U \to X)\}_{i\in I}$ of $X_{étale}$ is a covering if and only if $\{U_i \to U\}_{i\in I}$ is a covering of $(Sch/S)_{étale}$.

A consequence of our choice is that the étale site of an algebraic space in general does not have a final object! On the other hand, if X happens to be a scheme, then the definition above agrees with Topologies, Definition 33.4.8.

There are several other choices we could have made here. For example we could have considered all *algebraic spaces* U which are étale over X, or we could have considered all *affine schemes* U which are étale over X. We decided not to do so, since we like to think of plain old schemes as the fundamental objects of algebraic geometry. On the other hand, we do need these notions also, since the small étale site of an algebraic space is not sufficiently flexible, especially when discussing functoriality of the small étale site, see Lemma 48.15.7 below.

Definition 48.15.2. Let S be a scheme. Let Sch_{fppf} be a big fppf site containing S, and let $Sch_{étale}$ be the corresponding big étale site (i.e., having the same underlying category). Let X be an algebraic space over S. The site $X_{spaces,étale}$ of X is defined as follows:

(1) An object of $X_{spaces,étale}$ is a morphism $\varphi : U \to X$ where U is an algebraic space over S and φ is an étale morphism of algebraic spaces over S,
(2) a morphism $(\varphi : U \to X) \to (\varphi' : U' \to X)$ of $X_{spaces,étale}$ is given by a morphism of algebraic spaces $\chi : U \to U'$ such that $\varphi = \varphi' \circ \chi$, and
(3) a family of morphisms $\{\varphi_i : (U_i \to X) \to (U \to X)\}_{i \in I}$ of $X_{spaces,étale}$ is a covering if and only if $|U| = \bigcup \varphi_i(|U_i|)$.

(As usual we choose a set of coverings of this type, including at least the coverings in $X_{étale}$, as in Sets, Lemma 3.11.1 to turn $X_{spaces,étale}$ into a site.)

Since the identity morphism of X is étale it is clear that $X_{spaces,étale}$ does have a final object. Let us show right away that the corresponding topos equals the small étale topos of X.

Lemma 48.15.3. *The functor*

$$X_{étale} \longrightarrow X_{spaces,étale}, \quad U/X \longmapsto U/X$$

is a special cocontinuous functor (Sites, Definition 7.28.2) and hence induces an equivalence of topoi $Sh(X_{étale}) \to Sh(X_{spaces,étale})$.

Proof. We have to show that the functor satisfies the assumptions (1) – (5) of Sites, Lemma 7.28.1. It is clear that the functor is continuous and cocontinuous, which proves assumptions (1) and (2). Assumptions (3) and (4) hold simply because the functor is fully faithful. Assumption (5) holds, because an algebraic space by definition has a covering by a scheme. □

Remark 48.15.4. Let us explain the meaning of Lemma 48.15.3. Let S be a scheme, and let X be an algebraic space over S. Let $\mathcal{F}$ be a sheaf on the small étale site $X_{étale}$ of X. The lemma says that there exists a unique sheaf $\mathcal{F}'$ on $X_{spaces,étale}$ which restricts back to $\mathcal{F}$ on the subcategory $X_{étale}$. If $U \to X$ is an étale morphism of algebraic spaces, then how do we compute $\mathcal{F}'(U)$? Well, by definition of an algebraic space there exists a scheme U' and a surjective étale morphism $U' \to U$. Then $\{U' \to U\}$ is a covering in $X_{spaces,étale}$ and hence we get an equalizer diagram

$$\mathcal{F}'(U) \longrightarrow \mathcal{F}(U') \rightrightarrows \mathcal{F}(U' \times_U U').$$

Note that $U' \times_U U'$ is a scheme, and hence we may write $\mathcal{F}$ and not $\mathcal{F}'$. Thus we see how to compute $\mathcal{F}'$ when given the sheaf $\mathcal{F}$.

Lemma 48.15.5. *Let S be a scheme. Let X be an algebraic space over S. Let $X_{affine,étale}$ denote the full subcategory of $X_{étale}$ whose objects are those $U/X \in \mathrm{Ob}(X_{étale})$ with U affine. A covering of $X_{affine,étale}$ will be a standard étale covering, see Topologies, Definition 33.4.5. Then restriction*

$$\mathcal{F} \longmapsto \mathcal{F}|_{X_{affine,étale}}$$

defines an equivalence of topoi $Sh(S_{étale}) \cong Sh(S_{affine,étale})$.

Proof. This you can show directly from the definitions, and is a good exercise. But it also follows immediately from Sites, Lemma 7.28.1 by checking that the inclusion functor $X_{affine,étale} \to X_{étale}$ is a special cocontinuous functor as in Sites, Definition 7.28.2. □

Definition 48.15.6. Let S be a scheme. Let X be an algebraic space over S. The *étale topos* of X, or more precisely the *small étale topos* of X is the category $Sh(X_{étale})$ of sheaves of sets on $X_{étale}$.

By Lemma 48.15.3 we have $Sh(X_{étale}) = Sh(X_{spaces,étale})$, so we can also think of this as the category of sheaves of sets on $X_{spaces,étale}$. Similarly, by Lemma 48.15.5 we see that $Sh(X_{étale}) = Sh(X_{affine,étale})$. It turns out that the topos is functorial with respect to morphisms of algebraic spaces. Here is a precise statement.

Lemma 48.15.7. *Let S be a scheme. Let $f : X \to Y$ be a morphism of algebraic spaces over S.*

(1) *The continuous functor*

$$Y_{spaces,étale} \longrightarrow X_{spaces,étale}, \quad V \longmapsto X \times_Y V$$

induces a morphism of sites

$$f_{spaces,étale} : X_{spaces,étale} \to Y_{spaces,étale}.$$

(2) *The rule $f \mapsto f_{spaces,étale}$ is compatible with compositions, in other words $(f \circ g)_{spaces,étale} = f_{spaces,étale} \circ g_{spaces,étale}$ (see Sites, Definition 7.15.4).*
(3) *The morphism of topoi associated to $f_{spaces,étale}$ induces, via Lemma 48.15.3, a morphism of topoi $f_{small} : Sh(X_{étale}) \to Sh(Y_{étale})$ whose construction is compatible with compositions.*
(4) *If f is a representable morphism of algebraic spaces, then f_{small} comes from a morphism of sites $X_{étale} \to Y_{étale}$, corresponding to the continuous functor $V \mapsto X \times_Y V$.*

Proof. Let us show that the functor described in (1) satisfies the assumptions of Sites, Proposition 7.15.6. Thus we have to show that $Y_{spaces,étale}$ has a final object (namely Y) and that the functor transforms this into a final object in $X_{spaces,étale}$ (namely X). This is clear as $X \times_Y Y = X$ in any category. Next, we have to show that $Y_{spaces,étale}$ has fibre products. This is true since the category of algebraic spaces has fibre products, and since $V \times_Y V'$ is étale over Y if V and V' are étale over Y (see Lemmas 48.13.4 and 48.13.5 above). OK, so the proposition applies and we see that we get a morphism of sites as described in (1).

Part (2) you get by unwinding the definitions. Part (3) is clear by using the equivalences for X and Y from Lemma 48.15.3 above. Part (4) follows, because if f is

representable, then the functors above fit into a commutative diagram

$$\begin{array}{ccc} X_{étale} & \longrightarrow & X_{spaces,étale} \\ \uparrow & & \uparrow \\ Y_{étale} & \longrightarrow & Y_{spaces,étale} \end{array}$$

of categories. □

We can do a little bit better than the lemma above in describing the relationship between sheaves on X and sheaves on Y. Namely, we can formulate this in turns of f-maps, compare Sheaves, Definition 6.21.7, as follows.

Definition 48.15.8. Let S be a scheme. Let $f : X \to Y$ be a morphism of algebraic spaces over S. Let $\mathcal{F}$ be a sheaf of sets on $X_{étale}$ and let $\mathcal{G}$ be a sheaf of sets on $Y_{étale}$. An *f-map* $\varphi : \mathcal{G} \to \mathcal{F}$ is a collection of maps $\varphi_{(U,V,g)} : \mathcal{G}(V) \to \mathcal{F}(U)$ indexed by commutative diagrams

$$\begin{array}{ccc} U & \longrightarrow & X \\ {\scriptstyle g}\downarrow & & \downarrow{\scriptstyle f} \\ V & \longrightarrow & Y \end{array}$$

where $U \in X_{étale}$, $V \in Y_{étale}$ such that whenever given an extended diagram

$$\begin{array}{ccccc} U' & \longrightarrow & U & \longrightarrow & X \\ {\scriptstyle g'}\downarrow & & {\scriptstyle g}\downarrow & & \downarrow{\scriptstyle f} \\ V' & \longrightarrow & V & \longrightarrow & Y \end{array}$$

with $V' \to V$ and $U' \to U$ étale morphisms of schemes the diagram

$$\begin{array}{ccc} \mathcal{G}(V) & \xrightarrow[\varphi_{(U,V,g)}]{} & \mathcal{F}(U) \\ {\scriptstyle \text{restriction of } \mathcal{G}}\downarrow & & \downarrow{\scriptstyle \text{restriction of } \mathcal{F}} \\ \mathcal{G}(V') & \xrightarrow{\varphi_{(U',V',g')}} & \mathcal{F}(U') \end{array}$$

commutes.

Lemma 48.15.9. *Let S be a scheme. Let $f : X \to Y$ be a morphism of algebraic spaces over S. Let $\mathcal{F}$ be a sheaf of sets on $X_{étale}$ and let $\mathcal{G}$ be a sheaf of sets on $Y_{étale}$. There are canonical bijections between the following three sets:*

1. *The set of maps $\mathcal{G} \to f_{small,*}\mathcal{F}$.*
2. *The set of maps $f_{small}^{-1}\mathcal{G} \to \mathcal{F}$.*
3. *The set of f-maps $\varphi : \mathcal{G} \to \mathcal{F}$.*

Proof. Note that (1) and (2) are the same because the functors $f_{small,*}$ and f_{small}^{-1} are a pair of adjoint functors. Suppose that $\alpha : f_{small}^{-1}\mathcal{G} \to \mathcal{F}$ is a map of sheaves on $Y_{étale}$. Let a diagram

$$\begin{array}{ccc} U & \xrightarrow[j_U]{} & X \\ {\scriptstyle g}\downarrow & & \downarrow{\scriptstyle f} \\ V & \xrightarrow{j_V} & Y \end{array}$$

as in Definition 48.15.8 be given. By the commutativity of the diagram we also get a map $g_{small}^{-1}(j_V)^{-1}\mathcal{G} \to (j_U)^{-1}\mathcal{F}$ (compare Sites, Section 7.24 for the description of the localization functors). Hence we certainly get a map $\varphi_{(V,U,g)} : \mathcal{G}(V) = (j_V)^{-1}\mathcal{G}(V) \to (j_U)^{-1}\mathcal{F}(U) = \mathcal{F}(U)$. We omit the verification that this rule is compatible with further restrictions and defines an f-map from $\mathcal{G}$ to $\mathcal{F}$.

Conversely, suppose that we are given an f-map $\varphi = (\varphi_{(U,V,g)})$. Let $\mathcal{G}'$ (resp. $\mathcal{F}'$) denote the extension of $\mathcal{G}$ (resp. $\mathcal{F}$) to $Y_{spaces,étale}$ (resp. $X_{spaces,étale}$), see Lemma 48.15.3. Then we have to construct a map of sheaves

$$\mathcal{G}' \longrightarrow (f_{spaces,étale})_*\mathcal{F}'$$

To do this, let $V \to Y$ be an étale morphism of algebraic spaces. We have to construct a map of sets

$$\mathcal{G}'(V) \to \mathcal{F}'(X \times_Y V)$$

Choose an étale surjective morphism $V' \to V$ with V' a scheme, and after that choose an étale surjective morphism $U' \to X \times_U V'$ with U' a scheme. We get a morphism of schemes $g' : U' \to V'$ and also a morphism of schemes

$$g'' : U' \times_{X \times_Y V} U' \longrightarrow V' \times_V V'$$

Consider the following diagram

$$\begin{array}{ccccc}
\mathcal{F}'(X \times_Y V) & \longrightarrow & \mathcal{F}(U') & \rightrightarrows & \mathcal{F}(U' \times_{X \times_Y V} U') \\
\uparrow & & \uparrow{\scriptstyle \varphi_{(U',V',g')}} & & \uparrow{\scriptstyle \varphi_{(U'',V'',g'')}} \\
\mathcal{G}'(X \times_Y V) & \longrightarrow & \mathcal{G}(V') & \rightrightarrows & \mathcal{G}(V' \times_V V')
\end{array}$$

The compatibility of the maps $\varphi_{...}$ with restriction shows that the two right squares commute. The definition of coverings in $X_{spaces,étale}$ shows that the horizontal rows are equalizer diagrams. Hence we get the dotted arrow. We leave it to the reader to show that these arrows are compatible with the restriction mappings. □

If the morphism of algebraic spaces $X \to Y$ is étale, then the morphism of topoi $Sh(X_{étale}) \to Sh(Y_{étale})$ is a localization. Here is a statement.

Lemma 48.15.10. *Let S be a scheme, and let $f : X \to Y$ be a morphism of algebraic spaces over S. Assume f is étale. In this case there is a functor*

$$j : X_{étale} \to Y_{étale}, \quad (\varphi : U \to X) \mapsto (f \circ \varphi : U \to Y)$$

which is cocontinuous. The morphism of topoi f_{small} is the morphism of topoi associated to j, see Sites, Lemma 7.20.1. Moreover, j is continuous as well, hence Sites, Lemma 7.20.5 applies. In particular $f_{small}^{-1}\mathcal{G}(U) = \mathcal{G}(jU)$ for all sheaves $\mathcal{G}$ on $Y_{étale}$.

Proof. Note that by our very definition of an étale morphism of algebraic spaces (Definition 48.13.2) it is indeed the case that the rule given defines a functor j as indicated. It is clear that j is cocontinuous and continuous, simply because a covering $\{U_i \to U\}$ of $j(\varphi : U \to X)$ in $Y_{étale}$ is the same thing as a covering of $(\varphi : U \to X)$ in $X_{étale}$. It remains to show that j induces the same morphism of

topoi as f_{small}. To see this we consider the diagram

$$\begin{array}{ccc} X_{étale} & \longrightarrow & X_{spaces,étale} \\ \downarrow{\scriptstyle j} & & \uparrow{\scriptstyle v:V\mapsto X\times_Y V} \\ Y_{étale} & \longrightarrow & Y_{spaces,étale} \end{array} \qquad j_{spaces}$$

of categories. Here the functor j_{spaces} is the obvious extension of j to the category $X_{spaces,étale}$. Thus the inner square is commutative. In fact j_{spaces} can be identified with the localization functor $j_X : Y_{spaces,étale}/X \to Y_{spaces,étale}$ discussed in Sites, Section 7.24. Hence, by Sites, Lemma 7.26.2 the cocontinuous functor j_{spaces} and the functor v of the diagram induce the same morphism of topoi. By Sites, Lemma 7.20.2 the commutativity of the inner square (consisting of cocontinuous functors between sites) gives a commutative diagram of associated morphisms of topoi. Hence, by the construction of f_{small} in Lemma 48.15.7 we win. □

The lemma above says that the pullback of $\mathcal{G}$ via an étale morphism $f : X \to Y$ of algebraic spaces is simply the restriction of $\mathcal{G}$ to the category $X_{étale}$. We will often use the short hand

$$\mathcal{G}|_{X_{étale}} = f_{small}^{-1}\mathcal{G} \tag{48.15.10.1}$$

to indicate this. Note that the functor $j : X_{étale} \to Y_{étale}$ of the lemma in this situation is faithful, but not fully faithful in general. We will discuss this in a more technical fashion in Section 48.25.

Lemma 48.15.11. *Let S be a scheme. Let*

$$\begin{array}{ccc} X' & \longrightarrow & X \\ {\scriptstyle f'}\downarrow & & \downarrow{\scriptstyle f} \\ Y' & \xrightarrow{g} & Y \end{array}$$

be a cartesian square of algebraic spaces over S. Let $\mathcal{F}$ be a sheaf on $X_{étale}$. If g is étale, then

1. *$f'_{small,*}(\mathcal{F}|_{X'}) = (f_{small,*}\mathcal{F})|_{Y'}$ in $Sh(Y'_{étale})$[5], and*
2. *if $\mathcal{F}$ is an abelian sheaf, then $R^i f'_{small,*}(\mathcal{F}|_{X'}) = (R^i f_{small,*}\mathcal{F})|_{Y'}$.*

Proof. Consider the following diagram of functors

$$\begin{array}{ccc} X'_{spaces,étale} & \xrightarrow[j]{} & X_{spaces,étale} \\ {\scriptstyle V'\mapsto V'\times_{Y'} X'}\uparrow & & \uparrow{\scriptstyle V\mapsto V\times_Y X} \\ Y'_{spaces,étale} & \xrightarrow{j} & Y_{spaces,étale} \end{array}$$

The horizontal arrows are localizations and the vertical arrows induce morphisms of sites. Hence the last statement of Sites, Lemma 7.27.1 gives (1). To see (2) apply (1) to an injective resolution of $\mathcal{F}$ and use that restriction is exact and preserves injectives (see Cohomology on Sites, Lemma 21.8.1). □

[5] Also $(f')^{-1}_{small}(\mathcal{G}|_{Y'}) = (f^{-1}_{small}\mathcal{G})|_{X'}$ because of commutativity of the diagram and (48.15.10.1)

The following lemma says that you can think of a sheaf on the small étale site of an algebraic space as a compatible collection of sheaves on the small étale sites of schemes étale over the space. Please note that all the comparison mappings c_f in the lemma are isomorphisms, which is compatible with Topologies, Lemma 33.4.18 and the fact that all morphisms between objects of $X_{étale}$ are étale.

Lemma 48.15.12. *Let S be a scheme. Let X be an algebraic space over S. A sheaf $\mathcal{F}$ on $X_{étale}$ is given by the following data:*

(1) *for every $U \in \mathrm{Ob}(X_{étale})$ a sheaf $\mathcal{F}_U$ on $U_{étale}$,*
(2) *for every $f : U' \to U$ in $X_{étale}$ an isomorphism $c_f : f_{small}^{-1}\mathcal{F}_U \to \mathcal{F}_{U'}$.*

These data are subject to the condition that given any $f : U' \to U$ and $g : U'' \to U'$ in $X_{étale}$ the composition $g_{small}^{-1}c_f \circ c_g$ is equal to $c_{f\circ g}$.

Proof. Given a sheaf $\mathcal{F}$ on $X_{étale}$ and an object $\varphi : U \to X$ of $X_{étale}$ we set $\mathcal{F}_U = \varphi_{small}^{-1}\mathcal{F}$. If $\varphi' : U' \to X$ is a second object of $X_{étale}$, and $f : U' \to U$ is a morphism between them, then the isomorphism c_f comes from the fact that $f_{small}^{-1} \circ \varphi_{small}^{-1} = (\varphi')_{small}^{-1}$, see Lemma 48.15.7. The condition on the transitivity of the isomorphisms c_f follows from the functoriality of the small étale sites also; verification omitted.

Conversely, suppose we are given a collection of data $(\mathcal{F}_U, c_f)$ as in the lemma. In this case we simply define $\mathcal{F}$ by the rule $U \mapsto \mathcal{F}_U(U)$. Details omitted. □

Let S be a scheme. Let X be an algebraic space over S. Let $X = U/R$ be a presentation of X coming from any surjective étale morphism $\varphi : U \to X$, see Spaces, Definition 47.9.3. In particular, we obtain a groupoid (U, R, s, t, c, e, i) such that $j = (t, s) : R \to U \times_S U$, see Groupoids, Lemma 38.11.3.

Lemma 48.15.13. *With S, $\varphi : U \to X$, and (U, R, s, t, c, e, i) as above. For any sheaf $\mathcal{F}$ on $X_{étale}$ the sheaf[6] $\mathcal{G} = \varphi^{-1}\mathcal{F}$ comes equipped with a canonical isomorphism*

$$\alpha : t^{-1}\mathcal{G} \longrightarrow s^{-1}\mathcal{G}$$

such that the diagram

$$\begin{array}{ccccc} & & pr_1^{-1}t^{-1}\mathcal{G} & \xrightarrow[pr_1^{-1}\alpha]{} & pr_1^{-1}s^{-1}\mathcal{G} \\ & \swarrow^{=} & & & \searrow^{=} \\ pr_0^{-1}s^{-1}\mathcal{G} & & & & c^{-1}s^{-1}\mathcal{G} \\ & \nwarrow^{pr_0^{-1}\alpha} & & \nearrow^{c^{-1}\alpha} & \\ & & pr_0^{-1}t^{-1}\mathcal{G} & = & c^{-1}t^{-1}\mathcal{G} \end{array}$$

is a commutative. The functor $\mathcal{F} \mapsto (\mathcal{G}, \alpha)$ defines an equivalence of categories between sheaves on $X_{étale}$ and pairs $(\mathcal{G}, \alpha)$ as above.

First proof of Lemma 48.15.13. Let $\mathcal{C} = X_{spaces,étale}$. By Lemma 48.15.10 and its proof we have $U_{spaces,étale} = \mathcal{C}/U$ and the pullback functor φ^{-1} is just the

[6] In this lemma and its proof we write simply φ^{-1} instead of φ_{small}^{-1} and similarly for all the other pullbacks.

restriction functor. Moreover, $\{U \to X\}$ is a covering of the site $\mathcal{C}$ and $R = U \times_X U$. The isomorphism α is just the canonical identification

$$\left(\mathcal{F}|_{\mathcal{C}/U}\right)|_{\mathcal{C}/U \times_X U} = \left(\mathcal{F}|_{\mathcal{C}/U}\right)|_{\mathcal{C}/U \times_X U}$$

and the commutativity of the diagram is the cocycle condition for glueing data. Hence this lemma is a special case of glueing of sheaves, see Sites, Section 7.25. □

Second proof of Lemma 48.15.13. The existence of α comes from the fact that $\varphi \circ t = \varphi \circ s$ and that pullback is functorial in the morphism, see Lemma 48.15.7. In exactly the same way, i.e., by functoriality of pullback, we see that the isomorphism α fits into the commutative diagram. The construction $\mathcal{F} \mapsto (\varphi^{-1}\mathcal{F}, \alpha)$ is clearly functorial in the sheaf $\mathcal{F}$. Hence we obtain the functor.

Conversely, suppose that $(\mathcal{G}, \alpha)$ is a pair. Let $V \to X$ be an object of $X_{étale}$. In this case the morphism $V' = U \times_X V \to V$ is a surjective étale morphism of schemes, and hence $\{V' \to V\}$ is an étale covering of V. Set $\mathcal{G}' = (V' \to V)^{-1}\mathcal{G}$. Since $R = U \times_X U$ with $t = \mathrm{pr}_0$ and $s = \mathrm{pr}_0$ we see that $V' \times_V V' = R \times_X V$ with projection maps $s', t' : V' \times_V V' \to V'$ equal to the pullbacks of t and s. Hence α pulls back to an isomorphism $\alpha' : (t')^{-1}\mathcal{G}' \to (s')^{-1}\mathcal{G}'$. Having said this we simply define

$$\mathcal{F}(V) = \text{Equalizer}(\mathcal{G}(V') \rightrightarrows \mathcal{G}(V' \times_V V')).$$

We omit the verification that this defines a sheaf. To see that $\mathcal{G}(V) = \mathcal{F}(V)$ if there exists a morphism $V \to U$ note that in this case the equalizer is $H^0(\{V' \to V\}, \mathcal{G}) = \mathcal{G}(V)$. □

48.16. Points of the small étale site

This section is the analogue of Étale Cohomology, Section 44.29.

Definition 48.16.1. Let S be a scheme. Let X be an algebraic space over S.

(1) A *geometric point* of X is a morphism $\overline{x} : \mathrm{Spec}(k) \to X$, where k is an algebraically closed field. We often abuse notation and write $\overline{x} = \mathrm{Spec}(k)$.
(2) For every geometric point $\overline{x}$ we have the corresponding "image" point $x \in |X|$. We say that $\overline{x}$ is a *geometric point lying over* x.

It turns out that we can take stalks of sheaves on $X_{étale}$ at geometric point exactly in the same way as was done in the case of the small étale site of a scheme. In order to do this we define the notion of an étale neighbourhood as follows.

Definition 48.16.2. Let S be a scheme. Let X be an algebraic space over S. Let $\overline{x}$ be a geometric point of X.

(1) An *étale neighborhood* of $\overline{x}$ of X is a commutative diagram

$$\begin{array}{ccc} & & U \\ & \overset{\bar{u}}{\nearrow} & \downarrow \varphi \\ \overline{x} & \xrightarrow{\bar{x}} & X \end{array}$$

where φ is an étale morphism of algebraic spaces over S. We will use the notation $\varphi : (U, \overline{u}) \to (X, \overline{x})$ to indicate this situation.
(2) A *morphism of étale neighborhoods* $(U, \overline{u}) \to (U', \overline{u}')$ is an X-morphism $h : U \to U'$ such that $\overline{u}' = h \circ \overline{u}$.

Note that we allow U to be an algebraic space. When we take stalks of a sheaf on $X_{étale}$ we have to restrict to those U which are in $X_{étale}$, and so in this case we will only consider the case where U is a scheme. Alternately we can work with the site $X_{space,étale}$ and consider all étale neighbourhoods. And there won't be any difference because of the last assertion in the following lemma.

Lemma 48.16.3. *Let S be a scheme. Let X be an algebraic space over S. Let $\overline{x}$ be a geometric point of X. The category of étale neighborhoods is cofiltered. More precisely:*

1. *Let $(U_i, \overline{u}_i)_{i=1,2}$ be two étale neighborhoods of $\overline{x}$ in X. Then there exists a third étale neighborhood $(U, \overline{u})$ and morphisms $(U, \overline{u}) \to (U_i, \overline{u}_i)$, $i = 1, 2$.*
2. *Let $h_1, h_2 : (U, \overline{u}) \to (U', \overline{u}')$ be two morphisms between étale neighborhoods of $\overline{s}$. Then there exist an étale neighborhood $(U'', \overline{u}'')$ and a morphism $h : (U'', \overline{u}'') \to (U, \overline{u})$ which equalizes h_1 and h_2, i.e., such that $h_1 \circ h = h_2 \circ h$.*

Moreover, given any étale neighbourhood $(U, \overline{u}) \to (X, \overline{x})$ there exists a morphism of étale neighbourhoods $(U', \overline{u}') \to (U, \overline{u})$ where U' is a scheme.

Proof. For part (1), consider the fibre product $U = U_1 \times_X U_2$. It is étale over both U_1 and U_2 because étale morphisms are preserved under base change and composition, see Lemmas 48.13.5 and 48.13.4. The map $\overline{u} \to U$ defined by $(\overline{u}_1, \overline{u}_2)$ gives it the structure of an étale neighborhood mapping to both U_1 and U_2.

For part (2), define U'' as the fibre product

$$\begin{array}{ccc} U'' & \longrightarrow & U \\ \downarrow & & \downarrow{\scriptstyle (h_1,h_2)} \\ U' & \xrightarrow{\Delta} & U' \times_X U'. \end{array}$$

Since $\overline{u}$ and $\overline{u}'$ agree over X with $\overline{x}$, we see that $\overline{u}'' = (\overline{u}, \overline{u}')$ is a geometric point of U''. In particular $U'' \neq \emptyset$. Moreover, since U' is étale over X, so is the fibre product $U' \times_X U'$ (as seen above in the case of $U_1 \times_X U_2$). Hence the vertical arrow (h_1, h_2) is étale by Lemma 48.13.6. Therefore U'' is étale over U' by base change, and hence also étale over X (because compositions of étale morphisms are étale). Thus $(U'', \overline{u}'')$ is a solution to the problem posed by (2).

To see the final assertion, choose any surjective étale morphism $U' \to U$ where U' is a scheme. Then $U' \times_U \overline{u}$ is a scheme surjective and étale over $\overline{u} = \mathrm{Spec}(k)$ with k algebraically closed. It follows (see Morphisms, Lemma 28.37.7) that $U' \times_U \overline{u} \to \overline{u}$ has a section which gives us the desired $\overline{u}'$. □

Lemma 48.16.4. *Let S be a scheme. Let X be an algebraic space over S. Let $\overline{x} : \mathrm{Spec}(k) \to X$ be a geometric point of X lying over $x \in |X|$. Let $\varphi : U \to X$ be an étale morphism of algebraic spaces and let $u \in |U|$ with $\varphi(u) = x$. Then there exists a geometric point $\overline{u} : \mathrm{Spec}(k) \to U$ lying over u with $\overline{x} = f \circ \overline{u}$.*

Proof. Choose an affine scheme U' with $u' \in U'$ and an étale morphism $U' \to U$ which maps u' to u. If we can prove the lemma for $(U', u') \to (X, x)$ then the lemma follows. Hence we may assume that U is a scheme, in particular that $U \to X$ is

representable. Then look at the cartesian diagram

$$\begin{array}{ccc} \mathrm{Spec}(k) \times_{\overline{x},X,\varphi} U & \xrightarrow{\mathrm{pr}_2} & U \\ \downarrow{\scriptstyle \mathrm{pr}_1} & & \downarrow{\scriptstyle \varphi} \\ \mathrm{Spec}(k) & \xrightarrow{\overline{x}} & X \end{array}$$

The projection pr_1 is the base change of an étale morphisms so it is étale, see Lemma 48.13.5. Therefore, the scheme $\mathrm{Spec}(k) \times_{\overline{x},X,\varphi} U$ is a disjoint union of finite separable extensions of k, see Morphisms, Lemma 28.37.7. But k is algebraically closed, so all these extensions are trivial, so $\mathrm{Spec}(k) \times_{\overline{x},X,\varphi} U$ is a disjoint union of copies of $\mathrm{Spec}(k)$ and each of these corresponds to a geometric point $\overline{u}$ with $f \circ \overline{u} = \overline{x}$. By Lemma 48.4.3 the map

$$|\mathrm{Spec}(k) \times_{\overline{x},X,\varphi} U| \longrightarrow |\mathrm{Spec}(k)| \times_{|X|} |U|$$

is surjective, hence we can pick $\overline{u}$ to lie over u. □

Lemma 48.16.5. *Let S be a scheme. Let X be an algebraic space over S. Let $\overline{x}$ be a geometric point of X. Let $(U, \overline{u})$ an étale neighborhood of $\overline{x}$. Let $\{\varphi_i : U_i \to U\}_{i \in I}$ be an étale covering in $X_{spaces,étale}$. Then there exist $i \in I$ and $\overline{u}_i : \overline{x} \to U_i$ such that $\varphi_i : (U_i, \overline{u}_i) \to (U, \overline{u})$ is a morphism of étale neighborhoods.*

Proof. Let $u \in |U|$ be the image of $\overline{u}$. As $|U| = \bigcup_{i \in I} \varphi_i(|U_i|)$ there exists an i and a point $u_i \in U_i$ mapping to x. Apply Lemma 48.16.4 to $(U_i, u_i) \to (U, u)$ and $\overline{u}$ to get the desired geometric point. □

Definition 48.16.6. Let S be a scheme. Let X be an algebraic space over S. Let $\mathcal{F}$ be a presheaf on $X_{étale}$. Let $\overline{x}$ be a geometric point of X. The *stalk* of $\mathcal{F}$ at $\overline{x}$ is

$$\mathcal{F}_{\overline{x}} = \mathrm{colim}_{(U,\overline{u})} \mathcal{F}(U)$$

where $(U, \overline{u})$ runs over all étale neighborhoods of $\overline{x}$ in X with $U \in \mathrm{Ob}(X_{étale})$.

By Lemma 48.16.3, this colimit is over a filtered index category, namely the opposite of the category of étale neighborhoods in $X_{étale}$. More precisely Lemma 48.16.3 says the opposite of the category of all étale neighbourhoods is filtered, and the full subcategory of those which are in $X_{étale}$ is a cofinal subcategory hence also filtered.

This means an element of $\mathcal{F}_{\overline{x}}$ can be thought of as a triple $(U, \overline{u}, \sigma)$ where $U \in \mathrm{Ob}(X_{étale})$ and $\sigma \in \mathcal{F}(U)$. Two triples $(U, \overline{u}, \sigma)$, $(U', \overline{u}', \sigma')$ define the same element of the stalk if there exists a third étale neighbourhood $(U'', \overline{u}'')$, $U'' \in \mathrm{Ob}(X_{étale})$ and morphisms of étale neighbourhoods $h : (U'', \overline{u}'') \to (U, \overline{u})$, $h' : (U'', \overline{u}'') \to (U', \overline{u}')$ such that $h^*\sigma = (h')^*\sigma'$ in $\mathcal{F}(U'')$. See Categories, Section 4.19.

This also implies that if $\mathcal{F}'$ is the sheaf on $X_{spaces,étale}$ corresponding to $\mathcal{F}$ on $X_{étale}$, then

$$\mathcal{F}_{\overline{x}} = \mathrm{colim}_{(U,\overline{u})} \mathcal{F}'(U) \tag{48.16.6.1}$$

where now the colimit is over all the étale neighbourhoods of $\overline{x}$. We will often jump between the point of view of using $X_{étale}$ and $X_{spaces,étale}$ without further mention.

In particular this means that if $\mathcal{F}$ is a presheaf of abelian groups, rings, etc then $\mathcal{F}_{\overline{x}}$ is an abelian group, ring, etc simply by the usual way of defining the group structure on a directed colimit of abelian groups, rings, etc.

Lemma 48.16.7. *Let S be a scheme. Let X be an algebraic space over S. Let $\overline{x}$ be a geometric point of X. Consider the functor*

$$u : X_{étale} \longrightarrow Sets, \quad U \longmapsto |U_{\overline{x}}|$$

Then u defines a point p of the site $X_{étale}$ (Sites, Definition 7.31.2) and its associated stalk functor $\mathcal{F} \mapsto \mathcal{F}_p$ (Sites, Equation 7.31.1.1) is the functor $\mathcal{F} \mapsto \mathcal{F}_{\overline{x}}$ defined above.

Proof. In the proof of Lemma 48.16.5 we have seen that the scheme $U_{\overline{x}}$ is a disjoint union of schemes isomorphic to $\overline{x}$. Thus we can also think of $|U_{\overline{x}}|$ as the set of geometric points of U lying over $\overline{x}$, i.e., as the collection of morphisms $\overline{u} : \overline{x} \to U$ fitting into the diagram of Definition 48.16.1. From this it follows that $u(X)$ is a singleton, and that $u(U \times_V W) = u(U) \times_{u(V)} u(W)$ whenever $U \to V$ and $W \to V$ are morphisms in $X_{étale}$. And, given a covering $\{U_i \to U\}_{i \in I}$ in $X_{étale}$ we see that $\coprod u(U_i) \to u(U)$ is surjective by Lemma 48.16.5. Hence Sites, Proposition 7.32.2 applies, so p is a point of the site $X_{étale}$. Finally, the our functor $\mathcal{F} \mapsto \mathcal{F}_{\overline{s}}$ is given by exactly the same colimit as the functor $\mathcal{F} \mapsto \mathcal{F}_p$ associated to p in Sites, Equation 7.31.1.1 which proves the final assertion. □

Lemma 48.16.8. *Let S be a scheme. Let X be an algebraic space over S. Let $\overline{x}$ be a geometric point of X.*

1. *The stalk functor $PAb(X_{étale}) \to Ab$, $\mathcal{F} \mapsto \mathcal{F}_{\overline{x}}$ is exact.*
2. *We have $(\mathcal{F}^\#)_{\overline{x}} = \mathcal{F}_{\overline{x}}$ for any presheaf of sets $\mathcal{F}$ on $X_{étale}$.*
3. *The functor $Ab(X_{étale}) \to Ab$, $\mathcal{F} \mapsto \mathcal{F}_{\overline{x}}$ is exact.*
4. *Similarly the functors $PSh(X_{étale}) \to Sets$ and $Sh(X_{étale}) \to Sets$ given by the stalk functor $\mathcal{F} \mapsto \mathcal{F}_{\overline{x}}$ are exact (see Categories, Definition 4.23.1) and commute with arbitrary colimits.*

Proof. This result follows from the general material in Modules on Sites, Section 18.35. This is true because $\mathcal{F} \mapsto \mathcal{F}_{\overline{x}}$ comes from a point of the small étale site of X, see Lemma 48.16.7. See the proof of Étale Cohomology, Lemma 44.29.9 for a direct proof of some of these statements in the setting of the small étale site of a scheme. □

We will see below that the stalk functor $\mathcal{F} \mapsto \mathcal{F}_{\overline{x}}$ is really the pullback along the morphism $\overline{x}$. In that sense the following lemma is a generalization of the lemma above.

Lemma 48.16.9. *Let S be a scheme. Let $f : X \to Y$ be a morphism of algebraic spaces over S.*

1. *The functor $f^{-1}_{small} : Ab(Y_{étale}) \to Ab(X_{étale})$ is exact.*
2. *The functor $f^{-1}_{small} : Sh(Y_{étale}) \to Sh(X_{étale})$ is exact, i.e., it commutes with finite limits and colimits, see Categories, Definition 4.23.1.*
3. *For any étale morphism $V \to Y$ of algebraic spaces we have $f^{-1}_{small} h_V = h_{X \times_Y V}$.*
4. *Let $\overline{x} \to X$ be a geometric point. Let $\mathcal{G}$ be a sheaf on $Y_{étale}$. Then there is a canonical identification*

$$(f^{-1}_{small}\mathcal{G})_{\overline{x}} = \mathcal{G}_{\overline{y}}.$$

where $\overline{y} = f \circ \overline{x}$.

Proof. Recall that f_{small} is defined via $f_{spaces,small}$ in Lemma 48.15.7. Parts (1), (2) and (3) are general consequences of the fact that $f_{spaces,étale} : X_{spaces,étale} \to Y_{spaces,étale}$ is a morphism of sites, see Sites, Definition 7.15.1 for (2), Modules on Sites, Lemma 18.30.2 for (1), and Sites, Lemma 7.14.5 for (3).

Proof of (4). This statement is a special case of Sites, Lemma 7.33.1 via Lemma 48.16.7. We also provide a direct proof. Note that by Lemma 48.16.8. taking stalks commutes with sheafification. Let $\mathcal{G}'$ be the sheaf on $Y_{spaces,étale}$ whose restriction to $Y_{étale}$ is $\mathcal{G}$. Recall that $f^{-1}_{spaces,étale}\mathcal{G}'$ is the sheaf associated to the presheaf

$$U \longrightarrow \operatorname{colim}_{U \to X \times_Y V} \mathcal{G}'(V),$$

see Sites, Sections 7.14 and 7.5. Thus we have

$$\begin{aligned}
(f^{-1}_{spaces,étale}\mathcal{G}')_{\overline{x}} &= \operatorname{colim}_{(U,\overline{u})} f^{-1}_{spaces,étale}\mathcal{G}'(U) \\
&= \operatorname{colim}_{(U,\overline{u})} \operatorname{colim}_{a:U \to X \times_Y V} \mathcal{G}'(V) \\
&= \operatorname{colim}_{(V,\overline{v})} \mathcal{G}'(V) \\
&= \mathcal{G}'_{\overline{y}}
\end{aligned}$$

in the third equality the pair $(U, \overline{u})$ and the map $a : U \to X \times_Y V$ corresponds to the pair $(V, a \circ \overline{u})$. Since the stalk of $\mathcal{G}'$ (resp. $f^{-1}_{spaces,étale}\mathcal{G}'$) agrees with the stalk of $\mathcal{G}$ (resp. $f^{-1}_{small}\mathcal{G}$), see Equation (48.16.6.1) the result follows. □

Remark 48.16.10. This remark is the analogue of Étale Cohomology, Remark 44.57.6. Let S be a scheme. Let X be an algebraic space over S. Let $\overline{x} : \operatorname{Spec}(k) \to X$ be a geometric point of X. By Étale Cohomology, Theorem 44.57.3 the category of sheaves on $\operatorname{Spec}(k)_{étale}$ is equivalent to the category of sets (by taking a sheaf to its global sections). Hence it follows from Lemma 48.16.9 part (4) applied to the morphism $\overline{x}$ that the functor

$$Sh(X_{étale}) \longrightarrow Sets, \quad \mathcal{F} \longmapsto \mathcal{F}_{\overline{x}}$$

is isomorphic to the functor

$$Sh(X_{étale}) \longrightarrow Sh(\operatorname{Spec}(k)_{étale}) = Sets, \quad \mathcal{F} \longmapsto \overline{x}^*\mathcal{F}$$

Hence we may view the stalk functors as pullback functors along geometric morphisms (and not just some abstract morphisms of topoi as in the result of Lemma 48.16.7).

Remark 48.16.11. Let S be a scheme. Let X be an algebraic space over S. Let $x \in |X|$. We claim that for any pair of geometric points $\overline{x}$ and $\overline{x}'$ lying over x the stalk functors are isomorphic. By definition of $|X|$ we can find a third geometric point $\overline{x}''$ so that there exists a commutative diagram

$$\begin{array}{ccc}
\overline{x}'' & \longrightarrow & \overline{x}' \\
\downarrow & \overset{\overline{x}''}{\searrow} & \downarrow{\scriptstyle \overline{x}'} \\
\overline{x} & \xrightarrow{\overline{x}} & X.
\end{array}$$

Since the stalk functor $\mathcal{F} \mapsto \mathcal{F}_{\overline{x}}$ is given by pullback along the morphism $\overline{x}$ (and similarly for the others) we conclude by functoriality of pullbacks.

The following theorem says that the small étale site of an algebraic space has enough points.

Theorem 48.16.12. *Let S be a scheme. Let X be an algebraic space over S. A map $a : \mathcal{F} \to \mathcal{G}$ of sheaves of sets is injective (resp. surjective) if and only if the map on stalks $a_{\overline{x}} : \mathcal{F}_{\overline{x}} \to \mathcal{G}_{\overline{x}}$ is injective (resp. surjective) for all geometric points of X. A sequence of abelian sheaves on $X_{étale}$ is exact if and only if it is exact on all stalks at geometric points of S.*

Proof. We know the theorem is true if X is a scheme, see Étale Cohomology, Theorem 44.29.10. Choose a surjective étale morphism $f : U \to X$ where U is a scheme. Since $\{U \to X\}$ is a covering (in $X_{spaces,étale}$) we can check whether a map of sheaves is injective, or surjective by restricting to U. Now if $\overline{u} : \mathrm{Spec}(k) \to U$ is a geometric point of U, then $(\mathcal{F}|_U)_{\overline{u}} = \mathcal{F}_{\overline{x}}$ where $\overline{x} = f \circ \overline{u}$. (This is clear from the colimits defining the stalks at $\overline{u}$ and $\overline{x}$, but it also follows from Lemma 48.16.9.) Hence the result for U implies the result for X and we win. □

The following lemma should be skipped on a first reading.

Lemma 48.16.13. *Let S be a scheme. Let X be an algebraic space over S. Let $p : Sh(pt) \to Sh(X_{étale})$ be a point of the small étale topos of X. Then there exists a geometric point $\overline{x}$ of X such that the stalk functor $\mathcal{F} \mapsto \mathcal{F}_p$ is isomorphic to the stalk functor $\mathcal{F} \mapsto \mathcal{F}_{\overline{x}}$.*

Proof. By Sites, Lemma 7.31.7 there is a one to one correspondence between points of the site and points of the associated topos. Hence we may assume that p is given by a functor $u : X_{étale} \to Sets$ which defines a point of the site $X_{étale}$. Let $U \in \mathrm{Ob}(X_{étale})$ be an object whose structure morphism $j : U \to X$ is surjective. Note that h_U is a sheaf which surjects onto the final sheaf. Since taking stalks is exact we see that $(h_U)_p = u(U)$ is not empty (use Sites, Lemma 7.31.3). Pick $x \in u(U)$. By Sites, Lemma 7.34.1 we obtain a point $q : Sh(pt) \to Sh(U_{étale})$ such that $p = j_{small} \circ q$, so that $\mathcal{F}_p = (\mathcal{F}|_U)_q$ functorially. By Étale Cohomology, Lemma 44.29.12 there is a geometric point $\overline{u}$ of U and a functorial isomorphism $\mathcal{G}_q = \mathcal{G}_{\overline{u}}$ for $\mathcal{G} \in Sh(U_{étale})$. Set $\overline{x} = j \circ \overline{u}$. Then we see that $\mathcal{F}_{\overline{x}} \cong (\mathcal{F}|_U)_{\overline{u}}$ functorially in $\mathcal{F}$ on $X_{étale}$ by Lemma 48.16.9 and we win. □

48.17. Supports of abelian sheaves

First we talk about supports of local sections.

Lemma 48.17.1. *Let S be a scheme. Let X be an algebraic space over S. Let $\mathcal{F}$ be a subsheaf of the final object of the étale topos of X (see Sites, Example 7.10.2). Then there exists a unique open $W \subset X$ such that $\mathcal{F} = h_W$.*

Proof. The condition means that $\mathcal{F}(U)$ is a singleton or empty for all $\varphi : U \to X$ in $\mathrm{Ob}(X_{spaces,étale})$. In particular local sections always glue. If $\mathcal{F}(U) \neq \emptyset$, then $\mathcal{F}(\varphi(U)) \neq \emptyset$ because $\varphi(U) \subset X$ is an open subspace (Lemma 48.13.7) and $\{\varphi : U \to \varphi(U)\}$ is a covering in $X_{spaces,étale}$. Take $W = \bigcup_{\varphi : U \to S, \mathcal{F}(U) \neq \emptyset} \varphi(U)$ to conclude. □

Lemma 48.17.2. *Let S be a scheme. Let X be an algebraic space over S. Let $\mathcal{F}$ be an abelian sheaf on $X_{spaces,étale}$. Let $\sigma \in \mathcal{F}(U)$ be a local section. There exists an open subspace $W \subset U$ such that*

(1) *$W \subset U$ is the largest open subspace of U such that $\sigma|_W = 0$,*

(2) *for every* $\varphi : V \to U$ *in* $X_{étale}$ *we have*

$$\sigma|_V = 0 \Leftrightarrow \varphi(V) \subset W,$$

(3) *for every geometric point* $\overline{u}$ *of* U *we have*

$$(U, \overline{u}, \sigma) = 0 \text{ in } \mathcal{F}_{\overline{s}} \Leftrightarrow \overline{u} \in W$$

where $\overline{s} = (U \to S) \circ \overline{u}$.

Proof. Since $\mathcal{F}$ is a sheaf in the étale topology the restriction of $\mathcal{F}$ to U_{Zar} is a sheaf on U in the Zariski topology. Hence there exists a Zariski open W having property (1), see Modules, Lemma 17.5.2. Let $\varphi : V \to U$ be an arrow of $X_{étale}$. Note that $\varphi(V) \subset U$ is an open subspace (Lemma 48.13.7) and that $\{V \to \varphi(V)\}$ is an étale covering. Hence if $\sigma|_V = 0$, then by the sheaf condition for $\mathcal{F}$ we see that $\sigma|_{\varphi(V)} = 0$. This proves (2). To prove (3) we have to show that if $(U, \overline{u}, \sigma)$ defines the zero element of $\mathcal{F}_{\overline{s}}$, then $\overline{u} \in W$. This is true because the assumption means there exists a morphism of étale neighbourhoods $(V, \overline{v}) \to (U, \overline{u})$ such that $\sigma|_V = 0$. Hence by (2) we see that $V \to U$ maps into W, and hence $\overline{u} \in W$. □

Let S be a scheme. Let X be an algebraic space over S. Let $x \in |X|$. Let $\mathcal{F}$ be a sheaf on $X_{étale}$. By Remark 48.16.11 the isomorphism class of the stalk of the sheaf $\mathcal{F}$ at a geometric points lying over x is well defined.

Definition 48.17.3. Let S be a scheme. Let X be an algebraic space over S. Let $\mathcal{F}$ be an abelian sheaf on $X_{étale}$.

(1) The *support of* $\mathcal{F}$ is the set of points $x \in |X|$ such that $\mathcal{F}_{\overline{x}} \neq 0$ for any (some) geometric point $\overline{x}$ lying over x.
(2) Let $\sigma \in \mathcal{F}(U)$ be a section. The *support of* σ is the closed subset $U \setminus W$, where $W \subset U$ is the largest open subset of U on which σ restricts to zero (see Lemma 48.17.2).

Lemma 48.17.4. *Let S be a scheme. Let X be an algebraic space over S. Let $\mathcal{F}$ be an abelian sheaf on $X_{étale}$. Let $U \in \mathrm{Ob}(X_{étale})$ and $\sigma \in \mathcal{F}(U)$.*

(1) *The support of σ is closed in $|X|$.*
(2) *The support of $\sigma + \sigma'$ is contained in the union of the supports of $\sigma, \sigma' \in \mathcal{F}(X)$.*
(3) *If $\varphi : \mathcal{F} \to \mathcal{G}$ is a map of abelian sheaves on $X_{étale}$, then the support of $\varphi(\sigma)$ is contained in the support of $\sigma \in \mathcal{F}(U)$.*
(4) *The support of $\mathcal{F}$ is the union of the images of the supports of all local sections of $\mathcal{F}$.*
(5) *If $\mathcal{F} \to \mathcal{G}$ is surjective then the support of $\mathcal{G}$ is a subset of the support of $\mathcal{F}$.*
(6) *If $\mathcal{F} \to \mathcal{G}$ is injective then the support of $\mathcal{F}$ is a subset of the support of $\mathcal{G}$.*

Proof. Part (1) holds by definition. Parts (2) and (3) hold because they holds for the restriction of $\mathcal{F}$ and $\mathcal{G}$ to U_{Zar}, see Modules, Lemma 17.5.2. Part (4) is a direct consequence of Lemma 48.17.2 part (3). Parts (5) and (6) follow from the other parts. □

Lemma 48.17.5. *The support of a sheaf of rings on the small étale site of an algebraic space is closed.*

Proof. This is true because (according to our conventions) a ring is 0 if and only if $1 = 0$, and hence the support of a sheaf of rings is the support of the unit section. □

48.18. The structure sheaf of an algebraic space

The structure sheaf of an algebraic space is the sheaf of rings of the following lemma.

Lemma 48.18.1. *Let S be a scheme. Let X be an algebraic space over S. The rule $U \mapsto \Gamma(U, \mathcal{O}_U)$ defines a sheaf of rings on $X_{étale}$.*

Proof. Immediate from the definition of a covering and Descent, Lemma 34.7.1. □

Definition 48.18.2. Let S be a scheme. Let X be an algebraic space over S. The *structure sheaf* of X is the sheaf of rings $\mathcal{O}_X$ on the small étale site $X_{étale}$ described in Lemma 48.18.1.

According to Lemma 48.15.12 the sheaf $\mathcal{O}_X$ corresponds to a system of étale sheaves $(\mathcal{O}_X)_U$ for U ranging through the objects of $X_{étale}$. It is clear from the proof of that lemma and our definition that we have simply $(\mathcal{O}_X)_U = \mathcal{O}_U$ where $\mathcal{O}_U$ is the structure sheaf of $U_{étale}$ as introduced in Descent, Definition 34.7.2. In particular, if X is a scheme we recover the sheaf $\mathcal{O}_X$ on the small étale site of X.

Via the equivalence $Sh(X_{étale}) = Sh(X_{spaces,étale})$ of Lemma 48.15.3 we may also think of $\mathcal{O}_X$ as a sheaf of rings on $X_{spaces,étale}$. It is explained in Remark 48.15.4 how to compute $\mathcal{O}_X(Y)$, and in particular $\mathcal{O}_X(X)$, when $Y \to X$ is an object of $X_{spaces,étale}$.

Lemma 48.18.3. *Let S be a scheme. Let $f : X \to Y$ be a morphism of algebraic spaces over S. Then there is a canonical map $f^\sharp : f_{small}^{-1}\mathcal{O}_Y \to \mathcal{O}_X$ such that*

$$(f_{small}, f^\sharp) : (X_{étale}, \mathcal{O}_X) \longrightarrow (Y_{étale}, \mathcal{O}_Y)$$

is a morphism of ringed topoi. Furthermore,

1. *The construction $f \mapsto (f_{small}, f^\sharp)$ is compatible with compositions.*
2. *If f is a morphism of schemes, then $f^\sharp$ is the map described in Descent, Remark 34.7.4.*

Proof. By Lemma 48.15.9 it suffices to give an f-map from $\mathcal{O}_Y$ to $\mathcal{O}_X$. In other words, for every commutative diagram

$$\begin{array}{ccc} U & \longrightarrow & X \\ {\scriptstyle g}\downarrow & & \downarrow{\scriptstyle f} \\ V & \longrightarrow & Y \end{array}$$

where $U \in X_{étale}$, $V \in Y_{étale}$ we have to give a map of rings $(f^\sharp)_{(U,V,g)} : \Gamma(V, \mathcal{O}_V) \to \Gamma(U, \mathcal{O}_U)$. Of course we just take $(f^\sharp)_{(U,V,g)} = g^\sharp$. It is clear that this is compatible with restriction mappings and hence indeed gives an f-map. We omit checking compatibility with compositions and agreement with the construction in Descent, Remark 34.7.4. □

48.19. Stalks of the structure sheaf

This section is the analogue of Étale Cohomology, Section 48.19.

Lemma 48.19.1. *Let S be a scheme. Let X be an algebraic space over S. Let $\overline{x}$ be a geometric point of X. Let $(U, \overline{u})$ be an étale neighbourhood of $\overline{x}$ where U is a scheme. Then we have*

$$\mathcal{O}_{X,\overline{x}} = \mathcal{O}_{U,\overline{u}} = \mathcal{O}^{sh}_{U,u}$$

where the left hand side is the stalk of the structure sheaf of X, and the right hand side is the strict henselization of the local ring of U at the point u at which $\overline{u}$ is centered.

Proof. We know that the structure sheaf $\mathcal{O}_U$ on $U_{étale}$ is the restriction of the structure sheaf of X. Hence the first equality follows from Lemma 48.16.9 part (4). The second equality is explained in Étale Cohomology, Lemma 44.33.1. □

Definition 48.19.2. Let S be a scheme. Let X be an algebraic space over S. Let $\overline{x}$ be a geometric point of X lying over the point $x \in |X|$.

(1) The *étale local ring of X at $\overline{x}$* is the stalk of the structure sheaf $\mathcal{O}_X$ on $X_{étale}$ at $\overline{x}$. Notation: $\mathcal{O}_{X,\overline{x}}$.
(2) The *strict henselization of X at $\overline{x}$* is the scheme $\mathrm{Spec}(\mathcal{O}_{X,\overline{x}})$.

The isomorphism type of the strict henselization of X at $\overline{x}$ (as a scheme over X) depends only on the point $x \in |X|$ and not on the choice of the geometric point lying over x, see Remark 48.16.11.

Lemma 48.19.3. *Let S be a scheme. Let X be an algebraic space over S. The small étale site $X_{étale}$ endowed with its structure sheaf $\mathcal{O}_X$ is a locally ringed site, see Modules on Sites, Definition 18.39.4.*

Proof. This follows because the stalks $\mathcal{O}_{X,\overline{x}}$ are local, and because $S_{étale}$ has enough points, see Lemmas 48.19.1 and Theorem 48.16.12. See Modules on Sites, Lemma 18.39.2 and 18.39.3 for the fact that this implies the small étale site is locally ringed. □

48.20. Dimension of local rings

It turns out the dimension of the local ring of an algebraic space is a well defined concept.

Lemma 48.20.1. *Let S be a scheme. Let X be an algebraic space over S. Let $x \in |X|$ be a point. Let $d \in \{0, 1, 2, \ldots, \infty\}$. The following are equivalent*

(1) *for some scheme U and étale morphism $a : U \to X$ and point $u \in U$ with $a(u) = x$ we have $\dim(\mathcal{O}_{U,u}) = d$,*
(2) *for any scheme U, any étale morphism $a : U \to X$, and any point $u \in U$ with $a(u) = x$ we have $\dim(\mathcal{O}_{U,u}) = d$,*
(3) *$\dim(\mathcal{O}_{X,\overline{x}}) = d$ for some geometric point $\overline{x}$ lying over x, and*
(4) *$\dim(\mathcal{O}_{X,\overline{x}}) = d$ for any geometric point $\overline{x}$ lying over x.*

Proof. The equivalence of (1) and (2) follows from a combination of Lemma 48.7.4 and Descent, Lemma 34.17.3. The equivalence of (3) and (4) follows from the fact that the isomorphism type of $\mathcal{O}_{X,\overline{x}}$ only depends on $x \in |X|$, see Remark 48.16.11.

Using Lemma 48.19.1 the equivalence of (1)+(2) and (3)+(4) comes down to the following statement: Given any local ring R we have $\dim(R) = \dim(R^{sh})$. This is More on Algebra, Lemma 15.32.7. □

Definition 48.20.2. Let S be a scheme. Let X be an algebraic space over S. Let $x \in |X|$ be a point. The *dimension of the local ring of X at x* is the element $d \in \{0, 1, 2, \ldots, \infty\}$ satisfying the equivalent conditions of Lemma 48.20.1.

48.21. Local irreducibility

A point on an algebraic space has a well defined étale local ring, which corresponds to the strict henselization of the local ring in the case of a scheme. In general we cannot see how many irreducible components of the algebraic space pass through the given point from the étale local ring. Here is something we can do.

Lemma 48.21.1. *Let S be a scheme. Let X be an algebraic space over S. Let $x \in |X|$ be a point. The following are equivalent*

1. *for any scheme U and étale morphism $a : U \to X$ and $u \in U$ with $a(u) = x$ the local ring $\mathcal{O}_{U,u}$ has a unique minimal prime,*
2. *for any scheme U and étale morphism $a : U \to X$ and $u \in U$ with $a(u) = x$ there is a unique irreducible component of U through u, and*
3. *$\mathcal{O}_{X,\overline{x}}$ has a unique minimal prime for any geometric point $\overline{x}$ lying over x.*

Proof. The equivalence of (1) and (2) follows from the fact that irreducible components of U passing through u are in $1-1$ correspondence with minimal primes of the local ring of U at u. Let $a : U \to X$ and $u \in U$ be as in (1). Then $\mathcal{O}_{U,u} \to \mathcal{O}_{X,\overline{x}}$ is flat in particular injective. Hence if $f, g \in \mathcal{O}_{U,u}$ are non-nilpotent elements such that $fg = 0$, then the same is true in $\mathcal{O}_{X,\overline{x}}$. Conversely, suppose that $f, g \in \mathcal{O}_{X,\overline{x}}$ are non-nilpotent such that $fg = 0$. Since $\mathcal{O}_{X,\overline{x}}$ is the filtered colimit of the rings $\mathcal{O}_{U,u}$ we see that f, g are the images of elements of $\mathcal{O}_{U,u}$ for some choice of $a : U \to X$. Hence we see that $\mathcal{O}_{U,u}$ doesn't have a unique minimal prime. In this way we see the equivalence of (1) and (3). □

Definition 48.21.2. Let S be a scheme. Let X be an algebraic space over S. Let $x \in |X|$. We say that X is *geometrically unibranch at x* if the equivalent conditions of Lemma 48.21.1 hold. We say that X is *geometrically unibranch* if X is geometrically unibranch at every $x \in |X|$.

To prove this is consistent with the definition of [**DG67**] for schemes we offer the following lemma (see [**Art66**, Lemma 2.2]).

Lemma 48.21.3. *Let A be a local ring. Let A^{sh} be a strict henselization of A. The following are equivalent*

1. *A^{sh} has a unique minimal prime, and*
2. *A has a unique minimal prime $\mathfrak{p}$ and the integral closure A' of $A/\mathfrak{p}$ in its fraction field is a local ring whose residue field is purely inseparable over the residue field of A.*

Proof. Denote $\mathfrak{m}$ the maximal ideal of the ring A. Denote κ, κ^{sh} the residue field of A, A^{sh}.

Assume (1). Let $\mathfrak{p}^{sh}$ be the unique minimal prime of A^{sh}. The flatness of $A \to A^{sh}$ implies that $\mathfrak{p} = A \cap \mathfrak{p}^{sh}$ is the unique minimal prime of A (by going down, see

Algebra, Lemma 10.36.17). Also, since $A^{sh}/\mathfrak{p}A^{sh} = (A/\mathfrak{p})^{sh}$ (see Algebra, Lemma 10.143.30) is reduced by More on Algebra, Lemma 15.32.4 we see that $\mathfrak{p}^{sh} = \mathfrak{p}A^{sh}$. Since $A \to A'$ is integral, every maximal ideal of A' lies over $\mathfrak{m}$ (by going up for integral ring maps, see Algebra, Lemma 10.33.20). If A' is not local, then we can find distinct maximal ideals $\mathfrak{m}_1$, $\mathfrak{m}_2$. Choosing elements $f_1, f_2 \in A'$ with $f_i \in \mathfrak{m}_i, f_i \notin \mathfrak{m}_{3-i}$ we find a finite subalgebra $B = A[f_1, f_2] \subset A'$ with distinct maximal ideals $B \cap \mathfrak{m}_i$, $i = 1, 2$. If A' is local with maximal ideal $\mathfrak{m}'$, but $A/\mathfrak{m} \subset A'/\mathfrak{m}'$ is not purely inseparable, then we can find a $f \in A'$ whose image in $A'/\mathfrak{m}'$ generates finite, not purely inseparable extension of $A/\mathfrak{m}$ and we find a finite local subalgebra $B = A[f] \subset A'$ whose residue field is not a purely inseparable extension of $A/\mathfrak{m}$. Note that the inclusions

$$A/\mathfrak{p} \subset B \subset \kappa(\mathfrak{p})$$

give, on tensoring with the flat ring map $A \to A^{sh}$ the inclusions

$$A^{sh}/\mathfrak{p}^{sh} \subset B \otimes_A A^{sh} \subset \kappa(\mathfrak{p}) \otimes_A A^{sh} \subset \kappa(\mathfrak{p}^{sh})$$

the last inclusion because $\kappa(\mathfrak{p}) \otimes_A A^{sh} = \kappa(\mathfrak{p}) \otimes_{A/\mathfrak{p}} A^{sh}/\mathfrak{p}^{sh}$ is a localization of the domain $A^{sh}/\mathfrak{p}^{sh}$. Note that $B \otimes_A \kappa^{sh}$ has at least two maximal ideals because $B/\mathfrak{m}B$ either has two maximal ideals or one whose residue field is not purely inseparable over κ, and because κ^{sh} is separably algebraically closed. Hence, as A^{sh} is strictly henselian we see that $B \otimes_A A^{sh}$ is a product of ≥ 2 local rings, see Algebra, Lemma 10.143.7. But we've just seen that $B \otimes_A A^{sh}$ is a subring of a domain and we get a contradiction.

Assume (2). Let $A \to B$ be a local map of local rings which is a localization of an étale A-algebra. In particular $\mathfrak{m}_B$ is the unique prime containing $\mathfrak{m}_A B$. Then $B' = A' \otimes_A B$ is integral over B and the assumption that $A \to A'$ is local with purely inseparable residue field extension implies that B' is local. On the other hand, $A' \to B'$ is the localization of an étale ring map, hence B' is normal, see Algebra, Lemma 10.149.7. Thus B' is a (local) normal domain. Finally, we have

$$B/\mathfrak{p}B \subset B \otimes_A \kappa(\mathfrak{p}) = B' \otimes_{A'} f.f.(A') \subset f.f.(B')$$

Hence $B/\mathfrak{p}B$ is a domain, which implies that B has a unique minimal prime (since by flatness of $A \to B$ these all have to lie over $\mathfrak{p}$). Hence, by Lemma 48.21.1 we see that A^{sh} has a unique minimal prime. □

48.22. Noetherian spaces

We have already defined locally Noetherian algebraic spaces in Section 48.7.

Definition 48.22.1. Let S be a scheme. Let X be an algebraic space over S. We say X is *Noetherian* if X is quasi-compact, quasi-separated and locally Noetherian.

Note that a Noetherian algebraic space X is not just quasi-compact and locally Noetherian, but also quasi-separated. This does not conflict with the definition of a Noetherian scheme, as a locally Noetherian scheme is quasi-separated, see Properties, Lemma 27.5.4. This does not hold for algebraic spaces. Namely, $X = \mathbf{A}^1_k/\mathbf{Z}$, see Spaces, Example 47.14.8 is locally Noetherian and quasi-compact but not quasi-separated (hence not Noetherian according to our definitions).

A consequence of the choice made above is that an algebraic space of finite type over a Noetherian algebraic space is not automatically Noetherian, i.e., the analogue of Morphisms, Lemma 28.16.6 does not hold. The correct statement is that an

algebraic space of finite presentation over a Noetherian algebraic space is Noetherian (see Morphisms of Spaces, Lemma 49.26.6).

A Noetherian algebraic space X is very close to being a scheme. In the rest of this section we collect some lemmas to illustrate this.

Lemma 48.22.2. *Let S be a scheme. Let X be an algebraic space over S.*

(1) *If X is locally Noetherian then $|X|$ is a locally Noetherian topological space.*
(2) *If X is quasi-compact and locally Noetherian, then $|X|$ is a Noetherian topological space.*

Proof. Assume X is locally Noetherian. Choose a scheme U and a surjective étale morphism $U \to X$. As X is locally Noetherian we see that U is locally Noetherian. By Properties, Lemma 27.5.5 this means that $|U|$ is a locally Noetherian topological space. Since $|U| \to |X|$ is open and surjective we conclude that $|X|$ is locally Noetherian by Topology, Lemma 5.8.3. This proves (1). If X is quasi-compact and locally Noetherian, then $|X|$ is quasi-compact and locally Noetherian. Hence $|X|$ is Noetherian by Topology, Lemma 5.11.14. □

Lemma 48.22.3. *Let S be a scheme. Let X be an algebraic space over S. If X is Noetherian, then $|X|$ is a sober Noetherian topological space.*

Proof. A quasi-separated algebraic space has an underlying sober topological space, see Lemma 48.12.4. It is Noetherian by Lemma 48.22.2. □

Lemma 48.22.4. *Let S be a scheme. Let X be a Noetherian algebraic space over S. Let $\overline{x}$ be a geometric point of X. Then $\mathcal{O}_{X,\overline{x}}$ is a Noetherian local ring.*

Proof. Choose an étale neighbourhood $(U, \overline{u})$ of $\overline{x}$ where U is a scheme. Then $\mathcal{O}_{X,\overline{x}}$ is the strict henselization of the local ring of U at u, see Lemma 48.19.1. By our definition of Noetherian spaces the scheme U is Noetherian. Hence we conclude by More on Algebra, Lemma 15.32.3. □

48.23. Regular algebraic spaces

We have already defined regular algebraic spaces in Section 48.7.

Lemma 48.23.1. *Let S be a scheme. Let X be a locally Noetherian algebraic space over S. The following are equivalent*

(1) *X is regular, and*
(2) *every étale local ring $\mathcal{O}_{X,\overline{x}}$ is regular.*

Proof. Let U be a scheme and let $U \to X$ be a surjective étale morphism. By assumption U is locally Noetherian. Moreover, every étale local ring $\mathcal{O}_{X,\overline{x}}$ is the strict henselization of a local ring on U and conversely, see Lemma 48.19.1. Thus by More on Algebra, Lemma 15.32.10 we see that (2) is equivalent to every local ring of U being regular, i.e., U being a regular scheme (see Properties, Lemma 27.9.2). This equivalent to (1) by Definition 48.7.2. □

48.24. Sheaves of modules on algebraic spaces

If X is an algebraic space, then a sheaf of modules on X is a sheaf of $\mathcal{O}_X$-modules on the small étale site of X where $\mathcal{O}_X$ is the structure sheaf of X. The category of sheaves of modules is denoted $\mathit{Mod}(\mathcal{O}_X)$.

Given a morphism $f : X \to Y$ of algebraic spaces, by Lemma 48.18.3 we get a morphism of ringed topoi and hence by Modules on Sites, Definition 18.13.1 we get well defined pullback and direct image functors

$$f^* : Mod(\mathcal{O}_Y) \longrightarrow Mod(\mathcal{O}_X), \quad f_* : Mod(\mathcal{O}_X) \longrightarrow Mod(\mathcal{O}_Y) \tag{48.24.0.1}$$

which are adjoint in the usual way. If $g : Y \to Z$ is another morphism of algebraic spaces over S, then we have $(g \circ f)^* = f^* \circ g^*$ and $(g \circ f)_* = g_* \circ f_*$ simply because the morphisms of ringed topoi compose in the corresponding way (by the lemma).

Lemma 48.24.1. *Let S be a scheme. Let $f : X \to Y$ be an étale morphism of algebraic spaces over S. Then $f^{-1}\mathcal{O}_Y = \mathcal{O}_X$, and $f^*\mathcal{G} = f^{-1}_{small}\mathcal{G}$ for any sheaf of $\mathcal{O}_Y$-modules $\mathcal{G}$. In particular, $f^* : Mod(\mathcal{O}_X) \to Mod(\mathcal{O}_Y)$ is exact.*

Proof. By the description of inverse image in Lemma 48.15.10 and the definition of the structure sheaves it is clear that $f^{-1}_{small}\mathcal{O}_Y = \mathcal{O}_X$. Since the pullback

$$f^*\mathcal{G} = f^{-1}_{small}\mathcal{G} \otimes_{f^{-1}_{small}\mathcal{O}_Y} \mathcal{O}_X$$

by definition we conclude that $f^*\mathcal{G} = f^{-1}_{small}\mathcal{G}$. The exactness is clear because f^{-1}_{small} is exact, as f_{small} is a morphism of topoi. □

We continue our abuse of notation introduced in Equation (48.15.10.1) by writing

$$\mathcal{G}|_{X_{étale}} = f^*\mathcal{G} = f^{-1}_{small}\mathcal{G} \tag{48.24.1.1}$$

in the situation of the lemma above. We will discuss this in a more technical fashion in Section 48.25.

Lemma 48.24.2. *Let S be a scheme. Let*

$$\begin{array}{ccc} X' & \longrightarrow & X \\ {\scriptstyle f'}\downarrow & & \downarrow{\scriptstyle f} \\ Y' & \xrightarrow{g} & Y \end{array}$$

be a cartesian square of algebraic spaces over S. Let $\mathcal{F} \in Mod(\mathcal{O}_X)$. If g is étale, then $f'_*(\mathcal{F}|_{X'}) = (f_*\mathcal{F})|_{Y'}$[7] and $R^i f'_*(\mathcal{F}|_{X'}) = (R^i f_*\mathcal{F})|_{Y'}$ in $Mod(\mathcal{O}_{Y'})$.

Proof. This is a reformulation of Lemma 48.15.11 in the case of modules. □

Lemma 48.24.3. *Let S be a scheme. Let X be an algebraic space over S. A sheaf $\mathcal{F}$ of $\mathcal{O}_X$-modules is given by the following data:*

1. *for every $U \in \mathrm{Ob}(X_{étale})$ a sheaf $\mathcal{F}_U$ of $\mathcal{O}_U$-modules on $U_{étale}$,*
2. *for every $f : U' \to U$ in $X_{étale}$ an isomorphism $c_f : f^*_{small}\mathcal{F}_U \to \mathcal{F}_{U'}$.*

These data are subject to the condition that given any $f : U' \to U$ and $g : U'' \to U'$ in $X_{étale}$ the composition $g^{-1}_{small}c_f \circ c_g$ is equal to $c_{f \circ g}$.

Proof. Combine Lemmas 48.24.1 and 48.15.12, and use the fact that any morphism between objects of $X_{étale}$ is an étale morphism of schemes. □

[7] Also $(f')^*(\mathcal{G}|_{Y'}) = (f^*\mathcal{G})|_{X'}$ by commutativity of the diagram and (48.24.1.1)

48.25. Étale localization

Reading this section should be avoided at all cost.

Let $X \to Y$ be an étale morphism of algebraic spaces. Then X is an object of $Y_{spaces,étale}$ and it is immediate from the definitions, see also the proof of Lemma 48.15.10, that

$$X_{spaces,étale} = Y_{spaces,étale}/X \tag{48.25.0.1}$$

where the right hand side is the localization of the site $Y_{spaces,étale}$ at the object X, see Sites, Definition 7.24.1. Moreover, this identification is compatible with the structure sheaves by Lemma 48.24.1. Hence the ringed site $(X_{spaces,étale}, \mathcal{O}_X)$ is identified with the localization of the ringed site $(Y_{spaces,étale}, \mathcal{O}_Y)$ at the object X:

$$(X_{spaces,étale}, \mathcal{O}_X) = (Y_{spaces,étale}/X, \mathcal{O}_Y|_{Y_{spaces,étale}/X}) \tag{48.25.0.2}$$

The localization of a ringed site used on the right hand side is defined in Modules on Sites, Definition 18.19.1.

Assume now $X \to Y$ is an étale morphism of algebraic spaces and X is a scheme. Then X is an object of $Y_{étale}$ and it follows that

$$X_{étale} = Y_{étale}/X \tag{48.25.0.3}$$

and

$$(X_{étale}, \mathcal{O}_X) = (Y_{étale}/X, \mathcal{O}_Y|_{Y_{étale}/X}) \tag{48.25.0.4}$$

as above.

Finally, if $X \to Y$ is an étale morphism of algebraic spaces and X is an affine scheme, then X is an object of $Y_{affine,étale}$ and

$$X_{affine,étale} = Y_{affine,étale}/X \tag{48.25.0.5}$$

and

$$(X_{affine,étale}, \mathcal{O}_X) = (Y_{affine,étale}/X, \mathcal{O}_Y|_{Y_{affine,étale}/X}) \tag{48.25.0.6}$$

as above.

Next, we show that these localizations are compatible with morphisms.

Lemma 48.25.1. *Let S be a scheme. Let*

$$\begin{array}{ccc} U & \xrightarrow{g} & V \\ {\scriptstyle p}\downarrow & & \downarrow{\scriptstyle q} \\ X & \xrightarrow{f} & Y \end{array}$$

be a commutative diagram of algebraic spaces over S with p and q étale. Via the identifications (48.25.0.2) for $U \to X$ and $V \to Y$ the morphism of ringed topoi

$$(g_{spaces,étale}, g^\sharp) : (Sh(U_{spaces,étale}), \mathcal{O}_U) \longrightarrow (Sh(V_{spaces,étale}), \mathcal{O}_V)$$

is 2-isomorphic to the morphism $(f_{spaces,étale,c}, f_c^\sharp)$ constructed in Modules on Sites, Lemma 18.20.2 starting with the morphism of ringed sites $(f_{spaces,étale}, f^\sharp)$ and the map $c : U \to V \times_Y X$ corresponding to g.

Proof. The morphism $(f_{spaces,étale,c}, f_c^\sharp)$ is defined as a composition $f' \circ j$ of a localization and a base change map. Similarly g is a composition $U \to V \times_Y X \to V$. Hence it suffices to prove the lemma in the following two cases: (1) $f = \mathrm{id}$, and (2) $U = X \times_Y V$. In case (1) the morphism $g : U \to V$ is étale, see Lemma 48.13.6. Hence $(g_{spaces,étale}, g^\sharp)$ is a localization morphism by the discussion surrounding Equations (48.25.0.1) and (48.25.0.2) which is exactly the content of the lemma in this case. In case (2) the morphism $g_{spaces,étale}$ comes from the morphism of ringed sites given by the functor $V_{spaces,étale} \to U_{spaces,étale}$, $V'/V \mapsto V' \times_V U/U$ which is also what the morphism f' is defined by, see Sites, Lemma 7.27.1. We omit the verification that $(f')^\sharp = g^\sharp$ in this case (both are the restriction of $f^\sharp$ to $U_{spaces,étale}$). □

Lemma 48.25.2. *Same notation and assumptions as in Lemma 48.25.1 except that we also assume U and V are schemes. Via the identifications (48.25.0.4) for $U \to X$ and $V \to Y$ the morphism of ringed topoi*

$$(g_{small}, g^\sharp) : (Sh(U_{étale}), \mathcal{O}_U) \longrightarrow (Sh(V_{étale}), \mathcal{O}_V)$$

is 2-isomorphic to the morphism $(f_{small,s}, f_s^\sharp)$ constructed in Modules on Sites, Lemma 18.22.3 starting with $(f_{small}, f^\sharp)$ and the map $s : h_U \to f_{small}^{-1} h_V$ corresponding to g.

Proof. Note that $(g_{small}, g^\sharp)$ is 2-isomorphic as a morphism of ringed topoi to the morphism of ringed topoi associated to the morphism of ringed sites $(g_{spaces,étale}, g^\sharp)$. Hence we conclude by Lemma 48.25.1 and Modules on Sites, Lemma 18.22.4. □

48.26. Recovering morphisms

In this section we prove that the rule which associates to an algebraic space its locally ringed small étale topos is fully faithful in a suitable sense, see Theorem 48.26.4.

Lemma 48.26.1. *Let S be a scheme. Let $f : X \to Y$ be a morphism of algebraic spaces over S. The morphism of ringed topoi $(f_{small}, f^\sharp)$ associated to f is a morphism of locally ringed topoi, see Modules on Sites, Definition 18.39.8.*

Proof. Note that the assertion makes sense since we have seen that $(X_{étale}, \mathcal{O}_{X_{étale}})$ and $(Y_{étale}, \mathcal{O}_{Y_{étale}})$ are locally ringed sites, see Lemma 48.19.3. Moreover, we know that $X_{étale}$ has enough points, see Theorem 48.16.12. Hence it suffices to prove that $(f_{small}, f^\sharp)$ satisfies condition (3) of Modules on Sites, Lemma 18.39.7. To see this take a point p of $X_{étale}$. By Lemma 48.16.13 p corresponds to a geometric point $\overline{x}$ of X. By Lemma 48.16.9 the point $q = f_{small} \circ p$ corresponds to the geometric point $\overline{y} = f \circ \overline{x}$ of Y. Hence the assertion we have to prove is that the induced map of étale local rings

$$\mathcal{O}_{Y,\overline{y}} \longrightarrow \mathcal{O}_{X,\overline{x}}$$

is a local ring map. You can prove this directly, but instead we deduce it from the corresponding result for schemes. To do this choose a commutative diagram

$$\begin{array}{ccc} U & \xrightarrow[\psi]{} & V \\ \downarrow & & \downarrow \\ X & \longrightarrow & Y \end{array}$$

where U and V are schemes, and the vertical arrows are surjective étale (see Spaces, Lemma 47.11.4). Choose a lift $\overline{u} : \overline{x} \to U$ (possible by Lemma 48.16.5). Set $\overline{v} = \psi \circ \overline{u}$. We obtain a commutative diagram of étale local rings

$$\begin{array}{ccc} \mathcal{O}_{U,\overline{u}} & \longleftarrow & \mathcal{O}_{V,\overline{v}} \\ \uparrow & & \uparrow \\ \mathcal{O}_{X,\overline{x}} & \longleftarrow & \mathcal{O}_{Y,\overline{y}}. \end{array}$$

By Étale Cohomology, Lemma 44.41.1 the top horizontal arrow is a local ring map. Finally by Lemma 48.19.1 the vertical arrows are isomorphisms. Hence we win. □

Lemma 48.26.2. *Let S be a scheme. Let X, Y be algebraic spaces over S. Let $f : X \to Y$ be a morphism of algebraic spaces over S. Let t be a 2-morphism from $(f_{small}, f^\sharp)$ to itself, see Modules on Sites, Definition 18.8.1. Then $t = id$.*

Proof. Let X', resp. Y' be X viewed as an algebraic space over $\mathrm{Spec}(\mathbf{Z})$, see Spaces, Definition 47.16.2. It is clear from the construction that $(X_{small}, \mathcal{O})$ is equal to $(X'_{small}, \mathcal{O})$ and similarly for Y. Hence we may work with X' and Y'. In other words we may assume that $S = \mathrm{Spec}(\mathbf{Z})$.

Assume $S = \mathrm{Spec}(\mathbf{Z})$, $f : X \to Y$ and t are as in the lemma. This means that $t : f_{small}^{-1} \to f_{small}^{-1}$ is a transformation of functors such that the diagram

$$\begin{array}{ccc} f_{small}^{-1}\mathcal{O}_Y & \xleftarrow{\quad t \quad} & f_{small}^{-1}\mathcal{O}_Y \\ & {\scriptstyle f^\sharp} \searrow \quad \swarrow {\scriptstyle f^\sharp} & \\ & \mathcal{O}_X & \end{array}$$

is commutative. Suppose $V \to Y$ is étale with V affine. Write $V = \mathrm{Spec}(B)$. Choose generators $b_j \in B$, $j \in J$ for B as a $\mathbf{Z}$-algebra. Set $T = \mathrm{Spec}(\mathbf{Z}[\{x_j\}_{j \in J}])$. In the following we will use that $\mathrm{Mor}_{Sch}(U, T) = \prod_{j \in J} \Gamma(U, \mathcal{O}_U)$ for any scheme U without further mention. The surjective ring map $\mathbf{Z}[x_j] \to B$, $x_j \mapsto b_j$ corresponds to a closed immersion $V \to T$. We obtain a monomorphism

$$i : V \longrightarrow T_Y = T \times Y$$

of algebraic spaces over Y. In terms of sheaves on $Y_{étale}$ the morphism i induces an injection $h_i : h_V \to \prod_{j \in J} \mathcal{O}_Y$ of sheaves. The base change $i' : X \times_Y V \to T_X$ of i to X is a monomorphism too (Spaces, Lemma 47.5.5). Hence $i' : X \times_Y V \to T_X$ is a monomorphism, which in turn means that $h_{i'} : h_{X \times_Y V} \to \prod_{j \in J} \mathcal{O}_X$ is an injection of sheaves. Via the identification $f_{small}^{-1} h_V = h_{X \times_Y V}$ of Lemma 48.16.9 the map $h_{i'}$ is equal to

$$f_{small}^{-1} h_V \xrightarrow{f^{-1} h_i} \prod_{j \in J} f_{small}^{-1} \mathcal{O}_Y \xrightarrow{\prod f^\sharp} \prod_{j \in J} \mathcal{O}_X$$

(verification omitted). This means that the map $t : f_{small}^{-1}h_V \to f_{small}^{-1}h_V$ fits into the commutative diagram

$$\begin{array}{ccccc} f_{small}^{-1}h_V & \xrightarrow{f^{-1}h_i} & \prod_{j\in J} f_{small}^{-1}\mathcal{O}_Y & \xrightarrow{\prod f^\sharp} & \prod_{j\in J}\mathcal{O}_X \\ \downarrow t & & \downarrow \prod t & & \downarrow \text{id} \\ f_{small}^{-1}h_V & \xrightarrow{f^{-1}h_i} & \prod_{j\in J} f_{small}^{-1}\mathcal{O}_Y & \xrightarrow{\prod f^\sharp} & \prod_{j\in J}\mathcal{O}_X \end{array}$$

The commutativity of the right square holds by our assumption on t explained above. Since the composition of the horizontal arrows is injective by the discussion above we conclude that the left vertical arrow is the identity map as well. Any sheaf of sets on $Y_{étale}$ admits a surjection from a (huge) coproduct of sheaves of the form h_V with V affine (combine Lemma 48.15.5 with Sites, Lemma 7.13.5). Thus we conclude that $t : f_{small}^{-1} \to f_{small}^{-1}$ is the identity transformation as desired. □

Lemma 48.26.3. *Let S be a scheme. Let X, Y be algebraic spaces over S. Any two morphisms $a, b : X \to Y$ of algebraic spaces over S for which there exists a 2-isomorphism $(a_{small}, a^\sharp) \cong (b_{small}, b^\sharp)$ in the 2-category of ringed topoi are equal.*

Proof. Let $t : a_{small}^{-1} \to b_{small}^{-1}$ be the 2-isomorphism. We may equivalently think of t as a transformation $t : a_{spaces,étale}^{-1} \to b_{spaces,étale}^{-1}$ since there is not difference between sheaves on $X_{étale}$ and sheaves on $X_{spaces,étale}$. Choose a commutative diagram

$$\begin{array}{ccc} U & \xrightarrow[\alpha]{} & V \\ {\scriptstyle p}\downarrow & & \downarrow{\scriptstyle q} \\ X & \xrightarrow{a} & Y \end{array}$$

where U and V are schemes, and p and q are surjective étale. Consider the diagram

$$\begin{array}{ccc} h_U & \xrightarrow[\alpha]{} & a_{spaces,étale}^{-1}h_V \\ \| & & \downarrow t \\ h_U & \dashrightarrow & b_{spaces,étale}^{-1}h_V \end{array}$$

Since the sheaf $b_{spaces,étale}^{-1}h_V$ is isomorphic to $h_{V\times_{Y,b}X}$ we see that the dotted arrow comes from a morphism of schemes $\beta : U \to V$ fitting into a commutative diagram

$$\begin{array}{ccc} U & \xrightarrow[\beta]{} & V \\ {\scriptstyle p}\downarrow & & \downarrow{\scriptstyle q} \\ X & \xrightarrow{b} & Y \end{array}$$

We claim that there exists a sequence of 2-isomorphisms

$$\begin{aligned} (\alpha_{small}, \alpha^\sharp) &\cong (\alpha_{spaces,étale}, \alpha^\sharp) \\ &\cong (a_{spaces,étale,c}, a_c^\sharp) \\ &\cong (b_{spaces,étale,d}, b_d^\sharp) \\ &\cong (\beta_{spaces,étale}, \beta^\sharp) \\ &\cong (\beta_{small}, \beta^\sharp) \end{aligned}$$

The first and the last 2-isomorphisms come from the identifications between sheaves on $U_{spaces,étale}$ and sheaves on $U_{étale}$ and similarly for V. The second and fourth 2-isomorphisms are those of Lemma 48.25.1 with $c : U \to X \times_{a,Y} V$ induced by α and $d : U \to X \times_{b,Y} V$ induced by β. The middle 2-isomorphism comes from the transformation t. Namely, the functor $a^{-1}_{spaces,étale,c}$ corresponds to the functor

$$(\mathcal{H} \to h_V) \longmapsto (a^{-1}_{spaces,étale}\mathcal{H} \times_{a^{-1}_{spaces,étale}h_V,\alpha} h_U \to h_U)$$

and similarly for $b^{-1}_{spaces,étale,d}$, see Sites, Lemma 7.27.3. This uses the identification of sheaves on $Y_{spaces,étale}/V$ as arrows $(\mathcal{H} \to h_V)$ in $Sh(Y_{spaces,étale})$ and similarly for U/X, see Sites, Lemma 7.24.4. Via this identification the structure sheaf $\mathcal{O}_V$ corresponds to the pair $(\mathcal{O}_Y \times h_V \to h_V)$ and similarly for $\mathcal{O}_U$, see Modules on Sites, Lemma 18.21.3. Since t switches α and β we see that t induces an isomorphism

$$t : a^{-1}_{spaces,étale}\mathcal{H} \times_{a^{-1}_{spaces,étale}h_V,\alpha} h_U \longrightarrow b^{-1}_{spaces,étale}\mathcal{H} \times_{b^{-1}_{spaces,étale}h_V,\beta} h_U$$

over h_U functorially in $(\mathcal{H} \to h_V)$. Also, t is compatible with $a^\sharp_c$ and $b^\sharp_d$ as t is compatible with $a^\sharp$ and $b^\sharp$ by our description of the structure sheaves $\mathcal{O}_U$ and $\mathcal{O}_V$ above. Hence, the morphisms of ringed topoi $(\alpha_{small}, \alpha^\sharp)$ and $(\beta_{small}, \beta^\sharp)$ are 2-isomorphic. By Étale Cohomology, Lemma 44.41.3 we conclude $\alpha = \beta$! Since $p : U \to X$ is a surjection of sheaves it follows that $a = b$. □

Here is the main result of this section.

Theorem 48.26.4. *Let X, Y be algebraic spaces over* $\mathrm{Spec}(\mathbf{Z})$. *Let*

$$(g, g^\sharp) : (Sh(X_{étale}), \mathcal{O}_X) \longrightarrow (Sh(Y_{étale}), \mathcal{O}_Y)$$

be a morphism of locally ringed topoi. Then there exists a unique morphism of algebraic spaces $f : X \to Y$ such that $(g, g^\sharp)$ is isomorphic to $(f_{small}, f^\sharp)$. In other words, the construction

$$Spaces/\mathrm{Spec}(\mathbf{Z}) \longrightarrow Locally\ ringed\ topoi, \quad X \longrightarrow (X_{étale}, \mathcal{O}_X)$$

is fully faithful (morphisms up to 2-isomorphisms on the right hand side).

Proof. The uniqueness we have seen in Lemma 48.26.3. Thus it suffices to prove existence. In this proof we will freely use the identifications of Equation (48.25.0.4) as well as the result of Lemma 48.25.2.

Let $U \in \mathrm{Ob}(X_{étale})$, let $V \in \mathrm{Ob}(Y_{étale})$ and let $s \in g^{-1}h_V(U)$ be a section. We may think of s as a map of sheaves $s : h_U \to g^{-1}h_V$. By Modules on Sites, Lemma 18.22.3 we obtain a commutative diagram of morphisms of ringed topoi

$$\begin{array}{ccc} (Sh(X_{étale}/U), \mathcal{O}_U) & \xrightarrow[(j,j^\sharp)]{} & (Sh(X_{étale}), \mathcal{O}_X) \\ \downarrow{\scriptstyle (g_s,g_s^\sharp)} & & \downarrow{\scriptstyle (g,g^\sharp)} \\ (Sh(V_{étale}), \mathcal{O}_V) & \longrightarrow & (Sh(Y_{étale}), \mathcal{O}_Y). \end{array}$$

By Étale Cohomology, Theorem 44.41.5 we obtain a unique morphism of schemes $f_s : U \to V$ such that $(g_s, g_s^\sharp)$ is 2-isomorphic to $(f_{s,small}, f_s^\sharp)$. The construction $(U, V, s) \rightsquigarrow f_s$ just explained satisfies the following functoriality property: Suppose

given morphisms $a : U' \to U$ in $X_{étale}$ and $b : V' \to V$ in $Y_{étale}$ and a map $s' : h_{U'} \to g^{-1}h_{V'}$ such that the diagram

$$\begin{array}{ccc} h_{U'} & \xrightarrow{s'} & g^{-1}h_{V'} \\ {\scriptstyle a}\downarrow & & \downarrow{\scriptstyle g^{-1}b} \\ h_U & \xrightarrow{s} & g^{-1}h_V \end{array}$$

commutes. Then the diagram

$$\begin{array}{ccc} U' & \xrightarrow{f_{s'}} & u(V') \\ {\scriptstyle a}\downarrow & & \downarrow{\scriptstyle u(b)} \\ U & \xrightarrow{f_s} & u(V) \end{array}$$

of schemes commutes. The reason this is true is that the same condition holds for the morphisms $(g_s, g_s^\sharp)$ constructed in Modules on Sites, Lemma 18.22.3 and the uniqueness in Étale Cohomology, Theorem 44.41.5.

The problem is to glue the morphisms f_s to a morphism of algebraic spaces. To do this first choose a scheme V and a surjective étale morphism $V \to Y$. This means that $h_V \to *$ is surjective and hence $g^{-1}h_V \to *$ is surjective too. This means there exists a scheme U and a surjective étale morphism $U \to X$ and a morphism $s : h_U \to g^{-1}h_V$. Next, set $R = V \times_Y V$ and $R' = U \times_X U$. Then we get $g^{-1}h_R = g^{-1}h_V \times g^{-1}h_V$ as g^{-1} is exact. Thus s induces a morphism $s \times s : h_{R'} \to g^{-1}h_R$. Applying the constructions above we see that we get a commutative diagram of morphisms of schemes

$$\begin{array}{ccc} R' & \xrightarrow{f_{s\times s}} & R \\ \downarrow\downarrow & & \downarrow\downarrow \\ U & \xrightarrow{f_s} & V \end{array}$$

Since we have $X = U/R'$ and $Y = V/R$ (see Spaces, Lemma 47.9.1) we conclude that this diagram defines a morphism of algebraic spaces $f : X \to Y$ fitting into an obvious commutative diagram. Now we still have to show that $(f_{small}, f^\sharp)$ is 2-isomorphic to $(g, g^\sharp)$. Let $t_V : f_{s,small}^{-1} \to g_s^{-1}$ and $t_R : f_{s\times s,small}^{-1} \to g_{s\times s}^{-1}$ be the 2-isomorphisms which are given to us by the construction above. Let $\mathcal{G}$ be a sheaf on $Y_{étale}$. Then we see that t_V defines an isomorphism

$$f_{small}^{-1}\mathcal{G}|_{U_{étale}} = f_{s,small}^{-1}\mathcal{G}|_{V_{étale}} \xrightarrow{t_V} g_s^{-1}\mathcal{G}|_{V_{étale}} = g^{-1}\mathcal{G}|_{U_{étale}}.$$

Moreover, this isomorphism pulled back to R' via either projection $R' \to U$ is the isomorphism

$$f_{small}^{-1}\mathcal{G}|_{R'_{étale}} = f_{s\times s,small}^{-1}\mathcal{G}|_{R_{étale}} \xrightarrow{t_R} g_{s\times s}^{-1}\mathcal{G}|_{R_{étale}} = g^{-1}\mathcal{G}|_{R'_{étale}}.$$

Since $\{U \to X\}$ is a covering in the site $X_{spaces,étale}$ this means the first displayed isomorphism descends to an isomorphism $t : f_{small}^{-1}\mathcal{G} \to g^{-1}\mathcal{G}$ of sheaves (small detail omitted). The isomorphism is functorial in $\mathcal{G}$ since t_V and t_R are transformations of functors. Finally, t is compatible with $f^\sharp$ and $g^\sharp$ as t_V and t_R are (some details omitted). This finishes the proof of the theorem. □

Lemma 48.26.5. *Let X, Y be algebraic spaces over $\mathbf{Z}$. If*

$$(g, g^\sharp) : (Sh(X_{étale}), \mathcal{O}_X) \longrightarrow (Sh(Y_{étale}), \mathcal{O}_Y)$$

is an isomorphism of ringed topoi, then there exists a unique morphism $f : X \to Y$ of algebraic spaces such that $(g, g^\sharp)$ is isomorphic to $(f_{small}, f^\sharp)$ and moreover f is an isomorphism of algebraic spaces.

Proof. By Theorem 48.26.4 it suffices to show that $(g, g^\sharp)$ is a morphism of locally ringed topoi. By Modules on Sites, Lemma 18.39.7 (and since the site $X_{étale}$ has enough points) it suffices to check that the map $\mathcal{O}_{Y,q} \to \mathcal{O}_{X,p}$ induced by $g^\sharp$ is a local ring map where $q = f \circ p$ and p is any point of $X_{étale}$. As it is an isomorphism this is clear. □

48.27. Quasi-coherent sheaves on algebraic spaces

In Descent, Section 34.7 we have seen that for a scheme U, there is no difference between a quasi-coherent $\mathcal{O}_U$-module on U, or a quasi-coherent $\mathcal{O}$-module on the small étale site of U. Hence the following definition is compatible with our original notion of a quasi-coherent sheaf on a scheme (Schemes, Section 25.24), when applied to a representable algebraic space.

Definition 48.27.1. Let S be a scheme. Let X be an algebraic space over S. A *quasi-coherent $\mathcal{O}_X$-module* is a quasi-coherent module on the ringed site $(X_{étale}, \mathcal{O}_X)$ in the sense of Modules on Sites, Definition 18.23.1. The category of quasi-coherent sheaves on X is denoted $QCoh(\mathcal{O}_X)$.

Note that as being quasi-coherent is an intrinsic notion (see Modules on Sites, Lemma 18.23.2) this is equivalent to saying that the corresponding $\mathcal{O}_X$-module on $X_{spaces,étale}$ is quasi-coherent.

As usual, quasi-coherent sheaves behave well with respect to pullback.

Lemma 48.27.2. *Let S be a scheme. Let $f : X \to Y$ be a morphism of algebraic spaces over S. The pullback functor $f^* : Mod(\mathcal{O}_Y) \to Mod(\mathcal{O}_X)$ preserves quasi-coherent sheaves.*

Proof. This is a general fact, see Modules on Sites, Lemma 18.23.4. □

Note that this pullback functor agrees with the usual pullback functor between quasi-coherent sheaves of modules if X and Y happen to be schemes, see Descent, Proposition 34.7.14. Here is the obligatory lemma comparing this with quasi-coherent sheaves on the objects of the small étale site of X.

Lemma 48.27.3. *Let S be a scheme. Let X be an algebraic space over S. A quasi-coherent $\mathcal{O}_X$-module $\mathcal{F}$ is given by the following data:*

1. *for every $U \in \mathrm{Ob}(X_{étale})$ a quasi-coherent $\mathcal{O}_U$-module $\mathcal{F}_U$ on $U_{étale}$,*
2. *for every $f : U' \to U$ in $X_{étale}$ an isomorphism $c_f : f_{small}^* \mathcal{F}_U \to \mathcal{F}_{U'}$.*

These data are subject to the condition that given any $f : U' \to U$ and $g : U'' \to U'$ in $X_{étale}$ the composition $g_{small}^{-1} c_f \circ c_g$ is equal to $c_{f \circ g}$.

Proof. Combine Lemmas 48.27.2 and 48.24.3. □

Lemma 48.27.4. *Let S be a scheme. Let X be an algebraic space over S. Let $\mathcal{F}$ be a quasi-coherent $\mathcal{O}_X$-module. Let $x \in |X|$ be a point and let $\overline{x}$ be a geometric point lying over x. Finally, let $\varphi : (U, \overline{u}) \to (X, \overline{x})$ be an étale neighbourhood where U is a scheme. Then*

$$(\varphi^*\mathcal{F})_u \otimes_{\mathcal{O}_{U,u}} \mathcal{O}_{X,\overline{x}} = \mathcal{F}_{\overline{x}}$$

where $u \in U$ is the image of $\overline{u}$.

Proof. Note that $\mathcal{O}_{X,\overline{x}} = \mathcal{O}^{sh}_{U,u}$ by Lemma 48.19.1 hence the tensor product makes sense. Moreover, from Definition 48.16.6 it is clear that

$$\mathcal{F}_{\overline{u}} = \operatorname{colim}(\varphi^*\mathcal{F})_u$$

where the colimit is over $\varphi : (U, \overline{u}) \to (X, \overline{x})$ as in the lemma. Hence there is a canonical map from left to right in the statement of the lemma. We have a similar colimit description for $\mathcal{O}_{X,\overline{x}}$ and by Lemma 48.27.3 we have

$$((\varphi')^*\mathcal{F})_{u'} = (\varphi^*\mathcal{F})_u \otimes_{\mathcal{O}_{U,u}} \mathcal{O}_{U',u'}$$

whenever $(U', \overline{u}') \to (U, \overline{u})$ is a morphism of étale neighbourhoods. To complete the proof we use that $\otimes$ commutes with colimits. □

Lemma 48.27.5. *Let S be a scheme. Let $f : X \to Y$ be a morphism of algebraic spaces over S. Let $\mathcal{G}$ be a quasi-coherent $\mathcal{O}_Y$-module. Let $\overline{x}$ be a geometric point of X and let $\overline{y} = f \circ \overline{x}$ be the image in Y. Then there is a canonical isomorphism*

$$(f^*\mathcal{G})_{\overline{x}} = \mathcal{G}_{\overline{y}} \otimes_{\mathcal{O}_{Y,\overline{y}}} \mathcal{O}_{X,\overline{x}}$$

of the stalk of the pullback with the tensor product of the stalk with the local ring of X at $\overline{x}$.

Proof. Since $f^*\mathcal{G} = f^{-1}_{small}\mathcal{G} \otimes_{f^{-1}_{small}\mathcal{O}_Y} \mathcal{O}_X$ this follows from the description of stalks of pullbacks in Lemma 48.16.9 and the fact that taking stalks commutes with tensor products. A more direct way to see this is as follows. Choose a commutative diagram

$$\begin{array}{ccc} U & \xrightarrow{\alpha} & V \\ {\scriptstyle p}\downarrow & & \downarrow{\scriptstyle q} \\ X & \xrightarrow{a} & Y \end{array}$$

where U and V are schemes, and p and q are surjective étale. By Lemma 48.16.4 we can choose a geometric point $\overline{u}$ of U such that $\overline{x} = p \circ \overline{u}$. Set $\overline{v} = \alpha \circ \overline{u}$. Then we see that

$$\begin{aligned} (f^*\mathcal{G})_{\overline{x}} &= (p^*f^*\mathcal{G})_u \otimes_{\mathcal{O}_{U,u}} \mathcal{O}_{X,\overline{x}} \\ &= (\alpha^*q^*\mathcal{G})_u \otimes_{\mathcal{O}_{U,u}} \mathcal{O}_{X,\overline{x}} \\ &= (q^*\mathcal{G})_v \otimes_{\mathcal{O}_{V,v}} \mathcal{O}_{U,u} \otimes_{\mathcal{O}_{U,u}} \mathcal{O}_{X,\overline{x}} \\ &= (q^*\mathcal{G})_v \otimes_{\mathcal{O}_{V,v}} \mathcal{O}_{X,\overline{x}} \\ &= (q^*\mathcal{G})_v \otimes_{\mathcal{O}_{V,v}} \mathcal{O}_{Y,\overline{y}} \otimes_{\mathcal{O}_{Y,\overline{y}}} \mathcal{O}_{X,\overline{x}} \\ &= \mathcal{G}_{\overline{y}} \otimes_{\mathcal{O}_{Y,\overline{y}}} \mathcal{O}_{X,\overline{x}} \end{aligned}$$

Here we have used Lemma 48.27.4 (twice) and the corresponding result for pullbacks of quasi-coherent sheaves on schemes, see Sheaves, Lemma 6.26.4. □

Lemma 48.27.6. *Let S be a scheme. Let X be an algebraic space over S. Let $\mathcal{F}$ be a sheaf of $\mathcal{O}_X$-modules. The following are equivalent*

(1) $\mathcal{F}$ *is a quasi-coherent* $\mathcal{O}_X$*-module,*
(2) *there exists an étale morphism* $f : Y \to X$ *of algebraic spaces over* S *with* $|f| : |Y| \to |X|$ *surjective such that* $f^*\mathcal{F}$ *is quasi-coherent on* Y*,*
(3) *there exists a scheme* U *and a surjective étale morphism* $\varphi : U \to X$ *such that* $\varphi^*\mathcal{F}$ *is a quasi-coherent* $\mathcal{O}_U$*-module, and*
(4) *for every affine scheme* U *and étale morphism* $\varphi : U \to X$ *the restriction* $\varphi^*\mathcal{F}$ *is a quasi-coherent* $\mathcal{O}_U$*-module.*

Proof. It is clear that (1) implies (2) by considering id_X. Assume $f : Y \to X$ is as in (2), and let $V \to Y$ be a surjective étale morphism from a scheme towards Y. Then the composition $V \to X$ is surjective étale as well and by Lemma 48.27.2 the pullback of $\mathcal{F}$ to V is quasi-coherent as well. Hence we see that (2) implies (3).

Let $U \to X$ be as in (3). Let us use the abuse of notation introduced in Equation (48.24.1.1). As $\mathcal{F}|_{U_{étale}}$ is quasi-coherent there exists an étale covering $\{U_i \to U\}$ such that $\mathcal{F}|_{U_{i,étale}}$ has a global presentation, see Modules on Sites, Definition 18.17.1 and Lemma 18.23.3. Let $V \to X$ be an object of $X_{étale}$. Since $U \to X$ is surjective and étale, the family of maps $\{U_i \times_X V \to V\}$ is an étale covering of V. Via the morphisms $U_i \times_X V \to U_i$ we can restrict the global presentations of $\mathcal{F}|_{U_{i,étale}}$ to get a global presentation of $\mathcal{F}|_{(U_i \times_X V)_{étale}}$ Hence the sheaf $\mathcal{F}$ on $X_{étale}$ satisfies the condition of Modules on Sites, Definition 18.23.1 and hence is quasi-coherent.

The equivalence of (3) and (4) comes from the fact that any scheme has an affine open covering. □

Lemma 48.27.7. *Let* S *be a scheme. Let* X *be an algebraic space over* S*. The category* $QCoh(\mathcal{O}_X)$ *of quasi-coherent sheaves on* X *has the following properties:*

(1) *Any direct sum of quasi-coherent sheaves is quasi-coherent.*
(2) *Any colimit of quasi-coherent sheaves is quasi-coherent.*
(3) *The kernel and cokernel of a morphism of quasi-coherent sheaves is quasi-coherent.*
(4) *Given a short exact sequence of* $\mathcal{O}_X$*-modules* $0 \to \mathcal{F}_1 \to \mathcal{F}_2 \to \mathcal{F}_3 \to 0$ *if two out of three are quasi-coherent so is the third.*
(5) *Given two quasi-coherent* $\mathcal{O}_X$*-modules the tensor product is quasi-coherent.*
(6) *Given two quasi-coherent* $\mathcal{O}_X$*-modules* $\mathcal{F}$*,* $\mathcal{G}$ *such that* $\mathcal{F}$ *is of finite presentation (see Section 48.28), then the internal hom* $\mathcal{H}om_{\mathcal{O}_X}(\mathcal{F}, \mathcal{G})$ *is quasi-coherent.*

Proof. Note that we have the corresponding result for quasi-coherent modules on schemes, see Schemes, Section 25.24. We will reduce the lemma to this case by étale localization. Choose a scheme U and a surjective étale morphism $\varphi : U \to X$. In order to formulate this proof correctly, we temporarily go back to making the (pedantic) distinction between a quasi-coherent sheaf $\mathcal{G}$ on the scheme U and the associated quasi-coherent sheaf $\mathcal{G}^a$ (see Descent, Definition 34.7.2) on $U_{étale}$ We have a commutative diagram

$$\begin{array}{ccc} QCoh(\mathcal{O}_X) & \longrightarrow & QCoh(\mathcal{O}_U) \\ \downarrow & & \downarrow \\ Mod(\mathcal{O}_X) & \longrightarrow & Mod(\mathcal{O}_U) \end{array}$$

The bottom horizontal arrow is the restriction functor (48.24.1.1) $\mathcal{G} \mapsto \mathcal{G}|_{U_{étale}}$. This functor has both a left adjoint and a right adjoint, see Modules on Sites, Section 18.19, hence commutes with all limits and colimits. Moreover, we know that an object of $Mod(\mathcal{O}_X)$ is in $QCoh(\mathcal{O}_X)$ if and only if its restriction to U is in $QCoh(\mathcal{O}_U)$, see Lemma 48.27.6. Let $\mathcal{F}_i$ be a family of quasi-coherent $\mathcal{O}_X$-modules. Then $\bigoplus \mathcal{F}_i$ is an $\mathcal{O}_X$-module whose restriction to U is the direct sum of the restrictions. Let $\mathcal{G}_i$ be a quasi-coherent sheaf on U with $\mathcal{F}_i|_{U_{étale}} = \mathcal{G}_i^a$. Combining the above with Descent, Lemma 34.7.13 we see that

$$\Big(\bigoplus \mathcal{F}_i\Big)|_{U_{étale}} = \bigoplus \mathcal{F}_i|_{U_{étale}} = \bigoplus \mathcal{G}_i^a = \Big(\bigoplus \mathcal{G}_i\Big)^a$$

hence $\bigoplus \mathcal{F}_i$ is quasi-coherent and (1) follows. The other statements are proved just so (using the same references). □

It is in general not the case that the pushforward of a quasi-coherent sheaf along a morphism of algebraic spaces is quasi-coherent. We will return to this issue in Morphisms of Spaces, Section 49.12.

48.28. Properties of modules

In Modules on Sites, Sections 18.17, 18.23, and Definition 18.28.1 we have defined a number of intrinsic properties of modules of $\mathcal{O}$-module on any ringed topos. If X is an algebraic space, we will apply these notions freely to modules on the ringed site $(X_{étale}, \mathcal{O}_X)$, or equivalently on the ringed site $(X_{spaces,étale}, \mathcal{O}_X)$.

Global properties $\mathcal{P}$:

(a) *free*,
(b) *finite free*,
(c) *generated by global sections*,
(d) *generated by finitely many global sections*,
(e) having a *global presentation*, and
(f) having a *global finite presentation*.

Local properties $\mathcal{P}$:

(g) *locally free*,
(f) *finite locally free*,
(h) *locally generated by sections*,
(i) *locally generated by r sections*,
(j) *finite type*,
(k) *quasi-coherent* (see Section 48.27),
(l) *of finite presentation*,
(m) *coherent*, and
(n) *flat*.

Here are some results which follow immediately from the definitions:

(1) In each case, except for $\mathcal{P}$ ="coherent", the property is preserved under pullback, see Modules on Sites, Lemmas 18.17.2, 18.23.4, and 18.38.3.
(2) Each of the properties above (including coherent) are preserved under pullbacks by étale morphisms of algebraic spaces (because in this case pullback is given by restriction, see Lemma 48.15.10).

(3) Assume $f : Y \to X$ is a surjective étale morphism of algebraic spaces. For each of the local properties (g) – (m), the fact that $f^*\mathcal{F}$ has $\mathcal{P}$ implies that $\mathcal{F}$ has $\mathcal{P}$. This follows as $\{Y \to X\}$ is a covering in $X_{spaces,étale}$ and Modules on Sites, Lemma 18.23.3.
(4) If X is a scheme, $\mathcal{F}$ is a quasi-coherent module on $X_{étale}$, and $\mathcal{P}$ any property except "coherent" or "locally free", then $\mathcal{P}$ for $\mathcal{F}$ on $X_{étale}$ is equivalent to the corresponding property for $\mathcal{F}|_{X_{Zar}}$, i.e., it corresponds to $\mathcal{P}$ for $\mathcal{F}$ when we think of it as a quasi-coherent sheaf on the scheme X. See Descent, Lemma 34.7.12.
(5) If X is a locally Noetherian scheme, $\mathcal{F}$ is a quasi-coherent module on $X_{étale}$, then $\mathcal{F}$ is coherent on $X_{étale}$ if and only if $\mathcal{F}|_{X_{Zar}}$ is coherent, i.e., it corresponds to the usual notion of a coherent sheaf on the scheme X being coherent. See Descent, Lemma 34.7.12.

48.29. Locally projective modules

Recall that in Properties, Section 27.19 we defined the notion of a locally projective quasi-coherent module.

Lemma 48.29.1. *Let S be a scheme. Let X be an algebraic space over S. Let $\mathcal{F}$ be a quasi-coherent $\mathcal{O}_X$-module. The following are equivalent*

(1) *for some scheme U and surjective étale morphism $U \to X$ the restriction $\mathcal{F}|_U$ is locally projective on U, and*
(2) *for any scheme U and any étale morphism $U \to X$ the restriction $\mathcal{F}|_U$ is locally projective on U.*

Proof. Let $U \to X$ be as in (1) and let $V \to X$ be étale where V is a scheme. Then $\{U \times_X V \to V\}$ is an fppf covering of schemes. Hence if $\mathcal{F}|_U$ is locally projective, then $\mathcal{F}|_{U \times_X V}$ is locally projective (see Properties, Lemma 27.19.3) and hence $\mathcal{F}|_V$ is locally projective, see Descent, Lemma 34.6.7. □

Definition 48.29.2. Let S be a scheme. Let X be an algebraic space over S. Let $\mathcal{F}$ be a quasi-coherent $\mathcal{O}_X$-module. We say $\mathcal{F}$ is *locally projective* if the equivalent conditions of Lemma 48.29.1 are satisfied.

Lemma 48.29.3. *Let S be a scheme. Let $f : X \to Y$ be a morphism of algebraic spaces over S. Let $\mathcal{G}$ be a quasi-coherent $\mathcal{O}_Y$-module. If $\mathcal{G}$ is locally projective on Y, then $f^*\mathcal{G}$ is locally projective on X.*

Proof. Choose a surjective étale morphism $V \to Y$ with V a scheme. Choose a surjective étale morphism $U \to V \times_Y X$ with U a scheme. Denote $\psi : U \to V$ the induced morphism. Then

$$f^*\mathcal{G}|_U = \psi^*(\mathcal{G}|_V)$$

Hence the lemma follows from the definition and the result in the case of schemes, see Properties, Lemma 27.19.3. □

48.30. Quasi-coherent sheaves and presentations

Let S be a scheme. Let X be an algebraic space over S. Let $X = U/R$ be a presentation of X coming from any surjective étale morphism $\varphi : U \to X$, see Spaces, Definition 47.9.3. In particular, we obtain a groupoid (U, R, s, t, c), such that $j = (t, s) : R \to U \times_S U$, see Groupoids, Lemma 38.11.3. In Groupoids,

Definition 38.12.1 we have the defined the notion of a quasi-coherent sheaf on an arbitrary groupoid. With these notions in place we have the following observation.

Proposition 48.30.1. *With S, $\varphi : U \to X$, and (U, R, s, t, c) as above. For any quasi-coherent $\mathcal{O}_X$-module $\mathcal{F}$ the sheaf $\varphi^*\mathcal{F}$ comes equipped with a canonical isomorphism*

$$\alpha : t^*\varphi^*\mathcal{F} \longrightarrow s^*\varphi^*\mathcal{F}$$

which satisfies the conditions of Groupoids, Definition 38.12.1 and therefore defines a quasi-coherent sheaf on (U, R, s, t, c). The functor $\mathcal{F} \mapsto (\varphi^*\mathcal{F}, \alpha)$ defines an equivalence of categories

$$\begin{array}{c} \textit{Quasi-coherent} \\ \mathcal{O}_X\textit{-modules} \end{array} \longleftrightarrow \begin{array}{c} \textit{Quasi-coherent modules} \\ \textit{on } (U, R, s, t, c) \end{array}$$

Proof. In the statement of the proposition, and in this proof we think of a quasi-coherent sheaf on a scheme as a quasi-coherent sheaf on the small étale site of that scheme. This is permissible by the results of Descent, Section 34.7.

The existence of α comes from the fact that $\varphi \circ t = \varphi \circ s$ and that pullback is functorial in the morphism, see discussion surrounding Equation (48.24.0.1). In exactly the same way, i.e., by functoriality of pullback, we see that the isomorphism α satisfies condition (1) of Groupoids, Definition 38.12.1. To see condition (2) of the definition it suffices to see that α is an isomorphism which is clear. The construction $\mathcal{F} \mapsto (\varphi^*\mathcal{F}, \alpha)$ is clearly functorial in the quasi-coherent sheaf $\mathcal{F}$. Hence we obtain the functor from left to right in the displayed formula of the lemma.

Conversely, suppose that $(\mathcal{F}, \alpha)$ is a quasi-coherent sheaf on (U, R, s, t, c). Let $V \to X$ be an object of $X_{étale}$. In this case the morphism $V' = U \times_X V \to V$ is a surjective étale morphism of schemes, and hence $\{V' \to V\}$ is an étale covering of V. Moreover, the quasi-coherent sheaf $\mathcal{F}$ pulls back to a quasi-coherent sheaf $\mathcal{F}'$ on V'. Since $R = U \times_X U$ with $t = \mathrm{pr}_0$ and $s = \mathrm{pr}_0$ we see that $V' \times_V V' = R \times_X V$ with projection maps $V' \times_V V' \to V'$ equal to the pullbacks of t and s. Hence α pulls back to an isomorphism $\alpha' : \mathrm{pr}_0^*\mathcal{F}' \to \mathrm{pr}_1^*\mathcal{F}'$, and the pair $(\mathcal{F}', \alpha')$ is a descend datum for quasi-coherent sheaves with respect to $\{V' \to V\}$. By Descent, Proposition 34.5.2 this descent datum is effective, and we obtain a quasi-coherent $\mathcal{O}_V$-module $\mathcal{F}_V$ on $V_{étale}$. To see that this gives a quasi-coherent sheaf on $X_{étale}$ we have to show (by Lemma 48.27.3) that for any morphism $f : V_1 \to V_2$ in $X_{étale}$ there is a canonical isomorphism $c_f : \mathcal{F}_{V_1} \to \mathcal{F}_{V_2}$ compatible with compositions of morphisms. We omit the verification. We also omit the verification that this defines a functor from the category on the right to the category on the left which is inverse to the functor described above. □

Proposition 48.30.2. *Let S be a scheme Let X be an algebraic space over S.*

1. *The category $QCoh(\mathcal{O}_X)$ is a Grothendieck abelian category. Consequently, $QCoh(\mathcal{O}_X)$ has enough injectives and all limits.*
2. *The inclusion functor $QCoh(\mathcal{O}_X) \to Mod(\mathcal{O}_X)$ has a right adjoint*

$$Q^8 : Mod(\mathcal{O}_X) \longrightarrow QCoh(\mathcal{O}_X)$$

such that for every quasi-coherent sheaf $\mathcal{F}$ the adjunction mapping $Q(\mathcal{F}) \to \mathcal{F}$ is an isomorphism.

[8] This functor is sometimes called the *coherator*.

Proof. This proof is a repeat of the proof in the case of schemes, see Properties, Proposition 27.21.4. We advise the reader to read that proof first.

Part (1) means $QCoh(\mathcal{O}_X)$ (a) has all colimits, (b) filtered colimits are exact, and (c) has a generator, see Injectives, Section 19.10. By Lemma 48.27.7 colimits in $QCoh(\mathcal{O}_X)$ exist and agree with colimits in $Mod(\mathcal{O}_X)$. By Modules on Sites, Lemma 18.14.2 filtered colimits are exact. Hence (a) and (b) hold.

To construct a generator, choose a presentation $X = U/R$ so that (U, R, s, t, c) is an étale groupoid scheme and in particular s and t are flat morphisms of schemes. Pick a cardinal κ as in Groupoids, Lemma 38.13.6. Pick a collection $(\mathcal{E}_t, \alpha_t)_{t \in T}$ of κ-generated quasi-coherent modules on (U, R, s, t, c) as in Groupoids, Lemma 38.13.5. Let $\mathcal{F}_t$ be the quasi-coherent module on X which corresponds to the quasi-coherent module $(\mathcal{E}_t, \alpha_t)$ via the equivalence of categories of Proposition 48.30.1. Then we see that every quasi-coherent module $\mathcal{H}$ is the directed colimit of its quasi-coherent submodules which are isomorphic to one of the $\mathcal{F}_t$. Thus $\bigoplus_t \mathcal{F}_t$ is a generator of $QCoh(\mathcal{O}_X)$ and we conclude that (c) holds. The assertions on limits and injectives hold in any Grothendieck abelian category, see Injectives, Theorem 19.11.6 and Lemma 19.13.2.

Proof of (2). To construct Q we use the following general procedure. Given an object $\mathcal{F}$ of $Mod(\mathcal{O}_X)$ we consider the functor

$$QCoh(\mathcal{O}_X)^{opp} \longrightarrow Sets, \quad \mathcal{G} \longmapsto \mathrm{Hom}_X(\mathcal{G}, \mathcal{F})$$

This functor transforms colimits into limits, hence is representable, see Injectives, Lemma 19.13.1. Thus there exists a quasi-coherent sheaf $Q(\mathcal{F})$ and a functorial isomorphism $\mathrm{Hom}_X(\mathcal{G}, \mathcal{F}) = \mathrm{Hom}_X(\mathcal{G}, Q(\mathcal{F}))$ for $\mathcal{G}$ in $QCoh(\mathcal{O}_X)$. By the Yoneda lemma (Categories, Lemma 4.3.5) the construction $\mathcal{F} \rightsquigarrow Q(\mathcal{F})$ is functorial in $\mathcal{F}$. By construction Q is a right adjoint to the inclusion functor. The fact that $Q(\mathcal{F}) \to \mathcal{F}$ is an isomorphism when $\mathcal{F}$ is quasi-coherent is a formal consequence of the fact that the inclusion functor $QCoh(\mathcal{O}_X) \to Mod(\mathcal{O}_X)$ is fully faithful. □

48.31. Morphisms towards schemes

Here is the analogue of Schemes, Lemma 25.6.4.

Lemma 48.31.1. *Let X be an algebraic space over $\mathbf{Z}$. Let T be an affine scheme. The map*

$$\mathrm{Mor}(X, T) \longrightarrow \mathrm{Hom}(\Gamma(T, \mathcal{O}_T), \Gamma(X, \mathcal{O}_X))$$

which maps f to $f^\sharp$ (on global sections) is bijective.

Proof. We construct the inverse of the map. Let $\varphi : \Gamma(T, \mathcal{O}_T) \to \Gamma(X, \mathcal{O}_X)$ be a ring map. Choose a presentation $X = U/R$, see Spaces, Definition 47.9.3. By Schemes, Lemma 25.6.4 the composition

$$\Gamma(T, \mathcal{O}_T) \to \Gamma(X, \mathcal{O}_X) \to \Gamma(U, \mathcal{O}_U)$$

corresponds to a unique morphism of schemes $g : U \to T$. By the same lemma the two compositions $R \to U \to T$ are equal. Hence we obtain a morphism $f : X =$

$U/R \to T$ such that $U \to X \to T$ equals g. By construction the diagram

$$\begin{array}{ccc} \Gamma(U, \mathcal{O}_U) & \xleftarrow{f^\sharp} & \Gamma(X, \mathcal{O}_X) \\ & \nwarrow^{g^\sharp} & \uparrow{\scriptstyle \varphi} \\ & & \Gamma(T, \mathcal{O}_T) \end{array}$$

commutes. Hence $f^\sharp$ equals φ because $U \to X$ is an étale covering and $\mathcal{O}_X$ is a sheaf on $X_{étale}$. The uniqueness of f follows from the uniqueness of g. □

48.32. Quotients by free actions

Let S be a scheme. Let X be an algebraic space over S. Let G be an abstract group. Let $a : G \to \mathrm{Aut}(X)$ be a homomorphism, i.e., a is an *action* of G on X. We will say the action is *free* if for every scheme T over S the map

$$G \times X(T) \longrightarrow X(T)$$

is free. (We cannot use a criterion as in Spaces, Lemma 47.14.3 because points may not have well defined residue fields.) In case the action is free we're going to construct the quotient X/G as an algebraic space. This is a special case of the general Bootstrap, Lemma 61.11.5 that we will prove later.

Lemma 48.32.1. *Let S be a scheme. Let X be an algebraic space over S. Let G be an abstract group with a free action on X. Then the quotient sheaf X/G is an algebraic space.*

Proof. The statement means that the sheaf F associated to the presheaf

$$T \longmapsto X(T)/G$$

is an algebraic space. To see this we will construct a presentation. Namely, choose a scheme U and a surjective étale morphism $\varphi : U \to X$. Set $V = \coprod_{g \in G} U$ and set $\psi : V \to X$ equal to $a(g) \circ \varphi$ on the component corresponding to $g \in G$. Let G act on V by permuting the components, i.e., $g_0 \in G$ maps the component corresponding to g to the component corresponding to $g_0 g$ via the identity morphism of U. Then ψ is a G-equivariant morphism, i.e., we reduce to the case dealt with in the next paragraph.

Assume that there exists a G-action on U and that $U \to X$ is surjective, étale and G-equivariant. In this case there is an induced action of G on $R = U \times_X U$ compatible with the projection mappings $t, s : R \to U$. Now we claim that

$$X/G = U/\coprod\nolimits_{g \in G} R$$

where the map

$$j : \coprod\nolimits_{g \in G} R \longrightarrow U \times_S U$$

is given by $(r, g) \mapsto (t(r), g(s(r)))$. Note that j is a monomorphism: If $(t(r), g(s(r))) = (t(r'), g'(s(r')))$, then $t(r) = t(r')$, hence r and r' have the same image in X under both s and t, hence $g = g'$ (as G acts freely on X), hence $s(r) = s(r')$, hence $r = r'$ (as R is an equivalence relation on U). Moreover j is an equivalence relation (details omitted). Both projections $\coprod_{g \in G} R \to U$ are étale, as s and t are étale. Thus

j is an étale equivalence relation and $U/\coprod_{g\in G} R$ is an algebraic space by Spaces, Theorem 47.10.5. There is a map

$$U/\coprod\nolimits_{g\in G} R \longrightarrow X/G$$

induced by the map $U \to X$. We omit the proof that it is an isomorphism of sheaves. □

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CHAPTER 49

Morphisms of Algebraic Spaces

49.1. Introduction

In this chapter we introduce some types of morphisms of algebraic spaces. A reference is [**Knu71**].

The goal is to extend the definition of each of the types of morphisms of schemes defined in the chapters on schemes, and on morphisms of schemes to the category of algebraic spaces. Each case is slightly different and it seems best to treat them all separately.

49.2. Conventions

The standing assumption is that all schemes are contained in a big fppf site Sch_{fppf}. And all rings A considered have the property that $\mathrm{Spec}(A)$ is (isomorphic) to an object of this big site.

Let S be a scheme and let X be an algebraic space over S. In this chapter and the following we will write $X \times_S X$ for the product of X with itself (in the category of algebraic spaces over S), instead of $X \times X$.

49.3. Properties of representable morphisms

Let S be a scheme. Let $f : X \to Y$ be a representable morphism of algebraic spaces. In Spaces, Section 47.5 we defined what it means for f to have property $\mathcal{P}$ in case $\mathcal{P}$ is a property of morphisms of schemes which

(1) is preserved under any base change, see Schemes, Definition 25.18.3, and
(2) is fppf local on the base, see Descent, Definition 34.18.1.

Namely, in this case we say f has property $\mathcal{P}$ if and only if for every scheme U and any morphism $U \to Y$ the morphism of schemes $X \times_Y U \to U$ has property $\mathcal{P}$.

According to the lists in Spaces, Section 47.4 this applies to the following properties: (1)(a) closed immersions, (1)(b) open immersions, (1)(c) quasi-compact immersions, (2) quasi-compact, (3) universally-closed, (4) (quasi-)separated, (5) monomorphism, (6) surjective, (7) universally injective, (8) affine, (9) quasi-affine, (10) (locally) of finite type, (11) (locally) quasi-finite, (12) (locally) of finite presentation, (13) locally of finite type of relative dimension d, (14) universally open, (15) flat, (16) syntomic, (17) smooth, (18) unramified (resp. G-unramified), (19) étale, (20) proper, (21) finite or integral, (22) finite locally free, and (23) immersion.

In this chapter we will redefine these notions for not necessarily representable morphisms of algebraic spaces. Whenever we do this we will make sure that the new definition agrees with the old one, in order to avoid ambiguity.

Note that the definition above applies whenever X is a scheme, since a morphism from a scheme to an algebraic space is representable. And in particular it applies when both X and Y are schemes. In Spaces, Lemma 47.5.3 we have seen that in this case the definitions match, and no ambiguity arise.

Furthermore, in Spaces, Lemma 47.5.5 we have seen that the property of representable morphisms of algebraic spaces so defined is stable under arbitrary base change by a morphism of algebraic spaces. And finally, in Spaces, Lemmas 47.5.4 and 47.5.7 we have seen that if $\mathcal{P}$ is stable under compositions, which holds for the properties (1)(a), (1)(b), (1)(c), (2) – (23), except (13) above, then taking products of representable morphisms preserves property $\mathcal{P}$ and compositions of representable morphisms preserves property $\mathcal{P}$.

We will use these facts below, and whenever we do we will simply refer to this section as a reference.

49.4. Immersions

Open, closed and locally closed immersions of algebraic spaces were defined in Spaces, Section 47.12. Namely, a morphism of algebraic spaces is a *closed immersion* (resp. *open immersion*, resp. *immersion*) if it is representable and a closed immersion (resp. open immersion, resp. immersion) in the sense of Section 49.3.

In particular these types of morphisms are stable under base change and compositions of morphisms in the category of algebraic spaces over S, see Spaces, Lemmas 47.12.2 and 47.12.3.

Lemma 49.4.1. *Let S be a scheme. Let $f : X \to Y$ be a morphism of algebraic spaces over S. The following are equivalent:*

1. *f is a closed immersion (resp. open immersion, resp. immersion),*
2. *for every scheme Z and any morphism $Z \to Y$ the morphism $Z \times_Y X \to Z$ is a closed immersion (resp. open immersion, resp. immersion),*
3. *for every affine scheme Z and any morphism $Z \to Y$ the morphism $Z \times_Y X \to Z$ is a closed immersion (resp. open immersion, resp. immersion),*
4. *there exists a scheme V and a surjective étale morphism $V \to Y$ such that $V \times_Y X \to V$ is a closed immersion (resp. open immersion, resp. immersion), and*
5. *there exists a Zariski covering $Y = \bigcup Y_i$ such that each of the morphisms $f^{-1}(Y_i) \to Y_i$ is a closed immersion (resp. open immersion, resp. immersion).*

Proof. Using that a base change of a closed immersion (resp. open immersion, resp. immersion) is another one it is clear that (1) implies (2) and (2) implies (3). Also (3) implies (4) since we can take V to be a disjoint union of affines, see Properties of Spaces, Lemma 48.6.1.

Assume $V \to Y$ is as in (4). Let $\mathcal{P}$ be the property closed immersion (resp. open immersion, resp. immersion) of morphisms of schemes. Note that property $\mathcal{P}$ is preserved under any base change and fppf local on the base (see Section 49.3). Moreover, morphisms of type $\mathcal{P}$ are separated and locally quasi-finite (in each of the three cases, see Schemes, Lemma 25.23.7, and Morphisms, Lemma 28.21.14). Hence by More on Morphisms, Lemma 36.37.1 the morphisms of type $\mathcal{P}$ satisfy

descent for fppf covering. Thus Spaces, Lemma 47.11.3 applies and we see that $X \to Y$ is representable and has property $\mathcal{P}$, in other words (1) holds.

The equivalence of (1) and (5) follows from the fact that $\mathcal{P}$ is Zariski local on the target (since we saw above that $\mathcal{P}$ is in fact fppf local on the target). □

Lemma 49.4.2. *Let S be a scheme. Let $i : Z \to X$ be an immersion of algebraic spaces over S. Then $|i|(|Z|) \subset |X|$ is a locally closed subset, and i is a closed immersion if and only if $|i|(|Z|) \subset |X|$ is a closed subset.*

Proof. Let U be a scheme and let $U \to X$ be a surjective étale morphism. By assumption $T = U \times_X Z$ is a scheme and the morphism $j : T \to U$ is an immersion of schemes. Moreover, note that $|j|(|T|) \subset |U|$ is the inverse image of $|i|(|Z|) \subset |X|$, see Properties of Spaces, Lemma 48.4.3. Recall that $|U| \to |X|$ is surjective and open, see Properties of Spaces, Lemma 48.4.6. Hence since $|T|$ is locally closed in $|U|$ it follows that $|i|(|Z|)$ is locally closed in $|X|$, see Topology, Lemma 5.5.4. And in the same way we see that $|i|(|Z|) \subset |X|$ is closed if and only if $T \subset U$ is closed, Thus we can combine Lemma 49.4.1 above and Schemes, Lemma 25.10.4 to finish the proof. □

Remark 49.4.3. Let S be a scheme. Let $i : Z \to X$ be an immersion of algebraic spaces over S. Since i is a monomorphism we may think of $|Z|$ as a subset of $|X|$; in the rest of this remark we do so. Let $\partial|Z|$ be the boundary of $|Z|$ in the topological space $|X|$. In a formula

$$\partial|Z| = \overline{|Z|} \setminus |Z|.$$

Let ∂Z be the reduced closed subspace of X with $|\partial Z| = \partial|Z|$ obtained by taking the reduced induced closed subspace structure, see Properties of Spaces, Definition 48.9.4. By construction we see that $|Z|$ is closed in $|X| \setminus |\partial Z| = |X \setminus \partial Z|$. Hence it is true that any immersion of algebraic spaces can be factored as a closed immersion followed by an open immersion (but not the other way in general, see Morphisms, Example 28.3.4).

Remark 49.4.4. Let S be a scheme. Let X be an algebraic space over S. Let $T \subset |X|$ be a locally closed subset. Let ∂T be the boundary of T in the topological space $|X|$. In a formula

$$\partial T = \overline{T} \setminus T.$$

Let $U \subset X$ be the open subspace of X with $|U| = |X| \setminus \partial T$, see Properties of Spaces, Lemma 48.4.8. Let Z be the reduced closed subspace of U with $|Z| = T$ obtained by taking the reduced induced closed subspace structure, see Properties of Spaces, Definition 48.9.4. By construction $Z \to U$ is a closed immersion of algebraic spaces and $U \to X$ is an open immersion, hence $Z \to X$ is an immersion of algebraic spaces over S (see Spaces, Lemma 47.12.2). Note that Z is a reduced algebraic space and that $|Z| = T$ as subsets of $|X|$. We sometimes say Z is the *reduced induced subspace structure* on T.

Lemma 49.4.5. *Let S be a scheme. Let $Z \to X$ be an immersion of algebraic spaces over S. Assume $Z \to X$ is quasi-compact. There exists a factorization $Z \to \overline{Z} \to X$ where $Z \to \overline{Z}$ is an open immersion and $\overline{Z} \to X$ is a closed immersion.*

Proof. Let U be a scheme and let $U \to X$ be surjective étale. As usual denote $R = U \times_X U$ with projections $s, t : R \to U$. Set $T = Z \times_U X$. Let $\overline{T} \subset U$ be the scheme theoretic image of $T \to U$. Note that $s^{-1}\overline{T} = t^{-1}\overline{T}$ as taking scheme

theoretic images of quasi-compact morphisms commute with flat base change, see Morphisms, Lemma 28.26.14. Hence we obtain a closed subspace $\overline{Z} \subset X$ whose pullback to U is $\overline{T}$, see Properties of Spaces, Lemma 48.9.1. By Morphisms, Lemma 28.7.7 the morphism $T \to \overline{T}$ is an open immersion. It follows that $Z \to \overline{Z}$ is an open immersion and we win. □

49.5. Separation axioms

It makes sense to list some a priori properties of the diagonal of a morphism of algebraic spaces.

Lemma 49.5.1. *Let S be a scheme contained in Sch_{fppf}. Let $f : X \to Y$ be a morphism of algebraic spaces over S. Let $\Delta_{X/Y} : X \to X \times_Y X$ be the diagonal morphism. Then*

(1) *$\Delta_{X/Y}$ is representable,*
(2) *$\Delta_{X/Y}$ is locally of finite type,*
(3) *$\Delta_{X/Y}$ is a monomorphism,*
(4) *$\Delta_{X/Y}$ is separated, and*
(5) *$\Delta_{X/Y}$ is locally quasi-finite.*

Proof. We are going to use the fact that $\Delta_{X/S}$ is representable (by definition of an algebraic space) and that it satisfies properties (2) – (5), see Spaces, Lemma 47.13.1. Note that we have a factorization

$$X \longrightarrow X \times_Y X \longrightarrow X \times_S X$$

of the diagonal $\Delta_{X/S} : X \to X \times_S X$. Since $X \times_Y X \to X \times_S X$ is a monomorphism, and since $\Delta_{X/S}$ is representable, it follows formally that $\Delta_{X/Y}$ is representable. In particular, the rest of the statements now make sense, see Section 49.3.

Choose a surjective étale morphism $U \to X$, with U a scheme. Consider the diagram

$$\begin{array}{ccccc} R = U \times_X U & \longrightarrow & U \times_Y U & \longrightarrow & U \times_S U \\ \downarrow & & \downarrow & & \downarrow \\ X & \longrightarrow & X \times_Y X & \longrightarrow & X \times_S X \end{array}$$

Both squares are cartesian, hence so is the outer rectangle. The top row consists of schemes, and the vertical arrows are surjective étale morphisms. By Spaces, Lemma 47.11.2 the properties (2) – (5) for $\Delta_{X/Y}$ are equivalent to those of $R \to U \times_Y U$. In the proof of Spaces, Lemma 47.13.1 we have seen that $R \to U \times_S U$ has properties (2) – (5). The morphism $U \times_Y U \to U \times_S U$ is a monomorphism of schemes. These facts imply that $R \to U \times_S U$ have properties (2) – (5).

Namely: For (3), note that $R \to U \times_Y U$ is a monomorphism as the composition $R \to U \times_S U$ is a monomorphism. For (2), note that $R \to U \times_Y U$ is locally of finite type, as the composition $R \to U \times_S U$ is locally of finite type (Morphisms, Lemma 28.16.8). A monomorphism which is locally of finite type is locally quasi-finite because it has finite fibres (Morphisms, Lemma 28.21.7), hence (5). A monomorphism is separated (Schemes, Lemma 25.23.3), hence (4). □

Definition 49.5.2. Let S be a scheme. Let $f : X \to Y$ be a morphism of algebraic spaces over S. Let $\Delta_{X/Y} : X \to X \times_Y X$ be the diagonal morphism.

(1) We say f is *separated* if $\Delta_{X/Y}$ is a closed immersion.
(2) We say f is *locally separated*[1] if $\Delta_{X/Y}$ is an immersion.
(3) We say f is *quasi-separated* if $\Delta_{X/Y}$ is quasi-compact.

This definition makes sense since $\Delta_{X/Y}$ is representable, and hence we know what it means for it to have one of the properties described in the definition. We will see below (Lemma 49.5.13) that this definition matches the ones we already have for morphisms of schemes and representable morphisms.

Lemma 49.5.3. *Let S be a scheme. Let $f : X \to Y$ be a morphism of algebraic spaces over S. If f is separated, then f is locally separated and f is quasi-separated.*

Proof. This is true, via the general principle Spaces, Lemma 47.5.8, because a closed immersion of schemes is an immersion and is quasi-compact. □

Lemma 49.5.4. *All of the separation axioms listed in Definition 49.5.2 are stable under base change.*

Proof. Let $f : X \to Y$ and $Y' \to Y$ be morphisms of algebraic spaces. Let $f' : X' \to Y'$ be the base change of f by $Y' \to Y$. Then $\Delta_{X'/Y'}$ is the base change of $\Delta_{X/Y}$ by the morphism $X' \times_{Y'} X' \to X \times_Y X$. By the results of Section 49.3 each of the properties of the diagonal used in Definition 49.5.2 is stable under base change. Hence the lemma is true. □

Lemma 49.5.5. *Let S be a scheme. Let $f : X \to Z$, $g : Y \to Z$ and $Z \to T$ be morphisms of algebraic spaces over S. Consider the induced morphism $i : X \times_Z Y \to X \times_T Y$. Then*

(1) *i is representable, locally of finite type, locally quasi-finite, separated and a monomorphism,*
(2) *if $Z \to T$ is locally separated, then i is an immersion,*
(3) *if $Z \to T$ is separated, then i is a closed immersion, and*
(4) *if $Z \to T$ is quasi-separated, then i is quasi-compact.*

Proof. By general category theory the following diagram

$$\begin{array}{ccc} X \times_Z Y & \xrightarrow[i]{} & X \times_T Y \\ \downarrow & & \downarrow \\ Z & \xrightarrow{\Delta_{Z/T}} & Z \times_T Z \end{array}$$

is a fibre product diagram. Hence i is the base change of the diagonal morphism $\Delta_{Z/T}$. Thus the lemma follows from Lemma 49.5.1, and the material in Section 49.3. □

Lemma 49.5.6. *Let S be a scheme. Let T be an algebraic space over S. Let $g : X \to Y$ be a morphism of algebraic spaces over T. Consider the graph $i : X \to X \times_T Y$ of g. Then*

(1) *i is representable, locally of finite type, locally quasi-finite, separated and a monomorphism,*
(2) *if $Y \to T$ is locally separated, then i is an immersion,*
(3) *if $Y \to T$ is separated, then i is a closed immersion, and*
(4) *if $Y \to T$ is quasi-separated, then i is quasi-compact.*

[1] In the literature this term often refers to quasi-separated and locally separated morphisms.

Proof. This is a special case of Lemma 49.5.5 applied to the morphism $X = X \times_Y Y \to X \times_T Y$. □

Lemma 49.5.7. *Let S be a schemes. Let $f : X \to T$ be a morphism of algebraic spaces over S. Let $s : T \to X$ be a section of f (in a formula $f \circ s = id_T$). Then*

1. *s is representable, locally of finite type, locally quasi-finite, separated and a monomorphism,*
2. *if f is locally separated, then s is an immersion,*
3. *if f is separated, then s is a closed immersion, and*
4. *if f is quasi-separated, then s is quasi-compact.*

Proof. This is a special case of Lemma 49.5.6 applied to $g = s$ so the morphism $i = s : T \to T \times_T X$. □

Lemma 49.5.8. *All of the separation axioms listed in Definition 49.5.2 are stable under composition of morphisms.*

Proof. Let $f : X \to Y$ and $g : Y \to Z$ be morphisms of algebraic spaces to which the axiom in question applies. The diagonal $\Delta_{X/Z}$ is the composition

$$X \longrightarrow X \times_Y X \longrightarrow X \times_Z X.$$

Our separation axiom is defined by requiring the diagonal to have some property $\mathcal{P}$. By Lemma 49.5.5 above we see that the second arrow also has this property. Hence the lemma follows since the composition of (representable) morphisms with property $\mathcal{P}$ also is a morphism with property $\mathcal{P}$, see Section 49.3. □

Lemma 49.5.9. *Let S be a scheme. Let $f : X \to Y$ be a morphism of algebraic spaces over S.*

1. *If Y is separated and f is separated, then X is separated.*
2. *If Y is quasi-separated and f is quasi-separated, then X is quasi-separated.*
3. *If Y is locally separated and f is locally separated, then X is locally separated.*
4. *If Y is separated over S and f is separated, then X is separated over S.*
5. *If Y is quasi-separated over S and f is quasi-separated, then X is quasi-separated over S.*
6. *If Y is locally separated over S and f is locally separated, then X is locally separated over S.*

Proof. Parts (4), (5), and (6) follow immediately from Lemma 49.5.8 and Spaces, Definition 47.13.2. Parts (1), (2), and (3) reduce to parts (4), (5), and (6) by thinking of X and Y as algebraic spaces over $\mathrm{Spec}(\mathbf{Z})$, see Properties of Spaces, Definition 48.3.1. □

Lemma 49.5.10. *Let S be a scheme. Let $f : X \to Y$ and $g : Y \to Z$ be morphisms of algebraic spaces over S.*

1. *If $g \circ f$ is separated then so is f.*
2. *If $g \circ f$ is locally separated then so is f.*
3. *If $g \circ f$ is quasi-separated then so is f.*

Proof. Consider the factorization

$$X \to X \times_Y X \to X \times_Z X$$

of the diagonal morphism of $g \circ f$. In any case the last morphism is a monomorphism. Hence for any scheme T and morphism $T \to X \times_Y X$ we have the equality

$$X \times_{(X \times_Y X)} T = X \times_{(X \times_Z X)} T.$$

Hence the result is clear. □

Lemma 49.5.11. *Let S be a scheme. Let X be an algebraic space over S.*

(1) *If X is separated then X is separated over S.*
(2) *If X is locally separated then X is locally separated over S.*
(3) *If X is quasi-separated then X is quasi-separated over S.*

Let $f : X \to Y$ be a morphism of algebraic spaces over S.

(4) *If X is separated over S then f is separated.*
(5) *If X is locally separated over S then f is locally separated.*
(6) *If X is quasi-separated over S then f is quasi-separated.*

Proof. Parts (4), (5), and (6) follow immediately from Lemma 49.5.10 and Spaces, Definition 47.13.2. Parts (1), (2), and (3) follow from parts (4), (5), and (6) by thinking of X and Y as algebraic spaces over $\mathrm{Spec}(\mathbf{Z})$, see Properties of Spaces, Definition 48.3.1. □

Lemma 49.5.12. *Let S be a scheme. Let $f : X \to Y$ be a morphism of algebraic spaces over S. Let $\mathcal{P}$ be any of the separation axioms of Definition 49.5.2. The following are equivalent*

(1) *f is $\mathcal{P}$,*
(2) *for every scheme Z and morphism $Z \to Y$ the base change $Z \times_Y X \to Z$ of f is $\mathcal{P}$,*
(3) *for every affine scheme Z and every morphism $Z \to Y$ the base change $Z \times_Y X \to Z$ of f is $\mathcal{P}$,*
(4) *for every affine scheme Z and every morphism $Z \to Y$ the algebraic space $Z \times_Y X$ is $\mathcal{P}$ (see Properties of Spaces, Definition 48.3.1),*
(5) *there exists a scheme V and a surjective étale morphism $V \to Y$ such that the base change $V \times_Y X \to V$ has $\mathcal{P}$, and*
(6) *there exists a Zariski covering $Y = \bigcup Y_i$ such that each of the morphisms $f^{-1}(Y_i) \to Y_i$ has $\mathcal{P}$.*

Proof. We will repeatedly use Lemma 49.5.4 without further mention. In particular, it is clear that (1) implies (2) and (2) implies (3).

Let us prove that (3) and (4) are equivalent. Note that if Z is an affine scheme, then the morphism $Z \to \mathrm{Spec}(\mathbf{Z})$ is a separated morphism as a morphism of algebraic spaces over $\mathrm{Spec}(\mathbf{Z})$. If $Z \times_Y X \to Z$ is $\mathcal{P}$, then $Z \times_Y X \to \mathrm{Spec}(\mathbf{Z})$ is $\mathcal{P}$ as a composition (see Lemma 49.5.8). Hence the algebraic space $Z \times_Y X$ is $\mathcal{P}$. Conversely, if the algebraic space $Z \times_Y X$ is $\mathcal{P}$, then $Z \times_Y X \to \mathrm{Spec}(\mathbf{Z})$ is $\mathcal{P}$, and hence by Lemma 49.5.10 we see that $Z \times_Y X \to Z$ is $\mathcal{P}$.

Let us prove that (3) implies (5). Assume (3). Let V be a scheme and let $V \to Y$ be étale surjective. We have to show that $V \times_Y X \to V$ has property $\mathcal{P}$. In other words, we have to show that the morphism

$$V \times_Y X \longrightarrow (V \times_Y X) \times_V (V \times_Y X) = V \times_Y X \times_Y X$$

has the corresponding property (i.e., is a closed immersion, immersion, or quasi-compact). Let $V = \bigcup V_j$ be an affine open covering of V. By assumption we know that each of the morphisms

$$V_j \times_Y X \longrightarrow V_j \times_Y X \times_Y X$$

does have the corresponding property. Since being a closed immersion, immersion, quasi-compact immersion, or quasi-compact is Zariski local on the target, and since the V_j cover V we get the desired conclusion.

Let us prove that (5) implies (1). Let $V \to Y$ be as in (5). Then we have the fibre product diagram

$$\begin{array}{ccc} V \times_Y X & \longrightarrow & X \\ \downarrow & & \downarrow \\ V \times_Y X \times_Y X & \longrightarrow & X \times_Y X \end{array}$$

By assumption the left vertical arrow is a closed immersion, immersion, quasi-compact immersion, or quasi-compact. It follows from Spaces, Lemma 47.5.6 that also the right vertical arrow is a closed immersion, immersion, quasi-compact immersion, or quasi-compact.

It is clear that (1) implies (6) by taking the covering $Y = Y$. Assume $Y = \bigcup Y_i$ is as in (6). Choose schemes V_i and surjective étale morphisms $V_i \to Y_i$. Note that the morphisms $V_i \times_Y X \to V_i$ have $\mathcal{P}$ as they are base changes of the morphisms $f^{-1}(Y_i) \to Y_i$. Set $V = \coprod V_i$. Then $V \to Y$ is a morphism as in (5) (details omitted). Hence (6) implies (5) and we are done. □

Lemma 49.5.13. *Let S be a scheme. Let $f : X \to Y$ be a representable morphism of algebraic spaces over S.*

1. *The morphism f is locally separated.*
2. *The morphism f is (quasi-)separated in the sense of Definition 49.5.2 above if and only if f is (quasi-)separated in the sense of Section 49.3.*

In particular, if $f : X \to Y$ is a morphism of schemes over S, then f is (quasi-)separated in the sense of Definition 49.5.2 if and only if f is (quasi-)separated as a morphism of schemes.

Proof. This is the equivalence of (1) and (2) of Lemma 49.5.12 combined with the fact that any morphism of schemes is locally separated, see Schemes, Lemma 25.21.2. □

49.6. Surjective morphisms

We have already defined in Section 49.3 what it means for a representable morphism of algebraic spaces to be surjective.

Lemma 49.6.1. *Let S be a scheme. Let $f : X \to Y$ be a representable morphism of algebraic spaces over S. Then f is surjective if and only if $|f| : |X| \to |Y|$ is surjective.*

Proof. Namely, if $f : X \to Y$ is representable, then it is surjective if and only if for every scheme T and every morphism $T \to Y$ the base change $f_T : T \times_Y X \to T$ of f is a surjective morphism of schemes, in other words, if and only if $|f_T|$ is surjective. By Properties of Spaces, Lemma 48.4.3 the map $|T \times_Y X| \to |T| \times_{|Y|} |X|$ is always

surjective. Hence $|f_T| : |T \times_Y X| \to |T|$ is surjective if $|f| : |X| \to |Y|$ is surjective. Conversely, if $|f_T|$ is surjective for every $T \to Y$ as above, then by taking T to be the spectrum of a field we conclude that $|X| \to |Y|$ is surjective. □

This clears the way for the following definition.

Definition 49.6.2. Let S be a scheme. Let $f : X \to Y$ be a morphism of algebraic spaces over S. We say f is *surjective* if the map $|f| : |X| \to |Y|$ of associated topological spaces is surjective.

Lemma 49.6.3. *Let S be a scheme. Let $f : X \to Y$ be a morphism of algebraic spaces over S. The following are equivalent:*

1. *f is surjective,*
2. *for every scheme Z and any morphism $Z \to Y$ the morphism $Z \times_Y X \to Z$ is surjective,*
3. *for every affine scheme Z and any morphism $Z \to Y$ the morphism $Z \times_Y X \to Z$ is surjective,*
4. *there exists a scheme V and a surjective étale morphism $V \to Y$ such that $V \times_Y X \to V$ is a surjective morphism,*
5. *there exists a scheme U and a surjective étale morphism $\varphi : U \to X$ such that the composition $f \circ \varphi$ is surjective,*
6. *there exists a commutative diagram*

$$\begin{array}{ccc} U & \longrightarrow & V \\ \downarrow & & \downarrow \\ X & \longrightarrow & Y \end{array}$$

   where U, V are schemes and the vertical arrows are surjective étale such that the top horizontal arrow is surjective, and
7. *there exists a Zariski covering $Y = \bigcup Y_i$ such that each of the morphisms $f^{-1}(Y_i) \to Y_i$ is surjective.*

Proof. Omitted. □

Lemma 49.6.4. *The composition of surjective morphisms is surjective.*

Proof. This is immediate from the definition. □

Lemma 49.6.5. *The base change of a surjective morphism is surjective.*

Proof. Follows immediately from Properties of Spaces, Lemma 48.4.3. □

49.7. Open morphisms

For a representable morphism of algebraic spaces we have already defined (in Section 49.3) what it means to be universally open. Hence before we give the natural definition we check that it agrees with this in the representable case.

Lemma 49.7.1. *Let S be a scheme. Let $f : X \to Y$ be a representable morphism of algebraic spaces over S. The following are equivalent*

1. *f is universally open, and*
2. *for every morphism of algebraic spaces $Z \to Y$ the morphism of topological spaces $|Z \times_Y X| \to |Z|$ is open.*

Proof. Assume (1), and let $Z \to Y$ be as in (2). Choose a scheme V and a surjective étale morphism $V \to Y$. By assumption the morphism of schemes $V \times_Y X \to V$ is universally open. By Properties of Spaces, Section 48.4 in the commutative diagram

$$\begin{array}{ccc} |V \times_Y X| & \longrightarrow & |Z \times_Y X| \\ \downarrow & & \downarrow \\ |V| & \longrightarrow & |Z| \end{array}$$

the horizontal arrows are open and surjective, and moreover

$$|V \times_Y X| \longrightarrow |V| \times_{|Z|} |Z \times_Y X|$$

is surjective. Hence as the left vertical arrow is open it follows that the right vertical arrow is open. This proves (2). The implication (2) $\Rightarrow$ (1) is immediate from the definitions. □

Thus we may use the following natural definition.

Definition 49.7.2. Let S be a scheme. Let $f : X \to Y$ be a morphism of algebraic spaces over S.

(1) We say f is *open* if the map of topological spaces $|f| : |X| \to |Y|$ is open.
(2) We say f is *universally open* if for every morphism of algebraic spaces $Z \to Y$ the morphism of topological spaces
$$|Z \times_Y X| \to |Z|$$
is open, i.e., the base change $Z \times_Y X \to Z$ is open.

Note that an étale morphism of algebraic spaces is universally open, see Properties of Spaces, Definition 48.13.2 and Lemmas 48.13.7 and 48.13.5.

Lemma 49.7.3. *The base change of a universally open morphism of algebraic spaces by any morphism of algebraic spaces is universally open.*

Proof. This is immediate from the definition. □

Lemma 49.7.4. *The composition of a pair of (universally) open morphisms of algebraic spaces is (universally) open.*

Proof. Omitted. □

Lemma 49.7.5. *Let S be a scheme. Let $f : X \to Y$ be a morphism of algebraic spaces over S. The following are equivalent*

(1) *f is universally open,*
(2) *for every scheme Z and every morphism $Z \to Y$ the projection $|Z \times_Y X| \to |Z|$ is open,*
(3) *for every affine scheme Z and every morphism $Z \to Y$ the projection $|Z \times_Y X| \to |Z|$ is open, and*
(4) *there exists a scheme V and a surjective étale morphism $V \to Y$ such that $V \times_Y X \to V$ is a universally open morphism of algebraic spaces, and*
(5) *there exists a Zariski covering $Y = \bigcup Y_i$ such that each of the morphisms $f^{-1}(Y_i) \to Y_i$ is universally open.*

Proof. We omit the proof that (1) implies (2), and that (2) implies (3).

Assume (3). Choose a surjective étale morphism $V \to Y$. We are going to show that $V \times_Y X \to V$ is a universally open morphism of algebraic spaces. Let $Z \to V$ be a morphism from an algebraic space to V. Let $W \to Z$ be a surjective étale morphism where $W = \coprod W_i$ is a disjoint union of affine schemes, see Properties of Spaces, Lemma 48.6.1. Then we have the following commutative diagram

$$\begin{array}{ccccccc} \coprod_i |W_i \times_Y X| & = & |W \times_Y X| & \longrightarrow & |Z \times_Y X| & = & |Z \times_V (V \times_Y X)| \\ \downarrow & & \downarrow & & \downarrow & \swarrow & \\ \coprod |W_i| & = & |W| & \longrightarrow & |Z| & & \end{array}$$

We have to show the south-east arrow is open. The middle horizontal arrows are surjective and open (Properties of Spaces, Lemma 48.13.7). By assumption (3), and the fact that W_i is affine we see that the left vertical arrows are open. Hence it follows that the right vertical arrow is open.

Assume $V \to Y$ is as in (4). We will show that f is universally open. Let $Z \to Y$ be a morphism of algebraic spaces. Consider the diagram

$$\begin{array}{ccccc} |(V \times_Y Z) \times_V (V \times_Y X)| & = & |V \times_Y X| & \longrightarrow & |Z \times_Y X| \\ & \searrow & \downarrow & & \downarrow \\ & & |V \times_Y Z| & \longrightarrow & |Z| \end{array}$$

The south-west arrow is open by assumption. The horizontal arrows are surjective and open because the corresponding morphisms of algebraic spaces are étale (see Properties of Spaces, Lemma 48.13.7). It follows that the right vertical arrow is open.

Of course (1) implies (5) by taking the covering $Y = Y$. Assume $Y = \bigcup Y_i$ is as in (5). Then for any $Z \to Y$ we get a corresponding Zariski covering $Z = \bigcup Z_i$ such that the base change of f to Z_i is open. By a simple topological argument this implies that $Z \times_Y X \to Z$ is open. Hence (1) holds. □

Lemma 49.7.6. *Let S be a scheme. Let $p : X \to \mathrm{Spec}(k)$ be a morphism of algebraic spaces over S where k is a field. Then $p : X \to \mathrm{Spec}(k)$ is universally open.*

Proof. Choose a scheme U and a surjective étale morphism $U \to X$. The composition $U \to \mathrm{Spec}(k)$ is universally open (as a morphism of schemes) by Morphisms, Lemma 28.24.4. Let $Z \to \mathrm{Spec}(k)$ be a morphism of schemes. Then $U \times_{\mathrm{Spec}(k)} Z \to X \times_{\mathrm{Spec}(k)} Z$ is surjective, see Lemma 49.6.5. Hence the first of the maps

$$|U \times_{\mathrm{Spec}(k)} Z| \to |X \times_{\mathrm{Spec}(k)} Z| \to |Z|$$

is surjective. Since the composition is open by the above we conclude that the second map is open as well. Whence p is universally open by Lemma 49.7.5. □

49.8. Submersive morphisms

Definition 49.8.1. Let S be a scheme. Let $f : X \to Y$ be a morphism of algebraic spaces over S.

(1) We say f is *submersive*[2] if the continuous map $|X| \to |Y|$ is submersive, see Topology, Definition 5.5.3.
(2) We say f is *universally submersive* if for every morphism of algebraic spaces $Y' \to Y$ the base change $Y' \times_Y X \to Y'$ is submersive.

We note that a submersive morphism is in particular surjective.

49.9. Quasi-compact morphisms

By Section 49.3 we know what it means for a representable morphism of algebraic spaces to be quasi-compact. In order to formulate the definition for a general morphism of algebraic spaces we make the following observation.

Lemma 49.9.1. *Let S be a scheme. Let $f : X \to Y$ be a representable morphism of algebraic spaces over S. The following are equivalent:*

(1) *f is quasi-compact, and*
(2) *for every quasi-compact algebraic space Z and any morphism $Z \to Y$ the algebraic space $Z \times_Y X$ is quasi-compact.*

Proof. Assume (1), and let $Z \to Y$ be a morphism of algebraic spaces with Z quasi-compact. By Properties of Spaces, Definition 48.5.1 there exists a quasi-compact scheme U and a surjective étale morphism $U \to Z$. Since f is representable and quasi-compact we see by definition that $U \times_Y X$ is a scheme, and that $U \times_Y X \to U$ is quasi-compact. Hence $U \times_Y X$ is a quasi-compact scheme. The morphism $U \times_Y X \to Z \times_Y X$ is étale and surjective (as the base change of the representable étale and surjective morphism $U \to Z$, see Section 49.3). Hence by definition $Z \times_Y X$ is quasi-compact.

Assume (2). Let $Z \to Y$ be a morphism, where Z is a scheme. We have to show that $p : Z \times_Y X \to Z$ is quasi-compact. Let $U \subset Z$ be affine open. Then $p^{-1}(U) = U \times_Y Z$ and the scheme $U \times_Y Z$ is quasi-compact by assumption (2). Hence p is quasi-compact, see Schemes, Section 25.19. □

This motivates the following definition.

Definition 49.9.2. Let S be a scheme. Let $f : X \to Y$ be a morphism of algebraic spaces over S. We say f is *quasi-compact* if for every quasi-compact algebraic space Z and morphism $Z \to Y$ the fibre product $Z \times_Y X$ is quasi-compact.

By Lemma 49.9.1 above this agrees with the already existing notion for representable morphisms of algebraic spaces.

Lemma 49.9.3. *The base change of a quasi-compact morphism of algebraic spaces by any morphism of algebraic spaces is quasi-compact.*

Proof. Omitted. □

Lemma 49.9.4. *The composition of a pair of quasi-compact morphisms of algebraic spaces is quasi-compact.*

[2] This is very different from the notion of a submersion of differential manifolds.

Proof. Omitted. □

Lemma 49.9.5. *Let S be a scheme.*

(1) *If $X \to Y$ is a surjective morphism of algebraic spaces over S, and X is quasi-compact then Y is quasi-compact.*
(2) *If*

$$\begin{array}{ccc} X & \xrightarrow{f} & Y \\ & {\scriptstyle p}\searrow \quad \swarrow{\scriptstyle q} & \\ & Z & \end{array}$$

is a commutative diagram of morphisms of algebraic spaces over S and f is surjective and p is quasi-compact, then q is quasi-compact.

Proof. Assume X is quasi-compact and $X \to Y$ is surjective. By Definition 49.6.2 the map $|X| \to |Y|$ is surjective, hence we see Y is quasi-compact by Properties of Spaces, Lemma 48.5.2 and the topological fact that the image of a quasi-compact space under a continuous map is quasi-compact, see Topology, Lemma 5.11.7. Let f, p, q be as in (2). Let $T \to Z$ be a morphism whose source is a quasi-compact algebraic space. By assumption $T \times_Z X$ is quasi-compact. By Lemma 49.6.5 the morphism $T \times_Z X \to T \times_Z Y$ is surjective. Hence by part (1) we see $T \times_Z Y$ is quasi-compact too. Thus q is quasi-compact. □

Lemma 49.9.6. *Let S be a scheme. Let $f : X \to Y$ be a morphism of algebraic spaces over S. Let $g : Y' \to Y$ be a universally open and surjective morphism of algebraic spaces such that the base change $f' : X' \to Y'$ is quasi-compact. Then f is quasi-compact.*

Proof. Let $Z \to Y$ be a morphism of algebraic spaces with Z quasi-compact. As g is universally open and surjective, we see that $Y' \times_Y Z \to Z$ is open and surjective. As every point of $|Y' \times_Y Z|$ has a fundamental system of quasi-compact open neighbourhoods (see Properties of Spaces, Lemma 48.5.5) we can find a quasi-compact open $W \subset |Y' \times_Y Z|$ which surjects onto Z. Denote $f'' : W \times_Y X \to W$ the base change of f' by $W \to Y'$. By assumption $W \times_Y X$ is quasi-compact. As $W \to Z$ is surjective we see that $W \times_Y X \to Z \times_Y X$ is surjective. Hence $Z \times_Y X$ is quasi-compact by Lemma 49.9.5. Thus f is quasi-compact. □

Lemma 49.9.7. *Let S be a scheme. Let $f : X \to Y$ be a morphism of algebraic spaces over S. The following are equivalent:*

(1) *f is quasi-compact,*
(2) *for every scheme Z and any morphism $Z \to Y$ the morphism of algebraic spaces $Z \times_Y X \to Z$ is quasi-compact,*
(3) *for every affine scheme Z and any morphism $Z \to Y$ the algebraic space $Z \times_Y X$ is quasi-compact,*
(4) *there exists a scheme V and a surjective étale morphism $V \to Y$ such that $V \times_Y X \to V$ is a quasi-compact morphism of algebraic spaces, and*
(5) *there exists a surjective étale morphism $Y' \to Y$ of algebraic spaces such that $Y' \times_Y X \to Y'$ is a quasi-compact morphism of algebraic spaces, and*
(6) *there exists a Zariski covering $Y = \bigcup Y_i$ such that each of the morphisms $f^{-1}(Y_i) \to Y_i$ is quasi-compact.*

Proof. We will use Lemma 49.9.3 without further mention. It is clear that (1) implies (2) and that (2) implies (3). Assume (3). Let Z be a quasi-compact algebraic space over S, and let $Z \to Y$ be a morphism. By Properties of Spaces, Lemma 48.6.3 there exists an affine scheme U and a surjective étale morphism $U \to Z$. Then $U \times_Y X \to Z \times_Y X$ is a surjective morphism of algebraic spaces, see Lemma 49.6.5. By assumption $|U \times_Y X|$ is quasi-compact. It surjects onto $|Z \times_Y X|$, hence we conclude that $|Z \times_Y X|$ is quasi-compact, see Topology, Lemma 5.11.7. This proves that (3) implies (1).

The implications (1) $\Rightarrow$ (4), (4) $\Rightarrow$ (5) are clear. The implication (5) $\Rightarrow$ (1) follows from Lemma 49.9.6 and the fact that an étale morphism of algebraic spaces is universally open (see discussion following Definition 49.7.2).

Of course (1) implies (6) by taking the covering $Y = Y$. Assume $Y = \bigcup Y_i$ is as in (6). Let Z be affine and let $Z \to Y$ be a morphism. Then there exists a finite standard affine covering $Z = Z_1 \cup \ldots \cup Z_n$ such that each $Z_j \to Y$ factors through Y_{i_j} for some i_j. Hence the algebraic space

$$Z_j \times_Y X = Z_j \times_{Y_{i_j}} f^{-1}(Y_{i_j})$$

is quasi-compact. Since $Z \times_Y X = \bigcup_{j=1,\ldots,n} Z_j \times_Y X$ is a Zariski covering we see that $|Z \times_Y X| = \bigcup_{j=1,\ldots,n} |Z_j \times_Y X|$ (see Properties of Spaces, Lemma 48.4.8) is a finite union of quasi-compact spaces, hence quasi-compact. Thus we see that (6) implies (3). □

The following (and the next) lemma guarantees in particular that a morphism $X \to \mathrm{Spec}(A)$ is quasi-compact as soon as X is a quasi-compact algebraic space

Lemma 49.9.8. *Let S be a scheme. Let $f : X \to Y$ and $g : Y \to Z$ be morphisms of algebraic spaces over S. If $g \circ f$ is quasi-compact and g is quasi-separated then f is quasi-compact.*

Proof. This is true because f equals the composition $(1, f) : X \to X \times_Z Y \to Y$. The first map is quasi-compact by Lemma 49.5.7 because it is a section of the quasi-separated morphism $X \times_Z Y \to X$ (a base change of g, see Lemma 49.5.4). The second map is quasi-compact as it is the base change of f, see Lemma 49.9.3. And compositions of quasi-compact morphisms are quasi-compact, see Lemma 49.9.4. □

Lemma 49.9.9. *Let $f : X \to Y$ be a morphism of algebraic spaces over a scheme S.*

1. *If X is quasi-compact and Y is quasi-separated, then f is quasi-compact.*
2. *If X is quasi-compact and quasi-separated and Y is quasi-separated, then f is quasi-compact and quasi-separated.*
3. *A fibre product of quasi-compact and quasi-separated algebraic spaces is quasi-compact and quasi-separated.*

Proof. Part (1) follows from Lemma 49.9.8 with $Z = S = \mathrm{Spec}(\mathbf{Z})$. Part (2) follows from (1) and Lemma 49.5.10. For (3) let $X \to Y$ and $Z \to Y$ be morphisms of quasi-compact and quasi-separated algebraic spaces. Then $X \times_Y Z \to Z$ is quasi-compact and quasi-separated as a base change of $X \to Y$ using (2) and Lemmas 49.9.3 and 49.5.4. Hence $X \times_Y Z$ is quasi-compact and quasi-separated as

an algebraic space quasi-compact and quasi-separated over Z, see Lemmas 49.5.9 and 49.9.4. □

49.10. Universally closed morphisms

For a representable morphism of algebraic spaces we have already defined (in Section 49.3) what it means to be universally closed. Hence before we give the natural definition we check that it agrees with this in the representable case.

Lemma 49.10.1. *Let S be a scheme. Let $f : X \to Y$ be a representable morphism of algebraic spaces over S. The following are equivalent*

1. *f is universally closed, and*
2. *for every morphism of algebraic spaces $Z \to Y$ the morphism of topological spaces $|Z \times_Y X| \to |Z|$ is closed.*

Proof. Assume (1), and let $Z \to Y$ be as in (2). Choose a scheme V and a surjective étale morphism $V \to Y$. By assumption the morphism of schemes $V \times_Y X \to V$ is universally closed. By Properties of Spaces, Section 48.4 in the commutative diagram

$$\begin{array}{ccc} |V \times_Y X| & \longrightarrow & |Z \times_Y X| \\ \downarrow & & \downarrow \\ |V| & \longrightarrow & |Z| \end{array}$$

the horizontal arrows are open and surjective, and moreover

$$|V \times_Y X| \longrightarrow |V| \times_{|Z|} |Z \times_Y X|$$

is surjective. Hence as the left vertical arrow is closed it follows that the right vertical arrow is closed. This proves (2). The implication (2) $\Rightarrow$ (1) is immediate from the definitions. □

Thus we may use the following natural definition.

Definition 49.10.2. Let S be a scheme. Let $f : X \to Y$ be a morphism of algebraic spaces over S.

1. We say f is *closed* if the map of topological spaces $|X| \to |Y|$ is closed.
2. We say f is *universally closed* if for every morphism of algebraic spaces $Z \to Y$ the morphism of topological spaces
$$|Z \times_Y X| \to |Z|$$
is closed, i.e., the base change $Z \times_Y X \to Z$ is closed.

Lemma 49.10.3. *The base change of a universally closed morphism of algebraic spaces by any morphism of algebraic spaces is universally closed.*

Proof. This is immediate from the definition. □

Lemma 49.10.4. *The composition of a pair of (universally) closed morphisms of algebraic spaces is (universally) closed.*

Proof. Omitted. □

Lemma 49.10.5. *Let S be a scheme. Let $f : X \to Y$ be a morphism of algebraic spaces over S. The following are equivalent*

(1) *f is universally closed,*
(2) *for every scheme Z and every morphism $Z \to Y$ the projection $|Z \times_Y X| \to |Z|$ is closed,*
(3) *for every affine scheme Z and every morphism $Z \to Y$ the projection $|Z \times_Y X| \to |Z|$ is closed,*
(4) *there exists a scheme V and a surjective étale morphism $V \to Y$ such that $V \times_Y X \to V$ is a universally closed morphism of algebraic spaces, and*
(5) *there exists a Zariski covering $Y = \bigcup Y_i$ such that each of the morphisms $f^{-1}(Y_i) \to Y_i$ is universally closed.*

Proof. We omit the proof that (1) implies (2), and that (2) implies (3).

Assume (3). Choose a surjective étale morphism $V \to Y$. We are going to show that $V \times_Y X \to V$ is a universally closed morphism of algebraic spaces. Let $Z \to V$ be a morphism from an algebraic space to V. Let $W \to Z$ be a surjective étale morphism where $W = \coprod W_i$ is a disjoint union of affine schemes, see Properties of Spaces, Lemma 48.6.1. Then we have the following commutative diagram

$$\begin{array}{ccccccc}
\coprod_i |W_i \times_Y X| & = & |W \times_Y X| & \longrightarrow & |Z \times_Y X| & = & |Z \times_V (V \times_Y X)| \\
\downarrow & & \downarrow & & \downarrow & \swarrow & \\
\coprod |W_i| & = & |W| & \longrightarrow & |Z| & &
\end{array}$$

We have to show the south-east arrow is closed. The middle horizontal arrows are surjective and open (Properties of Spaces, Lemma 48.13.7). By assumption (3), and the fact that W_i is affine we see that the left vertical arrows are closed. Hence it follows that the right vertical arrow is closed.

Assume (4). We will show that f is universally closed. Let $Z \to Y$ be a morphism of algebraic spaces. Consider the diagram

$$\begin{array}{ccccc}
|(V \times_Y Z) \times_V (V \times_Y X)| & = & |V \times_Y X| & \longrightarrow & |Z \times_Y X| \\
& \searrow & \downarrow & & \downarrow \\
& & |V \times_Y Z| & \longrightarrow & |Z|
\end{array}$$

The south-west arrow is closed by assumption. The horizontal arrows are surjective and open because the corresponding morphisms of algebraic spaces are étale (see Properties of Spaces, Lemma 48.13.7). It follows that the right vertical arrow is closed.

Of course (1) implies (5) by taking the covering $Y = Y$. Assume $Y = \bigcup Y_i$ is as in (5). Then for any $Z \to Y$ we get a corresponding Zariski covering $Z = \bigcup Z_i$ such that the base change of f to Z_i is closed. By a simple topological argument this implies that $Z \times_Y X \to Z$ is closed. Hence (1) holds. □

Example 49.10.6. Strange example of a universally closed morphism. Let $\mathbf{Q} \subset k$ be a field of characteristic zero. Let $X = \mathbf{A}^1_k/\mathbf{Z}$ as in Spaces, Example 47.14.8. We claim the structure morphism $p : X \to \mathrm{Spec}(k)$ is universally closed. Namely, if Z/k is a scheme, and $T \subset |X \times_k Z|$ is closed, then T corresponds to a $\mathbf{Z}$-invariant closed subset of $T' \subset |\mathbf{A}^1 \times Z|$. It is easy to see that this implies that T' is the inverse image of a subset T'' of Z. By Morphisms, Lemma 28.26.10 we have that

$T'' \subset Z$ is closed. Of course T'' is the image of T. Hence p is universally closed by Lemma 49.10.5.

Lemma 49.10.7. *Let S be a scheme. A universally closed morphism of algebraic spaces over S is quasi-compact.*

Proof. This proof is a repeat of the proof in the case of schemes, see Morphisms, Lemma 28.42.9. Let $f : X \to Y$ be a morphism of algebraic spaces over S. Assume that f is not quasi-compact. Our goal is to show that f is not universally closed. By Lemma 49.9.7 there exists an affine scheme Z and a morphism $Z \to Y$ such that $Z \times_Y X \to Z$ is not quasi-compact. To achieve our goal it suffices to show that $Z \times_Y X \to Z$ is not universally closed, hence we may assume that $Y = \mathrm{Spec}(B)$ for some ring B.

Write $X = \bigcup_{i \in I} X_i$ where the X_i are quasi-compact open subspaces of X. For example, choose a surjective étale morphism $U \to X$ where U is a scheme, choose an affine open covering $U = \bigcup U_i$ and let $X_i \subset X$ be the image of U_i. We will use later that the morphisms $X_i \to Y$ are quasi-compact, see Lemma 49.9.8. Let $T = \mathrm{Spec}(B[a_i; i \in I])$. Let $T_i = D(a_i) \subset T$. Let $Z \subset T \times_Y X$ be the reduced closed subspace whose underlying closed set of points is $|T \times_Y Z| \setminus \bigcup_{i \in I} |T_i \times_Y X_i|$, see Properties of Spaces, Lemma 48.9.2. (Note that by the results of Section 49.4 the algebraic spaces $T_i \times_Y X_i$ are open subspaces of $T \times_Y X$.) Here is a diagram

$$\begin{array}{ccccc} Z & \longrightarrow & T \times_Y X & \xrightarrow[q]{} & X \\ & \searrow & \downarrow{\scriptstyle f_T} & & \downarrow{\scriptstyle f} \\ & & T & \xrightarrow{p} & Y \end{array}$$

It suffices to prove that the image $f_T(|Z|)$ is not closed in $|T|$.

We claim there exists a point $y \in Y$ such that there is no affine open neighborhood V of y in Y such that X_V is quasi-compact. If not then we can cover Y with finitely many such V and for each V the morphism $Y_V \to V$ is quasi-compact by Lemma 49.9.8 and then Lemma 49.9.7 implies f quasi-compact, a contradiction. Fix a $y \in Y$ as in the claim.

Let $t \in T$ be the point lying over y with $\kappa(t) = \kappa(y)$ such that $a_i = 1$ in $\kappa(t)$ for all i. Suppose $z \in |Z|$ with $f_T(z) = t$. Then $q(t) \in X_i$ for some i. Hence $f_T(z) \notin T_i$ by construction of Z, which contradicts the fact that $t \in T_i$ by construction. Hence we see that $t \in |T| \setminus f_T(|Z|)$.

Assume $f_T(|Z|)$ is closed in $|T|$. Then there exists an element $g \in B[a_i; i \in I]$ with $f_T(|Z|) \subset V(g)$ but $t \notin V(g)$. Hence the image of g in $\kappa(t)$ is nonzero. In particular some coefficient of g has nonzero image in $\kappa(y)$. Hence this coefficient is invertible on some affine open neighborhood V of y. Let J be the finite set of $j \in I$ such that the variable a_j appears in g. Since X_V is not quasi-compact and each $X_{i,V}$ is quasi-compact, we may choose a point $x \in |X_V| \setminus \bigcup_{j \in J} |X_{j,V}|$. In other words, $x \in |X| \setminus \bigcup_{j \in J} |X_j|$ and x lies above some $v \in V$. Since g has a coefficient that is invertible on V, we can find a point $t' \in T$ lying above v such that $t' \notin V(g)$ and $t' \in V(a_i)$ for all $i \notin J$. This is true because $V(a_i; i \in I \setminus J) = \mathrm{Spec}(B[a_j; j \in J])$ and the set of points of this scheme lying over v is bijective with $\mathrm{Spec}(\kappa(v)[a_j; j \in J])$ and g restricts to a nonzero element of this polynomial ring by construction. In other words $t' \notin T_i$ for each $i \notin J$. By Properties of Spaces, Lemma 48.4.3 we can

find a point z of $X \times_Y T$ mapping to $x \in X$ and to $t' \in T$. Since $x \notin |X_j|$ for $j \in J$ and $t' \notin T_i$ for $i \in I \setminus J$ we see that $z \in |Z|$. On the other hand $f_T(z) = t' \notin V(g)$ which contradicts $f_T(Z) \subset V(g)$. Thus the assumption "$f_T(|Z|)$ closed" is wrong and we conclude indeed that f_T is not closed as desired. □

The target of a separated algebraic space under a surjective universally closed morphism is separated.

Lemma 49.10.8. *Let S be a scheme. Let B be an algebraic space over S. Let $f : X \to Y$ be a surjective universally closed morphism of algebraic spaces over B.*

(1) *If X is quasi-separated, then Y is quasi-separated.*
(2) *If X is separated, then Y is separated.*
(3) *If X is quasi-separated over B, then Y is quasi-separated over B.*
(4) *If X is separated over B, then Y is separated over B.*

Proof. Parts (1) and (2) are a consequence of (3) and (4) for $S = B = \mathrm{Spec}(\mathbf{Z})$ (see Properties of Spaces, Definition 48.3.1). Consider the commutative diagram

$$\begin{array}{ccc} X & \xrightarrow{\Delta_{X/B}} & X \times_B X \\ \downarrow & & \downarrow \\ Y & \xrightarrow{\Delta_{Y/B}} & Y \times_B Y \end{array}$$

The left vertical arrow is surjective (i.e., universally surjective). The right vertical arrow is universally closed as a composition of the universally closed morphisms $X \times_B X \to X \times_B Y \to Y \times_B Y$. Hence it is also quasi-compact, see Lemma 49.10.7.

Assume X is quasi-separated over B, i.e., $\Delta_{X/B}$ is quasi-compact. Then if Z is quasi-compact and $Z \to Y \times_B Y$ is a morphism, then $Z \times_{Y \times_B Y} X \to Z \times_{Y \times_B Y} Y$ is surjective and $Z \times_{Y \times_B Y} X$ is quasi-compact by our remarks above. We conclude that $\Delta_{Y/B}$ is quasi-compact, i.e., Y is quasi-separated over B.

Assume X is separated over B, i.e., $\Delta_{X/B}$ is a closed immersion. Then if Z is affine, and $Z \to Y \times_B Y$ is a morphism, then $Z \times_{Y \times_B Y} X \to Z \times_{Y \times_B Y} Y$ is surjective and $Z \times_{Y \times_B Y} X \to Z$ is universally closed by our remarks above. We conclude that $\Delta_{Y/B}$ is universally closed. It follows that $\Delta_{Y/B}$ is representable, locally of finite type, a monomorphism (see Lemma 49.5.1) and universally closed, hence a closed immersion, see Étale Morphisms, Lemma 40.7.2 (and also the abstract principle Spaces, Lemma 47.5.8). Thus Y is separated over B. □

49.11. Monomorphisms

A representable morphism $X \to Y$ of algebraic spaces is a monomorphism according to Section 49.3 if for every scheme Z and morphism $Z \to Y$ the morphism $Z \times_Y X \to Z$ is representable by a monomorphism of schemes. This means exactly that $Z \times_Y X \to Z$ is an injective map of sheaves on $(Sch/S)_{fppf}$. Since this is supposed to hold for all Z and all maps $Z \to Y$ this is in turn equivalent to the map $X \to Y$ being an injective map of sheaves on $(Sch/S)_{fppf}$. Thus we may define a monomorphism of a (possibly nonrepresentable[3]) morphism of algebraic spaces as follows.

[3]We do not know whether or not every monomorphism of algebraic spaces is representable. If you do, please email stacks.project@gmail.com.

Definition 49.11.1. Let S be a scheme. A morphism of algebraic spaces over S is called a *monomorphism* if it is an injective map of sheaves, i.e., a monomorphism in the category of sheaves on $(Sch/S)_{fppf}$.

The following lemma shows that this also means that it is a monomorphism in the category of algebraic spaces over S.

Lemma 49.11.2. *Let S be a scheme. Let $j : X \to Y$ be a morphism of algebraic spaces over S. The following are equivalent:*

1. *j is a monomorphism (as in Definition 49.11.1),*
2. *j is a monomorphism in the category of algebraic spaces over S, and*
3. *the diagonal morphism $\Delta_{X/Y} : X \to X \times_Y X$ is an isomorphism.*

Proof. Note that $X \times_Y X$ is both the fibre product in the category of sheaves on $(Sch/S)_{fppf}$ and the fibre product in the category of algebraic spaces over S, see Spaces, Lemma 47.7.3. The equivalence of (1) and (3) is a general characterization of injective maps of sheaves on any site. The equivalence of (2) and (3) is a characterization of monomorphisms in any category with fibre products. □

Lemma 49.11.3. *A monomorphism of algebraic spaces is separated.*

Proof. This is true because an isomorphism is a closed immersion, and Lemma 49.11.2 above. □

Lemma 49.11.4. *A composition of monomorphisms is a monomorphism.*

Proof. True because a composition of injective sheaf maps is injective. □

Lemma 49.11.5. *The base change of a monomorphism is a monomorphism.*

Proof. This is a general fact about fibre products in a category of sheaves. □

Lemma 49.11.6. *Let S be a scheme. Let $f : X \to Y$ be a morphism of algebraic spaces over S. The following are equivalent*

1. *f is a monomorphism,*
2. *for every scheme Z and morphism $Z \to Y$ the base change $Z \times_Y X \to Z$ of f is a monomorphism,*
3. *for every affine scheme Z and every morphism $Z \to Y$ the base change $Z \times_Y X \to Z$ of f is a monomorphism,*
4. *there exists a scheme V and a surjective étale morphism $V \to Y$ such that the base change $V \times_Y X \to V$ is a monomorphism, and*
5. *there exists a Zariski covering $Y = \bigcup Y_i$ such that each of the morphisms $f^{-1}(Y_i) \to Y_i$ is a monomorphism.*

Proof. We will use without further mention that a base change of a monomorphism is a monomorphism, see Lemma 49.11.5. In particular it is clear that (1) $\Rightarrow$ (2) $\Rightarrow$ (3) $\Rightarrow$ (4) (by taking V to be a disjoint union of affine schemes étale over Y, see Properties of Spaces, Lemma 48.6.1). Let V be a scheme, and let $V \to Y$ be a surjective étale morphism. If $V \times_Y X \to V$ is a monomorphism, then it follows that $X \to Y$ is a monomorphism. Namely, given any cartesian diagram of sheaves

$$\begin{array}{ccc} \mathcal{F} & \xrightarrow{a} & \mathcal{G} \\ {\scriptstyle b}\downarrow & & \downarrow{\scriptstyle c} \\ \mathcal{H} & \xrightarrow{d} & \mathcal{I} \end{array} \qquad \mathcal{F} = \mathcal{H} \times_{\mathcal{I}} \mathcal{G}$$

if c is a surjection of sheaves, and a is injective, then also d is injective. Thus (4) implies (1). Proof of the equivalence of (5) and (1) is omitted. □

Lemma 49.11.7. *An immersion of algebraic spaces is a monomorphism. In particular, any immersion is separated.*

Proof. Let $f : X \to Y$ be an immersion of algebraic spaces. For any morphism $Z \to Y$ with Z representable the base change $Z \times_Y X \to Z$ is an immersion of schemes, hence a monomorphism, see Schemes, Lemma 25.23.7. Hence f is representable, and a monomorphism. □

We will improve on the following lemma in Decent Spaces, Lemma 50.15.1.

Lemma 49.11.8. *Let S be a scheme. Let k be a field and let $Z \to \mathrm{Spec}(k)$ be a monomorphism of algebraic spaces over S. Then either $Z = \emptyset$ or $Z = \mathrm{Spec}(k)$.*

Proof. By Lemmas 49.11.3 and 49.5.9 we see that Z is a separated algebraic space. Hence there exists an open dense subspace $Z' \subset Z$ which is a scheme, see Properties of Spaces, Proposition 48.10.3. By Schemes, Lemma 25.23.10 we see that either $Z' = \emptyset$ or $Z' \cong \mathrm{Spec}(k)$. In the first case we conclude that $Z = \emptyset$ and in the second case we conclude that $Z' = Z = \mathrm{Spec}(k)$ as $Z \to \mathrm{Spec}(k)$ is a monomorphism which is an isomorphism over Z'. □

Lemma 49.11.9. *Let S be a scheme. If $X \to Y$ is a monomorphism of algebraic spaces over S, then $|X| \to |Y|$ is injective.*

Proof. Immediate from the definitions. □

49.12. Pushforward of quasi-coherent sheaves

We first prove a simple lemma that relates pushforward of sheaves of modules for a morphism of algebraic spaces to pushforward of sheaves of modules for a morphism of schemes.

Lemma 49.12.1. *Let S be a scheme. Let $f : X \to Y$ be a morphism of algebraic spaces over S. Let $U \to X$ be a surjective étale morphism from a scheme to X. Set $R = U \times_X U$ and denote $t, s : R \to U$ the projection morphisms as usual. Denote $a : U \to Y$ and $b : R \to Y$ the induced morphisms. For any object $\mathcal{F}$ of $Mod(\mathcal{O}_X)$ there exists an exact sequence*

$$0 \to f_*\mathcal{F} \to a_*(\mathcal{F}|_U) \to b_*(\mathcal{F}|_R)$$

where the second arrow is the difference $t^* - s^*$.

Proof. We denote $\mathcal{F}$ also its extension to a sheaf of modules on $X_{spaces,étale}$, see Properties of Spaces, Remark 48.15.4. Let $V \to Y$ be an object of $Y_{étale}$. Then $V \times_Y X$ is an object of $X_{spaces,étale}$, and by definition $f_*\mathcal{F}(V) = \mathcal{F}(V \times_Y X)$. Since $U \to X$ is surjective étale, we see that $\{V \times_Y U \to V \times_Y X\}$ is a covering. Also, we have $(V \times_Y U) \times_X (V \times_Y U) = V \times_Y R$. Hence, by the sheaf condition of $\mathcal{F}$ on $X_{spaces,étale}$ we have a short exact sequence

$$0 \to \mathcal{F}(V \times_Y X) \to \mathcal{F}(V \times_Y U) \to \mathcal{F}(V \times_Y R)$$

where the second arrow is the difference of restricting via t or s. This exact sequence is functorial in V and hence we obtain the lemma. □

Let S be a scheme. Let $f : X \to Y$ be a quasi-compact and quasi-separated morphism of representable algebraic spaces X and Y over S. By Descent, Proposition 34.7.14 the functor $f_* : QCoh(\mathcal{O}_X) \to QCoh(\mathcal{O}_Y)$ agrees with the usual functor if we think of X and Y as schemes.

More generally, suppose $f : X \to Y$ is a representable, quasi-compact, and quasi-separated morphism of algebraic spaces over S. Let V be a scheme and let $V \to Y$ be an étale surjective morphism. Let $U = V \times_Y X$ and let $f' : U \to V$ be the base change of f. Then for any quasi-coherent $\mathcal{O}_X$-module $\mathcal{F}$ we have

$$f'_*(\mathcal{F}|_U) = (f_*\mathcal{F})|_V, \tag{49.12.1.1}$$

see Properties of Spaces, Lemma 48.24.2. And because $f' : U \to V$ is a quasi-compact and quasi-separated morphism of schemes, by the remark of the preceding paragraph we may compute $f'_*(\mathcal{F}|_U)$ by thinking of $\mathcal{F}|_U$ as a quasi-coherent sheaf on the scheme U, and f' as a morphism of schemes. We will frequently use this without further mention.

The next level of generality is to consider an arbitrary quasi-compact and quasi-separated morphism of algebraic spaces.

Lemma 49.12.2. *Let S be a scheme. Let $f : X \to Y$ be a morphism of algebraic spaces over S. If f is quasi-compact and quasi-separated, then f_* transforms quasi-coherent $\mathcal{O}_X$-modules into quasi-coherent $\mathcal{O}_Y$-modules.*

Proof. Let $\mathcal{F}$ be a quasi-coherent sheaf on X. We have to show that $f_*\mathcal{F}$ is a quasi-coherent sheaf on Y. For this it suffices to show that for any affine scheme V and étale morphism $V \to Y$ the restriction of $f_*\mathcal{F}$ to V is quasi-coherent, see Properties of Spaces, Lemma 48.27.6. Let $f' : V \times_Y X \to V$ be the base change of f by $V \to Y$. Note that f' is also quasi-compact and quasi-separated, see Lemmas 49.9.3 and 49.5.4. By (49.12.1.1) we know that the restriction of $f_*\mathcal{F}$ to V is f'_* of the restriction of $\mathcal{F}$ to $V \times_Y X$. Hence we may replace f by f', and assume that Y is an affine scheme.

Assume Y is an affine scheme. Since f is quasi-compact we see that X is quasi-compact. Thus we may choose an affine scheme U and a surjective étale morphism $U \to X$, see Properties of Spaces, Lemma 48.6.3. By Lemma 49.12.1 we get an exact sequence

$$0 \to f_*\mathcal{F} \to a_*(\mathcal{F}|_U) \to b_*(\mathcal{F}|_R).$$

where $R = U \times_X U$. As $X \to Y$ is quasi-separated we see that $R \to U \times_Y U$ is a quasi-compact monomorphism. This implies that R is a quasi-compact separated scheme (as U and Y are affine at this point). Hence $a : U \to Y$ and $b : R \to Y$ are quasi-compact and quasi-separated morphisms of schemes. Thus by Descent, Proposition 34.7.14 the sheaves $a_*(\mathcal{F}|_U)$ and $b_*(\mathcal{F}|_R)$ are quasi-coherent (see also the discussion preceding this lemma). This implies that $f_*\mathcal{F}$ is a kernel of quasi-coherent modules, and hence itself quasi-coherent, see Properties of Spaces, Lemma 48.27.7. □

Higher direct images are discussed in Cohomology of Spaces, Section 51.3.

49.13. Closed immersions

In this section we elucidate some of the results obtained previously on immersions of algebraic spaces. See Spaces, Section 47.12 and Section 49.4 in this chapter. This section is the analogue of Morphisms, Section 28.2 for algebraic spaces.

Lemma 49.13.1. *Let S be a scheme. Let X be an algebraic space over S. For every closed immersion $i : Z \to X$ the sheaf $i_*\mathcal{O}_Z$ is a quasi-coherent $\mathcal{O}_X$-module, the map $i^\sharp : \mathcal{O}_X \to i_*\mathcal{O}_Z$ is surjective and its kernel is a quasi-coherent sheaf of ideals. The rule $Z \mapsto Ker(\mathcal{O}_X \to i_*\mathcal{O}_Z)$ defines an inclusion reversing bijection*

$$\begin{array}{c} \text{closed subspaces} \\ Z \subset X \end{array} \longrightarrow \begin{array}{c} \text{quasi-coherent sheaves} \\ \text{of ideals } \mathcal{I} \subset \mathcal{O}_X \end{array}$$

Moreover, given a closed subscheme Z corresponding to the quasi-coherent sheaf of ideals $\mathcal{I} \subset \mathcal{O}_X$ a morphism of algebraic spaces $h : Y \to X$ factors through Z if and only if the map $h^*\mathcal{I} \to h^*\mathcal{O}_X = \mathcal{O}_Y$ is zero.

Proof. Let $U \to X$ be a surjective étale morphism whose source is a scheme. Consider the diagram

$$\begin{array}{ccc} U \times_X Z & \longrightarrow & Z \\ {\scriptstyle i'}\downarrow & & \downarrow{\scriptstyle i} \\ U & \longrightarrow & X \end{array}$$

By Lemma 49.4.1 we see that i is a closed immersion if and only if i' is a closed immersion. By Properties of Spaces, Lemma 48.24.2 we see that $i'_*\mathcal{O}_{U\times_X Z}$ is the restriction of $i_*\mathcal{O}_Z$ to U. Hence the assertions on $\mathcal{O}_X \to i_*\mathcal{O}_Z$ are equivalent to the corresponding assertions on $\mathcal{O}_U \to i'_*\mathcal{O}_{U\times_X Z}$. And since i' is a closed immersion of schemes, these results follow from Morphisms, Lemma 28.2.1.

Let us prove that given a quasi-coherent sheaf of ideals $\mathcal{I} \subset \mathcal{O}_X$ the formula

$$Z(T) = \{h : T \to X \mid h^*\mathcal{I} \to \mathcal{O}_T \text{ is zero}\}$$

defines a closed subspace of X. It is clearly a subfunctor of X. To show that $Z \to X$ is representable by closed immersions, let $\varphi : U \to X$ be a morphism from a scheme towards X. Then $Z \times_X U$ is represented by the analogous subfunctor of U corresponding to the sheaf of ideals $\mathrm{Im}(\varphi^*\mathcal{I} \to \mathcal{O}_U)$. By Properties of Spaces, Lemma 48.27.2 the $\mathcal{O}_U$-module $\varphi^*\mathcal{I}$ is quasi-coherent on on U, and hence $\mathrm{Im}(\varphi^*\mathcal{I} \to \mathcal{O}_U)$ is a quasi-coherent sheaf of ideals on U. By Schemes, Lemma 25.4.6 we conclude that $Z \times_X U$ is represented by the closed subscheme of U associated to $\mathrm{Im}(\varphi^*\mathcal{I} \to \mathcal{O}_U)$. Thus Z is a closed subspace of X.

In the formula for Z above the inputs T are schemes since algebraic spaces are sheaves on $(Sch/S)_{fppf}$. We omit the verification that the same formula remains true if T is an algebraic space. □

Definition 49.13.2. Let S be a scheme. Let $f : Y \to X$ be a morphism of algebraic spaces over S. Let $Z \subset X$ be a closed subspace. The *inverse image $f^{-1}(Z)$ of the closed subspace Z* is the closed subspace $Z \times_X Y$ of Y.

This definition makes sense by Lemma 49.4.1. If $\mathcal{I} \subset \mathcal{O}_X$ is the quasi-coherent sheaf of ideals corresponding to Z via Lemma 49.13.1 then $f^{-1}\mathcal{I}\mathcal{O}_Y = \mathrm{Im}(f^*\mathcal{I} \to \mathcal{O}_Y)$ is the sheaf of ideals corresponding to $f^{-1}(Z)$.

Lemma 49.13.3. *A closed immersion of algebraic spaces is quasi-compact.*

Proof. This follows from Schemes, Lemma 25.19.5 by general principles, see Spaces, Lemma 47.5.8. □

Lemma 49.13.4. *A closed immersion of algebraic spaces is separated.*

Proof. This follows from Schemes, Lemma 25.23.7 by general principles, see Spaces, Lemma 47.5.8. □

Lemma 49.13.5. *Let S be a scheme. Let $i : Z \to X$ be a closed immersion of algebraic spaces over S.*

(1) *The functor*
$$i_{small,*} : Sh(Z_{étale}) \longrightarrow Sh(X_{étale})$$
is fully faithful and its essential image is those sheaves of sets $\mathcal{F}$ on $X_{étale}$ whose restriction to $X \setminus Z$ is isomorphic to $*$, and

(2) *the functor*
$$i_{small,*} : Ab(Z_{étale}) \longrightarrow Ab(X_{étale})$$
is fully faithful and its essential image is those abelian sheaves on $X_{étale}$ whose support is contained in Z.

In both cases i_{small}^{-1} is a left inverse to the functor $i_{small,*}$.

Proof. Let U be a scheme and let $U \to X$ be surjective étale. Set $V = Z \times_X U$. Then V is a scheme and $i' : V \to U$ is a closed immersion of schemes. By Properties of Spaces, Lemma 48.15.11 for any sheaf $\mathcal{G}$ on Z we have
$$(i_{small}^{-1} i_{small,*}\mathcal{G})|_V = (i')_{small}^{-1} i'_{small,*}(\mathcal{G}|_V)$$
By Étale Cohomology, Proposition 44.47.4 the map $(i')_{small}^{-1} i'_{small,*}(\mathcal{G}|_V) \to \mathcal{G}|_V$ is an isomorphism. Since $V \to Z$ is surjective and étale this implies that $i_{small}^{-1} i_{small,*}\mathcal{G} \to \mathcal{G}$ is an isomorphism. This clearly implies that $i_{small,*}$ is fully faithful, see Sites, Lemma 7.40.1. To prove the statement on the essential image, consider a sheaf of sets $\mathcal{F}$ on $X_{étale}$ whose restriction to $X \setminus Z$ is isomorphic to $*$. As in the proof of Étale Cohomology, Proposition 44.47.4 we consider the adjunction mapping
$$\mathcal{F} \longrightarrow i_{small,*} i_{small}^{-1} \mathcal{F}.$$
As in the first part we see that the restriction of this map to U is an isomorphism by the corresponding result for the case of schemes. Since U is an étale covering of X we conclude it is an isomorphism. □

The following lemma holds more generally in the setting of a closed immersion of topoi (insert future reference here).

Lemma 49.13.6. *Let S be a scheme. Let $i : Z \to X$ be a closed immersion of algebraic spaces over S. Let $\mathcal{A}$ be a sheaf of rings on $X_{étale}$. Let $\mathcal{B}$ be a sheaf of rings on $Z_{étale}$. Let $\varphi : \mathcal{A} \to i_{small,*}\mathcal{B}$ (or what is the same thing $\varphi : i_{small}^{-1}\mathcal{A} \to \mathcal{B}$) be a homomorphism of sheaves of rings. Then for any sheaf of $\mathcal{A}$-modules $\mathcal{F}$ the adjunction mapping $\mathcal{F} \to i_{small,*} i_{small}^{-1} \mathcal{F}$ induces an isomorphism*
$$\mathcal{F} \otimes_{\mathcal{A}} i_{small,*}\mathcal{B} \longrightarrow i_{small,*}(i_{small}^{-1}\mathcal{F} \otimes_{i_{small}^{-1}\mathcal{A}} \mathcal{B}).$$

Proof. During this proof we drop the subscript $_{small}$ from the notation. There is a map $i^{-1}\mathcal{F} \to i^{-1}\mathcal{F} \otimes_{i^{-1}\mathcal{A}} \mathcal{B}$ to which we can apply i_* and compose with the adjunction map:

$$\mathcal{F} \longrightarrow i_*(i^{-1}\mathcal{F}) \longrightarrow i_*(i^{-1}\mathcal{F} \otimes_{i^{-1}\mathcal{A}} \mathcal{B}).$$

The composition is $\mathcal{A}$-linear where $\mathcal{A}$ acts on the target via φ. Note that this target $i_*(i^{-1}\mathcal{F} \otimes_{i^{-1}\mathcal{A}} \mathcal{B})$ has a canonical $i_*\mathcal{B}$-module structure. Hence by the universal property of tensor product we obtain a map as in the lemma.

Let $\mathcal{G}$ be a sheaf of $i_*\mathcal{B}$-modules on $X_{étale}$. Since the support of the sheaf of rings $i_*\mathcal{B}$ is contained in Z we see that the support of $\mathcal{G}$ is contained in Z. Hence by Lemma 49.13.5 we conclude that there exists a unique sheaf of $\mathcal{B}$-modules $\mathcal{H}$ and an isomorphism $i_*\mathcal{H} = \mathcal{G}$ as $i_*\mathcal{B}$-modules. To show that the map of the lemma is an isomorphism we show that the right hand side of the arrow satisfies the universal property enjoyed by the tensor product on the left (i.e., we will use Yoneda's lemma, see Categories, Lemma 4.3.5). To see this we have to show that maps into $\mathcal{G}$ agree. This can be seen using the following sequence of canonical isomorphisms

$$\begin{aligned}
\mathrm{Hom}_{i_*\mathcal{B}}(\mathcal{F} \otimes_{\mathcal{A}} i_*\mathcal{B}, \mathcal{G}) &= \mathrm{Hom}_{\mathcal{A}}(\mathcal{F}, \mathcal{G}) \\
&= \mathrm{Hom}_{\mathcal{A}}(\mathcal{F}, i_*(\mathcal{H})) \\
&= \mathrm{Hom}_{i^{-1}\mathcal{A}}(i^{-1}\mathcal{F}, \mathcal{H}) \\
&= \mathrm{Hom}_{\mathcal{B}}(i^{-1}\mathcal{F} \otimes_{i^{-1}\mathcal{A}} \mathcal{B}, \mathcal{H}) \\
&= \mathrm{Hom}_{i_*\mathcal{B}}(i_*(i^{-1}\mathcal{F} \otimes_{i^{-1}\mathcal{A}} \mathcal{B}), i_*\mathcal{H}) \\
&= \mathrm{Hom}_{i_*\mathcal{B}}(i_*(i^{-1}\mathcal{F} \otimes_{i^{-1}\mathcal{A}} \mathcal{B}), \mathcal{G})
\end{aligned}$$

The fifth equality holds because of the equivalence of categories in Lemma 49.13.5. □

49.14. Closed immersions and quasi-coherent sheaves

This section is the analogue of Morphisms, Section 28.4.

Lemma 49.14.1. *Let S be a scheme. Let $i : Z \to X$ be a closed immersion of algebraic spaces over S. Let $\mathcal{I} \subset \mathcal{O}_X$ be the quasi-coherent sheaf of ideals cutting out Z.*

1. *For any $\mathcal{O}_X$-module $\mathcal{F}$ the adjunction map $\mathcal{F} \to i_*i^*\mathcal{F}$ induces an isomorphism $\mathcal{F}/\mathcal{I}\mathcal{F} \cong i_*i^*\mathcal{F}$.*
2. *The functor i^* is a left inverse to i_*, i.e., for any $\mathcal{O}_Z$-module $\mathcal{G}$ the adjunction map $i^*i_*\mathcal{G} \to \mathcal{G}$ is an isomorphism.*
3. *The functor*
$$i_* : QCoh(\mathcal{O}_Z) \longrightarrow QCoh(\mathcal{O}_X)$$
is exact, fully faithful, with essential image those quasi-coherent $\mathcal{O}_X$-modules $\mathcal{F}$ such that $\mathcal{I}\mathcal{F} = 0$.

Proof. During this proof we work exclusively with sheaves on the small étale sites, and we use $i_*, i^{-1}, \ldots$ to denote pushforward and pullback of sheaves of abelian groups instead of $i_{small,*}, i^{-1}_{small}$.

Let $\mathcal{F}$ be an $\mathcal{O}_X$-module. By Lemma 49.13.6 we see that $i_*i^*\mathcal{F} = \mathcal{F} \otimes_{\mathcal{O}_X} \mathcal{O}_Z$. By Lemma 49.13.1 we see that we have a short exact sequence

$$0 \to \mathcal{I} \to \mathcal{O}_X \to i_*\mathcal{O}_Z \to 0$$

It follows from properties of the tensor product that $\mathcal{F} \otimes_{\mathcal{O}_X} i_*\mathcal{O}_Z = \mathcal{F}/\mathcal{I}\mathcal{F}$. This proves (1) (except that we omit the verification that the map is induced by the adjunction mapping).

Let $\mathcal{G}$ be any $\mathcal{O}_Z$-module. By Lemma 49.13.5 we see that $i^{-1}i_*\mathcal{G} = \mathcal{G}$. Hence to prove (2) we have to show that the canonical map $\mathcal{G} \otimes_{i^{-1}\mathcal{O}_X} \mathcal{O}_Z \to \mathcal{G}$ is an isomorphism. This follows from general properties of tensor products if we can show that $i^{-1}\mathcal{O}_X \to \mathcal{O}_Z$ is surjective. By Lemma 49.13.5 it suffices to prove that $i_*i^{-1}\mathcal{O}_X \to i_*\mathcal{O}_Z$ is surjective. Since the surjective map $\mathcal{O}_X \to i_*\mathcal{O}_Z$ factors through this map we see that (2) holds.

Finally we prove the most interesting part of the lemma, namely part (3). A closed immersion is quasi-compact and separated, see Lemmas 49.13.3 and 49.13.4. Hence Lemma 49.12.2 applies and the pushforward of a quasi-coherent sheaf on Z is indeed a quasi-coherent sheaf on X. Thus we obtain our functor $i_*^{QCoh} : QCoh(\mathcal{O}_Z) \to QCoh(\mathcal{O}_X)$. It is clear from part (2) that i_*^{QCoh} is fully faithful since it has a left inverse, namely i^*.

Now we turn to the description of the essential image of the functor i_*. It is clear that $\mathcal{I}(i_*\mathcal{G}) = 0$ for any $\mathcal{O}_Z$-module, since $\mathcal{I}$ is the kernel of the map $\mathcal{O}_X \to i_*\mathcal{O}_Z$ which is the map we use to put an $\mathcal{O}_X$-module structure on $i_*\mathcal{G}$. Next, suppose that $\mathcal{F}$ is any quasi-coherent $\mathcal{O}_X$-module such that $\mathcal{I}\mathcal{F} = 0$. Then we see that $\mathcal{F}$ is an $i_*\mathcal{O}_Z$-module because $i_*\mathcal{O}_Z = \mathcal{O}_X/\mathcal{I}$. Hence in particular its support is contained in Z. We apply Lemma 49.13.5 to see that $\mathcal{F} \cong i_*\mathcal{G}$ for some $\mathcal{O}_Z$-module $\mathcal{G}$. The only small detail left over is to see why $\mathcal{G}$ is quasi-coherent. This is true because $\mathcal{G} \cong i^*\mathcal{F}$ by part (2) and Properties of Spaces, Lemma 48.27.2. □

Let $i : Z \to X$ be a closed immersion of algebraic spaces. Because of the lemma above we often, by abuse of notation, denote $\mathcal{F}$ the sheaf $i_*\mathcal{F}$ on X.

Lemma 49.14.2. *Let S be a scheme. Let X be an algebraic space over S. Let $\mathcal{F}$ be a quasi-coherent $\mathcal{O}_X$-module. Let $\mathcal{G} \subset \mathcal{F}$ be a $\mathcal{O}_X$-submodule. There exists a unique quasi-coherent $\mathcal{O}_X$-submodule $\mathcal{G}' \subset \mathcal{G}$ with the following property: For every quasi-coherent $\mathcal{O}_X$-module $\mathcal{H}$ the map*

$$\mathrm{Hom}_{\mathcal{O}_X}(\mathcal{H}, \mathcal{G}') \longrightarrow \mathrm{Hom}_{\mathcal{O}_X}(\mathcal{H}, \mathcal{G})$$

is bijective. In particular $\mathcal{G}'$ is the largest quasi-coherent $\mathcal{O}_X$-submodule of $\mathcal{F}$ contained in $\mathcal{G}$.

Proof. Let $\mathcal{G}_a$, $a \in A$ be the set of quasi-coherent $\mathcal{O}_X$-submodules contained in $\mathcal{G}$. Then the image $\mathcal{G}'$ of

$$\bigoplus\nolimits_{a \in A} \mathcal{G}_a \longrightarrow \mathcal{F}$$

is quasi-coherent as the image of a map of quasi-coherent sheaves on X is quasi-coherent and since a direct sum of quasi-coherent sheaves is quasi-coherent, see Properties of Spaces, Lemma 48.27.7. The module $\mathcal{G}'$ is contained in $\mathcal{G}$. Hence this is the largest quasi-coherent $\mathcal{O}_X$-module contained in $\mathcal{G}$.

To prove the formula, let $\mathcal{H}$ be a quasi-coherent $\mathcal{O}_X$-module and let $\alpha : \mathcal{H} \to \mathcal{G}$ be an $\mathcal{O}_X$-module map. The image of the composition $\mathcal{H} \to \mathcal{G} \to \mathcal{F}$ is quasi-coherent as the image of a map of quasi-coherent sheaves. Hence it is contained in $\mathcal{G}'$. Hence α factors through $\mathcal{G}'$ as desired. □

Lemma 49.14.3. *Let S be a scheme. Let $i : Z \to X$ be a closed immersion of algebraic spaces over S. There is a functor[4] $i^! : QCoh(\mathcal{O}_X) \to QCoh(\mathcal{O}_Z)$ which is a right adjoint to i_*. (Compare Modules, Lemma 17.6.3.)*

Proof. Given quasi-coherent $\mathcal{O}_X$-module $\mathcal{G}$ we consider the subsheaf $\mathcal{H}_Z(\mathcal{G})$ of $\mathcal{G}$ of local sections annihilated by $\mathcal{I}$. By Lemma 49.14.2 there is a canonical largest quasi-coherent $\mathcal{O}_X$-submodule $\mathcal{H}_Z(\mathcal{G})'$. By construction we have

$$\mathrm{Hom}_{\mathcal{O}_X}(i_*\mathcal{F}, \mathcal{H}_Z(\mathcal{G})') = \mathrm{Hom}_{\mathcal{O}_X}(i_*\mathcal{F}, \mathcal{G})$$

for any quasi-coherent $\mathcal{O}_Z$-module $\mathcal{F}$. Hence we can set $i^!\mathcal{G} = i^*(\mathcal{H}_Z(\mathcal{G})')$. Details omitted. □

49.15. Supports of modules

In this section we collect some elementary results on supports of quasi-coherent modules on algebraic spaces. Let X be an algebraic space. The support of an abelian sheaf on $X_{étale}$ has been defined in Properties of Spaces, Section 48.17. We use the same definition for supports of modules. The following lemma tells us this agrees with the notion as defined for quasi-coherent modules on schemes.

Lemma 49.15.1. *Let S be a scheme. Let X be an algebraic space over S. Let $\mathcal{F}$ be a quasi-coherent $\mathcal{O}_X$-module. Let U be a scheme and let $\varphi : U \to X$ be an étale morphism. Then*

$$Supp(\varphi^*\mathcal{F}) = |\varphi|^{-1}(Supp(\mathcal{F}))$$

where the left hand side is the support of $\varphi^*\mathcal{F}$ as a quasi-coherent module on the scheme U.

Proof. Let $u \in U$ be a (usual) point and let $\overline{x}$ be a geometric point lying over u. By Properties of Spaces, Lemma 48.27.4 we have $(\varphi^*\mathcal{F})_u \otimes_{\mathcal{O}_{U,u}} \mathcal{O}_{X,\overline{x}} = \mathcal{F}_{\overline{x}}$. Since $\mathcal{O}_{U,u} \to \mathcal{O}_{X,\overline{x}}$ is the strict henselization by Properties of Spaces, Lemma 48.19.1 we see that it is faithfully flat (see More on Algebra, Lemma 15.32.1). Thus we see that $(\varphi^*\mathcal{F})_u = 0$ if and only if $\mathcal{F}_{\overline{x}} = 0$. This proves the lemma. □

For finite type quasi-coherent modules the support is closed, can be checked on fibres, and commutes with base change.

Lemma 49.15.2. *Let S be a scheme. Let X be an algebraic space over S. Let $\mathcal{F}$ be a finite type quasi-coherent $\mathcal{O}_X$-module. Then*

1. *The support of $\mathcal{F}$ is closed.*
2. *For a geometric point $\overline{x}$ lying over $x \in |X|$ we have*
$$x \in Supp(\mathcal{F}) \Leftrightarrow \mathcal{F}_{\overline{x}} \neq 0 \Leftrightarrow \mathcal{F}_{\overline{x}} \otimes_{\mathcal{O}_{X,\overline{x}}} \kappa(\overline{x}) \neq 0.$$
3. *For any morphism of algebraic spaces $f : Y \to X$ the pullback $f^*\mathcal{F}$ is of finite type as well and we have $Supp(f^*\mathcal{F}) = f^{-1}(Supp(\mathcal{F}))$.*

Proof. Choose a scheme U and a surjective étale morphism $\varphi : U \to X$. By Lemma 49.15.1 the inverse image of the support of $\mathcal{F}$ is the support of $\varphi^*\mathcal{F}$ which is closed by Morphisms, Lemma 28.5.3. Thus (1) follows from the definition of the topology on $|X|$.

The first equivalence in (2) is the definition of support. The second equivalence follows from Nakayama's lemma, see Algebra, Lemma 10.18.1.

[4] This is likely nonstandard notation.

Let $f : Y \to X$ be as in (3). Note that $f^*\mathcal{F}$ is of finite type by Properties of Spaces, Section 48.28. For the final assertion, let $\overline{y}$ be a geometric point of Y mapping to the geometric point $\overline{x}$ on X. Recall that

$$(f^*\mathcal{F})_{\overline{y}} = \mathcal{F}_{\overline{x}} \otimes_{\mathcal{O}_{X,\overline{x}}} \mathcal{O}_{Y,\overline{y}},$$

see Properties of Spaces, Lemma 48.27.5. Hence $(f^*\mathcal{F})_{\overline{y}} \otimes \kappa(\overline{y})$ is nonzero if and only if $\mathcal{F}_{\overline{x}} \otimes \kappa(\overline{x})$ is nonzero. By (2) this implies $x \in \mathrm{Supp}(\mathcal{F})$ if and only if $y \in \mathrm{Supp}(f^*\mathcal{F})$, which is the content of assertion (3). □

Our next task is to show that the scheme theoretic support of a finite type quasi-coherent module (see Morphisms, Definition 28.5.5) also makes sense for finite type quasi-coherent modules on algebraic spaces.

Lemma 49.15.3. *Let S be a scheme. Let X be an algebraic space over S. Let $\mathcal{F}$ be a finite type quasi-coherent $\mathcal{O}_X$-module. There exists a smallest closed subspace $i : Z \to X$ such that there exists a quasi-coherent $\mathcal{O}_Z$-module $\mathcal{G}$ with $i_*\mathcal{G} \cong \mathcal{F}$. Moreover:*

1. *If U is a scheme and $\varphi : U \to X$ is an étale morphism then $Z \times_X U$ is the scheme theoretic support of $\varphi^*\mathcal{F}$.*
2. *The quasi-coherent sheaf $\mathcal{G}$ is unique up to unique isomorphism.*
3. *The quasi-coherent sheaf $\mathcal{G}$ is of finite type.*
4. *The support of $\mathcal{G}$ and of $\mathcal{F}$ is Z.*

Proof. Choose a scheme U and a surjective étale morphism $\varphi : U \to X$. Let $R = U \times_X U$ with projections $s, t : R \to U$. Let $i' : Z' \to U$ be the scheme theoretic support of $\varphi^*\mathcal{F}$ and let $\mathcal{G}'$ be the (unique up to unique isomorphism) finite type quasi-coherent $\mathcal{O}_{Z'}$-module with $i'_*\mathcal{G}' = \varphi^*\mathcal{F}$, see Morphisms, Lemma 28.5.4. As $s^*\varphi^*\mathcal{F} = t^*\varphi^*\mathcal{F}$ we see that $R' = s^{-1}Z' = t^{-1}Z'$ as closed subschemes of R by Morphisms, Lemma 28.26.12. Thus we may apply Properties of Spaces, Lemma 48.9.1 to find a closed subspace $i : Z \to X$ whose pullback to U is Z'. Writing $s', t' : R' \to Z'$ the projections and $j' : R' \to R$ the given closed immersion, we see that

$$j'_*(s')^*\mathcal{G}' = s^*i'_*\mathcal{G}' = s^*\varphi^*\mathcal{F} = t^*\varphi^*\mathcal{F} = t^*i'_*\mathcal{G}' = j'_*(t')^*\mathcal{G}'$$

(the first and the last equality by Cohomology of Schemes, Lemma 29.5.2). Hence the uniqueness of Morphisms, Lemma 28.26.12 applied to $R' \to R$ gives an isomorphism $\alpha : (t')^*\mathcal{G}' \to (s')^*\mathcal{G}'$ compatible with the canonical isomorphism $t^*\varphi^*\mathcal{F} = s^*\varphi^*\mathcal{F}$ via j'_*. Clearly α satisfies the cocycle condition, hence we may apply Properties of Spaces, Proposition 48.30.1 to obtain a quasi-coherent module $\mathcal{G}$ on Z whose restriction to Z' is $\mathcal{G}'$ compatible with α. Again using the equivalence of the proposition mentioned above (this time for X) we conclude that $i_*\mathcal{G} \cong \mathcal{F}$.

This proves existence. The other properties of the lemma follow by comparing with the result for schemes using Lemma 49.15.1. Detailed proofs omitted. □

Definition 49.15.4. Let S be a scheme. Let X be an algebraic space over S. Let $\mathcal{F}$ be a finite type quasi-coherent $\mathcal{O}_X$-module. The *scheme theoretic support of $\mathcal{F}$* is the closed subspace $Z \subset X$ constructed in Lemma 49.15.3.

In this situation we often think of $\mathcal{F}$ as a quasi-coherent sheaf of finite type on Z (via the equivalence of categories of Lemma 49.14.1).

49.16. Scheme theoretic image

Caution: Some of the material in this section is ultra-general and behaves differently from what you might expect.

Lemma 49.16.1. *Let S be a scheme. Let $f : X \to Y$ be a morphism of algebraic spaces over S. There exists a closed subspace $Z \subset Y$ such that f factors through Z and such that for any other closed subspace $Z' \subset Y$ such that f factors through Z' we have $Z \subset Z'$.*

Proof. Let $\mathcal{I} = \mathrm{Ker}(\mathcal{O}_Y \to f_*\mathcal{O}_X)$. If $\mathcal{I}$ is quasi-coherent then we just take Z to be the closed subscheme determined by $\mathcal{I}$, see Lemma 49.13.1. In general the lemma requires us to show that there exists a largest quasi-coherent sheaf of ideals $\mathcal{I}'$ contained in $\mathcal{I}$. This follows from Lemma 49.14.2. □

Suppose that in the situation of Lemma 49.16.1 above X and Y are representable. Then the closed subspace $Z \subset Y$ found in the lemma agrees with the closed subscheme $Z \subset Y$ found in Morphisms, Lemma 28.6.1. The reason is that closed subspaces (or subschemes) are in a inclusion reversing correspondence with quasi-coherent ideal sheaves on $X_{étale}$ and X. As the category of quasi-coherent modules on $X_{étale}$ and X are the same (Properties of Spaces, Section 48.27) we conclude. Thus the following definition agrees with the earlier definition for morphisms of schemes.

Definition 49.16.2. Let S be a scheme. Let $f : X \to Y$ be a morphism of algebraic spaces over S. The *scheme theoretic image* of f is the smallest closed subspace $Z \subset Y$ through which f factors, see Lemma 49.16.1 above.

We often just denote $f : X \to Z$ the factorization of f. If the morphism f is not quasi-compact, then (in general) the construction of the scheme theoretic image does not commute with restriction to open subspaces of Y.

Lemma 49.16.3. *Let S be a scheme. Let $f : X \to Y$ be a morphism of algebraic spaces over S. Let $Z \subset Y$ be the scheme theoretic image of f. If f is quasi-compact then*

1. *the sheaf of ideals $\mathcal{I} = Ker(\mathcal{O}_Y \to f_*\mathcal{O}_X)$ is quasi-coherent,*
2. *the scheme theoretic image Z is the closed subspace corresponding to $\mathcal{I}$,*
3. *for any étale morphism $V \to Y$ the scheme theoretic image of $X \times_Y V \to V$ is equal to $Z \times_Y V$, and*
4. *the image $|f|(|X|) \subset |Z|$ is a dense subset of $|Z|$.*

Proof. Part (4) follows from part (3) and the construction of the topology on $|Z| \subset |X|$. To show (3) it suffices to prove (1) since the formation of $\mathcal{I}$ commutes with étale localization. If (1) holds then in the proof of Lemma 49.16.1 we showed (2). Thus it suffices to prove that $\mathcal{I}$ is quasi-coherent. Since the property of being quasi-coherent is étale local we may assume Y is an affine scheme. As f is quasi-compact, we can find an affine scheme U and a surjective étale morphism $U \to X$. Denote f' the composition $U \to X \to Y$. Then $f_*\mathcal{O}_X$ is a subsheaf of $f'_*\mathcal{O}_U$, and hence $\mathcal{I} = \mathrm{Ker}(\mathcal{O}_Y \to \mathcal{O}_{X'})$. By Lemma 49.12.2 the sheaf $f'_*\mathcal{O}_{X'}$ is quasi-coherent on Y. Hence we win. □

Lemma 49.16.4. *Let S be a scheme. Let $f : X \to Y$ be a morphism of algebraic spaces over S. Assume X is reduced. Then*

(1) *the scheme theoretic image Z of f is the reduced induced algebraic space structure on $\overline{|f|(|X|)}$, and*
(2) *for any étale morphism $V \to Y$ the scheme theoretic image of $X \times_Y V \to V$ is equal to $Z \times_Y V$.*

Proof. Part (1) is true because the reduced induced algebraic space structure on $\overline{|f|(|X|)}$ is the smallest closed subspace of Y through which f factors, see Properties of Spaces, Lemma 48.9.3. Part (2) follows from (1), the fact that $|V| \to |Y|$ is open, and the fact that being reduced is preserved under étale localization. □

Lemma 49.16.5. *Let S be a scheme. Let $f : X \to Y$ be a quasi-compact morphism of algebraic spaces over S. Let Z be the scheme theoretic image of f. Let $z \in |Z|$. There exists a valuation ring A with fraction field K and a commutative diagram*

$$\begin{array}{ccccc} \mathrm{Spec}(K) & \longrightarrow & & & X \\ \downarrow & & & \swarrow & \downarrow \\ \mathrm{Spec}(A) & \longrightarrow & Z & \longrightarrow & Y \end{array}$$

such that the closed point of $\mathrm{Spec}(A)$ *maps to z.*

Proof. Choose a scheme V and a surjective étale morphism $V \to Y$. Let $Z' \subset V$ be the scheme theoretic image of $X \times_Y V \to V$. By Lemma 49.16.3 $Z' = Z \times_Y V$. Thus we can choose a point $z' \in Z'$ mapping to z. By Morphisms, Lemma 28.6.5 we can choose a diagram

$$\begin{array}{ccccc} \mathrm{Spec}(K) & \longrightarrow & & & X \times_Y V \\ \downarrow & & & \swarrow & \downarrow \\ \mathrm{Spec}(A) & \longrightarrow & Z' & \longrightarrow & V \end{array}$$

such that the closed point of $\mathrm{Spec}(A)$ maps to z'. Composing with the projections $Z' \to Z$ and $X \times_Y V \to X$ we obtain a solution. □

49.17. Scheme theoretic closure and density

This section is the analogue of Morphisms, Section 28.7.

Lemma 49.17.1. *Let S be a scheme. Let $W \subset S$ be a scheme theoretically dense open subscheme (Morphisms, Definition 28.7.1). Let $f : X \to S$ be a morphism of schemes which is flat, locally of finite presentation, and locally quasi-finite. Then $f^{-1}(W)$ is scheme theoretically dense in X.*

Proof. We will use the characterization of Morphisms, Lemma 28.7.5. Assume $V \subset X$ is an open and $g \in \Gamma(V, \mathcal{O}_V)$ is a function which restricts to zero on $f^{-1}(W) \cap V$. We have to show that $g = 0$. Assume $g \neq 0$ to get a contradiction. By More on Morphisms, Lemma 36.31.11 we may shrink V, find an open $U \subset S$ fitting into a commutative diagram

$$\begin{array}{ccc} V & \longrightarrow & X \\ {\scriptstyle \pi}\downarrow & & \downarrow{\scriptstyle f} \\ U & \longrightarrow & S, \end{array}$$

a quasi-coherent subsheaf $\mathcal{F} \subset \mathcal{O}_U$, an integer $r > 0$, and an injective $\mathcal{O}_U$-module map $\mathcal{F}^{\oplus r} \to \pi_*\mathcal{O}_V$ whose image contains $g|_V$. Say $(g_1, \ldots, g_r) \in \Gamma(U, \mathcal{F}^{\oplus r})$ maps to g. Then we see that $g_i|_{W \cap U} = 0$ because $g|_{f^{-1}W \cap V} = 0$. Hence $g_i = 0$ because $\mathcal{F} \subset \mathcal{O}_U$ and W is scheme theoretically dense in S. This implies $g = 0$ which is the desired contradiction. □

Lemma 49.17.2. *Let S be a scheme. Let X be an algebraic space over S. Let $U \subset X$ be an open subspace. The following are equivalent*

(1) *for every étale morphism $\varphi : V \to X$ (of algebraic spaces) the scheme theoretic closure of $\varphi^{-1}(U)$ in V is equal to V,*
(2) *there exists a scheme V and a surjective étale morphism $\varphi : V \to X$ such that the scheme theoretic closure of $\varphi^{-1}(U)$ in V is equal to V,*

Proof. Observe that if $V \to V'$ is a morphism of algebraic spaces étale over X, and $Z \subset V$, resp. $Z' \subset V'$ is the scheme theoretic closure of $U \times_X V$, resp. $U \times_X V'$ in V, resp. V', then Z maps into Z'. Thus if $V \to V'$ is surjective and étale then $Z = V$ implies $Z' = V'$. Next, note that an étale morphism is flat, locally of finite presentation, and locally quasi-finite (see Morphisms, Section 28.37). Thus Lemma 49.17.1 implies that if V and V' are schemes, then $Z' = V'$ implies $Z = V$. A formal argument using that every algebraic space has an étale covering by a scheme shows that (1) and (2) are equivalent. □

It follows from Lemma 49.17.2 that the following definition is compatible with the definition in the case of schemes.

Definition 49.17.3. Let S be a scheme. Let X be an algebraic space over S. Let $U \subset X$ be an open subspace.

(1) The scheme theoretic image of the morphism $U \to X$ is called the *scheme theoretic closure of U in X*.
(2) We say U is *scheme theoretically dense in X* if the equivalent conditions of Lemma 49.17.2 are satisfied.

With this definition it is **not** the case that U is scheme theoretically dense in X if and only if the scheme theoretic closure of U is X. This is somewhat inelegant. But with suitable finiteness conditions we will see that it does hold.

Lemma 49.17.4. *Let S be a scheme. Let X be an algebraic space over S. Let $U \subset X$ be an open subspace. If $U \to X$ is quasi-compact, then U is scheme theoretically dense in X if and only if the scheme theoretic closure of U in X is X.*

Proof. Follows from Lemma 49.16.3 part (3). □

Lemma 49.17.5. *Let S be a scheme. Let $j : U \to X$ be an open immersion of algebraic spaces over S. Then U is scheme theoretically dense in X if and only if $\mathcal{O}_X \to j_*\mathcal{O}_U$ is injective.*

Proof. If $\mathcal{O}_X \to j_*\mathcal{O}_U$ is injective, then the same is true when restricted to any algebraic space V étale over X. Hence the scheme theoretic closure of $U \times_X V$ in V is equal to V, see proof of Lemma 49.16.1. Conversely, assume the scheme theoretic closure of $U \times_X V$ is equal to V for all V étale over X. Suppose that $\mathcal{O}_X \to j_*\mathcal{O}_U$ is not injective. Then we can find an affine, say $V = \mathrm{Spec}(A)$, étale over X and a nonzero element $f \in A$ such that f maps to zero in $\Gamma(V \times_X U, \mathcal{O})$. In this case

the scheme theoretic closure of $V \times_X U$ in V is clearly contained in $\mathrm{Spec}(A/(f))$ a contradiction. □

Lemma 49.17.6. *Let S be a scheme. Let X be an algebraic space over S. If U, V are scheme theoretically dense open subspaces of X, then so is $U \cap V$.*

Proof. Let $W \to X$ be any étale morphism. Consider the map $\mathcal{O}(W) \to \mathcal{O}(W \times_X V) \to \mathcal{O}(W \times_X (V \cap U))$. By Lemma 49.17.5 both maps are injective. Hence the composite is injective. Hence by Lemma 49.17.5 $U \cap V$ is scheme theoretically dense in X. □

Lemma 49.17.7. *Let S be a scheme. Let $h : Z \to X$ be an immersion of algebraic spaces over S. Assume either $Z \to X$ is quasi-compact or Z is reduced. Let $\overline{Z} \subset X$ be the scheme theoretic image of h. Then the morphism $Z \to \overline{Z}$ is an open immersion which identifies Z with a scheme theoretically dense open subspace of $\overline{Z}$. Moreover, Z is topologically dense in $\overline{Z}$.*

Proof. In both cases the formation of Z commutes with étale localization, see Lemmas 49.16.3 and 49.16.4. Hence this lemma follows from the case of schemes, see Morphisms, Lemma 28.7.7. □

Lemma 49.17.8. *Let S be a scheme. Let B be an algebraic space over S. Let $f, g : X \to Y$ be morphisms of algebraic spaces over B. Let $U \subset X$ be an open subspace such that $f|_U = g|_U$. If the scheme theoretic closure of U in X is X and $Y \to B$ is separated, then $f = g$.*

Proof. As $Y \to B$ is separated the fibre product $Y \times_{\Delta, Y \times_B Y, (f,g)} X$ is a closed subspace $Z \subset X$. As $f|_U = g|_U$ we see that $U \subset Z$. Hence $Z = X$ as U is assumed scheme theoretically dense in X. □

49.18. Universally injective morphisms

We have already defined in Section 49.3 what it means for a representable morphism of algebraic spaces to be universally injective. For a field K over S (recall this means that we are given a structure morphism $\mathrm{Spec}(K) \to S$) and an algebraic space X over S we write $X(K) = \mathrm{Mor}_S(\mathrm{Spec}(K), X)$. We first translate the condition for representable morphisms into a condition on the functor of points.

Lemma 49.18.1. *Let S be a scheme. Let $f : X \to Y$ be a representable morphism of algebraic spaces over S. Then f is universally injective if and only if for all fields K the map $X(K) \to Y(K)$ is injective.*

Proof. We are going to use Morphisms, Lemma 28.12.2 without further mention. Suppose that f is universally injective. Then for any field K and any morphism $\mathrm{Spec}(K) \to Y$ the morphism of schemes $\mathrm{Spec}(K) \times_Y X \to \mathrm{Spec}(K)$ is universally injective. Hence there exists at most one section of the morphism $\mathrm{Spec}(K) \times_Y X \to \mathrm{Spec}(K)$. Hence the map $X(K) \to Y(K)$ is injective. Conversely, suppose that for every field K the map $X(K) \to Y(K)$ is injective. Let $T \to Y$ be a morphism from a scheme into Y, and consider the base change $f_T : T \times_Y X \to T$. For any field K we have

$$(T \times_Y X)(K) = T(K) \times_{Y(K)} X(K)$$

by definition of the fibre product, and hence the injectivity of $X(K) \to Y(K)$ guarantees the injectivity of $(T \times_Y X)(K) \to T(K)$ which means that f_T is universally injective as desired. □

Next, we translate the property that the transformation between field valued points is injective into something more geometric.

Lemma 49.18.2. *Let S be a scheme. Let $f : X \to Y$ be a morphism of algebraic spaces over S. The following are equivalent:*

1. *the map $X(K) \to Y(K)$ is injective for every field K over S*
2. *for every morphism $Y' \to Y$ of algebraic spaces over S the induced map $|Y' \times_Y X| \to |Y'|$ is injective, and*
3. *the diagonal morphism $X \to X \times_Y X$ is surjective.*

Proof. Assume (1). Let $g : Y' \to Y$ be a morphism of algebraic spaces, and denote $f' : Y' \times_Y X \to Y'$ the base change of f. Let K_i, $i = 1, 2$ be fields and let $\varphi_i : \mathrm{Spec}(K_i) \to Y' \times_Y X$ be morphisms such that $f' \circ \varphi_1$ and $f' \circ \varphi_2$ define the same element of $|Y'|$. By definition this means there exists a field Ω and embeddings $\alpha_i : K_i \subset \Omega$ such that the two morphisms $f' \circ \varphi_i \circ \alpha_i : \mathrm{Spec}(\Omega) \to Y'$ are equal. Here is the corresponding commutative diagram

$$\begin{array}{ccccccc}
\mathrm{Spec}(\Omega) & \xrightarrow[\alpha_2]{} & \mathrm{Spec}(K_2) & & & & \\
& \searrow^{\alpha_1} & & \searrow^{\varphi_2} & & & \\
& & \mathrm{Spec}(K_1) & \xrightarrow{\varphi_1} & Y' \times_Y X & \xrightarrow{g'} & X \\
& & & & \downarrow{\scriptstyle f'} & & \downarrow{\scriptstyle f} \\
& & & & Y' & \xrightarrow{g} & Y.
\end{array}$$

In particular the compositions $g \circ f' \circ \varphi_i \circ \alpha_i$ are equal. By assumption (1) this implies that the morphism $g' \circ \varphi_i \circ \alpha_i$ are equal, where $g' : Y' \times_Y X \to X$ is the projection. By the universal property of the fibre product we conclude that the morphisms $\varphi_i \circ \alpha_i : \mathrm{Spec}(\Omega) \to Y' \times_Y X$ are equal. In other words φ_1 and φ_2 define the same point of $Y' \times_Y X$. We conclude that (2) holds.

Assume (2). Let K be a field over S, and let $a, b : \mathrm{Spec}(K) \to X$ be two morphisms such that $f \circ a = f \circ b$. Denote $c : \mathrm{Spec}(K) \to Y$ the common value. By assumption $|\mathrm{Spec}(K) \times_{c,Y} X| \to |\mathrm{Spec}(K)|$ is injective. This means there exists a field Ω and embeddings $\alpha_i : K \to \Omega$ such that

$$\begin{array}{ccc}
\mathrm{Spec}(\Omega) & \xrightarrow[\alpha_1]{} & \mathrm{Spec}(K) \\
{\scriptstyle \alpha_2}\downarrow & & \downarrow{\scriptstyle a} \\
\mathrm{Spec}(K) & \xrightarrow{b} & \mathrm{Spec}(K) \times_{c,Y} X
\end{array}$$

is commutative. Composing with the projection to $\mathrm{Spec}(K)$ we see that $\alpha_1 = \alpha_2$. Denote the common value α. Then we see that $\{\alpha : \mathrm{Spec}(\Omega) \to \mathrm{Spec}(K)\}$ is a fpqc covering of $\mathrm{Spec}(K)$ such that the two morphisms a, b become equal on the members of the covering. By Properties of Spaces, Lemma 48.14.1 we conclude that $a = b$. We conclude that (1) holds.

Assume (3). Let $x, x' \in |X|$ be a pair of points such that $f(x) = f(x')$ in $|Y|$. By Properties of Spaces, Lemma 48.4.3 we see there exists a $x'' \in |X \times_Y X|$ whose projections are x and x'. By assumption and Properties of Spaces, Lemma 48.4.4 there exists a $x''' \in |X|$ with $\Delta_{X/Y}(x''') = x''$. Thus $x = x'$. In other words f is injective. Since condition (3) is stable under base change we see that f satisfies (2).

Assume (2). Then in particular $|X \times_Y X| \to |X|$ is injective which implies immediately that $|\Delta_{X/Y}| : |X| \to |X \times_Y X|$ is surjective, which implies that $\Delta_{X/Y}$ is surjective by Properties of Spaces, Lemma 48.4.4. □

By the two lemmas above the following definition does not conflict with the already defined notion of a universally injective representable morphism of algebraic spaces.

Definition 49.18.3. Let S be a scheme. Let $f : X \to Y$ be a morphism of algebraic spaces over S. We say f is *universally injective* if for every morphism $Y' \to Y$ the induced map $|Y' \times_Y X| \to |Y'|$ is injective.

To be sure this means that any or all of the equivalent conditions of Lemma 49.18.2 hold.

Remark 49.18.4. A universally injective morphism of schemes is separated, see Morphisms, Lemma 28.12.3. This is not the case for morphisms of algebraic spaces. Namely, the algebraic space $X = \mathbf{A}^1_k/\{x \sim -x \mid x \neq 0\}$ constructed in Spaces, Example 47.14.1 comes equipped with a morphism $X \to \mathbf{A}^1_k$ which maps the point with coordinate x to the point with coordinate x^2. This is an isomorphism away from 0, and there is a unique point of X lying above 0. As X isn't separated this is a universally injective morphism of algebraic spaces which is not separated.

Lemma 49.18.5. *The base change of a universally injective morphism is universally injective.*

Proof. Omitted. Hint: This is formal. □

Lemma 49.18.6. *Let S be a scheme. Let $f : X \to Y$ be a morphism of algebraic spaces over S. The following are equivalent:*

1. *f is universally injective,*
2. *for every scheme Z and any morphism $Z \to Y$ the morphism $Z \times_Y X \to Z$ is universally injective,*
3. *for every affine scheme Z and any morphism $Z \to Y$ the morphism $Z \times_Y X \to Z$ is universally injective,*
4. *there exists a scheme Z and a surjective morphism $Z \to Y$ such that $Z \times_Y X \to Z$ is universally injective, and*
5. *there exists a Zariski covering $Y = \bigcup Y_i$ such that each of the morphisms $f^{-1}(Y_i) \to Y_i$ is universally injective.*

Proof. We will use that being universally injective is preserved under base change (Lemma 49.18.5) without further mention in this proof. It is clear that (1) $\Rightarrow$ (2) $\Rightarrow$ (3) $\Rightarrow$ (4).

Assume $g : Z \to Y$ as in (4). Let $y : \mathrm{Spec}(K) \to Y$ be a morphism from the spectrum of a field into Y. By assumption we can find an extension field $\alpha : K \subset K'$ and a morphism $z : \mathrm{Spec}(K') \to Z$ such that $y \circ \alpha = g \circ z$ (with obvious abuse of notation). By assumption the morphism $Z \times_Y X \to Z$ is universally injective, hence there is at most one lift of $g \circ z : \mathrm{Spec}(K') \to Y$ to a morphism into X. Since $\{\alpha : \mathrm{Spec}(K') \to \mathrm{Spec}(K)\}$ is a fpqc covering this implies there is at most one lift of $y : \mathrm{Spec}(K) \to Y$ to a morphism into X, see Properties of Spaces, Lemma 48.14.1. Thus we see that (1) holds.

We omit the verification that (5) is equivalent to (1). □

Lemma 49.18.7. *A composition of universally injective morphisms is universally injective.*

Proof. Omitted. □

49.19. Affine morphisms

We have already defined in Section 49.3 what it means for a representable morphism of algebraic spaces to be affine.

Lemma 49.19.1. *Let S be a scheme. Let $f : X \to Y$ be a representable morphism of algebraic spaces over S. Then f is affine if and only if for all affine schemes Z and morphisms $Z \to Y$ the scheme $X \times_Y Z$ is affine.*

Proof. This follows directly from the definition of an affine morphism of schemes (Morphisms, Definition 28.13.1). □

This clears the way for the following definition.

Definition 49.19.2. Let S be a scheme. Let $f : X \to Y$ be a morphism of algebraic spaces over S. We say f is *affine* if for every affine scheme Z and morphism $Z \to Y$ the algebraic space $X \times_Y Z$ is representable by an affine scheme.

Lemma 49.19.3. *Let S be a scheme. Let $f : X \to Y$ be a morphism of algebraic spaces over S. The following are equivalent:*

1. *f is representable and affine,*
2. *f is affine,*
3. *for every affine scheme V and étale morphism $V \to Y$ the scheme $X \times_Y V$ is affine,*
4. *there exists a scheme V and a surjective étale morphism $V \to Y$ such that $V \times_Y X \to V$ is affine, and*
5. *there exists a Zariski covering $Y = \bigcup Y_i$ such that each of the morphisms $f^{-1}(Y_i) \to Y_i$ is affine.*

Proof. It is clear that (1) implies (2), that (2) implies (3), and that (3) implies (4) by taking V to be a disjoint union of affines étale over Y, see Properties of Spaces, Lemma 48.6.1. Assume $V \to Y$ is as in (4). Then for every affine open W of V we see that $W \times_Y X$ is an affine open of $V \times_Y X$. Hence by Properties of Spaces, Lemma 48.10.1 we conclude that $V \times_Y X$ is a scheme. Moreover the morphism $V \times_Y X \to V$ is affine. This means we can apply Spaces, Lemma 47.11.3 because the class of affine morphisms satisfies all the required properties (see Morphisms, Lemmas 28.13.8 and Descent, Lemmas 34.19.16 and 34.33.1). The conclusion of applying this lemma is that f is representable and affine, i.e., (1) holds.

The equivalence of (1) and (5) follows from the fact that being affine is Zariski local on the target (the reference above shows that being affine is in fact fpqc local on the target). □

Lemma 49.19.4. *The composition of affine morphisms is affine.*

Proof. Omitted. □

Lemma 49.19.5. *The base change of an affine morphism is affine.*

Proof. Omitted. □

Lemma 49.19.6. *A closed immersion is affine.*

Proof. Follows immediately from the corresponding statement for morphisms of schemes, see Morphisms, Lemma 28.13.9. □

Lemma 49.19.7. *Let S be a scheme. Let X be an algebraic space over S. There is an anti-equivalence of categories*

$$\begin{matrix} \textit{algebraic space} \\ \textit{affine over } X \end{matrix} \longleftrightarrow \begin{matrix} \textit{quasi-coherent sheaves} \\ \textit{of } \mathcal{O}_X\textit{-algebras} \end{matrix}$$

which associates to $f : Y \to X$ the sheaf $f_*\mathcal{O}_Y$. Moreover, this equivalence is compatible with arbitrary base change.

Proof. This lemma is the analogue of Morphisms, Lemma 28.13.5. Let $\mathcal{A}$ be a quasi-coherent sheaf of $\mathcal{O}_X$-algebras. We will construct an affine morphism of algebraic spaces $\pi : Y = \underline{\mathrm{Spec}}_X(\mathcal{A}) \to X$ with $\pi_*\mathcal{O}_Y \cong \mathcal{A}$. To do this, choose a scheme U and a surjective étale morphism $\varphi : U \to X$. As usual denote $R = U \times_X U$ with projections $s, t : R \to U$. Denote $\psi : R \to X$ the composition $\psi = \varphi \circ s = \varphi \circ t$. By the aforementioned lemma there exists an affine morphisms of schemes $\pi_0 : V \to U$ and $\pi_1 : W \to R$ with $\pi_{0,*}\mathcal{O}_V \cong \varphi^*\mathcal{A}$ and $\pi_{1,*}\mathcal{O}_W \cong \psi^*\mathcal{A}$. Since the construction is compatible with base change there exist morphisms $s', t' : W \to V$ such that the diagrams

$$\begin{array}{ccc} W & \xrightarrow{s'} & V \\ \downarrow & & \downarrow \\ R & \xrightarrow{s} & U \end{array} \quad \text{and} \quad \begin{array}{ccc} W & \xrightarrow{t'} & V \\ \downarrow & & \downarrow \\ R & \xrightarrow{t} & U \end{array}$$

are cartesian. It follows that s', t' are étale. It is a formal consequence of the above that $(t', s') : W \to V \times_S V$ is a monomorphism. We omit the verification that $W \to V \times_S V$ is an equivalence relation (hint: think about the pullback of $\mathcal{A}$ to $U \times_X U \times_X U = R \times_{s,U,t} R$). The quotient sheaf $Y = V/W$ is an algebraic space, see Spaces, Theorem 47.10.5. By Groupoids, Lemma 38.18.7 we see that $Y \times_X U \cong V$. Hence $Y \to X$ is affine by Lemma 49.19.3. Finally, the isomorphism of

$$(Y \times_X U \to U)_*\mathcal{O}_{Y \times_X U} = \pi_{0,*}\mathcal{O}_V \cong \varphi^*\mathcal{A}$$

is compatible with glueing isomorphisms, whence $(Y \to X)_*\mathcal{O}_Y \cong \mathcal{A}$ by Properties of Spaces, Proposition 48.30.1. We omit the verification that this construction is compatible with base change. □

Definition 49.19.8. Let S be a scheme. Let X be an algebraic space over S. Let $\mathcal{A}$ be a quasi-coherent sheaf of $\mathcal{O}_X$-algebras. The *relative spectrum of $\mathcal{A}$ over X*, or simply the *spectrum of $\mathcal{A}$ over X* is the affine morphism $\underline{\mathrm{Spec}}(\mathcal{A}) \to X$ corresponding to $\mathcal{A}$ under the equivalence of categories of Lemma 49.19.7.

Forming the relative spectrum commutes with arbitrary base change.

Remark 49.19.9. Let S be a scheme. Let $f : Y \to X$ be a quasi-compact and quasi-separated morphism of algebraic spaces over S. Then f has a canonical factorization

$$Y \longrightarrow \underline{\mathrm{Spec}}_X(f_*\mathcal{O}_Y) \longrightarrow X$$

This makes sense because $f_*\mathcal{O}_Y$ is quasi-coherent by Lemma 49.12.2. The morphism $Y \to \underline{\mathrm{Spec}}_X(f_*\mathcal{O}_Y)$ comes from the canonical $\mathcal{O}_Y$-algebra map $f^*f_*\mathcal{O}_Y \to \mathcal{O}_Y$

which corresponds to a canonical morphism $Y \to Y \times_X \underline{\mathrm{Spec}}_X(f_*\mathcal{O}_Y)$ over Y (see Lemma 49.19.7) whence a factorization of f as above.

Lemma 49.19.10. *Let S be a scheme. Let $f : Y \to X$ be an affine morphism of algebraic spaces over S. Let $\mathcal{A} = f_*\mathcal{O}_Y$. The functor $\mathcal{F} \mapsto f_*\mathcal{F}$ induces an equivalence of categories*

$$\left\{\begin{array}{c}\text{category of quasi-coherent} \\ \mathcal{O}_Y\text{-modules}\end{array}\right\} \longrightarrow \left\{\begin{array}{c}\text{category of quasi-coherent} \\ \mathcal{A}\text{-modules}\end{array}\right\}$$

Moreover, an $\mathcal{A}$-module is quasi-coherent as an $\mathcal{O}_X$-module if and only if it is quasi-coherent as an $\mathcal{A}$-module.

Proof. Omitted. □

Lemma 49.19.11. *Let S be a scheme. Let B be an algebraic space over S. Suppose $g : X \to Y$ is a morphism of algebraic spaces over B.*

1. *If X is affine over B and $\Delta : Y \to Y \times_B Y$ is affine, then g is affine.*
2. *If X is affine over B and Y is separated over B, then g is affine.*
3. *A morphism from an affine scheme to an algebraic space with affine diagonal is affine.*
4. *A morphism from an affine scheme to a separated algebraic space is affine.*

Proof. Proof of (1). The base change $X \times_B Y \to Y$ is affine by Lemma 49.19.5. The morphism $(1, g) : X \to X \times_B Y$ is the base change of $Y \to Y \times_B Y$ by the morphism $X \times_B Y \to Y \times_B Y$. Hence it is affine by Lemma 49.19.5. The composition of affine morphisms is affine (see Lemma 49.19.4) and (1) follows. Part (2) follows from (1) as a closed immersion is affine (see Lemma 49.19.6) and Y/B separated means Δ is a closed immersion. Parts (3) and (4) are special cases of (1) and (2). □

Lemma 49.19.12. *Let S be a scheme. Let X be a quasi-separated algebraic space over S. Let A be an Artinian ring. Any morphism* $\mathrm{Spec}(A) \to X$ *is affine.*

Proof. Let $U \to X$ be an étale morphism with U affine. To prove the lemma we have to show that $\mathrm{Spec}(A) \times_X U$ is affine, see Lemma 49.19.3. Since X is quasi-separated the scheme $\mathrm{Spec}(A) \times_X U$ is quasi-compact. Moreover, the projection morphism $\mathrm{Spec}(A) \times_X U \to \mathrm{Spec}(A)$ is étale. Hence this morphism has finite discrete fibers and moreover the topology on $\mathrm{Spec}(A)$ is discrete. Thus $\mathrm{Spec}(A) \times_X U$ is a scheme whose underlying topological space is a finite discrete set. We are done by Schemes, Lemma 25.11.7. □

49.20. Quasi-affine morphisms

We have already defined in Section 49.3 what it means for a representable morphism of algebraic spaces to be quasi-affine.

Lemma 49.20.1. *Let S be a scheme. Let $f : X \to Y$ be a representable morphism of algebraic spaces over S. Then f is quasi-affine if and only if for all affine schemes Z and morphisms $Z \to Y$ the scheme $X \times_Y Z$ is quasi-affine.*

Proof. This follows directly from the definition of a quasi-affine morphism of schemes (Morphisms, Definition 28.14.1). □

This clears the way for the following definition.

Definition 49.20.2. Let S be a scheme. Let $f : X \to Y$ be a morphism of algebraic spaces over S. We say f is *quasi-affine* if for every affine scheme Z and morphism $Z \to Y$ the algebraic space $X \times_Y Z$ is representable by a quasi-affine scheme.

Lemma 49.20.3. *Let S be a scheme. Let $f : X \to Y$ be a morphism of algebraic spaces over S. The following are equivalent:*

(1) *f is representable and quasi-affine,*
(2) *f is quasi-affine,*
(3) *there exists a scheme V and a surjective étale morphism $V \to Y$ such that $V \times_Y X \to V$ is quasi-affine, and*
(4) *there exists a Zariski covering $Y = \bigcup Y_i$ such that each of the morphisms $f^{-1}(Y_i) \to Y_i$ is quasi-affine.*

Proof. It is clear that (1) implies (2) and that (2) implies (3) by taking V to be a disjoint union of affines étale over Y, see Properties of Spaces, Lemma 48.6.1. Assume $V \to Y$ is as in (3). Then for every affine open W of V we see that $W \times_Y X$ is a quasi-affine open of $V \times_Y X$. Hence by Properties of Spaces, Lemma 48.10.1 we conclude that $V \times_Y X$ is a scheme. Moreover the morphism $V \times_Y X \to V$ is quasi-affine. This means we can apply Spaces, Lemma 47.11.3 because the class of quasi-affine morphisms satisfies all the required properties (see Morphisms, Lemmas 28.14.5 and Descent, Lemmas 34.19.18 and 34.34.1). The conclusion of applying this lemma is that f is representable and quasi-affine, i.e., (1) holds.

The equivalence of (1) and (4) follows from the fact that being quasi-affine is Zariski local on the target (the reference above shows that being quasi-affine is in fact fpqc local on the target). □

Lemma 49.20.4. *The composition of quasi-affine morphisms is quasi-affine.*

Proof. Omitted. □

Lemma 49.20.5. *The base change of a quasi-affine morphism is quasi-affine.*

Proof. Omitted. □

Lemma 49.20.6. *Let S be a scheme. A quasi-compact and quasi-separated morphism of algebraic spaces $f : Y \to X$ is quasi-affine if and only if the canonical factorization $Y \to \underline{\mathrm{Spec}}_X(f_*\mathcal{O}_Y)$ (Remark 49.19.9) is an open immersion.*

Proof. Let $U \to X$ be a surjective morphism where U is a scheme. Since we may check whether f is quasi-affine after base change to U (Lemma 49.20.3), since $f_*\mathcal{O}_Y|_V$ is equal to $(Y \times_X U \to U)_*\mathcal{O}_{Y \times_X U}$ (Properties of Spaces, Lemma 48.24.2), and since formation of relative spectrum commutes with base change (Lemma 49.19.7), we see that the assertion reduces to the case that X is a scheme. If X is a scheme and either f is quasi-affine or $Y \to \underline{\mathrm{Spec}}_X(f_*\mathcal{O}_Y)$ is an open immersion, then Y is a scheme as well. Thus we reduce to Morphisms, Lemma 28.14.3. □

49.21. Types of morphisms étale local on source-and-target

Given a property of morphisms of schemes which is *étale local on the source-and-target*, see Descent, Definition 34.28.3 we may use it to define a corresponding property of morphisms of algebraic spaces, namely by imposing either of the equivalent conditions of the lemma below.

Lemma 49.21.1. *Let $\mathcal{P}$ be a property of morphisms of schemes which is étale local on the source-and-target. Let S be a scheme. Let $f : X \to Y$ be a morphism of algebraic spaces over S. Consider commutative diagrams*

$$\begin{array}{ccc} U & \xrightarrow{h} & V \\ {\scriptstyle a}\downarrow & & \downarrow{\scriptstyle b} \\ X & \xrightarrow{f} & Y \end{array}$$

where U and V are schemes and the vertical arrows are étale. The following are equivalent

1. *for any diagram as above the morphism h has property $\mathcal{P}$, and*
2. *for some diagram as above with $a : U \to X$ surjective the morphism h has property $\mathcal{P}$.*

If X and Y are representable, then this is also equivalent to f (as a morphism of schemes) having property $\mathcal{P}$. If $\mathcal{P}$ is also preserved under any base change, and fppf local on the base, then for representable morphisms f this is also equivalent to f having property $\mathcal{P}$ in the sense of Section 49.3.

Proof. Let us prove the equivalence of (1) and (2). The implication (1) $\Rightarrow$ (2) is immediate (taking into account Spaces, Lemma 47.11.4). Assume

$$\begin{array}{ccc} U & \xrightarrow{h} & V \\ \downarrow & & \downarrow \\ X & \xrightarrow{f} & Y \end{array} \qquad \begin{array}{ccc} U' & \xrightarrow{h'} & V' \\ \downarrow & & \downarrow \\ X & \xrightarrow{f} & Y \end{array}$$

are two diagrams as in the lemma. Assume $U \to X$ is surjective and h has property $\mathcal{P}$. To show that (2) implies (1) we have to prove that h' has $\mathcal{P}$. To do this consider the diagram

$$\begin{array}{ccccc} U & \longleftarrow & U \times_X U' & \longrightarrow & U' \\ {\scriptstyle h}\downarrow & & \downarrow{\scriptstyle (h,h')} & & \downarrow{\scriptstyle h'} \\ V & \longleftarrow & V \times_Y V' & \longrightarrow & V' \end{array}$$

By Descent, Lemma 34.28.5 we see that h has $\mathcal{P}$ implies (h, h') has $\mathcal{P}$ and since $U \times_X U' \to U'$ is surjective this implies (by the same lemma) that h' has $\mathcal{P}$.

If X and Y are representable, then Descent, Lemma 34.28.5 applies which shows that (1) and (2) are equivalent to f having $\mathcal{P}$.

Finally, suppose f is representable, and U, V, a, b, h are as in part (2) of the lemma, and that $\mathcal{P}$ is preserved under arbitrary base change. We have to show that for any scheme Z and morphism $Z \to X$ the base change $Z \times_Y X \to Z$ has property $\mathcal{P}$. Consider the diagram

$$\begin{array}{ccc} Z \times_Y U & \longrightarrow & Z \times_Y V \\ \downarrow & & \downarrow \\ Z \times_Y X & \longrightarrow & Z \end{array}$$

Note that the top horizontal arrow is a base change of h and hence has property $\mathcal{P}$. The left vertical arrow is étale and surjective and the right vertical arrow is étale.

Thus Descent, Lemma 34.28.5 once again kicks in and shows that $Z \times_Y X \to Z$ has property $\mathcal{P}$. □

Definition 49.21.2. Let S be a scheme. Let $\mathcal{P}$ be a property of morphisms of schemes which is étale local on the source-and-target. We say a morphism $f : X \to Y$ of algebraic spaces over S *has property* $\mathcal{P}$ if the equivalent conditions of Lemma 49.21.1 hold.

Given a property of morphisms of germs of schemes which is *étale local on the source-and-target*, see Descent, Definition 34.29.1 we may use it to define a corresponding property of morphisms of algebraic spaces at a point, namely by imposing either of the equivalent conditions of the lemma below.

Lemma 49.21.3. *Let $\mathcal{Q}$ be a property of morphisms of germs which is étale local on the source-and-target. Let S be a scheme. Let $f : X \to Y$ be a morphism of algebraic spaces over S. Let $x \in |X|$ be a point of X. Consider the diagrams*

$$\begin{array}{ccc} U & \xrightarrow{h} & V \\ {\scriptstyle a}\downarrow & & \downarrow{\scriptstyle b} \\ X & \xrightarrow{f} & Y \end{array} \qquad \begin{array}{ccc} u & \longrightarrow & v \\ \downarrow & & \downarrow \\ x & \longrightarrow & y \end{array}$$

where U and V are schemes, a, b are étale, and u, v, x, y are points of the corresponding spaces. The following are equivalent

1. *for any diagram as above we have $\mathcal{Q}((U, u) \to (V, v))$, and*
2. *for some diagram as above we have $\mathcal{Q}((U, u) \to (V, v))$.*

If X and Y are representable, then this is also equivalent to $\mathcal{Q}((X, x) \to (Y, y))$.

Proof. Omitted. Hint: Very similar to the proof of Lemma 49.21.1. □

Definition 49.21.4. Let $\mathcal{Q}$ be a property of morphisms of germs of schemes which is étale local on the source-and-target. Let S be a scheme. Given a morphism $f : X \to Y$ of algebraic spaces over S and a point $x \in |X|$ we say that f *has property $\mathcal{Q}$ at x* if the equivalent conditions of Lemma 49.21.3 hold.

The following lemma should not be used blindly to go from a property of morphisms to a property of morphisms at a point. For example if $\mathcal{P}$ is the property of being flat, then the property Q in the following lemma means "f is flat in an open neighbourhood of x" which is not the same as "f is flat at x".

Lemma 49.21.5. *Let $\mathcal{P}$ be a property of morphisms of schemes which is étale local on the source-and-target. Consider the property $\mathcal{Q}$ of morphisms of germs associated to $\mathcal{P}$ in Descent, Lemma 34.29.2. Then*

1. *$\mathcal{Q}$ is étale local on the source-and-target.*
2. *given a morphism of algebraic spaces $f : X \to Y$ and $x \in |X|$ the following are equivalent*
   (a) *f has $\mathcal{Q}$ at x, and*
   (b) *there is an open neighbourhood $X' \subset X$ of x such that $X' \to Y$ has $\mathcal{P}$.*
3. *given a morphism of algebraic spaces $f : X \to Y$ the following are equivalent:*
   (a) *f has $\mathcal{P}$,*
   (b) *for every $x \in |X|$ the morphism f has $\mathcal{Q}$ at x.*

Proof. See Descent, Lemma 34.29.2 for (1). The implication (1)(a) $\Rightarrow$ (2)(b) follows on letting $X' = a(U) \subset X$ given a diagram as in Lemma 49.21.3. The implication (2)(b) $\Rightarrow$ (1)(a) is clear. The equivalence of (3)(a) and (3)(b) follows from the corresponding result for morphisms of schemes, see Descent, Lemma 34.29.3. □

Remark 49.21.6. We will apply Lemma 49.21.5 above to all cases listed in Descent, Remark 34.28.7 except "flat". In each case we will do this by defining f to have property $\mathcal{P}$ at x if f has $\mathcal{P}$ in a neighbourhood of x.

49.22. Morphisms of finite type

The property "locally of finite type" of morphisms of schemes is étale local on the source-and-target, see Descent, Remark 34.28.7. It is also stable under base change and fpqc local on the target, see Morphisms, Lemma 28.16.4, and Descent, Lemmas 34.19.8. Hence, by Lemma 49.21.1 above, we may define what it means for a morphism of algebraic spaces to be locally of finite type as follows and it agrees with the already existing notion defined in Section 49.3 when the morphism is representable.

Definition 49.22.1. Let S be a scheme. Let $f : X \to Y$ be a morphism of algebraic spaces over S.

(1) We say f *locally of finite type* if the equivalent conditions of Lemma 49.21.1 hold with $\mathcal{P} =$ locally of finite type.
(2) Let $x \in |X|$. We say f is of *finite type at* x if there exists an open neighbourhood $X' \subset X$ of x such that $f|_{X'} : X' \to Y$ is locally of finite type.
(3) We say f is *of finite type* if it is locally of finite type and quasi-compact.

Consider the algebraic space $\mathbf{A}^1_k/\mathbf{Z}$ of Spaces, Example 47.14.8. The morphism $\mathbf{A}^1_k/\mathbf{Z} \to \mathrm{Spec}(k)$ is of finite type.

Lemma 49.22.2. *The composition of finite type morphisms is of finite type. The same holds for locally of finite type.*

Proof. Omitted. □

Lemma 49.22.3. *A base change of a finite type morphism is finite type. The same holds for locally of finite type.*

Proof. Omitted. □

Lemma 49.22.4. *Let S be a scheme. Let $f : X \to Y$ be a morphism of algebraic spaces over S. The following are equivalent:*

(1) *f is locally of finite type,*
(2) *for every $x \in |X|$ the morphism f is of finite type at x,*
(3) *for every scheme Z and any morphism $Z \to Y$ the morphism $Z \times_Y X \to Z$ is locally of finite type,*
(4) *for every affine scheme Z and any morphism $Z \to Y$ the morphism $Z \times_Y X \to Z$ is locally of finite type,*
(5) *there exists a scheme V and a surjective étale morphism $V \to Y$ such that $V \times_Y X \to V$ is locally of finite type,*
(6) *there exists a scheme U and a surjective étale morphism $\varphi : U \to X$ such that the composition $f \circ \varphi$ is locally of finite type,*

(7) *for every commutative diagram*

$$\begin{array}{ccc} U & \longrightarrow & V \\ \downarrow & & \downarrow \\ X & \longrightarrow & Y \end{array}$$

where U, V are schemes and the vertical arrows are étale the top horizontal arrow is locally of finite type,

(8) *there exists a commutative diagram*

$$\begin{array}{ccc} U & \longrightarrow & V \\ \downarrow & & \downarrow \\ X & \longrightarrow & Y \end{array}$$

where U, V are schemes, the vertical arrows are étale, $U \to X$ is surjective, and the top horizontal arrow is locally of finite type, and

(9) *there exist Zariski coverings $Y = \bigcup_{i \in I} Y_i$, and $f^{-1}(Y_i) = \bigcup X_{ij}$ such that each morphism $X_{ij} \to Y_i$ is locally of finite type.*

Proof. Omitted. □

Lemma 49.22.5. *Let S be a scheme. Let $f : X \to Y$ be a morphism of algebraic spaces over S. If f is locally of finite type and Y is locally Noetherian, then X is locally Noetherian.*

Proof. Let

$$\begin{array}{ccc} U & \longrightarrow & V \\ \downarrow & & \downarrow \\ X & \longrightarrow & Y \end{array}$$

be a commutative diagram where U, V are schemes and the vertical arrows are surjective étale. If f is locally of finite type, then $U \to V$ is locally of finite type. If Y is locally Noetherian, then V is locally Noetherian. By Morphisms, Lemma 28.16.6 we see that U is locally Noetherian, which means that X is locally Noetherian. □

Lemma 49.22.6. *Let S be a scheme. Let $f : X \to Y$, $g : Y \to Z$ be morphisms of algebraic spaces over S. If $g \circ f : X \to Z$ is locally of finite type, then $f : X \to Y$ is locally of finite type.*

Proof. We can find a diagram

$$\begin{array}{ccccc} U & \longrightarrow & V & \longrightarrow & W \\ \downarrow & & \downarrow & & \downarrow \\ X & \longrightarrow & Y & \longrightarrow & Z \end{array}$$

where U, V, W are schemes, the vertical arrows are étale and surjective, see Spaces, Lemma 47.11.4. At this point we can use Lemma 49.22.4 and Morphisms, Lemma 28.16.8 to conclude. □

Lemma 49.22.7. *An immersion is locally of finite type.*

Proof. Follows from the general principle Spaces, Lemma 47.5.8 and Morphisms, Lemmas 28.16.5. □

49.23. Points and geometric points

In this section we make some remarks on points and geometric points (see Properties of Spaces, Definition 48.16.1). One way to think about a geometric point of X is to consider a geometric point $\overline{s} : \mathrm{Spec}(k) \to S$ of S and a lift of $\overline{s}$ to a morphism $\overline{x}$ into X. Here is a diagram

$$\begin{array}{ccc} \mathrm{Spec}(k) & \xrightarrow{\overline{x}} & X \\ & \searrow^{\overline{s}} & \downarrow \\ & & S. \end{array}$$

We often say "let k be an algebraically closed field over S" to indicate that $\mathrm{Spec}(k)$ comes equipped with a morphism $\mathrm{Spec}(k) \to S$. In this situation we write

$$X(k) = \mathrm{Mor}_S(\mathrm{Spec}(k), X) = \{\overline{x} \in X \text{ lying over } \overline{s}\}$$

for the set of k-valued points of X. In this case the map $X(k) \to |X|$ maps into the subset $|X_s| \subset |X|$. Here $X_s = \mathrm{Spec}(\kappa(s)) \times_S X$, where $s \in S$ is the point corresponding to $\overline{s}$. As $\mathrm{Spec}(\kappa(s)) \to S$ is a monomorphism, also the base change $X_s \to X$ is a monomorphism, and $|X_s|$ is indeed a subset of $|X|$.

Lemma 49.23.1. *Let S be a scheme. Let $f : X \to Y$ be a morphism of algebraic spaces over S. Assume f is locally of finite type. The following are equivalent:*

1. *f is surjective, and*
2. *for every algebraically closed field k over S the induced map $X(k) \to Y(k)$ is surjective.*

Proof. Choose a diagram

$$\begin{array}{ccc} U & \longrightarrow & V \\ \downarrow & & \downarrow \\ X & \longrightarrow & Y \end{array}$$

with U, V schemes over S and vertical arrows surjective and étale, see Spaces, Lemma 47.11.4. Since f is locally of finite type we see that $U \to V$ is locally of finite type.

Assume (1) and let $\overline{y} \in Y(k)$. Then $U \to Y$ is surjective and locally of finite type by Lemmas 49.6.4 and 49.22.2. Let $Z = U \times_{Y,\overline{y}} \mathrm{Spec}(k)$. This is a scheme. The projection $Z \to \mathrm{Spec}(k)$ is surjective and locally of finite type by Lemmas 49.6.5 and 49.22.3. It follows from Varieties, Lemma 32.12.1 that Z has a k valued point $\overline{z}$. The image $\overline{x} \in X(k)$ of $\overline{z}$ maps to $\overline{y}$ as desired.

Assume (2). By Properties of Spaces, Lemma 48.4.4 it suffices to show that $|X| \to |Y|$ is surjective. Let $y \in |Y|$. Choose a $u \in U$ mapping to y. Let $k \supset \kappa(u)$ be an algebraic closure. Denote $\overline{u} \in U(k)$ the corresponding point and $\overline{y} \in Y(k)$ its image. By assumption there exists a $\overline{x} \in X(k)$ mapping to $\overline{y}$. Then it is clear that the image $x \in |X|$ of $\overline{x}$ maps to y. □

In order to state the next lemma we introduce the following notation. Given a scheme T we denote

$$\lambda(T) = \sup\{\aleph_0, |\kappa(t)|; t \in T\}.$$

In words $\lambda(T)$ is the smallest infinite cardinal bounding all the cardinalities of residue fields ot T. Note that if R is a ring then the cardinality of any residue field $\kappa(\mathfrak{p})$ of R is bounded by the cardinality of R (details omitted). This implies that $\lambda(T) \leq \text{size}(T)$ where $\text{size}(T)$ is the size of the scheme T as introduced in Sets, Section 3.9. If $K \subset L$ is a finitely generated field extension then $|K| \leq |L| \leq \max\{\aleph_0, |K|\}$. It follows that if $T' \to T$ is a morphism of schemes which is locally of finite type then $\lambda(T') \leq \lambda(T)$, and if $T' \to T$ is also surjective then equality holds. Next, suppose that S is a scheme and that X is an algebraic space over S. In this case we define

$$\lambda(X) := \lambda(U)$$

where U is any scheme over S which has a surjective étale morphism towards X. The reason that this is independent of the choice of U is that given a pair of such schemes U and U' the fibre product $U \times_X U'$ is a scheme which admits a surjective étale morphism to both U and U', whence $\lambda(U) = \lambda(U \times_X U') = \lambda(U')$ by the discussion above.

Lemma 49.23.2. *Let S be a scheme. Let X, Y be algebraic spaces over S.*

1. *As k ranges over all algebraically closed fields over S the collection of geometric points $\overline{y} \in Y(k)$ cover all of $|Y|$.*
2. *As k ranges over all algebraically closed fields over S with $|k| \geq \lambda(Y)$ and $|k| > \lambda(X)$ the geometric points $\overline{y} \in Y(k)$ cover all of $|Y|$.*
3. *For any geometric point $\overline{s} : \text{Spec}(k) \to S$ where k has cardinality $> \lambda(X)$ the map*
$$X(k) \longrightarrow |X_s|$$
is surjective.
4. *Let $X \to Y$ be a morphism of algebraic spaces over S. For any geometric point $\overline{s} : \text{Spec}(k) \to S$ where k has cardinality $> \lambda(X)$ the map*
$$X(k) \longrightarrow |X| \times_{|Y|} Y(k)$$
is surjective.
5. *Let $X \to Y$ be a morphism of algebraic spaces over S. The following are equivalent:*
   (a) *the map $X \to Y$ is surjective,*
   (b) *for all algebraically closed fields k over S with $|k| > \lambda(X)$, and $|k| \geq \lambda(Y)$ the map $X(k) \to Y(k)$ is surjective.*

Proof. To prove part (1) choose a surjective étale morphism $V \to Y$ where V is a scheme. For each $v \in V$ choose an algebraic closure $\kappa(v) \subset k_v$. Consider the morphisms $\overline{x} : \text{Spec}(k_v) \to V \to Y$. By construction of $|Y|$ these cover $|Y|$.

To prove part (2) we will use the following two facts whose proofs we omit: (i) If K is a field and $\overline{K}$ is algebraic closure then $|\overline{K}| \leq \max\{\aleph_0, |K|\}$. (ii) For any algebraically closed field k and any cardinal $\aleph$, $\aleph \geq |k|$ there exists an extension of algebraically closed fields $k \subset k'$ with $|k'| = \aleph$. Now we set $\aleph = \max\{\lambda(X), \lambda(Y)\}^+$. Here $\lambda^+ > \lambda$ indicates the next bigger cardinal, see Sets, Section 3.6. Now (i) implies that the fields k_u constructed in the first paragraph of the proof all have cardinality bounded by $\lambda(X)$. Hence by (ii) we can find extensions $k_u \subset k'_u$ such that $|k'_u| = \aleph$. The

morphisms $\overline{x}' : \mathrm{Spec}(k'_u) \to X$ cover $|X|$ as desired. To really finish the proof of (2) we need to show that the schemes $\mathrm{Spec}(k'_u)$ are (isomorphic to) objects of Sch_{fppf} because our conventions are that all schemes are objects of Sch_{fppf}; the rest of this paragraph should be skipped by anyone who is not interested in set theoretical considerations. By construction there exists an object T of Sch_{fppf} such that $\lambda(X)$ and $\lambda(Y)$ are bounded by $\mathrm{size}(T)$. By our construction of the category Sch_{fppf} in Topologies, Definitions 33.7.6 as the category Sch_α constructed in Sets, Lemma 3.9.2 we see that any scheme whose size is $\leq \mathrm{size}(T)^+$ is isomorphic to an object of Sch_{fppf}. See the expression for the function $Bound$ in Sets, Equation (3.9.1.1). Since $\aleph \leq \mathrm{size}(T)^+$ we conclude.

The notation X_s in part (3) means the fibre product $\mathrm{Spec}(\kappa(s)) \times_S X$, where $s \in S$ is the point corresponding to $\overline{s}$. Hence part (2) follows from (4) with $Y = \mathrm{Spec}(\kappa(s))$.

Let us prove (4). Let $X \to Y$ be a morphism of algebraic spaces over S. Let k be an algebraically closed field over S of cardinality $> \lambda(X)$. Let $\overline{y} \in Y(k)$ and $x \in |X|$ which map to the same element y of $|Y|$. We have to find $\overline{x} \in X(k)$ mapping to x and $\overline{y}$. Choose a commutative diagram

$$\begin{array}{ccc} U & \longrightarrow & V \\ \downarrow & & \downarrow \\ X & \longrightarrow & Y \end{array}$$

with U, V schemes over S and vertical arrows surjective and étale, see Spaces, Lemma 47.11.4. Choose a $u \in |U|$ which maps to x, and denote $v \in |V|$ the image. We will think of $u = \mathrm{Spec}(\kappa(u))$ and $v = \mathrm{Spec}(\kappa(v))$ as schemes. Note that $V \times_Y \mathrm{Spec}(k)$ is a scheme étale over k. Hence it is a disjoint union of spectra of finite separable extensions of k, see Morphisms, Lemma 28.37.7. As v maps to y we see that $v \times_Y \mathrm{Spec}(k)$ is a nonempty scheme. As $v \to V$ is a monomorphism, we see that $v \times_Y \mathrm{Spec}(k) \to V \times_Y \mathrm{Spec}(k)$ is a monomorphism. Hence $v \times_Y \mathrm{Spec}(k)$ is a disjoint union of spectra of finite separable extensions of k, by Schemes, Lemma 25.23.10. We conclude that the morphism $v \times_Y \mathrm{Spec}(k) \to \mathrm{Spec}(k)$ has a section, i.e., we can find a morphism $\overline{v} : \mathrm{Spec}(k) \to V$ lying over v and over $\overline{y}$. Finally we consider the scheme

$$u \times_{V,\overline{v}} \mathrm{Spec}(k) = \mathrm{Spec}(\kappa(u) \otimes_{\kappa(v)} k)$$

where $\kappa(v) \to k$ is the field map defining the morphism $\overline{v}$. Since the cardinality of k is larger than the cardinality of $\kappa(u)$ by assumption we may apply Algebra, Lemma 10.32.12 to see that any maximal ideal $\mathfrak{m} \subset \kappa(u) \otimes_{\kappa(v)} k$ has a residue field which is algebraic over k and hence equal to k. Such a maximal ideal will hence produce a morphism $\overline{u} : \mathrm{Spec}(k) \to U$ lying over u and mapping to $\overline{v}$. The composition $\mathrm{Spec}(k) \to U \to X$ will be the desired geometric point $\overline{x} \in X(k)$. This concludes the proof of part (4).

Part (5) is a formal consequence of parts (2) and (4) and Properties of Spaces, Lemma 48.4.4. □

49.24. Points of finite type

Let S be a scheme. Let X be an algebraic space over S. A finite type point $x \in |X|$ is a point which can be represented by a morphism $\mathrm{Spec}(k) \to X$ which is locally of finite type. Finite type points are a suitable replacement of closed points

for algebraic spaces and algebraic stacks. There are always "enough of them" for example.

Lemma 49.24.1. *Let S be a scheme. Let X be an algebraic space over S. Let $x \in |X|$. The following are equivalent:*

1. *There exists a morphism $\mathrm{Spec}(k) \to X$ which is locally of finite type and represents x.*
2. *There exists a scheme U, a closed point $u \in U$, and an étale morphism $\varphi : U \to X$ such that $\varphi(u) = x$.*

Proof. Let $u \in U$ and $U \to X$ be as in (2). Then $\mathrm{Spec}(\kappa(u)) \to U$ is of finite type, and $U \to X$ is representable and locally of finite type (by the general principle Spaces, Lemma 47.5.8 and Morphisms, Lemmas 28.37.11 and 28.22.8). Hence we see (1) holds by Lemma 49.22.2.

Conversely, assume $\mathrm{Spec}(k) \to X$ is locally of finite type and represents x. Let $U \to X$ be a surjective étale morphism where U is a scheme. By assumption $U \times_X \mathrm{Spec}(k) \to U$ is locally of finite type. Pick a finite type point v of $U \times_X \mathrm{Spec}(k)$ (there exists at least one, see Morphisms, Lemma 28.17.4). By Morphisms, Lemma 28.17.5 the image $u \in U$ of v is a finite type point of U. Hence by Morphisms, Lemma 28.17.4 after shrinking U we may assume that u is a closed point of U, i.e., (2) holds. □

Definition 49.24.2. Let S be a scheme. Let X be an algebraic space over S. We say a point $x \in |X|$ is a *finite type point*[5] if the equivalent conditions of Lemma 49.24.1 are satisfied. We denote $X_{\text{ft-pts}}$ the set of finite type points of X.

We can describe the set of finite type points as follows.

Lemma 49.24.3. *Let S be a scheme. Let X be an algebraic space over S. We have*

$$X_{ft\text{-}pts} = \bigcup\nolimits_{\varphi : U \to X \text{ étale}} |\varphi|(U_0)$$

where U_0 is the set of closed points of U. Here we may let U range over all schemes étale over X or over all affine schemes étale over X.

Proof. Immediate from Lemma 49.24.1. □

Lemma 49.24.4. *Let S be a scheme. Let $f : X \to Y$ be a morphism of algebraic spaces over S. If f is locally of finite type, then $f(X_{ft\text{-}pts}) \subset Y_{ft\text{-}pts}$.*

Proof. Take $x \in X_{\text{ft-pts}}$. Represent x by a locally finite type morphism $x : \mathrm{Spec}(k) \to X$. Then $f \circ x$ is locally of finite type by Lemma 49.22.2. Hence $f(x) \in Y_{\text{ft-pts}}$. □

Lemma 49.24.5. *Let S be a scheme. Let $f : X \to Y$ be a morphism of algebraic spaces over S. If f is locally of finite type and surjective, then $f(X_{ft\text{-}pts}) = Y_{ft\text{-}pts}$.*

Proof. We have $f(X_{\text{ft-pts}}) \subset Y_{\text{ft-pts}}$ by Lemma 49.24.4. Let $y \in |Y|$ be a finite type point. Represent y by a morphism $\mathrm{Spec}(k) \to Y$ which is locally of finite type. As f is surjective the algebraic space $X_k = \mathrm{Spec}(k) \times_Y X$ is nonempty, therefore has a finite type point $x \in |X_k|$ by Lemma 49.24.3. Now $X_k \to X$ is a morphism which

[5] This is a slight abuse of language as it would perhaps be more correct to say "locally finite type point".

is locally of finite type as a base change of $\mathrm{Spec}(k) \to Y$ (Lemma 49.22.3). Hence the image of x in X is a finite type point by Lemma 49.24.4 which maps to y by construction. □

Lemma 49.24.6. *Let S be a scheme. Let X be an algebraic space over S. For any locally closed subset $T \subset |X|$ we have*

$$T \neq \emptyset \Rightarrow T \cap X_{ft\text{-}pts} \neq \emptyset.$$

In particular, for any closed subset $T \subset |X|$ we see that $T \cap X_{ft\text{-}pts}$ is dense in T.

Proof. Let $i : Z \to X$ be the reduced induce subspace structure on T, see Remark 49.4.4. Any immersion is locally of finite type, see Lemma 49.22.7. Hence by Lemma 49.24.4 we see $Z_{\text{ft-pts}} \subset X_{\text{ft-pts}} \cap T$. Finally, any nonempty affine scheme U with an étale morphism towards Z has at least one closed point. Hence Z has at least one finite type point by Lemma 49.24.3. The lemma follows. □

Here is another, more technical, characterization of a finite type point on an algebraic space.

Lemma 49.24.7. *Let S be a scheme. Let X be an algebraic space over S. Let $x \in |X|$. The following are equivalent:*

1. *x is a finite type point,*
2. *there exists an algebraic space Z whose underlying topological space $|Z|$ is a singleton, and a morphism $f : Z \to X$ which is locally of finite type such that $\{x\} = |f|(|Z|)$, and*
3. *there exists an algebraic space Z and a morphism $f : Z \to X$ with the following properties:*
   (a) *there is a surjective étale morphism $z : \mathrm{Spec}(k) \to Z$ where k is a field,*
   (b) *f is locally of finite type,*
   (c) *f is a monomorphism, and*
   (d) *$x = f(z)$.*

Proof. Assume x is a finite type point. Choose an affine scheme U, a closed point $u \in U$, and an étale morphism $\varphi : U \to X$ with $\varphi(u) = x$, see Lemma 49.24.3. Set $u = \mathrm{Spec}(\kappa(u))$ as usual. The projection morphisms $u \times_X u \to u$ are the compositions

$$u \times_X u \to u \times_X U \to u \times_X X = u$$

where the first arrow is a closed immersion (a base change of $u \to U$) and the second arrow is étale (a base change of the étale morphism $U \to X$). Hence $u \times_X U$ is a disjoint union of spectra of finite separable extensions of k (see Morphisms, Lemma 28.37.7) and therefore the closed subscheme $u \times_X u$ is a disjoint union of finite separable extension of k, i.e., $u \times_X u \to u$ is étale. By Spaces, Theorem 47.10.5 we see that $Z = u/u \times_X u$ is an algebraic space. By construction the diagram

$$\begin{array}{ccc} u & \longrightarrow & U \\ \downarrow & & \downarrow \\ Z & \longrightarrow & X \end{array}$$

is commutative with étale vertical arrows. Hence $Z \to X$ is locally of finite type (see Lemma 49.22.4). By construction the morphism $Z \to X$ is a monomorphism and the image of z is x. Thus (3) holds.

It is clear that (3) implies (2). If (2) holds then x is a finite type point of X by Lemma 49.24.4 (and Lemma 49.24.6 to see that $Z_{\text{ft-pts}}$ is nonempty, i.e., the unique point of Z is a finite type point of Z). □

49.25. Quasi-finite morphisms

The property "locally quasi-finite" of morphisms of schemes is étale local on the source-and-target, see Descent, Remark 34.28.7. It is also stable under base change and fpqc local on the target, see Morphisms, Lemma 28.21.13, and Descent, Lemma 34.19.22. Hence, by Lemma 49.21.1 above, we may define what it means for a morphism of algebraic spaces to be locally quasi-finite as follows and it agrees with the already existing notion defined in Section 49.3 when the morphism is representable.

Definition 49.25.1. Let S be a scheme. Let $f : X \to Y$ be a morphism of algebraic spaces over S.

(1) We say f is *locally quasi-finite* if the equivalent conditions of Lemma 49.21.1 hold with $\mathcal{P} =$ locally quasi-finite.
(2) Let $x \in |X|$. We say f is *quasi-finite at* x if there exists an open neighbourhood $X' \subset X$ of x such that $f|_{X'} : X' \to Y$ is locally quasi-finite.
(3) A morphism of algebraic spaces $f : X \to Y$ is *quasi-finite* if it is locally quasi-finite and quasi-compact.

The last part is compatible with the notion of quasi-finiteness for morphisms of schemes by Morphisms, Lemma 28.21.9.

Lemma 49.25.2. *The composition of quasi-finite morphisms is quasi-finite. The same holds for locally quasi-finite.*

Proof. Omitted. □

Lemma 49.25.3. *A base change of a quasi-finite morphism is quasi-finite. The same holds for locally quasi-finite.*

Proof. Omitted. □

The following lemma characterizes locally quasi-finite morphisms as those morphisms which are locally of finite type and have "discrete fibres". However, it isn't enough to assume that $|X| \to |Y|$ has discrete fibres as the discussion in Examples, Section 78.36 shows.

Lemma 49.25.4. *Let S be a scheme. Let $f : X \to Y$ be a morphism of algebraic spaces. Assume f is locally of finite type. The following are equivalent*

(1) *f is locally quasi-finite,*
(2) *for every morphism $\mathrm{Spec}(k) \to Y$ where k is a field the space $|X_k|$ is discrete. Here $X_k = \mathrm{Spec}(k) \times_Y X$.*

Proof. Assume f is locally quasi-finite. Let $\mathrm{Spec}(k) \to Y$ be as in (2). Choose a surjective étale morphism $U \to X$ where U is a scheme. Then $U_k = \mathrm{Spec}(k) \times_Y U \to X_k$ is an étale morphism of algebraic spaces by Properties of Spaces, Lemma 48.13.5.

By Lemma 49.25.3 we see that $X_k \to \mathrm{Spec}(k)$ is locally quasi-finite. By definition this means that $U_k \to \mathrm{Spec}(k)$ is locally quasi-finite. Hence $|U_k|$ is discrete by Morphisms, Lemma 28.21.8. Since $|U_k| \to |X_k|$ is surjective and open we conclude that $|X_k|$ is discrete.

Conversely, assume (2). Choose a surjective étale morphism $V \to Y$ where V is a scheme. Choose a surjective étale morphism $U \to V \times_Y X$ where U is a scheme. Note that $U \to V$ is locally of finite type as f is locally of finite type. Picture

$$\begin{array}{ccccc} U & \longrightarrow & X \times_Y V & \longrightarrow & V \\ & \searrow & \downarrow & & \downarrow \\ & & X & \longrightarrow & Y \end{array}$$

If f is not locally quasi-finite then $U \to V$ is not locally quasi-finite. Hence there exists a specialization $u \rightsquigarrow u'$ for some $u, u' \in U$ lying over the same point $v \in V$, see Morphisms, Lemma 28.21.6. We claim that u, u' do not have the same image in $X_v = \mathrm{Spec}(\kappa(v)) \times_Y X$ which will contradict the assumption that $|X_v|$ is discrete as desired. Let $d = \mathrm{trdeg}_{\kappa(v)}(\kappa(u))$ and $d' = \mathrm{trdeg}_{\kappa(v)}(\kappa(u'))$. Then we see that $d > d'$ by Morphisms, Lemma 28.29.6. Note that U_v (the fibre of $U \to V$ over v) is the fibre product of U and X_v over $X \times_Y V$, hence $U_v \to X_v$ is étale (as a base change of the étale morphism $U \to X \times_Y V$). If $u, u' \in U_v$ map to the same element of $|X_v|$ then there exists a point $r \in R_v = U_v \times_{X_v} U_v$ with $t(r) = u$ and $s(r) = u'$, see Properties of Spaces, Lemma 48.4.3. Note that $s, t : R_v \to U_v$ are étale morphisms of schemes over $\kappa(v)$, hence $\kappa(u) \subset \kappa(r) \supset \kappa(u')$ are finite separable extensions of fields over $\kappa(v)$ (see Morphisms, Lemma 28.37.7). We conclude that the transcendence degrees are equal. This contradiction finishes the proof. □

Lemma 49.25.5. *Let S be a scheme. Let $f : X \to Y$ be a morphism of algebraic spaces over S. The following are equivalent:*

1. *f is locally quasi-finite,*
2. *for every $x \in |X|$ the morphism f is quasi-finite at x,*
3. *for every scheme Z and any morphism $Z \to Y$ the morphism $Z \times_Y X \to Z$ is locally quasi-finite,*
4. *for every affine scheme Z and any morphism $Z \to Y$ the morphism $Z \times_Y X \to Z$ is locally quasi-finite,*
5. *there exists a scheme V and a surjective étale morphism $V \to Y$ such that $V \times_Y X \to V$ is locally quasi-finite,*
6. *there exists a scheme U and a surjective étale morphism $\varphi : U \to X$ such that the composition $f \circ \varphi$ is locally quasi-finite,*
7. *for every commutative diagram*

$$\begin{array}{ccc} U & \longrightarrow & V \\ \downarrow & & \downarrow \\ X & \longrightarrow & Y \end{array}$$

where U, V are schemes and the vertical arrows are étale the top horizontal arrow is locally quasi-finite,

(8) *there exists a commutative diagram*

$$\begin{array}{ccc} U & \longrightarrow & V \\ \downarrow & & \downarrow \\ X & \longrightarrow & Y \end{array}$$

where U, V are schemes, the vertical arrows are étale, and $U \to X$ is surjective such that the top horizontal arrow is locally quasi-finite, and

(9) *there exist Zariski coverings $Y = \bigcup_{i \in I} Y_i$, and $f^{-1}(Y_i) = \bigcup X_{ij}$ such that each morphism $X_{ij} \to Y_i$ is locally quasi-finite.*

Proof. Omitted. □

Lemma 49.25.6. *An immersion is locally quasi-finite.*

Proof. Omitted. □

Lemma 49.25.7. *Let S be a scheme. Let $X \to Y \to Z$ be morphisms of algebraic spaces over S. If $X \to Z$ is locally quasi-finite, then $X \to Y$ is locally quasi-finite.*

Proof. Choose a commutative diagram

$$\begin{array}{ccccc} U & \longrightarrow & V & \longrightarrow & W \\ \downarrow & & \downarrow & & \downarrow \\ X & \longrightarrow & Y & \longrightarrow & Z \end{array}$$

with vertical arrows étale and surjective. (See Spaces, Lemma 47.11.4.) Apply Morphisms, Lemma 28.21.15 to the top row. □

Lemma 49.25.8. *Let S be a scheme. Let $f : X \to Y$ be a morphism of algebraic spaces over S. If f is locally of finite type and a monomorphism, then f is separated and locally quasi-finite.*

Proof. A monomorphism is separated, see Lemma 49.11.3. By Lemma 49.25.5 it suffices to prove the lemma after performing a base change by $Z \to Y$ with Z affine. Hence we may assume that Y is an affine scheme. Choose an affine scheme U and an étale morphism $U \to X$. Since $X \to Y$ is locally of finite type the morphism of affine schemes $U \to Y$ is of finite type. Since $X \to Y$ is a monomorphism we have $U \times_X U = U \times_Y U$. In particular the maps $U \times_Y U \to U$ are étale. Let $y \in Y$. Then either U_y is empty, or $\mathrm{Spec}(\kappa(u)) \times_{\mathrm{Spec}(\kappa(y))} U_y$ is isomorphic to the fibre of $U \times_Y U \to U$ over u for some $u \in U$ lying over y. This implies that the fibres of $U \to Y$ are finite discrete sets (as $U \times_Y U \to U$ is an étale morphism of affine schemes, see Morphisms, Lemma 28.37.7). Hence $U \to Y$ is quasi-finite, see Morphisms, Lemma 28.21.6. As $U \to X$ was an arbitrary étale morphism with U affine this implies that $X \to Y$ is locally quasi-finite. □

49.26. Morphisms of finite presentation

The property "locally of finite presentation" of morphisms of schemes is étale local on the source-and-target, see Descent, Remark 34.28.7. It is also stable under base change and fpqc local on the target, see Morphisms, Lemma 28.22.4, and Descent, Lemma 34.19.9. Hence, by Lemma 49.21.1 above, we may define what it means for a morphism of algebraic spaces to be locally of finite presentation as follows and it

agrees with the already existing notion defined in Section 49.3 when the morphism is representable.

Definition 49.26.1. Let S be a scheme. Let $X \to Y$ be a morphism of algebraic spaces over S.

(1) We say f is *locally of finite presentation* if the equivalent conditions of Lemma 49.21.1 hold with $\mathcal{P}$ ="locally of finite presentation".
(2) Let $x \in |X|$. We say f is of *finite presentation at* x if there exists an open neighbourhood $X' \subset X$ of x such that $f|_{X'} : X' \to Y$ is locally of finite presentation[6].
(3) A morphism of algebraic spaces $f : X \to Y$ is *of finite presentation* if it is locally of finite presentation, quasi-compact and quasi-separated.

Note that a morphism of finite presentation is **not** just a quasi-compact morphism which is locally of finite presentation.

Lemma 49.26.2. *The composition of morphisms of finite presentation is of finite presentation. The same holds for locally of finite presentation.*

Proof. Omitted. □

Lemma 49.26.3. *A base change of a morphism of finite presentation is of finite presentation The same holds for locally of finite presentation.*

Proof. Omitted. □

Lemma 49.26.4. *Let S be a scheme. Let $f : X \to Y$ be a morphism of algebraic spaces over S. The following are equivalent:*

(1) *f is locally of finite presentation,*
(2) *for every $x \in |X|$ the morphism f is of finite presentation at x,*
(3) *for every scheme Z and any morphism $Z \to Y$ the morphism $Z \times_Y X \to Z$ is locally of finite presentation,*
(4) *for every affine scheme Z and any morphism $Z \to Y$ the morphism $Z \times_Y X \to Z$ is locally of finite presentation,*
(5) *there exists a scheme V and a surjective étale morphism $V \to Y$ such that $V \times_Y X \to V$ is locally of finite presentation,*
(6) *there exists a scheme U and a surjective étale morphism $\varphi : U \to X$ such that the composition $f \circ \varphi$ is locally of finite presentation,*
(7) *for every commutative diagram*

$$\begin{array}{ccc} U & \longrightarrow & V \\ \downarrow & & \downarrow \\ X & \longrightarrow & Y \end{array}$$

where U, V are schemes and the vertical arrows are étale the top horizontal arrow is locally of finite presentation,

[6]It seems awkward to use "locally of finite presentation at x", but the current terminology may be misleading in the sense that "of finite presentation at x" does **not** mean that there is an open neighbourhood $X' \subset X$ such that $f|_{X'}$ is of finite presentation.

(8) *there exists a commutative diagram*

$$\begin{array}{ccc} U & \longrightarrow & V \\ \downarrow & & \downarrow \\ X & \longrightarrow & Y \end{array}$$

where U, V are schemes, the vertical arrows are étale, and $U \to X$ is surjective such that the top horizontal arrow is locally of finite presentation, and

(9) *there exist Zariski coverings $Y = \bigcup_{i \in I} Y_i$, and $f^{-1}(Y_i) = \bigcup X_{ij}$ such that each morphism $X_{ij} \to Y_i$ is locally of finite presentation.*

Proof. Omitted. □

Lemma 49.26.5. *A morphism which is locally of finite presentation is locally of finite type. A morphism of finite presentation is of finite type.*

Proof. Let $f : X \to Y$ be a morphism of algebraic spaces which is locally of finite presentation. This means there exists a diagram as in Lemma 49.21.1 with h locally of finite presentation and surjective vertical arrow a. By Morphisms, Lemma 28.22.8 h is locally of finite type. Hence $X \to Y$ is locally of finite type by definition. If f is of finite presentation then it is quasi-compact and it follows that f is of finite type. □

Lemma 49.26.6. *Let S be a scheme. Let $f : X \to Y$ be a morphism of algebraic spaces over S. If f is of finite presentation and Y is Noetherian, then X is Noetherian.*

Proof. Assume f is of finite presentation and Y Noetherian. By Lemmas 49.26.5 and 49.22.5 we see that X is locally Noetherian. As f is quasi-compact and Y is quasi-compact we see that X is quasi-compact. As f is of finite presentation it is quasi-separated (see Definition 49.26.1) and as Y is Noetherian it is quasi-separated (see Properties of Spaces, Definition 48.22.1). Hence X is quasi-separated by Lemma 49.5.9. Hence we have checked all three conditions of Properties of Spaces, Definition 48.22.1 and we win. □

Lemma 49.26.7. *Let S be a scheme. Let $f : X \to Y$ be a morphism of algebraic spaces over S.*

(1) *If Y is locally Noetherian and f locally of finite type then f is locally of finite presentation.*

(2) *If Y is locally Noetherian and f of finite type and quasi-separated then f is of finite presentation.*

Proof. Assume $f : X \to Y$ locally of finite type and Y locally Noetherian. This means there exists a diagram as in Lemma 49.21.1 with h locally of finite type and surjective vertical arrow a. By Morphisms, Lemma 28.22.9 h is locally of finite presentation. Hence $X \to Y$ is locally of finite presentation by definition. This proves (1). If f is of finite type and quasi-separated then it is also quasi-compact and quasi-separated and (2) follows immediately. □

Lemma 49.26.8. *Let S be a scheme. Let Y be an algebraic space over S which is quasi-compact and quasi-separated. If X is of finite presentation over Y, then X is quasi-compact and quasi-separated.*

Proof. Omitted. □

Lemma 49.26.9. *Let S be a scheme. Let $f : X \to Y$ and $Y \to Z$ be morphisms of algebraic spaces over S. If X is locally of finite presentation over Z, and Y is locally of finite type over Z, then f is locally of finite presentation.*

Proof. Choose a scheme W and a surjective étale morphism $W \to Z$. Then choose a scheme V and a surjective étale morphism $V \to W \times_Z Y$. Finally choose a scheme U and a surjective étale morphism $U \to V \times_Y X$. By definition U is locally of finite presentation over W and V is locally of finite type over W. By Morphisms, Lemma 28.22.11 the morphism $U \to V$ is locally of finite presentation. Hence f is locally of finite presentation. □

Lemma 49.26.10. *Let S be a scheme. Let $f : X \to Y$ be a morphism of algebraic spaces over S with diagonal $\Delta : X \to X \times_Y X$. If f is locally of finite type then Δ is locally of finite presentation. If f is quasi-separated and locally of finite type, then Δ is of finite presentation.*

Proof. Note that Δ is a morphism over X (via the second projection $X \times_Y X \to X$). Assume f is locally of finite type. Note that X is of finite presentation over X and $X \times_Y X$ is of finite type over X (by Lemma 49.22.3). Thus the first statement holds by Lemma 49.26.9. The second statement follows from the first, the definitions, and the fact that a diagonal morphism is separated (Lemma 49.5.1). □

Lemma 49.26.11. *An open immersion of algebraic spaces is locally of finite presentation.*

Proof. An open immersion is by definition representable, hence we can use the general principle Spaces, Lemma 47.5.8 and Morphisms, Lemma 28.22.5. □

Lemma 49.26.12. *A closed immersion $i : Z \to X$ is of finite presentation if and only if the associated quasi-coherent sheaf of ideals $\mathcal{I} = Ker(\mathcal{O}_X \to i_*\mathcal{O}_Z)$ is of finite type (as an $\mathcal{O}_X$-module).*

Proof. Let U be a scheme and let $U \to X$ be a surjective étale morphism. By Lemma 49.26.4 we see that $i' : Z \times_X U \to U$ is of finite presentation if and only if i is. By Properties of Spaces, Section 48.28 we see that $\mathcal{I}$ is of finite type if and only if $\mathcal{I}|_U = \mathrm{Ker}(\mathcal{O}_U \to i'_*\mathcal{O}_{Z \times_X U})$ is. Hence the result follows from the case of schemes, see Morphisms, Lemma 28.22.7. □

49.27. Flat morphisms

The property "flat" of morphisms of schemes is étale local on the source-and-target, see Descent, Remark 34.28.7. It is also stable under base change and fpqc local on the target, see Morphisms, Lemma 28.26.7 and Descent, Lemma 34.19.13. Hence, by Lemma 49.21.1 above, we may define the notion of a flat morphism of algebraic spaces as follows and it agrees with the already existing notion defined in Section 49.3 when the morphism is representable.

Definition 49.27.1. Let S be a scheme. Let $f : X \to Y$ be a morphism of algebraic spaces over S.

(1) We say f is *flat* if the equivalent conditions of Lemma 49.21.1 with $\mathcal{P}$ = "flat".
(2) Let $x \in |X|$. We say f is *flat at* x if the equivalent conditions of Lemma 49.21.3 holds with $\mathcal{Q}$ = "induced map local rings is flat".

Note that the second part makes sense by Descent, Lemma 34.29.4.

We do a quick sanity check.

Lemma 49.27.2. *Let S be a scheme. Let $f : X \to Y$ be a morphism of algebraic spaces over S. Then f is flat if and only if f is flat at all points of $|X|$.*

Proof. Choose a commutative diagram

$$\begin{array}{ccc} U & \xrightarrow{h} & V \\ {\scriptstyle a}\downarrow & & \downarrow{\scriptstyle b} \\ X & \xrightarrow{f} & Y \end{array}$$

where U and V are schemes, the vertical arrows are étale, and a is surjective. By definition f is flat if and only h is flat (Definition 49.21.2). By definition f is flat at $x \in |X|$ if and only if h is flat at some (equivalently any) $u \in U$ which maps to x (Definition 49.21.4). Thus the lemma follows from the fact that a morphism of schemes is flat if and only if it is flat at all points of the source (Morphisms, Definition 28.26.1). □

Lemma 49.27.3. *The composition of flat morphisms is flat.*

Proof. Omitted. □

Lemma 49.27.4. *The base change of a flat morphism is flat.*

Proof. Omitted. □

Lemma 49.27.5. *Let S be a scheme. Let $f : X \to Y$ be a morphism of algebraic spaces over S. The following are equivalent:*

1. *f is flat,*
2. *for every $x \in |X|$ the morphism f is flat at x,*
3. *for every scheme Z and any morphism $Z \to Y$ the morphism $Z \times_Y X \to Z$ is flat,*
4. *for every affine scheme Z and any morphism $Z \to Y$ the morphism $Z \times_Y X \to Z$ is flat,*
5. *there exists a scheme V and a surjective étale morphism $V \to Y$ such that $V \times_Y X \to V$ is flat,*
6. *there exists a scheme U and a surjective étale morphism $\varphi : U \to X$ such that the composition $f \circ \varphi$ is flat,*
7. *for every commutative diagram*

$$\begin{array}{ccc} U & \longrightarrow & V \\ \downarrow & & \downarrow \\ X & \longrightarrow & Y \end{array}$$

   where U, V are schemes and the vertical arrows are étale the top horizontal arrow is flat,
8. *there exists a commutative diagram*

$$\begin{array}{ccc} U & \longrightarrow & V \\ \downarrow & & \downarrow \\ X & \longrightarrow & Y \end{array}$$

where U, V are schemes, the vertical arrows are étale, and $U \to X$ is surjective such that the top horizontal arrow is flat, and

(9) *there exists a Zariski coverings $Y = \bigcup Y_i$ and $f^{-1}(Y_i) = \bigcup X_{ij}$ such that each morphism $X_{ij} \to Y_i$ is flat.*

Proof. Omitted. □

Lemma 49.27.6. *A flat morphism locally of finite presentation is universally open.*

Proof. Let $f : X \to Y$ be a flat morphism locally of finite presentation of algebraic spaces over S. Choose a diagram

$$\begin{array}{ccc} U & \xrightarrow[\alpha]{} & V \\ \downarrow & & \downarrow \\ X & \longrightarrow & Y \end{array}$$

where U and V are schemes and the vertical arrows are surjective and étale, see Spaces, Lemma 47.11.4. By Lemmas 49.27.5 and 49.26.4 the morphism α is flat and locally of finite presentation. Hence by Morphisms, Lemma 28.26.9 we see that α is universally open. Hence $X \to Y$ is universally open according to Lemma 49.7.5. □

Lemma 49.27.7. *Let S be a scheme. Let $f : X \to Y$ be a flat, quasi-compact, surjective morphism of algebraic spaces over S. A subset $T \subset |Y|$ is open (resp. closed) if and only $f^{-1}(|T|)$ is open (resp. closed) in $|X|$. In other words f is submersive, and in fact universally submersive.*

Proof. Choose affine schemes V_i and étale morphisms $V_i \to Y$ such that $V = \coprod V_i \to Y$ is surjective, see Properties of Spaces, Lemma 48.6.1. For each i the algebraic space $V_i \times_Y X$ is quasi-compact. Hence we can find an affine scheme U_i and a surjective étale morphism $U_i \to V_i \times_Y X$, see Properties of Spaces, Lemma 48.6.3. Then the composition $U_i \to V_i \times_Y X \to V_i$ is a surjective, flat morphism of affines. Of course then $U = \coprod U_i \to X$ is surjective and étale and $U = V \times_Y X$. Moreover, the morphism $U \to V$ is the disjoint union of the morphisms $U_i \to V_i$. Hence $U \to V$ is surjective, quasi-compact and flat. Consider the diagram

$$\begin{array}{ccc} U & \longrightarrow & X \\ \downarrow & & \downarrow \\ V & \longrightarrow & Y \end{array}$$

By definition of the topology on $|Y|$ the set T is closed (resp. open) if and only if $g^{-1}(T) \subset |V|$ is closed (resp. open). The same holds for $f^{-1}(T)$ and its inverse image in $|U|$. Since $U \to V$ is quasi-compact, surjective, and flat we win by Morphisms, Lemma 28.26.10. □

Lemma 49.27.8. *Let S be a scheme. Let $f : X \to Y$ be a morphism of algebraic spaces over S. Let $\overline{x}$ be a geometric point of X lying over the point $x \in |X|$. Let $\overline{y} = f \circ \overline{x}$. The following are equivalent*

(1) *f is flat at x, and*
(2) *the map on étale local rings $\mathcal{O}_{Y,\overline{y}} \to \mathcal{O}_{X,\overline{x}}$ is flat.*

Proof. Choose a commutative diagram

$$\begin{array}{ccc} U & \xrightarrow{h} & V \\ {\scriptstyle a}\downarrow & & \downarrow{\scriptstyle b} \\ X & \xrightarrow{f} & Y \end{array}$$

where U and V are schemes, a, b are étale, and $u \in U$ mapping to x. We can find a geometric point $\overline{u} : \mathrm{Spec}(k) \to U$ lying over u with $\overline{x} = a \circ \overline{u}$, see Properties of Spaces, Lemma 48.16.4. Set $\overline{v} = h \circ \overline{u}$ with image $v \in V$. We know that

$$\mathcal{O}_{X,\overline{x}} = \mathcal{O}^{sh}_{U,u} \quad \text{and} \quad \mathcal{O}_{Y,\overline{y}} = \mathcal{O}^{sh}_{V,v}$$

see Properties of Spaces, Lemma 48.19.1. We obtain a commutative diagram

$$\begin{array}{ccc} \mathcal{O}_{U,u} & \longrightarrow & \mathcal{O}_{X,\overline{x}} \\ \uparrow & & \uparrow \\ \mathcal{O}_{V,v} & \longrightarrow & \mathcal{O}_{Y,\overline{y}} \end{array}$$

of local rings with flat horizontal arrows. We have to show that the left vertical arrow is flat if and only if the right vertical arrow is. Algebra, Lemma 10.36.8 tells us $\mathcal{O}_{U,u}$ is flat over $\mathcal{O}_{V,v}$ if and only if $\mathcal{O}_{X,\overline{x}}$ is flat over $\mathcal{O}_{V,v}$. Hence the result follows from More on Flatness, Lemma 37.2.5. □

Lemma 49.27.9. *Let S be a scheme. Let $f : X \to Y$ be a morphism of algebraic spaces over S. Then f is flat if and only if the morphism of sites $(f_{small}, f^\sharp) : (X_{étale}, \mathcal{O}_X) \to (Y_{étale}, \mathcal{O}_Y)$ associated to f is flat.*

Proof. Flatness of $(f_{small}, f^\sharp)$ is defined in terms of flatness of $\mathcal{O}_X$ as a $f^{-1}_{small}\mathcal{O}_Y$-module. This can be checked at stalks, see Modules on Sites, Lemma 18.38.2 and Properties of Spaces, Theorem 48.16.12. But we've already seen that flatness of f can be checked on stalks, see Lemma 49.27.8. □

Lemma 49.27.10. *Let S be a scheme. Let $f : Y \to X$ be a morphism of algebraic spaces over S. Let $\mathcal{F}$ be a finite type quasi-coherent $\mathcal{O}_X$-module with scheme theoretic support $Z \subset X$. If f is flat, then $f^{-1}(Z)$ is the scheme theoretic support of $f^*\mathcal{F}$.*

Proof. Using the characterization of the scheme theoretic support as given in Lemma 49.15.3 and using the characterization of flat morphisms in terms of étale coverings in Lemma 49.27.5 we reduce to the case of schemes which is Morphisms, Lemma 28.26.12. □

Lemma 49.27.11. *Let S be a scheme. Let $f : X \to Y$ be a flat morphism of algebraic spaces over S. Let $V \to Y$ be a quasi-compact open immersion. If V is scheme theoretically dense in Y, then $f^{-1}V$ is scheme theoretically dense in X.*

Proof. Using the characterization of scheme theoretically dense opens in Lemma 49.17.2 and using the characterization of flat morphisms in terms of étale coverings in Lemma 49.27.5 we reduce to the case of schemes which is Morphisms, Lemma 28.26.13. □

Lemma 49.27.12. *Let S be a scheme. Let $f : X \to Y$ be a flat morphism of algebraic spaces over S. Let $g : V \to Y$ be a quasi-compact morphism of algebraic spaces. Let $Z \subset Y$ be the scheme theoretic image of g and let $Z' \subset X$ be the scheme theoretic image of the base change $V \times_Y X \to X$. Then $Z' = f^{-1}Z$.*

Proof. Let $Y' \to Y$ be a surjective étale morphism such that Y' is a disjoint union of affine schemes (Properties of Spaces, Lemma 48.6.1). Let $X' \to X \times_Y Y'$ be a surjective étale morphism such that X' is a disjoint union of affine schemes. By Lemma 49.27.5 the morphism $X' \to Y'$ is flat. Set $V' = V \times_Y Y'$. By Lemma 49.16.3 the inverse image of Z in Y' is the scheme theoretic image of $V' \to Y'$ and the inverse image of Z' in X' is the scheme theoretic image of $V' \times_{Y'} X' \to X'$. Since $X' \to X$ is surjective étale, it suffices to prove the result in the case of the morphisms $X' \to Y'$ and $V' \to Y'$. Thus we may assume X and Y are affine schemes. In this case V is a quasi-compact algebraic space. Choose an affine scheme W and a surjective étale morphism $W \to V$ (Properties of Spaces, Lemma 48.6.3). It is clear that the scheme theoretic image of $V \to Y$ agrees with the scheme theoretic image of $W \to Y$ and similarly for $V \times_Y X \to Y$ and $W \times_Y X \to X$. Thus we reduce to the case of schemes which is Morphisms, Lemma 28.26.14. □

49.28. Flat modules

In this section we define what it means for a module to be flat at a point. To do this we will use the notion of the stalk of a sheaf on the small étale site $X_{étale}$ of an algebraic space, see Properties of Spaces, Definition 48.16.6.

Lemma 49.28.1. *Let S be a scheme. Let $f : X \to Y$ be a morphism of algebraic spaces over S. Let $\mathcal{F}$ be a quasi-coherent sheaf on X. Let $x \in |X|$. The following are equivalent*

(1) *for some commutative diagram*

$$\begin{array}{ccc} U & \xrightarrow{h} & V \\ {\scriptstyle a}\downarrow & & \downarrow{\scriptstyle b} \\ X & \xrightarrow{f} & Y \end{array}$$

where U and V are schemes, a, b are étale, and $u \in U$ mapping to x the module $a^*\mathcal{F}$ is flat at u over V,

(2) *the stalk $\mathcal{F}_{\overline{x}}$ is flat over the étale local ring $\mathcal{O}_{Y,\overline{y}}$ where $\overline{x}$ is any geometric point lying over x and $\overline{y} = f \circ \overline{x}$.*

Proof. During this proof we fix a geometric proof $\overline{x} : \mathrm{Spec}(k) \to X$ over x and we denote $\overline{y} = f \circ \overline{x}$ its image in Y. Given a diagram as in (1) we can find a geometric point $\overline{u} : \mathrm{Spec}(k) \to U$ lying over u with $\overline{x} = a \circ \overline{u}$, see Properties of Spaces, Lemma 48.16.4. Set $\overline{v} = h \circ \overline{u}$ with image $v \in V$. We know that

$$\mathcal{O}_{X,\overline{x}} = \mathcal{O}^{sh}_{U,u} \quad \text{and} \quad \mathcal{O}_{Y,\overline{y}} = \mathcal{O}^{sh}_{V,v}$$

see Properties of Spaces, Lemma 48.19.1. We obtain a commutative diagram

$$\begin{array}{ccc} \mathcal{O}_{U,u} & \longrightarrow & \mathcal{O}_{X,\overline{x}} \\ \uparrow & & \uparrow \\ \mathcal{O}_{V,v} & \longrightarrow & \mathcal{O}_{Y,\overline{y}} \end{array}$$

of local rings. Finally, we have

$$\mathcal{F}_{\overline{x}} = (\varphi^*\mathcal{F})_u \otimes_{\mathcal{O}_{U,u}} \mathcal{O}_{X,\overline{x}}$$

by Properties of Spaces, Lemma 48.27.4. Thus Algebra, Lemma 10.36.8 tells us $(\varphi^*\mathcal{F})_u$ is flat over $\mathcal{O}_{V,v}$ if and only if $\mathcal{F}_{\overline{x}}$ is flat over $\mathcal{O}_{V,v}$. Hence the result follows from More on Flatness, Lemma 37.2.5. □

Definition 49.28.2. Let S be a scheme. Let $f : X \to Y$ be a morphism of algebraic spaces over S. Let $\mathcal{F}$ be a quasi-coherent sheaf on X.

(1) Let $x \in |X|$. We say $\mathcal{F}$ is *flat at x over Y* if the equivalent conditions of Lemma 49.28.1 hold.
(2) We say $\mathcal{F}$ is *flat over Y* if $\mathcal{F}$ is flat over Y at all $x \in |X|$.

Having defined this we have the obligatory base change lemma. This lemma implies that formation of the flat locus of a quasi-coherent sheaf commutes with flat base change.

Lemma 49.28.3. *Let S be a scheme. Let*

$$\begin{array}{ccc} X' & \xrightarrow{g'} & X \\ {\scriptstyle f'}\downarrow & & \downarrow{\scriptstyle f} \\ Y' & \xrightarrow{g} & Y \end{array}$$

be a cartesian diagram of algebraic spaces over S. Let $x' \in |X'|$ with image $x \in |X|$. Let $\mathcal{F}$ be a quasi-coherent sheaf on X and denote $\mathcal{F}' = (g')^*\mathcal{F}$.

(1) *If $\mathcal{F}$ is flat at x over Y then $\mathcal{F}'$ is flat at x' over Y'.*
(2) *If g is flat at $f'(x')$ and $\mathcal{F}'$ is flat at x' over Y', then $\mathcal{F}$ is flat at x over Y.*

In particular, if $\mathcal{F}$ is flat over Y, then $\mathcal{F}'$ is flat over Y'.

Proof. Choose a scheme V and a surjective étale morphism $V \to Y$. Choose a scheme U and a surjective étale morphism $U \to V \times_Y X$. Choose a scheme V' and a surjective étale morphism $V' \to V \times_Y Y'$. Then $U' = V' \times_V U$ is a scheme endowed with a surjective étale morphism $U' = V' \times_V U \to Y' \times_Y X = X'$. Pick $u' \in U'$ mapping to $x' \in |X'|$. Then we can check flatness of $\mathcal{F}'$ at x' over Y' in terms of flatness of $\mathcal{F}'|_{U'}$ at u' over V'. Hence the lemma follows from More on Morphisms, Lemma 36.12.2. □

The following lemma discusses "composition" of flat morphisms in terms of modules. It also shows that flatness satisfies a kind of top down descent.

Lemma 49.28.4. *Let S be a scheme. Let $X \to Y \to Z$ be morphisms of algebraic spaces over S. Let $\mathcal{F}$ be a quasi-coherent sheaf on X. Let $x \in |X|$ with image $y \in |Y|$.*

(1) *If $\mathcal{F}$ is flat at x over Y and Y is flat at y over Z, then $\mathcal{F}$ is flat at x over Z.*
(2) *Let $x : \mathrm{Spec}(K) \to X$ be a representative of x. If*
   (a) *$\mathcal{F}$ is flat at x over Y,*
   (b) *$x^*\mathcal{F} \neq 0$, and*
   (c) *$\mathcal{F}$ is flat at x over Z,*

   then Y is flat at y over Z.

(3) *Let $\overline{x}$ be a geometric point of X lying over x with image $\overline{y}$ in Y. If $\mathcal{F}_{\overline{x}}$ is a faithfully flat $\mathcal{O}_{Y,\overline{y}}$-module and $\mathcal{F}$ is flat at x over Z, then Y is flat at y over Z.*

Proof. Pick $\overline{x}$ and $\overline{y}$ as in part (3) and denote $\overline{z}$ the induced geometric point of Z. Via the characterization of flatness in Lemmas 49.28.1 and 49.27.8 the lemma reduces to a purely algebraic question on the local ring map $\mathcal{O}_{Z,\overline{z}} \to \mathcal{O}_{Y,\overline{y}}$ and the module $\mathcal{F}_{\overline{x}}$. Part (1) follows from Algebra, Lemma 10.36.3. We remark that condition (2)(b) guarantees that $\mathcal{F}_{\overline{x}}/\mathfrak{m}_{\overline{y}}\mathcal{F}_{\overline{x}}$ is nonzero. Hence (2)(a) + (2)(b) imply that $\mathcal{F}_{\overline{x}}$ is a faithfully flat $\mathcal{O}_{Y,\overline{y}}$-module, see Algebra, Lemma 10.36.14. Thus (2) is a special case of (3). Finally, (3) follows from Algebra, Lemma 10.36.9. □

Sometimes the base change happens "up on top". Here is a precise statement.

Lemma 49.28.5. *Let S be a scheme. Let $f : X \to Y$, $g : Y \to Z$ be morphisms of algebraic spaces over S. Let $\mathcal{G}$ be a quasi-coherent sheaf on Y. Let $x \in |X|$ with image $y \in |Y|$. If f is flat at x, then*

$$\mathcal{G} \text{ flat over } Z \text{ at } y \Leftrightarrow f^*\mathcal{G} \text{ flat over } Z \text{ at } x.$$

In particular: If f is surjective and flat, then $\mathcal{G}$ is flat over Z, if and only if $f^*\mathcal{G}$ is flat over Z.

Proof. Pick a geometric point $\overline{x}$ of X and denote $\overline{y}$ the image in Y and $\overline{z}$ the image in Z. Via the characterization of flatness in Lemmas 49.28.1 and 49.27.8 and the description of the stalk of $f^*\mathcal{G}$ at $\overline{x}$ of Properties of Spaces, Lemma 48.27.5 the lemma reduces to a purely algebraic question on the local ring maps $\mathcal{O}_{Z,\overline{z}} \to \mathcal{O}_{Y,\overline{y}} \to \mathcal{O}_{X,\overline{x}}$ and the module $\mathcal{G}_{\overline{y}}$. This algebraic statement is Algebra, Lemma 10.36.8. □

49.29. Generic flatness

This section is the analogue of Morphisms, Section 28.28.

Proposition 49.29.1. *Let S be a scheme. Let $f : X \to Y$ be a morphism of algebraic spaces over S. Let $\mathcal{F}$ be a quasi-coherent sheaf of $\mathcal{O}_X$-modules. Assume*

(1) *Y is reduced,*
(2) *f is of finite type, and*
(3) *$\mathcal{F}$ is a finite type $\mathcal{O}_X$-module.*

Then there exists an open dense subspace $W \subset Y$ such that the base change $X_W \to W$ of f is flat, locally of finite presentation, and quasi-compact and such that $\mathcal{F}|_{X_W}$ is flat over W and of finite presentation over $\mathcal{O}_{X_W}$.

Proof. Let V be a scheme and let $V \to Y$ be a surjective étale morphism. Let $X_V = V \times_Y X$ and let $\mathcal{F}_V$ be the restriction of $\mathcal{F}$ to X_V. Suppose that the result holds for the morphism $X_V \to V$ and the sheaf $\mathcal{F}_V$. Then there exists an open subscheme $V' \subset V$ such that $X_{V'} \to V'$ is flat and of finite presentation and $\mathcal{F}_{V'}$ is an $\mathcal{O}_{X_{V'}}$-module of finite presentation flat over V'. Let $W \subset Y$ be the image of the étale morphism $V' \to Y$, see Properties of Spaces, Lemma 48.4.10. Then $V' \to W$ is a surjective étale morphism, hence we see that $X_W \to W$ is flat, locally of finite presentation, and quasi-compact by Lemmas 49.26.4, 49.27.5, and 49.9.7. By the discussion in Properties of Spaces, Section 48.28 we see that $\mathcal{F}_W$ is of finite presentation as a $\mathcal{O}_{X_W}$-module and by Lemma 49.28.3 we see that $\mathcal{F}_W$ is flat over W. This argument reduces the proposition to the case where Y is a scheme.

Suppose we can prove the proposition when Y is an affine scheme. Let $f : X \to Y$ be a finite type morphism of algebraic spaces over S with Y a scheme, and let $\mathcal{F}$ be a finite type, quasi-coherent $\mathcal{O}_X$-module. Choose an affine open covering $Y = \bigcup V_j$. By assumption we can find dense open $W_j \subset V_j$ such that $X_{W_j} \to W_j$ is flat, locally of finite presentation, and quasi-compact and such that $\mathcal{F}|_{X_{W_j}}$ is flat over W_j and of finite presentation as an $\mathcal{O}_{X_{W_j}}$-module. In this situation we simply take $W = \bigcup W_j$ and we win. Hence we reduce the proposition to the case where Y is an affine scheme.

Let Y be an affine scheme over S, let $f : X \to Y$ be a finite type morphism of algebraic spaces over S, and let $\mathcal{F}$ be a finite type, quasi-coherent $\mathcal{O}_X$-module. Since f is of finite type it is quasi-compact, hence X is quasi-compact. Thus we can find an affine scheme U and a surjective étale morphism $U \to X$, see Properties of Spaces, Lemma 48.6.3. Note that $U \to Y$ is of finite type (this is what it means for f to be of finite type in this case). Hence we can apply Morphisms, Proposition 28.28.2 to see that there exists a dense open $W \subset Y$ such that $U_W \to W$ is flat and of finite presentation and such that $\mathcal{F}|_{U_W}$ is flat over W and of finite presentation as an $\mathcal{O}_{U_W}$-module. According to our definitions this means that the base change $X_W \to W$ of f is flat, locally of finite presentation, and quasi-compact and $\mathcal{F}|_{X_W}$ is flat over W and of finite presentation over $\mathcal{O}_{X_W}$. □

We cannot improve the result of the lemma above to requiring $X_W \to W$ to be of finite presentation as $\mathbf{A}^1_{\mathbf{Q}}/\mathbf{Z} \to \mathrm{Spec}(\mathbf{Q})$ gives a counter example. The problem is that the diagonal morphism $\Delta_{X/Y}$ may not be quasi-compact, i.e., f may not be quasi-separated. Clearly, this is also the only problem.

Proposition 49.29.2. *Let S be a scheme. Let $f : X \to Y$ be a morphism of algebraic spaces over S. Let $\mathcal{F}$ be a quasi-coherent sheaf of $\mathcal{O}_X$-modules. Assume*

(1) *Y is reduced,*
(2) *f is quasi-separated,*
(3) *f is of finite type, and*
(4) *$\mathcal{F}$ is a finite type $\mathcal{O}_X$-module.*

Then there exists an open dense subspace $W \subset Y$ such that the base change $X_W \to W$ of f is flat and of finite presentation and such that $\mathcal{F}|_{X_W}$ is flat over W and of finite presentation over $\mathcal{O}_{X_W}$.

Proof. This follows immediately from Proposition 49.29.1 and the fact that "of finite presentation" = "locally of finite presentation" + "quasi-compact" + "quasi-separated". □

49.30. Relative dimension

In this section we define the relative dimension of a morphism of algebraic spaces at a point, and some closely related properties.

Definition 49.30.1. Let S be a scheme. Let $f : X \to Y$ be a morphism of algebraic spaces over S. Let $x \in |X|$. Let $d, r \in \{0, 1, 2, \ldots, \infty\}$.

(1) We say the *dimension of the local ring of the fibre of f at x* is d if the equivalent conditions of Lemma 49.21.3 hold for the property $\mathcal{P}_d$ described in Descent, Lemma 34.29.6.

(2) We say the *transcendence degree of $x/f(x)$* is r if the equivalent conditions of Lemma 49.21.3 hold for the property $\mathcal{P}_r$ described in Descent, Lemma 34.29.7.
(3) We say the *f has relative dimension d at x* if the equivalent conditions of Lemma 49.21.3 hold for the property $\mathcal{P}_d$ described in Descent, Lemma 34.29.8.

Let us spell out what this means. Namely, choose some diagrams

$$\begin{array}{ccc} U & \xrightarrow{h} & V \\ {\scriptstyle a}\downarrow & & \downarrow{\scriptstyle b} \\ X & \xrightarrow{f} & Y \end{array} \qquad \begin{array}{ccc} u & \longrightarrow & v \\ \downarrow & & \downarrow \\ x & \longrightarrow & y \end{array}$$

as in Lemma 49.21.3. Then we have

$$\begin{array}{rcc} \text{relative dimension of } f \text{ at } x & = & \dim_u(U_v) \\ \text{dimension of local ring of the fibre of } f \text{ at } x & = & \dim(\mathcal{O}_{U_v,u}) \\ \text{transcendence degree of } x/f(x) & = & \operatorname{trdeg}_{\kappa(v)}(\kappa(u)) \end{array}$$

Note that if $Y = \operatorname{Spec}(k)$ is the spectrum of a field, then the relative dimension of X/Y at x is the same as $\dim_x(X)$, the transcendence degree of $x/f(x)$ is the transcendence degree over k, and the dimension of the local ring of the fibre of f at x is just the dimension of the local ring at x, i.e., the relative notions become absolute notions in that case.

Definition 49.30.2. Let S be a scheme. Let $f : X \to Y$ be a morphism of algebraic spaces over S. Let $d \in \{0, 1, 2, \ldots\}$.

(1) We say f has *relative dimension $\leq d$* if f has relative dimension $\leq d$ at all $x \in |X|$.
(2) We say f has *relative dimension d* if f has relative dimension d at all $x \in |X|$.

Having relative dimension *equal* to d means roughly speaking that all nonempty fibres are equidimensional of dimension d.

Lemma 49.30.3. *Let S be a scheme. Let $X \to Y \to Z$ be morphisms of algebraic spaces over S. Let $x \in |X|$ and let $y \in |Y|$, $z \in |Z|$ be the images. Assume $X \to Y$ is locally quasi-finite and $Y \to Z$ locally of finite type. Then the transcendence degree of x/z is equal to the transcendence degree of y/z.*

Proof. We can choose commutative diagrams

$$\begin{array}{ccccc} U & \longrightarrow & V & \longrightarrow & W \\ \downarrow & & \downarrow & & \downarrow \\ X & \longrightarrow & Y & \longrightarrow & Z \end{array} \qquad \begin{array}{ccccc} u & \longrightarrow & v & \longrightarrow & w \\ \downarrow & & \downarrow & & \downarrow \\ x & \longrightarrow & y & \longrightarrow & z \end{array}$$

where U, V, W are schemes and the vertical arrows are étale. By definition the morphism $U \to V$ is locally quasi-finite which implies that $\kappa(v) \subset \kappa(u)$ is finite, see Morphisms, Lemma 28.21.5. Hence the result is clear. □

49.31. Morphisms and dimensions of fibres

This section is the analogue of Morphisms, Section 28.29. The formulations in this section are a bit awkward since we do not have local rings of algebraic spaces at points.

Lemma 49.31.1. *Let S be a scheme. Let $f : X \to Y$ be a morphism of algebraic spaces over S. Let $x \in |X|$. Assume f is locally of finite type. Then we have*

$$\begin{array}{c} \textit{relative dimension of } f \textit{ at } x \\ = \\ \textit{dimension of local ring of the fibre of } f \textit{ at } x \\ + \\ \textit{transcendence degree of } x/f(x) \end{array}$$

where the notation is as in Definition 49.30.1.

Proof. This follows immediately from Morphisms, Lemma 28.29.1 applied to $h : U \to V$ and $u \in U$ as in Lemma 49.21.3. □

Lemma 49.31.2. *Let S be a scheme. Let $f : X \to Y$ and $g : Y \to Z$ be morphisms of algebraic spaces over S. Let $x \in |X|$ and set $y = f(x)$. Assume f and g locally of finite type. Then*

$$\begin{array}{c} \textit{relative dimension of } g \circ f \textit{ at } x \\ \leq \\ \textit{relative dimension of } f \textit{ at } x \\ + \\ \textit{relative dimension of } g \textit{ at } y \end{array}$$

Moreover, equality holds if for some morphism $\mathrm{Spec}(k) \to Z$ from the spectrum of a field in the class of $g(f(x)) = g(y)$ The morphism $X_k \to Y_k$ is flat at x. This holds for example if f is flat at x.

Proof. Choose a diagram

$$\begin{array}{ccccc} U & \longrightarrow & V & \longrightarrow & W \\ \downarrow & & \downarrow & & \downarrow \\ X & \longrightarrow & Y & \longrightarrow & Z \end{array}$$

with U, V, W schemes and vertical arrows étale and surjective. (See Spaces, Lemma 47.11.4.) Choose $u \in U$ mapping to x. Set v, w equal to the images of u in V, W. Apply Morphisms, Lemma 28.29.2 to the top row and the points u, v, w. Details omitted. □

Lemma 49.31.3. *Let S be a scheme. Let*

$$\begin{array}{ccc} X' & \xrightarrow[g']{} & X \\ {\scriptstyle f'}\downarrow & & \downarrow{\scriptstyle f} \\ Y' & \xrightarrow{g} & Y \end{array}$$

be a fibre product diagram of algebraic spaces over S. Let $x' \in |X'|$. Set $x = g'(x')$. Assume f locally of finite type. Then we have

$$\begin{array}{c} \textit{relative dimension of } f \textit{ at } x \\ = \\ \textit{relative dimension of } f' \textit{ at } x' \end{array}$$

Proof. Choose a surjective étale morphism $V \to Y$ with V a scheme. By Spaces, Lemma 47.11.4 we may choose morphisms of schemes $V' \to V$ lifting the morphism g and $U \to V$ lifting the morphism f such that $V' \to Y'$ and $U \to X$ are also surjective and étale. Set $U' = V' \times_V U$. Then the induced morphism $U' \to X'$ is also surjective and étale (argument omitted). Hence we can choose a $u' \in U'$ mapping to x'. At this point the result follows by applying Morphisms, Lemma 28.29.3 to the diagram of schemes involving U', U, V', V and the point u'. □

Lemma 49.31.4. *Let S be a scheme. Let $f : X \to Y$ be a morphism of algebraic spaces over S. Let $n \geq 0$. Assume f is locally of finite type. The set*

$$W_n = \{x \in |X| \text{ such that the relative dimension of } f \text{ at } x \leq n\}$$

is open in $|X|$.

Proof. Choose a diagram

$$\begin{array}{ccc} U & \xrightarrow{h} & V \\ {\scriptstyle a}\downarrow & & \downarrow \\ X & \longrightarrow & Y \end{array}$$

where U and V are schemes and the vertical arrows are surjective and étale, see Spaces, Lemma 47.11.4. By Morphisms, Lemma 28.29.4 the set U_n of points where h has relative dimension $\leq n$ is open in U. By our definition of relative dimension for morphisms of algebraic spaces at points we see that $U_n = a^{-1}(W_n)$. The lemma follows by definition of the topology on $|X|$. □

Lemma 49.31.5. *Let S be a scheme. Let $f : X \to Y$ be a morphism of algebraic spaces over S Let $n \geq 0$. Assume f is locally of finite presentation. The open*

$$W_n = \{x \in |X| \text{ such that the relative dimension of } f \text{ at } x \leq n\}$$

of Lemma 49.31.4 is retrocompact in $|X|$. (See Topology, Definition 5.11.1.)

Proof. Choose a diagram

$$\begin{array}{ccc} U & \xrightarrow{h} & V \\ {\scriptstyle a}\downarrow & & \downarrow \\ X & \longrightarrow & Y \end{array}$$

where U and V are schemes and the vertical arrows are surjective and étale, see Spaces, Lemma 47.11.4. In the proof of Lemma 49.31.4 we have seen that $a^{-1}(W_n) = U_n$ is the corresponding set for the morphism h. By Morphisms, Lemma 28.29.5 we see that U_n is retrocompact in U. The lemma follows by definition of the topology on $|X|$, compare with Properties of Spaces, Lemma 48.5.5 and its proof. □

Lemma 49.31.6. *Let S be a scheme. Let $f : X \to Y$ be a morphism of algebraic spaces over S. Assume f is locally of finite type. Then f is locally quasi-finite if and only if f has relative dimension 0 at each $x \in |X|$.*

Proof. Choose a diagram

$$\begin{array}{ccc} U & \xrightarrow[h]{} & V \\ {\scriptstyle a}\downarrow & & \downarrow \\ X & \longrightarrow & Y \end{array}$$

where U and V are schemes and the vertical arrows are surjective and étale, see Spaces, Lemma 47.11.4. The definitions imply that h is locally quasi-finite if and only if f is locally quasi-finite, and that f has relative dimension 0 at all $x \in |X|$ if and only if h has relative dimension 0 at all $u \in U$. Hence the result follows from the result for h which is Morphisms, Lemma 28.30.5. □

Lemma 49.31.7. *Let S be a scheme. Let $f : X \to Y$ be a morphism of algebraic spaces over S. Assume f is locally of finite type. Then there exists a canonical open subspace $X' \subset X$ such that $f|_{X'} : X' \to Y$ is locally quasi-finite, and such that the relative dimension of f at any $x \in |X|$, $x \notin |X'|$ is ≥ 1. Formation of X' commutes with arbitrary base change.*

Proof. Combine Lemmas 49.31.4, 49.31.6, and 49.31.3. □

Lemma 49.31.8. *Let S be a scheme. Consider a cartesian diagram*

$$\begin{array}{ccc} X & \xleftarrow[p]{} & F \\ \downarrow & & \downarrow \\ Y & \longleftarrow & \mathrm{Spec}(k) \end{array}$$

where $X \to Y$ is a morphism of algebraic spaces over S which is locally of finite type and where k is a field over S. Let $z \in |F|$ be such that $\dim_z(F) = 0$. Then, after replacing X by an open subspace containing $p(z)$, the morphism

$$X \longrightarrow Y$$

is locally quasi-finite.

Proof. Let $X' \subset X$ be the open subspace over which f is locally quasi-finite found in Lemma 49.31.7. Since the formation of X' commutes with arbitrary base change we see that $z \in X' \times_Y \mathrm{Spec}(k)$. Hence the lemma is clear. □

49.32. Syntomic morphisms

The property "syntomic" of morphisms of schemes is étale local on the source-and-target, see Descent, Remark 34.28.7. It is also stable under base change and fpqc local on the target, see Morphisms, Lemma 28.32.4 and Descent, Lemma 34.19.24. Hence, by Lemma 49.21.1 above, we may define the notion of a syntomic morphism of algebraic spaces as follows and it agrees with the already existing notion defined in Section 49.3 when the morphism is representable.

Definition 49.32.1. Let S be a scheme. Let $f : X \to Y$ be a morphism of algebraic spaces over S.

(1) We say f is *syntomic* if the equivalent conditions of Lemma 49.21.1 hold with $\mathcal{P}$ ="syntomic".
(2) Let $x \in |X|$. We say f is *syntomic at x* if there exists an open neighbourhood $X' \subset X$ of x such that $f|_{X'} : X' \to Y$ is syntomic.

Lemma 49.32.2. *The composition of syntomic morphisms is syntomic.*

Proof. Omitted. □

Lemma 49.32.3. *The base change of a syntomic morphism is syntomic.*

Proof. Omitted. □

Lemma 49.32.4. *Let S be a scheme. Let $f : X \to Y$ be a morphism of algebraic spaces over S. The following are equivalent:*

(1) *f is syntomic,*
(2) *for every $x \in |X|$ the morphism f is syntomic at x,*
(3) *for every scheme Z and any morphism $Z \to Y$ the morphism $Z \times_Y X \to Z$ is syntomic,*
(4) *for every affine scheme Z and any morphism $Z \to Y$ the morphism $Z \times_Y X \to Z$ is syntomic,*
(5) *there exists a scheme V and a surjective étale morphism $V \to Y$ such that $V \times_Y X \to V$ is a syntomic morphism,*
(6) *there exists a scheme U and a surjective étale morphism $\varphi : U \to X$ such that the composition $f \circ \varphi$ is syntomic,*
(7) *for every commutative diagram*

$$\begin{array}{ccc} U & \longrightarrow & V \\ \downarrow & & \downarrow \\ X & \longrightarrow & Y \end{array}$$

where U, V are schemes and the vertical arrows are étale the top horizontal arrow is syntomic,
(8) *there exists a commutative diagram*

$$\begin{array}{ccc} U & \longrightarrow & V \\ \downarrow & & \downarrow \\ X & \longrightarrow & Y \end{array}$$

where U, V are schemes, the vertical arrows are étale, and $U \to X$ is surjective such that the top horizontal arrow is syntomic, and
(9) *there exist Zariski coverings $Y = \bigcup_{i \in I} Y_i$, and $f^{-1}(Y_i) = \bigcup X_{ij}$ such that each morphism $X_{ij} \to Y_i$ is syntomic.*

Proof. Omitted. □

49.33. Smooth morphisms

The property "syntomic" of morphisms of schemes is étale local on the source-and-target, see Descent, Remark 34.28.7. It is also stable under base change and fpqc local on the target, see Morphisms, Lemma 28.35.5 and Descent, Lemma 34.19.25. Hence, by Lemma 49.21.1 above, we may define the notion of a smooth morphism of algebraic spaces as follows and it agrees with the already existing notion defined in Section 49.3 when the morphism is representable.

Definition 49.33.1. Let S be a scheme. Let $f : X \to Y$ be a morphism of algebraic spaces over S.

(1) We say f is *smooth* if the equivalent conditions of Lemma 49.21.1 hold with $\mathcal{P}$ ="smooth".
(2) Let $x \in |X|$. We say f is *smooth at* x if there exists an open neighbourhood $X' \subset X$ of x such that $f|_{X'} : X' \to Y$ is smooth.

Lemma 49.33.2. *The composition of smooth morphisms is smooth.*

Proof. Omitted. □

Lemma 49.33.3. *The base change of a smooth morphism is smooth.*

Proof. Omitted. □

Lemma 49.33.4. *Let S be a scheme. Let $f : X \to Y$ be a morphism of algebraic spaces over S. The following are equivalent:*

(1) *f is smooth,*
(2) *for every $x \in |X|$ the morphism f is smooth at x,*
(3) *for every scheme Z and any morphism $Z \to Y$ the morphism $Z \times_Y X \to Z$ is smooth,*
(4) *for every affine scheme Z and any morphism $Z \to Y$ the morphism $Z \times_Y X \to Z$ is smooth,*
(5) *there exists a scheme V and a surjective étale morphism $V \to Y$ such that $V \times_Y X \to V$ is a smooth morphism,*
(6) *there exists a scheme U and a surjective étale morphism $\varphi : U \to X$ such that the composition $f \circ \varphi$ is smooth,*
(7) *for every commutative diagram*
$$\begin{array}{ccc} U & \longrightarrow & V \\ \downarrow & & \downarrow \\ X & \longrightarrow & Y \end{array}$$
where U, V are schemes and the vertical arrows are étale the top horizontal arrow is smooth,
(8) *there exists a commutative diagram*
$$\begin{array}{ccc} U & \longrightarrow & V \\ \downarrow & & \downarrow \\ X & \longrightarrow & Y \end{array}$$
where U, V are schemes, the vertical arrows are étale, and $U \to X$ is surjective such that the top horizontal arrow is smooth, and
(9) *there exist Zariski coverings $Y = \bigcup_{i \in I} Y_i$, and $f^{-1}(Y_i) = \bigcup X_{ij}$ such that each morphism $X_{ij} \to Y_i$ is smooth.*

Proof. Omitted. □

Lemma 49.33.5. *A smooth morphism of algebraic spaces is locally of finite presentation.*

Proof. Let $X \to Y$ be a smooth morphism of algebraic spaces. By definition this means there exists a diagram as in Lemma 49.21.1 with h smooth and surjective vertical arrow a. By Morphisms, Lemma 28.35.8 h is locally of finite presentation. Hence $X \to Y$ is locally of finite presentation by definition. □

Lemma 49.33.6. *A smooth morphism of algebraic spaces is locally of finite type.*

Proof. Combine Lemmas 49.33.5 and 49.26.5. □

Lemma 49.33.7. *A smooth morphism of algebraic spaces is flat.*

Proof. Let $X \to Y$ be a smooth morphism of algebraic spaces. By definition this means there exists a diagram as in Lemma 49.21.1 with h smooth and surjective vertical arrow a. By Morphisms, Lemma 28.35.8 h is flat. Hence $X \to Y$ is flat by definition. □

Lemma 49.33.8. *A smooth morphism of algebraic spaces is syntomic.*

Proof. Let $X \to Y$ be a smooth morphism of algebraic spaces. By definition this means there exists a diagram as in Lemma 49.21.1 with h smooth and surjective vertical arrow a. By Morphisms, Lemma 28.35.7 h is syntomic. Hence $X \to Y$ is syntomic by definition. □

49.34. Unramified morphisms

The property "unramified" (resp. "G-unramified") of morphisms of schemes is étale local on the source-and-target, see Descent, Remark 34.28.7. It is also stable under base change and fpqc local on the target, see Morphisms, Lemma 28.36.5 and Descent, Lemma 34.19.26. Hence, by Lemma 49.21.1 above, we may define the notion of an unramified morphism (resp. G-unramified morphism) of algebraic spaces as follows and it agrees with the already existing notion defined in Section 49.3 when the morphism is representable.

Definition 49.34.1. Let S be a scheme. Let $f : X \to Y$ be a morphism of algebraic spaces over S.

(1) We say f is *unramified* if the equivalent conditions of Lemma 49.21.1 hold with $\mathcal{P} =$ unramified.
(2) Let $x \in |X|$. We say f is *unramified at* x if there exists an open neighbourhood $X' \subset X$ of x such that $f|_{X'} : X' \to Y$ is unramified.
(3) We say f is *G-unramified* if the equivalent conditions of Lemma 49.21.1 hold with $\mathcal{P} =$ G-unramified.
(4) Let $x \in |X|$. We say f is *G-unramified at* x if there exists an open neighbourhood $X' \subset X$ of x such that $f|_{X'} : X' \to Y$ is G-unramified.

Because of the following lemma, from here on we will only develop theory for unramified morphisms, and whenever we want to use a G-unramified morphism we will simply say "an unramified morphism locally of finite presentation".

Lemma 49.34.2. *Let S be a scheme. Let $f : X \to Y$ be a morphism of algebraic spaces over S. Then f is G-unramified if and only if f is unramified and locally of finite presentation.*

Proof. Consider any diagram as in Lemma 49.21.1. Then all we are saying is that the morphism h is G-unramified if and only if it is unramified and locally of finite presentation. This is clear from Morphisms, Definition 28.36.1. □

Lemma 49.34.3. *The composition of unramified morphisms is unramified.*

Proof. Omitted. □

Lemma 49.34.4. *The base change of an unramified morphism is unramified.*

Proof. Omitted. □

Lemma 49.34.5. *Let S be a scheme. Let $f : X \to Y$ be a morphism of algebraic spaces over S. The following are equivalent:*

(1) *f is unramified,*
(2) *for every $x \in |X|$ the morphism f is unramified at x,*
(3) *for every scheme Z and any morphism $Z \to Y$ the morphism $Z \times_Y X \to Z$ is unramified,*
(4) *for every affine scheme Z and any morphism $Z \to Y$ the morphism $Z \times_Y X \to Z$ is unramified,*
(5) *there exists a scheme V and a surjective étale morphism $V \to Y$ such that $V \times_Y X \to V$ is an unramified morphism,*
(6) *there exists a scheme U and a surjective étale morphism $\varphi : U \to X$ such that the composition $f \circ \varphi$ is unramified,*
(7) *for every commutative diagram*

$$\begin{array}{ccc} U & \longrightarrow & V \\ \downarrow & & \downarrow \\ X & \longrightarrow & Y \end{array}$$

where U, V are schemes and the vertical arrows are étale the top horizontal arrow is unramified,
(8) *there exists a commutative diagram*

$$\begin{array}{ccc} U & \longrightarrow & V \\ \downarrow & & \downarrow \\ X & \longrightarrow & Y \end{array}$$

where U, V are schemes, the vertical arrows are étale, and $U \to X$ is surjective such that the top horizontal arrow is unramified, and
(9) *there exist Zariski coverings $Y = \bigcup_{i \in I} Y_i$, and $f^{-1}(Y_i) = \bigcup X_{ij}$ such that each morphism $X_{ij} \to Y_i$ is unramified.*

Proof. Omitted. □

Lemma 49.34.6. *An unramified morphism of algebraic spaces is locally of finite type.*

Proof. Via a diagram as in Lemma 49.21.1 this translates into Morphisms, Lemma 28.36.9. □

Lemma 49.34.7. *If f is unramified at x then f is quasi-finite at x. In particular, an unramified morphism is locally quasi-finite.*

Proof. Via a diagram as in Lemma 49.21.1 this translates into Morphisms, Lemma 28.36.10. □

Lemma 49.34.8. *An immersion of algebraic spaces is unramified.*

Proof. Let $i : X \to Y$ be an immersion of algebraic spaces. Choose a scheme V and a surjective étale morphism $V \to Y$. Then $V \times_Y X \to V$ is an immersion of schemes, hence unramified (see Morphisms, Lemmas 28.36.7 and 28.36.8). Thus by definition i is unramified. □

Lemma 49.34.9. *Let S be a scheme. Let $f : X \to Y$ be a morphism of algebraic spaces over S.*

(1) *If f is unramified, then the diagonal morphism $\Delta_{X/Y} : X \to X \times_Y X$ is an open immersion.*
(2) *If f is locally of finite type and $\Delta_{X/Y}$ is an open immersion, then f is unramified.*

Proof. We know in any case that $\Delta_{X/Y}$ is a representable monomorphism, see Lemma 49.5.1. Choose a scheme V and a surjective étale morphism $V \to Y$. Choose a scheme U and a surjective étale morphism $U \to X \times_Y V$. Consider the commutative diagram

$$\begin{array}{ccccc} U & \xrightarrow{\Delta_{U/V}} & U \times_V U & \longrightarrow & V \\ \downarrow & & \downarrow & & \downarrow{\scriptstyle \Delta_{V/Y}} \\ X & \xrightarrow{\Delta_{X/Y}} & X \times_Y X & \longrightarrow & V \times_Y V \end{array}$$

with cartesian right square. The left vertical arrow is surjective étale. The right vertical arrow is étale as a morphism between schemes étale over Y, see Properties of Spaces, Lemma 48.13.6. Hence the middle vertical arrow is étale too (but it need not be surjective).

Assume f is unramified. Then $U \to V$ is unramified, hence $\Delta_{U/V}$ is an open immersion by Morphisms, Lemma 28.36.13. Looking at the left square of the diagram above we conclude that $\Delta_{X/Y}$ is an étale morphism, see Properties of Spaces, Lemma 48.13.3. Hence $\Delta_{X/Y}$ is a representable étale monomorphism, which implies that it is an open immersion by Étale Morphisms, Theorem 40.14.1. (See also Spaces, Lemma 47.5.8 for the translation from schemes language into the language of functors.)

Assume that f is locally of finite type and that $\Delta_{X/Y}$ is an open immersion. This implies that $U \to V$ is locally of finite type too (by definition of a morphism of algebraic spaces which is locally of finite type). Looking at the displayed diagram above we conclude that $\Delta_{U/V}$ is étale as a morphism between schemes étale over $X \times_Y X$, see Properties of Spaces, Lemma 48.13.6. But since $\Delta_{U/V}$ is the diagonal of a morphism between schemes we see that it is in any case an immersion, see Schemes, Lemma 25.21.2. Hence it is an open immersion, and we conclude that $U \to V$ is unramified by Morphisms, Lemma 28.36.13. This in turn means that f is unramified by definition. □

Lemma 49.34.10. *Let S be a scheme. Consider a commutative diagram*

$$\begin{array}{ccc} X & \xrightarrow{f} & Y \\ & {\scriptstyle p} \searrow \quad \swarrow {\scriptstyle q} & \\ & Z & \end{array}$$

of algebraic spaces over S. Assume that $X \to Z$ is locally of finite type. Then there exists an open subspace $U(f) \subset X$ such that $|U(f)| \subset |X|$ is the set of points where f is unramified. Moreover, for any morphism of algebraic spaces $Z' \to Z$, if $f' : X' \to Y'$ is the base change of f by $Z' \to Z$, then $U(f')$ is the inverse image of $U(f)$ under the projection $X' \to X$.

Proof. This lemma is the analogue of Morphisms, Lemma 28.36.15 and in fact we will deduce the lemma from it. By Definition 49.34.1 the set $\{x \in |X| : f \text{ is unramified at } x\}$ is open in X. Hence we only need to prove the final statement. By Lemma 49.22.6 the morphism $X \to Y$ is locally of finite type. By Lemma 49.22.3 the morphism $X' \to Y'$ is locally of finite type.

Choose a scheme W and a surjective étale morphism $W \to Z$. Choose a scheme V and a surjective étale morphism $V \to W \times_Z Y$. Choose a scheme U and a surjective étale morphism $U \to V \times_Y X$. Finally, choose a scheme W' and a surjective étale morphism $W' \to W \times_Z Z'$. Set $V' = W' \times_W V$ and $U' = W' \times_W U$, so that we obtain surjective étale morphisms $V' \to Y'$ and $U' \to X'$. We will use without further mention an étale morphism of algebraic spaces induces an open map of associated topological spaces (see Properties of Spaces, Lemma 48.13.7). This combined with Lemma 49.34.5 implies that $U(f)$ is the image in $|X|$ of the set T of points in U where the morphism $U \to V$ is unramified. Similarly, $U(f')$ is the image in $|X'|$ of the set T' of points in U' where the morphism $U' \to V'$ is unramified. Now, by construction the diagram

$$\begin{array}{ccc} U' & \longrightarrow & U \\ \downarrow & & \downarrow \\ V' & \longrightarrow & V \end{array}$$

is cartesian (in the category of schemes). Hence the aforementioned Morphisms, Lemma 28.36.15 applies to show that T' is the inverse image of T. Since $|U'| \to |X'|$ is surjective this implies the lemma. □

Lemma 49.34.11. *Let S be a scheme. Let $X \to Y \to Z$ be morphisms of algebraic spaces over S. If $X \to Z$ is unramified, then $X \to Y$ is unramified.*

Proof. Choose a commutative diagram

$$\begin{array}{ccccc} U & \longrightarrow & V & \longrightarrow & W \\ \downarrow & & \downarrow & & \downarrow \\ X & \longrightarrow & Y & \longrightarrow & Z \end{array}$$

with vertical arrows étale and surjective. (See Spaces, Lemma 47.11.4.) Apply Morphisms, Lemma 28.36.16 to the top row. □

49.35. Étale morphisms

The notion of an étale morphism of algebraic spaces was defined in Properties of Spaces, Definition 48.13.2. Here is what it means for a morphism to be étale at a point.

Definition 49.35.1. Let S be a scheme. Let $f : X \to Y$ be a morphism of algebraic spaces over S. Let $x \in |X|$. We say f is *étale at x* if there exists an open neighbourhood $X' \subset X$ of x such that $f|_{X'} : X' \to Y$ is étale

Lemma 49.35.2. *Let S be a scheme. Let $f : X \to Y$ be a morphism of algebraic spaces over S. The following are equivalent:*

(1) *f is étale,*
(2) *for every $x \in |X|$ the morphism f is étale at x,*
(3) *for every scheme Z and any morphism $Z \to Y$ the morphism $Z \times_Y X \to Z$ is étale,*
(4) *for every affine scheme Z and any morphism $Z \to Y$ the morphism $Z \times_Y X \to Z$ is étale,*
(5) *there exists a scheme V and a surjective étale morphism $V \to Y$ such that $V \times_Y X \to V$ is an étale morphism,*
(6) *there exists a scheme U and a surjective étale morphism $\varphi : U \to X$ such that the composition $f \circ \varphi$ is étale,*
(7) *for every commutative diagram*
$$\begin{array}{ccc} U & \longrightarrow & V \\ \downarrow & & \downarrow \\ X & \longrightarrow & Y \end{array}$$
where U, V are schemes and the vertical arrows are étale the top horizontal arrow is étale,
(8) *there exists a commutative diagram*
$$\begin{array}{ccc} U & \longrightarrow & V \\ \downarrow & & \downarrow \\ X & \longrightarrow & Y \end{array}$$
where U, V are schemes, the vertical arrows are étale, and $U \to X$ surjective such that the top horizontal arrow is étale, and
(9) *there exist Zariski coverings $Y = \bigcup Y_i$ and $f^{-1}(Y_i) = \bigcup X_{ij}$ such that each morphism $X_{ij} \to Y_i$ is étale.*

Proof. Combine Properties of Spaces, Lemmas 48.13.3, 48.13.5 and 48.13.4. Some details omitted. □

Lemma 49.35.3. *The composition of two étale morphisms of algebraic spaces is étale.*

Proof. This is a copy of Properties of Spaces, Lemma 48.13.4. □

Lemma 49.35.4. *The base change of an étale morphism of algebraic spaces by any morphism of algebraic spaces is étale.*

Proof. This is a copy of Properties of Spaces, Lemma 48.13.5. □

Lemma 49.35.5. *An étale morphism of algebraic spaces is locally quasi-finite.*

Proof. Let $X \to Y$ be an étale morphism of algebraic spaces, see Properties of Spaces, Definition 48.13.2. By Properties of Spaces, Lemma 48.13.3 we see this means there exists a diagram as in Lemma 49.21.1 with h étale and surjective vertical arrow a. By Morphisms, Lemma 28.37.6 h is locally quasi-finite. Hence $X \to Y$ is locally quasi-finite by definition. □

Lemma 49.35.6. *An étale morphism of algebraic spaces is smooth.*

Proof. The proof is identical to the proof of Lemma 49.35.5. It uses the fact that an étale morphism of schemes is smooth (by definition of an étale morphism of schemes). □

Lemma 49.35.7. *An étale morphism of algebraic spaces is flat.*

Proof. The proof is identical to the proof of Lemma 49.35.5. It uses Morphisms, Lemma 28.37.12. □

Lemma 49.35.8. *An étale morphism of algebraic spaces is locally of finite presentation.*

Proof. The proof is identical to the proof of Lemma 49.35.5. It uses Morphisms, Lemma 28.37.11. □

Lemma 49.35.9. *An étale morphism of algebraic spaces is locally of finite type.*

Proof. An étale morphism is locally of finite presentation and a morphism locally of finite presentation is locally of finite type, see Lemmas 49.35.8 and 49.26.5. □

Lemma 49.35.10. *An étale morphism of algebraic spaces is unramified.*

Proof. The proof is identical to the proof of Lemma 49.35.5. It uses Morphisms, Lemma 28.37.5. □

Lemma 49.35.11. *Let S be a scheme. Let X, Y be algebraic spaces étale over an algebraic space Z. Any morphism $X \to Y$ over Z is étale.*

Proof. This is a copy of Properties of Spaces, Lemma 48.13.6. □

Lemma 49.35.12. *A locally finitely presented, flat, unramified morphism of algebraic spaces is étale.*

Proof. Let $X \to Y$ be a locally finitely presented, flat, unramified morphism of algebraic spaces. By Properties of Spaces, Lemma 48.13.3 we see this means there exists a diagram as in Lemma 49.21.1 with h locally finitely presented, flat, unramified and surjective vertical arrow a. By Morphisms, Lemma 28.37.16 h is étale. Hence $X \to Y$ is étale by definition. □

49.36. Proper morphisms

The notion of a proper morphism plays an important role in algebraic geometry. Here is the definition of a proper morphism of algebraic spaces.

Definition 49.36.1. Let S be a scheme. Let $f : X \to Y$ be a morphism of algebraic spaces over S. We say f is *proper* if f is separated, finite type, and universally closed.

Lemma 49.36.2. *Let S be a scheme. Let $f : X \to Y$ be a morphism of algebraic spaces over S. The following are equivalent*

(1) *f is proper,*
(2) *for every scheme Z and every morphism $Z \to Y$ the projection $Z \times_Y X \to Z$ is proper,*
(3) *for every affine scheme Z and every morphism $Z \to Y$ the projection $Z \times_Y X \to Z$ is proper,*

(4) *there exists a scheme V and a surjective étale morphism $V \to Y$ such that $V \times_Y X \to V$ is proper, and*
(5) *there exists a Zariski covering $Y = \bigcup Y_i$ such that each of the morphisms $f^{-1}(Y_i) \to Y_i$ is proper.*

Proof. Combine Lemmas 49.5.12, 49.22.4, 49.9.7, and 49.10.5. □

Lemma 49.36.3. *A base change of a proper morphism is proper.*

Proof. See Lemmas 49.5.4, 49.22.3, and 49.10.3. □

Lemma 49.36.4. *A composition of proper morphisms is proper.*

Proof. See Lemmas 49.5.8, 49.22.2, and 49.10.4. □

Lemma 49.36.5. *A closed immersion of algebraic spaces is a proper morphism of algebraic spaces.*

Proof. As a closed immersion is by definition representable this follows from Spaces, Lemma 47.5.8 and the corresponding result for morphisms of schemes, see Morphisms, Lemma 28.42.6. □

Lemma 49.36.6. *Let S be a scheme. Consider a commutative diagram of algebraic spaces*

$$\begin{array}{ccc} X & \longrightarrow & Y \\ & \searrow \quad \swarrow & \\ & B & \end{array}$$

over S.

(1) *If $X \to B$ is universally closed and $Y \to B$ is separated, then the morphism $X \to Y$ is universally closed. In particular, the image of $|X|$ in $|Y|$ is closed.*
(2) *If $X \to B$ is proper and $Y \to B$ is separated, then the morphism $X \to Y$ is proper.*

Proof. Assume $X \to B$ is universally closed and $Y \to B$ is separated. We factor the morphism as $X \to X \times_B Y \to Y$. The first morphism is a closed immersion, see Lemma 49.5.6 hence universally closed. The projection $X \times_B Y \to Y$ is the base change of a universally closed morphism and hence universally closed, see Lemma 49.10.3. Thus $X \to Y$ is universally closed as the composition of universally closed morphisms, see Lemma 49.10.4. This proves (1). To deduce (2) combine (1) with Lemmas 49.5.10, 49.9.8, and 49.22.6. □

Lemma 49.36.7. *Let S be a scheme. Let B be an algebraic space over S. Let $f : X \to Y$ be a morphism of algebraic spaces over B. If X is universally closed over B and f is surjective then Y is universally closed over B. In particular, if also Y is separated and of finite type over B, then Y is proper over B.*

Proof. Assume X is universally closed and f surjective. Denote $p : X \to B$, $q : Y \to B$ the structure morphisms. Let $B' \to B$ be a morphism of algebraic spaces over S. The base change $f' : X_{B'} \to Y_{B'}$ is surjective (Lemma 49.6.5), and the base change $p' : X_{B'} \to B'$ is closed. If $T \subset Y_{B'}$ is closed, then $(f')^{-1}(T) \subset X_{B'}$ is closed, hence $p'((f')^{-1}(T)) = q'(T)$ is closed. So q' is closed. □

Lemma 49.36.8. *Let S be a scheme. Let $f : X \to Y$ be a morphism of algebraic spaces over S. The following are equivalent:*

(1) *f is separated,*
(2) *$\Delta_{X/Y} : X \to X \times_Y X$ is universally closed, and*
(3) *$\Delta_{X/Y} : X \to X \times_Y X$ is proper.*

Proof. The implication (1) $\Rightarrow$ (3) follows from Lemma 49.36.5. We will use Spaces, Lemma 47.5.8 without further mention in the rest of the proof. Recall that $\Delta_{X/Y}$ is a representable monomorphism which is locally of finite type, see Lemma 49.5.1. Since proper $\Rightarrow$ universally closed for morphisms of schemes we conclude that (3) implies (2). If $\Delta_{X/Y}$ is universally closed then Étale Morphisms, Lemma 40.7.2 implies that it is a closed immersion. Thus (2) $\Rightarrow$ (1) and we win. □

49.37. Valuative criteria

The formulation of the existence part of the valuative criterion is slightly different for morphisms of algebraic spaces, since it may be necessary to extend the fraction field of the valuation ring. See Example 49.37.4. Here is the definition.

Definition 49.37.1. Let S be a scheme. Let $f : X \to Y$ be a morphism of algebraic spaces over S. We say *f satisfies the uniqueness part of the valuative criterion* if given any commutative solid diagram

$$\begin{array}{ccc} \mathrm{Spec}(K) & \longrightarrow & X \\ \downarrow & \nearrow & \downarrow \\ \mathrm{Spec}(A) & \longrightarrow & Y \end{array}$$

where A is a valuation ring with field of fractions K, there exists at most one dotted arrow (without requiring existence). We say *f satisfies the existence part of the valuative criterion* if given any solid diagram as above there exists an extension $K \subset K'$ of fields, a valuation ring $A' \subset K'$ dominating A and a morphism $\mathrm{Spec}(A') \to X$ such that the following diagram commutes

$$\begin{array}{ccccc} \mathrm{Spec}(K') & \longrightarrow & \mathrm{Spec}(K) & \longrightarrow & X \\ \downarrow & & & \nearrow & \downarrow \\ \mathrm{Spec}(A') & \longrightarrow & \mathrm{Spec}(A) & \longrightarrow & Y \end{array}$$

We say *f satisfies the valuative criterion* if f satisfies both the existence and uniqueness part.

It turns out that for algebraic spaces, it always sufffices to take a finite separable extension $K \subset K'$ above. See Lemma 49.37.3. Before we prove it we show that the criterion is identical to the criterion as formulated for morphisms of schemes in case the morphism of algebraic spaces is representable.

Lemma 49.37.2. *Let S be a scheme. Let $f : X \to Y$ be a morphism of algebraic spaces over S. Assume f is representable. The following are equivalent*

(1) *f satisfies the existence part of the valuation criterion as in Definition 49.37.1 above, and*

(2) *given any commutative solid diagram*

$$\begin{array}{ccc} \mathrm{Spec}(K) & \longrightarrow & X \\ \downarrow & \nearrow & \downarrow \\ \mathrm{Spec}(A) & \longrightarrow & Y \end{array}$$

where A is a valuation ring with field of fractions K, there exists a dotted arrow, i.e., f satisfies the existence part of the valuative criterion as in Schemes, Definition 25.20.3.

Proof. It suffices to show that given a commutative diagram of the form

$$\begin{array}{ccccc} \mathrm{Spec}(K') & \longrightarrow & \mathrm{Spec}(K) & \longrightarrow & X \\ \downarrow & & & \nearrow_{\varphi} & \downarrow \\ \mathrm{Spec}(A') & \longrightarrow & \mathrm{Spec}(A) & \longrightarrow & Y \end{array}$$

as in Definition 49.37.1, then we can find a morphism $\mathrm{Spec}(A) \to X$ fitting into the diagram too. Set $X_A = \mathrm{Spec}(A) \times_Y X$. As f is representable we see that X_A is a scheme. The morphism φ gives a morphism $\varphi' : \mathrm{Spec}(A') \to X_A$. Let $x \in X_A$ be the image of the closed point of $\varphi' : \mathrm{Spec}(A') \to X_A$. Then we have the following commutative diagram of rings

$$\begin{array}{ccccc} K' & \longleftarrow & K & \longleftarrow & \mathcal{O}_{X_A,x} \\ \uparrow & & & \swarrow & \uparrow \\ A' & \longleftarrow & A & \longleftarrow & A \end{array}$$

Since A is a valuation ring, and since A' dominates A, we see that $K \cap A' = A$. Hence the ring map $\mathcal{O}_{X_A,x} \to K$ has image contained in A. Whence a morphism $\mathrm{Spec}(A) \to X_A$ (see Schemes, Section 25.13) as desired. □

Lemma 49.37.3. *Let S be a scheme. Let $f : X \to Y$ be a morphism of algebraic spaces over S. The following are equivalent*

(1) *f satisfies the existence part of the valuation criterion as in Definition 49.37.1, and*
(2) *f satisfies the existence part of the valuation criterion as in Definition 49.37.1 modified by requiring the extension $K \subset K'$ to be finite separable.*

Proof. We have to show that (1) implies (2). Suppose given a diagram

$$\begin{array}{ccccc} \mathrm{Spec}(K') & \longrightarrow & \mathrm{Spec}(K) & \longrightarrow & X \\ \downarrow & & & \nearrow & \downarrow \\ \mathrm{Spec}(A') & \longrightarrow & \mathrm{Spec}(A) & \longrightarrow & Y \end{array}$$

as in Definition 49.37.1 with $K \subset K'$ arbitrary. Choose a scheme U and a surjective étale morphism $U \to X$. Then

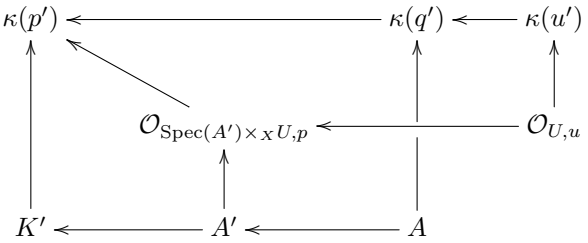
$$\mathrm{Spec}(A') \times_X U \longrightarrow \mathrm{Spec}(A')$$

is surjective étale. Let p be a point of $\mathrm{Spec}(A') \times_X U$ mapping to the closed point of $\mathrm{Spec}(A')$. Let $p' \rightsquigarrow p$ be a generalization of p mapping to the generic point

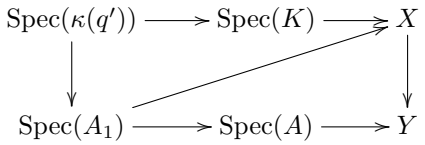
of $\mathrm{Spec}(A')$. Such a generalization exists because generalizations lift along flat morphisms of schemes, see Morphisms, Lemma 28.26.8. Then p' corresponds to a point of the scheme $\mathrm{Spec}(K') \times_X U$. Note that

$$\mathrm{Spec}(K') \times_X U = \mathrm{Spec}(K') \times_{\mathrm{Spec}(K)} (\mathrm{Spec}(K) \times_X U)$$

Hence p' maps to a point $q' \in \mathrm{Spec}(K) \times_X U$ whose residue field is a finite separable extension of K. Finally, $p' \rightsquigarrow p$ maps to a specialization $u' \rightsquigarrow u$ on the scheme U. With all this notation we get the following diagram of rings



This means that the ring $B \subset \kappa(q')$ generated by the images of A and $\mathcal{O}_{U,u}$ maps to a subring of $\kappa(p')$ contained in the image B' of $\mathcal{O}_{\mathrm{Spec}(A') \times_X U, p} \to \kappa(p')$. Note that B' is a local ring. Let $\mathfrak{m} \subset B$ be the maximal ideal. By construction $A \cap \mathfrak{m}$, (resp. $\mathcal{O}_{U,u} \cap \mathfrak{m}$, resp. $A' \cap \mathfrak{m}$) is the maximal ideal of A (resp. $\mathcal{O}_{U,u}$, resp. A'). Set $\mathfrak{q} = B \cap \mathfrak{m}$. This is a prime ideal such that $A \cap \mathfrak{q}$ is the maximal ideal of A. Hence $B_{\mathfrak{q}} \subset \kappa(q')$ is a local ring dominating A. By Algebra, Lemma 10.46.2 we can find a valuation ring $A_1 \subset \kappa(q')$ with field of fractions $\kappa(q')$ dominating $B_{\mathfrak{q}}$. The (local) ring map $\mathcal{O}_{U,u} \to A_1$ gives a morphism $\mathrm{Spec}(A_1) \to U \to X$ such that the diagram



is commutative. Since $f.f.(A_1) = \kappa(q') \supset K$ is finite separable by construction the lemma is proved. □

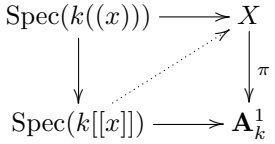
Example 49.37.4. Consider the algebraic space X constructed in Spaces, Example 47.14.2. Recall that it is the affine line with zero doubled in a Galois twisted relative to a degree two Galois extension $k \subset k'$. As such it comes with a morphism

$$\pi : X \longrightarrow S = \mathbf{A}^1_k$$

which is quasi-compact. We claim that π is universally closed. Namely, after base change by $\mathrm{Spec}(k') \to \mathrm{Spec}(k)$ the morphism π is identified with the morphism

$$\text{affine line with zero doubled} \longrightarrow \text{affine line}$$

which is universally closed (some details omitted). Since the morphism $\mathrm{Spec}(k') \to \mathrm{Spec}(k)$ is universally closed and surjective, a diagram chase shows that π is universally closed. On the other hand, consider the diagram



Since the unique point of X above $0 \in \mathbf{A}^1_k$ corresponds to a monomorphism $\mathrm{Spec}(k') \to X$ it is clear there cannot exist a dotted arrow! This shows that a finite separable field extension is needed in general.

Lemma 49.37.5. *The base change of a morphism of algebraic spaces which satisfies the existence part of (resp. uniqueness part of) the valuative criterion by any morphism of algebraic spaces satisfies the existence part of (resp. uniqueness part of) the valuative criterion.*

Proof. Let $f : X \to Y$ be a morphism of algebraic spaces over the scheme S. Let $Z \to Y$ be any morphism of algebraic spaces over S. Consider a solid commutative diagram of the following shape

$$\begin{array}{ccccc} \mathrm{Spec}(K) & \longrightarrow & Z \times_Y X & \longrightarrow & X \\ \downarrow & \nearrow & \downarrow & \nearrow & \downarrow \\ \mathrm{Spec}(A) & \longrightarrow & Z & \longrightarrow & Y \end{array}$$

Then the set of north-west dotted arrows making the diagram commute is in 1-1 correspondence with the set of west-north-west dotted arrows making the diagram commute. This proves the lemma in the case of "uniqueness". For the existence part, assume f satisfies the existence part of the valuative criterion. If we are given a solid commutative diagram as above, then by assumption there exists an extension $K \subset K'$ of fields and a valuation ring $A' \subset K'$ dominating A and a morphism $\mathrm{Spec}(A') \to X$ fitting into the following commutative diagram

$$\begin{array}{ccccccc} \mathrm{Spec}(K') & \longrightarrow & \mathrm{Spec}(K) & \longrightarrow & Z \times_Y X & \longrightarrow & X \\ \downarrow & & & & & & \downarrow \\ \mathrm{Spec}(A') & \longrightarrow & \mathrm{Spec}(A) & \longrightarrow & Z & \longrightarrow & Y \end{array}$$

And by the remarks above the skew arrow corresponds to an arrow $\mathrm{Spec}(A') \to Z \times_Y X$ as desired. □

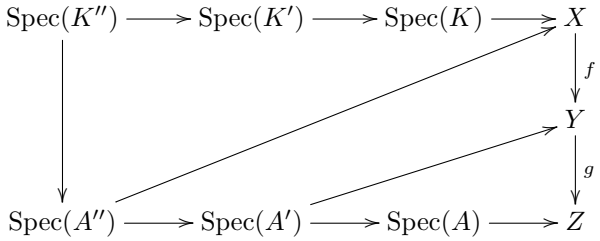
Lemma 49.37.6. *The composition of two morphisms of algebraic spaces which satisfy the (existence part of, resp. uniqueness part of) the valuative criterion satisfies the (existence part of, resp. uniqueness part of) the valuative criterion.*

Proof. Let $f : X \to Y$, $g : Y \to Z$ be morphisms of algebraic spaces over the scheme S. Consider a solid commutative diagram of the following shape

$$\begin{array}{ccc} \mathrm{Spec}(K) & \longrightarrow & X \\ \downarrow & \nearrow & \downarrow f \\ & & Y \\ & \nearrow & \downarrow g \\ \mathrm{Spec}(A) & \longrightarrow & Z \end{array}$$

If we have the uniqueness part for g, then there exists at most one north-west dotted arrow making the diagram commute. If we also have the uniqueness part for f, then we have at most one north-north-west dotted arrow making the diagram

commute. The proof in the existence case comes from contemplating the following diagram



Namely, the existence part for g gives us the extension K', the valuation ring A' and the arrow $\mathrm{Spec}(A') \to Y$, whereupon the existence part for f gives us the extension K'', the valuation ring A'' and the arrow $\mathrm{Spec}(A'') \to X$. □

49.38. Valuative criterion for universal closedness

The existence part of the valuative criterion implies universal closedness for quasi-compact morphisms, see Lemma 49.38.1. In the case of schemes, this is an "if and only if" statement, but for morphisms of algebraic spaces this is wrong. Example 49.10.6 shows that $\mathbf{A}^1_k/\mathbf{Z} \to \mathrm{Spec}(k)$ is universally closed, but it is easy to see that the existence part of the valuative criterion fails. We revisit this topic in Decent Spaces, Section 50.13 and show the converse holds if the source of the morphism is a decent space. See also Decent Spaces, Lemma 50.14.8.

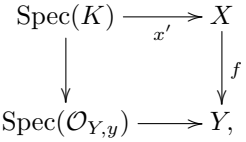
Lemma 49.38.1. *Let S be a scheme. Let $f : X \to Y$ be a morphism of algebraic spaces over S. Assume*

(1) *f is quasi-compact, and*
(2) *f satisfies the existence part of the valuative criterion.*

Then f is universally closed.

Proof. By Lemmas 49.9.3 and 49.37.5 properties (1) and (2) are preserved under any base change. By Lemma 49.10.5 we only have to show that $T \times_Y X \to T$ is universally closed, whenever T is an affine scheme over S mapping into Y. Hence it suffices to prove: If Y is an affine scheme, $f : X \to Y$ is quasi-compact and satisfies the existence part of the valuative criterion, then $f : |X| \to |Y|$ is closed. In this situation X is a quasi-compact algebraic space. By Properties of Spaces, Lemma 48.6.3 there exists an affine scheme U and a surjective étale morphism $\varphi : U \to X$. Let $T \subset |X|$ closed. The inverse image $\varphi^{-1}(T) \subset U$ is closed, and hence is the set of points of an affine closed subscheme $Z \subset U$. Thus, by Algebra, Lemma 10.38.5 we see that $f(T) = f(\varphi(|Z|)) \subset |Y|$ is closed if it is closed under specialization.

Let $y' \rightsquigarrow y$ be a specialization in Y with $y' \in f(T)$. Choose a point $x' \in T \subset |X|$ mapping to y' under f. We may represent x' by a morphism $\mathrm{Spec}(K) \to X$ for some field K. Thus we have the following diagram



see Schemes, Section 25.13 for the existence of the left vertical map. Choose a valuation ring $A \subset K$ dominating the image of the ring map $\mathcal{O}_{Y,y} \to K$ (this is

possible since the image is a local ring and not a field as $y' \neq y$, see Algebra, Lemma 10.46.2). By assumption there exists a field extension $K \subset K'$ and a valuation ring $A' \subset K'$ dominating A, and a morphism $\mathrm{Spec}(A') \to X$ fitting into the commutative diagram. Since A' dominates A, and A dominates $\mathcal{O}_{Y,y}$ we see that the closed point of $\mathrm{Spec}(A')$ maps to a point $x \in X$ with $f(x) = y$ which is a specialization of x'. Hence $x \in T$ as T is closed, and hence $y \in f(T)$ as desired. □

Lemma 49.38.2. *Let S be a scheme. Let $f : X \to Y$ be a flat morphism. Let $\mathrm{Spec}(A) \to Y$ be a morphism where A is a discrete valuation ring. If the closed point of $\mathrm{Spec}(A)$ maps to a point of $|Y|$ in the image of $|X| \to |Y|$, then there exists a commutative diagram*

$$\begin{array}{ccc} \mathrm{Spec}(A') & \longrightarrow & X \\ \downarrow & & \downarrow \\ \mathrm{Spec}(A) & \longrightarrow & Y \end{array}$$

where $A \to A'$ is a local ring map of valuation rings.

Proof. Omitted. □

Lemma 49.38.3. *Let S be a scheme. Let $f : X \to Y$ and $h : U \to Y$ be morphisms of algebraic spaces over S. If*

(1) *f and h are quasi-compact,*
(2) *$|h|(|U|)$ is dense in $|X|$, and*

given any commutative solid diagram

$$\begin{array}{ccccc} \mathrm{Spec}(K) & \longrightarrow & U & \longrightarrow & X \\ \downarrow & \nearrow & & & \downarrow \\ \mathrm{Spec}(A) & & \longrightarrow & & Y \end{array}$$

where A is a valuation ring with field of fractions K

(3) *there exists at most one dotted arrow making the diagram commute, and*
(4) *there exists an extension $K \subset K'$ of fields, a valuation ring $A' \subset K'$ dominating A and a morphism $\mathrm{Spec}(A') \to X$ such that the following diagram commutes*

$$\begin{array}{ccccccc} \mathrm{Spec}(K') & \longrightarrow & \mathrm{Spec}(K) & \longrightarrow & U & \longrightarrow & X \\ \downarrow & & & & & & \downarrow \\ \mathrm{Spec}(A') & \longrightarrow & \mathrm{Spec}(A) & & \longrightarrow & & Y \end{array}$$

then f is universally closed. If moreover

(5) *f is quasi-separated*

then f is separated and universally closed.

Proof. Assume (1), (2), (3), and (4). We will verify the existence part of the valuative criterion for f which will imply f is universally closed by Lemma 49.38.1.

To do this, consider a commutative diagram

$$\begin{array}{ccc} \mathrm{Spec}(K) & \longrightarrow & X \\ \downarrow & & \downarrow \\ \mathrm{Spec}(A) & \longrightarrow & Y \end{array} \tag{49.38.3.1}$$

where A is a valuation ring and K is the fraction field of A. Note that since valuation rings and fields are reduced, we may replace U, X, and S by their respective reductions by Properties of Spaces, Lemma 48.9.3. In this case the assumption that $h(U)$ is dense means that the scheme theoretic image of $h : U \to X$ is X, see Lemma 49.16.4.

Reduction to the case Y affine. Choose an étale morphism $\mathrm{Spec}(R) \to Y$ such that the closed point of $\mathrm{Spec}(A)$ maps to an element of $\mathrm{Im}(|\mathrm{Spec}(R)| \to |Y|)$. By Lemma 49.38.2 we can find a local ring map $A \to A'$ of valuation rings and a morphism $\mathrm{Spec}(A') \to \mathrm{Spec}(R)$ fitting into a commutative diagram

$$\begin{array}{ccc} \mathrm{Spec}(A') & \longrightarrow & \mathrm{Spec}(R) \\ \downarrow & & \downarrow \\ \mathrm{Spec}(A) & \longrightarrow & Y \end{array}$$

Since in Definition 49.37.1 we allow for extensions of valuation rings it is clear that we may replace A by A', Y by $\mathrm{Spec}(R)$, X by $X \times_Y \mathrm{Spec}(R)$ and U by $U \times_Y \mathrm{Spec}(R)$.

From now on we assume that $Y = \mathrm{Spec}(R)$ is an affine scheme. Let $\mathrm{Spec}(B) \to X$ be an étale morphism from an affine scheme such that the morphism $\mathrm{Spec}(K) \to X$ is in the image of $|\mathrm{Spec}(B)| \to |X|$. Since we may replace K by an extension $K' \supset K$ and A by a valuation ring $A' \subset K'$ dominating A (which exists by Algebra, Lemma 10.46.2), we may assume the morphism $\mathrm{Spec}(K) \to X$ factors through $\mathrm{Spec}(B)$ (by definition of $|X|$). In other words, we may think of K as a B-algebra. Choose a polynomial algebra P over B and a B-algebra surjection $P \to K$. Then $\mathrm{Spec}(P) \to X$ is flat as a composition $\mathrm{Spec}(P) \to \mathrm{Spec}(B) \to X$. Hence the scheme theoretic image of the morphism $U \times_X \mathrm{Spec}(P) \to \mathrm{Spec}(P)$ is $\mathrm{Spec}(P)$ by Lemma 49.27.12. By Lemma 49.16.5 we can find a commutative diagram

$$\begin{array}{ccc} \mathrm{Spec}(K') & \longrightarrow & U \times_X \mathrm{Spec}(P) \\ \downarrow & & \downarrow \\ \mathrm{Spec}(A') & \longrightarrow & \mathrm{Spec}(P) \end{array}$$

where A' is a valuation ring and K' is the fraction field of A' such that the closed point of $\mathrm{Spec}(A')$ maps to $\mathrm{Spec}(K) \subset \mathrm{Spec}(P)$. In other words, there is a B-algebra map $\varphi : K \to A'/\mathfrak{m}_{A'}$. Choose a valuation ring $A'' \subset A'/\mathfrak{m}_{A'}$ dominating $\varphi(A)$ with field of fractions $K'' = A'/\mathfrak{m}_{A'}$ (Algebra, Lemma 10.46.2). We set

$$C = \{\lambda \in A' \mid \lambda \bmod \mathfrak{m}_{A'} \in A''\}.$$

which is a valuation ring by Algebra, Lemma 10.46.7. As C is an R-algebra with fraction field K', we obtain a solid commutative diagram

$$\begin{array}{ccccccc} \mathrm{Spec}(K_1') & \dashrightarrow & \mathrm{Spec}(K') & \longrightarrow & U & \twoheadrightarrow & X \\ \vdots & & \downarrow & \nearrow & & & \downarrow \\ \mathrm{Spec}(C_1) & \dashrightarrow & \mathrm{Spec}(C) & & \longrightarrow & & Y \end{array}$$

as in the statement of the lemma. Thus assumption (4) produces $C \to C_1$ and the dotted arrows making the diagram commute. Let $A_1' = (C_1)_{\mathfrak{p}}$ be the localization of C_1 at a prime $\mathfrak{p} \subset C_1$ lying over $\mathfrak{m}_{A'} \subset C$. Since $C \to C_1$ is flat by More on Algebra, Lemma 15.15.4 such a prime $\mathfrak{p}$ exists by Algebra, Lemmas 10.36.16 and 10.36.15. Note that A' is the localization of C at $\mathfrak{m}_{A'}$ and that A_1' is a valuation ring (Algebra, Lemma 10.46.6). In other words, $A' \to A_1'$ is a local ring map of valuation rings. Assumption (3) implies

$$\begin{array}{ccccc} \mathrm{Spec}(A_1') & \longrightarrow & \mathrm{Spec}(C_1) & \longrightarrow & X \\ \downarrow & & & & \uparrow \\ \mathrm{Spec}(A') & \longrightarrow & \mathrm{Spec}(P) & \longrightarrow & \mathrm{Spec}(B) \end{array}$$

commutes. Hence the restriction of the morphism $\mathrm{Spec}(C_1) \to X$ to $\mathrm{Spec}(C_1/\mathfrak{p})$ restricts to the composition

$$\mathrm{Spec}(\kappa(\mathfrak{p})) \to \mathrm{Spec}(A'/\mathfrak{m}_{A'}) = \mathrm{Spec}(K'') \to \mathrm{Spec}(K) \to X$$

on the generic point of $\mathrm{Spec}(C_1/\mathfrak{p})$. Moreover, $C_1/\mathfrak{p}$ is a valuation ring (Algebra, Lemma 10.46.6) dominating A'' which dominates A. Thus the morphism $\mathrm{Spec}(C_1/\mathfrak{p}) \to X$ witnesses the existence part of the valuative criterion for the diagram (49.38.3.1) as desired.

Next, suppose that (5) is satisfied as well, i.e., the morphism $\Delta : X \to X \times_S X$ is quasi-compact. In this case assumptions (1) – (4) hold for h and Δ. Hence the first part of the proof shows that Δ is universally closed. By Lemma 49.36.8 we conclude that f is separated. □

49.39. Valuative criterion of separatedness

Lemma 49.39.1. *Let S be a scheme. Let $f : X \to Y$ be a morphism of algebraic spaces over S. If f is separated, then f satisfies the uniqueness part of the valuative criterion.*

Proof. Let a diagram as in Definition 49.37.1 be given. Suppose there are two distinct morphisms $a, b : \mathrm{Spec}(A) \to X$ fitting into the diagram. Let $Z \subset \mathrm{Spec}(A)$ be the equalizer of a and b. Then $Z = \mathrm{Spec}(A) \times_{(a,b),X \times_Y X,\Delta} X$. If f is separated, then Δ is a closed immersion, and this is a closed subscheme of $\mathrm{Spec}(A)$. By assumption it contains the generic point of $\mathrm{Spec}(A)$. Since A is a domain this implies $Z = \mathrm{Spec}(A)$. Hence $a = b$ as desired. □

Lemma 49.39.2 (Valuative criterion separatedness)**.** *Let S be a scheme. Let $f : X \to Y$ be a morphism of algebraic spaces over S. Assume*

(1) *the morphism f is quasi-separated, and*
(2) *the morphism f satisfies the uniqueness part of the valuative criterion.*

Then f is separated.

Proof. Assumption (1) means $\Delta_{X/Y}$ is quasi-compact. We claim the morphism $\Delta_{X/Y} : X \to X \times_Y X$ satisfies the existence part of the valuative criterion. Let a solid commutative diagram

$$\begin{array}{ccc} \mathrm{Spec}(K) & \longrightarrow & X \\ \downarrow & \nearrow & \downarrow \\ \mathrm{Spec}(A) & \longrightarrow & X \times_Y X \end{array}$$

be given. The lower right arrow corresponds to a pair of morphisms $a, b : \mathrm{Spec}(A) \to X$ over Y. By assumption (2) we see that $a = b$. Hence using a as the dotted arrow works. Hence Lemma 49.38.1 applies, and we see that $\Delta_{X/Y}$ is universally closed. Since always $\Delta_{X/Y}$ is locally of finite type and separated, we conclude from More on Morphisms, Lemma 36.31.5 that $\Delta_{X/Y}$ is a finite morphism (also, use the general principle of Spaces, Lemma 47.5.8). At this point $\Delta_{X/Y}$ is a representable, finite monomorphism, hence a closed immersion by Morphisms, Lemma 28.44.13. □

49.40. Integral and finite morphisms

We have already defined in Section 49.3 what it means for a representable morphism of algebraic spaces to be integral (resp. finite).

Lemma 49.40.1. *Let S be a scheme. Let $f : X \to Y$ be a representable morphism of algebraic spaces over S. Then f is integral (resp. finite) if and only if for all affine schemes Z and morphisms $Z \to Y$ the scheme $X \times_Y Z$ is affine and integral (resp. finite) over Z.*

Proof. This follows directly from the definition of an integral (resp. finite) morphism of schemes (Morphisms, Definition 28.44.1). □

This clears the way for the following definition.

Definition 49.40.2. Let S be a scheme. Let $f : X \to Y$ be a morphism of algebraic spaces over S.

(1) We say that f is *integral* if for every affine scheme Z and morphisms $Z \to Y$ the algebraic space $X \times_Y Z$ is representable by an affine scheme integral over Z.
(2) We say that f is *finite* if for every affine scheme Z and morphisms $Z \to Y$ the algebraic space $X \times_Y Z$ is representable by an affine scheme finite over Z.

Lemma 49.40.3. *Let S be a scheme. Let $f : X \to Y$ be a morphism of algebraic spaces over S. The following are equivalent:*

(1) *f is representable and integral (resp. finite),*
(2) *f is integral (resp. finite),*
(3) *there exists a scheme V and a surjective étale morphism $V \to Y$ such that $V \times_Y X \to V$ is integral (resp. finite), and*
(4) *there exists a Zariski covering $Y = \bigcup Y_i$ such that each of the morphisms $f^{-1}(Y_i) \to Y_i$ is integral (resp. finite).*

Proof. It is clear that (1) implies (2) and that (2) implies (3) by taking V to be a disjoint union of affines étale over Y, see Properties of Spaces, Lemma 48.6.1. Assume $V \to Y$ is as in (3). Then for every affine open W of V we see that $W \times_Y X$ is an affine open of $V \times_Y X$. Hence by Properties of Spaces, Lemma 48.10.1 we conclude that $V \times_Y X$ is a scheme. Moreover the morphism $V \times_Y X \to V$ is affine. This means we can apply Spaces, Lemma 47.11.3 because the class of integral (resp. finite) morphisms satisfies all the required properties (see Morphisms, Lemmas 28.44.6 and Descent, Lemmas 34.19.20, 34.19.21, and 34.33.1). The conclusion of applying this lemma is that f is representable and integral (resp. finite), i.e., (1) holds.

The equivalence of (1) and (4) follows from the fact that being integral (resp. finite) is Zariski local on the target (the reference above shows that being integral or finite is in fact fpqc local on the target). □

Lemma 49.40.4. *The composition of integral (resp. finite) morphisms is integral (resp. finite).*

Proof. Omitted. □

Lemma 49.40.5. *The base change of an integral (resp. finite) morphism is integral (resp. finite).*

Proof. Omitted. □

Lemma 49.40.6. *A finite morphism of algebraic spaces is integral. An integral morphism of algebraic spaces which is locally of finite type is finite.*

Proof. In both cases the morphism is representable, and you can check the condition after a base change by an affine scheme mapping into Y, see Lemmas 49.40.3. Hence this lemma follows from the same lemma for the case of schemes, see Morphisms, Lemma 28.44.4. □

Lemma 49.40.7. *Let S be a scheme. Let $f : X \to Y$ be a morphism of algebraic spaces over S. The following are equivalent*

(1) *f is integral, and*
(2) *f is affine and universally closed.*

Proof. In both cases the morphism is representable, and you can check the condition after a base change by an affine scheme mapping into Y, see Lemmas 49.40.3, 49.19.3, and 49.10.5. Hence the result follows from Morphisms, Lemma 28.44.7. □

Lemma 49.40.8. *A finite morphism of algebraic spaces is quasi-finite.*

Proof. Let $f : X \to Y$ be a morphism of algebraic spaces. By Definition 49.40.2 and Lemmas 49.9.7 and 49.25.5 both properties may be checked after base change to an affine over Y, i.e., we may assume Y affine. If f is finite then X is a scheme. Hence the result follows from the corresponding result for schemes, see Morphisms, Lemma 28.44.9. □

Lemma 49.40.9. *A finite morphism of algebraic spaces is proper.*

Proof. Let $f : X \to Y$ be a morphism of algebraic spaces. We think of proper as synonymous to "finite type, separated, and universally closed". By Definition 49.40.2 and Lemmas 49.22.4, 49.5.12, and 49.10.5 both properties may be checked

after base change to an affine over Y, i.e., we may assume Y affine. If f is finite then X is a scheme. Hence the result follows from the corresponding result for schemes, see Morphisms, Lemma 28.44.10. □

Lemma 49.40.10. *A closed immersion is finite (and a fortiori integral).*

Proof. Omitted. □

Lemma 49.40.11. *Let S be a scheme. Let $f : X \to Y$ and $g : Y \to Z$ be morphisms of algebraic spaces over S.*

(1) *If $g \circ f$ is finite and g separated then f is finite.*
(2) *If $g \circ f$ is integral and g separated then f is integral.*

Proof. Assume $g \circ f$ is finite (resp. integral) and g separated. The base change $X \times_Z Y \to Y$ is finite (resp. integral) by Lemma 49.40.5. The morphism $X \to X \times_Z Y$ is a closed immersion as $Y \to Z$ is separated, see Lemma 49.5.7. A closed immersion is finite (resp. integral), see Lemma 49.40.10. The composition of finite (resp. integral) morphisms is finite (resp. integral), see Lemma 49.40.4. Thus we win. □

49.41. Finite locally free morphisms

We have already defined in Section 49.3 what it means for a representable morphism of algebraic spaces to be finite locally free.

Lemma 49.41.1. *Let S be a scheme. Let $f : X \to Y$ be a representable morphism of algebraic spaces over S. Then f is finite locally free if and only if f is affine and the sheaf $f_*\mathcal{O}_X$ is a finite locally free $\mathcal{O}_Y$-module.*

Proof. Assume f is finite locally free (as defined in Section 49.3). This means that for every morphism $V \to Y$ whose source is a scheme the base change $f' : V \times_Y X \to V$ is a finite locally free morphism of schemes. This in turn means (by the definition of a finite locally free morphism of schemes) that $f'_*\mathcal{O}_{V \times_Y X}$ is a finite locally free $\mathcal{O}_V$-module. We may choose $V \to Y$ to be surjective and étale. By Properties of Spaces, Lemma 48.24.2 we conclude the restriction of $f_*\mathcal{O}_X$ to V is finite locally free. Hence by Modules on Sites, Lemma 18.23.3 applied to the sheaf $f_*\mathcal{O}_X$ on $Y_{spaces,étale}$ we conclude that $f_*\mathcal{O}_X$ is finite locally free.

Conversely, assume f is affine and that $f_*\mathcal{O}_X$ is a finite locally free $\mathcal{O}_Y$-module. Let V be a scheme, and let $V \to Y$ be a surjective étale morphism. Again by Properties of Spaces, Lemma 48.24.2 we see that $f'_*\mathcal{O}_{V \times_Y X}$ is finite locally free. Hence $f' : V \times_Y X \to V$ is finite locally free (as it is also affine). By Spaces, Lemma 47.11.3 we conclude that f is finite locally free (use Morphisms, Lemma 28.46.4 Descent, Lemmas 34.19.28 and 34.33.1). Thus we win. □

This clears the way for the following definition.

Definition 49.41.2. Let S be a scheme. Let $f : X \to Y$ be a morphism of algebraic spaces over S. We say that f is *finite locally free* if f is affine and $f_*\mathcal{O}_X$ is a finite locally free $\mathcal{O}_Y$-module. In this case we say f is has *rank* or *degree* d if the sheaf $f_*\mathcal{O}_X$ is finite locally free of rank d.

Lemma 49.41.3. *Let S be a scheme. Let $f : X \to Y$ be a morphism of algebraic spaces over S. The following are equivalent:*

(1) *f is representable and finite locally free,*
(2) *f is finite locally free,*
(3) *there exists a scheme V and a surjective étale morphism $V \to Y$ such that $V \times_Y X \to V$ is finite locally free, and*
(4) *there exists a Zariski covering $Y = \bigcup Y_i$ such that each morphism $f^{-1}(Y_i) \to Y_i$ is finite locally free.*

Proof. It is clear that (1) implies (2) and that (2) implies (3) by taking V to be a disjoint union of affines étale over Y, see Properties of Spaces, Lemma 48.6.1. Assume $V \to Y$ is as in (3). Then for every affine open W of V we see that $W \times_Y X$ is an affine open of $V \times_Y X$. Hence by Properties of Spaces, Lemma 48.10.1 we conclude that $V \times_Y X$ is a scheme. Moreover the morphism $V \times_Y X \to V$ is affine. This means we can apply Spaces, Lemma 47.11.3 because the class of finite locally free morphisms satisfies all the required properties (see Morphisms, Lemma 28.46.4 Descent, Lemmas 34.19.28 and 34.33.1). The conclusion of applying this lemma is that f is representable and finite locally free, i.e., (1) holds.

The equivalence of (1) and (4) follows from the fact that being finite locally free is Zariski local on the target (the reference above shows that being finite locally free is in fact fpqc local on the target). □

Lemma 49.41.4. *The composition of finite locally free morphisms is finite locally free.*

Proof. Omitted. □

Lemma 49.41.5. *The base change of a finite locally free morphism is finite locally free.*

Proof. Omitted. □

Lemma 49.41.6. *Let S be a scheme. Let $f : X \to Y$ be a morphism of algebraic spaces over S. The following are equivalent:*

(1) *f is finite locally free,*
(2) *f is finite, flat, and locally of finite presentation.*

If Y is locally Noetherian these are also equivalent to

(3) *f is finite and flat.*

Proof. In each of the three cases the morphism is representable and you can check the property after base change by a surjective étale morphism $V \to Y$, see Lemmas 49.40.3, 49.41.3, 49.27.5, and 49.26.4. If Y is locally Noetherian, then V is locally Noetherian. Hence the result follows from the corresponding result in the schemes case, see Morphisms, Lemma 28.46.2. □

49.42. Normalization of algebraic spaces

This section is the analogue of Morphisms, Section 28.48.

Lemma 49.42.1. *Let S be a scheme. Let X be an algebraic space over S. Let $\mathcal{A}$ be a quasi-coherent sheaf of $\mathcal{O}_X$-algebras. There exists a quasi-coherent sheaf of $\mathcal{O}_X$-algebras $\mathcal{A}' \subset \mathcal{A}$ such that for any affine object U of $X_{étale}$ the ring $\mathcal{A}'(U) \subset \mathcal{A}(U)$ is the integral closure of $\mathcal{O}_X(U)$ in $\mathcal{A}(U)$.*

Proof. By Properties of Spaces, Lemma 48.15.5 it suffices to prove that the rule given above defines a quasi-coherent module on $X_{affine,étale}$. To see this it suffices to show the following: Let $U_1 \to U_2$ be a morphism of affine objects of $X_{étale}$. Say $U_i = \mathrm{Spec}(R_i)$. Say $\mathcal{A}|_{(U_1)_{étale}}$ is the quasi-coherent sheaf associated to the R_2-algebra A. Let $A' \subset A$ be the integral closure of R_2 in A. Then $A' \otimes_{R_2} R_1$ is the integral closure of R_1 in $A \otimes_{R_2} R_1$. This is Algebra, Lemma 10.138.2. □

Definition 49.42.2. Let S be a scheme. Let X be an algebraic space over S. Let $\mathcal{A}$ be a quasi-coherent sheaf of $\mathcal{O}_X$-algebras. The *integral closure of $\mathcal{O}_X$ in $\mathcal{A}$* is the quasi-coherent $\mathcal{O}_X$-subalgebra $\mathcal{A}' \subset \mathcal{A}$ constructed in Lemma 49.42.1 above.

We will apply this in particular when $\mathcal{A} = f_*\mathcal{O}_Y$ for a quasi-compact and quasi-separated morphism of algebraic spaces $f : Y \to X$ (see Lemma 49.12.2). We can then take the relative spectrum of the quasi-coherent $\mathcal{O}_X$-algebra (Lemma 49.19.7) to obtain the normalization of X in Y.

Definition 49.42.3. Let S be a scheme. Let $f : Y \to X$ be a quasi-compact and quasi-separated morphism of algebraic spaces over S. Let $\mathcal{O}'$ be the integral closure of $\mathcal{O}_X$ in $f_*\mathcal{O}_Y$. The *normalization of X in Y* is the morphism of algebraic spaces

$$\nu : X' = \underline{\mathrm{Spec}}_X(\mathcal{O}') \to X$$

over S. It comes equipped with a natural factorization

$$Y \xrightarrow{f'} X' \xrightarrow{\nu} X$$

of the initial morphism f.

To get the factorization, use Remark 49.19.9 and functoriality of the $\underline{\mathrm{Spec}}$ construction. Here is a characterization of this construction.

Lemma 49.42.4. *Let S be a scheme. Let $f : Y \to X$ be a quasi-compact and quasi-separated morphism of algebraic spaces over S. The factorization $f = \nu \circ f'$, where $\nu : X' \to X$ is the normalization of X in Y is characterized by the following two properties:*

1. *the morphism ν is integral, and*
2. *for any factorization $f = \pi \circ g$, with $\pi : Z \to X$ integral, there exists a commutative diagram*

$$\begin{array}{ccc} Y & \xrightarrow{g} & Z \\ {\scriptstyle f'}\downarrow & \overset{h}{\nearrow} & \downarrow{\scriptstyle \pi} \\ X' & \xrightarrow{\nu} & X \end{array}$$

for a unique morphism $h : X' \to Z$.

Moreover, in (2) the morphism $h : X' \to Z$ is the normalization of Z in Y.

Proof. Let $\mathcal{O}' \subset f_*\mathcal{O}_Y$ be the integral closure of $\mathcal{O}_X$ as in Definition 49.42.3. The morphism ν is integral by construction, which proves (1). Assume given a factorization $f = \pi \circ g$ with $\pi : Z \to X$ integral as in (2). By Definition 49.40.2 π is affine, and hence Z is the relative spectrum of a quasi-coherent sheaf of $\mathcal{O}_X$-algebras $\mathcal{B}$. The morphism $g : X \to Z$ corresponds to a map of $\mathcal{O}_X$-algebras $\chi : \mathcal{B} \to f_*\mathcal{O}_Y$. Since $\mathcal{B}(U)$ is integral over $\mathcal{O}_X(U)$ for every affine U étale over X (by Definition 49.40.2) we see from Lemma 49.42.1 that $\chi(\mathcal{B}) \subset \mathcal{O}'$. By the

functoriality of the relative spectrum Lemma 49.19.7 this provides us with a unique morphism $h : X' \to Z$. We omit the verification that the diagram commutes.

It is clear that (1) and (2) characterize the factorization $f = \nu \circ f'$ since it characterizes it as an initial object in a category. The morphism h in (2) is integral by Lemma 49.40.11. Given a factorization $g = \pi' \circ g'$ with $\pi' : Z' \to Z$ integral, we get a factorization $f = (\pi \circ \pi') \circ g'$ and we get a morphism $h' : X' \to Z'$. Uniqueness implies that $\pi' \circ h' = h$. Hence the characterization (1), (2) applies to the morphism $h : X' \to Z$ which gives the last statement of the lemma. □

Lemma 49.42.5. *Let S be a scheme. Let $f : Y \to X$ be a quasi-compact and quasi-separated morphism of algebraic spaces over S. Suppose that $Y = Y_1 \coprod Y_2$ is a disjoint union of two algebraic spaces. Write $f_i = f|_{Y_i}$. Let X_i' be the normalization of X in Y_i. Then $X_1' \coprod X_2'$ is the normalization of X in Y.*

Proof. Omitted. □

Lemma 49.42.6. *Let S be a scheme. Let $f : Y \to X$ be an integral morphism of algebraic spaces over S. Then the integral closure of X in Y is equal to Y.*

Proof. Omitted. □

Lemma 49.42.7. *Let S be a scheme. Let X be an algebraic space over S. Let U be a scheme and let $U \to X$ be a surjective étale morphism. Assume that every quasi-compact open of U has finitely many irreducible components. Then there exists an integral morphism of algebraic spaces*

$$X^\nu \longrightarrow X$$

such that $X^\nu \times_X U$ is the normalization of U.

Proof. Set $R = U \times_X U$ with projections $s, t : R \to U$ and $j = (t, s) : R \to U \times_S U$ so that $X = U/R$, see Spaces, Lemma 47.9.1. The assumption on U means that the normalization U^ν of U is defined, see Morphisms, Definition 28.48.12. By More on Morphisms, Lemma 36.14.3 taking normalization commutes with étale morphisms of schemes. Thus we see that the normalization R^ν of R is isomorphic to both $R \times_{s,U} U^\nu$ and $U^\nu \times_{U,t} R$. Thus we obtain two étale morphisms $s^\nu : R^\nu \to U^\nu$ and $t^\nu : R^\nu \to U^\nu$ of schemes. The induced morphism $j^\nu : R^\nu \to U^\nu \times_S U^\nu$ is a monomorphism as R^ν is a subscheme of the restriction of R to U^ν. A formal computation with fibre products shows that $R^\nu \times_{s^\nu, U^\nu, t^\nu} R^\nu$ is the normalization of $R \times_{s,U,t} R$. Hence the (étale) morphism $c : R \times_{s,U,t} R \to R$ extends to c^ν as well. Combined we see that we obtain an étale equivalence relation. Setting $X^\nu = U^\nu/R^\nu$ (Spaces, Theorem 47.10.5) we win by Groupoids, Lemma 38.18.7. □

49.43. Separated, locally quasi-finite morphisms

We prove a result that is so interesting it deserves its own section. The result is that an algebraic space which is locally quasi-finite and separated over a scheme is a scheme. It implies that a separated and locally quasi-finite morphism is representable. But first... a lemma (which will be obsoleted by Proposition 49.43.2).

Lemma 49.43.1. *Let S be a scheme. Consider a commutative diagram*

$$\begin{array}{ccccc} V' & \longrightarrow & T' \times_T X & \longrightarrow & X \\ & \searrow & \downarrow & & \downarrow \\ & & T' & \longrightarrow & T \end{array}$$

Assume

(1) *$T' \to T$ is an étale morphism of affine schemes,*
(2) *X is an algebraic space,*
(3) *$X \to T$ is a separated, locally quasi-finite morphism,*
(4) *V' is an open subspace of $T' \times_T X$, and*
(5) *$V' \to T'$ is quasi-affine.*

In this situation the image U of V' in X is a quasi-compact open subspace of X which is representable.

Proof. We first make some trivial observations. Note that V' is representable by Lemma 49.20.3. It is also quasi-compact (as a quasi-affine scheme over an affine scheme, see Morphisms, Lemma 28.14.2). Since $T' \times_T X \to X$ is étale (Properties of Spaces, Lemma 48.13.5) the map $|T' \times_T X| \to |X|$ is open, see Properties of Spaces, Lemma 48.13.7. Let $U \subset X$ be the open subspace corresponding to the image of $|V'|$, see Properties of Spaces, Lemma 48.4.8. As $|V'|$ is quasi-compact we see that $|U|$ is quasi-compact, hence U is a quasi-compact algebraic space, by Properties of Spaces, Lemma 48.5.2.

By Morphisms, Lemma 28.50.8 the morphism $T' \to T$ is universally bounded. Hence we can do induction on the integer n bounding the degree of the fibres of $T' \to T$, see Morphisms, Lemma 28.50.7 for a description of this integer in the case of an étale morphism. If $n = 1$, then $T' \to T$ is an open immersion (see Étale Morphisms, Theorem 40.14.1), and the result is clear. Assume $n > 1$.

Consider the affine scheme $T'' = T' \times_T T'$. As $T' \to T$ is étale we have a decomposition (into open and closed affine subschemes) $T'' = \Delta(T') \amalg T^*$. Namely $\Delta = \Delta_{T'/T}$ is open by Morphisms, Lemma 28.36.13 and closed because $T' \to T$ is separated as a morphism of affines. As a base change the degrees of the fibres of the second projection $\mathrm{pr}_1 : T' \times_T T' \to T'$ are bounded by n, see Morphisms, Lemma 28.50.4. On the other hand, $\mathrm{pr}_1|_{\Delta(T')} : \Delta(T') \to T'$ is an isomorphism and every fibre has exactly one point. Thus, on applying Morphisms, Lemma 28.50.7 we conclude the degrees of the fibres of the restriction $\mathrm{pr}_1|_{T^*} : T^* \to T'$ are bounded by $n - 1$. Hence the induction hypothesis applied to the diagram

$$\begin{array}{ccccc} p_0^{-1}(V') \cap X^* & \longrightarrow & X^* & \xrightarrow[p_1|_{X^*}]{} & X' \\ & \searrow & \downarrow & & \downarrow \\ & & T^* & \xrightarrow{\mathrm{pr}_1|_{T^*}} & T' \end{array}$$

gives that $p_1(p_0^{-1}(V') \cap X^*)$ is a quasi-compact scheme. Here we set $X'' = T'' \times_T X$, $X^* = T^* \times_T X$, and $X' = T' \times_T X$, and $p_0, p_1 : X'' \to X'$ are the base changes of $\mathrm{pr}_0, \mathrm{pr}_1$. Most of the hypotheses of the lemma imply by base change the corresponding hypothesis for the diagram above. For example $p_0^{-1}(V') = T'' \times_{T'} V'$ is a scheme quasi-affine over T'' as a base change. Some verifications omitted.

By Properties of Spaces, Lemma 48.10.1 we conclude that

$$p_1(p_0^{-1}(V')) = V' \cup p_1(p_0^{-1}(V') \cap X^*)$$

is a quasi-compact scheme. Moreover, it is clear that $p_1(p_0^{-1}(V'))$ is the inverse image of the quasi-compact open subspace $U \subset X$ discussed in the first paragraph of the proof. In other words, $T' \times_T U$ is a scheme! Note that $T' \times_T U$ is quasi-compact and separated and locally quasi-finite over T', as $T' \times_T X \to T'$ is locally quasi-finite and separated being a base change of the original morphism $X \to T$ (see Lemmas 49.5.4 and 49.25.3). This implies by More on Morphisms, Lemma 36.31.2 that $T' \times_T U \to T'$ is quasi-affine.

By Descent, Lemma 34.35.1 this gives a descent datum on $T' \times_T U/T'$ relative to the étale covering $\{T' \to W\}$, where $W \subset T$ is the image of the morphism $T' \to T$. Because U' is quasi-affine over T' we see from Descent, Lemma 34.34.1 that this datum is effective, and by the last part of Descent, Lemma 34.35.1 this implies that U is a scheme as desired. Some minor details omitted. □

Proposition 49.43.2. *Let S be a scheme. Let $f : X \to T$ be a morphism of algebraic spaces. Assume*

(1) *T is representable,*
(2) *f is locally quasi-finite, and*
(3) *f is separated.*

Then X is representable.

Proof. Let $T = \bigcup T_i$ be an affine open covering of the scheme T. If we can show that the open subspaces $X_i = f^{-1}(T_i)$ are representable, then X is representable, see Properties of Spaces, Lemma 48.10.1. Note that $X_i = T_i \times_T X$ and that locally quasi-finite and separated are both stable under base change, see Lemmas 49.5.4 and 49.25.3. Hence we may assume T is an affine scheme.

By Properties of Spaces, Lemma 48.6.2 there exists a Zariski covering $X = \bigcup X_i$ such that each X_i has a surjective étale covering by an affine scheme. By Properties of Spaces, Lemma 48.10.1 again it suffices to prove the proposition for each X_i. Hence we may assume there exists an affine scheme U and a surjective étale morphism $U \to X$. This reduces us to the situation in the next paragraph.

Assume we have

$$U \longrightarrow X \longrightarrow T$$

where U and T are affine schemes, $U \to X$ is étale surjective, and $X \to T$ is separated and locally quasi-finite. By Lemmas 49.35.5 and 49.25.2 the morphism $U \to T$ is locally quasi-finite. Since U and T are affine it is quasi-finite. Set $R = U \times_X U$. Then $X = U/R$, see Spaces, Lemma 47.9.1. As $X \to T$ is separated the morphism $R \to U \times_T U$ is a closed immersion, see Lemma 49.5.5. In particular R is an affine scheme also. As $U \to X$ is étale the projection morphisms $t, s : R \to U$ are étale as well. In particular s and t are quasi-finite, flat and of finite presentation (see Morphisms, Lemmas 28.37.6, 28.37.12 and 28.37.11).

Let (U, R, s, t, c) be the groupoid associated to the étale equivalence relation R on U. Let $u \in U$ be a point, and denote $p \in T$ its image. We are going to use More on Groupoids, Lemma 39.12.2 for the groupoid (U, R, s, t, c) over the scheme T with points p and u as above. By the discussion in the previous paragraph

all the assumptions (1) – (7) of that lemma are satisfied. Hence we get an étale neighbourhood $(T', p') \to (T, p)$ and disjoint union decompositions

$$U_{T'} = U' \amalg W, \quad R_{T'} = R' \amalg W'$$

and $u' \in U'$ satisfying conclusions (a), (b), (c), (d), (e), (f), (g), and (h) of the aforementioned More on Groupoids, Lemma 39.12.2. We may and do assume that T' is affine (after possibly shrinking T'). Conclusion (h) implies that $R' = U' \times_{X_{T'}} U'$ with projection mappings identified with the restrictions of s' and t'. Thus $(U', R', s'|_{R'}, t'|_{R'}, c'|_{R' \times_{t', U', s'} R'})$ of conclusion (g) is an étale equivalence relation. By Spaces, Lemma 47.10.2 we conclude that U'/R' is an open subspace of $X_{T'}$. By conclusion (d) the schemes U', R' are affine and the morphisms $s'|_{R'}, t'|_{R'}$ are finite étale. Hence Groupoids, Proposition 38.20.8 kicks in and we see that U'/R' is an affine scheme.

We conclude that for every pair of points (u, p) as above we can find an étale neighbourhood $(T', p') \to (T, p)$ with $\kappa(p) = \kappa(p')$ and a point $u' \in U_{T'}$ mapping to u such that the image x' of u' in $|X_{T'}|$ has an open neighbourhood V' in $X_{T'}$ which is an affine scheme. We apply Lemma 49.43.1 to obtain an open subspace $W \subset X$ which is a scheme, and which contains x (the image of u in $|X|$). Since this works for every x we see that X is a scheme by Properties of Spaces, Lemma 48.10.1. This ends the proof. □

49.44. Applications

Here is another formulation of the result above.

Lemma 49.44.1. *Let S be a scheme. Let $f : X \to Y$ be a morphism of algebraic spaces over S. If f is locally quasi-finite and separated, then f is representable.*

Proof. This is immediate from Proposition 49.43.2 and the fact that being locally quasi-finite and separated is preserved under any base change, see Lemmas 49.25.3 and 49.5.4. □

Lemma 49.44.2. *Let S be a scheme. Let $f : X \to Y$ be an étale and universally injective morphism of algebraic spaces over S. Then f is an open immersion.*

Proof. Let $T \to Y$ be a morphism from a scheme into Y. If we can show that $X \times_Y T \to T$ is an open immersion, then we are done. Since being étale and being universally injective are properties of morphisms stable under base change (see Lemmas 49.35.4 and 49.18.5) we may assume that Y is a scheme. Note that the diagonal $\Delta_{X/Y} : X \to X \times_Y X$ is étale, a monomorphism, and surjective by Lemma 49.18.2. Hence we see that $\Delta_{X/Y}$ is an isomorphism (see Spaces, Lemma 47.5.9), in particular we see that X is separated over Y. It follows that X is a scheme too, by Proposition 49.43.2. Finally, $X \to Y$ is an open immersion by the fundamental theorem for étale morphisms of schemes, see Étale Morphisms, Theorem 40.14.1. □

49.45. Universal homeomorphisms

In Morphisms, Section 28.45 we have shown that a morphism of schemes is a universal homeomorphism if and only if it is integral, universally injective, and surjective. In particular the class of universal homeomorphisms of schemes is closed under composition and arbitrary base change and is fppf local on the base (as this is true

for integral, universally injective, and surjective morphisms). Thus, if we apply the discussion in Section 49.3 to this notion we see that we know what it means for a representable morphism of algebraic spaces to be a universal homeomorphism.

Lemma 49.45.1. *Let S be a scheme. Let $f : X \to Y$ be a representable morphism of algebraic spaces over S. Then f is a universal homeomorphism (as discussed above) if and only if for every morphism of algebraic spaces $Z \to Y$ the base change map $Z \times_Y X \to Z$ induces a homeomorphism $|Z \times_Y X| \to |Z|$.*

Proof. If for every morphism of algebraic spaces $Z \to Y$ the base change map $Z \times_Y X \to Z$ induces a homeomorphism $|Z \times_Y X| \to |Z|$, then the same is true whenever Z is a scheme, which formally implies that f is a universal homeomorphism in the sense of Section 49.3. Conversely, if f is a universal homeomorphism in the sense of Section 49.3 then $X \to Y$ is integral, universally injective and surjective (by Spaces, Lemma 47.5.8 and Morphisms, Lemma 28.45.3). Hence f is universally closed, see Lemma 49.40.7 and universally injective and (universally) surjective, i.e., f is a universal homeomorphism. □

Definition 49.45.2. Let S be a scheme. A morphisms $f : X \to Y$ of algebraic spaces over S is called a *universal homeomorphism* if and only if for every morphism of algebraic spaces $Z \to Y$ the base change $Z \times_Y X \to Z$ induces a homeomorphism $|Z \times_Y X| \to |Z|$.

This definition does not clash with the pre-existing definition for representable morphisms of algebraic spaces by our Lemma 49.45.1. For morphisms of algebraic spaces it is not the case that universal homeomorphisms are always integral.

Example 49.45.3. This is a continuation of Remark 49.18.4. Consider the algebraic space $X = \mathbf{A}^1_k/\{x \sim -x \mid x \neq 0\}$. There are morphisms

$$\mathbf{A}^1_k \longrightarrow X \longrightarrow \mathbf{A}^1_k$$

such that the first arrow is étale surjective, the second arrow is universally injective, and the composition is the map $x \mapsto x^2$. Hence the composition is universally closed. Thus it follows that the map $X \to \mathbf{A}^1_k$ is a universal homeomorphism, but $X \to \mathbf{A}^1_k$ is not separated.

Let S be a scheme. Let $f : X \to Y$ be a universal homeomorphism of algebraic spaces over S. Then f is universally closed, hence is quasi-compact, see Lemma 49.10.7. But f need not be separated (see example above), and not even quasi-separated: an example is to take infinite dimensional affine space $\mathbf{A}^\infty = \mathrm{Spec}(k[x_1, x_2, \ldots])$ modulo the equivalence relation given by flipping finitely many signs of nonzero coordinates (details omitted).

Lemma 49.45.4. *Let S be a scheme. Let X be an algebraic space over S. The canonical closed immersion $X_{red} \to X$ (see Properties of Spaces, Definition 48.9.4) is a universal homeomorphism.*

Proof. Omitted. □

49.46. Other chapters

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CHAPTER 50

Decent Algebraic Spaces

50.1. Introduction

In this chapter we talk study "local" properties of general algebraic spaces, i.e., those algebraic spaces which aren't quasi-separated. Quasi-separated algebraic spaces are studied in [**Kol96**]. It turns out that essentially new phenomena happen, especially regarding points and specializations of points, on more general algebraic spaces. On the other hand, for most basic results on algebraic spaces, one needn't worry about these phenomena, which is why we have decided to have this material in a separate chapter following the standard development of the theory.

50.2. Conventions

The standing assumption is that all schemes are contained in a big fppf site Sch_{fppf}. And all rings A considered have the property that $\mathrm{Spec}(A)$ is (isomorphic) to an object of this big site.

Let S be a scheme and let X be an algebraic space over S. In this chapter and the following we will write $X \times_S X$ for the product of X with itself (in the category of algebraic spaces over S), instead of $X \times X$.

50.3. Universally bounded fibres

We briefly discuss what it means for a morphism from a scheme to an algebraic space to have universally bounded fibres. Please refer to Morphisms, Section 28.50 for similar definitions and results on morphisms of schemes.

Definition 50.3.1. Let S be a scheme. Let X be an algebraic space over S, and let U be a scheme over S. Let $f : U \to X$ be a morphism over S. We say the *fibres of f are universally bounded*[1] if there exists an integer n such that for all fields k and all morphisms $\mathrm{Spec}(k) \to X$ the fibre product $\mathrm{Spec}(k) \times_X U$ is a finite scheme over k whose degree over k is $\leq n$.

This definition makes sense because the fibre product $\mathrm{Spec}(k) \times_Y X$ is a scheme. Moreover, if Y is a scheme we recover the notion of Morphisms, Definition 28.50.1 by virtue of Morphisms, Lemma 28.50.2.

Lemma 50.3.2. *Let S be a scheme. Let X be an algebraic space over S. Let $V \to U$ be a morphism of schemes over S, and let $U \to X$ be a morphism from U to X. If the fibres of $V \to U$ and $U \to X$ are universally bounded, then so are the fibres of $V \to X$.*

[1] This is probably nonstandard notation.

Proof. Let n be an integer which works for $V \to U$, and let m be an integer which works for $U \to X$ in Definition 50.3.1. Let $\mathrm{Spec}(k) \to X$ be a morphism, where k is a field. Consider the morphisms

$$\mathrm{Spec}(k) \times_X V \longrightarrow \mathrm{Spec}(k) \times_X U \longrightarrow \mathrm{Spec}(k).$$

By assumption the scheme $\mathrm{Spec}(k) \times_X U$ is finite of degree at most m over k, and n is an integer which bounds the degree of the fibres of the first morphism. Hence by Morphisms, Lemma 28.50.3 we conclude that $\mathrm{Spec}(k) \times_X V$ is finite over k of degree at most nm. □

Lemma 50.3.3. *Let S be a scheme. Let $Y \to X$ be a representable morphism of algebraic spaces over S. Let $U \to X$ be a morphism from a scheme to X. If the fibres of $U \to X$ are universally bounded, then the fibres of $U \times_X Y \to Y$ are universally bounded.*

Proof. This is clear from the definition, and properties of fibre products. (Note that $U \times_X Y$ is a scheme as we assumed $Y \to X$ representable, so the definition applies.) □

Lemma 50.3.4. *Let S be a scheme. Let $g : Y \to X$ be a representable morphism of algebraic spaces over S. Let $f : U \to X$ be a morphism from a scheme towards X. Let $f' : U \times_X Y \to Y$ be the base change of f. If*

$$Im(|f| : |U| \to |X|) \subset Im(|g| : |Y| \to |X|)$$

and f' has universally bounded fibres, then f has universally bounded fibres.

Proof. Let $n \geq 0$ be an integer bounding the degrees of the fibre products $\mathrm{Spec}(k) \times_Y (U \times_X Y)$ as in Definition 50.3.1 for the morphism f'. We claim that n works for f also. Namely, suppose that $x : \mathrm{Spec}(k) \to X$ is a morphism from the spectrum of a field. Then either $\mathrm{Spec}(k) \times_X U$ is empty (and there is nothing to prove), or x is in the image of $|f|$. By Properties of Spaces, Lemma 48.4.3 and the assumption of the lemma we see that this means there exists a field extension $k \subset k'$ and a commutative diagram

$$\begin{array}{ccc} \mathrm{Spec}(k') & \longrightarrow & Y \\ \downarrow & & \downarrow \\ \mathrm{Spec}(k) & \longrightarrow & X \end{array}$$

Hence we see that

$$\mathrm{Spec}(k') \times_Y (U \times_X Y) = \mathrm{Spec}(k') \times_{\mathrm{Spec}(k)} (\mathrm{Spec}(k) \times_X U)$$

Since the scheme $\mathrm{Spec}(k') \times_Y (U \times_X Y)$ is assumed finite of degree $\leq n$ over k' it follows that also $\mathrm{Spec}(k) \times_X U$ is finite of degree $\leq n$ over k as desired. (Some details omitted.) □

Lemma 50.3.5. *Let S be a scheme. Let X be an algebraic space over S. Consider a commutative diagram*

$$\begin{array}{ccc} U & \xrightarrow{f} & V \\ & {\scriptstyle g}\searrow \quad \swarrow{\scriptstyle h} & \\ & X & \end{array}$$

where U and V are schemes. If g has universally bounded fibres, and f is surjective and flat, then also h has universally bounded fibres.

Proof. Assume g has universally bounded fibres, and f is surjective and flat. Say $n \geq 0$ is an integer which bounds the degrees of the schemes $\mathrm{Spec}(k) \times_X U$ as in Definition 50.3.1. We claim n also works for h. Let $\mathrm{Spec}(k) \to X$ be a morphism from the spectrum of a field to X. Consider the morphism of schemes

$$\mathrm{Spec}(k) \times_X V \longrightarrow \mathrm{Spec}(k) \times_X U$$

It is flat and surjective. By assumption the scheme on the left is finite of degree $\leq n$ over $\mathrm{Spec}(k)$. It follows from Morphisms, Lemma 28.50.9 that the degree of the scheme on the right is also bounded by n as desired. □

Lemma 50.3.6. *Let S be a scheme. Let X be an algebraic space over S, and let U be a scheme over S. Let $\varphi : U \to X$ be a morphism over S. If the fibres of φ are universally bounded, then there exists an integer n such that each fibre of $|U| \to |X|$ has at most n elements.*

Proof. The integer n of Definition 50.3.1 works. Namely, pick $x \in |X|$. Represent x by a morphism $x : \mathrm{Spec}(k) \to X$. Then we get a commutative diagram

$$\begin{array}{ccc} \mathrm{Spec}(k) \times_X U & \longrightarrow & U \\ \downarrow & & \downarrow \\ \mathrm{Spec}(k) & \xrightarrow{x} & X \end{array}$$

which shows (via Properties of Spaces, Lemma 48.4.3) that the inverse image of x in $|U|$ is the image of the top horizontal arrow. Since $\mathrm{Spec}(k) \times_X U$ is finite of degree $\leq n$ over k it has at most n points. □

50.4. Finiteness conditions and points

In this section we elaborate on the question of when points can be represented by monomorphisms from spectra of fields into the space.

Remark 50.4.1. Before we give the proof of the next lemma let us recall some facts about étale morphisms of schemes:

(1) An étale morphism is flat and hence generalizations lift along an étale morphism (Morphisms, Lemmas 28.37.12 and 28.26.8).
(2) An étale morphism is unramified, an unramified morphism is locally quasi-finite, hence fibres are discrete (Morphisms, Lemmas 28.37.16, 28.36.10, and 28.21.6).
(3) A quasi-compact étale morphism is quasi-finite and in particular has finite fibres (Morphisms, Lemmas 28.21.9 and 28.21.10).
(4) An étale scheme over a field k is a disjoint union of spectra of finite separable field extension of k (Morphisms, Lemma 28.37.7).

For a general discussion of étale morphisms, please see Étale Morphisms, Section 40.11.

Lemma 50.4.2. *Let S be a scheme. Let X be an algebraic space over S. Let $x \in |X|$. The following are equivalent:*

(1) *there exists a family of schemes U_i and étale morphisms $\varphi_i : U_i \to X$ such that $\coprod \varphi_i : \coprod U_i \to X$ is surjective, and such that for each i the fibre of $|U_i| \to |X|$ over x is finite, and*
(2) *for every affine scheme U and étale morphism $\varphi : U \to X$ the fibre of $|U| \to |X|$ over x is finite.*

Proof. The implication (2) $\Rightarrow$ (1) is trivial. Let $\varphi_i : U_i \to X$ be a family of étale morphisms as in (1). Let $\varphi : U \to X$ be an étale morphism from an affine scheme towards X. Consider the fibre product diagrams

$$\begin{array}{ccc} U \times_X U_i & \xrightarrow[p_i]{} & U_i \\ {\scriptstyle q_i}\downarrow & & \downarrow{\scriptstyle \varphi_i} \\ U & \xrightarrow{\varphi} & X \end{array} \qquad \begin{array}{ccc} \coprod U \times_X U_i & \xrightarrow[\coprod p_i]{} & \coprod U_i \\ {\scriptstyle \coprod q_i}\downarrow & & \downarrow{\scriptstyle \coprod \varphi_i} \\ U & \xrightarrow{\varphi} & X \end{array}$$

Since q_i is étale it is open (see Remark 50.4.1). Moreover, the morphism $\coprod q_i$ is surjective. Hence there exist finitely many indices $i_1, \ldots, i_n$ and a quasi-compact opens $W_{i_j} \subset U \times_X U_{i_j}$ which surject onto U. The morphism p_i is étale, hence locally quasi-finite (see remark on étale morphisms above). Thus we may apply Morphisms, Lemma 28.50.8 to see the fibres of $p_{i_j}|_{W_{i_j}} : W_{i_j} \to U_i$ are finite. Hence by Properties of Spaces, Lemma 48.4.3 and the assumption on φ_i we conclude that the fibre of φ over x is finite. In other words (2) holds. □

Lemma 50.4.3. *Let S be a scheme. Let X be an algebraic space over S. Let $x \in |X|$. The following are equivalent:*

(1) *there exists a scheme U, an étale morphism $\varphi : U \to X$, and points $u, u' \in U$ mapping to x such that setting $R = U \times_X U$ the fibre of*

$$|R| \to |U| \times_{|X|} |U|$$

over (u, u') is finite,
(2) *for every scheme U, étale morphism $\varphi : U \to X$ and any points $u, u' \in U$ mapping to x setting $R = U \times_X U$ the fibre of*

$$|R| \to |U| \times_{|X|} |U|$$

over (u, u') is finite,
(3) *there exists a morphism $\mathrm{Spec}(k) \to X$ with k a field in the equivalence class of x such that the projections $\mathrm{Spec}(k) \times_X \mathrm{Spec}(k) \to \mathrm{Spec}(k)$ are étale and quasi-compact, and*
(4) *there exists a monomorphism $\mathrm{Spec}(k) \to X$ with k a field in the equivalence class of x.*

Proof. Assume (1), i.e., let $\varphi : U \to X$ be an étale morphism from a scheme towards X, and let u, u' be points of U lying over x such that the fibre of $|R| \to |U| \times_{|X|} |U|$ over (u, u') is a finite set. In this proof we think of a point $u = \mathrm{Spec}(\kappa(u))$ as a scheme. Note that $u \to U$, $u' \to U$ are monomorphisms (see Schemes, Lemma 25.23.6), hence $u \times_X u' \to R = U \times_X U$ is a monomorphism. In this language the assumption really means that $u \times_X u'$ is a scheme whose underlying topological space has finitely many points. Let $\psi : W \to X$ be an étale morphism from a scheme towards X. Let $w, w' \in W$ be points of W mapping to

x. We have to show that $w \times_X w'$ is a scheme whose underlying topological space has finitely many points. Consider the fibre product diagram

$$\begin{array}{ccc} W \times_X U & \xrightarrow{p} & U \\ {\scriptstyle q}\downarrow & & \downarrow{\scriptstyle \varphi} \\ W & \xrightarrow{\psi} & X \end{array}$$

As x is the image of u and u' we may pick points $\tilde{w}, \tilde{w}'$ in $W \times_X U$ with $q(\tilde{w}) = w$, $q(\tilde{w}') = w'$, $u = p(\tilde{w})$ and $u' = p(\tilde{w}')$, see Properties of Spaces, Lemma 48.4.3. As p, q are étale the field extensions $\kappa(w) \subset \kappa(\tilde{w}) \supset \kappa(u)$ and $\kappa(w') \subset \kappa(\tilde{w}') \supset \kappa(u')$ are finite separable, see Remark 50.4.1. Then we get a commutative diagram

$$\begin{array}{ccccc} w \times_X w' & \longleftarrow & \tilde{w} \times_X \tilde{w}' & \longrightarrow & u \times_X u' \\ \downarrow & & \downarrow & & \downarrow \\ w \times_X w' & \longleftarrow & \tilde{w} \times_S \tilde{w}' & \longrightarrow & u \times_S u' \end{array}$$

where the squares are fibre product squares. The lower horizontal morphisms are étale and quasi-compact, as any scheme of the form $\mathrm{Spec}(k) \times_S \mathrm{Spec}(k')$ is affine, and by our observations about the field extensions above. Thus we see that the top horizontal arrows are étale and quasi-compact and hence have finite fibres. We have seen above that $|u \times_X u'|$ is finite, so we conclude that $|w \times_X w'|$ is finite. In other words, (2) holds.

Assume (2). Let $U \to X$ be an étale morphism from a scheme U such that x is in the image of $|U| \to |X|$. Let $u \in U$ be a point mapping to x. Then we have seen in the previous paragraph that $u = \mathrm{Spec}(\kappa(u)) \to X$ has the property that $u \times_X u$ has a finite underlying topological space. On the other hand, the projection maps $u \times_X u \to u$ are the composition

$$u \times_X u \longrightarrow u \times_X U \longrightarrow u \times_X X = u,$$

i.e., the composition of a monomorphism (the base change of the monomorphism $u \to U$) by an étale morphism (the base change of the étale morphism $U \to X$). Hence $u \times_X U$ is a disjoint union of spectra of fields finite separable over $\kappa(u)$ (see Remark 50.4.1). Since $u \times_X u$ is finite the image of it in $u \times_X U$ is a finite disjoint union of spectra of fields finite separable over $\kappa(u)$. By Schemes, Lemma 25.23.10 we conclude that $u \times_X u$ is a finite disjoint union of spectra of fields finite separable over $\kappa(u)$. In other words, we see that $u \times_X u \to u$ is quasi-compact and étale. This means that (3) holds.

Let us prove that (3) implies (4). Let $\mathrm{Spec}(k) \to X$ be a morphism from the spectrum of a field into X, in the equivalence class of x such that the two projections $t, s : R = \mathrm{Spec}(k) \times_X \mathrm{Spec}(k) \to \mathrm{Spec}(k)$ are quasi-compact and étale. This means in particular that R is an étale equivalence relation on $\mathrm{Spec}(k)$. By Spaces, Theorem 47.10.5 we know that the quotient sheaf $X' = \mathrm{Spec}(k)/R$ is an algebraic space. By Groupoids, Lemma 38.18.6 the map $X' \to X$ is a monomorphism. Since s, t are quasi-compact, we see that R is quasi-compact and hence Properties of Spaces, Lemma 48.12.1 applies to X', and we see that $X' = \mathrm{Spec}(k')$ for some field k'. Hence we get a factorization

$$\mathrm{Spec}(k) \longrightarrow \mathrm{Spec}(k') \longrightarrow X$$

which shows that $\mathrm{Spec}(k') \to X$ is a monomorphism mapping to $x \in |X|$. In other words (4) holds.

Finally, we prove that (4) implies (1). Let $\mathrm{Spec}(k) \to X$ be a monomorphism with k a field in the equivalence class of x. Let $U \to X$ be a surjective étale morphism from a scheme U to X. Let $u \in U$ be a point over x. Since $\mathrm{Spec}(k) \times_X u$ is nonempty, and since $\mathrm{Spec}(k) \times_X u \to u$ is a monomorphism we conclude that $\mathrm{Spec}(k) \times_X u = u$ (see Schemes, Lemma 25.23.10). Hence $u \to U \to X$ factors through $\mathrm{Spec}(k) \to X$, here is a picture

$$\begin{array}{ccc} u & \longrightarrow & U \\ \downarrow & & \downarrow \\ \mathrm{Spec}(k) & \longrightarrow & X \end{array}$$

Since the right vertical arrow is étale this implies that $k \subset \kappa(u)$ is a finite separable extension. Hence we conclude that

$$u \times_X u = u \times_{\mathrm{Spec}(k)} u$$

is a finite scheme, and we win by the discussion of the meaning of property (1) in the first paragraph of this proof. □

Lemma 50.4.4. *Let S be a scheme. Let X be an algebraic space over S. Let $x \in |X|$. Let U be a scheme and let $\varphi : U \to X$ be an étale morphism. The following are equivalent:*

(1) *x is in the image of $|U| \to |X|$, and setting $R = U \times_X U$ the fibres of both*
$$|U| \longrightarrow |X| \quad \text{and} \quad |R| \longrightarrow |X|$$
over x are finite,

(2) *there exists a monomorphism $\mathrm{Spec}(k) \to X$ with k a field in the equivalence class of x, and the fibre product $\mathrm{Spec}(k) \times_X U$ is a finite nonempty scheme over k.*

Proof. Assume (1). This clearly implies the first condition of Lemma 50.4.3 and hence we obtain a monomorphism $\mathrm{Spec}(k) \to X$ in the class of x. Taking the fibre product we see that $\mathrm{Spec}(k) \times_X U \to \mathrm{Spec}(k)$ is a scheme étale over $\mathrm{Spec}(k)$ with finitely many points, hence a finite nonempty scheme over k, i.e., (2) holds.

Assume (2). By assumption x is in the image of $|U| \to |X|$. The finiteness of the fibre of $|U| \to |X|$ over x is clear since this fibre is equal to $|\mathrm{Spec}(k) \times_X U|$ by Properties of Spaces, Lemma 48.4.3. The finiteness of the fibre of $|R| \to |X|$ above x is also clear since it is equal to the set underlying the scheme

$$(\mathrm{Spec}(k) \times_X U) \times_{\mathrm{Spec}(k)} (\mathrm{Spec}(k) \times_X U)$$

which is finite over k. Thus (1) holds. □

Lemma 50.4.5. *Let S be a scheme. Let X be an algebraic space over S. Let $x \in |X|$. The following are equivalent:*

(1) *for every affine scheme U, any étale morphism $\varphi : U \to X$ setting $R = U \times_X U$ the fibres of both*
$$|U| \longrightarrow |X| \quad \text{and} \quad |R| \longrightarrow |X|$$
over x are finite,

(2) *there exist schemes U_i and étale morphisms $U_i \to X$ such that $\coprod U_i \to X$ is surjective and for each i, setting $R_i = U_i \times_X U_i$ the fibres of both*
$$|U_i| \longrightarrow |X| \quad \text{and} \quad |R_i| \longrightarrow |X|$$
over x are finite,

(3) *there exists a monomorphism $\mathrm{Spec}(k) \to X$ with k a field in the equivalence class of x, and for any affine scheme U and étale morphism $U \to X$ the fibre product $\mathrm{Spec}(k) \times_X U$ is a finite scheme over k, and*

(4) *there exists a quasi-compact monomorphism $\mathrm{Spec}(k) \to X$ with k a field in the equivalence class of x.*

Proof. The equivalence of (1) and (3) follows on applying Lemma 50.4.4 to every étale morphism $U \to X$ with U affine. It is clear that (3) implies (2). Assume $U_i \to X$ and R_i are as in (2). We conclude from Lemma 50.4.2 that for any affine scheme U and étale morphism $U \to X$ the fibre of $|U| \to |X|$ over x is finite. Say this fibre is $\{u_1, \ldots, u_n\}$. Then, as Lemma 50.4.3 (1) applies to $U_i \to X$ for some i such that x is in the image of $|U_i| \to |X|$, we see that the fibre of $|R = U \times_X U| \to |U| \times_{|X|} |U|$ is finite over (u_a, u_b), $a, b \in \{1, \ldots, n\}$. Hence the fibre of $|R| \to |X|$ over x is finite. In this way we see that (1) holds. At this point we know that (1), (2), and (3) are equivalent.

If (4) holds, then for any affine scheme U and étale morphism $U \to X$ the scheme $\mathrm{Spec}(k) \times_X U$ is on the one hand étale over k (hence a disjoint union of spectra of finite separable extensions of k by Remark 50.4.1) and on the other hand quasi-compact over U (hence quasi-compact). Thus we see that (3) holds. Conversely, if $U_i \to X$ is as in (2) and $\mathrm{Spec}(k) \to X$ is a monomorphism as in (3), then
$$\coprod \mathrm{Spec}(k) \times_X U_i \longrightarrow \coprod U_i$$
is quasi-compact (because over each U_i we see that $\mathrm{Spec}(k) \times_X U_i$ is a finite disjoint union spectra of fields). Thus $\mathrm{Spec}(k) \to X$ is quasi-compact by Morphisms of Spaces, Lemma 49.9.7. □

Lemma 50.4.6. *Let S be a scheme. Let X be an algebraic space over S. The following are equivalent:*

(1) *there exist schemes U_i and étale morphisms $U_i \to X$ such that $\coprod U_i \to X$ is surjective and each $U_i \to X$ has universally bounded fibres, and*

(2) *for every affine scheme U and étale morphism $\varphi : U \to X$ the fibres of $U \to X$ are universally bounded.*

Proof. The implication (2) $\Rightarrow$ (1) is trivial. Assume (1). Let $(\varphi_i : U_i \to X)_{i \in I}$ be a collection of étale morphisms from schemes towards X, covering X, such that each φ_i has universally bounded fibres. Let $\psi : U \to X$ be an étale morphism from an affine scheme towards X. For each i consider the fibre product diagram
$$\begin{array}{ccc} U \times_X U_i & \xrightarrow{p_i} & U_i \\ {\scriptstyle q_i}\downarrow & & \downarrow{\scriptstyle \varphi_i} \\ U & \xrightarrow{\psi} & X \end{array}$$
Since q_i is étale it is open (see Remark 50.4.1). Moreover, we have $U = \bigcup \mathrm{Im}(q_i)$, since the family $(\varphi_i)_{i \in I}$ is surjective. Since U is affine, hence quasi-compact we can finite finitely many $i_1, \ldots, i_n \in I$ and quasi-compact opens $W_j \subset U \times_X U_{i_j}$

such that $U = \bigcup p_{i_j}(W_j)$. The morphism p_{i_j} is étale, hence locally quasi-finite (see remark on étale morphisms above). Thus we may apply Morphisms, Lemma 28.50.8 to see the fibres of $p_{i_j}|_{W_j} : W_j \to U_{i_j}$ are universally bounded. Hence by Lemma 50.3.2 we see that the fibres of $W_j \to X$ are universally bounded. Thus also $\coprod_{j=1,\ldots,n} W_j \to X$ has universally bounded fibres. Since $\coprod_{j=1,\ldots,n} W_j \to X$ factors through the surjective étale map $\coprod q_{i_j}|_{W_j} : \coprod_{j=1,\ldots,n} W_j \to U$ we see that the fibres of $U \to X$ are universally bounded by Lemma 50.3.5. In other words (2) holds. □

Lemma 50.4.7. *Let S be a scheme. Let X be an algebraic space over S. The following are equivalent:*

(1) *there exists a Zariski covering $X = \bigcup X_i$ and for each i a scheme U_i and a quasi-compact surjective étale morphism $U_i \to X_i$, and*
(2) *there exist schemes U_i and étale morphisms $U_i \to X$ such that the projections $U_i \times_X U_i \to U_i$ are quasi-compact and $\coprod U_i \to X$ is surjective.*

Proof. If (1) holds then the morphisms $U_i \to X_i \to X$ are étale (combine Morphisms, Lemma 28.37.3 and Spaces, Lemmas 47.5.4 and 47.5.3). Moreover, as $U_i \times_X U_i = U_i \times_{X_i} U_i$, both projections $U_i \times_X U_i \to U_i$ are quasi-compact.

If (2) holds then let $X_i \subset X$ be the open subspace corresponding to the image of the open map $|U_i| \to |X|$, see Properties of Spaces, Lemma 48.4.10. The morphisms $U_i \to X_i$ are surjective. Hence $U_i \to X_i$ is surjective étale, and the projections $U_i \times_{X_i} U_i \to U_i$ are quasi-compact, because $U_i \times_{X_i} U_i = U_i \times_X U_i$. Thus by Spaces, Lemma 47.11.2 the morphisms $U_i \to X_i$ are quasi-compact. □

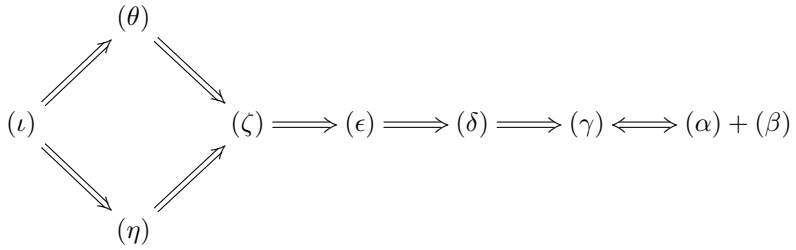
50.5. Conditions on algebraic spaces

In this section we discuss the relationship between various natural conditions on algebraic spaces we have seen above. Please read Section 50.6 to get a feeling for the meaning of these conditions.

Lemma 50.5.1. *Let S be a scheme. Let X be an algebraic space over S. Consider the following conditions on X:*

(α) *For every $x \in |X|$, the equivalent conditions of Lemma 50.4.2 hold.*
(β) *For every $x \in |X|$, the equivalent conditions of Lemma 50.4.3 hold.*
(γ) *For every $x \in |X|$, the equivalent conditions of Lemma 50.4.5 hold.*
(δ) *The equivalent conditions of Lemma 50.4.6 hold.*
(ϵ) *The equivalent conditions of Lemma 50.4.7 hold.*
(ζ) *The space X is Zariski locally quasi-separated.*
(η) *The space X is quasi-separated*
(θ) *The space X is representable, i.e., X is a scheme.*
(ι) *The space X is a quasi-separated scheme.*

We have



Proof. The implication $(\gamma) \Leftrightarrow (\alpha) + (\beta)$ is immediate. The implications in the diamond on the left are clear from the definitions.

Assume (ζ), i.e., that X is Zariski locally quasi-separated. Then (ϵ) holds by Properties of Spaces, Lemma 48.6.6.

Assume (ϵ). By Lemma 50.4.7 there exists a Zariski open covering $X = \bigcup X_i$ such that for each i there exists a scheme U_i and a quasi-compact surjective étale morphism $U_i \to X_i$. Choose an i and an affine open subscheme $W \subset U_i$. It suffices to show that $W \to X$ has universally bounded fibres, since then the family of all these morphisms $W \to X$ covers X. To do this we consider the diagram

$$\begin{array}{ccc} W \times_X U_i & \xrightarrow{p} & U_i \\ {\scriptstyle q}\downarrow & & \downarrow \\ W & \longrightarrow & X \end{array}$$

Since $W \to X$ factors through X_i we see that $W \times_X U_i = W \times_{X_i} U_i$, and hence q is quasi-compact. Since W is affine this implies that the scheme $W \times_X U_i$ is quasi-compact. Thus we may apply Morphisms, Lemma 28.50.8 and we conclude that p has universally bounded fibres. From Lemma 50.3.4 we conclude that $W \to X$ has universally bounded fibres as well.

Assume (δ). Let U be an affine scheme, and let $U \to X$ be an étale morphism. By assumption the fibres of the morphism $U \to X$ are universally bounded. Thus also the fibres of both projections $R = U \times_X U \to U$ are universally bounded, see Lemma 50.3.3. And by Lemma 50.3.2 also the fibres of $R \to X$ are universally bounded. Hence for any $x \in X$ the fibres of $|U| \to |X|$ and $|R| \to |X|$ over x are finite, see Lemma 50.3.6. In other words, the equivalent conditions of Lemma 50.4.5 hold. This proves that $(\delta) \Rightarrow (\gamma)$. □

Lemma 50.5.2. *Let S be a scheme. Let $\mathcal{P}$ be one of the properties (α), (β), (γ), (δ), (ϵ), (ζ), or (θ) of algebraic spaces listed in Lemma 50.5.1. Then if X is an algebraic space over S, and $X = \bigcup X_i$ is a Zariski open covering such that each X_i has $\mathcal{P}$, then X has $\mathcal{P}$.*

Proof. Let X be an algebraic space over S, and let $X = \bigcup X_i$ is a Zariski open covering such that each X_i has $\mathcal{P}$.

The case $\mathcal{P} = (\alpha)$. The condition (α) for X_i means that for every $x \in |X_i|$ and every affine scheme U, and étale morphism $\varphi : U \to X_i$ the fibre of $\varphi : |U| \to |X_i|$ over x is finite. Consider $x \in X$, an affine scheme U and an étale morphism $U \to X$. Since $X = \bigcup X_i$ is a Zariski open covering there exits a finite affine open covering $U = U_1 \cup \ldots \cup U_n$ such that each $U_j \to X$ factors through some X_{i_j}. By assumption

the fibres of $|U_j| \to |X_{i_j}|$ over x are finite for $j = 1, \ldots, n$. Clearly this means that the fibre of $|U| \to |X|$ over x is finite. This proves the result for (α).

The case $\mathcal{P} = (\beta)$. The condition (β) for X_i means that every $x \in |X_i|$ is represented by a monomorphism from the spectrum of a field towards X_i. Hence the same follows for X as $X_i \to X$ is a monomorphism and $X = \bigcup X_i$.

The case $\mathcal{P} = (\gamma)$. Note that $(\gamma) = (\alpha) + (\beta)$ by Lemma 50.5.1 hence the lemma for (γ) follows from the cases treated above.

The case $\mathcal{P} = (\delta)$. The condition (δ) for X_i means there exist schemes U_{ij} and étale morphisms $U_{ij} \to X_i$ with universally bounded fibres which cover X_i. These schemes also give an étale surjective morphism $\coprod U_{ij} \to X$ and $U_{ij} \to X$ still has universally bounded fibres.

The case $\mathcal{P} = (\epsilon)$. The condition (ϵ) for X_i means we can find a set J_i and morphisms $\varphi_{ij} : U_{ij} \to X_i$ such that each φ_{ij} is étale, both projections $U_{ij} \times_{X_i} U_{ij} \to U_{ij}$ are quasi-compact, and $\coprod_{j \in J_i} U_{ij} \to X_i$ is surjective. In this case the compositions $U_{ij} \to X_i \to X$ are étale (combine Morphisms, Lemmas 28.37.3 and 28.37.9 and Spaces, Lemmas 47.5.4 and 47.5.3). Since $X_i \subset X$ is a subspace we see that $U_{ij} \times_{X_i} U_{ij} = U_{ij} \times_X U_{ij}$, and hence the condition on fibre products is preserved. And clearly $\coprod_{i,j} U_{ij} \to X$ is surjective. Hence X satisfies (ϵ).

The case $\mathcal{P} = (\zeta)$. The condition (ζ) for X_i means that X_i is Zariski locally quasi-separated. It is immediately clear that this means X is Zariski locally quasi-separated.

For (θ), see Properties of Spaces, Lemma 48.10.1. □

Lemma 50.5.3. *Let S be a scheme. Let $\mathcal{P}$ be one of the properties (β), (γ), (δ), (ϵ), or (θ) of algebraic spaces listed in Lemma 50.5.1. Let X, Y be algebraic spaces over S. Let $X \to Y$ be a representable morphism. If Y has property $\mathcal{P}$, so does X.*

Proof. Assume $f : X \to Y$ is a representable morphism of algebraic spaces, and assume that Y has $\mathcal{P}$. Let $x \in |X|$, and set $y = f(x) \in |Y|$.

The case $\mathcal{P} = (\beta)$. Condition (β) for Y means there exists a monomorphism $\mathrm{Spec}(k) \to Y$ representing y. The fibre product $X_y = \mathrm{Spec}(k) \times_Y X$ is a scheme, and x corresponds to a point of X_y, i.e., to a monomorphism $\mathrm{Spec}(k') \to X_y$. As $X_y \to X$ is a monomorphism also we see that x is represented by the monomorphism $\mathrm{Spec}(k') \to X_y \to X$. In other words (β) holds for X.

The case $\mathcal{P} = (\gamma)$. Since $(\gamma) \Rightarrow (\beta)$ we have seen in the preceding paragraph that y and x can be represented by monomorphisms as in the following diagram

$$\begin{array}{ccc} \mathrm{Spec}(k') & \xrightarrow{x} & X \\ \downarrow & & \downarrow \\ \mathrm{Spec}(k) & \xrightarrow{y} & Y \end{array}$$

Also, by definition of property (γ) via Lemma 50.4.5 (2) there exist schemes V_i and étale morphisms $V_i \to Y$ such that $\coprod V_i \to Y$ is surjective and for each i, setting $R_i = V_i \times_Y V_i$ the fibres of both

$$|V_i| \longrightarrow |Y| \quad \text{and} \quad |R_i| \longrightarrow |Y|$$

over y are finite. This means that the schemes $(V_i)_y$ and $(R_i)_y$ are finite schemes over $y = \mathrm{Spec}(k)$. As $X \to Y$ is representable, the fibre products $U_i = V_i \times_Y X$ are schemes. The morphisms $U_i \to X$ are étale, and $\coprod U_i \to X$ is surjective. Finally, for each i we have

$$(U_i)_x = (V_i \times_Y X)_x = (V_i)_y \times_{\mathrm{Spec}(k)} \mathrm{Spec}(k')$$

and

$$(U_i \times_X U_i)_x = ((V_i \times_Y X) \times_X (V_i \times_Y X))_x = (R_i)_y \times_{\mathrm{Spec}(k)} \mathrm{Spec}(k')$$

hence these are finite over k' as base changes of the finite schemes $(V_i)_y$ and $(R_i)_y$. This implies that (γ) holds for X, again via the second condition of Lemma 50.4.5.

The case $\mathcal{P} = (\delta)$. Let $V \to Y$ be an étale morphism with V an affine scheme. Since Y has property (δ) this morphism has universally bounded fibres. By Lemma 50.3.3 the base change $V \times_Y X \to X$ also has universally bounded fibres. Hence the first part of Lemma 50.4.6 applies and we see that Y also has property (δ).

The case $\mathcal{P} = (\epsilon)$. We will repeatedly use Spaces, Lemma 47.5.5. Let $V_i \to Y$ be as in Lemma 50.4.7 (2). Set $U_i = X \times_Y V_i$. The morphisms $U_i \to X$ are étale, and $\coprod U_i \to X$ is surjective. Because $U_i \times_X U_i = X \times_Y (V_i \times_Y V_i)$ we see that the projections $U_i \times_Y U_i \to U_i$ are base changes of the projections $V_i \times_Y V_i \to V_i$, and so quasi-compact as well. Hence X satisfies Lemma 50.4.7 (2).

The case $\mathcal{P} = (\theta)$. In this case the result is Categories, Lemma 4.8.3. □

50.6. Reasonable and decent algebraic spaces

In Lemma 50.5.1 we have seen a number of conditions on algebraic spaces related to the behaviour of étale morphisms from affine schemes into X and related to the existence of special étale coverings of X by schemes. We tabulate the different types of conditions here:

(α)	fibres of etale morphisms from affines are finite
(β)	points come from monomorphisms of spectra of fields
(γ)	points come from quasi-compact monomorphisms of spectra of fields
(δ)	fibres of etale morphisms from affines are universally bounded
(ϵ)	cover by etale morphisms from schemes quasi-compact onto their image

The conditions in the following definition are not exactly conditions on the diagonal of X, but they are in some sense separation conditions on X.

Definition 50.6.1. Let S be a scheme. Let X be an algebraic space over S.

(1) We say X is *decent* if for every point $x \in X$ the equivalent conditions of Lemma 50.4.5 hold, in other words property (γ) of Lemma 50.5.1 holds.
(2) We say X is *reasonable* if the equivalent conditions of Lemma 50.4.6 hold, in other words property (δ) of Lemma 50.5.1 holds.
(3) We say X is *very reasonable* if the equivalent conditions of Lemma 50.4.7 hold, i.e., property (ϵ) of Lemma 50.5.1 holds.

In particular we have

$$\text{very reasonable} \Rightarrow \text{reasonable} \Rightarrow \text{decent}.$$

The notion of a very reasonable algebraic space is obsolete. It was introduced because the assumption was needed to prove some results which are now proven for the class of decent spaces. The class of decent spaces is the largest class of spaces X where one has a good relationship between the topology of $|X|$ and properties of X itself.

Example 50.6.2. The algebraic space $\mathbf{A}^1_{\mathbf{Q}}/\mathbf{Z}$ constructed in Spaces, Example 47.14.8 is not decent as its "generic point" cannot be represented by a monomorphism from the spectrum of a point.

Remark 50.6.3. Reasonable algebraic spaces are technically easier to work with than very reasonable algebraic spaces. For example, if $X \to Y$ is a quasi-compact étale surjective morphism of algebraic spaces and X is reasonable, then so is Y, see Lemma 50.14.5 but we don't know if this is true for the property "very reasonable". Below we give another technical property enjoyed by reasonable algebraic spaces.

Lemma 50.6.4. *Let S be a scheme. Let X be a quasi-compact reasonable algebraic space. Then there exists a directed system of quasi-compact and quasi-separated algebraic spaces X_i such that $X = \operatorname{colim}_i X_i$ (colimit in the category of sheaves).*

Proof. We sketch the proof. By Properties of Spaces, Lemma 48.6.3 we have $X = U/R$ with U affine. In this case, reasonable means $U \to X$ is universally bounded. Hence there exists an integer N such that the "fibres" of $U \to X$ have degree at most N, see Definition 50.3.1. Denote $s, t : R \to U$ and $c : R \times_{s,U,t} R \to R$ the groupoid structural maps.

Claim: for every quasi-compact open $A \subset R$ there exists an open $R' \subset R$ such that

(1) $A \subset R'$,
(2) R' is quasi-compact, and
(3) $(U, R', s|_{R'}, t|_{R'}, c|_{R' \times_{s,U,t} R'})$ is a groupoid scheme.

Note that $e : U \to R$ is open as it is a section of the étale morphism $s : R \to U$, see Étale Morphisms, Proposition 40.6.1. Moreover U is affine hence quasi-compact. Hence we may replace A by $A \cup e(U) \subset R$, and assume that A contains $e(U)$. Next, we define inductively $A^1 = A$, and

$$A^n = c(A^{n-1} \times_{s,U,t} A) \subset R$$

for $n \geq 2$. Arguing inductively, we see that A^n is quasi-compact for all $n \geq 2$, as the image of the quasi-compact fibre product $A^{n-1} \times_{s,U,t} A$. If k is an algebraically closed field over S, and we consider k-points then

$$A^n(k) = \left\{ (u, u') \in U(k) : \begin{array}{c} \text{there exist } u = u_1, u_2, \ldots, u_n \in U(k) \text{ with} \\ (u_i, u_{i+1}) \in A \text{ for all } i = 1, \ldots, n-1. \end{array} \right\}$$

But as the fibres of $U(k) \to X(k)$ have size at most N we see that if $n > N$ then we get a repeat in the sequence above, and we can shorten it proving $A^N = A^n$ for all $n \geq N$. This implies that $R' = A^N$ gives a groupoid scheme $(U, R', s|_{R'}, t|_{R'}, c|_{R' \times_{s,U,t} R'})$, proving the claim above.

Consider the map of sheaves on $(Sch/S)_{fppf}$

$$\operatorname{colim}_{R' \subset R} U/R' \longrightarrow U/R$$

where $R' \subset R$ runs over the quasi-compact open subschemes of R which give étale equivalence relations as above. Each of the quotients U/R' is an algebraic space (see

Spaces, Theorem 47.10.5). Since R' is quasi-compact, and U affine the morphism $R' \to U \times_{\mathrm{Spec}(\mathbf{Z})} U$ is quasi-compact, and hence U/R' is quasi-separated. Finally, if T is a quasi-compact scheme, then

$$\mathrm{colim}_{R' \subset R} U(T)/R'(T) \longrightarrow U(T)/R(T)$$

is a bijection, since every morphism from T into R ends up in one of the open subrelations R' by the claim above. This clearly implies that the colimit of the sheaves U/R' is U/R. In other words the algebraic space $X = U/R$ is the colimit of the quasi-separated algebraic spaces U/R'. □

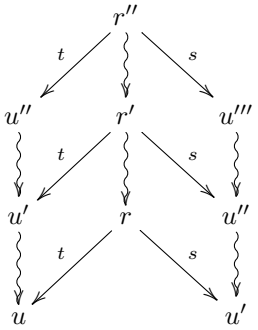
50.7. Points and specializations

There exists an étale morphism of algebraic spaces $f : X \to Y$ and a nontrivial specializations between points in a fibre of $|f| : |X| \to |Y|$, see Examples, Lemma 78.36.1. If the source of the morphism is a scheme we can avoid this by imposing condition (α) on Y.

Lemma 50.7.1. *Let S be a scheme. Let X be an algebraic space over S. Let $U \to X$ be an étale morphism from a scheme to X. Assume $u, u' \in |U|$ map to the same point x of $|X|$, and $u' \rightsquigarrow u$. If the pair (X, x) satisfies the equivalent conditions of Lemma 50.4.2 then $u = u'$.*

Proof. Assume the pair (X, x) satisfies the equivalent conditions for Lemma 50.4.2. Let U be a scheme, $U \to X$ étale, and let $u, u' \in |U|$ map to x of $|X|$, and $u' \rightsquigarrow u$. We may and do replace U by an affine neighbourhood of u. Let $t, s : R = U \times_X U \to U$ be the étale projection maps.

Pick a point $r \in R$ with $t(r) = u$ and $s(r) = u'$. This is possible by Properties of Spaces, Lemma 48.4.5. Because generalizations lift along the étale morphism t (Remark 50.4.1) we can find a specialization $r' \rightsquigarrow r$ with $t(r') = u'$. Set $u'' = s(r')$. Then $u'' \rightsquigarrow u'$. Thus we may repeat and find $r'' \rightsquigarrow r'$ with $t(r'') = u''$. Set $u''' = s(r'')$, and so on. Here is a picture:



In Remark 50.4.1 we have seen that there are no specializations among points in the fibres of the étale morphism s. Hence if $u^{(n+1)} = u^{(n)}$ for some n, then also $r^{(n)} = r^{(n-1)}$ and hence also (by taking t) $u^{(n)} = u^{(n-1)}$. This then forces the whole tower to collapse, in particular $u = u'$. Thus we see that if $u \neq u'$, then all the specializations are strict and $\{u, u', u'', \ldots\}$ is an infinite set of points in U which map to the point x in $|X|$. As we chose U affine this contradicts the second part of Lemma 50.4.2, as desired. □

Lemma 50.7.2. *Let S be an algebraic space. Let X be an algebraic space over S. Let $x, x' \in |X|$ and assume $x' \rightsquigarrow x$, i.e., x is a specialization of x'. Assume the pair (X, x') satisfies the equivalent conditions of Lemma 50.4.5. Then for every étale morphism $\varphi : U \to X$ from a scheme U and any $u \in U$ with $\varphi(u) = x$, exists a point $u' \in U$, $u' \rightsquigarrow u$ with $\varphi(u') = x'$.*

Proof. We may replace U by an affine open neighbourhood of u. Hence we may assume that U is affine. As x is in the image of the open map $|U| \to |X|$, so is x'. Thus we may replace X by the Zariski open subspace corresponding to the image of $|U| \to |X|$, see Properties of Spaces, Lemma 48.4.10. In other words we may assume that $U \to X$ is surjective and étale. Let $s, t : R = U \times_X U \to U$ be the projections. By our assumption that (X, x') satisfies the equivalent conditions of Lemma 50.4.5 we see that the fibres of $|U| \to |X|$ and $|R| \to |X|$ over x' are finite. Say $\{u'_1, \ldots, u'_n\} \subset U$ and $\{r'_1, \ldots, r'_m\} \subset R$ form the complete inverse image of $\{x'\}$. Consider the closed sets

$$T = \overline{\{u'_1\}} \cup \ldots \cup \overline{\{u'_n\}} \subset |U|, \quad T' = \overline{\{r'_1\}} \cup \ldots \cup \overline{\{r'_m\}} \subset |R|.$$

Trivially we have $s(T') \subset T$. Because R is an equivalence relation we also have $t(T') = s(T')$ as the set $\{r'_j\}$ is invariant under the inverse of R by construction. Let $w \in T$ be any point. Then $u'_i \rightsquigarrow w$ for some i. Choose $r \in R$ with $s(r) = w$. Since generalizations lift along $s : R \to U$, see Remark 50.4.1, we can find $r' \rightsquigarrow r$ with $s(r') = u'_i$. Then $r' = r'_j$ for some j and we conclude that $w \in s(T')$. Hence $T = s(T') = t(T')$ is an $|R|$-invariant closed set in $|U|$. This means T is the inverse image of a closed (!) subset $T'' = \varphi(T)$ of $|X|$, see Properties of Spaces, Lemmas 48.4.5 and 48.4.6. Hence $T'' = \overline{\{x'\}}$. Thus T contains some point u_1 mapping to x as $x \in T''$. I.e., we see that for some i there exists a specialization $u'_i \rightsquigarrow u_1$ which maps to the given specialization $x' \rightsquigarrow x$.

To finish the proof, choose a point $r \in R$ such that $s(r) = u$ and $t(r) = u_1$ (using Properties of Spaces, Lemma 48.4.3). As generalizations lift along t, and $u'_i \rightsquigarrow u_1$ we can find a specialization $r' \rightsquigarrow r$ such that $t(r') = u'_i$. Set $u' = s(r')$. Then $u' \rightsquigarrow u$ and $\varphi(u') = x'$ as desired. □

50.8. Schematic locus

In this section we prove that a decent algebraic space has a dense open subspace which is a scheme. We prove some results on stratifications by subschemes. We first prove this for reasonable algebraic spaces.

Lemma 50.8.1. *Let S be a scheme. Let $W \to X$ be a morphism of a scheme W to an algebraic space X which is flat, locally of finite presentation, separated, locally quasi-finite with universally bounded fibres. There exist reduced closed subspaces*

$$\emptyset = Z_{-1} \subset Z_0 \subset Z_1 \subset Z_2 \subset \ldots \subset Z_n = X$$

such that with $X_r = Z_r \setminus Z_{r-1}$ the stratification $X = \coprod_{r=0,\ldots,n} X_r$ is characterized by the following universal property: Given $g : T \to X$ the projection $W \times_X T \to T$ is finite locally free of degree r if and only if $g(|T|) \subset |X_r|$.

Proof. Let n be an integer bounding the degrees of the fibres of $W \to X$. Choose a scheme U and a surjective étale morphism $U \to X$. Apply More on Morphisms, Lemma 36.31.8 to $W \times_X U \to U$. We obtain closed subsets

$$\emptyset = Y_{-1} \subset Y_0 \subset Y_1 \subset Y_2 \subset \ldots \subset Y_n = U$$

characterized by the property stated in the lemma for the morphism $W \times_X U \to U$. Clearly, the formation of these closed subsets commutes with base change. Setting $R = U \times_X U$ with projection maps $s, t : R \to U$ we conclude that

$$s^{-1}(Y_r) = t^{-1}(Y_r)$$

as closed subsets of R. In other words the closed subsets $Y_r \subset U$ are R-invariant. This means that $|Y_r|$ is the inverse image of a closed subset $Z_r \subset |X|$. Denote $Z_r \subset X$ also the reduced induced algebraic space structure, see Properties of Spaces, Definition 48.9.4.

Let $g : T \to X$ be a morphism of algebraic spaces. Choose a scheme V and a surjective étale morphism $V \to T$. To prove the final assertion of the lemma it suffices to prove the assertion for the composition $V \to X$ (by our definition of finite locally free morphisms, see Morphisms of Spaces, Section 49.41). Similarly, the morphism of schemes $W \times_X V \to V$ is finite locally free of degree r if and only if the morphism of schemes

$$W \times_X (U \times_X V) \longrightarrow U \times_X V$$

is finite locally free of degree r (see Descent, Lemma 34.19.28). By construction this happens if and only if $|U \times_X V| \to |U|$ maps into $|Y_r|$, which is true if and only if $|V| \to |X|$ maps into $|Z_r|$. □

Lemma 50.8.2. *Let S be a scheme. Let $W \to X$ be a morphism of a scheme W to an algebraic space X which is flat, locally of finite presentation, separated, and locally quasi-finite. Then there exist open subspaces*

$$X = X_0 \supset X_1 \supset X_2 \supset \ldots$$

such that a morphism $\mathrm{Spec}(k) \to X$ factors through X_d if and only if $W \times_X \mathrm{Spec}(k)$ has degree $\geq d$ over k.

Proof. Choose a scheme U and a surjective étale morphism $U \to X$. Apply More on Morphisms, Lemma 36.31.10 to $W \times_X U \to U$. We obtain open subschemes

$$U = U_0 \supset U_1 \supset U_2 \supset \ldots$$

characterized by the property stated in the lemma for the morphism $W \times_X U \to U$. Clearly, the formation of these closed subsets commutes with base change. Setting $R = U \times_X U$ with projection maps $s, t : R \to U$ we conclude that

$$s^{-1}(U_d) = t^{-1}(U_d)$$

as open subschemes of R. In other words the open subschemes $U_d \subset U$ are R-invariant. This means that U_d is the inverse image of an open subspace $X_d \subset X$ (Properties of Spaces, Lemma 48.9.1). □

Proposition 50.8.3. *Let S be a scheme. Let X be an algebraic space over S. If X is reasonable, then there exists a dense open subspace of X which is a scheme.*

Proof. By Properties of Spaces, Lemma 48.10.1 the question is local on X. Hence we may assume there exists an affine scheme U and a surjective étale morphism $U \to X$ (Properties of Spaces, Lemma 48.6.1). Let n be an integer bounding the degrees of the fibres of $U \to X$ which exists as X is reasonable, see Definition 50.6.1. We will argue by induction on n that whenever

(1) $U \to X$ is a surjective étale morphism whose fibres have degree $\leq n$, and

(2) U is isomorphic to a locally closed subscheme of an affine scheme

then the schematic locus is dense in X.

Let $X_n \subset X$ be the open subspace which is the complement of the closed subspace $Z_{n-1} \subset X$ constructed in Lemma 50.8.1 using the morphism $U \to X$. Let $U_n \subset U$ be the inverse image of X_n. Then $U_n \to X_n$ is finite locally free of degree n. Hence X_n is a scheme by Properties of Spaces, Proposition 48.11.1 (and the fact that any finite set of points of U_n is contained in an affine open of U_n, see Properties, Lemma 27.27.5).

Let $X' \subset X$ be the open subspace such that $|X'|$ is the interior of $|Z_{n-1}|$ in $|X|$ (see Topology, Definition 5.20.1). Let $U' \subset U$ be the inverse image. Then $U' \to X'$ is surjective étale and has degrees of fibres bounded by $n-1$. By induction we see that the schematic locus of X' is an open dense $X'' \subset X'$. By elementary topology we see that $X'' \cup X_n \subset X$ is open and dense and we win. □

Theorem 50.8.4 (David Rydh)**.** *Let S be a scheme. Let X be an algebraic space over S. If X is decent, then there exists a dense open subspace of X which is a scheme.*

Proof. Assume X is a decent algebraic space for which the theorem is false. By Properties of Spaces, Lemma 48.10.1 there exists a largest open subspace $X' \subset X$ which is a scheme. Since X' is not dense in X, there exists an open subspace $X'' \subset X$ such that $|X''| \cap |X'| = \emptyset$. Replacing X by X'' we get a nonempty decent algebraic space X which does not contain *any* open subspace which is a scheme.

Choose a nonempty affine scheme U and an étale morphism $U \to X$. We may and do replace X by the open subscheme corresponding to the image of $|U| \to |X|$. Consider the sequence of open subspaces

$$X = X_0 \supset X_1 \supset X_2 \ldots$$

constructed in Lemma 50.8.2 for the morphism $U \to X$. Note that $X_0 = X_1$ as $U \to X$ is surjective. Let $U = U_0 = U_1 \supset U_2 \ldots$ be the induced sequence of open subschemes of U.

Choose a nonempty open affine $V_1 \subset U_1$ (for example $V_1 = U_1$). By induction we will construct a sequence of nonempty affine opens $V_1 \supset V_2 \supset \ldots$ with $V_n \subset U_n$. Namely, having constructed $V_1, \ldots, V_{n-1}$ we can always choose V_n unless $V_{n-1} \cap U_n = \emptyset$. But if $V_{n-1} \cap U_n = \emptyset$, then the open subspace $X' \subset X$ with $|X'| = \mathrm{Im}(|V_{n-1}| \to |X|)$ is contained in $|X| \setminus |X_n|$. Hence $V_{n-1} \to X'$ is an étale morphism whose fibres have degree bounded by $n-1$. In other words, X' is reasonable (by definition), hence X' contains a nonempty open subscheme by Proposition 50.8.3. This is a contradiction which shows that we can pick V_n.

By Limits, Lemma 31.3.4 the limit $V_\infty = \lim V_n$ is a nonempty scheme. Pick a morphism $\mathrm{Spec}(k) \to V_\infty$. The composition $\mathrm{Spec}(k) \to V_\infty \to U \to X$ has image contained in all X_d by construction. In other words, the fibred $U \times_X \mathrm{Spec}(k)$ has infinite degree which contradicts the definition of a decent space. This contradiction finishes the proof of the theorem. □

Lemma 50.8.5. *Let S be a scheme. Let X be a quasi-compact, reasonable algebraic space over S. There exist an integer n and open subspaces*

$$\emptyset = U_{n+1} \subset U_n \subset U_{n-1} \subset \ldots \subset U_1 = X$$

with the following property: setting $T_p = U_p \setminus U_{p+1}$ (with reduced induced subspace structure) there exists a separated scheme V_p and a surjective étale morphism $f_p : V_p \to U_p$ such that $f_p^{-1}(T_p) \to T_p$ is an isomorphism.

Proof. By Properties of Spaces, Lemma 48.6.3 we can choose an affine scheme U and a surjective étale morphism $U \to X$. Let n be an integer bounding the degrees of the fibres of $U \to X$ which exists as X is reasonable, see Definition 50.6.1. For $p \geq 0$ set

$$W_p = U \times_X \ldots \times_X U \setminus \text{all diagonals}$$

where the fibre product has p factors. Since U is separated, the morphism $U \to X$ is separated and all fibre products $U \times_X \ldots \times_X U$ are separated schemes. Since $U \to X$ is separated the diagonal $U \to U \times_X U$ is a closed immersion. Since $U \to X$ is étale the diagonal $U \to U \times_X U$ is an open immersion, see Morphisms of Spaces, Lemmas 49.35.10 and 49.34.9. Similarly, all the diagonal morphisms are open and closed immersions and W_p is an open and closed subscheme of $U \times_X \ldots \times_X U$. Moreover, the morphism

$$U \times_X \ldots \times_X U \longrightarrow U \times_{\mathrm{Spec}(\mathbf{Z})} \ldots \times_{\mathrm{Spec}(\mathbf{Z})} U$$

is locally quasi-finite and separated (Morphisms of Spaces, Lemma 49.5.5) and its target is an affine scheme. Hence every finite set of points of $U \times_X \ldots \times_X U$ is contained in an affine open, see More on Morphisms, Lemma 36.31.12. Therefore, the same is true for W_p. There is a free action of the symmetric group S_p on W_p over X (because we threw out the fix point locus from $U \times_X \ldots \times_X U$). By the above and Properties of Spaces, Proposition 48.11.1 the quotient $V_p = W_p/S_p$ is a scheme. Since the action of S_p on W_p was over X, there is a morphism $V_p \to X$. Since $W_p \to X$ is étale and since $W_p \to V_p$ is surjective étale, it follows that also $V_p \to X$ is étale, see Properties of Spaces, Lemma 48.13.3.

We let $U_p \subset X$ be the open subspace which is the image of $V_p \to X$. By construction morphism $\mathrm{Spec}(k) \to X$ with k algebraically closed, factors through U_p if and only if $U \times_X \mathrm{Spec}(k)$ has $\geq p$ points. It follows that the U_p give a filtration of X as stated in the lemma. Moreover, $\mathrm{Spec}(k) \to X$ factors through T_p if and only if $U \times_X \mathrm{Spec}(k)$ has exactly p points. In this case we see that $V_p \times_X \mathrm{Spec}(k)$ has exactly one point. Set $Z_p = f_p^{-1}(T_p) \subset V_p$. This is a closed subscheme of V_p. Then $Z_p \to T_p$ is an étale morphism between algebraic spaces which induces a bijection on k-valued points for any algebraically closed field k. To be sure this implies that $Z_p \to T_p$ is universally injective, whence an open immersion by Morphisms of Spaces, Lemma 49.44.2 hence an isomorphism and we win. □

Lemma 50.8.6. *Let S be a scheme. Let X be a quasi-compact, reasonable algebraic space over S. There exist an integer n and open subspaces*

$$\emptyset = U_{n+1} \subset U_n \subset U_{n-1} \subset \ldots \subset U_1 = X$$

such that each $T_p = U_p \setminus U_{p+1}$ (with reduced induced subspace structure) is a scheme.

Proof. Immediate consequence of Lemma 50.8.5. □

50.9. Points on spaces

In this section we prove some properties of points on decent algebraic spaces.

Lemma 50.9.1. *Let S be a scheme. Let X be an algebraic space over S. Consider the map*

$$\{\mathrm{Spec}(k) \to X \text{ monomorphism}\} \longrightarrow |X|$$

This map is always injective. If X is decent then this map is a bijection.

Proof. We have seen in Properties of Spaces, Lemma 48.4.11 that the map is an injection in general. By Lemma 50.5.1 it is surjective when X is decent (actually one can say this is part of the definition of being decent). □

The following lemma is a tiny bit stronger than Properties of Spaces, Lemma 48.12.1. We will improve this lemma in Lemma 50.11.1.

Lemma 50.9.2. *Let S be a scheme. Let k be a field. Let X be an algebraic space over S and assume that there exists a surjective étale morphism $\mathrm{Spec}(k) \to X$. If X is decent, then $X \cong \mathrm{Spec}(k')$ where $k' \subset k$ is a finite separable extension.*

Proof. The assumption implies that $|X| = \{x\}$ is a singleton. Since X is decent we can find a quasi-compact monomorphism $\mathrm{Spec}(k') \to X$ whose image is x. Then the projection $U = \mathrm{Spec}(k') \times_X \mathrm{Spec}(k) \to \mathrm{Spec}(k)$ is a monomorphism, whence $U = \mathrm{Spec}(k)$, see Schemes, Lemma 25.23.10. Hence the projection $\mathrm{Spec}(k) = U \to \mathrm{Spec}(k')$ is étale and we win. □

The following lemma shows that specialization of points behaves well on decent algebraic spaces. Spaces, Example 47.14.9 shows that this is **not** true in general.

Lemma 50.9.3. *Let S be a scheme. Let X be a decent algebraic space over S. Let $U \to X$ be an étale morphism from a scheme to X. If $u, u' \in |U|$ map to the same point of $|X|$, and $u' \rightsquigarrow u$, then $u = u'$.*

Proof. Combine Lemmas 50.5.1 and 50.7.1. □

Lemma 50.9.4. *Let S be an algebraic space. Let X be a decent algebraic space over S. Let $x, x' \in |X|$ and assume $x' \rightsquigarrow x$, i.e., x is a specialization of x'. Then for every étale morphism $\varphi : U \to X$ from a scheme U and any $u \in U$ with $\varphi(u) = x$, exists a point $u' \in U$, $u' \rightsquigarrow u$ with $\varphi(u') = x'$.*

Proof. Combine Lemmas 50.5.1 and 50.7.2. □

Lemma 50.9.5. *Let S be a scheme. Let X be a decent algebraic space over S. Then $|X|$ is Kolmogorov (see Topology, Definition 5.7.4).*

Proof. Let $x_1, x_2 \in |X|$ with $x_1 \rightsquigarrow x_2$ and $x_2 \rightsquigarrow x_1$. We have to show that $x_1 = x_2$. Pick a scheme U and an étale morphism $U \to X$ such that x_1, x_2 are both in the image of $|U| \to |X|$. By Lemma 50.9.4 we can find a specialization $u_1 \rightsquigarrow u_2$ in U mapping to $x_1 \rightsquigarrow x_2$. By Lemma 50.9.4 we can find $u_2' \rightsquigarrow u_1$ mapping to $x_2 \rightsquigarrow x_1$. This means that $u_2' \rightsquigarrow u_2$ is a specialization between points of U mapping to the same point of X, namely x_2. This is not possible, unless $u_2' = u_2$, see Lemma 50.9.3. Hence also $u_1 = u_2$ as desired. □

Proposition 50.9.6. *Let S be a scheme. Let X be a decent algebraic space over S. Then the topological space $|X|$ is sober (see Topology, Definition 5.7.4).*

Proof. We have seen in Lemma 50.9.5 that $|X|$ is Kolmogorov. Hence it remains to show that every irreducible closed subset $T \subset |X|$ has a generic point. By Properties of Spaces, Lemma 48.9.2 there exists a closed subspace $Z \subset X$ with $|Z| = |T|$. By definition this means that $Z \to X$ is a representable morphism of algebraic spaces. Hence Z is a decent algebraic space by Lemma 50.5.3. By Theorem 50.8.4 we see that there exists an open dense subspace $Z' \subset Z$ which is a scheme. This means that $|Z'| \subset T$ is open dense. Hence the topological space $|Z'|$ is irreducible, which means that Z' is an irreducible scheme. By Schemes, Lemma 25.11.1 we conclude that $|Z'|$ is the closure of a single point $\eta \in T$ and hence also $T = \overline{\{\eta\}}$, and we win. □

50.10. Reduced singleton spaces

A *singleton* space is an algebraic space X such that $|X|$ is a singleton. It turns out that these can be more interesting than just being the spectrum of a field, see Spaces, Example 47.14.7. We develop a tiny bit of machinery to be able to talk about these.

Lemma 50.10.1. *Let S be a scheme. Let Z be an algebraic space over S. Let k be a field and let $\mathrm{Spec}(k) \to Z$ be surjective and flat. Then any morphism $\mathrm{Spec}(k') \to Z$ where k' is a field is surjective and flat.*

Proof. Consider the fibre square

$$\begin{array}{ccc} T & \longrightarrow & \mathrm{Spec}(k) \\ \downarrow & & \downarrow \\ \mathrm{Spec}(k') & \longrightarrow & Z \end{array}$$

Note that $T \to \mathrm{Spec}(k')$ is flat and surjective hence T is not empty. On the other hand $T \to \mathrm{Spec}(k)$ is flat as k is a field. Hence $T \to Z$ is flat and surjective. It follows from Morphisms of Spaces, Lemma 49.28.5 that $\mathrm{Spec}(k') \to Z$ is flat. It is surjective as by assumption $|Z|$ is a singleton. □

Lemma 50.10.2. *Let S be a scheme. Let Z be an algebraic space over S. The following are equivalent*

1. *Z is reduced and $|Z|$ is a singleton,*
2. *there exists a surjective flat morphism $\mathrm{Spec}(k) \to Z$ where k is a field, and*
3. *there exists a locally of finite type, surjective, flat morphism $\mathrm{Spec}(k) \to Z$ where k is a field.*

Proof. Assume (1). Let W be a scheme and let $W \to Z$ be a surjective étale morphism. Then W is a reduced scheme. Let $\eta \in W$ be a generic point of an irreducible component of W. Since W is reduced we have $\mathcal{O}_{W,\eta} = \kappa(\eta)$. It follows that the canonical morphism $\eta = \mathrm{Spec}(\kappa(\eta)) \to W$ is flat. We see that the composition $\eta \to Z$ is flat (see Morphisms of Spaces, Lemma 49.27.3). It is also surjective as $|Z|$ is a singleton. In other words (2) holds.

Assume (2). Let W be a scheme and let $W \to Z$ be a surjective étale morphism. Choose a field k and a surjective flat morphism $\mathrm{Spec}(k) \to Z$. Then $W \times_Z \mathrm{Spec}(k)$ is a scheme étale over k. Hence $W \times_Z \mathrm{Spec}(k)$ is a disjoint union of spectra of fields

(see Remark 50.4.1), in particular reduced. Since $W \times_Z \mathrm{Spec}(k) \to W$ is surjective and flat we conclude that W is reduced (Descent, Lemma 34.15.1). In other words (1) holds.

It is clear that (3) implies (2). Finally, assume (2). Pick a nonempty affine scheme W and an étale morphism $W \to Z$. Pick a closed point $w \in W$ and set $k = \kappa(w)$. The composition

$$\mathrm{Spec}(k) \xrightarrow{w} W \longrightarrow Z$$

is locally of finite type by Morphisms of Spaces, Lemmas 49.22.2 and 49.35.9. It is also flat and surjective by Lemma 50.10.1. Hence (3) holds. □

The following lemma singles out a slightly better class of singleton algebraic spaces than the preceding lemma.

Lemma 50.10.3. *Let S be a scheme. Let Z be an algebraic space over S. The following are equivalent*

(1) *Z is reduced, locally Noetherian, and $|Z|$ is a singleton, and*
(2) *there exists a locally finitely presented, surjective, flat morphism $\mathrm{Spec}(k) \to Z$ where k is a field.*

Proof. Assume (2) holds. By Lemma 50.10.2 we see that Z is reduced and $|Z|$ is a singleton. Let W be a scheme and let $W \to Z$ be a surjective étale morphism. Choose a field k and a locally finitely presented, surjective, flat morphism $\mathrm{Spec}(k) \to Z$. Then $W \times_Z \mathrm{Spec}(k)$ is a scheme étale over k, hence a disjoint union of spectra of fields (see Remark 50.4.1), hence locally Noetherian. Since $W \times_Z \mathrm{Spec}(k) \to W$ is flat, surjective, and locally of finite presentation, we see that $\{W \times_Z \mathrm{Spec}(k) \to W\}$ is an fppf covering and we conclude that W is locally Noetherian (Descent, Lemma 34.12.1). In other words (1) holds.

Assume (1). Pick a nonempty affine scheme W and an étale morphism $W \to Z$. Pick a closed point $w \in W$ and set $k = \kappa(w)$. Because W is locally Noetherian the morphism $w : \mathrm{Spec}(k) \to W$ is of finite presentation, see Morphisms, Lemma 28.22.7. Hence the composition

$$\mathrm{Spec}(k) \xrightarrow{w} W \longrightarrow Z$$

is locally of finite presentation by Morphisms of Spaces, Lemmas 49.26.2 and 49.35.8. It is also flat and surjective by Lemma 50.10.1. Hence (2) holds. □

Lemma 50.10.4. *Let S be a scheme. Let $Z' \to Z$ be a monomorphism of algebraic spaces over S. Assume there exists a field k and a locally finitely presented, surjective, flat morphism $\mathrm{Spec}(k) \to Z$. Then either Z' is empty or $Z' = Z$.*

Proof. We may assume that Z' is nonempty. In this case the fibre product $T = Z' \times_Z \mathrm{Spec}(k)$ is nonempty, see Properties of Spaces, Lemma 48.4.3. Now T is an algebraic space and the projection $T \to \mathrm{Spec}(k)$ is a monomorphism. Hence $T = \mathrm{Spec}(k)$, see Morphisms of Spaces, Lemma 49.11.8. We conclude that $\mathrm{Spec}(k) \to Z$ factors through Z'. But as $\mathrm{Spec}(k) \to Z$ is surjective, flat and locally of finite presentation, we see that $\mathrm{Spec}(k) \to Z$ is surjective as a map of sheaves on $(Sch/S)_{fppf}$ (see Spaces, Remark 47.5.2) and we conclude that $Z' = Z$. □

The following lemma says that to each point of an algebraic space we can associate a canonical reduced, locally Noetherian singleton algebraic space.

Lemma 50.10.5. *Let S be a scheme. Let X be an algebraic space over S. Let $x \in |X|$. Then there exists a unique monomorphism $Z \to X$ of algebraic spaces over S such that Z is an algebraic space which satisfies the equivalent conditions of Lemma 50.10.3 and such that the image of $|Z| \to |X|$ is $\{x\}$.*

Proof. Choose a scheme U and a surjective étale morphism $U \to X$. Set $R = U \times_X U$ so that $X = U/R$ is a presentation (see Spaces, Section 47.9). Set

$$U' = \coprod\nolimits_{u \in U \text{ lying over } x} \operatorname{Spec}(\kappa(u)).$$

The canonical morphism $U' \to U$ is a monomorphism. Let

$$R' = U' \times_X U' = R \times_{(U \times_S U)} (U' \times_S U').$$

Because $U' \to U$ is a monomorphism we see that the projections $s', t' : R' \to U'$ factor as a monomorphism followed by an étale morphism. Hence, as U' is a disjoint union of spectra of fields, using Remark 50.4.1, and using Schemes, Lemma 25.23.10 we conclude that R' is a disjoint union of spectra of fields and that the morphisms $s', t' : R' \to U'$ are étale. Hence $Z = U'/R'$ is an algebraic space by Spaces, Theorem 47.10.5. As R' is the restriction of R by $U' \to U$ we see $Z \to X$ is a monomorphism by Groupoids, Lemma 38.18.6. Since $Z \to X$ is a monomorphism we see that $|Z| \to |X|$ is injective, see Morphisms of Spaces, Lemma 49.11.9. By Properties of Spaces, Lemma 48.4.3 we see that

$$|U'| = |Z \times_X U'| \to |Z| \times_{|X|} |U'|$$

is surjective which implies (by our choice of U') that $|Z| \to |X|$ has image $\{x\}$. We conclude that $|Z|$ is a singleton. Finally, by construction U' is locally Noetherian and reduced, i.e., we see that Z satisfies the equivalent conditions of Lemma 50.10.3.

Let us prove uniqueness of $Z \to X$. Suppose that $Z' \to X$ is a second such monomorphism of algebraic spaces. Then the projections

$$Z' \longleftarrow Z' \times_X Z \longrightarrow Z$$

are monomorphisms. The algebraic space in the middle is nonempty by Properties of Spaces, Lemma 48.4.3. Hence the two projections are isomorphisms by Lemma 50.10.4 and we win. □

We introduce the following terminology which foreshadows the residual gerbes we will introduce later, see Properties of Stacks, Definition 73.11.8.

Definition 50.10.6. Let S be a scheme. Let X be an algebraic space over S. Let $x \in |X|$. The *residual space of X at x*[2] is the monomorphism $Z_x \to X$ constructed in Lemma 50.10.5.

In particular we know that Z_x is a locally Noetherian, reduced, singleton algebraic space and that there exists a field and a surjective, flat, locally finitely presented morphism

$$\operatorname{Spec}(k) \longrightarrow Z_x.$$

It turns out that Z_x is a regular algebraic space as follows from the following lemma.

Lemma 50.10.7. *A reduced, locally Noetherian singleton algebraic space Z is regular.*

[2]This is nonstandard notation.

Proof. Let Z be a reduced, locally Noetherian singleton algebraic space over a scheme S. Let $W \to Z$ be a surjective étale morphism where W is a scheme. Let k be a field and let $\mathrm{Spec}(k) \to Z$ be surjective, flat, and locally of finite presentation (see Lemma 50.10.3). The scheme $T = W \times_Z \mathrm{Spec}(k)$ is étale over k in particular regular, see Remark 50.4.1. Since $T \to W$ is locally of finite presentation, flat, and surjective it follows that W is regular, see Descent, Lemma 34.15.2. By definition this means that Z is regular. □

50.11. Decent spaces

In this section we collect some useful facts on decent spaces.

Lemma 50.11.1. *Let S be a scheme. Let X be a decent algebraic space over S.*

(1) *If $|X|$ is a singleton then X is a scheme.*
(2) *If $|X|$ is a singleton and X is reduced, then $X \cong \mathrm{Spec}(k)$ for some field k.*

Proof. Assume $|X|$ is a singleton. Choose an affine scheme U and a surjective étale morphism $U \to X$. Set $R = U \times_X U$. Then U and R have finitely many points by Lemma 50.4.5 (and the definition of a decent space). All of these points are closed in U and R by Lemma 50.9.3. It follows that U and R are affine schemes. We may shrink U to a singleton space. Then U is the spectrum of a henselian local ring, see Algebra, Lemma 10.143.11. The projections $R \to U$ are étale, hence finite étale because U is the spectrum of a 0-dimensional henselian local ring, see Algebra, Lemma 10.143.3. It follows that X is a scheme by Groupoids, Proposition 38.20.8.

Part (2) follows from (1) and the fact that a reduced singleton scheme is the spectrum of a field. □

Remark 50.11.2. We will see in Limits of Spaces, Lemma 52.14.3 that an algebraic space whose reduction is a scheme is a scheme.

Lemma 50.11.3. *Let S be a scheme. Let X be a decent algebraic space over S. Consider a commutative diagram*

$$\begin{array}{ccc} \mathrm{Spec}(k) & \longrightarrow & X \\ & \searrow \quad \swarrow & \\ & S & \end{array}$$

Assume that the image point $s \in S$ of $\mathrm{Spec}(k) \to S$ is a closed point and that $\kappa(s) \subset k$ is algebraic. Then the image x of $\mathrm{Spec}(k) \to X$ is a closed point of $|X|$.

Proof. Suppose that $x \rightsquigarrow x'$ for some $x' \in |X|$. Choose an étale morphism $U \to X$ where U is a scheme and a point $u' \in U'$ mapping to x'. Choose a specialization $u \rightsquigarrow u'$ in U with u mapping to x in X, see Lemma 50.9.4. Then u is the image of a point w of the scheme $W = \mathrm{Spec}(k) \times_X U$. Since the projection $W \to \mathrm{Spec}(k)$ is étale we see that $\kappa(w) \supset k$ is finite. Hence $\kappa(w) \supset \kappa(s)$ is algebraic. Hence $\kappa(u) \supset \kappa(s)$ is algebraic. Thus u is a closed point of U by Morphisms, Lemma 28.21.2. Thus $u = u'$, whence $x = x'$. □

Lemma 50.11.4. *Let S be a scheme. Let X be a decent algebraic space over S. Consider a commutative diagram*

$$\begin{array}{ccc} \mathrm{Spec}(k) & \longrightarrow & X \\ & \searrow \quad \swarrow & \\ & S & \end{array}$$

Assume that the image point $s \in S$ of $\mathrm{Spec}(k) \to S$ is a closed point and that $\kappa(s) \subset k$ is finite. Then $\mathrm{Spec}(k) \to X$ is finite morphism. If $\kappa(s) = k$ then $\mathrm{Spec}(k) \to X$ is closed immersion.

Proof. By Lemma 50.11.3 the image point $x \in |X|$ is closed. Let $Z \subset X$ be the reduced closed subspace with $|Z| = \{x\}$ (Properties of Spaces, Lemma 48.9.2). Note that Z is a decent algebraic space by Lemma 50.5.3. By Lemma 50.11.1 we see that $Z = \mathrm{Spec}(k')$ for some field k'. Of course $k \supset k' \supset \kappa(s)$. Then $\mathrm{Spec}(k) \to Z$ is a finite morphism of schemes and $Z \to X$ is a finite morphism as it is a closed immersion. Hence $\mathrm{Spec}(k) \to X$ is finite (Morphisms of Spaces, Lemma 49.40.4). If $k = \kappa(s)$, then $\mathrm{Spec}(k) = Z$ and $\mathrm{Spec}(k) \to X$ is a closed immersion. □

50.12. Locally separated spaces

It turns out that a locally separated algebraic space is decent.

Lemma 50.12.1. *Let A be a ring. Let k be a field. Let $\mathfrak{p}_n$, $n \geq 1$ be a sequence of pairwise distinct primes of A. Moreover, for each n let $k \to \kappa(\mathfrak{p})$ be an embedding. Then the closure of the image of*

$$\coprod\nolimits_{n \neq m} \mathrm{Spec}(\kappa(\mathfrak{p}_n) \otimes_k \kappa(\mathfrak{p}_m)) \longrightarrow \mathrm{Spec}(A \otimes A)$$

meets the diagonal.

Proof. Set $k_n = \kappa(\mathfrak{p}_n)$. We may assume that $A = \prod k_n$. Denote $x_n = \mathrm{Spec}(k_n)$ the open and closed point corresponding to $A \to k_n$. Then $\mathrm{Spec}(A) = Z \amalg \{x_n\}$ where Z is a nonempty closed subset. Namely, $Z = V(e_n; n \geq 1)$ where e_n is the idempotent of A corresponding to the factor k_n and Z is nonempty as the ideal generated by the e_n is not equal to A. We will show that the closure of the image contains $\Delta(Z)$. The kernel of the map

$$(\prod k_n) \otimes_k (\prod k_m) \longrightarrow \prod\nolimits_{n \neq m} k_n \otimes_k k_m$$

is the ideal generated by $e_n \otimes e_n$, $n \geq 1$. Hence the closure of the image of the map on spectra is $V(e_n \otimes e_n; n \geq 1)$ whose intersection with $\Delta(\mathrm{Spec}(A))$ is $\Delta(Z)$. Thus it suffices to show that

$$\coprod\nolimits_{n \neq m} \mathrm{Spec}(k_n \otimes_k k_m) \longrightarrow \mathrm{Spec}(\prod\nolimits_{n \neq m} k_n \otimes_k k_m)$$

has dense image. This follows as the family of ring maps $\prod_{n \neq m} k_n \otimes_k k_m \to k_n \otimes_k k_m$ is jointly injective. □

Lemma 50.12.2 (David Rydh)**.** *A locally separated algebraic space is decent.*

Proof. Let S be a base scheme. Let X be a locally separated algebraic space over S. Let $x \in |X|$. Choose a scheme U, an étale morphism $U \to X$, and a point

$u \in U$ mapping to x in $|X|$. As usual we identify $u = \mathrm{Spec}(\kappa(u))$. As X is locally separated the morphism

$$u \times_X u \to u \times u$$

is an immersion (Morphisms of Spaces, Lemma 49.5.5). Hence More on Groupoids, Lemma 39.10.5 tells us that it is a closed immersion (use Schemes, Lemma 25.10.4). As $u \times_X u \to u \times_X U$ is a monomorphism (base change of $u \to U$) and as $u \times_X U \to u$ is étale we conclude that $u \times_X u$ is a disjoint union of spectra of fields (see Remark 50.4.1 and Schemes, Lemma 25.23.10). Since it is also closed in the affine scheme $u \times u$ we conclude $u \times_X u$ is a finite disjoint union of spectra of fields. Thus x can be represented by a monomorphism $\mathrm{Spec}(k) \to X$ where k is a field, see Lemma 50.4.3.

Next, let $U = \mathrm{Spec}(A)$ be an affine scheme and let $U \to X$ be an étale morphism. To finish the proof it suffices to show that $F = U \times_X \mathrm{Spec}(k)$ is finite. Write $F = \coprod_{i \in I} \mathrm{Spec}(k_i)$ as the disjoint union of finite separable extensions of k. We have to show that I is finite. Set $R = U \times_X U$. As X is locally separated, the morphism $j : R \to U \times U$ is an immersion. Let $e : U \to R$ be the diagonal map. Using that e is a morphism between étale schemes over U such that $\Delta = j \circ e$ is a closed immersion, we conclude that $R = e(U) \coprod W$ for some open and closed subscheme $W \subset R$. Since j is an immersion and $j|_{e(U)}$ is a closed immersion we conclude that $\overline{j(W)} \cap \Delta(U) = \emptyset$ in $U \times U$. Note that W contains $\mathrm{Spec}(k_i \otimes_k k_{i'})$ for all $i \neq i'$, $i, i' \in I$. By Lemma 50.12.1 we conclude that I is finite as desired. □

50.13. Valuative criterion

For a quasi-compact morphism from a decent space the valuative criterion is necessary in order for the morphism to be universally closed.

Proposition 50.13.1. *Let S be a scheme. Let $f : X \to Y$ be a morphism of algebraic spaces over S. Assume*

(1) *f is quasi-compact, and*
(2) *X is decent.*

Then f is universally closed if and only if the existence part of the valuative criterion holds.

Proof. In Morphisms of Spaces, Lemma 49.38.1 we have seen one of the implications. To prove the other, assume that f is universally closed. Let

$$\begin{array}{ccc} \mathrm{Spec}(K) & \longrightarrow & X \\ \downarrow & & \downarrow \\ \mathrm{Spec}(A) & \longrightarrow & Y \end{array}$$

be a diagram as in Morphisms of Spaces, Definition 49.37.1. Let $X_A = \mathrm{Spec}(A) \times_Y X$, so that we have

$$\begin{array}{ccc} \mathrm{Spec}(K) & \longrightarrow & X_A \\ & \searrow & \downarrow \\ & & \mathrm{Spec}(A) \end{array}$$

By Morphisms of Spaces, Lemma 49.9.3 we see that $X_A \to \mathrm{Spec}(A)$ is quasi-compact. Since $X_A \to X$ is representable, we see that X_A has is decent also, see Lemma 50.5.3. Moreover, as f is universally closed, we see that $X_A \to \mathrm{Spec}(A)$ is universally closed. Hence we may and do replace X by X_A and Y by $\mathrm{Spec}(A)$.

Let $x' \in |X|$ be the equivalence class of $\mathrm{Spec}(K) \to X$. Let $y \in |Y| = |\mathrm{Spec}(A)|$ be the closed point. Set $y' = f(x')$; it is the generic point of $\mathrm{Spec}(A)$. Since f is universally closed we see that $f(\overline{\{x'\}})$ contains $\overline{\{y'\}}$, and hence contains y. Let $x \in \overline{\{x'\}}$ be a point such that $f(x) = y$. Let U be a scheme, and $\varphi : U \to X$ an étale morphism such that there exists a $u \in U$ with $\varphi(u) = x$. By Lemma 50.7.2 and our assumption that X is decent there exists a specialization $u' \rightsquigarrow u$ on U with $\varphi(u') = x'$. This means that there exists a common field extension $K \subset K' \supset \kappa(u')$ such that

$$\begin{array}{ccc} \mathrm{Spec}(K') & \longrightarrow & U \\ \downarrow & & \downarrow \\ \mathrm{Spec}(K) & \longrightarrow & X \\ & \searrow & \downarrow \\ & & \mathrm{Spec}(A) \end{array}$$

is commutative. This gives the following commutative diagram of rings

$$\begin{array}{ccc} K' & \longleftarrow & \mathcal{O}_{U,u} \\ \uparrow & & \uparrow \\ K & & \\ & \nwarrow & \\ & & A \end{array}$$

By Algebra, Lemma 10.46.2 we can find a valuation ring $A' \subset K'$ dominating the image of $\mathcal{O}_{U,u}$ in K'. Since by construction $\mathcal{O}_{U,u}$ dominates A we see that A' dominates A also. Hence we obtain a diagram resembling the second diagram of Morphisms of Spaces, Definition 49.37.1 and the proposition is proved. □

50.14. Relative conditions

This is a (yet another) technical section dealing with conditions on algebraic spaces having to do with points. It is probably a good idea to skip this section.

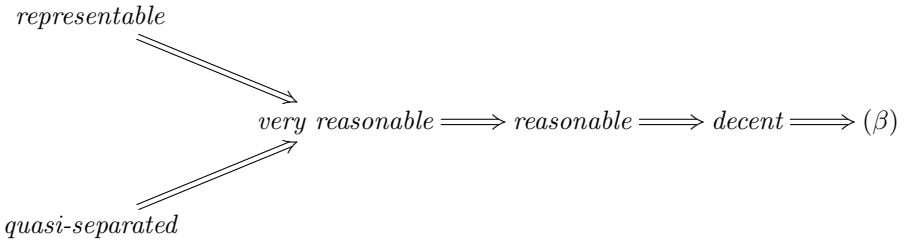
Definition 50.14.1. Let S be a scheme. We say an algebraic space X over S *has property* (β) if X has the corresponding property of Lemma 50.5.1. Let $f : X \to Y$ be a morphism of algebraic spaces over S.

(1) We say f *has property* (β) if for any scheme T and morphism $T \to Y$ the fibre product $T \times_Y X$ has property (β).
(2) We say f is *decent* if for any scheme T and morphism $T \to Y$ the fibre product $T \times_Y X$ is a decent algebraic space.
(3) We say f is *reasonable* if for any scheme T and morphism $T \to Y$ the fibre product $T \times_Y X$ is a reasonable algebraic space.

(4) We say f is *very reasonable* if for any scheme T and morphism $T \to Y$ the fibre product $T \times_Y X$ is a very reasonable algebraic space.

We refer to Remark 50.14.7 for an informal discussion. It will turn out that the class of very reasonable morphisms is not so useful, but that the classes of decent and reasonable morphisms are useful.

Lemma 50.14.2. *Let S be a scheme. Let $f : X \to Y$ be a morphism of algebraic spaces over S. We have the following implications among the conditions on f:*



Proof. This is clear from the definitions, Lemma 50.5.1 and Morphisms of Spaces, Lemma 49.5.12. □

Lemma 50.14.3. *Having property (β), (γ), (δ), or (ϵ) is preserved under arbitrary base change.*

Proof. Omitted. □

Lemma 50.14.4. *Having property (β), being decent, or being reasonable is preserved under compositions.*

Proof. Let $\omega \in \{\beta, decent, reasonable\}$. Let $f : X \to Y$ and $g : Y \to Z$ be morphisms of algebraic spaces over the scheme S. Assume f and g both have property (ω). Then we have to show that for any scheme T and morphism $T \to Z$ the space $T \times_Z X$ has (ω). By Lemma 50.14.3 this reduces us to the following claim: Suppose that Y is an algebraic space having property (ω), and that $f : X \to Y$ is a morphism with (ω). Then X has (ω).

Let us prove the claim in case $\omega = \beta$. In this case we have to show that any $x \in |X|$ is represented by a monomorphism from the spectrum of a field into X. Let $y = f(x) \in |Y|$. By assumption there exists a field k and a monomorphism $\mathrm{Spec}(k) \to Y$ representing y. Then x corresponds to a point x' of $\mathrm{Spec}(k) \times_Y X$. By assumption x' is represented by a monomorphism $\mathrm{Spec}(k') \to \mathrm{Spec}(k) \times_Y X$. Clearly the composition $\mathrm{Spec}(k') \to X$ is a monomorphism representing x.

Let us prove the claim in case $\omega = decent$. Let $x \in |X|$ and $y = f(x) \in |Y|$. By the result of the preceding paragraph we can choose a diagram

$$\begin{array}{ccc} \mathrm{Spec}(k') & \xrightarrow{x} & X \\ \downarrow & & \downarrow f \\ \mathrm{Spec}(k) & \xrightarrow{y} & Y \end{array}$$

whose horizontal arrows monomorphisms. As Y is decent the morphism y is quasi-compact. As f is decent the algebraic space $\mathrm{Spec}(k) \times_Y X$ is decent. Hence the

monomorphism $\mathrm{Spec}(k') \to \mathrm{Spec}(k) \times_Y X$ is quasi-compact. Then the monomorphism $x : \mathrm{Spec}(k') \to X$ is quasi-compact as a composition of quasi-compact morphisms (use Morphisms of Spaces, Lemmas 49.9.3 and 49.9.4). As the point x was arbitrary this implies X is decent.

Let us prove the claim in case $\omega = reasonable$. Choose $V \to Y$ étale with V an affine scheme. Choose $U \to V \times_Y X$ étale with U an affine scheme. By assumption $V \to Y$ has universally bounded fibres. By Lemma 50.3.3 the morphism $V \times_Y X \to X$ has universally bounded fibres. By assumption on f we see that $U \to V \times_Y X$ has universally bounded fibres. By Lemma 50.3.2 the composition $U \to X$ has universally bounded fibres. Hence there exists sufficiently many étale morphisms $U \to X$ from schemes with universally bounded fibres, and we conclude that X is reasonable. □

Lemma 50.14.5. *Let S be a scheme. Let $f : X \to Y$ be a morphism of algebraic spaces over S. Let $\mathcal{P} \in \{(\beta), decent, reasonable\}$. Assume*

1. *f is quasi-compact,*
2. *f is étale,*
3. *$|f| : |X| \to |Y|$ is surjective, and*
4. *the algebraic space X has property $\mathcal{P}$.*

Then Y has property $\mathcal{P}$.

Proof. Let us prove this in case $\mathcal{P} = (\beta)$. Let $y \in |Y|$ be a point. We have to show that y can be represented by a monomorphism from a field. Choose a point $x \in |X|$ with $f(x) = y$. By assumption we may represent x by a monomorphism $\mathrm{Spec}(k) \to X$, with k a field. By Lemma 50.4.3 it suffices to show that the projections $\mathrm{Spec}(k) \times_Y \mathrm{Spec}(k) \to \mathrm{Spec}(k)$ are étale and quasi-compact. We can factor the first projection as

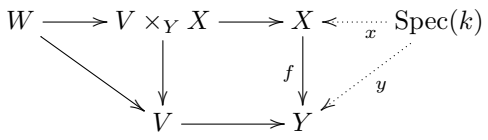
$$\mathrm{Spec}(k) \times_Y \mathrm{Spec}(k) \longrightarrow \mathrm{Spec}(k) \times_Y X \longrightarrow \mathrm{Spec}(k)$$

The first morphism is a monomorphism, and the second is étale and quasi-compact. By Properties of Spaces, Lemma 48.13.8 we see that $\mathrm{Spec}(k) \times_Y X$ is a scheme. Hence it is a finite disjoint union of spectra of finite separable field extensions of k. By Schemes, Lemma 25.23.10 we see that the first arrow identifies $\mathrm{Spec}(k) \times_Y \mathrm{Spec}(k)$ with a finite disjoint union of spectra of finite separable field extensions of k. Hence the projection morphism is étale and quasi-compact.

Let us prove this in case $\mathcal{P} = decent$. We have already seen in the first paragraph of the proof that this implies that every $y \in |Y|$ can be represented by a monomorphism $y : \mathrm{Spec}(k) \to Y$. Pick such a y. Pick an affine scheme U and an étale morphism $U \to X$ such that the image of $|U| \to |Y|$ contains y. By Lemma 50.4.5 it suffices to show that U_y is a finite scheme over k. The fibre product $X_y = \mathrm{Spec}(k) \times_Y X$ is a quasi-compact étale algebraic space over k. Hence by Properties of Spaces, Lemma 48.13.8 it is a scheme. So it is a finite disjoint union of spectra of finite separable extensions of k. Say $X_y = \{x_1, \ldots, x_n\}$ so x_i is given by $x_i : \mathrm{Spec}(k_i) \to X$ with $[k_i : k] < \infty$. By assumption X is decent, so the schemes $U_{x_i} = \mathrm{Spec}(k_i) \times_X U$ are finite over k_i. Finally, we note that $U_y = \coprod U_{x_i}$ as a scheme and we conclude that U_y is finite over k as desired.

Let us prove this in case $\mathcal{P} = reasonable$. Pick an affine scheme V and an étale morphism $V \to Y$. We have the show the fibres of $V \to Y$ are universally bounded.

The algebraic space $V \times_Y X$ is quasi-compact. Thus we can find an affine scheme W and a surjective étale morphism $W \to V \times_Y X$, see Properties of Spaces, Lemma 48.6.3. Here is a picture (solid diagram)



The morphism $W \to X$ is universally bounded by our assumption that the space X is reasonable. Let n be an integer bounding the degrees of the fibres of $W \to X$. We claim that the same integer works for bounding the fibres of $V \to Y$. Namely, suppose $y \in |Y|$ is a point. Then there exists a $x \in |X|$ with $f(x) = y$ (see above). This means we can find a field k and morphisms x, y given as dotted arrows in the diagram above. In particular we get a surjective étale morphism

$$\mathrm{Spec}(k) \times_{x,X} W \to \mathrm{Spec}(k) \times_{x,X} (V \times_Y X) = \mathrm{Spec}(k) \times_{y,Y} V$$

which shows that the degree of $\mathrm{Spec}(k) \times_{y,Y} V$ over k is less than or equal to the degree of $\mathrm{Spec}(k) \times_{x,X} W$ over k, i.e., $\leq n$, and we win. (This last part of the argument is the same as the argument in the proof of Lemma 50.3.4. Unfortunately that lemma is not general enough because it only applies to representable morphisms.) □

Lemma 50.14.6. *Let S be a scheme. Let $f : X \to Y$ be a morphism of algebraic spaces over S. Let $\mathcal{P} \in \{(\beta), decent, reasonable, very\ reasonable\}$. The following are equivalent*

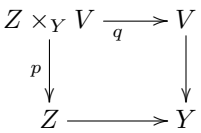
1. *f is $\mathcal{P}$,*
2. *for every affine scheme Z and every morphism $Z \to Y$ the base change $Z \times_Y X \to Z$ of f is $\mathcal{P}$,*
3. *for every affine scheme Z and every morphism $Z \to Y$ the algebraic space $Z \times_Y X$ is $\mathcal{P}$, and*
4. *there exists a Zariski covering $Y = \bigcup Y_i$ such that each morphism $f^{-1}(Y_i) \to Y_i$ has $\mathcal{P}$.*

If $\mathcal{P} \in \{(\beta), decent, reasonable\}$, then this is also equivalent to

5. *there exists a scheme V and a surjective étale morphism $V \to Y$ such that the base change $V \times_Y X \to V$ has $\mathcal{P}$.*

Proof. The implications (1) ⇒ (2) ⇒ (3) ⇒ (4) are trivial. The implication (3) ⇒ (1) can be seen as follows. Let $Z \to Y$ be a morphism whose source is a scheme over S. Consider the algebraic space $Z \times_Y X$. If we assume (3), then for any affine open $W \subset Z$, the open subspace $W \times_Y X$ of $Z \times_Y X$ has property $\mathcal{P}$. Hence by Lemma 50.5.2 the space $Z \times_Y X$ has property $\mathcal{P}$, i.e., (1) holds. A similar argument (omitted) shows that (4) implies (1).

The implication (1) ⇒ (5) is trivial. Let $V \to Y$ be an étale morphism from a scheme as in (5). Let Z be an affine scheme, and let $Z \to Y$ be a morphism. Consider the diagram



Since p is étale, and hence open, we can choose finitely many affine open subschemes $W_i \subset Z \times_Y V$ such that $Z = \bigcup p(W_i)$. Consider the commutative diagram

$$\begin{array}{ccccc} V \times_Y X & \longleftarrow & (\coprod W_i) \times_Y X & \longrightarrow & Z \times_Y X \\ \downarrow & & \downarrow & & \downarrow \\ V & \longleftarrow & \coprod W_i & \longrightarrow & Z \end{array}$$

We know $V \times_Y X$ has property $\mathcal{P}$. By Lemma 50.5.3 we see that $(\coprod W_i) \times_Y X$ has property $\mathcal{P}$. Note that the morphism $(\coprod W_i) \times_Y X \to Z \times_Y X$ is étale and quasi-compact as the base change of $\coprod W_i \to Z$. Hence by Lemma 50.14.5 we conclude that $Z \times_Y X$ has property $\mathcal{P}$. □

Remark 50.14.7. An informal description of the properties (β), decent, reasonable, very reasonable was given in Section 50.6. A morphism has one of these properties if (very) loosely speaking the fibres of the morphism have the corresponding properties. Being decent is useful to prove things about specializations of points on $|X|$. Being reasonable is a bit stronger and technically quite easy to work with.

Here is a lemma we promised earlier which uses decent morphisms.

Lemma 50.14.8. *Let S be a scheme. Let $f : X \to Y$ be a morphism of algebraic spaces over S. Assume f is quasi-compact and decent. (For example if f is representable, or quasi-separated, see Lemma 50.14.2.) Then f is universally closed if and only if the existence part of the valuative criterion holds.*

Proof. In Morphisms of Spaces, Lemma 49.38.1 we proved that any quasi-compact morphism which satisfies the existence part of the valuative criterion is universally closed. To prove the other, assume that f is universally closed. In the proof of Proposition 50.13.1 we have seen that it suffices to show, for any valuation ring A, and any morphism $\mathrm{Spec}(A) \to Y$, that the base change $f_A : X_A \to \mathrm{Spec}(A)$ satisfies the existence part of the valuative criterion. By definition the algebraic space X_A has property (γ) and hence Proposition 50.13.1 applies to the morphism f_A and we win. □

50.15. Monomorphisms

Here is another case where monomorphisms are representable.

Lemma 50.15.1. *Let S be a scheme. Let Y be a disjoint union of spectra of zero dimensional local rings over S. Let $f : X \to Y$ be a monomorphism of algebraic spaces over S. Then f is representable, i.e., X is a scheme.*

Proof. This immediately reduces to the case $Y = \mathrm{Spec}(A)$ where A is a zero dimensional local ring, i.e., $\mathrm{Spec}(A) = \{\mathfrak{m}_A\}$ is a singleton. If $X = \emptyset$, then there is nothing to prove. If not, choose a nonempty affine scheme $U = \mathrm{Spec}(B)$ and an étale morphism $U \to X$. As $|X|$ is a singleton (as a subset of $|Y|$, see Morphisms of Spaces, Lemma 49.11.9) we see that $U \to X$ is surjective. Note that $U \times_X U = U \times_Y U = \mathrm{Spec}(B \otimes_A B)$. Thus we see that the ring maps $B \to B \otimes_A B$ are étale. Since

$$(B \otimes_A B)/\mathfrak{m}_A(B \otimes_A B) = (B/\mathfrak{m}_A B) \otimes_{A/\mathfrak{m}_A} (B/\mathfrak{m}_A B)$$

we see that $B/\mathfrak{m}_A B \to (B \otimes_A B)/\mathfrak{m}_A(B \otimes_A B)$ is flat and in fact free of rank equal to the dimension of $B/\mathfrak{m}_A B$ as a $A/\mathfrak{m}_A$-vector space. Since $B \to B \otimes_A B$ is étale, this can only happen if this dimension is finite (see for example Morphisms, Lemmas 28.50.7 and 28.50.8). Every prime of B lies over $\mathfrak{m}_A$ (the unique prime of A). Hence $\mathrm{Spec}(B) = \mathrm{Spec}(B/\mathfrak{m}_A)$ as a topological space, and this space is a finite discrete set as $B/\mathfrak{m}_A B$ is an Artinian ring, see Algebra, Lemmas 10.49.2 and 10.49.8. Hence all prime ideals of B are maximal and $B = B_1 \times \ldots \times B_n$ is a product of finitely many local rings of dimension zero, see Algebra, Lemma 10.49.7. Thus $B \to B \otimes_A B$ is finite étale as all the local rings B_i are henselian by Algebra, Lemma 10.143.11. Thus X is an affine scheme by Groupoids, Proposition 38.20.8. □

50.16. Other chapters

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CHAPTER 51

Cohomology of Algebraic Spaces

51.1. Introduction

In this chapter we write about cohomology of algebraic spaces. This mean in particular cohomology of quasi-coherent sheaves, i.e., we prove analogues of the results in the chapter "Cohomology of Schemes". Some of the results in this chapter can be found in [**Knu71**].

51.2. Conventions

The standing assumption is that all schemes are contained in a big fppf site Sch_{fppf}. And all rings A considered have the property that $\mathrm{Spec}(A)$ is (isomorphic) to an object of this big site.

Let S be a scheme and let X be an algebraic space over S. In this chapter and the following we will write $X \times_S X$ for the product of X with itself (in the category of algebraic spaces over S), instead of $X \times X$.

51.3. Higher direct images

Let S be a scheme. Let X be a representable algebraic space over S. Let $\mathcal{F}$ be a quasi-coherent module on X (see Properties of Spaces, Section 48.27). By Descent, Proposition 34.7.10 the cohomology groups $H^i(X, \mathcal{F})$ agree with the usual cohomology group computed in the Zariski topology of the corresponding quasi-coherent module on the scheme representing X.

More generally, let $f : X \to Y$ be a quasi-compact and quasi-separated morphism of representable algebraic spaces X and Y. Let $\mathcal{F}$ be a quasi-coherent module on X. By Descent, Lemma 34.7.15 the sheaf $R^i f_* \mathcal{F}$ agrees with the usual higher direct image computed for the Zariski topology of the quasi-coherent module on the scheme representing X mapping to the scheme representing Y.

More generally still, suppose $f : X \to Y$ is a representable, quasi-compact, and quasi-separated morphism of algebraic spaces over S. Let V be a scheme and let $V \to Y$ be an étale surjective morphism. Let $U = V \times_Y X$ and let $f' : U \to V$ be the base change of f. Then for any quasi-coherent $\mathcal{O}_X$-module $\mathcal{F}$ we have

$$R^i f'_*(\mathcal{F}|_U) = (R^i f_* \mathcal{F})|_V, \tag{51.3.0.1}$$

see Properties of Spaces, Lemma 48.24.2. And because $f' : U \to V$ is a quasi-compact and quasi-separated morphism of schemes, by the remark of the preceding paragraph we may compute $R^i f'_*(\mathcal{F}|_U)$ by thinking of $\mathcal{F}|_U$ as a quasi-coherent sheaf on the scheme U, and f' as a morphism of schemes. We will frequently use this without further mention.

Next, we prove that higher direct images of quasi-coherent sheaves are quasi-coherent for any quasi-compact and quasi-separated morphism of algebraic spaces. In the proof we use a trick; a "better" proof would use a relative Cech complex, as discussed in Sheaves on Stacks, Sections 69.17 and 69.18 ff.

Lemma 51.3.1. *Let S be a scheme. Let $f : X \to Y$ be a morphism of algebraic spaces over S. If f is quasi-compact and quasi-separated, then $R^i f_*$ transforms quasi-coherent $\mathcal{O}_X$-modules into quasi-coherent $\mathcal{O}_Y$-modules.*

Proof. Let $V \to Y$ be an étale morphism where V is an affine scheme. Set $U = V \times_Y X$ and denote $f' : U \to V$ the induced morphism. Let $\mathcal{F}$ be a quasi-coherent $\mathcal{O}_X$-module. By Properties of Spaces, Lemma 48.24.2 we have $R^i f'_*(\mathcal{F}|_U) = (R^i f_* \mathcal{F})|_V$. Since the property of being a quasi-coherent module is local in the étale topology on Y (see Properties of Spaces, Lemma 48.27.6) we may replace Y by V, i.e., we may assume Y is an affine scheme.

Assume Y is affine. Since f is quasi-compact we see that X is quasi-compact. Thus we may choose an affine scheme U and a surjective étale morphism $g : U \to X$, see Properties of Spaces, Lemma 48.6.3. Picture

$$\begin{array}{ccc} U & \xrightarrow{g} & X \\ & \searrow^{f \circ g} & \downarrow f \\ & & Y \end{array}$$

The morphism $g : U \to X$ is representable, separated and quasi-compact because X is quasi-separated. Hence the lemma holds for g (by the discussion above the lemma). It also holds for $f \circ g : U \to Y$ (as this is a morphism of affine schemes).

In the situation described in the previous paragraph we will show by induction on n that IH_n: for any quasi-coherent sheaf $\mathcal{F}$ on X the sheaves $R^i f \mathcal{F}$ are quasi-coherent for $i \leq n$. The case $n = 0$ follows from Morphisms of Spaces, Lemma 49.12.2. Assume IH_n. In the rest of the proof we show that IH_{n+1} holds.

Let $\mathcal{H}$ be a quasi-coherent $\mathcal{O}_U$-module. Consider the Leray spectral sequence

$$E_2^{p,q} = R^p f_* R^q g_* \mathcal{H} \Rightarrow R^{p+q}(f \circ g)_* \mathcal{H}$$

Cohomology on Sites, Lemma 21.14.7. As $R^q g_* \mathcal{H}$ is quasi-coherent by IH_n all the sheaves $R^p f_* R^q g_* \mathcal{H}$ are quasi-coherent for $p \leq n$. The sheaves $R^{p+q}(f \circ g)_* \mathcal{H}$ are all quasi-coherent (in fact zero for $p + q > 0$ but we do not need this). Looking in degrees $\leq n + 1$ the only module which we do not yet know is quasi-coherent is $E_2^{n+1,0} = R^{n+1} f_* g_* \mathcal{H}$. Moreover, the differentials $d_r^{n+1,0} : E_r^{n+1,0} \to E_r^{n+1+r,1-r}$ are zero as the target is zero. Using that $QCoh(\mathcal{O}_X)$ is a weak Serre subcategory of $Mod(\mathcal{O}_X)$ (Properties of Spaces, Lemma 48.27.7) it follows that $R^{n+1} f_* g_* \mathcal{H}$ is quasi-coherent (details omitted).

Let $\mathcal{F}$ be a quasi-coherent $\mathcal{O}_X$-module. Set $\mathcal{H} = g^* \mathcal{F}$. The adjunction mapping $\mathcal{F} \to g_* g^* \mathcal{F} = g_* \mathcal{H}$ is injective as $U \to X$ is surjective étale. Consider the exact sequence

$$0 \to \mathcal{F} \to g_* \mathcal{H} \to \mathcal{G} \to 0$$

where $\mathcal{G}$ is the cokernel of the first map and in particular quasi-coherent. Applying the long exact cohomology sequence we obtain

$$R^n f_* g_* \mathcal{H} \to R^n f_* \mathcal{G} \to R^{n+1} f_* \mathcal{F} \to R^{n+1} f_* g_* \mathcal{H} \to R^{n+1} f_* \mathcal{G}$$

The cokernel of the first arrow is quasi-coherent and we have seen above that $R^{n+1}f_*g_*\mathcal{H}$ is quasi-coherent. Thus $R^{n+1}f_*\mathcal{F}$ has a 2-step filtration where the first step is quasi-coherent and the second a submodule of a quasi-coherent sheaf. Since $\mathcal{F}$ is an arbitrary quasi-coherent $\mathcal{O}_X$-module, this result also holds for $\mathcal{G}$. Thus we can choose an exact sequence $0 \to \mathcal{A} \to R^{n+1}f_*\mathcal{G} \to \mathcal{B}$ with $\mathcal{A}$, $\mathcal{B}$ quasi-coherent $\mathcal{O}_Y$-modules. Then the kernel $\mathcal{K}$ of $R^{n+1}f_*g_*\mathcal{H} \to R^{n+1}f_*\mathcal{G} \to \mathcal{B}$ is quasi-coherent, whereupon we obtain a map $\mathcal{K} \to \mathcal{A}$ whose kernel $\mathcal{K}'$ is quasi-coherent too. Hence $R^{n+1}f_*\mathcal{F}$ sits in an exact sequence

$$R^n f_* g_* \mathcal{H} \to R^n f_* \mathcal{G} \to R^{n+1} f_* \mathcal{F} \to \mathcal{K}' \to 0$$

with all modules quasi-coherent except for possibly $R^{n+1}f_*\mathcal{F}$. We conclude that $R^{n+1}f_*\mathcal{F}$ is quasi-coherent, i.e., IH_{n+1} holds as desired. □

Lemma 51.3.2. *Let S be a scheme. Let $f : X \to Y$ be a quasi-separated and quasi-compact morphism of algebraic spaces over S. For any quasi-coherent $\mathcal{O}_X$-module $\mathcal{F}$ and any affine object V of $Y_{étale}$ we have*

$$H^q(V \times_Y X, \mathcal{F}) = H^0(V, R^q f_* \mathcal{F})$$

for all $q \in \mathbf{Z}$.

Proof. Since formation of Rf_* commutes with étale localization (Properties of Spaces, Lemma 48.24.2) we may replace Y by V and assume $Y = V$ is affine. Consider the Leray spectral sequence $E_2^{p,q} = H^p(Y, R^q f_*\mathcal{F})$ converging to $H^{p+q}(X, \mathcal{F})$, see Cohomology on Sites, Lemma 21.14.5. By Lemma 51.3.1 we see that the sheaves $R^q f_*\mathcal{F}$ are quasi-coherent. By Cohomology of Schemes, Lemma 29.2.2 we see that $E_2^{p,q} = 0$ when $p > 0$. Hence the spectral sequence degenerates at E_2 and we win. □

51.4. Colimits and cohomology

The following lemma in particular applies to diagrams of quasi-coherent sheaves.

Lemma 51.4.1. *Let S be a scheme. Let X be an algebraic space over S. If X is quasi-compact and quasi-separated, then*

$$\operatorname{colim}_i H^p(X, \mathcal{F}_i) \longrightarrow H^p(X, \operatorname{colim}_i \mathcal{F}_i)$$

for every filtered diagram of abelian sheaves on $X_{étale}$.

Proof. This follows from Cohomology on Sites, Lemma 21.16.1. Namely, let $\mathcal{B} \subset \mathrm{Ob}(X_{spaces,étale})$ be the set of quasi-compact and quasi-separated spaces étale over X. Note that if $U \in \mathcal{B}$ then, because U is quasi-compact, the collection of finite coverings $\{U_i \to U\}$ with $U_i \in \mathcal{B}$ is cofinal in the set of coverings of U in $X_{étale}$. By Morphisms of Spaces, Lemma 49.9.9 the set $\mathcal{B}$ satisfies all the assumptions of Cohomology on Sites, Lemma 21.16.1. Since $X \in \mathcal{B}$ we win. □

Lemma 51.4.2. *Let S be a scheme. Let $f : X \to Y$ be a quasi-compact and quasi-separated morphism of algebraic spaces over S. Let $\mathcal{F} = \operatorname{colim} \mathcal{F}_i$ be a filtered colimit of abelian sheaves on $X_{étale}$. Then for any $p \geq 0$ we have*

$$R^p f_* \mathcal{F} = \operatorname{colim} R^p f_* \mathcal{F}_i.$$

Proof. Recall that $R^pf_*\mathcal{F}$ is the sheaf on $Y_{spaces,étale}$ associated to $V \mapsto H^p(V \times_Y X, \mathcal{F})$, see Cohomology on Sites, Lemma 21.8.4 and Properties of Spaces, Lemma 48.15.7. Recall that the colimit is the sheaf associated to the presheaf colimit. Hence we can apply Lemma 51.4.1 to $H^p(V \times_Y X, -)$ where V is affine to conclude (because when V is affine, then $V \times_Y X$ is quasi-compact and quasi-separated). Strictly speaking this also uses Properties of Spaces, Lemma 48.15.5 to see that there exist enough affine objects. □

The following lemma tells us that finitely presented modules behave as expected in quasi-compact and quasi-separated algebraic spaces.

Lemma 51.4.3. *Let S be a scheme. Let X be a quasi-compact and quasi-separated algebraic space over S. Let I be a partially ordered set and let $(\mathcal{F}_i, \varphi_{ii'})$ be a system over I of quasi-coherent $\mathcal{O}_X$-modules. Let $\mathcal{G}$ be an $\mathcal{O}_X$-module of finite presentation. Then we have*

$$\operatorname{colim}_i \operatorname{Hom}_X(\mathcal{G}, \mathcal{F}_i) = \operatorname{Hom}_X(\mathcal{G}, \operatorname{colim}_i \mathcal{F}_i).$$

Proof. Choose an affine scheme U and a surjective étale morphism $U \to X$. Set $R = U \times_X U$. Note that R is a quasi-compact (as X is quasi-separated and U quasi-compact) and separated (as U is separated) scheme. Hence we have

$$\operatorname{colim}_i \operatorname{Hom}_U(\mathcal{G}|_U, \mathcal{F}_i|_U) = \operatorname{Hom}_U(\mathcal{G}|_U, \operatorname{colim}_i \mathcal{F}_i|_U).$$

by Modules, Lemma 17.11.6 (and the material on restriction to schemes étale over X, see Properties of Spaces, Sections 48.27 and 48.28). Similarly for R. Since $QCoh(\mathcal{O}_X) = QCoh(U, R, s, t, c)$ (see Properties of Spaces, Proposition 48.30.1) the result follows formally. □

51.5. The alternating Čech complex

Let S be a scheme. Let $f : U \to X$ be an étale morphism of algebraic spaces over S. The functor

$$j : U_{spaces,étale} \longrightarrow X_{spaces,étale}, \quad V/U \longmapsto V/X$$

induces an equivalence of $U_{spaces,étale}$ with the localization $X_{spaces,étale}/U$, see Properties of Spaces, Section 48.25. Hence there exist functors

$$f_! : Ab(U_{étale}) \longrightarrow Ab(X_{étale}), \quad f_! : Mod(\mathcal{O}_U) \longrightarrow Mod(\mathcal{O}_X),$$

which are left adjoint to

$$f^{-1} : Ab(X_{étale}) \longrightarrow Ab(U_{étale}), \quad f^* : Mod(\mathcal{O}_X) \longrightarrow Mod(\mathcal{O}_U)$$

see Modules on Sites, Section 18.19. Warning: This functor, a priori, has nothing to do with cohomology with compact supports! We dubbed this functor "extension by zero" in the reference above. Note that the two versions of $f_!$ agree as $f^* = f^{-1}$ for sheaves of $\mathcal{O}_X$-modules.

As we are going to use this construction below let us recall some of its properties. Given an abelian sheaf $\mathcal{G}$ on $U_{étale}$ the sheaf $f_!\mathcal{G}$ is the sheafification of the presheaf

$$V/X \longmapsto f_!\mathcal{G}(V) = \bigoplus\nolimits_{\varphi \in \operatorname{Mor}_X(V,U)} \mathcal{G}(V \xrightarrow{\varphi} U),$$

see Modules on Sites, Lemma 18.19.2. Moreover, if $\mathcal{G}$ is an $\mathcal{O}_U$-module, then $f_!\mathcal{G}$ is the sheafification of the exact same presheaf of abelian groups which is endowed

with an $\mathcal{O}_X$-module structure in an obvious way (see loc. cit.). Let $\overline{x} : \mathrm{Spec}(k) \to X$ be a geometric point. Then there is a canonical identification

$$(f_!\mathcal{G})_{\overline{x}} = \bigoplus\nolimits_{\overline{u}} \mathcal{G}_{\overline{u}}$$

where the sum is over all $\overline{u} : \mathrm{Spec}(k) \to U$ such that $f \circ \overline{u} = \overline{x}$, see Modules on Sites, Lemma 18.37.1 and Properties of Spaces, Lemma 48.16.13. In the following we are going to study the sheaf $f_!\underline{\mathbf{Z}}$. Here $\underline{\mathbf{Z}}$ denotes the constant sheaf on $X_{étale}$ or $U_{étale}$.

Lemma 51.5.1. *Let S be a scheme. Let $f_i : U_i \to X$ be étale morphisms of algebraic spaces over S. Then there are isomorphisms*

$$f_{1,!}\underline{\mathbf{Z}} \otimes_{\mathbf{Z}} f_{2,!}\underline{\mathbf{Z}} \longrightarrow f_{12,!}\underline{\mathbf{Z}}$$

where $f_{12} : U_1 \times_X U_2 \to X$ is the structure morphism and

$$(f_1 \amalg f_2)_!\underline{\mathbf{Z}} \longrightarrow f_{1,!}\underline{\mathbf{Z}} \oplus f_{2,!}\underline{\mathbf{Z}}$$

Proof. Once we have defined the map it will be an isomorphism by our description of stalks above. To define the map it suffices to work on the level of presheaves. Thus we have to define a map

$$\left(\bigoplus\nolimits_{\varphi_1 \in \mathrm{Mor}_X(V,U_1)} \mathbf{Z}\right) \otimes_{\mathbf{Z}} \left(\bigoplus\nolimits_{\varphi_2 \in \mathrm{Mor}_X(V,U_2)} \mathbf{Z}\right) \longrightarrow \bigoplus\nolimits_{\varphi \in \mathrm{Mor}_X(V,U_1 \times_X U_2)} \mathbf{Z}$$

We map the element $1_{\varphi_1} \otimes 1_{\varphi_2}$ to the element $1_{\varphi_1 \times \varphi_2}$ with obvious notation. We omit the proof of the second equality. □

Another important feature is the trace map

$$\mathrm{Tr}_f : f_!\underline{\mathbf{Z}} \longrightarrow \underline{\mathbf{Z}}.$$

The trace map is adjoint to the map $\mathbf{Z} \to f^{-1}\underline{\mathbf{Z}}$ (which is an isomorphism). If $\overline{x}$ is above, then Tr_f on stalks at $\overline{x}$ is the map

$$(\mathrm{Tr}_f)_{\overline{x}} : (f_!\underline{\mathbf{Z}})_{\overline{x}} = \bigoplus\nolimits_{\overline{u}} \mathbf{Z} \longrightarrow \mathbf{Z} = \underline{\mathbf{Z}}_{\overline{x}}$$

which sums the given integers. This is true because it is adjoint to the map $1 : \mathbf{Z} \to f^{-1}\underline{\mathbf{Z}}$. In particular, if f is surjective as well as étale then Tr_f is surjective.

Assume that $f : U \to X$ is a surjective étale morphism of algebraic spaces. Consider the *Koszul complex* associated to the trace map we discussed above

$$\ldots \to \wedge^3 f_!\underline{\mathbf{Z}} \to \wedge^2 f_!\underline{\mathbf{Z}} \to f_!\underline{\mathbf{Z}} \to \underline{\mathbf{Z}} \to 0$$

Here the exterior powers are over the sheaf of rings $\underline{\mathbf{Z}}$. The maps are defined by the rule

$$e_1 \wedge \ldots \wedge e_n \longmapsto \sum\nolimits_{i=1,\ldots,n} (-1)^{i+1} \mathrm{Tr}_f(e_i) e_1 \wedge \ldots \wedge \widehat{e_i} \wedge \ldots \wedge e_n$$

where $e_1, \ldots, e_n$ are local sections of $f_!\underline{\mathbf{Z}}$. Let $\overline{x}$ be a geometric point of X and set $M_{\overline{x}} = (f_!\underline{\mathbf{Z}})_{\overline{x}} = \bigoplus_{\overline{u}} \mathbf{Z}$. Then the stalk of the complex above at $\overline{x}$ is the complex

$$\ldots \to \wedge^3 M_{\overline{x}} \to \wedge^2 M_{\overline{x}} \to M_{\overline{x}} \to \mathbf{Z} \to 0$$

which is exact because $M_{\overline{x}} \to \mathbf{Z}$ is surjective, see More on Algebra, Lemma 15.19.5. Hence if we let $K^\bullet = K^\bullet(f)$ be the complex with $K^i = \wedge^{i+1} f_!\underline{\mathbf{Z}}$, then we obtain a quasi-isomorphism

$$K^\bullet \longrightarrow \underline{\mathbf{Z}}[0] \tag{51.5.1.1}$$

We use the complex $K^\bullet$ to define what we call the alternating Čech complex associated to $f : U \to X$.

Definition 51.5.2. Let S be a scheme. Let $f : U \to X$ be a surjective étale morphism of algebraic spaces over S. Let $\mathcal{F}$ be an object of $Ab(X_{étale})$. The *alternating Čech complex*[1] $\check{\mathcal{C}}^\bullet_{alt}(f, \mathcal{F})$ associated to $\mathcal{F}$ and f is the complex

$$\mathrm{Hom}(K^0, \mathcal{F}) \to \mathrm{Hom}(K^1, \mathcal{F}) \to \mathrm{Hom}(K^2, \mathcal{F}) \to \ldots$$

with Hom groups computed in $Ab(X_{étale})$.

The reader may verify that if $U = \coprod U_i$ and $f|_{U_i} : U_i \to X$ is the open immersion of a subspace, then $\check{\mathcal{C}}^\bullet_{alt}(f, \mathcal{F})$ agrees with the complex introduced in Cohomology, Section 20.22 for the Zariski covering $X = \bigcup U_i$ and the restriction of $\mathcal{F}$ to the Zariski site of X. What is more important however, is to relate the cohomology of the alternating Čech complex to the cohomology.

Lemma 51.5.3. *Let S be a scheme. Let $f : U \to X$ be a surjective étale morphism of algebraic spaces over S. Let $\mathcal{F}$ be an object of $Ab(X_{étale})$. There exists a canonical map*

$$\check{\mathcal{C}}^\bullet_{alt}(f, \mathcal{F}) \longrightarrow R\Gamma(X, \mathcal{F})$$

in $D(Ab)$. Moreover, there is a spectral sequence with E_1-page

$$E_1^{p,q} = Ext^q_{Ab(X_{étale})}(K^p, \mathcal{F})$$

converging to $H^{p+q}(X, \mathcal{F})$ where $K^p = \wedge^{p+1} f_! \underline{\mathbf{Z}}$.

Proof. Recall that we have the quasi-isomorphism $K^\bullet \to \underline{\mathbf{Z}}[0]$, see (51.5.1.1). Choose an injective resolution $\mathcal{F} \to \mathcal{I}^\bullet$ in $Ab(X_{étale})$. Consider the double complex $A^{\bullet,\bullet}$ with terms

$$A^{p,q} = \mathrm{Hom}(K^p, \mathcal{I}^q)$$

where the differential $d_1^{p,q} : A^{p,q} \to A^{p+1,q}$ is the one coming from the differential $K^{p+1} \to K^p$ and the differential $d_2^{p,q} : A^{p,q} \to A^{p,q+1}$ is the one coming from the differential $\mathcal{I}^q \to \mathcal{I}^{q+1}$. Denote $sA^\bullet$ the total complex associated to the double complex $A^{\bullet,\bullet}$. We will use the two spectral sequences $({}'E_r, {}'d_r)$ and $({}''E_r, {}''d_r)$ associated to this double complex, see Homology, Section 12.22.

Because $K^\bullet$ is a resolution of $\underline{\mathbf{Z}}$ we see that the complexes

$$A^{\bullet,q} : \mathrm{Hom}(K^0, \mathcal{I}^q) \to \mathrm{Hom}(K^1, \mathcal{I}^q) \to \mathrm{Hom}(K^2, \mathcal{I}^q) \to \ldots$$

are acyclic in positive degrees and have H^0 equal to $\Gamma(X, \mathcal{I}^q)$. Hence by Homology, Lemma 12.22.6 and its proof the spectral sequence $({}''E_r, {}''d_r)$ degenerates, and the natural map

$$\mathcal{I}^\bullet(X) \longrightarrow sA^\bullet$$

is a quasi-isomorphism of complexes of abelian groups. In particular we conclude that $H^n(sA^\bullet) = H^n(X, \mathcal{F})$.

The map $\check{\mathcal{C}}^\bullet_{alt}(f, \mathcal{F}) \to R\Gamma(X, \mathcal{F})$ of the lemma is the composition of $\check{\mathcal{C}}^\bullet_{alt}(f, \mathcal{F}) \to SA^\bullet$ with the inverse of the displayed quasi-isomorphism.

Finally, consider the spectral sequence $({}'E_r, {}'d_r)$. We have

$$E_1^{p,q} = q\text{th cohomology of } \mathrm{Hom}(K^p, \mathcal{I}^0) \to \mathrm{Hom}(K^p, \mathcal{I}^1) \to \mathrm{Hom}(K^p, \mathcal{I}^2) \to \ldots$$

This proves the lemma. □

[1] This may be nonstandard notation

It follows from the lemma that it is important to understand the ext groups $\mathrm{Ext}_{Ab(X_{étale})}(K^p, \mathcal{F})$, i.e., the right derived functors of $\mathcal{F} \mapsto \mathrm{Hom}(K^p, \mathcal{F})$.

Lemma 51.5.4. *Let S be a scheme. Let $f : U \to X$ be a surjective, étale, and separated morphism of algebraic spaces over S. For $p \geq 0$ set*

$$W_p = U \times_X \ldots \times_X U \setminus \textit{all diagonals}$$

where the fibre product has $p+1$ factors. There is a free action of S_{p+1} on W_p over X and

$$\mathrm{Hom}(K^p, \mathcal{F}) = S_{p+1}\textit{-anti-invariant elements of } \mathcal{F}(W_p)$$

functorially in $\mathcal{F}$ where $K^p = \wedge^{p+1} f_!\underline{\mathbf{Z}}$.

Proof. Because $U \to X$ is separated the diagonal $U \to U \times_X U$ is a closed immersion. Since $U \to X$ is étale the diagonal $U \to U \times_X U$ is an open immersion, see Morphisms of Spaces, Lemmas 49.35.10 and 49.34.9. Hence W_p is an open and closed subspace of $U^{p+1} = U \times_X \ldots \times_X U$. The action of S_{p+1} on W_p is free as we've thrown out the fixed points of the action. By Lemma 51.5.1 we see that

$$(f_!\underline{\mathbf{Z}})^{\otimes p+1} = f_!^{p+1}\underline{\mathbf{Z}} = (W_p \to X)_!\underline{\mathbf{Z}} \oplus \textit{Rest}$$

where $f^{p+1} : U^{p+1} \to X$ is the structure morphism. Looking at stalks over a geometric point $\overline{x}$ of X we see that

$$\Big(\bigoplus\nolimits_{\overline{u} \mapsto \overline{x}} \mathbf{Z}\Big)^{\otimes p+1} \longrightarrow (W_p \to X)_!\underline{\mathbf{Z}}_{\overline{x}}$$

is the quotient whose kernel is generated by all tensors $1_{\overline{u}_0} \otimes \ldots \otimes 1_{\overline{u}_p}$ where $\overline{u}_i = \overline{u}_j$ for some $i \neq j$. Thus the quotient map

$$(f_!\underline{\mathbf{Z}})^{\otimes p+1} \longrightarrow \wedge^{p+1} f_!\underline{\mathbf{Z}}$$

factors through $(W_p \to X)_!\underline{\mathbf{Z}}$, i.e., we get

$$(f_!\underline{\mathbf{Z}})^{\otimes p+1} \longrightarrow (W_p \to X)_!\underline{\mathbf{Z}} \longrightarrow \wedge^{p+1} f_!\underline{\mathbf{Z}}$$

This already proves that $\mathrm{Hom}(K^p, \mathcal{F})$ is (functorially) a subgroup of

$$\mathrm{Hom}((W_p \to X)_!\underline{\mathbf{Z}}, \mathcal{F}) = \mathcal{F}(W_p)$$

To identify it with the S_{p+1}-anti-invariants we have to prove that the surjection $(W_p \to X)_!\underline{\mathbf{Z}} \to \wedge^{p+1} f_!\underline{\mathbf{Z}}$ is the maximal S_{p+1}-anti-invariant quotient. In other words, we have to show that $\wedge^{p+1} f_!\underline{\mathbf{Z}}$ is the quotient of $(W_p \to X)_!\underline{\mathbf{Z}}$ by the subsheaf generated by the local sections $s - \mathrm{sign}(\sigma)\sigma(s)$ where s is a local section of $(W_p \to X)_!\underline{\mathbf{Z}}$. This can be checked on the stacks, where it is clear. □

Lemma 51.5.5. *Let S be a scheme. Let W be an algebraic space over S. Let G be a finite group acting freely on W. Let $U = W/G$, see Properties of Spaces, Lemma 48.32.1. Let $\chi : G \to \{+1, -1\}$ be a character. Then there exists a rank 1 locally free sheaf of $\mathbf{Z}$-modules $\underline{\mathbf{Z}}(\chi)$ on $U_{étale}$ such that for every abelian sheaf $\mathcal{F}$ on $U_{étale}$ we have*

$$H^0(W, \mathcal{F}|_W)^\chi = H^0(U, \mathcal{F} \otimes_{\mathbf{Z}} \underline{\mathbf{Z}}(\chi))$$

Proof. The quotient morphism $q : W \to U$ is a G-torsor, i.e., there exists a surjective étale morphism $U' \to U$ such that $W \times_U U' = \coprod_{g \in G} U'$ as spaces with G-action over U'. (Namely, $U' = W$ works.) Hence $q_*\underline{\mathbf{Z}}$ is a finite locally free

$\mathbf{Z}$-module with an action of G. For any geometric point $\overline{u}$ of U, then we get G-equivariant isomorphisms

$$(q_*\underline{\mathbf{Z}})_{\overline{u}} = \bigoplus\nolimits_{\overline{w}\mapsto\overline{u}} \mathbf{Z} = \bigoplus\nolimits_{g\in G} \mathbf{Z} = \mathbf{Z}[G]$$

where the second $=$ uses a geometric point $\overline{w}_0$ lying over $\overline{u}$ and maps the summand corresponding to $g \in G$ to the summand corresponding to $g(\overline{w}_0)$. We have

$$H^0(W, \mathcal{F}|_W) = H^0(U, \mathcal{F} \otimes_{\mathbf{Z}} q_*\underline{\mathbf{Z}})$$

because $q_*\mathcal{F}|_W = \mathcal{F} \otimes_{\mathbf{Z}} q_*\underline{\mathbf{Z}}$ as one can check by restricting to U'. Let

$$\underline{\mathbf{Z}}(\chi) = (q_*\underline{\mathbf{Z}})^\chi \subset q_*\underline{\mathbf{Z}}$$

be the subsheaf of sections that transform according to χ. For any geometric point $\overline{u}$ of U we have

$$\underline{\mathbf{Z}}(\chi)_{\overline{u}} = \mathbf{Z} \cdot \sum\nolimits_g \chi(g)g \subset \mathbf{Z}[G] = (q_*\underline{\mathbf{Z}})_{\overline{u}}$$

It follows that $\underline{\mathbf{Z}}(\chi)$ is locally free of rank 1 (more precisely, this should be checked after restricting to U'). Note that for any $\mathbf{Z}$-module M the χ-semi-invariants of $M[G]$ are the elements of the form $m \cdot \sum_g \chi(g)g$. Thus we see that for any abelian sheaf $\mathcal{F}$ on U we have

$$(\mathcal{F} \otimes_{\mathbf{Z}} q_*\underline{\mathbf{Z}})^\chi = \mathcal{F} \otimes_{\mathbf{Z}} \underline{\mathbf{Z}}(\chi)$$

because we have equality at all stalks. The result of the lemma follows by taking global sections. □

Now we can put everything together and obtain the following pleasing result.

Lemma 51.5.6. *Let S be a scheme. Let $f : U \to X$ be a surjective, étale, and separated morphism of algebraic spaces over S. For $p \geq 0$ set*

$$W_p = U \times_X \ldots \times_X U \setminus \textit{all diagonals}$$

(with $p+1$ factors) as in Lemma 51.5.4. Let $\chi_p : S_{p+1} \to \{+1, -1\}$ be the sign character. Let $U_p = W_p/S_{p+1}$ and $\underline{\mathbf{Z}}(\chi_p)$ be as in Lemma 51.5.5. Then the spectral sequence of Lemma 51.5.3 has E_1-page

$$E_1^{p,q} = H^q(U_p, \mathcal{F}|_{U_p} \otimes_{\mathbf{Z}} \underline{\mathbf{Z}}(\chi_p))$$

and converges to $H^{p+q}(X, \mathcal{F})$.

Proof. Note that since the action of S_{p+1} on W_p is over X we do obtain a morphism $U_p \to X$. Since $W_p \to X$ is étale and since $W_p \to U_p$ is surjective étale, it follows that also $U_p \to X$ is étale, see Morphisms of Spaces, Lemma 49.35.2. Therefore an injective object of $Ab(X_{étale})$ restricts to an injective object of $Ab(U_{p,étale})$, see Cohomology on Sites, Lemma 21.8.1. Moreover, the functor $\mathcal{G} \mapsto \mathcal{G} \otimes_{\mathbf{Z}} \underline{\mathbf{Z}}(\chi_p))$ is an auto-equivalence of $Ab(U_p)$, whence transforms injective objects into injective objects and is exact (because $\underline{\mathbf{Z}}(\chi_p)$ is an invertible $\underline{\mathbf{Z}}$-module). Thus given an injective resolution $\mathcal{F} \to \mathcal{I}^\bullet$ in $Ab(X_{étale})$ the complex

$$\Gamma(U_p, \mathcal{I}^0|_{U_p} \otimes_{\mathbf{Z}} \underline{\mathbf{Z}}(\chi_p)) \to \Gamma(U_p, \mathcal{I}^1|_{U_p} \otimes_{\mathbf{Z}} \underline{\mathbf{Z}}(\chi_p)) \to \Gamma(U_p, \mathcal{I}^2|_{U_p} \otimes_{\mathbf{Z}} \underline{\mathbf{Z}}(\chi_p)) \to \ldots$$

computes $H^*(U_p, \mathcal{F}|_{U_p} \otimes_{\mathbf{Z}} \underline{\mathbf{Z}}(\chi_p))$. On the other hand, by Lemma 51.5.5 it is equal to the complex of S_{p+1}-anti-invariants in

$$\Gamma(W_p, \mathcal{I}^0) \to \Gamma(W_p, \mathcal{I}^1) \to \Gamma(W_p, \mathcal{I}^2) \to \ldots$$

which by Lemma 51.5.4 is equal to the complex

$$\mathrm{Hom}(K^p, \mathcal{I}^0) \to \mathrm{Hom}(K^p, \mathcal{I}^1) \to \mathrm{Hom}(K^p, \mathcal{I}^2) \to \ldots$$

which computes $\mathrm{Ext}^*_{Ab(X_{étale})}(K^p, \mathcal{F})$. Putting everything together we win. □

51.6. Higher vanishing for quasi-coherent sheaves

In this section we show that given a quasi-compact and quasi-separated algebraic space X there exists an integer $n = n(X)$ such that the cohomology of any quasi-coherent sheaf on X vanishes beyond degree n.

Lemma 51.6.1. *With S, W, G, U, χ as in Lemma 51.5.5. If $\mathcal{F}$ is a quasi-coherent $\mathcal{O}_U$-module, then so is $\mathcal{F} \otimes_{\mathbf{Z}} \underline{\mathbf{Z}}(\chi)$.*

Proof. The $\mathcal{O}_U$-module structure is clear. To check that $\mathcal{F} \otimes_{\mathbf{Z}} \underline{\mathbf{Z}}(\chi)$ is quasi-coherent it suffices to check étale locally. Hence the lemma follows as $\underline{\mathbf{Z}}(\chi)$ is finite locally free as a $\underline{\mathbf{Z}}$-module. □

The following proposition is interesting even if X is a scheme. It is the natural generalization of Cohomology of Schemes, Lemma 29.4.2. Before we state it, observe that given an étale morphism $f : U \to X$ from an affine scheme towards a quasi-separated algebraic space X the fibres of f are universally bounded, in particular there exists an integer d such that the fibres of $|U| \to |X|$ all have size at most d; this is the implication $(\eta) \Rightarrow (\delta)$ of Decent Spaces, Lemma 50.5.1.

Proposition 51.6.2. *Let S be a scheme. Let X be an algebraic space over S. Assume X is quasi-compact and separated. Let U be an affine scheme, and let $f : U \to X$ be a surjective étale morphism. Let d be an upper bound for the size of the fibres of $|U| \to |X|$. Then for any quasi-coherent $\mathcal{O}_X$-module $\mathcal{F}$ we have $H^q(X, \mathcal{F}) = 0$ for $q \geq d$.*

Proof. We will use the spectral sequence of Lemma 51.5.6. The lemma applies since f is separated as U is separated, see Morphisms of Spaces, Lemma 49.5.10. Since X is separated the scheme $U \times_X \ldots \times_X U$ is a closed subscheme of $U \times_{\mathrm{Spec}(\mathbf{Z})} \ldots \times_{\mathrm{Spec}(\mathbf{Z})} U$ hence is affine. Thus W_p is affine. Hence $U_p = W_p/S_{p+1}$ is an affine scheme by Groupoids, Proposition 38.20.8. The discussion in Section 51.3 shows that cohomology of quasi-coherent sheaves on W_p (as an algebraic space) agrees with the cohomology of the corresponding quasi-coherent sheaf on the underlying affine scheme, hence vanishes in positive degrees by Cohomology of Schemes, Lemma 29.2.2. By Lemma 51.6.1 the sheaves $\mathcal{F}|_{U_p} \otimes_{\mathbf{Z}} \underline{\mathbf{Z}}(\chi_p)$ are quasi-coherent. Hence $H^q(W_p, \mathcal{F}|_{U_p} \otimes_{\mathbf{Z}} \underline{\mathbf{Z}}(\chi_p))$ is zero when $q > 0$. By our definition of the integer d we see that $W_p = \emptyset$ for $p \geq d$. Hence also $H^0(W_p, \mathcal{F}|_{U_p} \otimes_{\mathbf{Z}} \underline{\mathbf{Z}}(\chi_p))$ is zero when $p \geq d$. This proves the proposition. □

In the following lemma we establish that a quasi-compact and quasi-separated algebraic space has finite cohomological dimension for quasi-coherent modules. We are explicit about the bound only because we will use it later to prove a similar result for higher direct images.

Lemma 51.6.3. *Let S be a scheme. Let X be an algebraic space over S. Assume X is quasi-compact and quasi-separated. Then we can choose*

1. *an affine scheme U,*
2. *a surjective étale morphism $f : U \to X$,*
3. *an integer d bounding the degrees of the fibres of $U \to X$,*
4. *for every $p = 0, 1, \ldots, d$ a surjective étale morphism $V_p \to U_p$ from an affine scheme V_p where U_p is as in Lemma 51.5.6, and*

(5) *an integer d_p bounding the degree of the fibres of $V_p \to U_p$.*

Moreover, whenever we have (1) – (5), then for any quasi-coherent $\mathcal{O}_X$-module $\mathcal{F}$ we have $H^q(X, \mathcal{F}) = 0$ for $q \geq \max(d_p + p)$.

Proof. Since X is quasi-compact we can find a surjective étale morphism $U \to X$ with U affine, see Properties of Spaces, Lemma 48.6.3. By Decent Spaces, Lemma 50.5.1 the fibres of f are universally bounded, hence we can find d. We have $U_p = W_p/S_{p+1}$ and $W_p \subset U \times_X \ldots \times_X U$ is open and closed. Since X is quasi-separated the schemes W_p are quasi-compact, hence U_p is quasi-compact. Since U is separated, the schemes W_p are separated, hence U_p is separated by (the absolute version of) Spaces, Lemma 47.14.5. By Properties of Spaces, Lemma 48.6.3 we can find the morphisms $V_p \to W_p$. By Decent Spaces, Lemma 50.5.1 we can find the integers d_p.

At this point the proof uses the spectral sequence

$$E_1^{p,q} = H^q(U_p, \mathcal{F}|_{U_p} \otimes_{\mathbf{Z}} \underline{\mathbf{Z}}(\chi_p)) \Rightarrow H^{p+q}(X, \mathcal{F})$$

see Lemma 51.5.6. By definition of the integer d we see that $U_p = 0$ for $p \geq d$. By Proposition 51.6.2 and Lemma 51.6.1 we see that $H^q(U_p, \mathcal{F}|_{U_p} \otimes_{\mathbf{Z}} \underline{\mathbf{Z}}(\chi_p))$ is zero for $q \geq d_p$ for $p = 0, \ldots, d$. Whence the lemma. □

51.7. Vanishing for higher direct images

We apply the results of Section 51.6 to obtain vanishing of higher direct images of quasi-coherent sheaves for quasi-compact and quasi-separated morphisms. This is useful because it allows one to argue by descending induction on the cohomological degree in certain situations.

Lemma 51.7.1. *Let S be a scheme. Let $f : X \to Y$ be a morphism of algebraic spaces over S. Assume that*

(1) *f is quasi-compact and quasi-separated, and*
(2) *Y is quasi-compact.*

Then there exists an integer $n(X \to Y)$ such that for any algebraic space Y', any morphism $Y' \to Y$ and any quasi-coherent sheaf $\mathcal{F}'$ on $X' = Y' \times_Y X$ the higher direct images $R^i f'_* \mathcal{F}'$ are zero for $i \geq n(X \to Y)$.

Proof. Let $V \to Y$ be a surjective étale morphism where V is an affine scheme, see Properties of Spaces, Lemma 48.6.3. Suppose we prove the result for the base change $f_V : V \times_Y X \to V$. Then the result holds for f with $n(X \to Y) = n(X_V \to V)$. Namely, if $Y' \to Y$ and $\mathcal{F}'$ are as in the lemma, then $R^i f'_* \mathcal{F}'|_{V \times_Y Y'}$ is equal to $R^i f'_{V,*} \mathcal{F}'|_{X'_V}$ where $f'_V : X'_V = V \times_Y Y' \times_Y X \to V \times_Y Y' = Y'_V$, see Properties of Spaces, Lemma 48.24.2. Thus we may assume that Y is an affine scheme.

Moreover, to prove the vanishing for all $Y' \to Y$ and $\mathcal{F}'$ it suffices to do so when Y' is an affine scheme. In this case, $R^i f'_* \mathcal{F}'$ is quasi-coherent by Lemma 51.3.1. Hence it suffices to prove that $H^i(X', \mathcal{F}') = 0$, because $H^i(X', \mathcal{F}') = H^0(Y', R^i f'_* \mathcal{F}')$ by Cohomology on Sites, Lemma 21.14.6 and the vanishing of higher cohomology of quasi-coherent sheaves on affine algebraic spaces (Proposition 51.6.2).

Choose $U \to X$, d, $V_p \to U_p$ and d_p as in Lemma 51.6.3. For any affine scheme Y' and morphism $Y' \to Y$ denote $X' = Y' \times_Y X$, $U' = Y' \times_Y U$, $V'_p = Y' \times_Y V_p$. Then $U' \to X'$, $d' = d$, $V'_p \to U'_p$ and $d'_p = d$ is a collection of choices as in Lemma 51.6.3

for the algebraic space X' (details omitted). Hence we see that $H^i(X', \mathcal{F}') = 0$ for $i \geq \max(p + d_p)$ and we win. □

Lemma 51.7.2. *Let S be a scheme. Let $f : X \to Y$ be an affine morphism of algebraic spaces over S. Then $R^i f_* \mathcal{F} = 0$ for $i > 0$ and any quasi-coherent $\mathcal{O}_X$-module $\mathcal{F}$.*

Proof. Recall that an affine morphism of algebraic spaces is representable. Hence this follows from (51.3.0.1) and Cohomology of Schemes, Lemma 29.2.3. □

51.8. Cohomology and base change, I

Let S be a scheme. Let $f : X \to Y$ be a morphism of algebraic spaces over S. Let $\mathcal{F}$ be a quasi-coherent sheaf on X. Suppose further that $g : Y' \to Y$ is a morphism of algebraic spaces over S. Denote $X' = X_{Y'} = Y' \times_Y X$ the base change of X and denote $f' : X' \to Y'$ the base change of f. Also write $g' : X' \to X$ the projection, and set $\mathcal{F}' = (g')^* \mathcal{F}$. Here is a diagram representing the situation:

$$\begin{array}{ccccc} \mathcal{F}' = (g')^*\mathcal{F} & & X' & \xrightarrow{g'} & X & & \mathcal{F} \\ & & \downarrow{\scriptstyle f'} & & \downarrow{\scriptstyle f} & & \\ Rf'_*\mathcal{F}' & & Y' & \xrightarrow{g} & Y & & Rf_*\mathcal{F} \end{array} \tag{51.8.0.1}$$

Here is the basic result for a flat base change.

Lemma 51.8.1. *In the situation above, assume that g is flat and that f is quasi-compact and quasi-separated. Then we have*

$$R^p f'_* \mathcal{F}' = g^* R^p f_* \mathcal{F}$$

for all $p \geq 0$ with notation as in (51.8.0.1).

Proof. The morphism g' is flat by Morphisms of Spaces, Lemma 49.27.4. Note that flatness of g and g' is equivalent to flatness of the morphisms of small étale ringed sites, see Morphisms of Spaces, Lemma 49.27.9. Hence we can apply Cohomology on Sites, Lemma 21.15.1 to obtain a base change map

$$g^* R^p f_* \mathcal{F} \longrightarrow R^p f'_* \mathcal{F}'$$

To prove this map is an isomorphism we can work locally in the étale topology on Y'. Thus we may assume that Y and Y' are affine schemes. Say $Y = \mathrm{Spec}(A)$ and $Y' = \mathrm{Spec}(B)$. In this case we are really trying to show that the map

$$H^p(X, \mathcal{F}) \otimes_A B \longrightarrow H^p(X_B, \mathcal{F}_B)$$

is an isomorphism where $X_B = \mathrm{Spec}(B) \times_{\mathrm{Spec}(A)} X$ and $\mathcal{F}_B$ is the pullback of $\mathcal{F}$ to X_B.

Fix $A \to B$ a flat ring map and let X be a quasi-compact and quasi-separated algebraic space over A. Note that $g' : X_B \to X$ is affine as a base change of $\mathrm{Spec}(B) \to \mathrm{Spec}(A)$. Hence the higher direct images $R^i(g')_* \mathcal{F}_B$ are zero by Lemma 51.7.2. Thus $H^p(X_B, \mathcal{F}_B) = H^p(X, g'_* \mathcal{F}_B)$, see Cohomology on Sites, Lemma 21.14.6. Moreover, we have

$$g'_* \mathcal{F}_B = \mathcal{F} \otimes_{\underline{A}} \underline{B}$$

where $\underline{A}$, $\underline{B}$ denotes the constant sheaf of rings with value A, B. Namely, it is clear that there is a map from right to left. For any affine scheme U étale over X we have

$$\begin{aligned} g'_*\mathcal{F}_B(U) &= \mathcal{F}_B(\mathrm{Spec}(B) \times_{\mathrm{Spec}(A)} U) \\ &= \Gamma(\mathrm{Spec}(B) \times_{\mathrm{Spec}(A)} U, (\mathrm{Spec}(B) \times_{\mathrm{Spec}(A)} U \to U)^*\mathcal{F}|_U) \\ &= B \otimes_A \mathcal{F}(U) \end{aligned}$$

hence the map is an isomorphism. Write $B = \mathrm{colim}\, M_i$ as a filtered colimit of finite free A-modules M_i using Lazard's theorem, see Algebra, Theorem 10.76.4. We deduce that

$$\begin{aligned} H^p(X, g'_*\mathcal{F}_B) &= H^p(X, \mathcal{F} \otimes_{\underline{A}} \underline{B}) \\ &= H^p(X, \mathrm{colim}_i\, \mathcal{F} \otimes_{\underline{A}} \underline{M_i}) \\ &= \mathrm{colim}_i\, H^p(X, \mathcal{F} \otimes_{\underline{A}} \underline{M_i}) \\ &= \mathrm{colim}_i\, H^p(X, \mathcal{F}) \otimes_A M_i \\ &= H^p(X, \mathcal{F}) \otimes_A \mathrm{colim}_i\, M_i \\ &= H^p(X, \mathcal{F}) \otimes_A B \end{aligned}$$

The first equality because $g'_*\mathcal{F}_B = \mathcal{F} \otimes_{\underline{A}} \underline{B}$ as seen above. The second because $\otimes$ commutes with colimits. The third equality because cohomology on X commutes with colimits (see Lemma 51.4.1). The fourth equality because M_i is finite free (i.e., because cohomology commutes with finite direct sums). The fifth because $\otimes$ commutes with colimits. The sixth by choice of our system. □

Lemma 51.8.2. *Let S be a scheme. Let $f : X \to Y$ be an affine morphism of algebraic spaces over S. Let $\mathcal{F}$ be a quasi-coherent $\mathcal{O}_X$-module. In this case $f_*\mathcal{F} \cong Rf_*\mathcal{F}$ is a quasi-coherent sheaf, and for every diagram (51.8.0.1) we have $g^*f_*\mathcal{F} = f'_*(g')^*\mathcal{F}$.*

Proof. By the discussion surrounding (51.3.0.1) this reduces to the case of an affine morphism of schemes which is treated in Cohomology of Schemes, Lemma 29.5.1. □

51.9. Coherent modules on locally Noetherian algebraic spaces

This section is the analogue of Cohomology of Schemes, Section 29.9. In Modules on Sites, Definition 18.23.1 we have defined coherent modules on any ringed topos. We use this notion to define coherent modules on locally Noetherian algebraic spaces. Although it is possible to work with coherent modules more generally we resist the urge to do so.

Definition 51.9.1. Let S be a scheme. Let X be a locally Noetherian algebraic space over S. A quasi-coherent module $\mathcal{F}$ on X is called *coherent* if $\mathcal{F}$ is a coherent $\mathcal{O}_X$-module on the site $X_{étale}$ in the sense of Modules on Sites, Definition 18.23.1.

Of course this definition is a bit hard to work with. We usually use the characterization given in the lemma below.

Lemma 51.9.2. *Let S be a scheme. Let X be a locally Noetherian algebraic space over S. Let $\mathcal{F}$ be an $\mathcal{O}_X$-module. The following are equivalent*

(1) *$\mathcal{F}$ is coherent,*

(2) $\mathcal{F}$ *is a quasi-coherent, finite type* $\mathcal{O}_X$*-module,*
(3) $\mathcal{F}$ *is a finitely presented* $\mathcal{O}_X$*-module,*
(4) *for any étale morphism* $\varphi : U \to X$ *where* U *is a scheme the pullback* $\varphi^*\mathcal{F}$ *is a coherent module on* U*, and*
(5) *there exists a surjective étale morphism* $\varphi : U \to X$ *where* U *is a scheme such that the pullback* $\varphi^*\mathcal{F}$ *is a coherent module on* U*.*

In particular $\mathcal{O}_X$ *is coherent, any invertible* $\mathcal{O}_X$*-module is coherent, and more generally any finite locally free* $\mathcal{O}_X$*-module is coherent.*

Proof. To be sure, if X is a locally Noetherian algebraic space and $U \to X$ is an étale morphism, then U is locally Noetherian, see Properties of Spaces, Section 48.7. The lemma then follows from the points (1) – (5) made in Properties of Spaces, Section 48.28 and the corresponding result for coherent modules on locally Noetherian schemes, see Cohomology of Schemes, Lemma 29.9.1. □

Lemma 51.9.3. *Let* S *be a scheme. Let* X *be a locally Noetherian algebraic space over* S*. The category of coherent* $\mathcal{O}_X$*-modules is abelian. More precisely, the kernel and cokernel of a map of coherent* $\mathcal{O}_X$*-modules are coherent. Any extension of coherent sheaves is coherent.*

Proof. Choose a scheme U and a surjective étale morphism $f : U \to X$. Pullback f^* is an exact functor as it equals a restriction functor, see Properties of Spaces, Equation (48.24.1.1). By Lemma 51.9.2 we can check whether an $\mathcal{O}_X$-module $\mathcal{F}$ is coherent by checking whether $f^*\mathcal{F}$ is coherent. Hence the lemma follows from the case of schemes which is Cohomology of Schemes, Lemma 29.9.2. □

Coherent modules form a Serre subcategory of the category of quasi-coherent $\mathcal{O}_X$-modules. This does not hold for modules on a general ringed topos.

Lemma 51.9.4. *Let* S *be a scheme. Let* X *be a locally Noetherian algebraic space over* S*. Let* $\mathcal{F}$ *be a coherent* $\mathcal{O}_X$*-module. Any quasi-coherent submodule of* $\mathcal{F}$ *is coherent. Any quasi-coherent quotient module of* $\mathcal{F}$ *is coherent.*

Proof. Choose a scheme U and a surjective étale morphism $f : U \to X$. Pullback f^* is an exact functor as it equals a restriction functor, see Properties of Spaces, Equation (48.24.1.1). By Lemma 51.9.2 we can check whether an $\mathcal{O}_X$-module $\mathcal{G}$ is coherent by checking whether $f^*\mathcal{H}$ is coherent. Hence the lemma follows from the case of schemes which is Cohomology of Schemes, Lemma 29.9.3. □

Lemma 51.9.5. *Let* S *be a scheme. Let* X *be a locally Noetherian algebraic space over* S*,. Let* $\mathcal{F}$*,* $\mathcal{G}$ *be coherent* $\mathcal{O}_X$*-modules. The* $\mathcal{O}_X$*-modules* $\mathcal{F} \otimes_{\mathcal{O}_X} \mathcal{G}$ *and* $\mathcal{H}om_{\mathcal{O}_X}(\mathcal{F}, \mathcal{G})$ *are coherent.*

Proof. Via Lemma 51.9.2 this follows from the result for schemes, see Cohomology of Schemes, Lemma 29.9.4. □

Lemma 51.9.6. *Let* S *be a scheme. Let* X *be a locally Noetherian algebraic space over* S*. Let* $\mathcal{F}$*,* $\mathcal{G}$ *be coherent* $\mathcal{O}_X$*-modules. Let* $\varphi : \mathcal{G} \to \mathcal{F}$ *be a homomorphism of* $\mathcal{O}_X$*-modules. Let* $\overline{x}$ *be a geometric point of* X *lying over* $x \in |X|$*.*

(1) *If* $\mathcal{F}_{\overline{x}} = 0$ *then there exists an open neighbourhood* $X' \subset X$ *of* x *such that* $\mathcal{F}|_{X'} = 0$*.*
(2) *If* $\varphi_{\overline{x}} : \mathcal{G}_{\overline{x}} \to \mathcal{F}_{\overline{x}}$ *is injective, then there exists an open neighbourhood* $X' \subset X$ *of* x *such that* $\varphi|_{X'}$ *is injective.*

(3) *If $\varphi_{\overline{x}} : \mathcal{G}_{\overline{x}} \to \mathcal{F}_{\overline{x}}$ is surjective, then there exists an open neighbourhood $X' \subset X$ of x such that $\varphi|_{X'}$ is surjective.*
(4) *If $\varphi_{\overline{x}} : \mathcal{G}_{\overline{x}} \to \mathcal{F}_{\overline{x}}$ is bijective, then there exists an open neighbourhood $X' \subset X$ of x such that $\varphi|_{X'}$ is an isomorphism.*

Proof. Let $\varphi : U \to X$ be an étale morphism where U is a scheme and let $u \in U$ be a point mapping to x. By Properties of Spaces, Lemmas 48.27.4 and 48.19.1 as well as More on Algebra, Lemma 15.32.1 we see that $\varphi_{\overline{x}}$ is injective, surjective, or bijective if and only if $\varphi_u : \varphi^*\mathcal{F}_u \to \varphi^*\mathcal{G}_u$ has the corresponding property. Thus we can apply the schemes version of this lemma to see that (after possibly shrinking U) the map $\varphi^*\mathcal{F} \to \varphi^*\mathcal{G}$ is injective, surjective, or an isomorphism. Let $X' \subset X$ be the open subspace corresponding to $|\varphi|(|U|) \subset |X|$, see Properties of Spaces, Lemma 48.4.8. Since $\{U \to X'\}$ is a covering for the étale topology, we conclude that $\varphi|_{X'}$ is injective, surjective, or an isomorphism as desired. Finally, observe that (1) follows from (2) by looking at the map $\mathcal{F} \to 0$. □

Lemma 51.9.7. *Let S be a scheme. Let X be a locally Noetherian algebraic space over S. Let $\mathcal{F}$ be a coherent $\mathcal{O}_X$-module. Let $i : Z \to X$ be the scheme theoretic support of $\mathcal{F}$ and $\mathcal{G}$ the quasi-coherent $\mathcal{O}_Z$-module such that $i_*\mathcal{G} = \mathcal{F}$, see Morphisms of Spaces, Definition 49.15.4. Then $\mathcal{G}$ is a coherent $\mathcal{O}_Z$-module.*

Proof. The statement of the lemma makes sense as a coherent module is in particular of finite type. Moreover, as $Z \to X$ is a closed immersion it is locally of finite type and hence Z is locally Noetherian, see Morphisms of Spaces, Lemmas 49.22.7 and 49.22.5. Finally, as $\mathcal{G}$ is of finite type it is a coherent $\mathcal{O}_Z$-module by Lemma 51.9.2 □

Lemma 51.9.8. *Let S be a scheme. Let $i : Z \to X$ be a closed immersion of locally Noetherian algebraic spaces over S. Let $\mathcal{I} \subset \mathcal{O}_X$ be the quasi-coherent sheaf of ideals cutting out Z. The functor i_* induces an equivalence between the category of coherent $\mathcal{O}_X$-modules annihilated by $\mathcal{I}$ and the category of coherent $\mathcal{O}_Z$-modules.*

Proof. The functor is fully faithful by Morphisms of Spaces, Lemma 49.14.1. Let $\mathcal{F}$ be a coherent $\mathcal{O}_X$-module annihilated by $\mathcal{I}$. By Morphisms of Spaces, Lemma 49.14.1 we can write $\mathcal{F} = i_*\mathcal{G}$ for some quasi-coherent sheaf $\mathcal{G}$ on Z. To check that $\mathcal{G}$ is coherent we can work étale locally (Lemma 51.9.2). Choosing an étale covering by a scheme we conclude that $\mathcal{G}$ is coherent by the case of schemes (Cohomology of Schemes, Lemma 29.9.8). Hence the functor is fully faithful and the proof is done. □

Lemma 51.9.9. *Let S be a scheme. Let $f : X \to Y$ be a finite morphism of algebraic spaces over S with Y locally Noetherian. Let $\mathcal{F}$ be a coherent $\mathcal{O}_X$-module. Assume f is finite and Y locally Noetherian. Then $R^p f_*\mathcal{F} = 0$ for $p > 0$ and $f_*\mathcal{F}$ is coherent.*

Proof. Choose a scheme V and a surjective étale morphism $V \to Y$. Then $V \times_Y X \to V$ is a finite morphism of locally Noetherian schemes. By (51.3.0.1) we reduce to the case of schemes which is Cohomology of Schemes, Lemma 29.9.9. □

51.10. Coherent sheaves on Noetherian spaces

In this section we mention some properties of coherent sheaves on Noetherian algebraic spaces.

Lemma 51.10.1. *Let S be a scheme. Let X be a Noetherian algebraic space over S. Let $\mathcal{F}$ be a coherent $\mathcal{O}_X$-module. The ascending chain condition holds for quasi-coherent submodules of $\mathcal{F}$. In other words, given any sequence*

$$\mathcal{F}_1 \subset \mathcal{F}_2 \subset \ldots \subset \mathcal{F}$$

of quasi-coherent submodules, then $\mathcal{F}_n = \mathcal{F}_{n+1} = \ldots$ for some $n \geq 0$.

Proof. Choose an affine scheme U and a surjective étale morphism $U \to X$ (see Properties of Spaces, Lemma 48.6.3). Then U is a Noetherian scheme (by Morphisms of Spaces, Lemma 49.22.5). If $\mathcal{F}_n|_U = \mathcal{F}_{n+1}|_U = \ldots$ then $\mathcal{F}_n = \mathcal{F}_{n+1} = \ldots$. Hence the result follows from the case of schemes, see Cohomology of Schemes, Lemma 29.10.1. □

Lemma 51.10.2. *Let S be a scheme. Let X be a Noetherian algebraic space over S. Let $\mathcal{F}$ be a coherent sheaf on X. Let $\mathcal{I} \subset \mathcal{O}_X$ be a quasi-coherent sheaf of ideals corresponding to a closed subspace $Z \subset X$. Then there is some $n \geq 0$ such that $\mathcal{I}^n\mathcal{F} = 0$ if and only if $\mathrm{Supp}(\mathcal{F}) \subset Z$ (set theoretically).*

Proof. Choose an affine scheme U and a surjective étale morphism $U \to X$ (see Properties of Spaces, Lemma 48.6.3). Then U is a Noetherian scheme (by Morphisms of Spaces, Lemma 49.22.5). Note that $\mathcal{I}^n\mathcal{F}|_U = 0$ if and only if $\mathcal{I}^n\mathcal{F} = 0$ and similarly for the condition on the support. Hence the result follows from the case of schemes, see Cohomology of Schemes, Lemma 29.10.2. □

Lemma 51.10.3 (Artin-Rees)**.** *Let S be a scheme. Let X be a Noetherian algebraic space over S. Let $\mathcal{F}$ be a coherent sheaf on X. Let $\mathcal{G} \subset \mathcal{F}$ be a quasi-coherent subsheaf. Let $\mathcal{I} \subset \mathcal{O}_X$ be a quasi-coherent sheaf of ideals. Then there exists a $c \geq 0$ such that for all $n \geq c$ we have*

$$\mathcal{I}^{n-c}(\mathcal{I}^c\mathcal{F} \cap \mathcal{G}) = \mathcal{I}^n\mathcal{F}$$

Proof. Choose an affine scheme U and a surjective étale morphism $U \to X$ (see Properties of Spaces, Lemma 48.6.3). Then U is a Noetherian scheme (by Morphisms of Spaces, Lemma 49.22.5). The equality of the lemma holds if and only if it holds after restricting to U. Hence the result follows from the case of schemes, see Cohomology of Schemes, Lemma 29.10.3. □

Lemma 51.10.4. *Let S be a scheme. Let X be a Noetherian algebraic space over S. Let $\mathcal{F}$ be a quasi-coherent $\mathcal{O}_X$-module. Let $\mathcal{G}$ be a coherent $\mathcal{O}_X$-module. Let $\mathcal{I} \subset \mathcal{O}_X$ be a quasi-coherent sheaf of ideals. Denote $Z \subset X$ the corresponding closed subspace and set $U = X \setminus Z$. There is a canonical isomorphism*

$$\mathrm{colim}_n \mathrm{Hom}_{\mathcal{O}_X}(\mathcal{I}^n\mathcal{G}, \mathcal{F}) \longrightarrow \mathrm{Hom}_{\mathcal{O}_U}(\mathcal{G}|_U, \mathcal{F}|_U).$$

In particular we have an isomorphism

$$\mathrm{colim}_n \mathrm{Hom}_{\mathcal{O}_X}(\mathcal{I}^n, \mathcal{F}) \longrightarrow \Gamma(U, \mathcal{F}).$$

Proof. Let W be an affine scheme and let $W \to X$ be a surjective étale morphism (see Properties of Spaces, Lemma 48.6.3). Set $R = W \times_X W$. Then W and R are Noetherian schemes, see Morphisms of Spaces, Lemma 49.22.5. Hence the result hold for the restrictions of $\mathcal{F}$, $\mathcal{G}$, and $\mathcal{I}$, U, Z to W and R by Cohomology of Schemes, Lemma 29.10.4. It follows formally that the result holds over X. □

51.11. Devissage of coherent sheaves

This section is the analogue of Cohomology of Schemes, Section 29.12.

Lemma 51.11.1. *Let S be a scheme. Let X be a Noetherian algebraic space over S. Let $\mathcal{F}$ be a coherent sheaf on X. Suppose that $Supp(\mathcal{F}) = Z \cup Z'$ with Z, Z' closed. Then there exists a short exact sequence of coherent sheaves*

$$0 \to \mathcal{G}' \to \mathcal{F} \to \mathcal{G} \to 0$$

with $Supp(\mathcal{G}') \subset Z'$ and $Supp(\mathcal{G}) \subset Z$.

Proof. Let $\mathcal{I} \subset \mathcal{O}_X$ be the sheaf of ideals defining the reduced induced closed subspace structure on Z, see Properties of Spaces, Lemma 48.9.2. Consider the subsheaves $\mathcal{G}'_n = \mathcal{I}^n\mathcal{F}$ and the quotients $\mathcal{G}_n = \mathcal{F}/\mathcal{I}^n\mathcal{F}$. For each n we have a short exact sequence

$$0 \to \mathcal{G}'_n \to \mathcal{F} \to \mathcal{G}_n \to 0$$

For every geometric point $\overline{x}$ of $Z' \setminus Z$ we have $\mathcal{I}_{\overline{x}} = \mathcal{O}_{X,\overline{x}}$ and hence $\mathcal{G}_{n,\overline{x}} = 0$. Thus we see that $\mathrm{Supp}(\mathcal{G}_n) \subset Z$. Note that $X \setminus Z'$ is a Noetherian algebraic space. Hence by Lemma 51.10.2 there exists an n such that $\mathcal{G}'_n|_{X\setminus Z'} = \mathcal{I}^n\mathcal{F}|_{X\setminus Z'} = 0$. For such an n we see that $\mathrm{Supp}(\mathcal{G}'_n) \subset Z'$. Thus setting $\mathcal{G}' = \mathcal{G}'_n$ and $\mathcal{G} = \mathcal{G}_n$ works. □

In the following we will freely use the scheme theoretic support of finite type modules as defined in Morphisms of Spaces, Definition 49.15.4.

Lemma 51.11.2. *Let S be a scheme. Let X be a Noetherian algebraic space over S. Let $\mathcal{F}$ be a coherent sheaf on X. Assume that the scheme theoretic support of $\mathcal{F}$ is a reduced $Z \subset X$ with $|Z|$ irreducible. Then there exist an integer $r > 0$, a nonzero sheaf of ideals $\mathcal{I} \subset \mathcal{O}_Z$, and an injective map of coherent sheaves*

$$i_*\left(\mathcal{I}^{\oplus r}\right) \to \mathcal{F}$$

whose cokernel is supported on a proper closed subspace of Z.

Proof. By assumption there exists a coherent $\mathcal{O}_Z$-module $\mathcal{G}$ with support Z and $\mathcal{F} \cong i_*\mathcal{G}$, see Lemma 51.9.7. Hence it suffices to prove the lemma for the case $Z = X$ and $i = \mathrm{id}$.

By Properties of Spaces, Proposition 48.10.3 there exists a dense open subspace $U \subset X$ which is a scheme. Note that U is a Noetherian integral scheme. After shrinking U we may assume that $\mathcal{F}|_U \cong \mathcal{O}_U^{\oplus r}$ (for example by Cohomology of Schemes, Lemma 29.12.2 or by a direct algebra argument). Let $\mathcal{I} \subset \mathcal{O}_X$ be a quasi-coherent sheaf of ideals whose associated closed subspace is the complement of U in X (see for example Properties of Spaces, Section 48.9). By Lemma 51.10.4 there exists an $n \geq 0$ and a morphism $\mathcal{I}^n(\mathcal{O}_X^{\oplus r}) \to \mathcal{F}$ which recovers our isomorphism over U. Since $\mathcal{I}^n(\mathcal{O}_X^{\oplus r}) = (\mathcal{I}^n)^{\oplus r}$ we get a map as in the lemma. It is injective: namely, if σ is a nonzero section of $\mathcal{I}^{\oplus r}$ over a scheme W étale over X, then because X hence W is reduced the support of σ contains a nonempty open of W. But the kernel of $(\mathcal{I}^n)^{\oplus r} \to \mathcal{F}$ is zero over a dense open, hence σ cannot be a section of the kernel. □

Lemma 51.11.3. *Let S be a scheme. Let X be a Noetherian algebraic space over S. Let $\mathcal{F}$ be a coherent sheaf on X. There exists a filtration*

$$0 = \mathcal{F}_0 \subset \mathcal{F}_1 \subset \ldots \subset \mathcal{F}_m = \mathcal{F}$$

by coherent subsheaves such that for each $j = 1, \ldots, m$ there exists a reduced closed subspace $Z_j \subset X$ with $|Z_j|$ irreducible and a sheaf of ideals $\mathcal{I}_j \subset \mathcal{O}_{Z_j}$ such that

$$\mathcal{F}_j/\mathcal{F}_{j-1} \cong (Z_j \to X)_*\mathcal{I}_j$$

Proof. Consider the collection

$$\mathcal{T} = \left\{ \begin{array}{c} T \subset |X| \text{ closed such that there exists a coherent sheaf } \mathcal{F} \\ \text{with } \mathrm{Supp}(\mathcal{F}) = T \text{ for which the lemma is wrong} \end{array} \right\}$$

We are trying to show that $\mathcal{T}$ is empty. If not, then because $|X|$ is Noetherian (Properties of Spaces, Lemma 48.22.2) we can choose a minimal element $T \in \mathcal{T}$. This means that there exists a coherent sheaf $\mathcal{F}$ on X whose support is T and for which the lemma does not hold. Clearly $T \neq \emptyset$ since the only sheaf whose support is empty is the zero sheaf for which the lemma does hold (with $m = 0$).

If T is not irreducible, then we can write $T = Z_1 \cup Z_2$ with Z_1, Z_2 closed and strictly smaller than T. Then we can apply Lemma 51.11.1 to get a short exact sequence of coherent sheaves

$$0 \to \mathcal{G}_1 \to \mathcal{F} \to \mathcal{G}_2 \to 0$$

with $\mathrm{Supp}(\mathcal{G}_i) \subset Z_i$. By minimality of T each of $\mathcal{G}_i$ has a filtration as in the statement of the lemma. By considering the induced filtration on $\mathcal{F}$ we arrive at a contradiction. Hence we conclude that T is irreducible.

Suppose T is irreducible. Let $\mathcal{J}$ be the sheaf of ideals defining the reduced induced closed subspace structure on T, see Properties of Spaces, Lemma 48.9.2. By Lemma 51.10.2 we see there exists an $n \geq 0$ such that $\mathcal{J}^n\mathcal{F} = 0$. Hence we obtain a filtration

$$0 = \mathcal{I}^n\mathcal{F} \subset \mathcal{I}^{n-1}\mathcal{F} \subset \ldots \subset \mathcal{I}\mathcal{F} \subset \mathcal{F}$$

each of whose successive subquotients is annihilated by $\mathcal{J}$. Hence if each of these subquotients has a filtration as in the statement of the lemma then also $\mathcal{F}$ does. In other words we may assume that $\mathcal{J}$ does annihilate $\mathcal{F}$.

Assume T is irreducible and $\mathcal{J}\mathcal{F} = 0$ where $\mathcal{J}$ is as above. Then the scheme theoretic support of $\mathcal{F}$ is T, see Morphisms of Spaces, Lemma 49.14.1. Hence we can apply Lemma 51.11.2. This gives a short exact sequence

$$0 \to i_*(\mathcal{I}^{\oplus r}) \to \mathcal{F} \to \mathcal{Q} \to 0$$

where the support of $\mathcal{Q}$ is a proper closed subset of T. Hence we see that $\mathcal{Q}$ has a filtration of the desired type by minimality of T. But then clearly $\mathcal{F}$ does too, which is our final contradiction. □

Lemma 51.11.4. *Let S be a scheme. Let X be a Noetherian algebraic space over S. Let $\mathcal{P}$ be a property of coherent sheaves on X. Assume*

(1) *For any short exact sequence of coherent sheaves*

$$0 \to \mathcal{F}_1 \to \mathcal{F} \to \mathcal{F}_2 \to 0$$

if $\mathcal{F}_i$, $i = 1, 2$ have property $\mathcal{P}$ then so does $\mathcal{F}$.

(2) *For every reduced closed subspace $Z \subset X$ with $|Z|$ irreducible and every quasi-coherent sheaf of ideals $\mathcal{I} \subset \mathcal{O}_Z$ we have $\mathcal{P}$ for $i_*\mathcal{I}$.*

Then property $\mathcal{P}$ holds for every coherent sheaf on X.

Proof. First note that if $\mathcal{F}$ is a coherent sheaf with a filtration

$$0 = \mathcal{F}_0 \subset \mathcal{F}_1 \subset \ldots \subset \mathcal{F}_m = \mathcal{F}$$

by coherent subsheaves such that each of $\mathcal{F}_i/\mathcal{F}_{i-1}$ has property $\mathcal{P}$, then so does $\mathcal{F}$. This follows from the property (1) for $\mathcal{P}$. On the other hand, by Lemma 51.11.3 we can filter any $\mathcal{F}$ with successive subquotients as in (2). Hence the lemma follows. □

Here is a more useful variant of the lemma above.

Lemma 51.11.5. *Let S be a scheme. Let X be a Noetherian algebraic space over S. Let $\mathcal{P}$ be a property of coherent sheaves on X. Assume*

(1) *For any short exact sequence of coherent sheaves*

$$0 \to \mathcal{F}_1 \to \mathcal{F} \to \mathcal{F}_2 \to 0$$

if $\mathcal{F}_i$, $i = 1, 2$ have property $\mathcal{P}$ then so does $\mathcal{F}$.

(2) *If $\mathcal{P}$ holds for a direct sum of coherent sheaves then it holds for both.*

(3) *For every reduced closed subspace $i : Z \to X$ with $|Z|$ irreducible there exists a coherent sheaf $\mathcal{G}$ on Z such that*
   (a) *$\mathit{Supp}(\mathcal{G}) = Z$,*
   (b) *for every nonzero quasi-coherent sheaf of ideals $\mathcal{I} \subset \mathcal{O}_Z$ there exists a quasi-coherent subsheaf $\mathcal{G}' \subset \mathcal{I}\mathcal{G}$ such that $\mathit{Supp}(\mathcal{G}/\mathcal{G}')$ is proper closed in Z and such that $\mathcal{P}$ holds for $i_*\mathcal{G}'$.*

Then property $\mathcal{P}$ holds for every coherent sheaf on X.

Proof. Consider the collection

$$\mathcal{T} = \left\{ \begin{array}{c} T \subset |X| \text{ closed such that there exists a coherent sheaf } \mathcal{F} \\ \text{with } \mathrm{Supp}(\mathcal{F}) = T \text{ for which the lemma is wrong} \end{array} \right\}$$

We are trying to show that $\mathcal{T}$ is empty. If not, then because $|X|$ is Noetherian (Properties of Spaces, Lemma 48.22.2) we can choose a minimal element $T \in \mathcal{T}$. This means that there exists a coherent sheaf $\mathcal{F}$ on X whose support is T and for which the lemma does not hold. Clearly $T \neq \emptyset$ because the only sheaf with support in $\emptyset$ for which $\mathcal{P}$ does hold (by property (2)).

If T is not irreducible, then we can write $T = Z_1 \cup Z_2$ with Z_1, Z_2 closed and strictly smaller than T. Then we can apply Lemma 51.11.1 to get a short exact sequence of coherent sheaves

$$0 \to \mathcal{G}_1 \to \mathcal{F} \to \mathcal{G}_2 \to 0$$

with $\mathrm{Supp}(\mathcal{G}_i) \subset Z_i$. By minimality of T each of $\mathcal{G}_i$ has $\mathcal{P}$. Hence $\mathcal{F}$ has property $\mathcal{P}$ by (1), a contradiction.

Suppose T is irreducible. Let $\mathcal{J}$ be the sheaf of ideals defining the reduced induced closed subspace structure on T, see Properties of Spaces, Lemma 48.9.2. By Lemma 51.10.2 we see there exists an $n \geq 0$ such that $\mathcal{J}^n\mathcal{F} = 0$. Hence we obtain a filtration

$$0 = \mathcal{I}^n\mathcal{F} \subset \mathcal{I}^{n-1}\mathcal{F} \subset \ldots \subset \mathcal{I}\mathcal{F} \subset \mathcal{F}$$

each of whose successive subquotients is annihilated by $\mathcal{J}$. Hence if each of these subquotients has a filtration as in the statement of the lemma then also $\mathcal{F}$ does. In other words we may assume that $\mathcal{J}$ does annihilate $\mathcal{F}$.

Assume T is irreducible and $\mathcal{J}\mathcal{F} = 0$ where $\mathcal{J}$ is as above. Denote $i : Z \to X$ the closed subspace corresponding to $\mathcal{J}$. Then $\mathcal{F} = i_*\mathcal{H}$ for some coherent $\mathcal{O}_Z$-module $\mathcal{H}$, see Morphisms of Spaces, Lemma 49.14.1 and Lemma 51.9.7. Let $\mathcal{G}$ be the

coherent sheaf on Z satisfying (3)(a) and (3)(b). We apply Lemma 51.11.2 to get injective maps

$$\mathcal{I}_1^{\oplus r_1} \to \mathcal{H} \quad \text{and} \quad \mathcal{I}_2^{\oplus r_2} \to \mathcal{G}$$

where the support of the cokernels are proper closed in Z. Hence we find an nonempty open $V \subset Z$ such that

$$\mathcal{H}_V^{\oplus r_2} \cong \mathcal{G}_V^{\oplus r_1}$$

Let $\mathcal{I} \subset \mathcal{O}_Z$ be a quasi-coherent ideal sheaf cutting out $Z \setminus V$ we obtain (Lemma 51.10.4) a map

$$\mathcal{I}^n \mathcal{G}^{\oplus r_1} \longrightarrow \mathcal{H}^{\oplus r_2}$$

which is an isomorphism over V. The kernel is supported on $Z \setminus V$ hence annihilated by some power of $\mathcal{I}$, see Lemma 51.10.2. Thus after increasing n we may assume the displayed map is injective, see Lemma 51.10.3. Applying (3)(b) we find $\mathcal{G}' \subset \mathcal{I}^n \mathcal{G}$ such that

$$(i_* \mathcal{G}')^{\oplus r_1} \longrightarrow i_* \mathcal{H}^{\oplus r_2} = \mathcal{F}^{\oplus r_2}$$

is injective with cokernel supported in a proper closed subset of Z and such that property $\mathcal{P}$ holds for $i_* \mathcal{G}'$. By (1) property $\mathcal{P}$ holds for $(i_* \mathcal{G}')^{\oplus r_1}$. By (1) and minimality of $T = |Z|$ property $\mathcal{P}$ holds for $\mathcal{F}^{\oplus r_2}$. And finally by (2) property $\mathcal{P}$ holds for $\mathcal{F}$ which is the desired contradiction. □

Lemma 51.11.6. *Let S be a scheme. Let X be a Noetherian algebraic space over S. Let $\mathcal{P}$ be a property of coherent sheaves on X. Assume*

1. *For any short exact sequence of coherent sheaves on X if two out of three have property $\mathcal{P}$ so does the third.*
2. *If $\mathcal{P}$ holds for a direct sum of coherent sheaves then it holds for both.*
3. *For every reduced closed subspace $i : Z \to X$ with $|Z|$ irreducible there exists a coherent sheaf $\mathcal{G}$ on X whose scheme theoretic support is Z such that $\mathcal{P}$ holds for $\mathcal{G}$.*

Then property $\mathcal{P}$ holds for every coherent sheaf on X.

Proof. We will show that conditions (1) and (2) of Lemma 51.11.4 hold. This is clear for condition (1). To show that (2) holds, let

$$\mathcal{T} = \left\{ \begin{array}{l} i : Z \to X \text{ reduced closed subspace with } |Z| \text{ irreducible such} \\ \text{that } i_* \mathcal{I} \text{ does not have } \mathcal{P} \text{ for some quasi-coherent } \mathcal{I} \subset \mathcal{O}_Z \end{array} \right\}$$

If $\mathcal{T}$ is nonempty, then since X is Noetherian, we can find an $i : Z \to X$ which is minimal in $\mathcal{T}$. We will show that this leads to a contradiction.

Let $\mathcal{G}$ be the sheaf whose scheme theoretic support is Z whose existence is assumed in assumption (3). Let $\varphi : i_* \mathcal{I}^{\oplus r} \to \mathcal{G}$ be as in Lemma 51.11.2. Let

$$0 = \mathcal{F}_0 \subset \mathcal{F}_1 \subset \ldots \subset \mathcal{F}_m = \text{Coker}(\varphi)$$

be a filtration as in Lemma 51.11.3. By minimality of Z and assumption (1) we see that $\text{Coker}(\varphi)$ has property $\mathcal{P}$. As φ is injective we conclude using assumption (1) once more that $i_* \mathcal{I}^{\oplus r}$ has property $\mathcal{P}$. Using assumption (2) we conclude that $i_* \mathcal{I}$ has property $\mathcal{P}$.

Finally, if $\mathcal{J} \subset \mathcal{O}_Z$ is a second quasi-coherent sheaf of ideals, set $\mathcal{K} = \mathcal{I} \cap \mathcal{J}$ and consider the short exact sequences

$$0 \to \mathcal{K} \to \mathcal{I} \to \mathcal{I}/\mathcal{K} \to 0 \quad \text{and} \quad 0 \to \mathcal{K} \to \mathcal{J} \to \mathcal{J}/\mathcal{K} \to 0$$

Arguing as above, using the minimality of Z, we see that $i_*\mathcal{I}/\mathcal{K}$ and $i_*\mathcal{J}/\mathcal{K}$ satisfy $\mathcal{P}$. Hence by assumption (1) we conclude that $i_*\mathcal{K}$ and then $i_*\mathcal{J}$ satisfy $\mathcal{P}$. In other words, Z is not an element of $\mathcal{T}$ which is the desired contradiction. □

51.12. Limits of coherent modules

A colimit of coherent modules (on a locally Noetherian algebraic space) is typically not coherent. But it is quasi-coherent as any colimit of quasi-coherent modules on an algebraic space is quasi-coherent, see Properties of Spaces, Lemma 48.27.7. Conversely, if the algebraic space is Noetherian, then every quasi-coherent module is a filtered colimit of coherent modules.

Lemma 51.12.1. *Let S be a scheme. Let X be a Noetherian algebraic space over S. Every quasi-coherent $\mathcal{O}_X$-module is the filtered colimit of its coherent submodules.*

Proof. Let $\mathcal{F}$ be a quasi-coherent $\mathcal{O}_X$-module. If $\mathcal{G}, \mathcal{H} \subset \mathcal{F}$ are coherent $\mathcal{O}_X$-submodules then the image of $\mathcal{G} \oplus \mathcal{H} \to \mathcal{F}$ is another coherent $\mathcal{O}_X$-submodule which contains both of them (see Lemmas 51.9.3 and 51.9.4). In this way we see that the system is directed. Hence it now suffices to show that $\mathcal{F}$ can be written as a filtered colimit of coherent modules, as then we can take the images of these modules in $\mathcal{F}$ to conclude there are enough of them.

Let U be an affine scheme and $U \to X$ a surjective étale morphism. Set $R = U \times_X U$ so that $X = U/R$ as usual. By Properties of Spaces, Proposition 48.30.1 we see that $QCoh(\mathcal{O}_X) = QCoh(U, R, s, t, c)$. Hence we reduce to showing the corresponding thing for $QCoh(U, R, s, t, c)$. Thus the result follows from the more general Groupoids, Lemma 38.13.3. □

Lemma 51.12.2. *Let S be a scheme. Let $f : X \to Y$ be an affine morphism of algebraic spaces over S with Y Noetherian. Then every quasi-coherent $\mathcal{O}_X$-module is a filtered colimit of finitely presented $\mathcal{O}_X$-modules.*

Proof. Let $\mathcal{F}$ be a quasi-coherent $\mathcal{O}_X$-module. Write $f_*\mathcal{F} = \operatorname{colim} \mathcal{H}_i$ with $\mathcal{H}_i$ a coherent $\mathcal{O}_Y$-module, see Lemma 51.12.1. By Lemma 51.9.2 the modules $\mathcal{H}_i$ are $\mathcal{O}_Y$-modules of finite presentation. Hence $f^*\mathcal{H}_i$ is an $\mathcal{O}_X$-module of finite presentation, see Properties of Spaces, Section 48.28. We claim the map

$$\operatorname{colim} f^*\mathcal{H}_i = f^*f_*\mathcal{F} \to \mathcal{F}$$

is surjective as f is assumed affine, Namely, choose a scheme V and a surjective étale morphism $V \to Y$. Set $U = X \times_Y V$. Then U is a scheme, $f' : U \to V$ is affine, and $U \to X$ is surjective étale. By Properties of Spaces, Lemma 48.24.2 we see that $f'_*(\mathcal{F}|_U) = f_*\mathcal{F}|_V$ and similarly for pullbacks. Thus the restriction of $f^*f_*\mathcal{F} \to \mathcal{F}$ to U is the map

$$f^*f_*\mathcal{F}|_U = (f')^*(f_*\mathcal{F})|_V) = (f')^*f'_*(\mathcal{F}|_U) \to \mathcal{F}|_U$$

which is surjective as f' is an affine morphism of schemes. Hence the claim holds.

We conclude that every quasi-coherent module on X is a quotient of a filtered colimit of finitely presented modules. In particular, we see that $\mathcal{F}$ is a cokernel of a map

$$\operatorname{colim}_{j \in J} \mathcal{G}_j \longrightarrow \operatorname{colim}_{i \in I} \mathcal{H}_i$$

with $\mathcal{G}_j$ and $\mathcal{H}_i$ finitely presented. Note that for every $j \in I$ there exist $i \in I$ and a morphism $\alpha : \mathcal{G}_j \to \mathcal{H}_i$ such that

$$\begin{array}{ccc} \mathcal{G}_j & \xrightarrow{\alpha} & \mathcal{H}_i \\ \downarrow & & \downarrow \\ \operatorname{colim}_{j \in J} \mathcal{G}_j & \longrightarrow & \operatorname{colim}_{i \in I} \mathcal{H}_i \end{array}$$

commutes, see Lemma 51.4.3. In this situation $\operatorname{Coker}(\alpha)$ is a finitely presented $\mathcal{O}_X$-module which comes endowed with a map $\operatorname{Coker}(\alpha) \to \mathcal{F}$. Consider the set K of triples (i, j, α) as above. We say that $(i, j, \alpha) \leq (i', j', \alpha')$ if and only if $i \leq i'$, $j \leq j'$, and the diagram

$$\begin{array}{ccc} \mathcal{G}_j & \xrightarrow{\alpha} & \mathcal{H}_i \\ \downarrow & & \downarrow \\ \mathcal{G}_{j'} & \xrightarrow{\alpha'} & \mathcal{H}_{i'} \end{array}$$

commutes. It follows from the above that K is a directed partially ordered set,

$$\mathcal{F} = \operatorname{colim}_{(i,j,\alpha) \in K} \operatorname{Coker}(\alpha),$$

and we win. □

51.13. Vanishing cohomology

In this section we show that a quasi-compact and quasi-separated algebraic space is affine if it has vanishing higher cohomology for all quasi-coherent sheaves. We do this in a sequence of lemmas all of which will become obsolete once we prove Proposition 51.13.9.

Situation 51.13.1. Here S is a scheme and X is a quasi-compact and quasi-separated algebraic space over S with the following property: For every quasi-coherent $\mathcal{O}_X$-module $\mathcal{F}$ we have $H^1(X, \mathcal{F}) = 0$. We set $A = \Gamma(X, \mathcal{O}_X)$.

We would like to show that the canonical morphism

$$p : X \to \operatorname{Spec}(A)$$

(see Properties of Spaces, Lemma 48.31.1) is an isomorphism. If M is an A-module we denote $M \otimes_A \mathcal{O}_X$ the quasi-coherent module $p^* \tilde{M}$.

Lemma 51.13.2. *In Situation 51.13.1 for an A-module M we have $p_*(M \otimes_A \mathcal{O}_X) = \widetilde{M}$ and $\Gamma(X, M \otimes_A \mathcal{O}_X) = M$.*

Proof. The equality $p_*(M \otimes_A \mathcal{O}_X) = \widetilde{M}$ follows from the equality $\Gamma(X, M \otimes_A \mathcal{O}_X) = M$ as $p_*(M \otimes_A \mathcal{O}_X)$ is a quasi-coherent module on $\operatorname{Spec}(A)$ by Morphisms of Spaces, Lemma 49.12.2. Observe that $\Gamma(X, \bigoplus_{i \in I} \mathcal{O}_X) = \bigoplus_{i \in I} A$ by Lemma 51.4.1. Hence the lemma holds for free modules. Choose a short exact sequence $F_1 \to F_0 \to M$ where F_0, F_1 are free A-modules. Since $H^1(X, -)$ is zero the global sections functor is right exact. Moreover the pullback p^* is right exact as well. Hence we see that

$$\Gamma(X, F_1 \otimes_A \mathcal{O}_X) \to \Gamma(X, F_0 \otimes_A \mathcal{O}_X) \to \Gamma(X, M \otimes_A \mathcal{O}_X) \to 0$$

is exact. The result follows. □

The following lemma shows that Situation 51.13.1 is preserved by base change of $X \to \mathrm{Spec}(A)$ by $\mathrm{Spec}(A') \to \mathrm{Spec}(A)$.

Lemma 51.13.3. *In Situation 51.13.1.*

(1) *Given an affine morphism $X' \to X$ of algebraic spaces, we have $H^1(X', \mathcal{F}') = 0$ for every quasi-coherent $\mathcal{O}_{X'}$-module $\mathcal{F}'$.*
(2) *Given an A-algebra A' setting $X' = X \times_{\mathrm{Spec}(A)} \mathrm{Spec}(A')$ the morphism $X' \to X$ is affine and $\Gamma(X', \mathcal{O}_{X'}) = A'$.*

Proof. Part (1) follows from Lemma 51.7.2 and the Leray spectral sequence (Cohomology on Sites, Lemma 21.14.5). Let $A \to A'$ be as in (2). Then $X' \to X$ is affine because affine morphisms are preserved under base change (Morphisms of Spaces, Lemma 49.19.5) and the fact that a morphism of affine schemes is affine. The equality $\Gamma(X', \mathcal{O}_{X'}) = A'$ follows as $(X' \to X)_*\mathcal{O}_{X'} = A' \otimes_A \mathcal{O}_X$ by Lemma 51.8.2 and thus

$$\Gamma(X', \mathcal{O}_{X'}) = \Gamma(X, (X' \to X)_*\mathcal{O}_{X'}) = \Gamma(X, A' \otimes_A \mathcal{O}_X) = A'$$

by Lemma 51.13.2. □

Lemma 51.13.4. *In Situation 51.13.1. Let $Z_0, Z_1 \subset |X|$ be disjoint closed subsets. Then there exists an $a \in A$ such that $Z_0 \subset V(a)$ and $Z_1 \subset V(a-1)$.*

Proof. We may and do endow Z_0, Z_1 with the reduced induced subspace structure (Properties of Spaces, Definition 48.9.4) and we denote $i_0 : Z_0 \to X$ and $i_1 : Z_1 \to X$ the corresponding closed immersions. Since $Z_0 \cap Z_1 = \emptyset$ we see that the canonical map of quasi-coherent $\mathcal{O}_X$-modules

$$\mathcal{O}_X \longrightarrow i_{0,*}\mathcal{O}_{Z_0} \oplus i_{1,*}\mathcal{O}_{Z_1}$$

is surjective (look at stalks at geometric points). Since $H^1(X, -)$ is zero on the kernel of this map the induced map of global sections is surjective. Thus we can find $a \in A$ which maps to the global section $(0, 1)$ of the right hand side. □

Lemma 51.13.5. *In Situation 51.13.1 the morphism $p : X \to \mathrm{Spec}(A)$ is surjective.*

Proof. Let $A \to k$ be a ring homomorphism where k is a field. It suffices to show that $X_k = \mathrm{Spec}(k) \times_{\mathrm{Spec}(A)} X$ is nonempty. By Lemma 51.13.3 we have $\Gamma(X_k, \mathcal{O}) = k$. Hence X_k is nonempty. □

Lemma 51.13.6. *In Situation 51.13.1 the morphism $p : X \to \mathrm{Spec}(A)$ is universally closed.*

Proof. Let $Z \subset |X|$ be a closed subset. We may and do endow Z with the reduced induced subspace structure (Properties of Spaces, Definition 48.9.4) and we denote $i : Z \to X$ the corresponding closed immersions. Then i is affine (Morphisms of Spaces, Lemma 49.19.6). Hence Z is another algebraic space as in Situation 51.13.1 by Lemma 51.13.3. Set $B = \Gamma(Z, \mathcal{O}_Z)$. Since $\mathcal{O}_X \to i_*\mathcal{O}_Z$ is surjective, we see that $A \to B$ is surjective by the vanishing of H^1 of the kernel. Consider the commutative diagram

$$\begin{array}{ccc} Z & \xrightarrow{i} & X \\ \downarrow & & \downarrow \\ \mathrm{Spec}(B) & \longrightarrow & \mathrm{Spec}(A) \end{array}$$

By Lemma 51.13.5 the map $Z \to \mathrm{Spec}(B)$ is surjective and by the above $\mathrm{Spec}(B) \to \mathrm{Spec}(A)$ is a closed immersion. Thus p is closed.

By Lemma 51.13.3 we see that the base change of p by $\mathrm{Spec}(A') \to \mathrm{Spec}(A)$ is closed for every ring map $A \to A'$. Hence p is universally closed by Morphisms of Spaces, Lemma 49.10.5. □

Lemma 51.13.7. *In Situation 51.13.1 the morphism $p : X \to \mathrm{Spec}(A)$ is universally injective.*

Proof. Let $A \to k$ be a ring homomorphism where k is a field. It suffices to show that $\mathrm{Spec}(k) \times_{\mathrm{Spec}(A)} X$ has at most one point (see Morphisms of Spaces, Lemma 49.18.6). Thus we may assume that A is a field and we have to show that $|X|$ has at most one point.

Let's think of X as an algebraic space over $\mathrm{Spec}(k)$ and let's use the notation $X(K)$ to denote K-valued points of X for any extension $k \subset K$, see Morphisms of Spaces, Section 49.23. If $k \subset K$ is an algebraically closed field extension of large transcendence degree, then we see that $X(K) \to |X|$ is surjective, see Morphisms of Spaces, Lemma 49.23.2. Hence, after replacing k by K, we see that it suffices to prove that $X(k)$ is a singleton (in the case $A = k$).

Let $x, x' \in X(k)$. By Decent Spaces, Lemma 50.11.3 we see that x and x' are closed points of $|X|$. Hence x and x' map to distinct points of $\mathrm{Spec}(k)$ if $x \neq x'$ by Lemma 51.13.4. We conclude that $x = x'$ as desired. □

Lemma 51.13.8. *In Situation 51.13.1 the morphism $p : X \to \mathrm{Spec}(A)$ is separated.*

Proof. We will use the results of Lemmas 51.13.2, 51.13.3 51.13.5, 51.13.6, and 51.13.7 without further mention. We will use the valuative criterion of separatedness, see Morphisms of Spaces, Lemma 49.39.2. Let R be a valuation ring over A with fraction field K. Let $\mathrm{Spec}(K) \to X$ be a morphism over $\mathrm{Spec}(A)$. We have to show that we can extend this to a morphism $\mathrm{Spec}(R) \to X$ in at most one way. We may replace A by R and X by $\mathrm{Spec}(R) \times_{\mathrm{Spec}(A)} X$. Hence we may assume that $A = R$ is a valuation ring with field of fractions K and that we have a K-point x in X.

It is clear that we may replace X by its reduction, see Properties of Spaces, Lemma 48.9.3. Since $X \to \mathrm{Spec}(A)$ is a universal homeomorphism we see that $|X|$ is the closure of $\{x\}$. For every nonzero $f \in A$ the kernel $\mathcal{I}_f$ of $f : \mathcal{O}_X \to \mathcal{O}_X$ is a quasi-coherent sheaf of ideals and any section $\mathrm{Spec}(A) \to X$ of p factors through the closed subscheme defined by $\mathcal{I}_f$. Hence we may also replace X by the closed subspace cut out by the quasi-coherent sheaf of ideals $\sum_{f \in A} \mathcal{I}_f$. In other words, we may assume that any nonzero $f \in A$ is a nonzerodivisor on $\mathcal{O}_X$.

Let U be an affine scheme and let $U \to X$ be a surjective étale morphism. Note that $U = \mathrm{Spec}(B)$ where B is a reduced A-algebra. Note that B is flat over A by More on Algebra, Lemma 15.15.4 and the fact $\mathcal{O}_X$ has no nonzero A-torsion. The fibre product $U \times_X \mathrm{Spec}(K) = \mathrm{Spec}(B \otimes_A K) = \coprod_{i=1,\ldots,n} \mathrm{Spec}(K_i)$ is a finite disjoint union of spectra of finite separable field extensions $K_i \supset K$ (for example because X is a decent space for which the general Decent Spaces, Lemma 50.4.5 holds). Choose a finite Galois extension $K \subset K'$ such that each K_i embeds into K' over K and choose a valuation ring $A' \subset K'$ dominating A (see Algebra, Lemma

10.46.2). After replacing A by A' and X by $\mathrm{Spec}(A') \times_{\mathrm{Spec}(A)} X$ we may assume that $K_i = K$ for all i (small detail omitted).

If X is normal then B is a finite product $B = B_1 \times \ldots \times B_n$ of normal domains (see Algebra, Lemma 10.34.14). Each of these has fraction field K by the above. One of these rings B_i, say B_1 has a prime ideal lying over $\mathfrak{m}_A$ because $X \to \mathrm{Spec}(A)$ is surjective. Then $A = B_1$ as A is a valuation ring. Thus we see that there exists an étale morphism $\mathrm{Spec}(A) \to X$! Of course this implies that $X = \mathrm{Spec}(A)$ (for example by Morphisms of Spaces, Lemma 49.44.2) and we win in the case that X is normal.

In the general (possibly nonnormal) case we see that $U = \mathrm{Spec}(B)$ has finitely many irreducible components (as all minimal primes of B lie over $(0) \subset A$ by flatness of $A \to B$). Thus we may consider the normalization $X^\nu \to X$ of X, see Morphisms of Spaces, Lemma 49.42.7. Note that $X^\nu \to X$ is integral hence affine and universally closed (see Morphisms of Spaces, Lemma 49.40.7). Note that $X^\nu \times_X U = U^\nu$, in particular $X^\nu \to \mathrm{Spec}(A)$ is flat (as the integral closure of B in its total quotient ring is torsion free over A hence flat). Set $A^\nu = \Gamma(X^\nu, \mathcal{O}_{X^\nu})$ and consider the diagram

$$\begin{array}{ccc} X^\nu & \longrightarrow & X \\ \downarrow & & \downarrow \\ \mathrm{Spec}(A^\nu) & \longrightarrow & \mathrm{Spec}(A) \end{array}$$

By the lemmas mentioned at the beginning of the proof, the left vertical arrow is (universally) surjective and the right vertical arrow is universally closed. Since the top horizontal arrow is universally closed by construction we conclude that $\mathrm{Spec}(A^\nu) \to \mathrm{Spec}(A)$ is universally closed. Hence $A \subset A^\nu$ is integral, see Morphisms, Lemma 28.44.7. Finally, A^ν is a torsion free A-algebra with $A^\nu \otimes_A K = K$ (as $\mathrm{Spec}(K)$ maps onto $X_K = X^\nu_K$). Hence $A = A^\nu$. Observe that $x : \mathrm{Spec}(K) \to X$ lifts to $x^\nu : \mathrm{Spec}(K) \to X^\nu$ and that

$$U^\nu \times_{X^\nu, x^\nu} \mathrm{Spec}(K) = X \times_{U,x} \mathrm{Spec}(K) = \coprod\nolimits_{i=1,\ldots,n} \mathrm{Spec}(K)$$

as normalization does not chance the scheme U over its generic points. Finally, as $X^\nu \to X$ is universally closed any morphism $\mathrm{Spec}(A) \to X$ extending x lifts to a morphism into X^ν extending x^ν (see Decent Spaces, Proposition 50.13.1). Thus it suffices there is at most one morphism $\mathrm{Spec}(A) \to X^\nu$ extending x^ν. This was proved above. □

Proposition 51.13.9. *Any algebraic space as in Situation 51.13.1 is an affine scheme.*

Proof. Choose an affine scheme $U = \mathrm{Spec}(B)$ and a surjective étale morphism $\varphi : U \to X$. Set $R = U \times_X U$. As p is separated (Lemma 51.13.8) we see that R is a closed subscheme of $U \times_{\mathrm{Spec}(A)} U = \mathrm{Spec}(B \otimes_A B)$. Hence $R = \mathrm{Spec}(C)$ is affine too and the ring map

$$B \otimes_A B \longrightarrow C$$

is surjective. Let us denote the two maps $s, t : B \to C$ as usual. Pick $g_1, \ldots, g_m \in B$ such that $s(g_1), \ldots, s(g_m)$ generate C over $t : B \to C$ (which is possible as $t : B \to C$

is of finite presentation and the displayed map is surjective). Then $g_1, \ldots, g_m$ give global sections of $\varphi_*\mathcal{O}_U$ and the map

$$\mathcal{O}_X[z_1, \ldots, z_n] \longrightarrow \varphi_*\mathcal{O}_U, \quad z_j \longmapsto g_j$$

is surjective: you can check this by restricting to U. Namely, $\varphi^*\varphi_*\mathcal{O}_U = t_*\mathcal{O}_R$ (by Lemma 51.8.1) hence you get exactly the condition that $s(g_i)$ generate C over $t : B \to C$. By the vanishing of H^1 of the kernel we see that

$$\Gamma(X, \mathcal{O}_X[x_1, \ldots, x_n]) = A[x_1, \ldots, x_n] \longrightarrow \Gamma(X, \varphi_*\mathcal{O}_U) = \Gamma(U, \mathcal{O}_U) = B$$

is surjective. Thus we conclude that B is a finite type A-algebra. Hence $X \to \mathrm{Spec}(A)$ is of finite type and separated. By Lemma 51.13.7 and Morphisms of Spaces, Lemma 49.25.4 it is also locally quasi-finite. Hence $X \to \mathrm{Spec}(A)$ is representable by Morphisms of Spaces, Lemma 49.44.1 and X is a scheme. Finally X is affine, hence equal to $\mathrm{Spec}(A)$, by an application of Cohomology of Schemes, Lemma 29.3.1. □

51.14. Finite morphisms and affines

This section is the analogue of Cohomology of Schemes, Section 29.13.

Lemma 51.14.1. *Let S be a scheme. Let $f : Y \to X$ be a morphism of algebraic spaces over S. Assume*

(1) *f finite,*
(2) *f surjective,*
(3) *Y affine, and*
(4) *X Noetherian.*

Then X is affine.

Proof. We will prove that under the assumptions of the lemma for any coherent $\mathcal{O}_X$-module $\mathcal{F}$ we have $H^1(X, \mathcal{F}) = 0$. This implies that $H^1(X, \mathcal{F}) = 0$ for every quasi-coherent $\mathcal{O}_X$-module $\mathcal{F}$ by Lemmas 51.12.1 and 51.4.1. Then it follows that X is affine from Proposition 51.13.9.

Let $\mathcal{P}$ be the property of coherent sheaves $\mathcal{F}$ on X defined by the rule

$$\mathcal{P}(\mathcal{F}) \Leftrightarrow H^1(X, \mathcal{F}) = 0.$$

We are going to apply Lemma 51.11.5. Thus we have to verify (1), (2) and (3) of that lemma for $\mathcal{P}$. Property (1) follows from the long exact cohomology sequence associated to a short exact sequence of sheaves. Property (2) follows since $H^1(X, -)$ is an additive functor. To see (3) let $i : Z \to X$ be a reduced closed subspace with $|Z|$ irreducible. Let $W = Z \times_X Y$ and denote $i' : W \to Y$ the corresponding closed immersion. Denote $f' : W \to Z$ the other projection which is a finite morphism of algebraic spaces. Since W is a closed subscheme of Y, it is affine. We claim that $\mathcal{G} = f_*i'_*\mathcal{O}_W = i_*f'_*\mathcal{O}_W$ satisfies properties (3)(a) and (3)(b) of Lemma 51.11.5 which will finish the proof. Property (3)(a) is clear as $W \to Z$ is surjective (because f is surjective). To see (3)(b) let $\mathcal{I}$ be a nonzero quasi-coherent sheaf of ideals on Z. We simply take $\mathcal{G}' = \mathcal{I}\mathcal{G}$. Namely, we have

$$\mathcal{I}\mathcal{G} = f'_*(\mathcal{I}')$$

where $\mathcal{I}' = \mathrm{Im}((f')^*\mathcal{I} \to \mathcal{O}_W)$. This is true because f' is a (representable) affine morphism of algebraic spaces and hence the result can be checked on an étale covering of Z by a scheme in which case the result is Cohomology of Schemes,

Lemma 29.13.2. Finally, f' is affine, hence $R^1 f'_* \mathcal{I}' = 0$ by Lemma 51.7.2. As W is affine we have $H^1(W, \mathcal{I}') = 0$ hence the Leray spectral sequence (in the form Cohomology on Sites, Lemma 21.14.6) implies that $H^1(Z, f'_* \mathcal{I}') = 0$. Since $i : Z \to X$ is affine we conclude that $R^1 i_* f'_* \mathcal{I}' = 0$ hence $H^1(X, i_* f'_* \mathcal{I}') = 0$ by Leray again and we win. □

51.15. A weak version of Chow's lemma

In this section we quickly prove the following lemma in order to help us prove the basic results on cohomology of coherent modules on proper algebraic spaces.

Lemma 51.15.1. *Let A be a ring. Let X be an algebraic space over $\mathrm{Spec}(A)$ whose structure morphism $X \to \mathrm{Spec}(A)$ is separated of finite type. Then there exists a proper surjective morphism $X' \to X$ where X' is a scheme which is H-quasi-projective over $\mathrm{Spec}(A)$.*

Proof. Let W be an affine scheme and let $f : W \to X$ be a surjective étale morphism. There exists an integer d such that all geometric fibres of f have $\leq d$ points (because X is a separated algebraic hence reasonable, see Decent Spaces, Lemma 50.5.1). Picking d minimal we get a nonempty open $U \subset X$ such that $f^{-1}(U) \to U$ is finite étale of degree d, see Decent Spaces, Lemma 50.8.1. Let

$$V \subset W \times_X W \times_X \ldots \times_X W$$

(d factors in the fibre product) be the complement of all the diagonals. Because $W \to X$ is separated the diagonal $W \to W \times_X W$ is a closed immersion. Since $W \to X$ is étale the diagonal $W \to W \times_X W$ is an open immersion, see Morphisms of Spaces, Lemmas 49.35.10 and 49.34.9. Hence the diagonals are open and closed subschemes of the quasi-compact scheme $W \times_X \ldots \times_X W$. In particular we conclude V is a quasi-compact scheme. Choose an open immersion $W \subset Y$ with Y H-projective over A (this is possible as W is affine and of finite type over A; for example we can use Morphisms, Lemmas 28.40.2 and 28.43.11). Let

$$Z \subset Y \times_A Y \times_A \ldots \times_A Y$$

be the scheme theoretic image of the composition $V \to W \times_X \ldots \times_X W \to Y \times_A \ldots \times_A Y$. Observe that this morphism is quasi-compact since V is quasi-compact and $Y \times_A \ldots \times_A Y$ is separated. Note that $V \to Z$ is an open immersion as $V \to Y \times_A \ldots \times_A Y$ is an immersion, see Morphisms, Lemma 28.7.7. The projection morphisms give d morphisms $g_i : Z \to Y$. These morphisms g_i are projective as Y is projective over A, see material in Morphisms, Section 28.43. We set

$$X' = \bigcup g_i^{-1}(W) \subset Z$$

There is a morphism $X' \to X$ whose restriction to $g_i^{-1}(W)$ is the composition $g_i^{-1}(W) \to W \to X$. Namely, these morphisms agree over V hence agree over $g_i^{-1}(W) \cap g_j^{-1}(W)$ by Morphisms of Spaces, Lemma 49.17.8. Claim: the morphism $X' \to X$ is proper.

If the claim holds, then the lemma follows by induction on d. Namely, by construction X' is H-quasi-projective over $\mathrm{Spec}(A)$. The image of $X' \to X$ contains the open U as V surjects onto U. Denote T the reduced induced algebraic space structure on $X \setminus U$. Then $T \times_X W$ is a closed subscheme of W, hence affine. Moreover, the morphism $T \times_X W \to T$ is étale and every geometric fibre has $< d$ points. By

induction hypothesis there exists a proper surjective morphism $T' \to T$ where T' is a scheme H-quasi-projective over $\mathrm{Spec}(A)$. Since T is a closed subspace of X we see that $T' \to X$ is a proper morphism. Thus the lemma follows by taking the proper surjective morphism $X' \amalg T' \to X$.

Proof of the claim. By construction the morphism $X' \to X$ is separated and of finite type. We will check conditions (1) – (4) of Morphisms of Spaces, Lemma 49.38.3 for the morphisms $V \to X'$ and $X' \to X$. Conditions (1) and (2) we have seen above. Condition (3) holds as $X' \to X$ is separated (as a morphism whose source is a separated algebraic space). Thus it suffices to check liftability to X' for diagrams

$$\begin{array}{ccc} \mathrm{Spec}(K) & \longrightarrow & V \\ \downarrow & & \downarrow \\ \mathrm{Spec}(R) & \longrightarrow & X \end{array}$$

where R is a valuation ring with fraction field K. Note that the top horizontal map is given by d pairwise distinct K-valued points $w_1, \ldots, w_d$ of W. In fact, this is a complete set of inverse images of the point $x \in X(K)$ coming from the diagram. Since $W \to X$ is surjective, we can, after possibly replacing R by an extension of valuation rings, lift the morphism $\mathrm{Spec}(R) \to X$ to a morphism $w : \mathrm{Spec}(R) \to W$, see Morphisms of Spaces, Lemma 49.38.2. Since $w_1, \ldots, w_d$ is a complete collection of inverse images of x we see that $w|_{\mathrm{Spec}(K)}$ is equal to one of them, say w_i. Thus we see that we get a commutative diagram

$$\begin{array}{ccc} \mathrm{Spec}(K) & \longrightarrow & Z \\ \downarrow & & \downarrow{\scriptstyle g_i} \\ \mathrm{Spec}(R) & \xrightarrow{w} & Y \end{array}$$

By the valuative criterion of properness for the projective morphism g_i we can lift w to $z : \mathrm{Spec}(R) \to Z$, see Morphisms, Lemma 28.43.5 and Schemes, Proposition 25.20.6. The image of z is in $g_i^{-1}(W) \subset X'$ and the proof is complete. □

51.16. Higher direct images of coherent sheaves

In this section we prove the fundamental fact that the higher direct images of a coherent sheaf under a proper morphism are coherent. First we prove a helper lemma.

Lemma 51.16.1. *Let S be a scheme. Consider a commutative diagram*

$$\begin{array}{ccc} X & \xrightarrow{i} & \mathbf{P}^n_Y \\ & {\scriptstyle f}\searrow & \downarrow \\ & & Y \end{array}$$

of algebraic spaces over S. Assume i is a closed immersion and Y Noetherian. Set $\mathcal{L} = i^*\mathcal{O}_{\mathbf{P}^n_Y}(1)$. Let $\mathcal{F}$ be a coherent module on X. Then there exists an integer d_0 such that for all $d \geq d_0$ we have $R^p f_*(\mathcal{F} \otimes_{\mathcal{O}_X} \mathcal{L}^{\otimes d}) = 0$ for all $p > 0$.

Proof. Checking whether $R^p f_*(\mathcal{F} \otimes \mathcal{L}^{\otimes d})$ is zero can be done étale locally on Y, see Equation (51.3.0.1). Hence we may assume Y is the spectrum of a Noetherian ring. In this case X is a scheme and the result follows from Cohomology of Schemes, Lemma 29.15.2. □

Lemma 51.16.2. *Let S be a scheme. Let $f : X \to Y$ be a proper morphism of algebraic spaces over S with Y locally Noetherian. Let $\mathcal{F}$ be a coherent $\mathcal{O}_X$-module. Then $R^i f_* \mathcal{F}$ is a coherent $\mathcal{O}_Y$-module for all $i \geq 0$.*

Proof. We first remark that X is a locally Noetherian algebraic space by Morphisms of Spaces, Lemma 49.22.5. Hence the statement of the lemma makes sense. Moreover, computing $R^i f_* \mathcal{F}$ commutes with étale localization on Y (Properties of Spaces, Lemma 48.24.2) and checking whether $R^i f_* \mathcal{F}$ coherent can be done étale locally on Y (Lemma 51.9.2). Hence we may assume that $Y = \mathrm{Spec}(A)$ is a Noetherian affine scheme.

Assume $Y = \mathrm{Spec}(A)$ is an affine scheme. Note that f is locally of finite presentation (Morphisms of Spaces, Lemma 49.26.7). Thus it is of finite presentation, hence X is Noetherian (Morphisms of Spaces, Lemma 49.26.6). Thus Lemma 51.11.6 applies to the category of coherent modules of X. For a coherent sheaf $\mathcal{F}$ on X we say $\mathcal{P}$ holds if and only if $R^i f_* \mathcal{F}$ is a coherent module on $\mathrm{Spec}(A)$. We will show that conditions (1), (2), and (3) of Lemma 51.11.6 hold for this property thereby finishing the proof of the lemma.

Verification of condition (1). Let

$$0 \to \mathcal{F}_1 \to \mathcal{F}_2 \to \mathcal{F}_3 \to 0$$

be a short exact sequence of coherent sheaves on X. Consider the long exact sequence of higher direct images

$$R^{p-1} f_* \mathcal{F}_3 \to R^p f_* \mathcal{F}_1 \to R^p f_* \mathcal{F}_2 \to R^p f_* \mathcal{F}_3 \to R^{p+1} f_* \mathcal{F}_1$$

Then it is clear that if 2-out-of-3 of the sheaves $\mathcal{F}_i$ have property $\mathcal{P}$, then the higher direct images of the third are sandwiched in this exact complex between two coherent sheaves. Hence these higher direct images are also coherent by Lemmas 51.9.3 and 51.9.4. Hence property $\mathcal{P}$ holds for the third as well.

Verification of condition (2). This follows immediately from the fact that $R^i f_*(\mathcal{F}_1 \oplus \mathcal{F}_2) = R^i f_* \mathcal{F}_1 \oplus R^i f_* \mathcal{F}_2$ and that a summand of a coherent module is coherent (see lemmas cited above).

Verification of condition (3). Let $i : Z \to X$ be a closed immersion with Z reduced and $|Z|$ irreducible. Set $g = f \circ i : Z \to \mathrm{Spec}(A)$. Let $\mathcal{G}$ be a coherent module on Z whose scheme theoretic support is equal to Z such that $R^p g_* \mathcal{G}$ is coherent for all p. Then $\mathcal{F} = i_* \mathcal{G}$ is a coherent module on X whose support scheme theoretic support is Z such that $R^p f_* \mathcal{F} = R^p g_* \mathcal{G}$. To see this use the Leray spectral sequence (Cohomology on Sites, Lemma 21.14.7) and the fact that $R^q i_* \mathcal{G} = 0$ for $q > 0$ by Lemma 51.7.2 and the fact that a closed immersion is affine. (Morphisms of Spaces, Lemma 49.19.6). Thus we reduce to finding a coherent sheaf $\mathcal{G}$ on Z with support equal to Z such that $R^p g_* \mathcal{G}$ is coherent for all p.

We apply Lemma 51.15.1 to the morphism $Z \to \mathrm{Spec}(A)$. Thus we get a diagram

$$\begin{array}{ccccc} Z & \xleftarrow{\pi} & Z' & \xrightarrow{i} & \mathbf{P}^n_A \\ & {\scriptstyle g}\searrow & \downarrow{\scriptstyle g'} & \swarrow & \\ & & \mathrm{Spec}(A) & & \end{array}$$

with $\pi : Z' \to Z$ proper surjective and i an immersion. Since $Z \to \mathrm{Spec}(A)$ is proper we conclude that g' is proper (Morphisms of Spaces, Lemma 49.36.4). Hence i is a closed immersion (Morphisms of Spaces, Lemmas 49.36.6 and 49.4.2). It follows that the morphism $i' = (i, \pi) : \mathbf{P}^n_A \times_{\mathrm{Spec}(A)} Z' = \mathbf{P}^n_Z$ is a closed immersion (Morphisms of Spaces, Lemma 49.5.6). Set

$$\mathcal{L} = i^*\mathcal{O}_{\mathbf{P}^n_A}(1) = (i')^*\mathcal{O}_{\mathbf{P}^n_Z}(1)$$

We may apply Lemma 51.16.1 to $\mathcal{L}$ and π as well as $\mathcal{L}$ and g'. Hence for all $d \gg 0$ we have $R^p\pi_*\mathcal{L}^{\otimes d} = 0$ for all $p > 0$ and $R^p(g')_*\mathcal{L}^{\otimes d} = 0$ for all $p > 0$. Set $\mathcal{G} = \pi_*\mathcal{L}^{\otimes d}$. By the Leray spectral sequence (Cohomology on Sites, Lemma 21.14.7) we have

$$E_2^{p,q} = R^p g_* R^q \pi_* \mathcal{L}^{\otimes d} \Rightarrow R^{p+q}(g')_*\mathcal{L}^{\otimes d}$$

and by choice of d the only nonzero terms in $E_2^{p,q}$ are those with $q = 0$ and the only nonzero terms of $R^{p+q}(g')_*\mathcal{L}^{\otimes d}$ are those with $p = q = 0$. This implies that $R^p g_*\mathcal{G} = 0$ for $p > 0$ and that $g_*\mathcal{G} = (g')_*\mathcal{L}^{\otimes n}$. Applying Cohomology of Schemes, Lemma 29.17.1 we see that $g_*\mathcal{G} = (g')_*\mathcal{L}^{\otimes d}$ is coherent.

We still have to check that the support of $\mathcal{G}$ is Z. This follows from the fact that $\mathcal{L}^{\otimes d}$ has lots of global sections. We spell it out here. Note that $\mathcal{L}^{\otimes d}$ is globally generated for all $d \geq 0$ because the same is true for $\mathcal{O}_{\mathbf{P}^n}(d)$. Pick a point $z \in Z'$ mapping to the generic point ξ of Z which we can do as π is surjective. (Observe that Z does indeed have a generic point as $|Z|$ is irreducible and Z is Noetherian, hence quasi-separated, hence $|Z|$ is a sober topological space by Properties of Spaces, Lemma 48.12.4.) Pick $s \in \Gamma(Z', \mathcal{L}^{\otimes d})$ which does not vanish at z. Since $\Gamma(Z, \mathcal{G}) = \Gamma(Z', \mathcal{L}^{\otimes d})$ we may think of s as a global section of $\mathcal{G}$. Choose a geometric point $\overline{z}$ of Z' lying over z and denote $\overline{\xi} = g' \circ \overline{z}$ the corresponding geometric point of Z. The adjunction map

$$(g')^*\mathcal{G} = (g')^* g'_* \mathcal{L}^{\otimes d} \longrightarrow \mathcal{L}^{\otimes d}$$

induces a map of stalks $\mathcal{G}_{\overline{\xi}} \to \mathcal{L}_{\overline{z}}$, see Properties of Spaces, Lemma 48.27.5. Moreover the adjunction map sends the pullback of s (viewed as a section of $\mathcal{G}$) to s (viewed as a section of $\mathcal{L}^{\otimes d}$). Thus the image of s in the vector space which is the source of the arrow

$$\mathcal{G}_{\overline{\xi}} \otimes \kappa(\overline{\xi}) \longrightarrow \mathcal{L}_{\overline{z}}^{\otimes d} \otimes \kappa(\overline{z})$$

isn't zero since by choice of s the image in the target of the arrow is nonzero. Hence ξ is in the support of $\mathcal{G}$ (Morphisms of Spaces, Lemma 49.15.2). Since $|Z|$ is irreducible and Z is reduced we conclude that the scheme theoretic support of $\mathcal{G}$ is all of Z as desired. □

Remark 51.16.3. Let S be a scheme. Let $f : X \to Y$ be a morphism of algebraic spaces over S. Assume f is locally of finite type and Y locally Noetherian. Then X is locally Noetherian (Morphisms of Spaces, Lemma 49.22.5). Let $\mathcal{F}$ be a coherent $\mathcal{O}_X$-module. Assume the scheme theoretic support Z of $\mathcal{F}$ is proper over Y. we claim $R^p f_*\mathcal{F}$ is a coherent $\mathcal{O}_Y$-module for all $p \geq 0$. Namely, Let $i : Z \to X$ be the closed immersion and write $\mathcal{F} = i_*\mathcal{G}$ for some coherent module $\mathcal{G}$ on Z (Lemma

51.9.7). Denoting $g : Z \to S$ the composition $f \circ i$ we see that $R^p g_* \mathcal{G}$ is coherent on S by Lemma 51.16.2. On the other hand, $R^q i_* \mathcal{G} = 0$ for $q > 0$ (Lemma 51.9.9). By Cohomology on Sites, Lemma 21.14.7 we get $R^p f_* \mathcal{F} = R^p g_* \mathcal{G}$ and the claim.

Lemma 51.16.4. *Let A be a Noetherian ring. Let $f : X \to \mathrm{Spec}(A)$ be a proper morphism of algebraic spaces. Let $\mathcal{F}$ be a coherent $\mathcal{O}_X$-module. Then $H^i(X, \mathcal{F})$ is finite A-module for all $i \geq 0$.*

Proof. This is just the affine case of Lemma 51.16.2. Namely, by Lemma 51.3.1 we know that $R^i f_* \mathcal{F}$ is a quasi-coherent sheaf. Hence it is the quasi-coherent sheaf associated to the A-module $\Gamma(\mathrm{Spec}(A), R^i f_* \mathcal{F}) = H^i(X, \mathcal{F})$. The equality holds by Cohomology on Sites, Lemma 21.14.6 and vanishing of higher cohomology groups of quasi-coherent modules on affine schemes (Cohomology of Schemes, Lemma 29.2.2). By Lemma 51.9.2 we see $R^i f_* \mathcal{F}$ is a coherent sheaf if and only if $H^i(X, \mathcal{F})$ is an A-module of finite type. Hence Lemma 51.16.2 gives us the conclusion. □

Lemma 51.16.5. *Let A be a Noetherian ring. Let B be a finitely generated graded A-algebra. Let $f : X \to \mathrm{Spec}(A)$ be a proper morphism of algebraic spaces. Set $\mathcal{B} = f^* \widetilde{B}$. Let $\mathcal{F}$ be a quasi-coherent graded $\mathcal{B}$-module of finite type. For every $p \geq 0$ the graded B-module $H^p(X, \mathcal{F})$ is a finite B-module.*

Proof. To prove this we consider the fibre product diagram

$$\begin{array}{ccc} X' = \mathrm{Spec}(B) \times_{\mathrm{Spec}(A)} X & \xrightarrow{\pi} & X \\ \downarrow{\scriptstyle f'} & & \downarrow{\scriptstyle f} \\ \mathrm{Spec}(B) & \longrightarrow & \mathrm{Spec}(A) \end{array}$$

Note that f' is a proper morphism, see Morphisms of Spaces, Lemma 49.36.3. Also, B is a finitely generated A-algebra, and hence Noetherian (Algebra, Lemma 10.29.1). This implies that X' is a Noetherian algebraic space (Morphisms of Spaces, Lemma 49.26.6). Note that X' is the relative spectrum of the quasi-coherent $\mathcal{O}_X$-algebra $\mathcal{B}$ by Morphisms of Spaces, Lemma 49.19.7. Since $\mathcal{F}$ is a quasi-coherent $\mathcal{B}$-module we see that there is a unique quasi-coherent $\mathcal{O}_{X'}$-module $\mathcal{F}'$ such that $\pi_* \mathcal{F}' = \mathcal{F}$, see Morphisms of Spaces, Lemma 49.19.10. Since $\mathcal{F}$ is finite type as a $\mathcal{B}$-module we conclude that $\mathcal{F}'$ is a finite type $\mathcal{O}_{X'}$-module (details omitted). In other words, $\mathcal{F}'$ is a coherent $\mathcal{O}_{X'}$-module (Lemma 51.9.2). Since the morphism $\pi : X' \to X$ is affine we have

$$H^p(X, \mathcal{F}) = H^p(X', \mathcal{F}')$$

by Lemma 51.7.2 and Cohomology on Sites, Lemma 21.14.6. Thus the lemma follows from Lemma 51.16.4. □

51.17. The theorem on formal functions

This section is the analogue of Cohomology of Schemes, Section 29.18. We encourage the reader to read that section first.

Situation 51.17.1. Here A is a Noetherian ring and $I \subset A$ is an ideal. Also, $f : X \to \mathrm{Spec}(A)$ is a proper morphism of algebraic spaces and $\mathcal{F}$ is a coherent sheaf on X.

In this situation we denote $I^n\mathcal{F}$ the quasi-coherent submodule of $\mathcal{F}$ generated as an $\mathcal{O}_X$-module by products of local sections of $\mathcal{F}$ and elements of I^n. In other words, it is the image of the map $f^*\widetilde{I} \otimes_{\mathcal{O}_X} \mathcal{F} \to \mathcal{F}$.

Lemma 51.17.2. *In Situation 51.17.1. Set $B = \bigoplus_{n\geq 0} I^n$. Then for every $p \geq 0$ the graded B-module $\bigoplus_{n\geq 0} H^p(X, I^n\mathcal{F})$ is a finite B-module.*

Proof. Let $\mathcal{B} = \bigoplus I^n\mathcal{O}_X = f^*\widetilde{B}$. Then $\bigoplus I^n\mathcal{F}$ is a finite type graded $\mathcal{B}$-module. Hence the result follows from Lemma 51.16.5. □

Lemma 51.17.3. *In Situation 51.17.1. For every $p \geq 0$ there exists an integer $c \geq 0$ such that*

(1) *the multiplication map $I^{n-c} \otimes H^p(X, I^c\mathcal{F}) \to H^p(X, I^n\mathcal{F})$ is surjective for all $n \geq c$, and*
(2) *the image of $H^p(X, I^{n+m}\mathcal{F}) \to H^p(X, I^n\mathcal{F})$ is contained in the submodule $I^{m-c}H^p(X, I^n\mathcal{F})$ for all $n \geq 0$, $m \geq c$.*

Proof. By Lemma 51.17.2 we can find $d_1, \ldots, d_t \geq 0$, and $x_i \in H^p(X, I^{d_i}\mathcal{F})$ such that $\bigoplus_{n\geq 0} H^p(X, I^n\mathcal{F})$ is generated by $x_1, \ldots, x_t$ over $B = \bigoplus_{n\geq 0} I^n$. Take $c = \max\{d_i\}$. It is clear that (1) holds. For (2) let $b = \max(0, n-c)$. Consider the commutative diagram of A-modules

$$\begin{array}{ccccc} I^{n+m-c-b} \otimes I^b \otimes H^p(X, I^c\mathcal{F}) & \longrightarrow & I^{n+m-c} \otimes H^p(X, I^c\mathcal{F}) & \longrightarrow & H^p(X, I^{n+m}\mathcal{F}) \\ \downarrow & & & & \downarrow \\ I^{n+m-c-b} \otimes H^p(X, I^n\mathcal{F}) & & \longrightarrow & & H^p(X, I^n\mathcal{F}) \end{array}$$

By part (1) of the lemma the composition of the horizontal arrows is surjective if $n + m \geq c$. On the other hand, it is clear that $n + m - c - b \geq m - c$. Hence part (2). □

Lemma 51.17.4. *In Situation 51.17.1. Fix $p \geq 0$.*

(1) *There exists a $c_1 \geq 0$ such that for all $n \geq c_1$ we have*
$$Ker(H^p(X, \mathcal{F}) \to H^p(X, \mathcal{F}/I^n\mathcal{F})) \subset I^{n-c_1}H^p(X, \mathcal{F}).$$
(2) *The inverse system*
$$(H^p(X, \mathcal{F}/I^n\mathcal{F}))_{n\in\mathbf{N}}$$
satisfies the Mittag-Leffler condition (see Homology, Definition 12.26.2).
(3) *In fact for any p and n there exists a $c_2(n) \geq n$ such that*
$$Im(H^p(X, \mathcal{F}/I^k\mathcal{F}) \to H^p(X, \mathcal{F}/I^n\mathcal{F})) = Im(H^p(X, \mathcal{F}) \to H^p(X, \mathcal{F}/I^n\mathcal{F}))$$
for all $k \geq c_2(n)$.

Proof. Let $c_1 = \max\{c_p, c_{p+1}\}$, where c_p, c_{p+1} are the integers found in Lemma 51.17.3 for H^p and H^{p+1}. We will use this constant in the proofs of (1), (2) and (3).

Let us prove part (1). Consider the short exact sequence
$$0 \to I^n\mathcal{F} \to \mathcal{F} \to \mathcal{F}/I^n\mathcal{F} \to 0$$
From the long exact cohomology sequence we see that
$$\mathrm{Ker}(H^p(X, \mathcal{F}) \to H^p(X, \mathcal{F}/I^n\mathcal{F})) = \mathrm{Im}(H^p(X, I^n\mathcal{F}) \to H^p(X, \mathcal{F}))$$

Hence by our choice of c_1 we see that this is contained in $I^{n-c_1}H^p(X, \mathcal{F})$ for $n \geq c_1$.

Note that part (3) implies part (2) by definition of the Mittag-Leffler condition.

Let us prove part (3). Fix an n throughout the rest of the proof. Consider the commutative diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & I^n\mathcal{F} & \longrightarrow & \mathcal{F} & \longrightarrow & \mathcal{F}/I^n\mathcal{F} & \longrightarrow & 0 \\ & & \uparrow & & \uparrow & & \uparrow & & \\ 0 & \longrightarrow & I^{n+m}\mathcal{F} & \longrightarrow & \mathcal{F} & \longrightarrow & \mathcal{F}/I^{n+m}\mathcal{F} & \longrightarrow & 0 \end{array}$$

This gives rise to the following commutative diagram

$$\begin{array}{ccccccc} H^p(X, I^n\mathcal{F}) & \longrightarrow & H^p(X, \mathcal{F}) & \longrightarrow & H^p(X, \mathcal{F}/I^n\mathcal{F}) & \xrightarrow{\delta} & H^{p+1}(X, I^n\mathcal{F}) \\ \uparrow & & \uparrow{\scriptstyle 1} & & \uparrow & & \uparrow{\scriptstyle a} \\ H^p(X, I^{n+m}\mathcal{F}) & \longrightarrow & H^p(X, \mathcal{F}) & \longrightarrow & H^p(X, \mathcal{F}/I^{n+m}\mathcal{F}) & \longrightarrow & H^{p+1}(X, I^{n+m}\mathcal{F}) \end{array}$$

If $m \geq c_1$ we see that the image of a is contained in $I^{m-c_1}H^{p+1}(X, I^n\mathcal{F})$. By the Artin-Rees lemma (see Algebra, Lemma 10.47.5) there exists an integer $c_3(n)$ such that

$$I^N H^{p+1}(X, I^n\mathcal{F}) \cap \mathrm{Im}(\delta) \subset \delta\left(I^{N-c_3(n)}H^p(X, \mathcal{F}/I^n\mathcal{F})\right)$$

for all $N \geq c_3(n)$. As $H^p(X, \mathcal{F}/I^n\mathcal{F})$ is annihilated by I^n, we see that if $m \geq c_3(n) + c_1 + n$, then

$$\mathrm{Im}(H^p(X, \mathcal{F}/I^{n+m}\mathcal{F}) \to H^p(X, \mathcal{F}/I^n\mathcal{F})) = \mathrm{Im}(H^p(X, \mathcal{F}) \to H^p(X, \mathcal{F}/I^n\mathcal{F}))$$

In other words, part (3) holds with $c_2(n) = c_3(n) + c_1 + n$. □

Theorem 51.17.5 (Theorem on formal functions)**.** *In Situation 51.17.1. Fix $p \geq 0$. The system of maps*

$$H^p(X, \mathcal{F})/I^n H^p(X, \mathcal{F}) \longrightarrow H^p(X, \mathcal{F}/I^n\mathcal{F})$$

define an isomorphism of limits

$$H^p(X, \mathcal{F})^\wedge \longrightarrow \lim_n H^p(X, \mathcal{F}/I^n\mathcal{F})$$

where the left hand side is the completion of the A-module $H^p(X, \mathcal{F})$ with respect to the ideal I, see Algebra, Section 10.91. Moreover, this is in fact a homeomorphism for the limit topologies.

Proof. In fact, this follows immediately from Lemma 51.17.4. We spell out the details. Set $M = H^p(X, \mathcal{F})$ and $M_n = H^p(X, \mathcal{F}/I^n\mathcal{F})$. Denote $N_n = \mathrm{Im}(M \to M_n)$. By the description of the limit in Homology, Section 12.26 we have

$$\lim_n M_n = \{(x_n) \in \prod M_n \mid \varphi_i(x_n) = x_{n-1},\ n = 2, 3, \ldots\}$$

Pick an element $x = (x_n) \in \lim_n M_n$. By Lemma 51.17.4 part (3) we have $x_n \in N_n$ for all n since by definition x_n is the image of some $x_{n+m} \in M_{n+m}$ for all m. By Lemma 51.17.4 part (1) we see that there exists a factorization

$$M \to N_n \to M/I^{n-c_1}M$$

of the reduction map. Denote $y_n \in M/I^{n-c_1}M$ the image of x_n for $n \geq c_1$. Since for $n' \geq n$ the composition $M \to M_{n'} \to M_n$ is the given map $M \to M_n$ we see that $y_{n'}$ maps to y_n under the canonical map $M/I^{n'-c_1}M \to M/I^{n-c_1}M$. Hence

$y = (y_{n+c_1})$ defines an element of $\lim_n M/I^nM$. We omit the verification that y maps to x under the map

$$M^\wedge = \lim_n M/I^nM \longrightarrow \lim_n M_n$$

of the lemma. We also omit the verification on topologies. □

Lemma 51.17.6. *Let A be a ring. Let $I \subset A$ be an ideal. Assume A is Noetherian and complete with respect to I. Let $f : X \to \mathrm{Spec}(A)$ be a proper morphism of algebraic spaces. Let $\mathcal{F}$ be a coherent sheaf on X. Then*

$$H^p(X, \mathcal{F}) = \lim_n H^p(X, \mathcal{F}/I^n\mathcal{F})$$

for all $p \geq 0$.

Proof. This is a reformulation of the theorem on formal functions (Theorem 51.17.5) in the case of a complete Noetherian base ring. Namely, in this case the A-module $H^p(X, \mathcal{F})$ is finite (Lemma 51.16.4) hence I-adically complete (Algebra, Lemma 10.91.2) and we see that completion on the left hand side is not necessary. □

Lemma 51.17.7. *Let S be a scheme. Let $f : X \to Y$ be a morphism of algebraic spaces over S and let $\mathcal{F}$ be a quasi-coherent sheaf on Y. Assume*

1. *Y locally Noetherian,*
2. *f proper, and*
3. *$\mathcal{F}$ coherent.*

Let $\overline{y}$ be a geometric point of Y. Consider the "infinitesimal neighbourhoods"

$$\begin{array}{ccc} X_n = \mathrm{Spec}(\mathcal{O}_{Y,\overline{y}}/\mathfrak{m}_{\overline{y}}^n) \times_Y X & \xrightarrow[i_n]{} & X \\ {\scriptstyle f_n}\downarrow & & \downarrow{\scriptstyle f} \\ \mathrm{Spec}(\mathcal{O}_{Y,\overline{y}}/\mathfrak{m}_{\overline{y}}^n) & \xrightarrow{c_n} & Y \end{array}$$

of the fibre X_1 and set $\mathcal{F}_n = i_n^*\mathcal{F}$. Then we have

$$(R^pf_*\mathcal{F})^\wedge_{\overline{y}} \cong \lim_n H^p(X_n, \mathcal{F}_n)$$

as $\mathcal{O}^\wedge_{Y,\overline{y}}$-modules.

Proof. This is just a reformulation of a special case of the theorem on formal functions, Theorem 51.17.5. Let us spell it out. Note that $\mathcal{O}_{Y,\overline{y}}$ is a Noetherian local ring, see Properties of Spaces, Lemma 48.22.4. Consider the canonical morphism $c : \mathrm{Spec}(\mathcal{O}_{Y,\overline{y}}) \to Y$. This is a flat morphism as it identifies local rings. Denote $f' : X' \to \mathrm{Spec}(\mathcal{O}_{Y,\overline{y}})$ the base change of f to this local ring. We see that $c^*R^pf_*\mathcal{F} = R^pf'_*\mathcal{F}'$ by Lemma 51.8.1. Moreover, we have canonical identifications $X_n = X'_n$ for all $n \geq 1$.

Hence we may assume that $Y = \mathrm{Spec}(A)$ is the spectrum of a strictly henselian Noetherian local ring A with maximal ideal $\mathfrak{m}$ and that $\overline{y} \to Y$ is equal to $\mathrm{Spec}(A/\mathfrak{m}) \to Y$. It follows that

$$(R^pf_*\mathcal{F})_{\overline{y}} = \Gamma(Y, R^pf_*\mathcal{F}) = H^p(X, \mathcal{F})$$

because $(Y, \overline{y})$ is an initial object in the category of étale neighbourhoods of $\overline{y}$. The morphisms c_n are each closed immersions. Hence their base changes i_n are closed

immersions as well. Note that $i_{n,*}\mathcal{F}_n = i_{n,*}i_n^*\mathcal{F} = \mathcal{F}/\mathfrak{m}^n\mathcal{F}$. By the Leray spectral sequence for i_n, and Lemma 51.9.9 we see that

$$H^p(X_n, \mathcal{F}_n) = H^p(X, i_{n,*}\mathcal{F}) = H^p(X, \mathcal{F}/\mathfrak{m}^n\mathcal{F})$$

Hence we may indeed apply the theorem on formal functions to compute the limit in the statement of the lemma and we win. □

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CHAPTER 52

Limits of Algebraic Spaces

52.1. Introduction

In this chapter we put material related to limits of algebraic spaces. A first topic is the characterization of algebraic spaces F locally of finite presentation over the base S as limit preserving functors. We continue with a study of limits of inverse systems over directed partially ordered sets with affine transition maps. We discuss absolute Noetherian approximation for quasi-compact and quasi-separated algebraic spaces following [**CLO09**]. Another approach is due to David Rydh (see [**Ryd08**]) whose results also cover absolute Noetherian approximation for certain algebraic stacks.

52.2. Conventions

The standing assumption is that all schemes are contained in a big fppf site Sch_{fppf}. And all rings A considered have the property that $\mathrm{Spec}(A)$ is (isomorphic) to an object of this big site.

Let S be a scheme and let X be an algebraic space over S. In this chapter and the following we will write $X \times_S X$ for the product of X with itself (in the category of algebraic spaces over S), instead of $X \times X$.

52.3. Morphisms of finite presentation

In this section we generalize Limits, Proposition 31.5.1 to morphisms of algebraic spaces. The motivation for the following definition comes from the proposition just cited.

Definition 52.3.1. Let S be a scheme.

(1) A functor $F : (Sch/S)^{opp}_{fppf} \to Sets$ is said to be *locally of finite presentation* or *limit preserving* if for every affine scheme T over S which is a limit $T = \lim T_i$ of a directed inverse system of affine schemes T_i over S, we have
$$F(T) = \operatorname{colim} F(T_i).$$
We sometimes say that F is *locally of finite presentation over* S.

(2) Let $F, G : (Sch/S)^{opp}_{fppf} \to Sets$. A transformation of functors $a : F \to G$ is *locally of finite presentation* if for every scheme T over S and every $y \in G(T)$ the functor
$$F_y : (Sch/T)^{opp}_{fppf} \longrightarrow Sets, \quad T'/T \longmapsto \{x \in F(T') \mid a(x) = y|_{T'}\}$$
is locally of finite presentation over T[1]. We sometimes say that F is *relatively limit preserving* over G.

[1]The characterization (2) in Lemma 52.3.2 may be easier to parse.

The functor F_y is in some sense the fiber of $a : F \to G$ over y, except that it is a presheaf on the big fppf site of T. A formula for this functor is:

$$F_y = F|_{(Sch/T)_{fppf}} \times_{G|_{(Sch/T)_{fppf}}} * \tag{52.3.1.1}$$

Here $*$ is the final object in the category of (pre)sheaves on $(Sch/T)_{fppf}$ (see Sites, Example 7.10.2) and the map $* \to G|_{(Sch/T)_{fppf}}$ is given by y. Note that if $j : (Sch/T)_{fppf} \to (Sch/S)_{fppf}$ is the localization functor, then the formula above becomes $F_y = j^{-1}F \times_{j^{-1}G} *$ and $j_!F_y$ is just the fiber product $F \times_{G,y} T$. (See Sites, Section 7.24, for information on localization, and especially Sites, Remark 7.24.9 for information on $j_!$ for presheaves.)

At this point we temporarily have two definitions of what it means for a morphism $X \to Y$ of algebraic spaces over S to be locally of finite presentation. Namely, one by Morphisms of Spaces, Definition 49.26.1 and one using that $X \to Y$ is a transformation of functors so that Definition 52.3.1 applies. We will show in Proposition 52.3.9 that these two definitions agree.

Lemma 52.3.2. *Let S be a scheme. Let $a : F \to G$ be a transformation of functors $(Sch/S)^{opp}_{fppf} \to Sets$. The following are equivalent*

1. *F is relatively limit preserving over G, and*
2. *for every every affine scheme T over S which is a limit $T = \lim T_i$ of a directed inverse system of affine schemes T_i over S the diagram of sets*

$$\begin{array}{ccc} \operatorname{colim}_i F(T_i) & \longrightarrow & F(T) \\ {\scriptstyle a}\downarrow & & \downarrow{\scriptstyle a} \\ \operatorname{colim}_i G(T_i) & \longrightarrow & G(T) \end{array}$$

is a fibre product diagram.

Proof. Assume (1). Consider $T = \lim_{i \in I} T_i$ as in (2). Let (y, x_T) be an element of the fibre product $\operatorname{colim}_i G(T_i) \times_{G(T)} F(T)$. Then y comes from $y_i \in G(T_i)$ for some i. Consider the functor F_{y_i} on $(Sch/T_i)_{fppf}$ as in Definition 52.3.1. We see that $x_T \in F_{y_i}(T)$. Moreover $T = \lim_{i' \geq i} T_{i'}$ is a directed system of affine schemes over T_i. Hence (1) implies that x_T the image of a unique element x of $\operatorname{colim}_{i' \geq i} F_{y_i}(T_{i'})$. Thus x is the unique element of $\operatorname{colim} F(T_i)$ which maps to the pair (y, x_T). This proves that (2) holds.

Assume (2). Let T be a scheme and $y_T \in G(T)$. We have to show that F_{y_T} is limit preserving. Let $T' = \lim_{i \in I} T'_i$ be an affine scheme over T which is the directed limit of affine scheme T'_i over T. Let $x_{T'} \in F_{y_T}$. Pick $i \in I$ which is possible as I is a directed partially ordered set. Denote $y_i \in F(T'_i)$ the image of $y_{T'}$. Then we see that $(y_i, x_{T'})$ is an element of the fibre product $\operatorname{colim}_i G(T'_i) \times_{G(T')} F(T')$. Hence by (2) we get a unique element x of $\operatorname{colim}_i F(T'_i)$ mapping to $(y_i, x_{T'})$. It is clear that x defines an element of $\operatorname{colim}_i F_y(T'_i)$ mapping to $x_{T'}$ and we win. □

Lemma 52.3.3. *Let S be a scheme contained in Sch_{fppf}. Let $F, G, H : (Sch/S)^{opp}_{fppf} \to Sets$. Let $a : F \to G$, $b : G \to H$ be transformations of functors. If a and b are locally of finite presentation, then*

$$b \circ a : F \longrightarrow H$$

is locally of finite presentation.

Proof. Let $T = \lim_{i \in I} T_i$ as in characterization (2) of Lemma 52.3.2. Consider the diagram

$$\begin{array}{ccc} \operatorname{colim}_i F(T_i) & \longrightarrow & F(T) \\ {\scriptstyle a}\downarrow & & \downarrow{\scriptstyle a} \\ \operatorname{colim}_i G(T_i) & \longrightarrow & G(T) \\ {\scriptstyle b}\downarrow & & \downarrow{\scriptstyle b} \\ \operatorname{colim}_i H(T_i) & \longrightarrow & H(T) \end{array}$$

By assumption the two squares are fibre product squares. Hence the outer rectangle is a fibre product diagram too which proves the lemma. □

Lemma 52.3.4. *Let S be a scheme contained in Sch_{fppf}. Let $F, G, H : (Sch/S)^{opp}_{fppf} \to Sets$. Let $a : F \to G$, $b : H \to G$ be transformations of functors. Consider the fibre product diagram*

$$\begin{array}{ccc} H \times_{b,G,a} F & \xrightarrow[b']{} & F \\ {\scriptstyle a'}\downarrow & & \downarrow{\scriptstyle a} \\ H & \xrightarrow{b} & G \end{array}$$

If a is locally of finite presentation, then the base change a' is locally of finite presentation.

Proof. Omitted. Hint: This is formal. □

Lemma 52.3.5. *Let T be an affine scheme which is written as a limit $T = \lim_{i \in I} T_i$ of a directed inverse system of affine schemes.*

1. *Let $\mathcal{V} = \{V_j \to T\}_{j=1,\ldots,m}$ be a standard fppf covering of T, see Topologies, Definition 33.7.5. Then there exists an index i and a standard fppf covering $\mathcal{V}_i = \{V_{i,j} \to T_i\}_{j=1,\ldots,m}$ whose base change $T \times_{T_i} \mathcal{V}_i$ to T is isomorphic to $\mathcal{V}$.*
2. *Let $\mathcal{V}_i$, $\mathcal{V}'_i$ be a pair of standard fppf coverings of T_i. If $f : T \times_{T_i} \mathcal{V} \to T \times_{T_i} \mathcal{V}'_i$ is a morphism of coverings of T, then there exists an index $i' \geq i$ and a morphism $f_{i'} : T_{i'} \times_{T_i} \mathcal{V} \to T_{i'} \times_{T_i} \mathcal{V}'_i$ whose base change to T is f.*
3. *If $f, g : \mathcal{V} \to \mathcal{V}'_i$ are morphisms of standard fppf coverings of T_i whose base changes f_T, g_T to T are equal then there exists an index $i' \geq i$ such that $f_{T_{i'}} = g_{T_{i'}}$.*

In other words, the category of standard fppf coverings of T is the colimit over I of the categories of standard fppf coverings of T_i

Proof. By Limits, Lemma 31.9.1 the category of schemes of finite presentation over T is the colimit over I of the categories of finite presentation over T_i. By Limits, Lemmas 31.7.2 and 31.7.6 the same is true for category of schemes which are affine, flat and of finite presentation over T. To finish the proof of the lemma it suffices to show that if $\{V_{j,i} \to T_i\}_{j=1,\ldots,m}$ is a finite family of flat finitely presented morphisms with $V_{j,i}$ affine, and the base change $\coprod_j T \times_{T_i} V_{j,i} \to T$ is surjective, then for some $i' \geq i$ the morphism $\coprod T_{i'} \times_{T_i} V_{j,i} \to T_{i'}$ is surjective. Denote $W_{i'} \subset T_{i'}$, resp. $W \subset T$ the image. Of course $W = T$ by assumption. Since the morphisms are flat and of finite presentation we see that W_i is a quasi-compact open of T_i, see Morphisms, Lemma 28.26.9. Moreover, $W = T \times_{T_i} W_i$ (formation of image

commutes with base change). Hence by Limits, Lemma 31.3.8 we conclude that $W_{i'} = T_{i'}$ for some large enough i' and we win. □

Lemma 52.3.6. *Let S be a scheme contained in Sch_{fppf}. Let $F : (Sch/S)^{opp}_{fppf} \to Sets$ be a functor. If F is locally of finite presentation over S then its sheafification $F^\#$ is locally of finite presentation over S.*

Proof. Assume F is locally of finite presentation. It suffices to show that F^+ is locally of finite presentation, since $F^\# = (F^+)^+$, see Sites, Theorem 7.10.10. Let T be an affine scheme over S, and let $T = \lim T_i$ be written as the directed limit of an inverse system of affine S schemes. Recall that $F^+(T)$ is the colimit of $\check{H}^0(\mathcal{V}, F)$ where the limit is over all coverings of T in $(Sch/S)_{fppf}$. Any fppf covering of an affine scheme can be refined by a standard fppf covering, see Topologies, Lemma 33.7.4. Hence we can write

$$F^+(T) = \text{colim}_{\mathcal{V} \text{ standard covering } T} \check{H}^0(\mathcal{V}, F).$$

By Lemma 52.3.5 we may rewrite this as

$$\text{colim}_{i \in I} \text{colim}_{\mathcal{V}_i \text{ standard covering } T_i} \check{H}^0(T \times_{T_i} \mathcal{V}_i, F).$$

(The order of the colimits is irrelevant by Categories, Lemma 4.14.9.) Given a standard fppf covering $\mathcal{V}_i = \{V_j \to T_i\}_{j=1,\ldots,m}$ of T_i we see that

$$T \times_{T_i} V_j = \lim_{i' \geq i} T_{i'} \times_T V_j$$

by Limits, Lemma 31.2.3, and similarly

$$T \times_{T_i} (V_j \times_{T_i} V_{j'}) = \lim_{i' \geq i} T_{i'} \times_T (V_j \times_{T_i} V_{j'}).$$

As the presheaf F is locally of finite presentation this means that

$$\check{H}^0(T \times_{T_i} \mathcal{V}_i, F) = \text{colim}_{i' \geq i} \check{H}^0(T_{i'} \times_{T_i} \mathcal{V}_i, F).$$

Hence the colimit expression for $F^+(T)$ above collapses to

$$\text{colim}_{i \in I} \text{colim}_{\mathcal{V} \text{ standard covering } T_i} \check{H}^0(\mathcal{V}, F). = \text{colim}_{i \in I} F^+(T_i).$$

In other words $F^+(T) = \text{colim}_i F^+(T_i)$ and hence the lemma holds. □

Lemma 52.3.7. *Let S be a scheme. Let $F : (Sch/S)^{opp}_{fppf} \to Sets$ be a functor. Assume that*

1. *F is a sheaf, and*
2. *there exists an fppf covering $\{U_j \to S\}_{j \in J}$ such that $F|_{(Sch/U_j)_{fppf}}$ is locally of finite presentation.*

Then F is locally of finite presentation.

Proof. Let T be an affine scheme over S. Let I be a directed partially ordered set, and let T_i be an inverse system of affine schemes over S such that $T = \lim T_i$. We have to show that the canonical map $\text{colim} F(T_i) \to F(T)$ is bijective.

Choose some $0 \in I$ and choose a standard fppf covering $\{V_{0,k} \to T_0\}_{k=1,\ldots,m}$ which refines the pullback $\{U_j \times_S T_0 \to T_0\}$ of the given fppf covering of S. For each $i \geq 0$ we set $V_{i,k} = T_i \times_{T_0} V_{0,k}$, and we set $V_k = T \times_{T_0} V_{0,k}$. Note that $V_k = \lim_{i \geq 0} V_{i,k}$, see Limits, Lemma 31.2.3.

Suppose that $x, x' \in \text{colim} F(T_i)$ map to the same element of $F(T)$. Say x, x' are given by elements $x_i, x'_i \in F(T_i)$ for some $i \in I$ (we may choose the same i for

both as I is directed). By assumption (2) and the fact that x_i, x_i' map to the same element of $F(T)$ this implies that

$$x_i|_{V_{i',k}} = x_i'|_{V_{i',k}}$$

for some suitably large $i' \in I$. We can choose the same i' for each k as $k \in \{1, \ldots, m\}$ ranges over a finite set. Since $\{V_{i',k} \to T_{i'}\}$ is an fppf covering and F is a sheaf this implies that $x_i|_{T_{i'}} = x_i'|_{T_{i'}}$ as desired. This proves that the map $\operatorname{colim} F(T_i) \to F(T)$ is injective.

To show surjectivity we argue in a similar fashion. Let $x \in F(T)$. By assumption (2) for each k we can choose a i such that $x|_{V_k}$ comes from an element $x_{i,k} \in F(V_{i,k})$. As before we may choose a single i which works for all k. By the injectivity proved above we see that

$$x_{i,k}|_{V_{i',k} \times_{T_{i'}} V_{i',l}} = x_{i,l}|_{V_{i',k} \times_{T_{i'}} V_{i',l}}$$

for some large enough i'. Hence by the sheaf condition of F the elements $x_{i,k}|_{V_{i',k}}$ glue to an element $x_{i'} \in F(T_{i'})$ as desired. □

Lemma 52.3.8. *Let S be a scheme contained in Sch_{fppf}. Let $F, G : (Sch/S)^{opp}_{fppf} \to Sets$ be functors. If $a : F \to G$ is a transformation which is locally of finite presentation, then the induced transformation of sheaves $F^\# \to G^\#$ is of finite presentation.*

Proof. Suppose that T is a scheme and $y \in G^\#(T)$. We have to show the functor $F^\#_y : (Sch/T)^{opp}_{fppf} \to Sets$ constructed from $F^\# \to G^\#$ and y as in Definition 52.3.1 is locally of finite presentation. By Equation (52.3.1.1) we see that $F^\#_y$ is a sheaf. Choose an fppf covering $\{V_j \to T\}_{j \in J}$ such that $y|_{V_j}$ comes from an element $y_j \in F(V_j)$. Note that the restriction of $F^\#$ to $(Sch/V_j)_{fppf}$ is just $F^\#_{y_j}$. If we can show that $F^\#_{y_j}$ is locally of finite presentation then Lemma 52.3.7 guarantees that $F^\#_y$ is locally of finite presentation and we win. This reduces us to the case $y \in G(T)$.

Let $y \in G(T)$. In this case we claim that $F^\#_y = (F_y)^\#$. This follows from Equation (52.3.1.1). Thus this case follows from Lemma 52.3.6. □

Proposition 52.3.9. *Let S be a scheme. Let $f : X \to Y$ be a morphism of algebraic spaces over S. The following are equivalent:*

1. *The morphism f is a morphism of algebraic spaces which is locally of finite presentation, see Morphisms of Spaces, Definition 49.26.1.*
2. *The morphism $f : X \to Y$ is locally of finite presentation as a transformation of functors, see Definition 52.3.1.*

Proof. Assume (1). Let T be a scheme and let $y \in Y(T)$. We have to show that $T \times_X Y$ is locally of finite presentation over T in the sense of Definition 52.3.1. Hence we are reduced to proving that if X is an algebraic space which is locally of finite presentation over S as an algebraic space, then it is locally of finite presentation as a functor $X : (Sch/S)^{opp}_{fppf} \to Sets$. To see this choose a presentation $X = U/R$, see Spaces, Definition 47.9.3. It follows from Morphisms of Spaces, Definition 49.26.1 that both U and R are schemes which are locally of finite presentation over S. Hence by Limits, Proposition 31.5.1 we have

$$U(T) = \operatorname{colim} U(T_i), \quad R(T) = \operatorname{colim} R(T_i)$$

whenever $T = \lim_i T_i$ in $(Sch/S)_{fppf}$. It follows that the presheaf

$$(Sch/S)^{opp}_{fppf} \longrightarrow Sets, \quad W \longmapsto U(W)/R(W)$$

is locally of finite presentation. Hence by Lemma 52.3.6 its sheafification $X = U/R$ is locally of finite presentation too.

Assume (2). Choose a scheme V and a surjective étale morphism $V \to Y$. Next, choose a scheme U and a surjective étale morphism $U \to V \times_Y X$. By Lemma 52.3.4 the transformation of functors $V \times_Y X \to V$ is locally of finite presentation. By Morphisms of Spaces, Lemma 49.35.8 the morphism of algebraic spaces $U \to V \times_Y X$ is locally of finite presentation, hence locally of finite presentation as a transformation of functors by the first part of the proof. By Lemma 52.3.3 the composition $U \to V \times_Y X \to V$ is locally of finite presentation as a transformation of functors. Hence the morphism of schemes $U \to V$ is locally of finite presentation by Limits, Proposition 31.5.1 (modulo a set theoretic remark, see last paragraph of the proof). This means, by definition, that (1) holds.

Set theoretic remark. Let $U \to V$ be a morphism of $(Sch/S)_{fppf}$. In the statement of Limits, Proposition 31.5.1 we characterize $U \to V$ as being locally of finite presentation if for *all* directed inverse systems $(T_i, f_{ii'})$ of affine schemes over V we have $U(T) = \operatorname{colim} V(T_i)$, but in the current setting we may only consider affine schemes T_i over V which are (isomorphic to) an object of $(Sch/S)_{fppf}$. So we have to make sure that there are enough affines in $(Sch/S)_{fppf}$ to make the proof work. Inspecting the proof of (2) $\Rightarrow$ (1) of Limits, Proposition 31.5.1 we see that the question reduces to the case that U and V are affine. Say $U = \operatorname{Spec}(A)$ and $V = \operatorname{Spec}(B)$. By construction of $(Sch/S)_{fppf}$ the spectrum of any ring of cardinality $\leq |B|$ is isomorphic to an object of $(Sch/S)_{fppf}$. Hence it suffices to observe that in the "only if" part of the proof of Algebra, Lemma 10.121.2 only A-algebras of cardinality $\leq |B|$ are used. □

Remark 52.3.10. Here is an important special case of Proposition 52.3.9. Let S be a scheme. Let X be an algebraic space over S. Then X is locally of finite presentation over S if and only if X, as a functor $(Sch/S)^{opp} \to Sets$, is limit preserving. Compare with Limits, Remark 31.5.2.

52.4. Limits of algebraic spaces

The following lemma explains how we think of limits of algebraic spaces in this chapter. We will use (without further mention) that the base change of an affine morphism of algebraic spaces is affine (see Morphisms of Spaces, Lemma 49.19.5).

Lemma 52.4.1. *Let S be a scheme. Let I be a directed partially ordered set. Let $(X_i, f_{ii'})$ be an inverse system over I in the category of algebraic spaces over S. If the morphisms $f_{ii'} : X_i \to X_{i'}$ are affine, then the limit $X = \lim_i X_i$ (as an fppf sheaf) is an algebraic space. Moreover,*

(1) *each of the morphisms $f_i : X \to X_i$ is affine,*
(2) *for any $i \in I$ and any morphism of algebraic spaces $T \to X_i$ we have*

$$X \times_{X_i} T = \lim_{i' \geq i} X_{i'} \times_{X_i} T.$$

as algebraic spaces over S.

Proof. Part (2) is a formal consequence of the existence of the limit $X = \lim X_i$ as an algebraic space over S. Choose an element $0 \in I$ (this is possible as a directed partially ordered set is nonempty). Choose a scheme U_0 and a surjective étale morphism $U_0 \to X_0$. Set $R_0 = U_0 \times_{X_0} U_0$ so that $X_0 = U_0/R_0$. For $i \geq 0$ set $U_i = X_i \times_{X_0} U_0$ and $R_i = X_i \times_{X_0} R_0 = U_i \times_{X_i} U_i$. By Limits, Lemma 31.2.2 we see that $U = \lim_{i \geq 0} U_i$ and $R = \lim_{i \geq 0} R_i$ are schemes. Moreover, the two morphisms $s, t : R \to U$ are the base change of the two projections $R_0 \to U_0$ by the morphism $U \to U_0$, in particular étale. The morphism $R \to U \times_S U$ defines an equivalence relation as directed a limit of equivalence relations is an equivalence relation. Hence the morphism $R \to U \times_S U$ is an étale equivalence relation. We claim that the natural map

$$U/R \longrightarrow \lim X_i \tag{52.4.1.1}$$

is an isomorphism of fppf sheaves on the category of schemes over S. The claim implies $X = \lim X_i$ is an algebraic space by Spaces, Theorem 47.10.5.

Let Z be a scheme and let $a : Z \to \lim X_i$ be a morphism. Then $a = (a_i)$ where $a_i : Z \to X_i$. Set $W_0 = Z \times_{a_0, X_0} U_0$. Note that $W_0 = Z \times_{a_i, X_i} U_i$ for all $i \geq 0$ by our choice of $U_i \to X_i$ above. Hence we obtain a morphism $W_0 \to \lim_{i \geq 0} U_i = U$. Since $W_0 \to Z$ is surjective and étale, we conclude that (52.4.1.1) is a surjective map of sheaves. Finally, suppose that Z is a scheme and that $a, b : Z \to U/R$ are two morphisms which are equalized by (52.4.1.1). We have to show that $a = b$. After replacing Z by the members of an fppf covering we may assume there exist morphisms $a', b' : Z \to U$ which give rise to a and b. The condition that a, b are equalized by (52.4.1.1) means that for each $i \geq 0$ the compositions $a'_i, b'_i : Z \to U \to U_i$ are equal as morphisms into $U_i/R_i = X_i$. Hence $(a'_i, b'_i) : Z \to U_i \times_S U_i$ factors through R_i, say by some morphism $c_i : Z \to R_i$. Since $R = \lim_{i \geq 0} R_i$ we see that $c = \lim c_i : Z \to R$ is a morphism which shows that a, b are equal as morphisms of Z into U/R.

Part (1) follows as we have seen above that $U_i \times_{X_i} X = U$ and $U \to U_i$ is affine by construction. □

Lemma 52.4.2. *Let S be a scheme. Let I be a directed partially ordered set. Let $(X_i, f_{ii'})$ be an inverse system over I of algebraic spaces over S with affine transition maps. Let $X = \lim_i X_i$. Let $0 \in I$. Suppose that $T \to X_0$ is a morphism of algebraic spaces. Then*

$$T \times_{X_0} X = \lim_{i \geq 0} T \times_{X_0} X_i$$

as algebraic spaces over S.

Proof. The limit X is an algebraic space by Lemma 52.4.1. The equality is formal, see Categories, Lemma 4.14.9. □

52.5. Descending properties

This section is the analogue of Limits, Section 31.3.

Situation 52.5.1. Let S be a scheme. Let $X = \lim_{i \in I} X_i$ be a limit of a directed system of algebraic spaces over S with affine transition morphisms (Lemma 52.4.1). We assume that X_i is quasi-compact and quasi-separated for all $i \in I$. We also choose an element $0 \in I$.

The following lemma holds a little bit more generally (namely when we just assume each X_i is a decent algebraic space).

Lemma 52.5.2. *In Situation 52.5.1 we have* $|X| = \lim |X_i|$.

Proof. There is a canonical map $|X| \to \lim |X_i|$. Choose an affine scheme U_0 and a surjective étale morphism $U_0 \to X_0$. Set $U_i = X_i \times_{X_0} U_0$ and $U = X \times_{X_0} U_0$. Set $R_i = U_i \times_{X_i} U_i$ and $R = U \times_X U$. Recall that $U = \lim U_i$ and $R = \lim R_i$, see proof of Lemma 52.4.1. Recall that $|X| = |U|/|R|$ and $|X_i| = |U_i|/|R_i|$. By Limits, Lemma 31.3.2 we have $|U| = \lim |U_i|$ and $|R| = \lim |R_i|$.

Surjectivity of $|X| \to \lim |X_i|$. Let $(x_i) \in \lim |X_i|$. Denote $S_i \subset |U_i|$ the inverse image of x_i. This is a finite nonempty set by Properties of Spaces, Lemma 48.12.3. Hence $\lim S_i$ is nonempty, see Categories, Lemma 4.21.5. Let $(u_i) \in \lim S_i \subset \lim |U_i|$. By the above this determines a point $u \in |U|$ which maps to an $x \in |X|$ mapping to the given element (x_i) of $\lim |X_i|$.

Injectivity of $|X| \to \lim |X_i|$. Suppose that $x, x' \in |X|$ map to the same point of $\lim |X_i|$. Choose lifts $u, u' \in |U|$ and denote $u_i, u_i' \in |U_i|$ the images. For each i let $T_i \subset |R_i|$ be the set of points mapping to $(u_i, u_i') \in |U_i| \times |U_i|$. This is a finite set by Properties of Spaces, Lemma 48.12.3 which is nonempty as we've assumed that x and x' map to the same point of X_i. Hence $\lim T_i$ is nonempty, see Categories, Lemma 4.21.5. As before let $r \in |R| = \lim |R_i|$ be a point corresponding to an element of $\lim T_i$. Then r maps to (u, u') in $|U| \times |U|$ by construction and we see that $x = x'$ in $|X|$ as desired. □

Lemma 52.5.3. *In Situation 52.5.1, if each X_i is nonempty, then $|X|$ is nonempty.*

Proof. Choose an affine scheme U_0 and a surjective étale morphism $U_0 \to X_0$. Set $U_i = X_i \times_{X_0} U_0$ and $U = X \times_{X_0} U_0$. Then each U_i is a nonempty affine scheme. Hence $U = \lim U_i$ is nonempty (Limits, Lemma 31.3.4) and thus X is nonempty. □

Lemma 52.5.4. *Notation and assumptions as in Situation 52.5.1. Suppose that $\mathcal{F}_0$ is a quasi-coherent sheaf on X_0. Set $\mathcal{F}_i = f_{0i}^*\mathcal{F}_0$ for $i \geq 0$ and set $\mathcal{F} = f_0^*\mathcal{F}_0$. Then*

$$\Gamma(X, \mathcal{F}) = \operatorname{colim}_{i \geq 0} \Gamma(X_i, \mathcal{F}_i)$$

Proof. Choose a surjective étale morphism $U_0 \to X_0$ where U_0 is an affine scheme (Properties of Spaces, Lemma 48.6.3). Set $U_i = X_i \times_{X_0} U_0$. Set $R_0 = U_0 \times_{X_0} U_0$ and $R_i = R_0 \times_{X_0} X_i$. In the proof of Lemma 52.4.1 we have seen that there exists a presentation $X = U/R$ with $U = \lim U_i$ and $R = \lim R_i$. Note that U_i and U are affine and that R_i and R are quasi-compact and separated (as X_i is quasi-separated). Hence Limits, Lemma 31.3.3 implies that

$$\mathcal{F}(U) = \operatorname{colim} \mathcal{F}_i(U_i) \quad \text{and} \quad \mathcal{F}(R) = \operatorname{colim} \mathcal{F}_i(R_i).$$

The lemma follows as $\Gamma(X, \mathcal{F}) = \operatorname{Ker}(\mathcal{F}(U) \to \mathcal{F}(R))$ and similarly $\Gamma(X_i, \mathcal{F}_i) = \operatorname{Ker}(\mathcal{F}_i(U_i) \to \mathcal{F}_i(R_i))$ □

Lemma 52.5.5. *Notation and assumptions as in Situation 52.5.1. For any quasi-compact open subspace $U \subset X$ there exists an i and a quasi-compact open $U_i \subset X_i$ whose inverse image in X is U.*

Proof. Follows formally from the construction of limits in Lemma 52.4.1 and the corresponding result for schemes: Limits, Lemma 31.3.8. □

The following lemma will be superseded by the stronger Lemma 52.6.9.

Lemma 52.5.6. *Notation and assumptions as in Situation 52.5.1. Let $f_0 : Y_0 \to Z_0$ be a morphism of algebraic spaces over X_0. Assume (a) $Y_0 \to X_0$ and $Z_0 \to X_0$ are representable, (b) Y_0, Z_0 quasi-compact and quasi-separated, (c) f_0 locally of finite presentation, and (d) $Y_0 \times_{X_0} X \to Z_0 \times_{X_0} X$ an isomorphism. Then there exists an $i \geq 0$ such that $Y_0 \times_{X_0} X_i \to Z_0 \times_{X_0} X_i$ is an isomorphism.*

Proof. Choose an affine scheme U_0 and a surjective étale morphism $U_0 \to X_0$. Set $U_i = U_0 \times_{X_0} X_i$ and $U = U_0 \times_{X_0} X$. Apply Limits, Lemma 31.7.9 to see that $Y_0 \times_{X_0} U_i \to Z_0 \times_{X_0} U_i$ is an isomorphism of schemes for some $i \geq 0$ (details omitted). As $U_i \to X_i$ is surjective étale, it follows that $Y_0 \times_{X_0} X_i \to Z_0 \times_{X_0} X_i$ is an isomorphism (details omitted). □

Lemma 52.5.7. *Notation and assumptions as in Situation 52.5.1. If X is separated, then X_i is separated for some $i \in I$.*

Proof. Choose an affine scheme U_0 and a surjective étale morphism $U_0 \to X_0$. For $i \geq 0$ set $U_i = U_0 \times_{X_0} X_i$ and set $U = U_0 \times_{X_0} X$. Note that U_i and U are affine schemes which come equipped with surjective étale morphisms $U_i \to X_i$ and $U \to X$. Set $R_i = U_i \times_{X_i} U_i$ and $R = U \times_X U$ with projections $s_i, t_i : R_i \to U_i$ and $s, t : R \to U$. Note that R_i and R are quasi-compact separated schemes (as the algebraic spaces X_i and X are quasi-separated). The maps $s_i : R_i \to U_i$ and $s : R \to U$ are of finite type. By definition X_i is separated if and only if $(t_i, s_i) : R_i \to U_i \times U_i$ is a closed immersion, and since X is separated by assumption, the morphism $(t, s) : R \to U \times U$ is a closed immersion. Since $R \to U$ is of finite type, there exists an i such that the morphism $R \to U_i \times U$ is a closed immersion (Limits, Lemma 31.3.13). Fix such an $i \in I$. Apply Limits, Lemma 31.7.4 to the system of morphisms $R_{i'} \to U_i \times U_{i'}$ for $i' \geq i$ (this is permissible as indeed $R_{i'} = R_i \times_{U_i \times U_i} U_i \times U_{i'}$) to see that $R_{i'} \to U_i \times U_{i'}$ is a closed immersion for i' sufficiently large. This implies immediately that $R_{i'} \to U_{i'} \times U_{i'}$ is a closed immersion finishing the proof of the lemma. □

Lemma 52.5.8. *Notation and assumptions as in Situation 52.5.1. If X is affine, then there exists an i such that X_i is affine.*

Proof. Choose $0 \in I$. Choose an affine scheme U_0 and a surjective étale morphism $U_0 \to X_0$. Set $U = U_0 \times_{X_0} X$ and $U_i = U_0 \times_{X_0} X_i$ for $i \geq 0$. Since the transition morphisms are affine, the algebraic spaces U_i and U are affine. Thus $U \to X$ is an étale morphism of affine schemes. Hence we can write $X = \mathrm{Spec}(A)$, $U = \mathrm{Spec}(B)$ and

$$B = A[x_1, \ldots, x_n]/(g_1, \ldots, g_n)$$

such that $\Delta = \det(\partial g_\lambda / \partial x_\mu)$ is invertible in B, see Algebra, Lemma 10.136.2. Set $A_i = \mathcal{O}_{X_i}(X_i)$. We have $A = \mathrm{colim}\, A_i$ by Lemma 52.5.4. After increasing 0 we may assume we have $g_{1,i}, \ldots, g_{n,i} \in A_i[x_1, \ldots, x_n]$ mapping to $g_1, \ldots, g_n$. Set

$$B_i = A_i[x_1, \ldots, x_n]/(g_{1,i}, \ldots, g_{n,i})$$

for all $i \geq 0$. Increasing 0 if necessary we may assume that $\Delta_i = \det(\partial g_{\lambda,i} / \partial x_\mu)$ is invertible in B_i for all $i \geq 0$. Thus $A_i \to B_i$ is an étale ring map. After increasing 0 we may assume also that $\mathrm{Spec}(B_i) \to \mathrm{Spec}(A_i)$ is surjective, see Limits, Lemma 31.7.11. Increasing 0 yet again we may choose elements $h_{1,i}, \ldots, h_{n,i} \in \mathcal{O}_{U_i}(U_i)$

which map to the classes of $x_1, \ldots, x_n$ in $B = \mathcal{O}_U(U)$ and such that $g_{\lambda,i}(h_{\nu,i}) = 0$ in $\mathcal{O}_{U_i}(U_i)$. Thus we obtain a commutative diagram

$$\begin{array}{ccc} X_i & \longleftarrow & U_i \\ \downarrow & & \downarrow \\ \mathrm{Spec}(A_i) & \longleftarrow & \mathrm{Spec}(B_i) \end{array} \tag{52.5.8.1}$$

By construction $B_i = B_0 \otimes_{A_0} A_i$ and $B = B_0 \otimes_{A_0} A$. Consider the morphism

$$f_0 : U_0 \longrightarrow X_0 \times_{\mathrm{Spec}(A_0)} \mathrm{Spec}(B_0)$$

This is a morphism of quasi-compact and quasi-separated algebraic spaces representable, separated and étale over X_0. The base change of f_0 to X is an isomorphism by our choices. Hence Lemma 52.5.6 guarantees that there exists an i such that the base change of f_0 to X_i is an isomorphism, in other words the diagram (52.5.8.1) is cartesian. Thus Descent, Lemma 34.35.1 applied to the fppf covering $\{\mathrm{Spec}(B_i) \to \mathrm{Spec}(A_i)\}$ combined with Descent, Lemma 34.33.1 give that $X_i \to \mathrm{Spec}(A_i)$ is representable by a scheme affine over $\mathrm{Spec}(A_i)$ as desired. (Of course it then also follows that $X_i = \mathrm{Spec}(A_i)$ but we don't need this.) □

Lemma 52.5.9. *Notation and assumptions as in Situation 52.5.1. If X is a scheme, then there exists an i such that X_i is a scheme.*

Proof. Choose a finite affine open covering $X = \bigcup W_j$. By Lemma 52.5.5 we can find an $i \in I$ and open subspaces $W_{j,i} \subset X_i$ whose base change to X is $W_j \to X$. By Lemma 52.5.8 we may assume that each $W_{j,i}$ is an affine scheme. This means that X_i is a scheme (see for example Properties of Spaces, Section 48.10). □

Lemma 52.5.10. *Let S be a scheme. Let B be an algebraic space over S. Let $X = \lim X_i$ be a directed limit of algebraic spaces over B with affine transition morphisms. Let $Y \to X$ be a morphism of algebraic spaces over B.*

1. *If $Y \to X$ is a closed immersion, X_i quasi-compact, and $Y \to B$ locally of finite type, then $Y \to X_i$ is a closed immersion for i large enough.*
2. *If $Y \to X$ is an immersion, X_i quasi-separated, $Y \to B$ locally of finite type, and Y quasi-compact, then $Y \to X_i$ is an immersion for i large enough.*
3. *If $Y \to X$ is an isomorphism, X_i quasi-compact, $X_i \to B$ locally of finite type, the transition morphisms $X_{i'} \to X_i$ are closed immersions, and $Y \to B$ is locally of finite presentation, then $Y \to X_i$ is an isomorphism for i large enough.*
4. *If $Y \to X$ is a monomorphism, X_i quasi-separated, $Y \to B$ locally of finite type, and Y quasi-compact, then $Y \to X_i$ is a monomorphism for i large enough.*

Proof. Proof of (1). Choose $0 \in I$. As X_0 is quasi-compact, we can choose an affine scheme W and an étale morphism $W \to B$ such that the image of $|X_0| \to |B|$ is contained in $|W| \to |B|$. Choose an affine scheme U_0 and an étale morphism $U_0 \to X_0 \times_B W$ such that $U_0 \to X_0$ is surjective. (This is possible by our choice of W and the fact that X_0 is quasi-compact; details omitted.) Let $V \to Y$, resp. $U \to X$, resp. $U_i \to X_i$ be the base change of $U_0 \to X_0$ (for $i \geq 0$). It suffices to prove that $V \to U_i$ is a closed immersion for i sufficiently large. Thus we reduce

to proving the result for $V \to U = \lim U_i$ over W. This follows from the case of schemes, which is Limits, Lemma 31.3.13.

Proof of (2). Choose $0 \in I$. Choose a quasi-compact open subspace $X_0' \subset X_0$ such that $Y \to X_0$ factors through X_0'. After replacing X_i by the inverse image of X_0' for $i \geq 0$ we may assume all X_i' are quasi-compact and quasi-separated. Let $U \subset X$ be a quasi-compact open such that $Y \to X$ factors through a closed immersion $Y \to U$ (U exists as Y is quasi-compact). By Lemma 52.5.5 we may assume that $U = \lim U_i$ with $U_i \subset X_i$ quasi-compact open. By part (1) we see that $Y \to U_i$ is a closed immersion for some i. Thus (2) holds.

Proof of (3). Choose $0 \in I$. Choose an affine scheme U_0 and a surjective étale morphism $U_0 \to X_0$. Set $U_i = X_i \times_{X_0} U_0$, $U = X \times_{X_0} U_0 = Y \times_{X_0} U_0$. Then $U = \lim U_i$ is a limit of affine schemes, the transition maps of the system are closed immersions, and $U \to U_0$ is of finite presentation (because $U \to B$ is locally of finite presentation and $U_0 \to B$ is locally of finite type and Morphisms of Spaces, Lemma 49.26.9). Thus we've reduced to the following algebra fact: If $A = \lim A_i$ is a directed colimit of R-algebras with surjective transition maps and A of finite presentation over A_0, then $A = A_i$ for some i. Namely, write $A = A_0/(f_1, \ldots, f_n)$. Pick i such that $f_1, \ldots, f_n$ map to zero under the surjective map $A_0 \to A_i$.

Proof of (4). Set $Z_i = Y \times_{X_i} Y$. As the transition morphisms $X_{i'} \to X_i$ are affine hence separated, the transition morphisms $Z_{i'} \to Z_i$ are closed immersions, see Morphisms of Spaces, Lemma 49.5.5. We have $\lim Z_i = Y \times_X Y = Y$ as $Y \to X$ is a monomorphism. Choose $0 \in I$. Since $Y \to X_0$ is locally of finite type (Morphisms of Spaces, Lemma 49.22.6) the morphism $Y \to Z_0$ is locally of finite presentation (Morphisms of Spaces, Lemma 49.26.10). The morphisms $Z_i \to Z_0$ are locally of finite type (they are closed immersions). Finally, $Z_i = Y \times_{X_i} Y$ is quasi-compact as X_i is quasi-separated and Y is quasi-compact. Thus part (3) applies to $Y = \lim_{i \geq 0} Z_i$ over Z_0 and we conclude $Y = Z_i$ for some i. This proves (4) and the lemma. □

Lemma 52.5.11. *Let S be a scheme. Let Y be an algebraic space over S. Let $X = \lim X_i$ be a directed limit of algebraic spaces over Y with affine transition morphisms. Assume*

(1) *Y is quasi-separated,*
(2) *X_i is quasi-compact and quasi-separated,*
(3) *the morphism $X \to Y$ is separated.*

Then $X_i \to Y$ is separated for all i large enough.

Proof. Let $0 \in I$. Choose an affine scheme W and an étale morphism $W \to Y$ such that the image of $|W| \to |Y|$ contains the image of $|X_0| \to |Y|$. This is possible as X_0 is quasi-compact. It suffices to check that $W \times_Y X_i \to W$ is separated for some $i \geq 0$ because the diagonal of $W \times_Y X_i$ over W is the base change of $X_i \to X_i \times_Y X_i$ by the surjective étale morphism $(X_i \times_Y X_i) \times_Y W \to X_i \times_Y X_i$. Since Y is quasi-separated the algebraic spaces $W \times_Y X_i$ are quasi-compact (as well as quasi-separated). Thus we may base change to W and assume Y is an affine scheme. When Y is an affine scheme, we have to show that X_i is a separated algebraic space for i large enough and we are given that X is a separated algebraic space. Thus this case follows from Lemma 52.5.7. □

52.6. Descending properties of morphisms

This section is the analogue of Section 52.5 for properties of morphisms. We will work in the following situation.

Situation 52.6.1. Let S be a scheme. Let $B = \lim B_i$ be a limit of a directed system of algebraic spaces over S with affine transition morphisms (Lemma 52.4.1). Let $0 \in I$ and let $f_0 : X_0 \to Y_0$ be a morphism of algebraic spaces over B_0. Assume B_0, X_0, Y_0 are quasi-compact and quasi-separated. Let $f_i : X_i \to Y_i$ be the base change of f_0 to B_i and let $f : X \to Y$ be the base change of f_0 to B.

Lemma 52.6.2. *With notation and assumptions as in Situation 52.6.1. If*

(1) *f is étale,*
(2) *f_0 is locally of finite presentation,*

then f_i is étale for some $i \geq 0$.

Proof. Choose an affine scheme V_0 and a surjective étale morphism $V_0 \to Y_0$. Choose an affine scheme U_0 and a surjective étale morphism $U_0 \to V_0 \times_{Y_0} X_0$. Diagram

$$\begin{array}{ccc} U_0 & \longrightarrow & V_0 \\ \downarrow & & \downarrow \\ X_0 & \longrightarrow & Y_0 \end{array}$$

The vertical arrows are surjective and étale by construction. We can base change this diagram to B_i or B to get

$$\begin{array}{ccc} U_i & \longrightarrow & V_i \\ \downarrow & & \downarrow \\ X_i & \longrightarrow & Y_i \end{array} \quad \text{and} \quad \begin{array}{ccc} U & \longrightarrow & V \\ \downarrow & & \downarrow \\ X & \longrightarrow & Y \end{array}$$

Note that U_i, V_i, U, V are affine schemes, the vertical morphisms are surjective étale, and the limit of the morphisms $U_i \to V_i$ is $U \to V$. Recall that $X_i \to Y_i$ is étale if and only if $U_i \to V_i$ is étale and similarly $X \to Y$ is étale if and only if $U \to V$ is étale (Morphisms of Spaces, Definition 49.35.1). Since f_0 is locally of finite presentation, so is the morphism $U_0 \to V_0$. Hence the lemma follows from Limits, Lemma 31.7.8. □

Lemma 52.6.3. *With notation and assumptions as in Situation 52.6.1. If*

(1) *f is surjective,*
(2) *f_0 is locally of finite presentation,*

then f_i is surjective for some $i \geq 0$.

Proof. Choose an affine scheme V_0 and a surjective étale morphism $V_0 \to Y_0$. Choose an affine scheme U_0 and a surjective étale morphism $U_0 \to V_0 \times_{Y_0} X_0$. Diagram

$$\begin{array}{ccc} U_0 & \longrightarrow & V_0 \\ \downarrow & & \downarrow \\ X_0 & \longrightarrow & Y_0 \end{array}$$

The vertical arrows are surjective and étale by construction. We can base change this diagram to B_i or B to get

$$\begin{array}{ccc} U_i & \longrightarrow & V_i \\ \downarrow & & \downarrow \\ X_i & \longrightarrow & Y_i \end{array} \quad \text{and} \quad \begin{array}{ccc} U & \longrightarrow & V \\ \downarrow & & \downarrow \\ X & \longrightarrow & Y \end{array}$$

Note that U_i, V_i, U, V are affine schemes, the vertical morphisms are surjective étale, the limit of the morphisms $U_i \to V_i$ is $U \to V$, and the morphisms $U_i \to X_i \times_{Y_i} V_i$ and $U \to X \times_Y V$ are surjective (as base changes of $U_0 \to X_0 \times_{Y_0} V_0$). In particular, we see that $X_i \to Y_i$ is surjective if and only if $U_i \to V_i$ is surjective and similarly $X \to Y$ is surjective if and only if $U \to V$ is surjective. Since f_0 is locally of finite presentation, so is the morphism $U_0 \to V_0$. Hence the lemma follows from the case of schemes (Limits, Lemma 31.7.11). □

Lemma 52.6.4. *Notation and assumptions as in Situation 52.6.1. If*

(1) *f is universally injective,*
(2) *f_0 is locally of finite type,*

then f_i is universally injective for some $i \geq 0$.

Proof. Recall that a morphism $X \to Y$ is universally injective if and only if the diagonal $X \to X \times_Y X$ is surjective (Morphisms of Spaces, Definition 49.18.3 and Lemma 49.18.2). Observe that $X_0 \to X_0 \times_{Y_0} X_0$ is of locally of finite presentation (Morphisms of Spaces, Lemma 49.26.10). Hence the lemma follows from Lemma 52.6.3 by considering the morphism $X_0 \to X_0 \times_{Y_0} X_0$. □

Lemma 52.6.5. *Notation and assumptions as in Situation 52.6.1. If f is affine, then f_i is affine for some $i \geq 0$.*

Proof. Choose an affine scheme V_0 and a surjective étale morphism $V_0 \to Y_0$. Set $V_i = V_0 \times_{Y_0} Y_i$ and $V = V_0 \times_{Y_0} Y$. Since f is affine we see that $V \times_Y X = \lim V_i \times_{Y_i} X_i$ is affine. By Lemma 52.5.8 we see that $V_i \times_{Y_i} X_i$ is affine for some $i \geq 0$. For this i the morphism f_i is affine (Morphisms of Spaces, Lemma 49.19.3). □

Lemma 52.6.6. *Notation and assumptions as in Situation 52.6.1. If*

(1) *f is finite,*
(2) *f_0 is locally of finite type,*

then f_i is finite for some $i \geq 0$.

Proof. Choose an affine scheme V_0 and a surjective étale morphism $V_0 \to Y_0$. Set $V_i = V_0 \times_{Y_0} Y_i$ and $V = V_0 \times_{Y_0} Y$. Since f is finite we see that $V \times_Y X = \lim V_i \times_{Y_i} X_i$ is a scheme finite over V. By Lemma 52.5.8 we see that $V_i \times_{Y_i} X_i$ is affine for some $i \geq 0$. Increasing i if necessary we find that $V_i \times_{Y_i} X_i \to V_i$ is finite by Limits, Lemma 31.7.3. For this i the morphism f_i is finite (Morphisms of Spaces, Lemma 49.40.3). □

Lemma 52.6.7. *Notation and assumptions as in Situation 52.6.1. If*

(1) *f is a closed immersion,*
(2) *f_0 is locally of finite type,*

then f_i is a closed immersion for some $i \geq 0$.

Proof. Choose an affine scheme V_0 and a surjective étale morphism $V_0 \to Y_0$. Set $V_i = V_0 \times_{Y_0} Y_i$ and $V = V_0 \times_{Y_0} Y$. Since f is a closed immersion we see that $V \times_Y X = \lim V_i \times_{Y_i} X_i$ is a closed subscheme of the affine scheme V. By Lemma 52.5.8 we see that $V_i \times_{Y_i} X_i$ is affine for some $i \geq 0$. Increasing i if necessary we find that $V_i \times_{Y_i} X_i \to V_i$ is a closed immersion by Limits, Lemma 31.7.4. For this i the morphism f_i is a closed immersion (Morphisms of Spaces, Lemma 49.40.3). □

Lemma 52.6.8. *Notation and assumptions as in Situation 52.6.1. If f is separated, then f_i is separated for some $i \geq 0$.*

Proof. Apply Lemma 52.6.7 to the diagonal morphism $\Delta_{X_0/Y_0} : X_0 \to X_0 \times_{Y_0} X_0$. (Diagonal morphisms are locally of finite type and the fibre product $X_0 \times_{Y_0} X_0$ is quasi-compact and quasi-separated. Some details omitted.) □

Lemma 52.6.9. *Notation and assumptions as in Situation 52.6.1. If*

(1) *f is a isomorphism,*
(2) *f_0 is locally of finite presentation,*

then f_i is a isomorphism for some $i \geq 0$.

Proof. Being an isomorphism is equivalent to being étale, universally injective, and surjective, see Morphisms of Spaces, Lemma 49.44.2. Thus the lemma follows from Lemmas 52.6.2, 52.6.3, and 52.6.4. □

Lemma 52.6.10. *Notation and assumptions as in Situation 52.6.1. If*

(1) *f is a monomorphism,*
(2) *f_0 is locally of finite type,*

then f_i is a monomorphism for some $i \geq 0$.

Proof. Recall that a morphism is a monomorphism if and only if the diagonal is an isomorphism. The morphism $X_0 \to X_0 \times_{Y_0} X_0$ is locally of finite presentation by Morphisms of Spaces, Lemma 49.26.10. Since $X_0 \times_{Y_0} X_0$ is quasi-compact and quasi-separated we conclude from Lemma 52.6.9 that $\Delta_i : X_i \to X_i \times_{Y_i} X_i$ is an isomorphism for some $i \geq 0$. For this i the morphism f_i is a monomorphism. □

Lemma 52.6.11. *Notation and assumptions as in Situation 52.6.1. Let $\mathcal{F}_0$ be a quasi-coherent $\mathcal{O}_{X_0}$-module and denote $\mathcal{F}_i$ the pullback to X_i and $\mathcal{F}$ the pullback to X. If*

(1) *$\mathcal{F}$ is flat over Y,*
(2) *$\mathcal{F}_0$ is of finite presentation, and*
(3) *f_0 is locally of finite presentation,*

then $\mathcal{F}_i$ is flat over Y_i for some $i \geq 0$. In particular, if f_0 is locally of finite presentation and f is flat, then f_i is flat for some $i \geq 0$.

Proof. Choose an affine scheme V_0 and a surjective étale morphism $V_0 \to Y_0$. Choose an affine scheme U_0 and a surjective étale morphism $U_0 \to V_0 \times_{Y_0} X_0$. Diagram

$$\begin{array}{ccc} U_0 & \longrightarrow & V_0 \\ \downarrow & & \downarrow \\ X_0 & \longrightarrow & Y_0 \end{array}$$

The vertical arrows are surjective and étale by construction. We can base change this diagram to B_i or B to get

$$\begin{array}{ccc} U_i & \longrightarrow & V_i \\ \downarrow & & \downarrow \\ X_i & \longrightarrow & Y_i \end{array} \quad \text{and} \quad \begin{array}{ccc} U & \longrightarrow & V \\ \downarrow & & \downarrow \\ X & \longrightarrow & Y \end{array}$$

Note that U_i, V_i, U, V are affine schemes, the vertical morphisms are surjective étale, and the limit of the morphisms $U_i \to V_i$ is $U \to V$. Recall that $\mathcal{F}_i$ is flat over Y_i if and only if $\mathcal{F}_i|_{U_i}$ is flat over V_i and similarly $\mathcal{F}$ is flat over Y if and only if $\mathcal{F}|_U$ is flat over V (Morphisms of Spaces, Definition 49.27.1). Since f_0 is locally of finite presentation, so is the morphism $U_0 \to V_0$. Hence the lemma follows from Limits, Lemma 31.9.3. □

Lemma 52.6.12. *Assumptions and notation as in Situation 52.6.1. If*

(1) *f is proper, and*
(2) *f_0 is locally of finite type,*

then there exists an i such that f_i is proper.

Proof. Choose an affine scheme V_0 and a surjective étale morphism $V_0 \to Y_0$. Set $V_i = Y_i \times_{Y_0} V_0$ and $V = Y \times_{Y_0} V_0$. It suffices to prove that the base change of f_i to V_i is proper, see Morphisms of Spaces, Lemma 49.36.2. Thus we may assume Y_0 is affine.

By Lemma 52.6.8 we see that f_i is separated for some $i \geq 0$. Replacing 0 by i we may assume that f_0 is separated. Observe that f_0 is quasi-compact. Thus f_0 is separated and of finite type. By Cohomology of Spaces, Lemma 51.15.1 we can choose a diagram

$$\begin{array}{ccccc} X_0 & \xleftarrow{\pi} & X_0' & \longrightarrow & \mathbf{P}^n_{Y_0} \\ & \searrow & \downarrow & \swarrow & \\ & & Y_0 & & \end{array}$$

where $X_0' \to \mathbf{P}^n_{Y_0}$ is an immersion, and $\pi : X_0' \to X_0$ is proper and surjective. Introduce $X' = X_0' \times_{Y_0} Y$ and $X_i' = X_0' \times_{Y_0} Y_i$. By Morphisms of Spaces, Lemmas 49.36.4 and 49.36.3 we see that $X' \to Y$ is proper. Hence $X' \to \mathbf{P}^n_Y$ is a closed immersion (Morphisms of Spaces, Lemma 49.36.6). By Morphisms of Spaces, Lemma 49.36.7 it suffices to prove that $X_i' \to Y_i$ is proper for some i. By Lemma 52.6.7 we find that $X_i' \to \mathbf{P}^n_{Y_i}$ is a closed immersion for i large enough. Then $X_i' \to Y_i$ is proper and we win. □

Recall the scheme theoretic support of a finite type quasi-coherent module, see Morphisms of Spaces, Definition 49.15.4.

Lemma 52.6.13. *Assumptions and notation as in Situation 52.6.1. Let $\mathcal{F}_0$ be a quasi-coherent $\mathcal{O}_{X_0}$-module. Denote $\mathcal{F}$ and $\mathcal{F}_i$ the pullbacks of $\mathcal{F}_0$ to X and X_i. Assume*

(1) *f_0 is locally of finite type,*
(2) *$\mathcal{F}_0$ is of finite type,*
(3) *the scheme theoretic support of $\mathcal{F}$ is proper over Y.*

Then the scheme theoretic support of $\mathcal{F}_i$ is proper over Y_i for some i.

Proof. We may replace X_0 by the scheme theoretic support of $\mathcal{F}_0$. By Morphisms of Spaces, Lemma 49.15.2 this guarantees that X_i is the support of $\mathcal{F}_i$ and X is the support of $\mathcal{F}$. Then, if $Z \subset X$ denotes the scheme theoretic support of $\mathcal{F}$, we see that $Z \to X$ is a universal homeomorphism. We conclude that $X \to Y$ is proper as this is true for $Z \to Y$ by assumption, see Morphisms, Lemma 28.42.8. By Lemma 52.6.12 we see that $X_i \to Y$ is proper for some i. Then it follows that the scheme theoretic support Z_i of $\mathcal{F}_i$ is proper over Y by Morphisms of Spaces, Lemmas 49.36.5 and 49.36.4. □

52.7. Descending relative objects

The following lemma is typical of the type of results in this section.

Lemma 52.7.1. *Let S be a scheme. Let I be a directed partially ordered set. Let $(X_i, f_{ii'})$ be an inverse system over I of algebraic spaces over S. Assume*

(1) *the morphisms $f_{ii'} : X_i \to X_{i'}$ are affine,*
(2) *the spaces X_i are quasi-compact and quasi-separated.*

Let $X = \lim_i X_i$. Then the category of algebraic spaces of finite presentation over X is the colimit over I of the categories of algebraic spaces of finite presentation over X_i.

Proof. Pick $0 \in I$. Choose a surjective étale morphism $U_0 \to X_0$ where U_0 is an affine scheme (Properties of Spaces, Lemma 48.6.3). Set $U_i = X_i \times_{X_0} U_0$. Set $R_0 = U_0 \times_{X_0} U_0$ and $R_i = R_0 \times_{X_0} X_i$. Denote $s_i, t_i : R_i \to U_i$ and $s, t : R \to U$ the two projections. In the proof of Lemma 52.4.1 we have seen that there exists a presentation $X = U/R$ with $U = \lim U_i$ and $R = \lim R_i$. Note that U_i and U are affine and that R_i and R are quasi-compact and separated (as X_i is quasi-separated). Let Y be an algebraic space over S and let $Y \to X$ be a morphism of finite presentation. Set $V = U \times_X Y$. This is an algebraic space of finite presentation over U. Choose an affine scheme W and a surjective étale morphism $W \to V$. Then $W \to Y$ is surjective étale as well. Set $R' = W \times_Y W$ so that $Y = W/R'$ (see Spaces, Section 47.9). Note that W is a scheme of finite presentation over U and that R' is a scheme of finite presentation over R (details omitted). By Limits, Lemma 31.9.1 we can find an index i and a morphism of schemes $W_i \to U_i$ of finite presentation whose base change to U gives $W \to U$. Similarly we can find, after possibly increasing i, a scheme R'_i of finite presentation over R_i whose base change to R is R'. The projection morphisms $s', t' : R' \to W$ are morphisms over the projection morphisms $s, t : R \to U$. Hence we can view s', resp. t' as a morphism between schemes of finite presentation over U (with structure morphism $R' \to U$ given by $R' \to R$ followed by s, resp. t). Hence we can apply Limits, Lemma 31.9.1 again to see that, after possibly increasing i, there exist morphisms $s'_i, t'_i : R'_i \to W_i$, whose base change to U is S', t'. By Limits, Lemmas 31.7.8 and 31.7.10 we may assume that s'_i, t'_i are étale and that $j'_i : R'_i \to W_i \times_{X_i} W_i$ is a monomorphism (here we view j'_i as a morphism of schemes of finite presentation over U_i via one of the projections – it doesn't matter which one). Setting $Y_i = W_i/R'_i$ (see Spaces, Theorem 47.10.5) we obtain an algebraic space of finite presentation over X_i whose base change to X is isomorphic to Y.

This shows that every algebraic space of finite presentation over X comes from an algebraic space of finite presentation over some X_i, i.e., it shows that the functor of the lemma is essentially surjective. To show that it is fully faithful, consider

an index $0 \in I$ and two algebraic spaces Y_0, Z_0 of finite presentation over X_0. Set $Y_i = X_i \times_{X_0} Y_0$, $Y = X \times_{X_0} Y_0$, $Z_i = X_i \times_{X_0} Z_0$, and $Z = X \times_{X_0} Z_0$. Let $\alpha : Y \to Z$ be a morphism of algebraic spaces over X. Choose a surjective étale morphism $V_0 \to Y_0$ where V_0 is an affine scheme. Set $V_i = V_0 \times_{Y_0} Y_i$ and $V = V_0 \times_{Y_0} Y$ which are affine schemes endowed with surjective étale morphisms to Y_i and Y. The composition $V \to Y \to Z \to Z_0$ comes from a (essentially unique) morphism $V_i \to Z_0$ for some $i \geq 0$ by Proposition 52.3.9 (applied to $Z_0 \to X_0$ which is of finite presentation by assumption). After increasing i the two compositions

$$V_i \times_{Y_i} V_i \to V_i \to Z_0$$

are equal as this is true in the limit. Hence we obtain a (essentially unique) morphism $Y_i \to Z_0$. Since this is a morphism over X_0 it induces a morphism into $Z_i = Z_0 \times_{X_0} X_i$ as desired. □

Lemma 52.7.2. *With notation and assumptions as in Lemma 52.7.1. The category of $\mathcal{O}_X$-modules of finite presentation is the colimit over I of the categories $\mathcal{O}_{X_i}$-modules of finite presentation.*

Proof. Choose $0 \in I$. Choose an affine scheme U_0 and a surjective étale morphism $U_0 \to X_0$. Set $U_i = X_i \times_{X_0} U_0$. Set $R_0 = U_0 \times_{X_0} U_0$ and $R_i = R_0 \times_{X_0} X_i$. Denote $s_i, t_i : R_i \to U_i$ and $s, t : R \to U$ the two projections. In the proof of Lemma 52.4.1 we have seen that there exists a presentation $X = U/R$ with $U = \lim U_i$ and $R = \lim R_i$. Note that U_i and U are affine and that R_i and R are quasi-compact and separated (as X_i is quasi-separated). Moreover, it is also true that $R \times_{s,U,t} R = \operatorname{colim} R_i \times_{s_i,U_i,t_i} R_i$. Thus we know that $QCoh(\mathcal{O}_U) = \operatorname{colim} QCoh(\mathcal{O}_{U_i})$, $QCoh(\mathcal{O}_R) = \operatorname{colim} QCoh(\mathcal{O}_{R_i})$, and $QCoh(\mathcal{O}_{R \times_{s,U,t} R}) = \operatorname{colim} QCoh(\mathcal{O}_{R_i \times_{s_i,U_i,t_i} R_i})$ by Limits, Lemma 31.9.2. We have $QCoh(\mathcal{O}_X) = QCoh(U, R, s, t, c)$ and $QCoh(\mathcal{O}_{X_i}) = QCoh(U_i, R_i, s_i, t_i, c_i)$, see Properties of Spaces, Proposition 48.30.1. Thus the result follows formally. □

52.8. Absolute Noetherian approximation

We use the following result which is almost identical to [**GR71**, Proposition 5.7.8].

Lemma 52.8.1. *Let X be a quasi-compact and quasi-separated algebraic space over* $\operatorname{Spec}(\mathbf{Z})$. *There exist an integer n and open subspaces*

$$\emptyset = U_{n+1} \subset U_n \subset U_{n-1} \subset \ldots \subset U_1 = X$$

with the following property: setting $T_p = U_p \setminus U_{p+1}$ (with reduced induced subspace structure) there exists a quasi-compact separated scheme V_p and a surjective étale morphism $f_p : V_p \to U_p$ such that $f_p^{-1}(T_p) \to T_p$ is an isomorphism.

Proof. The proof of this lemma is identical to the proof of Decent Spaces, Lemma 50.8.5. Observe that a quasi-separated space is reasonable, see Decent Spaces, Lemma 50.5.1 and Decent Spaces, Definition 50.6.1. At the end of the argument we add that since X is quasi-separated the schemes $V \times_X \ldots \times_X V$ are all quasi-compact. Hence the schemes W_p are quasi-compact. Hence the schemes $V_p = W_p/S_p$ are quasi-compact. □

Proposition 52.8.2. *Let X be a quasi-compact and quasi-separated algebraic space over* $\operatorname{Spec}(\mathbf{Z})$. *There exist a directed partially ordered set I and an inverse system of algebraic spaces $(X_i, f_{ii'})$ over I such that*

(1) *the transition morphisms $f_{ii'}$ are affine*
(2) *each X_i is quasi-separated and of finite type over $\mathbf{Z}$, and*
(3) *$X = \lim X_i$.*

Proof. Following [**CLO09**]. We apply Lemma 52.8.1 to get open subspaces $U_p \subset X$, schemes V_p, and morphisms $f_p : V_p \to U_p$ with properties as stated. Note that $f_n : V_n \to U_n$ is an étale morphism of algebraic spaces whose restriction to the inverse image of $T_n = (V_n)_{red}$ is an isomorphism. Hence f_n is an isomorphism, for example by Morphisms of Spaces, Lemma 49.44.2. In particular U_n is a quasi-compact and separated scheme. Thus we can write $U_n = \lim U_{n,i}$ as a directed limit of schemes of finite type over $\mathbf{Z}$ with affine transition morphisms, see Limits, Proposition 31.4.4. Thus, applying descending induction on p, we see that we have reduced to the problem posed in the following paragraph.

Here we have $U \subset X$, $U = \lim U_i$, $Z \subset X$, and $f : V \to X$ with the following properties

(1) X is a quasi-compact and quasi-separated algebraic space,
(2) V is a quasi-compact and separated scheme,
(3) $U \subset X$ is a quasi-compact open subspace,
(4) $(U_i, g_{ii'})$ is a directed system of quasi-separated algebraic spaces of finite type over $\mathbf{Z}$ with affine transition morphisms whose limit is U,
(5) $Z \subset X$ is a closed subspace such that $|X| = |U| \amalg |Z|$,
(6) $f : V \to X$ is a surjective étale morphism such that $f^{-1}(Z) \to Z$ is an isomorphism.

Problem: Show that the conclusion of the proposition holds for X.

Note that $W = f^{-1}(U) \subset V$ is a quasi-compact open subscheme étale over U. Hence we may apply Lemmas 52.7.1 and 52.6.2 to find an index $0 \in I$ and an étale morphism $W_0 \to U_0$ of finite presentation whose base change to U produces W. Setting $W_i = W_0 \times_{U_0} U_i$ we see that $W = \lim_{i \geq 0} W_i$. After increasing 0 we may assume the W_i are schemes, see Lemma 52.5.9. Moreover, W_i is of finite type over $\mathbf{Z}$.

Apply Limits, Lemma 31.4.3 to $W = \lim_{i \geq 0} W_i$ and the inclusion $W \subset V$. Replace I by the directed partially ordered set J found in that lemma. This allows us to write V as a directed limit $V = \lim V_i$ of finite type schemes over $\mathbf{Z}$ with affine transition maps such that each V_i contains W_i as an open subscheme (compatible with transition morphisms). For each i we can form the push out

$$\begin{array}{ccc} W_i & \longrightarrow & V_i \\ {\scriptstyle\Delta}\downarrow & & \downarrow \\ W_i \times_{U_i} W_i & \longrightarrow & R_i \end{array}$$

in the category of schemes. Namely, the left vertical and upper horizontal arrows are open immersions of schemes. In other words, we can construct R_i as the glueing of V_i and $W_i \times_{U_i} W_i$ along the common open W_i (see Schemes, Section 25.14). Note that the étale projection maps $W_i \times_{U_i} W_i \to W_i$ extend to étale morphisms $s_i, t_i : R_i \to V_i$. It is clear that the morphism $j_i = (t_i, s_i) : R_i \to V_i \times V_i$ is an étale equivalence relation on V_i. Note that $W_i \times_{U_i} W_i$ is quasi-compact (as

U_i is quasi-separated and W_i quasi-compact) and V_i is quasi-compact, hence R_i is quasi-compact. For $i \geq i'$ the diagram

$$\begin{array}{ccc} R_i & \longrightarrow & R_{i'} \\ {\scriptstyle s_i}\downarrow & & \downarrow{\scriptstyle s_{i'}} \\ V_i & \longrightarrow & V_{i'} \end{array} \tag{52.8.2.1}$$

is cartesian because

$$(W_{i'} \times_{U_{i'}} W_{i'}) \times_{U_{i'}} U_i = W_{i'} \times_{U_{i'}} U_i \times_{U_i} U_i \times_{U_{i'}} W_{i'} = W_i \times_{U_i} W_i.$$

Consider the algebraic space $X_i = V_i/R_i$ (see Spaces, Theorem 47.10.5). As V_i is of finite type over $\mathbf{Z}$ and R_i is quasi-compact we see that X_i is quasi-separated and of finite type over $\mathbf{Z}$ (see Properties of Spaces, Lemma 48.6.5 and Morphisms of Spaces, Lemmas 49.9.5 and 49.22.4). As the construction of R_i above is compatible with transition morphisms, we obtain morphisms of algebraic spaces $X_i \to X_{i'}$ for $i \geq i'$. The commutative diagrams

$$\begin{array}{ccc} V_i & \longrightarrow & V_{i'} \\ \downarrow & & \downarrow \\ X_i & \longrightarrow & X_{i'} \end{array}$$

are cartesian as (52.8.2.1) is cartesian, see Groupoids, Lemma 38.18.7. Since $V_i \to V_{i'}$ is affine, this implies that $X_i \to X_{i'}$ is affine, see Morphisms of Spaces, Lemma 49.19.3. Thus we can form the limit $X' = \lim X_i$ by Lemma 52.4.1. We claim that $X \cong X'$ which finishes the proof of the proposition.

Proof of the claim. Set $R = \lim R_i$. By construction the algebraic space X' comes equipped with a surjective étale morphism $V \to X'$ such that

$$V \times_{X'} V \cong R$$

(use Lemma 52.4.1). By construction $\lim W_i \times_{U_i} W_i = W \times_U W$ and $V = \lim V_i$ so that R is the union of $W \times_U W$ and V glued along W. Property (6) implies the projections $V \times_X V \to V$ are isomorphisms over $f^{-1}(Z) \subset V$. Hence the scheme $V \times_X V$ is the union of the opens $\Delta_{V/X}(V)$ and $W \times_U W$ which intersect along $\Delta_{W/X}(W)$. We conclude that there exists a unique isomorphism $R \cong V \times_X V$ compatible with the projections to V. Since $V \to X$ and $V \to X'$ are surjective étale we see that

$$X = V/V \times_X V = V/R = V/V \times_{X'} V = X'$$

by Spaces, Lemma 47.9.1 and we win. □

52.9. Applications

The following lemma can also be deduced directly from Lemma 52.8.1 without passing through absolute Noetherian approximation.

Lemma 52.9.1. *Let S be a scheme. Let X be a quasi-compact and quasi-separated algebraic space over S. Every quasi-coherent $\mathcal{O}_X$-module is a filtered colimit of finitely presented $\mathcal{O}_X$-modules.*

Proof. We may view as an algebraic space over $\mathrm{Spec}(\mathbf{Z})$, see Spaces, Definition 47.16.2 and Properties of Spaces, Definition 48.3.1. Thus we may apply Proposition 52.8.2 and write $X = \lim X_i$ with X_i of finite presentation over $\mathbf{Z}$. Thus X_i is a Noetherian algebraic space, see Morphisms of Spaces, Lemma 49.26.6. The morphism $X \to X_i$ is affine, see Lemma 52.4.1. Conclusion by Cohomology of Spaces, Lemma 51.12.2. □

The rest of this section consists of straightforward applications of Lemma 52.9.1.

Lemma 52.9.2. *Let S be a scheme. Let X be a quasi-compact and quasi-separated algebraic space over S. Let $\mathcal{F}$ be a quasi-coherent $\mathcal{O}_X$-module. Then $\mathcal{F}$ is the directed colimit of its finite type quasi-coherent submodules.*

Proof. If $\mathcal{G}, \mathcal{H} \subset \mathcal{F}$ are finite type quasi-coherent $\mathcal{O}_X$-submodules then the image of $\mathcal{G} \oplus \mathcal{H} \to \mathcal{F}$ is another finite type quasi-coherent $\mathcal{O}_X$-submodule which contains both of them. In this way we see that the system is directed. To show that $\mathcal{F}$ is the colimit of this system, write $\mathcal{F} = \mathrm{colim}_i \mathcal{F}_i$ as a directed colimit of finitely presented quasi-coherent sheaves as in Lemma 52.9.1. Then the images $\mathcal{G}_i = \mathrm{Im}(\mathcal{F}_i \to \mathcal{F})$ are finite type quasi-coherent subsheaves of $\mathcal{F}$. Since $\mathcal{F}$ is the colimit of these the result follows. □

Lemma 52.9.3. *Let S be a scheme. Let X be a quasi-compact and quasi-separated algebraic space over S. Let $\mathcal{F}$ be a finite type quasi-coherent $\mathcal{O}_X$-module. Then we can write $\mathcal{F} = \lim \mathcal{F}_i$ where each $\mathcal{F}_i$ is an $\mathcal{O}_X$-module of finite presentation and all transition maps $\mathcal{F}_i \to \mathcal{F}_{i'}$ surjective.*

Proof. Write $\mathcal{F} = \mathrm{colim}\, \mathcal{G}_i$ as a filtered colimit of finitely presented $\mathcal{O}_X$-modules (Lemma 52.9.1). We claim that $\mathcal{G}_i \to \mathcal{F}$ is surjective for some i. Namely, choose an étale surjection $U \to X$ where U is an affine scheme. Choose finitely many sections $s_k \in \mathcal{F}(U)$ generating $\mathcal{F}|_U$. Since U is affine we see that s_k is in the image of $\mathcal{G}_i \to \mathcal{F}$ for i large enough. Hence $\mathcal{G}_i \to \mathcal{F}$ is surjective for i large enough. Choose such an i and let $\mathcal{K} \subset \mathcal{G}_i$ be the kernel of the map $\mathcal{G}_i \to \mathcal{F}$. Write $\mathcal{K} = \mathrm{colim}\, \mathcal{K}_a$ as the filtered colimit of its finite type quasi-coherent submodules (Lemma 52.9.2). Then $\mathcal{F} = \mathrm{colim}\, \mathcal{G}_i/\mathcal{K}_a$ is a solution to the problem posed by the lemma. □

Let X be an algebraic space. In the following lemma we use the notion of a *finitely presented quasi-coherent $\mathcal{O}_X$-algebra $\mathcal{A}$*. This means that for every affine $U = \mathrm{Spec}(R)$ étale over X we have $\mathcal{A}|_U = \widetilde{A}$ where A is a (commutative) R-algebra which is of finite presentation as an R-algebra.

Lemma 52.9.4. *Let S be a scheme. Let X be a quasi-compact and quasi-separated algebraic space over S. Let $\mathcal{A}$ be a quasi-coherent $\mathcal{O}_X$-algebra. Then $\mathcal{A}$ is a directed colimit of finitely presented quasi-coherent $\mathcal{O}_X$-algebras.*

Proof. First we write $\mathcal{A} = \mathrm{colim}_i \mathcal{F}_i$ as a directed colimit of finitely presented quasi-coherent sheaves as in Lemma 52.9.1. For each i let $\mathcal{B}_i = \mathrm{Sym}(\mathcal{F}_i)$ be the symmetric algebra on $\mathcal{F}_i$ over $\mathcal{O}_X$. Write $\mathcal{I}_i = \mathrm{Ker}(\mathcal{B}_i \to \mathcal{A})$. Write $\mathcal{I}_i = \mathrm{colim}_j \mathcal{F}_{i,j}$ where $\mathcal{F}_{i,j}$ is a finite type quasi-coherent submodule of $\mathcal{I}_i$, see Lemma 52.9.2. Set $\mathcal{I}_{i,j} \subset \mathcal{I}_i$ equal to the $\mathcal{B}_i$-ideal generated by $\mathcal{F}_{i,j}$. Set $\mathcal{A}_{i,j} = \mathcal{B}_i/\mathcal{I}_{i,j}$. Then $\mathcal{A}_{i,j}$ is a quasi-coherent finitely presented $\mathcal{O}_X$-algebra. Define $(i,j) \leq (i',j')$ if $i \leq i'$ and the map $\mathcal{B}_i \to \mathcal{B}_{i'}$ maps the ideal $\mathcal{I}_{i,j}$ into the ideal $\mathcal{I}_{i',j'}$. Then it is clear that $\mathcal{A} = \mathrm{colim}_{i,j} \mathcal{A}_{i,j}$. □

Let X be an algebraic space. In the following lemma we use the notion of a *quasi-coherent $\mathcal{O}_X$-algebra $\mathcal{A}$ of finite type*. This means that for every affine $U = \mathrm{Spec}(R)$ étale over X we have $\mathcal{A}|_U = \widetilde{A}$ where A is a (commutative) R-algebra which is of finite type as an R-algebra.

Lemma 52.9.5. *Let S be a scheme. Let X be a quasi-compact and quasi-separated algebraic space over S. Let $\mathcal{A}$ be a quasi-coherent $\mathcal{O}_X$-algebra. Then $\mathcal{A}$ is the directed colimit of its finite type quasi-coherent $\mathcal{O}_X$-subalgebras.*

Proof. Omitted. Hint: Compare with the proof of Lemma 52.9.2. □

Let X be an algebraic space. In the following lemma we use the notion of a *finite (resp. integral) quasi-coherent $\mathcal{O}_X$-algebra $\mathcal{A}$*. This means that for every affine $U = \mathrm{Spec}(R)$ étale over X we have $\mathcal{A}|_U = \widetilde{A}$ where A is a (commutative) R-algebra which is finite (resp. integral) as an R-algebra.

Lemma 52.9.6. *Let S be a scheme. Let X be a quasi-compact and quasi-separated algebraic space over S. Let $\mathcal{A}$ be a finite quasi-coherent $\mathcal{O}_X$-algebra. Then $\mathcal{A} = \mathrm{colim}\, \mathcal{A}_i$ is a directed colimit of finite and finitely presented quasi-coherent $\mathcal{O}_X$-algebras with surjective transition maps.*

Proof. By Lemma 52.9.3 there exists a finitely presented $\mathcal{O}_X$-module $\mathcal{F}$ and a surjection $\mathcal{F} \to \mathcal{A}$. Using the algebra structure we obtain a surjection

$$\mathrm{Sym}^*_{\mathcal{O}_X}(\mathcal{F}) \longrightarrow \mathcal{A}$$

Denote $\mathcal{J}$ the kernel. Write $\mathcal{J} = \mathrm{colim}\, \mathcal{E}_i$ as a filtered colimit of finite type $\mathcal{O}_X$-submodules $\mathcal{E}_i$ (Lemma 52.9.2). Set

$$\mathcal{A}_i = \mathrm{Sym}^*_{\mathcal{O}_X}(\mathcal{F})/(\mathcal{E}_i)$$

where $(\mathcal{E}_i)$ indicates the ideal sheaf generated by the image of $\mathcal{E}_i \to \mathrm{Sym}^*_{\mathcal{O}_X}(\mathcal{F})$. Then each $\mathcal{A}_i$ is a finitely presented $\mathcal{O}_X$-algebra, the transition maps are surjective, and $\mathcal{A} = \mathrm{colim}\, \mathcal{A}_i$. To finish the proof we still have to show that $\mathcal{A}_i$ is a finite $\mathcal{O}_X$-algebra for i sufficiently large. To do this we choose an étale surjective map $U \to X$ where U is an affine scheme. Take generators $f_1, \ldots, f_m \in \Gamma(U, \mathcal{F})$. As $\mathcal{A}(U)$ is a finite $\mathcal{O}_X(U)$-algebra we see that for each j there exists a monic polynomial $P_j \in \mathcal{O}(U)[T]$ such that $P_j(f_j)$ is zero in $\mathcal{A}(U)$. Since $\mathcal{A} = \mathrm{colim}\, \mathcal{A}_i$ by construction, we have $P_j(f_j) = 0$ in $\mathcal{A}_i(U)$ for all sufficiently large i. For such i the algebras $\mathcal{A}_i$ are finite. □

Lemma 52.9.7. *Let S be a scheme. Let X be a quasi-compact and quasi-separated algebraic space over S. Let $\mathcal{A}$ be an integral quasi-coherent $\mathcal{O}_X$-algebra. Then*

1. *$\mathcal{A}$ is the directed colimit of its finite quasi-coherent $\mathcal{O}_X$-subalgebras, and*
2. *$\mathcal{A}$ is a directed colimit of finite and finitely presented $\mathcal{O}_X$-algebras.*

Proof. By Lemma 52.9.5 we have $\mathcal{A} = \mathrm{colim}\, \mathcal{A}_i$ where $\mathcal{A}_i \subset \mathcal{A}$ runs through the quasi-coherent $\mathcal{O}_X$-sub algebras of finite type. Any finite type quasi-coherent $\mathcal{O}_X$-subalgebra of $\mathcal{A}$ is finite (use Algebra, Lemma 10.33.5 on affine schemes étale over X). This proves (1).

To prove (2), write $\mathcal{A} = \mathrm{colim}\, \mathcal{F}_i$ as a colimit of finitely presented $\mathcal{O}_X$-modules using Lemma 52.9.1. For each i, let $\mathcal{J}_i$ be the kernel of the map

$$\mathrm{Sym}^*_{\mathcal{O}_X}(\mathcal{F}_i) \longrightarrow \mathcal{A}$$

For $i' \geq i$ there is an induced map $\mathcal{J}_i \to \mathcal{J}_{i'}$ and we have $\mathcal{A} = \operatorname{colim} \operatorname{Sym}^*_{\mathcal{O}_X}(\mathcal{F}_i)/\mathcal{J}_i$. Moreover, the quasi-coherent $\mathcal{O}_X$-algebras $\operatorname{Sym}^*_{\mathcal{O}_X}(\mathcal{F}_i)/\mathcal{J}_i$ are finite (see above). Write $\mathcal{J}_i = \operatorname{colim} \mathcal{E}_{ik}$ as a colimit of finitely presented $\mathcal{O}_X$-modules. Given $i' \geq i$ and k there exists a k' such that we have a map $\mathcal{E}_{ik} \to \mathcal{E}_{i'k'}$ making

$$\begin{array}{ccc} \mathcal{J}_i & \longrightarrow & \mathcal{J}_{i'} \\ \uparrow & & \uparrow \\ \mathcal{E}_{ik} & \longrightarrow & \mathcal{E}_{i'k'} \end{array}$$

commute. This follows from Cohomology of Spaces, Lemma 51.4.3. This induces a map

$$\mathcal{A}_{ik} = \operatorname{Sym}^*_{\mathcal{O}_X}(\mathcal{F}_i)/(\mathcal{E}_{ik}) \longrightarrow \operatorname{Sym}^*_{\mathcal{O}_X}(\mathcal{F}_{i'})/(\mathcal{E}_{i'k'}) = \mathcal{A}_{i'k'}$$

where $(\mathcal{E}_{ik})$ denotes the ideal generated by $\mathcal{E}_{ik}$. The quasi-coherent $\mathcal{O}_X$-algebras $\mathcal{A}_{ki}$ are of finite presentation and finite for k large enough (see proof of Lemma 52.9.6). Finally, we have

$$\operatorname{colim} \mathcal{A}_{ik} = \operatorname{colim} \mathcal{A}_i = \mathcal{A}$$

Namely, the first equality was shown in the proof of Lemma 52.9.6 and the second equality because $\mathcal{A}$ is the colimit of the modules $\mathcal{F}_i$. □

Lemma 52.9.8. *Let S be a scheme. Let X be a quasi-compact and quasi-separated algebraic space over S. Let $U \subset X$ be a quasi-compact open. Let $\mathcal{F}$ be a quasi-coherent $\mathcal{O}_X$-module. Let $\mathcal{G} \subset \mathcal{F}|_U$ be a quasi-coherent $\mathcal{O}_U$-submodule which is of finite type. Then there exists a quasi-coherent submodule $\mathcal{G}' \subset \mathcal{F}$ which is of finite type such that $\mathcal{G}'|_U = \mathcal{G}$.*

Proof. Denote $j : U \to X$ the inclusion morphism. As X is quasi-separated and U quasi-compact, the morphism j is quasi-compact. Hence $j_*\mathcal{G} \subset j_*\mathcal{F}|_U$ are quasi-coherent modules on X (Morphisms of Spaces, Lemma 49.12.2). Let $\mathcal{H} = \operatorname{Ker}(j_*\mathcal{G} \oplus \mathcal{F} \to j_*\mathcal{F}|_U)$. Then $\mathcal{H}|_U = \mathcal{G}$. By Lemma 52.9.2 we can find a finite type quasi-coherent submodule $\mathcal{H}' \subset \mathcal{H}$ such that $\mathcal{H}'|_U = \mathcal{H}|_U = \mathcal{G}$. Set $\mathcal{G}' = \operatorname{Im}(\mathcal{H}' \to \mathcal{F})$ to conclude. □

52.10. Relative approximation

The title of this section refers to the following result.

Lemma 52.10.1. *Let S be a scheme. Let $f : X \to Y$ be a morphism of algebraic spaces over S. Assume that*

1. *X is quasi-compact and quasi-separated, and*
2. *Y is quasi-separated.*

Then $X = \lim X_i$ is a limit of a directed system of algebraic spaces X_i of finite presentation over Y with affine transition morphisms over Y.

Proof. Since $|f|(|X|)$ is quasi-compact we may replace Y by a quasi-compact open subspace whose set of points contains $|f|(|X|)$. Hence we may assume Y is quasi-compact as well. Write $X = \lim X_a$ and $Y = \lim Y_b$ as in Proposition 52.8.2, i.e., with X_a and Y_b of finite type over $\mathbf{Z}$ and with affine transition morphisms.

By Proposition 52.3.9 we find that for each b there exists an a and a morphism $f_{a,b} : X_a \to Y_b$ making the diagram

$$\begin{array}{ccc} X & \longrightarrow & Y \\ \downarrow & & \downarrow \\ X_a & \longrightarrow & Y_b \end{array}$$

commute. Moreover the same proposition implies that, given a second triple $(a', b', f_{a',b'})$, there exists an $a'' \geq a'$ such that the compositions $X_{a''} \to X_a \to X_b$ and $X_{a''} \to X_{a'} \to X_{b'} \to X_b$ are equal. Consider the set of triples $(a, b, f_{a,b})$ endowed with the partial ordering

$$(a, b, f_{a,b}) \geq (a', b', f_{a',b'}) \Leftrightarrow a \geq a',\ b' \geq b, \text{ and } f_{a',b'} \circ h_{a,a'} = g_{b',b} \circ f_{a,b}$$

where $h_{a,a'} : X_a \to X_{a'}$ and $g_{b',b} : Y_{b'} \to Y_b$ are the transition morphisms. The remarks above show that this system is directed. It follows formally from the equalities $X = \lim X_a$ and $Y = \lim Y_b$ that

$$X = \lim_{(a,b,f_{a,b})} X_a \times_{f_{a,b}, Y_b} Y.$$

where the limit is over our directed system above. The transition morphisms $X_a \times_{Y_b} Y \to X_{a'} \times_{Y_{b'}} Y$ are affine as the composition

$$X_a \times_{Y_b} Y \to X_a \times_{Y_{b'}} Y \to X_{a'} \times_{Y_{b'}} Y$$

where the first morphism is a closed immersion (by Morphisms of Spaces, Lemma 49.5.5) and the second is a base change of an affine morphism (Morphisms of Spaces, Lemma 49.19.5) and the composition of affine morphisms is affine (Morphisms of Spaces, Lemma 49.19.4). The morphisms $f_{a,b}$ are of finite presentation (Morphisms of Spaces, Lemmas 49.26.7 and 49.26.9) and hence the base changes $X_a \times_{f_{a,b}, S_b} S \to S$ are of finite presentation (Morphisms of Spaces, Lemma 49.26.3). □

52.11. Finite type closed in finite presentation

This section is the analogue of Limits, Section 31.8.

Lemma 52.11.1. *Let S be a scheme. Let $f : X \to Y$ be an affine morphism of algebraic spaces over S. If Y quasi-compact and quasi-separated, then X is a directed limit $X = \lim X_i$ with each X_i affine and of finite presentation over Y.*

Proof. Consider the quasi-coherent $\mathcal{O}_Y$-module $\mathcal{A} = f_*\mathcal{O}_X$. By Lemma 52.9.4 we can write $\mathcal{A} = \operatorname{colim} \mathcal{A}_i$ as a directed colimit of finitely presented $\mathcal{O}_Y$-algebras $\mathcal{A}_i$. Set $X_i = \underline{\operatorname{Spec}}_Y(\mathcal{A}_i)$, see Morphisms of Spaces, Definition 49.19.8. By construction $X_i \to Y$ is affine and of finite presentation and $X = \lim X_i$. □

Lemma 52.11.2. *Let S be a scheme. Let $f : X \to Y$ be an integral morphism of algebraic spaces over S. Assume Y quasi-compact and quasi-separated. Then X can be written as a directed limit $X = \lim X_i$ where X_i are finite and of finite presentation over Y.*

Proof. Consider the finite quasi-coherent $\mathcal{O}_Y$-module $\mathcal{A} = f_*\mathcal{O}_X$. By Lemma 52.9.7 we can write $\mathcal{A} = \operatorname{colim} \mathcal{A}_i$ as a directed colimit of finite and finitely presented $\mathcal{O}_Y$-algebras $\mathcal{A}_i$. Set $X_i = \underline{\operatorname{Spec}}_Y(\mathcal{A}_i)$, see Morphisms of Spaces, Definition 49.19.8. By construction $X_i \to Y$ is finite and of finite presentation and $X = \lim X_i$. □

Lemma 52.11.3. *Let S be a scheme. Let $f : X \to Y$ be a finite morphism of algebraic spaces over S. Assume Y quasi-compact and quasi-separated. Then X can be written as a directed limit $X = \lim X_i$ where the transition maps are closed immersions and the objects X_i are finite and of finite presentation over Y.*

Proof. Consider the finite quasi-coherent $\mathcal{O}_Y$-module $\mathcal{A} = f_*\mathcal{O}_X$. By Lemma 52.9.6 we can write $\mathcal{A} = \operatorname{colim} \mathcal{A}_i$ as a directed colimit of finite and finitely presented $\mathcal{O}_Y$-algebras $\mathcal{A}_i$ with surjective transition maps. Set $X_i = \underline{\operatorname{Spec}}_Y(\mathcal{A}_i)$, see Morphisms of Spaces, Definition 49.19.8. By construction $X_i \to Y$ is finite and of finite presentation, the transition maps are closed immersions, and $X = \lim X_i$. □

Lemma 52.11.4. *Let S be a scheme. Let $f : X \to Y$ be a morphism of algebraic spaces over S. Assume*

(1) *f is locally of finite type and quasi-affine, and*
(2) *Y is quasi-compact and quasi-separated.*

Then there exists a morphism of finite presentation $f' : X' \to Y$ and a closed immersion $X \to X'$ over Y.

Proof. By Morphisms of Spaces, Lemma 49.20.6 we can find a factorization $X \to Z \to Y$ where $X \to Z$ is a quasi-compact open immersion and $Z \to Y$ is affine. Write $Z = \lim Z_i$ with Z_i affine and of finite presentation over Y (Lemma 52.11.1). For some $0 \in I$ we can find a quasi-compact open $U_0 \subset Z_0$ such that X is isomorphic to the inverse image of U_0 in Z (Lemma 52.5.5). Let U_i be the inverse image of U_0 in Z_i, so $U = \lim U_i$. By Lemma 52.5.10 we see that $X \to U_i$ is a closed immersion for some i large enough. Setting $X' = U_i$ finishes the proof. □

Lemma 52.11.5. *Let S be a scheme. Let $f : X \to Y$ be a morphism of algebraic spaces over S. Assume:*

(1) *f is of locally of finite type.*
(2) *X is quasi-compact and quasi-separated, and*
(3) *Y is quasi-compact and quasi-separated.*

Then there exists a morphism of finite presentation $f' : X' \to Y$ and a closed immersion $X \to X'$ of algebraic spaces over Y.

Proof. By Proposition 52.8.2 we can write $X = \lim_i X_i$ with X_i quasi-separated of finite type over $\mathbf{Z}$ and with transition morphisms $f_{ii'} : X_i \to X_{i'}$ affine. Consider the commutative diagram

$$\begin{array}{ccccc} X & \longrightarrow & X_{i,Y} & \longrightarrow & X_i \\ & \searrow & \downarrow & & \downarrow \\ & & Y & \longrightarrow & \operatorname{Spec}(\mathbf{Z}) \end{array}$$

Note that X_i is of finite presentation over $\operatorname{Spec}(\mathbf{Z})$, see Morphisms of Spaces, Lemma 49.26.7. Hence the base change $X_{i,Y} \to Y$ is of finite presentation by Morphisms of Spaces, Lemma 49.26.3. Observe that $\lim X_{i,Y} = X \times Y$ and that $X \to X \times Y$ is a monomorphism. By Lemma 52.5.10 we see that $X \to X_{i,Y}$ is a monomorphism for i large enough. Fix such an i. Note that $X \to X_{i,Y}$ is locally of finite type (Morphisms of Spaces, Lemma 49.22.6) and a monomorphism, hence separated and locally quasi-finite (Morphisms of Spaces, Lemma 49.25.8). Hence $X \to X_{i,Y}$ is representable. Hence $X \to X_{i,Y}$ is quasi-affine because we can use the principle

Spaces, Lemma 47.5.8 and the result for morphisms of schemes More on Morphisms, Lemma 36.31.2. Thus Lemma 52.11.4 gives a factorization $X \to X' \to X_{i,Y}$ with $X \to X'$ a closed immersion and $X' \to X_{i,Y}$ of finite presentation. Finally, $X' \to Y$ is of finite presentation as a composition of morphisms of finite presentation (Morphisms of Spaces, Lemma 49.26.2). □

Proposition 52.11.6. *Let S be a scheme. $f : X \to Y$ be a morphism of algebraic spaces over S. Assume*

(1) *f is of finite type and separated, and*
(2) *Y is quasi-compact and quasi-separated.*

Then there exists a separated morphism of finite presentation $f' : X' \to Y$ and a closed immersion $X \to X'$ over Y.

Proof. By Lemma 52.11.5 there is a closed immersion $X \to Z$ with Z/Y of finite presentation. Let $\mathcal{I} \subset \mathcal{O}_Z$ be the quasi-coherent sheaf of ideals defining X as a closed subscheme of Y. By Lemma 52.9.2 we can write $\mathcal{I}$ as a directed colimit $\mathcal{I} = \operatorname{colim}_{a \in A} \mathcal{I}_a$ of its quasi-coherent sheaves of ideals of finite type. Let $X_a \subset Z$ be the closed subspace defined by $\mathcal{I}_a$. These form an inverse system indexed by A. The transition morphisms $X_a \to X_{a'}$ are affine because they are closed immersions. Each X_a is quasi-compact and quasi-separated since it is a closed subspace of Z and Z is quasi-compact and quasi-separated by our assumptions. We have $X = \lim_a X_a$ as follows directly from the fact that $\mathcal{I} = \operatorname{colim}_{a \in A} \mathcal{I}_a$. Each of the morphisms $X_a \to Z$ is of finite presentation, see Morphisms, Lemma 28.22.7. Hence the morphisms $X_a \to Y$ are of finite presentation. Thus it suffices to show that $X_a \to Y$ is separated for some $a \in A$. This follows from Lemma 52.5.11 as we have assumed that $X \to Y$ is separated. □

52.12. Embedding into affine space

Some technical lemmas to be used in the proof of Chow's lemma later.

Lemma 52.12.1. *Let S be a scheme. Let $f : U \to X$ be a morphism of algebraic spaces over S. Assume U is an affine scheme, f is locally of finite type, and X quasi-separated and locally separated. Then there exists an immersion $U \to \mathbf{A}^n_X$ over X.*

Proof. Say $U = \operatorname{Spec}(A)$. Write $A = \operatorname{colim} A_i$ as a filtered colimit of finite type $\mathbf{Z}$-subalgebras. For each i the morphism $U \to U_i = \operatorname{Spec}(A_i)$ induces a morphism

$$U \longrightarrow X \times U_i$$

over X. In the limit the morphism $U \to X \times U$ is an immersion as X is locally separated, see Morphisms of Spaces, Lemma 49.5.6. By Lemma 52.5.10 we see that $U \to X \times U_i$ is an immersion for some i. Since U_i is isomorphic to a closed subscheme of $\mathbf{A}^n_{\mathbf{Z}}$ the lemma follows. □

Remark 52.12.2. We have seen in Examples, Section 78.16 that Lemma 52.12.1 does not hold if we drop the assumption that X be locally separated. This raises the question: Does Lemma 52.12.1 hold if we drop the assumption that X be quasi-separated? If you know the answer, please email stacks.project@gmail.com.

Lemma 52.12.3. *Let S be a scheme. Let $f : Y \to X$ be a morphism of algebraic spaces over S. Assume X Noetherian and f of finite presentation. Then there exists a dense open $V \subset Y$ and an immersion $V \to \mathbf{A}^n_X$.*

Proof. The assumptions imply that Y is Noetherian (Morphisms of Spaces, Lemma 49.26.6). Then Y is quasi-separated, hence has a dense open subscheme (Properties of Spaces, Proposition 48.10.3). Thus we may assume that Y is a Noetherian scheme. By removing intersections of irreducible components of Y (use Topology, Lemma 5.8.2 and Properties, Lemma 27.5.5) we may assume that Y is a disjoint union of irreducible Noetherian schemes. Since there is an immersion

$$\mathbf{A}^n_X \amalg \mathbf{A}^m_X \longrightarrow \mathbf{A}^{\max(n,m)+1}_X$$

(details omitted) we see that it suffices to prove the result in case Y is irreducible.

Assume Y is an irreducible scheme. Let $T \subset |X|$ be the closure of the image of $f : Y \to X$. Note that since $|Y|$ and $|X|$ are sober topological spaces (Properties of Spaces, Lemma 48.12.4) T is irreducible with a unique generic point ξ which is the image of the generic point η of Y. Let $\mathcal{I} \subset X$ be a quasi-coherent sheaf of ideals cutting out the reduced induced space structure on T (Properties of Spaces, Definition 48.9.4). Since $\mathcal{O}_{Y,\eta}$ is an Artinian local ring we see that for some $n > 0$ we have $f^{-1}\mathcal{I}^n\mathcal{O}_{Y,\eta} = 0$. As $f^{-1}\mathcal{I}\mathcal{O}_Y$ is a finite type quasi-coherent ideal we conclude that $f^{-1}\mathcal{I}^n\mathcal{O}_V = 0$ for some nonempty open $V \subset Y$. Let $Z \subset X$ be the closed subspace cut out by $\mathcal{I}^n$. By construction $V \to Y \to X$ factors through Z. Because $\mathbf{A}^n_Z \to \mathbf{A}^n_X$ is an immersion, we may replace X by Z and Y by V. Hence we reach the situation where Y and X are irreducible and $Y \to X$ maps the generic point of Y onto the generic point of X.

Assume Y and X are irreducible, Y is a scheme, and $Y \to X$ maps the generic point of Y onto the generic point of X. By Properties of Spaces, Proposition 48.10.3 X has a dense open subscheme $U \subset X$. Choose a nonempty affine open $V \subset Y$ whose image in X is contained in U. By Morphisms, Lemma 28.40.2 we may factor $V \to U$ as $V \to \mathbf{A}^n_U \to U$. Composing with $\mathbf{A}^n_U \to \mathbf{A}^n_X$ we obtain the desired immersion. □

52.13. Sections with support in a closed subset

This section is the analogue of Properties, Section 27.22.

Lemma 52.13.1. *Let S be a scheme. Let X be a quasi-compact and quasi-separated algebraic space. Let $U \subset X$ be an open subspace. The following are equivalent:*

1. *$U \to X$ is quasi-compact,*
2. *U is quasi-compact, and*
3. *there exists a finite type quasi-coherent sheaf of ideals $\mathcal{I} \subset \mathcal{O}_X$ such that $|X| \setminus |U| = |V(\mathcal{I})|$.*

Proof. Let W be an affine scheme and let $\varphi : W \to X$ be a surjective étale morphism, see Properties of Spaces, Lemma 48.6.3. If (1) holds, then $\varphi^{-1}(U) \to W$ is quasi-compact, hence $\varphi^{-1}(U)$ is quasi-compact, hence U is quasi-compact (as $|\varphi^{-1}(U)| \to |U|$ is surjective). If (2) holds, then $\varphi^{-1}(U)$ is quasi-compact because φ is quasi-compact since X is quasi-separated (Morphisms of Spaces, Lemma 49.9.9). Hence $\varphi^{-1}(U) \to W$ is a quasi-compact morphism of schemes by Properties, Lemma 27.22.1. It follows that $U \to X$ is quasi-compact by Morphisms of Spaces, Lemma 49.9.7. Thus (1) and (2) are equivalent.

Assume (1) and (2). By Properties of Spaces, Lemma 48.9.2 there exists a unique quasi-coherent sheaf of ideals $\mathcal{J}$ cutting out the reduced induced closed subspace

structure on $|X| \setminus |U|$. Note that $\mathcal{J}|_U = \mathcal{O}_U$ which is an $\mathcal{O}_U$-modules of finite type. As U is quasi-compact it follows from Lemma 52.9.2 that there exists a quasi-coherent subsheaf $\mathcal{I} \subset \mathcal{J}$ which is of finite type and has the property that $\mathcal{I}|_U = \mathcal{J}|_U$. Then $|X| \setminus |U| = |V(\mathcal{I})|$ and we obtain (3). Conversely, if $\mathcal{I}$ is as in (3), then $\varphi^{-1}(U) \subset W$ is a quasi-compact open by the lemma for schemes (Properties, Lemma 27.22.1) applied to $\varphi^{-1}\mathcal{I}$ on W. Thus (2) holds. □

Lemma 52.13.2. *Let S be a scheme. Let X be an algebraic space over S. Let $\mathcal{I} \subset \mathcal{O}_X$ be a quasi-coherent sheaf of ideals. Let $\mathcal{F}$ be a quasi-coherent $\mathcal{O}_X$-module. Consider the sheaf of $\mathcal{O}_X$-modules $\mathcal{F}'$ which associates to every object U of $X_{étale}$ the module*

$$\mathcal{F}'(U) = \{s \in \mathcal{F}(U) \mid \mathcal{I}s = 0\}$$

Assume $\mathcal{I}$ is of finite type. Then

1. *$\mathcal{F}'$ is a quasi-coherent sheaf of $\mathcal{O}_X$-modules,*
2. *for affine U in $X_{étale}$ we have $\mathcal{F}'(U) = \{s \in \mathcal{F}(U) \mid \mathcal{I}(U)s = 0\}$, and*
3. *$\mathcal{F}'_x = \{s \in \mathcal{F}_x \mid \mathcal{I}_x s = 0\}$.*

Proof. It is clear that the rule defining $\mathcal{F}'$ gives a subsheaf of $\mathcal{F}$. Hence we may work étale locally on X to verify the other statements. Thus the lemma reduces to the case of schemes which is Properties, Lemma 27.22.2. □

Definition 52.13.3. Let S be a scheme. Let X be an algebraic space over S. Let $\mathcal{I} \subset \mathcal{O}_X$ be a quasi-coherent sheaf of ideals of finite type. Let $\mathcal{F}$ be a quasi-coherent $\mathcal{O}_X$-module. The subsheaf $\mathcal{F}' \subset \mathcal{F}$ defined in Lemma 52.13.2 above is called the *subsheaf of sections annihilated by $\mathcal{I}$*.

Lemma 52.13.4. *Let S be a scheme. Let $f : X \to Y$ be a quasi-compact and quasi-separated morphism of algebraic spaces over S. Let $\mathcal{I} \subset \mathcal{O}_Y$ be a quasi-coherent sheaf of ideals of finite type. Let $\mathcal{F}$ be a quasi-coherent $\mathcal{O}_X$-module. Let $\mathcal{F}' \subset \mathcal{F}$ be the subsheaf of sections annihilated by $f^{-1}\mathcal{I}\mathcal{O}_X$. Then $f_*\mathcal{F}' \subset f_*\mathcal{F}$ is the subsheaf of sections annihilated by $\mathcal{I}$.*

Proof. Omitted. Hint: The assumption that f is quasi-compact and quasi-separated implies that $f_*\mathcal{F}$ is quasi-coherent (Morphisms of Spaces, Lemma 49.12.2) so that Lemma 52.13.2 applies to $\mathcal{I}$ and $f_*\mathcal{F}$. □

Next we come to the sheaf of sections supported in a closed subset. Again this isn't always a quasi-coherent sheaf, but if the complement of the closed is "retrocompact" in the given algebraic space, then it is.

Lemma 52.13.5. *Let S be a scheme. Let X be an algebraic space over S. Let $T \subset |X|$ be a closed subset and let $U \subset X$ be the open subspace such that $T \amalg |U| = |X|$. Let $\mathcal{F}$ be a quasi-coherent $\mathcal{O}_X$-module. Consider the sheaf of $\mathcal{O}_X$-modules $\mathcal{F}'$ which associates to every object $\varphi : W \to X$ of $X_{étale}$ the module*

$$\mathcal{F}'(W) = \{s \in \mathcal{F}(W) \mid \text{the support of } s \text{ is contained in } |\varphi|^{-1}(T)\}$$

If $U \to X$ is quasi-compact, then

1. *for W affine there exist a finitely generated ideal $I \subset \mathcal{O}_X(W)$ such that $|\varphi|^{-1}(T) = V(I)$,*
2. *for W and I as in (1) we have $\mathcal{F}'(W) = \{x \in \mathcal{F}(W) \mid I^n x = 0 \text{ for some } n\}$,*
3. *$\mathcal{F}'$ is a quasi-coherent sheaf of $\mathcal{O}_X$-modules.*

Proof. It is clear that the rule defining $\mathcal{F}'$ gives a subsheaf of $\mathcal{F}$. Hence we may work étale locally on X to verify the other statements. Thus the lemma reduces to the case of schemes which is Properties, Lemma 27.22.5. □

Definition 52.13.6. Let S be a scheme. Let X be an algebraic space over S. Let $T \subset |X|$ be a closed subset whose complement corresponds to an open subspace $U \subset X$ with quasi-compact inclusion morphism $U \to X$. Let $\mathcal{F}$ be a quasi-coherent $\mathcal{O}_X$-module. The quasi-coherent subsheaf $\mathcal{F}' \subset \mathcal{F}$ defined in Lemma 52.13.5 above is called the *subsheaf of sections supported on* T.

Lemma 52.13.7. *Let S be a scheme. Let $f : X \to Y$ be a quasi-compact and quasi-separated morphism of algebraic spaces over S. Let $T \subset |Y|$ be a closed subset. Assume $|Y| \setminus T$ corresponds to an open subspace $V \subset Y$ such that $V \to Y$ is quasi-compact. Let $\mathcal{F}$ be a quasi-coherent $\mathcal{O}_X$-module. Let $\mathcal{F}' \subset \mathcal{F}$ be the subsheaf of sections supported on $|f|^{-1}T$. Then $f_*\mathcal{F}' \subset f_*\mathcal{F}$ is the subsheaf of sections supported on T.*

Proof. Omitted. Hints: $|X| \setminus |f|^{-1}T$ is the support of the open subspace $U = f^{-1}V \subset X$. Since $V \to Y$ is quasi-compact, so is $U \to X$ (by base change). The assumption that f is quasi-compact and quasi-separated implies that $f_*\mathcal{F}$ is quasi-coherent. Hence Lemma 52.13.5 applies to T and $f_*\mathcal{F}$ as well as to $|f|^{-1}T$ and $\mathcal{F}$. The equality of the given quasi-coherent modules is immediate from the definitions. □

52.14. Characterizing affine spaces

This section is the analogue of Limits, Section 31.10.

Lemma 52.14.1. *Let S be a scheme. Let $f : X \to Y$ be a morphism of algebraic spaces over S. Assume that f is surjective and finite, and assume that X is affine. Then Y is affine.*

Proof. We may and do view $f : X \to Y$ as a morphism of algebraic space over $\mathrm{Spec}(\mathbf{Z})$ (see Spaces, Definition 47.16.2). Note that a finite morphism is affine and universally closed, see Morphisms of Spaces, Lemma 49.40.7. By Morphisms of Spaces, Lemma 49.10.8 we see that Y is a separated algebraic space. As f is surjective and X is quasi-compact we see that Y is quasi-compact.

By Lemma 52.11.3 we can write $X = \lim X_a$ with each $X_a \to Y$ finite and of finite presentation. By Lemma 52.5.8 we see that X_a is affine for a large enough. Hence we may and do assume that $f : X \to Y$ is finite, surjective, and of finite presentation.

By Proposition 52.8.2 we may write $Y = \lim Y_i$ as a directed limit of algebraic spaces of finite presentation over $\mathbf{Z}$. By Lemma 52.7.1 we can find $0 \in I$ and a morphism $X_0 \to Y_0$ of finite presentation such that $X_i = X_0 \times_{Y_0} Y_i$ for $i \geq 0$ and such that $X = \lim_i X_i$. By Lemma 52.6.6 we see that $X_i \to Y_i$ is finite for i large enough. By Lemma 52.6.3 we see that $X_i \to Y_i$ is surjective for i large enough. By Lemma 52.5.8 we see that X_i is affine for i large enough. Hence for i large enough we can apply Cohomology of Spaces, Lemma 51.14.1 to conclude that Y_i is affine. This implies that Y is affine and we conclude. □

Proposition 52.14.2. *Let S be a scheme. Let $f : X \to Y$ be a morphism of algebraic spaces over S. Assume that f is surjective and integral, and assume that X is affine. Then Y is affine.*

Proof. We may and do view $f : X \to Y$ as a morphism of algebraic space over $\mathrm{Spec}(\mathbf{Z})$ (see Spaces, Definition 47.16.2). Note that integral morphisms are affine and universally closed, see Morphisms of Spaces, Lemma 49.40.7. By Morphisms of Spaces, Lemma 49.10.8 we see that Y is a separated algebraic space. As f is surjective and X is quasi-compact we see that Y is quasi-compact.

Consider the sheaf $\mathcal{A} = f_*\mathcal{O}_X$. This is a quasi-coherent sheaf of $\mathcal{O}_Y$-algebras, see Morphisms of Spaces, Lemma 49.12.2. By Lemma 52.9.1 we can write $\mathcal{A} = \mathrm{colim}_i \mathcal{F}_i$ as a filtered colimit of finite type $\mathcal{O}_Y$-modules. Let $\mathcal{A}_i \subset \mathcal{A}$ be the $\mathcal{O}_Y$-subalgebra generated by $\mathcal{F}_i$. Since the map of algebras $\mathcal{O}_Y \to \mathcal{A}$ is integral, we see that each $\mathcal{A}_i$ is a finite quasi-coherent $\mathcal{O}_Y$-algebra. Hence

$$X_i = \underline{\mathrm{Spec}}_Y(\mathcal{A}_i) \longrightarrow Y$$

is a finite morphism of algebraic spaces. (Insert future reference to $\underline{\mathrm{Spec}}$ construction for algebraic spaces here.) It is clear that $X = \lim_i X_i$. Hence by Lemma 52.5.8 we see that for i sufficiently large the scheme X_i is affine. Moreover, since $X \to Y$ factors through each X_i we see that $X_i \to Y$ is surjective. Hence we conclude that Y is affine by Lemma 52.14.1. □

The following corollary of the result above can be found in [**CLO09**].

Lemma 52.14.3. *Let S be a scheme. Let X be an algebraic space over S. If X_{red} is a scheme, then X is a scheme.*

Proof. Let $U' \subset X_{red}$ be an open affine subscheme. Let $U \subset X$ be the open subspace corresponding to the open $|U'| \subset |X_{red}| = |X|$. Then $U' \to U$ is surjective and integral. Hence U is affine by Proposition 52.14.2. Thus every point is contained in an open subscheme of X, i.e., X is a scheme. □

Lemma 52.14.4. *Let S be a scheme. Let $f : X \to Y$ be a morphism of algebraic spaces over S. Assume f is integral and induces a bijection $|X| \to |Y|$. Then X is a scheme if and only if Y is a scheme.*

Proof. An integral morphism is representable by definition, hence if Y is a scheme, so is X. Conversely, assume that X is a scheme. Let $U \subset X$ be an affine open. An integral morphism is closed and $|f|$ is bijective, hence $|f|(|U|) \subset |Y|$ is open as the complement of $|f|(|X| \setminus |U|)$. Let $V \subset Y$ be the open subspace with $|V| = |f|(|U|)$, see Properties of Spaces, Lemma 48.4.8. Then $U \to V$ is integral and surjective, hence V is an affine scheme by Proposition 52.14.2. This concludes the proof. □

Lemma 52.14.5. *Let S be a scheme. Let $f : X \to B$ and $B' \to B$ be morphisms of algebraic spaces over S. Assume*

1. *$B' \to B$ is a closed immersion,*
2. *$|B'| \to |B|$ is bijective,*
3. *$X \times_B B' \to B'$ is a closed immersion, and*
4. *$X \to B$ is of finite type or $B' \to B$ is of finite presentation.*

Then $f : X \to B$ is a closed immersion.

Proof. Assumptions (1) and (2) imply that $B_{red} = B'_{red}$. Set $X' = X \times_B B'$. Then $X' \to X$ is closed immersion and $X'_{red} = X_{red}$. Let $U \to B$ be an étale morphism with U affine. Then $X' \times_B U \to X \times_B U$ is a closed immersion of algebraic spaces inducing an isomorphism on underlying reduced spaces. Since $X' \times_B U$ is a scheme (as $B' \to B$ and $X' \to B'$ are representable) so is $X \times_B U$ by Lemma 52.14.3. Hence $X \to B$ is representable too. Thus we reduce to the case of schemes, see Morphisms, Lemma 28.45.5. □

52.15. Other chapters

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CHAPTER 53

Divisors on Algebraic Spaces

53.1. Introduction

In this chapter we study divisors on algebraic spaces and related topics. A basic reference for algebraic spaces is [**Knu71**].

53.2. Effective Cartier divisors

For some reason it seem convenient to define the notion of an effective Cartier divisor before anything else. Note that in Morphisms of Spaces, Section 49.13 we discussed the correspondence between closed subspaces and quasi-coherent sheaves of ideals. Moreover, in Properties of Spaces, Section 48.28, we discussed properties of quasi-coherent modules, in particular "locally generated by 1 element". These references show that the following definition is compatible with the definition for schemes.

Definition 53.2.1. Let S be a scheme. Let X be an algebraic space over S.

(1) A *locally principal closed subspace* of X is a closed subspace whose sheaf of ideals is locally generated by 1 element.
(2) An *effective Cartier divisor* on X is a closed subspace $D \subset X$ such that the ideal sheaf $\mathcal{I}_D \subset \mathcal{O}_X$ is an invertible $\mathcal{O}_X$-module.

Thus an effective Cartier divisor is a locally principal closed subspace, but the converse is not always true. Effective Cartier divisors are closed subspaces of pure codimension 1 in the strongest possible sense. Namely they are locally cut out by a single element which is not a zerodivisor. In particular they are nowhere dense.

Lemma 53.2.2. *Let S be a scheme. Let X be an algebraic space over S. Let $D \subset X$ be a closed subspace. The following are equivalent:*

(1) *The subspace D is an effective Cartier divisor on X.*
(2) *For some scheme U and surjective étale morphism $U \to X$ the inverse image $D \times_X U$ is an effective Cartier divisor on U.*
(3) *For every scheme U and every étale morphism $U \to X$ the inverse image $D \times_X U$ is an effective Cartier divisor on U.*
(4) *For every $x \in |D|$ there exists an étale morphism $(U, u) \to (X, x)$ of pointed algebraic spaces such that $U = \mathrm{Spec}(A)$ and $D \times_X U = \mathrm{Spec}(A/(f))$ with $f \in A$ not a zerodivisor.*

Proof. The equivalence of (1) – (3) follows from Definition 53.2.1 and the references preceding it. Assume (1) and let $x \in |D|$. Choose a scheme W and a surjective étale morphism $W \to X$. Choose $w \in D \times_X W$ mapping to x. By (3) $D \times_X W$ is an effective Cartier divisor on W. Hence we can find affine étale neighbourhood U by choosing an affine open neighbourhood of w in W as in Divisors, Lemma 30.9.2.

Assume (2). Then we see that $\mathcal{I}_D|_U$ is invertible by Divisors, Lemma 30.9.2. Since we can find an étale covering of X by the collection of all such U and $X \setminus D$, we conclude that $\mathcal{I}_D$ is an invertible $\mathcal{O}_X$-module. □

Lemma 53.2.3. *Let S be a scheme. Let X be an algebraic space over S. Let $Z \subset X$ be a locally principal closed subspace. Let $U = X \setminus Z$. Then $U \to X$ is an affine morphism.*

Proof. The question is étale local on X, see Morphisms of Spaces, Lemmas 49.19.3 and Lemma 53.2.2. Thus this follows from the case of schemes which is Divisors, Lemma 30.9.3. □

Lemma 53.2.4. *Let S be a scheme. Let X be an algebraic space over S. Let $D \subset X$ be an effective Cartier divisor. Let $U = X \setminus D$. Then $U \to X$ is an affine morphism and U is scheme theoretically dense in X.*

Proof. Affineness is Lemma 53.2.3. The density question is étale local on X by Morphisms of Spaces, Definition 49.17.3. Thus this follows from the case of schemes which is Divisors, Lemma 30.9.4. □

Lemma 53.2.5. *Let S be a scheme. Let X be an algebraic space over S. Let $D \subset X$ be an effective Cartier divisor. Let $x \in |D|$. If $\dim_x(X) < \infty$, then $\dim_x(D) < \dim_x(X)$.*

Proof. Both the definition of an effective Cartier divisor and of the dimension of an an algebraic space at a point (Properties of Spaces, Definition 48.8.1) are étale local. Hence this lemma follows from the case of schemes which is Divisors, Lemma 30.9.5. □

Definition 53.2.6. Let S be a scheme. Let X be an algebraic space over S. Given effective Cartier divisors D_1, D_2 on X we set $D = D_1 + D_2$ equal to the closed subspace of X corresponding to the quasi-coherent sheaf of ideals $\mathcal{I}_{D_1}\mathcal{I}_{D_2} \subset \mathcal{O}_S$. We call this the *sum of the effective Cartier divisors D_1 and D_2*.

It is clear that we may define the sum $\sum n_i D_i$ given finitely many effective Cartier divisors D_i on X and nonnegative integers n_i.

Lemma 53.2.7. *The sum of two effective Cartier divisors is an effective Cartier divisor.*

Proof. Omitted. Étale locally this reduces to the following simple algebra fact: if $f_1, f_2 \in A$ are nonzerodivisors of a ring A, then $f_1 f_2 \in A$ is a nonzerodivisor. □

Lemma 53.2.8. *Let S be a scheme. Let X be an algebraic space over S. Let Z, Y be two closed subspaces of X with ideal sheaves $\mathcal{I}$ and $\mathcal{J}$. If $\mathcal{I}\mathcal{J}$ defines an effective Cartier divisor $D \subset X$, then Z and Y are effective Cartier divisors and $D = Z + Y$.*

Proof. By Lemma 53.2.2 this reduces to the case of schemes which is Divisors, Lemma 30.9.9. □

Recall that we have defined the inverse image of a closed subspace under any morphism of algebraic spaces in Morphisms of Spaces, Definition 49.13.2.

Lemma 53.2.9. *Let S be a scheme. Let $f : X' \to X$ be a morphism of algebraic spaces over S. Let $Z \subset X$ be a locally principal closed subspace. Then the inverse image $f^{-1}(Z)$ is a locally principal closed subspace of X'.*

Proof. Omitted. □

Definition 53.2.10. Let S be a scheme. Let $f : X' \to X$ be a morphism of algebraic spaces over S. Let $D \subset X$ be an effective Cartier divisor. We say the *pullback of D by f is defined* if the closed subspace $f^{-1}(D) \subset X'$ is an effective Cartier divisor. In this case we denote it either f^*D or $f^{-1}(D)$ and we call it the *pullback of the effective Cartier divisor*.

The condition that $f^{-1}(D)$ is an effective Cartier divisor is often satisfied in practice.

Lemma 53.2.11. *Let S be a scheme. Let $f : X \to Y$ be a morphism of algebraic spaces over S. Let $D \subset Y$ be an effective Cartier divisor. The pullback of D by f is defined in each of the following cases:*

(1) *f is flat, and*
(2) *add more here as needed.*

Proof. Omitted. □

Lemma 53.2.12. *Let S be a scheme. Let $f : X' \to X$ be a morphism of algebraic spaces over S. Let D_1, D_2 be effective Cartier divisors on X. If the pullbacks of D_1 and D_2 are defined then the pullback of $D = D_1 + D_2$ is defined and $f^*D = f^*D_1 + f^*D_2$.*

Proof. Omitted. □

Definition 53.2.13. Let S be a scheme. Let X be an algebraic space over S and let $D \subset X$ be an effective Cartier divisor. The *invertible sheaf $\mathcal{O}_X(D)$ associated to D* is given by

$$\mathcal{O}_X(D) := \mathcal{H}om_{\mathcal{O}_X}(\mathcal{I}_D, \mathcal{O}_X) = \mathcal{I}_D^{\otimes -1}.$$

The canonical section, usually denoted 1 or 1_D, is the global section of $\mathcal{O}_X(D)$ corresponding to the inclusion mapping $\mathcal{I}_D \to \mathcal{O}_X$.

Lemma 53.2.14. *Let S be a scheme. Let X be an algebraic space over S. Let D_1, D_2 be effective Cartier divisors on X. Let $D = D_1 + D_2$. Then there is a unique isomorphism*

$$\mathcal{O}_X(D_1) \otimes_{\mathcal{O}_X} \mathcal{O}_X(D_2) \longrightarrow \mathcal{O}_X(D)$$

which maps $1_{D_1} \otimes 1_{D_2}$ to 1_D.

Proof. Omitted. □

Definition 53.2.15. Let S be a scheme. Let X be an algebraic space over S. Let $\mathcal{L}$ be an invertible sheaf on X. A global section $s \in \Gamma(X, \mathcal{L})$ is called a *regular section* if the map $\mathcal{O}_X \to \mathcal{L}$, $f \mapsto fs$ is injective.

Lemma 53.2.16. *Let S be a scheme. Let X be an algebraic space over S. Let $f \in \Gamma(X, \mathcal{O}_X)$. The following are equivalent:*

(1) *f is a regular section, and*
(2) *for any $x \in X$ the image $f \in \mathcal{O}_{X,\overline{x}}$ is not a zerodivisor.*
(3) *for any affine $U = \mathrm{Spec}(A)$ étale over X the restriction $f|_U$ is a nonzerodivisor of A, and*
(4) *there exists a scheme U and a surjective étale morphism $U \to X$ such that $f|_U$ is a regular section of $\mathcal{O}_U$.*

Proof. Omitted. □

Note that a global section s of an invertible $\mathcal{O}_X$-module $\mathcal{L}$ may be seen as an $\mathcal{O}_X$-module map $s : \mathcal{O}_X \to \mathcal{L}$. Its dual is therefore a map $s : \mathcal{L}^{\otimes -1} \to \mathcal{O}_X$. (See Modules on Sites, Lemma 18.31.2 for the dual invertible sheaf.)

Definition 53.2.17. Let S be a scheme. Let X be an algebraic space over S. Let $\mathcal{L}$ be an invertible sheaf. Let $s \in \Gamma(X, \mathcal{L})$. The *zero scheme* of s is the closed subspace $Z(s) \subset X$ defined by the quasi-coherent sheaf of ideals $\mathcal{I} \subset \mathcal{O}_X$ which is the image of the map $s : \mathcal{L}^{\otimes -1} \to \mathcal{O}_X$.

Lemma 53.2.18. *Let S be a scheme. Let X be an algebraic space over S. Let $\mathcal{L}$ be an invertible $\mathcal{O}_X$-module. Let $s \in \Gamma(X, \mathcal{L})$.*

1. *Consider closed immersions $i : Z \to X$ such that $i^*s \in \Gamma(Z, i^*\mathcal{L}))$ is zero ordered by inclusion. The zero scheme $Z(s)$ is the maximal element of this ordered set.*
2. *For any morphism of algebraic spaces $f : Y \to X$ over S we have $f^*s = 0$ in $\Gamma(Y, f^*\mathcal{L})$ if and only if f factors through $Z(s)$.*
3. *The zero scheme $Z(s)$ is a locally principal closed subspace of X.*
4. *The zero scheme $Z(s)$ is an effective Cartier divisor on X if and only if s is a regular section of $\mathcal{L}$.*

Proof. Omitted. □

Lemma 53.2.19. *Let S be a scheme. Let X be an algebraic space over S.*

1. *If $D \subset X$ is an effective Cartier divisor, then the canonical section 1_D of $\mathcal{O}_X(D)$ is regular.*
2. *Conversely, if s is a regular section of the invertible sheaf $\mathcal{L}$, then there exists a unique effective Cartier divisor $D = Z(s) \subset X$ and a unique isomorphism $\mathcal{O}_X(D) \to \mathcal{L}$ which maps 1_D to s.*

The constructions $D \mapsto (\mathcal{O}_X(D), 1_D)$ and $(\mathcal{L}, s) \mapsto Z(s)$ give mutually inverse maps

$$\left\{\textit{effective Cartier divisors on } X\right\} \leftrightarrow \left\{\begin{array}{c}\textit{pairs } (\mathcal{L}, s) \textit{ consisting of an invertible} \\ \mathcal{O}_X\textit{-module and a regular global section}\end{array}\right\}$$

Proof. Omitted. □

53.3. Relative Proj

This section revisits the construction of the relative proj in the setting of algebraic spaces. The material in this section corresponds to the material in Constructions, Section 26.16 and Divisors, Section 30.16 in the case of schemes.

Situation 53.3.1. Here S is a scheme, X is an algebraic space over S, and $\mathcal{A}$ is a quasi-coherent graded $\mathcal{O}_X$-algebra.

In Situation 53.3.1 we are going to define a functor $F : (Sch/S)^{opp}_{fppf} \to Sets$ which will turn out to be an algebraic space. We will follow (mutatis mutandis) the procedure of Constructions, Section 26.16. First, given a scheme T over S we define a *quadruple over T* to be a system $(d, f : T \to S, \mathcal{L}, \psi)$

1. $d \geq 1$ is an integer,
2. $f : T \to X$ is a morphism over S,
3. $\mathcal{L}$ is an invertible $\mathcal{O}_T$-module, and

(4) $\psi : f^*\mathcal{A}^{(d)} \to \bigoplus_{n\geq 0} \mathcal{L}^{\otimes n}$ is a homomorphism of graded $\mathcal{O}_T$-algebras such that $f^*\mathcal{A}_d \to \mathcal{L}$ is surjective.

We say two quadruples $(d, f, \mathcal{L}, \psi)$ and $(d', f', \mathcal{L}', \psi')$ are *equivalent*[1] if and only if we have $f = f'$ and for some positive integer $m = ad = a'd'$ there exists an isomorphism $\beta : \mathcal{L}^{\otimes a} \to (\mathcal{L}')^{\otimes a'}$ with the property that $\beta \circ \psi|_{f^*\mathcal{A}^{(m)}}$ and $\psi'|_{f^*\mathcal{A}^{(m)}}$ agree as graded ring maps $f^*\mathcal{A}^{(m)} \to \bigoplus_{n\geq 0}(\mathcal{L}')^{\otimes mn}$. Given a quadruple $(d, f, \mathcal{L}, \psi)$ and a morphism $h : T' \to T$ we have the pullback $(d, f \circ h, h^*\mathcal{L}, h^*\psi)$. Pullback preserves the equivalence relation. Finally, for a *quasi-compact* scheme T over S we set

$$F(T) = \text{the set of equivalence classes of quadruples over } T$$

and for an arbitrary scheme T over S we set

$$F(T) = \lim_{V \subset T \text{ quasi-compact open}} F(V).$$

In other words, an element ξ of $F(T)$ corresponds to a compatible system of choices of elements $\xi_V \in F(V)$ where V ranges over the quasi-compact opens of T. Thus we have defined our functor

$$F : Sch^{opp} \longrightarrow Sets \tag{53.3.1.1}$$

There is a morphism $F \to X$ of functors sending the quadruple $(d, f, \mathcal{L}, \psi)$ to f.

Lemma 53.3.2. *In Situation 53.3.1. The functor F above is an algebraic space. For any morphism $g : Z \to X$ where Z is a scheme there is a canonical isomorphism $\underline{Proj}_Z(g^*\mathcal{A}) = Z \times_X F$ compatible with further base change.*

Proof. It suffices to prove the second assertion, see Spaces, Lemma 47.11.1. Let $g : Z \to X$ be a morphism where Z is a scheme. Let F' be the functor of quadruples associated to the graded quasi-coherent $\mathcal{O}_Z$-algebra $g^*\mathcal{A}$. Then there is a canonical isomorphism $F' = Z \times_X F$, sending a quadruple $(d, f : T \to Z, \mathcal{L}, \psi)$ for F' to $(d, g \circ f, \mathcal{L}, \psi)$ (details omitted, see proof of Constructions, Lemma 26.16.1). By Constructions, Lemmas 26.16.4, 26.16.5, and 26.16.6 and Definition 26.16.7 we see that F' is representable by $\underline{\mathrm{Proj}}_Z(g^*\mathcal{A})$. □

The lemma above tells us the following definition makes sense.

Definition 53.3.3. Let S be a scheme. Let X be an algebraic space over S. Let $\mathcal{A}$ be a quasi-coherent sheaf of graded $\mathcal{O}_X$-algebras. The *relative homogeneous spectrum of $\mathcal{A}$ over X*, or the *homogeneous spectrum of $\mathcal{A}$ over X*, or the *relative Proj of $\mathcal{A}$ over X* is the algebraic space F over X of Lemma 53.3.2. We denote it $\pi : \underline{\mathrm{Proj}}_X(\mathcal{A}) \to X$.

In particular the structure morphism of the relative Proj is representable by construction. We can also think about the relative Proj via glueing. Let $\varphi : U \to X$ be a surjective étale morphism, where U is a scheme. Set $R = U \times_X U$ with projection morphisms $s, t : R \to U$. By Lemma 53.3.2 there exists a canonical isomorphism

$$\gamma : \underline{\mathrm{Proj}}_U(\varphi^*\mathcal{A}) \longrightarrow \underline{\mathrm{Proj}}_X(\mathcal{A}) \times_X U$$

[1] This definition is motivated by Constructions, Lemma 26.16.4. The advantage of choosing this one is that it clearly defines an equivalence relation.

over U. Let $\alpha : t^*\varphi^*\mathcal{A} \to s^*\varphi^*\mathcal{A}$ be the canonical isomorphism of Properties of Spaces, Proposition 48.30.1. Then the diagram

$$\begin{array}{ccc} & \underline{\mathrm{Proj}}_U(\varphi^*\mathcal{A}) \times_{U,s} R = \underline{\mathrm{Proj}}_R(s^*\varphi^*\mathcal{A}) \\ \nearrow^{s^*\gamma} & & \\ \underline{\mathrm{Proj}}_X(\mathcal{A}) \times_X R & & \text{induced by } \alpha \downarrow \\ \searrow_{t^*\gamma} & & \\ & \underline{\mathrm{Proj}}_U(\varphi^*\mathcal{A}) \times_{U,t} R = \underline{\mathrm{Proj}}_R(t^*\varphi^*\mathcal{A}) \end{array}$$

is commutative (the equal signs come from Constructions, Lemma 26.16.10). Thus, if we denote $\mathcal{A}_U$, $\mathcal{A}_R$ the pullback of $\mathcal{A}$ to U, R, then $P = \underline{\mathrm{Proj}}_X(\mathcal{A})$ has an étale covering by the scheme $P_U = \underline{\mathrm{Proj}}_U(\mathcal{A}_U)$ and $P_U \times_P P_U$ is equal to $P_R = \underline{\mathrm{Proj}}_R(\mathcal{A}_R)$. Using these remarks we can argue in the usual fashion using étale localization to transfer results on the relative proj from the case of schemes to the case of algebraic spaces.

Lemma 53.3.4. *In Situation 53.3.1. The relative Proj comes equipped with a quasi-coherent sheaf of $\mathbf{Z}$-graded algebras $\bigoplus_{n\in\mathbf{Z}} \mathcal{O}_{\underline{Proj}_X(\mathcal{A})}(n)$ and a canonical homomorphism of graded algebras*

$$\psi : \pi^*\mathcal{A} \longrightarrow \bigoplus\nolimits_{n\geq 0} \mathcal{O}_{\underline{Proj}_X(\mathcal{A})}(n)$$

whose base change to any scheme over X agrees with Constructions, Lemma 26.15.5.

Proof. As in the discussion following Definition 53.3.3 choose a scheme U and a surjective étale morphism $U \to X$, set $R = U \times_X U$ with projections $s, t : R \to U$, $\mathcal{A}_U = \mathcal{A}|_U$, $\mathcal{A}_R = \mathcal{A}|_R$, and $\pi : P = \underline{\mathrm{Proj}}_X(\mathcal{A}) \to X$, $\pi_U : P_U = \underline{\mathrm{Proj}}_U(\mathcal{A}_U)$ and $\pi_R : P_R = \underline{\mathrm{Proj}}_U(\mathcal{A}_R)$. By the Constructions, Lemma 26.15.5 we have a quasi-coherent sheaf of $\mathbf{Z}$-graded $\mathcal{O}_{P_U}$-algebras $\bigoplus_{n\in\mathbf{Z}} \mathcal{O}_{P_U}(n)$ and a canonical map $\psi_U : \pi_U^*\mathcal{A}_U \to \bigoplus_{n\geq 0} \mathcal{O}_{P_U}(n)$ and similarly for P_R. By Constructions, Lemma 26.16.10 the pullback of $\mathcal{O}_{P_U}(n)$ and ψ_U by either projection $P_R \to P_U$ is equal to $\mathcal{O}_{P_R}(n)$ and ψ_R. By Properties of Spaces, Proposition 48.30.1 we obtain $\mathcal{O}_P(n)$ and ψ. We omit the verification of compatibility with pullback to arbitrary schemes over X. □

Having constructed the relative Proj we turn to some basic properties.

Lemma 53.3.5. *Let S be a scheme. Let $g : X' \to X$ be a morphism of algebraic spaces over S and let $\mathcal{A}$ be a quasi-coherent sheaf of graded $\mathcal{O}_X$-algebras. Then there is a canonical isomorphism*

$$r : \underline{Proj}_{X'}(g^*\mathcal{A}) \longrightarrow X' \times_X \underline{Proj}_X(\mathcal{A})$$

as well as a corresponding isomorphism

$$\theta : r^*pr_2^*\left(\bigoplus\nolimits_{d\in\mathbf{Z}} \mathcal{O}_{\underline{Proj}_X(\mathcal{A})}(d)\right) \longrightarrow \bigoplus\nolimits_{d\in\mathbf{Z}} \mathcal{O}_{\underline{Proj}_{X'}(g^*\mathcal{A})}(d)$$

of $\mathbf{Z}$-graded $\mathcal{O}_{\underline{Proj}_{X'}(g^*\mathcal{A})}$-algebras.

Proof. Let F be the functor (53.3.1.1) and let F' be the corresponding functor defined using $g^*\mathcal{A}$ on X'. We claim there is a canonical isomorphism $r : F' \to X' \times_X F$ of functors (and of course r is the isomorphism of the lemma). It suffices to construct the bijection $r : F'(T) \to X'(T) \times_{X(T)} F(T)$ for quasi-compact schemes T over S. First, if $\xi = (d', f', \mathcal{L}', \psi')$ is a quadruple over T for F', then we can set $r(\xi) = (f', (d', g \circ f', \mathcal{L}', \psi'))$. This makes sense as $(g \circ f')^*\mathcal{A}^{(d)} = (f')^*(g^*\mathcal{A})^{(d)}$. The inverse map sends the pair $(f', (d, f, \mathcal{L}, \psi))$ to the quadruple $(d, f', \mathcal{L}, \psi)$. We omit the proof of the final assertion (hint: reduce to the case of schemes by étale localization and apply Constructions, Lemma 26.16.10). □

Lemma 53.3.6. *In Situation 53.3.1 the morphism $\pi : \underline{Proj}_X(\mathcal{A}) \to X$ is separated.*

Proof. By Morphisms of Spaces, Lemma 49.5.12 and the construction of the relative Proj this follows from the case of schemes which is Constructions, Lemma 26.16.9. □

Lemma 53.3.7. *In Situation 53.3.1. If one of the following holds*

(1) *$\mathcal{A}$ is of finite type as a sheaf of $\mathcal{A}_0$-algebras,*
(2) *$\mathcal{A}$ is generated by $\mathcal{A}_1$ as an $\mathcal{A}_0$-algebra and $\mathcal{A}_1$ is a finite type $\mathcal{A}_0$-module,*
(3) *there exists a finite type quasi-coherent $\mathcal{A}_0$-submodule $\mathcal{F} \subset \mathcal{A}_+$ such that $\mathcal{A}_+/\mathcal{F}\mathcal{A}$ is a locally nilpotent sheaf of ideals of $\mathcal{A}/\mathcal{F}\mathcal{A}$,*

then $\pi : \underline{Proj}_X(\mathcal{A}) \to X$ is quasi-compact.

Proof. By Morphisms of Spaces, Lemma 49.9.7 and the construction of the relative Proj this follows from the case of schemes which is Divisors, Lemma 30.16.1. □

Lemma 53.3.8. *In Situation 53.3.1. If $\mathcal{A}$ is of finite type as a sheaf of $\mathcal{O}_X$-algebras, then $\pi : \underline{Proj}_X(\mathcal{A}) \to X$ is of finite type.*

Proof. By Morphisms of Spaces, Lemma 49.22.4 and the construction of the relative Proj this follows from the case of schemes which is Divisors, Lemma 30.16.2. □

Lemma 53.3.9. *In Situation 53.3.1. If $\mathcal{O}_X \to \mathcal{A}_0$ is an integral algebra map[2] and $\mathcal{A}$ is of finite type as an $\mathcal{A}_0$-algebra, then $\pi : \underline{Proj}_X(\mathcal{A}) \to X$ is universally closed.*

Proof. By Morphisms of Spaces, Lemma 49.10.5 and the construction of the relative Proj this follows from the case of schemes which is Divisors, Lemma 30.16.3. □

Lemma 53.3.10. *In Situation 53.3.1. The following conditions are equivalent*

(1) *$\mathcal{A}_0$ is a finite type $\mathcal{O}_X$-module and $\mathcal{A}$ is of finite type as an $\mathcal{A}_0$-algebra,*
(2) *$\mathcal{A}_0$ is a finite type $\mathcal{O}_X$-module and $\mathcal{A}$ is of finite type as an $\mathcal{O}_X$-algebra.*

If these conditions hold, then $\pi : \underline{Proj}_X(\mathcal{A}) \to X$ is proper.

Proof. By Morphisms of Spaces, Lemma 49.36.2 and the construction of the relative Proj this follows from the case of schemes which is Divisors, Lemma 30.16.3. □

Lemma 53.3.11. *Let S be a scheme. Let X be an algebraic space over S. Let $\mathcal{A}$ be a quasi-coherent sheaf of graded $\mathcal{O}_X$-modules generated as an $\mathcal{A}_0$-algebra by $\mathcal{A}_1$. With $P = \underline{Proj}_X(\mathcal{A})$ we have*

[2] In other words, the integral closure of $\mathcal{O}_X$ in $\mathcal{A}_0$, see Morphisms of Spaces, Definition 49.42.2, equals $\mathcal{A}_0$.

(1) *P represents the functor F_1 which associates to T over S the set of isomorphism classes of triples $(f, \mathcal{L}, \psi)$, where $f : T \to X$ is a morphism over S, $\mathcal{L}$ is an invertible $\mathcal{O}_T$-module, and $\psi : f^*\mathcal{A} \to \bigoplus_{n\geq 0} \mathcal{L}^{\otimes n}$ is a map of graded $\mathcal{O}_T$-algebras inducing a surjection $f^*\mathcal{A}_1 \to \mathcal{L}$,*
(2) *the canonical map $\pi^*\mathcal{A}_1 \to \mathcal{O}_P(1)$ is surjective, and*
(3) *each $\mathcal{O}_P(n)$ is invertible and the multiplication maps induce isomorphisms $\mathcal{O}_P(n) \otimes_{\mathcal{O}_P} \mathcal{O}_P(m) = \mathcal{O}_P(n+m)$.*

Proof. Omitted. See Constructions, Lemma 26.16.11 for the case of schemes. □

53.4. Functoriality of relative proj

This section is the analogue of Constructions, Section 26.18.

Lemma 53.4.1. *Let S be a scheme. Let X be an algebraic space over S. Let $\psi : \mathcal{A} \to \mathcal{B}$ be a map of quasi-coherent graded $\mathcal{O}_X$-algebras. Set $P = \underline{Proj}_X(\mathcal{A}) \to X$ and $Q = \underline{Proj}_X(\mathcal{B}) \to X$. There is a canonical open subspace $U(\psi) \subset Q$ and a canonical morphism of algebraic spaces*

$$r_\psi : U(\psi) \longrightarrow P$$

over X and a map of $\mathbf{Z}$-graded $\mathcal{O}_{U(\psi)}$-algebras

$$\theta = \theta_\psi : r_\psi^* \left(\bigoplus\nolimits_{d\in\mathbf{Z}} \mathcal{O}_P(d) \right) \longrightarrow \bigoplus\nolimits_{d\in\mathbf{Z}} \mathcal{O}_{U(\psi)}(d).$$

The triple $(U(\psi), r_\psi, \theta)$ is characterized by the property that for any scheme W étale over X the triple

$$(U(\psi) \times_X W, \quad r_\psi|_{U(\psi)\times_X W} : U(\psi) \times_X W \to P \times_X W, \quad \theta|_{U(\psi)\times_X W})$$

is equal to the triple associated to $\psi : \mathcal{A}|_W \to \mathcal{B}|_W$ of Constructions, Lemma 26.18.1.

Proof. This lemma follows from étale localization and the case of schemes, see discussion following Definition 53.3.3. Details omitted. □

Lemma 53.4.2. *Let S be a scheme. Let X be an algebraic space over S. Let $\mathcal{A}$, $\mathcal{B}$, and $\mathcal{C}$ be quasi-coherent graded $\mathcal{O}_X$-algebras. Set $P = \underline{Proj}_X(\mathcal{A})$, $Q = \underline{Proj}_X(\mathcal{B})$ and $R = \underline{Proj}_X(\mathcal{C})$. Let $\varphi : \mathcal{A} \to \mathcal{B}$, $\psi : \mathcal{B} \to \mathcal{C}$ be graded $\mathcal{O}_X$-algebra maps. Then we have*

$$U(\psi \circ \varphi) = r_\varphi^{-1}(U(\psi)) \quad \text{and} \quad r_{\psi\circ\varphi} = r_\varphi \circ r_\psi|_{U(\psi\circ\varphi)}.$$

In addition we have

$$\theta_\psi \circ r_\psi^* \theta_\varphi = \theta_{\psi\circ\varphi}$$

with obvious notation.

Proof. Omitted. □

Lemma 53.4.3. *With hypotheses and notation as in Lemma 53.4.1 above. Assume $\mathcal{A}_d \to \mathcal{B}_d$ is surjective for $d \gg 0$. Then*

(1) *$U(\psi) = Q$,*
(2) *$r_\psi : Q \to R$ is a closed immersion, and*
(3) *the maps $\theta : r_\psi^*\mathcal{O}_P(n) \to \mathcal{O}_Q(n)$ are surjective but not isomorphisms in general (even if $\mathcal{A} \to \mathcal{B}$ is surjective).*

Proof. Follows from the case of schemes (Constructions, Lemma 26.18.3) by étale localization. □

Lemma 53.4.4. *With hypotheses and notation as in Lemma 53.4.1 above. Assume $\mathcal{A}_d \to \mathcal{B}_d$ is an isomorphism for all $d \gg 0$. Then*

(1) *$U(\psi) = Q$,*
(2) *$r_\psi : Q \to P$ is an isomorphism, and*
(3) *the maps $\theta : r_\psi^* \mathcal{O}_P(n) \to \mathcal{O}_Q(n)$ are isomorphisms.*

Proof. Follows from the case of schemes (Constructions, Lemma 26.18.4) by étale localization. □

Lemma 53.4.5. *With hypotheses and notation as in Lemma 53.4.1 above. Assume $\mathcal{A}_d \to \mathcal{B}_d$ is surjective for $d \gg 0$ and that $\mathcal{A}$ is generated by $\mathcal{A}_1$ over $\mathcal{A}_0$. Then*

(1) *$U(\psi) = Q$,*
(2) *$r_\psi : Q \to P$ is a closed immersion, and*
(3) *the maps $\theta : r_\psi^* \mathcal{O}_P(n) \to \mathcal{O}_Q(n)$ are isomorphisms.*

Proof. Follows from the case of schemes (Constructions, Lemma 26.18.5) by étale localization. □

53.5. Closed subspaces of relative proj

Some auxiliary lemmas about closed subspaces of relative proj. This section is the analogue of Divisors, Section 30.17.

Lemma 53.5.1. *Let S be a scheme. Let X be an algebraic space over S. Let $\mathcal{A}$ be a quasi-coherent graded $\mathcal{O}_X$-algebra. Let $\pi : P = \underline{Proj}_X(\mathcal{A}) \to X$ be the relative Proj of $\mathcal{A}$. Let $i : Z \to P$ be a closed subspace. Denote $\mathcal{I} \subset \mathcal{A}$ the kernel of the canonical map*

$$\mathcal{A} \longrightarrow \bigoplus\nolimits_{d \geq 0} \pi_* \left((i_* \mathcal{O}_Z)(d) \right)$$

If π is quasi-compact, then there is an isomorphism $Z = \underline{Proj}_X(\mathcal{A}/\mathcal{I})$.

Proof. The morphism π is separated by Lemma 53.3.6. As π is quasi-compact, π_* transforms quasi-coherent modules into quasi-coherent modules, see Morphisms of Spaces, Lemma 49.12.2. Hence $\mathcal{I}$ is a quasi-coherent $\mathcal{O}_X$-module. In particular, $\mathcal{B} = \mathcal{A}/\mathcal{I}$ is a quasi-coherent graded $\mathcal{O}_X$-algebra. The functoriality morphism $Z' = \underline{\mathrm{Proj}}_X(\mathcal{B}) \to \underline{\mathrm{Proj}}_X(\mathcal{A})$ is everywhere defined and a closed immersion, see Lemma 53.4.3. Hence it suffices to prove $Z = Z'$ as closed subspaces of P.

Having said this, the question is étale local on the base and we reduce to the case of schemes (Divisors, Lemma 30.17.1) by étale localization. □

In case the closed subspace is locally cut out by finitely many equations we can define it by a finite type ideal sheaf of $\mathcal{A}$.

Lemma 53.5.2. *Let S be a scheme. Let X be a quasi-compact and quasi-separated algebraic space over S. Let $\mathcal{A}$ be a quasi-coherent graded $\mathcal{O}_X$-algebra. Let $\pi : P = \underline{Proj}_X(\mathcal{A}) \to X$ be the relative Proj of $\mathcal{A}$. Let $i : Z \to P$ be a closed subscheme. If π is quasi-compact and i of finite presentation, then there exists a $d > 0$ and a quasi-coherent finite type $\mathcal{O}_X$-submodule $\mathcal{F} \subset \mathcal{A}_d$ such that $Z = \underline{Proj}_X(\mathcal{A}/\mathcal{F}\mathcal{A})$.*

Proof. The reader can redo the arguments used in the case of schemes. However, we will show the lemma follows from the case of schemes by a trick. Let $\mathcal{I} \subset \mathcal{A}$ be the quasi-coherent graded ideal cutting out Z of Lemma 53.5.1. Choose an affine scheme U and a surjective étale morphism $U \to X$, see Properties of Spaces, Lemma 48.6.3. By the case of schemes (Divisors, Lemma 30.17.2) there exists a $d > 0$ and a quasi-coherent finite type $\mathcal{O}_U$-submodule $\mathcal{F}' \subset \mathcal{I}_d|_U \subset \mathcal{A}_d|_U$ such that $Z \times_X U$ is equal to $\underline{\mathrm{Proj}}_U(\mathcal{A}|_U/\mathcal{F}'\mathcal{A}|_U)$. By Limits of Spaces, Lemma 52.9.2 we can find a finite type quasi-coherent submodule $\mathcal{F} \subset \mathcal{I}_d$ such that $\mathcal{F}' \subset \mathcal{F}|_U$. Let $Z' = \underline{\mathrm{Proj}}_X(\mathcal{A}/\mathcal{F}\mathcal{A})$. Then $Z' \to P$ is a closed immersion (Lemma 53.4.5) and $Z \subset Z'$ as $\mathcal{F}\mathcal{A} \subset \mathcal{I}$. On the other hand, $Z' \times_X U \subset Z \times_X U$ by our choice of $\mathcal{F}$. Thus $Z = Z'$ as desired. □

Lemma 53.5.3. *Let S be a scheme. Let X be a quasi-compact and quasi-separated algebraic space over S. Let $\mathcal{A}$ be a quasi-coherent graded $\mathcal{O}_X$-algebra. Let $\pi : P = \underline{Proj}_X(\mathcal{A}) \to X$ be the relative Proj of $\mathcal{A}$. Let $i : Z \to X$ be a closed subspace. Let $U \subset X$ be an open. Assume that*

1. *π is quasi-compact,*
2. *i of finite presentation,*
3. *$|U| \cap |\pi|(|i|(|Z|)) = \emptyset$,*
4. *U is quasi-compact,*
5. *$\mathcal{A}_n$ is a finite type $\mathcal{O}_X$-module for all n.*

Then there exists a $d > 0$ and a quasi-coherent finite type $\mathcal{O}_X$-submodule $\mathcal{F} \subset \mathcal{A}_d$ with (a) $Z = \underline{Proj}_X(\mathcal{A}/\mathcal{F}\mathcal{A})$ and (b) the support of $\mathcal{A}_d/\mathcal{F}$ is disjoint from U.

Proof. We use the same trick as in the proof of Lemma 53.5.2 to reduce to the case of schemes. Let $\mathcal{I} \subset \mathcal{A}$ be the quasi-coherent graded ideal cutting out Z of Lemma 53.5.1. Choose an affine scheme W and a surjective étale morphism $W \to X$, see Properties of Spaces, Lemma 48.6.3. By the case of schemes (Divisors, Lemma 30.17.3) there exists a $d > 0$ and a quasi-coherent finite type $\mathcal{O}_W$-submodule $\mathcal{F}' \subset \mathcal{I}_d|_W \subset \mathcal{A}_d|_W$ such that (a) $Z \times_X W$ is equal to $\underline{\mathrm{Proj}}_W(\mathcal{A}|_W/\mathcal{F}'\mathcal{A}|_W)$ and (b) the support of $\mathcal{A}_d|_W/\mathcal{F}'$ is disjoint from $U \times_X W$. By Limits of Spaces, Lemma 52.9.2 we can find a finite type quasi-coherent submodule $\mathcal{F} \subset \mathcal{I}_d$ such that $\mathcal{F}' \subset \mathcal{F}|_W$. Let $Z' = \underline{\mathrm{Proj}}_X(\mathcal{A}/\mathcal{F}\mathcal{A})$. Then $Z' \to P$ is a closed immersion (Lemma 53.4.5) and $Z \subset Z'$ as $\mathcal{F}\mathcal{A} \subset \mathcal{I}$. On the other hand, $Z' \times_X W \subset Z \times_X W$ by our choice of $\mathcal{F}$. Thus $Z = Z'$. Finally, we see that $\mathcal{A}_d/\mathcal{F}$ is supported on $X \setminus U$ as $\mathcal{A}_d|_W/\mathcal{F}|_W$ is a quotient of $\mathcal{A}_d|_W/\mathcal{F}'$ which is supported on $W \setminus U \times_X W$. Thus the lemma follows. □

53.6. Blowing up

Blowing up is an important tool in algebraic geometry.

Definition 53.6.1. Let S be a scheme. Let X be an algebraic space over S. Let $\mathcal{I} \subset \mathcal{O}_X$ be a quasi-coherent sheaf of ideals, and let $Z \subset X$ be the closed subscheme corresponding to $\mathcal{I}$ (Morphisms of Spaces, Lemma 49.13.1). The *blowing up of X along Z*, or the *blowing up of X in the ideal sheaf $\mathcal{I}$* is the morphism

$$b : \underline{\mathrm{Proj}}_X\left(\bigoplus\nolimits_{n\geq 0} \mathcal{I}^n\right) \longrightarrow X$$

The *exceptional divisor* of the blow up is the inverse image $b^{-1}(Z)$. Sometimes Z is called the *center* of the blowup.

We will see later that the exceptional divisor is an effective Cartier divisor. Moreover, the blowing up is characterized as the "smallest" algebraic space over X such that the inverse image of Z is an effective Cartier divisor.

If $b : X' \to X$ is the blow up of X in Z, then we often denote $\mathcal{O}_{X'}(n)$ the twists of the structure sheaf. Note that these are invertible $\mathcal{O}_{X'}$-modules and that $\mathcal{O}_{X'}(n) = \mathcal{O}_{X'}(1)^{\otimes n}$ because X' is the relative Proj of a quasi-coherent graded $\mathcal{O}_X$-algebra which is generated in degree 1, see Lemma 53.3.11.

Lemma 53.6.2. *Let S be a scheme. Let X be an algebraic space over S. Let $\mathcal{I} \subset \mathcal{O}_X$ be a quasi-coherent sheaf of ideals. Let $U = \mathrm{Spec}(A)$ be an affine scheme étale over X and let $I \subset A$ be the ideal corresponding to $\mathcal{I}|_U$. If $X' \to X$ is the blow up of X in $\mathcal{I}$, then there is a canonical isomorphism*

$$U \times_X X' = \mathit{Proj}(\bigoplus\nolimits_{d \geq 0} I^d)$$

of schemes over U, where the right hand side is the homogeneous spectrum of the Rees algebra of I in A. Moreover, $U \times_X X'$ has an affine open covering by spectra of the affine blowup algebras $A[\frac{I}{a}]$.

Proof. Note that the restriction $\mathcal{I}|_U$ is equal to the pullback of $\mathcal{I}$ via the morphism $U \to X$, see Properties of Spaces, Section 48.24. Thus the lemma follows on combining Lemma 53.3.2 with Divisors, Lemma 30.18.2. □

Lemma 53.6.3. *Let S be a scheme. Let $X_1 \to X_2$ be a flat morphism of algebraic spaces over S. Let $Z_2 \subset X_2$ be a closed subspace. Let Z_1 be the inverse image of Z_2 in X_1. Let X_i' be the blow up of Z_i in X_i. Then there exists a cartesian diagram*

$$\begin{array}{ccc} X_1' & \longrightarrow & X_2' \\ \downarrow & & \downarrow \\ X_1 & \longrightarrow & X_2 \end{array}$$

of algebraic spaces over S.

Proof. Let $\mathcal{I}_2$ be the ideal sheaf of Z_2 in X_2. Denote $g : X_1 \to X_2$ the given morphism. Then the ideal sheaf $\mathcal{I}_1$ of Z_1 is the image of $g^*\mathcal{I}_2 \to \mathcal{O}_{X_1}$ (see Morphisms of Spaces, Definition 49.13.2 and discussion following the definition). By Lemma 53.3.5 we see that $X_1 \times_{X_2} X_2'$ is the relative Proj of $\bigoplus_{n \geq 0} g^*\mathcal{I}_2^n$. Because g is flat the map $g^*\mathcal{I}_2^n \to \mathcal{O}_{X_1}$ is injective with image $\mathcal{I}_1^n$. Thus we see that $X_1 \times_{X_2} X_2' = X_1'$. □

Lemma 53.6.4. *Let S be a scheme. Let X be an algebraic space over S. Let $Z \subset X$ be a closed subspace. The blowing up $b : X' \to X$ of Z in X has the following properties:*

1. *$b|_{b^{-1}(X \setminus Z)} : b^{-1}(X \setminus Z) \to X \setminus Z$ is an isomorphism,*
2. *the exceptional divisor $E = b^{-1}(Z)$ is an effective Cartier divisor on X',*
3. *there is a canonical isomorphism $\mathcal{O}_{X'}(-1) = \mathcal{O}_{X'}(E)$*

Proof. Let U be a scheme and let $U \to X$ be a surjective étale morphism. As blowing up commutes with flat base change (Lemma 53.6.3) we can prove each of these statements after base change to U. This reduces us to the case of schemes. In this case the result is Divisors, Lemma 30.18.4. □

Lemma 53.6.5 (Universal property blowing up)**.** *Let S be a scheme. Let X be an algebraic space over S. Let $Z \subset X$ be a closed subspace. Let $\mathcal{C}$ be the full subcategory of $(Spaces/X)$ consisting of $Y \to X$ such that the inverse image of Z is an effective Cartier divisor on Y. Then the blowing up $b : X' \to X$ of Z in X is a final object of $\mathcal{C}$.*

Proof. We see that $b : X' \to X$ is an object of $\mathcal{C}$ according to Lemma 53.6.4. Let $f : Y \to X$ be an object of $\mathcal{C}$. We have to show there exists a unique morphism $Y \to X'$ over X. Let $D = f^{-1}(Z)$. Let $\mathcal{I} \subset \mathcal{O}_X$ be the ideal sheaf of Z and let $\mathcal{I}_D$ be the ideal sheaf of D. Then $f^*\mathcal{I} \to \mathcal{I}_D$ is a surjection to an invertible $\mathcal{O}_Y$-module. This extends to a map $\psi : \bigoplus f^*\mathcal{I}^d \to \bigoplus \mathcal{I}_D^d$ of graded $\mathcal{O}_Y$-algebras. (We observe that $\mathcal{I}_D^d = \mathcal{I}_D^{\otimes d}$ as D is an effective Cartier divisor.) By Lemma 53.3.11. the triple $(f : Y \to X, \mathcal{I}_D, \psi)$ defines a morphism $Y \to X'$ over X. The restriction

$$Y \setminus D \longrightarrow X' \setminus b^{-1}(Z) = X \setminus Z$$

is unique. The open $Y \setminus D$ is scheme theoretically dense in Y according to Lemma 53.2.4. Thus the morphism $Y \to X'$ is unique by Morphisms of Spaces, Lemma 49.17.8 (also b is separated by Lemma 53.3.6). □

Lemma 53.6.6. *Let S be a scheme. Let X be an algebraic space over S. Let $Z \subset X$ be an effective Cartier divisor. The blowup of X in Z is the identity morphism of X.*

Proof. Immediate from the universal property of blowups (Lemma 53.6.5). □

Lemma 53.6.7. *Let S be a scheme. Let X be an algebraic space over S. Let $\mathcal{I} \subset \mathcal{O}_X$ be a quasi-coherent sheaf of ideals. If X is reduced, then the blow up X' of X in $\mathcal{I}$ is reduced.*

Proof. Let U be a scheme and let $U \to X$ be a surjective étale morphism. As blowing up commutes with flat base change (Lemma 53.6.3) we can prove each of these statements after base change to U. This reduces us to the case of schemes. In this case the result is Divisors, Lemma 30.18.8. □

Lemma 53.6.8. *Let S be a scheme. Let X be an algebraic space over S. Let $b : X' \to X$ be a blow up of X in a closed subspace. For any effective Cartier divisor D on X the pullback $b^{-1}D$ is defined (see Definition 53.2.10).*

Proof. By Lemmas 53.6.2 and 53.2.2 this reduces to the following algebra fact: Let A be a ring, $I \subset A$ an ideal, $a \in I$, and $x \in A$ a nonzerodivisor. Then the image of x in $A[\frac{I}{a}]$ is a nonzerodivisor. Namely, suppose that $x(y/a^n) = 0$ in $A[\frac{I}{a}]$. Then $a^m xy = 0$ in A for some m. Hence $a^m y = 0$ as x is a nonzerodivisor. Whence y/a^n is zero in $A[\frac{I}{a}]$ as desired. □

Lemma 53.6.9. *Let S be a scheme. Let X be an algebraic space over S. Let $\mathcal{I} \subset \mathcal{O}_X$ and $\mathcal{J}$ be quasi-coherent sheaves of ideals. Let $b : X' \to X$ be the blowing up of X in $\mathcal{I}$. Let $b' : X'' \to X'$ be the blowing up of X' in $b^{-1}\mathcal{J}\mathcal{O}_{X'}$. Then $X'' \to X$ is canonically isomorphic to the blowing up of X in $\mathcal{I}\mathcal{J}$.*

Proof. Let $E \subset X'$ be the exceptional divisor of b which is an effective Cartier divisor by Lemma 53.6.4. Then $(b')^{-1}E$ is an effective Cartier divisor on X'' by Lemma 53.6.8. Let $E' \subset X''$ be the exceptional divisor of b' (also an effective Cartier divisor). Consider the effective Cartier divisor $E'' = E' + (b')^{-1}E$. By construction

the ideal of E'' is $(b \circ b')^{-1}\mathcal{I}(b \circ b')^{-1}\mathcal{J}\mathcal{O}_{X''}$. Hence according to Lemma 53.6.5 there is a canonical morphism from X'' to the blowup $c : Y \to X$ of X in $\mathcal{I}\mathcal{J}$. Conversely, as $\mathcal{I}\mathcal{J}$ pulls back to an invertible ideal we see that $c^{-1}\mathcal{I}\mathcal{O}_Y$ defines an effective Cartier divisor, see Lemma 53.2.8. Thus a morphism $c' : Y \to X'$ over X by Lemma 53.6.5. Then $(c')^{-1}b^{-1}\mathcal{J}\mathcal{O}_Y = c^{-1}\mathcal{J}\mathcal{O}_Y$ which also defines an effective Cartier divisor. Thus a morphism $c'' : Y \to X''$ over X'. We omit the verification that this morphism is inverse to the morphism $X'' \to Y$ constructed earlier. □

Lemma 53.6.10. *Let S be a scheme. Let X be an algebraic space over S. Let $\mathcal{I} \subset \mathcal{O}_X$ be a quasi-coherent sheaf of ideals. Let $b : X' \to X$ be the blowing up of X in the ideal sheaf $\mathcal{I}$. If $\mathcal{I}$ is of finite type, then $b : X' \to X$ is a proper morphism.*

Proof. Let U be a scheme and let $U \to X$ be a surjective étale morphism. As blowing up commutes with flat base change (Lemma 53.6.3) we can prove each of these statements after base change to U (see Morphisms of Spaces, Lemma 49.36.2). This reduces us to the case of schemes. In this case the morphism b is projective by Divisors, Lemma 30.18.11 hence proper by Morphisms, Lemma 28.43.5. □

Lemma 53.6.11. *Let S be a scheme and let X be an algebraic space over S. Assume X is quasi-compact and quasi-separated. Let $Z \subset X$ be a closed subspace of finite presentation. Let $b : X' \to X$ be the blowing up with center Z. Let $Z' \subset X'$ be a closed subspace of finite presentation. Let $X'' \to X'$ be the blowing up with center Z'. There exists a closed subspace $Y \subset X$ of finite presentation, such that*

(1) *$|Y| = |Z| \cup |b|(|Z'|)$, and*
(2) *the composition $X'' \to X$ is isomorphic to the blowing up of X in Y.*

Proof. The condition that $Z \to X$ is of finite presentation means that Z is cut out by a finite type quasi-coherent sheaf of ideals $\mathcal{I} \subset \mathcal{O}_X$, see Morphisms of Spaces, Lemma 49.26.12. Write $\mathcal{A} = \bigoplus_{n \geq 0} \mathcal{I}^n$ so that $X' = \underline{\mathrm{Proj}}(\mathcal{A})$. Note that $X \setminus Z$ is a quasi-compact open subspace of X by Limits of Spaces, Lemma 52.13.1. Since $b^{-1}(X \setminus Z) \to X \setminus Z$ is an isomorphism (Lemma 53.6.4) the same result shows that $b^{-1}(X \setminus Z) \setminus Z'$ is quasi-compact open subspace in X'. Hence $U = X \setminus (Z \cup b(Z'))$ is quasi-compact open subspace in X. By Lemma 53.5.3 there exist a $d > 0$ and a finite type $\mathcal{O}_X$-submodule $\mathcal{F} \subset \mathcal{I}^d$ such that $Z' = \underline{\mathrm{Proj}}(\mathcal{A}/\mathcal{F}\mathcal{A})$ and such that the support of $\mathcal{I}^d/\mathcal{F}$ is contained in $X \setminus U$.

Since $\mathcal{F} \subset \mathcal{I}^d$ is an $\mathcal{O}_X$-submodule we may think of $\mathcal{F} \subset \mathcal{I}^d \subset \mathcal{O}_X$ as a finite type quasi-coherent sheaf of ideals on X. Let's denote this $\mathcal{J} \subset \mathcal{O}_X$ to prevent confusion. Since $\mathcal{I}^d/\mathcal{J}$ and $\mathcal{O}/\mathcal{I}^d$ are supported on $|X| \setminus |U|$ we see that $|V(\mathcal{J})|$ is contained in $|X| \setminus |U|$. Conversely, as $\mathcal{J} \subset \mathcal{I}^d$ we see that $|Z| \subset |V(\mathcal{J})|$. Over $X \setminus Z \cong X' \setminus b^{-1}(Z)$ the sheaf of ideals $\mathcal{J}$ cuts out Z' (see displayed formula below). Hence $|V(\mathcal{J})|$ equals $|Z| \cup |b|(|Z'|)$. It follows that also $|V(\mathcal{I}\mathcal{J})| = |Z| \cup |b|(|Z'|)$. Moreover, $\mathcal{I}\mathcal{J}$ is an ideal of finite type as a product of two such. We claim that $X'' \to X$ is isomorphic to the blowing up of X in $\mathcal{I}\mathcal{J}$ which finishes the proof of the lemma by setting $Y = V(\mathcal{I}\mathcal{J})$.

First, recall that the blow up of X in $\mathcal{I}\mathcal{J}$ is the same as the blow up of X' in $b^{-1}\mathcal{J}\mathcal{O}_{X'}$, see Lemma 53.6.9. Hence it suffices to show that the blow up of X' in $b^{-1}\mathcal{J}\mathcal{O}_{X'}$ agrees with the blow up of X' in Z'. We will show that

$$b^{-1}\mathcal{J}\mathcal{O}_{X'} = \mathcal{I}_E^d \mathcal{I}_{Z'}$$

as ideal sheaves on X''. This will prove what we want as $\mathcal{I}_E^d$ cuts out the effective Cartier divisor dE and we can use Lemmas 53.6.6 and 53.6.9.

To see the displayed equality of the ideals we may work locally. With notation A, I, $a \in I$ as in Lemma 53.6.2 we see that $\mathcal{F}$ corresponds to an R-submodule $M \subset I^d$ mapping isomorphically to an ideal $J \subset R$. The condition $Z' = \underline{\mathrm{Proj}}(\mathcal{A}/\mathcal{F}\mathcal{A})$ means that $Z' \cap \mathrm{Spec}(A[\frac{I}{a}])$ is cut out by the ideal generated by the elements m/a^d, $m \in M$. Say the element $m \in M$ corresponds to the function $f \in J$. Then in the affine blowup algebra $A' = A[\frac{I}{a}]$ we see that $f = (a^d m)/a^d = a^d(m/a^d)$. Thus the equality holds. □

53.7. Strict transform

This section is the analogue of Divisors, Section 30.19. Let S be a scheme, let B be an algebraic space over S, and let $Z \subset B$ be a closed subspace. Let $b : B' \to B$ be the blowing up of B in Z and denote $E \subset B'$ the exceptional divisor $E = b^{-1}Z$. In the following we will often consider an algebraic space X over B and form the cartesian diagram

$$\begin{array}{ccccc} \mathrm{pr}_{B'}^{-1}E & \longrightarrow & X \times_B B' & \xrightarrow{\mathrm{pr}_X} & X \\ \downarrow & & \downarrow{\scriptstyle \mathrm{pr}_{B'}} & & \downarrow{\scriptstyle f} \\ E & \longrightarrow & B' & \longrightarrow & B \end{array}$$

Since E is an effective Cartier divisor (Lemma 53.6.4) we see that $\mathrm{pr}_{B'}^{-1}E \subset X \times_B B'$ is locally principal (Lemma 53.2.9). Thus the inclusion morphism of the complement of $\mathrm{pr}_{B'}^{-1}E$ in $X \times_B B'$ is affine and in particular quasi-compact (Lemma 53.2.3). Consequently, for a quasi-coherent $\mathcal{O}_{X \times_B B'}$-module $\mathcal{G}$ the subsheaf of sections supported on $|\mathrm{pr}_{B'}^{-1}E|$ is a quasi-coherent submodule, see Limits of Spaces, Lemma 52.13.5. If $\mathcal{G}$ is a quasi-coherent sheaf of algebras, e.g., $\mathcal{G} = \mathcal{O}_{X \times_B B'}$, then this subsheaf is an ideal of $\mathcal{G}$.

Definition 53.7.1. With $Z \subset B$ and $f : X \to B$ as above.

(1) Given a quasi-coherent $\mathcal{O}_X$-module $\mathcal{F}$ the *strict transform* of $\mathcal{F}$ with respect to the blowup of B in Z is the quotient $\mathcal{F}'$ of $\mathrm{pr}_X^*\mathcal{F}$ by the submodule of sections supported on $|\mathrm{pr}_{B'}^{-1}E|$.
(2) The *strict transform* of X is the closed subscheme $X' \subset X \times_B B'$ cut out by the quasi-coherent ideal of sections of $\mathcal{O}_{X \times_B B'}$ supported on $|\mathrm{pr}_{B'}^{-1}E|$.

Note that taking the strict transform along a blowup depends on the closed subspace used for the blowup (and not just on the morphism $B' \to B$).

Lemma 53.7.2 (Étale localization and strict transform)**.** *In the situation of Definition 53.7.1. Let*

$$\begin{array}{ccc} U & \longrightarrow & X \\ \downarrow & & \downarrow \\ V & \longrightarrow & B \end{array}$$

be a commutative diagram of morphisms with U and V schemes and étale horizontal arrows. Let $V' \to V$ be the blowup of V in $Z \times_B V$. Then

(1) *$V' = V \times_B B'$ and the maps $V' \to B'$ and $U \times_V V' \to X \times_B B'$ are étale,*

(2) *the strict transform U' of U relative to $V' \to V$ is equal to $X' \times_X U$ where X' is the strict transform of X relative to $B' \to B$, and*
(3) *for a quasi-coherent $\mathcal{O}_X$-module $\mathcal{F}$ the restriction of the strict transform $\mathcal{F}'$ to $U \times_V V'$ is the strict transform of $\mathcal{F}|_U$ relative to $V' \to V$.*

Proof. Part (1) follows from the fact that blowup commutes with flat base change (Lemma 53.6.3), the fact that étale morphisms are flat, and that the base change of an étale morphism is étale. Part (3) then follows from the fact that taking the sheaf of sections supported on a closed commutes with pullback by étale morphisms, see Limits of Spaces, Lemma 52.13.5. Part (2) follows from (3) applied to $\mathcal{F} = \mathcal{O}_X$. □

Lemma 53.7.3. *In the situation of Definition 53.7.1.*

(1) *The strict transform X' of X is the blowup of X in the closed subspace $f^{-1}Z$ of X.*
(2) *For a quasi-coherent $\mathcal{O}_X$-module $\mathcal{F}$ the strict transform $\mathcal{F}'$ is canonically isomorphic to the pushforward along $X' \to X \times_B B'$ of the strict transform of $\mathcal{F}$ relative to the blowing up $X' \to X$.*

Proof. Let $X'' \to X$ be the blowup of X in $f^{-1}Z$. By the universal property of blowing up (Lemma 53.6.5) there exists a commutative diagram

$$\begin{array}{ccc} X'' & \longrightarrow & X \\ \downarrow & & \downarrow \\ B' & \longrightarrow & B \end{array}$$

whence a morphism $i : X'' \to X \times_B B'$. The first assertion of the lemma is that i is a closed immersion with image X'. The second assertion of the lemma is that $\mathcal{F}' = i_*\mathcal{F}''$ where $\mathcal{F}''$ is the strict transform of $\mathcal{F}$ with respect to the blowing up $X'' \to X$. We can check these assertions étale locally on X, hence we reduce to the case of schemes (Divisors, Lemma 30.19.2). Some details omitted. □

Lemma 53.7.4. *In the situation of Definition 53.7.1.*

(1) *If X is flat over B at all points lying over Z, then the strict transform of X is equal to the base change $X \times_B B'$.*
(2) *Let $\mathcal{F}$ be a quasi-coherent $\mathcal{O}_X$-module. If $\mathcal{F}$ is flat over B at all points lying over Z, then the strict transform $\mathcal{F}'$ of $\mathcal{F}$ is equal to the pullback $pr_X^*\mathcal{F}$.*

Proof. Omitted. Hint: Follows from the case of schemes (Divisors, Lemma 30.19.3) by étale localization (Lemma 53.7.2). □

Lemma 53.7.5. *Let S be a scheme. Let B be an algebraic space over S. Let $Z \subset B$ be a closed subspace. Let $b : B' \to B$ be the blowing up of Z in B. Let $g : X \to Y$ be an affine morphism of spaces over B. Let $\mathcal{F}$ be a quasi-coherent sheaf on X. Let $g' : X \times_B B' \to Y \times_B B'$ be the base change of g. Let $\mathcal{F}'$ be the strict transform of $\mathcal{F}$ relative to b. Then $g'_*\mathcal{F}'$ is the strict transform of $g_*\mathcal{F}$.*

Proof. Omitted. Hint: Follows from the case of schemes (Divisors, Lemma 30.19.4) by étale localization (Lemma 53.7.2). □

Lemma 53.7.6. *Let S be a scheme. Let B be an algebraic space over S. Let $Z \subset B$ be a closed subspace. Let $D \subset B$ be an effective Cartier divisor. Let $Z' \subset B$*

be the closed subspace cut out by the product of the ideal sheaves of Z and D. Let $B' \to B$ be the blowup of B in Z.

(1) *The blowup of B in Z' is isomorphic to $B' \to B$.*
(2) *Let $f : X \to B$ be a morphism of algebraic spaces and let $\mathcal{F}$ be a quasi-coherent $\mathcal{O}_X$-module. If the subsheaf of $\mathcal{F}$ of sections supported on $|f^{-1}D|$ is zero, then the strict transform of $\mathcal{F}$ relative to the blowing up in Z agrees with the strict transform of $\mathcal{F}$ relative to the blowing up of B in Z'.*

Proof. Omitted. Hint: Follows from the case of schemes (Divisors, Lemma 30.19.5) by étale localization (Lemma 53.7.2). □

Lemma 53.7.7. *Let S be a scheme. Let B be an algebraic space over S. Let $Z \subset B$ be a closed subspace. Let $b : B' \to B$ be the blowing up with center Z. Let $Z' \subset B'$ be a closed subspace. Let $B'' \to B'$ be the blowing up with center Z'. Let $Y \subset B$ be a closed subscheme such that $|Y| = |Z| \cup |b|(|Z'|)$ and the composition $B'' \to B$ is isomorphic to the blowing up of B in Y. In this situation, given any scheme X over B and $\mathcal{F} \in QCoh(\mathcal{O}_X)$ we have*

(1) *the strict transform of $\mathcal{F}$ with respect to the blowing up of B in Y is equal to the strict transform with respect to the blowup $B'' \to B'$ in Z' of the strict transform of $\mathcal{F}$ with respect to the blowup $B' \to B$ of B in Z, and*
(2) *the strict transform of X with respect to the blowing up of B in Y is equal to the strict transform with respect to the blowup $B'' \to B'$ in Z' of the strict transform of X with respect to the blowup $B' \to B$ of B in Z.*

Proof. Omitted. Hint: Follows from the case of schemes (Divisors, Lemma 30.19.6) by étale localization (Lemma 53.7.2). □

Lemma 53.7.8. *In the situation of Definition 53.7.1. Suppose that*

$$0 \to \mathcal{F}_1 \to \mathcal{F}_2 \to \mathcal{F}_3 \to 0$$

is an exact sequence of quasi-coherent sheaves on X which remains exact after any base change $T \to B$. Then the strict transforms of $\mathcal{F}_i'$ relative to any blowup $B' \to B$ form a short exact sequence $0 \to \mathcal{F}_1' \to \mathcal{F}_2' \to \mathcal{F}_3' \to 0$ too.

Proof. Omitted. Hint: Follows from the case of schemes (Divisors, Lemma 30.19.7) by étale localization (Lemma 53.7.2). □

53.8. Admissible blowups

To have a bit more control over our blowups we introduce the following standard terminology.

Definition 53.8.1. Let S be a scheme. Let X be an algebraic space over S. Let $U \subset X$ be an open subspace. A morphism $X' \to X$ is called a *U-admissible blowup* if there exists a closed immersion $Z \to X$ of finite presentation with Z disjoint from U such that X' is isomorphic to the blow up of X in Z.

We recall that $Z \to X$ is of finite presentation if and only if the ideal sheaf $\mathcal{I}_Z \subset \mathcal{O}_X$ is of finite type, see Morphisms of Spaces, Lemma 49.26.12. In particular, a U-admissible blowup is a proper morphism, see Lemma 53.6.10. Note that there can be multiple centers which give rise to the same morphism. Hence the requirement is just the existence of some center disjoint from U which produces X'. Finally, as

the morphism $b : X' \to X$ is an isomorphism over U (see Lemma 53.6.4) we will often abuse notation and think of U as an open subspace of X' as well.

Lemma 53.8.2. *Let S be a scheme. Let X be a quasi-compact and quasi-separated algebraic space over S. Let $U \subset X$ be a quasi-compact open subspace. Let $b : X' \to X$ be a U-admissible blowup. Let $X'' \to X'$ be a U-admissible blowup. Then the composition $X'' \to X$ is a U-admissible blowup.*

Proof. Immediate from the more precise Lemma 53.6.11. □

Lemma 53.8.3. *Let S be a scheme. Let X be a quasi-compact and quasi-separated algebraic space. Let $U, V \subset X$ be quasi-compact open subspaces. Let $b : V' \to V$ be a $U \cap V$-admissible blowup. Then there exists a U-admissible blowup $X' \to X$ whose restriction to V is V'.*

Proof. Let $\mathcal{I} \subset \mathcal{O}_V$ be the finite type quasi-coherent sheaf of ideals such that $V(\mathcal{I})$ is disjoint from $U \cap V$ and such that V' is isomorphic to the blow up of V in $\mathcal{I}$. Let $\mathcal{I}' \subset \mathcal{O}_{U \cup V}$ be the quasi-coherent sheaf of ideals whose restriction to U is $\mathcal{O}_U$ and whose restriction to V is $\mathcal{I}$. By Limits of Spaces, Lemma 52.9.8 there exists a finite type quasi-coherent sheaf of ideals $\mathcal{J} \subset \mathcal{O}_X$ whose restriction to $U \cup V$ is $\mathcal{I}'$. The lemma follows. □

Lemma 53.8.4. *Let S be a scheme. Let X be a quasi-compact and quasi-separated algebraic space over S. Let $U \subset X$ be a quasi-compact open subspace. Let $b_i : X_i \to X$, $i = 1, \ldots, n$ be U-admissible blowups. There exists a U-admissible blowup $b : X' \to X$ such that (a) b factors as $X' \to X_i \to X$ for $i = 1, \ldots, n$ and (b) each of the morphisms $X' \to X_i$ is a U-admissible blowup.*

Proof. Let $\mathcal{I}_i \subset \mathcal{O}_X$ be the finite type quasi-coherent sheaf of ideals such that $V(\mathcal{I}_i)$ is disjoint from U and such that X_i is isomorphic to the blow up of X in $\mathcal{I}_i$. Set $\mathcal{I} = \mathcal{I}_1 \cdot \ldots \cdot \mathcal{I}_n$ and let X' be the blowup of X in $\mathcal{I}$. Then $X' \to X$ factors through b_i by Lemma 53.6.9. □

Lemma 53.8.5. *Let S be a scheme. Let X be a quasi-compact and quasi-separated algebraic space over S. Let U, V be quasi-compact disjoint open subspaces of X. Then there exist a $U \cup V$-admissible blowup $b : X' \to X$ such that X' is a disjoint union of open subspaces $X' = X'_1 \amalg X'_2$ with $b^{-1}(U) \subset X'_1$ and $b^{-1}(V) \subset X'_2$.*

Proof. Choose a finite type quasi-coherent sheaf of ideals $\mathcal{I}$, resp. $\mathcal{J}$ such that $X \setminus U = V(\mathcal{I})$, resp. $X \setminus V = V(\mathcal{J})$, see Limits of Spaces, Lemma 52.13.1. Then $|V(\mathcal{I}\mathcal{J})| = |X|$. Hence $\mathcal{I}\mathcal{J}$ is a locally nilpotent sheaf of ideals. Since $\mathcal{I}$ and $\mathcal{J}$ are of finite type and X is quasi-compact there exists an $n > 0$ such that $\mathcal{I}^n\mathcal{J}^n = 0$. We may and do replace $\mathcal{I}$ by $\mathcal{I}^n$ and $\mathcal{J}$ by $\mathcal{J}^n$. Whence $\mathcal{I}\mathcal{J} = 0$. Let $b : X' \to X$ be the blowing up in $\mathcal{I} + \mathcal{J}$. This is $U \cup V$-admissible as $|V(\mathcal{I} + \mathcal{J})| = |X| \setminus |U| \cup |V|$. We will show that X' is a disjoint union of open subspaces $X' = X'_1 \amalg X'_2$ as in the statement of the lemma.

Since $|V(\mathcal{I} + \mathcal{J})|$ is the complement of $|U \cup V|$ we conclude that $V \cup U$ is scheme theoretically dense in X', see Lemmas 53.6.4 and 53.2.4. Thus if such a decomposition $X' = X'_1 \amalg X'_2$ into open and closed subspaces exists, then X'_1 is the scheme theoretic closure of U in X' and similarly X'_2 is the scheme theoretic closure of V in X'. Since $U \to X'$ and $V \to X'$ are quasi-compact taking scheme theoretic closures commutes with étale localization (Morphisms of Spaces, Lemma 49.16.3).

Hence to verify the existence of X_1' and X_2' we may work étale locally on X. This reduces us to the case of schemes which is treated in the proof of Divisors, Lemma 30.20.5. □

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CHAPTER 54

Topologies on Algebraic Spaces

54.1. Introduction

In this chapter we introduce some topologies on the category of algebraic spaces. Compare with the material in [**Gro71**], [**BLR90**], [**LMB00**] and [**Knu71**]. Before doing so we would like to point out that there are many different choices of sites (as defined in Sites, Definition 7.6.2) which give rise to the same notion of sheaf on the underlying category. Hence our choices may be slightly different from those in the references but ultimately lead to the same cohomology groups, etc.

54.2. The general procedure

In this section we explain a general procedure for producing the sites we will be working with. This discussion will make little or no sense unless the reader has read Topologies, Section 33.2.

Let S be a base scheme. Take any category Sch_α constructed as in Sets, Lemma 3.9.2 starting with S and any set of schemes over S you want to be included. Choose any set of coverings Cov_{fppf} on Sch_α as in Sets, Lemma 3.11.1 starting with the category Sch_α and the class of fppf coverings. Let Sch_{fppf} denote the big fppf site so obtained, and let $(Sch/S)_{fppf}$ denote the corresponding big fppf site of S. (The above is entirely as prescribed in Topologies, Section 33.7.)

Given choices as above the category of algebraic spaces over S has a set of isomorphism classes. One way to see this is to use the fact that any algebraic space over S is of the form U/R for some étale equivalence relation $j : R \to U \times_S U$ with $U, R \in \mathrm{Ob}((Sch/S)_{fppf})$, see Spaces, Lemma 47.9.1. Hence we can find a full subcategory $Spaces/S$ of the category of algebraic spaces over S which has a set of objects such that each algebraic space is isomorphic to an object of $Spaces/S$. We fix a choice of such a category.

In the sections below, given a topology τ, the big site $(Spaces/S)_\tau$ (resp. the big site $(Spaces/X)_\tau$ of an algebraic space X over S) has as underlying category the category $Spaces/S$ (resp. the subcategory $Spaces/X$ of $Spaces/S$, see Categories, Example 4.2.13). The procedure for turning this into a site is as usual by defining a class of τ-coverings and using Sets, Lemma 3.11.1 to choose a sufficiently large set of coverings which defines the topology.

We point out that the *small étale site* $X_{étale}$ *of an algebraic space* X has already been defined in Properties of Spaces, Definition 48.15.1. Its objects are schemes étale over X, of which there are plenty by definition of an algebraic spaces. However, a more natural site, from the perspective of this chapter (compare Topologies, Definition 33.4.8) is the site $X_{spaces,étale}$ of Properties of Spaces, Definition 48.15.2.

These two sites define the same topos, see Properties of Spaces, Lemma 48.15.3. We will not redefine these in this chapter; instead we will simply use them.

Finally, we intend not to define the Zariski sites, since these do not seem particularly useful (although the Zariski topology is occasionally useful).

54.3. Fpqc topology

We briefly discuss the notion of an fpqc covering of algebraic spaces. Please compare with Topologies, Section 33.8. We will show in Descent on Spaces, Proposition 55.4.1 that quasi-coherent sheaves descent along these.

Definition 54.3.1. Let S be a scheme, and let X be an algebraic space over S. An *fpqc covering of X* is a family of morphisms $\{f_i : X_i \to X\}_{i\in I}$ of algebraic spaces such that each f_i is flat and such that for every affine scheme Z and morphism $h : Z \to X$ there exists a standard fpqc covering $\{g_j : Z_j \to Z\}_{j=1,\ldots,n}$ which refines the family $\{X_i \times_X Z \to Z\}_{i\in I}$.

In other words, there exists indices $i_1, \ldots, i_n \in I$ and morphisms $h_j : U_j \to X_{i_j}$ such that $f_{i_j} \circ h_j = h \circ g_j$. Note that if X and all X_i are representable, this is the same as a fpqc covering of schemes by Topologies, Lemma 33.8.11.

Lemma 54.3.2. *Let S be a scheme. Let X be an algebraic space over S.*

1. *If $X' \to X$ is an isomorphism then $\{X' \to X\}$ is an fpqc covering of X.*
2. *If $\{X_i \to X\}_{i\in I}$ is an fpqc covering and for each i we have an fpqc covering $\{X_{ij} \to X_i\}_{j\in J_i}$, then $\{X_{ij} \to X\}_{i\in I, j\in J_i}$ is an fpqc covering.*
3. *If $\{X_i \to X\}_{i\in I}$ is an fpqc covering and $X' \to X$ is a morphism of algebraic spaces then $\{X' \times_X X_i \to X'\}_{i\in I}$ is an fpqc covering.*

Proof. Part (1) is clear. Consider $g : X' \to X$ and $\{X_i \to X\}_{i\in I}$ an fpqc covering as in (3). By Morphisms of Spaces, Lemma 49.27.4 the morphisms $X' \times_X X_i \to X'$ are flat. If $h' : Z \to X'$ is a morphism from an affine scheme towards X', then set $h = g \circ h' : Z \to X$. The assumption on $\{X_i \to X\}_{i\in I}$ means there exists a standard fpqc covering $\{Z_j \to Z\}_{j=1,\ldots,n}$ and morphisms $Z_j \to X_{i(j)}$ covering h for certain $i(j) \in I$. By the universal property of the fibre product we obtain morphisms $Z_j \to X' \times_X X_{i(j)}$ over h' also. Hence $\{X' \times_X X_i \to X'\}_{i\in I}$ is an fpqc covering. This proves (3).

Let $\{X_i \to X\}_{i\in I}$ and $\{X_{ij} \to X_i\}_{j\in J_i}$ be as in (2). Let $h : Z \to X$ be a morphism from an affine scheme towards X. By assumption there exists a standard fpqc covering $\{Z_j \to Z\}_{j=1,\ldots,n}$ and morphisms $h_j : Z_j \to X_{i(j)}$ covering h for some indices $i(j) \in I$. By assumption there exist standard fpqc coverings $\{Z_{j,l} \to Z_j\}_{l=1,\ldots,n(j)}$ and morphisms $Z_{j,l} \to X_{i(j)j(l)}$ covering h_j for some indices $j(l) \in J_{i(j)}$. By Topologies, Lemma 33.8.10 the family $\{Z_{j,l} \to Z\}$ is a standard fpqc covering. Hence we conclude that $\{X_{ij} \to X\}_{i\in I, j\in J_i}$ is an fpqc covering. □

Lemma 54.3.3. *Let S be a scheme, and let X be an algebraic space over S. Suppose that $\{f_i : X_i \to X\}_{i\in I}$ is a family of morphisms of algebraic spaces with target X. Let $U \to X$ be a surjective étale morphism from a scheme towards X. Then $\{f_i : X_i \to X\}_{i\in I}$ is an fpqc covering of X if and only if $\{U \times_X X_i \to U\}_{i\in I}$ is an fpqc covering of U.*

Proof. If $\{X_i \to X\}_{i\in I}$ is an fpqc covering, then so is $\{U \times_X X_i \to U\}_{i\in I}$ by Lemma 54.3.2. Assume that $\{U \times_X X_i \to U\}_{i\in I}$ is an fpqc covering. Let $h : Z \to X$ be a morphism from an affine scheme towards X. Then we see that $U \times_X Z \to Z$ is a surjective étale morphism of schemes, in particular open. Hence we can find finitely many affine opens $W_1, \ldots, W_t$ of $U \times_X Z$ whose images cover Z. For each j we may apply the condition that $\{U \times_X X_i \to U\}_{i\in I}$ is an fpqc covering to the morphism $W_j \to U$, and obtain a standard fpqc covering $\{W_{jl} \to W_j\}$ which refines $\{W_j \times_X X_i \to W_j\}_{i\in I}$. Hence $\{W_{jl} \to Z\}$ is a standard fpqc covering of Z (see Topologies, Lemma 33.8.10) which refines $\{Z \times_X X_i \to X\}$ and we win. □

Lemma 54.3.4. *Let S be a scheme, and let X be an algebraic space over S. Suppose that $\mathcal{U} = \{f_i : X_i \to X\}_{i\in I}$ is an fpqc covering of X. Then there exists a refinement $\mathcal{V} = \{g_i : T_i \to X\}$ of $\mathcal{U}$ which is an fpqc covering such that each T_i is a scheme.*

Proof. Omitted. Hint: For each i choose a scheme T_i and a surjective étale morphism $T_i \to X_i$. Then check that $\{T_i \to X\}$ is an fpqc covering. □

To be continued...

54.4. Fppf topology

In this section we discuss the notion of an fppf covering of algebraic spaces, and we define the big fppf site of an algebraic space. Please compare with Topologies, Section 33.7.

Definition 54.4.1. Let S be a scheme, and let X be an algebraic space over S. An *fppf covering of X* is a family of morphisms $\{f_i : X_i \to X\}_{i\in I}$ of algebraic spaces such that each f_i is flat and locally of finite presentation and such that $\bigcup_{i\in I} f_i(X_i) = X$.

This is exactly the same as Topologies, Definition 33.7.1. In particular, if X and all the X_i are schemes, then we recover the usual notion of an fppf covering of schemes.

Lemma 54.4.2. *Let S be a scheme. Let X be an algebraic space over S.*

1. *If $X' \to X$ is an isomorphism then $\{X' \to X\}$ is an fppf covering of X.*
2. *If $\{X_i \to X\}_{i\in I}$ is an fppf covering and for each i we have an fppf covering $\{X_{ij} \to X_i\}_{j\in J_i}$, then $\{X_{ij} \to X\}_{i\in I, j\in J_i}$ is an fppf covering.*
3. *If $\{X_i \to X\}_{i\in I}$ is an fppf covering and $X' \to X$ is a morphism of algebraic spaces then $\{X' \times_X X_i \to X'\}_{i\in I}$ is an fppf covering.*

Proof. Omitted. □

Lemma 54.4.3. *Let S be a scheme, and let X be an algebraic space over S. Suppose that $\mathcal{U} = \{f_i : X_i \to X\}_{i\in I}$ is an fppf covering of X. Then there exists a refinement $\mathcal{V} = \{g_i : T_i \to X\}$ of $\mathcal{U}$ which is an fppf covering such that each T_i is a scheme.*

Proof. Omitted. Hint: For each i choose a scheme T_i and a surjective étale morphism $T_i \to X_i$. Then check that $\{T_i \to X\}$ is an fppf covering. □

Lemma 54.4.4. *Let S be a scheme. Let $\{f_i : X_i \to X\}_{i\in I}$ be an fppf covering of algebraic spaces over S. Then the map of sheaves*

$$\coprod X_i \longrightarrow X$$

is surjective.

Proof. Omitted. See Spaces, Remark 47.5.2 if you are confused about the meaning of this simple lemma. □

To be continued...

54.5. Syntomic topology

In this section we discuss the notion of a syntomic covering of algebraic spaces, and we define the big syntomic site of an algebraic space. Please compare with Topologies, Section 33.6.

Definition 54.5.1. Let S be a scheme, and let X be an algebraic space over S. A *syntomic covering of* X is a family of morphisms $\{f_i : X_i \to X\}_{i \in I}$ of algebraic spaces such that each f_i is syntomic and such that $\bigcup_{i \in I} f_i(X_i) = X$.

This is exactly the same as Topologies, Definition 33.6.1. In particular, if X and all the X_i are schemes, then we recover the usual notion of a syntomic covering of schemes.

Lemma 54.5.2. *Let S be a scheme. Let X be an algebraic space over S.*

1. *If $X' \to X$ is an isomorphism then $\{X' \to X\}$ is a syntomic covering of X.*
2. *If $\{X_i \to X\}_{i \in I}$ is a syntomic covering and for each i we have a syntomic covering $\{X_{ij} \to X_i\}_{j \in J_i}$, then $\{X_{ij} \to X\}_{i \in I, j \in J_i}$ is a syntomic covering.*
3. *If $\{X_i \to X\}_{i \in I}$ is a syntomic covering and $X' \to X$ is a morphism of algebraic spaces then $\{X' \times_X X_i \to X'\}_{i \in I}$ is a syntomic covering.*

Proof. Omitted. □

To be continued...

54.6. Smooth topology

In this section we discuss the notion of a smooth covering of algebraic spaces, and we define the big smooth site of an algebraic space. Please compare with Topologies, Section 33.5.

Definition 54.6.1. Let S be a scheme, and let X be an algebraic space over S. A *smooth covering of* X is a family of morphisms $\{f_i : X_i \to X\}_{i \in I}$ of algebraic spaces such that each f_i is smooth and such that $\bigcup_{i \in I} f_i(X_i) = X$.

This is exactly the same as Topologies, Definition 33.5.1. In particular, if X and all the X_i are schemes, then we recover the usual notion of a smooth covering of schemes.

Lemma 54.6.2. *Let S be a scheme. Let X be an algebraic space over S.*

1. *If $X' \to X$ is an isomorphism then $\{X' \to X\}$ is a smooth covering of X.*
2. *If $\{X_i \to X\}_{i \in I}$ is a smooth covering and for each i we have a smooth covering $\{X_{ij} \to X_i\}_{j \in J_i}$, then $\{X_{ij} \to X\}_{i \in I, j \in J_i}$ is a smooth covering.*
3. *If $\{X_i \to X\}_{i \in I}$ is a smooth covering and $X' \to X$ is a morphism of algebraic spaces then $\{X' \times_X X_i \to X'\}_{i \in I}$ is a smooth covering.*

Proof. Omitted. □

To be continued...

54.7. Étale topology

In this section we discuss the notion of a étale covering of algebraic spaces, and we define the big étale site of an algebraic space. Please compare with Topologies, Section 33.4.

Definition 54.7.1. Let S be a scheme, and let X be an algebraic space over S. A *étale covering of X* is a family of morphisms $\{f_i : X_i \to X\}_{i \in I}$ of algebraic spaces such that each f_i is étale and such that $\bigcup_{i \in I} f_i(X_i) = X$.

This is exactly the same as Topologies, Definition 33.4.1. In particular, if X and all the X_i are schemes, then we recover the usual notion of a étale covering of schemes.

Lemma 54.7.2. *Let S be a scheme. Let X be an algebraic space over S.*

1. *If $X' \to X$ is an isomorphism then $\{X' \to X\}$ is a étale covering of X.*
2. *If $\{X_i \to X\}_{i \in I}$ is a étale covering and for each i we have a étale covering $\{X_{ij} \to X_i\}_{j \in J_i}$, then $\{X_{ij} \to X\}_{i \in I, j \in J_i}$ is a étale covering.*
3. *If $\{X_i \to X\}_{i \in I}$ is a étale covering and $X' \to X$ is a morphism of algebraic spaces then $\{X' \times_X X_i \to X'\}_{i \in I}$ is a étale covering.*

Proof. Omitted. □

To be continued...

54.8. Zariski topology

In Spaces, Section 47.12 we introduced the notion of a Zariski covering of an algebraic space by open subspaces. Here is the corresponding notion with open subspaces replaces by open immersions.

Definition 54.8.1. Let S be a scheme, and let X be an algebraic space over S. A *Zariski covering of X* is a family of morphisms $\{f_i : X_i \to X\}_{i \in I}$ of algebraic spaces such that each f_i is an open immersion and such that $\bigcup_{i \in I} f_i(X_i) = X$.

Although Zariski coverings are occasionally useful the corresponding topology on the category of algebraic spaces is really too coarse, and not particularly useful. Still, it does define a site.

Lemma 54.8.2. *Let S be a scheme. Let X be an algebraic space over S.*

1. *If $X' \to X$ is an isomorphism then $\{X' \to X\}$ is a Zariski covering of X.*
2. *If $\{X_i \to X\}_{i \in I}$ is a Zariski covering and for each i we have a Zariski covering $\{X_{ij} \to X_i\}_{j \in J_i}$, then $\{X_{ij} \to X\}_{i \in I, j \in J_i}$ is a Zariski covering.*
3. *If $\{X_i \to X\}_{i \in I}$ is a Zariski covering and $X' \to X$ is a morphism of algebraic spaces then $\{X' \times_X X_i \to X'\}_{i \in I}$ is a Zariski covering.*

Proof. Omitted. □

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CHAPTER 55

Descent and Algebraic Spaces

55.1. Introduction

In the chapter on topologies on algebraic spaces (see Topologies on Spaces, Section 54.1) we introduced étale, fppf, smooth, syntomic and fpqc coverings of algebraic spaces. In this chapter we discuss what kind of structures over algebraic spaces can be descended through such coverings. See for example [**Gro95a**], [**Gro95b**], [**Gro95e**], [**Gro95f**], [**Gro95c**], and [**Gro95d**].

55.2. Conventions

The standing assumption is that all schemes are contained in a big fppf site Sch_{fppf}. And all rings A considered have the property that $\mathrm{Spec}(A)$ is (isomorphic) to an object of this big site.

Let S be a scheme and let X be an algebraic space over S. In this chapter and the following we will write $X \times_S X$ for the product of X with itself (in the category of algebraic spaces over S), instead of $X \times X$.

55.3. Descent data for quasi-coherent sheaves

This section is the analogue of Descent, Section 34.2 for algebraic spaces. It makes sense to read that section first.

Definition 55.3.1. Let S be a scheme. Let $\{f_i : X_i \to X\}_{i \in I}$ be a family of morphisms of algebraic spaces over S with fixed target X.

(1) A *descent datum* $(\mathcal{F}_i, \varphi_{ij})$ *for quasi-coherent sheaves* with respect to the given family is given by a quasi-coherent sheaf $\mathcal{F}_i$ on X_i for each $i \in I$, an isomorphism of quasi-coherent $\mathcal{O}_{X_i \times_X X_j}$-modules $\varphi_{ij} : \mathrm{pr}_0^*\mathcal{F}_i \to \mathrm{pr}_1^*\mathcal{F}_j$ for each pair $(i, j) \in I^2$ such that for every triple of indices $(i, j, k) \in I^3$ the diagram

$$\begin{array}{ccc} \mathrm{pr}_0^*\mathcal{F}_i & \xrightarrow{\mathrm{pr}_{02}^*\varphi_{ik}} & \mathrm{pr}_2^*\mathcal{F}_k \\ & {\scriptstyle \mathrm{pr}_{01}^*\varphi_{ij}} \searrow \quad \nearrow {\scriptstyle \mathrm{pr}_{12}^*\varphi_{jk}} & \\ & \mathrm{pr}_1^*\mathcal{F}_j & \end{array}$$

of $\mathcal{O}_{X_i \times_X X_j \times_X X_k}$-modules commutes. This is called the *cocycle condition*.

(2) A *morphism* $\psi : (\mathcal{F}_i, \varphi_{ij}) \to (\mathcal{F}'_i, \varphi'_{ij})$ *of descent data* is given by a family $\psi = (\psi_i)_{i \in I}$ of morphisms of $\mathcal{O}_{X_i}$-modules $\psi_i : \mathcal{F}_i \to \mathcal{F}'_i$ such that all the

diagrams

$$\begin{array}{ccc} \mathrm{pr}_0^*\mathcal{F}_i & \xrightarrow{\varphi_{ij}} & \mathrm{pr}_1^*\mathcal{F}_j \\ \downarrow{\scriptstyle \mathrm{pr}_0^*\psi_i} & & \downarrow{\scriptstyle \mathrm{pr}_1^*\psi_j} \\ \mathrm{pr}_0^*\mathcal{F}'_i & \xrightarrow{\varphi'_{ij}} & \mathrm{pr}_1^*\mathcal{F}'_j \end{array}$$

commute.

Lemma 55.3.2. *Let S be a scheme. Let $\mathcal{U} = \{U_i \to U\}_{i\in I}$ and $\mathcal{V} = \{V_j \to V\}_{j\in J}$ be families of morphisms of algebraic spaces over S with fixed targets. Let $(g, \alpha : I \to J, (g_i)) : \mathcal{U} \to \mathcal{V}$ be a morphism of families of maps with fixed target, see Sites, Definition 7.8.1. Let $(\mathcal{F}_j, \varphi_{jj'})$ be a descent datum for quasi-coherent sheaves with respect to the family $\{V_j \to V\}_{j\in J}$. Then*

(1) *The system*
$$\left(g_i^*\mathcal{F}_{\alpha(i)}, (g_i \times g_{i'})^*\varphi_{\alpha(i)\alpha(i')}\right)$$
is a descent datum with respect to the family $\{U_i \to U\}_{i\in I}$.
(2) *This construction is functorial in the descent datum $(\mathcal{F}_j, \varphi_{jj'})$.*
(3) *Given a second morphism $(g', \alpha' : I \to J, (g'_i))$ of families of maps with fixed target with $g = g'$ there exists a functorial isomorphism of descent data*
$$(g_i^*\mathcal{F}_{\alpha(i)}, (g_i \times g_{i'})^*\varphi_{\alpha(i)\alpha(i')}) \cong ((g'_i)^*\mathcal{F}_{\alpha'(i)}, (g'_i \times g'_{i'})^*\varphi_{\alpha'(i)\alpha'(i')}).$$

Proof. Omitted. Hint: The maps $g_i^*\mathcal{F}_{\alpha(i)} \to (g'_i)^*\mathcal{F}_{\alpha'(i)}$ which give the isomorphism of descent data in part (3) are the pullbacks of the maps $\varphi_{\alpha(i)\alpha'(i)}$ by the morphisms $(g_i, g'_i) : U_i \to V_{\alpha(i)} \times_V V_{\alpha'(i)}$. □

Let $g : U \to V$ be a morphism of algebraic spaces. The lemma above tells us that there is a well defined pullback functor between the categories of descent data relative to families of maps with target V and U provided there is a morphism between those families of maps which "lives over g".

Definition 55.3.3. Let S be a scheme. Let $\{U_i \to U\}_{i\in I}$ be a family of morphisms of algebraic spaces over S with fixed target.

(1) Let $\mathcal{F}$ be a quasi-coherent $\mathcal{O}_U$-module. We call the unique descent on $\mathcal{F}$ datum with respect to the covering $\{U \to U\}$ the *trivial descent datum*.
(2) The pullback of the trivial descent datum to $\{U_i \to U\}$ is called the *canonical descent datum*. Notation: $(\mathcal{F}|_{U_i}, can)$.
(3) A descent datum $(\mathcal{F}_i, \varphi_{ij})$ for quasi-coherent sheaves with respect to the given family is said to be *effective* if there exists a quasi-coherent sheaf $\mathcal{F}$ on U such that $(\mathcal{F}_i, \varphi_{ij})$ is isomorphic to $(\mathcal{F}|_{U_i}, can)$.

Lemma 55.3.4. *Let S be a scheme. Let U be an algebraic space over S. Let $\{U_i \to U\}$ be a Zariski covering of U, see Topologies on Spaces, Definition 54.8.1. Any descent datum on quasi-coherent sheaves for the family $\mathcal{U} = \{U_i \to U\}$ is effective. Moreover, the functor from the category of quasi-coherent $\mathcal{O}_U$-modules to the category of descent data with respect to $\{U_i \to U\}$ is fully faithful.*

Proof. Omitted. □

55.4. Fpqc descent of quasi-coherent sheaves

The main application of flat descent for modules is the corresponding descent statement for quasi-coherent sheaves with respect to fpqc-coverings.

Proposition 55.4.1. *Let S be a scheme. Let $\{X_i \to X\}$ be an fpqc covering of algebraic spaces over S, see Topologies on Spaces, Definition 54.3.1. Any descent datum on quasi-coherent sheaves for $\{X_i \to X\}$ is effective. Moreover, the functor from the category of quasi-coherent $\mathcal{O}_X$-modules to the category of descent data with respect to $\{X_i \to X\}$ is fully faithful.*

Proof. This is more or less a formal consequence of the corresponding result for schemes, see Descent, Proposition 34.5.2. Here is a strategy for a proof:

(1) The fact that $\{X_i \to X\}$ is a refinement of the trivial covering $\{X \to X\}$ gives, via Lemma 55.3.2, a functor $QCoh(\mathcal{O}_X) \to DD(\{X_i \to X\})$ from the category of quasi-coherent $\mathcal{O}_X$-modules to the category of descent data for the given family.
(2) In order to prove the proposition we will construct a quasi-inverse functor $back : DD(\{X_i \to X\}) \to QCoh(\mathcal{O}_X)$.
(3) Applying again Lemma 55.3.2 we see that there is a functor $DD(\{X_i \to X\}) \to DD(\{T_j \to X\})$ if $\{T_j \to X\}$ is a refinement of the given family. Hence in order to construct the functor *back* we may assume that each X_i is a scheme, see Topologies on Spaces, Lemma 54.3.4. This reduces us to the case where all the X_i are schemes.
(4) A quasi-coherent sheaf on X is by definition a quasi-coherent $\mathcal{O}_X$-module on $X_{étale}$. Now for any $U \in \mathrm{Ob}(X_{étale})$ we get an fppf covering $\{U_i \times_X X_i \to U\}$ by schemes and a morphism $g : \{U_i \times_X X_i \to U\} \to \{X_i \to X\}$ of coverings lying over $U \to X$. Given a descent datum $\xi = (\mathcal{F}_i, \varphi_{ij})$ we obtain a quasi-coherent $\mathcal{O}_U$-module $\mathcal{F}_{\xi,U}$ corresponding to the pullback $g^*\xi$ of Lemma 55.3.2 to the covering of U and using effectivity for fppf covering of schemes, see Descent, Proposition 34.5.2.
(5) Check that $\xi \mapsto \mathcal{F}_{\xi,U}$ is functorial in ξ. Omitted.
(6) Check that $\xi \mapsto \mathcal{F}_{\xi,U}$ is compatible with morphisms $U \to U'$ of the site $X_{étale}$, so that the system of sheaves $\mathcal{F}_{\xi,U}$ corresponds to a quasi-coherent $\mathcal{F}_\xi$ on $X_{étale}$, see Properties of Spaces, Lemma 48.27.3. Details omitted.
(7) Check that $back : \xi \mapsto \mathcal{F}_\xi$ is quasi-inverse to the functor constructed in (1). Omitted.

This finishes the proof. □

55.5. Descent of finiteness properties of modules

This section is the analogue for the case of algebraic spaces of Descent, Section 34.6. The goal is to show that one can check a quasi-coherent module has a certain finiteness conditions by checking on the members of a covering. We will repeatedly use the following proof scheme. Suppose that X is an algebraic space, and that $\{X_i \to X\}$ is a fppf (resp. fpqc) covering. Let $U \to X$ be a surjective étale morphism such that U is a scheme. Then there exists an fppf (resp. fpqc) covering $\{Y_j \to X\}$ such that

(1) $\{Y_j \to X\}$ is a refinement of $\{X_i \to X\}$,
(2) each Y_j is a scheme, and

(3) each morphism $Y_j \to X$ factors though U, and
(4) $\{Y_j \to U\}$ is an fppf (resp. fpqc) covering of U.

Namely, first refine $\{X_i \to X\}$ by an fppf (resp. fpqc) covering such that each X_i is a scheme, see Topologies on Spaces, Lemma 54.4.3, resp. Lemma 54.3.4. Then set $Y_i = U \times_X X_i$. A quasi-coherent $\mathcal{O}_X$-module $\mathcal{F}$ is of finite type, of finite presentation, etc if and only if the quasi-coherent $\mathcal{O}_U$-module $\mathcal{F}|_U$ is of finite type, of finite presentation, etc. Hence we can use the existence of the refinement $\{Y_j \to X\}$ to reduce the proof of the following lemmas to the case of schemes. We will indicate this by saying that "*the result follows from the case of schemes by étale localization*".

Lemma 55.5.1. *Let X be an algebraic space over a scheme S. Let $\mathcal{F}$ be a quasi-coherent $\mathcal{O}_X$-module. Let $\{f_i : X_i \to X\}_{i \in I}$ be an fpqc covering such that each $f_i^*\mathcal{F}$ is a finite type $\mathcal{O}_{X_i}$-module. Then $\mathcal{F}$ is a finite type $\mathcal{O}_X$-module.*

Proof. This follows from the case of schemes, see Descent, Lemma 34.6.1, by étale localization. □

Lemma 55.5.2. *Let X be an algebraic space over a scheme S. Let $\mathcal{F}$ be a quasi-coherent $\mathcal{O}_X$-module. Let $\{f_i : X_i \to X\}_{i \in I}$ be an fpqc covering such that each $f_i^*\mathcal{F}$ is an $\mathcal{O}_{X_i}$-module of finite presentation. Then $\mathcal{F}$ is an $\mathcal{O}_X$-module of finite presentation.*

Proof. This follows from the case of schemes, see Descent, Lemma 34.6.3, by étale localization. □

Lemma 55.5.3. *Let X be an algebraic space over a scheme S. Let $\mathcal{F}$ be a quasi-coherent $\mathcal{O}_X$-module. Let $\{f_i : X_i \to X\}_{i \in I}$ be an fpqc covering such that each $f_i^*\mathcal{F}$ is a flat $\mathcal{O}_{X_i}$-module. Then $\mathcal{F}$ is a flat $\mathcal{O}_X$-module.*

Proof. This follows from the case of schemes, see Descent, Lemma 34.6.5, by étale localization. □

Lemma 55.5.4. *Let X be an algebraic space over a scheme S. Let $\mathcal{F}$ be a quasi-coherent $\mathcal{O}_X$-module. Let $\{f_i : X_i \to X\}_{i \in I}$ be an fpqc covering such that each $f_i^*\mathcal{F}$ is a finite locally free $\mathcal{O}_{X_i}$-module. Then $\mathcal{F}$ is a finite locally free $\mathcal{O}_X$-module.*

Proof. This follows from the case of schemes, see Descent, Lemma 34.6.6, by étale localization. □

The definition of a locally projective quasi-coherent sheaf can be found in Properties of Spaces, Section 48.29. It is also proved there that this notion is preserved under pullback.

Lemma 55.5.5. *Let X be an algebraic space over a scheme S. Let $\mathcal{F}$ be a quasi-coherent $\mathcal{O}_X$-module. Let $\{f_i : X_i \to X\}_{i \in I}$ be an fpqc covering such that each $f_i^*\mathcal{F}$ is a locally projective $\mathcal{O}_{X_i}$-module. Then $\mathcal{F}$ is a locally projective $\mathcal{O}_X$-module.*

Proof. This follows from the case of schemes, see Descent, Lemma 34.6.7, by étale localization. □

We also add here two results which are related to the results above, but are of a slightly different nature.

Lemma 55.5.6. *Let S be a scheme. Let $f : X \to Y$ be a morphism of algebraic spaces over S. Let $\mathcal{F}$ be a quasi-coherent $\mathcal{O}_X$-module. Assume f is a finite morphism. Then $\mathcal{F}$ is an $\mathcal{O}_X$-module of finite type if and only if $f_*\mathcal{F}$ is an $\mathcal{O}_Y$-module of finite type.*

Proof. As f is finite it is representable. Choose a scheme V and a surjective étale morphism $V \to Y$. Then $U = V \times_Y X$ is a scheme with a surjective étale morphism towards X and a finite morphism $\psi : U \to V$ (the base change of f). Since $\psi_*(\mathcal{F}|_U) = f_*\mathcal{F}|_V$ the result of the lemma follows immediately from the schemes version which is Descent, Lemma 34.6.9. □

Lemma 55.5.7. *Let S be a scheme. Let $f : X \to Y$ be a morphism of algebraic spaces over S. Let $\mathcal{F}$ be a quasi-coherent $\mathcal{O}_X$-module. Assume f is finite and of finite presentation. Then $\mathcal{F}$ is an $\mathcal{O}_X$-module of finite presentation if and only if $f_*\mathcal{F}$ is an $\mathcal{O}_Y$-module of finite presentation.*

Proof. As f is finite it is representable. Choose a scheme V and a surjective étale morphism $V \to Y$. Then $U = V \times_Y X$ is a scheme with a surjective étale morphism towards X and a finite morphism $\psi : U \to V$ (the base change of f). Since $\psi_*(\mathcal{F}|_U) = f_*\mathcal{F}|_V$ the result of the lemma follows immediately from the schemes version which is Descent, Lemma 34.6.10. □

55.6. Fpqc coverings

This section is the analogue of Descent, Section 34.9. At the moment we do not know if all of the material for fpqc coverings of schemes holds also for algebraic spaces.

Lemma 55.6.1. *Let S be a scheme. Let $\{f_i : T_i \to T\}_{i \in I}$ be an fpqc covering of algebraic spaces over S. Suppose that for each i we have an open subspace $W_i \subset T_i$ such that for all $i, j \in I$ we have $pr_0^{-1}(W_i) = pr_1^{-1}(W_j)$ as open subspaces of $T_i \times_T T_j$. Then there exists a unique open subspace $W \subset T$ such that $W_i = f_i^{-1}(W)$ for each i.*

Proof. By Topologies on Spaces, Lemma 54.3.4 we may assume each T_i is a scheme. Choose a scheme U and a surjective étale morphism $U \to T$. Then $\{T_i \times_T U \to U\}$ is an fpqc covering of U and $T_i \times_T U$ is a scheme for each i. Hence we see that the collection of opens $W_i \times_T U$ comes from a unique open subscheme $W' \subset U$ by Descent, Lemma 34.9.2. As $U \to X$ is open we can define $W \subset X$ the Zariski open which is the image of W', see Properties of Spaces, Section 48.4. We omit the verification that this works, i.e., that W_i is the inverse image of W for each i. □

We do not know whether the following lemma holds for fpqc instead of fppf, see Properties of Spaces, Remark 48.14.4.

Lemma 55.6.2. *Let S be a scheme. Let $\{T_i \to T\}$ be an fppf covering of algebraic spaces over S, see Topologies on Spaces, Definition 54.4.1. Then given an algebraic space B over S the sequence*

$$\mathrm{Mor}_S(T, B) \longrightarrow \prod_i \mathrm{Mor}_S(T_i, B) \rightrightarrows \prod_{i,j} \mathrm{Mor}_S(T_i \times_T T_j, B)$$

is an equalizer diagram. In other words, every representable functor on the category of algebraic spaces over S satisfies the sheaf condition for the fppf topology.

Proof. We have seen in Topologies on Spaces, Lemma 54.4.4 that $\coprod T_i \to T$ is surjective as a map of sheaves. Also note that $T_i \times_T T_j$ is the fibre product as sheaves. Since we have

$$\mathrm{Mor}_S(T, B) = \mathrm{Mor}_{Sh((Sch/S)_{fppf})}(T, B)$$

by definition the lemma follows formally. □

55.7. Descent of finiteness properties of morphisms

The following type of lemma is occasionally useful.

Lemma 55.7.1. *Let S be a scheme. Let $X \to Y \to Z$ be morphism of algebraic spaces. Let P be one of the following properties of morphisms of algebraic spaces over S: flat, locally finite type, locally finite presentation. Assume that $X \to Z$ has P and that $X \to Y$ is a surjection of sheaves on $(Sch/S)_{fppf}$. Then $Y \to Z$ is P.*

Proof. Choose a scheme W and a surjective étale morphism $W \to Z$. Choose a scheme V and a surjective étale morphism $V \to W \times_Z Y$. Choose a scheme U and a surjective étale morphism $U \to V \times_Y X$. By assumption we can find an fppf covering $\{V_i \to V\}$ and lifts $V_i \to X$ of the morphism $V_i \to Y$. Since $U \to X$ is surjective étale we see that over the members of the fppf covering $\{V_i \times_X U \to V\}$ we have lifts into U. Hence $U \to V$ induces a surjection of sheaves on $(Sch/S)_{fppf}$. By our definition of what it means to have property P for a morphism of algebraic spaces (see Morphisms of Spaces, Definition 49.27.1, Definition 49.22.1, and Definition 49.26.1) we see that $U \to W$ has P and we have to show $V \to W$ has P. Thus we reduce the question to the case of morphisms of schemes which is treated in Descent, Lemma 34.10.8. □

55.8. Descending properties of spaces

In this section we put some results of the following kind.

Lemma 55.8.1. *Let S be a scheme. Let $f : X \to Y$ be a morphism of algebraic spaces over S. Let $x \in |X|$. If f is flat at x and X is geometrically unibranch at x, then Y is geometrically unibranch at $f(x)$.*

Proof. Consider the map of étale local rings $\mathcal{O}_{Y,f(\overline{x})} \to \mathcal{O}_{X,\overline{x}}$. By Morphisms of Spaces, Lemma 49.27.8 this is flat. Hence if $\mathcal{O}_{X,\overline{x}}$ has a unique minimal prime, so does $\mathcal{O}_{Y,f(\overline{x})}$ (by going down, see Algebra, Lemma 10.36.17). □

Lemma 55.8.2. *Let S be a scheme. Let $f : X \to Y$ be a morphism of algebraic spaces over S. If f is flat and surjective and X is reduced, then Y is reduced.*

Proof. Choose a scheme V and a surjective étale morphism $V \to Y$. Choose a scheme U and a surjective étale morphism $U \to X \times_Y V$. As f is surjective and flat, the morphism of schemes $U \to V$ is surjective and flat. In this way we reduce the problem to the case of schemes (as reducedness of X and Y is defined in terms of reducedness of U and V, see Properties of Spaces, Section 48.7). The case of schemes is Descent, Lemma 34.15.1. □

Lemma 55.8.3. *Let $f : X \to Y$ be a morphism of algebraic spaces. If f is locally of finite presentation, flat, and surjective and X is locally Noetherian, then Y is locally Noetherian.*

Proof. Choose a scheme V and a surjective étale morphism $V \to Y$. Choose a scheme U and a surjective étale morphism $U \to X \times_Y V$. As f is surjective, flat, and locally of finite presentation the morphism of schemes $U \to V$ is surjective, flat, and locally of finite presentation. In this way we reduce the problem to the case of schemes (as being locally Noetherian for X and Y is defined in terms of being locally Noetherian of U and V, see Properties of Spaces, Section 48.7). In the case of schemes the result follows from Descent, Lemma 34.12.1. □

Lemma 55.8.4. *Let $f : X \to Y$ be a morphism of algebraic spaces. If f is locally of finite presentation, flat, and surjective and X is regular, then Y is regular.*

Proof. By Lemma 55.8.3 we know that Y is locally Noetherian. Choose a scheme V and a surjective étale morphism $V \to Y$. It suffices to prove that the local rings of V are all regular local rings, see Properties, Lemma 27.9.2. Choose a scheme U and a surjective étale morphism $U \to X \times_Y V$. As f is surjective and flat the morphism of schemes $U \to V$ is surjective and flat. By assumption U is a regular scheme in particular all of its local rings are regular (by the lemma above). Hence the lemma follows from Algebra, Lemma 10.105.8. □

55.9. Descending properties of morphisms

In this section we introduce the notion of when a property of morphisms of algebraic spaces is local on the target in a topology. Please compare with Descent, Section 34.18.

Definition 55.9.1. Let S be a scheme. Let $\mathcal{P}$ be a property of morphisms of algebraic spaces over S. Let $\tau \in \{fpqc, fppf, syntomic, smooth, étale\}$. We say $\mathcal{P}$ is *τ local on the base*, or *τ local on the target*, or *local on the base for the τ-topology* if for any τ-covering $\{Y_i \to Y\}_{i \in I}$ of algebraic spaces and any morphism of algebraic spaces $f : X \to Y$ we have

$$f \text{ has } \mathcal{P} \Leftrightarrow \text{each } Y_i \times_Y X \to Y_i \text{ has } \mathcal{P}.$$

To be sure, since isomorphisms are always coverings we see (or require) that property $\mathcal{P}$ holds for $X \to Y$ if and only if it holds for any arrow $X' \to Y'$ isomorphic to $X \to Y$. If a property is τ-local on the target then it is preserved by base changes by morphisms which occur in τ-coverings. Here is a formal statement.

Lemma 55.9.2. *Let S be a scheme. Let $\tau \in \{fpqc, fppf, syntomic, smooth, étale\}$. Let $\mathcal{P}$ be a property of morphisms of algebraic spaces over S which is τ local on the target. Let $f : X \to Y$ have property $\mathcal{P}$. For any morphism $Y' \to Y$ which is flat, resp. flat and locally of finite presentation, resp. syntomic, resp. étale, the base change $f' : Y' \times_Y X \to Y'$ of f has property $\mathcal{P}$.*

Proof. This is true because we can fit $Y' \to Y$ into a family of morphisms which forms a τ-covering. □

A simple often used consequence of the above is that if $f : X \to Y$ has property $\mathcal{P}$ which is τ-local on the target and $f(X) \subset V$ for some open subspace $V \subset Y$, then also the induced morphism $X \to V$ has $\mathcal{P}$. Proof: The base change f by $V \to Y$ gives $X \to V$.

Lemma 55.9.3. *Let S be a scheme. Let $\tau \in \{fppf, syntomic, smooth, étale\}$. Let $\mathcal{P}$ be a property of morphisms of algebraic spaces over S which is τ local on the*

target. For any morphism of algebraic spaces $f : X \to Y$ over S there exists a largest open subspace $W(f) \subset Y$ such that the restriction $X_{W(f)} \to W(f)$ has $\mathcal{P}$. Moreover,

1. *if $g : Y' \to Y$ is a morphism of algebraic spaces which is flat and locally of finite presentation, syntomic, smooth, or étale and the base change $f' : X_{Y'} \to Y'$ has $\mathcal{P}$, then g factors through $W(f)$,*
2. *if $g : Y' \to Y$ is flat and locally of finite presentation, syntomic, smooth, or étale, then $W(f') = g^{-1}(W(f))$, and*
3. *if $\{g_i : Y_i \to Y\}$ is a τ-covering, then $g_i^{-1}(W(f)) = W(f_i)$, where f_i is the base change of f by $Y_i \to Y$.*

Proof. Consider the union $W_{set} \subset |Y|$ of the images $g(|Y'|) \subset |Y|$ of morphisms $g : Y' \to Y$ with the properties:

1. g is flat and locally of finite presentation, syntomic, smooth, or étale, and
2. the base change $Y' \times_{g,Y} X \to Y'$ has property $\mathcal{P}$.

Since such a morphism g is open (see Morphisms of Spaces, Lemma 49.27.6) we see that W_{set} is an open subset of $|Y|$. Denote $W \subset Y$ the open subspace whose underlying set of points is W_{set}, see Properties of Spaces, Lemma 48.4.8. Since $\mathcal{P}$ is local in the τ topology the restriction $X_W \to W$ has property $\mathcal{P}$ because we are given a covering $\{Y' \to W\}$ of W such that the pullbacks have $\mathcal{P}$. This proves the existence and proves that $W(f)$ has property (1). To see property (2) note that $W(f') \supset g^{-1}(W(f))$ because $\mathcal{P}$ is stable under base change by flat and locally of finite presentation, syntomic, smooth, or étale morphisms, see Lemma 55.9.2. On the other hand, if $Y'' \subset Y'$ is an open such that $X_{Y''} \to Y''$ has property $\mathcal{P}$, then $Y'' \to Y$ factors through W by construction, i.e., $Y'' \subset g^{-1}(W(f))$. This proves (2). Assertion (3) follows from (2) because each morphism $Y_i \to Y$ is flat and locally of finite presentation, syntomic, smooth, or étale by our definition of a τ-covering. □

Lemma 55.9.4. *Let S be a scheme. Let $\mathcal{P}$ be a property of morphisms of algebraic spaces over S. Assume*

1. *if $X_i \to Y_i$, $i = 1, 2$ have property $\mathcal{P}$ so does $X_1 \amalg X_2 \to Y_1 \amalg Y_2$,*
2. *a morphism of algebraic spaces $f : X \to Y$ has property $\mathcal{P}$ if and only if for every affine scheme Z and morphism $Z \to Y$ the base change $Z \times_Y X \to Z$ of f has property $\mathcal{P}$, and*
3. *for any surjective flat morphism of affine schemes $Z' \to Z$ over S and a morphism $f : X \to Z$ from an algebraic space to Z we have*

$$f' : Z' \times_Z X \to Z' \text{ has } \mathcal{P} \Rightarrow f \text{ has } \mathcal{P}.$$

Then $\mathcal{P}$ is fpqc local on the base.

Proof. If $\mathcal{P}$ has property (2), then it is automatically stable under any base change. Hence the direct implication in Definition 55.9.1.

Let $\{Y_i \to Y\}_{i \in I}$ be an fpqc covering of algebraic spaces over S. Let $f : X \to Y$ be a morphism of algebraic spaces over S. Assume each base change $f_i : Y_i \times_Y X \to Y_i$ has property $\mathcal{P}$. Our goal is to show that f has $\mathcal{P}$. Let Z be an affine scheme, and let $Z \to Y$ be a morphism. By (2) it suffices to show that the morphism of algebraic spaces $Z \times_Y X \to Z$ has $\mathcal{P}$. Since $\{Y_i \to Y\}_{i \in I}$ is an fpqc covering we know there

exists a standard fpqc covering $\{Z_j \to Z\}_{j=1,\ldots,n}$ and morphisms $Z_j \to Y_{i_j}$ over Y for suitable indices $i_j \in I$. Since f_{i_j} has $\mathcal{P}$ we see that

$$Z_j \times_Y X = Z_j \times_{Y_{i_j}} (Y_{i_j} \times_Y X) \longrightarrow Z_j$$

has $\mathcal{P}$ as a base change of f_{i_j} (see first remark of the proof). Set $Z' = \coprod_{j=1,\ldots,n} Z_j$, so that $Z' \to Z$ is a flat and surjective morphism of affine schemes over S. By (1) we conclude that $Z' \times_Y X \to Z'$ has property $\mathcal{P}$. Since this is the base change of the morphism $Z \times_Y X \to Z$ by the morphism $Z' \to Z$ we conclude that $Z \times_Y X \to Z$ has property $\mathcal{P}$ as desired. □

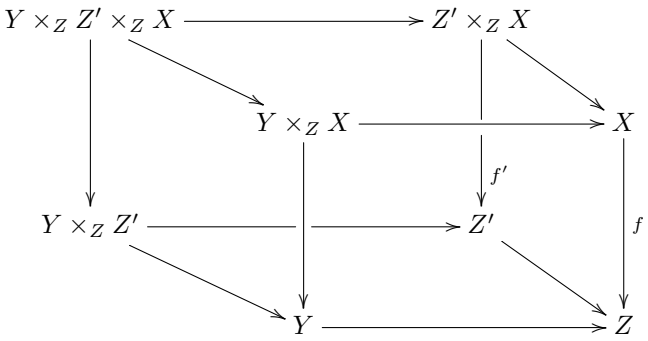
55.10. Descending properties of morphisms in the fpqc topology

In this section we find a large number of properties of morphisms of algebraic spaces which are local on the base in the fpqc topology. Please compare with Descent, Section 34.19 for the case of morphisms of schemes.

Lemma 55.10.1. *Let S be a scheme. The property $\mathcal{P}(f) =$ "f is quasi-compact" is fpqc local on the base on algebraic spaces over S.*

Proof. We will use Lemma 55.9.4 to prove this. Assumptions (1) and (2) of that lemma follow from Morphisms of Spaces, Lemma 49.9.7. Let $Z' \to Z$ be a surjective flat morphism of affine schemes over S. Let $f : X \to Z$ be a morphism of algebraic spaces, and assume that the base change $f' : Z' \times_Z X \to Z'$ is quasi-compact. We have to show that f is quasi-compact. To see this, using Morphisms of Spaces, Lemma 49.9.7 again, it is enough to show that for every affine scheme Y and morphism $Y \to Z$ the fibre product $Y \times_Z X$ is quasi-compact. Here is a picture:

(55.10.1.1)



Note that all squares are cartesian and the bottom square consists of affine schemes. The assumption that f' is quasi-compact combined with the fact that $Y \times_Z Z'$ is affine implies that $Y \times_Z Z' \times_Z X$ is quasi-compact. Since

$$Y \times_Z Z' \times_Z X \longrightarrow Y \times_Z X$$

is surjective as a base change of $Z' \to Z$ we conclude that $Y \times_Z X$ is quasi-compact, see Morphisms of Spaces, Lemma 49.9.5. This finishes the proof. □

Lemma 55.10.2. *Let S be a scheme. The property $\mathcal{P}(f) =$ "f is quasi-separated" is fpqc local on the base on algebraic spaces over S.*

Proof. A base change of a quasi-separated morphism is quasi-separated, see Morphisms of Spaces, Lemma 49.5.4. Hence the direct implication in Definition 55.9.1.

Let $\{Y_i \to Y\}_{i\in I}$ be an fpqc covering of algebraic spaces over S. Let $f : X \to Y$ be a morphism of algebraic spaces over S. Assume each base change $X_i := Y_i \times_Y X \to Y_i$ is quasi-separated. This means that each of the morphisms

$$\Delta_i : X_i \longrightarrow X_i \times_{Y_i} X_i = Y_i \times_Y (X \times_Y X)$$

is quasi-compact. The base change of a fpqc covering is an fpqc covering, see Topologies on Spaces, Lemma 54.3.2 hence $\{Y_i \times_Y (X \times_Y X) \to X \times_Y X\}$ is an fpqc covering of algebraic spaces. Moreover, each Δ_i is the base change of the morphism $\Delta : X \to X \times_Y X$. Hence it follows from Lemma 55.10.1 that Δ is quasi-compact, i.e., f is quasi-separated. □

Lemma 55.10.3. *Let S be a scheme. The property $\mathcal{P}(f) =$ "f is universally closed" is fpqc local on the base on algebraic spaces over S.*

Proof. We will use Lemma 55.9.4 to prove this. Assumptions (1) and (2) of that lemma follow from Morphisms of Spaces, Lemma 49.10.5. Let $Z' \to Z$ be a surjective flat morphism of affine schemes over S. Let $f : X \to Z$ be a morphism of algebraic spaces, and assume that the base change $f' : Z' \times_Z X \to Z'$ is universally closed. We have to show that f is universally closed. To see this, using Morphisms of Spaces, Lemma 49.10.5 again, it is enough to show that for every affine scheme Y and morphism $Y \to Z$ the map $|Y \times_Z X| \to |Y|$ is closed. Consider the cube (55.10.1.1). The assumption that f' is universally closed implies that $|Y \times_Z Z' \times_Z X| \to |Y \times_Z Z'|$ is closed. As $Y \times_Z Z' \to Y$ is surjective and flat as a base change of $Z' \to Z$ we see the map $|Y \times_Z Z'| \to |Y|$ is submersive, see Morphisms, Lemma 28.26.10. Moreover the map

$$|Y \times_Z Z' \times_Z X| \longrightarrow |Y \times_Z Z'| \times_{|Y|} |Y \times_Z X|$$

is surjective, see Properties of Spaces, Lemma 48.4.3. It follows by elementary topology that $|Y \times_Z X| \to |Y|$ is closed. □

Lemma 55.10.4. *Let S be a scheme. The property $\mathcal{P}(f) =$ "f is universally open" is fpqc local on the base on algebraic spaces over S.*

Proof. The proof is the same as the proof of Lemma 55.10.3. □

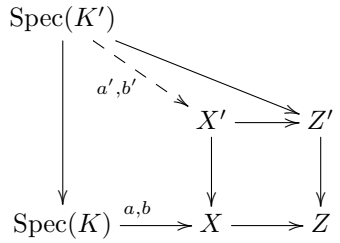
Lemma 55.10.5. *The property $\mathcal{P}(f) =$ "f is surjective" is fpqc local on the base.*

Proof. Omitted. (Hint: Use Properties of Spaces, Lemma 48.4.3.) □

Lemma 55.10.6. *The property $\mathcal{P}(f) =$ "f is universally injective" is fpqc local on the base.*

Proof. We will use Lemma 55.9.4 to prove this. Assumptions (1) and (2) of that lemma follow from Morphisms of Spaces, Lemma 49.10.5. Let $Z' \to Z$ be a flat surjective morphism of affine schemes over S and let $f : X \to Z$ be a morphism from an algebraic space to Z. Assume that the base change $f' : X' \to Z'$ is universally injective. Let K be a field, and let $a, b : \mathrm{Spec}(K) \to X$ be two morphisms such that $f \circ a = f \circ b$. As $Z' \to Z$ is surjective there exists a field extension $K \subset K'$ and a

morphism $\mathrm{Spec}(K') \to Z'$ such that the following solid diagram commutes



As the square is cartesian we get the two dotted arrows a', b' making the diagram commute. Since $X' \to Z'$ is universally injective we get $a' = b'$. This forces $a = b$ as $\{\mathrm{Spec}(K') \to \mathrm{Spec}(K)\}$ is an fpqc covering, see Properties of Spaces, Lemma 48.14.1. Hence f is universally injective as desired. □

Lemma 55.10.7. *The property $\mathcal{P}(f) =$ "f is locally of finite type" is fpqc local on the base.*

Proof. We will use Lemma 55.9.4 to prove this. Assumptions (1) and (2) of that lemma follow from Morphisms of Spaces, Lemma 49.22.4. Let $Z' \to Z$ be a surjective flat morphism of affine schemes over S. Let $f : X \to Z$ be a morphism of algebraic spaces, and assume that the base change $f' : Z' \times_Z X \to Z'$ is locally of finite type. We have to show that f is locally of finite type. Let U be a scheme and let $U \to X$ be surjective and étale. By Morphisms of Spaces, Lemma 49.22.4 again, it is enough to show that $U \to Z$ is locally of finite type. Since f' is locally of finite type, and since $Z' \times_Z U$ is a scheme étale over $Z' \times_Z X$ we conclude (by the same lemma again) that $Z' \times_Z U \to Z'$ is locally of finite type. As $\{Z' \to Z\}$ is an fpqc covering we conclude that $U \to Z$ is locally of finite type by Descent, Lemma 34.19.8 as desired. □

Lemma 55.10.8. *The property $\mathcal{P}(f) =$ "f is locally of finite presentation" is fpqc local on the base.*

Proof. We will use Lemma 55.9.4 to prove this. Assumptions (1) and (2) of that lemma follow from Morphisms of Spaces, Lemma 49.26.4. Let $Z' \to Z$ be a surjective flat morphism of affine schemes over S. Let $f : X \to Z$ be a morphism of algebraic spaces, and assume that the base change $f' : Z' \times_Z X \to Z'$ is locally of finite presentation. We have to show that f is locally of finite presentation. Let U be a scheme and let $U \to X$ be surjective and étale. By Morphisms of Spaces, Lemma 49.26.4 again, it is enough to show that $U \to Z$ is locally of finite presentation. Since f' is locally of finite presentation, and since $Z' \times_Z U$ is a scheme étale over $Z' \times_Z X$ we conclude (by the same lemma again) that $Z' \times_Z U \to Z'$ is locally of finite presentation. As $\{Z' \to Z\}$ is an fpqc covering we conclude that $U \to Z$ is locally of finite presentation by Descent, Lemma 34.19.9 as desired. □

Lemma 55.10.9. *The property $\mathcal{P}(f) =$ "f is of finite type" is fpqc local on the base.*

Proof. Combine Lemmas 55.10.1 and 55.10.7. □

Lemma 55.10.10. *The property $\mathcal{P}(f) =$ "f is of finite presentation" is fpqc local on the base.*

Proof. Combine Lemmas 55.10.1, 55.10.2 and 55.10.8. □

Lemma 55.10.11. *The property $\mathcal{P}(f) =$ "f is flat" is fpqc local on the base.*

Proof. We will use Lemma 55.9.4 to prove this. Assumptions (1) and (2) of that lemma follow from Morphisms of Spaces, Lemma 49.27.5. Let $Z' \to Z$ be a surjective flat morphism of affine schemes over S. Let $f : X \to Z$ be a morphism of algebraic spaces, and assume that the base change $f' : Z' \times_Z X \to Z'$ is flat. We have to show that f is flat. Let U be a scheme and let $U \to X$ be surjective and étale. By Morphisms of Spaces, Lemma 49.27.5 again, it is enough to show that $U \to Z$ is flat. Since f' is flat, and since $Z' \times_Z U$ is a scheme étale over $Z' \times_Z X$ we conclude (by the same lemma again) that $Z' \times_Z U \to Z'$ is flat. As $\{Z' \to Z\}$ is an fpqc covering we conclude that $U \to Z$ is flat by Descent, Lemma 34.19.13 as desired. □

Lemma 55.10.12. *The property $\mathcal{P}(f) =$ "f is an open immersion" is fpqc local on the base.*

Proof. We will use Lemma 55.9.4 to prove this. Assumptions (1) and (2) of that lemma follow from Morphisms of Spaces, Lemma 49.4.1. Consider a cartesian diagram

$$\begin{array}{ccc} X' & \longrightarrow & X \\ \downarrow & & \downarrow \\ Z' & \longrightarrow & Z \end{array}$$

of algebraic spaces over S where $Z' \to Z$ is a surjective flat morphism of affine schemes, and $X' \to Z'$ is an open immersion. We have to show that $X \to Z$ is an open immersion. Note that $|X'| \subset |Z'|$ corresponds to an open subscheme $U' \subset Z'$ (isomorphic to X') with the property that $\mathrm{pr}_0^{-1}(U') = \mathrm{pr}_1^{-1}(U')$ as open subschemes of $Z' \times_Z Z'$. Hence there exists an open subscheme $U \subset Z$ such that $X' = (Z' \to Z)^{-1}(U)$, see Descent, Lemma 34.9.2. As $X' \to Z'$ is quasi-separated also $X \to Z$ is quasi-separated, by Lemma 55.10.2. Hence X is quasi-separated over S (since Z is affine, hence separated, hence $Z \to S$ is separated, hence $X \to Z \to S$ is quasi-separated by Morphisms of Spaces, Lemma 49.5.8). Thus by Properties of Spaces, Lemma 48.14.3 we see that X satisfies the sheaf condition for the fpqc topology. Now we have the fpqc covering $\mathcal{U} = \{U' \to U\}$ and the element $U' \to X' \to X \in \check{H}^0(\mathcal{U}, X)$. By the sheaf condition we obtain a morphism $U \to X$ such that

$$\begin{array}{ccc} U' & \longrightarrow & U \\ \downarrow \cong & & \downarrow \\ X' & \longrightarrow & X \\ \downarrow & & \downarrow \\ Z' & \longrightarrow & Z \end{array}$$

is commutative. On the other hand, we know that for any scheme T pver S and T-valued point $T \to X$ the composition $T \to X \to Z$ is a morphism such that $Z' \times_Z T \to Z'$ factors through U'. Clearly this means that $T \to Z$ factors through U. In other words the map of sheaves $U \to X$ is bijective and we win. □

Lemma 55.10.13. *The property $\mathcal{P}(f) =$ "f is an isomorphism" is fpqc local on the base.*

Proof. Combine Lemmas 55.10.5 and 55.10.12. □

Lemma 55.10.14. *The property $\mathcal{P}(f) =$ "f is affine" is fpqc local on the base.*

Proof. We will use Lemma 55.9.4 to prove this. Assumptions (1) and (2) of that lemma follow from Morphisms of Spaces, Lemma 49.19.3. Let $Z' \to Z$ be a surjective flat morphism of affine schemes over S. Let $f : X \to Z$ be a morphism of algebraic spaces, and assume that the base change $f' : Z' \times_Z X \to Z'$ is affine. This implies that $Z' \times_Z X$ is representable, and that $Z' \times_Z X \to Z'$ is quasi-separated. Hence we conclude that $X \to Z$ is quasi-separated by Lemma 55.10.2. In particular X is quasi-separated over S, and satisfies the sheaf condition for the fpqc topology, see Properties of Spaces, Lemma 48.14.3.

Let X' be a scheme representing $Z' \times_Z X$. We obtain a canonical isomorphism

$$\varphi : X' \times_Z Z' \longrightarrow Z' \times_Z X'$$

since both schemes represent the algebraic space $Z' \times_Z Z' \times_Z X$. This is a descent datum for $X'/Z'/Z$, see Descent, Definition 34.30.1 (verification omitted, compare with Descent, Lemma 34.35.1). Since $X' \to Z'$ is affine this descent datum is effective, see Descent, Lemma 34.33.1. Thus there exists a scheme $Y \to Z$ over Z and an isomorphism $\psi : Z' \times_Z Y \to X'$ compatible with descent data. Of course $Y \to Z$ is affine (by construction or by Descent, Lemma 34.19.16). Note that $\mathcal{Y} = \{Z' \times_Z Y \to Y\}$ is a fpqc covering, and interpreting ψ as an element of $X(Z' \times_Z Y)$ we see that $\psi \in \check{H}^0(\mathcal{Y}, X)$. By the sheaf condition for X (see above) we obtain a morphism $Y \to X$. By construction the base change of this to Z' is an isomorphism, hence an isomorphism by Lemma 55.10.13. This proves that X is representable by an affine scheme and we win. □

Lemma 55.10.15. *The property $\mathcal{P}(f) =$ "f is a closed immersion" is fpqc local on the base.*

Proof. We will use Lemma 55.9.4 to prove this. Assumptions (1) and (2) of that lemma follow from Morphisms of Spaces, Lemma 49.4.1. Consider a cartesian diagram

$$\begin{array}{ccc} X' & \longrightarrow & X \\ \downarrow & & \downarrow \\ Z' & \longrightarrow & Z \end{array}$$

of algebraic spaces over S where $Z' \to Z$ is a surjective flat morphism of affine schemes, and $X' \to Z'$ is a closed immersion. We have to show that $X \to Z$ is a closed immersion. The morphism $X' \to Z'$ is affine. Hence by Lemma 55.10.14 we see that X is a scheme and $X \to Z$ is affine. It follows from Descent, Lemma 34.19.17 that $X \to Z$ is a closed immersion as desired. □

Lemma 55.10.16. *The property $\mathcal{P}(f) =$ "f is separated" is fpqc local on the base.*

Proof. A base change of a separated morphism is separated, see Morphisms of Spaces, Lemma 49.5.4. Hence the direct implication in Definition 55.9.1.

Let $\{Y_i \to Y\}_{i\in I}$ be an fpqc covering of algebraic spaces over S. Let $f : X \to Y$ be a morphism of algebraic spaces over S. Assume each base change $X_i := Y_i \times_Y X \to Y_i$ is separated. This means that each of the morphisms

$$\Delta_i : X_i \longrightarrow X_i \times_{Y_i} X_i = Y_i \times_Y (X \times_Y X)$$

is a closed immersion. The base change of a fpqc covering is an fpqc covering, see Topologies on Spaces, Lemma 54.3.2 hence $\{Y_i \times_Y (X \times_Y X) \to X \times_Y X\}$ is an fpqc covering of algebraic spaces. Moreover, each Δ_i is the base change of the morphism $\Delta : X \to X \times_Y X$. Hence it follows from Lemma 55.10.15 that Δ is a closed immersion, i.e., f is separated. □

Lemma 55.10.17. *The property $\mathcal{P}(f) =$ "f is proper" is fpqc local on the base.*

Proof. The lemma follows by combining Lemmas 55.10.3, 55.10.16 and 55.10.9. □

Lemma 55.10.18. *The property $\mathcal{P}(f) =$ "f is quasi-affine" is fpqc local on the base.*

Proof. We will use Lemma 55.9.4 to prove this. Assumptions (1) and (2) of that lemma follow from Morphisms of Spaces, Lemma 49.20.3. Let $Z' \to Z$ be a surjective flat morphism of affine schemes over S. Let $f : X \to Z$ be a morphism of algebraic spaces, and assume that the base change $f' : Z' \times_Z X \to Z'$ is quasi-affine. This implies that $Z' \times_Z X$ is representable, and that $Z' \times_Z X \to Z'$ is quasi-separated. Hence we conclude that $X \to Z$ is quasi-separated by Lemma 55.10.2. In particular X is quasi-separated over S, and satisfies the sheaf condition for the fpqc topology, see Properties of Spaces, Lemma 48.14.3.

Let X' be a scheme representing $Z' \times_Z X$. We obtain a canonical isomorphism

$$\varphi : X' \times_Z Z' \longrightarrow Z' \times_Z X'$$

since both schemes represent the algebraic space $Z' \times_Z Z' \times_Z X$. This is a descent datum for $X'/Z'/Z$, see Descent, Definition 34.30.1 (verification omitted, compare with Descent, Lemma 34.35.1). Since $X' \to Z'$ is quasi-affine this descent datum is effective, see Descent, Lemma 34.34.1. Thus there exists a scheme $Y \to Z$ over Z and an isomorphism $\psi : Z' \times_Z Y \to X'$ compatible with descent data. Of course $Y \to Z$ is quasi-affine (by construction or by Descent, Lemma 34.19.18). Note that $\mathcal{Y} = \{Z' \times_Z Y \to Y\}$ is a fpqc covering, and interpreting ψ as an element of $X(Z' \times_Z Y)$ we see that $\psi \in \check{H}^0(\mathcal{Y}, X)$. By the sheaf condition for X (see above) we obtain a morphism $Y \to X$. By construction the base change of this to Z' is an isomorphism, hence an isomorphism by Lemma 55.10.13. This proves that X is representable by a quasi-affine scheme and we win. □

Lemma 55.10.19. *The property $\mathcal{P}(f) =$ "f is a quasi-compact immersion" is fpqc local on the base.*

Proof. We will use Lemma 55.9.4 to prove this. Assumptions (1) and (2) of that lemma follow from Morphisms of Spaces, Lemmas 49.4.1 and 49.9.7. Consider a cartesian diagram

$$\begin{array}{ccc} X' & \longrightarrow & X \\ \downarrow & & \downarrow \\ Z' & \longrightarrow & Z \end{array}$$

of algebraic spaces over S where $Z' \to Z$ is a surjective flat morphism of affine schemes, and $X' \to Z'$ is a quasi-compact immersion. We have to show that $X \to Z$ is a closed immersion. The morphism $X' \to Z'$ is quasi-affine. Hence by Lemma 55.10.18 we see that X is a scheme and $X \to Z$ is quasi-affine. It follows from Descent, Lemma 34.19.19 that $X \to Z$ is a quasi-compact immersion as desired. □

Lemma 55.10.20. *The property $\mathcal{P}(f) =$ "f is integral" is fpqc local on the base.*

Proof. An integral morphism is the same thing as an affine, universally closed morphism. See Morphisms of Spaces, Lemma 49.40.7. Hence the lemma follows on combining Lemmas 55.10.3 and 55.10.14. □

Lemma 55.10.21. *The property $\mathcal{P}(f) =$ "f is finite" is fpqc local on the base.*

Proof. An finite morphism is the same thing as an integral, morphism which is locally of finite type. See Morphisms of Spaces, Lemma 49.40.6. Hence the lemma follows on combining Lemmas 55.10.7 and 55.10.20. □

Lemma 55.10.22. *The properties $\mathcal{P}(f) =$ "f is locally quasi-finite" and $\mathcal{P}(f) =$ "f is quasi-finite" are fpqc local on the base.*

Proof. We have already seen that "quasi-compact" is fpqc local on the base, see Lemma 55.10.1. Hence it is enough to prove the lemma for "locally quasi-finite". We will use Lemma 55.9.4 to prove this. Assumptions (1) and (2) of that lemma follow from Morphisms of Spaces, Lemma 49.25.5. Let $Z' \to Z$ be a surjective flat morphism of affine schemes over S. Let $f : X \to Z$ be a morphism of algebraic spaces, and assume that the base change $f' : Z' \times_Z X \to Z'$ is locally quasi-finite. We have to show that f is locally quasi-finite. Let U be a scheme and let $U \to X$ be surjective and étale. By Morphisms of Spaces, Lemma 49.25.5 again, it is enough to show that $U \to Z$ is locally quasi-finite. Since f' is locally quasi-finite, and since $Z' \times_Z U$ is a scheme étale over $Z' \times_Z X$ we conclude (by the same lemma again) that $Z' \times_Z U \to Z'$ is locally quasi-finite. As $\{Z' \to Z\}$ is an fpqc covering we conclude that $U \to Z$ is locally quasi-finite by Descent, Lemma 34.19.22 as desired. □

Lemma 55.10.23. *The property $\mathcal{P}(f) =$ "f is syntomic" is fpqc local on the base.*

Proof. We will use Lemma 55.9.4 to prove this. Assumptions (1) and (2) of that lemma follow from Morphisms of Spaces, Lemma 49.32.4. Let $Z' \to Z$ be a surjective flat morphism of affine schemes over S. Let $f : X \to Z$ be a morphism of algebraic spaces, and assume that the base change $f' : Z' \times_Z X \to Z'$ is syntomic. We have to show that f is syntomic. Let U be a scheme and let $U \to X$ be surjective and étale. By Morphisms of Spaces, Lemma 49.32.4 again, it is enough to show that $U \to Z$ is syntomic. Since f' is syntomic, and since $Z' \times_Z U$ is a scheme étale over $Z' \times_Z X$ we conclude (by the same lemma again) that $Z' \times_Z U \to Z'$ is syntomic. As $\{Z' \to Z\}$ is an fpqc covering we conclude that $U \to Z$ is syntomic by Descent, Lemma 34.19.24 as desired. □

Lemma 55.10.24. *The property $\mathcal{P}(f) =$ "f is smooth" is fpqc local on the base.*

Proof. We will use Lemma 55.9.4 to prove this. Assumptions (1) and (2) of that lemma follow from Morphisms of Spaces, Lemma 49.33.4. Let $Z' \to Z$ be a surjective flat morphism of affine schemes over S. Let $f : X \to Z$ be a morphism of algebraic spaces, and assume that the base change $f' : Z' \times_Z X \to Z'$ is smooth.

We have to show that f is smooth. Let U be a scheme and let $U \to X$ be surjective and étale. By Morphisms of Spaces, Lemma 49.33.4 again, it is enough to show that $U \to Z$ is smooth. Since f' is smooth, and since $Z' \times_Z U$ is a scheme étale over $Z' \times_Z X$ we conclude (by the same lemma again) that $Z' \times_Z U \to Z'$ is smooth. As $\{Z' \to Z\}$ is an fpqc covering we conclude that $U \to Z$ is smooth by Descent, Lemma 34.19.25 as desired. □

Lemma 55.10.25. *The property $\mathcal{P}(f) =$ "f is unramified" is fpqc local on the base.*

Proof. We will use Lemma 55.9.4 to prove this. Assumptions (1) and (2) of that lemma follow from Morphisms of Spaces, Lemma 49.34.5. Let $Z' \to Z$ be a surjective flat morphism of affine schemes over S. Let $f : X \to Z$ be a morphism of algebraic spaces, and assume that the base change $f' : Z' \times_Z X \to Z'$ is unramified. We have to show that f is unramified. Let U be a scheme and let $U \to X$ be surjective and étale. By Morphisms of Spaces, Lemma 49.34.5 again, it is enough to show that $U \to Z$ is unramified. Since f' is unramified, and since $Z' \times_Z U$ is a scheme étale over $Z' \times_Z X$ we conclude (by the same lemma again) that $Z' \times_Z U \to Z'$ is unramified. As $\{Z' \to Z\}$ is an fpqc covering we conclude that $U \to Z$ is unramified by Descent, Lemma 34.19.26 as desired. □

Lemma 55.10.26. *The property $\mathcal{P}(f) =$ "f is étale" is fpqc local on the base.*

Proof. We will use Lemma 55.9.4 to prove this. Assumptions (1) and (2) of that lemma follow from Morphisms of Spaces, Lemma 49.35.2. Let $Z' \to Z$ be a surjective flat morphism of affine schemes over S. Let $f : X \to Z$ be a morphism of algebraic spaces, and assume that the base change $f' : Z' \times_Z X \to Z'$ is étale. We have to show that f is étale. Let U be a scheme and let $U \to X$ be surjective and étale. By Morphisms of Spaces, Lemma 49.35.2 again, it is enough to show that $U \to Z$ is étale. Since f' is étale, and since $Z' \times_Z U$ is a scheme étale over $Z' \times_Z X$ we conclude (by the same lemma again) that $Z' \times_Z U \to Z'$ is étale. As $\{Z' \to Z\}$ is an fpqc covering we conclude that $U \to Z$ is étale by Descent, Lemma 34.19.27 as desired. □

Lemma 55.10.27. *The property $\mathcal{P}(f) =$ "f is finite locally free" is fpqc local on the base.*

Proof. Being finite locally free is equivalent to being finite, flat and locally of finite presentation (Morphisms of Spaces, Lemma 49.41.6). Hence this follows from Lemmas 55.10.21, 55.10.11, and 55.10.8. □

Lemma 55.10.28. *The property $\mathcal{P}(f) =$ "f is a monomorphism" is fpqc local on the base.*

Proof. Let $f : X \to Y$ be a morphism of algebraic spaces. Let $\{Y_i \to Y\}$ be an fpqc covering, and assume each of the base changes $f_i : X_i \to Y_i$ of f is a monomorphism. We have to show that f is a monomorphism.

First proof. Note that f is a monomorphism if and only if $\Delta : X \to X \times_Y X$ is an isomorphism. By applying this to f_i we see that each of the morphisms

$$\Delta_i : X_i \longrightarrow X_i \times_{Y_i} X_i = Y_i \times_Y (X \times_Y X)$$

is an isomorphism. The base change of an fpqc covering is an fpqc covering, see Topologies on Spaces, Lemma 54.3.2 hence $\{Y_i \times_Y (X \times_Y X) \to X \times_Y X\}$ is an

fpqc covering of algebraic spaces. Moreover, each Δ_i is the base change of the morphism $\Delta : X \to X \times_Y X$. Hence it follows from Lemma 55.10.13 that Δ is an isomorphism, i.e., f is a monomorphism.

Second proof. Let V be a scheme, and let $V \to Y$ be a surjective étale morphism. If we can show that $V \times_Y X \to V$ is a monomorphism, then it follows that $X \to Y$ is a monomorphism. Namely, given any cartesian diagram of sheaves

$$\begin{array}{ccc} \mathcal{F} & \xrightarrow{a} & \mathcal{G} \\ {\scriptstyle b}\downarrow & & \downarrow{\scriptstyle c} \\ \mathcal{H} & \xrightarrow{d} & \mathcal{I} \end{array} \qquad \mathcal{F} = \mathcal{H} \times_{\mathcal{I}} \mathcal{G}$$

if c is a surjection of sheaves, and a is injective, then also d is injective. This reduces the problem to the case where Y is a scheme. Moreover, in this case we may assume that the algebraic spaces Y_i are schemes also, since we can always refine the covering to place ourselves in this situation, see Topologies on Spaces, Lemma 54.3.4.

Assume $\{Y_i \to Y\}$ is an fpqc covering of schemes. Let $a, b : T \to X$ be two morphisms such that $f \circ a = f \circ b$. We have to show that $a = b$. Since f_i is a monomorphism we see that $a_i = b_i$, where $a_i, b_i : Y_i \times_Y T \to X_i$ are the base changes. In particular the compositions $Y_i \times_Y T \to T \to X$ are equal. Since $\{Y_i \times_Y T \to T\}$ is an fpqc covering we deduce that $a = b$ from Properties of Spaces, Lemma 48.14.1. □

55.11. Descending properties of morphisms in the fppf topology

In this section we find some properties of morphisms of algebraic spaces for which we could not (yet) show they are local on the base in the fpqc topology which, however, are local on the base in the fppf topology.

Lemma 55.11.1. *The property $\mathcal{P}(f) =$ "f is an immersion" is fppf local on the base.*

Proof. Let $f : X \to Y$ be a morphism of algebraic spaces. Let $\{Y_i \to Y\}_{i \in I}$ be an fppf covering of Y. Let $f_i : X_i \to Y_i$ be the base change of f.

If f is an immersion, then each f_i is an immersion by Spaces, Lemma 47.12.3. This proves the direct implication in Definition 55.9.1.

Conversely, assume each f_i is an immersion. By Morphisms of Spaces, Lemma 49.11.7 this implies each f_i is separated. By Morphisms of Spaces, Lemma 49.25.6 this implies each f_i is locally quasi-finite. Hence we see that f is locally quasi-finite and separated, by applying Lemmas 55.10.16 and 55.10.22. By Morphisms of Spaces, Lemma 49.44.1 this implies that f is representable!

By Morphisms of Spaces, Lemma 49.4.1 it suffices to show that for every scheme Z and morphism $Z \to Y$ the base change $Z \times_Y X \to Z$ is an immersion. By Topologies on Spaces, Lemma 54.4.3 we can find an fppf covering $\{Z_i \to Z\}$ by schemes which refines the pullback of the covering $\{Y_i \to Y\}$ to Z. Hence we see that $Z \times_Y X \to Z$ (which is a morphism of schemes according to the result of the preceding paragraph) becomes an immersion after pulling back to the members of an fppf (by schemes) of Z. Hence $Z \times_Y X \to Z$ is an immersion by the result for schemes, see Descent, Lemma 34.20.1. □

Lemma 55.11.2. *The property $\mathcal{P}(f) =$ "f is locally separated" is fppf local on the base.*

Proof. A base change of a locally separated morphism is locally separated, see Morphisms of Spaces, Lemma 49.5.4. Hence the direct implication in Definition 55.9.1.

Let $\{Y_i \to Y\}_{i\in I}$ be an fppf covering of algebraic spaces over S. Let $f : X \to Y$ be a morphism of algebraic spaces over S. Assume each base change $X_i := Y_i \times_Y X \to Y_i$ is locally separated. This means that each of the morphisms

$$\Delta_i : X_i \longrightarrow X_i \times_{Y_i} X_i = Y_i \times_Y (X \times_Y X)$$

is an immersion. The base change of a fppf covering is an fppf covering, see Topologies on Spaces, Lemma 54.4.2 hence $\{Y_i \times_Y (X \times_Y X) \to X \times_Y X\}$ is an fppf covering of algebraic spaces. Moreover, each Δ_i is the base change of the morphism $\Delta : X \to X \times_Y X$. Hence it follows from Lemma 55.11.1 that Δ is a immersion, i.e., f is locally separated. □

55.12. Properties of morphisms local on the source

In this section we define what it means for a property of morphisms of algebraic spaces to be local on the source. Please compare with Descent, Section 34.22.

Definition 55.12.1. Let S be a scheme. Let $\mathcal{P}$ be a property of morphisms of algebraic spaces over S. Let $\tau \in \{fpqc, fppf, syntomic, smooth, étale\}$. We say $\mathcal{P}$ is *τ local on the source*, or *local on the source for the τ-topology* if for any morphism $f : X \to Y$ of algebraic spaces over S, and any τ-covering $\{X_i \to X\}_{i\in I}$ of algebraic spaces we have

$$f \text{ has } \mathcal{P} \Leftrightarrow \text{each } X_i \to Y \text{ has } \mathcal{P}.$$

To be sure, since isomorphisms are always coverings we see (or require) that property $\mathcal{P}$ holds for $X \to Y$ if and only if it holds for any arrow $X' \to Y'$ isomorphic to $X \to Y$. If a property is τ-local on the source then it is preserved by precomposing with morphisms which occur in τ-coverings. Here is a formal statement.

Lemma 55.12.2. *Let S be a scheme. Let $\tau \in \{fpqc, fppf, syntomic, smooth, étale\}$. Let $\mathcal{P}$ be a property of morphisms of algebraic spaces over S which is τ local on the source. Let $f : X \to Y$ have property $\mathcal{P}$. For any morphism $a : X' \to X$ which is flat, resp. flat and locally of finite presentation, resp. syntomic, resp. smooth, resp. étale, the composition $f \circ a : X' \to Y$ has property $\mathcal{P}$.*

Proof. This is true because we can fit $X' \to X$ into a family of morphisms which forms a τ-covering. □

Lemma 55.12.3. *Let S be a scheme. Let $\tau \in \{fpqc, fppf, syntomic, smooth, étale\}$. Suppose that $\mathcal{P}$ is a property of morphisms of schemes over S which is étale local on the source-and-target. Denote $\mathcal{P}_{spaces}$ the corresponding property of morphisms of algebraic spaces over S, see Morphisms of Spaces, Definition 49.21.2. If $\mathcal{P}$ is local on the source for the τ-topology, then $\mathcal{P}_{spaces}$ is local on the source for the τ-topology.*

Proof. Let $f : X \to Y$ be a morphism of of algebraic spaces over S. Let $\{X_i \to X\}_{i\in I}$ be a τ-covering of algebraic spaces. Choose a scheme V and a surjective étale morphism $V \to Y$. Choose a scheme U and a surjective étale morphism

$U \to X \times_Y V$. For each i choose a scheme U_i and a surjective étale morphism $U_i \to X_i \times_X U$.

Note that $\{X_i \times_X U \to U\}_{i \in I}$ is a τ-covering. Note that each $\{U_i \to X_i \times_X U\}$ is an étale covering, hence a τ-covering. Hence $\{U_i \to U\}_{i \in I}$ is a τ-covering of algebraic spaces over S. But since U and each U_i is a scheme we see that $\{U_i \to U\}_{i \in I}$ is a τ-covering of schemes over S.

Now we have

$$\begin{aligned} f \text{ has } \mathcal{P}_{spaces} &\Leftrightarrow U \to V \text{ has } \mathcal{P} \\ &\Leftrightarrow \text{each } U_i \to V \text{ has } \mathcal{P} \\ &\Leftrightarrow \text{each } X_i \to Y \text{ has } \mathcal{P}_{spaces}. \end{aligned}$$

the first and last equivalence by the definition of $\mathcal{P}_{spaces}$ the middle equivalence because we assumed $\mathcal{P}$ is local on the source in the τ-topology. □

55.13. Properties of morphisms local in the fpqc topology on the source

Here are some properties of morphisms that are fpqc local on the source.

Lemma 55.13.1. *The property $\mathcal{P}(f) =$ "f is flat" is fpqc local on the source.*

Proof. Follows from Lemma 55.12.3 using Morphisms of Spaces, Definition 49.27.1 and Descent, Lemma 34.23.1. □

55.14. Properties of morphisms local in the fppf topology on the source

Here are some properties of morphisms that are fppf local on the source.

Lemma 55.14.1. *The property $\mathcal{P}(f) =$ "f is locally of finite presentation" is fppf local on the source.*

Proof. Follows from Lemma 55.12.3 using Morphisms of Spaces, Definition 49.26.1 and Descent, Lemma 34.24.1. □

Lemma 55.14.2. *The property $\mathcal{P}(f) =$ "f is locally of finite type" is fppf local on the source.*

Proof. Follows from Lemma 55.12.3 using Morphisms of Spaces, Definition 49.22.1 and Descent, Lemma 34.24.2. □

Lemma 55.14.3. *The property $\mathcal{P}(f) =$ "f is open" is fppf local on the source.*

Proof. Follows from Lemma 55.12.3 using Morphisms of Spaces, Definition 49.7.2 and Descent, Lemma 34.24.3. □

Lemma 55.14.4. *The property $\mathcal{P}(f) =$ "f is universally open" is fppf local on the source.*

Proof. Follows from Lemma 55.12.3 using Morphisms of Spaces, Definition 49.7.2 and Descent, Lemma 34.24.4. □

55.15. Properties of morphisms local in the syntomic topology on the source

Here are some properties of morphisms that are syntomic local on the source.

Lemma 55.15.1. *The property $\mathcal{P}(f) =$"f is syntomic" is syntomic local on the source.*

Proof. Follows from Lemma 55.12.3 using Morphisms of Spaces, Definition 49.32.1 and Descent, Lemma 34.25.1. □

55.16. Properties of morphisms local in the smooth topology on the source

Here are some properties of morphisms that are smooth local on the source.

Lemma 55.16.1. *The property $\mathcal{P}(f) =$"f is smooth" is smooth local on the source.*

Proof. Follows from Lemma 55.12.3 using Morphisms of Spaces, Definition 49.33.1 and Descent, Lemma 34.26.1. □

55.17. Properties of morphisms local in the étale topology on the source

Here are some properties of morphisms that are étale local on the source.

Lemma 55.17.1. *The property $\mathcal{P}(f) =$"f is étale" is étale local on the source.*

Proof. Follows from Lemma 55.12.3 using Morphisms of Spaces, Definition 49.35.1 and Descent, Lemma 34.27.1. □

Lemma 55.17.2. *The property $\mathcal{P}(f) =$"f is locally quasi-finite" is étale local on the source.*

Proof. Follows from Lemma 55.12.3 using Morphisms of Spaces, Definition 49.25.1 and Descent, Lemma 34.27.2. □

Lemma 55.17.3. *The property $\mathcal{P}(f) =$"f is unramified" is étale local on the source.*

Proof. Follows from Lemma 55.12.3 using Morphisms of Spaces, Definition 49.34.1 and Descent, Lemma 34.27.3. □

55.18. Properties of morphisms smooth local on source-and-target

Let $\mathcal{P}$ be a property of morphisms of algebraic spaces. There is an intuitive meaning to the phrase "$\mathcal{P}$ is smooth local on the source and target". However, it turns out that this notion is not the same as asking $\mathcal{P}$ to be both smooth local on the source and smooth local on the target. We have discussed a similar phenomenon (for the étale topology and the category of schemes) in great detail in Descent, Section 34.28 (for a quick overview take a look at Descent, Remark 34.28.8). However, there is an important difference between the case of the smooth and the étale topology. To see this difference we encourage the reader to ponder the difference between Descent, Lemma 34.28.4 and Lemma 55.18.2 as well as the difference between Descent, Lemma 34.28.5 and Lemma 55.18.3. Namely, in the étale setting the choice of the étale "covering" of the target is immaterial, whereas in the smooth setting it is not.

Definition 55.18.1. Let S be a scheme. Let $\mathcal{P}$ be a property of morphisms of algebraic spaces over S. We say $\mathcal{P}$ is *smooth local on source-and-target* if

(1) (stable under precomposing with smooth maps) if $f : X \to Y$ is smooth and $g : Y \to Z$ has $\mathcal{P}$, then $g \circ f$ has $\mathcal{P}$,
(2) (stable under smooth base change) if $f : X \to Y$ has $\mathcal{P}$ and $Y' \to Y$ is smooth, then the base change $f' : Y' \times_Y X \to Y'$ has $\mathcal{P}$, and
(3) (locality) given a morphism $f : X \to Y$ the following are equivalent
   (a) f has $\mathcal{P}$,
   (b) for every $x \in |X|$ there exists a commutative diagram

$$\begin{array}{ccc} U & \xrightarrow[h]{} & V \\ {\scriptstyle a}\downarrow & & \downarrow{\scriptstyle b} \\ X & \xrightarrow{f} & Y \end{array}$$

   with smooth vertical arrows and $u \in |U|$ with $a(u) = x$ such that h has $\mathcal{P}$.

The above serves as our definition. In the lemmas below we will show that this is equivalent to $\mathcal{P}$ being local on the target, local on the source, and stable under post-composing by smooth morphisms.

Lemma 55.18.2. *Let S be a scheme. Let $\mathcal{P}$ be a property of morphisms of algebraic spaces over S which is smooth local on source-and-target. Then*

(1) *$\mathcal{P}$ is smooth local on the source,*
(2) *$\mathcal{P}$ is smooth local on the target,*
(3) *$\mathcal{P}$ is stable under postcomposing with smooth morphisms: if $f : X \to Y$ has $\mathcal{P}$ and $g : Y \to Z$ is smooth, then $g \circ f$ has $\mathcal{P}$, and*

Proof. We write everything out completely.

Proof of (1). Let $f : X \to Y$ be a morphism of algebraic spaces over S. Let $\{X_i \to X\}_{i \in I}$ be a smooth covering of X. If each composition $h_i : X_i \to Y$ has $\mathcal{P}$, then for each $|x| \in X$ we can find an $i \in I$ and a point $x_i \in |X_i|$ mapping to x. Then $(X_i, x_i) \to (X, x)$ is a smooth morphism of pairs, and $\mathrm{id}_Y : Y \to Y$ is a smooth morphism, and h_i is as in part (3) of Definition 55.18.1. Thus we see that f has $\mathcal{P}$. Conversely, if f has $\mathcal{P}$ then each $X_i \to Y$ has $\mathcal{P}$ by Definition 55.18.1 part (1).

Proof of (2). Let $f : X \to Y$ be a morphism of algebraic spaces over S. Let $\{Y_i \to Y\}_{i \in I}$ be a smooth covering of Y. Write $X_i = Y_i \times_Y X$ and $h_i : X_i \to Y_i$ for the base change of f. If each $h_i : X_i \to Y_i$ has $\mathcal{P}$, then for each $x \in |X|$ we pick an $i \in I$ and a point $x_i \in |X_i|$ mapping to x. Then $(X_i, x_i) \to (X, x)$ is a smooth morphism of pairs, $Y_i \to Y$ is smooth, and h_i is as in part (3) of Definition 55.18.1. Thus we see that f has $\mathcal{P}$. Conversely, if f has $\mathcal{P}$, then each $X_i \to Y_i$ has $\mathcal{P}$ by Definition 55.18.1 part (2).

Proof of (3). Assume $f : X \to Y$ has $\mathcal{P}$ and $g : Y \to Z$ is smooth. For every $x \in |X|$ we can think of $(X, x) \to (X, x)$ as a smooth morphism of pairs, $Y \to Z$ is a smooth morphism, and $h = f$ is as in part (3) of Definition 55.18.1. Thus we see that $g \circ f$ has $\mathcal{P}$. □

The following lemma is the analogue of Morphisms, Lemma 28.15.4.

Lemma 55.18.3. *Let S be a scheme. Let $\mathcal{P}$ be a property of morphisms of algebraic spaces over S which is smooth local on source-and-target. Let $f : X \to Y$ be a morphism of algebraic spaces over S. The following are equivalent:*

(a) *f has property $\mathcal{P}$,*
(b) *for every $x \in |X|$ there exists a smooth morphism of pairs $a : (U, u) \to (X, x)$, a smooth morphism $b : V \to Y$, and a morphism $h : U \to V$ such that $f \circ a = b \circ h$ and h has $\mathcal{P}$,*
(c) *for some commutative diagram*

$$\begin{array}{ccc} U & \xrightarrow{h} & V \\ {\scriptstyle a}\downarrow & & \downarrow{\scriptstyle b} \\ X & \xrightarrow{f} & Y \end{array}$$

with a, b smooth and a surjective the morphism h has $\mathcal{P}$,
(d) *for any commutative diagram*

$$\begin{array}{ccc} U & \xrightarrow{h} & V \\ {\scriptstyle a}\downarrow & & \downarrow{\scriptstyle b} \\ X & \xrightarrow{f} & Y \end{array}$$

with b smooth and $U \to X \times_Y V$ smooth the morphism h has $\mathcal{P}$,
(e) *there exists a smooth covering $\{Y_i \to Y\}_{i \in I}$ such that each base change $Y_i \times_Y X \to Y_i$ has $\mathcal{P}$,*
(f) *there exists a smooth covering $\{X_i \to X\}_{i \in I}$ such that each composition $X_i \to Y$ has $\mathcal{P}$,*
(g) *there exists a smooth covering $\{Y_i \to Y\}_{i \in I}$ and for each $i \in I$ a smooth covering $\{X_{ij} \to Y_i \times_Y X\}_{j \in J_i}$ such that each morphism $X_{ij} \to Y_i$ has $\mathcal{P}$.*

Proof. The equivalence of (a) and (b) is part of Definition 55.18.1. The equivalence of (a) and (e) is Lemma 55.18.2 part (2). The equivalence of (a) and (f) is Lemma 55.18.2 part (1). As (a) is now equivalent to (e) and (f) it follows that (a) equivalent to (g).

It is clear that (c) implies (b). If (b) holds, then for any $x \in |X|$ we can choose a smooth morphism of pairs $a_x : (U_x, u_x) \to (X, x)$, a smooth morphism $b_x : V_x \to Y$, and a morphism $h_x : U_x \to V_x$ such that $f \circ a_x = b_x \circ h_x$ and h_x has $\mathcal{P}$. Then $h = \coprod h_x : \coprod U_x \to \coprod V_x$ with $a = \coprod a_x$ and $b = \coprod b_x$ is a diagram as in (c). (Note that h has property $\mathcal{P}$ as $\{V_x \to \coprod V_x\}$ is a smooth covering and $\mathcal{P}$ is smooth local on the target.) Thus (b) is equivalent to (c).

Now we know that (a), (b), (c), (e), (f), and (g) are equivalent. Suppose (a) holds. Let U, V, a, b, h be as in (d). Then $X \times_Y V \to V$ has $\mathcal{P}$ as $\mathcal{P}$ is stable under smooth base change, whence $U \to V$ has $\mathcal{P}$ as $\mathcal{P}$ is stable under precomposing with smooth morphisms. Conversely, if (d) holds, then setting $U = X$ and $V = Y$ we see that f has $\mathcal{P}$. □

Lemma 55.18.4. *Let S be a scheme. Let $\mathcal{P}$ be a property of morphisms of algebraic spaces over S. Assume*

(1) *$\mathcal{P}$ is smooth local on the source,*
(2) *$\mathcal{P}$ is smooth local on the target, and*

(3) *$\mathcal{P}$ is stable under postcomposing with smooth morphisms: if $f : X \to Y$ has $\mathcal{P}$ and $Y \subset Z$ is a smooth morphism then $X \to Z$ has $\mathcal{P}$.*

Then $\mathcal{P}$ is smooth local on the source-and-target.

Proof. Let $\mathcal{P}$ be a property of morphisms of algebraic spaces which satisfies conditions (1), (2) and (3) of the lemma. By Lemma 55.12.2 we see that $\mathcal{P}$ is stable under precomposing with smooth morphisms. By Lemma 55.9.2 we see that $\mathcal{P}$ is stable under smooth base change. Hence it suffices to prove part (3) of Definition 55.18.1 holds.

More precisely, suppose that $f : X \to Y$ is a morphism of algebraic spaces over S which satisfies Definition 55.18.1 part (3)(b). In other words, for every $x \in X$ there exists a smooth morphism $a_x : U_x \to X$, a point $u_x \in |U_x|$ mapping to x, a smooth morphism $b_x : V_x \to Y$, and a morphism $h_x : U_x \to V_x$ such that $f \circ a_x = b_x \circ h_x$ and h_x has $\mathcal{P}$. The proof of the lemma is complete once we show that f has $\mathcal{P}$. Set $U = \coprod U_x$, $a = \coprod a_x$, $V = \coprod V_x$, $b = \coprod b_x$, and $h = \coprod h_x$. We obtain a commutative diagram

$$\begin{array}{ccc} U & \xrightarrow{h} & V \\ {\scriptstyle a}\downarrow & & \downarrow{\scriptstyle b} \\ X & \xrightarrow{f} & Y \end{array}$$

with a, b smooth, a surjective. Note that h has $\mathcal{P}$ as each h_x does and $\mathcal{P}$ is smooth local on the target. Because a is surjective and $\mathcal{P}$ is smooth local on the source, it suffices to prove that $b \circ h$ has $\mathcal{P}$. This follows as we assumed that $\mathcal{P}$ is stable under postcomposing with a smooth morphism and as b is smooth. □

Remark 55.18.5. Using Lemma 55.18.4 and the work done in the earlier sections of this chapter it is easy to make a list of types of morphisms which are smooth local on the source-and-target. In each case we list the lemma which implies the property is smooth local on the source and the lemma which implies the property is smooth local on the target. In each case the third assumption of Lemma 55.18.4 is trivial to check, and we omit it. Here is the list:

(1) flat, see Lemmas 55.13.1 and 55.10.11,
(2) locally of finite presentation, see Lemmas 55.14.1 and 55.10.8,
(3) locally finite type, see Lemmas 55.14.2 and 55.10.7,
(4) universally open, see Lemmas 55.14.4 and 55.10.4,
(5) syntomic, see Lemmas 55.15.1 and 55.10.23,
(6) smooth, see Lemmas 55.16.1 and 55.10.24,
(7) add more here as needed.

55.19. Other chapters

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Schemes

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CHAPTER 56

Derived Categories of Spaces

56.1. Introduction

In this chapter we discuss derived categories of modules on algebraic spaces. There do not seem to be good introductory references addressing this topic; it is covered in the literature by referring to papers dealing with derived categories of modules on algebraic stacks, for example see [**Ols07b**].

56.2. Conventions

If $\mathcal{A}$ is an abelian category and M is an object of $\mathcal{A}$ then we also denote M the object of $K(\mathcal{A})$ and/or $D(\mathcal{A})$ corresponding to the complex which has M in degree 0 and is zero in all other degrees.

If we have a ring A, then $K(A)$ denotes the homotopy category of complexes of A-modules and $D(A)$ the associated derived category. Similarly, if we have a ringed space $(X, \mathcal{O}_X)$ the symbol $K(\mathcal{O}_X)$ denotes the homotopy category of complexes of $\mathcal{O}_X$-modules and $D(\mathcal{O}_X)$ the associated derived category.

56.3. Generalities

In this section we put some general results on cohomology of unbounded complexes of modules on algebraic spaces.

Lemma 56.3.1. *Let S be a scheme. Let $f : X \to Y$ be a morphism of algebraic spaces over S. Given an étale morphism $V \to Y$, set $U = V \times_Y X$ and denote $g : U \to V$ the projection morphism. Then $(Rf_*E)|_V = Rg_*(E|_U)$ for E in $D(\mathcal{O}_X)$.*

Proof. Represent E by a K-injective complex $\mathcal{I}^\bullet$ of $\mathcal{O}_X$-modules. Then $Rf_*(E) = f_*\mathcal{I}^\bullet$ and $Rg_*(E|_U) = g_*(\mathcal{I}^\bullet|_U)$ by Cohomology on Sites, Lemma 21.20.1. Hence the result follows from Properties of Spaces, Lemma 48.24.2. □

Definition 56.3.2. Let S be a scheme. Let X be an algebraic space over S. Let E be an object of $D(\mathcal{O}_X)$. Let $T \subset |X|$ be a closed subset. We say E is *supported on T* if the cohomology sheaves $H^i(E)$ are supported on T.

56.4. Derived category of quasi-coherent modules on the small étale site

Let X be a scheme. In this section we show that $D_{QCoh}(\mathcal{O}_X)$ can be defined in terms of the small étale site $X_{étale}$ of X. Denote $\mathcal{O}_{étale}$ the structure sheaf on $X_{étale}$. Consider the morphism of ringed sites

$$\epsilon : (X_{étale}, \mathcal{O}_{étale}) \longrightarrow (X_{Zar}, \mathcal{O}_X). \tag{56.4.0.1}$$

denoted $\mathrm{id}_{small,étale,Zar}$ in Descent, Lemma 34.7.5.

Lemma 56.4.1. *The morphism ϵ of (56.4.0.1) is a flat morphism of ringed sites. In particular the functor $\epsilon^* : Mod(\mathcal{O}_X) \to Mod(\mathcal{O}_{étale})$ is exact. Moreover, if $\epsilon^*\mathcal{F} = 0$, then $\mathcal{F} = 0$.*

Proof. The second assertion follows from the first by Modules on Sites, Lemma 18.30.2. To prove the first assertion we have to show that $\mathcal{O}_{étale}$ is a flat $\epsilon^{-1}\mathcal{O}_X$-module. To do this it suffices to check $\mathcal{O}_{X,x} \to \mathcal{O}_{étale,\overline{x}}$ is flat for any geometric point $\overline{x}$ of X, see Modules on Sites, Lemma 18.38.2, Sites, Lemma 7.33.1, and Étale Cohomology, Remarks 44.29.11. By Étale Cohomology, Lemma 44.33.1 we see that $\mathcal{O}_{étale,\overline{x}}$ is the strict henselization of $\mathcal{O}_{X,x}$. Thus $\mathcal{O}_{X,x} \to \mathcal{O}_{étale,\overline{x}}$ is faithfully flat by More on Algebra, Lemma 15.32.1. The final statement follows also: if $\epsilon^*\mathcal{F} = 0$, then

$$0 = \epsilon^*\mathcal{F}_{\overline{x}} = \mathcal{F}_x \otimes_{\mathcal{O}_{X,x}} \mathcal{O}_{étale}$$

(Modules on Sites, Lemma 18.35.4) for all geometric points $\overline{x}$. By faithful flatness of $\mathcal{O}_{X,x} \to \mathcal{O}_{étale,\overline{x}}$ we conclude $\mathcal{F}_x = 0$ for all $x \in X$. □

Let X be a scheme. Notation as in (56.4.0.1). Recall that $\epsilon^* : QCoh(\mathcal{O}_X) \to QCoh(\mathcal{O}_{étale})$ is an equivalence by Descent, Proposition 34.7.11 and Remark 34.7.6. Moreover, $QCoh(\mathcal{O}_{étale})$ forms a Serre subcategory of $Mod(\mathcal{O}_{étale})$ by Descent, Lemma 34.7.13. Hence we can let $D_{QCoh}(\mathcal{O}_{étale})$ be the triangulated subcategory of $D(\mathcal{O}_{étale})$ whose objects are the complexes with quasi-coherent cohomology sheaves, see Derived Categories, Section 13.12. The functor ϵ^* is exact (Lemma 56.4.1) hence induces $\epsilon^* : D(\mathcal{O}_X) \to D(\mathcal{O}_{étale})$ and since pullbacks of quasi-coherent modules are quasi-coherent also $\epsilon^* : D_{QCoh}(\mathcal{O}_X) \to D_{QCoh}(\mathcal{O}_{étale})$.

Lemma 56.4.2. *Let X be a scheme. The functor $\epsilon^* : D_{QCoh}(\mathcal{O}_X) \to D_{QCoh}(\mathcal{O}_{étale})$ defined above is an equivalence.*

Proof. We will prove this by showing the functor $R\epsilon_* : D(\mathcal{O}_{étale}) \to D(\mathcal{O}_X)$ induces a quasi-inverse.

Every quasi-coherent $\mathcal{O}_{étale}$-module $\mathcal{H}$ is of the form $\epsilon^*\mathcal{F}$ for some quasi-coherent $\mathcal{O}_X$-module $\mathcal{F}$, see Descent, Proposition 34.7.11. Since $\mathcal{F} = \epsilon_*\mathcal{H}$ in this case (as ϵ_* is the restriction to $X_{Zar} \subset X_{étale}$) we conclude that the adjunction map $\epsilon^*\epsilon_*\mathcal{H} \to \mathcal{H}$ is an isomorphism for all quasi-coherent $\mathcal{O}_{étale}$-modules $\mathcal{H}$.

Let E be an object of $D_{QCoh}(\mathcal{O}_{étale})$ and denote $\mathcal{H}^i = H^i(E)$ its ith cohomology sheaf. Let $\mathcal{B}$ be the set of affine objects of $X_{étale}$. Then $H^p(U, \mathcal{H}^i) = 0$ for all $p > 0$, all $i \in \mathbf{Z}$, and all $U \in \mathcal{B}$, see Descent, Proposition 34.7.10 and Cohomology of Schemes, Lemma 29.2.2. According to Cohomology on Sites, Lemma 21.22.2 this implies E is represented by a K-injective complex $\mathcal{I}^\bullet$ and $\mathcal{I}^\bullet = \lim \mathcal{I}^\bullet_n$ where each $\mathcal{I}^\bullet_n$ is a bounded below complex of injectives, the maps in the system $\ldots \to \mathcal{I}^\bullet_2 \to \mathcal{I}^\bullet_1$ are termwise split surjections, and each $\mathcal{I}^\bullet_n$ is quasi-isomorphic to $\tau_{\geq -n}E$. In particular,

$$R\epsilon_*E = \epsilon_*\mathcal{I}^\bullet = \lim \epsilon_*\mathcal{I}^\bullet_n$$

For every $U \in \mathcal{B}$ we have

$$H^m(\mathcal{I}^\bullet_n(U)) = \begin{cases} \mathcal{H}^m(U) & \text{if} \quad m \geq -n \\ 0 & \text{if} \quad m < n \end{cases}$$

by the vanishing of $H^p(U, \mathcal{H}^i)$ for $p > 0$, the spectral sequence Derived Categories, Lemma 13.20.3, and the fact that $\tau_{\geq -n}E \cong \mathcal{I}^\bullet_n$. Hence we can apply Homology,

Lemma 12.26.7 to the sequence of complexes

$$\lim_n \mathcal{I}_n^{m-2}(U) \to \lim_n \mathcal{I}_n^{m-1}(U) \to \lim_n \mathcal{I}_n^m(U) \to \lim_n \mathcal{I}_n^{m+1}(U)$$

to conclude that $H^m(\mathcal{I}^\bullet(U)) = \mathcal{H}^m(U)$ for $U \in \mathcal{B}$. Since ϵ_* is restriction to X_{Zar} we see, on applying the above to $U \subset X$ affine open, that $H^m(\epsilon_*\mathcal{I}^\bullet) = \epsilon_*\mathcal{H}^m$. Thus $R\epsilon_*$ indeed gives rise to a functor

$$R\epsilon_* : D_{QCoh}(\mathcal{O}_{étale}) \longrightarrow D_{QCoh}(\mathcal{O}_X)$$

For our object E of $D_{QCoh}(\mathcal{O}_{étale})$ above the adjunction map $\epsilon^* R\epsilon_* E \to E$ is an isomorphism as we've seen that the cohomology sheaves of $R\epsilon_* E$ are $\epsilon_*\mathcal{H}^m$ and we have $\epsilon^*\epsilon_*\mathcal{H}^m = \mathcal{H}^m$ (see above). For $F \in D_{QCoh}(\mathcal{O}_X)$ the adjunction map $F \to R\epsilon_*\epsilon^* F$ is an isomorphism for the same reason, i.e., because the cohomology sheaves of $R\epsilon_*\epsilon^* F$ are isomorphic to $\epsilon_* H^m(\epsilon^* F) = \epsilon_*\epsilon^* H^m(F) = H^m(F)$. □

56.5. Derived category of quasi-coherent modules

Let S be a scheme. Lemma 56.4.2 shows that the category $D_{QCoh}(\mathcal{O}_S)$ can be defined in terms of complexes of $\mathcal{O}_S$-modules on the scheme S or by complexes of $\mathcal{O}$-modules on the small étale site of S. Hence the following definition is compatible with the definition in the case of schemes.

Definition 56.5.1. Let S be a scheme. Let X be an algebraic space over S. The *derived category of $\mathcal{O}_X$-modules with quasi-coherent cohomology sheaves* is denoted $D_{QCoh}(\mathcal{O}_X)$.

This makes sense by Properties of Spaces, Lemma 48.27.7 and Derived Categories, Section 13.12. Thus we obtain a canonical functor

$$D(QCoh(\mathcal{O}_X)) \longrightarrow D_{QCoh}(\mathcal{O}_X) \tag{56.5.1.1}$$

see Derived Categories, Equation (13.12.1.1).

Observe that a flat morphism $f : Y \to X$ of algebraic spaces induces an exact functor $f^* : Mod(\mathcal{O}_X) \to Mod(\mathcal{O}_Y)$, see Morphisms of Spaces, Lemma 49.27.9 and Modules on Sites, Lemma 18.30.2. In particular $Lf^* : D(\mathcal{O}_X) \to D(\mathcal{O}_Y)$ is computed on any representative complex (Derived Categories, Lemma 13.16.8). We will write $Lf^* = f^*$ when f is flat and we have $H^i(f^*E) = f^*H^i(E)$ for E in $D(\mathcal{O}_X)$ in this case. We will use this often when f is étale. Of course in the étale case the pullback functor is just the restriction to $Y_{étale}$, see Properties of Spaces, Equation (48.24.1.1).

Lemma 56.5.2. *Let S be a scheme. Let X be an algebraic space over S. Let E be an object of $D(\mathcal{O}_X)$. The following are equivalent*

1. *E is in $D_{QCoh}(\mathcal{O}_X)$,*
2. *for every étale morphism $\varphi : U \to X$ where U is an affine scheme φ^*E is an object of $D_{QCoh}(\mathcal{O}_U)$,*
3. *for every étale morphism $\varphi : U \to X$ where U is a scheme φ^*E is an object of $D_{QCoh}(\mathcal{O}_U)$,*
4. *there exists a surjective étale morphism $\varphi : U \to X$ where U is a scheme such that φ^*E is an object of $D_{QCoh}(\mathcal{O}_U)$, and*
5. *there exists a surjective étale morphism of algebraic spaces $f : Y \to X$ such that Lf^*E is an object of $D_{QCoh}(\mathcal{O}_Y)$.*

Proof. This follows immediately from the discussion preceding the lemma and Properties of Spaces, Lemma 48.27.6. □

Lemma 56.5.3. *Let S be a scheme. Let X be an algebraic space over S. Then $D_{QCoh}(\mathcal{O}_X)$ has direct sums.*

Proof. By Injectives, Lemma 19.13.4 the derived category $D(\mathcal{O}_X)$ has direct sums and they are computed by taking termwise direct sums of any representatives. Thus it is clear that the cohomology sheaf of a direct sum is the direct sum of the cohomology sheaves as taking direct sums is an exact functor (in any grothendieck abelian category). The lemma follows as the direct sum of quasi-coherent sheaves is quasi-coherent, see Properties of Spaces, Lemma 48.27.7. □

Lemma 56.5.4. *Let S be a scheme. Let $f : Y \to X$ be a morphism of algebraic spaces over S. The functor Lf^* sends $D_{QCoh}(\mathcal{O}_X)$ into $D_{QCoh}(\mathcal{O}_Y)$.*

Proof. Choose a diagram

$$\begin{array}{ccc} U & \xrightarrow{h} & V \\ {\scriptstyle a}\downarrow & & \downarrow{\scriptstyle b} \\ X & \xrightarrow{f} & Y \end{array}$$

where U and V are schemes, the vertical arrows are étale, and a is surjective. Since $a^* \circ Lf^* = Lh^* \circ b^*$ the result follows from Lemma 56.5.2 and the case of schemes which is Derived Categories of Schemes, Lemma 35.3.6. □

Lemma 56.5.5. *Let S be a scheme. Let X be an algebraic space over S. For objects K, L of $D_{QCoh}(\mathcal{O}_X)$ the derived tensor product $K \otimes^{\mathbf{L}} L$ is in $D_{QCoh}(\mathcal{O}_X)$.*

Proof. Let $\varphi : U \to X$ be a surjective étale morphism from a scheme U. Since $\varphi^*(K \otimes^{\mathbf{L}}_{\mathcal{O}_X} L) = \varphi^*K \otimes^{\mathbf{L}}_{\mathcal{O}_U} \varphi^*L$ we see from Lemma 56.5.2 that this follows from the case of schemes which is Derived Categories of Schemes, Lemma 35.3.7. □

The following lemma will help us to "compute" a right derived functor on an object of $D_{QCoh}(\mathcal{O}_X)$.

Lemma 56.5.6. *Let S be a scheme. Let X be an algebraic space over S. Let E be an object of $D_{QCoh}(\mathcal{O}_X)$. Then there exists an inverse system $\mathcal{I}_n^\bullet$ of complexes of $\mathcal{O}_X$-modules such that*

1. *$\mathcal{I}^\bullet = \lim_n \mathcal{I}_n^\bullet$ represents E,*
2. *$\mathcal{I}_n^\bullet$ is a bounded below complex of injectives,*
3. *$\mathcal{I}^\bullet \to \mathcal{I}_n^\bullet$ induces an identification $\tau_{\geq -n}E \to \mathcal{I}_n^\bullet$ in $D(\mathcal{O}_X)$,*
4. *the transition maps $\mathcal{I}_{n+1}^\bullet \to \mathcal{I}_n^\bullet$ are termwise split surjections, and*
5. *$\mathcal{I}^\bullet$ is a K-injective complex of $\mathcal{O}_X$-modules.*

Moreover, E is the derived limit of the inverse system of its canonical truncations $\tau_{\geq -n}E$.

Proof. Denote $\mathcal{H}^i = H^i(E)$ the ith cohomology sheaf of E. Let $\mathcal{B}$ be the set of affine objects of $X_{étale}$. Then $H^p(U, \mathcal{H}^i) = 0$ for all $p > 0$, all $i \in \mathbf{Z}$, and all $U \in \mathcal{B}$ as U is an affine scheme. See discussion in Cohomology of Spaces, Section 51.3 and Cohomology of Schemes, Lemma 29.2.2. Thus the lemma follows from Cohomology on Sites, Lemmas 21.22.2 and 21.22.3. □

Lemma 56.5.7. *Let S be a scheme. Let X be an algebraic space over S. Let $F : Mod(\mathcal{O}_X) \to Ab$ be a functor and $N \geq 0$ an integer. Assume that*

(1) *F is left exact,*
(2) *F commutes with countable direct products,*
(3) *$R^pF(\mathcal{F}) = 0$ for all $p \geq N$ and $\mathcal{F}$ quasi-coherent.*

Then for $E \in D_{QCoh}(\mathcal{O}_X)$ the maps $R^pF(E) \to R^pF(\tau_{\geq p-N+1}E)$ are isomorphisms.

Proof. Let E be an object of $D_{QCoh}(\mathcal{O}_X)$. By shifting the complex we see it suffices to prove the assertion for $p = 0$. Choose $\mathcal{I}^\bullet = \lim \mathcal{I}_n^\bullet$ as in Lemma 56.5.6. As $\mathcal{I}^\bullet$ is K-injective $RF(E)$ is represented by $F(\mathcal{I}^\bullet)$. As F commutes with countable direct products, and since the maps $\mathcal{I}_n^m \to \mathcal{I}_{n-1}^m$ are split surjections, we get $F(\mathcal{I}^\bullet) = \lim F(\mathcal{I}_n^\bullet)$. The cohomology of

$$F(\mathcal{I}_n^{-2}) \to F(\mathcal{I}_n^{-1}) \to F(\mathcal{I}_n^0) \to F(\mathcal{I}_n^1) \tag{56.5.7.1}$$

in degree 0, resp. -1 is equal to $R^0F(\tau_{\geq -n}E)$, resp. $R^{-1}F(\tau_{\geq -n}E)$ because $\mathcal{I}_n^\bullet$ is a bounded below complex of injectives representing $\tau_{\geq -n}E$. We have a distinguished triangle

$$H^{-n}(E)[n] \to \tau_{\geq -n}E \to \tau_{\geq -n+1}E \to H^{-n}(E)[n+1]$$

(Derived Categories, Remark 13.11.4) in $D(\mathcal{O}_X)$. Since $H^{-n}(E)$ is quasi-coherent we have

$$R^pF(H^{-n}(E)[n]) = R^{p+n}F(H^{-n}(E)) = 0$$

for $p + n \geq N$ and

$$R^pF(H^{-n}(E)[n+1]) = R^{p+n+1}F(H^{-n}(E)) = 0$$

for $p + n + 1 \geq N$. We conclude that

$$R^pF(\tau_{\geq -n}E) \to R^pF(\tau_{\geq -n+1}E)$$

is an isomorphism for all $n \gg p$ and an isomorphism for $n \geq N$ for $p = 0$. Thus Homology, Lemma 12.26.7 applies to the system of sequences (56.5.7.1) and we conclude that $R^0F(E) = \lim R^0F(\tau_{\geq -n}E)$. By the above the system $R^0F(\tau_{\geq -n}E)$ is constant starting with $n = N - 1$ as desired. □

56.6. Total direct image

The following lemma is the analogue of Cohomology of Spaces, Lemma 51.7.1.

Lemma 56.6.1. *Let S be a scheme. Let $f : X \to Y$ be a quasi-separated and quasi-compact morphism of algebraic spaces over S.*

(1) *The functor Rf_* sends $D_{QCoh}(\mathcal{O}_X)$ into $D_{QCoh}(\mathcal{O}_Y)$.*
(2) *If Y is quasi-compact, there exists an integer $N = N(X, Y, f)$ such that for an object E of $D_{QCoh}(\mathcal{O}_X)$ with $H^m(E) = 0$ for $m > 0$ we have $H^m(Rf_*E) = 0$ for $m > N$.*
(3) *In fact, if Y is quasi-compact we can find $N = N(X, Y, f)$ such that for every morphism of algebraic spaces $Y' \to Y$ the same conclusion holds for the functor $R(f')_*$ where $f' : X' \to Y'$ is the base change of f.*

Proof. Let E be an object of $D_{QCoh}(\mathcal{O}_X)$. To prove (1) we have to show that Rf_*E has quasi-coherent cohomology sheaves. This question is local on Y, hence we may assume Y is quasi-compact. Pick $N = N(X, Y, f)$ as in Cohomology of Spaces, Lemma 51.7.1. Thus $R^pf_*\mathcal{F} = 0$ for all quasi-coherent $\mathcal{O}_X$-modules $\mathcal{F}$ and all

$p \geq N$. In particular, for any affine object V of $Y_{étale}$ we have $H^p(V \times_Y X, \mathcal{F}) = 0$ for $p \geq N$, see Cohomology of Spaces, Lemma 51.3.2.

Let E be an object of $D_{QCoh}(\mathcal{O}_X)$. Choose $\mathcal{I}^\bullet = \lim \mathcal{I}_n^\bullet$ as in Lemma 56.5.6. As $\mathcal{I}^\bullet$ is K-injective Rf_*E is represented by $f_*\mathcal{I}^\bullet = \lim f_*\mathcal{I}_n^\bullet$. Let V be an affine object of $Y_{étale}$. The cohomology $H^m(f_*\mathcal{I}_n^\bullet(V))$ of

$$f_*\mathcal{I}_n^{m-1}(V) \to f_*\mathcal{I}_n^m(V) \to f_*\mathcal{I}_n^{m+1}(V)$$

is equal to $H^m(V \times_Y X, \tau_{\geq -n}E)$ because $\mathcal{I}_n^\bullet$ is a bounded below complex of injectives representing $\tau_{\geq -n}E$. We have a distinguished triangle

$$H^{-n}(E)[n] \to \tau_{\geq -n}E \to \tau_{\geq -n+1}E \to H^{-n}(E)[n+1]$$

in $D(\mathcal{O}_X)$. Since $H^{-n}(E)$ is quasi-coherent we have $H^m(V \times_Y X, H^{-n}(E)[n]) = 0$ for $n + m \geq N$ by our choice of N. Similarly, $H^m(V \times_Y X, H^{-n}(E)[n+1]) = 0$ for $n + m + 1 \geq N$. We conclude that

$$H^m(f_*\mathcal{I}_n^\bullet(V)) \to H^m(f_*\mathcal{I}_{n-1}^\bullet(V))$$

is an isomorphism for all $n \geq N - m$. Thus Cohomology on Sites, Lemma 21.22.1 applies to show that the mth cohomology sheaf of $\lim f_*\mathcal{I}_n^\bullet$ agrees with the mth cohomology sheaf of $f_*\mathcal{I}_n^\bullet$ for $n \geq N - m$. Since these cohomology sheaves are quasi-coherent by Cohomology of Spaces, Lemma 51.3.1 we get (1).

Finally, we show that (2) and (3) hold with our choice of N. Namely, the stabilization proven above gives that $H^m(Rf_*E)$ is equal to $H^m(Rf_*(\tau_{\geq -n}E))$ for all n large enough which means we can work with objects in $D^+(\mathcal{O}_X)$ in order to prove (2) and (3). In this case we can for example use the spectral sequence

$$R^p f_* H^q(E) \Rightarrow R^{p+q} f_* E$$

(Derived Categories, Lemma 13.20.3) and the vanishing of $R^p f_* H^q(E)$ for $p \geq N$ to conclude. Some details omitted. □

Lemma 56.6.2. *Let S be a scheme. Let $f : X \to Y$ be a quasi-separated and quasi-compact morphism of algebraic spaces over S. Then $Rf_* : D_{QCoh}(\mathcal{O}_X) \to D_{QCoh}(\mathcal{O}_S)$ commutes with direct sums.*

Proof. Let E_i be a family of objects of $D_{QCoh}(\mathcal{O}_X)$ and set $E = \bigoplus E_i$. We want to show that the map

$$\bigoplus Rf_*E_i \longrightarrow Rf_*E$$

is an isomorphism. We will show it induces an isomorphism on cohomology sheaves in degree 0 which will imply the lemma. Choose an integer N as in Lemma 56.6.1. Then $R^0 f_* E = R^0 f_* \tau_{\geq -N} E$ and $R^0 f_* E_i = R^0 f_* \tau_{\geq -N} E_i$ by the lemma cited. Observe that $\tau_{\geq -N} E = \bigoplus \tau_{\geq -N} E_i$. Thus we may assume all of the E_i have vanishing cohomology sheaves in degrees $< -N$. Next we use the spectral sequences

$$R^p f_* H^q(E) \Rightarrow R^{p+q} f_* E \quad \text{and} \quad R^p f_* H^q(E_i) \Rightarrow R^{p+q} f_* E_i$$

(Derived Categories, Lemma 13.20.3) to reduce to the case of a direct sum of quasi-coherent sheaves. This case is handled by Cohomology of Spaces, Lemma 51.4.2. □

Remark 56.6.3. Let S be a scheme. Let $f : X \to Y$ be a quasi-compact and quasi-separated morphism of representable algebraic spaces X and Y over S. Let $f_0 : X_0 \to Y_0$ be a morphism of schemes representing f (awkward but temporary notation). Then we claim the diagrams

$$\begin{array}{ccc} D_{QCoh}(\mathcal{O}_{X_0}) & \xlongequal[\text{Lemma 56.4.2}]{} & D_{QCoh}(\mathcal{O}_X) \\ \downarrow{\scriptstyle Rf_{0,*}} & & \downarrow{\scriptstyle Rf_*} \\ D_{QCoh}(\mathcal{O}_{Y_0}) & \xlongequal{\text{Lemma 56.4.2}} & D_{QCoh}(\mathcal{O}_Y) \end{array}$$

(Lemma 56.6.1 and Derived Categories of Schemes, Lemma 35.4.1) and

$$\begin{array}{ccc} D_{QCoh}(\mathcal{O}_{X_0}) & \xlongequal[\text{Lemma 56.4.2}]{} & D_{QCoh}(\mathcal{O}_X) \\ \uparrow{\scriptstyle Lf_0^*} & & \uparrow{\scriptstyle Lf^*} \\ D_{QCoh}(\mathcal{O}_{Y_0}) & \xlongequal{\text{Lemma 56.4.2}} & D_{QCoh}(\mathcal{O}_Y) \end{array}$$

(Lemma 56.5.4 and Derived Categories of Schemes, Lemma 35.3.6) are commutative. The result for Lf^* and Lf_0^* follows as the equivalences $D_{QCoh}(\mathcal{O}_{X_0}) \to D_{QCoh}(\mathcal{O}_X)$ and $D_{QCoh}(\mathcal{O}_{Y_0}) \to D_{QCoh}(\mathcal{O}_Y)$ of Lemma 56.4.2 come from pulling back by the (flat) morphisms of ringed sites $\epsilon : X_{étale} \to X_{0,Zar}$ and $\epsilon : Y_{étale} \to Y_{0,Zar}$ and the diagram of ringed sites

$$\begin{array}{ccc} X_{0,Zar} & \xleftarrow{\epsilon} & X_{étale} \\ \downarrow{\scriptstyle f_0} & & \downarrow{\scriptstyle f} \\ Y_{0,Zar} & \xleftarrow{\epsilon} & Y_{étale} \end{array}$$

is commutative (details omitted). In fact the commutativity of the first diagram also follows as the proof of Lemma 56.4.2 shows that the functor $R\epsilon_*$ gives the equivalences $D_{QCoh}(\mathcal{O}_X) \to D_{QCoh}(\mathcal{O}_{X_0})$ and $D_{QCoh}(\mathcal{O}_Y) \to D_{QCoh}(\mathcal{O}_{Y_0})$.

Lemma 56.6.4. *Let S be a scheme. Let $f : X \to Y$ be an affine morphism of algebraic spaces over S. Then $Rf_* : D_{QCoh}(\mathcal{O}_X) \to D_{QCoh}(\mathcal{O}_Y)$ reflects isomorphisms.*

Proof. The statement means that a morphism $\alpha : E \to F$ of $D_{QCoh}(\mathcal{O}_X)$ is an isomorphism if $Rf_*\alpha$ is an isomorphism. We may check this on cohomology sheaves. In particular, the question is étale local on Y. Hence we may assume Y and therefore X is affine. In this case the problem reduces to the case of schemes (Derived Categories of Schemes, Lemma 35.4.3) via Lemma 56.4.2 and Remark 56.6.3. □

Lemma 56.6.5. *Let S be a scheme. Let $f : X \to Y$ be an affine morphism of algebraic spaces over S. For E in $D_{QCoh}(\mathcal{O}_Y)$ we have $Rf_*Lf^*E = E \otimes^{\mathbf{L}}_{\mathcal{O}_Y} f_*\mathcal{O}_X$.*

Proof. Since f is affine the map $f_*\mathcal{O}_X \to Rf_*\mathcal{O}_X$ is an isomorphism (Cohomology of Spaces, Lemma 51.7.2). There is a canonical map $E \otimes^{\mathbf{L}} f_*\mathcal{O}_X = E \otimes^{\mathbf{L}} Rf_*\mathcal{O}_X \to Rf_*Lf^*E$ adjoint to the map

$$Lf^*(E \otimes^{\mathbf{L}} Rf_*\mathcal{O}_X) = Lf^*E \otimes^{\mathbf{L}} Lf^*Rf_*\mathcal{O}_X \longrightarrow Lf^*E \otimes^{\mathbf{L}} \mathcal{O}_X = Lf^*E$$

coming from $1 : Lf^*E \to Lf^*E$ and the canonical map $Lf^*Rf_*\mathcal{O}_X \to \mathcal{O}_X$. To check the map so constructed is an isomorphism we may work locally on Y. Hence

we may assume Y and therefore X is affine. In this case the problem reduces to the case of schemes (Derived Categories of Schemes, Lemma 35.4.4) via Lemma 56.4.2 and Remark 56.6.3. □

56.7. Derived category of coherent modules

Let S be a scheme. Let X be a locally Noetherian algebraic space over S. In this case the category $Coh(\mathcal{O}_X) \subset Mod(\mathcal{O}_X)$ of coherent $\mathcal{O}_X$-modules is a weak Serre subcategory, see Homology, Section 12.9 and Cohomology of Spaces, Lemma 51.9.3. Denote

$$D_{Coh}(\mathcal{O}_X) \subset D(\mathcal{O}_X)$$

the subcategory of complexes whose cohomology sheaves are coherent, see Derived Categories, Section 13.12. Thus we obtain a canonical functor

$$D(Coh(\mathcal{O}_X)) \longrightarrow D_{Coh}(\mathcal{O}_X) \tag{56.7.0.1}$$

see Derived Categories, Equation (13.12.1.1).

Lemma 56.7.1. *Let S be a scheme. Let $f : X \to Y$ be a morphism of algebraic spaces over S. Assume f is locally of finite type and Y is Noetherian. Let E be an object of $D^b_{Coh}(\mathcal{O}_X)$ such that the scheme theoretic support of $H^i(E)$ is proper over Y for all i. Then Rf_*E is an object of $D^b_{Coh}(\mathcal{O}_Y)$.*

Proof. Consider the spectral sequence

$$R^p f_* H^q(E) \Rightarrow R^{p+q} f_* E$$

see Derived Categories, Lemma 13.20.3. By assumption and Cohomology of Spaces, Remark 51.16.3 the sheaves $R^p f_* H^q(E)$ are coherent. Hence $R^{p+q} f_* E$ is coherent, i.e., $E \in D_{Coh}(\mathcal{O}_S)$. Boundedness from below is trivial. Boundedness from above follows from Cohomology of Spaces, Lemma 51.7.1 or from Lemma 56.6.1. □

56.8. Induction principle

In this section we discuss an induction principle for algebraic spaces analogues to what is Cohomology of Schemes, Lemma 56.8.3 for schemes. To formulate it we introduce the notion of an *elementary distinguished square*; this terminology is borrowed from [**MV99**]. The principle as formulated here is implicit in the paper [**GR71**] by Raynaud and Gruson. A related principle for algebraic stacks is [**Ryd10**, Theorem D] by David Rydh.

Definition 56.8.1. Let S be a scheme. A commutative diagram

$$\begin{array}{ccc} U \times_W V & \longrightarrow & V \\ \downarrow & & \downarrow f \\ U & \xrightarrow{j} & W \end{array}$$

of algebraic spaces over S is called an *elementary distinguished square* if

(1) U is an open subspace of W and j is the inclusion morphism,
(2) f is étale, and
(3) setting $T = W \setminus U$ (with reduced induced subspace structure) the morphism $f^{-1}(T) \to T$ is an isomorphism.

We will indicate this by saying: "Let $(U \subset W, f : V \to W)$ be an elementary distinguished square."

Note that if $(U \subset W, f : V \to W)$ is an elementary distinguished square, then we have $W = U \cup f(V)$. Thus $\{U \to W, V \to W\}$ is an étale covering of W. It turns out that these étale coverings have nice properties and that in some sense there are "enough" of them.

Lemma 56.8.2. *Let S be a scheme. Let $(U \subset W, f : V \to W)$ be an elementary distinguished square of algebraic spaces over S.*

1. *If $V' \subset V$ and $U \subset U' \subset W$ are open subspaces and $W' = U' \cup f(V')$ then $(U' \subset W', f|_{V'} : V' \to W')$ is an elementary distinguished square.*
2. *If $p : W' \to W$ is a morphism of algebraic spaces, then $(p^{-1}(U) \subset W', V \times_W W' \to W')$ is an elementary distinguished square.*

Proof. Omitted. □

Lemma 56.8.3. *Let S be a scheme. Let X be a quasi-compact and quasi-separated algebraic space over S. Let P be a property of the quasi-compact and quasi-separated objects of $X_{spaces,étale}$. Assume that*

1. *P holds for every affine object of $X_{spaces,étale}$,*
2. *for every elementary distinguished square $(U \subset W, f : V \to W)$ such that*
   (a) *W is a quasi-compact and quasi-separated object of $X_{spaces,étale}$,*
   (b) *U is quasi-compact,*
   (c) *V is affine, and*
   (d) *P holds for U, V, and $U \times_W V$,*
   then P holds for W.

Then P holds for every quasi-compact and quasi-separated object of $X_{spaces,étale}$ and in particular for X.

Proof. We first claim that P holds for every representable quasi-compact and quasi-separated object of $X_{spaces,étale}$. Namely, suppose that $U \to X$ is étale and U is a quasi-compact and quasi-separated scheme. By assumption (1) property P holds for every affine open of U. Moreover, if $W, V \subset U$ are quasi-compact open with V affine and P holds for W, V, and $W \cap V$, then P holds for $W \cup V$ by (2) (as the pair $(W \subset W \cup V, V \to W \cup V)$ is an elementary distinguished square). Thus P holds for U by the induction principle for schemes, see Cohomology of Schemes, Lemma 29.4.1.

To finish the proof it suffices to prove P holds for X (because we can simply replace X by any quasi-compact and quasi-separated object of $X_{spaces,étale}$ we want to prove the result for). We will use the filtration

$$\emptyset = U_{n+1} \subset U_n \subset U_{n-1} \subset \ldots \subset U_1 = X$$

and the morphisms $f_p : V_p \to U_p$ of Limits of Spaces, Lemma 52.8.1. We will prove that P holds for U_p by descending induction on p. Note that P holds for U_{n+1} by (1) as an empty algebraic space is affine. Assume P holds for U_{p+1}. Note that $(U_{p+1} \subset U_p, f_p : V_p \to U_p)$ is an elementary distinguished square, but (2) may not apply as V_p may not be affine. However, as V_p is a quasi-compact scheme we may choose a finite affine open covering $V_p = V_{p,1} \cup \ldots \cup V_{p,m}$. Set $W_{p,0} = U_{p+1}$ and

$$W_{p,i} = U_{p+1} \cup f_p(V_{p,1} \cup \ldots \cup V_{p,i})$$

for $i = 1, \ldots, m$. These are quasi-compact open subspaces of X. Then we have

$$U_{p+1} = W_{p,0} \subset W_{p,1} \subset \ldots \subset W_{p,m} = U_p$$

and the pairs

$$(W_{p,0} \subset W_{p,1}, f_p|_{V_{p,1}}), (W_{p,1} \subset W_{p,2}, f_p|_{V_{p,2}}), \ldots, (W_{p,m-1} \subset W_{p,m}, f_p|_{V_{p,m}})$$

are elementary distinguished squares by Lemma 56.8.2. Note that P holds for each $V_{p,1}$ (as affine schemes) and for $W_{p,i} \times_{W_{p,i+1}} V_{p,i+1}$ as this is a quasi-compact open of $V_{p,i+1}$ and hence P holds for it by the first paragraph of this proof. Thus (2) applies to each of these and we inductively conclude P holds for $W_{p,1}, \ldots, W_{p,m} = U_p$. □

Lemma 56.8.4. *Let S be a scheme. Let X be a quasi-compact and quasi-separated algebraic space over S. Let $\mathcal{B} \subset \mathrm{Ob}(X_{spaces,étale})$. Let P be a property of the elements of $\mathcal{B}$. Assume that*

1. *every $W \in \mathcal{B}$ is quasi-compact and quasi-separated,*
2. *if $W \in \mathcal{B}$ and $U \subset W$ is quasi-compact open, then $U \in \mathcal{B}$,*
3. *if $V \in \mathrm{Ob}(X_{spaces,étale})$ is affine, then (a) $V \in \mathcal{B}$ and (b) P holds for V,*
4. *for every elementary distinguished square $(U \subset W, f : V \to W)$ such that*
   (a) *$W \in \mathcal{B}$,*
   (b) *U is quasi-compact,*
   (c) *V is affine, and*
   (d) *P holds for U, V, and $U \times_W V$,*
   then P holds for W.

Then P holds for every $W \in \mathcal{B}$.

Proof. This is proved in exactly the same manner as the proof of Lemma 56.8.3. (We remark that (4)(d) makes sense as $U \times_W V$ is a quasi-compact open of V hence an element of $\mathcal{B}$ by conditions (2) and (3).) □

Remark 56.8.5. How to choose the collection $\mathcal{B}$ in Lemma 56.8.4? Here are some examples:

1. If X is quasi-compact and separated, then we can choose $\mathcal{B}$ to be the set of quasi-compact and separated objects of $X_{spaces,étale}$. Then $X \in \mathcal{B}$ and $\mathcal{B}$ satisfies (1), (2), and (3)(a). With this choice of $\mathcal{B}$ Lemma 56.8.4 reproduces Lemma 56.8.3.
2. If X is quasi-compact with affine diagonal, then we can choose $\mathcal{B}$ to be the set of objects of $X_{spaces,étale}$ which are quasi-compact and have affine diagonal. Again $X \in \mathcal{B}$ and $\mathcal{B}$ satisfies (1), (2), and (3)(a).
3. If X is quasi-compact and quasi-separated, then the smallest subset $\mathcal{B}$ which contains X and satisfies (1), (2), and (3)(a) is given by the rule $W \in \mathcal{B}$ if and only if either W is a quasi-compact open subspace of X, or W is a quasi-compact open of an affine object of $X_{spaces,étale}$.

Here is a variant where we extend the truth from an open to larger opens.

Lemma 56.8.6. *Let S be a scheme. Let X be a quasi-compact and quasi-separated algebraic space over S. Let $W \subset X$ be a quasi-compact open subspace. Let P be a property of quasi-compact open subspaces of X. Assume that*

1. *P holds for W, and*
2. *for every elementary distinguished square $(W_1 \subset W_2, f : V \to W_2)$ where such that*
   (a) *W_1, W_2 are quasi-compact open subspaces of X,*
   (b) *$W \subset W_1$,*
   (c) *V is affine, and*

(d) *P holds for W_1,*
then P holds for W_2.

Then P holds for X.

Proof. We can deduce this from Lemma 56.8.4, but instead we will give a direct argument by eplicitly redoing the proof of Lemma 56.8.3. We will use the filtration

$$\emptyset = U_{n+1} \subset U_n \subset U_{n-1} \subset \ldots \subset U_1 = X$$

and the morphisms $f_p : V_p \to U_p$ of Limits of Spaces, Lemma 52.8.1. We will prove that P holds for $W_p = W \cup U_p$ by descending induction on p. This will finish the proof as $W_1 = X$. Note that P holds for $W_{n+1} = W \cap U_{n+1} = W$ by (1). Assume P holds for W_{p+1}. Observe that $W_p \setminus W_{p+1}$ (with reduced induced subspace structure) is a closed subspace of $U_p \setminus U_{p+1}$. Since $(U_{p+1} \subset U_p, f_p : V_p \to U_p)$ is an elementary distinguished square, the same is true for $(W_{p+1} \subset W_p, f_p : V_p \to W_p)$. However (2) may not apply as V_p may not be affine. However, as V_p is a quasi-compact scheme we may choose a finite affine open covering $V_p = V_{p,1} \cup \ldots \cup V_{p,m}$. Set $W_{p,0} = W_{p+1}$ and

$$W_{p,i} = W_{p+1} \cup f_p(V_{p,1} \cup \ldots \cup V_{p,i})$$

for $i = 1, \ldots, m$. These are quasi-compact open subspaces of X containing W. Then we have

$$W_{p+1} = W_{p,0} \subset W_{p,1} \subset \ldots \subset W_{p,m} = W_p$$

and the pairs

$$(W_{p,0} \subset W_{p,1}, f_p|_{V_{p,1}}), (W_{p,1} \subset W_{p,2}, f_p|_{V_{p,2}}), \ldots, (W_{p,m-1} \subset W_{p,m}, f_p|_{V_{p,m}})$$

are elementary distinguished squares by Lemma 56.8.2. Now (2) applies to each of these and we inductively conclude P holds for $W_{p,1}, \ldots, W_{p,m} = W_p$. □

56.9. Mayer-Vietoris

In this section we prove that an elementary distinguished triangle gives rise to various Mayer-Vietoris sequences.

Let S be a scheme. Let $U \to X$ be an étale morphism of algebraic spaces over S. In Properties of Spaces, Section 48.25 it was shown that $U_{spaces,étale} = X_{spaces,étale}/U$ compatible with structure sheaves. Hence in this situation we often think of the morphism $j_U : U \to X$ as a localization morphism (see Modules on Sites, Definition 18.19.1). In particular we think of pullback j_U^* as restriction to U and we often denote it by $|_U$; this is compatible with Properties of Spaces, Equation (48.24.1.1). In particular we see that

$$(\mathcal{F}|_U)_{\overline{u}} = \mathcal{F}_{\overline{x}} \tag{56.9.0.1}$$

if $\overline{u}$ is a geometric point of U and $\overline{x}$ the image of $\overline{u}$ in X. Moreover, restriction has an exact left adjoint $j_{U!}$, see Modules on Sites, Lemmas 18.19.2 and 18.19.3. Finally, recall that if $\mathcal{G}$ is an $\mathcal{O}_X$-module, then

$$(j_{U!}\mathcal{G})_{\overline{x}} = \bigoplus\nolimits_{\overline{u}} \mathcal{G}_{\overline{u}} \tag{56.9.0.2}$$

for any geometric point $\overline{x} : \mathrm{Spec}(k) \to X$ where the direct sum is over those morphism $\overline{u} : \mathrm{Spec}(k) \to U$ such that $j_U \circ \overline{u} = \overline{x}$, see Modules on Sites, Lemma 18.37.1 and Properties of Spaces, Lemma 48.16.13.

Lemma 56.9.1. *Let S be a scheme. Let $(U \subset X, V \to X)$ be an elementary distinguished square of algebraic spaces over S.*

(1) *For a sheaf of $\mathcal{O}_X$-modules $\mathcal{F}$ we have a short exact sequence*
$$0 \to j_{U \times_X V!}\mathcal{F}|_{U \times_X V} \to j_{U!}\mathcal{F}|_U \oplus j_{V!}\mathcal{F}|_V \to \mathcal{F} \to 0$$
(2) *For an object E of $D(\mathcal{O}_X)$ we have a distinguished triangle*
$$j_{U \times_X V!}E|_{U \times_X V} \to j_{U!}E|_U \oplus j_{V!}E|_V \to E \to j_{U \times_X V!}E|_{U \times_X V}[1]$$
in $D(\mathcal{O}_X)$.

Proof. To show the sequence of (1) is exact we may check on stalks at geometric points by Properties of Spaces, Theorem 48.16.12. Let $\overline{x}$ be a geometric point of X. By Equations (56.9.0.1) and (56.9.0.2) taking stalks at $\overline{x}$ we obtain the sequence
$$0 \to \bigoplus\nolimits_{(\overline{u},\overline{v})} \mathcal{F}_{\overline{x}} \to \bigoplus\nolimits_{\overline{u}} \mathcal{F}_{\overline{x}} \oplus \bigoplus\nolimits_{\overline{v}} \mathcal{F}_{\overline{x}} \to \mathcal{F}_{\overline{x}} \to 0$$
This sequence is exact because for every $\overline{x}$ there either is exactly one $\overline{u}$ mapping to $\overline{x}$, or there is no $\overline{u}$ and exactly one $\overline{v}$ mapping to $\overline{x}$.

Proof of (2). We have seen in Cohomology on Sites, Section 21.20 that the restriction functors and the extension by zero functors on derived categories are computed by just applying the functor to any complex. Let $\mathcal{E}^\bullet$ be a complex of $\mathcal{O}_X$-modules representing E. The distinguished triangle of the lemma is the distinguished triangle associated (by Derived Categories, Section 13.11 and especially Lemma 13.11.1) to the short exact sequence of complexes of $\mathcal{O}_X$-modules
$$0 \to j_{U \times_X V!}\mathcal{E}^\bullet|_{U \times_X V} \to j_{U!}\mathcal{E}^\bullet|_U \oplus j_{V!}\mathcal{E}^\bullet|_V \to \mathcal{E}^\bullet \to 0$$
which is short exact by (1). □

Lemma 56.9.2. *Let S be a scheme. Let $(U \subset X, V \to X)$ be an elementary distinguished square of algebraic spaces over S.*

(1) *For every sheaf of $\mathcal{O}_X$-modules $\mathcal{F}$ we have a short exact sequence*
$$0 \to \mathcal{F} \to j_{U,*}\mathcal{F}|_U \oplus j_{V,*}\mathcal{F}|_V \to j_{U \times_X V,*}\mathcal{F}|_{U \times_X V} \to 0$$
(2) *For any object E of $D(\mathcal{O}_X)$ we have a distinguished triangle*
$$E \to Rj_{U,*}E|_U \oplus Rj_{V,*}E|_V \to Rj_{U \times_X V,*}E|_{U \times_X V} \to E[1]$$
in $D(\mathcal{O}_X)$.

Proof. Let W be an object of $X_{étale}$. We claim the sequence
$$0 \to \mathcal{F}(W) \to \mathcal{F}(W \times_X U) \oplus \mathcal{F}(W \times_X V) \to \mathcal{F}(W \times_X U \times_X V)$$
is exact and that an element of the last group can locally on W be lifted to the middle one. By Lemma 56.8.2 the pair $(W \times_X U \subset W, V \times_X W \to W)$ is an elementary distinguished square. Thus we may assume $W = X$ and it suffices to prove the same thing for
$$0 \to \mathcal{F}(X) \to \mathcal{F}(U) \oplus \mathcal{F}(V) \to \mathcal{F}(U \times_X V)$$
We have seen that
$$0 \to j_{U \times_X V!}\mathcal{O}_{U \times_X V} \to j_{U!}\mathcal{O}_U \oplus j_{V!}\mathcal{O}_V \to \mathcal{O}_X \to 0$$
is a exact sequence of $\mathcal{O}_X$-modules in Lemma 56.9.1 and applying the right exact functor $\mathrm{Hom}_{\mathcal{O}_X}(-, \mathcal{F})$ gives the sequence above. This also means that the

obstruction to lifting $s \in \mathcal{F}(U \times_X V)$ to an element of $\mathcal{F}(U) \oplus \mathcal{F}(V)$ lies in $\mathrm{Ext}^1_{\mathcal{O}_X}(\mathcal{O}_X, \mathcal{F}) = H^1(X, \mathcal{F})$. By locality of cohomology (Cohomology on Sites, Lemma 21.8.3) this obstruction vanishes étale locally on X and the proof of (1) is complete.

Proof of (2). Choose a K-injective complex $\mathcal{I}^\bullet$ representing E whose terms $\mathcal{I}^n$ are injective objects of $Mod(\mathcal{O}_X)$, see Injectives, Theorem 19.12.6. Then $\mathcal{I}^\bullet|U$ is a K-injective complex (Cohomology on Sites, Lemma 21.20.1). Hence $Rj_{U,*}E|_U$ is represented by $j_{U,*}\mathcal{I}^\bullet|_U$. Similarly for V and $U \times_X V$. Hence the distinguished triangle of the lemma is the distinguished triangle associated (by Derived Categories, Section 13.11 and especially Lemma 13.11.1) to the short exact sequence of complexes

$$0 \to \mathcal{I}^\bullet \to j_{U,*}\mathcal{I}^\bullet|_U \oplus j_{V,*}\mathcal{I}^\bullet|_V \to j_{U\times_X V,*}\mathcal{I}^\bullet|_{U\times_X V} \to 0.$$

This sequence is exact by (1). □

Lemma 56.9.3. *Let S be a scheme. Let $f : X \to Y$ be a morphism of algebraic spaces over S. Let $(U \subset X, V \to X)$ be an elementary distinguished square. Denote $a = f|_U : U \to Y$, $b = f|_V : V \to Y$, and $c = f|_{U\times_X V} : U \times_X V \to Y$ the restrictions. For every object E of $D(\mathcal{O}_X)$ there exists a distinguished triangle*

$$Rf_*E \to Ra_*(E|_U) \oplus Rb_*(E|_V) \to Rc_*(E|_{U\times_X V}) \to Rf_*E[1]$$

in $D(\mathcal{O}_Y)$. This triangle is functorial in E.

Proof. Choose a K-injective complex $\mathcal{I}^\bullet$ representing E. We may assume $\mathcal{I}^n$ is an injective object of $Mod(\mathcal{O}_X)$ for all n, see Injectives, Theorem 19.12.6. Then Rf_*E is computed by $f_*\mathcal{I}^\bullet$. Similarly for U, V, and $U \cap V$ by Cohomology on Sites, Lemma 21.20.1. Hence the distinguished triangle of the lemma is the distinguished triangle associated (by Derived Categories, Section 13.11 and especially Lemma 13.11.1) to the short exact sequence of complexes

$$0 \to f_*\mathcal{I}^\bullet \to a_*\mathcal{I}^\bullet|_U \oplus b_*\mathcal{I}^\bullet|_V \to c_*\mathcal{I}^\bullet|_{U\times_X V} \to 0.$$

To see this is a short exact sequence of complexes we argue as follows. Pick an injective object $\mathcal{I}$ of $Mod(\mathcal{O}_X)$. Apply f_* to the short exact sequence

$$0 \to \mathcal{I} \to j_{U,*}\mathcal{I}|_U \oplus j_{V,*}\mathcal{I}|_V \to j_{U\times_X V,*}\mathcal{I}|_{U\times_X V} \to 0$$

of Lemma 56.9.2 and use that $R^1 f_*\mathcal{I} = 0$ to get a short exact sequence

$$0 \to f_*\mathcal{I} \to f_*j_{U,*}\mathcal{I}|_U \oplus f_*j_{V,*}\mathcal{I}|_V \to f_*j_{U\times_X V,*}\mathcal{I}|_{U\times_X V} \to 0$$

The proof is finished by observing that $a_* = f_*j_{U,*}$ and similarly for b_* and c_*. □

Lemma 56.9.4. *Let S be a scheme. Let $(U \subset X, V \to X)$ be an elementary distinguished square of algebraic spaces over S. For objects E, F of $D(\mathcal{O}_X)$ we have a Mayer-Vietoris sequence*

$$\ldots \longrightarrow Ext^{-1}(E_{U\times_X V}, F_{U\times_X V})$$

$$\mathrm{Hom}(E, F) \longrightarrow \mathrm{Hom}(E_U, F_U) \oplus \mathrm{Hom}(E_V, F_V) \longrightarrow \mathrm{Hom}(E_{U\times_X V}, F_{U\times_X V})$$

where the subscripts denote restrictions to the relevant opens and the Hom*'s are taken in the relevant derived categories.*

Proof. Use the distinguished triangle of Lemma 56.9.1 to obtain a long exact sequence of Hom's (from Derived Categories, Lemma 13.4.2) and use that $\mathrm{Hom}(j_{U!}E|_U, F) = \mathrm{Hom}(E|_U, F|_U)$ by Cohomology on Sites, Lemma 21.20.2. □

Lemma 56.9.5. *Let S be a scheme. Let $j : U \to X$ be a étale morphism of algebraic spaces over S. Given an étale morphism $V \to Y$, set $W = V \times_X U$ and denote $j_W : W \to V$ the projection morphism. Then $(j_!E)|_V = j_{W!}(E|_W)$ for E in $D(\mathcal{O}_U)$.*

Proof. This is true because $(j_!\mathcal{F})|_V = j_{W!}(\mathcal{F}|_W)$ for an $\mathcal{O}_X$-module $\mathcal{F}$ as follows immediately from the construction of the functors $j_!$ and $j_{W!}$, see Modules on Sites, Lemma 18.19.2. □

Lemma 56.9.6. *Let S be a scheme. Let $(U \subset X, j : V \to X)$ be an elementary distinguished square of algebraic spaces over S. Set $T = |X| \setminus |U|$.*

1. *If E is an object of $D(\mathcal{O}_X)$ supported on T, then (a) $E \to Rj_*(E|_V)$ and (b) $j_!(E|_V) \to E$ are isomorphisms.*
2. *If F is an object of $D(\mathcal{O}_V)$ supported on $j^{-1}T$, then (a) $F \to (j_!F)|_V$, (b) $(Rj_*F)|_V \to F$, and (c) $j_!F \to Rj_*F$ are isomorphisms.*

Proof. Let E be an object of $D(\mathcal{O}_X)$ whose cohomology sheaves are supported on T. Then we see that $E|_U = 0$ and $E|_{U \times_X V} = 0$ as T doesn't meet U and $j^{-1}T$ doesn't meet $U \times_X V$. Thus (1)(a) follows from Lemma 56.9.2. In exactly the same way (1)(b) follows from Lemma 56.9.1.

Let F be an object of $D(\mathcal{O}_V)$ whose cohomology sheaves are supported on $j^{-1}T$. By Lemma 56.3.1 we have $(Rj_*F)|_U = Rj_{W,*}(F|_W) = 0$ because $F|_W = 0$ by our assumption. Similarly $(j_!F)|_U = j_{W!}(F|_W) = 0$ by Lemma 56.9.5. Thus $j_!F$ and Rj_*F are supported on T and $(j_!F)|_V$ and $(Rj_*F)|_V$ are supported on $j^{-1}(T)$. To check that the maps (2)(a), (b), (c) are isomorphisms in the derived category, it suffices to check that these map induce isomorphisms on stalks of cohomology sheaves at geometric points of T and $j^{-1}(T)$ by Properties of Spaces, Theorem 48.16.12. This we may do after replacing X by V, U by $U \times_X V$, V by $V \times_X V$ and F by $F|_{V \times_X V}$ (restriction via first projection), see Lemmas 56.3.1, 56.9.5, and 56.8.2. Since $V \times_X V \to V$ has a section this reduces (2) to the case that $j : V \to X$ has a section.

Assume j has a section $\sigma : X \to V$. Set $V' = \sigma(X)$. This is an open subspace of V. Set $U' = j^{-1}(U)$. This is another open subspace of V. Then $(U' \subset V, V' \to V)$ is an elementary distinguished square. Observe that $F|_{U'} = 0$ and $F|_{V' \cap U'} = 0$ because F is supported on $j^{-1}(T)$. Denote $j' : V' \to V$ the open immersion and $j_{V'} : V' \to X$ the composition $V' \to V \to X$ which is the inverse of σ. Set $F' = \sigma^*F$. The distinguished triangles of Lemmas 56.9.1 and 56.9.2 show that $F = j'_!(F|_{V'})$ and $F = Rj'_*(F|_{V'})$. It follows that $j_!F = j_!j'_!(F|_{V'}) = j_{V'!}F = F'$ because $j_{V'} : V' \to X$ is an isomorphism and the inverse of σ. Similarly, $Rj_*F = Rj_*Rj'_*F = Rj_{V',*}F = F'$. This proves (2)(c). To prove (2)(a) and (2)(b) it suffices to show that $F = F'|_V$. This is clear because both F and $F'|_V$ restrict to zero on U' and $U' \cap V'$ and the same object on V'. □

We can glue complexes!

Lemma 56.9.7. *Let S be a scheme. Let $(U \subset X, V \to X)$ be an elementary distinguished square of algebraic spaces over S. Suppose given*

(1) *an object* E *of* $D(\mathcal{O}_X)$,
(2) *a morphism* $a : A \to E|_U$ *of* $D(\mathcal{O}_U)$,
(3) *a morphism* $b : B \to E|_V$ *of* $D(\mathcal{O}_V)$,
(4) *an isomorphism* $c : A|_{U \times_X V} \to B|_{U \times_X V}$

such that

$$a|_{U \times_X V} = b|_{U \times_X V} \circ c.$$

Then there exists a morphism $F \to E$ *in* $D(\mathcal{O}_X)$ *whose restriction to* U *is isomorphic to* a *and whose restriction to* V *is isomorphic to* b.

Proof. Denote j_U, j_V, $j_{U \times_X V}$ the corresponding morphisms towards X. Choose a distinguished triangle

$$F \to Rj_{U,*}A \oplus Rj_{V,*}B \to Rj_{U \times_X V,*}(B|_{U \times_X V}) \to F[1]$$

Here the map $Rj_{V,*}B \to Rj_{U \times_X V,*}(B|_{U \times_X V})$ is the obvious one. The map $Rj_{U,*}A \to Rj_{U \times_X V,*}(B|_{U \times_X V})$ is the composition of $Rj_{U,*}A \to Rj_{U \times_X V,*}(A|_{U \times_X V})$ with $Rj_{U \times_X V,*}c$. Restricting to U we obtain

$$F|_U \to A \oplus (Rj_{V,*}B)|_U \to (Rj_{U \times_X V,*}(B|_{U \times_X V}))|_U \to F|_U[1]$$

Denote $j : U \times_X V \to U$. Compatibility of restriction and total direct image (Lemma 56.3.1) shows that both $(Rj_{V,*}B)|_U$ and $(Rj_{U \times_X V,*}(B|_{U \times_X V}))|_U$ are canonically isomorphic to $Rj_*(B|_{U \times_X V})$. Hence the second arrow of the last displayed equation has a section, and we conclude that the morphism $F|_U \to A$ is an isomorphism.

To see that the morphism $F|_V \to B$ is an isomorphism we will use a trick. Namely, choose a distinguished triangle

$$F|_V \to B \to B' \to F[1]|_V$$

in $D(\mathcal{O}_V)$. Since $F|_U \to A$ is an isomorphism, and since we have the isomorphism $c : A|_{U \times_X V} \to B|_{U \times_X V}$ the restriction of $F|_V \to B$ is an isomorphism over $U \times_X V$. Thus B' is supported on $j_V^{-1}(T)$ where $T = |X| \setminus |U|$. On the other hand, there is a morphism of distinguished triangles

$$\begin{array}{ccccccc}
F & \longrightarrow & Rj_{U,*}F|_U \oplus Rj_{V,*}F|_V & \longrightarrow & Rj_{U \times_X V,*}F|_{U \times_X V} & \longrightarrow & F[1] \\
\downarrow & & \downarrow & & \downarrow & & \downarrow \\
F & \longrightarrow & Rj_{U,*}A \oplus Rj_{V,*}B & \longrightarrow & Rj_{U \times_X V,*}(B|_{U \times_X V}) & \longrightarrow & F[1]
\end{array}$$

The all of the vertical maps in this diagram are isomorphisms, except for the map $Rj_{V,*}F|_V \to Rj_{V,*}B$, hence that is an isomorphism too (Derived Categories, Lemma 13.4.3). This implies that $Rj_{V,*}B' = 0$. Hence $B' = 0$ by Lemma 56.9.6.

The existence of the morphism $F \to E$ follows from the Mayer-Vietoris sequence for Hom, see Lemma 56.9.4. □

56.10. The coherator

Let S be a scheme. Let X be an algebraic space over S. The *coherator* is a functor

$$Q_X : \mathit{Mod}(\mathcal{O}_X) \longrightarrow \mathit{QCoh}(\mathcal{O}_X)$$

which is right adjoint to the inclusion functor $\mathit{QCoh}(\mathcal{O}_X) \to \mathit{Mod}(\mathcal{O}_X)$. It exists for any algebraic space X and moreover the adjunction mapping $Q_X(\mathcal{F}) \to \mathcal{F}$

is an isomorphism for every quasi-coherent module $\mathcal{F}$, see Properties of Spaces, Proposition 48.30.2. Since Q_X is left exact (as a right adjoint) we can consider its right derived extension

$$RQ_X : D(\mathcal{O}_X) \longrightarrow D(QCoh(\mathcal{O}_X)).$$

As this functor is constructed by applying Q_X to a K-injective replacement we see that RQ_X is a right adjoint to the canonical functor $D(QCoh(\mathcal{O}_X)) \to D(\mathcal{O}_X)$.

Lemma 56.10.1. *Let S be a scheme. Let $f : X \to Y$ be an affine morphism of algebraic spaces over S. Then f_* defines a derived functor $f_* : D(QCoh(\mathcal{O}_X)) \to D(QCoh(\mathcal{O}_Y))$. This functor has the property that*

$$\begin{array}{ccc} D(QCoh(\mathcal{O}_X)) & \longrightarrow & D_{QCoh}(\mathcal{O}_X) \\ {\scriptstyle f_*}\downarrow & & \downarrow{\scriptstyle Rf_*} \\ D(QCoh(\mathcal{O}_Y)) & \longrightarrow & D_{QCoh}(\mathcal{O}_Y) \end{array}$$

commutes.

Proof. The functor $f_* : QCoh(\mathcal{O}_X) \to QCoh(\mathcal{O}_Y)$ is exact, see Cohomology of Spaces, Lemma 51.7.2. Hence f_* defines a derived functor $f_* : D(QCoh(\mathcal{O}_X)) \to D(QCoh(\mathcal{O}_Y))$ by simply applying f_* to any representative complex, see Derived Categories, Lemma 13.16.8. For any complex of $\mathcal{O}_X$-modules $\mathcal{F}^\bullet$ there is a canonical map $f_*\mathcal{F}^\bullet \to Rf_*\mathcal{F}^\bullet$. To finish the proof we show this is a quasi-isomorphism when $\mathcal{F}^\bullet$ is a complex with each $\mathcal{F}^n$ quasi-coherent. The statement is étale local on Y hence we may assume Y affine. As an affine morphism is representable we reduce to the case of schemes by the compatibility of Remark 56.6.3. The case of schemes is Derived Categories of Schemes, Lemma 35.6.1. □

Lemma 56.10.2. *Let S be a scheme. Let $f : X \to Y$ be a morphism of algebraic spaces over S. Assume that*

(1) *f is quasi-compact, quasi-separated, and flat, and*
(2) *denoting*
$$\Phi : D(QCoh(\mathcal{O}_X)) \to D(QCoh(\mathcal{O}_Y))$$
the right derived functor of $f_* : QCoh(\mathcal{O}_X) \to QCoh(\mathcal{O}_Y)$ the diagram
$$\begin{array}{ccc} D(QCoh(\mathcal{O}_X)) & \longrightarrow & D_{QCoh}(\mathcal{O}_X) \\ {\scriptstyle \Phi}\downarrow & & \downarrow{\scriptstyle Rf_*} \\ D(QCoh(\mathcal{O}_Y)) & \longrightarrow & D_{QCoh}(\mathcal{O}_Y) \end{array}$$
commutes.

Then $RQ_Y \circ Rf_* = \Phi \circ RQ_X$.

Proof. Since f is quasi-compact and quasi-separated, we see that f_* preserve quasi-coherence, see Morphisms of Spaces, Lemma 49.12.2. Recall that $QCoh(\mathcal{O}_X)$ is a Grothendieck abelian category (Properties of Spaces, Proposition 48.30.2). Hence any K in $D(QCoh(\mathcal{O}_X))$ can be represented by a K-injective complex $\mathcal{I}^\bullet$ of $QCoh(\mathcal{O}_X)$, see Injectives, Theorem 19.12.6. Then we can define $\Phi(K) = f_*\mathcal{I}^\bullet$.

Since f is flat, the functor f^* is exact. Hence f^* defines $f^* : D(\mathcal{O}_Y) \to D(\mathcal{O}_X)$ and also $f^* : D(QCoh(\mathcal{O}_Y)) \to D(QCoh(\mathcal{O}_X))$. The functor $f^* = Lf^* : D(\mathcal{O}_Y) \to$

$D(\mathcal{O}_X)$ is left adjoint to $Rf_* : D(\mathcal{O}_X) \to D(\mathcal{O}_Y)$, see Cohomology on Sites, Lemma 21.19.1. Similarly, the functor $f^* : D(QCoh(\mathcal{O}_Y)) \to D(QCoh(\mathcal{O}_X))$ is left adjoint to $\Phi : D(QCoh(\mathcal{O}_X)) \to D(QCoh(\mathcal{O}_Y))$ by Derived Categories, Lemma 13.27.4.

Let A be an object of $D(QCoh(\mathcal{O}_Y))$ and E an object of $D(\mathcal{O}_X)$. Then

$$\begin{aligned}
\mathrm{Hom}_{D(QCoh(\mathcal{O}_Y))}(A, RQ_Y(Rf_*E)) &= \mathrm{Hom}_{D(\mathcal{O}_Y)}(A, Rf_*E) \\
&= \mathrm{Hom}_{D(\mathcal{O}_X)}(f^*A, E) \\
&= \mathrm{Hom}_{D(QCoh(\mathcal{O}_X))}(f^*A, RQ_X(E)) \\
&= \mathrm{Hom}_{D(QCoh(\mathcal{O}_Y))}(A, \Phi(RQ_X(E)))
\end{aligned}$$

This implies what we want. □

Lemma 56.10.3. *Let S be a scheme. Let X be an affine algebraic space over S. Set $A = \Gamma(X, \mathcal{O}_X)$. Then*

1. *$Q_X : Mod(\mathcal{O}_X) \to QCoh(\mathcal{O}_X)$ is the functor which sends $\mathcal{F}$ to the quasi-coherent $\mathcal{O}_X$-module associated to the A-module $\Gamma(X, \mathcal{F})$,*
2. *$RQ_X : D(\mathcal{O}_X) \to D(QCoh(\mathcal{O}_X))$ is the functor which sends E to the complex of quasi-coherent $\mathcal{O}_X$-modules associated to the object $R\Gamma(X, E)$ of $D(A)$,*
3. *restricted to $D_{QCoh}(\mathcal{O}_X)$ the functor RQ_X defines a quasi-inverse to (56.5.1.1).*

Proof. Let $X_0 = \mathrm{Spec}(A)$ be the affine scheme representing X. Recall that there is a morphism of ringed sites $\epsilon : X_{étale} \to X_{0,Zar}$ which induces equivalences

$$QCoh(\mathcal{O}_X) \underset{\epsilon^*}{\overset{\epsilon_*}{\rightleftarrows}} QCoh(\mathcal{O}_{X_0}),$$

see Lemma 56.4.2. Hence we see that $Q_X = \epsilon^* \circ Q_{X_0} \circ \epsilon_*$ by uniqueness of adjoint functors. Hence (1) follows from the description of Q_{X_0} in Derived Categories of Schemes, Lemma 35.6.3 and the fact that $\Gamma(X_0, \epsilon_*\mathcal{F}) = \Gamma(X, \mathcal{F})$. Part (2) follows from (1) and the fact that the functor from A-modules to quasi-coherent $\mathcal{O}_X$-modules is exact. The third assertion now follows from the result for schemes (Derived Categories of Schemes, Lemma 35.6.3) and Lemma 56.4.2. □

Proposition 56.10.4. *Let S be a scheme. Let X be a quasi-compact algebraic space with affine diagonal. Then the functor (56.5.1.1)*

$$D(QCoh(\mathcal{O}_X)) \longrightarrow D_{QCoh}(\mathcal{O}_X)$$

is an equivalence with quasi-inverse given by RQ_X.

Proof. We first use the induction principle to prove i_X is fully faithfu. Let $\mathcal{B} \subset \mathrm{Ob}(X_{spaces,étale})$ be the set of objects which are quasi-compact and have affine diagonal. For $U \in \mathcal{B}$ let $P(U) =$ "the functor $i_U : D(QCoh(\mathcal{O}_U)) \to D_{QCoh}(\mathcal{O}_U)$ is fully faithful". By Remark 56.8.5 conditions (1), (2), and (3)(a) of Lemma 56.8.4 hold and we are left with proving (3)(b) and (4). Condition (3)(b) holds by Lemma 56.10.3.

Let $(U \subset W, V \to W)$ be an elementary distinguished square with V affine. Assume that P holds for U, V, and $U \times_W V$. We have to show that P holds for W. We may replace X by W, i.e., we may assume $W = X$ (we do this just to simplify the notation).

Suppose that A, B are objects of $D(QCoh(\mathcal{O}_X))$. We want to show that

$$\mathrm{Hom}_{D(QCoh(\mathcal{O}_X))}(A, B) \longrightarrow \mathrm{Hom}_{D(\mathcal{O}_X)}(i_X(A), i_X(B))$$

is bijective. Let $T = |X| \setminus |U|$.

Assume first $i_X(B)$ is supported on T. In this case the map

$$i_X(B) \to Rj_{V,*}(i_X(B)|_V) = Rj_{V,*}(i_V(B|_V))$$

is a quasi-isomorphism (Lemma 56.9.6). The morphism $V \to X$ is affine as V is affine and X has affine diagonal (Morphisms of Spaces, Lemma 49.19.11). Thus we have an object $j_{V,*}(B|_V)$ in $QCoh(\mathcal{O}_X)$ and an isomorphism $i_X(j_{V,*}(B|_V)) \to Rj_{V,*}(i_V(B|_V))$ in $D(\mathcal{O}_X)$ (Lemma 56.10.1). Moreover, $j_{V,*}$ and $-|_V$ are adjoint functors on the derived categories of quasi-coherent modules, see proof Lemma 56.10.2. The adjunction map $B \to j_{V,*}(B|_V)$ becomes an isomorphism after applying i_X, whence is an isomorphism in $D(QCoh(\mathcal{O}_X))$. Hence

$$\begin{aligned}
\mathrm{Mor}_{D(QCoh(\mathcal{O}_X))}(A, B) &= \mathrm{Mor}_{D(QCoh(\mathcal{O}_X))}(A, j_{V,*}(B|_V)) \\
&= \mathrm{Mor}_{D(QCoh(\mathcal{O}_V))}(A|_V, B|_V) \\
&= \mathrm{Mor}_{D(\mathcal{O}_V)}(i_V(A|_V), i_V(B|_V)) \\
&= \mathrm{Mor}_{D(\mathcal{O}_X)}(i_X(A), Rj_{V,*}(i_V(B|_V))) \\
&= \mathrm{Mor}_{D(\mathcal{O}_X)}(i_X(A), i_X(B))
\end{aligned}$$

as desired.

In general, choose any complex $\mathcal{B}^\bullet$ of quasi-coherent $\mathcal{O}_X$-modules representing B. Next, choose any quasi-isomorphism $s : \mathcal{B}^\bullet|_U \to \mathcal{C}^\bullet$ of complexes of quasi-coherent modules on U. As $j_U : U \to X$ is quasi-compact and quasi-separated the functor $j_{U,*}$ transforms quasi-coherent modules into quasi-coherent modules (Morphisms of Spaces, Lemma 49.12.2). Thus there is a canonical map $\mathcal{B}^\bullet \to j_{U,*}(\mathcal{B}^\bullet|_U) \to j_{U,*}\mathcal{C}^\bullet$ of complexes of quasi-coherent modules on X. Set $B'' = j_{U,*}\mathcal{C}^\bullet$ in $D(QCoh(\mathcal{O}_X))$ and choose a distinguished triangle

$$B \to B'' \to B' \to B^\bullet[1]$$

in $D(QCoh(\mathcal{O}_X))$. Since the first arrow of the triangle restricts to an isomorphism over U we see that B' is supported on T. Hence in the diagram

$$\begin{array}{ccc}
\mathrm{Hom}_{D(QCoh(\mathcal{O}_X))}(A, B'[-1]) & \longrightarrow & \mathrm{Hom}_{D(\mathcal{O}_X)}(i_X(A), i_X(B')[-1]) \\
\downarrow & & \downarrow \\
\mathrm{Hom}_{D(QCoh(\mathcal{O}_X))}(A, B) & \longrightarrow & \mathrm{Hom}_{D(\mathcal{O}_X)}(i_X(A), i_X(B)) \\
\downarrow & & \downarrow \\
\mathrm{Hom}_{D(QCoh(\mathcal{O}_X))}(A, B'') & \longrightarrow & \mathrm{Hom}_{D(\mathcal{O}_X)}(i_X(A), i_X(B'')) \\
\downarrow & & \downarrow \\
\mathrm{Hom}_{D(QCoh(\mathcal{O}_X))}(A, B') & \longrightarrow & \mathrm{Hom}_{D(\mathcal{O}_X)}(i_X(A), i_X(B'))
\end{array}$$

we have exact columns and the top and bottom horizontal arrows are bijective. Finally, choose a complex $\mathcal{A}^\bullet$ of quasi-coherent modules representing A.

Let $\alpha : i_X(A) \to i_X(B)$ be a morphism between in $D(\mathcal{O}_X)$. The restriction $\alpha|_U$ comes from a morphism in $D(QCoh(\mathcal{O}_U))$ by assumption. Hence there exists a choice of $s : \mathcal{B}^\bullet|_U \to \mathcal{C}^\bullet$ as above such that $\alpha|_U$ is represented by an actual map of complexes $\mathcal{A}^\bullet|_U \to \mathcal{C}^\bullet$. This corresponds to a map of complexes $\mathcal{A} \to j_{U,*}\mathcal{C}^\bullet$. In other words, the image of α in $\mathrm{Hom}_{D(\mathcal{O}_X)}(i_X(A), i_X(B''))$ comes from an element of $\mathrm{Hom}_{D(QCoh(\mathcal{O}_X))}(A, B'')$. A diagram chase then shows that α comes from a morphism $A \to B$ in $D(QCoh(\mathcal{O}_X))$. Finally, suppose that $a : A \to B$ is a morphism of $D(QCoh(\mathcal{O}_X))$ which becomes zero in $D(\mathcal{O}_X)$. After choosing $\mathcal{B}^\bullet$ suitably, we may assume a is represented by a morphism of complexes $a^\bullet : \mathcal{A}^\bullet \to \mathcal{B}^\bullet$. Since P holds for U the restriction $a^\bullet|_U$ is zero in $D(QCoh(\mathcal{O}_U))$. Thus we can choose s such that $s \circ a^\bullet|_U : \mathcal{A}^\bullet|_U \to \mathcal{C}^\bullet$ is homotopic to zero. Applying the functor $j_{U,*}$ we conclude that $\mathcal{A}^\bullet \to j_{U,*}\mathcal{C}^\bullet$ is homotopic to zero. Thus a maps to zero in $\mathrm{Hom}_{D(QCoh(\mathcal{O}_X))}(A, B'')$. Thus we may assume that a is the image of an element of $b \in \mathrm{Hom}_{D(QCoh(\mathcal{O}_X))}(A, B'[-1])$. The image of b in $\mathrm{Hom}_{D(\mathcal{O}_X)}(i_X(A), i_X(B')[-1])$ comes from a $\gamma \in \mathrm{Hom}_{D(\mathcal{O}_X)}(A, B''[-1])$ (as a maps to zero in the group on the right). Since we've seen above the horizontal arrows are surjective, we see that γ comes from a c in $\mathrm{Hom}_{D(QCoh(\mathcal{O}_X))}(A, B''[-1])$ which implies $a = 0$ as desired.

Since i_X is fully faithful with right adjoint RQ_X we see that $RQ_X \circ i_X = \mathrm{id}$ (Categories, Lemma 4.24.2). To finish the proof we show that for any E in $D_{QCoh}(\mathcal{O}_X)$ the map $i_X(RQ_X(E)) \to E$ is an isomorphism. Choose a distinguished triangle

$$i_X(RQ_X(E)) \to E \to E' \to i_X(RQ_X(E))[1]$$

in $D_{QCoh}(\mathcal{O}_X)$. A formal argument using the above shows that $i_X(RQ_X(E')) = 0$. Thus it suffices to prove that for $E \in D_{QCoh}(\mathcal{O}_X)$ the condition $i_X(RQ_X(E)) = 0$ implies that $E = 0$. Consider an étale morphism $j : V \to X$ with V affine. By Lemmas 56.10.3, 56.10.1, and 56.10.2 we have

$$Rj_*(E|_V) = Rj_*(i_V(RQ_V(E|_V))) = i_X(j_*(RQ_V(E|_V))) = i_X(RQ_X(Rj_*(E|_V)))$$

Choose a distinguished triangle

$$E \to Rj_*(E|_V) \to E' \to E[1]$$

Apply RQ_X to get a distinguished triangle

$$0 \to RQ_X(Rj_*(E|_V)) \to RQ_X(E') \to 0[1]$$

in other words the map in the middle is an isomorphism. Combined with the string of equalities above we find that our first distinghuished triangle becomes a distinguished triangle

$$E \to i_X(RQ_X(E')) \to E' \to E[1]$$

where the middle morphism is the adjunction map. However, the composition $E \to E'$ is zero, hence $E \to i_X(RQ_X(E'))$ is zero by adjunction! Since this morphism is isomorphic to the morphism $E \to Rj_*(E|_V)$ adjoint to $\mathrm{id} : E|_V \to E|_V$ we conclude that $E|_V$ is zero. Since this holds for all affine V étale over X we conlude E is zero as desired. □

Remark 56.10.5. Analyzing the proof of Proposition 56.10.4 we see that we have shown the following. Let X be a quasi-compact and quasi-separated scheme. Suppose that for every étale morphism $j : V \to X$ with V affine the right derived functor

$$\Phi : D(QCoh(\mathcal{O}_U)) \to D(QCoh(\mathcal{O}_X))$$

of the left exact functor $j_* : QCoh(\mathcal{O}_V) \to QCoh(\mathcal{O}_X)$ fits into a commutative diagram

$$\begin{array}{ccc} D(QCoh(\mathcal{O}_V)) & \xrightarrow[i_V]{} & D_{QCoh}(\mathcal{O}_V) \\ \Phi \downarrow & & \downarrow Rj_* \\ D(QCoh(\mathcal{O}_X)) & \xrightarrow{i_X} & D_{QCoh}(\mathcal{O}_X) \end{array}$$

Then the functor (56.5.1.1)

$$D(QCoh(\mathcal{O}_X)) \longrightarrow D_{QCoh}(\mathcal{O}_X)$$

is an equivalence with quasi-inverse given by RQ_X.

56.11. The coherator for Noetherian spaces

We need a little bit more about injective modules to treat the case of a Noetherian algebraic space.

Lemma 56.11.1. *Let S be a Noetherian affine scheme. Every injective object of $QCoh(\mathcal{O}_S)$ is a filtered colimit $\operatorname{colim}_i \mathcal{F}_i$ of quasi-coherent sheaves of the form*

$$\mathcal{F}_i = (Z_i \to S)_* \mathcal{G}_i$$

where Z_i is the spectrum of an Artinian ring and $\mathcal{G}_i$ is a coherent module on Z_i.

Proof. Let $S = \operatorname{Spec}(A)$. Let $\mathcal{J}$ be an injective object of $QCoh(\mathcal{O}_S)$. Since $QCoh(\mathcal{O}_S)$ is equivalent to the category of A-modules we see that $\mathcal{J}$ is equal to $\widetilde{J}$ for some injective A-module J. By Dualizing Complexes, Proposition 43.5.9 we can write $J = \bigoplus E_\alpha$ with E_α indecomposable and therefore isomorphic to the injective hull of a reside field at a point. Thus (because finite disjoint unions of Artinian schemes are Artinian) we may assume that J is the injective hull of $\kappa(\mathfrak{p})$ for some prime $\mathfrak{p}$ of A. Then $J = \bigcup J[\mathfrak{p}^n]$ where $J[\mathfrak{p}^n]$ is the injective hull of $\kappa(\mathfrak{p})$ over $A/\mathfrak{p}^n A_\mathfrak{p}$, see Dualizing Complexes, Lemma 43.7.3. Thus $\widetilde{J}$ is the colimit of the sheaves $(Z_n \to X)_* \mathcal{G}_n$ where $Z_n = \operatorname{Spec}(A_\mathfrak{p}/\mathfrak{p}^n A_\mathfrak{p})$ and $\mathfrak{G}_n$ the coherent sheaf associated to the finite $A/\mathfrak{p}^n A_\mathfrak{p}$-module $J[\mathfrak{p}^n]$. Finiteness follows from Dualizing Complexes, Lemma 43.6.1. □

Lemma 56.11.2. *Let S be an affine scheme. Let X be a Noetherian algebraic space over S. Every injective object of $QCoh(\mathcal{O}_X)$ is a direct summand of a filtered colimit $\operatorname{colim}_i \mathcal{F}_i$ of quasi-coherent sheaves of the form*

$$\mathcal{F}_i = (Z_i \to X)_* \mathcal{G}_i$$

where Z_i is the spectrum of an Artinian ring and $\mathcal{G}_i$ is a coherent module on Z_i.

Proof. Choose an affine scheme U and a surjective étale morphism $j : U \to X$ (Properties of Spaces, Lemma 48.6.3). Then U is a Noetherian affine scheme. Choose an injective object $\mathcal{J}'$ of $QCoh(\mathcal{O}_U)$ such that there exists an injection $\mathcal{J}|_U \to \mathcal{J}'$. Then

$$\mathcal{J} \to j_* \mathcal{J}'$$

is an injective morphism in $QCoh(\mathcal{O}_X)$, hence identifies $\mathcal{J}$ as a direct summand of $j_* \mathcal{J}'$. Thus the result follows from the corresponding result for $\mathcal{J}'$ proved in Lemma 56.11.1. □

Lemma 56.11.3. *Let S be a scheme. Let $f : X \to Y$ be a flat, quasi-compact, and quasi-separated morphism of algebraic spaces over S. If $\mathcal{J}$ is an injective object of $QCoh(\mathcal{O}_X)$, then $f_*\mathcal{J}$ is an injective object of $QCoh(\mathcal{O}_Y)$.*

Proof. Since f is quasi-compact and quasi-separated, the functor f_* transforms quasi-coherent sheaves into quasi-coherent sheaves (Morphisms of Spaces, Lemma 49.12.2). The functor f^* is a left adjoint to f_* which transforms injections into injections. Hence the result follows from Homology, Lemma 12.25.1 □

Lemma 56.11.4. *Let S be a scheme. Let X be a Noetherian algebraic space over S. If $\mathcal{J}$ is an injective object of $QCoh(\mathcal{O}_X)$, then*

1. *$H^p(U, \mathcal{J}|_U) = 0$ for $p > 0$ and for every quasi-compact and quasi-separated algebraic space U étale over X,*
2. *for any morphism $f : X \to Y$ of algebraic spaces over S with Y quasi-separated we have $R^p f_*\mathcal{J} = 0$ for $p > 0$.*

Proof. Proof of (1). Write $\mathcal{J}$ as a direct summand of $\operatorname{colim} \mathcal{F}_i$ with $\mathcal{F}_i = (Z_i \to X)_*\mathcal{G}_i$ as in Lemma 56.11.2. It is clear that it suffices to prove the vanishing for $\operatorname{colim} \mathcal{F}_i$. Since pullback commutes with colimits and since U is quasi-compact and quasi-separated, it suffices to prove $H^p(U, \mathcal{F}_i|_U) = 0$ for $p > 0$, see Cohomology of Spaces, Lemma 51.4.1. Observe that $Z_i \to X$ is an affine morphism, see Morphisms of Spaces, Lemma 49.19.12. Thus

$$\mathcal{F}_i|_U = (Z_i \times_X U \to U)_*\mathcal{G}'_i = R(Z_i \times_X U \to U)_*\mathcal{G}'_i$$

where $\mathcal{G}'_i$ is the pullback of $\mathcal{G}_i$ to $Z_i \times_X U$, see Cohomology of Spaces, Lemma 51.8.2. Since $Z_i \times_X U$ is affine we conlude that $\mathcal{G}'_i$ has no higher cohomology on $Z_i \times_X U$. By the Leray spectral sequence we conclude the same thing is true for $\mathcal{F}_i|_U$ (Cohomology on Sites, Lemma 21.14.6).

Proof of (2). Let $f : X \to Y$ be a morphism of algebraic spaces over S. Let $V \to Y$ be an étale morphism with V affine. Then $V \times_Y X \to X$ is an étale morphism and $V \times_Y X$ is a quasi-compact and quasi-separated algebraic space étale over X (details omitted). Hence $H^p(V \times_Y X, \mathcal{J})$ is zero by part (1). Since $R^p f_*\mathcal{J}$ is the sheaf associated to the presheaf $V \mapsto H^p(V \times_Y X, \mathcal{J})$ the result is proved. □

Lemma 56.11.5. *Let S be a scheme. Let $f : X \to Y$ be a morphism of Noetherian algebraic spaces over S. Then f_* on quasi-coherent sheaves has a right derived extension $\Phi : D(QCoh(\mathcal{O}_X)) \to D(QCoh(\mathcal{O}_Y))$ such that the diagram*

$$\begin{array}{ccc} D(QCoh(\mathcal{O}_X)) & \longrightarrow & D_{QCoh}(\mathcal{O}_X) \\ \downarrow{\scriptstyle \Phi} & & \downarrow{\scriptstyle Rf_*} \\ D(QCoh(\mathcal{O}_Y)) & \longrightarrow & D_{QCoh}(\mathcal{O}_Y) \end{array}$$

commutes.

Proof. Since X and Y are Noetherian the morphism is quasi-compact and quasi-separated (see Morphisms of Spaces, Lemma 49.9.9). Thus f_* preserve quasi-coherence, see Morphisms of Spaces, Lemma 49.12.2. Next, Let K be an object of $D(QCoh(\mathcal{O}_X))$. Since $QCoh(\mathcal{O}_X)$ is a Grothendieck abelian category (Properties of Spaces, Proposition 48.30.2), we can represent K by a K-injective complex $\mathcal{I}^\bullet$ such

that each $\mathcal{I}^n$ is an injective object of $QCoh(\mathcal{O}_X)$, see Injectives, Theorem 19.12.6. Thus we see that the functor Φ is defined by setting

$$\Phi(K) = f_*\mathcal{I}^\bullet$$

where the right hand side is viewed as an object of $D(QCoh(\mathcal{O}_Y))$. To finish the proof of the lemma it suffices to show that the canonical map

$$f_*\mathcal{I}^\bullet \longrightarrow Rf_*\mathcal{I}^\bullet$$

is an isomorphism in $D(\mathcal{O}_Y)$. To see this it suffices to prove the map induces an isomorphism on cohomology sheaves. Pick any $m \in \mathbf{Z}$. Let $N = N(X, Y, f)$ be as in Lemma 56.6.1. Consider the short exact sequence

$$0 \to \sigma_{\geq m-N-1}\mathcal{I}^\bullet \to \mathcal{I}^\bullet \to \sigma_{\leq m-N-2}\mathcal{I}^\bullet \to 0$$

of complexes of quasi-coherent sheaves on X. By Lemma 56.6.1 we see that the cohomology sheaves of $Rf_*\sigma_{\leq m-N-2}\mathcal{I}^\bullet$ are zero in degrees $\geq m-1$. Thus we see that $R^m f_*\mathcal{I}^\bullet$ is isomorphic to $R^m f_*\sigma_{\geq m-N-1}\mathcal{I}^\bullet$. In other words, we may assume that $\mathcal{I}^\bullet$ is a bounded below complex of injective objects of $QCoh(\mathcal{O}_X)$. This case follows from Leray's acyclicity lemma (Derived Categories, Lemma 13.16.7) with required vanishing because of Lemma 56.11.4. □

Proposition 56.11.6. *Let S be a scheme. Let X be a Noetherian algebraic space over S. Then the functor (56.5.1.1)*

$$D(QCoh(\mathcal{O}_X)) \longrightarrow D_{QCoh}(\mathcal{O}_X)$$

is an equivalence with quasi-inverse given by RQ_X.

Proof. This follows using the exact same argument as in the proof of Proposition 56.10.4 using Lemma 56.11.5. See discussion in Remark 56.10.5. □

56.12. Pseudo-coherent and perfect complexes

In this section we study the general notions defined in Cohomology on Sites, Sections 21.32, 21.33, 21.34, and 21.35 for the étale site of an algebraic space. In particular we match this with what happens for schemes.

First we compare the notion of a pseudo-coherent complex on a scheme and on its associated small étale site.

Lemma 56.12.1. *Let X be a scheme. Let $\mathcal{F}$ be an $\mathcal{O}_X$-module. The following are equivalent*

(1) *$\mathcal{F}$ is of finite type as an $\mathcal{O}_X$-module, and*
(2) *$\epsilon^*\mathcal{F}$ is of finite type as an $\mathcal{O}_{étale}$-module on the small étale site of X.*

Here ϵ is as in (56.4.0.1).

Proof. The implication (1) $\Rightarrow$ (2) is a general fact, see Modules on Sites, Lemma 18.23.4. Assume (2). By assumption there exists an étale covering $\{f_i : X_i \to X\}$ such that $\epsilon^*\mathcal{F}|_{(X_i)_{étale}}$ is generated by finitely many sections. Let $x \in X$. We will show that $\mathcal{F}$ is generated by finitely many sections in a neighbourhood of x. Say x is in the image of $X_i \to X$ and denote $X' = X_i$. Let $s_1, \ldots, s_n \in \Gamma(X', \epsilon^*\mathcal{F}|_{X'_{étale}})$ be generating sections. As $\epsilon^*\mathcal{F} = \epsilon^{-1}\mathcal{F} \otimes_{\epsilon^{-1}\mathcal{O}_X} \mathcal{O}_{étale}$ we can find an étale morphism $X'' \to X'$ such that x is in the image of $X' \to X$ and such that $s_i|_{X''} = \sum s_{ij} \otimes a_{ij}$ for some sections $s_{ij} \in \epsilon^{-1}\mathcal{F}(X'')$ and $a_{ij} \in \mathcal{O}_{étale}(X'')$. Denote $U \subset X$ the image of $X'' \to X$. This is an open subscheme as $f'' : X'' \to X$ is étale (Morphisms,

Lemma 28.37.13). After possibly shrinking X'' more we may assume s_{ij} come from elements $t_{ij} \in \mathcal{F}(U)$ as follows from the construction of the inverse image functor ϵ^{-1}. Now we claim that t_{ij} generate $\mathcal{F}|_U$ which finishes the proof of the lemma. Namely, the corresponding map $\mathcal{O}_U^{\oplus N} \to \mathcal{F}|_U$ has the property that its pullback by f'' to X'' is surjective. Since $f'' : X'' \to U$ is a surjective flat morphism of schemes, this implies that $\mathcal{O}_U^{\oplus N} \to \mathcal{F}|_U$ is surjective by looking at stalks and using that $\mathcal{O}_{U,f''(z)} \to \mathcal{O}_{X'',z}$ is faithfully flat for all $z \in X''$. □

In the situation above the morphism of sites ϵ is flat hence defines a pullback on complexes of modules.

Lemma 56.12.2. *Let X be a scheme. Let E be an object of $D(\mathcal{O}_X)$. The following are equivalent*

(1) *E is m-pseudo-coherent, and*
(2) *$\epsilon^* E$ is m-pseudo-coherent on the small étale site of X.*

Here ϵ is as in (56.4.0.1).

Proof. The implication (1) $\Rightarrow$ (2) is a general fact, see Cohomology on Sites, Lemma 21.33.3. Assume $\epsilon^* E$ is m-pseudo-coherent. We will use without further mention that ϵ^* is an exact functor and that therefore

$$\epsilon^* H^i(E) = H^i(\epsilon^* E).$$

To show that E is m-pseudo-coherent we may work locally on X, hence we may assume that X is quasi-compact (for example affine). Since X is quasi-compact every étale covering $\{U_i \to X\}$ has a finite refinement. Thus we see that $\epsilon^* E$ is an object of $D^-(\mathcal{O}_{étale})$, see comments following Cohomology on Sites, Definition 21.33.1. By Lemma 56.4.1 it follows that E is an object of $D^-(\mathcal{O}_X)$.

Let $n \in \mathbf{Z}$ be the largest integer such that $H^n(E)$ is nonzero; then n is also the largest integer such that $H^n(\epsilon^* E)$ is nonzero. We will prove the lemma by induction on $n - m$. If $n < m$, then the lemma is clearly true. If $n \geq m$, then $H^n(\epsilon^* E)$ is a finite $\mathcal{O}_{étale}$-module, see Cohomology on Sites, Lemma 21.33.7. Hence $H^n(E)$ is a finite $\mathcal{O}_X$-module, see Lemma 56.12.1. After replacing X by the members of an open covering, we may assume there exists a surjection $\mathcal{O}_X^{\oplus t} \to H^n(E)$. We may locally on X lift this to a map of complexes $\alpha : \mathcal{O}_X^{\oplus t}[-n] \to E$ (details omitted). Choose a distinguished triangle

$$\mathcal{O}_X^{\oplus t}[-n] \to E \to C \to \mathcal{O}_X^{\oplus t}[-n+1]$$

Then C has vanishing cohomology in degrees $\geq n$. On the other hand, the complex $\epsilon^* C$ is m-pseudo-coherent, see Cohomology on Sites, Lemma 21.33.4. Hence by induction we see that C is m-pseudo-coherent. Applying Cohomology on Sites, Lemma 21.33.4 once more we conclude. □

Lemma 56.12.3. *Let X be a scheme. Let E be an object of $D(\mathcal{O}_X)$. Then*

(1) *E has tor amplitude in $[a, b]$ if and only if $\epsilon^* E$ has tor amplitude in $[a, b]$.*
(2) *E has finite tor dimension if and only if $\epsilon^* E$ has finite tor dimension.*

Here ϵ is as in (56.4.0.1).

Proof. The easy implication follows from the general result contained in Cohomology on Sites, Lemma 21.34.4 (and the fact that the small étale site of X has enough points, see Étale Cohomology, Remarks 44.29.11). For the converse, assume that

ϵ^*E has tor amplitude in $[a, b]$ Let $\mathcal{F}$ be an $\mathcal{O}_X$-module. As ϵ is a flat morphism of ringed sites (Lemma 56.4.1) we have

$$\epsilon^*(E \otimes^{\mathbf{L}}_{\mathcal{O}_X} \mathcal{F}) = \epsilon^*E \otimes^{\mathbf{L}}_{\mathcal{O}_{étale}} \epsilon^*\mathcal{F}$$

Thus the (assumed) vanishing of cohomology sheaves on the right hand side implies the desired vanishing of the cohomology sheaves of $E \otimes^{\mathbf{L}}_{\mathcal{O}_X} \mathcal{F}$ via Lemma 56.4.1. □

Lemma 56.12.4. *Let X be a scheme. Let E be an object of $D(\mathcal{O}_X)$. Then E is a perfect object of $D(\mathcal{O}_X)$ if and only if ϵ^*E is a perfect object of $D(\mathcal{O}_{étale})$. Here ϵ is as in (56.4.0.1).*

Proof. The easy implication follows from the general result contained in Cohomology on Sites, Lemma 21.35.5 (and the fact that the small étale site of X has enough points, see Étale Cohomology, Remarks 44.29.11). For the converse, we can use the equivalence of Cohomology on Sites, Lemma 21.35.4 and the corresponding results for pseudo-coherent and complexes of finite tor dimension, namely Lemmas 56.12.2 and 56.12.3. Some details omitted. □

Lemma 56.12.5. *Let S be a scheme. Let X be an algebraic space over S. If E is an m-pseudo-coherent object of $D(\mathcal{O}_X)$, then $H^i(E)$ is a quasi-coherent $\mathcal{O}_X$-module for $i > m$. If E is pseudo-coherent, then E is an object of $D_{QCoh}(\mathcal{O}_X)$.*

Proof. Locally $H^i(E)$ is isomorphic to $H^i(\mathcal{E}^\bullet)$ with $\mathcal{E}^\bullet$ strictly perfect. The sheaves $\mathcal{E}^i$ are direct summands of finite free modules, hence quasi-coherent. The lemma follows. □

Lemma 56.12.6. *Let S be a scheme. Let X be a Noetherian algebraic space over S. Let E be an object of $D_{QCoh}(\mathcal{O}_X)$. For $m \in \mathbf{Z}$ the following are equivalent*

1. *$H^i(E)$ is coherent for $i \geq m$ and zero for $i \gg 0$, and*
2. *E is m-pseudo-coherent.*

In particular, E is pseudo-coherent if and only if E is an object of $D^-_{Coh}(\mathcal{O}_X)$.

Proof. As X is quasi-compact we can find an affine scheme U and a surjective étale morphism $U \to X$ (Properties of Spaces, Lemma 48.6.3). Observe that U is Noetherian. Note that E is m-pseudo-coherent if and only if $E|_U$ is m-pseudo-coherent (follows from the definition or from Cohomology on Sites, Lemma 21.33.2). Similarly, $H^i(E)$ is coherent if and only if $H^i(E)|_U = H^i(E|_U)$ is coherent (see Cohomology of Spaces, Lemma 51.9.2). Thus we may assume that X is representable.

If X is representable by a scheme X_0 then (Lemma 56.4.2) we can write $E = \epsilon^*E_0$ where E_0 is an object of $D_{QCoh}(\mathcal{O}_{X_0})$ and $\epsilon : X_{étale} \to (X_0)_{Zar}$ is as in (56.4.0.1). In this case E is m-pseudo-coherent if and only if E_0 is by Lemma 56.12.2. Similarly, $H^i(E_0)$ is of finite type (i.e., coherent) if and only if $H^i(E)$ is by Lemma 56.12.1. Finally, $H^i(E_0) = 0$ if and only if $H^i(E) = 0$ by Lemma 56.4.1. Thus we reduce to the case of schemes which is Derived Categories of Schemes, Lemma 35.9.4. □

Lemma 56.12.7. *Let S be a scheme. Let X be a quasi-separated algebraic space over S. Let E be an object of $D_{QCoh}(\mathcal{O}_X)$. Let $a \leq b$. The following are equivalent*

1. *E has tor amplitude in $[a, b]$, and*
2. *for all $\mathcal{F}$ in $QCoh(\mathcal{O}_X)$ we have $H^i(E \otimes^{\mathbf{L}}_{\mathcal{O}_X} \mathcal{F}) = 0$ for $i \notin [a, b]$.*

Proof. It is clear that (1) implies (2). Assume (2). Let $j : U \to X$ be an étale morphism with U affine. As X is quasi-separated $j : U \to X$ is quasi-compact and separated, hence j_* transforms quasi-coherent modules into quasi-coherent modules (Morphisms of Spaces, Lemma 49.12.2). Thus the functor $QCoh(\mathcal{O}_X) \to QCoh(\mathcal{O}_U)$ is essentially surjective. It follows that condition (2) implies the vanishing of $H^i(E|_U \otimes^{\mathbf{L}}_{\mathcal{O}_U} \mathcal{G})$ for $i \notin [a, b]$ for all quasi-coherent $\mathcal{O}_U$-modules $\mathcal{G}$. Since it suffices to prove that $E|_U$ has tor amplitude in $[a, b]$ we reduce to the case where X is representable.

If X is representable by a scheme X_0 then (Lemma 56.4.2) we can write $E = \epsilon^* E_0$ where E_0 is an object of $D_{QCoh}(\mathcal{O}_{X_0})$ and $\epsilon : X_{étale} \to (X_0)_{Zar}$ is as in (56.4.0.1). For every quasi-coherent module $\mathcal{F}_0$ on X_0 the module $\epsilon^*\mathcal{F}_0$ is quasi-coherent on X and

$$H^i(E \otimes^{\mathbf{L}}_{\mathcal{O}_X} \epsilon^*\mathcal{F}_0) = \epsilon^* H^i(E_0 \otimes^{\mathbf{L}}_{\mathcal{O}_{X_0}} \mathcal{F}_0)$$

as ϵ is flat (Lemma 56.4.1). Moreover, the vanishing of these sheaves for $i \notin [a, b]$ implies the same thing for $H^i(E_0 \otimes^{\mathbf{L}}_{\mathcal{O}_{X_0}} \mathcal{F}_0)$ by the same lemma. Thus we've reduced the problem to the case of schemes which is treated in Derived Categories of Schemes, Lemma 35.9.6. □

56.13. Approximation by perfect complexes

In this section we continue the discussion started in Derived Categories of Schemes, Section 35.12.

Definition 56.13.1. Let S be a scheme. Let X be an algebraic space over S. Consider triples (T, E, m) where

(1) $T \subset |X|$ is a closed subset,
(2) E is an object of $D_{QCoh}(\mathcal{O}_X)$, and
(3) $m \in \mathbf{Z}$.

We say *approximation holds for the triple* (T, E, m) if there exists a perfect object P of $D(\mathcal{O}_X)$ supported on T and a map $\alpha : P \to E$ which induces isomorphisms $H^i(P) \to H^i(E)$ for $i > m$ and a surjection $H^m(P) \to H^m(E)$.

Approximation cannot hold for every triple. Please read the remarks following Derived Categories of Schemes, Definition 35.12.1 to see why.

Definition 56.13.2. Let S be a scheme. Let X be an algebraic space over S. We say *approximation by perfect complexes holds* on X if for any closed subset $T \subset |X|$ such that the morphism $X \setminus T \to X$ is quasi-compact there exists an integer r such that for every triple (T, E, m) as in Definition 56.13.1 with

(1) E is $(m - r)$-pseudo-coherent, and
(2) $H^i(E)$ is supported on T for $i \geq m - r$

approximation holds.

Lemma 56.13.3. *Let S be a scheme. Let $(U \subset X, j : V \to X)$ be an elementary distinguished square of algebraic space over S. Let E be a perfect object of $D(\mathcal{O}_V)$ supported on $j^{-1}(T)$ where $T = |X| \setminus |U|$. Then Rj_*E is a perfect object of $D(\mathcal{O}_X)$.*

Proof. Being perfect is local on $X_{étale}$. Thus it suffices to check that Rj_*E is perfect when restricted to U and V. We have $Rj_*E|_V = E$ by Lemma 56.9.6 which is perfect. We have $Rj_*E|_U = 0$ because $E|_{V \setminus j^{-1}(T)} = 0$ (use Lemma 56.3.1). □

Lemma 56.13.4. *Let S be a scheme. Let $(U \subset X, j : V \to X)$ be an elementary distinguished square of algebraic spaces over S. Let T be a closed subset of $|X| \setminus |U|$ and let (T, E, m) be a triple as in Definition 56.13.1. If*

(1) *approximation holds for $(j^{-1}T, E|_V, m)$, and*
(2) *the sheaves $H^i(E)$ for $i \geq m$ are supported on T,*

then approximation holds for (T, E, m).

Proof. Let $P \to E|_V$ be an approximation of the triple $(j^{-1}T, E|_V, m)$ over V. Then Rj_*P is a perfect object of $D(\mathcal{O}_X)$ by Lemma 56.13.3. On the other hand, $Rj_*P = j_!P$ by Lemma 56.9.6. We see that $j_!P$ is supported on T for example by (56.9.0.2). Hence we obtain an approximation $Rj_*P = j_!P \to j_!(E|_V) \to E$. □

Lemma 56.13.5. *Let S be a scheme. Let X be an algebraic space over S which is representable by an affine scheme. Then approximation holds for every triple (T, E, m) as in Definition 56.13.1 such that there exists an integer $r \geq 0$ with*

(1) *E is m-pseudo-coherent,*
(2) *$H^i(E)$ is supported on T for $i \geq m - r + 1$,*
(3) *$X \setminus T$ is the union of r affine opens.*

In particular, approximation by perfect complexes holds for affine schemes.

Proof. Let X_0 be an affine scheme representing X. Let $T_0 \subset X_0$ by the closed subset corresponding to T. Let $\epsilon : X_{étale} \to X_{0,Zar}$ be the morphism (56.4.0.1). We may write $E = \epsilon^*E_0$ for some object E_0 of $D_{QCoh}(\mathcal{O}_{X_0})$, see Lemma 56.4.2. Then E_0 is m-pseudo-coherent, see Lemma 56.12.2. Comparing stalks of cohomology sheaves (see proof of Lemma 56.4.1) we see that $H^i(E_0)$ is supported on T_0 for $i \geq m - r + 1$. By Derived Categories of Schemes, Lemma 35.12.4 there exists an approximation $P_0 \to E_0$ of (T_0, E_0, m). By Lemma 56.12.4 we see that $P = \epsilon^*P_0$ is a perfect object of $D(\mathcal{O}_X)$. Pulling back we obtain an approximation $P = \epsilon^*P_0 \to \epsilon^*E_0 = E$ as desired. □

Lemma 56.13.6. *Let S be a scheme. Let $(U \subset X, j : V \to X)$ be an elementary distinguished square of algebraic spaces over S. Assume U quasi-compact, V affine, and $U \times_X V$ quasi-compact. If approximation by perfect complexes holds on U, then approximation by perfect complexes holds on X.*

Proof. Let $T \subset |X|$ be a closed subset with $X \setminus T \to X$ quasi-compact. Let r_U be the integer of Definition 56.13.2 adapted to the pair $(U, T \cap |U|)$. Set $T' = T \setminus |U|$. Endow T' with the induced reduced subspace structure. Since $|T'|$ is contained in $|X| \setminus |U|$ we see that $j^{-1}(T') \to T'$ is an isomorphism. Moreover, $V \setminus j^{-1}(T')$ is quasi-compact as it is the fibre product of $U \times_X V$ with $X \setminus T$ over X and we've assumed $U \times_X V$ quasi-compact and $X \setminus T \to X$ quasi-compact. Let r' be the number of affines needed to cover $V \setminus j^{-1}(T')$. We claim that $r = \max(r_U, r')$ works for the pair (X, T).

To see this choose a triple (T, E, m) such that E is $(m - r)$-pseudo-coherent and $H^i(E)$ is supported on T for $i \geq m - r$. Let t be the largest integer such that $H^t(E)|_U$ is nonzero. (Such an integer exists as U is quasi-compact and $E|_U$ is $(m-r)$-pseudo-coherent.) We will prove that E can be approximated by induction on t.

Base case: $t \leq m - r'$. This means that $H^i(E)$ is supported on T' for $i \geq m - r'$. Hence Lemma 56.13.5 guarantees the existence of an approximation $P \to E|_V$

of $(T', E|_V, m)$ on V. Applying Lemma 56.13.4 we see that (T', E, m) can be approximated. Such an approximation is also an approximation of (T, E, m).

Induction step. Choose an approximation $P \to E|_U$ of $(T \cap |U|, E|_U, m)$. This in particular gives a surjection $H^t(P) \to H^t(E|_U)$. In the rest of the proof we will use the equivalence of Lemma 56.4.2 (and the compatibilities of Remark 56.6.3) for the representable algebraic spaces V and $U \times_X V$. We will also use the fact that $(m-r)$-pseudo-coherence, resp. perfectness on the Zariski site and étale site agree, see Lemmas 56.12.2 and 56.12.4. Thus we can use the results of Derived Categories of Schemes, Section 35.11 for the open immersion $U \times_X V \subset V$. In this way Derived Categories of Schemes, Lemma 35.11.8 implies there exists a perfect object Q in $D(\mathcal{O}_V)$ supported on $j^{-1}(T)$ and an isomorphism $Q|_{U\times_X V} \to (P \oplus P[1])|_{U\times_X V}$. By Derived Categories of Schemes, Lemma 35.11.5 we can replace Q by $Q \otimes^{\mathbf{L}} I$ and assume that the map

$$Q|_{U\times_X V} \longrightarrow (P \oplus P[1])|_{U\times_X V} \longrightarrow P|_{U\times_X V} \longrightarrow E|_{U\times_X V}$$

lifts to $Q \to E|_V$. By Lemma 56.9.7 we find an morphism $a : R \to E$ of $D(\mathcal{O}_X)$ such that $a|_U$ is isomorphic to $P \oplus P[1] \to E|_U$ and $a|_V$ isomorphic to $Q \to E|_V$. Thus R is perfect and supported on T and the map $H^t(R) \to H^t(E)$ is surjective on restriction to U. Choose a distinguised triangle

$$R \to E \to E' \to R[1]$$

Then E' is $(m-r)$-pseudo-coherent (Cohomology on Sites, Lemma 21.33.4), $H^i(E')|_U = 0$ for $i \geq t$, and $H^i(E')$ is supported on T for $i \geq m - r$. By induction we find an approximation $R' \to E'$ of (T, E', m). Fit the composition $R' \to E' \to R[1]$ into a distringuished triangle $R \to R'' \to R' \to R[1]$ and extend the morphisms $R' \to E'$ and $R[1] \to R[1]$ into a morphism of distinguished triangles

$$\begin{array}{ccccccc} R & \longrightarrow & R'' & \longrightarrow & R' & \longrightarrow & R[1] \\ \downarrow & & \downarrow & & \downarrow & & \downarrow \\ R & \longrightarrow & E & \longrightarrow & E' & \longrightarrow & R[1] \end{array}$$

using TR3. Then R'' is a perfect complex (Cohomology on Sites, Lemma 21.35.6) supported on T. An easy diagram chase shows that $R'' \to E$ is the desired approximation. □

Theorem 56.13.7. *Let S be a scheme. Let X be a quasi-compact and quasi-separated algebraic space over S. Then approximation by perfect complexes holds on X.*

Proof. This follows from the induction principle of Lemma 56.8.3 and Lemmas 56.13.6 and 56.13.5. □

56.14. Generating derived categories

This section is the analogue of Derived Categories of Schemes, Section 35.13. However, we first prove the following lemma which is the analogue of Derived Categories of Schemes, Lemma 35.11.9.

Lemma 56.14.1. *Let S be a scheme. Let X be a quasi-compact and quasi-separated algebraic space over S. Let $W \subset X$ be a quasi-compact open. Let $T \subset |X|$ be a closed subset such that $X \setminus T \to X$ is a quasi-compact morphism. Let E be an*

object of $D_{QCoh}(\mathcal{O}_X)$. Let $\alpha : P \to E|_W$ be a map where P is a perfect object of $D(\mathcal{O}_W)$ supported on $T \cap W$. Then there exists a map $\beta : R \to E$ where R is a perfect object of $D(\mathcal{O}_X)$ supported on T such that P is a direct summand of $R|_W$ in $D(\mathcal{O}_W)$ compatible α and $\beta|_W$.

Proof. We will use the induction principle of Lemma 56.8.6 to prove this. Thus we immediately reduce to the case where we have an elementary distinguished square $(W \subset X, f : V \to X)$ with V affine and $P \to E|_W$ as in the statement of the lemma. In the rest of the proof we will use Lemma 56.4.2 (and the compatibilities of Remark 56.6.3) for the representable algebraic spaces V and $W \times_X V$. We will also use the fact that perfectness on the Zariski site and étale site agree, see Lemma 56.12.4.

By Derived Categories of Schemes, Lemma 35.11.8 we can choose a perfect object Q in $D(\mathcal{O}_V)$ supported on $f^{-1}T$ and an isomorphism $Q|_{W\times_X V} \to (P \oplus P[1])|_{W\times_X V}$. By Derived Categories of Schemes, Lemma 35.11.5 we can replace Q by $Q \otimes^{\mathbf{L}} I$ (still supported on $f^{-1}T$) and assume that the map

$$Q|_{W\times_X V} \to (P \oplus P[1])|_{W\times V} \longrightarrow P|_{W\times_X V} \longrightarrow E|_{W\times_X V}$$

lifts to $Q \to E|_V$. By Lemma 56.9.7 we find an morphism $a : R \to E$ of $D(\mathcal{O}_X)$ such that $a|_W$ is isomorphic to $P \oplus P[1] \to E|_W$ and $a|_V$ isomorphic to $Q \to E|_V$. Thus R is perfect and supported on T as desired. □

Remark 56.14.2. The proof of Lemma 56.14.1 shows that

$$R|_W = P \oplus P^{\oplus n_1}[1] \oplus \ldots \oplus P^{\oplus n_m}[m]$$

for some $m \geq 0$ and $n_j \geq 0$. Thus the highest degree cohomology sheaf of $R|_W$ equals that of P. By repeating the construction for the map $P^{\oplus n_1}[1] \oplus \ldots \oplus P^{\oplus n_m}[m] \to R|_W$, taking cones, and using induction we can achieve equality of cohomology sheaves of $R|_W$ and P above any given degree.

Lemma 56.14.3. *Let S be a scheme. Let X be a quasi-compact and quasi-separated algebraic space over S. Let W be a quasi-compact open subspace of X. Let P be a perfect object of $D(\mathcal{O}_W)$. Then P is a direct summand of the restriction of a perfect object of $D(\mathcal{O}_X)$.*

Proof. Special case of Lemma 56.14.1. □

Theorem 56.14.4. *Let S be a scheme. Let X be a quasi-compact and quasi-separated algebraic space over S. The category $D_{QCoh}(\mathcal{O}_X)$ can be generated by a single perfect object. More precisely, there exists a perfect object P of $D(\mathcal{O}_X)$ such that for $E \in D_{QCoh}(\mathcal{O}_X)$ the following are equivalent*

1. *$E = 0$, and*
2. *$\mathrm{Hom}_{D(\mathcal{O}_X)}(P[n], E) = 0$ for all $n \in \mathbf{Z}$.*

Proof. We will prove this using the induction principle of Lemma 56.8.3

If X is affine, then $\mathcal{O}_X$ is a perfect generator. This follows from Lemma 56.4.2 and Derived Categories of Schemes, Lemma 35.3.4.

Assume that $(U \subset X, f : V \to X)$ is an elementary distinguished square with U quasi-compact such that the theorem holds for U and V is an affine scheme. Let P be a perfect object of $D(\mathcal{O}_U)$ which is a generator for $D_{QCoh}(\mathcal{O}_U)$. Using Lemma 56.14.3 we may choose a perfect object Q of $D(\mathcal{O}_X)$ whose restriction to

U is a direct sum one of whose summands is P. Say $V = \mathrm{Spec}(A)$. Let $Z \subset V$ be the reduced closed subscheme which is the inverse image of $X \setminus U$ and maps isomorphically to it (see Definition 56.8.1). This is a retrocompact closed subset of V. Choose $f_1, \ldots, f_r \in A$ such that $Z = V(f_1, \ldots, f_r)$. Let $K \in D(\mathcal{O}_V)$ be the perfect object corresponding to the Koszul complex on $f_1, \ldots, f_r$ over A. Note that since K is supported on Z, the pushforward $K' = Rf_*K$ is a perfect object of $D(\mathcal{O}_X)$ whose restriction to V is K (see Lemmas 56.13.3 and 56.9.6). We claim that $Q \oplus K'$ is a generator for $D_{QCoh}(\mathcal{O}_X)$.

Namely, by Lemma 56.9.6 we have $K' = f_!K$ and hence

$$\mathrm{Hom}_{D(\mathcal{O}_X)}(K'[n], E) = \mathrm{Hom}_{D(\mathcal{O}_V)}(K[n], E|_V)$$

Thus by Derived Categories of Schemes, Lemma 35.13.2 (using also Lemma 56.4.2) the vanishing of these groups implies that $E|_V$ is isomorphic to $R(U \times_X V \to V)_* E|_{U \times_X V}$. This implies that $E = R(U \to X)_* E|_U$ (small detail omitted). If this is the case then

$$\mathrm{Hom}_{D(\mathcal{O}_X)}(Q[n], E) = \mathrm{Hom}_{D(\mathcal{O}_U)}(Q|_U[n], E|_U)$$

which contains $\mathrm{Hom}_{D(\mathcal{O}_U)}(P[n], E|_U)$ as a direct summand. Thus by our choice of P the vanishing of these groups implies that $E|_U$ is zero. Whence E is zero. □

56.15. Compact and perfect objects

This section is the analogue of Derived Categories of Schemes, Section 35.14.

Proposition 56.15.1. *Let S be a scheme. Let X be a quasi-compact and quasi-separated algebraic space over S. An object of $D_{QCoh}(\mathcal{O}_X)$ is compact if and only if it is perfect.*

Proof. By Cohomology on Sites, Lemma 21.38.1 the perfect objects even define compact objects of $D(\mathcal{O}_X)$. Conversely, let K be a compact object of $D_{QCoh}(\mathcal{O}_X)$. To show that K is perfect, it suffices to show that $K|_U$ is perfect for every affine scheme U étale over X, see Cohomology on Sites, Lemma 21.35.2. Observe that $j : U \to X$ is a quasi-compact and separated morphism. Hence $Rj_* : D_{QCoh}(\mathcal{O}_U) \to D_{QCoh}(\mathcal{O}_X)$ commutes with direct sums, see Lemma 56.6.2. Thus the adjointness of restriction to U and Rj_* implies that $K|_U$ is a perfect object of $D_{QCoh}(\mathcal{O}_U)$. Hence we reduce to the case that X is affine, in particular a quasi-compact and quasi-separated scheme. Via Lemma 56.4.2 and 56.12.4 we reduce to the case of schemes, i.e., to Derived Categories of Schemes, Proposition 35.14.1. □

56.16. Derived categories as module categories

The section is the analogue of Derived Categories of Schemes, Section 35.15.

Lemma 56.16.1. *Let S be a scheme. Let X be an algebraic space over S. Let $K^\bullet$ be a complex of $\mathcal{O}_X$-modules whose cohomology sheaves are quasi-coherent. Let $(E, d) = \mathrm{Hom}_{Comp^{dg}(\mathcal{O}_X)}(K^\bullet, K^\bullet)$ be the endomorphism differential graded algebra. Then the functor*

$$- \otimes^{\mathbf{L}}_E K^\bullet : D(E, d) \longrightarrow D(\mathcal{O}_X)$$

of Differential Graded Algebra, Lemma 22.25.3 has image contained in $D_{QCoh}(\mathcal{O}_X)$.

Proof. Let P be a differential graded E-module with property P. Let $F_\bullet$ be a filtration on P as in Differential Graded Algebra, Section 22.13. Then we have

$$P \otimes_E K^\bullet = \text{hocolim } F_i P \otimes_E K^\bullet$$

Each of the $F_i P$ has a finite filtration whose graded pieces are direct sums of $E[k]$. The result follows easily. □

The following lemma can be strengthened (there is a uniformity in the vanishing over all L with nonzero cohomology sheaves only in a fixed range).

Lemma 56.16.2. *Let S be a scheme. Let X be a quasi-compact and quasi-separated algebraic space over S. Let K, L be objects of $D(\mathcal{O}_X)$ with K perfect and L in $D^b_{QCoh}(\mathcal{O}_X)$. Then $Ext^n_{D(\mathcal{O}_X)}(K, L)$ is nonzero for only a finite number of n.*

Proof. Since K is perfect we have

$$\text{Ext}^i_{D(\mathcal{O}_X)}(K, L) = H^i(X, K^\wedge \otimes^{\mathbf{L}}_{\mathcal{O}_X} L)$$

where $K^\wedge$ is the "dual" perfect complex to K, see Cohomology on Sites, Lemma 21.35.9. Note that $P = K^\wedge \otimes^{\mathbf{L}}_{\mathcal{O}_X} L$ is in $D_{QCoh}(X)$ by Lemmas 56.5.5 and 56.12.5 (to see that a perfect complex has quasi-coherent cohomology sheaves). On the other hand, the spectral sequence

$$E_1^{p,q} = H^p(K^\wedge \otimes^{\mathbf{L}}_{\mathcal{O}_X} H^q(L)) \Rightarrow H^{p+q}(K^\wedge \otimes^{\mathbf{L}}_{\mathcal{O}_X} L) = H^{p+q}(P),$$

the boundedness of L, and the finite tor amplitude of $K^\wedge$ show that P has only finitely many nonzero cohomology sheaves. It follows that $H^n(X, P) = 0$ for $n \ll 0$. But also $H^n(X, P) = 0$ for $n \gg 0$ by Cohomology of Spaces, Lemma 51.6.3 and the spectral sequence expressing $H^n(X, P^\bullet)$ in terms of $H^p(X, H^q(P^\bullet))$ using that the cohomology sheaves of P are quasi-coherent. □

The following is the analogue of Derived Categories of Schemes, Theorem 35.15.3.

Theorem 56.16.3. *Let S be a scheme. Let X be a quasi-compact and quasi-separated algebraic space over S. Then there exist a differential graded algebra (E, d) with only a finite number of nonzero cohomology groups $H^i(E)$ such that $D_{QCoh}(\mathcal{O}_X)$ is equivalent to $D(E, d)$.*

Proof. Let $K^\bullet$ be a K-injective complex of $\mathcal{O}$-modules which is perfect and generates $D_{QCoh}(\mathcal{O}_X)$. Such a thing exists by Theorem 56.14.4 and the existence of K-injective resolutions. We will show the theorem holds with

$$(E, \mathrm{d}) = \text{Hom}_{\text{Comp}^{dg}(\mathcal{O}_X)}(K^\bullet, K^\bullet)$$

where $\text{Comp}^{dg}(\mathcal{O}_X)$ is the differential graded category of complexes of $\mathcal{O}$-modules. Please see Differential Graded Algebra, Section 22.25. Since $K^\bullet$ is K-injective we have

$$H^n(E) = \text{Ext}^n_{D(\mathcal{O}_X)}(K^\bullet, K^\bullet) \tag{56.16.3.1}$$

for all $n \in \mathbf{Z}$. Only a finite number of these Exts are nonzero by Lemma 56.16.2. Consider the functor

$$- \otimes^{\mathbf{L}}_E K^\bullet : D(E, \mathrm{d}) \longrightarrow D(\mathcal{O}_X)$$

of Differential Graded Algebra, Lemma 22.25.3. Since $K^\bullet$ is perfect, it defines a compact object of $D(\mathcal{O}_X)$, see Proposition 56.15.1. Combined with (56.16.3.1) the

functor above is fully faithful as follows from Differential Graded Algebra, Lemmas 22.25.5. It has a right adjoint

$$R\mathrm{Hom}(K^\bullet, -) : D(\mathcal{O}_X) \longrightarrow D(E, \mathrm{d})$$

by Differential Graded Algebra, Lemmas 22.25.4 which is a left quasi-inverse functor by generalities on adjoint functors. On the other hand, it follows from Lemma 56.16.1 that we obtain

$$- \otimes^{\mathbf{L}}_E K^\bullet : D(E, \mathrm{d}) \longrightarrow D_{QCoh}(\mathcal{O}_X)$$

and by our choice of $K^\bullet$ as a generator of $D_{QCoh}(\mathcal{O}_X)$ the kernel of the adjoint restricted to $D_{QCoh}(\mathcal{O}_X)$ is zero. A formal argument shows that we obtain the desired equivalence, see Derived Categories, Lemma 13.6.12. □

56.17. Cohomology and base change, IV

This section is the analogue of Derived Categories of Schemes, Section 35.16.

Lemma 56.17.1. *Let S be a scheme. Let $f : X \to Y$ be a quasi-compact and quasi-separated morphism of algebraic spaces over S. For E in $D_{QCoh}(\mathcal{O}_X)$ and K in $D_{QCoh}(\mathcal{O}_Y)$ we have*

$$Rf_*(E) \otimes^{\mathbf{L}}_{\mathcal{O}_Y} K = Rf_*(E \otimes^{\mathbf{L}}_{\mathcal{O}_X} Lf^*K)$$

Proof. Without any assumptions there is a map $Rf_*(E) \otimes^{\mathbf{L}}_{\mathcal{O}_Y} K \to Rf_*(E \otimes^{\mathbf{L}}_{\mathcal{O}_X} Lf^*K)$. Namely, it is the adjoint to the canonical map

$$Lf^*(Rf_*(E) \otimes^{\mathbf{L}}_{\mathcal{O}_Y} K) = Lf^*(Rf_*(E)) \otimes^{\mathbf{L}}_{\mathcal{O}_X} Lf^*K \longrightarrow E \otimes^{\mathbf{L}}_{\mathcal{O}_X} Lf^*K$$

coming from the map $Lf^*Rf_*E \to E$. See Cohomology on Sites, Lemmas 21.18.4 and 21.19.1. To check it is an isomorphism we may work étale locally on Y. Hence we reduce to the case that Y is an affine scheme.

Suppose that $K = \bigoplus K_i$ is a direct sum of some complexes $K_i \in D_{QCoh}(\mathcal{O}_Y)$. If the statement holds for each K_i, then it holds for K. Namely, the functors Lf^* and $\otimes^{\mathbf{L}}$ preserve direct sums by construction and Rf_* commutes with direct sums (for complexes with quasi-coherent cohomology sheaves) by Lemma 56.6.2. Moreover, suppose that $K \to L \to M \to K[1]$ is a distinguished triangle in $D_{QCoh}(Y)$. Then if the statement of the lemma holds for two of K, L, M, then it holds for the third (as the functors involved are exact functors of triangulated categories).

Assume Y affine, say $Y = \mathrm{Spec}(A)$. The functor $\widetilde{\ } : D(A) \to D_{QCoh}(\mathcal{O}_Y)$ is an equivalence by Lemma 56.4.2 and Derived Categories of Schemes, Lemma 35.3.4. Let T be the property for $K \in D(A)$ that the statement of the lemma holds for $\widetilde{K}$. The discussion above and More on Algebra, Remark 15.43.11 shows that it suffices to prove T holds for $A[k]$. This finishes the proof, as the statement of the lemma is clear for shifts of the structure sheaf. □

Definition 56.17.2. Let S be a scheme. Let B be an algebraic space over S. Let X, Y be algebraic spaces over B. We say X and Y are *Tor independent over B* if and only if for every commutative diagram

$$\begin{array}{ccc} \mathrm{Spec}(k) & \xrightarrow{\overline{x}} & X \\ {\scriptstyle \overline{y}}\downarrow & \swarrow{\scriptstyle \overline{b}} & \downarrow \\ Y & \longrightarrow & B \end{array}$$

of geometric points the rings $\mathcal{O}_{X,\overline{x}}$ and $\mathcal{O}_{Y,\overline{y}}$ are Tor independent over $\mathcal{O}_{B,\overline{b}}$ (see More on Algebra, Definition 15.45.1).

The following lemma shows in particular that this definition agrees with our definition in the case of representable algebraic spaces.

Lemma 56.17.3. *Let S be a scheme. Let B be an algebraic space over S. Let X, Y be algebraic spaces over B. The following are equivalent*

(1) *X and Y are Tor independent over B,*
(2) *for every commutative diagram*

$$\begin{array}{ccccc} U & \longrightarrow & W & \longleftarrow & V \\ \downarrow & & \downarrow & & \downarrow \\ X & \longrightarrow & B & \longleftarrow & Y \end{array}$$

with étale vertical arrows U and V are Tor independent over W,
(3) *for some commutative diagram as in (2) with (a) $W \to B$ étale surjective, (b) $U \to X \times_B W$ étale surjective, (c) $V \to Y \times_B W$ étale surjective, the spaces U and V are Tor independent over W, and*
(4) *for some commutative diagram as in (3) with U, V, W schemes, the schemes U and V are Tor independent over W in the sense of Derived Categories of Schemes, Definition 35.16.2.*

Proof. For an étale morphism $\varphi : U \to X$ of algebraic spaces and geometric point $\overline{u}$ the map of local rings $\mathcal{O}_{X,\varphi(\overline{u})} \to \mathcal{O}_{U,\overline{u}}$ is an isomorphism. Hence the equivalence of (1) and (2) follows. So does the implication (1) $\Rightarrow$ (3). Assume (3) and pick a diagram of geometric points as in Definition 56.17.2. The assumptions imply that we can first lift $\overline{b}$ to a geometric point $\overline{w}$ of W, then lift the geometric point $(\overline{x}, \overline{b})$ to a geometric point $\overline{u}$ of U, and finally lift the geometric point $(\overline{y}, \overline{b})$ to a geometric point $\overline{v}$ of V. Use Properties of Spaces, Lemma 48.16.4 to find the lifts. Using the remark on local rings above we conclude that the condition of the definition is satisfied for the given diagram.

Having made these initial points, it is clear that (4) comes down to the statement that Definition 56.17.2 agrees with Derived Categories of Schemes, Definition 35.16.2 when X, Y, and B are schemes.

Let $\overline{x}, \overline{b}, \overline{y}$ be as in Definition 56.17.2 lying over the points x, y, b. Recall that $\mathcal{O}_{X,\overline{x}} = \mathcal{O}^{sh}_{X,x}$ (Properties of Spaces, Lemma 48.19.1) and similarly for the other two. By Algebra, Lemma 10.143.28 we see that $\mathcal{O}_{X,\overline{x}}$ is a strict henselization of $\mathcal{O}_{X,x} \otimes_{\mathcal{O}_{B,b}} \mathcal{O}_{B,\overline{b}}$. In particular, the ring map

$$\mathcal{O}_{X,x} \otimes_{\mathcal{O}_{B,b}} \mathcal{O}_{B,\overline{b}} \longrightarrow \mathcal{O}_{X,\overline{x}}$$

is flat (More on Algebra, Lemma 15.32.1). By More on Algebra, Lemma 15.45.3 we see that

$$\mathrm{Tor}_i^{\mathcal{O}_{B,b}}(\mathcal{O}_{X,x}, \mathcal{O}_{Y,y}) \otimes_{\mathcal{O}_{X,x} \otimes_{\mathcal{O}_{B,b}} \mathcal{O}_{Y,y}} (\mathcal{O}_{X,\overline{x}} \otimes_{\mathcal{O}_{B,\overline{b}}} \mathcal{O}_{Y,\overline{y}}) = \mathrm{Tor}_i^{\mathcal{O}_{B,\overline{b}}}(\mathcal{O}_{X,\overline{x}}, \mathcal{O}_{Y,\overline{y}})$$

Hence it follows that if X and Y are Tor independent over B as schemes, then X and Y are Tor independent as algebraic spaces over B.

For the converse, we may assume X, Y, and B are affine. Observe that the ring map

$$\mathcal{O}_{X,x} \otimes_{\mathcal{O}_{B,b}} \mathcal{O}_{Y,y} \longrightarrow \mathcal{O}_{X,\overline{x}} \otimes_{\mathcal{O}_{B,\overline{b}}} \mathcal{O}_{Y,\overline{y}}$$

is flat by the observations given above. Moreover, the image of the map on spectra includes all primes $\mathfrak{s} \subset \mathcal{O}_{X,x} \otimes_{\mathcal{O}_{B,b}} \mathcal{O}_{Y,y}$ lying over $\mathfrak{m}_x$ and $\mathfrak{m}_y$. Hence from this and the displayed formula of Tor's above we see that if X and Y are Tor independent over B as algebraic spaces, then

$$\mathrm{Tor}_i^{\mathcal{O}_{B,b}}(\mathcal{O}_{X,x}, \mathcal{O}_{Y,y})_{\mathfrak{s}} = 0$$

for all $i > 0$ and all $\mathfrak{s}$ as above. By More on Algebra, Lemma 15.45.4 applied to the ring maps $\Gamma(B, \mathcal{O}_B) \to \Gamma(X, \mathcal{O}_X)$ and $\Gamma(B, \mathcal{O}_B) \to \Gamma(X, \mathcal{O}_X)$ this implies that X and Y are Tor independent over B. □

Lemma 56.17.4. *Let S be a scheme. Let $g : Y' \to Y$ be a morphism of algebraic spaces over S. Let $f : X \to Y$ be a quasi-compact and quasi-separated morphism of algebraic spaces over S. Consider the base change diagram*

$$\begin{array}{ccc} X' & \xrightarrow{h} & X \\ {\scriptstyle f'}\downarrow & & \downarrow{\scriptstyle f} \\ Y' & \xrightarrow{g} & Y \end{array}$$

If X and Y' are Tor independent over Y, then for all $E \in D_{QCoh}(\mathcal{O}_X)$ we have $Rf'_*Lh^*E = Lg^*Rf_*E$.

Proof. For any object E of $D(\mathcal{O}_X)$ we can use Cohomology on Sites, Remark 21.19.2 to get a canonical base change map $Lg^*Rf_*E \to Rf'_*Lh^*E$. To check this is an isomorphism we may work étale locally on Y'. Hence we may assume $g : Y' \to Y$ is a morphism of affine schemes. In particular, g is affine and it suffices to show that

$$Rg_*Lg^*Rf_*E \to Rg_*Rf'_*Lh^*E = Rf_*(Rh_*Lh^*E)$$

is an isomorphism, see Lemma 56.6.4 (and use Lemmas 56.5.4, 56.5.5, and 56.6.1 to see that the objects Rf'_*Lh^*E and Lg^*Rf_*E have quasi-coherent cohomology sheaves). Note that h is affine as well (Morphisms of Spaces, Lemma 49.19.5). By Lemma 56.6.5 the map becomes a map

$$Rf_*E \otimes^{\mathbf{L}}_{\mathcal{O}_Y} g_*\mathcal{O}_{Y'} \longrightarrow Rf_*(E \otimes^{\mathbf{L}}_{\mathcal{O}_X} h_*\mathcal{O}_{X'})$$

Observe that $h_*\mathcal{O}_{X'} = f^*g_*\mathcal{O}_{Y'}$. Thus by Lemma 56.17.1 it suffices to prove that $Lf^*g_*\mathcal{O}_{Y'} = f^*g_*\mathcal{O}_{Y'}$. This follows from our assumption that X and Y' are Tor independent over Y. Namely, to check it we may work étale locally on X, hence we may also assume X is affine. Say $X = \mathrm{Spec}(A)$, $Y = \mathrm{Spec}(R)$ and $Y' = \mathrm{Spec}(R')$. Our assumption implies that A and R' are Tor independent over R (see Lemma 56.17.3 and More on Algebra, Lemma 15.45.4), i.e., $\mathrm{Tor}_i^R(A, R') = 0$ for $i > 0$. In other words $A \otimes^{\mathbf{L}}_R R' = A \otimes_R R'$ which exactly means that $Lf^*g_*\mathcal{O}_{Y'} = f^*g_*\mathcal{O}_{Y'}$. □

Lemma 56.17.5. *Let S be a scheme. Let Y be a Noetherian algebraic space over S. Let $f : X \to Y$ be a morphism of algebraic spaces which is locally of finite type and quasi-separated. Let $E \in D(\mathcal{O}_X)$ such that*

1. *$E \in D^b_{Coh}(\mathcal{O}_X)$,*
2. *the scheme theoretic support of $H^i(E)$ is proper over Y for all i,*

(3) *E has finite tor dimension as an object of $D(f^{-1}\mathcal{O}_Y)$.*

Then Rf_*E is a perfect object of $D(\mathcal{O}_Y)$.

Proof. By Lemma 56.7.1 we see that Rf_*E is an object of $D^b_{Coh}(\mathcal{O}_Y)$. Hence Rf_*E is pseudo-coherent (Lemma 56.12.6). Hence it suffices to show that Rf_*E has finite tor dimension, see Cohomology on Sites, Lemma 21.35.4. By Lemma 56.12.7 it suffices to check that $Rf_*(E) \otimes^{\mathbf{L}}_{\mathcal{O}_Y} \mathcal{F}$ has universally bounded cohomology for all quasi-coherent sheaves $\mathcal{F}$ on Y. Bounded from above is clear as $Rf_*(E)$ is bounded from above. Let $T \subset X$ be the union of the supports of $H^i(E)$ for all i. Then T is proper over Y by assumptions (1) and (2). In particular there exists a quasi-compact open subspace $X' \subset X$ containing T. Setting $f' = f|_{X'}$ we have $Rf_*(E) = Rf'_*(E|_{X'})$ because E restricts to zero on $X \setminus T$. Thus we may replace X by X' and assume f is quasi-compact. We have assumed f is quasi-separated. Thus

$$Rf_*(E) \otimes^{\mathbf{L}}_{\mathcal{O}_Y} \mathcal{F} = Rf_* \left(E \otimes^{\mathbf{L}}_{\mathcal{O}_X} Lf^*\mathcal{F}\right) = Rf_* \left(E \otimes^{\mathbf{L}}_{f^{-1}\mathcal{O}_Y} f^{-1}\mathcal{F}\right)$$

by Lemma 56.17.1 and Cohomology on Sites, Lemma 21.18.5. By assumption (3) the complex $E \otimes^{\mathbf{L}}_{f^{-1}\mathcal{O}_Y} f^{-1}\mathcal{F}$ has cohomology sheaves in a given finite range, say $[a, b]$. Then Rf_* of it has cohomology in the range $[a, \infty)$ and we win. □

56.18. Computing Ext groups and base change

The results in this section will be used to verify one of Artin's criteria for Quot functors, Hilbert schemes, and other moduli problems.

Lemma 56.18.1. *Let S be a scheme. Let B be a Noetherian algebraic space over S. Let $f : X \to B$ be a morphism of algebraic spaces which is locally of finite type and quasi-separated. Let $E \in D(\mathcal{O}_X)$ and $\mathcal{G}$ an $\mathcal{O}_X$-module. Assume*

(1) *E is perfect, and*
(2) *$\mathcal{G}$ is a coherent $\mathcal{O}_X$-module flat over B with scheme theoretic support proper over B.*

Then

$$K = Rf_*R\mathcal{H}om(E, \mathcal{G})$$

is a perfect object of $D(\mathcal{O}_B)$ and there are functorial isomorphisms

$$H^i(B, K \otimes^{\mathbf{L}}_{\mathcal{O}_B} \mathcal{F}) \longrightarrow Ext^i_{\mathcal{O}_X}(E, \mathcal{G} \otimes_{\mathcal{O}_X} f^*\mathcal{F})$$

for $\mathcal{F}$ quasi-coherent on B compatible with boundary maps (see proof).

Proof. Since E is a perfect complex there exists a dual perfect complex E', see Cohomology on Sites, Lemma 21.35.9. Observe that

$$\mathrm{Ext}^i_{\mathcal{O}_X}(E, \mathcal{G} \otimes_{\mathcal{O}_X} f^*\mathcal{F}) = H^i(X, E' \otimes^{\mathbf{L}}_{\mathcal{O}_X} (\mathcal{G} \otimes_{\mathcal{O}_X} f^*\mathcal{F}))$$

by construction of E'. We have

$$\mathcal{G} \otimes^{\mathbf{L}}_{\mathcal{O}_X} Lf^*\mathcal{F} = \mathcal{G} \otimes^{\mathbf{L}}_{f^{-1}\mathcal{O}_B} f^{-1}\mathcal{F} = \mathcal{G} \otimes_{f^{-1}\mathcal{O}_B} f^{-1}\mathcal{F} = \mathcal{G} \otimes_{\mathcal{O}_X} f^*\mathcal{F}$$

the first equality by Cohomology on Sites, Lemma 21.18.5, the second as $\mathcal{G}$ is a flat $f^{-1}\mathcal{O}_B$-module, and the third by definition of pullbacks. Hence we obtain

$$\begin{aligned} H^i(X, E' \otimes^{\mathbf{L}}_{\mathcal{O}_X} (\mathcal{G} \otimes_{\mathcal{O}_X} f^*\mathcal{F})) &= H^i(X, E' \otimes^{\mathbf{L}}_{\mathcal{O}_X} \mathcal{G} \otimes^{\mathbf{L}}_{\mathcal{O}_X} Lf^*\mathcal{F}) \\ &= H^i(B, Rf_*(E' \otimes^{\mathbf{L}}_{\mathcal{O}_X} \mathcal{G} \otimes^{\mathbf{L}}_{\mathcal{O}_X} Lf^*\mathcal{F})) \\ &= H^i(B, Rf_*(E' \otimes^{\mathbf{L}}_{\mathcal{O}_X} \mathcal{G}) \otimes^{\mathbf{L}}_{\mathcal{O}_B} \mathcal{F}) \end{aligned}$$

The first equality by the above, the second by Leray (Cohomology on Sites, Remark 21.14.4), and the third equality by Lemma 56.17.1. The object

$$K = Rf_*(E' \otimes^{\mathbf{L}}_{\mathcal{O}_X} \mathcal{G}) = Rf_* R\mathcal{H}om(E, \mathcal{G})$$

(second equality by Cohomology on Sites, Lemma 21.35.9) is perfect by Lemma 56.17.5. This lemma applies. Namely, the hypotheses on the cohomology sheaves $H^i(E' \otimes^{\mathbf{L}}_{\mathcal{O}_X} \mathcal{G})$ are clear and the hypothesis on finite tor dimension follows as $\mathcal{G}$ is flat over $f^{-1}\mathcal{O}_B$ and locally E' is isomorphic to a finite complex of free $\mathcal{O}_X$-modules.

The statement on boundary maps means the following: Given a short exact sequence $0 \to \mathcal{F}_1 \to \mathcal{F}_2 \to \mathcal{F}_3 \to 0$ then the isomorphisms fit into commutative diagrams

$$\begin{array}{ccc} H^i(B, K \otimes^{\mathbf{L}}_{\mathcal{O}_B} \mathcal{F}_3) & \longrightarrow & \mathrm{Ext}^i_{\mathcal{O}_X}(E, \mathcal{G} \otimes_{\mathcal{O}_X} f^*\mathcal{F}_3) \\ \downarrow{\scriptstyle \delta} & & \downarrow{\scriptstyle \delta} \\ H^{i+1}(B, K \otimes^{\mathbf{L}}_{\mathcal{O}_B} \mathcal{F}_1) & \longrightarrow & \mathrm{Ext}^{i+1}_{\mathcal{O}_X}(E, \mathcal{G} \otimes_{\mathcal{O}_X} f^*\mathcal{F}_1) \end{array}$$

where the boundary maps come from the distinguished triangle

$$K \otimes^{\mathbf{L}}_{\mathcal{O}_B} \mathcal{F}_1 \to K \otimes^{\mathbf{L}}_{\mathcal{O}_B} \mathcal{F}_2 \to K \otimes^{\mathbf{L}}_{\mathcal{O}_B} \mathcal{F}_3 \to K \otimes^{\mathbf{L}}_{\mathcal{O}_B} \mathcal{F}_1[1]$$

and the distinguished triangle in $D(\mathcal{O}_X)$ associated to the short exact sequence

$$0 \to \mathcal{G} \otimes_{\mathcal{O}_X} f^*\mathcal{F}_1 \to \mathcal{G} \otimes_{\mathcal{O}_X} f^*\mathcal{F}_2 \to \mathcal{G} \otimes_{\mathcal{O}_X} f^*\mathcal{F}_3 \to 0$$

This sequence is exact because $\mathcal{G}$ is flat over B. We omit the verification of the commutativity of the displayed diagram. □

Lemma 56.18.2. *Let X be a scheme. Let E, F be objects of $D(\mathcal{O}_X)$. If E is pseudo-coherent and F lies in $D^+(\mathcal{O}_X)$, then there is a canonical isomorphism*

$$\epsilon^* R\mathcal{H}om(E, F) \longrightarrow R\mathcal{H}om(\epsilon^* E, \epsilon^* F)$$

Here ϵ is as in (56.4.0.1).

Proof. Recall that ϵ is flat (Lemma 56.4.1) and hence $\epsilon^* = L\epsilon^*$. There is a canonical map from left to right by Cohomology on Sites, Remark 21.25.6. To see this is an isomorphism we can work locally, i.e., we may assume X is an affine scheme. In this case we can represent E by a bounded above complex $\mathcal{E}^\bullet$ of finite free $\mathcal{O}_X$-modules, see Derived Categories of Schemes, Lemma 35.11.2. We may also represent F by a bounded below complex $\mathcal{F}^\bullet$ of $\mathcal{O}_X$-modules. Applying Cohomology, Lemma 20.32.10 we see that $R\mathcal{H}om(E, F)$ is represented by the complex with terms

$$\bigoplus\nolimits_{n=-p+q} \mathcal{H}om_{\mathcal{O}_X}(\mathcal{E}^p, \mathcal{F}^q)$$

Applying Cohomology on Sites, Lemma 21.32.10 we see that $R\mathcal{H}om(\epsilon^* E, \epsilon^* F)$ is represented by the complex with terms

$$\bigoplus\nolimits_{n=-p+q} \mathcal{H}om_{\mathcal{O}_{étale}}(\epsilon^*\mathcal{E}^p, \epsilon^*\mathcal{F}^q)$$

Thus the statement of the lemma boils down to the true fact that the canonical map

$$\epsilon^* \mathcal{H}om_{\mathcal{O}_X}(\mathcal{E}, \mathcal{F}) \longrightarrow \mathcal{H}om_{\mathcal{O}_{étale}}(\epsilon^*\mathcal{E}, \epsilon^*\mathcal{F})$$

is an isomorphism for any $\mathcal{O}_X$-module $\mathcal{F}$ and finite free $\mathcal{O}_X$-module $\mathcal{E}$. □

Lemma 56.18.3. *Let S be a scheme. Let $f : X \to Y$ be a quasi-compact and quasi-separated morphism of algebraic spaces over S. Let $E \in D(\mathcal{O}_X)$ be pseudo-coherent. Let $\mathcal{G}$ be a quasi-coherent $\mathcal{O}_X$-module flat over Y. Then formation of*

$$Rf_*R\mathcal{H}om(E, \mathcal{G})$$

commutes with arbitrary base change (see proof for precise statement).

Proof. The statement means the following. Let $g : Y' \to Y$ be a morphism of algebraic spaces and consider the base change diagram

$$\begin{array}{ccc} X' & \xrightarrow{h} & X \\ {\scriptstyle f'}\downarrow & & \downarrow{\scriptstyle f} \\ Y' & \xrightarrow{g} & Y \end{array}$$

in other words $X' = Y' \times_Y X$. Set $E' = Lh^*E$ and $\mathcal{G}' = h^*\mathcal{G}$ (here we do **not** use the derived pullback). The lemma asserts that we have

$$Lg^*Rf_*R\mathcal{H}om(E, \mathcal{G}) = Rf'_*R\mathcal{H}om(E', \mathcal{G}')$$

To prove this, note that in Cohomology on Sites, Remark 21.25.7 we have constructed an arrow

$$Lg^*Rf_*R\mathcal{H}om(E, \mathcal{G}) \longrightarrow R(f')_*R\mathcal{H}om(Lh^*E, Lh^*\mathcal{G})$$

which we can compose with the map $Lh^*\mathcal{G} \to h^*\mathcal{G}$ to get a canonical map

$$Lg^*Rf_*R\mathcal{H}om(E, \mathcal{G}) \to Rf'_*R\mathcal{H}om(E', \mathcal{G}')$$

To check this map is an isomorphism we may work étale locally on Y'. Hence we may assume $g : Y' \to Y$ is a morphism of affine schemes. In this case, we will use the induction principle to prove this map is always an isomorphism for any quasi-compact and quasi-separated algebraic space X over Y (Lemma 56.8.3).

If X is a scheme (for example affine), then the result holds. Namely, $E \in D_{QCoh}(\mathcal{O}_X)$ by Lemma 56.12.5. Hence E comes from an object of the derived category of the underlying scheme by Lemma 56.4.2. This complex is pseudo-coherent by Lemma 56.12.2. Furthermore, the constructions $R\mathcal{H}om$ (internal hom), Rf_* (derived push-forward), and Lg^* (derived pullback) are (in the current situation) compatible with pulling back from the Zariski site (Lemma 56.18.2 and Remark 56.6.3). Thus in this case the result follows from the case of schemes which is Derived Categories of Schemes, Lemma 35.17.2.

The induction step. Let $(U \subset X, f : V \to X)$ be an elementary distinguished square with U, V, $U \times_X V$ quasi-compact such that the result holds for the restriction of E and $\mathcal{G}$ to U, V, and $U \times_X V$. Denote $a = f|_U$, $b = f|_V$ and $c = f|_{U \times_X V}$. Let $a' : U' \to Y'$, $b' : V' \to Y'$ and $c' : U' \times_{X'} V' \to Y'$ be the base changes of a, b, and c. Note that formation of $R\mathcal{H}om$ commutes with restriction (Cohomology on Sites, Lemma 21.25.3). Set $H = R\mathcal{H}om(E, \mathcal{G})$ and $H' = R\mathcal{H}om(E', \mathcal{G}')$. Using the distinguished triangles from relative Mayer-Vietoris (Lemma 56.9.3) we obtain

a commutative diagram

$$\begin{array}{ccc}
Lg^*Rf_*H & \longrightarrow & Rf'_*H' \\
\downarrow & & \downarrow \\
Lg^*Ra_*H|_U \oplus Lg^*Rb_*H|_V & \longrightarrow & Ra'_*H'|_{U'} \oplus Rb'_*H'|_{V'} \\
\downarrow & & \downarrow \\
Lg^*Rc_*H|_{U\times_X V} & \longrightarrow & Rc'_*H'|_{U'\times_{X'} V'} \\
\downarrow & & \downarrow \\
Lg^*Rf_*H[1] & \longrightarrow & Rf'_*H'[1]
\end{array}$$

Since the 2nd and 3rd horizontal arrows are isomorphisms so is the first (Derived Categories, Lemma 13.4.3) and the proof of the lemma is finished. □

Lemma 56.18.4. *Let S be a scheme. Let B be a Noetherian algebraic space over S. Let $f : X \to B$ be a morphism of algebraic spaces which is locally of finite type and quasi-separated. Let $E \in D(\mathcal{O}_X)$ and $\mathcal{G}$ an $\mathcal{O}_X$-module. Assume*

(1) *$E \in D^-_{Coh}(\mathcal{O}_X)$, and*
(2) *$\mathcal{G}$ is a coherent $\mathcal{O}_X$-module flat over B with scheme theoretic support proper over B.*

Then for every $m \in \mathbf{Z}$ there exists a perfect object K of $D(\mathcal{O}_B)$ and functorial maps

$$\alpha^i_{\mathcal{F}} : Ext^i_{\mathcal{O}_X}(E, \mathcal{G} \otimes_{\mathcal{O}_X} f^*\mathcal{F}) \longrightarrow H^i(B, K \otimes^{\mathbf{L}}_{\mathcal{O}_B} \mathcal{F})$$

for $\mathcal{F}$ quasi-coherent on B compatible with boundary maps (see proof) such that $\alpha^i_{\mathcal{F}}$ is an isomorphism for $i \leq m$.

Proof. We may replace X by a quasi-compact open neighbourhood of the support of $\mathcal{G}$, hence we may assume X is Noetherian. In this case X and f are quasi-compact and quasi-separated. Choose an approximation $P \to E$ by a perfect complex P of $(X, E, -m-1)$ (possible by Theorem 56.13.7). Then the induced map

$$\mathrm{Ext}^i_{\mathcal{O}_X}(E, \mathcal{G} \otimes_{\mathcal{O}_X} f^*\mathcal{F}) \longrightarrow \mathrm{Ext}^i_{\mathcal{O}_X}(P, \mathcal{G} \otimes_{\mathcal{O}_X} f^*\mathcal{F})$$

is an isomorphism for $i \leq m$. Namely, the kernel, resp. cokernel of this map is a quotient, resp. submodule of

$$\mathrm{Ext}^i_{\mathcal{O}_X}(C, \mathcal{G} \otimes_{\mathcal{O}_X} f^*\mathcal{F}) \quad \text{resp.} \quad \mathrm{Ext}^{i+1}_{\mathcal{O}_X}(C, \mathcal{G} \otimes_{\mathcal{O}_X} f^*\mathcal{F})$$

where C is the cone of $P \to E$. Since C has vanishing cohomology sheaves in degrees $\geq -m-1$ these Ext-groups are zero for $i \leq m+1$ by Derived Categories, Lemma 13.26.3. This reduces us to the case that E is a perfect complex which is Lemma 56.18.1.

The statement on boundaries is explained in the proof of Lemma 56.18.1. □

56.19. Limits and derived categories

In this section we collect some results about the derived category of an algebraic space which is the limit of an inverse system of algebraic spaces. More precisely, we will work in the following setting.

Situation 56.19.1. Let S be a scheme. Let $X = \lim_{i \in I} X_i$ be a limit of a directed system of algebraic spaces over S with affine transition morphisms $f_{i'i} : X_{i'} \to X_i$. We denote $f_i : X \to X_i$ the projection. We assume that X_i is quasi-compact and quasi-separated for all $i \in I$. We also choose an element $0 \in I$.

Lemma 56.19.2. *In Situation 56.19.1. Let E_0 and K_0 be objects of $D(\mathcal{O}_{X_0})$. Set $E_i = Lf_{i0}^* E_0$ and $K_i = Lf_{i0}^* K_0$ for $i \geq 0$ and set $E = Lf_0^* E_0$ and $K = Lf_0^* K_0$. Then the map*

$$\operatorname{colim}_{i \geq 0} \operatorname{Hom}_{D(\mathcal{O}_{X_i})}(E_i, K_i) \longrightarrow \operatorname{Hom}_{D(\mathcal{O}_X)}(E, K)$$

is an isomorphism if either

(1) *E_0 is perfect and $K_0 \in D_{QCoh}(\mathcal{O}_{X_0})$, or*
(2) *E_0 is pseudo-coherent and $K_0 \in D_{QCoh}(\mathcal{O}_{X_0})$ has finite tor dimension.*

Proof. For every quasi-compact and quasi-separated object U_0 of $(X_0)_{spaces,étale}$ consider the condition P that the canonical map

$$\operatorname{colim}_{i \geq 0} \operatorname{Hom}_{D(\mathcal{O}_{U_i})}(E_i|_{U_i}, K_i|_{U_i}) \longrightarrow \operatorname{Hom}_{D(\mathcal{O}_U)}(E|_U, K|_U)$$

is an isomorphism, where $U = X \times_{X_0} U_0$ and $U_i = X_i \times_{X_0} U_0$. We will prove P holds for each U_0 by the induction principle of Lemma 56.8.3. Condition (2) of this lemma follows immediately from Mayer-Vietoris for hom in the derived category, see Lemma 56.9.4. Thus it suffices to prove the lemma when X_0 is affine.

If X_0 is affine, then the result follows from the case of schemes, see Derived Categories of Schemes, Lemma 35.18.2. To see this use the equivalence of Lemma 56.4.2 and use the translation of properties explained in Lemmas 56.12.2, 56.12.3, and 56.12.4. □

Lemma 56.19.3. *In Situation 56.19.1 the category of perfect objects of $D(\mathcal{O}_X)$ is the colimit of the categories of perfect objects of $D(\mathcal{O}_{X_i})$.*

Proof. For every quasi-compact and quasi-separated object U_0 of $(X_0)_{spaces,étale}$ consider the condition P that the functor

$$\operatorname{colim}_{i \geq 0} D_{perf}(\mathcal{O}_{U_i}) \longrightarrow D_{perf}(\mathcal{O}_U)$$

is an equivalence where $_{perf}$ indicates the full subcategory of perfect objects and where $U = X \times_{X_0} U_0$ and $U_i = X_i \times_{X_0} U_0$. We will prove P holds for every U_0 by the induction principle of Lemma 56.8.3. First, we observe that we already know the functor is fully faithful by Lemma 56.19.2. Thus it suffices to prove essential surjectivity.

We first check condition (2) of the induction principle. Thus suppose that we have an elementary distinguished square $(U_0 \subset X_0, V_0 \to X_0)$ and that P holds for U_0, V_0, and $U_0 \times_{X_0} V_0$. Let E be a perfect object of $D(\mathcal{O}_X)$. We can find $i \geq 0$ and $E_{U,i}$ perfect on U_i and $E_{V,i}$ perfect on V_i whose pullback to U and V are isomorphic to $E|_U$ and $E|_V$. Denote

$$a : E_{U,i} \to (R(X \to X_i)_* E)|_{U_i} \quad \text{and} \quad b : E_{V,i} \to (R(X \to X_i)_* E)|_{V_i}$$

the maps adjoint to the isomorphisms $L(U \to U_i)^* E_{U,i} \to E|_U$ and $L(V \to V_i)^* E_{V,i} \to E|_V$. By fully faithfulness, after increasing i, we can find an isomorphism $c : E_{U,i}|_{U_i \times_{X_i} V_i} \to E_{V,i}|_{U_i \times_{X_i} V_i}$ which pulls back to the identifications

$$L(U \to U_i)^* E_{U,i}|_{U \times_X V} \to E|_{U \times_X V} \to L(V \to V_i)^* E_{V,i}|_{U \times_X V}.$$

Apply Lemma 56.9.7 to get an object E_i on X_i and a map $d : E_i \to R(X \to X_i)_* E$ which restricts to the maps a and b over U_i and V_i. Then it is clear that E_i is perfect and that d is adjoint to an isomorphism $L(X \to X_i)^* E_i \to E$.

Finally, we check condition (1) of the induction principle, in other words, we check the lemma holds when X_0 is affine. This follows from the case of schemes, see Derived Categories of Schemes, Lemma 35.18.3. To see this use the equivalence of Lemma 56.4.2 and use the translation of Lemma 56.12.4. □

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CHAPTER 57

More on Morphisms of Spaces

57.1. Introduction

In this chapter we continue our study of properties of morphisms of algebraic spaces. A fundamental reference is [**Knu71**].

57.2. Conventions

The standing assumption is that all schemes are contained in a big fppf site Sch_{fppf}. And all rings A considered have the property that $\mathrm{Spec}(A)$ is (isomorphic) to an object of this big site.

Let S be a scheme and let X be an algebraic space over S. In this chapter and the following we will write $X \times_S X$ for the product of X with itself (in the category of algebraic spaces over S), instead of $X \times X$.

57.3. Radicial morphisms

It turns out that a radicial morphism is not the same thing as a universally injective morphism, contrary to what happens with morphisms of schemes. In fact it is a bit stronger.

Definition 57.3.1. Let S be a scheme. Let $f : X \to Y$ be a morphism of algebraic spaces over S. We say f is *radicial* if for any morphism $\mathrm{Spec}(K) \to Y$ where K is a field the reduction $(\mathrm{Spec}(K) \times_Y X)_{red}$ is either empty or representable by the spectrum of a purely inseparable field extension of K.

Lemma 57.3.2. *A radicial morphism of algebraic spaces is universally injective.*

Proof. Let S be a scheme. Let $f : X \to Y$ be a radicial morphism of algebraic spaces over S. It is clear from the definition that given a morphism $\mathrm{Spec}(K) \to Y$ there is at most one lift of this morphism to a morphism into X. Hence we conclude that f is universally injective by Morphisms of Spaces, Lemma 49.18.2. □

Example 57.3.3. It is no longer true that universally injective is equivalent to radicial. For example the morphism

$$X = [\mathrm{Spec}(\overline{\mathbf{Q}})/\mathrm{Gal}(\overline{\mathbf{Q}}/\mathbf{Q})] \longrightarrow S = \mathrm{Spec}(\mathbf{Q})$$

of Spaces, Example 47.14.7 is universally injective, but is not radicial in the sense above.

Nonetheless it is often the case that the reverse implication holds.

Lemma 57.3.4. *Let S be a scheme. Let $f : X \to Y$ be a universally injective morphism of algebraic spaces over S.*

(1) *If f is decent then f is radicial.*

(2) *If f is quasi-separated then f is radicial.*
(3) *If f is locally separated then f is radicial.*

Proof. Let $\mathcal{P}$ be a property of morphisms of algebraic spaces which is stable under base change and composition and holds for closed immersions. Assume $f : X \to Y$ has $\mathcal{P}$ and is universally injective. Then, in the situation of Definition 57.3.1 the morphism $(\mathrm{Spec}(K) \times_Y X)_{red} \to \mathrm{Spec}(K)$ is universally injective and has $\mathcal{P}$. This reduces the problem of proving

$$\mathcal{P} + \text{universally injective} \Rightarrow \text{radicial}$$

to the problem of proving that any nonempty reduced algebraic space X over field whose structure morphism $X \to \mathrm{Spec}(K)$ is universally injective and $\mathcal{P}$ is representable by the spectrum of a field. Namely, then $X \to \mathrm{Spec}(K)$ will be a morphism of schemes and we conclude by the equivalence of radicial and universally injective for morphisms of schemes, see Morphisms, Lemma 28.12.2.

Let us prove (1). Assume f is decent and universally injective. By Decent Spaces, Lemmas 50.14.3, 50.14.4, and 50.14.2 (to see that an immersion is decent) we see that the discussion in the first paragraph applies. Let X be a nonempty decent reduced algebraic space universally injective over a field K. In particular we see that $|X|$ is a singleton. By Decent Spaces, Lemma 50.11.1 we conclude that $X \cong \mathrm{Spec}(L)$ for some extension $K \subset L$ as desired.

A quasi-separated morphism is decent, see Decent Spaces, Lemma 50.14.2. Hence (1) implies (2).

Let us prove (3). Recall that the separation axioms are stable under base change and composition and that closed immersions are separated, see Morphisms of Spaces, Lemmas 49.5.4, 49.5.8, and 49.11.7. Thus the discussion in the first paragraph of the proof applies. Let X be a reduced algebraic space universally injective and locally separated over a field K. In particular $|X|$ is a singleton hence X is quasi-compact, see Properties of Spaces, Lemma 48.5.2. We can find a surjective étale morphism $U \to X$ with U affine, see Properties of Spaces, Lemma 48.6.3. Consider the morphism of schemes

$$j : U \times_X U \longrightarrow U \times_{\mathrm{Spec}(K)} U$$

As $X \to \mathrm{Spec}(K)$ is universally injective j is surjective, and as $X \to \mathrm{Spec}(K)$ is locally separated j is an immersion. A surjective immersion is a closed immersion, see Schemes, Lemma 25.10.4. Hence $R = U \times_X U$ is affine as a closed subscheme of an affine scheme. In particular R is quasi-compact. It follows that $X = U/R$ is quasi-separated, and the result follows from (2). □

Remark 57.3.5. Let $X \to Y$ be a morphism of algebraic spaces. For some applications (of radicial morphisms) it is enough to require that for every $\mathrm{Spec}(K) \to Y$ where K is a field

(1) the space $|\mathrm{Spec}(K) \times_Y X|$ is a singleton,
(2) there exists a monomorphism $\mathrm{Spec}(L) \to \mathrm{Spec}(K) \times_Y X$, and
(3) $K \subset L$ is purely inseparable.

If needed later we will may call such a morphism *weakly radicial*. For example if $X \to Y$ is a surjective weakly radicial morphism then $X(k) \to Y(k)$ is surjective for every algebraically closed field k. Note that the base change $X_{\overline{\mathbf{Q}}} \to \mathrm{Spec}(\overline{\mathbf{Q}})$ of the morphism in Example 57.3.3 is weakly radicial, but not radicial. The analogue

of Lemma 57.3.4 is that if $X \to Y$ has property (β) and is universally injective, then it is weakly radicial (proof omitted).

57.4. Conormal sheaf of an immersion

Let S be a scheme. Let $i : Z \to X$ be a closed immersion of algebraic spaces over S. Let $\mathcal{I} \subset \mathcal{O}_X$ be the corresponding quasi-coherent sheaf of ideals, see Morphisms of Spaces, Lemma 49.13.1. Consider the short exact sequence

$$0 \to \mathcal{I}^2 \to \mathcal{I} \to \mathcal{I}/\mathcal{I}^2 \to 0$$

of quasi-coherent sheaves on X. Since the sheaf $\mathcal{I}/\mathcal{I}^2$ is annihilated by $\mathcal{I}$ it corresponds to a sheaf on Z by Morphisms of Spaces, Lemma 49.14.1. This quasi-coherent $\mathcal{O}_Z$-module is the *conormal sheaf of Z in X* and is often denoted $\mathcal{I}/\mathcal{I}^2$ by the abuse of notation mentioned in Morphisms of Spaces, Section 49.14.

In case $i : Z \to X$ is a (locally closed) immersion we define the conormal sheaf of i as the conormal sheaf of the closed immersion $i : Z \to X \setminus \partial Z$, see Morphisms of Spaces, Remark 49.4.3. It is often denoted $\mathcal{I}/\mathcal{I}^2$ where $\mathcal{I}$ is the ideal sheaf of the closed immersion $i : Z \to X \setminus \partial Z$.

Definition 57.4.1. Let $i : Z \to X$ be an immersion. The *conormal sheaf $\mathcal{C}_{Z/X}$ of Z in X* or the *conormal sheaf of i* is the quasi-coherent $\mathcal{O}_Z$-module $\mathcal{I}/\mathcal{I}^2$ described above.

In [**DG67**, IV Definition 16.1.2] this sheaf is denoted $\mathcal{N}_{Z/X}$. We will not follow this convention since we would like to reserve the notation $\mathcal{N}_{Z/X}$ for the *normal sheaf of the immersion*. It is defined as

$$\mathcal{N}_{Z/X} = \mathcal{H}om_{\mathcal{O}_Z}(\mathcal{C}_{Z/X}, \mathcal{O}_Z) = \mathcal{H}om_{\mathcal{O}_Z}(\mathcal{I}/\mathcal{I}^2, \mathcal{O}_Z)$$

provided the conormal sheaf is of finite presentation (otherwise the normal sheaf may not even be quasi-coherent). We will come back to the normal sheaf later (insert future reference here).

Lemma 57.4.2. *Let S be a scheme. Let $i : Z \to X$ be an immersion. Let $\varphi : U \to X$ be an étale morphism where U is a scheme. Set $Z_U = U \times_X Z$ which is a locally closed subscheme of U. Then*

$$\mathcal{C}_{Z/X}|_{Z_U} = \mathcal{C}_{Z_U/U}$$

canonically and functorially in U.

Proof. Let $T \subset X$ be a closed subspace such that i defines a closed immersion into $X \setminus T$. Let $\mathcal{I}$ be the quasi-coherent sheaf of ideals on $X \setminus T$ defining Z. Then the lemma just states that $\mathcal{I}|_{U \setminus \varphi^{-1}(T)}$ is the sheaf of ideals of the immersion $Z_U \to U \setminus \varphi^{-1}(T)$. This is clear from the construction of $\mathcal{I}$ in Morphisms of Spaces, Lemma 49.13.1. □

Lemma 57.4.3. *Let S be a scheme. Let*

$$\begin{array}{ccc} Z & \xrightarrow{i} & X \\ {\scriptstyle f}\downarrow & & \downarrow{\scriptstyle g} \\ Z' & \xrightarrow{i'} & X' \end{array}$$

be a commutative diagram of algebraic spaces over S. Assume i, i' immersions. There is a canonical map of $\mathcal{O}_Z$-modules

$$f^*\mathcal{C}_{Z'/X'} \longrightarrow \mathcal{C}_{Z/X}$$

Proof. First find open subspaces $U' \subset X'$ and $U \subset X$ such that $g(U) \subset U'$ and such that $i(Z) \subset U$ and $i(Z') \subset U'$ are closed (proof existence omitted). Replacing X by U and X' by U' we may assume that i and i' are closed immersions. Let $\mathcal{I}' \subset \mathcal{O}_{X'}$ and $\mathcal{I} \subset \mathcal{O}_X$ be the quasi-coherent sheaves of ideals associated to i' and i, see Morphisms of Spaces, Lemma 49.13.1. Consider the composition

$$g^{-1}\mathcal{I}' \to g^{-1}\mathcal{O}_{X'} \xrightarrow{g^\sharp} \mathcal{O}_X \to \mathcal{O}_X/\mathcal{I} = i_*\mathcal{O}_Z$$

Since $g(i(Z)) \subset Z'$ we conclude this composition is zero (see statement on factorizations in Morphisms of Spaces, Lemma 49.13.1). Thus we obtain a commutative diagram

$$\begin{array}{ccccccccc}
0 & \longrightarrow & \mathcal{I} & \longrightarrow & \mathcal{O}_X & \longrightarrow & i_*\mathcal{O}_Z & \longrightarrow & 0 \\
 & & \uparrow & & \uparrow & & \uparrow & & \\
0 & \longrightarrow & g^{-1}\mathcal{I}' & \longrightarrow & g^{-1}\mathcal{O}_{X'} & \longrightarrow & g^{-1}i'_*\mathcal{O}_{Z'} & \longrightarrow & 0
\end{array}$$

The lower row is exact since g^{-1} is an exact functor. By exactness we also see that $(g^{-1}\mathcal{I}')^2 = g^{-1}((\mathcal{I}')^2)$. Hence the diagram induces a map $g^{-1}(\mathcal{I}'/(\mathcal{I}')^2) \to \mathcal{I}/\mathcal{I}^2$. Pulling back (using i^{-1} for example) to Z we obtain $i^{-1}g^{-1}(\mathcal{I}'/(\mathcal{I}')^2) \to \mathcal{C}_{Z/X}$. Since $i^{-1}g^{-1} = f^{-1}(i')^{-1}$ this gives a map $f^{-1}\mathcal{C}_{Z'/X'} \to \mathcal{C}_{Z/X}$, which induces the desired map. □

Lemma 57.4.4. *Let S be a scheme. The conormal sheaf of Definition 57.4.1, and its functoriality of Lemma 57.4.3 satisfy the following properties:*

1. *If $Z \to X$ is an immersion of schemes over S, then the conormal sheaf agrees with the one from Morphisms, Definition 28.33.1.*
2. *If in Lemma 57.4.3 all the spaces are schemes, then the map $f^*\mathcal{C}_{Z'/X'} \to \mathcal{C}_{Z/X}$ is the same as the one constructed in Morphisms, Lemma 28.33.3.*
3. *Given a commutative diagram*

$$\begin{array}{ccc}
Z & \xrightarrow[i]{} & X \\
{\scriptstyle f}\downarrow & & \downarrow{\scriptstyle g} \\
Z' & \xrightarrow{i'} & X' \\
{\scriptstyle f'}\downarrow & & \downarrow{\scriptstyle g'} \\
Z'' & \xrightarrow{i''} & X''
\end{array}$$

then the map $(f' \circ f)^*\mathcal{C}_{Z''/X''} \to \mathcal{C}_{Z/X}$ is the same as the composition of $f^*\mathcal{C}_{Z'/X'} \to \mathcal{C}_{Z/X}$ with the pullback by f of $(f')^*\mathcal{C}_{Z''/X''} \to \mathcal{C}_{Z'/X'}$

Proof. Omitted. Note that Part (1) is a special case of Lemma 57.4.2. □

Lemma 57.4.5. *Let S be a scheme. Let*

$$\begin{array}{ccc}
Z & \xrightarrow[i]{} & X \\
{\scriptstyle f}\downarrow & & \downarrow{\scriptstyle g} \\
Z' & \xrightarrow{i'} & X'
\end{array}$$

be a fibre product diagram of algebraic spaces over S. Assume i, i' immersions. Then the canonical map $f^*\mathcal{C}_{Z'/X'} \to \mathcal{C}_{Z/X}$ of Lemma 57.4.3 is surjective. If g is flat, then it is an isomorphism.

Proof. Choose a commutative diagram

$$\begin{array}{ccc} U & \longrightarrow & X \\ \downarrow & & \downarrow \\ U' & \longrightarrow & X' \end{array}$$

where U, U' are schemes and the horizontal arrows are surjective and étale, see Spaces, Lemma 47.11.4. Then using Lemmas 57.4.2 and 57.4.4 we see that the question reduces to the case of a morphism of schemes. In the schemes case this is Morphisms, Lemma 28.33.4. □

Lemma 57.4.6. *Let S be a scheme. Let $Z \to Y \to X$ be immersions of algebraic spaces. Then there is a canonical exact sequence*

$$i^*\mathcal{C}_{Y/X} \to \mathcal{C}_{Z/X} \to \mathcal{C}_{Z/Y} \to 0$$

where the maps come from Lemma 57.4.3 and $i : Z \to Y$ is the first morphism.

Proof. Let U be a scheme and let $U \to X$ be a surjective étale morphism. Via Lemmas 57.4.2 and 57.4.4 the exactness of the sequence translates immediately into the exactness of the corresponding sequence for the immersions of schemes $Z \times_X U \to Y \times_X U \to U$. Hence the lemma follows from Morphisms, Lemma 28.33.5. □

57.5. The normal cone of an immersion

Let S be a scheme. Let $i : Z \to X$ be a closed immersion of algebraic spaces over S. Let $\mathcal{I} \subset \mathcal{O}_X$ be the corresponding quasi-coherent sheaf of ideals, see Morphisms of Spaces, Lemma 49.13.1. Consider the quasi-coherent sheaf of graded $\mathcal{O}_X$-algebras $\bigoplus_{n\geq 0} \mathcal{I}^n/\mathcal{I}^{n+1}$. Since the sheaves $\mathcal{I}^n/\mathcal{I}^{n+1}$ are each annihilated by $\mathcal{I}$ this graded algebra corresponds to a quasi-coherent sheaf of graded $\mathcal{O}_Z$-algebras by Morphisms of Spaces, Lemma 49.14.1. This quasi-coherent graded $\mathcal{O}_Z$-algebra is called the *conormal algebra of Z in X* and is often simply denoted $\bigoplus_{n\geq 0} \mathcal{I}^n/\mathcal{I}^{n+1}$ by the abuse of notation mentioned in Morphisms of Spaces, Section 49.14.

In case $i : Z \to X$ is a (locally closed) immersion we define the conormal algebra of i as the conormal algebra of the closed immersion $i : Z \to X \setminus \partial Z$, see Morphisms of Spaces, Remark 49.4.3. It is often denoted $\bigoplus_{n\geq 0} \mathcal{I}^n/\mathcal{I}^{n+1}$ where $\mathcal{I}$ is the ideal sheaf of the closed immersion $i : Z \to X \setminus \partial Z$.

Definition 57.5.1. Let $i : Z \to X$ be an immersion. The *conormal algebra $\mathcal{C}_{Z/X,*}$ of Z in X* or the *conormal algebra of i* is the quasi-coherent sheaf of graded $\mathcal{O}_Z$-algebras $\bigoplus_{n\geq 0} \mathcal{I}^n/\mathcal{I}^{n+1}$ described above.

Thus $\mathcal{C}_{Z/X,1} = \mathcal{C}_{Z/X}$ is the conormal sheaf of the immersion. Also $\mathcal{C}_{Z/X,0} = \mathcal{O}_Z$ and $\mathcal{C}_{Z/X,n}$ is a quasi-coherent $\mathcal{O}_Z$-module characterized by the property

$$i_*\mathcal{C}_{Z/X,n} = \mathcal{I}^n/\mathcal{I}^{n+1} \tag{57.5.1.1}$$

where $i : Z \to X \setminus \partial Z$ and $\mathcal{I}$ is the ideal sheaf of i as above. Finally, note that there is a canonical surjective map

$$\mathrm{Sym}^*(\mathcal{C}_{Z/X}) \longrightarrow \mathcal{C}_{Z/X,*} \tag{57.5.1.2}$$

of quasi-coherent graded $\mathcal{O}_Z$-algebras which is an isomorphism in degrees 0 and 1.

Lemma 57.5.2. *Let S be a scheme. Let $i : Z \to X$ be an immersion of algebraic spaces over S. Let $\varphi : U \to X$ be an étale morphism where U is a scheme. Set $Z_U = U \times_X Z$ which is a locally closed subscheme of U. Then*

$$\mathcal{C}_{Z/X,*}|_{Z_U} = \mathcal{C}_{Z_U/U,*}$$

canonically and functorially in U.

Proof. Let $T \subset X$ be a closed subspace such that i defines a closed immersion into $X \setminus T$. Let $\mathcal{I}$ be the quasi-coherent sheaf of ideals on $X \setminus T$ defining Z. Then the lemma follows from the fact that $\mathcal{I}|_{U \setminus \varphi^{-1}(T)}$ is the sheaf of ideals of the immersion $Z_U \to U \setminus \varphi^{-1}(T)$. This is clear from the construction of $\mathcal{I}$ in Morphisms of Spaces, Lemma 49.13.1. □

Lemma 57.5.3. *Let S be a scheme. Let*

$$\begin{array}{ccc} Z & \xrightarrow{i} & X \\ {\scriptstyle f}\downarrow & & \downarrow{\scriptstyle g} \\ Z' & \xrightarrow{i'} & X' \end{array}$$

be a commutative diagram of algebraic spaces over S. Assume i, i' immersions. There is a canonical map of graded $\mathcal{O}_Z$-algebras

$$f^*\mathcal{C}_{Z'/X',*} \longrightarrow \mathcal{C}_{Z/X,*}$$

Proof. First find open subspaces $U' \subset X'$ and $U \subset X$ such that $g(U) \subset U'$ and such that $i(Z) \subset U$ and $i(Z') \subset U'$ are closed (proof existence omitted). Replacing X by U and X' by U' we may assume that i and i' are closed immersions. Let $\mathcal{I}' \subset \mathcal{O}_{X'}$ and $\mathcal{I} \subset \mathcal{O}_X$ be the quasi-coherent sheaves of ideals associated to i' and i, see Morphisms of Spaces, Lemma 49.13.1. Consider the composition

$$g^{-1}\mathcal{I}' \to g^{-1}\mathcal{O}_{X'} \xrightarrow{g^\sharp} \mathcal{O}_X \to \mathcal{O}_X/\mathcal{I} = i_*\mathcal{O}_Z$$

Since $g(i(Z)) \subset Z'$ we conclude this composition is zero (see statement on factorizations in Morphisms of Spaces, Lemma 49.13.1). Thus we obtain a commutative diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \mathcal{I} & \longrightarrow & \mathcal{O}_X & \longrightarrow & i_*\mathcal{O}_Z & \longrightarrow & 0 \\ & & \uparrow & & \uparrow & & \uparrow & & \\ 0 & \longrightarrow & g^{-1}\mathcal{I}' & \longrightarrow & g^{-1}\mathcal{O}_{X'} & \longrightarrow & g^{-1}i'_*\mathcal{O}_{Z'} & \longrightarrow & 0 \end{array}$$

The lower row is exact since g^{-1} is an exact functor. By exactness we also see that $(g^{-1}\mathcal{I}')^n = g^{-1}((\mathcal{I}')^n)$ for all $n \geq 1$. Hence the diagram induces a map $g^{-1}((\mathcal{I}')^n/(\mathcal{I}')^{n+1}) \to \mathcal{I}^n/\mathcal{I}^{n+1}$. Pulling back (using i^{-1} for example) to Z we obtain $i^{-1}g^{-1}((\mathcal{I}')^n/(\mathcal{I}')^{n+1}) \to \mathcal{C}_{Z/X,n}$. Since $i^{-1}g^{-1} = f^{-1}(i')^{-1}$ this gives maps $f^{-1}\mathcal{C}_{Z'/X',n} \to \mathcal{C}_{Z/X,n}$, which induce the desired map. □

Lemma 57.5.4. *Let S be a scheme. Let*

$$\begin{array}{ccc} Z & \xrightarrow{i} & X \\ {\scriptstyle f}\downarrow & & \downarrow{\scriptstyle g} \\ Z' & \xrightarrow{i'} & X' \end{array}$$

be a cartesion square of algebraic spaces over S with i, i' immersions. Then the canonical map $f^*\mathcal{C}_{Z'/X',*} \to \mathcal{C}_{Z/X,*}$ of Lemma 57.5.3 is surjective. If g is flat, then it is an isomorphism.

Proof. We may check the statement after étale localizing X'. In this case we may assume $X' \to X$ is a morphism of schemes, hence Z and Z' are schemes and the result follows from the case of schemes, see Divisors, Lemma 30.11.4. □

We use the same conventions for cones and vector bundles over algebraic spaces as we do for schemes (where we use the conventions of EGA), see Constructions, Sections 26.7 and 26.6. In particular, a vector bundle is a very general gadget (and not locally isomorphic to an affine space bundle).

Definition 57.5.5. Let S be a scheme. Let $i : Z \to X$ be an immersion of algebraic spaces over S. The *normal cone* $C_Z X$ of Z in X is

$$C_Z X = \underline{\mathrm{Spec}}_Z(\mathcal{C}_{Z/X,*})$$

see Morphisms of Spaces, Definition 49.19.8. The *normal bundle* of Z in X is the vector bundle

$$N_Z X = \underline{\mathrm{Spec}}_Z(\mathrm{Sym}(\mathcal{C}_{Z/X}))$$

Thus $C_Z X \to Z$ is a cone over Z and $N_Z X \to Z$ is a vector bundle over Z. Moreover, the canonical surjection (57.5.1.2) of graded algebras defines a canonical closed immersion

$$C_Z X \longrightarrow N_Z X \tag{57.5.5.1}$$

of cones over Z.

57.6. Sheaf of differentials of a morphism

We suggest the reader take a look at the corresponding section in the chapter on commutative algebra (Algebra, Section 10.125), the corresponding section in the chapter on morphism of schemes (Morphisms, Section 28.34) as well as Modules on Sites, Section 18.32. We first show that the notion of sheaf of differentials for a morphism of schemes agrees with the corresponding morphism of small étale (ringed) sites.

To clearly state the following lemma we temporarily go back to denoting $\mathcal{F}^a$ the sheaf of $\mathcal{O}_{X_{étale}}$-modules associated to a quasi-coherent $\mathcal{O}_X$-module $\mathcal{F}$ on the scheme X, see Descent, Definition 34.7.2.

Lemma 57.6.1. *Let $f : X \to Y$ be a morphism of schemes. Let $f_{small} : X_{étale} \to Y_{étale}$ be the associated morphism of small étale sites, see Descent, Remark 34.7.4. Then there is a canonical isomorphism*

$$(\Omega_{X/Y})^a = \Omega_{X_{étale}/Y_{étale}}$$

compatible with universal derivations. Here the first module is the sheaf on $X_{étale}$ associated to the quasi-coherent $\mathcal{O}_X$-module $\Omega_{X/Y}$, see Morphisms, Definition 28.34.1, and the second module is the one from Modules on Sites, Definition 18.32.3.

Proof. Let $h : U \to X$ be an étale morphism. In this case the natural map $h^*\Omega_{X/Y} \to \Omega_{U/Y}$ is an isomorphism, see More on Morphisms, Lemma 36.7.7. This means that there is a natural $\mathcal{O}_{Y_{étale}}$-derivation

$$\mathrm{d}^a : \mathcal{O}_{X_{étale}} \longrightarrow (\Omega_{X/Y})^a$$

since we have just seen that the value of $(\Omega_{X/Y})^a$ on any object U of $X_{étale}$ is canonically identified with $\Gamma(U, \Omega_{U/Y})$. By the universal property of $\mathrm{d}_{X/Y} : \mathcal{O}_{X_{étale}} \to \Omega_{X_{étale}/Y_{étale}}$ there is a unique $\mathcal{O}_{X_{étale}}$-linear map $c : \Omega_{X_{étale}/Y_{étale}} \to (\Omega_{X/Y})^a$ such that $\mathrm{d}^a = c \circ \mathrm{d}_{X/Y}$.

Conversely, suppose that $\mathcal{F}$ is an $\mathcal{O}_{X_{étale}}$-module and $D : \mathcal{O}_{X_{étale}} \to \mathcal{F}$ is a $\mathcal{O}_{Y_{étale}}$-derivation. Then we can simply restrict D to the small Zariski site X_{Zar} of X. Since sheaves on X_{Zar} agree with sheaves on X, see Descent, Remark 34.7.3, we see that $D|_{X_{Zar}} : \mathcal{O}_X \to \mathcal{F}|_{X_{Zar}}$ is just a "usual" Y-derivation. Hence we obtain a map $\psi : \Omega_{X/Y} \longrightarrow \mathcal{F}|_{X_{Zar}}$ such that $D|_{X_{Zar}} = \psi \circ \mathrm{d}$. In particular, if we apply this with $\mathcal{F} = \Omega_{X_{étale}/Y_{étale}}$ we obtain a map

$$c' : \Omega_{X/Y} \longrightarrow \Omega_{X_{étale}/Y_{étale}}|_{X_{Zar}}$$

Consider the morphism of ringed sites $\mathrm{id}_{small,étale,Zar} : X_{étale} \to X_{Zar}$ discussed in Descent, Remark 34.7.4 and Lemma 34.7.5. Since the restriction functor $\mathcal{F} \mapsto \mathcal{F}|_{X_{Zar}}$ is equal to $\mathrm{id}_{small,étale,Zar,*}$, since $\mathrm{id}^*_{small,étale,Zar}$ is left adjoint to $\mathrm{id}_{small,étale,Zar,*}$ and since $(\Omega_{X/Y})^a = \mathrm{id}^*_{small,étale,Zar}\Omega_{X/Y}$ we see that c' is adjoint to a map

$$c'' : (\Omega_{X/Y})^a \longrightarrow \Omega_{X_{étale}/Y_{étale}}.$$

We claim that c'' and c' are mutually inverse. This claim finishes the proof of the lemma. To see this it is enough to show that $c''(\mathrm{d}(f)) = \mathrm{d}_{X/Y}(f)$ and $c(\mathrm{d}_{X/Y}(f)) = \mathrm{d}(f)$ if f is a local section of $\mathcal{O}_X$ over an open of X. We omit the verification. □

This clears the way for the following definition. For an alternative, see Remark 57.6.5.

Definition 57.6.2. Let S be a scheme. Let $f : X \to Y$ be a morphism of algebraic spaces over S. The *sheaf of differentials* $\Omega_{X/Y}$ *of* X *over* Y is sheaf of differentials (Modules on Sites, Definition 18.32.10) for the morphism of ringed topoi

$$(f_{small}, f^\sharp) : (X_{étale}, \mathcal{O}_X) \to (Y_{étale}, \mathcal{O}_Y)$$

of Properties of Spaces, Lemma 48.18.3. The *universal* Y*-derivation* will be denoted $\mathrm{d}_{X/Y} : \mathcal{O}_X \to \Omega_{X/Y}$.

By Lemma 57.6.1 this does not conflict with the already existing notion in case X and Y are representable. From now on, if X and Y are representable, we no longer distinguish between the sheaf of differentials defined above and the one defined in Morphisms, Definition 28.34.1. We want to relate this to the usual modules of differentials for morphisms of schemes. Here is the key lemma.

Lemma 57.6.3. *Let S be a scheme. Let $f : X \to Y$ be a morphism of algebraic spaces over S. Consider any commutative diagram*

$$\begin{array}{ccc} U & \xrightarrow{\psi} & V \\ {\scriptstyle a}\downarrow & & \downarrow{\scriptstyle b} \\ X & \xrightarrow{f} & Y \end{array}$$

where the vertical arrows are étale morphisms of algebraic spaces. Then

$$\Omega_{X/Y}|_{U_{étale}} = \Omega_{U/V}$$

In particular, if U, V are schemes, then this is equal to the usual sheaf of differentials of the morphism of schemes $U \to V$.

Proof. By Properties of Spaces, Lemma 48.15.10 and Equation (48.15.10.1) we may think of the restriction of a sheaf on $X_{étale}$ to $U_{étale}$ as the pullback by a_{small}. Similarly for b. By Modules on Sites, Lemma 18.32.6 we have

$$\Omega_{X/Y}|_{U_{étale}} = \Omega_{\mathcal{O}_{U_{étale}}/a_{small}^{-1}f_{small}^{-1}\mathcal{O}_{Y_{étale}}}$$

Since $a_{small}^{-1}f_{small}^{-1}\mathcal{O}_{Y_{étale}} = \psi_{small}^{-1}b_{small}^{-1}\mathcal{O}_{Y_{étale}} = \psi_{small}^{-1}\mathcal{O}_{V_{étale}}$ we see that the lemma holds. □

Lemma 57.6.4. *Let S be a scheme. Let $f : X \to Y$ be a morphism of algebraic spaces over S. Then $\Omega_{X/Y}$ is a quasi-coherent $\mathcal{O}_X$-module.*

Proof. Choose a diagram as in Lemma 57.6.3 with a and b surjective and U and V schemes. Then we see that $\Omega_{X/Y}|_U = \Omega_{U/V}$ which is quasi-coherent (for example by Morphisms, Lemma 28.34.7). Hence we conclude that $\Omega_{X/Y}$ is quasi-coherent by Properties of Spaces, Lemma 48.27.6. □

Remark 57.6.5. Now that we know that $\Omega_{X/Y}$ is quasi-coherent we can attempt to construct it in another manner. For example we can use the result of Properties of Spaces, Section 48.30 to construct the sheaf of differentials by glueing. For example if Y is a scheme and if $U \to X$ is a surjective étale morphism from a scheme towards X, then we see that $\Omega_{U/Y}$ is a quasi-coherent $\mathcal{O}_U$-module, and since $s, t : R \to U$ are étale we get an isomorphism

$$\alpha : s^*\Omega_{U/Y} \to \Omega_{R/Y} \to t^*\Omega_{U/Y}$$

by using Morphisms, Lemma 28.35.16. You check that this satisfies the cocycle condition and you're done. If Y is not a scheme, then you define $\Omega_{U/Y}$ as the cokernel of the map $(U \to Y)^*\Omega_{Y/S} \to \Omega_{U/S}$, and proceed as before. This two step process is a little bit ugly. Another possibility is to glue the sheaves $\Omega_{U/V}$ for any diagram as in Lemma 57.6.3 but this is not very elegant either. Both approaches will work however, and will give a slightly more elementary construction of the sheaf of differentials.

Lemma 57.6.6. *Let S be a scheme. Let*

$$\begin{array}{ccc} X' & \xrightarrow{f} & X \\ \downarrow & & \downarrow \\ Y' & \longrightarrow & Y \end{array}$$

be a commutative diagram of algebraic spaces. The map $f^\sharp : \mathcal{O}_X \to f_*\mathcal{O}_{X'}$ composed with the map $f_* d_{X'/Y'} : f_*\mathcal{O}_{X'} \to f_*\Omega_{X'/Y'}$ is a Y-derivation. Hence we obtain a canonical map of $\mathcal{O}_X$-modules $\Omega_{X/Y} \to f_*\Omega_{X'/Y'}$, and by adjointness of f_* and f^* a canonical $\mathcal{O}_{X'}$-module homomorphism

$$c_f : f^*\Omega_{X/Y} \longrightarrow \Omega_{X'/Y'}.$$

It is uniquely characterized by the property that $f^* d_{X/Y}(t)$ mapsto $d_{X'/Y'}(f^*t)$ for any local section t of $\mathcal{O}_X$.

Proof. This is a special case of Modules on Sites, Lemma 18.32.11. □

Lemma 57.6.7. *Let S be a scheme. Let*

$$\begin{array}{ccccc} X'' & \xrightarrow{g} & X' & \xrightarrow{f} & X \\ \downarrow & & \downarrow & & \downarrow \\ Y'' & \longrightarrow & Y' & \longrightarrow & Y \end{array}$$

be a commutative diagram of algebraic spaces over S. Then we have

$$c_{f\circ g} = c_g \circ g^* c_f$$

as maps $(f \circ g)^*\Omega_{X/Y} \to \Omega_{X''/Y''}$.

Proof. Omitted. Hint: Use the characterization of $c_f, c_g, c_{f\circ g}$ in terms of the effect these maps have on local sections. □

Lemma 57.6.8. *Let S be a scheme. Let $f : X \to Y$, $g : Y \to B$ be morphisms of algebraic spaces over S. Then there is a canonical exact sequence*

$$f^*\Omega_{Y/B} \to \Omega_{X/B} \to \Omega_{X/Y} \to 0$$

where the maps come from applications of Lemma 57.6.6.

Proof. Follows from the schemes version, see Morphisms, Lemma 28.34.9, of this result via étale localization, see Lemma 57.6.3. □

Lemma 57.6.9. *Let S be a scheme. If $X \to Y$ is an immersion of algebraic spaces over S then $\Omega_{X/S}$ is zero.*

Proof. Follows from the schemes version, see Morphisms, Lemma 28.34.14, of this result via étale localization, see Lemma 57.6.3. □

Lemma 57.6.10. *Let S be a scheme. Let B be an algebraic space over S. Let $i : Z \to X$ be an immersion of algebraic spaces over B. There is a canonical exact sequence*

$$\mathcal{C}_{Z/X} \to i^*\Omega_{X/B} \to \Omega_{Z/B} \to 0$$

where the first arrow is induced by $d_{X/B}$ and the second arrow comes from Lemma 57.6.6.

Proof. This is the algebraic spaces version of Morphisms, Lemma 28.34.15 and will be a consequence of that lemma by étale localization, see Lemmas 57.6.3 and 57.4.2. However, we should make sure we can define the first arrow globally. Hence we explain the meaning of "induced by $\mathrm{d}_{X/B}$" here. Namely, we may assume that i is a closed immersion after replacing X by an open subspace. Let $\mathcal{I} \subset \mathcal{O}_X$ be the quasi-coherent sheaf of ideals corresponding to $Z \subset X$. Then $\mathrm{d}_{X/S} : \mathcal{I} \to \Omega_{X/S}$ maps the subsheaf $\mathcal{I}^2 \subset \mathcal{I}$ to $\mathcal{I}\Omega_{X/S}$. Hence it induces a map $\mathcal{I}/\mathcal{I}^2 \to \Omega_{X/S}/\mathcal{I}\Omega_{X/S}$

which is $\mathcal{O}_X/\mathcal{I}$-linear. By Morphisms of Spaces, Lemma 49.14.1 this corresponds to a map $\mathcal{C}_{Z/X} \to i^*\Omega_{X/S}$ as desired. □

Lemma 57.6.11. *Let S be a scheme. Let B be an algebraic space over S. Let $i : Z \to X$ be an immersion of schemes over B, and assume i (étale locally) has a left inverse. Then the canonical sequence*

$$0 \to \mathcal{C}_{Z/X} \to i^*\Omega_{X/B} \to \Omega_{Z/B} \to 0$$

of Lemma 57.6.10 is (étale locally) split exact.

Proof. Clarification: we claim that if $g : X \to Z$ is a left inverse of i, then i^*c_g is a right inverse of the map $i^*\Omega_{X/B} \to \Omega_{Z/B}$. Having said this, the result follows from the corresponding result for morphisms of schemes by étale localization, see Lemmas 57.6.3 and 57.4.2. □

Lemma 57.6.12. *Let S be a scheme. Let $X \to Y$ be a morphism of algebraic spaces over S. Let $g : Y' \to Y$ be a morphism of algebraic spaces over S. Let $X' = X_{Y'}$ be the base change of X. Denote $g' : X' \to X$ the projection. Then the map*

$$(g')^*\Omega_{X/Y} \to \Omega_{X'/Y'}$$

of Lemma 57.6.6 is an isomorphism.

Proof. Follows from the schemes version, see Morphisms, Lemma 28.34.10 and étale localization, see Lemma 57.6.3. □

Lemma 57.6.13. *Let S be a scheme. Let $f : X \to B$ and $g : Y \to B$ be morphisms of algebraic spaces over S with the same target. Let $p : X \times_B Y \to X$ and $q : X \times_B Y \to Y$ be the projection morphisms. The maps from Lemma 57.6.6*

$$p^*\Omega_{X/S} \oplus q^*\Omega_{Y/S} \longrightarrow \Omega_{X \times_S Y/S}$$

give an isomorphism.

Proof. Follows from the schemes version, see Morphisms, Lemma 28.34.11 and étale localization, see Lemma 57.6.3. □

Lemma 57.6.14. *Let S be a scheme. Let $f : X \to Y$ be a morphism of algebraic spaces over S. If f is locally of finite type, then $\Omega_{X/Y}$ is a finite type $\mathcal{O}_X$-module.*

Proof. Follows from the schemes version, see Morphisms, Lemma 28.34.12 and étale localization, see Lemma 57.6.3. □

Lemma 57.6.15. *Let S be a scheme. Let $f : X \to Y$ be a morphism of algebraic spaces over S. If f is locally of finite type, then $\Omega_{X/Y}$ is an $\mathcal{O}_X$-module of finite presentation.*

Proof. Follows from the schemes version, see Morphisms, Lemma 28.34.13 and étale localization, see Lemma 57.6.3. □

57.7. Topological invariance of the étale site

We show that the site $X_{spaces,étale}$ is a "topological invariant". It then follows that $X_{étale}$, which consists of the representable objects in $X_{spaces,étale}$, is a topological invariant too, see Lemma 57.7.2.

Theorem 57.7.1. *Let S be a scheme. Let $f : X \to Y$ be a morphism of algebraic spaces over S. Assume f is integral, universally injective and surjective. The functor*

$$V \longmapsto V_X = X \times_Y V$$

defines an equivalence of categories $Y_{spaces,étale} \to X_{spaces,étale}$.

Proof. The morphism f is representable and a universal homeomorphism, see Morphisms of Spaces, Section 49.45.

We first prove that the functor is faithful. Suppose that V', V are objects of $Y_{spaces,étale}$ and that $a, b : V' \to V$ are distinct morphisms over Y. Since V', V are étale over Y the equalizer

$$E = V' \times_{(a,b),V \times_Y V,\Delta_{V/Y}} V$$

of a, b is étale over Y also. Hence $E \to V'$ is an étale monomorphism (i.e., an open immersion) which is an isomorphism if and only if it is surjective. Since $X \to Y$ is a universal homeomorphism we see that this is the case if and only if $E_X = V'_X$, i.e., if and only if $a_X = b_X$.

Next, we prove that the functor is fully faithful. Suppose that V', V are objects of $Y_{spaces,étale}$ and that $c : V'_X \to V_X$ is a morphism over X. We want to construct a morphism $a : V' \to V$ over Y such that $a_X = c$. Let $a' : V'' \to V'$ be a surjective étale morphism such that V'' is a separated algebraic space. If we can construct a morphism $a'' : V'' \to V$ such that $a''_X = c \circ a'_X$, then the two compositions

$$V'' \times_{V'} V'' \xrightarrow{\mathrm{pr}_i} V'' \xrightarrow{a''} V$$

will be equal by the faithfulness of the functor proved in the first paragraph. Hence a'' will factor through a unique morphism $a : V' \to V$ as V' is (as a sheaf) the quotient of V'' by the equivalence relation $V'' \times_{V'} V''$. Hence we may assume that V' is separated. In this case the graph

$$\Gamma_c \subset (V' \times_Y V)_X$$

is open and closed (details omitted). Since $X \to Y$ is a universal homeomorphism, there exists an open and closed subspace $\Gamma \subset V' \times_Y V$ such that $\Gamma_X = \Gamma_c$. The projection $\Gamma \to V'$ is an étale morphism whose base change to X is an isomorphism. Hence $\Gamma \to V'$ is étale, universally injective, and surjective, so an isomorphism by Morphisms of Spaces, Lemma 49.44.2. Thus Γ is the graph of a morphism $a : V' \to V$ as desired.

Finally, we prove that the functor is essentially surjective. Suppose that U is an object of $X_{spaces,étale}$. We have to find an object V of $Y_{spaces,étale}$ such that $V_X \cong U$. Let $U' \to U$ be a surjective étale morphism such that $U' \cong V'_X$ and $U' \times_U U' \cong V''_X$ for some objects V'', V' of $Y_{spaces,étale}$. Then by fully faithfulness of the functor we obtain morphisms $s, t : V'' \to V'$ with $t_X = \mathrm{pr}_0$ and $s_X = \mathrm{pr}_1$ as morphisms $U' \times_U U' \to U'$. Using that $(\mathrm{pr}_0, \mathrm{pr}_1) : U' \times_U U' \to U' \times_S U'$ is an étale equivalence relation, and that $U' \to V'$ and $U' \times_U U' \to V''$ are universally injective

and surjective we deduce that $(t, s) : V'' \to V' \times_S V'$ is an étale equivalence relation. Then the quotient $V = V'/V''$ (see Spaces, Theorem 47.10.5) is an algebraic space V over Y. There is a morphism $V' \to V$ such that $V'' = V' \times_V V'$. Thus we obtain a morphism $V \to Y$ (see Descent on Spaces, Lemma 55.6.2). On base change to X we see that we have a morphism $U' \to V_X$ and a compatible isomorphism $U' \times_{V_X} U' = U' \times_U U'$, which implies that $V_X \cong U$ (by the lemma just cited once more).

Pick a scheme W and a surjective étale morphism $W \to Y$. Pick a scheme U' and a surjective étale morphism $U' \to U \times_X W_X$. Note that U' and $U' \times_U U'$ are schemes étale over X whose structure morphism to X factors through the scheme W_X. Hence by Étale Cohomology, Theorem 44.46.1 there exist schemes V', V'' étale over W whose base change to W_X is isomorphic to respectively U' and $U' \times_U U'$. This finishes the proof. □

Lemma 57.7.2. *With assumption and notation as in Theorem 57.7.1 the equivalence of categories $Y_{spaces,étale} \to X_{spaces,étale}$ restricts to an equivalence of categories $Y_{étale} \to X_{étale}$.*

Proof. This is just the statement that given an object $V \in Y_{spaces,étale}$ we have V is a scheme if and only if $V \times_Y X$ is a scheme. Since $V \times_Y X \to V$ is integral, universally injective, and surjective (as a base change of $X \to Y$) this follows from Limits of Spaces, Lemma 52.14.4. □

Remark 57.7.3. A universal homeomorphism of algebraic spaces need not be representable, see Morphisms of Spaces, Example 49.45.3. The argument in the proof of Theorem 57.7.1 above cannot be used in this case. In fact we do not know whether given a universal homeomorphism of algebraic spaces $f : X \to Y$ the categories $X_{spaces,étale}$ and $Y_{spaces,étale}$ are equivalent. If you do, please email stacks.project@gmail.com.

57.8. Thickenings

The following terminology may not be completely standard, but it is convenient.

Definition 57.8.1. Thickenings. Let S be a scheme.

(1) We say an algebraic space X' is a *thickening* of an algebraic space X if X is a closed subspace of X' and the associated topological spaces are equal.
(2) We say X' is a *first order thickening* of X if X is a closed subspace of X' and the quasi-coherent sheaf of ideals $\mathcal{I} \subset \mathcal{O}_{X'}$ defining X has square zero.
(3) Given two thickenings $X \subset X'$ and $Y \subset Y'$ a *morphism of thickenings* is a morphism $f' : X' \to Y'$ such that $f(X) \subset Y$, i.e., such that $f'|_X$ factors through the closed subspace Y. In this situation we set $f = f'|_X : X \to Y$ and we say that $(f, f') : (X \subset X') \to (Y \subset Y')$ is a morphism of thickenings.
(4) Let B be an algebraic space. We similarly define *thickenings over B*, and *morphisms of thickenings over B*. This means that the spaces X, X', Y, Y' above are algebraic spaces endowed with a structure morphism to B, and that the morphisms $X \to X'$, $Y \to Y'$ and $f' : X' \to Y'$ are morphisms over B.

The fundamental equivalence. Note that if $X \subset X'$ is a thickening, then $X \to X'$ is integral and universally bijective. This implies that

$$X_{spaces,étale} = X'_{spaces,étale} \tag{57.8.1.1}$$

via the pullback functor, see Theorem 57.7.1. Hence we may think of $\mathcal{O}_{X'}$ as a sheaf on $X_{spaces,étale}$. Thus a canonical equivalence of locally ringed topoi

$$(Sh(X'_{spaces,étale}), \mathcal{O}_{X'}) \cong (Sh(X_{spaces,étale}), \mathcal{O}_{X'}) \tag{57.8.1.2}$$

Below we will frequently combine this with the fully faithfulness result of Properties of Spaces, Theorem 48.26.4. For example the closed immersion $i_X : X \to X'$ corresponds to the surjective map $i_X^\sharp : \mathcal{O}_{X'} \to \mathcal{O}_X$.

Let S be a scheme, and let B be an algebraic space over S. Let $(f, f') : (X \subset X') \to (Y \subset Y')$ be a morphism of thickenings over B. Note that the diagram of continuous functors

$$\begin{array}{ccc} X_{spaces,étale} & \longleftarrow & Y_{spaces,étale} \\ \uparrow & & \uparrow \\ X'_{spaces,étale} & \longleftarrow & Y'_{spaces,étale} \end{array}$$

is commutative and the vertical arrows are equivalences. Hence $f_{spaces,étale}$, f_{small}, $f'_{spaces,étale}$, and f'_{small} all define the same morphism of topoi. Thus we may think of

$$(f')^\sharp : f^{-1}_{spaces,étale}\mathcal{O}_{Y'} \longrightarrow \mathcal{O}_{X'}$$

as a map of sheaves of $\mathcal{O}_B$-algebras fitting into the commutative diagram

$$\begin{array}{ccc} f^{-1}_{spaces,étale}\mathcal{O}_Y & \xrightarrow{f^\sharp} & \mathcal{O}_X \\ {\scriptstyle i_Y^\sharp}\uparrow & & \uparrow{\scriptstyle i_X^\sharp} \\ f^{-1}_{spaces,étale}\mathcal{O}_{Y'} & \xrightarrow{(f')^\sharp} & \mathcal{O}_{X'} \end{array}$$

Here $i_X : X \to X'$ and $i_Y : Y \to Y'$ are the names of the given closed immersions.

Lemma 57.8.2. *Let S be a scheme. Let B be an algebraic space over S. Let $X \subset X'$ and $Y \subset Y'$ be thickenings of algebraic spaces over B. Let $f : X \to Y$ be a morphism of algebraic spaces over B. Given any map of $\mathcal{O}_B$-algebras*

$$\alpha : f^{-1}_{spaces,étale}\mathcal{O}_{Y'} \to \mathcal{O}_{X'}$$

such that

$$\begin{array}{ccc} f^{-1}_{spaces,étale}\mathcal{O}_Y & \xrightarrow{f^\sharp} & \mathcal{O}_X \\ {\scriptstyle i_Y^\sharp}\uparrow & & \uparrow{\scriptstyle i_X^\sharp} \\ f^{-1}_{spaces,étale}\mathcal{O}_{Y'} & \xrightarrow{\alpha} & \mathcal{O}_{X'} \end{array}$$

commutes, there exists a unique morphism of (f, f') of thickenings over B such that $\alpha = (f')^\sharp$.

Proof. To find f', by Properties of Spaces, Theorem 48.26.4, all we have to do is show that the morphism of ringed topoi

$$(f_{spaces,étale}, \alpha) : (Sh(X_{spaces,étale}), \mathcal{O}_{X'}) \longrightarrow (Sh(Y_{spaces,étale}), \mathcal{O}_{Y'})$$

is a morphism of locally ringed topoi. This follows directly from the definition of morphisms of locally ringed topoi (Modules on Sites, Definition 18.39.8), the fact that $(f, f^\sharp)$ is a morphism of locally ringed topoi (Properties of Spaces, Lemma 48.26.1), that α fits into the given commutative diagram, and the fact that the kernels of $i_X^\sharp$ and $i_Y^\sharp$ are locally nilpotent. Finally, the fact that $f' \circ i_X = i_Y \circ f$ follows from the commutativity of the diagram and another application of Properties of Spaces, Theorem 48.26.4. We omit the verification that f' is a morphism over B. □

Lemma 57.8.3. *Let S be a scheme. Let $X \subset X'$ be a thickening of algebraic spaces over S. For any open subspace $U \subset X$ there exists a unique open subspace $U' \subset X'$ such that $U = X \times_{X'} U'$.*

Proof. Let $U' \to X'$ be the object of $X'_{spaces,étale}$ corresponding to the object $U \to X$ of $X_{spaces,étale}$ via (57.8.1.1). The morphism $U' \to X'$ is étale and universally injective, hence an open immersion, see Morphisms of Spaces, Lemma 49.44.2. □

Finite order thickenings. Let $i_X : X \to X'$ be a thickening of algebraic spaces. Any local section of the kernel $\mathcal{I} = \mathrm{Ker}(i_X^\sharp) \subset \mathcal{O}_{X'}$ is locally nilpotent. Let us say that $X \subset X'$ is a *finite order thickening* if the ideal sheaf $\mathcal{I}$ is "globally" nilpotent, i.e., if there exists an $n \geq 0$ such that $\mathcal{I}^{n+1} = 0$. Technically the class of finite order thickenings $X \subset X'$ is much easier to handle than the general case. Namely, in this case we have a filtration

$$0 \subset \mathcal{I}^n \subset \mathcal{I}^{n-1} \subset \ldots \subset \mathcal{I} \subset \mathcal{O}_{X'}$$

and we see that X' is filtered by closed subspaces

$$X = X_0 \subset X_1 \subset \ldots \subset X_{n-1} \subset X_{n+1} = X'$$

such that each pair $X_i \subset X_{i+1}$ is a first order thickening over B. Using simple induction arguments many results proved for first order thickenings can be rephrased as results on finite order thickenings.

Lemma 57.8.4. *Let S be a scheme. Let $X \subset X'$ be a thickening of algebraic spaces over S. Let U be an affine object of $X_{spaces,étale}$. Then*

$$\Gamma(U, \mathcal{O}_{X'}) \to \Gamma(U, \mathcal{O}_X)$$

is surjective where we think of $\mathcal{O}_{X'}$ as a sheaf on $X_{spaces,étale}$ via (57.8.1.2).

Proof. Let $U' \to X'$ be the étale morphism of algebraic spaces such that $U = X \times_{X'} U'$, see Theorem 57.7.1. By Limits of Spaces, Lemma 52.14.1 we see that U' is an affine scheme. Hence $\Gamma(U, \mathcal{O}_{X'}) = \Gamma(U', \mathcal{O}_{U'}) \to \Gamma(U, \mathcal{O}_U)$ is surjective as $U \to U'$ is a closed immersion of affine schemes. Below we give a direct proof for finite order thickenings which is the case most used in practice. □

Proof for finite order thickenings. We may assume that $X \subset X'$ is a first order thickening by the principle explained above. Denote $\mathcal{I}$ the kernel of the surjection $\mathcal{O}_{X'} \to \mathcal{O}_X$. As $\mathcal{I}$ is a quasi-coherent $\mathcal{O}_{X'}$-module and since $\mathcal{I}^2 = 0$ by the definition of a first order thickening we may apply Morphisms of Spaces, Lemma 49.14.1 to see that $\mathcal{I}$ is a quasi-coherent $\mathcal{O}_X$-module. Hence the lemma follows from the long exact cohomology sequence associated to the short exact sequence

$$0 \to \mathcal{I} \to \mathcal{O}_{X'} \to \mathcal{O}_X \to 0$$

and the fact that $H^1_{étale}(U, \mathcal{I}) = 0$ as $\mathcal{I}$ is quasi-coherent, see Descent, Proposition 34.7.10 and Cohomology of Schemes, Lemma 29.2.2. □

Lemma 57.8.5. *Let S be a scheme. Let $X \subset X'$ be a thickening of algebraic spaces over S. If X is (representable by) a scheme, then so is X'.*

Proof. Note that $X'_{red} = X_{red}$. Hence if X is a scheme, then X'_{red} is a scheme. Thus the result follows from Limits of Spaces, Lemma 52.14.3. Below we give a direct proof for finite order thickenings which is the case most often used in practice. □

Proof for finite order thickenings. It suffices to prove this when X' is a first order thickening of X. By Properties of Spaces, Lemma 48.10.1 there is a largest open subspace of X' which is a scheme. Thus we have to show that every point x of $|X'| = |X|$ is contained in an open subspace of X' which is a scheme. Using Lemma 57.8.3 we may replace $X \subset X'$ by $U \subset U'$ with $x \in U$ and U an affine scheme. Hence we may assume that X is affine. Thus we reduce to the case discussed in the next paragraph.

Assume $X \subset X'$ is a first order thickening where X is an affine scheme. Set $A = \Gamma(X, \mathcal{O}_X)$ and $A' = \Gamma(X', \mathcal{O}_{X'})$. By Lemma 57.8.4 the map $A \to A'$ is surjective. The kernel I is an ideal of square zero. By Properties of Spaces, Lemma 48.31.1 we obtain a canonical morphism $f : X' \to \mathrm{Spec}(A')$ which fits into the following commutative diagram

$$\begin{array}{ccc} X & \longrightarrow & X' \\ \| & & \downarrow f \\ \mathrm{Spec}(A) & \longrightarrow & \mathrm{Spec}(A') \end{array}$$

Because the horizontal arrows are thickenings it is clear that f is universally injective and surjective. Hence it suffices to show that f is étale, since then Morphisms of Spaces, Lemma 49.44.2 will imply that f is an isomorphism.

To prove that f is étale choose an affine scheme U' and an étale morphism $U' \to X'$. It suffices to show that $U' \to X' \to \mathrm{Spec}(A')$ is étale, see Properties of Spaces, Definition 48.13.2. Write $U' = \mathrm{Spec}(B')$. Set $U = X \times_{X'} U'$. Since U is a closed subspace of U', it is a closed subscheme, hence $U = \mathrm{Spec}(B)$ with $B' \to B$ surjective. Denote $J = \mathrm{Ker}(B' \to B)$ and note that $J = \Gamma(U, \mathcal{I})$ where $\mathcal{I} = \mathrm{Ker}(\mathcal{O}_{X'} \to \mathcal{O}_X)$ on $X_{spaces,étale}$ as in the proof of Lemma 57.8.4. The morphism $U' \to X' \to \mathrm{Spec}(A')$ induces a commutative diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & J & \longrightarrow & B' & \longrightarrow & B & \longrightarrow & 0 \\ & & \uparrow & & \uparrow & & \uparrow & & \\ 0 & \longrightarrow & I & \longrightarrow & A' & \longrightarrow & A & \longrightarrow & 0 \end{array}$$

Now, since $\mathcal{I}$ is a quasi-coherent $\mathcal{O}_X$-module we have $\mathcal{I} = (\widetilde{I})^a$, see Descent, Definition 34.7.2 for notation and Descent, Proposition 34.7.11 for why this is true. Hence we see that $J = I \otimes_A B$. Finally, note that $A \to B$ is étale as $U \to X$ is étale as the base change of the étale morphism $U' \to X'$. We conclude that $A' \to B'$ is étale by Algebra, Lemma 10.136.12. □

Lemma 57.8.6. *Let S be a scheme. Let $X \subset X'$ be a thickening of algebraic spaces over S. The functor*

$$V' \longmapsto V = X \times_{X'} V'$$

defines an equivalence of categories $X'_{\acute{e}tale} \to X_{\acute{e}tale}$.

Proof. The functor $V' \mapsto V$ defines an equivalence of categories $X'_{spaces,\acute{e}tale} \to X_{spaces,\acute{e}tale}$, see Theorem 57.7.1. Thus it suffices to show that V is a scheme if and only if V' is a scheme. This is the content of Lemma 57.8.5. □

First order thickening are described as follows.

Lemma 57.8.7. *Let S be a scheme. Let $f : X \to B$ be a morphism of algebraic spaces over S. Consider a short exact sequence*

$$0 \to \mathcal{I} \to \mathcal{A} \to \mathcal{O}_X \to 0$$

of sheaves on $X_{\acute{e}tale}$ where $\mathcal{A}$ is a sheaf of $f^{-1}\mathcal{O}_B$-algebras, $\mathcal{A} \to \mathcal{O}_X$ is a surjection of sheaves of $f^{-1}\mathcal{O}_B$-algebras, and $\mathcal{I}$ is its kernel. If

(1) *$\mathcal{I}$ is an ideal of square zero in $\mathcal{A}$, and*
(2) *$\mathcal{I}$ is quasi-coherent as an $\mathcal{O}_X$-module*

then there exists a first order thickening $X \subset X'$ over B and an isomorphism $\mathcal{O}_{X'} \to \mathcal{A}$ of $f^{-1}\mathcal{O}_B$-algebras compatible with the surjections to $\mathcal{O}_X$.

Proof. In this proof we redo some of the arguments used in the proofs of Lemmas 57.8.4 and 57.8.5. We first handle the case $B = S = \mathrm{Spec}(\mathbf{Z})$. Let U be an affine scheme, and let $U \to X$ be étale. Then

$$0 \to \mathcal{I}(U) \to \mathcal{A}(U) \to \mathcal{O}_X(U) \to 0$$

is exact as $H^1(U_{\acute{e}tale}, \mathcal{I}) = 0$ as $\mathcal{I}$ is quasi-coherent, see Descent, Proposition 34.7.10 and Cohomology of Schemes, Lemma 29.2.2. If $V \to U$ is a morphism of affine objects of $X_{spaces,\acute{e}tale}$ then

$$\mathcal{I}(V) = \mathcal{I}(U) \otimes_{\mathcal{O}_X(U)} \mathcal{O}_X(V)$$

since $\mathcal{I}$ is a quasi-coherent $\mathcal{O}_X$-module, see Descent, Proposition 34.7.11. Hence $\mathcal{A}(U) \to \mathcal{A}(V)$ is an étale ring map, see Algebra, Lemma 10.136.12. Hence we see that

$$U \longmapsto U' = \mathrm{Spec}(\mathcal{A}(U))$$

is a functor from $X_{affine,\acute{e}tale}$ to the category of affine schemes and étale morphisms. In fact, we claim that this functor can be extended to a functor $U \mapsto U'$ on all of $X_{\acute{e}tale}$. To see this, if U is an object of $X_{\acute{e}tale}$, note that

$$0 \to \mathcal{I}|_{U_{Zar}} \to \mathcal{A}|_{U_{Zar}} \to \mathcal{O}_X|_{U_{Zar}} \to 0$$

and $\mathcal{I}|_{U_{Zar}}$ is a quasi-coherent sheaf on U, see Descent, Proposition 34.7.14. Hence by More on Morphisms, Lemma 36.2.2 we obtain a first order thickening $U \subset U'$ of schemes such that $\mathcal{O}_{U'}$ is isomorphic to $\mathcal{A}|_{U_{Zar}}$. It is clear that this construction is compatible with the construction for affines above.

Choose a presentation $X = U/R$, see Spaces, Definition 47.9.3 so that $s, t : R \to U$ define an étale equivalence relation. Applying the functor above we obtain an étale equivalence relation $s', t' : R' \to U'$ in schemes. Consider the algebraic space $X' = U'/R'$ (see Spaces, Theorem 47.10.5). The morphism $X = U/R \to U'/R' = X'$ is a

first order thickening. Consider $\mathcal{O}_{X'}$ viewed as a sheaf on $X_{étale}$. By construction we have an isomorphism

$$\gamma : \mathcal{O}_{X'}|_{U_{étale}} \longrightarrow \mathcal{A}|_{U_{étale}}$$

such that $s^{-1}\gamma$ agrees with $t^{-1}\gamma$ on $R_{étale}$. Hence by Properties of Spaces, Lemma 48.15.13 this implies that γ comes from a unique isomorphism $\mathcal{O}_{X'} \to \mathcal{A}$ as desired.

To handle the case of a general base algebraic space B, we first construct X' as an algebraic space over $\mathbf{Z}$ as above. Then we use the isomorphism $\mathcal{O}_{X'} \to \mathcal{A}$ to define $f^{-1}\mathcal{O}_B \to \mathcal{O}_{X'}$. According to Lemma 57.8.2 this defines a morphism $X' \to B$ compatible with the given morphism $X \to B$ and we are done. □

Lemma 57.8.8. *Let S be a scheme. Let $Y \subset Y'$ be a thickening of algebraic spaces over S. Let $X' \to Y'$ be a morphism and set $X = Y \times_{Y'} X'$. Then $(X \subset X') \to (Y \subset Y')$ is a morphism of thickenings. If $Y \subset Y'$ is a first (resp. finite order) thickening, then $X \subset X'$ is a first (resp. finite order) thickening.*

Proof. Omitted. □

Lemma 57.8.9. *Let S be a scheme. Let $(f, f') : (X \subset X') \to (Y \subset Y')$ be a morphism of thickenings of algebraic spaces over S. Then*

1. *f is an affine morphism if and only if f' is an affine morphism,*
2. *f is a surjective morphism if and only if f' is a surjective morphism,*
3. *f is quasi-compact if and only if f' quasi-compact,*
4. *f is universally closed if and only if f' is universally closed,*
5. *f is integral if and only if f' is integral,*
6. *f is (quasi-)separated if and only if f' is (quasi-)separated,*
7. *f is universally injective if and only if f' is universally injective,*
8. *f is universally open if and only if f' is universally open, and*
9. *add more here.*

Proof. Observe that $Y \to Y'$ and $X \to X'$ are integral and universal homeomorphisms. This immediately implies parts (2), (3), (4), (7), and (8). Part (1) follows from Limits of Spaces, Proposition 52.14.2 which tells us that there is a 1-to-1 correspondence between affine schemes étale over X and X' and between affine schemes étale over Y and Y'. Part (5) follows from (1) and (4) by Morphisms of Spaces, Lemma 49.40.7. Finally, note that

$$X \times_Y X = X \times_{Y'} X \to X \times_{Y'} X' \to X' \times_{Y'} X'$$

is a thickening (the two arrows are thickenings by Lemma 57.8.8). Hence applying (3) and (4) to the morphism $(X \subset X') \to (X \times_Y X \to X' \times_{Y'} X')$ we obtain (6). □

Lemma 57.8.10. *Let $(f, f') : (X \subset X') \to (S \subset S')$ be a morphism of thickenings such that $X = S \times_{S'} X'$. If $S \subset S'$ is a finite order thickening, then*

1. *f is a closed immersion if and only if f' is a closed immersion,*
2. *f is locally of finite type if and only if f' is locally of finite type,*
3. *f is locally quasi-finite if and only if f' is locally quasi-finite,*
4. *f is locally of finite type of relative dimension d if and only if f' is locally of finite type of relative dimension d,*
5. *f is unramified if and only if f' is unramified,*
6. *f is proper if and only if f' is proper,*

(7) *f is a finite morphism if and only if f' is an finite morphism, and*
(8) *add more here.*

Proof. Choose a scheme V' and a surjective étale morphism $V' \to Y'$. Choose a scheme U' and a surjective étale morphism $U' \to X' \times_{Y'} V'$. Set $V = Y \times_{Y'} V'$ and $U = X \times_{X'} U'$. Then for étale local properties of morphisms we can reduce to the morphism of thickenings of schemes $(U \subset U') \to (V \subset V')$ and apply More on Morphisms, Lemma 36.2.5. This proves (2), (3), (4), and (5).

The properties of morphisms in (1), (6), (7) are stable under base change, hence if f' has property $\mathcal{P}$, then so does f. See Spaces, Lemma 47.12.3, and Morphisms of Spaces, Lemmas 49.36.3, 49.40.5.

The interesting direction in (1), (6), (7) to assume that f has the property and deduce that f' has it too. By induction on the order of the thickening we may assume that $S \subset S'$ is a first order thickening, see discussion on finite order thickenings above.

Proof of (1). Choose a scheme V' and a surjective étale morphism $V' \to Y'$. Set $V = Y \times_{Y'} V'$, $U' = X' \times_{Y'} V'$ and $U = X \times_Y V$. Then $U \to V$ is a closed immersion, which implies that U is a scheme, which in turn implies that U' is a scheme (Lemma 57.8.5). Thus we can apply the lemma in the case of schemes to $(U \subset U') \to (V \subset V')$ to conclude.

Proof of (6). Follows by combining (2) with results of Lemma 57.8.9 and the fact that proper equals quasi-compact + separated + locally of finite type + universally closed.

Proof of (7). Follows by combining (2) with results of Lemma 57.8.9 and using the fact that finite equals integral + locally of finite type (Morphisms, Lemma 28.44.4). □

57.9. First order infinitesimal neighbourhood

A natural construction of first order thickenings is the following. Suppose that $i : Z \to X$ be an immersion of algebraic spaces. Choose an open subspace $U \subset X$ such that i identifies Z with a closed subspace $Z \subset U$ (see Morphisms of Spaces, Remark 49.4.3). Let $\mathcal{I} \subset \mathcal{O}_U$ be the quasi-coherent sheaf of ideals defining Z in U, see Morphisms of Spaces, Lemma 49.13.1. Then we can consider the closed subspace $Z' \subset U$ defined by the quasi-coherent sheaf of ideals $\mathcal{I}^2$.

Definition 57.9.1. Let $i : Z \to X$ be an immersion of algebraic spaces. The *first order infinitesimal neighbourhood* of Z in X is the first order thickening $Z \subset Z'$ over X described above.

This thickening has the following universal property (which will assuage any fears that the construction above depends on the choice of the open U).

Lemma 57.9.2. *Let $i : Z \to X$ be an immersion of algebraic spaces. The first order infinitesimal neighbourhood Z' of Z in X has the following universal property: Given any commutative diagram*

$$\begin{array}{ccc} Z & \xleftarrow{a} & T \\ {\scriptstyle i}\downarrow & & \downarrow \\ X & \xleftarrow{b} & T' \end{array}$$

where $T \subset T'$ is a first order thickening over X, there exists a unique morphism $(a', a) : (T \subset T') \to (Z \subset Z')$ of thickenings over X.

Proof. Let $U \subset X$ be the open subspace used in the construction of Z', i.e., an open such that Z is identified with a closed subspace of U cut out by the quasi-coherent sheaf of ideals $\mathcal{I}$. Since $|T| = |T'|$ we see that $|b|(|T'|) \subset |U|$. Hence we can think of b as a morphism into U, see Properties of Spaces, Lemma 48.4.9. Let $\mathcal{J} \subset \mathcal{O}_{T'}$ be the square zero quasi-coherent sheaf of ideals cutting out T. By the commutativity of the diagram we have $b|_T = i \circ a$ where $i : Z \to U$ is the closed immersion. We conclude that $b^\sharp(b^{-1}\mathcal{I}) \subset \mathcal{J}$ by Morphisms of Spaces, Lemma 49.13.1. As T' is a first order thickening of T we see that $\mathcal{J}^2 = 0$ hence $b^\sharp(b^{-1}(\mathcal{I}^2)) = 0$. By Morphisms of Spaces, Lemma 49.13.1 this implies that b factors through Z'. Letting $a' : T' \to Z'$ be this factorization we win. □

Lemma 57.9.3. *Let $i : Z \to X$ be an immersion of algebraic spaces. Let $Z \subset Z'$ be the first order infinitesimal neighbourhood of Z in X. Then the diagram*

$$\begin{array}{ccc} Z & \longrightarrow & Z' \\ \downarrow & & \downarrow \\ Z & \longrightarrow & X \end{array}$$

induces a map of conormal sheaves $\mathcal{C}_{Z/X} \to \mathcal{C}_{Z/Z'}$ by Lemma 57.4.3. This map is an isomorphism.

Proof. This is clear from the construction of Z' above. □

57.10. Formally smooth, étale, unramified transformations

Recall that a ring map $R \to A$ is called *formally smooth*, resp. *formally étale*, resp. *formally unramified* (see Algebra, Definition 10.131.1, resp. Definition 10.141.1, resp. Definition 10.139.1) if for every commutative solid diagram

$$\begin{array}{ccc} A & \longrightarrow & B/I \\ \uparrow & \nearrow & \uparrow \\ R & \longrightarrow & B \end{array}$$

where $I \subset B$ is an ideal of square zero, there exists a, resp. exists a unique, resp. exists at most one dotted arrow which makes the diagram commute. This motivates the following analogue for morphisms of algebraic spaces, and more generally functors.

Definition 57.10.1. Let S be a scheme. Let $a : F \to G$ be a transformation of functors $F, G : (Sch/S)^{opp}_{fppf} \to Sets$. Consider commutative solid diagrams of the form

$$\begin{array}{ccc} F & \longleftarrow & T \\ {\scriptstyle a}\downarrow & \nwarrow & \downarrow{\scriptstyle i} \\ G & \longleftarrow & T' \end{array}$$

where T and T' are affine schemes and i is a closed immersion defined by an ideal of square zero.

(1) We say a is *formally smooth* if given any solid diagram as above there exists a dotted arrow making the diagram commute[1].
(2) We say a is *formally étale* if given any solid diagram as above there exists exactly one dotted arrow making the diagram commute.
(3) We say a is *formally unramified* if given any solid diagram as above there exists at most one dotted arrow making the diagram commute.

Lemma 57.10.2. *Let S be a scheme. Let $a : F \to G$ be a transformation of functors $F, G : (Sch/S)^{opp}_{fppf} \to Sets$. Then a is formally étale if and only if a is both formally smooth and formally unramified.*

Proof. Formal from the definition. □

Lemma 57.10.3. *Composition.*

(1) *A composition of formally smooth transformations of functors is formally smooth.*
(2) *A composition of formally étale transformations of functors is formally étale.*
(3) *A composition of formally unramified transformations of functors is formally unramified.*

Proof. This is formal. □

Lemma 57.10.4. *Let S be a scheme contained in Sch_{fppf}. Let $F, G, H : (Sch/S)^{opp}_{fppf} \to Sets$. Let $a : F \to G$, $b : H \to G$ be transformations of functors. Consider the fibre product diagram*

$$\begin{array}{ccc} H \times_{b,G,a} F & \xrightarrow{b'} & F \\ {\scriptstyle a'}\downarrow & & \downarrow{\scriptstyle a} \\ H & \xrightarrow{b} & G \end{array}$$

(1) *If a is formally smooth, then the base change a' is formally smooth.*
(2) *If a is formally étale, then the base change a' is formally étale.*
(3) *If a is formally unramified, then the base change a' is formally unramified.*

Proof. This is formal. □

Lemma 57.10.5. *Let S be a scheme. Let $F, G : (Sch/S)^{opp}_{fppf} \to Sets$. Let $a : F \to G$ be a representable transformation of functors.*

(1) *If a is smooth then a is formally smooth.*
(2) *If a is étale, then a is formally étale.*
(3) *If a is unramified, then a is formally unramified.*

Proof. Consider a solid commutative diagram

$$\begin{array}{ccc} F & \longleftarrow & T \\ {\scriptstyle a}\downarrow & \nwarrow & \downarrow{\scriptstyle i} \\ G & \longleftarrow & T' \end{array}$$

[1]This is just one possible definition that one can make here. Another slightly weaker condition would be to require that the dotted arrow exists fppf locally on T'. This weaker notion has in some sense better formal properties.

as in Definition 57.10.1. Then $F \times_G T'$ is a scheme smooth (resp. étale, resp. unramified) over T'. Hence by More on Morphisms, Lemma 36.9.7 (resp. Lemma 36.6.9, resp. Lemma 36.4.8) we can fill in (resp. uniquely fill in, resp. fill in in at most one way) the dotted arrow in the diagram

$$\begin{array}{ccc} F \times_G T' & \longleftarrow & T \\ \downarrow & \nearrow & \downarrow i \\ T' & \longleftarrow & T' \end{array}$$

an hence we also obtain the corresponding assertion in the first diagram. □

Lemma 57.10.6. *Let S be a scheme contained in Sch_{fppf}. Let $F, G, H : (Sch/S)^{opp}_{fppf} \to Sets$. Let $a : F \to G$, $b : G \to H$ be transformations of functors. Assume that a is representable, surjective, and étale.*

(1) *If b is formally smooth, then $b \circ a$ is formally smooth.*
(2) *If b is formally étale, then $b \circ a$ is formally étale.*
(3) *If b is formally unramified, then $b \circ a$ is formally unramified.*

Conversely, consider a solid commutative diagram

$$\begin{array}{ccc} G & \longleftarrow & T \\ {\scriptstyle b}\downarrow & \nearrow & \downarrow{\scriptstyle i} \\ H & \longleftarrow & T' \end{array}$$

with T' an affine scheme over S and $i : T \to T'$ a closed immersion defined by an ideal of square zero.

(4) *If $b \circ a$ is formally smooth, then for every $t \in T$ there exists an étale morphism of affines $U' \to T'$ and a morphism $U' \to G$ such that*

$$\begin{array}{ccccc} G & \longleftarrow & T & \longleftarrow & T \times_{T'} U' \\ {\scriptstyle b}\downarrow & \swarrow & & & \downarrow \\ H & \longleftarrow & T' & \longleftarrow & U' \end{array}$$

commutes and t is in the image of $U' \to T'$.
(5) *If $b \circ a$ is formally unramified, then there exists at most one dotted arrow in the diagram above, i.e., b is formally unramified.*
(6) *If $b \circ a$ is formally étale, then there exists exactly one dotted arrow in the diagram above, i.e., b is formally étale.*

Proof. Assume b is formally smooth (resp. formally étale, resp. formally unramified). Since an étale morphism is both smooth and unramified we see that a is representable and smooth (resp. étale, resp. unramified). Hence parts (1), (2) and (3) follow from a combination of Lemma 57.10.5 and Lemma 57.10.3.

Assume that $b \circ a$ is formally smooth. Consider a diagram as in the statement of the lemma. Let $W = F \times_G T$. By assumption W is a scheme surjective étale over T. By Étale Morphisms, Theorem 40.15.2 there exists a scheme W' étale over T' such that $W = T \times_{T'} W'$. Choose an affine open subscheme $U' \subset W'$ such that t

is in the image of $U' \to T'$. Because $b \circ a$ is formally smooth we see that the exist morphisms $U' \to F$ such that

$$\begin{array}{ccccc} F & \longleftarrow & W & \longleftarrow & T \times_{T'} U' \\ {\scriptstyle b\circ a}\downarrow & \nwarrow & & & \downarrow \\ H & \longleftarrow & T' & \longleftarrow & U' \end{array}$$

commutes. Taking the composition $U' \to F \to G$ gives a map as in part (5) of the lemma.

Assume that $f, g : T' \to G$ are two dotted arrows fitting into the diagram of the lemma. Let $W = F \times_G T$. By assumption W is a scheme surjective étale over T. By Étale Morphisms, Theorem 40.15.2 there exists a scheme W' étale over T' such that $W = T \times_{T'} W'$. Since a is formally étale the compositions

$$W' \to T' \xrightarrow{f} G \quad \text{and} \quad W' \to T' \xrightarrow{g} G$$

lift to morphisms $f', g' : W' \to F$ (lift on affine opens and glue by uniqueness). Now if $b \circ a : F \to H$ is formally unramified, then $f' = g'$ and hence $f = g$ as $W' \to T'$ is an étale covering. This proves part (6) of the lemma.

Assume that $b \circ a$ is formally étale. Then by part (4) we can étale locally on T' find a dotted arrow fitting into the diagram and by part (5) this dotted arrow is unique. Hence we may glue the local solutions to get assertion (6). Some details omitted. □

Remark 57.10.7. It is tempting to think that in the situation of Lemma 57.10.6 we have "b formally smooth" $\Leftrightarrow$ "$b \circ a$ formally smooth". However, this is likely not true in general.

Lemma 57.10.8. *Let S be a scheme. Let $F, G, H : (Sch/S)^{opp}_{fppf} \to Sets$. Let $a : F \to G$, $b : G \to H$ be transformations of functors. Assume b is formally unramified.*

(1) *If $b \circ a$ is formally unramified then a is formally unramified.*
(2) *If $b \circ a$ is formally étale then a is formally étale.*
(3) *If $b \circ a$ is formally smooth then a is formally smooth.*

Proof. Let $T \subset T'$ be a closed immersion of affine schemes defined by an ideal of square zero. Let $g' : T' \to G$ and $f : T \to F$ be given such that $g'|_T = a \circ f$. Because b is formally unramified, there is a one to one correspondence between

$$\{f' : T' \to F \mid f = f'|_T \text{ and } a \circ f' = g'\}$$

and

$$\{f' : T' \to F \mid f = f'|_T \text{ and } b \circ a \circ f' = b \circ g'\}.$$

From this the lemma follows formally. □

57.11. Formally unramified morphisms

In this section we work out what it means that a morphism of algebraic spaces is formally unramified.

Definition 57.11.1. Let S be a scheme. A morphism $f : X \to Y$ of algebraic spaces over S is said to be *formally unramified* if it is formally unramified as a transformation of functors as in Definition 57.10.1.

We will not restate the results proved in the more general setting of formally unramified transformations of functors in Section 57.10. It turns out we can characterize this property in terms of vanishing of the module of relative differentials, see Lemma 57.11.6.

Lemma 57.11.2. *Let S be a scheme. Let $f : X \to Y$ be a morphism of algebraic spaces over S. The following are equivalent:*

(1) *f is formally unramified,*
(2) *for every diagram*

$$\begin{array}{ccc} U & \xrightarrow{\psi} & V \\ \downarrow & & \downarrow \\ X & \xrightarrow{f} & Y \end{array}$$

where U and V are schemes and the vertical arrows are étale the morphism of schemes ψ is formally unramified (as in More on Morphisms, Definition 36.4.1), and
(3) *for one such diagram with surjective vertical arrows the morphism ψ is formally unramified.*

Proof. Assume f is formally unramified. By Lemma 57.10.5 the morphisms $U \to X$ and $V \to Y$ are formally unramified. Thus by Lemma 57.10.3 the composition $U \to Y$ is formally unramified. Then it follows from Lemma 57.10.8 that $U \to V$ is formally unramified. Thus (1) implies (2). And (2) implies (3) trivially

Assume given a diagram as in (3). By Lemma 57.10.5 the morphism $V \to Y$ is formally unramified. Thus by Lemma 57.10.3 the composition $U \to Y$ is formally unramified. Then it follows from Lemma 57.10.6 that $X \to Y$ is formally unramified, i.e., (1) holds. □

Lemma 57.11.3. *Let S be a scheme. If $f : X \to Y$ is a formally unramified morphism of algebraic spaces over S, then given any solid commutative diagram*

$$\begin{array}{ccc} X & \longleftarrow & T \\ {\scriptstyle f}\downarrow & \nwarrow & \downarrow{\scriptstyle i} \\ S & \longleftarrow & T' \end{array}$$

where $T \subset T'$ is a first order thickening of algebraic spaces over S there exists at most one dotted arrow making the diagram commute. In other words, in Definition 57.11.1 the condition that T be an affine scheme may be dropped.

Proof. This is true because there exists a surjective étale morphism $U' \to T'$ where U' is a disjoint union of affine schemes (see Properties of Spaces, Lemma 48.6.1) and a morphism $T' \to X$ is determined by its restriction to U'. □

Lemma 57.11.4. *A composition of formally unramified morphisms is formally unramified.*

Proof. This is formal. □

Lemma 57.11.5. *A base change of a formally unramified morphism is formally unramified.*

Proof. This is formal. □

Lemma 57.11.6. *Let S be a scheme. Let $f : X \to Y$ be a morphism of algebraic spaces over S. The following are equivalent:*

(1) *f is formally unramified, and*
(2) *$\Omega_{X/Y} = 0$.*

Proof. This is a combination of Lemma 57.11.2, More on Morphisms, Lemma 36.4.7, and Lemma 57.6.3. □

Lemma 57.11.7. *Let S be a scheme. Let $f : X \to Y$ be a morphism of algebraic spaces over S. The following are equivalent:*

(1) *The morphism f is unramified,*
(2) *the morphism f is locally of finite type and $\Omega_{X/Y} = 0$, and*
(3) *the morphism f is locally of finite type and formally unramified.*

Proof. Choose a diagram

$$\begin{array}{ccc} U & \xrightarrow{\psi} & V \\ \downarrow & & \downarrow \\ X & \xrightarrow{f} & Y \end{array}$$

where U and V are schemes and the vertical arrows are étale and surjective. Then we see

$$\begin{aligned} f \text{ unramified} &\Leftrightarrow \psi \text{ unramified} \\ &\Leftrightarrow \psi \text{ locally finite type and } \Omega_{U/V} = 0 \\ &\Leftrightarrow f \text{ locally finite type and } \Omega_{X/Y} = 0 \\ &\Leftrightarrow f \text{ locally finite type and formally unramified} \end{aligned}$$

Here we have used Morphisms, Lemma 28.36.2 and Lemma 57.11.6. □

Lemma 57.11.8. *Let S be a scheme. Let $f : X \to Y$ be a morphism of algebraic spaces over S. The following are equivalent:*

(1) *f is unramified and a monomorphism,*
(2) *f is unramified and universally injective,*
(3) *f is locally of finite type and a monomorphism,*
(4) *f is universally injective, locally of finite type, and formally unramified.*

Moreover, in this case f is also representable, separated, and locally quasi-finite.

Proof. We have seen in Lemma 57.11.7 that being formally unramified and locally of finite type is the same thing as being unramified. Hence (4) is equivalent to (2). A monomorphism is certainly formally unramified hence (3) implies (4). It is clear that (1) implies (3). Finally, if (2) holds, then $\Delta : X \to X \times_Y X$ is both an open immersion (Morphisms of Spaces, Lemma 49.34.9) and surjective (Morphisms of Spaces, Lemma 49.18.2) hence an isomorphism, i.e., f is a monomorphism. In this way we see that (2) implies (1). Finally, we see that f is representable, separated, and locally quasi-finite by Morphisms of Spaces, Lemmas 49.25.8 and 49.44.1. □

Lemma 57.11.9. *Let S be a scheme. Let $f : X \to Y$ be a morphism of algebraic spaces over S. The following are equivalent:*

(1) *f is a closed immersion,*
(2) *f is universally closed, unramified, and a monomorphism,*
(3) *f is universally closed, unramified, and universally injective,*

(4) *f is universally closed, locally of finite type, and a monomorphism,*
(5) *f is universally closed, universally injective, locally of finite type, and formally unramified.*

Proof. The equivalence of (2) – (5) follows immediately from Lemma 57.11.8. Moreover, if (2) – (5) are satisfied then f is representable. Similarly, if (1) is satisfied then f is representable. Hence the result follows from the case of schemes, see Étale Morphisms, Lemma 40.7.2. □

57.12. Universal first order thickenings

Let S be a scheme. Let $h : Z \to X$ be a morphism of algebraic spaces over S. A *universal first order thickening* of Z over X is a first order thickening $Z \subset Z'$ over X such that given any first order thickening $T \subset T'$ over X and a solid commutative diagram

(57.12.0.1)
$$\begin{array}{ccccc} Z & \xleftarrow{a} & & & T \\ & \searrow & & \swarrow & \\ & & Z' & & T' \\ & & & \searrow \quad \swarrow & \\ & & & X & \end{array}$$

there exists a unique dotted arrow making the diagram commute. Note that in this situation $(a, a') : (T \subset T') \to (Z \subset Z')$ is a morphism of thickenings over X. Thus if a universal first order thickening exists, then it is unique up to unique isomorphism. In general a universal first order thickening does not exist, but if h is formally unramified then it does. Before we prove this, let us show that a universal first order thickening in the category of schemes is a universal first order thickening in the category of algebraic spaces.

Lemma 57.12.1. *Let S be a scheme. Let $h : Z \to X$ be a morphism of algebraic spaces over S. Let $Z \subset Z'$ be a first order thickening over X. The following are equivalent*

(1) *$Z \subset Z'$ is a universal first order thickening,*
(2) *for any diagram (57.12.0.1) with T' a scheme a unique dotted arrow exists making the diagram commute, and*
(3) *for any diagram (57.12.0.1) with T' an affine scheme a unique dotted arrow exists making the diagram commute.*

Proof. The implications (1) $\Rightarrow$ (2) $\Rightarrow$ (3) are formal. Assume (3) a assume given an arbitrary diagram (57.12.0.1). Choose a presentation $T' = U'/R'$, see Spaces, Definition 47.9.3. We may assume that $U' = \coprod U'_i$ is a disjoint union of affines, so $R' = U' \times_{T'} U' = \coprod_{i,j} U'_i \times'_T U'_j$. For each pair (i, j) choose an affine open covering $U'_i \times'_T U'_j = \bigcup_k R'_{ijk}$. Denote U_i, R_{ijk} the fibre products with T over T'. Then each $U_i \subset U'_i$ and $R_{ijk} \subset R'_{ijk}$ is a first order thickening of affine schemes. Denote $a_i : U_i \to Z$, resp. $a_{ijk} : R_{ijk} \to Z$ the composition of $a : T \to Z$ with the morphism $U_i \to T$, resp. $R_{ijk} \to T$. By (3) applied to $a_i : U_i \to Z$ we obtain unique morphisms $a'_i : U'_i \to Z'$. By (3) applied to a_{ijk} we see that the two compositions $R'_{ijk} \to R'_i \to Z'$ and $R'_{ijk} \to R'_j \to Z'$ are equal. Hence

$a' = \coprod a'_i : U' = \coprod U'_i \to Z'$ descends to the quotient sheaf $T' = U'/R'$ and we win. □

Lemma 57.12.2. *Let S be a scheme. Let $Z \to Y \to X$ be morphisms of algebraic spaces over S. If $Z \subset Z'$ is a universal first order thickening of Z over Y and $Y \to X$ is formally étale, then $Z \subset Z'$ is a universal first order thickening of Z over X.*

Proof. This is formal. Namely, by Lemma 57.12.1 it suffices to consider solid commutative diagrams (57.12.0.1) with T' an affine scheme. The composition $T \to Z \to Y$ lifts uniquely to $T' \to Y$ as $Y \to X$ is assumed formally étale. Hence the fact that $Z \subset Z'$ is a universal first order thickening over Y produces the desired morphism $a' : T' \to Z'$. □

Lemma 57.12.3. *Let S be a scheme. Let $Z \to Y \to X$ be morphisms of algebraic spaces over S. Assume $Z \to Y$ is étale.*

1. *If $Y \subset Y'$ is a universal first order thickening of Y over X, then the unique étale morphism $Z' \to Y'$ such that $Z = Y \times_{Y'} Z'$ (see Theorem 57.7.1) is a universal first order thickening of Z over X.*
2. *If $Z \to Y$ is surjective and $(Z \subset Z') \to (Y \subset Y')$ is an étale morphism of first order thickenings over X and Z' is a universal first order thickening of Z over X, then Y' is a universal first order thickening of Y over X.*

Proof. Proof of (1). By Lemma 57.12.1 it suffices to consider solid commutative diagrams (57.12.0.1) with T' an affine scheme. The composition $T \to Z \to Y$ lifts uniquely to $T' \to Y'$ as Y' is the universal first order thickening. Then the fact that $Z' \to Y'$ is étale implies (see Lemma 57.10.5) that $T' \to Y'$ lifts to the desired morphism $a' : T' \to Z'$.

Proof of (2). Let $T \subset T'$ be a first order thickening over X and let $a : T \to Y$ be a morphism. Set $W = T \times_Y Z$ and denote $c : W \to Z$ the projection Let $W' \to T'$ be the unique étale morphism such that $W = T \times_{T'} W'$, see Theorem 57.7.1. Note that $W' \to T'$ is surjective as $Z \to Y$ is surjective. By assumption we obtain a unique morphism $c' : W' \to Z'$ over X restricting to c on W. By uniqueness the two restrictions of c' to $W' \times_{T'} W'$ are equal (as the two restrictions of c to $W \times_T W$ are equal). Hence c' descends to a unique morphism $a' : T' \to Y'$ and we win. □

Lemma 57.12.4. *Let S be a scheme. Let $h : Z \to X$ be a formally unramified morphism of algebraic spaces over S. There exists a universal first order thickening $Z \subset Z'$ of Z over X.*

Proof. Choose any commutative diagram

$$\begin{array}{ccc} V & \longrightarrow & U \\ \downarrow & & \downarrow \\ Z & \longrightarrow & X \end{array}$$

where V and U are schemes and the vertical arrows are étale. Note that $V \to U$ is a formally unramified morphism of schemes, see Lemma 57.11.2. Combining Lemma 57.12.1 and More on Morphisms, Lemma 36.5.1 we see that a universal first order thickening $V \subset V'$ of V over U exists. By Lemma 57.12.2 part (1) V' is a universal first order thickening of V over X.

Fix a scheme U and a surjective étale morphism $U \to X$. The argument above shows that for any $V \to Z$ étale with V a scheme such that $V \to Z \to X$ factors through U a universal first order thickening $V \subset V'$ of V over X exists (but does not depend on the chosen factorization of $V \to X$ through U). Now we may choose V such that $V \to Z$ is surjective étale (see Spaces, Lemma 47.11.4). Then $R = V \times_Z V$ a scheme étale over Z such that $R \to X$ factors through U also. Hence we obtain universal first order thickenings $V \subset V'$ and $R \subset R'$ over X. As $V \subset V'$ is a universal first order thickening, the two projections $s, t : R \to V$ lift to morphisms $s', t' : R' \to V'$. By Lemma 57.12.3 as R' is the universal first order thickening of R over X these morphisms are étale. Then $(t', s') : R' \to V'$ is an étale equivalence relation and we can set $Z' = V'/R'$. Since $V' \to Z'$ is surjective étale and v' is the universal first order thickening of V over X we conclude from Lemma 57.12.2 part (2) that Z' is a universal first order thickening of Z over X. □

Definition 57.12.5. Let S be a scheme. Let $h : Z \to X$ be a formally unramified morphism of algebraic spaces over S.

(1) The *universal first order thickening* of Z over X is the thickening $Z \subset Z'$ constructed in Lemma 57.12.4.
(2) The *conormal sheaf of Z over X* is the conormal sheaf of Z in its universal first order thickening Z' over X.

We often denote the conormal sheaf $\mathcal{C}_{Z/X}$ in this situation.

Thus we see that there is a short exact sequence of sheaves

$$0 \to \mathcal{C}_{Z/X} \to \mathcal{O}_{Z'} \to \mathcal{O}_Z \to 0$$

on $Z_{étale}$ and $\mathcal{C}_{Z/X}$ is a quasi-coherent $\mathcal{O}_Z$-module. The following lemma proves that there is no conflict between this definition and the definition in case $Z \to X$ is an immersion.

Lemma 57.12.6. *Let S be a scheme. Let $i : Z \to X$ be an immersion of algebraic spaces over S. Then*

(1) *i is formally unramified,*
(2) *the universal first order thickening of Z over X is the first order infinitesimal neighbourhood of Z in X of Definition 57.9.1,*
(3) *the conormal sheaf of i in the sense of Definition 57.4.1 agrees with the conormal sheaf of i in the sense of Definition 57.12.5.*

Proof. An immersion of algebraic spaces is by definition a representable morphism. Hence by Morphisms, Lemmas 28.36.7 and 28.36.8 an immersion is unramified (via the abstract principle of Spaces, Lemma 47.5.8). Hence it is formally unramified by Lemma 57.11.7. The other assertions follow by combining Lemmas 57.9.2 and 57.9.3 and the definitions. □

Lemma 57.12.7. *Let S be a scheme. Let $Z \to X$ be a formally unramified morphism of algebraic spaces over S. Then the universal first order thickening Z' is formally unramified over X.*

Proof. Let $T \subset T'$ be a first order thickening of affine schemes over X. Let

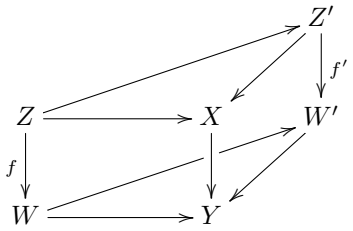
$$\begin{array}{ccc} Z' & \xleftarrow{c} & T \\ \downarrow & \nwarrow^{a,b} & \downarrow \\ X & \longleftarrow & T' \end{array}$$

be a commutative diagram. Set $T_0 = c^{-1}(Z) \subset T$ and $T'_a = a^{-1}(Z)$ (scheme theoretically). Since Z' is a first order thickening of Z, we see that T' is a first order thickening of T'_a. Moreover, since $c = a|_T$ we see that $T_0 = T \cap T'_a$ (scheme theoretically). As T' is a first order thickening of T it follows that T'_a is a first order thickening of T_0. Now $a|_{T'_a}$ and $b|_{T'_a}$ are morphisms of T'_a into Z' over X which agree on T_0 as morphisms into Z. Hence by the universal property of Z' we conclude that $a|_{T'_a} = b|_{T'_a}$. Thus a and b are morphism from the first order thickening T' of T'_a whose restrictions to T'_a agree as morphisms into Z. Thus using the universal property of Z' once more we conclude that $a = b$. In other words, the defining property of a formally unramified morphism holds for $Z' \to X$ as desired. □

Lemma 57.12.8. *Let S be a scheme Consider a commutative diagram of algebraic spaces over S*

$$\begin{array}{ccc} Z & \xrightarrow{h} & X \\ {\scriptstyle f}\downarrow & & \downarrow{\scriptstyle g} \\ W & \xrightarrow{h'} & Y \end{array}$$

with h and h' formally unramified. Let $Z \subset Z'$ be the universal first order thickening of Z over X. Let $W \subset W'$ be the universal first order thickening of W over Y. There exists a canonical morphism $(f, f') : (Z, Z') \to (W, W')$ of thickenings over Y which fits into the following commutative diagram



In particular the morphism (f, f') of thickenings induces a morphism of conormal sheaves $f^*\mathcal{C}_{W/Y} \to \mathcal{C}_{Z/X}$.

Proof. The first assertion is clear from the universal property of W'. The induced map on conormal sheaves is the map of Lemma 57.4.3 applied to $(Z \subset Z') \to (W \subset W')$. □

Lemma 57.12.9. *Let S be a scheme. Let*

$$\begin{array}{ccc} Z & \xrightarrow{h} & X \\ {\scriptstyle f}\downarrow & & \downarrow{\scriptstyle g} \\ W & \xrightarrow{h'} & Y \end{array}$$

be a fibre product diagram of algebraic spaces over S with h' formally unramified. Then h is formally unramified and if $W \subset W'$ is the universal first order thickening

of W over Y, then $Z = X \times_Y W \subset X \times_Y W'$ is the universal first order thickening of Z over X. In particular the canonical map $f^*\mathcal{C}_{W/Y} \to \mathcal{C}_{Z/X}$ of Lemma 57.12.8 is surjective.

Proof. The morphism h is formally unramified by Lemma 57.11.5. It is clear that $X \times_Y W'$ is a first order thickening. It is straightforward to check that it has the universal property because W' has the universal property (by mapping properties of fibre products). See Lemma 57.4.5 for why this implies that the map of conormal sheaves is surjective. □

Lemma 57.12.10. *Let S be a scheme. Let*

$$\begin{array}{ccc} Z & \xrightarrow{h} & X \\ {\scriptstyle f}\downarrow & & \downarrow{\scriptstyle g} \\ W & \xrightarrow{h'} & Y \end{array}$$

be a fibre product diagram of algebraic spaces over S with h' formally unramified and g flat. In this case the corresponding map $Z' \to W'$ of universal first order thickenings is flat, and $f^*\mathcal{C}_{W/Y} \to \mathcal{C}_{Z/X}$ is an isomorphism.

Proof. Flatness is preserved under base change, see Morphisms of Spaces, Lemma 49.27.4. Hence the first statement follows from the description of W' in Lemma 57.12.9. It is clear that $X \times_Y W'$ is a first order thickening. It is straightforward to check that it has the universal property because W' has the universal property (by mapping properties of fibre products). See Lemma 57.4.5 for why this implies that the map of conormal sheaves is an isomorphism. □

Lemma 57.12.11. *Taking the universal first order thickenings commutes with étale localization. More precisely, let $h : Z \to X$ be a formally unramified morphism of algebraic spaces over a base scheme S. Let*

$$\begin{array}{ccc} V & \longrightarrow & U \\ \downarrow & & \downarrow \\ Z & \longrightarrow & X \end{array}$$

be a commutative diagram with étale vertical arrows. Let Z' be the universal first order thickening of Z over X. Then $V \to U$ is formally unramified and the universal first order thickening V' of V over U is étale over Z'. In particular, $\mathcal{C}_{Z/X}|_V = \mathcal{C}_{V/U}$.

Proof. The first statement is Lemma 57.11.2. The compatibility of universal first order thickenings is a consequence of Lemmas 57.12.2 and 57.12.3. □

Lemma 57.12.12. *Let S be a scheme. Let B be an algebraic space over S. Let $h : Z \to X$ be a formally unramified morphism of algebraic spaces over B. Let $Z \subset Z'$ be the universal first order thickening of Z over X with structure morphism $h' : Z' \to X$. The canonical map*

$$dh' : (h')^*\Omega_{X/B} \to \Omega_{Z'/B}$$

induces an isomorphism $h^*\Omega_{X/B} \to \Omega_{Z'/B} \otimes \mathcal{O}_Z$.

Proof. The map $c_{h'}$ is the map defined in Lemma 57.6.6. If $i : Z \to Z'$ is the given closed immersion, then $i^*c_{h'}$ is a map $h^*\Omega_{X/S} \to \Omega_{Z'/S} \otimes \mathcal{O}_Z$. Checking that it is an isomorphism reduces to the case of schemes by étale localization, see Lemma 57.12.11 and Lemma 57.6.3. In this case the result is More on Morphisms, Lemma 36.5.9. □

Lemma 57.12.13. *Let S be a scheme. Let B be an algebraic space over S. Let $h : Z \to X$ be a formally unramified morphism of algebraic spaces over B. There is a canonical exact sequence*

$$\mathcal{C}_{Z/X} \to h^*\Omega_{X/B} \to \Omega_{Z/B} \to 0.$$

The first arrow is induced by $d_{Z'/B}$ where Z' is the universal first order neighbourhood of Z over X.

Proof. We know that there is a canonical exact sequence

$$\mathcal{C}_{Z/Z'} \to \Omega_{Z'/S} \otimes \mathcal{O}_Z \to \Omega_{Z/S} \to 0.$$

see Lemma 57.6.10. Hence the result follows on applying Lemma 57.12.12. □

Lemma 57.12.14. *Let S be a scheme. Let*

$$\begin{array}{ccc} Z & \xrightarrow{i} & X \\ & \searrow^{j} & \downarrow \\ & & Y \end{array}$$

be a commutative diagram of algebraic spaces over S where i and j are formally unramified. Then there is a canonical exact sequence

$$\mathcal{C}_{Z/Y} \to \mathcal{C}_{Z/X} \to i^*\Omega_{X/Y} \to 0$$

where the first arrow comes from Lemma 57.12.8 and the second from Lemma 57.12.13.

Proof. Since the maps have been defined, checking the sequence is exact reduces to the case of schemes by étale localization, see Lemma 57.12.11 and Lemma 57.6.3. In this case the result is More on Morphisms, Lemma 36.5.11. □

Lemma 57.12.15. *Let S be a scheme. Let $Z \to Y \to X$ be formally unramified morphisms of algebraic spaces over S.*

(1) *If $Z \subset Z'$ is the universal first order thickening of Z over X and $Y \subset Y'$ is the universal first order thickening of Y over X, then there is a morphism $Z' \to Y'$ and $Y \times_{Y'} Z'$ is the universal first order thickening of Z over Y.*
(2) *There is a canonical exact sequence*

$$i^*\mathcal{C}_{Y/X} \to \mathcal{C}_{Z/X} \to \mathcal{C}_{Z/Y} \to 0$$

where the maps come from Lemma 57.12.8 and $i : Z \to Y$ is the first morphism.

Proof. The map $h : Z' \to Y'$ in (1) comes from Lemma 57.12.8. The assertion that $Y \times_{Y'} Z'$ is the universal first order thickening of Z over Y is clear from the universal properties of Z' and Y'. By Lemma 57.4.6 we have an exact sequence

$$(i')^*\mathcal{C}_{Y \times_{Y'} Z'/Z'} \to \mathcal{C}_{Z/Z'} \to \mathcal{C}_{Z/Y \times_{Y'} Z'} \to 0$$

where $i' : Z \to Y \times_{Y'} Z'$ is the given morphism. By Lemma 57.4.5 there exists a surjection $h^*\mathcal{C}_{Y/Y'} \to \mathcal{C}_{Y \times_{Y'} Z'/Z'}$. Combined with the equalities $\mathcal{C}_{Y/Y'} = \mathcal{C}_{Y/X}$, $\mathcal{C}_{Z/Z'} = \mathcal{C}_{Z/X}$, and $\mathcal{C}_{Z/Y \times_{Y'} Z'} = \mathcal{C}_{Z/Y}$ this proves the lemma. □

57.13. Formally étale morphisms

In this section we work out what it means that a morphism of algebraic spaces is formally étale.

Definition 57.13.1. Let S be a scheme. A morphism $f : X \to Y$ of algebraic spaces over S is said to be *formally étale* if it is formally étale as a transformation of functors as in Definition 57.10.1.

We will not restate the results proved in the more general setting of formally étale transformations of functors in Section 57.10.

Lemma 57.13.2. *Let S be a scheme. Let $f : X \to Y$ be a morphism of algebraic spaces over S. The following are equivalent:*

(1) *f is formally étale,*
(2) *for every diagram*

$$\begin{array}{ccc} U & \xrightarrow{\psi} & V \\ \downarrow & & \downarrow \\ X & \xrightarrow{f} & Y \end{array}$$

where U and V are schemes and the vertical arrows are étale the morphism of schemes ψ is formally étale (as in More on Morphisms, Definition 36.6.1), and
(3) *for one such diagram with surjective vertical arrows the morphism ψ is formally étale.*

Proof. Assume f is formally étale. By Lemma 57.10.5 the morphisms $U \to X$ and $V \to Y$ are formally étale. Thus by Lemma 57.10.3 the composition $U \to Y$ is formally étale. Then it follows from Lemma 57.10.8 that $U \to V$ is formally étale. Thus (1) implies (2). And (2) implies (3) trivially

Assume given a diagram as in (3). By Lemma 57.10.5 the morphism $V \to Y$ is formally étale. Thus by Lemma 57.10.3 the composition $U \to Y$ is formally étale. Then it follows from Lemma 57.10.6 that $X \to Y$ is formally étale, i.e., (1) holds. □

Lemma 57.13.3. *Let S be a scheme. Let $f : X \to Y$ be a formally étale morphism of algebraic spaces over S. Then given any solid commutative diagram*

$$\begin{array}{ccc} X & \xleftarrow{a} & T \\ {\scriptstyle f}\downarrow & \nwarrow & \downarrow{\scriptstyle i} \\ Y & \xleftarrow{} & T' \end{array}$$

where $T \subset T'$ is a first order thickening of algebraic spaces over Y there exists exactly one dotted arrow making the diagram commute. In other words, in Definition 57.13.1 the condition that T be affine may be dropped.

Proof. Let $U' \to T'$ be a surjective étale morphism where $U' = \coprod U'_i$ is a disjoint union of affine schemes. Let $U_i = T \times_{T'} U'_i$. Then we get morphisms $a'_i : U'_i \to X$ such that $a'_i|_{U_i}$ equals the composition $U_i \to T \to X$. By uniqueness (see Lemma 57.11.3) we see that a'_i and a'_j agree on the fibre product $U'_i \times_{T'} U'_j$. Hence $\coprod a'_i : U' \to X$ descends to give a unique morphism $a' : T' \to X$. □

Lemma 57.13.4. *A composition of formally étale morphisms is formally étale.*

Proof. This is formal. □

Lemma 57.13.5. *A base change of a formally étale morphism is formally étale.*

Proof. This is formal. □

Lemma 57.13.6. *Let S be a scheme. Let $f : X \to Y$ be a morphism of algebraic spaces over S The following are equivalent:*

1. *f is formally étale,*
2. *f is formally unramified and the universal first order thickening of X over Y is equal to X,*
3. *f is formally unramified and $\mathcal{C}_{X/Y} = 0$, and*
4. *$\Omega_{X/Y} = 0$ and $\mathcal{C}_{X/Y} = 0$.*

Proof. Actually, the last assertion only make sense because $\Omega_{X/Y} = 0$ implies that $\mathcal{C}_{X/Y}$ is defined via Lemma 57.11.6 and Definition 57.12.5. This also makes it clear that (3) and (4) are equivalent.

Either of the assumptions (1), (2), and (3) imply that f is formally unramified. Hence we may assume f is formally unramified. The equivalence of (1), (2), and (3) follow from the universal property of the universal first order thickening X' of X over S and the fact that $X = X' \Leftrightarrow \mathcal{C}_{X/Y} = 0$ since after all by definition $\mathcal{C}_{X/Y} = \mathcal{C}_{X/X'}$ is the ideal sheaf of X in X'. □

Lemma 57.13.7. *An unramified flat morphism is formally étale.*

Proof. Follows from the case of schemes, see More on Morphisms, Lemma 36.6.7 and étale localization, see Lemmas 57.11.2 and 57.13.2 and Morphisms of Spaces, Lemma 49.27.5. □

Lemma 57.13.8. *Let S be a scheme. Let $f : X \to Y$ be a morphism of algebraic spaces over S. The following are equivalent:*

1. *The morphism f is étale, and*
2. *the morphism f is locally of finite presentation and formally étale.*

Proof. Follows from the case of schemes, see More on Morphisms, Lemma 36.6.9 and étale localization, see Lemma 57.13.2 and Morphisms of Spaces, Lemmas 49.26.4 and 49.35.2. □

57.14. Infinitesimal deformations of maps

In this section we explain how a derivation can be used to infinitesimally move a map. Throughout this section we use that a sheaf on a thickening X' of X can be seen as a sheaf on X, see Equations (57.8.1.1) and (57.8.1.2).

Lemma 57.14.1. *Let S be a scheme. Let B be an algebraic space over S. Let $X \subset X'$ and $Y \subset Y'$ be two first order thickenings of algebraic spaces over B. Let $(a, a'), (b, b') : (X \subset X') \to (Y \subset Y')$ be two morphisms of thickenings over B. Assume that*

(1) *$a = b$, and*
(2) *the two maps $a^*\mathcal{C}_{Y/Y'} \to \mathcal{C}_{X/X'}$ (Lemma 57.4.3) are equal.*

Then the map $(a')^\sharp - (b')^\sharp$ factors as

$$\mathcal{O}_{Y'} \to \mathcal{O}_Y \xrightarrow{D} a_*\mathcal{C}_{X/X'} \to a_*\mathcal{O}_{X'}$$

where D is an $\mathcal{O}_B$-derivation.

Proof. Instead of working on Y we work on X. The advantage is that the pullback functor a^{-1} is exact. Using (1) and (2) we obtain a commutive diagram with exact rows

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \mathcal{C}_{X/X'} & \longrightarrow & \mathcal{O}_{X'} & \longrightarrow & \mathcal{O}_X & \longrightarrow & 0 \\ & & \uparrow & & \uparrow{\scriptstyle (a')^\sharp} \uparrow{\scriptstyle (b')^\sharp} & & \uparrow & & \\ 0 & \longrightarrow & a^{-1}\mathcal{C}_{Y/Y'} & \longrightarrow & a^{-1}\mathcal{O}_{Y'} & \longrightarrow & a^{-1}\mathcal{O}_Y & \longrightarrow & 0 \end{array}$$

Now it is a general fact that in such a situation the difference of the $\mathcal{O}_B$-algebra maps $(a')^\sharp$ and $(b')^\sharp$ is an $\mathcal{O}_B$-derivation from $a^{-1}\mathcal{O}_Y$ to $\mathcal{C}_{X/X'}$. By adjointness of the functors a^{-1} and a_* this is the same thing as an $\mathcal{O}_B$-derivation from $\mathcal{O}_Y$ into $a_*\mathcal{C}_{X/X'}$. Some details omitted. □

Note that in the situation of the lemma above we may write D as

$$D = \mathrm{d}_{Y/B} \circ \theta \tag{57.14.1.1}$$

where θ is an $\mathcal{O}_Y$-linear map $\theta : \Omega_{Y/B} \to a_*\mathcal{C}_{X/X'}$. Of course, then by adjunction again we may view θ as an $\mathcal{O}_X$-linear map $\theta : a^*\Omega_{Y/B} \to \mathcal{C}_{X/X'}$.

Lemma 57.14.2. *Let S be a scheme. Let B be an algebraic space over S. Let $(a, a') : (X \subset X') \to (Y \subset Y')$ be a morphism of first order thickenings over B. Let*

$$\theta : a^*\Omega_{Y/B} \to \mathcal{C}_{X/X'}$$

be an $\mathcal{O}_X$-linear map. Then there exists a unique morphism of pairs $(b, b') : (X \subset X') \to (Y \subset Y')$ such that (1) and (2) of Lemma 57.14.1 hold and the derivation D and θ are related by Equation (57.14.1.1).

Proof. Consider the map

$$\alpha = (a')^\sharp + D : a^{-1}\mathcal{O}_{Y'} \to \mathcal{O}_{X'}$$

where D is as in Equation (57.14.1.1). As D is an $\mathcal{O}_B$-derivation it follows that α is a map of sheaves of $\mathcal{O}_B$-algebras. By construction we have $i_X^\sharp \circ \alpha = a^\sharp \circ i_Y^\sharp$ where $i_X : X \to X'$ and $i_Y : Y \to Y'$ are the given closed immersions. By Lemma 57.8.2 we obtain a unique morphism $(a, b') : (X \subset X') \to (Y \subset Y')$ of thickenings over B such that $\alpha = (b')^\sharp$. Setting $b = a$ we win. □

Lemma 57.14.3. *Let S be a scheme. Let B be an algebraic space over S. Let $X \subset X'$ and $Y \subset Y'$ be first order thickenings over B. Assume given a morphism $a : X \to Y$ and a map $A : a^*\mathcal{C}_{Y/Y'} \to \mathcal{C}_{X/X'}$ of $\mathcal{O}_X$-modules. For an object U' of $(X')_{spaces,étale}$ with $U = X \times_{X'} U'$ consider morphisms $a' : U' \to Y'$ such that*

(1) a' *is a morphism over* B,
(2) $a'|_U = a|_U$, *and*
(3) *the induced map* $a^*\mathcal{C}_{Y/Y'}|_U \to \mathcal{C}_{X/X'}|_U$ *is the restriction of* A *to* U.

Then the rule

$$U' \mapsto \{a' : U' \to Y' \text{ such that (1), (2), (3) hold.}\} \tag{57.14.3.1}$$

defines a sheaf of sets on $(X')_{spaces,étale}$.

Proof. Denote $\mathcal{F}$ the rule of the lemma. The restriction mapping $\mathcal{F}(U') \to \mathcal{F}(V')$ for $V' \subset U' \subset X'$ of $\mathcal{F}$ is really the restriction map $a' \mapsto a'|_{V'}$. With this definition in place it is clear that $\mathcal{F}$ is a sheaf since morphisms of algebraic spaces satisfy étale descent, see Descent on Spaces, Lemma 55.6.2. □

Lemma 57.14.4. *Same notation and assumptions as in Lemma 57.14.3. We identify sheaves on* X *and* X' *via (57.8.1.1). There is an action of the sheaf*

$$\mathcal{H}om_{\mathcal{O}_X}(a^*\Omega_{Y/B}, \mathcal{C}_{X/X'})$$

on the sheaf (57.14.3.1). Moreover, the action is simply transitive for any object U' *of* $(X')_{spaces,étale}$ *over which the sheaf (57.14.3.1) has a section.*

Proof. This is a combination of Lemmas 57.14.1, 57.14.2, and 57.14.3. □

Remark 57.14.5. A special case of Lemmas 57.14.1, 57.14.2, 57.14.3, and 57.14.4 is where $Y = Y'$. In this case the map A is always zero. The sheaf of Lemma 57.14.3 is just given by the rule

$$U' \mapsto \{a' : U' \to Y \text{ over } S \text{ with } a'|_U = a|_U\}$$

and we act on this by the sheaf $\mathcal{H}om_{\mathcal{O}_X}(a^*\Omega_{Y/B}, \mathcal{C}_{X/X'})$. The action of a local section θ on a' is sometimes indicated by $\theta \cdot a'$. Note that this means nothing else than the fact that $(a')^\sharp$ and $(\theta \cdot a')^\sharp$ differ by a derivation D which is related to θ by Equation (57.14.1.1).

57.15. Infinitesimal deformations of algebraic spaces

The following simple lemma is often a convenient tool to check whether an infinitesimal deformation of a map is flat.

Lemma 57.15.1. *Let* S *be a scheme. Let* $(f, f') : (X \subset X') \to (Y \subset Y')$ *be a morphism of first order thickenings of algebraic spaces over* S. *Assume that* f *is flat. Then the following are equivalent*

(1) f' *is flat and* $X = Y \times_{Y'} X'$, *and*
(2) *the canonical map* $f^*\mathcal{C}_{Y/Y'} \to \mathcal{C}_{X/X'}$ *is an isomorphism.*

Proof. Choose a scheme V' and a surjective étale morphism $V' \to Y'$. Choose a scheme U' and a surjective étale morphism $U' \to X' \times_{Y'} V'$. Set $U = X \times_{X'} U'$ and $V = Y \times_{Y'} V'$. According to our definition of a flat morphism of algebraic spaces we see that the induced map $g : U \to V$ is a flat morphism of schemes and that f' is flat if and only if the corresponding morphism $g' : U' \to V'$ is flat. Also, $X = Y \times_{Y'} X'$ if and only if $U = V \times_{V'} V'$. Finally, the map $f^*\mathcal{C}_{Y/Y'} \to \mathcal{C}_{X/X'}$ is an isomorphism if and only if $g^*\mathcal{C}_{V/V'} \to \mathcal{C}_{U/U'}$ is an isomorphism. Hence the lemma follows from its analogue for morphisms of schemes, see More on Morphisms, Lemma 36.8.1. □

57.16. Formally smooth morphisms

In this section we introduce the notion of a formally smooth morphism $X \to Y$ of algebraic spaces. Such a morphism is characterized by the property that T-valued points of X lift to infinitesimal thickenings of T provided T is affine. The main result is that a morphism which is formally smooth and locally of finite presentation is smooth, see Lemma 57.16.6. It turns out that this criterion is often easier to use than the Jacobian criterion.

Definition 57.16.1. Let S be a scheme. A morphism $f : X \to Y$ of algebraic spaces over S is said to be *formally smooth* if it is formally smooth as a transformation of functors as in Definition 57.10.1.

In the cases of formally unramified and formally étale morphisms the condition that T' be affine could be dropped, see Lemmas 57.11.3 and 57.13.3. This is no longer true in the case of formally smooth morphisms. In fact, a slightly more natural condition would be that we should be able to fill in the dotted arrow étale locally on T'. In fact, analyzing the proof of Lemma 57.16.6 shows that this would be equivalent to the definition as it currently stands. It is also true that requiring the existence of the dotted arrow fppf locally on T' would be sufficient, but that is slightly more difficult to prove.

We will not restate the results proved in the more general setting of formally smooth transformations of functors in Section 57.10.

Lemma 57.16.2. *A composition of formally smooth morphisms is formally smooth.*

Proof. Omitted. □

Lemma 57.16.3. *A base change of a formally smooth morphism is formally smooth.*

Proof. Omitted, but see Algebra, Lemma 10.131.2 for the algebraic version. □

Lemma 57.16.4. *Let $f : X \to S$ be a morphism of schemes. Then f is formally étale if and only if f is formally smooth and formally unramified.*

Proof. Omitted. □

Here is a helper lemma which will be superseded by Lemma 57.16.9.

Lemma 57.16.5. *Let S be a scheme. Let*

$$\begin{array}{ccc} U & \xrightarrow{\psi} & V \\ \downarrow & & \downarrow \\ X & \xrightarrow{f} & Y \end{array}$$

be a commutative diagram of morphisms of algebraic spaces over S. If the vertical arrows are étale and f is formally smooth, then ψ is formally smooth.

Proof. By Lemma 57.10.5 the morphisms $U \to X$ and $V \to Y$ are formally étale. By Lemma 57.10.3 the composition $U \to Y$ is formally smooth. By Lemma 57.10.8 we see $\psi : U \to V$ is formally smooth. □

The following lemma is the main result of this section. It implies, combined with Limits of Spaces, Proposition 52.3.9, that we can recognize whether a morphism of algebraic spaces $f : X \to Y$ is smooth in terms of "simple" properties of the transformation of functors $X \to Y$.

Lemma 57.16.6 (Infinitesimal lifting criterion)**.** *Let S be a scheme. Let $f : X \to Y$ be a morphism of algebraic spaces over S. The following are equivalent:*

(1) *The morphism f is smooth.*
(2) *The morphism f is locally of finite presentation, and formally smooth.*

Proof. Assume $f : X \to S$ is locally of finite presentation and formally smooth. Consider a commutative diagram

$$\begin{array}{ccc} U & \xrightarrow{\psi} & V \\ \downarrow & & \downarrow \\ X & \xrightarrow{f} & Y \end{array}$$

where U and V are schemes and the vertical arrows are étale and surjective. By Lemma 57.16.5 we see $\psi : U \to V$ is formally smooth. By Morphisms of Spaces, Lemma 49.26.4 the morphism ψ is locally of finite presentation. Hence by the case of schemes the morphism ψ is smooth, see More on Morphisms, Lemma 36.9.7. Hence f is smooth, see Morphisms of Spaces, Lemma 49.33.4.

Conversely, assume that $f : X \to Y$ is smooth. Consider a solid commutative diagram

$$\begin{array}{ccc} X & \xleftarrow{a} & T \\ {\scriptstyle f}\downarrow & \nwarrow & \downarrow{\scriptstyle i} \\ Y & \xleftarrow{} & T' \end{array}$$

as in Definition 57.16.1. We will show the dotted arrow exists thereby proving that f is formally smooth. Let $\mathcal{F}$ be the sheaf of sets on $(T')_{spaces,étale}$ of Lemma 57.14.3, see also Remark 57.14.5. Let

$$\mathcal{H} = \mathcal{H}om_{\mathcal{O}_T}(a^*\Omega_{X/Y}, \mathcal{C}_{T/T'})$$

be the sheaf of $\mathcal{O}_T$-modules on $T_{étale}$ introduced in Lemma 57.14.4. The action $\mathcal{H} \times \mathcal{F} \to \mathcal{F}$ turns $\mathcal{F}$ into a pseudo $\mathcal{H}$-torsor, see Cohomology on Sites, Definition 21.5.1. Our goal is to show that $\mathcal{F}$ is a trivial $\mathcal{H}$-torsor. There are two steps: (I) To show that $\mathcal{F}$ is a torsor we have to show that $\mathcal{F}$ has étale locally a section. (II) To show that $\mathcal{F}$ is the trivial torsor it suffices to show that $H^1(T_{étale}, \mathcal{H}) = 0$, see Cohomology on Sites, Lemma 21.5.3.

First we prove (I). To see this choose a commutative diagram

$$\begin{array}{ccc} U & \xrightarrow{\psi} & V \\ \downarrow & & \downarrow \\ X & \xrightarrow{f} & Y \end{array}$$

where U and V are schemes and the vertical arrows are étale and surjective. As f is assumed smooth we see that ψ is smooth and hence formally smooth by Lemma 57.10.5. By the same lemma the morphism $V \to Y$ is formally étale. Thus by Lemma 57.10.3 the composition $U \to Y$ is formally smooth. Then (I) follows from Lemma 57.10.6 part (4).

Finally we prove (II). By Lemma 57.6.15 we see that $\Omega_{X/S}$ is of finite presentation. Hence $a^*\Omega_{X/S}$ is of finite presentation (see Properties of Spaces, Section 48.28). Hence the sheaf $\mathcal{H} = \mathcal{H}om_{\mathcal{O}_T}(a^*\Omega_{X/Y}, \mathcal{C}_{T/T'})$ is quasi-coherent by Properties of

Spaces, Lemma 48.27.7. Thus by Descent, Proposition 34.7.10 and Cohomology of Schemes, Lemma 29.2.2 we have

$$H^1(T_{spaces,étale}, \mathcal{H}) = H^1(T_{étale}, \mathcal{H}) = H^1(T, \mathcal{H}) = 0$$

as desired. □

We do a bit more work to show that being formally smooth is étale local on the source. To begin we show that a formally smooth morphism has a nice sheaf of differentials. The notion of a locally projective quasi-coherent module is defined in Properties of Spaces, Section 48.29.

Lemma 57.16.7. *Let S be a scheme. Let $f : X \to Y$ be a formally smooth morphism of algebraic spaces over S. Then $\Omega_{X/Y}$ is locally projective on X.*

Proof. Choose a diagram

$$\begin{array}{ccc} U & \xrightarrow{\psi} & V \\ \downarrow & & \downarrow \\ X & \xrightarrow{f} & Y \end{array}$$

where U and V are affine(!) schemes and the vertical arrows are étale. By Lemma 57.16.5 we see $\psi : U \to V$ is formally smooth. Hence $\Gamma(V, \mathcal{O}_V) \to \Gamma(U, \mathcal{O}_U)$ is a formally smooth ring map, see More on Morphisms, Lemma 36.9.6. Hence by Algebra, Lemma 10.131.7 the $\Gamma(U, \mathcal{O}_U)$-module $\Omega_{\Gamma(U,\mathcal{O}_U)/\Gamma(V,\mathcal{O}_V)}$ is projective. Hence $\Omega_{U/V}$ is locally projective, see Properties, Section 27.19. Since $\Omega_{X/Y}|_U = \Omega_{U/V}$ we see that $\Omega_{X/Y}$ is locally projective too. (Because we can find an étale covering of X by the affine U's fitting into diagrams as above – details omitted.) □

Lemma 57.16.8. *Let T be an affine scheme. Let $\mathcal{F}$, $\mathcal{G}$ be quasi-coherent $\mathcal{O}_T$-modules on $T_{étale}$. Consider the internal hom sheaf $\mathcal{H} = \mathcal{H}om_{\mathcal{O}_T}(\mathcal{F}, \mathcal{G})$ on $T_{étale}$. If $\mathcal{F}$ is locally projective, then $H^1(T_{étale}, \mathcal{H}) = 0$.*

Proof. By the definition of a locally projective sheaf on an algebraic space (see Properties of Spaces, Definition 48.29.2) we see that $\mathcal{F}_{Zar} = \mathcal{F}|_{T_{Zar}}$ is a locally projective sheaf on the scheme T. Thus $\mathcal{F}_{Zar}$ is a direct summand of a free $\mathcal{O}_{T_{Zar}}$-module. Whereupon we conclude (as $\mathcal{F} = (\mathcal{F}_{Zar})^a$, see Descent, Proposition 34.7.11) that $\mathcal{F}$ is a direct summand of a free $\mathcal{O}_T$-module on $T_{étale}$. Hence we may assume that $\mathcal{F} = \bigoplus_{i \in I} \mathcal{O}_T$ is a free module. In this case $\mathcal{H} = \prod_{i \in I} \mathcal{G}$ is a product of quasi-coherent modules. By Cohomology on Sites, Lemma 21.12.5 we conclude that $H^1 = 0$ because the cohomology of a quasi-coherent sheaf on an affine scheme is zero, see Descent, Proposition 34.7.10 and Cohomology of Schemes, Lemma 29.2.2. □

Lemma 57.16.9. *Let S be a scheme. Let $f : X \to Y$ be a morphism of algebraic spaces over S. The following are equivalent:*

1. *f is formally smooth,*
2. *for every diagram*

$$\begin{array}{ccc} U & \xrightarrow{\psi} & V \\ \downarrow & & \downarrow \\ X & \xrightarrow{f} & Y \end{array}$$

where U and V are schemes and the vertical arrows are étale the morphism of schemes ψ is formally smooth (as in More on Morphisms, Definition 36.4.1), and

(3) *for one such diagram with surjective vertical arrows the morphism ψ is formally smooth.*

Proof. We have seen that (1) implies (2) and (3) in Lemma 57.16.5. Assume (3). The proof that f is formally smooth is entirely similar to the proof of (1) $\Rightarrow$ (2) of Lemma 57.16.6.

Consider a solid commutative diagram

$$\begin{array}{ccc} X & \xleftarrow{a} & T \\ {\scriptstyle f}\downarrow & \nwarrow & \downarrow{\scriptstyle i} \\ Y & \longleftarrow & T' \end{array}$$

as in Definition 57.16.1. We will show the dotted arrow exists thereby proving that f is formally smooth. Let $\mathcal{F}$ be the sheaf of sets on $(T')_{spaces,étale}$ of Lemma 57.14.3, see also Remark 57.14.5. Let

$$\mathcal{H} = \mathcal{H}om_{\mathcal{O}_T}(a^*\Omega_{X/Y}, \mathcal{C}_{T/T'})$$

be the sheaf of $\mathcal{O}_T$-modules on $T_{étale}$ introduced in Lemma 57.14.4. The action $\mathcal{H} \times \mathcal{F} \to \mathcal{F}$ turns $\mathcal{F}$ into a pseudo $\mathcal{H}$-torsor, see Cohomology on Sites, Definition 21.5.1. Our goal is to show that $\mathcal{F}$ is a trivial $\mathcal{H}$-torsor. There are two steps: (I) To show that $\mathcal{F}$ is a torsor we have to show that $\mathcal{F}$ has étale locally a section. (II) To show that $\mathcal{F}$ is the trivial torsor it suffices to show that $H^1(T_{étale}, \mathcal{H}) = 0$, see Cohomology on Sites, Lemma 21.5.3.

First we prove (I). To see this consider a diagram (which exists because we are assuming (3))

$$\begin{array}{ccc} U & \xrightarrow{\psi} & V \\ \downarrow & & \downarrow \\ X & \xrightarrow{f} & Y \end{array}$$

where U and V are schemes, the vertical arrows are étale and surjective, and ψ is formally smooth. By Lemma 57.10.5 the morphism $V \to Y$ is formally étale. Thus by Lemma 57.10.3 the composition $U \to Y$ is formally smooth. Then (I) follows from Lemma 57.10.6 part (4).

Finally we prove (II). By Lemma 57.16.7 we see that $\Omega_{U/V}$ locally projective. Hence $\Omega_{X/Y}$ is locally projective, see Descent on Spaces, Lemma 55.5.5. Hence $a^*\Omega_{X/Y}$ is locally projective, see Properties of Spaces, Lemma 48.29.3. Hence

$$H^1(T_{étale}, \mathcal{H}) = H^1(T_{étale}, \mathcal{H}om_{\mathcal{O}_T}(a^*\Omega_{X/Y}, \mathcal{C}_{T/T'}) = 0$$

by Lemma 57.16.8 as desired. □

Lemma 57.16.10. *The property $\mathcal{P}(f) =$ "f is formally smooth" is fpqc local on the base.*

Proof. Let $f : X \to Y$ be a morphism of algebraic spaces over a scheme S. Choose an index set I and diagrams

$$\begin{array}{ccc} U_i & \xrightarrow[\psi_i]{} & V_i \\ \downarrow & & \downarrow \\ X & \xrightarrow{f} & Y \end{array}$$

with étale vertical arrows and U_i, V_i affine schemes. Moreover, assume that $\coprod U_i \to X$ and $\coprod V_i \to Y$ are surjective, see Properties of Spaces, Lemma 48.6.1. By Lemma 57.16.9 we see that f is formally smooth if and only if each of the morphisms ψ_i are formally smooth. Hence we reduce to the case of a morphism of affine schemes. In this case the result follows from Algebra, Lemma 10.131.15. Some details omitted. □

Lemma 57.16.11. *Let S be a scheme. Let $f : X \to Y$, $g : Y \to Z$ be morphisms of algebraic spaces over S. Assume f is formally smooth. Then*

$$0 \to f^*\Omega_{Y/Z} \to \Omega_{X/Z} \to \Omega_{X/Z} \to 0$$

Lemma 57.6.8 is short exact.

Proof. Follows from the case of schemes, see More on Morphisms, Lemma 36.9.9, by étale localization, see Lemmas 57.16.9 and 57.6.3. □

Lemma 57.16.12. *Let S be a scheme. Let B be an algebraic space over S. Let $h : Z \to X$ be a formally unramified morphism of algebraic spaces over B. Assume that Z is formally smooth over B. Then the canonical exact sequence*

$$0 \to \mathcal{C}_{Z/X} \to i^*\Omega_{X/B} \to \Omega_{Z/B} \to 0$$

of Lemma 57.12.13 is short exact.

Proof. Let $Z \to Z'$ be the universal first order thickening of Z over X. From the proof of Lemma 57.12.13 we see that our sequence is identified with the sequence

$$\mathcal{C}_{Z/Z'} \to \Omega_{Z'/B} \otimes \mathcal{O}_Z \to \Omega_{Z/B} \to 0.$$

Since $Z \to S$ is formally smooth we can étale locally on Z' find a left inverse $Z' \to Z$ over B to the inclusion map $Z \to Z'$. Thus the sequence is étale locally split, see Lemma 57.6.11. □

Lemma 57.16.13. *Let S be a scheme. Let*

$$\begin{array}{ccc} Z & \xrightarrow[i]{} & X \\ & \searrow^{j} & \downarrow f \\ & & Y \end{array}$$

be a commutative diagram of algebraic spaces over S where i and j are formally unramified and f is formally smooth. Then the canonical exact sequence

$$0 \to \mathcal{C}_{Z/Y} \to \mathcal{C}_{Z/X} \to i^*\Omega_{X/Y} \to 0$$

of Lemma 57.12.14 is exact and locally split.

Proof. Denote $Z \to Z'$ the universal first order thickening of Z over X. Denote $Z \to Z''$ the universal first order thickening of Z over Y. By Lemma 57.12.13 here is a canonical morphism $Z' \to Z''$ so that we have a commutative diagram

$$\begin{array}{ccccc} Z & \longrightarrow & Z' & \xrightarrow{a} & X \\ & \searrow^{i'} & \downarrow k & & \downarrow f \\ & \searrow_{j'} & Z'' & \xrightarrow{b} & Y \end{array}$$

The sequence above is identified with the sequence

$$\mathcal{C}_{Z/Z''} \to \mathcal{C}_{Z/Z'} \to (i')^*\Omega_{Z'/Z''} \to 0$$

via our definitions concerning conormal sheaves of formally unramified morphisms. Let $U'' \to Z''$ be an étale morphism with U'' affine. Denote $U \to Z$ and $U' \to Z'$ the corresponding affine schemes étale over Z and Z'. As f is formally smooth there exists a morphism $h : U'' \to X$ which agrees with i on U and such that $f \circ h$ equals $b|_{U''}$. Since Z' is the universal first order thickening we obtain a unique morphism $g : U'' \to Z'$ such that $g = a \circ h$. The universal property of Z'' implies that $k \circ g$ is the inclusion map $U'' \to Z''$. Hence g is a left inverse to k. Picture

$$\begin{array}{ccc} U & \longrightarrow & Z' \\ \downarrow & \overset{g}{\nearrow} & \downarrow k \\ U'' & \longrightarrow & Z'' \end{array}$$

Thus g induces a map $\mathcal{C}_{Z/Z'}|_U \to \mathcal{C}_{Z/Z''}|_U$ which is a left inverse to the map $\mathcal{C}_{Z/Z''} \to \mathcal{C}_{Z/Z'}$ over U. □

57.17. Pushouts in the category of algebraic spaces

This section is analogue of More on Morphisms, Section 36.11. We first prove a general result on colimits and algebraic spaces. To do this we discuss a bit of notation. Let S be a scheme. Let $\mathcal{I} \to (Sch/S)_{fppf}$, $i \mapsto X_i$ be a diagram (see Categories, Section 4.14). For each i we may consider the small étale site $X_{i,étale}$. For each morphism $i \to j$ of $\mathcal{I}$ we have the morphism $X_i \to X_j$ and hence a pullback functor $X_{j,étale} \to X_{i,étale}$. Hence we obtain a pseudo functor from $\mathcal{I}^{opp}$ into the 2-category of categories. Denote

$$\lim_i X_{i,étale}$$

the 2-limit (see insert future reference here). What does this mean concretely? An object of this limit is a system of étale morphisms $U_i \to X_i$ over $\mathcal{I}$ such that for each $i \to j$ in $\mathcal{I}$ the diagram

$$\begin{array}{ccc} U_i & \longrightarrow & U_j \\ \downarrow & & \downarrow \\ X_i & \longrightarrow & X_j \end{array}$$

is cartesian. Morphisms between objects are defined in the obvious manner. Suppose that $f_i : X_i \to T$ is a family of morphisms such that for each $i \to j$ the composition $X_i \to X_j \to T$ is equal to f_i. Then we get a functor $T_{étale} \to \lim X_{i,étale}$. With this notation in hand we can formulate our lemma.

Lemma 57.17.1. *Let S be a scheme. Let $\mathcal{I} \to (Sch/S)_{fppf}$, $i \mapsto X_i$ be a diagram as above. Assume that*

(1) *$X = \operatorname{colim} X_i$ exists in the category of schemes,*
(2) *$\coprod X_i \to X$ is surjective,*
(3) *if $U \to X$ is étale and $U_i = X_i \times_X U$, then $U = \operatorname{colim} U_i$ in the category of schemes, and*
(4) *the functor $X_{étale} \to \lim X_{i,étale}$ is an equivalence.*

Then $X = \operatorname{colim} X_i$ in the category of algebraic spaces over S also.

Proof. Let Z be an algebraic space over S. Suppose that $f_i : X_i \to Z$ is a family of morphisms such that for each $i \to j$ the composition $X_i \to X_j \to Z$ is equal to f_i. We have to construct a morphism of algebraic spaces $f : X \to Z$ such that we can recover f_i as the composition $X_i \to X \to Z$. Let $W \to Z$ be a surjective étale morphism of a scheme to Z. For each i set $U_i = W \times_{Z,f_i} X_i$ and denote $h_i : U_i \to W$ the projection. Then $U_i \to X_i$ forms an object of $\lim X_{i,étale}$. By assumption (4) we can find an étale morphism $U \to X$ and (functorial) isomorphisms $U_i = X_i \times_X U$. By assumption (3) there exists a morphism $h : U \to W$ such that the compositions $U_i \to U \to W$ are h_i. Let $g : U \to Z$ be the composition of h with the map $W \to Z$. To finish the proof we have to show that $g : U \to Z$ descends to a morphism $X \to Z$. To do this, consider the morphism $(h, h) : U \times_X U \to W \times_Z W$. Composing with $U_i \times_{X_i} U_i \to U \times_X U$ we obtain (h_i, h_i) which factors through $W \times_Z W$. Since $U \times_X U$ is the colimit of the schemes $U_i \times_{X_i} U_i$ by (3) we see that (h, h) factors through $W \times_Z W$. Hence the two compositions $U \times_X U \to U \to W \to Z$ are equal. Because each $U_i \to X_i$ is surjective and assumption (2) we see that $U \to X$ is surjective. As Z is a sheaf for the étale topology, we conclude that $g : U \to Z$ descends to $f : X \to Z$ as desired. □

Lemma 57.17.2. *Let S be a scheme. Let $X \to X'$ be a thickening of schemes over S and let $X \to Y$ be an affine morphism of schemes over S. Let $Y' = Y \amalg_X X'$ be the pushout in the category of schemes (see More on Morphisms, Lemma 36.11.1). Then Y' is also a pushout in the category of algebraic spaces over S.*

Proof. This is an immediate consequence of Lemma 57.17.1 and More on Morphisms, Lemmas 36.11.1, 36.11.2, and 36.11.4. □

Lemma 57.17.3. *Let S be a scheme. Let $X \to X'$ be a thickening of algebraic spaces over S and let $X \to Y$ be an affine morphism of algebraic spaces over S. Then there exists a pushout*

$$\begin{array}{ccc} X & \longrightarrow & X' \\ {\scriptstyle f}\downarrow & & \downarrow{\scriptstyle f'} \\ Y & \longrightarrow & Y \amalg_X X' \end{array}$$

in the category of algebraic spaces over S. Moreover $Y' = Y \amalg_X X'$ is a thickening of Y and

$$\mathcal{O}_{Y'} = \mathcal{O}_Y \times_{f_*\mathcal{O}_X} f'_*\mathcal{O}_{X'}$$

as sheaves on $Y_{étale} = (Y')_{étale}$.

Proof. Choose a scheme V and a surjective étale morphism $V \to Y$. Set $U = V \times_Y X$. This is a scheme affine over V with a surjective étale morphism $U \to X$.

By Lemma 57.8.6 there exists a $U' \to X'$ surjective étale with $U = U' \times_{X'} X$. In particular the morphism of schemes $U \to U'$ is a thickening too. Apply More on Morphisms, Lemma 36.11.1 to obtain a pushout $V' = V \amalg_U U'$ in the category of schemes.

We repeat this procedure to construct a pushout

$$\begin{array}{ccc} U \times_X U & \longrightarrow & U' \times_{X'} U' \\ \downarrow & & \downarrow \\ V \times_Y V & \longrightarrow & R' \end{array}$$

in the category of schemes. Consider the morphisms

$$U \times_X U \to U \to V', \quad U' \times_{X'} U' \to U' \to V', \quad V \times_Y V \to V \to V'$$

where we use the first projection in each case. Clearly these glue to give a morphism $t' : R' \to V'$ which is étale by More on Morphisms, Lemma 36.11.4. Similarly, we obtain $s' : R' \to V'$ étale. The morphism $j' = (t', s') : R' \to V' \times_S V'$ is unramified (as t' is étale) and a monomorphism when restricted to the closed subscheme $V \times_Y V \subset R'$. As $V \times_Y V \subset R'$ is a thickening it follows that j' is a monomorphism too. Finally, j' is an equivalence relation as we can use the functoriality of pushouts of schemes to construct a morphism $c' : R' \times_{s',V',t'} R' \to R'$ (details omitted). At this point we set $Y' = U'/R'$, see Spaces, Theorem 47.10.5.

We have morphisms $X' = U'/U' \times_{X'} U' \to V'/R' = Y'$ and $Y = V/V \times_Y V \to V'/R' = Y'$. By construction these fit into the commutative diagram

$$\begin{array}{ccc} X & \longrightarrow & X' \\ {\scriptstyle f}\downarrow & & \downarrow{\scriptstyle f'} \\ Y & \longrightarrow & Y' \end{array}$$

Since $Y \to Y'$ is a thickening we have $Y_{étale} = (Y')_{étale}$, see Lemma 57.8.6. The commutativity of the diagram gives a map of sheaves

$$\mathcal{O}_{Y'} \longrightarrow \mathcal{O}_Y \times_{f_*\mathcal{O}_X} f'_*\mathcal{O}_{X'}$$

on this set. By More on Morphisms, Lemma 36.11.1 this map is an isomorphism when we restrict to the scheme V', hence it is an isomorphism.

To finish the proof we show that the diagram above is a pushout in the category of algebraic spaces. To see this, let Z be an algebraic space and let $a' : X' \to Z$ and $b : Y \to Z$ be morphisms of algebraic spaces. By Lemma 57.17.2 we obtain a unique morphism $h : V' \to Z$ fitting into the commutative diagrams

$$\begin{array}{ccc} U' & \longrightarrow & V' \\ \downarrow & & \downarrow{\scriptstyle h} \\ X' & \xrightarrow{a'} & Z \end{array} \quad \text{and} \quad \begin{array}{ccc} V & \longrightarrow & V' \\ \downarrow & & \downarrow{\scriptstyle h} \\ Y & \xrightarrow{b} & Z \end{array}$$

The uniqueness shows that $h \circ t' = h \circ s'$. Hence h factors uniquely as $V' \to Y' \to Z$ and we win. □

In the following lemma we use the fibre product of categories as defined in Categories, Example 4.29.3.

Lemma 57.17.4. *Let S be a base scheme. Let $X \to X'$ be a thickening of algebraic spaces over S and let $X \to Y$ be an affine morphism of algebraic spaces over S. Let $Y' = Y \amalg_X X'$ be the pushout (see Lemma 57.17.3). Base change gives a functor*

$$F : (Spaces/Y') \longrightarrow (Spaces/Y) \times_{(Spaces/Y')} (Spaces/X')$$

given by $V' \longmapsto (V' \times_{Y'} Y, V' \times_{Y'} X', 1)$ which sends (Sch/Y') into $(Sch/Y) \times_{(Sch/Y')} (Sch/X')$. The functor F has a left adjoint

$$G : (Spaces/Y) \times_{(Spaces/Y')} (Spaces/X') \longrightarrow (Spaces/Y')$$

which sends the triple (V, U', φ) to the pushout $V \amalg_{(V \times_Y X)} U'$ in the category of algebraic spaces over S. The functor G sends $(Sch/Y) \times_{(Sch/Y')} (Sch/X')$ into (Sch/Y').

Proof. The proof is completely formal. Since the morphisms $X \to X'$ and $X \to Y$ are representable it is clear that F sends (Sch/Y') into $(Sch/Y) \times_{(Sch/Y')} (Sch/X')$.

Let us construct G. Let (V, U', φ) be an object of the fibre product category. Set $U = U' \times_{X'} X$. Note that $U \to U'$ is a thickening. Since $\varphi : V \times_Y X \to U' \times_{X'} X = U$ is an isomorphism we have a morphism $U \to V$ over $X \to Y$ which identifies U with the fibre product $X \times_Y V$. In particular $U \to V$ is affine, see Morphisms of Spaces, Lemma 49.19.5. Hence we can apply Lemma 57.17.3 to get a pushout $V' = V \amalg_U U'$. Denote $V' \to Y'$ the morphism we obtain in virtue of the fact that V' is a pushout and because we are given morphisms $V \to Y$ and $U' \to X'$ agreeing on U as morphisms into Y'. Setting $G(V, U', \varphi) = V'$ gives the functor G.

If (V, U', φ) is an object of $(Sch/Y) \times_{(Sch/Y')} (Sch/X')$ then $U = U' \times_{X'} X$ is a scheme too and we can form the pushout $V' = V \amalg_U U'$ in the category of schemes by More on Morphisms, Lemma 36.11.1. By Lemma 57.17.2 this is also a pushout in the category of schemes, hence G sends $(Sch/Y) \times_{(Sch/Y')} (Sch/X')$ into (Sch/Y').

Let us prove that G is a left adjoint to F. Let Z be an algebraic space over Y'. We have to show that

$$\mathrm{Mor}(V', Z) = \mathrm{Mor}((V, U', \varphi), F(Z))$$

where the morphism sets are taking in their respective categories. Let $g' : V' \to Z$ be a morphism. Denote $\tilde{g}$, resp. $\tilde{f}'$ the composition of g' with the morphism $V \to V'$, resp. $U' \to V'$. Base change $\tilde{g}$, resp. $\tilde{f}'$ by $Y \to Y'$, resp. $X' \to Y'$ to get a morphism $g : V \to Z \times_{Y'} Y$, resp. $f' : U' \to Z \times_{Y'} X'$. Then (g, f') is an element of the right hand side of the equation above (details omitted). Conversely, suppose that $(g, f') : (V, U', \varphi) \to F(Z)$ is an element of the right hand side. We may consider the composition $\tilde{g} : V \to Z$, resp. $\tilde{f}' : U' \to Z$ of g, resp. f by $Z \times_{Y'} X' \to Z$, resp. $Z \times_{Y'} Y \to Z$. Then $\tilde{g}$ and $\tilde{f}'$ agree as morphism from U to Z. By the universal property of pushout, we obtain a morphism $g' : V' \to Z$, i.e., an element of the left hand side. We omit the verification that these constructions are mutually inverse. □

Lemma 57.17.5. *Let S be a scheme. Let*

$$\begin{array}{ccccc} A & \longrightarrow & C & \longrightarrow & E \\ \downarrow & & \downarrow & & \downarrow \\ B & \longrightarrow & D & \longrightarrow & F \end{array}$$

be a commutative diagram of algebraic spaces over S. Assume that A, B, C, D and A, B, E, F form cartesian squares and that $B \to D$ is surjective étale. Then C, D, E, F is a cartesian square.

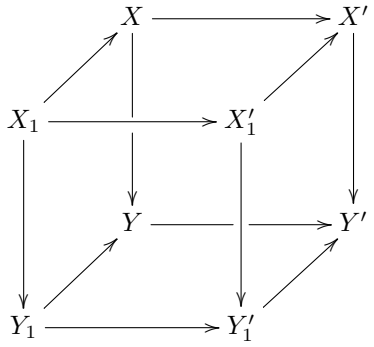
Proof. This is formal. □

Lemma 57.17.6. *In the situation of Lemma 57.17.4 the functor $F \circ G$ is isomorphic to the identity functor.*

Proof. We will prove that $F \circ G$ is isomorphic to the identity by reducing this to the corresponding statement of More on Morphisms, Lemma 36.11.2.

Choose a scheme Y_1 and a surjective étale morphism $Y_1 \to Y$. Set $X_1 = Y_1 \times_Y X$. This is a scheme affine over Y_1 with a surjective étale morphism $X_1 \to X$. By Lemma 57.8.6 there exists a $X_1' \to X'$ surjective étale with $X_1 = X_1' \times_{X'} X$. In particular the morphism of schemes $X_1 \to X_1'$ is a thickening too. Apply More on Morphisms, Lemma 36.11.1 to obtain a pushout $Y_1' = Y_1 \amalg_{X_1} X_1'$ in the category of schemes. In the proof of Lemma 57.17.3 we constructed Y' as a quotient of an étale equivalence relation on Y_1' such that we get a commutative diagram

(57.17.6.1)



where all squares except the front and back squares are cartesian (the front and back squares are pushouts) and the northeast arrows are surjective étale. Denote F_1, G_1 the functors constructed in More on Morphisms, Lemma 36.11.2 for the front square. Then the diagram of categories

$$\begin{array}{ccc} (Sch/Y_1') & \underset{F_1}{\overset{G_1}{\leftarrow\!\!\rightarrow}} & (Sch/Y_1) \times_{(Sch/Y_1')} (Sch/X_1') \\ \downarrow & & \downarrow \\ (Spaces/Y') & \underset{F}{\overset{G}{\leftarrow\!\!\rightarrow}} & (Spaces/Y) \times_{(Spaces/Y')} (Spaces/X') \end{array}$$

is commutative by simple considerations regarding base change functors and the agreement of pushouts in schemes with pushouts in spaces of Lemma 57.17.2.

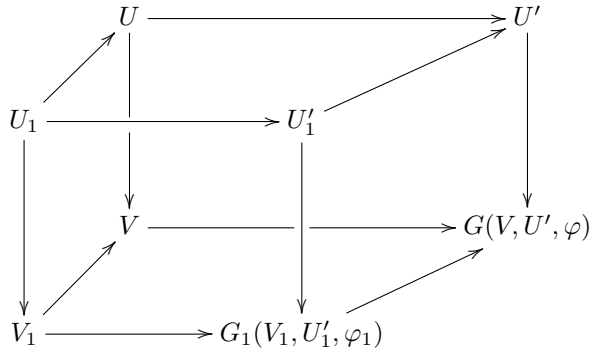
Let (V, U', φ) be an object of $(Spaces/Y) \times_{(Spaces/Y')} (Spaces/X')$. Denote $U = U' \times_{X'} X$ so that $G(V, U', \varphi) = V \amalg_U U'$. Choose a scheme V_1 and a surjective étale morphism $V_1 \to Y_1 \times_Y V$. Set $U_1 = V_1 \times_Y X$. Then

$$U_1 = V_1 \times_Y X \longrightarrow (Y_1 \times_Y V) \times_Y X = X_1 \times_Y V = X_1 \times_X X \times_Y V = X_1 \times_X U$$

is surjective étale too. By Lemma 57.8.6 there exists a thickening $U_1 \to U_1'$ and a surjective étale morphism $U_1' \to X_1' \times_{X'} U'$ whose base change to $X_1 \times_X U$ is the

displayed morphism. At this point (V_1, U'_1, φ_1) is an object of $(Sch/Y_1) \times_{(Sch/Y'_1)} (Sch/X'_1)$. In the proof of Lemma 57.17.3 we constructed $G(V, U', \varphi) = V \amalg_U U'$ as a quotient of an étale equivalence relation on $G_1(V_1, U'_1, \varphi_1) = V_1 \amalg_{U_1} U'_1$ such that we get a commutative diagram

(57.17.6.2)



where all squares except the front and back squares are cartesian (the front and back squares are pushouts) and the northeast arrows are surjective étale. In particular

$$G_1(V_1, U'_1, \varphi_1) \to G(V, U', \varphi)$$

is surjective étale.

Finally, we come to the proof of the lemma. We have to show that the adjunction mapping $(V, U', \varphi) \to F(G(V, U', \varphi))$ is an isomorphism. We know $(V_1, U'_1, \varphi_1) \to F_1(G_1(V_1, U'_1, \varphi_1))$ is an isomorphism by More on Morphisms, Lemma 36.11.2. Recall that F and F_1 are given by base change. Using the properties of (57.17.6.2) and Lemma 57.17.5 we see that $V \to G(V, U', \varphi) \times_{Y'} Y$ and $U' \to G(V, U', \varphi) \times_{Y'} X'$ are isomorphisms, i.e., $(V, U', \varphi) \to F(G(V, U', \varphi))$ is an isomorphism. □

Lemma 57.17.7. *Let S be a base scheme. Let $X \to X'$ be a thickening of algebraic spaces over S and let $X \to Y$ be an affine morphism of algebraic spaces over S. Let $Y' = Y \amalg_X X'$ be the pushout (see Lemma 57.17.3). Let $V' \to Y'$ be a morphism of algebraic spaces over S. Set $V = Y \times_{Y'} V'$, $U' = X' \times_{Y'} V'$, and $U = X \times_{Y'} V'$. There is an equivalence of categories between*

1. *quasi-coherent $\mathcal{O}_{V'}$-modules flat over Y', and*
2. *the category of triples $(\mathcal{G}, \mathcal{F}', \varphi)$ where*
   (a) *$\mathcal{G}$ is a quasi-coherent $\mathcal{O}_V$-module flat over Y,*
   (b) *$\mathcal{F}'$ is a quasi-coherent $\mathcal{O}_{U'}$-module flat over X, and*
   (c) *$\varphi : (U \to V)^*\mathcal{G} \to (U \to U')^*\mathcal{F}'$ is an isomorphism of $\mathcal{O}_U$-modules.*

The equivalence maps $\mathcal{G}'$ to $((V \to V')^*\mathcal{G}', (U' \to V')^*\mathcal{G}', can)$. Suppose $\mathcal{G}'$ corresponds to the triple $(\mathcal{G}, \mathcal{F}', \varphi)$. Then

(a) *$\mathcal{G}'$ is a finite type $\mathcal{O}_{V'}$-module if and only if $\mathcal{G}$ and $\mathcal{F}'$ are finite type $\mathcal{O}_Y$ and $\mathcal{O}_{U'}$-modules.*
(b) *if $V' \to Y'$ is locally of finite presentation, then $\mathcal{G}'$ is an $\mathcal{O}_{V'}$-module of finite presentation if and only if $\mathcal{G}$ and $\mathcal{F}'$ are $\mathcal{O}_Y$ and $\mathcal{O}_{U'}$-modules of finite presentation.*

Proof. A quasi-inverse functor assigns to the triple $(\mathcal{G}, \mathcal{F}', \varphi)$ the fibre product

$$(V \to V')_*\mathcal{G} \times_{(U \to V')_*\mathcal{F}} (U' \to V')_*\mathcal{F}'$$

where $\mathcal{F} = (U \to U')^*\mathcal{F}'$. This works, because on affines étale over V' and Y' we recover the equivalence of More on Algebra, Lemma 15.4.12. Details omitted.

Parts (a) and (b) reduce by étale localization (Properties of Spaces, Section 48.28) to the case where V' and Y' are affine in which case the result follows from More on Algebra, Lemmas 15.4.11 and 15.4.13. □

Lemma 57.17.8. *In the situation of Lemma 57.17.6. If $V' = G(V, U', \varphi)$ for some triple (V, U', φ), then*

1. *$V' \to Y'$ is locally of finite type if and only if $V \to Y$ and $U' \to X'$ are locally of finite type,*
2. *$V' \to Y'$ is flat if and only if $V \to Y$ and $U' \to X'$ are flat,*
3. *$V' \to Y'$ is flat and locally of finite presentation if and only if $V \to Y$ and $U' \to X'$ are flat and locally of finite presentation,*
4. *$V' \to Y'$ is smooth if and only if $V \to Y$ and $U' \to X'$ are smooth,*
5. *$V' \to Y'$ is étale if and only if $V \to Y$ and $U' \to X'$ are étale, and*
6. *add more here as needed.*

If W' is flat over Y', then the adjunction mapping $G(F(W')) \to W'$ is an isomorphism. Hence F and G define mutually quasi-inverse functors between the category of spaces flat over Y' and the category of triples (V, U', φ) with $V \to Y$ and $U' \to X'$ flat.

Proof. Choose a diagram (57.17.6.1) as in the proof of Lemma 57.17.6.

Proof of (1) – (5). Let (V, U', φ) be an object of $(\mathit{Spaces}/Y) \times_{(\mathit{Spaces}/Y')} (\mathit{Spaces}/X')$. Construct a diagram (57.17.6.2) as in the proof of Lemma 57.17.6. Then the base change of $G(V, U', \varphi) \to Y'$ to Y'_1 is $G_1(V_1, U'_1, \varphi_1) \to Y'_1$. Hence (1) – (5) follow immediately from the corresponding statements of More on Morphisms, Lemma 36.11.4 for schemes.

Suppose that $W' \to Y'$ is flat. Choose a scheme W'_1 and a surjective étale morphism $W'_1 \to Y'_1 \times_{Y'} W'$. Observe that $W'_1 \to W'$ is surjective étale as a composition of surjective étale morphisms. We know that $G_1(F_1(W'_1)) \to W'_1$ is an isomorphism by More on Morphisms, Lemma 36.11.4 applied to W'_1 over Y'_1 and the front of the diagram (with functors G_1 and F_1 as in the proof of Lemma 57.17.6). Then the construction of $G(F(W'))$ (as a pushout, i.e., as constructed in Lemma 57.17.3) shows that $G_1(F_1(W'_1)) \to G(F(W))$ is surjective étale. Whereupon we conclude that $G(F(W)) \to W$ is étale, see for example Properties of Spaces, Lemma 48.13.3. But $G(F(W)) \to W$ is an isomorphism on underlying reduced algebraic spaces (by construction), hence it is an isomorphism. □

57.18. Openness of the flat locus

This section is analogue of More on Morphisms, Section 36.12. Note that we have defined the notion of flatness for quasi-coherent modules on algebraic spaces in Morphisms of Spaces, Section 49.28.

Theorem 57.18.1. *Let S be a scheme. Let $f : X \to Y$ be a morphism of algebraic spaces over S. Let $\mathcal{F}$ be a quasi-coherent sheaf on X. Assume f is locally of finite presentation and that $\mathcal{F}$ is an $\mathcal{O}_X$-module which is locally of finite presentation. Then*

$$\{x \in |X| : \mathcal{F} \text{ is flat over } Y \text{ at } x\}$$

is open in $|X|$.

Proof. Choose a commutative diagram

$$\begin{array}{ccc} U & \xrightarrow{\alpha} & V \\ {\scriptstyle p}\downarrow & & \downarrow{\scriptstyle q} \\ X & \xrightarrow{a} & Y \end{array}$$

with U, V schemes and p, q surjective and étale as in Spaces, Lemma 47.11.4. By More on Morphisms, Theorem 36.12.1 the set $U' = \{u \in |U| : p^*\mathcal{F} \text{ is flat over } V \text{ at } u\}$ is open in U. By Morphisms of Spaces, Definition 49.28.2 the image of U' in $|X|$ is the set of the theorem. Hence we are done because the map $|U| \to |X|$ is open, see Properties of Spaces, Lemma 48.4.6. □

Lemma 57.18.2. *Let S be a scheme. Let*

$$\begin{array}{ccc} X' & \xrightarrow{g'} & X \\ {\scriptstyle f'}\downarrow & & \downarrow{\scriptstyle f} \\ Y' & \xrightarrow{g} & Y \end{array}$$

be a cartesian diagram of algebraic spaces over S. Let $\mathcal{F}$ be a quasi-coherent $\mathcal{O}_X$-module. Assume g is flat, f is locally of finite presentation, and $\mathcal{F}$ is locally of finite presentation. Then

$$\{x' \in |X'| : (g')^*\mathcal{F} \text{ is flat over } Y' \text{ at } x'\}$$

is the inverse image of the open subset of Theorem 57.18.1 under the continuous map $|g'| : |X'| \to |X|$.

Proof. This follows from Morphisms of Spaces, Lemma 49.28.3. □

57.19. Critère de platitude par fibres

Let S be a scheme. Consider a commutative diagram of algebraic spaces over S

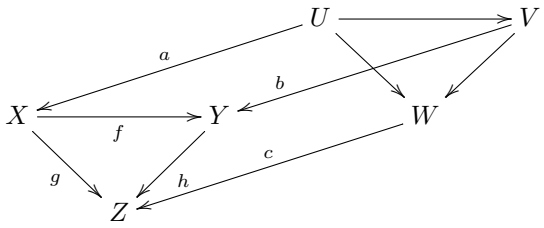
$$\begin{array}{ccc} X & \xrightarrow{f} & Y \\ & {\scriptstyle g}\searrow \quad \swarrow{\scriptstyle h} & \\ & Z & \end{array}$$

and a quasi-coherent $\mathcal{O}_X$-module $\mathcal{F}$. Given a point $x \in |X|$ we consider the question as to whether $\mathcal{F}$ is flat over Y at x. If $\mathcal{F}$ is flat over Z at x, then the theorem below states this question is intimately related to the question of whether the restriction of $\mathcal{F}$ to the fibre of $X \to Z$ over $g(x)$ is flat over the fibre of $Y \to Z$ over $g(x)$. To make sense out of this we offer the following preliminary lemma.

Lemma 57.19.1. *In the situation above the following are equivalent*

1. *Pick a geometric point $\overline{x}$ of X lying over x. Set $\overline{y} = f \circ \overline{x}$ and $\overline{z} = g \circ \overline{x}$. Then the module $\mathcal{F}_{\overline{x}}/\mathfrak{m}_{\overline{z}}\mathcal{F}_{\overline{x}}$ is flat over $\mathcal{O}_{Y,\overline{y}}/\mathfrak{m}_{\overline{z}}\mathcal{O}_{Y,\overline{y}}$.*
2. *Pick a morphism $x : \mathrm{Spec}(K) \to X$ in the equivalence class of x. Set $z = g \circ x$, $X_z = \mathrm{Spec}(K) \times_{z,Z} X$, $Y_z = \mathrm{Spec}(K) \times_{z,Z} Y$, and $\mathcal{F}_z$ the pullback of $\mathcal{F}$ to X_z. Then $\mathcal{F}_z$ is flat at x over Y_z (as defined in Morphisms of Spaces, Definition 49.28.2).*

(3) *Pick a commutative diagram*



where U, V, W are schemes, and a, b, c are étale, and a point $u \in U$ mapping to x. Let $w \in W$ be the image of u. Let $\mathcal{F}_w$ be the pullback of $\mathcal{F}$ to the fibre U_w of $U \to W$ at w. Then $\mathcal{F}_w$ is flat over V_w at u.

Proof. Note that in (2) the morphism $x : \mathrm{Spec}(K) \to X$ defines a K-rational point of X_z, hence the statement makes sense. Moreover, note that we can always choose a diagram as in (3) by: first choosing a scheme W and a surjective étale morphism $W \to Z$, then choosing a scheme V and a surjective étale morphism $V \to W \times_Z Y$, and finally choosing a scheme U and a surjective étale morphism $U \to V \times_Y X$. Having made these choices we set $U \to W$ equal to the composition $U \to V \to W$ and we can pick a point $u \in U$ mapping to x because the morphism $U \to X$ is surjective.

Suppose given both a diagram as in (3) and a geometric point $\overline{x} : \mathrm{Spec}(k) \to X$ as in (1). By Properties of Spaces, Lemma 48.16.4 we can choose a geometric point $\overline{u} : \mathrm{Spec}(k) \to U$ lying over u such that $\overline{x} = a \circ \overline{u}$. Denote $\overline{v} : \mathrm{Spec}(k) \to V$ and $\overline{w} : \mathrm{Spec}(k) \to W$ the induced geometric points of V and W. In this setting we know that $\mathcal{O}_{X,\overline{x}} = \mathcal{O}^{sh}_{U,u}$ and similarly for Y and Z, see Properties of Spaces, Lemma 48.19.1. In the same vein we have

$$\mathcal{F}_{\overline{x}} = (a^*\mathcal{F})_u \otimes_{\mathcal{O}_{U,u}} \mathcal{O}^{sh}_{U,u}$$

see Properties of Spaces, Lemma 48.27.4. Note that the stalk of $\mathcal{F}_w$ at u is given by

$$(\mathcal{F}_w)_u = (a^*\mathcal{F})_u / \mathfrak{m}_w (a^*\mathcal{F})_u$$

and the local ring of V_w at v is given by

$$\mathcal{O}_{V_w,v} = \mathcal{O}_{V,v}/\mathfrak{m}_w \mathcal{O}_{V,v}.$$

Since $\mathfrak{m}_{\overline{z}} = \mathfrak{m}_w \mathcal{O}_{Z,\overline{z}} = \mathfrak{m}_w \mathcal{O}^{sh}_{W,w}$ we see that

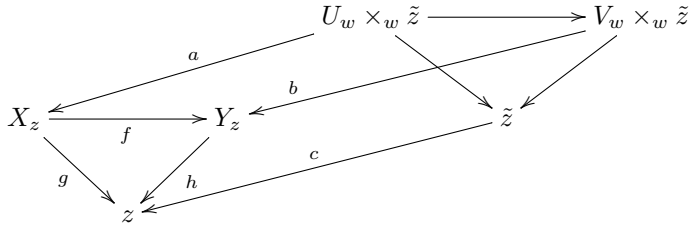
$$\begin{aligned}
\mathcal{F}_{\overline{x}}/\mathfrak{m}_{\overline{z}}\mathcal{F}_{\overline{x}} &= (a^*\mathcal{F})_u \otimes_{\mathcal{O}_{U,u}} \mathcal{O}_{X,\overline{x}}/\mathfrak{m}_{\overline{z}}\mathcal{O}_{X,\overline{x}} \\
&= (\mathcal{F}_w)_u \otimes_{\mathcal{O}_{U_w,u}} \mathcal{O}^{sh}_{U,u}/\mathfrak{m}_w \mathcal{O}^{sh}_{U,u} \\
&= (\mathcal{F}_w)_u \otimes_{\mathcal{O}_{U_w,u}} \mathcal{O}^{sh}_{U_w,\overline{u}} \\
&= (\mathcal{F}_w)_{\overline{u}}
\end{aligned}$$

the penultimate equality by Algebra, Lemma 10.143.30 and the last equality by Properties of Spaces, Lemma 48.27.4. The same arguments applied to the structure sheaves of V and Y show that

$$\mathcal{O}^{sh}_{V_w,\overline{v}} = \mathcal{O}^{sh}_{V,v}/\mathfrak{m}_w \mathcal{O}^{sh}_{V,v} = \mathcal{O}_{Y,\overline{y}}/\mathfrak{m}_{\overline{z}}\mathcal{O}_{Y,\overline{y}}.$$

OK, and now we can use Morphisms of Spaces, Lemma 49.28.1 to see that (1) is equivalent to (3).

Finally we prove the equivalence of (2) and (3). To do this we pick a field extension $\tilde{K}$ of K and and a morphism $\tilde{x} : \mathrm{Spec}(\tilde{K}) \to U$ which lies over u (this is possible because $u \times_{X,x} \mathrm{Spec}(K)$ is a nonempty scheme). Set $\tilde{z} : \mathrm{Spec}(\tilde{K}) \to U \to W$ be the composition. We obtain a commutative diagram



where $z = \mathrm{Spec}(K)$ and $w = \mathrm{Spec}(\kappa(w))$. Now it is clear that $\mathcal{F}_w$ and $\mathcal{F}_z$ pull back to the same module on $U_w \times_w \tilde{z}$. This leads to a commutative diagram

$$\begin{array}{ccccc} X_z & \longleftarrow & U_w \times_w \tilde{z} & \longrightarrow & U_w \\ \downarrow & & \downarrow & & \downarrow \\ Y_z & \longleftarrow & V_w \times_w \tilde{z} & \longrightarrow & V_w \end{array}$$

both of whose squares are cartesian and whose bottom horizontal arrows are flat: the lower left horizontal arrow is the composition of the morphism $Y \times_Z \tilde{z} \to Y \times_Z z = Y_z$ (base change of a flat morphism), the étale morphism $V \times_Z \tilde{z} \to Y \times_Z \tilde{z}$, and the étale morphism $V \times_W \tilde{z} \to V \times_Z \tilde{z}$. Thus it follows from Morphisms of Spaces, Lemma 49.28.3 that

$$\mathcal{F}_z \text{ flat at } x \text{ over } Y_z \Leftrightarrow \mathcal{F}|_{U_w \times_w \tilde{z}} \text{ flat at } \tilde{x} \text{ over } V_w \times_w \tilde{z} \Leftrightarrow \mathcal{F}_w \text{ flat at } u \text{ over } V_w$$

and we win. □

Definition 57.19.2. Let S be a scheme. Let $X \to Y \to Z$ be morphisms of algebraic spaces over S. Let $\mathcal{F}$ be a quasi-coherent $\mathcal{O}_X$-module. Let $x \in |X|$ be a point and denote $z \in |Z|$ its image.

(1) We say *the restriction of $\mathcal{F}$ to its fibre over z is flat at x over the fibre of Y over z* if the equivalent conditions of Lemma 57.19.1 are satisfied.
(2) We say *the fibre of X over z is flat at x over the fibre of Y over z* if the equivalent conditions of Lemma 57.19.1 holds with $\mathcal{F} = \mathcal{O}_X$.
(3) We say *the fibre of X over z is flat over the fibre of Y over z* if for all $x \in |X|$ lying over z the fibre of X over z is flat at x over the fibre of Y over z

With this definition in hand we can state the criterion as follows. (We leave the Noetherian version for later; insert future reference here.)

Theorem 57.19.3. *Let S be a scheme. Let $f : X \to Y$ and $Y \to Z$ be a morphisms of algebraic spaces over S. Let $\mathcal{F}$ be a quasi-coherent $\mathcal{O}_X$-module. Assume*

(1) *X is locally of finite presentation over Z,*
(2) *$\mathcal{F}$ an $\mathcal{O}_X$-module of finite presentation, and*
(3) *Y is locally of finite type over Z.*

Let $x \in |X|$ and let $y \in |Y|$ and $z \in |Z|$ be the images of x. If $\mathcal{F}_{\overline{x}} \neq 0$, then the following are equivalent:

(1) *$\mathcal{F}$ is flat over Z at x and the restriction of $\mathcal{F}$ to its fibre over z is flat at x over the fibre of Y over z, and*
(2) *Y is flat over Z at y and $\mathcal{F}$ is flat over Y at x.*

Moreover, the set of points x where (1) and (2) hold is open in $\mathrm{Supp}(\mathcal{F})$.

Proof. Choose a diagram as in Lemma 57.19.1 part (3). It follows from the definitions that this reduces to the corresponding theorem for the morphisms of schemes $U \to V \to W$, the quasi-coherent sheaf $a^*\mathcal{F}$, and the point $u \in U$. Thus the theorem follows from the corresponding result for schemes which is More on Morphisms, Theorem 36.13.2. □

Lemma 57.19.4. *Let S be a scheme. Let $f : X \to Y$ and $Y \to Z$ be a morphism of algebraic spaces over S. Assume*

(1) *X is locally of finite presentation over Z,*
(2) *X is flat over Z,*
(3) *for every $z \in |Z|$ the fibre of X over z is flat over the fibre of Y over z, and*
(4) *Y is locally of finite type over Z.*

Then f is flat. If f is also surjective, then Y is flat over Z.

Proof. This is a special case of Theorem 57.19.3. □

Lemma 57.19.5. *Let S be a scheme. Let $f : X \to Y$ and $Y \to Z$ be morphisms of algebraic spaces over S. Let $\mathcal{F}$ be a quasi-coherent $\mathcal{O}_X$-module. Assume*

(1) *X is locally of finite presentation over Z,*
(2) *$\mathcal{F}$ an $\mathcal{O}_X$-module of finite presentation,*
(3) *$\mathcal{F}$ is flat over Z, and*
(4) *Y is locally of finite type over Z.*

Then the set

$$A = \{x \in |X| : \mathcal{F} \text{ flat at } x \text{ over } Y\}.$$

is open in $|X|$ and its formation commutes with arbitrary base change: If $Z' \to Z$ is a morphism of algebraic spaces, and A' is the set of points of $X' = X \times_Z Z'$ where $\mathcal{F}' = \mathcal{F} \times_Z Z'$ is flat over $Y' = Y \times_Z Z'$, then A' is the inverse image of A under the continuous map $|X'| \to |X|$.

Proof. One way to prove this is to translate the proof as given in More on Morphisms, Lemma 36.13.4 into the category of algebraic spaces. Instead we will prove this by reducing to the case of schemes instead. Namely, choose a diagram as in Lemma 57.19.1 part (3) such that a, b, and c are surjective. It follows from the definitions that this reduces to the corresponding theorem for the morphisms of schemes $U \to V \to W$, the quasi-coherent sheaf $a^*\mathcal{F}$, and the point $u \in U$. The only minor point to make is that given a morphism of algebraic spaces $Z' \to Z$ we choose a scheme W' and a surjective étale morphism $W' \to W \times_Z Z'$. Then we set $U' = W' \times_W U$ and $V' = W' \times_W V$. We write a', b', c' for the morphisms from U', V', W' to X', Y', Z'. In this case A, resp. A' are images of the open subsets of U, resp. U' associated to $a^*\mathcal{F}$, resp. $(a')^*\mathcal{F}'$. This indeed does reduce the lemma to More on Morphisms, Lemma 36.13.4. □

Lemma 57.19.6. *Let S be a scheme. Let $f : X \to Y$ and $Y \to Z$ be a morphism of algebraic spaces over S. Assume*

(1) X *is locally of finite presentation over* Z,
(2) X *is flat over* Z, *and*
(3) Y *is locally of finite type over* Z.

Then the set

$$\{x \in |X| : X \text{ flat at } x \text{ over } Y\}.$$

is open in $|X|$ *and its formation commutes with arbitrary base change* $Z' \to Z$.

Proof. This is a special case of Lemma 57.19.5. □

57.20. Flatness over a Noetherian base

Just like for checking smoothness, if the base is Noetherian it suffices to check flatness over Artinian rings. Here is a sample statement.

Lemma 57.20.1. *Let* A *be a Noetherian ring. Let* $I \subset A$ *be an ideal. Let* X *be an algebraic space locally of finite presentation over* $S = \mathrm{Spec}(A)$. *For* $n \geq 1$ *set* $S_n = \mathrm{Spec}(A/I^n)$ *and* $X_n = S_n \times_S X$. *Let* $\mathcal{F}$ *be coherent* $\mathcal{O}_X$*-module. If for every* $n \geq 1$ *the pullback* $\mathcal{F}_n$ *of* $\mathcal{F}$ *to* X *is flat over* S_n, *then the (open) locus where* $\mathcal{F}$ *is flat over* X *contains the inverse image of* $V(I)$ *under* $X \to S$.

Proof. The locus where $\mathcal{F}$ is flat over S is open in $|X|$ by Theorem 57.18.1. The statement is insensitive to replacing X by the members of an étale covering, hence we may assume X is an affine scheme. In this case the result follows immediately from Algebra, Lemma 10.93.10. Some details omitted. □

57.21. Normalization revisited

Normalization commutes with smooth base change.

Lemma 57.21.1. *Let* S *be a scheme. Let* $f : Y \to X$ *be a smooth morphism of algebraic spaces over* S. *Let* $\mathcal{A}$ *be a quasi-coherent sheaf of* $\mathcal{O}_X$*-algebras. The integral closure of* $\mathcal{O}_Y$ *in* $f^*\mathcal{A}$ *is equal to* $f^*\mathcal{A}'$ *where* $\mathcal{A}' \subset \mathcal{A}$ *is the integral closure of* $\mathcal{O}_X$ *in* $\mathcal{A}$.

Proof. By our construction of the integral closure, see Morphisms of Spaces, Definition 49.42.2, this reduces immediately to the case where X and Y are affine. In this case the result is Algebra, Lemma 10.138.4. □

Lemma 57.21.2 (Normalization commutes with smooth base change)**.** *Let* S *be a scheme. Let*

$$\begin{array}{ccc} Y_2 & \longrightarrow & Y_1 \\ \downarrow & & \downarrow f \\ X_2 & \xrightarrow{\varphi} & X_1 \end{array}$$

be a fibre square of algebraic spaces over S. *Assume* f *is quasi-compact and quasi-separated and* φ *is smooth. Let* $Y_i \to X_i' \to X_i$ *be the normalization of* X_i *in* Y_i. *Then* $X_2' \cong X_2 \times_{X_1} X_1'$.

Proof. The base change of the factorization $Y_1 \to X_1' \to X_1$ to X_2 is a factorization $Y_2 \to X_2 \times_{X_1} X_1' \to X_1$ and $X_2 \times_{X_1} X_1' \to X_1$ is integral (Morphisms of Spaces, Lemma 49.40.5). Hence we get a morphism $h : X_2' \to X_2 \times_{X_1} X_1'$ by the universal property of Morphisms of Spaces, Lemma 49.42.4. Observe that X_2' is the relative spectrum of the integral closure of $\mathcal{O}_{X_2}$ in $f_{2,*}\mathcal{O}_{Y_2}$. If $\mathcal{A}' \subset f_{1,*}\mathcal{O}_{Y_1}$ denotes the

integral closure of $\mathcal{O}_{X_2}$, then $X_2 \times_{X_1} X_1'$ is the relative spectrum of $\varphi^*\mathcal{A}'$ as the construction of the relative spectrum commutes with arbitrary base change. By Cohomology of Spaces, Lemma 51.8.1 we know that $f_{2,*}\mathcal{O}_{Y_2} = \varphi^* f_{1,*}\mathcal{O}_{Y_1}$. Hence the result follows from Lemma 57.21.1. □

57.22. Slicing Cohen-Macaulay morphisms

Let S be a scheme. Let X be an algebraic space over S. Let $f_1, \ldots, f_r \in \Gamma(X, \mathcal{O}_X)$. In this case we denote $V(f_1, \ldots, f_r)$ the *closed subspace of X cut out by $f_1, \ldots, f_r$*. More precisely, we can define $V(f_1, \ldots, f_r)$ as the closed subspace of X corresponding to the quasi-coherent sheaf of ideals generated by $f_1, \ldots, f_r$, see Morphisms of Spaces, Lemma 49.13.1. Alternatively, we can choose a presentation $X = U/R$ and consider the closed subscheme $Z \subset U$ cut out by $f_1|U, \ldots, f_r|_U$. It is clear that Z is an R-invariant (see Groupoids, Definition 38.17.1) closed subscheme and we may set $V(f_1, \ldots, f_r) = Z/R_Z$.

Lemma 57.22.1. *Let S be a scheme. Consider a cartesian diagram*

$$\begin{array}{ccc} X & \xleftarrow{p} & F \\ \downarrow & & \downarrow \\ Y & \longleftarrow & \mathrm{Spec}(k) \end{array}$$

where $X \to Y$ is a morphism of algebraic spaces over S which is flat and locally of finite presentation, and where k is a field over S. Let $f_1, \ldots, f_r \in \Gamma(X, \mathcal{O}_X)$ and $z \in |F|$ such that $f_1, \ldots, f_r$ map to a regular sequence in the local ring $\mathcal{O}_{F,\overline{z}}$. Then, after replacing X by an open subspace containing $p(z)$, the morphism

$$V(f_1, \ldots, f_r) \longrightarrow Y$$

is flat and locally of finite presentation.

Proof. Set $Z = V(f_1, \ldots, f_r)$. It is clear that $Z \to X$ is locally of finite presentation, hence the composition $Z \to Y$ is locally of finite presentation, see Morphisms of Spaces, Lemma 49.26.2. Hence it suffices to show that $Z \to Y$ is flat in a neighbourhood of $p(z)$. Let $k \subset k'$ be an extension field. Then $F' = F \times_{\mathrm{Spec}(k)} \mathrm{Spec}(k')$ is surjective and flat over F, hence we can find a point $z' \in |F'|$ mapping to z and the local ring map $\mathcal{O}_{F,\overline{z}} \to \mathcal{O}_{F',\overline{z}'}$ is flat, see Morphisms of Spaces, Lemma 49.27.8. Hence the image of $f_1, \ldots, f_r$ in $\mathcal{O}_{F',\overline{z}'}$ is a regular sequence too, see Algebra, Lemma 10.65.7. Thus, during the proof we may replace k by an extension field. In particular, we may assume that $z \in |F|$ comes from a section $z : \mathrm{Spec}(k) \to F$ of the structure morphism $F \to \mathrm{Spec}(k)$.

Choose a scheme V and a surjective étale morphism $V \to Y$. Choose a scheme U and a surjective étale morphism $U \to X \times_Y V$. After possibly enlarging k once more we may assume that $\mathrm{Spec}(k) \to F \to X$ factors through U (as $U \to X$ is surjective). Let $u : \mathrm{Spec}(k) \to U$ be such a factorization and denote $v \in V$ the image of u. Note that the morphisms

$$U_v \times_{\mathrm{Spec}(\kappa(v))} \mathrm{Spec}(k) = U \times_V \mathrm{Spec}(k) \to U \times_Y \mathrm{Spec}(k) \to F$$

are étale (the first as the base change of $V \to V \times_Y V$ and the second as the base change of $U \to X$). Moreover, by construction the point $u : \mathrm{Spec}(k) \to U$ gives a point of the left most space which maps to z on the right. Hence the elements

$f_1, \ldots, f_r$ map to a regular sequence in the local ring on the right of the following map

$$\mathcal{O}_{U_v,u} \longrightarrow \mathcal{O}_{U_v \times_{\mathrm{Spec}(\kappa(v))} \mathrm{Spec}(k), \overline{u}} = \mathcal{O}_{U \times_V \mathrm{Spec}(k), \overline{u}}.$$

But since the displayed arrow is flat (combine More on Flatness, Lemma 37.2.5 and Morphisms of Spaces, Lemma 49.27.8) we see from Algebra, Lemma 10.65.7 that $f_1, \ldots, f_r$ maps to a regular sequence in $\mathcal{O}_{U_v,u}$. By More on Morphisms, Lemma 36.18.2 we conclude that the morphism of schemes

$$V(f_1, \ldots, f_r) \times_X U = V(f_1|_U, \ldots, f_r|_U) \to V$$

is flat in an open neighbourhood U' of u. Let $X' \subset X$ be the open subspace corresponding to the image of $|U'| \to |X|$ (see Properties of Spaces, Lemmas 48.4.6 and 48.4.8). We conclude that $V(f_1, \ldots, f_r) \cap X' \to Y$ is flat (see Morphisms of Spaces, Definition 49.27.1) as we have the commutative diagram

$$\begin{array}{ccc} V(f_1, \ldots, f_r) \times_X U' & \longrightarrow & V \\ \downarrow{\scriptstyle a} & & \downarrow{\scriptstyle b} \\ V(f_1, \ldots, f_r) \cap X' & \longrightarrow & Y \end{array}$$

with a, b étale and a surjective. □

57.23. Étale localization of morphisms

The section is the analogue of More on Morphisms, Section 36.30.

Lemma 57.23.1. *Let S be a scheme. Let $f : X \to Y$ be a morphism of algebraic spaces over S. Let $y \in |Y|$. Let $x_1, \ldots, x_n \in |X|$ mapping to y. Assume that*

(1) *f is locally of finite type,*
(2) *f is separated, and*
(3) *f is quasi-finite at $x_1, \ldots, x_n$.*

Then there exists an étale morphism $(U, u) \to (Y, y)$ of pointed algebraic spaces and a decomposition

$$U \times_Y X = W \amalg V$$

into open and closed subspaces such that the morphism $V \to U$ is finite, every point of the fibre of $|V| \to |U|$ over u maps to an x_i, and the fibre of $|W| \to |U|$ over u contains no point mapping to an x_i.

Proof. We may assume Y is an affine scheme as the question is étale local on Y. Let $X' \subset X$ be an open subspace containing $x_1, \ldots, x_n$. Then it suffices to prove the result for $X' \to Y$. Thus we may assume X is quasi-compact and separated. Choose an affine scheme Z and surjective étale morphism $Z \to X$ (Properties of Spaces, Lemma 48.6.3). The fibres of $|Z| \to |X|$ are finite, see Properties of Spaces, Lemma 48.12.3 (or the more general discussion in Decent Spaces, Section 50.6). Let $\{z_1, \ldots, z_m\} \subset |Z|$ be the preimage of $\{x_1, \ldots, x_n\}$. By More on Morphisms, Lemma 36.30.4, after replacing (Y, y) by an étale neighbourhood, we may assume that $Z = Z_1 \amalg Z_2$ with $Z_1 \to Y$ finite and $(Z_1)_y = \{z_1, \ldots, z_m\}$. Since $X \to Y$ is separated, the image V of $Z_1 \to X$ is both open and closed (Morphisms of Spaces, Lemma 49.36.6). Set $W = X \setminus V$ to get a decomposition as in the lemma. To finish the proof we have to show that $V \to Y$ is finite. It is clear that $V \to Y$ is separated (by assumption) and locally quasi-finite (as Z_1 is finite over Y and an étale cover

of V). Hence V is a scheme, see Morphisms of Spaces, Proposition 49.43.2. By Morphisms, Lemma 28.42.8 we conclude that $V \to Y$ is proper. An application of More on Morphisms, Lemma 36.31.5 shows $V \to Y$ is finite, finishing the proof. $\square$

57.24. Zariski's Main Theorem

In this section we apply the results of the previous section to prove Zariski's main theorem for morphisms of algebraic spaces. This section is the analogue of More on Morphisms, Section 36.31.

Lemma 57.24.1. *Let S be a scheme. Let $f : X \to Y$ be a morphism of algebraic spaces over S which is of finite type and separated. Let Y' be the normalization of Y in X. Picture:*

$$\begin{array}{ccc} X & \xrightarrow{\quad} & Y' \\ & {\scriptstyle f'} \searrow \quad \swarrow {\scriptstyle \nu} & \\ & {\scriptstyle f} \searrow \quad \swarrow & \\ & Y & \end{array}$$

Then there exists an open subspace $U' \subset Y'$ such that

(1) *$(f')^{-1}(U') \to U'$ is an isomorphism, and*
(2) *$(f')^{-1}(U') \subset X$ is the set of points at which f is quasi-finite.*

Proof. By Morphisms of Spaces, Lemma 49.31.7 there is an open subspace $U \subset X$ corresponding to the points of $|X|$ where f is quasi-finite. We have to prove

(a) the image of $|U| \to |Y'|$ is $|U'|$ for some open subspace U' of Y',
(b) $U = f^{-1}(U')$, and
(c) $U \to U'$ is an isomorphism.

Since formation of U commutes with arbitrary base change (Morphisms of Spaces, Lemma 49.31.7), since formation of the normalization Y' commutes with smooth base change (Lemma 57.21.2), since étale morphisms are open, and since "being an isomorphism" is fpqc local on the base (Descent on Spaces, Lemma 55.10.13), it suffices to prove (a), (b), (c) étale locally on Y (some details omitted). Thus we may assume Y is an affine scheme. This implies that Y' is an (affine) scheme as well.

Let $x \in |U|$ be arbitrary. We claim that there exists an open neighbourhood $f'(x) \in V \subset Y'$ such that $(f')^{-1}V \to V$ is an isomorphism. We first prove the claim implies the lemma. Namely, then $(f')^{-1}V \cong V$ is a scheme (as an open of Y'), locally of finite type over Y (as an open subspace of X), and for $v \in V$ the residue field extension $\kappa(v) \supset \kappa(\nu(v))$ is algebraic (as $V \subset Y'$ and Y' is integral over Y). Hence the fibres of $V \to Y$ are discrete (Morphisms, Lemma 28.21.2) and $(f')^{-1}V \to Y$ is locally quasi-finite (Morphisms, Lemma 28.21.8). This implies $(f')^{-1}V \subset U$ and $V \subset U'$. Since x was arbitrary we see that (a), (b), and (c) are true.

Let $y = f(x) \in |Y|$. Let $(T, t) \to (Y, y)$ be an étale morphism of pointed schemes. Denote by a subscript $_T$ the base change to T. Let $z \in X_T$ be a point in the fibre X_t lying over x. Note that $U_T \subset X_T$ is the set of points where f_T is quasi-finite, see Morphisms of Spaces, Lemma 49.31.7. Note that

$$X_T \xrightarrow{f'_T} Y'_T \xrightarrow{\nu_T} T$$

is the normalization of T in X_T, see Lemma 57.21.2. Suppose that the claim holds for $z \in U_T \subset X_T \to Y'_T \to T$, i.e., suppose that we can find an open neighbourhood $f'_T(z) \in V' \subset Y'_T$ such that $(f'_T)^{-1}V' \to V'$ is an isomorphism. The morphism $Y'_T \to Y'$ is étale hence the image $V \subset Y'$ of V' is open. Observe that $f'(x) \in V$ as $f'_T(z) \in V'$. Observe that

$$\begin{array}{ccc} (f'_T)^{-1}V' & \longrightarrow & (f')^{-1}(V) \\ \downarrow & & \downarrow \\ V' & \longrightarrow & V \end{array}$$

is a fibre square (as $Y'_T \times_{Y'} X = X_T$). Since the left vertical arrow is an isomorphism and $\{V' \to V\}$ is a étale covering, we conclude that the right vertical arrow is an isomorphism by Descent on Spaces, Lemma 55.10.13. In other words, the claim holds for $x \in U \subset X \to Y' \to Y$.

By the result of the previous paragraph we may replace Y by an étale neighbourhood of $y = f(x)$ in order to prove the claim. Thus we may assume there is a decomposition

$$X = V \amalg W$$

into open and closed subspaces where $V \to Y$ is finite and $x \in V$, see Lemma 57.23.1. Since X is a disjoint union of V and W over Y and since $V \to Y$ is finite we see that the normalization of Y in X is the morphism

$$X = V \amalg W \longrightarrow V \amalg W' \longrightarrow S$$

where W' is the normalization of Y in W, see Morphisms of Spaces, Lemmas 49.42.5, 49.40.6, and 49.42.6. The claim follows and we win. □

Lemma 57.24.2. *Let S be a scheme. Let $f : X \to Y$ be a morphism of algebraic spaces over S. Assume f is quasi-finite and separated. Let Y' be the normalization of Y in X. Picture:*

$$\begin{array}{ccc} X & \xrightarrow{f'} & Y' \\ & {\scriptstyle f}\searrow \quad \swarrow{\scriptstyle \nu} & \\ & Y & \end{array}$$

Then f' is a quasi-compact open immersion and ν is integral. In particular f is quasi-affine.

Proof. This follows from Lemma 57.24.1. Namely, by that lemma there exists an open subspace $U' \subset Y'$ such that $(f')^{-1}(U') = X$ (!) and $X \to U'$ is an isomorphism! In other words, f' is an open immersion. Note that f' is quasi-compact as f is quasi-compact and $\nu : Y' \to Y$ is separated (Morphisms of Spaces, Lemma 49.9.8). Hence for every affine scheme Z and morphism $Z \to Y$ the fibre product $Z \times_Y X$ is a quasi-compact open subscheme of the affine scheme $Z \times_Y Y'$. Hence f is quasi-affine by definition. □

Lemma 57.24.3. *Let S be a scheme. Let $f : X \to Y$ be a morphism of algebraic spaces over S. Assume f is quasi-finite and separated and assume that Y is quasi-compact and quasi-separated. Then there exists a factorization*

$$\begin{array}{ccc} X & \xrightarrow{j} & T \\ & {\scriptstyle f}\searrow \quad \swarrow{\scriptstyle \pi} & \\ & Y & \end{array}$$

where j is a quasi-compact open immersion and π is finite.

Proof. Let $X \to Y' \to Y$ be as in the conclusion of Lemma 57.24.2. By Limits of Spaces, Lemma 52.9.7 we can write $\nu_*\mathcal{O}_{Y'} = \operatorname{colim}_{i\in I} \mathcal{A}_i$ as a directed colimit of finite quasi-coherent $\mathcal{O}_X$-algebras $\mathcal{A}_i \subset \nu_*\mathcal{O}_{Y'}$. Then $\pi_i : T_i = \underline{\operatorname{Spec}}_Y(\mathcal{A}_i) \to Y$ is a finite morphism for each i. Note that the transition morphisms $T_{i'} \to T_i$ are affine and that $Y' = \lim T_i$.

By Limits of Spaces, Lemma 52.5.5 there exists an i and a quasi-compact open $U_i \subset T_i$ whose inverse image in Y' equals $f'(X)$. For $i' \geq i$ let $U_{i'}$ be the inverse image of U_i in $T_{i'}$. Then $X \cong f'(X) = \lim_{i' \geq i} U_{i'}$, see Limits of Spaces, Lemma 52.4.1. By Limits of Spaces, Lemma 52.5.10 we see that $X \to U_{i'}$ is a closed immersion for some $i' \geq i$. (In fact $X \cong U_{i'}$ for sufficiently large i' but we don't need this.) Hence $X \to T_{i'}$ is an immersion. By Morphisms of Spaces, Lemma 49.4.5 we can factor this as $X \to T \to T_{i'}$ where the first arrow is an open immersion and the second a closed immersion. Thus we win. □

Lemma 57.24.4. *With notation and hypotheses as in Lemma 57.24.3. Assume moreover that f is locally of finite presentation. Then we can choose the factorization such that T is finite and of finite presentation over Y.*

Proof. By Limits of Spaces, Lemma 52.11.3 we can write $T = \lim T_i$ where all T_i are finite and of finite presentation over Y and the transition morphisms $T_{i'} \to T_i$ are closed immersions. By Limits of Spaces, Lemma 52.5.5 there exists an i and an open subscheme $U_i \subset T_i$ whose inverse image in T is X. By Limits of Spaces, Lemma 52.5.10 we see that $X \cong U_i$ for large enough i. Replacing T by T_i finishes the proof. □

Lemma 57.24.5. *Let S be a scheme. Let $U \to X$ be a surjective étale morphism of algebraic spaces over S. Assume X is a quasi-compact and quasi-separated algebraic space. Then there exists a surjective integral morphism $Y \to X$ where Y is a scheme, such that for every $y \in Y$ there is an open neighbourhood $V \subset Y$ such that $V \to X$ factors through U.*

Proof. Choose a surjective étale morphism $U' \to U$ where U' is a scheme. It is clear that we may replace U by U' and hence we may assume U is a scheme. Since X is quasi-compact, there exist finitely many affine opens $U_i \subset U$ such that $U' = \coprod U_i \to X$ is surjective. After replacing U by U' again, we see that we may assume U is affine. Since X is quasi-separated, hence reasonable, there exists an integer d bounding the degree of the geometric fibres of $U \to X$ (see Decent Spaces, Lemma 50.5.1). We will prove the lemma by induction on d for all quasi-compact and separated schemes U mapping surjective and étale onto X. If $d = 1$, then $U = X$ and the result holds with $Y = U$. Assume $d > 1$.

We apply Lemma 57.24.2 and we obtain a factorization

$$\begin{array}{ccc} U & \xrightarrow{\quad j \quad} & Y \\ & \searrow \quad \swarrow_{\pi} & \\ & X & \end{array}$$

with π integral and j a quasi-compact open immersion. We may and do assume that $j(U)$ is scheme theoretically dense in Y. Note that

$$U \times_X Y = U \amalg W$$

where the first summand is the image of $U \to U \times_X Y$ (which is closed by Morphisms of Spaces, Lemma 49.5.6 and open because it is étale as a morphism between algebraic spaces étale over Y) and the second summand is the (open and closed) complement. The image $V \subset Y$ of W is an open subspace containing $Y \setminus U$.

The étale morphism $W \to Y$ has geometric fibres of cardinality $< d$. Namely, this is clear for geometric points of $U \subset Y$ by inspection. Since $|U| \subset |Y|$ is dense, it holds for all geometric points of Y for example by Decent Spaces, Lemma 50.8.1 (the degree of the fibres of a quasi-compact étale morphism does not go up under specialization). Thus we may apply the induction hypothesis to $W \to V$ and find a surjective integral morphism $Z \to V$ with Z a scheme, which Zariski locally factors through W. Choose a factorization $Z \to Z' \to Y$ with $Z' \to Y$ integral and $Z \to Z'$ open immersion (Lemma 57.24.2). After replacing Z' by the scheme theoretic closure of Z in Z' we may assume that Z is scheme theoretically dense in Z'. After doing this we have $Z' \times_Y V = Z$. Finally, let $T \subset Y$ be the induced closed subspace structure on $Y \setminus V$. Consider the morphism

$$Z' \amalg T \longrightarrow X$$

This is a surjective integral morphism by construction. Since $T \subset U$ it is clear that the morphism $T \to X$ factors through U. On the other hand, let $z \in Z'$ be a point. If $z \notin Z$, then z maps to a point of $Y \setminus V \subset U$ and we find a neighbourhood of z on which the morphism factors through U. If $z \in Z$, then we have a neighbourhood $V \subset Z$ which factors through $W \subset U \times_X Y$ and hence through U. □

Lemma 57.24.6. *In the the situation of Lemma 57.24.5 we may choose Y to be finite and of finite presentation over X.*

Proof. Choose a finite affine open covering $Y = \bigcup V_j$ such that $V_j \to X$ factors through U. We can write $Y = \lim Y_i$ with $Y_i \to X$ finite and of finite presentation, see Limits of Spaces, Lemma 52.11.2. For large enough i the algebraic space Y_i is a scheme, see Limits of Spaces, Lemma 52.5.9. For large enough i we can find affine opens $V_{i,j} \subset Y_i$ whose inverse image in Y recovers V_j, see Limits of Spaces, Lemma 52.5.5. For even larger i the morphisms $V_j \to U$ over X come from morphisms $V_{i,j} \to U$ over X, see Limits of Spaces, Proposition 52.3.9. This finishes the proof. □

57.25. Extending properties from an open

In this section we collect a number of results of the form: If $f : X \to Y$ is a flat morphism of algebraic spaces and f satisfies some property over a dense open of Y, then f satisfies the same property over all of Y.

Lemma 57.25.1. *Let S be a scheme. Let $f : X \to Y$ be a morphism of algebraic spaces over S. Let $\mathcal{F}$ be a quasi-coherent $\mathcal{O}_X$-module. Let $V \subset Y$ be an open subspace. Assume*

(1) *f is locally of finite presentation,*
(2) *$\mathcal{F}$ is of finite type and flat over Y,*
(3) *$V \to Y$ is quasi-compact and scheme theoretically dense,*
(4) *$\mathcal{F}|_{f^{-1}V}$ is of finite presentation.*

Then $\mathcal{F}$ is of finite presentation.

Proof. It suffices to prove the pullback of $\mathcal{F}$ to a scheme surjective and étale over X is of finite presentation. Hence we may assume X is a scheme. Similarly, we can replace Y by a scheme surjective and étale and over Y (the inverse image of V in this scheme is scheme theoretically dense, see Morphisms of Spaces, Section 49.17). Thus we reduce to the case of schemes which is More on Flatness, Lemma 37.10.11. □

Lemma 57.25.2. *Let S be a scheme. Let $f : X \to Y$ be a morphism of algebraic spaces over S. Let $V \subset Y$ be an open subspace. Assume*

(1) *f is locally of finite type and flat,*
(2) *$V \to Y$ is quasi-compact and scheme theoretically dense,*
(3) *$f|_{f^{-1}V} : f^{-1}V \to V$ is locally of finite presentation.*

Then f is of locally of finite presentation.

Proof. The proof is identical to the proof of Lemma 57.25.1 except one uses More on Flatness, Lemma 37.10.12. □

Lemma 57.25.3. *Let S be a scheme. Let $f : X \to Y$ be a morphism of algebraic spaces over S. Let $V \subset Y$ be an open subspace. Let $d \geq 0$. Assume*

(1) *f is flat and locally of finite presentation,*
(2) *$V \subset Y$ is scheme theoretically dense, and*
(3) *$f|_{f^{-1}V} : f^{-1}V \to V$ has relative dimension $\leq d$.*

Then $f : X \to Y$ has relative dimension $\leq d$.

Proof. By definition the property of having relative dimension $\leq d$ can be checked on an étale covering, see Morphisms of Spaces, Sections 49.30. Thus it suffices to prove f has relative dimension $\leq d$ after replacing X by a scheme surjective and étale over X. Similarly, we can replace Y by a scheme surjective and étale and over Y. The inverse image of V in this scheme is scheme theoretically dense, see Morphisms of Spaces, Section 49.17. Since a scheme theoretically dense open of a scheme is in particular dense, we reduce to the case of schemes which is More on Morphisms, Lemma 36.17.7. □

Lemma 57.25.4. *Let S be a scheme. Let $f : X \to Y$ be a morphism of algebraic spaces over S. Let $V \subset Y$ be an open subspace. If*

(1) *f is separated, locally of finite type, and flat,*
(2) *$f^{-1}(V) \to V$ is an isomorphism, and*
(3) *$V \to Y$ is quasi-compact and scheme theoretically dense,*

then f is an open immersion.

Proof. Applying Lemma 57.25.2 we see that f is locally of finite presentation. Applying Lemma 57.25.3 we see that f has relative dimension ≤ 0. By Morphisms of Spaces, Lemma 49.31.6 this implies that f is locally quasi-finite. By Morphisms of Spaces, Lemma 49.44.1 this implies that f is representable. By Descent on Spaces, Lemma 55.10.12 we can check whether f is an open immersion étale locally on Y. Hence we may assume that Y is a scheme. Since f is representable, we reduce to the case of schemes which is More on Morphisms, Lemma 36.31.4. □

57.26. Blowing up and flatness

Instead of redoing the work in More on Flatness, Section 37.26 we prove an analogue of More on Flatness, Lemma 37.26.6 which tells us that the problem of finding a suitable blowup is often étale local on the base.

Lemma 57.26.1. *Let S be a scheme. Let X be a quasi-compact and quasi-separated algebraic space over S. Let $\varphi : W \to X$ be a quasi-compact separated étale morphism. Let $U \subset X$ be a quasi-compact open subspace. Let $\mathcal{I} \subset \mathcal{O}_U$ be a finite type quasi-coherent sheaf of ideals such that $V(\mathcal{I}) \cap \varphi^{-1}(U) = \emptyset$. Then there exists a finite type quasi-coherent sheaf of ideals $\mathcal{J} \subset \mathcal{O}_X$ such that*

(1) *$V(\mathcal{J}) \cap U = \emptyset$, and*
(2) *$\varphi^{-1}(\mathcal{J})\mathcal{O}_W = \mathcal{I}\mathcal{I}'$ for some finite type quasi-coherent ideal $\mathcal{I}' \subset \mathcal{O}_W$.*

Proof. Choose a factorization $W \to Y \to X$ where $j : W \to Y$ is a quasi-compact open immersion and $\pi : Y \to X$ is a finite morphism of finite presentation (Lemma 57.24.4). Let $V = j(W) \cup \pi^{-1}(U) \subset Y$. Note that $\mathcal{I}$ on $W \cong j(W)$ and $\mathcal{O}_{\pi^{-1}(U)}$ glue to a finite type quasi-coherent sheaf of ideals $\mathcal{I}_1 \subset \mathcal{O}_V$. By Limits of Spaces, Lemma 52.9.8 there exists a finite type quasi-coherent sheaf of ideals $\mathcal{I}_2 \subset \mathcal{O}_Y$ such that $\mathcal{I}_2|_V = \mathcal{I}_1$. In other words, $\mathcal{I}_2 \subset \mathcal{O}_Y$ is a finite type quasi-coherent sheaf of ideals such that $V(\mathcal{I}_2)$ is disjoint from $\pi^{-1}(U)$ and $j^{-1}\mathcal{I}_2 = \mathcal{I}$. Denote $i : Z \to Y$ the corresponding closed immersion which is of finite presentation (Morphisms of Spaces, Lemma 49.26.12). In particular the composition $\tau = \pi \circ i : Z \to X$ is finite and of finite presentation (Morphisms of Spaces, Lemmas 49.26.2 and 49.40.4).

Let $\mathcal{F} = \tau_*\mathcal{O}_Z$ which we think of as a quasi-coherent $\mathcal{O}_X$-module. By Descent on Spaces, Lemma 55.5.7 we see that $\mathcal{F}$ is a finitely presented $\mathcal{O}_X$-module. Let $\mathcal{J} = \mathrm{Fit}_0(\mathcal{F})$. (Insert reference to fitting modules on ringed topoi here.) This is a finite type quasi-coherent sheaf of ideals on X (as $\mathcal{F}$ is of finite presentation, see More on Algebra, Lemma 15.5.4). Part (1) of the lemma holds because $|\tau|(|Z|) \cap |U| = \emptyset$ by our choice of $\mathcal{I}_2$ and because the 0th fitting ideal of the trivial module equals the structure sheaf. To prove (2) note that $\varphi^{-1}(\mathcal{J})\mathcal{O}_W = \mathrm{Fit}_0(\varphi^*\mathcal{F})$ because taking fitting ideals commutes with base change. On the other hand, as $\varphi : W \to X$ is separated and étale we see that $(1, j) : W \to W \times_X Y$ is an open and closed immersion. Hence $W \times_Y Z = V(\mathcal{I}) \amalg Z'$ for some finite and finitely presented morphism of algebraic spaces $\tau' : Z' \to W$. Thus we see that

$$\begin{aligned} \mathrm{Fit}_0(\varphi^*\mathcal{F}) &= \mathrm{Fit}_0((W \times_Y Z \to W)_*\mathcal{O}_{W \times_Y Z}) \\ &= \mathrm{Fit}_0(\mathcal{O}_W/\mathcal{I}) \cdot \mathrm{Fit}_0(\tau'_*\mathcal{O}_{Z'}) \\ &= \mathcal{I} \cdot \mathrm{Fit}_0(\tau'_*\mathcal{O}_{Z'}) \end{aligned}$$

the second equality by More on Algebra, Lemma 15.5.4 translated in sheaves on ringed topoi. Setting $\mathcal{I}' = \mathrm{Fit}_0(\tau'_*\mathcal{O}_{Z'})$ finishes the proof of the lemma. □

Theorem 57.26.2. *Let S be a scheme. Let B be a quasi-compact and quasi-separated algebraic space over S. Let X be an algebraic space over B. Let $\mathcal{F}$ be a quasi-coherent module on X. Let $U \subset B$ be a quasi-compact open subspace. Assume*

(1) *X is quasi-compact,*
(2) *X is locally of finite presentation over B,*
(3) *$\mathcal{F}$ is a module of finite type,*
(4) *$\mathcal{F}_U$ is of finite presentation, and*
(5) *$\mathcal{F}_U$ is flat over U.*

Then there exists a U-admissible blowup $B' \to B$ such that the strict transform $\mathcal{F}'$ of $\mathcal{F}$ is an $\mathcal{O}_{X \times_B B'}$-module of finite presentation and flat over B'.

Proof. Choose an affine scheme V and a surjective étale morphism $V \to X$. Because strict transform commutes with étale localization (Divisors on Spaces, Lemma 53.7.2) it suffices to prove the result with X replaced by V. Hence we may assume that $X \to B$ is representable (in addition to the hypotheses of the lemma).

Assume that $X \to B$ is representable. Choose an affine scheme W and a surjective étale morphism $\varphi : W \to B$. Note that $X \times_B W$ is a scheme. By the case of schemes (More on Flatness, Theorem 37.26.8) we can find a finite type quasi-coherent sheaf of ideals $\mathcal{I} \subset \mathcal{O}_W$ such that (a) $|V(\mathcal{I})| \cap |\varphi^{-1}(U)| = \emptyset$ and (b) the strict transform of $\mathcal{F}|_{X \times_B W}$ with respect to the blowing up $W' \to W$ in $\mathcal{I}$ becomes flat over W' and is a module of finite presentation. Choose a finite type sheaf of ideals $\mathcal{J} \subset \mathcal{O}_B$ as in Lemma 57.26.1. Let $B' \to B$ be the blowing up of $\mathcal{J}$. We claim that this blow up works. Namely, it is clear that $B' \to B$ is U-admissible by our choice of ideal $\mathcal{J}$. Moreover, the base change $B' \times_B W \to W$ is the blowup of W in $\varphi^{-1}\mathcal{J} = \mathcal{I}\mathcal{I}'$ (compatibility of blowup with flat base change, see Divisors on Spaces, Lemma 53.6.3). Hence there is a factorization

$$W \times_B B' \to W' \to W$$

where the first morphism is a blowup as well, see Divisors on Spaces, Lemma 53.6.9). The restriction of $\mathcal{F}'$ (which lives on $B' \times_B X$) to $W \times_B B' \times_B X$ is the strict transform of $\mathcal{F}|_{X \times_B W}$ (Divisors on Spaces, Lemma 53.7.2) and hence is the twice repeated strict transform of $\mathcal{F}|_{X \times_B W}$ by the two blowups displayed above (Divisors on Spaces, Lemma 53.7.7). After the first blow up our sheaf is already flat over the base and of finite presentation (by construction). Whence this holds after the second strict transform as well (since this is a pullback by Divisors on Spaces, Lemma 53.7.4). Thus we see that the restriction of $\mathcal{F}'$ to an étale cover of $B' \times_B X$ has the desired properties and the theorem is proved. □

57.27. Applications

In this section we apply the result on flattening by blowing up.

Lemma 57.27.1. *Let S be a scheme. Let B be a quasi-compact and quasi-separated algebraic space over S. Let X be an algebraic space over B. Let $U \subset B$ be a quasi-compact open subspace. Assume*

(1) *$X \to B$ is of finite type and quasi-separated, and*
(2) *$X_U \to U$ is flat and locally of finite presentation.*

Then there exists a U-admissible blowup $B' \to B$ such that the strict transform of X is flat and of finite presentation over B'.

Proof. Let $B' \to B$ be a U-admissible blowup. Note that the strict transform of X is quasi-compact and quasi-separated over B' as X is quasi-compact and quasi-separated over B. Hence we only need to worry about finding a U-admissible blowup such that the strict transform becomes flat and locally of finite presentation. We cannot directly apply Theorem 57.26.2 because X is not locally of finite presentation over B.

Choose an affine scheme V and a surjective étale morphism $V \to X$. (This is possible as X is quasi-compact as a finite type space over the quasi-compact space B.) Then it suffices to show the result for the morphism $V \to B$ (as strict transform commutes with étale localization, see Divisors on Spaces, Lemma 53.7.2). Hence we may assume that $X \to B$ is separated as well as finite type. In this case we can find a closed immersion $i : X \to Y$ with $Y \to B$ separated and of finite presentation, see Limits of Spaces, Proposition 52.11.6.

Apply Theorem 57.26.2 to $\mathcal{F} = i_*\mathcal{O}_X$ on Y/B. We find a U-admissible blowup $B' \to B$ such that that strict transform of $\mathcal{F}$ is flat over B' and of finite presentation. Let X' be the strict transform of X under the blowup $B' \to B$. Let $i' : X' \to Y \times_B B'$ be the induced morphism. Since taking strict transform commutes with pushforward along affine morphisms (Divisors on Spaces, Lemma 53.7.5), we see that $i'_*\mathcal{O}_{X'}$ is flat over B' and of finite presentation as a $\mathcal{O}_{Y \times_B B'}$-module. Thus $X' \to B'$ is flat and locally of finite presentation. This implies the lemma by our earlier remarks. □

Lemma 57.27.2. *Let S be a scheme. Let $\varphi : X \to B$ be a morphism of algebraic spaces over S. Assume φ is of finite type with B quasi-compact and quasi-separated. Let $U \subset B$ be a quasi-compact open subspace such that $\varphi^{-1}U \to U$ is an isomorphism. Then there exists a U-admissible blowup $B' \to B$ such that U is scheme theoretically dense in B' and such that the strict transform X' of X is isomorphic to an open subspace of B'.*

Proof. As the composition of U-admissible blowups is U-admissible (Divisors on Spaces, Lemma 53.8.2) we can proceed in stages. Pick a finite type quasi-coherent sheaf of ideals $\mathcal{I} \subset \mathcal{O}_B$ with $|B| \setminus |U| = |V(\mathcal{I})|$. Replace B by the blowup of B in $\mathcal{I}$ and X by the strict transform of X. After this replacement $B \setminus U$ is the support of an effective Cartier divisor D (Divisors on Spaces, Lemma 53.6.4). In particular U is scheme theoretically dense in B (Divisors on Spaces, Lemma 53.2.4). Next, we do another U-admissible blowup to get to the situation where $X \to B$ is flat and of finite presentation, see Lemma 57.27.1. Note that U is still scheme theoretically dense in B. Hence $X \to B$ is an open immersion by Lemma 57.25.4. □

The following lemma says that a proper modification can be dominated by a blowup.

Lemma 57.27.3. *Let S be a scheme. Let $\varphi : X \to B$ be a proper morphism of algebraic spaces over S. Assume B quasi-compact and quasi-separated. Let $U \subset B$ be a quasi-compact open subspace such that $\varphi^{-1}U \to U$ is an isomorphism. Then there exists a U-admissible blowup $B' \to B$ which dominates X, i.e., such that there exists a factorization $B' \to X \to B$ of the blowup morphism.*

Proof. By Lemma 57.27.2 we may find a U-admissible blowup $B' \to B$ such that the strict transform X' is an open subspace of B' and U is scheme theoretically dense in B'. Since $X' \to B'$ is proper we see that $|X'|$ is closed in $|B'|$. As $U \subset B'$ is dense $X' = B'$. □

57.28. Chow's lemma

In this section we prove some variants of Chow's lemma. Since we have yet to define projective morphisms of algebraic spaces, the statements will involve representable proper morphisms, rather than projective ones.

Lemma 57.28.1. *Let S be a scheme. Let Y be a quasi-compact and quasi-separated algebraic space over S. Let $U \to X_1$ and $U \to X_2$ be open immersions of algebraic spaces over Y and assume U, X_1, X_2 of finite type and separated over Y. Then there exists a commutative diagram*

$$\begin{array}{ccccc} X_1' & \longrightarrow & X & \longleftarrow & X_2' \\ \downarrow & \nwarrow & \uparrow & \nearrow & \downarrow \\ X_1 & \longleftarrow & U & \longrightarrow & X_2 \end{array}$$

of algebraic spaces over Y where $X_i' \to X_i$ is a U-admissible blowup, $X_i' \to X$ is an open immersion, and X is separated and finite type over Y.

Proof. Throughout the proof all the algebraic spaces will be separated of finite type over Y. This in particular implies these algebraic spaces and the morphisms between them will be quasi-compact and quasi-separated. We will use that if $U \to W$ is an immersion of such spaces over Y, then the scheme theoretic image Z of U in W is a closed subspace of W and $U \to Z$ is an open immersion, $U \subset Z$ is scheme theoretically dense, and $|U| \subset |Z|$ is dense. See Morphisms of Spaces, Lemma 49.17.7.

Let $X_{12} \subset X_1 \times_Y X_2$ be the scheme theoretic image of $U \to X_1 \times_Y X_2$. We claim the projections $p_i : X_{12} \to X_i$ induce isomorphisms $p_i^{-1}(U) \to U$. Namely, $p_i : X_{12} \to X_i$ is separated and $U \to X_{12}$ is a section of p_i. Hence $U \to p_i^{-1}(U)$ is a closed immersion (Morphisms of Spaces, Lemma 49.5.6) as well as scheme theoretically dense whence an isomorphism. Choose a U-admissible blowup $X_i^i \to X_i$ such that the strict transform X_{12}^i of X_{12} is isomorphic to an open subspace of X_i^i, see Lemma 57.27.2. Let $\mathcal{I}_i \subset \mathcal{O}_{X_i}$ be the corresponding finite type quasi-coherent sheaf of ideals. Recall that $X_{12}^i \to X_{12}$ is the blowup in $p_i^{-1}\mathcal{I}_i\mathcal{O}_{X_{12}}$. Let X_{12}' be the blowup of X_{12} in $p_1^{-1}\mathcal{I}_1 p_2^{-1}\mathcal{I}_2\mathcal{O}_{X_{12}}$. We obtain a commutative diagram

$$\begin{array}{ccc} X_{12}' & \longrightarrow & X_{12}^2 \\ \downarrow & & \downarrow \\ X_{12}^1 & \longrightarrow & X_{12} \end{array}$$

where all the morphisms are U-admissible blowing ups. Choose a finite type quasi-coherent sheaf of ideals $\mathcal{J}_i$ on X_i^i extending the pull back of $\mathcal{I}_{1-i}$ to X_{12}^i (see Limits of Spaces, Lemma 52.9.8). Let $X_i' \to X_i^i$ be the blowing up in $\mathcal{J}_i$. By construction $X_{12}' \subset X_i'$ is an open subspace and the diagram

$$\begin{array}{ccc} X_{12}' & \longrightarrow & X_i' \\ \downarrow & & \downarrow \\ X_{12}^i & \longrightarrow & X_i^i \end{array}$$

is commutative with vertical arrows blowing ups and horizontal arrows open immersions. Note that $X'_{12} \to X'_1 \times_Y X'_2$ is an immersion and proper (use that $X'_{12} \to X_{12}$ is proper and $X_{12} \to X_1 \times_Y X_2$ is closed and $X'_1 \times_Y X'_2 \to X_1 \times_Y X_2$ is separated and apply Morphisms of Spaces, Lemma 49.36.6). Thus $X'_{12} \to X'_1 \times_Y X'_2$ is a closed immersion. It follows that if we define X by glueing X'_1 and X'_2 along the common open subspace X'_{12}, then $X \to Y$ is of finite type and separated (details omitted). As compositions of U-admissible blowups are U-admissible blowups (Divisors on Spaces, Lemma 53.8.2) the lemma is proved. □

Lemma 57.28.2. *Let S be a scheme. Let $f : X \to Y$ be a morphism of algebraic spaces over S. Let $U \subset X$ be an open subscheme. Assume*

(1) *U is quasi-compact,*
(2) *Y is quasi-compact and quasi-separated,*
(3) *there exists an immersion $U \to \mathbf{P}^n_Y$ over Y,*
(4) *f is of finite type and separated.*

Then there exists a commutative diagram

$$\begin{array}{ccc} X' & \longrightarrow & \overline{X}' \\ \downarrow & & \downarrow \\ X & \longrightarrow & Y \end{array}$$

where $X' \to X$ is a U-admissible blowup, $X' \to \overline{X}'$ is an open immersion, and $\overline{X}' \to Y$ is a proper and representable morphism of algebraic spaces.

Proof. Let $Z \subset \mathbf{P}^n_Y$ be the scheme theoretic image of the immersion $U \to \mathbf{P}^n_Y$. Since $U \to \mathbf{P}^n_Y$ is quasi-compact we see that $U \subset Z$ is a (scheme theoretically) dense open subspace (Morphisms of Spaces, Lemma 49.17.7). Apply Lemma 57.28.1 to find a diagram

$$\begin{array}{ccccc} X' & \longrightarrow & \overline{X}' & \longleftarrow & Z' \\ \downarrow & \nwarrow & \uparrow & \nearrow & \downarrow \\ X & \longleftarrow & U & \longrightarrow & Z \end{array}$$

with properties as listed in the statement of that lemma. Since $Z' \to Z \to Y$ is proper we see that $Z' \subset \overline{X}'$ is closed (see Morphisms of Spaces, Lemma 49.36.6). After replacing $\overline{X}'$ by a further U-admissible blowup we may assume that U is scheme theoretically dense in $\overline{X}'$ (details omitted; use Divisors on Spaces, Lemmas 53.6.4 and 53.2.4). It follows that $Z' = \overline{X}'$ and the lemma is proved. □

Lemma 57.28.3. *Let S be a scheme. Let $f : X \to Y$ be a morphism of algebraic spaces over S. Assume f separated, of finite type, and Y Noetherian. Then there exists a commutative diagram*

$$\begin{array}{ccc} X' & \longrightarrow & \overline{X}' \\ \downarrow & & \downarrow \\ X & \longrightarrow & Y \end{array}$$

where $X' \to X$ is a U-admissible blowup for some dense open $U \subset X$, the morphism $X' \to \overline{X}'$ is an open immersion, and $\overline{X}' \to Y$ is a proper and representable morphism of algebraic spaces.

Proof. By Limits of Spaces, Lemma 52.12.3 there exists a dense open subspace $U \subset X$ and an immersion $U \to \mathbf{A}^n_Y$ over Y. Composing with the open immersion $\mathbf{A}^n_Y \to \mathbf{P}^n_Y$ we obtain a situation as in Lemma 57.28.2 and the result follows. □

Remark 57.28.4. In Lemma 57.28.2 the morphism $\overline{X}' \to Y$ is a composition

$$\overline{X}' \to Z \to \mathbf{P}^n_Y \to Y$$

where $b : \overline{X}' \to Z$ is a U-admissible blowing up (in particular $b|_U : U \to b(U)$ is an isomorphism onto an open subspace of Z) and where $Z \to \mathbf{P}^n_Y$ is a closed immersion. This is immediate from the proof. It follows that the morphism $\overline{X}' \to Y$ obtained in the statement of Lemma 57.28.3 has a factorization of this type as well.

The following result is [**Knu71**, IV Theorem 3.1]. Note that the immersion $X' \to \mathbf{P}^n_Y$ is quasi-compact, hence can be factored as $X' \to \overline{X}' \to \mathbf{P}^n_Y$ where the first morphism is an open immersion and the second morphism a closed immersion (Morphisms of Spaces, Lemma 49.17.7).

Lemma 57.28.5 (Chow's lemma)**.** *Let S be a scheme. Let $f : X \to Y$ be a morphism of algebraic spaces over S. Assume f separated of finite type, and Y separated and Noetherian. Then there exists a commutative diagram*

$$\begin{array}{ccc} X' & \longrightarrow & \mathbf{P}^n_Y \\ \downarrow & & \downarrow \\ X & \longrightarrow & Y \end{array}$$

where $X' \to X$ is a U-admissible blowup for some dense open $U \subset X$ and the morphism $X' \to \mathbf{P}^n_Y$ is an immersion.

Proof. In this first paragraph of the proof we reduce the lemma to the case where Y is of finite type over $\mathrm{Spec}(\mathbf{Z})$. We may and do replace the base scheme S by $\mathrm{Spec}(\mathbf{Z})$. We can write $Y = \lim Y_i$ as a directed limit of separated algebraic spaces of finite type over $\mathrm{Spec}(\mathbf{Z})$, see Limits of Spaces, Proposition 52.8.2 and Lemma 52.5.7. For all i sufficiently large we can find a separated finite type morphism $X_i \to Y_i$ such that $X = Y \times_{Y_i} X_i$, see Limits of Spaces, Lemmas 52.7.1 and 52.6.8. Let $\eta_1, \ldots, \eta_n$ be the generic points of the irreducible components of $|X|$ (X is Noetherian as a finite type separated algebraic space over the Noetherian algebraic space Y and therefore $|X|$ is a Noetherian topological space). By Limits of Spaces, Lemma 52.5.2 we find that the images of $\eta_1, \ldots, \eta_n$ in $|X_i|$ are distinct for i large enough. We may replace X_i by the scheme theoretic image of the (quasi-compact, in fact affine) morphism $X \to X_i$. After this replacement we see that the images of $\eta_1, \ldots, \eta_n$ in $|X_i|$ are the generic points of the irreducible components of $|X_i|$, see Morphisms of Spaces, Lemma 49.16.3. Having said this, suppose we can find a diagram

$$\begin{array}{ccc} X'_i & \longrightarrow & \mathbf{P}^n_{Y_i} \\ \downarrow & & \downarrow \\ X_i & \longrightarrow & Y \end{array}$$

where $X_i' \to X_i$ is a U_i-admissible blowup for some dense open $U_i \subset X_i$ and the morphism $X_i' \to \mathbf{P}^n_{Y_i}$ is an immersion. Then the strict transform $X' \to X$ of X relative to $X_i' \to X_i$ is a U-admissible blowing up where $U \subset X$ is the inverse image of U_i in X. Because of our carefuly chosen index i it follows that $\eta_1, \ldots, \eta_n \in |U|$ and $U \subset X$ is dense. Moreover, $X' \to \mathbf{P}^n_Y$ is an immersion as X' is closed in $X_i' \times_{X_i} X = X_i' \times_{Y_i} Y$ which comes with an immersion into $\mathbf{P}^n_Y$. Thus we have reduced to the situation of the following paragraph.

Assume that Y is separated of finite type over $\mathrm{Spec}(\mathbf{Z})$. Then $X \to \mathrm{Spec}(\mathbf{Z})$ is separated of finite type as well. We apply Lemma 57.28.3 to find a diagram

$$\begin{array}{ccc} X' & \longrightarrow & \overline{X}' \\ \downarrow & & \downarrow \\ X & \longrightarrow & \mathrm{Spec}(\mathbf{Z}) \end{array}$$

where $X' \to X$ is a U-admissible blowup for some dense open $U \subset X$ and $X' \to \overline{X}'$ is an open immersion and $\overline{X}' \to \mathrm{Spec}(\mathbf{Z})$ is representable and proper. In fact, by Remark 57.28.4 we see that $\overline{X}' \to \mathrm{Spec}(\mathbf{Z})$ can be factored as

$$\overline{X}' \to Z \to \mathbf{P}^n_{\mathbf{Z}} \to \mathrm{Spec}(\mathbf{Z}).$$

where the first morphism is a U-admissible blowing up, the second morphism is a closed immersion, and the third morphism is the structure morphism. Note that Z has an ample invertible sheaf, namely $\mathcal{O}_{\mathbf{P}^n}(1)|_Z$. Hence $\overline{X}' \to Z$ is a H-projective morphism by Morphisms, Lemma 28.43.13. It follows that $\overline{X}' \to \mathrm{Spec}(\mathbf{Z})$ is H-projective by Morphisms, Lemma 28.43.7. Thus there exists a closed immersion $\overline{X}' \to \mathbf{P}^n_{\mathrm{Spec}(\mathbf{Z})}$. It follows that the diagonal map $X' \to Y \times \mathbf{P}^n_{\mathrm{Spec}(\mathbf{Z})} = \mathbf{P}^n_Y$ is an immersion and we win. □

57.29. Variants of Chow's Lemma

In this section we prove a number of variants of Chow's lemma dealing with morphisms between non-Noetherian algebraic spaces. The Noetherian versions are Lemma 57.28.3 and Lemma 57.28.5.

Lemma 57.29.1. *Let S be a scheme. Let Y be a quasi-compact and quasi-separated algebraic space over S. Let $f : X \to Y$ be a separated morphism of finite type. Then there exists a commutative diagram*

$$\begin{array}{ccc} X' & \longrightarrow & \overline{X}' \\ \downarrow & & \downarrow \\ X & \longrightarrow & Y \end{array}$$

where $X' \to X$ is proper surjective, $X' \to \overline{X}'$ is an open immersion, and $\overline{X}' \to Y$ is proper and representable morphism of algebraic spaces.

Proof. By Limits of Spaces, Proposition 52.11.6 we can find a closed immersion $X \to X_1$ where X_1 is separated and of finite presentation over Y. Clearly, if we prove the assertion for $X_1 \to Y$, then the result follows for X. Hence we may assume that X is of finite presentation over Y.

We may and do replace the base scheme S by $\mathrm{Spec}(\mathbf{Z})$. Write $Y = \lim_i Y_i$ as a directed limit of quasi-separated algebraic spaces of finite type over $\mathrm{Spec}(\mathbf{Z})$, see Limits of Spaces, Proposition 52.8.2. By Limits of Spaces, Lemma 52.7.1 we can find an index $i \in I$ and a scheme $X_i \to Y_i$ of finite presentation so that $X = Y \times_{Y_i} X_i$. By Limits of Spaces, Lemma 52.6.8 we may assume that $X_i \to Y_i$ is separated. Clearly, if we prove the assertion for X_i over Y_i, then the assertion holds for X. The case $X_i \to Y_i$ is treated by Lemma 57.28.3. □

Lemma 57.29.2. *Let S be a scheme. Let $f : X \to Y$ be a morphism of algebraic spaces over S. Assume f separated of finite type, and Y separated and quasi-compact. Then there exists a commutative diagram*

$$\begin{array}{ccc} X' & \longrightarrow & \mathbf{P}^n_Y \\ \downarrow & & \downarrow \\ X & \longrightarrow & Y \end{array}$$

where $X' \to X$ is proper surjective morphism and the morphism $X' \to \mathbf{P}^n_Y$ is an immersion.

Proof. By Limits of Spaces, Proposition 52.11.6 we can find a closed immersion $X \to X_1$ where X_1 is separated and of finite presentation over Y. Clearly, if we prove the assertion for $X_1 \to Y$, then the result follows for X. Hence we may assume that X is of finite presentation over Y.

We may and do replace the base scheme S by $\mathrm{Spec}(\mathbf{Z})$. Write $Y = \lim_i Y_i$ as a directed limit of quasi-separated algebraic spaces of finite type over $\mathrm{Spec}(\mathbf{Z})$, see Limits of Spaces, Proposition 52.8.2. By Limits of Spaces, Lemma 52.5.7 we may assume that Y_i is separated for all i. By Limits of Spaces, Lemma 52.7.1 we can find an index $i \in I$ and a scheme $X_i \to Y_i$ of finite presentation so that $X = Y \times_{Y_i} X_i$. By Limits of Spaces, Lemma 52.6.8 we may assume that $X_i \to Y_i$ is separated. Clearly, if we prove the assertion for X_i over Y_i, then the assertion holds for X. The case $X_i \to Y_i$ is treated by Lemma 57.28.5. □

57.30. Grothendieck's existence theorem

In this section we discuss Grothendieck's existence theorem for algebraic spaces. Instead of developing a theory of "formal algebraic spaces" we temporarily develop a bit of language that replaces the notion of a "coherent module on a Noetherian adic formal space".

Let S be a scheme. Let X be a Noetherian algebraic space over S. Let $\mathcal{I} \subset \mathcal{O}_X$ be a quasi-coherent sheaf of ideals. Below we will consider inverse systems $(\mathcal{F}_n)$ of coherent $\mathcal{O}_X$-modules such that

(1) $\mathcal{F}_n$ is annihilated by $\mathcal{I}^n$, and
(2) the transition maps induce isomorphisms $\mathcal{F}_{n+1}/\mathcal{I}^n\mathcal{F}_{n+1} \to \mathcal{F}_n$.

A morphism $\alpha : (\mathcal{F}_n) \to (\mathcal{G}_n)$ of such inverse systems is simply a compatible system of morphisms $\alpha_n : \mathcal{F}_n \to \mathcal{G}_n$. Let us denote the category of these inverse systems with $Coh(X, \mathcal{I})$. We will develop some theory regarding these systems that will parallel to the corresponding results in the case of schemes, see Cohomology of Schemes, Sections 29.21, 29.22, and 29.23.

Functoriality. Let $f : X \to Y$ be a morphism of Noetherian algebraic spaces over a scheme S, and let $\mathcal{J} \subset \mathcal{O}_Y$ be a quasi-coherent sheaf of ideals. Set $\mathcal{I} = f^{-1}\mathcal{J}\mathcal{O}_X$. In this situation there is a functor

$$f^* : Coh(Y, \mathcal{J}) \longrightarrow Coh(X, \mathcal{I})$$

which sends $(\mathcal{G}_n)$ to $(f^*\mathcal{G}_n)$. Compare with Cohomology of Schemes, Lemma 29.22.1. If f is étale, then we may think of this as simply the restriction of the system to X, see Properties of Spaces, Equation 48.24.1.1.

Étale descent. Let S be a scheme. Let $U_0 \to X$ be a surjective étale morphism of Noetherian algebraic spaces. Set $U_1 = U_0 \times_X U_0$ and $U_2 = U_0 \times_X U_0 \times_X U_0$. Let $\mathcal{I} \subset \mathcal{O}_X$ be a quasi-coherent sheaf of ideals. Set $\mathcal{I}_i = \mathcal{I}|_{U_i}$. In this situation we obtain a diagram of categories

$$Coh(X, \mathcal{I}) \longrightarrow Coh(U_0, \mathcal{I}_0) \rightrightarrows Coh(U_1, \mathcal{I}_1) \mathrel{\substack{\rightarrow\\[-1ex]\rightarrow\\[-1ex]\rightarrow}} Coh(U_2, \mathcal{I}_2)$$

an the first arrow presents $Coh(X, \mathcal{I})$ as the homotopy limit of the right part of the diagram. More precisely, given a *descent datum*, i.e., a pair $((\mathcal{G}_n), \varphi)$ where $(\mathcal{G}_n)$ is an object of $Coh(U_0, \mathcal{I}_0)$ and $\varphi : \mathrm{pr}_0^*(\mathcal{G}_n) \to \mathrm{pr}_1^*(\mathcal{G}_n)$ is an isomorphism in $Coh(U_1, \mathcal{I}_1)$ satisfying the cocycle condition in $Coh(U_2, \mathcal{I}_2)$, then there exists a unique object $(\mathcal{F}_n)$ of $Coh(X, \mathcal{I})$ whose associated canonical descent datum is isomorphic to $((\mathcal{G}_n), \varphi)$. Compare with Descent on Spaces, Definition 55.3.3. The proof of this statement follows immediately by applying Descent on Spaces, Proposition 55.4.1 to the descent data $(\mathcal{G}_n, \varphi_n)$ for varying n.

Lemma 57.30.1. *Let S be a scheme. Let X be a Noetherian algebraic space over S and let $\mathcal{I} \subset \mathcal{O}_X$ be a quasi-coherent sheaf of ideals.*

1. *The category $Coh(X, \mathcal{I})$ is abelian.*
2. *Exactness in $Coh(X, \mathcal{I})$ can be checked étale locally.*
3. *For any flat morphism $f : X' \to X$ of Noetherian algebraic spaces the functor $f^* : Coh(X, \mathcal{I}) \to Coh(X', f^{-1}\mathcal{I}\mathcal{O}_{X'})$ is exact.*

Proof. Proof of (1). Choose an affine scheme U_0 and a surjective étale morphism $U_0 \to X$. Set $U_1 = U_0 \times_X U_0$ and $U_2 = U_0 \times_X U_0 \times_X U_0$ as in our discussion of étale descent above. The categories $Coh(U_i, \mathcal{I}_i)$ are abelian (Cohomology of Schemes, Lemma 29.21.2) and the pullback functors are exact functors $\mathrm{Coh}(U_0, \mathcal{I}_0) \to \mathrm{Coh}(U_1, \mathcal{I}_1)$ and $\mathrm{Coh}(U_1, \mathcal{I}_1) \to \mathrm{Coh}(U_2, \mathcal{I}_2)$ (Cohomology of Schemes, Lemma 29.22.1). The lemma then follows formally from the description of $\mathrm{Coh}(X, \mathcal{I})$ as a category of descent data. Some details omitted; compare with the proof of Groupoids, Lemma 38.12.6.

Part (2) follows immediately from the discussion in the previous paragraph. In the situation of (3) choose a commutative diagram

$$\begin{array}{ccc} U' & \longrightarrow & U \\ \downarrow & & \downarrow \\ X' & \longrightarrow & X \end{array}$$

where U' and U are affine schemes and the vertical morphisms are surjective étale. Then $U' \to U$ is a flat morphism of Noetherian schemes (Morphisms of Spaces, Lemma 49.27.5) whence the pullback functor $Coh(U, \mathcal{I}\mathcal{O}_U) \to Coh(U', \mathcal{I}\mathcal{O}_{U'})$ is

exact by Cohomology of Schemes, Lemma 29.22.1. Since we can check exactness in $Coh(X, \mathcal{O}_X)$ on U and similarly for X', U' the assertion follows. □

Lemma 57.30.2. *Let S be a scheme. Let X be a Noetherian algebraic space over S and let $\mathcal{I} \subset \mathcal{O}_X$ be a quasi-coherent sheaf of ideals. A map $(\mathcal{F}_n) \to (\mathcal{G}_n)$ is surjective in $Coh(X, \mathcal{I})$ if and only if $\mathcal{F}_1 \to \mathcal{G}_1$ is surjective.*

Proof. We can check on an affine étale cover of X by Lemma 57.30.1. Thus we reduce to the case of schemes which is Cohomology of Schemes, Lemma 29.21.3. □

Let S be a scheme. Let X be a Noetherian algebraic space over S and let $\mathcal{I} \subset \mathcal{O}_X$ be a quasi-coherent sheaf of ideals. There is a functor

$$Coh(\mathcal{O}_X) \longrightarrow Coh(X, \mathcal{I}), \quad \mathcal{F} \longmapsto \mathcal{F}^\wedge \tag{57.30.2.1}$$

which associates to the coherent $\mathcal{O}_X$-module $\mathcal{F}$ the object $\mathcal{F}^\wedge = (\mathcal{F}/\mathcal{I}^n\mathcal{F})$ of $Coh(X, \mathcal{I})$.

Lemma 57.30.3. *The functor (57.30.2.1) is exact.*

Proof. It suffices to check this étale locally on X, see Lemma 57.30.1. Thus we reduce to the case of schemes which is Cohomology of Schemes, Lemma 29.21.5. □

Lemma 57.30.4. *Let S be a scheme. Let X be a Noetherian algebraic space over S and let $\mathcal{I} \subset \mathcal{O}_X$ be a quasi-coherent sheaf of ideals. Let $\mathcal{F}$, $\mathcal{G}$ be coherent $\mathcal{O}_X$-modules. Set $\mathcal{H} = \mathcal{H}om_{\mathcal{O}_X}(\mathcal{F}, \mathcal{G})$. Then*

$$\lim H^0(X, \mathcal{H}/\mathcal{I}^n\mathcal{H}) = \mathrm{Mor}_{Coh(X,\mathcal{I})}(\mathcal{F}^\wedge, \mathcal{G}^\wedge).$$

Proof. Since $\mathcal{H}$ is a sheaf on $X_{étale}$ and since we have étale descent for objects of $Coh(X, \mathcal{I})$ it suffices to prove this étale locally. Thus we reduce to the case of schemes which is Cohomology of Schemes, Lemma 29.21.6. □

We introduce the setting that we will focus on throughout the rest of this section.

Situation 57.30.5. Here A is a Noetherian ring complete with respect to an ideal I. Also $f : X \to \mathrm{Spec}(A)$ is a finite type separated morphism of algebraic spaces and $\mathcal{I} = I\mathcal{O}_X$.

In this situation we denote

$$Coh_{\text{support proper over } A}(\mathcal{O}_X)$$

be the full subcategory of $Coh(\mathcal{O}_X)$ consisting of those coherent $\mathcal{O}_X$-modules whose scheme theoretic support is proper over $\mathrm{Spec}(A)$. Similarly, we let

$$Coh_{\text{support proper over } A}(X, \mathcal{I})$$

be the full subcategory of $Coh(X, \mathcal{I})$ consisting of those objects $(\mathcal{F}_n)$ such that the scheme theoretic support of $\mathcal{F}_1$ is proper over $\mathrm{Spec}(A)$. Since the support of a quotient module is contained in the support of the module, it follows that (57.30.2.1) induces a functor

$$Coh_{\text{support proper over } A}(\mathcal{O}_X) \longrightarrow Coh_{\text{support proper over } A}(X, \mathcal{I}) \tag{57.30.5.1}$$

Our first result is that this functor is fully faithful.

Lemma 57.30.6. *In Situation 57.30.5. Let $\mathcal{F}$, $\mathcal{G}$ be coherent $\mathcal{O}_X$-modules. Assume that the intersection of the scheme theoretic supports of $\mathcal{F}$ and $\mathcal{G}$ is proper over* $\mathrm{Spec}(A)$. *Then the map*

$$\mathrm{Mor}_{Coh(\mathcal{O}_X)}(\mathcal{F}, \mathcal{G}) \longrightarrow \mathrm{Mor}_{Coh(X,\mathcal{I})}(\mathcal{F}^\wedge, \mathcal{G}^\wedge)$$

coming from (57.30.2.1) is a bijection. In particular, (57.30.5.1) is fully faithful.

Proof. Let $\mathcal{H} = \mathcal{H}om_{\mathcal{O}_X}(\mathcal{G}, \mathcal{F})$. This is a coherent $\mathcal{O}_X$-module because its restriction of schemes étale over X is coherent by Modules, Lemma 17.19.4. By Lemma 57.30.4 the map

$$\lim_n H^0(X, \mathcal{H}/\mathcal{I}^n\mathcal{H}) \to \mathrm{Mor}_{Coh(X,\mathcal{I})}(\mathcal{G}^\wedge, \mathcal{F}^\wedge)$$

is bijective. Let $i : Z \to X$ be the scheme theoretic support of $\mathcal{H}$. It is clear that Z is a closed subspace contained in the intersection of the scheme theoretic supports of $\mathcal{F}$ and $\mathcal{G}$. Hence $Z \to \mathrm{Spec}(A)$ is proper by assumption. Write $\mathcal{H} = i_*\mathcal{H}'$ for some coherent $\mathcal{O}_Z$-module $\mathcal{H}'$. We have $i_*(\mathcal{H}'/I^n\mathcal{H}') = \mathcal{H}/I^n\mathcal{H}$. Hence we obtain

$$\begin{aligned}
\lim_n H^0(X, \mathcal{H}/\mathcal{I}^n\mathcal{H}) &= \lim_n H^0(Z, \mathcal{H}'/\mathcal{I}^n\mathcal{H}') \\
&= H^0(Z, \mathcal{H}') \\
&= H^0(X, \mathcal{H}) \\
&= \mathrm{Mor}_{Coh(\mathcal{O}_X)}(\mathcal{F}, \mathcal{G})
\end{aligned}$$

the second equality by the theorem on formal functions functions (Cohomology of Spaces, Lemma 51.17.6). This proves the lemma. □

Remark 57.30.7. Let S be a scheme. Let X be a Noetherian algebraic space over S and let $\mathcal{I}, \mathcal{K} \subset \mathcal{O}_X$ be quasi-coherent sheaves of ideals. Let $\alpha : (\mathcal{F}_n) \to (\mathcal{G}_n)$ be a morphism of $Coh(X, \mathcal{I})$. Given an affine scheme $U = \mathrm{Spec}(A)$ and a surjective étale morphism $U \to X$ denote $I, K \subset A$ the ideals corresponding to the restrictions $\mathcal{I}|_U, \mathcal{K}|_U$. Denote $\alpha_U : M \to N$ of finite $A^\wedge$-modules which corresponds to $\alpha|_U$ via Cohomology of Schemes, Lemma 29.21.1. We claim the following are equivalent

(1) there exists an integer $t \geq 1$ such that $\mathrm{Ker}(\alpha_n)$ and $\mathrm{Coker}(\alpha_n)$ are annihilated by $\mathcal{K}^t$ for all $n \geq 1$,
(2) for any (or some) affine open $\mathrm{Spec}(A) = U \subset X$ as above the modules $\mathrm{Ker}(\alpha_U)$ and $\mathrm{Coker}(\alpha_U)$ are annihilated by K^t for some integer $t \geq 1$.

If these equivalent conditions hold we will say that α is a *map whose kernel and cokernel are annihilated by a power of* $\mathcal{K}$. To see the equivalence we refer to Cohomology of Schemes, Remark 29.22.2.

Lemma 57.30.8. *Let S be a scheme. Let X be a Noetherian algebraic space over S and let $\mathcal{I} \subset \mathcal{O}_X$ be a quasi-coherent sheaf of ideals. Let $\mathcal{G}$ be a coherent $\mathcal{O}_X$-module, $(\mathcal{F}_n)$ an object of $Coh(X, \mathcal{I})$, and $\alpha : (\mathcal{F}_n) \to \mathcal{G}^\wedge$ a map whose kernel and cokernel are annihilated by a power of $\mathcal{I}$. Then there exists a unique (up to unique isomorphism) triple $(\mathcal{F}, a, \beta)$ where*

(1) *$\mathcal{F}$ is a coherent $\mathcal{O}_X$-module,*
(2) *$a : \mathcal{F} \to \mathcal{G}$ is an $\mathcal{O}_X$-module map whose kernel and cokernel are annihilated by a power of $\mathcal{I}$,*
(3) *$\beta : (\mathcal{F}_n) \to \mathcal{F}^\wedge$ is an isomorphism, and*
(4) *$\alpha = a^\wedge \circ \beta$.*

Proof. The uniqueness and étale descent for objects of $Coh(X, \mathcal{I})$ and $Coh(\mathcal{O}_X)$ implies it suffices to construct $(\mathcal{F}, a, \beta)$ étale locally on X. Thus we reduce to the case of schemes which is Cohomology of Schemes, Lemma 29.22.3. □

Lemma 57.30.9. *In Situation 57.30.5. Let $\mathcal{K} \subset \mathcal{O}_X$ be a quasi-coherent sheaf of ideals. Let $X_e \subset X$ be the closed subspace cut out by $\mathcal{K}^e$. Let $\mathcal{I}_e = \mathcal{I}\mathcal{O}_{X_e}$. Let $(\mathcal{F}_n)$ be an object of $Coh_{\text{support proper over } A}(X, \mathcal{I})$. Assume*

1. *the functor $Coh_{\text{support proper over } A}(\mathcal{O}_{X_e}) \to Coh_{\text{support proper over } A}(X_e, \mathcal{I}_e)$ is an equivalence for all $e \geq 1$, and*
2. *there exists an object $\mathcal{H}$ of $Coh_{\text{support proper over } A}(\mathcal{O}_X)$ and a map $\alpha : (\mathcal{F}_n) \to \mathcal{H}^\wedge$ whose kernel and cokernel are annihilated by a power of $\mathcal{K}$.*

Then $(\mathcal{F}_n)$ is in the essential image of (57.30.5.1).

Proof. During this proof we will use without further mention that for a closed immersion $i : Z \to X$ the functor i_* gives an equivalence between the category of coherent modules on Z and coherent modules on X annihilated by the ideal sheaf of Z, see Cohomology of Spaces, Lemma 51.9.8. In particular we think of

$$Coh_{\text{support proper over } A}(\mathcal{O}_{X_e}) \subset Coh_{\text{support proper over } A}(\mathcal{O}_X)$$

as the full subcategory of consisting of modules annihilated by $\mathcal{K}^e$ and

$$Coh_{\text{support proper over } A}(X_e, \mathcal{I}_e) \subset Coh_{\text{support proper over } A}(X, \mathcal{I})$$

as the full subcategory of of objects annihilated by $\mathcal{K}^e$. Moreover (1) tells us these two categories are equivalent under the completion functor (57.30.5.1).

Applying this equivalence we get a coherent $\mathcal{O}_X$-module $\mathcal{G}_e$ annihilated by $\mathcal{K}^e$ corresponding to the system $(\mathcal{F}_n/\mathcal{K}^e\mathcal{F}_n)$ of $Coh_{\text{support proper over } A}(X, \mathcal{I})$. The maps $\mathcal{F}_n/\mathcal{K}^{e+1}\mathcal{F}_n \to \mathcal{F}_n/\mathcal{K}^e\mathcal{F}_n$ correspond to canonical maps $\mathcal{G}_{e+1} \to \mathcal{G}_e$ which induce isomorphisms $\mathcal{G}_{e+1}/\mathcal{K}^e\mathcal{G}_{e+1} \to \mathcal{G}_e$. We obtain an object $(\mathcal{G}_e)$ of the category $Coh_{\text{support proper over } A}(X, \mathcal{K})$. The map α induces a system of maps

$$\mathcal{F}_n/\mathcal{K}^e\mathcal{F}_n \longrightarrow \mathcal{H}/(\mathcal{I}^n + \mathcal{K}^e)\mathcal{H}$$

whence maps $\mathcal{G}_e \to \mathcal{H}/\mathcal{K}^e\mathcal{H}$ (by the equivalence of categories again). Let $t \geq 1$ be an integer, which exists by assumption (2), such that $\mathcal{K}^t$ annihilates the kernel and cokernel of all the maps $\mathcal{F}_n \to \mathcal{H}/\mathcal{I}^n\mathcal{H}$. Then $\mathcal{K}^{2t}$ annihilates the kernel and cokernel of the maps $\mathcal{F}_n/\mathcal{K}^e\mathcal{F}_n \to \mathcal{H}/(\mathcal{I}^n + \mathcal{K}^e)\mathcal{H}$ (details omitted; see Cohomology of Schemes, Remark 29.22.2). Whereupon we conclude that $\mathcal{K}^{4t}$ annihilates the kernel and the cokernel of the maps

$$\mathcal{G}_e \longrightarrow \mathcal{H}/\mathcal{K}^e\mathcal{H},$$

(details omitted; see Cohomology of Schemes, Remark 29.22.2). We apply Lemma 57.30.8 to obtain a coherent $\mathcal{O}_X$-module $\mathcal{F}$, a map $a : \mathcal{F} \to \mathcal{H}$ and an isomorphism $\beta : (\mathcal{G}_e) \to (\mathcal{F}/\mathcal{K}^e\mathcal{F})$ in $Coh(X, \mathcal{K})$. Working backwards, for a given n the triple $(\mathcal{F}/\mathcal{I}^n\mathcal{F}, a \bmod \mathcal{I}^n, \beta \bmod \mathcal{I}^n)$ is a triple as in the lemma for the morphism $\alpha_n \bmod \mathcal{K}^e : (\mathcal{F}_n/\mathcal{K}^e\mathcal{F}_n) \to (\mathcal{H}/(\mathcal{I}^n + \mathcal{K}^e)\mathcal{H})$ of $Coh(X, \mathcal{K})$. Thus the uniqueness in Lemma 57.30.8 gives a canonical isomorphism $\mathcal{F}/\mathcal{I}^n\mathcal{F} \to \mathcal{F}_n$ compatible with all the morphisms in sight.

To finish the proof of the lemma we still have to show that the scheme theoretic support of $\mathcal{F}$ is proper over A. By construction the kernel of $a : \mathcal{F} \to \mathcal{H}$ is annihilated by a power of $\mathcal{K}$. Hence the support of this kernel is contained in the support of $\mathcal{G}_1$. Since $\mathcal{G}_1$ is an object of $Coh_{\text{support proper over } A}(\mathcal{O}_{X_1})$ we see this is

proper over A. Combined with the fact that the support of $\mathcal{H}$ is proper over A we conclude that the support of $\mathcal{F}$ is proper over A (some details omitted). □

Lemma 57.30.10. *Let S be a scheme. Let $f : X \to Y$ be a representable proper morphism of Noetherian algebraic spaces over S. Let $\mathcal{J}, \mathcal{K} \subset \mathcal{O}_Y$ be quasi-coherent sheaves of ideals. Assume f is an isomorphism over $V = Y \setminus V(\mathcal{K})$. Set $\mathcal{I} = f^{-1}\mathcal{J}\mathcal{O}_X$. Let $(\mathcal{G}_n)$ be an object of $Coh(Y, \mathcal{J})$, let $\mathcal{F}$ be a coherent $\mathcal{O}_X$-module, and let $\beta : (f^*\mathcal{G}_n) \to \mathcal{F}^\wedge$ be an isomorphism in $Coh(X, \mathcal{I})$. Then there exists a map*

$$\alpha : (\mathcal{G}_n) \longrightarrow (f_*\mathcal{F})^\wedge$$

in $Coh(Y, \mathcal{J})$ whose kernel and cokernel are annihilated by a power of $\mathcal{K}$.

Proof. Since f is a proper morphism we see that $f_*\mathcal{F}$ is a coherent $\mathcal{O}_Y$-module (Cohomology of Spaces, Lemma 51.16.2). Thus the statement of the lemma makes sense. Consider the compositions

$$\gamma_n : \mathcal{G}_n \to f_*f^*\mathcal{G}_n \to f_*(\mathcal{F}/\mathcal{I}^n\mathcal{F}).$$

Here the first map is the adjunction map and the second is $f_*\beta_n$. We claim that there exists a unique α as in the lemma such that the compositions

$$\mathcal{G}_n \xrightarrow{\alpha_n} f_*\mathcal{F}/\mathcal{J}^n f_*\mathcal{F} \to f_*(\mathcal{F}/\mathcal{I}^n\mathcal{F})$$

equal γ_n for all n. Because of the uniqueness and étale descent for $Coh(Y, \mathcal{J})$ it suffices to prove this étale locally on Y. Thus we may assume Y is the spectrum of a Noetherian ring. As f is representable we see that X is a scheme as well. Thus we reduce to the case of schemes, see proof of Cohomology of Schemes, Lemma 29.22.5. □

Theorem 57.30.11 (Grothendieck's existence theorem)**.** *In Situation 57.30.5 the functor (57.30.5.1) is an equivalence.*

Proof. We will use the equivalence of categories of Cohomology of Spaces, Lemma 51.9.8 without further mention in the proof of the theorem. By Lemma 57.30.6 the functor is fully faithful. Thus we need to prove the functor is essentially surjective.

Consider the collection Ξ of quasi-coherent sheaves of ideals $\mathcal{K} \subset \mathcal{O}_X$ such that the statement holds for every object $(\mathcal{F}_n)$ of $Coh_{\text{support proper over } A}(X, \mathcal{I})$ annihilated by $\mathcal{K}$. We want to show (0) is in Ξ. If not, then since X is Noetherian there exists a maximal quasi-coherent sheaf of ideals $\mathcal{K}$ not in Ξ, see Cohomology of Spaces, Lemma 51.10.1. After replacing X by the closed subscheme of X corresponding to $\mathcal{K}$ we may assume that every nonzero $\mathcal{K}$ is in Ξ. Let $(\mathcal{F}_n)$ be an object of $Coh_{\text{support proper over } A}(X, \mathcal{I})$. We will show that this object is in the essential image, thereby completing the proof of the theorem.

Apply Chow's lemma (Lemma 57.28.5) to find a proper surjective morphism $f : Y \to X$ which is an isomorphism over a dense open $U \subset X$ such that Y is H-quasi-projective over A. Note that Y is a scheme and f representable. Choose an open immersion $j : Y \to Y'$ with Y' projective over A, see Morphisms, Lemma 28.43.11. Let T_n be the scheme theoretic support of $\mathcal{F}_n$. Note that $|T_n| = |T_1|$, hence T_n is proper over A for all n (Morphisms of Spaces, Lemma 49.36.7). Then $f^*\mathcal{F}_n$ is supported on the closed subscheme $f^{-1}T_n$ which is proper over A (by Morphisms of Spaces, Lemma 49.36.4 and properness of f). In particular, the composition $f^{-1}T_n \to Y \to Y'$ is closed (Morphisms, Lemma 28.42.7). Let $T'_n \subset Y'$ be the corresponding closed subscheme; it is contained in the open subscheme Y and equal

to $f^{-1}T_n$ as a closed subscheme of Y. Let $\mathcal{F}'_n$ be the coherent $\mathcal{O}_{Y'}$-module corresponding to $f^*\mathcal{F}_n$ viewed as a coherent module on Y' via the closed immersion $f^{-1}T_n = T'_n \subset Y'$. Then $(\mathcal{F}'_n)$ is an object of $Coh(Y', I\mathcal{O}_{Y'})$. By the projective case of Grothendieck's existence theorem (Cohomology of Schemes, Lemma 29.21.9) there exists a coherent $\mathcal{O}_{Y'}$-module $\mathcal{F}'$ and an isomorphism $(\mathcal{F}')^\wedge \cong (\mathcal{F}'_n)$ in $Coh(Y', I\mathcal{O}_{Y'})$. Let $Z' \subset Y'$ be the scheme theoretic support of $\mathcal{F}'$. Since $\mathcal{F}'/I\mathcal{F}' = \mathcal{F}'_1$ we see that $Z' \cap V(I\mathcal{O}_{Y'}) = T'_1$ set-theoretically. The structure morphism $p' : Y' \to \mathrm{Spec}(A)$ is proper, hence $p'(Z' \cap (Y' \setminus Y))$ is closed in $\mathrm{Spec}(A)$. If nonempty, then it would contain a point of $V(I)$ as I is contained in the radical of A (Algebra, Lemma 10.91.11). But we've seen above that $Z' \cap (p')^{-1}V(I) = T'_1 \subset Y$ hence we conclude that $Z' \subset Y$. Thus $\mathcal{F}'|_Y$ is supported on a closed subscheme of Y proper over A.

Let $\mathcal{K}$ be the quasi-coherent sheaf of ideals cutting out the reduced complement $X \setminus U$. By Cohomology of Spaces, Lemma 51.16.2 the $\mathcal{O}_X$-module $\mathcal{H} = f_*\mathcal{F}'$ is coherent and by Lemma 57.30.10 there exists a morphism $\alpha : (\mathcal{F}_n) \to \mathcal{H}^\wedge$ in the category $Coh_{\text{support proper over } A}(X, \mathcal{I})$ whose kernel and cokernel are annihilated by a power of $\mathcal{K}$. Let $Z_0 \subset X$ be the scheme theoretic support of $\mathcal{H}$. It is clear that $|Z_0| \subset f(|Z'|)$. Hence $Z_0 \to \mathrm{Spec}(A)$ is proper (Morphisms of Spaces, Lemma 49.36.7). Thus $\mathcal{H}$ is an object of $Coh_{\text{support proper over } A}(\mathcal{O}_X)$. Since each of the sheaves of ideals $\mathcal{K}^e$ is an element of Ξ we see that the assumptions of Lemma 57.30.9 are satisfied and we conclude. □

Remark 57.30.12 (Unwinding Grothendieck's existence theorem)**.** Let A be a Noetherian ring complete with respect to an ideal I. Write $S = \mathrm{Spec}(A)$ and $S_n = \mathrm{Spec}(A/I^n)$. Let $X \to S$ be a morphism of algebraic spaces that is separated and of finite type. For $n \geq 1$ we set $X_n = X \times_S S_n$. Picture:

$$\begin{array}{ccccccccc} X_1 & \xrightarrow[i_1]{} & X_2 & \xrightarrow[i_2]{} & X_3 & \longrightarrow & \ldots & \qquad & X \\ \downarrow & & \downarrow & & \downarrow & & & & \downarrow \\ S_1 & \longrightarrow & S_2 & \longrightarrow & S_3 & \longrightarrow & \ldots & & S \end{array}$$

In this situation we consider systems $(\mathcal{F}_n, \varphi_n)$ where

(1) $\mathcal{F}_n$ is a coherent $\mathcal{O}_{X_n}$-module,
(2) $\varphi_n : i_n^*\mathcal{F}_{n+1} \to \mathcal{F}_n$ is an isomorphism, and
(3) $\mathrm{Supp}(\mathcal{F}_1)$ is proper over S_1.

Theorem 57.30.11 says that the completion functor

$$\begin{array}{c} \text{coherent } \mathcal{O}_X\text{-modules } \mathcal{F} \\ \text{with support proper over } A \end{array} \longrightarrow \begin{array}{c} \text{systems } (\mathcal{F}_n) \\ \text{as above} \end{array}$$

is an equivalence of categories. In the special case that X is proper over A we can omit the conditions on the supports.

57.31. Grothendieck's algebraization theorem

This section is the analogue of Cohomology of Schemes, Section 29.23. However, this section is missing the result on algebraization of deformations of proper algebraic spaces endowed with ample invertible sheaves, as a proper algebraic space

which comes with an ample invertible sheaf is a scheme. Our first result is a translation of Grothendieck's existence theorem in terms of closed subschemes and finite morphisms.

Lemma 57.31.1. *Let A be a Noetherian ring complete with respect to an ideal I. Write $S = \mathrm{Spec}(A)$ and $S_n = \mathrm{Spec}(A/I^n)$. Let $X \to S$ be a morphism of algebraic spaces that is separated and of finite type. For $n \geq 1$ we set $X_n = X \times_S S_n$. Suppose given a commutative diagram*

$$\begin{array}{ccccccc} Z_1 & \longrightarrow & Z_2 & \longrightarrow & Z_3 & \longrightarrow & \ldots \\ \downarrow & & \downarrow & & \downarrow & & \\ X_1 & \xrightarrow{i_1} & X_2 & \xrightarrow{i_2} & X_3 & \longrightarrow & \ldots \end{array}$$

of algebraic spaces with cartesian squares. Assume that

(1) *$Z_1 \to X_1$ is a closed immersion, and*
(2) *$Z_1 \to S_1$ is proper.*

Then there exists a closed immersion of algebraic spaces $Z \to X$ such that $Z_n = Z \times_S S_n$ for all $n \geq 1$.

Proof. Let's write $j_n : Z_n \to X_n$ for the vertical morphisms. As the squares in the statement are cartesian we see that the base change of j_n to X_1 is j_1. Thus Limits of Spaces, Lemma 52.14.5 shows that j_n is a closed immersion. Set $\mathcal{F}_n = j_{n,*}\mathcal{O}_{Z_n}$, so that $j_n^\sharp$ is a surjection $\mathcal{O}_{X_n} \to \mathcal{F}_n$. Again using that the squares are cartesian we see that the pullback of $\mathcal{F}_{n+1}$ to X_n is $\mathcal{F}_n$. Hence Grothendieck's existence theorem, as reformulated in Remark 57.30.12, tells us there exists a map $\mathcal{O}_X \to \mathcal{F}$ of coherent $\mathcal{O}_X$-modules whose restriction to X_n recovers $\mathcal{O}_{X_n} \to \mathcal{F}_n$. As the completion functor is exact (Lemma 57.30.3) we see that $\mathcal{O}_X \to \mathcal{F}$ is surjective. Thus $\mathcal{F} = \mathcal{O}_X/\mathcal{J}$ for some quasi-coherent sheaf of ideals $\mathcal{J}$. Setting $Z = V(\mathcal{J})$ finishes the proof. □

Lemma 57.31.2. *Let A be a Noetherian ring complete with respect to an ideal I. Write $S = \mathrm{Spec}(A)$ and $S_n = \mathrm{Spec}(A/I^n)$. Let $X \to S$ be a morphism of algebraic spaces that is separated and of finite type. For $n \geq 1$ we set $X_n = X \times_S S_n$. Suppose given a commutative diagram*

$$\begin{array}{ccccccc} Y_1 & \longrightarrow & Y_2 & \longrightarrow & Y_3 & \longrightarrow & \ldots \\ \downarrow & & \downarrow & & \downarrow & & \\ X_1 & \xrightarrow{i_1} & X_2 & \xrightarrow{i_2} & X_3 & \longrightarrow & \ldots \end{array}$$

of algebraic spaces with cartesian squares. Assume that

(1) *$Y_1 \to X_1$ is a finite morphism, and*
(2) *$Y_1 \to S_1$ is proper.*

Then there exists a finite morphism of algebraic spaces $Y \to X$ such that $Y_n = Y \times_S S_n$ for all $n \geq 1$.

Proof. Let's write $f_n : Y_n \to X_n$ for the vertical morphisms. As the squares in the statement are cartesian we see that the base change of f_n to X_1 is f_1. Thus Lemma 57.8.10 shows that f_n is a finite morphism. Set $\mathcal{F}_n = f_{n,*}\mathcal{O}_{Y_n}$. Using that the squares are cartesian we see that the pullback of $\mathcal{F}_{n+1}$ to X_n is $\mathcal{F}_n$. Hence Grothendieck's existence theorem, as reformulated in Remark 57.30.12, tells us

there exists a coherent $\mathcal{O}_X$-module $\mathcal{F}$ whose restriction to X_n recovers $\mathcal{F}_n$. As the completion functor is fuly faithful (Theorem 57.30.11) we see that the multiplication maps $\mathcal{F}_n \otimes_{\mathcal{O}_{X_n}} \mathcal{F}_n \to \mathcal{F}_n$ fit together to give an algebra structure on $\mathcal{F}$. Setting $Y = \underline{\mathrm{Spec}}_X(\mathcal{F})$ finishes the proof. □

57.32. Regular immersions

This section is the analogue of Divisors, Section 30.13 for morphisms of algebraic spaces. The reader is encouraged to read up on regular immersions of schemes in that section first.

In Divisors, Section 30.13 we defined four types of regular immersions for morphisms of schemes. Of these only three are (as far as we know) local on the target for the étale topology; as usual plain old regular immersions aren't. This is why for morphisms of algebraic spaces we cannot actually define regular immersions. (These kinds of annoyances prompted Grothendieck and his school to replace original notion of a regular immersion by a Koszul-regular immersions, see [**BGI71**, Exposee VII, Definition 1.4].) But we can define Koszul-regular, H_1-regular, and quasi-regular immersions. Another remark is that since Koszul-regular immersions are not preserved by arbitrary base change, we cannot use the strategy of Morphisms of Spaces, Section 49.3 to define them. Similarly, as Koszul-regular immersions are not étale local on the source, we cannot use Morphisms of Spaces, Lemma 49.21.1 to define them either. We replace this lemma instead by the following.

Lemma 57.32.1. *Let $\mathcal{P}$ be a property of morphisms of schemes which is étale local on the target. Let S be a scheme. Let $f : X \to Y$ be a representable morphism of algebraic spaces over S. Consider commutative diagrams*

$$\begin{array}{ccc} X \times_Y V & \longrightarrow & V \\ \downarrow & & \downarrow \\ X & \xrightarrow{f} & Y \end{array}$$

where V is a scheme and $V \to Y$ is étale. The following are equivalent

1. *for any diagram as above the projection $X \times_Y V \to V$ has property $\mathcal{P}$, and*
2. *for some diagram as above with $V \to Y$ surjective the projection $X \times_Y V \to V$ has property $\mathcal{P}$.*

If X and Y are representable, then this is also equivalent to f (as a morphism of schemes) having property $\mathcal{P}$.

Proof. Let us prove the equivalence of (1) and (2). The implication (1) $\Rightarrow$ (2) is immediate. Assume

$$\begin{array}{ccc} X \times_Y V & \longrightarrow & V \\ \downarrow & & \downarrow \\ X & \xrightarrow{f} & Y \end{array} \qquad \begin{array}{ccc} X \times_Y V' & \longrightarrow & V' \\ \downarrow & & \downarrow \\ X & \xrightarrow{f} & Y \end{array}$$

are two diagrams as in the lemma. Assume $V \to Y$ is surjective and $X \times_Y V \to V$ has property $\mathcal{P}$. To show that (2) implies (1) we have to prove that $X \times_Y V' \to V'$

has $\mathcal{P}$. To do this consider the diagram

$$\begin{array}{ccccc} X \times_Y V & \longleftarrow & (X \times_Y V) \times_X (X \times_Y V') & \longrightarrow & X \times_Y V' \\ \downarrow & & \downarrow & & \downarrow \\ V & \longleftarrow & V \times_Y V' & \longrightarrow & V' \end{array}$$

By our assumption that $\mathcal{P}$ is étale local on the source, we see that $\mathcal{P}$ is preserved under étale base change, see Descent, Lemma 34.18.2. Hence if the left vertical arrow has $\mathcal{P}$ the so does the middle vertical arrow. Since $U \times_X U' \to U'$ is surjective and étale (hence defines an étale covering of U') this implies (as $\mathcal{P}$ is assumed local for the étale topology on the target) that the left vertical arrow has $\mathcal{P}$.

If X and Y are representable, then we can take $\mathrm{id}_Y : Y \to Y$ as our étale covering to see the final statement of the lemma is true. □

Note that "being a Koszul-regular (resp. H_1-regular, resp. quasi-regular) immersion" is a property of morphisms of schemes which is fpqc local on the target, see Descent, Lemma 34.19.30. Hence the following definition now makes sense.

Definition 57.32.2. Let S be a scheme. Let $i : X \to Y$ be a morphism of algebraic spaces over S.

(1) We say i is a *Koszul-regular immersion* if i is representable and the equivalent conditions of Lemma 57.32.1 hold with $\mathcal{P}(f) =$"f is a Koszul-regular immersion".
(2) We say i is an *H_1-regular immersion* if i is representable and the equivalent conditions of Lemma 57.32.1 hold with $\mathcal{P}(f) =$"f is an H_1-regular immersion".
(3) We say i is a *quasi-regular immersion* if i is representable and the equivalent conditions of Lemma 57.32.1 hold with $\mathcal{P}(f) =$"f is a quasi-regular immersion".

Lemma 57.32.3. *Let S be a scheme. Let $i : Z \to X$ be an immersion of algebraic spaces over S. We have the following implications: i is Koszul-regular $\Rightarrow$ i is H_1-regular $\Rightarrow$ i is quasi-regular.*

Proof. Via the definition this lemma immediately reduces to Divisors, Lemma 30.13.2. □

Lemma 57.32.4. *Let S be a scheme. Let $i : Z \to X$ be an immersion of algebraic spaces over S. Assume X is locally Noetherian. Then i is Koszul-regular $\Leftrightarrow$ i is H_1-regular $\Leftrightarrow$ i is quasi-regular.*

Proof. Via Definition 57.32.2 (and the definition of a locally Noetherian algebraic space in Properties of Spaces, Section 48.7) this immediately translates to the case of schemes which is Divisors, Lemma 30.13.3. □

Lemma 57.32.5. *Let S be a scheme. Let $i : Z \to X$ be a Koszul-regular, H_1-regular, or quasi-regular immersion of algebraic spaces over S. Let $X' \to X$ be a flat morphism of algebraic spaces over S. Then the base change $i' : Z \times_X X' \to X'$ is a Koszul-regular, H_1-regular, or quasi-regular immersion.*

Proof. Via Definition 57.32.2 (and the definition of a flat morphism of algebraic spaces in Morphisms of Spaces, Section 49.27) this lemma reduces to the case of schemes, see Divisors, Lemma 30.13.4. □

Lemma 57.32.6. *Let S be a scheme. Let $i : Z \to X$ be an immersion of algebraic spaces over S. Then i is a quasi-regular immersion if and only if the following conditions are satisfied*

(1) *i is locally of finite presentation,*
(2) *the conormal sheaf $\mathcal{C}_{Z/X}$ is finite locally free, and*
(3) *the map (57.5.1.2) is an isomorphism.*

Proof. Follows from the case of schemes (Divisors, Lemma 30.13.5) via étale localization (use Definition 57.32.2 and Lemma 57.5.2). □

Lemma 57.32.7. *Let S be a scheme. Let $Z \to Y \to X$ be immersions of algebraic spaces over S. Assume that $Z \to Y$ is H_1-regular. Then the canonical sequence of Lemma 57.4.6*

$$0 \to i^*\mathcal{C}_{Y/X} \to \mathcal{C}_{Z/X} \to \mathcal{C}_{Z/Y} \to 0$$

is exact and (étale) locally split.

Proof. Since $\mathcal{C}_{Z/Y}$ is finite locally free (see Lemma 57.32.6 and Lemma 57.32.3) it suffices to prove that the sequence is exact. It suffices to show that the first map is injective as the sequence is already right exact in general. After étale localization on X this reduces to the case of schemes, see Divisors, Lemma 30.13.6. □

A composition of quasi-regular immersions may not be quasi-regular, see Algebra, Remark 10.66.8. The other types of regular immersions are preserved under composition.

Lemma 57.32.8. *Let S be a scheme. Let $i : Z \to Y$ and $j : Y \to X$ be immersions of algebraic spaces over S.*

(1) *If i and j are Koszul-regular immersions, so is $j \circ i$.*
(2) *If i and j are H_1-regular immersions, so is $j \circ i$.*
(3) *If i is an H_1-regular immersion and j is a quasi-regular immersion, then $j \circ i$ is a quasi-regular immersion.*

Proof. Immediate from the case of schemes, see Divisors, Lemma 30.13.7. □

Lemma 57.32.9. *Let S be a scheme. Let $i : Z \to Y$ and $j : Y \to X$ be immersions of algebraic spaces over S. Assume that the sequence*

$$0 \to i^*\mathcal{C}_{Y/X} \to \mathcal{C}_{Z/X} \to \mathcal{C}_{Z/Y} \to 0$$

of Lemma 57.4.6 is exact and locally split.

(1) *If $j \circ i$ is a quasi-regular immersion, so is i.*
(2) *If $j \circ i$ is a H_1-regular immersion, so is i.*
(3) *If both j and $j \circ i$ are Koszul-regular immersions, so is i.*

Proof. Immediate from the case of schemes, see Divisors, Lemma 30.13.8. □

Lemma 57.32.10. *Let S be a scheme. Let $i : Z \to Y$ and $j : Y \to X$ be immersions of algebraic spaces over S. Assume X is locally Noetherian. The following are equivalent*

(1) *i and j are Koszul regular immersions,*

(2) *i and $j \circ i$ are Koszul regular immersions,*
(3) *$j \circ i$ is a Koszul regular immersion and the conormal sequence*

$$0 \to i^*\mathcal{C}_{Y/X} \to \mathcal{C}_{Z/X} \to \mathcal{C}_{Z/Y} \to 0$$

is exact and locally split.

Proof. Immediate from the case of schemes, see Divisors, Lemma 30.13.9. □

57.33. Pseudo-coherent morphisms

This section is the analogue of More on Morphisms, Section 36.40 for morphisms of schemes. The reader is encouraged to read up on pseudo-coherent morphisms of schemes in that section first.

The property "pseudo-coherent" of morphisms of schemes is étale local on the source-and-target. To see this use More on Morphisms, Lemmas 36.40.9 and 36.40.12 and Descent, Lemma 34.28.6. By Morphisms of Spaces, Lemma 49.21.1 we may define the notion of a pseudo-coherent morphism of algebraic spaces as follows and it agrees with the already existing notion defined in More on Morphisms, Section 36.40 when the algebraic spaces in question are representable.

Definition 57.33.1. Let S be a scheme. Let $f : X \to Y$ be a morphism of algebraic spaces over S.

(1) We say f is *pseudo-coherent* if the equivalent conditions of Morphisms of Spaces, Lemma 49.21.1 hold with $\mathcal{P}$ ="pseudo-coherent".
(2) Let $x \in |X|$. We say f is *pseudo-coherent at x* if there exists an open neighbourhood $X' \subset X$ of x such that $f|_{X'} : X' \to Y$ is pseudo-coherent.

Beware that a base change of a pseudo-coherent morphism is not pseudo-coherent in general.

Lemma 57.33.2. *A flat base change of a pseudo-coherent morphism is pseudo-coherent.*

Proof. Omitted. Hint: Use the schemes version of this lemma, see More on Morphisms, Lemma 36.40.3. □

Lemma 57.33.3. *A composition of pseudo-coherent morphisms is pseudo-coherent.*

Proof. Omitted. Hint: Use the schemes version of this lemma, see More on Morphisms, Lemma 36.40.4. □

Lemma 57.33.4. *A pseudo-coherent morphism is locally of finite presentation.*

Proof. Immediate from the definitions. □

Lemma 57.33.5. *A flat morphism which is locally of finite presentation is pseudo-coherent.*

Proof. Omitted. Hint: Use the schemes version of this lemma, see More on Morphisms, Lemma 36.40.6. □

Lemma 57.33.6. *Let $f : X \to Y$ be a morphism of algebraic spaces pseudo-coherent over a base algebraic space B. Then f is pseudo-coherent.*

Proof. Omitted. Hint: Use the schemes version of this lemma, see More on Morphisms, Lemma 36.40.7. □

Lemma 57.33.7. *Let S be a scheme. Let $f : X \to Y$ be a morphism of algebraic spaces over S. If Y is locally Noetherian, then f is pseudo-coherent if and only if f is locally of finite type.*

Proof. Omitted. Hint: Use the schemes version of this lemma, see More on Morphisms, Lemma 36.40.8. □

57.34. Perfect morphisms

This section is the analogue of More on Morphisms, Section 36.41 for morphisms of schemes. The reader is encouraged to read up on perfect morphisms of schemes in that section first.

The property "perfect" of morphisms of schemes is étale local on the source-and-target. To see this use More on Morphisms, Lemmas 36.41.10 and 36.41.12 and Descent, Lemma 34.28.6. By Morphisms of Spaces, Lemma 49.21.1 we may define the notion of a perfect morphism of algebraic spaces as follows and it agrees with the already existing notion defined in More on Morphisms, Section 36.41 when the algebraic spaces in question are representable.

Definition 57.34.1. Let S be a scheme. Let $f : X \to Y$ be a morphism of algebraic spaces over S.

(1) We say f is *perfect* if the equivalent conditions of Morphisms of Spaces, Lemma 49.21.1 hold with $\mathcal{P}$ ="perfect".
(2) Let $x \in |X|$. We say f is *perfect at x* if there exists an open neighbourhood $X' \subset X$ of x such that $f|_{X'} : X' \to Y$ is perfect.

Note that a perfect morphism is pseudo-coherent, hence locally of finite presentation. Beware that a base change of a perfect morphism is not perfect in general.

Lemma 57.34.2. *A flat base change of a perfect morphism is perfect.*

Proof. Omitted. Hint: Use the schemes version of this lemma, see More on Morphisms, Lemma 36.41.3. □

Lemma 57.34.3. *A composition of perfect morphisms is perfect.*

Proof. Omitted. Hint: Use the schemes version of this lemma, see More on Morphisms, Lemma 36.41.4. □

Lemma 57.34.4. *Let S be a scheme. Let $f : X \to Y$ be a morphism of algebraic spaces over S. The following are equivalent*

(1) *f is flat and perfect, and*
(2) *f is flat and locally of finite presentation.*

Proof. Omitted. Hint: Use the schemes version of this lemma, see More on Morphisms, Lemma 36.41.5. □

57.35. Local complete intersection morphisms

This section is the analogue of More on Morphisms, Section 36.42 for morphisms of schemes. The reader is encouraged to read up on local complete intersection morphisms of schemes in that section first.

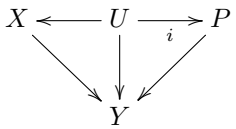
The property "being a local complete intersection morphism" of morphisms of schemes is étale local on the source-and-target. To see this use More on Morphisms, Lemmas 36.42.12 and 36.42.13 and Descent, Lemma 34.28.6. By Morphisms of Spaces, Lemma 49.21.1 we may define the notion of a local complete intersection morphism of algebraic spaces as follows and it agrees with the already existing notion defined in More on Morphisms, Section 36.42 when the algebraic spaces in question are representable.

Definition 57.35.1. Let S be a scheme. Let $f : X \to Y$ be a morphism of algebraic spaces over S.

(1) We say f is a *Koszul morphism*, or that f is a *local complete intersection morphism* if the equivalent conditions of Morphisms of Spaces, Lemma 49.21.1 hold with $\mathcal{P}(f) =$"f is a local complete intersection morphism".
(2) Let $x \in |X|$. We say f is *Koszul at* x if there exists an open neighbourhood $X' \subset X$ of x such that $f|_{X'} : X' \to Y$ is a local complete intersection morphism.

In some sense the defining property of a local complete intersection morphism is the result of the following lemma.

Lemma 57.35.2. *Let S be a scheme. Let $f : X \to Y$ be a local complete intersection morphism of algebraic spaces over S. Let P be an algebraic space smooth over Y. Let $U \to X$ be an étale morphism of algebraic spaces and let $i : U \to P$ an immersion of algebraic spaces over Y. Picture:*



Then i is a Koszul-regular immersion of algebraic spaces.

Proof. Choose a scheme V and a surjective étale morphism $V \to Y$. Choose a scheme W and a surjective étale morphism $W \to P \times_Y V$. Set $U' = U \times_P W$, which is a scheme étale over U. We have to show that $U' \to W$ is a Koszul-regular immersion of schemes, see Definition 57.32.2. By Definition 57.35.1 above the morphism of schemes $U' \to V$ is a local complete intersection morphism. Hence the result follows from More on Morphisms, Lemma 36.42.3. □

It seems like a good idea to collect here some properties in common with all Koszul morphisms.

Lemma 57.35.3. *Let S be a scheme. Let $f : X \to Y$ be a local complete intersection morphism of algebraic spaces over S. Then*

(1) *f is locally of finite presentation,*
(2) *f is pseudo-coherent, and*
(3) *f is perfect.*

Proof. Omitted. Hint: Use the schemes version of this lemma, see More on Morphisms, Lemma 36.42.4. □

Beware that a base change of a Koszul morphism is not Koszul in general.

Lemma 57.35.4. *A flat base change of a local complete intersection morphism is a local complete intersection morphism.*

Proof. Omitted. Hint: Use the schemes version of this lemma, see More on Morphisms, Lemma 36.42.6. □

Lemma 57.35.5. *A composition of local complete intersection morphisms is a local complete intersection morphism.*

Proof. Omitted. Hint: Use the schemes version of this lemma, see More on Morphisms, Lemma 36.42.7. □

Lemma 57.35.6. *Let S be a scheme. Let $f : X \to Y$ be a morphism of algebraic spaces over S. The following are equivalent*

(1) *f is flat and a local complete intersection morphism, and*
(2) *f is syntomic.*

Proof. Omitted. Hint: Use the schemes version of this lemma, see More on Morphisms, Lemma 36.42.8. □

Lemma 57.35.7. *Let S be a scheme. Consider a commutative diagram*

$$\begin{array}{ccc} X & \xrightarrow{f} & Y \\ & {\scriptstyle p}\searrow \quad \swarrow{\scriptstyle q} & \\ & Z & \end{array}$$

of algebraic spaces over S. Assume that both p and q are flat and locally of finite presentation. Then there exists an open subspace $U(f) \subset X$ such that $|U(f)| \subset |X|$ is the set of points where f is Koszul. Moreover, for any morphism of algebraic spaces $Z' \to Z$, if $f' : X' \to Y'$ is the base change of f by $Z' \to Z$, then $U(f')$ is the inverse image of $U(f)$ under the projection $X' \to X$.

Proof. This lemma is the analogue of More on Morphisms, Lemma 36.42.14 and in fact we will deduce the lemma from it. By Definition 57.35.1 the set $\{x \in |X| : f \text{ is Koszul at } x\}$ is open in $|X|$ hence by Properties of Spaces, Lemma 48.4.8 it corresponds to an open subspace $U(f)$ of X. Hence we only need to prove the final statement.

Choose a scheme W and a surjective étale morphism $W \to Z$. Choose a scheme V and a surjective étale morphism $V \to W \times_Z Y$. Choose a scheme U and a surjective étale morphism $U \to V \times_Y X$. Finally, choose a scheme W' and a surjective étale morphism $W' \to W \times_Z Z'$. Set $V' = W' \times_W V$ and $U' = W' \times_W U$, so that we obtain surjective étale morphisms $V' \to Y'$ and $U' \to X'$. We will use without further mention an étale morphism of algebraic spaces induces an open map of associated topological spaces (see Properties of Spaces, Lemma 48.13.7). Note that by definition $U(f)$ is the image in $|X|$ of the set T of points in U where the morphism of schemes $U \to V$ is Koszul. Similarly, $U(f')$ is the image in $|X'|$ of the set T' of points in U' where the morphism of schemes $U' \to V'$ is Koszul. Now, by construction the diagram

$$\begin{array}{ccc} U' & \longrightarrow & U \\ \downarrow & & \downarrow \\ V' & \longrightarrow & V \end{array}$$

is cartesian (in the category of schemes). Hence the aforementioned More on Morphisms, Lemma 36.42.14 applies to show that T' is the inverse image of T. Since $|U'| \to |X'|$ is surjective this implies the lemma. □

Lemma 57.35.8. *Let S be a scheme. Let $f : X \to Y$ be a local complete intersection morphism of algebraic spaces over S. Then f is unramified if and only if f is formally unramified and in this case the conormal sheaf $\mathcal{C}_{X/Y}$ is finite locally free on X.*

Proof. This follows from the corresponding result for morphisms of schemes, see More on Morphisms, Lemma 36.42.15, by étale localization, see Lemma 57.12.11. (Note that in the situation of this lemma the morphism $V \to U$ is unramified and a local complete intersection morphism by definition.) □

Lemma 57.35.9. *Let S be a scheme. Let $Z \to Y \to X$ be formally unramified morphisms of algebraic spaces over S. Assume that $Z \to Y$ is a local complete intersection morphism. The exact sequence*

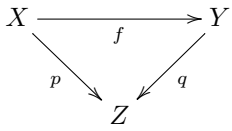
$$0 \to i^*\mathcal{C}_{Y/X} \to \mathcal{C}_{Z/X} \to \mathcal{C}_{Z/Y} \to 0$$

of Lemma 57.4.6 is short exact.

Proof. Choose a scheme U and a surjective étale morphism $U \to X$. Choose a scheme V and a surjective étale morphism $V \to U \times_X Y$. Choose a scheme W and a surjective étale morphism $W \to V \times_Y Z$. By Lemma 57.12.11 the morphisms $W \to V$ and $V \to U$ are formally unramified. Moreover the sequence $i^*\mathcal{C}_{Y/X} \to \mathcal{C}_{Z/X} \to \mathcal{C}_{Z/Y} \to 0$ restricts to the corresponding sequence $i^*\mathcal{C}_{V/U} \to \mathcal{C}_{W/U} \to \mathcal{C}_{W/V} \to 0$ for $W \to V \to U$. Hence the result follows from the result for schemes (More on Morphisms, Lemma 36.42.16) as by definition the morphism $W \to V$ is a local complete intersection morphism. □

57.36. When is a morphism an isomorphism?

More generally we can ask: "When does a morphism have property $\mathcal{P}$?" A more precise question is the following. Suppose given a commutative diagram



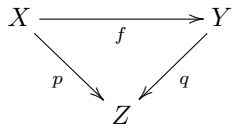
of algebraic spaces. Does there exist a monomorphism of algebraic spaces $W \to Z$ with the following two properties:

(1) the base change $f_W : X_W \to Y_W$ has property $\mathcal{P}$, and
(2) any morphism $Z' \to Z$ of algebraic spaces factors through W if and only if the base change $f_{Z'} : X_{Z'} \to Y_{Z'}$ has property $\mathcal{P}$.

In many cases, if $W \to Z$ exists, then it is an immersion, open immersion, or closed immersion.

The answer to this question may depend on auxiliary properties of the morphisms f, p, and q. An example is $\mathcal{P}(f) =$"f is flat" which we have discussed for morphisms of schemes in the case $Y = S$ in great detail in the chapter "More on Flatness", starting with More on Flatness, Section 37.19.

Lemma 57.36.1. *Consider a commutative diagram*



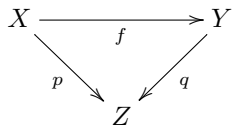
of algebraic spaces. Assume that p is locally of finite type and closed. Then there exists an open subspace $W \subset Z$ such that a morphism $Z' \to Z$ factors through W if and only if the base change $f_{Z'} : X_{Z'} \to Y_{Z'}$ is unramified.

Proof. By Morphisms of Spaces, Lemma 49.34.10 there exists an open subspace $U(f) \subset X$ which is the set of points where f is unramified. Moreover, formation of $U(f)$ commutes with arbitrary base change. Let $W \subset Z$ be the open subspace (see Properties of Spaces, Lemma 48.4.8) with underlying set of points

$$|W| = |Z| \setminus |p| \left(|X| \setminus |U(f)| \right)$$

i.e., $z \in |Z|$ is a point of W if and only if f is unramified at every point of X above z. Note that this is open because we assumed that p is closed. Since the formation of $U(f)$ commutes with arbitrary base change we immediately see (using Properties of Spaces, Lemma 48.4.9) that W has the desired universal property. □

Lemma 57.36.2. *Consider a commutative diagram*



of algebraic spaces. Assume that

(1) *p is locally of finite type,*
(2) *p is closed, and*
(3) *$p_2 : X \times_Y X \to Z$ is closed.*

Then there exists an open subspace $W \subset Z$ such that a morphism $Z' \to Z$ factors through W if and only if the base change $f_{Z'} : X_{Z'} \to Y_{Z'}$ is unramified and universally injective.

Proof. After replacing Z by the open subspace found in Lemma 57.36.1 we may assume that f is already unramified; note that this does not destroy assumption (2) or (3). By Morphisms of Spaces, Lemma 49.34.9 we see that $\Delta_{X/Y} : X \to X \times_Y X$ is an open immersion. This remains true after any base change. Hence by Morphisms of Spaces, Lemma 49.18.2 we see that $f_{Z'}$ is universally injective if and only if the base change of the diagonal $X_{Z'} \to (X \times_Y X)_{Z'}$ is an isomorphism. Let $W \subset Z$ be the open subspace (see Properties of Spaces, Lemma 48.4.8) with underlying set of points

$$|W| = |Z| \setminus |p_2| \left(|X \times_Y X| \setminus \mathrm{Im}(|\Delta_{X/Y}|) \right)$$

i.e., $z \in |Z|$ is a point of W if and only if the fibre of $|X \times_Y X| \to |Z|$ over z is in the image of $|X| \to |X \times_Y X|$. Then it is clear from the discussion above that the restriction $p^{-1}(W) \to q^{-1}(W)$ of f is unramified and universally injective.

Conversely, suppose that $f_{Z'}$ is unramified and universally injective. In order to show that $Z' \to Z$ factors through W it suffices to show that $|Z'| \to |Z|$ has image contained in $|W|$, see Properties of Spaces, Lemma 48.4.9. Hence it suffices to

prove the result when Z' is the spectrum of a field. Denote $z \in |Z|$ the image of $|Z'| \to |Z|$. The discussion above shows that

$$|X_{Z'}| \longrightarrow |(X \times_Y X)_{Z'}|$$

is surjective. By Properties of Spaces, Lemma 48.4.3 in the commutative diagram

$$\begin{array}{ccc} |X_{Z'}| & \longrightarrow & |(X \times_Y X)_{Z'}| \\ \downarrow & & \downarrow \\ |p|^{-1}(\{z\}) & \longrightarrow & |p_2|^{-1}(\{z\}) \end{array}$$

the vertical arrows are surjective. It follows that $z \in |W|$ as desired. □

Lemma 57.36.3. *Consider a commutative diagram*

$$\begin{array}{ccc} X & \xrightarrow{f} & Y \\ & {\scriptstyle p} \searrow \quad \swarrow {\scriptstyle q} & \\ & Z & \end{array}$$

of algebraic spaces. Assume that

(1) *p is locally of finite type,*
(2) *p is universally closed, and*
(3) *$q : Y \to Z$ is separated.*

Then there exists an open subspace $W \subset Z$ such that a morphism $Z' \to Z$ factors through W if and only if the base change $f_{Z'} : X_{Z'} \to Y_{Z'}$ is a closed immersion.

Proof. We will use the characterization of closed immersions as universally closed, unramified, and universally injective morphisms, see Lemma 57.11.9. First, note that since p is universally closed and q is separated, we see that f is universally closed, see Morphisms of Spaces, Lemma 49.36.6. It follows that any base change of f is universally closed, see Morphisms of Spaces, Lemma 49.10.3. Thus to finish the proof of the lemma it suffices to prove that the assumptions of Lemma 57.36.2 are satisfied. The projection $\mathrm{pr}_0 : X \times_Y X \to X$ is universally closed as a base change of f, see Morphisms of Spaces, Lemma 49.10.3. Hence $X \times_Y X \to Z$ is universally closed as a composition of universally closed morphisms (see Morphisms of Spaces, Lemma 49.10.4). This finishes the proof of the lemma. □

Lemma 57.36.4. *Consider a commutative diagram*

$$\begin{array}{ccc} X & \xrightarrow{f} & Y \\ & {\scriptstyle p} \searrow \quad \swarrow {\scriptstyle q} & \\ & Z & \end{array}$$

of algebraic spaces. Assume that

(1) *p is locally of finite presentation,*
(2) *p is flat,*
(3) *p is closed, and*
(4) *q is locally of finite type.*

Then there exists an open subspace $W \subset Z$ such that a morphism $Z' \to Z$ factors through W if and only if the base change $f_{Z'} : X_{Z'} \to Y_{Z'}$ is flat.

Proof. By Lemma 57.19.6 the set

$$A = \{x \in |X| : X \text{ flat at } x \text{ over } Y\}.$$

is open in $|X|$ and its formation commutes with arbitrary base change. Let $W \subset Z$ be the open subspace (see Properties of Spaces, Lemma 48.4.8) with underlying set of points

$$|W| = |Z| \setminus |p|(|X| \setminus A)$$

i.e., $z \in |Z|$ is a point of W if and only if the whole fibre of $|X| \to |Z|$ over z is contained in A. This is open because p is closed. Since the formation of A commutes with arbitrary base change it follows that W works. □

Lemma 57.36.5. *Consider a commutative diagram*

$$\begin{array}{ccc} X & \xrightarrow{f} & Y \\ & {\scriptstyle p}\searrow \quad \swarrow{\scriptstyle q} & \\ & Z & \end{array}$$

of algebraic spaces. Assume that

(1) *p is locally of finite presentation,*
(2) *p is flat,*
(3) *p is closed,*
(4) *q is locally of finite type, and*
(5) *q is closed.*

Then there exists an open subspace $W \subset Z$ such that a morphism $Z' \to Z$ factors through W if and only if the base change $f_{Z'} : X_{Z'} \to Y_{Z'}$ is surjective and flat.

Proof. By Lemma 57.36.4 we may assume that f is flat. Note that f is locally of finite presentation by Morphisms of Spaces, Lemma 49.26.9. Hence f is open, see Morphisms of Spaces, Lemma 49.27.6. Let $W \subset Z$ be the open subspace (see Properties of Spaces, Lemma 48.4.8) with underlying set of points

$$|W| = |Z| \setminus |q|(|Y| \setminus |f|(|X|)).$$

in other words for $z \in |Z|$ we have $z \in |W|$ if and only if the whole fibre of $|Y| \to |Z|$ over z is in the image of $|X| \to |Y|$. Since q is closed this set is open in $|Z|$. The morphism $X_W \to Y_W$ is surjective by construction. Finally, suppose that $X_{Z'} \to Y_{Z'}$ is surjective. In order to show that $Z' \to Z$ factors through W it suffices to show that $|Z'| \to |Z|$ has image contained in $|W|$, see Properties of Spaces, Lemma 48.4.9. Hence it suffices to prove the result when Z' is the spectrum of a field. Denote $z \in |Z|$ the image of $|Z'| \to |Z|$. By Properties of Spaces, Lemma 48.4.3 in the commutative diagram

$$\begin{array}{ccc} |X_{Z'}| & \longrightarrow & |Y_{Z'}| \\ \downarrow & & \downarrow \\ |p|^{-1}(\{z\}) & \longrightarrow & |q|^{-1}(\{z\}) \end{array}$$

the vertical arrows are surjective. It follows that $z \in |W|$ as desired. □

Lemma 57.36.6. *Consider a commutative diagram*

$$\begin{array}{ccc} X & \xrightarrow{f} & Y \\ & {\scriptstyle p}\searrow \quad \swarrow{\scriptstyle q} & \\ & Z & \end{array}$$

of algebraic spaces. Assume that

(1) *p is locally of finite presentation,*
(2) *p is flat,*
(3) *p is universally closed,*
(4) *q is locally of finite type,*
(5) *q is closed, and*
(6) *q is separated.*

Then there exists an open subspace $W \subset Z$ such that a morphism $Z' \to Z$ factors through W if and only if the base change $f_{Z'} : X_{Z'} \to Y_{Z'}$ is an isomorphism.

Proof. By Lemma 57.36.5 there exists an open subspace $W_1 \subset Z$ such that $f_{Z'}$ is surjective and flat if and only if $Z' \to Z$ factors through W_1. By Lemma 57.36.3 there exists an open subspace $W_2 \subset Z$ such that $f_{Z'}$ is a closed immersion if and only if $Z' \to Z$ factors through W_2. We claim that $W = W_1 \cap W_2$ works. Certainly, if $f_{Z'}$ is an isomorphism, then $Z' \to Z$ factors through W. Hence it suffices to show that f_W is an isomorphism. By construction f_W is a surjective flat closed immersion. In particular f_W is representable. Since a surjective flat closed immersion of schemes is an isomorphism (see Morphisms, Lemma 28.27.1) we win. (Note that actually f_W is locally of finite presentation, whence open, so you can avoid the use of this lemma if you like.) □

Lemma 57.36.7. *Consider a commutative diagram*

$$\begin{array}{ccc} X & \xrightarrow{f} & Y \\ & {\scriptstyle p}\searrow \quad \swarrow{\scriptstyle q} & \\ & Z & \end{array}$$

of algebraic spaces. Assume that

(1) *p is flat and locally of finite presentation,*
(2) *p is closed, and*
(3) *q is flat and locally of finite presentation,*

Then there exists an open subspace $W \subset Z$ such that a morphism $Z' \to Z$ factors through W if and only if the base change $f_{Z'} : X_{Z'} \to Y_{Z'}$ is a local complete intersection morphism.

Proof. By Lemma 57.35.7 there exists an open subspace $U(f) \subset X$ which is the set of points where f is Koszul. Moreover, formation of $U(f)$ commutes with arbitrary base change. Let $W \subset Z$ be the open subspace (see Properties of Spaces, Lemma 48.4.8) with underlying set of points

$$|W| = |Z| \setminus |p| (|X| \setminus |U(f)|)$$

i.e., $z \in |Z|$ is a point of W if and only if f is Koszul at every point of X above z. Note that this is open because we assumed that p is closed. Since the formation of

$U(f)$ commutes with arbitrary base change we immediately see (using Properties of Spaces, Lemma 48.4.9) that W has the desired universal property. □

57.37. Exact sequences of differentials and conormal sheaves

In this section we collect some results on exact sequences of conormal sheaves and sheaves of differentials. In some sense these are all realizations of the triangle of cotangent complexes associated to composable morphisms of algebraic spaces.

In the sequences below each of the maps are as constructed in either Lemma 57.6.6 or Lemma 57.12.8. Let S be a scheme. Let $g : Z \to Y$ and $f : Y \to X$ be morphisms of algebraic spaces over S.

(1) There is a canonical exact sequence
$$g^*\Omega_{Y/X} \to \Omega_{Z/X} \to \Omega_{Z/Y} \to 0,$$
see Lemma 57.6.8. If $g : Z \to Y$ is formally smooth, then this sequence is a short exact sequence, see Lemma 57.16.11.

(2) If g is formally unramified, then there is a canonical exact sequence
$$\mathcal{C}_{Z/Y} \to g^*\Omega_{Y/X} \to \Omega_{Z/X} \to 0,$$
see Lemma 57.12.13. If $f \circ g : Z \to X$ is formally smooth, then this sequence is a short exact sequence, see Lemma 57.16.12.

(3) if g and $f \circ g$ are formally unramified, then there is a canonical exact sequence
$$\mathcal{C}_{Z/X} \to \mathcal{C}_{Z/Y} \to g^*\Omega_{Y/X} \to 0,$$
see Lemma 57.12.14. If $f : Y \to X$ is formally smooth, then this sequence is a short exact sequence, see Lemma 57.16.13.

(4) if g and f are formally unramified, then there is a canonical exact sequence
$$g^*\mathcal{C}_{Y/X} \to \mathcal{C}_{Z/X} \to \mathcal{C}_{Z/Y} \to 0.$$
see Lemma 57.12.15. If $g : Z \to Y$ is a local complete intersection morphism, then this sequence is a short exact sequence, see Lemma 57.35.9.

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CHAPTER 58

Algebraic Spaces over Fields

58.1. Introduction

This chapter is the analogue of the chapter on varieties in the setting of algebraic spaces. A reference for algebraic spaces is [**Kol96**].

58.2. Conventions

The standing assumption is that all schemes are contained in a big fppf site Sch_{fppf}. And all rings A considered have the property that $\mathrm{Spec}(A)$ is (isomorphic) to an object of this big site.

Let S be a scheme and let X be an algebraic space over S. In this chapter and the following we will write $X \times_S X$ for the product of X with itself (in the category of algebraic spaces over S), instead of $X \times X$.

58.3. Geometric components

Lemma 58.3.1. *Let k be an algebraically closed field. Let A, B be strictly henselian local k-algebras with residue field equal to k. Let C be the strict henselization of $A \otimes_k B$ at the maximal ideal $\mathfrak{m}_A \otimes_k B + A \otimes_k \mathfrak{m}_B$. Then the minimal primes of C correspond 1-to-1 to pairs of minimal primes of A and B.*

Proof. First note that a minimal prime $\mathfrak{r}$ of C maps to a minimal prime $\mathfrak{p}$ in A and to a minimal prime $\mathfrak{q}$ of B because the ring maps $A \to C$ and $B \to C$ are flat (by going down for flat ring map Algebra, Lemma 10.36.17). Hence it suffices to show that the strict henselization of $(A/\mathfrak{p} \otimes_k B/\mathfrak{q})_{\mathfrak{m}_A \otimes_k B + A \otimes_k \mathfrak{m}_B}$ has a unique minimal prime ideal. By Algebra, Lemma 10.143.30 the rings $A/\mathfrak{p}$, $B/\mathfrak{q}$ are strictly henselian. Hence we may assume that A and B are strictly henselian local domains and our goal is to show that C has a unique minimal prime. By Properties of Spaces, Lemma 48.21.3. we see that the integral closure A' of A in its fraction field is a normal local domain with residue field k and similarly for the integral closure B' of B into its fraction field. By Algebra, Lemma 10.151.4 we see that $A' \otimes_k B'$ is a normal ring. Hence its localization

$$R = (A' \otimes_k B')_{\mathfrak{m}_{A'} \otimes_k B' + A' \otimes_k \mathfrak{m}_{B'}}$$

is a normal local domain. Note that $A \otimes_k B \to A' \otimes_k B'$ is integral (hence gong up holds – Algebra, Lemma 10.33.20) and that $\mathfrak{m}_{A'} \otimes_k B' + A' \otimes_k \mathfrak{m}_{B'}$ is the unique maximal ideal of $A' \otimes_k B'$ lying over $\mathfrak{m}_A \otimes_k B + A \otimes_k \mathfrak{m}_B$. Hence we see that

$$R = (A' \otimes_k B')_{\mathfrak{m}_A \otimes_k B + A \otimes_k \mathfrak{m}_B}$$

by Algebra, Lemma 10.38.11. It follows that

$$(A \otimes_k B)_{\mathfrak{m}_A \otimes_k B + A \otimes_k \mathfrak{m}_B} \longrightarrow R$$

is integral. We conclude that R is the integral closure of $(A \otimes_k B)_{\mathfrak{m}_A \otimes_k B + A \otimes_k \mathfrak{m}_B}$ in its fraction field, and by Properties of Spaces, Lemma 48.21.3 once again we conclude that C has a unique prime ideal. □

58.4. Schematic locus

Lemma 58.4.1. *Let k be a field. Let X be an algebraic space over $\mathrm{Spec}(k)$. If X is locally of finite type over k and has dimension 0, then X is a scheme.*

Proof. Let U be an affine scheme and let $U \to X$ be an étale morphism. Set $R = U \times_X U$. Note that the two projection morphisms $s, t : R \to U$ are étale morphisms of schemes. By Properties of Spaces, Definition 48.8.2 we see that $\dim(U) = 0$ and similarly $\dim(R) = 0$. On the other hand, the morphism $U \to \mathrm{Spec}(k)$ is locally of finite type as the composition of the étale morphism $U \to X$ and $X \to \mathrm{Spec}(k)$, see Morphisms of Spaces, Lemmas 49.22.2 and 49.35.9. Similarly, $R \to \mathrm{Spec}(k)$ is locally of finite type. Hence by Varieties, Lemma 32.13.2 we see that U and R are disjoint unions of spectra of local Artinian k-algebras A finite over k. In particular, as

$$R = U \times_X U \longrightarrow U \times_{\mathrm{Spec}(k)} U$$

is a monomorphism, we see that R is a finite union of spectra of finite k-algebras. It follows that R is affine, see Schemes, Lemma 25.6.8. Applying Varieties, Lemma 32.13.2 once more we see that R is finite over k. Hence s, t are finite, see Morphisms, Lemma 28.44.12. Thus Groupoids, Proposition 38.20.8 shows that the open subspace U/R of X is an affine scheme. Since the schematic locus of X is an open subspace (see Properties of Spaces, Lemma 48.10.1), and since $U \to X$ was an arbitrary étale morphism from an affine scheme we conclude that X is a scheme. □

Lemma 58.4.2. *Let k be a field. Let X be an algebraic space over k. The following are equivalent*

1. *X is locally quasi-finite over k,*
2. *X is locally of finite type over k and has dimension 0,*
3. *X is a scheme and is locally quasi-finite over k,*
4. *X is a scheme and is locally of finite type over k and has dimension 0, and*
5. *X is a disjoint union of spectra of Artinian local k-algebras A over k with $\dim_k(A) < \infty$.*

Proof. Because we are over a field relative dimension of X/k is the same as the dimension of X. Hence by Morphisms of Spaces, Lemma 49.31.6 we see that (1) and (2) are equivalent. Hence it follows from Lemma 58.4.1 (and trivial implications) that (1) – (4) are equivalent. Finally, Varieties, Lemma 32.13.2 shows that (1) – (4) are equivalent with (5). □

Lemma 58.4.3. *Let k be a field. Let $f : X \to Y$ be a monomorphism of algebraic spaces over k. If Y is locally quasi-finite over k so is X.*

Proof. Assume Y is locally quasi-finite over k. By Lemma 58.4.2 we see that $Y = \coprod \mathrm{Spec}(A_i)$ where each A_i is an Artinian local ring finite over k. By Decent Spaces, Lemma 50.15.1 we see that X is a scheme. Consider $X_i = f^{-1}(\mathrm{Spec}(A_i))$. Then X_i has either one or zero points. If X_i has zero points there is nothing to

prove. If X_i has one point, then $X_i = \mathrm{Spec}(B_i)$ with B_i a zero dimensional local ring and $A_i \to B_i$ is an epimorphism of rings. In particular $A_i/\mathfrak{m}_{A_i} = B_i/\mathfrak{m}_{A_i}B_i$ and we see that $A_i \to B_i$ is surjective by Nakayama's lemma, Algebra, Lemma 10.18.1 (because $\mathfrak{m}_{A_i}$ is a nilpotent ideal!). Thus B_i is a finite local k-algebra, and we conclude by Lemma 58.4.2 that $X \to \mathrm{Spec}(k)$ is locally quasi-finite. □

58.5. Spaces smooth over fields

Lemma 58.5.1. *Let k be a field. Let X be an algebraic space smooth over k. Then X is a regular algebraic space.*

Proof. Choose a scheme U and a surjective étale morphism $U \to X$. The morphism $U \to \mathrm{Spec}(k)$ is smooth as a composition of an étale (hence smooth) morphism and a smooth morphism (see Morphisms of Spaces, Lemmas 49.35.6 and 49.33.2). Hence U is regular by Varieties, Lemma 32.15.3. By Properties of Spaces, Definition 48.7.2 this means that X is regular. □

Lemma 58.5.2. *Let k be a field. Let X be an algebraic space smooth over* $\mathrm{Spec}(k)$. *The set of $x \in |X|$ which are image of morphisms* $\mathrm{Spec}(k') \to X$ *with $k' \supset k$ finite separable is dense in $|X|$.*

Proof. Choose a scheme U and a surjective étale morphism $U \to X$. The morphism $U \to \mathrm{Spec}(k)$ is smooth as a composition of an étale (hence smooth) morphism and a smooth morphism (see Morphisms of Spaces, Lemmas 49.35.6 and 49.33.2). Hence we can apply Varieties, Lemma 32.15.6 to see that the closed points of U whose residue fields are finite separable over k are dense. This implies the lemma by our definition of the topology on $|X|$. □

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CHAPTER 59

Groupoids in Algebraic Spaces

59.1. Introduction

This chapter is devoted to generalities concerning groupoids in algebraic spaces. We recommend reading the beautiful paper [**KM97**] by Keel and Mori.

A lot of what we say here is a repeat of what we said in the chapter on groupoid schemes, see Groupoids, Section 38.1. The discussion of quotient stacks is new here.

59.2. Conventions

The standing assumption is that all schemes are contained in a big fppf site Sch_{fppf}. And all rings A considered have the property that $\mathrm{Spec}(A)$ is (isomorphic) to an object of this big site.

Let S be a scheme and let X be an algebraic space over S. In this chapter and the following we will write $X \times_S X$ for the product of X with itself (in the category of algebraic spaces over S), instead of $X \times X$.

We continue our convention to label projection maps starting with index 0, so we have $\mathrm{pr}_0 : X \times_S Y \to X$ and $\mathrm{pr}_1 : X \times_S Y \to Y$.

59.3. Notation

Let S be a scheme; this will be our base scheme and all algebraic spaces will be over S. Let B be an algebraic space over S; this will be our base algebraic space, and often other algebraic spaces, and schemes will be over B. If we say that X is an algebraic space over B, then we mean that X is an algebraic space over S which comes equipped with structure morphism $X \to B$. Moreover, we try to reserve the letter T to denote a "test" scheme over B. In other words T is a scheme which comes equipped with a structure morphism $T \to B$. In this situation we denote $X(T)$ for the set of T-valued points of X *over* B. In a formula:

$$X(T) = \mathrm{Mor}_B(T, X).$$

Similarly, given a second algebraic space Y over B we set

$$X(Y) = \mathrm{Mor}_B(Y, X).$$

Suppose we are given algebraic spaces X, Y over B as above and a morphism $f : X \to Y$ over B. For any scheme T over B we get an induced map of sets

$$f : X(T) \longrightarrow Y(T)$$

which is functorial in the scheme T over B. As f is a map of sheaves on $(Sch/S)_{fppf}$ over the sheaf B it is clear that f determines and is determined by this rule. More generally, we use the same notation for maps between fibre products. For example, if X, Y, Z are algebraic spaces over B, and if $m : X \times_B Y \to Z \times_B Z$ is a morphism

of algebraic spaces over B, then we think of m as corresponding to a collection of maps between T-valued points

$$X(T) \times Y(T) \longrightarrow Z(T) \times Z(T).$$

And so on and so forth.

Finally, given two maps $f, g : X \to Y$ of algebraic spaces over B, if the induced maps $f, g : X(T) \to Y(T)$ are equal for every scheme T over B, then $f = g$, and hence also $f, g : X(Z) \to Y(Z)$ are equal for every third algebraic space Z over B. Hence, for example, to check the axioms for an group algebraic space G over B, it suffices to check commutativity of diagram on T-valued points where T is a scheme over B as we do in Definition 59.5.1 below.

59.4. Equivalence relations

Please refer to Groupoids, Section 38.3 for notation.

Definition 59.4.1. Let $B \to S$ as in Section 59.3. Let U be an algebraic space over B.

(1) A *pre-relation* on U over B is any morphism $j : R \to U \times_B U$ of algebraic spaces over B. In this case we set $t = \mathrm{pr}_0 \circ j$ and $s = \mathrm{pr}_1 \circ j$, so that $j = (t, s)$.
(2) A *relation* on U over B is a monomorphism $j : R \to U \times_B U$ of algebraic spaces over B.
(3) A *pre-equivalence relation* is a pre-relation $j : R \to U \times_B U$ such that the image of $j : R(T) \to U(T) \times U(T)$ is an equivalence relation for all schemes T over B.
(4) We say a morphism $R \to U \times_B U$ of algebraic spaces over B is an *equivalence relation on U over B* if and only if for every T over B the T-valued points of R define an equivalence relation on the set of T-valued points of U.

In other words, an equivalence relation is a pre-equivalence relation such that j is a relation.

Lemma 59.4.2. *Let $B \to S$ as in Section 59.3. Let U be an algebraic space over B. Let $j : R \to U \times_B U$ be a pre-relation. Let $g : U' \to U$ be a morphism of algebraic spaces over B. Finally, set*

$$R' = (U' \times_B U') \times_{U \times_B U} R \xrightarrow{j'} U' \times_B U'$$

Then j' is a pre-relation on U' over B. If j is a relation, then j' is a relation. If j is a pre-equivalence relation, then j' is a pre-equivalence relation. If j is an equivalence relation, then j' is an equivalence relation.

Proof. Omitted. □

Definition 59.4.3. Let $B \to S$ as in Section 59.3. Let U be an algebraic space over B. Let $j : R \to U \times_B U$ be a pre-relation. Let $g : U' \to U$ be a morphism of algebraic spaces over B. The pre-relation $j' : R' \to U' \times_B U'$ is called the *restriction*, or *pullback* of the pre-relation j to U'. In this situation we sometimes write $R' = R|_{U'}$.

Lemma 59.4.4. *Let $B \to S$ as in Section 59.3. Let $j : R \to U \times_B U$ be a pre-relation of algebraic spaces over B. Consider the relation on $|U|$ defined by the rule*

$$x \sim y \Leftrightarrow \exists\ r \in |R| : t(r) = x, s(r) = y.$$

If j is a pre-equivalence relation then this is an equivalence relation.

Proof. Suppose that $x \sim y$ and $y \sim z$. Pick $r \in |R|$ with $t(r) = x$, $s(r) = y$ and pick $r' \in |R|$ with $t(r') = y$, $s(r') = z$. We may pick a field K such that r and r' can be represented by morphisms $r, r' : \mathrm{Spec}(K) \to R$ with $s \circ r = t \circ r'$. Denote $x = t \circ r$, $y = s \circ r = t \circ r'$, and $z = s \circ r'$, so $x, y, z : \mathrm{Spec}(K) \to U$. By construction $(x, y) \in j(R(K))$ and $(y, z) \in j(R(K))$. Since j is a pre-equivalence relation we see that also $(x, z) \in j(R(K))$. This clearly implies that $x \sim z$.

The proof that $\sim$ is reflexive and symmetric is omitted. □

59.5. Group algebraic spaces

Please refer to Groupoids, Section 38.4 for notation.

Definition 59.5.1. Let $B \to S$ as in Section 59.3.

(1) A *group algebraic space over B* is a pair (G, m), where G is an algebraic space over B and $m : G \times_B G \to G$ is a morphism of algebraic spaces over B with the following property: For every scheme T over B the pair $(G(T), m)$ is a group.
(2) A *morphism $\psi : (G, m) \to (G', m')$ of group algebraic spaces over B* is a morphism $\psi : G \to G'$ of algebraic spaces over B such that for every T/B the induced map $\psi : G(T) \to G'(T)$ is a homomorphism of groups.

Let (G, m) be a group algebraic space over the algebraic space B. By the discussion in Groupoids, Section 38.4 we obtain morphisms of algebraic spaces over B (identity) $e : B \to G$ and (inverse) $i : B \to B$ such that for every T the quadruple $(G(T), m, e, i)$ satisfies the axioms of a group.

Let (G, m), (G', m') be group algebraic spaces over B. Let $f : G \to G'$ be a morphism of algebraic spaces over B. It follows from the definition that f is a morphism of group algebraic spaces over B if and only if the following diagram is commutative:

$$\begin{array}{ccc} G \times_B G & \xrightarrow[f \times f]{} & G' \times_B G' \\ {\scriptstyle m}\downarrow & & \downarrow{\scriptstyle m} \\ G & \xrightarrow{f} & G' \end{array}$$

Lemma 59.5.2. *Let $B \to S$ as in Section 59.3. Let (G, m) be a group algebraic space over B. Let $B' \to B$ be a morphism of algebraic spaces. The pullback $(G_{B'}, m_{B'})$ is a group algebraic space over B'.*

Proof. Omitted. □

59.6. Properties of group algebraic spaces

In this section we collect some simple properties of group algebraic spaces which hold over any base.

Lemma 59.6.1. *Let S be a scheme. Let B be an algebraic space over S. Let G be a group algebraic space over B. Then $G \to B$ is separated (resp. quasi-separated, resp. locally separated) if and only if the identity morphism $e : B \to G$ is a closed immersion (resp. quasi-compact, resp. an immersion).*

Proof. We recall that by Morphisms of Spaces, Lemma 49.5.7 we have that e is a closed immersion (resp. quasi-compact, resp. an immersion) if $G \to B$ is separated (resp. quasi-separated, resp. locally separated). For the converse, consider the diagram

$$\begin{array}{ccc} G & \xrightarrow[\Delta_{G/B}]{} & G \times_B G \\ \downarrow & & \downarrow{\scriptstyle (g,g') \mapsto m(i(g),g')} \\ B & \xrightarrow{e} & G \end{array}$$

It is an exercise in the functorial point of view in algebraic geometry to show that this diagram is cartesian. In other words, we see that $\Delta_{G/B}$ is a base change of e. Hence if e is a closed immersion (resp. quasi-compact, resp. an immersion) so is $\Delta_{G/B}$, see Spaces, Lemma 47.12.3 (resp. Morphisms of Spaces, Lemma 49.9.3, resp. Spaces, Lemma 47.12.3). □

59.7. Examples of group algebraic spaces

If $G \to S$ is a group scheme over the base scheme S, then the base change G_B to any algebraic space B over S is an group algebraic space over B by Lemma 59.5.2. We will frequently use this in the examples below.

Example 59.7.1 (Multiplicative group algebraic space)**.** Let $B \to S$ as in Section 59.3. Consider the functor which associates to any scheme T over B the group $\Gamma(T, \mathcal{O}_T^*)$ of units in the global sections of the structure sheaf. This is representable by the group algebraic space

$$\mathbf{G}_{m,B} = B \times_S \mathbf{G}_{m,S}$$

over B. Here $\mathbf{G}_{m,S}$ is the multiplicative group scheme over S, see Groupoids, Example 38.5.1.

Example 59.7.2 (Roots of unity as a group algebraic space)**.** Let $B \to S$ as in Section 59.3. Let $n \in \mathbf{N}$. Consider the functor which associates to any scheme T over B the subgroup of $\Gamma(T, \mathcal{O}_T^*)$ consisting of nth roots of unity. This is representable by the group algebraic space

$$\mu_{n,B} = B \times_S \mu_{n,S}$$

over B. Here $\mu_{n,S}$ is the group scheme of nth roots of unity over S, see Groupoids, Example 38.5.2.

Example 59.7.3 (Additive group algebraic space)**.** Let $B \to S$ as in Section 59.3. Consider the functor which associates to any scheme T over B the group $\Gamma(T, \mathcal{O}_T)$ of global sections of the structure sheaf. This is representable by the group algebraic space

$$\mathbf{G}_{a,B} = B \times_S \mathbf{G}_{a,S}$$

over B. Here $\mathbf{G}_{a,S}$ is the additive group scheme over S, see Groupoids, Example 38.5.3.

Example 59.7.4 (General linear group algebraic space)**.** Let $B \to S$ as in Section 59.3. Let $n \geq 1$. Consider the functor which associates to any scheme T over B the group

$$\mathrm{GL}_n(\Gamma(T, \mathcal{O}_T))$$

of invertible $n \times n$ matrices over the global sections of the structure sheaf. This is representable by the group algebraic space

$$\mathrm{GL}_{n,B} = B \times_S \mathrm{GL}_{n,S}$$

over B. Here $\mathbf{G}_{m,S}$ is the general linear group scheme over S, see Groupoids, Example 38.5.4.

Example 59.7.5. Let $B \to S$ as in Section 59.3. Let $n \geq 1$. The determinant defines a morphisms of group algebraic spaces

$$\det : \mathrm{GL}_{n,B} \longrightarrow \mathbf{G}_{m,B}$$

over B. It is the base change of the determinant morphism over S from Groupoids, Example 38.5.5.

Example 59.7.6 (Constant group algebraic space)**.** Let $B \to S$ as in Section 59.3. Let G be an abstract group. Consider the functor which associates to any scheme T over B the group of locally constant maps $T \to G$ (where T has the Zariski topology and G the discrete topology). This is representable by the group algebraic space

$$G_B = B \times_S G_S$$

over B. Here G_S is the constant group scheme introduced in Groupoids, Example 38.5.6.

59.8. Actions of group algebraic spaces

Please refer to Groupoids, Section 38.8 for notation.

Definition 59.8.1. Let $B \to S$ as in Section 59.3. Let (G, m) be a group algebraic space over B. Let X be an algebraic space over B.

(1) An *action of G on the algebraic space X/B* is a morphism $a : G \times_B X \to X$ over B such that for every scheme T over B the map $a : G(T) \times X(T) \to X(T)$ defines the structure of a $G(T)$-set on $X(T)$.
(2) Suppose that X, Y are algebraic spaces over B each endowed with an action of G. An *equivariant* or more precisely a *G-equivariant* morphism $\psi : X \to Y$ is a morphism of algebraic spaces over B such that for every T over B the map $\psi : X(T) \to Y(T)$ is a morphism of $G(T)$-sets.

In situation (1) this means that the diagrams

(59.8.1.1)
$$\begin{array}{ccc} G \times_B G \times_B X & \xrightarrow[1_G \times a]{} & G \times_B X \\ {\scriptstyle m \times 1_X}\downarrow & & \downarrow{\scriptstyle a} \\ G \times_B X & \xrightarrow{a} & X \end{array} \qquad \begin{array}{ccc} G \times_B X & \xrightarrow{a} & X \\ {\scriptstyle e \times 1_X}\uparrow & \nearrow_{1_X} & \\ X & & \end{array}$$

are commutative. In situation (2) this just means that the diagram

$$\begin{array}{ccc} G \times_B X & \xrightarrow[\mathrm{id} \times f]{} & G \times_B Y \\ {\scriptstyle a}\downarrow & & \downarrow{\scriptstyle a} \\ X & \xrightarrow{f} & Y \end{array}$$

commutes.

Definition 59.8.2. Let $B \to S$, $G \to B$, and $X \to B$ as in Definition 59.8.1. Let $a : G \times_B X \to X$ be an action of G on X/B. We say the action is *free* if for every scheme T over B the action $a : G(T) \times X(T) \to X(T)$ is a free action of the group $G(T)$ on the set $X(T)$.

Lemma 59.8.3. *Situation as in Definition 59.8.2, The action a is free if and only if*

$$G \times_B X \to X \times_B X, \quad (g, x) \mapsto (a(g, x), x)$$

is a monomorphism of algebraic spaces.

Proof. Immediate from the definitions. □

59.9. Principal homogeneous spaces

This section is the analogue of Groupoids, Section 38.9. We suggest reading that section first.

Definition 59.9.1. Let S be a scheme. Let B be an algebraic space over S. Let (G, m) be a group algebraic space over B. Let X be an algebraic space over B, and let $a : G \times_B X \to X$ be an action of G on X.

(1) We say X is a *pseudo G-torsor* or that X is *formally principally homogeneous under G* if the induced morphism $G \times_B X \to X \times_B X$, $(g, x) \mapsto (a(g, x), x)$ is an isomorphism.
(2) A pseudo G-torsor X is called *trivial* if there exists an G-equivariant isomorphism $G \to X$ over B where G acts on G by left multiplication.

It is clear that if $B' \to B$ is a morphism of algebraic spaces then the pullback $X_{B'}$ of a pseudo G-torsor over B is a pseudo $G_{B'}$-torsor over B'.

Lemma 59.9.2. *In the situation of Definition 59.9.1.*

(1) *The algebraic space X is a pseudo G-torsor if and only if for every scheme T over B the set $X(T)$ is either empty or the action of the group $G(T)$ on $X(T)$ is simply transitive.*
(2) *A pseudo G-torsor X is trivial if and only if the morphism $X \to B$ has a section.*

Proof. Omitted. □

Definition 59.9.3. Let S be a scheme. Let B be an algebraic space over S. Let (G, m) be a group algebraic space over B. Let X be a pseudo G-torsor over B.

(1) We say X is a *principal homogeneous space*, or more precisely a *principal homogeneous G-space over B* if there exists a fpqc covering[1] $\{B_i \to B\}_{i \in I}$ such that each $X_{B_i} \to B_i$ has a section (i.e., is a trivial pseudo G_{B_i}-torsor).
(2) Let $\tau \in \{Zariski, étale, smooth, syntomic, fppf\}$. We say X is a *G-torsor in the τ topology*, or a *τ G-torsor*, or simply a *τ torsor* if there exists a τ covering $\{B_i \to B\}_{i \in I}$ such that each $X_{B_i} \to B_i$ has a section.

[1] The default type of torsor in Groupoids, Definition 38.9.3 is a pseudo torsor which is trivial on an fpqc covering. Since G, as an algebraic space, can be seen a sheaf of groups there already is a notion of a G-torsor which corresponds to fppf-torsor, see Lemma 59.9.4. Hence we use "principal homogeneous space" for a pseudo torsor which is fpqc locally trivial, and we try to avoid using the word torsor in this situation.

(3) If X is a G-torsor, then we say that it is *quasi-isotrivial* if it is a torsor for the étale topology.
(4) If X is a G-torsor, then we say that it is *locally trivial* if it is a torsor for the Zariski topology.

We sometimes say "let X be a G-principal homogeneous space over B" to indicate that X is an algebraic space over B equipped with an action of G which turns it into a principal homogeneous space over B. Next we show that this agrees with the notation introduced earlier when both apply.

Lemma 59.9.4. *Let S be a scheme. Let (G, m) be a group algebraic space over S. Let X be an algebraic space over S, and let $a : G \times_S X \to X$ be an action of G on X. Then X is a G-torsor in the $fppf$-topology in the sense of Definition 59.9.3 if and only if X is a G-torsor on $(Sch/S)_{fppf}$ in the sense of Cohomology on Sites, Definition 21.5.1.*

Proof. Omitted. □

59.10. Equivariant quasi-coherent sheaves

Please compare with Groupoids, Section 38.10.

Definition 59.10.1. Let $B \to S$ as in Section 59.3. Let (G, m) be a group algebraic space over B, and let $a : G \times_B X \to X$ be an action of G on the algebraic space X over B. An *G-equivariant quasi-coherent $\mathcal{O}_X$-module*, or simply a *equivariant quasi-coherent $\mathcal{O}_X$-module*, is a pair $(\mathcal{F}, \alpha)$, where $\mathcal{F}$ is a quasi-coherent $\mathcal{O}_X$-module, and α is a $\mathcal{O}_{G \times_B X}$-module map

$$\alpha : a^*\mathcal{F} \longrightarrow \mathrm{pr}_1^*\mathcal{F}$$

where $\mathrm{pr}_1 : G \times_B X \to X$ is the projection such that

(1) the diagram

$$\begin{array}{ccc} (1_G \times a)^*\mathrm{pr}_2^*\mathcal{F} & \xrightarrow{\mathrm{pr}_{12}^*\alpha} & \mathrm{pr}_2^*\mathcal{F} \\ {\scriptstyle (1_G \times a)^*\alpha}\uparrow & & \uparrow{\scriptstyle (m \times 1_X)^*\alpha} \\ (1_G \times a)^*a^*\mathcal{F} & = & (m \times 1_X)^*a^*\mathcal{F} \end{array}$$

is a commutative in the category of $\mathcal{O}_{G \times_B G \times_B X}$-modules, and

(2) the pullback

$$(e \times 1_X)^*\alpha : \mathcal{F} \longrightarrow \mathcal{F}$$

is the identity map.

For explanation compare with the relevant diagrams of Equation (59.8.1.1).

Note that the commutativity of the first diagram guarantees that $(e \times 1_X)^*\alpha$ is an idempotent operator on $\mathcal{F}$, and hence condition (2) is just the condition that it is an isomorphism.

Lemma 59.10.2. *Let $B \to S$ as in Section 59.3. Let G be a group algebraic space over B. Let $f : X \to Y$ be a G-equivariant morphism between algebraic spaces over B endowed with G-actions. Then pullback f^* given by $(\mathcal{F}, \alpha) \mapsto (f^*\mathcal{F}, (1_G \times f)^*\alpha)$ defines a functor from the category of G-equivariant sheaves on X to the category of quasi-coherent G-equivariant sheaves on Y.*

Proof. Omitted. □

59.11. Groupoids in algebraic spaces

Please refer to Groupoids, Section 38.11 for notation.

Definition 59.11.1. Let $B \to S$ as in Section 59.3.

(1) A *groupoid in algebraic spaces over B* is a quintuple (U, R, s, t, c) where U and R are algebraic spaces over B, and $s, t : R \to U$ and $c : R \times_{s,U,t} R \to R$ are morphisms of algebraic spaces over B with the following property: For any scheme T over B the quintuple

$$(U(T), R(T), s, t, c)$$

is a groupoid category.

(2) A *morphism $f : (U, R, s, t, c) \to (U', R', s', t', c')$ of groupoids in algebraic spaces over B* is given by morphisms of algebraic spaces $f : U \to U'$ and $f : R \to R'$ over B with the following property: For any scheme T over B the maps f define a functor from the groupoid category $(U(T), R(T), s, t, c)$ to the groupoid category $(U'(T), R'(T), s', t', c')$.

Let (U, R, s, t, c) be a groupoid in algebraic spaces over B. Note that there are unique morphisms of algebraic spaces $e : U \to R$ and $i : R \to R$ over B such that for every scheme T over B the induced map $e : U(T) \to R(T)$ is the identity, and $i : R(T) \to R(T)$ is the inverse of the groupoid category. The septuple (U, R, s, t, c, e, i) satisfies commutative diagrams corresponding to each of the axioms (1), (2)(a), (2)(b), (3)(a) and (3)(b) of Groupoids, Section 38.11. Conversely given a septuple with this property the quintuple (U, R, s, t, c) is a groupoid in algebraic spaces over B. Note that i is an isomorphism, and e is a section of both s and t. Moreover, given a groupoid in algebraic spaces over B we denote

$$j = (t, s) : R \longrightarrow U \times_B U$$

which is compatible with our conventions in Section 59.4 above. We sometimes say "let (U, R, s, t, c, e, i) be a groupoid in algebraic spaces over B" to stress the existence of identity and inverse.

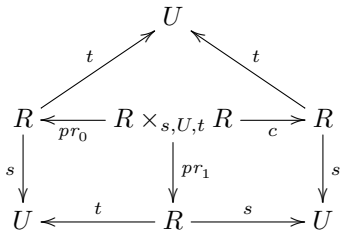
Lemma 59.11.2. *Let $B \to S$ as in Section 59.3. Given a groupoid in algebraic spaces (U, R, s, t, c) over B the morphism $j : R \to U \times_B U$ is a pre-equivalence relation.*

Proof. Omitted. This is a nice exercise in the definitions. □

Lemma 59.11.3. *Let $B \to S$ as in Section 59.3. Given an equivalence relation $j : R \to U$ over B there is a unique way to extend it to a groupoid in algebraic spaces (U, R, s, t, c) over B.*

Proof. Omitted. This is a nice exercise in the definitions. □

Lemma 59.11.4. *Let $B \to S$ as in Section 59.3. Let (U, R, s, t, c) be a groupoid in algebraic spaces over B. In the commutative diagram*

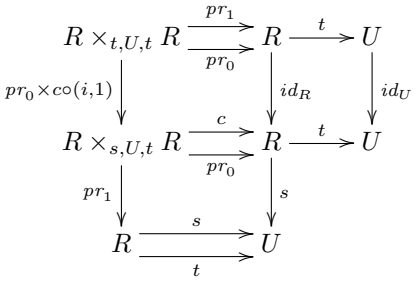


the two lower squares are fibre product squares. Moreover, the triangle on top (which is really a square) is also cartesian.

Proof. Omitted. Exercise in the definitions and the functorial point of view in algebraic geometry. □

Lemma 59.11.5. *Let $B \to S$ be as in Section 59.3. Let (U, R, s, t, c, e, i) be a groupoid in algebraic spaces over B. The diagram*

(59.11.5.1)



is commutative. The two top rows are isomorphic via the vertical maps given. The two lower left squares are cartesian.

Proof. The commutativity of the diagram follows from the axioms of a groupoid. Note that, in terms of groupoids, the top left vertical arrow assigns to a pair of morphisms (α, β) with the same target, the pair of morphisms $(\alpha, \alpha^{-1} \circ \beta)$. In any groupoid this defines a bijection between Arrows $\times_{t,\mathrm{Ob},t}$ Arrows and Arrows $\times_{s,\mathrm{Ob},t}$ Arrows. Hence the second assertion of the lemma. The last assertion follows from Lemma 59.11.4. □

59.12. Quasi-coherent sheaves on groupoids

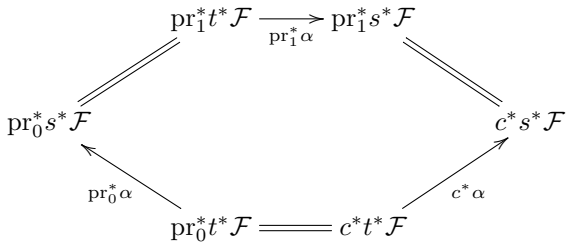
Please compare with Groupoids, Section 38.12.

Definition 59.12.1. Let $B \to S$ as in Section 59.3. Let (U, R, s, t, c) be a groupoid in algebraic spaces over B. A *quasi-coherent module on* (U, R, s, t, c) is a pair $(\mathcal{F}, \alpha)$, where $\mathcal{F}$ is a quasi-coherent $\mathcal{O}_U$-module, and α is a $\mathcal{O}_R$-module map

$$\alpha : t^*\mathcal{F} \longrightarrow s^*\mathcal{F}$$

such that

(1) the diagram



is a commutative in the category of $\mathcal{O}_{R\times_{s,U,t}R}$-modules, and

(2) the pullback

$$e^*\alpha : \mathcal{F} \longrightarrow \mathcal{F}$$

is the identity map.

Compare with the commutative diagrams of Lemma 59.11.4.

The commutativity of the first diagram forces the operator $e^*\alpha$ to be idempotent. Hence the second condition can be reformulated as saying that $e^*\alpha$ is an isomorphism. In fact, the condition implies that α is an isomorphism.

Lemma 59.12.2. *Let S be a scheme, let (U, R, s, t, c) be a groupoid scheme over S. If $(\mathcal{F}, \alpha)$ is a quasi-coherent module on (U, R, s, t, c) then α is an isomorphism.*

Proof. Pull back the commutative diagram of Definition 59.12.1 by the morphism $(i, 1) : R \to R \times_{s,U,t} R$. Then we see that $i^*\alpha \circ \alpha = s^*e^*\alpha$. Pulling back by the morphism $(1, i)$ we obtain the relation $\alpha \circ i^*\alpha = t^*e^*\alpha$. By the second assumption these morphisms are the identity. Hence $i^*\alpha$ is an inverse of α. □

Lemma 59.12.3. *Let $B \to S$ as in Section 59.3. Consider a morphism $f : (U, R, s, t, c) \to (U', R', s', t', c')$ of groupoid in algebraic spaces over B. Then pullback f^* given by*

$$(\mathcal{F}, \alpha) \mapsto (f^*\mathcal{F}, f^*\alpha)$$

defines a functor from the category of quasi-coherent sheaves on (U', R', s', t', c') to the category of quasi-coherent sheaves on (U, R, s, t, c).

Proof. Omitted. □

Lemma 59.12.4. *Let $B \to S$ be as in Section 59.3. Let (U, R, s, t, c) be a groupoid in algebraic spaces over B. The category of quasi-coherent modules on (U, R, s, t, c) has colimits.*

Proof. Let $i \mapsto (\mathcal{F}_i, \alpha_i)$ be a diagram over the index category $\mathcal{I}$. We can form the colimit $\mathcal{F} = \operatorname{colim} \mathcal{F}_i$ which is a quasi-coherent sheaf on U, see Properties of Spaces, Lemma 48.27.7. Since colimits commute with pullback we see that $s^*\mathcal{F} = \operatorname{colim} s^*\mathcal{F}_i$ and similarly $t^*\mathcal{F} = \operatorname{colim} t^*\mathcal{F}_i$. Hence we can set $\alpha = \operatorname{colim} \alpha_i$. We omit the proof that $(\mathcal{F}, \alpha)$ is the colimit of the diagram in the category of quasi-coherent modules on (U, R, s, t, c). □

Lemma 59.12.5. *Let $B \to S$ as in Section 59.3. Let (U, R, s, t, c) be a groupoid in algebraic spaces over B. If s, t are flat, then the category of quasi-coherent modules on (U, R, s, t, c) is abelian.*

Proof. Let $\varphi : (\mathcal{F}, \alpha) \to (\mathcal{G}, \beta)$ be a homomorphism of quasi-coherent modules on (U, R, s, t, c). Since s is flat we see that

$$0 \to s^*\mathrm{Ker}(\varphi) \to s^*\mathcal{F} \to s^*\mathcal{G} \to s^*\mathrm{Coker}(\varphi) \to 0$$

is exact and similarly for pullback by t. Hence α and β induce isomorphisms $\kappa : t^*\mathrm{Ker}(\varphi) \to s^*\mathrm{Ker}(\varphi)$ and $\lambda : t^*\mathrm{Coker}(\varphi) \to s^*\mathrm{Coker}(\varphi)$ which satisfy the cocycle condition. Then it is straightforward to verify that $(\mathrm{Ker}(\varphi), \kappa)$ and $(\mathrm{Coker}(\varphi), \lambda)$ are a kernel and cokernel in the category of quasi-coherent modules on (U, R, s, t, c). Moreover, the condition $\mathrm{Coim}(\varphi) = \mathrm{Im}(\varphi)$ follows because it holds over U. □

59.13. Crystals in quasi-coherent sheaves

Let (I, Φ, j) be a pair consisting of a set I and a pre-relation $j : \Phi \to I \times I$. Assume given for every $i \in I$ a scheme X_i and for every $\phi \in \Phi$ a morphisms of schemes $f_\phi : X_{i'} \to X_i$ where $j(\phi) = (i, i')$. Set $X = (\{X_i\}_{i\in I}, \{f_\phi\}_{\phi\in\Phi})$. Define a *crystal in quasi-coherent modules on* X as a rule which associates to every $i \in \mathrm{Ob}(\mathcal{I})$ a quasi-coherent sheaf $\mathcal{F}_i$ on X_i and for every $\phi \in \Phi$ with $j(\phi) = (i, i')$ an isomorphism

$$\alpha_\phi : f_\phi^*\mathcal{F}_i \longrightarrow \mathcal{F}_{i'}$$

of quasi-coherent sheaves on $X_{i'}$. These crystals in quasi-coherent modules form an additive category $CQC(X)$[2]. This category has colimits (proof is the same as the proof of Lemma 59.12.4). If all the morphisms f_ϕ are flat, then $CQC(X)$ is abelian (proof is the same as the proof of Lemma 59.12.5). Let κ be a cardinal. We say that a crystal in quasi-coherent modules $\mathcal{F}$ on X is *κ-generated* if each $\mathcal{F}_i$ is κ-generated (see Properties, Definition 27.21.1).

Lemma 59.13.1. *In the situation above, if all the morphisms f_ϕ are flat, then there exists a cardinal κ such that every object $(\{\mathcal{F}_i\}_{i\in I}, \{\alpha_\phi\}_{\phi\in\Phi})$ of $CQC(X)$ is the directed colimit of its κ-generated submodules.*

Proof. In the lemma and in this proof a *submodule* of $(\{\mathcal{F}_i\}_{i\in I}, \{\alpha_\phi\}_{\phi\in\Phi})$ means the data of a quasi-coherent submodule $\mathcal{G}_i \subset \mathcal{F}_i$ for all i such that $\alpha_\phi(f_\phi^*\mathcal{G}_i) = \mathcal{G}_{i'}$ as subsheaves of $\mathcal{F}_{i'}$ for all $\phi \in \Phi$. This makes sense because since f_ϕ is flat the pullback f_ϕ^* is exact, i.e., preserves subsheaves. The proof will be a variant to the proof of Properties, Lemma 27.21.3. We urge the reader to read that proof first.

We claim that it suffices to prove the lemma in case all the schemes X_i are affine. To see this let

$$J = \coprod\nolimits_{i\in I} \{U \subset X_i \text{ affine open}\}$$

and let

$$\Psi = \coprod\nolimits_{\phi\in\Phi} \{(U, V) \mid U \subset X_i, V \subset X_{i'} \text{ affine open with } f_\phi(U) \subset V\}$$
$$\coprod\coprod\nolimits_{i\in I} \{(U, U') \mid U, U' \subset X_i \text{ affine open with } U \subset U'\}$$

endowed with the obvious map $\Psi \to J \times J$. Then our $(\mathcal{F}, \alpha)$ induces a crystal in quasi-coherent sheaves $(\{\mathcal{H}_j\}_{j\in J}, \{\beta_\psi\}_{\psi\in\Psi})$ on $Y = (J, \Psi)$ by setting $\mathcal{H}_{(i,U)} = \mathcal{F}_i|_U$

[2]We could single out a set of triples $\phi, \phi', \phi'' \in \Phi$ with $j(\phi) = (i, i')$, $j(\phi') = (i', i'')$, and $j(\phi'') = (i, i'')$ such that $f_{\phi''} = f_\phi \circ f_{\phi'}$ and require that $\alpha_{\phi'} \circ f_{\phi'}^*\alpha_\phi = \alpha_{\phi''}$ for these triples. This would define an additive subcategory. For example the data (I, Φ) could be the set of objects and arrows of an index category and X could be a diagram of schemes over this index category. The result of Lemma 59.13.1 immediately gives the corresponding result in the subcategory.

for $(i, U) \in J$ and setting β_ψ for $\psi \in \Psi$ equal to the restriction of α_ϕ to U if $\psi = (\phi, U, V)$ and equal to $\mathrm{id} : (\mathcal{F}_i|_{U'})|_U \to \mathcal{F}_i|_U$ when $\psi = (i, U, U')$. Moreover, submodules of $(\{\mathcal{H}_j\}_{j\in J}, \{\beta_\psi\}_{\psi\in\Psi})$ correspond 1-to-1 with submodules of $(\{\mathcal{F}_i\}_{i\in I}, \{\alpha_\phi\}_{\phi\in\Phi})$. We omit the proof (hint: use Sheaves, Section 6.30). Moreover, it is clear that if κ works for Y, then the same κ works for X (by the definition of κ-generated modules). Hence it suffices to proof the lemma for crystals in quasi-coherent sheaves on Y.

Assume that all the schemes X_i are affine. Let κ be an infinite cardinal larger than the cardinality of I or Φ. Let $(\{\mathcal{F}_i\}_{i\in I}, \{\alpha_\phi\}_{\phi\in\Phi})$ be an object of $CQC(X)$. For each i write $X_i = \mathrm{Spec}(A_i)$ and $M_i = \Gamma(X_i, \mathcal{F}_i)$. For every $\phi \in \Phi$ with $j(\phi) = (i, i')$ the map α_ϕ translates into an $A_{i'}$-module isomorphism

$$\alpha_\phi : M_i \otimes_{A_i} A_{i'} \longrightarrow M_{i'}$$

Using the axiom of choice choose a rule

$$(\phi, m) \longmapsto S(\phi, m')$$

where the source is the collection of pairs (ϕ, m') such that $\phi \in \Phi$ with $j(\phi) = (i, i')$ and $m' \in M_{i'}$ and where the output is a finite subset $S(\phi, m') \subset M_i$ so that

$$m' = \alpha_\phi\left(\sum\nolimits_{m\in S(\phi,m')} m \otimes a'_m\right)$$

for some $a'_m \in A_{i'}$.

Having made these choices we claim that any section of any $\mathcal{F}_i$ over any X_i is in a κ-generated submodule. To see this suppose that we are given a collection $\mathcal{S} = \{S_i\}_{i\in I}$ of subsets $S_i \subset M_i$ each with cardinality at most κ. Then we define a new collection $\mathcal{S}' = \{S'_i\}_{i\in I}$ with

$$S'_i = S_i \cup \bigcup\nolimits_{(\phi,m'),\ j(\phi)=(i,i'),\ m'\in S_{i'}} S(\phi, m')$$

Note that each S'_i still has cardinality at most κ. Set $\mathcal{S}^{(0)} = \mathcal{S}$, $\mathcal{S}^{(1)} = \mathcal{S}'$ and by induction $\mathcal{S}^{(n+1)} = (\mathcal{S}^{(n)})'$. Then set $S_i^{(\infty)} = \bigcup_{n\geq 0} S_i^{(n)}$ and $\mathcal{S}^{(\infty)} = \{S_i^{(\infty)}\}_{i\in I}$. By construction, for every $\phi \in \Phi$ with $j(\phi) = (i, i')$ and every $m' \in S_{i'}^{(\infty)}$ we can write m' as a finite linear combination of images $\alpha_\phi(m \otimes 1)$ with $m \in S_i^{(\infty)}$. Thus we see that setting N_i equal to the A_i-submodule of M_i generated by $S_i^{(\infty)}$ the corresponding quasi-coherent submodules $\widetilde{N_i} \subset \mathcal{F}_i$ form a κ-generated submodule. This finishes the proof. □

Lemma 59.13.2. *Let $B \to S$ as in Section 59.3. Let (U, R, s, t, c) be a groupoid in algebraic spaces over B. If s, t are flat, then there exists a set T and a family of objects $(\mathcal{F}_t, \alpha_t)_{t\in T}$ of $QCoh(U, R, s, t, c)$ such that every object $(\mathcal{F}, \alpha)$ is the directed colimit of its submodules isomorphic to one of the objects $(\mathcal{F}_t, \alpha_t)$.*

Proof. This lemma is a generalization of Groupoids, Lemma 38.13.6 which deals with the case of a groupoid in schemes. We can't quite use the same argument, so we use the material on "crystals of quasi-coherent sheaves" we developed above.

Choose a scheme W and a surjective étale morphism $W \to U$. Choose a scheme V and a surjective étale morphism $V \to W \times_{U,s} R$. Choose a scheme V' and a surjective étale morphism $V' \to R \times_{t,U} W$. Consider the collection of schemes

$$I = \{W, W \times_U W, V, V', V \times_R V'\}$$

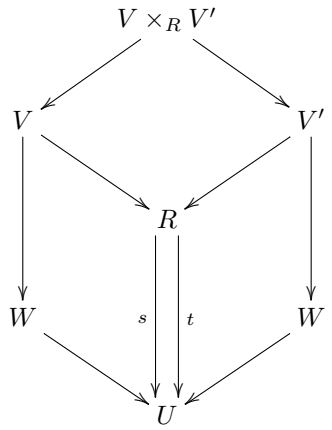
and the set of morphisms of schemes

$$\Phi = \{\mathrm{pr}_i : W \times_U W \to W, V \to W, V' \to W, V \times_R V' \to V, V \times_R V' \to V'\}$$

Set $X = (I, \Phi)$. Recall that we have defined a category $CQC(X)$ of crystals of quasi-coherent sheaves on X. There is a functor

$$QCoh(U, R, s, t, c) \longrightarrow CQC(X)$$

which assigns to $(\mathcal{F}, \alpha)$ the sheaf $\mathcal{F}|_W$ on W, the sheaf $\mathcal{F}|_{W \times_U W}$ on $W \times_U W$, the pullback of $\mathcal{F}$ via $V \to W \times_{U,s} R \to W \to U$ on V, the pullback of $\mathcal{F}$ via $V' \to R \times_{t,U} W \to W \to U$ on V', and finally the pullback of $\mathcal{F}$ via $V \times_R V' \to V \to W \times_{U,s} R \to W \to U$ on $V \times_R V'$. As comparison maps $\{\alpha_\phi\}_{\phi \in \Phi}$ we use the obvious ones (coming from associativity of pullbacks) except for the map $\phi = \mathrm{pr}_{V'} : V \times_R V' \to V'$ we use the pullback of $\alpha : t^*\mathcal{F} \to s^*\mathcal{F}$ to $V \times_R V'$. This makes sense because of the following commutative diagram



The functor displayed above isn't an equivalence of categories. However, since $W \to U$ is surjective étale it is faithful[3]. Since all the morphisms in the diagram above are flat we see that it is an exact functor of abelian categories. Moreover, we claim that given $(\mathcal{F}, \alpha)$ with image $(\{\mathcal{F}_i\}_{i \in I}, \{\alpha_\phi\}_{\phi \in \Phi})$ there is a 1-to-1 correspondence between quasi-coherent submodules of $(\mathcal{F}, \alpha)$ and $(\{\mathcal{F}_i\}_{i \in I}, \{\alpha_\phi\}_{\phi \in \Phi})$. Namely, given a submodule of $(\{\mathcal{F}_i\}_{i \in I}, \{\alpha_\phi\}_{\phi \in \Phi})$ compatibility of the submodule over W with the projection maps $W \times_U W \to W$ will guarantee the submodule comes from a quasi-coherent submodule of $\mathcal{F}$ (by Properties of Spaces, Proposition 48.30.1) and compatibility with $\alpha_{\mathrm{pr}_{V'}}$ will insure this subsheaf is compatible with α (details omitted).

Choose a cardinal κ as in Lemma 59.13.1 for the system $X = (I, \Phi)$. It is clear from Properties, Lemma 27.21.2 that there is a set of isomorphism classes of κ-generated crystals in quasi-coherent sheaves on X. Hence the result is clear. □

59.14. Groupoids and group spaces

Please compare with Groupoids, Section 38.14.

[3]In fact the functor is fully faithful, but we won't need this.

Lemma 59.14.1. *Let $B \to S$ as in Section 59.3. Let (G, m) be a group algebraic space over B with identity e_G and inverse i_G. Let X be an algebraic space over B and let $a : G \times_B X \to X$ be an action of G on X over B. Then we get a groupoid in algebraic spaces (U, R, s, t, c, e, i) over B in the following manner:*

1. *We set $U = X$, and $R = G \times_B X$.*
2. *We set $s : R \to U$ equal to $(g, x) \mapsto x$.*
3. *We set $t : R \to U$ equal to $(g, x) \mapsto a(g, x)$.*
4. *We set $c : R \times_{s,U,t} R \to R$ equal to $((g, x), (g', x')) \mapsto (m(g, g'), x')$.*
5. *We set $e : U \to R$ equal to $x \mapsto (e_G(x), x)$.*
6. *We set $i : R \to R$ equal to $(g, x) \mapsto (i_G(g), a(g, x))$.*

Proof. Omitted. Hint: It is enough to show that this works on the set level. For this use the description above the lemma describing g as an arrow from v to $a(g, v)$. □

Lemma 59.14.2. *Let $B \to S$ as in Section 59.3. Let (G, m) be a group algebraic space over B. Let X be an algebraic space over B and let $a : G \times_B X \to X$ be an action of G on X over B. Let (U, R, s, t, c) be the groupoid in algebraic spaces constructed in Lemma 59.14.1. The rule $(\mathcal{F}, \alpha) \mapsto (\mathcal{F}, \alpha)$ defines an equivalence of categories between G-equivariant $\mathcal{O}_X$-modules and the category of quasi-coherent modules on (U, R, s, t, c).*

Proof. The assertion makes sense because $t = a$ and $s = \mathrm{pr}_1$ as morphisms $R = G \times_B X \to X$, see Definitions 59.10.1 and 59.12.1. Using the translation in Lemma 59.14.1 the commutativity requirements of the two definitions match up exactly. □

59.15. The stabilizer group algebraic space

Please compare with Groupoids, Section 38.15. Given a groupoid in algebraic spaces we get a group algebraic space as follows.

Lemma 59.15.1. *Let $B \to S$ as in Section 59.3. Let (U, R, s, t, c) be a groupoid in algebraic spaces over B. The algebraic space G defined by the cartesian square*

$$\begin{array}{ccc} G & \longrightarrow & R \\ \downarrow & & \downarrow{\scriptstyle j=(t,s)} \\ U & \xrightarrow{\Delta} & U \times_B U \end{array}$$

is a group algebraic space over U with composition law m induced by the composition law c.

Proof. This is true because in a groupoid category the set of self maps of any object forms a group. □

Since Δ is a monomorphism we see that $G = j^{-1}(\Delta_{U/B})$ is a subsheaf of R. Thinking of it in this way, the structure morphism $G = j^{-1}(\Delta_{U/B}) \to U$ is induced by either s or t (it is the same), and m is induced by c.

Definition 59.15.2. Let $B \to S$ as in Section 59.3. Let (U, R, s, t, c) be a groupoid in algebraic spaces over B. The group algebraic space $j^{-1}(\Delta_{U/B}) \to U$ is called the *stabilizer of the groupoid in algebraic spaces* (U, R, s, t, c).

In the literature the stabilizer group algebraic space is often denoted S (because the word stabilizer starts with an "s" presumably); we cannot do this since we have already used S for the base scheme.

Lemma 59.15.3. *Let $B \to S$ as in Section 59.3. Let (U, R, s, t, c) be a groupoid in algebraic spaces over B, and let G/U be its stabilizer. Denote R_t/U the algebraic space R seen as an algebraic space over U via the morphism $t : R \to U$. There is a canonical left action*

$$a : G \times_U R_t \longrightarrow R_t$$

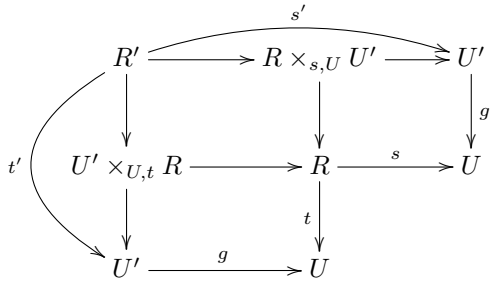
induced by the composition law c.

Proof. In terms of points over T/B we define $a(g, r) = c(g, r)$. □

59.16. Restricting groupoids

Please refer to Groupoids, Section 38.16 for notation.

Lemma 59.16.1. *Let $B \to S$ as in Section 59.3. Let (U, R, s, t, c) be a groupoid in algebraic spaces over B. Let $g : U' \to U$ be a morphism of algebraic spaces. Consider the following diagram*



where all the squares are fibre product squares. Then there is a canonical composition law $c' : R' \times_{s',U',t'} R' \to R'$ such that (U', R', s', t', c') is a groupoid in algebraic spaces over B and such that $U' \to U$, $R' \to R$ defines a morphism $(U', R', s', t', c') \to (U, R, s, t, c)$ of groupoids in algebraic spaces over B. Moreover, for any scheme T over B the functor of groupoids

$$(U'(T), R'(T), s', t', c') \to (U(T), R(T), s, t, c)$$

is the restriction (see Groupoids, Section 38.16) of $(U(T), R(T), s, t, c)$ via the map $U'(T) \to U(T)$.

Proof. Omitted. □

Definition 59.16.2. Let $B \to S$ as in Section 59.3. Let (U, R, s, t, c) be a groupoid in algebraic spaces over B. Let $g : U' \to U$ be a morphism of algebraic spaces over B. The morphism of groupoids in algebraic spaces $(U', R', s', t', c') \to (U, R, s, t, c)$ constructed in Lemma 59.16.1 is called the *restriction of (U, R, s, t, c) to U'*. We sometime use the notation $R' = R|_{U'}$ in this case.

Lemma 59.16.3. *The notions of restricting groupoids and (pre-)equivalence relations defined in Definitions 59.16.2 and 59.4.3 agree via the constructions of Lemmas 59.11.2 and 59.11.3.*

Proof. What we are saying here is that R' of Lemma 59.16.1 is also equal to

$$R' = (U' \times_B U') \times_{U \times_B U} R \longrightarrow U' \times_B U'$$

In fact this might have been a clearer way to state that lemma. □

59.17. Invariant subspaces

In this section we discuss briefly the notion of an invariant subspace.

Definition 59.17.1. Let $B \to S$ as in Section 59.3. Let (U, R, s, t, c) be a groupoid in algebraic spaces over the base B.

(1) We say an open subspace $W \subset U$ is *R-invariant* if $t(s^{-1}(W)) \subset W$.
(2) A locally closed subspace $Z \subset U$ is called *R-invariant* if $t^{-1}(Z) = s^{-1}(Z)$ as locally closed subspaces of R.
(3) A monomorphism of algebraic spaces $T \to U$ is *R-invariant* if $T \times_{U,t} R = R \times_{s,U} T$ as algebraic spaces over R.

For an open subspace $W \subset U$ the R-invariance is also equivalent to requiring that $s^{-1}(W) = t^{-1}(W)$. If $W \subset U$ is R-invariant then the restriction of R to W is just $R_W = s^{-1}(W) = t^{-1}(W)$. Similarly, if $Z \subset U$ is an R-invariant locally closed subspace, then the restriction of R to Z is just $R_Z = s^{-1}(Z) = t^{-1}(Z)$.

Lemma 59.17.2. *Let $B \to S$ as in Section 59.3. Let (U, R, s, t, c) be a groupoid in algebraic spaces over B.*

(1) *If s and t are open, then for every open $W \subset U$ the open $s(t^{-1}(W))$ is R-invariant.*
(2) *If s and t are open and quasi-compact, then U has an open covering consisting of R-invariant quasi-compact open subspaces.*

Proof. Assume s and t open and $W \subset U$ open. Since s is open we see that $W' = s(t^{-1}(W))$ is an open subspace of U. Now it is quite easy to using the functorial point of view that this is an R-invariant open subset of U, but we are going to argue this directly by some diagrams, since we think it is instructive. Note that $t^{-1}(W')$ is the image of the morphism

$$A := t^{-1}(W) \times_{s|_{t^{-1}(W)}, U, t} R \xrightarrow{\mathrm{pr}_1} R$$

and that $s^{-1}(W')$ is the image of the morphism

$$B := R \times_{s, U, s|_{t^{-1}(W)}} t^{-1}(W) \xrightarrow{\mathrm{pr}_0} R.$$

The algebraic spaces A, B on the left of the arrows above are open subspaces of $R \times_{s,U,t} R$ and $R \times_{s,U,s} R$ respectively. By Lemma 59.11.4 the diagram

$$\begin{array}{ccc} R \times_{s,U,t} R & \xrightarrow[(\mathrm{pr}_1, c)]{} & R \times_{s,U,s} R \\ & {\scriptstyle \mathrm{pr}_1} \searrow \quad \swarrow {\scriptstyle \mathrm{pr}_0} & \\ & R & \end{array}$$

is commutative, and the horizontal arrow is an isomorphism. Moreover, it is clear that $(\mathrm{pr}_1, c)(A) = B$. Hence we conclude $s^{-1}(W') = t^{-1}(W')$, and W' is R-invariant. This proves (1).

Assume now that s, t are both open and quasi-compact. Then, if $W \subset U$ is a quasi-compact open, then also $W' = s(t^{-1}(W))$ is a quasi-compact open, and invariant

by the discussion above. Letting W range over images of affines étale over U we see (2). □

59.18. Quotient sheaves

Let S be a scheme, and let B be an algebraic space over S. Let $j : R \to U \times_B U$ be a pre-relation over B. For each scheme S' over S we can take the equivalence relation $\sim_{S'}$ generated by the image of $j(S') : R(S') \to U(S') \times U(S')$. Hence we get a presheaf

$$\begin{array}{ccc} (Sch/S)^{opp}_{fppf} & \longrightarrow & Sets, \\ S' & \longmapsto & U(S')/\sim_{S'} \end{array} \tag{59.18.0.1}$$

Note that since j is a morphism of algebraic spaces over B and into $U \times_B U$ there is a canonical transformation of presheaves from the presheaf (59.18.0.1) to B.

Definition 59.18.1. Let $B \to S$ and the pre-relation $j : R \to U \times_B U$ be as above. In this setting the *quotient sheaf* U/R associated to j is the sheafification of the presheaf (59.18.0.1) on $(Sch/S)_{fppf}$. If $j : R \to U \times_B U$ comes from the action of a group algebraic space G over B on U as in Lemma 59.14.1 then we denote the quotient sheaf U/G.

This means exactly that the diagram

$$R \rightrightarrows U \longrightarrow U/R$$

is a coequalizer diagram in the category of sheaves of sets on $(Sch/S)_{fppf}$. Again there is a canonical map of sheaves $U/R \to B$ as j is a morphism of algebraic spaces over B into $U \times_B U$.

Remark 59.18.2. A variant of the construction above would have been to sheafify the functor

$$\begin{array}{ccc} (Spaces/B)^{opp}_{fppf} & \longrightarrow & Sets, \\ X & \longmapsto & U(X)/\sim_X \end{array}$$

where now $\sim_X \subset U(X) \times U(X)$ is the equivalence relation generated by the image of $j : R(X) \to U(X) \times U(X)$. Here of course $U(X) = \mathrm{Mor}_B(X, U)$ and $R(X) = \mathrm{Mor}_B(X, R)$. In fact, the result would have been the same, via the identifications of (insert future reference in Topologies of Spaces here).

Definition 59.18.3. In the situation of Definition 59.18.1. We say that the pre-relation j has a *quotient representable by an algebraic space* if the sheaf U/R is an algebraic space. We say that the pre-relation j has a *representable quotient* if the sheaf U/R is representable by a scheme. We will say a groupoid in algebraic spaces (U, R, s, t, c) over B has a *representable quotient* (resp. *quotient representable by an algebraic space* if the quotient U/R with $j = (t, s)$ is representable (resp. an algebraic space).

If the quotient U/R is representable by M (either a scheme or an algebraic space over S), then it comes equipped with a canonical structure morphism $M \to B$ as we've seen above.

The following lemma characterizes M representing the quotient. It applies for example if $U \to M$ is flat, of finite presentation and surjective, and $R \cong U \times_M U$.

Lemma 59.18.4. *In the situation of Definition 59.18.1. Assume there is an algebraic space M over S, and a morphism $U \to M$ such that*

(1) *the morphism $U \to M$ equalizes s, t,*
(2) *the map $U \to M$ is a surjection of sheaves, and*
(3) *the induced map $(t, s) : R \to U \times_M U$ is a surjection of sheaves.*

In this case M represents the quotient sheaf U/R.

Proof. Condition (1) says that $U \to M$ factors through U/R. Condition (2) says that $U/R \to M$ is surjective as a map of sheaves. Condition (3) says that $U/R \to M$ is injective as a map of sheaves. Hence the lemma follows. □

The following lemma is wrong if we do not require j to be a pre-equivalence relation (but just a pre-relation say).

Lemma 59.18.5. *Let S be a scheme. Let B be an algebraic space over S. Let $j : R \to U \times_B U$ be a pre-equivalence relation over B. For a scheme S' over S and $a, b \in U(S')$ the following are equivalent:*

(1) *a and b map to the same element of $(U/R)(S')$, and*
(2) *there exists an fppf covering $\{f_i : S_i \to S'\}$ of S' and morphisms $r_i : S_i \to R$ such that $a \circ f_i = s \circ r_i$ and $b \circ f_i = t \circ r_i$.*

In other words, in this case the map of sheaves

$$R \longrightarrow U \times_{U/R} U$$

is surjective.

Proof. Omitted. Hint: The reason this works is that the presheaf (59.18.0.1) in this case is really given by $T \mapsto U(T)/j(R(T))$ as $j(R(T)) \subset U(T) \times U(T)$ is an equivalence relation, see Definition 59.4.1. □

Lemma 59.18.6. *Let S be a scheme. Let B be an algebraic space over S. Let $j : R \to U \times_B U$ be a pre-relation over B and $g : U' \to U$ a morphism of algebraic spaces over B. Let $j' : R' \to U' \times_B U'$ be the restriction of j to U'. The map of quotient sheaves*

$$U'/R' \longrightarrow U/R$$

is injective. If $U' \to U$ is surjective as a map of sheaves, for example if $\{g : U' \to U\}$ is an fppf covering (see Topologies on Spaces, Definition 54.4.1), then $U'/R' \to U/R$ is an isomorphism of sheaves.

Proof. Suppose $\xi, \xi' \in (U'/R')(S')$ are sections which map to the same section of U/R. Then we can find an fppf covering $\mathcal{S} = \{S_i \to S'\}$ of S' such that $\xi|_{S_i}, \xi'|_{S_i}$ are given by $a_i, a_i' \in U'(S_i)$. By Lemma 59.18.5 and the axioms of a site we may after refining $\mathcal{T}$ assume there exist morphisms $r_i : S_i \to R$ such that $g \circ a_i = s \circ r_i$, $g \circ a_i' = t \circ r_i$. Since by construction $R' = R \times_{U \times_S U} (U' \times_S U')$ we see that $(r_i, (a_i, a_i')) \in R'(S_i)$ and this shows that a_i and a_i' define the same section of U'/R' over S_i. By the sheaf condition this implies $\xi = \xi'$.

If $U' \to U$ is a surjective map of sheaves, then $U'/R' \to U/R$ is surjective also. Finally, if $\{g : U' \to U\}$ is a fppf covering, then the map of sheaves $U' \to U$ is surjective, see Topologies on Spaces, Lemma 54.4.4. □

Lemma 59.18.7. *Let S be a scheme. Let B be an algebraic space over S. Let (U, R, s, t, c) be a groupoid in algebraic spaces over B. Let $g : U' \to U$ a morphism*

of algebraic spaces over B. Let (U', R', s', t', c') be the restriction of (U, R, s, t, c) to U'. The map of quotient sheaves

$$U'/R' \longrightarrow U/R$$

is injective. If the composition

$$U' \times_{g,U,t} R \xrightarrow[pr_1]{} R \xrightarrow[s]{} U, \quad \text{with } h : R \to U$$

is a surjection of fppf sheaves then the map is bijective. This holds for example if $\{h : U' \times_{g,U,t} R \to U\}$ is an $fppf$-covering, or if $U' \to U$ is a surjection of sheaves, or if $\{g : U' \to U\}$ is a covering in the fppf topology.

Proof. Injectivity follows on combining Lemmas 59.11.2 and 59.18.6. To see surjectivity (see Sites, Section 7.12 for a characterization of surjective maps of sheaves) we argue as follows. Suppose that T is a scheme and $\sigma \in U/R(T)$. There exists a covering $\{T_i \to T\}$ such that $\sigma|_{T_i}$ is the image of some element $f_i \in U(T_i)$. Hence we may assume that σ if the image of $f \in U(T)$. By the assumption that h is a surjection of sheaves, we can find an fppf covering $\{\varphi_i : T_i \to T\}$ and morphisms $f_i : T_i \to U' \times_{g,U,t} R$ such that $f \circ \varphi_i = h \circ f_i$. Denote $f'_i = \mathrm{pr}_0 \circ f_i : T_i \to U'$. Then we see that $f'_i \in U'(T_i)$ maps to $g \circ f'_i \in U(T_i)$ and that $g \circ f'_i \sim_{T_i} h \circ f_i = f \circ \varphi_i$ notation as in (59.18.0.1). Namely, the element of $R(T_i)$ giving the relation is $\mathrm{pr}_1 \circ f_i$. This means that the restriction of σ to T_i is in the image of $U'/R'(T_i) \to U/R(T_i)$ as desired.

If $\{h\}$ is an fppf covering, then it induces a surjection of sheaves, see Topologies on Spaces, Lemma 54.4.4. If $U' \to U$ is surjective, then also h is surjective as s has a section (namely the neutral element e of the groupoid scheme). □

59.19. Quotient stacks

In this section and the next few sections we describe a kind of generalization of Section 59.18 above and Groupoids, Section 38.18. It is different in the following way: We are going to take quotient stacks instead of quotient sheaves.

Let us assume we have a scheme S, and algebraic space B over S and a groupoid in algebraic spaces (U, R, s, t, c) over B. Given these data we consider the functor

$$\begin{array}{ccc} (Sch/S)^{opp}_{fppf} & \longrightarrow & Groupoids \\ S' & \longmapsto & (U(S'), R(S'), s, t, c) \end{array} \tag{59.19.0.1}$$

By Categories, Example 4.35.1 this "presheaf in groupoids" corresponds to a category fibred in groupoids over $(Sch/S)_{fppf}$. In this chapter we will denote this

$$[U/_pR] \to (Sch/S)_{fppf}$$

where the subscript $_p$ is there to distinguish from the quotient stack.

Definition 59.19.1. Quotient stacks. Let $B \to S$ be as above.

(1) Let (U, R, s, t, c) be a groupoid in algebraic spaces over B. The *quotient stack*

$$p : [U/R] \longrightarrow (Sch/S)_{fppf}$$

of (U, R, s, t, c) is the stackification (see Stacks, Lemma 8.9.1) of the category fibred in groupoids $[U/_pR]$ over $(Sch/S)_{fppf}$ associated to (59.19.0.1).

(2) Let (G, m) be a group algebraic space over B. Let $a : G \times_B X \to X$ be an action of G on an algebraic space over B. The *quotient stack*

$$p : [X/G] \longrightarrow (Sch/S)_{fppf}$$

is the quotient stack associated to the groupoid in algebraic spaces $(X, G \times_B X, s, t, c)$ over B of Lemma 59.14.1.

Thus $[U/R]$ and $[X/G]$ are stacks in groupoids over $(Sch/S)_{fppf}$. These stacks will be very important later on and hence it makes sense to give a detailed description. Recall that given an algebraic space X over S we use the notation $\mathcal{S}_X \to (Sch/S)_{fppf}$ to denote the stack in sets associated to the sheaf X, see Categories, Lemma 4.36.6 and Stacks, Lemma 8.6.2.

Lemma 59.19.2. *Assume $B \to S$ and (U, R, s, t, c) as in Definition 59.19.1 (1). There are canonical 1-morphisms $\pi : \mathcal{S}_U \to [U/R]$, and $[U/R] \to \mathcal{S}_B$ of stacks in groupoids over $(Sch/S)_{fppf}$. The composition $\mathcal{S}_U \to \mathcal{S}_B$ is the 1-morphism associated to the structure morphism $U \to B$.*

Proof. During this proof let us denote $[U/_pR]$ the category fibred in groupoids associated to the presheaf in groupoids (59.19.0.1). By construction of the stackification there is a 1-morphism $[U/_pR] \to [U/R]$. The 1-morphism $\mathcal{S}_U \to [U/R]$ is simply the composition $\mathcal{S}_U \to [U/_pR] \to [U/R]$, where the first arrow associates to the scheme S'/S and morphism $x : S' \to U$ over S the object $x \in U(S')$ of the fibre category of $[U/_pR]$ over S'.

To construct the 1-morphism $[U/R] \to \mathcal{S}_B$ it is enough to construct the 1-morphism $[U/_pR] \to \mathcal{S}_B$, see Stacks, Lemma 8.9.2. On objects over S'/S we just use the map

$$U(S') \longrightarrow B(S')$$

coming from the structure morphism $U \to B$. And clearly, if $a \in R(S')$ is an "arrow" with source $s(a) \in U(S')$ and target $t(a) \in U(S')$, then since s and t are morphisms *over* B these both map to the same element $\overline{a}$ of $B(S')$. Hence we can map an arrow $a \in R(S')$ to the identity morphism of $\overline{a}$. (This is good because the fibre category $(\mathcal{S}_B)_{S'}$ only contains identities.) We omit the verification that this rule is compatible with pullback on these split fibred categories, and hence defines a 1-morphism $[U/_pR] \to \mathcal{S}_B$ as desired.

We omit the verification of the last statement. □

Lemma 59.19.3. *Assumptions and notation as in Lemma 59.19.2. There exists a canonical 2-morphism $\alpha : \pi \circ s \to \pi \circ t$ making the diagram*

$$\begin{array}{ccc} \mathcal{S}_R & \xrightarrow[s]{} & \mathcal{S}_U \\ {\scriptstyle t}\downarrow & & \downarrow{\scriptstyle \pi} \\ \mathcal{S}_U & \xrightarrow{\pi} & [U/R] \end{array}$$

2-commutative.

Proof. Let S' be a scheme over S. Let $r : S' \to R$ be a morphism over S. Then $r \in R(S')$ is an isomorphism between the objects $s \circ r, t \circ r \in U(S')$. Moreover, this construction is compatible with pullbacks. This gives a canonical 2-morphism

$\alpha_p : \pi_p \circ s \to \pi_p \circ t$ where $\pi_p : \mathcal{S}_U \to [U/_pR]$ is as in the proof of Lemma 59.19.2. Thus even the diagram

$$\begin{array}{ccc} \mathcal{S}_R & \xrightarrow{s} & \mathcal{S}_U \\ \downarrow{\scriptstyle t} & & \downarrow{\scriptstyle \pi_p} \\ \mathcal{S}_U & \xrightarrow{\pi_p} & [U/_pR] \end{array}$$

is 2-commutative. Thus a fortiori the diagram of the lemma is 2-commutative. □

Remark 59.19.4. In future chapters we will use the ambiguous notation where instead of writing $\mathcal{S}_X$ for the stack in sets associated to X we simply write X. Using this notation the diagram of Lemma 59.19.3 becomes the familiar diagram

$$\begin{array}{ccc} R & \xrightarrow{s} & U \\ \downarrow{\scriptstyle t} & & \downarrow{\scriptstyle \pi} \\ U & \xrightarrow{\pi} & [U/R] \end{array}$$

In the following sections we will show that this diagram has many good properties. In particular we will show that it is a 2-fibre product (Section 59.21) and that it is close to being a 2-coequalizer of s and t (Section 59.22).

59.20. Functoriality of quotient stacks

A morphism of groupoids in algebraic spaces gives an associated morphism of quotient stacks.

Lemma 59.20.1. *Let S be a scheme. Let B be an algebraic space over S. Let $f : (U, R, s, t, c) \to (U', R', s', t', c')$ be a morphism of groupoids in algebraic spaces over B. Then f induces a canonical 1-morphism of quotient stacks*

$$[f] : [U/R] \longrightarrow [U'/R'].$$

Proof. Denote $[U/_pR]$ and $[U'/_pR']$ the categories fibred in groupoids over the base site $(Sch/S)_{fppf}$ associated to the functors (59.19.0.1). It is clear that f defines a 1-morphism $[U/_pR] \to [U'/_pR']$ which we can compose with the stackyfication functor for $[U'/R']$ to get $[U/_pR] \to [U'/R']$. Then, by the universal property of the stackyfication functor $[U/_pR] \to [U/R]$, see Stacks, Lemma 8.9.2 we get $[U/R] \to [U'/R']$. □

Let $B \to S$ and $f : (U, R, s, t, c) \to (U', R', s', t', c')$ be as in Lemma 59.20.1. In this situation, we define a third groupoid in algebraic spaces over B as follows, using the language of T-valued points where T is a (varying) scheme over B:

(1) $U'' = U \times_{f,U',t'} R'$ so that a T-valued point is a pair (u, r') with $f(u) = t'(r')$,
(2) $R'' = R \times_{f \circ s, U', t'} R'$ so that a T-valued point is a pair (r, r') with $f(s(r)) = t'(r')$,
(3) $s'' : R'' \to U''$ is given by $s''(r, r') = (s(r), r')$,
(4) $t'' : R'' \to U''$ is given by $t''(r, r') = (t(r), c'(f(r), r'))$,
(5) $c'' : R'' \times_{s'', U'', t''} R'' \to R''$ is given by $c''((r_1, r_1'), (r_2, r_2')) = (c(r_1, r_2), r_2')$.

The formula for c'' makes sense as $s''(r_1, r_1') = t''(r_2, r_2')$. It is clear that c'' is associative. The identity e'' is given by $e''(u, r) = (e(u), r)$. The inverse of (r, r') is given by $(i(r), c'(f(r), r'))$. Thus we do indeed get a groupoid in algebraic spaces over B.

Clearly the maps $U'' \to U$ and $R'' \to R$ define a morphism $g : (U'', R'', s'', t'', c'') \to (U, R, s, t, c)$ of groupoids in algebraic spaces over B. Moreover, the maps $U'' \to U'$, $(u, r') \mapsto s'(r')$ and $R'' \to U'$, $(r, r') \mapsto s'(r')$ show that in fact $(U'', R'', s'', t'', c'')$ is a groupoid in algebraic spaces over U'.

Lemma 59.20.2. *Notation and assumption as in Lemma 59.20.1. Let $(U'', R'', s'', t'', c'')$ be the groupoid in algebraic spaces over B constructed above. There is a 2-commutative square*

$$\begin{array}{ccc} [U''/R''] & \xrightarrow[[g]]{} & [U/R] \\ \downarrow & & \downarrow{\scriptstyle [f]} \\ \mathcal{S}_{U'} & \longrightarrow & [U'/R'] \end{array}$$

which identifies $[U''/R'']$ with the 2-fibre product.

Proof. The maps $[f]$ and $[g]$ come from an application of Lemma 59.20.1 and the other two maps come from Lemma 59.19.2 (and the fact that $(U'', R'', s'', t'', c'')$ lives over U'). To show the 2-fibre product property, it suffices to prove the lemma for the diagram

$$\begin{array}{ccc} [U''/_pR''] & \xrightarrow[[g]]{} & [U/_pR] \\ \downarrow & & \downarrow{\scriptstyle [f]} \\ \mathcal{S}_{U'} & \longrightarrow & [U'/_pR'] \end{array}$$

of categories fibred in groupoids, see Stacks, Lemma 8.9.3. In other words, it suffices to show that an object of the 2-fibre product $\mathcal{S}_U \times_{[U'/_pR']} [U/_pR]$ over T corresponds to a T-valued point of U'' and similarly for morphisms. And of course this is exactly how we constructed U'' and R'' in the first place.

In detail, an object of $\mathcal{S}_U \times_{[U'/_pR']} [U/_pR]$ over T is a triple (u', u, r') where u' is a T-valued point of U', u is a T-valued point of U, and r' is a morphism from u' to $f(u)$ in $[U'/R']_T$, i.e., r' is a T-valued point of R with $s'(r') = u'$ and $t'(r') = f(u)$. Clearly we can forget about u' without losing information and we see that these objects are in one-to-one correspondence with T-valued points of R''.

Similarly for morphisms: Let (u_1', u_1, r_1') and (u_2', u_2, r_2') be two objects of the fibre product over T. Then a morphism from (u_2', u_2, r_2') to (u_1', u_1, r_1') is given by $(1, r)$ where $1 : u_1' \to u_2'$ means simply $u_1' = u_2'$ (this is so because $\mathcal{S}_U$ is fibred in sets), and r is a T-valued point of R with $s(r) = u_2$, $t(r) = u_1$ and moreover $c'(f(r), r_2') = r_1'$. Hence the arrow

$$(1, r) : (u_2', u_2, r_2') \to (u_1', u_1, r_1')$$

is completely determined by knowing the pair (r, r_2'). Thus the functor of arrows is represented by R'', and moreover the morphisms s'', t'', and c'' clearly correspond to source, target and composition in the 2-fibre product $\mathcal{S}_U \times_{[U'/_pR']} [U/_pR]$. $\square$

59.21. The 2-cartesian square of a quotient stack

In this section we compute the *Isom*-sheaves for a quotient stack and we deduce that the defining diagram of a quotient stack is a 2-fibre product.

Lemma 59.21.1. *Assume $B \to S$, (U, R, s, t, c) and $\pi : \mathcal{S}_U \to [U/R]$ are as in Lemma 59.19.2. Let S' be a scheme over S. Let $x, y \in \mathrm{Ob}([U/R]_{S'})$ be objects of the quotient stack over S'. If $x = \pi(x')$ and $y = \pi(y')$ for some morphisms $x', y' : S' \to U$, then*

$$Isom(x, y) = S' \times_{(y', x'), U \times_S U} R$$

as sheaves over S'.

Proof. Let $[U/_pR]$ be the category fibred in groupoids associated to the presheaf in groupoids (59.19.0.1) as in the proof of Lemma 59.19.2. By construction the sheaf $Isom(x, y)$ is the sheaf associated to the presheaf $Isom(x', y')$. On the other hand, by definition of morphisms in $[U/_pR]$ we have

$$Isom(x', y') = S' \times_{(y', x'), U \times_S U} R$$

and the right hand side is an algebraic space, therefore a sheaf. □

Lemma 59.21.2. *Assume $B \to S$, (U, R, s, t, c), and $\pi : \mathcal{S}_U \to [U/R]$ are as in Lemma 59.19.2. The 2-commutative square*

$$\begin{array}{ccc} \mathcal{S}_R & \xrightarrow{s} & \mathcal{S}_U \\ {\scriptstyle t}\downarrow & & \downarrow{\scriptstyle \pi} \\ \mathcal{S}_U & \xrightarrow{\pi} & [U/R] \end{array}$$

of Lemma 59.19.3 is a 2-fibre product of stacks in groupoids of $(Sch/S)_{fppf}$.

Proof. According to Stacks, Lemma 8.5.6 the lemma makes sense. It also tells us that we have to show that the functor

$$\mathcal{S}_R \longrightarrow \mathcal{S}_U \times_{[U/R]} \mathcal{S}_U$$

which maps $r : T \to R$ to $(T, t(r), s(r), \alpha(r))$ is an equivalence, where the right hand side is the 2-fibre product as described in Categories, Lemma 4.30.3. This is, after spelling out the definitions, exactly the content of Lemma 59.21.1. (Alternative proof: Work out the meaning of Lemma 59.20.2 in this situation will give you the result also.) □

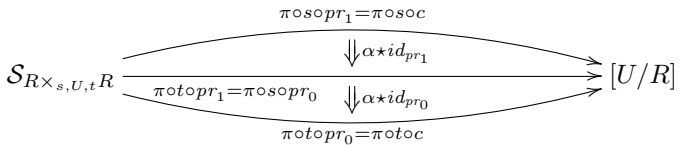
Lemma 59.21.3. *Assume $B \to S$ and (U, R, s, t, c) are as in Definition 59.19.1 (1). For any scheme T over S and objects x, y of $[U/R]$ over T the sheaf $Isom(x, y)$ on $(Sch/T)_{fppf}$ has the following property: There exists a fppf covering $\{T_i \to T\}_{i \in I}$ such that $Isom(x, y)|_{(Sch/T_i)_{fppf}}$ is representable by an algebraic space.*

Proof. Follows immediately from Lemma 59.21.1 and the fact that both x and y locally in the fppf topology come from objects of $\mathcal{S}_U$ by construction of the quotient stack. □

59.22. The 2-coequalizer property of a quotient stack

On a groupoid we have the composition, which leads to a cocycle condition for the canonical 2-morphism of the lemma above. To give the precise formulation we will use the notation introduced in Categories, Sections 4.26 and 4.27.

Lemma 59.22.1. *Assumptions and notation as in Lemmas 59.19.2 and 59.19.3. The vertical composition of*



is the 2-morphism $\alpha \star id_c$. In a formula $\alpha \star id_c = (\alpha \star id_{pr_0}) \circ (\alpha \star id_{pr_1})$.

Proof. We make two remarks:

(1) The formula $\alpha \star \mathrm{id}_c = (\alpha \star \mathrm{id}_{\mathrm{pr}_0}) \circ (\alpha \star \mathrm{id}_{\mathrm{pr}_1})$ only makes sense if you realize the *equalities* $\pi \circ s \circ \mathrm{pr}_1 = \pi \circ s \circ c$, $\pi \circ t \circ \mathrm{pr}_1 = \pi \circ s \circ \mathrm{pr}_0$, and $\pi \circ t \circ \mathrm{pr}_0 = \pi \circ t \circ c$. Namely, the second one implies the vertical composition $\circ$ makes sense, and the other two guarantee the two sides of the formula are 2-morphisms with the same source and target.

(2) The reason the lemma holds is that composition in the category fibred in groupoids $[U/_pR]$ associated to the presheaf in groupoids (59.19.0.1) comes from the composition law $c : R \times_{s,U,t} R \to R$.

We omit the proof of the lemma. □

Note that, in the situation of the lemma, we actually have the equalities $s\circ\mathrm{pr}_1 = s\circ c$, $t \circ \mathrm{pr}_1 = s \circ \mathrm{pr}_0$, and $t \circ \mathrm{pr}_0 = t \circ c$ before composing with π. Hence the formula in the lemma below makes sense in exactly the same way that the formula in the lemma above makes sense.

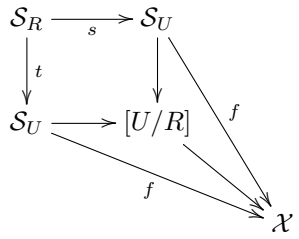
Lemma 59.22.2. *Assumptions and notation as in Lemmas 59.19.2 and 59.19.3. The 2-commutative diagram of Lemma 59.19.3 is a 2-coequalizer in the following sense: Given*

(1) *a stack in groupoids $\mathcal{X}$ over $(Sch/S)_{fppf}$,*
(2) *a 1-morphism $f : \mathcal{S}_U \to \mathcal{X}$, and*
(3) *a 2-arrow $\beta : f \circ s \to f \circ t$*

such that

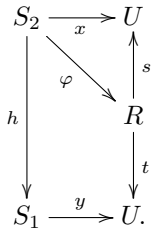
$$\beta \star id_c = (\beta \star id_{pr_0}) \circ (\beta \star id_{pr_1})$$

then there exists a 1-morphism $[U/R] \to \mathcal{X}$ which makes the diagram

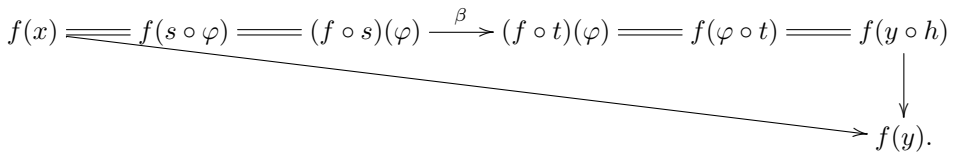


2-commute.

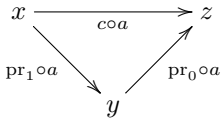
Proof. Suppose given $\mathcal{X}$, f and β as in the lemma. By Stacks, Lemma 8.9.2 it suffices to construct a 1-morphism $g : [U/_pR] \to \mathcal{X}$. First we note that the 1-morphism $\mathcal{S}_U \to [U/_pR]$ is bijective on objects. Hence on objects we can set $g(x) = f(x)$ for $x \in \mathrm{Ob}(\mathcal{S}_U) = \mathrm{Ob}([U/_pR])$. A morphism $\varphi : x \to y$ of $[U/_pR]$ arises from a commutative diagram



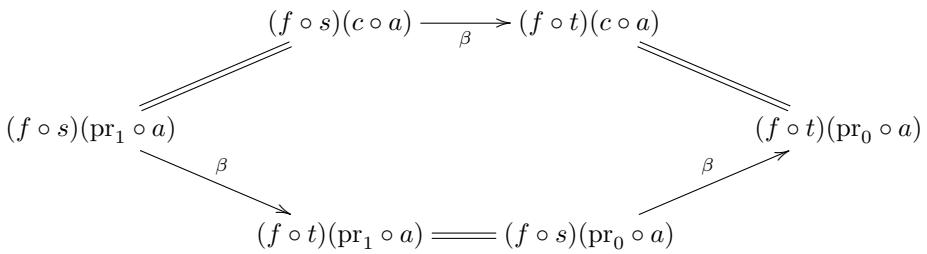
Thus we can set $g(\varphi)$ equal to the composition



The vertical arrow is the result of applying the functor f to the canonical morphism $y \circ h \to y$ in $\mathcal{S}_U$ (namely, the strongly cartesian morphism lifting h with target y). Let us verify that f so defined is compatible with composition, at least on fibre categories. So let S' be a scheme over S, and let $a : S' \to R \times_{s,U,t} R$ be a morphism. In this situation we set $x = s \circ \mathrm{pr}_1 \circ a = s \circ c \circ a$, $y = t \circ \mathrm{pr}_1 \circ a = s \circ \mathrm{pr}_0 \circ a$, and $z = t \circ \mathrm{pr}_0 \circ a = t \circ \mathrm{pr}_0 \circ c$ to get a commutative diagram



in the fibre category $[U/_pR]_{S'}$. Moreover, any commutative triangle in this fibre category has this form. Then we see by our definitions above that f maps this to a commutative diagram if and only if the diagram



is commutative which is exactly the condition expressed by the formula in the lemma. We omit the verification that f maps identities to identities and is compatible with composition for arbitrary morphisms. □

59.23. Explicit description of quotient stacks

In order to formulate the result we need to introduce some notation. Assume $B \to S$ and (U, R, s, t, c) are as in Definition 59.19.1 (1). Let T be a scheme over S. Let

$\mathcal{T} = \{T_i \to T\}_{i\in I}$ be an fppf covering. A $[U/R]$-*descent datum* relative to $\mathcal{T}$ is given by a system (u_i, r_{ij}) where

(1) for each i a morphism $u_i : T_i \to U$, and
(2) for each i, j a morphism $r_{ij} : T_i \times_T T_j \to R$

such that

(a) as morphisms $T_i \times_T T_j \to U$ we have
$$s \circ r_{ij} = u_i \circ \mathrm{pr}_0 \quad \text{and} \quad t \circ r_{ij} = u_j \circ \mathrm{pr}_1,$$
(b) as morphisms $T_i \times_T T_j \times_T T_k \to R$ we have
$$c \circ (r_{jk} \circ \mathrm{pr}_{12}, r_{ij} \circ \mathrm{pr}_{01}) = r_{ik} \circ \mathrm{pr}_{02}.$$

A *morphism* $(u_i, r_{ij}) \to (u'_i, r'_{ij})$ *between two* $[U/R]$-*descent data* over the same covering $\mathcal{T}$ is a collection $(r_i : T_i \to R)$ such that

(α) as morphisms $T_i \to U$ we have
$$u_i = s \circ r_i \quad \text{and} \quad u'_i = t \circ r_i$$
(β) as morphisms $T_i \times_T T_j \to R$ we have
$$c \circ (r'_{ij}, r_i \circ \mathrm{pr}_0) = c \circ (r_j \circ \mathrm{pr}_1, r_{ij}).$$

There is a natural composition law on morphisms of descent data relative to a fixed covering and we obtain a category of descent data. This category is a groupoid. Finally, if $\mathcal{T}' = \{T'_j \to T\}_{j\in J}$ is a second fppf covering which refines $\mathcal{T}$ then there is a notion of pullback of descent data. This is particularly easy to describe explicitly in this case. Namely, if $\alpha : J \to I$ and $\varphi_j : T'_j \to T_{\alpha(i)}$ is the morphism of coverings, then the pullback of the descent datum $(u_i, r_{ii'})$ is simply
$$(u_{\alpha(i)} \circ \varphi_j, r_{\alpha(j)\alpha(j')} \circ \varphi_j \times \varphi_{j'}).$$
Pullback defined in this manner defines a functor from the category of descent data over $\mathcal{T}$ to the category of descend data over $\mathcal{T}'$.

Lemma 59.23.1. *Assume $B \to S$ and (U, R, s, t, c) are as in Definition 59.19.1 (1). Let $\pi : \mathcal{S}_U \to [U/R]$ be as in Lemma 59.19.2. Let T be a scheme over S.*

(1) *for every object x of the fibre category $[U/R]_T$ there exists an fppf covering $\{f_i : T_i \to T\}_{i\in I}$ such that $f_i^* x \cong \pi(u_i)$ for some $u_i \in U(T_i)$,*
(2) *the composition of the isomorphisms*
$$\pi(u_i \circ pr_0) = pr_0^*\pi(u_i) \cong pr_0^* f_i^* x \cong pr_1^* f_j^* x \cong pr_1^*\pi(u_j) = \pi(u_j \circ pr_1)$$
are of the form $\pi(r_{ij})$ for certain morphisms $r_{ij} : T_i \times_T T_j \to R$,
(3) *the system (u_i, r_{ij}) forms a $[U/R]$-descent datum as defined above,*
(4) *any $[U/R]$-descent datum (u_i, r_{ij}) arises in this manner,*
(5) *if x corresponds to (u_i, r_{ij}) as above, and $y \in \mathrm{Ob}([U/R]_T)$ corresponds to (u'_i, r'_{ij}) then there is a canonical bijection*
$$\mathrm{Mor}_{[U/R]_T}(x, y) \longleftrightarrow \left\{ \begin{array}{c} \text{morphisms } (u_i, r_{ij}) \to (u'_i, r'_{ij}) \\ \text{of } [U/R]\text{-descent data} \end{array} \right\}$$
(6) *this correspondence is compatible with refinements of fppf coverings.*

Proof. Statement (1) is part of the construction of the stackyfication. Part (2) follows from Lemma 59.21.1. We omit the verification of (3). Part (4) is a translation of the fact that in a stack all descent data are effective. We omit the verifications of (5) and (6). □

59.24. Restriction and quotient stacks

In this section we study what happens to the quotient stack when taking a restriction.

Lemma 59.24.1. *Notation and assumption as in Lemma 59.20.1. The morphism of quotient stacks*

$$[f] : [U/R] \longrightarrow [U'/R']$$

is fully faithful if and only if R is the restriction of R via the morphism $f : U \to U'$.

Proof. Let x, y be objects of $[U/R]$ over a scheme T/S. Let x', y' be the images of x, y in the category $[U'/'R]_T$. The functor $[f]$ is fully faithful if and only if the map of sheaves

$$Isom(x, y) \longrightarrow Isom(x', y')$$

is an isomorphism for every T, x, y. We may test this locally on T (in the fppf topology). Hence, by Lemma 59.23.1 we may assume that x, y come from $a, b \in U(T)$. In that case we see that x', y' correspond to $f \circ a, f \circ b$. By Lemma 59.21.1 the displayed map of sheaves in this case becomes

$$T \times_{(a,b),U \times_B U} R \longrightarrow T \times_{f \circ a, f \circ b, U' \times_B U'} R'.$$

This is an isomorphism if R is the restriction, because in that case $R = (U \times_B U) \times_{U' \times_B U'} R'$, see Lemma 59.16.3 and its proof. Conversely, if the last displayed map is an isomorphism for all T, a, b, then it follows that $R = (U \times_B U) \times_{U' \times_B U'} R'$, i.e., R is the restriction of R'. □

Lemma 59.24.2. *Notation and assumption as in Lemma 59.20.1. The morphism of quotient stacks*

$$[f] : [U/R] \longrightarrow [U'/R']$$

is an equivalence if and only if

(1) *(U, R, s, t, c) is the restriction of (U', R', s', t', c') via $f : U \to U'$, and*
(2) *the map*

$$U \times_{f,U',t'} R' \xrightarrow{pr_1} R' \xrightarrow{s'} U', \qquad h : U \times_{f,U',t'} R' \to U'$$

is a surjection of sheaves.

Part (2) holds for example if $\{h : U \times_{f,U',t'} R' \to U'\}$ is an fppf covering, or if $f : U \to U'$ is a surjection of sheaves, or if $\{f : U \to U'\}$ is an fppf covering.

Proof. We already know that part (1) is equivalent to fully faithfulness by Lemma 59.24.1. Hence we may assume that (1) holds and that $[f]$ is fully faithful. Our goal is to show, under these assumptions, that $[f]$ is an equivalence if and only if (2) holds. We may use Stacks, Lemma 8.4.8 which characterizes equivalences.

Assume (2). We will use Stacks, Lemma 8.4.8 to prove $[f]$ is an equivalence. Suppose that T is a scheme and $x' \in \mathrm{Ob}([U'/R']_T)$. There exists a covering $\{g_i : T_i \to T\}$ such that $g_i^* x'$ is the image of some element $a_i' \in U'(T_i)$, see Lemma 59.23.1. Hence we may assume that x' is the image of $a' \in U'(T)$. By the assumption that h is a surjection of sheaves, we can find an fppf covering $\{\varphi_i : T_i \to T\}$ and morphisms $b_i : T_i \to U \times_{g,U',t'} R'$ such that $a' \circ \varphi_i = h \circ b_i$. Denote $a_i = \mathrm{pr}_0 \circ b_i : T_i \to U$. Then we see that $a_i \in U(T_i)$ maps to $f \circ a_i \in U'(T_i)$

and that $f \circ a_i \cong_{T_i} h \circ b_i = a' \circ \varphi_i$, where $\cong_{T_i}$ denotes isomorphism in the fibre category $[U'/R']_{T_i}$. Namely, the element of $R'(T_i)$ giving the isomorphism is $\mathrm{pr}_1 \circ b_i$. This means that the restriction of x to T_i is in the essential image of the functor $[U/R]_{T_i} \to [U'/R']_{T_i}$ as desired.

Assume $[f]$ is an equivalence. Let $\xi' \in [U'/R']_{U'}$ denote the object corresponding to the identity morphism of U'. Applying Stacks, Lemma 8.4.8 we see there exists an fppf covering $\mathcal{U}' = \{g'_i : U'_i \to U'\}$ such that $(g'_i)^*\xi' \cong [f](\xi_i)$ for some ξ_i in $[U/R]_{U'_i}$. After refining the covering $\mathcal{U}'$ (using Lemma 59.23.1) we may assume ξ_i comes from a morphism $a_i : U'_i \to U$. The fact that $[f](\xi_i) \cong (g'_i)^*\xi'$ means that, after possibly refining the covering $\mathcal{U}'$ once more, there exist morphisms $r'_i : U'_i \to R'$ with $t' \circ r'_i = f \circ a_i$ and $s' \circ r'_i = \mathrm{id}_{U'} \circ g'_i$. Picture

$$\begin{array}{ccccc} U & & \xleftarrow{a_i} & & U'_i \\ \downarrow{\scriptstyle f} & & & \swarrow{\scriptstyle r'_i} & \downarrow{\scriptstyle g'_i} \\ U' & \xleftarrow{t'} & R' & \xrightarrow{s'} & U' \end{array}$$

Thus $(a_i, r'_i) : U'_i \to U \times_{g,U',t'} R'$ are morphisms such that $h \circ (a_i, r'_i) = g'_i$ and we conclude that $\{h : U \times_{g,U',t'} R' \to U'\}$ can be refined by the fppf covering $\mathcal{U}'$ which means that h induces a surjection of sheaves, see Topologies on Spaces, Lemma 54.4.4.

If $\{h\}$ is an fppf covering, then it induces a surjection of sheaves, see Topologies on Spaces, Lemma 54.4.4. If $U' \to U$ is surjective, then also h is surjective as s has a section (namely the neutral element e of the groupoid in algebraic spaces). □

Lemma 59.24.3. *Notation and assumption as in Lemma 59.20.1. Assume that*

$$\begin{array}{ccc} R & \xrightarrow{f} & R' \\ {\scriptstyle s}\downarrow & & \downarrow{\scriptstyle s'} \\ U & \xrightarrow{f} & U' \end{array}$$

is cartesian. Then

$$\begin{array}{ccc} \mathcal{S}_U & \longrightarrow & [U/R] \\ \downarrow & & \downarrow{\scriptstyle [f]} \\ \mathcal{S}_{U'} & \longrightarrow & [U'/R'] \end{array}$$

is a 2-fibre product square.

Proof. Applying the inverse isomorphisms $i : R \to R$ and $i' : R' \to R'$ to the (first) cartesian diagram of the statement of the lemma we see that

$$\begin{array}{ccc} R & \xrightarrow{f} & R' \\ {\scriptstyle t}\downarrow & & \downarrow{\scriptstyle t'} \\ U & \xrightarrow{f} & U' \end{array}$$

is cartesian as well. By Lemma 59.20.2 we have a 2-fibre square

$$\begin{array}{ccc} [U''/R''] & \longrightarrow & [U/R] \\ \downarrow & & \downarrow \\ \mathcal{S}_{U'} & \longrightarrow & [U'/R'] \end{array}$$

where $U'' = U \times_{f,U',t'} R'$ and $R'' = R \times_{f \circ s,U',t'} R'$. By the above we see that $(t, f) : R \to U''$ is an isomorphism, and that

$$R'' = R \times_{f \circ s,U',t'} R' = R \times_{s,U} U \times_{f,U',t'} R' = R \times_{s,U,t} \times R.$$

Explicitly the isomorphism $R \times_{s,U,t} R \to R''$ is given by the rule $(r_0, r_1) \mapsto (r_0, f(r_1))$. Moreover, s'', t'', c'' translate into the maps

$$R \times_{s,U,t} R \to R, \quad s''(r_0, r_1) = r_1, \quad t''(r_0, r_1) = c(r_0, r_1)$$

and

$$\begin{array}{rccc} c'' : & (R \times_{s,U,t} R) \times_{s'',R,t''} (R \times_{s,U,t} R) & \longrightarrow & R \times_{s,U,t} R, \\ & ((r_0, r_1), (r_2, r_3)) & \longmapsto & (c(r_0, r_2), r_3). \end{array}$$

Precomposing with the isomorphism

$$R \times_{s,U,s} R \longrightarrow R \times_{s,U,t} R, \quad (r_0, r_1) \longmapsto (c(r_0, i(r_1)), r_1)$$

we see that t'' and s'' turn into pr_0 and pr_1 and that c'' turns into $\mathrm{pr}_{02} : R \times_{s,U,s} R \times_{s,U,s} R \to R \times_{s,U,s} R$. Hence we see that there is an isomorphism $[U''/R''] \cong [R/R \times_{s,U,s} R]$ where as a groupoid in algebraic spaces $(R, R \times_{s,U,s} R, s'', t'', c'')$ is the restriction of the trivial groupoid $(U, U, \mathrm{id}, \mathrm{id}, \mathrm{id})$ via $s : R \to U$. Since $s : R \to U$ is a surjection of fppf sheaves (as it has a right inverse) the morphism

$$[U''/R''] \cong [R/R \times_{s,U,s} R] \longrightarrow [U/U] = \mathcal{S}_U$$

is an equivalence by Lemma 59.24.2. This proves the lemma. □

59.25. Inertia and quotient stacks

The (relative) inertia stack of a stack in groupoids is defined in Stacks, Section 8.7. The actual construction, in the setting of fibred categories, and some of its properties is in Categories, Section 4.32.

Lemma 59.25.1. *Assume $B \to S$ and (U, R, s, t, c) as in Definition 59.19.1 (1). Let G/U be the stabilizer group algebraic space of the groupoid (U, R, s, t, c, e, i), see Definition 59.15.2. Set $R' = R \times_{s,U} G$ and set*

1. *$s' : R' \to G$, $(r, g) \mapsto g$,*
2. *$t' : R' \to G$, $(r, g) \mapsto c(r, c(g, i(r)))$,*
3. *$c' : R' \times_{s',G,t'} R' \to R'$, $((r_1, g_1), (r_2, g_2) \mapsto (c(r_1, r_2), g_1)$.*

Then (G, R', s', t', c') is a groupoid in algebraic spaces over B and

$$\mathcal{I}_{[U/R]} = [G/R'].$$

i.e., the associated quotient stack is the inertia stack of $[U/R]$.

Proof. By Stacks, Lemma 8.8.5 it suffices to prove that $\mathcal{I}_{[U/_pR]} = [G/_pR']$. Let T be a scheme over S. Recall that an object of the inertia fibred category of $[U/_pR]$ over T is given by a pair (x, g) where x is an object of $[U/_pR]$ over T and g is an automorphism of x in its fibre category over T. In other words, $x : T \to U$ and $g : T \to R$ such that $x = s \circ g = t \circ g$. This means exactly that $g : T \to G$.

A morphism in the inertia fibred category from $(x, g) \to (y, h)$ over T is given by $r : T \to R$ such that $s(r) = x$, $t(r) = y$ and $c(r, g) = c(h, r)$, see the commutative diagram in Categories, Lemma 4.32.1. In a formula

$$h = c(r, c(g, i(r))) = c(c(r, g), i(r)).$$

The notation $s(r)$, etc is a short hand for $s \circ r$, etc. The composition of $r_1 : (x_2, g_2) \to (x_1, g_1)$ and $r_2 : (x_1, g_1) \to (x_2, g_2)$ is $c(r_1, r_2) : (x_1, g_1) \to (x_3, g_3)$.

Note that in the above we could have written g in stead of (x, g) for an object of $\mathcal{I}_{[U/_pR]}$ over T as x is the image of g under the structure morphism $G \to U$. Then the morphisms $g \to h$ in $\mathcal{I}_{[U/_pR]}$ over T correspond exactly to morphisms $r' : T \to R'$ with $s'(r') = g$ and $t'(r') = h$. Moreover, the composition corresponds to the rule explained in (3). Thus the lemma is proved. □

Lemma 59.25.2. *Assume $B \to S$ and (U, R, s, t, c) as in Definition 59.19.1 (1). Let G/U be the stabilizer group algebraic space of the groupoid (U, R, s, t, c, e, i), see Definition 59.15.2. There is a canonical 2-cartesian diagram*

$$\begin{array}{ccc} \mathcal{S}_G & \longrightarrow & \mathcal{S}_U \\ \downarrow & & \downarrow \\ \mathcal{I}_{[U/R]} & \longrightarrow & [U/R] \end{array}$$

of stacks in groupoids of $(Sch/S)_{fppf}$.

Proof. By Lemma 59.24.3 it suffices to prove that the morphism $s' : R' \to G$ of Lemma 59.25.1 isomorphic to the base change of s by the structure morphism $G \to U$. This base change property is clear from the construction of s'. □

59.26. Gerbes and quotient stacks

In this section we relate quotient stacks to the discussion Stacks, Section 8.11 and especially gerbes as defined in Stacks, Definition 8.11.4. The stacks in groupoids occurring in this section are generally speaking not algebraic stacks!

Lemma 59.26.1. *Notation and assumption as in Lemma 59.20.1. The morphism of quotient stacks*

$$[f] : [U/R] \longrightarrow [U'/R']$$

turns $[U/R]$ into a gerbe over $[U'/R']$ if $f : U \to U'$ and $R \to R'|_U$ are surjective maps of fppf sheaves. Here $R'|_U$ is the restriction of R' to U via $f : U \to U'$.

Proof. We will verify that Stacks, Lemma 8.11.3 properties (2) (a) and (2) (b) hold. Property (2)(a) holds because $U \to U'$ is a surjective map of sheaves (use Lemma 59.23.1 to see that objects in $[U'/R']$ locally come from U'). To prove (2)(b) let x, y be objects of $[U/R]$ over a scheme T/S. Let x', y' be the images of x, y in the category $[U'/'R]_T$. Condition (2)(b) requires us to check the map of sheaves

$$Isom(x, y) \longrightarrow Isom(x', y')$$

on $(Sch/T)_{fppf}$ is surjective. To see this we may work fppf locally on T and assume that come from $a, b \in U(T)$. In that case we see that x', y' correspond to $f \circ a, f \circ b$. By Lemma 59.21.1 the displayed map of sheaves in this case becomes

$$T \times_{(a,b),U \times_B U} R \longrightarrow T \times_{f \circ a, f \circ b, U' \times_B U'} R' = T \times_{(a,b),U \times_B U} R'|_U.$$

Hence the assumption that $R \to R'|_U$ is a surjective map of fppf sheaves on $(Sch/S)_{fppf}$ implies the desired surjectivity. □

Lemma 59.26.2. *Let S be a scheme. Let B be an algebraic space over S. Let G be a group algebraic space over B. Endow B with the trivial action of G. The morphism*

$$[B/G] \longrightarrow \mathcal{S}_B$$

(Lemma 59.19.2) turns $[B/G]$ into a gerbe over B.

Proof. Immediate from Lemma 59.26.1 as the morphisms $B \to B$ and $B \times_B G \to B$ are surjective as morphisms of sheaves. □

59.27. Quotient stacks and change of big site

We suggest skipping this section on a first reading. Pullbacks of stacks are defined in Stacks, Section 8.12.

Lemma 59.27.1. *Suppose given big sites Sch_{fppf} and Sch'_{fppf}. Assume that Sch_{fppf} is contained in Sch'_{fppf}, see Topologies, Section 33.10. Let $S \in \mathrm{Ob}(Sch_{fppf})$. Let $B, U, R \in Sh((Sch/S)_{fppf})$ be algebraic spaces, and let (U, R, s, t, c) be a groupoid in algebraic spaces over B. Let $f : (Sch'/S)_{fppf} \to (Sch/S)_{fppf}$ the morphism of sites corresponding to the inclusion functor $u : Sch_{fppf} \to Sch'_{fppf}$. Then we have a canonical equivalence*

$$[f^{-1}U/f^{-1}R] \longrightarrow f^{-1}[U/R]$$

of stacks in groupoids over $(Sch'/S)_{fppf}$.

Proof. Note that $f^{-1}B, f^{-1}U, f^{-1}R \in Sh((Sch'/S)_{fppf})$ are algebraic spaces by Spaces, Lemma 47.15.1 and hence $(f^{-1}U, f^{-1}R, f^{-1}s, f^{-1}t, f^{-1}c)$ is a groupoid in algebraic spaces over $f^{-1}B$. Thus the statement makes sense.

The category $u_p[U/_pR]$ is the localization of the category $u_{pp}[U/_pR]$ at right multiplicative system I of morphisms. An object of $u_{pp}[U/_pR]$ is a triple

$$(T', \phi : T' \to T, x)$$

where $T' \in \mathrm{Ob}((Sch'/S)_{fppf})$, $T \in \mathrm{Ob}((Sch/S)_{fppf})$, ϕ is a morphism of schemes over S, and $x : T \to U$ is a morphism of sheaves on $(Sch/S)_{fppf}$. Note that the morphism of schemes $\phi : T' \to T$ is the same thing as a morphism $\phi : T' \to u(T)$, and since $u(T)$ represents $f^{-1}T$ it is the same thing as a morphism $T' \to f^{-1}T$. Moreover, as f^{-1} on algebraic spaces is fully faithful, see Spaces, Lemma 47.15.2, we may think of x as a morphism $x : f^{-1}T \to f^{-1}U$ as well. From now on we will make such identifications without further mention. A morphism

$$(a, a', \alpha) : (T'_1, \phi_1 : T'_1 \to T_1, x_1) \longrightarrow (T'_2, \phi_2 : T'_2 \to T_2, x_2)$$

of $u_{pp}[U/_pR]$ is a commutative diagram

$$\begin{array}{ccccc} & & & & U \\ & & & \overset{x_1}{\nearrow} & \uparrow{\scriptstyle s} \\ T'_1 & \xrightarrow[\phi_1]{} & T_1 & \xrightarrow[\alpha]{} & R \\ {\scriptstyle a'}\downarrow & & {\scriptstyle a}\downarrow & & \downarrow{\scriptstyle t} \\ T'_2 & \xrightarrow{\phi_2} & T_2 & \xrightarrow{x_2} & U \end{array}$$

and such a morphism is an element of I if and only if $T_1' = T_2'$ and $a' = \mathrm{id}$. We define a functor

$$u_{pp}[U/_pR] \longrightarrow [f^{-1}U/_pf^{-1}R]$$

by the rules

$$(T', \phi : T' \to T, x) \longmapsto (x \circ \phi : T' \to f^{-1}U)$$

on objects and

$$(a, a', \alpha) \longmapsto (\alpha \circ \phi_1 : T_1' \to f^{-1}R)$$

on morphisms as above. It is clear that elements of I are transformed into isomorphisms as $(f^{-1}U, f^{-1}R, f^{-1}s, f^{-1}t, f^{-1}c)$ is a groupoid in algebraic spaces over $f^{-1}B$. Hence this functor factors in a canonical way through a functor

$$u_p[U/_pR] \longrightarrow [f^{-1}U/_pf^{-1}R]$$

Applying stackification we obtain a functor of stacks

$$f^{-1}[U/R] \longrightarrow [f^{-1}U/f^{-1}R]$$

over $(Sch'/S)_{fppf}$, as by Stacks, Lemma 8.12.11 the stack $f^{-1}[U/R]$ is the stackification of $u_p[U/_pR]$.

At this point we have a morphism of stacks, and to verify that it is an equivalence it suffices to show that it is fully faithful and that objects are locally in the essential image, see Stacks, Lemmas 8.4.7 and 8.4.8. The statement on objects holds as $f^{-1}R$ admits a surjective étale morphism $f^{-1}W \to f^{-1}R$ for some object W of $(Sch/S)_{fppf}$. To show that the functor is "full", it suffices to show that morphisms are locally in the image of the functor which holds as $f^{-1}U$ admits a surjective étale morphism $f^{-1}W \to f^{-1}U$ for some object W of $(Sch/S)_{fppf}$. We omit the proof that the functor is faithful. □

59.28. Separation conditions

This really means conditions on the morphism $j : R \to U \times_B U$ when given a groupoid in algebraic spaces (U, R, s, t, c) over B. As in the previous section we first formulate the corresponding diagram.

Lemma 59.28.1. *Let $B \to S$ be as in Section 59.3. Let (U, R, s, t, c) be a groupoid in algebraic spaces over B. Let $G \to U$ be the stabilizer group algebraic space. The commutative diagram*

$$\begin{array}{ccccc} R & \xrightarrow{f \mapsto (f, s(f))} & R \times_{s,U} U & \longrightarrow & U \\ \downarrow{\scriptstyle \Delta_{R/U \times_B U}} & & \downarrow & & \downarrow \\ R \times_{(U \times_B U)} R & \xrightarrow{(f,g) \mapsto (f, f^{-1} \circ g)} & R \times_{s,U} G & \longrightarrow & G \end{array}$$

the two left horizontal arrows are isomorphisms and the right square is a fibre product square.

Proof. Omitted. Exercise in the definitions and the functorial point of view in algebraic geometry. □

Lemma 59.28.2. *Let $B \to S$ be as in Section 59.3. Let (U, R, s, t, c) be a groupoid in algebraic spaces over B. Let $G \to U$ be the stabilizer group algebraic space.*

(1) *The following are equivalent*
   (a) *$j : R \to U \times_B U$ is separated,*

(b) $G \to U$ *is separated, and*
(c) $e : U \to G$ *is a closed immersion.*

(2) *The following are equivalent*
(a) $j : R \to U \times_B U$ *is locally separated,*
(b) $G \to U$ *is locally separated, and*
(c) $e : U \to G$ *is an immersion.*

(3) *The following are equivalent*
(a) $j : R \to U \times_B U$ *is quasi-separated,*
(b) $G \to U$ *is quasi-separated, and*
(c) $e : U \to G$ *is quasi-compact.*

Proof. The group algebraic space $G \to U$ is the base change of $R \to U \times_B U$ by the diagonal morphism $U \to U \times_B U$, see Lemma 59.15.1. Hence if j is separated (resp. locally separated, resp. quasi-separated), then $G \to U$ is separated (resp. locally separated, resp. quasi-separated). See Morphisms of Spaces, Lemma 49.5.4. Thus (a) $\Rightarrow$ (b) in (1), (2), and (3).

Conversely, if $G \to U$ is separated (resp. locally separated, resp. quasi-separated), then the morphism $e : U \to G$, as a section of the structure morphism $G \to U$ is a closed immersion (resp. an immersion, resp. quasi-compact), see Morphisms of Spaces, Lemma 49.5.7. Thus (b) $\Rightarrow$ (c) in (1), (2), and (3).

If e is a closed immersion (resp. an immersion, resp. quasi-compact) then by the result of Lemma 59.28.1 (and Spaces, Lemma 47.12.3, and Morphisms of Spaces, Lemma 49.9.3) we see that $\Delta_{R/U \times_B U}$ is a closed immersion (resp. an immersion, resp. quasi-compact). Thus (c) $\Rightarrow$ (a) in (1), (2), and (3). □

59.29. Other chapters

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(7) Sites and Sheaves
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Schemes

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CHAPTER 60

More on Groupoids in Spaces

60.1. Introduction

This chapter is devoted to advanced topics on groupoids in algebraic spaces. Even though the results are stated in terms of groupoids in algebraic spaces, the reader should keep in mind the 2-cartesian diagram

$$\begin{array}{ccc} R & \longrightarrow & U \\ \downarrow & & \downarrow \\ U & \longrightarrow & [U/R] \end{array} \tag{60.1.0.1}$$

where $[U/R]$ is the quotient stack, see Groupoids in Spaces, Remark 59.19.4. Many of the results are motivated by thinking about this diagram. See for example the beautiful paper [**KM97**] by Keel and Mori.

60.2. Notation

We continue to abide by the conventions and notation introduced in Groupoids in Spaces, Section 59.3.

60.3. Useful diagrams

We briefly restate the results of Groupoids in Spaces, Lemmas 59.11.4 and 59.11.5 for easy reference in this chapter. Let S be a scheme. Let B be an algebraic space over S. Let (U, R, s, t, c) be a groupoid in algebraic spaces over B. In the commutative diagram

$$\begin{array}{ccccc} & & U & & \\ & \overset{t}{\nearrow} & & \overset{t}{\nwarrow} & \\ R & \xleftarrow[\mathrm{pr}_0]{} & R \times_{s,U,t} R & \xrightarrow[c]{} & R \\ \downarrow{\scriptstyle s} & & \downarrow{\scriptstyle \mathrm{pr}_1} & & \downarrow{\scriptstyle s} \\ U & \xleftarrow{t} & R & \xrightarrow{s} & U \end{array} \tag{60.3.0.2}$$

the two lower squares are fibre product squares. Moreover, the triangle on top (which is really a square) is also cartesian.

The diagram

$$
\begin{array}{ccccc}
R \times_{t,U,t} R & \xrightarrow[\mathrm{pr}_0]{\mathrm{pr}_1} & R & \xrightarrow{t} & U \\
\downarrow{\scriptstyle \mathrm{pr}_0 \times c \circ (i,1)} & & \downarrow{\scriptstyle \mathrm{id}_R} & & \downarrow{\scriptstyle \mathrm{id}_U} \\
R \times_{s,U,t} R & \xrightarrow[\mathrm{pr}_0]{c} & R & \xrightarrow{t} & U \\
\downarrow{\scriptstyle \mathrm{pr}_1} & & \downarrow{\scriptstyle s} & & \\
R & \xrightarrow[t]{s} & U & &
\end{array}
\tag{60.3.0.3}
$$

is commutative. The two top rows are isomorphic via the vertical maps given. The two lower left squares are cartesian.

60.4. Properties of groupoids

This section is the analogue of More on Groupoids, Section 39.5. The reader is strongly encouraged to read that section first.

The following lemma is the analogue of More on Groupoids, Lemma 39.5.4.

Lemma 60.4.1. *Let $B \to S$ be as in Section 60.2. Let (U, R, s, t, c) be a groupoid in algebraic spaces over B. Let $\tau \in \{fppf, étale, smooth, syntomic\}$. Let $\mathcal{P}$ be a property of morphisms of algebraic spaces which is τ-local on the target (Descent on Spaces, Definition 55.9.1). Assume $\{s : R \to U\}$ and $\{t : R \to U\}$ are coverings for the τ-topology. Let $W \subset U$ be the maximal open subspace such that $s^{-1}(W) \to W$ has property $\mathcal{P}$. Then W is R-invariant (Groupoids in Spaces, Definition 59.17.1).*

Proof. The existence and properties of the open $W \subset U$ are described in Descent on Spaces, Lemma 55.9.3. In Diagram (60.3.0.2) let $W_1 \subset R$ be the maximal open subscheme over which the morphism $\mathrm{pr}_1 : R \times_{s,U,t} R \to R$ has property $\mathcal{P}$. It follows from the aforementioned Descent on Spaces, Lemma 55.9.3 and the assumption that $\{s : R \to U\}$ and $\{t : R \to U\}$ are coverings for the τ-topology that $t^{-1}(W) = W_1 = s^{-1}(W)$ as desired. □

Lemma 60.4.2. *Let $B \to S$ be as in Section 60.2. Let (U, R, s, t, c) be a groupoid in algebraic spaces over B. Let $G \to U$ be its stabilizer group algebraic space. Let $\tau \in \{fppf, étale, smooth, syntomic\}$. Let $\mathcal{P}$ be a property of morphisms of algebraic spaces which is τ-local on the target. Assume $\{s : R \to U\}$ and $\{t : R \to U\}$ are coverings for the τ-topology. Let $W \subset U$ be the maximal open subspace such that $G_W \to W$ has property $\mathcal{P}$. Then W is R-invariant (see Groupoids in Spaces, Definition 59.17.1).*

Proof. The existence and properties of the open $W \subset U$ are described in Descent on Spaces, Lemma 55.9.3. The morphism

$$G \times_{U,t} R \longrightarrow R \times_{s,U} G, \quad (g, r) \longmapsto (r, r^{-1} \circ g \circ r)$$

is an isomorphism of algebraic spaces over R (where $\circ$ denotes composition in the groupoid). Hence $s^{-1}(W) = t^{-1}(W)$ by the properties of W proved in the aforementioned Descent on Spaces, Lemma 55.9.3. □

60.5. Comparing fibres

This section is the analogue of More on Groupoids, Section 39.6. The reader is strongly encouraged to read that section first.

Lemma 60.5.1. *Let $B \to S$ be as in Section 60.2. Let (U, R, s, t, c) be a groupoid in algebraic spaces over B. Let K be a field and let $r, r' : \mathrm{Spec}(K) \to R$ be morphisms such that $t \circ r = t \circ r' : \mathrm{Spec}(K) \to U$. Set $u = s \circ r$, $u' = s \circ r'$ and denote $F_u = \mathrm{Spec}(K) \times_{u,U,s} R$ and $F_{u'} = \mathrm{Spec}(K) \times_{u',U,s} R$ the fibre products. Then $F_u \cong F_{u'}$ as algebraic spaces over K.*

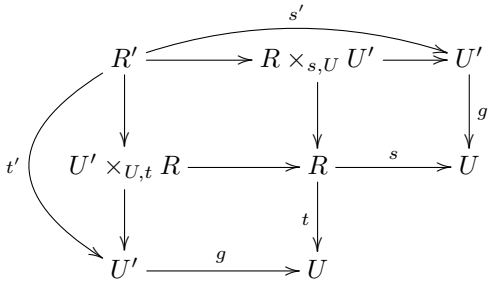
Proof. We use the properties and the existence of Diagram (60.3.0.2). There exists a morphism $\xi : \mathrm{Spec}(K) \to R \times_{s,U,t} R$ with $\mathrm{pr}_0 \circ \xi = r$ and $c \circ \xi = r'$. Let $\tilde{r} = \mathrm{pr}_1 \circ \xi : \mathrm{Spec}(K) \to R$. Then looking at the bottom two squares of Diagram (60.3.0.2) we see that both F_u and $F_{u'}$ are identified with the algebraic space $\mathrm{Spec}(K) \times_{\tilde{r},R,\mathrm{pr}_1} (R \times_{s,U,t} R)$. □

Actually, in the situation of the lemma the morphisms of pairs $s : (R, r) \to (U, u)$ and $s : (R, r') \to (U, u')$ are locally isomorphic in the τ-topology, provided $\{s : R \to U\}$ is a τ-covering. We will insert a precise statement here if needed.

60.6. Restricting groupoids

In this section we collect a bunch of lemmas on properties of groupoids which are inherited by restrictions. Most of these lemmas can be proved by contemplating the defining diagram

(60.6.0.1)



of a restriction. See Groupoids in Spaces, Lemma 59.16.1.

Lemma 60.6.1. *Let S be a scheme. Let B be an algebraic space over S. Let (U, R, s, t, c) be a groupoid in algebraic spaces over B. Let $g : U' \to U$ be a morphism of algebraic spaces over B. Let (U', R', s', t', c') be the restriction of (U, R, s, t, c) via g.*

1. *If s, t are locally of finite type and g is locally of finite type, then s', t' are locally of finite type.*
2. *If s, t are locally of finite presentation and g is locally of finite presentation, then s', t' are locally of finite presentation.*
3. *If s, t are flat and g is flat, then s', t' are flat.*
4. *Add more here.*

Proof. The property of being locally of finite type is stable under composition and arbitrary base change, see Morphisms of Spaces, Lemmas 49.22.2 and 49.22.3. Hence (1) is clear from Diagram (60.6.0.1). For the other cases, see Morphisms of Spaces, Lemmas 49.26.2, 49.26.3, 49.27.3, and 49.27.4. □

60.7. Properties of groups over fields and groupoids on fields

The reader is advised to first look at the corresponding sections for groupoid schemes, see Groupoids, Section 38.7 and More on Groupoids, Section 39.9.

Situation 60.7.1. Here S is a scheme, k is a field over S, and (G, m) is a group algebraic spaces over $\mathrm{Spec}(k)$.

Situation 60.7.2. Here S is a scheme, B is an algebraic space, and (U, R, s, t, c) is a groupoid in algebraic spaces over B with $U = \mathrm{Spec}(k)$ for some field k.

Note that in Situation 60.7.1 we obtain a groupoid in algebraic spaces

$$(\mathrm{Spec}(k), G, p, p, m) \tag{60.7.2.1}$$

where $p : G \to \mathrm{Spec}(k)$ is the structure morphism of G, see Groupoids in Spaces, Lemma 59.14.1. This is a situation as in Situation 60.7.2. We will use this without further mention in the rest of this section.

Lemma 60.7.3. *In Situation 60.7.2 the composition morphism $c : R \times_{s,U,t} R \to R$ is flat and universally open. In Situation 60.7.1 the group law $m : G \times_k G \to G$ is flat and universally open.*

Proof. The composition is isomorphic to the projection map $\mathrm{pr}_1 : R \times_{t,U,t} R \to R$ by Diagram (60.3.0.3). The projection is flat as a base change of the flat morphism t and open by Morphisms of Spaces, Lemma 49.7.6. The second assertion follows immediately from the first because m matches c in (60.7.2.1). □

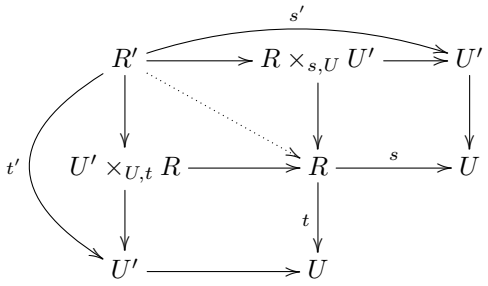
Note that the following lemma applies in particular when working with either quasi-separated or locally separated algebraic spaces (Decent Spaces, Lemma 50.12.2).

Lemma 60.7.4. *In Situation 60.7.2 assume R is a decent space. Then R is a separated algebraic space. In Situation 60.7.1 assume that G is a decent algebraic space. Then G is separated algebraic space.*

Proof. We first prove the second assertion. By Groupoids in Spaces, Lemma 59.6.1 we have to show that $e : S \to G$ is a closed immersion. This follows from Decent Spaces, Lemma 50.11.4.

Next, we prove the second assertion. To do this we may replace B by S. By the paragraph above the stabilizer group scheme $G \to U$ is separated. By Groupoids in Spaces, Lemma 59.28.2 the morphism $j = (t, s) : R \to U \times_S U$ is separated. As U is the spectrum of a field the scheme $U \times_S U$ is affine (by the construction of fibre products in Schemes, Section 25.17). Hence R is separated, see Morphisms of Spaces, Lemma 49.5.9. □

Lemma 60.7.5. *In Situation 60.7.2. Let $k \subset k'$ be a field extension, $U' = \mathrm{Spec}(k')$ and let (U', R', s', t', c') be the restriction of (U, R, s, t, c) via $U' \to U$. In the defining diagram*



all the morphisms are surjective, flat, and universally open. The dotted arrow $R' \to R$ is in addition affine.

Proof. The morphism $U' \to U$ equals $\mathrm{Spec}(k') \to \mathrm{Spec}(k)$, hence is affine, surjective and flat. The morphisms $s, t : R \to U$ and the morphism $U' \to U$ are universally open by Morphisms, Lemma 28.24.4. Since R is not empty and U is the spectrum of a field the morphisms $s, t : R \to U$ are surjective and flat. Then you conclude by using Morphisms of Spaces, Lemmas 49.6.5, 49.6.4, 49.7.4, 49.19.5, 49.19.4, 49.27.4, and 49.27.3. □

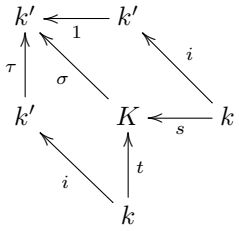
Lemma 60.7.6. *In Situation 60.7.2. For any point $r \in |R|$ there exist*

1. *a field extension $k \subset k'$ with k' algebraically closed,*
2. *a point $r' : \mathrm{Spec}(k') \to R'$ where (U', R', s', t', c') is the restriction of (U, R, s, t, c) via $\mathrm{Spec}(k') \to \mathrm{Spec}(k)$*

such that

1. *the point r' maps to r under the morphism $R' \to R$, and*
2. *the maps $s' \circ r', t' \circ r' : \mathrm{Spec}(k') \to \mathrm{Spec}(k')$ are automorphisms.*

Proof. Let's represent r by a morphism $r : \mathrm{Spec}(K) \to R$ for some field K. To prove the lemma we have to find an algebraically closed field k' and a commutative diagram



where $s, t : k \to K$ are the field maps coming from $s \circ r$ and $t \circ r$. In the proof of More on Groupoids, Lemma 39.9.5 it is shown how to construct such a diagram. □

Lemma 60.7.7. *In Situation 60.7.2. If $r : \mathrm{Spec}(k) \to R$ is a morphism such that $s \circ r, t \circ r$ are automorphisms of $\mathrm{Spec}(k)$, then the map*

$$R \longrightarrow R, \quad x \longmapsto c(r, x)$$

is an automorphism $R \to R$ which maps e to r.

Proof. Proof is identical to the proof of More on Groupoids, Lemma 39.9.6. □

Lemma 60.7.8. *In Situation 60.7.2 the algebraic space R is geometrically unibranch. In Situation 60.7.1 the algebraic space G is geometrically unibranch.*

Proof. Let $r \in |R|$. We have to show that R is geometrically unibranch at r. Combining Lemma 60.7.5 with Descent on Spaces, Lemma 55.8.1 we see that it suffices to prove this in case k is algebraically closed and r comes from a morphism $r : \mathrm{Spec}(k) \to R$ such that $s \circ r$ and $t \circ r$ are automorphisms of $\mathrm{Spec}(k)$. By Lemma 60.7.7 we reduce to the case that $r = e$ is the identity of R and k is algebraically closed.

Assume $r = e$ and k is algebraically closed. Let $A = \mathcal{O}_{R,e}$ be the étale local ring of R at e and let $C = \mathcal{O}_{R \times_{s,U,t} R,(e,e)}$ be the étale local ring of $R \times_{s,U,t} R$ at (e, e). By Spaces over Fields, Lemma 58.3.1 the minimal prime ideals $\mathfrak{q}$ of C correspond 1-to-1 to pairs of minimal primes $\mathfrak{p}, \mathfrak{p}' \subset A$. On the other hand, the composition law induces a flat ring map

$$\begin{array}{ccc} A & \xrightarrow{c^\sharp} & C \\ & & \uparrow \\ & & A \otimes_{s^\sharp,k,t^\sharp} A \end{array} \qquad \begin{array}{c} \mathfrak{q} \\ \top \\ \mathfrak{p} \otimes A + A \otimes \mathfrak{p}' \end{array}$$

Note that $(c^\sharp)^{-1}(\mathfrak{q})$ contains both $\mathfrak{p}$ and $\mathfrak{p}'$ as the diagrams

$$\begin{array}{ccc} A & \xrightarrow{c^\sharp} & C \\ \uparrow & & \uparrow \\ A \otimes_{s^\sharp,k} k & \xleftarrow{1 \otimes e^\sharp} & A \otimes_{s^\sharp,k,t^\sharp} A \end{array} \qquad \begin{array}{ccc} A & \xrightarrow{c^\sharp} & C \\ \uparrow & & \uparrow \\ k \otimes_{k,t^\sharp} A & \xleftarrow{e^\sharp \otimes 1} & A \otimes_{s^\sharp,k,t^\sharp} A \end{array}$$

commute by (60.3.0.2). Since $c^\sharp$ is flat (as c is a flat morphism by Lemma 60.7.3), we see that $(c^\sharp)^{-1}(\mathfrak{q})$ is a minimal prime of A. Hence $\mathfrak{p} = (c^\sharp)^{-1}(\mathfrak{q}) = \mathfrak{p}'$. □

In the following lemma we use dimension of algebraic spaces (at a point) as defined in Properties of Spaces, Section 48.8. We also use the dimension of the local ring defined in Properties of Spaces, Section 48.20 and transcendence degree of points, see Morphisms of Spaces, Section 49.30.

Lemma 60.7.9. *In Situation 60.7.2 assume s, t are locally of finite type. For all $r \in |R|$*

1. $\dim(R) = \dim_r(R)$,
2. *the transcendence degree of r over $\mathrm{Spec}(k)$ via s equals the transcendence degree of r over $\mathrm{Spec}(k)$ via t, and*
3. *if the transcendence degree mentioned in (2) is 0, then $\dim(R) = \dim(\mathcal{O}_{R,\overline{r}})$.*

Proof. Let $r \in |R|$. Denote $\mathrm{trdeg}(r/_s k)$ the transcendence degree of r over $\mathrm{Spec}(k)$ via s. Choose an étale morphism $\varphi : V \to R$ where V is a scheme and $v \in V$ mapping to r. Using the definitions mentioned above the lemma we see that

$$\dim_r(R) = \dim_v(V) = \dim(\mathcal{O}_{V,v}) + \mathrm{trdeg}_{s(k)}(\kappa(v)) = \dim(\mathcal{O}_{R,\overline{r}}) + \mathrm{trdeg}(r/_s k)$$

and similarly for t (the second equality by Morphisms, Lemma 28.29.1). Hence we see that $\mathrm{trdeg}(r/_s k) = \mathrm{trdeg}(r/_t k)$, i.e., (2) holds.

Let $k \subset k'$ be a field extension. Note that the restriction R' of R to $\mathrm{Spec}(k')$ (see Lemma 60.7.5) is obtained from R by two base changes by morphisms of fields.

Thus Morphisms of Spaces, Lemma 49.31.3 shows the dimension of R at a point is unchanged by this operation. Hence in order to prove (1) we may assume, by Lemma 60.7.6, that r is represented by a morphism $r : \mathrm{Spec}(k) \to R$ such that both $s \circ r$ and $t \circ r$ are automorphisms of $\mathrm{Spec}(k)$. In this case there exists an automorphism $R \to R$ which maps r to e (Lemma 60.7.7). Hence we see that $\dim_r(R) = \dim_e(R)$ for any r. By definition this means that $\dim_r(R) = \dim(R)$.

Part (3) is a formal consequence of the results obtained in the discussion above. □

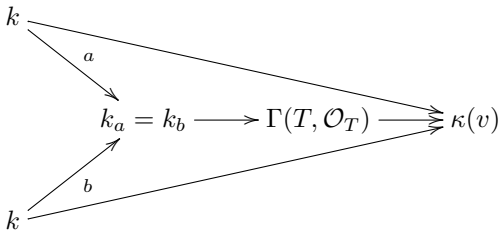
Lemma 60.7.10. *In Situation 60.7.1 assume G locally of finite type. For all $g \in |G|$*

(1) $\dim(G) = \dim_g(G)$,
(2) *if the transcendence degree of g over k is 0, then* $\dim(G) = \dim(\mathcal{O}_{G,\overline{g}})$.

Proof. Immediate from Lemma 60.7.9 via (60.7.2.1). □

Lemma 60.7.11. *In Situation 60.7.2 assume s, t are locally of finite type. Let $G = \mathrm{Spec}(k) \times_{\Delta, \mathrm{Spec}(k) \times_B \mathrm{Spec}(k), t \times s} R$ be the stabilizer group algebraic space. Then we have* $\dim(R) = \dim(G)$.

Proof. Since G and R are equidimensional (see Lemmas 60.7.9 and 60.7.10) it suffices to prove that $\dim_e(R) = \dim_e(G)$. Let V be an affine scheme, $v \in V$, and let $\varphi : V \to R$ be an étale morphism of schemes such that $\varphi(v) = e$. Note that V is a Noetherian scheme as $s \circ \varphi$ is locally of finite type as a composition of morphisms locally of finite type and as V is quasi-compact (use Morphisms of Spaces, Lemmas 49.22.2, 49.35.8, and 49.26.5 and Morphisms, Lemma 28.16.6). Hence V is locally connected (see Properties, Lemma 27.5.5 and Topology, Lemma 5.8.6). Thus we may replace V by the connected component containing v (it is still affine as it is an open and closed subscheme of V). Set $T = V_{red}$ equal to the reduction of V. Consider the two morphisms $a, b : T \to \mathrm{Spec}(k)$ given by $a = s \circ \varphi|_T$ and $b = t \circ \varphi|_T$. Note that a, b induce the same field map $k \to \kappa(v)$ because $\varphi(v) = e$! Let $k_a \subset \Gamma(T, \mathcal{O}_T)$ be the integral closure of $a^\sharp(k) \subset \Gamma(T, \mathcal{O}_T)$. Similarly, let $k_b \subset \Gamma(T, \mathcal{O}_T)$ be the integral closure of $b^\sharp(k) \subset \Gamma(T, \mathcal{O}_T)$. By Varieties, Proposition 32.18.1 we see that $k_a = k_b$. Thus we obtain the following commutative diagram



As discussed above the long arrows are equal. Since $k_a = k_b \to \kappa(v)$ is injective we conclude that the two morphisms a and b agree. Hence $T \to R$ factors through G. It follows that $R_{red} = G_{red}$ in an open neighbourhood of e which certainly implies that $\dim_e(R) = \dim_e(G)$. □

60.8. The finite part of a morphism

Let S be a scheme. Let $f : X \to Y$ be a morphism of algebraic spaces over S. For an algebraic space or a scheme T over S consider pairs (a, Z) where

$$\begin{array}{l} a : T \to Y \text{ is a morphism over } S, \\ Z \subset T \times_Y X \text{ is an open subspace} \\ \text{such that } \mathrm{pr}_0|_Z : Z \to T \text{ is finite.} \end{array} \tag{60.8.0.1}$$

Suppose $h : T' \to T$ is a morphism of algebraic spaces over S and (a, Z) is a pair over T. Set $a' = a \circ h$ and $Z' = (h \times \mathrm{id}_X)^{-1}(Z) = T' \times_T Z$. Then the pair (a', Z') satisfies (1), (2) over T'. This follows as finite morphisms are preserved under base change, see Morphisms of Spaces, Lemma 49.40.5. Thus we obtain a functor

$$\begin{array}{rccc} (X/Y)_{fin} : & (Sch/S)^{opp} & \longrightarrow & Sets \\ & T & \longmapsto & \{(a, Z) \text{ as above}\} \end{array} \tag{60.8.0.2}$$

For applications we are mainly interested in this functor $(X/Y)_{fin}$ when f is separated and locally of finite type. To get an idea of what this is all about, take a look at Remark 60.8.6.

Lemma 60.8.1. *Let S be a scheme. Let $f : X \to Y$ be a morphism of algebraic spaces over S. Then we have*

1. *The presheaf $(X/Y)_{fin}$ satisfies the sheaf condition for the fppf topology.*
2. *If T is an algebraic space over S, then there is a canonical bijection*

$$\mathrm{Mor}_{Sh((Sch/S)_{fppf})}(T, (X/Y)_{fin}) = \{(a, Z) \text{ satisfying 60.8.0.1}\}$$

Proof. Let T be an algebraic space over S. Let $\{T_i \to T\}$ be an fppf covering (by algebraic spaces). Let $s_i = (a_i, Z_i)$ be pairs over T_i satisfying 60.8.0.1 such that we have $s_i|_{T_i \times_T T_j} = s_j|_{T_i \times_T T_j}$. First, this implies in particular that a_i and a_j define the same morphism $T_i \times_T T_j \to Y$. By Descent on Spaces, Lemma 55.6.2 we deduce that there exists a unique morphism $a : T \to Y$ such that a_i equals the composition $T_i \to T \to Y$. Second, this implies that $Z_i \subset T_i \times_Y X$ are open subspaces whose inverse images in $(T_i \times_T T_j) \times_Y X$ are equal. Since $\{T_i \times_Y X \to T \times_Y X\}$ is an fppf covering we deduce that there exists a unique open subspace $Z \subset T \times_Y X$ which restricts back to Z_i over T_i, see Descent on Spaces, Lemma 55.6.1. We claim that the projection $Z \to T$ is finite. This follows as being finite is local for the fpqc topology, see Descent on Spaces, Lemma 55.10.21.

Note that the result of the preceding paragraph in particular implies (1).

Let T be an algebraic space over S. In order to prove (2) we will construct mutually inverse maps between the displayed sets. In the following when we say "pair" we mean a pair satisfying conditions 60.8.0.1.

Let $v : T \to (X/Y)_{fin}$ be a natural transformation. Choose a scheme U and a surjective étale morphism $p : U \to T$. Then $v(p) \in (X/Y)_{fin}(U)$ corresponds to a pair (a_U, Z_U) over U. Let $R = U \times_T U$ with projections $t, s : R \to U$. As v is a transformation of functors we see that the pullbacks of (a_U, Z_U) by s and t agree. Hence, since $\{U \to T\}$ is an fppf covering, we may apply the result of the first paragraph that deduce that there exists a unique pair (a, Z) over T.

Conversely, let (a, Z) be a pair over T. Let $U \to T$, $R = U \times_T U$, and $t, s : R \to U$ be as above. Then the restriction $(a, Z)|_U$ gives rise to a transformation of functors $v : h_U \to (X/Y)_{fin}$ by the Yoneda lemma (Categories, Lemma 4.3.5). As the

two pullbacks $s^*(a, Z)|_U$ and $t^*(a, Z)|_U$ are equal, we see that v coequalizes the two maps $h_t, h_s : h_R \to h_U$. Since $T = U/R$ is the fppf quotient sheaf by Spaces, Lemma 47.9.1 and since $(X/Y)_{fin}$ is an fppf sheaf by (1) we conclude that v factors through a map $T \to (X/Y)_{fin}$.

We omit the verification that the two constructions above are mutually inverse. □

Lemma 60.8.2. *Let S be a scheme. Consider a commutative diagram*

$$\begin{array}{ccc} X' & \xrightarrow{j} & X \\ & \searrow \quad \swarrow & \\ & Y & \end{array}$$

of algebraic spaces over S. If j is an open immersion, then there is a canonical injective map of sheaves $j : (X'/Y)_{fin} \to (X/Y)_{fin}$.

Proof. If (a, Z) is a pair over T for X'/Y, then $(a, j(Z))$ is a pair over T for X/Y. □

Lemma 60.8.3. *Let S be a scheme. Let $f : X \to Y$ be a morphism of algebraic spaces over S which is locally of finite type. Let $X' \subset X$ be the maximal open subspace over which f is locally quasi-finite, see Morphisms of Spaces, Lemma 49.31.7. Then $(X/Y)_{fin} = (X'/Y)_{fin}$.*

Proof. Lemma 60.8.2 gives us an injective map $(X'/Y)_{fin} \to (X/Y)_{fin}$. Morphisms of Spaces, Lemma 49.31.7 assures us that formation of X' commutes with base change. Hence everything comes down to proving that if $Z \subset X$ is a open subspace such that $f|_Z : Z \to Y$ is finite, then $Z \subset X'$. This is true because a finite morphism is locally quasi-finite, see Morphisms of Spaces, Lemma 49.40.8. □

Lemma 60.8.4. *Let S be a scheme. Let $f : X \to Y$ be a morphism of algebraic spaces over S. Let T be an algebraic space over S, and let (a, Z) be a pair as in 60.8.0.1. If f is separated, then Z is closed in $T \times_Y X$.*

Proof. A finite morphism of algebraic spaces is universally closed by Morphisms of Spaces, Lemma 49.40.9. Since f is separated so is the morphism $T \times_Y X \to T$, see Morphisms of Spaces, Lemma 49.5.4. Thus the closedness of Z follows from Morphisms of Spaces, Lemma 49.36.6. □

Remark 60.8.5. Let $f : X \to Y$ be a separated morphism of algebraic spaces. The sheaf $(X/Y)_{fin}$ comes with a natural map $(X/Y)_{fin} \to Y$ by mapping the pair $(a, Z) \in (X/Y)_{fin}(T)$ to the element $a \in Y(T)$. We can use Lemma 60.8.4 to define operations

$$\star_i : (X/Y)_{fin} \times_Y (X/Y)_{fin} \longrightarrow (X/Y)_{fin}$$

by the rules

$$\begin{aligned} \star_1 : ((a, Z_1), (a, Z_2)) &\longmapsto (a, Z_1 \cup Z_2) \\ \star_2 : ((a, Z_1), (a, Z_2)) &\longmapsto (a, Z_1 \cap Z_2) \\ \star_3 : ((a, Z_1), (a, Z_2)) &\longmapsto (a, Z_1 \setminus Z_2) \\ \star_4 : ((a, Z_1), (a, Z_2)) &\longmapsto (a, Z_2 \setminus Z_1). \end{aligned}$$

The reason this works is that $Z_1 \cap Z_2$ is both open and closed inside Z_1 and Z_2 (which also implies that $Z_1 \cup Z_2$ is the disjoint union of the other three pieces).

Thus we can think of $(X/Y)_{fin}$ as an $\mathbf{F}_2$-algebras (without unit) over Y with multiplication given by $ss' = \star_2(s, s')$, and addition given by

$$s + s' = \star_1(\star_3(s, s'), \star_4(s, s'))$$

which boils down to taking the symmetric difference. Note that in this sheaf of algebras $0 = (1_Y, \emptyset)$ and that indeed $s + s = 0$ for any local section s. If $f : X \to Y$ is finite, then this algebra has a unit namely $1 = (1_Y, X)$ and $\star_3(s, s') = s(1 + s')$, and $\star_4(s, s') = (1 + s)s'$.

Remark 60.8.6. Let $f : X \to Y$ be a separated, locally quasi-finite morphism of schemes. In this case the sheaf $(X/Y)_{fin}$ is closely related to the sheaf $f_!\mathbf{F}_2$ (insert future reference here) on $Y_{étale}$. Namely, if $V \to Y$ is étale, and $s \in \Gamma(V, f_!\mathbf{F}_2)$, then $s \in \Gamma(V \times_Y X, \mathbf{F}_2)$ is a section with proper support $Z = \mathrm{Supp}(s)$ over V. Since f is als locally quasi-finite we see that the projection $Z \to V$ is actually finite. Since the support of a section of a constant abelian sheaf is open we see that the pair $(V \to Y, \mathrm{Supp}(s))$ satisfies 60.8.0.1. In fact, $f_!\mathbf{F}_2 \cong (X/Y)_{fin}|_{Y_{étale}}$ in this case which also explains the $\mathbf{F}_2$-algebra structure introduced in Remark 60.8.5.

Lemma 60.8.7. *Let S be a scheme. Let $f : X \to Y$ be a morphism of algebraic spaces over S. The diagonal of $(X/Y)_{fin} \to Y$*

$$(X/Y)_{fin} \longrightarrow (X/Y)_{fin} \times_Y (X/Y)_{fin}$$

is representable (by schemes) and an open immersion and the "absolute" diagonal

$$(X/Y)_{fin} \longrightarrow (X/Y)_{fin} \times (X/Y)_{fin}$$

is representable (by schemes).

Proof. The second statement follows from the first as the absolute diagonal is the composition of the relative diagonal and a base change of the diagonal of Y (which is representable by schemes), see Spaces, Section 47.3. To prove the first assertion we have to show the following: Given a scheme T and two pairs (a, Z_1) and (a, Z_2) over T with identical first component satisfying 60.8.0.1 there is an open subscheme $V \subset T$ with the following property: For any morphism of schemes $h : T' \to T$ we have

$$h(T') \subset V \Leftrightarrow \Big(T' \times_T Z_1 = T' \times_T Z_2 \text{ as subspaces of } T' \times_Y X\Big)$$

Let us construct V. Note that $Z_1 \cap Z_2$ is open in Z_1 and in Z_2. Since $\mathrm{pr}_0|_{Z_i} : Z_i \to T$ is finite, hence proper (see Morphisms of Spaces, Lemma 49.40.9) we see that

$$E = \mathrm{pr}_0|_{Z_1}(Z_1 \setminus Z_1 \cap Z_2)) \cup \mathrm{pr}_0|_{Z_2}(Z_2 \setminus Z_1 \cap Z_2))$$

is closed in T. Now it is clear that $V = T \setminus E$ works. □

Lemma 60.8.8. *Let S be a scheme. Let $f : X \to Y$ be a morphism of algebraic spaces over S. Suppose that U is a scheme, $U \to Y$ is an étale morphism and $Z \subset U \times_Y X$ is an open subspace finite over U. Then the induced morphism $U \to (X/Y)_{fin}$ is étale.*

Proof. This is formal from the description of the diagonal in Lemma 60.8.7 but we write it out since it is an important step in the development of the theory. We have to check that for any scheme T over S and a morphism $T \to (X/Y)_{fin}$ the projection map

$$T \times_{(X/Y)_{fin}} U \longrightarrow T$$

is étale. Note that

$$T \times_{(X/Y)_{fin}} U = (X/Y)_{fin} \times_{((X/Y)_{fin} \times_Y (X/Y)_{fin})} (T \times_Y U)$$

Applying the result of Lemma 60.8.7 we see that $T \times_{(X/Y)_{fin}} U$ is represented by an open subscheme of $T \times_Y U$. As the projection $T \times_Y U \to T$ is étale by Morphisms of Spaces, Lemma 49.35.4 we conclude. □

Lemma 60.8.9. *Let S be a scheme. Let*

$$\begin{array}{ccc} X' & \longrightarrow & X \\ \downarrow & & \downarrow \\ Y' & \longrightarrow & Y \end{array}$$

be a fibre product square of algebraic spaces over S. Then

$$\begin{array}{ccc} (X'/Y')_{fin} & \longrightarrow & (X/Y)_{fin} \\ \downarrow & & \downarrow \\ Y' & \longrightarrow & Y \end{array}$$

is a fibre product square of sheaves on $(Sch/S)_{fppf}$.

Proof. It follows immediately from the definitions that the sheaf $(X'/Y')_{fin}$ is equal to the sheaf $Y' \times_Y (X/Y)_{fin}$. □

Lemma 60.8.10. *Let S be a scheme. Let $f : X \to Y$ be a morphism of algebraic spaces over S. If f is separated and locally quasi-finite, then there exists a scheme U étale over Y and a surjective étale morphism $U \to (X/Y)_{fin}$ over Y.*

Proof. Note that the assertion makes sense by the result of Lemma 60.8.7 on the diagonal of $(X/Y)_{fin}$, see Spaces, Lemma 47.5.10. Let V be a scheme and let $V \to Y$ be a surjective étale morphism. By Lemma 60.8.9 the morphism $(V \times_Y X/V)_{fin} \to (X/Y)_{fin}$ is a base change of the map $V \to Y$ and hence is surjective and étale, see Spaces, Lemma 47.5.5. Hence it suffices to prove the lemma for $(V \times_Y X/V)_{fin}$. (Here we implicitly use that the composition of representable, surjective, and étale transformations of functors is again representable, surjective, and étale, see Spaces, Lemmas 47.3.2 and 47.5.4, and Morphisms, Lemmas 28.11.2 and 28.37.3.) Note that the properties of being separated and locally quasi-finite are preserved under base change, see Morphisms of Spaces, Lemmas 49.5.4 and 49.25.3. Hence $V \times_Y X \to V$ is separated and locally quasi-finite as well, and by Morphisms of Spaces, Proposition 49.43.2 we see that $V \times_Y X$ is a scheme as well. Thus we may assume that $f : X \to Y$ is a separated and locally quasi-finite morphism of schemes.

Pick a point $y \in Y$. Pick $x_1, \ldots, x_n \in X$ points lying over y. Pick an étale neighbourhood $a : (U, u) \to (Y, y)$ and a decomposition

$$U \times_S X = W \coprod \coprod_{i=1,\ldots,n} \coprod_{j=1,\ldots,m_j} V_{i,j}$$

as in More on Morphisms, Lemma 36.30.5. Pick any subset

$$I \subset \{(i,j) \mid 1 \leq i \leq n,\ 1 \leq j \leq m_i\}.$$

Given these choices we obtain a pair (a, Z) with $Z = \bigcup_{(i,j) \in I} V_{i,j}$ which satisfies conditions 60.8.0.1. In other words we obtain a morphism $U \to (X/Y)_{fin}$. The

construction of this morphism depends on all the things we picked above, so we should really write

$$U(y, n, x_1, \ldots, x_n, a, I) \longrightarrow (X/Y)_{fin}$$

This morphism is étale by Lemma 60.8.8.

Claim: The disjoint union of all of these is surjective onto $(X/Y)_{fin}$. It is clear that if the claim holds, then the lemma is true.

To show surjectivity we have to show the following (see Spaces, Remark 47.5.2): Given a scheme T over S, a point $t \in T$, and a map $T \to (X/Y)_{fin}$ we can find a datum $(y, n, x_1, \ldots, x_n, a, I)$ as above such that t is in the image of the projection map

$$U(y, n, x_1, \ldots, x_n, a, I) \times_{(X/Y)_{fin}} T \longrightarrow T.$$

To prove this we may clearly replace T by $\mathrm{Spec}(\overline{\kappa(t)})$ and $T \to (X/Y)_{fin}$ by the composition $\mathrm{Spec}(\overline{\kappa(t)}) \to T \to (X/Y)_{fin}$. In other words, we may assume that T is the spectrum of an algebraically closed field.

Let $T = \mathrm{Spec}(k)$ be the spectrum of an algebraically closed field k. The morphism $T \to (X/Y)_{fin}$ is given by a pair $(T \to Y, Z)$ satisfying conditions 60.8.0.1. Here is a picture:

$$\begin{array}{ccccc} & & Z & \longrightarrow & X \\ & & \downarrow & & \downarrow \\ \mathrm{Spec}(k) & = & T & \longrightarrow & Y \end{array}$$

Let $y \in Y$ be the image point of $T \to Y$. Since Z is finite over k it has finitely many points. Thus there exist finitely many points $x_1, \ldots, x_n \in X$ such that the image of Z in X is contained in $\{x_1, \ldots, x_n\}$. Choose $a : (U, u) \to (Y, y)$ adapted to y and $x_1, \ldots, x_n$ as above, which gives the diagram

$$\begin{array}{ccc} W \coprod \coprod_{i=1,\ldots,n} \coprod_{j=1,\ldots,m_j} V_{i,j} & \longrightarrow & X \\ \downarrow & & \downarrow \\ U & \longrightarrow & Y. \end{array}$$

Since k is algebraically closed and $\kappa(y) \subset \kappa(u)$ is finite separable we may factor the morphism $T = \mathrm{Spec}(k) \to Y$ through the morphism $u = \mathrm{Spec}(\kappa(u)) \to \mathrm{Spec}(\kappa(y)) = y \subset Y$. With this choice we obtain the commutative diagram:

$$\begin{array}{ccccc} Z & \longrightarrow & W \coprod \coprod_{i=1,\ldots,n} \coprod_{j=1,\ldots,m_j} V_{i,j} & \longrightarrow & X \\ \downarrow & & \downarrow & & \downarrow \\ \mathrm{Spec}(k) & \longrightarrow & U & \longrightarrow & Y \end{array}$$

We know that the image of the left upper arrow ends up in $\coprod V_{i,j}$. Recall also that Z is an open subscheme of $\mathrm{Spec}(k) \times_Y X$ by definition of $(X/Y)_{fin}$ and that the right hand square is a fibre product square. Thus we see that

$$Z \subset \coprod\nolimits_{i=1,\ldots,n} \coprod\nolimits_{j=1,\ldots,m_j} \mathrm{Spec}(k) \times_U V_{i,j}$$

is an open subscheme. By construction (see More on Morphisms, Lemma 36.30.5) each $V_{i,j}$ has a unique point $v_{i,j}$ lying over u with purely inseparable residue field

extension $\kappa(u) \subset \kappa(v_{i,j})$. Hence each scheme $\mathrm{Spec}(k) \times_U V_{i,j}$ has exactly one point. Thus we see that

$$Z = \coprod_{(i,j)\in I} \mathrm{Spec}(k) \times_U V_{i,j}$$

for a unique subset $I \subset \{(i,j) \mid 1 \leq i \leq n,\ 1 \leq j \leq m_i\}$. Unwinding the definitions this shows that

$$U(y, n, x_1, \ldots, x_n, a, I) \times_{(X/Y)_{fin}} T$$

with I as found above is nonempty as desired. □

Proposition 60.8.11. *Let S be a scheme. Let $f : X \to Y$ be a morphism of algebraic spaces over S which is separated and locally of finite type. Then $(X/Y)_{fin}$ is an algebraic space. Moreover, the morphism $(X/Y)_{fin} \to Y$ is étale.*

Proof. By Lemma 60.8.3 we may replace X by the open subscheme which is locally quasi-finite over Y. Hence we may assume that f is separated and locally quasi-finite. We will check the three conditions of Spaces, Definition 47.6.1. Condition (1) follows from Lemma 60.8.1. Condition (2) follows from Lemma 60.8.7. Finally, condition (3) follows from Lemma 60.8.10. Thus $(X/Y)_{fin}$ is an algebraic space. Moreover, that lemma shows that there exists a commutative diagram

$$\begin{array}{ccc} U & \longrightarrow & (X/Y)_{fin} \\ & \searrow \quad \swarrow & \\ & Y & \end{array}$$

with horizontal arrow surjective and étale and south-east arrow étale. By Properties of Spaces, Lemma 48.13.3 this implies that the south-west arrow is étale as well. □

Remark 60.8.12. The condition that f be separated cannot be dropped from Proposition 60.8.11. An example is to take X the affine line with zero doubled, see Schemes, Example 25.14.3, $Y = \mathbf{A}^1_k$ the affine line, and $X \to Y$ the obvious map. Recall that over $0 \in Y$ there are two points 0_1 and 0_2 in X. Thus $(X/Y)_{fin}$ has four points over 0, namely $\emptyset, \{0_1\}, \{0_2\}, \{0_1, 0_2\}$. Of these four points only three can be lifted to an open subscheme of $U \times_Y X$ finite over U for $U \to Y$ étale, namely $\emptyset, \{0_1\}, \{0_2\}$. This shows that $(X/Y)_{fin}$ if representable by an algebraic space is not étale over Y. Similar arguments show that $(X/Y)_{fin}$ is really not an algebraic space. Details omitted.

Remark 60.8.13. Let $Y = \mathbf{A}^1_{\mathbf{R}}$ be the affine line over the real numbers, and let $X = \mathrm{Spec}(\mathbf{C})$ mapping to the $\mathbf{R}$-rational point 0 in Y. In this case the morphism $f : X \to Y$ is finite, but it is not the case that $(X/Y)_{fin}$ is a scheme. Namely, one can show that in this case the algebraic space $(X/Y)_{fin}$ is isomorphic to the algebraic space of Spaces, Example 47.14.2 associated to the extension $\mathbf{R} \subset \mathbf{C}$. Thus it is really necessary to leave the category of schemes in order to represent the sheaf $(X/Y)_{fin}$, even when f is a finite morphism.

Lemma 60.8.14. *Let S be a scheme. Let $f : X \to Y$ be a morphism of algebraic spaces over S which is separated, flat, and locally of finite presentation. In this case*

(1) *$(X/Y)_{fin} \to Y$ is separated, representable, and étale, and*
(2) *if Y is a scheme, then $(X/Y)_{fin}$ is (representable by) a scheme.*

Proof. Since f is in particular separated and locally of finite type (see Morphisms of Spaces, Lemma 49.26.5) we see that $(X/Y)_{fin}$ is an algebraic space by Proposition 60.8.11. To prove that $(X/Y)_{fin} \to Y$ is separated we have to show the following: Given a scheme T and two pairs (a, Z_1) and (a, Z_2) over T with identical first component satisfying 60.8.0.1 there is a closed subscheme $V \subset T$ with the following property: For any morphism of schemes $h : T' \to T$ we have

$$h \text{ factors through } V \Leftrightarrow \left(T' \times_T Z_1 = T' \times_T Z_2 \text{ as subspaces of } T' \times_Y X\right)$$

In the proof of Lemma 60.8.7 we have seen that $V = T' \setminus E$ is an open subscheme of T' with closed complement

$$E = \mathrm{pr}_0|_{Z_1}(Z_1 \setminus Z_1 \cap Z_2)) \cup \mathrm{pr}_0|_{Z_2}(Z_2 \setminus Z_1 \cap Z_2)).$$

Thus everything comes down to showing that E is also open. By Lemma 60.8.4 we see that Z_1 and Z_2 are closed in $T' \times_Y X$. Hence $Z_1 \setminus Z_1 \cap Z_2$ is open in Z_1. As f is flat and locally of finite presentation, so is $\mathrm{pr}_0|_{Z_1}$. This is true as Z_1 is an open subspace of the base change $T' \times_Y X$, and Morphisms of Spaces, Lemmas 49.26.3 and Lemmas 49.27.4. Hence $\mathrm{pr}_0|_{Z_1}$ is open, see Morphisms of Spaces, Lemma 49.27.6. Thus $\mathrm{pr}_0|_{Z_1}(Z_1 \setminus Z_1 \cap Z_2))$ is open and it follows that E is open as desired.

We have already seen that $(X/Y)_{fin} \to Y$ is étale, see Proposition 60.8.11. Hence now we know it is locally quasi-finite (see Morphisms of Spaces, Lemma 49.35.5) and separated, hence representable by Morphisms of Spaces, Lemma 49.44.1. The final assertion is clear (if you like you can use Morphisms of Spaces, Proposition 49.43.2). □

Variant: Let S be a scheme. Let $f : X \to Y$ be a morphism of algebraic spaces over S. Let $\sigma : Y \to X$ be a section of f. For an algebraic space or a scheme T over S consider pairs (a, Z) where

$$\begin{array}{c} a : T \to Y \text{ is a morphism over } S, \\ Z \subset T \times_Y X \text{ is an open subspace} \\ \text{such that } \mathrm{pr}_0|_Z : Z \to T \text{ is finite and} \\ (1_T, \sigma \circ a) : T \to T \times_Y X \text{ factors through } Z. \end{array} \tag{60.8.14.1}$$

We will denote $(X/Y, \sigma)_{fin}$ the subfunctor of $(X/Y)_{fin}$ parametrizing these pairs.

Lemma 60.8.15. *Let S be a scheme. Let $f : X \to Y$ be a morphism of algebraic spaces over S. Let $\sigma : Y \to X$ be a section of f. Consider the transformation of functors*

$$t : (X/Y, \sigma)_{fin} \longrightarrow (X/Y)_{fin}.$$

defined above. Then

1. *t is representable by open immersions,*
2. *if f is separated, then t is representable by open and closed immersions,*
3. *if $(X/Y)_{fin}$ is an algebraic space, then $(X/Y, \sigma)_{fin}$ is an algebraic space and an open subspace of $(X/Y)_{fin}$, and*
4. *if $(X/Y)_{fin}$ is a scheme, then $(X/Y, \sigma)_{fin}$ is an open subscheme of it.*

Proof. Omitted. Hint: Given a pair (a, Z) over T as in (60.8.0.1) the inverse image of Z by $(1_T, \sigma \circ a) : T \to T \times_Y X$ is the open subscheme of T we are looking for. □

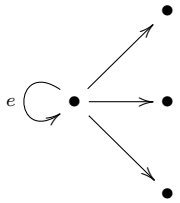
60.9. Finite collections of arrows

Let $\mathcal{C}$ be a groupoid, see Categories, Definition 4.2.5. As discussed in Groupoids, Section 38.11 this corresponds to a septuple $(\text{Ob}, \text{Arrows}, s, t, c, e, i)$.

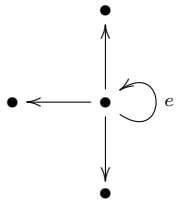
Using this data we can make another groupoid $\mathcal{C}_{fin}$ as follows:

(1) An object of $\mathcal{C}_{fin}$ consists of a finite subset $Z \subset \text{Arrows}$ with the following properties:
   (a) $s(Z) = \{u\}$ is a singleton, and
   (b) $e(u) \in Z$.
(2) A morphism of $\mathcal{C}_{fin}$ consists of a pair (Z, z), where Z is an object of $\mathcal{C}_{fin}$ and $z \in Z$.
(3) The source of (Z, z) is Z.
(4) The target of (Z, z) is $t(Z, z) = \{z' \circ z^{-1}; z' \in Z\}$.
(5) Given (Z_1, z_1), (Z_2, z_2) such that $s(Z_1, z_1) = t(Z_2, z_2)$ the composition $(Z_1, z_1) \circ (Z_2, z_2)$ is $(Z_2, z_1 \circ z_2)$.

We omit the verification that this defines a groupoid. Pictorially an object of $\mathcal{C}_{fin}$ can be viewed as a diagram



To make a morphism of $\mathcal{C}_{fin}$ you pick one of the arrows and you precompose the other arrows by its inverse. For example if we pick the middle horizontal arrow then the target is the picture



Note that the cardinalities of $s(Z, z)$ and $t(Z, z)$ are equal. So $\mathcal{C}_{fin}$ is really a countable disjoint union of groupoids.

60.10. The finite part of a groupoid

In this section we are going to use the idea explained in Section 60.9 to take the finite part of a groupoid in algebraic spaces.

Let S be a scheme. Let B be an algebraic space over S. Let (U, R, s, t, c, e, i) be a groupoid in algebraic spaces over B. Assumption: The morphisms s, t are separated and locally of finite type. This notation and assumption will we be fixed throughout this section.

Denote R_s the algebraic space R seen as an algebraic space over U via s. Let $U' = (R_s/U, e)_{fin}$. Since s is separated and locally of finite type, by Proposition 60.8.11 and Lemma 60.8.15, we see that U' is an algebraic space endowed with an

étale morphism $g : U' \to U$. Moreover, by Lemma 60.8.1 there exists a universal open subspace $Z_{univ} \subset R \times_{s,U,g} U'$ which is finite over U' and such that $(1_{U'}, e \circ g) : U' \to R \times_{s,U,g} U'$ factors through Z_{univ}. Moreover, by Lemma 60.8.4 the open subspace Z_{univ} is also closed in $R \times_{s,U',g} U$. Picture so far:

$$\begin{array}{ccc} Z_{univ} & & \\ \downarrow & \searrow & \\ R \times_{s,U,g} U' & \longrightarrow & U' \\ \downarrow & & \downarrow g \\ R & \xrightarrow{s} & U \end{array}$$

Let T be a scheme over B. We see that a T-valued point of Z_{univ} may be viewed as a triple (u, Z, z) where

(1) $u : T \to U$ is a T-valued point of U,
(2) $Z \subset R \times_{s,U,u} T$ is an open and closed subspace finite over T such that $(e \circ u, 1_T)$ factors through it, and
(3) $z : T \to R$ is a T-valued point of R with $s \circ z = u$ and such that $(z, 1_T)$ factors through Z.

Having said this, it is morally clear from the discussion in Section 60.9 that we can turn (Z_{univ}, U') into a groupoid in algebraic spaces over B. To make sure will define the morphisms s', t', c', e', i' one by one using the functorial point of view. (Please don't read this before reading and understanding the simple construction in Section 60.9.)

The morphism $s' : Z_{univ} \to U'$ corresponds to the rule

$$s' : (u, Z, z) \mapsto (u, Z).$$

The morphism $t' : Z_{univ} \to U'$ is given by the rule

$$t' : (u, Z, z) \mapsto (t \circ z, c(Z, i \circ z)).$$

The entry $c(Z, i \circ z)$ makes sense as the map $c(-, i \circ z) : R \times_{s,U,u} T \to R \times_{s,U,t \circ z} T$ is an isomorphism with inverse $c(-, z)$. The morphism $e' : U' \to Z_{univ}$ is given by the rule

$$e' : (u, Z) \mapsto (u, Z, (e \circ u, 1_T)).$$

Note that this makes sense by the requirement that $(e \circ u, 1_T)$ factors through Z. The morphism $i' : Z_{univ} \to Z_{univ}$ is given by the rule

$$i' : (u, Z, z) \mapsto (t \circ z, c(Z, i \circ z), i \circ z).$$

Finally, composition is defined by the rule

$$c' : ((u_1, Z_1, z_1), (u_2, Z_2, z_2)) \mapsto (u_2, Z_2, z_1 \circ z_2).$$

We omit the verification that the axioms of a groupoid in algebraic spaces hold for $(U', Z_{univ}, s', t', c', e', i')$.

A final piece of information is that there is a canonical morphism of groupoids

$$(U', Z_{univ}, s', t', c', e', i') \longrightarrow (U, R, s, t, c, e, i)$$

Namely, the morphism $U' \to U$ is the morphism $g : U' \to U$ which is defined by the rule $(u, Z) \mapsto u$. The morphism $Z_{univ} \to R$ is defined by the rule $(u, Z, z) \mapsto z$. This finishes the construction. Let us summarize our findings as follows.

Lemma 60.10.1. *Let S be a scheme. Let B be an algebraic space over S. Let (U, R, s, t, c, e, i) be a groupoid in algebraic spaces over B. Assume the morphisms s, t are separated and locally of finite type. There exists a canonical morphism*

$$(U', Z_{univ}, s', t', c', e', i') \longrightarrow (U, R, s, t, c, e, i)$$

of groupoids in algebraic spaces over B where

1. *$g : U' \to U$ is identified with $(R_s/U, e)_{fin} \to U$, and*
2. *$Z_{univ} \subset R \times_{s,U,g} U'$ is the universal open (and closed) subspace finite over U' which contains the base change of the unit e.*

Proof. See discussion above. □

60.11. Étale localization of groupoid schemes

In this section we prove results similar to [**KM97**, Proposition 4.2]. We try to be a bit more general, and we try to avoid using Hilbert schemes by using the finite part of a morphism instead. The goal is to "split" a groupoid in algebraic spaces over a point after étale localization. Here is the definition (very similar to [**KM97**, Definition 4.1]).

Definition 60.11.1. Let S be a scheme. Let B be an algebraic space over S Let (U, R, s, t, c) be a groupoid in algebraic spaces over B. Let $u \in |U|$ be a point.

1. We say R is *split over u* if there exists an open subspace $P \subset R$ such that
   (a) $(U, P, s|_P, t|_P, c|_{P \times_{s,U,t} P})$ is a groupoid in algebraic spaces over B,
   (b) $s|_P$, $t|_P$ are finite, and
   (c) $\{r \in |R| : s(r) = u, t(r) = u\} \subset P$.
   The choice of such a P will be called a *splitting of R over u*.
2. We say R is *quasi-split over u* if there exists an open subspace $P \subset R$ such that
   (a) $(U, P, s|_P, t|_P, c|_{P \times_{s,U,t} P})$ is a groupoid in algebraic spaces over B,
   (b) $s|_P$, $t|_P$ are finite, and
   (c) $e(u) \in |P|$[1].
   The choice of such a P will be called a *quasi-splitting of R over u*.

Note the similarity of the conditions on P to the conditions on pairs in (60.8.0.1). In particular, if s, t are separated, then P is also closed in R (see Lemma 60.8.4).

Suppose we start with a groupoid in algebraic spaces (U, R, s, t, c) over B and a point $u \in |U|$. Since the goal is to split the groupoid after étale localization we may as well replace U by an affine scheme (what we mean is that this is harmless for any possible application). Moreover, the additional hypotheses we are going to have to impose will force R to be a scheme at least in a neighbourhood of $\{r \in |R| : s(r) = u, t(r) = u\}$ or $e(u)$. This is why we start with a groupoid scheme as described below. However, our technique of proof leads us outside of the category of schemes, which is why we have formulated a splitting for the case of groupoids in algebraic spaces above. On the other hand, we know of no applications but the

[1] This condition is implied by (a).

case where the morphisms s, t are also flat and of finite presentation, in which case we end up back in the category of schemes.

Situation 60.11.2. (Assumptions for splitting.) Let S be a scheme. Let (U, R, s, t, c) be a groupoid scheme over S. Let $u \in U$ be a point. Assume that

(1) $s, t : R \to U$ are separated,
(2) s, t are locally of finite type,
(3) the set $\{r \in R : s(r) = u, t(r) = u\}$ is finite, and
(4) s is quasi-finite at each point of the set in (3).

Note that assumptions (3) and (4) are implied by the assumption that the fibre $s^{-1}(\{u\})$ is finite, see Morphisms, Lemma 28.21.7.

Situation 60.11.3. (Assumptions for quasi-splitting.) Let S be a scheme. Let (U, R, s, t, c) be a groupoid scheme over S. Let $u \in U$ be a point. Assume that

(1) $s, t : R \to U$ are separated,
(2) s, t are locally of finite type, and
(3) s is quasi-finite at $e(u)$.

It turns out that for applications to the existence theorems for algebraic spaces the case of quasi-splittings is sufficient. In fact, it is for us somehow a more natural case to consider, as in the stacks project there are no finiteness conditions on the diagonal of an algebraic space, hence the assumption that $\{r \in R : s(r) = u, t(r) = u\}$ is finite need not hold even for a presentation $X = U/R$ of an algebraic space X.

Lemma 60.11.4. *Assumptions and notation as in Situation 60.11.2. Then there exists an algebraic space U', an étale morphism $U' \to U$, and a point $u' : \mathrm{Spec}(\kappa(u)) \to U'$ lying over $u : \mathrm{Spec}(\kappa(u)) \to U$ such that the restriction $R' = R|_{U'}$ of R to U' splits over u'.*

Proof. Let $f : (U', Z_{univ}, s', t', c') \to (U, R, s, t, c)$ be as constructed in Lemma 60.10.1. Recall that $R' = R \times_{(U \times_S U)} (U' \times_S U')$. Thus we get a morphism $(f, t', s') : Z_{univ} \to R'$ of groupoids in algebraic spaces

$$(U', Z_{univ}, s', t', c') \to (U', R', s', t', c')$$

(by abuse of notation we indicate the morphisms in the two groupoids by the same symbols). Now, as $Z \subset R \times_{s,U,g} U'$ is open and $R' \to R \times_{s,U,g} U'$ is étale (as a base change of $U' \to U$) we see that $Z_{univ} \to R'$ is an open immersion. By construction the morphisms $s', t' : Z_{univ} \to U'$ are finite. It remains to find the point u' of U'.

We think of u as a morphism $\mathrm{Spec}(\kappa(u)) \to U$ as in the statement of the lemma. Set $F_u = R \times_{s,U} \mathrm{Spec}(\kappa(u))$. The set $\{r \in R : s(r) = u, t(r) = u\}$ is finite by assumption and $F_u \to \mathrm{Spec}(\kappa(u))$ is quasi-finite at each of its elements. Hence we can find a decomposition into open and closed subschemes

$$F_u = Z_u \coprod Rest$$

for some scheme Z_u finite over $\kappa(u)$ whose support is $\{r \in R : s(r) = u, t(r) = u\}$. Note that $e(u) \in Z_u$. Hence by the construction of U' in Section 60.10 (u, Z_u) defines a $\mathrm{Spec}(\kappa(u))$-valued point u' of U'.

We still have to show that the set $\{r' \in |R'| : s'(r') = u', t'(r') = u'\}$ is contained in $|Z_{univ}|$. Pick any point r' in this set and represent it by a morphism $r' : \mathrm{Spec}(k) \to R'$. Denote $z : \mathrm{Spec}(k) \to R$ the composition of r' with the map $R' \to R$. Since

$\kappa(u) = \kappa(u')$, and since $s'(r') = u'$, $t(r') = u'$ no information is lost by considering the point z rather than the point r', i.e., we can recover r' from the point z. For example z is an element of the set $\{r \in R : s(r) = u, t(r) = u\}$ by our assumption on r'. The composition $s \circ z : \mathrm{Spec}(k) \to U$ factors through u, so we may think of $s \circ z$ as a morphism $\mathrm{Spec}(k) \to \mathrm{Spec}(\kappa(u))$. Hence we can consider the triple

$$(s \circ z, Z_u \times_{\mathrm{Spec}(\kappa(u)), s \circ z} \mathrm{Spec}(k), z)$$

where Z_u is as above. This defines a $\mathrm{Spec}(k)$-valued point of Z_{univ} above whose image under the map $Z_{univ} \to R'$ is the point r' by the relationship between z and r' mentioned above. This finishes the proof. □

Lemma 60.11.5. *Assumptions and notation as in Situation 60.11.3. Then there exists an algebraic space U', an étale morphism $U' \to U$, and a point $u' : \mathrm{Spec}(\kappa(u)) \to U'$ lying over $u : \mathrm{Spec}(\kappa(u)) \to U$ such that the restriction $R' = R|_{U'}$ of R to U' is quasi-split over u'.*

Proof. The proof is almost exactly the same as the proof of Lemma 60.11.4. Let $f : (U', Z_{univ}, s', t', c') \to (U, R, s, t, c)$ be as constructed in Lemma 60.10.1. Recall that $R' = R \times_{(U \times_S U)} (U' \times_S U')$. Thus we get a morphism $(f, t', s') : Z_{univ} \to R'$ of groupoids in algebraic spaces

$$(U', Z_{univ}, s', t', c') \to (U', R', s', t', c')$$

(by abuse of notation we indicate the morphisms in the two groupoids by the same symbols). Now, as $Z \subset R \times_{s,U,g} U'$ is open and $R' \to R \times_{s,U,g} U'$ is étale (as a base change of $U' \to U$) we see that $Z_{univ} \to R'$ is an open immersion. By construction the morphisms $s', t' : Z_{univ} \to U'$ are finite. It remains to find the point u' of U'.

We think of u as a morphism $\mathrm{Spec}(\kappa(u)) \to U$ as in the statement of the lemma. Set $F_u = R \times_{s,U} \mathrm{Spec}(\kappa(u))$. The morphism $F_u \to \mathrm{Spec}(\kappa(u))$ is quasi-finite at $e(u)$ by assumption. Hence we can find a decomposition into open and closed subschemes

$$F_u = Z_u \coprod Rest$$

for some scheme Z_u finite over $\kappa(u)$ whose support is $e(u)$. Hence by the construction of U' in Section 60.10 (u, Z_u) defines a $\mathrm{Spec}(\kappa(u))$-valued point u' of U'. To finish the proof we have to show that $e'(u') \in Z_{univ}$ which is clear. □

Finally, when we add additional assumptions we obtain schemes.

Lemma 60.11.6. *Assumptions and notation as in Situation 60.11.2. Assume in addition that s, t are flat and locally of finite presentation. Then there exists a scheme U', a separated étale morphism $U' \to U$, and a point $u' \in U'$ lying over u with $\kappa(u) = \kappa(u')$ such that the restriction $R' = R|_{U'}$ of R to U' splits over u'.*

Proof. This follows from the construction of U' in the proof of Lemma 60.11.4 because in this case $U' = (R_s/U, e)_{fin}$ is a scheme separated over U by Lemmas 60.8.14 and 60.8.15. □

Lemma 60.11.7. *Assumptions and notation as in Situation 60.11.3. Assume in addition that s, t are flat and locally of finite presentation. Then there exists a scheme U', a separated étale morphism $U' \to U$, and a point $u' \in U'$ lying over u with $\kappa(u) = \kappa(u')$ such that the restriction $R' = R|_{U'}$ of R to U' is quasi-split over u'.*

Proof. This follows from the construction of U' in the proof of Lemma 60.11.5 because in this case $U' = (R_s/U, e)_{fin}$ is a scheme separated over U by Lemmas 60.8.14 and 60.8.15. □

In fact we can obtain affine schemes by applying an earlier result on finite locally free groupoids.

Lemma 60.11.8. *Assumptions and notation as in Situation 60.11.2. Assume in addition that s, t are flat and locally of finite presentation and that U is affine. Then there exists an affine scheme U', an étale morphism $U' \to U$, and a point $u' \in U'$ lying over u with $\kappa(u) = \kappa(u')$ such that the restriction $R' = R|_{U'}$ of R to U' splits over u'.*

Proof. Let $U' \to U$ and $u' \in U'$ be the étale morphism of schemes we found in Lemma 60.11.6. Let $P \subset R'$ be the splitting of R' over u'. By More on Groupoids, Lemma 39.8.1 the morphisms $s', t' : R' \to U'$ are flat and locally of finite presentation. They are finite by assumption. Hence s', t' are finite locally free, see Morphisms, Lemma 28.46.2. In particular $t(s^{-1}(u'))$ is a finite set of points $\{u'_1, u'_2, \ldots, u'_n\}$ of U'. Choose a quasi-compact open $W \subset U'$ containing each u'_i. As U is affine the morphism $W \to U$ is quasi-compact (see Schemes, Lemma 25.19.2). The morphism $W \to U$ is also locally quasi-finite (see Morphisms, Lemma 28.37.6) and separated. Hence by More on Morphisms, Lemma 36.31.2 (a version of Zariski's Main Theorem) we conclude that W is quasi-affine. By Properties, Lemma 27.27.5 we see that $\{u'_1, \ldots, u'_n\}$ are contained in an affine open of U'. Thus we may apply Groupoids, Lemma 38.21.1 to conclude that there exists an affine P-invariant open $U'' \subset U'$ which contains u'.

To finish the proof denote $R'' = R|_{U''}$ the restriction of R to U''. This is the same as the restriction of R' to U''. As $P \subset R'$ is an open and closed subscheme, so is $P|_{U''} \subset R''$. By construction the open subscheme $U'' \subset U'$ is P-invariant which means that $P|_{U''} = (s'|_P)^{-1}(U'') = (t'|_P)^{-1}(U'')$ (see discussion in Groupoids, Section 38.17) so the restrictions of s'' and t'' to $P|_{U''}$ are still finite. The sub groupoid scheme $P|_{U''}$ is still a splitting of R'' over u''; above we verified (a), (b) and (c) holds as $\{r' \in R' : t'(r') = u', s'(r') = u'\} = \{r'' \in R'' : t''(r'') = u', s''(r'') = u'\}$ trivially. The lemma is proved. □

Lemma 60.11.9. *Assumptions and notation as in Situation 60.11.3. Assume in addition that s, t are flat and locally of finite presentation and that U is affine. Then there exists an affine scheme U', an étale morphism $U' \to U$, and a point $u' \in U'$ lying over u with $\kappa(u) = \kappa(u')$ such that the restriction $R' = R|_{U'}$ of R to U' is quasi-split over u'.*

Proof. The proof of this lemma is literally the same as the proof of Lemma 60.11.8 except that "splitting" needs to be replaced by "quasi-splitting" (2 times) and that the reference to Lemma 60.11.6. needs to be replaced by a reference to Lemma 60.11.7. □

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CHAPTER 61

Bootstrap

61.1. Introduction

In this chapter we use the material from the preceding sections to give criteria under which a presheaf of sets on the category of schemes is an algebraic space. Some of this material comes from the work of Artin, see [**Art69b**], [**Art70**], [**Art73**], [**Art71b**], [**Art71a**], [**Art69a**], [**Art69c**], and [**Art74**]. However, our method will be to use as much as possible arguments similar to those of the paper by Keel and Mori, see [**KM97**].

61.2. Conventions

The standing assumption is that all schemes are contained in a big fppf site Sch_{fppf}. And all rings A considered have the property that $\mathrm{Spec}(A)$ is (isomorphic) to an object of this big site.

Let S be a scheme and let X be an algebraic space over S. In this chapter and the following we will write $X \times_S X$ for the product of X with itself (in the category of algebraic spaces over S), instead of $X \times X$.

61.3. Morphisms representable by algebraic spaces

Here we define the notion of one presheaf being relatively representable by algebraic spaces over another, and we prove some properties of this notion.

Definition 61.3.1. Let S be a scheme contained in Sch_{fppf}. Let F, G be presheaves on Sch_{fppf}/S. We say a morphism $a : F \to G$ is *representable by algebraic spaces* if for every $U \in \mathrm{Ob}((Sch/S)_{fppf})$ and any $\xi : U \to G$ the fiber product $U \times_{\xi,G} F$ is an algebraic space.

Here is a sanity check.

Lemma 61.3.2. *Let S be a scheme. Let $f : X \to Y$ be a morphism of algebraic spaces over S. Then f is representable by algebraic spaces.*

Proof. This is formal. It relies on the fact that the category of algebraic spaces over S has fibre products, see Spaces, Lemma 47.7.3. □

Lemma 61.3.3. *Let S be a scheme. Let*

$$\begin{array}{ccc} G' \times_G F & \longrightarrow & F \\ \downarrow{\scriptstyle a'} & & \downarrow{\scriptstyle a} \\ G' & \longrightarrow & G \end{array}$$

be a fibre square of presheaves on $(Sch/S)_{fppf}$. If a is representable by algebraic spaces so is a'.

Proof. Omitted. Hint: This is formal. □

Lemma 61.3.4. *Let S be a scheme contained in Sch_{fppf}. Let $F, G : (Sch/S)^{opp}_{fppf} \to Sets$. Let $a : F \to G$ be representable by algebraic spaces. If G is a sheaf, then so is F.*

Proof. (Same as the proof of Spaces, Lemma 47.3.5.) Let $\{\varphi_i : T_i \to T\}$ be a covering of the site $(Sch/S)_{fppf}$. Let $s_i \in F(T_i)$ which satisfy the sheaf condition. Then $\sigma_i = a(s_i) \in G(T_i)$ satisfy the sheaf condition also. Hence there exists a unique $\sigma \in G(T)$ such that $\sigma_i = \sigma|_{T_i}$. By assumption $F' = h_T \times_{\sigma,G,a} F$ is a sheaf. Note that $(\varphi_i, s_i) \in F'(T_i)$ satisfy the sheaf condition also, and hence come from some unique $(\mathrm{id}_T, s) \in F'(T)$. Clearly s is the section of F we are looking for. □

Lemma 61.3.5. *Let S be a scheme contained in Sch_{fppf}. Let $F, G : (Sch/S)^{opp}_{fppf} \to Sets$. Let $a : F \to G$ be representable by algebraic spaces. Then $\Delta_{F/G} : F \to F \times_G F$ is representable by algebraic spaces.*

Proof. (Same as the proof of Spaces, Lemma 47.3.6.) Let U be a scheme. Let $\xi = (\xi_1, \xi_2) \in (F \times_G F)(U)$. Set $\xi' = a(\xi_1) = a(\xi_2) \in G(U)$. By assumption there exist an algebraic space V and a morphism $V \to U$ representing the fibre product $U \times_{\xi',G} F$. In particular, the elements ξ_1, ξ_2 give morphisms $f_1, f_2 : U \to V$ over U. Because V represents the fibre product $U \times_{\xi',G} F$ and because $\xi' = a \circ \xi_1 = a \circ \xi_2$ we see that if $g : U' \to U$ is a morphism then

$$g^*\xi_1 = g^*\xi_2 \Leftrightarrow f_1 \circ g = f_2 \circ g.$$

In other words, we see that $U \times_{\xi, F \times_G F} F$ is represented by $V \times_{\Delta, V \times V, (f_1, f_2)} U$ which is an algebraic space. □

The proof of Lemma 61.3.6 below is actually slightly tricky. Namely, we cannot use the argument of the proof of Spaces, Lemma 47.11.1 because we do not yet know that a composition of transformations representable by algebraic spaces is representable by algebraic spaces. In fact, we will use this lemma to prove that statement.

Lemma 61.3.6. *Let S be a scheme contained in Sch_{fppf}. Let $F, G : (Sch/S)^{opp}_{fppf} \to Sets$. Let $a : F \to G$ be representable by algebraic spaces. If G is an algebraic space, then so is F.*

Proof. We have seen in Lemma 61.3.4 that F is a sheaf.

Let U be a scheme and let $U \to G$ be a surjective étale morphism. In this case $U \times_G F$ is an algebraic space. Let W be a scheme and let $W \to U \times_G F$ be a surjective étale morphism.

First we claim that $W \to F$ is representable. To see this let X be a scheme and let $X \to F$ be a morphism. Then

$$W \times_F X = W \times_{U \times_G F} U \times_G F \times_F X = W \times_{U \times_G F} (U \times_G X)$$

Since both $U \times_G F$ and G are algebraic spaces we see that this is a scheme.

Next, we claim that $W \to F$ is surjective and étale (this makes sense now that we know it is representable). This follows from the formula above since both $W \to U \times_G F$ and $U \to G$ are étale and surjective, hence $W \times_{U \times_G F} (U \times_G X) \to U \times_G X$ and $U \times_G X \to X$ are surjective and étale, and the composition of surjective étale morphisms is surjective and étale.

Set $R = W \times_F W$. By the above R is a scheme and the projections $t, s : R \to W$ are étale. It is clear that R is an equivalence relation, and $W \to F$ is a surjection of sheaves. Hence R is an étale equivalence relation and $F = W/R$. Hence F is an algebraic space by Spaces, Theorem 47.10.5. □

Lemma 61.3.7. *Let S be a scheme. Let $a : F \to G$ be a map of presheaves on $(Sch/S)_{fppf}$. Suppose $a : F \to G$ is representable by algebraic spaces. If X is an algebraic space over S, and $X \to G$ is a map of presheaves then $X \times_G F$ is an algebraic space.*

Proof. By Lemma 61.3.3 the transformation $X \times_G F \to X$ is representable by algebraic spaces. Hence it is an algebraic space by Lemma 61.3.6. □

Lemma 61.3.8. *Let S be a scheme. Let*

$$F \xrightarrow{a} G \xrightarrow{b} H$$

be maps of presheaves on $(Sch/S)_{fppf}$. If a and b are representable by algebraic spaces, so is $b \circ a$.

Proof. Let T be a scheme over S, and let $T \to H$ be a morphism. By assumption $T \times_H G$ is an algebraic space. Hence by Lemma 61.3.7 we see that $T \times_H F = (T \times_H G) \times_G F$ is an algebraic space as well. □

Lemma 61.3.9. *Let S be a scheme. Let $F_i, G_i : (Sch/S)^{opp}_{fppf} \to Sets$, $i = 1, 2$. Let $a_i : F_i \to G_i$, $i = 1, 2$ be representable by algebraic spaces. Then*

$$a_1 \times a_2 : F_1 \times F_2 \longrightarrow G_1 \times G_2$$

is a representable by algebraic spaces.

Proof. Write $a_1 \times a_2$ as the composition $F_1 \times F_2 \to G_1 \times F_2 \to G_1 \times G_2$. The first arrow is the base change of a_1 by the map $G_1 \times F_2 \to G_1$, and the second arrow is the base change of a_2 by the map $G_1 \times G_2 \to G_2$. Hence this lemma is a formal consequence of Lemmas 61.3.8 and 61.3.3. □

Lemma 61.3.10. *Let $S \in \mathrm{Ob}(Sch_{fppf})$. Let F be a presheaf of sets on $(Sch/S)_{fppf}$. Assume*

1. *F is a sheaf for the Zariski topology on $(Sch/S)_{fppf}$,*
2. *there exists an index set I and subfunctors $F_i \subset F$ such that*
   (a) *each F_i is an fppf sheaf,*
   (b) *each $F_i \to F$ is representable by algebraic spaces,*
   (c) *$\coprod F_i \to F$ becomes surjective after fppf sheafification.*

Then F is an fppf sheaf.

Proof. Let $T \in \mathrm{Ob}((Sch/S)_{fppf})$ and let $s \in F(T)$. By (2)(c) there exists an fppf covering $\{T_j \to T\}$ such that $s|_{T_j}$ is a section of $F_{\alpha(j)}$ for some $\alpha(j) \in I$. Let $W_j \subset T$ be the image of $T_j \to T$ which is an open subscheme Morphisms, Lemma 28.26.9. By (2)(b) we see $F_{\alpha(j)} \times_{F, s|_{W_j}} W_j \to W_j$ is a monomorphism of algebraic spaces through which T_j factors. Since $\{T_j \to W_j\}$ is an fppf covering, we conclude that $F_{\alpha(j)} \times_{F, s|_{W_j}} W_j = W_j$, in other words $s|_{W_j} \in F_{\alpha(j)}(W_j)$. Hence we conclude that $\coprod F_i \to F$ is surjective for the Zariski topology.

Let $\{T_j \to T\}$ be an fppf covering in $(Sch/S)_{fppf}$. Let $s, s' \in F(T)$ with $s|_{T_j} = s'|_{T_j}$ for all j. We want to show that s, s' are equal. As F is a Zariski sheaf by (1) we

may work Zariski locally on T. By the result of the previous paragraph we may assume there exist i such that $s \in F_i(T)$. Then we see that $s'|_{T_j}$ is a section of F_i. By (2)(b) we see $F_i \times_{F,s'} T \to T$ is a monomorphism of algebraic spaces through which all of the T_j factor. Hence we conclude that $s' \in F_i(T)$. Since F_i is a sheaf for the fppf topology we conclude that $s = s'$.

Let $\{T_j \to T\}$ be an fppf covering in $(Sch/S)_{fppf}$ and let $s_j \in F(T_j)$ such that $s_j|_{T_j \times_T T_{j'}} = s_{j'}|_{T_j \times_T T_{j'}}$. By assumption (2)(b) we may refine the covering and assume that $s_j \in F_{\alpha(j)}(T_j)$ for some $\alpha(j) \in I$. Let $W_j \subset T$ be the image of $T_j \to T$ which is an open subscheme Morphisms, Lemma 28.26.9. Then $\{T_j \to W_j\}$ is an fppf covering. Since $F_{\alpha(j)}$ is a sub presheaf of F we see that the two restrictions of s_j to $T_j \times_{W_j} T_j$ agree as elements of $F_{\alpha(j)}(T_j \times_{W_j} T_j)$. Hence, the sheaf condition for $F_{\alpha(j)}$ implies there exists a $s'_j \in F_{\alpha(j)}(W_j)$ whose restriction to T_j is s_j. For a pair of indices j and j' the sections $s'_j|_{W_j \cap W_{j'}}$ and $s'_{j'}|_{W_j \cap W_{j'}}$ of F agree by the result of the previous paragraph. This finishes the proof by the fact that F is a Zariski sheaf. □

61.4. Properties of maps of presheaves representable by algebraic spaces

Here is the definition that makes this work.

Definition 61.4.1. Let S be a scheme. Let $a : F \to G$ be a map of presheaves on $(Sch/S)_{fppf}$ which is representable by algebraic spaces. Let $\mathcal{P}$ be a property of morphisms of algebraic spaces which

(1) is preserved under any base change, and
(2) is fppf local on the base, see Descent on Spaces, Definition 55.9.1.

In this case we say that a has *property* $\mathcal{P}$ if for every scheme U and $\xi : U \to G$ the resulting morphism of algebraic spaces $U \times_G F \to U$ has property $\mathcal{P}$.

It is important to note that we will only use this definition for properties of morphisms that are stable under base change, and local in the fppf topology on the base. This is not because the definition doesn't make sense otherwise; rather it is because we may want to give a different definition which is better suited to the property we have in mind.

The definition above applies for example to the properties of being "surjective", "flat", "étale", and "locally of finite presentation". In other words, a is *surjective* (resp. *étale*, *flat*, *locally of finite presentation*) if for every scheme T and map $\xi : T \to G$ the morphism of algebraic spaces $T \times_{\xi,G} F \to T$ is surjective (resp. étale, flat, locally of finite presentation).

By Lemma 61.3.2 any morphism between algebraic spaces over S is representable by algebraic spaces. And by Morphisms of Spaces, Lemma 49.6.3 (resp. 49.35.2, 49.27.5, 49.26.4) the definition of surjective (resp. étale, flat, locally of finite presentation) above agrees with the already existing definition of morphisms of algebraic spaces.

Some formal lemmas follow.

Lemma 61.4.2. *Let S be a scheme. Let $\mathcal{P}$ be a property as in Definition 61.4.1. Let*

$$\begin{array}{ccc} G' \times_G F & \longrightarrow & F \\ \downarrow{\scriptstyle a'} & & \downarrow{\scriptstyle a} \\ G' & \longrightarrow & G \end{array}$$

be a fibre square of presheaves on $(Sch/S)_{fppf}$. If a is representable by algebraic spaces and has $\mathcal{P}$ so does a'.

Proof. Omitted. Hint: This is formal. □

Lemma 61.4.3. *Let S be a scheme. Let $\mathcal{P}$ be a property as in Definition 61.4.1, and assume $\mathcal{P}$ is stable under composition. Let*

$$F \xrightarrow{a} G \xrightarrow{b} H$$

be maps of presheaves on $(Sch/S)_{fppf}$. If a, b are representable by algebraic spaces and has $\mathcal{P}$ so does $b \circ a$.

Proof. Omitted. Hint: See Lemma 61.3.8 and use stability under composition. □

Lemma 61.4.4. *Let S be a scheme. Let $F_i, G_i : (Sch/S)^{opp}_{fppf} \to Sets$, $i = 1, 2$. Let $a_i : F_i \to G_i$, $i = 1, 2$ be representable by algebraic spaces. Let $\mathcal{P}$ be a property as in Definition 61.4.1 which is stable under composition. If a_1 and a_2 have property $\mathcal{P}$ so does $a_1 \times a_2 : F_1 \times F_2 \longrightarrow G_1 \times G_2$.*

Proof. Note that the lemma makes sense by Lemma 61.3.9. Proof omitted. □

Lemma 61.4.5. *Let S be a scheme. Let $F, G : (Sch/S)^{opp}_{fppf} \to Sets$ be sheaves. Let $a : F \to G$ be representable by algebraic spaces, flat, locally of finite presentation, and surjective. Then $a : F \to G$ is surjective as a map of sheaves.*

Proof. Let T be a scheme over S and let $g : T \to G$ be a T-valued point of G. By assumption $T' = F \times_G T$ is an algebraic space and the morphism $T' \to T$ is a flat, locally of finite presentation, and surjective morphism of algebraic spaces. Let $U \to T'$ be a surjective étale morphism, where U is a scheme. Then by the definition of flat morphisms of algebraic spaces the morphism of schemes $U \to T$ is flat. Similarly for "locally of finite presentation". The morphism $U \to T$ is surjective also, see Morphisms of Spaces, Lemma 49.6.3. Hence we see that $\{U \to T\}$ is an fppf covering such that $g|_U \in G(U)$ comes from an element of $F(U)$, namely the map $U \to T' \to F$. This proves the map is surjective as a map of sheaves, see Sites, Definition 7.12.1. □

61.5. Bootstrapping the diagonal

Lemma 61.5.1. *Let S be a scheme. If F is a presheaf on $(Sch/S)_{fppf}$. The following are equivalent:*

1. *$\Delta_F : F \to F \times F$ is representable by algebraic spaces,*
2. *for every scheme T any map $T \to F$ is representable by algebraic spaces, and*
3. *for every algebraic space X any map $X \to F$ is representable by algebraic spaces.*

Proof. Assume (1). Let $X \to F$ be as in (3). Let T be a scheme, and let $T \to F$ be a morphism. Then we have

$$T \times_F X = (T \times_S X) \times_{F \times F, \Delta} F$$

which is an algebraic space by Lemma 61.3.7 and (1). Hence $X \to F$ is representable, i.e., (3) holds. The implication (3) $\Rightarrow$ (2) is trivial. Assume (2). Let T be a scheme, and let $(a, b) : T \to F \times F$ be a morphism. Then

$$F \times_{\Delta_F, F \times F} T = T \times_{a, F, b} T$$

which is an algebraic space by assumption. Hence Δ_F is representable by algebraic spaces, i.e., (1) holds. □

In particular if F is a presheaf satisfying the equivalent conditions of the lemma, then for any morphism $X \to F$ where X is an algebraic space it makes sense to say that $X \to F$ is surjective (resp. étale, flat, locally of finite presentation) by using Definition 61.4.1.

Before we actually do the bootstrap we prove a fun lemma.

Lemma 61.5.2. *Let S be a scheme. Let*

$$\begin{array}{ccc} E & \xrightarrow{a} & F \\ {\scriptstyle f}\downarrow & & \downarrow{\scriptstyle g} \\ H & \xrightarrow{b} & G \end{array}$$

be a cartesian diagram of sheaves on $(Sch/S)_{fppf}$, so $E = H \times_G F$. If

(1) *g is representable by algebraic spaces, surjective, flat, and locally of finite presentation, and*
(2) *a is representable by algebraic spaces, separated, and locally quasi-finite*

then b is representable (by schemes) as well as separated and locally quasi-finite.

Proof. Let T be a scheme, and let $T \to G$ be a morphism. We have to show that $T \times_G H$ is an algebraic space, and that the morphism $T \times_G H \to T$ is separated and locally quasi-finite. Thus we may base change the whole diagram to T and assume that G is a scheme. In this case F is an algebraic space. Let U be a scheme, and let $U \to F$ be a surjective étale morphism. Then $U \to F$ is representable, surjective, flat and locally of finite presentation by Morphisms of Spaces, Lemmas 49.35.7 and 49.35.8. By Lemma 61.3.8 $U \to G$ is surjective, flat and locally of finite presentation also. Note that the base change $E \times_F U \to U$ of a is still separated and locally quasi-finite (by Lemma 61.4.2). Hence we may replace the upper part of the diagram of the lemma by $E \times_F U \to U$. In other words, we may assume that $F \to G$ is a surjective, flat morphism of schemes which is locally of finite presentation. In particular, $\{F \to G\}$ is an fppf covering of schemes. By Morphisms of Spaces, Proposition 49.43.2 we conclude that E is a scheme also. By Descent, Lemma 34.35.1 the fact that $E = H \times_G F$ means that we get a descent datum on E relative to the fppf covering $\{F \to G\}$. By More on Morphisms, Lemma 36.37.1 this descent datum is effective. By Descent, Lemma 34.35.1 again this implies that H is a scheme. By Descent, Lemmas 34.19.5 and 34.19.22 it now follows that b is separated and locally quasi-finite. □

Here is the result that the section title refers to.

Lemma 61.5.3. *Let S be a scheme. Let $F : (Sch/S)^{opp}_{fppf} \to Sets$ be a functor. Assume that*

(1) *the presheaf F is a sheaf,*
(2) *there exists an algebraic space X and a map $X \to F$ which is representable by algebraic spaces, surjective, flat and locally of finite presentation.*

Then Δ_F is representable (by schemes).

Proof. Let $U \to X$ be a surjective étale morphism from a scheme towards X. Then $U \to X$ is representable, surjective, flat and locally of finite presentation by Morphisms of Spaces, Lemmas 49.35.7 and 49.35.8. By Lemma 61.4.3 the composition $U \to F$ is representable by algebraic spaces, surjective, flat and locally of finite presentation also. Thus we see that $R = U \times_F U$ is an algebraic space, see Lemma 61.3.7. The morphism of algebraic spaces $R \to U \times_S U$ is a monomorphism, hence separated (as the diagonal of a monomorphism is an isomorphism, see Morphisms of Spaces, Lemma 49.11.2). Since $U \to F$ is locally of finite presentation, both morphisms $R \to U$ are locally of finite presentation, see Lemma 61.4.2. Hence $R \to U \times_S U$ is locally of finite type (use Morphisms of Spaces, Lemmas 49.26.5 and 49.22.6). Altogether this means that $R \to U \times_S U$ is a monomorphism which is locally of finite type, hence a separated and locally quasi-finite morphism, see Morphisms of Spaces, Lemma 49.25.8.

Now we are ready to prove that Δ_F is representable. Let T be a scheme, and let $(a, b) : T \to F \times F$ be a morphism. Set

$$T' = (U \times_S U) \times_{F \times F} T.$$

Note that $U \times_S U \to F \times F$ is representable by algebraic spaces, surjective, flat and locally of finite presentation by Lemma 61.4.4. Hence T' is an algebraic space, and the projection morphism $T' \to T$ is surjective, flat, and locally of finite presentation. Consider $Z = T \times_{F \times F} F$ (this is a sheaf) and

$$Z' = T' \times_{U \times_S U} R = T' \times_T Z.$$

We see that Z' is an algebraic space, and $Z' \to T'$ is separated and locally quasi-finite by the discussion in the first paragraph of the proof which showed that R is an algebraic space and that the morphism $R \to U \times_S U$ has those properties. Hence we may apply Lemma 61.5.2 to the diagram

$$\begin{array}{ccc} Z' & \longrightarrow & T' \\ \downarrow & & \downarrow \\ Z & \longrightarrow & T \end{array}$$

and we conclude. □

61.6. Bootstrap

We warn the reader right away that the result of this section will be superseded by the stronger Theorem 61.10.1. On the other hand, the theorem in this section is quite a bit easier to prove and still provides quite a bit of insight into how things work, especially for those readers mainly interested in Deligne-Mumford stacks.

In Spaces, Section 47.6 we defined an algebraic space as a sheaf in the fppf topology whose diagonal is representable, and such that there exist a surjective étale

morphism from a scheme towards it. In this section we show that a sheaf in the fppf topology whose diagonal is representable by algebraic spaces and which has an étale surjective covering by an algebraic space is also an algebraic space. In other words, the category of algebraic spaces is an enlargement of the category of schemes by those fppf sheaves F which have a representable diagonal and an étale covering by a scheme. The result of this section says that doing the same process again starting with the category of algebraic spaces, does not lead to yet another category.

Another motivation for the material in this section is that it will guarantee later that a Deligne-Mumford stack whose inertia stack is trivial is equivalent to an algebraic space, see Algebraic Stacks, Lemma 67.13.2.

Here is the main result of this section (as we mentioned above this will be superseded by the stronger Theorem 61.10.1).

Theorem 61.6.1. *Let S be a scheme. Let $F : (Sch/S)^{opp}_{fppf} \to Sets$ be a functor. Assume that*

1. *the presheaf F is a sheaf,*
2. *the diagonal morphism $F \to F \times F$ is representable by algebraic spaces, and*
3. *there exists an algebraic space X and a map $X \to F$ which is surjective, and étale.*

Then F is an algebraic space.

Proof. We will use the remarks directly below Definition 61.4.1 without further mention. In the situation of the theorem, let $U \to X$ be a surjective étale morphism from a scheme towards X. By Lemma 61.3.8 $U \to F$ is surjective and étale also. Hence the theorem boils down to proving that Δ_F is representable. This follows immediately from Lemma 61.5.3. On the other hand we can circumvent this lemma and show directly F is an algebraic space as in the next paragraph.

Let U be a scheme, and let $U \to F$ be surjective and étale. Set $R = U \times_F U$, which is an algebraic space (see Lemma 61.5.1). The morphism of algebraic spaces $R \to U \times_S U$ is a monomorphism, hence separated (as the diagonal of a monomorphism is an isomorphism). Moreover, since $U \to F$ is étale, we see that $R \to U$ is étale, by Lemma 61.4.2. In particular, we see that $R \to U$ is locally quasi-finite, see Morphisms of Spaces, Lemma 49.35.5. We conclude that also $R \to U \times_S U$ is locally quasi-finite by Morphisms of Spaces, Lemma 49.25.7. Hence Morphisms of Spaces, Proposition 49.43.2 applies and R is a scheme. Hence $F = U/R$ is an algebraic space according to Spaces, Theorem 47.10.5. □

61.7. Finding opens

First we prove a lemma which is a slight improvement and generalization of Spaces, Lemma 47.10.2 to quotient sheaves associated to groupoids.

Lemma 61.7.1. *Let S be a scheme. Let (U, R, s, t, c) be a groupoid scheme over S. Let $g : U' \to U$ be a morphism. Assume*

(1) *the composition*

$$U' \times_{g,U,t} R \xrightarrow{h} R \xrightarrow[pr_1]{} R \xrightarrow[s]{} U$$

has an open image $W \subset U$, and
(2) *the resulting map $h : U' \times_{g,U,t} R \to W$ defines a surjection of sheaves in the fppf topology.*

Let $R' = R|_{U'}$ be the restriction of R to U. Then the map of quotient sheaves

$$R'/U' \to R/U$$

in the fppf topology is representable, and is an open immersion.

Proof. Note that W is an R-invariant open subscheme of U. This is true because the set of points of W is the set of points of U which are equivalent in the sense of Groupoids, Lemma 38.3.4 to a point of $g(U') \subset U$ (the lemma applies as $j : R \to U \times_S U$ is a pre-equivalence relation by Groupoids, Lemma 38.11.2). Also $g : U' \to U$ factors through W. Let $R|_W$ be the restriction of R to W. Then it follows that R' is also the restriction of $R|_W$ to U'. Hence we can factor the map of sheaves of the lemma as

$$U'/R' \longrightarrow W/R|_W \longrightarrow U/R$$

By Groupoids, Lemma 38.18.6 we see that the first arrow is an isomorphism of sheaves. Hence it suffices to show the lemma in case g is the immersion of an R-invariant open into U.

Assume $U' \subset U$ is an R-invariant open and g is the inclusion morphism. Set $F = U/R$ and $F' = U'/R'$. By Groupoids, Lemma 38.18.5 or 38.18.6 the map $F' \to F$ is injective. Let $\xi \in F(T)$. We have to show that $T \times_{\xi,F} F'$ is representable by an open subscheme of T. There exists an fppf covering $\{f_i : T_i \to T\}$ such that $\xi|_{T_i}$ is the image via $U \to U/R$ of a morphism $a_i : T_i \to U$. Set $V_i = s_i^{-1}(U')$. We claim that $V_i \times_T T_j = T_i \times_T V_j$ as open subschemes of $T_i \times_T T_j$.

As $a_i \circ \mathrm{pr}_0$ and $a_j \circ \mathrm{pr}_1$ are morphisms $T_i \times_T T_j \to U$ which both map to the section $\xi|_{T_i \times_T T_j} \in F(T_i \times_T T_j)$ we can find an fppf covering $\{f_{ijk} : T_{ijk} \to T_i \times_T T_j\}$ and morphisms $r_{ijk} : T_{ijk} \to R$ such that

$$a_i \circ \mathrm{pr}_0 \circ f_{ijk} = s \circ r_{ijk}, \quad a_j \circ \mathrm{pr}_1 \circ f_{ijk} = t \circ r_{ijk},$$

see Groupoids, Lemma 38.18.4. Since U' is R-invariant we have $s^{-1}(U') = t^{-1}(U')$ and hence $f_{ijk}^{-1}(V_i \times_T T_j) = f_{ijk}^{-1}(T_i \times_T V_j)$. As $\{f_{ijk}\}$ is surjective this implies the claim above. Hence by Descent, Lemma 34.9.2 there exists an open subscheme $V \subset T$ such that $f_i^{-1}(V) = V_i$. We claim that V represents $T \times_{\xi,F} F'$.

As a first step, we will show that $\xi|_V$ lies in $F'(V) \subset F(V)$. Namely, the family of morphisms $\{V_i \to V\}$ is an fppf covering, and by construction we have $\xi|_{V_i} \in F'(V_i)$. Hence by the sheaf property of F' we get $\xi|_V \in F'(V)$. Finally, let $T' \to T$ be a morphism of schemes and that $\xi|_{T'} \in F'(T')$. To finish the proof we have to show that $T' \to T$ factors through V. We can find a fppf covering $\{T'_j \to T'\}_{j \in J}$ and morphisms $b_j : T'_j \to U'$ such that $\xi|_{T'_j}$ is the image via $U' \to U/R$ of b_j. Clearly, it is enough to show that the compositions $T'_j \to T$ factor through V. Hence we may assume that $\xi|_{T'}$ is the image of a morphism $b : T' \to U'$. Now, it is enough to show that $T' \times_T T_i \to T_i$ factors through V_i. Over the scheme $T' \times_T T_i$ the

restriction of ξ is the image of two elements of $(U/R)(T' \times_T T_i)$, namely $a_i \circ \mathrm{pr}_1$, and $b \circ \mathrm{pr}_0$, the second of which factors through the R-invariant open U'. Hence by Groupoids, Lemma 38.18.4 there exists a covering $\{h_k : Z_k \to T' \times_T T_i\}$ and morphisms $r_k : Z_k \to R$ such that $a_i \circ \mathrm{pr}_1 \circ h_k = s \circ r_k$ and $b \circ \mathrm{pr}_0 \circ h_k = t \circ r_k$. As U' is an R-invariant open the fact that b has image in U' then implies that each $a_i \circ \mathrm{pr}_1 \circ h_k$ has image in U'. It follows from this that $T' \times_T T_i \to T_i$ has image in V_i by definition of V_i which concludes the proof. □

61.8. Slicing equivalence relations

In this section we explain how to "improve" a given equivalence relation by slicing. This is not a kind of "étale slicing" that you may be used to but a much coarser kind of slicing.

Lemma 61.8.1. *Let S be a scheme. Let $j : R \to U \times_S U$ be an equivalence relation on schemes over S. Assume $s, t : R \to U$ are flat and locally of finite presentation. Then there exists an equivalence relation $j' : R' \to U' \times_S U'$ on schemes over S, and an isomorphism*

$$U'/R' \longrightarrow U/R$$

induced by a morphism $U' \to U$ which maps R' into R such that $s', t' : R \to U$ are flat, locally of finite presentation and locally quasi-finite.

Proof. We will prove this lemma in several steps. We will use without further mention that an equivalence relation gives rise to a groupoid scheme and that the restriction of an equivalence relation is an equivalence relation, see Groupoids, Lemmas 38.3.2, 38.11.3, and 38.16.3.

Step 1: We may assume that $s, t : R \to U$ are locally of finite presentation and Cohen-Macaulay morphisms. Namely, as in More on Groupoids, Lemma 39.7.1 let $g : U' \to U$ be the open subscheme such that $t^{-1}(U') \subset R$ is the maximal open over which $s : R \to U$ is Cohen-Macaulay, and denote R' the restriction of R to U'. By the lemma cited above we see that

$$t^{-1}(U') = U' \times_{g,U,t} R \xrightarrow[\mathrm{pr}_1]{} R \xrightarrow[s]{} U, \quad \text{with } h : U' \times_{g,U,t} R \to U$$

is surjective. Since h is flat and locally of finite presentation, we see that $\{h\}$ is a fppf covering. Hence by Groupoids, Lemma 38.18.6 we see that $U'/R' \to U/R$ is an isomorphism. By the construction of U' we see that s', t' are Cohen-Macaulay and locally of finite presentation.

Step 2. Assume s, t are Cohen-Macaulay and locally of finite presentation. Let $u \in U$ be a point of finite type. By More on Groupoids, Lemma 39.11.4 there exists an affine scheme U' and a morphism $g : U' \to U$ such that

(1) g is an immersion,
(2) $u \in U'$,
(3) g is locally of finite presentation,
(4) h is flat, locally of finite presentation and locally quasi-finite, and
(5) the morphisms $s', t' : R' \to U'$ are flat, locally of finite presentation and locally quasi-finite.

Here we have used the notation introduced in More on Groupoids, Situation 39.11.1.

Step 3. For each point $u \in U$ which is of finite type choose a $g_u : U'_u \to U$ as in Step 2 and denote R'_u the restriction of R to U'_u. Denote $h_u = s \circ \mathrm{pr}_1 : U'_u \times_{g_u,U,t} R \to U$. Set $U' = \coprod_{u \in U} U'_u$, and $g = \coprod g_u$. Let R' be the restriction of R to U as above. We claim that the pair (U', g) works[1]. Note that

$$\begin{aligned} R' &= \coprod\nolimits_{u_1,u_2 \in U} (U'_{u_1} \times_{g_{u_1},U,t} R) \times_R (R \times_{s,U,g_{u_2}} U'_{u_2}) \\ &= \coprod\nolimits_{u_1,u_2 \in U} (U'_{u_1} \times_{g_{u_1},U,t} R) \times_{h_{u_1},U,g_{u_2}} U'_{u_2} \end{aligned}$$

Hence the projection $s' : R' \to U' = \coprod U'_{u_2}$ is flat, locally of finite presentation and locally quasi-finite as a base change of $\coprod h_{u_1}$. Finally, by construction the morphism $h : U' \times_{g,U,t} R \to U$ is equal to $\coprod h_u$ hence its image contains all points of finite type of U. Since each h_u is flat and locally of finite presentation we conclude that h is flat and locally of finite presentation. In particular, the image of h is open (see Morphisms, Lemma 28.26.9) and since the set of points of finite type is dense (see Morphisms, Lemma 28.17.7) we conclude that the image of h is U. This implies that $\{h\}$ is an fppf covering. By Groupoids, Lemma 38.18.6 this means that $U'/R' \to U/R$ is an isomorphism. This finishes the proof of the lemma. □

61.9. Quotient by a subgroupoid

We need one more lemma before we can do our final bootstrap. Let us discuss what is going on in terms of "plain" groupoids before embarking on the scheme theoretic version.

Let $\mathcal{C}$ be a groupoid, see Categories, Definition 4.2.5. As discussed in Groupoids, Section 38.11 this corresponds to a quintuple $(\mathrm{Ob}, \mathrm{Arrows}, s, t, c)$. Suppose we are given a subset $P \subset \mathrm{Arrows}$ such that $(\mathrm{Ob}, P, s|_P, t|_P, c|_P)$ is also a groupoid and such that there are no nontrivial automorphisms in P. Then we can construct the quotient groupoid $(\overline{\mathrm{Ob}}, \overline{\mathrm{Arrows}}, \overline{s}, \overline{t}, \overline{c})$ as follows:

(1) $\overline{\mathrm{Ob}} = \mathrm{Ob}/P$ is the set of P-isomorphism classes,
(2) $\overline{\mathrm{Arrows}} = P\backslash\mathrm{Arrows}/P$ is the set of arrows in $\mathcal{C}$ up to pre-composing and post-composing by arrows of P,
(3) the source and target maps $\overline{s}, \overline{t} : P\backslash\mathrm{Arrows}/P \to \mathrm{Ob}/P$ are induced by s, t,
(4) composition is defined by the rule $\overline{c}(\overline{a}, \overline{b}) = \overline{c(a, b)}$ which is well defined.

In fact, it turns out that the original groupoid $(\mathrm{Ob}, \mathrm{Arrows}, s, t, c)$ is canonically isomorphic to the restriction (see discussion in Groupoids, Section 38.16) of the groupoid $(\overline{\mathrm{Ob}}, \overline{\mathrm{Arrows}}, \overline{s}, \overline{t}, \overline{c})$ via the quotient map $g : \mathrm{Ob} \to \overline{\mathrm{Ob}}$. Recall that this means that

$$\mathrm{Arrows} = \mathrm{Ob} \times_{g,\overline{\mathrm{Ob}},\overline{t}} \overline{\mathrm{Arrows}} \times_{\overline{s},\overline{\mathrm{Ob}},g} \mathrm{Ob}$$

which holds as P has no nontrivial automorphisms. We omit the details.

[1]Here we should check that U' is not too large, i.e., that it is isomorphic to an object of the category Sch_{fppf}, see Section 61.2. This is a purely set theoretical matter; let us use the notion of size of a scheme introduced in Sets, Section 3.9. Note that each U'_u has size at most the size of U and that the cardinality of the index set is at most the cardinality of $|U|$ which is bounded by the size of U. Hence U' is isomorphic to an object of Sch_{fppf} by Sets, Lemma 3.9.9 part (6).

The following lemma holds in much greater generality, but this is the version we use in the proof of the final bootstrap (after which we can more easily prove the more general versions of this lemma).

Lemma 61.9.1. *Let S be a scheme. Let (U, R, s, t, c) be a groupoid scheme over S. Let $P \to R$ be monomorphism of schemes. Assume that*

(1) *$(U, P, s|_P, t|_P, c|_{P \times_{s,U,t} P})$ is a groupoid scheme,*
(2) *$s|_P, t|_P : P \to U$ are finite locally free,*
(3) *$j|_P : P \to U \times_S U$ is a monomorphism.*
(4) *U is affine, and*
(5) *$j : R \to U \times_S U$ is separated and locally quasi-finite,*

Then U/P is representable by an affine scheme $\overline{U}$, the quotient morphism $U \to \overline{U}$ is finite locally free, and $P = U \times_{\overline{U}} U$. Moreover, R is the restriction of a groupoid scheme $(\overline{U}, \overline{R}, \overline{s}, \overline{t}, \overline{c})$ on $\overline{U}$ via the quotient morphism $U \to \overline{U}$.

Proof. Conditions (1), (2), (3), and (4) and Groupoids, Proposition 38.20.8 imply the affine scheme $\overline{U}$ representing U/P exists, the morphism $U \to \overline{U}$ is finite locally free, and $P = U \times_{\overline{U}} U$. The identification $P = U \times_{\overline{U}} U$ is such that $t|_P = \mathrm{pr}_0$ and $s|_P = \mathrm{pr}_1$, and such that composition is equal to $\mathrm{pr}_{02} : U \times_{\overline{U}} U \times_{\overline{U}} U \to U \times_{\overline{U}} U$. A product of finite locally free morphisms is finite locally free (see Spaces, Lemma 47.5.7 and Morphisms, Lemmas 28.46.4 and 28.46.3). To get $\overline{R}$ we are going to descend the scheme R via the finite locally free morphism $U \times_S U \to \overline{U} \times_S \overline{U}$. Namely, note that

$$(U \times_S U) \times_{(\overline{U} \times_S \overline{U})} (U \times_S U) = P \times_S P$$

by the above. Thus giving a descent datum (see Descent, Definition 34.30.1) for $R/U \times_S U/\overline{U} \times_S \overline{U}$ consists of an isomorphism

$$\varphi : R \times_{(U \times_S U), t \times t} (P \times_S P) \longrightarrow (P \times_S P) \times_{s \times s, (U \times_S U)} R$$

over $P \times_S P$ satisfying a cocycle condition. We define φ on T-valued points by the rule

$$\varphi : (r, (p, p')) \longmapsto ((p, p'), p^{-1} \circ r \circ p')$$

where the composition is taken in the groupoid category $(U(T), R(T), s, t, c)$. This makes sense because for $(r, (p, p'))$ to be a T-valued point of the source of φ it needs to be the case that $t(r) = t(p)$ and $s(r) = t(p')$. Note that this map is an isomorphism with inverse given by $((p, p'), r') \mapsto (p \circ r' \circ (p')^{-1}, (p, p'))$. To check the cocycle condition we have to verify that $\varphi_{02} = \varphi_{12} \circ \varphi_{01}$ as maps over

$$(U \times_S U) \times_{(\overline{U} \times_S \overline{U})} (U \times_S U) \times_{(\overline{U} \times_S \overline{U})} (U \times_S U) = (P \times_S P) \times_{s \times s, (U \times_S U), t \times t} (P \times_S P)$$

By explicit calculation we see that

$$\begin{array}{lllll} \varphi_{02} & (r, (p_1, p_1'), (p_2, p_2')) & \mapsto & ((p_1, p_1'), (p_2, p_2'), (p_1 \circ p_2)^{-1} \circ r \circ (p_1' \circ p_2')) \\ \varphi_{01} & (r, (p_1, p_1'), (p_2, p_2')) & \mapsto & ((p_1, p_1'), p_1^{-1} \circ r \circ p_1', (p_2, p_2')) \\ \varphi_{12} & ((p_1, p_1'), r, (p_2, p_2')) & \mapsto & ((p_1, p_1'), (p_2, p_2'), p_2^{-1} \circ r \circ p_2') \end{array}$$

(with obvious notation) which implies what we want. As j is separated and locally quasi-finite by (5) we may apply More on Morphisms, Lemma 36.37.1 to get a scheme $\overline{R} \to \overline{U} \times_S \overline{U}$ and an isomorphism

$$R \to \overline{R} \times_{(\overline{U} \times_S \overline{U})} (U \times_S U)$$

which identifies the descent datum φ with the canonical descent datum on $\overline{R} \times_{(\overline{U} \times_S \overline{U})} (U \times_S U)$, see Descent, Definition 34.30.10.

Since $U \times_S U \to \overline{U} \times_S \overline{U}$ is finite locally free we conclude that $R \to \overline{R}$ is finite locally free as a base change. Hence $R \to \overline{R}$ is surjective as a map of sheaves on $(Sch/S)_{fppf}$. Our choice of φ implies that given T-valued points $r, r' \in R(T)$ these have the same image in $\overline{R}$ if and only if $p^{-1} \circ r \circ p'$ for some $p, p' \in P(T)$. Thus $\overline{R}$ represents the sheaf

$$T \longmapsto \overline{R(T)} = P(T) \backslash R(T) / P(T)$$

with notation as in the discussion preceding the lemma. Hence we can define the groupoid structure on $(\overline{U} = U/P, \overline{R} = P \backslash R / P)$ exactly as in the discussion of the "plain" groupoid case. It follows from this that (U, R, s, t, c) is the pullback of this groupoid structure via the morphism $U \to \overline{U}$. This concludes the proof. □

61.10. Final bootstrap

The results in this section go quite a bit beyond the earlier results.

Theorem 61.10.1. *Let S be a scheme. Let $F : (Sch/S)^{opp}_{fppf} \to Sets$ be a functor. Any one of the following conditions implies that F is an algebraic space:*

1. *$F = U/R$ where (U, R, s, t, c) is a groupoid in algebraic spaces over S such that s, t are flat and locally of finite presentation, and $j = (t, s) : R \to U \times_S U$ is an equivalence relation,*
2. *$F = U/R$ where (U, R, s, t, c) is a groupoid scheme over S such that s, t are flat and locally of finite presentation, and $j = (t, s) : R \to U \times_S U$ is an equivalence relation,*
3. *F is a sheaf and there exists an algebraic space U and a morphism $U \to F$ which is which is representable by algebraic spaces, surjective, flat and locally of finite presentation,*
4. *F is a sheaf and there exists a scheme U and a morphism $U \to F$ which is which is representable (by algebraic spaces or schemes), surjective, flat and locally of finite presentation,*
5. *F is a sheaf, Δ_F is representable by algebraic spaces, and there exists an algebraic space U and a morphism $U \to F$ which is surjective, flat, and locally of finite presentation, or*
6. *F is a sheaf, Δ_F is representable, and there exists a scheme U and a morphism $U \to F$ which is surjective, flat, and locally of finite presentation.*

Proof. Trivial observations: (6) is a special case of (5) and (4) is a special case of (3). We first prove that cases (5) and (3) reduce to case (1). Namely, by bootstrapping the diagonal Lemma 61.5.3 we see that (3) implies (5). In case (5) we set $R = U \times_F U$ which is an algebraic space by assumption. Moreover, by assumption both projections $s, t : R \to U$ are surjective, flat and locally of finite presentation. The map $j : R \to U \times_S U$ is clearly an equivalence relation. By Lemma 61.4.5 the map $U \to F$ is a surjection of sheaves. Thus $F = U/R$ which reduces us to case (1).

Next, we show that (1) reduces to (2). Namely, let (U, R, s, t, c) be a groupoid in algebraic spaces over S such that s, t are flat and locally of finite presentation, and $j = (t, s) : R \to U \times_S U$ is an equivalence relation. Choose a scheme U' and a surjective étale morphism $U' \to U$. Let $R' = R|_{U'}$ be the restriction of R

to U'. By Groupoids in Spaces, Lemma 59.18.6 we see that $U/R = U'/R'$. Since $s', t' : R' \to U'$ are also flat and locally of finite presentation (see More on Groupoids in Spaces, Lemma 60.6.1) this reduces us to the case where U is a scheme. As j is an equivalence relation we see that j is a monomorphism. As $s : R \to U$ is locally of finite presentation we see that $j : R \to U \times_S U$ is locally of finite type, see Morphisms of Spaces, Lemma 49.22.6. By Morphisms of Spaces, Lemma 49.25.8 we see that j is locally quasi-finite and separated. Hence if U is a scheme, then R is a scheme by Morphisms of Spaces, Proposition 49.43.2. Thus we reduce to proving the theorem in case (2).

Assume $F = U/R$ where (U, R, s, t, c) is a groupoid scheme over S such that s, t are flat and locally of finite presentation, and $j = (t, s) : R \to U \times_S U$ is an equivalence relation. By Lemma 61.8.1 we reduce to that case where s, t are flat, locally of finite presentation, and locally quasi-finite. Let $U = \bigcup_{i \in I} U_i$ be an affine open covering (with index set I of cardinality $\leq$ than the size of U to avoid set theoretic problems later – most readers can safely ignore this remark). Let $(U_i, R_i, s_i, t_i, c_i)$ be the restriction of R to U_i. It is clear that s_i, t_i are still flat, locally of finite presentation, and locally quasi-finite as R_i is the open subscheme $s^{-1}(U_i) \cap t^{-1}(U_i)$ of R and s_i, t_i are the restrictions of s, t to this open. By Lemma 61.7.1 (or the simpler Spaces, Lemma 47.10.2) the map $U_i/R_i \to U/R$ is representable by open immersions. Hence if we can show that $F_i = U_i/R_i$ is an algebraic space, then $\coprod_{i \in I} F_i$ is an algebraic space by Spaces, Lemma 47.8.3. As $U = \bigcup U_i$ is an open covering it is clear that $\coprod F_i \to F$ is surjective. Thus it follows that U/R is an algebraic space, by Spaces, Lemma 47.8.4. In this way we reduce to the case where U is affine and s, t are flat, locally of finite presentation, and locally quasi-finite and j is an equivalence.

Assume (U, R, s, t, c) is a groupoid scheme over S, with U affine, such that s, t are flat, locally of finite presentation, and locally quasi-finite, and j is an equivalence relation. Choose $u \in U$. We apply More on Groupoids in Spaces, Lemma 60.11.9 to $u \in U, R, s, t, c$. We obtain an affine scheme U', an étale morphism $g : U' \to U$, a point $u' \in U'$ with $\kappa(u) = \kappa(u')$ such that the restriction $R' = R|_{U'}$ is quasi-split over u'. Note that the image $g(U')$ is open as g is étale and contains u'. Hence, repeatedly applying the lemma, we can find finitely many points $u_i \in U$, $i = 1, \ldots, n$, affine schemes U'_i, étale morphisms $g_i : U'_i \to U$, points $u'_i \in U'_i$ with $g(u'_i) = u_i$ such that (a) each restriction R'_i is quasi-split over some point in U'_i and (b) $U = \bigcup_{i=1,\ldots,n} g_i(U'_i)$. Now we rerun the last part of the argument in the preceding paragraph: Using Lemma 61.7.1 (or the simpler Spaces, Lemma 47.10.2) the map $U'_i/R'_i \to U/R$ is representable by open immersions. If we can show that $F_i = U'_i/R'_i$ is an algebraic space, then $\coprod_{i \in I} F_i$ is an algebraic space by Spaces, Lemma 47.8.3. As $\{g_i : U'_i \to U\}$ is an étale covering it is clear that $\coprod F_i \to F$ is surjective. Thus it follows that U/R is an algebraic space, by Spaces, Lemma 47.8.4. In this way we reduce to the case where U is affine and s, t are flat, locally of finite presentation, and locally quasi-finite, j is an equivalence, and R is quasi-split over u for some $u \in U$.

Assume (U, R, s, t, c) is a groupoid scheme over S, with U affine, $u \in U$ such that s, t are flat, locally of finite presentation, and locally quasi-finite and $j = (t, s) : R \to U \times_S U$ is an equivalence relation and R is quasi-split over u. Let $P \subset R$ be a quasi-splitting of R over u. By Lemma 61.9.1 we see that (U, R, s, t, c) is the

restriction of a groupoid $(\overline{U}, \overline{R}, \overline{s}, \overline{t}, \overline{c})$ by a surjective finite locally free morphism $U \to \overline{U}$ such that $P = U \times_{\overline{U}} U$. Note that s, t are the base changes of the morphisms $\overline{s}, \overline{t}$ by $U \to \overline{U}$. As $\{U \to \overline{U}\}$ is an fppf covering we conclude $\overline{s}, \overline{t}$ are flat, locally of finite presentation, and locally quasi-finite, see Descent, Lemmas 34.19.13, 34.19.9, and 34.19.22. Consider the commutative diagram

$$\begin{array}{ccccc} U \times_{\overline{U}} U & = & P & \longrightarrow & R \\ & \searrow & \downarrow & & \downarrow \\ & & \overline{U} & \xrightarrow{\overline{e}} & \overline{R} \end{array}$$

It is a general fact about restrictions that the outer four corners form a cartesian diagram. By the equality we see the inner square is cartesian. Since P is open in R (by definition of a quasi-splitting) we conclude that $\overline{e}$ is an open immersion by Descent, Lemma 34.19.14. An application of Groupoids, Lemma 38.18.5 shows that $U/R = \overline{U}/\overline{R}$. Hence we have reduced to the case where (U, R, s, t, c) is a groupoid scheme over S, with U affine, $u \in U$ such that s, t are flat, locally of finite presentation, and locally quasi-finite and $j = (t, s) : R \to U \times_S U$ is an equivalence relation and $e : U \to R$ is an open immersion!

But of course, if e is an open immersion and s, t are flat and locally of finite presentation then the morphisms t, s are étale. For example you can see this by applying More on Groupoids, Lemma 39.4.1 which shows that $\Omega_{R/U} = 0$ which in turn implies that $s, t : R \to U$ is G-unramified (see Morphisms, Lemma 28.36.2), which in turn implies that s, t are étale (see Morphisms, Lemma 28.37.16). And if s, t are étale then finally U/R is an algebraic space by Spaces, Theorem 47.10.5. □

61.11. Applications

As a first application we obtain the following fundamental fact:

A sheaf which is fppf locally an algebraic space is an algebraic space.

This is the content of the following lemma. Note that assumption (2) is equivalent to the condition that $F|_{(Sch/S_i)_{fppf}}$ is an algebraic space, see Spaces, Lemma 47.16.4. Assumption (3) is a set theoretic condition which may be ignored by those not worried about set theoretic questions.

Lemma 61.11.1. *Let S be a scheme. Let $F : (Sch/S)^{opp}_{fppf} \to Sets$ be a functor. Let $\{S_i \to S\}_{i \in I}$ be a covering of $(Sch/S)_{fppf}$. Assume that*

(1) *F is a sheaf,*
(2) *each $F_i = h_{S_i} \times F$ is an algebraic space, and*
(3) *$\coprod_{i \in I} F_i$ is an algebraic space (see Spaces, Lemma 47.8.3).*

Then F is an algebraic space.

Proof. Consider the morphism $\coprod F_i \to F$. This is the base change of $\coprod S_i \to S$ via $F \to S$. Hence it is representable, locally of finite presentation, flat and surjective by our definition of an fppf covering and Lemma 61.4.2. Thus Theorem 61.10.1 applies to show that F is an algebraic space. □

Here is a special case where we do not need to worry about set theoretical issues.

Lemma 61.11.2. *Let S be a scheme. Let $F : (Sch/S)^{opp}_{fppf} \to Sets$ be a functor. Let $\{S_i \to S\}_{i \in I}$ be a covering of $(Sch/S)_{fppf}$. Assume that*

(1) *F is a sheaf,*
(2) *each $F_i = h_{S_i} \times F$ is an algebraic space, and*
(3) *the morphisms $F_i \to S_i$ are of finite type.*

Then F is an algebraic space.

Proof. We will use Lemma 61.11.1 above. To do this we will show that the assumption that F_i is of finite type over S_i to prove that the set theoretic condition in the lemma is satisfied (after perhaps refining the given covering of S a bit). We suggest the reader skip the rest of the proof.

If $S'_i \to S_i$ is a morphism of schemes then

$$h_{S'_i} \times F = h_{S'_i} \times_{h_{S_i}} h_{S_i} \times F = h_{S'_i} \times_{h_{S_i}} F_i$$

is an algebraic space of finite type over S'_i, see Spaces, Lemma 47.7.3 and Morphisms of Spaces, Lemma 49.22.3. Thus we may refine the given covering. After doing this we may assume: (a) each S_i is affine, and (b) the cardinality of I is at most the cardinality of the set of points of S. (Since to cover all of S it is enough that each point is in the image of $S_i \to S$ for some i.)

Since each S_i is affine and each F_i of finite type over S_i we conclude that F_i is quasi-compact. Hence by Properties of Spaces, Lemma 48.6.3 we can find an affine $U_i \in \mathrm{Ob}((Sch/S)_{fppf})$ and a surjective étale morphism $U_i \to F_i$. The fact that $F_i \to S_i$ is locally of finite type then implies that $U_i \to S_i$ is locally of finite type, and in particular $U_i \to S$ is locally of finite type. By Sets, Lemma 3.9.7 we conclude that $\mathrm{size}(U_i) \leq \mathrm{size}(S)$. Since also $|I| \leq \mathrm{size}(S)$ we conclude that $\coprod_{i \in I} U_i$ is isomorphic to an object of $(Sch/S)_{fppf}$ by Sets, Lemma 3.9.5 and the construction of *Sch*. This implies that $\coprod F_i$ is an algebraic space by Spaces, Lemma 47.8.3 and we win. □

Lemma 61.11.3. *Assume $B \to S$ and (U, R, s, t, c) are as in Groupoids in Spaces, Definition 59.19.1 (1). For any scheme T over S and objects x, y of $[U/R]$ over T the sheaf $Isom(x, y)$ on $(Sch/T)_{fppf}$ is an algebraic space.*

Proof. By Groupoids in Spaces, Lemma 59.21.3 there exists an fppf covering $\{T_i \to T\}_{i \in I}$ such that $Isom(x, y)|_{(Sch/T_i)_{fppf}}$ is an algebraic space for each i. By Spaces, Lemma 47.16.4 this means that each $F_i = h_{S_i} \times Isom(x, y)$ is an algebraic space. Thus to prove the lemma we only have to verify the set theoretic condition that $\coprod F_i$ is an algebraic space of Lemma 61.11.1 above to conclude. To do this we use Spaces, Lemma 47.8.3 which requires showing that I and the F_i are not "too large". We suggest the reader skip the rest of the proof.

Choose $U' \in \mathrm{Ob}(Sch/S)_{fppf}$ and a surjective étale morphism $U' \to U$. Let R' be the restriction of R to U'. Since $[U/R] = [U'/R']$ we may, after replacing U by U', assume that U is a scheme. (This step is here so that the fibre products below are over a scheme.)

Note that if we refine the covering $\{T_i \to T\}$ then it remains true that each F_i is an algebraic space. Hence we may assume that each T_i is affine. Since $T_i \to T$ is locally of finite presentation, this then implies that $\mathrm{size}(T_i) \leq \mathrm{size}(T)$, see Sets, Lemma 3.9.7. We may also assume that the cardinality of the index set I is at most the cardinality of the set of points of T since to get a covering it suffices to check that each point of T is in the image. Hence $|I| \leq \mathrm{size}(T)$. Choose $W \in \mathrm{Ob}((Sch/S)_{fppf})$ and a surjective étale morphism $W \to R$. Note that in the proof of Groupoids in

Spaces, Lemma 59.21.3 we showed that F_i is representable by $T_i \times_{(y_i,x_i),U\times_B U} R$ for some $x_i, y_i : T_i \to U$. Hence now we see that $V_i = T_i \times_{(y_i,x_i),U\times_B U} W$ is a scheme which comes with an étale surjection $V_i \to F_i$. By Sets, Lemma 3.9.6 we see that

$$\text{size}(V_i) \leq \max\{\text{size}(T_i), \text{size}(W)\} \leq \max\{\text{size}(T), \text{size}(W)\}$$

Hence, by Sets, Lemma 3.9.5 we conclude that

$$\text{size}(\coprod\nolimits_{i\in I} V_i) \leq \max\{|I|, \text{size}(T), \text{size}(W)\}.$$

Hence we conclude by our construction of *Sch* that $\coprod_{i\in I} V_i$ is isomorphic to an object V of $(Sch/S)_{fppf}$. This verifies the hypothesis of Spaces, Lemma 47.8.3 and we win. □

Lemma 61.11.4. *Let S be a scheme. Consider an algebraic space F of the form $F = U/R$ where (U, R, s, t, c) is a groupoid in algebraic spaces over S such that s, t are flat and locally of finite presentation, and $j = (t, s) : R \to U \times_S U$ is an equivalence relation. Then $U \to F$ is surjective, flat, and locally of finite presentation.*

Proof. This is almost but not quite a triviality. Namely, by Groupoids in Spaces, Lemma 59.18.5 and the fact that j is a monomorphism we see that $R = U \times_F U$. Choose a scheme W and a surjective étale morphism $W \to F$. As $U \to F$ is a surjection of sheaves we can find an fppf covering $\{W_i \to W\}$ and maps $W_i \to U$ lifting the morphisms $W_i \to F$. Then we see that

$$W_i \times_F U = W_i \times_U U \times_F U = W_i \times_{U,t} R$$

and the projection $W_i \times_F U \to W_i$ is the base change of $t : R \to U$ hence flat and locally of finite presentation, see Morphisms of Spaces, Lemmas 49.27.4 and 49.26.3. Hence by Descent on Spaces, Lemmas 55.10.11 and 55.10.8 we see that $U \to F$ is flat and locally of finite presentation. It is surjective by Spaces, Remark 47.5.2. □

Lemma 61.11.5. *Let S be a scheme. Let $X \to B$ be a morphism of algebraic spaces over S. Let G be a group algebraic space over B and let $a : G \times_B X \to X$ be an action of G on X/B. If*

1. *a is a free action, and*
2. *$G \to B$ is flat and locally of finite presentation,*

then X/G (see Groupoids in Spaces, Definition 59.18.1) is an algebraic space and $X \to X/G$ is surjective, flat, and locally of finite presentation.

Proof. The fact that X/G is an algebraic space is immediate from Theorem 61.10.1 and the definitions. Namely, $X/G = X/R$ where $R = G \times_B X$. The morphisms $s, t : G \times_B X \to X$ are flat and locally of finite presentation (clear for s as a base change of $G \to B$ and by symmetry using the inverse it follows for t) and the morphism $j : G \times_B X \to X \times_B X$ is a monomorphism by Groupoids in Spaces, Lemma 59.8.3 as the action is free. The assertions about the morphism $X \to X/G$ follow from Lemma 61.11.4. □

Lemma 61.11.6. *Let $\{S_i \to S\}_{i\in I}$ be a covering of $(Sch/S)_{fppf}$. Let G be a group algebraic space over S, and denote $G_i = G_{S_i}$ the base changes. Suppose given*

1. *for each $i \in I$ an fppf G_i-torsor X_i over S_i, and*
2. *for each $i, j \in I$ a $G_{S_i\times_S S_j}$-equivariant isomorphism $\varphi_{ij} : X_i \times_S S_j \to S_i \times_S X_j$ satisfying the cocycle condition over every $S_i \times_S S_j \times_S S_j$.*

Then there exists an fppf G-torsor X over S whose base change to S_i is isomorphic to X_i such that we recover the descent datum φ_{ij}.

Proof. We may think of X_i as a sheaf on $(Sch/S_i)_{fppf}$, see Spaces, Section 47.16. By Sites, Section 7.25 the descent datum (X_i, φ_{ij}) is effective in the sense that there exists a unique sheaf X on $(Sch/S)_{fppf}$ which recovers the algebraic spaces X_i after restricting back to $(Sch/S_i)_{fppf}$. Hence we see that $X_i = h_{S_i} \times X$. By Lemma 61.11.1 we see that X is an algebraic space, modulo verifying that $\coprod X_i$ is an algebraic space which we do at the end of the proof. By the equivalence of categories in Sites, Lemma 7.25.3 the action maps $G_i \times_{S_i} X_i \to X_i$ glue to give a map $a : G \times_S X \to X$. Now we have to show that a is an action and that X is a pseudo-torsor, and fppf locally trivial (see Groupoids in Spaces, Definition 59.9.3). These may be checked fppf locally, and hence follow from the corresponding properties of the actions $G_i \times_{S_i} X_i \to X_i$. Hence the lemma is true.

We suggest the reader skip the rest of the proof, which is purely set theoretical. Pick coverings $\{S_{ij} \to S_j\}_{j \in J_i}$ of $(Sch/S)_{fppf}$ which trivialize the G_i torsors X_i (possible by assumption, and Topologies, Lemma 33.7.7 part (1)). Then $\{S_{ij} \to S\}_{i \in I, j \in J_i}$ is a covering of $(Sch/S)_{fppf}$ and hence we may assume that each X_i is the trivial torsor! Of course we may also refine the covering further, hence we may assume that each S_i is affine and that the index set I has cardinality bounded by the cardinality of the set of points of S. Choose $U \in \mathrm{Ob}((Sch/S)_{fppf})$ and a surjective étale morphism $U \to G$. Then we see that $U_i = U \times_S S_i$ comes with an étale surjective morphism to $X_i \cong G_i$. By Sets, Lemma 3.9.6 we see $\mathrm{size}(U_i) \leq \max\{\mathrm{size}(U), \mathrm{size}(S_i)\}$. By Sets, Lemma 3.9.7 we have $\mathrm{size}(S_i) \leq \mathrm{size}(S)$. Hence we see that $\mathrm{size}(U_i) \leq \max\{\mathrm{size}(U), \mathrm{size}(S)\}$ for all $i \in I$. Together with the bound on $|I|$ we found above we conclude from Sets, Lemma 3.9.5 that $\mathrm{size}(\coprod U_i) \leq \max\{\mathrm{size}(U), \mathrm{size}(S)\}$. Hence Spaces, Lemma 47.8.3 applies to show that $\coprod X_i$ is an algebraic space which is what we had to prove. □

61.12. Algebraic spaces in the étale topology

Let S be a scheme. Instead of working with sheaves over the big fppf site $(Sch/S)_{fppf}$ we could work with sheaves over the big étale site $(Sch/S)_{étale}$. All of the material in Algebraic Spaces, Sections 47.3 and 47.5 makes sense for sheaves over $(Sch/S)_{étale}$. Thus we get a second notion of algebraic spaces by working in the étale topology. This notion is (a priori) weaker then the notion introduced in Algebraic Spaces, Definition 47.6.1 since a sheaf in the fppf topology is certainly a sheaf in the étale topology. However, the notions are equivalent as is shown by the following lemma.

Lemma 61.12.1. *Denote the common underlying category of Sch_{fppf} and $Sch_{étale}$ by Sch_α (see Topologies, Remark 33.9.1). Let S be an object of Sch_α.*

$$F : (Sch_\alpha/S)^{opp} \longrightarrow Sets$$

be a presheaf with the following properties:

1. *F is a sheaf for the étale topology,*
2. *the diagonal $\Delta : F \to F \times F$ is representable, and*
3. *there exists $U \in \mathrm{Ob}(Sch_\alpha/S)$ and $U \to F$ which is surjective and étale.*

Then F is an algebraic space in the sense of Algebraic Spaces, Definition 47.6.1.

Proof. Note that properties (2) and (3) of the lemma and the corresponding properties (2) and (3) of Algebraic Spaces, Definition 47.6.1 are independent of the topology. This is true because these properties involve only the notion of a fibre product of presheaves, maps of presheaves, the notion of a representable transformation of functors, and what it means for such a transformation to be surjective and étale. Thus all we have to prove is that an étale sheaf F with properties (2) and (3) is also an fppf sheaf.

To do this, let $R = U \times_F U$. By (2) the presheaf R is representable by a scheme and by (3) the projections $R \to U$ are étale. Thus $j : R \to U \times_S U$ is an étale equivalence relation. Moreover $U \to F$ identifies F as the quotient of U by R for the étale topology: (a) if $T \to F$ is a morphism, then $\{T \times_F U \to T\}$ is an étale covering, hence $U \to F$ is a surjection of sheaves for the étale topology, (b) if $a, b : T \to U$ map to the same section of F, then $(a, b) : T \to R$ hence a and b have the same image in the quotient of U by R for the étale topology. Next, let U/R denote the quotient sheaf in the fppf topology which is an algebraic space by Spaces, Theorem 47.10.5. Thus we have morphisms (transformations of functors)

$$U \to F \to U/R.$$

By the aforementioned Spaces, Theorem 47.10.5 the composition is representable, surjective, and étale. Hence for any scheme T and morphism $T \to U/R$ the fibre product $V = T \times_{U/R} U$ is a scheme surjective and étale over T. In other words, $\{V \to U\}$ is an étale covering. This proves that $U \to U/R$ is surjective as a map of sheaves in the étale topology. It follows that $F \to U/R$ is surjective as a map of sheaves in the étale topology. On the other hand, the map $F \to U/R$ is injective (as a map of presheaves) since $R = U \times_{U/R} U$ again by Spaces, Theorem 47.10.5. It follows that $F \to U/R$ is an isomorphism of étale sheaves, see Sites, Lemma 7.12.2 which concludes the proof. □

In fact, it suffices to have a smooth cover by a scheme and it suffices to assume the diagonal is representable by algebraic spaces.

Lemma 61.12.2. *Denote the common underlying category of Sch_{fppf} and $Sch_{étale}$ by Sch_α (see Topologies, Remark 33.9.1). Let S be an object of Sch_α.*

$$F : (Sch_\alpha/S)^{opp} \longrightarrow Sets$$

be a presheaf with the following properties:

1. *F is a sheaf for the étale topology,*
2. *the diagonal $\Delta : F \to F \times F$ is representable by algebraic spaces, and*
3. *there exists $U \in \mathrm{Ob}(Sch_\alpha/S)$ and $U \to F$ which is surjective and smooth.*

Then F is an algebraic space in the sense of Algebraic Spaces, Definition 47.6.1.

Proof. The proof mirrors the proof of Lemma 61.12.1. Let $R = U \times_F U$. By (2) the presheaf R is an algebraic space and by (3) the projections $R \to U$ are smooth and surjective. Denote (U, R, s, t, c) the groupoid associated to the equivalence relation $j : R \to U \times_S U$ (see Groupoids in Spaces, Lemma 59.11.3). By Theorem 61.10.1 we see that $X = U/R$ (quotient in the fppf-topology) is an algebraic space. Using that the smooth topology and the étale topology have the same sheaves (by More on Morphisms, Lemma 36.28.7) we see the map $U \to F$ identifies F as the quotient

of U by R for the smooth topology (details omitted). Thus we have morphisms (transformations of functors)

$$U \to F \to X.$$

By Lemma 61.11.4 we see that $U \to X$ is surjective, flat and locally of finite presentation. By Groupoids in Spaces, Lemma 59.18.5 (and the fact that j is a monomorphism) we have $R = U \times_X U$. By Descent on Spaces, Lemma 55.10.24 we conclude that $U \to X$ is smooth and surjective (as the projections $R \to U$ are smooth and surjective and $\{U \to X\}$ is an fppf covering). Hence for any scheme T and morphism $T \to X$ the fibre product $T \times_X U$ is an algebraic space surjective and smooth over T. Choose a scheme V and a surjective étale morphism $V \to T \times_X U$. Then $\{V \to T\}$ is a smooth covering such that $V \to T \to X$ lifts to a morphism $V \to U$. This proves that $U \to X$ is surjective as a map of sheaves in the smooth topology. It follows that $F \to X$ is surjective as a map of sheaves in the smooth topology. On the other hand, the map $F \to X$ is injective (as a map of presheaves) since $R = U \times_X U$. It follows that $F \to X$ is an isomorphism of smooth (= étale) sheaves, see Sites, Lemma 7.12.2 which concludes the proof. □

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CHAPTER 62

Quotients of Groupoids

62.1. Introduction

This chapter is devoted to generalities concerning groupoids and their quotients (as far as they exist). There is a lot of literature on this subject, see for example [**MFK94**], [**Ses72**], [**Kol97**], [**KM97**], [**Kol08**] and many more.

62.2. Conventions and notation

In this chapter the conventions and notation are those introduced in Groupoids in Spaces, Sections 59.2 and 59.3.

62.3. Invariant morphisms

Definition 62.3.1. Let S be a scheme, and let B be an algebraic space over S. Let $j = (t, s) : R \to U \times_B U$ be a pre-relation of algebraic spaces over B. We say a morphism $\phi : U \to X$ of algebraic spaces over B is *R-invariant* if the diagram

$$\begin{array}{ccc} R & \xrightarrow[s]{} & U \\ {\scriptstyle t}\downarrow & & \downarrow{\scriptstyle \phi} \\ U & \xrightarrow{\phi} & X \end{array}$$

is commutative. If $j : R \to U \times_B U$ comes from the action of a group algebraic space G on U over B as in Groupoids in Spaces, Lemma 59.14.1, then we say that ϕ is *G-invariant*.

In other words, a morphism $U \to X$ is R-invariant if it equalizes s and t. We can reformulate this in terms of associated quotient sheaves as follows.

Lemma 62.3.2. *Let S be a scheme, and let B be an algebraic space over S. Let $j = (t, s) : R \to U \times_B U$ be a pre-relation of algebraic spaces over B. A morphism of algebraic spaces $\phi : U \to X$ is R-invariant if and only if it factors as $U \to U/R \to X$.*

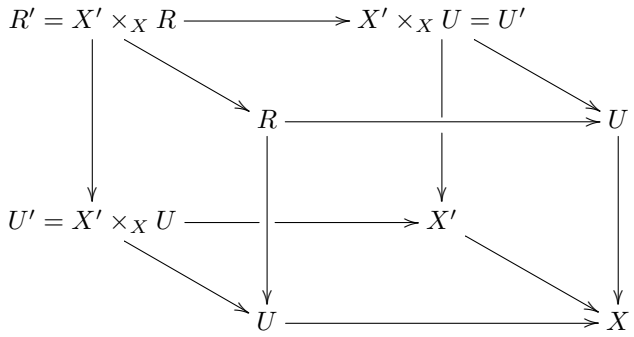
Proof. This is clear from the definition of the quotient sheaf in Groupoids in Spaces, Section 59.18. □

Lemma 62.3.3. *Let S be a scheme, and let B be an algebraic space over S. Let $j = (t, s) : R \to U \times_B U$ be a pre-relation of algebraic spaces over B. Let $U \to X$ be an R-invariant morphism of algebraic spaces over B. Let $X' \to X$ be any morphism of algebraic spaces.*

(1) *Setting $U' = X' \times_X U$, $R' = X' \times_X R$ we obtain a pre-relation $j' : R' \to U' \times_B U'$.*

(2) *The pre-relation $j' : R' \to U' \times_B U'$ is the restriction of $j : R \to U \times_B U$ via $U' \to U$, see Groupoids in Spaces, Definition 59.4.3.*
(3) *If j is a relation, then j' is a relation.*
(4) *If j is a pre-equivalence relation, then j' is a pre-equivalence relation.*
(5) *If j is an equivalence relation, then j' is an equivalence relation.*
(6) *If j comes from a groupoid in algebraic spaces (U, R, s, t, c) over B, then j' comes from the restriction of this groupoid to U', see Groupoids in Spaces, Definition 59.16.2.*
(7) *If j comes from the action of a group algebraic space G/B on U as in Groupoids in Spaces, Lemma 59.14.1 then j' comes from the induced action of G on U'.*

Proof. Omitted. Hint: Functorial point of view combined with the picture:



□

Definition 62.3.4. In the situation of Lemma 62.3.3 we call $j' : R' \to U' \times_B U'$ the *pullback* of the pre-relation j to X'. We say it is a *flat pullback* if $X' \to X$ is a flat morphism of algebraic spaces.

62.4. Categorical quotients

This is the most basic kind of quotient one can consider.

Definition 62.4.1. Let S be a scheme, and let B be an algebraic space over S. Let $j = (t, s) : R \to U \times_B U$ be pre-relation in algebraic spaces over B.

(1) We say a morphism $\phi : U \to X$ of algebraic spaces over B is a *categorical quotient* if it is R-invariant, and for every R-invariant morphism $\psi : U \to Y$ of algebraic spaces over B there exists a unique morphism $\chi : X \to Y$ such that $\psi = \phi \circ \chi$.
(2) Let $\mathcal{C}$ be a full subcategory of the category of algebraic spaces over B. Assume U, R are objects of $\mathcal{C}$. In this situation we say a morphism $\phi : U \to X$ of algebraic spaces over B is a *categorical quotient in $\mathcal{C}$* if $X \in \mathrm{Ob}(\mathcal{C})$, and ϕ is R-invariant, and for every R-invariant morphism $\psi : U \to Y$ with $Y \in \mathrm{Ob}(\mathcal{C})$ there exists a unique morphism $\chi : X \to Y$ such that $\psi = \phi \circ \chi$.
(3) If $B = S$ and $\mathcal{C}$ is the category of schemes over S, then we say $U \to X$ is a *categorical quotient in the category of schemes*, or simply a *categorical quotient in schemes*.

We often single out a category $\mathcal{C}$ of algebraic spaces over B by some separation axiom, see Example 62.4.3 for some standard cases. Note that if $\phi : U \to X$ is a categorical quotient if and only if $U \to X$ is a coequalizer for the morphisms $t, s : R \to U$ in the category. Hence we immediately deduce the following lemma.

Lemma 62.4.2. *Let S be a scheme, and let B be an algebraic space over S. Let $j : R \to U \times_B U$ be a pre-relation in algebraic spaces over B. If a categorical quotient in the category of algebraic spaces over B exists, then it is unique up to unique isomorphism. Similarly for categorical quotients in full subcategories of Spaces/B.*

Proof. See Categories, Section 4.11. □

Example 62.4.3. Let S be a scheme, and let B be an algebraic space over S. Here are some standard examples of categories $\mathcal{C}$ that we often come up when applying Definition 62.4.1:

(1) $\mathcal{C}$ is the category of all algebraic spaces over B,
(2) B is separated and $\mathcal{C}$ is the category of all separated algebraic spaces over B,
(3) B is quasi-separated and $\mathcal{C}$ is the category of all quasi-separated algebraic spaces over B,
(4) B is locally separated and $\mathcal{C}$ is the category of all locally separated algebraic spaces over B,
(5) B is decent and $\mathcal{C}$ is the category of all decent algebraic spaces over B, and
(6) $S = B$ and $\mathcal{C}$ is the category of schemes over S.

In this case, if $\phi : U \to X$ is a categorical quotient then we say $U \to X$ is (1) a *categorical quotient*, (2) a *categorical quotient in separated algebraic spaces*, (3) a *categorical quotient in quasi-separated algebraic spaces*, (4) a *categorical quotient in locally separated algebraic spaces*, (5) a *categorical quotient in decent algebraic spaces*, (6) a *categorical quotient in schemes*.

Definition 62.4.4. Let S be a scheme, and let B be an algebraic space over S. Let $\mathcal{C}$ be a full subcategory of the category of algebraic spaces over B closed under fibre products. Let $j = (t, s) : R \to U \times_B U$ be pre-relation in $\mathcal{C}$, and let $U \to X$ be an R-invariant morphism with $X \in \mathrm{Ob}(\mathcal{C})$.

(1) We say $U \to X$ is a *universal categorical quotient* in $\mathcal{C}$ if for every morphism $X' \to X$ in $\mathcal{C}$ the morphism $U' = X' \times_X U \to X'$ is the categorical quotient in $\mathcal{C}$ of the pullback $j' : R' \to U'$ of j.
(2) We say $U \to X$ is a *uniform categorical quotient* in $\mathcal{C}$ if for every flat morphism $X' \to X$ in $\mathcal{C}$ the morphism $U' = X' \times_X U \to X'$ is the categorical quotient in $\mathcal{C}$ of the pullback $j' : R' \to U'$ of j.

Lemma 62.4.5. *In the situation of Definition 62.4.1. If $\phi : U \to X$ is a categorical quotient and U is reduced, then X is reduced. The same holds for categorical quotients in a category of spaces $\mathcal{C}$ listed in Example 62.4.3.*

Proof. Let X_{red} be the reduction of the algebraic space X. Since U is reduced the morphism $\phi : U \to X$ factors through $i : X_{red} \to X$ (insert future reference here). Denote this morphism by $\phi_{red} : U \to X_{red}$. Since $\phi \circ s = \phi \circ t$ we see that also $\phi_{red} \circ s = \phi_{red} \circ t$ (as $i : X_{red} \to X$ is a monomorphism). Hence by the universal

property of ϕ there exists a morphism $\chi : X \to X_{red}$ such that $\phi_{red} = \phi \circ \chi$. By uniqueness we see that $i \circ \chi = \mathrm{id}_X$ and $\chi \circ i = \mathrm{id}_{X_{red}}$. Hence i is an isomorphism and X is reduced.

To show that this argument works in a category $\mathcal{C}$ one just needs to show that the reduction of an object of $\mathcal{C}$ is an object of $\mathcal{C}$. We omit the verification that this holds for each of the standard examples. □

62.5. Quotients as orbit spaces

Let $j = (t, s) : R \to U \times_B U$ be a pre-relation. If j is a pre-equivalence relation, then loosely speaking the "orbits" of R on U are the subsets $t(s^{-1}(\{u\}))$ of U. However, if j is just a pre-relation, then we need to take the equivalence relation generated by R.

Definition 62.5.1. Let S be a scheme, and let B be an algebraic space over S. Let $j : R \to U \times_B U$ be a pre-relation over B. If $u \in |U|$, then the *orbit*, or more precisely the *R-orbit* of u is

$$O_u = \left\{ u' \in |U| \ : \begin{array}{c} \exists n \geq 1,\ \exists u_0, \ldots, u_n \in |U| \text{ such that} \\ \text{for all } i \in \{0, \ldots, n-1\} \text{ either} \\ u_i = u_{i+1} \text{ or} \\ \exists r \in |R|,\ s(r) = u_i, t(r) = u_{i+1} \text{ or} \\ \exists r \in |R|,\ t(r) = u_i, s(r) = u_{i+1} \end{array} \right\}$$

It is clear that these are the equivalence classes of an equivalence relation, i.e., we have $u' \in O_u$ if and only if $u \in O_{u'}$. The following lemma is a reformulation of Groupoids in Spaces, Lemma 59.4.4.

Lemma 62.5.2. *Let $B \to S$ as in Section 62.2. Let $j : R \to U \times_B U$ be a pre-equivalence relation of algebraic spaces over B. Then*

$$O_u = \{u' \in |U| \text{ such that } \exists r \in |R|,\ s(r) = u,\ t(r) = u'\}.$$

Proof. By the aforementioned Groupoids in Spaces, Lemma 59.4.4 we see that the orbits O_u as defined in the lemma give a disjoint union decomposition of $|U|$. Thus we see they are equal to the orbits as defined in Definition 62.5.1. □

Lemma 62.5.3. *In the situation of Definition 62.5.1. Let $\phi : U \to X$ be an R-invariant morphism of algebraic spaces over B. Then $|\phi| : |U| \to |X|$ is constant on the orbits.*

Proof. To see this we just have to show that $\phi(u) = \phi(u')$ for all $u, u' \in |U|$ such that there exists an $r \in |R|$ such that $s(r) = u$ and $t(r) = u'$. And this is clear since ϕ equalizes s and t. □

There are several problems with considering the orbits $O_u \subset |U|$ as a tool for singling out properties of quotient maps. One issue is the following. Suppose that $\mathrm{Spec}(k) \to B$ is a geometric point of B. Consider the canonical map

$$U(k) \longrightarrow |U|.$$

Then it is usually not the case that the equivalence classes of the equivalence relation generated by $j(R(k)) \subset U(k) \times U(k)$ are the inverse images of the orbits $O_u \subset |U|$. A silly example is to take $S = B = \mathrm{Spec}(\mathbf{Z})$, $U = R = \mathrm{Spec}(k)$ with $s = t = \mathrm{id}_k$. Then $|U| = |R|$ is a single point but $U(k)/R(k)$ is enormous. A more interesting

example is to take $S = B = \mathrm{Spec}(\mathbf{Q})$, choose some of number fields $K \subset L$, and set $U = \mathrm{Spec}(L)$ and $R = \mathrm{Spec}(L \otimes_K L)$ with obvious maps $s, t : R \to U$. In this case $|U|$ still has just one point, but the quotient

$$U(k)/R(k) = \mathrm{Hom}(K, k)$$

consists of more than one element. We conclude from both examples that if $U \to X$ is an R-invariant map and if we want it to "separate orbits" we get a much stronger and interesting notion by considering the induced maps $U(k) \to X(k)$ and ask that those maps separate orbits.

There is an issue with this too. Namely, suppose that $S = B = \mathrm{Spec}(\mathbf{R})$, $U = \mathrm{Spec}(\mathbf{C})$, and $R = \mathrm{Spec}(\mathbf{C}) \amalg \mathrm{Spec}(K)$ for some field extension $\sigma : \mathbf{C} \to K$. Let the maps s, t be given by the identity on the component $\mathrm{Spec}(\mathbf{C})$, but by $\sigma, \sigma \circ \tau$ on the second component where τ is complex conjugation. If K is a nontrivial extension of $\mathbf{C}$, then the two points $1, \tau \in U(\mathbf{C})$ are not equivalent under $j(R(\mathbf{C}))$. But after choosing an extension $\mathbf{C} \subset \Omega$ of sufficiently large cardinality (for example larger than the cardinality of K) then the images of $1, \tau \in U(\mathbf{C})$ in $U(\Omega)$ do become equivalent! It seems intuitively clear that this happens either because $s, t : R \to U$ are not locally of finite type or because the cardinality of the field k is not large enough.

Keeping this in mind we make the following definition.

Definition 62.5.4. Let S be a scheme, and let B be an algebraic space over S. Let $j : R \to U \times_B U$ be a pre-relation over B. Let $\mathrm{Spec}(k) \to B$ be a geometric point of B.

(1) We say $\overline{u}, \overline{u}' \in U(k)$ are *weakly R-equivalent* if they are in the same equivalence class for the equivalence relation generated by the relation $j(R(k)) \subset U(k) \times U(k)$.
(2) We say $\overline{u}, \overline{u}' \in U(k)$ are *R-equivalent* if for some overfield $k \subset \Omega$ the images in $U(\Omega)$ are weakly R-equivalent.
(3) The *weak orbit*, or more precisely the *weak R-orbit* of $\overline{u} \in U(k)$ is set of all elements of $U(k)$ which are weakly R-equivalent to $\overline{u}$.
(4) The *orbit*, or more precisely the *R-orbit* of $\overline{u} \in U(k)$ is set of all elements of $U(k)$ which are R-equivalent to $\overline{u}$.

It turns out that in good cases orbits and weak orbits agree, see Lemma 62.5.7. The following lemma illustrates the difference in the special case of a pre-equivalence relation.

Lemma 62.5.5. *Let S be a scheme, and let B be an algebraic space over S. Let $\mathrm{Spec}(k) \to B$ be a geometric point of B. Let $j : R \to U \times_B U$ be a pre-equivalence relation over B. In this case the weak orbit of $\overline{u} \in U(k)$ is simply*

$$\{\overline{u}' \in U(k) \text{ such that } \exists \overline{r} \in R(k),\ s(\overline{r}) = \overline{u},\ t(\overline{r}) = \overline{u}'\}$$

and the orbit of $\overline{u} \in U(k)$ is

$$\{\overline{u}' \in U(k) : \exists \text{ field extension } k \subset K,\ \exists\ r \in R(K),\ s(r) = \overline{u},\ t(r) = \overline{u}'\}$$

Proof. This is true because by definition of a pre-equivalence relation the image $j(R(k)) \subset U(k) \times U(k)$ is an equivalence relation. □

Let us describe the recipe for turning any pre-relation into a pre-equivalence relation. We will use the morphisms

$$\begin{array}{llcccccc} & j_{diag} & : & U & \longrightarrow & U \times_B U, & u & \longmapsto & (u, u) \\ (62.5.5.1) & j_{flip} & : & R & \longrightarrow & U \times_B U, & r & \longmapsto & (s(r), t(r)) \\ & j_{comp} & : & R \times_{s,U,t} R & \longrightarrow & U \times_B U, & (r, r') & \longmapsto & (t(r), s(r')) \end{array}$$

We define $j_1 = (t_1, s_1) : R_1 \to U \times_B U$ to be the morphism

$$j \amalg j_{diag} \amalg j_{flip} : R \amalg U \amalg R \longrightarrow U \times_B U$$

with notation as in Equation (62.5.5.1). For $n > 1$ we set

$$j_n = (t_n, s_n) : R_n = R_1 \times_{s_1, U, t_{n-1}} R_{n-1} \longrightarrow U \times_B U$$

where t_n comes from t_1 precomposed with projection onto R_1 and s_n comes from s_{n-1} precomposed with projection onto R_{n-1}. Finally, we denote

$$j_\infty = (t_\infty, s_\infty) : R_\infty = \coprod\nolimits_{n \geq 1} R_n \longrightarrow U \times_B U.$$

Lemma 62.5.6. *Let S be a scheme, and let B be an algebraic space over S. Let $j : R \to U \times_B U$ be a pre-relation over B. Then $j_\infty : R_\infty \to U \times_B U$ is a pre-equivalence relation over B. Moreover*

1. *$\phi : U \to X$ is R-invariant if and only if it is R_∞-invariant,*
2. *the canonical map of quotient sheaves $U/R \to U/R_\infty$ (see Groupoids in Spaces, Section 59.18) is an isomorphism,*
3. *weak R-orbits agree with weak R_∞-orbits,*
4. *R-orbits agree with R_∞-orbits,*
5. *if s, t are locally of finite type, then s_∞, t_∞ are locally of finite type,*
6. *add more here as needed.*

Proof. Omitted. Hint for (5): Any property of s, t which is stable under composition and stable under base change, and Zariski local on the source will be inherited by s_∞, t_∞. □

Lemma 62.5.7. *Let S be a scheme, and let B be an algebraic space over S. Let $j : R \to U \times_B U$ be a pre-relation over B. Let $\mathrm{Spec}(k) \to B$ be a geometric point of B.*

1. *If $s, t : R \to U$ are locally of finite type then weak R-equivalence on $U(k)$ agrees with R-equivalence, and weak R-orbits agree with R-orbits on $U(k)$.*
2. *If k has sufficiently large cardinality then weak R-equivalence on $U(k)$ agrees with R-equivalence, and weak R-orbits agree with R-orbits on $U(k)$.*

Proof. We first prove (1). Assume s, t locally of finite type. By Lemma 62.5.6 we may assume that R is a pre-equivalence relation. Let k be an algebraically closed field over B. Suppose $\overline{u}, \overline{u}' \in U(k)$ are R-equivalent. Then for some extension field $k \subset \Omega$ there exists a point $\overline{r} \in R(\Omega)$ mapping to $(\overline{u}, \overline{u}') \in (U \times_B U)(\Omega)$, see Lemma 62.5.5. Hence

$$Z = R \times_{j, U \times_B U, (\overline{u}, \overline{u}')} \mathrm{Spec}(k)$$

is nonempty. As s is locally of finite type we see that also j is locally of finite type, see Morphisms of Spaces, Lemma 49.22.6. This implies Z is a nonempty algebraic space locally of finite type over the algebraically closed field k (use Morphisms of Spaces, Lemma 49.22.3). Thus Z has a k-valued point, see Morphisms of Spaces,

Lemma 49.23.1. Hence we conclude there exists a $\overline{r} \in R(k)$ with $j(\overline{r}) = (\overline{u}, \overline{u}')$, and we conclude that $\overline{u}, \overline{u}'$ are R-equivalent as desired.

The proof of part (2) is the same, except that it uses Morphisms of Spaces, Lemma 49.23.2 instead of Morphisms of Spaces, Lemma 49.23.1. This shows that the assertion holds as soon as $|k| > \lambda(R)$ with $\lambda(R)$ as introduced just above Morphisms of Spaces, Lemma 49.23.1. □

In the following definition we use the terminology "k is a field over B" to mean that $\mathrm{Spec}(k)$ comes equipped with a morphism $\mathrm{Spec}(k) \to B$.

Definition 62.5.8. Let S be a scheme, and let B be an algebraic space over S. Let $j : R \to U \times_B U$ be a pre-relation over B.

(1) We say $\phi : U \to X$ is *set-theoretically R-invariant* if and only if the map $U(k) \to X(k)$ equalizes the two maps $s, t : R(k) \to U(k)$ for every algebraically closed field k over B.
(2) We say $\phi : U \to X$ *separates orbits*, or *separates R-orbits* if it is set-theoretically R-invariant and $\phi(\overline{u}) = \phi(\overline{u}')$ in $X(k)$ implies that $\overline{u}, \overline{u}' \in U(k)$ are in the same orbit for every algebraically closed field k over B.

In Example 62.5.12 we show that being set-theoretically invariant is "too weak" a notion in the category of algebraic spaces. A more geometric reformulation of what it means to be set-theoretically invariant or to separate orbits is in Lemma 62.5.17.

Lemma 62.5.9. *In the situation of Definition 62.5.8. A morphism $\phi : U \to X$ is set-theoretically R-invariant if and only if for any algebraically closed field k over B the map $U(k) \to X(k)$ is constant on orbits.*

Proof. This is true because the condition is supposed to hold for all algebraically closed fields over B. □

Lemma 62.5.10. *In the situation of Definition 62.5.8. An invariant morphism is set-theoretically invariant.*

Proof. This is immediate from the definitions. □

Lemma 62.5.11. *In the situation of Definition 62.5.8. Let $\phi : U \to X$ be a morphism of algebraic spaces over B. Assume*

(1) *ϕ is set-theoretically R-invariant,*
(2) *R is reduced, and*
(3) *X is locally separated over B.*

Then ϕ is R-invariant.

Proof. Consider the equalizer

$$Z = R \times_{(\phi,\phi)\circ j, X \times_B X, \Delta_{X/B}} X$$

algebraic space. Then $Z \to R$ is an immersion by assumption (3). By assumption (1) $|Z| \to |R|$ is surjective. This implies that $Z \to R$ is a bijective closed immersion (use Schemes, Lemma 25.10.4) and by assumption (2) we conclude that $Z = R$. □

Example 62.5.12. There exist reduced quasi-separated algebraic spaces X, Y and a pair of morphisms $a, b : Y \to X$ which agree on all k-valued points but are not equal. To get an example take $Y = \mathrm{Spec}(k[[x]])$ and

$$X = \mathbf{A}^1_k \Big/ \big(\Delta \amalg \{(x, -x) \mid x \neq 0\}\big)$$

the algebraic space of Spaces, Example 47.14.1. The two morphisms $a, b : Y \to X$ come from the two maps $x \mapsto x$ and $x \mapsto -x$ from Y to $\mathbf{A}^1_k = \mathrm{Spec}(k[x])$. On the generic point the two maps are the same because on the open part $x \neq 0$ of the space X the functions x and $-x$ are equal. On the closed point the maps are obviously the same. It is also true that $a \neq b$. This implies that Lemma 62.5.11 does not hold with assumption (3) replaced by the assumption that X be quasi-separated. Namely, consider the diagram

$$\begin{array}{ccc} Y & \xrightarrow[1]{} & Y \\ {\scriptstyle -1}\downarrow & & \downarrow{\scriptstyle a} \\ Y & \xrightarrow{a} & X \end{array}$$

then the composition $a \circ (-1) = b$. Hence we can set $R = Y$, $U = Y$, $s = 1$, $t = -1$, $\phi = a$ to get an example of a set-theoretically invariant morphism which is not invariant.

The example above is instructive because the map $Y \to X$ even separates orbits. It shows that in the category of algebraic spaces there are simply too many set-theoretically invariant morphisms lying around. Next, let us define what it means for R to be a set-theoretic equivalence relation, while remembering that we need to allow for field extensions to make this work correctly.

Definition 62.5.13. Let S be a scheme, and let B be an algebraic space over S. Let $j : R \to U \times_B U$ be a pre-relation over B.

(1) We say j is a *set-theoretic pre-equivalence relation* if

$$\overline{u} \sim_R \overline{u}' \Leftrightarrow \begin{array}{c} \exists \text{ field extension } k \subset K,\ \exists\ r \in R(K), \\ s(r) = \overline{u},\ t(r) = \overline{u}' \end{array}$$

defines an equivalence relation on $U(k)$ for all algebraically closed fields k over B.

(2) We say j is a *set-theoretic equivalence relation* if j is universally injective and a set-theoretic pre-equivalence relation.

Let us reformulate this in more geometric terms.

Lemma 62.5.14. *In the situation of Definition 62.5.13. The following are equivalent:*

(1) *The morphism j is a set-theoretic pre-equivalence relation.*
(2) *The subset $j(|R|) \subset |U \times_B U|$ contains the image of $|j'|$ for any of the morphisms j' as in Equation (62.5.5.1).*
(3) *For every algebraically closed field k over B of sufficiently large cardinality the subset $j(R(k)) \subset U(k) \times U(k)$ is an equivalence relation.*

If s, t are locally of finite type these are also equivalent to

(4) *For every algebraically closed field k over B the subset $j(R(k)) \subset U(k) \times U(k)$ is an equivalence relation.*

Proof. Assume (2). Let k be an algebraically closed field over B. We are going to show that $\sim_R$ is an equivalence relation. Suppose that $\overline{u}_i : \mathrm{Spec}(k) \to U$, $i = 1, 2$ are k-valued points of U. Suppose that $(\overline{u}_1, \overline{u}_2)$ is the image of a K-valued point

$r \in R(K)$. Consider the solid commutative diagram

$$\begin{array}{ccccc} \mathrm{Spec}(K') & \dashrightarrow & \mathrm{Spec}(k) & \longleftarrow & \mathrm{Spec}(K) \\ \downarrow & & \downarrow{\scriptstyle (\overline{u}_2, \overline{u}_1)} & & \downarrow \\ R & \xrightarrow{j} & U \times_B U & \xleftarrow{j_{flip}} & R \end{array}$$

We also denote $r \in |R|$ the image of r. By assumption the image of $|j_{flip}|$ is contained in the image of $|j|$, in other words there exists a $r' \in |R|$ such that $|j|(r') = |j_{flip}|(r)$. But note that $(\overline{u}_2, \overline{u}_1)$ is in the equivalence class that defines $|j|(r')$ (by the commutativity of the solid part of the diagram). This means there exists a field extension $k \subset K'$ and a morphism $r' : \mathrm{Spec}(K) \to R$ (abusively denoted r' as well) with $j \circ r' = (\overline{u}_2, \overline{u}_1) \circ i$ where $i : \mathrm{Spec}(K') \to \mathrm{Spec}(K)$ is the obvious map. In other words the dotted part of the diagram commutes. This proves that $\sim_R$ is a symmetric relation on $U(k)$. In the similar way, using that the image of $|j_{diag}|$ is contained in the image of $|j|$ we see that $\sim_R$ is reflexive (details omitted).

To show that $\sim_R$ is transitive assume given $\overline{u}_i : \mathrm{Spec}(k) \to U$, $i = 1, 2, 3$ and field extensions $k \subset K_i$ and points $r_i : \mathrm{Spec}(K_i) \to R$, $i = 1, 2$ such that $j(r_1) = (\overline{u}_1, \overline{u}_2)$ and $j(r_1) = (\overline{u}_2, \overline{u}_3)$. Then we may choose a commutative diagram of fields

$$\begin{array}{ccc} K & \longleftarrow & K_2 \\ \uparrow & & \uparrow \\ K_1 & \longleftarrow & k \end{array}$$

and we may think of $r_1, r_2 \in R(K)$. We consider the commutative solid diagram

$$\begin{array}{ccccc} \mathrm{Spec}(K') & \dashrightarrow & \mathrm{Spec}(k) & \longleftarrow & \mathrm{Spec}(K) \\ \downarrow & & \downarrow{\scriptstyle (\overline{u}_1, \overline{u}_3)} & & \downarrow{\scriptstyle (r_1, r_2)} \\ R & \xrightarrow{j} & U \times_B U & \xleftarrow{j_{comp}} & R \times_{s,U,t} R \end{array}$$

By exactly the same reasoning as in the first part of the proof, but this time using that $|j_{comp}|((r_1, r_2))$ is in the image of $|j|$, we conclude that a field K' and dotted arrows exist making the diagram commute. This proves that $\sim_R$ is transitive and concludes the proof that (2) implies (1).

Assume (1) and let k be an algebraically closed field over B whose cardinality is larger than $\lambda(R)$, see Morphisms of Spaces, Lemma 49.23.2. Suppose that $\overline{u} \sim_R \overline{u}'$ with $\overline{u}, \overline{u}' \in U(k)$. By assumption there exists a point in $|R|$ mapping to $(\overline{u}, \overline{u}') \in |U \times_B U|$. Hence by Morphisms of Spaces, Lemma 49.23.2 we conclude there exists an $\overline{r} \in R(k)$ with $j(\overline{r}) = (\overline{u}, \overline{u}')$. In this way we see that (1) implies (3).

Assume (3). Let us show that $\mathrm{Im}(|j_{comp}|) \subset \mathrm{Im}(|j|)$. Pick any point $c \in |R \times_{s,U,t} R|$. We may represent this by a morphism $\overline{c} : \mathrm{Spec}(k) \to R \times_{s,U,t} R$, with k over B having sufficiently large cardinality. By assumption we see that $j_{comp}(\overline{c}) \in U(k) \times U(k) = (U \times_B U)(k)$ is also the image $j(\overline{r})$ for some $\overline{r} \in R(k)$. Hence $j_{comp}(c) = j(r)$ in $|U \times_B U|$ as desired (with $r \in |R|$ the equivalence class of $\overline{r}$). The same argument shows also that $\mathrm{Im}(|j_{diag}|) \subset \mathrm{Im}(|j|)$ and $\mathrm{Im}(|j_{flip}|) \subset \mathrm{Im}(|j|)$ (details omitted). In this way we see that (3) implies (2). At this point we have shown that (1), (2) and (3) are all equivalent.

It is clear that (4) implies (3) (without any assumptions on s, t). To finish the proof of the lemma we show that (1) implies (4) if s, t are locally of finite type. Namely, let k be an algebraically closed field over B. Suppose that $\overline{u} \sim_R \overline{u}'$ with $\overline{u}, \overline{u}' \in U(k)$. By assumption the algebraic space $Z = R \times_{j, U \times_B U, (\overline{u}, \overline{u}')} \mathrm{Spec}(k)$ is nonempty. On the other hand, since $j = (t, s)$ is locally of finite type the morphism $Z \to \mathrm{Spec}(k)$ is locally of finite type as well (use Morphisms of Spaces, Lemmas 49.22.6 and 49.22.3). Hence Z has a k point by Morphisms of Spaces, Lemma 49.23.1 and we conclude that $(\overline{u}, \overline{u}') \in j(R(k))$ as desired. This finishes the proof of the lemma. □

Lemma 62.5.15. *In the situation of Definition 62.5.13. The following are equivalent:*

(1) *The morphism j is a set-theoretic equivalence relation.*
(2) *The morphism j is universally injective and $j(|R|) \subset |U \times_B U|$ contains the image of $|j'|$ for any of the morphisms j' as in Equation (62.5.5.1).*
(3) *For every algebraically closed field k over B of sufficiently large cardinality the map $j : R(k) \to U(k) \times U(k)$ is injective and its image is an equivalence relation.*

If j is decent, or locally separated, or quasi-separated these are also equivalent to

(4) *For every algebraically closed field k over B the map $j : R(k) \to U(k) \times U(k)$ is injective and its image is an equivalence relation.*

Proof. The implications (1) $\Rightarrow$ (2) and (2) $\Rightarrow$ (3) follow from Lemma 62.5.14 and the definitions. The same lemma shows that (3) implies j is a set-theoretic pre-equivalence relation. But of course condition (3) also implies that j is universally injective, see Morphisms of Spaces, Lemma 49.18.2, so that j is indeed a set-theoretic equivalence relation. At this point we know that (1), (2), (3) are all equivalent.

Condition (4) implies (3) without any further hypotheses on j. Assume j is decent, or locally separated, or quasi-separated and the equivalent conditions (1), (2), (3) hold. By More on Morphisms of Spaces, Lemma 57.3.4 we see that j is radicial. Let k be any algebraically closed field over B. Let $\overline{u}, \overline{u}' \in U(k)$ with $\overline{u} \sim_R \overline{u}'$. We see that $R \times_{U \times_B U, (\overline{u}, \overline{u}')} \mathrm{Spec}(k)$ is nonempty. Hence, as j is radicial, its reduction is the spectrum of a field purely inseparable over k. As $k = \overline{k}$ we see that it is the spectrum of k. Whence a point $\overline{r} \in R(k)$ with $t(\overline{r}) = \overline{u}$ and $s(\overline{r}) = \overline{u}'$ as desired. □

Lemma 62.5.16. *Let S be a scheme, and let B be an algebraic space over S. Let $j : R \to U \times_B U$ be a pre-relation over B.*

(1) *If j is a pre-equivalence relation, then j is a set-theoretic pre-equivalence relation. This holds in particular when j comes from a groupoid in algebraic spaces, or from an action of a group algebraic space on U.*
(2) *If j is an equivalence relation, then j is a set-theoretic equivalence relation.*

Proof. Omitted. □

Lemma 62.5.17. *Let $B \to S$ be as in Section 62.2. Let $j : R \to U \times_B U$ be a pre-relation. Let $\phi : U \to X$ be a morphism of algebraic spaces over B. Consider*

the diagram

$$\begin{array}{ccc} (U \times_X U) \times_{(U \times_B U)} R & \xrightarrow{p} & R \\ \downarrow{\scriptstyle q} & & \downarrow{\scriptstyle j} \\ U \times_X U & \xrightarrow{c} & U \times_B U \end{array}$$

Then we have:

(1) *The morphism ϕ is set-theoretically invariant if and only if p is surjective.*
(2) *If j is a set-theoretic pre-equivalence relation then ϕ separates orbits if and only if p and q are surjective.*
(3) *If p and q are surjective, then j is a set-theoretic pre-equivalence relation (and ϕ separates orbits).*
(4) *If ϕ is R-invariant and j is a set-theoretic pre-equivalence relation, then ϕ separates orbits if and only if the induced morphism $R \to U \times_X U$ is surjective.*

Proof. Assume ϕ is set-theoretically invariant. This means that for any algebraically closed field k over B and any $\overline{r} \in R(k)$ we have $\phi(s(\overline{r})) = \phi(t(\overline{r}))$. Hence $((\phi(t(\overline{r})), \phi(s(\overline{r}))), \overline{r})$ defines a point in the fibre product mapping to $\overline{r}$ via p. This shows that p is surjective. Conversely, assume p is surjective. Pick $\overline{r} \in R(k)$. As p is surjective, we can find a field extension $k \subset K$ and a K-valued point $\tilde{r}$ of the fibre product with $p(\tilde{r}) = \overline{r}$. Then $q(\tilde{r}) \in U \times_X U$ maps to $(t(\overline{r}), s(\overline{r}))$ in $U \times_B U$ and we conclude that $\phi(s(\overline{r})) = \phi(t(\overline{r}))$. This proves that ϕ is set-theoretically invariant.

The proofs of (2), (3), and (4) are omitted. Hint: Assume k is an algebraically closed field over B of large cardinality. Consider the associated diagram of sets

$$\begin{array}{ccc} (U(k) \times_{X(k)} U(k)) \times_{U(k) \times U(k)} R(k) & \xrightarrow{p} & R(k) \\ \downarrow{\scriptstyle q} & & \downarrow{\scriptstyle j} \\ U(k) \times_{X(k)} U(k) & \xrightarrow{c} & U(k) \times U(k) \end{array}$$

By the lemmas above the equivalences posed in (2), (3), and (4) become set-theoretic questions related to the diagram we just displayed, using that surjectivity translates into surjectivity on k-valued points by Morphisms of Spaces, Lemma 49.23.2. □

Because we have seen above that the notion of a set-theoretically invariant morphism is a rather weak one in the category of algebraic spaces, we define an orbit space for a pre-relation as follows.

Definition 62.5.18. Let $B \to S$ as in Section 62.2. Let $j : R \to U \times_B U$ be a pre-relation. We say $\phi : U \to X$ is an *orbit space for* R if

(1) ϕ is R-invariant,
(2) ϕ separates R-orbits, and
(3) ϕ is surjective.

The definition of separating R-orbits involves a discussion of points with values in algebraically closed fields. But as we've seen in many cases this just corresponds to the surjectivity of certain canonically associated morphisms of algebraic spaces. We summarize some of the discussion above in the following characterization of orbit spaces.

Lemma 62.5.19. *Let $B \to S$ as in Section 62.2. Let $j : R \to U \times_B U$ be a set-theoretic pre-equivalence relation. A morphism $\phi : U \to X$ is an orbit space for R if and only if*

(1) *$\phi \circ s = \phi \circ t$, i.e., ϕ is invariant,*
(2) *the induced morphism $(t, s) : R \to U \times_X U$ is surjective, and*
(3) *the morphism $\phi : U \to X$ is surjective.*

This characterization applies for example if j is a pre-equivalence relation, or comes from a groupoid in algebraic spaces over B, or comes from the action of a group algebraic space over B on U.

Proof. Follows immediately from Lemma 62.5.17 part (4). □

In the following lemma it is (probably) not good enough to assume just that the morphisms s, t are locally of finite type. The reason is that it may happen that some map $\phi : U \to X$ is an orbit space, yet is not locally of finite type. In that case $U(k) \to X(k)$ may not be surjective for all algebraically closed fields k over B.

Lemma 62.5.20. *Let $B \to S$ as in Section 62.2. Let $j = (t, s) : R \to U \times_B U$ be a pre-relation. Assume R, U are locally of finite type over B. Let $\phi : U \to X$ be an R-invariant morphism of algebraic spaces over B. Then ϕ is an orbit space for R if and only if the natural map*

$$U(k)/\big(\text{equivalence relation generated by } j(R(k))\big) \longrightarrow X(k)$$

is bijective for all algebraically closed fields k over B.

Proof. Note that since U, R are locally of finite type over B all of the morphisms s, t, j, ϕ are locally of finite type, see Morphisms of Spaces, Lemma 49.22.6. We will also use without further mention Morphisms of Spaces, Lemma 49.23.1. Assume ϕ is an orbit space. Let k be any algebraically closed field over B. Let $\overline{x} \in X(k)$. Consider $U \times_{\phi, X, \overline{x}} \mathrm{Spec}(k)$. This is a nonempty algebraic space which is locally of finite type over k. Hence it has a k-valued point. This shows the displayed map of the lemma is surjective. Suppose that $\overline{u}, \overline{u}' \in U(k)$ map to the same element of $X(k)$. By Definition 62.5.8 this means that $\overline{u}, \overline{u}'$ are in the same R-orbit. By Lemma 62.5.7 this means that they are equivalent under the equivalence relation generated by $j(R(k))$. Thus the displayed morphism is injective.

Conversely, assume the displayed map is bijective for all algebraically closed fields k over B. This condition clearly implies that ϕ is surjective. We have already assumed that ϕ is R-invariant. Finally, the injectivity of all the displayed maps implies that ϕ separates orbits. Hence ϕ is an orbit space. □

62.6. Coarse quotients

We only add this here so that we can later say that coarse quotients correspond to coarse moduli spaces (or moduli schemes).

Definition 62.6.1. Let S be a scheme and B an algebraic space over S. Let $j : R \to U \times_B U$ be a pre-relation. A morphism $\phi : U \to X$ of algebraic spaces over B is called a *coarse quotient* if

(1) ϕ is a categorical quotient, and
(2) ϕ is an orbit space.

If $S = B$, U, R are all schemes, then we say a morphism of schemes $\phi : U \to X$ is a *coarse quotient in schemes* if

(1) ϕ is a categorical quotient in schemes, and
(2) ϕ is an orbit space.

In many situations the algebraic spaces R and U are locally of finite type over B and the orbit space condition simply means that

$$U(k)/\big(\text{equivalence relation generated by } j(R(k))\big) \cong X(k)$$

for all algebraically closed fields k. See Lemma 62.5.20. If j is also a (set-theoretic) pre-equivalence relation, then the condition is simply equivalent to $U(k)/j(R(k)) \to X(k)$ being bijective for all algebraically closed fields k.

62.7. Topological properties

Let S be a scheme and B an algebraic space over S. Let $j : R \to U \times_B U$ be a pre-relation. We say a subset $T \subset |U|$ is *R-invariant* if $s^{-1}(T) = t^{-1}(T)$ as subsets of $|R|$. Note that if T is closed, then it may not be the case that the corresponding reduced closed subspace of U is R-invariant (as in Groupoids in Spaces, Definition 59.17.1) because the pullbacks $s^{-1}(T)$, $t^{-1}(T)$ may not be reduced. Here are some conditions that we can consider for an invariant morphism $\phi : U \to X$.

Definition 62.7.1. Let S be a scheme and B an algebraic space over S. Let $j : R \to U \times_B U$ be a pre-relation. Let $\phi : U \to X$ be an R-invariant morphism of algebraic spaces over B.

(62.7.1.1) The morphism ϕ is submersive.
(62.7.1.2) For any R-invariant closed subset $Z \subset |U|$ the image $\phi(Z)$ is closed in $|X|$.
(62.7.1.3) Condition (62.7.1.2) holds and for any pair of R-invariant closed subsets $Z_1, Z_2 \subset |U|$ we have
$$\phi(Z_1 \cap Z_2) = \phi(Z_1) \cap \phi(Z_2)$$
(62.7.1.4) The morphism $(t, s) : R \to U \times_X U$ is universally submersive.

For each of these properties we can also require them to hold after any flat pullback, or after any pullback, see Definition 62.3.4. In this case we say condition (62.7.1.1), (62.7.1.2), (62.7.1.3), or (62.7.1.4) holds *uniformly* or *universally*.

62.8. Invariant functions

In some cases it is convenient to pin down the structure sheaf of a quotient by requiring any invariant function to be a local section of the structure sheaf of the quotient.

Definition 62.8.1. Let S be a scheme and B an algebraic space over S. Let $j : R \to U \times_B U$ be a pre-relation. Let $\phi : U \to X$ be an R-invariant morphism. Denote $\phi' = \phi \circ s = \phi \circ t : R \to X$.

(1) We denote $(\phi_*\mathcal{O}_U)^R$ the $\mathcal{O}_X$-sub-algebra of $\phi_*\mathcal{O}_U$ which is the equalizer of the two maps
$$\phi_*\mathcal{O}_U \xrightarrow[\phi_* t^\sharp]{\phi_* s^\sharp} \phi'_*\mathcal{O}_R$$

on $X_{étale}$. We sometimes call this the *sheaf of R-invariant functions on X*.
(2) We say *the functions on X are the R-invariant functions on U* if the natural map $\mathcal{O}_X \to (\phi_*\mathcal{O}_U)^R$ is an isomorphism.

Of course we can require this property holds after any (flat or any) pullback, leading to a (uniform or) universal notion. This condition is often thrown in with other conditions in order to obtain a (more) unique quotient. And of course a good deal of motivation for the whole subject comes from the following special case: $U = \mathrm{Spec}(A)$ is an affine scheme over a field $S = B = \mathrm{Spec}(k)$ and where $R = G \times U$, with G an affine group scheme over k. In this case you have the option of taking for the quotient:

$$X = \mathrm{Spec}(A^G)$$

so that at least the condition of the definition above is satisfied. Even though this is a nice thing you can do it is often not the right quotient; for example if $U = \mathrm{GL}_{n,k}$ and G is the group of upper triangular matrices, then the above gives $X = \mathrm{Spec}(k)$, whereas a much better quotient (namely the flag variety) exists.

62.9. Good quotients

Especially when taking quotients by group actions the following definition is useful.

Definition 62.9.1. Let S be a scheme and B an algebraic space over S. Let $j : R \to U \times_B U$ be a pre-relation. A morphism $\phi : U \to X$ of algebraic spaces over B is called a *good quotient* if

(1) ϕ is invariant,
(2) ϕ is affine,
(3) ϕ is surjective,
(4) condition (62.7.1.3) holds universally, and
(5) the functions on X are the R-invariant functions on U.

In [**Ses72**] Seshadri gives almost the same definition, except that instead of (4) he simply requires the condition (62.7.1.3) to hold – he does not require it to hold universally.

62.10. Geometric quotients

This is Mumford's definition of a geometric quotient (at least the definition from the first edition of GIT; as far as we can tell later editions changed "universally submersive" to "submersive").

Definition 62.10.1. Let S be a scheme and B an algebraic space over S. Let $j : R \to U \times_B U$ be a pre-relation. A morphism $\phi : U \to X$ of algebraic spaces over B is called a *geometric quotient* if

(1) ϕ is an orbit space,
(2) condition (62.7.1.1) holds universally, i.e., ϕ is universally submersive, and
(3) the functions on X are the R-invariant functions on U.

62.11. Other chapters

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CHAPTER 63

Simplicial Spaces

63.1. Introduction

This chapter develops some theory concerning simplicial topological spaces, simplicial ringed spaces, simplicial schemes, and simplicial algebraic spaces. The theory of simplicial spaces sometimes allows one to prove local to global principles which appear difficult to prove in other ways. Some example applications can be found in the papers [**Fal03**], [**Kie72**], and [**Del74b**].

We assume throughout that the reader is familiar with the basic concepts and results of the chapter Simplical Methods, see Simplicial, Section 14.1. In particular, we continue to write X and not $X_\bullet$ for a simplicial object.

63.2. Simplicial topological spaces

A *simplicial space* is a simplicial object in the category of topological spaces where morphisms are continuous maps of topological spaces. (We will use "simplicial algebraic space" to refer to simplicial objects in the category of algebraic spaces.) We may picture a simplicial space X as follows

$$X_2 \;\substack{\longrightarrow\\ \longleftarrow\\ \longrightarrow\\ \longleftarrow\\ \longrightarrow}\; X_1 \;\substack{\longrightarrow\\ \longleftarrow\\ \longleftarrow}\; X_0$$

Here there are two morphisms $d_0^1, d_1^1 : X_1 \to X_0$ and a single morphism $s_0^0 : X_0 \to X_1$, etc. It is important to keep in mind that $d_i^n : X_n \to X_{n-1}$ should be thought of as a "projection forgetting the ith coordinate" and $s_j^n : X_n \to X_{n+1}$ as the diagonal map repeating the jth coordinate.

Let X be a simplicial space. We associate a site X_{Zar}[1] to X as follows.

(1) An object of X_{Zar} is an open U of X_n for some n,
(2) a morphism $U \to V$ of X_{Zar} is given by a $\varphi : [m] \to [n]$ where n, m are such that $U \subset X_n$, $V \subset X_m$ and φ is such that $X(\varphi)(U) \subset V$, and
(3) a covering $\{U_i \to U\}$ in X_{Zar} means that $U, U_i \subset X_n$ are open, the maps $U_i \to U$ are given by $\mathrm{id} : [n] \to [n]$, and $U = \bigcup U_i$.

Note that in particular, if $U \to V$ is a morphism of X_{Zar} give by φ, then $X(\varphi) : X_n \to X_m$ does in fact induce a continuous map $U \to V$ of topological spaces.
It is clear that the above is a special case of a construction that associates to any diagram of topological spaces a site. We formulate the obligatory lemma.

Lemma 63.2.1. *Let X be a simplicial space. Then X_{Zar} as defined above is a site.*

Proof. Omitted. □

[1] This notation is similar to the notation in Sites, Example 7.6.4 and Topologies, Definition 33.3.7.

Let X be a simplicial space. Let $\mathcal{F}$ be a sheaf on X_{Zar}. It is clear from the definition of coverings, that the restriction of $\mathcal{F}$ to the opens of X_n defines a sheaf $\mathcal{F}_n$ on the topological space X_n. For every $\varphi : [m] \to [n]$ the restriction maps of $\mathcal{F}$ for pairs $U \subset X_n$, $V \subset X_m$ with $X(\varphi)(U) \subset V$, define an $X(\varphi)$-map $\mathcal{F}(\varphi) : \mathcal{F}_m \to \mathcal{F}_n$, see Sheaves, Definition 6.21.7. Moreover, given $\varphi : [m] \to [n]$ and $\psi : [l] \to [m]$ we have

$$\mathcal{F}(\psi) \circ \mathcal{F}(\varphi) = \mathcal{F}(\varphi \circ \psi)$$

(LHS uses composition of f-maps, see Sheaves, Definition 6.21.9). Clearly, the converse is true as well: if we have a system $(\{\mathcal{F}_n\}_{n\geq 0}, \{\mathcal{F}(\varphi)\}_{\varphi \in \text{Arrows}(\Delta)})$ as above, satisfying the displayed equalities, then we obtain a sheaf on X_{Zar}.

Lemma 63.2.2. *Let X be a simplicial space. There is an equivalence of categories between*

(1) *$Sh(X_{Zar})$, and*
(2) *category of systems $(\mathcal{F}_n, \mathcal{F}(\varphi))$ described above.*

Proof. See discussion above. □

Lemma 63.2.3. *Let $f : Y \to X$ be a morphism of simplicial spaces. Then the functor $u : X_{Zar} \to Y_{Zar}$ which associates to the open $U \subset X_n$ the open $f_n^{-1}(U) \subset Y_n$ defines a morphism of sites $f_{Zar} : Y_{Zar} \to X_{Zar}$.*

Proof. It is clear that u is a continuous functor. Hence we obtain functors $f_{Zar,*} = u^s$ and $f_{Zar}^{-1} = u_s$, see Sites, Section 7.15. To see that we obtain a morphism of sites we have to show that u^s is exact. We will use Sites, Lemma 7.15.5 to see this. Let $V \subset Y_n$ be an open subset. The category $\mathcal{I}_V^u$ (see Sites, Section 7.5) consists of pairs (U, φ) where $\varphi : [m] \to [n]$ and $U \subset X_m$ open such that $Y(\varphi)(V) \subset f_m^{-1}(U)$. Moreover, a morphism $(U, \varphi) \to (U', \varphi')$ is given by a $\psi : [m'] \to [m]$ such that $X(\psi)(U) \subset U'$ and $\varphi \circ \psi = \varphi'$. It is our task to show that $\mathcal{I}_V^u$ is cofiltered.

We verify the conditions of Categories, Definition 4.20.1. Condition (1) holds because $(X_n, \text{id}_{[n]})$ is an object. Let (U, φ) be an object. The condition $Y(\varphi)(V) \subset f_m^{-1}(U)$ is equivalent to $V \subset f_n^{-1}(X(\varphi)^{-1}(U))$. Hence we obtain a morphism $(X(\varphi)^{-1}(U), \text{id}_{[n]}) \to (U, \varphi)$ given by setting $\psi = \varphi$. Moreover, given a pair of objects of the form $(U, \text{id}_{[n]})$ and $(U', \text{id}_{[n]})$ we see there exists an object, namely $(U \cap U', \text{id}_{[n]})$, which maps to both of them. Thus condition (2) holds. To verify condition (3) suppose given two morphisms $a, a' : (U, \varphi) \to (U', \varphi')$ given by $\psi, \psi' : [m'] \to [m]$. Then precomposing with the morphism $(X(\varphi)^{-1}(U), \text{id}_{[n]}) \to (U, \varphi)$ given by φ equalizes a, a' because $\varphi \circ \psi = \varphi' = \varphi \circ \psi'$. This finishes the proof. □

Lemma 63.2.4. *Let $f : Y \to X$ be a morphism of simplicial spaces. In terms of the description of sheaves in Lemma 63.2.2 the morphism f_{Zar} of Lemma 63.2.3 can be described as follows.*

(1) *If $\mathcal{G}$ is a sheaf on Y, then $(f_{Zar,*}\mathcal{G})_n = f_{n,*}\mathcal{G}_n$.*
(2) *If $\mathcal{F}$ is a sheaf on X, then $(f_{Zar}^{-1}\mathcal{F})_n = f_n^{-1}\mathcal{F}_n$.*

Proof. The first part is immediate from the definitions. For the second part, note that in the proof of Lemma 63.2.3 we have shown that for a $V \subset Y_n$ open the category $(\mathcal{I}_V^u)^{opp}$ contains as a cofinal subcategory the category of opens $U \subset X_n$ with $f_n^{-1}(U) \supset V$ and morphisms given by inclusions. Hence we see that the restriction of $u_p\mathcal{F}$ to opens of Y_n is the presheaf $f_{n,p}\mathcal{F}_n$ as defined in Sheaves, Lemma 6.21.3. Since $f_{Zar}^{-1}\mathcal{F} = u_s\mathcal{F}$ is the sheafification of $u_p\mathcal{F}$ and since sheafification uses

only coverings and since coverings in Y_{Zar} use only inclusions between opens on the same Y_n, the result follows from the fact that $f_n^{-1}\mathcal{F}_n$ is (correspondingly) the sheafification of $f_{n,p}\mathcal{F}_n$, see Sheaves, Section 6.21. □

Let X be a topological space. In Sites, Example 7.6.4 we denoted X_{Zar} the site consisting of opens of X with inclusions as morphisms and coverings given by open coverings. We identify the topos $Sh(X_{Zar})$ with the category of sheaves on X.

Lemma 63.2.5. *Let X be a simplicial space. The functor $X_{n,Zar} \to X_{Zar}$, $U \mapsto U$ is continuous and cocontinuous. The associated morphism of topoi $g : Sh(X_n) \to Sh(X_{Zar})$ satisfies*

1. *g^{-1} associates to the sheaf $\mathcal{F}$ on X the sheaf $\mathcal{F}_n$ on X_n,*
2. *g^{-1} has a left adjoint $g_!^{Sh}$ which commutes with finite connected limits,*
3. *$g^{-1} : Ab(X_{Zar}) \to Ab(X_n)$ has a left adjoint $g_! : Ab(X_n) \to Ab(X_{Zar})$ which is exact.*

Proof. Besides the properties of our functor mentioned in the statement, the category $X_{n,Zar}$ has fibre products and equalizers and the functor commutes with them (beware that X_{Zar} does not have all fibre products). Hence the lemma follows from the discussion in Sites, Sections 7.19 and 7.20 and Modules on Sites, Section 18.16. More precisely, Sites, Lemmas 7.20.1, 7.20.5, and 7.20.6 and Modules on Sites, Lemmas 18.16.2 and 18.16.3. □

Lemma 63.2.6. *Let X be a simplicial space. If $\mathcal{I}$ is an injective abelian sheaf on X_{Zar}, then $\mathcal{I}_n$ is an injective abelian sheaf on X_n.*

Proof. This follows from Homology, Lemma 12.25.1 and Lemma 63.2.5. □

Lemma 63.2.7. *Let $f : Y \to X$ be a morphism of simplicial spaces. Then*

$$\begin{array}{ccc} Sh(Y_n) & \xrightarrow{f_n} & Sh(X_n) \\ \downarrow & & \downarrow \\ Sh(Y_{Zar}) & \xrightarrow{f_{Zar}} & Sh(X_{Zar}) \end{array}$$

is a commutative diagram of topoi.

Proof. Direct from the description of pullback functors in Lemmas 63.2.4 and 63.2.5. □

Let X be a topological space. Denote $X_\bullet$ the constant simplicial topological space with value X. By Lemma 63.2.2 a sheaf on $X_{\bullet,Zar}$ is the same thing as a cosimplicial object in the category of sheaves on X.

Lemma 63.2.8. *Let X be a topological space. Let $X_\bullet$ be the constant simplical topological space with value X. The functor*

$$X_{\bullet,Zar} \longrightarrow X_{Zar}, \quad U \longmapsto U$$

is continuous and cocontinuous and defines a morphism of topoi $g : Sh(X_{\bullet,Zar}) \to Sh(X)$ as well as a left adjoint $g_!$ to g^{-1}. We have

1. *g^{-1} associates to a sheaf on X the constant cosimplicial sheaf on X,*
2. *$g_!$ associates to a sheaf $\mathcal{F}$ on $X_{\bullet,Zar}$ the sheaf $\mathcal{F}_0$, and*
3. *g_* associates to a sheaf $\mathcal{F}$ on $X_{\bullet,Zar}$ the equalizer of the two maps $\mathcal{F}_0 \to \mathcal{F}_1$.*

Proof. The statements about the functor are straightforward to verify. The existence of g and $g_!$ follow from Sites, Lemmas 7.20.1 and 7.20.5. The description of g^{-1} is immediate from Sites, Lemma 7.20.5. The description of g_* and $g_!$ follows as the functors given are right and left adjoint to g^{-1}. □

Lemma 63.2.9. *Let Y be a simplicial space and X a topological space. Let $a : Y \to X$ be an augmentation (Simplicial, Definition 14.18.1). There is a canonical morphism of topoi*

$$a : Sh(Y_{Zar}) \to Sh(X)$$

which comes from composing the morphism $a_{Zar} : Sh(Y_{Zar}) \to Sh(X_{\bullet,Zar})$ of Lemma 63.2.3 with the morphism $g : Sh(X_{\bullet,Zar}) \to Sh(X)$ of Lemma 63.2.8.

Proof. This lemma proves itself. □

Lemma 63.2.10. *Let X be a simplicial topological space. The complex of abelian presheaves on X_{Zar}*

$$\ldots \to \mathbf{Z}_{X_2} \to \mathbf{Z}_{X_1} \to \mathbf{Z}_{X_0}$$

with boundary $\sum(-1)^i d_i^n$ is a resolution of the constant presheaf $\mathbf{Z}$.

Proof. Let $U \subset X_m$ be an object of X_{Zar}. Then the value of the complex above on U is the complex of abelian groups

$$\ldots \to \mathbf{Z}[\mathrm{Mor}_\Delta([2],[m])] \to \mathbf{Z}[\mathrm{Mor}_\Delta([1],[m])] \to \mathbf{Z}[\mathrm{Mor}_\Delta([0],[m])]$$

In other words, this is the complex associated to the free abelian group on the simplicial set $\Delta[m]$, see Simplicial, Example 14.11.2. Since $\Delta[m]$ is homotopy equivalent to $\Delta[0]$, see Simplicial, Example 14.24.7, and since "taking free abelian groups" is a functor, we see that the complex above is homotopy equivalent to the free abelian group on $\Delta[0]$ (Simplicial, Remark 14.24.4 and Lemma 14.25.2). This complex is acyclic in positive degrees and equal to $\mathbf{Z}$ in degree 0. □

Lemma 63.2.11. *Let X be a simplicial topological space. Let $\mathcal{F}$ be an abelian sheaf on X. There is a spectral sequence $(E_r, d_r)_{r\geq 0}$ with*

$$E_1^{p,q} = H^q(X_p, \mathcal{F}_p)$$

converging to $H^{p+q}(X_{Zar}, \mathcal{F})$. This spectral sequence is functorial in $\mathcal{F}$.

Proof. Let $\mathcal{F} \to \mathcal{I}^\bullet$ be an injective resolution. Consider the double complex with terms

$$A^{p,q} = \mathcal{I}^q(X_p)$$

and first differential given by the alternating sum along the maps d_i^{p+1}-maps $\mathcal{I}_p^q \to \mathcal{I}_{p+1}^q$, see Lemma 63.2.2. Note that

$$A^{p,q} = \Gamma(X_p, \mathcal{I}_p^q) = \mathrm{Mor}_{PSh}(h_{X_p}, \mathcal{I}^q) = \mathrm{Mor}_{PAb}(\mathbf{Z}_{X_p}, \mathcal{I}^q)$$

Hence it follows from Lemma 63.2.10 and Cohomology on Sites, Lemma 21.11.1 that the rows of the double complex are exact in positive degrees and evaluate to $\Gamma(X_{Zar}, \mathcal{I}^q)$ in degree 0. On the other hand, since restriction is exact (Lemma 63.2.5) the map

$$\mathcal{F}_p \to \mathcal{I}_p^\bullet$$

is a resolution. The sheaves $\mathcal{I}_p^q$ are injective abelian sheaves on X_p (Lemma 63.2.6). Hence the cohomology of the columns computes the groups $H^q(X_p, \mathcal{F}_p)$. We conclude by applying Homology, Lemmas 12.22.5 and 12.22.6. □

63.3. Simplicial sites and topoi

It seems natural to define a *simplicial site* as a simplicial object in the (big) category whose objects are sites and whose morphisms are morphisms of sites. See Sites, Definitions 7.6.2 and 7.15.1 with composition of morphisms as in Sites, Lemma 7.15.3. But here are some variants one might want to consider: (a) we could work with cocontinuous functors (see Sites, Sections 7.19 and 7.20) between sites instead, (b) we could work in a suitable 2-category of sites where one introduces the notion of a 2-morphism between morphisms of sites, (c) we could work in a 2-category constructed out of cocontinuous functors. Instead of picking one of these variants as a definition we will simply develop theory as needed.

Certainly a *simplicial topos* should probably be defined as a pseudo-functor from Δ^{opp} into the 2-category of topoi. See Categories, Definition 4.27.5 and Sites, Section 7.16 and 7.35. We will try to avoid working with such a beast if possible.

Let $\mathcal{C}$ be a simplicial object in the category whose objects are sites and whose morphisms are morphisms of sites. This means that for every morphism $\varphi : [m] \to [n]$ of Δ we have a morphism of sites $f_\varphi : \mathcal{C}_n \to \mathcal{C}_m$. This morphism is given by a continuous functor in the oppsite direction which we will denote $u_\varphi : \mathcal{C}_m \to \mathcal{C}_n$.

Lemma 63.3.1. *Let $\mathcal{C}$ be a simplicial object in the category of sites. With notation as above we construct a site $\mathcal{C}_{site}$ as follows.*

1. *An object of $\mathcal{C}_{site}$ is an object U of $\mathcal{C}_n$ for some n,*
2. *a morphism $(\varphi, f) : U \to V$ of $\mathcal{C}_{site}$ is given by a map $\varphi : [m] \to [n]$ with $U \in \mathrm{Ob}(\mathcal{C}_n)$, $V \in \mathrm{Ob}(\mathcal{C}_m)$ and a morphism $f : U \to u_\varphi(V)$ of $\mathcal{C}_n$, and*
3. *a covering $\{(id, f_i) : U_i \to U\}$ in $\mathcal{C}_{site}$ is given by an n and a covering $\{f_i : U_i \to U\}$ of $\mathcal{C}_n$.*

Proof. Composition of $(\varphi, f) : U \to V$ with $(\psi, g) : V \to W$ is given by $(\varphi \circ \psi, u_\varphi(g) \circ f)$. This uses that $u_\varphi \circ u_\psi = u_{\varphi\circ\psi}$.

Let $\{(\mathrm{id}, f_i) : U_i \to U\}$ be a covering as in (3) and let $(\varphi, g) : W \to U$ be a morphism with $W \in \mathrm{Ob}(\mathcal{C}_m)$. We claim that

$$W \times_{(\varphi,g),U,(\mathrm{id},f_i)} U_i = W \times_{g,u_\varphi(U),u_\varphi(f_i)} u_\varphi(U_i)$$

in the category $\mathcal{C}_{site}$. This makes sense as by our definition of morphisms of sites, the required fibre products in $\mathcal{C}_m$ exist since u_φ transforms coverings into coverings. The same reasoning implies the claim (details omitted). Thus we see that the collection of coverings is stable under base change. The other axioms of a site are immediate. □

Let $\mathcal{C}$ be a simplicial object in the category whose objects are sites and whose morphisms are cocontinuous functors. This means that for every morphism $\varphi : [m] \to [n]$ of Δ we have a cocontinuous functor denoted $u_\varphi : \mathcal{C}_n \to \mathcal{C}_m$.

Lemma 63.3.2. *Let $\mathcal{C}$ be a simplicial object in the category whose objects are sites and whose morphisms are cocontinuous functors. With notation as above, assume the functors $u_\varphi : \mathcal{C}_n \to \mathcal{C}_m$ have property P of Sites, Remark 7.19.5. Then we can construct a site $\mathcal{C}_{site}$ as follows.*

1. *An object of $\mathcal{C}_{site}$ is an object U of $\mathcal{C}_n$ for some n,*
2. *a morphism $(\varphi, f) : U \to V$ of $\mathcal{C}_{site}$ is given by a map $\varphi : [m] \to [n]$ with $U \in \mathrm{Ob}(\mathcal{C}_n)$, $V \in \mathrm{Ob}(\mathcal{C}_m)$ and a morphism $f : u_\varphi(U) \to V$ of $\mathcal{C}_m$, and*

(3) *a covering* $\{(id, f_i) : U_i \to U\}$ *in* $\mathcal{C}_{site}$ *is given by an* n *and a covering* $\{f_i : U_i \to U\}$ *of* $\mathcal{C}_n$.

Proof. Composition of $(\varphi, f) : U \to V$ with $(\psi, g) : V \to W$ is given by $(\varphi \circ \psi, g \circ u_\psi(f))$. This uses that $u_\psi \circ u_\varphi = u_{\varphi\circ\psi}$.

Let $\{(\mathrm{id}, f_i) : U_i \to U\}$ be a covering as in (3) and let $(\varphi, g) : W \to U$ be a morphism with $W \in \mathrm{Ob}(\mathcal{C}_m)$. We claim that

$$W \times_{(\varphi,g),U,(\mathrm{id},f_i)} U_i = W \times_{g,U,f_i} U_i$$

in the category $\mathcal{C}_{site}$ where the right hand side is the object of $\mathcal{C}_m$ defined in Sites, Remark 7.19.5 which exists by property P. Compatibility of this type of fibre product with compositions of functors implies the claim (details omitted). Since the family $\{W \times_{g,U,f_i} U_i \to W\}$ is a covering of $\mathcal{C}_m$ by property P we see that the collection of coverings is stable under base change. The other axioms of a site are immediate. □

Situation 63.3.3. Here we have one of the following two cases

(A) $\mathcal{C}$ is a simplicial object in the category whose objects are sites and whose morphisms are morphisms of sites. For every morphism $\varphi : [m] \to [n]$ of Δ we have a morphism of sites $f_\varphi : \mathcal{C}_n \to \mathcal{C}_m$ given by a continuous functor $u_\varphi : \mathcal{C}_m \to \mathcal{C}_n$.

(B) $\mathcal{C}$ is a simplicial object in the category whose objects are sites and whose morphisms are cocontinuous functors having property P of Sites, Remark 7.19.5. For every morphism $\varphi : [m] \to [n]$ of Δ we have a cocontinuous functor $u_\varphi : \mathcal{C}_n \to \mathcal{C}_m$ which induces a morphism of topoi $f_\varphi : Sh(\mathcal{C}_n) \to Sh(\mathcal{C}_m)$.

As usual we will denote f_φ^{-1} and $f_{\varphi,*}$ the pullback and pushforward. We let $\mathcal{C}_{site}$ denote the site defined in Lemma 63.3.1 (case A) or Lemma 63.3.2 (case B).

Let $\mathcal{C}$ be as in Situation 63.3.3. Let $\mathcal{F}$ be a sheaf on $\mathcal{C}_{site}$. It is clear from the definition of coverings, that the restriction of $\mathcal{F}$ to the objects of $\mathcal{C}_n$ defines a sheaf $\mathcal{F}_n$ on the site $\mathcal{C}_n$. For every $\varphi : [m] \to [n]$ the restriction maps of $\mathcal{F}$ along the morphisms $(\varphi, f) : U \to V$ with $U \in \mathrm{Ob}(\mathcal{C}_n)$ and $V \in \mathrm{Ob}(\mathcal{C}_m)$ define an element $\mathcal{F}(\varphi)$ of

$$\mathrm{Mor}_{Sh(\mathcal{C}_m)}(\mathcal{F}_m, f_{\varphi,*}\mathcal{F}_n) = \mathrm{Mor}_{Sh(\mathcal{C}_n)}(f_\varphi^{-1}\mathcal{F}_m, \mathcal{F}_n)$$

Moreover, given $\varphi : [m] \to [n]$ and $\psi : [l] \to [m]$ we have

$$f_\varphi^{-1}\mathcal{F}(\psi) \circ \mathcal{F}(\varphi) = \mathcal{F}(\varphi \circ \psi)$$

Clearly, the converse is true as well: if we have a system $(\{\mathcal{F}_n\}_{n\geq 0}, \{\mathcal{F}(\varphi)\}_{\varphi\in \mathrm{Arrows}(\Delta)})$ as above, satisfying the displayed equalities, then we obtain a sheaf on $\mathcal{C}_{site}$.

Lemma 63.3.4. *In Situation 63.3.3 there is an equivalence of categories between*

(1) $Sh(\mathcal{C}_{site})$, *and*

(2) *category of systems* $(\mathcal{F}_n, \mathcal{F}(\varphi))$ *described above.*

In particular, the topos $Sh(\mathcal{C}_{site})$ *only depends on the topoi* $Sh(\mathcal{C}_n)$ *and the morphisms of topoi* f_φ.

Proof. See discussion above. □

Lemma 63.3.5. *In Situation 63.3.3 the functor* $\mathcal{C}_n \to \mathcal{C}_{site}$, $U \mapsto U$ *is continuous and cocontinuous. The associated morphism of topoi* $g : Sh(\mathcal{C}_n) \to Sh(\mathcal{C}_{site})$ *satisfies*

(1) g^{-1} *associates to the sheaf* $\mathcal{F}$ *on* $\mathcal{C}_{site}$ *the sheaf* $\mathcal{F}_n$ *on* $\mathcal{C}_n$,
(2) g^{-1} *has a left adjoint* $g_!^{Sh}$ *which commutes with finite connected limits, and*
(3) $g^{-1} : Ab(\mathcal{C}_{site}) \to Ab(\mathcal{C}_n)$ *has a left adjoint* $g_! : Ab(\mathcal{C}_n) \to Ab(\mathcal{C}_{site})$ *which is exact.*

Proof. It is clear that functor $\mathcal{C}_n \to \mathcal{C}_{site}$ is continuous and cocontinuous. Hence part (1) and the existence of $g_!^{Sh}$ and $g_!$ follows from Sites, Lemmas 7.20.1 and 7.20.5 and Modules on Sites, Lemmas 18.16.2 and 18.16.4.

Next, we come to the exactness properties of $g_!^{Sh}$ and $g_!$. Perhaps the most straightforward way to prove this is to give a formula for these functors. If $\mathcal{G}$ is a sheaf on $\mathcal{C}_n$, then we claim $\mathcal{H} = g_!^{Sh}\mathcal{G}$ is the sheaf on $\mathcal{C}_{site}$ whose degree m part is the sheaf

$$\mathcal{H}_m = \coprod\nolimits_{\varphi:[n]\to[m]} f_\varphi^{-1}\mathcal{G}$$

Given a map $\psi : [m] \to [m']$ the map $\mathcal{H}(\psi) : f_\psi^{-1}\mathcal{H}_m \to \mathcal{H}_{m'}$ is given on components by the identifications

$$f_\psi^{-1} f_\varphi^{-1}\mathcal{G} \to f_{\psi\circ\varphi}^{-1}\mathcal{G}$$

Observe that given a map $a : \mathcal{H} \to \mathcal{F}$ of sheaves on $\mathcal{C}_{site}$ we obtain a map $\mathcal{G} \to \mathcal{F}_n$ corresponding to the restriction of a_n to the component $\mathcal{G}$ in $\mathcal{H}_n$. Conversely, given $b : \mathcal{G} \to \mathcal{H}_n$ we can define $a : \mathcal{H} \to \mathcal{F}$ by letting a_m be the map which on components

$$f_\varphi^{-1}\mathcal{G} \to \mathcal{F}_m$$

uses the maps adjoint to $\mathcal{F}(\varphi) \circ f_\varphi^{-1} b$. We omit the arguments showing these two constructions give mutually inverse maps

$$\mathrm{Mor}_{Sh(\mathcal{C}_n)}(\mathcal{G}, \mathcal{F}_n) = \mathrm{Mor}_{Sh(\mathcal{C}_{site})}(\mathcal{H}, \mathcal{F})$$

thus verifying the claim above. If $\mathcal{G}$ is an abelian sheaf on $\mathcal{C}_n$, then $g_!\mathcal{G}$ is the abelian sheaf on $\mathcal{C}_{site}$ whose degree m part is the sheaf

$$\bigoplus\nolimits_{\varphi:[n]\to[m]} f_\varphi^{-1}\mathcal{G}$$

with transition maps defined exactly as above. By definition of the site $\mathcal{C}_{site}$ we see that these functors have the desired exactness properties and we conclude. □

Lemma 63.3.6. *In Situation 63.3.3. If* $\mathcal{I}$ *is an injective abelian sheaf on* $\mathcal{C}_{site}$*, then* $\mathcal{I}_n$ *is an injective abelian sheaf on* $\mathcal{C}_n$*.*

Proof. This follows from Homology, Lemma 12.25.1 and Lemma 63.3.5. □

Let $\mathcal{C}$ be as in Situation 63.3.3. In statement of the following lemmas we will let $g_n : \mathcal{C}_n \to \mathcal{C}_{site}$ be the functor of Lemma 63.3.5. If $\varphi : [m] \to [n]$ is a morphism of Δ, then the diagram of topoi

$$\begin{array}{ccc} Sh(\mathcal{C}_n) & \xrightarrow{f_\varphi} & Sh(\mathcal{C}_m) \\ & {\scriptstyle g_n}\searrow \quad \swarrow{\scriptstyle g_m} & \\ & Sh(\mathcal{C}_{site}) & \end{array}$$

is not commutative, but there is a 2-morphism $g_n \to g_m \circ f_\varphi$ coming from the maps $\mathcal{F}(\varphi) : f_\varphi^{-1}\mathcal{F}_m \to \mathcal{F}_n$. See Sites, Section 7.35.

Lemma 63.3.7. *In Situation 63.3.3 and with notation as above there is a complex*

$$\ldots \to g_{2!}\mathbf{Z} \to g_{1!}\mathbf{Z} \to g_{0!}\mathbf{Z}$$

of abelian sheaves on $\mathcal{C}_{site}$ which forms a resolution of the constant sheaf with value $\mathbf{Z}$ on $\mathcal{C}_{site}$.

Proof. We will use the description of the functors $g_{n!}$ in the proof of Lemma 63.3.5 without further mention. As maps of the complex we take $\sum(-1)^i d_i^n$ where $d_i^n : g_{n!}\mathbf{Z} \to g_{n-1!}\mathbf{Z}$ is the adjoint to the map $\mathbf{Z} \to \bigoplus_{[n-1]\to[n]} \mathbf{Z} = g_n^{-1}g_{n-1!}\mathbf{Z}$ corresponding to the factor labeled with $\delta_i^n : [n-1] \to [n]$. Then g_m^{-1} applied to the complex gives the complex

$$\ldots \to \bigoplus\nolimits_{\alpha\in \mathrm{Mor}_\Delta([2],[m])} \mathbf{Z} \to \bigoplus\nolimits_{\alpha\in \mathrm{Mor}_\Delta([1],[m])} \mathbf{Z} \to \bigoplus\nolimits_{\alpha\in \mathrm{Mor}_\Delta([0],[m])} \mathbf{Z}$$

on $\mathcal{C}_m$. In other words, this is the complex associated to the free abelian sheaf on the simplicial set $\Delta[m]$, see Simplicial, Example 14.11.2. Since $\Delta[m]$ is homotopy equivalent to $\Delta[0]$, see Simplicial, Example 14.24.7, and since "taking free abelian sheaf on" is a functor, we see that the complex above is homotopy equivalent to the free abelian sheaf on $\Delta[0]$ (Simplicial, Remark 14.24.4 and Lemma 14.25.2). This complex is acyclic in positive degrees and equal to $\mathbf{Z}$ in degree 0. □

Lemma 63.3.8. *In Situation 63.3.3. Let $\mathcal{F}$ be an abelian sheaf on $\mathcal{C}_{site}$. There is a spectral sequence $(E_r, d_r)_{r\geq 0}$ with*

$$E_1^{p,q} = H^q(\mathcal{C}_p, \mathcal{F}_p)$$

converging to $H^{p+q}(\mathcal{C}_{site}, \mathcal{F})$. This spectral sequence is functorial in $\mathcal{F}$.

Proof. Let $\mathcal{F} \to \mathcal{I}^\bullet$ be an injective resolution. Consider the double complex with terms

$$A^{p,q} = \Gamma(\mathcal{C}_p, \mathcal{I}_p^q)$$

and first differential given by the alternating sum along the maps d_i^{p+1}-maps $\mathcal{I}_p^q \to \mathcal{I}_{p+1}^q$, see Lemma 63.3.4. Note that

$$A^{p,q} = \Gamma(\mathcal{C}_p, \mathcal{I}_p^q) = \mathrm{Mor}_{Ab(\mathcal{C}_{site})}(g_{p!}\mathbf{Z}, \mathcal{I}^q)$$

Hence it follows from Lemma 63.3.7 that the rows of the double complex are exact in positive degrees and evaluate to $\Gamma(\mathcal{C}_{site}, \mathcal{I}^q)$ in degree 0. On the other hand, since restriction is exact (Lemma 63.3.5) the map

$$\mathcal{F}_p \to \mathcal{I}_p^\bullet$$

is a resolution. The sheaves $\mathcal{I}_p^q$ are injective abelian sheaves on $\mathcal{C}_p$ (Lemma 63.3.6). Hence the cohomology of the columns computes the groups $H^q(\mathcal{C}_p, \mathcal{F}_p)$. We conclude by applying Homology, Lemmas 12.22.5 and 12.22.6. □

63.4. Simplicial semi-representable objects

Let $\mathcal{C}$ be a site. Recall that $\mathrm{SR}(\mathcal{C})$ denotes the category of semi-representable objects of $\mathcal{C}$. See Hypercoverings, Definition 24.2.1. For an object $K = \{U_i\}_{i\in I}$ of $\mathrm{SR}(\mathcal{C})$ we will use the notation

$$\mathcal{C}/K = \coprod\nolimits_{i\in I} \mathcal{C}/U_i$$

and we will call it the *localization of $\mathcal{C}$ at K*. There is a natural structure of a site on this category, with coverings inherited from the localizations $\mathcal{C}/U_i$ (and whence

from $\mathcal{C}$). If $f : K \to L$ is a morphism of $\mathrm{SR}(\mathcal{C})$, then we obtain a cocontinuous functor

$$f : \mathcal{C}/K \longrightarrow \mathcal{C}/L$$

by applying the construction of Sites, Lemma 7.24.7 to the components. More precisely, if $f = (\alpha, f_i)$ where $K = \{U_i\}_{i \in I}$, $L = \{V_j\}_{j \in J}$, $\alpha : I \to J$, and $f_i : U_i \to V_{\alpha(i)}$ then f maps the component $\mathcal{C}/U_i$ into the component $\mathcal{C}/V_{\alpha(i)}$ via the construction of the aforementioned lemma.

Let K be a simplicial object of $\mathrm{SR}(\mathcal{C})$. By the construction above we obtain a simplicial object $n \mapsto \mathcal{C}/K_n$ in the category whose objects are sites and whose morphisms are cocontinuous functors of sites. Since these localization functors satisfy the assumption of Lemma 63.3.2 by Sites, Remark 7.24.10 we obtain a site $(\mathcal{C}/K)_{site}$.

We can describe this site explicitly as follows. Say $K_n = \{U_{n,i}\}_{i \in I_n}$ and that for $\varphi : [m] \to [n]$ the morphism $K(\varphi) : K_n \to K_m$ is given by $a(\varphi) : I_n \to I_m$ and $f_{\varphi,i} : U_{n,i} \to U_{m,a(\varphi)(i)}$ for $i \in I_n$. Then we have

(1) an object of $\mathcal{C}/K$ corresponds to an object $(U/U_{n,i})$ of $\mathcal{C}/U_{n,i}$ for some n and some $i \in I_n$,
(2) a morphism between U and V is a pair (φ, f) where $\varphi : [m] \to [n]$ with $U/U_{n,i}$ and $V/U_{m,a(\varphi)(i)}$ and $f : U \to V$ is a morphism of $\mathcal{C}$ such that

$$\begin{array}{ccc} U & \xrightarrow{f} & V \\ \downarrow & & \downarrow \\ U_{n,i} & \xrightarrow{f_{\varphi,i}} & U_{m,a(\varphi)(i)} \end{array}$$

is commutative, and
(3) a covering $\{(\mathrm{id}, f_j) : U_j \to U\}$ is given by an n and $i \in I_n$ and objects $U/U_{n,i}$, $U_j/U_{n,i}$ such that $\{f_j : U_j \to U\}$ is a covering of $\mathcal{C}$.

Lemma 63.4.1. *Let $\mathcal{C}$ be a site. Let K be a simplicial object of $SR(\mathcal{C})$. If $\mathcal{C}$ has fibre products, then $\mathcal{C}/K$ can also be viewed as a simplicial object in the category whose objects are sites and whose morphisms are morphisms of sites. The construction of Lemma 63.3.1 then produces the same site as the construction above.*

Proof. Given a morphism of objects $U \to V$ of $\mathcal{C}$ the localization morphism $j : \mathcal{C}/U \to \mathcal{C}/U$ is a left adjoint to the base change functor $\mathcal{C}/V \to \mathcal{C}/U$. The base change functor is continuous and induces the same morphism of topoi as j. See Sites, Lemma 7.26.3. Argueing as above we can use this to define a morphism of sites $\mathcal{C}/A \to \mathcal{C}/B$ given any morphism $A \to B$ of $\mathrm{SR}(\mathcal{C})$. Applying this to the morphisms of the simplicial object K we obtain simplicial object $(\mathcal{C}/K)'$ in the category of sites with morphisms of sites. Let $(\mathcal{C}/K)'_{site}$ be the site constructed in Lemma 63.3.1. Since the base change functors are adjoint to the localization functors, we find that $(\mathcal{C}/K)'_{site}$ is the same as the category $(\mathcal{C}/K)_{site}$. Equality of the sets of coverings is immediate from the definitions. □

Let $\mathcal{C}$ be a site. Let $L = \{V_i\}$ be an object of $\mathrm{SR}(\mathcal{C})$. There is a continuous and cocontinuous localization functor $j : \mathcal{C}/K \to \mathcal{C}$ which is the product of the localization functors $\mathcal{C}/V_i \to \mathcal{C}$. We obtain functors j^{-1}, j_*, $j_!^{Sh}$, and $j_!$ exactly as in Sites, Section 7.24 and Modules on Sites, Section 18.19. Given a simplicial object K of $\mathrm{SR}(\mathcal{C})$ we obtain a family of localization functors $j_n : \mathcal{C}/K_n \to \mathcal{C}$.

Lemma 63.4.2. *Let $\mathcal{C}$ be a site. Let K be a simplicial object of $SR(\mathcal{C})$. The forgetful functor $(\mathcal{C}/K)_{site} \to \mathcal{C}$ is continuous and cocontinuous and induces a morphism of topoi*

$$g : Sh((\mathcal{C}/K)_{site}) \longrightarrow Sh(\mathcal{C})$$

as well as functors $g_!^{Sh}$ and $g_!$ left adjoint to g^{-1} on sheaves of sets and abelian groups with the following properties:

1. *the functor g^{-1} associates to a sheaf $\mathcal{F}$ on $\mathcal{C}$ the sheaf on $(\mathcal{C}/K)_{site}$ wich in degree n is equal to $j_n^{-1}\mathcal{F}$,*
2. *the functor g_* associates to a sheaf $\mathcal{G}$ on $(\mathcal{C}/K)_{site}$ the equalizer of the two maps $j_{0,*}\mathcal{G}_0 \to j_{1,*}\mathcal{G}_1$,*

Proof. The functor is continuous and cocontinuous by our choice of coverings and our description of (certain) fibre products in $(\mathcal{C}/K)_{site}$ in the proof of Lemma 63.3.2. Details omitted. Thus we obtain a morphism of topoi and functors $g_!^{Sh}$ and $g_!$, see Sites, Section 7.20 and Modules on Sites, Section 18.16. The description of g^{-1} is immediate from the definition as the compostion $\mathcal{C}/K_n \to \mathcal{C}/K \to \mathcal{C}$ is the localization morphism j_n.

Proof of (2). Let $\mathcal{F}$ be a sheaf on $\mathcal{C}$ and let $\mathcal{G}$ be a sheaf on $(\mathcal{C}/K)_{site}$. A map $a : g^{-1}\mathcal{F} \to \mathcal{G}$ corresponds to a system of maps $a_n : j_n^{-1}\mathcal{F} \to \mathcal{G}_n$ on $\mathcal{C}/K_n$ by Lemma 63.3.4. Taking $n = 0$ we get a map $j_0^{-1}\mathcal{F} \to \mathcal{G}_0$ which is adjoint to a map $a_0 : \mathcal{F} \to j_{0,*}\mathcal{G}_0$. Since a_0 is compatible with a_1 via the two maps $j_{0,*}\mathcal{G}_0 \to j_{1,*}\mathcal{G}_1$ we see that a_0 maps into the equalizer. Conversely, given a map $a_0 : \mathcal{F} \to j_{0,*}\mathcal{G}_0$ into the equalizer we can pick, for any n, one of the maps $j_{0,*}\mathcal{G}_0 \to j_{n,*}\mathcal{G}_n$ and compose to get a well defined map $a_n : \mathcal{F} \to j_{n,*}\mathcal{G}_n$. These fit together to define a map of sheaves $g^{-1}\mathcal{F} \to \mathcal{G}$. □

Lemma 63.4.3. *Let $\mathcal{C}$ be a site with equalizers and fibre products. Let $\mathcal{G}$ be a presheaf of sets on $\mathcal{C}$. Let K be a hypercovering of $\mathcal{G}$, see Hypercoverings, Definition 24.5.1. Then we have a canonical isomorphism*

$$R\Gamma(\mathcal{G}, E) = R\Gamma((\mathcal{C}/K)_{site}, g^{-1}E)$$

for $E \in D^+(\mathcal{C})$. If K is a hypercovering, then $R\Gamma(E) = R\Gamma((\mathcal{C}/K)_{site}, g^{-1}E)$.

Proof. First, let $\mathcal{I}$ be an injective abelian sheaf on $\mathcal{C}$. Then the spectral sequence of Lemma 63.3.8 for the sheaf $g^{-1}\mathcal{I}$ degenerates as $(g^{-1}\mathcal{I})_p$ is the restriction of $\mathcal{I}$ to $\mathcal{C}/K_p$ which is injective by Cohomology on Sites, Lemma 21.8.1 (extended in the obvious manner to localization at semi-representable objects of $\mathcal{C}$). Thus we see that the complex

$$\mathcal{I}(K_0) \to \mathcal{I}(K_1) \to \mathcal{I}(K_2) \to \ldots$$

computes $R\Gamma((\mathcal{C}/K)_{site}, g^{-1}\mathcal{I})$. This is exactly the Čech complex of $\mathcal{I}$ with respect to the simplicial object K of $\mathrm{SR}(\mathcal{C})$ as defined in Hypercoverings, Section 24.4. Thus Hypercoverings, Lemma 24.5.3 shows that this complex computes $R\Gamma(\mathcal{G}, \mathcal{I})$ (which has zero higher cohomology groups as $\mathcal{I}$ is injective). In other words, we have $H^0(\mathcal{G}, \mathcal{I}) = H^0((\mathcal{C}/K)_{site}, \mathcal{I})$ and $H^p(\mathcal{G}, \mathcal{I}) = H^p((\mathcal{C}/K)_{site}, \mathcal{I}) = 0$ for all $p > 0$.

The lemma now follows formally. Namely, let $A \in D^+(\mathcal{C})$ be arbitrary. We can represent A by a bounded below complex $\mathcal{I}^\bullet$ of injective abelian sheaves. By Leray's

acyclicity lemma (Derived Categories, Lemma 13.16.7) $R\Gamma((\mathcal{C}/K)_{site}, A)$ is computed by the complex $\Gamma((\mathcal{C}/K)_{site}, g^{-1}\mathcal{I}^\bullet)$ and $R\Gamma(\mathcal{G}, A)$ is computed by $\Gamma(\mathcal{G}, \mathcal{I}^\bullet)$. Since these complexes are the same we obtain the conclusion.

The final statement refers to the special case where $\mathcal{G} = *$ is the final object in the category of presheaves on $\mathcal{C}$. □

Lemma 63.4.4. *Let $\mathcal{C}$ be a site with fibre products. Let X be an object of $\mathcal{C}$. Let K be a hypercovering of X, see Hypercoverings, Definition 24.2.6. Then we have a canonical isomorphism*

$$R\Gamma(X, E) = R\Gamma((\mathcal{C}/K)_{site}, g^{-1}E)$$

for $E \in D^+(\mathcal{C})$.

Proof. If $\mathcal{C}$ also has equalizers, then this is a special case of Lemma 63.4.3 because a hypercovering of X is a hypercovering of h_X by Hypercoverings, Lemma 24.2.10. This also uses that $H^q(h_X, \mathcal{F}) = H^q(h_X^\#, \mathcal{F}) = H^q(X, \mathcal{F})$, see discussion in Hypercoverings, Section 24.5 and Cohomology on Sites, Section 21.13. In general (when $\mathcal{C}$ does not have equalizers) one proves this using *exactly* the same argument as in the proof of Lemma 63.4.3 but substituting Hypercoverings, Lemma 24.4.2 for Hypercoverings, Lemma 24.5.3. □

63.5. Hypercovering in a site

In the previous section we worked out, in great generality, how hypercoverings give rise to simplicial sites and how cohomology of (say) constant sheaves on this site computes the cohomology of the object the hypercovering is augmented towards. In this section we explain what this means in a special case.

Let $\mathcal{C}$ be a site with fibre products. Let X be an object of $\mathcal{C}$ and let $X_\bullet$ be a simplicial object of $\mathcal{C}$. Assume we have an augmentation

$$a : X_\bullet \to X$$

The discussion above turns this into a morphism of topoi

$$g : (\mathcal{C}/X_\bullet)_{site} \longrightarrow \mathcal{C}/X$$

Here an object of the site $(\mathcal{C}/X_\bullet)_{site}$ is given by a U/X_n and a morphism $(\varphi, f) : U/X_n \to V/X_m$ is given by a morphism $\varphi : [m] \to [n]$ in Δ and a morphism $f : U \to V$ such that the diagram

$$\begin{array}{ccc} U & \xrightarrow{f} & V \\ \downarrow & & \downarrow \\ X_n & \xrightarrow{\varphi} & X_m \end{array}$$

is commutative. The morphism of topoi g is given by the cocontinuous functor $U/X_n \mapsto U/X$. That's all folks!

Thus we may translate some of the results above to this setting. For example, let us say that the augmentation is a *hypercovering* if the following hold

(1) $\{X_0 \to X\}$ is a covering of $\mathcal{C}$,
(2) $\{X_1 \to X_0 \times_X X_0\}$ is a covering of $\mathcal{C}$,
(3) $\{X_{n+1} \to (\mathrm{cosk}_n \mathrm{sk}_n X_\bullet)_{n+1}\}$ is a covering of $\mathcal{C}$ for $n \geq 1$.

The category $\mathcal{C}/X$ has all finite limits, hence the coskelets used in the formulation above exist.

Lemma 63.5.1. *In the situation above assume that $X_\bullet$ is a hypercovering of X. Then we have a canonical isomorphism*

$$R\Gamma(X, E) = R\Gamma((\mathcal{C}/X_\bullet)_{site}, g^{-1}E)$$

for $E \in D^+(\mathcal{C}/X)$.

Proof. This is a special case of Lemma 63.4.4. □

63.6. Proper hypercoverings in topology

Let's work in the category LC of Hausdorff and locally quasi-compact topological spaces and continuous maps, see Cohomology on Sites, Section 21.23. Let X be an object of LC and let $X_\bullet$ be a simplicial object of LC. Assume we have an augmentation

$$a : X_\bullet \to X$$

We say that $X_\bullet$ is a *proper hypercovering* of X if

(1) $X_0 \to X$ is a proper surjective map,
(2) $X_1 \to X_0 \times_X X_0$ is a proper surjective map,
(3) $X_{n+1} \to (\mathrm{cosk}_n \mathrm{sk}_n X_\bullet)_{n+1}$ is a proper surjective map for $n \geq 1$.

The category LC has all finite limits, hence the coskelets used in the formulation above exist.

Principle: Proper hypercoverings can be used to compute cohomology.

A key idea behind the proof of the principle is to find a topology on LC which is stronger than the usual one such that (A) a surjective proper map defines a covering, and (B) cohomology of usual sheaves with respect to this stronger topology agrees with the usual cohomology. Properties (A) and (B) hold for the qc topology, see Cohomology on Sites, Section 21.23. Once we have (A) and (B) we deduce the principle via a combination of the spectral sequences of Hypercoverings, Lemma 24.4.3 and Lemma 63.2.11. The following lemma is just a first step.

Lemma 63.6.1. *In the situation above, let $\mathcal{F}$ be an abelian sheaf on X. Let $\mathcal{F}_n$ be the pullback to X_n. If $X_\bullet$ is a proper hypercovering of X, then there exists a canonical spectral sequence*

$$E_1^{p,q} = H^q(X_p, \mathcal{F}_p)$$

converging to $H^{p+q}(X, \mathcal{F})$.

Proof. By Cohomology on Sites, Lemma 21.23.6 we have

$$H^*(X, \mathcal{F}) = H^*(LC_{qc}/X, \epsilon^{-1}\pi^{-1}\mathcal{F}).$$

Since a proper surjective map defines a qc covering (Cohomology on Sites, Lemma 21.23.7) we see that $X_\bullet \to X$ is a hypercovering in the site LC_{qc} as in Section 63.5. Thus we have

$$R\Gamma(X, \mathcal{F}) = R\Gamma(LC_{qc}/X, \epsilon^{-1}\pi^{-1}\mathcal{F}) = R\Gamma((LC/X_\bullet)_{site}, g^{-1}\epsilon^{-1}\pi^{-1}\mathcal{F})$$

by Lemma 63.5.1. By Lemma 63.3.8 there is a spectral sequence with

$$E_1^{p,q} = H^q(LC_{qc}/X_p, (g^{-1}\epsilon^{-1}\pi^{-1}\mathcal{F})_p)$$

converging to the cohomology of $g^{-1}\epsilon^{-1}\pi^{-1}\mathcal{F}$. Finally, the restriction $(g^{-1}\epsilon^{-1}\pi^{-1}\mathcal{F})_p$ is just the restriction to LC_{qc}/X_p of $\epsilon^{-1}\pi^{-1}\mathcal{F}$ which by Cohomology on Sites, Lemma 21.23.5 is the pullback of $\mathcal{F}_p$ to LC_{qc}/X_p. By Cohomology on Sites, Lemma 21.23.6 again we conclude that

$$H^q(LC_{qc}/X_p, (g^{-1}\epsilon^{-1}\pi^{-1}\mathcal{F})_p) = H^q(X_p, \mathcal{F}_p)$$

and the proof is finished. □

Lemma 63.6.2. *In the situation above, let $\mathcal{F}$ be an abelian sheaf on X. Let $\mathcal{F}_\bullet$ be the pullback of $\mathcal{F}$ via $a : X_\bullet \to X$. If $X_\bullet$ is a proper hypercovering of X, then*

$$H^*(X, \mathcal{F}) = H^*((X_\bullet)_{Zar}, \mathcal{F}_\bullet)$$

Proof. Consider the continuous functor

$$(X_\bullet)_{Zar} \longrightarrow (LC_{qc}/X_\bullet)_{site}, \quad U \longmapsto U$$

We obtain a commutative diagram of topoi

$$\begin{array}{ccc} Sh((LC_{qc}/X_\bullet)_{site}) & \longrightarrow & Sh((X_\bullet)_{Zar}) \\ \downarrow{\scriptstyle g} & & \downarrow{\scriptstyle g} \\ Sh(LC_{qc}/X) & \xrightarrow{\pi\circ\epsilon} & Sh(X_{Zar}) \end{array}$$

Thus our sheaf $\mathcal{F}$ gives rise to a compatible collection of abelian sheaves in each topos. In the proof of Lemma 63.6.1 we have seen that the sheaf $\mathcal{F}$ has the same cohomology as the sheaf $\epsilon^{-1}\pi^{-1}\mathcal{F}$ and $g^{-1}\epsilon^{-1}\pi^{-1}\mathcal{F}$. On the other hand, the terms of the spectral sequence of Lemma 63.2.11 for $\mathcal{F}_\bullet$ are the same as those in the statement and proof of Lemma 63.6.1. A simple argument with spectral sequences then shows that the map

$$R\Gamma((X_\bullet)_{Zar}, \mathcal{F}_\bullet) \longrightarrow R\Gamma((LC_{qc}/X_\bullet)_{site}, g^{-1}\epsilon^{-1}\pi^{-1}\mathcal{F})$$

is an isomorphism. Some details omitted. □

Lemma 63.6.3. *In the situation above, assume $a : X_\bullet \to X$ gives a proper hypercovering of X. Then for all $K \in D^+(X)$*

$$K \to Ra_*(a^{-1}K)$$

is an isomorphism where $a : Sh((X_\bullet)_{Zar}) \to Sh(X)$ is as in Lemma 63.2.9.

Proof. Observe that for any abelian sheaf $\mathcal{F}$ on X the sheaf $R^q a_*(a^{-1}\mathcal{F})$ is the sheaf associated to the presheaf

$$U \mapsto H^q((U_\bullet)_{Zar}, a^{-1}\mathcal{F}) = H^q(U, \mathcal{F})$$

where $U_\bullet = a^{-1}(U)$. The last equality holds by Lemma 63.6.2. Thus $R^q a_*(a^{-1}\mathcal{F})$ is zero for $q > 0$ and equal to $\mathcal{F}$ for $q = 0$. This proves the result in case K consists of a single abelian sheaf in a single degree. The general case follows from this immediately. □

63.7. Simplicial schemes

A *simplicial scheme* is a simplicial object in the category of schemes, see Simplicial, Definition 14.3.1. Recall that a simplicial scheme looks like

$$X_2 \;\substack{\longrightarrow\\ \longrightarrow\\ \longrightarrow\\ \longrightarrow}\; X_1 \;\substack{\longrightarrow\\ \longleftarrow\\ \longrightarrow}\; X_0$$

Here there are two morphisms $d_0^1, d_1^1 : X_1 \to X_0$ and a single morphism $s_0^0 : X_0 \to X_1$, etc. It is important to keep in mind that $d_i^n : X_n \to X_{n-1}$ should be thought of as a "projection forgetting the ith coordinate" and $s_j^n : X_n \to X_{n+1}$ as the diagonal map repeating the jth coordinate.

63.8. Descent in terms of simplicial schemes

Cartesian morphisms are defined as follows.

Definition 63.8.1. Let $a : Y \to X$ be a morphism of simplicial schemes. We say a is *cartesian*, or that Y *is cartesian over* X, if for every morphism $\varphi : [n] \to [m]$ of Δ the corresponding diagram

$$\begin{array}{ccc} Y_m & \xrightarrow{a} & X_m \\ {\scriptstyle Y(\varphi)}\downarrow & & \downarrow{\scriptstyle X(\varphi)} \\ Y_n & \xrightarrow{a} & X_n \end{array}$$

is a fibre square in the category of schemes.

Cartesian morphisms are related to descent data. First we prove a general lemma describing the category of cartesian simplicial schemes over a fixed simplicial scheme. In this lemma we denote $f^* : Sch/X \to Sch/Y$ the base change functor associated to a morphism of schemes $Y \to X$.

Lemma 63.8.2. *Let X be a simplicial scheme. The category of simplicial schemes cartesian over X is equivalent to the category of pairs (V, φ) where V is a scheme over X_0 and*

$$\varphi : V \times_{X_0, d_1^1} X_1 \longrightarrow X_1 \times_{d_0^1, X_0} V$$

is an isomorphism over X_1 such that $(s_0^0)^*\varphi = id_V$ and such that

$$(d_1^2)^*\varphi = (d_0^2)^*\varphi \circ (d_2^2)^*\varphi$$

as morphisms of schemes over X_2.

Proof. The statement of the displayed equality makes sense because $d_1^1 \circ d_2^2 = d_1^1 \circ d_1^2$, $d_1^1 \circ d_0^2 = d_0^1 \circ d_2^2$, and $d_0^1 \circ d_0^2 = d_0^1 \circ d_1^2$ as morphisms $X_2 \to X_0$, see Simplicial, Remark 14.3.3 hence we can picture these maps as follows

$$\begin{array}{ccccc} & & X_2 \times_{d_1^1 \circ d_0^2, X_0} V & \xrightarrow[(d_0^2)^*\varphi]{} & X_2 \times_{d_0^1 \circ d_0^2, X_0} V & & \\ & \swarrow & & & & \searrow & \\ X_2 \times_{d_0^1 \circ d_2^2, X_0} V & & & & & & X_2 \times_{d_0^1 \circ d_1^2, X_0} V \\ & {\scriptstyle (d_2^2)^*\varphi}\nwarrow & & & & \nearrow{\scriptstyle (d_1^2)^*\varphi} & \\ & & X_2 \times_{d_1^1 \circ d_2^2, X_0} V & = & X_2 \times_{d_1^1 \circ d_1^2, X_0} V & & \end{array}$$

and the condition signifies the diagram is commutative. It is clear that given a simplicial scheme Y cartesian over X we can set $V = Y_0$ and φ equal to the composition

$$V \times_{X_0, d_1^1} X_1 = Y_0 \times_{X_0, d_1^1} X_1 = Y_1 = X_1 \times_{X_0, d_0^1} Y_0 = X_1 \times_{X_0, d_0^1} V$$

of identifications given by the cartesian structure. To prove this functor is an equivalence we construct a quasi-inverse. The construction of the quasi-inverse is analogous to the construction discussed in Descent, Section 34.3 from which we borrow the notation $\tau_i^n : [0] \to [n]$, $0 \mapsto i$ and $\tau_{ij}^n : [1] \to [n]$, $0 \mapsto i$, $1 \mapsto j$. Namely, given a pair (V, φ) as in the lemma we set $Y_n = X_n \times_{X(\tau_n^n), X_0} V$. Then given $\beta : [n] \to [m]$ we define $V(\beta) : Y_m \to Y_n$ as the pullback by $X(\tau^m_{\beta(n)m})$ of the map φ postcomposed by the projection $X_m \times_{X(\beta), X_n} Y_n \to Y_n$. This makes sense because

$$X_m \times_{X(\tau^m_{\beta(n)m}), X_1} X_1 \times_{d_1^1, X_0} V = X_m \times_{X(\tau_m^m), X_0} V = Y_m$$

and

$$X_m \times_{X(\tau^m_{\beta(n)m}), X_1} X_1 \times_{d_0^1, X_0} V = X_m \times_{X(\tau^m_{\beta(n)}), X_0} V = X_m \times_{X(\beta), X_n} Y_n.$$

We omit the verification that the commutativity of the displayed diagram above implies the maps compose correctly. We also omit the verification that the two functors are quasi-inverse to each other. □

Definition 63.8.3. Let $f : X \to S$ be a morphism of schemes. The *simplicial scheme associated to* f, denoted $(X/S)_\bullet$, is the functor $\Delta^{opp} \to Sch$, $[n] \mapsto X \times_S \ldots \times_S X$ described in Simplicial, Example 14.3.5.

Thus $(X/S)_n$ is the $(n+1)$-fold fibre product of X over S. The morphism $d_0^1 : X \times_S X \to X$ is the map $(x_0, x_1) \mapsto x_1$ and the morphism d_1^1 is the other projection. The morphism s_0^0 is the diagonal morphism $X \to X \times_S X$.

Lemma 63.8.4. *Let $f : X \to S$ be a morphism of schemes. Let $\pi : Y \to (X/S)_\bullet$ be a cartesian morphism of simplicial schemes. Set $V = Y_0$ considered as a scheme over X. The morphisms $d_0^1, d_1^1 : Y_1 \to Y_0$ and the morphism $\pi_1 : Y_1 \to X \times_S X$ induce isomorphisms*

$$V \times_S X \xleftarrow{(d_1^1, pr_1 \circ \pi_1)} Y_1 \xrightarrow{(pr_0 \circ \pi_1, d_0^1)} X \times_S V.$$

Denote $\varphi : V \times_S X \to X \times_S V$ the resulting isomorphism. Then the pair (V, φ) is a descent datum relative to $X \to S$.

Proof. This is a special case of (part of) Lemma 63.8.2 as the displayed equation of that lemma is equivalent to the cocycle condition of Descent, Definition 34.30.1. □

Lemma 63.8.5. *Let $f : X \to S$ be a morphism of schemes. The construction*

$$\begin{array}{c} \textit{category of cartesian} \\ \textit{schemes over } (X/S)_\bullet \end{array} \longrightarrow \begin{array}{c} \textit{category of descent data} \\ \textit{relative to } X/S \end{array}$$

of Lemma 63.8.4 is an equivalence of categories.

Proof. The functor from left to right is given in Lemma 63.8.4. Hence this is a special case of Lemma 63.8.2. □

We may reinterpret the pullback of Descent, Lemma 34.30.6 as follows. Suppose given a morphism of simplicial schemes $f : X' \to X$ and a cartesian morphism of simplicial schemes $Y \to X$. Then the fibre product (viewed as a "pullback")

$$f^*Y = Y \times_X X'$$

of simplicial schemes is a simplicial scheme cartesian over X'. Suppose given a commutative diagram of morphisms of schemes

$$\begin{array}{ccc} X' & \xrightarrow{f} & X \\ \downarrow & & \downarrow \\ S' & \longrightarrow & S. \end{array}$$

This gives rise to a morphism of simplicial schemes

$$f_\bullet : (X'/S')_\bullet \longrightarrow (X/S)_\bullet.$$

We claim that the "pullback" $f_\bullet^*$ along the morphism $f_\bullet : (X'/S')_\bullet \to (X/S)_\bullet$ corresponds via Lemma 63.8.5 with the pullback defined in terms of descent data in the aforementioned Descent, Lemma 34.30.6.

63.9. Quasi-coherent modules on simplicial schemes

In the following definition we make use of the description of sheaves on a simplicial space given in Lemma 63.2.2.

Definition 63.9.1. Let S be a scheme. Let U be a simplicial scheme over S.

(1) A *quasi-coherent sheaf* on U is given by a sheaf of $\mathcal{O}_U$-modules $\mathcal{F}$ such that $\mathcal{F}_n$ is quasi-coherent for all $n \geq 0$.
(2) A quasi-coherent sheaf $\mathcal{F}$ on U is *cartesian* if and only if all the maps $\mathcal{F}(\varphi) : \mathcal{F}_n \to \mathcal{F}_m$ induce isomorphisms $U(\varphi)^*\mathcal{F}_n \to \mathcal{F}_m$.

The property on pullbacks needs only be checked for the degeneracies.

Lemma 63.9.2. *Let S be a scheme. Let U be a simplicial scheme over S. Let $\mathcal{F}$ be a quasi-coherent module on U. Then $\mathcal{F}$ is cartesian if and only if the induced maps $(d_j^n)^*\mathcal{F}_{n-1} \to \mathcal{F}_n$ are isomorphisms.*

Proof. The category Δ is generated by the morphisms the morphisms δ_j^n and σ_j^n, see Simplicial, Lemma 14.2.2. Hence we only need to check the maps $(d_j^n)^*\mathcal{F}_{n-1} \to \mathcal{F}_n$ and $(s_j^n)^*\mathcal{F}_{n+1} \to \mathcal{F}_n$ are isomorphisms, see Simplicial, Lemma 14.3.2 for notation. But $d_j^{n+1} \circ s_j^n = \mathrm{id}_{U_n}$ so it the result for d_j^{n+1} implies the result for s_j^n. □

Lemma 63.9.3. *Let S be a scheme. Let U be a simplicial scheme over S. The category of cartesian quasi-coherent modules over U is equivalent to the category of pairs $(\mathcal{F}, \alpha)$ where $\mathcal{F}$ is a quasi-coherent module over U_0 and*

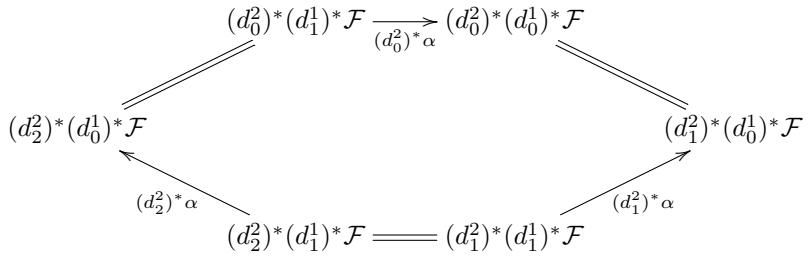
$$\alpha : (d_1^1)^*\mathcal{F} \longrightarrow (d_0^1)^*\mathcal{F}$$

is an isomorphism such that $(s_0^0)^*\alpha = id_{\mathcal{F}}$ and such that

$$(d_1^2)^*\alpha = (d_0^2)^*\alpha \circ (d_2^2)^*\alpha$$

on X_2.

Proof. The statement of the displayed equality makes sense because $d^1_1 \circ d^2_2 = d^1_1 \circ d^2_1$, $d^1_1 \circ d^2_0 = d^1_0 \circ d^2_2$, and $d^1_0 \circ d^2_0 = d^1_0 \circ d^2_1$ as morphisms $X_2 \to X_0$, see Simplicial, Remark 14.3.3 hence we can picture these maps as follows



and the condition signifies the diagram is commutative. It is clear that given a cartesian quasi-coherent sheaf $\mathcal{F}$ we can set $\mathcal{F} = \mathcal{F}_0$ and α equal to the composition

$$(d^0_1)^*\mathcal{F}_0 = \mathcal{F}_1 = (d^0_0)^*\mathcal{F}_0$$

of identifications given by the cartesian structure. To prove this functor is an equivalence we construct a quasi-inverse. The construction of the quasi-inverse is analogous to the construction discussed in Descent, Section 34.3 from which we borrow the notation $\tau^n_i : [0] \to [n]$, $0 \mapsto i$ and $\tau^n_{ij} : [1] \to [n]$, $0 \mapsto i$, $1 \mapsto j$. Namely, given a pair $(\mathcal{F}, \alpha)$ as in the lemma we set $\mathcal{F}_n = X(\tau^n_n)^*\mathcal{F}$. Then given $\beta : [n] \to [m]$ we define $\mathcal{F}(\beta) : \mathcal{F}_n \to \mathcal{F}_m$ as the pullback by $X(\tau^m_{\beta(n)m})$ of the map α precomposed with the canonical $X(\beta)$-map $\mathcal{F}_n \to X(\beta)^*\mathcal{F}_n$. We omit the verification that the commutativity of the displayed diagram above implies the maps compose correctly. We also omit the verification that the two functors are quasi-inverse to each other. □

Lemma 63.9.4. *Let $f : V \to U$ be a morphism of simplicial schemes. Given a cartesian quasi-coherent module $\mathcal{F}$ on U the pullback $f^*\mathcal{F}$ is a cartesian quasi-coherent module on V.*

Proof. This is immediate from the definitions. □

Lemma 63.9.5. *Let $f : V \to U$ be a cartesian morphism of simplicial schemes. Assume the morphisms $d^n_j : U_n \to U_{n-1}$ are flat and the morphisms $V_n \to U_n$ are quasi-compact and quasi-separated. For a cartesian quasi-coherent module $\mathcal{G}$ on V the pushforward $f_*\mathcal{G}$ is a cartesian quasi-coherent module on U.*

Proof. If $\mathcal{F} = f_*\mathcal{G}$, then $\mathcal{F}_n = f_{n,*}\mathcal{G}_n$ and the maps $\mathcal{F}(\varphi)$ are defined using the base change maps, see Cohomology, Section 20.18. The sheaves $\mathcal{F}_n$ are quasi-coherent by Schemes, Lemma 25.24.1. The base change maps along the degeneracies d^n_j are isomorphisms by Cohomology of Schemes, Lemma 29.5.2. Hence we are done by Lemma 63.9.2. □

Lemma 63.9.6. *Let $f : V \to U$ be a cartesian morphism of simplicial schemes. Assume the morphisms $d^n_j : U_n \to U_{n-1}$ are flat and the morphisms $V_n \to U_n$ are quasi-compact and quasi-separated. Then f^* and f_* form an adjoint pair of functors between the categories of cartesian quasi-coherent modules on U and V.*

Proof. We have seen in Lemmas 63.9.4 and 63.9.5 that the statement makes sense. The adjointness property follows immediately from the fact that each f^*_n is adjoint to $f_{n,*}$. □

Lemma 63.9.7. *Let $f : X \to S$ be a morphism of schemes which has a section*[2]*. Let $(X/S)_\bullet$ be the simplicial scheme associated to $X \to S$, see Definition 63.8.3. Then pullback defines an equivalence between the category of quasi-coherent $\mathcal{O}_S$-modules and the category of cartesian quasi-coherent modules on $(X/S)_\bullet$.*

Proof. Let $\sigma : S \to X$ be a section of f. Let $(\mathcal{F}, \alpha)$ be a pair as in Lemma 63.9.3. Set $\mathcal{G} = \sigma^*\mathcal{F}$. Consider the diagram

$$\begin{array}{ccccc} X & \xrightarrow[(\sigma\circ f,1)]{} & X \times_S X & \xrightarrow[\mathrm{pr}_1]{} & X \\ {\scriptstyle f}\downarrow & & \downarrow{\scriptstyle \mathrm{pr}_0} & & \\ S & \xrightarrow{\sigma} & X & & \end{array}$$

Note that $\mathrm{pr}_0 = d_1^1$ and $\mathrm{pr}_1 = d_0^1$. Hence we see that $(\sigma \circ f, 1)^*\alpha$ defines an isomorphism

$$f^*\mathcal{G} = (\sigma \circ f, 1)^*\mathrm{pr}_0^*\mathcal{F} \longrightarrow (\sigma \circ f, 1)^*\mathrm{pr}_1^*\mathcal{F} = \mathcal{F}$$

We omit the verification that this isomorphism is compatible with α and the canonical isomorphism $\mathrm{pr}_0^*f^*\mathcal{G} \to \mathrm{pr}_1^*f^*\mathcal{G}$. □

63.10. Groupoids and simplicial schemes

Given a groupoid in schemes we can build a simplicial scheme. It will turn out that the category of quasi-coherent sheaves on a groupoid is equivalent to the category of cartesian quasi-coherent sheaves on the associated simplicial scheme.

Lemma 63.10.1. *Let (U, R, s, t, c, e, i) be a groupoid scheme over S. There exists a simplicial scheme X over S with the following properties*

(1) $X_0 = U$, $X_1 = R$, $X_2 = R \times_{s,U,t} R$,
(2) $s_0^0 = e : X_0 \to X_1$,
(3) $d_0^1 = s : X_1 \to X_0$, $d_1^1 = t : X_1 \to X_0$,
(4) $s_0^1 = (e \circ t, 1) : X_1 \to X_2$, $s_1^1 = (1, e \circ t) : X_1 \to X_2$,
(5) $d_0^2 = pr_1 : X_2 \to X_1$, $d_1^2 = c : X_2 \to X_1$, $d_2^2 = pr_0$, *and*
(6) $X = cosk_2 sk_2 X$.

For all n we have $X_n = R \times_{s,U,t} \ldots \times_{s,U,t} R$ with n factors. The map $d_j^n : X_n \to X_{n-1}$ is given on functors of points by

$$(r_1, \ldots, r_n) \longmapsto (r_1, \ldots, c(r_j, r_{j+1}), \ldots, r_n)$$

for $1 \leq j \leq n-1$ whereas $d_0^n(r_1, \ldots, r_n) = (r_2, \ldots, r_n)$ and $d_n^n(r_1, \ldots, r_n) = (r_1, \ldots, r_{n-1})$.

Proof. We only have to verify that the rules prescribed in (1), (2), (3), (4), (5) define a 2-truncated simplicial scheme U' over S, since then (6) allows us to set $X = \mathrm{cosk}_2 U'$, see Simplicial, Lemma 14.17.3. Using the functor of points approach,

[2] In fact, it would be enough to assume that f has fpqc locally on S a section, since we have descent of quasi-coherent modules by Descent, Section 34.5.

all we have to verify is that if $(\mathrm{Ob}, \mathrm{Arrows}, s, t, c, e, i)$ is a groupoid, then

$$\begin{array}{ccccc} \mathrm{Arrows} \times_{s,\mathrm{Ob},t} \mathrm{Arrows} & & & & \\ \downarrow{\scriptstyle \mathrm{pr}_1} \quad \uparrow{\scriptstyle 1,e} & & \downarrow{\scriptstyle c} & & \uparrow{\scriptstyle e,1} \quad \downarrow{\scriptstyle \mathrm{pr}_0} \\ \mathrm{Arrows} & & & & \\ \downarrow{\scriptstyle s} \quad \uparrow{\scriptstyle e} \quad \downarrow{\scriptstyle t} & & & & \\ \mathrm{Ob} & & & & \end{array}$$

is a 2-truncated simplicial set. We omit the details.

Finally, the description of X_n for $n > 2$ follows by induction from the description of X_0, X_1, X_2, and Simplicial, Remark 14.17.10 and Lemma 14.17.7. Alternately, one shows that cosk_2 applied to the 2-truncated simplicial set displayed above gives a simplicial set whose nth term equals $\mathrm{Arrows} \times_{s,\mathrm{Ob},t} \ldots \times_{s,\mathrm{Ob},t} \mathrm{Arrows}$ with n factors and degeneracy maps as given in the lemma. Some details omitted. □

Lemma 63.10.2. *Let S be a scheme. Let (U, R, s, t, c) be a groupoid scheme over S. Let X be the simplicial scheme over S constructed in Lemma 63.10.1. Then the category of quasi-coherent modules on (U, R, s, t, c) is equivalent to the category of cartesian quasi-coherent modules on X.*

Proof. This is clear from Lemma 63.9.3 and Groupoids, Definition 38.12.1. □

In the following lemma we will use the concept of a cartesian morphism $V \to U$ of simplicial schemes as defined in Definition 63.8.1.

Lemma 63.10.3. *Let (U, R, s, t, c) be a groupoid scheme over a scheme S. Let X be the simplicial scheme over S constructed in Lemma 63.10.1. Let $(R/U)_\bullet$ be the simplicial scheme associated to $s : R \to U$, see Definition 63.8.3. There exists a cartesian morphism $t_\bullet : (R/U)_\bullet \to X$ of simplicial schemes with low degree morphisms given by*

$$\begin{array}{ccccc} R \times_{s,U,s} R \times_{s,U,s} R & \overset{pr_{12}}{\underset{pr_{01}}{\overset{pr_{02}}{\rightrightarrows}}} & R \times_{s,U,s} R & \overset{pr_1}{\underset{pr_0}{\rightrightarrows}} & R \\ \downarrow{\scriptstyle (r_0,r_1,r_2)\mapsto(r_0\circ r_1^{-1}, r_1\circ r_2^{-1})} & & \downarrow{\scriptstyle (r_0,r_1)\mapsto r_0\circ r_1^{-1}} & & \downarrow{\scriptstyle t} \\ R \times_{s,U,t} R & \overset{pr_1}{\underset{pr_0}{\overset{c}{\rightrightarrows}}} & R & \overset{s}{\underset{t}{\rightrightarrows}} & U \end{array}$$

Proof. For arbitrary n we define $(R/U)_\bullet \to X_n$ by the rule

$$(r_0, \ldots, r_n) \longrightarrow (r_0 \circ r_1^{-1}, \ldots, r_{n-1} \circ r_n^{-1})$$

Compatibility with degeneracy maps is clear from the description of the degeneracies in Lemma 63.10.1. We omit the verification that the maps respect the morphisms s_j^n. Groupoids, Lemma 38.11.5 (with the roles of s and t reversed) shows that the two right squares are cartesian. In exactly the same manner one shows all the other squares are cartesian too. Hence the morphism is cartesian. □

63.11. Descent data give equivalence relations

In Section 63.8 we saw how descent data relative to $X \to S$ can be formulated in terms of cartesian simplicial schemes over $(X/S)_\bullet$. Here we link this to equivalence relations as follows.

Lemma 63.11.1. *Let $f : X \to S$ be a morphism of schemes. Let $\pi : Y \to (X/S)_\bullet$ be a cartesian morphism of simplicial schemes, see Definitions 63.8.1 and 63.8.3. Then the morphism*

$$j = (d_1^1, d_0^1) : Y_1 \to Y_0 \times_S Y_0$$

defines an equivalence relation on Y_0 over S, see Groupoids, Definition 38.3.1.

Proof. Note that j is a monomorphism. Namely the composition $Y_1 \to Y_0 \times_S Y_0 \to Y_0 \times_S X$ is an isomorphism as π is cartesian.

Consider the morphism

$$(d_2^2, d_0^2) : Y_2 \to Y_1 \times_{d_0^1, Y_0, d_1^1} Y_1.$$

This works because $d_0 \circ d_2 = d_1 \circ d_0$, see Simplicial, Remark 14.3.3. Also, it is a morphism over $(X/S)_2$. It is an isomorphism because $Y \to (X/S)_\bullet$ is cartesian. Note for example that the right hand side is isomorphic to $Y_0 \times_{\pi_0, X, \mathrm{pr}_1} (X \times_S X \times_S X) = X \times_S Y_0 \times_S X$ because π is cartesian. Details omitted.

As in Groupoids, Definition 38.3.1 we denote $t = \mathrm{pr}_0 \circ j = d_1^1$ and $s = \mathrm{pr}_1 \circ j = d_0^1$. The isomorphism above, combined with the morphism $d_1^2 : Y_2 \to Y_1$ give us a composition morphism

$$c : Y_1 \times_{s, Y_0, t} Y_1 \longrightarrow Y_1$$

over $Y_0 \times_S Y_0$. This immediately implies that for any scheme T/S the relation $Y_1(T) \subset Y_0(T) \times Y_0(T)$ is transitive.

Reflexivity follows from the fact that the restriction of the morphism j to the diagonal $\Delta : X \to X \times_S X$ is an isomorphism (again use the cartesian property of π).

To see symmetry we consider the morphism

$$(d_2^2, d_1^2) : Y_2 \to Y_1 \times_{d_1^1, Y_0, d_1^1} Y_1.$$

This works because $d_1 \circ d_2 = d_1 \circ d_1$, see Simplicial, Remark 14.3.3. It is an isomorphism because $Y \to (X/S)_\bullet$ is cartesian. Note for example that the right hand side is isomorphic to $Y_0 \times_{\pi_0, X, \mathrm{pr}_0} (X \times_S X \times_S X) = Y_0 \times_S X \times_S X$ because π is cartesian. Details omitted.

Let T/S be a scheme. Let $a \sim b$ for $a, b \in Y_0(T)$ be synonymous with $(a, b) \in Y_1(T)$. The isomorphism (d_2^2, d_1^2) above implies that if $a \sim b$ and $a \sim c$, then $b \sim c$. Combined with reflexivity this shows that $\sim$ is an equivalence relation. □

63.12. An example case

In this section we show that disjoint unions of spectra of Artinian rings can be descended along a quasi-compact surjective flat morphism of schemes.

Lemma 63.12.1. *Let $X \to S$ be a morphism of schemes. Suppose $Y \to (X/S)_\bullet$ is a cartesian morphism of simplicial schemes. For $y \in Y_0$ a point define*

$$T_y = \{y' \in Y_0 \mid \exists\ y_1 \in Y_1 : d_1^1(y_1) = y, d_0^1(y_1) = y'\}$$

as a subset of Y_0. Then $y \in T_y$ and $T_y \cap T_{y'} \neq \emptyset \Rightarrow T_y = T_{y'}$.

Proof. Combine Lemma 63.11.1 and Groupoids, Lemma 38.3.4. □

Lemma 63.12.2. *Let $X \to S$ be a morphism of schemes. Suppose $Y \to (X/S)_\bullet$ is a cartesian morphism of simplicial schemes. Let $y \in Y_0$ be a point. If $X \to S$ is quasi-compact, then*

$$T_y = \{y' \in Y_0 \mid \exists\ y_1 \in Y_1 : d_1^1(y_1) = y, d_0^1(y_1) = y'\}$$

is a quasi-compact subset of Y_0.

Proof. Let F_y be the scheme theoretic fibre of $d_1^1 : Y_1 \to Y_0$ at y. Then we see that T_y is the image of the morphism

$$\begin{array}{ccccc} F_y & \longrightarrow & Y_1 & \xrightarrow{d_0^1} & Y_0 \\ \downarrow & & \downarrow{\scriptstyle d_1^1} & & \\ y & \longrightarrow & Y_0 & & \end{array}$$

Note that F_y is quasi-compact. This proves the lemma. □

Lemma 63.12.3. *Let $X \to S$ be a quasi-compact flat surjective morphism. Let (V, φ) be a descent datum relative to $X \to S$. If V is a disjoint union of spectra of Artinian rings, then (V, φ) is effective.*

Proof. Let $Y \to (X/S)_\bullet$ be the cartesian morphism of simplicial schemes corresponding to (V, φ) by Lemma 63.8.5. Observe that $Y_0 = V$. Write $V = \coprod_{i \in I} \mathrm{Spec}(A_i)$ with each A_i local Artinian. Moreover, let $v_i \in V$ be the unique closed point of $\mathrm{Spec}(A_i)$ for all $i \in I$. Write $i \sim j$ if and only if $v_i \in T_{v_j}$ with notation as in Lemma 63.12.1 above. By Lemmas 63.12.1 and 63.12.2 this is an equivalence relation with finite equivalence classes. Let $\overline{I} = I/\sim$. Then we can write $V = \coprod_{\bar{i} \in \overline{I}} V_{\bar{i}}$ with $V_{\bar{i}} = \coprod_{i \in \bar{i}} \mathrm{Spec}(A_i)$. By construction we see that $\varphi : V \times_S X \to X \times_S V$ maps the open and closed subspaces $V_{\bar{i}} \times_S X$ into the open and closed subspaces $X \times_S V_{\bar{i}}$. In other words, we get descent data $(V_{\bar{i}}, \varphi_{\bar{i}})$, and (V, φ) is the coproduct of them in the category of descent data. Since each of the $V_{\bar{i}}$ is a finite union of spectra of Artinian local rings the morphism $V_{\bar{i}} \to X$ is affine, see Morphisms, Lemma 28.13.13. Since $\{X \to S\}$ is an fpqc covering we see that all the descent data $(V_{\bar{i}}, \varphi_{\bar{i}})$ are effective by Descent, Lemma 34.33.1. □

To be sure, the lemma above has very limited applicability!

63.13. Other chapters

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CHAPTER 64

Formal Deformation Theory

64.1. Introduction

This chapter develops formal deformation theory in a form applicable later in the stacks project, closely following Rim [**GRR72**, Exposee VI] and Schlessinger [**Sch68**]. We strongly encourage the reader new to this topic to read the paper by Schlessinger first, as it is sufficiently general for most applications, and Schlessinger's results are indeed used in most papers that use this kind of formal deformation theory.

Let Λ be a complete Noetherian local ring with residue field k, and let $\mathcal{C}_\Lambda$ denote the category of Artinian local Λ-algebras with residue field k. Given a functor $F : \mathcal{C}_\Lambda \to \mathit{Sets}$ such that $F(k)$ is a one element set, Schlessinger's paper introduced conditions (H1)-(H4) such that:

(1) F has a "hull" if and only if (H1)-(H3) hold.
(2) F is prorepresentable if and only (H1)-(H4) hold.

The purpose of this chapter is to generalize these results in two ways exactly as is done in Rim's paper:

(A) The functor F is replaced by a category $\mathcal{F}$ cofibered in groupoids over $\mathcal{C}_\Lambda$, see Section 64.3.
(B) We let Λ be a Noetherian ring and $\Lambda \to k$ a finite ring map to a field. The category $\mathcal{C}_\Lambda$ is the category of Artinian local Λ-algebras A endowed with a given identification $A/\mathfrak{m}_A = k$.

The analogue of the condition that $F(k)$ is a one element set is that $\mathcal{F}(k)$ is the trivial groupoid. If $\mathcal{F}$ satisfies this condition then we say it is a *predeformation category*, but in general we do not make this assumption. Rim's paper [**GRR72**, Exposee VI] is the original source for the results in this document. We also mention the useful paper [**TV10**], which discusses deformation theory with groupoids but in less generality than we do here.

An important role is played by the "completion" $\widehat{\mathcal{C}}_\Lambda$ of the category $\mathcal{C}_\Lambda$. An object of $\widehat{\mathcal{C}}_\Lambda$ is a Noetherian complete local Λ-algebra R whose residue field is identified with k, see Section 64.4. On the one hand $\mathcal{C}_\Lambda \subset \widehat{\mathcal{C}}_\Lambda$ is a strictly full subcategory and on the other hand $\widehat{\mathcal{C}}_\Lambda$ is a full subcategory of the category of pro-objects of $\mathcal{C}_\Lambda$. A functor $\mathcal{C}_\Lambda \to \mathit{Sets}$ is *prorepresentable* if it is isomorphic to the restriction of a representable functor $\underline{R} = \mathrm{Mor}_{\widehat{\mathcal{C}}_\Lambda}(R, -)$ to $\mathcal{C}_\Lambda$ where $R \in \mathrm{Ob}(\widehat{\mathcal{C}}_\Lambda)$.

Categories cofibred in groupoids are dual to categories fibred in groupoids; we introduced them in Section 64.5. A *smooth* morphism of categories cofibred in groupoids over $\mathcal{C}_\Lambda$ is one that satisfies the infinitesimal lifting criterion for objects, see Section 64.8. This is analogous to the definition of a formally smooth ring map, see

Algebra, Definition 10.131.1 and is exactly dual to the notion in Criteria for Representability, Section 70.6. This is an important notion as we eventually want to prove that certain kinds of categories cofibred in groupoids have a smooth prorepresentable presentation, much like the characterization of algebraic stacks in Algebraic Stacks, Sections 67.16 and 67.17. A *versal formal object* of a category $\mathcal{F}$ cofibred in groupoids over $\mathcal{C}_\Lambda$ is an object $\xi \in \widehat{\mathcal{F}}(R)$ of the completion such that the associated morphism $\underline{\xi} : \underline{R} \to \mathcal{F}$ is smooth.

In Section 64.9, we define conditions (S1) and (S2) on $\mathcal{F}$ generalizing Schlessinger's (H1) and (H2). The analogue of Schlessinger's (H3)—the condition that $\mathcal{F}$ has finite dimensional tangent space—is not given a name. A key step in the development of the theory is the existence of versal formal objects for predeformation categories satisfying (S1), (S2) and (H3), see Lemma 64.12.4. Schlessinger's notion of a *hull* for a functor $F : \mathcal{C}_\Lambda \to \mathit{Sets}$ is, in our terminology, a versal formal object $\xi \in \widehat{F}(R)$ such that the induced map of tangent spaces $d\underline{\xi} : T\underline{R} \to TF$ is an isomorphism. In the literature a hull is often called a "miniversal" object. We do not do so, and here is why. It can happen that a functor has a versal formal object without having a hull. Moreover, we show in Section 64.13 that if a predeformation category has a versal formal object, then it always has a *minimal* one (as defined in Definition 64.13.4) which is unique up to isomorphism, see Lemma 64.13.5. But it can happen that the minimal versal formal object does not induce an isomorphism on tangent spaces! (See Examples 64.14.3 and 64.14.8.)

Keeping in mind the differences pointed out above, Theorem 64.14.5 is the direct generalization of (1) above: it recovers Schlessinger's result in the case that $\mathcal{F}$ is a functor and it characterizes minimal versal formal objects, in the presence of conditions (S1) and (S2), in terms of the map $d\underline{\xi} : T\underline{R} \to TF$ on tangent spaces.

In Section 64.15, we define Rim's condition (RS) on $\mathcal{F}$ generalizing Schlessinger's (H4). A *deformation category* is defined as a predeformation category satisfying (RS). The analogue to prorepresentable functors are the categories cofibred in groupoids over $\mathcal{C}_\Lambda$ which have a *presentation by a smooth prorepresentable groupoid in functors* on $\mathcal{C}_\Lambda$, see Definitions 64.19.1, 64.20.1, and 64.21.1. This notion of a presentation takes into account the groupoid structure of the fibers of $\mathcal{F}$. In Theorem 64.24.4 we prove that $\mathcal{F}$ has a presentation by a smooth prorepresentable groupoid in functors if and only if $\mathcal{F}$ has a finite dimensional tangent space and finite dimensional infinitesimal automorphism space. This is the generalization of (2) above: it reduces to Schlessinger's result in the case that $\mathcal{F}$ is a functor. There is a final Section 64.25 where we discuss how to use minimal versal formal objects to produce a (unique up to isomorphism) minimal presentation by a smooth prorepresentable groupoid in functors.

We also find the following conceptual explanation for Schlessinger's conditions. If a predeformation category $\mathcal{F}$ satisfies (RS), then the associated functor of isomorphism classes $\overline{\mathcal{F}} : \mathcal{C}_\Lambda \to \mathit{Sets}$ satisfies (H1) and (H2) (Lemmas 64.15.6 and 64.9.5). Conversely, if a functor $F : \mathcal{C}_\Lambda \to \mathit{Sets}$ arises naturally as the functor of isomorphism classes of a category $\mathcal{F}$ cofibered in groupoids, then it seems to happen in practice that an argument showing F satisfies (H1) and (H2) will also show $\mathcal{F}$ satisfies (RS) (see Artin's Axioms, Section 71.22 for examples). Moreover, if $\mathcal{F}$ satisfies (RS),

then condition (H4) for $\overline{\mathcal{F}}$ has a simple interpretation in terms of extending automorphisms of objects of $\mathcal{F}$ (Lemma 64.15.7). These observations suggest that (RS) should be regarded as the fundamental deformation theoretic glueing condition.

64.2. Notation and Conventions

A ring is commutative with 1. The maximal ideal of a local ring A is denoted by $\mathfrak{m}_A$. The set of positive integers is denoted by $\mathbf{N} = \{1, 2, 3, \ldots\}$. If U is an object of a category $\mathcal{C}$, we denote by $\underline{U}$ the functor $\mathrm{Mor}_{\mathcal{C}}(U, -) : \mathcal{C} \to \mathit{Sets}$, see Remarks 64.5.2 (12). Warning: this may conflict with the notation in other chapters where sometimes use $\underline{U}$ to denote $h_U(-) = \mathrm{Mor}_{\mathcal{C}}(-, U)$.

Throughout this chapter Λ is a Noetherian ring and $\Lambda \to k$ is a finite ring map from Λ to a field. The kernel of this map is denoted $\mathfrak{m}_\Lambda$ and the image $k' \subset k$. It turns out that $\mathfrak{m}_\Lambda$ is a maximal ideal, $k' = \Lambda/\mathfrak{m}_\Lambda$ is a field, and the extension $k' \subset k$ is finite. See discussion surrounding (64.3.3.1).

64.3. The base category

Motivation. An important application of formal deformation theory is to criteria for representability by algebraic spaces. Suppose given a locally Noetherian base change S and a functor $F : (\mathit{Sch}/S)^{opp}_{fppf} \to \mathit{Sets}$. Let k be a finite type field over S, i.e., we are given a finite type morphism $\mathrm{Spec}(k) \to S$. One of Artin's criteria is that for any element $x \in F(\mathrm{Spec}(k))$ the predeformation functor associated to the triple (S, k, x) should be prorepresentable. By Morphisms, Lemma 28.17.1 the condition that k is of finite type over S means that there exists an affine open $\mathrm{Spec}(\Lambda) \subset S$ such that k is a finite Λ-algebra. This motivates why we work throughout this chapter with a base category as follows.

Definition 64.3.1. Let Λ be a Noetherian ring and let $\Lambda \to k$ be a finite ring map where k is a field. We define $\mathcal{C}_\Lambda$ to be the category with

(1) objects are pairs (A, φ) where A is an Artinian local Λ-algebra and where $\varphi : A/\mathfrak{m}_A \to k$ is a Λ-algebra isomorphism, and
(2) morphisms $f : (B, \psi) \to (A, \varphi)$ are local Λ-algebra homomorphisms such that $\varphi \circ (f \bmod \mathfrak{m}) = \psi$.

We say we are in the *classical case* if Λ is a Noetherian complete local ring and k is its residue field.

Note that if $\Lambda \to k$ is surjective and if A is an Artinian local Λ-algebra, then the identification φ, if it exists, is unique. Moreover, in this case any Λ-algebra map $A \to B$ is going to be compatible with the identifications. Hence in this case $\mathcal{C}_\Lambda$ is just the category of local Artinian Λ-algebras whose residue field "is" k. By abuse of notation we also denote objects of $\mathcal{C}_\Lambda$ simply A in the general case. Moreover, we will often write $A/\mathfrak{m} = k$, i.e., we will pretend all rings in $\mathcal{C}_\Lambda$ have residue field k (since all ring maps in $\mathcal{C}_\Lambda$ are compatible with the given identifications this should never cause any problems). Throughout the rest of this chapter the base ring Λ and the field k are fixed. The category $\mathcal{C}_\Lambda$ will be the base category for the cofibered categories considered below.

Definition 64.3.2. Let $f : B \to A$ be a ring map in $\mathcal{C}_\Lambda$. We say f is a *small extension* if it is surjective and $\ker(f)$ is a nonzero principal ideal which is annihilated by $\mathfrak{m}_B$.

By the following lemma we can often reduce arguments involving surjective ring maps in $\mathcal{C}_\Lambda$ to the case of small extensions.

Lemma 64.3.3. *Let $f : B \to A$ be a surjective ring map in $\mathcal{C}_\Lambda$. Then f can be factored as a composition of small extensions.*

Proof. Let I be the kernel of f. The maximal ideal $\mathfrak{m}_B$ is nilpotent since B is Artinian, say $\mathfrak{m}_B^n = 0$. Hence we get a factorization

$$B = B/I\mathfrak{m}_B^{n-1} \to B/I\mathfrak{m}_B^{n-2} \to \cdots \to B/I \cong A$$

of f into a composition of surjective maps whose kernels are annihilated by the maximal ideal. Thus it suffices to prove the lemma when f itself is such a map, i.e. when I is annihilated by $\mathfrak{m}_B$. In this case I is a k-vector space, which has finite dimension, see Algebra, Lemma 10.49.8. Take a basis $x_1, \ldots, x_n$ of I as a k-vector space to get a factorization

$$B \to B/(x_1) \to \cdots \to B/(x_1, \ldots, x_n) \cong A$$

of f into a composition of small extensions. □

The next lemma says that we can compute the length of a module over a local Λ-algebra with residue field k in terms of the length over Λ. To explain the notation in the statement, let $k' \subset k$ be the image of our fixed finite ring map $\Lambda \to k$. Note that k/k' is a finite extension of rings. Hence k' is a field and k'/k is a finite extension, see Algebra, Lemma 10.33.16. Moreover, as $\Lambda \to k'$ is surjective we see that its kernel is a maximal ideal $\mathfrak{m}_\Lambda$. Thus

$$[k : k'] = [k : \Lambda/\mathfrak{m}_\Lambda] < \infty \tag{64.3.3.1}$$

and in the classical case we have $k = k'$. The notation $k' = \Lambda/\mathfrak{m}_\Lambda$ will be fixed throughout this chapter.

Lemma 64.3.4. *Let A be a local Λ-algebra with residue field k. Let M be an A-module. Then $[k : k']\text{length}_A(M) = \text{length}_\Lambda(M)$. In the classical case we have $\text{length}_A(M) = \text{length}_\Lambda(M)$.*

Proof. If M is a simple A-module then $M \cong k$ as an A-module, see Algebra, Lemma 10.48.10. In this case $\text{length}_A(M) = 1$ and $\text{length}_\Lambda(M) = [k' : k]$, see Algebra, Lemma 10.48.6. If $\text{length}_A(M)$ is finite, then the result follows on choosing a filtration of M by A-submodules with simple quotients using additivity, see Algebra, Lemma 10.48.3. If $\text{length}_A(M)$ is infinite, the result follows from the obvious inequality $\text{length}_A(M) \leq \text{length}_\Lambda(M)$. □

Lemma 64.3.5. *Let $A \to B$ be a ring map in $\mathcal{C}_\Lambda$. The following are equivalent*

1. *f is surjective,*
2. *$\mathfrak{m}_A/\mathfrak{m}_A^2 \to \mathfrak{m}_B/\mathfrak{m}_B^2$ is surjective, and*
3. *$\mathfrak{m}_A/(\mathfrak{m}_\Lambda A + \mathfrak{m}_A^2) \to \mathfrak{m}_B/(\mathfrak{m}_\Lambda B + \mathfrak{m}_B^2)$ is surjective.*

Proof. For any ring map $f : A \to B$ in $\mathcal{C}_\Lambda$ we have $f(\mathfrak{m}_A) \subset \mathfrak{m}_B$ for example because $\mathfrak{m}_A$, $\mathfrak{m}_B$ is the set of nilpotent elements of A, B. Suppose f is surjective. Let $y \in \mathfrak{m}_B$. Choose $x \in A$ with $f(x) = y$. Since f induces an isomorphism $A/\mathfrak{m}_A \to B/\mathfrak{m}_B$ we see that $x \in \mathfrak{m}_A$. Hence the induced map $\mathfrak{m}_A/\mathfrak{m}_A^2 \to \mathfrak{m}_B/\mathfrak{m}_B^2$ is surjective. In this way we see that (1) implies (2).

It is clear that (2) implies (3). The map $A \to B$ gives rise to a canonical commutative diagram

$$\begin{array}{ccccccc} \mathfrak{m}_\Lambda/\mathfrak{m}_\Lambda^2 \otimes_{k'} k & \longrightarrow & \mathfrak{m}_A/\mathfrak{m}_A^2 & \longrightarrow & \mathfrak{m}_A/(\mathfrak{m}_\Lambda A + \mathfrak{m}_A^2) & \longrightarrow & 0 \\ \downarrow & & \downarrow & & \downarrow & & \\ \mathfrak{m}_\Lambda/\mathfrak{m}_\Lambda^2 \otimes_{k'} k & \longrightarrow & \mathfrak{m}_B/\mathfrak{m}_B^2 & \longrightarrow & \mathfrak{m}_B/(\mathfrak{m}_\Lambda B + \mathfrak{m}_B^2) & \longrightarrow & 0 \end{array}$$

with exact rows. Hence if (3) holds, then so does (2).

Assume (2). To show that $A \to B$ is surjective it suffices by Nakayama's lemma (Algebra, Lemma 10.18.1) to show that $A/\mathfrak{m}_A \to B/\mathfrak{m}_A B$ is surjective. (Note that $\mathfrak{m}_A$ is a nilpotent ideal.) As $k = A/\mathfrak{m}_A = B/\mathfrak{m}_B$ it suffices to show that $\mathfrak{m}_A B \to \mathfrak{m}_B$ is surjective. Applying Nakayama's lemma once more we see that it suffices to see that $\mathfrak{m}_A B/\mathfrak{m}_A\mathfrak{m}_B \to \mathfrak{m}_B/\mathfrak{m}_B^2$ is surjective which is what we assumed. □

If $A \to B$ is a ring map in $\mathcal{C}_\Lambda$, then the map $\mathfrak{m}_A/(\mathfrak{m}_\Lambda A + \mathfrak{m}_A^2) \to \mathfrak{m}_B/(\mathfrak{m}_\Lambda B + \mathfrak{m}_B^2)$ is the map on relative cotangent spaces. Here is a formal definition.

Definition 64.3.6. Let $R \to S$ be a local homomorphism of local rings. The *relative cotangent space*[1] of R over S is the $S/\mathfrak{m}_S$-vector space $\mathfrak{m}_S/(\mathfrak{m}_R S + \mathfrak{m}_S^2)$.

If $f_1 : A_1 \to A$ and $f_2 : A_2 \to A$ are two ring maps, then the fiber product $A_1 \times_A A_2$ is the subring of $A_1 \times A_2$ consisting of elements whose two projections to A are equal. Throughout this chapter we will be considering conditions involving such a fiber product when f_1 and f_2 are in $\mathcal{C}_\Lambda$. It isn't always the case that the fibre product is an object of $\mathcal{C}_\Lambda$.

Example 64.3.7. Let p be a prime number and let $n \in \mathbf{N}$. Let $\Lambda = \mathbf{F}_p(t_1, t_2, \ldots, t_n)$ and let $k = \mathbf{F}_p(x_1, \ldots, x_n)$ with map $\Lambda \to k$ given by $t_i \mapsto x_i^p$. Let $A = k[\epsilon] = k[x]/(x^2)$. Then A is an object of $\mathcal{C}_\Lambda$. Suppose that $D : k \to k$ is a derivation of k over Λ, for example $D = \partial/\partial x_i$. Then the map

$$f_D : k \longrightarrow k[\epsilon], \quad a \mapsto a + D(a)\epsilon$$

is a morphism of $\mathcal{C}_\Lambda$. Set $A_1 = A_2 = k$ and set $f_1 = f_{\partial/\partial x_1}$ and $f_2(a) = a$. Then $A_1 \times_A A_2 = \{a \in k \mid \partial/\partial x_1(a) = 0\}$ which does not surject onto k. Hence the fibre product isn't an object of $\mathcal{C}_\Lambda$.

It turns out that this problem can only occur if the residue field extension $k' \subset k$ (64.3.3.1) is inseparable and neither f_1 nor f_2 is surjective.

Lemma 64.3.8. *Let $f_1 : A_1 \to A$ and $f_2 : A_2 \to A$ be ring maps in $\mathcal{C}_\Lambda$. Then:*

1. *If f_1 or f_2 is surjective, then $A_1 \times_A A_2$ is in $\mathcal{C}_\Lambda$.*
2. *If f_2 is a small extension, then so is $A_1 \times_A A_2 \to A_1$.*
3. *If the field extension $k' \subset k$ is separable, then $A_1 \times_A A_2$ is in $\mathcal{C}_\Lambda$.*

Proof. The ring $A_1 \times_A A_2$ is a Λ-algebra via the map $\Lambda \to A_1 \times_A A_2$ induced by the maps $\Lambda \to A_1$ and $\Lambda \to A_2$. It is a local ring with unique maximal ideal

$$\mathfrak{m}_{A_1} \times_{\mathfrak{m}_A} \mathfrak{m}_{A_2} = \mathrm{Ker}(A_1 \times_A A_2 \longrightarrow k)$$

[1]Caution: We will see later that in our general setting the tangent space of an object $A \in \mathcal{C}_\Lambda$ over Λ should not be defined simply as the k-linear dual of the relative cotangent space. In fact, the correct definition of the relative cotangent space is $\Omega_{S/R} \otimes_S S/\mathfrak{m}_S$.

A ring is Artinian if and only if it has finite length as a module over itself, see Algebra, Lemma 10.49.8. Since A_1 and A_2 are Artinian, Lemma 64.3.4 implies $\text{length}_\Lambda(A_1)$ and $\text{length}_\Lambda(A_2)$, and hence $\text{length}_\Lambda(A_1 \times A_2)$, are all finite. As $A_1 \times_A A_2 \subset A_1 \times A_2$ is a Λ-submodule, this implies $\text{length}_{A_1 \times_A A_2}(A_1 \times_A A_2) \leq \text{length}_\Lambda(A_1 \times_A A_2)$ is finite. So $A_1 \times_A A_2$ is Artinian. Thus the only thing that is keeping $A_1 \times_A A_2$ from being an object of $\mathcal{C}_\Lambda$ is the possibility that its residue field maps to a proper subfield of k via the map $A_1 \times_A A_2 \to A \to A/\mathfrak{m}_A = k$ above.

Proof of (1). If f_2 is surjective, then the projection $A_1 \times_A A_2 \to A_1$ is surjective. Hence the composition $A_1 \times_A A_2 \to A_1 \to A_1/\mathfrak{m}_{A_1} = k$ is surjective and we conclude that $A_1 \times_A A_2$ is an object of $\mathcal{C}_\Lambda$.

Proof of (2). If f_2 is a small extension then $A_2 \to A$ and $A_1 \times_A A_2 \to A_1$ are both surjective with the same kernel. Hence the kernel of $A_1 \times_A A_2 \to A_1$ is a 1-dimensional k-vector space and we see that $A_1 \times_A A_2 \to A_1$ is a small extension.

Proof of (3). Choose $\overline{x} \in k$ such that $k = k'(\overline{x})$ (see Fields, Lemma 9.18.1). Let $P'(T) \in k'[T]$ be the minimal polynomial of $\overline{x}$ over k'. Since k/k' is separable we see that $\mathrm{d}P/\mathrm{d}T(\overline{x}) \neq 0$. Choose a monic $P \in \Lambda[T]$ which maps to P' under the surjective map $\Lambda[T] \to k'[T]$. Because A, A_1, A_2 are henselian, see Algebra, Lemma 10.143.11, we can find $x, x_1, x_2 \in A, A_1, A_2$ with $P(x) = 0, P(x_1) = 0, P(x_2) = 0$ and such that the image of x, x_1, x_2 in k is $\overline{x}$. Then $(x_1, x_2) \in A_1 \times_A A_2$ because x_1, x_2 map to $x \in A$ by uniqueness, see Algebra, Lemma 10.143.2. Hence the residue field of $A_1 \times_A A_2$ contains a generator of k over k' and we win. □

Next we define essential surjections in $\mathcal{C}_\Lambda$. A necessary and sufficient condition for a surjection in $\mathcal{C}_\Lambda$ to be essential is given in Lemma 64.3.12.

Definition 64.3.9. Let $f : B \to A$ be a ring map in $\mathcal{C}_\Lambda$. We say f is an *essential surjection* if it has the following properties:

(1) f is surjective.
(2) If $g : C \to B$ is a ring map in $\mathcal{C}_\Lambda$ such that $f \circ g$ is surjective, then g is surjective.

Using Lemma 64.3.5, we can characterize essential surjections in $\mathcal{C}_\Lambda$ as follows.

Lemma 64.3.10. *Let $f : B \to A$ be a ring map in $\mathcal{C}_\Lambda$. The following are equivalent*

(1) *f is an essential surjection,*
(2) *the map $B/\mathfrak{m}_B^2 \to A/\mathfrak{m}_A^2$ is an essential surjection, and*
(3) *the map $B/(\mathfrak{m}_\Lambda B + \mathfrak{m}_B^2) \to A/(\mathfrak{m}_\Lambda A + \mathfrak{m}_A^2)$ is an essential surjection.*

Proof. Assume (3). Let $C \to B$ be a ring map in $\mathcal{C}_\Lambda$ such that $C \to A$ is surjective. Then $C \to A/(\mathfrak{m}_\Lambda A + \mathfrak{m}_A^2)$ is surjective too. We conclude that $C \to B/(\mathfrak{m}_\Lambda B + \mathfrak{m}_B^2)$ is surjective by our assumption. Hence $C \to B$ is surjective by applying Lemma 64.3.5 (2 times).

Assume (1). Let $C \to B/(\mathfrak{m}_\Lambda B + \mathfrak{m}_B^2)$ be a morphism of $\mathcal{C}_\Lambda$ such that $C \to A/(\mathfrak{m}_\Lambda A + \mathfrak{m}_A^2)$ is surjective. Set $C' = C \times_{B/(\mathfrak{m}_\Lambda B + \mathfrak{m}_B^2)} B$ which is an object of $\mathcal{C}_\Lambda$ by Lemma 64.3.8. Note that $C' \to A/(\mathfrak{m}_\Lambda A + \mathfrak{m}_A^2)$ is still surjective, hence $C' \to A$ is surjective by Lemma 64.3.5. Thus $C' \to B$ is surjective by our assumption. This implies that $C' \to B/(\mathfrak{m}_\Lambda B + \mathfrak{m}_B^2)$ is surjective, which implies by the construction of C' that $C \to B/(\mathfrak{m}_\Lambda B + \mathfrak{m}_B^2)$ is surjective.

In the first paragraph we proved (3) $\Rightarrow$ (1) and in the second paragraph we proved (1) $\Rightarrow$ (3). The equivalence of (2) and (3) is a special case of the equivalence of (1) and (3), hence we are done. $\square$

To analyze essential surjections in $\mathcal{C}_\Lambda$ a bit more we introduce some notation. Suppose that A is an object of $\mathcal{C}_\Lambda$. There is a canonical exact sequence

$$\mathfrak{m}_A/\mathfrak{m}_A^2 \xrightarrow{\mathrm{d}_A} \Omega_{A/\Lambda} \otimes_A k \to \Omega_{k/\Lambda} \to 0 \tag{64.3.10.1}$$

see Algebra, Lemma 10.125.9. Note that $\Omega_{k/\Lambda} = \Omega_{k/k'}$ with k' as in (64.3.3.1). Let $H_1(L_{k/\Lambda})$ be the first homology module of the naive cotangent complex of k over Λ, see Algebra, Definition 10.127.1. Then we can extend (64.3.10.1) to the exact sequence

$$H_1(L_{k/\Lambda}) \to \mathfrak{m}_A/\mathfrak{m}_A^2 \xrightarrow{\mathrm{d}_A} \Omega_{A/\Lambda} \otimes_A k \to \Omega_{k/\Lambda} \to 0, \tag{64.3.10.2}$$

see Algebra, Lemma 10.127.4. If $B \to A$ is a ring map in $\mathcal{C}_\Lambda$ then we obtain a commutative diagram

$$\begin{array}{ccccccc} H_1(L_{k/\Lambda}) & \longrightarrow & \mathfrak{m}_B/\mathfrak{m}_B^2 & \xrightarrow[\mathrm{d}_B]{} & \Omega_{B/\Lambda} \otimes_B k & \longrightarrow & \Omega_{k/\Lambda} \longrightarrow 0 \\ \| & & \downarrow & & \downarrow & & \| \\ H_1(L_{k/\Lambda}) & \longrightarrow & \mathfrak{m}_A/\mathfrak{m}_A^2 & \xrightarrow{\mathrm{d}_A} & \Omega_{A/\Lambda} \otimes_A k & \longrightarrow & \Omega_{k/\Lambda} \longrightarrow 0 \end{array} \tag{64.3.10.3}$$

with exact rows.

Lemma 64.3.11. *There is a canonical map*

$$\mathfrak{m}_\Lambda/\mathfrak{m}_\Lambda^2 \longrightarrow H_1(L_{k/\Lambda}).$$

If $k' \subset k$ is separable (for example if the characteristic of k is zero), then this map induces an isomorphism $\mathfrak{m}_\Lambda/\mathfrak{m}_\Lambda^2 \otimes_{k'} k = H_1(L_{k/\Lambda})$. If $k = k'$ (for example in the classical case), then $\mathfrak{m}_\Lambda/\mathfrak{m}_\Lambda^2 = H_1(L_{k/\Lambda})$. The composition

$$\mathfrak{m}_\Lambda/\mathfrak{m}_\Lambda^2 \longrightarrow H_1(L_{k/\Lambda}) \longrightarrow \mathfrak{m}_A/\mathfrak{m}_A^2$$

comes from the canonical map $\mathfrak{m}_\Lambda \to \mathfrak{m}_A$.

Proof. Note that $H_1(L_{k'/\Lambda}) = \mathfrak{m}_\Lambda/\mathfrak{m}_\Lambda^2$ as $\Lambda \to k'$ is surjective with kernel $\mathfrak{m}_\Lambda$. The map arises from functoriality of the naive cotangent complex. If $k' \subset k$ is separable, then $k' \to k$ is an étale ring map, see Algebra, Lemma 10.136.4. Thus its naive cotangent complex has trivial homology groups, see Algebra, Definition 10.136.1. Then Algebra, Lemma 10.127.4 applied to the ring maps $\Lambda \to k' \to k$ implies that $\mathfrak{m}_\Lambda/\mathfrak{m}_\Lambda^2 \otimes_{k'} k = H_1(L_{k/\Lambda})$. We omit the proof of the final statement. $\square$

Lemma 64.3.12. *Let $f : B \to A$ be a ring map in $\mathcal{C}_\Lambda$. Notation as in (64.3.10.3).*

1. *The equivalent conditions of Lemma 64.3.10 characterizing when f is surjective are also equivalent to*
   (a) *$Im(d_B) \to Im(d_A)$ is surjective, and*
   (b) *the map $\Omega_{B/\Lambda} \otimes_B k \to \Omega_{A/\Lambda} \otimes_A k$ is surjective.*
2. *The following are equivalent*
   (a) *f is an essential surjection,*
   (b) *the map $Im(d_B) \to Im(d_A)$ is an isomorphism, and*
   (c) *the map $\Omega_{B/\Lambda} \otimes_B k \to \Omega_{A/\Lambda} \otimes_A k$ is an isomorphism.*

(3) *If k/k' is separable, then f is an essential surjection if and only if the map $\mathfrak{m}_B/(\mathfrak{m}_\Lambda B + \mathfrak{m}_B^2) \to \mathfrak{m}_A/(\mathfrak{m}_\Lambda A + \mathfrak{m}_A^2)$ is an isomorphism.*
(4) *If f is a small extension, then f is not essential if and only if f has a section $s : A \to B$ in $\mathcal{C}_\Lambda$ with $f \circ s = id_A$.*

Proof. Proof of (1). It follows from (64.3.10.3) that (1)(a) and (1)(b) are equivalent. Also, if $A \to B$ is surjective, then (1)(a) and (1)(b) hold. Assume (1)(a). Since the kernel of d_A is the image of $H_1(L_{k/\Lambda})$ which also maps to $\mathfrak{m}_B/\mathfrak{m}_B^2$ we conclude that $\mathfrak{m}_B/\mathfrak{m}_B^2 \to \mathfrak{m}_A/\mathfrak{m}_A^2$ is surjective. Hence $B \to A$ is surjective by Lemma 64.3.5. This finishes the proof of (1).

Proof of (2). The equivalence of (2)(b) and (2)(c) is immediate from (64.3.10.3).

Assume (2)(b). Let $g : C \to B$ be a ring map in $\mathcal{C}_\Lambda$ such that $f \circ g$ is surjective. We conclude that $\mathfrak{m}_C/\mathfrak{m}_C^2 \to \mathfrak{m}_A/\mathfrak{m}_A^2$ is surjective by Lemma 64.3.5. Hence $\mathrm{Im}(\mathrm{d}_C) \to \mathrm{Im}(\mathrm{d}_A)$ is surjective and by the assumption we see that $\mathrm{Im}(\mathrm{d}_C) \to \mathrm{Im}(\mathrm{d}_B)$ is surjective. It follows that $C \to B$ is surjective by (1).

Assume (2)(a). Then f is surjective and we see that $\Omega_{B/\Lambda} \otimes_B k \to \Omega_{A/\Lambda} \otimes_A k$ is surjective. Let K be the kernel. Note that $K = \mathrm{d}_B(\mathrm{Ker}(\mathfrak{m}_B/\mathfrak{m}_B^2 \to \mathfrak{m}_A/\mathfrak{m}_A^2))$ by (64.3.10.3). Choose a splitting

$$\Omega_{B/\Lambda} \otimes_B k = \Omega_{A/\Lambda} \otimes_A k \oplus K$$

of k-vector space. The map $\mathrm{d} : B \to \Omega_{B/\Lambda}$ induces via the projection onto K a map $D : B \to K$. Set $C = \{b \in B \mid D(b) = 0\}$. The Leibniz rule shows that this is a Λ-subalgebra of B. Let $\overline{x} \in k$. Choose $x \in B$ mapping to $\overline{x}$. If $D(x) \neq 0$, then we can find an element $y \in \mathfrak{m}_B$ such that $D(y) = D(x)$. Hence $x - y \in C$ is an element which maps to $\overline{x}$. Thus $C \to k$ is surjective and C is an object of $\mathcal{C}_\Lambda$. Similarly, pick $\omega \in \mathrm{Im}(\mathrm{d}_A)$. We can find $x \in \mathfrak{m}_B$ such that $\mathrm{d}_B(x)$ maps to ω by (1). If $D(x) \neq 0$, then we can find an element $y \in \mathfrak{m}_B$ which maps to zero in $\mathfrak{m}_A/\mathfrak{m}_A^2$ such that $D(y) = D(x)$. Hence $z = x - y$ is an element of $\mathfrak{m}_C$ whose image $\mathrm{d}_C(z) \in \Omega_{C/k} \otimes_C k$ maps to ω. Hence $\mathrm{Im}(\mathrm{d}_C) \to \mathrm{Im}(\mathrm{d}_A)$ is surjective. We conclude that $C \to A$ is surjective by (1). Hence $C \to B$ is surjective by assumption. Hence $D = 0$, i.e., $K = 0$, i.e., (2)(c) holds. This finishes the proof of (2).

Proof of (3). If k'/k is separable, then $H_1(L_{k/\Lambda}) = \mathfrak{m}_\Lambda/\mathfrak{m}_\Lambda^2 \otimes_{k'} k$, see Lemma 64.3.11. Hence $\mathrm{Im}(\mathrm{d}_A) = \mathfrak{m}_A/(\mathfrak{m}_\Lambda A + \mathfrak{m}_A^2)$ and similarly for B. Thus (3) follows from (2).

Proof of (4). A section s of f is not surjective (by definition a small extension has nontrivial kernel), hence f is not essentially surjective. Conversely, assume f is a small surjection but not an essential surjection. Choose a ring map $C \to B$ in $\mathcal{C}_\Lambda$ which is not surjective, such that $C \to A$ is surjective. Let $C' \subset B$ be the image of $C \to B$. Then $C' \neq B$ but C' surjects onto A. Since $f : B \to A$ is a small extension, $\mathrm{length}_C(B) = \mathrm{length}_C(A) + 1$. Thus $\mathrm{length}_C(C') \leq \mathrm{length}_C(A)$ since C' is a proper subring of B. But $C' \to A$ is surjective, so in fact we must have $\mathrm{length}_C(C') = \mathrm{length}_C(A)$ and $C' \to A$ is an isomorphism which gives us our section. □

Example 64.3.13. Let $\Lambda = k[[x]]$ be the power series ring in 1 variable over k. Set $A = k$ and $B = \Lambda/(x^2)$. Then $B \to A$ is an essential surjection by Lemma 64.3.12

because it is a small extension and the map $B \to A$ does not have a right inverse (in the category $\mathcal{C}_\Lambda$). But the map

$$k \cong \mathfrak{m}_B/\mathfrak{m}_B^2 \longrightarrow \mathfrak{m}_A/\mathfrak{m}_A^2 = 0$$

is not an isomorphism. Thus in Lemma 64.3.12 (3) it is necessary to consider the map of relative cotangent spaces $\mathfrak{m}_B/(\mathfrak{m}_\Lambda B + \mathfrak{m}_B^2) \to \mathfrak{m}_A/(\mathfrak{m}_\Lambda A + \mathfrak{m}_A^2)$.

64.4. The completed base category

The following "completion" of the category $\mathcal{C}_\Lambda$ will serve as the base category of the completion of a category cofibered in groupoids over $\mathcal{C}_\Lambda$ (Section 64.7).

Definition 64.4.1. Let Λ be a Noetherian ring and let $\Lambda \to k$ be a finite ring map where k is a field. We define $\widehat{\mathcal{C}}_\Lambda$ to be the category with

(1) objects are pairs (R, φ) where R is a Noetherian complete local Λ-algebra and where $\varphi : R/\mathfrak{m}_R \to k$ is a Λ-algebra isomorphism, and
(2) morphisms $f : (S, \psi) \to (R, \varphi)$ are local Λ-algebra homomorphisms such that $\varphi \circ (f \bmod \mathfrak{m}) = \psi$.

As in the discussion following Definition 64.3.1 we will usually denote an object of $\widehat{\mathcal{C}}_\Lambda$ simply R, with the identification $R/\mathfrak{m}_R = k$ understood. In this section we discuss some basic properties of objects and morphisms of the category $\widehat{\mathcal{C}}_\Lambda$ paralleling our discussion of the category $\mathcal{C}_\Lambda$ in the previous section.

Our first observation is that any object $A \in \mathcal{C}_\Lambda$ is an object of $\widehat{\mathcal{C}}_\Lambda$ as an Artinian local ring is always Noetherian and complete with respect to its maximal ideal (which is after all a nilpotent ideal). Moreover, it is clear from the definitions that $\mathcal{C}_\Lambda \subset \widehat{\mathcal{C}}_\Lambda$ is the strictly full subcategory consisting of all Artinian rings. As it turns out, conversely every object of $\widehat{\mathcal{C}}_\Lambda$ is a limit of objects of $\mathcal{C}_\Lambda$.

Suppose that R is an object of $\widehat{\mathcal{C}}_\Lambda$. Consider the rings $R_n = R/\mathfrak{m}_R^n$ for $n \in \mathbf{N}$. These are Noetherian local rings with a unique nilpotent prime ideal, hence Artinian, see Algebra, Proposition 10.57.6. The ring maps

$$\ldots \to R_{n+1} \to R_n \to \ldots \to R_2 \to R_1 = k$$

are all surjective. Completeness of R by definition means that $R = \lim R_n$. If $f : R \to S$ is a ring map in $\widehat{\mathcal{C}}_\Lambda$ then we obtain a system of ring maps $f_n : R_n \to S_n$ whose limit is the given map.

Lemma 64.4.2. *Let $f : R \to S$ be a ring map in $\widehat{\mathcal{C}}_\Lambda$. The following are equivalent*

(1) *f is surjective,*
(2) *the map $\mathfrak{m}_R/\mathfrak{m}_R^2 \to \mathfrak{m}_S/\mathfrak{m}_S^2$ is surjective, and*
(3) *the map $\mathfrak{m}_R/(\mathfrak{m}_\Lambda R + \mathfrak{m}_R^2) \to \mathfrak{m}_S/(\mathfrak{m}_\Lambda S + \mathfrak{m}_S^2)$ is surjective.*

Proof. Note that for $n \geq 2$ we have the equality of relative cotangent spaces

$$\mathfrak{m}_R/(\mathfrak{m}_\Lambda R + \mathfrak{m}_R^2) = \mathfrak{m}_{R_n}/(\mathfrak{m}_\Lambda R_n + \mathfrak{m}_{R_n}^2)$$

and similarly for S. Hence by Lemma 64.3.5 we see that $R_n \to S_n$ is surjective for all n. Now let K_n be the kernel of $R_n \to S_n$. Then the sequences

$$0 \to K_n \to R_n \to S_n \to 0$$

form an exact sequence of directed inverse systems. The system (K_n) is Mittag-Leffler since each K_n is Artinian. Hence by Algebra, Lemma 10.81.4 taking limits preserves exactness. So $\lim R_n \to \lim S_n$ is surjective, i.e., f is surjective. □

Lemma 64.4.3. *The category $\widehat{\mathcal{C}}_\Lambda$ admits pushouts.*

Proof. Let $R \to S_1$ and $R \to S_2$ be morphisms of $\widehat{\mathcal{C}}_\Lambda$. Consider the ring $C = S_1 \otimes_R S_2$. This ring has a finitely generated maximal ideal $\mathfrak{m} = \mathfrak{m}_{S_1} \otimes S_2 + S_1 \otimes \mathfrak{m}_{S_2}$ with residue field k. Set $C^\wedge$ equal to the completion of C with respect to $\mathfrak{m}$. Then $C^\wedge$ is a Noetherian ring complete with respect to the maximal ideal $\mathfrak{m}^\wedge = \mathfrak{m}C^\wedge$ whose residue field is identified with k, see Algebra, Lemma 10.91.9. Hence $C^\wedge$ is an object of $\widehat{\mathcal{C}}_\Lambda$. Then $S_1 \to C^\wedge$ and $S_2 \to C^\wedge$ turn $C^\wedge$ into a pushout over R in $\widehat{\mathcal{C}}_\Lambda$ (details omitted). □

We will not need the following lemma.

Lemma 64.4.4. *The category $\widehat{\mathcal{C}}_\Lambda$ admits coproducts of pairs of objects.*

Proof. Let R and S be objects of $\widehat{\mathcal{C}}_\Lambda$. Consider the ring $C = R \otimes_\Lambda S$. There is a canonical surjective map $C \to R \otimes_\Lambda S \to k \otimes_\Lambda k \to k$ where the last map is the multiplication map. The kernel of $C \to k$ is a maximal ideal $\mathfrak{m}$. Note that $\mathfrak{m}$ is generated by $\mathfrak{m}_R C$, $\mathfrak{m}_S C$ and finitely many elements of C which map to generators of the kernel of $k \otimes_\Lambda k \to k$. Hence $\mathfrak{m}$ is a finitely generated ideal. Set $C^\wedge$ equal to the completion of C with respect to $\mathfrak{m}$. Then $C^\wedge$ is a Noetherian ring complete with respect to the maximal ideal $\mathfrak{m}^\wedge = \mathfrak{m}C^\wedge$ with residue field k, see Algebra, Lemma 10.91.9. Hence $C^\wedge$ is an object of $\widehat{\mathcal{C}}_\Lambda$. Then $R \to C^\wedge$ and $S \to C^\wedge$ turn $C^\wedge$ into a coproduct in $\widehat{\mathcal{C}}_\Lambda$ (details omitted). □

An empty coproduct in a category is an initial object of the category. In the classical case $\widehat{\mathcal{C}}_\Lambda$ has an initial object, namely Λ itself. More generally, if $k' = k$, then the completion $\Lambda^\wedge$ of Λ with respect to $\mathfrak{m}_\Lambda$ is an initial object. More generally still, if $k' \subset k$ is separable, then $\widehat{\mathcal{C}}_\Lambda$ has an initial object too. Namely, choose a monic polynomial $P \in \Lambda[T]$ such that $k \cong k'[T]/(P')$ where $p' \in k'[T]$ is the image of P. Then $R = \Lambda^\wedge[T]/(P)$ is an initial object, see proof of Lemma 64.3.8.

If R is an initial object as above, then we have $\mathcal{C}_\Lambda = \mathcal{C}_R$ and $\widehat{\mathcal{C}}_\Lambda = \widehat{\mathcal{C}}_R$ which effectively brings the whole discussion in this chapter back to the classical case. But, if $k' \subset k$ is inseparable, then an initial object does not exist.

Lemma 64.4.5. *Let S be an object of $\widehat{\mathcal{C}}_\Lambda$. Then $\dim_k Der_\Lambda(S, k) < \infty$.*

Proof. Let $x_1, \ldots, x_n \in \mathfrak{m}_S$ map to a k-basis for the relative cotangent space $\mathfrak{m}_S/(\mathfrak{m}_\Lambda S + \mathfrak{m}_S^2)$. Choose $y_1, \ldots, y_m \in S$ whose images in k generate k over k'. We claim that $\dim_k \mathrm{Der}_\Lambda(S, k) \leq n+m$. To see this it suffices to prove that if $D(x_i) = 0$ and $D(y_j) = 0$, then $D = 0$. Let $a \in S$. We can find a polynomial $P = \sum \lambda_J y^J$ with $\lambda_J \in \Lambda$ whose image in k is the same as the image of a in k. Then we see that $D(a - P) = D(a) - D(P) = D(a)$ by our assumption that $D(y_j) = 0$ for all j. Thus we may assume $a \in \mathfrak{m}_S$. Write $a = \sum a_i x_i$ with $a_i \in S$. By the Leibniz rule

$$D(a) = \sum x_i D(a_i) + \sum a_i D(x_i) = \sum x_i D(a_i)$$

as we assumed $D(x_i) = 0$. We have $\sum x_i D(a_i) = 0$ as multiplication by x_i is zero on k. □

Lemma 64.4.6. *Let $f : R \to S$ be a morphism of $\widehat{\mathcal{C}}_\Lambda$. If $Der_\Lambda(S, k) \to Der_\Lambda(R, k)$ is injective, then f is surjective.*

Proof. If f is not surjective, then $\mathfrak{m}_S/(\mathfrak{m}_R S + \mathfrak{m}_S^2)$ is nonzero by Lemma 64.4.2. Then also $Q = S/(f(R)+\mathfrak{m}_R S+\mathfrak{m}_S^2)$ is nonzero. Note that Q is a $k = R/\mathfrak{m}_R$-vector space via f. We turn Q into an S-module via $S \to k$. The quotient map $D : S \to Q$ is an R-derivation: if $a_1, a_2 \in S$, we can write $a_1 = f(b_1) + a_1'$ and $a_2 = f(b_2) + a_2'$ for some $b_1, b_2 \in R$ and $a_1', a_2' \in \mathfrak{m}_S$. Then b_i and a_i have the same image in k for $i = 1, 2$ and

$$\begin{aligned} a_1a_2 &= (f(b_1) + a_1')(f(b_2) + a_2') \\ &= f(b_1)a_2' + f(b_2)a_1' \\ &= f(b_1)(f(b_2) + a_2') + f(b_2)(f(b_1) + a_1') \\ &= f(b_1)a_2 + f(b_2)a_1 \end{aligned}$$

in Q which proves the Leibnize rule. Hence $D : S \to Q$ is a Λ-derivation which is zero on composing with $R \to S$. Since $Q \neq 0$ there also exist derivations $D : S \to k$ which are zero on composing with $R \to S$, i.e., $\mathrm{Der}_\Lambda(S, k) \to \mathrm{Der}_\Lambda(R, k)$ is not injective. □

Lemma 64.4.7. *Let R be an object of $\widehat{\mathcal{C}}_\Lambda$. Let (J_n) be a decreasing sequence of ideals such that $\mathfrak{m}_R^n \subset J_n$. Set $J = \bigcap J_n$. Then the sequence (J_n/J) defines the $\mathfrak{m}_{R/J}$-adic topology on R/J.*

Proof. It is clear that $\mathfrak{m}_{R/J}^n \subset J_n/J$. Thus it suffices to show that for every n there exists an N such that $J_N/J \subset \mathfrak{m}_{R/J}^n$. This is equivalent to $J_N \subset \mathfrak{m}_R^n + J$. For each n the ring $R/\mathfrak{m}_R^n$ is Artinian, hence there exists a N_n such that

$$J_{N_n} + \mathfrak{m}_R^n = J_{N_n+1} + \mathfrak{m}_R^n = \ldots$$

Set $E_n = (J_{N_n} + \mathfrak{m}_R^n)/\mathfrak{m}_R^n$. Set $E = \lim E_n \subset \lim R/\mathfrak{m}_R^n = R$. Note that $E \subset J$ as for any $f \in E$ and any m we have $f \in J_m + \mathfrak{m}_R^n$ for all $n \gg 0$, so $f \in J_m$ by Artin-Rees, see Algebra, Lemma 10.47.6. Since the transition maps $E_n \to E_{n-1}$ are all surjective, we see that J surjects onto E_n. Hence for $N = N_n$ works. □

Lemma 64.4.8. *Let $\ldots \to A_3 \to A_2 \to A_1$ be a sequence of surjective ring maps in $\mathcal{C}_\Lambda$. If $\dim_k(\mathfrak{m}_{A_n}/\mathfrak{m}_{A_n}^2)$ is bounded, then $S = \lim A_n$ is an object in $\widehat{\mathcal{C}}_\Lambda$ and the ideals $I_n = Ker(S \to A_n)$ define the $\mathfrak{m}_S$-adic topology on S.*

Proof. We will use freely that the maps $S \to A_n$ are surjective for all n. Note that the maps $\mathfrak{m}_{A_{n+1}}/\mathfrak{m}_{A_{n+1}}^2 \to \mathfrak{m}_{A_n}/\mathfrak{m}_{A_n}^2$ are surjective, see Lemma 64.4.2. Hence for n sufficiently large the dimension $\dim_k(\mathfrak{m}_{A_n}/\mathfrak{m}_{A_n}^2)$ stabilizes to an integer, say r. Thus we can find $x_1, \ldots, x_r \in \mathfrak{m}_S$ whose images in A_n generate $\mathfrak{m}_{A_n}$. Moreover, pick $y_1, \ldots, y_t \in S$ whose images in k generate k over Λ. Then we get a ring map $P = \Lambda[z_1, \ldots, z_{r+t}] \to S$, $z_i \mapsto x_i$ and $z_{r+j} \mapsto y_j$ such that the composition $P \to S \to A_n$ is surjective for all n. Let $\mathfrak{m} \subset P$ be the kernel of $P \to k$. Let $R = P^\wedge$ be the $\mathfrak{m}$-adic completion of P; this is an object of $\widehat{\mathcal{C}}_\Lambda$. Since we still have the compatible system of (surjective) maps $R \to A_n$ we get a map $R \to S$. Set $J_n = \mathrm{Ker}(R \to A_n)$. Set $J = \bigcap J_n$. By Lemma 64.4.7 we see that $R/J = \lim R/J_n = \lim A_n = S$ and that the ideals $J_n/J = I_n$ define the $\mathfrak{m}$-adic topology. (Note that for each n we have $\mathfrak{m}_R^{N_n} \subset J_n$ for some N_n and not necessarily $N_n = n$, so a renumbering of the ideals J_n may be necessary before applying the lemma.) □

Lemma 64.4.9. *Let $R', R \in \mathrm{Ob}(\widehat{\mathcal{C}}_\Lambda)$. Suppose that $R = R' \oplus I$ for some ideal I of R. Let $x_1, \ldots, x_r \in I$ map to a basis of $I/\mathfrak{m}_R I$. Set $S = R'[[X_1, \ldots, X_r]]$ and consider the R'-algebra map $S \to R$ mapping X_i to x_i. Assume that for every $n \gg 0$ the map $S/\mathfrak{m}_S^n \to R/\mathfrak{m}_R^n$ has a left inverse in $\mathcal{C}_\Lambda$. Then $S \to R$ is an isomorphism.*

Proof. As $R = R' \oplus I$ we have

$$\mathfrak{m}_R/\mathfrak{m}_R^2 = \mathfrak{m}_{R'}/\mathfrak{m}_{R'}^2 \oplus I/\mathfrak{m}_R I$$

and similarly

$$\mathfrak{m}_R/\mathfrak{m}_R^2 = \mathfrak{m}_{R'}/\mathfrak{m}_{R'}^2 \oplus \bigoplus kX_i$$

Hence for $n > 1$ the map $S/\mathfrak{m}_S^n \to R/\mathfrak{m}_R^n$ induces an isomorphism on cotangent spaces. Thus a left inverse $h_n : R/\mathfrak{m}_R^n \to S/\mathfrak{m}_S^n$ is surjective by Lemma 64.4.2. Since h_n is injective as a left inverse it is an isomorphism. Thus the canonical surjections $S/\mathfrak{m}_S^n \to R/\mathfrak{m}_R^n$ are all isomorphisms and we win. □

64.5. Categories cofibered in groupoids

In developing the theory we work with categories *cofibered* in groupoids. We assume as known the definition and basic properties of categories *fibered* in groupoids, see Categories, Section 4.33.

Definition 64.5.1. Let $\mathcal{C}$ be a category. A *category cofibered in groupoids over $\mathcal{C}$* is a category $\mathcal{F}$ equipped with a functor $p : \mathcal{F} \to \mathcal{C}$ such that $\mathcal{F}^{opp}$ is a category fibered in groupoids over $\mathcal{C}^{opp}$ via $p^{opp} : \mathcal{F}^{opp} \to \mathcal{C}^{opp}$.

Explicitly, $p : \mathcal{F} \to \mathcal{C}$ is cofibered in groupoids if the following two conditions hold:

(1) For every morphism $f : U \to V$ in $\mathcal{C}$ and every object x lying over U, there is a morphism $x \to y$ of $\mathcal{F}$ lying over f.
(2) For every pair of morphisms $a : x \to y$ and $b : x \to z$ of $\mathcal{F}$ and any morphism $f : p(y) \to p(z)$ such that $p(b) = f \circ p(a)$, there exists a unique morphism $c : y \to z$ of $\mathcal{F}$ lying over f such that $b = c \circ a$.

Remarks 64.5.2. Everything about categories fibered in groupoids translates directly to the cofibered setting. The following remarks are meant to fix notation. Let $\mathcal{C}$ be a category.

(1) We often omit the functor $p : \mathcal{F} \to \mathcal{C}$ from the notation.
(2) The fiber category over an object U in $\mathcal{C}$ is denoted by $\mathcal{F}(U)$. Its objects are those of $\mathcal{F}$ lying over U and its morphisms are those of $\mathcal{F}$ lying over id_U. If x, y are objects of $\mathcal{F}(U)$, we sometimes write $\mathrm{Mor}_U(x, y)$ for $\mathrm{Mor}_{\mathcal{F}(U)}(x, y)$.
(3) The fibre categories $\mathcal{F}(U)$ are groupoids, see Categories, Lemma 4.33.2. Hence the morphisms in $\mathcal{F}(U)$ are all isomorphisms. We sometimes write $\mathrm{Aut}_U(x)$ for $\mathrm{Mor}_{\mathcal{F}(U)}(x, x)$.
(4) Let $\mathcal{F}$ be a category cofibered in groupoids over $\mathcal{C}$, let $f : U \to V$ be a morphism in $\mathcal{C}$, and let $x \in \mathrm{Ob}(\mathcal{F}(U))$. A *pushforward* of x along f is a morphism $x \to y$ of $\mathcal{F}$ lying over f. A pushforward is unique up to unique isomorphism (see the discussion following Categories, Definition 4.31.1). We sometimes write $x \to f_* x$ for "the" pushforward of x along f.

(5) A *choice of pushforwards for* $\mathcal{F}$ is the choice of a pushforward of x along f for every pair (x, f) as above. We can make such a choice of pushforwards for $\mathcal{F}$ by the axiom of choice.
(6) Let $\mathcal{F}$ be a category cofibered in groupoids over $\mathcal{C}$. Given a choice of pushforwards for $\mathcal{F}$, there is an associated pseudo-functor $\mathcal{C} \to \mathit{Groupoids}$. We will never use this construction so we give no details.
(7) A morphism of categories cofibered in groupoids over $\mathcal{C}$ is a functor commuting with the projections to $\mathcal{C}$. If $\mathcal{F}$ and $\mathcal{F}'$ are categories cofibered in groupoids over $\mathcal{C}$, we denote the morphisms from $\mathcal{F}$ to $\mathcal{F}'$ by $\mathrm{Mor}_{\mathcal{C}}(\mathcal{F}, \mathcal{F}')$.
(8) Categories cofibered in groupoids form a $(2,1)$-category $\mathrm{Cof}(\mathcal{C})$. Its 1-morphisms are the morphisms described in (7). If $p : \mathcal{F} \to C$ and $p' : \mathcal{F}' \to \mathcal{C}$ are categories cofibered in groupoids and $\varphi, \psi : \mathcal{F} \to \mathcal{F}'$ are 1-morphisms, then a 2-morphism $t : \varphi \to \psi$ is a morphism of functors such that $p'(t_x) = \mathrm{id}_{p(x)}$ for all $x \in \mathrm{Ob}(\mathcal{F})$.
(9) Let $F : \mathcal{C} \to \mathit{Groupoids}$ be a functor. There is a category cofibered in groupoids $\mathcal{F} \to \mathcal{C}$ associated to F as follows. An object of $\mathcal{F}$ is a pair (U, x) where $U \in \mathrm{Ob}(\mathcal{C})$ and $x \in \mathrm{Ob}(F(U))$. A morphism $(U, x) \to (V, y)$ is a pair (f, a) where $f \in \mathrm{Mor}_{\mathcal{C}}(U, V)$ and $a \in \mathrm{Mor}_{F(V)}(F(f)(x), y)$. The functor $\mathcal{F} \to \mathcal{C}$ sends (U, x) to U. See Categories, Section 4.35.
(10) Let $\mathcal{F}$ be cofibered in groupoids over $\mathcal{C}$. For $U \in \mathrm{Ob}(\mathcal{C})$ set $\overline{\mathcal{F}}(U)$ equal to the set of isomorphisms classes of the category $\mathcal{F}(U)$. If $f : U \to V$ is a morphism of $\mathcal{C}$, then we obtain a map of sets $\overline{\mathcal{F}}(U) \to \overline{\mathcal{F}}(V)$ by mapping the isomorphism class of x to the isomorphism class of a pushforward f_*x of x see (4). Then $\overline{\mathcal{F}} : \mathcal{C} \to \mathit{Sets}$ is a functor. Similarly, if $\varphi : \mathcal{F} \to \mathcal{G}$ is a morphism of cofibered categories, we denote by $\overline{\varphi} : \overline{\mathcal{F}} \to \overline{\mathcal{G}}$ the associated morphism of functors.
(11) Let $F : \mathcal{C} \to \mathit{Sets}$ be a functor. We can think of a set as a discrete category, i.e., as a groupoid with only identity morphisms. Then the construction (9) associates to F a category cofibered in sets. This defines a fully faithful embedding of the category of functors $\mathcal{C} \to \mathit{Sets}$ to the category of categories cofibered in groupoids over $\mathcal{C}$. We identify the category of functors with its image under this embedding. Hence if $F : \mathcal{C} \to \mathit{Sets}$ is a functor, we denote the associated category cofibered in sets also by F; and if $\varphi : F \to G$ is a morphism of functors, we denote still by φ the corresponding morphism of categories cofibered in sets, and vice-versa. See Categories, Section 4.36.
(12) Let U be an object of $\mathcal{C}$. We write $\underline{U}$ for the functor $\mathrm{Mor}_{\mathcal{C}}(U, -) : \mathcal{C} \to \mathit{Sets}$. This defines a fully faithful embedding of $\mathcal{C}^{opp}$ into the category of functors $\mathcal{C} \to \mathit{Sets}$. Hence, if $f : U \to V$ is a morphism, we are justified in denoting still by f the induced morphism $\underline{V} \to \underline{U}$, and vice-versa.
(13) Fiber products of categories cofibered in groupoids: If $\mathcal{F} \to \mathcal{H}$ and $\mathcal{G} \to \mathcal{H}$ are morphisms of categories cofibered in groupoids over $\mathcal{C}_\Lambda$, then a construction of their 2-fiber product is given by the construction for their 2-fiber product as categories over $\mathcal{C}_\Lambda$, as described in Categories, Lemma 4.30.3.
(14) Restricting the base category: Let $p : \mathcal{F} \to \mathcal{C}$ be a category cofibered in groupoids, and let $\mathcal{C}'$ be a full subcategory of $\mathcal{C}$. The restriction $\mathcal{F}|_{\mathcal{C}'}$ is the

full subcategory of $\mathcal{F}$ whose objects lie over objects of $\mathcal{C}'$. It is a category cofibered in groupoids via the functor $p|_{\mathcal{C}'} : \mathcal{F}|_{\mathcal{C}'} \to \mathcal{C}'$.

64.6. Prorepresentable functors and predeformation categories

Our basic goal is to understand categories cofibered in groupoids over $\mathcal{C}_\Lambda$ and $\widehat{\mathcal{C}_\Lambda}$. Since $\mathcal{C}_\Lambda$ is a full subcategory of $\widehat{\mathcal{C}_\Lambda}$ we can restrict categories cofibred in groupoids over $\widehat{\mathcal{C}_\Lambda}$ to $\mathcal{C}_\Lambda$, see Remarks 64.5.2 (14). In particular we can do this with functors, in particular with representable functors. The functors on $\mathcal{C}_\Lambda$ one obtains in this way are called prorepresentable functors.

Definition 64.6.1. Let $F : \mathcal{C}_\Lambda \to \mathit{Sets}$ be a functor. We say F is *prorepresentable* if there exists an isomorphism $F \cong \underline{R}|_{\mathcal{C}_\Lambda}$ of functors for some $R \in \mathrm{Ob}(\widehat{\mathcal{C}_\Lambda})$.

Note that if $F : \mathcal{C}_\Lambda \to \mathit{Sets}$ is prorepresentable by $R \in \mathrm{Ob}(\widehat{\mathcal{C}_\Lambda})$, then

$$F(k) = \mathrm{Mor}_{\widehat{\mathcal{C}_\Lambda}}(R, k) = \{*\}$$

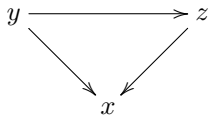
is a singleton. The categories cofibered in groupoids over $\mathcal{C}_\Lambda$ that are arise in deformation theory will often satisfy an analogous condition.

Definition 64.6.2. A *predeformation category* $\mathcal{F}$ is a category cofibered in groupoids over $\mathcal{C}_\Lambda$ such that $\mathcal{F}(k)$ is equivalent to a category with a single object and a single morphism, i.e., $\mathcal{F}(k)$ contains at least one object and there is a unique morphism between any two objects. A *morphism of predeformation categories* is a morphism of categories cofibered in groupoids over $\mathcal{C}_\Lambda$.

A feature of a predeformation category is the following. Let $x_0 \in \mathrm{Ob}(\mathcal{F}(k))$. Then every object of $\mathcal{F}$ comes equipped with a unique morphism to x_0. Namely, if x is an object of $\mathcal{F}$ over A, then we can choose a pushforward $x \to q_*x$ where $q : A \to k$ is the quotient map. There is a unique isomorphism $q_*x \to x_0$ and the composition $x \to q_*x \to x_0$ is the desired morphism.

Remark 64.6.3. We say that a functor $F : \mathcal{C}_\Lambda \to \mathit{Sets}$ is a *predeformation functor* if the associated cofibered set is a predeformation category, i.e. if $F(k)$ is a one element set. Thus if $\mathcal{F}$ is a predeformation category, then $\overline{\mathcal{F}}$ is a predeformation functor.

Remark 64.6.4. Let $p : \mathcal{F} \to \mathcal{C}_\Lambda$ be a category cofibered in groupoids, and let $x \in \mathrm{Ob}(\mathcal{F}(k))$. We denote by $\mathcal{F}_x$ the category of objects over x. An object of $\mathcal{F}_x$ is an arrow $y \to x$. A morphism $(y \to x) \to (z \to x)$ in $\mathcal{F}_x$ is a commutative diagram



There is a forgetful functor $\mathcal{F}_x \to \mathcal{F}$. We define the functor $p_x : \mathcal{F}_x \to \mathcal{C}_\Lambda$ as the composition $\mathcal{F}_x \to \mathcal{F} \xrightarrow{p} \mathcal{C}_\Lambda$. Then $p_x : \mathcal{F}_x \to \mathcal{C}_\Lambda$ is a predeformation category (proof omitted). In this way we can pass from an arbitrary category cofibered in groupoids over $\mathcal{C}_\Lambda$ to a predeformation category at any $x \in \mathrm{Ob}(\mathcal{F}(k))$.

64.7. Formal objects and completion categories

In this section we discuss how to go between categories cofibred in groupoids over $\mathcal{C}_\Lambda$ to categories cofibred in groupoids over $\widehat{\mathcal{C}}_\Lambda$ and vice versa.

Definition 64.7.1. Let $\mathcal{F}$ be a category cofibered in groupoids over $\mathcal{C}_\Lambda$. The *category $\widehat{\mathcal{F}}$ of formal objects of $\mathcal{F}$* is the category with the following objects and morphisms.

(1) A *formal object $\xi = (R, \xi_n, f_n)$ of $\mathcal{F}$* consists of an object R of $\widehat{\mathcal{C}}_\Lambda$, and a collection indexed by $n \in \mathbf{N}$ of objects ξ_n of $\mathcal{F}(R/\mathfrak{m}_R^n)$ and morphisms $f_n : \xi_{n+1} \to \xi_n$ lying over the projection $R/\mathfrak{m}_R^{n+1} \to R/\mathfrak{m}_R^n$.

(2) Let $\xi = (R, \xi_n, f_n)$ and $\eta = (S, \eta_n, g_n)$ be formal objects of $\mathcal{F}$. A *morphism $a : \xi \to \eta$ of formal objects* consists of a map $a_0 : R \to S$ in $\widehat{\mathcal{C}}_\Lambda$ and a collection $a_n : \xi_n \to \eta_n$ of morphisms of $\mathcal{F}$ lying over $R/\mathfrak{m}_R^n \to S/\mathfrak{m}_S^n$, such that for every n the diagram

$$\begin{array}{ccc} \xi_{n+1} & \xrightarrow[f_n]{} & \xi_n \\ {\scriptstyle a_{n+1}}\downarrow & & \downarrow{\scriptstyle a_n} \\ \eta_{n+1} & \xrightarrow{g_n} & \eta_n \end{array}$$

commutes.

The category of formal objects comes with a functor $\widehat{p} : \widehat{\mathcal{F}} \to \widehat{\mathcal{C}}_\Lambda$ which sends an object (R, ξ_n, f_n) to R and a morphism $(R, \xi_n, f_n) \to (S, \eta_n, g_n)$ to the map $R \to S$.

Lemma 64.7.2. *Let $p : \mathcal{F} \to \mathcal{C}_\Lambda$ be a category cofibered in groupoids. Then $\widehat{p} : \widehat{\mathcal{F}} \to \widehat{\mathcal{C}}_\Lambda$ is a category cofibered in groupoids.*

Proof. Let $R \to S$ be a ring map in $\widehat{\mathcal{C}}_\Lambda$. Let (R, ξ_n, f_n) be an object of $\widehat{\mathcal{F}}$. For each n choose a pushforward $\xi_n \to \eta_n$ of ξ_n along $R/\mathfrak{m}_R^n \to S/\mathfrak{m}_S^n$. For each n there exists a unique morphism $g_n : \eta_{n+1} \to \eta_n$ in $\mathcal{F}$ lying over $S/\mathfrak{m}_S^{n+1} \to S/\mathfrak{m}_S^n$ such that

$$\begin{array}{ccc} \xi_{n+1} & \xrightarrow[f_n]{} & \xi_n \\ \downarrow & & \downarrow \\ \eta_{n+1} & \xrightarrow{g_n} & \eta_n \end{array}$$

commutes (by the first axiom of a category cofibred in groupoids). Hence we obtain a morphism $(R, \xi_n, f_n) \to (S, \eta_n, g_n)$ lying over $R \to S$, i.e., the first axiom of a category cofibred in groupoids holds for $\widehat{\mathcal{F}}$. To see the second axiom suppose that we have morphisms $a : (R, \xi_n, f_n) \to (S, \eta_n, g_n)$ and $b : (R, \xi_n, f_n) \to (T, \theta_n, h_n)$ in $\widehat{\mathcal{F}}$ and a morphism $c_0 : S \to T$ in $\widehat{\mathcal{C}}_\Lambda$ such that $c_0 \circ a_0 = b_0$. By the second axiom of a category cofibred in groupoids for $\mathcal{F}$ we obtain unique maps $c_n : \eta_n \to \theta_n$ lying over $S/\mathfrak{m}_S^n \to T/\mathfrak{m}_T^n$ such that $c_n \circ a_n = b_n$. Setting $c = (c_n)_{n \geq 0}$ gives the desired morphism $c : (S, \eta_n, g_n) \to (T, \theta_n, h_n)$ in $\widehat{\mathcal{F}}$ (we omit the verification that $h_n \circ c_{n+1} = c_n \circ g_n$). □

Definition 64.7.3. Let $p : \mathcal{F} \to \mathcal{C}_\Lambda$ be a category cofibered in groupoids. The category cofibered in groupoids $\widehat{p} : \widehat{\mathcal{F}} \to \widehat{\mathcal{C}}_\Lambda$ is called the *completion of $\mathcal{F}$*.

If $\mathcal{F}$ is a category cofibered in groupoids over $\mathcal{C}_\Lambda$, we have defined $\widehat{\mathcal{F}}(R)$ for $R \in \mathrm{Ob}(\widehat{\mathcal{C}_\Lambda})$ in terms of the filtration of R by powers of its maximal ideal. But suppose $\mathcal{I} = (I_n)$ is a filtration of R by ideals inducing the $\mathfrak{m}_R$-adic topology. We define $\widehat{\mathcal{F}}_\mathcal{I}(R)$ to be the category with the following objects and morphisms:

(1) An object is a collection $(\xi_n, f_n)_{n\in\mathbf{N}}$ of objects ξ_n of $\mathcal{F}(R/I_n)$ and morphisms $f_n : \xi_{n+1} \to \xi_n$ lying over the projections $R/I_{n+1} \to R/I_n$.
(2) A morphism $a : (\xi_n, f_n) \to (\eta_n, g_n)$ consists of a collection $a_n : \xi_n \to \eta_n$ of morphisms in $\mathcal{F}(R/I_n)$, such that for every n the diagram

$$\begin{array}{ccc} \xi_{n+1} & \xrightarrow{f_n} & \xi_n \\ {\scriptstyle a_{n+1}}\downarrow & & \downarrow{\scriptstyle a_n} \\ \eta_{n+1} & \xrightarrow{g_n} & \eta_n \end{array}$$

commutes.

Lemma 64.7.4. *In the situation above, $\widehat{\mathcal{F}}_\mathcal{I}(R)$ is equivalent to the category $\widehat{\mathcal{F}}(R)$.*

Proof. An equivalence $\widehat{\mathcal{F}}_\mathcal{I}(R) \to \widehat{\mathcal{F}}(R)$ can be defined as follows. For each n, let $m(n)$ be the least m that $I_m \subset \mathfrak{m}_R^n$. Given an object (ξ_n, f_n) of $\widehat{\mathcal{F}}_\mathcal{I}(R)$, let η_n be the pushforward of $\xi_{m(n)}$ along $R/I_{m(n)} \to R/\mathfrak{m}_R^n$. Let $g_n : \eta_{n+1} \to \eta_n$ be the unique morphism of $\mathcal{F}$ lying over $R/\mathfrak{m}_R^{n+1} \to R/\mathfrak{m}_R^n$ such that

$$\begin{array}{ccc} \xi_{m(n+1)} & \xrightarrow[f_{m(n)}\circ\ldots\circ f_{m(n+1)-1}]{} & \xi_{m(n)} \\ \downarrow & & \downarrow \\ \eta_{n+1} & \xrightarrow{g_n} & \eta_n \end{array}$$

commutes (existence and uniqueness is guaranteed by the axioms of a cofibred category). The functor $\widehat{\mathcal{F}}_\mathcal{I}(R) \to \widehat{\mathcal{F}}(R)$ sends (ξ_n, f_n) to (R, η_n, g_n). We omit the verification that this is indeed an equivalence of categories. □

Remark 64.7.5. Let $p : \mathcal{F} \to \mathcal{C}_\Lambda$ be a category cofibered in groupoids. Suppose that for each $R \in \mathrm{Ob}(\widehat{\mathcal{C}_\Lambda})$ we are given a filtration $\mathcal{I}_R$ of R by ideals. If $\mathcal{I}_R$ induces the $\mathfrak{m}_R$-adic topology on R for all R, then one can define a category $\widehat{\mathcal{F}}_\mathcal{I}$ by mimicking the definition of $\widehat{\mathcal{F}}$. This category comes equipped with a morphism $\widehat{p}_\mathcal{I} : \widehat{\mathcal{F}}_\mathcal{I} \to \widehat{\mathcal{C}}_\Lambda$ making it into a category cofibered in groupoids such that $\widehat{\mathcal{F}}_\mathcal{I}(R)$ is isomorphic to $\widehat{\mathcal{F}}_{\mathcal{I}_R}(R)$ as defined above. The categories cofibered in groupoids $\widehat{\mathcal{F}}_\mathcal{I}$ and $\widehat{\mathcal{F}}$ are equivalent, by using over an object $R \in \mathrm{Ob}(\widehat{\mathcal{C}_\Lambda})$ the equivalence of Lemma 64.7.4.

Remark 64.7.6. Let $F : \mathcal{C}_\Lambda \to \mathit{Sets}$ be a functor. Identifying functors with cofibered sets, the completion of F is the functor $\widehat{F} : \widehat{\mathcal{C}}_\Lambda \to \mathit{Sets}$ given by $\widehat{F}(S) = \lim F(S/\mathfrak{m}_S^n)$. This agrees with the definition in Schlessinger's paper [**Sch68**].

Remark 64.7.7. Let $\mathcal{F}$ be a category cofibred in groupoids over $\mathcal{C}_\Lambda$. We claim that there is a canonical equivalence

$$can : \widehat{\mathcal{F}}|_{\mathcal{C}_\Lambda} \longrightarrow \mathcal{F}.$$

Namely, let $A \in \mathrm{Ob}(\mathcal{C}_\Lambda)$ and let (A, ξ_n, f_n) be an object of $\widehat{\mathcal{F}}|_{\mathcal{C}_\Lambda}(A)$. Since A is Artinian there is a minimal $m \in \mathbf{N}$ such that $\mathfrak{m}_A^m = 0$. Then *can* sends (A, ξ_n, f_n)

to ξ_m. This functor is an equivalence of categories cofibered in groupoids by Categories, Lemma 4.33.8 because it is an equivalence on all fibre categories by Lemma 64.7.4 and the fact that the $\mathfrak{m}_A$-adic topology on a local Artinian ring A comes from the zero ideal. We will frequently identify $\mathcal{F}$ with a full subcategory of $\widehat{\mathcal{F}}$ via a quasi-inverse to the functor *can*.

Remark 64.7.8. Let $\varphi : \mathcal{F} \to \mathcal{G}$ be a morphism of categories cofibered in groupoids over $\mathcal{C}_\Lambda$. Then there is an induced morphism $\widehat{\varphi} : \widehat{\mathcal{F}} \to \widehat{\mathcal{G}}$ of categories cofibered in groupoids over $\widehat{\mathcal{C}}_\Lambda$. It sends an object $\xi = (R, \xi_n, f_n)$ of $\widehat{\mathcal{F}}$ to $(R, \varphi(\xi_n), \varphi(f_n))$, and it sends a morphism $(a_0 : R \to S, a_n : \xi_n \to \eta_n)$ between objects ξ and η of $\widehat{\mathcal{F}}$ to $(a_0 : R \to S, \varphi(a_n) : \varphi(\xi_n) \to \varphi(\eta_n))$. Finally, if $t : \varphi \to \varphi'$ is a 2-morphism between 1-morphisms $\varphi, \varphi' : \mathcal{F} \to \mathcal{G}$ of categories cofibred in groupoids, then we obtain a 2-morphism $\widehat{t} : \widehat{\varphi} \to \widehat{\varphi}'$. Namely, for $\xi = (R, \xi_n, f_n)$ as above we set $\widehat{t}_\xi = (t_{\varphi(\xi_n)})$. Hence completion defines a functor between 2-categories

$$\widehat{\ } : \mathrm{Cof}(\mathcal{C}_\Lambda) \longrightarrow \mathrm{Cof}(\widehat{\mathcal{C}}_\Lambda)$$

from the 2-category of categories cofibred in groupoids over $\mathcal{C}_\Lambda$ to the 2-category of categories cofibred in groupoids over $\widehat{\mathcal{C}}_\Lambda$.

Remark 64.7.9. We claim the completion functor of Remark 64.7.8 and the restriction functor $|_{\mathcal{C}_\Lambda} : \mathrm{Cof}(\widehat{\mathcal{C}}_\Lambda) \to \mathrm{Cof}(\mathcal{C}_\Lambda)$ of Remarks 64.5.2 (14) are "2-adjoint" in the following precise sense. Let $\mathcal{F} \in \mathrm{Ob}(\mathrm{Cof}(\mathcal{C}_\Lambda))$ and let $\mathcal{G} \in \mathrm{Ob}(\mathrm{Cof}(\widehat{\mathcal{C}}_\Lambda))$. Then there is an equivalence of categories

$$\Phi : \mathrm{Mor}_{\mathcal{C}_\Lambda}(\mathcal{G}|_{\mathcal{C}_\Lambda}, \mathcal{F}) \longrightarrow \mathrm{Mor}_{\widehat{\mathcal{C}}_\Lambda}(\mathcal{G}, \widehat{\mathcal{F}})$$

To describe this equivalence, we define canonical morphisms $\mathcal{G} \to \widehat{\mathcal{G}|_{\mathcal{C}_\Lambda}}$ and $\widehat{\mathcal{F}}|_{\mathcal{C}_\Lambda} \to \mathcal{F}$ as follows

(1) Let $R \in \mathrm{Ob}(\widehat{\mathcal{C}}_\Lambda))$ and let ξ be an object of the fiber category $\mathcal{G}(R)$. Choose a pushforward $\xi \to \xi_n$ of ξ to $R/\mathfrak{m}_R^n$ for each $n \in \mathbf{N}$, and let $f_n : \xi_{n+1} \to \xi_n$ be the induced morphism. Then $\mathcal{G} \to \widehat{\mathcal{G}|_{\mathcal{C}_\Lambda}}$ sends ξ to (R, ξ_n, f_n).
(2) This is the equivalence $can : \widehat{\mathcal{F}}|_{\mathcal{C}_\Lambda} \to \mathcal{F}$ of Remark 64.7.7.

Having said this, the equivalence $\Phi : \mathrm{Mor}_{\mathcal{C}_\Lambda}(\mathcal{G}|_{\mathcal{C}_\Lambda}, \mathcal{F}) \to \mathrm{Mor}_{\widehat{\mathcal{C}}_\Lambda}(\mathcal{G}, \widehat{\mathcal{F}})$ sends a morphism $\varphi : \mathcal{G}|_{\mathcal{C}_\Lambda} \to \mathcal{F}$ to

$$\mathcal{G} \to \widehat{\mathcal{G}|_{\mathcal{C}_\Lambda}} \xrightarrow{\widehat{\varphi}} \widehat{\mathcal{F}}$$

There is a quasi-inverse $\Psi : \mathrm{Mor}_{\widehat{\mathcal{C}}_\Lambda}(\mathcal{G}, \widehat{\mathcal{F}}) \to \mathrm{Mor}_{\mathcal{C}_\Lambda}(\mathcal{G}|_{\mathcal{C}_\Lambda}, \mathcal{F})$ to Φ which sends $\psi : \mathcal{G} \to \widehat{\mathcal{F}}$ to

$$\mathcal{G}|_{\mathcal{C}_\Lambda} \xrightarrow{\psi|_{\mathcal{C}_\Lambda}} \widehat{\mathcal{F}}|_{\mathcal{C}_\Lambda} \to \mathcal{F}.$$

We omit the verification that Φ and Ψ are quasi-inverse. We also do not address functoriality of Φ (because it would lead into 3-category territory which we want to avoid at all cost).

Remark 64.7.10. For a category $\mathcal{C}$ we denote by $\mathrm{CofSet}(\mathcal{C})$ the category of cofibered sets over $\mathcal{C}$. It is a 1-category isomorphic the category of functors $\mathcal{C} \to \mathit{Sets}$. See Remarks 64.5.2 (11). The completion and restriction functors restrict to functors $\widehat{\ } : \mathrm{CofSet}(\mathcal{C}_\Lambda) \to \mathrm{CofSet}(\widehat{\mathcal{C}}_\Lambda)$ and $|_{\mathcal{C}_\Lambda} : \mathrm{CofSet}(\widehat{\mathcal{C}}_\Lambda) \to \mathrm{CofSet}(\mathcal{C}_\Lambda)$ which we denote by the same symbols. As functors on the categories of cofibered sets,

completion and restriction are adjoints in the usual 1-categorical sense: the same construction as in Remark 64.7.9 defines a functorial bijection

$$\mathrm{Mor}_{\mathcal{C}_\Lambda}(G|_{\mathcal{C}_\Lambda}, F) \longrightarrow \mathrm{Mor}_{\widehat{\mathcal{C}}_\Lambda}(G, \widehat{F})$$

for $F \in \mathrm{Ob}(\mathrm{CofSet}(\mathcal{C}_\Lambda))$ and $G \in \mathrm{Ob}(\mathrm{CofSet}(\widehat{\mathcal{C}}_\Lambda))$. Again the map $\widehat{F}|_{\mathcal{C}_\Lambda} \to F$ is an isomorphism.

Remark 64.7.11. Let $G : \widehat{\mathcal{C}}_\Lambda \to \mathit{Sets}$ be a functor that commutes with limits. Then the map $G \to \widehat{G|_{\mathcal{C}_\Lambda}}$ described in Remark 64.7.9 is an isomorphism. Indeed, if S is an object of $\widehat{\mathcal{C}}_\Lambda$, then we have canonical bijections

$$\widehat{G|_{\mathcal{C}_\Lambda}}(S) = \lim_n G(S/\mathfrak{m}_S^n) = G(\lim_n S/\mathfrak{m}_S^n) = G(S).$$

In particular, if R is an object of $\widehat{\mathcal{C}}_\Lambda$ then $\underline{R} = \widehat{\underline{R}|_{\mathcal{C}_\Lambda}}$ because the representable functor $\underline{R}$ commutes with limits by definition of limits.

Remark 64.7.12. Let R be an object of $\widehat{\mathcal{C}}_\Lambda$. It defines a functor $\underline{R} : \widehat{\mathcal{C}}_\Lambda \to \mathit{Sets}$ as described in Remarks 64.5.2 (12). As usual we identify this functor with the associated cofibered set. If $\mathcal{F}$ is a cofibered category over $\mathcal{C}_\Lambda$, then there is an equivalence of categories

$$\mathrm{Mor}_{\mathcal{C}_\Lambda}(\underline{R}|_{\mathcal{C}_\Lambda}, \mathcal{F}) \longrightarrow \widehat{\mathcal{F}}(R). \tag{64.7.12.1}$$

It is given by the composition

$$\mathrm{Mor}_{\mathcal{C}_\Lambda}(\underline{R}|_{\mathcal{C}_\Lambda}, \mathcal{F}) \xrightarrow{\Phi} \mathrm{Mor}_{\widehat{\mathcal{C}}_\Lambda}(\underline{R}, \widehat{\mathcal{F}}) \xrightarrow{\sim} \widehat{\mathcal{F}}(R)$$

where Φ is as in Remark 64.7.9 and the second equivalence comes from the 2-Yoneda lemma (the cofibered analogue of Categories, Lemma 4.39.1). Explicitly, the equivalence sends a morphism $\varphi : \underline{R}|_{\mathcal{C}_\Lambda} \to \mathcal{F}$ to the formal object $(R, \varphi(R \to R/\mathfrak{m}_R^n), \varphi(f_n))$ in $\widehat{\mathcal{F}}(R)$, where $f_n : R/\mathfrak{m}_R^{n+1} \to R/\mathfrak{m}_R^n$ is the projection.

Assume a choice of pushforwards for $\mathcal{F}$ has been made. Given any $\xi \in \mathrm{Ob}(\widehat{\mathcal{F}}(R))$ we construct an explicit $\underline{\xi} : \underline{R}|_{\mathcal{C}_\Lambda} \to \mathcal{F}$ which maps to ξ under (64.7.12.1). Namely, say $\xi = (R, \xi_n, f_n)$. An object α in $\underline{R}|_{\mathcal{C}_\Lambda}$ is the same thing as a morphism $\alpha : R \to A$ of $\widehat{\mathcal{C}}_\Lambda$ with A Artinian. Let $m \in \mathbf{N}$ be minimal such that $\mathfrak{m}_A^m = 0$. Then α factors through a unique $\alpha_m : R/\mathfrak{m}_R^m \to A$ and we can set $\underline{\xi}(\alpha) = \alpha_{m,*}\xi_m$. We omit the description of $\underline{\xi}$ on morphisms and we omit the proof that $\underline{\xi}$ maps to ξ via (64.7.12.1).

Assume a choice of pushforwards for $\widehat{\mathcal{F}}$ has been made. In this case the proof of Categories, Lemma 4.39.1 gives an explicit quasi-inverse

$$\iota : \widehat{\mathcal{F}}(R) \longrightarrow \mathrm{Mor}_{\widehat{\mathcal{C}}_\Lambda}(\underline{R}, \widehat{\mathcal{F}})$$

to the 2-Yoneda equivalence which takes ξ to the morphism $\iota(\xi) : \underline{R} \to \widehat{\mathcal{F}}$ sending $f \in \underline{R}(S) = \mathrm{Mor}_{\mathcal{C}_\Lambda}(R, S)$ to $f_*\xi$. A quasi-inverse to (64.7.12.1) is then

$$\widehat{\mathcal{F}}(R) \xrightarrow{\iota} \mathrm{Mor}_{\widehat{\mathcal{C}}_\Lambda}(\underline{R}, \widehat{\mathcal{F}}) \xrightarrow{\Psi} \mathrm{Mor}_{\mathcal{C}_\Lambda}(\underline{R}|_{\mathcal{C}_\Lambda}, \mathcal{F})$$

where Ψ is as in Remark 64.7.9. Given $\xi \in \mathrm{Ob}(\widehat{\mathcal{F}}(R))$ we have $\Psi(\iota(\xi)) \cong \underline{\xi}$ where $\underline{\xi}$ is as in the previous paragraph, because both are mapped to ξ under the equivalence of categories (64.7.12.1). Using $\underline{R} = \widehat{\underline{R}|_{\mathcal{C}_\Lambda}}$ (see Remark 64.7.11) and unwinding the definitions of Φ and Ψ we conclude that $\iota(\xi)$ is isomorphic to the completion of $\underline{\xi}$.

Remark 64.7.13. Let $\mathcal{F}$ be a category cofibred in groupoids over $\mathcal{C}_\Lambda$. Let $\xi = (R, \xi_i, f_n)$ and $\eta = (S, \eta_n, g_n)$ be formal objects of $\mathcal{F}$. Let $a = (a_n) : \xi \to \eta$ be a morphism of formal objects, i.e., a morphism of $\widehat{\mathcal{F}}$. Let $f = \widehat{p}(a) = a_0 : R \to S$ be the projection of a in $\widehat{\mathcal{C}}_\Lambda$. Then we obtain a 2-commutative diagram

$$\begin{array}{ccc} \underline{R}|_{\mathcal{C}_\Lambda} & \xleftarrow{\quad f \quad} & \underline{S}|_{\mathcal{C}_\Lambda} \\ & {\scriptstyle \underline{\xi}} \searrow \quad \swarrow {\scriptstyle \underline{\eta}} & \\ & \mathcal{F} & \end{array}$$

where $\underline{\xi}$ and $\underline{\eta}$ are the morphisms constructed in Remark 64.7.12. To see this let $\alpha : S \to A$ be an object of $\underline{S}|_{\mathcal{C}_\Lambda}$ (see loc. cit.). Let $m \in \mathbf{N}$ be minimal such that $\mathfrak{m}_A^m = 0$. We get a commutative diagram

$$\begin{array}{ccccc} R & \longrightarrow & R/\mathfrak{m}_R^m & & \\ \downarrow{\scriptstyle f} & & \downarrow{\scriptstyle f_m} & \searrow{\scriptstyle \beta_m} & \\ S & \longrightarrow & S/\mathfrak{m}_S^m & \xrightarrow{\alpha_m} & A \end{array}$$

such that the bottom arrows compose to give α. Then $\underline{\eta}(\alpha) = \alpha_{m,*}\eta_m$ and $\underline{\xi}(\alpha \circ f) = \beta_{m,*}\xi_m$. The morphism $a_m : \xi_m \to \eta_m$ lies over f_m hence we obtain a canonical morphism

$$\underline{\xi}(\alpha \circ f) = \beta_{m,*}\xi_m \longrightarrow \underline{\eta}(\alpha) = \alpha_{m,*}\eta_m$$

lying over id_A such that

$$\begin{array}{ccc} \xi_m & \longrightarrow & \beta_{m,*}\xi_m \\ \downarrow{\scriptstyle a_m} & & \downarrow \\ \eta_m & \longrightarrow & \alpha_{m,*}\eta_m \end{array}$$

commutes by the axioms of a category cofibred in groupoids. This defines a transformation of functors $\underline{\xi} \circ f \to \underline{\eta}$ which witnesses the 2-commutativity of the first diagram of this remark.

Remark 64.7.14. According to Remark 64.7.12, giving a formal object ξ of $\mathcal{F}$ is equivalent to giving a prorepresentable functor $U : \mathcal{C}_\Lambda \to \mathit{Sets}$ and a morphism $U \to \mathcal{F}$.

64.8. Smooth morphisms

In this section we discuss smooth morphisms of categories cofibered in groupoids over $\mathcal{C}_\Lambda$.

Definition 64.8.1. Let $\varphi : \mathcal{F} \to \mathcal{G}$ be a morphism of categories cofibered in groupoids over $\mathcal{C}_\Lambda$. We say φ is *smooth* if it satisfies the following condition: Let $B \to A$ be a surjective ring map in $\mathcal{C}_\Lambda$. Let $y \in \mathrm{Ob}(\mathcal{G}(B)), x \in \mathrm{Ob}(\mathcal{F}(A))$, and $y \to \varphi(x)$ be a morphism lying over $B \to A$. Then there exists $x' \in \mathrm{Ob}(\mathcal{F}(B))$, a morphism $x' \to x$ lying over $B \to A$, and a morphism $\varphi(x') \to y$ lying over

$\mathrm{id} : B \to B$, such that the diagram

$$\begin{array}{ccc} \varphi(x') & \longrightarrow & y \\ & \searrow & \downarrow \\ & & \varphi(x) \end{array}$$

commutes.

Lemma 64.8.2. *Let $\varphi : \mathcal{F} \to \mathcal{G}$ be a morphism of categories cofibered in groupoids over $\mathcal{C}_\Lambda$. Then φ is smooth if the condition in Definition 64.8.1 is assumed to hold only for small extensions $B \to A$.*

Proof. Let $B \to A$ be a surjective ring map in $\mathcal{C}_\Lambda$. Let $y \in \mathrm{Ob}(\mathcal{G}(B))$, $x \in \mathrm{Ob}(\mathcal{F}(A))$, and $y \to \varphi(x)$ be a morphism lying over $B \to A$. By Lemma 64.3.3 we can factor $B \to A$ into small extensions $B = B_n \to B_{n-1} \to \ldots \to B_0 = A$. We argue by induction on n. If $n = 1$ the result is true by assumption. If $n > 1$, then denote $f : B = B_n \to B_{n-1}$ and denote $g : B_{n-1} \to B_0 = A$. Choose a pushforward $y \to f_*y$ of y along f, so that the morphism $y \to \varphi(x)$ factors as $y \to f_*y \to \varphi(x)$. By the induction hypothesis we can find $x_{n-1} \to x$ lying over $g : B_{n-1} \to A$ and $a : \varphi(x_{n-1}) \to f_*y$ lying over $\mathrm{id} : B_{n-1} \to B_{n-1}$ such that

$$\begin{array}{ccc} \varphi(x_{n-1}) & \xrightarrow[a]{} & f_*y \\ & \searrow & \downarrow \\ & & \varphi(x) \end{array}$$

commutes. We can apply the assumption to the composition $y \to \varphi(x_{n-1})$ of $y \to f_*y$ with $a^{-1} : f_*y \to \varphi(x_{n-1})$. We obtain $x_n \to x_{n-1}$ lying over $B_n \to B_{n-1}$ and $\varphi(x_n) \to y$ lying over $\mathrm{id} : B_n \to B_n$ so that the diagram

$$\begin{array}{ccc} \varphi(x_n) & \longrightarrow & y \\ \downarrow & & \downarrow \\ \varphi(x_{n-1}) & \xrightarrow{a} & f_*y \\ & \searrow & \downarrow \\ & & \varphi(x) \end{array}$$

commutes. Then the composition $x_n \to x_{n-1} \to x$ and $\varphi(x_n) \to y$ are the morphisms required by the definition of smoothness. □

Remark 64.8.3. Let $\varphi : \mathcal{F} \to \mathcal{G}$ be a morphism of categories cofibered in groupoids over $\mathcal{C}_\Lambda$. Let $B \to A$ be a ring map in $\mathcal{C}_\Lambda$. Choices of pushforwards along $B \to A$ for objects in the fiber categories $\mathcal{F}(B)$ and $\mathcal{G}(B)$ determine functors $\mathcal{F}(B) \to \mathcal{F}(A)$ and $\mathcal{G}(B) \to \mathcal{G}(A)$ fitting into a 2-commutative diagram

$$\begin{array}{ccc} \mathcal{F}(B) & \xrightarrow{\varphi} & \mathcal{G}(B) \\ \downarrow & & \downarrow \\ \mathcal{F}(A) & \xrightarrow{\varphi} & \mathcal{G}(A). \end{array}$$

Hence there is an induced functor $\mathcal{F}(B) \to \mathcal{F}(A) \times_{\mathcal{G}(A)} \mathcal{G}(B)$. Unwinding the definitions shows that $\varphi : \mathcal{F} \to \mathcal{G}$ is smooth if and only if this induced functor is essentially surjective whenever $B \to A$ is surjective (or equivalently, by Lemma 64.8.2, whenever $B \to A$ is a small extension).

Remark 64.8.4. The characterization of smooth morphisms in Remark 64.8.3 is analogous to Schlessinger's notion of a smooth morphism of functors, cf. [**Sch68**, Definition 2.2.]. In fact, when $\mathcal{F}$ and $\mathcal{G}$ are cofibered in sets then our notion is equivalent to Schlessinger's. Namely, in this case let $F, G : \mathcal{C}_\Lambda \to \mathit{Sets}$ be the corresponding functors, see Remarks 64.5.2 (11). Then $F \to G$ is smooth if and only if for every surjection of rings $B \to A$ in $\mathcal{C}_\Lambda$ the map $F(B) \to F(A) \times_{G(A)} G(B)$ is surjective.

Remark 64.8.5. Let $\mathcal{F}$ be a category cofibered in groupoids over $\mathcal{C}_\Lambda$. Then the morphism $\mathcal{F} \to \overline{\mathcal{F}}$ is smooth.

If $R \to S$ is a ring map $\widehat{\mathcal{C}}_\Lambda$, then there is an induced morphism $\underline{S} \to \underline{R}$ between the functors $\underline{S}, \underline{R} : \widehat{\mathcal{C}}_\Lambda \to \mathit{Sets}$. In this situation, smoothness of the restriction $\underline{S}|_{\mathcal{C}_\Lambda} \to \underline{R}|_{\mathcal{C}_\Lambda}$ is a familiar notion:

Lemma 64.8.6. *Let $R \to S$ be a ring map in $\widehat{\mathcal{C}}_\Lambda$. Then the induced morphism $\underline{S}|_{\mathcal{C}_\Lambda} \to \underline{R}|_{\mathcal{C}_\Lambda}$ is smooth if and only if S is a power series ring over R.*

Proof. Assume S is a power series ring over R. Say $S = R[[x_1, \ldots, x_n]]$. Smoothness of $\underline{S}|_{\mathcal{C}_\Lambda} \to \underline{R}|_{\mathcal{C}_\Lambda}$ means the following (see Remark 64.8.4): Given a surjective ring map $B \to A$ in $\mathcal{C}_\Lambda$, a ring map $R \to B$, a ring map $S \to A$ such that the solid diagram

$$\begin{array}{ccc} S & \longrightarrow & A \\ \uparrow & \nearrow & \uparrow \\ R & \longrightarrow & B \end{array}$$

is commutative then a dotted arrow exists making the diagram commute. (Note the similarity with Algebra, Definition 10.131.1.) To construct the dotted arrow choose elements $b_i \in B$ whose images in A are equal to the images of x_i in A. Note that $b_i \in \mathfrak{m}_B$ as x_i maps to an element of $\mathfrak{m}_A$. Hence there is a unique R-algebra map $R[[x_1, \ldots, x_n]] \to B$ which maps x_i to b_i and which can serve as our dotted arrow.

Conversely, assume $\underline{S}|_{\mathcal{C}_\Lambda} \to \underline{R}|_{\mathcal{C}_\Lambda}$ is smooth. Let $x_1, \ldots, x_n \in S$ be elements whose images form a basis in the relative cotangent space $\mathfrak{m}_S/(\mathfrak{m}_R S + \mathfrak{m}_S^2)$ of S over R. Set $T = R[[X_1, \ldots, X_n]]$. Note that both

$$S/(\mathfrak{m}_R S + \mathfrak{m}_S^2) \cong R/\mathfrak{m}_R[x_1, \ldots, x_n]/(x_i x_j)$$

and

$$T/(\mathfrak{m}_R T + \mathfrak{m}_T^2) \cong R/\mathfrak{m}_R[X_1, \ldots, X_n]/(X_i X_j).$$

Let $S/(\mathfrak{m}_R S + \mathfrak{m}_S^2) \to T/(\mathfrak{m}_R T + \mathfrak{m}_T^2)$ be the local R-algebra isomorphism given by mapping the class of x_i to the class of X_i. Let $f_1 : S \to T/(\mathfrak{m}_R T + \mathfrak{m}_T^2)$ be the composition $S \to S/(\mathfrak{m}_R S + \mathfrak{m}_S^2) \to T/(\mathfrak{m}_R T + \mathfrak{m}_T^2)$. The assumption that $\underline{S}|_{\mathcal{C}_\Lambda} \to \underline{R}|_{\mathcal{C}_\Lambda}$ is smooth means we can lift f_1 to a map $f_2 : S \to T/\mathfrak{m}_T^2$, then to a map $f_3 : S \to T/\mathfrak{m}_T^3$, and so on, for all $n \geq 1$. Thus we get an induced map $f : S \to T = \lim T/\mathfrak{m}_T^n$ of local R-algebras. By our choice of f_1, the map f

induces an isomorphism $\mathfrak{m}_S/(\mathfrak{m}_R S + \mathfrak{m}_S^2) \to \mathfrak{m}_T/(\mathfrak{m}_R T + \mathfrak{m}_T^2)$ of relative cotangent spaces. Hence f is surjective by Lemma 64.4.2 (where we think of f as a map in $\widehat{\mathcal{C}}_R$). Choose preimages $y_i \in S$ of $X_i \in T$ under f. As T is a power series ring over R there exists a local R-algebra homomorphism $s : T \to S$ mapping X_i to y_i. By construction $f \circ s = \mathrm{id}$. Then s is injective. But s induces an isomorphism on relative cotangent spaces since f does, so it is also surjective by Lemma 64.4.2 again. Hence s and f are isomorphisms. □

Smooth morphisms satisfy the following functorial properties.

Lemma 64.8.7. *Let $\varphi : \mathcal{F} \to \mathcal{G}$ and $\psi : \mathcal{G} \to \mathcal{H}$ be morphisms of categories cofibered in groupoids over $\mathcal{C}_\Lambda$.*

1. *If φ and ψ are smooth, then $\psi \circ \varphi$ is smooth.*
2. *If φ is essentially surjective and $\psi \circ \varphi$ is smooth, then ψ is smooth.*
3. *If $\mathcal{G}' \to \mathcal{G}$ is a morphism of categories cofibered in groupoids and φ is smooth, then $\mathcal{F} \times_{\mathcal{G}} \mathcal{G}' \to \mathcal{G}'$ is smooth.*

Proof. Statements (1) and (2) follow immediately from the definitions. Proof of (3) omitted. Hints: use the formulation of smoothness given in Remark 64.8.3 and use that $\mathcal{F} \times_{\mathcal{G}} \mathcal{G}'$ is the 2-fibre product, see Remarks 64.5.2 (13). □

Lemma 64.8.8. *Let $\varphi : \mathcal{F} \to \mathcal{G}$ be a smooth morphism of categories cofibered in groupoids over $\mathcal{C}_\Lambda$. Assume $\varphi : \mathcal{F}(k) \to \mathcal{G}(k)$ is essentially surjective. Then $\varphi : \mathcal{F} \to \mathcal{G}$ and $\widehat{\varphi} : \widehat{\mathcal{F}} \to \widehat{\mathcal{G}}$ are essentially surjective.*

Proof. Let y be an object of $\mathcal{G}$ lying over $A \in \mathrm{Ob}(\mathcal{C}_\Lambda)$. Let $y \to y_0$ be a pushforward of y along $A \to k$. By the assumption on essential surjectivity of $\varphi : \mathcal{F}(k) \to \mathcal{G}(k)$ there exist an object x_0 of $\mathcal{F}$ lying over k and an isomorphism $y_0 \to \varphi(x_0)$. Smoothness of φ implies there exists an object x of $\mathcal{F}$ over A whose image $\varphi(x)$ is isomorphic to y. Thus $\varphi : \mathcal{F} \to \mathcal{G}$ is essentially surjective.

Let $\eta = (R, \eta_n, g_n)$ be an object of $\widehat{\mathcal{G}}$. We construct an object ξ of $\widehat{\mathcal{F}}$ with an isomorphism $\eta \to \varphi(\xi)$. By the assumption on essential surjectivity of $\varphi : \mathcal{F}(k) \to \mathcal{G}(k)$, there exists a morphism $\eta_1 \to \varphi(\xi_1)$ in $\mathcal{G}(k)$ for some $\xi_1 \in \mathrm{Ob}(\mathcal{F}(k))$. The morphism $\eta_2 \xrightarrow{g_1} \eta_1 \to \varphi(\xi_1)$ lies over the surjective ring map $R/\mathfrak{m}_R^2 \to k$, hence by smoothness of φ there exists $\xi_2 \in \mathrm{Ob}(\mathcal{F}(R/\mathfrak{m}_R^2))$, a morphism $f_1 : \xi_2 \to \xi_1$ lying over $R/\mathfrak{m}_R^2 \to k$, and a morphism $\eta_2 \to \varphi(\xi_2)$ such that

$$\begin{array}{ccc} \varphi(\xi_2) & \xrightarrow{\varphi(f_1)} & \varphi(\xi_1) \\ \uparrow & & \uparrow \\ \eta_2 & \xrightarrow{g_1} & \eta_1 \end{array}$$

commutes. Continuing in this way we construct an object $\xi = (R, \xi_n, f_n)$ of $\widehat{\mathcal{F}}$ and a morphism $\eta \to \varphi(\xi) = (R, \varphi(\xi_n), \varphi(f_n))$ in $\widehat{\mathcal{G}}(R)$. □

Remark 64.8.9. Let $p : \mathcal{F} \to \mathcal{C}_\Lambda$ be a category cofibered in groupoids. We can consider $\mathcal{C}_\Lambda$ as the trivial category cofibered in groupoids over $\mathcal{C}_\Lambda$, and then p is a morphism of categories cofibered in groupoids over $\mathcal{C}_\Lambda$. We say $\mathcal{F}$ is *smooth* if its structure morphism p is smooth. This is the "absolute" notion of smoothness for a category cofibered in groupoids over $\mathcal{C}_\Lambda$.

Example 64.8.10. Let $R \in \mathrm{Ob}(\widehat{\mathcal{C}}_\Lambda)$. When is $\underline{R}|_{\mathcal{C}_\Lambda}$ smooth? In the classical case this means that R is a power series ring over Λ, see Lemma 64.8.6. (Strictly speaking this uses that $\underline{\Lambda}|_{\mathcal{C}_\Lambda} = \mathcal{C}_\Lambda$ because Λ is an initial object of $\widehat{\mathcal{C}}_\Lambda$ in the classical case.) In the general case we can construct examples as follows. Pick an integer $n \geq 0$ and a maximal ideal $\mathfrak{m} \subset \Lambda[x_1, \ldots, x_n]$ lying over $\mathfrak{m}_\Lambda$ so that

$$k' = \Lambda/\mathfrak{m}_\Lambda \longrightarrow \Lambda[x_1, \ldots, x_n]/\mathfrak{m}$$

is isomorphic to $k' \to k$. Fix such an identification $k = \Lambda[x_1, \ldots, x_n]/\mathfrak{m}$. Set $R = \Lambda[x_1, \ldots, x_n]^\wedge$ equal to the $\mathfrak{m}$-adic completion of $\Lambda[x_1, \ldots, x_n]$. Then R is an object of $\widehat{\mathcal{C}}_\Lambda$. Namely, it is a complete local Noetherian ring (see Algebra, Lemma 10.91.10) and its residue field is identified with k. We claim that $\underline{R}|_{\mathcal{C}_\Lambda}$ is smooth. To see this we have to show: Given a surjection $B \to A$ in $\mathcal{C}_\Lambda$ and a map $R \to A$ there exists a lift of this map to B. This is clear as we can first lift the composition $\Lambda[x_1, \ldots, x_n] \to R \to A$ to a map $\Lambda[x_1, \ldots, x_n] \to B$ and then observe that this latter map factors through the completion R as B is complete (being Artinian). In fact, it turns out that whenever $\underline{R}|_{\mathcal{C}_\Lambda}$ is smooth, then R is isomorphic to a completion of a smooth algebra over Λ, but we won't use this.

Example 64.8.11. Here is a more explicit example of an R as in Example 64.8.10. Let p be a prime number and let $n \in \mathbf{N}$. Let $\Lambda = \mathbf{F}_p(t_1, t_2, \ldots, t_n)$ and let $k = \mathbf{F}_p(x_1, \ldots, x_n)$ with map $\Lambda \to k$ given by $t_i \mapsto x_i^p$. Then we can take

$$R = \Lambda[x_1, \ldots, x_n]^\wedge_{(x_1^p - t_1, \ldots, x_n^p - t_n)}$$

We cannot do "better" in this example, i.e., we cannot approximate $\mathcal{C}_\Lambda$ by a smaller smooth object of $\widehat{\mathcal{C}}_\Lambda$ (one can argue that the dimension of R has to be at least n since the map $\Omega_{R/\Lambda} \otimes_R k \to \Omega_{k/\Lambda}$ is surjective). We will discuss this phenomenon later in more detail.

Remark 64.8.12. Suppose $\mathcal{F}$ is a predeformation category admitting a smooth morphism $\varphi : \mathcal{U} \to \mathcal{F}$ from a predeformation category $\mathcal{U}$. Then by Lemma 64.8.8 φ is essentially surjective, so by Lemma 64.8.7 $p : \mathcal{F} \to \mathcal{C}_\Lambda$ is smooth if and only if the composition $\mathcal{U} \xrightarrow{\varphi} \mathcal{F} \xrightarrow{p} \mathcal{C}_\Lambda$ is smooth, i.e. $\mathcal{F}$ is smooth if and only if $\mathcal{U}$ is smooth.

Later we are interested in producing smooth morphisms from prorepresentable functors to predeformation categories $\mathcal{F}$. By the discussion in Remark 64.7.12 these morphisms correspond to certain formal objects of $\mathcal{F}$ More precisely, these are the so-called versal formal objects of $\mathcal{F}$.

Definition 64.8.13. Let $\mathcal{F}$ be a category cofibered in groupoids. Let ξ be a formal object of $\mathcal{F}$ lying over $R \in \mathrm{Ob}(\widehat{\mathcal{C}}_\Lambda)$. We say ξ is *versal* if the corresponding morphism $\underline{\xi} : \underline{R}|_{\mathcal{C}_\Lambda} \to \mathcal{F}$ of Remark 64.7.12 is smooth.

Remark 64.8.14. Let $\mathcal{F}$ be a category cofibered in groupoids over $\mathcal{C}_\Lambda$, and let ξ be a formal object of $\mathcal{F}$. It follows from the definition of smoothness that versality of ξ is equivalent to the following condition: If

$$\begin{array}{ccc} & & y \\ & & \downarrow \\ \xi & \longrightarrow & x \end{array}$$

is a diagram in $\widehat{\mathcal{F}}$ such that $y \to x$ lies over a surjective map $B \to A$ of Artinian rings (we may assume it is a small extension), then there exists a morphism $\xi \to y$ such that

$$\begin{array}{ccc} & & y \\ & \nearrow & \downarrow \\ \xi & \longrightarrow & x \end{array}$$

commutes. In particular, the condition that ξ be versal does not depend on the choices of pushforwards made in the construction of $\underline{\xi} : \underline{R}|_{\mathcal{C}_\Lambda} \to \mathcal{F}$ in Remark 64.7.12.

Lemma 64.8.15. *Let $\mathcal{F}$ be a predeformation category. Let ξ be a versal formal object of $\mathcal{F}$. For any formal object η of $\widehat{\mathcal{F}}$, there exists a morphism $\xi \to \eta$.*

Proof. By assumption the morphism $\underline{\xi} : \underline{R}|_{\mathcal{C}_\Lambda} \to \mathcal{F}$ is smooth. Then $\iota(\xi) : \underline{R} \to \widehat{\mathcal{F}}$ is the completion of $\underline{\xi}$, see Remark 64.7.12. By Lemma 64.8.8 there exists an object f of $\underline{R}$ such that $\iota(\xi)(f) = \eta$. Then f is a ring map $f : R \to S$ in $\widehat{\mathcal{C}_\Lambda}$. And $\iota(\xi)(f) = \eta$ means that $f_*\xi \cong \eta$ which means exactly that there is a morphism $\xi \to \eta$ lying over f. □

64.9. Schlessinger's conditions

In the following we often consider fibre products $A_1 \times_A A_2$ of rings in the category $\mathcal{C}_\Lambda$. We have seen in Example 64.3.7 that such a fibre product may not always be an object of $\mathcal{C}_\Lambda$. However, in virtually all cases below one of the two maps $A_i \to A$ is surjective and $A_1 \times_A A_2$ will be an object of $\mathcal{C}_\Lambda$ by Lemma 64.3.8. We will use this result without further mention.

We denote by $k[\epsilon]$ the ring of dual numbers over k. More generally, for a k-vector space V, we denote by $k[V]$ the k-algebra whose underlying vector space is $k \oplus V$ and whose multiplication is given by $(a, v) \cdot (a', v') = (aa', av' + a'v)$. When $V = k$, $k[V]$ is the ring of dual numbers over k. For any finite dimensional k-vector space V the ring $k[V]$ is in $\mathcal{C}_\Lambda$.

Definition 64.9.1. Let $\mathcal{F}$ be a category cofibered in groupoids over $\mathcal{C}_\Lambda$. We define *conditions (S1) and (S2)* on $\mathcal{F}$ as follows:

(S1) Every diagram in $\mathcal{F}$

$$\begin{array}{ccc} & & x_2 \\ & & \downarrow \\ x_1 & \longrightarrow & x \end{array} \quad \text{lying over} \quad \begin{array}{ccc} & & A_2 \\ & & \downarrow \\ A_1 & \longrightarrow & A \end{array}$$

in $\mathcal{C}_\Lambda$ with $A_2 \to A$ surjective can be completed to a commutative diagram

$$\begin{array}{ccc} y & \longrightarrow & x_2 \\ \downarrow & & \downarrow \\ x_1 & \longrightarrow & x \end{array} \quad \text{lying over} \quad \begin{array}{ccc} A_1 \times_A A_2 & \longrightarrow & A_2 \\ \downarrow & & \downarrow \\ A_1 & \longrightarrow & A. \end{array}$$

(S2) The condition of (S1) holds for diagrams in $\mathcal{F}$ lying over a diagram in $\mathcal{C}_\Lambda$ of the form

$$\begin{array}{ccc} & & k[\epsilon] \\ & & \downarrow \\ A & \longrightarrow & k. \end{array}$$

Moreover, if we have two commutative diagrams in $\mathcal{F}$

$$\begin{array}{ccc} y & \xrightarrow[c]{} & x_\epsilon \\ {\scriptstyle a}\downarrow & & \downarrow{\scriptstyle e} \\ x & \xrightarrow{d} & x_0 \end{array} \quad \text{and} \quad \begin{array}{ccc} y' & \xrightarrow[c']{} & x_\epsilon \\ {\scriptstyle a'}\downarrow & & \downarrow{\scriptstyle e} \\ x & \xrightarrow{d} & x_0 \end{array} \quad \text{lying over} \quad \begin{array}{ccc} A \times_k k[\epsilon] & \longrightarrow & k[\epsilon] \\ \downarrow & & \downarrow \\ A & \longrightarrow & k \end{array}$$

then there exists a morphism $b : y \to y'$ in $\mathcal{F}(A \times_k k[\epsilon])$ such that $a = a' \circ b$.

We can partly explain the meaning of conditions (S1) and (S2) in terms of fibre categories. Suppose that $f_1 : A_1 \to A$ and $f_2 : A_2 \to A$ are ring maps in $\mathcal{C}_\Lambda$ with f_2 surjective. Denote $p_i : A_1 \times_A A_2 \to A_i$ the projection maps. Assume a choice of pushforwards for $\mathcal{F}$ has been made. Then the commutative diagram of rings translates into a 2-commutative diagram

$$\begin{array}{ccc} \mathcal{F}(A_1 \times_A A_2) & \xrightarrow[p_{2,*}]{} & \mathcal{F}(A_2) \\ {\scriptstyle p_{1,*}}\downarrow & & \downarrow{\scriptstyle f_{2,*}} \\ \mathcal{F}(A_1) & \xrightarrow{f_{1,*}} & \mathcal{F}(A) \end{array}$$

of fibre categories whence a functor

$$\mathcal{F}(A_1 \times_A A_2) \to \mathcal{F}(A_1) \times_{\mathcal{F}(A)} \mathcal{F}(A_2) \tag{64.9.1.1}$$

into the 2-fibre product of categories. Condition (S1) requires that this functor be essentially surjective. The first part of condition (S2) requires that this functor be a essentially surjective if f_2 equals the map $k[\epsilon] \to k$. Moreover in this case, the second part of (S2) implies that two objects which become isomorphic in the target are isomorphic in the source (but it is *not* equivalent to this statement). The advantage of stating the conditions as in the definition is that no choices have to be made.

Lemma 64.9.2. *Let $\mathcal{F}$ be a category cofibered in groupoids over $\mathcal{C}_\Lambda$. Then $\mathcal{F}$ satisfies (S1) if the condition of (S1) is assumed to hold only when $A_2 \to A$ is a small extension.*

Proof. Proof omitted. Hints: apply Lemma 64.3.3 and use induction similar to the proof of Lemma 64.8.2. □

Remark 64.9.3. When $\mathcal{F}$ is cofibered in sets, conditions (S1) and (S2) are exactly conditions (H1) and (H2) from Schlessinger's paper [**Sch68**]. Namely, for a functor $F : \mathcal{C}_\Lambda \to \mathit{Sets}$, conditions (S1) and (S2) state:

(S1) If $A_1 \to A$ and $A_2 \to A$ are maps in $\mathcal{C}_\Lambda$ with $A_2 \to A$ surjective, then the induced map $F(A_1 \times_A A_2) \to F(A_1) \times_{F(A)} F(A_2)$ is surjective.

(S2) If $A \to k$ is a map in $\mathcal{C}_\Lambda$, then the induced map $F(A \times_k k[\epsilon]) \to F(A) \times_{F(k)} F(k[\epsilon])$ is bijective.

The injectivity of the map $F(A \times_k k[\epsilon]) \to F(A) \times_{F(k)} F(k[\epsilon])$ comes from the second part of condition (S2) and the fact that morphisms are identities.

Lemma 64.9.4. *Let $\mathcal{F}$ be a category cofibred in groupoids over $\mathcal{C}_\Lambda$. If $\mathcal{F}$ satisfies (S2), then the condition of (S2) also holds when $k[\epsilon]$ is replaced by $k[V]$ for any finite dimensional k-vector space V.*

Proof. In the case that $\mathcal{F}$ is cofibred in sets, i.e., corresponds to a functor $F : \mathcal{C}_\Lambda \to Sets$ this follows from the description of (S2) for F in Remark 64.9.3 and the fact that $k[V] \cong k[\epsilon] \times_k \ldots \times_k k[\epsilon]$ with $\dim_k V$ factors. The case of functors is what we will use in the rest of this chapter.

We prove the general case by induction on $\dim(V)$. If $\dim(V) = 1$, then $k[V] \cong k[\epsilon]$ and the result holds by assumption. If $\dim(V) > 1$ we write $V = V' \oplus k\epsilon$. Pick a diagram

$$\begin{array}{ccc} & & x_V \\ & & \downarrow \\ x & \longrightarrow & x_0 \end{array} \quad \text{lying over} \quad \begin{array}{ccc} & & k[V] \\ & & \downarrow \\ A & \longrightarrow & k \end{array}$$

Choose a morphism $x_V \to x_{V'}$ lying over $k[V] \to k[V']$ and a morphism $x_V \to x_\epsilon$ lying over $k[V] \to k[\epsilon]$. Note that the morphism $x_V \to x_0$ factors as $x_V \to x_{V'} \to x_0$ and as $x_V \to x_\epsilon \to x_0$. By induction hypothesis we can find a diagram

$$\begin{array}{ccc} y' & \longrightarrow & x_{V'} \\ \downarrow & & \downarrow \\ x & \longrightarrow & x_0 \end{array} \quad \text{lying over} \quad \begin{array}{ccc} A \times_k k[V'] & \longrightarrow & k[V'] \\ \downarrow & & \downarrow \\ A & \longrightarrow & k \end{array}$$

This gives us a commutative diagram

$$\begin{array}{ccc} & & x_\epsilon \\ & & \downarrow \\ y' & \longrightarrow & x_0 \end{array} \quad \text{lying over} \quad \begin{array}{ccc} & & k[\epsilon] \\ & & \downarrow \\ A \times_k k[V'] & \longrightarrow & k \end{array}$$

Hence by (S2) we get a commutative diagram

$$\begin{array}{ccc} y & \longrightarrow & x_\epsilon \\ \downarrow & & \downarrow \\ y' & \longrightarrow & x_0 \end{array} \quad \text{lying over} \quad \begin{array}{ccc} (A \times_k k[V']) \times_k k[\epsilon] & \longrightarrow & k[\epsilon] \\ \downarrow & & \downarrow \\ A \times_k k[V'] & \longrightarrow & k \end{array}$$

Note that $(A \times_k k[V']) \times_k k[\epsilon] = A \times_k k[V' \oplus k\epsilon] = A \times_k k[V]$. We claim that y fits into the correct commutative diagram. To see this we let $y \to y_V$ be a morphism lying over $A \times_k k[V] \to k[V]$. We can factor the morphisms $y \to y' \to x_{V'}$ and $y \to x_\epsilon$ through the morphism $y \to y_V$ (by the axioms of categories cofibred in groupoids). Hence we see that both y_V and x_V fit into commutative diagrams

$$\begin{array}{ccc} y_V & \longrightarrow & x_\epsilon \\ \downarrow & & \downarrow \\ x_{V'} & \longrightarrow & x_0 \end{array} \quad \text{and} \quad \begin{array}{ccc} x_V & \longrightarrow & x_\epsilon \\ \downarrow & & \downarrow \\ x_{V'} & \longrightarrow & x_0 \end{array}$$

and hence by the second part of (S2) there exists an isomorphism $y_V \to x_V$ compatible with $y_V \to x_{V'}$ and $x_V \to x_{V'}$ and in particular compatible with the maps to x_0. The composition $y \to y_V \to x_V$ then fits into the required commutative diagram

$$\begin{array}{ccc} y & \longrightarrow & x_V \\ \downarrow & & \downarrow \\ x & \longrightarrow & x_0 \end{array} \quad \text{lying over} \quad \begin{array}{ccc} A \times_k k[V] & \longrightarrow & k[V] \\ \downarrow & & \downarrow \\ A & \longrightarrow & k \end{array}$$

In this way we see that the first part of $(S2)$ holds with $k[\epsilon]$ replaced by $k[V]$.

To prove the second part suppose given two commutative diagrams

$$\begin{array}{ccc} y & \longrightarrow & x_V \\ \downarrow & & \downarrow \\ x & \longrightarrow & x_0 \end{array} \quad \text{and} \quad \begin{array}{ccc} y' & \longrightarrow & x_V \\ \downarrow & & \downarrow \\ x & \longrightarrow & x_0 \end{array} \quad \text{lying over} \quad \begin{array}{ccc} A \times_k k[V] & \longrightarrow & k[V] \\ \downarrow & & \downarrow \\ A & \longrightarrow & k \end{array}$$

We will use the morphisms $x_V \to x_{V'} \to x_0$ and $x_V \to x_\epsilon \to x_0$ introduced in the first paragraph of the proof. Choose morphisms $y \to y_{V'}$ and $y' \to y'_{V'}$ lying over $A \times_k k[V] \to A \times_k k[V']$. The axioms of a cofibred category imply we can find commutative diagrams

$$\begin{array}{ccc} y_{V'} & \longrightarrow & x_{V'} \\ \downarrow & & \downarrow \\ x & \longrightarrow & x_0 \end{array} \quad \text{and} \quad \begin{array}{ccc} y'_{V'} & \longrightarrow & x_{V'} \\ \downarrow & & \downarrow \\ x & \longrightarrow & x_0 \end{array} \quad \text{lying over} \quad \begin{array}{ccc} A \times_k k[V'] & \longrightarrow & k[V'] \\ \downarrow & & \downarrow \\ A & \longrightarrow & k \end{array}$$

By induction hypothesis we obtain an isomorphism $b : y_{V'} \to y'_{V'}$ compatible with the morphisms $y_{V'} \to x$ and $y'_{V'} \to x$, in particular compatible with the morphisms to x_0. Then we have commutative diagrams

$$\begin{array}{ccc} y & \longrightarrow & x_\epsilon \\ \downarrow & & \downarrow \\ y'_{V'} & \longrightarrow & x_0 \end{array} \quad \text{and} \quad \begin{array}{ccc} y' & \longrightarrow & x_\epsilon \\ \downarrow & & \downarrow \\ y'_{V'} & \longrightarrow & x_0 \end{array} \quad \text{lying over} \quad \begin{array}{ccc} A \times_k k[\epsilon] & \longrightarrow & k[\epsilon] \\ \downarrow & & \downarrow \\ A & \longrightarrow & k \end{array}$$

where the morphism $y \to y'_{V'}$ is the composition $y \to y_{V'} \xrightarrow{b} y'_{V'}$ and where the morphisms $y \to x_\epsilon$ and $y' \to x_\epsilon$ are the compositions of the maps $y \to x_V$ and $y' \to x_V$ with the morphism $x_V \to x_\epsilon$. Then the second part of (S2) guarantees the existence of an isomorphism $y \to y'$ compatible with the maps to $y'_{V'}$, in particular compatible with the maps to x (because b was compatible with the maps to x). □

Lemma 64.9.5. *Let $\mathcal{F}$ be a category cofibered in groupoids over $\mathcal{C}_\Lambda$.*

1. *If $\mathcal{F}$ satisfies (S1), then so does $\overline{\mathcal{F}}$.*
2. *If $\mathcal{F}$ satisfies (S2), then so does $\overline{\mathcal{F}}$ provided at least one of the following conditions is satisfied*
   (a) *$\mathcal{F}$ is a predeformation category,*
   (b) *the category $\mathcal{F}(k)$ is a set or a setoid, or*
   (c) *for any morphism $x_\epsilon \to x_0$ of $\mathcal{F}$ lying over $k[\epsilon] \to k$ the pushforward map $Aut_{k[\epsilon]}(x_\epsilon) \to Aut_k(x_0)$ is surjective.*

Proof. Assume $\mathcal{F}$ has (S1). Suppose we have ring maps $f_i : A_i \to A$ in $\mathcal{C}_\Lambda$ with f_2 surjective. Let $x_i \in \mathcal{F}(A_i)$ such that the pushforwards $f_{1,*}(x_1)$ and $f_{2,*}(x_2)$ are isomorphic. Then we can denote x an object of $\mathcal{F}$ over A isomorphic to both of these and we obtain a diagram as in (S1). Hence we find an object y of $\mathcal{F}$ over $A_1 \times_A A_2$ whose pushforward to A_1, resp. A_2 is isomorphic to x_1, resp. x_2. In this way we see that (S1) holds for $\overline{\mathcal{F}}$.

Assume $\mathcal{F}$ has (S2). The first part of (S2) for $\overline{\mathcal{F}}$ follows as in the argument above. The second part of (S2) for $\overline{\mathcal{F}}$ signifies that the map

$$\overline{\mathcal{F}}(A \times_k k[\epsilon]) \to \overline{\mathcal{F}}(A) \times_{\overline{\mathcal{F}}(k)} \overline{\mathcal{F}}(k[\epsilon])$$

is injective for any ring A in $\mathcal{C}_\Lambda$. Suppose that $y, y' \in \mathcal{F}(A \times_k k[\epsilon])$. Using the axioms of cofibred categories we can choose commutative diagrams

$$\begin{array}{ccc} y & \xrightarrow[c]{} & x_\epsilon \\ {\scriptstyle a}\downarrow & & \downarrow{\scriptstyle e} \\ x & \xrightarrow[d]{} & x_0 \end{array} \quad \text{and} \quad \begin{array}{ccc} y' & \xrightarrow[c']{} & x'_\epsilon \\ {\scriptstyle a'}\downarrow & & \downarrow{\scriptstyle e'} \\ x' & \xrightarrow[d']{} & x'_0 \end{array} \quad \text{lying over} \quad \begin{array}{ccc} A \times_k k[\epsilon] & \longrightarrow & k[\epsilon] \\ \downarrow & & \downarrow \\ A & \longrightarrow & k \end{array}$$

Assume that there exist isomorphisms $\alpha : x \to x'$ in $\mathcal{F}(A)$ and $\beta : x_\epsilon \to x'_\epsilon$ in $\mathcal{F}(k[\epsilon])$. This also means there exists an isomorphism $\gamma : x_0 \to x'_0$ compatible with α. To prove (S2) for $\overline{\mathcal{F}}$ we have to show that there exists an isomorphism $y \to y'$ in $\mathcal{F}(A \times_k k[\epsilon])$. By (S2) for $\mathcal{F}$ such a morphism will exist if we can choose the isomorphisms α and β and γ such that

$$\begin{array}{ccccc} x & \longrightarrow & x_0 & \xleftarrow[e]{} & x_\epsilon \\ \downarrow{\scriptstyle \alpha} & & \downarrow{\scriptstyle \gamma} & & \downarrow{\scriptstyle \beta} \\ x' & \longrightarrow & x'_0 & \xleftarrow[e']{} & x'_\epsilon \end{array}$$

is commutative (because then we can replace x by x' and x_ϵ by x'_ϵ in the previous displayed diagram). The left hand square commutes by our choice of γ. We can factor $e' \circ \beta$ as $\gamma' \circ e$ for some second map $\gamma' : x_0 \to x'_0$. Now the question is whether we can arrange it so that $\gamma = \gamma'$? This is clear if $\mathcal{F}(k)$ is a set, or a setoid. Moreover, if $\mathrm{Aut}_{k[\epsilon]}(x_\epsilon) \to \mathrm{Aut}_k(x_0)$ is surjective, then we can adjust the choice of β by precomposing with an automorphism of x_ϵ whose image is $\gamma^{-1} \circ \gamma'$ to make things work. □

Lemma 64.9.6. *Let $\mathcal{F}$ be a category cofibered in groupoids over $\mathcal{C}_\Lambda$. Let $x_0 \in \mathrm{Ob}(\mathcal{F}(k))$. Let $\mathcal{F}_{x_0}$ be the category cofibred in groupoids over $\mathcal{C}_\Lambda$ constructed in Remark 64.6.4.*

(1) *If $\mathcal{F}$ satisfies (S1), then so does $\mathcal{F}_{x_0}$.*
(2) *If $\mathcal{F}$ satisfies (S2), then so does $\mathcal{F}_{x_0}$.*

Proof. Any diagram as in Definition 64.9.1 in $\mathcal{F}_{x_0}$ gives rise to a diagram in $\mathcal{F}$ and the output of condition (S1) or (S2) for this diagram in $\mathcal{F}$ can be viewed as an output for $\mathcal{F}_{x_0}$ as well. □

Lemma 64.9.7. *Let $p : \mathcal{F} \to \mathcal{C}_\Lambda$ be a category cofibered in groupoids. Consider a diagram of $\mathcal{F}$*

$$\begin{array}{ccc} y & \longrightarrow & x_\epsilon \\ {\scriptstyle a}\downarrow & & \downarrow{\scriptstyle e} \\ x & \xrightarrow{d} & x_0 \end{array} \quad \text{lying over} \quad \begin{array}{ccc} A \times_k k[\epsilon] & \longrightarrow & k[\epsilon] \\ \downarrow & & \downarrow \\ A & \longrightarrow & k. \end{array}$$

in $\mathcal{C}_\Lambda$. Assume $\mathcal{F}$ satisfies (S2). Then there exists a morphism $s : x \to y$ with $a \circ s = id_x$ if and only if there exists a morphism $s_\epsilon : x \to x_\epsilon$ with $e \circ s_\epsilon = d$.

Proof. The "only if" direction is clear. Conversely, assume there exists a morphism $s_\epsilon : x \to x_\epsilon$ with $e \circ s_\epsilon = d$. Note that $p(s_\epsilon) : A \to k[\epsilon]$ is a ring map compatible with the map $A \to k$. Hence we obtain

$$\sigma = (\mathrm{id}_A, p(s_\epsilon)) : A \to A \times_k k[\epsilon].$$

Choose a pushforward $x \to \sigma_* x$. By construction we can factor s_ϵ as $x \to \sigma_* x \to x_\epsilon$. Moreover, as σ is a section of $A \times_k k[\epsilon] \to A$, we get a morphism $\sigma_* x \to x$ such that $x \to \sigma_* x \to x$ is id_x. Because $e \circ s_\epsilon = d$ we find that the diagram

$$\begin{array}{ccc} \sigma_* x & \longrightarrow & x_\epsilon \\ \downarrow & & \downarrow{\scriptstyle e} \\ x & \xrightarrow{d} & x_0 \end{array}$$

is commutative. Hence by (S2) we obtain a morphism $\sigma_* x \to y$ such that $\sigma_* x \to y \to x$ is the given map $\sigma_* x \to x$. The solution to the problem is now to take $a : x \to y$ equal to the composition $x \to \sigma_* x \to y$. □

Lemma 64.9.8. *Consider a commutative diagram in a predeformation category $\mathcal{F}$*

$$\begin{array}{ccc} y & \longrightarrow & x_2 \\ \downarrow & & \downarrow{\scriptstyle a_2} \\ x_1 & \xrightarrow{a_1} & x \end{array} \quad \text{lying over} \quad \begin{array}{ccc} A_1 \times_A A_2 & \longrightarrow & A_2 \\ \downarrow & & \downarrow{\scriptstyle f_2} \\ A_1 & \xrightarrow{f_1} & A \end{array}$$

in $\mathcal{C}_\Lambda$ where $f_2 : A_2 \to A$ is a small extension. Assume there is a map $h : A_1 \to A_2$ such that $f_2 = f_1 \circ h$. Let $I = Ker(f_2)$. Consider the ring map

$$g : A_1 \times_A A_2 \longrightarrow k[I] = k \oplus I, \quad (u, v) \longmapsto \overline{u} \oplus (v - h(u))$$

Choose a pushforward $y \to g_* y$. Assume $\mathcal{F}$ satisfies (S2). If there exists a morphism $x_1 \to g_* y$, then there exists a morphism $b : x_1 \to x_2$ such that $a_1 = a_2 \circ b$.

Proof. Note that $\mathrm{id}_{A_1} \times g : A_1 \times_A A_2 \to A_1 \times_k k[I]$ is an isomorphism and that $k[I] \cong k[\epsilon]$. Hence we have a diagram

$$\begin{array}{ccc} y & \longrightarrow & g_* y \\ \downarrow & & \downarrow \\ x_1 & \longrightarrow & x_0 \end{array} \quad \text{lying over} \quad \begin{array}{ccc} A_1 \times_k k[\epsilon] & \longrightarrow & k[\epsilon] \\ \downarrow & & \downarrow \\ A_1 & \longrightarrow & k. \end{array}$$

where x_0 is an object of $\mathcal{F}$ lying over k (every object of $\mathcal{F}$ has a unique morphism to x_0, see discussion following Definition 64.6.2). If we have a morphism $x_1 \to g_* y$ then Lemma 64.9.7 provides us with a section $s : x_1 \to y$ of the map $y \to x_1$. Composing this with the map $y \to x_2$ we obtain $b : x_1 \to x_2$ which has the

property that $a_1 = a_2 \circ b$ because the diagram of the lemma commutes and because s is a section. □

64.10. Tangent spaces of functors

Let R be a ring. We write Mod_R for the category of R-modules and Mod_R^{fg} for the category of finitely generated R-modules.

Definition 64.10.1. Let $L : \mathrm{Mod}_R^{fg} \to \mathrm{Mod}_R$, resp. $L : \mathrm{Mod}_R \to \mathrm{Mod}_R$ be a functor. We say that L is *R-linear* if for every pair of objects M, N of Mod_R^{fg}, resp. Mod_R the map

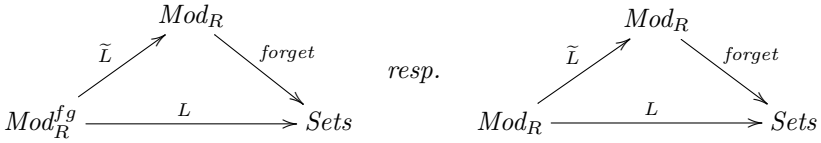
$$L : \mathrm{Hom}_R(M, N) \longrightarrow \mathrm{Hom}_R(L(M), L(N))$$

is a map of R-modules.

Remark 64.10.2. One can define the notion of an R-linearity for any functor between categories enriched over Mod_R. We made the definition specifically for functors $L : \mathrm{Mod}_R^{fg} \to \mathrm{Mod}_R$ and $L : \mathrm{Mod}_R \to \mathrm{Mod}_R$ because these are the cases that we have needed so far.

Remark 64.10.3. If $L : \mathrm{Mod}_R^{fg} \to \mathrm{Mod}_R$ is an R-linear functor, then L preserves finite products and sends the zero module to the zero module, see Homology, Lemma 12.3.7. On the other hand, if a functor $\mathrm{Mod}_R^{fg} \to \mathit{Sets}$ preserves finite products and sends the zero module to a one element set, then it has a unique lift to a R-linear functor, see Lemma 64.10.4.

Lemma 64.10.4. *Let $L : Mod_R^{fg} \to Sets$, resp. $L : Mod_R \to Sets$ be a functor. Suppose $L(0)$ is a one element set and L preserves finite products. Then there exists a unique R-linear functor $\widetilde{L} : Mod_R^{fg} \to Mod_R$, resp. $\widetilde{L} : Mod_R^{fg} \to Mod_R$, such that*



commutes.

Proof. We only prove this in case $L : \mathrm{Mod}_R^{fg} \to \mathit{Sets}$. Let M be a finitely generated R-module. We define $\widetilde{L}(M)$ to be the set $L(M)$ with the following R-module structure.

Multiplication: If $r \in R$, multiplication by r on $L(M)$ is defined to be the map $L(M) \to L(M)$ induced by the multiplication map $r\cdot : M \to M$.

Addition: The sum map $M \times M \to M : (m_1, m_2) \mapsto m_1 + m_2$ induces a map $L(M \times M) \to L(M)$. By assumption $L(M \times M)$ is canonically isomorphic to $L(M) \times L(M)$. Addition on $L(M)$ is defined by the map $L(M) \times L(M) \cong L(M \times M) \to L(M)$.

Zero: There is a unique map $0 \to M$. The zero element of $L(M)$ is the image of $L(0) \to L(M)$.

We omit the verification that this defines an R-module $\widetilde{L}(M)$, the unique such that is R-linearly functorial in M. □

Lemma 64.10.5. *Let $L_1, L_2 : Mod_R^{fg} \to Sets$ be functors that take 0 to a one element set and preserve finite products. Let $t : L_1 \to L_2$ be a morphism of functors. Then t induces a morphism $\widetilde{t} : \widetilde{L}_1 \to \widetilde{L}_2$ between the functors guaranteed by Lemma 64.10.4, which is given simply by $\widetilde{t}_M = t_M : \widetilde{L}_1(M) \to \widetilde{L}_2(M)$ for each $M \in \mathrm{Ob}(Mod_R^{fg})$. In other words, $t_M : \widetilde{L}_1(M) \to \widetilde{L}_2(M)$ is a map of R-modules.*

Proof. Omitted. □

In the case $R = K$ is a field, a K-linear functor $L : \mathrm{Mod}_K^{fg} \to \mathrm{Mod}_K$ is determined by its value $L(K)$.

Lemma 64.10.6. *Let K be a field. Let $L : Mod_K^{fg} \to Mod_K$ be a K-linear functor. Then L is isomorphic to the functor $L(K) \otimes_K - : Mod_K^{fg} \to Mod_K$.*

Proof. For $V \in \mathrm{Ob}(\mathrm{Mod}_K^{fg})$, the isomorphism $L(K) \otimes_K V \to L(V)$ is given on pure tensors by $x \otimes v \mapsto L(f_v)(x)$, where $f_v : K \to V$ is the K-linear map sending $1 \mapsto v$. When $V = K$, this is the isomorphism $L(K) \otimes_K K \to L(K)$ given by multiplication by K. For general V, it is an isomorphism by the case $V = K$ and the fact that L commutes with finite products (Remark 64.10.3). □

For a ring R and an R-module M, let $R[M]$ be the R-algebra whose underlying R-module is $R \oplus M$ and whose multiplication is given by $(r, m) \cdot (r', m') = (rr', rm' + r'm)$. When $M = R$ this is the ring of dual numbers over R, which we denote by $R[\epsilon]$.

Now let S be a ring and assume R is an S-algebra. Then the assignment $M \mapsto R[M]$ determines a functor $\mathrm{Mod}_R \to S\text{-Alg}/R$, where $S\text{-Alg}/R$ denotes the category of S-algebras over R. Note that $S\text{-Alg}/R$ admits finite products: if $A_1 \to R$ and $A_2 \to R$ are two objects, then $A_1 \times_R A_2$ is a product.

Lemma 64.10.7. *Let R be an S-algebra. Then the functor $Mod_R \to S\text{-}Alg/R$ described above preserves finite products.*

Proof. This is merely the statement that if M and N are R-modules, then the map $R[M \times N] \to R[M] \times_R R[N]$ is an isomorphism in $S\text{-Alg}/R$. □

Lemma 64.10.8. *Let R be an S-algebra, and let $\mathcal{C}$ be a strictly full subcategory of $S\text{-}Alg/R$ containing $R[M]$ for all $M \in \mathrm{Ob}(Mod_R^{fg})$. Let $F : \mathcal{C} \to Sets$ be a functor. Suppose that $F(R)$ is a one element set and that for any $M, N \in \mathrm{Ob}(Mod_R^{fg})$, the induced map*

$$F(R[M] \times_R R[N]) \to F(R[M]) \times F(R[N])$$

is a bijection. Then $F(R[M])$ has a natural R-module structure for any $M \in \mathrm{Ob}(Mod_R^{fg})$.

Proof. Note that $R \cong R[0]$ and $R[M] \times_R R[N] \cong R[M \times N]$ hence R and $R[M] \times_R R[N]$ are objects of $\mathcal{C}$ by our assumptions on $\mathcal{C}$. Thus the conditions on F make sense. The functor $\mathrm{Mod}_R \to S\text{-Alg}/R$ of Lemma 64.10.7 restricts to a functor $\mathrm{Mod}_R^{fg} \to \mathcal{C}$ by the assumption on $\mathcal{C}$. Let L be the composition $\mathrm{Mod}_R^{fg} \to \mathcal{C} \to Sets$, i.e., $L(M) = F(R[M])$. Then L preserves finite products by Lemma 64.10.7 and the assumption on F. Hence Lemma 64.10.4 shows that $L(M) = F(R[M])$ has a natural R-module structure for any $M \in \mathrm{Ob}(\mathrm{Mod}_R^{fg})$. □

Definition 64.10.9. Let $\mathcal{C}$ be a category as in Lemma 64.10.8. Let $F : \mathcal{C} \to \mathit{Sets}$ be a functor such that $F(R)$ is a one element set. The *tangent space* TF *of* F is $F(R[\epsilon])$.

When $F : \mathcal{C} \to \mathit{Sets}$ satisfies the hypotheses of Lemma 64.10.8, the tangent space TF has a natural R-module structure.

Example 64.10.10. Since $\mathcal{C}_\Lambda$ contains all $k[V]$ for finite dimensional vector spaces V we see that Definition 64.10.9 applies with $S = \Lambda$, $R = k$, $\mathcal{C} = \mathcal{C}_\Lambda$, and $F : \mathcal{C}_\Lambda \to \mathit{Sets}$ a predeformation functor. The tangent space is $TF = F(k[\epsilon])$.

Example 64.10.11. Let us work out the tangent space of Example 64.10.10 when $F : \mathcal{C}_\Lambda \to \mathit{Sets}$ is a prorepresentable functor, say $F = \underline{S}|_{\mathcal{C}_\Lambda}$ for $S \in \mathrm{Ob}(\widehat{\mathcal{C}}_\Lambda)$. Then F commutes with arbitrary limits and thus satisfies the hypotheses of Lemma 64.10.8. We compute

$$TF = F(k[\epsilon]) = \mathrm{Mor}_{\mathcal{C}_\Lambda}(S, k[\epsilon]) = \mathrm{Der}_\Lambda(S, k)$$

and more generally for a finite dimensional k-vector space V we have

$$F(k[V]) = \mathrm{Mor}_{\mathcal{C}_\Lambda}(S, k[V]) = \mathrm{Der}_\Lambda(S, V).$$

Explicitly, a Λ-algebra map $f : S \to k[V]$ compatible with the augmentations $q : S \to k$ and $k[V] \to k$ corresponds to the derivation D defined by $s \mapsto f(s) - q(s)$. Conversely, a Λ-derivation $D : S \to V$ corresponds to $f : S \to k[V]$ in $\mathcal{C}_\Lambda$ defined by the rule $f(s) = q(s) + D(s)$. Since these identifications are functorial we see that the k-vector spaces structures on TF and $\mathrm{Der}_\Lambda(S, k)$ correspond (see Lemma 64.10.5). It follows that $\dim_k TF$ is finite by Lemma 64.4.5.

Example 64.10.12. The computation of Example 64.10.11 simplifies in the classical case. Namely, in this case the tangent space of the functor $F = \underline{S}|_{\mathcal{C}_\Lambda}$ is simply the relative cotangent space of S over Λ, in a formula $TF = T_{S/\Lambda}$. In fact, this works more generally when the field extension $k' \subset k$ is separable. See Exercises, Exercise 79.28.2.

Lemma 64.10.13. *Let $F, G : \mathcal{C} \to \mathit{Sets}$ be functors satisfying the hypotheses of Lemma 64.10.8. Let $t : F \to G$ be a morphism of functors. For any $M \in \mathrm{Ob}(\mathit{Mod}_R^{fg})$, the map $t_{R[M]} : F(R[M]) \to G(R[M])$ is a map of R-modules, where $F(R[M])$ and $G(R[M])$ are given the R-module structure from Lemma 64.10.8. In particular, $t_{R[\epsilon]} : TF \to TG$ is a map of R-modules.*

Proof. Follows from Lemma 64.10.5. □

Example 64.10.14. Suppose that $f : R \to S$ is a ring map in $\widehat{\mathcal{C}}_\Lambda$. Set $F = \underline{R}|_{\mathcal{C}_\Lambda}$ and $G = \underline{S}|_{\mathcal{C}_\Lambda}$. The ring map f induces a transformation of functors $G \to F$. By Lemma 64.10.13 we get a k-linear map $TG \to TF$. This is the map

$$TG = \mathrm{Der}_\Lambda(S, k) \longrightarrow \mathrm{Der}_\Lambda(R, k) = TF$$

as follows from the canonical identifications $F(k[V]) = \mathrm{Der}_\Lambda(R, V)$ and $G(k[V]) = \mathrm{Der}_\Lambda(S, V)$ of Example 64.10.11 and the rule for computing the map on tangent spaces.

Lemma 64.10.15. *Let $F : \mathcal{C} \to \mathit{Sets}$ be a functor satisfying the hypotheses of Lemma 64.10.8. Assume $R = K$ is a field. Then $F(K[V]) \cong TF \otimes_K V$ for any finite dimensional K-vector space V.*

Proof. Follows from Lemma 64.10.6. □

64.11. Tangent spaces of predeformation categories

We will define tangent spaces of predeformation functors using the general Definition 64.10.9. We have spelled this out in Example 64.10.10. It applies to predeformation categories by looking at the associated functor of isomorphism classes.

Definition 64.11.1. Let $\mathcal{F}$ be a predeformation category. The *tangent space* $T\mathcal{F}$ *of* $\mathcal{F}$ is the set $\overline{\mathcal{F}}(k[\epsilon])$ of isomorphism classes of objects in the fiber category $\mathcal{F}(k[\epsilon])$.

Thus $T\mathcal{F}$ is nothing but the tangent space of the associated functor $\overline{\mathcal{F}} : \mathcal{C}_\Lambda \to \mathit{Sets}$. It has a natural vector space structure when $\mathcal{F}$ satisfies (S2), or, in fact, as long as $\overline{\mathcal{F}}$ does.

Lemma 64.11.2. *Let $\mathcal{F}$ be a predeformation category such that $\overline{\mathcal{F}}$ satisfies (S2). Then $T\mathcal{F}$ has a natural k-vector space structure. For any finite dimensional vector space V we have $\overline{\mathcal{F}}(k[V]) = T\mathcal{F} \otimes_k V$ functorially in V.*

Proof. Let us write $F = \overline{\mathcal{F}} : \mathcal{C}_\Lambda \to \mathit{Sets}$. This is a predeformation functor and F satisfies (S2). By Lemma 64.9.4 (and the translation of Remark 64.9.3) we see that

$$F(A \times_k k[V]) \longrightarrow F(A) \times F(k[V])$$

is a bijection for every finite dimensional vector space V and every $A \in \mathrm{Ob}(\mathcal{C}_\Lambda)$. In particular, if $A = k[W]$ then we see that $F(k[W] \times_k k[V]) = F(k[W]) \times F(k[V])$. In other words, the hypotheses of Lemma 64.10.8 hold and we see that $TF = T\mathcal{F}$ has a natural k-vector space structure. The final assertion follows from Lemma 64.10.15. □

A morphism of predeformation categories induces a map on tangent spaces.

Definition 64.11.3. Let $\varphi : \mathcal{F} \to \mathcal{G}$ be a morphism predeformation categories. The *differential* $d\varphi : T\mathcal{F} \to T\mathcal{G}$ *of* φ is the map obtained by evaluating the morphism of functors $\overline{\varphi} : \overline{\mathcal{F}} \to \overline{\mathcal{G}}$ at $A = k[\epsilon]$.

Lemma 64.11.4. *Let $\varphi : \mathcal{F} \to \mathcal{G}$ be a morphism predeformation categories. Assume $\overline{\mathcal{F}}$ and $\overline{\mathcal{G}}$ both satisfy (S2). Then $d\varphi : T\mathcal{F} \to T\mathcal{G}$ is k-linear.*

Proof. In the proof of Lemma 64.11.2 we have seen that $\overline{\mathcal{F}}$ and $\overline{\mathcal{G}}$ satisfy the hypotheses of Lemma 64.10.8. Hence the lemma follows from Lemma 64.10.13. □

Remark 64.11.5. We can globalize the notions of tangent space and differential to arbitrary categories cofibered in groupoids as follows. Let $\mathcal{F}$ be a category cofibered in groupoids over $\mathcal{C}_\Lambda$, and let $x \in \mathrm{Ob}(\mathcal{F}(k))$. As in Remark 64.6.4, we get a predeformation category $\mathcal{F}_x$. We define the *tangent space* $T_x\mathcal{F}$ *of* $\mathcal{F}$ *at* x to be the tangent space $T\mathcal{F}_x$ of $\mathcal{F}_x$. Similarly, if $\varphi : \mathcal{F} \to \mathcal{G}$ is a morphism of categories cofibered in groupoids over $\mathcal{C}_\Lambda$ and $x \in \mathrm{Ob}(\mathcal{F}(k))$, then there is an induced morphism $\varphi_x : \mathcal{F}_x \to \mathcal{G}_{\varphi(x)}$. We define the *differential* $d_x\varphi : T_x\mathcal{F} \to T_{\varphi(x)}\mathcal{G}$ *of* φ *at* x to be the map $d\varphi_x : T\mathcal{F}_x \to T\mathcal{G}_{\varphi(x)}$. If both $\mathcal{F}$ and $\mathcal{G}$ satisfy (S2) then all of these tangent spaces have a natural k-vector space structure and all the differentials $d_x\varphi : T_x\mathcal{F} \to T_{\varphi(x)}\mathcal{G}$ are k-linear (use Lemmas 64.9.6 and 64.11.4).

The following observations are uninteresting in the classical case or when $k' \subset k$ is a separable field extension, because then $\mathrm{Der}_\Lambda(k, k)$ and $\mathrm{Der}_\Lambda(V, k)$ are zero. There is a canonical identification

$$\mathrm{Mor}_{\mathcal{C}_\Lambda}(k, k[\epsilon]) = \mathrm{Der}_\Lambda(k, k).$$

Namely, for $D \in \mathrm{Der}_\Lambda(k,k)$ let $f_D : k \to k[\epsilon]$ be the map $a \mapsto a + D(a)\epsilon$. More generally, given a finite dimensional vector space V over k we have

$$\mathrm{Mor}_{\mathcal{C}_\Lambda}(k, k[V]) = \mathrm{Der}_\Lambda(k, V)$$

and we will use the same notation f_D for the map associated to the derivation D. We also have

$$\mathrm{Mor}_{\mathcal{C}_\Lambda}(k[W], k[V]) = \mathrm{Hom}_k(V, W) \oplus \mathrm{Der}_\Lambda(k, V)$$

where (φ, D) corresponds to the map $f_{\varphi,D} : a + w \mapsto a + \varphi(w) + D(a)$. We will sometimes write $f_{1,D} : a+v \to a+v+D(a)$ for the automorphism of $k[V]$ determined by the derivation $D : k \to V$. Note that $f_{1,D} \circ f_{1,D'} = f_{1,D+D'}$.

Let $\mathcal{F}$ be a predeformation category over $\mathcal{C}_\Lambda$. Let $x_0 \in \mathrm{Ob}(\mathcal{F}(k))$. By the above there is a canonical map

$$\gamma_V : \mathrm{Der}_\Lambda(k, V) \longrightarrow \overline{\mathcal{F}}(k[V])$$

defined by $D \mapsto f_{D,*}(x_0)$. Moreover, there is an action

$$a_V : \mathrm{Der}_\Lambda(k, V) \times \overline{\mathcal{F}}(k[V]) \longrightarrow \overline{\mathcal{F}}(k[V])$$

defined by $(D, x) \mapsto f_{1,D,*}(x)$. These two maps are compatible, i.e., $f_{1,D,*}f_{D',*}x_0 = f_{D+D',*}x_0$ as follows from a computation of the compositions of these maps. Note that the maps γ_V and a_V are independent of the choice of x_0 as there is a unique x_0 up to isomorphism.

Lemma 64.11.6. *Let $\mathcal{F}$ be a predeformation category over $\mathcal{C}_\Lambda$. If $\overline{\mathcal{F}}$ has (S2) then the maps γ_V are k-linear and we have $a_V(D, x) = x + \gamma_V(D)$.*

Proof. In the proof of Lemma 64.11.2 we have seen that the functor $V \mapsto \overline{\mathcal{F}}(k[V])$ transforms 0 to a singleton and products to products. The same is true of the functor $V \mapsto \mathrm{Der}_\Lambda(k, V)$. Hence γ_V is linear by Lemma 64.10.5. Let $D : k \to V$ be a Λ-derivation. Set $D_1 : k \to V^{\oplus 2}$ equal to $a \mapsto (D(a), 0)$. Then

$$\begin{array}{ccc} k[V \times V] & \xrightarrow{+} & k[V] \\ \downarrow{\scriptstyle f_{1,D_1}} & & \downarrow{\scriptstyle f_{1,D}} \\ k[V \times V] & \xrightarrow{+} & k[V] \end{array}$$

commutes. Unwinding the definitions and using that $\overline{F}(V \times V) = \overline{F}(V) \times \overline{F}(V)$ this means that $a_D(x_1) + x_2 = a_D(x_1 + x_2)$ for all $x_1, x_2 \in \overline{F}(V)$. Thus it suffices to show that $a_V(D, 0) = 0 + \gamma_V(D)$ where $0 \in \overline{F}(V)$ is the zero vector. By definition this is the element $f_{0,*}(x_0)$. Since $f_D = f_{1,D} \circ f_0$ the desired result follows. □

A special case of the constructions above are the map

$$\gamma : \mathrm{Der}_\Lambda(k, k) \longrightarrow T\mathcal{F} \tag{64.11.6.1}$$

and the action

$$a : \mathrm{Der}_\Lambda(k, k) \times T\mathcal{F} \longrightarrow T\mathcal{F} \tag{64.11.6.2}$$

defined for any predeformation category $\mathcal{F}$. Note that if $\varphi : \mathcal{F} \to \mathcal{G}$ is a morphism of predeformation categories, then we get commutative diagrams

$$\begin{array}{ccc} \mathrm{Der}_\Lambda(k,k) & \xrightarrow{\gamma} & T\mathcal{F} \\ & \searrow^{\gamma} & \downarrow{\scriptstyle d\varphi} \\ & & T\mathcal{G} \end{array} \quad \text{and} \quad \begin{array}{ccc} \mathrm{Der}_\Lambda(k,k) \times T\mathcal{F} & \xrightarrow{a} & T\mathcal{F} \\ \downarrow{\scriptstyle 1\times d\varphi} & & \downarrow{\scriptstyle d\varphi} \\ \mathrm{Der}_\Lambda(k,k) \times T\mathcal{G} & \xrightarrow{a} & T\mathcal{G} \end{array}$$

64.12. Versal formal objects

The existence of a versal formal object forces $\mathcal{F}$ to have property (S1).

Lemma 64.12.1. *Let $\mathcal{F}$ be a predeformation category. Assume $\mathcal{F}$ has a versal formal object. Then $\mathcal{F}$ satisfies (S1).*

Proof. Let ξ be a versal formal object of $\mathcal{F}$. Let

$$\begin{array}{ccc} & & x_2 \\ & & \downarrow \\ x_1 & \longrightarrow & x \end{array}$$

be a diagram in $\mathcal{F}$ such that $x_2 \to x$ lies over a surjective ring map. Since the natural morphism $\widehat{\mathcal{F}}|_{\mathcal{C}_\Lambda} \xrightarrow{\sim} \mathcal{F}$ is an equivalence (see Remark 64.7.7), we can consider this diagram also as a diagram in $\widehat{\mathcal{F}}$. By Lemma 64.8.15 there exists a morphism $\xi \to x_1$, so by Remark 64.8.14 we also get a morphism $\xi \to x_2$ making the diagram

$$\begin{array}{ccc} \xi & \longrightarrow & x_2 \\ \downarrow & & \downarrow \\ x_1 & \longrightarrow & x \end{array}$$

commute. If $x_1 \to x$ and $x_2 \to x$ lie above ring maps $A_1 \to A$ and $A_2 \to A$ then taking the pushforward of ξ to $A_1 \times_A A_2$ gives an object y as required by (S1). □

In the case that our cofibred category satisfies (S1) and (S2) we can characterize the versal formal objects as follows.

Lemma 64.12.2. *Let $\mathcal{F}$ be a predeformation category satisfying (S1) and (S2). Let ξ be a formal object of $\mathcal{F}$ corresponding to $\underline{\xi} : \underline{R}|_{\mathcal{C}_\Lambda} \to \mathcal{F}$, see Remark 64.7.12. Then ξ is versal if and only if the following two conditions hold:*

(1) *the map $d\underline{\xi} : T\underline{R}|_{\mathcal{C}_\Lambda} \to T\mathcal{F}$ on tangent spaces is surjective, and*
(2) *given a diagram in $\widehat{\mathcal{F}}$*

$$\begin{array}{ccc} & & y \\ & & \downarrow \\ \xi & \longrightarrow & x \end{array} \quad \textit{lying over} \quad \begin{array}{ccc} & & B \\ & & \downarrow{\scriptstyle f} \\ R & \longrightarrow & A \end{array}$$

in $\widehat{\mathcal{C}}_\Lambda$ with $B \to A$ a small extension of Artinian rings, then there exists a ring map $R \to B$ such that

$$\begin{array}{ccc} & & B \\ & \nearrow & \downarrow f \\ R & \longrightarrow & A \end{array}$$

commutes.

Proof. If ξ is versal then (1) holds by Lemma 64.8.8 and (2) holds by Remark 64.8.14. Assume (1) and (2) hold. By Remark 64.8.14 we must show that given a diagram in $\widehat{\mathcal{F}}$ as in (2), there exists $\xi \to y$ such that

$$\begin{array}{ccc} & & y \\ & \nearrow & \downarrow \\ \xi & \longrightarrow & x \end{array}$$

commutes. Let $b : R \to B$ be the map guaranteed by (2). Denote $y' = b_*\xi$ and choose a factorization $\xi \to y' \to x$ lying over $R \to B \to A$ of the given morphism $\xi \to x$. By (S1) we obtain a commutative diagram

$$\begin{array}{ccc} z & \longrightarrow & y \\ \downarrow & & \downarrow \\ y' & \longrightarrow & x \end{array} \quad \text{lying over} \quad \begin{array}{ccc} B \times_A B & \longrightarrow & B \\ \downarrow & & \downarrow f \\ B & \xrightarrow{f} & A. \end{array}$$

Set $I = \mathrm{Ker}(k)$. Let $\overline{g} : B \times_A B \to k[I]$ be the ring map $(u, v) \mapsto \overline{u} \oplus (v - u)$, cf. Lemma 64.9.8. By (1) there exists a morphism $\xi \to \overline{g}_* z$ which lies over a ring map $i : R \to k[\epsilon]$. Choose an Artinian quotient $b_1 : R \to B_1$ such that both $b : R \to B$ and $i : R \to k[\epsilon]$ factor through $R \to B_1$, i.e., giving $h : B_1 \to B$ and $i' : B_1 \to k[\epsilon]$. Choose a pushforward $y_1 = b_{1,*}\xi$, a factorization $\xi \to y_1 \to y'$ lying over $R \to B_1 \to B$ of $\xi \to y'$, and a factorization $\xi \to y_1 \to \overline{g}_* z$ lying over $R \to B_1 \to k[\epsilon]$ of $\xi \to \overline{g}_* z$. Applying (S1) once more we obtain

$$\begin{array}{ccccc} z_1 & \longrightarrow & z & \longrightarrow & y \\ \downarrow & & \downarrow & & \downarrow \\ y_1 & \longrightarrow & y' & \longrightarrow & x \end{array} \quad \text{lying over} \quad \begin{array}{ccccc} B_1 \times_A B & \longrightarrow & B \times_A B & \longrightarrow & B \\ \downarrow & & \downarrow & & \downarrow f \\ B_1 & \longrightarrow & B & \longrightarrow & A. \end{array}$$

Note that the map $g : B_1 \times_A B \to k[I]$ of Lemma 64.9.8 (defined using h) is the composition of $B_1 \times_A B \to B \times_A B$ and the map $\overline{g}$ above. By construction there exists a morphism $y_1 \to g_* z_1 \cong \overline{g}_* z$! Hence Lemma 64.9.8 applies (to the outer rectangles in the diagrams above) to give a morphism $y_1 \to y$ and precomposing with $\xi \to y_1$ gives the desired morphism $\xi \to y$. □

If $\mathcal{F}$ has property (S1) then the "largest quotient where a lift exists" exists. Here is a precise statement.

Lemma 64.12.3. *Let $\mathcal{F}$ be a category cofibred in groupoids over $\mathcal{C}_\Lambda$ which has (S1). Let $B \to A$ be a surjection in $\mathcal{C}_\Lambda$ with kernel I annihilated by $\mathfrak{m}_B$. Let $x \in \mathcal{F}(A)$. The set of ideals*

$$\mathcal{J} = \{J \subset I \mid \text{there exists an } y \to x \text{ lying over } B/J \to A\}$$

has a smallest element.

Proof. Note that $\mathcal{J}$ is nonempty as $I \in \mathcal{J}$. Also, if $J \in \mathcal{J}$ and $J \subset J' \subset I$ then $J' \in \mathcal{J}$ because we can pushforward the object y to an object y' over B/J'. Let J and K be elements of the displayed set. We claim that $J \cap K \in \mathcal{J}$ which will prove the lemma. Since I is a k-vector space we can find an ideal $J \subset J' \subset I$ such that $J \cap K = J' \cap K$ and such that $J' + K = I$. By the above we may replace J by J' and assume that $J + K = I$. In this case

$$A/(J \cap K) = A/J \times_{A/I} A/K.$$

Hence the existence of an element $z \in \mathcal{F}(A/(J \cap K))$ mapping to x follows, via (S1), from the existence of the elements we have assumed exist over A/J and A/K. $\square$

We will improve on the following result later.

Lemma 64.12.4. *Let $\mathcal{F}$ be a category cofibred in groupoids over $\mathcal{C}_\Lambda$. Assume the following conditions hold:*

(1) *$\mathcal{F}$ is a predeformation category.*
(2) *$\mathcal{F}$ satisfies (S1).*
(3) *$\mathcal{F}$ satisfies (S2).*
(4) *$\dim_k T\mathcal{F}$ is finite.*

Then $\mathcal{F}$ has a versal formal object.

Proof. Assume (1), (2), (3), and (4) hold. Choose an object $R \in \mathrm{Ob}(\widehat{\mathcal{C}}_\Lambda)$ such that $\underline{R}|_{\mathcal{C}_\Lambda}$ is smooth, see Example 64.8.10. Let $r = \dim_k T\mathcal{F}$ and put $S = R[[X_1, \ldots, X_r]]$.

We are going to inductively construct for $n \geq 2$ pairs $(J_n, f_{n-1} : \xi_n \to \xi_{n-1})$ where $J_n \subset S$ is an decreasing sequence of ideals and $f_{n-1} : \xi_n \to \xi_{n-1}$ is a morphism of $\mathcal{F}$ lying over the projection $S/J_n \to S/J_{n-1}$.

Step 1. Let $J_1 = \mathfrak{m}_S$. Let ξ_1 be the unique (up to unique isomorphism) object of $\mathcal{F}$ over $k = S/J_1 = S/\mathfrak{m}_S$

Step 2. Let $J_2 = \mathfrak{m}_S^2 + \mathfrak{m}_R S$. Then $S/J_2 = k[V]$ with $V = kX_1 \oplus \ldots \oplus kX_r$ By (S2) for $\overline{\mathcal{F}}$ we get a bijection

$$\overline{\mathcal{F}}(S/J_2) \longrightarrow T\mathcal{F} \otimes_k V,$$

see Lemmas 64.9.5 and 64.11.2. Choose a basis $\theta_1, \ldots, \theta_r$ for $T\mathcal{F}$ and set $\xi_2 = \sum \theta_i \otimes X_i \in \mathrm{Ob}(\mathcal{F}(S/J_2))$. The point of this choice is that

$$d\xi_2 : \mathrm{Mor}_{\mathcal{C}_\Lambda}(S/J_2, k[\epsilon]) \longrightarrow T\mathcal{F}$$

is surjective. Let $f_1 : \xi_2 \to \xi_1$ be the unique morphism.

Induction step. Assume $(J_n, f_{n-1} : \xi_n \to \xi_{n-1})$ has been constructed for some $n \geq 2$. There is a minimal element J_{n+1} of the set of ideals $J \subset S$ satisfying: (a) $\mathfrak{m}_S J_n \subset J \subset J_n$ and (b) there exists a morphism $\xi_{n+1} \to \xi_n$ lying over $S/J \to S/J_n$, see Lemma 64.12.3. Let $f_n : \xi_{n+1} \to \xi_n$ be any morphism of $\mathcal{F}$ lying over $S/J_{n+1} \to S/J_n$.

Set $J = \bigcap J_n$. Set $\overline{S} = S/J$. Set $\overline{J}_n = J_n/J$. By Lemma 64.4.7 the sequence of ideals $(\overline{J}_n)$ induces the $\mathfrak{m}_{\overline{S}}$-adic topology on $\overline{S}$. Since (ξ_n, f_n) is an object of $\widehat{\mathcal{F}}_{\mathcal{I}}(\overline{S})$, where $\mathcal{I}$ is the filtration $(\overline{J}_n)$ of $\overline{S}$, we see that (ξ_n, f_n) induces an object ξ of $\widehat{\mathcal{F}}(\overline{S})$. see Lemma 64.7.4.

We prove ξ is versal. For versality it suffices to check conditions (1) and (2) of Lemma 64.12.2. Condition (1) follows from our choice of ξ_2 in Step 2 above. Suppose given a diagram in $\widehat{\mathcal{F}}$

$$\begin{array}{ccc} & & y \\ & & \downarrow \\ \eta & \longrightarrow & x \end{array} \quad \text{lying over} \quad \begin{array}{ccc} B & & \\ \downarrow f & & \\ \overline{S} & \longrightarrow & A \end{array}$$

in $\widehat{\mathcal{C}}_\Lambda$ with $f : B \to A$ a small extension of Artinian rings. We have to show there is a map $\overline{S} \to B$ fitting into the diagram on the right. Choose n such that $\overline{S} \to A$ factors through $\overline{S} \to S/J_n$. This is possible as the sequence $(\overline{J}_n)$ induces the $\mathfrak{m}_{\overline{S}}$-adic topology as we saw above. The pushforward of ξ along $\overline{S} \to S/J_n$ is ξ_n. We may factor $\xi \to x$ as $\xi \to \xi_n \to x$ hence we get a diagram in $\mathcal{F}$

$$\begin{array}{ccc} & & y \\ & & \downarrow \\ \xi_n & \longrightarrow & x \end{array} \quad \text{lying over} \quad \begin{array}{ccc} B & & \\ \downarrow f & & \\ S/J_n & \longrightarrow & A. \end{array}$$

To check condition (2) of Lemma 64.12.2 it suffices to complete the diagram

$$\begin{array}{ccc} S/J_{n+1} & \dashrightarrow & B \\ \downarrow & & \downarrow f \\ S/J_n & \longrightarrow & A \end{array}$$

or equivalently, to complete the diagram

$$\begin{array}{ccc} & \nearrow & S/J_n \times_A B \\ & \swarrow & \downarrow p_1 \\ S/J_{n+1} & \longrightarrow & S/J_n. \end{array}$$

If p_1 has a section we are done. If not, by Lemma 64.3.8 (2) p_1 is a small extension, so by Lemma 64.3.12 (4) p_1 is an essential surjection. Recall that $S = R[[X_1, \ldots, X_r]]$ and that we chose R such that $\underline{R}|_{\mathcal{C}_\Lambda}$ is smooth. Hence there exists a map $h : R \to B$ lifting the map $R \to S \to S/J_n \to A$. By the universal property of a power series ring there is an R-algebra map $h : S = R[[X_1, \ldots, X_2]] \to B$ lifting the given map $S \to S/J_n \to A$. This induces a map $g : S \to S/J_n \times_A B$ making the solid square in the diagram

$$\begin{array}{ccccc} S & \xrightarrow[g]{} & S/J_n \times_A B & & \\ \downarrow & \nearrow & \downarrow p_1 & & \\ S/J_{n+1} & \longrightarrow & S/J_n & & \end{array}$$

commute. Then g is a surjection since p_1 is an essential surjection. We claim the ideal $K = \mathrm{Ker}(g)$ of S satisfies conditions (a) and (b) of the construction of J_{n+1} in the induction step above. Namely, $K \subset J_n$ is clear and $\mathfrak{m}_S J_n \subset K$ as p_1 is a

small extension; this proves (a). By (S1) applied to

$$\begin{array}{ccc} & & y \\ & & \downarrow \\ \xi_n & \longrightarrow & x, \end{array}$$

there exists a lifting of ξ_n to $S/K \cong S/J_n \times_A B$, so (b) holds. Since J_{n+1} was the minimal ideal with properties (a) and (b) this implies $J_{n+1} \subset K$. Thus the desired map $S/J_{n+1} \to S/K \cong S/J_n \times_A B$ exists. □

64.13. Minimal versal formal objects

We do a little bit of work to try and understand (non)uniqueness of versal formal objects. It turns out that if a predeformation category has a versal formal object, then it has a minimal versal formal object and any two such are isomorphic. Moreover, all versal formal objects are "more or less" the same up to replacing the base ring by a power series extension.

Let $\mathcal{F}$ be a category cofibred in groupoids over $\mathcal{C}_\Lambda$. For every object x of $\mathcal{F}$ lying over $A \in \mathrm{Ob}(\mathcal{C}_\Lambda)$ consider the category $\mathcal{S}_x$ with objects

$$\mathrm{Ob}(\mathcal{S}_x) = \{x' \to x \mid x' \to x \text{ lies over } A' \subset A\}$$

and morphisms are morphisms over x. For every $y \to x$ in $\mathcal{F}$ lying over $f : B \to A$ in $\mathcal{C}_\Lambda$ there is a functor $f_* : \mathcal{S}_y \to \mathcal{S}_x$ defined as follows: Given $y' \to y$ lying over $B' \subset B$ set $A' = f(B')$ and let $y' \to x'$ be over $B' \to f(B')$ be the pushforward of y'. By the axioms of a category cofibred in groupoids we obtain a unique morphism $x' \to x$ lying over $f(B') \to A$ such that

$$\begin{array}{ccc} y' & \longrightarrow & x' \\ \downarrow & & \downarrow \\ y & \longrightarrow & x \end{array}$$

commutes. Then $x' \to x$ is an object of $\mathcal{S}_x$. We say an object $x' \to x$ of $\mathcal{S}_x$ is *minimal* if any morphism $(x'_1 \to x) \to (x' \to x)$ in $\mathcal{S}_x$ is an isomorphism, i.e., x' and x'_1 are defined over the same subring of A. Since A has finite length as a Λ-module we see that minimal objects always exist.

Lemma 64.13.1. *Let $\mathcal{F}$ be a category cofibred in groupoids over $\mathcal{C}_\Lambda$ which has (S1).*

1. *For $y \to x$ in $\mathcal{F}$ a minimal object in $\mathcal{S}_y$ maps to a minimal object of $\mathcal{S}_x$.*
2. *For $y \to x$ in $\mathcal{F}$ lying over a surjection $f : B \to A$ in $\mathcal{C}_\Lambda$ every minimal object of $\mathcal{S}_x$ is the image of a minimal object of $\mathcal{S}_y$.*

Proof. Proof of (1). Say $y \to x$ lies over $f : B \to A$. Let $y' \to y$ lying over $B' \subset B$ be a minimal object of $\mathcal{S}_y$. Let

$$\begin{array}{ccc} y' & \longrightarrow & x' \\ \downarrow & & \downarrow \\ y & \longrightarrow & x \end{array} \quad \text{lying over} \quad \begin{array}{ccc} B' & \longrightarrow & f(B') \\ \downarrow & & \downarrow \\ B & \longrightarrow & A \end{array}$$

be as in the construction of f_* above. Suppose that $(x'' \to x) \to (x' \to x)$ is a morphism of $\mathcal{S}_x$ with $x'' \to x'$ lying over $A'' \subset f(B')$. By (S1) there exists

$y'' \to y'$ lying over $B' \times_{f(B')} A'' \to B'$. Since $y' \to y$ is minimal we conclude that $B' \times_{f(B')} A'' \to B'$ is an isomorphism, which implies that $A'' = f(B')$, i.e., $x' \to x$ is minimal.

Proof of (2). Suppose $f : B \to A$ is surjective and $y \to x$ lies over f. Let $x' \to x$ be a minimal object of $\mathcal{S}_x$ lying over $A' \subset A$. By (S1) there exists $y' \to y$ lying over $B' = f^{-1}(A') = B \times_A A' \to B$ whose image in $\mathcal{S}_x$ is $x' \to x$. So $f_*(y' \to y) = x' \to x$. Choose a morphism $(y'' \to y) \to (y' \to y)$ in $\mathcal{S}_y$ with $y'' \to y$ a minimal object (this is possible by the remark on lengths above the lemma). Then $f_*(y'' \to y)$ is an object of $\mathcal{S}_x$ which maps to $x' \to x$ (by functoriality of f_*) hence is isomorphic to $x' \to x$ by minimality of $x' \to x$. □

Lemma 64.13.2. *Let $\mathcal{F}$ be a category cofibred in groupoids over $\mathcal{C}_\Lambda$ which has (S1). Let ξ be a versal formal object of $\mathcal{F}$ lying over R. There exists a morphism $\xi' \to \xi$ lying over $R' \subset R$ with the following minimality properties*

(1) *for every $f : R \to A$ with $A \in \mathrm{Ob}(\mathcal{C}_\Lambda)$ the pushforwards*

$$\begin{array}{ccc} \xi' & \longrightarrow & x' \\ \downarrow & & \downarrow \\ \xi & \longrightarrow & x \end{array} \quad \text{lying over} \quad \begin{array}{ccc} R' & \longrightarrow & f(R') \\ \downarrow & & \downarrow \\ R & \longrightarrow & A \end{array}$$

produce a minimal object $x' \to x$ of $\mathcal{S}_x$, and

(2) *for any morphism of formal objects $\xi'' \to \xi'$ the corresponding morphism $R'' \to R'$ is surjective.*

Proof. Write $\xi = (R, \xi_n, f_n)$. Set $R'_1 = k$ and $\xi'_1 = \xi_1$. Suppose that we have constructed minimal objects $\xi'_m \to \xi_m$ of $\mathcal{S}_{\xi_m}$ lying over $R'_m \subset R/\mathfrak{m}_R^m$ for $m \leq n$ and morphisms $f'_m : \xi'_{m+1} \to \xi'_m$ compatible with f_m for $m \leq n-1$. By Lemma 64.13.1 (2) there exists a minimal object $\xi'_{n+1} \to \xi_{n+1}$ lying over $R'_{n+1} \subset R/\mathfrak{m}_R^{n+1}$ whose image is $\xi'_n \to \xi_n$ over $R'_n \subset R/\mathfrak{m}_R^n$. This produces the commutative diagram

$$\begin{array}{ccc} \xi'_{n+1} & \xrightarrow[f'_n]{} & \xi'_n \\ \downarrow & & \downarrow \\ \xi_{n+1} & \xrightarrow{f_n} & \xi_n \end{array}$$

by construction. Moreover the ring map $R'_{n+1} \to R'_n$ is surjective. Set $R' = \lim_n R'_n$. Then $R' \to R$ is injective.

However, it isn't a priori clear that R' is Noetherian. To prove this we use that ξ is versal. Namely, versality implies that there exists a morphism $\xi \to \xi'_n$ in $\widehat{\mathcal{F}}$, see Lemma 64.8.15. The corresponding map $R \to R'_n$ has to be surjective (as $\xi'_n \to \xi_n$ is minimal in $\mathcal{S}_{\xi_n}$). Thus the dimensions of the cotangent spaces are bounded and Lemma 64.4.8 implies R' is Noetherian, i.e., an object of $\widehat{\mathcal{C}}_\Lambda$. By Lemma 64.7.4 (plus the result on filtrations of Lemma 64.4.8) the sequence of elements ξ'_n defines a formal object ξ' over R' and we have a map $\xi' \to \xi$.

By construction (1) holds for $R \to R/\mathfrak{m}_R^n$ for each n. Since each $R \to A$ as in (1) factors through $R \to R/\mathfrak{m}_R^n \to A$ we see that (1) for $x' \to x$ over $f(R) \subset A$ follows from the minimality of $\xi'_n \to \xi_n$ over $R'_n \to R/\mathfrak{m}_R^n$ by Lemma 64.13.1 (1).

If $R'' \to R'$ as in (2) is not surjective, then $R'' \to R' \to R'_n$ would not be surjective for some n and $\xi'_n \to \xi_n$ wouldn't be minimal, a contradiction. This contradiction proves (2). $\square$

Lemma 64.13.3. *Let $\mathcal{F}$ be a category cofibred in groupoids over $\mathcal{C}_\Lambda$ which has (S1). Let ξ be a versal formal object of $\mathcal{F}$ lying over R. Let $\xi' \to \xi$ be a morphism of formal objects lying over $R' \subset R$ as constructed in Lemma 64.13.2. Then*

$$R \cong R'[[x_1, \ldots, x_r]]$$

is a power series ring over R'. Moreover, ξ' is a versal formal object too.

Proof. By Lemma 64.8.15 there exists a morphism $\xi \to \xi'$. By Lemma 64.13.2 the corresponding map $f : R \to R'$ induces a surjection $f|_{R'} : R' \to R'$. This is an isomorphism by Algebra, Lemma 10.29.8. Hence $I = \mathrm{Ker}(f)$ is an ideal of R such that $R = R' \oplus I$. Let $x_1, \ldots, x_n \in I$ be elements which form a basis for $I/\mathfrak{m}_R I$. Consider the map $S = R'[[X_1, \ldots, X_r]] \to R$ mapping X_i to x_i. For every $n \geq 1$ we get a surjection of Artinian R'-algebras $B = S/\mathfrak{m}_S^n \to R/\mathfrak{m}_R^n = A$. Denote $y \in \mathrm{Ob}(\mathcal{F}(B)$, resp. $x \in \mathrm{Ob}(\mathcal{F}(A))$ the pushforward of ξ' along $R' \to S \to B$, resp. $R' \to S \to A$. Note that x is also the pushforward of ξ along $R \to A$ as ξ is the pushforward of ξ' along $R' \to R$. Thus we have a solid diagram

$$\begin{array}{ccc} & & y \\ & \nearrow & \downarrow \\ \xi & \longrightarrow & x \end{array} \quad \text{lying over} \quad \begin{array}{ccc} & & S/\mathfrak{m}_S^n \\ & \nearrow & \downarrow \\ R & \longrightarrow & R/\mathfrak{m}_R^n \end{array}$$

Because ξ is versal, using Remark 64.8.14 we obtain the dotted arrows fitting into these diagrams. In particular, the maps $S/\mathfrak{m}_S^n \to R/\mathfrak{m}_R^n$ have sections $h_n : R/\mathfrak{m}_R^n \to S/\mathfrak{m}_S^n$. It follows from Lemma 64.4.9 that $S \to R$ is an isomorphism.

As ξ is a pushforward of ξ' along $R' \to R$ we obtain from Remark 64.7.13 a commutative diagram

$$\begin{array}{ccc} \underline{R}|_{\mathcal{C}_\Lambda} & \longrightarrow & \underline{R'}|_{\mathcal{C}_\Lambda} \\ & {\scriptstyle \underline{\xi}} \searrow \quad \swarrow {\scriptstyle \underline{\xi'}} & \\ & \mathcal{F} & \end{array}$$

Since $R' \to R$ has a left inverse (namely $R \to R/I = R'$) we see that $\underline{R}|_{\mathcal{C}_\Lambda} \to \underline{R'}|_{\mathcal{C}_\Lambda}$ is essentially surjective. Hence by Lemma 64.8.7 we see that $\underline{\xi'}$ is smooth, i.e., ξ' is a versal formal object. $\square$

Motivated by the preceding lemmas we make the following definition.

Definition 64.13.4. Let $\mathcal{F}$ be a predeformation category. We say a versal formal object ξ of $\mathcal{F}$ is *minimal*[2] if for any morphism of formal objects $\xi' \to \xi$ the underlying map on rings is surjective. Sometimes a minimal versal formal object is called *miniversal*.

[2]This may be nonstandard terminology. Many authors tie this notion in with properties of tangent spaces. We will make the link in Section 64.14.

The work in this section shows this definition is reasonable. First of all, the existence of a versal formal object implies that $\mathcal{F}$ has (S1). Then the preceding lemmas show there exists a minimal versal formal object. Finally, any two minimal versal formal objects are isomorphic. Here is a summary of our results (with detailed proofs).

Lemma 64.13.5. *Let $\mathcal{F}$ be a predeformation category which has a versal formal object. Then*

(1) *$\mathcal{F}$ has a minimal versal formal object,*
(2) *minimal versal objects are unique up to isomorphism, and*
(3) *any versal object is the pushforward of a minimal versal object along a power series ring extension.*

Proof. Suppose $\mathcal{F}$ has a versal formal object ξ over R. Then it satisfies (S1), see Lemma 64.12.1. Let $\xi' \to \xi$ over $R' \subset R$ be any of the morphisms constructed in Lemma 64.13.2. By Lemma 64.13.3 we see that ξ' is versal, hence it is a minimal versal formal object (by construction). This proves (1). Also, $R \cong R'[[x_1, \ldots, x_n]]$ which proves (3).

Suppose that ξ_i/R_i are two minimal versal formal objects. By Lemma 64.8.15 there exist morphisms $\xi_1 \to \xi_2$ and $\xi_2 \to \xi_1$. The corresponding ring maps $f : R_1 \to R_2$ and $g : R_2 \to R_1$ are surjective by minimality. Hence the compositions $g \circ f : R_1 \to R_1$ and $f \circ g : R_2 \to R_2$ are isomorphisms by Algebra, Lemma 10.29.8. Thus f and g are isomorphisms whence the maps $\xi_1 \to \xi_2$ and $\xi_2 \to \xi_1$ are isomorphisms (because $\widehat{\mathcal{F}}$ is cofibred in groupoids by Lemma 64.7.2). This proves (2) and finishes the proof of the lemma. □

64.14. Miniversal formal objects and tangent spaces

The general notion of minimality introduced in Definition 64.13.4 can sometimes be deduced from the behaviour on tangent spaces. Let ξ be a formal object of the predeformation category $\mathcal{F}$ and let $\underline{\xi} : \underline{R}|_{\mathcal{C}_\Lambda} \to \mathcal{F}$ be the corresponding morphism. Then we can consider the following two conditions

$$d\underline{\xi} : \mathrm{Der}_\Lambda(R, k) \to T\mathcal{F} \text{ is bijective} \tag{64.14.0.1}$$

$$d\underline{\xi} : \mathrm{Der}_\Lambda(R, k) \to T\mathcal{F} \text{ is bijective on } \mathrm{Der}_\Lambda(k, k)\text{-orbits.} \tag{64.14.0.2}$$

Here we are using the identification $T\underline{R}|_{\mathcal{C}_\Lambda} = \mathrm{Der}_\Lambda(R, k)$ of Example 64.10.11 and the action (64.11.6.2) of derivations on the tangent spaces. If $k' \subset k$ is separable, then $\mathrm{Der}_\Lambda(k, k) = 0$ and the two conditions are equivalent. It turns out that, in the presence of condition (S2) a versal formal object is minimal if and only if $\underline{\xi}$ satisfies (64.14.0.2). Moreover, if $\underline{\xi}$ satisfies (64.14.0.1), then $\mathcal{F}$ satisfies (S2).

Lemma 64.14.1. *Let $\mathcal{F}$ be a predeformation category. Let ξ be a versal formal object of $\mathcal{F}$ such that (64.14.0.2) holds. Then ξ is a minimal versal formal object. In particular, such ξ are unique up to isomorphism.*

Proof. If ξ is not minimal, then there exists a morphism $\xi' \to \xi$ lying over $R' \to R$ such that $R = R'[[x_1, \ldots, x_n]]$ with $n > 0$, see Lemma 64.13.5. Thus $d\underline{\xi}$ factors as

$$\mathrm{Der}_\Lambda(R, k) \to \mathrm{Der}_\Lambda(R', k) \to T\mathcal{F}$$

and we see that (64.14.0.2) cannot hold because $D : f \mapsto \partial/\partial x_1(f) \bmod \mathfrak{m}_R$ is an element of the kernel of the first arrow which is not in the image of $\mathrm{Der}_\Lambda(k, k) \to \mathrm{Der}_\Lambda(R, k)$. □

Lemma 64.14.2. *Let $\mathcal{F}$ be a predeformation category. Let ξ be a versal formal object of $\mathcal{F}$ such that (64.14.0.1) holds. Then*

(1) *$\mathcal{F}$ satisfies (S1).*
(2) *$\mathcal{F}$ satisfies (S2).*
(3) *$\dim_k T\mathcal{F}$ is finite.*

Proof. Condition (S1) holds by Lemma 64.12.1. The first part of (S2) holds since (S1) holds. Let

$$\begin{array}{ccc} y & \xrightarrow{c} & x_\epsilon \\ {\scriptstyle a}\downarrow & & \downarrow{\scriptstyle e} \\ x & \xrightarrow{d} & x_0 \end{array} \quad \text{and} \quad \begin{array}{ccc} y' & \xrightarrow{c'} & x_\epsilon \\ {\scriptstyle a'}\downarrow & & \downarrow{\scriptstyle e} \\ x & \xrightarrow{d} & x_0 \end{array} \quad \text{lying over} \quad \begin{array}{ccc} A \times_k k[\epsilon] & \longrightarrow & k[\epsilon] \\ \downarrow & & \downarrow \\ A & \longrightarrow & k \end{array}$$

be diagrams as in the second part of (S2). As above we can find morphisms $b : \xi \to y$ and $b' : \xi \to y'$ such that

$$\begin{array}{ccc} \xi & \xrightarrow{b'} & y' \\ {\scriptstyle b}\downarrow & & \downarrow{\scriptstyle a'} \\ y & \xrightarrow{a} & x \end{array}$$

commutes. Let $p : \mathcal{F} \to \mathcal{C}_\Lambda$ denote the structure morphism. Say $\widehat{p}(\xi) = R$, i.e., ξ lies over $R \in \mathrm{Ob}(\widehat{\mathcal{C}}_\Lambda)$. We see that the pushforward of ξ via $p(c) \circ p(b)$ is x_ϵ and that the pushforward of ξ via $p(c') \circ p(b')$ is x_ϵ. Since ξ satisfies (64.14.0.1), we see that $p(c) \circ p(b) = p(c') \circ p(b')$ as maps $R \to k[\epsilon]$. Hence $p(b) = p(b')$ as maps from $R \to A \times_k k[\epsilon]$. Thus we see that y and y' are isomorphic to the pushforward of ξ along this map and we get a unique morphism $y \to y'$ over $A \times_k k[\epsilon]$ compatible with b and b' as desired.

Finally, by Example 64.10.11 we see $\dim_k T\mathcal{F} = \dim_k T\underline{R}|_{\mathcal{C}_\Lambda}$ is finite. □

Example 64.14.3. There exist predeformation categories which have a versal formal object satisfying (64.14.0.2) but which do not satisfy (S2). A quick example is to take $F = \underline{k[\epsilon]}/G$ where $G \subset \mathrm{Aut}_{\mathcal{C}_\Lambda}(k[\epsilon])$ is a finite nontrivial subgroup. Namely, the map $\underline{k[\epsilon]} \to F$ is smooth, but the tangent space of F does not have a natural k-vector space structure (as it is a quotient of a k-vector space by a finite group).

Lemma 64.14.4. *Let $\mathcal{F}$ be a predeformation category satisfying (S2) which has a versal formal object. Then its minimal versal formal object satisfies (64.14.0.2).*

Proof. Let ξ be a minimal versal formal object for $\mathcal{F}$, see Lemma 64.13.5. Say ξ lies over $R \in \mathrm{Ob}(\widehat{\mathcal{C}}_\Lambda)$. In order to parse (64.14.0.2) we point out that $T\mathcal{F}$ has a natural k-vector space structure (see Lemma 64.11.2), that $d\xi : \mathrm{Der}_\Lambda(R, k) \to T\mathcal{F}$ is linear (see Lemma 64.11.4), and that the action of $\mathrm{Der}_\Lambda(k, k)$ is given by addition

(see Lemma 64.11.6). Consider the diagram

$$\begin{array}{ccccc} & & \mathrm{Hom}_k(\mathfrak{m}_R/\mathfrak{m}_R^2, k) & & \\ & & \uparrow & & \\ K & \longrightarrow & \mathrm{Der}_\Lambda(R,k) & \xrightarrow{d\underline{\xi}} & T\mathcal{F} \\ & & \uparrow & \nearrow & \\ & & \mathrm{Der}_\Lambda(k,k) & & \end{array}$$

The vector space K is the kernel of $d\underline{\xi}$. Note that the middle column is exact in the middle as it is dual to the sequence (64.3.10.1). If (64.14.0.2) fails, then we can find a nonzero element $D \in K$ which does not map to zero in $\mathrm{Hom}_k(\mathfrak{m}_R/\mathfrak{m}_R^2, k)$. This means there exists an $t \in \mathfrak{m}_R$ such that $D(t) = 1$. Set $R' = \{a \in R \mid D(a) = 0\}$. As D is a derivation this is a subring of R. Since $D(t) = 1$ we see that $R' \to k$ is surjective (compare with the proof of Lemma 64.3.12). Note that $\mathfrak{m}_{R'} = \mathrm{Ker}(D : \mathfrak{m}_R \to k)$ is an ideal of R and $\mathfrak{m}_R^2 \subset \mathfrak{m}_{R'}$. Hence

$$\mathfrak{m}_R/\mathfrak{m}_R^2 = \mathfrak{m}_{R'}/\mathfrak{m}_R^2 + k\bar{t}$$

which implies that the map

$$R'/\mathfrak{m}_R^2 \times_k k[\epsilon] \to R/\mathfrak{m}_R^2$$

sending ϵ to $\bar{t}$ is an isomorphism. In particular there is a map $R/\mathfrak{m}_R^2 \to R'/\mathfrak{m}_R^2$.

Let $\xi \to y$ be a morphism lying over $R \to R/\mathfrak{m}_R^2$. Let $y \to x$ be a morphism lying over $R/\mathfrak{m}_R^2 \to R'/\mathfrak{m}_R^2$. Let $y \to x_\epsilon$ be a morphism lying over $R/\mathfrak{m}_R^2 \to k[\epsilon]$. Let x_0 be the unique (up to unique isomorphism) object of $\mathcal{F}$ over k. By the axioms of a category cofibred in groupoids we obtain a commutative diagram

$$\begin{array}{ccc} y & \longrightarrow & x_\epsilon \\ \downarrow & & \downarrow \\ x & \longrightarrow & x_0 \end{array} \quad \text{lying over} \quad \begin{array}{ccc} R'/\mathfrak{m}_R^2 \times_k k[\epsilon] & \longrightarrow & k[\epsilon] \\ \downarrow & & \downarrow \\ R'/\mathfrak{m}_R^2 & \longrightarrow & k. \end{array}$$

Because $D \in K$ we see that x_ϵ is isomorphic to $0 \in \mathcal{F}(k[\epsilon])$, i.e., x_ϵ is the pushforward of x_0 via $k \to k[\epsilon], a \mapsto a$. Hence by Lemma 64.9.7 we see that there exists a morphism $x \to y$. Since $\mathrm{length}_\Lambda(R'/\mathfrak{m}_R^2) < \mathrm{length}_\Lambda(R/\mathfrak{m}_R^2)$ the corresponding ring map $R'/\mathfrak{m}_R^2 \to R/\mathfrak{m}_R^2$ is not surjective. This contradicts the minimality of ξ/R, see part (1) of Lemma 64.13.2. This contradiction shows that such a D cannot exist, hence we win. □

Theorem 64.14.5. *Let $\mathcal{F}$ be a predeformation category. Consider the following conditions*

1. *$\mathcal{F}$ has a minimal versal formal object satisfying (64.14.0.1),*
2. *$\mathcal{F}$ has a minimal versal formal object satisfying (64.14.0.2),*
3. *the following conditions hold:*
   (a) *$\mathcal{F}$ satisfies (S1).*
   (b) *$\mathcal{F}$ satisfies (S2).*
   (c) *$\dim_k T\mathcal{F}$ is finite.*

We always have

$$(1) \Rightarrow (3) \Rightarrow (2).$$

If $k' \subset k$ is separable, then all three are equivalent.

Proof. Lemma 64.14.2 shows that (1) $\Rightarrow$ (3). Lemmas 64.12.4 and 64.14.4 show that (3) $\Rightarrow$ (2). If $k' \subset k$ is separable then $\mathrm{Der}_\Lambda(k, k) = 0$ and we see that (64.14.0.1) = (64.14.0.2), i.e., (1) is the same as (2).

An alternative proof of (3) $\Rightarrow$ (1) in the classical case is to add a few words to the proof of Lemma 64.12.4 to see that one can right away construct a versal object which satisfies (64.14.0.1) in this case. This avoids the use of Lemma 64.12.4 in the classical case. Details omitted. □

Remark 64.14.6. When $\mathcal{F}$ is a predeformation functor, the condition $\dim_k T\mathcal{F} < \infty$ is precisely condition (H3) from Schlessinger's paper. In the classical case (or the case where $k' \subset k$ is separable), Theorem 64.14.5 recovers Schlessinger's theorem on the existence of "hulls". In our terminology a *hull* is a versal formal object ξ for a predeformation functor such that $d\xi$ is an isomorphism.

Remark 64.14.7. Let $\mathcal{F}$ be a predeformation category satisfying (S1), (S2), and $\dim_k T\mathcal{F} < \infty$. Then $\overline{\mathcal{F}}$ also satisfies (S1), (S2), and $\dim_k T\overline{\mathcal{F}} < \infty$, see Lemma 64.9.5. Thus, if $k' \subset k$ is separable, then $\overline{\mathcal{F}}$ has a hull (see Remark 64.14.6). In fact, if ξ is a minimal versal object for $\mathcal{F}$ lying over R, then the composition

$$\underline{R}|_{\mathcal{C}_\Lambda} \longrightarrow \mathcal{F} \longrightarrow \overline{\mathcal{F}}$$

is smooth and identifies tangent spaces, i.e., the image $\overline{\xi}$ of ξ in $\overline{\mathcal{F}}$ is a hull. This follows from the fact that $\mathcal{F} \to \overline{\mathcal{F}}$ identifies tangent spaces.

Example 64.14.8. In Example 64.8.10 we constructed objects $R \in \widehat{\mathcal{C}}_\Lambda$ such that $\underline{R}|_{\mathcal{C}_\Lambda}$ is smooth. We can reformulate this as follows. Let $\mathcal{F} = \mathcal{C}_\Lambda$ considered as cofibred in groupoids via the identity functor. In other words, $\mathcal{F}$ is the category cofibred in sets corresponding to the functor $F : A \mapsto \{*\}$ (this is the final object in the category of functors $\mathcal{C}_\Lambda \to \mathit{Sets}$). The condition that $\underline{R}|_{\mathcal{C}_\Lambda}$ is smooth means exactly that $\underline{R}|_{\mathcal{C}_\Lambda} \to F$ is smooth, i.e., that $\xi = *$ is a formal versal object of $\mathcal{F}$ over R. Hence $\mathcal{F}$ has a versal formal object. In fact, it is easy to see that $\mathcal{F}$ satisfies condition (3) of Theorem 64.14.5. The theorem implies that (2) holds. This means we can find a minimal versal formal object $* \in \widehat{\mathcal{F}}(S)$ over some $S \in \widehat{\mathcal{C}}_\Lambda$ such that $d* : \mathrm{Der}_\Lambda(S, k) \to 0$ is bijective on $\mathrm{Der}_\Lambda(k, k)$-orbits. Clearly this means that the injection $\mathrm{Der}_\Lambda(k, k) \to \mathrm{Der}_\Lambda(S, k)$ is also surjective. In other words, the exact sequence (64.3.10.2) turns into a pair of isomorphisms

$$H_1(L_{k/\Lambda}) = \mathfrak{m}_S/\mathfrak{m}_S^2 \quad \text{and} \quad \Omega_{S/\Lambda} \otimes_S k = \Omega_{k/\Lambda}.$$

(The first arrow is injective because of the formal smoothness of S over Λ; details omitted.) Of course the existence of such a ring S can be proved directly by judiciously slicing the ring R constructed in Example 64.8.10.

64.15. Rim-Schlessinger conditions and deformation categories

There is a very natural property of categories fibred in groupoids over $\mathcal{C}_\Lambda$ which is easy to check in practice and which implies Schlessinger's properties (S1) and (S2) we have introduced earlier.

Definition 64.15.1. Let $\mathcal{F}$ be a category cofibered in groupoids over $\mathcal{C}_\Lambda$. We say that $\mathcal{F}$ satisfies *condition (RS)* if for every diagram in $\mathcal{F}$

$$\begin{array}{ccc} & & x_2 \\ & & \downarrow \\ x_1 & \longrightarrow & x \end{array} \quad \text{lying over} \quad \begin{array}{ccc} & & A_2 \\ & & \downarrow \\ A_1 & \longrightarrow & A \end{array}$$

in $\mathcal{C}_\Lambda$ with $A_2 \to A$ surjective, there exists a fiber product $x_1 \times_x x_2$ in $\mathcal{F}$ such that the diagram

$$\begin{array}{ccc} x_1 \times_x x_2 & \longrightarrow & x_2 \\ \downarrow & & \downarrow \\ x_1 & \longrightarrow & x \end{array} \quad \text{lies over} \quad \begin{array}{ccc} A_1 \times_A A_2 & \longrightarrow & A_2 \\ \downarrow & & \downarrow \\ A_1 & \longrightarrow & A. \end{array}$$

Lemma 64.15.2. *Let $\mathcal{F}$ be a category cofibered in groupoids over $\mathcal{C}_\Lambda$ satisfying (RS). Given a commutative diagram in $\mathcal{F}$*

$$\begin{array}{ccc} y & \longrightarrow & x_2 \\ \downarrow & & \downarrow \\ x_1 & \longrightarrow & x \end{array} \quad \textit{lying over} \quad \begin{array}{ccc} A_1 \times_A A_2 & \longrightarrow & A_2 \\ \downarrow & & \downarrow \\ A_1 & \longrightarrow & A. \end{array}$$

with $A_2 \to A$ surjective, then it is a fiber square.

Proof. Since $\mathcal{F}$ satisfies (RS), there exists a fiber product diagram

$$\begin{array}{ccc} x_1 \times_x x_2 & \longrightarrow & x_2 \\ \downarrow & & \downarrow \\ x_1 & \longrightarrow & x \end{array} \quad \text{lying over} \quad \begin{array}{ccc} A_1 \times_A A_2 & \longrightarrow & A_2 \\ \downarrow & & \downarrow \\ A_1 & \longrightarrow & A. \end{array}$$

The induced map $y \to x_1 \times_x x_2$ lies over $\mathrm{id} : A_1 \times_A A_1 \to A_1 \times_A A_1$, hence it is an isomorphism. □

Lemma 64.15.3. *Let $\mathcal{F}$ be a category cofibered in groupoids over $\mathcal{C}_\Lambda$. Then $\mathcal{F}$ satisfies (RS) if the condition in Definition 64.15.1 is assumed to hold only when $A_2 \to A$ is a small extension.*

Proof. Apply Lemma 64.3.3. The proof is similar to that of Lemma 64.8.2. □

Lemma 64.15.4. *Let $\mathcal{F}$ be a category cofibered in groupoids over $\mathcal{C}_\Lambda$. The following are equivalent*

1. *$\mathcal{F}$ satisfies (RS),*
2. *the functor $\mathcal{F}(A_1 \times_A A_2) \to \mathcal{F}(A_1) \times_{\mathcal{F}(A)} \mathcal{F}(A_2)$ see (64.9.1.1) is an equivalence of categories whenever $A_2 \to A$ is surjective, and*
3. *same as in (2) whenever $A_2 \to A$ is a small extension.*

Proof. Assume (1). By Lemma 64.15.2 we see that every object of $\mathcal{F}(A_1 \times_A A_2)$ is of the form $x_1 \times_x x_2$. Moreover

$$\mathrm{Mor}_{A_1 \times_A A_2}(x_1 \times_x x_2, y_1 \times_y y_2) = \mathrm{Mor}_{A_1}(x_1, y_1) \times_{\mathrm{Mor}_A(x,y)} \mathrm{Mor}_{A_2}(x_2, y_2).$$

Hence we see that $\mathcal{F}(A_1 \times_A A_2)$ is a 2-fibre product of $\mathcal{F}(A_1)$ with $\mathcal{F}(A_2)$ over $\mathcal{F}(A)$ by Categories, Remark 4.29.5. In other words, we see that (2) holds.

The implication (2) $\Rightarrow$ (3) is immediate.

Assume (3). Let $q_1 : A_1 \to A$ and $q_2 : A_2 \to A$ be given with q_2 a small extension. We will use the description of the 2-fibre product $\mathcal{F}(A_1) \times_{\mathcal{F}(A)} \mathcal{F}(A_2)$ from Categories, Remark 4.29.5. Hence let $y \in \mathcal{F}(A_1 \times_A A_2)$ correspond to $(x_1, x_2, x, a_1 : x_1 \to x, a_2 : x_2 \to x)$. Let z be an object of $\mathcal{F}$ lying over C. Then

$$\begin{aligned}
\mathrm{Mor}_{\mathcal{F}}(z, y) &= \{(f, \alpha) \mid f : C \to A_1 \times_A A_2, \alpha : f_* z \to y\} \\
&= \{(f_1, f_2, \alpha_1, \alpha_2) \mid f_i : C \to A_i,\ \alpha_i : f_{i,*} z \to x_i, \\
&\qquad q_1 \circ f_1 = q_2 \circ f_2,\ q_{1,*}\alpha_1 = q_{2,*}\alpha_2\} \\
&= \mathrm{Mor}_{\mathcal{F}}(z, x_1) \times_{\mathrm{Mor}_{\mathcal{F}}(z,x)} \mathrm{Mor}_{\mathcal{F}}(z, x_2)
\end{aligned}$$

whence y is a fibre product of x_1 and x_2 over x. Thus we see that $\mathcal{F}$ satisfies (RS) in case $A_2 \to A$ is a small extension. Hence (RS) holds by Lemma 64.15.3. □

Remark 64.15.5. When $\mathcal{F}$ is cofibered in sets, condition (RS) is exactly condition (H4) from Schlessinger's paper [**Sch68**, Theorem 2.11]. Namely, for a functor $F : \mathcal{C}_\Lambda \to \mathit{Sets}$, condition (RS) states: If $A_1 \to A$ and $A_2 \to A$ are maps in $\mathcal{C}_\Lambda$ with $A_2 \to A$ surjective, then the induced map $F(A_1 \times_A A_2) \to F(A_1) \times_{F(A)} F(A_2)$ is bijective.

Lemma 64.15.6. *Let $\mathcal{F}$ be a category cofibered in groupoids over $\mathcal{C}_\Lambda$. The condition (RS) for $\mathcal{F}$ implies both (S1) and (S2) for $\mathcal{F}$.*

Proof. Using the reformulation of Lemma 64.15.4 and the explanation of (S1) following Definition 64.9.1 it is immediate that (RS) implies (S1). This proves the first part of (S2). The second part of (S2) follows because Lemma 64.15.2 tells us that $y = x_1 \times_{d, x_0, e} x_2 = y'$ if y, y' are as in the second part of the definition of (S2) in Definition 64.9.1. (In fact the morphism $y \to y'$ is compatible with both a, a' and c, c'!) □

The following lemma is the analogue of Lemma 64.9.5. Recall that if $\mathcal{F}$ is a category cofibred in groupoids over $\mathcal{C}_\Lambda$ and x is an object of $\mathcal{F}$ lying over A, then we denote $\mathrm{Aut}_A(x) = \mathrm{Mor}_A(x, x) = \mathrm{Mor}_{\mathcal{F}(A)}(x, x)$. If $x' \to x$ is a morphism of $\mathcal{F}$ lying over $A' \to A$ then there is a well defined map of groups $\mathrm{Aut}_{A'}(x') \to \mathrm{Aut}_A(x)$.

Lemma 64.15.7. *Let $\mathcal{F}$ be a category cofibered in groupoids over $\mathcal{C}_\Lambda$ satisfying (RS). The following conditions are equivalent:*

1. *$\overline{\mathcal{F}}$ satisfies (RS).*
2. *Let $f_1 : A_1 \to A$ and $f_2 : A_2 \to A$ be ring maps in $\mathcal{C}_\Lambda$ with f_2 surjective. The induced map of sets of isomorphism classes*
$$\overline{\mathcal{F}(A_1) \times_{\mathcal{F}(A)} \mathcal{F}(A_2)} \to \overline{\mathcal{F}}(A_1) \times_{\overline{\mathcal{F}}(A)} \overline{\mathcal{F}}(A_2)$$
is injective.
3. *For every morphism $x' \to x$ in $\mathcal{F}$ lying over a surjective ring map $A' \to A$, the map $Aut_{A'}(x') \to Aut_A(x)$ is surjective.*
4. *For every morphism $x' \to x$ in $\mathcal{F}$ lying over a small extension $A' \to A$, the map $Aut_{A'}(x') \to Aut_A(x)$ is surjective.*

Proof. We prove that (1) is equivalent to (2) and (2) is equivalent to (3). The equivalence of (3) and (4) follows from Lemma 64.3.3.

Let $f_1 : A_1 \to A$ and $f_2 : A_2 \to A$ be ring maps in $\mathcal{C}_\Lambda$ with f_2 surjective. By Remark 64.15.5 we see $\overline{\mathcal{F}}$ satisfies (RS) if and only if the map

$$\overline{\mathcal{F}}(A_1 \times_A A_2) \to \overline{\mathcal{F}}(A_1) \times_{\overline{\mathcal{F}}(A)} \overline{\mathcal{F}}(A_2)$$

is bijective for any such f_1, f_2. This map is at least surjective since that is the condition of (S1) and $\overline{\mathcal{F}}$ satisfies (S1) by Lemmas 64.15.6 and 64.9.5. Moreover, this map factors as

$$\overline{\mathcal{F}}(A_1 \times_A A_2) \longrightarrow \overline{\mathcal{F}(A_1) \times_{\mathcal{F}(A)} \mathcal{F}(A_2)} \longrightarrow \overline{\mathcal{F}}(A_1) \times_{\overline{\mathcal{F}}(A)} \overline{\mathcal{F}}(A_2),$$

where the first map is a bijection since

$$\mathcal{F}(A_1 \times_A A_2) \longrightarrow \mathcal{F}(A_1) \times_{\mathcal{F}(A)} \mathcal{F}(A_2)$$

is an equivalence by (RS) for $\mathcal{F}$. Hence (1) is equivalent to (2).

Assume (2) holds. Let $x' \to x$ be a morphism in $\mathcal{F}$ lying over a surjective ring map $f : A' \to A$. Let $a \in \mathrm{Aut}_A(x)$. The objects

$$(x', x', a : x \to x),\ (x', x', \mathrm{id} : x \to x)$$

of $\mathcal{F}(A') \times_{\mathcal{F}(A)} \mathcal{F}(A')$ have the same image in $\overline{\mathcal{F}}(A') \times_{\overline{\mathcal{F}}(A)} \overline{\mathcal{F}}(A')$. By (2) there exists maps $b_1, b_2 : x' \to x'$ such that

$$\begin{array}{ccc} x & \xrightarrow[a]{} & x \\ {\scriptstyle f_*b_1}\downarrow & & \downarrow{\scriptstyle f_*b_2} \\ x & \xrightarrow{\mathrm{id}} & x \end{array}$$

commutes. Hence $b_2^{-1} \circ b_1 \in \mathrm{Aut}_{A'}(x')$ has image $a \in \mathrm{Aut}_A(x)$. Hence (3) holds.

Assume (3) holds. Suppose

$$(x_1, x_2, a : (f_1)_*x_1 \to (f_2)_*x_2),\ (x_1', x_2', a' : (f_1)_*x_1' \to (f_2)_*x_2')$$

are objects of $\mathcal{F}(A_1) \times_{\mathcal{F}(A)} \mathcal{F}(A_2)$ with the same image in $\overline{\mathcal{F}}(A_1) \times_{\overline{\mathcal{F}}(A)} \overline{\mathcal{F}}(A_2)$. Then there are morphisms $b_1 : x_1 \to x_1'$ in $\mathcal{F}(A_1)$ and $b_2 : x_2 \to x_2'$ in $\mathcal{F}(A_2)$. By (3) we can modify b_2 by an automorphism of x_2 over A_2 so that the diagram

$$\begin{array}{ccc} (f_1)_*x_1 & \xrightarrow[a]{} & (f_2)_*x_2 \\ {\scriptstyle (f_1)_*b_1}\downarrow & & \downarrow{\scriptstyle (f_2)_*b_2} \\ (f_1)_*x_1' & \xrightarrow{a'} & (f_2)_*x_2'. \end{array}$$

commutes. This proves $(x_1, x_2, a) \cong (x_1', x_2', a')$ in $\overline{\mathcal{F}(A_1) \times_{\mathcal{F}(A)} \mathcal{F}(A_2)}$. Hence (2) holds. □

Finally we define the notion of a deformation category.

Definition 64.15.8. A *deformation category* is a predeformation category $\mathcal{F}$ satisfying (RS). A morphism of deformation categories is a morphism of categories over $\mathcal{C}_\Lambda$.

Remark 64.15.9. We say that a functor $F : \mathcal{C}_\Lambda \to \mathit{Sets}$ is a *deformation functor* if the associated cofibered set is a deformation category, i.e. if $F(k)$ is a one element set and F satisfies (RS). If $\mathcal{F}$ is a deformation category, then $\overline{\mathcal{F}}$ is a predeformation functor but not necessarily a deformation functor, as Lemma 64.15.7 shows.

Example 64.15.10. A prorepresentable functor F is a deformation functor. Namely, suppose $R \in \mathrm{Ob}(\widehat{\mathcal{C}}_\Lambda)$ and $F(A) = \mathrm{Mor}_{\widehat{\mathcal{C}}_\Lambda}(R, A)$. There is a unique morphism $R \to k$, so $F(k)$ is a one element set. Since

$$\mathrm{Hom}_\Lambda(R, A_1 \times_A A_2) = \mathrm{Hom}_\Lambda(R, A_1) \times_{\mathrm{Hom}_\Lambda(R,A)} \mathrm{Hom}_\Lambda(R, A_2)$$

the same is true for maps in $\widehat{\mathcal{C}}_\Lambda$ and we see that F has (RS).

The following is one of our typical remarks on passing from a category cofibered in groupoids to the predeformation category at a point over k: it says that this process preserves (RS).

Lemma 64.15.11. *Let $\mathcal{F}$ be a category cofibered in groupoids over $\mathcal{C}_\Lambda$. Let $x_0 \in \mathrm{Ob}(\mathcal{F}(k))$. Let $\mathcal{F}_{x_0}$ be the category cofibred in groupoids over $\mathcal{C}_\Lambda$ constructed in Remark 64.6.4. If $\mathcal{F}$ satisfies (RS), then so does $\mathcal{F}_{x_0}$. In particular, $\mathcal{F}_{x_0}$ is a deformation category.*

Proof. Any diagram as in Definition 64.15.1 in $\mathcal{F}_{x_0}$ gives rise to a diagram in $\mathcal{F}$ and the output of (RS) for this diagram in $\mathcal{F}$ can be viewed as an output for $\mathcal{F}_{x_0}$ as well. □

The following lemma is the analogue of the fact that 2-fibre products of algebraic stacks are algebraic stacks.

Lemma 64.15.12. *Let*

$$\begin{array}{ccc} \mathcal{H} \times_{\mathcal{F}} \mathcal{G} & \longrightarrow & \mathcal{G} \\ \downarrow & & \downarrow{\scriptstyle g} \\ \mathcal{H} & \xrightarrow{f} & \mathcal{F} \end{array}$$

be 2-fibre product of categories cofibered in groupoids over $\mathcal{C}_\Lambda$. If $\mathcal{F}, \mathcal{G}, \mathcal{H}$ all satisfy (RS), then $\mathcal{H} \times_{\mathcal{F}} \mathcal{G}$ satisfies (RS).

Proof. If A is an object of $\mathcal{C}_\Lambda$, then an object of the fiber category of $\mathcal{H} \times_{\mathcal{F}} \mathcal{G}$ over A is a triple (u, v, a) where $u \in \mathcal{H}(A)$, $v \in \mathcal{G}(A)$, and $a : f(u) \to g(v)$ is a morphism in $\mathcal{F}(A)$. Consider a diagram in $\mathcal{H} \times_{\mathcal{F}} \mathcal{G}$

$$\begin{array}{ccc} & & (u_2, v_2, a_2) \\ & & \downarrow \\ (u_1, v_1, a_1) & \longrightarrow & (u, v, a) \end{array} \quad \text{lying over} \quad \begin{array}{ccc} & & A_2 \\ & & \downarrow \\ A_1 & \longrightarrow & A \end{array}$$

in $\mathcal{C}_\Lambda$ with $A_2 \to A$ surjective. Since $\mathcal{H}$ and $\mathcal{G}$ satisfy (RS), there are fiber products $u_1 \times_u u_2$ and $v_1 \times_v v_2$ lying over $A_1 \times_A A_2$. Since $\mathcal{F}$ satisfies (RS), Lemma 64.15.2 shows

$$\begin{array}{ccc} f(u_1 \times_u u_2) & \longrightarrow & f(u_2) \\ \downarrow & & \downarrow \\ f(u_1) & \longrightarrow & f(u) \end{array} \quad \text{and} \quad \begin{array}{ccc} g(v_1 \times_v v_2) & \longrightarrow & g(v_2) \\ \downarrow & & \downarrow \\ g(v_1) & \longrightarrow & g(v) \end{array}$$

are both fiber squares in $\mathcal{F}$. Thus we can view $a_1 \times_a a_2$ as a morphism from $f(u_1 \times_u u_2)$ to $g(v_1 \times_v v_2)$ over $A_1 \times_A A_2$. It follows that

$$\begin{array}{ccc} (u_1 \times_u u_2, v_1 \times_v v_2, a_1 \times_a a_2) & \longrightarrow & (u_2, v_2, a_2) \\ \downarrow & & \downarrow \\ (u_1, v_1, a_1) & \longrightarrow & (u, v, a) \end{array}$$

is a fiber square in $\mathcal{H} \times_{\mathcal{F}} \mathcal{G}$ as desired. □

64.16. Lifts of objects

The content of this section is that the tangent space has a principal homogeneous action on the set of lifts along a small surjection in the case of a deformation category.

Definition 64.16.1. Let $\mathcal{F}$ be a category cofibered in groupoids over $\mathcal{C}_\Lambda$. Let $f : A' \to A$ be a map in $\mathcal{C}_\Lambda$. Let $x \in \mathcal{F}(A)$. The category $\mathit{Lift}(x, f)$ of lifts of x along f is the category with the following objects and morphisms.

(1) Objects: A *lift of x along f* is a morphism $x' \to x$ lying over f.
(2) Morphisms: A *morphism of lifts* from $a_1 : x'_1 \to x$ to $a_2 : x'_2 \to x$ is a morphism $b : x'_1 \to x'_2$ in $\mathcal{F}(A')$ such that $a_2 = a_1 \circ b$.

The set $\mathrm{Lift}(x, f)$ of lifts of x along f is the set of isomorphism classes of $\mathit{Lift}(x, f)$.

Remark 64.16.2. When the map $f : A' \to A$ is clear from the context, we may write $\mathit{Lift}(x, A')$ and $\mathrm{Lift}(x, A')$ in place of $\mathit{Lift}(x, f)$ and $\mathrm{Lift}(x, f)$.

Remark 64.16.3. Let $\mathcal{F}$ be a category cofibred in groupoids over $\mathcal{C}_\Lambda$. Let $x_0 \in \mathrm{Ob}(\mathcal{F}(k))$. Let V be a finite dimensional vector space. Then $\mathrm{Lift}(x_0, k[V])$ is the set of isomorphism classes of $\mathcal{F}_{x_0}(k[V])$ where $\mathcal{F}_{x_0}$ is the predeformation category of objects in $\mathcal{F}$ lying over x_0, see Remark 64.6.4. Hence if $\mathcal{F}$ satisfies (S2), then so does $\mathcal{F}_{x_0}$ (see Lemma 64.9.6) and by Lemma 64.11.2 we see that

$$\mathrm{Lift}(x_0, k[V]) = T\mathcal{F}_{x_0} \otimes_k V$$

as k-vector spaces.

Remark 64.16.4. Let $\mathcal{F}$ be a category cofibered in groupoids over $\mathcal{C}_\Lambda$ satisfying (RS). Let

$$\begin{array}{ccc} A_1 \times_A A_2 & \longrightarrow & A_2 \\ \downarrow & & \downarrow \\ A_1 & \longrightarrow & A \end{array}$$

be a fibre square in $\mathcal{C}_\Lambda$ such that either $A_1 \to A$ or $A_2 \to A$ is surjective. Let $x \in \mathrm{Ob}(\mathcal{F}(A))$. Given lifts $x_1 \to x$ and $x_2 \to x$ of x to A_1 and A_2, we get by (RS) a lift $x_1 \times_x x_2 \to x$ of x to $A_1 \times_A A_2$. Conversely, by Lemma 64.15.2 any lift of x to $A_1 \times_A A_2$ is of this form. Hence a bijection

$$\mathrm{Lift}(x, A_1) \times \mathrm{Lift}(x, A_2) \longrightarrow \mathrm{Lift}(x, A_1 \times_A A_2).$$

Similarly, if $x_1 \to x$ is a fixed lifting of x to A_1, then there is a bijection

$$\mathrm{Lift}(x_1, A_1 \times_A A_2) \longrightarrow \mathrm{Lift}(x, A_2).$$

Now let

$$\begin{array}{ccccc} A'_1 \times_A A_2 & \longrightarrow & A_1 \times_A A_2 & \longrightarrow & A_2 \\ \downarrow & & \downarrow & & \downarrow \\ A'_1 & \longrightarrow & A_1 & \longrightarrow & A \end{array}$$

be a composition of fibre squares in $\mathcal{C}_\Lambda$ with both $A'_1 \to A_1$ and $A_1 \to A$ surjective. Let $x_1 \to x$ be a morphism lying over $A_1 \to A$. Then by the above we have bijections

$$\begin{aligned} \text{Lift}(x_1, A'_1 \times_A A_2) &= \text{Lift}(x_1, A'_1) \times \text{Lift}(x_1, A_1 \times_A A_2) \\ &= \text{Lift}(x_1, A'_1) \times \text{Lift}(x, A_2). \end{aligned}$$

Lemma 64.16.5. *Let $\mathcal{F}$ be a deformation category. Let $A' \to A$ be a surjective ring map in $\mathcal{C}_\Lambda$ whose kernel I is annihilated by $\mathfrak{m}_{A'}$. Let $x \in \text{Ob}(\mathcal{F}(A))$. If Lift$(x, A')$ is nonempty, then there is a free and transitive action of $T\mathcal{F} \otimes_k I$ on Lift(x, A').*

Proof. Consider the ring map $g : A' \times_A A' \to k[I]$ defined by the rule $g(a_1, a_2) = \overline{a_1} \oplus a_2 - a_1$ (compare with Lemma 64.9.8). There is an isomorphism

$$A' \times_A A' \xrightarrow{\sim} A' \times_k k[I]$$

given by $(a_1, a_2) \mapsto (a_1, g(a_1, a_2))$. This isomorphism commutes with the projections to A' on the first factor, and hence with the projections of $A' \times_A A'$ and $A' \times_k k[I]$ to A. Thus there is a bijection

$$\text{Lift}(x, A' \times_A A') \longrightarrow \text{Lift}(x, A' \times_k k[I]) \tag{64.16.5.1}$$

By Remark 64.16.4 there is a bijection

$$\text{Lift}(x, A') \times \text{Lift}(x, A') \longrightarrow \text{Lift}(x, A' \times_A A') \tag{64.16.5.2}$$

There is a commutative diagram

$$\begin{array}{ccccc} A' \times_k k[I] & \longrightarrow & A \times_k k[I] & \longrightarrow & k[I] \\ \downarrow & & \downarrow & & \downarrow \\ A' & \longrightarrow & A & \longrightarrow & k. \end{array}$$

Thus if we choose a pushforward $x \to x_0$ of x along $A \to k$, we obtain by the end of Remark 64.16.4 a bijection

$$\text{Lift}(x, A' \times_k k[I]) \longrightarrow \text{Lift}(x, A') \times \text{Lift}(x_0, k[I]) \tag{64.16.5.3}$$

Composing (64.16.5.2), (64.16.5.1), and (64.16.5.3) we get a bijection

$$\Phi : \text{Lift}(x, A') \times \text{Lift}(x, A') \longrightarrow \text{Lift}(x, A') \times \text{Lift}(x_0, k[I]).$$

This bijection commutes with the projections on the first factors. By Remark 64.16.3 we see that $\text{Lift}(x_0, k[I]) = T\mathcal{F} \otimes_k I$. If pr_2 is the second projection of $\text{Lift}(x, A') \times \text{Lift}(x, A')$, then we get a map

$$a = \text{pr}_2 \circ \Phi^{-1} : \text{Lift}(x, A') \times (T\mathcal{F} \otimes_k I) \longrightarrow \text{Lift}(x, A').$$

Unwinding all the above we see that $a(x' \to x, \theta)$ is the unique lift $x'' \to x$ such that $g_*(x', x'') = \theta$ in $\text{Lift}(x_0, k[I]) = T\mathcal{F} \otimes_k I$. To see this is an action of $T\mathcal{F} \otimes_k I$ on $\text{Lift}(x, A')$ we have to show the following: if x', x'', x''' are lifts of x and $g_*(x', x'') =$

θ, $g_*(x'', x''') = \theta'$, then $g_*(x', x''') = \theta + \theta'$. This follows from the commutative diagram

$$\begin{array}{ccc} A' \times_A A' \times_A A' & \xrightarrow{(a_1,a_2,a_3)\mapsto(g(a_1,a_2),g(a_2,a_3))} & k[I] \times_k k[I] = k[I \times I] \\ & \searrow^{(a_1,a_2,a_3)\mapsto g(a_1,a_3)} & \downarrow + \\ & & k[I] \end{array}$$

The action is free and transitive because Φ is bijective. □

Remark 64.16.6. The action of Lemma 64.16.5 is functorial. Let $\varphi : \mathcal{F} \to \mathcal{G}$ be a morphism of deformation categories. Let $A' \to A$ be a surjective ring map whose kernel I is annihilated by $\mathfrak{m}_{A'}$. Let $x \in \mathrm{Ob}(\mathcal{F}(A))$. In this situation φ induces the vertical arrows in the following commutative diagram

$$\begin{array}{ccc} \mathrm{Lift}(x, A') \times (T\mathcal{F} \otimes_k I) & \longrightarrow & \mathrm{Lift}(x, A') \\ \downarrow{\scriptstyle (\varphi, d\varphi \otimes \mathrm{id}_I)} & & \downarrow{\scriptstyle \varphi} \\ \mathrm{Lift}(\varphi(x), A') \times (T\mathcal{G} \otimes_k I) & \longrightarrow & \mathrm{Lift}(\varphi(x), A') \end{array}$$

The commutativity follows as each of the maps (64.16.5.2), (64.16.5.1), and (64.16.5.3) of the proof of Lemma 64.16.5 gives rise to a similar commutative diagram.

64.17. Schlessinger's theorem on prorepresentable functors

We deduce Schlessinger's theorem characterizing prorepresentable functors on $\mathcal{C}_\Lambda$.

Lemma 64.17.1. *Let $F, G : \mathcal{C}_\Lambda \to Sets$ be deformation functors. Let $\varphi : F \to G$ be a smooth morphism which induces an isomorphism $d\varphi : TF \to TG$ of tangent spaces. Then φ is an isomorphism.*

Proof. We prove $F(A) \to G(A)$ is a bijection for all $A \in \mathrm{Ob}(\mathcal{C}_\Lambda)$ by induction on $\mathrm{length}_A(A)$. For $A = k$ the statement follows from the assumption that F and G are deformation functors. Suppose that the statement holds for rings of length less than n and let A' be a ring of length n. Choose a small extension $f : A' \to A$. We have a commutative diagram

$$\begin{array}{ccc} F(A') & \longrightarrow & G(A') \\ \downarrow{\scriptstyle F(f)} & & \downarrow{\scriptstyle G(f)} \\ F(A) & \xrightarrow{\sim} & G(A) \end{array}$$

where the map $F(A) \to G(A)$ is a bijection. By smoothness of $F \to G$, $F(A') \to G(A')$ is surjective (Lemma 64.8.8). Thus we can check bijectivity by checking it on fibers $F(f)^{-1}(x) \to G(f)^{-1}(\varphi(x))$ for $x \in F(A)$ such that $F(f)^{-1}(x)$ is nonempty. These fibers are precisely $\mathrm{Lift}(x, A')$ and $\mathrm{Lift}(\varphi(x), A')$ and by assumption we have an isomorphism $d\varphi \otimes \mathrm{id} : TF \otimes_k \mathrm{Ker}(f) \to TG \otimes_k \mathrm{Ker}(f)$. Thus, by Lemma 64.16.5 and Remark 64.16.6, for $x \in F(A)$ such that $F(f)^{-1}(x)$ is nonempty the map $F(f)^{-1}(x) \to G(f)^{-1}(\varphi(x))$ is a map of sets commuting with free transitive actions by $TF \otimes_k \mathrm{Ker}(f)$. Hence it is bijective. □

Note that in case $k' \subset k$ is separable condition (c) in the theorem below is empty.

Theorem 64.17.2. *Let $F : \mathcal{C}_\Lambda \to Sets$ be a functor. Then F is prorepresentable if and only if (a) F is a deformation functor, (b) $\dim_k TF$ is finite, and (c) $\gamma : Der_\Lambda(k,k) \to TF$ is injective.*

Proof. Assume F is prorepresentable by $R \in \widehat{\mathcal{C}_\Lambda}$. We see F is a deformation functor by Example 64.15.10. We see $\dim_k TF$ is finite by Example 64.10.11. Finally, $\mathrm{Der}_\Lambda(k,k) \to TF$ is identified with $\mathrm{Der}_\Lambda(k,k) \to \mathrm{Der}_\Lambda(R,k)$ by Example 64.10.14 which is injective because $R \to k$ is surjective.

Conversely, assume (a), (b), and (c) hold. By Lemma 64.15.6 we see that (S1) and (S2) hold. Hence by Theorem 64.14.5 there exists a minimal versal formal object ξ of F such that (64.14.0.2) holds. Say ξ lies over R. The map

$$d\underline{\xi} : \mathrm{Der}_\Lambda(R,k) \to T\mathcal{F}$$

is bijective on $\mathrm{Der}_\Lambda(k,k)$-orbits. Since the action of $\mathrm{Der}_\Lambda(k,k)$ on the left hand side is free by (c) and Lemma 64.11.6 we see that the map is bijective. Thus we see that $\underline{\xi}$ is an isomorphism by Lemma 64.17.1. □

64.18. Infinitesimal automorphisms

Let $\mathcal{F}$ be a category cofibered in groupoids over $\mathcal{C}_\Lambda$. Given a morphism $x' \to x$ in $\mathcal{F}$ lying over $A' \to A$, there is an induced homomorphism

$$\mathrm{Aut}_{A'}(x') \to \mathrm{Aut}_A(x).$$

Lemma 64.15.7 says that the cokernel of this homomorphism determines whether condition (RS) on $\mathcal{F}$ passes to $\overline{\mathcal{F}}$. In this section we study the kernel of this homomorphism. We will see that it also gives a measure of how far $\mathcal{F}$ is from $\overline{\mathcal{F}}$.

Definition 64.18.1. Let $\mathcal{F}$ be a category cofibered in groupoids over $\mathcal{C}_\Lambda$. Let $x' \to x$ be a morphism in $\mathcal{F}$ lying over $A' \to A$. The *group of infinitesimal automorphisms of x' over x* is the kernel of $\mathrm{Aut}_{A'}(x') \to \mathrm{Aut}_A(x)$. Notation $\mathrm{Inf}(x'/x) = \mathrm{Ker}(\mathrm{Aut}_{A'}(x') \to \mathrm{Aut}_A(x))$.

Definition 64.18.2. Let $\mathcal{F}$ be a category cofibered in groupoids over $\mathcal{C}_\Lambda$. Let $x_0 \in \mathrm{Ob}(\mathcal{F}(k))$. Assume a choice of pushforward $x_0 \to x_0'$ of x_0 along the map $k \to k[\epsilon], a \mapsto a$ has been made. Then there is a unique map $x_0' \to x_0$ such that $x_0 \to x_0' \to x_0$ is the identity on x_0. The *group of infinitesimal automorphisms of* x_0 is $\mathrm{Inf}_{x_0}(\mathcal{F}) := \mathrm{Inf}(x_0'/x_0)$.

Remark 64.18.3. Up to isomorphism, $\mathrm{Inf}_{x_0}(\mathcal{F})$ does not depend on the choice of pushforward $x_0 \to x_0'$. Moreover, if $y_0 \in \mathcal{F}(k)$ and $x_0 \cong y_0$ in $\mathcal{F}(k)$, then $\mathrm{Inf}_{x_0}(\mathcal{F}) \cong \mathrm{Inf}_{y_0}(\mathcal{F})$.

Remark 64.18.4. When $\mathcal{F}$ is a predeformation category, $\mathrm{Aut}_k(x_0)$ is trivial and hence $\mathrm{Inf}_{x_0}(\mathcal{F}) = \mathrm{Aut}_{k[\epsilon]}(x_0')$.

We will see that $\mathrm{Inf}_{x_0}(\mathcal{F})$ has a natural k-vector space structure when $\mathcal{F}$ satisfies (RS). At the same time, we will see that if $\mathcal{F}$ satisfies (RS), then the infinitesimal automorphisms $\mathrm{Inf}(x'/x)$ of a morphism $x' \to x$ lying over a small extension are governed by $\mathrm{Inf}_{x_0}(\mathcal{F})$, where x_0 is a pushforward of x to $\mathcal{F}(k)$. In order to do this, we introduce the automorphism functor for any object $x \in \mathrm{Ob}(\mathcal{F})$ as follows.

Definition 64.18.5. Let $p : \mathcal{F} \to \mathcal{C}$ be a category cofibered in groupoids over an arbitrary base category $\mathcal{C}$. Assume a choice of pushforwards has been made. Let $x \in \mathrm{Ob}(\mathcal{F})$ and let $U = p(x)$. Let $U/\mathcal{C}$ denote the category of objects under U. The *automorphism functor of* x is the functor $Aut(x) : U/\mathcal{C} \to Sets$ sending an object $f : U \to V$ to $\mathrm{Aut}_V(f_*x)$ and sending a morphism

$$\begin{array}{ccc} V' & \longrightarrow & V \\ & {\scriptstyle f'} \nwarrow \quad \nearrow {\scriptstyle f} & \\ & U & \end{array}$$

to the homomorphism $\mathrm{Aut}_{V'}(f'_*x) \to \mathrm{Aut}_V(f_*x)$ coming from the unique morphism $f'_*x \to f_*x$ lying over $V' \to V$ and compatible with $x \to f'_*x$ and $x \to f_*x$.

We will be concerned with the automorphism functors of objects in a category cofibered in groupoids $\mathcal{F}$ over $\mathcal{C}_\Lambda$. If $A \in \mathrm{Ob}(\mathcal{C}_\Lambda)$, then the category $A/\mathcal{C}_\Lambda$ is nothing but the category $\mathcal{C}_A$, i.e. the category defined in Section 64.3 where we take $\Lambda = A$ and $k = A/\mathfrak{m}_A$. Hence the automorphism functor of an object $x \in \mathrm{Ob}(\mathcal{F}(A))$ is a functor $Aut(x) : \mathcal{C}_A \to Sets$.

The following lemma could be deduced from Lemma 64.15.12 by thinking about the "inertia" of a category cofibred in groupoids, see for example Stacks, Section 8.7 and Categories, Section 4.32. However, it is easier to see it directly.

Lemma 64.18.6. *Let $\mathcal{F}$ be a category cofibered in groupoids over $\mathcal{C}_\Lambda$ satisfying (RS). Let $x \in \mathrm{Ob}(\mathcal{F}(A))$. Then $Aut(x) : \mathcal{C}_A \to Sets$ satisfies (RS).*

Proof. It follows that $Aut(x)$ satisfies (RS) from the fully faithfulness of the functor $\mathcal{F}(A_1 \times_A A_2) \to \mathcal{F}(A_1) \times_{\mathcal{F}(A)} \mathcal{F}(A_2)$ in Lemma 64.15.4. □

Lemma 64.18.7. *Let $\mathcal{F}$ be a category cofibered in groupoids over $\mathcal{C}_\Lambda$ satisfying (RS). Let $x \in \mathrm{Ob}(\mathcal{F}(A))$. Let x_0 be a pushforward of x to $\mathcal{F}(k)$.*

1. *$T_{id_{x_0}} Aut(x)$ has a natural k-vector space structure such that addition agrees with composition in $T_{id_{x_0}} Aut(x)$. In particular, composition in $T_{id_{x_0}} Aut(x)$ is commutative.*
2. *There is a canonical isomorphism $T_{id_{x_0}} Aut(x) \to T_{id_{x_0}} Aut(x_0)$ of k-vector spaces.*

Proof. We apply Remark 64.6.4 to the functor $Aut(x) : \mathcal{C}_A \to Sets$ and the element $\mathrm{id}_{x_0} \in Aut(x)(k)$ to get a predeformation functor $F = Aut(x)_{\mathrm{id}_{x_0}}$. By Lemmas 64.18.6 and 64.15.11 F is a deformation functor. By definition $T_{\mathrm{id}_{x_0}} Aut(x) = TF = F(k[\epsilon])$ which has a natural k-vector space structure specified by Lemma 64.10.8.

Addition is defined as the composition

$$F(k[\epsilon]) \times F(k[\epsilon]) \longrightarrow F(k[\epsilon] \times_k k[\epsilon]) \longrightarrow F(k[\epsilon])$$

where the first map is the inverse of the bijection guaranteed by (RS) and the second is induced by the k-algebra map $k[\epsilon] \times_k k[\epsilon] \to k[\epsilon]$ which maps $(\epsilon, 0)$ and $(0, \epsilon)$ to ϵ. If $A \to B$ is a ring map in $\mathcal{C}_\Lambda$, then $F(A) \to F(B)$ is a homomorphism where $F(A) = Aut(x)_{\mathrm{id}_{x_0}}(A)$ and $F(B) = Aut(x)_{\mathrm{id}_{x_0}}(B)$ are groups under composition. We conclude that $+ : F(k[\epsilon]) \times F(k[\epsilon]) \to F(k[\epsilon])$ is a homomorphism where $F(k[\epsilon])$ is regarded as a group under composition. With $\mathrm{id} \in F(k[\epsilon])$ the unit element we see

that $+(v, \mathrm{id}) = +(\mathrm{id}, v) = v$ for any $v \in F(k[\epsilon])$ because (id, v) is the pushforward of v along the ring map $k[\epsilon] \to k[\epsilon] \times_k k[\epsilon]$ with $\epsilon \mapsto (\epsilon, 0)$. In general, given a group G with multiplication $\circ$ and $+ : G \times G \to G$ is a homomorphism such that $+(g, 1) = +(1, g) = g$, where 1 is the identity of G, then $+ = \circ$. This shows addition in the k-vector space structure on $F(k[\epsilon])$ agrees with composition.

Finally, (2) is a matter of unwinding the definitions. Namely $T_{\mathrm{id}_{x_0}} Aut(x)$ is the set of automorphisms α of the pushforward of x along $A \to k \to k[\epsilon]$ which are trivial modulo ϵ. On the other hand $T_{\mathrm{id}_{x_0}} Aut(x_0)$ is the set of automorphisms of the pushforward of x_0 along $k \to k[\epsilon]$ which are trivial modulo ϵ. Since x_0 is the pushforward of x along $A \to k$ the result is clear. □

Remark 64.18.8. We point out some basic relationships between infinitesimal automorphism groups, liftings, and tangent spaces to automorphism functors. Let $\mathcal{F}$ be a category cofibered in groupoids over $\mathcal{C}_\Lambda$. Let $x' \to x$ be a morphism lying over a ring map $A' \to A$. Let x_0 be a pushforward of x to $\mathcal{F}(k)$. Then from the definitions we have an equality

$$\mathrm{Inf}(x'/x) = \mathrm{Lift}(\mathrm{id}_x, A')$$

where the liftings are of id_x as an object of $Aut(x')$. If $x_0 \in \mathrm{Ob}(\mathcal{F}(k))$ and x_0' is the pushforward to $\mathcal{F}(k[\epsilon])$, then applying this to $x_0' \to x_0$ we get

$$\mathrm{Inf}_{x_0}(\mathcal{F}) = \mathrm{Lift}(\mathrm{id}_{x_0}, k[\epsilon]) = T_{\mathrm{id}_{x_0}} Aut(x_0),$$

the last equality following directly from the definitions.

Lemma 64.18.9. *Let $\mathcal{F}$ be a category cofibered in groupoids over $\mathcal{C}_\Lambda$ satisfying (RS). Let $x_0 \in \mathrm{Ob}(\mathcal{F}(k))$. Then $Inf_{x_0}(\mathcal{F})$ is equal as a set to $T_{id_{x_0}} Aut(x_0)$, and so has a natural k-vector space structure such that addition agrees with composition of automorphisms.*

Proof. The equality of sets is as in the end of Remark 64.18.8 and the statement about the vector space structure follows from Lemma 64.18.7. □

Lemma 64.18.10. *Let $\varphi : \mathcal{F} \to \mathcal{G}$ be a morphism of categories cofibred in groupoids over $\mathcal{C}_\Lambda$ satisfying (RS). Let $x_0 \in \mathrm{Ob}(\mathcal{F}(k))$. Then φ induces a k-linear map $Inf_{x_0}(\mathcal{F}) \to Inf_{\varphi(x_0)}(\mathcal{G})$.*

Proof. It is clear that φ induces a morphism from $Aut(x_0) \to Aut(\varphi(x_0))$ which maps the identity to the identity. Hence this follows from the result for tangent spaces, see Lemma 64.11.4. □

Lemma 64.18.11. *Let $\mathcal{F}$ be a category cofibered in groupoids over $\mathcal{C}_\Lambda$ satisfying (RS). Let $x' \to x$ be a morphism lying over a surjective ring map $A' \to A$ with kernel I annihilated by $\mathfrak{m}_{A'}$. Let x_0 be a pushforward of x to $\mathcal{F}(k)$. Then $Inf(x'/x)$ has a free and transitive action by $T_{id_{x_0}} Aut(x') \otimes_k I = Inf_{x_0}(\mathcal{F}) \otimes_k I$.*

Proof. This is just the analogue of Lemma 64.16.5 in the setting of automorphism sheaves. To be precise, we apply Remark 64.6.4 to the functor $Aut(x') : \mathcal{C}_{A'} \to Sets$ and the element $\mathrm{id}_{x_0} \in Aut(x)(k)$ to get a predeformation functor $F = Aut(x')_{\mathrm{id}_{x_0}}$. By Lemmas 64.18.6 and 64.15.11 F is a deformation functor. Hence Lemma 64.16.5 gives a free and transitive action of $TF \otimes_k I$ on $\mathrm{Lift}(\mathrm{id}_x, A')$, because as $\mathrm{Lift}(\mathrm{id}_x, A')$ is a group it is always nonempty. Note that we have equalities of vector spaces

$$TF = T_{\mathrm{id}_{x_0}} Aut(x') \otimes_k I = \mathrm{Inf}_{x_0}(\mathcal{F}) \otimes_k I$$

by Lemma 64.18.7. The equality $\mathrm{Inf}(x'/x) = \mathrm{Lift}(\mathrm{id}_x, A')$ of Remark 64.18.8 finishes the proof. □

Lemma 64.18.12. *Let $\mathcal{F}$ be a category cofibered in groupoids over $\mathcal{C}_\Lambda$ satisfying (RS). Let $x' \to x$ be a morphism in $\mathcal{F}$ lying over a surjective ring map. Let x_0 be a pushforward of x to $\mathcal{F}(k)$. If $Inf_{x_0}(\mathcal{F}) = 0$ then $Inf(x'/x) = 0$.*

Proof. Follows from Lemmas 64.3.3 and 64.18.11. □

Lemma 64.18.13. *Let $\mathcal{F}$ be a category cofibered in groupoids over $\mathcal{C}_\Lambda$ satisfying (RS). Let $x_0 \in \mathrm{Ob}(\mathcal{F}(k))$. Then $Inf_{x_0}(\mathcal{F}) = 0$ if and only if the natural morphism $\mathcal{F}_{x_0} \to \overline{\mathcal{F}_{x_0}}$ of categories cofibered in groupoids is an equivalence.*

Proof. The morphism $\mathcal{F}_{x_0} \to \overline{\mathcal{F}_{x_0}}$ is an equivalence if and only if $\mathcal{F}_{x_0}$ is fibered in setoids, cf. Categories, Section 4.37 (a setoid is by definition a groupoid in which the only automorphism of any object is the identity). We prove that $\mathrm{Inf}_{x_0}(\mathcal{F}) = 0$ if and only if this condition holds for $\mathcal{F}_{x_0}$. Obviously if $\mathcal{F}_{x_0}$ is fibered in setoids then $\mathrm{Inf}_{x_0}(\mathcal{F}) = 0$. Conversely assume $\mathrm{Inf}_{x_0}(\mathcal{F}) = 0$. Let A be an object of $\mathcal{C}_\Lambda$. Then by Lemma 64.18.12, $\mathrm{Inf}(x/x_0) = 0$ for any object $x \to x_0$ of $\mathcal{F}_{x_0}(A)$. Since by definition $\mathrm{Inf}(x/x_0)$ equals the group of automorphisms of $x \to x_0$ in $\mathcal{F}_{x_0}(A)$, this proves $\mathcal{F}_{x_0}(A)$ is a setoid. □

Lemma 64.18.14. *Let $f : \mathcal{H} \to \mathcal{F}$ and $g : \mathcal{G} \to \mathcal{F}$ be 1-morphisms of deformation categories. Then*

1. *$\mathcal{W} = \mathcal{H} \times_{\mathcal{F}} \mathcal{G}$ is a deformation category*
2. *let $w_0 \in \mathrm{Ob}(\mathcal{W}(k))$ and let x_0, y_0, z_0 be the image of w_0 in $\mathcal{F}, \mathcal{H}, \mathcal{G}$. Then we have a 6-term exact sequence of vector spaces*

$$0 \to Inf_{w_0}(\mathcal{W}) \to Inf_{y_0}(\mathcal{H}) \oplus Inf_{z_0}(\mathcal{G}) \to Inf_{x_0}(\mathcal{F}) \to T\mathcal{W} \to T\mathcal{H} \oplus T\mathcal{G} \to T\mathcal{F}$$

Proof. Part (1) follows from Lemma 64.15.12 and the fact that $\mathcal{W}(k)$ is the fibre product of two setoids with a unique isomorphism class over a setoid with a unique isomorphism class.

Part (2). We apply Lemmas 64.11.4 and 64.18.10 to get all the linear maps except for the "boundary map" $\delta : \mathrm{Inf}_{x_0}(\mathcal{F}) \to T\mathcal{W}$. We will insert suitable signs later.

Construction of δ. Choose a pushforward $w_0 \to w_0'$ along $k \to k[\epsilon]$. Denote x_0', y_0', z_0' the images of w_0' in $\mathcal{F}, \mathcal{H}, \mathcal{G}$. In particular we obtain isomorphisms $b' : f(y_0') \to x_0'$ and $c' : x_0' \to g(z_0')$. Denote $b : f(y_0) \to x_0$ and $c : x_0 \to g(z_0)$ the pushforwards along $k[\epsilon] \to k$. Observe that this means $w_0' = (k[\epsilon], y_0', z_0', c' \circ b')$ and $w_0 = (k, y_0, z_0, c \circ b)$ in terms of the explicit form of the fibre product of categories, see Remarks 64.5.2 (13). Given $\alpha : x_0' \to x_0'$ we set $\delta(\alpha) = (k[\epsilon], y_0', z_0', c' \circ \alpha \circ b')$ which is indeed an object of $\mathcal{W}$ over $k[\epsilon]$ and comes with a morphism $(k[\epsilon], y_0', z_0', c' \circ \alpha \circ b') \to w_0$ over $k[\epsilon] \to k$ as α pushes forward to the identity over k. More generally, for any k-vector space V we can define a map

$$\mathrm{Lift}(\mathrm{id}_{x_0}, k[V]) \longrightarrow \mathrm{Lift}(w_0, k[V])$$

using exactly the same formulae. This construction is functorial in the vector space V (details omitted). Hence δ is k-linear by an application of Lemma 64.10.5.

Having constructed these maps it is straightforward to show the sequence is exact. Injectivity of the first map comes from the fact that $f \times g : \mathcal{W} \to \mathcal{F} \times \mathcal{G}$ is faithful. If $(\beta, \gamma) \in \mathrm{Inf}_{y_0}(\mathcal{H}) \oplus \mathrm{Inf}_{z_0}(\mathcal{G})$ map to the same element of $\mathrm{Inf}_{x_0}(\mathcal{F})$ then (β, γ) defines

an automorphism of $w_0' = (k[\epsilon], y_0', z_0', c' \circ b')$ whence exactness at the second spot. If α as above gives the trivial deformation $(k[\epsilon], y_0', z_0', c' \circ \alpha \circ b')$ of w_0, then the isomorphism $w_0' = (k[\epsilon], y_0', z_0', c' \circ b') \to (k[\epsilon], y_0', z_0', c' \circ \alpha \circ b')$ produces a pair (β, γ) which is a preimage of α. If $w = (k[\epsilon], y, z, \phi)$ is a deformation of w_0 such that $y_0' \cong y$ and $z \cong z_0'$ then the map

$$f(y_0') \to f(y) \xrightarrow{\phi} g(z) \to g(z_0')$$

is an α which maps to w under δ. Finally, if y and z are deformations of y_0 and z_0 and there exists an isomorphism $\phi : f(y) \to g(z)$ of deformations of $f(y_0) = x_0 = g(z_0)$ then we get a preimage $w = (k[\epsilon], y, z, \phi)$ of (x, y) in $T\mathcal{W}$. This finishes the proof. □

64.19. Groupoids in functors on an arbitrary category

We begin with generalities on groupoids in functors on an arbitrary category. In the next section we will pass to the category $\mathcal{C}_\Lambda$. For clarity we shall sometimes refer to an ordinary groupoid, i.e., a category whose morphisms are all isomorphisms, as a groupoid category.

Definition 64.19.1. Let $\mathcal{C}$ be a category. The *category of groupoids in functors on* $\mathcal{C}$ is the category with the following objects and morphisms.

(1) Objects: A *groupoid in functors on* $\mathcal{C}$ is a quintuple (U, R, s, t, c) where $U, R : \mathcal{C} \to \mathit{Sets}$ are functors and $s, t : R \to U$ and $c : R \times_{s,U,t} R \to R$ are morphisms with the following property: For any object T of $\mathcal{C}$, the quintuple
$$(U(T), R(T), s, t, c)$$
is a groupoid category.

(2) Morphisms: A *morphism* $(U, R, s, t, c) \to (U', R', s', t', c')$ *of groupoids in functors on* $\mathcal{C}$ consists of morphisms $U \to U'$ and $R \to R'$ with the following property: For any object T of $\mathcal{C}$, the induced maps $U(T) \to U'(T)$ and $R(T) \to R'(T)$ define a functor between groupoid categories
$$(U(T), R(T), s, t, c) \to (U'(T), R'(T), s', t', c').$$

Remark 64.19.2. A groupoid in functors on $\mathcal{C}$ amounts to the data of a functor $\mathcal{C} \to \mathit{Groupoids}$, and a morphism of groupoids in functors on $\mathcal{C}$ amounts to a morphism of the corresponding functors $\mathcal{C} \to \mathit{Groupoids}$ (where *Groupoids* is regarded as a 1-category). However, for our purposes it is more convenient to use the terminology of groupoids in functors. In fact, thinking of a groupoid in functors as the corresponding functor $\mathcal{C} \to \mathit{Groupoids}$, or equivalently as the category cofibered in groupoids associated to that functor, can lead to confusion (Remark 64.21.2).

Remark 64.19.3. Let (U, R, s, t, c) be a groupoid in functors on a category $\mathcal{C}$. There are unique morphisms $e : U \to R$ and $i : R \to R$ such that for every object T of $\mathcal{C}$, $e : U(T) \to R(T)$ sends $x \in U(T)$ to the identity morphism on x and $i : R(T) \to R(T)$ sends $a \in U(T)$ to the inverse of a in the groupoid category $(U(T), R(T), s, t, c)$. We will sometimes refer to s, t, c, e, and i as "source", "target", "composition", "identity", and "inverse".

Definition 64.19.4. Let $\mathcal{C}$ be a category. A groupoid in functors on $\mathcal{C}$ is *representable* if it is isomorphic to one of the form $(\underline{U}, \underline{R}, s, t, c)$ where U and R are objects of $\mathcal{C}$ and the pushout $R \amalg_{s,U,t} R$ exists.

Remark 64.19.5. Hence a representable groupoid in functors on $\mathcal{C}$ is given by objects U and R of $\mathcal{C}$ and morphisms $s, t : U \to R$ and $c : R \to R \amalg_{s,U,t} R$ such that $(\underline{U}, \underline{R}, s, t, c)$ satisfies the condition of Definition 64.19.1. The reason for requiring the existence of the pushout $R \coprod_{s,U,t} R$ is so that the composition morphism c is defined at the level of morphisms in $\mathcal{C}$. This requirement will always be satisfied below when we consider representable groupoids in functors on $\widehat{\mathcal{C}}_\Lambda$, since by Lemma 64.4.3 the category $\widehat{\mathcal{C}}_\Lambda$ admits pushouts.

Remark 64.19.6. We will say "*let* $(\underline{U}, \underline{R}, s, t, c)$ *be a groupoid in functors on* $\mathcal{C}$" to mean that we have a representable groupoid in functors. Thus this means that U and R are objects of $\mathcal{C}$, there are morphisms $s, t : U \to R$, the pushout $R \amalg_{s,U,t} R$ exists, there is a morphism $c : R \to R \amalg_{s,U,t} R$, and $(\underline{U}, \underline{R}, s, t, c)$ is a groupoid in functors on $\mathcal{C}$.

We introduce notation for restriction of groupoids in functors. This will be relevant below in situations where we restrict from $\widehat{\mathcal{C}}_\Lambda$ to $\mathcal{C}_\Lambda$.

Definition 64.19.7. Let (U, R, s, t, c) be a groupoid in functors on a category $\mathcal{C}$. Let $\mathcal{C}'$ be a subcategory of $\mathcal{C}$. The *restriction* $(U, R, s, t, c)|_{\mathcal{C}'}$ *of* (U, R, s, t, c) *to* $\mathcal{C}'$ is the groupoid in functors on $\mathcal{C}'$ given by $(U|_{\mathcal{C}'}, R|_{\mathcal{C}'}, s|_{\mathcal{C}'}, t|_{\mathcal{C}'}, c|_{\mathcal{C}'})$.

Remark 64.19.8. In the situation of Definition 64.19.7, we often denote $s|_{\mathcal{C}'}, t|_{\mathcal{C}'}, c|_{\mathcal{C}'}$ simply by s, t, c.

Definition 64.19.9. Let (U, R, s, t, c) be a groupoid in functors on a category $\mathcal{C}$.

(1) The assignment $T \mapsto (U(T), R(T), s, t, c)$ determines a functor $\mathcal{C} \to \mathit{Groupoids}$. The *quotient category cofibered in groupoids* $[U/R] \to \mathcal{C}$ is the category cofibered in groupoids over $\mathcal{C}$ associated to this functor (as in Remarks 64.5.2 (9)).
(2) The *quotient morphism* $U \to [U/R]$ is the morphism of categories cofibered in groupoids over $\mathcal{C}$ defined by the rules
   (a) $x \in U(T)$ maps to the object $(T, x) \in \mathrm{Ob}([U/R](T))$, and
   (b) $x \in U(T)$ and $f : T \to T'$ give rise to the morphism $(f, \mathrm{id}_{U(f)(x)}) : (T, x) \to (T, U(f)(x))$ lying over $f : T \to T'$.

64.20. Groupoids in functors on the base category

In this section we discuss groupoids in functors on $\mathcal{C}_\Lambda$. Our eventual goal is to show that prorepresentable groupoids in functors on $\mathcal{C}_\Lambda$ serve as "presentations" for well-behaved deformation categories in the same way that smooth groupoids in algebraic spaces serve as presentations for algebraic stacks, cf. Algebraic Stacks, Section 67.16.

Definition 64.20.1. A groupoid in functors on $\mathcal{C}_\Lambda$ is *prorepresentable* if it is isomorphic to $(\underline{R_0}, \underline{R_1}, s, t, c)|_{\mathcal{C}_\Lambda}$ for some representable groupoid in functors $(\underline{R_0}, \underline{R_1}, s, t, c)$ on the category $\widehat{\mathcal{C}}_\Lambda$.

Let (U, R, s, t, c) be a groupoid in functors on $\mathcal{C}_\Lambda$. Taking completions, we get a quintuple $(\widehat{U}, \widehat{R}, \widehat{s}, \widehat{t}, \widehat{c})$. By Remark 64.7.10 completion as a functor on $\mathrm{CofSet}(\mathcal{C}_\Lambda)$ is a right adjoint, so it commutes with limits. In particular, there is a canonical isomorphism

$$\widehat{R \times_{s,U,t} R} \longrightarrow \widehat{R} \times_{\widehat{s},\widehat{U},\widehat{t}} \widehat{R},$$

so $\widehat{c}$ can be regarded as a functor $\widehat{R} \times_{\widehat{s},\widehat{U},\widehat{t}} \widehat{R} \to \widehat{R}$. Then $(\widehat{U}, \widehat{R}, \widehat{s}, \widehat{t}, \widehat{c})$ is a groupoid in functors on $\widehat{\mathcal{C}}_\Lambda$, with identity and inverse morphisms being the completions of those of (U, R, s, t, c).

Definition 64.20.2. Let (U, R, s, t, c) be a groupoid in functors on $\mathcal{C}_\Lambda$. The *completion* $(U, R, s, t, c)^\wedge$ *of* (U, R, s, t, c) is the groupoid in functors $(\widehat{U}, \widehat{R}, \widehat{s}, \widehat{t}, \widehat{c})$ on $\widehat{\mathcal{C}}_\Lambda$ described above.

Remark 64.20.3. Let (U, R, s, t, c) be a groupoid in functors on $\mathcal{C}_\Lambda$. Then there is a canonical isomorphism $(U, R, s, t, c)^\wedge|_{\mathcal{C}_\Lambda} \cong (U, R, s, t, c)$, see Remark 64.7.7. On the other hand, let (U, R, s, t, c) be a groupoid in functors on $\widehat{\mathcal{C}}_\Lambda$ such that $U, R : \widehat{\mathcal{C}}_\Lambda \to Sets$ both commute with limits, e.g. if U, R are representable. Then there is a canonical isomorphism $((U, R, s, t, c)|_{\mathcal{C}_\Lambda})^\wedge \cong (U, R, s, t, c)$. This follows from Remark 64.7.11.

Lemma 64.20.4. *Let (U, R, s, t, c) be a groupoid in functors on $\mathcal{C}_\Lambda$.*

1. *(U, R, s, t, c) is prorepresentable if and only if its completion is representable as a groupoid in functors on $\widehat{\mathcal{C}}_\Lambda$.*
2. *(U, R, s, t, c) is prorepresentable if and only if U and R are prorepresentable.*

Proof. Part (1) follows from Remark 64.20.3. For (2), the "only if" direction is clear from the definition of a prorepresentable groupoid in functors. Conversely, assume U and R are prorepresentable, say $U \cong \underline{R_0}|_{\mathcal{C}_\Lambda}$ and $R \cong \underline{R_1}|_{\mathcal{C}_\Lambda}$ for objects R_0 and R_1 of $\widehat{\mathcal{C}}_\Lambda$. Since $\underline{R_0} \cong \widehat{\underline{R_0}|_{\mathcal{C}_\Lambda}}$ and $\underline{R_1} \cong \widehat{\underline{R_1}|_{\mathcal{C}_\Lambda}}$ by Remark 64.7.11 we see that the completion $(U, R, s, t, c)^\wedge$ is a groupoid in functors of the form $(\underline{R_0}, \underline{R_1}, \widehat{s}, \widehat{t}, \widehat{c})$. By Lemma 64.4.3 the pushout $\underline{R_1} \times_{\widehat{s},\underline{R_1},\widehat{t}} \underline{R_1}$ exists. Hence $(\underline{R_0}, \underline{R_1}, \widehat{s}, \widehat{t}, \widehat{c})$ is a representable groupoid in functors on $\widehat{\mathcal{C}}_\Lambda$. Finally, the restriction $(\underline{R_0}, \underline{R_1}, s, t, c)|_{\mathcal{C}_\Lambda}$ gives back (U, R, s, t, c) by Remark 64.20.3 hence (U, R, s, t, c) is prorepresentable by definition. □

64.21. Smooth groupoids in functors on the base category

The notion of smoothness for groupoids in functors on $\mathcal{C}_\Lambda$ is defined as follows.

Definition 64.21.1. Let (U, R, s, t, c) be a groupoid in functors on $\mathcal{C}_\Lambda$. We say (U, R, s, t, c) is *smooth* if $s, t : R \to U$ are smooth.

Remark 64.21.2. We note that this terminology is potentially confusing: if (U, R, s, t, c) is a smooth groupoid in functors, then the quotient $[U/R]$ need not be a smooth category cofibred in groupoids as defined in Remark 64.8.9. However smoothness of (U, R, s, t, c) does imply (in fact is equivalent to) smoothness of the quotient morphism $U \to [U/R]$ as we shall see in Lemma 64.21.4.

Remark 64.21.3. Let $(\underline{R_0}, \underline{R_1}, s, t, c)|_{\mathcal{C}_\Lambda}$ be a prorepresentable groupoid in functors on $\mathcal{C}_\Lambda$. Then $(\underline{R_0}, \underline{R_1}, s, t, c)|_{\mathcal{C}_\Lambda}$ is smooth if and only if R_1 is a power series over R_0 via both s and t. This follows from Lemma 64.8.6.

Lemma 64.21.4. *Let (U, R, s, t, c) be a groupoid in functors on $\mathcal{C}_\Lambda$. The following are equivalent:*

1. *The groupoid in functors (U, R, s, t, c) is smooth.*
2. *The morphism $s : R \to U$ is smooth.*

(3) *The morphism $t : R \to U$ is smooth.*
(4) *The quotient morphism $U \to [U/R]$ is smooth.*

Proof. Statement (2) is equivalent to (3) since the inverse $i : R \to R$ of (U, R, s, t, c) is an isomorphism and $t = s \circ i$. By definition (1) is equivalent to (2) and (3) together, hence it is equivalent to either of them individually.

Finally we prove (2) is equivalent to (4). Unwinding the definitions:

(2) Smoothness of $s : R \to U$ amounts to the following condition: If $f : B \to A$ is a surjective ring map in $\mathcal{C}_\Lambda$, $a \in R(A)$, and $y \in U(B)$ such that $s(a) = U(f)(y)$, then there exists $a' \in R(B)$ such that $R(f)(a') = a$ and $s(a') = y$.
(4) Smoothness of $U \to [U/R]$ amounts to the following condition: If $f : B \to A$ is a surjective ring map in $\mathcal{C}_\Lambda$ and $(f, a) : (B, y) \to (A, x)$ is a morphism of $[U/R]$, then there exists $x' \in U(B)$ and $b \in R(B)$ with $s(b) = x'$, $t(b) = y$ such that $c(a, R(f)(b)) = e(x)$. Here $e : U \to R$ denotes the identity and the notation (f, a) is as in Remarks 64.5.2 (9); in particular $a \in R(A)$ with $s(a) = U(f)(y)$ and $t(a) = x$.

If (4) holds and f, a, y as in (2) are given, let $x = t(a)$ so that we have a morphism $(f, a) : (B, y) \to (A, x)$. Then (4) produces x' and b, and $a' = i(b)$ satisfies the requirements of (2). Conversely, assume (2) holds and let $(f, a) : (B, y) \to (A, x)$ as in (4) be given. Then (2) produces $a' \in R(B)$, and $x' = t(a')$ and $b = i(a')$ satisfy the requirements of (4). □

64.22. Deformation categories as quotients of groupoids in functors

We discuss conditions on a groupoid in functors on $\mathcal{C}_\Lambda$ which guarantee that the quotient is a deformation category, and we calculate the tangent and infinitesimal automorphism spaces of such a quotient.

Lemma 64.22.1. *Let (U, R, s, t, c) be a smooth groupoid in functors on $\mathcal{C}_\Lambda$. Assume U and R satisfy (RS). Then $[U/R]$ satisfies (RS).*

Proof. Let

$$\begin{array}{ccc} & & (A_2, x_2) \\ & & \downarrow{\scriptstyle (f_2, a_2)} \\ (A_1, x_1) & \xrightarrow{(f_1, a_1)} & (A, x) \end{array}$$

be a diagram in $[U/R]$ such that $f_2 : A_2 \to A$ is surjective. The notation is as in Remarks 64.5.2 (9). Hence $f_1 : A_1 \to A$, $f_2 : A_2 \to A$ are maps in $\mathcal{C}_\Lambda$, $x \in U(A)$, $x_1 \in U(A_1)$, $x_2 \in U(A_2)$, and $a_1, a_2 \in R(A)$ with $s(a_1) = U(f_1)(x_1)$, $t(a_1) = x$ and $s(a_2) = U(f_2)(x_2)$, $t(a_2) = x$. We construct a fiber product lying over $A_1 \times_A A_2$ for this diagram in $[U/R]$ as follows.

Let $a = c(i(a_1), a_2)$, where $i : R \to R$ is the inverse morphism. Then $a \in R(A)$, $x_2 \in U(A_2)$ and $s(a) = U(f_2)(x_2)$. Hence an element $(a, x_2) \in R(A) \times_{s, U(A), U(f_2)} U(A_2)$. By smoothness of $s : R \to U$ there is an element $\widetilde{a} \in R(A_2)$ with $R(f_2)(\widetilde{a}) = a$ and $s(\widetilde{a}) = x_2$. In particular $U(f_2)(t(\widetilde{a})) = t(a) = U(f_1)(x_1)$. Thus x_1 and $t(\widetilde{a})$ define an element

$$(x_1, t(\widetilde{a})) \in U(A_1) \times_{U(A)} U(A_2).$$

By the assumption that U satisfies (RS), we have an identification $U(A_1) \times_{U(A)} U(A_2) = U(A_1 \times_A A_2)$. Let us denote $x_1 \times t(\widetilde{a}) \in U(A_1 \times_A A_2)$ the element corresponding to $(x_1, t(\widetilde{a})) \in U(A_1) \times_{U(A)} U(A_2)$. Let p_1, p_2 be the projections of $A_1 \times_A A_2$. We claim

$$\begin{array}{ccc} (A_1 \times_A A_2, x_1 \times t(\widetilde{a})) & \xrightarrow[(p_2, i(\widetilde{a}))]{} & (A_2, x_2) \\ \downarrow{\scriptstyle (p_1, e(x_1))} & & \downarrow{\scriptstyle (f_2, a_2)} \\ (A_1, x_1) & \xrightarrow{(f_1, a_1)} & (A, x) \end{array}$$

is a fiber square in $[U/R]$. (Note $e : U \to R$ denotes the identity.)

The diagram is commutative because $c(a_2, R(f_2)(i(\widetilde{a}))) = c(a_2, i(a)) = a_1$. To check it is a fiber square, let

$$\begin{array}{ccc} (B, z) & \xrightarrow[(g_2, b_2)]{} & (A_2, x_2) \\ \downarrow{\scriptstyle (g_1, b_1)} & & \downarrow{\scriptstyle (f_2, a_2)} \\ (A_1, x_1) & \xrightarrow{(f_1, a_1)} & (A, x) \end{array}$$

be a commutative diagram in $[U/R]$. We will show there is a unique morphism $(g, b) : (B, z) \to (A_1 \times_A A_2, x_1 \times t(\widetilde{a}))$ compatible with the morphisms to (A_1, x_1) and (A_2, x_2). We must take $g = (g_1, g_2) : B \to A_1 \times_A A_2$. Since by assumption R satisfies (RS), we have an identification $R(A_1 \times_A A_2) = R(A_1) \times_{R(A)} R(A_2)$. Hence we can write $b = (b_1', b_2')$ for some $b_1' \in R(A_1)$, $b_2' \in R(A_2)$ which agree in $R(A)$. Then $((g_1, g_2), (b_1', b_2')) : (B, z) \to (A_1 \times_A A_2, x_1 \times t(\widetilde{a}))$ will commute with the projections if and only if $b_1' = b_1$ and $b_2' = c(\widetilde{a}, b_2)$ proving unicity and existence. □

Lemma 64.22.2. *Let (U, R, s, t, c) be a smooth groupoid in functors on $\mathcal{C}_\Lambda$. Assume U and R are deformation functors. Then:*

(1) *The quotient $[U/R]$ is a deformation category.*
(2) *The tangent space of $[U/R]$ is*
$$T[U/R] = Coker(ds - dt : TR \to TU).$$
(3) *Let x_0 be the unique object of $[U/R](k)$. The space of infinitesimal automorphisms of $[U/R]$ is*
$$Inf_{x_0}([U/R]) = Ker(ds \oplus dt : TR \to TU \oplus TU).$$

Proof. Since U and R are deformation functors $[U/R]$ is a predeformation category. Since (RS) holds for deformation functors by definition we see that (RS) holds for [U/R] by Lemma 64.22.1. Hence $[U/R]$ is a deformation category. Statements (2) and (3) follow directly from the definitions. □

64.23. Presentations of categories cofibered in groupoids

A presentation is defined as follows.

Definition 64.23.1. Let $\mathcal{F}$ be a category cofibered in groupoids over a category $\mathcal{C}$. Let (U, R, s, t, c) be a groupoid in functors on $\mathcal{C}$. A *presentation of $\mathcal{F}$ by (U, R, s, t, c)* is an equivalence $\varphi : [U/R] \to \mathcal{F}$ of categories cofibered in groupoids over $\mathcal{C}$.

The following two general lemmas will be used to get presentations.

Lemma 64.23.2. *Let $\mathcal{F}$ be category cofibered in groupoids over a category $\mathcal{C}$. Let $U : \mathcal{C} \to Sets$ be a functor. Let $f : U \to \mathcal{F}$ be a morphism of categories cofibered in groupoids over $\mathcal{C}$. Define R, s, t, c as follows:*

(1) *$R : \mathcal{C} \to Sets$ is the functor $U \times_{f,\mathcal{F},f} U$.*
(2) *$t, s : R \to U$ are the first and second projections, respectively.*
(3) *$c : R \times_{s,U,t} R \to R$ is the morphism given by projection onto the first and last factors of $U \times_{f,\mathcal{F},f} U \times_{f,\mathcal{F},f} U$ under the canonical isomorphism $R \times_{s,U,t} R \to U \times_{f,\mathcal{F},f} U \times_{f,\mathcal{F},f} U$.*

Then (U, R, s, t, c) is a groupoid in functors on $\mathcal{C}$.

Proof. Omitted. □

Lemma 64.23.3. *Let $\mathcal{F}$ be category cofibered in groupoids over a category $\mathcal{C}$. Let $U : \mathcal{C} \to Sets$ be a functor. Let $f : U \to \mathcal{F}$ be a morphism of categories cofibered in groupoids over $\mathcal{C}$. Let (U, R, s, t, c) be the groupoid in functors on $\mathcal{C}$ constructed from $f : U \to \mathcal{F}$ in Lemma 64.23.2. Then there is a natural morphism $[f] : [U/R] \to \mathcal{F}$ such that:*

(1) *$[f] : [U/R] \to \mathcal{F}$ is fully faithful.*
(2) *$[f] : [U/R] \to \mathcal{F}$ is an equivalence if and only if $f : U \to \mathcal{F}$ is essentially surjective.*

Proof. Omitted. □

64.24. Presentations of deformation categories

According to the next lemma, a smooth morphism from a predeformation functor to a predeformation category $\mathcal{F}$ gives rise to a presentation of $\mathcal{F}$ by a smooth groupoid in functors.

Lemma 64.24.1. *Let $\mathcal{F}$ be a category cofibered in groupoids over $\mathcal{C}_\Lambda$. Let $U : \mathcal{C}_\Lambda \to Sets$ be a functor. Let $f : U \to \mathcal{F}$ be a smooth morphism of categories cofibered in groupoids. Then:*

(1) *If (U, R, s, t, c) is the groupoid in functors on $\mathcal{C}_\Lambda$ constructed from $f : U \to \mathcal{F}$ in Lemma 64.23.2, then (U, R, s, t, c) is smooth.*
(2) *If $f : U(k) \to \mathcal{F}(k)$ is essentially surjective, then the morphism $[f] : [U/R] \to \mathcal{F}$ of Lemma 64.23.3 is an equivalence.*

Proof. From the construction of Lemma 64.23.2 we have a commutative diagram

$$\begin{array}{ccc} R = U \times_{f,\mathcal{F},f} U & \xrightarrow[s]{} & U \\ \downarrow{\scriptstyle t} & & \downarrow{\scriptstyle f} \\ U & \xrightarrow{f} & \mathcal{F} \end{array}$$

where t, s are the first and second projections. So t, s are smooth by Lemma 64.8.7. Hence (1) holds.

If the assumption of (2) holds, then by Lemma 64.8.8 the morphism $f : U \to \mathcal{F}$ is essentially surjective. Hence by Lemma 64.23.3 the morphism $[f] : [U/R] \to \mathcal{F}$ is an equivalence. □

Lemma 64.24.2. *Let $\mathcal{F}$ be a deformation category. Let $U : \mathcal{C}_\Lambda \to Sets$ be a deformation functor. Let $f : U \to \mathcal{F}$ be a morphism of categories cofibered in groupoids. Let u_0 be the unique element of $U(k)$. Then $U \times_{f,\mathcal{F},f} U$ is a deformation functor with tangent space fitting into an exact sequence of k-vector spaces*

$$0 \to Inf_{f(u_0)}(\mathcal{F}) \to T(U \times_{f,\mathcal{F},f} U) \to TU \oplus TU$$

Proof. Follows from Lemma 64.18.14 and the fact that $\mathrm{Inf}_{u_0}(U) = (0)$. □

Lemma 64.24.3. *Let $\mathcal{F}$ be a deformation category. Let $U : \mathcal{C}_\Lambda \to Sets$ be a prorepresentable functor. Let $f : U \to \mathcal{F}$ be a morphism of categories cofibered in groupoids. Let (U, R, s, t, c) be the groupoid in functors on $\mathcal{C}_\Lambda$ constructed from $f : U \to \mathcal{F}$ in Lemma 64.23.2. Assume $\dim_k Inf_{x_0}(\mathcal{F})$ is finite for $x_0 \in \mathrm{Ob}(\mathcal{F}(k))$. Then (U, R, s, t, c) is prorepresentable.*

Proof. Note that U is a deformation functor by Example 64.15.10. By Lemma 64.24.2 we see that $R = U \times_{f,\mathcal{F},f} U$ is a deformation functor whose tangent space $TR = T(U \times_{f,\mathcal{F},f} U)$ sits in an exact sequence $0 \to \mathrm{Inf}_{x_0}(\mathcal{F}) \to TR \to TU \oplus TU$. Since we have assumed the first space has finite dimension and since TU has finite dimension by Example 64.10.11 we see that $\dim TR < \infty$. The map $\gamma : \mathrm{Der}_\Lambda(k, k) \to TR$ see (64.11.6.1) is injective because its composition with $TR \to TU$ is injective by Theorem 64.17.2 for the prorepresentable functor U. Thus R is prorepresentable by Theorem 64.17.2. It follows from Lemma 64.20.4 that (U, R, s, t, c) is prorepresentable. □

Theorem 64.24.4. *Let $\mathcal{F}$ be a category cofibered in groupoids over $\mathcal{C}_\Lambda$. Then $\mathcal{F}$ admits a presentation by a smooth prorepresentable groupoid in functors on $\mathcal{C}_\Lambda$ if and only if the following conditions hold:*

1. *$\mathcal{F}$ is a deformation category.*
2. *$\dim_k T\mathcal{F}$ is finite.*
3. *$\dim_k Inf_{x_0}(\mathcal{F})$ is finite for some $x_0 \in \mathrm{Ob}(\mathcal{F}(k))$.*

Proof. Recall that a prorepresentable functor is a deformation functor, see Example 64.15.10. Thus if $\mathcal{F}$ is equivalent to a smooth prorepresentable groupoid in functors, then conditions (1), (2), and (3) follow from Lemma 64.22.2 (1), (2), and (3).

Conversely, assume conditions (1), (2), and (3) hold. Condition (1) implies that (S1) and (S2) are satisfied, see Lemma 64.15.6. By Lemma 64.12.4 there exists a versal formal object ξ. Setting $U = \underline{R}|_{\mathcal{C}_\Lambda}$ the associated map $\underline{\xi} : U \to \mathcal{F}$ is smooth (this is the definition of a versal formal object). Let (U, R, s, t, c) be the groupoid in functors constructed in Lemma 64.23.2 from the map $\underline{\xi}$. By Lemma 64.24.1 we see that (U, R, s, t, c) is a smooth groupoid in functors and that $[U/R] \to \mathcal{F}$ is an equivalence. By Lemma 64.24.3 we see that (U, R, s, t, c) is prorepresentable. Hence $[U/R] \to \mathcal{F}$ is the desired presentation of $\mathcal{F}$. □

64.25. Remarks regarding minimality

The main theorem of this chapter is Theorem 64.24.4 above. It describes completely those categories cofibred in groupoids over $\mathcal{C}_\Lambda$ which have a presentation by a smooth prorepresentable groupoid in functors. In this section we briefly discuss how the minimality discussed in Sections 64.13 and 64.14 can be used to obtain a "minimal" smooth prorepresentable presentation.

Definition 64.25.1. Let (U, R, s, t, c) be a smooth prorepresentable groupoid in functors on $\mathcal{C}_\Lambda$.

(1) We say (U, R, s, t, c) is *normalized* if the groupoid $(U(k[\epsilon]), R(k[\epsilon]), s, t, c)$ is totally disconnected, i.e., there are no morphisms between distinct objects.
(2) We say (U, R, s, t, c) is *minimal* if the $U \to [U/R]$ is given by a minimal versal formal object of $[U/R]$.

The difference between the two notions is related to the difference between conditions (64.14.0.1) and (64.14.0.2) and disappears when $k' \subset k$ is separable. Also a normalized smooth prorepresentable groupoid in functors is minimal as the following lemma shows. Here is a precise statement.

Lemma 64.25.2. *Let (U, R, s, t, c) be a smooth prorepresentable groupoid in functors on $\mathcal{C}_\Lambda$.*

(1) *(U, R, s, t, c) is normalized if and only if the morphism $U \to [U/R]$ induces an isomorphism on tangent spaces, and*
(2) *(U, R, s, t, c) is minimal if and only if the kernel of $TU \to T[U/R]$ is contained in the image of $Der_\Lambda(k, k) \to TU$.*

Proof. Part (1) follows immediately from the definitions. To see part (2) set $\mathcal{F} = [U/R]$. Since $\mathcal{F}$ has a presentation it is a deformation category, see Theorem 64.24.4. In particular it satisfies (RS), (S1), and (S2), see Lemma 64.15.6. Recall that minimal versal formal objects are unique up to isomorphism, see Lemma 64.13.5. By Theorem 64.14.5 a minimal versal object induces a map $\underline{\xi} : \underline{R}|_{\mathcal{C}_\Lambda} \to \mathcal{F}$ satisfying (64.14.0.2). Since $U \cong \underline{R}|_{\mathcal{C}_\Lambda}$ over $\mathcal{F}$ we see that $TU \to T\mathcal{F} = T[U/R]$ satisfies the property as stated in the lemma. □

The quotient of a minimal prorepresentable groupoid in functors on $\mathcal{C}_\Lambda$ does not admit autoequivalences which are not automorphisms. To prove this, we first note the following lemma.

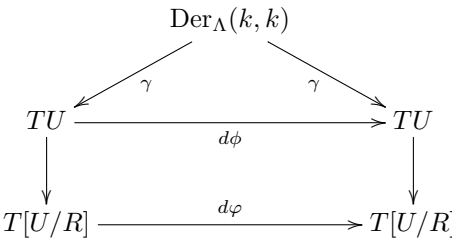
Lemma 64.25.3. *Let $U : \mathcal{C}_\Lambda \to Sets$ be a prorepresentable functor. Let $\varphi : U \to U$ be a morphism such that $d\varphi : TU \to TU$ is an isomorphism. Then φ is an isomorphism.*

Proof. If $U \cong \underline{R}|_{\mathcal{C}_\Lambda}$ for some $R \in \mathrm{Ob}(\widehat{\mathcal{C}_\Lambda})$, then completing φ gives a morphism $\underline{R} \to \underline{R}$. If $f : R \to R$ is the corresponding morphism in $\widehat{\mathcal{C}_\Lambda}$, then f induces an isomorphism $\mathrm{Der}_\Lambda(R, k) \to \mathrm{Der}_\Lambda(R, k)$, see Example 64.10.14. In particular f is a surjection by Lemma 64.4.6. As a surjective endomorphism of a Noetherian ring is an isomorphism (see Algebra, Lemma 10.29.8) we conclude f, hence $\underline{R} \to \underline{R}$, hence $\varphi : U \to U$ is an isomorphism. □

Lemma 64.25.4. *Let (U, R, s, t, c) be a minimal smooth prorepresentable groupoid in functors on $\mathcal{C}_\Lambda$. If $\varphi : [U/R] \to [U/R]$ is an equivalence of categories cofibered in groupoids, then φ is an isomorphism.*

Proof. A morphism $\varphi : [U/R] \to [U/R]$ is the same thing as a morphism $\varphi : (U, R, s, t, c) \to (U, R, s, t, c)$ of groupoids in functors over $\mathcal{C}_\Lambda$ as defined in Definition 64.19.1. Denote $\phi : U \to U$ and $\psi : R \to R$ the corresponding morphisms. Because

the diagram



is commutative, since $d\varphi$ is bijective, and since we have the characterization of minimality in Lemma 64.25.2 we conclude that $d\phi$ is injective (hence bijective by dimension reasons). Thus $\phi : U \to U$ is an isomorphism by Lemma 64.25.3. We can use a similar argument, using the exact sequence

$$0 \to \mathrm{Inf}_{x_0}([U/R]) \to TR \to TU \oplus TU$$

of Lemma 64.24.2 to prove that $\psi : R \to R$ is an isomorphism. But is also a consequence of the fact that $R = U \times_{[U/R]} U$ and that φ and ϕ are isomorphisms. □

Lemma 64.25.5. *Let (U, R, s, t, c) and (U', R', s', t', c') be minimal smooth prorepresentable groupoids in functors on $\mathcal{C}_\Lambda$. If $\varphi : [U/R] \to [U'/R']$ is an equivalence of categories cofibered in groupoids, then φ is an isomorphism.*

Proof. Let $\psi : [U'/R'] \to [U/R]$ be a quasi-inverse to φ. Then $\psi \circ \varphi$ and $\varphi \circ \psi$ are isomorphisms by Lemma 64.25.4, hence φ and ψ are isomorphisms. □

The following lemma summarizes some of the things we have seen earlier in this chapter.

Lemma 64.25.6. *Let $\mathcal{F}$ be a deformation category such that $\dim_k T\mathcal{F} < \infty$ and $\dim_k Inf_{x_0}(\mathcal{F}) < \infty$ for some $x_0 \in \mathrm{Ob}(\mathcal{F}(k))$. Then there exists a minimal versal formal object ξ of $\mathcal{F}$. Say ξ lies over $R \in \mathrm{Ob}(\widehat{\mathcal{C}_\Lambda})$. Let $U = \underline{R}|_{\mathcal{C}_\Lambda}$. Let $f = \underline{\xi} : U \to \mathcal{F}$ be the associated morphism. Let (U, R, s, t, c) be the groupoid in functors on $\mathcal{C}_\Lambda$ constructed from $f : U \to \mathcal{F}$ in Lemma 64.23.2. Then (U, R, s, t, c) is a minimal smooth prorepresentable groupoid in functors on $\mathcal{C}_\Lambda$ and there is an equivalence $[U/R] \to \mathcal{F}$.*

Proof. As $\mathcal{F}$ is a deformation category it satisfies (S1) and (S2), see Lemma 64.15.6. By Lemma 64.12.4 there exists a versal formal object. By Lemma 64.13.5 there exists a minimal versal formal object ξ/R as in the statement of the lemma. Setting $U = \underline{R}|_{\mathcal{C}_\Lambda}$ the associated map $\underline{\xi} : U \to \mathcal{F}$ is smooth (this is the definition of a versal formal object). Let (U, R, s, t, c) be the groupoid in functors constructed in Lemma 64.23.2 from the map $\underline{\xi}$. By Lemma 64.24.1 we see that (U, R, s, t, c) is a smooth groupoid in functors and that $[U/R] \to \mathcal{F}$ is an equivalence. By Lemma 64.24.3 we see that (U, R, s, t, c) is prorepresentable. Finally, (U, R, s, t, c) is minimal because $U \to [U/R] = \mathcal{F}$ corresponds to the minimal versal formal object ξ. □

Presentations by minimal prorepresentable groupoids in functors satisfy the following uniqueness property.

Lemma 64.25.7. *Let $\mathcal{F}$ be category cofibered in groupoids over $\mathcal{C}_\Lambda$. Assume there exist presentations of $\mathcal{F}$ by minimal smooth prorepresentable groupoids in functors*

(U, R, s, t, c) and (U', R', s', t', c'). Then (U, R, s, t, c) and (U', R', s', t', c') are isomorphic.

Proof. Follows from Lemma 64.25.5 and the observation that a morphism $[U/R] \to [U'/R']$ is the same thing as a morphism of groupoids in functors (by our explicit construction of $[U/R]$ in Definition 64.19.9). □

In summary we have proved the following theorem.

Theorem 64.25.8. *Let $\mathcal{F}$ be a category cofibered in groupoids over $\mathcal{C}_\Lambda$. Consider the following conditions*

1. *$\mathcal{F}$ admits a presentation by a normalized smooth prorepresentable groupoid in functors on $\mathcal{C}_\Lambda$,*
2. *$\mathcal{F}$ admits a presentation by a smooth prorepresentable groupoid in functors on $\mathcal{C}_\Lambda$,*
3. *$\mathcal{F}$ admits a presentation by a minimal smooth prorepresentable groupoid in functors on $\mathcal{C}_\Lambda$, and*
4. *$\mathcal{F}$ satisfies the following conditions*
   (a) *$\mathcal{F}$ is a deformation category.*
   (b) *$\dim_k T\mathcal{F}$ is finite.*
   (c) *$\dim_k Inf_{x_0}(\mathcal{F})$ is finite for some $x_0 \in \mathrm{Ob}(\mathcal{F}(k))$.*

Then (2), (3), (4) are equivalent and are implied by (1). If $k' \subset k$ is separable, then (1), (2), (3), (4) are all equivalent. Furthermore, the minimal smooth prorepresentable groupoids in functors which provide a presentation of $\mathcal{F}$ are unique up to isomorphism.

Proof. We see that (1) implies (3) and is equivalent to (3) if $k' \subset k$ is separable from Lemma 64.25.2. It is clear that (3) implies (2). We see that (2) implies (4) by Theorem 64.24.4. We see that (4) implies (3) by Lemma 64.25.6. This proves all the implications. The final uniqueness statement follows from Lemma 64.25.7. □

64.26. Change of residue field

In this section we quickly discuss what happens if we replace the residue field k by a finite extension. Let Λ be a Noetherian ring and let $\Lambda \to k$ be a finite ring map where k is a field. Throughout this whole chapter we have used $\mathcal{C}_\Lambda$ to denote the category of Artinian local Λ-algebras whose residue field is identified with k, see Definition 64.3.1. However, since in this section we will discuss what happen when we change k we will instead use the notation $\mathcal{C}_{\Lambda,k}$ to indicate the dependence on k.

Situation 64.26.1. Let Λ be a Noetherian ring and let $\Lambda \to k \to l$ be a finite ring maps where k and l are fields. Thus $k \subset l$ is a finite extensions of fields. A typical object of $\mathcal{C}_{\Lambda,l}$ will be denoted B and a typical object of $\mathcal{C}_{\Lambda,k}$ will be denoted A. We define

$$\mathcal{C}_{\Lambda,l} \longrightarrow \mathcal{C}_{\Lambda,k}, \quad B \longmapsto B \times_l k \tag{64.26.1.1}$$

Given a category cofibred in groupoids $p : \mathcal{F} \to \mathcal{C}_{\Lambda,k}$ we obtain an associated category cofibred in groupoids

$$p_{l/k} : \mathcal{F}_{l/k} \longrightarrow \mathcal{C}_{\Lambda,l}$$

by setting $\mathcal{F}_{l/k}(B) = \mathcal{F}(B \times_l k)$.

The functor (64.26.1.1) makes sense: because $B \times_l k \subset B$ we have

$$\begin{aligned}[k : k'] \operatorname{length}_{B \times_l k}(B \times_l k) &= \operatorname{length}_\Lambda(B \times_l k) \\ &\leq \operatorname{length}_\Lambda(B) \\ &= [l : k'] \operatorname{length}_B(B) < \infty\end{aligned}$$

(see Lemma 64.3.4) hence $B \times_l k$ is Artinian (see Algebra, Lemma 10.49.8). Thus $B \times_l k$ is an Artinian local ring with residue field k. Note that (64.26.1.1) commutes with fibre products

$$(B_1 \times_B B_2) \times_l k = (B_1 \times_l k) \times_{(B \times_l k)} (B_2 \times_l k)$$

and transforms surjective ring maps into surjective ring maps. We use the "expensive" notation $\mathcal{F}_{l/k}$ to prevent confusion with the construction of Remark 64.6.4. Here are some elementary observations.

Lemma 64.26.2. *With notation and assumptions as in Situation 64.26.1.*

(1) *We have* $\overline{\mathcal{F}_{l/k}} = (\overline{\mathcal{F}})_{l/k}$.
(2) *If* $\mathcal{F}$ *is a predeformation category, then* $\mathcal{F}_{l/k}$ *is a predeformation category.*
(3) *If* $\mathcal{F}$ *satisfies (S1), then* $\mathcal{F}_{l/k}$ *satisfies (S1).*
(4) *If* $\mathcal{F}$ *satisfies (S2), then* $\mathcal{F}_{l/k}$ *satisfies (S2).*
(5) *If* $\mathcal{F}$ *satisfies (RS), then* $\mathcal{F}_{l/k}$ *satisfies (RS).*

Proof. Part (1) is immediate from the definitions.

Since $\mathcal{F}_{l/k}(l) = \mathcal{F}(k)$ part (2) follows from the definition, see Definition 64.6.2.

Part (3) follows as the functor (64.26.1.1) commutes with fibre products and transforms surjective maps into surjective maps, see Definition 64.9.1.

Part (4). To see this consider a diagram

$$\begin{array}{ccc} & & l[\epsilon] \\ & & \downarrow \\ B & \longrightarrow & l \end{array}$$

in $\mathcal{C}_{\Lambda,l}$ as in Definition 64.9.1. Applying the functor (64.26.1.1) we obtain

$$\begin{array}{ccc} & & k[l\epsilon] \\ & & \downarrow \\ B \times_l k & \longrightarrow & k \end{array}$$

where $l\epsilon$ denotes the finite dimensional k-vector space $l\epsilon \subset l[\epsilon]$. According to Lemma 64.9.4 the condition of (S2) for $\mathcal{F}$ also holds for this diagram. Hence (S2) holds for $\mathcal{F}_{l/k}$.

Part (5) follows from the characterization of (RS) in Lemma 64.15.4 part (2) and the fact that (64.26.1.1) commutes with fibre products. □

The following lemma applies in particular when $\mathcal{F}$ satisfies (S2) and is a predeformation category, see Lemma 64.9.5.

Lemma 64.26.3. *With notation and assumptions as in Situation 64.26.1. Assume $\mathcal{F}$ is a predeformation category and $\overline{\mathcal{F}}$ satisfies (S2). Then there is a canonical l-vector space isomorphism*

$$T\mathcal{F} \otimes_k l \longrightarrow T\mathcal{F}_{l/k}$$

of tangent spaces.

Proof. By Lemma 64.26.2 we may replace $\mathcal{F}$ by $\overline{\mathcal{F}}$. Moreover we see that $T\mathcal{F}$, resp. $T\mathcal{F}_{l/k}$ has a canonical k-vector space structure, resp. l-vector space structure, see Lemma 64.11.2. Then

$$T\mathcal{F}_{l/k} = \mathcal{F}_{l/k}(l[\epsilon]) = \mathcal{F}(k[l\epsilon]) = T\mathcal{F} \otimes_k l$$

the last equality by Lemma 64.11.2. More generally, given a finite dimensional l-vector space V we have

$$\mathcal{F}_{l/k}(l[V]) = \mathcal{F}(k[V_k]) = T\mathcal{F} \otimes_k V_k$$

where V_k denotes V seen as a k-vector space. We conclude that the functors $V \mapsto \mathcal{F}_{l/k}(l[V])$ and $V \mapsto T\mathcal{F} \otimes_k V_k$ are canonically identified as functors to the category of sets. By Lemma 64.10.4 we see there is at most one way to turn either functor into an l-linear functor. Hence the isomorphisms are compatible with the l-vector space structures and we win. □

Lemma 64.26.4. *With notation and assumptions as in Situation 64.26.1. Assume $\mathcal{F}$ is a deformation category. Let $x_0 \in \mathrm{Ob}(\mathcal{F}(k))$ and denote $x_{l,0}$ the corresponding object of $\mathcal{F}_{l/k}$ over l. Then there is a canonical l-vector space isomorphism*

$$Inf_{x_0}(\mathcal{F}) \otimes_k l \longrightarrow Inf_{x_{l,0}}(\mathcal{F}_{l/k})$$

of infinitesimal automorphism spaces.

Proof. Recall that the vector space structure on $\mathrm{Inf}_{x_0}(\mathcal{F})$ comes from identifying it with the tangent space of the functor $Aut(x_0)$ which is defined on the category $\mathcal{C}_{k,k}$ of Artinian local k-algebras with residue field k. Similarly, $\mathrm{Inf}_{x_{l,0}}(\mathcal{F}_{l/k})$ is the tangent space of $Aut(x_{l,0})$ which is defined on the category $\mathcal{C}_{l,l}$ of Artinian local l-algebras with residue field l. Unwinding the definitions we see that $Aut(x_{l,0})$ is the restriction of $Aut(x_0)_{l/k}$ (which lives on $\mathcal{C}_{k,l}$) to $\mathcal{C}_{l,l}$. Since there is no difference between the tangent space of $Aut(x_0)_{l/k}$ seen as a functor on $\mathcal{C}_{k,l}$ or $\mathcal{C}_{l,l}$, the lemma follows from Lemma 64.26.3 and the fact that $Aut(x_0)$ satisfies (RS) by Lemma 64.18.6 (whence we have (S2) by Lemma 64.15.6). □

Lemma 64.26.5. *With notation and assumptions as in Situation 64.26.1. If $\mathcal{F} \to \mathcal{G}$ is a smooth morphism of categories cofibred in groupoids over $\mathcal{C}_{\Lambda,k}$, then $\mathcal{F}_{l/k} \to \mathcal{G}_{l/k}$ is a smooth morphism of categories cofibred in groupoids over $\mathcal{C}_{\Lambda,l}$.*

Proof. This follows immediately from the definitions and the fact that (64.26.1.1) preserves surjections. □

There are many more things you can say about the relationship between $\mathcal{F}$ and $\mathcal{F}_{l/k}$ (in particular about the relationship between versal deformations) and we will add these here as needed.

64.27. Other chapters

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CHAPTER 65

Deformation Theory

65.1. Introduction

The goal of this chapter is to give a (relatively) gentle introduction to deformation theory of modules, morphisms, etc. In this chapter we deal with those results that can be proven using the naive cotangent complex. In the chapter on the cotangent complex we will extend these results a little bit. The advanced reader may wish to consult the treatise by Illusie on this subject, see [**Ill72**].

65.2. Deformations of rings and the naive cotangent complex

In this section we use the naive cotangent complex to do a little bit of deformation theory. We start with a surjective ring map $A' \to A$ whose kernel is an ideal I of square zero. Moreover we assume given a ring map $A \to B$, a B-module N, and an A-module map $c : I \to N$. In this section we ask ourselves whether we can find the question mark fitting into the following diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & N & \longrightarrow & ? & \longrightarrow & B & \longrightarrow & 0 \\ & & \uparrow{\scriptstyle c} & & \uparrow & & \uparrow & & \\ 0 & \longrightarrow & I & \longrightarrow & A' & \longrightarrow & A & \longrightarrow & 0 \end{array} \tag{65.2.0.1}$$

and moreover how unique the solution is (if it exists). More precisely, we look for a surjection of A'-algebras $B' \to B$ whose kernel is identified with N such that $A' \to B'$ induces the given map c. We will say B' is a *solution* to (65.2.0.1).

Lemma 65.2.1. *Given a commutative diagram*

$$\begin{array}{ccccccccc} & & 0 & \longrightarrow & N_2 & \longrightarrow & B_2' & \longrightarrow & B_2 & \longrightarrow & 0 \\ & & & & \uparrow{\scriptstyle c_2} & & \uparrow & & \uparrow & & \\ & & 0 & \longrightarrow & I_2 & \longrightarrow & A_2' & \longrightarrow & A_2 & \longrightarrow & 0 \\ & & & \nearrow & \uparrow & \nearrow & \uparrow & \nearrow & \uparrow & & \\ 0 & \longrightarrow & N_1 & \longrightarrow & B_1' & \longrightarrow & B_1 & \longrightarrow & 0 & & \\ & & \uparrow{\scriptstyle c_1} & \nearrow & \uparrow & \nearrow & \uparrow & & & & \\ 0 & \longrightarrow & I_1 & \longrightarrow & A_1' & \longrightarrow & A_1 & \longrightarrow & 0 & & \end{array}$$

with front and back solutions to (65.2.0.1) we have

(1) *There exist a canonical element in $Ext^1_{B_1}(NL_{B_1/A_1}, N_2)$ whose vanishing is a necessary and sufficient condition for the existence of a ring map $B_1' \to B_2'$ fitting into the diagram.*

(2) *If there exists a map $B_1' \to B_2'$ fitting into the diagram the set of all such maps is a principal homogeneous space under* $\mathrm{Hom}_{B_1}(\Omega_{B_1/A_1}, N_2)$.

Proof. Let $E = B_1$ viewed as a set. Consider the surjection $A_1[E] \to B_1$ with kernel J used to define the naive cotangent complex by the formula

$$NL_{B_1/A_1} = (J/J^2 \to \Omega_{A_1[E]/A_1} \otimes_{A_1[E]} B_1)$$

in Algebra, Section 10.127. Since $\Omega_{A_1[E]/A_1} \otimes B_1$ is a free B_1-module we have

$$\mathrm{Ext}^1_{B_1}(NL_{B_1/A_1}, N_2) = \frac{\mathrm{Hom}_{B_1}(J/J^2, N_2)}{\mathrm{Hom}_{B_1}(\Omega_{A_1[E]/A_1} \otimes B_1, N_2)}$$

We will construct an obstruction in the module on the right. Let $J' = \mathrm{Ker}(A_1'[E] \to B_1)$. Note that there is a surjection $J' \to J$ whose kernel is $I_1 A_1[E]$. For every $e \in E$ denote $x_e \in A_1[E]$ the corresponding variable. Choose a lift $y_e \in B_1'$ of the image of x_e in B_1 and a lift $z_e \in B_2'$ of the image of x_e in B_2. These choices determine A_1'-algebra maps

$$A_1'[E] \to B_1' \quad \text{and} \quad A_1'[E] \to B_2'$$

The first of these gives a map $J' \to N_1$, $f' \mapsto f'(y_e)$ and the second gives a map $J' \to N_2$, $f' \mapsto f'(z_e)$. A calculation shows that these maps annihilate $(J')^2$. Because the left square of the diagram (involving c_1 and c_2) commutes we see that these maps agree on $I_1 A_1[E]$ as maps into N_2. Observe that B_1' is the pushout of $J' \to A_1'[B_1]$ and $J' \to N_1$. Thus, if the maps $J' \to N_1 \to N_2$ and $J' \to N_2$ agree, then we obtain a map $B_1' \to B_2'$ fitting into the diagram. Thus we let the obstruction be the class of the map

$$J/J^2 \to N_2, \quad f \mapsto f'(z_e) - \nu(f'(y_e))$$

where $\nu : N_1 \to N_2$ is the given map and where $f' \in J'$ is a lift of f. This is well defined by our remarks above. Note that we have the freedom to modify our choices of z_e into $z_e + \delta_{2,e}$ and y_e into $y_e + \delta_{1,e}$ for some $\delta_{i,e} \in N_i$. This will modify the map above into

$$f \mapsto f'(z_e + \delta_{2,e}) - \nu(f'(y_e + \delta_{1,e})) = f'(z_e) - \nu(f'(z_e)) + \sum (\delta_{2,e} - \nu(\delta_{1,e})) \frac{\partial f}{\partial x_e}$$

This means exactly that we are modifying the map $J/J^2 \to N_2$ by the composition $J/J^2 \to \Omega_{A_1[E]/A_1} \otimes B_1 \to N_2$ where the second map sends $\mathrm{d}x_e$ to $\delta_{2,e} - \nu(\delta_{1,e})$. Thus our obstruction is well defined and is zero if and only if a lift exists.

Part (2) comes from the observation that given two maps $\varphi, \psi : B_1' \to B_2'$ fitting into the diagram, then $\varphi - \psi$ factors through a map $D : B_1 \to N_2$ which is an A_1-derivation:

$$\begin{aligned} D(fg) &= \varphi(f'g') - \psi(f'g') \\ &= \varphi(f')\varphi(g') - \psi(f')\psi(g') \\ &= (\varphi(f') - \psi(f'))\varphi(g') + \psi(f')(\varphi(g') - \psi(g')) \\ &= gD(f) + fD(g) \end{aligned}$$

Thus D corresponds to a unique B_1-linear map $\Omega_{B_1/A_1} \to N_2$. Conversely, given such a linear map we get a derivation D and given a ring map $\psi : B_1' \to B_2'$ fitting into the diagram the map $\psi + D$ is another ring map fitting into the diagram. $\square$

The naive cotangent complex isn't good enough to contain all information regarding obstructions to finding solutions to (65.2.0.1). However, if the ring map is a local complete intersection, then the obstruction vanishes. This is a kind of lifting result; observe that for syntomic ring maps we have proved a rather strong lifting result in Smoothing Ring Maps, Proposition 16.4.2.

Lemma 65.2.2. *If $A \to B$ is a local complete intersection ring map, then there exists a solution to (65.2.0.1).*

Proof. Write $B = A[x_1, \ldots, x_n]/J$. Let $J' \subset A'[x_1, \ldots, x_n]$ be the inverse image of J. Denote $I[x_1, \ldots, x_n]$ the kernel of $A'[x_1, \ldots, x_n] \to A[x_1, \ldots, x_n]$. By More on Algebra, Lemma 15.21.5 we have $I[x_1, \ldots, x_n] \cap (J')^2 = J'I[x_1, \ldots, x_n] = JI[x_1, \ldots, x_n]$. Hence we obtain a short exact sequence

$$0 \to I \otimes_A B \to J'/(J')^2 \to J/J^2 \to 0$$

Since J/J^2 is projective (More on Algebra, Lemma 15.21.3) we can choose a splitting of this sequence

$$J'/(J')^2 = I \otimes_A B \oplus J/J^2$$

Let $(J')^2 \subset J'' \subset J'$ be the elements which map to the second summand in the decomposition above. Then

$$0 \to I \otimes_A B \to A'[x_1, \ldots, x_n]/J'' \to B \to 0$$

is a solution to (65.2.0.1) with $N = I \otimes_A B$. The general case is obtained by doing a pushout along the given map $I \otimes_A B \to N$. □

Lemma 65.2.3. *If there exists a solution to (65.2.0.1), then the set of isomorphism classes of solutions is principal homogeneous under $Ext^1_B(NL_{B/A}, N)$.*

Proof. We observe right away that given two solutions B'_1 and B'_2 to (65.2.0.1) we obtain by Lemma 65.2.1 an obstruction element $o(B'_1, B'_2) \in \mathrm{Ext}^1_B(NL_{B/A}, N)$ to the existence of a map $B'_1 \to B'_2$. Clearly, this element is the obstruction to the existence of an isomorphism, hence separates the isomorphism classes. To finish the proof it therefore suffices to show that given a solution B' and an element $\xi \in \mathrm{Ext}^1_B(NL_{B/A}, N)$ we can find a second solution B'_ξ such that $o(B', B'_\xi) = \xi$.

Let $E = B$ viewed as a set. Consider the surjection $A[E] \to B$ with kernel J used to define the naive cotangent complex by the formula

$$NL_{B/A} = (J/J^2 \to \Omega_{A[E]/A} \otimes_{A[E]} B)$$

in Algebra, Section 10.127. Since $\Omega_{A[E]/A} \otimes B$ is a free B-module we have

$$\mathrm{Ext}^1_B(NL_{B/A}, N) = \frac{\mathrm{Hom}_B(J/J^2, N)}{\mathrm{Hom}_B(\Omega_{A[E]/A} \otimes B, N)}$$

Thus we may represent ξ as the class of a morphism $\delta : J/J^2 \to N$.

For every $e \in E$ denote $x_e \in A[E]$ the corresponding variable. Choose a lift $y_e \in B'$ of the image of x_e in B. These choices determine an A'-algebra map $\varphi : A'[E] \to B'$. Let $J' = \mathrm{Ker}(A'[E] \to B)$. Observe that φ induces a map $\varphi|_{J'} : J' \to N$ and that

B' is the pushout, as in the following diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & N & \longrightarrow & B' & \longrightarrow & B & \longrightarrow & 0 \\ & & \uparrow{\scriptstyle \varphi|_{J'}} & & \uparrow & & \uparrow{\scriptstyle =} & & \\ 0 & \longrightarrow & J' & \longrightarrow & A'[E] & \longrightarrow & B & \longrightarrow & 0 \end{array}$$

Let $\psi : J' \to N$ be the sum of the map $\varphi|_{J'}$ and the composition

$$J' \to J'/(J')^2 \to J/J^2 \xrightarrow{\delta} N.$$

Then the pushout along ψ is an other ring extension B'_ξ fitting into a diagram as above. A calculation shows that $o(B', B'_\xi) = \xi$ as desired. □

Lemma 65.2.4. *Let A be a ring and let I be an A-module.*

(1) *The set of extensions of rings $0 \to I \to A' \to A \to 0$ where I is an ideal of square zero is canonically bijective to $Ext^1_A(NL_{A/\mathbf{Z}}, I)$.*
(2) *Given a ring map $A \to B$, a B-module N, an A-module map $c : I \to N$, and given extensions of rings with square zero kernels:*
   (a) *$0 \to I \to A' \to A \to 0$ corresponding to $\alpha \in Ext^1_A(NL_{A/\mathbf{Z}}, I)$, and*
   (b) *$0 \to N \to B' \to B \to 0$ corresponding to $\beta \in Ext^1_B(NL_{B/\mathbf{Z}}, N)$*
   then there is a map $A' \to B'$ fitting into a diagram (65.2.0.1) if and only if β and α map to the same element of $Ext^1_A(NL_{A/\mathbf{Z}}, N)$.

Proof. To prove this we apply the previous results where we work over $0 \to 0 \to \mathbf{Z} \to \mathbf{Z} \to 0$, in order words, we work over the extension of $\mathbf{Z}$ by 0. Part (1) follows from Lemma 65.2.3 and the fact that there exists a solution, namely $I \oplus A$. Part (2) follows from Lemma 65.2.1 and a compatibility between the constructions in the proofs of Lemmas 65.2.3 and 65.2.1 whose statement and proof we omit. □

65.3. Thickenings of ringed spaces

In the following few sections we will use the following notions:

(1) A sheaf of ideals $\mathcal{I} \subset \mathcal{O}_{X'}$ on a ringed space $(X', \mathcal{O}_{X'})$ is *locally nilpotent* if any local section of $\mathcal{I}$ is locally nilpotent. Compare with Algebra, Item 29.
(2) A *thickening* of ringed spaces is a morphism $i : (X, \mathcal{O}_X) \to (X', \mathcal{O}_{X'})$ of ringed spaces such that
   (a) i induces a homeomorphism $X \to X'$,
   (b) the map $i^\sharp : \mathcal{O}_{X'} \to i_*\mathcal{O}_X$ is surjective, and
   (c) the kernel of $i^\sharp$ is a locally nilpotent sheaf of ideals.
(3) A *first order thickening* of ringed spaces is a thickening $i : (X, \mathcal{O}_X) \to (X', \mathcal{O}_{X'})$ of ringed spaces such that $\mathrm{Ker}(i^\sharp)$ has square zero.
(4) It is clear how to define *morphisms of thickenings*, *morphisms of thickenings over a base ringed space*, etc.

If $i : (X, \mathcal{O}_X) \to (X', \mathcal{O}_{X'})$ is a thickening of ringed spaces then we identify the underlying topological spaces and think of $\mathcal{O}_X$, $\mathcal{O}_{X'}$, and $\mathcal{I} = \mathrm{Ker}(i^\sharp)$ as sheaves on $X = X'$. We obtain a short exact sequence

$$0 \to \mathcal{I} \to \mathcal{O}_{X'} \to \mathcal{O}_X \to 0$$

of $\mathcal{O}_{X'}$-modules. By Modules, Lemma 17.13.4 the category of $\mathcal{O}_X$-modules is equivalent to the category of $\mathcal{O}_{X'}$-modules annihilated by $\mathcal{I}$. In particular, if i is a first order thickening, then $\mathcal{I}$ is a $\mathcal{O}_X$-module.

Situation 65.3.1. A morphism of thickenings (f, f') is given by a commutative diagram

$$\begin{array}{ccc} (X, \mathcal{O}_X) & \xrightarrow[i]{} & (X', \mathcal{O}_{X'}) \\ {\scriptstyle f}\downarrow & & \downarrow{\scriptstyle f'} \\ (S, \mathcal{O}_S) & \xrightarrow{t} & (S', \mathcal{O}_{S'}) \end{array} \tag{65.3.1.1}$$

of ringed spaces whose horizontal arrows are thickenings. In this situation we set $\mathcal{I} = \mathrm{Ker}(i^\sharp) \subset \mathcal{O}_{X'}$ and $\mathcal{J} = \mathrm{Ker}(t^\sharp) \subset \mathcal{O}_{S'}$. As $f = f'$ on underlying topological spaces we will identify the (topological) pullback functors f^{-1} and $(f')^{-1}$. Observe that $(f')^\sharp : f^{-1}\mathcal{O}_{S'} \to \mathcal{O}_{X'}$ induces in particular a map $f^{-1}\mathcal{J} \to \mathcal{I}$ and therefore a map of $\mathcal{O}_{X'}$-modules

$$(f')^*\mathcal{J} \longrightarrow \mathcal{I}$$

If i and t are first order thickenings, then $(f')^*\mathcal{J} = f^*\mathcal{J}$ and the map above becomes a map $f^*\mathcal{J} \to \mathcal{I}$.

Definition 65.3.2. In Situation 65.3.1 we say that (f, f') is a *strict morphism of thickenings* if the map $(f')^*\mathcal{J} \longrightarrow \mathcal{I}$ is surjective.

The following lemma in particular shows that a morphism $(f, f') : (X \subset X') \to (S \subset S')$ of thickenings of schemes is strict if and only if $X = S \times_{S'} X'$.

Lemma 65.3.3. *In Situation 65.3.1 the morphism (f, f') is a strict morphism of thickenings if and only if (65.3.1.1) is cartesian in the category of ringed spaces.*

Proof. Omitted. □

65.4. Modules on first order thickenings of ringed spaces

In this section we discuss some preliminaries to the deformation theory of modules. Let $i : (X, \mathcal{O}_X) \to (X', \mathcal{O}_{X'})$ be a first order thickening of ringed spaces. We will freely use the notation introduced in Section 65.3, in particular we will identify the underlying topological spaces. In this section we consider short exact sequences

$$0 \to \mathcal{K} \to \mathcal{F}' \to \mathcal{F} \to 0 \tag{65.4.0.1}$$

of $\mathcal{O}_{X'}$-modules, where $\mathcal{F}$, $\mathcal{K}$ are $\mathcal{O}_X$-modules and $\mathcal{F}'$ is an $\mathcal{O}_{X'}$-module. In this situation we have a canonical $\mathcal{O}_X$-module map

$$c_{\mathcal{F}'} : \mathcal{I} \otimes_{\mathcal{O}_X} \mathcal{F} \longrightarrow \mathcal{K}$$

where $\mathcal{I} = \mathrm{Ker}(i^\sharp)$. Namely, given local sections f of $\mathcal{I}$ and s of $\mathcal{F}$ we set $c_{\mathcal{F}'}(f \otimes s) = fs'$ where s' is a local section of $\mathcal{F}'$ lifting s.

Lemma 65.4.1. *Let $i : (X, \mathcal{O}_X) \to (X', \mathcal{O}_{X'})$ be a first order thickening of ringed spaces. Assume given extensions*

$$0 \to \mathcal{K} \to \mathcal{F}' \to \mathcal{F} \to 0 \quad \text{and} \quad 0 \to \mathcal{L} \to \mathcal{G}' \to \mathcal{G} \to 0$$

as in (65.4.0.1) and maps $\varphi : \mathcal{F} \to \mathcal{G}$ and $\psi : \mathcal{K} \to \mathcal{L}$.

(1) *If there exists an $\mathcal{O}_{X'}$-module map $\varphi' : \mathcal{F}' \to \mathcal{G}'$ compatible with φ and ψ, then the diagram*

$$\begin{array}{ccc} \mathcal{I} \otimes_{\mathcal{O}_X} \mathcal{F} & \xrightarrow{c_{\mathcal{F}'}} & \mathcal{K} \\ {\scriptstyle 1\otimes\varphi}\downarrow & & \downarrow{\scriptstyle \psi} \\ \mathcal{I} \otimes_{\mathcal{O}_X} \mathcal{G} & \xrightarrow{c_{\mathcal{G}'}} & \mathcal{L} \end{array}$$

is commutative.

(2) *The set of $\mathcal{O}_{X'}$-module maps $\varphi' : \mathcal{F}' \to \mathcal{G}'$ compatible with φ and ψ is, if nonempty, a principal homogeneous space under* $\mathrm{Hom}_{\mathcal{O}_X}(\mathcal{F}, \mathcal{L})$.

Proof. Part (1) is immediate from the description of the maps. For (2), if φ' and φ'' are two maps $\mathcal{F}' \to \mathcal{G}'$ compatible with φ and ψ, then $\varphi' - \varphi''$ factors as

$$\mathcal{F}' \to \mathcal{F} \to \mathcal{L} \to \mathcal{G}'$$

The map in the middle comes from a unique element of $\mathrm{Hom}_{\mathcal{O}_X}(\mathcal{F}, \mathcal{L})$ by Modules, Lemma 17.13.4. Conversely, given an element α of this group we can add the composition (as displayed above with α in the middle) to φ'. Some details omitted. □

Lemma 65.4.2. *Let $i : (X, \mathcal{O}_X) \to (X', \mathcal{O}_{X'})$ be a first order thickening of ringed spaces. Assume given extensions*

$$0 \to \mathcal{K} \to \mathcal{F}' \to \mathcal{F} \to 0 \quad \text{and} \quad 0 \to \mathcal{L} \to \mathcal{G}' \to \mathcal{G} \to 0$$

as in (65.4.0.1) and maps $\varphi : \mathcal{F} \to \mathcal{G}$ and $\psi : \mathcal{K} \to \mathcal{L}$. Assume the diagram

$$\begin{array}{ccc} \mathcal{I} \otimes_{\mathcal{O}_X} \mathcal{F} & \xrightarrow{c_{\mathcal{F}'}} & \mathcal{K} \\ {\scriptstyle 1\otimes\varphi}\downarrow & & \downarrow{\scriptstyle \psi} \\ \mathcal{I} \otimes_{\mathcal{O}_X} \mathcal{G} & \xrightarrow{c_{\mathcal{G}'}} & \mathcal{L} \end{array}$$

is commutative. Then there exists an element

$$o(\varphi, \psi) \in Ext^1_{\mathcal{O}_X}(\mathcal{F}, \mathcal{L})$$

whose vanishing is a necessary and sufficient condition for the existence of a map $\varphi' : \mathcal{F}' \to \mathcal{G}'$ compatible with φ and ψ.

Proof. We can construct explicitly an extension

$$0 \to \mathcal{L} \to \mathcal{H} \to \mathcal{F} \to 0$$

by taking $\mathcal{H}$ to be the cohomology of the complex

$$\mathcal{K} \xrightarrow{1,-\psi} \mathcal{F}' \oplus \mathcal{G}' \xrightarrow{\varphi,1} \mathcal{G}$$

in the middle (with obvious notation). A calculation with local sections using the assumption that the diagram of the lemma commutes shows that $\mathcal{H}$ is annihilated by $\mathcal{I}$. Hence $\mathcal{H}$ defines a class in

$$\mathrm{Ext}^1_{\mathcal{O}_X}(\mathcal{F}, \mathcal{L}) \subset \mathrm{Ext}^1_{\mathcal{O}_{X'}}(\mathcal{F}, \mathcal{L})$$

Finally, the class of $\mathcal{H}$ is the difference of the pushout of the extension $\mathcal{F}'$ via ψ and the pullback of the extension $\mathcal{G}'$ via φ (calculations omitted). Thus the vanishing

of the class of $\mathcal{H}$ is equivalent to the existence of a commutative diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \mathcal{K} & \longrightarrow & \mathcal{F}' & \longrightarrow & \mathcal{F} & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle \psi} & & \downarrow{\scriptstyle \varphi'} & & \downarrow{\scriptstyle \varphi} & & \\ 0 & \longrightarrow & \mathcal{L} & \longrightarrow & \mathcal{G}' & \longrightarrow & \mathcal{G} & \longrightarrow & 0 \end{array}$$

as desired. □

Lemma 65.4.3. *Let $i : (X, \mathcal{O}_X) \to (X', \mathcal{O}_{X'})$ be a first order thickening of ringed spaces. Assume given $\mathcal{O}_X$-modules $\mathcal{F}$, $\mathcal{K}$ and an $\mathcal{O}_X$-linear map $c : \mathcal{I} \otimes_{\mathcal{O}_X} \mathcal{F} \to \mathcal{K}$. If there exists a sequence (65.4.0.1) with $c_{\mathcal{F}'} = c$ then the set of isomorphism classes of these extensions is principal homogeneous under $Ext^1_{\mathcal{O}_X}(\mathcal{F}, \mathcal{K})$.*

Proof. Assume given extensions

$$0 \to \mathcal{K} \to \mathcal{F}'_1 \to \mathcal{F} \to 0 \quad \text{and} \quad 0 \to \mathcal{K} \to \mathcal{F}'_2 \to \mathcal{F} \to 0$$

with $c_{\mathcal{F}'_1} = c_{\mathcal{F}'_2} = c$. Then the difference (in the extension group, see Homology, Section 12.6) is an extension

$$0 \to \mathcal{K} \to \mathcal{E} \to \mathcal{F} \to 0$$

where $\mathcal{E}$ is annihilated by $\mathcal{I}$ (local computation omitted). Hence the sequence is an extension of $\mathcal{O}_X$-modules, see Modules, Lemma 17.13.4. Conversely, given such an extension $\mathcal{E}$ we can add the extension $\mathcal{E}$ to the $\mathcal{O}_{X'}$-extension $\mathcal{F}'$ without affecting the map $c_{\mathcal{F}'}$. Some details omitted. □

Lemma 65.4.4. *Let $i : (X, \mathcal{O}_X) \to (X', \mathcal{O}_{X'})$ be a first order thickening of ringed spaces. Assume given $\mathcal{O}_X$-modules $\mathcal{F}$, $\mathcal{K}$ and an $\mathcal{O}_X$-linear map $c : \mathcal{I} \otimes_{\mathcal{O}_X} \mathcal{F} \to \mathcal{K}$. Then there exists an element*

$$o(\mathcal{F}, \mathcal{K}, c) \in Ext^2_{\mathcal{O}_X}(\mathcal{F}, \mathcal{K})$$

whose vanishing is a necessary and sufficient condition for the existence of a sequence (65.4.0.1) with $c_{\mathcal{F}'} = c$.

Proof. We first show that if $\mathcal{K}$ is an injective $\mathcal{O}_X$-module, then there does exist a sequence (65.4.0.1) with $c_{\mathcal{F}'} = c$. To do this, choose a flat $\mathcal{O}_{X'}$-module $\mathcal{H}'$ and a surjection $\mathcal{H}' \to \mathcal{F}$ (Modules, Lemma 17.16.6). Let $\mathcal{J} \subset \mathcal{H}'$ be the kernel. Since $\mathcal{H}'$ is flat we have

$$\mathcal{I} \otimes_{\mathcal{O}_{X'}} \mathcal{H}' = \mathcal{I}\mathcal{H}' \subset \mathcal{J} \subset \mathcal{H}'$$

Observe that the map

$$\mathcal{I}\mathcal{H}' = \mathcal{I} \otimes_{\mathcal{O}_{X'}} \mathcal{H}' \longrightarrow \mathcal{I} \otimes_{\mathcal{O}_{X'}} \mathcal{F} = \mathcal{I} \otimes_{\mathcal{O}_X} \mathcal{F}$$

annihilates $\mathcal{I}\mathcal{J}$. Namely, if f is a local section of $\mathcal{I}$ and s is a local section of $\mathcal{H}$, then fs is mapped to $f \otimes \overline{s}$ where $\overline{s}$ is the image of s in $\mathcal{F}$. Thus we obtain

$$\begin{array}{ccc} \mathcal{I}\mathcal{H}'/\mathcal{I}\mathcal{J} & \hookrightarrow & \mathcal{J}/\mathcal{I}\mathcal{J} \\ \downarrow & & \downarrow{\scriptstyle \gamma} \\ \mathcal{I} \otimes_{\mathcal{O}_X} \mathcal{F} & \xrightarrow{c} & \mathcal{K} \end{array}$$

a diagram of $\mathcal{O}_X$-modules. If $\mathcal{K}$ is injective as an $\mathcal{O}_X$-module, then we obtain the dotted arrow. Denote $\gamma' : \mathcal{J} \to \mathcal{K}$ the composition of γ with $\mathcal{J} \to \mathcal{J}/\mathcal{I}\mathcal{J}$. A local calculation shows the pushout

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \mathcal{J} & \longrightarrow & \mathcal{H}' & \longrightarrow & \mathcal{F} & \longrightarrow & 0 \\ & & {\scriptstyle\gamma'}\downarrow & & \downarrow & & \| & & \\ 0 & \longrightarrow & \mathcal{K} & \longrightarrow & \mathcal{F}' & \longrightarrow & \mathcal{F} & \longrightarrow & 0 \end{array}$$

is a solution to the problem posed by the lemma.

General case. Choose an embedding $\mathcal{K} \subset \mathcal{K}'$ with $\mathcal{K}'$ an injective $\mathcal{O}_X$-module. Let $\mathcal{Q}$ be the quotient, so that we have an exact sequence

$$0 \to \mathcal{K} \to \mathcal{K}' \to \mathcal{Q} \to 0$$

Denote $c' : \mathcal{I} \otimes_{\mathcal{O}_X} \mathcal{F} \to \mathcal{K}'$ be the composition. By the paragraph above there exists a sequence

$$0 \to \mathcal{K}' \to \mathcal{E}' \to \mathcal{F} \to 0$$

as in (65.4.0.1) with $c_{\mathcal{E}'} = c'$. Note that c' composed with the map $\mathcal{K}' \to \mathcal{Q}$ is zero, hence the pushout of $\mathcal{E}'$ by $\mathcal{K}' \to \mathcal{Q}$ is an extension

$$0 \to \mathcal{Q} \to \mathcal{D}' \to \mathcal{F} \to 0$$

as in (65.4.0.1) with $c_{\mathcal{D}'} = 0$. This means exactly that $\mathcal{D}'$ is annihilated by $\mathcal{I}$, in other words, the $\mathcal{D}'$ is an extension of $\mathcal{O}_X$-modules, i.e., defines an element

$$o(\mathcal{F}, \mathcal{K}, c) \in \mathrm{Ext}^1_{\mathcal{O}_X}(\mathcal{F}, \mathcal{Q}) = \mathrm{Ext}^2_{\mathcal{O}_X}(\mathcal{F}, \mathcal{K})$$

(the equality holds by the long exact cohomology sequence associated to the exact sequence above and the vanishing of higher ext groups into the injective module $\mathcal{K}'$). If $o(\mathcal{F}, \mathcal{K}, c) = 0$, then we can choose a splitting $s : \mathcal{F} \to \mathcal{D}'$ and we can set

$$\mathcal{F}' = \mathrm{Ker}(\mathcal{E}' \to \mathcal{D}'/s(\mathcal{F}))$$

so that we obtain the following diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \mathcal{K} & \longrightarrow & \mathcal{F}' & \longrightarrow & \mathcal{F} & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \| & & \\ 0 & \longrightarrow & \mathcal{K}' & \longrightarrow & \mathcal{E}' & \longrightarrow & \mathcal{F} & \longrightarrow & 0 \end{array}$$

with exact rows which shows that $c_{\mathcal{F}'} = c$. Conversely, if $\mathcal{F}'$ exists, then the pushout of $\mathcal{F}'$ by the map $\mathcal{K} \to \mathcal{K}'$ is isomorphic to $\mathcal{E}'$ by Lemma 65.4.3 and the vanishing of higher ext groups into the injective module $\mathcal{K}'$. This gives a diagram as above, which implies that $\mathcal{D}'$ is split as an extension, i.e., the class $o(\mathcal{F}, \mathcal{K}, c)$ is zero. □

Remark 65.4.5. Let $(X, \mathcal{O}_X)$ be a ringed space. A first order thickening $i : (X, \mathcal{O}_X) \to (X', \mathcal{O}_{X'})$ is said to be *trivial* if there exists a morphism of ringed spaces $\pi : (X', \mathcal{O}_{X'}) \to (X, \mathcal{O}_X)$ which is a left inverse to i. The choice of such a morphism π is called a *trivialization* of the first order thickening. Given π we obtain a splitting

$$\mathcal{O}_{X'} = \mathcal{O}_X \oplus \mathcal{I} \tag{65.4.5.1}$$

as sheaves of algebras on X by using $\pi^\sharp$ to split the surjection $\mathcal{O}_{X'} \to \mathcal{O}_X$. Conversely, such a splitting determines a morphism π. The category of trivialized first order thickenings of $(X, \mathcal{O}_X)$ is equivalent to the category of $\mathcal{O}_X$-modules.

Remark 65.4.6. Let $i : (X, \mathcal{O}_X) \to (X', \mathcal{O}_{X'})$ be a trivial first order thickening of ringed spaces and let $\pi : (X', \mathcal{O}_{X'}) \to (X, \mathcal{O}_X)$ be a trivialization. Then given any triple $(\mathcal{F}, \mathcal{K}, c)$ consisting of a pair of $\mathcal{O}_X$-modules and a map $c : \mathcal{I} \otimes_{\mathcal{O}_X} \mathcal{F} \to \mathcal{K}$ we may set

$$\mathcal{F}'_{c,triv} = \mathcal{F} \oplus \mathcal{K}$$

and use the splitting (65.4.5.1) associated to π and the map c to define the $\mathcal{O}_{X'}$-module structure and obtain an extension (65.4.0.1). We will call $\mathcal{F}'_{c,triv}$ the *trivial extension* of $\mathcal{F}$ by $\mathcal{K}$ corresponding to c and the trivialization π. Given any extension $\mathcal{F}'$ as in (65.4.0.1) we can use $\pi^\sharp : \mathcal{O}_X \to \mathcal{O}_{X'}$ to think of $\mathcal{F}'$ as an $\mathcal{O}_X$-module extension, hence a class $\xi_{\mathcal{F}'}$ in $\mathrm{Ext}^1_{\mathcal{O}_X}(\mathcal{F}, \mathcal{K})$. Lemma 65.4.3 assures that $\mathcal{F}' \mapsto \xi_{\mathcal{F}'}$ induces a bijection

$$\left\{ \begin{array}{c} \text{isomorphism classes of extensions} \\ \mathcal{F}' \text{ as in (65.4.0.1) with } c = c_{\mathcal{F}'} \end{array} \right\} \longrightarrow \mathrm{Ext}^1_{\mathcal{O}_X}(\mathcal{F}, \mathcal{K})$$

Moreover, the trivial extension $\mathcal{F}'_{c,triv}$ maps to the zero class.

Remark 65.4.7. Let $(X, \mathcal{O}_X)$ be a ringed space. Let $(X, \mathcal{O}_X) \to (X'_i, \mathcal{O}_{X'_i})$, $i = 1, 2$ be first order thickenings with ideal sheaves $\mathcal{I}_i$. Let $h : (X'_1, \mathcal{O}_{X'_1}) \to (X'_2, \mathcal{O}_{X'_2})$ be a morphism of first order thickenings of $(X, \mathcal{O}_X)$. Picture

$$\begin{array}{ccc} & (X, \mathcal{O}_X) & \\ \swarrow & & \searrow \\ (X'_1, \mathcal{O}_{X'_1}) & \xrightarrow{\quad h \quad} & (X'_2, \mathcal{O}_{X'_2}) \end{array}$$

Observe that $h^\sharp : \mathcal{O}_{X'_2} \to \mathcal{O}_{X'_1}$ in particular induces an $\mathcal{O}_X$-module map $\mathcal{I}_2 \to \mathcal{I}_1$. Let $\mathcal{F}$ be an $\mathcal{O}_X$-module. Let $(\mathcal{K}_i, c_i)$, $i = 1, 2$ be a pair consisting of an $\mathcal{O}_X$-module $\mathcal{K}_i$ and a map $c_i : \mathcal{I}_i \otimes_{\mathcal{O}_X} \mathcal{F} \to \mathcal{K}_i$. Assume furthermore given a map of $\mathcal{O}_X$-modules $\mathcal{K}_2 \to \mathcal{K}_1$ such that

$$\begin{array}{ccc} \mathcal{I}_2 \otimes_{\mathcal{O}_X} \mathcal{F} & \xrightarrow[c_2]{} & \mathcal{K}_2 \\ \downarrow & & \downarrow \\ \mathcal{I}_1 \otimes_{\mathcal{O}_X} \mathcal{F} & \xrightarrow{c_1} & \mathcal{K}_1 \end{array}$$

is commutative. Then there is a canonical functoriality

$$\left\{ \begin{array}{c} \mathcal{F}'_2 \text{ as in (65.4.0.1) with} \\ c_2 = c_{\mathcal{F}'_2} \text{ and } \mathcal{K} = \mathcal{K}_2 \end{array} \right\} \longrightarrow \left\{ \begin{array}{c} \mathcal{F}'_1 \text{ as in (65.4.0.1) with} \\ c_1 = c_{\mathcal{F}'_1} \text{ and } \mathcal{K} = \mathcal{K}_1 \end{array} \right\}$$

Namely, thinking of all sheaves $\mathcal{O}_X$, $\mathcal{O}_{X'_i}$, $\mathcal{F}$, $\mathcal{K}_i$, etc as sheaves on X, we set given $\mathcal{F}'_2$ the sheaf $\mathcal{F}'_1$ equal to the pushout, i.e., fitting into the following diagram of extensions

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \mathcal{K}_2 & \longrightarrow & \mathcal{F}'_2 & \longrightarrow & \mathcal{F} & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \| & & \\ 0 & \longrightarrow & \mathcal{K}_1 & \longrightarrow & \mathcal{F}'_1 & \longrightarrow & \mathcal{F} & \longrightarrow & 0 \end{array}$$

We omit the construction of the $\mathcal{O}_{X'_1}$-module structure on the pushout (this uses the commutativity of the diagram involving c_1 and c_2).

Remark 65.4.8. Let $(X, \mathcal{O}_X)$, $(X, \mathcal{O}_X) \to (X'_i, \mathcal{O}_{X'_i})$, $\mathcal{I}_i$, and $h : (X'_1, \mathcal{O}_{X'_1}) \to (X'_2, \mathcal{O}_{X'_2})$ be as in Remark 65.4.7. Assume that we are given given trivializations $\pi_i : X'_i \to X$ such that $\pi_1 = h \circ \pi_2$. In other words, assume h is a morphism of trivialized first order thickening of $(X, \mathcal{O}_X)$. Let $(\mathcal{K}_i, c_i)$, $i = 1, 2$ be a pair consisting of an $\mathcal{O}_X$-module $\mathcal{K}_i$ and a map $c_i : \mathcal{I}_i \otimes_{\mathcal{O}_X} \mathcal{F} \to \mathcal{K}_i$. Assume furthermore given a map of $\mathcal{O}_X$-modules $\mathcal{K}_2 \to \mathcal{K}_1$ such that

$$\begin{array}{ccc} \mathcal{I}_2 \otimes_{\mathcal{O}_X} \mathcal{F} & \xrightarrow[c_2]{} & \mathcal{K}_2 \\ \downarrow & & \downarrow \\ \mathcal{I}_1 \otimes_{\mathcal{O}_X} \mathcal{F} & \xrightarrow{c_1} & \mathcal{K}_1 \end{array}$$

is commutative. In this situation the construction of Remark 65.4.6 induces a commutative diagram

$$\begin{array}{ccc} \{\mathcal{F}'_2 \text{ as in (65.4.0.1) with } c_2 = c_{\mathcal{F}'_2} \text{ and } \mathcal{K} = \mathcal{K}_2\} & \longrightarrow & \mathrm{Ext}^1_{\mathcal{O}_X}(\mathcal{F}, \mathcal{K}_2) \\ \downarrow & & \downarrow \\ \{\mathcal{F}'_1 \text{ as in (65.4.0.1) with } c_1 = c_{\mathcal{F}'_1} \text{ and } \mathcal{K} = \mathcal{K}_1\} & \longrightarrow & \mathrm{Ext}^1_{\mathcal{O}_X}(\mathcal{F}, \mathcal{K}_1) \end{array}$$

where the vertical map on the right is given by functoriality of Ext and the map $\mathcal{K}_2 \to \mathcal{K}_1$ and the vertical map on the left is the one from Remark 65.4.7.

Remark 65.4.9. Let $(X, \mathcal{O}_X)$ be a ringed space. We define a sequence of morphisms of first order thickenings

$$(X'_1, \mathcal{O}_{X'_1}) \to (X'_2, \mathcal{O}_{X'_2}) \to (X'_3, \mathcal{O}_{X'_3})$$

of $(X, \mathcal{O}_X)$ to be a *complex* if the corresponding maps between the ideal sheaves $\mathcal{I}_i$ give a complex of $\mathcal{O}_X$-modules $\mathcal{I}_3 \to \mathcal{I}_2 \to \mathcal{I}_1$ (i.e., the composition is zero). In this case the composition $(X'_1, \mathcal{O}_{X'_1}) \to (X'_3, \mathcal{O}_{X'_3})$ factors through $(X, \mathcal{O}_X) \to (X'_3, \mathcal{O}_{X'_3})$, i.e., the first order thickening $(X'_1, \mathcal{O}_{X'_1})$ of $(X, \mathcal{O}_X)$ is trivial and comes with a canonical trivialization $\pi : (X'_1, \mathcal{O}_{X'_1}) \to (X, \mathcal{O}_X)$.

We say a sequence of morphisms of first order thickenings

$$(X'_1, \mathcal{O}_{X'_1}) \to (X'_2, \mathcal{O}_{X'_2}) \to (X'_3, \mathcal{O}_{X'_3})$$

of $(X, \mathcal{O}_X)$ is *a short exact sequence* if the corresponding maps between ideal sheaves is a short exact sequence

$$0 \to \mathcal{I}_3 \to \mathcal{I}_2 \to \mathcal{I}_1 \to 0$$

of $\mathcal{O}_X$-modules.

Remark 65.4.10. Let $(X, \mathcal{O}_X)$ be a ringed space. Let $\mathcal{F}$ be an $\mathcal{O}_X$-module. Let

$$(X'_1, \mathcal{O}_{X'_1}) \to (X'_2, \mathcal{O}_{X'_2}) \to (X'_3, \mathcal{O}_{X'_3})$$

be a complex first order thickenings of $(X, \mathcal{O}_X)$, see Remark 65.4.9. Let $(\mathcal{K}_i, c_i)$, $i = 1, 2, 3$ be pairs consisting of an $\mathcal{O}_X$-module $\mathcal{K}_i$ and a map $c_i : \mathcal{I}_i \otimes_{\mathcal{O}_X} \mathcal{F} \to \mathcal{K}_i$. Assume given a short exact sequence of $\mathcal{O}_X$-modules

$$0 \to \mathcal{K}_3 \to \mathcal{K}_2 \to \mathcal{K}_1 \to 0$$

such that

$$\begin{array}{ccc} \mathcal{I}_2 \otimes_{\mathcal{O}_X} \mathcal{F} & \xrightarrow{c_2} & \mathcal{K}_2 \\ \downarrow & & \downarrow \\ \mathcal{I}_1 \otimes_{\mathcal{O}_X} \mathcal{F} & \xrightarrow{c_1} & \mathcal{K}_1 \end{array} \quad \text{and} \quad \begin{array}{ccc} \mathcal{I}_3 \otimes_{\mathcal{O}_X} \mathcal{F} & \xrightarrow{c_3} & \mathcal{K}_3 \\ \downarrow & & \downarrow \\ \mathcal{I}_2 \otimes_{\mathcal{O}_X} \mathcal{F} & \xrightarrow{c_2} & \mathcal{K}_2 \end{array}$$

are commutative. Finally, assume given an extension

$$0 \to \mathcal{K}_2 \to \mathcal{F}_2' \to \mathcal{F} \to 0$$

as in (65.4.0.1) with $\mathcal{K} = \mathcal{K}_2$ of $\mathcal{O}_{X_2'}$-modules with $c_{\mathcal{F}_2'} = c_2$. In this situation we can apply the functoriality of Remark 65.4.7 to obtain an extension $\mathcal{F}_1'$ on X_1' (we'll describe $\mathcal{F}_1'$ in this special case below). By Remark 65.4.6 using the canonical splitting $\pi : (X_1', \mathcal{O}_{X_1'}) \to (X, \mathcal{O}_X)$ of Remark 65.4.9 we obtain $\xi_{\mathcal{F}_1'} \in \mathrm{Ext}^1_{\mathcal{O}_X}(\mathcal{F}, \mathcal{K}_1)$. Finally, we have the obstruction

$$o(\mathcal{F}, \mathcal{K}_3, c_3) \in \mathrm{Ext}^2_{\mathcal{O}_X}(\mathcal{F}, \mathcal{K}_3)$$

see Lemma 65.4.4. In this situation we **claim** that the canonical map

$$\partial : \mathrm{Ext}^1_{\mathcal{O}_X}(\mathcal{F}, \mathcal{K}_1) \longrightarrow \mathrm{Ext}^2_{\mathcal{O}_X}(\mathcal{F}, \mathcal{K}_3)$$

coming from the short exact sequence $0 \to \mathcal{K}_3 \to \mathcal{K}_2 \to \mathcal{K}_1 \to 0$ sends $\xi_{\mathcal{F}_1'}$ to the obstruction class $o(\mathcal{F}, \mathcal{K}_3, c_3)$.

To prove this claim choose an embedding $j : \mathcal{K}_3 \to \mathcal{K}$ where $\mathcal{K}$ is an injective $\mathcal{O}_X$-module. We can lift j to a map $j' : \mathcal{K}_2 \to \mathcal{K}$. Set $\mathcal{E}_2' = j'_* \mathcal{F}_2'$ equal to the pushout of $\mathcal{F}_2'$ by j' so that $c_{\mathcal{E}_2'} = j' \circ c_2$. Picture:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \mathcal{K}_2 & \longrightarrow & \mathcal{F}_2' & \longrightarrow & \mathcal{F} & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle j'} & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & \mathcal{K} & \longrightarrow & \mathcal{E}_2' & \longrightarrow & \mathcal{F} & \longrightarrow & 0 \end{array}$$

Set $\mathcal{E}_3' = \mathcal{E}_2'$ but viewed as an $\mathcal{O}_{X_3'}$-module via $\mathcal{O}_{X_3'} \to \mathcal{O}_{X_2'}$. Then $c_{\mathcal{E}_3'} = j \circ c_3$. The proof of Lemma 65.4.4 constructs $o(\mathcal{F}, \mathcal{K}_3, c_3)$ as the boundary of the class of the extension of $\mathcal{O}_X$-modules

$$0 \to \mathcal{K}/\mathcal{K}_3 \to \mathcal{E}_3'/\mathcal{K}_3 \to \mathcal{F} \to 0$$

On the other hand, note that $\mathcal{F}_1' = \mathcal{F}_2'/\mathcal{K}_3$ hence the class $\xi_{\mathcal{F}_1'}$ is the class of the extension

$$0 \to \mathcal{K}_2/\mathcal{K}_3 \to \mathcal{F}_2'/\mathcal{K}_3 \to \mathcal{F} \to 0$$

seen as a sequence of $\mathcal{O}_X$-modules using $\pi^\sharp$ where $\pi : (X_1', \mathcal{O}_{X_1'}) \to (X, \mathcal{O}_X)$ is the canonical splitting. Thus finally, the claim follows from the fact that we have a commutative diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \mathcal{K}_2/\mathcal{K}_3 & \longrightarrow & \mathcal{F}_2'/\mathcal{K}_3 & \longrightarrow & \mathcal{F} & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & \mathcal{K}/\mathcal{K}_3 & \longrightarrow & \mathcal{E}_3'/\mathcal{K}_3 & \longrightarrow & \mathcal{F} & \longrightarrow & 0 \end{array}$$

which is $\mathcal{O}_X$-linear (with the $\mathcal{O}_X$-module structures given above).

65.5. Infinitesimal deformations of modules on ringed spaces

Let $i : (X, \mathcal{O}_X) \to (X', \mathcal{O}_{X'})$ be a first order thickening of ringed spaces. We freely use the notation introduced in Section 65.3. Let $\mathcal{F}'$ be an $\mathcal{O}_{X'}$-module and set $\mathcal{F} = i^*\mathcal{F}'$. In this situation we have a short exact sequence

$$0 \to \mathcal{I}\mathcal{F}' \to \mathcal{F}' \to \mathcal{F} \to 0$$

of $\mathcal{O}_{X'}$-modules. Since $\mathcal{I}^2 = 0$ the $\mathcal{O}_{X'}$-module structure on $\mathcal{I}\mathcal{F}'$ comes from a unique $\mathcal{O}_X$-module structure. Thus the sequence above is an extension as in (65.4.0.1). As a special case, if $\mathcal{F}' = \mathcal{O}_{X'}$ we have $i^*\mathcal{O}_{X'} = \mathcal{O}_X$ and $\mathcal{I}\mathcal{O}_{X'} = \mathcal{I}$ and we recover the sequence of structure sheaves

$$0 \to \mathcal{I} \to \mathcal{O}_{X'} \to \mathcal{O}_X \to 0$$

Lemma 65.5.1. *Let $i : (X, \mathcal{O}_X) \to (X', \mathcal{O}_{X'})$ be a first order thickening of ringed spaces. Let $\mathcal{F}'$, $\mathcal{G}'$ be $\mathcal{O}_{X'}$-modules. Set $\mathcal{F} = i^*\mathcal{F}'$ and $\mathcal{G} = i^*\mathcal{G}'$. Let $\varphi : \mathcal{F} \to \mathcal{G}$ be an $\mathcal{O}_X$-linear map. The set of lifts of φ to an $\mathcal{O}_{X'}$-linear map $\varphi' : \mathcal{F}' \to \mathcal{G}'$ is, if nonempty, a principal homogeneous space under* $\mathrm{Hom}_{\mathcal{O}_X}(\mathcal{F}, \mathcal{I}\mathcal{G}')$.

Proof. This is a special case of Lemma 65.4.1 but we also give a direct proof. We have short exact sequences of modules

$$0 \to \mathcal{I} \to \mathcal{O}_{X'} \to \mathcal{O}_X \to 0 \quad \text{and} \quad 0 \to \mathcal{I}\mathcal{G}' \to \mathcal{G}' \to \mathcal{G} \to 0$$

and similarly for $\mathcal{F}'$. Since $\mathcal{I}$ has square zero the $\mathcal{O}_{X'}$-module structure on $\mathcal{I}$ and $\mathcal{I}\mathcal{G}'$ comes from a unique $\mathcal{O}_X$-module structure. It follows that

$$\mathrm{Hom}_{\mathcal{O}_{X'}}(\mathcal{F}', \mathcal{I}\mathcal{G}') = \mathrm{Hom}_{\mathcal{O}_X}(\mathcal{F}, \mathcal{I}\mathcal{G}') \quad \text{and} \quad \mathrm{Hom}_{\mathcal{O}_{X'}}(\mathcal{F}', \mathcal{G}) = \mathrm{Hom}_{\mathcal{O}_X}(\mathcal{F}, \mathcal{G})$$

The lemma now follows from the exact sequence

$$0 \to \mathrm{Hom}_{\mathcal{O}_{X'}}(\mathcal{F}', \mathcal{I}\mathcal{G}') \to \mathrm{Hom}_{\mathcal{O}_{X'}}(\mathcal{F}', \mathcal{G}') \to \mathrm{Hom}_{\mathcal{O}_{X'}}(\mathcal{F}', \mathcal{G})$$

see Homology, Lemma 12.5.8. □

Lemma 65.5.2. *Let (f, f') be a morphism of first order thickenings of ringed spaces as in Situation 65.3.1. Let $\mathcal{F}'$ be an $\mathcal{O}_{X'}$-module and set $\mathcal{F} = i^*\mathcal{F}'$. Assume that $\mathcal{F}$ is flat over S and that (f, f') is a strict morphism of thickenings (Definition 65.3.2). Then the following are equivalent*

1. *$\mathcal{F}'$ is flat over S', and*
2. *the canonical map $f^*\mathcal{J} \otimes_{\mathcal{O}_X} \mathcal{F} \to \mathcal{I}\mathcal{F}'$ is an isomorphism.*

Moreover, in this case the maps

$$f^*\mathcal{J} \otimes_{\mathcal{O}_X} \mathcal{F} \to \mathcal{I} \otimes_{\mathcal{O}_X} \mathcal{F} \to \mathcal{I}\mathcal{F}'$$

are isomorphisms.

Proof. The map $f^*\mathcal{J} \to \mathcal{I}$ is surjective as (f, f') is a strict morphism of thickenings. Hence the final statement is a consequence of (2).

Proof of the equivalence of (1) and (2). We may check these conditions at stalks. Let $x \in X \subset X'$ be a point with image $s = f(x) \in S \subset S'$. Set $A' = \mathcal{O}_{S',s}$, $B' = \mathcal{O}_{X',x}$, $A = \mathcal{O}_{S,s}$, and $B = \mathcal{O}_{X,x}$. Then $A = A'/J$ and $B = B'/I$ for some square zero ideals. Since (f, f') is a strict morphism of thickenings we have $I = JB'$. Let $M' = \mathcal{F}'_x$ and $M = \mathcal{F}_x$. Then M' is a B'-module and M is a B-module. Since

$\mathcal{F} = i^*\mathcal{F}'$ we see that the kernel of the surjection $M' \to M$ is $IM' = JM'$. Thus we have a short exact sequence

$$0 \to JM' \to M' \to M \to 0$$

Using Sheaves, Lemma 6.26.4 and Modules, Lemma 17.15.1 to identify stalks of pullbacks and tensor products we see that the stalk at x of the canonical map of the lemma is the map

$$(J \otimes_A B) \otimes_B M = J \otimes_A M = J \otimes_{A'} M' \longrightarrow JM'$$

The assumption that $\mathcal{F}$ is flat over S signifies that M is a flat A-module.

Assume (1). Flatness implies $\mathrm{Tor}_1^{A'}(M', A) = 0$ by Algebra, Lemma 10.70.7. This means $J \otimes_{A'} M' \to M'$ is injective by Algebra, Remark 10.70.8. Hence $J \otimes_A M \to JM'$ is an isomorphism.

Assume (2). Then $J \otimes_{A'} M' \to M'$ is injective. Hence $\mathrm{Tor}_1^{A'}(M', A) = 0$ by Algebra, Remark 10.70.8. Hence M' is flat over A' by Algebra, Lemma 10.93.8. □

Lemma 65.5.3. *Let (f, f') be a morphism of first order thickenings as in Situation 65.3.1. Let $\mathcal{F}'$, $\mathcal{G}'$ be $\mathcal{O}_{X'}$-modules and set $\mathcal{F} = i^*\mathcal{F}'$ and $\mathcal{G} = i^*\mathcal{G}'$. Let $\varphi : \mathcal{F} \to \mathcal{G}$ be an $\mathcal{O}_X$-linear map. Assume that $\mathcal{G}'$ is flat over S' and that (f, f') is a strict morphism of thickenings. The set of lifts of φ to an $\mathcal{O}_{X'}$-linear map $\varphi' : \mathcal{F}' \to \mathcal{G}'$ is, if nonempty, a principal homogeneous space under*

$$\mathrm{Hom}_{\mathcal{O}_X}(\mathcal{F}, \mathcal{G} \otimes_{\mathcal{O}_X} f^*\mathcal{J})$$

Proof. Combine Lemmas 65.5.1 and 65.5.2. □

Lemma 65.5.4. *Let $i : (X, \mathcal{O}_X) \to (X', \mathcal{O}_{X'})$ be a first order thickening of ringed spaces. Let $\mathcal{F}'$, $\mathcal{G}'$ be $\mathcal{O}_{X'}$-modules and set $\mathcal{F} = i^*\mathcal{F}'$ and $\mathcal{G} = i^*\mathcal{G}'$. Let $\varphi : \mathcal{F} \to \mathcal{G}$ be an $\mathcal{O}_X$-linear map. There exists an element*

$$o(\varphi) \in Ext^1_{\mathcal{O}_X}(Li^*\mathcal{F}', \mathcal{I}\mathcal{G}')$$

whose vanishing is a necessary and sufficient condition for the existence of a lift of φ to an $\mathcal{O}_{X'}$-linear map $\varphi' : \mathcal{F}' \to \mathcal{G}'$.

Proof. It is clear from the proof of Lemma 65.5.1 that the vanishing of the boundary of φ via the map

$$\mathrm{Hom}_{\mathcal{O}_X}(\mathcal{F}, \mathcal{G}) = \mathrm{Hom}_{\mathcal{O}_{X'}}(\mathcal{F}', \mathcal{G}) \longrightarrow \mathrm{Ext}^1_{\mathcal{O}_{X'}}(\mathcal{F}', \mathcal{I}\mathcal{G}')$$

is a necessary and sufficient condition for the existence of a lift. We conclude as

$$\mathrm{Ext}^1_{\mathcal{O}_{X'}}(\mathcal{F}', \mathcal{I}\mathcal{G}') = \mathrm{Ext}^1_{\mathcal{O}_X}(Li^*\mathcal{F}', \mathcal{I}\mathcal{G}')$$

the adjointness of $i_* = Ri_*$ and Li^* on the derived category (Cohomology, Lemma 20.27.1). □

Lemma 65.5.5. *Let (f, f') be a morphism of first order thickenings as in Situation 65.3.1. Let $\mathcal{F}'$, $\mathcal{G}'$ be $\mathcal{O}_{X'}$-modules and set $\mathcal{F} = i^*\mathcal{F}'$ and $\mathcal{G} = i^*\mathcal{G}'$. Let $\varphi : \mathcal{F} \to \mathcal{G}$ be an $\mathcal{O}_X$-linear map. Assume that $\mathcal{F}'$ and $\mathcal{G}'$ are flat over S' and that (f, f') is a strict morphism of thickenings. There exists an element*

$$o(\varphi) \in Ext^1_{\mathcal{O}_X}(\mathcal{F}, \mathcal{G} \otimes_{\mathcal{O}_X} f^*\mathcal{J})$$

whose vanishing is a necessary and sufficient condition for the existence of a lift of φ to an $\mathcal{O}_{X'}$-linear map $\varphi' : \mathcal{F}' \to \mathcal{G}'$.

First proof. This follows from Lemma 65.5.4 as we claim that under the assumptions of the lemma we have

$$\mathrm{Ext}^1_{\mathcal{O}_X}(Li^*\mathcal{F}', \mathcal{I}\mathcal{G}') = \mathrm{Ext}^1_{\mathcal{O}_X}(\mathcal{F}, \mathcal{G} \otimes_{\mathcal{O}_X} f^*\mathcal{J})$$

Namely, we have $\mathcal{I}\mathcal{G}' = \mathcal{G} \otimes_{\mathcal{O}_X} f^*\mathcal{J}$ by Lemma 65.5.2. On the other hand, observe that

$$H^{-1}(Li^*\mathcal{F}') = \mathrm{Tor}_1^{\mathcal{O}_{X'}}(\mathcal{F}', \mathcal{O}_X)$$

(local computation omitted). Using the short exact sequence

$$0 \to \mathcal{I} \to \mathcal{O}_{X'} \to \mathcal{O}_X \to 0$$

we see that this Tor_1 is computed by the kernel of the map $\mathcal{I} \otimes_{\mathcal{O}_X} \mathcal{F} \to \mathcal{I}\mathcal{F}'$ which is zero by the final assertion of Lemma 65.5.2. Thus $\tau_{\geq -1}Li^*\mathcal{F}' = \mathcal{F}$. On the other hand, we have

$$\mathrm{Ext}^1_{\mathcal{O}_X}(Li^*\mathcal{F}', \mathcal{I}\mathcal{G}') = \mathrm{Ext}^1_{\mathcal{O}_X}(\tau_{\geq -1}Li^*\mathcal{F}', \mathcal{I}\mathcal{G}')$$

by the dual of Derived Categories, Lemma 13.16.1. □

Second proof. We can apply Lemma 65.4.2 as follows. Note that $\mathcal{K} = \mathcal{I} \otimes_{\mathcal{O}_X} \mathcal{F}$ and $\mathcal{L} = \mathcal{I} \otimes_{\mathcal{O}_X} \mathcal{G}$ by Lemma 65.5.2, that $c_{\mathcal{F}'} = 1 \otimes 1$ and $c_{\mathcal{G}'} = 1 \otimes 1$ and taking $\psi = 1 \otimes \varphi$ the diagram of the lemma commutes. Thus $o(\varphi) = o(\varphi, 1 \otimes \varphi)$ works. □

Lemma 65.5.6. *Let (f, f') be a morphism of first order thickenings as in Situation 65.3.1. Let $\mathcal{F}$ be an $\mathcal{O}_X$-module. Assume (f, f') is a strict morphism of thickenings and $\mathcal{F}$ flat over S. If there exists a pair $(\mathcal{F}', \alpha)$ consisting of an $\mathcal{O}_{X'}$-module $\mathcal{F}'$ flat over S' and an isomorphism $\alpha : i^*\mathcal{F}' \to \mathcal{F}$, then the set of isomorphism classes of such pairs is principal homogeneous under $Ext^1_{\mathcal{O}_X}(\mathcal{F}, \mathcal{I} \otimes_{\mathcal{O}_X} \mathcal{F})$.*

Proof. If we assume there exists one such module, then the canonical map

$$f^*\mathcal{J} \otimes_{\mathcal{O}_X} \mathcal{F} \to \mathcal{I} \otimes_{\mathcal{O}_X} \mathcal{F}$$

is an isomorphism by Lemma 65.5.2. Apply Lemma 65.4.3 with $\mathcal{K} = \mathcal{I} \otimes_{\mathcal{O}_X} \mathcal{F}$ and $c = 1$. By Lemma 65.5.2 the corresponding extensions $\mathcal{F}'$ are all flat over S'. □

Lemma 65.5.7. *Let (f, f') be a morphism of first order thickenings as in Situation 65.3.1. Let $\mathcal{F}$ be an $\mathcal{O}_X$-module. Assume (f, f') is a strict morphism of thickenings and $\mathcal{F}$ flat over S. There exists an $\mathcal{O}_{X'}$-module $\mathcal{F}'$ flat over S' with $i^*\mathcal{F}' \cong \mathcal{F}$, if and only if*

1. *the canonical map $f^*\mathcal{J} \otimes_{\mathcal{O}_X} \mathcal{F} \to \mathcal{I} \otimes_{\mathcal{O}_X} \mathcal{F}$ is an isomorphism, and*
2. *the class $o(\mathcal{F}, \mathcal{I} \otimes_{\mathcal{O}_X} \mathcal{F}, 1) \in Ext^2_{\mathcal{O}_X}(\mathcal{F}, \mathcal{I} \otimes_{\mathcal{O}_X} \mathcal{F})$ of Lemma 65.4.4 is zero.*

Proof. This follows immediately from the characterization of $\mathcal{O}_{X'}$-modules flat over S' of Lemma 65.5.2 and Lemma 65.4.4. □

65.6. Application to flat modules on flat thickenings of ringed spaces

Consider a commutative diagram

$$\begin{array}{ccc} (X, \mathcal{O}_X) & \xrightarrow[i]{} & (X', \mathcal{O}_{X'}) \\ {\scriptstyle f}\downarrow & & \downarrow{\scriptstyle f'} \\ (S, \mathcal{O}_S) & \xrightarrow{t} & (S', \mathcal{O}_{S'}) \end{array}$$

of ringed spaces whose horizontal arrows are first order thickenings as in Situation 65.3.1. Set $\mathcal{I} = \mathrm{Ker}(i^\sharp) \subset \mathcal{O}_{X'}$ and $\mathcal{J} = \mathrm{Ker}(t^\sharp) \subset \mathcal{O}_{S'}$. Let $\mathcal{F}$ be an $\mathcal{O}_X$-module. Assume that

(1) (f, f') is a strict morphism of thickenings,
(2) f' is flat, and
(3) $\mathcal{F}$ is flat over S.

Note that (1) + (2) imply that $\mathcal{I} = f^*\mathcal{J}$ (apply Lemma 65.5.2 to $\mathcal{O}_{X'}$). The theory of the preceding section is especially nice under these assumptions. We summarize the results already obtained in the following lemma.

Lemma 65.6.1. *In the situation above.*

(1) *There exists an $\mathcal{O}_{X'}$-module $\mathcal{F}'$ flat over S' with $i^*\mathcal{F}' \cong \mathcal{F}$, if and only if the class $o(\mathcal{F}, f^*\mathcal{J} \otimes_{\mathcal{O}_X} \mathcal{F}, 1) \in Ext^2_{\mathcal{O}_X}(\mathcal{F}, f^*\mathcal{J} \otimes_{\mathcal{O}_X} \mathcal{F})$ of Lemma 65.4.4 is zero.*
(2) *If such a module exists, then the set of isomorphism classes of lifts is principal homogeneous under $Ext^1_{\mathcal{O}_X}(\mathcal{F}, f^*\mathcal{J} \otimes_{\mathcal{O}_X} \mathcal{F})$.*
(3) *Given a lift $\mathcal{F}'$, the set of automorphisms of $\mathcal{F}'$ which pull back to $id_{\mathcal{F}}$ is canonically isomorphic to $Ext^0_{\mathcal{O}_X}(\mathcal{F}, f^*\mathcal{J} \otimes_{\mathcal{O}_X} \mathcal{F})$.*

Proof. Part (1) follows from Lemma 65.5.7 as we have seen above that $\mathcal{I} = f^*\mathcal{J}$. Part (2) follows from Lemma 65.5.6. Part (3) follows from Lemma 65.5.3. □

Situation 65.6.2. Let $f : (X, \mathcal{O}_X) \to (S, \mathcal{O}_S)$ be a morphism of ringed spaces. Consider a commutative diagram

$$\begin{array}{ccccc} (X'_1, \mathcal{O}'_1) & \xrightarrow{h} & (X'_2, \mathcal{O}'_2) & \longrightarrow & (X'_3, \mathcal{O}'_3) \\ \downarrow{\scriptstyle f'_1} & & \downarrow{\scriptstyle f'_2} & & \downarrow{\scriptstyle f'_3} \\ (S'_1, \mathcal{O}_{S'_1}) & \longrightarrow & (S'_2, \mathcal{O}_{S'_2}) & \longrightarrow & (S'_3, \mathcal{O}_{S'_3}) \end{array}$$

where (a) the top row is a short exact sequence of first order thickenings of X, (b) the lower row is a short exact sequence of first order thickenings of S, (c) each f'_i restricts to f, (d) each pair (f, f'_i) is a strict morphism of thickenings, and (e) each f'_i is flat. Finally, let $\mathcal{F}'_2$ be an $\mathcal{O}'_2$-module flat over S'_2 and set $\mathcal{F} = \mathcal{F}'_2|_X$. Let $\pi : X'_1 \to X$ be the canonical splitting (Remark 65.4.9).

Lemma 65.6.3. *In Situation 65.6.2 the modules $\pi^*\mathcal{F}$ and $h^*\mathcal{F}'_2$ are $\mathcal{O}'_1$-modules flat over S'_1 restricting to $\mathcal{F}$ on X. Their difference (Lemma 65.6.1) is an element θ of $Ext^1_{\mathcal{O}_X}(\mathcal{F}, f^*\mathcal{J}_1 \otimes_{\mathcal{O}_X} \mathcal{F})$ whose boundary in $Ext^2_{\mathcal{O}_X}(\mathcal{F}, f^*\mathcal{J}_3 \otimes_{\mathcal{O}_X} \mathcal{F})$ equals the obstruction (Lemma 65.6.1) to lifting $\mathcal{F}$ to an $\mathcal{O}'_3$-module flat over S'_3.*

Proof. Note that both $\pi^*\mathcal{F}$ and $h^*\mathcal{F}'_2$ restrict to $\mathcal{F}$ on X and that the kernels of $\pi^*\mathcal{F} \to \mathcal{F}$ and $h^*\mathcal{F}'_2 \to \mathcal{F}$ are given by $f^*\mathcal{J}_1 \otimes_{\mathcal{O}_X} \mathcal{F}$. Hence flatness by Lemma 65.5.2. Taking the boundary makes sense as the sequence of modules

$$0 \to f^*\mathcal{J}_3 \otimes_{\mathcal{O}_X} \mathcal{F} \to f^*\mathcal{J}_2 \otimes_{\mathcal{O}_X} \mathcal{F} \to f^*\mathcal{J}_1 \otimes_{\mathcal{O}_X} \mathcal{F} \to 0$$

is short exact due to the assumptions in Situation 65.6.2 and the fact that $\mathcal{F}$ is flat over S. The statement on the obstruction class is a direct translation of the result of Remark 65.4.10 to this particular situation. □

65.7. Deformations of ringed spaces and the naive cotangent complex

In this section we use the naive cotangent complex to do a little bit of deformation theory. We start with a first order thickening $t : (S, \mathcal{O}_S) \to (S', \mathcal{O}_{S'})$ of ringed spaces. We denote $\mathcal{J} = \mathrm{Ker}(t^\sharp)$ and we identify the underlying topological spaces of S and S'. Moreover we assume given a morphism of ringed spaces $f : (X, \mathcal{O}_X) \to (S, \mathcal{O}_S)$, an $\mathcal{O}_X$-module $\mathcal{G}$, and an f-map $c : \mathcal{J} \to \mathcal{G}$ of sheaves of modules (Sheaves, Definition 6.21.7 and Section 6.26). In this section we ask ourselves whether we can find the question mark fitting into the following diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \mathcal{G} & \longrightarrow & ? & \longrightarrow & \mathcal{O}_X & \longrightarrow & 0 \\ & & \uparrow{\scriptstyle c} & & \uparrow & & \uparrow & & \\ 0 & \longrightarrow & \mathcal{J} & \longrightarrow & \mathcal{O}_{S'} & \longrightarrow & \mathcal{O}_S & \longrightarrow & 0 \end{array} \tag{65.7.0.1}$$

(where the vertical arrows are f-maps) and moreover how unique the solution is (if it exists). More precisely, we look for a first order thickening $i : (X, \mathcal{O}_X) \to (X', \mathcal{O}_{X'})$ and a morphism of thickenings (f, f') as in (65.3.1.1) where $\mathrm{Ker}(i^\sharp)$ is identified with $\mathcal{G}$ such that $(f')^\sharp$ induces the given map c. We will say X' is a *solution* to (65.7.0.1).

Lemma 65.7.1. *Assume given a commutative diagram of morphisms ringed spaces*

$$\begin{array}{ccccc} & & (X_2, \mathcal{O}_{X_2}) & \xrightarrow[i_2]{} & (X_2', \mathcal{O}_{X_2'}) \\ & \swarrow^{g} & \downarrow{\scriptstyle f_2} & & \downarrow{\scriptstyle f_2'} \\ & & (S_2, \mathcal{O}_{S_2}) & \xrightarrow{t_2} & (S_2', \mathcal{O}_{S_2'}) \\ (X_1, \mathcal{O}_{X_1}) & \xrightarrow[i_1]{} & (X_1', \mathcal{O}_{X_1'}) & \swarrow & \\ \downarrow{\scriptstyle f_1} & \swarrow & \downarrow{\scriptstyle f_1'} & & \\ (S_1, \mathcal{O}_{S_1}) & \xrightarrow{t_1} & (S_1', \mathcal{O}_{S_1'}) & & \end{array} \tag{65.7.1.1}$$

whose horizontal arrows are first order thickenings. Set $\mathcal{G}_j = Ker(i_j^\sharp)$ and assume given a g-map $\nu : \mathcal{G}_1 \to \mathcal{G}_2$ of modules giving rise to the commutative diagram

$$\begin{array}{ccccccccccc} 0 & \longrightarrow & \mathcal{G}_2 & \longrightarrow & \mathcal{O}_{X_2'} & \longrightarrow & \mathcal{O}_{X_2} & \longrightarrow & 0 & & \\ & & \uparrow{\scriptstyle c_2} & & \uparrow & & \uparrow & & & & \\ 0 & \longrightarrow & \mathcal{J}_2 & \longrightarrow & \mathcal{O}_{S_2'} & \longrightarrow & \mathcal{O}_{S_2} & \longrightarrow & 0 & & \\ & \nearrow & \uparrow & \nearrow & \uparrow & \nearrow & \uparrow & & & & \\ 0 & \longrightarrow & \mathcal{G}_1 & \longrightarrow & \mathcal{O}_{X_1'} & \longrightarrow & \mathcal{O}_{X_1} & \longrightarrow & 0 & & \\ & & \uparrow{\scriptstyle c_1} & \nearrow & \uparrow & \nearrow & \uparrow & \nearrow & & & \\ 0 & \longrightarrow & \mathcal{J}_1 & \longrightarrow & \mathcal{O}_{S_1'} & \longrightarrow & \mathcal{O}_{S_1} & \longrightarrow & 0 & & \end{array} \tag{65.7.1.2}$$

with front and back solutions to (65.7.0.1).

(1) *There exist a canonical element in* $Ext^1_{\mathcal{O}_{X_2}}(Lg^* NL_{X_1/S_1}, \mathcal{G}_2)$ *whose vanishing is a necessary and sufficient condition for the existence of a morphism of ringed spaces* $X_2' \to X_1'$ *fitting into (65.7.1.1) compatibly with* ν.
(2) *If there exists a morphism* $X_2' \to X_1'$ *fitting into (65.7.1.1) compatibly with* ν *the set of all such morphisms is a principal homogeneous space under*

$$\mathrm{Hom}_{\mathcal{O}_{X_1}}(\Omega_{X_1/S_1}, g_*\mathcal{G}_2) = \mathrm{Hom}_{\mathcal{O}_{X_2}}(g^*\Omega_{X_1/S_1}, \mathcal{G}_2) = Ext^0_{\mathcal{O}_{X_2}}(Lg^* NL_{X_1/S_1}, \mathcal{G}_2).$$

Proof. The equalities in the last statement of the lemma are immediate from the definitions. Thus we will work with the groups $\mathrm{Ext}^k_{\mathcal{O}_{X_2}}(Lg^* NL_{X_1/S_1}, \mathcal{G}_2)$, $k = 0, 1$ in the rest of the proof. We first argue that we can reduce to the case where the underlying topological spaces of all ringed spaces in the lemma is the same.

To do this, observe that $g^{-1} NL_{X_1/S_1}$ is equal to the naive cotangent complex of the homomorphism of sheaves of rings $g^{-1}f_1^{-1}\mathcal{O}_{S_1} \to g^{-1}\mathcal{O}_{X_1}$, see Modules, Lemma 17.23.6. Moreover, the degree 0 term of NL_{X_1/S_1} is a flat $\mathcal{O}_{X_1}$-module, hence the canonical map

$$Lg^* NL_{X_1/S_1} \longrightarrow g^{-1} NL_{X_1/S_1} \otimes_{g^{-1}\mathcal{O}_{X_1}} \mathcal{O}_{X_2}$$

induces an isomorphism on cohomology sheaves in degrees 0 and -1. Thus we may replace the Ext groups of the lemma with

$$\mathrm{Ext}^k_{g^{-1}\mathcal{O}_{X_1}}(g^{-1} NL_{X_1/S_1}, \mathcal{G}_2) = \mathrm{Ext}^k_{g^{-1}\mathcal{O}_{X_1}}(NL_{g^{-1}\mathcal{O}_{X_1}/g^{-1}f_1^{-1}\mathcal{O}_{S_1}}, \mathcal{G}_2)$$

The set of morphism of ringed spaces $X_2' \to X_1'$ fitting into (65.7.1.1) compatibly with ν is in one-to-one bijection with the set of homomorphisms of $g^{-1}f_1^{-1}\mathcal{O}_{S_1'}$-algebras $g^{-1}\mathcal{O}_{X_1'} \to \mathcal{O}_{X_2'}$ which are compatible with $f^\sharp$ and ν. In this way we see that we may assume we have a diagram (65.7.1.2) of sheaves on X and we are looking to find a homomorphism of sheaves of rings $\mathcal{O}_{X_1'} \to \mathcal{O}_{X_2'}$ fitting into it.

In the rest of the proof of the lemma we assume all underlying topological spaces are the same, i.e., we have a diagram (65.7.1.2) of sheaves on a space X and we are looking for homomorphisms of sheaves of rings $\mathcal{O}_{X_1'} \to \mathcal{O}_{X_2'}$ fitting into it. As ext groups we will use $\mathrm{Ext}^k_{\mathcal{O}_{X_1}}(NL_{\mathcal{O}_{X_1}/\mathcal{O}_{S_1}}, \mathcal{G}_2)$, $k = 0, 1$.

Step 1. Construction of the obstruction class. Consider the sheaf of sets

$$\mathcal{E} = \mathcal{O}_{X_1'} \times_{\mathcal{O}_{X_2}} \mathcal{O}_{X_2'}$$

This comes with a surjective map $\alpha : \mathcal{E} \to \mathcal{O}_{X_1}$ and hence we can use $NL(\alpha)$ instead of $NL_{\mathcal{O}_{X_1}/\mathcal{O}_{S_1}}$, see Modules, Lemma 17.24.2. Set

$$\mathcal{I}' = \mathrm{Ker}(\mathcal{O}_{S_1'}[\mathcal{E}] \to \mathcal{O}_{X_1}) \quad \text{and} \quad \mathcal{I} = \mathrm{Ker}(\mathcal{O}_{S_1}[\mathcal{E}] \to \mathcal{O}_{X_1})$$

There is a surjection $\mathcal{I}' \to \mathcal{I}$ whose kernel is $\mathcal{J}_1\mathcal{O}_{S_1'}[\mathcal{E}]$. We obtain two homomorphisms of $\mathcal{O}_{S_2'}$-algebras

$$a : \mathcal{O}_{S_1'}[\mathcal{E}] \to \mathcal{O}_{X_1'} \quad \text{and} \quad b : \mathcal{O}_{S_1'}[\mathcal{E}] \to \mathcal{O}_{X_2'}$$

which induce maps $a|_{\mathcal{I}'} : \mathcal{I}' \to \mathcal{G}_1$ and $b|_{\mathcal{I}'} : \mathcal{I}' \to \mathcal{G}_2$. Both a and b annihilate $(\mathcal{I}')^2$. Moreover a and b agree on $\mathcal{J}_1\mathcal{O}_{S_1'}[\mathcal{E}]$ as maps into $\mathcal{G}_2$ because the left hand square of (65.7.1.2) is commutative. Thus the difference $b|_{\mathcal{I}'} - \nu \circ a|_{\mathcal{I}'}$ induces a well defined $\mathcal{O}_{X_1}$-linear map

$$\xi : \mathcal{I}/\mathcal{I}^2 \longrightarrow \mathcal{G}_2$$

which sends the class of a local section f of $\mathcal{I}$ to $a(f') - \nu(b(f'))$ where f' is a lift of f to a local section of $\mathcal{I}'$. We let $[\xi] \in \mathrm{Ext}^1_{\mathcal{O}_{X_1}}(NL(\alpha), \mathcal{G}_2)$ be the image (see below).

Step 2. Vanishing of $[\xi]$ is necessary. Let us write $\Omega = \Omega_{\mathcal{O}_{S_1}[\mathcal{E}]/\mathcal{O}_{S_1}} \otimes_{\mathcal{O}_{S_1}[\mathcal{E}]} \mathcal{O}_{X_1}$. Observe that $NL(\alpha) = (\mathcal{I}/\mathcal{I}^2 \to \Omega)$ fits into a distinguished triangle

$$\Omega[0] \to NL(\alpha) \to \mathcal{I}/\mathcal{I}^2[1] \to \Omega[1]$$

Thus we see that $[\xi]$ is zero if and only if ξ is a composition $\mathcal{I}/\mathcal{I}^2 \to \Omega \to \mathcal{G}_2$ for some map $\Omega \to \mathcal{G}_2$. Suppose there exists a homomorphisms of sheaves of rings $\varphi : \mathcal{O}_{X'_1} \to \mathcal{O}_{X'_2}$ fitting into (65.7.1.2). In this case consider the map $\mathcal{O}_{S'_1}[\mathcal{E}] \to \mathcal{G}_2$, $f' \mapsto b(f') - \varphi(a(f'))$. A calculation shows this annihilates $\mathcal{J}_1\mathcal{O}_{S'_1}[\mathcal{E}]$ and induces a derivation $\mathcal{O}_{S_1}[\mathcal{E}] \to \mathcal{G}_2$. The resulting linear map $\Omega \to \mathcal{G}_2$ witnesses the fact that $[\xi] = 0$ in this case.

Step 3. Vanishing of $[\xi]$ is sufficient. Let $\theta : \Omega \to \mathcal{G}_2$ be a $\mathcal{O}_{X_1}$-linear map such that ξ is equal to $\theta \circ (\mathcal{I}/\mathcal{I}^2 \to \Omega)$. Then a calculation shows that

$$b + \theta \circ d : \mathcal{O}_{S'_1}[\mathcal{E}] \to \mathcal{O}_{X'_2}$$

annihilates $\mathcal{I}'$ and hence defines a map $\mathcal{O}_{X'_1} \to \mathcal{O}_{X'_2}$ fitting into (65.7.1.2).

Proof of (2) in the special case above. Omitted. Hint: This is exactly the same as the proof of (2) of Lemma 65.2.1. □

Lemma 65.7.2. *Let X be a topological space. Let $\mathcal{A} \to \mathcal{B}$ be a homomorphism of sheaves of rings. Let $\mathcal{G}$ be a $\mathcal{B}$-module. Let $\xi \in Ext^1_{\mathcal{B}}(NL_{\mathcal{B}/\mathcal{A}}, \mathcal{G})$. There exists a map of sheaves of sets $\alpha : \mathcal{E} \to \mathcal{B}$ such that $\xi \in Ext^1_{\mathcal{B}}(NL(\alpha), \mathcal{G})$ is the class of a map $\mathcal{I}/\mathcal{I}^2 \to \mathcal{G}$ (see proof for notation).*

Proof. Recall that given $\alpha : \mathcal{E} \to \mathcal{B}$ such that $\mathcal{A}[\mathcal{E}] \to \mathcal{B}$ is surjective with kernel $\mathcal{I}$ the complex $NL(\alpha) = (\mathcal{I}/\mathcal{I}^2 \to \Omega_{\mathcal{A}[\mathcal{E}]/\mathcal{A}} \otimes_{\mathcal{A}[\mathcal{E}]} \mathcal{B})$ is canonically isomorphic to $NL_{\mathcal{B}/\mathcal{A}}$, see Modules, Lemma 17.24.2. Observe moreover, that $\Omega = \Omega_{\mathcal{A}[\mathcal{E}]/\mathcal{A}} \otimes_{\mathcal{A}[\mathcal{E}]} \mathcal{B}$ is the sheaf associated to the presheaf $U \mapsto \bigoplus_{e \in \mathcal{E}(U)} \mathcal{B}(U)$. In other words, Ω is the free $\mathcal{B}$-module on the sheaf of sets $\mathcal{E}$ and in particular there is a canonical map $\mathcal{E} \to \Omega$.

Having said this, pick some $\mathcal{E}$ (for example $\mathcal{E} = \mathcal{B}$ as in the definition of the naive cotangent complex). The obstruction to writing ξ as the class of a map $\mathcal{I}/\mathcal{I}^2 \to \mathcal{G}$ is an element in $\mathrm{Ext}^1_{\mathcal{B}}(\Omega, \mathcal{G})$. Say this is represented by the extension $0 \to \mathcal{G} \to \mathcal{H} \to \Omega \to 0$ of $\mathcal{B}$-modules. Consider the sheaf of sets $\mathcal{E}' = \mathcal{E} \times_\Omega \mathcal{H}$ which comes with an induced map $\alpha' : \mathcal{E}' \to \mathcal{B}$. Let $\mathcal{I}' = \mathrm{Ker}(\mathcal{A}[\mathcal{E}'] \to \mathcal{B})$ and $\Omega' = \Omega_{\mathcal{A}[\mathcal{E}']/\mathcal{A}} \otimes_{\mathcal{A}[\mathcal{E}']} \mathcal{B}$. The pullback of ξ under the quasi-isomorphism $NL(\alpha') \to NL(\alpha)$ maps to zero in $\mathrm{Ext}^1_{\mathcal{B}}(\Omega', \mathcal{G})$ because the pullback of the extension $\mathcal{H}$ by the map $\Omega' \to \Omega$ is split as Ω' is the free $\mathcal{B}$-module on the sheaf of sets $\mathcal{E}'$ and since by construction there is a commutative diagram

$$\begin{array}{ccc} \mathcal{E}' & \longrightarrow & \mathcal{E} \\ \downarrow & & \downarrow \\ \mathcal{H} & \longrightarrow & \Omega \end{array}$$

This finishes the proof. □

Lemma 65.7.3. *If there exists a solution to (65.7.0.1), then the set of isomorphism classes of solutions is principal homogeneous under $Ext^1_{\mathcal{O}_X}(NL_{X/S}, \mathcal{G})$.*

Proof. We observe right away that given two solutions X'_1 and X'_2 to (65.7.0.1) we obtain by Lemma 65.7.1 an obstruction element $o(X'_1, X'_2) \in \mathrm{Ext}^1_{\mathcal{O}_X}(NL_{X/S}, \mathcal{G})$ to the existence of a map $X'_1 \to X'_2$. Clearly, this element is the obstruction to the existence of an isomorphism, hence separates the isomorphism classes. To finish the proof it therefore suffices to show that given a solution X' and an element $\xi \in \mathrm{Ext}^1_{\mathcal{O}_X}(NL_{X/S}, \mathcal{G})$ we can find a second solution X'_ξ such that $o(X', X'_\xi) = \xi$.

Pick $\alpha : \mathcal{E} \to \mathcal{O}_X$ as in Lemma 65.7.2 for the class ξ. Consider the surjection $f^{-1}\mathcal{O}_S[\mathcal{E}] \to \mathcal{O}_X$ with kernel $\mathcal{I}$ and corresponding naive cotangent complex $NL(\alpha) = (\mathcal{I}/\mathcal{I}^2 \to \Omega_{f^{-1}\mathcal{O}_S[\mathcal{E}]/f^{-1}\mathcal{O}_S} \otimes_{f^{-1}\mathcal{O}_S[\mathcal{E}]} \mathcal{O}_X)$. By the lemma ξ is the class of a morphism $\delta : \mathcal{I}/\mathcal{I}^2 \to \mathcal{G}$. After replacing $\mathcal{E}$ by $\mathcal{E} \times_{\mathcal{O}_X} \mathcal{O}_{X'}$ we may also assume that α factors through a map $\alpha' : \mathcal{E} \to \mathcal{O}_{X'}$.

These choices determine an $f^{-1}\mathcal{O}_{S'}$-algebra map $\varphi : \mathcal{O}_{S'}[\mathcal{E}] \to \mathcal{O}_{X'}$. Let $\mathcal{I}' = \mathrm{Ker}(\varphi)$. Observe that φ induces a map $\varphi|_{\mathcal{I}'} : \mathcal{I}' \to \mathcal{G}$ and that $\mathcal{O}_{X'}$ is the pushout, as in the following diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \mathcal{G} & \longrightarrow & \mathcal{O}_{X'} & \longrightarrow & \mathcal{O}_X & \longrightarrow & 0 \\ & & \uparrow{\scriptstyle \varphi|_{\mathcal{I}'}} & & \uparrow & & \uparrow{\scriptstyle =} & & \\ 0 & \longrightarrow & \mathcal{I}' & \longrightarrow & f^{-1}\mathcal{O}_{S'}[\mathcal{E}] & \longrightarrow & \mathcal{O}_X & \longrightarrow & 0 \end{array}$$

Let $\psi : \mathcal{I}' \to \mathcal{G}$ be the sum of the map $\varphi|_{\mathcal{I}'}$ and the composition

$$\mathcal{I}' \to \mathcal{I}'/(\mathcal{I}')^2 \to \mathcal{I}/\mathcal{I}^2 \xrightarrow{\delta} \mathcal{G}.$$

Then the pushout along ψ is an other ring extension $\mathcal{O}_{X'_\xi}$ fitting into a diagram as above. A calculation (omitted) shows that $o(X', X'_\xi) = \xi$ as desired. □

Lemma 65.7.4. *Let $(S, \mathcal{O}_S)$ be a ringed space and let $\mathcal{J}$ be an $\mathcal{O}_S$-module.*

1. *The set of extensions of sheaves of rings $0 \to \mathcal{J} \to \mathcal{O}_{S'} \to \mathcal{O}_S \to 0$ where $\mathcal{J}$ is an ideal of square zero is canonically bijective to $Ext^1_{\mathcal{O}_S}(NL_{S/\mathbf{Z}}, \mathcal{J})$.*
2. *Given a morphism of ringed spaces $f : (X, \mathcal{O}_X) \to (S, \mathcal{O}_S)$, an $\mathcal{O}_X$-module $\mathcal{G}$, an f-map $c : \mathcal{J} \to \mathcal{G}$, and given extensions of sheaves of rings with square zero kernels:*
   (a) *$0 \to \mathcal{J} \to \mathcal{O}_{S'} \to \mathcal{O}_S \to 0$ corresponding to $\alpha \in Ext^1_{\mathcal{O}_S}(NL_{S/\mathbf{Z}}, \mathcal{J})$,*
   (b) *$0 \to \mathcal{G} \to \mathcal{O}_{X'} \to \mathcal{O}_X \to 0$ corresponding to $\beta \in Ext^1_{\mathcal{O}_X}(NL_{X/\mathbf{Z}}, \mathcal{G})$*
   then there is a morphism $X' \to S'$ fitting into a diagram (65.7.0.1) if and only if β and α map to the same element of $Ext^1_{\mathcal{O}_X}(Lf^* NL_{S/\mathbf{Z}}, \mathcal{G})$.

Proof. To prove this we apply the previous results where we work over the base ringed space $(*, \mathbf{Z})$ with trivial thickening. Part (1) follows from Lemma 65.7.3 and the fact that there exists a solution, namely $\mathcal{J} \oplus \mathcal{O}_S$. Part (2) follows from Lemma 65.7.1 and a compatibility between the constructions in the proofs of Lemmas 65.7.3 and 65.7.1 whose statement and proof we omit. □

65.8. Thickenings of ringed topoi

This section is the analogue of Section 65.3 for ringed topoi. In the following few sections we will use the following notions:

(1) A sheaf of ideals $\mathcal{I} \subset \mathcal{O}'$ on a ringed topos $(Sh(\mathcal{D}), \mathcal{O}')$ is *locally nilpotent* if any local section of $\mathcal{I}$ is locally nilpotent.
(2) A *thickening* of ringed topoi is a morphism $i : (Sh(\mathcal{C}), \mathcal{O}) \to (Sh(\mathcal{D}), \mathcal{O}')$ of ringed topoi such that
   (a) i_* is an equivalence $Sh(\mathcal{C}) \to Sh(\mathcal{D})$,
   (b) the map $i^\sharp : \mathcal{O}' \to i_*\mathcal{O}$ is surjective, and
   (c) the kernel of $i^\sharp$ is a locally nilpotent sheaf of ideals.
(3) A *first order thickening* of ringed topoi is a thickening $i : (Sh(\mathcal{C}), \mathcal{O}) \to (Sh(\mathcal{D}), \mathcal{O}')$ of ringed topoi such that $\mathrm{Ker}(i^\sharp)$ has square zero.
(4) It is clear how to define *morphisms of thickenings of ringed topoi*, *morphisms of thickenings of ringed topoi over a base ringed topos*, etc.

If $i : (Sh(\mathcal{C}), \mathcal{O}) \to (Sh(\mathcal{D}), \mathcal{O}')$ is a thickening of ringed topoi then we identify the underlying topoi and think of $\mathcal{O}$, $\mathcal{O}'$, and $\mathcal{I} = \mathrm{Ker}(i^\sharp)$ as sheaves on $\mathcal{C}$. We obtain a short exact sequence

$$0 \to \mathcal{I} \to \mathcal{O}' \to \mathcal{O} \to 0$$

of $\mathcal{O}'$-modules. By Modules on Sites, Lemma 18.25.1 the category of $\mathcal{O}$-modules is equivalent to the category of $\mathcal{O}'$-modules annihilated by $\mathcal{I}$. In particular, if i is a first order thickening, then $\mathcal{I}$ is a $\mathcal{O}$-module.

Situation 65.8.1. A morphism of thickenings of ringed topoi (f, f') is given by a commutative diagram

$$\begin{array}{ccc} (Sh(\mathcal{C}), \mathcal{O}) & \xrightarrow{i} & (Sh(\mathcal{D}), \mathcal{O}') \\ {\scriptstyle f}\downarrow & & \downarrow{\scriptstyle f'} \\ (Sh(\mathcal{B}), \mathcal{O}_\mathcal{B}) & \xrightarrow{t} & (Sh(\mathcal{B}'), \mathcal{O}_{\mathcal{B}'}) \end{array} \tag{65.8.1.1}$$

of ringed topoi whose horizontal arrows are thickenings. In this situation we set $\mathcal{I} = \mathrm{Ker}(i^\sharp) \subset \mathcal{O}'$ and $\mathcal{J} = \mathrm{Ker}(t^\sharp) \subset \mathcal{O}_{\mathcal{B}'}$. As $f = f'$ on underlying topoi we will identify the pullback functors f^{-1} and $(f')^{-1}$. Observe that $(f')^\sharp : f^{-1}\mathcal{O}_{\mathcal{B}'} \to \mathcal{O}'$ induces in particular a map $f^{-1}\mathcal{J} \to \mathcal{I}$ and therefore a map of $\mathcal{O}'$-modules

$$(f')^*\mathcal{J} \longrightarrow \mathcal{I}$$

If i and t are first order thickenings, then $(f')^*\mathcal{J} = f^*\mathcal{J}$ and the map above becomes a map $f^*\mathcal{J} \to \mathcal{I}$.

Definition 65.8.2. In Situation 65.8.1 we say that (f, f') is a *strict morphism of thickenings* if the map $(f')^*\mathcal{J} \longrightarrow \mathcal{I}$ is surjective.

65.9. Modules on first order thickenings of ringed topoi

In this section we discuss some preliminaries to the deformation theory of modules. Let $i : (Sh(\mathcal{C}, \mathcal{O}) \to (Sh(\mathcal{D}), \mathcal{O}')$ be a first order thickening of ringed topoi. We will freely use the notation introduced in Section 65.8, in particular we will identify the underlying topological topoi. In this section we consider short exact sequences

$$0 \to \mathcal{K} \to \mathcal{F}' \to \mathcal{F} \to 0 \tag{65.9.0.1}$$

of $\mathcal{O}'$-modules, where $\mathcal{F}$, $\mathcal{K}$ are $\mathcal{O}$-modules and $\mathcal{F}'$ is an $\mathcal{O}'$-module. In this situation we have a canonical $\mathcal{O}$-module map

$$c_{\mathcal{F}'} : \mathcal{I} \otimes_\mathcal{O} \mathcal{F} \longrightarrow \mathcal{K}$$

where $\mathcal{I} = \mathrm{Ker}(i^\sharp)$. Namely, given local sections f of $\mathcal{I}$ and s of $\mathcal{F}$ we set $c_{\mathcal{F}'}(f \otimes s) = fs'$ where s' is a local section of $\mathcal{F}'$ lifting s.

Lemma 65.9.1. *Let $i : (Sh(\mathcal{C}), \mathcal{O}) \to (Sh(\mathcal{D}), \mathcal{O}')$ be a first order thickening of ringed topoi. Assume given extensions*

$$0 \to \mathcal{K} \to \mathcal{F}' \to \mathcal{F} \to 0 \quad \text{and} \quad 0 \to \mathcal{L} \to \mathcal{G}' \to \mathcal{G} \to 0$$

as in (65.9.0.1) and maps $\varphi : \mathcal{F} \to \mathcal{G}$ and $\psi : \mathcal{K} \to \mathcal{L}$.

(1) *If there exists an $\mathcal{O}'$-module map $\varphi' : \mathcal{F}' \to \mathcal{G}'$ compatible with φ and ψ, then the diagram*

$$\begin{array}{ccc} \mathcal{I} \otimes_{\mathcal{O}} \mathcal{F} & \xrightarrow[c_{\mathcal{F}'}]{} & \mathcal{K} \\ {\scriptstyle 1\otimes\varphi}\downarrow & & \downarrow{\scriptstyle \psi} \\ \mathcal{I} \otimes_{\mathcal{O}} \mathcal{G} & \xrightarrow{c_{\mathcal{G}'}} & \mathcal{L} \end{array}$$

is commutative.

(2) *The set of $\mathcal{O}'$-module maps $\varphi' : \mathcal{F}' \to \mathcal{G}'$ compatible with φ and ψ is, if nonempty, a principal homogeneous space under $\mathrm{Hom}_{\mathcal{O}}(\mathcal{F}, \mathcal{L})$.*

Proof. Part (1) is immediate from the description of the maps. For (2), if φ' and φ'' are two maps $\mathcal{F}' \to \mathcal{G}'$ compatible with φ and ψ, then $\varphi' - \varphi''$ factors as

$$\mathcal{F}' \to \mathcal{F} \to \mathcal{L} \to \mathcal{G}'$$

The map in the middle comes from a unique element of $\mathrm{Hom}_{\mathcal{O}}(\mathcal{F}, \mathcal{L})$ by Modules on Sites, Lemma 18.25.1. Conversely, given an element α of this group we can add the composition (as displayed above with α in the middle) to φ'. Some details omitted. □

Lemma 65.9.2. *Let $i : (Sh(\mathcal{C}), \mathcal{O}) \to (Sh(\mathcal{D}), \mathcal{O}')$ be a first order thickening of ringed topoi. Assume given extensions*

$$0 \to \mathcal{K} \to \mathcal{F}' \to \mathcal{F} \to 0 \quad \text{and} \quad 0 \to \mathcal{L} \to \mathcal{G}' \to \mathcal{G} \to 0$$

as in (65.9.0.1) and maps $\varphi : \mathcal{F} \to \mathcal{G}$ and $\psi : \mathcal{K} \to \mathcal{L}$. Assume the diagram

$$\begin{array}{ccc} \mathcal{I} \otimes_{\mathcal{O}} \mathcal{F} & \xrightarrow[c_{\mathcal{F}'}]{} & \mathcal{K} \\ {\scriptstyle 1\otimes\varphi}\downarrow & & \downarrow{\scriptstyle \psi} \\ \mathcal{I} \otimes_{\mathcal{O}} \mathcal{G} & \xrightarrow{c_{\mathcal{G}'}} & \mathcal{L} \end{array}$$

is commutative. Then there exists an element

$$o(\varphi, \psi) \in Ext^1_{\mathcal{O}}(\mathcal{F}, \mathcal{L})$$

whose vanishing is a necessary and sufficient condition for the existence of a map $\varphi' : \mathcal{F}' \to \mathcal{G}'$ compatible with φ and ψ.

Proof. We can construct explicitly an extension

$$0 \to \mathcal{L} \to \mathcal{H} \to \mathcal{F} \to 0$$

by taking $\mathcal{H}$ to be the cohomology of the complex

$$\mathcal{K} \xrightarrow{1,-\psi} \mathcal{F}' \oplus \mathcal{G}' \xrightarrow{\varphi,1} \mathcal{G}$$

in the middle (with obvious notation). A calculation with local sections using the assumption that the diagram of the lemma commutes shows that $\mathcal{H}$ is annihilated by $\mathcal{I}$. Hence $\mathcal{H}$ defines a class in

$$\mathrm{Ext}^1_{\mathcal{O}}(\mathcal{F}, \mathcal{L}) \subset \mathrm{Ext}^1_{\mathcal{O}'}(\mathcal{F}, \mathcal{L})$$

Finally, the class of $\mathcal{H}$ is the difference of the pushout of the extension $\mathcal{F}'$ via ψ and the pullback of the extension $\mathcal{G}'$ via φ (calculations omitted). Thus the vanishing of the class of $\mathcal{H}$ is equivalent to the existence of a commutative diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \mathcal{K} & \longrightarrow & \mathcal{F}' & \longrightarrow & \mathcal{F} & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle \psi} & & \downarrow{\scriptstyle \varphi'} & & \downarrow{\scriptstyle \varphi} & & \\ 0 & \longrightarrow & \mathcal{L} & \longrightarrow & \mathcal{G}' & \longrightarrow & \mathcal{G} & \longrightarrow & 0 \end{array}$$

as desired. □

Lemma 65.9.3. *Let $i : (Sh(\mathcal{C}), \mathcal{O}) \to (Sh(\mathcal{D}), \mathcal{O}')$ be a first order thickening of ringed topoi. Assume given $\mathcal{O}$-modules $\mathcal{F}$, $\mathcal{K}$ and an $\mathcal{O}$-linear map $c : \mathcal{I} \otimes_{\mathcal{O}} \mathcal{F} \to \mathcal{K}$. If there exists a sequence (65.9.0.1) with $c_{\mathcal{F}'} = c$ then the set of isomorphism classes of these extensions is principal homogeneous under $Ext^1_{\mathcal{O}}(\mathcal{F}, \mathcal{K})$.*

Proof. Assume given extensions

$$0 \to \mathcal{K} \to \mathcal{F}'_1 \to \mathcal{F} \to 0 \quad \text{and} \quad 0 \to \mathcal{K} \to \mathcal{F}'_2 \to \mathcal{F} \to 0$$

with $c_{\mathcal{F}'_1} = c_{\mathcal{F}'_2} = c$. Then the difference (in the extension group, see Homology, Section 12.6) is an extension

$$0 \to \mathcal{K} \to \mathcal{E} \to \mathcal{F} \to 0$$

where $\mathcal{E}$ is annihilated by $\mathcal{I}$ (local computation omitted). Hence the sequence is an extension of $\mathcal{O}$-modules, see Modules on Sites, Lemma 18.25.1. Conversely, given such an extension $\mathcal{E}$ we can add the extension $\mathcal{E}$ to the $\mathcal{O}'$-extension $\mathcal{F}'$ without affecting the map $c_{\mathcal{F}'}$. Some details omitted. □

Lemma 65.9.4. *Let $i : (Sh(\mathcal{C}), \mathcal{O}) \to (Sh(\mathcal{D}), \mathcal{O}')$ be a first order thickening of ringed topoi. Assume given $\mathcal{O}$-modules $\mathcal{F}$, $\mathcal{K}$ and an $\mathcal{O}$-linear map $c : \mathcal{I} \otimes_{\mathcal{O}} \mathcal{F} \to \mathcal{K}$. Then there exists an element*

$$o(\mathcal{F}, \mathcal{K}, c) \in Ext^2_{\mathcal{O}}(\mathcal{F}, \mathcal{K})$$

whose vanishing is a necessary and sufficient condition for the existence of a sequence (65.9.0.1) with $c_{\mathcal{F}'} = c$.

Proof. We first show that if $\mathcal{K}$ is an injective $\mathcal{O}$-module, then there does exist a sequence (65.9.0.1) with $c_{\mathcal{F}'} = c$. To do this, choose a flat $\mathcal{O}'$-module $\mathcal{H}'$ and a surjection $\mathcal{H}' \to \mathcal{F}$ (Modules on Sites, Lemma 18.28.6). Let $\mathcal{J} \subset \mathcal{H}'$ be the kernel. Since $\mathcal{H}'$ is flat we have

$$\mathcal{I} \otimes_{\mathcal{O}'} \mathcal{H}' = \mathcal{I}\mathcal{H}' \subset \mathcal{J} \subset \mathcal{H}'$$

Observe that the map

$$\mathcal{I}\mathcal{H}' = \mathcal{I} \otimes_{\mathcal{O}'} \mathcal{H}' \longrightarrow \mathcal{I} \otimes_{\mathcal{O}'} \mathcal{F} = \mathcal{I} \otimes_{\mathcal{O}} \mathcal{F}$$

annihilates $\mathcal{I}\mathcal{J}$. Namely, if f is a local section of $\mathcal{I}$ and s is a local section of $\mathcal{H}$, then fs is mapped to $f \otimes \overline{s}$ where $\overline{s}$ is the image of s in $\mathcal{F}$. Thus we obtain

$$\begin{array}{ccc} \mathcal{I}\mathcal{H}'/\mathcal{I}\mathcal{J} & \hookrightarrow & \mathcal{J}/\mathcal{I}\mathcal{J} \\ \downarrow & & \downarrow \gamma \\ \mathcal{I} \otimes_{\mathcal{O}} \mathcal{F} & \xrightarrow{c} & \mathcal{K} \end{array}$$

a diagram of $\mathcal{O}$-modules. If $\mathcal{K}$ is injective as an $\mathcal{O}$-module, then we obtain the dotted arrow. Denote $\gamma' : \mathcal{J} \to \mathcal{K}$ the composition of γ with $\mathcal{J} \to \mathcal{J}/\mathcal{I}\mathcal{J}$. A local calculation shows the pushout

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \mathcal{J} & \longrightarrow & \mathcal{H}' & \longrightarrow & \mathcal{F} & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle \gamma'} & & \downarrow & & \| & & \\ 0 & \longrightarrow & \mathcal{K} & \longrightarrow & \mathcal{F}' & \longrightarrow & \mathcal{F} & \longrightarrow & 0 \end{array}$$

is a solution to the problem posed by the lemma.

General case. Choose an embedding $\mathcal{K} \subset \mathcal{K}'$ with $\mathcal{K}'$ an injective $\mathcal{O}$-module. Let $\mathcal{Q}$ be the quotient, so that we have an exact sequence

$$0 \to \mathcal{K} \to \mathcal{K}' \to \mathcal{Q} \to 0$$

Denote $c' : \mathcal{I} \otimes_{\mathcal{O}} \mathcal{F} \to \mathcal{K}'$ be the composition. By the paragraph above there exists a sequence

$$0 \to \mathcal{K}' \to \mathcal{E}' \to \mathcal{F} \to 0$$

as in (65.9.0.1) with $c_{\mathcal{E}'} = c'$. Note that c' composed with the map $\mathcal{K}' \to \mathcal{Q}$ is zero, hence the pushout of $\mathcal{E}'$ by $\mathcal{K}' \to \mathcal{Q}$ is an extension

$$0 \to \mathcal{Q} \to \mathcal{D}' \to \mathcal{F} \to 0$$

as in (65.9.0.1) with $c_{\mathcal{D}'} = 0$. This means exactly that $\mathcal{D}'$ is annihilated by $\mathcal{I}$, in other words, the $\mathcal{D}'$ is an extension of $\mathcal{O}$-modules, i.e., defines an element

$$o(\mathcal{F}, \mathcal{K}, c) \in \mathrm{Ext}^1_{\mathcal{O}}(\mathcal{F}, \mathcal{Q}) = \mathrm{Ext}^2_{\mathcal{O}}(\mathcal{F}, \mathcal{K})$$

(the equality holds by the long exact cohomology sequence associated to the exact sequence above and the vanishing of higher ext groups into the injective module $\mathcal{K}'$). If $o(\mathcal{F}, \mathcal{K}, c) = 0$, then we can choose a splitting $s : \mathcal{F} \to \mathcal{D}'$ and we can set

$$\mathcal{F}' = \mathrm{Ker}(\mathcal{E}' \to \mathcal{D}'/s(\mathcal{F}))$$

so that we obtain the following diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \mathcal{K} & \longrightarrow & \mathcal{F}' & \longrightarrow & \mathcal{F} & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \| & & \\ 0 & \longrightarrow & \mathcal{K}' & \longrightarrow & \mathcal{E}' & \longrightarrow & \mathcal{F} & \longrightarrow & 0 \end{array}$$

with exact rows which shows that $c_{\mathcal{F}'} = c$. Conversely, if $\mathcal{F}'$ exists, then the pushout of $\mathcal{F}'$ by the map $\mathcal{K} \to \mathcal{K}'$ is isomorphic to $\mathcal{E}'$ by Lemma 65.9.3 and the vanishing of higher ext groups into the injective module $\mathcal{K}'$. This gives a diagram as above, which implies that $\mathcal{D}'$ is split as an extension, i.e., the class $o(\mathcal{F}, \mathcal{K}, c)$ is zero. $\square$

Remark 65.9.5. Let $(Sh(\mathcal{C}), \mathcal{O})$ be a ringed topos. A first order thickening $i : (Sh(\mathcal{C}), \mathcal{O}) \to (Sh(\mathcal{D}), \mathcal{O}')$ is said to be *trivial* if there exists a morphism of ringed topoi $\pi : (Sh(\mathcal{D}), \mathcal{O}') \to (Sh(\mathcal{C}), \mathcal{O})$ which is a left inverse to i. The choice of such a morphism π is called a *trivialization* of the first order thickening. Given π we obtain a splitting

$$\mathcal{O}' = \mathcal{O} \oplus \mathcal{I} \tag{65.9.5.1}$$

as sheaves of algebras on $\mathcal{C}$ by using $\pi^\sharp$ to split the surjection $\mathcal{O}' \to \mathcal{O}$. Conversely, such a splitting determines a morphism π. The category of trivialized first order thickenings of $(Sh(\mathcal{C}), \mathcal{O})$ is equivalent to the category of $\mathcal{O}$-modules.

Remark 65.9.6. Let $i : (Sh(\mathcal{C}), \mathcal{O}) \to (Sh(\mathcal{D}), \mathcal{O}')$ be a trivial first order thickening of ringed topoi and let $\pi : (Sh(\mathcal{D}), \mathcal{O}') \to (Sh(\mathcal{C}), \mathcal{O})$ be a trivialization. Then given any triple $(\mathcal{F}, \mathcal{K}, c)$ consisting of a pair of $\mathcal{O}$-modules and a map $c : \mathcal{I} \otimes_{\mathcal{O}} \mathcal{F} \to \mathcal{K}$ we may set

$$\mathcal{F}'_{c,triv} = \mathcal{F} \oplus \mathcal{K}$$

and use the splitting (65.9.5.1) associated to π and the map c to define the $\mathcal{O}'$-module structure and obtain an extension (65.9.0.1). We will call $\mathcal{F}'_{c,triv}$ the *trivial extension* of $\mathcal{F}$ by $\mathcal{K}$ corresponding to c and the trivialization π. Given any extension $\mathcal{F}'$ as in (65.9.0.1) we can use $\pi^\sharp : \mathcal{O} \to \mathcal{O}'$ to think of $\mathcal{F}'$ as an $\mathcal{O}$-module extension, hence a class $\xi_{\mathcal{F}'}$ in $\mathrm{Ext}^1_{\mathcal{O}}(\mathcal{F}, \mathcal{K})$. Lemma 65.9.3 assures that $\mathcal{F}' \mapsto \xi_{\mathcal{F}'}$ induces a bijection

$$\left\{ \begin{array}{c} \text{isomorphism classes of extensions} \\ \mathcal{F}' \text{ as in (65.9.0.1) with } c = c_{\mathcal{F}'} \end{array} \right\} \longrightarrow \mathrm{Ext}^1_{\mathcal{O}}(\mathcal{F}, \mathcal{K})$$

Moreover, the trivial extension $\mathcal{F}'_{c,triv}$ maps to the zero class.

Remark 65.9.7. Let $(Sh(\mathcal{C}), \mathcal{O})$ be a ringed topos. Let $(Sh(\mathcal{C}), \mathcal{O}) \to (Sh(\mathcal{D}_i), \mathcal{O}'_i)$, $i = 1, 2$ be first order thickenings with ideal sheaves $\mathcal{I}_i$. Let $h : (Sh(\mathcal{D}_1), \mathcal{O}'_1) \to (Sh(\mathcal{D}_2), \mathcal{O}'_2)$ be a morphism of first order thickenings of $(Sh(\mathcal{C}), \mathcal{O})$. Picture

$$\begin{array}{ccc} & (Sh(\mathcal{C}), \mathcal{O}) & \\ \swarrow & & \searrow \\ (Sh(\mathcal{D}_1), \mathcal{O}'_1) & \xrightarrow{h} & (Sh(\mathcal{D}_2), \mathcal{O}'_2) \end{array}$$

Observe that $h^\sharp : \mathcal{O}'_2 \to \mathcal{O}'_1$ in particular induces an $\mathcal{O}$-module map $\mathcal{I}_2 \to \mathcal{I}_1$. Let $\mathcal{F}$ be an $\mathcal{O}$-module. Let $(\mathcal{K}_i, c_i)$, $i = 1, 2$ be a pair consisting of an $\mathcal{O}$-module $\mathcal{K}_i$ and a map $c_i : \mathcal{I}_i \otimes_{\mathcal{O}} \mathcal{F} \to \mathcal{K}_i$. Assume furthermore given a map of $\mathcal{O}$-modules $\mathcal{K}_2 \to \mathcal{K}_1$ such that

$$\begin{array}{ccc} \mathcal{I}_2 \otimes_{\mathcal{O}} \mathcal{F} & \xrightarrow[c_2]{} & \mathcal{K}_2 \\ \downarrow & & \downarrow \\ \mathcal{I}_1 \otimes_{\mathcal{O}} \mathcal{F} & \xrightarrow{c_1} & \mathcal{K}_1 \end{array}$$

is commutative. Then there is a canonical functoriality

$$\left\{ \begin{array}{c} \mathcal{F}'_2 \text{ as in (65.9.0.1) with} \\ c_2 = c_{\mathcal{F}'_2} \text{ and } \mathcal{K} = \mathcal{K}_2 \end{array} \right\} \longrightarrow \left\{ \begin{array}{c} \mathcal{F}'_1 \text{ as in (65.9.0.1) with} \\ c_1 = c_{\mathcal{F}'_1} \text{ and } \mathcal{K} = \mathcal{K}_1 \end{array} \right\}$$

Namely, thinking of all sheaves $\mathcal{O}$, $\mathcal{O}'_i$, $\mathcal{F}$, $\mathcal{K}_i$, etc as sheaves on $\mathcal{C}$, we set given $\mathcal{F}'_2$ the sheaf $\mathcal{F}'_1$ equal to the pushout, i.e., fitting into the following diagram of

extensions

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \mathcal{K}_2 & \longrightarrow & \mathcal{F}_2' & \longrightarrow & \mathcal{F} & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \| & & \\ 0 & \longrightarrow & \mathcal{K}_1 & \longrightarrow & \mathcal{F}_1' & \longrightarrow & \mathcal{F} & \longrightarrow & 0 \end{array}$$

We omit the construction of the $\mathcal{O}_1'$-module structure on the pushout (this uses the commutativity of the diagram involving c_1 and c_2).

Remark 65.9.8. Let $(Sh(\mathcal{C}), \mathcal{O})$, $(Sh(\mathcal{C}), \mathcal{O}) \to (Sh(\mathcal{D}_i), \mathcal{O}_i')$, $\mathcal{I}_i$, and $h : (Sh(\mathcal{D}_1), \mathcal{O}_1') \to (Sh(\mathcal{D}_2), \mathcal{O}_2')$ be as in Remark 65.9.7. Assume that we are given given trivializations $\pi_i : (Sh(\mathcal{D}_i), \mathcal{O}_i') \to (Sh(\mathcal{C}), \mathcal{O})$ such that $\pi_1 = h \circ \pi_2$. In other words, assume h is a morphism of trivialized first order thickenings of $(Sh(\mathcal{C}), \mathcal{O})$. Let $(\mathcal{K}_i, c_i)$, $i = 1, 2$ be a pair consisting of an $\mathcal{O}$-module $\mathcal{K}_i$ and a map $c_i : \mathcal{I}_i \otimes_{\mathcal{O}} \mathcal{F} \to \mathcal{K}_i$. Assume furthermore given a map of $\mathcal{O}$-modules $\mathcal{K}_2 \to \mathcal{K}_1$ such that

$$\begin{array}{ccc} \mathcal{I}_2 \otimes_{\mathcal{O}} \mathcal{F} & \xrightarrow[c_2]{} & \mathcal{K}_2 \\ \downarrow & & \downarrow \\ \mathcal{I}_1 \otimes_{\mathcal{O}} \mathcal{F} & \xrightarrow{c_1} & \mathcal{K}_1 \end{array}$$

is commutative. In this situation the construction of Remark 65.9.6 induces a commutative diagram

$$\begin{array}{ccc} \{\mathcal{F}_2' \text{ as in (65.9.0.1) with } c_2 = c_{\mathcal{F}_2'} \text{ and } \mathcal{K} = \mathcal{K}_2\} & \longrightarrow & \mathrm{Ext}^1_{\mathcal{O}}(\mathcal{F}, \mathcal{K}_2) \\ \downarrow & & \downarrow \\ \{\mathcal{F}_1' \text{ as in (65.9.0.1) with } c_1 = c_{\mathcal{F}_1'} \text{ and } \mathcal{K} = \mathcal{K}_1\} & \longrightarrow & \mathrm{Ext}^1_{\mathcal{O}}(\mathcal{F}, \mathcal{K}_1) \end{array}$$

where the vertical map on the right is given by functoriality of Ext and the map $\mathcal{K}_2 \to \mathcal{K}_1$ and the vertical map on the left is the one from Remark 65.9.7.

Remark 65.9.9. Let $(Sh(\mathcal{C}), \mathcal{O})$ be a ringed topos. We define a sequence of morphisms of first order thickenings

$$(Sh(\mathcal{D}_1), \mathcal{O}_1') \to (Sh(\mathcal{D}_2), \mathcal{O}_2') \to (Sh(\mathcal{D}_3), \mathcal{O}_3')$$

of $(Sh(\mathcal{C}), \mathcal{O})$ to be a *complex* if the corresponding maps between the ideal sheaves $\mathcal{I}_i$ give a complex of $\mathcal{O}$-modules $\mathcal{I}_3 \to \mathcal{I}_2 \to \mathcal{I}_1$ (i.e., the composition is zero). In this case the composition $(Sh(\mathcal{D}_1), \mathcal{O}_1') \to (Sh(\mathcal{D}_3), \mathcal{O}_3')$ factors through $(Sh(\mathcal{C}), \mathcal{O}) \to (Sh(\mathcal{D}_3), \mathcal{O}_3')$, i.e., the first order thickening $(Sh(\mathcal{D}_1), \mathcal{O}_1')$ of $(Sh(\mathcal{C}), \mathcal{O})$ is trivial and comes with a canonical trivialization $\pi : (Sh(\mathcal{D}_1), \mathcal{O}_1') \to (Sh(\mathcal{C}), \mathcal{O})$.

We say a sequence of morphisms of first order thickenings

$$(Sh(\mathcal{D}_1), \mathcal{O}_1') \to (Sh(\mathcal{D}_2), \mathcal{O}_2') \to (Sh(\mathcal{D}_3), \mathcal{O}_3')$$

of $(Sh(\mathcal{C}), \mathcal{O})$ is *a short exact sequence* if the corresponding maps between ideal sheaves is a short exact sequence

$$0 \to \mathcal{I}_3 \to \mathcal{I}_2 \to \mathcal{I}_1 \to 0$$

of $\mathcal{O}$-modules.

Remark 65.9.10. Let $(Sh(\mathcal{C}), \mathcal{O})$ be a ringed topos. Let $\mathcal{F}$ be an $\mathcal{O}$-module. Let

$$(Sh(\mathcal{D}_1), \mathcal{O}'_1) \to (Sh(\mathcal{D}_2), \mathcal{O}'_2) \to (Sh(\mathcal{D}_3), \mathcal{O}'_3)$$

be a complex first order thickenings of $(Sh(\mathcal{C}), \mathcal{O})$, see Remark 65.9.9. Let $(\mathcal{K}_i, c_i)$, $i = 1, 2, 3$ be pairs consisting of an $\mathcal{O}$-module $\mathcal{K}_i$ and a map $c_i : \mathcal{I}_i \otimes_{\mathcal{O}} \mathcal{F} \to \mathcal{K}_i$. Assume given a short exact sequence of $\mathcal{O}$-modules

$$0 \to \mathcal{K}_3 \to \mathcal{K}_2 \to \mathcal{K}_1 \to 0$$

such that

$$\begin{array}{ccc} \mathcal{I}_2 \otimes_{\mathcal{O}} \mathcal{F} & \xrightarrow[c_2]{} & \mathcal{K}_2 \\ \downarrow & & \downarrow \\ \mathcal{I}_1 \otimes_{\mathcal{O}} \mathcal{F} & \xrightarrow{c_1} & \mathcal{K}_1 \end{array} \quad \text{and} \quad \begin{array}{ccc} \mathcal{I}_3 \otimes_{\mathcal{O}} \mathcal{F} & \xrightarrow[c_3]{} & \mathcal{K}_3 \\ \downarrow & & \downarrow \\ \mathcal{I}_2 \otimes_{\mathcal{O}} \mathcal{F} & \xrightarrow{c_2} & \mathcal{K}_2 \end{array}$$

are commutative. Finally, assume given an extension

$$0 \to \mathcal{K}_2 \to \mathcal{F}'_2 \to \mathcal{F} \to 0$$

as in (65.9.0.1) with $\mathcal{K} = \mathcal{K}_2$ of $\mathcal{O}'_2$-modules with $c_{\mathcal{F}'_2} = c_2$. In this situation we can apply the functoriality of Remark 65.9.7 to obtain an extension $\mathcal{F}'_1$ of $\mathcal{O}'_1$-modules (we'll describe $\mathcal{F}'_1$ in this special case below). By Remark 65.9.6 using the canonical splitting $\pi : (Sh(\mathcal{D}_1), \mathcal{O}'_1) \to (Sh(\mathcal{C}), \mathcal{O})$ of Remark 65.9.9 we obtain $\xi_{\mathcal{F}'_1} \in \mathrm{Ext}^1_{\mathcal{O}}(\mathcal{F}, \mathcal{K}_1)$. Finally, we have the obstruction

$$o(\mathcal{F}, \mathcal{K}_3, c_3) \in \mathrm{Ext}^2_{\mathcal{O}}(\mathcal{F}, \mathcal{K}_3)$$

see Lemma 65.9.4. In this situation we **claim** that the canonical map

$$\partial : \mathrm{Ext}^1_{\mathcal{O}}(\mathcal{F}, \mathcal{K}_1) \longrightarrow \mathrm{Ext}^2_{\mathcal{O}}(\mathcal{F}, \mathcal{K}_3)$$

coming from the short exact sequence $0 \to \mathcal{K}_3 \to \mathcal{K}_2 \to \mathcal{K}_1 \to 0$ sends $\xi_{\mathcal{F}'_1}$ to the obstruction class $o(\mathcal{F}, \mathcal{K}_3, c_3)$.

To prove this claim choose an embedding $j : \mathcal{K}_3 \to \mathcal{K}$ where $\mathcal{K}$ is an injective $\mathcal{O}$-module. We can lift j to a map $j' : \mathcal{K}_2 \to \mathcal{K}$. Set $\mathcal{E}'_2 = j'_* \mathcal{F}'_2$ equal to the pushout of $\mathcal{F}'_2$ by j' so that $c_{\mathcal{E}'_2} = j' \circ c_2$. Picture:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \mathcal{K}_2 & \longrightarrow & \mathcal{F}'_2 & \longrightarrow & \mathcal{F} & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle j'} & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & \mathcal{K} & \longrightarrow & \mathcal{E}'_2 & \longrightarrow & \mathcal{F} & \longrightarrow & 0 \end{array}$$

Set $\mathcal{E}'_3 = \mathcal{E}'_2$ but viewed as an $\mathcal{O}'_3$-module via $\mathcal{O}'_3 \to \mathcal{O}'_2$. Then $c_{\mathcal{E}'_3} = j \circ c_3$. The proof of Lemma 65.9.4 constructs $o(\mathcal{F}, \mathcal{K}_3, c_3)$ as the boundary of the class of the extension of $\mathcal{O}$-modules

$$0 \to \mathcal{K}/\mathcal{K}_3 \to \mathcal{E}'_3/\mathcal{K}_3 \to \mathcal{F} \to 0$$

On the other hand, note that $\mathcal{F}'_1 = \mathcal{F}'_2/\mathcal{K}_3$ hence the class $\xi_{\mathcal{F}'_1}$ is the class of the extension

$$0 \to \mathcal{K}_2/\mathcal{K}_3 \to \mathcal{F}'_2/\mathcal{K}_3 \to \mathcal{F} \to 0$$

seen as a sequence of $\mathcal{O}$-modules using $\pi^\sharp$ where $\pi : (Sh(\mathcal{D}_1), \mathcal{O}'_1) \to (Sh(\mathcal{C}), \mathcal{O})$ is the canonical splitting. Thus finally, the claim follows from the fact that we have a commutative diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \mathcal{K}_2/\mathcal{K}_3 & \longrightarrow & \mathcal{F}'_2/\mathcal{K}_3 & \longrightarrow & \mathcal{F} & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & \mathcal{K}/\mathcal{K}_3 & \longrightarrow & \mathcal{E}'_3/\mathcal{K}_3 & \longrightarrow & \mathcal{F} & \longrightarrow & 0 \end{array}$$

which is $\mathcal{O}$-linear (with the $\mathcal{O}$-module structures given above).

65.10. Infinitesimal deformations of modules on ringed topi

Let $i : (Sh(\mathcal{C}), \mathcal{O}) \to (Sh(\mathcal{D}), \mathcal{O}')$ be a first order thickening of ringed topoi. We freely use the notation introduced in Section 65.8. Let $\mathcal{F}'$ be an $\mathcal{O}'$-module and set $\mathcal{F} = i^*\mathcal{F}'$. In this situation we have a short exact sequence

$$0 \to \mathcal{I}\mathcal{F}' \to \mathcal{F}' \to \mathcal{F} \to 0$$

of $\mathcal{O}'$-modules. Since $\mathcal{I}^2 = 0$ the $\mathcal{O}'$-module structure on $\mathcal{I}\mathcal{F}'$ comes from a unique $\mathcal{O}$-module structure. Thus the sequence above is an extension as in (65.9.0.1). As a special case, if $\mathcal{F}' = \mathcal{O}'$ we have $i^*\mathcal{O}' = \mathcal{O}$ and $\mathcal{I}\mathcal{O}' = \mathcal{I}$ and we recover the sequence of structure sheaves

$$0 \to \mathcal{I} \to \mathcal{O}' \to \mathcal{O} \to 0$$

Lemma 65.10.1. *Let $i : (Sh(\mathcal{C}), \mathcal{O}) \to (Sh(\mathcal{D}), \mathcal{O}')$ be a first order thickening of ringed topoi. Let $\mathcal{F}'$, $\mathcal{G}'$ be $\mathcal{O}'$-modules. Set $\mathcal{F} = i^*\mathcal{F}'$ and $\mathcal{G} = i^*\mathcal{G}'$. Let $\varphi : \mathcal{F} \to \mathcal{G}$ be an $\mathcal{O}$-linear map. The set of lifts of φ to an $\mathcal{O}'$-linear map $\varphi' : \mathcal{F}' \to \mathcal{G}'$ is, if nonempty, a principal homogeneous space under* $\mathrm{Hom}_{\mathcal{O}}(\mathcal{F}, \mathcal{I}\mathcal{G}')$.

Proof. This is a special case of Lemma 65.9.1 but we also give a direct proof. We have short exact sequences of modules

$$0 \to \mathcal{I} \to \mathcal{O}' \to \mathcal{O} \to 0 \quad \text{and} \quad 0 \to \mathcal{I}\mathcal{G}' \to \mathcal{G}' \to \mathcal{G} \to 0$$

and similarly for $\mathcal{F}'$. Since $\mathcal{I}$ has square zero the $\mathcal{O}'$-module structure on $\mathcal{I}$ and $\mathcal{I}\mathcal{G}'$ comes from a unique $\mathcal{O}$-module structure. It follows that

$$\mathrm{Hom}_{\mathcal{O}'}(\mathcal{F}', \mathcal{I}\mathcal{G}') = \mathrm{Hom}_{\mathcal{O}}(\mathcal{F}, \mathcal{I}\mathcal{G}') \quad \text{and} \quad \mathrm{Hom}_{\mathcal{O}'}(\mathcal{F}', \mathcal{G}) = \mathrm{Hom}_{\mathcal{O}}(\mathcal{F}, \mathcal{G})$$

The lemma now follows from the exact sequence

$$0 \to \mathrm{Hom}_{\mathcal{O}'}(\mathcal{F}', \mathcal{I}\mathcal{G}') \to \mathrm{Hom}_{\mathcal{O}'}(\mathcal{F}', \mathcal{G}') \to \mathrm{Hom}_{\mathcal{O}'}(\mathcal{F}', \mathcal{G})$$

see Homology, Lemma 12.5.8. □

Lemma 65.10.2. *Let (f, f') be a morphism of first order thickenings of ringed topoi as in Situation 65.8.1. Let $\mathcal{F}'$ be an $\mathcal{O}'$-module and set $\mathcal{F} = i^*\mathcal{F}'$. Assume that $\mathcal{F}$ is flat over $\mathcal{O}_{\mathcal{B}}$ and that (f, f') is a strict morphism of thickenings (Definition 65.8.2). Then the following are equivalent*

(1) *$\mathcal{F}'$ is flat over $\mathcal{O}_{\mathcal{B}'}$, and*
(2) *the canonical map $f^*\mathcal{J} \otimes_{\mathcal{O}} \mathcal{F} \to \mathcal{I}\mathcal{F}'$ is an isomorphism.*

Moreover, in this case the maps

$$f^*\mathcal{J} \otimes_{\mathcal{O}} \mathcal{F} \to \mathcal{I} \otimes_{\mathcal{O}} \mathcal{F} \to \mathcal{I}\mathcal{F}'$$

are isomorphisms.

Proof. The map $f^*\mathcal{J} \to \mathcal{I}$ is surjective as (f, f') is a strict morphism of thickenings. Hence the final statement is a consequence of (2).

Proof of the equivalence of (1) and (2). By definition flatness over $\mathcal{O}_\mathcal{B}$ means flatness over $f^{-1}\mathcal{O}_\mathcal{B}$. Similarly for flatness over $f^{-1}\mathcal{O}_{\mathcal{B}'}$. Note that the strictness of (f, f') and the assumption that $\mathcal{F} = i^*\mathcal{F}'$ imply that

$$\mathcal{F} = \mathcal{F}'/(f^{-1}\mathcal{J})\mathcal{F}'$$

as sheaves on $\mathcal{C}$. Moreover, observe that $f^*\mathcal{J} \otimes_\mathcal{O} \mathcal{F} = f^{-1}\mathcal{J} \otimes_{f^{-1}\mathcal{O}_\mathcal{B}} \mathcal{F}$. Hence the equivalence of (1) and (2) follows from Modules on Sites, Lemma 18.28.13. □

Lemma 65.10.3. *Let (f, f') be a morphism of first order thickenings of ringed topoi as in Situation 65.8.1. Let $\mathcal{F}'$ be an $\mathcal{O}'$-module and set $\mathcal{F} = i^*\mathcal{F}'$. Assume that $\mathcal{F}'$ is flat over $\mathcal{O}_{\mathcal{B}'}$ and that (f, f') is a strict morphism of thickenings. Then the following are equivalent*

(1) *$\mathcal{F}'$ is an $\mathcal{O}'$-module of finite presentation, and*
(2) *$\mathcal{F}$ is an $\mathcal{O}$-module of finite presentation.*

Proof. The implication (1) $\Rightarrow$ (2) follows from Modules on Sites, Lemma 18.23.4. For the converse, assume $\mathcal{F}$ of finite presentation. We may and do assume that $\mathcal{C} = \mathcal{C}'$. By Lemma 65.10.2 we have a short exact sequence

$$0 \to \mathcal{I} \otimes_{\mathcal{O}_X} \mathcal{F} \to \mathcal{F}' \to \mathcal{F} \to 0$$

Let U be an object of $\mathcal{C}$ such that $\mathcal{F}|_U$ has a presentation

$$\mathcal{O}_U^{\oplus m} \to \mathcal{O}_U^{\oplus n} \to \mathcal{F}|_U \to 0$$

After replacing U by the members of a covering we may assume the map $\mathcal{O}_U^{\oplus n} \to \mathcal{F}|_U$ lifts to a map $(\mathcal{O}'_U)^{\oplus n} \to \mathcal{F}'|_U$. The induced map $\mathcal{I}^{\oplus n} \to \mathcal{I} \otimes \mathcal{F}$ is surjective by right exactness of $\otimes$. Thus after replacing U by the members of a covering we can find a lift $(\mathcal{O}'|_U)^{\oplus m} \to (\mathcal{O}'|_U)^{\oplus n}$ of the given map $\mathcal{O}_U^{\oplus m} \to \mathcal{O}_U^{\oplus n}$ such that

$$(\mathcal{O}'_U)^{\oplus m} \to (\mathcal{O}'_U)^{\oplus n} \to \mathcal{F}'|_U \to 0$$

is a complex. Using right exactness of $\otimes$ once more it is seen that this complex is exact. □

Lemma 65.10.4. *Let (f, f') be a morphism of first order thickenings as in Situation 65.8.1. Let $\mathcal{F}'$, $\mathcal{G}'$ be $\mathcal{O}'$-modules and set $\mathcal{F} = i^*\mathcal{F}'$ and $\mathcal{G} = i^*\mathcal{G}'$. Let $\varphi : \mathcal{F} \to \mathcal{G}$ be an $\mathcal{O}$-linear map. Assume that $\mathcal{G}'$ is flat over $\mathcal{O}_{\mathcal{B}'}$ and that (f, f') is a strict morphism of thickenings. The set of lifts of φ to an $\mathcal{O}'$-linear map $\varphi' : \mathcal{F}' \to \mathcal{G}'$ is, if nonempty, a principal homogeneous space under*

$$\mathrm{Hom}_\mathcal{O}(\mathcal{F}, \mathcal{G} \otimes_\mathcal{O} f^*\mathcal{J})$$

Proof. Combine Lemmas 65.10.1 and 65.10.2. □

Lemma 65.10.5. *Let $i : (Sh(\mathcal{C}), \mathcal{O}) \to (Sh(\mathcal{D}), \mathcal{O}')$ be a first order thickening of ringed topoi. Let $\mathcal{F}'$, $\mathcal{G}'$ be $\mathcal{O}'$-modules and set $\mathcal{F} = i^*\mathcal{F}'$ and $\mathcal{G} = i^*\mathcal{G}'$. Let $\varphi : \mathcal{F} \to \mathcal{G}$ be an $\mathcal{O}$-linear map. There exists an element*

$$o(\varphi) \in Ext^1_\mathcal{O}(Li^*\mathcal{F}', \mathcal{I}\mathcal{G}')$$

whose vanishing is a necessary and sufficient condition for the existence of a lift of φ to an $\mathcal{O}'$-linear map $\varphi' : \mathcal{F}' \to \mathcal{G}'$.

Proof. It is clear from the proof of Lemma 65.10.1 that the vanishing of the boundary of φ via the map

$$\mathrm{Hom}_{\mathcal{O}}(\mathcal{F}, \mathcal{G}) = \mathrm{Hom}_{\mathcal{O}'}(\mathcal{F}', \mathcal{G}) \longrightarrow \mathrm{Ext}^1_{\mathcal{O}'}(\mathcal{F}', \mathcal{I}\mathcal{G}')$$

is a necessary and sufficient condition for the existence of a lift. We conclude as

$$\mathrm{Ext}^1_{\mathcal{O}'}(\mathcal{F}', \mathcal{I}\mathcal{G}') = \mathrm{Ext}^1_{\mathcal{O}}(Li^*\mathcal{F}', \mathcal{I}\mathcal{G}')$$

the adjointness of $i_* = Ri_*$ and Li^* on the derived category (Cohomology on Sites, Lemma 21.19.1). □

Lemma 65.10.6. *Let (f, f') be a morphism of first order thickenings as in Situation 65.8.1. Let $\mathcal{F}'$, $\mathcal{G}'$ be $\mathcal{O}'$-modules and set $\mathcal{F} = i^*\mathcal{F}'$ and $\mathcal{G} = i^*\mathcal{G}'$. Let $\varphi : \mathcal{F} \to \mathcal{G}$ be an $\mathcal{O}$-linear map. Assume that $\mathcal{F}'$ and $\mathcal{G}'$ are flat over $\mathcal{O}_{\mathcal{B}'}$ and that (f, f') is a strict morphism of thickenings. There exists an element*

$$o(\varphi) \in Ext^1_{\mathcal{O}}(\mathcal{F}, \mathcal{G} \otimes_{\mathcal{O}} f^*\mathcal{J})$$

whose vanishing is a necessary and sufficient condition for the existence of a lift of φ to an $\mathcal{O}'$-linear map $\varphi' : \mathcal{F}' \to \mathcal{G}'$.

First proof. This follows from Lemma 65.10.5 as we claim that under the assumptions of the lemma we have

$$\mathrm{Ext}^1_{\mathcal{O}}(Li^*\mathcal{F}', \mathcal{I}\mathcal{G}') = \mathrm{Ext}^1_{\mathcal{O}}(\mathcal{F}, \mathcal{G} \otimes_{\mathcal{O}} f^*\mathcal{J})$$

Namely, we have $\mathcal{I}\mathcal{G}' = \mathcal{G} \otimes_{\mathcal{O}} f^*\mathcal{J}$ by Lemma 65.10.2. On the other hand, observe that

$$H^{-1}(Li^*\mathcal{F}') = \mathrm{Tor}_1^{\mathcal{O}'}(\mathcal{F}', \mathcal{O})$$

(local computation omitted). Using the short exact sequence

$$0 \to \mathcal{I} \to \mathcal{O}' \to \mathcal{O} \to 0$$

we see that this Tor_1 is computed by the kernel of the map $\mathcal{I} \otimes_{\mathcal{O}} \mathcal{F} \to \mathcal{I}\mathcal{F}'$ which is zero by the final assertion of Lemma 65.10.2. Thus $\tau_{\geq -1}Li^*\mathcal{F}' = \mathcal{F}$. On the other hand, we have

$$\mathrm{Ext}^1_{\mathcal{O}}(Li^*\mathcal{F}', \mathcal{I}\mathcal{G}') = \mathrm{Ext}^1_{\mathcal{O}}(\tau_{\geq -1}Li^*\mathcal{F}', \mathcal{I}\mathcal{G}')$$

by the dual of Derived Categories, Lemma 13.16.1. □

Second proof. We can apply Lemma 65.9.2 as follows. Note that $\mathcal{K} = \mathcal{I} \otimes_{\mathcal{O}} \mathcal{F}$ and $\mathcal{L} = \mathcal{I} \otimes_{\mathcal{O}} \mathcal{G}$ by Lemma 65.10.2, that $c_{\mathcal{F}'} = 1 \otimes 1$ and $c_{\mathcal{G}'} = 1 \otimes 1$ and taking $\psi = 1 \otimes \varphi$ the diagram of the lemma commutes. Thus $o(\varphi) = o(\varphi, 1 \otimes \varphi)$ works. □

Lemma 65.10.7. *Let (f, f') be a morphism of first order thickenings as in Situation 65.8.1. Let $\mathcal{F}$ be an $\mathcal{O}$-module. Assume (f, f') is a strict morphism of thickenings and $\mathcal{F}$ flat over $\mathcal{O}_{\mathcal{B}}$. If there exists a pair $(\mathcal{F}', \alpha)$ consisting of an $\mathcal{O}'$-module $\mathcal{F}'$ flat over $\mathcal{O}_{\mathcal{B}'}$ and an isomorphism $\alpha : i^*\mathcal{F}' \to \mathcal{F}$, then the set of isomorphism classes of such pairs is principal homogeneous under $Ext^1_{\mathcal{O}}(\mathcal{F}, \mathcal{I} \otimes_{\mathcal{O}} \mathcal{F})$.*

Proof. If we assume there exists one such module, then the canonical map

$$f^*\mathcal{J} \otimes_{\mathcal{O}} \mathcal{F} \to \mathcal{I} \otimes_{\mathcal{O}} \mathcal{F}$$

is an isomorphism by Lemma 65.10.2. Apply Lemma 65.9.3 with $\mathcal{K} = \mathcal{I} \otimes_{\mathcal{O}} \mathcal{F}$ and $c = 1$. By Lemma 65.10.2 the corresponding extensions $\mathcal{F}'$ are all flat over $\mathcal{O}_{\mathcal{B}'}$. □

Lemma 65.10.8. *Let (f, f') be a morphism of first order thickenings as in Situation 65.8.1. Let $\mathcal{F}$ be an $\mathcal{O}$-module. Assume (f, f') is a strict morphism of thickenings and $\mathcal{F}$ flat over $\mathcal{O}_\mathcal{B}$. There exists an $\mathcal{O}'$-module $\mathcal{F}'$ flat over $\mathcal{O}_{\mathcal{B}'}$ with $i^*\mathcal{F}' \cong \mathcal{F}$, if and only if*

1. *the canonical map $f^*\mathcal{J} \otimes_\mathcal{O} \mathcal{F} \to \mathcal{I} \otimes_\mathcal{O} \mathcal{F}$ is an isomorphism, and*
2. *the class $o(\mathcal{F}, \mathcal{I} \otimes_\mathcal{O} \mathcal{F}, 1) \in Ext^2_\mathcal{O}(\mathcal{F}, \mathcal{I} \otimes_\mathcal{O} \mathcal{F})$ of Lemma 65.9.4 is zero.*

Proof. This follows immediately from the characterization of $\mathcal{O}'$-modules flat over $\mathcal{O}_{\mathcal{B}'}$ of Lemma 65.10.2 and Lemma 65.9.4. □

65.11. Application to flat modules on flat thickenings of ringed topoi

Consider a commutative diagram

$$\begin{array}{ccc} (Sh(\mathcal{C}), \mathcal{O}) & \xrightarrow[i]{} & (Sh(\mathcal{D}), \mathcal{O}') \\ \downarrow{\scriptstyle f} & & \downarrow{\scriptstyle f'} \\ (Sh(\mathcal{B}), \mathcal{O}_\mathcal{B}) & \xrightarrow{t} & (Sh(\mathcal{B}'), \mathcal{O}_{\mathcal{B}'}) \end{array}$$

of ringed topoi whose horizontal arrows are first order thickenings as in Situation 65.8.1. Set $\mathcal{I} = \mathrm{Ker}(i^\sharp) \subset \mathcal{O}'$ and $\mathcal{J} = \mathrm{Ker}(t^\sharp) \subset \mathcal{O}_{\mathcal{B}'}$. Let $\mathcal{F}$ be an $\mathcal{O}$-module. Assume that

1. (f, f') is a strict morphism of thickenings,
2. f' is flat, and
3. $\mathcal{F}$ is flat over $\mathcal{O}_\mathcal{B}$.

Note that (1) + (2) imply that $\mathcal{I} = f^*\mathcal{J}$ (apply Lemma 65.10.2 to $\mathcal{O}'$). The theory of the preceding section is especially nice under these assumptions. We summarize the results already obtained in the following lemma.

Lemma 65.11.1. *In the situation above.*

1. *There exists an $\mathcal{O}'$-module $\mathcal{F}'$ flat over $\mathcal{O}_{\mathcal{B}'}$ with $i^*\mathcal{F}' \cong \mathcal{F}$, if and only if the class $o(\mathcal{F}, f^*\mathcal{J} \otimes_\mathcal{O} \mathcal{F}, 1) \in Ext^2_\mathcal{O}(\mathcal{F}, f^*\mathcal{J} \otimes_\mathcal{O} \mathcal{F})$ of Lemma 65.9.4 is zero.*
2. *If such a module exists, then the set of isomorphism classes of lifts is principal homogeneous under $Ext^1_\mathcal{O}(\mathcal{F}, f^*\mathcal{J} \otimes_\mathcal{O} \mathcal{F})$.*
3. *Given a lift $\mathcal{F}'$, the set of automorphisms of $\mathcal{F}'$ which pull back to $id_\mathcal{F}$ is canonically isomorphic to $Ext^0_\mathcal{O}(\mathcal{F}, f^*\mathcal{J} \otimes_\mathcal{O} \mathcal{F})$.*

Proof. Part (1) follows from Lemma 65.10.8 as we have seen above that $\mathcal{I} = f^*\mathcal{J}$. Part (2) follows from Lemma 65.10.7. Part (3) follows from Lemma 65.10.4. □

Situation 65.11.2. Let $f : (Sh(\mathcal{C}), \mathcal{O}) \to (Sh(\mathcal{B}), \mathcal{O}_\mathcal{B})$ be a morphism of ringed topoi. Consider a commutative diagram

$$\begin{array}{ccccc} (Sh(\mathcal{C}'_1), \mathcal{O}'_1) & \xrightarrow[h]{} & (Sh(\mathcal{C}'_2), \mathcal{O}'_2) & \longrightarrow & (Sh(\mathcal{C}'_3), \mathcal{O}'_3) \\ \downarrow{\scriptstyle f'_1} & & \downarrow{\scriptstyle f'_2} & & \downarrow{\scriptstyle f'_3} \\ (Sh(\mathcal{B}'_1), \mathcal{O}_{\mathcal{B}'_1}) & \longrightarrow & (Sh(\mathcal{B}'_2), \mathcal{O}_{\mathcal{B}'_2}) & \longrightarrow & (Sh(\mathcal{B}'_3), \mathcal{O}_{\mathcal{B}'_3}) \end{array}$$

where (a) the top row is a short exact sequence of first order thickenings of $(Sh(\mathcal{C}), \mathcal{O})$, (b) the lower row is a short exact sequence of first order thickenings of $(Sh(\mathcal{B}), \mathcal{O}_\mathcal{B})$, (c) each f'_i restricts to f, (d) each pair (f, f'_i) is a strict morphism of thickenings,

and (e) each f_i' is flat. Finally, let $\mathcal{F}_2'$ be an $\mathcal{O}_2'$-module flat over $\mathcal{O}_{\mathcal{B}_2'}$ and set $\mathcal{F} = \mathcal{F}_2' \otimes \mathcal{O}$. Let $\pi : (Sh(\mathcal{C}_1'), \mathcal{O}_1') \to (Sh(\mathcal{C}), \mathcal{O})$ be the canonical splitting (Remark 65.9.9).

Lemma 65.11.3. *In Situation 65.11.2 the modules $\pi^*\mathcal{F}$ and $h^*\mathcal{F}_2'$ are $\mathcal{O}_1'$-modules flat over $\mathcal{O}_{\mathcal{B}_1'}$ restricting to $\mathcal{F}$ on $(Sh(\mathcal{C}), \mathcal{O})$. Their difference (Lemma 65.11.1) is an element θ of $Ext^1_{\mathcal{O}}(\mathcal{F}, f^*\mathcal{J}_1 \otimes_{\mathcal{O}} \mathcal{F})$ whose boundary in $Ext^2_{\mathcal{O}}(\mathcal{F}, f^*\mathcal{J}_3 \otimes_{\mathcal{O}} \mathcal{F})$ equals the obstruction (Lemma 65.11.1) to lifting $\mathcal{F}$ to an $\mathcal{O}_3'$-module flat over $\mathcal{O}_{\mathcal{B}_3'}$.*

Proof. Note that both $\pi^*\mathcal{F}$ and $h^*\mathcal{F}_2'$ restrict to $\mathcal{F}$ on $(Sh(\mathcal{C}), \mathcal{O})$ and that the kernels of $\pi^*\mathcal{F} \to \mathcal{F}$ and $h^*\mathcal{F}_2' \to \mathcal{F}$ are given by $f^*\mathcal{J}_1 \otimes_{\mathcal{O}} \mathcal{F}$. Hence flatness by Lemma 65.10.2. Taking the boundary makes sense as the sequence of modules

$$0 \to f^*\mathcal{J}_3 \otimes_{\mathcal{O}} \mathcal{F} \to f^*\mathcal{J}_2 \otimes_{\mathcal{O}} \mathcal{F} \to f^*\mathcal{J}_1 \otimes_{\mathcal{O}} \mathcal{F} \to 0$$

is short exact due to the assumptions in Situation 65.11.2 and the fact that $\mathcal{F}$ is flat over $\mathcal{O}_{\mathcal{B}}$. The statement on the obstruction class is a direct translation of the result of Remark 65.9.10 to this particular situation. □

65.12. Deformations of ringed topoi and the naive cotangent complex

In this section we use the naive cotangent complex to do a little bit of deformation theory. We start with a first order thickening $t : (Sh(\mathcal{B}), \mathcal{O}_{\mathcal{B}}) \to (Sh(\mathcal{B}'), \mathcal{O}_{\mathcal{B}'})$ of ringed topoi. We denote $\mathcal{J} = \mathrm{Ker}(t^\sharp)$ and we identify the underlying topoi of $\mathcal{B}$ and $\mathcal{B}'$. Moreover we assume given a morphism of ringed topoi $f : (Sh(\mathcal{C}), \mathcal{O}) \to (Sh(\mathcal{B}), \mathcal{O}_{\mathcal{B}})$, an $\mathcal{O}$-module $\mathcal{G}$, and a map $f^{-1}\mathcal{J} \to \mathcal{G}$ of sheaves of $f^{-1}\mathcal{O}_{\mathcal{B}}$-modules. In this section we ask ourselves whether we can find the question mark fitting into the following diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \mathcal{G} & \longrightarrow & ? & \longrightarrow & \mathcal{O} & \longrightarrow & 0 \\ & & \uparrow{\scriptstyle c} & & \uparrow & & \uparrow & & \\ 0 & \longrightarrow & f^{-1}\mathcal{J} & \longrightarrow & f^{-1}\mathcal{O}_{\mathcal{B}'} & \longrightarrow & f^{-1}\mathcal{O}_{\mathcal{B}} & \longrightarrow & 0 \end{array} \tag{65.12.0.1}$$

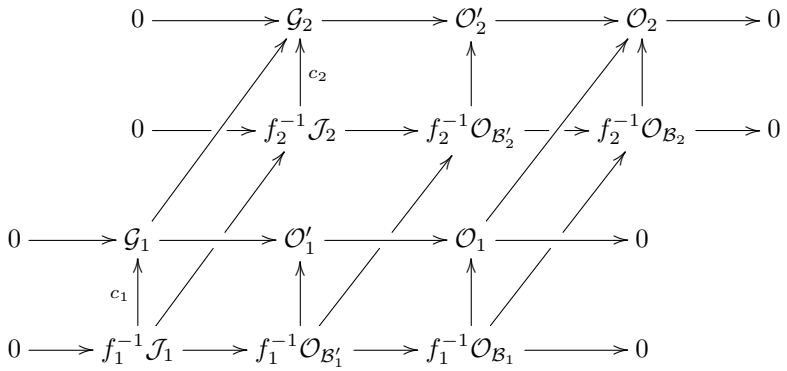
and moreover how unique the solution is (if it exists). More precisely, we look for a first order thickening $i : (Sh(\mathcal{C}), \mathcal{O}) \to (Sh(\mathcal{C}'), \mathcal{O}')$ and a morphism of thickenings (f, f') as in (65.8.1.1) where $\mathrm{Ker}(i^\sharp)$ is identified with $\mathcal{G}$ such that $(f')^\sharp$ induces the given map c. We will say $(Sh(\mathcal{C}'), \mathcal{O}')$ is a *solution* to (65.12.0.1).

Lemma 65.12.1. *Assume given a commutative diagram of morphisms ringed topoi*

$$\begin{array}{ccccc} & & (Sh(\mathcal{C}_2), \mathcal{O}_2) & \xrightarrow[i_2]{} & (Sh(\mathcal{C}_2'), \mathcal{O}_2') \\ & \swarrow^{g} & \downarrow{\scriptstyle f_2} & & \downarrow{\scriptstyle f_2'} \\ & & (Sh(\mathcal{B}_2), \mathcal{O}_{\mathcal{B}_2}) & \xrightarrow{t_2} & (Sh(\mathcal{B}_2'), \mathcal{O}_{\mathcal{B}_2'}) \\ (Sh(\mathcal{C}_1), \mathcal{O}_1) & \xrightarrow[i_1]{} & (Sh(\mathcal{C}_1'), \mathcal{O}_1') & \swarrow & \\ \downarrow{\scriptstyle f_1} & \swarrow & \downarrow{\scriptstyle f_1'} & & \\ (Sh(\mathcal{B}_1), \mathcal{O}_{\mathcal{B}_1}) & \xrightarrow{t_1} & (Sh(\mathcal{B}_1'), \mathcal{O}_{\mathcal{B}_1'}) & & \end{array} \tag{65.12.1.1}$$

whose horizontal arrows are first order thickenings. Set $\mathcal{G}_j = Ker(i_j^\sharp)$ *and assume given a map of* $g^{-1}\mathcal{O}_1$*-modules* $\nu : g^{-1}\mathcal{G}_1 \to \mathcal{G}_2$ *giving rise to the commutative diagram*

(65.12.1.2)



with front and back solutions to (65.12.0.1). (The north-north-west arrows are maps on $\mathcal{C}_2$ *after applying* g^{-1} *to the source.)*

(1) *There exist a canonical element in* $Ext^1_{\mathcal{O}_2}(Lg^* NL_{\mathcal{O}_1/\mathcal{O}_{\mathcal{B}_1}}, \mathcal{G}_2)$ *whose vanishing is a necessary and sufficient condition for the existence of a morphism of ringed topoi* $(Sh(\mathcal{C}'_2), \mathcal{O}'_2) \to (Sh(\mathcal{C}'_1), \mathcal{O}'_1)$ *fitting into (65.12.1.1) compatibly with* ν*.*

(2) *If there exists a morphism* $(Sh(\mathcal{C}'_2), \mathcal{O}'_2) \to (Sh(\mathcal{C}'_1), \mathcal{O}'_1)$ *fitting into (65.12.1.1) compatibly with* ν *the set of all such morphisms is a principal homogeneous space under*

$$\mathrm{Hom}_{\mathcal{O}_1}(\Omega_{\mathcal{O}_1/\mathcal{O}_{\mathcal{B}_1}}, g_*\mathcal{G}_2) = \mathrm{Hom}_{\mathcal{O}_2}(g^*\Omega_{\mathcal{O}_1/\mathcal{O}_{\mathcal{B}_1}}, \mathcal{G}_2) = Ext^0_{\mathcal{O}_2}(Lg^* NL_{\mathcal{O}_1/\mathcal{O}_{\mathcal{B}_1}}, \mathcal{G}_2).$$

Proof. The proof of this lemma is identical to the proof of Lemma 65.7.1. We urge the reader to read that proof instead of this one. We will identify the underlying topoi for every thickening in sight (we have already used this convention in the statement). The equalities in the last statement of the lemma are immediate from the definitions. Thus we will work with the groups $\mathrm{Ext}^k_{\mathcal{O}_2}(Lg^* NL_{\mathcal{O}_1/\mathcal{O}_{\mathcal{B}_1}}, \mathcal{G}_2)$, $k = 0, 1$ in the rest of the proof. We first argue that we can reduce to the case where the underlying topos of all ringed topoi in the lemma is the same.

To do this, observe that $g^{-1} NL_{\mathcal{O}_1/\mathcal{O}_{\mathcal{B}_1}}$ is equal to the naive cotangent complex of the homomorphism of sheaves of rings $g^{-1}f_1^{-1}\mathcal{O}_{\mathcal{B}_1} \to g^{-1}\mathcal{O}_1$, see Modules on Sites, Lemma 18.32.5. Moreover, the degree 0 term of $NL_{\mathcal{O}_1/\mathcal{O}_{\mathcal{B}_1}}$ is a flat $\mathcal{O}_1$-module, hence the canonical map

$$Lg^* NL_{\mathcal{O}_1/\mathcal{O}_{\mathcal{B}_1}} \longrightarrow g^{-1} NL_{\mathcal{O}_1/\mathcal{O}_{\mathcal{B}_1}} \otimes_{g^{-1}\mathcal{O}_1} \mathcal{O}_2$$

induces an isomorphism on cohomology sheaves in degrees 0 and -1. Thus we may replace the Ext groups of the lemma with

$$\mathrm{Ext}^k_{g^{-1}\mathcal{O}_1}(g^{-1} NL_{\mathcal{O}_1/\mathcal{O}_{\mathcal{B}_1}}, \mathcal{G}_2) = \mathrm{Ext}^k_{g^{-1}\mathcal{O}_1}(NL_{g^{-1}\mathcal{O}_1/g^{-1}f_1^{-1}\mathcal{O}_{\mathcal{B}_1}}, \mathcal{G}_2)$$

The set of morphism of ringed topoi $(Sh(\mathcal{C}'_2), \mathcal{O}'_2) \to (Sh(\mathcal{C}'_1), \mathcal{O}'_1)$ fitting into (65.12.1.1) compatibly with ν is in one-to-one bijection with the set of homomorphisms of $g^{-1}f_1^{-1}\mathcal{O}_{\mathcal{B}'_1}$-algebras $g^{-1}\mathcal{O}'_1 \to \mathcal{O}'_2$ which are compatible with $f^\sharp$ and ν. In this way we see that we may assume we have a diagram (65.12.1.2) of sheaves on a site $\mathcal{C}$

(with $f_1 = f_2 = \text{id}$ on underlying topoi) and we are looking to find a homomorphism of sheaves of rings $\mathcal{O}'_1 \to \mathcal{O}'_2$ fitting into it.

In the rest of the proof of the lemma we assume all underlying topological spaces are the same, i.e., we have a diagram (65.12.1.2) of sheaves on a site $\mathcal{C}$ (with $f_1 = f_2 = \text{id}$ on underlying topoi) and we are looking for homomorphisms of sheaves of rings $\mathcal{O}'_1 \to \mathcal{O}'_2$ fitting into it. As ext groups we will use $\text{Ext}^k_{\mathcal{O}_1}(NL_{\mathcal{O}_1/\mathcal{O}_{\mathcal{B}_1}}, \mathcal{G}_2)$, $k = 0, 1$.

Step 1. Construction of the obstruction class. Consider the sheaf of sets

$$\mathcal{E} = \mathcal{O}'_1 \times_{\mathcal{O}_2} \mathcal{O}'_2$$

This comes with a surjective map $\alpha : \mathcal{E} \to \mathcal{O}_1$ and hence we can use $NL(\alpha)$ instead of $NL_{\mathcal{O}_1/\mathcal{O}_{\mathcal{B}_1}}$, see Modules on Sites, Lemma 18.34.2. Set

$$\mathcal{I}' = \text{Ker}(\mathcal{O}_{\mathcal{B}'_1}[\mathcal{E}] \to \mathcal{O}_1) \quad \text{and} \quad \mathcal{I} = \text{Ker}(\mathcal{O}_{\mathcal{B}_1}[\mathcal{E}] \to \mathcal{O}_1)$$

There is a surjection $\mathcal{I}' \to \mathcal{I}$ whose kernel is $\mathcal{J}_1\mathcal{O}_{\mathcal{B}'_1}[\mathcal{E}]$. We obtain two homomorphisms of $\mathcal{O}_{\mathcal{B}'_2}$-algebras

$$a : \mathcal{O}_{\mathcal{B}'_1}[\mathcal{E}] \to \mathcal{O}'_1 \quad \text{and} \quad b : \mathcal{O}_{\mathcal{B}'_1}[\mathcal{E}] \to \mathcal{O}'_2$$

which induce maps $a|_{\mathcal{I}'} : \mathcal{I}' \to \mathcal{G}_1$ and $b|_{\mathcal{I}'} : \mathcal{I}' \to \mathcal{G}_2$. Both a and b annihilate $(\mathcal{I}')^2$. Moreover a and b agree on $\mathcal{J}_1\mathcal{O}_{\mathcal{B}'_1}[\mathcal{E}]$ as maps into $\mathcal{G}_2$ because the left hand square of (65.12.1.2) is commutative. Thus the difference $b|_{\mathcal{I}'} - \nu \circ a|_{\mathcal{I}'}$ induces a well defined $\mathcal{O}_1$-linear map

$$\xi : \mathcal{I}/\mathcal{I}^2 \longrightarrow \mathcal{G}_2$$

which sends the class of a local section f of $\mathcal{I}$ to $a(f') - \nu(b(f'))$ where f' is a lift of f to a local section of $\mathcal{I}'$. We let $[\xi] \in \text{Ext}^1_{\mathcal{O}_1}(NL(\alpha), \mathcal{G}_2)$ be the image (see below).

Step 2. Vanishing of $[\xi]$ is necessary. Let us write $\Omega = \Omega_{\mathcal{O}_{\mathcal{B}_1}[\mathcal{E}]/\mathcal{O}_{\mathcal{B}_1}} \otimes_{\mathcal{O}_{\mathcal{B}_1}[\mathcal{E}]} \mathcal{O}_1$. Observe that $NL(\alpha) = (\mathcal{I}/\mathcal{I}^2 \to \Omega)$ fits into a distinguished triangle

$$\Omega[0] \to NL(\alpha) \to \mathcal{I}/\mathcal{I}^2[1] \to \Omega[1]$$

Thus we see that $[\xi]$ is zero if and only if ξ is a composition $\mathcal{I}/\mathcal{I}^2 \to \Omega \to \mathcal{G}_2$ for some map $\Omega \to \mathcal{G}_2$. Suppose there exists a homomorphisms of sheaves of rings $\varphi : \mathcal{O}'_1 \to \mathcal{O}'_2$ fitting into (65.12.1.2). In this case consider the map $\mathcal{O}'_1[\mathcal{E}] \to \mathcal{G}_2$, $f' \mapsto b(f') - \varphi(a(f'))$. A calculation shows this annihilates $\mathcal{J}_1\mathcal{O}_{\mathcal{B}'_1}[\mathcal{E}]$ and induces a derivation $\mathcal{O}_{\mathcal{B}_1}[\mathcal{E}] \to \mathcal{G}_2$. The resulting linear map $\Omega \to \mathcal{G}_2$ witnesses the fact that $[\xi] = 0$ in this case.

Step 3. Vanishing of $[\xi]$ is sufficient. Let $\theta : \Omega \to \mathcal{G}_2$ be a $\mathcal{O}_1$-linear map such that ξ is equal to $\theta \circ (\mathcal{I}/\mathcal{I}^2 \to \Omega)$. Then a calculation shows that

$$b + \theta \circ d : \mathcal{O}_{\mathcal{B}'_1}[\mathcal{E}] \longrightarrow \mathcal{O}'_2$$

annihilates $\mathcal{I}'$ and hence defines a map $\mathcal{O}'_1 \to \mathcal{O}'_2$ fitting into (65.12.1.2).

Proof of (2) in the special case above. Omitted. Hint: This is exactly the same as the proof of (2) of Lemma 65.2.1. □

Lemma 65.12.2. *Let $\mathcal{C}$ be a site. Let $\mathcal{A} \to \mathcal{B}$ be a homomorphism of sheaves of rings on $\mathcal{C}$. Let $\mathcal{G}$ be a $\mathcal{B}$-module. Let $\xi \in Ext^1_{\mathcal{B}}(NL_{\mathcal{B}/\mathcal{A}}, \mathcal{G})$. There exists a map of sheaves of sets $\alpha : \mathcal{E} \to \mathcal{B}$ such that $\xi \in Ext^1_{\mathcal{B}}(NL(\alpha), \mathcal{G})$ is the class of a map $\mathcal{I}/\mathcal{I}^2 \to \mathcal{G}$ (see proof for notation).*

Proof. Recall that given $\alpha : \mathcal{E} \to \mathcal{B}$ such that $\mathcal{A}[\mathcal{E}] \to \mathcal{B}$ is surjective with kernel $\mathcal{I}$ the complex $NL(\alpha) = (\mathcal{I}/\mathcal{I}^2 \to \Omega_{\mathcal{A}[\mathcal{E}]/\mathcal{A}} \otimes_{\mathcal{A}[\mathcal{E}]} \mathcal{B})$ is canonically isomorphic to $NL_{\mathcal{B}/\mathcal{A}}$, see Modules on Sites, Lemma 18.34.2. Observe moreover, that $\Omega = \Omega_{\mathcal{A}[\mathcal{E}]/\mathcal{A}} \otimes_{\mathcal{A}[\mathcal{E}]} \mathcal{B}$ is the sheaf associated to the presheaf $U \mapsto \bigoplus_{e \in \mathcal{E}(U)} \mathcal{B}(U)$. In other words, Ω is the free $\mathcal{B}$-module on the sheaf of sets $\mathcal{E}$ and in particular there is a canonical map $\mathcal{E} \to \Omega$.

Having said this, pick some $\mathcal{E}$ (for example $\mathcal{E} = \mathcal{B}$ as in the definition of the naive cotangent complex). The obstruction to writing ξ as the class of a map $\mathcal{I}/\mathcal{I}^2 \to \mathcal{G}$ is an element in $\mathrm{Ext}^1_{\mathcal{B}}(\Omega, \mathcal{G})$. Say this is represented by the extension $0 \to \mathcal{G} \to \mathcal{H} \to \Omega \to 0$ of $\mathcal{B}$-modules. Consider the sheaf of sets $\mathcal{E}' = \mathcal{E} \times_\Omega \mathcal{H}$ which comes with an induced map $\alpha' : \mathcal{E}' \to \mathcal{B}$. Let $\mathcal{I}' = \mathrm{Ker}(\mathcal{A}[\mathcal{E}'] \to \mathcal{B})$ and $\Omega' = \Omega_{\mathcal{A}[\mathcal{E}']/\mathcal{A}} \otimes_{\mathcal{A}[\mathcal{E}']} \mathcal{B}$. The pullback of ξ under the quasi-isomorphism $NL(\alpha') \to NL(\alpha)$ maps to zero in $\mathrm{Ext}^1_{\mathcal{B}}(\Omega', \mathcal{G})$ because the pullback of the extension $\mathcal{H}$ by the map $\Omega' \to \Omega$ is split as Ω' is the free $\mathcal{B}$-module on the sheaf of sets $\mathcal{E}'$ and since by construction there is a commutative diagram

$$\begin{array}{ccc} \mathcal{E}' & \longrightarrow & \mathcal{E} \\ \downarrow & & \downarrow \\ \mathcal{H} & \longrightarrow & \Omega \end{array}$$

This finishes the proof. □

Lemma 65.12.3. *If there exists a solution to (65.12.0.1), then the set of isomorphism classes of solutions is principal homogeneous under $Ext^1_{\mathcal{O}}(NL_{\mathcal{O}/\mathcal{O}_\mathcal{B}}, \mathcal{G})$.*

Proof. We observe right away that given two solutions $\mathcal{O}'_1$ and $\mathcal{O}'_2$ to (65.12.0.1) we obtain by Lemma 65.12.1 an obstruction element $o(\mathcal{O}'_1, \mathcal{O}'_2) \in \mathrm{Ext}^1_{\mathcal{O}}(NL_{\mathcal{O}/\mathcal{O}_\mathcal{B}}, \mathcal{G})$ to the existence of a map $\mathcal{O}'_1 \to \mathcal{O}'_2$. Clearly, this element is the obstruction to the existence of an isomorphism, hence separates the isomorphism classes. To finish the proof it therefore suffices to show that given a solution $\mathcal{O}'$ and an element $\xi \in \mathrm{Ext}^1_{\mathcal{O}}(NL_{\mathcal{O}/\mathcal{O}_\mathcal{B}}, \mathcal{G})$ we can find a second solution $\mathcal{O}'_\xi$ such that $o(\mathcal{O}', \mathcal{O}'_\xi) = \xi$.

Pick $\alpha : \mathcal{E} \to \mathcal{O}$ as in Lemma 65.12.2 for the class ξ. Consider the surjection $f^{-1}\mathcal{O}_\mathcal{B}[\mathcal{E}] \to \mathcal{O}$ with kernel $\mathcal{I}$ and corresponding naive cotangent complex $NL(\alpha) = (\mathcal{I}/\mathcal{I}^2 \to \Omega_{f^{-1}\mathcal{O}_\mathcal{B}[\mathcal{E}]/f^{-1}\mathcal{O}_\mathcal{B}} \otimes_{f^{-1}\mathcal{O}_\mathcal{B}[\mathcal{E}]} \mathcal{O})$. By the lemma ξ is the class of a morphism $\delta : \mathcal{I}/\mathcal{I}^2 \to \mathcal{G}$. After replacing $\mathcal{E}$ by $\mathcal{E} \times_\mathcal{O} \mathcal{O}'$ we may also assume that α factors through a map $\alpha' : \mathcal{E} \to \mathcal{O}'$.

These choices determine an $f^{-1}\mathcal{O}_{\mathcal{B}'}$-algebra map $\varphi : \mathcal{O}_{\mathcal{B}'}[\mathcal{E}] \to \mathcal{O}'$. Let $\mathcal{I}' = \mathrm{Ker}(\varphi)$. Observe that φ induces a map $\varphi|_{\mathcal{I}'} : \mathcal{I}' \to \mathcal{G}$ and that $\mathcal{O}'$ is the pushout, as in the following diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \mathcal{G} & \longrightarrow & \mathcal{O}' & \longrightarrow & \mathcal{O} & \longrightarrow & 0 \\ & & \uparrow{\scriptstyle \varphi|_{\mathcal{I}'}} & & \uparrow & & \uparrow{\scriptstyle =} & & \\ 0 & \longrightarrow & \mathcal{I}' & \longrightarrow & f^{-1}\mathcal{O}_{\mathcal{B}'}[\mathcal{E}] & \longrightarrow & \mathcal{O} & \longrightarrow & 0 \end{array}$$

Let $\psi : \mathcal{I}' \to \mathcal{G}$ be the sum of the map $\varphi|_{\mathcal{I}'}$ and the composition

$$\mathcal{I}' \to \mathcal{I}'/(\mathcal{I}')^2 \to \mathcal{I}/\mathcal{I}^2 \xrightarrow{\delta} \mathcal{G}.$$

Then the pushout along ψ is an other ring extension $\mathcal{O}'_\xi$ fitting into a diagram as above. A calculation (omitted) shows that $o(\mathcal{O}', \mathcal{O}'_\xi) = \xi$ as desired. □

Lemma 65.12.4. *Let $(Sh(\mathcal{B}), \mathcal{O}_\mathcal{B})$ be a ringed topos and let $\mathcal{J}$ be an $\mathcal{O}_\mathcal{B}$-module.*

(1) *The set of extensions of sheaves of rings $0 \to \mathcal{J} \to \mathcal{O}_{\mathcal{B}'} \to \mathcal{O}_\mathcal{B} \to 0$ where $\mathcal{J}$ is an ideal of square zero is canonically bijective to $Ext^1_{\mathcal{O}_\mathcal{B}}(NL_{\mathcal{O}_\mathcal{B}/\mathbf{Z}}, \mathcal{J})$.*

(2) *Given a morphism of ringed topoi $f : (Sh(\mathcal{C}), \mathcal{O}) \to (Sh(\mathcal{B}), \mathcal{O}_\mathcal{B})$, an $\mathcal{O}$-module $\mathcal{G}$, an $f^{-1}\mathcal{O}_\mathcal{B}$-module map $c : f^{-1}\mathcal{J} \to \mathcal{G}$, and given extensions of sheaves of rings with square zero kernels:*
   (a) *$0 \to \mathcal{J} \to \mathcal{O}_{\mathcal{B}'} \to \mathcal{O}_\mathcal{B} \to 0$ corresponding to $\alpha \in Ext^1_{\mathcal{O}_\mathcal{B}}(NL_{\mathcal{O}_\mathcal{B}/\mathbf{Z}}, \mathcal{J})$,*
   (b) *$0 \to \mathcal{G} \to \mathcal{O}' \to \mathcal{O} \to 0$ corresponding to $\beta \in Ext^1_\mathcal{O}(NL_{\mathcal{O}/\mathbf{Z}}, \mathcal{G})$*
   then there is a morphism $(Sh(\mathcal{C}), \mathcal{O}') \to (Sh(\mathcal{B}, \mathcal{O}_{\mathcal{B}'})$ fitting into a diagram (65.12.0.1) if and only if β and α map to the same element of $Ext^1_\mathcal{O}(Lf^* NL_{\mathcal{O}_\mathcal{B}/\mathbf{Z}}, \mathcal{G})$.

Proof. To prove this we apply the previous results where we work over the base ringed topos $(Sh(*), \mathbf{Z})$ with trivial thickening. Part (1) follows from Lemma 65.12.3 and the fact that there exists a solution, namely $\mathcal{J} \oplus \mathcal{O}_\mathcal{B}$. Part (2) follows from Lemma 65.12.1 and a compatibility between the constructions in the proofs of Lemmas 65.12.3 and 65.12.1 whose statement and proof we omit. □

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CHAPTER 66

The Cotangent Complex

66.1. Introduction

The goal of this chapter is to construct the cotangent complex of a ring map, of a morphism of schemes, and of a morphism of algebraic spaces. Some references are the notes [**Qui**], the paper [**Qui70**], and the books [**And67**] and [**Ill72**].

66.2. Advice for the reader

In writing this chapter we have tried to minimize the use of simplicial techniques. We view the choice of a *resolution* $P_\bullet$ of a ring B over a ring A as a tool to calculating the *homology* of abelian sheaves on the category $\mathcal{C}_{B/A}$, see Remark 66.5.5. This is similar to the role played by a "good cover" to compute cohomology using the Čech complex. To read a bit on homology on categories, please visit Cohomology on Sites, Section 21.28. The derived lower shriek functor $L\pi_!$ is to homology what $R\Gamma(\mathcal{C}_{B/A}, -)$ is to cohomology. The category $\mathcal{C}_{B/A}$, studied in Section 66.4, is the opposite of the category of factorizations $A \to P \to B$ where P is a polynomial algebra over A. This category comes with maps of sheaves of rings

$$\underline{A} \longrightarrow \mathcal{O} \longrightarrow \underline{B}$$

where over the object $U = (P \to B)$ we have $\mathcal{O}(U) = P$. It turns out that we obtain the cotangent complex of B over A as

$$L_{B/A} = L\pi_!(\Omega_{\mathcal{O}/\underline{A}} \otimes_{\mathcal{O}} \underline{B})$$

see Lemma 66.4.3. We have consistently tried to use this point of view to prove the basic properties of cotangent complexes of ring maps. In particular, all of the results can be proven without relying on the existence of standard resolutions, although we have not done so. The theory is quite satisfactory, except that perhaps the proof of the fundamental triangle (Proposition 66.7.4) uses just a little bit more theory on derived lower shriek functors. To provide the reader with an alternative, we give a rather complete sketch of an approach to this result based on simple properties of standard resolutions in Remarks 66.7.5 and 66.7.6.

Our approach to the cotangent complex for morphisms of ringed topoi, morphisms of schemes, morphisms of algebraic spaces, etc is to deduce as much as possible from the case of "plain ring maps" discussed above.

66.3. The cotangent complex of a ring map

Let A be a ring. Let Alg_A be the category of A-algebras. Consider the pair of adjoint functors (F, i) where $i : Alg_A \to Sets$ is the forgetful functor and $F : Sets \to Alg_A$ assigns to a set E the polynomial algebra $A[E]$ on E over A. Let $X_\bullet$ be the simplicial object of $\mathrm{Fun}(Alg_A, Alg_A)$ constructed in Simplicial, Section 14.31.

Consider an A-algebra B. Denote $P_\bullet = X_\bullet(B)$ the resulting simplicial A-algebra. Recall that $P_0 = A[B]$, $P_1 = A[A[B]]$, and so on. In particular each term P_n is a polynomial A-algebra. Recall also that there is an augmentation

$$\epsilon : P_\bullet \longrightarrow B$$

where we view B as a constant simplicial A-algebra.

Definition 66.3.1. Let $A \to B$ be a ring map. The *standard resolution of B over A* is the augmentation $\epsilon : P_\bullet \to A$ with terms

$$P_0 = A[B], \quad P_1 = A[A[B]], \quad \ldots$$

and maps as constructed above.

It will turn out that we can use the standard resolution to compute left derived functors in certain settings.

Definition 66.3.2. The *cotangent complex* $L_{B/A}$ of a ring map $A \to B$ is the complex of B-modules associated to the simplicial B-module

$$\Omega_{P_\bullet/A} \otimes_{P_\bullet, \epsilon} B$$

where $\epsilon : P_\bullet \to B$ is the standard resolution of B over A.

In Simplicial, Section 14.21 we associate a chain complex to a simplicial module, but here we work with cochain complexes. Thus the term $L^{-n}_{B/A}$ in degree $-n$ is the B-module $\Omega_{P_n/A} \otimes_{P_n, \epsilon_n} B$ and $L^m_{B/A} = 0$ for $m > 0$.

Remark 66.3.3. Let $A \to B$ be a ring map. Let $\mathcal{A}$ be the category of arrows $\psi : C \to B$ of A-algebras and let $\mathcal{S}$ be the category of maps $E \to B$ where E is a set. There are adjoint functors $i : \mathcal{A} \to \mathcal{S}$ (the forgetful functor) and $F : \mathcal{S} \to \mathcal{A}$ which sends $E \to B$ to $A[E] \to B$. Let $X_\bullet$ be the simplicial object of $\mathrm{Fun}(\mathcal{A}, \mathcal{A})$ constructed in Simplicial, Section 14.31. The diagram

$$\begin{array}{ccc} \mathcal{A} & \rightleftarrows & \mathcal{S} \\ \downarrow & & \downarrow \\ Alg_A & \rightleftarrows & Sets \end{array}$$

commutes. It follows that $X_\bullet(\mathrm{id}_B : B \to B)$ is equal to the standard resolution of B over A.

Lemma 66.3.4. *Let $A_i \to B_i$ be a system of ring maps over a directed index set I. Then* $\operatorname{colim} L_{A_i/B_i} = L_{\operatorname{colim} A_i / \operatorname{colim} B_i}$.

Proof. This is true because the forgetful functor $i : A\text{-}Alg \to Sets$ and its adjoint $F : Sets \to A\text{-}Alg$ commute with filtered colimits. Moreover, the functor $B/A \mapsto \Omega_{B/A}$ does as well (Algebra, Lemma 10.125.4). □

66.4. Simplicial resolutions and derived lower shriek

Let $A \to B$ be a ring map. Consider the category of A-algebra maps $\alpha : P \to B$ where P is a polynomial algebra over A (in some set[1] of variables). Let $\mathcal{C} = \mathcal{C}_{B/A}$

[1] It suffices to consider sets of cardinality at most the cardinality of B.

denote the **opposite** of this category. The reason for taking the opposite is that we want to think of objects (P, α) as corresponding to the diagram of affine schemes

$$\begin{array}{ccc} \mathrm{Spec}(B) & \longrightarrow & \mathrm{Spec}(P) \\ & \searrow & \downarrow \\ & & \mathrm{Spec}(A) \end{array}$$

We endow $\mathcal{C}$ with the chaotic topology (Sites, Example 7.6.6), i.e., we endow $\mathcal{C}$ with the structure of a site where coverings are given by identities so that all presheaves are sheaves. Moreover, we endow $\mathcal{C}$ with two sheaves of rings. The first is the sheaf $\mathcal{O}$ which sends to object (P, α) to P. Then second is the constant sheaf B, which we will denote $\underline{B}$. We obtain the following diagram of morphisms of ringed topoi

$$\begin{array}{ccc} (Sh(\mathcal{C}), \underline{B}) & \xrightarrow[i]{} & (Sh(\mathcal{C}), \mathcal{O}) \\ \downarrow{\scriptstyle \pi} & & \\ (Sh(*), B) & & \end{array} \tag{66.4.0.1}$$

The morphism i is the identity on underlying topoi and $i^\sharp : \mathcal{O} \to \underline{B}$ is the obvious map. The map π is as in Cohomology on Sites, Example 21.28.1. An important role will be played in the following by the derived functors $Li^* : D(\mathcal{O}) \longrightarrow D(\underline{B})$ left adjoint to $Ri_* = i_* : D(\underline{B}) \to D(\mathcal{O})$ and $L\pi_! : D(\underline{B}) \longrightarrow D(B)$ left adjoint to $\pi^* = \pi^{-1} : D(B) \to D(\underline{B})$.

Lemma 66.4.1. *With notation as above let $P_\bullet$ be a simplicial A-algebra endowed with an augmentation $\epsilon : P_\bullet \to B$. Assume each P_n is a polynomial algebra over A and ϵ is a trivial Kan fibration on underlying simplicial sets. Then*

$$L\pi_!(\mathcal{F}) = \mathcal{F}(P_\bullet, \epsilon)$$

in $D(Ab)$, resp. $D(B)$ functorially in $\mathcal{F}$ in $Ab(\mathcal{C})$, resp. $Mod(\underline{B})$.

Proof. We will use the criterion of Cohomology on Sites, Lemma 21.28.7 to prove this. Given an object $U = (Q, \beta)$ of $\mathcal{C}$ we have to show that

$$S_\bullet = \mathrm{Mor}_{\mathcal{C}}((Q, \beta), (P_\bullet, \epsilon))$$

is homotopy equivalent to a singleton. Write $Q = A[E]$ for some set E (this is possible by our choice of the category $\mathcal{C}$). We see that

$$S_\bullet = \mathrm{Mor}_{Sets}((E, \beta|_E), (P_\bullet, \epsilon))$$

Let $*$ be the constant simplicial set on a singleton. For $b \in B$ let $F_{b,\bullet}$ be the simplicial set defined by the cartesian diagram

$$\begin{array}{ccc} F_{b,\bullet} & \longrightarrow & P_\bullet \\ \downarrow & & \downarrow{\scriptstyle \epsilon} \\ * & \xrightarrow{b} & B \end{array}$$

With this notation $S_\bullet = \prod_{e \in E} F_{\beta(e),\bullet}$. Since we assumed ϵ is a trivial Kan fibration we see that $F_{b,\bullet} \to *$ is a trivial Kan fibration (Simplicial, Lemma 14.28.3). Thus $S_\bullet \to *$ is a trivial Kan fibration (Simplicial, Lemma 14.28.6). Therefore $S_\bullet$ is homotopy equivalent to $*$ (Simplicial, Lemma 14.28.8). $\square$

In particular, we can use the standard resolution of B over A to compute derived lower shriek.

Lemma 66.4.2. *Let $A \to B$ be a ring map. Let $\epsilon : P_\bullet \to B$ be the standard resolution of B over A. Let π be as in (66.4.0.1). Then*

$$L\pi_!(\mathcal{F}) = \mathcal{F}(P_\bullet, \epsilon)$$

in $D(Ab)$, resp. $D(B)$ functorially in $\mathcal{F}$ in $Ab(\mathcal{C})$, resp. $Mod(\underline{B})$.

First proof. We will apply Lemma 66.4.1. Since the terms P_n are polynomial algebras we see the first assumption of that lemma is satisfied. The second assumption is proved as follows. By Simplicial, Lemma 14.31.5 the map ϵ is a homotopy equivalence of underlying simplicial sets. By Simplicial, Lemma 14.29.9 this implies ϵ induces a quasi-isomorphism of associated complexes of abelian groups. By Simplicial, Lemma 14.29.8 this implies that ϵ is a trivial Kan fibration of underlying simplicial sets. □

Second proof. We will use the criterion of Cohomology on Sites, Lemma 21.28.7. Let $U = (Q, \beta)$ be an object of $\mathcal{C}$. We have to show that

$$S_\bullet = \mathrm{Mor}_{\mathcal{C}}((Q, \beta), (P_\bullet, \epsilon))$$

is homotopy equivalent to a singleton. Write $Q = A[E]$ for some set E (this is possible by our choice of the category $\mathcal{C}$). Using the notation of Remark 66.3.3 we see that

$$S_\bullet = \mathrm{Mor}_{\mathcal{S}}((E \to B), i(P_\bullet \to B))$$

By Simplicial, Lemma 14.31.5 the map $i(P_\bullet \to B) \to i(B \to B)$ is a homotopy equivalence in $\mathcal{S}$. Hence $S_\bullet$ is homotopy equivalent to

$$\mathrm{Mor}_{\mathcal{S}}((E \to B), (B \to B)) = \{*\}$$

as desired. □

Lemma 66.4.3. *Let $A \to B$ be a ring map. Let π and i be as in (66.4.0.1). There is a canonical isomorphism*

$$L_{B/A} = L\pi_!(Li^*\Omega_{\mathcal{O}/A}) = L\pi_!(i^*\Omega_{\mathcal{O}/A}) = L\pi_!(\Omega_{\mathcal{O}/A} \otimes_{\mathcal{O}} \underline{B})$$

in $D(B)$.

Proof. For an object $\alpha : P \to B$ of the category $\mathcal{C}$ the module $\Omega_{P/A}$ is a free P-module. Thus $\Omega_{\mathcal{O}/A}$ is a flat $\mathcal{O}$-module. Hence $Li^*\Omega_{\mathcal{O}/A} = i^*\Omega_{\mathcal{O}/A}$ is the sheaf of $\underline{B}$-modules which associates to $\alpha : P \to A$ the B-module $\Omega_{P/A} \otimes_{P,\alpha} B$. By Lemma 66.4.2 we see that the right hand side is computed by the value of this sheaf on the standard resolution which is our definition of the left hand side (Definition 66.3.2). □

Lemma 66.4.4. *If $A \to B$ is a ring map, then $L\pi_!(\pi^{-1}M) = M$ with π as in (66.4.0.1).*

Proof. This follows from Lemma 66.4.1 which tells us $L\pi_!(\pi^{-1}M)$ is computed by $(\pi^{-1}M)(P_\bullet, \epsilon)$ which is the constant simplicial object on M. □

Lemma 66.4.5. *If $A \to B$ is a ring map, then $H^0(L_{B/A}) = \Omega_{B/A}$.*

Proof. We will prove this by a direct calculation. We will use the identification of Lemma 66.4.3. There is clearly a map from $\Omega_{\mathcal{O}/A} \otimes \underline{B}$ to the constant sheaf with value $\Omega_{B/A}$. Thus this map induces a map

$$H^0(L_{B/A}) = H^0(L\pi_!(\Omega_{\mathcal{O}/A} \otimes \underline{B})) = \pi_!(\Omega_{\mathcal{O}/A} \otimes \underline{B}) \to \Omega_{B/A}$$

By choosing an object $P \to B$ of $\mathcal{C}_{B/A}$ with $P \to B$ surjective we see that this map is surjective (by Algebra, Lemma 10.125.6). To show that it is injective, suppose that $P \to B$ is an object of $\mathcal{C}_{B/A}$ and that $\xi \in \Omega_{P/A} \otimes_P B$ is an element which maps to zero in $\Omega_{B/A}$. We first choose factorization $P \to P' \to B$ such that $P' \to B$ is surjective and P' is a polynomial algebra over A. We may replace P by P'. If $B = P/I$, then the kernel $\Omega_{P/A} \otimes_P B \to \Omega_{B/A}$ is the image of I/I^2 (Algebra, Lemma 10.125.9). Say ξ is the image of $f \in I$. Then we consider the two maps $a, b : P' = P[x] \to P$, the first of which maps x to 0 and the second of which maps x to f (in both cases $P[x] \to B$ maps x to zero). We see that ξ and 0 are the image of $\mathrm{d}x \otimes 1$ in $\Omega_{P'/A} \otimes_{P'} B$. Thus ξ and 0 have the same image in the colimit (see Cohomology on Sites, Example 21.28.1) $\pi_!(\Omega_{\mathcal{O}/A} \otimes \underline{B})$ as desired. □

Lemma 66.4.6. *If B is a polynomial algebra over the ring A, then with π as in (66.4.0.1) we have that $\pi_!$ is exact and $\pi_!\mathcal{F} = \mathcal{F}(B \to B)$.*

Proof. This follows from Lemma 66.4.1 which tells us the constant simplicial algebra on B can be used to compute $L\pi_!$. □

Lemma 66.4.7. *If B is a polynomial algebra over the ring A, then $L_{B/A}$ is quasi-isomorphic to $\Omega_{B/A}[0]$.*

Proof. Immediate from Lemmas 66.4.3 and 66.4.6. □

66.5. Constructing a resolution

In the Noetherian finite type case we can construct a "small" simplicial resolution for finite type ring maps.

Lemma 66.5.1. *Let A be a Noetherian ring. Let $A \to B$ be a finite type ring map. Let $\mathcal{A}$ be the category of A-algebra maps $C \to B$. Let $n \geq 0$ and let $P_\bullet$ be a simplicial object of $\mathcal{A}$ such that*

1. *$P_\bullet \to B$ is a trivial Kan fibration of simplicial sets,*
2. *P_k is finite type over A for $k \leq n$,*
3. *$P_\bullet = cosk_n sk_n P_\bullet$ as simplicial objects of $\mathcal{A}$.*

Then P_{n+1} is a finite type A-algebra.

Proof. Although the proof we give of this lemma is straightforward, it is a bit messy. To clarify the idea we explain what happens for low n before giving the proof in general. For example, if $n = 0$, then (3) means that $P_1 = P_0 \times_B P_0$. Since the ring map $P_0 \to B$ is surjective, this is of finite type over A by More on Algebra, Lemma 15.4.1.

If $n = 1$, then (3) means that

$$P_2 = \{(f_0, f_1, f_2) \in P_1^3 \mid d_0 f_0 = d_0 f_1,\ d_1 f_0 = d_0 f_2,\ d_1 f_1 = d_1 f_2\}$$

where the equalities take place in P_0. Observe that the triple

$$(d_0 f_0, d_1 f_0, d_1 f_1) = (d_0 f_1, d_0 f_2, d_1 f_2)$$

is an element of the fibre product $P_0 \times_B P_0 \times_B P_0$ over B because the maps $d_i : P_1 \to P_0$ are morphisms over B. Thus we get a map

$$\psi : P_2 \longrightarrow P_0 \times_B P_0 \times_B P_0$$

The fibre of ψ over an element $(g_0, g_1, g_2) \in P_0 \times_B P_0 \times_B P_0$ is the set of triples (f_0, f_1, f_2) of 1-simplices with $(d_0, d_1)(f_0) = (g_0, g_1)$, $(d_0, d_1)(f_1) = (g_0, g_2)$, and $(d_0, d_1)(f_2) = (g_1, g_2)$. As $P_\bullet \to B$ is a trivial Kan fibration the map $(d_0, d_1) : P_1 \to P_0 \times_B P_0$ is surjective. Thus we see that P_2 fits into the cartesian diagram

$$\begin{array}{ccc} P_2 & \longrightarrow & P_1^3 \\ \downarrow & & \downarrow \\ P_0 \times_B P_0 \times_B P_0 & \longrightarrow & (P_0 \times_B P_0)^3 \end{array}$$

By More on Algebra, Lemma 15.4.2 we conclude. The general case is similar, but requires a bit more notation.

The case $n > 1$. By Simplicial, Lemma 14.17.15 the condition $P_\bullet = \mathrm{cosk}_n \mathrm{sk}_n P_\bullet$ implies the same thing is true in the category of simplicial A-algebras and hence in the category of sets (as the forgetful functor from A-algebras to sets commutes with limits). Thus

$$P_{n+1} = \mathrm{Mor}(\Delta[n+1], P_\bullet) = \mathrm{Mor}(\mathrm{sk}_n \Delta[n+1], \mathrm{sk}_n P_\bullet)$$

by Simplicial, Lemma 14.11.3 and Equation (14.17.1.1). We will prove by induction on $1 \leq k < m \leq n+1$ that the ring

$$Q_{k,m} = \mathrm{Mor}(\mathrm{sk}_k \Delta[m], \mathrm{sk}_k P_\bullet)$$

is of finite type over A. The case $k = 1$, $1 < m \leq n+1$ is entirely similar to the discussion above in the case $n = 1$. Namely, there is a cartesian diagram

$$\begin{array}{ccc} Q_{1,m} & \longrightarrow & P_1^N \\ \downarrow & & \downarrow \\ P_0 \times_B \ldots \times_B P_0 & \longrightarrow & (P_0 \times_B P_0)^N \end{array}$$

where $N = \binom{m+1}{2}$. We conclude as before.

Let $1 \leq k_0 \leq n$ and assume $Q_{k,m}$ is of finite type over A for all $1 \leq k \leq k_0$ and $k < m \leq n+1$. For $k_0 + 1 < m \leq n+1$ we claim there is a cartesian square

$$\begin{array}{ccc} Q_{k_0+1,m} & \longrightarrow & P_{k_0+1}^N \\ \downarrow & & \downarrow \\ Q_{k_0,m} & \longrightarrow & Q_{k_0,k_0+1}^N \end{array}$$

where N is the number of nondegenerate (k_0+1)-simplices of $\Delta[m]$. Namely, to see this is true, think of an element of $Q_{k_0+1,m}$ as a function f from the (k_0+1)-skeleton of $\Delta[m]$ to $P_\bullet$. We can restrict f to the k_0-skeleton which gives the left vertical map of the diagram. We can also restrict to each nondegenerate (k_0+1)-simplex which gives the top horizontal arrow. Moreover, to give such an f is the same thing as giving its restriction to k_0-skeleton and to each nondegenerate (k_0+1)-face,

provided these agree on the overlap, and this is exactly the content of the diagram. Moreover, the fact that $P_\bullet \to B$ is a trivial Kan fibration implies that the map

$$P_{k_0} \to Q_{k_0,k_0+1} = \mathrm{Mor}(\partial\Delta[k_0+1], P_\bullet)$$

is surjective as every map $\partial\Delta[k_0+1] \to B$ can be extended to $\Delta[k_0+1] \to B$ for $k_0 \geq 1$ (small argument about constant simplicial sets omitted). Since by induction hypothesis the rings $Q_{k_0,m}$, Q_{k_0,k_0+1} are finite type A-algebras, so is $Q_{k_0+1,m}$ by More on Algebra, Lemma 15.4.2 once more. □

Proposition 66.5.2. *Let A be a Noetherian ring. Let $A \to B$ be a finite type ring map. There exists a simplicial A-algebra $P_\bullet$ with an augmentation $\epsilon : P_\bullet \to B$ such that each P_n is a polynomial algebra of finite type over A and such that ϵ is a trivial Kan fibration of simplicial sets.*

Proof. Let $\mathcal{A}$ be the category of A-algebra maps $C \to B$. In this proof our simplicial objects and skelet and coskelet functors will be taken in this category.

Choose a polynomial algebra P_0 of finite type over A and a surjection $P_0 \to B$. As a first approximation we take $P_\bullet = \mathrm{cosk}_0(P_0)$. In other words, $P_\bullet$ is the simplicial A-algebra with terms $P_n = P_0 \times_A \ldots \times_A P_0$. (In the final paragraph of the proof this simplicial object will be denoted $P^0_\bullet$.) By Simplicial, Lemma 14.30.3 the map $P_\bullet \to B$ is a trivial Kan fibration of simplicial sets. Also, observe that $P_\bullet = \mathrm{cosk}_0\mathrm{sk}_0 P_\bullet$.

Suppose for some $n \geq 0$ we have constructed $P_\bullet$ (in the final paragraph of the proof this will be $P^n_\bullet$) such that

(a) $P_\bullet \to B$ is a trivial Kan fibration of simplicial sets,
(b) P_k is a finitely generated polynomial algebra for $0 \leq k \leq n$, and
(c) $P_\bullet = \mathrm{cosk}_n\mathrm{sk}_n P_\bullet$

By Lemma 66.5.1 we can find a finitely generated polynomial algebra Q over A and a surjection $Q \to P_{n+1}$. Since P_n is a polynomial algebra the A-algebra maps $s_i : P_n \to P_{n+1}$ lift to maps $s'_i : P_n \to Q$. Set $d'_j : Q \to P_n$ equal to the composition of $Q \to P_{n+1}$ and $d_j : P_{n+1} \to P_n$. We obtain a truncated simplicial object $P'_\bullet$ of $\mathcal{A}$ by setting $P'_k = P_k$ for $k \leq n$ and $P'_{n+1} = Q$ and morphisms $d'_i = d_i$ and $s'_i = s_i$ in degrees $k \leq n-1$ and using the morphisms d'_j and s'_i in degree n. Extend this to a full simplicial object $P'_\bullet$ of $\mathcal{A}$ using cosk_{n+1}. By functoriality of the coskelet functors there is a morphism $P'_\bullet \to P_\bullet$ of simplicial objects extending the given morphism of $(n+1)$-truncated simplicial objects. (This morphism will be denoted $P^{n+1}_\bullet \to P^n_\bullet$ in the final paragraph of the proof.)

Note that conditions (b) and (c) are satisfied for $P'_\bullet$ with n replaced by $n+1$. We claim the map $P'_\bullet \to P_\bullet$ satisfies assumptions (1), (2), (3), and (4) of Simplicial, Lemmas 14.30.1 with $n+1$ instead of n. Conditions (1) and (2) hold by construction. By Simplicial, Lemma 14.17.15 we see that we have $P_\bullet = \mathrm{cosk}_{n+1}\mathrm{sk}_{n+1}P_\bullet$ and $P'_\bullet = \mathrm{cosk}_{n+1}\mathrm{sk}_{n+1}P'_\bullet$ not only in $\mathcal{A}$ but also in the category of A-algebras, whence in the category of sets (as the forgetful functor from A-algebras to sets commutes with all limits). This proves (3) and (4). Thus the lemma applies and $P'_\bullet \to P_\bullet$ is a trivial Kan fibration. By Simplicial, Lemma 14.28.4 we conclude that $P'_\bullet \to B$ is a trivial Kan fibration and (a) holds as well.

To finish the proof we take the inverse limit $P_\bullet = \lim P^n_\bullet$ of the sequence of simplicial algebras

$$\ldots \to P^2_\bullet \to P^1_\bullet \to P^0_\bullet$$

constructed above. The map $P_\bullet \to B$ is a trivial Kan fibration by Simplicial, Lemma 14.28.5. However, the construction above stabilizes in each degree to a fixed finitely generated polynomial algebra as desired. □

Lemma 66.5.3. *Let A be a Noetherian ring. Let $A \to B$ be a finite type ring map. Let π, $\underline{B}$ be as in (66.4.0.1). If $\mathcal{F}$ is an $\underline{B}$-module such that $\mathcal{F}(P, \alpha)$ is a finite B-module for all $\alpha : P = A[x_1, \ldots, x_n] \to B$, then the cohomology modules of $L\pi_!(\mathcal{F})$ are finite B-modules.*

Proof. By Lemma 66.4.1 and Proposition 66.5.2 we can compute $L\pi_!(\mathcal{F})$ by a complex constructed out of the values of $\mathcal{F}$ on finite type polynomial algebras. □

Lemma 66.5.4. *Let A be a Noetherian ring. Let $A \to B$ be a finite type ring map. Then $H^n(L_{B/A})$ is a finite B-module for all $n \in \mathbf{Z}$.*

Proof. Apply Lemmas 66.4.3 and 66.5.3. □

Remark 66.5.5 (Resolutions)**.** Let $A \to B$ be any ring map. Let us call an augmented simplicial A-algebra $\epsilon : P_\bullet \to B$ a *resolution of B over A* if each P_n is a polynomial algebra and ϵ is a trivial Kan fibration of simplicial sets. If $P_\bullet \to B$ is an augmentation of a simplicial A-algebra with each P_n a polynomial algebra surjecting onto B, then the following are equivalent

(1) $\epsilon : P_\bullet \to B$ is a resolution of B over A,
(2) $\epsilon : P_\bullet \to B$ is a quasi-isomorphism on associated complexes,
(3) $\epsilon : P_\bullet \to B$ induces a homotopy equivalence of simplicial sets.

To see this use Simplicial, Lemmas 14.28.8, 14.29.9, and 14.29.8. A resolution $P_\bullet$ of B over A gives a cosimplicial object $U_\bullet$ of $\mathcal{C}_{B/A}$ as in Cohomology on Sites, Lemma 21.28.7 and it follows that

$$L\pi_!\mathcal{F} = \mathcal{F}(P_\bullet)$$

functorially in $\mathcal{F}$, see Lemma 66.4.1. The (formal part of the) proof of Proposition 66.5.2 shows that resolutions exist. We also have seen in the first proof of Lemma 66.4.2 that the standard resolution of B over A is a resolution (so that this terminology doesn't lead to a conflict). However, the argument in the proof of Proposition 66.5.2 shows the existence of resolutions without appealing to the simplicial computations in Simplicial, Section 14.31. Moreover, for *any* choice of resolution we have a canonical isomorphism

$$L_{B/A} = \Omega_{P_\bullet/A} \otimes_{P_\bullet, \epsilon} B$$

in $D(B)$ by Lemma 66.4.3. The freedom to choose an arbitrary resolution can be quite useful.

Lemma 66.5.6. *Let $A \to B$ be a ring map. Let π, $\mathcal{O}$, $\underline{B}$ be as in (66.4.0.1). For any $\mathcal{O}$-module $\mathcal{F}$ we have*

$$L\pi_!(\mathcal{F}) = L\pi_!(Li^*\mathcal{F}) = L\pi_!(\mathcal{F} \otimes^{\mathbf{L}}_{\mathcal{O}} \underline{B})$$

in $D(Ab)$.

Proof. It suffices to verify the assumptions of Cohomology on Sites, Lemma 21.28.12 hold for $\mathcal{O} \to \underline{B}$ on $\mathcal{C}_{B/A}$. We will use the results of Remark 66.5.5 without further mention. Choose a resolution $P_\bullet$ of B over A to get a suitable cosimplicial object $U_\bullet$ of $\mathcal{C}_{B/A}$. Since $P_\bullet \to B$ induces a quasi-isomorphism on associated complexes of abelian groups we see that $L\pi_!\mathcal{O} = B$. On the other hand $L\pi_!\underline{B}$ is computed by

$\underline{B}(U_\bullet) = B$. This verifies the second assumption of Cohomology on Sites, Lemma 21.28.12 and we are done with the proof. □

Lemma 66.5.7. *Let $A \to B$ be a ring map. Let π, $\mathcal{O}$, $\underline{B}$ be as in (66.4.0.1). We have*

$$L\pi_!(\mathcal{O}) = L\pi_!(\underline{B}) = B \quad \text{and} \quad L_{B/A} = L\pi_!(\Omega_{\mathcal{O}/A} \otimes_{\mathcal{O}} \underline{B}) = L\pi_!(\Omega_{\mathcal{O}/A})$$

in $D(Ab)$.

Proof. This is just an application of Lemma 66.5.6 (and the first equality on the right is Lemma 66.4.3). □

Here is a special case of the fundamental triangle that is easy to prove.

Lemma 66.5.8. *Let $A \to B \to C$ be ring maps. If B is a polynomial algebra over A, then there is a distinguished triangle $L_{B/A} \otimes^{\mathbf{L}}_B C \to L_{C/A} \to L_{C/B} \to L_{B/A} \otimes^{\mathbf{L}}_B C[1]$ in $D(C)$.*

Proof. We will use the observations of Remark 66.5.5 without further mention. Choose a resolution $\epsilon : P_\bullet \to C$ of C over B (for example the standard resolution). Since B is a polynomial algebra over A we see that $P_\bullet$ is also a resolution of C over A. Hence $L_{C/A}$ is computed by $\Omega_{P_\bullet/A} \otimes_{P_\bullet,\epsilon} C$ and $L_{C/B}$ is computed by $\Omega_{P_\bullet/B} \otimes_{P_\bullet,\epsilon} C$. Since for each n we have the short exact sequence $0 \to \Omega_{B/A} \otimes_B P_n \to \Omega_{P_n/A} \to \Omega_{P_n/B}$ (Algebra, Lemma 10.131.9) and since $L_{B/A} = \Omega_{B/A}[0]$ (Lemma 66.4.7) we obtain the result. □

Example 66.5.9. Let $A \to B$ be a ring map. In this example we will construct an "explicit" resolution $P_\bullet$ of B over A of length 2. To do this we follow the procedure of the proof of Proposition 66.5.2, see also the discussion in Remark 66.5.5.

We choose a surjection $P_0 = A[u_i] \to B$ where u_i is a set of variables. Choose generators $f_t \in P_0$, $t \in T$ of the ideal $\mathrm{Ker}(P_0 \to B)$. We choose $P_1 = A[u_i, x_t]$ with face maps d_0 and d_1 the unique A-algebra maps with $d_j(u_i) = u_i$ and $d_0(x_t) = 0$ and $d_1(x_t) = f_t$. The map $s_0 : P_0 \to P_1$ is the unique A-algebra map with $s_0(u_i) = u_i$. It is clear that

$$P_1 \xrightarrow{d_0 - d_1} P_0 \to B \to 0$$

is exact, in particular the map $(d_0, d_1) : P_1 \to P_0 \times_B P_0$ is surjective. Thus, if $P_\bullet$ denotes the 1-truncated simplicial A-algebra given by P_0, P_1, d_0, d_1, and s_0, then the augmentation $\mathrm{cosk}_1(P_\bullet) \to B$ is a trivial Kan fibration. The next step of the procedure in the proof of Proposition 66.5.2 is to choose a polynomial algebra P_2 and a surjection

$$P_2 \longrightarrow \mathrm{cosk}_1(P_\bullet)_2$$

Recall that

$$\mathrm{cosk}_1(P_\bullet)_2 = \{(g_0, g_1, g_2) \in P_1^3 \mid d_0(g_0) = d_0(g_1), d_1(g_0) = d_0(g_2), d_1(g_1) = d_1(g_2)\}$$

Thinking of $g_i \in P_1$ as a polynomial in x_t the conditions are

$$g_0(0) = g_1(0), \quad g_0(f_t) = g_2(0), \quad g_1(f_t) = g_2(f_t)$$

Thus $\mathrm{cosk}_1(P_\bullet)_2$ contains the elements $y_t = (x_t, x_t, f_t)$ and $z_t = (0, x_t, x_t)$. Every element G in $\mathrm{cosk}_1(P_\bullet)_2$ is of the form $G = H + (0, 0, g)$ where H is in the image of $A[u_i, y_t, z_t] \to \mathrm{cosk}_1(P_\bullet)_2$. Here $g \in P_1$ is a polynomial with vanishing constant term such that $g(f_t) = 0$ in P_0. Observe that

(1) $g = x_t x_{t'} - f_t x_{t'}$ and
(2) $g = \sum r_t x_t$ with $r_t \in P_0$ if $\sum r_t f_t = 0$ in P_0

are elements of P_1 of the desired form. Let

$$Rel = \mathrm{Ker}(\bigoplus\nolimits_{t \in T} P_0 \longrightarrow P_0), \quad (r_t) \longmapsto \sum r_t f_t$$

We set $P_2 = A[u_i, y_t, z_t, v_r, w_{t,t'}]$ where $r = (r_t) \in Rel$, with map

$$P_2 \longrightarrow \mathrm{cosk}_1(P_\bullet)_2$$

given by $y_t \mapsto (x_t, x_t, f_t)$, $z_t \mapsto (0, x_t, x_t)$, $v_r \mapsto (0, 0, \sum r_t x_t)$, and $w_{t,t'} \mapsto (0, 0, x_t x_{t'} - f_t x_{t'})$. A calculation (omitted) shows that this map is surjective. Our choice of the map displayed above determines the maps $d_0, d_1, d_2 : P_2 \to P_1$. Finally, the procedure in the proof of Proposition 66.5.2 tells us to choose the maps $s_0, s_1 : P_1 \to P_2$ lifting the two maps $P_1 \to \mathrm{cosk}_1(P_\bullet)_2$. It is clear that we can take s_i to be the unique A-algebra maps determined by $s_0(x_t) = y_t$ and $s_1(x_t) = z_t$.

66.6. Functoriality

In this section we consider a commutative square

$$\begin{array}{ccc} B & \longrightarrow & B' \\ \uparrow & & \uparrow \\ A & \longrightarrow & A' \end{array} \tag{66.6.0.1}$$

of ring maps. We claim there is a canonical B-linear map of complexes

$$L_{B/A} \longrightarrow L_{B'/A'}$$

associated to this diagram. Namely, if $P_\bullet \to B$ is the standard resolution of B over A and $P'_\bullet \to B'$ is the standard resolution of B' over A', then there is a canonical map $P_\bullet \to P'_\bullet$ of simplicial A-algebras compatible with the augmentations $P_\bullet \to B$ and $P'_\bullet \to B'$. This can be seen in terms of the construction of standard resolutions in Simplicial, Section 14.31 but in the special case at hand it probably suffices to say simply that the maps

$$P_0 = A[B] \longrightarrow A'[B'] = P'_0, \quad P_1 = A[A[B]] \longrightarrow A'[A'[B']] = P'_1,$$

and so on are given by the given maps $A \to A'$ and $B \to B'$. The desired map $L_{B/A} \to L_{B'/A'}$ then comes from the associated maps $\Omega_{P_n/A} \to \Omega_{P'_n/A'}$.

Another description of the functoriality map can be given as follows. Let $\mathcal{C} = \mathcal{C}_{B/A}$ and $\mathcal{C}' = \mathcal{C}'_{B'/A}$ be the categories considered in Section 66.4. There is a functor

$$u : \mathcal{C} \longrightarrow \mathcal{C}', \quad (P, \alpha) \longmapsto (P \otimes_A A', c \circ (\alpha \otimes 1))$$

where $c : B \otimes_A A' \to B'$ is the obvious map. As discussed in Cohomology on Sites, Example 21.28.3 we obtain a morphism of topoi $g : Sh(\mathcal{C}) \to Sh(\mathcal{C}')$ and a commutative diagram of maps of ringed topoi

$$\begin{array}{ccccc} (Sh(\mathcal{C}), \underline{B}) & \xleftarrow{h} & (Sh(\mathcal{C}'), \underline{B'}) & \xrightarrow{g} & (Sh(\mathcal{C}), \underline{B'}) \\ \downarrow{\scriptstyle \pi} & & \downarrow{\scriptstyle \pi} & & \downarrow{\scriptstyle \pi'} \\ (Sh(*), B) & \xleftarrow{f} & (Sh(*), B') & \longrightarrow & (Sh(*), B') \end{array} \tag{66.6.0.2}$$

Here h is the identity on underlying topoi and given by the ring map $B \to B'$ on sheaves of rings. By Cohomology on Sites, Remark 21.27.7 given $\mathcal{F}$ on $\mathcal{C}$ and $\mathcal{F}'$ on $\mathcal{C}'$ and a transformation $t : \mathcal{F} \to g^{-1}\mathcal{F}'$ we obtain a canonical map $L\pi_!(\mathcal{F}) \to L\pi'_!(\mathcal{F}')$. If we apply this to the sheaves

$$\mathcal{F} : (P, \alpha) \mapsto \Omega_{P/A} \otimes_P B, \quad \mathcal{F}' : (P', \alpha') \mapsto \Omega_{P'/A'} \otimes_{P'} B',$$

and the transformation t given by the canonical maps

$$\Omega_{P/A} \otimes_P B \longrightarrow \Omega_{P \otimes_A A'/A'} \otimes_{P \otimes_A A'} B'$$

to get a canonical map

$$L\pi_!(\Omega_{\mathcal{O}/A} \otimes_{\mathcal{O}} \underline{B}) \longrightarrow L\pi'_!(\Omega_{\mathcal{O}'/A'} \otimes_{\mathcal{O}'} \underline{B'})$$

By Lemma 66.4.3 this gives $L_{B/A} \to L_{B'/A'}$. We omit the verification that this map agrees with the map defined above in terms of simplicial resolutions.

Lemma 66.6.1. *Assume (66.6.0.1) induces a quasi-isomorphism $B \otimes^{\mathbf{L}}_A A' = B'$. Then, with notation as in (66.6.0.2) and $\mathcal{F}' \in Ab(\mathcal{C}')$, we have $L\pi_!(g^{-1}\mathcal{F}') = L\pi'_!(\mathcal{F}')$.*

Proof. We use the results of Remark 66.5.5 without further mention. We will apply Cohomology on Sites, Lemma 21.28.8. Let $P_\bullet \to B$ be a resolution. If we can show that $u(P_\bullet) = P_\bullet \otimes_A A' \to B'$ is a quasi-isomorphism, then we are done. The complex of A-modules $s(P_\bullet)$ associated to $P_\bullet$ (viewed as a simplicial A-module) is a free A-module resolution of B. Namely, P_n is a free A-module and $s(P_\bullet) \to B$ is a quasi-isomorphism. Thus $B \otimes^{\mathbf{L}}_A A'$ is computed by $s(P_\bullet) \otimes_A A' = s(P_\bullet \otimes_A A')$. Therefore the assumption of the lemma signifies that $\epsilon' : P_\bullet \otimes_A A' \to B'$ is a quasi-isomorphism. □

The following lemma in particular applies when $A \to A'$ is flat and $B' = B \otimes_A A'$ (flat base change).

Lemma 66.6.2. *If (66.6.0.1) induces a quasi-isomorphism $B \otimes^{\mathbf{L}}_A A' = B'$, then the functoriality map induces an isomorphism*

$$L_{B/A} \otimes^{\mathbf{L}}_B B' \longrightarrow L_{B'/A'}$$

Proof. We will use the notation introduced in Equation (66.6.0.2). We have

$$L_{B/A} \otimes^{\mathbf{L}}_B B' = L\pi_!(\Omega_{\mathcal{O}/A} \otimes_{\mathcal{O}} \underline{B}) \otimes^{\mathbf{L}}_B B' = L\pi_!(Lh^*(\Omega_{\mathcal{O}/A} \otimes_{\mathcal{O}} \underline{B}))$$

the first equality by Lemma 66.4.3 and the second by Cohomology on Sites, Lemma 21.28.6. Since $\Omega_{\mathcal{O}/A}$ is a flat $\mathcal{O}$-module, we see that $\Omega_{\mathcal{O}/A} \otimes_{\mathcal{O}} \underline{B}$ is a flat $\underline{B}$-module. Thus $Lh^*(\Omega_{\mathcal{O}/A} \otimes_{\mathcal{O}} \underline{B}) = \Omega_{\mathcal{O}/A} \otimes_{\mathcal{O}} \underline{B'}$ which is equal to $g^{-1}(\Omega_{\mathcal{O}'/A'} \otimes_{\mathcal{O}'} \underline{B'})$ by inspection. we conclude by Lemma 66.6.1 and the fact that $L_{B'/A'}$ is computed by $L\pi'_!(\Omega_{\mathcal{O}'/A'} \otimes_{\mathcal{O}'} \underline{B'})$. □

Remark 66.6.3. Suppose that we are given a square (66.6.0.1) such that there exists an arrow $\kappa : B \to A'$ making the diagram commute:

$$\begin{array}{ccc} B & \xrightarrow{\beta} & B' \\ \uparrow & \overset{\kappa}{\searrow} & \uparrow \\ A & \xrightarrow{\alpha} & A' \end{array}$$

In this case we claim the functoriality map $P_\bullet \to P'_\bullet$ is homotopic to the composition $P_\bullet \to B \to A' \to P'_\bullet$. Namely, using κ the functoriality map factors as

$$P_\bullet \to P_{A'/A',\bullet} \to P'_\bullet$$

where $P_{A'/A',\bullet}$ is the standard resolution of A' over A'. Since A' is the polynomial algebra on the empty set over A' we see from Simplicial, Lemma 14.31.5 that the augmentation $\epsilon_{A'/A'} : P_{A'/A',\bullet} \to A'$ is a homotopy equivalence of simplicial rings. Observe that the homotopy inverse map $c : A' \to P_{A'/A',\bullet}$ constructed in the proof of that lemma is just the structure morphism, hence we conclude what we want because the two compositions

$$P_\bullet \longrightarrow P_{A'/A',\bullet} \underset{c \circ \epsilon_{A'/A'}}{\overset{\mathrm{id}}{\rightrightarrows}} P_{A'/A',\bullet} \longrightarrow P'_\bullet$$

are the two maps discussed above and these are homotopic (Simplicial, Remark 14.24.5). Since the second map $P_\bullet \to P'_\bullet$ induces the zero map $\Omega_{P_\bullet/A} \to \Omega_{P'_\bullet/A'}$ we conclude that the functoriality map $L_{B/A} \to L_{B'/A'}$ is homotopic to zero in this case.

Lemma 66.6.4. *Let $A \to B$ and $A \to C$ be ring maps. Then the map $L_{B\times C/A} \to L_{B/A} \oplus L_{C/A}$ is an isomorphism in $D(B \times C)$.*

Proof. Although this lemma can be deduced from the fundamental triangle we will give a direct and elementary proof of this now. Factor the ring map $A \to B \times C$ as $A \to A[x] \to B \times C$ where $x \mapsto (1, 0)$. By Lemma 66.5.8 we have a distinguished triangle

$$L_{A[x]/A} \otimes^{\mathbf{L}}_{A[x]} (B \times C) \to L_{B\times C/A} \to L_{B\times C/A[x]} \to L_{A[x]/A} \otimes^{\mathbf{L}}_{A[x]} (B \times C)[1]$$

in $D(B \times C)$. Similarly we have the distinguished triangles

$$\begin{aligned} L_{A[x]/A} \otimes^{\mathbf{L}}_{A[x]} B \to L_{B/A} \to L_{B/A[x]} \to L_{A[x]/A} \otimes^{\mathbf{L}}_{A[x]} B[1] \\ L_{A[x]/A} \otimes^{\mathbf{L}}_{A[x]} C \to L_{C/A} \to L_{C/A[x]} \to L_{A[x]/A} \otimes^{\mathbf{L}}_{A[x]} C[1] \end{aligned}$$

Thus it suffices to prove the result for $B\times C$ over $A[x]$. Note that $A[x] \to A[x, x^{-1}]$ is flat, that $(B\times C)\otimes_{A[x]} A[x, x^{-1}] = B\otimes_{A[x]} A[x, x^{-1}]$, and that $C\otimes_{A[x]} A[x, x^{-1}] = 0$. Thus by base change (Lemma 66.6.2) the map $L_{B\times C/A[x]} \to L_{B/A[x]} \oplus L_{C/A[x]}$ becomes an isomorphism after inverting x. In the same way one shows that the map becomes an isomorphism after inverting $x - 1$. This proves the lemma. □

66.7. The fundamental triangle

In this section we consider a sequence of ring maps $A \to B \to C$. It is our goal to show that this triangle gives rise to a distinguished triangle

$$L_{B/A} \otimes^{\mathbf{L}}_B C \to L_{C/A} \to L_{C/B} \to L_{B/A} \otimes^{\mathbf{L}}_B C[1] \tag{66.7.0.1}$$

in $D(C)$. This will be proved in Proposition 66.7.4. For an alternative approach see Remark 66.7.5.

Consider the category $\mathcal{C}_{C/B/A}$ wich is the **opposite** of the category whose objects are $(P \to B, Q \to C)$ where

(1) P is a polynomial algebra over A,
(2) $P \to B$ is an A-algebra homomorphism,
(3) Q is a polynomial algebra over P, and

(4) $Q \to C$ is a P-algebra-homomorphism.

We take the opposite as we want to think of $(P \to B, Q \to C)$ as corresponding to the commutative diagram

$$\begin{array}{ccc} \mathrm{Spec}(C) & \longrightarrow & \mathrm{Spec}(Q) \\ \downarrow & & \downarrow \\ \mathrm{Spec}(B) & \longrightarrow & \mathrm{Spec}(P) \\ \downarrow & \swarrow & \\ \mathrm{Spec}(A) & & \end{array}$$

Let $\mathcal{C}_{B/A}$, $\mathcal{C}_{C/A}$, $\mathcal{C}_{C/B}$ be the categories considered in Section 66.4. There are functors

$$\begin{aligned} u_1 : \mathcal{C}_{C/B/A} \to \mathcal{C}_{B/A}, & \quad (P \to B, Q \to C) \mapsto (P \to B) \\ u_2 : \mathcal{C}_{C/B/A} \to \mathcal{C}_{C/A}, & \quad (P \to B, Q \to C) \mapsto (Q \to C) \\ u_3 : \mathcal{C}_{C/B/A} \to \mathcal{C}_{C/B}, & \quad (P \to B, Q \to C) \mapsto (Q \otimes_P B \to C) \end{aligned}$$

These functors induce corresponding morphisms of topoi g_i. Let us denote $\mathcal{O}_i = g_i^{-1}\mathcal{O}$ so that we get morphisms of ringed topoi

$$\begin{aligned} g_1 : (Sh(\mathcal{C}_{C/B/A}), \mathcal{O}_1) & \longrightarrow (Sh(\mathcal{C}_{B/A}), \mathcal{O}) \\ g_2 : (Sh(\mathcal{C}_{C/B/A}), \mathcal{O}_2) & \longrightarrow (Sh(\mathcal{C}_{C/A}), \mathcal{O}) \\ g_3 : (Sh(\mathcal{C}_{C/B/A}), \mathcal{O}_3) & \longrightarrow (Sh(\mathcal{C}_{C/B}), \mathcal{O}) \end{aligned} \tag{66.7.0.2}$$

Let us denote $\pi : Sh(\mathcal{C}_{C/B/A}) \to Sh(*)$, $\pi_1 : Sh(\mathcal{C}_{B/A}) \to Sh(*)$, $\pi_2 : Sh(\mathcal{C}_{C/A}) \to Sh(*)$, and $\pi_3 : Sh(\mathcal{C}_{C/B}) \to Sh(*)$, so that $\pi = \pi_i \circ g_i$. We will obtain our distinguished triangle from the identification of the cotangent complex in Lemma 66.4.3 and the following lemmas.

Lemma 66.7.1. *With notation as in (66.7.0.2) set*

$$\begin{aligned} \Omega_1 &= \Omega_{\mathcal{O}/A} \otimes_{\mathcal{O}} \underline{B} \text{ on } \mathcal{C}_{B/A} \\ \Omega_2 &= \Omega_{\mathcal{O}/A} \otimes_{\mathcal{O}} \underline{C} \text{ on } \mathcal{C}_{C/A} \\ \Omega_3 &= \Omega_{\mathcal{O}/B} \otimes_{\mathcal{O}} \underline{C} \text{ on } \mathcal{C}_{C/B} \end{aligned}$$

Then we have a canonical short exact sequence of sheaves of $\underline{C}$-modules

$$0 \to g_1^{-1}\Omega_1 \otimes_{\underline{B}} \underline{C} \to g_2^{-1}\Omega_2 \to g_3^{-1}\Omega_3 \to 0$$

on $\mathcal{C}_{C/B/A}$.

Proof. Recall that g_i^{-1} is gotten by simply precomposing with u_i. Given an object $U = (P \to B, Q \to C)$ we have a split short exact sequence

$$0 \to \Omega_{P/A} \otimes Q \to \Omega_{Q/A} \to \Omega_{Q/P} \to 0$$

for example by Algebra, Lemma 10.131.9. Tensoring with C over Q we obtain a short exact sequence

$$0 \to \Omega_{P/A} \otimes C \to \Omega_{Q/A} \otimes C \to \Omega_{Q/P} \otimes C \to 0$$

We have $\Omega_{P/A} \otimes C = \Omega_{P/A} \otimes B \otimes C$ whence this is the value of $g_1^{-1}\Omega_1 \otimes_{\underline{B}} \underline{C}$ on U. The module $\Omega_{Q/A} \otimes C$ is the value of $g_2^{-1}\Omega_2$ on U. We have $\Omega_{Q/P} \otimes C = \Omega_{Q \otimes_P B/B} \otimes C$ by Algebra, Lemma 10.125.12 hence this is the value of $g_3^{-1}\Omega_3$ on U. Thus the short exact sequence of the lemma comes from assigning to U the last displayed short exact sequence. $\square$

Lemma 66.7.2. *With notation as in (66.7.0.2) suppose that C is a polynomial algebra over B. Then $L\pi_!(g_3^{-1}\mathcal{F}) = L\pi_{3,!}\mathcal{F} = \pi_{3,!}\mathcal{F}$ for any abelian sheaf $\mathcal{F}$ on $\mathcal{C}_{C/B}$*

Proof. Write $C = B[E]$ for some set E. Choose a resolution $P_\bullet \to B$ of B over A. For every n consider the object $U_n = (P_n \to B, P_n[E] \to C)$ of $\mathcal{C}_{C/B/A}$. Then $U_\bullet$ is a cosimplicial object of $\mathcal{C}_{C/B/A}$. Note that $u_3(U_\bullet)$ is the constant cosimplicial object of $\mathcal{C}_{C/B}$ with value $(C \to C)$. We will prove that the object $U_\bullet$ of $\mathcal{C}_{C/B/A}$ satisfies the hypotheses of Cohomology on Sites, Lemma 21.28.7. This implies the lemma as it shows that $L\pi_!(g_3^{-1}\mathcal{F})$ is computed by the constant simplicial abelian group $\mathcal{F}(C \to C)$ which is the value of $L\pi_{3,!}\mathcal{F} = \pi_{3,!}\mathcal{F}$ by Lemma 66.4.6.

Let $U = (\beta : P \to B, \gamma : Q \to C)$ be an object of $\mathcal{C}_{C/B/A}$. We may write $P = A[S]$ and $Q = A[S \amalg T]$ by the definition of our category $\mathcal{C}_{C/B/A}$. We have to show that

$$\mathrm{Mor}_{\mathcal{C}_{C/B/A}}(U_\bullet, U)$$

is homotopy equivalent to a singleton simplicial set $*$. Observe that this simplicial set is the product

$$\prod\nolimits_{s\in S} F_s \times \prod\nolimits_{t\in T} F'_t$$

where F_s is the corresponding simplicial set for $U_s = (A[\{s\}] \to B, A[\{s\}] \to C)$ and F'_t is the corresponding simplicial set for $U_t = (A \to B, A[\{t\}] \to C)$. Namely, the object U is the product $\prod U_s \times \prod U_t$ in $\mathcal{C}_{C/B/A}$. It suffices each F_s and F'_t is homotopy equivalent to $*$, see Simplicial, Lemma 14.24.10. The case of F_s follows as $P_\bullet \to B$ is a trivial Kan fibration (as a resolution) and F_s is the fibre of this map over $\beta(s)$. (Use Simplicial, Lemmas 14.28.3 and 14.28.8). The case of F'_t is more interesting. Here we are saying that the fibre of

$$P_\bullet[E] \longrightarrow C = B[E]$$

over $\gamma(t) \in C$ is homotopy equivalent to a point. In fact we will show this map is a trivial Kan fibration. Namely, $P_\bullet \to B$ is a trivial can fibration. For any ring R we have

$$R[E] = \mathrm{colim}_{\Sigma\subset \mathrm{Map}(E,\mathbf{Z}_{\geq 0}) \text{ finite}} \prod\nolimits_{I\in\Sigma} R$$

(filtered colimit). Thus the displayed map of simplicial sets is a filtered colimit of trivial Kan fibrations, whence a trivial Kan fibration by Simplicial, Lemma 14.28.7. □

Lemma 66.7.3. *With notation as in (66.7.0.2) we have $Lg_{i,!} \circ g_i^{-1} = id$ for $i = 1, 2, 3$ and hence also $L\pi_! \circ g_i^{-1} = L\pi_{i,!}$ for $i = 1, 2, 3$.*

Proof. Proof for $i = 1$. We claim the functor $\mathcal{C}_{C/B/A}$ is a fibred category over $\mathcal{C}_{B/A}$ Namely, suppose given $(P \to B, Q \to C)$ and a morphism $(P' \to B) \to (P \to B)$ of $\mathcal{C}_{B/A}$. Recall that this means we have an A-algebra homomorphism $P \to P'$ compatible with maps to B. Then we set $Q' = Q \otimes_P P'$ with induced map to C and the morphism

$$(P' \to B, Q' \to C) \longrightarrow (P \to B, Q \to C)$$

in $\mathcal{C}_{C/B/A}$ (note reversal arrows again) is strongly cartesian in $\mathcal{C}_{C/B/A}$ over $\mathcal{C}_{B/A}$. Moreover, observe that the fibre category of u_1 over $P \to B$ is the category $\mathcal{C}_{C/P}$. Let $\mathcal{F}$ be an abelian sheaf on $\mathcal{C}_{B/A}$. Since we have a fibred category we may apply Cohomology on Sites, Lemma 21.29.2. Thus $L_n g_{1,!} g_1^{-1}\mathcal{F}$ is the (pre)sheaf which

assigns to $U \in \mathrm{Ob}(\mathcal{C}_{B/A})$ the nth homology of $g_1^{-1}\mathcal{F}$ restricted to the fibre category over U. Since these restrictions are constant the desired result follows from Lemma 66.4.4 via our identifications of fibre categories above.

The case $i = 2$. We claim $\mathcal{C}_{C/B/A}$ is a fibred category over $\mathcal{C}_{C/A}$ is a fibred category. Namely, suppose given $(P \to B, Q \to C)$ and a morphism $(Q' \to C) \to (Q \to C)$ of $\mathcal{C}_{C/A}$. Recall that this means we have a B-algebra homomorphism $Q \to Q'$ compatible with maps to C. Then

$$(P \to B, Q' \to C) \longrightarrow (P \to B, Q \to C)$$

is strongly cartesian in $\mathcal{C}_{C/B/A}$ over $\mathcal{C}_{C/A}$. Note that the fibre category of u_2 over $Q \to C$ has an final (beware reversal arrows) object, namely, $(A \to B, Q \to C)$. Let $\mathcal{F}$ be an abelian sheaf on $\mathcal{C}_{C/A}$. Since we have a fibred category we may apply Cohomology on Sites, Lemma 21.29.2. Thus $L_n g_{2,!} g_2^{-1}\mathcal{F}$ is the (pre)sheaf which assigns to $U \in \mathrm{Ob}(\mathcal{C}_{C/A})$ the nth homology of $g_1^{-1}\mathcal{F}$ restricted to the fibre category over U. Since these restrictions are constant the desired result follows from Cohomology on Sites, Lemma 21.28.5 because the fibre categories all have final objects.

The case $i = 3$. In this case we will apply Cohomology on Sites, Lemma 21.29.3 to $u = u_3 : \mathcal{C}_{C/B/A} \to \mathcal{C}_{C/B}$ and $\mathcal{F}' = g_3^{-1}\mathcal{F}$ for some abelian sheaf $\mathcal{F}$ on $\mathcal{C}_{C/B}$. Suppose $U = (\overline{Q} \to C)$ is an object of $\mathcal{C}_{C/B}$. Then $\mathcal{I}_U = \mathcal{C}_{\overline{Q}/B/A}$ (again beware of reversal of arrows). The sheaf $\mathcal{F}'_U$ is given by the rule $(P \to B, Q \to \overline{Q}) \mapsto \mathcal{F}(Q \otimes_P B \to C)$. In other words, this sheaf is the pullback of a sheaf on $\mathcal{C}_{\overline{Q}/C}$ via the morphism $Sh(\mathcal{C}_{\overline{Q}/B/A}) \to Sh(\mathcal{C}_{\overline{Q}/B})$. Thus Lemma 66.7.2 shows that $H_n(\mathcal{I}_U, \mathcal{F}'_U) = 0$ for $n > 0$ and equal to $\mathcal{F}(\overline{Q} \to C)$ for $n = 0$. The aforementioned Cohomology on Sites, Lemma 21.29.3 implies that $Lg_{3,!}(g_3^{-1}\mathcal{F}) = \mathcal{F}$ and the proof is done. □

Proposition 66.7.4. *Let $A \to B \to C$ be ring maps. There is a canonical distinguished triangle*

$$L_{B/A} \otimes^{\mathbf{L}}_B C \to L_{C/A} \to L_{C/B} \to L_{B/A} \otimes^{\mathbf{L}}_B C[1]$$

in $D(C)$.

Proof. Consider the short exact sequence of sheaves of Lemma 66.7.1 and apply the derived functor $L\pi_!$ to obtain a distinguished triangle

$$L\pi_!(g_1^{-1}\Omega_1 \otimes_{\underline{B}} \underline{C}) \to L\pi_!(g_2^{-1}\Omega_2) \to L\pi_!(g_3^{-1}\Omega_3) \to L\pi_!(g_1^{-1}\Omega_1 \otimes_{\underline{B}} \underline{C})[1]$$

in $D(C)$. Using Lemmas 66.7.3 and 66.4.3 we see that the second and third terms agree with $L_{C/A}$ and $L_{C/B}$ and the first one equals

$$L\pi_{1,!}(\Omega_1 \otimes_{\underline{B}} \underline{C}) = L\pi_{1,!}(\Omega_1) \otimes^{\mathbf{L}}_B C = L_{B/A} \otimes^{\mathbf{L}}_B C$$

The first equality by Cohomology on Sites, Lemma 21.28.6 (and flatness of Ω_1 as a sheaf of modules over $\underline{B}$) and the second by Lemma 66.4.3. □

Remark 66.7.5. We sketch an alternative, perhaps simpler, proof of the existence of the fundamental triangle. Let $A \to B \to C$ be ring maps and assume that $B \to C$ is injective. Let $P_\bullet \to B$ be the standard resolution of B over A and let

$Q_\bullet \to C$ be the standard resolution of C over B. Picture

$$\begin{array}{lccccc} P_\bullet: & A[A[A[B]]] & \rightrightarrows & A[A[B]] & \rightrightarrows & A[B] \longrightarrow B \\ & \downarrow & & \downarrow & & \downarrow \\ Q_\bullet: & A[A[A[C]]] & \rightrightarrows & A[A[C]] & \rightrightarrows & A[C] \longrightarrow C \end{array}$$

Observe that since $B \to C$ is injective, the ring Q_n is a polynomial algebra over P_n for all n. Hence we obtain a cosimplicial object in $\mathcal{C}_{C/B/A}$ (beware reversal arrows). Now set $\overline{Q}_\bullet = Q_\bullet \otimes_{P_\bullet} B$. The key to the proof of Proposition 66.7.4 is to show that $\overline{Q}_\bullet$ is a resolution of C over B. This follows from Cohomology on Sites, Lemma 21.28.12 applied to $\mathcal{C} = \Delta$, $\mathcal{O} = P_\bullet$, $\mathcal{O}' = B$, and $\mathcal{F} = Q_\bullet$ (this uses that Q_n is flat over P_n; see Cohomology on Sites, Remark 21.28.11 to relate simplicial modules to sheaves). The key fact implies that the distinguished triangle of Proposition 66.7.4 is the distinguished triangle associated to the short exact sequence of simplicial C-modules

$$0 \to \Omega_{P_\bullet/A} \otimes_{P_\bullet} C \to \Omega_{Q_\bullet/A} \otimes_{Q_\bullet} C \to \Omega_{\overline{Q}_\bullet/B} \otimes_{\overline{Q}_\bullet} C \to 0$$

which is deduced from the short exact sequences $0 \to \Omega_{P_n/A} \otimes_{P_n} Q_n \to \Omega_{Q_n/A} \to \Omega_{Q_n/P_n} \to 0$ of Algebra, Lemma 10.131.9. Namely, by Remark 66.5.5 and the key fact the complex on the right hand side represents $L_{C/B}$ in $D(C)$.

If $B \to C$ is not injective, then we can use the above to get a fundamental triangle for $A \to B \to B \times C$. Since $L_{B\times C/B} \to L_{B/B} \oplus L_{C/B}$ and $L_{B\times C/A} \to L_{B/A} \oplus L_{C/A}$ are quasi-isomorphism in $D(B \times C)$ (Lemma 66.6.4) this induces the desired distinguished triangle in $D(C)$ by tensoring with the flat ring map $B \times C \to C$.

Remark 66.7.6. Let $A \to B \to C$ be ring maps with $B \to C$ injective. Recall the notation $P_\bullet$, $Q_\bullet$, $\overline{Q}_\bullet$ of Remark 66.7.5. Let $R_\bullet$ be the standard resolution of C over B. In this remark we explain how to get the canonical identification of $\Omega_{\overline{Q}_\bullet/B} \otimes_{\overline{Q}_\bullet} C$ with $L_{C/B} = \Omega_{R_\bullet/B} \otimes_{R_\bullet} C$. Let $S_\bullet \to B$ be the standard resolution of B over B. Note that the functoriality map $S_\bullet \to R_\bullet$ identifies R_n as a polynomial algebra over S_n because $B \to C$ is injective. For example in degree 0 we have the map $B[B] \to B[C]$, in degree 1 the map $B[B[B]] \to B[B[C]]$, and so on. Thus $\overline{R}_\bullet = R_\bullet \otimes_{S_\bullet} B$ is a simplicial polynomial algebra over B as well and it follows (as in Remark 66.7.5) from Cohomology on Sites, Lemma 21.28.12 that $\overline{R}_\bullet \to C$ is a resolution. Since we have a commutative diagram

$$\begin{array}{ccccc} Q_\bullet & \longrightarrow & R_\bullet & & \\ \uparrow & & \uparrow & & \\ P_\bullet & \longrightarrow & S_\bullet & \longrightarrow & B \end{array}$$

we obtain a canonical map $\overline{Q}_\bullet = Q_\bullet \otimes_{P_\bullet} B \to \overline{R}_\bullet$. Thus the maps

$$L_{C/B} = \Omega_{R_\bullet/B} \otimes_{R_\bullet} C \longrightarrow \Omega_{\overline{R}_\bullet/B} \otimes_{\overline{R}_\bullet} C \longleftarrow \Omega_{\overline{Q}_\bullet/B} \otimes_{\overline{Q}_\bullet} C$$

are quasi-isomorphisms (Remark 66.5.5) and composing one with the inverse of the other gives the desired identification.

66.8. Localization and étale ring maps

In this section we study what happens if we localize our rings. Let $A \to A' \to B$ be ring maps such that $B = B \otimes^{\mathbf{L}}_A A'$. This happens for example if $A' = S^{-1}A$ is the localization of A at a multiplicative subset $S \subset A$. In this case for an abelian sheaf $\mathcal{F}'$ on $\mathcal{C}_{B/A'}$ the homology of $g^{-1}\mathcal{F}'$ over $\mathcal{C}_{B/A}$ agrees with the homology of $\mathcal{F}'$ over $\mathcal{C}_{B/A'}$, see Lemma 66.6.1 for a precise statement.

Lemma 66.8.1. *Let $A \to A' \to B$ be ring maps such that $B = B \otimes^{\mathbf{L}}_A A'$. Then $L_{B/A} = L_{B/A'}$ in $D(B)$.*

Proof. According to the discussion above (i.e., using Lemma 66.6.1) and Lemma 66.4.3 we have to show that the sheaf given by the rule $(P \to B) \mapsto \Omega_{P/A} \otimes_P B$ on $\mathcal{C}_{B/A}$ is the pullback of the sheaf given by the rule $(P \to B) \mapsto \Omega_{P/A'} \otimes_P B$. The pullback functor g^{-1} is given by precomposing with the functor $u : \mathcal{C}_{B/A} \to \mathcal{C}_{B/A'}$, $(P \to B) \mapsto (P \otimes_A A' \to B)$. Thus we have to show that

$$\Omega_{P/A} \otimes_P B = \Omega_{P \otimes_A A'/A'} \otimes_{(P \otimes_A A')} B$$

By Algebra, Lemma 10.125.12 the right hand side is equal to

$$(\Omega_{P/A} \otimes_A A') \otimes_{(P \otimes_A A')} B$$

Since P is a polynomial algebra over A the module $\Omega_{P/A}$ is free and the equality is obvious. □

Lemma 66.8.2. *Let $A \to B$ be a ring map such that $B = B \otimes^{\mathbf{L}}_A B$. Then $L_{B/A} = 0$ in $D(B)$.*

Proof. This is true because $L_{B/A} = L_{B/B} = 0$ by Lemmas 66.8.1 and 66.4.7. □

Lemma 66.8.3. *Let $A \to B$ be a ring map such that $Tor^A_i(B, B) = 0$ for $i > 0$ and such that $L_{B/B \otimes_A B} = 0$. Then $L_{B/A} = 0$ in $D(B)$.*

Proof. By Lemma 66.6.2 we see that $L_{B/A} \otimes^{\mathbf{L}}_B (B \otimes_A B) = L_{B \otimes_A B/B}$. Now we use the distinguished triangle (66.7.0.1)

$$L_{B \otimes_A B/B} \otimes^{\mathbf{L}}_{(B \otimes_A B)} B \to L_{B/B} \to L_{B/B \otimes_A B} \to L_{B \otimes_A B/B} \otimes^{\mathbf{L}}_{(B \otimes_A B)} B[1]$$

associated to the ring maps $B \to B \otimes_A B \to B$ and the vanishing of $L_{B/B}$ (Lemma 66.4.7) and $L_{B/B \otimes_A B}$ (assumed) to see that

$$0 = L_{B \otimes_A B/B} \otimes^{\mathbf{L}}_{(B \otimes_A B)} B = L_{B/A} \otimes^{\mathbf{L}}_B (B \otimes_A B) \otimes^{\mathbf{L}}_{(B \otimes_A B)} B = L_{B/A}$$

as desired. □

Lemma 66.8.4. *The cotangent complex $L_{B/A}$ is zero in each of the following cases:*

1. *$A \to B$ and $B \otimes_A B \to B$ are flat, i.e., $A \to B$ is weakly étale (More on Algebra, Definition 15.60.1),*
2. *$A \to B$ is a flat epimorphism of rings,*
3. *$B = S^{-1}A$ for some multiplicative subset $S \subset A$,*
4. *$A \to B$ is unramified and flat,*
5. *$A \to B$ is étale,*
6. *$A \to B$ is a filtered colimit of ring maps for which the cotangent complex vanishes,*
7. *B is a henselization of a local ring of A,*
8. *B is a strict henselization of a local ring of A, and*

(9) *add more here.*

Proof. In case (1) we may apply Lemma 66.8.2 to the surjective flat ring map $B \otimes_A B \to B$ to conclude that $L_{B \otimes_A B/B} = 0$ and then we use Lemma 66.8.3 to conclude. The cases (2) – (5) are each special cases of (1). Part (6) follows from Lemma 66.3.4. Parts (7) and (8) follows from the fact that (strict) henselizations are filtered colimits of étale ring extensions of A, see Algebra, Lemmas 10.143.21 and 10.143.27. □

Lemma 66.8.5. *Let $A \to B \to C$ be ring maps such that $L_{C/B} = 0$. Then $L_{C/A} = L_{B/A} \otimes^{\mathbf{L}}_B C$.*

Proof. This is a trivial consequence of the distinguished triangle (66.7.0.1). □

Lemma 66.8.6. *Let $A \to B$ be ring maps and $S \subset A$, $T \subset B$ multiplicative subsets such that S maps into T. Then $L_{T^{-1}B/S^{-1}A} = L_{B/A} \otimes_B T^{-1}B$ in $D(T^{-1}B)$.*

Proof. Lemma 66.8.5 shows that $L_{T^{-1}B/A} = L_{B/A} \otimes_B T^{-1}B$ and Lemma 66.8.1 shows that $L_{T^{-1}B/A} = L_{T^{-1}B/S^{-1}A}$. □

Lemma 66.8.7. *Let $A \to B$ be a local ring homomorphism of local rings. Let $A^h \to B^h$, resp. $A^{sh} \to B^{sh}$ be the induced maps of henselizations, resp. strict henselizations. Then*

$$L_{B^h/A^h} = L_{B^h/A} = L_{B/A} \otimes^{\mathbf{L}}_B B^h \quad \text{resp.} \quad L_{B^{sh}/A^{sh}} = L_{B^{sh}/A} = L_{B/A} \otimes^{\mathbf{L}}_B B^{sh}$$

in $D(B^h)$, resp. $D(B^{sh})$.

Proof. The complexes $L_{A^h/A}$, $L_{A^{sh}/A}$, $L_{B^h/B}$, and $L_{B^{sh}/B}$ are all zero by Lemma 66.8.4. Using the fundamental distinguished triangle (66.7.0.1) for $A \to B \to B^h$ we obtain $L_{B^h/A} = L_{B/A} \otimes^{\mathbf{L}}_B B^h$. Using the fundamental triangle for $A \to A^h \to B^h$ we obtain $L_{B^h/A^h} = L_{B^h/A}$. Similarly for strict henselizations. □

66.9. Smooth ring maps

Let $C \to B$ be a surjection of rings with kernel I. Let us call such a ring map "weakly quasi-regular" if I/I^2 is a flat B-module and $\mathrm{Tor}^C_*(B, B)$ is the exterior algebra on I/I^2. The generalization to "smooth ring maps" of what is done in Lemma 66.8.4 for "étale ring maps" is to look at flat ring maps $A \to B$ such that the multiplication map $B \otimes_A B \to B$ is weakly quasi-regular. For the moment we just stick to smooth ring maps.

Lemma 66.9.1. *If $A \to B$ is a smooth ring map, then $L_{B/A} = \Omega_{B/A}[0]$.*

Proof. We have the agreement in cohomological degree 0 by Lemma 66.4.5. Thus it suffices to prove the other cohomology groups are zero. It suffices to prove this locally on $\mathrm{Spec}(B)$ as $L_{B_g/A} = (L_{B/A})_g$ for $g \in B$ by Lemma 66.8.5. Thus we may assume that $A \to B$ is standard smooth (Algebra, Lemma 10.130.10), i.e., that we can factor $A \to B$ as $A \to A[x_1, \ldots, x_n] \to B$ with $A[x_1, \ldots, x_n] \to B$ étale. In this case Lemmas 66.8.4 and Lemma 66.8.5 show that $L_{B/A} = L_{A[x_1,\ldots,x_n]/A} \otimes B$ whence the conclusion by Lemma 66.4.7. □

66.10. Comparison with the naive cotangent complex

The naive cotangent complex was introduced in Algebra, Section 10.127.

Remark 66.10.1. Let $A \to B$ be a ring map. Working on $\mathcal{C}_{B/A}$ as in Section 66.4 let $\mathcal{J} \subset \mathcal{O}$ be the kernel of $\mathcal{O} \to \underline{B}$. Note that $L\pi_!(\mathcal{J}) = 0$ by Lemma 66.5.7. Set $\Omega = \Omega_{\mathcal{O}/A} \otimes_{\mathcal{O}} \underline{B}$ so that $L_{B/A} = L\pi_!(\Omega)$ by Lemma 66.4.3. It follows that $L\pi_!(\mathcal{J} \to \Omega) = L\pi_!(\Omega) = L_{B/A}$. Thus, for any object $U = (P \to B)$ of $\mathcal{C}_{B/A}$ we obtain a map

$$(J \to \Omega_{P/A} \otimes_P B) \longrightarrow L_{B/A} \tag{66.10.1.1}$$

where $J = \mathrm{Ker}(P \to B)$ in $D(A)$, see Cohomology on Sites, Remark 21.28.4. Continuing in this manner, note that $L\pi_!(\mathcal{J} \otimes^{\mathbf{L}}_{\mathcal{O}} \underline{B}) = L\pi_!(\mathcal{J}) = 0$ by Lemma 66.5.6. Since $\mathrm{Tor}^{\mathcal{O}}_0(\mathcal{J}, \underline{B}) = \mathcal{J}/\mathcal{J}^2$ the spectral sequence

$$H_p(\mathcal{C}_{B/A}, \mathrm{Tor}^{\mathcal{O}}_q(\mathcal{J}, \underline{B})) \Rightarrow H_{p+q}(\mathcal{C}_{B/A}, \mathcal{J} \otimes^{\mathbf{L}}_{\mathcal{O}} \underline{B}) = 0$$

(dual of Derived Categories, Lemma 13.20.3) implies that $H_0(\mathcal{C}_{B/A}, \mathcal{J}/\mathcal{J}^2) = 0$ and $H_1(\mathcal{C}_{B/A}, \mathcal{J}/\mathcal{J}^2) = 0$. It follows that the complex of $\underline{B}$-modules $\mathcal{J}/\mathcal{J}^2 \to \Omega$ satisfies $\tau_{\geq -1} L\pi_!(\mathcal{J}/\mathcal{J}^2 \to \Omega) = \tau_{\geq -1} L_{B/A}$. Thus, for any object $U = (P \to B)$ of $\mathcal{C}_{B/A}$ we obtain a map

$$(J/J^2 \to \Omega_{P/A} \otimes_P B) \longrightarrow \tau_{\geq -1} L_{B/A} \tag{66.10.1.2}$$

in $D(B)$, see Cohomology on Sites, Remark 21.28.4.

The first case is where we have a surjection of rings.

Lemma 66.10.2. *Let $A \to B$ be a surjective ring map with kernel I. Then $H^0(L_{B/A}) = 0$ and $H^{-1}(L_{B/A}) = I/I^2$. This isomorphism comes from the map (66.10.1.2) for the object $(A \to B)$ of $\mathcal{C}_{B/A}$.*

Proof. We will show below (using the surjectivity of $A \to B$) that there exists a short exact sequence

$$0 \to \pi^{-1}(I/I^2) \to \mathcal{J}/\mathcal{J}^2 \to \Omega \to 0$$

of sheaves on $\mathcal{C}_{B/A}$. Taking $L\pi_!$ and the associated long exact sequence of homology, and using the vanishing of $H_1(\mathcal{C}_{B/A}, \mathcal{J}/\mathcal{J}^2)$ and $H_0(\mathcal{C}_{B/A}, \mathcal{J}/\mathcal{J}^2)$ shown in Remark 66.10.1 we obtain what we want using Lemma 66.4.4.

What is left is to verify the local statement mentioned above. For every object $U = (P \to B)$ of $\mathcal{C}_{B/A}$ we can choose an isomorphism $P = A[E]$ such that the map $P \to B$ maps each $e \in E$ to zero. Then $J = \mathcal{J}(U) \subset P = \mathcal{O}(U)$ is equal to $J = IP + (e; e \in E)$. The value on U of the short sequence of sheaves above is the sequence

$$0 \to I/I^2 \to J/J^2 \to \Omega_{P/A} \otimes_P B \to 0$$

Verification omitted (hint: the only tricky point is that $IP \cap J^2 = IJ$; which follows for example from More on Algebra, Lemma 15.20.8). □

Lemma 66.10.3. *Let $A \to B$ be a ring map. Then $\tau_{\geq -1} L_{B/A}$ is canonically quasi-isomorphic to the naive cotangent complex.*

Proof. Consider $P = A[B] \to B$ with kernel I. The naive cotangent complex $NL_{B/A}$ of B over A is the complex $I/I^2 \to \Omega_{P/A} \otimes_P B$, see Algebra, Definition 10.127.1. Observe that in (66.10.1.2) we have already constructed a canonical map

$$c : NL_{B/A} \longrightarrow \tau_{\geq -1} L_{B/A}$$

Consider the distinguished triangle (66.7.0.1)

$$L_{P/A} \otimes^{\mathbf{L}}_P B \to L_{B/A} \to L_{B/P} \to (L_{P/A} \otimes^{\mathbf{L}}_P B)[1]$$

associated to the ring maps $A \to A[B] \to B$. We know that $L_{P/A} = \Omega_{P/A}[0] = NL_{P/A}$ in $D(P)$ (Lemma 66.4.7 and Algebra, Lemma 10.127.3) and that $\tau_{\geq -1} L_{B/P} = I/I^2[1] = NL_{B/P}$ in $D(B)$ (Lemma 66.10.2 and Algebra, Lemma 10.127.6). To show c is a quasi-isomorphism it suffices by Algebra, Lemma 10.127.4 and the long exact cohomology sequence associated to the distinguished triangle to show that the maps $L_{P/A} \to L_{B/A} \to L_{B/P}$ are compatible on cohomology groups with the corresponding maps $NL_{P/A} \to NL_{B/A} \to NL_{B/P}$ of the naive cotangent complex. We omit the verification. □

Remark 66.10.4. We can make the comparison map of Lemma 66.10.3 explicit in the following way. Let $P_\bullet$ be the standard resolution of B over A. Let $I = \mathrm{Ker}(A[B] \to B)$. Recall that $P_0 = A[B]$. The map of the lemma is given by the commutative diagram

$$\begin{array}{ccccccc} L_{B/A} & \ldots \longrightarrow & \Omega_{P_2/A} \otimes_{P_2} B & \longrightarrow & \Omega_{P_1/A} \otimes_{P_1} B & \longrightarrow & \Omega_{P_0/A} \otimes_{P_0} B \\ \downarrow & & \downarrow & & \downarrow & & \downarrow \\ NL_{B/A} & \ldots \longrightarrow & 0 & \longrightarrow & I/I^2 & \longrightarrow & \Omega_{P_0/A} \otimes_{P_0} B \end{array}$$

We construct the downward arrow with target I/I^2 by sending $\mathrm{d}f \otimes b$ to the class of $(d_0(f) - d_1(f))b$ in I/I^2. Here $d_i : P_1 \to P_0$, $i = 0, 1$ are the two face maps of the simplicial structure. This makes sense as $d_0 - d_1$ maps P_1 into $I = \mathrm{Ker}(P_0 \to B)$. We omit the verification that this rule is well defined. Our map is compatible with the differential $\Omega_{P_1/A} \otimes_{P_1} B \to \Omega_{P_0/A} \otimes_{P_0} B$ as this differential maps $\mathrm{d}f \otimes b$ to $\mathrm{d}(d_0(f) - d_1(f)) \otimes b$. Moreover, the differential $\Omega_{P_2/A} \otimes_{P_2} B \to \Omega_{P_1/A} \otimes_{P_1} B$ maps $\mathrm{d}f \otimes b$ to $\mathrm{d}(d_0(f) - d_1(f) + d_2(f)) \otimes b$ which are annihilated by our downward arrow. Hence a map of complexes. We omit the verification that this is the same as the map of Lemma 66.10.3.

Remark 66.10.5. Adopt notation as in Remark 66.10.1. The arguments given there show that the differential

$$H_2(\mathcal{C}_{B/A}, \mathcal{J}/\mathcal{J}^2) \longrightarrow H_0(\mathcal{C}_{B/A}, \mathrm{Tor}_1^{\mathcal{O}}(\mathcal{J}, \underline{B}))$$

of the spectral sequence is an isomorphism. Let $\mathcal{C}'_{B/A}$ denote the full subcategory of $\mathcal{C}_{B/A}$ consisting of surjective maps $P \to B$. The agreement of the cotangent complex with the naive cotangent complex (Lemma 66.10.3) shows that we have an exact sequence of sheaves

$$0 \to \underline{H_1(L_{B/A})} \to \mathcal{J}/\mathcal{J}^2 \xrightarrow{\mathrm{d}} \Omega \to \underline{H_2(L_{B/A})} \to 0$$

on $\mathcal{C}'_{B/A}$. It follows that $\mathrm{Ker}(d)$ and $\mathrm{Coker}(d)$ on the whole category $\mathcal{C}_{B/A}$ have vanishing higher homology groups, since these are computed by the homology groups

of constant simplicial abelian groups by Lemma 66.4.1. Hence we conclude that

$$H_n(\mathcal{C}_{B/A}, \mathcal{J}/\mathcal{J}^2) \to H_n(L_{B/A})$$

is an isomorphism for all $n \geq 2$. Combined with the remark above we obtain the formula $H_2(L_{B/A}) = H_0(\mathcal{C}_{B/A}, \mathrm{Tor}_1^{\mathcal{O}}(\mathcal{J}, \underline{B}))$.

66.11. A spectral sequence of Quillen

In this section we discuss a spectral sequence relating derived tensor product to the cotangent complex.

Lemma 66.11.1. *Notation and assumptions as in Cohomology on Sites, Example 21.28.1. Assume $\mathcal{C}$ has a cosimplicial object as in Cohomology on Sites, Lemma 21.28.7. Let $\mathcal{F}$ be a flat $\underline{B}$-module such that $H_0(\mathcal{C}, \mathcal{F}) = 0$. Then $H_l(\mathcal{C}, Sym^k_{\underline{B}}(\mathcal{F})) = 0$ for $l < k$.*

Proof. We drop the subscript $\underline{B}$ from tensor products, wedge powers, and symmetric powers. We will prove the lemma by induction on k. The cases $k = 0, 1$ follow from the assumptions. If $k > 1$ consider the exact complex

$$\ldots \to \wedge^2\mathcal{F} \otimes \mathrm{Sym}^{k-2}\mathcal{F} \to \mathcal{F} \otimes \mathrm{Sym}^{k-1}\mathcal{F} \to \mathrm{Sym}^k\mathcal{F} \to 0$$

with differentials as in the Koszul complex. If we think of this as a resolution of $\mathrm{Sym}^k\mathcal{F}$, then this gives a first quadrant spectral sequence

$$E_1^{p,q} = H_p(\mathcal{C}, \wedge^{q+1}\mathcal{F} \otimes \mathrm{Sym}^{k-q-1}\mathcal{F}) \Rightarrow H_{p+q}(\mathcal{C}, \mathrm{Sym}^k(\mathcal{F}))$$

By Cohomology on Sites, Lemma 21.28.10 we have

$$L\pi_!(\wedge^{q+1}\mathcal{F} \otimes \mathrm{Sym}^{k-q-1}\mathcal{F}) = L\pi_!(\wedge^{q+1}\mathcal{F}) \otimes^{\mathbf{L}}_B L\pi_!(\mathrm{Sym}^{k-q-1}\mathcal{F}))$$

It follows (from the construction of derived tensor products) that the induction hypothesis combined with the vanishing of $H_0(\mathcal{C}, \wedge^{q+1}(\mathcal{F})) = 0$ will prove what we want. This is true because $\wedge^{q+1}(\mathcal{F})$ is a quotient of $\mathcal{F}^{\otimes q+1}$ and $H_0(\mathcal{C}, \mathcal{F}^{\otimes q+1})$ is a quotient of $H_0(\mathcal{C}, \mathcal{F})^{\otimes q+1}$ which is zero. □

Remark 66.11.2. In the situation of Lemma 66.11.1 one can show that $H_k(\mathcal{C}, \mathrm{Sym}^k(\mathcal{F})) = \wedge^k_B(H_1(\mathcal{C}, \mathcal{F}))$. Namely, it can be deduced from the proof that $H_k(\mathcal{C}, \mathrm{Sym}^k(\mathcal{F}))$ is the S_k-coinvariants of

$$H^{-k}(L\pi_!(\mathcal{F}) \otimes^{\mathbf{L}}_B L\pi_!(\mathcal{F}) \otimes^{\mathbf{L}}_B \ldots \otimes^{\mathbf{L}}_B L\pi_!(\mathcal{F})) = H_1(\mathcal{C}, \mathcal{F})^{\otimes k}$$

Thus our claim is that this action is given by the usual action of S_k on the tensor product multiplied by the sign character. To prove this one has to work through the sign conventions in the definition of the total complex associated to a multicomplex. We omit the verification.

Lemma 66.11.3. *Let A be a ring. Let $P = A[E]$ be a polynomial ring. Set $I = (e; e \in E) \subset P$. The maps $Tor_i^P(A, I^{n+1}) \to Tor_i^P(A, I^n)$ are zero for all i and n.*

Proof. Denote $x_e \in P$ the variable corresponding to $e \in E$. A free resolution of A over P is given by the Koszul complex $K_\bullet$ on the x_e. Here K_i has basis given by wedges $e_1 \wedge \ldots \wedge e_i$, $e_1, \ldots, e_i \in E$ and $d(e) = x_e$. Thus $K_\bullet \otimes_P I^n = I^n K_\bullet$ computes $\mathrm{Tor}_i^P(A, I^n)$. Observe that everything is graded with $\deg(x_e) = 1$, $\deg(e) = 1$, and $\deg(a) = 0$ for $a \in A$. Suppose $\xi \in I^{n+1}K_i$ is a cocycle homogeneous of degree m. Note that $m \geq i + 1 + n$. Then $\xi = \mathrm{d}\eta$ for some $\eta \in K_{i+1}$ as $K_\bullet$ is exact in degrees

> 0. (The case $i = 0$ is left to the reader.) Now $\deg(\eta) = m \geq i + 1 + n$. Hence writing η in terms of the basis we see the coordinates are in I^n. Thus ξ maps to zero in the homology of $I^n K_\bullet$ as desired. $\square$

Theorem 66.11.4 (Quillen spectral sequence)**.** *Let $A \to B$ be a surjective ring map. Consider the sheaf $\Omega = \Omega_{\mathcal{O}/A} \otimes_{\mathcal{O}} \underline{B}$ of $\underline{B}$-modules on $\mathcal{C}_{B/A}$, see Section 66.4. Then there is a spectral sequence with E_1-page*

$$E_1^{p,q} = H_{-p-q}(\mathcal{C}_{B/A}, Sym^p_{\underline{B}}(\Omega)) \Rightarrow Tor^A_{-p-q}(B, B)$$

with d_r of bidegree $(r, -r+1)$. Moreover, $H_i(\mathcal{C}_{B/A}, Sym^k_{\underline{B}}(\Omega)) = 0$ for $i < k$.

Proof. Let $I \subset A$ be the kernel of $\underline{A} \to B$. Let $\mathcal{J} \subset \mathcal{O}$ be the kernel of $\mathcal{O} \to \underline{B}$. Then $I\mathcal{O} \subset \mathcal{J}$. Set $\mathcal{K} = \mathcal{J}/I\mathcal{O}$ and $\overline{\mathcal{O}} = \mathcal{O}/I\mathcal{O}$.

For every object $U = (P \to B)$ of $\mathcal{C}_{B/A}$ we can choose an isomorphism $P = A[E]$ such that the map $P \to B$ maps each $e \in E$ to zero. Then $J = \mathcal{J}(U) \subset P = \mathcal{O}(U)$ is equal to $J = IP + (e; e \in E)$. Moreover $\overline{\mathcal{O}}(U) = B[E]$ and $K = \mathcal{K}(U) = (e; e \in E)$ is the ideal generated by the variables in the polynomial ring $B[E]$. In particular it is clear that

$$K/K^2 \xrightarrow{\mathrm{d}} \Omega_{P/A} \otimes_P B$$

is a bijection. In other words, $\Omega = \mathcal{K}/\mathcal{K}^2$ and $\mathrm{Sym}^k_B(\Omega) = \mathcal{K}^k/\mathcal{K}^{k+1}$. Note that $\pi_!(\Omega) = \Omega_{B/A} = 0$ (Lemma 66.4.5) as $A \to B$ is surjective (Algebra, Lemma 10.125.5). By Lemma 66.11.1 we conclude that

$$H_i(\mathcal{C}_{B/A}, \mathcal{K}^k/\mathcal{K}^{k+1}) = H_i(\mathcal{C}_{B/A}, \mathrm{Sym}^k_{\underline{B}}(\Omega)) = 0$$

for $i < k$. This proves the final statement of the theorem.

The approach to the theorem is to note that

$$B \otimes^{\mathbf{L}}_A B = L\pi_!(\mathcal{O}) \otimes^{\mathbf{L}}_A B = L\pi_!(\mathcal{O} \otimes^{\mathbf{L}}_{\underline{A}} \underline{B}) = L\pi_!(\overline{\mathcal{O}})$$

The first equality by Lemma 66.5.7, the second equality by Cohomology on Sites, Lemma 21.28.6, and the third equality as $\mathcal{O}$ is flat over $\underline{A}$. The sheaf $\overline{\mathcal{O}}$ has a filtration

$$\ldots \subset \mathcal{K}^3 \subset \mathcal{K}^2 \subset \mathcal{K} \subset \overline{\mathcal{O}}$$

This induces a filtration F on a complex C representing $L\pi_!(\overline{\mathcal{O}})$ with F^pC representing $L\pi_!(\mathcal{K}^p)$ (construction of C and F omitted). Consider the spectral sequence of Homology, Section 12.21 associated to (C, F). It has E_1-page

$$E_1^{p,q} = H_{-p-q}(\mathcal{C}_{B/A}, \mathcal{K}^p/\mathcal{K}^{p+1}) \quad \Rightarrow \quad H_{-p-q}(\mathcal{C}_{B/A}, \overline{\mathcal{O}}) = \mathrm{Tor}^A_{-p-q}(B, B)$$

and differentials $E_r^{p,q} \to E_r^{p+r,q-r+1}$. To show convergence we will show that for every k there exists a c such that $H_i(\mathcal{C}_{B/A}, \mathcal{K}^n) = 0$ for $i < k$ and $n > c$[2].

Given $k \geq 0$ set $c = k^2$. We claim that

$$H_i(\mathcal{C}_{B/A}, \mathcal{K}^{n+c}) \to H_i(\mathcal{C}_{B/A}, \mathcal{K}^n)$$

is zero for $i < k$ and all $n \geq 0$. Note that $\mathcal{K}^n/\mathcal{K}^{n+c}$ has a finite filtration whose successive quotients $\mathcal{K}^m/\mathcal{K}^{m+1}$, $n \leq m < n + c$ have $H_i(\mathcal{C}_{B/A}, \mathcal{K}^m/\mathcal{K}^{m+1}) = 0$ for $i < n$ (see above). Hence the claim implies $H_i(\mathcal{C}_{B/A}, \mathcal{K}^{n+c}) = 0$ for $i < k$ and all $n \geq k$ which is what we need to show.

[2] A posteriori the "correct" vanishing $H_i(\mathcal{C}_{B/A}, \mathcal{K}^n) = 0$ for $i < n$ can be concluded.

Proof of the claim. Recall that for any $\mathcal{O}$-module $\mathcal{F}$ the map $\mathcal{F} \to \mathcal{F} \otimes^{\mathbf{L}}_{\mathcal{O}} B$ induces an isomorphism on applying $L\pi_!$, see Lemma 66.5.6. Consider the map

$$\mathcal{K}^{n+k} \otimes^{\mathbf{L}}_{\mathcal{O}} B \longrightarrow \mathcal{K}^n \otimes^{\mathbf{L}}_{\mathcal{O}} B$$

We claim that this map induces the zero map on cohomology sheaves in degrees $0, -1, \ldots, -k+1$. If this second claim holds, then the k-fold composition

$$\mathcal{K}^{n+c} \otimes^{\mathbf{L}}_{\mathcal{O}} B \longrightarrow \mathcal{K}^n \otimes^{\mathbf{L}}_{\mathcal{O}} B$$

factors through $\tau_{\leq -k}\mathcal{K}^n \otimes^{\mathbf{L}}_{\mathcal{O}} B$ hence induces zero on $H_i(\mathcal{C}_{B/A}, -) = L_i\pi_!(-)$ for $i < k$, see Derived Categories, Lemma 13.11.5. By the remark above this means the same thing is true for $H_i(\mathcal{C}_{B/A}, \mathcal{K}^{n+c}) \to H_i(\mathcal{C}_{B/A}, \mathcal{K}^n)$ which proves the (first) claim.

Proof of the second claim. The statement is local, hence we may work over an object $U = (P \to B)$ as above. We have to show the maps

$$\mathrm{Tor}^P_i(B, K^{n+k}) \to \mathrm{Tor}^P_i(B, K^n)$$

are zero for $i < k$. There is a spectral sequence

$$\mathrm{Tor}^P_a(P/IP, \mathrm{Tor}^{P/IP}_b(B, K^n)) \Rightarrow \mathrm{Tor}^P_{a+b}(B, K^n),$$

see More on Algebra, Example 15.46.2. Thus it suffices to prove the maps

$$\mathrm{Tor}^{P/IP}_i(B, K^{n+1}) \to \mathrm{Tor}^{P/IP}_i(B, K^n)$$

are zero for all i. This is Lemma 66.11.3. □

Remark 66.11.5. In the situation of Theorem 66.11.4 let $I = \mathrm{Ker}(A \to B)$. Then $H^{-1}(L_{B/A}) = H_1(\mathcal{C}_{B/A}, \Omega) = I/I^2$, see Lemma 66.10.2. Hence $H_k(\mathcal{C}_{B/A}, \mathrm{Sym}^k(\Omega)) = \wedge^k_B(I/I^2)$ by Remark 66.11.2. Thus the E_1-page looks like

$$\begin{array}{llll}
B & & & \\
0 & & & \\
0 & I/I^2 & & \\
0 & H^{-2}(L_{B/A}) & & \\
0 & H^{-3}(L_{B/A}) & \wedge^2(I/I^2) & \\
0 & H^{-4}(L_{B/A}) & H_3(\mathcal{C}_{B/A}, \mathrm{Sym}^2(\Omega)) & \\
0 & H^{-5}(L_{B/A}) & H_4(\mathcal{C}_{B/A}, \mathrm{Sym}^2(\Omega)) & \wedge^3(I/I^2)
\end{array}$$

with horizontal differential. Thus we obtain edge maps $\mathrm{Tor}^A_i(B, B) \to H^{-i}(L_{B/A})$, $i > 0$ and $\wedge^i_B(I/I^2) \to \mathrm{Tor}^A_i(B, B)$. Finally, we have $\mathrm{Tor}^A_1(B, B) = I/I^2$ and there is a five term exact sequence

$$\mathrm{Tor}^A_3(B, B) \to H^{-3}(L_{B/A}) \to \wedge^2_B(I/I^2) \to \mathrm{Tor}^A_2(B, B) \to H^{-2}(L_{B/A}) \to 0$$

of low degree terms.

Remark 66.11.6. Let $A \to B$ be a ring map. Let $P_\bullet$ be a resolution of B over A (Remark 66.5.5). Set $J_n = \mathrm{Ker}(P_n \to B)$. Note that

$$\mathrm{Tor}^{P_n}_2(B, B) = \mathrm{Tor}^{P_n}_1(J_n, B) = \mathrm{Ker}(J_n \otimes_{P_n} J_n \to J_n^2).$$

Hence $H_2(L_{B/A})$ is canonically equal to

$$\mathrm{Coker}(\mathrm{Tor}^{P_1}_2(B, B) \to \mathrm{Tor}^{P_0}_2(B, B))$$

by Remark 66.10.5. To make this more explicit we choose P_2, P_1, P_0 as in Example 66.5.9. We claim that

$$\mathrm{Tor}_2^{P_1}(B,B) = \wedge^2(\bigoplus\nolimits_{t\in T} B) \ \oplus \ \bigoplus\nolimits_{t\in T} J_0 \ \oplus \ \mathrm{Tor}_2^{P_0}(B,B)$$

Namely, the basis elements $x_t \wedge x_{t'}$ of the first summand corresponds to the element $x_t \otimes x_{t'} - x_{t'} \otimes x_t$ of $J_1 \otimes_{P_1} J_1$. For $f \in J_0$ the element $x_t \otimes f$ of the second summand corresponds to the element $x_t \otimes s_0(f) - s_0(f) \otimes x_t$ of $J_1 \otimes_{P_1} J_1$. Finally, the map $\mathrm{Tor}_2^{P_0}(B,B) \to \mathrm{Tor}_2^{P_1}(B,B)$ is given by s_0. The map $d_0 - d_1 : \mathrm{Tor}_2^{P_1}(B,B) \to \mathrm{Tor}_2^{P_0}(B,B)$ is zero on the last summand, maps $x_t \otimes f$ to $f \otimes f_t - f_t \otimes f$, and maps $x_t \wedge x_{t'}$ to $f_t \otimes f_{t'} - f_{t'} \otimes f_t$. All in all we conclude that there is an exact sequence

$$\wedge^2_B(J_0/J_0^2) \to \mathrm{Tor}_2^{P_0}(B,B) \to H^{-2}(L_{B/A}) \to 0$$

In this way we obtain a direct proof of a consequence of Quillen's spectral sequence discussed in Remark 66.11.5.

66.12. Comparison with Lichtenbaum-Schlessinger

Let $A \to B$ be a ring map. In [**LS67**] there is a fairly explicit determination of $\tau_{\geq -2} L_{B/A}$ which is often used in calculations of versal deformation spaces of singularities. The construction follows. Choose a polynomial algebra P over A and a surjection $P \to B$ with kernel I. Choose generators f_t, $t \in T$ for I which induces a surjection $F = \bigoplus_{t\in T} P \to I$ with F a free P algebra. Let $Rel \subset F$ be the kernel of $F \to I$, in other words Rel is the set of relations among the f_t. Let $TrivRel \subset Rel$ be the submodule of trivial relations, i.e., the submodule of Rel generated by the elements $(\ldots, f_{t'}, 0, \ldots, 0, -f_t, 0, \ldots)$. Consider the complex of B-modules

$$Rel/TrivRel \longrightarrow F \otimes_P B \longrightarrow \Omega_{P/A} \otimes_P B \tag{66.12.0.1}$$

where the last term is placed in degree 0. The first map is the obvious one and the second map sends the basis element corresponding to $t \in T$ to $\mathrm{d}f_t \otimes 1$.

Definition 66.12.1. Let $A \to B$ be a ring map. Let M be a (B,B)-bimodule over A. An *A-biderivation* is an A-linear map $\lambda : B \to M$ such that $\lambda(xy) = x\lambda(y) + \lambda(x)y$.

For a polynomial algebra the biderivations are easy to describe.

Lemma 66.12.2. *Let $P = A[S]$ be a polynomial ring over A. Let M be a (P,P)-bimodule over A. Given $m_s \in M$ for $s \in S$, there exists a unique A-biderivation $\lambda : P \to M$ mapping s to m_s for $s \in S$.*

Proof. We set

$$\lambda(s_1 \ldots s_t) = \sum s_1 \ldots s_{i-1} m_{s_i} s_{i+1} \ldots s_t$$

in M. Extending by A-linearity we obtain a biderivation. □

Here is the comparison statement. The reader may also read about this in [**And74**, page 206, Proposition 12] or in the paper [**DRGV92**] which extends the complex (66.12.0.1) by one term and the comparison to $\tau_{\geq -3}$.

Lemma 66.12.3. *In the situation above denote L the complex (66.12.0.1). There is a canonical map $L_{B/A} \to L$ in $D(A)$ which induces an isomorphism $\tau_{\geq -2} L_{B/A} \to L$ in $D(B)$.*

Proof. Let $P_\bullet \to B$ be a resolution of B over A (Remark 66.5.5). We will identify $L_{B/A}$ with $\Omega_{P_\bullet/A} \otimes B$. To construct the map we make some choices.

Choose an A-algebra map $\psi : P_0 \to P$ compatible with the given maps $P_0 \to B$ and $P \to B$.

Write $P_1 = A[S]$ for some set S. For $s \in S$ we may write

$$\psi(d_0(s) - d_1(s)) = \sum p_{s,t} f_t$$

for some $p_{s,t} \in P$. Think of $F = \bigoplus_{t \in T} P$ as a (P_1, P_1)-bimodule via the maps $(\psi \circ d_0, \psi \circ d_1)$. By Lemma 66.12.2 we obtain a unique A-biderivation $\lambda : P_1 \to F$ mapping s to the vector with coordinates $p_{s,t}$. By construction the composition

$$P_1 \longrightarrow F \longrightarrow P$$

sends $f \in P_1$ to $\psi(d_0(f) - d_1(f))$ because the map $f \mapsto \psi(d_0(f) - d_1(f))$ is an A-biderivation agreeing with the composition on generators.

For $g \in P_2$ we claim that $\lambda(d_0(g) - d_1(g) + d_2(g))$ is an element of Rel. Namely, by the last remark of the previous paragraph the image of $\lambda(d_0(g) - d_1(g) + d_2(g))$ in P is

$$\psi((d_0 - d_1)(d_0(g) - d_1(g) + d_2(g)))$$

which is zero by Simplicial, Section 14.21).

The choice of ψ determines a map

$$\mathrm{d}\psi \otimes 1 : \Omega_{P_0/A} \otimes B \longrightarrow \Omega_{P/A} \otimes B$$

Composing λ with the map $F \to F \otimes B$ gives a usual A-derivation as the two P_1-module structures on $F \otimes B$ agree. Thus λ determines a map

$$\overline{\lambda} : \Omega_{P_1/A} \otimes B \longrightarrow F \otimes B$$

Finally, We obtain a B-linear map

$$q : \Omega_{P_2/A} \otimes B \longrightarrow Rel/TrivRel$$

by mapping $\mathrm{d}g$ to the class of $\lambda(d_0(g) - d_1(g) + d_2(g))$ in the quotient.

The diagram

$$\begin{array}{ccccccc}
\Omega_{P_3/A} \otimes B & \longrightarrow & \Omega_{P_2/A} \otimes B & \longrightarrow & \Omega_{P_1/A} \otimes B & \longrightarrow & \Omega_{P_0/A} \otimes B \\
\downarrow & & \downarrow{\scriptstyle q} & & \downarrow{\scriptstyle \overline{\lambda}} & & \downarrow{\scriptstyle \mathrm{d}\psi \otimes 1} \\
0 & \longrightarrow & Rel/TrivRel & \longrightarrow & F \otimes B & \longrightarrow & \Omega_{P/A} \otimes B
\end{array}$$

commutes (calculation omitted) and we obtain the map of the lemma. By Remark 66.10.4 and Lemma 66.10.3 we see that this map induces isomorphisms $H_1(L_{B/A}) \to H_1(L)$ and $H_0(L_{B/A}) \to H_0(L)$.

It remains to see that our map $L_{B/A} \to L$ induces an isomorphism $H_2(L_{B/A}) \to H_2(L)$. Choose a resolution of B over A with $P_0 = P = A[u_i]$ and then P_1 and P_2 as in Example 66.5.9. In Remark 66.11.6 we have constructed an exact sequence

$$\wedge^2_B(J_0/J_0^2) \to \mathrm{Tor}_2^{P_0}(B, B) \to H^{-2}(L_{B/A}) \to 0$$

where $P_0 = P$ and $J_0 = \mathrm{Ker}(P \to B) = I$. Calculating the Tor group using the short exact sequences $0 \to I \to P \to B \to 0$ and $0 \to Rel \to F \to I \to 0$ we find that $\mathrm{Tor}_2^P(B, B) = \mathrm{Ker}(Rel \otimes B \to F \otimes B)$. The image of the map $\wedge^2_B(I/I^2) \to$

$\mathrm{Tor}_2^P(B,B)$ under this identification is exactly the image of $TrivRel \otimes B$. Thus we see that $H_2(L_{B/A}) \cong H_2(L)$.

Finally, we have to check that our map $L_{B/A} \to L$ actually induces this isomorphism. We will use the notation and results discussed in Example 66.5.9 and Remarks 66.11.6 and 66.10.5 without further mention. Pick an element ξ of $\mathrm{Tor}_2^{P_0}(B,B) = \mathrm{Ker}(I \otimes_P I \to I^2)$. Write $\xi = \sum h_{t',t} f_{t'} \otimes f_t$ for some $h_{t',t} \in P$. Tracing through the exact sequences above we find that ξ corresponds to the image in $Rel \otimes B$ of the element $r \in Rel \subset F = \bigoplus_{t \in T} P$ with tth coordinate $r_t = \sum_{t' \in T} h_{t',t} f_{t'}$. On the other hand, ξ corresponds to the element of $H_2(L_{B/A}) = H_2(\Omega)$ which is the image via $\mathrm{d} : H_2(\mathcal{J}/\mathcal{J}^2) \to H_2(\Omega)$ of the boundary of ξ under the 2-extension

$$0 \to \mathrm{Tor}_2^{\mathcal{O}}(\underline{B}, \underline{B}) \to \mathcal{J} \otimes_{\mathcal{O}} \mathcal{J} \to \mathcal{J} \to \mathcal{J}/\mathcal{J}^2 \to 0$$

We compute the successive transgressions of our element. First we have

$$\xi = (d_0 - d_1)(-\sum s_0(h_{t',t} f_{t'}) \otimes x_t)$$

and next we have

$$\sum s_0(h_{t',t} f_{t'}) x_t = d_0(v_r) - d_1(v_r) + d_2(v_r)$$

by our choice of the variables v in Example 66.5.9. We may choose our map λ above such that $\lambda(u_i) = 0$ and $\lambda(x_t) = -e_t$ where $e_t \in F$ denotes the basis vector corresponding to $t \in T$. Hence the construction of our map q above sends $\mathrm{d}v_r$ to

$$\lambda(\sum s_0(h_{t',t} f_{t'}) x_t) = \sum\nolimits_t \left(\sum\nolimits_{t'} h_{t',t} f_{t'} \right) e_t$$

matching the image of ξ in $Rel \otimes B$ (the two minus signs we found above cancel out). This agreement finishes the proof. □

Remark 66.12.4 (Functoriality of the Lichtenbaum-Schlessinger complex)**.** Consider a commutative square

$$\begin{array}{ccc} A' & \longrightarrow & B' \\ \uparrow & & \uparrow \\ A & \longrightarrow & B \end{array}$$

of ring maps. Choose a factorization

$$\begin{array}{ccccc} A' & \longrightarrow & P' & \longrightarrow & B' \\ \uparrow & & \uparrow & & \uparrow \\ A & \longrightarrow & P & \longrightarrow & B \end{array}$$

with P a polynomial algebra over A and P' a polynomial algebra over A'. Choose generators f_t, $t \in T$ for $\mathrm{Ker}(P \to B)$. For $t \in T$ denote f'_t the image of f_t in P'. Choose $f'_s \in P'$ such that the elements f'_t for $t \in T' = T \amalg S$ generate the kernel of $P' \to B'$. Set $F = \bigoplus_{t \in T} P$ and $F' = \bigoplus_{t' \in T'} P'$. Let $Rel = \mathrm{Ker}(F \to P)$ and $Rel' = \mathrm{Ker}(F' \to P')$ where the maps are given by multiplication by f_t, resp. f'_t on the coordinates. Finally, set $TrivRel$, resp. $TrivRel'$ equal to the submodule of Rel, resp. $TrivRel$ generated by the elements $(\ldots, f_{t'}, 0, \ldots, 0, -f_t, 0, \ldots)$ for

$t, t' \in T$, resp. T'. Having made these choices we obtain a canonical commutative diagram

$$\begin{array}{ccccc} L': & Rel'/TrivRel' & \longrightarrow & F' \otimes_{P'} B' & \longrightarrow & \Omega_{P'/A'} \otimes_{P'} B' \\ \uparrow & \uparrow & & \uparrow & & \uparrow \\ L: & Rel/TrivRel & \longrightarrow & F \otimes_P B & \longrightarrow & \Omega_{P/A} \otimes_P B \end{array}$$

Moreover, tracing through the choices made in the proof of Lemma 66.12.3 the reader sees that one obtains a commutative diagram

$$\begin{array}{ccc} L_{B'/A'} & \longrightarrow & L' \\ \uparrow & & \uparrow \\ L_{B/A} & \longrightarrow & L \end{array}$$

66.13. The cotangent complex of a local complete intersection

If $A \to B$ is a local complete intersection map, then $L_{B/A}$ is a perfect complex. The key to proving this is the following lemma.

Lemma 66.13.1. *Let $A = \mathbf{Z}[x] \to B = \mathbf{Z}$ be the ring map which sends x to 0. Let $I = (x) \subset A$. Then $L_{B/A}$ is quasi-isomorphic to $I/I^2[1]$.*

Proof. There are several ways to prove this. For example one can explicitly construct a resolution of B over A and compute. Or one can use the spectral sequence of Quillen (Theorem 66.11.4) and the vanishing of $\mathrm{Tor}_i^A(B, B)$ for $i > 1$. Finally, one can use (66.7.0.1) which is what we will do here. Namely, consider the distinguished triangle

$$L_{\mathbf{Z}[x]/\mathbf{Z}} \otimes_{\mathbf{Z}[x]} \mathbf{Z} \to L_{\mathbf{Z}/\mathbf{Z}} \to L_{\mathbf{Z}/\mathbf{Z}[x]} \to L_{\mathbf{Z}[x]/\mathbf{Z}} \otimes_{\mathbf{Z}[x]} \mathbf{Z}[1]$$

The complex $L_{\mathbf{Z}[x]/\mathbf{Z}}$ is quasi-isomorphic to $\Omega_{\mathbf{Z}[x]/\mathbf{Z}}$ by Lemma 66.4.7. The complex $L_{\mathbf{Z}/\mathbf{Z}}$ is zero in $D(\mathbf{Z})$ by Lemma 66.8.4. Thus we see that $L_{B/A}$ has only one nonzero cohomology group which is as described in the lemma by Lemma 66.10.2. □

Lemma 66.13.2. *Let $A \to B$ be a surjective ring map whose kernel I is generated by a regular sequence. Then $L_{B/A}$ is quasi-isomorphic to $I/I^2[1]$.*

Proof. This is true if $I = (0)$. If $I = (f)$ is generated by a single nonzerodivisor, then consider the ring map $\mathbf{Z}[x] \to A$ which sends x to f. By assumption we have $B = A \otimes^{\mathbf{L}}_{\mathbf{Z}[x]} \mathbf{Z}$. Thus we obtain $L_{B/A} = I/I^2[1]$ from Lemmas 66.6.2 and 66.13.1.

We prove the general case by induction. Suppose that we have $I = (f_1, \ldots, f_r)$ where $f_1, \ldots, f_r$ is a regular sequence. Set $C = A/(f_1, \ldots, f_{r-1})$. By induction the result is true for $A \to C$ and $C \to B$. We have a distinguished triangle (66.7.0.1)

$$L_{C/A} \otimes^{\mathbf{L}}_C B \to L_{B/A} \to L_{B/C} \to L_{C/A} \otimes^{\mathbf{L}}_C B[1]$$

which shows that $L_{B/A}$ has only one nonzero cohomology group which is as described in the lemma by Lemma 66.10.2. □

Lemma 66.13.3. *Let $A \to B$ be a surjective ring map whose kernel I is Koszul. Then $L_{B/A}$ is quasi-isomorphic to $I/I^2[1]$.*

Proof. Flat locally on $\mathrm{Spec}(A)$ the ideal I is generated by a regular sequence, see More on Algebra, Lemma 15.20.17. Hence this follows from Lemma 66.6.2 and flat descent. □

Proposition 66.13.4. *Let $A \to B$ be a local complete intersection map. Then $L_{B/A}$ is a perfect complex with tor amplitude in $[-1, 0]$.*

Proof. Choose a surjection $P = A[x_1, \ldots, x_n] \to B$ with kernel J. By Lemma 66.10.3 we see that $J/J^2 \to \bigoplus B\mathrm{d}x_i$ is quasi-isomorphic to $\tau_{\geq -1}L_{B/A}$. Note that J/J^2 is finite projective (More on Algebra, Lemma 15.21.3), hence $\tau_{\geq -1}L_{B/A}$ is a perfect complex with tor amplitude in $[-1, 0]$. Thus it suffices to show that $H^i(L_{B/A}) = 0$ for $i \notin [-1, 0]$. This follows from (66.7.0.1)

$$L_{P/A} \otimes^{\mathbf{L}}_P B \to L_{B/A} \to L_{B/P} \to L_{P/A} \otimes^{\mathbf{L}}_P B[1]$$

and Lemma 66.13.3 to see that $H^i(L_{B/P})$ is zero unless $i \in \{-1, 0\}$. (We also use Lemma 66.4.7 for the term on the left.) □

66.14. Tensor products and the cotangent complex

Let R be a ring and let A, B be R-algebras. In this section we discuss $L_{A\otimes_R B/R}$. Most of the information we want is contained in the following diagram (66.14.0.1)

$$\begin{array}{ccccc}
L_{A/R} \otimes^{\mathbf{L}}_A (A \otimes_R B) & \longrightarrow & L_{A\otimes_R B/B} & \longrightarrow & E \\
\| & & \uparrow & & \uparrow \\
L_{A/R} \otimes^{\mathbf{L}}_A (A \otimes_R B) & \longrightarrow & L_{A\otimes_R B/R} & \longrightarrow & L_{A\otimes_R B/A} \\
 & & \uparrow & & \uparrow \\
 & & L_{B/R} \otimes^{\mathbf{L}}_B (A \otimes_R B) & = & L_{B/R} \otimes^{\mathbf{L}}_B (A \otimes_R B)
\end{array}$$

Explanation: The middle row is the fundamental triangle (66.7.0.1) for the ring maps $R \to A \to A \otimes_R B$. The middle column is the fundamental triangle (66.7.0.1) for the ring maps $R \to B \to A \otimes_R B$. Next, E is an object of $D(A \otimes_R B)$ which "fits" into the upper right corner, i.e., which turns both the top row and the right column into distinguished triangles. Such an E exists by Derived Categories, Proposition 13.4.21 applied to the lower left square (with 0 placed in the missing spot). To be more explicit, we could for example define E as the cone (Derived Categories, Definition 13.8.1) of the map of complexes

$$L_{A/R} \otimes^{\mathbf{L}}_A (A \otimes_R B) \oplus L_{B/R} \otimes^{\mathbf{L}}_B (A \otimes_R B) \longrightarrow L_{A\otimes_R B/R}$$

and get the two maps with target E by an application of TR3. In the Tor independent case the object E is zero.

Lemma 66.14.1. *If A and B are Tor independent R-algebras, then the object E in (66.14.0.1) is zero. In this case we have*

$$L_{A\otimes_R B/R} = L_{A/R} \otimes^{\mathbf{L}}_A (A \otimes_R B) \oplus L_{B/R} \otimes^{\mathbf{L}}_B (A \otimes_R B)$$

which is represented by the complex $L_{A/R} \otimes_R B \oplus L_{B/R} \otimes_R A$ of $A \otimes_R B$-modules.

Proof. The first two statements are immediate from Lemma 66.6.2. The last statement follows as $L_{A/R}$ is a complex of free A-modules, hence $L_{A/R} \otimes^{\mathbf{L}}_A (A \otimes_R B)$ is represented by $L_{A/R} \otimes_A (A \otimes_R B) = L_{A/R} \otimes_R B$ □

In general we can say this about the object E.

Lemma 66.14.2. *Let R be a ring and let A, B be R-algebras. The object E in (66.14.0.1) satisfies*

$$H^i(E) = \begin{cases} 0 & \text{if} \quad i \geq -1 \\ Tor_1^R(A, B) & \text{if} \quad i = -2 \end{cases}$$

Proof. We use the description of E as the cone on $L_{B/R} \otimes^{\mathbf{L}}_B (A \otimes_R B) \to L_{A \otimes_R B/A}$. By Lemma 66.12.3 the canonical truncations $\tau_{\geq -2} L_{B/R}$ and $\tau_{\geq -2} L_{A \otimes_R B/A}$ are computed by the Lichtenbaum-Schlessinger complex (66.12.0.1). These isomorphisms are compatible with functoriality (Remark 66.12.4). Thus in this proof we work with the Lichtenbaum-Schlessinger complexes.

Choose a polynomial algebra P over R and a surjection $P \to B$. Choose generators $f_t \in P$, $t \in T$ of the kernel of this surjection. Let $Rel \subset F = \bigoplus_{t \in T} P$ be the kernel of the map $F \to P$ which maps the basis vector corresponding to t to f_t. Set $P_A = A \otimes_R P$ and $F_A = A \otimes_R F = P_A \otimes_P F$. Let Rel_A be the kernel of the map $F_A \to P_A$. Using the exact sequence

$$0 \to Rel \to F \to P \to B \to 0$$

and standard short exact sequences for Tor we obtain an exact sequence

$$A \otimes_R Rel \to Rel_A \to \mathrm{Tor}_1^R(A, B) \to 0$$

Note that $P_A \to A \otimes_R B$ is a surjection whose kernel is generated by the elements $1 \otimes f_t$ in P_A. Denote $TrivRel_A \subset Rel_A$ the P_A-submodule generated by the elements $(\ldots, 1 \otimes f_{t'}, 0, \ldots, 0, -1 \otimes f_t \otimes 1, 0, \ldots)$. Since $TrivRel \otimes_R A \to TrivRel_A$ is surjective, we find a canonical exact sequence

$$A \otimes_R (Rel/TrivRel) \to Rel_A/TrivRel_A \to \mathrm{Tor}_1^R(A, B) \to 0$$

The map of Lichtenbaum-Schlessinger complexes is given by the diagram

$$\begin{array}{ccccc} Rel_A/TrivRel_A & \longrightarrow & F_A \otimes_{P_A} (A \otimes_R B) & \longrightarrow & \Omega_{P_A/A \otimes_R B} \otimes_{P_A} (A \otimes_R B) \\ \uparrow{\scriptstyle -2} & & \uparrow{\scriptstyle -1} & & \uparrow{\scriptstyle 0} \\ Rel/TrivRel & \longrightarrow & F \otimes_P B & \longrightarrow & \Omega_{P/A} \otimes_P B \end{array}$$

Note that vertical maps -1 and -0 induce an isomorphism after applying the functor $A \otimes_R - = P_A \otimes_P -$ to the source and the vertical map -2 gives exactly the map whose cokernel is the desired Tor module as we saw above. □

66.15. Deformations of ring maps and the cotangent complex

This section is the continuation of Deformation Theory, Section 65.2 which we urge the reader to read first. We start with a surjective ring map $A' \to A$ whose kernel is an ideal I of square zero. Moreover we assume given a ring map $A \to B$, a

B-module N, and an A-module map $c : I \to N$. In this section we ask ourselves whether we can find the question mark fitting into the following diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & N & \longrightarrow & ? & \longrightarrow & B & \longrightarrow & 0 \\ & & \uparrow{\scriptstyle c} & & \uparrow & & \uparrow & & \\ 0 & \longrightarrow & I & \longrightarrow & A' & \longrightarrow & A & \longrightarrow & 0 \end{array} \tag{66.15.0.1}$$

and moreover how unique the solution is (if it exists). More precisely, we look for a surjection of A'-algebras $B' \to B$ whose kernel is identified with N such that $A' \to B'$ induces the given map c. We will say B' is a *solution* to (66.15.0.1).

Lemma 66.15.1. *In the situation above we have*

(1) *There is a canonical element $\xi \in Ext^2_B(L_{B/A}, N)$ whose vanishing is a sufficient and necessary condition for the existence of a solution to (66.15.0.1).*
(2) *If there exists a solution, then then the set of isomorphism classes of solutions is principal homogeneous under $Ext^1_B(L_{B/A}, N)$.*
(3) *Given a solution B', the set of automorphisms of B' fitting into (66.15.0.1) is canonically isomorphic to $Ext^0_B(L_{B/A}, N)$.*

Proof. Via the identifications $NL_{B/A} = \tau_{\geq -1} L_{B/A}$ (Lemma 66.10.3) and $H^0(L_{B/A}) = \Omega_{B/A}$ (Lemma 66.4.5) we have seen parts (2) and (3) in Deformation Theory, Lemmas 65.2.1 and 65.2.3.

Proof of (1). We will use the results of Deformation Theory, Lemma 65.2.4 without further mention. Let $\alpha \in \mathrm{Ext}^1_A(NL_{A/\mathbf{Z}}, I)$ be the element corresponding to the isomorphism class of A'. The existence of B' corresponds to an element $\beta \in \mathrm{Ext}^1_B(NL_{B/\mathbf{Z}}, N)$ which maps to the image of α in $\mathrm{Ext}^1_A(NL_{A/\mathbf{Z}}, N)$. Note that

$$\mathrm{Ext}^1_A(NL_{A/\mathbf{Z}}, N) = \mathrm{Ext}^1_A(L_{A/\mathbf{Z}}, N) = \mathrm{Ext}^1_B(L_{A/\mathbf{Z}} \otimes^{\mathbf{L}}_A B, N)$$

and

$$\mathrm{Ext}^1_B(NL_{B/\mathbf{Z}}, N) = \mathrm{Ext}^1_B(L_{B/\mathbf{Z}}, N)$$

by Lemma 66.10.3. Since the distinguished triangle (66.7.0.1) for $\mathbf{Z} \to A \to B$ gives rise to a long exact sequence

$$\ldots \to \mathrm{Ext}^1_B(L_{B/\mathbf{Z}}, N) \to \mathrm{Ext}^1_B(L_{A/\mathbf{Z}} \otimes^{\mathbf{L}}_A B, N) \to \mathrm{Ext}^2_B(L_{B/A}, N) \to \ldots$$

we obtain the result with ξ the image of α. □

66.16. The Atiyah class of a module

Let $A \to B$ be a ring map. Let M be a B-module. Let $P \to B$ be an object of $\mathcal{C}_{B/A}$ (Section 66.4). Consider the extension of principal parts

$$0 \to \Omega_{P/A} \otimes_P M \to P^1_{P/A}(M) \to M \to 0$$

see Algebra, Lemma 10.126.6. This sequence is functorial in P by Algebra, Remark 10.126.7. Thus we obtain a short exact sequence of sheaves of $\mathcal{O}$-modules

$$0 \to \Omega_{\mathcal{O}/\underline{A}} \otimes_{\mathcal{O}} \underline{M} \to P^1_{\mathcal{O}/\underline{A}}(M) \to \underline{M} \to 0$$

on $\mathcal{C}_{B/A}$. We have $L\pi_!(\Omega_{\mathcal{O}/\underline{A}} \otimes_{\mathcal{O}} \underline{M}) = L_{B/A} \otimes_B M = L_{B/A} \otimes^{\mathbf{L}}_B M$ by Lemma 66.4.2 and the flatness of the terms of $L_{B/A}$. We have $L\pi_!(\underline{M}) = M$ by Lemma 66.4.4. Thus a distinguished triangle

$$L_{B/A} \otimes^{\mathbf{L}}_B M \to L\pi_!\left(P^1_{\mathcal{O}/\underline{A}}(M)\right) \to M \to L_{B/A} \otimes^{\mathbf{L}}_B M[1] \tag{66.16.0.1}$$

in $D(B)$. Here we use Cohomology on Sites, Remark 21.28.13 to get a distinguished triangle in $D(B)$ and not just in $D(A)$.

Definition 66.16.1. Let $A \to B$ be a ring map. Let M be a B-module. The map $M \to L_{B/A} \otimes^{\mathbf{L}}_B M[1]$ in (66.16.0.1) is called the *Atiyah class* of M.

66.17. The cotangent complex

In this section we discuss the cotangent complex of a map of sheaves of rings on a site. In later sections we specialize this to obtain the cotangent complex of a morphism of ringed topoi, a morphism of ringed spaces, a morphism of schemes, a morphism of algebraic space, etc.

Let $\mathcal{C}$ be a site and let $Sh(\mathcal{C})$ denote the associated topos. Let $\mathcal{A}$ denote a sheaf of rings on $\mathcal{C}$. Let $\mathcal{A}$-Alg be the category of $\mathcal{A}$-algebras. Consider the pair of adjoint functors (F, i) where $i : \mathcal{A}$-$Alg \to Sh(\mathcal{C})$ is the forgetful functor and $F : Sh(\mathcal{C}) \to \mathcal{A}$-$Alg$ assigns to a sheaf of sets $\mathcal{E}$ the polynomial algebra $\mathcal{A}[\mathcal{E}]$ on $\mathcal{E}$ over $\mathcal{A}$. Let $X_\bullet$ be the simplicial object of $\mathrm{Fun}(\mathcal{A}$-$Alg, \mathcal{A}$-$Alg)$ constructed in Simplicial, Section 14.31.

Now assume that $\mathcal{A} \to \mathcal{B}$ is a homomorphism of sheaves of rings. Then $\mathcal{B}$ is an object of the category $\mathcal{A}$-Alg. Denote $\mathcal{P}_\bullet = X_\bullet(\mathcal{B})$ the resulting simplicial $\mathcal{A}$-algebra. Recall that $\mathcal{P}_0 = \mathcal{A}[\mathcal{B}]$, $\mathcal{P}_1 = \mathcal{A}[\mathcal{A}[\mathcal{B}]]$, and so on. Recall also that there is an augmentation

$$\epsilon : \mathcal{P}_\bullet \longrightarrow \mathcal{B}$$

where we view $\mathcal{B}$ as a constant simplicial $\mathcal{A}$-algebra.

Definition 66.17.1. Let $\mathcal{C}$ be a site. Let $\mathcal{A} \to \mathcal{B}$ be a homomorphism of sheaves of rings on $\mathcal{C}$. The *standard resolution of $\mathcal{B}$ over $\mathcal{A}$* is the augmentation $\epsilon : \mathcal{P}_\bullet \to \mathcal{B}$ with terms

$$\mathcal{P}_0 = \mathcal{A}[\mathcal{B}], \quad \mathcal{P}_1 = \mathcal{A}[\mathcal{A}[\mathcal{B}]], \quad \ldots$$

and maps as constructed above.

With this definition in hand the cotangent complex of a map of sheaves of rings is defined as follows. We will use the module of differentials as defined in Modules on Sites, Section 18.32.

Definition 66.17.2. Let $\mathcal{C}$ be a site. Let $\mathcal{A} \to \mathcal{B}$ be a homomorphism of sheaves of rings on $\mathcal{C}$. The *cotangent complex* $L_{\mathcal{B}/\mathcal{A}}$ is the complex of $\mathcal{B}$-modules associated to the simplicial module

$$\Omega_{\mathcal{P}_\bullet/\mathcal{A}} \otimes_{\mathcal{P}_\bullet, \epsilon} \mathcal{B}$$

where $\epsilon : \mathcal{P}_\bullet \to \mathcal{B}$ is the standard resolution of $\mathcal{B}$ over $\mathcal{A}$. We usually think of $L_{\mathcal{B}/\mathcal{A}}$ as an object of $D(\mathcal{B})$.

These constructions satisfy a functoriality similar to that discussed in Section 66.6. Namely, given a commutative diagram

$$\begin{array}{ccc} \mathcal{B} & \longrightarrow & \mathcal{B}' \\ \uparrow & & \uparrow \\ \mathcal{A} & \longrightarrow & \mathcal{A}' \end{array} \tag{66.17.2.1}$$

of sheaves of rings on $\mathcal{C}$ there is a canonical $\mathcal{B}$-linear map of complexes

$$L_{\mathcal{B}/\mathcal{A}} \longrightarrow L_{\mathcal{B}'/\mathcal{A}'}$$

constructed as follows. If $\mathcal{P}_\bullet \to \mathcal{B}$ is the standard resolution of $\mathcal{B}$ over $\mathcal{A}$ and $\mathcal{P}'_\bullet \to \mathcal{B}'$ is the standard resolution of $\mathcal{B}'$ over $\mathcal{A}'$, then there is a canonical map $\mathcal{P}_\bullet \to \mathcal{P}'_\bullet$ of simplicial $\mathcal{A}$-algebras compatible with the augmentations $\mathcal{P}_\bullet \to \mathcal{B}$ and $\mathcal{P}'_\bullet \to \mathcal{B}'$. The maps

$$\mathcal{P}_0 = \mathcal{A}[\mathcal{B}] \longrightarrow \mathcal{A}'[\mathcal{B}'] = \mathcal{P}'_0, \quad \mathcal{P}_1 = \mathcal{A}[\mathcal{A}[\mathcal{B}]] \longrightarrow \mathcal{A}'[\mathcal{A}'[\mathcal{B}']] = \mathcal{P}'_1$$

and so on are given by the given maps $\mathcal{A} \to \mathcal{A}'$ and $\mathcal{B} \to \mathcal{B}'$. The desired map $L_{\mathcal{B}/\mathcal{A}} \to L_{\mathcal{B}'/\mathcal{A}'}$ then comes from the associated maps on sheaves of differentials.

Lemma 66.17.3. *Let $f : Sh(\mathcal{D}) \to Sh(\mathcal{C})$ be a morphism of topoi. Let $\mathcal{A} \to \mathcal{B}$ be a homomorphism of sheaves of rings on $\mathcal{C}$. Then $f^{-1}L_{\mathcal{B}/\mathcal{A}} = L_{f^{-1}\mathcal{B}/f^{-1}\mathcal{A}}$.*

Proof. The diagram

$$\begin{array}{ccc} \mathcal{A}\text{-}Alg & \rightleftarrows & Sh(\mathcal{C}) \\ {\scriptstyle f^{-1}}\downarrow & & \downarrow{\scriptstyle f^{-1}} \\ f^{-1}\mathcal{A}\text{-}Alg & \rightleftarrows & Sh(\mathcal{D}) \end{array}$$

commutes. □

Lemma 66.17.4. *Let $\mathcal{C}$ be a site. Let $\mathcal{A} \to \mathcal{B}$ be a homomorphism of sheaves of rings on $\mathcal{C}$. Then $H^i(L_{\mathcal{B}/\mathcal{A}})$ is the sheaf associated to the presheaf $U \mapsto H^i(L_{\mathcal{B}(U)/\mathcal{A}(U)})$.*

Proof. Let $\mathcal{C}'$ be the site we get by endowing $\mathcal{C}$ with the chaotic topology (presheaves are sheaves). There is a morphism of topoi $f : Sh(\mathcal{C}) \to Sh(\mathcal{C}')$ where f_* is the inclusion of sheaves into presheaves and f^{-1} is sheafification. By Lemma 66.17.3 it suffices to prove the result for $\mathcal{C}'$, i.e., in case $\mathcal{C}$ has the chaotic topology.

If $\mathcal{C}$ carries the chaotic topology, then $L_{\mathcal{B}/\mathcal{A}}(U)$ is equal to $L_{\mathcal{B}(U)/\mathcal{A}(U)}$ because

$$\begin{array}{ccc} \mathcal{A}\text{-}Alg & \rightleftarrows & Sh(\mathcal{C}) \\ {\scriptstyle \text{sections over } U}\downarrow & & \downarrow{\scriptstyle \text{sections over } U} \\ \mathcal{A}(U)\text{-}Alg & \rightleftarrows & Sets \end{array}$$

commutes. □

Remark 66.17.5. It is clear from the proof of Lemma 66.17.4 that for any $U \in \mathrm{Ob}(\mathcal{C})$ there is a canonical map $L_{\mathcal{B}(U)/\mathcal{A}(U)} \to L_{\mathcal{B}/\mathcal{A}}(U)$ of complexes of $\mathcal{B}(U)$-modules. Moreover, these maps are compatible with restriction maps and the complex $L_{\mathcal{B}/\mathcal{A}}$ is the sheafification of the rule $U \mapsto L_{\mathcal{B}(U)/\mathcal{A}(U)}$.

Lemma 66.17.6. *Let $\mathcal{C}$ be a site. Let $\mathcal{A} \to \mathcal{B}$ be a homomorphism of sheaves of rings on $\mathcal{C}$. Then $H^0(L_{\mathcal{B}/\mathcal{A}}) = \Omega_{\mathcal{B}/\mathcal{A}}$.*

Proof. Follows from Lemmas 66.17.4 and 66.4.5 and Modules on Sites, Lemma 18.32.4. □

Lemma 66.17.7. *Let $\mathcal{C}$ be a site. Let $\mathcal{A} \to \mathcal{B}$ and $\mathcal{A} \to \mathcal{B}'$ be homomorphisms of sheaves of rings on $\mathcal{C}$. Then*

$$L_{\mathcal{B}\times\mathcal{B}'/\mathcal{A}} \longrightarrow L_{\mathcal{B}/\mathcal{A}} \oplus L_{\mathcal{B}'/\mathcal{A}}$$

is an isomorphism in $D(\mathcal{B} \times \mathcal{B}')$.

Proof. By Lemma 66.17.4 it suffices to prove this for ring maps. In the case of rings this is Lemma 66.6.4. □

The fundamental triangle for the cotangent complex of sheaves of rings is an easy consequence of the result for homomorphisms of rings.

Lemma 66.17.8. *Let $\mathcal{D}$ be a site. Let $\mathcal{A} \to \mathcal{B} \to \mathcal{C}$ be homomorphisms of sheaves of rings on $\mathcal{D}$. There is a canonical distinguished triangle*

$$L_{\mathcal{B}/\mathcal{A}} \otimes^{\mathbf{L}}_{\mathcal{B}} \mathcal{C} \to L_{\mathcal{C}/\mathcal{A}} \to L_{\mathcal{C}/\mathcal{B}} \to L_{\mathcal{B}/\mathcal{A}} \otimes^{\mathbf{L}}_{\mathcal{B}} \mathcal{C}[1]$$

in $D(\mathcal{C})$.

Proof. We will use the method described in Remarks 66.7.5 and 66.7.6 to construct the triangle; we will freely use the results mentioned there. As in those remarks we first construct the triangle in case $\mathcal{B} \to \mathcal{C}$ is an injective map of sheaves of rings. In this case we set

(1) $\mathcal{P}_\bullet$ is the standard resolution of $\mathcal{B}$ over $\mathcal{A}$,
(2) $\mathcal{Q}_\bullet$ is the standard resolution of $\mathcal{C}$ over $\mathcal{A}$,
(3) $\mathcal{R}_\bullet$ is the standard resolution of $\mathcal{C}$ over $\mathcal{B}$,
(4) $\mathcal{S}_\bullet$ is the standard resolution of $\mathcal{B}$ over $\mathcal{B}$,
(5) $\overline{\mathcal{Q}}_\bullet = \mathcal{Q}_\bullet \otimes_{\mathcal{P}_\bullet} \mathcal{B}$, and
(6) $\overline{\mathcal{R}}_\bullet = \mathcal{R}_\bullet \otimes_{\mathcal{S}_\bullet} \mathcal{B}$.

The distinguished triangle is the distinguished triangle associated to the short exact sequence of simplicial $\mathcal{C}$-modules

$$0 \to \Omega_{\mathcal{P}_\bullet/\mathcal{A}} \otimes_{\mathcal{P}_\bullet} \mathcal{C} \to \Omega_{\mathcal{Q}_\bullet/\mathcal{A}} \otimes_{\mathcal{Q}_\bullet} \mathcal{C} \to \Omega_{\overline{\mathcal{Q}}_\bullet/\mathcal{B}} \otimes_{\overline{\mathcal{Q}}_\bullet} \mathcal{C} \to 0$$

The first two terms are equal to the first two terms of the triangle of the statement of the lemma. The identification of the last term with $L_{\mathcal{C}/\mathcal{B}}$ uses the quasi-isomorphisms of complexes

$$L_{\mathcal{C}/\mathcal{B}} = \Omega_{\mathcal{R}_\bullet/\mathcal{B}} \otimes_{\mathcal{R}_\bullet} \mathcal{C} \longrightarrow \Omega_{\overline{\mathcal{R}}_\bullet/\mathcal{B}} \otimes_{\overline{\mathcal{R}}_\bullet} \mathcal{C} \longleftarrow \Omega_{\overline{\mathcal{Q}}_\bullet/\mathcal{B}} \otimes_{\overline{\mathcal{Q}}_\bullet} \mathcal{C}$$

All the constructions used above can first be done on the level of presheaves and then sheafified. Hence to prove sequences are exact, or that map are quasi-isomorphisms it suffices to prove the corresponding statement for the ring maps $\mathcal{A}(U) \to \mathcal{B}(U) \to \mathcal{C}(U)$ which are known. This finishes the proof in the case that $\mathcal{B} \to \mathcal{C}$ is injective.

In general, we reduce to the case where $\mathcal{B} \to \mathcal{C}$ is injective by replacing $\mathcal{C}$ by $\mathcal{B} \times \mathcal{C}$ if necessary. This is possible by the argument given in Remark 66.7.5 by Lemma 66.17.7. □

Lemma 66.17.9. *Let $\mathcal{C}$ be a site. Let $\mathcal{A} \to \mathcal{B}$ be a homomorphism of sheaves of rings on $\mathcal{C}$. If p is a point of $\mathcal{C}$, then $(L_{\mathcal{B}/\mathcal{A}})_p = L_{\mathcal{B}_p/\mathcal{A}_p}$.*

Proof. This is a special case of Lemma 66.17.3. □

For the construction of the naive cotangent complex and its properties we refer to Modules on Sites, Section 18.34.

Lemma 66.17.10. *Let $\mathcal{C}$ be a site. Let $\mathcal{A} \to \mathcal{B}$ be a homomorphism of sheaves of rings on $\mathcal{C}$. There is a canonical map $L_{\mathcal{B}/\mathcal{A}} \to NL_{\mathcal{B}/\mathcal{A}}$ which identifies the naive cotangent complex with the truncation $\tau_{\geq -1}L_{\mathcal{B}/\mathcal{A}}$.*

Proof. Let $\mathcal{P}_\bullet$ be the standard resolution of $\mathcal{B}$ over $\mathcal{A}$. Let $\mathcal{I} = \mathrm{Ker}(\mathcal{A}[\mathcal{B}] \to \mathcal{B})$. Recall that $\mathcal{P}_0 = \mathcal{A}[\mathcal{B}]$. The map of the lemma is given by the commutative diagram

$$\begin{array}{ccccccccc} L_{\mathcal{B}/\mathcal{A}} & & \ldots & \longrightarrow & \Omega_{\mathcal{P}_2/\mathcal{A}} \otimes_{\mathcal{P}_2} \mathcal{B} & \longrightarrow & \Omega_{\mathcal{P}_1/\mathcal{A}} \otimes_{\mathcal{P}_1} \mathcal{B} & \longrightarrow & \Omega_{\mathcal{P}_0/\mathcal{A}} \otimes_{\mathcal{P}_0} \mathcal{B} \\ \downarrow & & & & \downarrow & & \downarrow & & \downarrow \\ NL_{\mathcal{B}/\mathcal{A}} & & \ldots & \longrightarrow & 0 & \longrightarrow & \mathcal{I}/\mathcal{I}^2 & \longrightarrow & \Omega_{\mathcal{P}_0/\mathcal{A}} \otimes_{\mathcal{P}_0} \mathcal{B} \end{array}$$

We construct the downward arrow with target $\mathcal{I}/\mathcal{I}^2$ by sending a local section $\mathrm{d}f \otimes b$ to the class of $(d_0(f) - d_1(f))b$ in $\mathcal{I}/\mathcal{I}^2$. Here $d_i : \mathcal{P}_1 \to \mathcal{P}_0$, $i = 0, 1$ are the two face maps of the simplicial structure. This makes sense as $d_0 - d_1$ maps $\mathcal{P}_1$ into $\mathcal{I} = \mathrm{Ker}(\mathcal{P}_0 \to \mathcal{B})$. We omit the verification that this rule is well defined. Our map is compatible with the differential $\Omega_{\mathcal{P}_1/\mathcal{A}} \otimes_{\mathcal{P}_1} \mathcal{B} \to \Omega_{\mathcal{P}_0/\mathcal{A}} \otimes_{\mathcal{P}_0} \mathcal{B}$ as this differential maps a local section $\mathrm{d}f \otimes b$ to $\mathrm{d}(d_0(f) - d_1(f)) \otimes b$. Moreover, the differential $\Omega_{\mathcal{P}_2/\mathcal{A}} \otimes_{\mathcal{P}_2} \mathcal{B} \to \Omega_{\mathcal{P}_1/\mathcal{A}} \otimes_{\mathcal{P}_1} \mathcal{B}$ maps a local section $\mathrm{d}f \otimes b$ to $\mathrm{d}(d_0(f) - d_1(f) + d_2(f)) \otimes b$ which are annihilated by our downward arrow. Hence a map of complexes.

To see that our map induces an isomorphism on the cohomology sheaves H^0 and H^{-1} we argue as follows. Let $\mathcal{C}'$ be the site with the same underlying category as $\mathcal{C}$ but endowed with the chaotic topology. Let $f : Sh(\mathcal{C}) \to Sh(\mathcal{C}')$ be the morphism of topoi whose pullback functor is sheafification. Let $\mathcal{A}' \to \mathcal{B}'$ be the given map, but thought of as a map of sheaves of rings on $\mathcal{C}'$. The construction above gives a map $L_{\mathcal{B}'/\mathcal{A}'} \to NL_{\mathcal{B}'/\mathcal{A}'}$ on $\mathcal{C}'$ whose value over any object U of $\mathcal{C}'$ is just the map

$$L_{\mathcal{B}(U)/\mathcal{A}(U)} \to NL_{\mathcal{B}(U)/\mathcal{A}(U)}$$

of Remark 66.10.4 which induces an isomorphism on H^0 and H^{-1}. Since $f^{-1}L_{\mathcal{B}'/\mathcal{A}'} = L_{\mathcal{B}/\mathcal{A}}$ (Lemma 66.17.3) and $f^{-1}\,NL_{\mathcal{B}'/\mathcal{A}'} = NL_{\mathcal{B}/\mathcal{A}}$ (Modules on Sites, Lemma 18.34.3) the lemma is proved. □

66.18. The Atiyah class of a sheaf of modules

Let $\mathcal{C}$ be a site. Let $\mathcal{A} \to \mathcal{B}$ be a homomorphism of sheaves of rings. Let $\mathcal{F}$ be a sheaf of $\mathcal{B}$-modules. Let $\mathcal{P}_\bullet \to \mathcal{B}$ be the standard resolution of $\mathcal{B}$ over $\mathcal{A}$ (Section 66.17). For every $n \geq 0$ consider the extension of principal parts

$$0 \to \Omega_{\mathcal{P}_n/\mathcal{A}} \otimes_{\mathcal{P}_n} \mathcal{F} \to \mathcal{P}^1_{\mathcal{P}_n/\mathcal{A}}(\mathcal{F}) \to \mathcal{F} \to 0 \tag{66.18.0.1}$$

see Modules on Sites, Lemma 18.33.6. The functoriality of this construction (Modules on Sites, Remark 18.33.7) tells us (66.18.0.1) is the degree n part of a short exact sequence of simplicial $\mathcal{P}_\bullet$-modules (Cohomology on Sites, Section 21.30). Using the functor $L\pi_! : D(\mathcal{P}_\bullet) \to D(\mathcal{B})$ of Cohomology on Sites, Remark 21.30.3 (here we use that $\mathcal{P}_\bullet \to \mathcal{A}$ is a resolution) we obtain a distinguished triangle

$$L_{\mathcal{B}/\mathcal{A}} \otimes^{\mathbf{L}}_{\mathcal{B}} \mathcal{F} \to L\pi_! \left(\mathcal{P}^1_{\mathcal{P}_\bullet/\mathcal{A}}(\mathcal{F}) \right) \to \mathcal{F} \to L_{\mathcal{B}/\mathcal{A}} \otimes^{\mathbf{L}}_{\mathcal{B}} \mathcal{F}[1] \tag{66.18.0.2}$$

in $D(\mathcal{B})$.

Definition 66.18.1. Let $\mathcal{C}$ be a site. Let $\mathcal{A} \to \mathcal{B}$ be a homomorphism of sheaves of rings. Let $\mathcal{F}$ be a sheaf of $\mathcal{B}$-modules. The map $\mathcal{F} \to L_{\mathcal{B}/\mathcal{A}} \otimes^{\mathbf{L}}_{\mathcal{B}} \mathcal{F}[1]$ in (66.18.0.2) is called the *Atiyah class* of $\mathcal{F}$.

66.19. The cotangent complex of a morphism of ringed spaces

The cotangent complex of a morphism of ringed spaces is defined in terms of the cotangent complex we defined above.

Definition 66.19.1. Let $f : (X, \mathcal{O}_X) \to (S, \mathcal{O}_S)$ be a morphism of ringed spaces. The *cotangent complex* L_f of f is $L_f = L_{\mathcal{O}_X/f^{-1}\mathcal{O}_S}$. We will also use the notation $L_f = L_{X/S} = L_{\mathcal{O}_X/\mathcal{O}_S}$.

More precisely, this means that we consider the cotangent complex (Definition 66.17.2) of the homomorphism $f^\sharp : f^{-1}\mathcal{O}_S \to \mathcal{O}_X$ of sheaves of rings on the site associated to the topological space X (Sites, Example 7.6.4).

Lemma 66.19.2. *Let $f : (X, \mathcal{O}_X) \to (S, \mathcal{O}_S)$ be a morphism of ringed spaces. Then $H^0(L_{X/S}) = \Omega_{X/S}$.*

Proof. Special case of Lemma 66.17.6. □

Lemma 66.19.3. *Let $f : X \to Y$ and $g : Y \to Z$ be morphisms of ringed spaces. Then there is a canonical distinguished triangle*

$$Lf^* L_{Y/Z} \to L_{X/Z} \to L_{X/Y} \to Lf^* L_{Y/Z}[1]$$

in $D(\mathcal{O}_X)$.

Proof. Set $h = g \circ f$ so that $h^{-1}\mathcal{O}_Z = f^{-1}g^{-1}\mathcal{O}_Z$. By Lemma 66.17.3 we have $f^{-1}L_{Y/Z} = L_{f^{-1}\mathcal{O}_Y/h^{-1}\mathcal{O}_Z}$ and this is a complex of flat $f^{-1}\mathcal{O}_Y$-modules. Hence the distinguished triangle above is an example of the distinguished triangle of Lemma 66.17.8 with $\mathcal{A} = h^{-1}\mathcal{O}_Z$, $\mathcal{B} = f^{-1}\mathcal{O}_Y$, and $\mathcal{C} = \mathcal{O}_X$. □

Lemma 66.19.4. *Let $f : (X, \mathcal{O}_X) \to (Y, \mathcal{O}_Y)$ be a morphism of ringed spaces. There is a canonical map $L_{X/Y} \to NL_{X/Y}$ which identifies the naive cotangent complex with the truncation $\tau_{\geq -1} L_{X/Y}$.*

Proof. Special case of Lemma 66.17.10. □

66.20. Deformations of ringed spaces and the cotangent complex

This section is the continuation of Deformation Theory, Section 65.7 which we urge the reader to read first. We briefly recall the setup. We have a first order thickening $t : (S, \mathcal{O}_S) \to (S', \mathcal{O}_{S'})$ of ringed spaces with $\mathcal{J} = \mathrm{Ker}(t^\sharp)$, a morphism of ringed spaces $f : (X, \mathcal{O}_X) \to (S, \mathcal{O}_S)$, an $\mathcal{O}_X$-module $\mathcal{G}$, and an f-map $c : \mathcal{J} \to \mathcal{G}$ of sheaves of modules. We ask whether we can find the question mark fitting into the following diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \mathcal{G} & \longrightarrow & ? & \longrightarrow & \mathcal{O}_X & \longrightarrow & 0 \\ & & \uparrow{\scriptstyle c} & & \uparrow & & \uparrow & & \\ 0 & \longrightarrow & \mathcal{J} & \longrightarrow & \mathcal{O}_{S'} & \longrightarrow & \mathcal{O}_S & \longrightarrow & 0 \end{array} \tag{66.20.0.1}$$

and moreover how unique the solution is (if it exists). More precisely, we look for a first order thickening $i : (X, \mathcal{O}_X) \to (X', \mathcal{O}_{X'})$ and a morphism of thickenings

(f, f') as in Deformation Theory, Equation (65.3.1.1) where $\mathrm{Ker}(i^\sharp)$ is identified with $\mathcal{G}$ such that $(f')^\sharp$ induces the given map c. We will say X' is a *solution* to (66.20.0.1).

Lemma 66.20.1. *In the situation above we have*

(1) *There is a canonical element $\xi \in Ext^2_{\mathcal{O}_X}(L_{X/S}, \mathcal{G})$ whose vanishing is a sufficient and necessary condition for the existence of a solution to (66.20.0.1).*
(2) *If there exists a solution, then then the set of isomorphism classes of solutions is principal homogeneous under $Ext^1_{\mathcal{O}_X}(L_{X/S}, \mathcal{G})$.*
(3) *Given a solution X', the set of automorphisms of X' fitting into (66.20.0.1) is canonically isomorphic to $Ext^0_{\mathcal{O}_X}(L_{X/S}, \mathcal{G})$.*

Proof. Via the identifications $NL_{X/S} = \tau_{\geq -1} L_{X/S}$ (Lemma 66.19.4) and $H^0(L_{X/S}) = \Omega_{X/S}$ (Lemma 66.19.2) we have seen parts (2) and (3) in Deformation Theory, Lemmas 65.7.1 and 65.7.3.

Proof of (1). We will use the results of Deformation Theory, Lemma 65.7.4 without further mention. Let $\alpha \in \mathrm{Ext}^1_{\mathcal{O}_S}(NL_{S/\mathbf{Z}}, \mathcal{J})$ be the element corresponding to the isomorphism class of S'. The existence of X' corresponds to an element $\beta \in \mathrm{Ext}^1_{\mathcal{O}_X}(NL_{X/\mathbf{Z}}, \mathcal{G})$ which maps to the image of α in $\mathrm{Ext}^1_{\mathcal{O}_X}(Lf^* NL_{S/\mathbf{Z}}, \mathcal{G})$. Note that

$$\mathrm{Ext}^1_{\mathcal{O}_X}(Lf^* NL_{S/\mathbf{Z}}, \mathcal{G}) = \mathrm{Ext}^1_{\mathcal{O}_X}(Lf^* L_{S/\mathbf{Z}}, \mathcal{G})$$

and

$$\mathrm{Ext}^1_{\mathcal{O}_X}(NL_{X/\mathbf{Z}}, \mathcal{G}) = \mathrm{Ext}^1_{\mathcal{O}_X}(L_{X/\mathbf{Z}}, \mathcal{G})$$

by Lemma 66.19.4. The distinguished triangle of Lemma 66.19.3 for $X \to S \to (*, \mathbf{Z})$ gives rise to a long exact sequence

$$\ldots \to \mathrm{Ext}^1_{\mathcal{O}_X}(L_{X/\mathbf{Z}}, \mathcal{G}) \to \mathrm{Ext}^1_{\mathcal{O}_X}(Lf^* L_{S/\mathbf{Z}}, \mathcal{G}) \to \mathrm{Ext}^2_{\mathcal{O}_X}(L_{X/S}, \mathcal{G}) \to \ldots$$

We obtain the result with ξ the image of α. □

66.21. The cotangent complex of a morphism of ringed topoi

The cotangent complex of a morphism of ringed topoi is defined in terms of the cotangent complex we defined above.

Definition 66.21.1. Let $(f, f^\sharp) : (Sh(\mathcal{C}), \mathcal{O}_\mathcal{C}) \to (Sh(\mathcal{D}), \mathcal{O}_\mathcal{D})$ be a morphism of ringed topoi. The *cotangent complex* L_f of f is $L_f = L_{\mathcal{O}_\mathcal{C}/f^{-1}\mathcal{O}_\mathcal{D}}$. We sometimes write $L_f = L_{\mathcal{O}_\mathcal{C}/\mathcal{O}_\mathcal{D}}$.

This definition applies to many situations, but it doesn't always produce the thing one expects. For example, if $f : X \to Y$ is a morphism of schemes, then f induces a morphism of big étale sites $f_{big} : (Sch/X)_{étale} \to (Sch/Y)_{étale}$ which is a morphism of ringed topoi (Descent, Remark 34.7.4). However, $L_{f_{big}} = 0$ since $(f_{big})^\sharp$ is an isomorphism. On the other hand, if we take L_f where we think of f as a morphism between the underlying Zariski ringed topoi, then L_f does agree with the cotangent complex $L_{X/Y}$ (as defined below) whose zeroth cohomology sheaf is $\Omega_{X/Y}$.

Lemma 66.21.2. *Let $f : (Sh(\mathcal{C}), \mathcal{O}) \to (Sh(\mathcal{B}), \mathcal{O}_\mathcal{B})$ be a morphism of ringed topoi. Then $H^0(L_f) = \Omega_f$.*

Proof. Special case of Lemma 66.17.6. □

Lemma 66.21.3. *Let $f : (Sh(\mathcal{C}_1), \mathcal{O}_1) \to (Sh(\mathcal{C}_2), \mathcal{O}_2)$ and $g : (Sh(\mathcal{C}_2), \mathcal{O}_2) \to (Sh(\mathcal{C}_3), \mathcal{O}_3)$ be morphisms of ringed topoi. Then there is a canonical distinguished triangle*

$$Lf^*L_g \to L_{g\circ f} \to L_f \to Lf^*L_g[1]$$

in $D(\mathcal{O}_1)$.

Proof. Set $h = g \circ f$ so that $h^{-1}\mathcal{O}_3 = f^{-1}g^{-1}\mathcal{O}_3$. By Lemma 66.17.3 we have $f^{-1}L_g = L_{f^{-1}\mathcal{O}_2/h^{-1}\mathcal{O}_3}$ and this is a complex of flat $f^{-1}\mathcal{O}_2$-modules. Hence the distinguished triangle above is an example of the distinguished triangle of Lemma 66.17.8 with $\mathcal{A} = h^{-1}\mathcal{O}_3$, $\mathcal{B} = f^{-1}\mathcal{O}_2$, and $\mathcal{C} = \mathcal{O}_1$. □

Lemma 66.21.4. *Let $f : (Sh(\mathcal{C}), \mathcal{O}) \to (Sh(\mathcal{B}), \mathcal{O}_\mathcal{B})$ be a morphism of ringed topoi. There is a canonical map $L_f \to NL_f$ which identifies the naive cotangent complex with the truncation $\tau_{\geq -1}L_f$.*

Proof. Special case of Lemma 66.17.10. □

66.22. Deformations of ringed topoi and the cotangent complex

This section is the continuation of Deformation Theory, Section 65.12 which we urge the reader to read first. We briefly recall the setup. We have a first order thickening $t : (Sh(\mathcal{B}), \mathcal{O}_\mathcal{B}) \to (Sh(\mathcal{B}'), \mathcal{O}_{\mathcal{B}'})$ of ringed topoi with $\mathcal{J} = \mathrm{Ker}(t^\sharp)$, a morphism of ringed topoi $f : (Sh(\mathcal{C}), \mathcal{O}) \to (Sh(\mathcal{B}), \mathcal{O}_\mathcal{B})$, an $\mathcal{O}$-module $\mathcal{G}$, and a map $f^{-1}\mathcal{J} \to \mathcal{G}$ of sheaves of $f^{-1}\mathcal{O}_\mathcal{B}$-modules. We ask whether we can find the question mark fitting into the following diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \mathcal{G} & \longrightarrow & ? & \longrightarrow & \mathcal{O}_X & \longrightarrow & 0 \\ & & \uparrow{\scriptstyle c} & & \uparrow & & \uparrow & & \\ 0 & \longrightarrow & f^{-1}\mathcal{J} & \longrightarrow & f^{-1}\mathcal{O}_{S'} & \longrightarrow & f^{-1}\mathcal{O}_S & \longrightarrow & 0 \end{array} \tag{66.22.0.1}$$

and moreover how unique the solution is (if it exists). More precisely, we look for a first order thickening $i : (Sh(\mathcal{C}), \mathcal{O}) \to (Sh(\mathcal{C}'), \mathcal{O}')$ and a morphism of thickenings (f, f') as in Deformation Theory, Equation (65.8.1.1) where $\mathrm{Ker}(i^\sharp)$ is identified with $\mathcal{G}$ such that $(f')^\sharp$ induces the given map c. We will say $(Sh(\mathcal{C}'), \mathcal{O}')$ is a *solution* to (66.22.0.1).

Lemma 66.22.1. *In the situation above we have*

1. *There is a canonical element $\xi \in Ext^2_\mathcal{O}(L_f, \mathcal{G})$ whose vanishing is a sufficient and necessary condition for the existence of a solution to (66.22.0.1).*
2. *If there exists a solution, then then the set of isomorphism classes of solutions is principal homogeneous under $Ext^1_\mathcal{O}(L_f, \mathcal{G})$.*
3. *Given a solution X', the set of automorphisms of X' fitting into (66.22.0.1) is canonically isomorphic to $Ext^0_\mathcal{O}(L_f, \mathcal{G})$.*

Proof. Via the identifications $NL_f = \tau_{\geq -1}L_f$ (Lemma 66.21.4) and $H^0(L_{X/S}) = \Omega_{X/S}$ (Lemma 66.21.2) we have seen parts (2) and (3) in Deformation Theory, Lemmas 65.12.1 and 65.12.3.

Proof of (1). We will use the results of Deformation Theory, Lemma 65.12.4 without further mention. Denote

$$p : (Sh(\mathcal{C}), \mathcal{O}) \to (Sh(*), \mathbf{Z}) \quad \text{and} \quad q : (Sh(\mathcal{B}), \mathcal{O}_\mathcal{B}) \to (Sh(*), \mathbf{Z}).$$

Let $\alpha \in \mathrm{Ext}^1_{\mathcal{O}_\mathcal{B}}(NL_q, \mathcal{J})$ be the element corresponding to the isomorphism class of $\mathcal{O}_{\mathcal{B}'}$. The existence of $\mathcal{O}'$ corresponds to an element $\beta \in \mathrm{Ext}^1_{\mathcal{O}}(NL_p, \mathcal{G})$ which maps to the image of α in $\mathrm{Ext}^1_{\mathcal{O}_X}(Lf^* NL_q, \mathcal{G})$. Note that

$$\mathrm{Ext}^1_{\mathcal{O}_X}(Lf^* NL_q, \mathcal{G}) = \mathrm{Ext}^1_{\mathcal{O}_X}(Lf^* L_q, \mathcal{G})$$

and

$$\mathrm{Ext}^1_{\mathcal{O}_X}(NL_p, \mathcal{G}) = \mathrm{Ext}^1_{\mathcal{O}_X}(L_p, \mathcal{G})$$

by Lemma 66.21.4. The distinguished triangle of Lemma 66.21.3 for $p = q \circ f$ gives rise to a long exact sequence

$$\ldots \to \mathrm{Ext}^1_{\mathcal{O}_X}(L_p, \mathcal{G}) \to \mathrm{Ext}^1_{\mathcal{O}_X}(Lf^* L_q, \mathcal{G}) \to \mathrm{Ext}^2_{\mathcal{O}_X}(L_f, \mathcal{G}) \to \ldots$$

We obtain the result with ξ the image of α. □

66.23. The cotangent complex of a morphism of schemes

As promised above we define the cotangent complex of a morphism of schemes as follows.

Definition 66.23.1. Let $f : X \to Y$ be a morphism of schemes. The *cotangent complex* $L_{X/Y}$ *of* X *over* Y is the cotangent complex of f as a morphism of ringed spaces (Definition 66.19.1).

In particular, the results of Section 66.19 apply to cotangent complexes of morphisms of schemes. The next lemma shows this definition is compatible with the definition for ring maps and it also implies that $L_{X/Y}$ is an object of $D_{QCoh}(\mathcal{O}_X)$.

Lemma 66.23.2. *Let $f : X \to Y$ be a morphism of schemes. Let $U = \mathrm{Spec}(A) \subset X$ and $V = \mathrm{Spec}(B) \subset Y$ be affine opens such that $f(U) \subset V$. There is a canonical map*

$$\widetilde{L_{B/A}} \longrightarrow L_{X/Y}|_U$$

of complexes which is an isomorphism in $D(\mathcal{O}_U)$. This map is compatible with restricting to smaller affine opens of X and Y.

Proof. By Remark 66.17.5 there is a canonical map of complexes $L_{\mathcal{O}_X(U)/f^{-1}\mathcal{O}_Y(U)} \to L_{X/Y}(U)$ of $B = \mathcal{O}_X(U)$-modules, which is compatible with further restrictions. Using the canonical map $A \to f^{-1}\mathcal{O}_Y(U)$ we obtain a canonical map $L_{B/A} \to L_{\mathcal{B}/\mathcal{A}}(U)$ of B-modules. Using the universal property of the $\widetilde{\ }$ functor (see Schemes, Lemma 25.7.1) we obtain a map as in the statement of the lemma. We may check this map is an isomorphism on cohomology sheaves by checking it induces isomorphisms on stalks. This follows immediately from Lemmas 66.17.9 and 66.8.6 (and the description of the stalks of $\mathcal{O}_X$ and $f^{-1}\mathcal{O}_Y$ at a point $\mathfrak{p} \in \mathrm{Spec}(B)$ as $B_\mathfrak{p}$ and $A_\mathfrak{q}$ where $\mathfrak{q} = A \cap \mathfrak{p}$; references used are Schemes, Lemma 25.5.4 and Sheaves, Lemma 6.21.5). □

Lemma 66.23.3. *Let Λ be a ring. Let X be a scheme over Λ. Then*

$$L_{X/\mathrm{Spec}(\Lambda)} = L_{\mathcal{O}_X/\underline{\Lambda}}$$

where $\underline{\Lambda}$ is the constant sheaf with value Λ on X.

Proof. Let $p : X \to \mathrm{Spec}(\Lambda)$ be the structure morphism. Let $q : \mathrm{Spec}(\Lambda) \to (*, \Lambda)$ be the obvious morphism. By the distinguished triangle of Lemma 66.19.3 it suffices to show that $L_q = 0$. To see this it suffices to show for $\mathfrak{p} \in \mathrm{Spec}(\Lambda)$ that

$$(L_q)_{\mathfrak{p}} = L_{\mathcal{O}_{\mathrm{Spec}(\Lambda),\mathfrak{p}}/\Lambda} = L_{\Lambda_{\mathfrak{p}}/\Lambda}$$

(Lemma 66.17.9) is zero which follows from Lemma 66.8.4. □

66.24. The cotangent complex of a scheme over a ring

Let Λ be a ring and let X be a scheme over Λ. Write $L_{X/\mathrm{Spec}(\Lambda)} = L_{X/\Lambda}$ which is justified by Lemma 66.23.3. In this section we give a description of $L_{X/\Lambda}$ similar to Lemma 66.4.3. Namely, we construct a category $\mathcal{C}_{X/\Lambda}$ fibred over X_{Zar} and endow it with a sheaf of (polynomial) Λ-algebras $\mathcal{O}$ such that

$$L_{X/\Lambda} = L\pi_!(\Omega_{\mathcal{O}/\underline{\Lambda}} \otimes_{\mathcal{O}} \underline{\mathcal{O}}_X).$$

We will later use the category $\mathcal{C}_{X/\Lambda}$ to construct a naive obstruction theory for the stack of coherent sheaves.

Let Λ be a ring. Let X be a scheme over Λ. Let $\mathcal{C}_{X/\Lambda}$ be the category whose objects are commutative diagrams

$$\begin{array}{ccc} X & \longleftarrow & U \\ \downarrow & & \downarrow \\ \mathrm{Spec}(\Lambda) & \longleftarrow & \mathbf{A} \end{array} \tag{66.24.0.1}$$

of schemes where

(1) U is an open subscheme of X,
(2) there exists an isomorphism $\mathbf{A} = \mathrm{Spec}(P)$ where P is a polynomial algebra over Λ (on some set of variables).

In other words, $\mathbf{A}$ is an (infinite dimensional) affine space over $\mathrm{Spec}(\Lambda)$. Morphisms are given by commutative diagrams. Recall that X_{Zar} denotes the small Zariski site X. There is a forgetful functor

$$u : \mathcal{C}_{X/\Lambda} \to X_{Zar},\ (U \to \mathbf{A}) \mapsto U$$

Observe that the fibre category over U is canonically equivalent to the category $\mathcal{C}_{\mathcal{O}_X(U)/\Lambda}$ introduced in Section 66.4.

Lemma 66.24.1. *In the situation above the category $\mathcal{C}_{X/\Lambda}$ is fibred over X_{Zar}.*

Proof. Given an object $U \to \mathbf{A}$ of $\mathcal{C}_{X/\Lambda}$ and a morphism $U' \to U$ of X_{Zar} consider the object $U' \to \mathbf{A}$ of $\mathcal{C}_{X/\Lambda}$ where $U' \to \mathbf{A}$ is the composition of $U \to \mathbf{A}$ and $U' \to U$. The morphism $(U' \to \mathbf{A}) \to (U \to \mathbf{A})$ of $\mathcal{C}_{X/\Lambda}$ is strongly cartesian over X_{Zar}. □

We endow $\mathcal{C}_{X/\Lambda}$ with the topology inherited from X_{Zar} (see Stacks, Section 8.10). The functor u defines a morphism of topoi $\pi : Sh(\mathcal{C}_{X/\Lambda}) \to Sh(X_{Zar})$. The site $\mathcal{C}_{X/\Lambda}$ comes with several sheaves of rings.

(1) The sheaf $\mathcal{O}$ given by the rule $(U \to \mathbf{A}) \mapsto \Gamma(\mathbf{A}, \mathcal{O}_{\mathbf{A}})$.
(2) The sheaf $\underline{\mathcal{O}}_X = \pi^{-1}\mathcal{O}_X$ given by the rule $(U \to \mathbf{A}) \mapsto \mathcal{O}_X(U)$.
(3) The constant sheaf $\underline{\Lambda}$.

We obtain morphisms of ringed topoi

$$\begin{array}{ccc} (Sh(\mathcal{C}_{X/\Lambda}), \underline{\mathcal{O}}_X) & \xrightarrow[i]{} & (Sh(\mathcal{C}_{X/\Lambda}), \mathcal{O}) \\ \downarrow{\scriptstyle \pi} & & \\ (Sh(X_{Zar}), \mathcal{O}_X) & & \end{array} \tag{66.24.1.1}$$

The morphism i is the identity on underlying topoi and $i^\sharp : \mathcal{O} \to \underline{\mathcal{O}}_X$ is the obvious map. The map π is a special case of Cohomology on Sites, Situation 21.27.1. An important role will be played in the following by the derived functors $Li^* : D(\mathcal{O}) \longrightarrow D(\underline{\mathcal{O}}_X)$ left adjoint to $Ri_* = i_* : D(\underline{\mathcal{O}}_X) \to D(\mathcal{O})$ and $L\pi_! : D(\underline{\mathcal{O}}_X) \longrightarrow D(\mathcal{O}_X)$ left adjoint to $\pi^* = \pi^{-1} : D(\mathcal{O}_X) \to D(\underline{\mathcal{O}}_X)$. We can compute $L\pi_!$ thanks to our earlier work.

Remark 66.24.2. In the situation above, for every $U \subset X$ open let $P_{\bullet,U}$ be the standard resolution of $\mathcal{O}_X(U)$ over Λ. Set $\mathbf{A}_{n,U} = \mathrm{Spec}(P_{n,U})$. Then $\mathbf{A}_{\bullet,U}$ is a cosimplicial object of the fibre category $\mathcal{C}_{\mathcal{O}_X(U)/\Lambda}$ of $\mathcal{C}_{X/\Lambda}$ over U. Moreover, as discussed in Remark 66.5.5 we have that $\mathbf{A}_{\bullet,U}$ is a cosimplicial object of $\mathcal{C}_{\mathcal{O}_X(U)/\Lambda}$ as in Cohomology on Sites, Lemma 21.28.7. Since the construction $U \mapsto \mathbf{A}_{\bullet,U}$ is functorial in U, given any (abelian) sheaf $\mathcal{F}$ on $\mathcal{C}_{X/\Lambda}$ we obtain a complex of presheaves

$$U \longmapsto \mathcal{F}(\mathbf{A}_{\bullet,U})$$

whose cohomology groups compute the homology of $\mathcal{F}$ on the fibre category. We conclude by Cohomology on Sites, Lemma 21.29.2 that the sheafification computes $L_n\pi_!(\mathcal{F})$. In other words, the complex of sheaves whose term in degree $-n$ is the sheafification of $U \mapsto \mathcal{F}(\mathbf{A}_{n,U})$ computes $L\pi_!(\mathcal{F})$.

With this remark out of the way we can state the main result of this section.

Lemma 66.24.3. *In the situation above there is a canonical isomorphism*

$$L_{X/\Lambda} = L\pi_!(Li^*\Omega_{\mathcal{O}/\underline{\Lambda}}) = L\pi_!(i^*\Omega_{\mathcal{O}/\underline{\Lambda}}) = L\pi_!(\Omega_{\mathcal{O}/\underline{\Lambda}} \otimes_{\mathcal{O}} \underline{\mathcal{O}}_X)$$

in $D(\mathcal{O}_X)$.

Proof. We first observe that for any object $(U \to \mathbf{A})$ of $\mathcal{C}_{X/\Lambda}$ the value of the sheaf $\mathcal{O}$ is a polynomial algebra over Λ. Hence $\Omega_{\mathcal{O}/\underline{\Lambda}}$ is a flat $\mathcal{O}$-module and we conclude the second and third equalities of the statement of the lemma hold.

By Remark 66.24.2 the object $L\pi_!(\Omega_{\mathcal{O}/\underline{\Lambda}} \otimes_{\mathcal{O}} \underline{\mathcal{O}}_X)$ is computed as the sheafification of the complex of presheaves

$$U \mapsto \left(\Omega_{\mathcal{O}/\underline{\Lambda}} \otimes_{\mathcal{O}} \underline{\mathcal{O}}_X\right)(\mathbf{A}_{\bullet,U}) = \Omega_{P_{\bullet,U}/\Lambda} \otimes_{P_{\bullet,U}} \mathcal{O}_X(U) = L_{\mathcal{O}_X(U)/\Lambda}$$

using notation as in Remark 66.24.2. Now Remark 66.17.5 shows that $L\pi_!(\Omega_{\mathcal{O}/\underline{\Lambda}} \otimes_{\mathcal{O}} \underline{\mathcal{O}}_X)$ computes the cotangent complex of the map of rings $\underline{\Lambda} \to \mathcal{O}_X$ on X. This is what we want by Lemma 66.23.3. □

66.25. The cotangent complex of a morphism of algebraic spaces

We define the cotangent complex of a morphism of algebraic spaces using the associated morphism between the small étale sites.

Definition 66.25.1. Let S be a scheme. Let $f : X \to Y$ be a morphism of algebraic spaces over S. The *cotangent complex* $L_{X/Y}$ *of* X *over* Y is the cotangent complex of the morphism of ringed topoi f_{small} between the small étale sites of X and Y (see Properties of Spaces, Lemma 48.18.3 and Definition 66.21.1).

In particular, the results of Section 66.21 apply to cotangent complexes of morphisms of algebraic spaces. The next lemmas show this definition is compatible with the definition for ring maps and for schemes and that $L_{X/Y}$ is an object of $D_{QCoh}(\mathcal{O}_X)$.

Lemma 66.25.2. *Let S be a scheme. Consider a commutative diagram*

$$\begin{array}{ccc} U & \xrightarrow{g} & V \\ {\scriptstyle p}\downarrow & & \downarrow{\scriptstyle q} \\ X & \xrightarrow{f} & Y \end{array}$$

of algebraic spaces over S with p and q étale. Then there is a canonical identification $L_{X/Y}|_{U_{étale}} = L_{U/V}$ in $D(\mathcal{O}_U)$.

Proof. Formation of the cotangent complex commutes with pullback (Lemma 66.17.3) and we have $p_{small}^{-1}\mathcal{O}_X = \mathcal{O}_U$ and $g_{small}^{-1}\mathcal{O}_{V_{étale}} = p_{small}^{-1}f_{small}^{-1}\mathcal{O}_{Y_{étale}}$ because $q_{small}^{-1}\mathcal{O}_{Y_{étale}} = \mathcal{O}_{V_{étale}}$ (Properties of Spaces, Lemma 48.24.1). Tracing through the definitions we conclude that $L_{X/Y}|_{U_{étale}} = L_{U/V}$. □

Lemma 66.25.3. *Let S be a scheme. Let $f : X \to Y$ be a morphism of algebraic spaces over S. Assume X and Y representable by schemes X_0 and Y_0. Then there is a canonical identification $L_{X/Y} = \epsilon^* L_{X_0/Y_0}$ in $D(\mathcal{O}_X)$ where ϵ is as in Derived Categories of Spaces, Section 56.4 and L_{X_0/Y_0} is as in Definition 66.23.1.*

Proof. Let $f_0 : X_0 \to Y_0$ be the morphism of schemes corresponding to f. There is a canonical map $\epsilon^{-1}f_0^{-1}\mathcal{O}_{Y_0} \to f_{small}^{-1}\mathcal{O}_Y$ compatible with $\epsilon^\sharp : \epsilon^{-1}\mathcal{O}_{X_0} \to \mathcal{O}_X$ because there is a commutative diagram

$$\begin{array}{ccc} X_{0,Zar} & \xleftarrow{\epsilon} & X_{étale} \\ {\scriptstyle f_0}\downarrow & & \downarrow{\scriptstyle f} \\ Y_{0,Zar} & \xleftarrow{\epsilon} & Y_{étale} \end{array}$$

see Derived Categories of Spaces, Remark 56.6.3. Thus we obtain a canonical map

$$\epsilon^{-1}L_{X_0/Y_0} = \epsilon^{-1}L_{\mathcal{O}_{X_0}/f_0^{-1}\mathcal{O}_{Y_0}} = L_{\epsilon^{-1}\mathcal{O}_{X_0}/\epsilon^{-1}f_0^{-1}\mathcal{O}_{Y_0}} \longrightarrow L_{\mathcal{O}_X/f_{small}^{-1}\mathcal{O}_Y} = L_{X/Y}$$

by the functoriality discussed in Section 66.17 and Lemma 66.17.3. To see that the induced map $\epsilon^* L_{X_0/Y_0} \to L_{X/Y}$ is an isomorphism we may check on stalks at geometric points (Properties of Spaces, Theorem 48.16.12). We will use Lemma 66.17.9 to compute the stalks. Let $\overline{x} : \mathrm{Spec}(k) \to X_0$ be a geometric point lying over $x \in X_0$, with $\overline{y} = f \circ \overline{x}$ lying over $y \in Y_0$. Then

$$L_{X/Y,\overline{x}} = L_{\mathcal{O}_{X,\overline{x}}/\mathcal{O}_{Y,\overline{y}}}$$

and

$$(\epsilon^* L_{X_0/Y_0})_{\overline{x}} = L_{X_0/Y_0,x} \otimes_{\mathcal{O}_{X_0,x}} \mathcal{O}_{X,\overline{x}} = L_{\mathcal{O}_{X_0,x}/\mathcal{O}_{Y_0,y}} \otimes_{\mathcal{O}_{X_0,x}} \mathcal{O}_{X,\overline{x}}$$

Some details omitted (hint: use that the stalk of a pullback is the stalk at the image point, see Sites, Lemma 7.33.1, as well as the corresponding result for modules, see

Modules on Sites, Lemma 18.35.4). Observe that $\mathcal{O}_{X,\overline{x}}$ is the strict henselization of $\mathcal{O}_{X_0,x}$ and similarly for $\mathcal{O}_{Y,\overline{y}}$ (Properties of Spaces, Lemma 48.19.1). Thus the result follows from Lemma 66.8.7. □

Lemma 66.25.4. *Let Λ be a ring. Let X be an algebraic space over Λ. Then*

$$L_{X/\operatorname{Spec}(\Lambda)} = L_{\mathcal{O}_X/\underline{\Lambda}}$$

where $\underline{\Lambda}$ is the constant sheaf with value Λ on $X_{étale}$.

Proof. Let $p : X \to \operatorname{Spec}(\Lambda)$ be the structure morphism. Let $q : \operatorname{Spec}(\Lambda)_{étale} \to (*, \Lambda)$ be the obvious morphism. By the distinguished triangle of Lemma 66.21.3 it suffices to show that $L_q = 0$. To see this it suffices to show (Properties of Spaces, Theorem 48.16.12) for a geometric point $\overline{t} : \operatorname{Spec}(k) \to \operatorname{Spec}(\Lambda)$ that

$$(L_q)_{\overline{t}} = L_{\mathcal{O}_{\operatorname{Spec}(\Lambda)_{étale},\overline{t}}/\Lambda}$$

(Lemma 66.17.9) is zero. Since $\mathcal{O}_{\operatorname{Spec}(\Lambda)_{étale},\overline{t}}$ is a strict henselization of a local ring of Λ (Properties of Spaces, Lemma 48.19.1) this follows from Lemma 66.8.4. □

66.26. The cotangent complex of an algebraic space over a ring

Let Λ be a ring and let X be an algebraic space over Λ. Write $L_{X/\operatorname{Spec}(\Lambda)} = L_{X/\Lambda}$ which is justified by Lemma 66.25.4. In this section we give a description of $L_{X/\Lambda}$ similar to Lemma 66.4.3. Namely, we construct a category $\mathcal{C}_{X/\Lambda}$ fibred over $X_{étale}$ and endow it with a sheaf of (polynomial) Λ-algebras $\mathcal{O}$ such that

$$L_{X/\Lambda} = L\pi_!(\Omega_{\mathcal{O}/\underline{\Lambda}} \otimes_{\mathcal{O}} \underline{\mathcal{O}}_X).$$

We will later use the category $\mathcal{C}_{X/\Lambda}$ to construct a naive obstruction theory for the stack of coherent sheaves.

Let Λ be a ring. Let X be an algebraic space over Λ. Let $\mathcal{C}_{X/\Lambda}$ be the category whose objects are commutative diagrams

$$\begin{array}{ccc} X & \longleftarrow & U \\ \downarrow & & \downarrow \\ \operatorname{Spec}(\Lambda) & \longleftarrow & \mathbf{A} \end{array} \tag{66.26.0.1}$$

of schemes where

(1) U is a scheme,
(2) $U \to X$ is étale,
(3) there exists an isomorphism $\mathbf{A} = \operatorname{Spec}(P)$ where P is a polynomial algebra over Λ (on some set of variables).

In other words, $\mathbf{A}$ is an (infinite dimensional) affine space over $\operatorname{Spec}(\Lambda)$. Morphisms are given by commutative diagrams. Recall that $X_{étale}$ denotes the small étale site of X whose objects are schemes étale over X. There is a forgetful functor

$$u : \mathcal{C}_{X/\Lambda} \to X_{étale}, \quad (U \to \mathbf{A}) \mapsto U$$

Observe that the fibre category over U is canonically equivalent to the category $\mathcal{C}_{\mathcal{O}_X(U)/\Lambda}$ introduced in Section 66.4.

Lemma 66.26.1. *In the situation above the category $\mathcal{C}_{X/\Lambda}$ is fibred over $X_{étale}$.*

Proof. Given an object $U \to \mathbf{A}$ of $\mathcal{C}_{X/\Lambda}$ and a morphism $U' \to U$ of $X_{étale}$ consider the object $U' \to \mathbf{A}$ of $\mathcal{C}_{X/\Lambda}$ where $U' \to \mathbf{A}$ is the composition of $U \to \mathbf{A}$ and $U' \to U$. The morphism $(U' \to \mathbf{A}) \to (U \to \mathbf{A})$ of $\mathcal{C}_{X/\Lambda}$ is strongly cartesian over $X_{étale}$. □

We endow $\mathcal{C}_{X/\Lambda}$ with the topology inherited from $X_{étale}$ (see Stacks, Section 8.10). The functor u defines a morphism of topoi $\pi : Sh(\mathcal{C}_{X/\Lambda}) \to Sh(X_{étale})$. The site $\mathcal{C}_{X/\Lambda}$ comes with several sheaves of rings.

(1) The sheaf $\mathcal{O}$ given by the rule $(U \to \mathbf{A}) \mapsto \Gamma(\mathbf{A}, \mathcal{O}_{\mathbf{A}})$.
(2) The sheaf $\underline{\mathcal{O}}_X = \pi^{-1}\mathcal{O}_X$ given by the rule $(U \to \mathbf{A}) \mapsto \mathcal{O}_X(U)$.
(3) The constant sheaf $\underline{\Lambda}$.

We obtain morphisms of ringed topoi

$$\begin{array}{ccc} (Sh(\mathcal{C}_{X/\Lambda}), \underline{\mathcal{O}}_X) & \xrightarrow[i]{} & (Sh(\mathcal{C}_{X/\Lambda}), \mathcal{O}) \\ \downarrow{\scriptstyle \pi} & & \\ (Sh(X_{étale}), \mathcal{O}_X) & & \end{array} \tag{66.26.1.1}$$

The morphism i is the identity on underlying topoi and $i^\sharp : \mathcal{O} \to \underline{\mathcal{O}}_X$ is the obvious map. The map π is a special case of Cohomology on Sites, Situation 21.27.1. An important role will be played in the following by the derived functors $Li^* : D(\mathcal{O}) \longrightarrow D(\underline{\mathcal{O}}_X)$ left adjoint to $Ri_* = i_* : D(\underline{\mathcal{O}}_X) \to D(\mathcal{O})$ and $L\pi_! : D(\underline{\mathcal{O}}_X) \longrightarrow D(\mathcal{O}_X)$ left adjoint to $\pi^* = \pi^{-1} : D(\mathcal{O}_X) \to D(\underline{\mathcal{O}}_X)$. We can compute $L\pi_!$ thanks to our earlier work.

Remark 66.26.2. In the situation above, for every object $U \to X$ of $X_{étale}$ let $P_{\bullet,U}$ be the standard resolution of $\mathcal{O}_X(U)$ over Λ. Set $\mathbf{A}_{n,U} = \mathrm{Spec}(P_{n,U})$. Then $\mathbf{A}_{\bullet,U}$ is a cosimplicial object of the fibre category $\mathcal{C}_{\mathcal{O}_X(U)/\Lambda}$ of $\mathcal{C}_{X/\Lambda}$ over U. Moreover, as discussed in Remark 66.5.5 we have that $\mathbf{A}_{\bullet,U}$ is a cosimplicial object of $\mathcal{C}_{\mathcal{O}_X(U)/\Lambda}$ as in Cohomology on Sites, Lemma 21.28.7. Since the construction $U \mapsto \mathbf{A}_{\bullet,U}$ is functorial in U, given any (abelian) sheaf $\mathcal{F}$ on $\mathcal{C}_{X/\Lambda}$ we obtain a complex of presheaves

$$U \longmapsto \mathcal{F}(\mathbf{A}_{\bullet,U})$$

whose cohomology groups compute the homology of $\mathcal{F}$ on the fibre category. We conclude by Cohomology on Sites, Lemma 21.29.2 that the sheafification computes $L_n\pi_!(\mathcal{F})$. In other words, the complex of sheaves whose term in degree $-n$ is the sheafification of $U \mapsto \mathcal{F}(\mathbf{A}_{n,U})$ computes $L\pi_!(\mathcal{F})$.

With this remark out of the way we can state the main result of this section.

Lemma 66.26.3. *In the situation above there is a canonical isomorphism*

$$L_{X/\Lambda} = L\pi_!(Li^*\Omega_{\mathcal{O}/\underline{\Lambda}}) = L\pi_!(i^*\Omega_{\mathcal{O}/\underline{\Lambda}}) = L\pi_!(\Omega_{\mathcal{O}/\underline{\Lambda}} \otimes_{\mathcal{O}} \underline{\mathcal{O}}_X)$$

in $D(\mathcal{O}_X)$.

Proof. We first observe that for any object $(U \to \mathbf{A})$ of $\mathcal{C}_{X/\Lambda}$ the value of the sheaf $\mathcal{O}$ is a polynomial algebra over Λ. Hence $\Omega_{\mathcal{O}/\underline{\Lambda}}$ is a flat $\mathcal{O}$-module and we conclude the second and third equalities of the statement of the lemma hold.

By Remark 66.26.2 the object $L\pi_!(\Omega_{\mathcal{O}/\underline{\Lambda}} \otimes_{\mathcal{O}} \underline{\mathcal{O}}_X)$ is computed as the sheafification of the complex of presheaves

$$U \mapsto \left(\Omega_{\mathcal{O}/\underline{\Lambda}} \otimes_{\mathcal{O}} \underline{\mathcal{O}}_X\right)(\mathbf{A}_{\bullet,U}) = \Omega_{P_{\bullet,U}/\Lambda} \otimes_{P_{\bullet,U}} \mathcal{O}_X(U) = L_{\mathcal{O}_X(U)/\Lambda}$$

using notation as in Remark 66.26.2. Now Remark 66.17.5 shows that $L\pi_!(\Omega_{\mathcal{O}/\underline{\Lambda}} \otimes_{\mathcal{O}} \underline{\mathcal{O}}_X)$ computes the cotangent complex of the map of rings $\underline{\Lambda} \to \mathcal{O}_X$ on $X_{étale}$. This is what we want by Lemma 66.25.4. □

66.27. Fibre products of algebraic spaces and the cotangent complex

Let S be a scheme. Let $X \to B$ and $Y \to B$ be morphisms of algebraic spaces pver S. Consider the fibre product $X \times_B Y$ with projection morphisms $p : X \times_B Y \to X$ and $q : X \times_B Y \to Y$. In this section we discuss $L_{X \times_B Y/B}$. Most of the information we want is contained in the following diagram

$$\begin{array}{ccccc} Lp^*L_{X/B} & \longrightarrow & L_{X\times_B Y/Y} & \longrightarrow & E \\ \| & & \uparrow & & \uparrow \\ Lp^*L_{X/B} & \longrightarrow & L_{X\times_B Y/B} & \longrightarrow & L_{X\times_B Y/X} \\ & & \uparrow & & \uparrow \\ & & Lq^*L_{Y/B} & = & Lq^*L_{Y/B} \end{array} \tag{66.27.0.1}$$

Explanation: The middle row is the fundamental triangle of Lemma 66.21.3 for the morphisms $X \times_B Y \to X \to B$. The middle column is the fundamental triangle for the morphisms $X \times_B Y \to Y \to B$. Next, E is an object of $D(\mathcal{O}_{X \times_B Y})$ which "fits" into the upper right corner, i.e., which turns both the top row and the right column into distinguished triangles. Such an E exists by Derived Categories, Proposition 13.4.21 applied to the lower left square (with 0 placed in the missing spot). To be more explicit, we could for example define E as the cone (Derived Categories, Definition 13.8.1) of the map of complexes

$$Lp^*L_{X/B} \oplus Lq^*L_{Y/B} \longrightarrow L_{X\times_B Y/B}$$

and get the two maps with target E by an application of TR3. In the Tor independent case the object E is zero.

Lemma 66.27.1. *In the situation above, if X and Y are Tor independent over B, then the object E in (66.27.0.1) is zero. In this case we have*

$$L_{X\times_B Y/B} = Lp^*L_{X/B} \oplus Lq^*L_{Y/B}$$

Proof. Choose a scheme W and a surjective étale morphsm $W \to B$. Choose a scheme U and a surjective étale morphism $U \to X \times_B W$. Choose a scheme V and a surjective étale morphism $V \to Y \times_B W$. Then $U \times_W V \to X \times_B Y$ is surjective étale too. Hence it suffices to prove that the restriction of E to $U \times_W V$ is zero. By Lemma 66.25.3 and Derived Categories of Spaces, Lemma 56.17.3 this reduces us to the case of schemes. Taking suitable affine opens we reduce to the case of affine schemes. Using Lemma 66.23.2 we reduce to the case of a tensor product of rings, i.e., to Lemma 66.14.1. □

In general we can say the following about the object E.

Lemma 66.27.2. *Let S be a scheme. Let $X \to B$ and $Y \to B$ be morphisms of algebraic spaces over S. The object E in (66.27.0.1) satisfies $H^i(E) = 0$ for $i = 0, -1$ and for a geometric point $(\overline{x}, \overline{y}) : \mathrm{Spec}(k) \to X \times_B Y$ we have*

$$H^{-2}(E)_{(\overline{x},\overline{y})} = Tor_1^R(A, B) \otimes_{A \otimes_R B} C$$

where $R = \mathcal{O}_{B,\overline{b}}$, $A = \mathcal{O}_{X,\overline{x}}$, $B = \mathcal{O}_{Y,\overline{y}}$, *and* $C = \mathcal{O}_{X\times_B Y,(\overline{x},\overline{y})}$.

Proof. The formation of the cotangent complex commutes with taking stalks and pullbacks, see Lemmas 66.17.9 and 66.17.3. Note that C is a henselization of $A \otimes_R B$. $L_{C/R} = L_{A\otimes_R B/R} \otimes_{A\otimes_R B} C$ by the results of Section 66.8. Thus the stalk of E at our geometric point is the cone of the map $L_{A/R} \otimes C \to L_{A\otimes_R B/R} \otimes C$. Therefore the results of the lemma follow from the case of rings, i.e., Lemma 66.14.2. □

66.28. Other chapters

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CHAPTER 67

Algebraic Stacks

67.1. Introduction

This is where we define algebraic stacks and make some very elementary observations. The general philosophy will be to have no separation conditions whatsoever and add those conditions necessary to make lemmas, propositions, theorems true/provable. Thus the notions discussed here differ slightly from those in other places in the literature, e.g., [**LMB00**].

This chapter is not an introduction to algebraic stacks. For an informal discussion of algebraic stacks, please take a look at Introducing Algebraic Stacks, Section 77.1.

67.2. Conventions

The conventions we use in this chapter are the same as those in the chapter on algebraic spaces. For convenience we repeat them here.

We work in a suitable big fppf site Sch_{fppf} as in Topologies, Definition 33.7.6. So, if not explicitly stated otherwise all schemes will be objects of Sch_{fppf}. We discuss what changes if you change the big fppf site in Section 67.18.

We will always work relative to a base S contained in Sch_{fppf}. And we will then work with the big fppf site $(Sch/S)_{fppf}$, see Topologies, Definition 33.7.8. The absolute case can be recovered by taking $S = \mathrm{Spec}(\mathbf{Z})$.

If U, T are schemes over S, then we denote $U(T)$ for the set of T-valued points *over* S. In a formula: $U(T) = \mathrm{Mor}_S(T, U)$.

Note that any fpqc covering is a universal effective epimorphism, see Descent, Lemma 34.9.3. Hence the topology on Sch_{fppf} is weaker than the canonical topology and all representable presheaves are sheaves.

67.3. Notation

We use the letters S, T, U, V, X, Y to indicate schemes. We use the letters $\mathcal{X}, \mathcal{Y}, \mathcal{Z}$ to indicate categories (fibred, fibred in groupoids, stacks, ...) over $(Sch/S)_{fppf}$. We use small case letters f, g for functors such as $f : \mathcal{X} \to \mathcal{Y}$ over $(Sch/S)_{fppf}$. We use capital F, G, H for algebraic spaces over S, and more generally for presheaves of sets on $(Sch/S)_{fppf}$. (In future chapters we will revert to using also X, Y, etc for algebraic spaces.)

The reason for these choices is that we want to clearly distinguish between the different types of objects in this chapter, to build the foundations.

67.4. Representable categories fibred in groupoids

Let S be a scheme contained in Sch_{fppf}. The basic object of study in this chapter will be a category fibred in groupoids $p : \mathcal{X} \to (Sch/S)_{fppf}$, see Categories, Definition 4.33.1. We will often simply say "let $\mathcal{X}$ be a category fibred in groupoids over $(Sch/S)_{fppf}$" to indicate this situation. A 1-morphism $\mathcal{X} \to \mathcal{Y}$ of categories in groupoids over $(Sch/S)_{fppf}$ will be a 1-morphism in the 2-category of categories fibred in groupoids over $(Sch/S)_{fppf}$, see Categories, Definition 4.33.6. It is simply a functor $\mathcal{X} \to \mathcal{Y}$ over $(Sch/S)_{fppf}$. We recall this is really a $(2,1)$-category and that all 2-fibre products exist.

Let $\mathcal{X}$ be a category fibred in groupoids over $(Sch/S)_{fppf}$. Recall that $\mathcal{X}$ is said to be *representable* if there exists a scheme $U \in \mathrm{Ob}((Sch/S)_{fppf})$ and an equivalence

$$j : \mathcal{X} \longrightarrow (Sch/U)_{fppf}$$

of categories over $(Sch/S)_{fppf}$, see Categories, Definition 4.38.1. We will sometimes say that $\mathcal{X}$ is *representable by a scheme* to distinguish from the case where $\mathcal{X}$ is representable by an algebraic space (see below).

If $\mathcal{X}, \mathcal{Y}$ are fibred in groupoids and representable by U, V, then we have

$$\mathrm{Mor}_{Cat/(Sch/S)_{fppf}}(\mathcal{X}, \mathcal{Y}) \Big/ 2\text{-isomorphism} = \mathrm{Mor}_{Sch/S}(U, V) \tag{67.4.0.1}$$

see Categories, Lemma 4.38.3. More precisely, any 1-morphism $\mathcal{X} \to \mathcal{Y}$ gives rise to a morphism $U \to V$. Conversely, given a morphism of schemes $U \to V$ over S there exists a 1-morphism $\phi : \mathcal{X} \to \mathcal{Y}$ which gives rise to $U \to V$ and which is unique up to unique 2-isomorphism.

67.5. The 2-Yoneda lemma

Let $U \in \mathrm{Ob}((Sch/S)_{fppf})$, and let $\mathcal{X}$ be a category fibred in groupoids over $(Sch/S)_{fppf}$. We will frequently use the 2-Yoneda lemma, see Categories, Lemma 4.39.1. Technically it says that there is an equivalence of categories

$$\mathrm{Mor}_{Cat/(Sch/S)_{fppf}}((Sch/U)_{fppf}, \mathcal{X}) \longrightarrow \mathcal{X}_U, \quad f \longmapsto f(U/U).$$

It says that 1-morphisms $(Sch/U)_{fppf} \to \mathcal{X}$ correspond to objects x of the fibre category $\mathcal{X}_U$. Namely, given a 1-morphism $f : (Sch/U)_{fppf} \to \mathcal{X}$ we obtain the object $x = f(U/U) \in \mathrm{Ob}(\mathcal{X}_U)$. Conversely, given a choice of pullbacks for $\mathcal{X}$ as in Categories, Definition 4.31.6, and an object x of $\mathcal{X}_U$, we obtain a functor $(Sch/U)_{fppf} \to \mathcal{X}$ defined by the rule

$$(\varphi : V \to U) \longmapsto \varphi^* x$$

on objects. By abuse of notation we use $x : (Sch/U)_{fppf} \to \mathcal{X}$ to indicate this functor. It indeed has the property that $x(U/U) = x$ and moreover, given any other functor f with $f(U/U) = x$ there exists a unique 2-isomorphism $x \to f$. In other words the functor x is well determined by the object x up to unique 2-isomorphism.

We will use this without further mention in the following.

67.6. Representable morphisms of categories fibred in groupoids

Let $\mathcal{X}$, $\mathcal{Y}$ be categories fibred in groupoids over $(Sch/S)_{fppf}$. Let $f : \mathcal{X} \to \mathcal{Y}$ be a *representable* 1*-morphism*, see Categories, Definition 4.39.5. This means that for every $U \in \mathrm{Ob}((Sch/S)_{fppf})$ and any $y \in \mathrm{Ob}(\mathcal{Y}_U)$ the 2-fibre product $(Sch/U)_{fppf} \times_{y,\mathcal{Y}} \mathcal{X}$ is representable. Choose a representing object V_y and an equivalence

$$(Sch/V_y)_{fppf} \longrightarrow (Sch/U)_{fppf} \times_{y,\mathcal{Y}} \mathcal{X}.$$

The projection $(Sch/V_y)_{fppf} \to (Sch/U)_{fppf} \times_{\mathcal{Y}} \mathcal{Y} \to (Sch/U)_{fppf}$ comes from a morphism of schemes $f_y : V_y \to U$, see Section 67.4. We represent this by the diagram

$$\begin{array}{ccccc} V_y & \rightsquigarrow & (Sch/V_y)_{fppf} & \longrightarrow & \mathcal{X} \\ {\scriptstyle f_y}\downarrow & & \downarrow & & \downarrow{\scriptstyle f} \\ U & \rightsquigarrow & (Sch/U)_{fppf} & \xrightarrow{y} & \mathcal{Y} \end{array} \tag{67.6.0.2}$$

where the squiggly arrows represent the 2-Yoneda embedding. Here are some lemmas about this notion that work in great generality (namely, they work for categories fibred in groupoids over any base category which has fibre products).

Lemma 67.6.1. *Let S, X, Y be objects of Sch_{fppf}. Let $f : X \to Y$ be a morphism of schemes. Then the* 1*-morphism induced by f*

$$(Sch/X)_{fppf} \longrightarrow (Sch/Y)_{fppf}$$

is a representable 1*-morphism.*

Proof. This is formal and relies only on the fact that the category $(Sch/S)_{fppf}$ has fibre products. □

Lemma 67.6.2. *Let S be an object of Sch_{fppf}. Consider a* 2*-commutative diagram*

$$\begin{array}{ccc} \mathcal{X}' & \longrightarrow & \mathcal{X} \\ {\scriptstyle f'}\downarrow & & \downarrow{\scriptstyle f} \\ \mathcal{Y}' & \longrightarrow & \mathcal{Y} \end{array}$$

of 1*-morphisms of categories fibred in groupoids over $(Sch/S)_{fppf}$. Assume the horizontal arrows are equivalences. Then f is representable if and only if f' is representable.*

Proof. Omitted. □

Lemma 67.6.3. *Let S be a scheme contained in Sch_{fppf}. Let $\mathcal{X}, \mathcal{Y}, \mathcal{Z}$ be categories fibred in groupoids over $(Sch/S)_{fppf}$ Let $f : \mathcal{X} \to \mathcal{Y}$, $g : \mathcal{Y} \to \mathcal{Z}$ be representable* 1*-morphisms. Then*

$$g \circ f : \mathcal{X} \longrightarrow \mathcal{Z}$$

is a representable 1*-morphism.*

Proof. This is entirely formal and works in any category. □

Lemma 67.6.4. *Let S be a scheme contained in Sch_{fppf}. Let $\mathcal{X}, \mathcal{Y}, \mathcal{Z}$ be categories fibred in groupoids over $(Sch/S)_{fppf}$ Let $f : \mathcal{X} \to \mathcal{Y}$ be a representable 1-morphism. Let $g : \mathcal{Z} \to \mathcal{Y}$ be any 1-morphism. Consider the fibre product diagram*

$$\begin{array}{ccc} \mathcal{Z} \times_{g,\mathcal{Y},f} \mathcal{X} & \xrightarrow{g'} & \mathcal{X} \\ {\scriptstyle f'}\downarrow & & \downarrow{\scriptstyle f} \\ \mathcal{Z} & \xrightarrow{g} & \mathcal{Y} \end{array}$$

Then the base change f' is a representable 1-morphism.

Proof. This is entirely formal and works in any category. □

Lemma 67.6.5. *Let S be a scheme contained in Sch_{fppf}. Let $\mathcal{X}_i, \mathcal{Y}_i$ be categories fibred in groupoids over $(Sch/S)_{fppf}$, $i = 1, 2$. Let $f_i : \mathcal{X}_i \to \mathcal{Y}_i$, $i = 1, 2$ be representable 1-morphisms. Then*

$$f_1 \times f_2 : \mathcal{X}_1 \times \mathcal{X}_2 \longrightarrow \mathcal{Y}_1 \times \mathcal{Y}_2$$

is a representable 1-morphism.

Proof. Write $f_1 \times f_2$ as the composition $\mathcal{X}_1 \times \mathcal{X}_2 \to \mathcal{Y}_1 \times \mathcal{X}_2 \to \mathcal{Y}_1 \times \mathcal{Y}_2$. The first arrow is the base change of f_1 by the map $\mathcal{Y}_1 \times \mathcal{X}_2 \to \mathcal{Y}_1$, and the second arrow is the base change of f_2 by the map $\mathcal{Y}_1 \times \mathcal{Y}_2 \to \mathcal{Y}_2$. Hence this lemma is a formal consequence of Lemmas 67.6.3 and 67.6.4. □

67.7. Split categories fibred in groupoids

Let S be a scheme contained in Sch_{fppf}. Recall that given a "presheaf of groupoids"

$$F : (Sch/S)^{opp}_{fppf} \longrightarrow Groupoids$$

we get a category fibred in groupoids $\mathcal{S}_F$ over $(Sch/S)_{fppf}$, see Categories, Example 4.35.1. Any category fibred in groupoids isomorphic (!) to one of these is called a *split category fibred in groupoids*. Any category fibred in groupoids is equivalent to a split one.

If F is a presheaf of sets then $\mathcal{S}_F$ is fibred in sets, see Categories, Definition 4.36.2, and Categories, Example 4.36.5. The rule $F \mapsto \mathcal{S}_F$ is in some sense fully faithful on presheaves, see Categories, Lemma 4.36.6. If F, G are presheaves, then

$$\mathcal{S}_{F \times G} = \mathcal{S}_F \times_{(Sch/S)_{fppf}} \mathcal{S}_G$$

and if $F \to H$ and $G \to H$ are maps of presheaves of sets, then

$$\mathcal{S}_{F \times_H G} = \mathcal{S}_F \times_{\mathcal{S}_H} \mathcal{S}_G$$

where the right hand sides are 2-fibre products. This is immediate from the definitions as the fibre categories of $\mathcal{S}_F, \mathcal{S}_G, \mathcal{S}_H$ have only identity morphisms.

An even more special case is where $F = h_X$ is a representable presheaf. In this case we have $\mathcal{S}_{h_X} = (Sch/X)_{fppf}$, see Categories, Example 4.36.7.

We will use the notation $\mathcal{S}_F$ without further mention in the following.

67.8. Categories fibred in groupoids representable by algebraic spaces

A slightly weaker notion than being representable is the notion of being representable by algebraic spaces which we discuss in this section. This discussion might have been avoided had we worked with some category $Spaces_{fppf}$ of algebraic spaces instead of the category Sch_{fppf}. However, it seems to us natural to consider the category of schemes as the natural collection of "test objects" over which the fibre categories of an algebraic stack are defined.

In analogy with Categories, Definitions 4.38.1 we make the following definition.

Definition 67.8.1. Let S be a scheme contained in Sch_{fppf}. A category fibred in groupoids $p : \mathcal{X} \to (Sch/S)_{fppf}$ is called *representable by an algebraic space over* S if there exists an algebraic space F over S and an equivalence $j : \mathcal{X} \to \mathcal{S}_F$ of categories over $(Sch/S)_{fppf}$.

We continue our abuse of notation in suppressing the equivalence j whenever we encounter such a situation. It follows formally from the above that if $\mathcal{X}$ is representable (by a scheme), then it is representable by an algebraic space. Here is the analogue of Categories, Lemma 4.38.2.

Lemma 67.8.2. *Let S be a scheme contained in Sch_{fppf}. Let $p : \mathcal{X} \to (Sch/S)_{fppf}$ be a category fibred in groupoids. Then $\mathcal{X}$ is representable by an algebraic space over S if and only if the following conditions are satisfied:*

(1) *$\mathcal{X}$ is fibred in setoids*[1]*, and*
(2) *the presheaf $U \mapsto \mathrm{Ob}(\mathcal{X}_U)/\cong$ is an algebraic space.*

Proof. Omitted, but see Categories, Lemma 4.38.2. □

If $\mathcal{X}, \mathcal{Y}$ are fibred in groupoids and representable by algebraic spaces F, G over S, then we have

$$\mathrm{Mor}_{Cat/(Sch/S)_{fppf}}(\mathcal{X}, \mathcal{Y}) \Big/ \text{2-isomorphism} = \mathrm{Mor}_{Sch/S}(F, G) \tag{67.8.2.1}$$

see Categories, Lemma 4.37.6. More precisely, any 1-morphism $\mathcal{X} \to \mathcal{Y}$ gives rise to a morphism $F \to G$. Conversely, give a morphism of sheaves $F \to G$ over S there exists a 1-morphism $\phi : \mathcal{X} \to \mathcal{Y}$ which gives rise to $F \to G$ and which is unique up to unique 2-isomorphism.

67.9. Morphisms representable by algebraic spaces

In analogy with Categories, Definition 4.39.5 we make the following definition.

Definition 67.9.1. Let S be a scheme contained in Sch_{fppf}. A 1-morphism $f : \mathcal{X} \to \mathcal{Y}$ of categories fibred in groupoids over $(Sch/S)_{fppf}$ is called *representable by algebraic spaces* if for any $U \in \mathrm{Ob}((Sch/S)_{fppf})$ and any $y : (Sch/U)_{fppf} \to \mathcal{Y}$ the category fibred in groupoids

$$(Sch/U)_{fppf} \times_{y,\mathcal{Y}} \mathcal{X}$$

over $(Sch/U)_{fppf}$ is representable by an algebraic space over U.

[1] This means that it is fibred in groupoids and objects in the fibre categories have no nontrivial automorphisms, see Categories, Definition 4.36.2.

Choose an algebraic space F_y over U which represents $(Sch/U)_{fppf} \times_{y,\mathcal{Y}} \mathcal{X}$. We may think of F_y as an algebraic space over S which comes equipped with a canonical morphism $f_y : F_y \to U$ over S, see Spaces, Section 47.16. Here is the diagram

$$\begin{array}{ccccc} F_y & \rightsquigarrow & (Sch/U)_{fppf} \times_{y,\mathcal{Y}} \mathcal{X} & \xrightarrow{\mathrm{pr}_1} & \mathcal{X} \\ \downarrow{\scriptstyle f_y} & & \downarrow{\scriptstyle \mathrm{pr}_0} & & \downarrow{\scriptstyle f} \\ U & \rightsquigarrow & (Sch/U)_{fppf} & \xrightarrow{y} & \mathcal{Y} \end{array} \tag{67.9.1.1}$$

where the squiggly arrows represent the construction which associates to a stack fibred in setoids its associated sheaf of isomorphism classes of objects. The right square is 2-commutative, and is a 2-fibre product square.

Here is the analogue of Categories, Lemma 4.39.7.

Lemma 67.9.2. *Let S be a scheme contained in Sch_{fppf}. Let $f : \mathcal{X} \to \mathcal{Y}$ be a 1-morphism of categories fibred in groupoids over $(Sch/S)_{fppf}$. The following are necessary and sufficient conditions for f to be representable by algebraic spaces:*

1. *for each scheme U/S the functor $f_U : \mathcal{X}_U \longrightarrow \mathcal{Y}_U$ between fibre categories is faithful, and*
2. *for each U and each $y \in \mathrm{Ob}(\mathcal{Y}_U)$ the presheaf*
$$(h : V \to U) \longmapsto \{(x, \phi) \mid x \in \mathrm{Ob}(\mathcal{X}_V), \phi : h^*y \to f(x)\}/\cong$$
is an algebraic space over U.

Here we have made a choice of pullbacks for $\mathcal{Y}$.

Proof. This follows from the description of fibre categories of the 2-fibre products $(Sch/U)_{fppf} \times_{y,\mathcal{Y}} \mathcal{X}$ in Categories, Lemma 4.39.3 combined with Lemma 67.8.2. □

Here are some lemmas about this notion that work in great generality.

Lemma 67.9.3. *Let S be an object of Sch_{fppf}. Consider a 2-commutative diagram*

$$\begin{array}{ccc} \mathcal{X}' & \longrightarrow & \mathcal{X} \\ \downarrow{\scriptstyle f'} & & \downarrow{\scriptstyle f} \\ \mathcal{Y}' & \longrightarrow & \mathcal{Y} \end{array}$$

of 1-morphisms of categories fibred in groupoids over $(Sch/S)_{fppf}$. Assume the horizontal arrows are equivalences. Then f is representable by algebraic spaces if and only if f' is representable by algebraic spaces.

Proof. Omitted. □

Lemma 67.9.4. *Let S be an object of Sch_{fppf}. Let $f : \mathcal{X} \to \mathcal{Y}$ be a 1-morphism of categories fibred in groupoids over S. If $\mathcal{X}$ and $\mathcal{Y}$ are representable by algebraic spaces over S, then the 1-morphism f is representable by algebraic spaces.*

Proof. Omitted. This relies only on the fact that the category of algebraic spaces over S has fibre products, see Spaces, Lemma 47.7.3. □

Lemma 67.9.5. *Let S be an object of Sch_{fppf}. Let $a : F \to G$ be a map of presheaves of sets on $(Sch/S)_{fppf}$. Denote $a' : \mathcal{S}_F \to \mathcal{S}_G$ the associated map of categories fibred in sets. Then a is representable by algebraic spaces (see Bootstrap, Definition 61.3.1) if and only if a' is representable by algebraic spaces.*

Proof. Omitted. □

Lemma 67.9.6. *Let S be an object of Sch_{fppf}. Let $f : \mathcal{X} \to \mathcal{Y}$ be a 1-morphism of categories fibred in setoids over $(Sch/S)_{fppf}$. Let F, resp. G be the presheaf which to T associates the set of isomorphism classes of objects of $\mathcal{X}_T$, resp. $\mathcal{Y}_T$. Let $a : F \to G$ be the map of presheaves corresponding to f. Then a is representable by algebraic spaces (see Bootstrap, Definition 61.3.1) if and only if f is representable by algebraic spaces.*

Proof. Omitted. Hint: Combine Lemmas 67.9.3 and 67.9.5. □

Lemma 67.9.7. *Let S be a scheme contained in Sch_{fppf}. Let $\mathcal{X}, \mathcal{Y}, \mathcal{Z}$ be categories fibred in groupoids over $(Sch/S)_{fppf}$. Let $f : \mathcal{X} \to \mathcal{Y}$ be a 1-morphism representable by algebraic spaces. Let $g : \mathcal{Z} \to \mathcal{Y}$ be any 1-morphism. Consider the fibre product diagram*

$$\begin{array}{ccc} \mathcal{Z} \times_{g,\mathcal{Y},f} \mathcal{X} & \xrightarrow{g'} & \mathcal{X} \\ {\scriptstyle f'}\downarrow & & \downarrow{\scriptstyle f} \\ \mathcal{Z} & \xrightarrow{g} & \mathcal{Y} \end{array}$$

Then the base change f' is a 1-morphism representable by algebraic spaces.

Proof. This is formal. □

Lemma 67.9.8. *Let S be a scheme contained in Sch_{fppf}. Let $\mathcal{X}, \mathcal{Y}, \mathcal{Z}$ be categories fibred in groupoids over $(Sch/S)_{fppf}$ Let $f : \mathcal{X} \to \mathcal{Y}$, $g : \mathcal{Z} \to \mathcal{Y}$ be 1-morphisms. Assume*

(1) *f is representable by algebraic spaces, and*
(2) *$\mathcal{Z}$ is representable by an algebraic space over S.*

Then the 2-fibre product $\mathcal{Z} \times_{g,\mathcal{Y},f} \mathcal{X}$ is representable by an algebraic space.

Proof. This is a reformulation of Bootstrap, Lemma 61.3.6. First note that $\mathcal{Z} \times_{g,\mathcal{Y},f} \mathcal{X}$ is fibred in setoids over $(Sch/S)_{fppf}$. Hence it is equivalent to $\mathcal{S}_F$ for some presheaf F on $(Sch/S)_{fppf}$, see Categories, Lemma 4.37.5. Moreover, let G be an algebraic space which represents $\mathcal{Z}$. The 1-morphism $\mathcal{Z} \times_{g,\mathcal{Y},f} \mathcal{X} \to \mathcal{Z}$ is representable by algebraic spaces by Lemma 67.9.7. And $\mathcal{Z} \times_{g,\mathcal{Y},f} \mathcal{X} \to \mathcal{Z}$ corresponds to a morphism $F \to G$ by Categories, Lemma 4.37.6. Then $F \to G$ is representable by algebraic spaces by Lemma 67.9.6. Hence Bootstrap, Lemma 61.3.6 implies that F is an algebraic space as desired. □

Let S, $\mathcal{X}$, $\mathcal{Y}$, $\mathcal{Z}$, f, g be as in Lemma 67.9.8. Let F and G be algebraic spaces over S such that F represents $\mathcal{Z} \times_{g,\mathcal{Y},f} \mathcal{X}$ and G represents $\mathcal{Z}$. The 1-morphism $f' : \mathcal{Z} \times_{g,\mathcal{Y},f} \mathcal{X} \to \mathcal{Z}$ corresponds to a morphism $f' : F \to G$ of algebraic spaces by (67.8.2.1). Thus we have the following diagram

$$\begin{array}{ccccc} F & \rightsquigarrow & \mathcal{Z} \times_{g,\mathcal{Y},f} \mathcal{X} & \longrightarrow & \mathcal{X} \\ {\scriptstyle f'}\downarrow & & \downarrow & & \downarrow{\scriptstyle f} \\ G & \rightsquigarrow & \mathcal{Z} & \xrightarrow{g} & \mathcal{Y} \end{array} \tag{67.9.8.1}$$

where the squiggly arrows represent the construction which associates to a stack fibred in setoids its associated sheaf of isomorphism classes of objects. The middle square is 2-commutative with equivalences as horizontal arrows.

Lemma 67.9.9. *Let S be a scheme contained in Sch_{fppf}. Let $\mathcal{X}, \mathcal{Y}, \mathcal{Z}$ be categories fibred in groupoids over $(Sch/S)_{fppf}$. If $f : \mathcal{X} \to \mathcal{Y}$, $g : \mathcal{Y} \to \mathcal{Z}$ are 1-morphisms representable by algebraic spaces, then*

$$g \circ f : \mathcal{X} \longrightarrow \mathcal{Z}$$

is a 1-morphism representable by algebraic spaces.

Proof. This follows from Lemma 67.9.8. Details omitted. □

Lemma 67.9.10. *Let S be a scheme contained in Sch_{fppf}. Let $\mathcal{X}_i, \mathcal{Y}_i$ be categories fibred in groupoids over $(Sch/S)_{fppf}$, $i = 1, 2$. Let $f_i : \mathcal{X}_i \to \mathcal{Y}_i$, $i = 1, 2$ be 1-morphisms representable by algebraic spaces. Then*

$$f_1 \times f_2 : \mathcal{X}_1 \times \mathcal{X}_2 \longrightarrow \mathcal{Y}_1 \times \mathcal{Y}_2$$

is a 1-morphism representable by algebraic spaces.

Proof. Write $f_1 \times f_2$ as the composition $\mathcal{X}_1 \times \mathcal{X}_2 \to \mathcal{Y}_1 \times \mathcal{X}_2 \to \mathcal{Y}_1 \times \mathcal{Y}_2$. The first arrow is the base change of f_1 by the map $\mathcal{Y}_1 \times \mathcal{X}_2 \to \mathcal{Y}_1$, and the second arrow is the base change of f_2 by the map $\mathcal{Y}_1 \times \mathcal{Y}_2 \to \mathcal{Y}_2$. Hence this lemma is a formal consequence of Lemmas 67.9.9 and 67.9.7. □

67.10. Properties of morphisms representable by algebraic spaces

Here is the definition that makes this work.

Definition 67.10.1. Let S be a scheme contained in Sch_{fppf}. Let $f : \mathcal{X} \to \mathcal{Y}$ be a 1-morphism of categories fibred in groupoids over $(Sch/S)_{fppf}$. Assume f is representable by algebraic spaces. Let $\mathcal{P}$ be a property of morphisms of algebraic spaces which

(1) is preserved under any base change, and
(2) is fppf local on the base, see Descent on Spaces, Definition 55.9.1.

In this case we say that f has *property* $\mathcal{P}$ if for every $U \in \mathrm{Ob}((Sch/S)_{fppf})$ and any $y \in \mathcal{Y}_U$ the resulting morphism of algebraic spaces $f_y : F_y \to U$, see diagram (67.9.1.1), has property $\mathcal{P}$.

It is important to note that we will only use this definition for properties of morphisms that are stable under base change, and local in the fppf topology on the target. This is not because the definition doesn't make sense otherwise; rather it is because we may want to give a different definition which is better suited to the property we have in mind.

Lemma 67.10.2. *Let S be an object of Sch_{fppf}. Let $\mathcal{P}$ be as in Definition 67.10.1. Consider a 2-commutative diagram*

$$\begin{array}{ccc} \mathcal{X}' & \longrightarrow & \mathcal{X} \\ {\scriptstyle f'}\downarrow & & \downarrow{\scriptstyle f} \\ \mathcal{Y}' & \longrightarrow & \mathcal{Y} \end{array}$$

of 1-morphisms of categories fibred in groupoids over $(Sch/S)_{fppf}$. Assume the horizontal arrows are equivalences and f (or equivalently f') is representably by algebraic spaces. Then f has $\mathcal{P}$ if and only if f' has $\mathcal{P}$.

Proof. Note that this makes sense by Lemma 67.9.3. Proof omitted. □

Here is a sanity check.

Lemma 67.10.3. *Let S be a scheme contained in Sch_{fppf}. Let $a : F \to G$ be a map of presheaves on $(Sch/S)_{fppf}$. Let $\mathcal{P}$ be as in Definition 67.10.1. Assume a is representable by algebraic spaces. Then $a : F \to G$ has property $\mathcal{P}$ (see Bootstrap, Definition 61.4.1) if and only if the corresponding morphism $\mathcal{S}_F \to \mathcal{S}_G$ of categories fibred in groupoids has property $\mathcal{P}$.*

Proof. Note that the lemma makes sense by Lemma 67.9.5. Proof omitted. □

Lemma 67.10.4. *Let S be an object of Sch_{fppf}. Let $\mathcal{P}$ be as in Definition 67.10.1. Let $f : \mathcal{X} \to \mathcal{Y}$ be a 1-morphism of categories fibred in setoids over $(Sch/S)_{fppf}$. Let F, resp. G be the presheaf which to T associates the set of isomorphism classes of objects of $\mathcal{X}_T$, resp. $\mathcal{Y}_T$. Let $a : F \to G$ be the map of presheaves corresponding to f. Then a has $\mathcal{P}$ if and only if f has $\mathcal{P}$.*

Proof. The lemma makes sense by Lemma 67.9.6. The lemma follows on combining Lemmas 67.10.2 and 67.10.3. □

Lemma 67.10.5. *Let S be a scheme contained in Sch_{fppf}. Let $\mathcal{X}$, $\mathcal{Y}$, $\mathcal{Z}$ be categories fibred in groupoids over $(Sch/S)_{fppf}$. Let $\mathcal{P}$ be a property as in Definition 67.10.1 which is stable under composition. Let $f : \mathcal{X} \to \mathcal{Y}$, $g : \mathcal{Y} \to \mathcal{Z}$ be 1-morphisms which are representable by algebraic spaces. If f and g have property $\mathcal{P}$ so does $g \circ f : \mathcal{X} \to \mathcal{Z}$.*

Proof. Note that the lemma makes sense by Lemma 67.9.9. Proof omitted. □

Lemma 67.10.6. *Let S be a scheme contained in Sch_{fppf}. Let $\mathcal{X}, \mathcal{Y}, \mathcal{Z}$ be categories fibred in groupoids over $(Sch/S)_{fppf}$. Let $\mathcal{P}$ be a property as in Definition 67.10.1. Let $f : \mathcal{X} \to \mathcal{Y}$ be a 1-morphism representable by algebraic spaces. Let $g : \mathcal{Z} \to \mathcal{Y}$ be any 1-morphism. Consider the 2-fibre product diagram*

$$\begin{array}{ccc} \mathcal{Z} \times_{g,\mathcal{Y},f} \mathcal{X} & \xrightarrow{g'} & \mathcal{X} \\ {\scriptstyle f'}\downarrow & & \downarrow{\scriptstyle f} \\ \mathcal{Z} & \xrightarrow{g} & \mathcal{Y} \end{array}$$

If f has $\mathcal{P}$, then the base change f' has $\mathcal{P}$.

Proof. The lemma makes sense by Lemma 67.9.7. Proof omitted. □

Lemma 67.10.7. *Let S be a scheme contained in Sch_{fppf}. Let $\mathcal{X}, \mathcal{Y}, \mathcal{Z}$ be categories fibred in groupoids over $(Sch/S)_{fppf}$. Let $\mathcal{P}$ be a property as in Definition 67.10.1. Let $f : \mathcal{X} \to \mathcal{Y}$ be a 1-morphism representable by algebraic spaces. Let $g : \mathcal{Z} \to \mathcal{Y}$ be any 1-morphism. Consider the fibre product diagram*

$$\begin{array}{ccc} \mathcal{Z} \times_{g,\mathcal{Y},f} \mathcal{X} & \xrightarrow{g'} & \mathcal{X} \\ {\scriptstyle f'}\downarrow & & \downarrow{\scriptstyle f} \\ \mathcal{Z} & \xrightarrow{g} & \mathcal{Y} \end{array}$$

Assume that for every scheme U and object x of $\mathcal{Y}_U$, there exists an fppf covering $\{U_i \to U\}$ such that $x|_{U_i}$ is in the essential image of the functor $g : \mathcal{Z}_{U_i} \to \mathcal{Y}_{U_i}$. In this case, if f' has $\mathcal{P}$, then f has $\mathcal{P}$.

Proof. Proof omitted. Hint: Compare with the proof of Spaces, Lemma 47.5.6. □

Lemma 67.10.8. *Let S be a scheme contained in Sch_{fppf}. Let $\mathcal{P}$ be a property as in Definition 67.10.1 which is stable under composition. Let $\mathcal{X}_i, \mathcal{Y}_i$ be categories fibred in groupoids over $(Sch/S)_{fppf}$, $i = 1, 2$. Let $f_i : \mathcal{X}_i \to \mathcal{Y}_i$, $i = 1, 2$ be 1-morphisms representable by algebraic spaces. If f_1 and f_2 have property $\mathcal{P}$ so does $f_1 \times f_2 : \mathcal{X}_1 \times \mathcal{X}_2 \to \mathcal{Y}_1 \times \mathcal{Y}_2$.*

Proof. The lemma makes sense by Lemma 67.9.10. Proof omitted. □

Lemma 67.10.9. *Let S be a scheme contained in Sch_{fppf}. Let $\mathcal{X}$, $\mathcal{Y}$ be categories fibred in groupoids over $(Sch/S)_{fppf}$. Let $f : \mathcal{X} \to \mathcal{Y}$ be a 1-morphism representable by algebraic spaces. Let $\mathcal{P}$, $\mathcal{P}'$ be properties as in Definition 67.10.1. Suppose that for any morphism of algebraic spaces $a : F \to G$ we have $\mathcal{P}(a) \Rightarrow \mathcal{P}'(a)$. If f has property $\mathcal{P}$ then f has property $\mathcal{P}'$.*

Proof. Formal. □

Lemma 67.10.10. *Let S be a scheme contained in Sch_{fppf}. Let $j : \mathcal{X} \to \mathcal{Y}$ be a 1-morphism of categories fibred in groupoids over $(Sch/S)_{fppf}$. Assume j is representable by algebraic spaces and a monomorphism (see Definition 67.10.1 and Descent on Spaces, Lemma 55.10.28). Then j is fully faithful on fibre categories.*

Proof. We have seen in Lemma 67.9.2 that j is faithful on fibre categories. Consider a scheme U, two objects u, v of $\mathcal{X}_U$, and an isomorphism $t : j(u) \to j(v)$ in $\mathcal{Y}_U$. We have to construct an isomorphism in $\mathcal{X}_U$ between u and v. By the 2-Yoneda lemma (see Section 67.5) we think of u, v as 1-morphisms $u, v : (Sch/U)_{fppf} \to \mathcal{X}$ and we consider the 2-fibre product

$$(Sch/U)_{fppf} \times_{j \circ v, \mathcal{Y}} \mathcal{X}.$$

By assumption this is representable by an algebraic space $F_{j \circ v}$, over U and the morphism $F_{j \circ v} \to U$ is a monomorphism. But since $(1_U, v, 1_{j(v)})$ gives a 1-morphism of $(Sch/U)_{fppf}$ into the displayed 2-fibre product, we see that $F_{j \circ v} = U$ (here we use that if $V \to U$ is a monomorphism of algebraic spaces which has a section, then $V = U$). Therefore the 1-morphism projecting to the first coordinate

$$(Sch/U)_{fppf} \times_{j \circ v, \mathcal{Y}} \mathcal{X} \to (Sch/U)_{fppf}$$

is an equivalence of fibre categories. Since $(1_U, u, t)$ and $(1_U, v, 1_{j(v)})$ give two objects in $((Sch/U)_{fppf} \times_{j \circ v, \mathcal{Y}} \mathcal{X})_U$ which have the same first coordinate, there must be a 2-morphism between them in the 2-fibre product. This is by definition a morphism $\tilde{t} : u \to v$ such that $j(\tilde{t}) = t$. □

Here is a characterization of those categories fibred in groupoids for which the diagonal is representable by algebraic spaces.

Lemma 67.10.11. *Let S be a scheme contained in Sch_{fppf}. Let $\mathcal{X}$ be a category fibred in groupoids over $(Sch/S)_{fppf}$. The following are equivalent:*

1. *the diagonal $\mathcal{X} \to \mathcal{X} \times \mathcal{X}$ is representable by algebraic spaces,*
2. *for every scheme U over S, and any $x, y \in \mathrm{Ob}(\mathcal{X}_U)$ the sheaf $Isom(x, y)$ is representable by an algebraic space over U,*
3. *for every scheme U over S, and any $x \in \mathrm{Ob}(\mathcal{X}_U)$ the associated 1-morphism $x : (Sch/U)_{fppf} \to \mathcal{X}$ is representable by algebraic spaces,*
4. *for every pair of schemes T_1, T_2 over S, and any $x_i \in \mathrm{Ob}(\mathcal{X}_{T_i})$, $i = 1, 2$ the 2-fibre product $(Sch/T_1)_{fppf} \times_{x_1, \mathcal{X}, x_2} (Sch/T_2)_{fppf}$ is representable by an algebraic space,*

(5) *for every representable category fibred in groupoids $\mathcal{U}$ over $(Sch/S)_{fppf}$ every 1-morphism $\mathcal{U} \to \mathcal{X}$ is representable by algebraic spaces,*
(6) *for every pair $\mathcal{T}_1, \mathcal{T}_2$ of representable categories fibred in groupoids over $(Sch/S)_{fppf}$ and any 1-morphisms $x_i : \mathcal{T}_i \to \mathcal{X}$, $i = 1, 2$ the 2-fibre product $\mathcal{T}_1 \times_{x_1, \mathcal{X}, x_2} \mathcal{T}_2$ is representable by an algebraic space,*
(7) *for every category fibred in groupoids $\mathcal{U}$ over $(Sch/S)_{fppf}$ which is representable by an algebraic space every 1-morphism $\mathcal{U} \to \mathcal{X}$ is representable by algebraic spaces,*
(8) *for every pair $\mathcal{T}_1, \mathcal{T}_2$ of categories fibred in groupoids over $(Sch/S)_{fppf}$ which are representable by algebraic spaces, and any 1-morphisms $x_i : \mathcal{T}_i \to \mathcal{X}$ the 2-fibre product $\mathcal{T}_1 \times_{x_1, \mathcal{X}, x_2} \mathcal{T}_2$ is representable by an algebraic space.*

Proof. The equivalence of (1) and (2) follows from Stacks, Lemma 8.2.5 and the definitions. Let us prove the equivalence of (1) and (3). Write $\mathcal{C} = (Sch/S)_{fppf}$ for the base category. We will use some of the observations of the proof of the similar Categories, Lemma 4.39.8. We will use the symbol $\cong$ to mean "equivalence of categories fibred in groupoids over $\mathcal{C} = (Sch/S)_{fppf}$". Assume (1). Suppose given U and x as in (3). For any scheme V and $y \in \mathrm{Ob}(\mathcal{X}_V)$ we see (compare reference above) that

$$\mathcal{C}/U \times_{x, \mathcal{X}, y} \mathcal{C}/V \cong (\mathcal{C}/U \times_S V) \times_{(x,y), \mathcal{X} \times \mathcal{X}, \Delta} \mathcal{X}$$

which is representable by an algebraic space by assumption. Conversely, assume (3). Consider any scheme U over S and a pair (x, x') of objects of $\mathcal{X}$ over U. We have to show that $\mathcal{X} \times_{\Delta, \mathcal{X} \times \mathcal{X}, (x,x')} U$ is representable by an algebraic space. This is clear because (compare reference above)

$$\mathcal{X} \times_{\Delta, \mathcal{X} \times \mathcal{X}, (x,x')} \mathcal{C}/U \cong (\mathcal{C}/U \times_{x, \mathcal{X}, x'} \mathcal{C}/U) \times_{\mathcal{C}/U \times_S U, \Delta} \mathcal{C}/U$$

and the right hand side is representable by an algebraic space by assumption and the fact that the category of algebraic spaces over S has fibre products and contains U and S.

The equivalences (3) $\Leftrightarrow$ (4), (5) $\Leftrightarrow$ (6), and (7) $\Leftrightarrow$ (8) are formal. The equivalences (3) $\Leftrightarrow$ (5) and (4) $\Leftrightarrow$ (6) follow from Lemma 67.9.3. Assume (3), and let $\mathcal{U} \to \mathcal{X}$ be as in (7). To prove (7) we have to show that for every scheme V and 1-morphism $y : (Sch/V)_{fppf} \to \mathcal{X}$ the 2-fibre product $(Sch/V)_{fppf} \times_{y, \mathcal{X}} \mathcal{U}$ is representable by an algebraic space. Property (3) tells us that y is representable by algebraic spaces hence Lemma 67.9.8 implies what we want. Finally, (7) directly implies (3). □

In the situation of the lemma, for any 1-morphism $x : (Sch/U)_{fppf} \to \mathcal{X}$ as in the lemma, it makes sense to say that x has property $\mathcal{P}$, for any property as in Definition 67.10.1. In particular this holds for $\mathcal{P}$ = "surjective", $\mathcal{P}$ = "smooth", and $\mathcal{P}$ = "étale", see Descent on Spaces, Lemmas 55.10.5, 55.10.24, and 55.10.26. We will use these three cases in the definitions of algebraic stacks below.

67.11. Stacks in groupoids

Let S be a scheme contained in Sch_{fppf}. Recall that a category $p : \mathcal{X} \to (Sch/S)_{fppf}$ over $(Sch/S)_{fppf}$ is said to be a *stack in groupoids* (see Stacks, Definition 8.5.1) if and only if

(1) $p : \mathcal{X} \to \mathcal{C}$ is fibred in groupoids over $(Sch/S)_{fppf}$,

(2) for all $U \in \mathrm{Ob}((Sch/S)_{fppf})$, for all $x, y \in \mathrm{Ob}(\mathcal{X}_U)$ the presheaf $Isom(x, y)$ is a sheaf on the site $(Sch/U)_{fppf}$, and
(3) for all coverings $\mathcal{U} = \{U_i \to U\}$ in $(Sch/S)_{fppf}$, all descent data (x_i, ϕ_{ij}) for $\mathcal{U}$ are effective.

For examples see Examples of Stacks, Section 68.8 ff.

67.12. Algebraic stacks

Here is the definition of an algebraic stack. We remark that condition (2) implies we can make sense out of the condition in part (3) that $(Sch/U)_{fppf} \to \mathcal{X}$ is smooth and surjective, see discussion following Lemma 67.10.11.

Definition 67.12.1. Let S be a base scheme contained in Sch_{fppf}. An *algebraic stack over* S is a category

$$p : \mathcal{X} \to (Sch/S)_{fppf}$$

over $(Sch/S)_{fppf}$ with the following properties:

(1) The category $\mathcal{X}$ is a stack in groupoids over $(Sch/S)_{fppf}$.
(2) The diagonal $\Delta : \mathcal{X} \to \mathcal{X} \times \mathcal{X}$ is representable by algebraic spaces.
(3) There exists a scheme $U \in \mathrm{Ob}((Sch/S)_{fppf})$ and a 1-morphism $(Sch/U)_{fppf} \to \mathcal{X}$ which is surjective and smooth[2].

There are some differences with other definitions found in the literature.

The first is that we require $\mathcal{X}$ to be a stack in groupoids in the fppf topology, whereas in many references the étale topology is used. It somehow seems to us that the fppf topology is the natural topology to work with. In the end the resulting 2-category of algebraic stacks ends up being the same. This is explained in Criteria for Representability, Section 70.19.

The second is that we only require the diagonal map of $\mathcal{X}$ to be representable by algebraic spaces, whereas in most references some other conditions are imposed. Our point of view is to try to prove a certain number of the results that follow only assuming that the diagonal of $\mathcal{X}$ be representable by algebraic spaces, and simply add an additional hypothesis wherever this is necessary. It has the added benefit that any algebraic space (as defined in Spaces, Definition 47.6.1) gives rise to an algebraic stack.

The third is that in some papers it is required that there exists a scheme U and a surjective and étale morphism $U \to \mathcal{X}$. In the groundbreaking paper [**DM69**] where algebraic stacks were first introduced Deligne and Mumford used this definition and showed that the moduli stack of stable genus $g > 1$ curves is an algebraic stack which has an étale covering by a scheme. Michael Artin, see [**Art74**], realized that many natural results on algebraic stacks generalize to the case where one only assume a smooth covering by a scheme. Hence our choice above. To distinguish the two cases one sees the terms "Deligne-Mumford stack" and "Artin stack" used in the literature. We will reserve the term "Artin stack" for later use (insert future reference here), and continue to use "algebraic stack", but we will use "Deligne-Mumford stack" to indicate those algebraic stacks which have an étale covering by a scheme.

[2] In future chapters we will denote this simply $U \to \mathcal{X}$ as is customary in the literature. Another good alternative would be to formulate this condition as the existence of a representable category fibred in groupoids $\mathcal{U}$ and a surjective smooth 1-morphism $\mathcal{U} \to \mathcal{X}$.

Definition 67.12.2. Let S be a scheme contained in Sch_{fppf}. Let $\mathcal{X}$ be an algebraic stack over S. We say $\mathcal{X}$ is a *Deligne-Mumford stack* if there exists a scheme U and a surjective étale morphism $(Sch/U)_{fppf} \to \mathcal{X}$.

We will compare our notion of a Deligne-Mumford stack with the notion as defined in the paper by Deligne and Mumford later (see insert future reference here).

The category of algebraic stacks over S forms a 2-category. Here is the precise definition.

Definition 67.12.3. Let S be a scheme contained in Sch_{fppf}. The 2-*category of algebraic stacks over* S is the sub 2-category of the 2-category of categories fibred in groupoids over $(Sch/S)_{fppf}$ (see Categories, Definition 4.33.6) defined as follows:

(1) Its objects are those categories fibred in groupoids over $(Sch/S)_{fppf}$ which are algebraic stacks over S.
(2) Its 1-morphisms $f : \mathcal{X} \to \mathcal{Y}$ are any functors of categories over $(Sch/S)_{fppf}$, as in Categories, Definition 4.30.1.
(3) Its 2-morphisms are transformations between functors over $(Sch/S)_{fppf}$, as in Categories, Definition 4.30.1.

In other words this 2-category is the full sub 2-category of $Cat/(Sch/S)_{fppf}$ whose objects are algebraic stacks. Note that every 2-morphism is automatically an isomorphism. Hence this is actually a $(2,1)$-category and not just a 2-category.

We will see later (insert future reference here) that this 2-category has 2-fibre products.

Similar to the remark above the 2-category of algebraic stacks over S is a full sub 2-category of the 2-category of categories fibred in groupoids over $(Sch/S)_{fppf}$. It turns out that it is closed under equivalences. Here is the precise statement.

Lemma 67.12.4. *Let S be a scheme contained in Sch_{fppf}. Let $\mathcal{X}$, $\mathcal{Y}$ be categories over $(Sch/S)_{fppf}$. Assume $\mathcal{X}$, $\mathcal{Y}$ are equivalent as categories over $(Sch/S)_{fppf}$. Then $\mathcal{X}$ is an algebraic stack if and only if $\mathcal{Y}$ is an algebraic stack. Similarly, $\mathcal{X}$ is a Deligne-Mumford stack if and only if $\mathcal{Y}$ is a Deligne-Mumford stack.*

Proof. Assume $\mathcal{X}$ is an algebraic stack (resp. a Deligne-Mumford stack). By Stacks, Lemma 8.5.4 this implies that $\mathcal{Y}$ is a stack in groupoids over Sch_{fppf}. Choose an equivalence $f : \mathcal{X} \to \mathcal{Y}$ over Sch_{fppf}. This gives a 2-commutative diagram

$$\begin{array}{ccc} \mathcal{X} & \xrightarrow{f} & \mathcal{Y} \\ {\scriptstyle \Delta_{\mathcal{X}}}\downarrow & & \downarrow{\scriptstyle \Delta_{\mathcal{Y}}} \\ \mathcal{X} \times \mathcal{X} & \xrightarrow{f \times f} & \mathcal{Y} \times \mathcal{Y} \end{array}$$

whose horizontal arrows are equivalences. This implies that $\Delta_{\mathcal{Y}}$ is representable by algebraic spaces according to Lemma 67.9.3. Finally, let U be a scheme over S, and let $x : (Sch/U)_{fppf} \to \mathcal{X}$ be a 1-morphism which is surjective and smooth (resp. étale). Considering the diagram

$$\begin{array}{ccc} (Sch/U)_{fppf} & \xrightarrow{\mathrm{id}} & (Sch/U)_{fppf} \\ {\scriptstyle x}\downarrow & & \downarrow{\scriptstyle f \circ x} \\ \mathcal{X} & \xrightarrow{f} & \mathcal{Y} \end{array}$$

and applying Lemma 67.10.2 we conclude that $x \circ f$ is surjective and smooth (resp. étale) as desired. □

67.13. Algebraic stacks and algebraic spaces

In this section we discuss some simple criteria which imply that an algebraic stack is an algebraic space. The main result is that this happens exactly when objects of fibre categories have no nontrivial automorphisms. This is not a triviality! Before we come to this we first do a sanity check.

Lemma 67.13.1. *Let S be a scheme contained in Sch_{fppf}.*

1. *A category fibred in groupoids $p : \mathcal{X} \to (Sch/S)_{fppf}$ which is representable by an algebraic space is a Deligne-Mumford stack.*
2. *If F is an algebraic space over S, then the associated category fibred in groupoids $p : \mathcal{S}_F \to (Sch/S)_{fppf}$ is a Deligne-Mumford stack.*
3. *If $X \in \mathrm{Ob}((Sch/S)_{fppf})$, then $(Sch/X)_{fppf} \to (Sch/S)_{fppf}$ is a Deligne-Mumford stack.*

Proof. It is clear that (2) implies (3). Parts (1) and (2) are equivalent by Lemma 67.12.4. Hence it suffices to prove (2). First, we note that $\mathcal{S}_F$ is stack in sets since F is a sheaf (Stacks, Lemma 8.6.3). A fortiori it is a stack in groupoids. Second the diagonal morphism $\mathcal{S}_F \to \mathcal{S}_F \times \mathcal{S}_F$ is the same as the morphism $\mathcal{S}_F \to \mathcal{S}_{F \times F}$ which comes from the diagonal of F. Hence this is representable by algebraic spaces according to Lemma 67.9.4. Actually it is even representable (by schemes), as the diagonal of an algebraic space is representable, but we do not need this. Let U be a scheme and let $h_U \to F$ be a surjective étale morphism. We may think of this a surjective étale morphism of algebraic spaces. Hence by Lemma 67.10.3 the corresponding 1-morphism $(Sch/U)_{fppf} \to \mathcal{S}_F$ is surjective and étale. □

The following result says that a Deligne-Mumford stack whose inertia is trivial "is" an algebraic space. This lemma will be obsoleted by the stronger Proposition 67.13.3 below which says that this holds more generally for algebraic stacks...

Lemma 67.13.2. *Let S be a scheme contained in Sch_{fppf}. Let $\mathcal{X}$ be an algebraic stack over S. The following are equivalent*

1. *$\mathcal{X}$ is a Deligne-Mumford stack and is a stack in setoids,*
2. *$\mathcal{X}$ is a Deligne-Mumford stack such that the canonical 1-morphism $\mathcal{I}_\mathcal{X} \to \mathcal{X}$ is an equivalence, and*
3. *$\mathcal{X}$ is representable by an algebraic space.*

Proof. The equivalence of (1) and (2) follows from Stacks, Lemma 8.7.2. The implication (3) $\Rightarrow$ (1) follows from Lemma 67.13.1. Finally, assume (1). By Stacks, Lemma 8.6.3 there exists a sheaf F on $(Sch/S)_{fppf}$ and an equivalence $j : \mathcal{X} \to \mathcal{S}_F$. By Lemma 67.9.5 the fact that $\Delta_\mathcal{X}$ is representable by algebraic spaces, means that $\Delta_F : F \to F \times F$ is representable by algebraic spaces. Let U be a scheme, and let $x : (Sch/U)_{fppf} \to \mathcal{X}$ be a surjective étale morphism. The composition $j \circ x : (Sch/U)_{fppf} \to \mathcal{S}_F$ corresponds to a morphism $h_U \to F$ of sheaves. By Bootstrap, Lemma 61.5.1 this morphism is representable by algebraic spaces. Hence by Lemma 67.10.4 we conclude that $h_U \to F$ is surjective and étale. Finally, we apply Bootstrap, Theorem 61.6.1 to see that F is an algebraic space. □

Proposition 67.13.3. *Let S be a scheme contained in Sch_{fppf}. Let $\mathcal{X}$ be an algebraic stack over S. The following are equivalent*

(1) *$\mathcal{X}$ is a stack in setoids,*
(2) *the canonical 1-morphism $\mathcal{I}_\mathcal{X} \to \mathcal{X}$ is an equivalence, and*
(3) *$\mathcal{X}$ is representable by an algebraic space.*

Proof. The equivalence of (1) and (2) follows from Stacks, Lemma 8.7.2. The implication (3) $\Rightarrow$ (1) follows from Lemma 67.13.2. Finally, assume (1). By Stacks, Lemma 8.6.3 there exists an equivalence $j : \mathcal{X} \to \mathcal{S}_F$ where F is a sheaf on $(Sch/S)_{fppf}$. By Lemma 67.9.5 the fact that $\Delta_\mathcal{X}$ is representable by algebraic spaces, means that $\Delta_F : F \to F \times F$ is representable by algebraic spaces. Let U be a scheme and let $x : (Sch/U)_{fppf} \to \mathcal{X}$ be a surjective smooth morphism. The composition $j \circ x : (Sch/U)_{fppf} \to \mathcal{S}_F$ corresponds to a morphism $h_U \to F$ of sheaves. By Bootstrap, Lemma 61.5.1 this morphism is representable by algebraic spaces. Hence by Lemma 67.10.4 we conclude that $h_U \to F$ is surjective and smooth. In particular it is surjective, flat and locally of finite presentation (by Lemma 67.10.9 and the fact that a smooth morphism of algebraic spaces is flat and locally of finite presentation, see Morphisms of Spaces, Lemmas 49.33.5 and 49.33.7). Finally, we apply Bootstrap, Theorem 61.10.1 to see that F is an algebraic space. □

67.14. 2-Fibre products of algebraic stacks

The 2-category of algebraic stacks has products and 2-fibre products. The first lemma is really a special case of Lemma 67.14.3 but its proof is slightly easier.

Lemma 67.14.1. *Let S be a scheme contained in Sch_{fppf}. Let $\mathcal{X}$, $\mathcal{Y}$ be algebraic stacks over S. Then $\mathcal{X} \times_{(Sch/S)_{fppf}} \mathcal{Y}$ is an algebraic stack, and is a product in the 2-category of algebraic stacks over S.*

Proof. An object of $\mathcal{X} \times_{(Sch/S)_{fppf}} \mathcal{Y}$ over T is just a pair (x, y) where x is an object of $\mathcal{X}_T$ and y is an object of $\mathcal{Y}_T$. Hence it is immediate from the definitions that $\mathcal{X} \times_{(Sch/S)_{fppf}} \mathcal{Y}$ is a stack in groupoids. If (x, y) and (x', y') are two objects of $\mathcal{X} \times_{(Sch/S)_{fppf}} \mathcal{Y}$ over T, then

$$Isom((x, y), (x', y')) = Isom(x, x') \times Isom(y, y').$$

Hence it follows from the equivalences in Lemma 67.10.11 and the fact that the category of algebraic spaces has products that the diagonal of $\mathcal{X} \times_{(Sch/S)_{fppf}} \mathcal{Y}$ is representable by algebraic spaces. Finally, suppose that $U, V \in \mathrm{Ob}((Sch/S)_{fppf})$, and let x, y be surjective smooth morphisms $x : (Sch/U)_{fppf} \to \mathcal{X}$, $y : (Sch/Y)_{fppf} \to \mathcal{Y}$. Note that

$$(Sch/U \times_S V)_{fppf} = (Sch/U)_{fppf} \times_{(Sch/S)_{fppf}} (Sch/V)_{fppf}.$$

The object $(\mathrm{pr}_U^* x, \mathrm{pr}_V^* y)$ of $\mathcal{X} \times_{(Sch/S)_{fppf}} \mathcal{Y}$ over $(Sch/U \times_S V)_{fppf}$ thus defines a 1-morphism

$$(Sch/U \times_S V)_{fppf} \longrightarrow \mathcal{X} \times_{(Sch/S)_{fppf}} \mathcal{Y}$$

which is the composition of base changes of x and y, hence is surjective and smooth, see Lemmas 67.10.6 and 67.10.5. We conclude that $\mathcal{X} \times_{(Sch/S)_{fppf}} \mathcal{Y}$ is indeed an algebraic stack. We omit the verification that it really is a product. □

Lemma 67.14.2. *Let S be a scheme contained in Sch_{fppf}. Let $\mathcal{Z}$ be a stack in groupoids over $(Sch/S)_{fppf}$ whose diagonal is representable by algebraic spaces. Let $\mathcal{X}$, $\mathcal{Y}$ be algebraic stacks over S. Let $f : \mathcal{X} \to \mathcal{Z}$, $g : \mathcal{Y} \to \mathcal{Z}$ be 1-morphisms of stacks in groupoids. Then the 2-fibre product $\mathcal{X} \times_{f,\mathcal{Z},g} \mathcal{Y}$ is an algebraic stack.*

Proof. We have to check conditions (1), (2), and (3) of Definition 67.12.1. The first condition follows from Stacks, Lemma 8.5.6.

The second condition we have to check is that the *Isom*-sheaves are representable by algebraic spaces. To do this, suppose that T is a scheme over S, and u, v are objects of $(\mathcal{X} \times_{f,\mathcal{Z},g} \mathcal{Y})_T$. By our construction of 2-fibre products (which goes all the way back to Categories, Lemma 4.30.3) we may write $u = (x, y, \alpha)$ and $v = (x', y', \alpha')$. Here $\alpha : f(x) \to g(y)$ and similarly for α'. Then it is clear that

$$\begin{array}{ccc} Isom(u, v) & \longrightarrow & Isom(y, y') \\ \downarrow & & \downarrow{\scriptstyle \phi \mapsto g(\phi) \circ \alpha} \\ Isom(x, x') & \xrightarrow{\psi \mapsto \alpha' \circ f(\psi)} & Isom(f(x), g(y')) \end{array}$$

is a cartesian diagram of sheaves on $(Sch/T)_{fppf}$. Since by assumption the sheaves $Isom(y, y')$, $Isom(x, x')$, $Isom(f(x), g(y'))$ are algebraic spaces (see Lemma 67.10.11) we see that $Isom(u, v)$ is an algebraic space.

Let $U, V \in \mathrm{Ob}((Sch/S)_{fppf})$, and let x, y be surjective smooth morphisms $x : (Sch/U)_{fppf} \to \mathcal{X}$, $y : (Sch/Y)_{fppf} \to \mathcal{Y}$. Consider the morphism

$$(Sch/U)_{fppf} \times_{f \circ x, \mathcal{Z}, g \circ y} (Sch/V)_{fppf} \longrightarrow \mathcal{X} \times_{f,\mathcal{Z},g} \mathcal{Y}.$$

As the diagonal of $\mathcal{Z}$ is representable by algebraic spaces the source of this arrow is representable by an algebraic space F, see Lemma 67.10.11. Moreover, the morphism is the composition of base changes of x and y, hence surjective and smooth, see Lemmas 67.10.6 and 67.10.5. Choosing a scheme W and a surjective étale morphism $W \to F$ we see that the composition of the displayed 1-morphism with the corresponding 1-morphism

$$(Sch/W)_{fppf} \longrightarrow (Sch/U)_{fppf} \times_{f \circ x, \mathcal{Z}, g \circ y} (Sch/V)_{fppf}$$

is surjective and smooth which proves the last condition. □

Lemma 67.14.3. *Let S be a scheme contained in Sch_{fppf}. Let $\mathcal{X}, \mathcal{Y}, \mathcal{Z}$ be algebraic stacks over S. Let $f : \mathcal{X} \to \mathcal{Z}$, $g : \mathcal{Y} \to \mathcal{Z}$ be 1-morphisms of algebraic stacks. Then the 2-fibre product $\mathcal{X} \times_{f,\mathcal{Z},g} \mathcal{Y}$ is an algebraic stack. It is also the 2-fibre product in the 2-category of algebraic stacks over $(Sch/S)_{fppf}$.*

Proof. The fact that $\mathcal{X} \times_{f,\mathcal{Z},g} \mathcal{Y}$ is an algebraic stack follows from the stronger Lemma 67.14.2. The fact that $\mathcal{X} \times_{f,\mathcal{Z},g} \mathcal{Y}$ is a 2-fibre product in the 2-category of algebraic stacks over S follows formally from the fact that the 2-category of algebraic stacks over S is a full sub 2-category of the 2-category of stacks in groupoids over $(Sch/S)_{fppf}$. □

67.15. Algebraic stacks, overhauled

Some basic results on algebraic stacks.

Lemma 67.15.1. *Let S be a scheme contained in Sch_{fppf}. Let $f : \mathcal{X} \to \mathcal{Y}$ be a 1-morphism of algebraic stacks over S. Let $V \in \mathrm{Ob}((Sch/S)_{fppf})$. Let $y : (Sch/V)_{fppf} \to \mathcal{Y}$ be surjective and smooth. Then there exists an object $U \in \mathrm{Ob}((Sch/S)_{fppf})$ and a 2-commutative diagram*

$$\begin{array}{ccc} (Sch/U)_{fppf} & \xrightarrow{a} & (Sch/V)_{fppf} \\ {\scriptstyle x}\downarrow & & \downarrow{\scriptstyle y} \\ \mathcal{X} & \xrightarrow{f} & \mathcal{Y} \end{array}$$

with x surjective and smooth.

Proof. First choose $W \in \mathrm{Ob}((Sch/S)_{fppf})$ and a surjective smooth 1-morphism $z : (Sch/W)_{fppf} \to \mathcal{X}$. As $\mathcal{Y}$ is an algebraic stack we may choose an equivalence

$$j : \mathcal{S}_F \longrightarrow (Sch/W)_{fppf} \times_{f \circ z, \mathcal{Y}, y} (Sch/V)_{fppf}$$

where F is an algebraic space. By Lemma 67.10.6 the morphism $\mathcal{S}_F \to (Sch/W)_{fppf}$ is surjective and smooth as a base change of y. Hence by Lemma 67.10.5 we see that $\mathcal{S}_F \to \mathcal{X}$ is surjective and smooth. Choose an object $U \in \mathrm{Ob}((Sch/S)_{fppf})$ and a surjective étale morphism $U \to F$. Then applying Lemma 67.10.5 once more we obtain the desired properties. □

This lemma is a generalization of Proposition 67.13.3.

Lemma 67.15.2. *Let S be a scheme contained in Sch_{fppf}. Let $f : \mathcal{X} \to \mathcal{Y}$ be a 1-morphism of algebraic stacks over S. The following are equivalent:*

1. *for $U \in \mathrm{Ob}((Sch/S)_{fppf})$ the functor $f : \mathcal{X}_U \to \mathcal{Y}_U$ is faithful,*
2. *the functor f is faithful, and*
3. *f is representable by algebraic spaces.*

Proof. Parts (1) and (2) are equivalent by general properties of 1-morphisms of categories fibred in groupoids, see Categories, Lemma 4.33.8. We see that (3) implies (2) by Lemma 67.9.2. Finally, assume (2). Let U be a scheme. Let $y \in \mathrm{Ob}(\mathcal{Y}_U)$. We have to prove that

$$\mathcal{W} = (Sch/U)_{fppf} \times_{y, \mathcal{Y}} \mathcal{X}$$

is representable by an algebraic space over U. Since $(Sch/U)_{fppf}$ is an algebraic stack we see from Lemma 67.14.3 that $\mathcal{W}$ is an algebraic stack. On the other hand the explicit description of objects of $\mathcal{W}$ as triples $(V, x, \alpha : y(V) \to f(x))$ and the fact that f is faithful, shows that the fibre categories of $\mathcal{W}$ are setoids. Hence Proposition 67.13.3 guarantees that $\mathcal{W}$ is representable by an algebraic space. □

Lemma 67.15.3. *Let S be a scheme contained in Sch_{fppf}. Let $u : \mathcal{U} \to \mathcal{X}$ be a 1-morphism of stacks in groupoids over $(Sch/S)_{fppf}$. If*

1. *$\mathcal{U}$ is representable by an algebraic space, and*
2. *u is representable by algebraic spaces, surjective and smooth,*

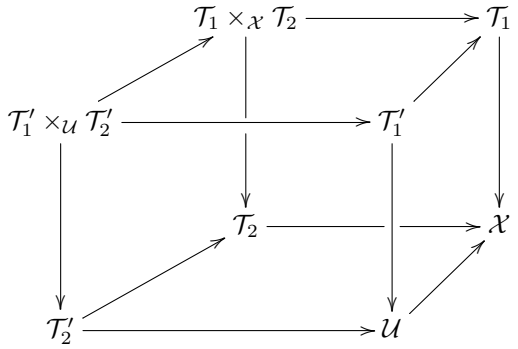
then $\mathcal{X}$ is an algebraic stack over S.

Proof. We have to show that $\Delta : \mathcal{X} \to \mathcal{X} \times \mathcal{X}$ is representable by algebraic spaces, see Definition 67.12.1. Given two schemes T_1, T_2 over S denote $\mathcal{T}_i = (Sch/T_i)_{fppf}$ the associated representable fibre categories. Suppose given 1-morphisms $f_i : \mathcal{T}_i \to \mathcal{X}$. According to Lemma 67.10.11 it suffices to prove that the 2-fibered product

$\mathcal{T}_1 \times_{\mathcal{X}} \mathcal{T}_2$ is representable by an algebraic space. By Stacks, Lemma 8.6.8 this is in any case a stack in setoids. Thus $\mathcal{T}_1 \times_{\mathcal{X}} \mathcal{T}_2$ corresponds to some sheaf F on $(Sch/S)_{fppf}$, see Stacks, Lemma 8.6.3. Let U be the algebraic space which represents $\mathcal{U}$. By assumption

$$\mathcal{T}_i' = \mathcal{U} \times_{u,\mathcal{X},f_i} \mathcal{T}_i$$

is representable by an algebraic space T_i' over S. Hence $\mathcal{T}_1' \times_{\mathcal{U}} \mathcal{T}_2'$ is representable by the algebraic space $T_1' \times_U T_2'$. Consider the commutative diagram



In this diagram the bottom square, the right square, the back square, and the front square are 2-fibre products. A formal argument then shows that $\mathcal{T}_1' \times_{\mathcal{U}} \mathcal{T}_2' \to \mathcal{T}_1 \times_{\mathcal{X}} \mathcal{T}_2$ is the "base change" of $\mathcal{U} \to \mathcal{X}$, more precisely the diagram

$$\begin{array}{ccc} \mathcal{T}_1' \times_{\mathcal{U}} \mathcal{T}_2' & \longrightarrow & \mathcal{U} \\ \downarrow & & \downarrow \\ \mathcal{T}_1 \times_{\mathcal{X}} \mathcal{T}_2 & \longrightarrow & \mathcal{X} \end{array}$$

is a 2-fibre square. Hence $T_1' \times_U T_2' \to F$ is representable by algebraic spaces, smooth, and surjective, see Lemmas 67.9.6, 67.9.7, 67.10.4, and 67.10.6. Therefore F is an algebraic space by Bootstrap, Theorem 61.10.1 and we win. □

An application of Lemma 67.15.3 is that something which is an algebraic space over an algebraic stack is an algebraic stack. This is the analogue of Bootstrap, Lemma 61.3.6. Actually, it suffices to assume the morphism $\mathcal{X} \to \mathcal{Y}$ is "algebraic", as we will see in Criteria for Representability, Lemma 70.8.2.

Lemma 67.15.4. *Let S be a scheme contained in Sch_{fppf}. Let $\mathcal{X} \to \mathcal{Y}$ be a morphism of stacks in groupoids over $(Sch/S)_{fppf}$. Assume that*

(1) *$\mathcal{X} \to \mathcal{Y}$ is representable by algebraic spaces, and*
(2) *$\mathcal{Y}$ is an algebraic stack over S.*

Then $\mathcal{X}$ is an algebraic stack over S.

Proof. Let $\mathcal{V} \to \mathcal{Y}$ be a surjective smooth 1-morphism from a representable stack in groupoids to $\mathcal{Y}$. This exists by Definition 67.12.1. Then the 2-fibre product $\mathcal{U} = \mathcal{V} \times_{\mathcal{Y}} \mathcal{X}$ is representable by an algebraic space by Lemma 67.9.8. The 1-morphism $\mathcal{U} \to \mathcal{X}$ is representable by algebraic spaces, smooth, and surjective, see Lemmas 67.9.7 and 67.10.6. By Lemma 67.15.3 we conclude that $\mathcal{X}$ is an algebraic stack. □

Lemma 67.15.5. *Let S be a scheme contained in Sch_{fppf}. Let $j : \mathcal{X} \to \mathcal{Y}$ be a 1 morphism of categories fibred in groupoids over $(Sch/S)_{fppf}$. Assume j is representable by algebraic spaces and a monomorphism[3]. Then, if $\mathcal{Y}$ is a stack in groupoids (resp. an algebraic stack), so is $\mathcal{X}$.*

Proof. We prove that $\mathcal{X}$ is a stack. The case of algebraic stacks will then follow from Lemma 67.15.4. It suffices to check effectiveness of descent for $\mathcal{X}$. Fix a scheme T and an fppf covering $\{f_i : T_i \to T\}$. Suppose we have objects x_i of the fibre categories $\mathcal{X}_{T_i}$ together with a descent datum. Then since $\mathcal{Y}$ is a stack, there exists an object y in the fibre category $\mathcal{Y}_T$ such that $f_i^*(y) \simeq j(x_i)$ in $\mathcal{Y}_{T_i}$. By hypothesis, the 2-fibered product

$$\mathcal{X} \times_{j,\mathcal{Y},y} (Sch/T)_{fppf}$$

is representable by an algebraic space U such that the induced morphism $U \to T$ is a monomorphism of algebraic spaces. By the universal property of the 2-fibre product and the fact that $f_i^*(y) \cong j(x_i)$, we have that $f_i : T_i \to T$ factors through $U \to T$ for all i. Hence $U \to T$ is a monomorphism of fppf sheaves, but also surjective as $\{f_i : T_i \to T\}$ is a covering. We conclude that $U = T$. Thus y comes from some object x of the fibre category $\mathcal{X}_T$. We have $f_i^* x \cong x_i$ in the fibre category $\mathcal{X}_{T_i}$ as the functor j is fully faithful on fibre categories, see Lemma 67.10.10. □

67.16. From an algebraic stack to a presentation

Given an algebraic stack over S we obtain a groupoid in algebraic spaces over S whose associated quotient stack is the algebraic stack.

Recall that if (U, R, s, t, c) is a groupoid in algebraic spaces over S then $[U/R]$ denotes the quotient stack associated to this datum, see Groupoids in Spaces, Definition 59.19.1. In general $[U/R]$ is **not** an algebraic stack. In particular the stack $[U/R]$ occurring in the following lemma is in general not algebraic.

Lemma 67.16.1. *Let S be a scheme contained in Sch_{fppf}. Let $\mathcal{X}$ be an algebraic stack over S. Let $\mathcal{U}$ be an algebraic stack over S which is representable by an algebraic space. Let $f : \mathcal{U} \to \mathcal{X}$ be a 1-morphism. Then*

1. *the 2-fibre product $\mathcal{R} = \mathcal{U} \times_{f,\mathcal{X},f} \mathcal{U}$ is representable by an algebraic space,*
2. *there is a canonical equivalence*
$$\mathcal{U} \times_{f,\mathcal{X},f} \mathcal{U} \times_{f,\mathcal{X},f} \mathcal{U} = \mathcal{R} \times_{pr_1,\mathcal{U},pr_0} \mathcal{R},$$
3. *the projection pr_{02} induces via (2) a 1-morphism*
$$pr_{02} : \mathcal{R} \times_{pr_1,\mathcal{U},pr_0} \mathcal{R} \longrightarrow \mathcal{R}$$
4. *let U, R be the algebraic spaces representing $\mathcal{U}, \mathcal{R}$ and $t, s : R \to U$ and $c : R \times_{s,U,t} R \to U$ are the morphisms corresponding to the 1-morphisms $pr_0, pr_1 : \mathcal{R} \to \mathcal{U}$ and $pr_{02} : \mathcal{R} \times_{pr_1,\mathcal{U},pr_0} \mathcal{R} \to \mathcal{R}$ above, then the quintuple (U, R, s, t, c) is a groupoid in algebraic spaces over S,*
5. *the morphism f induces a canonical 1-morphism $f_{can} : [U/R] \to \mathcal{X}$ of stacks in groupoids over $(Sch/S)_{fppf}$, and*
6. *the 1-morphism $f_{can} : [U/R] \to \mathcal{X}$ is fully faithful.*

[3] For example an open immersion.

Proof. Proof of (1). By definition $\Delta_{\mathcal{X}}$ is representable by algebraic spaces so Lemma 67.10.11 applies to show that $\mathcal{U} \to \mathcal{X}$ is representable by algebraic spaces. Hence the result follows from Lemma 67.9.8.

Let T be a scheme over S. By construction of the 2-fibre product (see Categories, Lemma 4.30.3) we see that the objects of the fibre category $\mathcal{R}_T$ are triples (a, b, α) where $a, b \in \mathrm{Ob}(\mathcal{U}_T)$ and $\alpha : f(a) \to f(b)$ is a morphism in the fibre category $\mathcal{X}_T$.

Proof of (2). The equivalence comes from repeatedly applying Categories, Lemmas 4.29.8 and 4.29.10. Let us identify $\mathcal{U} \times_{\mathcal{X}} \mathcal{U} \times_{\mathcal{X}} \mathcal{U}$ with $(\mathcal{U} \times_{\mathcal{X}} \mathcal{U}) \times_{\mathcal{X}} \mathcal{U}$. If T is a scheme over S, then on fibre categories over T this equivalence maps the object $((a, b, \alpha), c, \beta)$ on the left hand side to the object $((a, b, \alpha), (b, c, \beta))$ of the right hand side.

Proof of (3). The 1-morphism pr_{02} is constructed in the proof of Categories, Lemma 4.29.9. In terms of the description of objects of the fibre category above we see that $((a, b, \alpha), (b, c, \beta))$ maps to $(a, c, \beta \circ \alpha)$.

Unfortunately, this is *not compatible* with our conventions on groupoids where we always have $j = (t, s) : R \to U$, and we "think" of a T-valued point r of R as a morphism $r : s(r) \to t(r)$. However, this does not affect the proof of (4), since the opposite of a groupoid is a groupoid. But in the proof of (5) it is responsible for the inverses in the displayed formula below.

Proof of (4). Recall that the sheaf U is isomorphic to the sheaf $T \mapsto \mathrm{Ob}(\mathcal{U}_T)/\cong$, and similarly for R, see Lemma 67.8.2. It follows from Categories, Lemma 4.37.8 that this description is compatible with 2-fibre products so we get a similar matching of $\mathcal{R} \times_{\mathrm{pr}_1, \mathcal{U}, \mathrm{pr}_0} \mathcal{R}$ and $R \times_{s,U,t} R$. The morphisms $t, s : R \to U$ and $c : R \times_{s,U,t} R \to R$ we get from the general equality (67.8.2.1). Explicitly these maps are the transformations of functors that come from letting pr_0, pr_0, pr_{02} act on isomorphism classes of objects of fibre categories. Hence to show that we obtain a groupoid in algebraic spaces it suffices to show that for every scheme T over S the structure

$$(\mathrm{Ob}(\mathcal{U}_T)/\cong, \mathrm{Ob}(\mathcal{R}_T)/\cong, \mathrm{pr}_1, \mathrm{pr}_0, \mathrm{pr}_{02})$$

is a groupoid which is clear from our description of objects of $\mathcal{R}_T$ above.

Proof of (5). We will eventually apply Groupoids in Spaces, Lemma 59.22.2 to obtain the functor $[U/R] \to \mathcal{X}$. Consider the 1-morphism $f : \mathcal{U} \to \mathcal{X}$. We have a 2-arrow $\tau : f \circ \mathrm{pr}_1 \to f \circ \mathrm{pr}_0$ by definition of $\mathcal{R}$ as the 2-fibre product. Namely, on an object (a, b, α) of $\mathcal{R}$ over T it is the map $\alpha^{-1} : b \to a$. We claim that

$$\tau \circ \mathrm{id}_{\mathrm{pr}_{02}} = (\tau \star \mathrm{id}_{\mathrm{pr}_0}) \circ (\tau \star \mathrm{id}_{\mathrm{pr}_1}).$$

This identity says that given an object $((a, b, \alpha), (b, c, \beta))$ of $\mathcal{R} \times_{\mathrm{pr}_1, \mathcal{U}, \mathrm{pr}_0} \mathcal{R}$ over T, then the composition of

$$c \xrightarrow{\beta^{-1}} b \xrightarrow{\alpha^{-1}} a$$

is the same as the arrow $(\beta \circ \alpha)^{-1} : a \to c$. This is clearly true, hence the claim holds. In this way we see that all the assumption of Groupoids in Spaces, Lemma 59.22.2 are satisfied for the structure $(\mathcal{U}, \mathcal{R}, \mathrm{pr}_0, \mathrm{pr}_1, \mathrm{pr}_{02})$ and the 1-morphism f and the 2-morphism τ. Except, to apply the lemma we need to prove this holds for the structure $(\mathcal{S}_U, \mathcal{S}_R, s, t, c)$ with suitable morphisms.

Now there should be some general abstract nonsense argument which transfer these data between the two, but it seems to be quite long. Instead, we use the following

trick. Pick a quasi-inverse $j^{-1} : \mathcal{S}_U \to \mathcal{U}$ of the canonical equivalence $j : \mathcal{U} \to \mathcal{S}_U$ which comes from $U(T) = \mathrm{Ob}(\mathcal{U}_T)/\cong$. This just means that for every scheme T/S and every object $a \in \mathcal{U}_T$ we have picked out a particular element of its isomorphism class, namely $j^{-1}(j(a))$. Using j^{-1} we may therefore see $\mathcal{S}_U$ as a subcategory of $\mathcal{U}$. Having chosen this subcategory we can consider those objects (a, b, α) of $\mathcal{R}_T$ such that a, b are objects of $(\mathcal{S}_U)_T$, i.e., such that $j^{-1}(j(a)) = a$ and $j^{-1}(j(b)) = b$. Then it is clear that this forms a subcategory of $\mathcal{R}$ which maps isomorphically to $\mathcal{S}_R$ via the canonical equivalence $\mathcal{R} \to \mathcal{S}_R$. Moreover, this is clearly compatible with forming the 2-fibre product $\mathcal{R} \times_{\mathrm{pr}_1, \mathcal{U}, \mathrm{pr}_0} \mathcal{R}$. Hence we see that we may simply restrict f to $\mathcal{S}_U$ and restrict τ to a transformation between functors $\mathcal{S}_R \to \mathcal{X}$. Hence it is clear that the displayed equality of Groupoids in Spaces, Lemma 59.22.2 holds since it holds even as an equality of transformations of functors $\mathcal{R} \times_{\mathrm{pr}_1, \mathcal{U}, \mathrm{pr}_0} \mathcal{R} \to \mathcal{X}$ before restricting to the subcategory $\mathcal{S}_{R \times_{s,U,t} R}$.

This proves that Groupoids in Spaces, Lemma 59.22.2 applies and we get our desired morphism of stacks $f_{can} : [U/R] \to \mathcal{X}$. We briefly spell out how f_{can} is defined in this special case. On an object a of $\mathcal{S}_U$ over T we have $f_{can}(a) = f(a)$, where we think of $\mathcal{S}_U \subset \mathcal{U}$ by the chosen embedding above. If a, b are objects of $\mathcal{S}_U$ over T, then a morphism $\varphi : a \to b$ in $[U/R]$ is by definition an object of the form $\varphi = (b, a, \alpha)$ of $\mathcal{R}$ over T. (Note switch.) And the rule in the proof of Groupoids in Spaces, Lemma 59.22.2 is that

$$f_{can}(\varphi) = \Big(f(a) \xrightarrow{\alpha^{-1}} f(b)\Big). \tag{67.16.1.1}$$

Proof of (6). Both $[U/R]$ and $\mathcal{X}$ are stacks. Hence given a scheme T/S and objects a, b of $[U/R]$ over T we obtain a transformation of fppf sheaves

$$Isom(a, b) \longrightarrow Isom(f_{can}(a), f_{can}(b))$$

on $(Sch/T)_{fppf}$. We have to show that this is an isomorphism. We may work fppf locally on T, hence we may assume that a, b come from morphisms $a, b : T \to U$. By the embedding $\mathcal{S}_U \subset \mathcal{U}$ above we may also think of a, b as objects of $\mathcal{U}$ over T. In Groupoids in Spaces, Lemma 59.21.1 we have seen that the left hand sheaf is represented by the algebraic space

$$R \times_{(t,s), U \times_S U, (b,a)} T$$

over T. On the other hand, the right hand side is by Stacks, Lemma 8.2.5 equal to the sheaf associated to the following stack in setoids:

$$\mathcal{X} \times_{\mathcal{X} \times \mathcal{X}, (f \circ b, f \circ a)} T = \mathcal{X} \times_{\mathcal{X} \times \mathcal{X}, (f,f)} (\mathcal{U} \times \mathcal{U}) \times_{\mathcal{U} \times \mathcal{U}, (b,a)} T = \mathcal{R} \times_{(\mathrm{pr}_0, \mathrm{pr}_1), \mathcal{U} \times \mathcal{U}, (b,a)} T$$

which is representable by the fibre product displayed above. At this point we have shown that the two *Isom*-sheaves are isomorphic. Our 1-morphism $f_{can} : [U/R] \to \mathcal{X}$ induces this isomorphism on *Isom*-sheaves by Equation (67.16.1.1). □

We can use the previous very abstract lemma to produce presentations.

Lemma 67.16.2. *Let S be a scheme contained in Sch_{fppf}. Let $\mathcal{X}$ be an algebraic stack over S. Let U be an algebraic space over S. Let $f : \mathcal{S}_U \to \mathcal{X}$ be a surjective smooth morphism. Let (U, R, s, t, c) be the groupoid in algebraic spaces and $f_{can} : [U/R] \to \mathcal{X}$ be the result of applying Lemma 67.16.1 to U and f. Then*

1. *the morphisms s, t are smooth, and*
2. *the 1-morphism $f_{can} : [U/R] \to \mathcal{X}$ is an equivalence.*

Proof. The morphisms s, t are smooth by Lemmas 67.10.2 and 67.10.3. As the 1-morphism f is smooth and surjective it is clear that given any scheme T and any object $a \in \mathrm{Ob}(\mathcal{X}_T)$ there exists a smooth and surjective morphism $T' \to T$ such that $a|'_T$ comes from an object of $[U/R]_{T'}$. Since $f_{can} : [U/R] \to \mathcal{X}$ is fully faithful, we deduce that $[U/R] \to \mathcal{X}$ is essentially surjective as descent data on objects are effective on both sides, see Stacks, Lemma 8.4.8. □

Remark 67.16.3. If the morphism $f : \mathcal{S}_U \to \mathcal{X}$ of Lemma 67.16.2 is only assumed surjective, flat and locally of finite presentation, then it will still be the case that $f_{can} : [U/R] \to \mathcal{X}$ is an equivalence. In this case the morphisms s, t will be flat and locally of finite presentation, but of course not smooth in general.

Lemma 67.16.2 suggests the following definitions.

Definition 67.16.4. Let S be a scheme. Let B be an algebraic space over S. Let (U, R, s, t, c) be a groupoid in algebraic spaces over B. We say (U, R, s, t, c) is a *smooth groupoid*[4] if $s, t : R \to U$ are smooth morphisms of algebraic spaces.

Definition 67.16.5. Let $\mathcal{X}$ be an algebraic stack over S. A *presentation* of $\mathcal{X}$ is given by a smooth groupoid (U, R, s, t, c) in algebraic spaces over S, and an equivalence $f : [U/R] \to \mathcal{X}$.

We have seen above that every algebraic stack has a presentation. Our next task is to show that every smooth groupoid in algebraic spaces over S gives rise to an algebraic stack.

67.17. The algebraic stack associated to a smooth groupoid

In this section we start with a smooth groupoid in algebraic spaces and we show that the associated quotient stack is an algebraic stack.

Lemma 67.17.1. *Let S be a scheme contained in Sch_{fppf}. Let (U, R, s, t, c) be a groupoid in algebraic spaces over S. Then the diagonal of $[U/R]$ is representable by algebraic spaces.*

Proof. It suffices to show that the *Isom*-sheaves are algebraic spaces, see Lemma 67.10.11. This follows from Bootstrap, Lemma 61.11.3. □

Lemma 67.17.2. *Let S be a scheme contained in Sch_{fppf}. Let (U, R, s, t, c) be a smooth groupoid in algebraic spaces over S. Then the morphism $\mathcal{S}_U \to [U/R]$ is smooth and surjective.*

Proof. Let T be a scheme and let $x : (Sch/T)_{fppf} \to [U/R]$ be a 1-morphism. We have to show that the projection

$$\mathcal{S}_U \times_{[U/R]} (Sch/T)_{fppf} \longrightarrow (Sch/T)_{fppf}$$

is surjective and smooth. We already know that the left hand side is representable by an algebraic space F, see Lemmas 67.17.1 and 67.10.11. Hence we have to show the corresponding morphism $F \to T$ of algebraic spaces is surjective and smooth. Since we are working with properties of morphisms of algebraic spaces which are local on the target in the fppf topology we may check this fppf locally on T. By construction, there exists an fppf covering $\{T_i \to T\}$ of T such that $x|_{(Sch/T_i)_{fppf}}$

[4] This terminology might be a bit confusing: it does not imply that $[U/R]$ is smooth over anything.

comes from a morphism $x_i : T_i \to U$. (Note that $F \times_T T_i$ represents the 2-fibre product $\mathcal{S}_U \times_{[U/R]} (Sch/T_i)_{fppf}$ so everything is compatible with the base change via $T_i \to T$.) Hence we may assume that x comes from $x : T \to U$. In this case we see that

$$\mathcal{S}_U \times_{[U/R]} (Sch/T)_{fppf} = (\mathcal{S}_U \times_{[U/R]} \mathcal{S}_U) \times_{\mathcal{S}_U} (Sch/T)_{fppf} = \mathcal{S}_R \times_{\mathcal{S}_U} (Sch/T)_{fppf}$$

The first equality by Categories, Lemma 4.29.10 and the second equality by Groupoids in Spaces, Lemma 59.21.2. Clearly the last 2-fibre product is represented by the algebraic space $F = R \times_{s,U,x} T$ and the projection $R \times_{s,U,x} T \to T$ is smooth as the base change of the smooth morphism of algebraic spaces $s : R \to U$. It is also surjective as s has a section (namely the identity $e : U \to R$ of the groupoid). This proves the lemma. □

Here is the main result of this section.

Theorem 67.17.3. *Let S be a scheme contained in Sch_{fppf}. Let (U, R, s, t, c) be a smooth groupoid in algebraic spaces over S. Then the quotient stack $[U/R]$ is an algebraic stack over S.*

Proof. We check the three conditions of Definition 67.12.1. By construction we have that $[U/R]$ is a stack in groupoids which is the first condition.

The second condition follows from the stronger Lemma 67.17.1.

Finally, we have to show there exists a scheme W over S and a surjective smooth 1-morphism $(Sch/W)_{fppf} \longrightarrow \mathcal{X}$. First choose $W \in \mathrm{Ob}((Sch/S)_{fppf})$ and a surjective étale morphism $W \to U$. Note that this gives a surjective étale morphism $\mathcal{S}_W \to \mathcal{S}_U$ of categories fibred in sets, see Lemma 67.10.3. Of course then $\mathcal{S}_W \to \mathcal{S}_U$ is also surjective and smooth, see Lemma 67.10.9. Hence $\mathcal{S}_W \to \mathcal{S}_U \to [U/R]$ is surjective and smooth by a combination of Lemmas 67.17.2 and 67.10.5. □

67.18. Change of big site

In this section we briefly discuss what happens when we change big sites. The upshot is that we can always enlarge the big site at will, hence we may assume any set of schemes we want to consider is contained in the big fppf site over which we consider our algebraic space. We encourage the reader to skip this section.

Pullbacks of stacks is defined in Stacks, Section 8.12.

Lemma 67.18.1. *Suppose given big sites Sch_{fppf} and Sch'_{fppf}. Assume that Sch_{fppf} is contained in Sch'_{fppf}, see Topologies, Section 33.10. Let S be an object of Sch_{fppf}. Let Let $f : (Sch'/S)_{fppf} \to (Sch/S)_{fppf}$ the morphism of sites corresponding to the inclusion functor $u : (Sch/S)_{fppf} \to (Sch'/S)_{fppf}$. Let $\mathcal{X}$ be a stack in groupoids over $(Sch/S)_{fppf}$.*

1. *if $\mathcal{X}$ is representable by some $X \in \mathrm{Ob}((Sch/S)_{fppf})$, then $f^{-1}\mathcal{X}$ is representable too, in fact it is representable by the same scheme X, now viewed as an object of $(Sch'/S)_{fppf}$,*
2. *if $\mathcal{X}$ is representable by $F \in Sh((Sch/S)_{fppf})$ which is an algebraic space, then $f^{-1}\mathcal{X}$ is representable by the algebraic space $f^{-1}F$,*
3. *if $\mathcal{X}$ is an algebraic stack, then $f^{-1}\mathcal{X}$ is an algebraic stack, and*
4. *if $\mathcal{X}$ is a Deligne-Mumford stack, then $f^{-1}\mathcal{X}$ is a Deligne-Mumford stack too.*

Proof. Let us prove (3). By Lemma 67.16.2 we may write $\mathcal{X} = [U/R]$ for some smooth groupoid in algebraic spaces (U, R, s, t, c). By Groupoids in Spaces, Lemma 59.27.1 we see that $f^{-1}[U/R] = [f^{-1}U/f^{-1}R]$. Of course $(f^{-1}U, f^{-1}R, f^{-1}s, f^{-1}t, f^{-1}c)$ is a smooth groupoid in algebraic spaces too. Hence (3) is proved.

Now the other cases (1), (2), (4) each mean that $\mathcal{X}$ has a presentation $[U/R]$ of a particular kind, and hence translate into the same kind of presentation for $f^{-1}\mathcal{X} = [f^{-1}U/f^{-1}R]$. Whence the lemma is proved. □

It is not true (in general) that the restriction of an algebraic space over the bigger site is an algebraic space over the smaller site (simply by reasons of cardinality). Hence we can only ever use a simple lemma of this kind to enlarge the base category and never to shrink it.

Lemma 67.18.2. *Suppose Sch_{fppf} is contained in Sch'_{fppf}. Let S be an object of Sch_{fppf}. Denote Algebraic-Stacks/S the 2-category of algebraic spaces over S defined using Sch_{fppf}. Similarly, denote Algebraic-Stacks$'$/S the 2-category of algebraic spaces over S defined using Sch'_{fppf}. The rule $\mathcal{X} \mapsto f^{-1}\mathcal{X}$ of Lemma 67.18.1 defines a functor of 2-categories*

$$\textit{Algebraic-Stacks}/S \longrightarrow \textit{Algebraic-Stacks}'/S$$

which defines equivalences of morphism categories

$$\mathrm{Mor}_{\textit{Algebraic-Stacks}/S}(\mathcal{X}, \mathcal{Y}) \longrightarrow \mathrm{Mor}_{\textit{Algebraic-Stacks}'/S}(f^{-1}\mathcal{X}, f^{-1}\mathcal{Y})$$

for every objects $\mathcal{X}, \mathcal{Y}$ of Algebraic-Stacks/S. An object $\mathcal{X}'$ of Algebraic-Stacks$'$/S is equivalence to $f^{-1}\mathcal{X}$ for some $\mathcal{X}$ in Algebraic-Stacks/S if and only if it has a presentation $\mathcal{X} = [U'/R']$ with U', R' isomorphic to $f^{-1}U$, $f^{-1}R$ for some $U, R \in$ Spaces/S.

Proof. The statement on morphism categories is a consequence of the more general Stacks, Lemma 8.12.12. The characterization of the "essential image" follows from the description of f^{-1} in the proof of Lemma 67.18.1. □

67.19. Change of base scheme

In this section we briefly discuss what happens when we change base schemes. The upshot is that given a morphism $S \to S'$ of base schemes, any algebraic stack over S can be viewed as an algebraic stack over S'.

Lemma 67.19.1. *Let Sch_{fppf} be a big fppf site. Let $S \to S'$ be a morphism of this site. The constructions A and B of Stacks, Section 8.13 above give isomorphisms of 2-categories*

$$\left\{\begin{array}{c}\text{2-category of algebraic}\\ \text{stacks } \mathcal{X} \text{ over } S\end{array}\right\} \leftrightarrow \left\{\begin{array}{c}\text{2-category of pairs } (\mathcal{X}', f) \text{ consisting of an}\\ \text{algebraic stack } \mathcal{X}' \text{ over } S' \text{ and a morphism}\\ f : \mathcal{X}' \to (Sch/S)_{fppf} \text{ of algebraic stacks over } S'\end{array}\right\}$$

Proof. The statement makes sense as the functor $j : (Sch/S)_{fppf} \to (Sch/S')_{fppf}$ is the localization functor associated to the object S/S' of $(Sch/S')_{fppf}$. By Stacks, Lemma 8.13.2 the only thing to show is that the constructions A and B preserve the subcategories of algebraic stacks. For example, if $\mathcal{X} = [U/R]$ then construction A applied to $\mathcal{X}$ just produces $\mathcal{X}' = \mathcal{X}$. Conversely, if $\mathcal{X}' = [U'/R']$ the morphism p induces morphisms of algebraic spaces $U' \to S$ and $R' \to S$, and then $\mathcal{X} = [U'/R']$ but now viewed as a stack over S. Hence the lemma is clear. □

Definition 67.19.2. Let Sch_{fppf} be a big fppf site. Let $S \to S'$ be a morphism of this site. If $p : \mathcal{X} \to (Sch/S)_{fppf}$ is an algebraic stack over S, then $\mathcal{X}$ *viewed as an algebraic stack over* S' is the algebraic stack

$$\mathcal{X} \longrightarrow (Sch/S')_{fppf}$$

gotten by applying construction A of Lemma 67.19.1 to $\mathcal{X}$.

Conversely, what if we start with an algebraic stack $\mathcal{X}'$ over S' and we want to get an algebraic stack over S? Well, then we consider the 2-fibre product

$$\mathcal{X}'_S = (Sch/S)_{fppf} \times_{(Sch/S')_{fppf}} \mathcal{X}'$$

which is an algebraic stack over S' according to Lemma 67.14.3. Moreover, it comes equipped with a natural 1-morphism $p : \mathcal{X}'_S \to (Sch/S)_{fppf}$ and hence by Lemma 67.19.1 it corresponds in a canonical way to an algebraic stack over S.

Definition 67.19.3. Let Sch_{fppf} be a big fppf site. Let $S \to S'$ be a morphism of this site. Let $\mathcal{X}'$ be an algebraic stack over S'. The *change of base of* $\mathcal{X}'$ is the algebraic space $\mathcal{X}'_S$ over S described above.

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CHAPTER 68

Examples of Stacks

68.1. Introduction

This is a discussion of examples of stacks in algebraic geometry. Some of them are algebraic stacks, some are not. We will discuss which are algebraic stacks in a later chapter. This means that in this chapter we mainly worry about the descent conditions. See [**Vis**] for example.

Some of the notation, conventions and terminology in this chapter is awkward and may seem backwards to the more experienced reader. This is intentional. Please see Quot, Section 72.1 for an explanation.

68.2. Notation

In this chapter we fix a suitable big fppf site Sch_{fppf} as in Topologies, Definition 33.7.6. So, if not explicitly stated otherwise all schemes will be objects of Sch_{fppf}. We will always work relative to a base S contained in Sch_{fppf}. And we will then work with the big fppf site $(Sch/S)_{fppf}$, see Topologies, Definition 33.7.8. The absolute case can be recovered by taking $S = \mathrm{Spec}(\mathbf{Z})$.

68.3. Examples of stacks

We first give some important examples of stacks over $(Sch/S)_{fppf}$.

68.4. Quasi-coherent sheaves

We define a category $QCoh$ as follows:

(1) An object of $QCoh$ is a pair $(X, \mathcal{F})$, where X/S is an object of $(Sch/S)_{fppf}$, and $\mathcal{F}$ is a quasi-coherent $\mathcal{O}_X$-module, and
(2) a morphism $(f, \varphi) : (Y, \mathcal{G}) \to (X, \mathcal{F})$ is a pair consisting of a morphism $f : Y \to X$ of schemes over S and an f-map (see Sheaves, Section 6.26) $\varphi : \mathcal{F} \to \mathcal{G}$.
(3) The composition of morphisms

$$(Z, \mathcal{H}) \xrightarrow{(g,\psi)} (Y, \mathcal{G}) \xrightarrow{(f,\phi)} (X, \mathcal{F})$$

is $(f \circ g, \psi \circ \phi)$ where $\psi \circ \phi$ is the composition of f-maps.

Thus $QCoh$ is a category and

$$p : QCoh \to (Sch/S)_{fppf}, \quad (X, \mathcal{F}) \mapsto X$$

is a functor. Note that the fibre category of $QCoh$ over a scheme X is just the category $QCoh(\mathcal{O}_X)$ of quasi-coherent $\mathcal{O}_X$-modules. We remark for later use that

given $(X, \mathcal{F}), (Y, \mathcal{G}) \in \mathrm{Ob}(QCoh)$ we have

$$\mathrm{Mor}_{QCoh}((Y, \mathcal{G}), (X, \mathcal{F})) = \coprod_{f \in \mathrm{Mor}_S(Y,X)} \mathrm{Mor}_{QCoh(\mathcal{O}_Y)}(f^*\mathcal{F}, \mathcal{G}) \tag{68.4.0.1}$$

See the discussion on f-maps of modules in Sheaves, Section 6.26.

The category $QCoh$ is not a stack over $(Sch/S)_{fppf}$ because its collection of objects is a proper class. On the other hand we will see that it does satisfy all the axioms of a stack. We will get around the set theoretical issue in Section 68.5.

Lemma 68.4.1. *A morphism $(f, \varphi) : (Y, \mathcal{G}) \to (X, \mathcal{F})$ of $QCoh$ is strongly cartesian if and only if the map φ induces an isomorphism $f^*\mathcal{F} \to \mathcal{G}$.*

Proof. Let $(X, \mathcal{F}) \in \mathrm{Ob}(QCoh)$. Let $f : Y \to X$ be a morphism of $(Sch/S)_{fppf}$. Note that there is a canonical f-map $c : \mathcal{F} \to f^*\mathcal{F}$ and hence we get a morphism $(f, c) : (Y, f^*\mathcal{F}) \to (X, \mathcal{F})$. We claim that (f, c) is strongly cartesian. Namely, for any object $(Z, \mathcal{H})$ of $QCoh$ we have

$$\begin{aligned}
\mathrm{Mor}_{QCoh}((Z, \mathcal{H}), (Y, f^*\mathcal{F})) &= \coprod_{g \in \mathrm{Mor}_S(Z,Y)} \mathrm{Mor}_{QCoh(\mathcal{O}_Z)}(g^*f^*\mathcal{F}, \mathcal{H}) \\
&= \coprod_{g \in \mathrm{Mor}_S(Z,Y)} \mathrm{Mor}_{QCoh(\mathcal{O}_Z)}((f \circ g)^*\mathcal{F}, \mathcal{H}) \\
&= \mathrm{Mor}_{QCoh}((Z, \mathcal{H}), (X, \mathcal{F})) \times_{\mathrm{Mor}_S(Z,X)} \mathrm{Mor}_S(Z, Y)
\end{aligned}$$

where we have used Equation (68.4.0.1) twice. This proves that the condition of Categories, Definition 4.31.1 holds for (f, c), and hence our claim is true. Now by Categories, Lemma 4.31.2 we see that isomorphisms are strongly cartesian and compositions of strongly cartesian morphisms are strongly cartesian which proves the "if" part of the lemma. For the converse, note that given $(X, \mathcal{F})$ and $f : Y \to X$, if there exists a strongly cartesian morphism lifting f with target $(X, \mathcal{F})$ then it has to be isomorphic to (f, c) (see discussion following Categories, Definition 4.31.1). Hence the "only if" part of the lemma holds. □

Lemma 68.4.2. *The functor $p : QCoh \to (Sch/S)_{fppf}$ satisfies conditions (1), (2) and (3) of Stacks, Definition 8.4.1.*

Proof. It is clear from Lemma 68.4.1 that $QCoh$ is a fibred category over $(Sch/S)_{fppf}$. Given covering $\mathcal{U} = \{X_i \to X\}_{i \in I}$ of $(Sch/S)_{fppf}$ the functor

$$QCoh(\mathcal{O}_T) \longrightarrow DD(\mathcal{U})$$

is fully faithful and essentially surjective, see Descent, Proposition 34.5.2. Hence Stacks, Lemma 8.4.2 applies to show that $QCoh$ satisfies all the axioms of a stack. □

68.5. The stack of finitely generated quasi-coherent sheaves

It turns out that we can get a stack of quasi-coherent sheaves if we only consider finite type quasi-coherent modules. Let us denote

$$p_{fg} : QCoh_{fg} \to (Sch/S)_{fppf}$$

the full subcategory of $QCoh$ over $(Sch/S)_{fppf}$ consisting of pairs $(T, \mathcal{F})$ such that $\mathcal{F}$ is a quasi-coherent $\mathcal{O}_T$-module of finite type.

Lemma 68.5.1. *The functor $p_{fg} : QCoh_{fg} \to (Sch/S)_{fppf}$ satisfies conditions (1), (2) and (3) of Stacks, Definition 8.4.1.*

Proof. We will verify assumptions (1), (2), (3) of Stacks, Lemma 8.4.3 to prove this. By Lemma 68.4.1 a morphism $(Y, \mathcal{G}) \to (X, \mathcal{F})$ is strongly cartesian if and only if it induces an isomorphism $f^*\mathcal{F} \to \mathcal{G}$. By Modules, Lemma 17.9.2 the pullback of a finite type $\mathcal{O}_X$-module is of finite type. Hence assumption (1) of Stacks, Lemma 8.4.3 holds. Assumption (2) holds trivially. Finally, to prove assumption (3) we have to show: If $\mathcal{F}$ is a quasi-coherent $\mathcal{O}_X$-module and $\{f_i : X_i \to X\}$ is an fppf covering such that each $f_i^*\mathcal{F}$ is of finite type, then $\mathcal{F}$ is of finite type. Considering the restriction of $\mathcal{F}$ to an affine open of X this reduces to the following algebra statement: Suppose that $R \to S$ is a finitely presented, faithfully flat ring map and M an R-module. If $M \otimes_R S$ is a finitely generated S-module, then M is a finitely generated R-module. A stronger form of the algebra fact can be found in Algebra, Lemma 10.78.2. □

Lemma 68.5.2. *Let $(X, \mathcal{O}_X)$ be a ringed space.*

(1) *The category of finite type $\mathcal{O}_X$-modules has a set of isomorphism classes.*
(2) *The category of finite type quasi-coherent $\mathcal{O}_X$-modules has a set of isomorphism classes.*

Proof. Part (2) follows from part (1) as the category in (2) is a full subcategory of the category in (1). Consider any open covering $\mathcal{U} : X = \bigcup_{i \in I} U_i$. Denote $j_i : U_i \to X$ the inclusion maps. Consider any map $r : I \to \mathbf{N}$. If $\mathcal{F}$ is an $\mathcal{O}_X$-module whose restriction to U_i is generated by at most $r(i)$ sections from $\mathcal{F}(U_i)$, then $\mathcal{F}$ is a quotient of the sheaf

$$\mathcal{H}_{\mathcal{U},r} = \bigoplus\nolimits_{i \in I} j_{i,!}\mathcal{O}_{U_i}^{\oplus r(i)}$$

By definition, if $\mathcal{F}$ is of finite type, then there exists some open covering with $\mathcal{U}$ whose index set is $I = X$ such that this condition is true. Hence it suffices to show that there is a set of possible choices for $\mathcal{U}$ (obvious), a set of possible choices for $r : I \to \mathbf{N}$ (obvious), and a set of possible quotient modules of $\mathcal{H}_{\mathcal{U},r}$ for each $\mathcal{U}$ and r. In other words, it suffices to show that given an $\mathcal{O}_X$-module $\mathcal{H}$ there is at most a set of isomorphism classes of quotients. This last assertion becomes obvious by thinking of the kernels of a quotient map $\mathcal{H} \to \mathcal{F}$ as being parametrized by a subset of the power set of $\prod_{U \subset X \text{ open}} \mathcal{H}(U)$. □

Lemma 68.5.3. *There exists a subcategory $QCoh_{fg,small} \subset QCoh_{fg}$ with the following properties:*

(1) *the inclusion functor $QCoh_{fg,small} \to QCoh_{fg}$ is fully faithful and essentially surjective, and*
(2) *the functor $p_{fg,small} : QCoh_{fg,small} \to (Sch/S)_{fppf}$ turns $QCoh_{fg,small}$ into a stack over $(Sch/S)_{fppf}$.*

Proof. We have seen in Lemmas 68.5.1 and 68.5.2 that $p_{fg} : QCoh_{fg} \to (Sch/S)_{fppf}$ satisfies (1), (2) and (3) of Stacks, Definition 8.4.1 as well as the additional condition (4) of Stacks, Remark 8.4.9. Hence we obtain $QCoh_{fg,small}$ from the discussion in that remark. □

We will often perform the replacement

$$QCoh_{fg} \rightsquigarrow QCoh_{fg,small}$$

without further remarking on it, and by abuse of notation we will simply denote $QCoh_{fg}$ this replacement.

Remark 68.5.4. Note that the whole discussion in this section works if we want to consider those quasi-coherent sheaves which are locally generated by at most κ sections, for some infinite cardinal κ, e.g., $\kappa = \aleph_0$.

68.6. Algebraic spaces

We define a category *Spaces* as follows:

(1) An object of *Spaces* is a morphism $X \to U$ of algebraic spaces over S, where U is representable by an object of $(Sch/S)_{fppf}$, and
(2) a morphism $(f, g) : (X \to U) \to (Y \to V)$ is a commutative diagram

$$\begin{array}{ccc} X & \xrightarrow{f} & Y \\ \downarrow & & \downarrow \\ U & \xrightarrow{g} & V \end{array}$$

of morphisms of algebraic spaces over S.

Thus *Spaces* is a category and

$$p : Spaces \to (Sch/S)_{fppf}, \quad (X \to U) \mapsto U$$

is a functor. Note that the fibre category of *Spaces* over a scheme U is just the category *Spaces*$/U$ of algebraic spaces over U (see Topologies on Spaces, Section 54.2). Hence we sometimes think of an object of *Spaces* as a pair X/U consisting of a scheme U and an algebraic space X over U. We remark for later use that given $(X/U), (Y/V) \in \mathrm{Ob}(Spaces)$ we have

$$\mathrm{Mor}_{Spaces}(X/U, Y/V) = \coprod\nolimits_{g \in \mathrm{Mor}_S(U,V)} \mathrm{Mor}_{Spaces/U}(X, U \times_{g,V} Y) \tag{68.6.0.1}$$

The category *Spaces* is almost, but not quite a stack over $(Sch/S)_{fppf}$. The problem is a set theoretical issue as we will explain below.

Lemma 68.6.1. *A morphism $(f, g) : X/U \to Y/V$ of Spaces is strongly cartesian if and only if the map f induces an isomorphism $X \to U \times_{g,V} Y$.*

Proof. Let $Y/V \in \mathrm{Ob}(Spaces)$. Let $g : U \to V$ be a morphism of $(Sch/S)_{fppf}$. Note that the projection $p : U \times_{g,V} Y \to Y$ gives rise a morphism $(p, g) : U \times_{g,V} Y/U \to Y/V$ of *Spaces*. We claim that (p, g) is strongly cartesian. Namely, for any object Z/W of *Spaces* we have

$$\begin{aligned} \mathrm{Mor}_{Spaces}(Z/W, U \times_{g,V} Y/U) &= \coprod\nolimits_{h \in \mathrm{Mor}_S(W,U)} \mathrm{Mor}_{Spaces/W}(Z, W \times_{h,U} U \times_{g,V} Y) \\ &= \coprod\nolimits_{h \in \mathrm{Mor}_S(W,U)} \mathrm{Mor}_{Spaces/W}(Z, W \times_{g \circ h,V} Y) \\ &= \mathrm{Mor}_{Spaces}(Z/W, Y/V) \times_{\mathrm{Mor}_S(W,V)} \mathrm{Mor}_S(W, U) \end{aligned}$$

where we have used Equation (68.6.0.1) twice. This proves that the condition of Categories, Definition 4.31.1 holds for (p, g), and hence our claim is true. Now by Categories, Lemma 4.31.2 we see that isomorphisms are strongly cartesian and compositions of strongly cartesian morphisms are strongly cartesian which proves the "if" part of the lemma. For the converse, note that given Y/V and $g : U \to V$, if there exists a strongly cartesian morphism lifting g with target Y/V then it has to be isomorphic to (p, g) (see discussion following Categories, Definition 4.31.1). Hence the "only if" part of the lemma holds. □

Lemma 68.6.2. *The functor $p : Spaces \to (Sch/S)_{fppf}$ satisfies conditions (1) and (2) of Stacks, Definition 8.4.1.*

Proof. It is follows from Lemma 68.6.1 that *Spaces* is a fibred category over $(Sch/S)_{fppf}$ which proves (1). Suppose that $\{U_i \to U\}_{i\in I}$ is a covering of $(Sch/S)_{fppf}$. Suppose that X, Y are algebraic spaces over U. Finally, suppose that $\varphi_i : X_{U_i} \to Y_{U_i}$ are morphisms of $Spaces/U_i$ such that φ_i and φ_j restrict to the same morphisms $X_{U_i\times_U U_j} \to Y_{U_i\times_U U_j}$ of algebraic spaces over $U_i \times_U U_j$. To prove (2) we have to show that there exists a unique morphism $\varphi : X \to Y$ over U whose base change to U_i is equal to φ_i. As a morphism from X to Y is the same thing as a map of sheaves this follows directly from Sites, Lemma 7.25.1. □

Remark 68.6.3. Ignoring set theoretical difficulties[1] *Spaces* also satisfies descent for objects and hence is a stack. Namely, we have to show that given

(1) an fppf covering $\{U_i \to U\}_{i\in I}$,
(2) for each $i \in I$ an algebraic space X_i/U_i, and
(3) for each $i, j \in I$ an isomorphism $\varphi_{ij} : X_i \times_U U_j \to U_i \times_U X_j$ of algebraic spaces over $U_i \times_U U_j$ satisfying the cocycle condition over $U_i \times_U U_j \times_U U_k$,

there exists an algebraic space X/U and isomorphisms $X_{U_i} \cong X_i$ over U_i recovering the isomorphisms φ_{ij}. First, note that by Sites, Lemma 7.25.2 there exists a sheaf X on $(Sch/U)_{fppf}$ recovering the X_i and the φ_{ij}. Then by Bootstrap, Lemma 61.11.1 we see that X is an algebraic space (if we ignore the set theoretic condition of that lemma). We will use this argument in the next section to show that if we consider only algebraic spaces of finite type, then we obtain a stack.

68.7. The stack of finite type algebraic spaces

It turns out that we can get a stack of spaces if we only consider spaces of finite type. Let us denote

$$p_{ft} : Spaces_{ft} \to (Sch/S)_{fppf}$$

the full subcategory of *Spaces* over $(Sch/S)_{fppf}$ consisting of pairs X/U such that $X \to U$ is a morphism of finite type.

Lemma 68.7.1. *The functor $p_{ft} : Spaces_{ft} \to (Sch/S)_{fppf}$ satisfies the conditions (1), (2) and (3) of Stacks, Definition 8.4.1.*

Proof. We are going to write this out in ridiculous detail (which may make it hard to see what is going on).

We have seen in Lemma 68.6.1 that a morphism $(f, g) : X/U \to Y/V$ of *Spaces* is strongly cartesian if the induced morphism $f : X \to U \times_V Y$ is an isomorphism. Note that if $Y \to V$ is of finite type then also $U \times_V Y \to U$ is of finite type, see Morphisms of Spaces, Lemma 49.22.3. So if $(f, g) : X/U \to Y/V$ of *Spaces* is strongly cartesian in *Spaces* and Y/V is an object of $Spaces_{ft}$ then automatically also X/U is an object of $Spaces_{ft}$, and of course (f, g) is also strongly cartesian in $Spaces_{ft}$. In this way we conclude that $Spaces_{ft}$ is a fibred category over $(Sch/S)_{fppf}$. This proves (1).

[1]The difficulty is not that *Spaces* is a proper class, since by our definition of an algebraic space over S there is only a set worth of isomorphism classes of algebraic spaces over S. It is rather that arbitrary disjoint unions of algebraic spaces may end up being too large, hence lie outside of our chosen "partial universe" of sets.

The argument above also shows that the inclusion functor $\mathit{Spaces}_{ft} \to \mathit{Spaces}$ transforms strongly cartesian morphisms into strongly cartesian morphisms. In other words $\mathit{Spaces}_{ft} \to \mathit{Spaces}$ is a 1-morphism of fibred categories over $(\mathit{Sch}/S)_{fppf}$.

Let $U \in \mathrm{Ob}((\mathit{Sch}/S)_{fppf})$. Let X, Y be algebraic spaces of finite type over U. By Stacks, Lemma 8.2.3 we obtain a map of presheaves

$$\mathit{Mor}_{\mathit{Spaces}_{ft}}(X, Y) \longrightarrow \mathit{Mor}_{\mathit{Spaces}}(X, Y)$$

which is an isomorphism as Spaces_{ft} is a full subcategory of Spaces. Hence the left hand side is a sheaf, because in Lemma 68.6.2 we showed the right hand side is a sheaf. This proves (2).

To prove condition (3) of Stacks, Definition 8.4.1 we have to show the following: Given

(1) a covering $\{U_i \to U\}_{i \in I}$ of $(\mathit{Sch}/S)_{fppf}$,
(2) for each $i \in I$ an algebraic space X_i of finite type over U_i, and
(3) for each $i, j \in I$ an isomorphism $\varphi_{ij} : X_i \times_U U_j \to U_i \times_U X_j$ of algebraic spaces over $U_i \times_U U_j$ satisfying the cocycle condition over $U_i \times_U U_j \times_U U_k$,

there exists an algebraic space X of finite type over U and isomorphisms $X_{U_i} \cong X_i$ over U_i recovering the isomorphisms φ_{ij}. By Sites, Lemma 7.25.2 there exists a sheaf X on $(\mathit{Sch}/U)_{fppf}$ recovering the X_i and the φ_{ij}. Then by Bootstrap, Lemma 61.11.2 we see that X is an algebraic space. By Descent on Spaces, Lemma 55.10.8 we see that $X \to U$ is of finite type which concludes the proof. □

Lemma 68.7.2. *There exists a subcategory $\mathit{Spaces}_{ft,small} \subset \mathit{Spaces}_{ft}$ with the following properties:*

(1) *the inclusion functor $\mathit{Spaces}_{ft,small} \to \mathit{Spaces}_{ft}$ is fully faithful and essentially surjective, and*
(2) *the functor $p_{ft,small} : \mathit{Spaces}_{ft,small} \to (\mathit{Sch}/S)_{fppf}$ turns $\mathit{Spaces}_{ft,small}$ into a stack over $(\mathit{Sch}/S)_{fppf}$.*

Proof. We have seen in Lemmas 68.7.1 that $p_{fg} : \mathit{QCoh}_{fg} \to (\mathit{Sch}/S)_{fppf}$ satisfies (1), (2) and (3) of Stacks, Definition 8.4.1. The additional condition (4) of Stacks, Remark 8.4.9 holds because every algebraic space X over S is of the form U/R for $U, R \in \mathrm{Ob}((\mathit{Sch}/S)_{fppf})$, see Spaces, Lemma 47.9.1. Thus there is only a set worth of isomorphism classes of objects. Hence we obtain $\mathit{Spaces}_{ft,small}$ from the discussion in that remark. □

We will often perform the replacement

$$\mathit{Spaces}_{ft} \rightsquigarrow \mathit{Spaces}_{ft,small}$$

without further remarking on it, and by abuse of notation we will simply denote Spaces_{ft} this replacement.

Remark 68.7.3. Note that the whole discussion in this section works if we want to consider those algebraic spaces X/U which are locally of finite type such that the inverse image in X of an affine open of U can be covered by countably many affines. If needed we can also introduce the notion of a morphism of κ-type (meaning some bound on the number of generators of ring extensions and some bound on the cardinality of the affines over a given affine in the base) where κ is a cardinal, and then we can produce a stack

$$\mathit{Spaces}_{\kappa} \longrightarrow (\mathit{Sch}/S)_{fppf}$$

in exactly the same manner as above (provided we make sure that Sch is large enough depending on κ).

68.8. Examples of stacks in groupoids

The examples above are examples of stacks which are not stacks in groupoids. In the rest of this chapter we give algebraic geometric examples of stacks in groupoids.

68.9. The stack associated to a sheaf

Let $F : (Sch/S)^{opp}_{fppf} \to Sets$ be a presheaf. We obtain a category fibred in sets

$$p_F : \mathcal{S}_F \to (Sch/S)_{fppf},$$

see Categories, Example 4.36.5. This is a stack in sets if and only if F is a sheaf, see Stacks, Lemma 8.6.3.

68.10. The stack in groupoids of finitely generated quasi-coherent sheaves

Let $p : QCoh_{fg} \to (Sch/S)_{fppf}$ be the stack introduced in Section 68.5 (using the abuse of notation introduced there). We can turn this into a stack in groupoids $p' : QCoh'_{fg} \to (Sch/S)_{fppf}$ by the procedure of Categories, Lemma 4.33.3, see Stacks, Lemma 8.5.3. In this particular case this simply means $QCoh'_{fg}$ has the same objects as $QCoh_{fg}$ but the morphisms are pairs $(f, g) : (U, \mathcal{F}) \to (U', \mathcal{F}')$ where g is an isomorphism $g : f^*\mathcal{F}' \to \mathcal{F}$.

68.11. The stack in groupoids of finite type algebraic spaces

Let $p : Spaces_{ft} \to (Sch/S)_{fppf}$ be the stack introduced in Section 68.7 (using the abuse of notation introduced there). We can turn this into a stack in groupoids $p' : Spaces'_{ft} \to (Sch/S)_{fppf}$ by the procedure of Categories, Lemma 4.33.3, see Stacks, Lemma 8.5.3. In this particular case this simply means $Spaces'_{ft}$ has the same objects as $Spaces_{ft}$, i.e., finite type morphisms $X \to U$ where X is an algebraic space over S and U is a scheme over S. But the morphisms $(f, g) : X/U \to Y/V$ are now commutative diagrams

$$\begin{array}{ccc} X & \xrightarrow{f} & Y \\ \downarrow & & \downarrow \\ U & \xrightarrow{g} & V \end{array}$$

which are cartesian.

68.12. Quotient stacks

Let (U, R, s, t, c) be a groupoid in algebraic spaces over S. In this case the quotient stack

$$[U/R] \longrightarrow (Sch/S)_{fppf}$$

is a stack in groupoids by construction, see Groupoids in Spaces, Definition 59.19.1. It is even the case that the $Isom$-sheaves are representable by algebraic spaces, see Bootstrap, Lemma 61.11.3. These quotient stacks are of fundamental importance to the theory of algebraic stacks.

A special case of the construction above is the quotient stack

$$[X/G] \longrightarrow (Sch/S)_{fppf}$$

associated to a datum $(B, G/B, m, X/B, a)$. Here

(1) B is an algebraic space over S,
(2) (G, m) is a group algebraic space over B,
(3) X is an algebraic space over B, and
(4) $a : G \times_B X \to X$ is an action of G on X over B.

Namely, by Groupoids in Spaces, Definition 59.19.1 the stack in groupoids $[X/G]$ is the quotient stack $[X/G \times_B X]$ given above. It behooves us to spell out what the category $[X/G]$ really looks like. We will do this in Section 68.14.

68.13. Classifying torsors

We want to carefuly explain a number of variants of what it could mean to study the stack of torsors for a group algebraic space G or a sheaf of groups $\mathcal{G}$.

68.13.1. Torsors for a sheaf of groups. Let $\mathcal{G}$ be a sheaf of groups on $(Sch/S)_{fppf}$. For $U \in \mathrm{Ob}((Sch/S)_{fppf})$ we denote $\mathcal{G}|_U$ the restriction of $\mathcal{G}$ to $(Sch/U)_{fppf}$. We define a category $\mathcal{G}\text{-}Torsors$ as follows:

(1) An object of $\mathcal{G}\text{-}Torsors$ is a pair $(U, \mathcal{F})$ where U is an object of $(Sch/S)_{fppf}$ and $\mathcal{F}$ is a $\mathcal{G}|_U$-torsor, see Cohomology on Sites, Definition 21.5.1.
(2) A morphism $(U, \mathcal{F}) \to (V, \mathcal{H})$ is given by a pair (f, α), where $f : U \to V$ is a morphism of schemes over S, and $\alpha : f^{-1}\mathcal{H} \to \mathcal{F}$ is an isomorphism of $\mathcal{G}|_U$-torsors.

Thus $\mathcal{G}\text{-}Torsors$ is a category and

$$p : \mathcal{G}\text{-}Torsors \longrightarrow (Sch/S)_{fppf}, \quad (U, \mathcal{F}) \longmapsto U$$

is a functor. Note that the fibre category of $\mathcal{G}\text{-}Torsors$ over U is the category of $\mathcal{G}|_U$-torsors which is a groupoid.

Lemma 68.13.2. *Up to a replacement as in Stacks, Remark 8.4.9 the functor*

$$p : \mathcal{G}\text{-}Torsors \longrightarrow (Sch/S)_{fppf}$$

defines a stack in groupoids over $(Sch/S)_{fppf}$.

Proof. The most difficult part of the proof is to show that we have descent for objects. Let $\{U_i \to U\}_{i \in I}$ be a covering of $(Sch/S)_{fppf}$. Suppose that for each i we are given a $\mathcal{G}|_{U_i}$-torsor $\mathcal{F}_i$, and for each $i, j \in I$ an isomorphism $\varphi_{ij} : \mathcal{F}_i|_{U_i \times_U U_j} \to \mathcal{F}_j|_{U_i \times_U U_j}$ of $\mathcal{G}|_{U_i \times_U U_j}$-torsors satisfying a suitable cocycle condition on $U_i \times_U U_j \times_U U_k$. Then by Sites, Section 7.25 we obtain a sheaf $\mathcal{F}$ on $(Sch/U)_{fppf}$ whose restriction to each U_i recovers $\mathcal{F}_i$ as well as recovering the descent data. By the equivalence of categories in Sites, Lemma 7.25.3 the action maps $\mathcal{G}|_{U_i} \times \mathcal{F}_i \to \mathcal{F}_i$ glue to give a map $a : \mathcal{G}|_U \times \mathcal{F} \to \mathcal{F}$. Now we have to show that a is an action and that $\mathcal{F}$ becomes a $\mathcal{G}|_U$-torsor. Both properties may be checked locally, and hence follow from the corresponding properties of the actions $\mathcal{G}|_{U_i} \times \mathcal{F}_i \to \mathcal{F}_i$. This proves that descent for objects holds in $\mathcal{G}\text{-}Torsors$. Some details omitted. □

68.13.3. Variant on torsors for a sheaf. The construction of Subsection 68.13.1 can be generalized slightly. Namely, let $\mathcal{G} \to \mathcal{B}$ be a map of sheaves on $(Sch/S)_{fppf}$ and let

$$m : \mathcal{G} \times_{\mathcal{B}} \mathcal{G} \longrightarrow \mathcal{G}$$

be a group law on $\mathcal{G}/\mathcal{B}$. In other words, the pair $(\mathcal{G}, m)$ is a group object of the topos $Sh((Sch/S)_{fppf})/\mathcal{B}$. See Sites, Section 7.29 for information regarding localizations of topoi. In this setting we can define a category $\mathcal{G}/\mathcal{B}$-*Torsors* as follows (where we use the Yoneda embedding to think of schemes as sheaves):

(1) An object of $\mathcal{G}/\mathcal{B}$-*Torsors* is a triple $(U, b, \mathcal{F})$ where
   (a) U is an object of $(Sch/S)_{fppf}$,
   (b) $b : U \to \mathcal{B}$ is a section of $\mathcal{B}$ over U, and
   (c) $\mathcal{F}$ is a $U \times_{b,\mathcal{B}} \mathcal{G}$-torsor over U.

(2) A morphism $(U, b, \mathcal{F}) \to (U', b', \mathcal{F}')$ is given by a pair (f, g), where $f : U \to U'$ is a morphism of schemes over S such that $b = b' \circ f$, and $g : f^{-1}\mathcal{F}' \to \mathcal{F}$ is an isomorphism of $U \times_{b,\mathcal{B}} \mathcal{G}$-torsors.

Thus $\mathcal{G}/\mathcal{B}$-*Torsors* is a category and

$$p : \mathcal{G}/\mathcal{B}\text{-}Torsors \longrightarrow (Sch/S)_{fppf}, \quad (U, b, \mathcal{F}) \longmapsto U$$

is a functor. Note that the fibre category of $\mathcal{G}/\mathcal{B}$-*Torsors* over U is the disjoint union over $b : U \to \mathcal{B}$ of the categories of $U \times_{b,\mathcal{B}} \mathcal{G}$-torsors, hence is a groupoid.

In the special case $\mathcal{B} = S$ we recover the category $\mathcal{G}$-*Torsors* introduced in Subsection 68.13.1.

Lemma 68.13.4. *Up to a replacement as in Stacks, Remark 8.4.9 the functor*

$$p : \mathcal{G}/\mathcal{B}\text{-}Torsors \longrightarrow (Sch/S)_{fppf}$$

defines a stack in groupoids over $(Sch/S)_{fppf}$.

Proof. This proof is a repeat of the proof of Lemma 68.13.2. The reader is encouraged to read that proof first since the notation is less cumbersome. The most difficult part of the proof is to show that we have descent for objects. Let $\{U_i \to U\}_{i \in I}$ be a covering of $(Sch/S)_{fppf}$. Suppose that for each i we are given a pair $(b_i, \mathcal{F}_i)$ consisting of a morphism $b_i : U_i \to \mathcal{B}$ and a $U_i \times_{b_i,\mathcal{B}} \mathcal{G}$-torsor $\mathcal{F}_i$, and for each $i, j \in I$ we have $b_i|_{U_i \times_U U_j} = b_j|_{U_i \times_U U_j}$ and we are given an isomorphism $\varphi_{ij} : \mathcal{F}_i|_{U_i \times_U U_j} \to \mathcal{F}_j|_{U_i \times_U U_j}$ of $(U_i \times_U U_j) \times_{\mathcal{B}} \mathcal{G}$-torsors satisfying a suitable cocycle condition on $U_i \times_U U_j \times_U U_k$. Then by Sites, Section 7.25 we obtain a sheaf $\mathcal{F}$ on $(Sch/U)_{fppf}$ whose restriction to each U_i recovers $\mathcal{F}_i$ as well as recovering the descent data. By the sheaf axiom for $\mathcal{B}$ the morphisms b_i come from a unique morphism $b : U \to \mathcal{B}$. By the equivalence of categories in Sites, Lemma 7.25.3 the action maps $(U_i \times_{b_i,\mathcal{B}} \mathcal{G}) \times_{U_i} \mathcal{F}_i \to \mathcal{F}_i$ glue to give a map $(U \times_{b,\mathcal{B}} \mathcal{G}) \times \mathcal{F} \to \mathcal{F}$. Now we have to show that this is an action and that $\mathcal{F}$ becomes a $U \times_{b,\mathcal{B}} \mathcal{G}$-torsor. Both properties may be checked locally, and hence follow from the corresponding properties of the actions on the $\mathcal{F}_i$. This proves that descent for objects holds in $\mathcal{G}/\mathcal{B}$-*Torsors*. Some details omitted. □

68.13.5. Principal homogeneous spaces. Let B be an algebraic space over S. Let G be a group algebraic space over B. We define a category G-*Principal* as follows:

(1) An object of G-*Principal* is a triple (U, b, X) where

(a) U is an object of $(Sch/S)_{fppf}$,
(b) $b : U \to B$ is a morphism over S, and
(c) X is a principal homogeneous G_U-space over U where $G_U = U \times_{b,B} G$.

See Groupoids in Spaces, Definition 59.9.3.

(2) A morphism $(U, b, X) \to (U', b', X')$ is given by a pair (f, g), where $f : U \to U'$ is a morphism of schemes over B, and $g : X \to U \times_{f,U'} X'$ is an isomorphism of principal homogeneous G_U-spaces.

Thus G-*Principal* is a category and

$$p : G\text{-}Principal \longrightarrow (Sch/S)_{fppf}, \quad (U, b, X) \longmapsto U$$

is a functor. Note that the fibre category of G-*Principal* over U is the disjoint union over $b : U \to B$ of the categories of principal homogeneous $U \times_{b,B} G$-spaces, hence is a groupoid.

In the special case $S = B$ the objects are simply pairs (U, X) where U is a scheme over S, and X is a principal homogeneous G_U-space over U. Moreover, morphisms are simply cartesian diagrams

$$\begin{array}{ccc} X & \xrightarrow{g} & X' \\ \downarrow & & \downarrow \\ U & \xrightarrow{f} & U' \end{array}$$

where g is G-equivariant.

Remark 68.13.6. We conjecture that up to a replacement as in Stacks, Remark 8.4.9 the functor

$$p : G\text{-}Principal \longrightarrow (Sch/S)_{fppf}$$

defines a stack in groupoids over $(Sch/S)_{fppf}$. This would follow if one could show that given

(1) a covering $\{U_i \to U\}_{i \in I}$ of $(Sch/S)_{fppf}$,
(2) an group algebraic space H over U,
(3) for every i a principal homogeneous H_{U_i}-space X_i over U_i, and
(4) H-equivariant isomorphisms $\varphi_{ij} : X_{i,U_i \times_U U_j} \to X_{j,U_i \times_U U_j}$ satisfying the cocycle condition,

there exists a principal homogeneous H-space X over U which recovers (X_i, φ_{ij}). The technique of the proof of Bootstrap, Lemma 61.11.6 reduces this to a set theoretical question, so the reader who ignores set theoretical questions will "know" that the result is true. In `http://math.columbia.edu/~dejong/wordpress/?p=591` there is a suggestion as to how to approach this problem.

68.13.7. Variant on principal homogeneous spaces. Let S be a scheme. Let $B = S$. Let G be a group scheme over $B = S$. In this setting we can define a full subcategory G-*Principal-Schemes* $\subset G$-*Principal* whose objects are pairs (U, X) where U is an object of $(Sch/S)_{fppf}$ and $X \to U$ is a principal homogeneous G-space over U which is representable, i.e., a scheme.

It is in general not the case that G-*Principal-Schemes* is a stack in groupoids over $(Sch/S)_{fppf}$. The reason is that in general there really do exist principal homogeneous spaces which are not schemes, hence descent for objects will not be satisfied in general.

68.13.8. Torsors in fppf topology. Let B be an algebraic space over S. Let G be a group algebraic space over B. We define a category $G\text{-}Torsors$ as follows:

(1) An object of $G\text{-}Torsors$ is a triple (U, b, X) where
   (a) U is an object of $(Sch/S)_{fppf}$,
   (b) $b : U \to B$ is a morphism, and
   (c) X is an fppf G_U-torsor over U where $G_U = U \times_{b,B} G$.
   See Groupoids in Spaces, Definition 59.9.3.
(2) A morphism $(U, b, X) \to (U', b', X')$ is given by a pair (f, g), where $f : U \to U'$ is a morphism of schemes over B, and $g : X \to U \times_{f,U'} X'$ is an isomorphism of G_U-torsors.

Thus $G\text{-}Torsors$ is a category and

$$p : G\text{-}Torsors \longrightarrow (Sch/S)_{fppf}, \quad (U, a, X) \longmapsto U$$

is a functor. Note that the fibre category of $G\text{-}Torsors$ over U is the disjoint union over $b : U \to B$ of the categories of fppf $U \times_{b,B} G$-torsors, hence is a groupoid.

In the special case $S = B$ the objects are simply pairs (U, X) where U is a scheme over S, and X is an fppf G_U-torsor over U. Moreover, morphisms are simply cartesian diagrams

$$\begin{array}{ccc} X & \xrightarrow{g} & X' \\ \downarrow & & \downarrow \\ U & \xrightarrow{f} & U' \end{array}$$

where g is G-equivariant.

Lemma 68.13.9. *Up to a replacement as in Stacks, Remark 8.4.9 the functor*

$$p : G\text{-}Torsors \longrightarrow (Sch/S)_{fppf}$$

defines a stack in groupoids over $(Sch/S)_{fppf}$.

Proof. The most difficult part of the proof is to show that we have descent for objects, which is Bootstrap, Lemma 61.11.6. Some details omitted. □

Lemma 68.13.10. *Let B be an algebraic space over S. Let G be a group algebraic space over B. Denote $\mathcal{G}$, resp. $\mathcal{B}$ the algebraic space G, resp. B seen as a sheaf on $(Sch/S)_{fppf}$. The functor*

$$G\text{-}Torsors \longrightarrow \mathcal{G}/\mathcal{B}\text{-}Torsors$$

which associates to a triple (U, b, X) the triple $(U, b, \mathcal{X})$ where $\mathcal{X}$ is X viewed as a sheaf is an equivalence of stacks in groupoids over $(Sch/S)_{fppf}$.

Proof. We will use the result of Stacks, Lemma 8.4.8 to prove this. The functor is fully faithful since the category of algebraic spaces over S is a full subcategory of the category of sheaves on $(Sch/S)_{fppf}$. Moreover, all objects (on both sides) are locally trivial torsors so condition (2) of the lemma referenced above holds. Hence the functor is an equivalence. □

68.13.11. Variant on torsors in fppf topology. Let S be a scheme. Let $B = S$. Let G be a group scheme over $B = S$. In this setting we can define a full subcategory *G-Torsors-Schemes* $\subset$ *G-Torsors* whose objects are pairs (U, X) where U is an object of $(Sch/S)_{fppf}$ and $X \to U$ is an fppf G-torsor over U which is representable, i.e., a scheme.

It is in general not the case that *G-Torsors-Schemes* is a stack in groupoids over $(Sch/S)_{fppf}$. The reason is that in general there really do exist fppf G-torsors which are not schemes, hence descent for objects will not be satisfied in general.

68.14. Quotients by group actions

At this point we have introduced enough notation that we can work out in more detail what the stacks $[X/G]$ of Section 68.12 look like.

Situation 68.14.1. Here

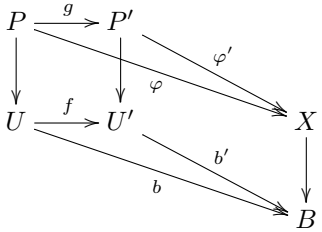
(1) S is a scheme contained in Sch_{fppf},
(2) B is an algebraic space over S,
(3) (G, m) is a group algebraic space over B,
(4) $\pi : X \to B$ is an algebraic space over B, and
(5) $a : G \times_B X \to X$ is an action of G on X over B.

In this situation we construct a category $[[X/G]]$[2] as follows:

(1) An object of $[[X/G]]$ consists of a quadruple $(U, b, P, \varphi : P \to X)$ where
   (a) U is an object of $(Sch/S)_{fppf}$,
   (b) $b : U \to B$ is a morphism over S,
   (c) P is an fppf G_U-torsor over U where $G_U = U \times_{b,B} G$, and
   (d) $\varphi : P \to X$ is a G-equivariant morphism fitting into the commutative diagram

$$\begin{array}{ccc} P & \xrightarrow{\varphi} & X \\ \downarrow & & \downarrow \\ U & \xrightarrow{b} & B \end{array}$$

(2) A morphism of $[[X/G]]$ is a pair $(f, g) : (U, b, P, \varphi) \to (U', b', P', \varphi')$ where $f : U \to U'$ is a morphism of schemes over B and $g : P \to P'$ is a G-equivariant morphism over f which induces an isomorphism $P \cong U \times_{f,U'} P'$, and has the property that $\varphi = \varphi' \circ g$. In other words (f, g) fits into the following commutative diagram



[2]The notation $[[X/G]]$ with double brackets serves to distinguish this category from the stack $[X/G]$ introduced earlier. In Proposition 68.14.4 we show that the two are canonically equivalent. Afterwards we will use the notation $[X/G]$ to indicate either.

Thus $[[X/G]]$ is a category and

$$p : [[X/G]] \longrightarrow (Sch/S)_{fppf}, \quad (U, b, P, \varphi) \longmapsto U$$

is a functor. Note that the fibre category of $[[X/G]]$ over U is the disjoint union over $b \in \mathrm{Mor}_S(U, B)$ of $U \times_{b,B} G$-torsors P endowed with a G-equivariant morphism to X. Hence the fibre categories of $[[X/G]]$ are groupoids.

Note that the functor

$$[[X/G]] \longrightarrow G\text{-}Torsors, \quad (U, b, P, \varphi) \longmapsto (U, b, P)$$

is a 1-morphism of categories over $(Sch/S)_{fppf}$.

Lemma 68.14.2. *Up to a replacement as in Stacks, Remark 8.4.9 the functor*

$$p : [[X/G]] \longrightarrow (Sch/S)_{fppf}$$

defines a stack in groupoids over $(Sch/S)_{fppf}$.

Proof. The most difficult part of the proof is to show that we have descent for objects. Suppose that $\{U_i \to U\}_{i \in I}$ is a covering in $(Sch/S)_{fppf}$. Let $\xi_i = (U_i, b_i, P_i, \varphi_i)$ be objects of $[[X/G]]$ over U_i, and let $\varphi_{ij} : \mathrm{pr}_0^*\xi_i \to \mathrm{pr}_1^*\xi_j$ be a descent datum. This in particular implies that we get a descent datum on the triples (U_i, b_i, P_i) for the stack in groupoids G-*Torsors* by applying the functor $[[X/G]] \to G$-*Torsors* above. Hence we may assume that $b_i = b|_{U_i}$ for some morphism $b : U \to B$, and that $P_i = U_i \times_U P$ for some fppf $G_U = U \times_{b,B} G$-torsor P over U. Then finally the morphisms φ_i are compatible with the canonical descent datum on the restrictions $U_i \times_U P$ and hence define a morphism $\varphi : P \to X$. (For example you can use Sites, Lemma 7.25.3 or you can use Descent on Spaces, Lemma 55.6.2 to get φ.) This proves descent for objects. We omit the proof of the other two defining properties of a stack in groupoids. □

Remark 68.14.3. Let S be a scheme. Let G be an abstract group. Let X be an algebraic space over S. Let $G \to \mathrm{Aut}_S(X)$ be a group homomorphism. In this setting we can define $[[X/G]]$ similarly to the above as follows:

(1) An object of $[[X/G]]$ consists of a triple $(U, P, \varphi : P \to X)$ where
   (a) U is an object of $(Sch/S)_{fppf}$,
   (b) P is a sheaf on $(Sch/U)_{fppf}$ which comes with an action of G that turns it into a torsor under the constant sheaf with value G, and
   (c) $\varphi : P \to X$ is a G-equivariant map of sheaves.

(2) A morphism $(f, g) : (U, P, \varphi) \to (U', P', \varphi')$ is given by a morphism of schemes $f : T \to T'$ and a G-equivariant isomorphism $g : P \to f^{-1}P'$ such that $\varphi = \varphi' \circ g$.

In exactly the same manner as above we obtain a functor

$$[[X/G]] \longrightarrow (Sch/S)_{fppf}$$

which turns $[[X/G]]$ into a stack in groupoids over $(Sch/S)_{fppf}$. The constant sheaf $\underline{G}$ is (provided the cardinality of G is not too large) representable by G_S on $(Sch/S)_{fppf}$ and this version of $[[X/G]]$ is equivalent to the stack $[[X/G_S]]$ introduced above.

Proposition 68.14.4. *In Situation 68.14.1 there exists a canonical equivalence*

$$[X/G] \longrightarrow [[X/G]]$$

of stacks in groupoids over $(Sch/S)_{fppf}$.

Proof. We write this out in detail, to make sure that all the definitions work out in exactly the correct manner. Recall that $[X/G]$ is the quotient stack associated to the groupoid in algebraic spaces $(X, G \times_B X, s, t, c)$, see Groupoids in Spaces, Definition 59.19.1. This means that $[X/G]$ is the stackification of the category fibred in groupoids $[X/_pG]$ associated to the functor

$$(Sch/S)_{fppf} \longrightarrow Groupoids, \quad U \longmapsto (X(U), G(U) \times_{B(U)} X(U), s, t, c)$$

where $s(g, x) = x$, $t(g, x) = a(g, x)$, and $c((g, x), (g', x')) = (m(g, g'), x')$. By the construction of Categories, Example 4.35.1 an object of $[X/_pG]$ is a pair (U, x) with $x \in X(U)$ and a morphism $(f, g) : (U, x) \to (U', x')$ of $[X/_pG]$ is given by a morphism of schemes $f : U \to U'$ and an element $g \in G(U)$ such that $a(g, x) = x' \circ f$. Hence we can define a 1-morphism of stacks in groupoids

$$F_p : [X/_pG] \longrightarrow [[X/G]]$$

by the following rules: On objects we set

$$F_p(U, x) = (U, \pi \circ x, G \times_{B,\pi\circ x} U, a \circ (\mathrm{id}_G \times x))$$

This makes sense because the diagram

$$\begin{array}{ccccc} G \times_{B,\pi\circ x} U & \xrightarrow[\mathrm{id}_G \times x]{} & G \times_{B,\pi} X & \xrightarrow[a]{} & X \\ \downarrow & & & & \downarrow \pi \\ U & & \xrightarrow{\quad\pi\circ x\quad} & & B \end{array}$$

commutes, and the two horizontal arrows are G-equivariant if we think of the fibre products as trivial G-torsors over U, resp. X. On morphisms $(f, g) : (U, x) \to (U', x')$ we set $F_p(f, g) = (f, R_g)$ where R_g denotes right translation by g. More precisely, the morphism of $F_p(f, g) : F_p(U, x) \to F_p(U', x')$ is given by the cartesian diagram

$$\begin{array}{ccc} G \times_{B,\pi\circ x} U & \xrightarrow[R_g]{} & G \times_{B,\pi\circ x'} U' \\ \downarrow & & \downarrow \\ U & \xrightarrow{\quad f \quad} & U' \end{array}$$

where R_g on T-valued points is given by

$$R_g(g', u) = (m(g', g), f(u))$$

To see that this works we have to verify that

$$a \circ (\mathrm{id}_G \times x) = a \circ (\mathrm{id}_G \times x') \circ R_g$$

which is true because the right hand side applied to the T-valued point (g', u) gives

$$\begin{aligned} a((\mathrm{id}_G \times x')(m(g', g), f(u))) &= a(m(g', g), x'(f(u))) \\ &= a(g', a(g, x'(f(u)))) \\ &= a(g', x(u)) \end{aligned}$$

because $a(g, x) = x' \circ f$ as desired.

By the universal property of stackification from Stacks, Lemma 8.9.2 we obtain a canonical extension $F : [X/G] \to [[X/G]]$ of the 1-morphism F_p above. We first prove that F is fully faithful. To do this, since both source and target are stacks in

groupoids, it suffices to prove that the *Isom*-sheaves are identified under F. Pick a scheme U and objects ξ, ξ' of $[X/G]$ over U. We want to show that

$$F : Isom_{[X/G]}(\xi, \xi') \longrightarrow Isom_{[[X/G]]}(F(\xi), F(\xi'))$$

is an isomorphism of sheaves. To do this it suffices to work locally on U, and hence we may assume that ξ, ξ' come from objects (U, x), (U, x') of $[X/_pG]$ over U; this follows directly from the construction of the stackification, and it is also worked out in detail in Groupoids in Spaces, Section 59.23. Either by directly using the description of morphisms in $[X/_pG]$ above, or using Groupoids in Spaces, Lemma 59.21.1 we see that in this case

$$Isom_{[X/G]}(\xi, \xi') = U \times_{(x,x'),X\times_S X,(s,t)} (G \times_B X)$$

A T-valued point of this fibre product corresponds to a pair (u, g) with $u \in U(T)$, and $g \in G(T)$ such that $a(g, x \circ u) = x' \circ u$. (Note that this implies $\pi \circ x \circ u = \pi \circ x' \circ u$.) On the other hand, a T-valued point of $Isom_{[[X/G]]}(F(\xi), F(\xi'))$ by definition corresponds to a morphism $u : T \to U$ such that $\pi \circ x \circ u = \pi \circ x' \circ u : T \to B$ and an isomorphism

$$R : G \times_{B,\pi\circ x\circ u} T \longrightarrow G \times_{B,\pi\circ x'\circ u} T$$

of trivial G_T-torsors compatible with the given maps to X. Since the torsors are trivial we see that $R = R_g$ (right multiplication) by some $g \in G(T)$. Compatibility with the maps $a \circ (1_G, x \circ u), a \circ (1_G, x' \circ u) : G \times_B T \to X$ is equivalent to the condition that $a(g, x \circ u) = x' \circ u$. Hence we obtain the desired equality of *Isom*-sheaves.

Now that we know that F is fully faithful we see that Stacks, Lemma 8.4.8 applies. Thus to show that F is an equivalence it suffices to show that objects of $[[X/G]]$ are fppf locally in the essential image of F. This is clear as fppf torsors are locally trivial, and hence we win. □

68.15. The Picard stack

Let S be a scheme. Let $\pi : X \to B$ be a morphism of algebraic spaces over S. We define a category $Pic_{X/B}$ as follows:

(1) An object is a triple $(U, b, \mathcal{L})$, where
   (a) U is an object of $(Sch/S)_{fppf}$,
   (b) $b : U \to B$ is a morphism over S, and
   (c) $\mathcal{L}$ is in invertible sheaf on the base change $X_U = U \times_{b,B} X$.

(2) A morphism $(f, g) : (U, b, \mathcal{L}) \to (U', b', \mathcal{L}')$ is given by a morphism of schemes $f : U \to U'$ over B and an isomorphism $g : f^*\mathcal{L}' \to \mathcal{L}$.

The composition of $(f, g) : (U, b, \mathcal{L}) \to (U', b', \mathcal{L}')$ with $(f', g') : (U', b', \mathcal{L}') \to (U'', b'', \mathcal{L}'')$ is given by $(f \circ f', g \circ f^*(g'))$. Thus we get a category $Pic_{X/B}$ and

$$p : Pic_{X/B} \longrightarrow (Sch/S)_{fppf}, \quad (U, b, \mathcal{L}) \longmapsto U$$

is a functor. Note that the fibre category of $Pic_{X/B}$ over U is the disjoint union over $b \in \mathrm{Mor}_S(U, B)$ of the categories of invertible sheaves on $X_U = U \times_{b,B} X$. Hence the fibre categories are groupoids.

Lemma 68.15.1. *Up to a replacement as in Stacks, Remark 8.4.9 the functor*

$$Pic_{X/B} \longrightarrow (Sch/S)_{fppf}$$

defines a stack in groupoids over $(Sch/S)_{fppf}$.

Proof. As usual, the hardest part is to show descent for objects. To see this let $\{U_i \to U\}$ be a covering of $(Sch/S)_{fppf}$. Let $\xi_i = (U_i, b_i, \mathcal{L}_i)$ be an object of $Pic_{X/B}$ lying over U, and let $\varphi_{ij} : \mathrm{pr}_0^*\xi_i \to \mathrm{pr}_1^*\xi_j$ be a descent datum. This implies in particular that the morphisms b_i are the restrictions of a morphism $b : U \to B$. Write $X_U = U \times_{b,B} X$ and $X_i = U_i \times_{b_i,B} X = U_i \times_U U \times_{b,B} X = U_i \times_U X_U$. Observe that $\mathcal{L}_i$ is an invertible $\mathcal{O}_{X_i}$-module. Note that $\{X_i \to X_U\}$ forms an fppf covering as well. Moreover, the descent datum φ_{ij} translates into a descent datum on the invertible sheaves $\mathcal{L}_i$ relative to the fppf covering $\{X_i \to X_U\}$. Hence by Descent on Spaces, Proposition 55.4.1 we obtain a unique invertible sheaf $\mathcal{L}$ on X_U which recovers $\mathcal{L}_i$ and the descent data over X_i. The triple $(U, b, \mathcal{L})$ is therefore the object of $Pic_{X/B}$ over U we were looking for. Details omitted. □

68.16. Examples of inertia stacks

Here are some examples of inertia stacks.

Example 68.16.1. Let S be a scheme. Let G be a commutative group. Let $X \to S$ be a scheme over S. Let $a : G \times X \to X$ be an action of G on X. For $g \in G$ we denote $g : X \to X$ the corresponding automorphism. In this case the inertia stack of $[X/G]$ (see Remark 68.14.3) is given by

$$I_{[X/G]} = \coprod\nolimits_{g \in G} [X^g/G],$$

where, given an element g of G, the symbol X^g denotes the scheme $X^g = \{x \in X \mid g(x) = x\}$. In a formula X^g is really the fibre product

$$X^g = X \times_{(1,1),X \times_S X,(g,1)} X.$$

Indeed, for any S-scheme T, a T-point on the inertia stack of $[X/G]$ consists of a triple $(P/T, \phi, \alpha)$ consisting of a G-torsor $P \to T$ together with a G-equivariant isomorphism $\phi : P \to X$, together with an automorphism α of $P \to T$ over T such that $\phi \circ \alpha = \phi$. Since G is a sheaf of *commutative* groups, α is, locally in the fppf topology over T, given by multiplication by some element g of G. The condition that $\phi \circ \alpha = \phi$ means that ϕ factors through the inclusion of X^g in X, i.e., ϕ is obtained by composing that inclusion with a morphism $P \to X^\gamma$. The above discussion allows us to define a morphism of fibred categories $I_{[X/G]} \to \coprod_{g \in G} [X^g/G]$ given on T-points by the discussion above. We omit showing that this is an equivalence.

Example 68.16.2. Let $X \to S$ be a morphism of schemes. Assume that for any $T \to S$ the base change $f_T : X_T \to T$ has the property that the map $\mathcal{O}_T \to f_{T,*}\mathcal{O}_{X_T}$ is an isomorphism. (This implies that f is *cohomologically flat in dimension* 0 (insert future reference here) but is stronger.) Consider the Picard stack $Pic_{X/S}$, see Section 68.15. The points of its inertia stack over an S-scheme T consist of pairs $(\mathcal{L}, \alpha)$ where $\mathcal{L}$ is a line bundle on X_T and α is an automorphism of that line bundle. I.e., we can think of α as an element of $H^0(X_T, \mathcal{O}_{X_T})^\times = H^0(T, \mathcal{O}_T^*)$ by our condition. Note that $H^0(T, \mathcal{O}_T^*) = \mathbf{G}_{m,S}(T)$, see Groupoids, Example 38.5.1. Hence the inertia stack of $Pic_{X/S}$ is

$$I_{Pic_{X/S}} = \mathbf{G}_{m,S} \times_S Pic_{X/S}.$$

as a stack over $(Sch/S)_{fppf}$.

68.17. Finite Hilbert stacks

We formulate this in somewhat greater generality than is perhaps strictly needed. Fix a 1-morphism

$$F : \mathcal{X} \longrightarrow \mathcal{Y}$$

of stacks in groupoids over $(Sch/S)_{fppf}$. For each integer $d \geq 1$ consider a category $\mathcal{H}_d(\mathcal{X}/\mathcal{Y})$ defined as follows:

(1) An object (U, Z, y, x, α) where U, Z are objects of in $(Sch/S)_{fppf}$ and Z is a finite locally free of degree d over U, where $y \in \mathrm{Ob}(\mathcal{Y}_U)$, $x \in \mathrm{Ob}(\mathcal{X}_Z)$ and $\alpha : y|_Z \to F(x)$ is an isomorphism[3].

(2) A morphism $(U, Z, y, x, \alpha) \to (U', Z', y', x', \alpha')$ is given by a morphism of schemes $f : U \to U'$, a morphism of schemes $g : Z \to Z'$ which induces an isomorphism $Z \to Z' \times_{U'} U$, and isomorphisms $b : y \to f^*y'$, $a : x \to g^*x'$ inducing a commutative diagram

$$\begin{array}{ccc} y|_Z & \xrightarrow{\alpha} & F(x) \\ {\scriptstyle b|_Z}\downarrow & & \downarrow{\scriptstyle F(a)} \\ f^*y'|_Z & \xrightarrow{\alpha'} & F(g^*x') \end{array}$$

It is clear from the definitions that there is a canonical forgetful functor

$$p : \mathcal{H}_d(\mathcal{X}/\mathcal{Y}) \longrightarrow (Sch/S)_{fppf}$$

which assigns to the quintuple (U, Z, y, x, α) the scheme U and to the morphism $(f, g, b, a) : (U, Z, y, x, \alpha) \to (U', Z', y', x', \alpha')$ the morphism $f : U \to U'$.

Lemma 68.17.1. *The category $\mathcal{H}_d(\mathcal{X}/\mathcal{Y})$ endowed with the functor p above defines a stack in groupoids over $(Sch/S)_{fppf}$.*

Proof. As usual, the hardest part is to show descent for objects. To see this let $\{U_i \to U\}$ be a covering of $(Sch/S)_{fppf}$. Let $\xi_i = (U_i, Z_i, y_i, x_i, \alpha_i)$ be an object of $\mathcal{H}_d(\mathcal{X}/\mathcal{Y})$ lying over U_i, and let $\varphi_{ij} : \mathrm{pr}_0^*\xi_i \to \mathrm{pr}_1^*\xi_j$ be a descent datum. First, observe that φ_{ij} induces a descent datum $(Z_i/U_i, \varphi_{ij})$ which is effective by Descent, Lemma 34.33.1 This produces a scheme Z/U which is finite locally free of degree d by Descent, Lemma 34.19.28. From now on we identify Z_i with $Z \times_U U_i$. Next, the objects y_i in the fibre categories $\mathcal{Y}_{U_i}$ descend to an object y in $\mathcal{Y}_U$ because $\mathcal{Y}$ is a

[3] This means the data gives rise, via the 2-Yoneda lemma (Categories, Lemma 4.39.1), to a 2-commutative diagram

$$\begin{array}{ccc} (Sch/Z)_{fppf} & \xrightarrow{x} & \mathcal{X} \\ \downarrow & & \downarrow{\scriptstyle F} \\ (Sch/U)_{fppf} & \xrightarrow{y} & \mathcal{Y} \end{array}$$

of stacks in groupoids over $(Sch/S)_{fppf}$. Alternatively, we may picture α as a 2-morphism

$$(Sch/Z)_{fppf} \xrightarrow[F\circ x]{y\circ(Z\to U)} \mathcal{Y}. \quad (\Downarrow \alpha)$$

stack in groupoids. Similarly the objects x_i in the fibre categories $\mathcal{X}_{Z_i}$ descend to an object x in $\mathcal{X}_Z$ because $\mathcal{X}$ is a stack in groupoids. Finally, the given isomorphisms

$$\alpha_i : (y|_Z)_{Z_i} = y_i|_{Z_i} \longrightarrow F(x_i) = F(x|_{Z_i})$$

glue to a morphism $\alpha : y|_Z \to F(x)$ as the $\mathcal{Y}$ is a stack and hence $Isom_{\mathcal{Y}}(y|_Z, F(x))$ is a sheaf. Details omitted. □

Definition 68.17.2. We will denote $\mathcal{H}_d(\mathcal{X}/\mathcal{Y})$ the *degree d finite Hilbert stack of $\mathcal{X}$ over $\mathcal{Y}$* constructed above. If $\mathcal{Y} = S$ we write $\mathcal{H}_d(\mathcal{X}) = \mathcal{H}_d(\mathcal{X}/\mathcal{Y})$. If $\mathcal{X} = \mathcal{Y} = S$ we denote it $\mathcal{H}_d$.

Note that given $F : \mathcal{X} \to \mathcal{Y}$ as above we have the following natural 1-morphisms of stacks in groupoids over $(Sch/S)_{fppf}$:

$$\begin{array}{ccccc} \mathcal{H}_d(\mathcal{X}) & \longleftarrow & \mathcal{H}_d(\mathcal{X}/\mathcal{Y}) & \longrightarrow & \mathcal{Y} \\ & \searrow & \downarrow & & \\ & & \mathcal{H}_d & & \end{array} \tag{68.17.2.1}$$

Each of the arrows is given by a "forgetful functor".

Lemma 68.17.3. *The 1-morphism $\mathcal{H}_d(\mathcal{X}/\mathcal{Y}) \to \mathcal{H}_d(\mathcal{X})$ is faithful.*

Proof. To check that $\mathcal{H}_d(\mathcal{X}/\mathcal{Y}) \to \mathcal{H}_d(\mathcal{X})$ is faithful it suffices to prove that it is faithful on fibre categories. Suppose that $\xi = (U, Z, y, x, \alpha)$ and $\xi' = (U, Z', y', x', \alpha')$ are two objects of $\mathcal{H}_d(\mathcal{X}/\mathcal{Y})$ over the scheme U. Let $(g, b, a), (g', b', a') : \xi \to \xi'$ be two morphisms in the fibre category of $\mathcal{H}_d(\mathcal{X}/\mathcal{Y})$ over U. The image of these morphisms in $\mathcal{H}_d(\mathcal{X})$ agree if and only if $g = g'$ and $a = a'$. Then the commutative diagram

$$\begin{array}{ccc} y|_Z & \xrightarrow{\alpha} & F(x) \\ {\scriptstyle b|_Z,\ b'|_Z}\downarrow & & \downarrow{\scriptstyle F(a)=F(a')} \\ y'|_Z & \xrightarrow{\alpha'} & F(g^*x') = F((g')^*x') \end{array}$$

implies that $b|_Z = b'|_Z$. Since $Z \to U$ is finite locally free of degree d we see $\{Z \to U\}$ is an fppf covering, hence $b = b'$. □

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CHAPTER 69

Sheaves on Algebraic Stacks

69.1. Introduction

There is a myriad of ways to think about sheaves on algebraic stacks. In this chapter we discuss one approach, which is particularly well adapted to our foundations for algebraic stacks. Whenever we introduce a type of sheaves we will indicate the precise relationship with similar notions in the literature. The goal of this chapter is to state those results that are either obviously true or straightforward to prove and leave more intricate constructions till later.

In fact, it turns out that to develop a fully fledged theory of constructible étale sheaves and/or an adequate discussion of derived categories of complexes $\mathcal{O}$-modules whose cohomology sheaves are quasi-coherent takes a significant amount of work, see [**Ols07b**]. We will return to this in Cohomology of Stacks, Section 75.1.

In the literature and in research papers on sheaves on algebraic stacks the lisse-étale site of an algebraic stack often plays a prominent role. However, it is a problematic beast, because it turns out that a morphism of algebraic stacks does not induce a morphism of lisse-étale topoi. We have therefore made the design decision to avoid any mention of the lisse-étale site as long as possible. Arguments that traditionally use the lisse-étale site will be replaced by an argument using a Čech covering in the site $\mathcal{X}_{smooth}$ defined below.

Some of the notation, conventions and terminology in this chapter is awkward and may seem backwards to the more experienced reader. This is intentional. Please see Quot, Section 72.1 for an explanation.

69.2. Conventions

The conventions we use in this chapter are the same as those in the chapter on algebraic stacks, see Algebraic Stacks, Section 67.2. For convenience we repeat them here.

We work in a suitable big fppf site Sch_{fppf} as in Topologies, Definition 33.7.6. So, if not explicitly stated otherwise all schemes will be objects of Sch_{fppf}. We record what changes if you change the big fppf site elsewhere (insert future reference here).

We will always work relative to a base S contained in Sch_{fppf}. And we will then work with the big fppf site $(Sch/S)_{fppf}$, see Topologies, Definition 33.7.8. The absolute case can be recovered by taking $S = \mathrm{Spec}(\mathbf{Z})$.

69.3. Presheaves

In this section we define presheaves on categories fibred in groupoids over $(Sch/S)_{fppf}$, but most of the discussion works for categories over any base category. This section also serves to introduce the notation we will use later on.

Definition 69.3.1. Let $p : \mathcal{X} \to (Sch/S)_{fppf}$ be a category fibred in groupoids.

(1) A *presheaf on* $\mathcal{X}$ is a presheaf on the underlying category of $\mathcal{X}$.
(2) A *morphism of presheaves on* $\mathcal{X}$ is a morphism of presheaves on the underlying category of $\mathcal{X}$.

We denote $PSh(\mathcal{X})$ the category of presheaves on $\mathcal{X}$.

This defines presheaves of sets. Of course we can also talk about presheaves of pointed sets, abelian groups, groups, monoids, rings, modules over a fixed ring, and lie algebras over a fixed field, etc. The category of *abelian presheaves*, i.e., presheaves of abelian groups, is denoted $PAb(\mathcal{X})$.

Let $f : \mathcal{X} \to \mathcal{Y}$ be a 1-morphism of categories fibred in groupoids over $(Sch/S)_{fppf}$. Recall that this means just that f is a functor over $(Sch/S)_{fppf}$. The material in Sites, Section 7.18 provides us with a pair of adjoint functors[1]

$$f^p : PSh(\mathcal{Y}) \longrightarrow PSh(\mathcal{X}) \quad \text{and} \quad {}_pf : PSh(\mathcal{X}) \longrightarrow PSh(\mathcal{Y}). \tag{69.3.1.1}$$

The adjointness is

$$\mathrm{Mor}_{PSh(\mathcal{X})}(f^p\mathcal{G}, \mathcal{F}) = \mathrm{Mor}_{PSh(\mathcal{Y})}(\mathcal{G}, {}_pf\mathcal{F})$$

where $\mathcal{F} \in \mathrm{Ob}(PSh(\mathcal{X}))$ and $\mathcal{G} \in \mathrm{Ob}(PSh(\mathcal{Y}))$. We call $f^p\mathcal{G}$ the *pullback* of $\mathcal{G}$. It follows from the definitions that

$$f^p\mathcal{G}(x) = \mathcal{G}(f(x))$$

for any $x \in \mathrm{Ob}(\mathcal{X})$. The presheaf ${}_pf\mathcal{F}$ is called the *pushforward* of $\mathcal{F}$. It is described by the formula

$$({}_pf\mathcal{F})(y) = \lim_{f(x) \to y} \mathcal{F}(x).$$

The rest of this section should probably be moved to the chapter on sites and in any case should be skipped on a first reading.

Lemma 69.3.2. *Let $f : \mathcal{X} \to \mathcal{Y}$ and $g : \mathcal{Y} \to \mathcal{Z}$ be 1-morphisms of categories fibred in groupoids over $(Sch/S)_{fppf}$. Then $(g \circ f)^p = f^p \circ g^p$ and there is a canonical isomorphism ${}_p(g \circ f) \to {}_pg \circ {}_pf$ compatible with with adjointness of $(f^p, {}_pf)$, $(g^p, {}_pg)$, and $((g \circ f)^p, {}_p(g \circ f))$.*

Proof. Let $\mathcal{H}$ be a presheaf on $\mathcal{Z}$. Then $(g \circ f)^p\mathcal{H} = f^p(g^p\mathcal{H})$ is given by the equalities

$$(g \circ f)^p\mathcal{H}(x) = \mathcal{H}((g \circ f)(x)) = \mathcal{H}(g(f(x))) = f^p(g^p\mathcal{H})(x).$$

We omit the verification that this is compatible with restriction maps.

Next, we define the transformation ${}_p(g \circ f) \to {}_pg \circ {}_pf$. Let $\mathcal{F}$ be a presheaf on $\mathcal{X}$. If z is an object of $\mathcal{Z}$ then we get a category $\mathcal{J}$ of quadruples $(x, f(x) \to y, y, g(y) \to z)$ and a category $\mathcal{I}$ of pairs $(x, g(f(x)) \to z)$. There is a canonical functor $\mathcal{J} \to \mathcal{I}$

[1] These functors will be denoted f^{-1} and f_* after Lemma 69.4.4 has been proved.

sending the object $(x, \alpha : f(x) \to y, y, \beta : g(y) \to z)$ to $(x, \beta \circ f(\alpha) : g(f(x)) \to z)$. This gives the arrow in

$$\begin{aligned}
({}_p(g \circ f)\mathcal{F})(z) &= \lim_{g(f(x)) \to z} \mathcal{F}(x) \\
&= \lim_{\mathcal{I}} \mathcal{F} \\
&\to \lim_{\mathcal{J}} \mathcal{F} \\
&= \lim_{g(y) \to z} \left(\lim_{f(x) \to y} \mathcal{F}(x) \right) \\
&= ({}_pg \circ {}_pf\mathcal{F})(x)
\end{aligned}$$

by Categories, Lemma 4.14.8. We omit the verification that this is compatible with restriction maps. An alternative to this direct construction is to define ${}_p(g \circ f) \cong {}_pg \circ {}_pf$ as the unique map compatible with the adjointness properties. This also has the advantage that one does not need to prove the compatibility.

Compatibility with adjointness of $(f^p, {}_pf)$, $(g^p, {}_pg)$, and $((g \circ f)^p, {}_p(g \circ f))$ means that given presheaves $\mathcal{H}$ and $\mathcal{F}$ as above we have a commutative diagram

$$\begin{array}{ccccc}
\mathrm{Mor}_{PSh(\mathcal{X})}(f^pg^p\mathcal{H}, \mathcal{F}) & = & \mathrm{Mor}_{PSh(\mathcal{Y})}(g^p\mathcal{H}, {}_pf\mathcal{F}) & = & \mathrm{Mor}_{PSh(\mathcal{Y})}(\mathcal{H}, {}_pg{}_pf\mathcal{F}) \\
\| & & & & \uparrow \\
\mathrm{Mor}_{PSh(\mathcal{X})}((g \circ f)^p\mathcal{G}, \mathcal{F}) & & = & & \mathrm{Mor}_{PSh(\mathcal{Y})}(\mathcal{G}, {}_p(g \circ f)\mathcal{F})
\end{array}$$

Proof omitted. □

Lemma 69.3.3. *Let $f, g : \mathcal{X} \to \mathcal{Y}$ be 1-morphisms of categories fibred in groupoids over $(Sch/S)_{fppf}$. Let $t : f \to g$ be a 2-morphism of categories fibred in groupoids over $(Sch/S)_{fppf}$. Assigned to t there are canonical isomorphisms of functors*

$$t^p : g^p \longrightarrow f^p \quad \text{and} \quad {}_pt : {}_pf \longrightarrow {}_pg$$

which compatible with adjointness of $(f^p, {}_pf)$ and $(g^p, {}_pg)$ and with vertical and horizontal composition of 2-morphisms.

Proof. Let $\mathcal{G}$ be a presheaf on $\mathcal{Y}$. Then $t^p : g^p\mathcal{G} \to f^p\mathcal{G}$ is given by the family of maps

$$g^p\mathcal{G}(x) = \mathcal{G}(g(x)) \xrightarrow{\mathcal{G}(t_x)} \mathcal{G}(f(x)) = f^p\mathcal{G}(x)$$

parametrized by $x \in \mathrm{Ob}(\mathcal{X})$. This makes sense as $t_x : f(x) \to g(x)$ and $\mathcal{G}$ is a contravariant functor. We omit the verification that this is compatible with restriction mappings.

To define the transformation ${}_pt$ for $y \in \mathrm{Ob}(\mathcal{Y})$ define ${}^f_y\mathcal{I}$, resp. ${}^g_y\mathcal{I}$ to be the category of pairs $(x, \psi : f(x) \to y)$, resp. $(x, \psi : g(x) \to y)$, see Sites, Section 7.18. Note that t defines a functor ${}_yt : {}^g_y\mathcal{I} \to {}^f_y\mathcal{I}$ given by the rule

$$(x, g(x) \to y) \longmapsto (x, f(x) \xrightarrow{t_x} g(x) \to y).$$

Note that for $\mathcal{F}$ a presheaf on $\mathcal{X}$ the composition of ${}_yt$ with $\mathcal{F} : {}^f_y\mathcal{I}^{opp} \to Sets$, $(x, f(x) \to y) \mapsto \mathcal{F}(x)$ is equal to $\mathcal{F} : {}^g_y\mathcal{I}^{opp} \to Sets$. Hence by Categories, Lemma 4.14.8 we get for every $y \in \mathrm{Ob}(\mathcal{Y})$ a canonical map

$$({}_pf\mathcal{F})(y) = \lim_{{}^f_y\mathcal{I}} \mathcal{F} \longrightarrow \lim_{{}^g_y\mathcal{I}} \mathcal{F} = ({}_pg\mathcal{F})(y)$$

We omit the verification that this is compatible with restriction mappings. An alternative to this direct construction is to define ${}_pt$ as the unique map compatible

with the adjointness properties of the pairs $(f^p, {}_pf)$ and $(g^p, {}_pg)$ (see below). This also has the advantage that one does not need to prove the compatibility.

Compatibility with adjointness of $(f^p, {}_pf)$ and $(g^p, {}_pg)$ means that given presheaves $\mathcal{G}$ and $\mathcal{F}$ as above we have a commutative diagram

$$\begin{array}{ccc} \mathrm{Mor}_{PSh(\mathcal{X})}(f^p\mathcal{G}, \mathcal{F}) & = & \mathrm{Mor}_{PSh(\mathcal{Y})}(\mathcal{G}, {}_pf\mathcal{F}) \\ {\scriptstyle -\circ t^p}\downarrow & & \downarrow{\scriptstyle {}_pt\circ -} \\ \mathrm{Mor}_{PSh(\mathcal{X})}(g^p\mathcal{G}, \mathcal{F}) & = & \mathrm{Mor}_{PSh(\mathcal{Y})}(\mathcal{G}, {}_pg\mathcal{F}) \end{array}$$

Proof omitted. Hint: Work through the proof of Sites, Lemma 7.18.2 and observe the compatibility from the explicit description of the horizontal and vertical maps in the diagram.

We omit the verification that this is compatible with vertical and horizontal compositions. Hint: The proof of this for t^p is straightforward and one can conclude that this holds for the ${}_pt$ maps using compatibility with adjointness. □

69.4. Sheaves

We first make an observation that is important and trivial (especially for those readers who do not worry about set theoretical issues).

Consider a big fppf site Sch_{fppf} as in Topologies, Definition 33.7.6 and denote its underlying category Sch_α. Besides being the underlying category of a fppf site, the category Sch_α can also can serve as the underlying category for a big Zariski site, a big étale site, a big smooth site, and a big syntomic site, see Topologies, Remark 33.9.1. We denote these sites Sch_{Zar}, $Sch_{étale}$, Sch_{smooth}, and $Sch_{syntomic}$. In this situation, since we have defined the big Zariski site $(Sch/S)_{Zar}$ of S, the big étale site $(Sch/S)_{étale}$ of S, the big smooth site $(Sch/S)_{smooth}$ of S, the big syntomic site $(Sch/S)_{syntomic}$ of S, and the big fppf site $(Sch/S)_{fppf}$ of S as the localizations (see Sites, Section 7.24) Sch_{Zar}/S, $Sch_{étale}/S$, Sch_{smooth}/S, $Sch_{syntomic}/S$, and Sch_{fppf}/S of these (absolute) big sites we see that all of these have the same underlying category, namely Sch_α/S.

It follows that if we have a category $p : \mathcal{X} \to (Sch/S)_{fppf}$ fibred in groupoids, then $\mathcal{X}$ inherits a Zariski, étale, smooth, syntomic, and fppf topology, see Stacks, Definition 8.10.2.

Definition 69.4.1. Let $\mathcal{X}$ be a category fibred in groupoids over $(Sch/S)_{fppf}$.

1. The *associated Zariski site*, denoted $\mathcal{X}_{Zar}$, is the structure of site on $\mathcal{X}$ inherited from $(Sch/S)_{Zar}$.
2. The *associated étale site*, denoted $\mathcal{X}_{étale}$, is the structure of site on $\mathcal{X}$ inherited from $(Sch/S)_{étale}$.
3. The *associated smooth site*, denoted $\mathcal{X}_{smooth}$, is the structure of site on $\mathcal{X}$ inherited from $(Sch/S)_{smooth}$.
4. The *associated syntomic site*, denoted $\mathcal{X}_{syntomic}$, is the structure of site on $\mathcal{X}$ inherited from $(Sch/S)_{syntomic}$.
5. The *associated fppf site*, denoted $\mathcal{X}_{fppf}$, is the structure of site on $\mathcal{X}$ inherited from $(Sch/S)_{fppf}$.

This definition makes sense by the discussion above. If $\mathcal{X}$ is an algebraic stack, the literature calls $\mathcal{X}_{fppf}$ (or a site equivalent to it) the *big fppf site* of $\mathcal{X}$ and similarly for the other ones. We may occasionally use this terminology to distinguish this construction from others.

Remark 69.4.2. We only use this notation when the symbol $\mathcal{X}$ refers to a category fibred in groupoids, and not a scheme, an algebraic space, etc. In this way we will avoid confusion with the small étale site of a scheme, or algebraic space which is denoted $X_{\acute{e}tale}$ (in which case we use a roman capital instead of a calligraphic one).

Now that we have these topologies defined we can say what it means to have a sheaf on $\mathcal{X}$, i.e., define the corresponding topoi.

Definition 69.4.3. Let $\mathcal{X}$ be a category fibred in groupoids over $(Sch/S)_{fppf}$. Let $\mathcal{F}$ be a presheaf on $\mathcal{X}$.

(1) We say $\mathcal{F}$ is a *Zariski sheaf*, or a *sheaf for the Zariski topology* if $\mathcal{F}$ is a sheaf on the associated Zariski site $\mathcal{X}_{Zar}$.
(2) We say $\mathcal{F}$ is an *étale sheaf*, or a *sheaf for the étale topology* if $\mathcal{F}$ is a sheaf on the associated étale site $\mathcal{X}_{\acute{e}tale}$.
(3) We say $\mathcal{F}$ is a *smooth sheaf*, or a *sheaf for the smooth topology* if $\mathcal{F}$ is a sheaf on the associated smooth site $\mathcal{X}_{smooth}$.
(4) We say $\mathcal{F}$ is a *syntomic sheaf*, or a *sheaf for the syntomic topology* if $\mathcal{F}$ is a sheaf on the associated syntomic site $\mathcal{X}_{syntomic}$.
(5) We say $\mathcal{F}$ is an *fppf sheaf*, or a *sheaf*, or a *sheaf for the fppf topology* if $\mathcal{F}$ is a sheaf on the associated fppf site $\mathcal{X}_{fppf}$.

A morphism of sheaves is just a morphism of presheaves. We denote these categories of sheaves $Sh(\mathcal{X}_{Zar})$, $Sh(\mathcal{X}_{\acute{e}tale})$, $Sh(\mathcal{X}_{smooth})$, $Sh(\mathcal{X}_{syntomic})$, and $Sh(\mathcal{X}_{fppf})$.

Of course we can also talk about sheaves of pointed sets, abelian groups, groups, monoids, rings, modules over a fixed ring, and lie algebras over a fixed field, etc. The category of *abelian sheaves*, i.e., sheaves of abelian groups, is denoted $Ab(\mathcal{X}_{fppf})$ and similarly for the other topologies. If $\mathcal{X}$ is an algebraic stack, then $Sh(\mathcal{X}_{fppf})$ is equivalent (modulo set theoretical problems) to what in the literature would be termed the *category of sheaves on the big fppf site of* $\mathcal{X}$. Similar for other topologies. We may occasionally use this terminology to distinguish this construction from others.

Since the topologies are listed in increasing order of strength we have the following strictly full inclusions

$$Sh(\mathcal{X}_{fppf}) \subset Sh(\mathcal{X}_{syntomic}) \subset Sh(\mathcal{X}_{smooth}) \subset Sh(\mathcal{X}_{\acute{e}tale}) \subset Sh(\mathcal{X}_{Zar}) \subset PSh(\mathcal{X})$$

We sometimes write $Sh(\mathcal{X}_{fppf}) = Sh(\mathcal{X})$ and $Ab(\mathcal{X}_{fppf}) = Ab(\mathcal{X})$ in accordance with our terminology that a sheaf on $\mathcal{X}$ is an fppf sheaf on $\mathcal{X}$.

With this setup functoriality of these topoi is straightforward, and moreover, is compatible with the inclusion functors above.

Lemma 69.4.4. *Let $f : \mathcal{X} \to \mathcal{Y}$ be a 1-morphism of categories fibred in groupoids over $(Sch/S)_{fppf}$. Let $\tau \in \{Zar, \acute{e}tale, smooth, syntomic, fppf\}$. The functors ${}_pf$ and f^p of (69.3.1.1) transform τ sheaves into τ sheaves and define a morphism of topoi $f : Sh(\mathcal{X}_\tau) \to Sh(\mathcal{Y}_\tau)$.*

Proof. This follows immediately from Stacks, Lemma 8.10.3. □

In other words, pushforward and pullback of presheaves as defined in Section 69.3 also produces *pushforward* and *pullback* of τ-sheaves. Having said all of the above we see that we can write $f^p = f^{-1}$ and ${}_pf = f_*$ without any possibility of confusion.

Definition 69.4.5. Let $f : \mathcal{X} \to \mathcal{Y}$ be a morphism of categories fibred in groupoids over $(Sch/S)_{fppf}$. We denote

$$f = (f^{-1}, f_*) : Sh(\mathcal{X}_{fppf}) \longrightarrow Sh(\mathcal{Y}_{fppf})$$

the *associated morphism of fppf topoi* constructed above. Similarly for the associated Zariski, étale, smooth, and syntomic topoi.

As discussed in Sites, Section 7.43 the same formula (on the underlying sheaf of sets) defines pushforward and pullback for sheaves (for one of our topologies) of pointed sets, abelian groups, groups, monoids, rings, modules over a fixed ring, and lie algebras over a fixed field, etc.

69.5. Computing pushforward

Let $f : \mathcal{X} \to \mathcal{Y}$ be a 1-morphism of categories fibred in groupoids over $(Sch/S)_{fppf}$. Let $\mathcal{F}$ be a presheaf on $\mathcal{X}$. Let $y \in \mathrm{Ob}(\mathcal{Y})$. We can compute $f_*\mathcal{F}(y)$ in the following way. Suppose that y lies over the scheme V and using the 2-Yoneda lemma think of y as a 1-morphism. Consider the projection

$$\mathrm{pr} : (Sch/V)_{fppf} \times_{y,\mathcal{Y}} \mathcal{X} \longrightarrow \mathcal{X}$$

Then we have a canonical identification

$$f_*\mathcal{F}(y) = \Gamma\Big((Sch/V)_{fppf} \times_{y,\mathcal{Y}} \mathcal{X},\ \mathrm{pr}^{-1}\mathcal{F}\Big) \tag{69.5.0.1}$$

Namely, objects of the 2-fibre product are triples $(h : U \to V, x, f(x) \to h^*y)$. Dropping the h from the notation we see that this is equivalent to the data of an object x of $\mathcal{X}$ and a morphism $\alpha : f(x) \to y$ of $\mathcal{Y}$. Since $f_*\mathcal{F}(y) = \lim_{f(x)\to y} \mathcal{F}(x)$ by definition the equality follows.

As a consequence we have the following "base change" result for pushforwards. This result is trivial and hinges on the fact that we are using "big" sites.

Lemma 69.5.1. *Let S be a scheme. Let*

$$\begin{array}{ccc} \mathcal{Y}' \times_{\mathcal{Y}} \mathcal{X} & \xrightarrow{g'} & \mathcal{X} \\ {\scriptstyle f'}\downarrow & & \downarrow{\scriptstyle f} \\ \mathcal{Y}' & \xrightarrow{g} & \mathcal{Y} \end{array}$$

be a 2-cartesian diagram of categories fibred in groupoids over S. Then we have a canonical isomorphism

$$g^{-1}f_*\mathcal{F} \longrightarrow f'_*(g')^{-1}\mathcal{F}$$

functorial in the presheaf $\mathcal{F}$ on $\mathcal{X}$.

Proof. Given an object y' of $\mathcal{Y}'$ over V there is an equivalence

$$(Sch/V)_{fppf} \times_{g(y'),\mathcal{Y}} \mathcal{X} = (Sch/V)_{fppf} \times_{y',\mathcal{Y}'} (\mathcal{Y}' \times_{\mathcal{Y}} \mathcal{X})$$

Hence by (69.5.0.1) a bijection $g^{-1}f_*\mathcal{F}(y') \to f'_*(g')^{-1}\mathcal{F}(y')$. We omit the verification that this is compatible with restriction mappings. □

In the case of a representable morphism of categories fibred in groupoids this formula (69.5.0.1) simplifies. We suggest the reader skip the rest of this section.

Lemma 69.5.2. *Let $f : \mathcal{X} \to \mathcal{Y}$ be a 1-morphism of categories fibred in groupoids over $(Sch/S)_{fppf}$. The following are equivalent*

(1) *f is representable, and*
(2) *for every $y \in \mathrm{Ob}(\mathcal{Y})$ the functor $\mathcal{X}^{opp} \to Sets$, $x \mapsto \mathrm{Mor}_{\mathcal{Y}}(f(x), y)$ is representable.*

Proof. According to the discussion in Algebraic Stacks, Section 67.6 we see that f is representable if and only if for every $y \in \mathrm{Ob}(\mathcal{Y})$ lying over U the 2-fibre product $(Sch/U)_{fppf} \times_{y,\mathcal{Y}} \mathcal{X}$ is representable, i.e., of the form $(Sch/V_y)_{fppf}$ for some scheme V_y over U. Objects in this 2-fibre products are triples $(h : V \to U, x, \alpha : f(x) \to h^*y)$ where α lies over id_V. Dropping the h from the notation we see that this is equivalent to the data of an object x of $\mathcal{X}$ and a morphism $f(x) \to y$. Hence the 2-fibre product is representable by V_y and $f(x_y) \to y$ where x_y is an object of $\mathcal{X}$ over V_y if and only if the functor in (2) is representable by x_y with universal object a map $f(x_y) \to y$. □

Let

$$\begin{array}{ccc} \mathcal{X} & \xrightarrow{\quad f \quad} & \mathcal{Y} \\ & {\scriptstyle p} \searrow \quad \swarrow {\scriptstyle q} & \\ & (Sch/S)_{fppf} & \end{array}$$

be a 1-morphism of categories fibred in groupoids. Assume f is representable. For every $y \in \mathrm{Ob}(\mathcal{Y})$ we choose an object $u(y) \in \mathrm{Ob}(\mathcal{X})$ representing the functor $x \mapsto \mathrm{Mor}_{\mathcal{Y}}(f(x), y)$ of Lemma 69.5.2 (this is possible by the axiom of choice). The objects come with canonical morphisms $f(u(y)) \to y$ by construction. For every morphism $\beta : y' \to y$ in $\mathcal{Y}$ we obtain a unique morphism $u(\beta) : u(y') \to u(y)$ in $\mathcal{X}$ such that the diagram

$$\begin{array}{ccc} f(u(y')) & \xrightarrow{f(u(\beta))} & f(u(y)) \\ \downarrow & & \downarrow \\ y' & \longrightarrow & y \end{array}$$

commutes. In other words, $u : \mathcal{Y} \to \mathcal{X}$ is a functor. In fact, we can say a little bit more. Namely, suppose that $V' = q(y')$, $V = q(y)$, $U' = p(u(y'))$ and $U = p(u(y))$. Then

$$\begin{array}{ccc} U' & \xrightarrow{p(u(\beta))} & U \\ \downarrow & & \downarrow \\ V' & \xrightarrow{q(\beta)} & V \end{array}$$

is a fibre product square. This is true because $U' \to U$ represents the base change $(Sch/V')_{fppf} \times_{y',\mathcal{Y}} \mathcal{X} \to (Sch/V)_{fppf} \times_{y,\mathcal{Y}} \mathcal{X}$ of $V' \to V$.

Lemma 69.5.3. *Let $f : \mathcal{X} \to \mathcal{Y}$ be a representable 1-morphism of categories fibred in groupoids over $(Sch/S)_{fppf}$. Let $\tau \in \{Zar, étale, smooth, syntomic, fppf\}$. Then the functor $u : \mathcal{Y}_\tau \to \mathcal{X}_\tau$ is continuous and defines a morphism of sites $\mathcal{X}_\tau \to \mathcal{Y}_\tau$ which induces the same morphism of topoi $Sh(\mathcal{X}_\tau) \to Sh(\mathcal{Y}_\tau)$ as the*

morphism f constructed in Lemma 69.4.4. Moreover, $f_*\mathcal{F}(y) = \mathcal{F}(u(y))$ for any presheaf $\mathcal{F}$ on $\mathcal{X}$.

Proof. Let $\{y_i \to y\}$ be a τ-covering in $\mathcal{Y}$. By definition this simply means that $\{q(y_i) \to q(y)\}$ is a τ-covering of schemes. By the final remark above the lemma we see that $\{p(u(y_i)) \to p(u(y))\}$ is the base change of the τ-covering $\{q(y_i) \to q(y)\}$ by $p(u(y)) \to q(y)$, hence is itself a τ-covering by the axioms of a site. Hence $\{u(y_i) \to u(y)\}$ is a τ-covering of $\mathcal{X}$. This proves that u is continuous.

Let's use the notation u_p, u_s, u^p, u^s of Sites, Sections 7.5 and 7.14. If we can show the final assertion of the lemma, then we see that $f_* = u^p = u^s$ (by continuity of u seen above) and hence by adjointness $f^{-1} = u_s$ which will prove u_s is exact, hence that u determines a morphism of sites, and the equality will be clear as well. To see that $f_*\mathcal{F}(y) = \mathcal{F}(u(y))$ note that by definition

$$f_*\mathcal{F}(y) = ({}_pf\mathcal{F})(y) = \lim\nolimits_{f(x)\to y} \mathcal{F}(x).$$

Since $u(y)$ is a final object in the category the limit is taken over we conclude. □

69.6. The structure sheaf

Let $\tau \in \{Zar, étale, smooth, syntomic, fppf\}$. Let $p : \mathcal{X} \to (Sch/S)_{fppf}$ be a category fibred in groupoids. The 2-category of categories fibred in groupoids over $(Sch/S)_{fppf}$ has a final object, namely, $\mathrm{id} : (Sch/S)_{fppf} \to (Sch/S)_{fppf}$ and p is a 1-morphism from $\mathcal{X}$ to this final object. Hence any presheaf $\mathcal{G}$ on $(Sch/S)_{fppf}$ gives a presheaf $p^{-1}\mathcal{G}$ on $\mathcal{X}$ defined by the rule $p^{-1}\mathcal{G}(x) = \mathcal{G}(p(x))$. Moreover, the discussion in Section 69.4 shows that $p^{-1}\mathcal{G}$ is a τ sheaf whenever $\mathcal{G}$ is a τ-sheaf.

Recall that the site $(Sch/S)_{fppf}$ is a ringed site with structure sheaf $\mathcal{O}$ defined by the rule

$$(Sch/S)^{opp} \longrightarrow Rings, \quad U/S \longmapsto \Gamma(U, \mathcal{O}_U)$$

see Descent, Definition 34.7.2.

Definition 69.6.1. Let $p : \mathcal{X} \to (Sch/S)_{fppf}$ be a category fibred in groupoids. The *structure sheaf of* $\mathcal{X}$ is the sheaf of rings $\mathcal{O}_\mathcal{X} = p^{-1}\mathcal{O}$.

For an object x of $\mathcal{X}$ lying over U we have $\mathcal{O}_\mathcal{X}(x) = \mathcal{O}(U) = \Gamma(U, \mathcal{O}_U)$. Needless to say $\mathcal{O}_\mathcal{X}$ is also a Zariski, étale, smooth, and syntomic sheaf, and hence each of the sites $\mathcal{X}_{Zar}$, $\mathcal{X}_{étale}$, $\mathcal{X}_{smooth}$, $\mathcal{X}_{syntomic}$, and $\mathcal{X}_{fppf}$ is a ringed site. This construction is functorial as well.

Lemma 69.6.2. *Let $f : \mathcal{X} \to \mathcal{Y}$ be a 1-morphism of categories fibred in groupoids over $(Sch/S)_{fppf}$. Let $\tau \in \{Zar, étale, smooth, syntomic, fppf\}$. There is a canonical identification $f^{-1}\mathcal{O}_\mathcal{X} = \mathcal{O}_\mathcal{Y}$ which turns $f : Sh(\mathcal{X}_\tau) \to Sh(\mathcal{Y}_\tau)$ into a morphism of ringed topoi.*

Proof. Denote $p : \mathcal{X} \to (Sch/S)_{fppf}$ and $q : \mathcal{Y} \to (Sch/S)_{fppf}$ the structural functors. Then $q = p \circ f$, hence $q^{-1} = f^{-1} \circ p^{-1}$ by Lemma 69.3.2. The result follows. □

Remark 69.6.3. In the situation of Lemma 69.6.2 the morphism of ringed topoi $f : Sh(\mathcal{X}_\tau) \to Sh(\mathcal{Y}_\tau)$ is flat as is clear from the equality $f^{-1}\mathcal{O}_\mathcal{X} = \mathcal{O}_\mathcal{Y}$. This is a bit counter intuitive, for example because a closed immersion of algebraic stacks is typically not flat (as a morphism of algebraic stacks). However, exactly the same

thing happens when taking a closed immersion $i : X \to Y$ of schemes: in this case the associated morphism of big τ-sites $i : (Sch/X)_\tau \to (Sch/Y)_\tau$ also is flat.

69.7. Sheaves of modules

Since we have a structure sheaf we have modules.

Definition 69.7.1. Let $\mathcal{X}$ be a category fibred in groupoids over $(Sch/S)_{fppf}$.

(1) A *presheaf of modules on* $\mathcal{X}$ is a presheaf of $\mathcal{O}_\mathcal{X}$-modules. The category of presheaves of modules is denoted $PMod(\mathcal{O}_\mathcal{X})$.
(2) We say a presheaf of modules $\mathcal{F}$ is an *$\mathcal{O}_\mathcal{X}$-module*, or more precisely a *sheaf of $\mathcal{O}_\mathcal{X}$-modules* if $\mathcal{F}$ is an fppf sheaf. The category of $\mathcal{O}_\mathcal{X}$-modules is denoted $Mod(\mathcal{O}_\mathcal{X})$.

These (pre)sheaves of modules occur in the literature as *(pre)sheaves of $\mathcal{O}_\mathcal{X}$-modules on the big fppf site of $\mathcal{X}$*. We will occasionally use this terminology if we want to distinguish these categories from others. We will also encounter presheaves of modules which are sheaves in the Zariski, étale, smooth, or syntomic topologies (without necessarily being sheaves). If need be these will be denoted $Mod(\mathcal{X}_{étale}, \mathcal{O}_\mathcal{X})$ and similarly for the other topologies.

Next, we address functoriality – first for presheaves of modules. Let

$$\begin{array}{ccc} \mathcal{X} & \xrightarrow{f} & \mathcal{Y} \\ & {\scriptstyle p} \searrow \quad \swarrow {\scriptstyle q} & \\ & (Sch/S)_{fppf} & \end{array}$$

be a 1-morphism of categories fibred in groupoids. The functors f^{-1}, f_* on abelian presheaves extend to functors

(69.7.1.1)
$$f^{-1} : PMod(\mathcal{O}_\mathcal{Y}) \longrightarrow PMod(\mathcal{O}_\mathcal{X}) \quad \text{and} \quad f_* : PMod(\mathcal{O}_\mathcal{Y}) \longrightarrow PMod(\mathcal{O}_\mathcal{X})$$

This is immediate for f^{-1} because $f^{-1}\mathcal{G}(x) = \mathcal{G}(f(x))$ which is a module over $\mathcal{O}_\mathcal{Y}(f(x)) = \mathcal{O}(q(f(x))) = \mathcal{O}(p(x)) = \mathcal{O}_\mathcal{X}(x)$. Alternatively it follows because $f^{-1}\mathcal{O}_\mathcal{Y} = \mathcal{O}_\mathcal{X}$ and because f^{-1} commutes with limits (on presheaves). Since f_* is a right adjoint it commutes with all limits (on presheaves) in particular products. Hence we can extend f_* to a functor on presheaves of modules as in the proof of Modules on Sites, Lemma 18.12.1. We claim that the functors (69.7.1.1) form an adjoint pair of functors:

$$\mathrm{Mor}_{PMod(\mathcal{O}_\mathcal{X})}(f^{-1}\mathcal{G}, \mathcal{F}) = \mathrm{Mor}_{PMod(\mathcal{O}_\mathcal{Y})}(\mathcal{G}, f_*\mathcal{F}).$$

As $f^{-1}\mathcal{O}_\mathcal{Y} = \mathcal{O}_\mathcal{X}$ this follows from Modules on Sites, Lemma 18.12.3 by endowing $\mathcal{X}$ and $\mathcal{Y}$ with the chaotic topology.

Next, we discuss functoriality for modules, i.e., for sheaves of modules in the fppf topology. Denote by f also the induced morphism of ringed topoi, see Lemma 69.6.2 (for the fppf topologies right now). Note that the functors f^{-1} and f_* of (69.7.1.1) preserve the subcategories of sheaves of modules, see Lemma 69.4.4. Hence it follows immediately that

(69.7.1.2)
$$f^{-1} : Mod(\mathcal{O}_\mathcal{Y}) \longrightarrow Mod(\mathcal{O}_\mathcal{X}) \quad \text{and} \quad f_* : Mod(\mathcal{O}_\mathcal{Y}) \longrightarrow Mod(\mathcal{O}_\mathcal{X})$$

form an adjoint pair of functors:

$$\mathrm{Mor}_{Mod(\mathcal{O}_\mathcal{X})}(f^{-1}\mathcal{G}, \mathcal{F}) = \mathrm{Mor}_{Mod(\mathcal{O}_\mathcal{Y})}(\mathcal{G}, f_*\mathcal{F}).$$

By uniqueness of adjoints we conclude that $f^* = f^{-1}$ where f^* is as defined in Modules on Sites, Section 18.13 for the morphism of ringed topoi f above. Of course we could have seen this directly because $f^*(-) = f^{-1}(-) \otimes_{f^{-1}\mathcal{O}_\mathcal{Y}} \mathcal{O}_\mathcal{X}$ and because $f^{-1}\mathcal{O}_\mathcal{Y} = \mathcal{O}_\mathcal{X}$.

Similarly for sheaves of modules in the Zariski, étale, smooth, syntomic topology.

69.8. Representable categories

In this short section we compare our definitions with what happens in case the algebraic stacks in question are representable.

Lemma 69.8.1. *Let S be a scheme. Let $\mathcal{X}$ be a category fibred in groupoids over (Sch/S). Assume $\mathcal{X}$ is representable by a scheme X. For $\tau \in \{Zar, étale, smooth, syntomic, fppf\}$ there is a canonical equivalence*

$$(\mathcal{X}_\tau, \mathcal{O}_\mathcal{X}) = ((Sch/X)_\tau, \mathcal{O}_X)$$

of ringed sites.

Proof. This follows by choosing an equivalence $(Sch/X)_\tau \to \mathcal{X}$ of categories fibred in groupoids over $(Sch/S)_{fppf}$ and using the functoriality of the construction $\mathcal{X} \rightsquigarrow \mathcal{X}_\tau$. □

Lemma 69.8.2. *Let S be a scheme. Let $f : \mathcal{X} \to \mathcal{Y}$ be a morphism of categories fibred in groupoids over S. Assume $\mathcal{X}$, $\mathcal{Y}$ are representable by schemes X, Y. Let $f : X \to Y$ be the morphism of schemes corresponding to f. For $\tau \in \{Zar, étale, smooth, syntomic, fppf\}$ the morphism of ringed topoi $f : (Sh(\mathcal{X}_\tau), \mathcal{O}_\mathcal{X}) \to (Sh(\mathcal{X}_\tau), \mathcal{O}_\mathcal{X})$ agrees with the morphisms of ringed topoi $f : (Sh((Sch/X)_\tau), \mathcal{O}_X) \to (Sh((Sch/Y)_\tau), \mathcal{O}_Y)$ via the identifications of Lemma 69.8.1.*

Proof. Follows by unwinding the definitions. □

69.9. Restriction

A trivial but useful observation is that the localization of a category fibred in groupoids at an object is equivalent to the big site of the scheme it lies over.

Lemma 69.9.1. *Let $p : \mathcal{X} \to (Sch/S)_{fppf}$ be a category fibred in groupoids. Let $\tau \in \{Zar, étale, smooth, syntomic, fppf\}$. Let $x \in \mathrm{Ob}(\mathcal{X})$ lying over $U = p(x)$. The functor p induces an equivalence of sites $\mathcal{X}_\tau/x \to (Sch/U)_\tau$.*

Proof. Note that $(Sch/U)_\tau$ is the localization of the site $(Sch/S)_{fppf}$ at the object U. It follows from Categories, Definition 4.33.1 that the rule $x'/x \mapsto p(x')/p(x)$ defines an equivalence of categories $\mathcal{X}_\tau/x \to (Sch/U)_\tau$. Whereupon it follows from Stacks, Definition 8.10.2 that coverings of x' in $\mathcal{X}_\tau/x$ are in bijective correspondence with coverings of $p(x')$ in $(Sch/U)_\tau$. □

We use the lemma above to talk about the pullback and the restriction of a (pre)sheaf to a scheme.

Definition 69.9.2. Let $p : \mathcal{X} \to (Sch/S)_{fppf}$ be a category fibred in groupoids. Let $x \in \mathrm{Ob}(\mathcal{X})$ lying over $U = p(x)$. Let $\mathcal{F}$ be a presheaf on $\mathcal{X}$.

(1) The *pullback* $x^{-1}\mathcal{F}$ *of* $\mathcal{F}$ is the restriction $\mathcal{F}|_{(\mathcal{X}/x)}$ viewed as a presheaf on $(Sch/U)_{fppf}$ via the equivalence $\mathcal{X}/x \to (Sch/U)_{fppf}$ of Lemma 69.9.1.
(2) The *restriction of* $\mathcal{F}$ *to* $U_{étale}$ is $x^{-1}\mathcal{F}|_{U_{étale}}$, abusively written $\mathcal{F}|_{U_{étale}}$.

This notation makes sense because to the object x the 2-Yoneda lemma, see Algebraic Stacks, Section 67.5 associates a 1-morphism $x : (Sch/U)_{fppf} \to \mathcal{X}/x$ which is quasi-inverse to $p : \mathcal{X}/x \to (Sch/U)_{fppf}$. Hence $x^{-1}\mathcal{F}$ truly is the pullback of $\mathcal{F}$ via this 1-morphism. In particular, by the material above, if $\mathcal{F}$ is a sheaf (or a Zariski, étale, smooth, syntomic sheaf), then $x^{-1}\mathcal{F}$ is a sheaf on $(Sch/U)_{fppf}$ (or on $(Sch/U)_{Zar}$, $(Sch/U)_{étale}$, $(Sch/U)_{smooth}$, $(Sch/U)_{syntomic}$).

Let $p : \mathcal{X} \to (Sch/S)_{fppf}$ be a category fibred in groupoids. Let $\varphi : x \to y$ be a morphism of $\mathcal{X}$ lying over the morphism of schemes $a : U \to V$. Recall that a induces a morphism of small étale sites $a_{small} : U_{étale} \to V_{étale}$, see Étale Cohomology, Section 44.34. Let $\mathcal{F}$ be a presheaf on $\mathcal{X}$. Let $\mathcal{F}|_{U_{étale}}$ and $\mathcal{F}|_{V_{étale}}$ be the restrictions of $\mathcal{F}$ via x and y. There is a natural *comparison* map

$$c_\varphi : \mathcal{F}|_{V_{étale}} \longrightarrow a_{small,*}(\mathcal{F}|_{U_{étale}}) \tag{69.9.2.1}$$

of presheaves on $U_{étale}$. Namely, if $V' \to V$ is étale, set $U' = V' \times_V U$ and define c_φ on sections over V' via

$$\begin{array}{ccccc} a_{small,*}(\mathcal{F}|_{U_{étale}})(V') & = & \mathcal{F}|_{U_{étale}}(U') & = & \mathcal{F}(x') \\ {\scriptstyle c_\varphi}\uparrow & & & & \uparrow{\scriptstyle \mathcal{F}(\varphi')} \\ \mathcal{F}|_{V_{étale}}(V') & = & = & = & \mathcal{F}(y') \end{array}$$

Here $\varphi' : x' \to y'$ is a morphism of $\mathcal{X}$ fitting into a commutative diagram

$$\begin{array}{ccc} x' & \longrightarrow & x \\ {\scriptstyle \varphi'}\downarrow & & \downarrow{\scriptstyle \varphi} \\ y' & \longrightarrow & y \end{array} \quad \text{lying over} \quad \begin{array}{ccc} U' & \longrightarrow & U \\ \downarrow & & \downarrow{\scriptstyle a} \\ V' & \longrightarrow & V \end{array}$$

The existence and uniqueness of φ' follow from the axioms of a category fibred in groupoids. We omit the verification that c_φ so defined is indeed a map of presheaves (i.e., compatible with restriction mappings) and that it is functorial in $\mathcal{F}$. In case $\mathcal{F}$ is a sheaf for the étale topology we obtain a *comparison* map

$$c_\varphi : a_{small}^{-1}(\mathcal{F}|_{V_{étale}}) \longrightarrow \mathcal{F}|_{U_{étale}} \tag{69.9.2.2}$$

which is also denoted c_φ as indicated (this is the customary abuse of notation in not distinguishing between adjoint maps).

Lemma 69.9.3. *Let $\mathcal{F}$ be an étale sheaf on $\mathcal{X} \to (Sch/S)_{fppf}$.*

(1) *If $\varphi : x \to y$ and $\psi : y \to z$ are morphisms of $\mathcal{X}$ lying over $a : U \to V$ and $b : V \to W$, then the composition*

$$a_{small}^{-1}(b_{small}^{-1}(\mathcal{F}|_{W_{étale}})) \xrightarrow{a_{small}^{-1}c_\psi} a_{small}^{-1}(\mathcal{F}|_{V_{étale}}) \xrightarrow{c_\varphi} \mathcal{F}|_{U_{étale}}$$

is equal to $c_{\psi\circ\varphi}$ via the identification

$$(b \circ a)_{small}^{-1}(\mathcal{F}|_{W_{étale}}) = a_{small}^{-1}(b_{small}^{-1}(\mathcal{F}|_{W_{étale}})).$$

(2) *If $\varphi : x \to y$ lies over an étale morphism of schemes $a : U \to V$, then (69.9.2.2) is an isomorphism.*

(3) *Suppose $f : \mathcal{Y} \to \mathcal{X}$ is a 1-morphism of categories fibred in groupoids over $(Sch/S)_{fppf}$ and y is an object of $\mathcal{Y}$ lying over the scheme U with image $x = f(y)$. Then there is a canonical identification $f^{-1}\mathcal{F}|_{U_{étale}} = \mathcal{F}|_{U_{étale}}$.*
(4) *Moreover, given $\psi : y' \to y$ in $\mathcal{Y}$ lying over $a : U' \to U$ the comparison map $c_\psi : a_{small}^{-1}(F^{-1}\mathcal{F}|_{U_{étale}}) \to F^{-1}\mathcal{F}|_{U'_{étale}}$ is equal to the comparison map $c_{f(\psi)} : a_{small}^{-1}\mathcal{F}|_{U_{étale}} \to \mathcal{F}|_{U'_{étale}}$ via the identifications in (3).*

Proof. The verification of these properties is omitted. □

Next, we turn to the restriction of (pre)sheaves of modules.

Lemma 69.9.4. *Let $p : \mathcal{X} \to (Sch/S)_{fppf}$ be a category fibred in groupoids. Let $\tau \in \{Zar, étale, smooth, syntomic, fppf\}$. Let $x \in \mathrm{Ob}(\mathcal{X})$ lying over $U = p(x)$. The equivalence of Lemma 69.9.1 extends to an equivalence of ringed sites $(\mathcal{X}_\tau/x, \mathcal{O}_\mathcal{X}|_x) \to ((Sch/U)_\tau, \mathcal{O})$.*

Proof. This is immediate from the construction of the structure sheaves. □

Let $\mathcal{X}$ be a category fibred in groupoids over $(Sch/S)_{fppf}$. Let $\mathcal{F}$ be a (pre)sheaf of modules on $\mathcal{X}$ as in Definition 69.7.1. Let x be an object of $\mathcal{X}$ lying over U. Then Lemma 69.9.4 guarantees that the restriction $x^{-1}\mathcal{F}$ is a (pre)sheaf of modules on $(Sch/U)_{fppf}$. We will sometimes write $x^*\mathcal{F} = x^{-1}\mathcal{F}$ in this case. Similarly, if $\mathcal{F}$ is a sheaf for the Zariski, étale, smooth, or syntomic topology, then $x^{-1}\mathcal{F}$ is as well. Moreover, the restriction $\mathcal{F}|_{U_{étale}} = x^{-1}\mathcal{F}|_{U_{étale}}$ to U is a presheaf of $\mathcal{O}_{U_{étale}}$-modules. If $\mathcal{F}$ is a sheaf for the étale topology, then $\mathcal{F}|_{U_{étale}}$ is a sheaf of modules. Moreover, if $\varphi : x \to y$ is a morphism of $\mathcal{X}$ lying over $a : U \to V$ then the comparison map (69.9.2.2) is compatible with $a_{small}^\sharp$ (see Descent, Remark 34.7.4) and induces a *comparison* map

$$c_\varphi : a_{small}^*(\mathcal{F}|_{V_{étale}}) \longrightarrow \mathcal{F}|_{U_{étale}} \tag{69.9.4.1}$$

of $\mathcal{O}_{U_{étale}}$-modules. Note that the properties (1), (2), (3), and (4) of Lemma 69.9.3 hold in the setting of étale sheaves of modules as well. We will use this in the following without further mention.

Lemma 69.9.5. *Let $p : \mathcal{X} \to (Sch/S)_{fppf}$ be a category fibred in groupoids. Let $\tau \in \{Zar, étale, smooth, syntomic, fppf\}$. The site $\mathcal{X}_\tau$ has enough points.*

Proof. By Sites, Lemma 7.37.5 we have to show that there exists a family of objects x of $\mathcal{X}$ such that $\mathcal{X}_\tau/x$ has enough points and such that the sheaves $h_x^\#$ cover the final object of the category of sheaves. By Lemma 69.9.1 and Étale Cohomology, Lemma 44.30.1 we see that $\mathcal{X}_\tau/x$ has enough points for every object x and we win. □

69.10. Restriction to algebraic spaces

In this section we consider sheaves on categories representable by algebraic spaces. The following lemma is the analogue of Topologies, Lemma 33.4.13 for algebraic spaces.

Lemma 69.10.1. *Let S be a scheme. Let $\mathcal{X} \to (Sch/S)_{fppf}$ be a category fibred in groupoids. Assume $\mathcal{X}$ is representably by an algebraic space F. Then there exists a continuous and cocontinuous functor $F_{étale} \to \mathcal{X}_{étale}$ which induces a morphism of ringed sites*

$$\pi_F : (\mathcal{X}_{étale}, \mathcal{O}_\mathcal{X}) \longrightarrow (F_{étale}, \mathcal{O}_F)$$

and a morphism of ringed topoi

$$i_F : (Sh(F_{étale}), \mathcal{O}_F) \longrightarrow (Sh(\mathcal{X}_{étale}), \mathcal{O}_{\mathcal{X}})$$

such that $\pi_F \circ i_F = id$. *Moreover* $\pi_{F,*} = i_F^{-1}$.

Proof. Choose an equivalence $j : \mathcal{S}_F \to \mathcal{X}$, see Algebraic Stacks, Sections 67.7 and 67.8. An object of $F_{étale}$ is a scheme U together with an étale morphism $\varphi : U \to F$. Then φ is an object of $\mathcal{S}_F$ over U. Hence $j(\varphi)$ is an object of $\mathcal{X}$ over U. In this way j induces a functor $u : F_{étale} \to \mathcal{X}$. It is clear that u is continuous and cocontinuous for the étale topology on $\mathcal{X}$. Since j is an equivalence, the functor u is fully faithful. Also, fibre products and equalizers exist in $F_{étale}$ and u commutes with them because these are computed on the level of underlying schemes in $F_{étale}$. Thus Sites, Lemmas 7.20.5, 7.20.6, and 7.20.7 apply. In particular u defines a morphism of topoi $i_F : Sh(F_{étale}) \to Sh(\mathcal{X}_{étale})$ and there exists a left adjoint $i_{F,!}$ of i_F^{-1} which commutes with fibre products and equalizers.

We claim that $i_{F,!}$ is exact. If this is true, then we can define π_F by the rules $\pi_F^{-1} = i_{F,!}$ and $\pi_{F,*} = i_F^{-1}$ and everything is clear. To prove the claim, note that we already know that $i_{F,!}$ is right exact and preserves fibre products. Hence it suffices to show that $i_{F,!}* = *$ where $*$ indicates the final object in the category of sheaves of sets. Let U be a scheme and let $\varphi : U \to F$ be surjective and étale. Set $R = U \times_F U$. Then

$$h_R \rightrightarrows h_U \longrightarrow *$$

is a coequalizer diagram in $Sh(F_{étale})$. Using the right exactness of $i_{F,!}$, using $i_{F,!} = (u_p)^\#$, and using Sites, Lemma 7.5.6 we see that

$$h_{u(R)} \rightrightarrows h_{u(U)} \longrightarrow i_{F,!}*$$

is a coequalizer diagram in $Sh(F_{étale})$. Using that j is an equivalence and that $F = U/R$ it follows that the coequalizer in $Sh(\mathcal{X}_{étale})$ of the two maps $h_{u(R)} \to h_{u(U)}$ is $*$. We omit the proof that these morphisms are compatible with structure sheaves. □

Assume $\mathcal{X}$ is an algebraic stack represented by the algebraic space F. Let $j : \mathcal{S}_F \to \mathcal{X}$ be an equivalence and denote $u : F_{étale} \to \mathcal{X}_{étale}$ the functor of the proof of Lemma 69.10.1 above. Given a sheaf $\mathcal{F}$ on $\mathcal{X}_{étale}$ we have

$$\pi_{F,*}\mathcal{F}(U) = i_F^{-1}\mathcal{F}(U) = \mathcal{F}(u(U)).$$

This is why we often think of i_F^{-1} as a *restriction functor* similarly to Definition 69.9.2 and to the restriction of a sheaf on the big étale site of a scheme to the small étale site of a scheme. We often use the notation

$$\mathcal{F}|_{F_{étale}} = i_F^{-1}\mathcal{F} = \pi_{F,*}\mathcal{F} \tag{69.10.1.1}$$

in this situation.

Lemma 69.10.2. *Let S be a scheme. Let $f : \mathcal{X} \to \mathcal{Y}$ be a morphism of categories fibred in groupoids over $(Sch/S)_{fppf}$. Assume $\mathcal{X}$, $\mathcal{Y}$ are representable by algebraic spaces F, G. Denote $f : F \to G$ the induced morphism of algebraic spaces, and*

$f_{small} : F_{étale} \to G_{étale}$ the corresponding morphism of ringed topoi. Then

$$\begin{array}{ccc} (Sh(\mathcal{X}_{étale}), \mathcal{O}_\mathcal{X}) & \xrightarrow{f} & (Sh(\mathcal{Y}_{étale}), \mathcal{O}_\mathcal{Y}) \\ \downarrow{\scriptstyle \pi_F} & & \downarrow{\scriptstyle \pi_G} \\ (Sh(F_{étale}), \mathcal{O}_F) & \xrightarrow{f_{small}} & (Sh(G_{étale}), \mathcal{O}_G) \end{array}$$

is a commutative diagram of ringed topoi.

Proof. This is similar to Topologies, Lemma 33.4.16 (3) but there is a small snag due to the fact that $F \to G$ may not be representable by schemes. In particular we don't get a commutative diagram of ringed sites, but only a commutative diagram of ringed topoi.

Before we start the proof proper, we choose equivalences $j : \mathcal{S}_F \to \mathcal{X}$ and $j' : \mathcal{S}_G \to \mathcal{Y}$ which induce functors $u : F_{étale} \to \mathcal{X}$ and $u' : G_{étale} \to \mathcal{Y}$ as in the proof of Lemma 69.10.1. Because of the 2-functoriality of sheaves on categories fibred in groupoids over Sch_{fppf} (see discussion in Section 69.3) we may assume that $\mathcal{X} = \mathcal{S}_F$ and $\mathcal{Y} = \mathcal{S}_G$ and that $f : \mathcal{S}_F \to \mathcal{S}_G$ is the functor associated to the morphism $f : F \to G$. Correspondingly we will omit u and u' from the notation, i.e., given an object $U \to F$ of $F_{étale}$ we denote U/F the corresponding object of $\mathcal{X}$. Similarly for G.

Let $\mathcal{G}$ be a sheaf on $\mathcal{X}_{étale}$. To prove (2) we compute $\pi_{G,*}f_*\mathcal{G}$ and $f_{small,*}\pi_{F,*}\mathcal{G}$. To do this let $V \to G$ be an object of $G_{étale}$. Then

$$\pi_{G,*}f_*\mathcal{G}(V) = f_*\mathcal{G}(V/G) = \Gamma\Big((Sch/V)_{fppf} \times_\mathcal{Y} \mathcal{X},\ \mathrm{pr}^{-1}\mathcal{G}\Big)$$

see (69.5.0.1). The fibre product in the formula is

$$(Sch/V)_{fppf} \times_\mathcal{Y} \mathcal{X} = (Sch/V)_{fppf} \times_{\mathcal{S}_G} \mathcal{S}_F = \mathcal{S}_{V \times_G F}$$

i.e., it is the split category fibred in groupoids associated to the algebraic space $V \times_G F$. And $\mathrm{pr}^{-1}\mathcal{G}$ is a sheaf on $\mathcal{S}_{V \times_G F}$ for the étale topology.

In particular, if $V \times_G F$ is representable, i.e., if it is a scheme, then $\pi_{G,*}f_*\mathcal{G}(V) = \mathcal{G}(V \times_G F/F)$ and also

$$f_{small,*}\pi_{F,*}\mathcal{G}(V) = \pi_{F,*}\mathcal{G}(V \times_G F) = \mathcal{G}(V \times_G F/F)$$

which proves the desired equality in this special case.

In general, choose a scheme U and a surjective étale morphism $U \to V \times_G F$. Set $R = U \times_{V \times_G F} U$. Then $U/V \times_G F$ and $R/V \times_G F$ are objects of the fibre product category above. Since $\mathrm{pr}^{-1}\mathcal{G}$ is a sheaf for the étale topology on $\mathcal{S}_{V \times_G F}$ the diagram

$$\Gamma\Big((Sch/V)_{fppf} \times_\mathcal{Y} \mathcal{X},\ \mathrm{pr}^{-1}\mathcal{G}\Big) \longrightarrow \mathrm{pr}^{-1}\mathcal{G}(U/V \times_G F) \rightrightarrows \mathrm{pr}^{-1}\mathcal{G}(R/V \times_G F)$$

is an equalizer diagram. Note that $\mathrm{pr}^{-1}\mathcal{G}(U/V \times_G F) = \mathcal{G}(U/F)$ and $\mathrm{pr}^{-1}\mathcal{G}(R/V \times_G F) = \mathcal{G}(R/F)$ by the definition of pullbacks. Moreover, by the material in Properties of Spaces, Section 48.15 (especially, Properties of Spaces, Remark 48.15.4 and Lemma 48.15.7) we see that there is an equalizer diagram

$$f_{small,*}\pi_{F,*}\mathcal{G}(V) \longrightarrow \pi_{F,*}\mathcal{G}(U/F) \rightrightarrows \pi_{F,*}\mathcal{G}(R/F)$$

Since we also have $\pi_{F,*}\mathcal{G}(U/F) = \mathcal{G}(U/F)$ and $\pi_{F,*}\mathcal{G}(U/F) = \mathcal{G}(U/F)$ we obtain a canonical identification $f_{small,*}\pi_{F,*}\mathcal{G}(V) = \pi_{G,*}f_*\mathcal{G}(V)$. We omit the proof that this is compatible with restriction mappings and that it is functorial in $\mathcal{G}$. □

Let $f : \mathcal{X} \to \mathcal{Y}$ and $f : F \to G$ be as in the second part of the lemma above. A consequence of the lemma, using (69.10.1.1), is that

$$(f_*\mathcal{F})|_{G_{étale}} = f_{small,*}(\mathcal{F}|_{F_{étale}}) \tag{69.10.2.1}$$

for any sheaf $\mathcal{F}$ on $\mathcal{X}_{étale}$. Moreover, if $\mathcal{F}$ is a sheaf of $\mathcal{O}$-modules, then (69.10.2.1) is an isomorphism of $\mathcal{O}_G$-modules on $G_{étale}$.

Finally, suppose that we have a 2-commutative diagram

$$\begin{array}{ccc} \mathcal{U} & \xrightarrow{a} & \mathcal{V} \\ & \swarrow^{f} \quad \swarrow\!\!\swarrow_{\varphi} & \downarrow g \\ & & \mathcal{X} \end{array}$$

of 1-morphisms of categories fibred in groupoids over $(Sch/S)_{fppf}$, that $\mathcal{F}$ is a sheaf on $\mathcal{X}_{étale}$, and that $\mathcal{U}, \mathcal{V}$ are representable by algebraic spaces U, V. Then we obtain a comparison map

$$c_\varphi : a_{small}^{-1}(g^{-1}\mathcal{F}|_{V_{étale}}) \longrightarrow f^{-1}\mathcal{F}|_{U_{étale}} \tag{69.10.2.2}$$

where $a : U \to V$ denotes the morphism of algebraic spaces corresponding to a. This is the analogue of (69.9.2.2). We define c_φ as the adjoint to the map

$$g^{-1}\mathcal{F}|_{V_{étale}} \longrightarrow a_{small,*}(f^{-1}\mathcal{F}|_{U_{étale}}) = (a_*f^{-1}\mathcal{F})|_{V_{étale}}$$

(equality by (69.10.2.1)) which is the restriction to V (69.10.1.1) of the map

$$g^{-1}\mathcal{F} \to a_*a^{-1}g^{-1}\mathcal{F} = a_*f^{-1}\mathcal{F}$$

where the last equality uses the 2-commutativity of the diagram above. In case $\mathcal{F}$ is a sheaf of $\mathcal{O}_\mathcal{X}$-modules c_φ induces a *comparison* map

$$c_\varphi : a_{small}^*(g^*\mathcal{F}|_{V_{étale}}) \longrightarrow f^*\mathcal{F}|_{U_{étale}} \tag{69.10.2.3}$$

of $\mathcal{O}_{U_{étale}}$-modules. Note that the properties (1), (2), (3), and (4) of Lemma 69.9.3 hold in this setting as well.

69.11. Quasi-coherent modules

At this point we can apply the general definition of a quasi-coherent module to the situation discussed in this chapter.

Definition 69.11.1. Let $p : \mathcal{X} \to (Sch/S)_{fppf}$ be a category fibred in groupoids. A *quasi-coherent module on* $\mathcal{X}$, or a *quasi-coherent* $\mathcal{O}_\mathcal{X}$*-module* is a quasi-coherent module on the ringed site $(\mathcal{X}_{fppf}, \mathcal{O}_\mathcal{X})$ as in Modules on Sites, Definition 18.23.1. The category of quasi-coherent sheaves on $\mathcal{X}$ is denoted $QCoh(\mathcal{O}_\mathcal{X})$.

If $\mathcal{X}$ is an algebraic stack, then this definition agrees with all definitions in the literature in the sense that $QCoh(\mathcal{O}_\mathcal{X})$ is equivalent (modulo set theoretic issues) to any variant of this category defined in the literature. For example, we will match our definition with the definition in [**Ols07b**, Definition 6.1] in Cohomology on Stacks, Lemma 69.11.5. We will also see alternative constructions of this category later on.

In general (as is the case for morphisms of schemes) the pushforward of quasi-coherent sheaf along a 1-morphism is not quasi-coherent. Pullback does preserve quasi-coherence.

Lemma 69.11.2. *Let $f : \mathcal{X} \to \mathcal{Y}$ be a 1-morphism of categories fibred in groupoids over $(Sch/S)_{fppf}$. The pullback functor $f^* = f^{-1} : Mod(\mathcal{O}_\mathcal{Y}) \to Mod(\mathcal{O}_\mathcal{X})$ preserves quasi-coherent sheaves.*

Proof. This is a general fact, see Modules on Sites, Lemma 18.23.4. □

It turns out that quasi-coherent sheaves have a very simple characterization in terms of their pullbacks. See also Lemma 69.11.5 for a characterization in terms of restrictions.

Lemma 69.11.3. *Let $p : \mathcal{X} \to (Sch/S)_{fppf}$ be a category fibred in groupoids. Let $\mathcal{F}$ be a sheaf of $\mathcal{O}_\mathcal{X}$-modules. Then $\mathcal{F}$ is quasi-coherent if and only if $x^*\mathcal{F}$ is a quasi-coherent sheaf on $(Sch/U)_{fppf}$ for every object x of $\mathcal{X}$ with $U = p(x)$.*

Proof. By Lemma 69.11.2 the condition is necessary. Conversely, since $x^*\mathcal{F}$ is just the restriction to $\mathcal{X}_{fppf}/x$ we see that it is sufficient directly from the definition of a quasi-coherent sheaf (and the fact that the notion of being quasi-coherent is an intrinsic property of sheaves of modules, see Modules on Sites, Section 18.18). □

Although there is a variant for the Zariski topology, it seems that the étale topology is the natural topology to use in the following definition.

Definition 69.11.4. Let $p : \mathcal{X} \to (Sch/S)_{fppf}$ be a category fibred in groupoids. Let $\mathcal{F}$ be a presheaf of $\mathcal{O}_\mathcal{X}$-modules. We say $\mathcal{F}$ is *locally quasi-coherent*[2] if $\mathcal{F}$ is a sheaf for the étale topology and for every object x of $\mathcal{X}$ the restriction $x^*\mathcal{F}|_{U_{étale}}$ is a quasi-coherent sheaf. Here $U = p(x)$.

We use $LQCoh(\mathcal{O}_\mathcal{X})$ to indicate the category of locally quasi-coherent modules. We now have the following diagram of categories of modules

$$\begin{array}{ccc} QCoh(\mathcal{O}_\mathcal{X}) & \longrightarrow & Mod(\mathcal{O}_\mathcal{X}) \\ \downarrow & & \downarrow \\ LQCoh(\mathcal{O}_\mathcal{X}) & \longrightarrow & Mod(\mathcal{X}_{étale}, \mathcal{O}_\mathcal{X}) \end{array}$$

where the arrows are strictly full embeddings. It turns out that many results for quasi-coherent sheaves have a counter part for locally quasi-coherent modules. Moreover, from many points of view (as we shall see later) this is a natural category to consider. For example the quasi-coherent sheaves are exactly those locally quasi-coherent modules that are "cartesian", i.e., satisfy the second condition of the lemma below.

Lemma 69.11.5. *Let $p : \mathcal{X} \to (Sch/S)_{fppf}$ be a category fibred in groupoids. Let $\mathcal{F}$ be a presheaf of $\mathcal{O}_\mathcal{X}$-modules. Then $\mathcal{F}$ is quasi-coherent if and only if the following two conditions hold*

1. *$\mathcal{F}$ is locally quasi-coherent, and*
2. *for any morphism $\varphi : x \to y$ of $\mathcal{X}$ lying over $f : U \to V$ the comparison map $c_\varphi : f^*_{small}\mathcal{F}|_{V_{étale}} \to \mathcal{F}|_{U_{étale}}$ of (69.9.4.1) is an isomorphism.*

[2] This is nonstandard notation.

Proof. Assume $\mathcal{F}$ is quasi-coherent. Then $\mathcal{F}$ is a sheaf for the fppf topology, hence a sheaf for the étale topology. Moreover, any pullback of $\mathcal{F}$ to a ringed topos is quasi-coherent, hence the restrictions $x^*\mathcal{F}|_{U_{étale}}$ are quasi-coherent. This proves $\mathcal{F}$ is locally quasi-coherent. Let y be an object of $\mathcal{X}$ with $V = p(y)$. We have seen that $\mathcal{X}/y = (Sch/V)_{fppf}$. By Descent, Proposition 34.7.11 it follows that $y^*\mathcal{F}$ is the quasi-coherent module associated to a (usual) quasi-coherent module $\mathcal{F}_V$ on the scheme V. Hence certainly the comparison maps (69.9.4.1) are isomorphisms.

Conversely, suppose that $\mathcal{F}$ satisfies (1) and (2). Let y be an object of $\mathcal{X}$ with $V = p(y)$. Denote $\mathcal{F}_V$ the quasi-coherent module on the scheme V corresponding to the restriction $y^*\mathcal{F}|_{V_{étale}}$ which is quasi-coherent by assumption (1), see Descent, Proposition 34.7.11. Condition (2) now signifies that the restrictions $x^*\mathcal{F}|_{U_{étale}}$ for x over y are each isomorphic to the (étale sheaf associated to the) pullback of $\mathcal{F}_V$ via the corresponding morphism of schemes $U \to V$. Hence $y^*\mathcal{F}$ is the sheaf on $(Sch/V)_{fppf}$ associated to $\mathcal{F}_V$. Hence it is quasi-coherent (by Descent, Proposition 34.7.11 again) and we see that $\mathcal{F}$ is quasi-coherent on $\mathcal{X}$ by Lemma 69.11.3. □

Lemma 69.11.6. *Let $f : \mathcal{X} \to \mathcal{Y}$ be a 1-morphism of categories fibred in groupoids over $(Sch/S)_{fppf}$. The pullback functor $f^* = f^{-1} : Mod(\mathcal{Y}_{étale}, \mathcal{O}_\mathcal{Y}) \to Mod(\mathcal{X}_{étale}, \mathcal{O}_\mathcal{X})$ preserves locally quasi-coherent sheaves.*

Proof. Let $\mathcal{G}$ be locally quasi-coherent on $\mathcal{Y}$. Choose an object x of $\mathcal{X}$ lying over the scheme U. The restriction $x^*f^*\mathcal{G}|_{U_{étale}}$ equals $(f \circ x)^*\mathcal{G}|_{U_{étale}}$ hence is a quasi-coherent sheaf by assumption on $\mathcal{G}$. □

Lemma 69.11.7. *Let $p : \mathcal{X} \to (Sch/S)_{fppf}$ be a category fibred in groupoids.*

1. *The category $LQCoh(\mathcal{O}_\mathcal{X})$ has colimits and they agree with colimits in the category $Mod(\mathcal{X}_{étale}, \mathcal{O}_\mathcal{X})$.*
2. *The category $LQCoh(\mathcal{O}_\mathcal{X})$ is abelian with kernels and cokernels computed in $Mod(\mathcal{X}_{étale}, \mathcal{O}_\mathcal{X})$, in other words the inclusion functor is exact.*
3. *Given a short exact sequence $0 \to \mathcal{F}_1 \to \mathcal{F}_2 \to \mathcal{F}_3 \to 0$ of $Mod(\mathcal{X}_{étale}, \mathcal{O}_\mathcal{X})$ if two out of three are locally quasi-coherent so is the third.*
4. *Given $\mathcal{F}, \mathcal{G}$ in $LQCoh(\mathcal{O}_\mathcal{X})$ the tensor product $\mathcal{F} \otimes_{\mathcal{O}_\mathcal{X}} \mathcal{G}$ in $Mod(\mathcal{X}_{étale}, \mathcal{O}_\mathcal{X})$ is an object of $LQCoh(\mathcal{O}_\mathcal{X})$.*
5. *Given $\mathcal{F}, \mathcal{G}$ in $LQCoh(\mathcal{O}_\mathcal{X})$ with $\mathcal{F}$ locally of finite presentation on $\mathcal{X}_{étale}$ the sheaf $\mathcal{H}om_{\mathcal{O}_\mathcal{X}}(\mathcal{F}, \mathcal{G})$ in $Mod(\mathcal{X}_{étale}, \mathcal{O}_\mathcal{X})$ is an object of $LQCoh(\mathcal{O}_\mathcal{X})$.*

Proof. Each of these statements follows from the corresponding statement of Descent, Lemma 34.7.13. For example, suppose that $\mathcal{I} \to LQCoh(\mathcal{O}_\mathcal{X})$, $i \mapsto \mathcal{F}_i$ is a diagram. Consider the object $\mathcal{F} = \operatorname{colim}_i \mathcal{F}_i$ of $Mod(\mathcal{X}_{étale}, \mathcal{O}_\mathcal{X})$. For any object x of $\mathcal{X}$ with $U = p(x)$ the pullback functor x^* commutes with all colimits as it is a left adjoint. Hence $x^*\mathcal{F} = \operatorname{colim}_i x^*\mathcal{F}_i$. Similarly we have $x^*\mathcal{F}|_{U_{étale}} = \operatorname{colim}_i x^*\mathcal{F}_i|_{U_{étale}}$. Now by assumption each $x^*\mathcal{F}_i|_{U_{étale}}$ is quasi-coherent, hence the colimit is quasi-coherent by the aforementioned Descent, Lemma 34.7.13. This proves (1).

It follows from (1) that cokernels exist in $LQCoh(\mathcal{O}_\mathcal{X})$ and agree with the cokernels computed in $Mod(\mathcal{X}_{étale}, \mathcal{O}_\mathcal{X})$. Let $\varphi : \mathcal{F} \to \mathcal{G}$ be a morphism of $LQCoh(\mathcal{O}_\mathcal{X})$ and let $\mathcal{K} = \operatorname{Ker}(\varphi)$ computed in $Mod(\mathcal{X}_{étale}, \mathcal{O}_\mathcal{X})$. If we can show that $\mathcal{K}$ is a locally quasi-coherent module, then the proof of (2) is complete. To see this, note that kernels are computed in the category of presheaves (no sheafification necessary). Hence $\mathcal{K}|_{U_{étale}}$ is the kernel of the map $\mathcal{F}|_{U_{étale}} \to \mathcal{G}|_{U_{étale}}$, i.e., is the kernel of a

map of quasi-coherent sheaves on $U_{étale}$ whence quasi-coherent by Descent, Lemma 34.7.13. This proves (2).

Parts (3), (4), and (5) follow in exactly the same way. Details omitted. □

In the generality discussed here we don't know how to prove that the category of quasi-coherent sheaves is abelian. Here is what we can prove without any further work.

Lemma 69.11.8. *Let $p : \mathcal{X} \to (Sch/S)_{fppf}$ be a category fibred in groupoids.*

1. *The category $QCoh(\mathcal{O}_\mathcal{X})$ has colimits and they agree with colimits in the category $Mod(\mathcal{O}_\mathcal{X})$ as well as with colimits in the category $LQCoh(\mathcal{O}_\mathcal{X})$.*
2. *Given $\mathcal{F}, \mathcal{G}$ in $QCoh(\mathcal{O}_\mathcal{X})$ the tensor product $\mathcal{F} \otimes_{\mathcal{O}_\mathcal{X}} \mathcal{G}$ in $Mod(\mathcal{O}_\mathcal{X})$ is an object of $QCoh(\mathcal{O}_\mathcal{X})$.*
3. *Given $\mathcal{F}, \mathcal{G}$ in $QCoh(\mathcal{O}_\mathcal{X})$ with $\mathcal{F}$ locally of finite presentation on $\mathcal{X}_{fppf}$ the sheaf $\mathcal{H}om_{\mathcal{O}_\mathcal{X}}(\mathcal{F}, \mathcal{G})$ in $Mod(\mathcal{O}_\mathcal{X})$ is an object of $QCoh(\mathcal{O}_\mathcal{X})$.*

Proof. Let $\mathcal{I} \to QCoh(\mathcal{O}_\mathcal{X})$, $i \mapsto \mathcal{F}_i$ be a diagram. Consider the object $\mathcal{F} = \operatorname{colim}_i \mathcal{F}_i$ of $Mod(\mathcal{O}_\mathcal{X})$. For any object x of $\mathcal{X}$ with $U = p(x)$ the pullback functor x^* commutes with all colimits as it is a left adjoint. Hence $x^*\mathcal{F} = \operatorname{colim}_i x^*\mathcal{F}_i$ in $Mod((Sch/U)_{fppf}, \mathcal{O})$. We conclude from Descent, Lemma 34.7.13 that $x^*\mathcal{F}$ is quasi-coherent, hence $\mathcal{F}$ is quasi-coherent, see Lemma 69.11.3. Thus we see that $QCoh(\mathcal{O}_\mathcal{X})$ has colimits and they agree with colimits in the category $Mod(\mathcal{O}_\mathcal{X})$. In particular the (fppf) sheaf $\mathcal{F}$ is also the colimit of the diagram in $Mod(\mathcal{X}_{étale}, \mathcal{O}_\mathcal{X})$, hence $\mathcal{F}$ is also the colimit in $LQCoh(\mathcal{O}_\mathcal{X})$. This proves (1).

Parts (2) and (3) are proved in the same way. Details omitted. □

69.12. Stackification and sheaves

It turns out that the category of sheaves on a category fibred in groupoids only "knows about" the stackification.

Lemma 69.12.1. *Let $f : \mathcal{X} \to \mathcal{Y}$ be a 1-morphism of categories fibred in groupoids over $(Sch/S)_{fppf}$. If f induces an equivalence of stackifications, then the morphism of topoi $f : Sh(\mathcal{X}_{fppf}) \to Sh(\mathcal{Y}_{fppf})$ is an equivalence.*

Proof. We may assume $\mathcal{Y}$ is the stackification of $\mathcal{X}$. We claim that $f : \mathcal{X} \to \mathcal{Y}$ is a special cocontinuous functor, see Sites, Definition 7.28.2 which will prove the lemma. By Stacks, Lemma 8.10.3 the functor f is continuous and cocontinuous. By Stacks, Lemma 8.8.1 we see that conditions (3), (4), and (5) of Sites, Lemma 7.28.1 hold. □

Lemma 69.12.2. *Let $f : \mathcal{X} \to \mathcal{Y}$ be a 1-morphism of categories fibred in groupoids over $(Sch/S)_{fppf}$. If f induces an equivalence of stackifications, then f^* induces equivalences $Mod(\mathcal{O}_\mathcal{X}) \to Mod(\mathcal{O}_\mathcal{Y})$ and $QCoh(\mathcal{O}_\mathcal{X}) \to QCoh(\mathcal{O}_\mathcal{Y})$.*

Proof. We may assume $\mathcal{Y}$ is the stackification of $\mathcal{X}$. The first assertion is clear from Lemma 69.12.1 and $\mathcal{O}_\mathcal{X} = f^{-1}\mathcal{O}_\mathcal{Y}$. Pullback of quasi-coherent sheaves are quasi-coherent, see Lemma 69.11.2. Hence it suffices to show that if $f^*\mathcal{G}$ is quasi-coherent, then $\mathcal{G}$ is. To see this, let y be an object of $\mathcal{Y}$. Translating the condition that $\mathcal{Y}$ is the stackification of $\mathcal{X}$ we see there exists an fppf covering $\{y_i \to y\}$ in $\mathcal{Y}$ such that $y_i \cong f(x_i)$ for some x_i object of $\mathcal{X}$. Say x_i and y_i lie over the scheme U_i. Then $f^*\mathcal{G}$ being quasi-coherent, means that $x_i^* f^*\mathcal{G}$ is quasi-coherent. Since $x_i^* f^*\mathcal{G}$

is isomorphic to $y_i^*\mathcal{G}$ (as sheaves on $(Sch/U_i)_{fppf}$ we see that $y_i^*\mathcal{G}$ is quasi-coherent. It follows from Modules on Sites, Lemma 18.23.3 that the restriction of $\mathcal{G}$ to $\mathcal{Y}/y$ is quasi-coherent. Hence $\mathcal{G}$ is quasi-coherent by Lemma 69.11.3. □

69.13. Quasi-coherent sheaves and presentations

In Groupoids in Spaces, Definition 59.12.1 we have the defined the notion of a quasi-coherent module on an arbitrary groupoid. The following (formal) proposition tells us that we can study quasi-coherent sheaves on quotient stacks in terms of quasi-coherent modules on presentations.

Proposition 69.13.1. *Let (U, R, s, t, c) be a groupoid in algebraic spaces over S. Let $\mathcal{X} = [U/R]$ be the quotient stack. The category of quasi-coherent modules on $\mathcal{X}$ is equivalent to the category of quasi-coherent modules on (U, R, s, t, c).*

Proof. Denote $QCoh(U, R, s, t, c)$ the category of quasi-coherent modules on the groupoid (U, R, s, t, c). We will construct quasi-inverse functors

$$QCoh(\mathcal{O}_\mathcal{X}) \longleftrightarrow QCoh(U, R, s, t, c).$$

According to Lemma 69.12.2 the stackification map $[U/_pR] \to [U/R]$ (see Groupoids in Spaces, Definition 59.19.1) induces an equivalence of categories of quasi-coherent sheaves. Thus it suffices to prove the lemma with $\mathcal{X} = [U/_pR]$.

Recall that an object $x = (T, u)$ of $\mathcal{X} = [U/_pR]$ is given by a scheme T and a morphism $u : T \to U$. A morphism $(T, u) \to (T', u')$ is given by a pair (f, r) where $f : T \to T'$ and $r : T \to R$ with $s \circ r = u$ and $t \circ r = u' \circ f$. Let us call a *special morphism* any morphism of the form $(f, e \circ u' \circ f) : (T, u' \circ f) \to (T', u')$. The category of (T, u) with special morphisms is just the category of schemes over U.

Let $\mathcal{F}$ be a quasi-coherent sheaf on $\mathcal{X}$. Then we obtain for every $x = (T, u)$ a quasi-coherent sheaf $\mathcal{F}_{(T,u)} = x^*\mathcal{F}|_{T_{étale}}$ on T. Moreover, for any morphism $(f, r) : x = (T, u) \to (T', u') = x'$ we obtain a comparison isomorphism

$$c_{(f,r)} : f_{small}^*\mathcal{F}_{(T',u')} \longrightarrow \mathcal{F}_{(T,u)}$$

see Lemma 69.11.5. Moreover, these isomorphisms are compatible with compositions, see Lemma 69.9.3. If U, R are schemes, then we can construct the quasi-coherent sheaf on the groupoid as follows: First the object (U, id) corresponds to a quasi-coherent sheaf $\mathcal{F}_{(U,\mathrm{id})}$ on U. Next, the isomorphism $\alpha : t_{small}^*\mathcal{F}_{(U,\mathrm{id})} \to s_{small}^*\mathcal{F}_{(U,\mathrm{id})}$ comes from

(1) the morphism $(R, \mathrm{id}_R) : (R, s) \to (R, t)$ in the category $[U/_pR]$ which produces an isomorphism $\mathcal{F}_{(R,t)} \to \mathcal{F}_{(R,s)}$,
(2) the special morphism $(R, s) \to (U, \mathrm{id})$ which produces an isomorphism $s_{small}^*\mathcal{F}_{(U,\mathrm{id})} \to \mathcal{F}_{(R,s)}$, and
(3) the special morphism $(R, t) \to (U, \mathrm{id})$ which produces an isomorphism $t_{small}^*\mathcal{F}_{(U,\mathrm{id})} \to \mathcal{F}_{(R,t)}$.

The cocycle condition for α follows from the condition that (U, R, s, t, c) is groupoid, i.e., that composition is associative (details omitted).

To do this in general, i.e., when U and R are algebraic spaces, it suffices to explain how to associate to an algebraic space (W, u) over U a quasi-coherent sheaf $\mathcal{F}_{(W,u)}$ and to construct the comparison maps for morphisms between these. We

set $\mathcal{F}_{(W,u)} = x^*\mathcal{F}|_{W_{étale}}$ where x is the 1-morphism $\mathcal{S}_W \to \mathcal{S}_U \to [U/_pR]$ and the comparison maps are explained in (69.10.2.3).

Conversely, suppose that $(\mathcal{G}, \alpha)$ is a quasi-coherent module on (U, R, s, t, c). We are going to define a presheaf of modules $\mathcal{F}$ on $\mathcal{X}$ as follows. Given an object (T, u) of $[U/_pR]$ we set

$$\mathcal{F}(T, u) := \Gamma(T, u^*_{small}\mathcal{G}).$$

Given a morphism $(f, r) : (T, u) \to (T', u')$ we get a map

$$\begin{aligned}
\mathcal{F}(T', u') &= \Gamma(T', (u')^*_{small}\mathcal{G}) \\
&\to \Gamma(T, f^*_{small}(u')^*_{small}\mathcal{G}) = \Gamma(T, (u' \circ f)^*_{small}\mathcal{G}) \\
&= \Gamma(T, (t \circ r)^*_{small}\mathcal{G}) = \Gamma(T, r^*_{small}t^*_{small}\mathcal{G}) \\
&\to \Gamma(T, r^*_{small}s^*_{small}\mathcal{G}) = \Gamma(T, (s \circ r)^*_{small}\mathcal{G}) \\
&= \Gamma(T, u^*_{small}\mathcal{G}) \\
&= \mathcal{F}(T, u)
\end{aligned}$$

where the first arrow is pullback along f and the second arrow is α. Note that if (T, r) is a special morphism, then this map is just pullback along f as $e^*_{small}\alpha = \mathrm{id}$ by the axioms of a sheaf of quasi-coherent modules on a groupoid. The cocycle condition implies that $\mathcal{F}$ is a presheaf of modules (details omitted). It is immediate from the definition that $\mathcal{F}$ is quasi-coherent when pulled back to $(Sch/T)_{fppf}$ (by the simple description of the restriction maps of $\mathcal{F}$ in case of a special morphism).

We omit the verification that the functors constructed above are quasi-inverse to each other. □

We finish this section with a technical lemma on maps out of quasi-coherent sheaves. It is an analogue of Schemes, Lemma 25.7.1. We will see later (Criteria for Representability, Theorem 70.17.2) that the assumptions on the groupoid imply that $\mathcal{X}$ is an algebraic stack.

Lemma 69.13.2. *Let (U, R, s, t, c) be a groupoid in algebraic spaces over S. Assume s, t are flat and locally of finite presentation. Let $\mathcal{X} = [U/R]$ be the quotient stack. Denote $\pi : \mathcal{S}_U \to \mathcal{X}$ the quotient map. Let $\mathcal{F}$ be a quasi-coherent $\mathcal{O}_\mathcal{X}$-module, and let $\mathcal{H}$ be any object of $Mod(\mathcal{O}_\mathcal{X})$. The map*

$$\mathrm{Hom}_{\mathcal{O}_\mathcal{X}}(\mathcal{F}, \mathcal{H}) \longrightarrow \mathrm{Hom}_{\mathcal{O}_U}(x^*\mathcal{F}|_{U_{étale}}, x^*\mathcal{H}|_{U_{étale}}), \quad \phi \longmapsto x^*\phi|_{U_{étale}}$$

is injective and its image consists of exactly those $\varphi : x^*\mathcal{F}|_{U_{étale}} \to x^*\mathcal{H}|_{U_{étale}}$ which give rise to a commutative diagram

$$\begin{array}{ccccc}
s^*_{small}(x^*\mathcal{F}|_{U_{étale}}) & \longrightarrow & (x \circ s)^*\mathcal{F}|_{R_{étale}} = (x \circ t)^*\mathcal{F}|_{R_{étale}} & \longleftarrow & t^*_{small}(x^*\mathcal{F}|_{U_{étale}}) \\
\downarrow{\scriptstyle s^*_{small}\varphi} & & & & \downarrow{\scriptstyle t^*_{small}\varphi} \\
s^*_{small}(x^*\mathcal{H}|_{U_{étale}}) & \longrightarrow & (x \circ s)^*\mathcal{H}|_{R_{étale}} = (x \circ t)^*\mathcal{H}|_{R_{étale}} & \longleftarrow & t^*_{small}(x^*\mathcal{H}|_{U_{étale}})
\end{array}$$

of modules on $R_{étale}$ where the horizontal arrows are the comparison maps (69.10.2.3).

Proof. According to Lemma 69.12.2 the stackification map $[U/_pR] \to [U/R]$ (see Groupoids in Spaces, Definition 59.19.1) induces an equivalence of categories of quasi-coherent sheaves and of fppf $\mathcal{O}$-modules. Thus it suffices to prove the lemma with $\mathcal{X} = [U/_pR]$. By Proposition 69.13.1 and its proof there exists a quasi-coherent module $(\mathcal{G}, \alpha)$ on (U, R, s, t, c) such that $\mathcal{F}$ is given by the rule $\mathcal{F}(T, u) = \Gamma(T, u^*\mathcal{G})$.

In particular $x^*\mathcal{F}|_{U_{étale}} = \mathcal{G}$ and it is clear that the map of the statement of the lemma is injective. Moreover, given a map $\varphi : \mathcal{G} \to x^*\mathcal{H}|_{U_{étale}}$ and given any object $y = (T, u)$ of $[U/_pR]$ we can consider the map

$$\mathcal{F}(y) = \Gamma(T, u^*\mathcal{G}) \xrightarrow{u^*_{small}\varphi} \Gamma(T, u^*_{small}x^*\mathcal{H}|_{U_{étale}}) \to \Gamma(T, y^*\mathcal{H}|_{T_{étale}}) = \mathcal{H}(y)$$

where the second arrow is the comparison map (69.9.4.1) for the sheaf $\mathcal{H}$. This assignment is compatible with the restriction mappings of the sheaves $\mathcal{F}$ and $\mathcal{G}$ for morphisms of $[U/_pR]$ if the cocycle condition of the lemma is satisfied. Proof omitted. Hint: the restriction maps of $\mathcal{F}$ are made explicit in terms of $(\mathcal{G}, \alpha)$ in the proof of Proposition 69.13.1. □

69.14. Quasi-coherent sheaves on algebraic stacks

Let $\mathcal{X}$ be an algebraic stack over S. By Algebraic Stacks, Lemma 67.16.2 we can find an equivalence $[U/R] \to \mathcal{X}$ where (U, R, s, t, c) is a smooth groupoid in algebraic spaces. Then

$$QCoh(\mathcal{O}_\mathcal{X}) \cong QCoh(\mathcal{O}_{[U/R]}) \cong QCoh(U, R, s, t, c)$$

where the second equivalence is Proposition 69.13.1. Hence the category of quasi-coherent sheaves on an algebraic stack is equivalent to the category of quasi-coherent modules on a smooth groupoid in algebraic spaces. In particular, by Groupoids in Spaces, Lemma 59.12.5 we see that $QCoh(\mathcal{O}_\mathcal{X})$ is abelian!

There is something slightly disconcerting about our current setup. It is that the fully faithful embedding

$$QCoh(\mathcal{O}_\mathcal{X}) \longrightarrow Mod(\mathcal{O}_\mathcal{X})$$

is in general **not** exact. However, exactly the same thing happens for schemes: for most schemes X the embedding

$$QCoh(\mathcal{O}_X) \cong QCoh((Sch/X)_{fppf}, \mathcal{O}_X) \longrightarrow Mod((Sch/X)_{fppf}, \mathcal{O}_X)$$

isn't exact, see Descent, Lemma 34.7.13. Parenthetically, the example in the proof of Descent, Lemma 34.7.13 shows that in general the strictly full embedding $QCoh(\mathcal{O}_\mathcal{X}) \to LQCoh(\mathcal{O}_\mathcal{X})$ isn't exact either.

We collect all the positive results obtained so far in a single statement.

Lemma 69.14.1. *Let $\mathcal{X}$ be an algebraic stack over S.*

1. *If $[U/R] \to \mathcal{X}$ is a presentation of $\mathcal{X}$ then there is a canonical equivalence $QCoh(\mathcal{O}_\mathcal{X}) \cong QCoh(U, R, s, t, c)$.*
2. *The category $QCoh(\mathcal{O}_\mathcal{X})$ is abelian.*
3. *The category $QCoh(\mathcal{O}_\mathcal{X})$ has colimits and they agree with colimits in the category $Mod(\mathcal{O}_\mathcal{X})$.*
4. *Given $\mathcal{F}, \mathcal{G}$ in $QCoh(\mathcal{O}_\mathcal{X})$ the tensor product $\mathcal{F} \otimes_{\mathcal{O}_\mathcal{X}} \mathcal{G}$ in $Mod(\mathcal{O}_\mathcal{X})$ is an object of $QCoh(\mathcal{O}_\mathcal{X})$.*
5. *Given $\mathcal{F}, \mathcal{G}$ in $QCoh(\mathcal{O}_\mathcal{X})$ with $\mathcal{F}$ locally of finite presentation on $\mathcal{X}_{fppf}$ the sheaf $\mathcal{H}om_{\mathcal{O}_\mathcal{X}}(\mathcal{F}, \mathcal{G})$ in $Mod(\mathcal{O}_\mathcal{X})$ is an object of $QCoh(\mathcal{O}_\mathcal{X})$.*

Proof. Properties (3), (4), and (5) were proven in Lemma 69.11.8. Part (1) is Proposition 69.13.1. Part (2) follows from Groupoids in Spaces, Lemma 59.12.5 as discussed above. □

Proposition 69.14.2. *Let $\mathcal{X}$ be an algebraic stack over S.*

(1) *The category $QCoh(\mathcal{O}_\mathcal{X})$ is a Grothendieck abelian category. Consequently, $QCoh(\mathcal{O}_\mathcal{X})$ has enough injectives and all limits.*
(2) *The inclusion functor $QCoh(\mathcal{O}_\mathcal{X}) \to Mod(\mathcal{O}_\mathcal{X})$ has a right adjoint*

$$Q^3 : Mod(\mathcal{O}_\mathcal{X}) \longrightarrow QCoh(\mathcal{O}_\mathcal{X})$$

such that for every quasi-coherent sheaf $\mathcal{F}$ the adjunction mapping $Q(\mathcal{F}) \to \mathcal{F}$ is an isomorphism.

Proof. This proof is a repeat of the proof in the case of schemes, see Properties, Proposition 27.21.4 and the case of algebraic spaces, see Properties of Spaces, Proposition 48.30.2. We advise the reader to read either of those proofs first.

Part (1) means $QCoh(\mathcal{O}_\mathcal{X})$ (a) has all colimits, (b) filtered colimits are exact, and (c) has a generator, see Injectives, Section 19.10. By Lemma 69.14.1 colimits in $QCoh(\mathcal{O}_X)$ exist and agree with colimits in $Mod(\mathcal{O}_X)$. By Modules on Sites, Lemma 18.14.2 filtered colimits are exact. Hence (a) and (b) hold.

Choose a presentation $\mathcal{X} = [U/R]$ so that (U, R, s, t, c) is a smooth groupoid in algebraic spaces and in particular s and t are flat morphisms of algebraic spaces. By Lemma 69.14.1 above we have $QCoh(\mathcal{O}_\mathcal{X}) = QCoh(U, R, s, t, c)$. By Groupoids in Spaces, Lemma 59.13.2 there exists a set T and a family $(\mathcal{F}_t)_{t\in T}$ of quasi-coherent sheaves on $\mathcal{X}$ such that every quasi-coherent sheaf on $\mathcal{X}$ is the directed colimit of its subsheaves which are isomorphic to one of the $\mathcal{F}_t$. Thus $\bigoplus_t \mathcal{F}_t$ is a generator of $QCoh(\mathcal{O}_X)$ and we conclude that (c) holds. The assertions on limits and injectives hold in any Grothendieck abelian category, see Injectives, Theorem 19.11.6 and Lemma 19.13.2.

Proof of (2). To construct Q we use the following general procedure. Given an object $\mathcal{F}$ of $Mod(\mathcal{O}_\mathcal{X})$ we consider the functor

$$QCoh(\mathcal{O}_\mathcal{X})^{opp} \longrightarrow Sets, \quad \mathcal{G} \longmapsto \mathrm{Hom}_\mathcal{X}(\mathcal{G}, \mathcal{F})$$

This functor transforms colimits into limits, hence is representable, see Injectives, Lemma 19.13.1. Thus there exists a quasi-coherent sheaf $Q(\mathcal{F})$ and a functorial isomorphism $\mathrm{Hom}_\mathcal{X}(\mathcal{G}, \mathcal{F}) = \mathrm{Hom}_\mathcal{X}(\mathcal{G}, Q(\mathcal{F}))$ for $\mathcal{G}$ in $QCoh(\mathcal{O}_\mathcal{X})$. By the Yoneda lemma (Categories, Lemma 4.3.5) the construction $\mathcal{F} \rightsquigarrow Q(\mathcal{F})$ is functorial in $\mathcal{F}$. By construction Q is a right adjoint to the inclusion functor. The fact that $Q(\mathcal{F}) \to \mathcal{F}$ is an isomorphism when $\mathcal{F}$ is quasi-coherent is a formal consequence of the fact that the inclusion functor $QCoh(\mathcal{O}_\mathcal{X}) \to Mod(\mathcal{O}_\mathcal{X})$ is fully faithful. □

69.15. Cohomology

Let S be a scheme and let $\mathcal{X}$ be a category fibred in groupoids over $(Sch/S)_{fppf}$. For any $\tau \in \{Zariski, étale, smooth, syntomic, fppf\}$ the categories $Ab(\mathcal{X}_\tau)$ and $Mod(\mathcal{X}_\tau, \mathcal{O}_\mathcal{X})$ have enough injectives, see Injectives, Theorems 19.7.4 and 19.8.4. Thus we can use the machinery of Cohomology on Sites, Section 21.3 to define the cohomology groups

$$H^p(\mathcal{X}_\tau, \mathcal{F}) = H^p_\tau(\mathcal{X}, \mathcal{F}) \quad \text{and} \quad H^p(x, \mathcal{F}) = H^p_\tau(x, \mathcal{F})$$

for any $x \in \mathrm{Ob}(\mathcal{X})$ and any object $\mathcal{F}$ of $Ab(\mathcal{X}_\tau)$ or $Mod(\mathcal{X}_\tau, \mathcal{O}_\mathcal{X})$. Moreover, if $f : \mathcal{X} \to \mathcal{Y}$ is a 1-morphism of categories fibred in groupoids over $(Sch/S)_{fppf}$, then we obtain the higher direct images $R^i f_* \mathcal{F}$ in $Ab(\mathcal{Y}_\tau)$ or $Mod(\mathcal{Y}_\tau, \mathcal{O}_\mathcal{Y})$. Of

[3] This functor is sometimes called the *coherator*.

course, as explained in Cohomology on Sites, Section 21.4 there are also derived versions of $H^p(-)$ and $R^i f_*$.

Lemma 69.15.1. *Let S be a scheme. Let $\mathcal{X}$ be a category fibred in groupoids over $(Sch/S)_{fppf}$. Let $\tau \in \{Zariski, étale, smooth, syntomic, fppf\}$. Let $x \in \mathrm{Ob}(\mathcal{X})$ be an object lying over the scheme U. Let $\mathcal{F}$ be an object of $Ab(\mathcal{X}_\tau)$ or $Mod(\mathcal{X}_\tau, \mathcal{O}_\mathcal{X})$. Then*

$$H^p_\tau(x, \mathcal{F}) = H^p((Sch/U)_\tau, x^{-1}\mathcal{F})$$

and if $\tau = étale$, then we also have

$$H^p_{étale}(x, \mathcal{F}) = H^p(U_{étale}, \mathcal{F}|_{U_{étale}}).$$

Proof. The first statement follows from Cohomology on Sites, Lemma 21.8.1 and the equivalence of Lemma 69.9.4. The second statement follows from the first combined with Étale Cohomology, Lemma 44.20.5. □

69.16. Injective sheaves

The pushforward of an injective abelian sheaf or module is injective.

Lemma 69.16.1. *Let $f : \mathcal{X} \to \mathcal{Y}$ be a 1-morphism of categories fibred in groupoids over $(Sch/S)_{fppf}$. Let $\tau \in \{Zar, étale, smooth, syntomic, fppf\}$.*

1. *$f_*\mathcal{I}$ is injective in $Ab(\mathcal{Y}_\tau)$ for $\mathcal{I}$ injective in $Ab(\mathcal{X}_\tau)$, and*
2. *$f_*\mathcal{I}$ is injective in $Mod(\mathcal{Y}_\tau, \mathcal{O}_\mathcal{Y})$ for $\mathcal{I}$ injective in $Mod(\mathcal{X}_\tau, \mathcal{O}_\mathcal{X})$.*

Proof. This follows formally from the fact that f^{-1} is an exact left adjoint of f_*, see Homology, Lemma 12.25.1. □

In the rest of this section we prove that pullback f^{-1} has a left adjoint $f_!$ on abelian sheaves and modules. If f is representable (by schemes or by algebraic spaces), then it will turn out that $f_!$ is exact and f^{-1} will preserve injectives. We first prove a few preliminary lemmas about fibre products and equalizers in categories fibred in groupoids and their behaviour with respect to morphisms.

Lemma 69.16.2. *Let $p : \mathcal{X} \to (Sch/S)_{fppf}$ be a category fibred in groupoids.*

1. *The category $\mathcal{X}$ has fibre products.*
2. *If the Isom-presheaves of $\mathcal{X}$ are representable by algebraic spaces, then $\mathcal{X}$ has equalizers.*
3. *If $\mathcal{X}$ is an algebraic stack (or more generally a quotient stack), then $\mathcal{X}$ has equalizers.*

Proof. Part (1) follows Categories, Lemma 4.33.14 as $(Sch/S)_{fppf}$ has fibre products.

Let $a, b : x \to y$ be morphisms of $\mathcal{X}$. Set $U = p(x)$ and $V = p(y)$. The category of schemes has equalizers hence we can let $W \to U$ be the equalizer of $p(a)$ and $p(b)$. Denote $c : z \to x$ a morphism of $\mathcal{X}$ lying over $W \to U$. The equalizer of a and b, if it exists, is the equalizer of $a \circ c$ and $b \circ c$. Thus we may assume that $p(a) = p(b) = f : U \to V$. As $\mathcal{X}$ is fibred in groupoids, there exists a unique automorphism $i : x \to x$ in the fibre category of $\mathcal{X}$ over U such that $a \circ i = b$. Again the equalizer of a and b is the equalizer of id_x and i. Recall that the $Isom_\mathcal{X}(x)$ is the presheaf on $(Sch/U)_{fppf}$ which to V/U associates the set of automorphisms of $x|_V$ in the fibre category of $\mathcal{X}$ over V, see Stacks, Definition 8.2.2. If $Isom_\mathcal{X}(x)$ is representable by an algebraic space $G \to U$, then we see that id_x and i define

morphisms $e, i : U \to G$ over U. Set $V = U \times_{e,G,i} U$, which by Morphisms of Spaces, Lemma 49.5.7 is a scheme. Then it is clear that $x|_V \to x$ is the equalizer of the maps id_x and i in $\mathcal{X}$. This proves (2).

If $\mathcal{X} = [U/R]$ for some groupoid in algebraic spaces (U, R, s, t, c) over S, then the hypothesis of (2) holds by Bootstrap, Lemma 61.11.3. If $\mathcal{X}$ is an algebraic stack, then we can choose a presentation $[U/R] \cong \mathcal{X}$ by Algebraic Stacks, Lemma 67.16.2. □

Lemma 69.16.3. *Let $f : \mathcal{X} \to \mathcal{Y}$ be a 1-morphism of categories fibred in groupoids over $(Sch/S)_{fppf}$.*

(1) *The functor f transforms fibre products into fibre products.*
(2) *If f is faithful, then f transforms equalizers into equalizers.*

Proof. By Categories, Lemma 4.33.14 we see that a fibre product in $\mathcal{X}$ is any commutative square lying over a fibre product diagram in $(Sch/S)_{fppf}$. Similarly for $\mathcal{Y}$. Hence (1) is clear.

Let $x \to x'$ be the equalizer of two morphisms $a, b : x' \to x''$ in $\mathcal{X}$. We will show that $f(x) \to f(x')$ is the equalizer of $f(a)$ and $f(b)$. Let $y \to f(x)$ be a morphism of $\mathcal{Y}$ equalizing $f(a)$ and $f(b)$. Say x, x', x'' lie over the schemes U, U', U'' and y lies over V. Denote $h : V \to U'$ the image of $y \to f(x)$ in the category of schemes. The morphism $y \to f(x)$ is isomorphic to $f(h^*x') \to f(x')$ by the axioms of fibred categories. Hence, as f is faithful, we see that $h^*x' \to x'$ equalizes a and b. Thus we obtain a unique morphism $h^*x' \to x$ whose image $y = f(h^*x') \to f(x)$ is the desired morphism in $\mathcal{Y}$. □

Lemma 69.16.4. *Let $f : \mathcal{X} \to \mathcal{Y}$, $g : \mathcal{Z} \to \mathcal{Y}$ be faithful 1-morphisms of categories fibred in groupoids over $(Sch/S)_{fppf}$.*

(1) *the functor $\mathcal{X} \times_{\mathcal{Y}} \mathcal{Z} \to \mathcal{Y}$ is faithful, and*
(2) *if $\mathcal{X}, \mathcal{Z}$ have equalizers, so does $\mathcal{X} \times_{\mathcal{Y}} \mathcal{Z}$.*

Proof. We think of objects in $\mathcal{X} \times_{\mathcal{Y}} \mathcal{Z}$ as quadruples (U, x, z, α) where $\alpha : f(x) \to g(z)$ is an isomorphism over U, see Categories, Lemma 4.30.3. A morphism $(U, x, z, \alpha) \to (U', x', z', \alpha')$ is a pair of morphisms $a : x \to x'$ and $b : z \to z'$ compatible with α and α'. Thus it is clear that if f and g are faithful, so is the functor $\mathcal{X} \times_{\mathcal{Y}} \mathcal{Z} \to \mathcal{Y}$. Now, suppose that $(a, b), (a', b') : (U, x, z, \alpha) \to (U', x', z', \alpha')$ are two morphisms of the 2-fibre product. Then consider the equalizer $x'' \to x$ of a and a' and the equalizer $z'' \to z$ of b and b'. Since f commutes with equalizers (by Lemma 69.16.3) we see that $f(x'') \to f(x)$ is the equalizer of $f(a)$ and $f(a')$. Similarly, $g(z'') \to g(z)$ is the equalizer of $g(b)$ and $g(b')$. Picture

$$\begin{array}{ccccc} f(x'') & \longrightarrow & f(x) & \overset{f(a)}{\underset{f(a')}{\rightrightarrows}} & f(x') \\ {\scriptstyle \alpha''}\downarrow & & {\scriptstyle \alpha}\downarrow & & \downarrow{\scriptstyle \alpha'} \\ g(z'') & \longrightarrow & g(z) & \overset{g(b)}{\underset{g(b')}{\rightrightarrows}} & g(z') \end{array}$$

It is clear that the dotted arrow exists and is an isomorphism. However, it is not a priori the case that the image of α'' in the category of schemes is the identity of its source. On the other hand, the existence of α'' means that we can assume that x'' and z'' are defined over the same scheme and that the morphisms $x'' \to x$

and $z'' \to z$ have the same image in the category of schemes. Redoing the diagram above we see that the dotted arrow now does project to an identity morphism and we win. Some details omitted. □

As we are working with big sites we have the following somewhat counter intuitive result (which also holds for morphisms of big sites of schemes). Warning: This result isn't true if we drop the hypothesis that f is faithful.

Lemma 69.16.5. *Let $f : \mathcal{X} \to \mathcal{Y}$ be a 1-morphism of categories fibred in groupoids over $(Sch/S)_{fppf}$. Let $\tau \in \{Zar, étale, smooth, syntomic, fppf\}$. The functor $f^{-1} : Ab(\mathcal{Y}_\tau) \to Ab(\mathcal{X}_\tau)$ has a left adjoint $f_! : Ab(\mathcal{X}_\tau) \to Ab(\mathcal{Y}_\tau)$. If f is faithful and $\mathcal{X}$ has equalizers, then*

(1) *$f_!$ is exact, and*
(2) *$f^{-1}\mathcal{I}$ is injective in $Ab(\mathcal{X}_\tau)$ for $\mathcal{I}$ injective in $Ab(\mathcal{Y}_\tau)$.*

Proof. By Stacks, Lemma 8.10.3 the functor f is continuous and cocontinuous. Hence by Modules on Sites, Lemma 18.16.2 the functor $f^{-1} : Ab(\mathcal{Y}_\tau) \to Ab(\mathcal{X}_\tau)$ has a left adjoint $f_! : Ab(\mathcal{X}_\tau) \to Ab(\mathcal{Y}_\tau)$. To see (1) we apply Modules on Sites, Lemma 18.16.3 and to see that the hypotheses of that lemma are satisfied use Lemmas 69.16.2 and 69.16.3 above. Part (2) follows from this formally, see Homology, Lemma 12.25.1. □

Lemma 69.16.6. *Let $f : \mathcal{X} \to \mathcal{Y}$ be a 1-morphism of categories fibred in groupoids over $(Sch/S)_{fppf}$. Let $\tau \in \{Zar, étale, smooth, syntomic, fppf\}$. The functor $f^* : Mod(\mathcal{Y}_\tau, \mathcal{O}_\mathcal{Y}) \to Mod(\mathcal{X}_\tau, \mathcal{O}_\mathcal{X})$ has a left adjoint $f_! : Mod(\mathcal{X}_\tau, \mathcal{O}_\mathcal{X}) \to Mod(\mathcal{Y}_\tau, \mathcal{O}_\mathcal{Y})$ which agrees with the functor $f_!$ of Lemma 69.16.5 on underlying abelian sheaves. If f is faithful and $\mathcal{X}$ has equalizers, then*

(1) *$f_!$ is exact, and*
(2) *$f^{-1}\mathcal{I}$ is injective in $Mod(\mathcal{X}_\tau, \mathcal{O}_\mathcal{X})$ for $\mathcal{I}$ injective in $Mod(\mathcal{Y}_\tau, \mathcal{O}_\mathcal{X})$.*

Proof. Recall that f is a continuous and cocontinuous functor of sites and that $f^{-1}\mathcal{O}_\mathcal{Y} = \mathcal{O}_\mathcal{X}$. Hence Modules on Sites, Lemma 18.40.1 implies f^* has a left adjoint $f_!^{Mod}$. Let x be an object of $\mathcal{X}$ lying over the scheme U. Then f induces an equivalence of ringed sites

$$\mathcal{X}/x \longrightarrow \mathcal{Y}/f(x)$$

as both sides are equivalent to $(Sch/U)_\tau$, see Lemma 69.9.4. Modules on Sites, Remark 18.40.2 shows that $f_!$ agrees with the functor on abelian sheaves.

Assume now that $\mathcal{X}$ has equalizers and that f is faithful. Lemma 69.16.5 tells us that $f_!$ is exact. Finally, Homology, Lemma 12.25.1 implies the statement on pullbacks of injective modules. □

69.17. The Čech complex

To compute the cohomology of a sheaf on an algebraic stack we compare it to the cohomology of the sheaf restricted to coverings of the given algebraic stack.

Throughout this section the situation will be as follows. We are given a 1-morphism of categories fibred in groupoids

$$\begin{array}{ccc} \mathcal{U} & \xrightarrow{\quad f \quad} & \mathcal{X} \\ & {\scriptstyle q} \searrow \quad \swarrow {\scriptstyle p} & \\ & (Sch/S)_{fppf} & \end{array} \tag{69.17.0.1}$$

We are going to think about $\mathcal{U}$ as a "covering" of $\mathcal{X}$. Hence we want to consider the simplicial object

$$\mathcal{U} \times_{\mathcal{X}} \mathcal{U} \times_{\mathcal{X}} \mathcal{U} \substack{\longrightarrow \\ \longrightarrow \\ \longrightarrow} \mathcal{U} \times_{\mathcal{X}} \mathcal{U} \substack{\longrightarrow \\ \longrightarrow} \mathcal{U}$$

in the category of categories fibred in groupoids over $(Sch/S)_{fppf}$. However, since this is a $(2,1)$-category and not a category, we should say explicitly what we mean. Namely, we let $\mathcal{U}_n$ be the category with objects $(u_0, \ldots, u_n, x, \alpha_0, \ldots, \alpha_n)$ where $\alpha_i : f(u_i) \to x$ is an isomorphism in $\mathcal{X}$. We denote $f_n : \mathcal{U}_n \to \mathcal{X}$ the 1-morphism which assigns to $(u_0, \ldots, u_n, x, \alpha_0, \ldots, \alpha_n)$ the object x. Note that $\mathcal{U}_0 = \mathcal{U}$ and $f_0 = f$. Given a map $\varphi : [m] \to [n]$ we consider the 1-morphism $\mathcal{U}_\varphi : \mathcal{U}_n \longrightarrow \mathcal{U}_n$ given by

$$(u_0, \ldots, u_n, x, \alpha_0, \ldots, \alpha_n) \longmapsto (u_{\varphi(0)}, \ldots, u_{\varphi(n)}, x, \alpha_{\varphi(0)}, \ldots, \alpha_{\varphi(n)})$$

on objects. All of these 1-morphisms compose correctly on the nose (no 2-morphisms required) and all of these 1-morphisms are 1-morphisms over $\mathcal{X}$. We denote $\mathcal{U}_\bullet$ this simplicial object. If $\mathcal{F}$ is a presheaf of sets on $\mathcal{X}$, then we obtain a cosimplicial set

$$\Gamma(\mathcal{U}_0, f_0^{-1}\mathcal{F}) \substack{\longrightarrow} \Gamma(\mathcal{U}_1, f_1^{-1}\mathcal{F}) \substack{\longrightarrow \\ \longrightarrow} \Gamma(\mathcal{U}_2, f_2^{-1}\mathcal{F})$$

Here the arrows are the pullback maps along the given morphisms of the simplicial object. If $\mathcal{F}$ is a presheaf of abelian groups, this is a cosimplicial abelian group.

Let $\mathcal{U} \to \mathcal{X}$ be as above and let $\mathcal{F}$ be an abelian presheaf on $\mathcal{X}$. The *Čech complex* associated to the situation is denoted $\check{\mathcal{C}}^\bullet(\mathcal{U} \to \mathcal{X}, \mathcal{F})$. It is the cochain complex associated to the cosimplicial abelian group above, see Simplicial, Section 14.23. It has terms

$$\check{\mathcal{C}}^n(\mathcal{U} \to \mathcal{X}, \mathcal{F}) = \Gamma(\mathcal{U}_n, f_n^{-1}\mathcal{F}).$$

The boundary maps are the maps

$$d^n = \sum\nolimits_{i=0}^{n+1} (-1)^i \delta_i^{n+1} : \Gamma(\mathcal{U}_n, f_n^{-1}\mathcal{F}) \longrightarrow \Gamma(\mathcal{U}_{n+1}, f_{n+1}^{-1}\mathcal{F})$$

where δ_i^{n+1} corresponds to the map $[n] \to [n+1]$ omitting the index i. Note that the map $\Gamma(\mathcal{X}, \mathcal{F}) \to \Gamma(\mathcal{U}_0, f_0^{-1}\mathcal{F}_0)$ is in the kernel of the differential d^0. Hence we define the *extended Čech complex* to be the complex

$$\ldots \to 0 \to \Gamma(\mathcal{X}, \mathcal{F}) \to \Gamma(\mathcal{U}_0, f_0^{-1}\mathcal{F}_0) \to \Gamma(\mathcal{U}_1, f_1^{-1}\mathcal{F}_1) \to \ldots$$

with $\Gamma(\mathcal{X}, \mathcal{F})$ placed in degree -1. The extended Čech complex is acyclic if and only if the canonical map

$$\Gamma(\mathcal{X}, \mathcal{F})[0] \longrightarrow \check{\mathcal{C}}^\bullet(\mathcal{U} \to \mathcal{X}, \mathcal{F})$$

is a quasi-isomorphism of complexes.

Lemma 69.17.1. *Generalities on Čech complexes.*

(1) *If*

$$\begin{array}{ccc} \mathcal{V} & \xrightarrow[h]{} & \mathcal{U} \\ {\scriptstyle g}\downarrow & & \downarrow{\scriptstyle f} \\ \mathcal{Y} & \xrightarrow{e} & \mathcal{X} \end{array}$$

is 2*-commutative diagram of categories fibred in groupoids over* $(Sch/S)_{fppf}$*, then there is a morphism of Čech complexes*

$$\check{\mathcal{C}}^\bullet(\mathcal{U} \to \mathcal{X}, \mathcal{F}) \longrightarrow \check{\mathcal{C}}^\bullet(\mathcal{V} \to \mathcal{Y}, e^{-1}\mathcal{F})$$

(2) *if* h *and* e *are equivalences, then the map of (1) is an isomorphism,*
(3) *if* $f, f' : \mathcal{U} \to \mathcal{X}$ *are* 2*-isomorphic, then the associated Čech complexes are isomorphic,*

Proof. In the situation of (1) let $t : f \circ h \to e \circ g$ be a 2-morphism. The map on complexes is given in degree n by pullback along the 1-morphisms $\mathcal{V}_n \to \mathcal{U}_n$ given by the rule

$$(v_0, \ldots, v_n, y, \beta_0, \ldots, \beta_n) \longmapsto (h(v_0), \ldots, h(v_n), e(y), e(\beta_0) \circ t_{v_0}, \ldots, e(\beta_n) \circ t_{v_n}).$$

For (2), note that pullback on global sections is an isomorphism for any presheaf of sets when the pullback is along an equivalence of categories. Part (3) follows on combining (1) and (2). □

Lemma 69.17.2. *If there exists a* 1*-morphism* $s : \mathcal{X} \to \mathcal{U}$ *such that* $f \circ s$ *is* 2*-isomorphic to* $id_\mathcal{X}$ *then the extended Čech complex is homotopic to zero.*

Proof. Set $\mathcal{U}' = \mathcal{U} \times_\mathcal{X} \mathcal{X}$ equal to the fibre product as described in Categories, Lemma 4.30.3. Set $f' : \mathcal{U}' \to \mathcal{X}$ equal to the second projection. Then $\mathcal{U} \to \mathcal{U}'$, $u \mapsto (u, f(x), 1)$ is an equivalence over $\mathcal{X}$, hence we may replace $(\mathcal{U}, f)$ by $(\mathcal{U}', f')$ by Lemma 69.17.1. The advantage of this is that now f' has a section s' such that $f' \circ s' = \mathrm{id}_\mathcal{X}$ on the nose. Namely, if $t : s \circ f \to \mathrm{id}_\mathcal{X}$ is a 2-isomorphism then we can set $s'(x) = (s(x), x, t_x)$. Thus we may assume that $f \circ s = \mathrm{id}_\mathcal{X}$.

In the case that $f \circ s = \mathrm{id}_\mathcal{X}$ the result follows from general principles. We give the homotopy explicitly. Namely, for $n \geq 0$ define $s_n : \mathcal{U}_n \to \mathcal{U}_{n+1}$ to be the 1-morphism defined by the rule on objects

$$(u_0, \ldots, u_n, x, \alpha_0, \ldots, \alpha_n) \longmapsto (u_0, \ldots, u_n, s(x), x, \alpha_0, \ldots, \alpha_n, \mathrm{id}_x).$$

Define

$$h^{n+1} : \Gamma(\mathcal{U}_{n+1}, f_{n+1}^{-1}\mathcal{F}) \longrightarrow \Gamma(\mathcal{U}_n, f_n^{-1}\mathcal{F})$$

as pullback along s_n. We also set $s_{-1} = s$ and $h^0 : \Gamma(\mathcal{U}_0, f_0^{-1}\mathcal{F}) \to \Gamma(\mathcal{X}, \mathcal{F})$ equal to pullback along s_{-1}. Then the family of maps $\{h^n\}_{n \geq 0}$ is a homotopy between 1 and 0 on the extended Čech complex. □

69.18. The relative Čech complex

Let $f : \mathcal{U} \to \mathcal{X}$ be a 1-morphism of categories fibred in groupoids over $(Sch/S)_{fppf}$ as in (69.17.0.1). Consider the associated simplicial object $\mathcal{U}_\bullet$ and the maps $f_n : \mathcal{U}_n \to \mathcal{X}$. Let $\tau \in \{Zar, étale, smooth, syntomic, fppf\}$. Finally, suppose that $\mathcal{F}$ is a sheaf (of sets) on $\mathcal{X}_\tau$. Then

$$f_{0,*}f_0^{-1}\mathcal{F} \rightrightarrows f_{1,*}f_1^{-1}\mathcal{F} \mathrel{\substack{\rightarrow\\[-1ex]\rightarrow\\[-1ex]\rightarrow}} f_{2,*}f_2^{-1}\mathcal{F}$$

is a cosimplicial sheaf on $\mathcal{X}_\tau$ where we use the pullback maps introduced in Sites, Section 7.44. If $\mathcal{F}$ is an abelian sheaf, then $f_{n,*}f_n^{-1}\mathcal{F}$ form a cosimplicial abelian sheaf on $\mathcal{X}_\tau$. The associated complex (see Simplicial, Section 14.23)

$$\ldots \to 0 \to f_{0,*}f_0^{-1}\mathcal{F} \to f_{1,*}f_1^{-1}\mathcal{F} \to f_{2,*}f_2^{-1}\mathcal{F} \to \ldots$$

is called the *relative Čech complex* associated to the situation. We will denote this complex $\mathcal{K}^\bullet(f, \mathcal{F})$. The *extended relative Čech complex* is the complex

$$\ldots \to 0 \to \mathcal{F} \to f_{0,*}f_0^{-1}\mathcal{F} \to f_{1,*}f_1^{-1}\mathcal{F} \to f_{2,*}f_2^{-1}\mathcal{F} \to \ldots$$

with $\mathcal{F}$ in degree -1. The extended relative Čech complex is acyclic if and only if the map $\mathcal{F}[0] \to \mathcal{K}^\bullet(f, \mathcal{F})$ is a quasi-isomorphism of complexes of sheaves.

Remark 69.18.1. We can define the complex $\mathcal{K}^\bullet(f, \mathcal{F})$ also if $\mathcal{F}$ is a presheaf, only we cannot use the reference to Sites, Section 7.44 to define the pullback maps. To explain the pullback maps, suppose given a commutative diagram

$$\begin{array}{ccc} \mathcal{V} & \xrightarrow{h} & \mathcal{U} \\ & {\scriptstyle g}\searrow \quad \swarrow{\scriptstyle f} & \\ & \mathcal{X} & \end{array}$$

of categories fibred in groupoids over $(Sch/S)_{fppf}$ and a presheaf $\mathcal{G}$ on $\mathcal{U}$ we can define the pullback map $f_*\mathcal{G} \to g_*h^{-1}\mathcal{G}$ as the composition

$$f_*\mathcal{G} \longrightarrow f_*h_*h^{-1}\mathcal{G} = g_*h^{-1}\mathcal{G}$$

where the map comes from the adjunction map $\mathcal{G} \to h_*h^{-1}\mathcal{G}$. This works because in our situation the functors h_* and h^{-1} are adjoint in presheaves (and agree with their counter parts on sheaves). See Sections 69.3 and 69.4.

Lemma 69.18.2. *Generalities on relative Čech complexes.*

(1) *If*

$$\begin{array}{ccc} \mathcal{V} & \xrightarrow{h} & \mathcal{U} \\ {\scriptstyle g}\downarrow & & \downarrow{\scriptstyle f} \\ \mathcal{Y} & \xrightarrow{e} & \mathcal{X} \end{array}$$

is a 2-commutative diagram of categories fibred in groupoids over $(Sch/S)_{fppf}$, then there is a morphism $e^{-1}\mathcal{K}^\bullet(f, \mathcal{F}) \to \mathcal{K}^\bullet(g, e^{-1}\mathcal{F})$.

(2) *if h and e are equivalences, then the map of (1) is an isomorphism,*

(3) *if $f, f' : \mathcal{U} \to \mathcal{X}$ are 2-isomorphic, then the associated relative Čech complexes are isomorphic,*

Proof. Literally the same as the proof of Lemma 69.17.1 using the pullback maps of Remark 69.18.1. □

Lemma 69.18.3. *If there exists a 1-morphism $s : \mathcal{X} \to \mathcal{U}$ such that $f \circ s$ is 2-isomorphic to $id_\mathcal{X}$ then the extended relative Čech complex is homotopic to zero.*

Proof. Literally the same as the proof of Lemma 69.17.2. □

Remark 69.18.4. Let us "compute" the value of the relative Čech complex on an object x of $\mathcal{X}$. Say $p(x) = U$. Consider the 2-fibre product diagram (which serves to introduce the notation $g : \mathcal{V} \to \mathcal{Y}$)

$$\begin{array}{ccccc} \mathcal{V} & = & (Sch/U)_{fppf} \times_{x,\mathcal{X}} \mathcal{U} & \longrightarrow & \mathcal{U} \\ {\scriptstyle g}\downarrow & & \downarrow & & \downarrow{\scriptstyle f} \\ \mathcal{Y} & = & (Sch/U)_{fppf} & \xrightarrow{x} & \mathcal{X} \end{array}$$

Note that the morphism $\mathcal{V}_n \to \mathcal{U}_n$ of the proof of Lemma 69.17.1 induces an equivalence $\mathcal{V}_n = (Sch/U)_{fppf} \times_{x,\mathcal{X}} \mathcal{U}_n$. Hence we see from (69.5.0.1) that

$$\Gamma(x, \mathcal{K}^\bullet(f, \mathcal{F})) = \check{\mathcal{C}}^\bullet(\mathcal{V} \to \mathcal{Y}, x^{-1}\mathcal{F})$$

In words: The value of the relative Čech complex on an object x of $\mathcal{X}$ is the Čech complex of the base change of f to $\mathcal{X}/x \cong (Sch/U)_{fppf}$. This implies for example that Lemma 69.17.2 implies Lemma 69.18.3 and more generally that results on the (usual) Čech complex imply results for the relative Čech complex.

Lemma 69.18.5. *Let*

$$\begin{array}{ccc} \mathcal{V} & \xrightarrow[h]{} & \mathcal{U} \\ {\scriptstyle g}\downarrow & & \downarrow{\scriptstyle f} \\ \mathcal{Y} & \xrightarrow{e} & \mathcal{X} \end{array}$$

be a 2-fibre product of categories fibred in groupoids over $(Sch/S)_{fppf}$ and let $\mathcal{F}$ be an abelian presheaf on $\mathcal{X}$. Then the map $e^{-1}\mathcal{K}^\bullet(f, \mathcal{F}) \to \mathcal{K}^\bullet(g, e^{-1}\mathcal{F})$ of Lemma 69.18.2 is an isomorphism of complexes of abelian presheaves.

Proof. Let y be an object of $\mathcal{Y}$ lying over the scheme T. Set $x = e(y)$. We are going to show that the map induces an isomorphism on sections over y. Note that

$$\Gamma(y, e^{-1}\mathcal{K}^\bullet(f, \mathcal{F})) = \Gamma(x, \mathcal{K}^\bullet(f, \mathcal{F})) = \check{\mathcal{C}}^\bullet((Sch/T)_{fppf} \times_{x,\mathcal{X}} \mathcal{U} \to (Sch/T)_{fppf}, x^{-1}\mathcal{F})$$

by Remark 69.18.4. On the other hand,

$$\Gamma(y, \mathcal{K}^\bullet(g, e^{-1}\mathcal{F})) = \check{\mathcal{C}}^\bullet((Sch/T)_{fppf} \times_{y,\mathcal{Y}} \mathcal{V} \to (Sch/T)_{fppf}, y^{-1}e^{-1}\mathcal{F})$$

also by Remark 69.18.4. Note that $y^{-1}e^{-1}\mathcal{F} = x^{-1}\mathcal{F}$ and since the diagram is 2-cartesian the 1-morphism

$$(Sch/T)_{fppf} \times_{y,\mathcal{Y}} \mathcal{V} \to (Sch/T)_{fppf} \times_{x,\mathcal{X}} \mathcal{U}$$

is an equivalence. Hence the map on sections over y is an isomorphism by Lemma 69.17.1. □

Exactness can be checked on a "covering".

Lemma 69.18.6. *Let $f : \mathcal{U} \to \mathcal{X}$ be a 1-morphism of categories fibred in groupoids over $(Sch/S)_{fppf}$. Let $\tau \in \{Zar, étale, smooth, syntomic, fppf\}$. Let*

$$\mathcal{F} \to \mathcal{G} \to \mathcal{H}$$

be a complex in $Ab(\mathcal{X}_\tau)$. Assume that

1. *for every object x of $\mathcal{X}$ there exists a covering $\{x_i \to x\}$ in $\mathcal{X}_\tau$ such that each x_i is isomorphic to $f(u_i)$ for some object u_i of $\mathcal{U}$, and*
2. *$f^{-1}\mathcal{F} \to f^{-1}\mathcal{G} \to f^{-1}\mathcal{H}$ is exact.*

Then the sequence $\mathcal{F} \to \mathcal{G} \to \mathcal{H}$ is exact.

Proof. Let x be an object of $\mathcal{X}$ lying over the scheme T. Consider the sequence $x^{-1}\mathcal{F} \to x^{-1}\mathcal{G} \to x^{-1}\mathcal{H}$ of abelian sheaves on $(Sch/T)_\tau$. It suffices to show this sequence is exact. By assumption there exists a τ-covering $\{T_i \to T\}$ such that $x|_{T_i}$ is isomorphic to $f(u_i)$ for some object u_i of $\mathcal{U}$ over T_i and moreover the sequence $u_i^{-1}f^{-1}\mathcal{F} \to u_i^{-1}f^{-1}\mathcal{G} \to u_i^{-1}f^{-1}\mathcal{H}$ of abelian sheaves on $(Sch/T_i)_\tau$ is exact. Since $u_i^{-1}f^{-1}\mathcal{F} = x^{-1}\mathcal{F}|_{(Sch/T_i)_\tau}$ we conclude that the sequence $x^{-1}\mathcal{F} \to x^{-1}\mathcal{G} \to x^{-1}\mathcal{H}$ become exact after localizing at each of the members of a covering, hence the sequence is exact. □

Proposition 69.18.7. *Let $f : \mathcal{U} \to \mathcal{X}$ be a 1-morphism of categories fibred in groupoids over $(Sch/S)_{fppf}$. Let $\tau \in \{Zar, étale, smooth, syntomic, fppf\}$. If*

1. *$\mathcal{F}$ is an abelian sheaf on $\mathcal{X}_\tau$, and*
2. *for every object x of $\mathcal{X}$ there exists a covering $\{x_i \to x\}$ in $\mathcal{X}_\tau$ such that each x_i is isomorphic to $f(u_i)$ for some object u_i of $\mathcal{U}$,*

then the extended relative Čech complex

$$\ldots \to 0 \to \mathcal{F} \to f_{0,*}f_0^{-1}\mathcal{F} \to f_{1,*}f_1^{-1}\mathcal{F} \to f_{2,*}f_2^{-1}\mathcal{F} \to \ldots$$

is exact in $Ab(\mathcal{X}_\tau)$.

Proof. By Lemma 69.18.6 it suffices to check exactness after pulling back to $\mathcal{U}$. By Lemma 69.18.5 the pullback of the extended relative Čech complex is isomorphic to the extend relative Čech complex for the morphism $\mathcal{U} \times_\mathcal{X} \mathcal{U} \to \mathcal{U}$ and an abelian sheaf on $\mathcal{U}_\tau$. Since there is a section $\Delta_{\mathcal{U}/\mathcal{X}} : \mathcal{U} \to \mathcal{U} \times_\mathcal{X} \mathcal{U}$ exactness follows from Lemma 69.18.3. □

Using this we can construct the Čech-to-cohomology spectral sequence as follows. We first give a technical, precise version. In the next section we give a version that applies only to algebraic stacks.

Lemma 69.18.8. *Let $f : \mathcal{U} \to \mathcal{X}$ be a 1-morphism of categories fibred in groupoids over $(Sch/S)_{fppf}$. Let $\tau \in \{Zar, étale, smooth, syntomic, fppf\}$. Assume*

1. *$\mathcal{F}$ is an abelian sheaf on $\mathcal{X}_\tau$,*
2. *for every object x of $\mathcal{X}$ there exists a covering $\{x_i \to x\}$ in $\mathcal{X}_\tau$ such that each x_i is isomorphic to $f(u_i)$ for some object u_i of $\mathcal{U}$,*
3. *the category $\mathcal{U}$ has equalizers, and*
4. *the functor f is faithful.*

Then there is a first quadrant spectral sequence of abelian groups

$$E_2^{p,q} = H^q((\mathcal{U}_p)_\tau, f_p^{-1}\mathcal{F}) \Rightarrow H^{p+q}(\mathcal{X}_\tau, \mathcal{F})$$

converging to the cohomology of $\mathcal{F}$ in the τ-topology.

Proof. Before we start the proof we make some remarks. By Lemma 69.16.4 (and induction) all of the categories fibred in groupoids $\mathcal{U}_p$ have equalizers and all of the morphisms $f_p : \mathcal{U}_p \to \mathcal{X}$ are faithful. Let $\mathcal{I}$ be an injective object of $Ab(\mathcal{X}_\tau)$. By Lemma 69.16.5 we see $f_p^{-1}\mathcal{I}$ is an injective object of $Ab((\mathcal{U}_p)_\tau)$. Hence $f_{p,*}f_p^{-1}\mathcal{I}$ is an injective object of $Ab(\mathcal{X}_\tau)$ by Lemma 69.16.1. Hence Proposition 69.18.7 shows that the extended relative Čech complex

$$\ldots \to 0 \to \mathcal{I} \to f_{0,*}f_0^{-1}\mathcal{I} \to f_{1,*}f_1^{-1}\mathcal{I} \to f_{2,*}f_2^{-1}\mathcal{I} \to \ldots$$

is an exact complex in $Ab(\mathcal{X}_\tau)$ all of whose terms are injective. Taking global sections of this complex is exact and we see that the Čech complex $\check{\mathcal{C}}^\bullet(\mathcal{U} \to \mathcal{X}, \mathcal{I})$ is quasi-isomorphic to $\Gamma(\mathcal{X}_\tau, \mathcal{I})[0]$.

With these preliminaries out of the way consider the two spectral sequences associated to the double complex (see Homology, Section 12.22)

$$\check{\mathcal{C}}^\bullet(\mathcal{U} \to \mathcal{X}, \mathcal{I}^\bullet)$$

where $\mathcal{F} \to \mathcal{I}^\bullet$ is an injective resolution in $Ab(\mathcal{X}_\tau)$. The discussion above shows that Homology, Lemma 12.22.6 applies which shows that $\Gamma(\mathcal{X}_\tau, \mathcal{I}^\bullet)$ is quasi-isomorphic to the total complex associated to the double complex. By our remarks above the complex $f_p^{-1}\mathcal{I}^\bullet$ is an injective resolution of $f_p^{-1}\mathcal{F}$. Hence the other spectral sequence is as indicated in the lemma. □

To be sure there is a version for modules as well.

Lemma 69.18.9. *Let $f : \mathcal{U} \to \mathcal{X}$ be a 1-morphism of categories fibred in groupoids over $(Sch/S)_{fppf}$. Let $\tau \in \{Zar, étale, smooth, syntomic, fppf\}$. Assume*

1. *$\mathcal{F}$ is an object of $Mod(\mathcal{X}_\tau, \mathcal{O}_\mathcal{X})$,*
2. *for every object x of $\mathcal{X}$ there exists a covering $\{x_i \to x\}$ in $\mathcal{X}_\tau$ such that each x_i is isomorphic to $f(u_i)$ for some object u_i of $\mathcal{U}$,*
3. *the category $\mathcal{U}$ has equalizers, and*
4. *the functor f is faithful.*

Then there is a first quadrant spectral sequence of $\Gamma(\mathcal{O}_\mathcal{X})$-modules

$$E_2^{p,q} = H^q((\mathcal{U}_p)_\tau, f_p^*\mathcal{F}) \Rightarrow H^{p+q}(\mathcal{X}_\tau, \mathcal{F})$$

converging to the cohomology of $\mathcal{F}$ in the τ-topology.

Proof. The proof of this lemma is identical to the proof of Lemma 69.18.8 except that it uses an injective resolution in $Mod(\mathcal{X}_\tau, \mathcal{O}_\mathcal{X})$ and it uses Lemma 69.16.6 instead of Lemma 69.16.5. □

Here is a lemma that translates a more usual kind of covering in the kinds of coverings we have encountered above.

Lemma 69.18.10. *Let $f : \mathcal{X} \to \mathcal{Y}$ be a 1-morphism of categories fibred in groupoids over $(Sch/S)_{fppf}$.*

1. *Assume that f is representable by algebraic spaces, surjective, flat, and locally of finite presentation. Then for any object y of $\mathcal{Y}$ there exists an fppf covering $\{y_i \to y\}$ and objects x_i of $\mathcal{X}$ such that $f(x_i) \cong y_i$ in $\mathcal{Y}$.*
2. *Assume that f is representable by algebraic spaces, surjective, and smooth. Then for any object y of $\mathcal{Y}$ there exists an étale covering $\{y_i \to y\}$ and objects x_i of $\mathcal{X}$ such that $f(x_i) \cong y_i$ in $\mathcal{Y}$.*

Proof. Proof of (1). Suppose that y lies over the scheme V. We may think of y as a morphism $(Sch/V)_{fppf} \to \mathcal{Y}$. By definition the 2-fibre product $\mathcal{X} \times_\mathcal{Y} (Sch/V)_{fppf}$ is representable by an algebraic space W and the morphism $W \to V$ is surjective, flat, and locally of finite presentation. Choose a scheme U and a surjective étale morphism $U \to W$. Then $U \to V$ is also surjective, flat, and locally of finite presentation (see Morphisms of Spaces, Lemmas 49.35.7, 49.35.8, 49.6.4, 49.26.2, and 49.27.3). Hence $\{U \to V\}$ is an fppf covering. Denote x the object of $\mathcal{X}$ over

U corresponding to the 1-morphism $(Sch/U)_{fppf} \to \mathcal{X}$. Then $\{f(x) \to y\}$ is the desired fppf covering of $\mathcal{Y}$.

Proof of (1). Suppose that y lies over the scheme V. We may think of y as a morphism $(Sch/V)_{fppf} \to \mathcal{Y}$. By definition the 2-fibre product $\mathcal{X} \times_{\mathcal{Y}} (Sch/V)_{fppf}$ is representable by an algebraic space W and the morphism $W \to V$ is surjective and smooth. Choose a scheme U and a surjective étale morphism $U \to W$. Then $U \to V$ is also surjective and smooth (see Morphisms of Spaces, Lemmas 49.35.6, 49.6.4, and 49.33.2). Hence $\{U \to V\}$ is a smooth covering. By More on Morphisms, Lemma 36.28.7 there exists an étale covering $\{V_i \to V\}$ such that each $V_i \to V$ factors through U. Denote x_i the object of $\mathcal{X}$ over V_i corresponding to the 1-morphism

$$(Sch/V_i)_{fppf} \to (Sch/U)_{fppf} \to \mathcal{X}.$$

Then $\{f(x_i) \to y\}$ is the desired étale covering of $\mathcal{Y}$. □

Lemma 69.18.11. *Let $f : \mathcal{U} \to \mathcal{X}$ and $g : \mathcal{X} \to \mathcal{Y}$ be composable 1-morphisms of categories fibred in groupoids over $(Sch/S)_{fppf}$. Let $\tau \in \{Zar, étale, smooth, syntomic, fppf\}$. Assume*

1. *$\mathcal{F}$ is an abelian sheaf on $\mathcal{X}_\tau$,*
2. *for every object x of $\mathcal{X}$ there exists a covering $\{x_i \to x\}$ in $\mathcal{X}_\tau$ such that each x_i is isomorphic to $f(u_i)$ for some object u_i of $\mathcal{U}$,*
3. *the category $\mathcal{U}$ has equalizers, and*
4. *the functor f is faithful.*

Then there is a first quadrant spectral sequence of abelian sheaves on $\mathcal{Y}_\tau$

$$E_2^{p,q} = R^q(g \circ f_p)_* f_p^{-1}\mathcal{F} \Rightarrow R^{p+q} g_* \mathcal{F}$$

where all higher direct images are computed in the τ-topology.

Proof. Note that the assumptions on $f : \mathcal{U} \to \mathcal{X}$ and $\mathcal{F}$ are identical to those in Lemma 69.18.8. Hence the preliminary remarks made in the proof of that lemma hold here also. These remarks imply in particular that

$$0 \to g_*\mathcal{I} \to (g \circ f_0)_* f_0^{-1}\mathcal{I} \to (g \circ f_1)_* f_1^{-1}\mathcal{I} \to \ldots$$

is exact if $\mathcal{I}$ is an injective object of $Ab(\mathcal{X}_\tau)$. Having said this, consider the two spectral sequences of Homology, Section 12.22 associated to the double complex $\mathcal{C}^{\bullet,\bullet}$ with terms

$$\mathcal{C}^{p,q} = (g \circ f_p)_* \mathcal{I}^q$$

where $\mathcal{F} \to \mathcal{I}^\bullet$ is an injective resolution in $Ab(\mathcal{X}_\tau)$. The first spectral sequence implies, via Homology, Lemma 12.22.6, that $g_*\mathcal{I}^\bullet$ is quasi-isomorphic to the total complex associated to $\mathcal{C}^{\bullet,\bullet}$. Since $f_p^{-1}\mathcal{I}^\bullet$ is an injective resolution of $f_p^{-1}\mathcal{F}$ (see Lemma 69.16.5) the second spectral sequence has terms $E_2^{p,q} = R^q(g \circ f_p)_* f_p^{-1}\mathcal{F}$ as in the statement of the lemma. □

Lemma 69.18.12. *Let $f : \mathcal{U} \to \mathcal{X}$ and $g : \mathcal{X} \to \mathcal{Y}$ be composable 1-morphisms of categories fibred in groupoids over $(Sch/S)_{fppf}$. Let $\tau \in \{Zar, étale, smooth, syntomic, fppf\}$. Assume*

1. *$\mathcal{F}$ is an object of $Mod(\mathcal{X}_\tau, \mathcal{O}_\mathcal{X})$,*
2. *for every object x of $\mathcal{X}$ there exists a covering $\{x_i \to x\}$ in $\mathcal{X}_\tau$ such that each x_i is isomorphic to $f(u_i)$ for some object u_i of $\mathcal{U}$,*
3. *the category $\mathcal{U}$ has equalizers, and*

(4) *the functor f is faithful.*

Then there is a first quadrant spectral sequence in $Mod(\mathcal{Y}_\tau, \mathcal{O}_\mathcal{Y})$

$$E_2^{p,q} = R^q(g \circ f_p)_* f_p^{-1}\mathcal{F} \Rightarrow R^{p+q} g_* \mathcal{F}$$

where all higher direct images are computed in the τ-topology.

Proof. The proof is identical to the proof of Lemma 69.18.11 except that it uses an injective resolution in $Mod(\mathcal{X}_\tau, \mathcal{O}_\mathcal{X})$ and it uses Lemma 69.16.6 instead of Lemma 69.16.5. □

69.19. Cohomology on algebraic stacks

Let $\mathcal{X}$ be an algebraic stack over S. In the sections above we have seen how to define sheaves for the étale, ..., fppf topologies on $\mathcal{X}$. In fact, we have constructed a site $\mathcal{X}_\tau$ for each $\tau \in \{Zar, étale, smooth, syntomic, fppf\}$. There is a notion of an abelian sheaf $\mathcal{F}$ on these sites. In the chapter on cohomology of sites we have explained how to define cohomology. Putting all of this together, let's define the *derived global sections*

$$R\Gamma_{Zar}(\mathcal{X}, \mathcal{F}), R\Gamma_{étale}(\mathcal{X}, \mathcal{F}), \ldots, R\Gamma_{fppf}(\mathcal{X}, \mathcal{F})$$

as $\Gamma(\mathcal{X}_\tau, \mathcal{I}^\bullet)$ where $\mathcal{F} \to \mathcal{I}^\bullet$ is an injective resolution in $Ab(\mathcal{X}_\tau)$. The ith cohomology group is the ith cohomology of the total derived cohomology. We will denote this

$$H^i_{Zar}(\mathcal{X}, \mathcal{F}), H^i_{étale}(\mathcal{X}, \mathcal{F}), \ldots, H^i_{fppf}(\mathcal{X}, \mathcal{F}).$$

It will turn out that $H^i_{étale} = H^i_{smooth}$ because of More on Morphisms, Lemma 36.28.7. If $\mathcal{F}$ is a presheaf of $\mathcal{O}_\mathcal{X}$-modules which is a sheaf in the τ-topology, then we use injective resolutions in $Mod(\mathcal{X}_\tau, \mathcal{O}_\mathcal{X})$ to compute total derived global sections and cohomology groups; of course the end result is quasi-isomorphic resp. isomorphic by the general fact Cohomology on Sites, Lemma 21.12.4.

So far our only tool to compute cohomology groups is the result on Čech complexes proved above. We rephrase it here in the language of algebraic stacks for the étale and the fppf topology. Let $f : \mathcal{U} \to \mathcal{X}$ be a 1-morphism of algebraic stacks. Recall that

$$f_p : \mathcal{U}_p = \mathcal{U} \times_\mathcal{X} \ldots \times_\mathcal{X} \mathcal{U} \longrightarrow \mathcal{X}$$

is the structure morphism where there are $(p+1)$-factors. Also, recall that a sheaf on $\mathcal{X}$ is a sheaf for the fppf topology. Note that if $\mathcal{U}$ is an algebraic space, then $f : \mathcal{U} \to \mathcal{X}$ is representable by algebraic spaces, see Algebraic Stacks, Lemma 67.10.11. Thus the proposition applies in particular to a smooth cover of the algebraic stack $\mathcal{X}$ by a scheme.

Proposition 69.19.1. *Let $f : \mathcal{U} \to \mathcal{X}$ be a 1-morphism of algebraic stacks.*

(1) *Let $\mathcal{F}$ be an abelian étale sheaf on $\mathcal{X}$. Assume that f is representable by algebraic spaces, surjective, and smooth. Then there is a spectral sequence*

$$E_2^{p,q} = H^q_{étale}(\mathcal{U}_p, f_p^{-1}\mathcal{F}) \Rightarrow H^{p+q}_{étale}(\mathcal{X}, \mathcal{F})$$

(2) *Let $\mathcal{F}$ be an abelian sheaf on $\mathcal{X}$. Assume that f is representable by algebraic spaces, surjective, flat, and locally of finite presentation. Then there is a spectral sequence*

$$E_2^{p,q} = H^q_{fppf}(\mathcal{U}_p, f_p^{-1}\mathcal{F}) \Rightarrow H^{p+q}_{fppf}(\mathcal{X}, \mathcal{F})$$

Proof. To see this we will check the hypotheses (1) – (4) of Lemma 69.18.8. The 1-morphism f is faithful by Algebraic Stacks, Lemma 67.15.2. This proves (4). Hypothesis (3) follows from the fact that $\mathcal{U}$ is an algebraic stack, see Lemma 69.16.2. To see (2) apply Lemma 69.18.10. Condition (1) is satisfied by fiat. □

69.20. Higher direct images and algebraic stacks

Let $g : \mathcal{X} \to \mathcal{Y}$ be a 1-morphism of algebraic stacks over S. In the sections above we have constructed a morphism of ringed topoi $g : Sh(\mathcal{X}_\tau) \to Sh(\mathcal{Y}_\tau)$ for each $\tau \in \{Zar, étale, smooth, syntomic, fppf\}$. In the chapter on cohomology of sites we have explained how to define higher direct images. Hence the *derived direct image* $Rg_*\mathcal{F}$ is defined as $g_*\mathcal{I}^\bullet$ where $\mathcal{F} \to \mathcal{I}^\bullet$ is an injective resolution in $Ab(\mathcal{X}_\tau)$. The ith higher direct image $R^ig_*\mathcal{F}$ is the ith cohomology of the derived direct image. Important: it matters which topology τ is used here!

If $\mathcal{F}$ is a presheaf of $\mathcal{O}_\mathcal{X}$-modules which is a sheaf in the τ-topology, then we use injective resolutions in $Mod(\mathcal{X}_\tau, \mathcal{O}_\mathcal{X})$ to compute derived direct image and higher direct images.

So far our only tool to compute the higher direct images of g_* is the result on Čech complexes proved above. This requires the choice of a "covering" $f : \mathcal{U} \to \mathcal{X}$. If $\mathcal{U}$ is an algebraic space, then $f : \mathcal{U} \to \mathcal{X}$ is representable by algebraic spaces, see Algebraic Stacks, Lemma 67.10.11. Thus the proposition applies in particular to a smooth cover of the algebraic stack $\mathcal{X}$ by a scheme.

Proposition 69.20.1. *Let $f : \mathcal{U} \to \mathcal{X}$ and $g : \mathcal{X} \to \mathcal{Y}$ be composable 1-morphisms of algebraic stacks.*

(1) *Assume that f is representable by algebraic spaces, surjective and smooth.*
   (a) *If $\mathcal{F}$ is in $Ab(\mathcal{X}_{étale})$ then there is a spectral sequence*
   $$E_2^{p,q} = R^q(g \circ f_p)_* f_p^{-1}\mathcal{F} \Rightarrow R^{p+q}g_*\mathcal{F}$$
   in $Ab(\mathcal{Y}_{étale})$ with higher direct images computed in the étale topology.
   (b) *If $\mathcal{F}$ is in $Mod(\mathcal{X}_{étale}, \mathcal{O}_\mathcal{X})$ then there is a spectral sequence*
   $$E_2^{p,q} = R^q(g \circ f_p)_* f_p^{-1}\mathcal{F} \Rightarrow R^{p+q}g_*\mathcal{F}$$
   in $Mod(\mathcal{Y}_{étale}, \mathcal{O}_\mathcal{Y})$.

(2) *Assume that f is representable by algebraic spaces, surjective, flat, and locally of finite presentation.*
   (a) *If $\mathcal{F}$ is in $Ab(\mathcal{X})$ then there is a spectral sequence*
   $$E_2^{p,q} = R^q(g \circ f_p)_* f_p^{-1}\mathcal{F} \Rightarrow R^{p+q}g_*\mathcal{F}$$
   in $Ab(\mathcal{Y})$ with higher direct images computed in the fppf topology.
   (b) *If $\mathcal{F}$ is in $Mod(\mathcal{O}_\mathcal{X})$ then there is a spectral sequence*
   $$E_2^{p,q} = R^q(g \circ f_p)_* f_p^{-1}\mathcal{F} \Rightarrow R^{p+q}g_*\mathcal{F}$$
   in $Mod(\mathcal{O}_\mathcal{Y})$.

Proof. To see this we will check the hypotheses (1) – (4) of Lemma 69.18.11 and Lemma 69.18.12. The 1-morphism f is faithful by Algebraic Stacks, Lemma 67.15.2. This proves (4). Hypothesis (3) follows from the fact that $\mathcal{U}$ is an algebraic stack, see Lemma 69.16.2. To see (2) apply Lemma 69.18.10. Condition (1) is satisfied by fiat in all four cases. □

Here is a description of higher direct images for a morphism of algebraic stacks.

Lemma 69.20.2. *Let S be a scheme. Let $f : \mathcal{X} \to \mathcal{Y}$ be a 1-morphism of algebraic stacks[4] over S. Let $\tau \in \{Zariski, étale, smooth, syntomic, fppf\}$. Let $\mathcal{F}$ be an object of $Ab(\mathcal{X}_\tau)$ or $Mod(\mathcal{X}_\tau, \mathcal{O}_\mathcal{X})$. Then the sheaf $R^i f_* \mathcal{F}$ is the sheaf associated to the presheaf*

$$y \longmapsto H^i_\tau\Big((Sch/V)_{fppf} \times_{y,\mathcal{Y}} \mathcal{X},\ pr^{-1}\mathcal{F}\Big)$$

Here y is a typical object of $\mathcal{Y}$ lying over the scheme V.

Proof. Choose an injective resolution $\mathcal{F}[0] \to \mathcal{I}^\bullet$. By the formula for pushforward (69.5.0.1) we see that $R^i f_* \mathcal{F}$ is the sheaf associated to the presheaf which associates to y the cohomology of the complex

$$\begin{array}{c} \Gamma\Big((Sch/V)_{fppf} \times_{y,\mathcal{Y}} \mathcal{X},\ \mathrm{pr}^{-1}\mathcal{I}^{i-1}\Big) \\ \downarrow \\ \Gamma\Big((Sch/V)_{fppf} \times_{y,\mathcal{Y}} \mathcal{X},\ \mathrm{pr}^{-1}\mathcal{I}^{i}\Big) \\ \downarrow \\ \Gamma\Big((Sch/V)_{fppf} \times_{y,\mathcal{Y}} \mathcal{X},\ \mathrm{pr}^{-1}\mathcal{I}^{i+1}\Big) \end{array}$$

Since pr^{-1} is exact, it suffices to show that pr^{-1} preserves injectives. This follows from Lemmas 69.16.5 and 69.16.6 as well as the fact that pr is a representable morphism of algebraic stacks (so that pr is faithful by Algebraic Stacks, Lemma 67.15.2 and that $(Sch/V)_{fppf} \times_{y,\mathcal{Y}} \mathcal{X}$ has equalizers by Lemma 69.16.2). □

Here is a trivial base change result.

Lemma 69.20.3. *Let S be a scheme. Let $\tau \in \{Zariski, étale, smooth, syntomic, fppf\}$. Let*

$$\begin{array}{ccc} \mathcal{Y}' \times_\mathcal{Y} \mathcal{X} & \xrightarrow{g'} & \mathcal{X} \\ {\scriptstyle f'}\downarrow & & \downarrow{\scriptstyle f} \\ \mathcal{Y}' & \xrightarrow{g} & \mathcal{Y} \end{array}$$

be a 2-cartesian diagram of algebraic stacks over S. Then the base change map is an isomorphism

$$g^{-1} R f_* \mathcal{F} \longrightarrow R f'_* (g')^{-1} \mathcal{F}$$

functorial for $\mathcal{F}$ in $Ab(\mathcal{X}_\tau)$ or $\mathcal{F}$ in $Mod(\mathcal{X}_\tau, \mathcal{O}_\mathcal{X})$.

Proof. The isomorphism $g^{-1} f_* \mathcal{F} = f'_* (g')^{-1} \mathcal{F}$ is Lemma 69.5.1 (and it holds for arbitrary presheaves). For the derived direct images, there is a base change map because the morphisms g and g' are flat, see Cohomology on Sites, Section 21.15. To see that this map is a quasi-isomorphism we can use that for an object y' of $\mathcal{Y}'$ over a scheme V there is an equivalence

$$(Sch/V)_{fppf} \times_{g(y'),\mathcal{Y}} \mathcal{X} = (Sch/V)_{fppf} \times_{y',\mathcal{Y}'} (\mathcal{Y}' \times_\mathcal{Y} \mathcal{X})$$

We conclude that the induced map $g^{-1} R^i f_* \mathcal{F} \to R^i f'_* (g')^{-1} \mathcal{F}$ is an isomorphism by Lemma 69.20.2. □

[4] This result should hold for any 1-morphism of categories fibred in groupoids over $(Sch/S)_{fppf}$.

69.21. Comparison

In this section we collect some results on comparing cohomology defined using stacks and using algebraic spaces.

Lemma 69.21.1. *Let S be a scheme. Let $\mathcal{X}$ be an algebraic stack over S representable by the algebraic space F.*

(1) *$\mathcal{I}|_{F_{étale}}$ is injective in $Ab(F_{étale})$ for $\mathcal{I}$ injective in $Ab(\mathcal{X}_{étale})$, and*
(2) *$\mathcal{I}|_{F_{étale}}$ is injective in $Mod(F_{étale}, \mathcal{O}_F)$ for $\mathcal{I}$ injective in $Mod(\mathcal{X}_{étale}, \mathcal{O})$.*

Proof. This follows formally from the fact that the restriction functor $\pi_{F,*} = i_F^{-1}$ (see Lemma 69.10.1) is an exact left adjoint of $i_{F,*}$, see Homology, Lemma 12.25.1. □

Lemma 69.21.2. *Let S be a scheme. Let $f : \mathcal{X} \to \mathcal{Y}$ be a morphism of algebraic stacks over S. Assume $\mathcal{X}$, $\mathcal{Y}$ are representable by algebraic spaces F, G. Denote $f : F \to G$ the induced morphism of algebraic spaces.*

(1) *For any $\mathcal{F} \in Ab(\mathcal{X}_{étale})$ we have*
$$(Rf_*\mathcal{F})|_{G_{étale}} = Rf_{small,*}(\mathcal{F}|_{F_{étale}})$$
in $D(G_{étale})$.
(2) *For any object $\mathcal{F}$ of $Mod(\mathcal{X}_{étale}, \mathcal{O}_\mathcal{X})$ we have*
$$(Rf_*\mathcal{F})|_{G_{étale}} = Rf_{small,*}(\mathcal{F}|_{F_{étale}})$$
in $D(\mathcal{O}_G)$.

Proof. Follows immediately from Lemma 69.21.1 and (69.10.2.1) on choosing an injective resolution of $\mathcal{F}$. □

Lemma 69.21.3. *Let S be a scheme. Consider a 2-fibre product square*
$$\begin{array}{ccc} \mathcal{X}' & \xrightarrow{g'} & \mathcal{X} \\ {\scriptstyle f'}\downarrow & & \downarrow{\scriptstyle f} \\ \mathcal{Y}' & \xrightarrow{g} & \mathcal{Y} \end{array}$$
of algebraic stacks over S. Assume that f is representable by algebraic spaces and that $\mathcal{Y}'$ is representable by an algebraic space G'. Then $\mathcal{X}'$ is representable by an algebraic space F' and denoting $f' : F' \to G'$ the induced morphism of algebraic spaces we have
$$g^{-1}(Rf_*\mathcal{F})|_{G'_{étale}} = Rf'_{small,*}((g')^{-1}\mathcal{F}|_{F'_{étale}})$$
for any $\mathcal{F}$ in $Ab(\mathcal{X}_{étale})$ or in $Mod(\mathcal{X}_{étale}, \mathcal{O}_\mathcal{X})$

Proof. Follows formally on combining Lemmas 69.20.3 and 69.21.2. □

69.22. Change of topology

Here is a technical lemma which tells us that the fppf cohomology of a locally quasi-coherent sheaf is equal to its étale cohomology provided the comparison maps are isomorphisms for morphisms of $\mathcal{X}$ lying over flat morphisms.

Lemma 69.22.1. *Let S be a scheme. Let $\mathcal{X}$ be an algebraic stack over S. Let $\mathcal{F}$ be a presheaf of $\mathcal{O}_\mathcal{X}$-modules. Assume*

(a) *$\mathcal{F}$ is locally quasi-coherent, and*

(b) *for any morphism $\varphi : x \to y$ of $\mathcal{X}$ which lies over a morphism of schemes $f : U \to V$ which is flat and locally of finite presentation the comparison map $c_\varphi : f^*_{small}\mathcal{F}|_{V_{étale}} \to \mathcal{F}|_{U_{étale}}$ of (69.9.4.1) is an isomorphism.*

Then $\mathcal{F}$ is a sheaf for the fppf topology.

Proof. Let $\{x_i \to x\}$ be an fppf covering of $\mathcal{X}$ lying over the fppf covering $\{f_i : U_i \to U\}$ of schemes over S. By assumption the restriction $\mathcal{G} = \mathcal{F}|_{U_{étale}}$ is quasi-coherent and the comparison maps $f^*_{i,small}\mathcal{G} \to \mathcal{F}|_{U_{i,étale}}$ are isomorphisms. Hence the sheaf condition for $\mathcal{F}$ and the covering $\{x_i \to x\}$ is equivalent to the sheaf condition for $\mathcal{G}^a$ on $(Sch/U)_{fppf}$ and the covering $\{U_i \to U\}$ which holds by Descent, Lemma 34.7.1. □

Lemma 69.22.2. *Let S be a scheme. Let $\mathcal{X}$ be an algebraic stack over S. Let $\mathcal{F}$ be a presheaf $\mathcal{O}_\mathcal{X}$-module such that*

(a) *$\mathcal{F}$ is locally quasi-coherent, and*
(b) *for any morphism $\varphi : x \to y$ of $\mathcal{X}$ which lies over a morphism of schemes $f : U \to V$ which is flat and locally of finite presentation, the comparison map $c_\varphi : f^*_{small}\mathcal{F}|_{V_{étale}} \to \mathcal{F}|_{U_{étale}}$ of (69.9.4.1) is an isomorphism.*

Then $\mathcal{F}$ is an $\mathcal{O}_\mathcal{X}$-module and we have the following

(1) *If $\epsilon : \mathcal{X}_{fppf} \to \mathcal{X}_{étale}$ is the comparison morphism, then $R\epsilon_*\mathcal{F} = \epsilon_*\mathcal{F}$.*
(2) *The cohomology groups $H^p_{fppf}(\mathcal{X}, \mathcal{F})$ are equal to the cohomology groups computed in the étale topology on $\mathcal{X}$. Similarly for the cohomology groups $H^p_{fppf}(x, \mathcal{F})$ and the derived versions $R\Gamma(\mathcal{X}, \mathcal{F})$ and $R\Gamma(x, \mathcal{F})$.*
(3) *If $f : \mathcal{X} \to \mathcal{Y}$ is a 1-morphism of categories fibred in groupoids over $(Sch/S)_{fppf}$ then $R^i f_*\mathcal{F}$ is equal to the fppf-sheafification of the higher direct image computed in the étale cohomology. Similarly for derived push-forward.*

Proof. The assertion that $\mathcal{F}$ is an $\mathcal{O}_\mathcal{X}$-module follows from Lemma 69.22.1. Note that ϵ is a morphism of sites given by the identity functor on $\mathcal{X}$. The sheaf $R^p\epsilon_*\mathcal{F}$ is therefore the sheaf associated to the presheaf $x \mapsto H^p_{fppf}(x, \mathcal{F})$, see Cohomology on Sites, Lemma 21.8.4. To prove (1) it suffices to show that $H^p_{fppf}(x, \mathcal{F}) = 0$ for $p > 0$ whenever x lies over an affine scheme U. By Lemma 69.15.1 we have $H^p_{fppf}(x, \mathcal{F}) = H^p((Sch/U)_{fppf}, x^{-1}\mathcal{F})$. Combining Descent, Lemma 34.8.4 with Cohomology of Schemes, Lemma 29.2.2 we see that these cohomology groups are zero.

We have seen above that $\epsilon_*\mathcal{F}$ and $\mathcal{F}$ are the sheaves on $\mathcal{X}_{étale}$ and $\mathcal{X}_{fppf}$ corresponding to the same presheaf on $\mathcal{X}$ (and this is true more generally for any sheaf in the fppf topology on $\mathcal{X}$). We often abusively identify $\mathcal{F}$ and $\epsilon_*\mathcal{F}$ and this is the sense in which parts (2) and (3) of the lemma should be understood. Thus part (2) follows formally from (1) and the Leray spectral sequence, see Cohomology on Sites, Lemma 21.14.6.

Finally we prove (3). The sheaf $R^i f_*\mathcal{F}$ (resp. $Rf_{étale,*}\mathcal{F}$) is the sheaf associated to the presheaf

$$y \longmapsto H^i_\tau\Big((Sch/V)_{fppf} \times_{y,\mathcal{Y}} \mathcal{X},\ \mathrm{pr}^{-1}\mathcal{F}\Big)$$

where τ is *fppf* (resp. *étale*), see Lemma 69.20.2. Note that $\mathrm{pr}^{-1}\mathcal{F}$ satisfies properties (a) and (b) also (by Lemmas 69.11.6 and 69.9.3), hence these two presheaves are equal by (2). This immediately implies (3). □

We will use the following lemma to compare étale cohomology of sheaves on algebraic stacks with cohomology on the lisse-étale topos.

Lemma 69.22.3. *Let S be a scheme. Let $\mathcal{X}$ be an algebraic stack over S. Let $\tau = étale$ (resp. $\tau = fppf$). Let $\mathcal{X}' \subset \mathcal{X}$ be a full subcategory with the following properties*

1. *if $x \to x'$ is a morphism of $\mathcal{X}$ which lies over a smooth (resp. flat and locally finitely presented) morphism of schemes and $x' \in \mathrm{Ob}(\mathcal{X}')$, then $x \in \mathrm{Ob}(\mathcal{X}')$, and*
2. *there exists an object $x \in \mathrm{Ob}(\mathcal{X}')$ lying over a scheme U such that the associated 1-morphism $x : (Sch/U)_{fppf} \to \mathcal{X}$ is smooth and surjective.*

We get a site $\mathcal{X}'_\tau$ by declaring a covering of $\mathcal{X}'$ to be any family of morphisms $\{x_i \to x\}$ in $\mathcal{X}'$ which is a covering in $\mathcal{X}_\tau$. Then the inclusion functor $\mathcal{X}' \to \mathcal{X}_\tau$ is fully faithful, cocontinuous, and continuous, whence defines a morphism of topoi

$$g : Sh(\mathcal{X}'_\tau) \longrightarrow Sh(\mathcal{X}_\tau)$$

and $H^p(\mathcal{X}'_\tau, g^{-1}\mathcal{F}) = H^p(\mathcal{X}_\tau, \mathcal{F})$ for all $p \geq 0$ and all $\mathcal{F} \in Ab(\mathcal{X}_\tau)$.

Proof. Note that assumption (1) implies that if $\{x_i \to x\}$ is a covering of $\mathcal{X}_\tau$ and $x \in \mathrm{Ob}(\mathcal{X}')$, then we have $x_i \in \mathrm{Ob}(\mathcal{X}')$. Hence we see that $\mathcal{X}' \to \mathcal{X}$ is continuous and cocontinuous as the coverings of objects of $\mathcal{X}'_\tau$ agree with their coverings seen as objects of $\mathcal{X}_\tau$. We obtain the morphism g and the functor g^{-1} is identified with the restriction functor, see Sites, Lemma 7.20.5.

In particular, if $\{x_i \to x\}$ is a covering in $\mathcal{X}'_\tau$, then for any abelian sheaf $\mathcal{F}$ on $\mathcal{X}$ then

$$\check{H}^p(\{x_i \to x\}, g^{-1}\mathcal{F}) = \check{H}^p(\{x_i \to x\}, \mathcal{F})$$

Thus if $\mathcal{I}$ is an injective abelian sheaf on $\mathcal{X}_\tau$ then we see that the higher Čech cohomology groups are zero (Cohomology on Sites, Lemma 21.11.2). Hence $H^p(x, g^{-1}\mathcal{I}) = 0$ for all objects x of $\mathcal{X}'$ (Cohomology on Sites, Lemma 21.11.8). In other words injective abelian sheaves on $\mathcal{X}_\tau$ are right acyclic for the functor $H^0(x, g^{-1}-)$. It follows that $H^p(x, g^{-1}\mathcal{F}) = H^p(x, \mathcal{F})$ for all $\mathcal{F} \in Ab(\mathcal{X})$ and all $x \in \mathrm{Ob}(\mathcal{X}')$.

Choose an object $x \in \mathcal{X}'$ lying over a scheme U as in assumption (2). In particular $\mathcal{X}/x \to \mathcal{X}$ is a morphism of algebraic stacks which representable by algebraic spaces, surjective, and smooth. (Note that $\mathcal{X}/x$ is equivalent to $(Sch/U)_{fppf}$, see Lemma 69.9.1.) The map of sheaves

$$h_x \longrightarrow *$$

in $Sh(\mathcal{X}_\tau)$ is surjective. Namely, for any object x' of $\mathcal{X}$ there exists a τ-covering $\{x'_i \to x'\}$ such that there exist morphisms $x'_i \to x$, see Lemma 69.18.10. Since g is exact, the map of sheaves

$$g^{-1}h_x \longrightarrow * = g^{-1}*$$

in $Sh(\mathcal{X}'_\tau)$ is surjective also. Let $h_{x,n}$ be the $(n+1)$-fold product $h_x \times \ldots \times h_x$. Then we have spectral sequences

$$E_1^{p,q} = H^q(h_{x,p}, \mathcal{F}) \Rightarrow H^{p+q}(\mathcal{X}_\tau, \mathcal{F}) \tag{69.22.3.1}$$

and

$$E_1^{p,q} = H^q(g^{-1}h_{x,p}, g^{-1}\mathcal{F}) \Rightarrow H^{p+q}(\mathcal{X}'_\tau, g^{-1}\mathcal{F}) \tag{69.22.3.2}$$

see Cohomology on Sites, Lemma 21.13.2.

Case I: $\mathcal{X}$ has a final object x which is also an object of $\mathcal{X}'$. This case follows immediately from the discussion in the second paragraph above.

Case II: $\mathcal{X}$ is representable by an algebraic space F. In this case the sheaves $h_{x,n}$ are representable by an object x_n in $\mathcal{X}$. (Namely, if $\mathcal{S}_F = \mathcal{X}$ and $x : U \to F$ is the given object, then $h_{x,n}$ is representable by the object $U \times_F \ldots \times_F U \to F$ of $\mathcal{S}_F$.) It follows that $H^q(h_{x,p}, \mathcal{F}) = H^q(x_p, \mathcal{F})$. The morphisms $x_n \to x$ lie over smooth morphisms of schemes, hence $x_n \in \mathcal{X}'$ for all n. Hence $H^q(g^{-1}h_{x,p}, g^{-1}\mathcal{F}) = H^q(x_p, g^{-1}\mathcal{F})$. Thus in the two spectral sequences (69.22.3.1) and (69.22.3.2) above the $E_1^{p,q}$ terms agree by the discussion in the second paragraph. The lemma follows in Case II as well.

Case III: $\mathcal{X}$ is an algebraic stack. We claim that in this case the cohomology groups $H^q(h_{x,p}, \mathcal{F})$ and $H^q(g^{-1}h_{x,n}, g^{-1}\mathcal{F})$ agree by Case II above. Once we have proved this the result will follow as before.

Namely, consider the category $\mathcal{X}/h_{x,n}$, see Sites, Lemma 7.29.3. Since $h_{x,n}$ is the $(n+1)$-fold product of h_x an object of this category is an $(n+2)$-tuple $(y, s_0, \ldots, s_n)$ where y is an object of $\mathcal{X}$ and each $s_i : y \to x$ is a morphism of $\mathcal{X}$. This is a category over $(Sch/S)_{fppf}$. There is an equivalence

$$\mathcal{X}/h_{x,n} \longrightarrow (Sch/U)_{fppf} \times_{\mathcal{X}} \ldots \times_{\mathcal{X}} (Sch/U)_{fppf} =: \mathcal{U}_n$$

over $(Sch/S)_{fppf}$. Namely, if $x : (Sch/U)_{fppf} \to \mathcal{X}$ also denotes the 1-morphism associated with x and $p : \mathcal{X} \to (Sch/S)_{fppf}$ the structure functor, then we can think of $(y, s_0, \ldots, s_n)$ as $(y, f_0, \ldots, f_n, \alpha_0, \ldots, \alpha_n)$ where y is an object of $\mathcal{X}$, $f_i : p(y) \to p(x)$ is a morphism of schemes, and $\alpha_i : y \to x(f_i)$ an isomorphism. The category of $2n+3$-tuples $(y, f_0, \ldots, f_n, \alpha_0, \ldots, \alpha_n)$ is an incarnation of the $(n+1)$-fold fibred product $\mathcal{U}_n$ of algebraic stacks displayed above, as we discussed in Section 69.17. By Cohomology on Sites, Lemma 21.13.3 we have

$$H^p(\mathcal{U}_n, \mathcal{F}|_{\mathcal{U}_n}) = H^p(\mathcal{X}/h_{x,n}, \mathcal{F}|_{\mathcal{X}/h_{x,n}}) = H^p(h_{x,n}, \mathcal{F}).$$

Finally, we discuss the "primed" analogue of this. Namely, $\mathcal{X}'/h_{x,n}$ corresponds, via the equivalence above to the full subcategory $\mathcal{U}'_n \subset \mathcal{U}_n$ consisting of those tuples $(y, f_0, \ldots, f_n, \alpha_0, \ldots, \alpha_n)$ with $y \in \mathcal{X}'$. Hence certainly property (1) of the statement of the lemma holds for the inclusion $\mathcal{U}'_n \subset \mathcal{U}_n$. To see property (2) choose an object $\xi = (y, s_0, \ldots, s_n)$ which lies over a scheme W such that $(Sch/W)_{fppf} \to \mathcal{U}_n$ is smooth and surjective (this is possible as $\mathcal{U}_n$ is an algebraic stack). Then $(Sch/W)_{fppf} \to \mathcal{U}_n \to (Sch/U)_{fppf}$ is smooth as a composition of base changes of the morphism $x : (Sch/U)_{fppf} \to \mathcal{X}$, see Algebraic Stacks, Lemmas 67.10.6 and 67.10.5. Thus axiom (1) for $\mathcal{X}$ implies that y is an object of $\mathcal{X}'$ whence ξ is an object of $\mathcal{U}'_n$. Using again

$$H^p(\mathcal{U}'_n, \mathcal{F}|_{\mathcal{U}'_n}) = H^p(\mathcal{X}'/h_{x,n}, \mathcal{F}|_{\mathcal{X}'/h_{x,n}}) = H^p(g^{-1}h_{x,n}, g^{-1}\mathcal{F}).$$

we now can use Case II for $\mathcal{U}'_n \subset \mathcal{U}_n$ to conclude. □

69.23. Other chapters

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(5) Topology
(6) Sheaves on Spaces
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Schemes

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CHAPTER 70

Criteria for Representability

70.1. Introduction

The purpose of this chapter is to find criteria guaranteeing that a stack in groupoids over the category of schemes with the fppf topology is an algebraic stack. Historically, this often involved proving that certain functors were representable, see Grothendieck's lectures [**Gro95a**], [**Gro95b**], [**Gro95e**], [**Gro95f**], [**Gro95c**], and [**Gro95d**]. This explains the title of this chapter. Another important source of this material comes from the work of Artin, see [**Art69b**], [**Art70**], [**Art73**], [**Art71b**], [**Art71a**], [**Art69a**], [**Art69c**], and [**Art74**].

Some of the notation, conventions and terminology in this chapter is awkward and may seem backwards to the more experienced reader. This is intentional. Please see Quot, Section 72.1 for an explanation.

70.2. Conventions

The conventions we use in this chapter are the same as those in the chapter on algebraic stacks, see Algebraic Stacks, Section 67.2.

70.3. What we already know

The analogue of this chapter for algebraic spaces is the chapter entitled "Bootstrap", see Bootstrap, Section 61.1. That chapter already contains some representability results. Moreover, some of the preliminary material treated there we already have worked out in the chapter on algebraic stacks. Here is a list:

(1) We discuss morphisms of presheaves representable by algebraic spaces in Bootstrap, Section 61.3. In Algebraic Stacks, Section 67.9 we discuss the notion of a 1-morphism of categories fibred in groupoids being representable by algebraic spaces.
(2) We discuss properties of morphisms of presheaves representable by algebraic spaces in Bootstrap, Section 61.4. In Algebraic Stacks, Section 67.10 we discuss the notion of a 1-morphism of categories fibred in groupoids being representable by algebraic spaces.
(3) We proved that if F is a sheaf whose diagonal is representable by algebraic spaces and which has an étale covering by an algebraic space, then F is an algebraic space, see Bootstrap, Theorem 61.6.1. (This is a weak version of the result in the next item on the list.)
(4) We proved that if F is a sheaf and if there exists an algebraic space U and a morphism $U \to F$ which is representable by algebraic spaces, surjective, flat, and locally of finite presentation, then F is an algebraic space, see Bootstrap, Theorem 61.10.1.

(5) We have also proved the "smooth" analogue of (4) for algebraic stacks: If $\mathcal{X}$ is a stack in groupoids over $(Sch/S)_{fppf}$ and if there exists a stack in groupoids $\mathcal{U}$ over $(Sch/S)_{fppf}$ which is representable by an algebraic space and a 1-morphism $u : \mathcal{U} \to \mathcal{X}$ which is representable by algebraic spaces, surjective, and smooth then $\mathcal{X}$ is an algebraic stack, see Algebraic Stacks, Lemma 67.15.3.

Our first task now is to prove the analogue of (4) for algebraic stacks in general; it is Theorem 70.16.1.

70.4. Morphisms of stacks in groupoids

This section is preliminary and should be skipped on a first reading.

Lemma 70.4.1. *Let $\mathcal{X} \to \mathcal{Y} \to \mathcal{Z}$ be 1-morphisms of categories fibred in groupoids over $(Sch/S)_{fppf}$. If $\mathcal{X} \to \mathcal{Z}$ and $\mathcal{Y} \to \mathcal{Z}$ are representable by algebraic spaces and étale so is $\mathcal{X} \to \mathcal{Y}$.*

Proof. Let $\mathcal{U}$ be a representable category fibred in groupoids over S. Let $f : \mathcal{U} \to \mathcal{Y}$ be a 1-morphism. We have to show that $\mathcal{X} \times_{\mathcal{Y}} \mathcal{U}$ is representable by an algebraic space and étale over $\mathcal{U}$. Consider the composition $h : \mathcal{U} \to \mathcal{Z}$. Then

$$\mathcal{X} \times_{\mathcal{Z}} \mathcal{U} \longrightarrow \mathcal{Y} \times_{\mathcal{Z}} \mathcal{U}$$

is a 1-morphism between categories fibres in groupoids which are both representable by algebraic spaces and both étale over $\mathcal{U}$. Hence by Properties of Spaces, Lemma 48.13.6 this is represented by an étale morphism of algebraic spaces. Finally, we obtain the result we want as the morphism f induces a morphism $\mathcal{U} \to \mathcal{Y} \times_{\mathcal{Z}} \mathcal{U}$ and we have

$$\mathcal{X} \times_{\mathcal{Y}} \mathcal{U} = (\mathcal{X} \times_{\mathcal{Z}} \mathcal{U}) \times_{(\mathcal{Y} \times_{\mathcal{Z}} \mathcal{U})} \mathcal{U}.$$

□

Lemma 70.4.2. *Let $\mathcal{X}, \mathcal{Y}, \mathcal{Z}$ be stacks in groupoids over $(Sch/S)_{fppf}$. Suppose that $\mathcal{X} \to \mathcal{Y}$ and $\mathcal{Z} \to \mathcal{Y}$ are 1-morphisms. If*

(1) *$\mathcal{Y}$, $\mathcal{Z}$ are representable by algebraic spaces Y, Z over S,*
(2) *the associated morphism of algebraic spaces $Y \to Z$ is surjective, flat and locally of finite presentation, and*
(3) *$\mathcal{Y} \times_{\mathcal{Z}} \mathcal{X}$ is a stack in setoids,*

then $\mathcal{X}$ is a stack in setoids.

Proof. This is a special case of Stacks, Lemma 8.6.10. □

The following lemma is the analogue of Algebraic Stacks, Lemma 67.15.3 and will be superseded by the stronger Theorem 70.16.1.

Lemma 70.4.3. *Let S be a scheme. Let $u : \mathcal{U} \to \mathcal{X}$ be a 1-morphism of stacks in groupoids over $(Sch/S)_{fppf}$. If*

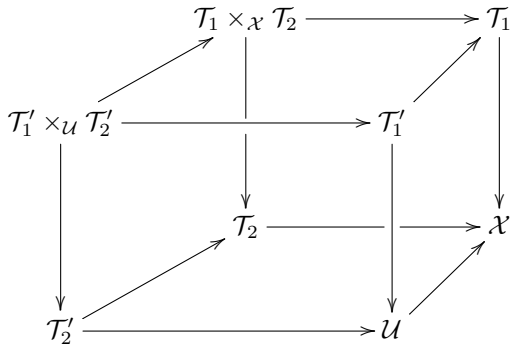
(1) *$\mathcal{U}$ is representable by an algebraic space, and*
(2) *u is representable by algebraic spaces, surjective, flat and locally of finite presentation,*

then $\Delta : \mathcal{X} \to \mathcal{X} \times \mathcal{X}$ representable by algebraic spaces.

Proof. Given two schemes T_1, T_2 over S denote $\mathcal{T}_i = (Sch/T_i)_{fppf}$ the associated representable fibre categories. Suppose given 1-morphisms $f_i : \mathcal{T}_i \to \mathcal{X}$. According to Algebraic Stacks, Lemma 67.10.11 it suffices to prove that the 2-fibered product $\mathcal{T}_1 \times_{\mathcal{X}} \mathcal{T}_2$ is representable by an algebraic space. By Stacks, Lemma 8.6.8 this is in any case a stack in setoids. Thus $\mathcal{T}_1 \times_{\mathcal{X}} \mathcal{T}_2$ corresponds to some sheaf F on $(Sch/S)_{fppf}$, see Stacks, Lemma 8.6.3. Let U be the algebraic space which represents $\mathcal{U}$. By assumption

$$\mathcal{T}_i' = \mathcal{U} \times_{u,\mathcal{X},f_i} \mathcal{T}_i$$

is representable by an algebraic space T_i' over S. Hence $\mathcal{T}_1' \times_{\mathcal{U}} \mathcal{T}_2'$ is representable by the algebraic space $T_1' \times_U T_2'$. Consider the commutative diagram



In this diagram the bottom square, the right square, the back square, and the front square are 2-fibre products. A formal argument then shows that $\mathcal{T}_1' \times_{\mathcal{U}} \mathcal{T}_2' \to \mathcal{T}_1 \times_{\mathcal{X}} \mathcal{T}_2$ is the "base change" of $\mathcal{U} \to \mathcal{X}$, more precisely the diagram

$$\begin{array}{ccc} \mathcal{T}_1' \times_{\mathcal{U}} \mathcal{T}_2' & \longrightarrow & \mathcal{U} \\ \downarrow & & \downarrow \\ \mathcal{T}_1 \times_{\mathcal{X}} \mathcal{T}_2 & \longrightarrow & \mathcal{X} \end{array}$$

is a 2-fibre square. Hence $T_1' \times_U T_2' \to F$ is representable by algebraic spaces, flat, locally of finite presentation and surjective, see Algebraic Stacks, Lemmas 67.9.6, 67.9.7, 67.10.4, and 67.10.6. Therefore F is an algebraic space by Bootstrap, Theorem 61.10.1 and we win. □

Lemma 70.4.4. *Let $\mathcal{X}$ be a category fibred in groupoids over $(Sch/S)_{fppf}$. The following are equivalent*

1. *$\Delta_\Delta : \mathcal{X} \to \mathcal{X} \times_{\mathcal{X}\times\mathcal{X}} \mathcal{X}$ is representable by algebraic spaces,*
2. *for every 1-morphism $\mathcal{V} \to \mathcal{X} \times \mathcal{X}$ with $\mathcal{V}$ representable (by a scheme) and the fibre product $\mathcal{Y} = \mathcal{X} \times_{\Delta,\mathcal{X}\times\mathcal{X}} \mathcal{V}$ has diagonal representable by algebraic spaces.*

Proof. Although this is a bit of a brain twister, it is completely formal. Namely, recall that $\mathcal{X} \times_{\mathcal{X}\times\mathcal{X}} \mathcal{X} = \mathcal{I}_{\mathcal{X}}$ is the inertia of $\mathcal{X}$ and that Δ_Δ is the identity section of $\mathcal{I}_{\mathcal{X}}$, see Categories, Section 4.32. Thus condition (1) says the following: Given a scheme V, an object x of $\mathcal{X}$ over V, and a morphism $\alpha : x \to x$ of $\mathcal{X}_V$ the condition "$\alpha = \mathrm{id}_x$" defines an algebraic space over V. (In other words, there exists a monomorphism of algebraic spaces $W \to V$ such that a morphism of schemes $f : T \to V$ factors through W if and only if $f^*\alpha = \mathrm{id}_{f^*x}$.)

On the other hand, let V be a scheme and let x, y be objects of $\mathcal{X}$ over V. Then (x, y) define a morphism $\mathcal{V} = (Sch/V)_{fppf} \to \mathcal{X} \times \mathcal{X}$. Next, let $h : V' \to V$ be a morphism of schemes and let $\alpha : h^*x \to h^*y$ and $\beta : h^*x \to h^*y$ be morphisms of $\mathcal{X}_{V'}$. Then (α, β) define a morphism $\mathcal{V}' = (Sch/V)_{fppf} \to \mathcal{Y} \times \mathcal{Y}$. Condition (2) now says that (with any choices as above) the condition "$\alpha = \beta$" defines an algebraic space over V.

To see the equivalence, given (α, β) as in (2) we see that (1) implies that "$\alpha^{-1} \circ \beta = \mathrm{id}_{h^*x}$" defines an algebraic space. The implication (2) $\Rightarrow$ (1) follows by taking $h = \mathrm{id}_V$ and $\beta = \mathrm{id}_x$. □

70.5. Limit preserving on objects

Let S be a scheme. Let $p : \mathcal{X} \to \mathcal{Y}$ be a 1-morphism of categories fibred in groupoids over $(Sch/S)_{fppf}$. We will say that p is *limit preserving on objects* if the following condition holds: Given any data consisting of

(1) an affine scheme $U = \lim_{i \in I} U_i$ which is written as the directed limit of affine schemes U_i over S,
(2) an object y_i of $\mathcal{Y}$ over U_i for some i,
(3) an object x of $\mathcal{X}$ over U, and
(4) an isomorphism $\gamma : p(x) \to y_i|_U$,

then there exists an $i' \geq i$, an object $x_{i'}$ of $\mathcal{X}$ over $U_{i'}$, an isomorphism $\beta : x_{i'}|_U \to x$, and an isomorphism $\gamma_{i'} : p(x_{i'}) \to y_i|_{U_{i'}}$ such that

$$\begin{array}{ccc} p(x_{i'}|_U) & \xrightarrow[\gamma_{i'}|_U]{} & (y_i|_{U_{i'}})|_U \\ {\scriptstyle p(\beta)}\downarrow & & \| \\ p(x) & \xrightarrow{\gamma} & y_i|_U \end{array} \tag{70.5.0.1}$$

commutes. In this situation we say that "$(i', x_{i'}, \beta, \gamma_{i'})$ is a *solution* to the problem posed by our data (1), (2), (3), (4)". The motivation for this definition comes from Limits of Spaces, Lemma 52.3.2.

Lemma 70.5.1. *Let $p : \mathcal{X} \to \mathcal{Y}$ and $q : \mathcal{Z} \to \mathcal{Y}$ be 1-morphisms of categories fibred in groupoids over $(Sch/S)_{fppf}$. If $p : \mathcal{X} \to \mathcal{Y}$ is limit preserving on objects, then so is the base change $p' : \mathcal{X} \times_{\mathcal{Y}} \mathcal{Z} \to \mathcal{Z}$ of p by q.*

Proof. This is formal. Let $U = \lim_{i \in I} U_i$ be the directed limit of affine schemes U_i over S, let z_i be an object of $\mathcal{Z}$ over U_i for some i, let w be an object of $\mathcal{X} \times_{\mathcal{Y}} \mathcal{Z}$ over U, and let $\delta : p'(w) \to z_i|_U$ be an isomorphism. We may write $w = (U, x, z, \alpha)$ for some object x of $\mathcal{X}$ over U and object z of $\mathcal{Z}$ over U and isomorphism $\alpha : p(x) \to q(z)$. Note that $p'(w) = z$ hence $\delta : z \to z_i|_U$. Set $y_i = q(z_i)$ and $\gamma = q(\delta) \circ \alpha : p(x) \to y_i|_U$. As p is limit preserving on objects there exists an $i' \geq i$ and an object $x_{i'}$ of $\mathcal{X}$ over $U_{i'}$ as well as isomorphisms $\beta : x_{i'}|_U \to x$ and $\gamma_{i'} : p(x_{i'}) \to y_i|_{U_{i'}}$ such that (70.5.0.1) commutes. Then we consider the object $w_{i'} = (U_{i'}, x_{i'}, z_i|_{U_{i'}}, \gamma_{i'})$ of $\mathcal{X} \times_{\mathcal{Y}} \mathcal{Z}$ over $U_{i'}$ and define isomorphisms

$$w_{i'}|_U = (U, x_{i'}|_U, z_i|_U, \gamma_{i'}|_U) \xrightarrow{(\beta, \delta^{-1})} (U, x, z, \alpha) = w$$

and

$$p'(w_{i'}) = z_i|_{U_{i'}} \xrightarrow{\mathrm{id}} z_i|_{U_{i'}}.$$

These combine to give a solution to the problem. □

Lemma 70.5.2. *Let $p : \mathcal{X} \to \mathcal{Y}$ and $q : \mathcal{Y} \to \mathcal{Z}$ be 1-morphisms of categories fibred in groupoids over $(Sch/S)_{fppf}$. If p and q are limit preserving on objects, then so is the composition $q \circ p$.*

Proof. This is formal. Let $U = \lim_{i \in I} U_i$ be the directed limit of affine schemes U_i over S, let z_i be an object of $\mathcal{Z}$ over U_i for some i, let x be an object of $\mathcal{X}$ over U, and let $\gamma : q(p(x)) \to z_i|_U$ be an isomorphism. As q is limit preserving on objects there exist an $i' \geq i$, an object $y_{i'}$ of $\mathcal{Y}$ over $U_{i'}$, an isomorphism $\beta : y_{i'}|_U \to p(x)$, and an isomorphism $\gamma_{i'} : q(y_{i'}) \to z_i|_{U_{i'}}$ such that (70.5.0.1) is commutative. As p is limit preserving on objects there exist an $i'' \geq i'$, an object $x_{i''}$ of $\mathcal{X}$ over $U_{i''}$, an isomorphism $\beta' : x_{i''}|_U \to x$, and an isomorphism $\gamma'_{i''} : p(x_{i''}) \to y_{i'}|_{U_{i''}}$ such that (70.5.0.1) is commutative. The solution is to take $x_{i''}$ over $U_{i''}$ with isomorphism

$$q(p(x_{i''})) \xrightarrow{q(\gamma'_{i''})} q(y_{i'})|_{U_{i''}} \xrightarrow{\gamma_{i'}|_{U_{i''}}} z_i|_{U_{i''}}$$

and isomorphism $\beta' : x_{i''}|_U \to x$. We omit the verification that (70.5.0.1) is commutative. □

Lemma 70.5.3. *Let $p : \mathcal{X} \to \mathcal{Y}$ be a 1-morphism of categories fibred in groupoids over $(Sch/S)_{fppf}$. If p is representable by algebraic spaces, then the following are equivalent:*

1. *p is limit preserving on objects, and*
2. *p is locally of finite presentation (see Algebraic Stacks, Definition 67.10.1).*

Proof. Assume (2). Let $U = \lim_{i \in I} U_i$ be the directed limit of affine schemes U_i over S, let y_i be an object of $\mathcal{Y}$ over U_i for some i, let x be an object of $\mathcal{X}$ over U, and let $\gamma : p(x) \to y_i|_U$ be an isomorphism. Let X_{y_i} denote an algebraic space over U_i representing the 2-fibre product

$$(Sch/U_i)_{fppf} \times_{y_i, \mathcal{Y}, p} \mathcal{X}.$$

Note that $\xi = (U, U \to U_i, x, \gamma^{-1})$ defines an object of this 2-fibre product over U. Via the 2-Yoneda lemma ξ corresponds to a morphism $f_\xi : U \to X_{y_i}$ over U_i. By Limits of Spaces, Proposition 52.3.9 there exists an $i' \geq i$ and a morphism $f_{i'} : U_{i'} \to X_{y_i}$ such that f_ξ is the composition of $f_{i'}$ and the projection morphism $U \to U_{i'}$. Also, the 2-Yoneda lemma tells us that $f_{i'}$ corresponds to an object $\xi_{i'} = (U_{i'}, U_{i'} \to U_i, x_{i'}, \alpha)$ of the displayed 2-fibre product over $U_{i'}$ whose restriction to U recovers ξ. In particular we obtain an isomorphism $\gamma : x_{i'}|U \to x$. Note that $\alpha : y_i|_{U_{i'}} \to p(x_{i'})$. Hence we see that taking $x_{i'}$, the isomorphism $\gamma : x_{i'}|U \to x$, and the isomorphism $\beta = \alpha^{-1} : p(x_{i'}) \to y_i|_{U_{i'}}$ is a solution to the problem.

Assume (1). Choose a scheme T and a 1-morphism $y : (Sch/T)_{fppf} \to \mathcal{Y}$. Let X_y be an algebraic space over T representing the 2-fibre product $(Sch/T)_{fppf} \times_{y, \mathcal{Y}, p} \mathcal{X}$. We have to show that $X_y \to T$ is locally of finite presentation. To do this we may use Limits of Spaces, Proposition 52.3.9 in the form described in Limits of Spaces, Remark 52.3.10. Hence it suffices to show that given an affine scheme $U = \lim_{i \in I} U_i$ written as the directed limit of affine schemes over T, then $X_y(U) = \operatorname{colim}_i X_y(U_i)$. Pick any $i \in I$ and set $y_i = y|_{U_i}$. Also denote i' an element of I which is bigger than or equal to i. By the 2-Yoneda lemma morphisms $U \to X_y$ over T correspond bijectively to isomorphism classes of pairs (x, α) where x is an object of $\mathcal{X}$ over U and $\alpha : y|_U \to p(x)$ is an isomorphism. Of course giving α is, up to an inverse,

the same thing as giving an isomorphism $\gamma : p(x) \to y_i|_U$. Similarly for morphisms $U_{i'} \to X_y$ over T. Hence (1) guarantees that

$$X_y(U) = \text{colim}_{i' \geq i} X_y(U_{i'})$$

in this situation and we win. □

Lemma 70.5.4. *Let $p : \mathcal{X} \to \mathcal{Y}$ be a 1-morphism of categories fibred in groupoids over $(Sch/S)_{fppf}$. Assume p is representable by algebraic spaces and an open immersion. Then p is limit preserving on objects.*

Proof. This follows from Lemma 70.5.3 and (via the general principle Algebraic Stacks, Lemma 67.10.9) from the fact that an open immersion of algebraic spaces is locally of finite presentation, see Morphisms of Spaces, Lemma 49.26.11. □

Let S be a scheme. In the following lemma we need the notion of the *size* of an algebraic space X over S. Namely, given a cardinal κ we will say X has size$(X) \leq \kappa$ if and only if there exists a scheme U with size$(U) \leq \kappa$ (see Sets, Section 3.9) and a surjective étale morphism $U \to X$.

Lemma 70.5.5. *Let S be a scheme. Let $\kappa = size(T)$ for some $T \in \text{Ob}((Sch/S)_{fppf})$. Let $f : \mathcal{X} \to \mathcal{Y}$ be a 1-morphism of categories fibred in groupoids over $(Sch/S)_{fppf}$ such that*

1. *$\mathcal{Y} \to (Sch/S)_{fppf}$ is limit preserving on objects,*
2. *for an affine scheme V locally of finite presentation over S and $y \in \text{Ob}(\mathcal{Y}_V)$ the fibre product $(Sch/V)_{fppf} \times_{y,\mathcal{Y}} \mathcal{X}$ is representable by an algebraic space of size $\leq \kappa$[1],*
3. *$\mathcal{X}$ and $\mathcal{Y}$ are stacks for the Zariski topology.*

Then f is representable by algebraic spaces.

Proof. Let V be a scheme over S and $y \in \mathcal{Y}_V$. We have to prove $(Sch/V)_{fppf} \times_{y,\mathcal{Y}} \mathcal{X}$ is representable by an algebraic space.

Case I: V is affine and maps into an affine open $\text{Spec}(\Lambda) \subset S$. Then we can write $V = \lim V_i$ with each V_i affine and of finite presentation over $\text{Spec}(\Lambda)$, see Algebra, Lemma 10.121.1. Then y comes from an object y_i over V_i for some i by assumption (1). By assumption (3) the fibre product $(Sch/V_i)_{fppf} \times_{y_i,\mathcal{Y}} \mathcal{X}$ is representable by an algebraic space Z_i. Then $(Sch/V)_{fppf} \times_{y,\mathcal{Y}} \mathcal{X}$ is representable by $Z \times_{V_i} V$.

Case II: V is general. Choose an affine open covering $V = \bigcup_{i \in I} V_i$ such that each V_i maps into an affine open of S. We first claim that $\mathcal{Z} = (Sch/V)_{fppf} \times_{y,\mathcal{Y}} \mathcal{X}$ is a stack in setoids for the Zariski topology. Namely, it is a stack in groupoids for the Zariski topology by Stacks, Lemma 8.5.6. Then suppose that z is an object of $\mathcal{Z}$ over a scheme T. Denote $g : T \to V$ the morphism corresponding to the projection of z in $(Sch/V)_{fppf}$. Consider the Zariski sheaf $I = Isom_{\mathcal{Z}}(z, z)$. By Case I we see that $I|_{g^{-1}(V_i)} = *$ (the singleton sheaf). Hence $\mathcal{I} = *$. Thus $\mathcal{Z}$ is fibred in setoids. To finish the proof we have to show that the Zariski sheaf $Z : T \mapsto \text{Ob}(\mathcal{Z}_T)/\cong$ is an algebraic space, see Algebraic Stacks, Lemma 67.8.2. There is a map $p : Z \to V$ (transformation of functors) and by Case I we know that $Z_i = p^{-1}(V_i)$ is an algebraic space. The morphisms $Z_i \to Z$ are representable by open immersions and $\coprod Z_i \to Z$ is surjective (in the Zariski topology). Hence Z is

[1]The condition on size can be dropped by those ignoring set theoretic issues.

a sheaf for the fppf topology by Bootstrap, Lemma 61.3.10. Thus Spaces, Lemma 47.8.4 applies and we conclude that Z is an algebraic space[2]. □

Lemma 70.5.6. *Let S be a scheme. Let $f : \mathcal{X} \to \mathcal{Y}$ be a 1-morphism of categories fibred in groupoids over $(Sch/S)_{fppf}$. Let $\mathcal{P}$ be a property of morphisms of algebraic spaces as in Algebraic Stacks, Definition 67.10.1. If*

1. *f is representable by algebraic spaces,*
2. *$\mathcal{Y} \to (Sch/S)_{fppf}$ is limit preserving on objects,*
3. *for an affine scheme V locally of finite presentation over S and $y \in \mathcal{Y}_V$ the resulting morphism of algebraic spaces $f_y : F_y \to V$, see Algebraic Stacks, Equation (67.9.1.1), has property $\mathcal{P}$.*

Then f has property $\mathcal{P}$.

Proof. Let V be a scheme over S and $y \in \mathcal{Y}_V$. We have to show that $F_y \to V$ has property $\mathcal{P}$. Since $\mathcal{P}$ is fppf local on the base we may assume that V is an affine scheme which maps into an affine open $\mathrm{Spec}(\Lambda) \subset S$. Thus we can write $V = \lim V_i$ with each V_i affine and of finite presentation over $\mathrm{Spec}(\Lambda)$, see Algebra, Lemma 10.121.1. Then y comes from an object y_i over V_i for some i by assumption (2). By assumption (3) the morphism $F_{y_i} \to V_i$ has property $\mathcal{P}$. As $\mathcal{P}$ is stable under arbitrary base change and since $F_y = F_{y_i} \times_{V_i} V$ we conclude that $F_y \to V$ has property $\mathcal{P}$ as desired. □

70.6. Formally smooth on objects

Let S be a scheme. Let $p : \mathcal{X} \to \mathcal{Y}$ be a 1-morphism of categories fibred in groupoids over $(Sch/S)_{fppf}$. We will say that p is *formally smooth on objects* if the following condition holds: Given any data consisting of

1. a first order thickening $U \subset U'$ of affine schemes over S,
2. an object y' of $\mathcal{Y}$ over U',
3. an object x of $\mathcal{X}$ over U, and
4. an isomorphism $\gamma : p(x) \to y'|_U$,

then there exists an object x' of $\mathcal{X}$ over U' with an isomorphism $\beta : x'|_U \to x$ and an isomorphism $\gamma' : p(x') \to y'$ such that

$$\begin{array}{ccc} p(x'|_U) & \xrightarrow{\gamma'|_U} & y'|_U \\ {\scriptstyle p(\beta)}\downarrow & & \| \\ p(x) & \xrightarrow{\gamma} & y'|_U \end{array} \tag{70.6.0.1}$$

commutes. In this situation we say that "(x', β, γ') is a *solution* to the problem posed by our data (1), (2), (3), (4)".

Lemma 70.6.1. *Let $p : \mathcal{X} \to \mathcal{Y}$ and $q : \mathcal{Z} \to \mathcal{Y}$ be 1-morphisms of categories fibred in groupoids over $(Sch/S)_{fppf}$. If $p : \mathcal{X} \to \mathcal{Y}$ is formally smooth on objects, then so is the base change $p' : \mathcal{X} \times_{\mathcal{Y}} \mathcal{Z} \to \mathcal{Z}$ of p by q.*

[2] To see that the set theoretic condition of that lemma is satisfied we argue as follows: First choose the open covering such that $|I| \leq \mathrm{size}(V)$. Next, choose schemes U_i of size $\leq \max(\kappa, \mathrm{size}(V))$ and surjective étale morphisms $U_i \to Z_i$; we can do this by assumption (2) and Sets, Lemma 3.9.6 (details omitted). Then Sets, Lemma 3.9.9 implies that $\coprod U_i$ is an object of $(Sch/S)_{fppf}$. Hence $\coprod Z_i$ is an algebraic space by Spaces, Lemma 47.8.3.

Proof. This is formal. Let $U \subset U'$ be a first order thickening of affine schemes over S, let z' be an object of $\mathcal{Z}$ over U', let w be an object of $\mathcal{X} \times_{\mathcal{Y}} \mathcal{Z}$ over U, and let $\delta : p'(w) \to z'|_U$ be an isomorphism. We may write $w = (U, x, z, \alpha)$ for some object x of $\mathcal{X}$ over U and object z of $\mathcal{Z}$ over U and isomorphism $\alpha : p(x) \to q(z)$. Note that $p'(w) = z$ hence $\delta : z \to z|_U$. Set $y' = q(z')$ and $\gamma = q(\delta) \circ \alpha : p(x) \to y'|_U$. As p is formally smooth on objects there exists an object x' of $\mathcal{X}$ over U' as well as isomorphisms $\beta : x'|_U \to x$ and $\gamma' : p(x') \to y'$ such that (70.6.0.1) commutes. Then we consider the object $w = (U', x', z', \gamma')$ of $\mathcal{X} \times_{\mathcal{Y}} \mathcal{Z}$ over U' and define isomorphisms

$$w'|_U = (U, x'|_U, z'|_U, \gamma'|_U) \xrightarrow{(\beta, \delta^{-1})} (U, x, z, \alpha) = w$$

and

$$p'(w') = z' \xrightarrow{\mathrm{id}} z'.$$

These combine to give a solution to the problem. □

Lemma 70.6.2. *Let $p : \mathcal{X} \to \mathcal{Y}$ and $q : \mathcal{Y} \to \mathcal{Z}$ be 1-morphisms of categories fibred in groupoids over $(Sch/S)_{fppf}$. If p and q are formally smooth on objects, then so is the composition $q \circ p$.*

Proof. This is formal. Let $U \subset U'$ be a first order thickening of affine schemes over S, let z' be an object of $\mathcal{Z}$ over U', let x be an object of $\mathcal{X}$ over U, and let $\gamma : q(p(x)) \to z'|_U$ be an isomorphism. As q is formally smooth on objects there exist an object y' of $\mathcal{Y}$ over U', an isomorphism $\beta : y'|_U \to p(x)$, and an isomorphism $\gamma' : q(y') \to z'$ such that (70.6.0.1) is commutative. As p is formally smooth on objects there exist an object x' of $\mathcal{X}$ over U', an isomorphism $\beta' : x'|_U \to x$, and an isomorphism $\gamma'' : p(x') \to y'$ such that (70.6.0.1) is commutative. The solution is to take x' over U' with isomorphism

$$q(p(x')) \xrightarrow{q(\gamma'')} q(y') \xrightarrow{\gamma'} z'$$

and isomorphism $\beta' : x'|_U \to x$. We omit the verification that (70.6.0.1) is commutative. □

Note that the class of formally smooth morphisms of algebraic spaces is stable under arbitrary base change and local on the target in the fpqc topology, see More on Morphisms of Spaces, Lemma 57.16.3 and 57.16.10. Hence condition (2) in the lemma below makes sense.

Lemma 70.6.3. *Let $p : \mathcal{X} \to \mathcal{Y}$ be a 1-morphism of categories fibred in groupoids over $(Sch/S)_{fppf}$. If p is representable by algebraic spaces, then the following are equivalent:*

1. *p is formally smooth on objects, and*
2. *p is formally smooth (see Algebraic Stacks, Definition 67.10.1).*

Proof. Assume (2). Let $U \subset U'$ be a first order thickening of affine schemes over S, let y' be an object of $\mathcal{Y}$ over U', let x be an object of $\mathcal{X}$ over U, and let $\gamma : p(x) \to y'|_U$ be an isomorphism. Let $X_{y'}$ denote an algebraic space over U' representing the 2-fibre product

$$(Sch/U')_{fppf} \times_{y', \mathcal{Y}, p} \mathcal{X}.$$

Note that $\xi = (U, U \to U', x, \gamma^{-1})$ defines an object of this 2-fibre product over U. Via the 2-Yoneda lemma ξ corresponds to a morphism $f_\xi : U \to X_{y'}$ over U'. As

$X_{y'} \to U'$ is formally smooth by assumption there exists a morphism $f' : U' \to X_{y'}$ such that f_ξ is the composition of f' and the morphism $U \to U'$. Also, the 2-Yoneda lemma tells us that f' corresponds to an object $\xi' = (U', U' \to U', x', \alpha)$ of the displayed 2-fibre product over U' whose restriction to U recovers ξ. In particular we obtain an isomorphism $\gamma : x'|U \to x$. Note that $\alpha : y' \to p(x')$. Hence we see that taking x', the isomorphism $\gamma : x'|U \to x$, and the isomorphism $\beta = \alpha^{-1} : p(x') \to y'$ is a solution to the problem.

Assume (1). Choose a scheme T and a 1-morphism $y : (Sch/T)_{fppf} \to \mathcal{Y}$. Let X_y be an algebraic space over T representing the 2-fibre product $(Sch/T)_{fppf} \times_{y,\mathcal{Y},p} \mathcal{X}$. We have to show that $X_y \to T$ is formally smooth. Hence it suffices to show that given a first order thickening $U \subset U'$ of affine schemes over T, then $X_y(U') \to X_y(U')$ is surjective (morphisms in the category of algebraic spaces over T). Set $y' = y|_{U'}$. By the 2-Yoneda lemma morphisms $U \to X_y$ over T correspond bijectively to isomorphism classes of pairs (x, α) where x is an object of $\mathcal{X}$ over U and $\alpha : y|_U \to p(x)$ is an isomorphism. Of course giving α is, up to an inverse, the same thing as giving an isomorphism $\gamma : p(x) \to y'|_U$. Similarly for morphisms $U' \to X_y$ over T. Hence (1) guarantees the surjectivity of $X_y(U') \to X_y(U')$ in this situation and we win. □

70.7. Surjective on objects

Let S be a scheme. Let $p : \mathcal{X} \to \mathcal{Y}$ be a 1-morphism of categories fibred in groupoids over $(Sch/S)_{fppf}$. We will say that p is *surjective on objects* if the following condition holds: Given any data consisting of

(1) a field k over S, and
(2) an object y of $\mathcal{Y}$ over $\mathrm{Spec}(k)$,

then there exists an extension $k \subset K$ of fields over S, an object x of $\mathcal{X}$ over $\mathrm{Spec}(K)$ such that $p(x) \cong y|_{\mathrm{Spec}(K)}$.

Lemma 70.7.1. *Let $p : \mathcal{X} \to \mathcal{Y}$ and $q : \mathcal{Z} \to \mathcal{Y}$ be 1-morphisms of categories fibred in groupoids over $(Sch/S)_{fppf}$. If $p : \mathcal{X} \to \mathcal{Y}$ is surjective on objects, then so is the base change $p' : \mathcal{X} \times_{\mathcal{Y}} \mathcal{Z} \to \mathcal{Z}$ of p by q.*

Proof. This is formal. Let z be an object of $\mathcal{Z}$ over a field k. As p is surjective on objects there exists an extension $k \subset K$ and an object x of $\mathcal{X}$ over K and an isomorphism $\alpha : p(x) \to q(z)|_{\mathrm{Spec}(K)}$. Then $w = (\mathrm{Spec}(K), x, z|_{\mathrm{Spec}(K)}, \alpha)$ is an object of $\mathcal{X} \times_{\mathcal{Y}} \mathcal{Z}$ over K with $p'(w) = z|_{\mathrm{Spec}(K)}$. □

Lemma 70.7.2. *Let $p : \mathcal{X} \to \mathcal{Y}$ and $q : \mathcal{Y} \to \mathcal{Z}$ be 1-morphisms of categories fibred in groupoids over $(Sch/S)_{fppf}$. If p and q are surjective on objects, then so is the composition $q \circ p$.*

Proof. This is formal. Let z be an object of $\mathcal{Z}$ over a field k. As q is surjective on objects there exists a field extension $k \subset K$ and an object y of $\mathcal{Y}$ over K such that $q(y) \cong x|_{\mathrm{Spec}(K)}$. As p is surjective on objects there exists a field extension $K \subset L$ and an object x of $\mathcal{X}$ over L such that $p(x) \cong y|_{\mathrm{Spec}(L)}$. Then the field extension $k \subset L$ and the object x of $\mathcal{X}$ over L satisfy $q(p(x)) \cong z|_{\mathrm{Spec}(L)}$ as desired. □

Lemma 70.7.3. *Let $p : \mathcal{X} \to \mathcal{Y}$ be a 1-morphism of categories fibred in groupoids over $(Sch/S)_{fppf}$. If p is representable by algebraic spaces, then the following are equivalent:*

(1) *p is surjective on objects, and*
(2) *p is surjective (see Algebraic Stacks, Definition 67.10.1).*

Proof. Assume (2). Let k be a field and let y be an object of $\mathcal{Y}$ over k. Let X_y denote an algebraic space over k representing the 2-fibre product

$$(Sch/\operatorname{Spec}(k))_{fppf} \times_{y,\mathcal{Y},p} \mathcal{X}.$$

As we've assumed that p is surjective we see that X_y is not empty. Hence we can find a field extension $k \subset K$ and a K-valued point x of X_y. Via the 2-Yoneda lemma this corresponds to an object x of $\mathcal{X}$ over K together with an isomorphism $p(x) \cong y|_{\operatorname{Spec}(K)}$ and we see that (1) holds.

Assume (1). Choose a scheme T and a 1-morphism $y : (Sch/T)_{fppf} \to \mathcal{Y}$. Let X_y be an algebraic space over T representing the 2-fibre product $(Sch/T)_{fppf} \times_{y,\mathcal{Y},p} \mathcal{X}$. We have to show that $X_y \to T$ is surjective. By Morphisms of Spaces, Definition 49.6.2 we have to show that $|X_y| \to |T|$ is surjective. This means exactly that given a field k over T and a morphism $t : \operatorname{Spec}(k) \to T$ there exists a field extension $k \subset K$ and a morphism $x : \operatorname{Spec}(K) \to X_y$ such that

$$\begin{array}{ccc} \operatorname{Spec}(K) & \xrightarrow{x} & X_y \\ \downarrow & & \downarrow \\ \operatorname{Spec}(k) & \xrightarrow{t} & T \end{array}$$

commutes. By the 2-Yoneda lemma this means exactly that we have to find $k \subset K$ and an object x of $\mathcal{X}$ over K such that $p(x) \cong t^*y|_{\operatorname{Spec}(K)}$. Hence (1) guarantees that this is the case and we win. □

70.8. Algebraic morphisms

The following notion is occasionally useful.

Definition 70.8.1. Let S be a scheme. Let $F : \mathcal{X} \to \mathcal{Y}$ be a 1-morphism of stacks in groupoids over $(Sch/S)_{fppf}$. We say that F is *algebraic* if for every scheme T and every object ξ of $\mathcal{Y}$ over T the 2-fibre product

$$(Sch/T)_{fppf} \times_{\xi,\mathcal{Y}} \mathcal{X}$$

is an algebraic stack over S.

With this terminology in place we have the following result that generalizes Algebraic Stacks, Lemma 67.15.4.

Lemma 70.8.2. *Let S be a scheme. Let $F : \mathcal{X} \to \mathcal{Y}$ be a 1-morphism of stacks in groupoids over $(Sch/S)_{fppf}$. If*

(1) *$\mathcal{Y}$ is an algebraic stack, and*
(2) *F is algebraic (see above),*

then $\mathcal{X}$ is an algebraic stack.

Proof. By assumption (1) there exists a scheme T and an object ξ of $\mathcal{Y}$ over T such that the corresponding 1-morphism $\xi : (Sch/T)_{fppf} \to \mathcal{Y}$ is smooth an surjective. Then $\mathcal{U} = (Sch/T)_{fppf} \times_{\xi,\mathcal{Y}} \mathcal{X}$ is is an algebraic stack by assumption (2). Choose a scheme U and a surjective smooth 1-morphism $(Sch/U)_{fppf} \to \mathcal{U}$. The projection $\mathcal{U} \longrightarrow \mathcal{X}$ is, as the base change of the morphism $\xi : (Sch/T)_{fppf} \to \mathcal{Y}$,

surjective and smooth, see Algebraic Stacks, Lemma 67.10.6. Then the composition $(Sch/U)_{fppf} \to \mathcal{U} \to \mathcal{X}$ is surjective and smooth as a composition of surjective and smooth morphisms, see Algebraic Stacks, Lemma 67.10.5. Hence $\mathcal{X}$ is an algebraic stack by Algebraic Stacks, Lemma 67.15.3. □

Lemma 70.8.3. *Let S be a scheme. Let $F : \mathcal{X} \to \mathcal{Y}$ be a 1-morphism of stacks in groupoids over $(Sch/S)_{fppf}$. If $\mathcal{X}$ is an algebraic stack and $\Delta : \mathcal{Y} \to \mathcal{Y} \times \mathcal{Y}$ is representable by algebraic spaces, then F is algebraic.*

Proof. Choose a representable stack in groupoids $\mathcal{U}$ and a surjective smooth 1-morphism $\mathcal{U} \to \mathcal{X}$. Let T be a scheme and let ξ be an object of $\mathcal{Y}$ over T. The morphism of 2-fibre products

$$(Sch/T)_{fppf} \times_{\xi,\mathcal{Y}} \mathcal{U} \longrightarrow (Sch/T)_{fppf} \times_{\xi,\mathcal{Y}} \mathcal{X}$$

is representable by algebraic spaces, surjective, and smooth as a base change of $\mathcal{U} \to \mathcal{X}$, see Algebraic Stacks, Lemmas 67.9.7 and 67.10.6. By our condition on the diagonal of $\mathcal{Y}$ we see that the source of this morphism is representable by an algebraic space, see Algebraic Stacks, Lemma 67.10.11. Hence the target is an algebraic stack by Algebraic Stacks, Lemma 67.15.3. □

70.9. Spaces of sections

Given morphisms $W \to Z \to U$ we can consider the functor that associates to a scheme U' over U the set of sections $\sigma : Z_{U'} \to W_{U'}$ of the base change $W_{U'} \to Z_{U'}$ of the morphism $W \to Z$. In this section we prove some preliminary lemmas on this functor.

Lemma 70.9.1. *Let $Z \to U$ be a finite morphism of schemes. Let W be an algebraic space and let $W \to Z$ be a surjective étale morphism. Then there exists a surjective étale morphism $U' \to U$ and a section*

$$\sigma : Z_{U'} \to W_{U'}$$

of the morphism $W_{U'} \to Z_{U'}$.

Proof. We may choose a separated scheme W' and a surjective étale morphism $W' \to W$. Hence after replacing W by W' we may assume that W is a separated scheme. Write $f : W \to Z$ and $\pi : Z \to U$. Note that $f \circ \pi : W \to U$ is separated as W is separated (see Schemes, Lemma 25.21.14). Let $u \in U$ be a point. Clearly it suffices to find an étale neighbourhood (U', u') of (U, u) such that a section σ exists over U'. Let $z_1, \ldots, z_r$ be the points of Z lying above u. For each i choose a point $w_i \in W$ which maps to z_i. We may pick an étale neighbourhood $(U', u') \to (U, u)$ such that the conclusions of More on Morphisms, Lemma 36.30.5 hold for both $Z \to U$ and the points $z_1, \ldots, z_r$ and $W \to U$ and the points $w_1, \ldots, w_r$. Hence, after replacing (U, u) by (U', u') and relabeling, we may assume that all the field extensions $\kappa(u) \subset \kappa(z_i)$ and $\kappa(u) \subset \kappa(w_i)$ are purely inseparable, and moreover that there exist disjoint union decompositions

$$Z = V_1 \amalg \ldots \amalg V_r \amalg A, \quad W = W_1 \amalg \ldots \amalg W_r \amalg B$$

by open and closed subschemes with $z_i \in V_i$, $w_i \in W_i$ and $V_i \to U$, $W_i \to U$ finite. After replacing U by $U \setminus \pi(A)$ we may assume that $A = \emptyset$, i.e., $Z = V_1 \amalg \ldots \amalg V_r$. After replacing W_i by $W_i \cap f^{-1}(V_i)$ and B by $B \cup \bigcup W_i \cap f^{-1}(Z \setminus V_i)$ we may assume that f maps W_i into V_i. Then $f_i = f|_{W_i} : W_i \to V_i$ is a morphism of

schemes finite over U, hence finite (see Morphisms, Lemma 28.44.12). It is also étale (by assumption), $f_i^{-1}(\{z_i\}) = w_i$, and induces an isomorphism of residue fields $\kappa(z_i) = \kappa(w_i)$ (because both are purely inseparable extensions of $\kappa(u)$ and $\kappa(z_i) \subset \kappa(w_i)$ is separable as f is étale). Hence by Étale Morphisms, Lemma 40.14.2 we see that f_i is an isomorphism in a neighbourhood V_i' of z_i. Since $\pi : Z \to U$ is closed, after shrinking U, we may assume that $W_i \to V_i$ is an isomorphism. This proves the lemma. □

Lemma 70.9.2. *Let $Z \to U$ be a finite locally free morphism of schemes. Let W be an algebraic space and let $W \to Z$ be an étale morphism. Then the functor*

$$F : (Sch/U)^{opp}_{fppf} \longrightarrow Sets,$$

defined by the rule

$$U' \longmapsto F(U') = \{\sigma : Z_{U'} \to W_{U'} \text{ section of } W_{U'} \to Z_{U'}\}$$

is an algebraic space and the morphism $F \to U$ is étale.

Proof. Assume first that $W \to Z$ is also separated. Let U' be a scheme over U and let $\sigma \in F(U')$. By Morphisms of Spaces, Lemma 49.5.7 the morphism σ is a closed immersion. Moreover, σ is étale by Properties of Spaces, Lemma 48.13.6. Hence σ is also an open immersion, see Morphisms of Spaces, Lemma 49.44.2. In other words, $Z_\sigma = \sigma(Z_{U'}) \subset W_{U'}$ is an open subspace such that the morphism $Z_\sigma \to Z_{U'}$ is an isomorphism. In particular, the morphism $Z_\sigma \to U'$ is finite. Hence we obtain a transformation of functors

$$F \longrightarrow (W/U)_{fin}, \quad \sigma \longmapsto (U' \to U, Z_\sigma)$$

where $(W/U)_{fin}$ is the finite part of the morphism $W \to U$ introduced in More on Groupoids in Spaces, Section 60.8. It is clear that this transformation of functors is injective (since we can recover σ from Z_σ as the inverse of the isomorphism $Z_\sigma \to Z_{U'}$). By More on Groupoids in Spaces, Proposition 60.8.11 we know that $(W/U)_{fin}$ is an algebraic space étale over U. Hence to finish the proof in this case it suffices to show that $F \to (W/U)_{fin}$ is representable and an open immersion. To see this suppose that we are given a morphism of schemes $U' \to U$ and an open subspace $Z' \subset W_{U'}$ such that $Z' \to U'$ is finite. Then it suffices to show that there exists an open subscheme $U'' \subset U'$ such that a morphism $T \to U'$ factors through U'' if and only if $Z' \times_{U'} T$ maps isomorphically to $Z \times_{U'} T$. This follows from More on Morphisms of Spaces, Lemma 57.36.6 (here we use that $Z \to B$ is flat and locally of finite presentation as well as finite). Hence we have proved the lemma in case $W \to Z$ is separated as well as étale.

In the general case we choose a separated scheme W' and a surjective étale morphism $W' \to W$. Note that the morphisms $W' \to W$ and $W \to Z$ are separated as their source is separated. Denote F' the functor associated to $W' \to Z \to U$ as in the lemma. In the first paragraph of the proof we showed that F' is representable by an algebraic space étale over U. By Lemma 70.9.1 the map of functors $F' \to F$ is surjective for the étale topology on Sch/U. Moreover, if U' and $\sigma : Z_{U'} \to W_{U'}$ define a point $\xi \in F(U')$, then the fibre product

$$F'' = F' \times_{F,\xi} U'$$

is the functor on Sch/U' associated to the morphisms

$$W'_{U'} \times_{W_{U'},\sigma} Z_{U'} \to Z_{U'} \to U'.$$

Since the first morphism is separated as a base change of a separated morphism, we see that F'' is an algebraic space étale over U' by the result of the first paragraph. It follows that $F' \to F$ is a surjective étale transformation of functors, which is representable by algebraic spaces. Hence F is an algebraic space by Bootstrap, Theorem 61.10.1. Since $F' \to F$ is an étale surjective morphism of algebraic spaces it follows that $F \to U$ is étale because $F' \to U$ is étale. □

70.10. Relative morphisms

Let S be a scheme. Let $Z \to B$ and $X \to B$ be morphisms of algebraic spaces over S. Given a scheme T we can consider pairs (a, b) where $a : T \to B$ is a morphism and $b : T \times_{a,B} Z \to T \times_{a,B} X$ is a morphism over T. Picture

(70.10.0.1)
$$\begin{array}{ccccc} T \times_{a,B} Z & \xrightarrow[b]{} & T \times_{a,B} X & \qquad Z & \qquad X \\ & \searrow & \swarrow & \qquad \searrow & \qquad \swarrow \\ & T & \xrightarrow{a} & B & \end{array}$$

Of course, we can also think of b as a morphism $b : T \times_{a,B} Z \to X$ such that

$$\begin{array}{ccccc} T \times_{a,B} Z & \longrightarrow & Z & \overset{b}{\longrightarrow} & X \\ \downarrow & & & \searrow & \uparrow \\ T & & \xrightarrow{a} & & B \end{array}$$

commutes. In this situation we can define a functor

$$Mor_B(Z, X) : (Sch/S)^{opp} \longrightarrow Sets, \quad T \longmapsto \{(a, b) \text{ as above}\} \tag{70.10.0.2}$$

Sometimes we think of this as a functor defined on the category of schemes over B, in which case we drop a from the notation.

Lemma 70.10.1. *Let S be a scheme. Let $Z \to B$ and $X \to B$ be morphisms of algebraic spaces over S. Then*

1. *$Mor_B(Z, X)$ is a sheaf on $(Sch/S)_{fppf}$.*
2. *If T is an algebraic space over S, then there is a canonical bijection*

$$\mathrm{Mor}_{Sh((Sch/S)_{fppf})}(T, Mor_B(Z, X)) = \{(a, b) \text{ as in } (70.10.0.1)\}$$

Proof. Let T be an algebraic space over S. Let $\{T_i \to T\}$ be an fppf covering of T (as in Topologies on Spaces, Section 54.4). Suppose that $(a_i, b_i) \in Mor_B(Z, X)(T_i)$ such that $(a_i, b_i)|_{T_i \times_T T_j} = (a_j, b_j)|_{T_i \times_T T_j}$ for all i, j. Then by Descent on Spaces, Lemma 55.6.2 there exists a unique morphism $a : T \to B$ such that a_i is the composition of $T_i \to T$ and a. Then $\{T_i \times_{a_i,B} Z \to T \times_{a,B} Z\}$ is an fppf covering too and the same lemma implies there exists a unique morphism $b : T \times_{a,B} Z \to T \times_{a,B} X$ such that b_i is the composition of $T_i \times_{a_i,B} Z \to T \times_{a,B} Z$ and b. Hence $(a, b) \in Mor_B(Z, X)(T)$ restricts to (a_i, b_i) over T_i for all i.

Note that the result of the preceding paragraph in particular implies (1).

Let T be an algebraic space over S. In order to prove (2) we will construct mutually inverse maps between the displayed sets. In the following when we say "pair" we mean a pair (a, b) fitting into (70.10.0.1).

Let $v : T \to Mor_B(Z, X)$ be a natural transformation. Choose a scheme U and a surjective étale morphism $p : U \to T$. Then $v(p) \in Mor_B(Z, X)(U)$ corresponds

to a pair (a_U, b_U) over U. Let $R = U \times_T U$ with projections $t, s : R \to U$. As v is a transformation of functors we see that the pullbacks of (a_U, b_U) by s and t agree. Hence, since $\{U \to T\}$ is an fppf covering, we may apply the result of the first paragraph that deduce that there exists a unique pair (a, b) over T.

Conversely, let (a, b) be a pair over T. Let $U \to T$, $R = U \times_T U$, and $t, s : R \to U$ be as above. Then the restriction $(a, b)|_U$ gives rise to a transformation of functors $v : h_U \to Mor_B(Z, X)$ by the Yoneda lemma (Categories, Lemma 4.3.5). As the two pullbacks $s^*(a, b)|_U$ and $t^*(a, b)|_U$ are equal, we see that v coequalizes the two maps $h_t, h_s : h_R \to h_U$. Since $T = U/R$ is the fppf quotient sheaf by Spaces, Lemma 47.9.1 and since $Mor_B(Z, X)$ is an fppf sheaf by (1) we conclude that v factors through a map $T \to Mor_B(Z, X)$.

We omit the verification that the two constructions above are mutually inverse. □

Lemma 70.10.2. *Let S be a scheme. Let $Z \to B$, $X \to B$, and $B' \to B$ be morphisms of algebraic spaces over S. Set $Z' = B' \times_B Z$ and $X' = B' \times_B X$. Then*

$$Mor_{B'}(Z', X') = B' \times_B Mor_B(Z, X)$$

in $Sh((Sch/S)_{fppf})$.

Proof. The equality as functors follows immediately from the definitions. The equality as sheaves follows from this because both sides are sheaves according to Lemma 70.10.1 and the fact that a fibre product of sheaves is the same as the corresponding fibre product of pre-sheaves (i.e., functors). □

Lemma 70.10.3. *Let S be a scheme. Let $Z \to B$ and $X' \to X \to B$ be morphisms of algebraic spaces over S. Assume*

(1) *$X' \to X$ is étale, and*
(2) *$Z \to B$ is finite locally free.*

Then $Mor_B(Z, X') \to Mor_B(Z, X)$ is representable by algebraic spaces and étale. If $X' \to X$ is also surjective, then $Mor_B(Z, X') \to Mor_B(Z, X)$ is surjective.

Proof. Let U be a scheme and let $\xi = (a, b)$ be an element of $Mor_B(Z, X)(U)$. We have to prove that the functor

$$h_U \times_{\xi, Mor_B(Z,X)} Mor_B(Z, X')$$

is representable by an algebraic space étale over U. Set $Z_U = U \times_{a,B} Z$ and $W = Z_U \times_{b,X} X'$. Then $W \to Z_U \to U$ is as in Lemma 70.9.2 and the sheaf F defined there is identified with the fibre product displayed above. Hence the first assertion of the lemma. The second assertion follows from this and Lemma 70.9.1 which guarantees that $F \to U$ is surjective in the situation above. □

Lemma 70.10.4. *Let $Z \to B$ and $X \to B$ be morphisms of affine schemes. Assume $\Gamma(Z, \mathcal{O}_Z)$ is a finite free $\Gamma(B, \mathcal{O}_B)$-module. Then $Mor_B(Z, X)$ is representable by an affine scheme over B.*

Proof. Write $B = \mathrm{Spec}(R)$. Choose a basis $\{e_1, \ldots, e_m\}$ for $\Gamma(Z, \mathcal{O}_Z)$. Finally, choose a presentation

$$\Gamma(X, \mathcal{O}_X) = R[\{x_i\}_{i \in I}]/(\{f_k\}_{k \in K}).$$

We will denote $\overline{x}_i$ the image of x_i in this quotient. Write

$$P = R[\{a_{ij}\}_{i \in I, 1 \leq j \leq m}].$$

Consider the R-algebra map

$$\Psi : R[\{x_i\}_{i\in I}] \longrightarrow P \otimes_R \Gamma(Z, \mathcal{O}_Z), \quad x_i \longmapsto \sum\nolimits_j a_{ij} \otimes e_j.$$

Write $\Psi(f_k) = \sum c_{kj} \otimes e_j$ with $c_{kj} \in P$. Finally, denote $J \subset P$ the ideal generated by the elements c_{kj}, $k \in K$, $1 \leq j \leq m$. We claim that $W = \mathrm{Spec}(P/J)$ represents the functor $Mor_B(Z, X)$.

First, note that by construction P/J is an R-algebra, hence a morphism $a_{univ} : W \to B$. Second, by construction the map Ψ factors through $\Gamma(X, \mathcal{O}_X)$, hence we obtain an P/J-algebra homomorphism

$$P/J \otimes_R \Gamma(X, \mathcal{O}_X) \longrightarrow P/J \otimes_R \Gamma(Z, \mathcal{O}_Z)$$

which determines a morphism $b_{univ} : W \times_{a_{univ},B} Z \to W \times_{a_{univ},B} X$. By the Yoneda lemma the pair (a_{univ}, b_{univ}) determines a transformation of functors $W \to Mor_B(Z, X)$ which we claim is an isomorphism. To show that it is an isomorphism it suffices to show that it induces a bijection of sets $W(T) \to Mor_B(Z, X)(T)$ over any affine scheme T.

Suppose $T = \mathrm{Spec}(R')$ is an affine scheme and $(a, b) \in Mor_B(Z, X)(T)$, then a defines an R-algebra structure on R' and b defines an R'-algebra map

$$b^\sharp : R' \otimes_R \Gamma(X, \mathcal{O}_X) \longrightarrow R' \otimes_R \Gamma(Z, \mathcal{O}_Z).$$

In particular we can write $b^\sharp(1 \otimes \overline{x}_i) = \sum \alpha_{ij} \otimes e_j$ for some $\alpha_{ij} \in R'$. This corresponds to an R-algebra map $P \to R'$ determined by the rule $a_{ij} \mapsto \alpha_{ij}$. This map factors through the quotient P/J by the construction of the ideal J to give a map $P/J \to R'$. This in turn corresponds to a morphism $T \to W$ such that (a, b) is the pullback of (a_{univ}, b_{univ}). Some details omitted. □

Proposition 70.10.5. *Let S be a scheme. Let $Z \to B$ and $X \to B$ be morphisms of algebraic spaces over S. If $Z \to B$ is finite locally free then $Mor_B(Z, X)$ is an algebraic space.*

Proof. Choose a scheme $B' = \coprod B'_i$ which is a disjoint union of affine schemes B'_i and an étale surjective morphism $B' \to B$. We may also assume that $B'_i \times_B Z$ is the spectrum of a ring which is finite free as a $\Gamma(B'_i, \mathcal{O}_{B'_i})$-module. By Lemma 70.10.2 and Spaces, Lemma 47.5.5 the morphism $Mor_{B'}(Z', X') \to Mor_B(Z, X)$ is surjective étale. Hence by Bootstrap, Theorem 61.10.1 it suffices to prove the proposition when $B = B'$ is a disjoint union of affine schemes B'_i so that each $B'_i \times_B Z$ is finite free over B'_i. Then it actually suffices to prove the result for the restriction to each B'_i. Thus we may assume that B is affine and that $\Gamma(Z, \mathcal{O}_Z)$ is a finite free $\Gamma(B, \mathcal{O}_B)$-module.

Choose a scheme X' which is a disjoint union of affine schemes and a surjective étale morphism $X' \to X$. By Lemma 70.10.3 the morphism $Mor_B(Z, X') \to Mor_B(Z, X)$ is representable by algebraic spaces, étale, and surjective. Hence by Bootstrap, Theorem 61.10.1 it suffices to prove the proposition when X is a disjoint union of affine schemes. This reduces us to the case discussed in the next paragraph.

Assume $X = \coprod_{i\in I} X_i$ is a disjoint union of affine schemes, B is affine, and that $\Gamma(Z, \mathcal{O}_Z)$ is a finite free $\Gamma(B, \mathcal{O}_B)$-module. For any finite subset $E \subset I$ set

$$F_E = Mor_B(Z, \coprod\nolimits_{i\in E} X_i).$$

By Lemma 70.10.4 we see that F_E is an algebraic space. Consider the morphism

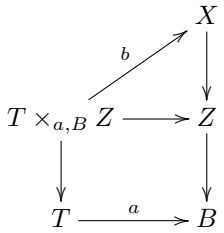
$$\coprod_{E \subset I \text{ finite}} F_E \longrightarrow Mor_B(Z, X)$$

Each of the morphisms $F_E \to Mor_B(Z, X)$ is an open immersion, because it is simply the locus parametrizing pairs (a, b) where b maps into the open subscheme $\coprod_{i \in E} X_i$ of X. Moreover, if T is quasi-compact, then for any pair (a, b) the image of b is contained in $\coprod_{i \in E} X_i$ for some $E \subset I$ finite. Hence the displayed arrow is in fact an open covering and we win[3] by Spaces, Lemma 47.8.4. □

70.11. Restriction of scalars

Suppose $X \to Z \to B$ are morphisms of algebraic spaces over S. Given a scheme T we can consider pairs (a, b) where $a : T \to B$ is a morphism and $b : T \times_{a,B} Z \to X$ is a morphism over Z. Picture

(70.11.0.1)



In this situation we can define a functor

$$\text{Res}_{Z/B}(X) : (Sch/S)^{opp} \longrightarrow Sets, \quad T \longmapsto \{(a, b) \text{ as above}\} \tag{70.11.0.2}$$

Sometimes we think of this as a functor defined on the category of schemes over B, in which case we drop a from the notation.

Lemma 70.11.1. *Let S be a scheme. Let $X \to Z \to B$ be morphisms of algebraic spaces over S. Then*

1. *$Res_{Z/B}(X)$ is a sheaf on $(Sch/S)_{fppf}$.*
2. *If T is an algebraic space over S, then there is a canonical bijection*

$$\text{Mor}_{Sh((Sch/S)_{fppf})}(T, Res_{Z/B}(X)) = \{(a, b) \text{ as in } (70.11.0.1)\}$$

Proof. Let T be an algebraic space over S. Let $\{T_i \to T\}$ be an fppf covering of T (as in Topologies on Spaces, Section 54.4). Suppose that $(a_i, b_i) \in \text{Res}_{Z/B}(X)(T_i)$ such that $(a_i, b_i)|_{T_i \times_T T_j} = (a_j, b_j)|_{T_i \times_T T_j}$ for all i, j. Then by Descent on Spaces, Lemma 55.6.2 there exists a unique morphism $a : T \to B$ such that a_i is the composition of $T_i \to T$ and a. Then $\{T_i \times_{a_i,B} Z \to T \times_{a,B} Z\}$ is an fppf covering too and the same lemma implies there exists a unique morphism $b : T \times_{a,B} Z \to X$ such that b_i is the composition of $T_i \times_{a_i,B} Z \to T \times_{a,B} Z$ and b. Hence $(a, b) \in \text{Res}_{Z/B}(X)(T)$ restricts to (a_i, b_i) over T_i for all i.

Note that the result of the preceding paragraph in particular implies (1).

Let T be an algebraic space over S. In order to prove (2) we will construct mutually inverse maps between the displayed sets. In the following when we say "pair" we mean a pair (a, b) fitting into (70.11.0.1).

[3] Modulo some set theoretic arguments. Namely, we have to show that $\coprod F_E$ is an algebraic space. This follows because $|I| \leq \text{size}(X)$ and $\text{size}(F_E) \leq \text{size}(X)$ as follows from the explicit description of F_E in the proof of Lemma 70.10.4. Some details omitted.

Let $v : T \to \mathrm{Res}_{Z/B}(X)$ be a natural transformation. Choose a scheme U and a surjective étale morphism $p : U \to T$. Then $v(p) \in \mathrm{Res}_{Z/B}(X)(U)$ corresponds to a pair (a_U, b_U) over U. Let $R = U \times_T U$ with projections $t, s : R \to U$. As v is a transformation of functors we see that the pullbacks of (a_U, b_U) by s and t agree. Hence, since $\{U \to T\}$ is an fppf covering, we may apply the result of the first paragraph that deduce that there exists a unique pair (a, b) over T.

Conversely, let (a, b) be a pair over T. Let $U \to T$, $R = U \times_T U$, and $t, s : R \to U$ be as above. Then the restriction $(a, b)|_U$ gives rise to a transformation of functors $v : h_U \to \mathrm{Res}_{Z/B}(X)$ by the Yoneda lemma (Categories, Lemma 4.3.5). As the two pullbacks $s^*(a, b)|_U$ and $t^*(a, b)|_U$ are equal, we see that v coequalizes the two maps $h_t, h_s : h_R \to h_U$. Since $T = U/R$ is the fppf quotient sheaf by Spaces, Lemma 47.9.1 and since $\mathrm{Res}_{Z/B}(X)$ is an fppf sheaf by (1) we conclude that v factors through a map $T \to \mathrm{Res}_{Z/B}(X)$.

We omit the verification that the two constructions above are mutually inverse. □

Of course the sheaf $\mathrm{Res}_{Z/B}(X)$ comes with a natural transformation of functors $\mathrm{Res}_{Z/B}(X) \to B$. We will use this without further mention in the following.

Lemma 70.11.2. *Let S be a scheme. Let $X \to Z \to B$ and $B' \to B$ be morphisms of algebraic spaces over S. Set $Z' = B' \times_B Z$ and $X' = B' \times_B X$. Then*

$$Res_{Z'/B'}(X') = B' \times_B Res_{Z/B}(X)$$

in $Sh((Sch/S)_{fppf})$.

Proof. The equality as functors follows immediately from the definitions. The equality as sheaves follows from this because both sides are sheaves according to Lemma 70.11.1 and the fact that a fibre product of sheaves is the same as the corresponding fibre product of pre-sheaves (i.e., functors). □

Lemma 70.11.3. *Let S be a scheme. Let $X' \to X \to Z \to B$ be morphisms of algebraic spaces over S. Assume*

(1) *$X' \to X$ is étale, and*
(2) *$Z \to B$ is finite locally free.*

Then $Res_{Z/B}(X') \to Res_{Z/B}(X)$ is representable by algebraic spaces and étale. If $X' \to X$ is also surjective, then $Res_{Z/B}(X') \to Res_{Z/B}(X)$ is surjective.

Proof. Let U be a scheme and let $\xi = (a, b)$ be an element of $\mathrm{Res}_{Z/B}(X)(U)$. We have to prove that the functor

$$h_U \times_{\xi, \mathrm{Res}_{Z/B}(X)} \mathrm{Res}_{Z/B}(X')$$

is representable by an algebraic space étale over U. Set $Z_U = U \times_{a,B} Z$ and $W = Z_U \times_{b,X} X'$. Then $W \to Z_U \to U$ is as in Lemma 70.9.2 and the sheaf F defined there is identified with the fibre product displayed above. Hence the first assertion of the lemma. The second assertion follows from this and Lemma 70.9.1 which guarantees that $F \to U$ is surjective in the situation above. □

At this point we can use the lemmas above to prove that $\mathrm{Res}_{Z/B}(X)$ is an algebraic space whenever $Z \to B$ is finite locally free in almost exactly the same way as in the proof that $Mor_B(Z, X)$ is an algebraic spaces, see Proposition 70.10.5. Instead we will directly deduce this result from the following lemma and the fact that $Mor_B(Z, X)$ is an algebraic space.

Lemma 70.11.4. *Let S be a scheme. Let $X \to Z \to B$ be morphisms of algebraic spaces over S. The following diagram*

$$\begin{array}{ccc} Mor_B(Z, X) & \longrightarrow & Mor_B(Z, Z) \\ \uparrow & & \uparrow{\scriptstyle id_Z} \\ Res_{Z/B}(X) & \longrightarrow & B \end{array}$$

is a cartesian diagram of sheaves on $(Sch/S)_{fppf}$.

Proof. Omitted. Hint: Exercise in the functorial point of view in algebraic geometry. □

Proposition 70.11.5. *Let S be a scheme. Let $X \to Z \to B$ be morphisms of algebraic spaces over S. If $Z \to B$ is finite locally free then $Res_{Z/B}(X)$ is an algebraic space.*

Proof. By Proposition 70.10.5 the functors $Mor_B(Z, X)$ and $Mor_B(Z, Z)$ are algebraic spaces. Hence this follows from the cartesian diagram of Lemma 70.11.4 and the fact that fibre products of algebraic spaces exist and are given by the fibre product in the underlying category of sheaves of sets (see Spaces, Lemma 47.7.2). □

70.12. Finite Hilbert stacks

In this section we prove some results concerning the finite Hilbert stacks $\mathcal{H}_d(\mathcal{X}/\mathcal{Y})$ introduced in Examples of Stacks, Section 68.17.

Lemma 70.12.1. *Consider a 2-commutative diagram*

$$\begin{array}{ccc} \mathcal{X}' & \xrightarrow[G]{} & \mathcal{X} \\ {\scriptstyle F'}\downarrow & & \downarrow{\scriptstyle F} \\ \mathcal{Y}' & \xrightarrow{H} & \mathcal{Y} \end{array}$$

of stacks in groupoids over $(Sch/S)_{fppf}$ with a given 2-isomorphism $\gamma : H \circ F' \to F \circ G$. In this situation we obtain a canonical 1-morphism $\mathcal{H}_d(\mathcal{X}'/\mathcal{Y}') \to \mathcal{H}_d(\mathcal{X}/\mathcal{Y})$. This morphism is compatible with the forgetful 1-morphisms of Examples of Stacks, Equation (68.17.2.1).

Proof. We map the object (U, Z, y', x', α') to the object $(U, Z, H(y'), G(x'), \gamma \star \mathrm{id}_H \star \alpha')$ where $\star$ denotes horizontal composition of 2-morphisms, see Categories, Definition 4.26.1. To a morphism $(f, g, b, a) : (U_1, Z_1, y'_1, x'_1, \alpha'_1) \to (U_2, Z_2, y'_2, x'_2, \alpha'_2)$ we assign $(f, g, H(b), G(a))$. We omit the verification that this defines a functor between categories over $(Sch/S)_{fppf}$. □

Lemma 70.12.2. *In the situation of Lemma 70.12.1 assume that the given square is 2-cartesian. Then the diagram*

$$\begin{array}{ccc} \mathcal{H}_d(\mathcal{X}'/\mathcal{Y}') & \longrightarrow & \mathcal{H}_d(\mathcal{X}/\mathcal{Y}) \\ \downarrow & & \downarrow \\ \mathcal{Y}' & \longrightarrow & \mathcal{Y} \end{array}$$

is 2-cartesian.

Proof. We get a 2-commutative diagram by Lemma 70.12.1 and hence we get a 1-morphism (i.e., a functor)

$$\mathcal{H}_d(\mathcal{X}'/\mathcal{Y}') \longrightarrow \mathcal{Y}' \times_{\mathcal{Y}} \mathcal{H}_d(\mathcal{X}/\mathcal{Y})$$

We indicate why this functor is essentially surjective. Namely, an object of the category on the right hand side is given by a scheme U over S, an object y' of $\mathcal{Y}'_U$, an object (U, Z, y, x, α) of $\mathcal{H}_d(\mathcal{X}/\mathcal{Y})$ over U and an isomorphism $H(y') \to y$ in $\mathcal{Y}_U$. The assumption means exactly that there exists an object x' of $\mathcal{X}'_Z$ such that there exist isomorphisms $G(x') \cong x$ and $\alpha' : y'|_Z \to F'(x')$ compatible with α. Then we see that (U, Z, y', x', α') is an object of $\mathcal{H}_d(\mathcal{X}'/\mathcal{Y}')$ over U. Details omitted. □

Lemma 70.12.3. *In the situation of Lemma 70.12.1 assume*

(1) $\mathcal{Y}' = \mathcal{Y}$ *and* $H = id_{\mathcal{Y}}$,
(2) G *is representable by algebraic spaces and étale.*

Then $\mathcal{H}_d(\mathcal{X}'/\mathcal{Y}) \to \mathcal{H}_d(\mathcal{X}/\mathcal{Y})$ *is representable by algebraic spaces and étale. If* G *is also surjective, then* $\mathcal{H}_d(\mathcal{X}'/\mathcal{Y}) \to \mathcal{H}_d(\mathcal{X}/\mathcal{Y})$ *is surjective.*

Proof. Let U be a scheme and let $\xi = (U, Z, y, x, \alpha)$ be an object of $\mathcal{H}_d(\mathcal{X}/\mathcal{Y})$ over U. We have to prove that the 2-fibre product

$$(Sch/U)_{fppf} \times_{\xi, \mathcal{H}_d(\mathcal{X}/\mathcal{Y})} \mathcal{H}_d(\mathcal{X}'/\mathcal{Y}) \tag{70.12.3.1}$$

is representable by an algebraic space étale over U. An object of this over U' corresponds to an object x' in the fibre category of $\mathcal{X}'$ over $Z_{U'}$ such that $G(x') \cong x|_{Z_{U'}}$. By assumption the 2-fibre product

$$(Sch/Z)_{fppf} \times_{x, \mathcal{X}} \mathcal{X}'$$

is representable by an algebraic space W such that the projection $W \to Z$ is étale. Then (70.12.3.1) is representable by the algebraic space F parametrizing sections of $W \to Z$ over U introduced in Lemma 70.9.2. Since $F \to U$ is étale we conclude that $\mathcal{H}_d(\mathcal{X}'/\mathcal{Y}) \to \mathcal{H}_d(\mathcal{X}/\mathcal{Y})$ is representable by algebraic spaces and étale. Finally, if $\mathcal{X}' \to \mathcal{X}$ is surjective also, then $W \to Z$ is surjective, and hence $F \to U$ is surjective by Lemma 70.9.1. Thus in this case $\mathcal{H}_d(\mathcal{X}'/\mathcal{Y}) \to \mathcal{H}_d(\mathcal{X}/\mathcal{Y})$ is also surjective. □

Lemma 70.12.4. *In the situation of Lemma 70.12.1. Assume that* G, H *are representable by algebraic spaces and étale. Then* $\mathcal{H}_d(\mathcal{X}'/\mathcal{Y}') \to \mathcal{H}_d(\mathcal{X}/\mathcal{Y})$ *is representable by algebraic spaces and étale. If also* H *is surjective and the induced functor* $\mathcal{X}' \to \mathcal{Y}' \times_{\mathcal{Y}} \mathcal{X}$ *is surjective, then* $\mathcal{H}_d(\mathcal{X}'/\mathcal{Y}') \to \mathcal{H}_d(\mathcal{X}/\mathcal{Y})$ *is surjective.*

Proof. Set $\mathcal{X}'' = \mathcal{Y}' \times_{\mathcal{Y}} \mathcal{X}$. By Lemma 70.4.1 the 1-morphism $\mathcal{X}' \to \mathcal{X}''$ is representable by algebraic spaces and étale (in particular the condition in the second statement of the lemma that $\mathcal{X}' \to \mathcal{X}''$ be surjective makes sense). We obtain a 2-commutative diagram

$$\begin{array}{ccccc} \mathcal{X}' & \longrightarrow & \mathcal{X}'' & \longrightarrow & \mathcal{X} \\ \downarrow & & \downarrow & & \downarrow \\ \mathcal{Y}' & \longrightarrow & \mathcal{Y}' & \longrightarrow & \mathcal{Y} \end{array}$$

It follows from Lemma 70.12.2 that $\mathcal{H}_d(\mathcal{X}''/\mathcal{Y}')$ is the base change of $\mathcal{H}_d(\mathcal{X}/\mathcal{Y})$ by $\mathcal{Y}' \to \mathcal{Y}$. In particular we see that $\mathcal{H}_d(\mathcal{X}''/\mathcal{Y}') \to \mathcal{H}_d(\mathcal{X}/\mathcal{Y})$ is representable by algebraic spaces and étale, see Algebraic Stacks, Lemma 67.10.6. Moreover, it is also surjective if H is. Hence if we can show that the result holds for the left square

in the diagram, then we're done. In this way we reduce to the case where $\mathcal{Y}' = \mathcal{Y}$ which is the content of Lemma 70.12.3. □

Lemma 70.12.5. *Let $F : \mathcal{X} \to \mathcal{Y}$ be a 1-morphism of stacks in groupoids over $(Sch/S)_{fppf}$. Assume that $\Delta : \mathcal{Y} \to \mathcal{Y} \times \mathcal{Y}$ is representable by algebraic spaces. Then*

$$\mathcal{H}_d(\mathcal{X}/\mathcal{Y}) \longrightarrow \mathcal{H}_d(\mathcal{X}) \times \mathcal{Y}$$

see Examples of Stacks, Equation (68.17.2.1) is representable by algebraic spaces.

Proof. Let U be a scheme and let $\xi = (U, Z, p, x, 1)$ be an object of $\mathcal{H}_d(\mathcal{X}) = \mathcal{H}_d(\mathcal{X}/S)$ over U. Here p is just the structure morphism of U. The fifth component 1 exists and is unique since everything is over S. Also, let y be an object of $\mathcal{Y}$ over U. We have to show the 2-fibre product

$$(Sch/U)_{fppf} \times_{\xi \times y, \mathcal{H}_d(\mathcal{X}) \times \mathcal{Y}} \mathcal{H}_d(\mathcal{X}/\mathcal{Y}) \tag{70.12.5.1}$$

is representable by an algebraic space. To explain why this is so we introduce

$$I = Isom_{\mathcal{Y}}(y|_Z, F(x))$$

which is an algebraic space over Z by assumption. Let $a : U' \to U$ be a scheme over U. What does it mean to give an object of the fibre category of (70.12.5.1) over U'? Well, it means that we have an object $\xi' = (U', Z', y', x', \alpha')$ of $\mathcal{H}_d(\mathcal{X}/\mathcal{Y})$ over U' and isomorphisms $(U', Z', p', x', 1) \cong (U, Z, p, x, 1)|_{U'}$ and $y' \cong y|_{U'}$. Thus ξ' is isomorphic to $(U', U' \times_{a,U} Z, a^*y, x|_{U' \times_{a,U} Z}, \alpha)$ for some morphism

$$\alpha : a^*y|_{U' \times_{a,U} Z} \longrightarrow F(x|_{U' \times_{a,U} Z})$$

in the fibre category of $\mathcal{Y}$ over $U' \times_{a,U} Z$. Hence we can view α as a morphism $b : U' \times_{a,U} Z \to I$. In this way we see that (70.12.5.1) is representable by $\mathrm{Res}_{Z/U}(I)$ which is an algebraic space by Proposition 70.11.5. □

The following lemma is a (partial) generalization of Lemma 70.12.3.

Lemma 70.12.6. *Let $F : \mathcal{X} \to \mathcal{Y}$ and $G : \mathcal{X}' \to \mathcal{X}$ be 1-morphisms of stacks in groupoids over $(Sch/S)_{fppf}$. If G is representable by algebraic spaces, then the 1-morphism*

$$\mathcal{H}_d(\mathcal{X}'/\mathcal{Y}) \longrightarrow \mathcal{H}_d(\mathcal{X}/\mathcal{Y})$$

is representable by algebraic spaces.

Proof. Let U be a scheme and let $\xi = (U, Z, y, x, \alpha)$ be an object of $\mathcal{H}_d(\mathcal{X}/\mathcal{Y})$ over U. We have to prove that the 2-fibre product

$$(Sch/U)_{fppf} \times_{\xi, \mathcal{H}_d(\mathcal{X}/\mathcal{Y})} \mathcal{H}_d(\mathcal{X}'/\mathcal{Y}) \tag{70.12.6.1}$$

is representable by an algebraic space étale over U. An object of this over $a : U' \to U$ corresponds to an object x' of $\mathcal{X}'$ over $U' \times_{a,U} Z$ such that $G(x') \cong x|_{U' \times_{a,U} Z}$. By assumption the 2-fibre product

$$(Sch/Z)_{fppf} \times_{x, \mathcal{X}} \mathcal{X}'$$

is representable by an algebraic space X over Z. It follows that (70.12.6.1) is representable by $\mathrm{Res}_{Z/U}(X)$, which is an algebraic space by Proposition 70.11.5. □

Lemma 70.12.7. *Let $F : \mathcal{X} \to \mathcal{Y}$ be a 1-morphism of stacks in groupoids over $(Sch/S)_{fppf}$. Assume F is representable by algebraic spaces and locally of finite presentation. Then*

$$p : \mathcal{H}_d(\mathcal{X}/\mathcal{Y}) \to \mathcal{Y}$$

is limit preserving on objects.

Proof. This means we have to show the following: Given

(1) an affine scheme $U = \lim_i U_i$ which is written as the directed limit of affine schemes U_i over S,
(2) an object y_i of $\mathcal{Y}$ over U_i for some i, and
(3) an object $\Xi = (U, Z, y, x, \alpha)$ of $\mathcal{H}_d(\mathcal{X}/\mathcal{Y})$ over U such that $y = y_i|_U$,

then there exists an $i' \geq i$ and an object $\Xi_{i'} = (U_{i'}, Z_{i'}, y_{i'}, x_{i'}, \alpha_{i'})$ of $\mathcal{H}_d(\mathcal{X}/\mathcal{Y})$ over $U_{i'}$ with $\Xi_{i'}|_U = \Xi$ and $y_{i'} = y_i|_{U_{i'}}$. Namely, the last two equalities will take care of the commutativity of (70.5.0.1).

Let $X_{y_i} \to U_i$ be an algebraic space representing the 2-fibre product

$$(Sch/U_i)_{fppf} \times_{y_i, \mathcal{Y}, F} \mathcal{X}.$$

Note that $X_{y_i} \to U_i$ is locally of finite presentation by our assumption on F. Write Ξ. It is clear that $\xi = (Z, Z \to U_i, x, \alpha)$ is an object of the 2-fibre product displayed above, hence ξ gives rise to a morphism $f_\xi : Z \to X_{y_i}$ of algebraic spaces over U_i (since X_{y_i} is the functor of isomorphisms classes of objects of $(Sch/U_i)_{fppf} \times_{y, \mathcal{Y}, F} \mathcal{X}$, see Algebraic Stacks, Lemma 67.8.2). By Limits, Lemmas 31.9.1 and 31.7.7 there exists an $i' \geq i$ and a finite locally free morphism $Z_{i'} \to U_{i'}$ of degree d whose base change to U is Z. By Limits of Spaces, Proposition 52.3.9 we may, after replacing i' by a bigger index, assume there exists a morphism $f_{i'} : Z_{i'} \to X_{y_i}$ such that

$$\begin{array}{ccccc} Z & \longrightarrow & Z_{i'} & \xrightarrow[f_{i'}]{} & X_{y_i} \\ \downarrow & & \downarrow & & \downarrow \\ U & \longrightarrow & U_{i'} & \longrightarrow & U_i \end{array} \qquad (f_\xi : Z \to X_{y_i})$$

is commutative. We set $\Xi_{i'} = (U_{i'}, Z_{i'}, y_{i'}, x_{i'}, \alpha_{i'})$ where

(1) $y_{i'}$ is the object of $\mathcal{Y}$ over $U_{i'}$ which is the pullback of y_i to $U_{i'}$,
(2) $x_{i'}$ is the object of $\mathcal{X}$ over $Z_{i'}$ corresponding via the 2-Yoneda lemma to the 1-morphism

$$(Sch/Z_{i'})_{fppf} \to \mathcal{S}_{X_{y_i}} \to (Sch/U_i)_{fppf} \times_{y_i, \mathcal{Y}, F} \mathcal{X} \to \mathcal{X}$$

where the middle arrow is the equivalence which defines X_{y_i} (notation as in Algebraic Stacks, Sections 67.8 and 67.7).
(3) $\alpha_{i'} : y_{i'}|_{Z_{i'}} \to F(x_{i'})$ is the isomorphism coming from the 2-commutativity of the diagram

$$\begin{array}{ccccc} (Sch/Z_{i'})_{fppf} & \longrightarrow & (Sch/U_i)_{fppf} \times_{y_i, \mathcal{Y}, F} \mathcal{X} & \longrightarrow & \mathcal{X} \\ & \searrow & \downarrow & & \downarrow F \\ & & (Sch/U_{i'})_{fppf} & \longrightarrow & \mathcal{Y} \end{array}$$

Recall that $f_\xi : Z \to X_{y_i}$ was the morphism corresponding to the object $\xi = (Z, Z \to U_i, x, \alpha)$ of $(Sch/U_i)_{fppf} \times_{y_i, \mathcal{Y}, F} \mathcal{X}$ over Z. By construction $f_{i'}$ is the morphism corresponding to the object $\xi_{i'} = (Z_{i'}, Z_{i'} \to U_i, x_{i'}, \alpha_{i'})$. As $f_\xi = f_{i'} \circ (Z \to Z_{i'})$ we see that the object $\xi_{i'} = (Z_{i'}, Z_{i'} \to U_i, x_{i'}, \alpha_{i'})$ pulls back to ξ over Z. Thus $x_{i'}$ pulls back to x and $\alpha_{i'}$ pulls back to α. This means that $\Xi_{i'}$ pulls back to Ξ over U and we win. □

70.13. The finite Hilbert stack of a point

Let $d \geq 1$ be an integer. In Examples of Stacks, Definition 68.17.2 we defined a stack in groupoids $\mathcal{H}_d$. In this section we prove that $\mathcal{H}_d$ is an algebraic stack. We will throughout assume that $S = \mathrm{Spec}(\mathbf{Z})$. The general case will follow from this by base change. Recall that the fibre category of $\mathcal{H}_d$ over a scheme T is the category of finite locally free morphisms $\pi : Z \to T$ of degree d. Instead of classifying these directly we first study the quasi-coherent sheaves of algebras $\pi_* \mathcal{O}_Z$.

Let R be a ring. Let us temporarily make the following definition: A *free d-dimensional algebra over R* is given by a commutative R-algebra structure m on $R^{\oplus d}$ such that $e_1 = (1, 0, \ldots, 0)$ is a unit[4]. We think of m as an R-linear map

$$m : R^{\oplus d} \otimes_R R^{\oplus d} \longrightarrow R^{\oplus d}$$

such that $m(e_1, x) = m(x, e_1) = x$ and such that m defines a commutative and associative ring structure. If we write $m(e_i, e_j) = \sum a_{ij}^k e_k$ then we see this boils down to the conditions

$$\begin{cases} \sum_l a_{ij}^l a_{lk}^m = \sum_l a_{il}^m a_{jk}^l & \forall i, j, k, m \\ a_{ij}^k = a_{ji}^k & \forall i, j, k \\ a_{i1}^j = \delta_{ij} & \forall i, j \end{cases}$$

where δ_{ij} is the Kronecker δ-function. OK, so let's define

$$R_{univ} = \mathbf{Z}[a_{ij}^k]/J$$

where the ideal J is the ideal generated by the relations displayed above. Denote

$$m_{univ} : R_{univ}^{\oplus d} \otimes_{R_{univ}} R_{univ}^{\oplus d} \longrightarrow R_{univ}^{\oplus d}$$

the free d-dimensional algebra m over R_{univ} whose structure constants are the classes of a_{ij}^k modulo J. Then it is clear that given any free d-dimensional algebra m over a ring R there exists a unique $\mathbf{Z}$-algebra homomorphism $\psi : R_{univ} \to R$ such that $\psi_* m_{univ} = m$ (this means that m is what you get by applying the base change functor $- \otimes_{R_{univ}} R$ to m_{univ}). In other words, setting $X = \mathrm{Spec}(R_{univ})$ we obtain a canonical identification

$$X(T) = \{\text{free } d\text{-dimensional algebras } m \text{ over } R\}$$

for varying $T = \mathrm{Spec}(R)$. By Zariski localization we obtain the following seemingly more general identification

$$X(T) = \{\text{free } d\text{-dimensional algebras } m \text{ over } \Gamma(T, \mathcal{O}_T)\} \tag{70.13.0.1}$$

for any scheme T.

[4]It may be better to think of this as a pair consisting of a multiplication map $m : R^{\oplus d} \otimes_R R^{\oplus d} \to R^{\oplus d}$ and a ring map $\psi : R \to R^{\oplus d}$ satisfying a bunch of axioms.

Next we talk a little bit about *isomorphisms of free d-dimensional R-algebras*. Namely, suppose that m, m' are two free d-dimensional algebras over a ring R. An *isomorphism from m to m'* is given by an invertible R-linear map

$$\varphi : R^{\oplus d} \longrightarrow R^{\oplus d}$$

such that $\varphi(e_1) = e_1$ and such that

$$m \circ \varphi \otimes \varphi = \varphi \circ m'.$$

Note that we can compose these so that the collection of free d-dimensional algebras over R becomes a category. In this way we obtain a functor

$$FA_d : Sch_{fppf}^{opp} \longrightarrow Groupoids \tag{70.13.0.2}$$

from the category of schemes to groupoids: to a scheme T we associate the set of free d-dimensional algebras over $\Gamma(T, \mathcal{O}_T)$ endowed with the structure of a category using the notion of isomorphisms just defined.

The above suggests we consider the functor G in groups which associates to any scheme T the group

$$G(T) = \{g \in \mathrm{GL}_d(\Gamma(T, \mathcal{O}_T)) \mid g(e_1) = e_1\}$$

It is clear that $G \subset \mathrm{GL}_d$ (see Groupoids, Example 38.5.4) is the closed subgroup scheme cut out by the equations $x_{11} = 1$ and $x_{i1} = 0$ for $i > 1$. Hence G is a smooth affine group scheme over $\mathrm{Spec}(\mathbf{Z})$. Consider the action

$$a : G \times_{\mathrm{Spec}(\mathbf{Z})} X \longrightarrow X$$

which associates to a T-valued point (g, m) with $T = \mathrm{Spec}(R)$ on the left hand side the free d-dimensional algebra over R given by

$$a(g, m) = g^{-1} \circ m \circ g \otimes g.$$

Note that this means that g defines an isomorphism $m \to a(g, m)$ of d-dimensional free R-algebras. We omit the verification that a indeed defines an action of the group scheme G on the scheme X.

Lemma 70.13.1. *The functor in groupoids FA_d defined in (70.13.0.2) is isomorphic (!) to the functor in groupoids which associates to a scheme T the category with*

1. *set of objects is $X(T)$,*
2. *set of morphisms is $G(T) \times X(T)$,*
3. *$s : G(T) \times X(T) \to X(T)$ is the projection map,*
4. *$t : G(T) \times X(T) \to X(T)$ is $a(T)$, and*
5. *composition $G(T) \times X(T) \times_{s, X(T), t} G(T) \times X(T) \to G(T) \times X(T)$ is given by $((g, m), (g', m')) \mapsto (gg', m')$.*

Proof. We have seen the rule on objects in (70.13.0.1). We have also seen above that $g \in G(T)$ can be viewed as a morphism from m to $a(g, m)$ for any free d-dimensional algebra m. Conversely, any morphism $m \to m'$ is given by an invertible linear map φ which corresponds to an element $g \in G(T)$ such that $m' = a(g, m)$. □

In fact the groupoid $(X, G \times X, s, t, c)$ described in the lemma above is the groupoid associated to the action $a : G \times X \to X$ as defined in Groupoids, Lemma 38.14.1. Since G is smooth over $\mathrm{Spec}(\mathbf{Z})$ we see that the two morphisms $s, t : G \times X \to X$ are smooth: by symmetry it suffices to prove that one of them is, and s is the base

change of $G \to \mathrm{Spec}(\mathbf{Z})$. Hence $(G \times X, X, s, t, c)$ is a smooth groupoid scheme, and the quotient stack $[X/G]$ is an algebraic stack by Algebraic Stacks, Theorem 67.17.3.

Proposition 70.13.2. *The stack $\mathcal{H}_d$ is equivalent to the quotient stack $[X/G]$ described above. In particular $\mathcal{H}_d$ is an algebraic stack.*

Proof. Note that by Groupoids in Spaces, Definition 59.19.1 the quotient stack $[X/G]$ is the stackification of the category fibred in groupoids associated to the "presheaf in groupoids" which associates to a scheme T the groupoid

$$(X(T), G(T) \times X(T), s, t, c).$$

Since this "presheaf in groupoids" is isomorphic to FA_d by Lemma 70.13.1 it suffices to prove that the $\mathcal{H}_d$ is the stackification of (the category fibred in groupoids associated to the "presheaf in groupoids") FA_d. To do this we first define a functor

$$\mathrm{Spec} : FA_d \longrightarrow \mathcal{H}_d$$

Recall that the fibre category of $\mathcal{H}_d$ over a scheme T is the category of finite locally free morphisms $Z \to T$ of degree d. Thus given a scheme T and a free d-dimensional $\Gamma(T, \mathcal{O}_T)$-algebra m we may assign to this the object

$$Z = \underline{\mathrm{Spec}}_T(\mathcal{A})$$

of $\mathcal{H}_{d,T}$ where $\mathcal{A} = \mathcal{O}_T^{\oplus d}$ endowed with a $\mathcal{O}_T$-algebra structure via m. Moreover, if m' is a second such free d-dimensional $\Gamma(T, \mathcal{O}_T)$-algebra and if $\varphi : m \to m'$ is an isomorphism of these, then the induced $\mathcal{O}_T$-linear map $\varphi : \mathcal{O}_T^{\oplus d} \to \mathcal{O}_T^{\oplus d}$ induces an isomorphism

$$\varphi : \mathcal{A}' \longrightarrow \mathcal{A}$$

of quasi-coherent $\mathcal{O}_T$-algebras. Hence

$$\underline{\mathrm{Spec}}_T(\varphi) : \underline{\mathrm{Spec}}_T(\mathcal{A}) \longrightarrow \underline{\mathrm{Spec}}_T(\mathcal{A}')$$

is a morphism in the fibre category $\mathcal{H}_{d,T}$. We omit the verification that this construction is compatible with base change so we get indeed a functor $\mathrm{Spec} : FA_d \to \mathcal{H}_d$ as claimed above.

To show that $\mathrm{Spec} : FA_d \to \mathcal{H}_d$ induces an equivalence between the stackification of FA_d and $\mathcal{H}_d$ it suffices to check that

(1) $Isom(m, m') = Isom(\mathrm{Spec}(m), \mathrm{Spec}(m'))$ for any $m, m' \in FA_d(T)$.
(2) for any scheme T and any object $Z \to T$ of $\mathcal{H}_{d,T}$ there exists a covering $\{T_i \to T\}$ such that $Z|_{T_i}$ is isomorphic to $\mathrm{Spec}(m)$ for some $m \in FA_d(T_i)$, and

see Stacks, Lemma 8.9.1. The first statement follows from the observation that any isomorphism

$$\underline{\mathrm{Spec}}_T(\mathcal{A}) \longrightarrow \underline{\mathrm{Spec}}_T(\mathcal{A}')$$

is necessarily given by a global invertible matrix g when $\mathcal{A} = \mathcal{A}' = \mathcal{O}_T^{\oplus d}$ as modules. To prove the second statement let $\pi : Z \to T$ be a finite locally free morphism of degree d. Then $\mathcal{A}$ is a locally free sheaf $\mathcal{O}_T$-modules of rank d. Consider the element $1 \in \Gamma(T, \mathcal{A})$. This element is nonzero in $\mathcal{A} \otimes_{\mathcal{O}_{T,t}} \kappa(t)$ for every $t \in T$ since the scheme $Z_t = \mathrm{Spec}(\mathcal{A} \otimes_{\mathcal{O}_{T,t}} \kappa(t))$ is nonempty being of degree $d > 0$ over $\kappa(t)$. Thus $1 : \mathcal{O}_T \to \mathcal{A}$ can locally be used as the first basis element (for example you can use Algebra, Lemma 10.74.3 parts (1) and (2) to see this). Thus, after localizing

on T we may assume that there exists an isomorphism $\varphi : \mathcal{A} \to \mathcal{O}_T^{\oplus d}$ such that $1 \in \Gamma(\mathcal{A})$ corresponds to the first basis element. In this situation the multiplication map $\mathcal{A} \otimes_{\mathcal{O}_T} \mathcal{A} \to \mathcal{A}$ translates via φ into a free d-dimensional algebra m over $\Gamma(T, \mathcal{O}_T)$. This finishes the proof. □

70.14. Finite Hilbert stacks of spaces

The finite Hilbert stack of an algebraic space is an algebraic stack.

Lemma 70.14.1. *Let S be a scheme. Let X be an algebraic space over S. Then $\mathcal{H}_d(X)$ is an algebraic stack.*

Proof. The 1-morphism

$$\mathcal{H}_d(X) \longrightarrow \mathcal{H}_d$$

is representable by algebraic spaces according to Lemma 70.12.6. The stack $\mathcal{H}_d$ is an algebraic stack according to Proposition 70.13.2. Hence $\mathcal{H}_d(X)$ is an algebraic stack by Algebraic Stacks, Lemma 67.15.4. □

This lemma allows us to bootstrap.

Lemma 70.14.2. *Let S be a scheme. Let $F : \mathcal{X} \to \mathcal{Y}$ be a 1-morphism of stacks in groupoids over $(Sch/S)_{fppf}$ such that*

1. *$\mathcal{X}$ is representable by an algebraic space, and*
2. *F is representable by algebraic spaces, surjective, flat, and locally of finite presentation.*

Then $\mathcal{H}_d(\mathcal{X}/\mathcal{Y})$ is an algebraic stack.

Proof. Choose a representable stack in groupoids $\mathcal{U}$ over S and a 1-morphism $f : \mathcal{U} \to \mathcal{H}_d(\mathcal{X})$ which is representable by algebraic spaces, smooth, and surjective. This is possible because $\mathcal{H}_d(\mathcal{X})$ is an algebraic stack by Lemma 70.14.1. Consider the 2-fibre product

$$\mathcal{W} = \mathcal{H}_d(\mathcal{X}/\mathcal{Y}) \times_{\mathcal{H}_d(\mathcal{X}), f} \mathcal{U}.$$

Since $\mathcal{U}$ is representable (in particular a stack in setoids) it follows from Examples of Stacks, Lemma 68.17.3 and Stacks, Lemma 8.6.7 that $\mathcal{W}$ is a stack in setoids. The 1-morphism $\mathcal{W} \to \mathcal{H}_d(\mathcal{X}/\mathcal{Y})$ is representable by algebraic spaces, smooth, and surjective as a base change of the morphism f (see Algebraic Stacks, Lemmas 67.9.7 and 67.10.6). Thus, if we can show that $\mathcal{W}$ is representable by an algebraic space, then the lemma follows from Algebraic Stacks, Lemma 67.15.3.

The diagonal of $\mathcal{Y}$ is representable by algebraic spaces according to Lemma 70.4.3. We may apply Lemma 70.12.5 to see that the 1-morphism

$$\mathcal{H}_d(\mathcal{X}/\mathcal{Y}) \longrightarrow \mathcal{H}_d(\mathcal{X}) \times \mathcal{Y}$$

is representable by algebraic spaces. Consider the 2-fibre product

$$\mathcal{V} = \mathcal{H}_d(\mathcal{X}/\mathcal{Y}) \times_{(\mathcal{H}_d(\mathcal{X}) \times \mathcal{Y}), f \times F} (\mathcal{U} \times \mathcal{X}).$$

The projection morphism $\mathcal{V} \to \mathcal{U} \times \mathcal{X}$ is representable by algebraic spaces as a base change of the last displayed morphism. Hence $\mathcal{V}$ is an algebraic space (see

Bootstrap, Lemma 61.3.6 or Algebraic Stacks, Lemma 67.9.8). The 1-morphism $\mathcal{V} \to \mathcal{U}$ fits into the following 2-cartesian diagram

$$\begin{array}{ccc} \mathcal{V} & \longrightarrow & \mathcal{X} \\ \downarrow & & \downarrow F \\ \mathcal{W} & \longrightarrow & \mathcal{Y} \end{array}$$

because

$$\mathcal{H}_d(\mathcal{X}/\mathcal{Y}) \times_{(\mathcal{H}_d(\mathcal{X})\times\mathcal{Y}),f\times F} (\mathcal{U} \times \mathcal{X}) = (\mathcal{H}_d(\mathcal{X}/\mathcal{Y}) \times_{\mathcal{H}_d(\mathcal{X}),f} \mathcal{U}) \times_{\mathcal{Y},F} \mathcal{X}.$$

Hence $\mathcal{V} \to \mathcal{W}$ is representable by algebraic spaces, surjective, flat, and locally of finite presentation as a base change of F. It follows that the same thing is true for the corresponding sheaves of sets associated to $\mathcal{V}$ and $\mathcal{W}$, see Algebraic Stacks, Lemma 67.10.4. Thus we conclude that the sheaf associated to $\mathcal{W}$ is an algebraic space by Bootstrap, Theorem 61.10.1. □

70.15. LCI locus in the Hilbert stack

Please consult Examples of Stacks, Section 68.17 for notation. Fix a 1-morphism $F : \mathcal{X} \longrightarrow \mathcal{Y}$ of stacks in groupoids over $(Sch/S)_{fppf}$. Assume that F is representable by algebraic spaces. Fix $d \geq 1$. Consider an object (U, Z, y, x, α) of $\mathcal{H}_d$. There is an induced 1-morphism

$$(Sch/Z)_{fppf} \longrightarrow (Sch/U)_{fppf} \times_{y,\mathcal{Y},F} \mathcal{X}$$

(by the universal property of 2-fibre products) which is representable by a morphism of algebraic spaces over U. Namely, since F is representable by algebraic spaces, we may choose an algebraic space X_y over U which represents the 2-fibre product $(Sch/U)_{fppf} \times_{y,\mathcal{Y},F} \mathcal{X}$. Since $\alpha : y|_Z \to F(x)$ is an isomorphism we see that $\xi = (Z, Z \to U, x, \alpha)$ is an object of the 2-fibre product $(Sch/U)_{fppf} \times_{y,\mathcal{Y},F} \mathcal{X}$ over Z. Hence ξ gives rise to a morphism $x_\alpha : Z \to X_y$ of algebraic spaces over U as X_y is the functor of isomorphisms classes of objects of $(Sch/U)_{fppf} \times_{y,\mathcal{Y},F} \mathcal{X}$, see Algebraic Stacks, Lemma 67.8.2. Here is a picture

(70.15.0.1)

$$\begin{array}{ccc} Z & \xrightarrow[x_\alpha]{} & X_y \\ & \searrow & \downarrow \\ & & U \end{array} \qquad \begin{array}{ccccc} (Sch/Z)_{fppf} & \xrightarrow[x,\alpha]{} & (Sch/U)_{fppf} \times_{y,\mathcal{Y},F} \mathcal{X} & \longrightarrow & \mathcal{X} \\ & \searrow & \downarrow & & \downarrow F \\ & & (Sch/U)_{fppf} & \xrightarrow{y} & \mathcal{Y} \end{array}$$

We remark that if $(f, g, b, a) : (U, Z, y, x, \alpha) \to (U', Z', y', x', \alpha')$ is a morphism between objects of $\mathcal{H}_d$, then the morphism $x'_{\alpha'} : Z' \to X'_{y'}$ is the base change of the morphism x_α by the morphism $g : U' \to U$ (details omitted).

Now assume moreover that F is flat and locally of finite presentation. In this situation we define a full subcategory

$$\mathcal{H}_{d,lci}(\mathcal{X}/\mathcal{Y}) \subset \mathcal{H}_d(\mathcal{X}/\mathcal{Y})$$

consisting of those objects (U, Z, y, x, α) of $\mathcal{H}_d(\mathcal{X}/\mathcal{Y})$ such that the corresponding morphism $x_\alpha : Z \to X_y$ is unramified and a local complete intersection morphism (see Morphisms of Spaces, Definition 49.34.1 and More on Morphisms of Spaces, Definition 57.35.1 for definitions).

Lemma 70.15.1. *Let S be a scheme. Fix a 1-morphism $F : \mathcal{X} \longrightarrow \mathcal{Y}$ of stacks in groupoids over $(Sch/S)_{fppf}$. Assume F is representable by algebraic spaces, flat, and locally of finite presentation. Then $\mathcal{H}_{d,lci}(\mathcal{X}/\mathcal{Y})$ is a stack in groupoids and the inclusion functor*

$$\mathcal{H}_{d,lci}(\mathcal{X}/\mathcal{Y}) \longrightarrow \mathcal{H}_d(\mathcal{X}/\mathcal{Y})$$

is representable and an open immersion.

Proof. Let $\Xi = (U, Z, y, x, \alpha)$ be an object of $\mathcal{H}_d$. It follows from the remark following (70.15.0.1) that the pullback of Ξ by $U' \to U$ belongs to $\mathcal{H}_{d,lci}(\mathcal{X}/\mathcal{Y})$ if and only if the base change of x_α is unramified and a local complete intersection morphism. Note that $Z \to U$ is finite locally free (hence flat, locally of finite presentation and universally closed) and that $X_y \to U$ is flat and locally of finite presentation by our assumption on F. Then More on Morphisms of Spaces, Lemmas 57.36.1 and 57.36.7 imply exists an open subscheme $W \subset U$ such that a morphism $U' \to U$ factors through W if and only if the base change of x_α via $U' \to U$ is unramified and a local complete intersection morphism. This implies that

$$(Sch/U)_{fppf} \times_{\Xi, \mathcal{H}_d(\mathcal{X}/\mathcal{Y})} \mathcal{H}_{d,lci}(\mathcal{X}/\mathcal{Y})$$

is representable by W. Hence the final statement of the lemma holds. The first statement (that $\mathcal{H}_{d,lci}(\mathcal{X}/\mathcal{Y})$ is a stack in groupoids) follows from this an Algebraic Stacks, Lemma 67.15.5. □

Local complete intersection morphisms are "locally unobstructed". This holds in much greater generality than the special case that we need in this chapter here.

Lemma 70.15.2. *Let $U \subset U'$ be a first order thickening of affine schemes. Let X' be an algebraic space flat over U'. Set $X = U \times_{U'} X'$. Let $Z \to U$ be finite locally free of degree d. Finally, let $f : Z \to X$ be unramified and a local complete intersection morphism. Then there exists a commutative diagram*

$$\begin{array}{ccc} (Z \subset Z') & \xrightarrow[(f,f')]{} & (X \subset X') \\ & \searrow \quad \swarrow & \\ & (U \subset U') & \end{array}$$

of algebraic spaces over U' such that $Z' \to U'$ is finite locally free of degree d and $Z = U \times_{U'} Z'$.

Proof. By More on Morphisms of Spaces, Lemma 57.35.8 the conormal sheaf $\mathcal{C}_{Z/X}$ of the unramified morphism $Z \to X$ is a finite locally free $\mathcal{O}_Z$-module and by More on Morphisms of Spaces, Lemma 57.35.9 we have an exact sequence

$$0 \to i^*\mathcal{C}_{X/X'} \to \mathcal{C}_{Z/X'} \to \mathcal{C}_{Z/X} \to 0$$

of conormal sheaves. Since Z is affine this sequence is split. Choose a splitting

$$\mathcal{C}_{Z/X'} = i^*\mathcal{C}_{X/X'} \oplus \mathcal{C}_{Z/X}$$

Let $Z \subset Z''$ be the universal first order thickening of of Z over X' (see More on Morphisms of Spaces, Section 57.12). Denote $\mathcal{I} \subset \mathcal{O}_{Z''}$ the quasi-coherent sheaf of ideals corresponding to $Z \subset Z''$. By definition we have $\mathcal{C}_{Z/X'}$ is $\mathcal{I}$ viewed as a sheaf on Z. Hence the splitting above determines a splitting

$$\mathcal{I} = i^*\mathcal{C}_{X/X'} \oplus \mathcal{C}_{Z/X}$$

Let $Z' \subset Z''$ be the closed subscheme cut out by $\mathcal{C}_{Z/X} \subset \mathcal{I}$ viewed as a quasi-coherent sheaf of ideals on Z''. It is clear that Z' is a first order thickening of Z and that we obtain a commutative diagram of first order thickenings as in the statement of the lemma.

Since $X' \to U'$ is flat and since $X = U \times_{U'} X'$ we see that $\mathcal{C}_{X/X'}$ is the pullback of $\mathcal{C}_{U/U'}$ to X, see More on Morphisms of Spaces, Lemma 57.15.1. Note that by construction $\mathcal{C}_{Z/Z'} = i^*\mathcal{C}_{X/X'}$ hence we conclude that $\mathcal{C}_{Z/Z'}$ is isomorphic to the pullback of $\mathcal{C}_{U/U'}$ to Z. Applying More on Morphisms of Spaces, Lemma 57.15.1 once again (or its analogue for schemes, see More on Morphisms, Lemma 36.8.1) we conclude that $Z' \to U'$ is flat and that $Z = U \times_{U'} Z'$. Finally, More on Morphisms, Lemma 36.8.3 shows that $Z' \to U'$ is finite locally free of degree d. □

Lemma 70.15.3. *Let $F : \mathcal{X} \to \mathcal{Y}$ be a 1-morphism of stacks in groupoids over $(Sch/S)_{fppf}$. Assume F is representable by algebraic spaces, flat, and locally of finite presentation. Then*

$$p : \mathcal{H}_{d,lci}(\mathcal{X}/\mathcal{Y}) \to \mathcal{Y}$$

is formally smooth on objects.

Proof. We have to show the following: Given

(1) an object (U, Z, y, x, α) of $\mathcal{H}_{d,lci}(\mathcal{X}/\mathcal{Y})$ over an affine scheme U,
(2) a first order thickening $U \subset U'$, and
(3) an object y' of $\mathcal{Y}$ over U' such that $y'|_U = y$,

then there exists an object $(U', Z', y', x', \alpha')$ of $\mathcal{H}_{d,lci}(\mathcal{X}/\mathcal{Y})$ over U' with $Z = U \times_{U'} Z'$, with $x = x'|_Z$, and with $\alpha = \alpha'|_U$. Namely, the last two equalities will take care of the commutativity of (70.6.0.1).

Consider the morphism $x_\alpha : Z \to X_y$ constructed in Equation (70.15.0.1). Denote similarly $X'_{y'}$ the algebraic space over U' representing the 2-fibre product $(Sch/U')_{fppf} \times_{y',\mathcal{Y},F} \mathcal{X}$. By assumption the morphism $X'_{y'} \to U'$ is flat (and locally of finite presentation). As $y'|_U = y$ we see that $X_y = U \times_{U'} X'_{y'}$. Hence we may apply Lemma 70.15.2 to find $Z' \to U'$ finite locally free of degree d with $Z = U \times_{U'} Z'$ and with $Z' \to X'_{y'}$ extending x_α. By construction the morphism $Z' \to X'_{y'}$ corresponds to a pair (x', α'). It is clear that $(U', Z', y', x', \alpha')$ is an object of $\mathcal{H}_d(\mathcal{X}/\mathcal{Y})$ over U' with $Z = U \times_{U'} Z'$, with $x = x'|_Z$, and with $\alpha = \alpha'|_U$. As we've seen in Lemma 70.15.1 that $\mathcal{H}_{d,lci}(\mathcal{X}/\mathcal{Y}) \subset \mathcal{H}_d(\mathcal{X}/\mathcal{Y})$ is an "open substack" it follows that $(U', Z', y', x', \alpha')$ is an object of $\mathcal{H}_{d,lci}(\mathcal{X}/\mathcal{Y})$ as desired. □

Lemma 70.15.4. *Let $F : \mathcal{X} \to \mathcal{Y}$ be a 1-morphism of stacks in groupoids over $(Sch/S)_{fppf}$. Assume F is representable by algebraic spaces, flat, surjective, and locally of finite presentation. Then*

$$\coprod\nolimits_{d \geq 1} \mathcal{H}_{d,lci}(\mathcal{X}/\mathcal{Y}) \longrightarrow \mathcal{Y}$$

is surjective on objects.

Proof. It suffices to prove the following: For any field k and object y of $\mathcal{Y}$ over $\mathrm{Spec}(k)$ there exists an integer $d \geq 1$ and an object (U, Z, y, x, α) of $\mathcal{H}_{d,lci}(\mathcal{X}/\mathcal{Y})$ with $U = \mathrm{Spec}(k)$. Namely, in this case we see that p is surjective on objects in the strong sense that an extension of the field is not needed.

Denote X_y the algebraic space over $U = \mathrm{Spec}(k)$ representing the 2-fibre product $(Sch/U')_{fppf} \times_{y',\mathcal{Y},F} \mathcal{X}$. By assumption the morphism $X_y \to \mathrm{Spec}(k)$ is surjective

and locally of finite presentation (and flat). In particular X_y is nonempty. Choose a nonempty affine scheme V and an étale morphism $V \to X_y$. Note that $V \to \mathrm{Spec}(k)$ is (flat), surjective, and locally of finite presentation (by Morphisms of Spaces, Definition 49.26.1). Pick a closed point $v \in V$ where $V \to \mathrm{Spec}(k)$ is Cohen-Macaulay (i.e., V is Cohen-Macaulay at v), see More on Morphisms, Lemma 36.17.4. Applying More on Morphisms, Lemma 36.18.4 we find a regular immersion $Z \to V$ with $Z = \{v\}$. This implies $Z \to V$ is a closed immersion. Moreover, it follows that $Z \to \mathrm{Spec}(k)$ is finite (for example by Algebra, Lemma 10.116.1). Hence $Z \to \mathrm{Spec}(k)$ is finite locally free of some degree d. Now $Z \to X_y$ is unramified as the composition of an closed immersion followed by an étale morphism (see Morphisms of Spaces, Lemmas 49.34.3, 49.35.10, and 49.34.8). Finally, $Z \to X_y$ is a local complete intersection morphism as a composition of a regular immersion of schemes and an étale morphism of algebraic spaces (see More on Morphisms, Lemma 36.42.9 and Morphisms of Spaces, Lemmas 49.35.6 and 49.33.8 and More on Morphisms of Spaces, Lemmas 57.35.6 and 57.35.5). The morphism $Z \to X_y$ corresponds to an object x of $\mathcal{X}$ over Z together with an isomorphism $\alpha : y|_Z \to F(x)$. We obtain an object (U, Z, y, x, α) of $\mathcal{H}_d(\mathcal{X}/\mathcal{Y})$. By what was said above about the morphism $Z \to X_y$ we see that it actually is an object of the subcategory $\mathcal{H}_{d,lci}(\mathcal{X}/\mathcal{Y})$ and we win. □

70.16. Bootstrapping algebraic stacks

The following theorem is one of the main results of this chapter.

Theorem 70.16.1. *Let S be a scheme. Let $F : \mathcal{X} \to \mathcal{Y}$ be a 1-morphism of stacks in groupoids over $(Sch/S)_{fppf}$. If*

1. *$\mathcal{X}$ is representable by an algebraic space, and*
2. *F is representable by algebraic spaces, surjective, flat and locally of finite presentation,*

then $\mathcal{Y}$ is an algebraic stack.

Proof. By Lemma 70.4.3 we see that the diagonal of $\mathcal{Y}$ is representable by algebraic spaces. Hence we only need to verify the existence of a 1-morphism $f : \mathcal{V} \to \mathcal{Y}$ of stacks in groupoids over $(Sch/S)_{fppf}$ with $\mathcal{V}$ representable and f surjective and smooth. By Lemma 70.14.2 we know that

$$\coprod_{d \geq 1} \mathcal{H}_d(\mathcal{X}/\mathcal{Y})$$

is an algebraic stack. It follows from Lemma 70.15.1 and Algebraic Stacks, Lemma 67.15.5 that

$$\coprod_{d \geq 1} \mathcal{H}_{d,lci}(\mathcal{X}/\mathcal{Y})$$

is an algebraic stack as well. Choose a representable stack in groupoids $\mathcal{V}$ over $(Sch/S)_{fppf}$ and a surjective and smooth 1-morphism

$$\mathcal{V} \longrightarrow \coprod_{d \geq 1} \mathcal{H}_{d,lci}(\mathcal{X}/\mathcal{Y}).$$

We claim that the composition

$$\mathcal{V} \longrightarrow \coprod_{d \geq 1} \mathcal{H}_{d,lci}(\mathcal{X}/\mathcal{Y}) \longrightarrow \mathcal{Y}$$

is smooth and surjective which finishes the proof of the theorem. In fact, the smoothness will be a consequence of Lemmas 70.12.7 and 70.15.3 and the surjectivity a consequence of Lemma 70.15.4. We spell out the details in the following paragraph.

By construction $\mathcal{V} \to \coprod_{d\geq 1} \mathcal{H}_{d,lci}(\mathcal{X}/\mathcal{Y})$ is representable by algebraic spaces, surjective, and smooth (and hence also locally of finite presentation and formally smooth by the general principle Algebraic Stacks, Lemma 67.10.9 and More on Morphisms of Spaces, Lemma 57.16.6). Applying Lemmas 70.5.3, 70.6.3, and 70.7.3 we see that $\mathcal{V} \to \coprod_{d\geq 1} \mathcal{H}_{d,lci}(\mathcal{X}/\mathcal{Y})$ is limit preserving on objects, formally smooth on objects, and surjective on objects. The 1-morphism $\coprod_{d\geq 1} \mathcal{H}_{d,lci}(\mathcal{X}/\mathcal{Y}) \to \mathcal{Y}$ is

(1) limit preserving on objects: this is Lemma 70.12.7 for $\mathcal{H}_d(\mathcal{X}/\mathcal{Y}) \to \mathcal{Y}$ and we combine it with Lemmas 70.15.1, 70.5.4, and 70.5.2 to get it for $\mathcal{H}_{d,lci}(\mathcal{X}/\mathcal{Y}) \to \mathcal{Y}$,
(2) formally smooth on objects by Lemma 70.15.3, and
(3) surjective on objects by Lemma 70.15.4.

Using Lemmas 70.5.2, 70.6.2, and 70.7.2 we conclude that the composition $\mathcal{V} \to \mathcal{Y}$ is limit preserving on objects, formally smooth on objects, and surjective on objects. Using Lemmas 70.5.3, 70.6.3, and 70.7.3 we see that $\mathcal{V} \to \mathcal{Y}$ is locally of finite presentation, formally smooth, and surjective. Finally, using (via the general principle Algebraic Stacks, Lemma 67.10.9) the infinitesimal lifting criterion (More on Morphisms of Spaces, Lemma 57.16.6) we see that $\mathcal{V} \to \mathcal{Y}$ is smooth and we win. □

70.17. Applications

Our first task is to show that the quotient stack $[U/R]$ associated to a "flat and locally finitely presented groupoid" is an algebraic stack. See Groupoids in Spaces, Definition 59.19.1 for the definition of the quotient stack. The following lemma is preliminary and is the analogue of Algebraic Stacks, Lemma 67.17.2.

Lemma 70.17.1. *Let S be a scheme contained in Sch_{fppf}. Let (U, R, s, t, c) be a groupoid in algebraic spaces over S. Assume s, t are flat and locally of finite presentation. Then the morphism $\mathcal{S}_U \to [U/R]$ is flat, locally of finite presentation, and surjective.*

Proof. Let T be a scheme and let $x : (Sch/T)_{fppf} \to [U/R]$ be a 1-morphism. We have to show that the projection

$$\mathcal{S}_U \times_{[U/R]} (Sch/T)_{fppf} \longrightarrow (Sch/T)_{fppf}$$

is surjective and smooth. We already know that the left hand side is representable by an algebraic space F, see Algebraic Stacks, Lemmas 67.17.1 and 67.10.11. Hence we have to show the corresponding morphism $F \to T$ of algebraic spaces is surjective, locally of finite presentation, and flat. Since we are working with properties of morphisms of algebraic spaces which are local on the target in the fppf topology we may check this fppf locally on T. By construction, there exists an fppf covering $\{T_i \to T\}$ of T such that $x|_{(Sch/T_i)_{fppf}}$ comes from a morphism $x_i : T_i \to U$. (Note that $F \times_T T_i$ represents the 2-fibre product $\mathcal{S}_U \times_{[U/R]} (Sch/T_i)_{fppf}$ so everything is compatible with the base change via $T_i \to T$.) Hence we may assume that x comes from $x : T \to U$. In this case we see that

$$\mathcal{S}_U \times_{[U/R]} (Sch/T)_{fppf} = (\mathcal{S}_U \times_{[U/R]} \mathcal{S}_U) \times_{\mathcal{S}_U} (Sch/T)_{fppf} = \mathcal{S}_R \times_{\mathcal{S}_U} (Sch/T)_{fppf}$$

The first equality by Categories, Lemma 4.29.10 and the second equality by Groupoids in Spaces, Lemma 59.21.2. Clearly the last 2-fibre product is represented by the algebraic space $F = R \times_{s,U,x} T$ and the projection $R \times_{s,U,x} T \to T$ is flat and locally of finite presentation as the base change of the flat locally finitely presented morphism of algebraic spaces $s : R \to U$. It is also surjective as s has a section (namely the identity $e : U \to R$ of the groupoid). This proves the lemma. □

Here is the first main result of this section.

Theorem 70.17.2. *Let S be a scheme contained in Sch_{fppf}. Let (U, R, s, t, c) be a groupoid in algebraic spaces over S. Assume s, t are flat and locally of finite presentation. Then the quotient stack $[U/R]$ is an algebraic stack over S.*

Proof. We check the two conditions of Theorem 70.16.1 for the morphism

$$(Sch/U)_{fppf} \longrightarrow [U/R].$$

The first is trivial (as U is an algebraic space). The second is Lemma 70.17.1. □

70.18. When is a quotient stack algebraic?

In Groupoids in Spaces, Section 59.19 we have defined the quotient stack $[U/R]$ associated to a groupoid (U, R, s, t, c) in algebraic spaces. Note that $[U/R]$ is a stack in groupoids whose diagonal is representable by algebraic spaces (see Bootstrap, Lemma 61.11.3 and Algebraic Stacks, Lemma 67.10.11) and such that there exists an algebraic space U and a 1-morphism $(Sch/U)_{fppf} \to [U/R]$ which is an "fppf surjection" in the sense that it induces a map on presheaves of isomorphism classes of objects which becomes surjective after sheafification. However, it is not the case that that $[U/R]$ is an algebraic stack in general. This is not a contradiction with Theorem 70.16.1 as the 1-morphism $(Sch/U)_{fppf} \to [U/R]$ is not representable by algebraic spaces in general, and if it is it may not be flat and locally of finite presentation.

The easiest way to make examples of non-algebraic quotient stacks is to look at quotients of the form $[S/G]$ where S is a scheme and G is a group scheme over S acting trivially on S. Namely, we will see below (Lemma 70.18.3) that if $[S/G]$ is algebraic, then $G \to S$ has to be flat and locally of finite presentation. An explicit example can be found in Examples, Section 78.38.

Lemma 70.18.1. *Let S be a scheme and let B be an algebraic space over S. Let (U, R, s, t, c) be a groupoid in algebraic spaces over B. The quotient stack $[U/R]$ is an algebraic stack if and only if there exists a morphism of algebraic spaces $g : U' \to U$ such that*

1. *the composition $U' \times_{g,U,t} R \to R \xrightarrow{s} U$ is a surjection of sheaves, and*
2. *the morphisms $s', t' : R' \to U'$ are flat and locally of finite presentation where (U', R', s', t', c') is the restriction of (U, R, s, t, c) via g.*

Proof. First, assume that $g : U' \to U$ satisfies (1) and (2). Property (1) implies that $[U'/R'] \to [U/R]$ is an equivalence, see Groupoids in Spaces, Lemma 59.24.2. By Theorem 70.17.2 the quotient stack $[U'/R']$ is an algebraic stack. Hence $[U/R]$ is an algebraic stack too, see Algebraic Stacks, Lemma 67.12.4.

Conversely, assume that $[U/R]$ is an algebraic stack. We may choose a scheme W and a surjective smooth 1-morphism

$$f : (Sch/W)_{fppf} \longrightarrow [U/R].$$

By the 2-Yoneda lemma (Algebraic Stacks, Section 67.5) this corresponds to an object ξ of $[U/R]$ over W. By the description of $[U/R]$ in Groupoids in Spaces, Lemma 59.23.1 we can find a surjective, flat, locally finitely presented morphism $b : U' \to W$ of schemes such that $\xi' = b^*\xi$ corresponds to a morphism $g : U' \to U$. Note that the 1-morphism

$$f' : (Sch/U')_{fppf} \longrightarrow [U/R].$$

corresponding to ξ' is surjective, flat, and locally of finite presentation, see Algebraic Stacks, Lemma 67.10.5. Hence $(Sch/U')_{fppf} \times_{[U/R]} (Sch/U')_{fppf}$ which is represented by the algebraic space

$$Isom_{[U/R]}(\mathrm{pr}_0^*\xi', \mathrm{pr}_1^*\xi') = (U' \times_S U') \times_{(g\circ\mathrm{pr}_0, g\circ\mathrm{pr}_1), U\times_S U} R = R'$$

(see Groupoids in Spaces, Lemma 59.21.1 for the first equality; the second is the definition of restriction) is flat and locally of finite presentation over U' via both s' and t' (by base change, see Algebraic Stacks, Lemma 67.10.6). By this description of R' and by Algebraic Stacks, Lemma 67.16.1 we obtain a canonical fully faithful 1-morphism $[U'/R'] \to [U/R]$. This 1-morphism is essentially surjective because f' is flat, locally of finite presentation, and surjective (see Stacks, Lemma 8.4.8); another way to prove this is to use Algebraic Stacks, Remark 67.16.3. Finally, we can use Groupoids in Spaces, Lemma 59.24.2 to conclude that the composition $U' \times_{g,U,t} R \to R \xrightarrow{s} U$ is a surjection of sheaves. □

Lemma 70.18.2. *Let S be a scheme and let B be an algebraic space over S. Let G be a group algebraic space over B. Let X be an algebraic space over B and let $a : G \times_B X \to X$ be an action of G on X over B. The quotient stack $[X/G]$ is an algebraic stack if and only if there exists a morphism of algebraic spaces $\varphi : X' \to X$ such that*

(1) *$G \times_B X' \to X$, $(g, x') \mapsto a(g, \varphi(x'))$ is a surjection of sheaves, and*
(2) *the two projections $X'' \to X'$ of the algebraic space X'' given by the rule*

$$T \longmapsto \{(x'_1, g, x'_2) \in (X' \times_B G \times_B X')(T) \mid \varphi(x'_1) = a(g, \varphi(x'_2))\}$$

are flat and locally of finite presentation.

Proof. This lemma is a special case of Lemma 70.18.1. Namely, the quotient stack $[X/G]$ is by Groupoids in Spaces, Definition 59.19.1 equal to the quotient stack $[X/G \times_B X]$ of the groupoid in algebraic spaces $(X, G \times_B X, s, t, c)$ associated to the group action in Groupoids in Spaces, Lemma 59.14.1. There is one small observation that is needed to get condition (1). Namely, the morphism $s : G \times_B X \to X$ is the second projection and the morphism $t : G \times_B X \to X$ is the action morphism a. Hence the morphism $h : U' \times_{g,U,t} R \to R \xrightarrow{s} U$ from Lemma 70.18.1 corresponds to the morphism

$$X' \times_{\varphi,X,a} (G \times_B X) \xrightarrow{\mathrm{pr}_1} X$$

in the current setting. However, because of the symmetry given by the inverse of G this morphism is isomorphic to the morphism

$$(G \times_B X) \times_{\mathrm{pr}_1,X,\varphi} X' \xrightarrow{a} X$$

of the statement of the lemma. Details omitted. □

Lemma 70.18.3. *Let S be a scheme and let B be an algebraic space over S. Let G be a group algebraic space over B. Endow B with the trivial action of G. Then the quotient stack $[B/G]$ is an algebraic stack if and only if G is flat and locally of finite presentation over B.*

Proof. If G is flat and locally of finite presentation over B, then $[B/G]$ is an algebraic stack by Theorem 70.17.2.

Conversely, assume that $[B/G]$ is an algebraic stack. By Lemma 70.18.2 and because the action is trivial, we see there exists an algebraic space B' and a morphism $B' \to B$ such that (1) $B' \to B$ is a surjection of sheaves and (2) the projections

$$B' \times_B G \times_B B' \to B'$$

are flat and locally of finite presentation. Note that the base change $B' \times_B G \times_B B' \to G \times_B B'$ of $B' \to B$ is a surjection of sheaves also. Thus it follows from Descent on Spaces, Lemma 55.7.1 that the projection $G \times_B B' \to B'$ is flat and locally of finite presentation. By (1) we can find an fppf covering $\{B_i \to B\}$ such that $B_i \to B$ factors through $B' \to B$. Hence $G \times_B B_i \to B_i$ is flat and locally of finite presentation by base change. By Descent on Spaces, Lemmas 55.10.11 and 55.10.8 we conclude that $G \to B$ is flat and locally of finite presentation. □

70.19. Algebraic stacks in the étale topology

Let S be a scheme. Instead of working with stacks in groupoids over the big fppf site $(Sch/S)_{fppf}$ we could work with stacks in groupoids over the big étale site $(Sch/S)_{étale}$. All of the material in Algebraic Stacks, Sections 67.4, 67.5, 67.6, 67.7, 67.8, 67.9, 67.10, and 67.11 makes sense for categories fibred in groupoids over $(Sch/S)_{étale}$. Thus we get a second notion of an algebraic stack by working in the étale topology. This notion is (a priori) weaker then the notion introduced in Algebraic Stacks, Definition 67.12.1 since a stack in the fppf topology is certainly a stack in the étale topology. However, the notions are equivalent as is shown by the following lemma.

Lemma 70.19.1. *Denote the common underlying category of Sch_{fppf} and $Sch_{étale}$ by Sch_α (see Sheaves on Stacks, Section 69.4 and Topologies, Remark 33.9.1). Let S be an object of Sch_α. Let*

$$p : \mathcal{X} \to Sch_\alpha/S$$

be a category fibred in groupoids with the following properties:

1. *$\mathcal{X}$ is a stack in groupoids over $(Sch/S)_{étale}$,*
2. *the diagonal $\Delta : \mathcal{X} \to \mathcal{X} \times \mathcal{X}$ is representable by algebraic spaces[5], and*
3. *there exists $U \in \mathrm{Ob}(Sch_\alpha/S)$ and a 1-morphism $(Sch/U)_{étale} \to \mathcal{X}$ which is surjective and smooth.*

Then $\mathcal{X}$ is an algebraic stack in the sense of Algebraic Stacks, Definition 67.12.1.

[5]Here we can either mean sheaves in the étale topology whose diagonal is representable and which have an étale surjective covering by a scheme or algebraic spaces as defined in Algebraic Spaces, Definition 47.6.1. Namely, by Bootstrap, Lemma 61.12.1 there is no difference.

Proof. Note that properties (2) and (3) of the lemma and the corresponding properties (2) and (3) of Algebraic Stacks, Definition 67.12.1 are independent of the topology. This is true because these properties involve only the notion of a 2-fibre product of categories fibred in groupoids, 1- and 2-morphisms of categories fibred in groupoids, the notion of a 1-morphism of categories fibred in groupoids representable by algebraic spaces, and what it means for such a 1-morphism to be surjective and smooth. Thus all we have to prove is that an étale stack in groupoids $\mathcal{X}$ with properties (2) and (3) is also an fppf stack in groupoids.

Using (2) let R be an algebraic space representing

$$(Sch_\alpha/U) \times_{\mathcal{X}} (Sch_\alpha/U)$$

By (3) the projections $s, t : R \to U$ are smooth. Exactly as in the proof of Algebraic Stacks, Lemma 67.16.1 there exists a groupoid in spaces (U, R, s, t, c) and a canonical fully faithful 1-morphism $[U/R]_{étale} \to \mathcal{X}$ where $[U/R]_{étale}$ is the étale stackification of presheaf in groupoids

$$T \longmapsto (U(T), R(T), s(T), t(T), c(T))$$

Claim: If $V \to T$ is a surjective smooth morphism from an algebraic space V to a scheme T, then there exists an étale covering $\{T_i \to T\}$ refining the covering $\{V \to T\}$. This follows from More on Morphisms, Lemma 36.28.7 or the more general Sheaves on Stacks, Lemma 69.18.10. Using the claim and arguing exactly as in Algebraic Stacks, Lemma 67.16.2 it follows that $[U/R]_{étale} \to \mathcal{X}$ is an equivalence.

Next, let $[U/R]$ denote the quotient stack in the fppf topology which is an algebraic stack by Algebraic Stacks, Theorem 67.17.3. Thus we have 1-morphisms

$$U \to [U/R]_{étale} \to [U/R].$$

Both $U \to [U/R]_{étale} \cong \mathcal{X}$ and $U \to [U/R]$ are surjective and smooth (the first by assumption and the second by the theorem) and in both cases the fibre product $U \times_{\mathcal{X}} U$ and $U \times_{[U/R]} U$ is representable by R. Hence the 1-morphism $[U/R]_{étale} \to [U/R]$ is fully faithful (since morphisms in the quotient stacks are given by morphisms into R, see Groupoids in Spaces, Section 59.23).

Finally, for any scheme T and morphism $t : T \to [U/R]$ the fibre product $V = T \times_{U/R} U$ is an algebraic space surjective and smooth over T. By the claim above there exists an étale covering $\{T_i \to T\}_{i \in I}$ and morphisms $T_i \to V$ over T. This proves that the object t of $[U/R]$ over T comes étale locally from U. We conclude that $[U/R]_{étale} \to [U/R]$ is an equivalence of stacks in groupoids over $(Sch/S)_{étale}$ by Stacks, Lemma 8.4.8. This concludes the proof. □

70.20. Other chapters

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CHAPTER 71

Artin's axioms

71.1. Introduction

In this chapter we discuss Artin's axioms for the representability of functors by algebraic spaces. As references we suggest the papers [**Art69b**], [**Art70**], [**Art74**].

Some of the notation, conventions and terminology in this chapter is awkward and may seem backwards to the more experienced reader. This is intentional. Please see Quot, Section 72.1 for an explanation.

71.2. Conventions

The conventions we use in this chapter are the same as those in the chapter on algebraic stacks, see Algebraic Stacks, Section 67.2. In this chapter the base scheme S will often be locally Noetherian (although we will always reiterate this condition when stating results).

71.3. Predeformation categories

Let S be a locally Noetherian base scheme. Let

$$p : \mathcal{X} \longrightarrow (Sch/S)_{fppf}$$

be a category fibred in groupoids. Let k be a field and let $\mathrm{Spec}(k) \to S$ be a morphism of finite type (see Morphisms, Lemma 28.17.1). We will sometimes simply say that k *is a field of finite type over* S. Let x_0 be an object of $\mathcal{X}$ lying over $\mathrm{Spec}(k)$. Given S, $\mathcal{X}$, k, and x_0 we will construct a predeformation category, as defined in Formal Deformation Theory, Definition 64.6.2. The construction will resemble to construction of Formal Deformation Theory, Remark 64.6.4.

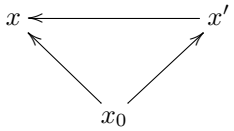
First, by Morphisms, Lemma 28.17.1 we may pick an affine open $\mathrm{Spec}(\Lambda) \subset S$ such that $\mathrm{Spec}(k) \to S$ factors through $\mathrm{Spec}(\Lambda)$ and the associated ring map $\Lambda \to k$ is finite. This provides us with the category $\mathcal{C}_\Lambda$, see Formal Deformation Theory, Definition 64.3.1. The category $\mathcal{C}_\Lambda$, up to canonical equivalence, does not depend on the choice of the affine open $\mathrm{Spec}(\Lambda)$ of S. Namely, $\mathcal{C}_\Lambda$ is equivalent to the opposite of the category of factorizations

$$\mathrm{Spec}(k) \to \mathrm{Spec}(A) \to S \tag{71.3.0.1}$$

of the structure morphism such that A is an Artinian local ring and such that $\mathrm{Spec}(k) \to \mathrm{Spec}(A)$ corresponds to a ring map $A \to k$ which identifies k with the residue field of A.

We let $\mathcal{F} = \mathcal{F}_{\mathcal{X},k,x_0}$ be the category whose

(1) objects are morphisms $x_0 \to x$ of $\mathcal{X}$ where $p(x) = \mathrm{Spec}(A)$ with A an Artinian local ring and $p(x_0) \to p(x) \to S$ a factorization as in (71.3.0.1), and
(2) morphisms $(x_0 \to x) \to (x_0 \to x')$ are commutative diagrams



in $\mathcal{X}$. (Note the reversal of arrows.)

If $x_0 \to x$ is an object of $\mathcal{F}$ then writing $p(x) = \mathrm{Spec}(A)$ we obtain an object A of $\mathcal{C}_\Lambda$. We often say that $x_0 \to x$ or x lies over A. A morphism of $\mathcal{F}$ between objects $x_0 \to x$ lying over A and $x_0 \to x'$ lying over A' corresponds to a morphism $x' \to x$ of $\mathcal{X}$, hence a morphism $p(x' \to x) : \mathrm{Spec}(A') \to \mathrm{Spec}(A)$ which in turn corresponds to a ring map $A \to A'$. As $\mathcal{X}$ is a category over the category of schemes over S we see that $A \to A'$ is Λ-algebra homomorphism. Thus we obtain a functor

$$p : \mathcal{F} = \mathcal{F}_{\mathcal{X},k,x_0} \longrightarrow \mathcal{C}_\Lambda. \tag{71.3.0.2}$$

We will use the notation $\mathcal{F}(A)$ to denote the fibre category over an object A of $\mathcal{C}_\Lambda$. An object of $\mathcal{F}(A)$ is simply a morphism $x_0 \to x$ of $\mathcal{X}$ such that x lies over $\mathrm{Spec}(A)$ and $x_0 \to x$ lies over $\mathrm{Spec}(k) \to \mathrm{Spec}(A)$.

Lemma 71.3.1. *The functor $p : \mathcal{F} \to \mathcal{C}_\Lambda$ defined above is a predeformation category.*

Proof. We have to show that $\mathcal{F}$ is (a) cofibred in groupoids over $\mathcal{C}_\Lambda$ and (b) that $\mathcal{F}(k)$ is a category equivalent to a category with a single object and a single morphism.

Proof of (a). The fibre categories of $\mathcal{F}$ over $\mathcal{C}_\Lambda$ are groupoids as the fibre categories of $\mathcal{X}$ are groupoids. Let $A \to A'$ be a morphism of $\mathcal{C}_\Lambda$ and let $x_0 \to x$ be an object of $\mathcal{F}(A)$. Because $\mathcal{X}$ is fibred in groupoids, we can find a morphism $x' \to x$ lying over $\mathrm{Spec}(A') \to \mathrm{Spec}(A)$. Since the composition $A \to A' \to k$ is equal the given map $A \to k$ we see (by uniqueness of pullbacks up to isomorphism) that the pullback via $\mathrm{Spec}(k) \to \mathrm{Spec}(A')$ of x' is x_0, i.e., that there exists a morphism $x_0 \to x'$ lying over $\mathrm{Spec}(k) \to \mathrm{Spec}(A')$ compatible with $x_0 \to x$ and $x' \to x$. This proves that $\mathcal{F}$ has pushforwards. We conclude by (the dual of) Categories, Lemma 4.33.2.

Proof of (b). If $A = k$, then $\mathrm{Spec}(k) = \mathrm{Spec}(A)$ and since $\mathcal{X}$ is fibred in groupoids over $(Sch/S)_{fppf}$ we see that given any object $x_0 \to x$ in $\mathcal{F}(k)$ the morphism $x_0 \to x$ is an isomorphism. Hence every object of $\mathcal{F}(k)$ is isomorphic to $x_0 \to x_0$. Clearly the only self morphism of $x_0 \to x_0$ in $\mathcal{F}$ is the identity. □

Let S be a locally Noetherian base scheme. Let $F : \mathcal{X} \to \mathcal{Y}$ be a 1-morphism between categories fibred in groupoids over $(Sch/S)_{fppf}$. Let k is a field of finite type over S. Let x_0 be an object of $\mathcal{X}$ lying over $\mathrm{Spec}(k)$. Set $y_0 = F(x_0)$ which is an object of $\mathcal{Y}$ lying over $\mathrm{Spec}(k)$. Then F induces a functor

$$F : \mathcal{F}_{\mathcal{X},k,x_0} \longrightarrow \mathcal{F}_{\mathcal{Y},k,y_0} \tag{71.3.1.1}$$

of categories cofibred over $\mathcal{C}_\Lambda$. Namely, to the object $x_0 \to x$ of $\mathcal{F}_{\mathcal{X},k,x_0}(A)$ we associate the object $F(x_0) \to F(x)$ of $\mathcal{F}_{\mathcal{Y},k,y_0}(A)$.

Lemma 71.3.2. *Let S be a locally Noetherian scheme. Let $F : \mathcal{X} \to \mathcal{Y}$ be a 1-morphism of categories fibred in groupoids over $(Sch/S)_{fppf}$ which is formally smooth on objects (see Criteria for Representability, Section 70.6). Then for every finite type field k over S and for every object x_0 of $\mathcal{X}$ over k the functor (71.3.1.1) is smooth in the sense of Formal Deformation Theory, Definition 64.8.1.*

Proof. This is a matter of unwinding the definitions. Details omitted. □

Lemma 71.3.3. *Let S be a locally Noetherian scheme. Let*

$$\begin{array}{ccc} \mathcal{W} & \longrightarrow & \mathcal{Z} \\ \downarrow & & \downarrow \\ \mathcal{X} & \longrightarrow & \mathcal{Y} \end{array}$$

be a 2-fibre product of categories fibred in groupoids over $(Sch/S)_{fppf}$. Let k be a finite type field over S and w_0 an object of $\mathcal{W}$ over k. Let x_0, z_0, y_0 be the images of w_0 under the morphisms in the diagram. Then

$$\begin{array}{ccc} \mathcal{F}_{\mathcal{W},k,w_0} & \longrightarrow & \mathcal{F}_{\mathcal{Z},k,z_0} \\ \downarrow & & \downarrow \\ \mathcal{F}_{\mathcal{X},k,x_0} & \longrightarrow & \mathcal{F}_{\mathcal{Y},k,y_0} \end{array}$$

is a fibre product of predeformation categories.

Proof. This is a matter of unwinding the definitions. Details omitted. □

71.4. Pushouts and stacks

In this section we show that algebraic stacks behave well with respect to certain pushouts. The results in this section hold over any base scheme.

The following lemma is also correct when Y, X', X, Y' are algebraic spaces, see (insert future reference here).

Lemma 71.4.1. *Let S be a scheme. Let*

$$\begin{array}{ccc} X & \longrightarrow & X' \\ \downarrow & & \downarrow \\ Y & \longrightarrow & Y' \end{array}$$

be a pushout in the category of schemes over S where $X \to X'$ is a thickening and $X \to Y$ is affine, see More on Morphisms, Lemma 36.11.1. Let $\mathcal{Z}$ be an algebraic stack over S. Then the functor of fibre categories

$$\mathcal{Z}_{Y'} \longrightarrow \mathcal{Z}_Y \times_{\mathcal{Z}_X} \mathcal{Z}_{X'}$$

is an equivalence of categories.

Proof. Let y' be an object of left hand side. The sheaf $Isom(y', y')$ on the category of schemes over Y' is representable by an algebraic space I over Y', see Algebraic Stacks, Lemma 67.10.11. We conclude that the functor of the lemma is fully faithful as Y' is the pushout in the category of algebraic spaces as well as the category of schemes, see More on Morphisms of Spaces, Lemma 57.17.2.

Let (y, x', f) be an object of the right hand side. Here $f : y|_X \to x'|_X$ is an isomorphism. To finish the proof we have to construct an object y' of $\mathcal{Z}_{Y'}$ whose restrictions to Y and X' agree with y and x' in a manner compatible with φ. In fact, it suffices to construct y' fppf locally on Y', see Stacks, Lemma 8.4.8. Choose a representable algebraic stack $\mathcal{W}$ and a surjective smooth morphism $\mathcal{W} \to \mathcal{Z}$. Then

$$(Sch/Y)_{fppf} \times_{y,\mathcal{Z}} \mathcal{W} \quad \text{and} \quad (Sch/X')_{fppf} \times_{x',\mathcal{Z}} \mathcal{W}$$

are algebraic stacks representable by algebraic spaces V and U' smooth over Y and X'. The isomorphism f induces an isomorphism $\varphi : V \times_Y X \to U' \times_{X'} X$ over X. By More on Morphisms of Spaces, Lemmas 57.17.3 and 57.17.8 we see that the pushout $V' = V \amalg_{V \times_Y X} U'$ is an algebraic space smooth over Y' whose base change to Y and X' recovers V and U' in a manner compatible with φ.

Let W be the algebraic space representing $\mathcal{W}$. The projections $V \to W$ and $U' \to W$ agree as morphisms over $V \times_Y X \cong U' \times_{X'} X$ hence the universal property of the pushout determines a morphism of algebraic spaces $V' \to W$. Choose a scheme Y'_1 and a surjective étale morphism $Y'_1 \to V'$. Set $Y_1 = Y \times_{Y'} Y'_1$, $X'_1 = X' \times_{Y'} Y'_1$, $X_1 = X \times_{Y'} Y'_1$. The composition

$$(Sch/Y'_1) \to (Sch/V') \to (Sch/W) = \mathcal{W} \to \mathcal{Z}$$

corresponds by the 2-Yoneda lemma to an object y'_1 of $\mathcal{Z}$ over Y'_1 whose restriction to Y_1 and X'_1 agrees with $y|_{Y_1}$ and $x'|_{X'_1}$ in a manner compatible with $f|_{X_1}$. Thus we have constructed our desired object smooth locally over Y' and we win. □

71.5. The Rim-Schlessinger condition

The motivation for the following definition comes from Lemma 71.4.1 and Formal Deformation Theory, Definition 64.15.1 and Lemma 64.15.4.

Definition 71.5.1. Let S be a locally Noetherian scheme. Let $\mathcal{Z}$ be a category fibred in groupoids over $(Sch/S)_{fppf}$. We say $\mathcal{Z}$ satisfies *condition (RS)* if for every pushout

$$\begin{array}{ccc} X & \longrightarrow & X' \\ \downarrow & & \downarrow \\ Y & \longrightarrow & Y' = Y \amalg_X X' \end{array}$$

in the category of schemes over S where

(1) X, X', Y, Y' are spectra of local Artinian rings,
(2) X, X', Y, Y' are of finite type over S, and
(3) $X \to X'$ (and hence $Y \to Y'$) is a closed immersion

the functor of fibre categories

$$\mathcal{Z}_{Y'} \longrightarrow \mathcal{Z}_Y \times_{\mathcal{Z}_X} \mathcal{Z}_{X'}$$

is an equivalence of categories.

If A is an Artinian local ring with residue field k, then any morphism $\mathrm{Spec}(A) \to S$ is affine and of finite type if and only if the induced morphism $\mathrm{Spec}(k) \to S$ is of finite type, see Morphisms, Lemmas 28.13.13 and 28.17.2.

Lemma 71.5.2. *Let $\mathcal{X}$ be an algebraic stack over a locally Noetherian base S. Then $\mathcal{X}$ satisfies (RS).*

Proof. Immediate from the definitions and Lemma 71.4.1. □

Lemma 71.5.3. *Let S be a scheme. Let $p : \mathcal{X} \to \mathcal{Y}$ and $q : \mathcal{Z} \to \mathcal{Y}$ be 1-morphisms of categories fibred in groupoids over $(Sch/S)_{fppf}$. If $\mathcal{X}$, $\mathcal{Y}$, and $\mathcal{Z}$ satisfy (RS), then so does $\mathcal{X} \times_{\mathcal{Y}} \mathcal{Z}$.*

Proof. This is formal. Let

$$\begin{array}{ccc} X & \longrightarrow & X' \\ \downarrow & & \downarrow \\ Y & \longrightarrow & Y' = Y \amalg_X X' \end{array}$$

be a diagram as in Definition 71.5.1. We have to show that

$$(\mathcal{X} \times_{\mathcal{Y}} \mathcal{Z})_{Y'} \longrightarrow (\mathcal{X} \times_{\mathcal{Y}} \mathcal{Z})_Y \times_{(\mathcal{X} \times_{\mathcal{Y}} \mathcal{Z})_X} (\mathcal{X} \times_{\mathcal{Y}} \mathcal{Z})_{X'}$$

is an equivalence. Using the definition of the 2-fibre product this becomes

$$\mathcal{X}_{Y'} \times_{\mathcal{Y}_{Y'}} \mathcal{Z}_{Y'} \longrightarrow (\mathcal{X}_Y \times_{\mathcal{Y}_Y} \mathcal{Z}_Y) \times_{(\mathcal{X}_X \times_{\mathcal{Y}_X} \mathcal{Z}_X)} (\mathcal{X}_{X'} \times_{\mathcal{Y}_{X'}} \mathcal{Z}_{X'}). \tag{71.5.3.1}$$

We are given that each of the functors

$$\mathcal{X}_{Y'} \to \mathcal{X}_Y \times_{\mathcal{Y}_Y} \mathcal{Z}_Y, \quad \mathcal{Y}_{Y'} \to \mathcal{X}_X \times_{\mathcal{Y}_X} \mathcal{Z}_X, \quad \mathcal{Z}_{Y'} \to \mathcal{X}_{X'} \times_{\mathcal{Y}_{X'}} \mathcal{Z}_{X'}$$

are equivalences. An object of the right hand side of (71.5.3.1) is a system

$$((x_Y, z_Y, \phi_Y), (x_{X'}, z_{X'}, \phi_{X'}), (\alpha, \beta)).$$

Then $(x_Y, x_{Y'}, \alpha)$ is isomorphic to the image of an object $x_{Y'}$ in $\mathcal{X}_{Y'}$ and $(z_Y, z_{Y'}, \beta)$ is isomorphic to the image of an object $z_{Y'}$ of $\mathcal{Z}_{Y'}$. The pair of morphisms $(\phi_Y, \phi_{X'})$ corresponds to a morphism ψ between the images of $x_{Y'}$ and $z_{Y'}$ in $\mathcal{Y}_{Y'}$. Then $(x_{Y'}, z_{Y'}, \psi)$ is an object of the left hand side of (71.5.3.1) mapping to the given object of the right hand side. This proves that (71.5.3.1) is essentially surjective. We omit the proof that it is fully faithful. □

71.6. Deformation categories

We match the notation introduced above with the notation from the chapter "Formal Deformation Theory".

Lemma 71.6.1. *Let S be a locally Noetherian scheme. Let $\mathcal{X}$ be a category fibred in groupoids over $(Sch/S)_{fppf}$ satisfying (RS). For any field k of finite type over S and any object x_0 of $\mathcal{X}$ lying over k the predeformation category $p : \mathcal{F}_{\mathcal{X},k,x_0} \to \mathcal{C}_\Lambda$ (71.3.0.2) is a deformation category, see Formal Deformation Theory, Definition 64.15.8.*

Proof. Set $\mathcal{F} = \mathcal{F}_{\mathcal{X},k,x_0}$. Let $f_1 : A_1 \to A$ and $f_2 : A_2 \to A$ be ring maps in $\mathcal{C}_\Lambda$ with f_2 surjective. We have to show that the functor

$$\mathcal{F}(A_1 \times_A A_2) \longrightarrow \mathcal{F}(A_1) \times_{\mathcal{F}(A)} \mathcal{F}(A_2)$$

is an equivalence, see Formal Deformation Theory, Lemma 64.15.4. Set $X = \mathrm{Spec}(A)$, $X' = \mathrm{Spec}(A_2)$, $Y = \mathrm{Spec}(A_1)$ and $Y' = \mathrm{Spec}(A_1 \times_A A_2)$. Note that

$Y' = Y \amalg_X X'$ in the category of schemes, see More on Morphisms, Lemma 36.11.1. We know that in the diagram of functors of fibre categories

$$\begin{array}{ccc} \mathcal{X}_{Y'} & \longrightarrow & \mathcal{X}_Y \times_{\mathcal{X}_X} \mathcal{X}_{X'} \\ \downarrow & & \downarrow \\ \mathcal{X}_{\mathrm{Spec}(k)} & = & \mathcal{X}_{\mathrm{Spec}(k)} \end{array}$$

the top horizontal arrow is an equivalence by Definition 71.5.1. Since $\mathcal{F}(B)$ is the category of objects of $\mathcal{X}_{\mathrm{Spec}(B)}$ with an identification with x_0 over k we win. □

Remark 71.6.2. Let S be a locally Noetherian scheme. Let $\mathcal{X}$ be fibred in groupoids over $(Sch/S)fppf$. Let k be a field of finite type over S and x_0 an object of $\mathcal{X}$ over k. Let $p : \mathcal{F} \to \mathcal{C}_\Lambda$ be as in (71.3.0.2). If $\mathcal{F}$ is a deformation category, i.e., if $\mathcal{F}$ satisfies the Rim-Schlessinger condition (RS), then we see that $\mathcal{F}$ satisfies Schlessingers conditions (S1) and (S2) by Formal Deformation Theory, Lemma 64.15.6. Let $\overline{\mathcal{F}}$ be the functor of isomorphism classes, see Formal Deformation Theory, Remarks 64.5.2 (10). Then $\overline{\mathcal{F}}$ satisfies (S1) and (S2) as well, see Formal Deformation Theory, Lemma 64.9.5. This holds in particular in the situation of Lemma 71.6.1.

71.7. Change of field

This section is the analogue of Formal Deformation Theory, Section 64.26. As pointed out there, to discuss what happens under change of field we need to write $\mathcal{C}_{\Lambda,k}$ instead of $\mathcal{C}_\Lambda$. In the following lemma we use the notation $\mathcal{F}_{l/k}$ introduced in Formal Deformation Theory, Situation 64.26.1.

Lemma 71.7.1. *Let S be a locally Noetherian scheme. Let $\mathcal{X}$ be a category fibred in groupoids over $(Sch/S)_{fppf}$. Let k be a field of finite type over S and let $k \subset l$ be a finite extension. Let x_0 be an object of $\mathcal{F}$ lying over $\mathrm{Spec}(k)$. Denote $x_{l,0}$ the restriction of x_0 to $\mathrm{Spec}(l)$. Then there is a canonical functor*

$$(\mathcal{F}_{\mathcal{X},k,x_0})_{l/k} \longrightarrow \mathcal{F}_{\mathcal{X},l,x_{l,0}}$$

of categories cofibred in groupoids over $\mathcal{C}_{\Lambda,l}$. If $\mathcal{X}$ satisfies (RS), then this functor is an equivalence.

Proof. Consider a factorization

$$\mathrm{Spec}(l) \to \mathrm{Spec}(B) \to S$$

as in (71.3.0.1). By definition we have

$$(\mathcal{F}_{\mathcal{X},k,x_0})_{l/k}(B) = \mathcal{F}_{\mathcal{X},k,x_0}(B \times_l k)$$

see Formal Deformation Theory, Situation 64.26.1. Thus an object of this is a morphism $x_0 \to x$ of $\mathcal{X}$ lying over the morphism $\mathrm{Spec}(k) \to \mathrm{Spec}(B \times_l k)$. Choosing pullback functor for $\mathcal{X}$ we can associate to $x_0 \to x$ the morphism $x_{l,0} \to x_B$ where x_B is the restriction of x to $\mathrm{Spec}(B)$ (via the morphism $\mathrm{Spec}(B) \to \mathrm{Spec}(B \times_l k)$ coming from $B \times_l k \subset B$). This construction is functorial in B and compatible with morphisms.

Next, assume $\mathcal{X}$ satisfies (RS). Consider the diagrams

$$\begin{array}{ccc} l & \longleftarrow & B \\ \uparrow & & \uparrow \\ k & \longleftarrow & B \times_l k \end{array} \quad \text{and} \quad \begin{array}{ccc} \mathrm{Spec}(l) & \longrightarrow & \mathrm{Spec}(B) \\ \downarrow & & \downarrow \\ \mathrm{Spec}(k) & \longrightarrow & \mathrm{Spec}(B \times_l k) \end{array}$$

The diagram on the left is a fibre product of rings. The diagram on the right is a pushout in the category of schemes, see More on Morphisms, Lemma 36.11.1. These schemes are all of finite type over S (see remarks following Definition 71.5.1). Hence (RS) kicks in to give an equivalence of fibre categories

$$\mathcal{X}_{\mathrm{Spec}(B \times_l k)} \longrightarrow \mathcal{X}_{\mathrm{Spec}(k)} \times_{\mathcal{X}_{\mathrm{Spec}(l)}} \mathcal{X}_{\mathrm{Spec}(B)}$$

This implies that the functor defined above gives an equivalence of fibre categories. Hence the functor is an equivalence on categories cofibred in groupoids by (the dual of) Categories, Lemma 4.33.8. □

71.8. Tangent spaces

Let S be a locally Noetherian scheme. Let $\mathcal{X}$ be a category fibred in groupoids over $(Sch/S)_{fppf}$. Let k be a field of finite type over S and let x_0 be an object of $\mathcal{X}$ over k. In Formal Deformation Theory, Section 64.11 we have defined the *tangent space*

$$T\mathcal{F}_{\mathcal{X},k,x_0} = \left\{ \begin{array}{l} \text{isomorphism classes of morphisms} \\ x_0 \to x \text{ over } \mathrm{Spec}(k) \to \mathrm{Spec}(k[\epsilon]) \end{array} \right\} \tag{71.8.0.1}$$

of the predeformation category $\mathcal{F}_{\mathcal{X},k,x_0}$. In Formal Deformation Theory, Section 64.18 we have defined

$$\mathrm{Inf}_{x_0}(\mathcal{F}_{\mathcal{X},k,x_0}) = \mathrm{Ker}\left(\mathrm{Aut}_{\mathrm{Spec}(k[\epsilon])}(x_0') \to \mathrm{Aut}_{\mathrm{Spec}(k)}(x_0)\right) \tag{71.8.0.2}$$

where x_0' is the pullback of x_0 to $\mathrm{Spec}(k[\epsilon])$. If $\mathcal{X}$ satisfies the Rim-Schlessinger condition (RS), then $T\mathcal{F}_{\mathcal{X},k,x_0}$ comes equipped with a natural k-vector space structure by Formal Deformation Theory, Lemma 64.11.2 (assumptions hold by Lemma 71.6.1 and Remark 71.6.2). Moreover, Formal Deformation Theory, Lemma 64.18.9 shows that $\mathrm{Inf}_{x_0}(\mathcal{F}_{\mathcal{X},k,x_0})$ has a natural k-vector space structure such that addition agrees with composition of automorphisms. A natural condition is to ask these vector spaces to have finite dimension.

The following lemma tells us this is true if $\mathcal{X}$ is locally of finite type over S (see Morphisms of Stacks, Section 74.13).

Lemma 71.8.1. *Let S be a locally Noetherian scheme. Assume*

1. *$\mathcal{X}$ is an algebraic stack,*
2. *U is a scheme locally of finite type over S, and*
3. *$(Sch/U)_{fppf} \to \mathcal{X}$ is a smooth surjective morphism.*

Then, for any $\mathcal{F} = \mathcal{F}_{\mathcal{X},k,x_0}$ as in Section 71.3 the tangent space $T\mathcal{F}$ and infinitesimal automorphism space $Inf_{x_0}(\mathcal{F})$ have finite dimension over k

Proof. Let us write $\mathcal{U} = (Sch/U)_{fppf}$. By our definition of algebraic stacks the 1-morphism $\mathcal{U} \to \mathcal{X}$ is representable by algebraic spaces. Hence in particular the 2-fibre product

$$\mathcal{U}_{x_0} = (Sch/\mathrm{Spec}(k))_{fppf} \times_{\mathcal{X}} \mathcal{U}$$

is representable by an algebraic space U_{x_0} over $\mathrm{Spec}(k)$. Then $U_{x_0} \to \mathrm{Spec}(k)$ is smooth and surjective (in particular U_{x_0} is nonempty). By Spaces over Fields, Lemma 58.5.2 we can find a finite extension $l \supset k$ and a point $\mathrm{Spec}(l) \to U_{x_0}$ over k. We have

$$(\mathcal{F}_{\mathcal{X},k,x_0})_{l/k} = \mathcal{F}_{\mathcal{X},l,x_{l,0}}$$

by Lemma 71.7.1 and the fact that $\mathcal{X}$ satisfies (RS). Thus we see that

$$T\mathcal{F} \otimes_k l \cong T\mathcal{F}_{\mathcal{X},l,x_{l,0}} \quad \text{and} \quad \mathrm{Inf}_{x_0}(\mathcal{F}) \otimes_k l \cong \mathrm{Inf}_{x_{l,0}}(\mathcal{F}_{\mathcal{X},l,x_{l,0}})$$

by Formal Deformation Theory, Lemmas 64.26.3 and 64.26.4 (these are applicable by Lemmas 71.5.2 and 71.6.1 and Remark 71.6.2). Hence it suffices to prove that $T\mathcal{F}_{\mathcal{X},l,x_{l,0}}$ and $\mathrm{Inf}_{x_{l,0}}(\mathcal{F}_{\mathcal{X},l,x_{l,0}})$ have finite dimension over l. Note that $x_{l,0}$ comes from a point u_0 of $\mathcal{U}$ over l.

We interrupt the flow of the argument to show that the lemma for infinitesimal automorphisms follows from the lemma for tangent spaces. Namely, let $\mathcal{R} = \mathcal{U} \times_{\mathcal{X}} \mathcal{U}$. Let r_0 be the l-valued point $(u_0, u_0, \mathrm{id}_{x_0})$ of $\mathcal{R}$. Combining Lemma 71.3.3 and Formal Deformation Theory, Lemma 64.24.2 we see that

$$\mathrm{Inf}_{x_{l,0}}(\mathcal{F}_{\mathcal{X},l,x_{l,0}}) \subset T\mathcal{F}_{\mathcal{R},k,r_0}$$

Note that $\mathcal{R}$ is an algebraic stack, see Algebraic Stacks, Lemma 67.14.2. Also, $\mathcal{R}$ is representably by an algebraic space R smooth over U (via either projection, see Algebraic Stacks, Lemma 67.16.2). Hence, choose an scheme U' and a surjective étale morphism $U' \to R$ we see that U' is smooth over U, hence locally of finite type over S. As $(Sch/U')_{fppf} \to \mathcal{R}$ is surjective and smooth, we have reduced the question to the case of tangent spaces.

The functor (71.3.1.1)

$$\mathcal{F}_{\mathcal{U},l,u_0} \longrightarrow \mathcal{F}_{\mathcal{X},l,x_{l,0}}$$

is smooth by Lemma 71.3.2 and Criteria for Representability, Lemma 70.6.3. The induced map on tangent spaces

$$T\mathcal{F}_{\mathcal{U},l,u_0} \longrightarrow T\mathcal{F}_{\mathcal{X},l,x_{l,0}}$$

is l-linear (by Formal Deformation Theory, Lemma 64.11.4) and surjective (as smooth maps of predeformation categories induce surjective maps on tangent spaces by Formal Deformation Theory, Lemma 64.8.8). Hence it suffices to prove that the tangent space of the deformation space associated to the representable algebraic stack $\mathcal{U}$ at the point u_0 is finite dimensional. Let $\mathrm{Spec}(R) \subset U$ be an affine open such that $u_0 : \mathrm{Spec}(l) \to U$ factors through $\mathrm{Spec}(R)$ and such that $\mathrm{Spec}(R) \to S$ factors through $\mathrm{Spec}(\Lambda) \subset S$. Let $\mathfrak{m}_R \subset R$ be the kernel of the Λ-algebra map $\varphi_0 : R \to l$ corresponding to u_0. Note that R, being of finite type over the Noetherian ring Λ, is a Noetherian ring. Hence $\mathfrak{m}_R = (f_1, \ldots, f_n)$ is a finitely generated ideal. We have

$$T\mathcal{F}_{\mathcal{U},l,u_0} = \{\varphi : R \to l[\epsilon] \mid \varphi \text{ is a } \Lambda\text{-algebra map and } \varphi \bmod \epsilon = \varphi_0\}$$

An element of the right hand side is determined by its values on $f_1, \ldots, f_n$ hence the dimension is at most n and we win. Some details omitted. □

Lemma 71.8.2. *Let S be a locally Noetherian scheme. Let $p : \mathcal{X} \to \mathcal{Y}$ and $q : \mathcal{Z} \to \mathcal{Y}$ be 1-morphisms of categories fibred in groupoids over $(Sch/S)_{fppf}$. Assume $\mathcal{X}$, $\mathcal{Y}$, $\mathcal{Z}$ satisfy (RS). Let k be a field of finite type over S and let w_0 be*

an object of $\mathcal{W} = \mathcal{X} \times_{\mathcal{Y}} \mathcal{Z}$ over k. Denote x_0, y_0, z_0 the objects of $\mathcal{X}, \mathcal{Y}, \mathcal{Z}$ you get from w_0. Then there is a 6-term exact sequence

$$0 \longrightarrow Inf_{w_0}(\mathcal{F}_{\mathcal{W},k,W_0}) \longrightarrow Inf_{x_0}(\mathcal{F}_{\mathcal{X},k,x_0}) \oplus Inf_{z_0}(\mathcal{F}_{\mathcal{Z},k,z_0}) \longrightarrow Inf_{y_0}(\mathcal{F}_{\mathcal{Y},k,y_0})$$
$$T\mathcal{F}_{\mathcal{W},k,w_0} \longrightarrow T\mathcal{F}_{\mathcal{X},k,x_0} \oplus T\mathcal{F}_{\mathcal{Z},k,z_0} \longrightarrow T\mathcal{F}_{\mathcal{Y},k,y_0}$$

of k-vector spaces.

Proof. Apply Lemmas 71.3.3 and 71.6.1 and Formal Deformation Theory, Lemma 64.18.14. □

71.9. Formal objects

In this section we transfer some of the notions already defined in the chapter "Formal Deformation Theory" to the current setting. In the following we will say "R is an S-algebra" to indicate that R is a ring endowed with a morphism of schemes $\mathrm{Spec}(R) \to S$.

Definition 71.9.1. Let S be a locally Noetherian scheme. Let $p : \mathcal{X} \to (Sch/S)_{fppf}$ be a category fibred in groupoids.

(1) A *formal object* $\xi = (R, \xi_n, f_n)$ of $\mathcal{X}$ consists of a Noetherian complete local S-algebra R, objects ξ_n of $\mathcal{X}$ lying over $\mathrm{Spec}(R/\mathfrak{m}_R^n)$, and morphisms $\xi_n \to \xi_{n+1}$ of $\mathcal{X}$ lying over $\mathrm{Spec}(R/\mathfrak{m}^n) \to \mathrm{Spec}(R/\mathfrak{m}^{n+1})$ such that $R/\mathfrak{m}$ is a field of finite type over S.

(2) A *morphism of formal objects* $a : \xi = (R, \xi_n, f_n) \to \eta = (T, \eta_n, g_n)$ is given by morphisms $a_n : \xi_n \to \eta_n$ such that for every n the diagram

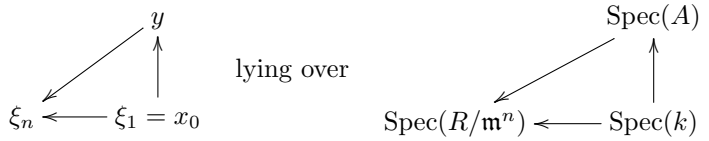
$$\begin{array}{ccc} \xi_{n+1} & \xrightarrow{f_n} & \xi_n \\ {\scriptstyle a_{n+1}}\downarrow & & \downarrow{\scriptstyle a_n} \\ \eta_{n+1} & \xrightarrow{g_n} & \eta_n \end{array}$$

is commutative. Applying the functor p we obtain a compatible collection of morphisms $\mathrm{Spec}(R/\mathfrak{m}_R^n) \to \mathrm{Spec}(T/\mathfrak{m}_T^n)$ and hence a morphism $a_0 : \mathrm{Spec}(R) \to \mathrm{Spec}(T)$ over S. We say that *a lies over a_0*.

Thus we obtain a category of formal object of $\mathcal{X}$. If $F : \mathcal{X} \to \mathcal{Y}$ is a 1-morphism of categories fibred in groupoids over $(Sch/S)_{fppf}$, then F induces a functor between categories of formal objects as well.

Given a formal object $\xi = (R, \xi_n, f_n)$, set $k = R/\mathfrak{m}$ and $x_0 = \xi_1$. Then the formal object ξ defines a formal object ξ of $\mathcal{F}_{\mathcal{X},k,x_0}$ as defined in Formal Deformation Theory, Definition 64.7.1. We will use the terminology introduced in that chapter. In particular, we will say that ξ is *versal* if ξ (as a formal object of $\mathcal{F}$) is versal in the sense of Formal Deformation Theory, Definition 64.8.13. We briefly spell out here what this means. Suppose given a morphism $x_0 \to y$ lying over a closed immersion $\mathrm{Spec}(k) \to \mathrm{Spec}(A)$ where A is an Artinian local ring with residue field

k. Then versality implies there exists an $n \geq 1$ and a commutative diagram



Please compare with Formal Deformation Theory, Remark 64.8.14.

Lemma 71.9.2. *Let S be a locally Noetherian scheme. Let $F : \mathcal{X} \to \mathcal{Y}$ be a 1-morphism of categories fibred in groupoids over $(Sch/S)_{fppf}$. Let $\eta = (R, \eta_n, g_n)$ be a formal object of $\mathcal{Y}$ and let ξ_1 be an object of $\mathcal{X}$ with $F(\xi_1) \cong \eta_1$. If F is formally smooth on objects (see Criteria for Representability, Section 70.6), then there exists a formal object $\xi = (R, \xi_n, f_n)$ of $\mathcal{X}$ such that $F(\xi) \cong \eta$.*

Proof. Note that each of the morphisms $\mathrm{Spec}(R/\mathfrak{m}^n) \to \mathrm{Spec}(R/\mathfrak{m}^{n+1})$ is a first order thickening of affine schemes over S. Hence the assumption on F means that we can successively lift ξ_1 to objects $\xi_2, \xi_3, \ldots$ of $\mathcal{X}$ endowed with compatible isomorphisms $\eta_n|_{\mathrm{Spec}(R/\mathfrak{m}^{n-1})} \cong \eta_{n-1}$ and $F(\eta_n) \cong \xi_n$. □

Let S be a locally Noetherian scheme. Let $p : \mathcal{X} \to (Sch/S)_{fppf}$ be a category fibred in groupoids. Suppose that x is an object of $\mathcal{X}$ over R, where R is a Noetherian complete local S-algebra with residue field of finite type over S. Then we can consider the system of restrictions $\xi_n = x|_{\mathrm{Spec}(R/\mathfrak{m}^n)}$ endowed with the natural morphisms $\xi_1 \to \xi_2 \to \ldots$ coming from transitivity of restriction. Thus $\xi = (R, \xi_n, \xi_n \to \xi_{n+1})$ is a formal object of $\mathcal{X}$. This construction is functorial in the object x. Thus we obtain a functor

(71.9.2.1)
$$\left\{\begin{array}{c}\text{objects } x \text{ of } \mathcal{X} \text{ such that } p(x) = \mathrm{Spec}(R) \\ \text{where } R \text{ is Noetherian complete local} \\ \text{with } R/\mathfrak{m} \text{ of finite type over } S\end{array}\right\} \longrightarrow \left\{\text{formal objects of } \mathcal{X}\right\}$$

To be precise the left hand side is the full subcategory of $\mathcal{X}$ consisting of objects as indicated and the right hand side is the category of formal objects of $\mathcal{X}$ as in Definition 71.9.1.

Definition 71.9.3. Let S be a locally Noetherian scheme. Let $\mathcal{X}$ be a category fibred in groupoids over $(Sch/S)_{fppf}$. A formal object $\xi = (R, \xi_n, f_n)$ of $\mathcal{X}$ is called *effective* if it is in the essential image of the functor (71.9.2.1).

If the category fibred in groupoids is an algebraic stack, then every formal object is effective as follows from the next lemma.

Lemma 71.9.4. *Let S be a locally Noetherian scheme. Let $\mathcal{X}$ be an algebraic stack over S. The functor (71.9.2.1) is an equivalence.*

Proof. Case I: $\mathcal{X}$ is representable (by a scheme). Say $\mathcal{X} = (Sch/X)_{fppf}$ for some scheme X over S. Unwinding the definitions we have to prove the following: Given a Noetherian complete local S-algebra R with $R/\mathfrak{m}$ of finite type over S we have

$$\mathrm{Mor}_S(\mathrm{Spec}(R), X) \longrightarrow \lim \mathrm{Mor}_S(\mathrm{Spec}(R/\mathfrak{m}^n, X)$$

is bijective. We can split both sides of the arrow into a disjoint union over $x \in X$ by looking at those morphisms (resp. systems of morphisms) mapping the closed point of $\mathrm{Spec}(R)$ (resp. the unique point of $\mathrm{Spec}(R/\mathfrak{m}^n)$) to x. Let $s \in S$ be the image of

x. Applying the discussion in Schemes, Section 25.13 we see that the arrow above is the disjoint union of the arrows

$$\mathrm{Hom}_{\mathcal{O}_{S,s}}(\mathcal{O}_{X,x}, R) \longrightarrow \lim \mathrm{Hom}_{\mathcal{O}_{S,s}}(\mathcal{O}_{X,x}, R/\mathfrak{m}^n)$$

By the universal property of limits each of these maps is bijective.

Case II. $\mathcal{X}$ is representable by an algebraic space. Say $\mathcal{X}$ is representable by X. Again we have to show that

$$\mathrm{Mor}_S(\mathrm{Spec}(R), X) \longrightarrow \lim \mathrm{Mor}_S(\mathrm{Spec}(R/\mathfrak{m}^n, X)$$

is bijective for R as above.

Injectivity: Let $x, x' : \mathrm{Spec}(R) \to X$ be two morphisms mapping to the same element in the right hand side. Consider the fibre product

$$T = \mathrm{Spec}(R) \times_{(x,x'),X\times_S X,\Delta} X$$

Then T is a scheme and $T \to \mathrm{Spec}(R)$ is locally of finite type, monomorphism, separated, and locally quasi-finite, see Morphisms of Spaces, Lemma 49.5.1. In particular T is locally Noetherian, see Morphisms, Lemma 28.16.6. Let $t \in T$ be the unique point mapping to the closed point of $\mathrm{Spec}(R)$ which exists as x and x' agree over $R/\mathfrak{m}$. Then $R \to \mathcal{O}_{T,t}$ is a local ring map of Noetherian rings such that $R/\mathfrak{m}^n \to \mathcal{O}_{T,t}/\mathfrak{m}^n\mathcal{O}_{T,t}$ is an isomorphism for all n (because x and x' agree over $\mathrm{Spec}(R/\mathfrak{m}^n)$ for all n). Since $\mathcal{O}_{T,t}$ maps injectively into its completion (see Algebra, Lemma 10.47.6) we conclude that $R = \mathcal{O}_{T,t}$. Hence x and x' agree over R.

Surjectivity: Let (x_n) be an element of the right hand side. Choose a scheme U and a surjective étale morphism $U \to X$. Denote $x_0 : \mathrm{Spec}(k) \to X$ the morphism induced on the residue field $k = R/\mathfrak{m}$. The morphism of schemes $U \times_{X,x_0} \mathrm{Spec}(k) \to \mathrm{Spec}(k)$ is surjective étale. Thus $U \times_{X,x_0} \mathrm{Spec}(k)$ is a nonempty disjoint union of spectra of finite separable field extensions of k, see Morphisms, Lemma 28.37.7. Hence we can find a finite separable field extension $k \subset k'$ and a k'-point $u_0 : \mathrm{Spec}(k') \to U$ such that

$$\begin{array}{ccc} \mathrm{Spec}(k') & \xrightarrow[u_0]{} & U \\ \downarrow & & \downarrow \\ \mathrm{Spec}(k) & \xrightarrow{x_0} & X \end{array}$$

commutes. Let $R \subset R'$ be the finite étale extension of Noetherian complete local rings which induces $k \subset k'$ on residue fields (see Algebra, Lemmas 10.143.8 and 10.143.10). Denote x'_n the restriction of x_n to $\mathrm{Spec}(R'/(\mathfrak{m}')^n)$. By Lemma 71.9.2 and Criteria for Representability, Lemma 70.6.3 we can find an element $(u'_n) \in \lim \mathrm{Hom}_S(\mathrm{Spec}(R'/(\mathfrak{m}')^n), U)$ mapping to (x'_n). By Case I the family (u'_n) comes from a unique morphism $u' : \mathrm{Spec}(R') \to U$. Denote $x' : \mathrm{Spec}(R') \to X$ the composition. Note that $R' \otimes_R R'$ is a finite product of spectra of Noetherian complete local rings to which our current discussion applies. Hence the diagram

$$\begin{array}{ccc} \mathrm{Spec}(R' \otimes_R R') & \longrightarrow & \mathrm{Spec}(R') \\ \downarrow & & \downarrow{\scriptstyle x'} \\ \mathrm{Spec}(R') & \xrightarrow{x'} & X \end{array}$$

is commutative by the injectivity shown above and the fact that x'_n is the restriction of x_n which is defined over $R/\mathfrak{m}^n$. Since $\{\mathrm{Spec}(R') \to \mathrm{Spec}(R)\}$ is an fppf covering we conclude that x' descends to a morphism $x : \mathrm{Spec}(R) \to X$. We omit the proof that x_n is the restriction of x to $\mathrm{Spec}(R/\mathfrak{m}^n)$.

Case III: General case of an algebraic stack. A general remark is that the left and right hand side of (71.9.2.1) are categories fibred in groupoids over the category of affine schemes over S which are spectra of Noetherian complete local rings with residue field of finite type over S. We will also see in the proof below that they form stacks for a certain topology on this category.

We first prove fully faithfulness. Let R be a Noetherian complete local S-algebra with $k = R/\mathfrak{m}$ of finite type over S. Let x, x' be objects of $\mathcal{X}$ over R. As $\mathcal{X}$ is an algebraic stack $Isom(x, x')$ is representable by an algebraic space I over $\mathrm{Spec}(R)$, see Algebraic Stacks, Lemma 67.10.11. Applying Case II to I over $\mathrm{Spec}(R)$ implies immediately that (71.9.2.1) is fully faithful on fibre categories over $\mathrm{Spec}(R)$. Hence the functor is fully faithful by Categories, Lemma 4.33.8.

Essential surjectivity. Let $\xi = (R, \xi_n, f_n)$ be a formal object of $\mathcal{X}$. Choose a scheme U over S and a surjective smooth morphism $f : (Sch/U)_{fppf} \to \mathcal{X}$. For every n consider the fibre product

$$(Sch/\mathrm{Spec}(R/\mathfrak{m}^n))_{fppf} \times_{\xi_n, \mathcal{X}, f} (Sch/U)_{fppf}$$

By assumption this is representable by an algebraic space V_n surjective and smooth over $\mathrm{Spec}(R/\mathfrak{m}^n)$. The morphisms $f_n : \xi_n \to \xi_{n+1}$ induce cartesian squares

$$\begin{array}{ccc} V_{n+1} & \longleftarrow & V_n \\ \downarrow & & \downarrow \\ \mathrm{Spec}(R/\mathfrak{m}^{n+1}) & \longleftarrow & \mathrm{Spec}(R/\mathfrak{m}^n) \end{array}$$

of algebraic spaces. By Spaces over Fields, Lemma 58.5.2 we can find a finite separable extension $k \subset k'$ and a point $v'_1 : \mathrm{Spec}(k') \to V_1$ over k. Let $R \subset R'$ be the finite étale extension whose residue field extension is $k \subset k'$ (exists and is unique by Algebra, Lemmas 10.143.8 and 10.143.10). By the infinitesimal lifting criterion of smoothness (see More on Morphisms of Spaces, Lemma 57.16.6) applied to $V_n \to \mathrm{Spec}(R/\mathfrak{m}^n)$ for $n = 2, 3, 4, \ldots$ we can successively find morphisms $v'_n : \mathrm{Spec}(R'/(\mathfrak{m}')^n) \to V_n$ over $\mathrm{Spec}(R/\mathfrak{m}^n)$ fitting into commutive diagrams

$$\begin{array}{ccc} \mathrm{Spec}(R'/(\mathfrak{m}')^{n+1}) & \longleftarrow & \mathrm{Spec}(R'/(\mathfrak{m}')^n) \\ {\scriptstyle v'_{n+1}}\downarrow & & \downarrow{\scriptstyle v'_n} \\ V_{n+1} & \longleftarrow & V_n \end{array}$$

Composing with the projection morphisms $V_n \to U$ we obtain a compatible system of morphisms $u'_n : \mathrm{Spec}(R'/(\mathfrak{m}')^n) \to U$. By Case I the family (u'_n) comes from a unique morphism $u' : \mathrm{Spec}(R') \to U$. Denote x' the object of $\mathcal{X}$ over $\mathrm{Spec}(R')$ we get by applying the 1-morphism f to u'. By construction, there exists a morphism of formal objects

$$(71.9.2.1)(x') = (R', x'|_{\mathrm{Spec}(R'/(\mathfrak{m}')^n)}, \ldots) \longrightarrow (R, \xi_n, f_n)$$

lying over $\mathrm{Spec}(R') \to \mathrm{Spec}(R)$. Note that $R' \otimes_R R'$ is a finite product of spectra of Noetherian complete local rings to which our current discussion applies. Denote $p_0, p_1 : \mathrm{Spec}(R' \otimes_R R') \to \mathrm{Spec}(R')$ the two projections. By the fully faithfulness shown above there exists a canonical isomorphism $\varphi : p_0^* x' \to p_1^* x'$ because we have such isomorphisms over $\mathrm{Spec}((R' \otimes_R R')/\mathfrak{m}^n(R' \otimes_R R'))$. We omit the proof that the isomorphism φ satisfies the cocycle condition (see Stacks, Definition 8.3.1). Since $\{\mathrm{Spec}(R') \to \mathrm{Spec}(R)\}$ is an fppf covering we conclude that x' descends to an object x of $\mathcal{X}$ over $\mathrm{Spec}(R)$. We omit the proof that x_n is the restriction of x to $\mathrm{Spec}(R/\mathfrak{m}^n)$. □

Lemma 71.9.5. *Let S be a scheme. Let $p : \mathcal{X} \to \mathcal{Y}$ and $q : \mathcal{Z} \to \mathcal{Y}$ be 1-morphisms of categories fibred in groupoids over $(Sch/S)_{fppf}$. If the functor (71.9.2.1) is an equivalence for $\mathcal{X}$, $\mathcal{Y}$, and $\mathcal{Z}$, then it is and equivalence for $\mathcal{X} \times_{\mathcal{Y}} \mathcal{Z}$.*

Proof. The left and the right hand side of (71.9.2.1) for $\mathcal{X} \times_{\mathcal{Y}} \mathcal{Z}$ are simply the 2-fibre products of the left and the right hand side of (71.9.2.1) for $\mathcal{X}$, $\mathcal{Z}$ over $\mathcal{Y}$. Hence the result follows as taking 2-fibre products is compatible with equivalences of categories, see Categories, Lemma 4.29.7. □

71.10. Approximation

A fundamental insight of Michael Artin is that you can approximate objects of a limit preserving stack. Namely, given an object x of the stack over a Noetherian complete local ring, you can find an object x_A over an algebraic ring which is "close to" x. Here an algebraic ring means a finite type S-algebra and close means adically close. In this section we present this in a simple, yet general form.

To formulate the result we need to pull together some definitions from different places in the stacks project. First, in Criteria for Representability, Section 70.5 we introduced *limit preserving on objects* for 1-morphisms of categories fibred in groupoids over the category of schemes. In More on Algebra, Definition 15.37.1 we defined the notion of a *G-ring*. Let S be a locally Noetherian scheme. Let A be an S-algebra. We say that A is *of finite type over S* or is a *finite type S-algebra* if $\mathrm{Spec}(A) \to S$ is of finite type. In this case A is a Noetherian ring. Finally, given a ring A and ideal I we denote $\mathrm{Gr}_I(A) = \bigoplus I^n/I^{n+1}$.

Lemma 71.10.1. *Let S be a locally Noetherian scheme. Let $p : \mathcal{X} \to (Sch/S)_{fppf}$ be a category fibred in groupoids. Let x be an object of $\mathcal{X}$ lying over $\mathrm{Spec}(R)$ where R is a Noetherian complete local ring with residue field k of finite type over S. Let $s \in S$ be the image of $\mathrm{Spec}(k) \to S$. Assume that (a) $\mathcal{O}_{S,s}$ is a G-ring and (b) p is limit preserving on objects. Then for every integer $N \geq 1$ there exist*

1. *a finite type S-algebra A,*
2. *a maximal ideal $\mathfrak{m}_A \subset A$,*
3. *an object x_A of $\mathcal{X}$ over $\mathrm{Spec}(A)$,*
4. *an S-isomorphism $R/\mathfrak{m}_R^N \cong A/\mathfrak{m}_A^N$,*
5. *an isomorphism $x|_{\mathrm{Spec}(R/\mathfrak{m}_R^N)} \cong x_A|_{\mathrm{Spec}(A/\mathfrak{m}_A^N)}$ compatible with (4), and*
6. *an isomorphism $Gr_{\mathfrak{m}_R}(R) \cong Gr_{\mathfrak{m}_A}(A)$ of graded k-algebras.*

Proof. Choose an affine open $\mathrm{Spec}(\Lambda) \subset S$ such that k is a finite Λ-algebra, see Morphisms, Lemma 28.17.1. We may and do replace S by $\mathrm{Spec}(\Lambda)$.

We may write R as a directed colimit $R = \mathrm{colim}\, C_j$ where each C_j is a finite type Λ-algebra (see Algebra, Lemma 10.121.1). By assumption (b) the object x is

isomorphic to the restriction of an object over one of the C_j. Hence we may choose a finite type Λ-algebra C, a Λ-algebra map $C \to R$, and an object x_C of $\mathcal{X}$ over $\mathrm{Spec}(C)$ such that $x = x_C|_{\mathrm{Spec}(R)}$. The choice of C is a bookkeeping device and could be avoided. For later use, let us write $C = \Lambda[y_1, \ldots, y_u]/(f_1, \ldots, f_v)$ and we denote $\overline{a}_i \in R$ the image of y_i under the map $C \to R$. Set $\mathfrak{m}_C = C \cap \mathfrak{m}_R$.

Choose a Λ-algebra surjection $\Lambda[x_1, \ldots, x_s] \to k$ and denote $\mathfrak{m}'$ the kernel. By the universal property of polynomial rings we may lift this to a Λ-algebra map $\Lambda[x_1, \ldots, x_s] \to R$. We add some variables (i.e., we increase s a bit) mapping to generators of $\mathfrak{m}_R$. Having done this we see that $\Lambda[x_1, \ldots, x_s] \to R/\mathfrak{m}_R^2$ is surjective. Then we see that

$$P = \Lambda[x_1, \ldots, x_s]^\wedge_{\mathfrak{m}'} \longrightarrow R \tag{71.10.1.1}$$

is a surjective map of Noetherian complete local rings, see for example Formal Deformation Theory, Lemma 64.4.2.

Choose lifts $a_i \in P$ of $\overline{a}_i$ we found above. Choose generators $b_1, \ldots, b_r \in P$ for the kernel of (71.10.1.1). Choose $c_{ji} \in P$ such that

$$f_j(a_1, \ldots, a_u) = \sum c_{ji} b_i$$

in P which is possible by the choices made so far. Choose generators

$$k_1, \ldots, k_t \in \mathrm{Ker}(P^{\oplus r} \xrightarrow{(b_1, \ldots, b_r)} P)$$

and write $k_i = (k_{i1}, \ldots, k_{ir})$ and $K = (k_{ij})$ so that

$$P^{\oplus t} \xrightarrow{K} P^{\oplus r} \xrightarrow{(b_1, \ldots, b_r)} P \to R \to 0$$

is an exact sequence of P-modules. In particular we have $\sum k_{ij} b_j = 0$. After possibly increasing N we may assume $N - 1$ works in the Artin-Rees lemma for the first two maps of this exact sequence (see More on Algebra, Section 15.3 for terminology).

By assumption $\mathcal{O}_{S,s} = \Lambda_{\Lambda \cap \mathfrak{m}'}$ is a G-ring. Hence by More on Algebra, Proposition 15.37.10 the ring $\Lambda[x_1, \ldots, x_s]_{\mathfrak{m}'}$ is a G-ring. Hence by Smoothing Ring Maps, Theorem 16.13.2 there exist an étale ring map

$$\Lambda[x_1, \ldots, x_s]_{\mathfrak{m}'} \to B,$$

a maximal ideal $\mathfrak{m}_B$ of B lying over $\mathfrak{m}'$, and elements $a'_i, b'_i, c'_{ij}, k'_{ij} \in B'$ such that

(1) $\kappa(\mathfrak{m}') = \kappa(\mathfrak{m}_B)$ which implies that $\Lambda[x_1, \ldots, x_s]_{\mathfrak{m}'} \subset B_{\mathfrak{m}_B} \subset P$ and P is identified with the completion of B at $\mathfrak{m}_B$, see remark preceding Smoothing Ring Maps, Theorem 16.13.2,
(2) $a_i - a'_i, b_i - b'_i, c_{ij} - c'_{ij}, k_{ij} - k'_{ij} \in (\mathfrak{m}')^N P$, and
(3) $f_j(a'_1, \ldots, a'_u) = \sum c'_{ji} b'_i$ and $\sum k'_{ij} b'_j = 0$.

Set $A = B/(b'_1, \ldots, b'_r)$ and denote $\mathfrak{m}_A$ the image of $\mathfrak{m}_B$ in A. (Note that A is essentially of finite type over Λ; at the end of the proof we will show how to obtain an A which is of finite type over Λ.) There is a ring map $C \to A$ sending $y_i \mapsto a'_i$ because the a'_i satisfy the desired equations modulo $(b'_1, \ldots, b'_r)$. Note that $A/\mathfrak{m}_A^N = R/\mathfrak{m}_R^N$ as quotients of $P = B^\wedge$ by property (2) above. Set $x_A = x_C|_{\mathrm{Spec}(A)}$. Since the maps

$$C \to A \to A/\mathfrak{m}_A^N \cong R/\mathfrak{m}_R^N \quad \text{and} \quad C \to R \to R/\mathfrak{m}_R^N$$

are equal we see that x_A and x agree modulo $\mathfrak{m}_R^N$ via the isomorphism $A/\mathfrak{m}_A^N = R/\mathfrak{m}_R^N$. At this point we have shown properties (1) – (5) of the statement of the lemma. To see (6) note that

$$P^{\oplus t} \xrightarrow{K} P^{\oplus r} \xrightarrow{(b_1,\ldots,b_r)} P \quad \text{and} \quad P^{\oplus t} \xrightarrow{K'} P^{\oplus r} \xrightarrow{(b'_1,\ldots,b'_r)} P$$

are two complexes of P-modules which are congruent modulo $(\mathfrak{m}')^N$ with the first one being exact. By our choice of N above we see from More on Algebra, Lemma 15.3.2 that $R = P/(b_1,\ldots,b_r)$ and $P/(b'_1,\ldots,b'_r) = B^\wedge/(b'_1,\ldots,b'_r) = A^\wedge$ have isomorphic associated graded algebras, which is what we wanted to show.

This last paragraph of the proof serves to clean up the issue that A is essentially of finite type over S and not yet of finite type. The construction above gives $A = B/(b'_1,\ldots,b'_r)$ and $\mathfrak{m}_A \subset A$ with B étale over $\Lambda[x_1,\ldots,x_s]_{\mathfrak{m}'}$. Hence A is of finite type over the Noetherian ring $\Lambda[x_1,\ldots,x_s]_{\mathfrak{m}'}$. Thus we can write $A = (A_0)_{\mathfrak{m}'}$ for some finite type $\Lambda[x_1,\ldots,x_n]$ algebra A_0. Then $A = \operatorname{colim}(A_0)_f$ where $f \in \Lambda[x_1,\ldots,x_n] \setminus \mathfrak{m}'$, see Algebra, Lemma 10.9.9. Because $p : \mathcal{X} \to (Sch/S)_{fppf}$ is limit preserving on objects, we see that x_A comes from some object $x_{(A_0)_f}$ over $\operatorname{Spec}((A_0)_f)$ for an f as above. After replacing A by $(A_0)_f$ and x_A by $x_{(A_0)_f}$ and $\mathfrak{m}_A$ by $(A_0)_f \cap \mathfrak{m}_A$ the proof is finished. □

71.11. Versality

In the previous section we explained how to approximate objects over complete local rings by algebraic objects. But in order to show that a stack $\mathcal{X}$ is an algebraic stack, we need to find smooth 1-morphisms from schemes towards $\mathcal{X}$. Since we are not going to assume a priori that $\mathcal{X}$ has a representable diagonal, we cannot even speak about smooth morphisms towards $\mathcal{X}$. Instead, borrowing terminology from deformation theory, we will introduce versal objects.

Let S be a locally Noetherian scheme. Let U be a scheme over S with structure morphism $U \to S$ locally of finite type. Let $u_0 \in U$ be a finite type point of U, see Morphisms, Definition 28.17.3. Set $k = \kappa(u_0)$. Note that the composition $\operatorname{Spec}(k) \to S$ is also of finite type, see Morphisms, Lemma 28.16.3. Let $p : \mathcal{X} \to (Sch/S)_{fppf}$ be a category fibred in groupoids. Let x be an object of $\mathcal{X}$ which lies over U. Denote x_0 the pullback of x by u_0. By the 2-Yoneda lemma x corresponds to a 1-morphism

$$x : (Sch/U)_{fppf} \longrightarrow \mathcal{X},$$

see Algebraic Stacks, Section 67.5. We obtain a morphism of predeformation categories

$$\hat{x} : \mathcal{F}_{(Sch/U)_{fppf},k,u_0} \longrightarrow \mathcal{F}_{\mathcal{X},k,x_0}, \tag{71.11.0.2}$$

over $\mathcal{C}_\Lambda$ see (71.3.1.1).

Definition 71.11.1. Let S be a locally Noetherian scheme. Let $\mathcal{X}$ be fibred in groupoids over $(Sch/S)_{fppf}$. Let U be a scheme locally of finite type over S. Let x be an object of $\mathcal{X}$ lying over U. Let u_0 be finite type point of U. We say x is *versal* at u_0 if the morphism $\hat{x}$ (71.11.0.2) is smooth, see Formal Deformation Theory, Definition 64.8.1.

This definition matches our notion of versality for formal objects of $\mathcal{X}$ in the following way. Observe that $\mathcal{O}_{U,u_0}$ is a Noetherian local S-algebra with residue field

k. Hence $R = \mathcal{O}_{U,u_0}^\wedge$ is an object of $\mathcal{C}_\Lambda^\wedge$, see Formal Deformation Theory, Definition 64.4.1. There is an identification of predeformation categories

$$\underline{R}|_{\mathcal{C}_\Lambda} = \mathcal{F}_{(Sch/U)_{fppf},k,u_0},$$

see Formal Deformation Theory, Remark 64.7.12 for notation. Namely, given an Artinian local S-algebra A with residue field identified with k we have

$$\mathrm{Mor}_{\mathcal{C}_\Lambda^\wedge}(R, A) = \{\varphi \in \mathrm{Mor}_S(\mathrm{Spec}(A), U) \mid \varphi|_{\mathrm{Spec}(k)} = u_0\}$$

Let ξ be the formal object of $\mathcal{X}$ over R associated to $x|_{\mathrm{Spec}(R)}$, see (71.9.2.1). Then

$$\underline{R}|_{\mathcal{C}_\Lambda} = \mathcal{F}_{(Sch/U)_{fppf},k,u_0} \xrightarrow{\hat{x}} \mathcal{F}_{\mathcal{X},k,x_0},$$

corresponds to ξ via the correspondence between formal objects and morphisms of Formal Deformation Theory, Equation (64.7.12.1). In other words, we see that

$$x \text{ is versal at } u_0 \Leftrightarrow \xi \text{ is versal}$$

It turns out that this notion is well behaved with respect to field extensions.

Lemma 71.11.2. *Let S, $\mathcal{X}$, U, x, u_0 be as in Definition 71.11.1. Let l be a field and let $u_{l,0} : \mathrm{Spec}(l) \to U$ be a morphism with image u_0 such that $k = \kappa(u_0) \subset l$ is finite. Set $x_{l,0} = x_0|_{\mathrm{Spec}(l)}$. If $\mathcal{X}$ satisfies (RS) and x is versal at u_0, then*

$$\mathcal{F}_{(Sch/U)_{fppf},l,u_{l,0}} \longrightarrow \mathcal{F}_{\mathcal{X},l,x_{l,0}}$$

is smooth.

Proof. Note that $(Sch/U)_{fppf}$ satisfies (RS) by Lemma 71.5.2. Hence the functor of the lemma is the functor

$$(\mathcal{F}_{(Sch/U)_{fppf},k,u_0})_{l/k} \longrightarrow (\mathcal{F}_{\mathcal{X},k,x_0})_{l/k}$$

associated to $\hat{x}$, see Lemma 71.7.1. Hence the lemma follows from Formal Deformation Theory, Lemma 64.26.5. □

We restate the approximation result in terms of versal objects.

Lemma 71.11.3. *Let S be a locally Noetherian scheme. Let $p : \mathcal{X} \to (Sch/S)_{fppf}$ be a category fibred in groupoids. Let $\xi = (R, \xi_n, f_n)$ be a formal object of $\mathcal{X}$ with ξ_1 lying over $\mathrm{Spec}(k) \to S$ with image $s \in S$. Assume*

1. *ξ is versal,*
2. *ξ is effective,*
3. *$\mathcal{O}_{S,s}$ is a G-ring, and*
4. *$p : \mathcal{X} \to (Sch/S)_{fppf}$ is limit preserving on objects.*

Then there exist a morphism of finite type $U \to S$, a finite type point $u_0 \in U$ with residue field k, and an object x of $\mathcal{X}$ over U such that x is versal at u_0 and such that $x|_{\mathrm{Spec}(k)} \cong \xi_1$.

Proof. Choose an object x_R of $\mathcal{X}$ lying over $\mathrm{Spec}(R)$ whose associated formal object is ξ. Let $N = 2$ and apply Lemma 71.10.1. We obtain $A, \mathfrak{m}_A, \xi_A, \ldots$. Let $\eta = (A^\wedge, \eta_n, g_n)$ be the formal object associated to $\xi_A|_{\mathrm{Spec}(A^\wedge)}$. We have a diagram

$$\begin{array}{ccc} \eta & & A^\wedge \\ \downarrow & \text{lying over} & \downarrow \\ \xi_2 = \eta_2 & & R/\mathfrak{m}_R^2 = A/\mathfrak{m}_A^2 \end{array}$$

The versality of ξ means exactly that we can find the dotted arrows in the diagrams, because we can successively find morphisms $\xi \to \eta_3$, $\xi \to \eta_4$, and so on by Formal Deformation Theory, Remark 64.8.14. The corresponding ring map $R \to A^\wedge$ is surjective by Formal Deformation Theory, Lemma 64.4.2. On the other hand, we have $\dim_k \mathfrak{m}_R^n/\mathfrak{m}_R^{n+1} = \dim_k \mathfrak{m}_A^n/\mathfrak{m}_A^{n+1}$ for all n by construction. Hence $R/\mathfrak{m}_R^n$ and $A/\mathfrak{m}_A^n$ have the same (finite) length as Λ-modules by additivity of length and Formal Deformation Theory, Lemma 64.3.4. It follows that $R/\mathfrak{m}_R^n \to A/\mathfrak{m}_A^n$ is an isomorphism for all n, hence $R \to A^\wedge$ is an isomorphism. Thus η is isomorphic to a versal object, hence versal itself. □

Example 71.11.4. In this example we show that the local ring $\mathcal{O}_{S,s}$ has to be a G-ring in order for the result of Lemma 71.11.3 to be true. Namely, let Λ be a Noetherian ring and let $\mathfrak{m}$ be a maximal ideal of Λ. Set $R = \Lambda_\mathfrak{m}^\wedge$. Let $\Lambda \to C \to R$ be a factorization with C of finite type over Λ. Set $S = \mathrm{Spec}(\Lambda)$, $U = S \setminus \{\mathfrak{m}\}$, and $S' = U \amalg \mathrm{Spec}(C)$. Consider the functor $F : (Sch/S)_{fppf}^{opp} \to Sets$ defined by the rule

$$F(T) = \begin{cases} * & \text{if } T \to S \text{ factors through } S' \\ \emptyset & \text{else} \end{cases}$$

Let $\mathcal{X} = \mathcal{S}_F$ is the category fibred in sets associated to F, see Algebraic Stacks, Section 67.7. Then $\mathcal{X} \to (Sch/S)_{fppf}$ is limit preserving on objects and there exists an effective, versal formal object ξ over R. Hence if the conclusion of Lemma 71.11.3 holds for $\mathcal{X}$, then there exists a finite type ring map $\Lambda \to A$ and a maximal ideal $\mathfrak{m}_A$ lying over $\mathfrak{m}$ such that

(1) $\kappa(\mathfrak{m}) = \kappa(\mathfrak{m}_A)$,
(2) $\Lambda \to A$ and $\mathfrak{m}_A$ satisfy condition (4) of Algebra, Lemma 10.134.2, and
(3) there exists a Λ-algebra map $C \to A$.

Thus $\Lambda \to A$ is smooth at $\mathfrak{m}_A$ by the lemma cited. Slicing A we may assume that $\Lambda \to A$ is étale at $\mathfrak{m}_A$, see for example More on Morphisms, Lemma 36.28.5 or argue directly. Write $C = \Lambda[y_1, \ldots, y_n]/(f_1, \ldots, f_m)$. Then $C \to R$ corresponds to a solution in R of the system of equations $f_1 = \ldots = f_m = 0$, see Smoothing Ring Maps, Section 16.13. Thus if the conclusion of Lemma 71.11.3 holds for every $\mathcal{X}$ as above, then a system of equations which has a solution in R has a solution in the henselization of $\Lambda_\mathfrak{m}$. In other words, the approximation property holds for $\Lambda_\mathfrak{m}^h$. This implies that $\Lambda_\mathfrak{m}^h$ is a G-ring (insert future reference here; see also discussion in Smoothing Ring Maps, Section 16.1) which in turn implies that $\Lambda_\mathfrak{m}$ is a G-ring.

71.12. Axioms

Let S be a locally Noetherian scheme. Let $p : \mathcal{X} \to (Sch/S)_{fppf}$ be a category fibred in groupoids. Here are the axioms we will consider on $\mathcal{X}$.

[-1] a set theoretic condition[1] to be ignored by readers who are not interested in set theoretical issues,
[0] $\mathcal{X}$ is a stack in groupoids for the étale topology,
[1] $\mathcal{X}$ is limit preserving,
[2] $\mathcal{X}$ satisfies the Rim-Schlessinger condition (RS),

[1]The condition is the following: the supremum of all the cardinalities $|\mathrm{Ob}(\mathcal{X}_{\mathrm{Spec}(k)})/\cong|$ and $|\mathrm{Arrows}(\mathcal{X}_{\mathrm{Spec}(k)})|$ where k runs over the finite type fields over S is $\leq$ than the size of some object of $(Sch/S)_{fppf}$.

[3] the spaces $T\mathcal{F}_{\mathcal{X},k,x_0}$ and $\mathrm{Inf}_{x_0}(\mathcal{F}_{\mathcal{X},k,x_0})$ are finite dimensional for every k and x_0, see (71.8.0.1) and (71.8.0.2),
[4] the functor (71.9.2.1) is an equivalence,
[5] $\mathcal{X}$ and $\Delta : \mathcal{X} \to \mathcal{X} \times \mathcal{X}$ satisfy openness of versality.

We still have to define the meaning of "limit preserving" and "openness of versality".

71.13. Limit preserving

The morphism $p : \mathcal{X} \to (Sch/S)_{fppf}$ is limit preserving on objects, as defined in Criteria for Representability, Section 70.5, if the functor of the definition below is essentially surjective. However, the example in Examples, Section 78.39 shows that this isn't equivalent to being limit preserving.

Definition 71.13.1. Let S be a scheme. Let $\mathcal{X}$ be a category fibred in groupoids over $(Sch/S)_{fppf}$. We say $\mathcal{X}$ is *limit preserving* if for every affine scheme T over S which is a limit $T = \lim T_i$ of a directed inverse system of affine schemes T_i over S, we have an equivalence

$$\operatorname{colim} \mathcal{X}_{T_i} \longrightarrow \mathcal{X}_T$$

of fibre categories.

We spell out what this means. First, given objects x, y of $\mathcal{X}$ over T_i we should have

$$\mathrm{Mor}_{\mathcal{X}_T}(x|_T, y|_T) = \mathrm{colim}_{i' \geq i} \mathrm{Mor}_{\mathcal{X}_{T_i'}}(x|_{T_i'}, y|_{T_i'})$$

and second every object of $\mathcal{X}_T$ is isomorphic to the restriction of an object over T_i for some i. Note that the first condition means that the presheaves $Isom_{\mathcal{X}}(x, y)$ (see Stacks, Definition 8.2.2) are limit preserving.

Lemma 71.13.2. *Let S be a scheme. Let $p : \mathcal{X} \to \mathcal{Y}$ and $q : \mathcal{Z} \to \mathcal{Y}$ be 1-morphisms of categories fibred in groupoids over $(Sch/S)_{fppf}$.*

(1) *If $\mathcal{X} \to (Sch/S)_{fppf}$ and $\mathcal{Z} \to (Sch/S)_{fppf}$ are limit preserving on objects and $\mathcal{Y}$ is limit preserving, then $\mathcal{X} \times_{\mathcal{Y}} \mathcal{Z} \to (Sch/S)_{fppf}$ is limit preserving on objects.*
(2) *If $\mathcal{X}$, $\mathcal{Y}$, and $\mathcal{Z}$ are limit preserving, then so is $\mathcal{X} \times_{\mathcal{Y}} \mathcal{Z}$.*

Proof. This is formal. Proof of (1). Let $T = \lim_{i \in I} T_i$ be the directed limit of affine schemes T_i over S. We will prove that the functor $\operatorname{colim} \mathcal{X}_{T_i} \to \mathcal{X}_T$ is essentially surjective. Recall that an object of the fibre product over T is a quadruple (T, x, z, α) where x is an object of $\mathcal{X}$ lying over T, z is an object of $\mathcal{Z}$ lying over T, and $\alpha : p(x) \to q(z)$ is a morphism in the fibre category of $\mathcal{Y}$ over T. By assumption on $\mathcal{X}$ and $\mathcal{Z}$ we can find an i and objects x_i and z_i over T_i such that $x_i|_T \cong T$ and $z_i|_T \cong z$. Then α corresponds to an isomorphism $p(x_i)|_T \to q(z_i)|_T$ which comes from an isomorphism $\alpha_{i'} : p(x_i)|_{T_{i'}} \to q(z_i)|_{T_{i'}}$ by our assumption on $\mathcal{Y}$. After replacing i by i', x_i by $x_i|_{T_{i'}}$, and z_i by $z_i|_{T_{i'}}$ we see that $(T_i, x_i, z_i, \alpha_i)$ is an object of the fibre product over T_i which restricts to an object isomorphic to (T, x, z, α) over T as desired.

We omit the arguments showing that $\operatorname{colim} \mathcal{X}_{T_i} \to \mathcal{X}_T$ is fully faithful in (2). □

Lemma 71.13.3. *Let S be a scheme. Let $\mathcal{X}$ be an algebraic stack over S. Then the following are equivalent*

(1) *$\mathcal{X}$ is a stack in setoids and $\mathcal{X} \to (Sch/S)_{fppf}$ is limit preserving on objects,*
(2) *$\mathcal{X}$ is a stack in setoids and limit preserving,*

(3) $\mathcal{X}$ *is representable by an algebraic space locally of finite presentation.*

Proof. Under each of the three assumptions $\mathcal{X}$ is representable by an algebraic space X over S, see Algebraic Stacks, Proposition 67.13.3. It is clear that (1) and (2) are equivalent as a functor between setoids is an equivalence if and only if it is surjective on isomorphism classes. Finally, (1) and (3) are equivalent by Limits of Spaces, Proposition 52.3.9. □

71.14. Openness of versality

Next, we come to openness of versality.

Definition 71.14.1. Let S be a locally Noetherian scheme.

(1) Let $\mathcal{X}$ be a category fibred in groupoids over $(Sch/S)_{fppf}$. We say $\mathcal{X}$ satisfies *openness of versality* if given a scheme U locally of finite type over S, an object x of $\mathcal{X}$ over U, and a finite type point $u_0 \in U$ such that x is versal at u_0, then there exists an open neighbourhood $u_0 \in U' \subset U$ such that x is versal at every finite type point of U'.

(2) Let $f : \mathcal{Y} \to \mathcal{X}$ be a 1-morphism of categories fibred in groupoids. We say f satisfies *openness of versality* if given a morphism of schemes $V \to U$ locally of finite type over S, an object x of $\mathcal{X}$ over U, an object y of $\mathcal{Y}$ over V, a morphism $\alpha : f(y) \to x$ of $\mathcal{X}$ over $V \to U$, and a finite type point v_0 of V such that (y, α) is versal at v_0 as an object of $\mathcal{Y} \times_{\mathcal{X}} (Sch/U)_{fppf}$, then there exists an open neighbourhood $v_0 \in V' \subset V$ such that (y, α) is versal at every finite type point of V'.

Openness of versality is often the hardest to check. The following example shows that requiring this is necessary however.

Example 71.14.2. Let k be a field and set $\Lambda = k[s, t]$. Consider the functor $F : \Lambda\text{-algebras} \longrightarrow Sets$ defined by the rule

$$F(A) = \begin{cases} * & \text{if there exist } f_1, \ldots, f_n \in A \text{ such that} \\ & \quad A = (s, t, f_1, \ldots, f_n) \text{ and } f_i s = 0 \ \forall i \\ \emptyset & \text{else} \end{cases}$$

Geometrically $F(A) = *$ means there exists a quasi-compact open neighbourhood W of $V(s, t) \subset \mathrm{Spec}(A)$ such that $s|_W = 0$. Let $\mathcal{X} \subset (Sch/\mathrm{Spec}(\Lambda))_{fppf}$ be the full subcategory consisting of schemes T which have an affine open covering $T = \bigcup \mathrm{Spec}(A_j)$ with $F(A_j) = *$ for all j. Then $\mathcal{X}$ satisfies [0], [1], [2], [3], and [4] but not [5]. Namely, over $U = \mathrm{Spec}(k[s, t]/(s))$ there exists an object x which is versal at $u_0 = (s, t)$ but not at any other point. Details omitted.

Let S be a locally Noetherian scheme. Let $f : \mathcal{X} \to \mathcal{Y}$ be a 1-morphism of categories fibred in groupoids over $(Sch/S)_{fppf}$. In the following two lemmas we will use the following property

(71.14.2.1) for all fields k of finite type over S and all $x_0 \in \mathrm{Ob}(\mathcal{X}_{\mathrm{Spec}(k)})$ the map $\mathcal{F}_{\mathcal{X},k,x_0} \to \mathcal{F}_{\mathcal{Y},k,f(x_0)}$ of predeformation categories is smooth

We formulate some lemmas around this concept. First we link it with (openness of) versality.

Lemma 71.14.3. *Let S be a locally Noetherian scheme. Let $\mathcal{X}$ be a category fibred in groupoids over $(Sch/S)_{fppf}$. Let U be a scheme locally of finite type over S. Let x be an object of $\mathcal{X}$ over U. Assume that x is versal at every finite type point of U and that $\mathcal{X}$ satisfies (RS). Then $x : (Sch/U)_{fppf} \to \mathcal{X}$ satisfies (71.14.2.1).*

Proof. Let $\mathrm{Spec}(l) \to U$ be a morphism with l of finite type over S. Then the image $u_0 \in U$ is a finite type point of U and $\kappa(u_0) \subset l$ is a finite extension, see discussion in Morphisms, Section 28.17. Hence we see that $\mathcal{F}_{\mathcal{X},k,u_{l,0}} \to \mathcal{F}_{\mathcal{Y},k,x_{l,0}}$ is smooth by Lemma 71.11.2. □

Lemma 71.14.4. *Let S be a locally Noetherian scheme. Let $f : \mathcal{X} \to \mathcal{Y}$ and $g : \mathcal{Y} \to \mathcal{Z}$ be composable 1-morphisms of categories fibred in groupoids over $(Sch/S)_{fppf}$. If f and g satisfy (71.14.2.1) so does $g \circ f$.*

Proof. This is formal. □

Lemma 71.14.5. *Let S be a locally Noetherian scheme. Let $f : \mathcal{X} \to \mathcal{Y}$ and $\mathcal{Z} \to \mathcal{Y}$ be 1-morphisms of categories fibred in groupoids over $(Sch/S)_{fppf}$. If f satisfies (71.14.2.1) so does the projection $\mathcal{X} \times_{\mathcal{Y}} \mathcal{Z} \to \mathcal{Z}$.*

Proof. Follows immediately from Lemma 71.3.3 and Formal Deformation Theory, Lemma 64.8.7. □

Lemma 71.14.6. *Let S be a locally Noetherian scheme. Let $f : \mathcal{X} \to \mathcal{Y}$ be a 1-morphisms of categories fibred in groupoids over $(Sch/S)_{fppf}$. If f is formally smooth on objects, then f satisfies (71.14.2.1). If f is representable by algebraic spaces and smooth, then f satisfies (71.14.2.1).*

Proof. The first statement is equivalent to Lemma 71.3.2. The second follows from this and Criteria for Representability, Lemma 70.6.3. □

Lemma 71.14.7. *Let S be a locally Noetherian scheme. Let $f : \mathcal{X} \to \mathcal{Y}$ be a 1-morphism of categories fibred in groupoids over $(Sch/S)_{fppf}$. Assume*

1. *f is representable by algebraic spaces,*
2. *f satisfies (71.14.2.1),*
3. *$\mathcal{X} \to (Sch/S)_{fppf}$ is limit preserving on objects, and*
4. *$\mathcal{Y}$ is limit preserving.*

Then f is smooth.

Proof. The key ingredient of the proof is More on Morphisms, Lemma 36.10.1 which (almost) says that a morphism of schemes of finite type over S satisfying (71.14.2.1) is a smooth morphism. The other arguments of the proof are essentially bookkeeping.

Let V be a scheme over S and let y be an object of $\mathcal{Y}$ over V. Let Z be an algebraic space representing the 2-fibre product $\mathcal{Z} = \mathcal{X} \times_{f,\mathcal{X},y} (Sch/V)_{fppf}$. We have to show that the projection morphism $Z \to V$ is smooth, see Algebraic Stacks, Definition 67.10.1. In fact, it suffices to do this when V is an affine scheme locally of finite presentation over S, see Criteria for Representability, Lemma 70.5.6. Then $(Sch/V)_{fppf}$ is limit preserving by Lemma 71.13.3. Hence $Z \to S$ is locally of finite presentation by Lemmas 71.13.2 and 71.13.3. Choose a scheme W and a surjective étale morphism $W \to Z$. Then W is locally of finite presentation over S.

Since f satisfies (71.14.2.1) we see that so does $\mathcal{Z} \to (Sch/V)_{fppf}$, see Lemma 71.14.5. Next, we see that $(Sch/W)_{fppf} \to \mathcal{Z}$ satisfies (71.14.2.1) by Lemma 71.14.6. Thus the composition

$$(Sch/W)_{fppf} \to \mathcal{Z} \to (Sch/V)_{fppf}$$

satisfies (71.14.2.1) by Lemma 71.14.4. More on Morphisms, Lemma 36.10.1 shows that the composition $W \to Z \to V$ is smooth at every finite type point w_0 of W. Since the smooth locus is open we conclude that $W \to V$ is a smooth morphism of schemes by Morphisms, Lemma 28.17.7. Thus we conclude that $Z \to V$ is a smooth morphism of algebraic spaces by definition. □

The lemma below is how we will use openness of versality.

Lemma 71.14.8. *Let S be a locally Noetherian scheme. Let $p : \mathcal{X} \to (Sch/S)_{fppf}$ be a category fibred in groupoids. Let k be a finite type field over S and let x_0 be an object of $\mathcal{X}$ over $\mathrm{Spec}(k)$ with image $s \in S$. Assume*

(1) *$\Delta : \mathcal{X} \to \mathcal{X} \times \mathcal{X}$ is representable by algebraic spaces,*
(2) *$\mathcal{X}$ satisfies axioms [1], [2], [3], [4], and openness of versality, and*
(3) *$\mathcal{O}_{S,s}$ is a G-ring.*

Then there exist a morphism of finite type $U \to S$ and an object x of $\mathcal{X}$ over U such that

$$x : (Sch/U)_{fppf} \longrightarrow \mathcal{X}$$

is smooth and such that there exists a finite type point $u_0 \in U$ whose residue field is k and such that $x|_{u_0} \cong x_0$.

Proof. By axiom [2], Lemma 71.6.1, and Remark 71.6.2 we see that $\mathcal{F}_{\mathcal{X},k,x_0}$ satisfies (S1) and (S2). Since also the tangent space has finite dimension by axiom [3] we deduce from Formal Deformation Theory, Lemma 64.12.4 that $\mathcal{F}_{\mathcal{X},k,x_0}$ has a versal formal object ξ. By axiom [4] ξ is effective. By axiom [1] and Lemma 71.11.3 there exists a morphism of finite type $U \to S$, an object x of $\mathcal{X}$ over U, and a finite type point u_0 of U with residue field k such that x is versal at u_0 and such that $x|_{\mathrm{Spec}(k)} \cong x_0$. By openness of versality we may shrink U and assume that x is versal at every finite type point u_0 of U. We claim that

$$x : (Sch/U)_{fppf} \longrightarrow \mathcal{X}$$

is smooth which proves the lemma. Namely, by Lemma 71.14.3 x satisfies (71.14.2.1) whereupon Lemma 71.14.7 finishes the proof. □

71.15. Axioms for functors

Let S be a scheme. Let $F : (Sch/S)^{opp}_{fppf} \to Sets$ be a functor. Denote $\mathcal{X} = \mathcal{S}_F$ the category fibred in sets associated to F, see Algebraic Stacks, Section 67.7. In this section we provide a translation between the material above as it applies to $\mathcal{X}$, to statements about F.

Let S be a locally Noetherian scheme. Let $F : (Sch/S)^{opp}_{fppf} \to Sets$ be a functor. Let k be a field of finite type over S. Let $x_0 \in F(\mathrm{Spec}(k))$. The associated predeformation category (71.3.0.2) corresponds to the functor

$$F_{k,x_0} : \mathcal{C}_\Lambda \longrightarrow Sets, \quad A \longmapsto \{x \in F(\mathrm{Spec}(A)) \mid x|_{\mathrm{Spec}(k)} = x_0\}.$$

Recall that we do not distinguish between categories cofibred in sets over $\mathcal{C}_\Lambda$ and functor $\mathcal{C}_\Lambda \to \mathit{Sets}$, see Formal Deformation Theory, Remarks 64.5.2 (11). Given a transformation of functors $a : F \to G$, setting $y_0 = a(x_0)$ we obtain a morphism

$$F_{k,x_0} \longrightarrow G_{k,y_0}$$

see (71.3.1.1). Lemma 71.3.2 tells us that if $a : F \to G$ is formally smooth (in the sense of More on Morphisms of Spaces, Definition 57.10.1), then $F_{k,x_0} \longrightarrow G_{k,y_0}$ is smooth as in Formal Deformation Theory, Remark 64.8.4.

Lemma 71.4.1 says that if $Y' = Y \amalg_X X'$ in the category of schemes over S where $X \to X'$ is a thickening and $X \to Y$ is affine, then the map

$$F(Y \amalg_X X') \to F(Y) \times_{F(X)} F(X')$$

is a bijection, provided that F is an algebraic space. We say a general functor F satisfies the *Rim-Schlessinger condition* or we say F *satisfies (RS)* if given any pushout $Y' = Y \amalg_X X'$ where Y, X, X' are spectra of Artinian local rings of finite type over S, then

$$F(Y \amalg_X X') \to F(Y) \times_{F(X)} F(X')$$

is a bijection. Thus every algebraic space satisfies (RS).

Lemma 71.6.1 says that given a functor F which satisfies (RS), then all F_{k,x_0} are deformation functors as in Formal Deformation Theory, Definition 64.15.8, i.e., they satisfy (RS) as in Formal Deformation Theory, Remark 64.15.5. In particular the tangent space

$$TF_{k,x_0} = \{x \in F(\mathrm{Spec}(k[\epsilon])) \mid x|_{\mathrm{Spec}(k)} = x_0\}$$

has the structure of a k-vector space by Formal Deformation Theory, Lemma 64.11.2.

Lemma 71.8.1 says that an algebraic space F locally of finite type over S gives rise to deformation functors F_{k,x_0} with finite dimensional tangent spaces TF_{k,x_0}.

A *formal object*[2] $\xi = (R, \xi_n)$ of F consists of a Noetherian complete local S-algebra R whose residue field is of finite type over S, together with elements $\xi_n \in F(\mathrm{Spec}(R/\mathfrak{m}^n))$ such that $\xi_{n+1}|_{\mathrm{Spec}(R/\mathfrak{m}^n)} = \xi_n$. A formal object ξ defines a formal object ξ of $F_{R/\mathfrak{m},\xi_1}$. We say ξ is *versal* if and only if it is versal in the sense of Formal Deformation Theory, Definition 64.8.13. A formal object $\xi = (R, \xi_n)$ is called *effective* if there exists an $x \in F(\mathrm{Spec}(R))$ such that $\xi_n = x|_{\mathrm{Spec}(R/\mathfrak{m}^n)}$ for all $n \geq 1$. Lemma 71.9.4 says that if F is an algebraic space, then every formal object is effective.

Let U be a scheme locally of finite type over S and let $x \in F(U)$. Let $u_0 \in U$ be a finite type point. We say that x is versal at u_0 if and only if $\xi = (\mathcal{O}^\wedge_{U,u_0}, x|_{\mathrm{Spec}(\mathcal{O}_{U,u_0}/\mathfrak{m}^n_{u_0})})$ is a versal formal object in the sense described above.

Let S be a locally Noetherian scheme. Let $F : (\mathit{Sch}/S)^{opp}_{fppf} \to \mathit{Sch}$ be a functor. Here are the axioms we will consider on F.

- [-1] a set theoretic condition[3] to be ignored by readers who are not interested in set theoretical issues,
- [0] F is a sheaf for the étale topology,

[2] This is what Artin calls a formal deformation.

[3] The condition is the following: the supremum of all the cardinalities $|F(\mathrm{Spec}(k))|$ where k runs over the finite type fields over S is $\leq$ than the size of some object of $(\mathit{Sch}/S)_{fppf}$.

[1] F is limit preserving,
[2] F satisfies the Rim-Schlessinger condition (RS),
[3] every tangent space TF_{k,x_0} is finite dimensional,
[4] every formal object is effective,
[5] F satisfies openness of versality.

Here *limit preserving* is the notion defined in Limits of Spaces, Definition 52.3.1 and *openness of versality* means the following: Given a scheme U locally of finite type over S, given $x \in F(U)$, and given a finite type point $u_0 \in U$ such that x is versal at u_0, then there exists an open neighbourhood $u_0 \in U' \subset U$ such that x is versal at every finite type point of U'.

71.16. Algebraic spaces

The following is our first main result on algebraic spaces.

Proposition 71.16.1. *Let S be a locally Noetherian scheme. Let $F : (Sch/S)^{opp}_{fppf} \to Sets$ be a functor. Assume that*

(1) *$\Delta : F \to F \times F$ is representable by algebraic spaces,*
(2) *F satisfies axioms [-1], [0], [1], [2], [3], [4], and [5],*
(3) *$\mathcal{O}_{S,s}$ is a G-ring for all finite type points s of S.*

Then F is an algebraic space.

Proof. Lemma 71.14.8 applies to F. Using this we choose, for every finite type field k over S and $x_0 \in F(\mathrm{Spec}(k))$, an affine scheme U_{k,x_0} of finite type over S and a smooth morphism $U_{k,x_0} \to F$ such that there exists a finite type point $u_{k,x_0} \in U_{k,x_0}$ with residue field k such that x_0 is the image of u_{k,x_0}. Then

$$U = \coprod\nolimits_{k,x_0} U_{k,x_0} \longrightarrow F$$

is smooth[4]. To finish the proof it suffices to show this map is surjective, see Bootstrap, Lemma 61.12.2 (this is where we use axiom [0]). By Criteria for Representability, Lemma 70.5.6 it suffices to show that $U \times_F V \to V$ is surjective for those $V \to F$ where V is an affine scheme locally of finite presentation over S. Since $U \times_F V \to V$ is smooth the image is open. Hence it suffices to show that the image of $U \times_F V \to V$ contains all finite type points of V, see Morphisms, Lemma 28.17.7. Let $v_0 \in V$ be a finite type point. Then $k = \kappa(v_0)$ is a finite type field over S. Denote x_0 the composition $\mathrm{Spec}(k) \xrightarrow{v_0} V \to F$. Then $(u_{k,x_0}, v_0) : \mathrm{Spec}(k) \to U \times_F V$ is a point mapping to v_0 and we win. □

Lemma 71.16.2. *Let S be a locally Noetherian scheme. Let $a : F \to G$ be a transformation of functors $F, G : (Sch/S)^{opp}_{fppf} \to Sets$. Assume that*

(1) *a is injective,*
(2) *F satisfies axioms [0], [1], [2], [4], and [5],*
(3) *$\mathcal{O}_{S,s}$ is a G-ring for all finite type points s of S,*
(4) *G is an algebraic space locally of finite type over S,*

Then F is an algebraic space.

[4]Set theoretical remark: This coproduct is (isomorphic) to an object of $(Sch/S)_{fppf}$ as we have a bound on the index set by axiom [-1], see Sets, Lemma 3.9.9.

Proof. By Lemma 71.8.1 the functor G satisfies [3]. As $F \to G$ is injective, we conclude that F also satisfies [3]. Moreover, as $F \to G$ is injective, we see that given schemes U, V and morphisms $U \to F$ and $V \to F$, then $U \times_F V = U \times_G V$. Hence $\Delta : F \to F \times F$ is representable (by schemes) as this holds for G by assumption. Thus Proposition 71.16.1 applies[5]. □

71.17. Algebraic stacks

Proposition 71.17.2 is our first main result on algebraic stacks.

Lemma 71.17.1. *Let S be a locally Noetherian scheme. Let $p : \mathcal{X} \to (Sch/S)^{opp}_{fppf}$ be a category fibred in groupoids. Assume that*

(1) *$\Delta : \mathcal{X} \to \mathcal{X} \times \mathcal{X}$ is representable by algebraic spaces,*
(2) *$\mathcal{X}$ satisfies axioms [-1], [0], [1], [2], [3], [4] (see Section 71.12),*
(3) *$\mathcal{X}$ satisfies openness of versality, and*
(4) *$\mathcal{O}_{S,s}$ is a G-ring for all finite type points s of S.*

Then $\mathcal{X}$ is an algebraic stack.

Proof. Lemma 71.14.8 applies to $\mathcal{X}$. Using this we choose, for every finite type field k over S and every isomorphism class of object $x_0 \in \mathrm{Ob}(\mathcal{X}_{\mathrm{Spec}(k)})$, an affine scheme U_{k,x_0} of finite type over S and a smooth morphism $(Sch/U_{k,x_0})_{fppf} \to \mathcal{X}$ such that there exists a finite type point $u_{k,x_0} \in U_{k,x_0}$ with residue field k such that x_0 is the image of u_{k,x_0}. Then

$$(Sch/U)_{fppf} \to \mathcal{X}, \quad \text{with} \quad U = \coprod\nolimits_{k,x_0} U_{k,x_0}$$

is smooth[6]. To finish the proof it suffices to show this map is surjective, see Criteria for Representability, Lemma 70.19.1 (this is where we use axiom [0]). By Criteria for Representability, Lemma 70.5.6 it suffices to show that $(Sch/U)_{fppf} \times_{\mathcal{X}} (Sch/V)_{fppf} \to (Sch/V)_{fppf}$ is surjective for those $y : (Sch/V)_{fppf} \to \mathcal{X}$ where V is an affine scheme locally of finite presentation over S. By assumption (1) the fibre product $(Sch/U)_{fppf} \times_{\mathcal{X}} (Sch/V)_{fppf}$ is representable by an algebraic space W. Then $W \to V$ is smooth, hence the image is open. Hence it suffices to show that the image of $W \to V$ contains all finite type points of V, see Morphisms, Lemma 28.17.7. Let $v_0 \in V$ be a finite type point. Then $k = \kappa(v_0)$ is a finite type field over S. Denote $x_0 = y|_{\mathrm{Spec}(k)}$ the pullback of y by v_0. Then (u_{k,x_0}, v_0) will give a morphism $\mathrm{Spec}(k) \to W$ whose composition with $W \to V$ is v_0 and we win. □

Proposition 71.17.2. *Let S be a locally Noetherian scheme. Let $p : \mathcal{X} \to (Sch/S)^{opp}_{fppf}$ be a category fibred in groupoids. Assume that*

(1) *$\Delta_\Delta : \mathcal{X} \to \mathcal{X} \times_{\mathcal{X} \times \mathcal{X}} \mathcal{X}$ is representable by algebraic spaces,*
(2) *$\mathcal{X}$ satisfies axioms [-1], [0], [1], [2], [3], [4], and [5] (see Section 71.12),*
(3) *$\mathcal{O}_{S,s}$ is a G-ring for all finite type points s of S.*

Then $\mathcal{X}$ is an algebraic stack.

Proof. We first prove that $\Delta : \mathcal{X} \to \mathcal{X} \times \mathcal{X}$ is representable by algebraic spaces. To do this it suffices to show that

$$\mathcal{Y} = \mathcal{X} \times_{\Delta, \mathcal{X} \times \mathcal{X}, y} (Sch/V)_{fppf}$$

[5] The set theoretic condition [-1] holds for F as it holds for G. Details omitted.

[6] Set theoretical remark: This coproduct is (isomorphic) to an object of $(Sch/S)_{fppf}$ as we have a bound on the index set by axiom [-1], see Sets, Lemma 3.9.9.

is representable by an algebraic space for any affine scheme V locally of finite presentation over S and object y of $\mathcal{X} \times \mathcal{X}$ over V, see Criteria for Representability, Lemma 70.5.5[7]. Observe that $\mathcal{Y}$ is fibred in setoids (Stacks, Lemma 8.2.5) and let $Y : (Sch/S)^{opp}_{fppf} \to Sets$, $T \mapsto \mathrm{Ob}(\mathcal{Y}_T)/\cong$ be the functor of isomorphism classes. We will apply Proposition 71.16.1 to see that Y is an algebraic space.

Note that $\Delta_{\mathcal{Y}} : \mathcal{Y} \to \mathcal{Y} \times \mathcal{Y}$ (and hence also $Y \to Y \times Y$) is representably by algebraic spaces by condition (1) and Criteria for Representability, Lemma 70.4.4. Observe that Y is a sheaf for the étale topology by Stacks, Lemmas 8.6.3 and 8.6.7, i.e., property [0] holds. Also Y is limit preserving by Lemma 71.13.2, i.e., we have [1]. Note that Y has (RS), i.e., axiom [2] holds, by Lemmas 71.5.2 and 71.5.3. Axiom [3] for Y follows from Lemmas 71.8.1 and 71.8.2. Axiom [4] follows from Lemmas 71.9.4 and 71.9.5. Axiom [5] for Y follows directly from openness of versality for $\Delta_{\mathcal{X}}$ which is part of axiom [5] for $\mathcal{X}$. Thus all the assumptions of Proposition 71.16.1 are satisfied and Y is an algebraic space.

At this point it follows from Lemma 71.17.1 that $\mathcal{X}$ is an algebraic stack. □

71.18. Infinitesimal deformations

In this section we discuss a generalization of the notion of the tangent space introduced in Section 71.8. To do this intelligently, we borrow some notation from Formal Deformation Theory, Sections 64.10, 64.16, and 64.18.

Let S be a scheme. Let $\mathcal{X}$ be a category fibred in groupoids over $(Sch/S)_{fppf}$. Given a homomorphism $A' \to A$ of S-algebras and an object x of $\mathcal{X}$ over $\mathrm{Spec}(A)$ we write $Lift(x, A')$ for the category of lifts of x to $\mathrm{Spec}(A')$. An object of $Lift(x, A')$ is a morphism $x \to x'$ of $\mathcal{X}$ lying over $\mathrm{Spec}(A) \to \mathrm{Spec}(A')$ and morphisms of $Lift(x, A')$ are defined as commutative diagrams. The set of isomorphism classes of $Lift(x, A')$ is denoted $\mathrm{Lift}(x, A')$. See Formal Deformation Theory, Definition 64.16.1 and Remark 64.16.2. If $A' \to A$ is surjective with locally nilpotent kernel we call an element x' of $\mathrm{Lift}(x, A')$ a *(infinitesimal) deformation* of x. In this case the *group of infinitesimal automorphisms of x' over x* is the kernel

$$\mathrm{Inf}(x'/x) = \mathrm{Ker}\left(\mathrm{Aut}_{\mathcal{X}_{\mathrm{Spec}(A')}}(x') \to \mathrm{Aut}_{\mathcal{X}_{\mathrm{Spec}(A)}}(x)\right)$$

Note that an element of $\mathrm{Inf}(x'/x)$ is the same thing as a lift of id_x over $\mathrm{Spec}(A')$ for (the category fibred in sets associated to) $Aut_{\mathcal{X}}(x')$. Compare with Formal Deformation Theory, Definition 64.18.1 and Formal Deformation Theory, Remark 64.18.8.

If M is an A-module we denote $A[M]$ the A-algebra whose underlying A-module is $A \oplus M$ and whose multiplication is given by $(a, m) \cdot (a', m') = (aa', am' + a'm)$. When $M = A$ this is the ring of dual numbers over A, which we denote $A[\epsilon]$ as is customary. There is an A-algebra map $A[M] \to A$. The pullback of x to $\mathrm{Spec}(A[M])$ is called the *trivial deformation* of x to $\mathrm{Spec}(A[M])$.

[7]The set theoretic condition in Criteria for Representability, Lemma 70.5.5 will hold: the size of the algebraic space Y representing $\mathcal{Y}$ is suitably bounded. Namely, $Y \to S$ will be locally of finite type and Y will satisfy axiom [-1]. Details omitted.

Lemma 71.18.1. *Let S be a scheme. Let $f : \mathcal{X} \to \mathcal{Y}$ be a 1-morphism of categories fibred in groupoids over $(Sch/S)_{fppf}$. Let*

$$\begin{array}{ccc} B' & \longrightarrow & B \\ \uparrow & & \uparrow \\ A' & \longrightarrow & A \end{array}$$

be a commutative diagram of S-algebras. Let x be an object of $\mathcal{X}$ over $\mathrm{Spec}(A)$, let y be an object of $\mathcal{Y}$ over $\mathrm{Spec}(B)$, and let $\phi : f(x)|_{\mathrm{Spec}(B)} \to y$ be a morphism of $\mathcal{Y}$ over $\mathrm{Spec}(B)$. Then there is a canonical functor

$$Lift(x, A') \longrightarrow Lift(y, B')$$

of categories of lifts induced by f and ϕ. The construction is compatible with compositions of 1-morphisms of categories fibred in groupoids in an obvious manner.

Proof. This lemma proves itself. □

In the rest of this chapter the following strictly stronger version of the Rim-Schlessinger conditions will play an important role.

Definition 71.18.2. Let S be a locally Noetherian scheme. Let $\mathcal{X}$ be a category fibred in groupoids over $(Sch/S)_{fppf}$. We say $\mathcal{X}$ satisfies *condition (RS*)* if given an affine open $\mathrm{Spec}(\Lambda) \subset S$ and a fibre product diagram

$$\begin{array}{ccc} B' & \longrightarrow & B \\ \uparrow & & \uparrow \\ A' = A \times_B B' & \longrightarrow & A \end{array}$$

of Λ-algebras, with $B' \to B$ surjective with square zero kernel, the functor of fibre categories

$$\mathcal{X}_{\mathrm{Spec}(A')} \longrightarrow \mathcal{X}_{\mathrm{Spec}(A)} \times_{\mathcal{X}_{\mathrm{Spec}(B)}} \mathcal{X}_{\mathrm{Spec}(B')}$$

is an equivalence of categories.

We make some observations: with $A \to B \leftarrow B'$ as in Definition 71.18.2

(1) if A, B, B' are of finite type over Λ and B is finite over A, then A' is of finite type over Λ, see More on Algebra, Lemma 15.4.1,
(2) we have $\mathrm{Spec}(A') = \mathrm{Spec}(A) \amalg_{\mathrm{Spec}(B)} \mathrm{Spec}(B')$ in the category of schemes, see More on Morphisms, Lemma 36.11.1,
(3) if $\mathcal{X}$ is an algebraic stack, then $\mathcal{X}$ satisfies (RS*) by Lemma 71.4.1, and
(4) if $\mathcal{X}$ satisfies (RS*), then $\mathcal{X}$ satisfies (RS) because (RS) covers exactly those cases of (RS*) where A, B, B' are Artinian local.

Let S be a locally Noetherian base. Let $\mathcal{X}$ be a category fibred in groupoids over $(Sch/S)_{fppf}$. We define a category whose objects are pairs $(x, A' \to A)$ where

(1) $A' \to A$ is a surjection of S-algebras whose kernel is an ideal of square zero such that $\mathrm{Spec}(A)$ maps into an affine open of S, and
(2) x is an object of $\mathcal{X}$ lying over $\mathrm{Spec}(A)$.

A morphism $(y, B' \to B) \to (x, A' \to A)$ is given by a commutative diagram

$$\begin{array}{ccc} B' & \longrightarrow & B \\ \uparrow & & \uparrow \\ A' & \longrightarrow & A \end{array}$$

of S-algebras together with a morphism $x|_{\mathrm{Spec}(B)} \to y$ over $\mathrm{Spec}(B)$. Let us call this the category of *deformation situations*.

Lemma 71.18.3. *Let S be a locally Noetherian scheme. Let $\mathcal{X}$ be a category fibred in groupoids over $(Sch/S)_{fppf}$. Assume $\mathcal{X}$ satisfies condition (RS*). Let A be an S-algebra such that $\mathrm{Spec}(A) \to S$ maps into an affine open and let x be an object of $\mathcal{X}$ over $\mathrm{Spec}(A)$.*

1. *There exists an A-linear functor $Inf_x : Mod_A \to Mod_A$ such that given a deformation situation $(x, A' \to A)$ and a lift x' there is an isomorphism $Inf_x(I) \to Inf(x'/x)$ where $I = Ker(A' \to A)$.*
2. *There exists an A-linear functor $T_x : Mod_A \to Mod_A$ such that*
   (a) *given M in Mod_A there is a bijection $T_x(M) \to Lift(x, A[M])$,*
   (b) *given a deformation situation $(x, A' \to A)$ there is an action*

$$T_x(I) \times Lift(x, A') \to Lift(x, A')$$

where $I = Ker(A' \to A)$. It is simply transitive if $Lift(x, A') \neq \emptyset$.

Proof. To define Inf_x, resp. T_x we consider the functors

$$\mathrm{Mod}_A \longrightarrow Sets, \quad M \longrightarrow \mathrm{Lift}(\mathrm{id}_x, A[M]), \quad \text{resp.} \quad M \longrightarrow \mathrm{Lift}(x, A[M])$$

(for the first consider lifts of id_x as automorphisms of the trivial deformation of x to $A[M]$) and we apply Formal Deformation Theory, Lemma 64.10.4. This lemma is applicable, since (RS*) tells us that

$$Lift(x, A[M \times N]) = Lift(x, A[M]) \times Lift(x, A[N])$$

as categories (and trivial deformations match up too).

Let $(x, A' \to A)$ be a deformation situation. Consider the ring map $g : A' \times_A A' \to A[I]$ defined by the rule $g(a_1, a_2) = \overline{a_1} \oplus a_2 - a_1$. There is an isomorphism

$$A' \times_A A' \longrightarrow A' \times_A A[I]$$

given by $(a_1, a_2) \mapsto (a_1, g(a_1, a_2))$. This isomorphism commutes with the projections to A' on the first factor, and hence with the projections to A. Thus applying (RS*) twice we find equivalences of categories

$$\begin{aligned} Lift(x, A') \times Lift(x, A') &= Lift(x, A' \times_A A') \\ &= Lift(x, A' \times_A A[I]) \\ &= Lift(x, A') \times Lift(x, A[I]) \end{aligned}$$

Using these maps and projection onto the last factor of the last product we see that we obtain "difference maps"

$$\mathrm{Inf}(x'/x) \times \mathrm{Inf}(x'/x) \longrightarrow \mathrm{Inf}_x(I) \quad \text{and} \quad \mathrm{Lift}(x, A') \times \mathrm{Lift}(x, A') \longrightarrow T_x(I)$$

These difference maps satisfy the transitivity rule "$(x'_1 - x'_2) + (x'_2 - x'_3) = x'_1 - x'_3$" because

$$\begin{array}{ccc} A' \times_A A' \times_A A' & \xrightarrow{(a_1,a_2,a_3)\mapsto(g(a_1,a_2),g(a_2,a_3))} & A[I] \times_A A[I] = A[I \times I] \\ & \searrow^{(a_1,a_2,a_3)\mapsto g(a_1,a_3)} & \downarrow + \\ & & A[I] \end{array}$$

is commutative. Inverting the string of equivalences above we obtain an action which is free and transitive provided $\mathrm{Inf}(x'/x)$, resp. $\mathrm{Lift}(x, A')$ is nonempty. Note that $\mathrm{Inf}(x'/x)$ is always nonempty as it is a group. □

Remark 71.18.4 (Functoriality)**.** Assumptions and notation as in Lemma 71.18.3. Suppose $A \to B$ is a ring map and $y = x|_{\mathrm{Spec}(B)}$. Let $M \in \mathrm{Mod}_A$, $N \in \mathrm{Mod}_B$ and let $M \to N$ an A-linear map. Then there are canonical maps $\mathrm{Inf}_x(M) \to \mathrm{Inf}_y(N)$ and $T_x(M) \to T_y(N)$ simply because there is a pullback functor

$$\mathit{Lift}(x, A[M]) \to \mathit{Lift}(y, B[N])$$

coming from the ring map $A[M] \to B[N]$. Similarly, given a morphism of deformation situations $(y, B' \to B) \to (x, A' \to A)$ we obtain a pullback functor $\mathit{Lift}(x, A') \to \mathit{Lift}(y, B')$. Since the construction of the action, the addition, and the scalar multiplication on Inf_x and T_x use only morphisms in the categories of lifts (see proof of Formal Deformation Theory, Lemma 64.10.4) we see that the constructions above are functorial. In other words we obtain A-linear maps

$$\mathrm{Inf}_x(M) \to \mathrm{Inf}_y(N) \quad \text{and} \quad T_x(M) \to T_y(N)$$

such that the diagrams

$$\begin{array}{ccc} \mathrm{Inf}_y(J) & \longrightarrow & \mathrm{Inf}(y'/y) \\ \uparrow & & \uparrow \\ \mathrm{Inf}_x(I) & \longrightarrow & \mathrm{Inf}(x'/x) \end{array} \quad \text{and} \quad \begin{array}{ccc} T_y(J) \times \mathrm{Lift}(y, B') & \longrightarrow & \mathrm{Lift}(y, B') \\ \uparrow & & \uparrow \\ T_x(I) \times \mathrm{Lift}(x, A') & \longrightarrow & \mathrm{Lift}(x, A') \end{array}$$

commute. Here $I = \mathrm{Ker}(A' \to A)$, $J = \mathrm{Ker}(B' \to B)$, x' is a lift of x to A' (which may not always exist) and $y' = x'|_{\mathrm{Spec}(B')}$.

Remark 71.18.5 (Automorphisms)**.** Assumptions and notation as in Lemma 71.18.3. Let x', x'' be lifts of x to A'. Then we have a composition map

$$\mathrm{Inf}(x''/x) \times \mathrm{Mor}_{\mathit{Lift}(x,A')}(x', x'') \times \mathrm{Inf}(x'/x) \longrightarrow \mathrm{Mor}_{\mathit{Lift}(x,A')}(x', x'').$$

Since $\mathit{Lift}(x, A')$ is a groupoid, if $\mathrm{Mor}_{\mathit{Lift}(x,A')}(x', x'')$ is nonempty, then this defines a simply transitive left action of $\mathrm{Inf}(x'/x)$ on $\mathrm{Mor}_{\mathit{Lift}(x,A')}(x', x'')$ and a simply transitive right action by $\mathrm{Inf}(x'/x)$. Now the lemma says that $\mathrm{Inf}(x'/x) = \mathrm{Inf}_x(I) = \mathrm{Inf}(x''/x)$. We claim that the two actions described above agree via these identifications. Namely, either $x' \ncong x''$ in which the claim is clear, or $x' \cong x''$ and in that case we may assume that $x'' = x'$ in which case the result follows from the fact that $\mathrm{Inf}(x'/x)$ is commutative. In particular, we obtain a well defined action

$$\mathrm{Inf}_x(I) \times \mathrm{Mor}_{\mathit{Lift}(x,A')}(x', x'') \longrightarrow \mathrm{Mor}_{\mathit{Lift}(x,A')}(x', x'')$$

which is simply transitive as soon as $\mathrm{Mor}_{\mathit{Lift}(x,A')}(x', x'')$ is nonempty.

Remark 71.18.6 (Canonical element)**.** Assumptions and notation as in Lemma 71.18.3. Choose an affine open $\mathrm{Spec}(\Lambda) \subset S$ such that $\mathrm{Spec}(A) \to S$ corresponds to a ring map $\Lambda \to A$. Consider the ring map

$$A \longrightarrow A[\Omega_{A/\Lambda}], \quad a \longmapsto (a, \mathrm{d}_{A/\Lambda}(a))$$

Pulling back x along the corresponding morphism $\mathrm{Spec}(A[\Omega_{A/\Lambda}]) \to \mathrm{Spec}(A)$ we obtain a deformation x_{can} of x over $A[\Omega_{A/\Lambda}]$. We call this the *canonical element*

$$x_{can} \in T_x(\Omega_{A/\Lambda}) = \mathrm{Lift}(x, A[\Omega_{A/\Lambda}]).$$

Next, assume that $\Lambda \to A$ is of finite type and let $k = \kappa(\mathfrak{p})$ be a residue field at a finite type point u_0 of $U = \mathrm{Spec}(A)$. Let $x_0 = x|_{u_0}$. By (RS*) and the fact that $A[k] = A \times_k k[k]$ the space $T_x(k)$ is the tangent space to the deformation functor $\mathcal{F}_{\mathcal{X},k,x_0}$. Via

$$T\mathcal{F}_{U,k,u_0} = \mathrm{Der}_\Lambda(A, k) = \mathrm{Hom}_A(\Omega_{A/\Lambda}, k)$$

(see Formal Deformation Theory, Example 64.10.11) and functoriality of T_x the canonical element produces the map on tangent spaces induced by the object x over U. Namely, $\theta \in T\mathcal{F}_{U,k,u_0}$ maps to $T_x(\theta)(x_{can})$ in $T_x(k) = T\mathcal{F}_{\mathcal{X},k,x_0}$.

Remark 71.18.7 (Canonical automorphism)**.** Let S be a locally Noetherian scheme. Let $\mathcal{X}$ be a category fibred in groupoids over $(Sch/S)_{fppf}$. Assume $\mathcal{X}$ satisfies condition (RS*). Let A be an S-algebra such that $\mathrm{Spec}(A) \to S$ maps into an affine open and let x, y be objects of $\mathcal{X}$ over $\mathrm{Spec}(A)$. Further, let $A \to B$ be a ring map and let $\alpha : x|_{\mathrm{Spec}(B)} \to y|_{\mathrm{Spec}(B)}$ be a morphism of $\mathcal{X}$ over $\mathrm{Spec}(B)$. Consider the ring map

$$B \longrightarrow B[\Omega_{B/A}], \quad b \longmapsto (b, \mathrm{d}_{B/A}(b))$$

Pulling back α along the corresponding morphism $\mathrm{Spec}(B[\Omega_{B/A}]) \to \mathrm{Spec}(B)$ we obtain a morphism α_{can} between the pullbacks of x and y over $B[\Omega_{B/A}]$. On the other hand, we can pullback α by the morphism $\mathrm{Spec}(B[\Omega_{B/A}]) \to \mathrm{Spec}(B)$ corresponding to the injection of B into the first summand of $B[\Omega_{B/A}]$. By the discussion of Remark 71.18.5 we can take the difference

$$\varphi(x, y, \alpha) = \alpha_{can} - \alpha|_{\mathrm{Spec}(B[\Omega_{B/A}])} \in \mathrm{Inf}_{x|_{\mathrm{Spec}(B)}}(\Omega_{B/A}).$$

We will call this the *canonical automorphism*. It depends on all the ingredients A, x, y, $A \to B$ and α.

Remark 71.18.8. Let S be a locally Noetherian scheme. Let $\mathcal{X}$ be a category fibred in groupoids over $(Sch/S)_{fppf}$. Let A be an S-algebra such that $\mathrm{Spec}(A) \to S$ maps into an affine open. There is a notion of a *short exact sequence*

$$(x, A'_1 \to A) \to (x, A'_2 \to A) \to (x, A'_3 \to A)$$

of deformation situations: we ask the corresponding maps between the kernels $I_i = \mathrm{Ker}(A'_i \to A)$ give a short exact sequence

$$0 \to I_3 \to I_2 \to I_1 \to 0$$

of A-modules. Note that in this case the map $A'_3 \to A'_1$ factors through A, hence there is a canonical isomorphism $A'_1 = A[I_1]$.

71.19. Obstruction theories

In this section we describe what an obstruction theory is. Contrary to the spaces of infinitesimal deformations and infinitesimal automorphisms, an obstruction theory is an additional piece of data. The formulation is motivated by the results of Lemma 71.18.3 and Remark 71.18.4.

Definition 71.19.1. Let S be a locally Noetherian base. Let $\mathcal{X}$ be a category fibred in groupoids over $(Sch/S)_{fppf}$. An *obstruction theory* is given by the following data

(1) for every S-algebra A such that $\mathrm{Spec}(A) \to S$ maps into an affine open and every object x of $\mathcal{X}$ over $\mathrm{Spec}(A)$ an A-linear functor
$$\mathcal{O}_x : \mathrm{Mod}_A \to \mathrm{Mod}_A$$
of *obstruction modules*,
(2) for (x, A) as in (1), a ring map $A \to B$, $M \in \mathrm{Mod}_A$, $N \in \mathrm{Mod}_B$, and an A-linear map $M \to N$ an induced A-linear map $\mathcal{O}_x(M) \to \mathcal{O}_y(N)$ where $y = x|_{\mathrm{Spec}(B)}$, and
(3) for every deformation situation $(x, A' \to A)$ an *obstruction* element $o_x(A') \in \mathcal{O}_x(I)$ where $I = \mathrm{Ker}(A' \to A)$.

These data are subject to the following conditions

(i) the functoriality maps turn the obstruction modules into a functor from the category of triples (x, A, M) to sets,
(ii) for every morphism of deformation situations $(y, B' \to B) \to (x, A' \to A)$ the element $o_x(A')$ maps to $o_y(B')$, and
(iii) we have
$$\mathrm{Lift}(x, A') \neq \emptyset \Leftrightarrow o_x(A') = 0$$
for every deformation situation $(x, A' \to A)$.

This last condition explains the terminology. The module $\mathcal{O}_x(A')$ is called the *obstruction module*. The element $o_x(A')$ is the *obstruction*. Most obstruction theories have additional properties, and in order to make them useful additional conditions are needed. Moreover, this is just a sample definition, for example in the definition we could consider only deformation situations of finite type over S.

One of the main reasons for introducing obstruction theories is to check openness of versality. The initial idea to do this is due to Artin, see the papers of Artin mentioned in the introduction. It has been taken up for example in the work by Flenner [**Fle81**], Hall [**Hal12**], Hall and Rydh [**HR12**], Olsson [**Ols06a**], Olsson and Starr [**OS03**], and Lieblich [**Lie06a**] (random order of references). Moreover, for particular categories fibred in groupoids, often authors develop a little bit of theory adapted to the problem at hand. We will develop this theory later (insert future reference here).

Example 71.19.2. Let $S = \mathrm{Spec}(\Lambda)$ for some Noetherian ring Λ. Let $W \to S$ be a morphism of schemes. Let $\mathcal{F}$ be a quasi-coherent $\mathcal{O}_W$-module flat over S. Consider the functor
$$F : (Sch/S)^{opp}_{fppf} \longrightarrow Sets, \quad T/S \longrightarrow H^0(W_T, \mathcal{F}_T)$$
where $W_T = T \times_S W$ is the base change and $\mathcal{F}_T$ is the pullback of $\mathcal{F}$ to W_T. If $T = \mathrm{Spec}(A)$ we will write $W_T = W_A$, etc. Let $\mathcal{X} \to (Sch/S)_{fppf}$ be the category fibred in groupoids associated to F. Then $\mathcal{X}$ has an obstruction theory. Namely,

(1) given A over Λ and $x \in H^0(W_A, \mathcal{F}_A)$ we set $\mathcal{O}_x(M) = H^1(W_A, \mathcal{F}_A \otimes_A M)$,
(2) given a deformation situation $(x, A' \to A)$ we let $o_x(A') \in \mathcal{O}_x(A)$ be the image of x under the boundary map

$$H^0(W_A, \mathcal{F}_A) \longrightarrow H^1(W_A, \mathcal{F}_A \otimes_A I)$$

coming from the short exact sequence of modules

$$0 \to \mathcal{F}_A \otimes_A I \to \mathcal{F}_{A'} \to \mathcal{F}_A \to 0.$$

We have omitted some details, in particular the construction of the short exact sequence above (it uses that W_A and $W_{A'}$ have the same underlying topological space) and the explanation for why flatness of $\mathcal{F}$ over S implies that the sequence above is short exact.

Example 71.19.3 (Key example)**.** Let $S = \mathrm{Spec}(\Lambda)$ for some Noetherian ring Λ. Say $\mathcal{X} = (Sch/X)_{fppf}$ with $X = \mathrm{Spec}(R)$ and $R = \Lambda[x_1, \ldots, x_n]/J$. The naive cotangent complex $NL_{R/\Lambda}$ is (canonically) homotopy equivalent to

$$J/J^2 \longrightarrow \bigoplus\nolimits_{i=1,\ldots,n} R\mathrm{d}x_i,$$

see Algebra, Lemma 10.127.2. Consider a deformation situation $(x, A' \to A)$. Denote I the kernel of $A' \to A$. The object x corresponds to $(a_1, \ldots, a_n)$ with $a_i \in A$ such that $f(a_1, \ldots, a_n) = 0$ in A for all $f \in J$. Set

$$\begin{aligned} \mathcal{O}_x(A') &= \mathrm{Hom}_R(J/J^2, I)/\mathrm{Hom}_R(R^{\oplus n}, I) \\ &= \mathrm{Ext}^1_R(NL_{R/\Lambda}, I) \\ &= \mathrm{Ext}^1_A(NL_{R/\Lambda} \otimes_R A, I). \end{aligned}$$

Choose lifts $a'_i \in A'$ of a_i in A. Then $o_x(A')$ is the class of the map $J/J^2 \to I$ defined by sending $f \in J$ to $f(a'_1, \ldots, a'_n) \in I$. We omit the verification that $o_x(A')$ is independent of choices. It is clear that if $o_x(A') = 0$ then the map lifts. Finally, functoriality is straightforward. Thus we obtain an obstruction theory. We observe that $o_x(A')$ can be described a bit more canonically as the composition

$$NL_{R/\Lambda} \to NL_{A/\Lambda} \to NL_{A/A'} = I[1]$$

in $D(A)$, see Algebra, Lemma 10.127.6 for the last identification.

71.20. Naive obstruction theories

The title of this section refers to the fact that we will use the naive cotangent complex in this section. Let $(x, A' \to A)$ be a deformation situation for a given category fibred in groupoids over a locally Noetherian scheme S. The key Example 71.19.3 suggests that any obstruction theory should be closely related to maps in $D(A)$ with target the naive cotangent complex of A. Working this out we find a criterion for versality in Lemma 71.20.3 which leads to a criterion for openness of versality in Lemma 71.20.4. We introduce a notion of a naive obstruction theory in Definition 71.20.5 to try to formalize the notion a bit further.

In the following we will use the naive cotangent complex as defined in Algebra, Section 10.127. In particular, if $A' \to A$ is a surjection of Λ-algebras with square zero kernel I, then there are maps

$$NL_{A'/\Lambda} \to NL_{A/\Lambda} \to NL_{A/A'}$$

whose composition is homotopy equivalent to zero (see Algebra, Remark 10.127.5). This doesn't form a distinguished triangle in general as we are using the naive cotangent complex and not the full one. There is a homotopy equivalence $NL_{A/A'} \to I[1]$ (the complex consisting of I placed in degree -1, see Algebra, Lemma 10.127.6). Finally, note that there is a canonical map $NL_{A/\Lambda} \to \Omega_{A/\Lambda}$.

Lemma 71.20.1. *Let $A \to k$ be a ring map with k a field. Let $E \in D^-(A)$. Then $Ext^i_A(E, k) = \mathrm{Hom}_k(H^{-i}(E \otimes^{\mathbf{L}} k), k)$.*

Proof. Omitted. Hint: Replace E by a bounded above complex of free A-modules and compute both sides. □

Lemma 71.20.2. *Let $\Lambda \to A \to k$ be finite type ring maps of Noetherian rings with $k = \kappa(\mathfrak{p})$ for some prime $\mathfrak{p}$ of A. Let $\xi : E \to NL_{A/\Lambda}$ be morphism of $D^-(A)$ such that $H^{-1}(\xi \otimes^{\mathbf{L}} k)$ is not surjective. Then there exists a surjection $A' \to A$ of Λ-algebras such that*

(a) *$I = Ker(A' \to A)$ has square zero and is isomorphic to k as an A-module,*
(b) *$\Omega_{A'/\Lambda} \otimes k = \Omega_{A/\Lambda} \otimes k$, and*
(c) *$E \to NL_{A/A'}$ is zero.*

Proof. Let $f \in A$, $f \notin \mathfrak{p}$. Suppose that $A'' \to A_f$ satisfies (a), (b), (c) for the induced map $E \otimes_A A_f \to NL_{A_f/\Lambda}$, see Algebra, Lemma 10.127.13. Then we can set $A' = A'' \times_{A_f} A$ and get a solution. Namely, it is clear that $A' \to A$ satisfies (a) because $\mathrm{Ker}(A' \to A) = \mathrm{Ker}(A'' \to A) = I$. Pick $f'' \in A''$ lifting f. Then the localization of A' at (f'', f) is isomorphic to A'' (for example by More on Algebra, Lemma 15.4.3). Thus (b) and (c) are clear for A' too. In this way we see that we may replace A by the localization A_f (finitely many times). In particular (after such a replacement) we may assume that $\mathfrak{p}$ is a maximal ideal of A, see Morphisms, Lemma 28.17.1.

Choose a presentation $A = \Lambda[x_1, \ldots, x_n]/J$. Then $NL_{A/\Lambda}$ is (canonically) homotopy equivalent to

$$J/J^2 \longrightarrow \bigoplus\nolimits_{i=1,\ldots,n} A\mathrm{d}x_i,$$

see Algebra, Lemma 10.127.2. After localizing if necessary (using Nakayama's lemma) we can choose generators $f_1, \ldots, f_m$ of J such that $f_j \otimes 1$ form a basis for $J/J^2 \otimes_A k$. Moreover, after renumbering, we can assume that the images of $\mathrm{d}f_1, \ldots, \mathrm{d}f_r$ form a basis for the image of $J/J^2 \otimes k \to \bigoplus k\mathrm{d}x_i$ and that $\mathrm{d}f_{r+1}, \ldots, \mathrm{d}f_m$ map to zero in $\bigoplus k\mathrm{d}x_i$. With these choices the space

$$H^{-1}(NL_{A/\Lambda} \otimes^{\mathbf{L}}_A k) = H^{-1}(NL_{A/\Lambda} \otimes_A k)$$

has basis $f_{r+1} \otimes 1, \ldots, f_m \otimes 1$. Changing basis once again we may assume that the image of $H^{-1}(\xi \otimes^{\mathbf{L}} k)$ is contained in the k-span of $f_{r+1} \otimes 1, \ldots, f_{m-1} \otimes 1$. Set

$$A' = \Lambda[x_1, \ldots, x_n]/(f_1, \ldots, f_{m-1}, \mathfrak{p}f_m)$$

By construction $A' \to A$ satisfies (a). Since $\mathrm{d}f_m$ maps to zero in $\bigoplus k\mathrm{d}x_i$ we see that (b) holds. Finally, by construction the induced map $E \to NL_{A/A'} = I[1]$ induces the zero map $H^{-1}(E \otimes^{\mathbf{L}}_A k) \to I \otimes_A k$. By Lemma 71.20.1 we see that the composition is zero. □

The following lemma is our key technical result.

Lemma 71.20.3. *Let S be a locally Noetherian scheme. Let $\mathcal{X}$ be a category fibred in groupoids over $(Sch/S)_{fppf}$ satisfying (RS*). Let $U = \mathrm{Spec}(A)$ be an affine scheme of finite type over S which maps into an affine open $\mathrm{Spec}(\Lambda)$. Let x be an object of $\mathcal{X}$ over U. Let $\xi : E \to NL_{A/\Lambda}$ be a morphism of $D^-(A)$. Assume*

(i) *for every deformation situation $(x, A' \to A)$ we have: x lifts to $\mathrm{Spec}(A')$ if and only if $E \to NL_{A/\Lambda} \to NL_{A/A'}$ is zero, and*
(ii) *there is an isomorphism of functors $T_x(-) \to Ext^0_A(E, -)$ such that $E \to NL_{A/\Lambda} \to \Omega^1_{A/\Lambda}$ corresponds to the canonical element (see Remark 71.18.6).*

Let $u_0 \in U$ be a finite type point with residue field $k = \kappa(u_0)$. Consider the following statements

(1) *x is versal at u_0, and*
(2) *$\xi : E \to NL_{A/\Lambda}$ induces a surjection $H^{-1}(E \otimes^{\mathbf{L}}_A k) \to H^{-1}(NL_{A/\Lambda} \otimes^{\mathbf{L}}_A k)$ and an injection $H^0(E \otimes^{\mathbf{L}}_A k) \to H^0(NL_{A/\Lambda} \otimes^{\mathbf{L}}_A k)$.*

Then we always have (2) $\Rightarrow$ (1) and we have (1) $\Rightarrow$ (2) if u_0 is a closed point.

Proof. Let $\mathfrak{p} = \mathrm{Ker}(A \to k)$ be the prime corresponding to u_0.

Assume that x versal at u_0 and that u_0 is a closed point of U. If $H^{-1}(\xi \otimes^{\mathbf{L}}_A k)$ is not surjective, then let $A' \to A$ be an extension with kernel I as in Lemma 71.20.2. Because u_0 is a closed point, we see that I is a finite A-module, hence that A' is a finite type Λ-algebra (this fails if u_0 is not closed). In particular A' is Noetherian. By property (c) for A' and (i) for ξ we see that x lifts to an object x' over A'. Let $\mathfrak{p}' \subset A'$ be kernel of the surjective map to k. By Artin-Rees (Algebra, Lemma 10.47.4) there exists an $n > 1$ such that $(\mathfrak{p}')^n \cap I = 0$. Then we see that

$$B' = A'/(\mathfrak{p}')^n \longrightarrow A/\mathfrak{p}^n = B$$

is a small, essential extension of local Artinian rings, see Formal Deformation Theory, Lemma 64.3.12. On the other hand, as x is versal at u_0 and as $x'|_{\mathrm{Spec}(B')}$ is a lift of $x|_{\mathrm{Spec}(B)}$, there exists an integer $m \geq n$ and a map $q : A/\mathfrak{p}^m \to B'$ such that the composition $A/\mathfrak{p}^m \to B' \to B$ is the quotient map. Since the maximal ideal of B' has nth power equal to zero, this q factors through B which contradicts the fact that $B' \to B$ is an essential surjection. This contradiction shows that $H^{-1}(\xi \otimes^{\mathbf{L}}_A k)$ is surjective.

Assume that x versal at u_0. By Lemma 71.20.1 the map $H^0(\xi \otimes^{\mathbf{L}}_A k)$ is dual to the map $\mathrm{Ext}^0_A(NL_{A/\Lambda}, k) \to \mathrm{Ext}^0_A(E, k)$. Note that

$$\mathrm{Ext}^0_A(NL_{A/\Lambda}, k) = \mathrm{Der}_\Lambda(A, k) \quad \text{and} \quad T_x(k) = \mathrm{Ext}^0_A(E, k)$$

Condition (ii) assures us the map $\mathrm{Ext}^0_A(NL_{A/\Lambda}, k) \to \mathrm{Ext}^0_A(E, k)$ sends a tangent vector θ to U at u_0 to the corresponding infinitesimal deformation of x_0, see Remark 71.18.6. Hence if x is versal, then this map is surjective, see Formal Deformation Theory, Lemma 64.12.2. Hence $H^0(\xi \otimes^{\mathbf{L}}_A k)$ is injective. This finishes the proof of (1) $\Rightarrow$ (2) in case u_0 is a closed point.

For the rest of the proof assume $H^{-1}(E \otimes^{\mathbf{L}}_A k) \to H^{-1}(NL_{A/\Lambda} \otimes^{\mathbf{L}}_A k)$ is surjective and $H^0(E \otimes^{\mathbf{L}}_A k) \to H^0(NL_{A/\Lambda} \otimes^{\mathbf{L}}_A k)$ injective. Set $R = A^\wedge_\mathfrak{p}$ and let η be the formal object over R associated to $x|_{\mathrm{Spec}(R)}$. The map $d\underline{\eta}$ on tangent spaces is surjective because it is identified with the dual of the injective map $H^0(E \otimes^{\mathbf{L}}_A k) \to H^0(NL_{A/\Lambda} \otimes^{\mathbf{L}}_A k)$ (see previous paragraph). According to Formal Deformation Theory, Lemma 64.12.2 it suffices to prove the following: Let $C' \to C$ be a small

extension of finite type Artinian local Λ-algebras with residue field k. Let $R \to C$ be a Λ-algebra map compatible with identifications of residue fields. Let $y = x|_{\mathrm{Spec}(C)}$ and let y' be a lift of y to C'. To show: we can lift the Λ-algebra map $R \to C$ to $R \to C'$.

Observe that it suffices to lift the Λ-algebra map $A \to C$. Let $I = \mathrm{Ker}(C' \to C)$. Note that I is a 1-dimensional k-vector space. The obstruction ob to lifting $A \to C$ is an element of $\mathrm{Ext}^1_A(NL_{A/\Lambda}, I)$, see Example 71.19.3. By Lemma 71.20.1 and our assumption the map ξ induces an injection

$$\mathrm{Ext}^1_A(NL_{A/\Lambda}, I) \longrightarrow \mathrm{Ext}^1_A(E, I)$$

By the construction of ob and (i) the image of ob in $\mathrm{Ext}^1_A(E, I)$ is the obstruction to lifting x to $A \times_C C'$. By (RS*) the fact that y/C lifts to y'/C' implies that x lifts to $A \times_C C'$. Hence $ob = 0$ and we are done. □

The key lemma above allows us to conclude that we have openness of versality in some cases.

Lemma 71.20.4. *Let S be a locally Noetherian scheme. Let $\mathcal{X}$ be a category fibred in groupoids over $(Sch/S)_{fppf}$ satisfying (RS*). Let $U = \mathrm{Spec}(A)$ be an affine scheme of finite type over S which maps into an affine open $\mathrm{Spec}(\Lambda)$. Let x be an object of $\mathcal{X}$ over U. Let $\xi : E \to NL_{A/\Lambda}$ be a morphism of $D^-(A)$. Assume*

(i) *for every deformation situation $(x, A' \to A)$ we have: x lifts to $\mathrm{Spec}(A')$ if and only if $E \to NL_{A/\Lambda} \to NL_{A/A'}$ is zero,*
(ii) *there is an isomorphism of functors $T_x(-) \to Ext^0_A(E, -)$ such that $E \to NL_{A/\Lambda} \to \Omega^1_{A/\Lambda}$ corresponds to the canonical element (see Remark 71.18.6),*
(iii) *the cohomology groups of E are finite A-modules.*

If x is versal at a closed point $u_0 \in U$, then there exists an open neighbourhood $u_0 \in U' \subset U$ such that x is versal at every finite type point of U'.

Proof. Let C be the cone of ξ so that we have a distinguished triangle

$$E \to NL_{A/\Lambda} \to C \to E[1]$$

in $D^-(A)$. By Lemma 71.20.3 the assumption that x is versal at u_0 implies that $H^{-1}(C \otimes^{\mathbf{L}} k) = 0$. By More on Algebra, Lemma 15.50.14 there exists an $f \in A$ not contained in the prime corresponding to u_0 such that $H^{-1}(C \otimes^{\mathbf{L}}_A M) = 0$ for any A_f-module M. Using Lemma 71.20.3 again we see that we have versality for all finite type points of the open $D(f) \subset U$. □

The technical lemmas above suggest the following definition.

Definition 71.20.5. Let S be a locally Noetherian base. Let $\mathcal{X}$ be a category fibred in groupoids over $(Sch/S)_{fppf}$. Assume that $\mathcal{X}$ satisfies (RS*). A *naive obstruction theory* is given by the following data

(1) for every S-algebra A such that $\mathrm{Spec}(A) \to S$ maps into an affine open $\mathrm{Spec}(\Lambda) \subset S$ and every object x of $\mathcal{X}$ over $\mathrm{Spec}(A)$ we are given an object $E_x \in D^-(A)$ and a map $\xi_x : E \to NL_{A/\Lambda}$,
(2) given (x, A) as in (1) there are transformations of functors

$$\mathrm{Inf}_x(-) \to \mathrm{Ext}^{-1}_A(E_x, -) \quad \text{and} \quad T_x(-) \to \mathrm{Ext}^0_A(E_x, -)$$

(3) for (x, A) as in (1) and a ring map $A \to B$ setting $y = x|_{\mathrm{Spec}(B)}$ there is a functoriality map $E_x \to E_y$ in $D(A)$.

These data are subject to the following conditions

(i) in the situation of (3) the diagram

$$\begin{array}{ccc} E_y & \xrightarrow[\xi_y]{} & NL_{B/\Lambda} \\ \uparrow & & \uparrow \\ E_x & \xrightarrow{\xi_x} & NL_{A/\Lambda} \end{array}$$

is commutative in $D(A)$,

(ii) given (x, A) as in (1) and $A \to B \to C$ setting $y = x|_{\mathrm{Spec}(B)}$ and $z = x|_{\mathrm{Spec}(C)}$ the composition of the functoriality maps $E_x \to E_y$ and $E_y \to E_z$ is the functoriality map $E_x \to E_z$,

(iii) the maps of (2) are isomorphisms compatible with the functoriality maps and the maps of Remark 71.18.4,

(iv) the composition $E_x \to NL_{A/\Lambda} \to \Omega_{A/\Lambda}$ corresponds to the canonical element of $T_x(\Omega_{A/\Lambda}) = \mathrm{Ext}^0(E_x, \Omega_{A/\Lambda})$, see Remark 71.18.6,

(v) given a deformation situation $(x, A' \to A)$ with $I = \mathrm{Ker}(A' \to A)$ the composition $E_x \to NL_{A/\Lambda} \to NL_{A/A'}$ is zero in

$$\mathrm{Hom}_A(E_x, NL_{A/\Lambda}) = \mathrm{Ext}^0_A(E_x, NL_{A/A'}) = \mathrm{Ext}^1_A(E_x, I)$$

if and only if x lifts to A'.

Thus we see in particular that we obtain an obstruction theory as in Section 71.19 by setting $\mathcal{O}_x(-) = \mathrm{Ext}^1_A(E_x, -)$.

Lemma 71.20.6. *Let S and $\mathcal{X}$ be as in Definition 71.20.5 and let $\mathcal{X}$ be endowed with a naive obstruction theory. Let $A \to B$ and $y \to x$ be as in (3). Let k be a B-algebra which is a field. Then the functoriality map $E_x \to E_y$ induces bijections*

$$H^i(E_x \otimes^{\mathbf{L}}_A k) \to H^i(E_y \otimes^{\mathbf{L}}_A k)$$

for $i = 0, 1$.

Proof. Let $z = x|_{\mathrm{Spec}(k)}$. Then (RS*) implies that

$$\mathit{Lift}(x, A[k]) = \mathit{Lift}(z, k[k]) \quad \text{and} \quad \mathit{Lift}(y, B[k]) = \mathit{Lift}(z, k[k])$$

because $A[k] = A \times_k k[k]$ and $B[k] = B \times_k k[k]$. Hence the properties of a naive obstruction theory imply that the functoriality map $E_x \to E_y$ induces bijections $\mathrm{Ext}^i_A(E_x, k) \to \mathrm{Ext}^i_B(E_y, k)$ for $i = -1, 0$. By Lemma 71.20.1 our maps $H^i(E_x \otimes^{\mathbf{L}}_A k) \to H^i(E_y \otimes^{\mathbf{L}}_A k)$, $i = 0, 1$ induce isomorphisms on dual vector spaces hence are isomorphisms. □

Lemma 71.20.7. *Let S be a locally Noetherian scheme. Let $p : \mathcal{X} \to (Sch/S)^{opp}_{fppf}$ be a category fibred in groupoids. Assume that $\mathcal{X}$ satisfies (RS*) and that $\mathcal{X}$ has a naive obstruction theory. Then openness of versality holds for $\mathcal{X}$ provided the complexes E_x of Definition 71.20.5 have finitely generated cohomology groups for pairs (A, x) where A is of finite type over S.*

Proof. Let U be a scheme locally of finite type over S, let x be an object of $\mathcal{X}$ over U, and let u_0 be a finite type point of U such that x is versal at u_0. We may first shrink U to an affine scheme such that u_0 is a closed point and such that $U \to S$ maps into an affine open $\mathrm{Spec}(\Lambda)$. Say $U = \mathrm{Spec}(A)$. Let $\xi_x : E_x \to NL_{A/\Lambda}$ be the obstruction map. At this point we may apply Lemma 71.20.4 to conclude. □

71.21. A dual notion

Let $(x, A' \to A)$ be a deformation situation for a given category $\mathcal{X}$ fibred in groupoids over a locally Noetherian scheme S. Assume $\mathcal{X}$ has an obstruction theory, see Definition 71.19.1. In practice one often has a complex $K^\bullet$ of A-modules and isomorphisms of functors

$$\mathrm{Inf}_x(-) \to H^0(K^\bullet \otimes^{\mathbf{L}}_A -), \quad T_x(-) \to H^1(K^\bullet \otimes^{\mathbf{L}}_A -), \quad \mathcal{O}_x(-) \to H^2(K^\bullet \otimes^{\mathbf{L}}_A -)$$

In this section we formalize this a little bit and show how this leads to a verification of openness of versality in some cases.

Example 71.21.1. Let $\Lambda, S, W, \mathcal{F}$ be as in Example 71.19.2. Assume that $W \to S$ is proper and $\mathcal{F}$ coherent. By Cohomology of Schemes, Remark 29.20.2 there exists a finite complex of finite projective Λ-modules $N^\bullet$ which universally computes the cohomology of $\mathcal{F}$. In particular the obstruction spaces from Example 71.19.2 are $\mathcal{O}_x(M) = H^1(N^\bullet \otimes_\Lambda M)$. Hence with $K^\bullet = N^\bullet \otimes_\Lambda A[-1]$ we see that $\mathcal{O}_x(M) = H^2(K^\bullet \otimes^{\mathbf{L}}_A M)$.

Situation 71.21.2. Let S be a locally Noetherian scheme. Let $\mathcal{X}$ be a category fibred in groupoids over $(Sch/S)_{fppf}$. Assume that $\mathcal{X}$ has (RS*) so that we can speak of the functor $T_x(-)$, see Lemma 71.18.3. Let $U = \mathrm{Spec}(A)$ be an affine scheme of finite type over S which maps into an affine open $\mathrm{Spec}(\Lambda)$. Let x be an object of $\mathcal{X}$ over U. Assume we are given

(1) a complex of A-modules $K^\bullet$,
(2) a transformation of functors $T_x(-) \to H^1(K^\bullet \otimes^{\mathbf{L}}_A -)$,
(3) for every deformation situation $(x, A' \to A)$ with kernel $I = \mathrm{Ker}(A' \to A)$ an element $o_x(A') \in H^2(K^\bullet \otimes^{\mathbf{L}}_A I)$

satisfying the following (minimal) conditions

(i) the transformation $T_x(-) \to H^1(K^\bullet \otimes^{\mathbf{L}}_A -)$ is an isomorphism,
(ii) given a morphism $(x, A'' \to A) \to (x, A' \to A)$ of deformation situations the element $o_x(A')$ maps to the element $o_x(A'')$ via the map $H^2(K^\bullet \otimes^{\mathbf{L}}_A I) \to H^2(K^\bullet \otimes^{\mathbf{L}}_A I')$ where $I' = \mathrm{Ker}(A'' \to A)$, and
(iii) x lifts to an object over $\mathrm{Spec}(A')$ if and only if $o_x(A') = 0$.

It is possible to incorporate infinitesimal automorphisms as well, but we refrain from doing so in order to get the sharpest possible result.

In Situation 71.21.2 an important role will be played by $K^\bullet \otimes^{\mathbf{L}}_A NL_{A/\Lambda}$. Suppose we are given an element $\xi \in H^1(K^\bullet \otimes^{\mathbf{L}}_A NL_{A/\Lambda})$. Then (1) for any surjection $A' \to A$ of Λ-algebras with kernel I of square zero the canonical map $NL_{A/\Lambda} \to NL_{A/A'} = I[1]$ sends ξ to an element $\xi_{A'} \in H^2(K^\bullet \otimes^{\mathbf{L}}_A I)$ and (2) the map $NL_{A/\Lambda} \to \Omega_{A/\Lambda}$ sends ξ to an element ξ_{can} of $H^1(K^\bullet \otimes^{\mathbf{L}}_A \Omega_{A/\Lambda})$.

Lemma 71.21.3. *In Situation 71.21.2. Assume furthermore that*

(iv) *given a short exact sequence of deformation situations as in Remark 71.18.8 and a lift $x'_2 \in Lift(x, A'_2)$ then $o_x(A'_3) \in H^2(K^\bullet \otimes^{\mathbf{L}}_A I_3)$ equals $\partial\theta$ where $\theta \in H^1(K^\bullet \otimes^{\mathbf{L}}_A I_1)$ is the element corresponding to $x'_2|_{\mathrm{Spec}(A'_1)}$ via $A'_1 = A[I_1]$ and the given map $T_x(-) \to H^1(K^\bullet \otimes^{\mathbf{L}}_A -)$.*

In this case there exists an element $\xi \in H^1(K^\bullet \otimes^{\mathbf{L}}_A NL_{A/\Lambda})$ such that

(1) *for every deformation situation $(x, A' \to A)$ we have $\xi_{A'} = o_x(A')$, and*

(2) *ξ_{can} matches the canonical element of Remark 71.18.6 via the given transformation $T_x(-) \to H^1(K^\bullet \otimes^{\mathbf{L}}_A -)$.*

Proof. Choose a $\alpha : \Lambda[x_1, \ldots, x_n] \to A$ with kernel J. Write $P = \Lambda[x_1, \ldots, x_n]$. In the rest of this proof we work with

$$NL(\alpha) = (J/J^2 \longrightarrow \bigoplus A\mathrm{d}x_i)$$

which is permissible by Algebra, Lemma 10.127.2 and More on Algebra, Lemma 15.43.1. Consider the element $o_x(P/J^2) \in H^2(K^\bullet \otimes^{\mathbf{L}}_A J/J^2)$ and consider the quotient

$$C = (P/J^2 \times \bigoplus A\mathrm{d}x_i)/(J/J^2)$$

where J/J^2 is embedded diagonally. Note that $C \to A$ is a surjection with kernel $\bigoplus A\mathrm{d}x_i$. Moreover there is a section $A \to C$ to $C \to A$ given by mapping the class of $f \in P$ to the class of $(f, \mathrm{d}f)$ in the pushout. For later use, denote x_C the pullback of x along the corresponding morphism $\mathrm{Spec}(C) \to \mathrm{Spec}(A)$. Thus we see that $o_x(C) = 0$. We conclude that $o_x(P/J^2)$ maps to zero in $H^2(K^\bullet \otimes^{\mathbf{L}}_A \bigoplus A\mathrm{d}x_i)$. It follows that there exists some element $\xi \in H^1(K^\bullet \otimes^{\mathbf{L}}_A NL(\alpha))$ mapping to $o_x(P/J^2)$.

Note that for any deformation situation $(x, A' \to A)$ there exists a Λ-algebra map $P/J^2 \to A'$ compatible with the augmentations to A. Hence the element ξ satisfies the first property of the lemma by construction and property (ii) of Situation 71.21.2.

Note that our choice of ξ was well defined up to the choice of an element of $H^1(K^\bullet \otimes^{\mathbf{L}}_A \bigoplus A\mathrm{d}x_i)$. We will show that after modifying ξ by an element of the aforementioned group we can arrange it so that the second assertion of the lemma is true. Let $C' \subset C$ be the image of P/J^2 under the Λ-algebra map $P/J^2 \to C$ (inclusion of first factor). Observe that $\mathrm{Ker}(C' \to A) = \mathrm{Im}(J/J^2 \to \bigoplus A\mathrm{d}x_i)$. Set $\overline{C} = A[\Omega_{A/\Lambda}]$. The map $P/J^2 \times \bigoplus A\mathrm{d}x_i \to \overline{C}$, $(f, \sum f_i \mathrm{d}x_i) \mapsto (f \bmod J, \sum f_i \mathrm{d}x_i)$ factors through a surjective map $C \to \overline{C}$. Then

$$(x, \overline{C} \to A) \to (x, C \to A) \to (x, C' \to A)$$

is a short exact sequence of deformation situations. The associated splitting $\overline{C} = A[\Omega_{A/\Lambda}]$ (from Remark 71.18.8) equals the given splitting above. Moreover, the section $A \to C$ composed with the map $C \to \overline{C}$ is the map $(1, \mathrm{d}) : A \to A[\Omega_{A/\Lambda}]$ of Remark 71.18.6. Thus x_C restricts to the canonical element x_{can} of $T_x(\Omega_{A/\Lambda}) = \mathrm{Lift}(x, A[\Omega_{A/\Lambda}])$. By condition (iv) we conclude that $o_x(P/J^2)$ maps to ∂x_{can} in

$$H^1(K^\bullet \otimes^{\mathbf{L}}_A \mathrm{Im}(J/J^2 \to \bigoplus A\mathrm{d}x_i))$$

By construction ξ maps to $o_x(P/J^2)$. It follows that x_{can} and ξ_{can} map to the same element in the displayed group which means (by the long exact cohomology sequence) that they differ by an element of $H^1(K^\bullet \otimes^{\mathbf{L}}_A \bigoplus A\mathrm{d}x_i)$ as desired. □

Lemma 71.21.4. *In Situation 71.21.2 assume that (iv) of Lemma 71.21.3 holds and that $K^\bullet$ is a perfect object of $D(A)$. In this case, if x is versal at a closed point $u_0 \in U$ then there exists an open neighbourhood $u_0 \in U' \subset U$ such that x is versal at every finite type point of U'.*

Proof. We may assume that $K^\bullet$ is a finite complex of finite projective A-modules. Thus the derived tensor product with $K^\bullet$ is the same as simply tensoring with $K^\bullet$. Let $E^\bullet$ be the dual perfect complex to $K^\bullet$, see More on Algebra, Lemma 15.50.19. (So $E^n = \mathrm{Hom}_A(K^{-n}, A)$ with differentials the transpose of the differentials of $K^\bullet$.) Let $E \in D^-(A)$ denote the object represented by the complex $E^\bullet[-1]$. Let $\xi \in H^1(\mathrm{Tot}(K^\bullet \otimes_A NL_{A/\Lambda}))$ be the element constructed in Lemma 71.21.3 and denote $\xi : E = E^\bullet[-1] \to NL_{A/\Lambda}$ the corresponding map (loc.cit.). We claim that the pair (E, ξ) satisfies all the assumptions of Lemma 71.20.4 which finishes the proof.

Namely, assumption (i) of Lemma 71.20.4 follows from conclusion (1) of Lemma 71.21.3 and the fact that $H^2(K^\bullet \otimes^{\mathbf{L}}_A -) = \mathrm{Ext}^1(E, -)$ by loc.cit. Assumption (ii) of Lemma 71.20.4 follows from conclusion (2) of Lemma 71.21.3 and the fact that $H^1(K^\bullet \otimes^{\mathbf{L}}_A -) = \mathrm{Ext}^0(E, -)$ by loc.cit. Assumption (iii) of Lemma 71.20.4 is clear. □

71.22. Examples of deformation problems

List of things that should go here:

(1) Describe the general outline of an example.
(2) Deformations of schemes:
  (a) The Rim-Schlessinger condition.
  (b) Computing the tangent space.
  (c) Computing the infinitesimal deformations.
  (d) The deformation category of an affine hypersurface.
(3) Deformations of representations of abstract groups.
(4) Deformations of representations of topological groups (e.g., profinite ones).
(5) Deformations of sheaves (for example fix X/S, a finite type point s of S, and a quasi-coherent sheaf $\mathcal{F}_s$ over X_s).
(6) Deformations of algebraic spaces (very similar to deformations of schemes; maybe even easier?).
(7) Deformations of maps (eg morphisms between schemes; you can fix both or one of the target and/or source).
(8) Add more here.

71.23. Other chapters

Preliminaries

(1) Introduction
(2) Conventions
(3) Set Theory
(4) Categories
(5) Topology
(6) Sheaves on Spaces
(7) Sites and Sheaves
(8) Stacks
(9) Fields
(10) Commutative Algebra
(11) Brauer Groups
(12) Homological Algebra
(13) Derived Categories
(14) Simplicial Methods
(15) More on Algebra
(16) Smoothing Ring Maps
(17) Sheaves of Modules
(18) Modules on Sites
(19) Injectives
(20) Cohomology of Sheaves
(21) Cohomology on Sites
(22) Differential Graded Algebra
(23) Divided Power Algebra
(24) Hypercoverings

Schemes

(25) Schemes
(26) Constructions of Schemes
(27) Properties of Schemes
(28) Morphisms of Schemes
(29) Cohomology of Schemes
(30) Divisors
(31) Limits of Schemes
(32) Varieties
(33) Topologies on Schemes
(34) Descent
(35) Derived Categories of Schemes
(36) More on Morphisms
(37) More on Flatness
(38) Groupoid Schemes
(39) More on Groupoid Schemes
(40) Étale Morphisms of Schemes

Topics in Scheme Theory

(41) Chow Homology
(42) Adequate Modules
(43) Dualizing Complexes
(44) Étale Cohomology
(45) Crystalline Cohomology
(46) Pro-étale Cohomology

Algebraic Spaces

(47) Algebraic Spaces
(48) Properties of Algebraic Spaces
(49) Morphisms of Algebraic Spaces
(50) Decent Algebraic Spaces
(51) Cohomology of Algebraic Spaces
(52) Limits of Algebraic Spaces
(53) Divisors on Algebraic Spaces
(54) Topologies on Algebraic Spaces
(55) Descent and Algebraic Spaces
(56) Derived Categories of Spaces
(57) More on Morphisms of Spaces
(58) Algebraic Spaces over Fields
(59) Groupoids in Algebraic Spaces
(60) More on Groupoids in Spaces
(61) Bootstrap
(62) Quotients of Groupoids
(63) Simplicial Spaces

Deformation Theory

(64) Formal Deformation Theory
(65) Deformation Theory
(66) The Cotangent Complex

Algebraic Stacks

(67) Algebraic Stacks
(68) Examples of Stacks
(69) Sheaves on Algebraic Stacks
(70) Criteria for Representability
(71) Artin's Axioms
(72) Quot and Hilbert Spaces
(73) Properties of Algebraic Stacks
(74) Morphisms of Algebraic Stacks
(75) Cohomology of Algebraic Stacks
(76) Derived Categories of Stacks
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Miscellany

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CHAPTER 72

Quot and Hilbert Spaces

72.1. Introduction

The purpose of this chapter is to write about Quot and Hilbert functors and to prove that these are algebraic spaces provided certain technical conditions are satisfied. In this chapter we will discuss this in the setting of algebraic space. A reference is Grothendieck's lectures, see [**Gro95a**], [**Gro95b**], [**Gro95e**], [**Gro95f**], [**Gro95c**], and [**Gro95d**]. Another reference is the paper [**OS03**]; this paper discusses the more general case of Quot and Hilbert spaces associated to a morphism of algebraic stacks which we will discuss in another chapter, see (insert future reference here).

In the case of Hilbert spaces there is a more general notion of "Hilbert stacks" which we will discuss in a separate chapter, see (insert future reference here).

We have intentionally placed this chapter, as well as the chapters "Examples of Stacks", "Sheaves on Algebraic Stacks", "Criteria for Representability", and "Artin's Axioms" before the general development of the theory of algebraic stacks. The reason for this is that starting with the next chapter (see Properties of Stacks, Section 73.2) we will no longer distinguish between a scheme and the algebraic stack it gives rise to. Thus our language will become more flexible and easier for a human to parse, but also less precise. These first few chapters, including the initial chapter "Algebraic Stacks", lay the groundwork that later allow us to ignore some of the very technical distinctions between different ways of thinking about algebraic stacks. But especially in the chapters "Artin's Axioms" and "Criteria of Representability" we need to be very precise about what objects exactly we are working with, as we are trying to show that certain constructions produce algebraic stacks or algebraic spaces.

Unfortunately, this means that some of the notation, conventions and terminology is awkward and may seem backwards to the more experienced reader. We hope the reader will forgive us!

72.2. Conventions

The standing assumption is that all schemes are contained in a big fppf site Sch_{fppf}. And all rings A considered have the property that $\mathrm{Spec}(A)$ is (isomorphic) to an object of this big site.

Let S be a scheme and let X be an algebraic space over S. In this chapter and the following we will write $X \times_S X$ for the product of X with itself (in the category of algebraic spaces over S), instead of $X \times X$.

72.3. The Hom functor

In this section we study the functor of homomorphisms defined below.

Situation 72.3.1. Let S be a scheme. Let $f : X \to B$ be a morphism of algebraic spaces over S. Let $\mathcal{F}$, $\mathcal{G}$ be quasi-coherent $\mathcal{O}_X$-modules. For any scheme T over B we will denote $\mathcal{F}_T$ and $\mathcal{G}_T$ the base changes of $\mathcal{F}$ and $\mathcal{G}$ to T, in other words, the pullbacks via the projection morphism $X_T = X \times_B T \to X$. We consider the functor

$$Hom(\mathcal{F}, \mathcal{G}) : (Sch/B)^{opp} \longrightarrow Sets, \quad T \longrightarrow \mathrm{Hom}_{\mathcal{O}_{X_T}}(\mathcal{F}_T, \mathcal{G}_T) \tag{72.3.1.1}$$

In Situation 72.3.1 we sometimes think of the functor $Hom(\mathcal{F}, \mathcal{G})$ as a functor

$$Hom(\mathcal{F}, \mathcal{G}) : (Sch/S)^{opp} \longrightarrow Sets$$

endowed with a morphism $Hom(\mathcal{F}, \mathcal{G}) \to B$. Namely, if T is a scheme over S, then an element of $Hom(\mathcal{F}, \mathcal{G})(T)$ consists of a pair (h, u), where h is a morphism $h : T \to B$ and $u : \mathcal{F}_T \to \mathcal{G}_T$ is an $\mathcal{O}_{X_T}$-module map where $X_T = T \times_{h,B} X$ and $\mathcal{F}_T$ and $\mathcal{G}_T$ are the pullbacks to X_T. In particular, when we say that $Hom(\mathcal{F}, \mathcal{G})$ is an algebraic space, we mean that the corresponding functor $(Sch/S)^{opp} \to Sets$ is an algebraic space.

Lemma 72.3.2. *In Situation 72.3.1 the functor $Hom(\mathcal{F}, \mathcal{G})$ satisfies the sheaf property for the fpqc topology.*

Proof. Let $\{T_i \to T\}_{i \in I}$ be an fpqc covering of schemes over B. Set $X_i = X_{T_i} = X \times_S T_i$ and $\mathcal{F}_i = u_{T_i}$ and $\mathcal{G}_i = \mathcal{G}_{T_i}$. Note that $\{X_i \to X_T\}_{i \in I}$ is an fpqc covering of X_T, see Topologies on Spaces, Lemma 54.3.2. Thus a family of maps $u_i : \mathcal{F}_i \to \mathcal{G}_i$ such that u_i and u_j restrict to the same map on $X_{T_i \times_T T_j}$ comes from a unique map $u : \mathcal{F}_T \to \mathcal{G}_T$ by descent (Descent on Spaces, Proposition 55.4.1). □

Remark 72.3.3. In Situation 72.3.1 let $B' \to B$ be a morphism of algebraic spaces over S. Set $X' = X \times_B B'$ and denote $\mathcal{F}'$, $\mathcal{G}'$ the pullback of $\mathcal{F}$, $\mathcal{G}$ to X'. Then we obtain a functor $Hom(\mathcal{F}', \mathcal{G}') : (Sch/B')^{opp} \to Sets$ associated to the base change $f' : X' \to B'$. For a scheme T over B' it is clear that we have

$$Hom(\mathcal{F}', \mathcal{G}')(T) = Hom(\mathcal{F}, \mathcal{G})(T)$$

where on the right hand side we think of T as a scheme over B via the composition $T \to B' \to B$. This trivial remark will occasionally be useful to change the base algebraic space.

Lemma 72.3.4. *In Situation 72.3.1 let $\{X_i \to X\}_{i \in I}$ be an fppf covering and for each $i, j \in I$ let $\{X_{ijk} \to X_i \times_X X_j\}$ be an fppf covering. Denote $\mathcal{F}_i$, resp. $\mathcal{F}_{ijk}$ the pullback of $\mathcal{F}$ to X_i, resp. X_{ijk}. Similarly define $\mathcal{G}_i$ and $\mathcal{G}_{ijk}$. For every scheme T over B the diagram*

$$Hom(\mathcal{F}, \mathcal{G})(T) \longrightarrow \prod_i Hom(\mathcal{F}_i, \mathcal{G}_i)(T) \underset{pr_1^*}{\overset{pr_0^*}{\rightrightarrows}} \prod_{i,j,k} Hom(\mathcal{F}_{ijk}, \mathcal{G}_{ijk})(T)$$

presents the first arrow as the equalizer of the other two.

Proof. Let $u_i : \mathcal{F}_{i,T} \to \mathcal{G}_{i,T}$ be an element in the equalizer of pr_0^* and pr_1^*. Since the base change of an fppf covering is an fppf covering (Topologies on Spaces, Lemma 54.4.2) we see that $\{X_{i,T} \to X_T\}_{i \in I}$ and $\{X_{ijk,T} \to X_{i,T} \times_{X_T} X_{j,T}\}$ are

fppf coverings. Applying Descent on Spaces, Proposition 55.4.1 we first conclude that u_i and u_j restrict to the same morphism over $X_{i,T} \times_{X_T} X_{j,T}$, whereupon a second application shows that there is a unique morphism $u : \mathcal{F}_T \to \mathcal{G}_T$ restricting to u_i for each i. This finishes the proof. □

Lemma 72.3.5. *In Situation 72.3.1. If $\mathcal{F}$ is of finite presentation and f is quasi-compact and quasi-separated, then $Hom(\mathcal{F}, \mathcal{G})$ is limit preserving.*

Proof. Let $T = \lim_{i \in I} T_i$ be a directed limit of affine B-schemes. We have to show that

$$Hom(\mathcal{F}, \mathcal{G})(T) = \operatorname{colim} Hom(\mathcal{F}, \mathcal{G})(T_i)$$

Pick $0 \in I$. We may replace B by T_0, X by X_{T_0}, $\mathcal{F}$ by $\mathcal{F}_{T_0}$, $\mathcal{G}$ by $\mathcal{G}_{T_0}$, and I by $\{i \in I \mid i \geq 0\}$. See Remark 72.3.3. Thus we may assume $B = \operatorname{Spec}(R)$ is affine.

When B is affine, then X is quasi-compact and quasi-separated. Choose a surjective étale morphism $U \to X$ where U is an affine scheme (Properties of Spaces, Lemma 48.6.3). Since X is quasi-separated, the scheme $U \times_X U$ is quasi-compact and we may choose a surjective étale morphism $V \to U \times_X U$ where V is an affine scheme. Applying Lemma 72.3.4 we see that $Hom(\mathcal{F}, \mathcal{G})$ is the equalizer of two maps between

$$Hom(\mathcal{F}|_U, \mathcal{G}|_U) \quad \text{and} \quad Hom(\mathcal{F}|_V, \mathcal{G}|_V)$$

This reduces us to the case that X is affine.

In the affine case the statement of the lemma reduces to the following problem: Given a ring map $R \to A$, two A-modules M, N and a directed system of R-algebras $C = \operatorname{colim} C_i$. When is it true that the map

$$\operatorname{colim} \operatorname{Hom}_{A \otimes_R C_i}(M \otimes_R C_i, N \otimes_R C_i) \longrightarrow \operatorname{Hom}_{A \otimes_R C}(M \otimes_R C, N \otimes_R C)$$

is bijective? By Algebra, Lemma 10.121.3 this holds if $M \otimes_R C$ is of finite presentation over $A \otimes_R C$, i.e., when M is of finite presentation over A. □

Lemma 72.3.6. *Let S be a scheme. Let B be an algebraic space over S. Let $i : X' \to X$ be a closed immersion of algebraic spaces over B. Let $\mathcal{F}$ be a quasi-coherent $\mathcal{O}_X$-module and let $\mathcal{G}'$ be a quasi-coherent $\mathcal{O}_{X'}$-module. Then*

$$Hom(\mathcal{F}, i_*\mathcal{G}') = Hom(i^*\mathcal{F}, \mathcal{G}')$$

as functors on (Sch/B).

Proof. Let $g : T \to B$ be a morphism where T is a scheme. Denote $i_T : X'_T \to X_T$ the base change of i. Denote $h : X_T \to X$ and $h' : X'_T \to X'$ the projections. Observe that $(h')^*i^*\mathcal{F} = i_T^*h^*\mathcal{F}$. As a closed immersion is affine (Morphisms of Spaces, Lemma 49.19.6) we have $h^*i_*\mathcal{G} = i_{T,*}(h')^*\mathcal{G}$ by Cohomology of Spaces, Lemma 51.8.2. Thus we have

$$\begin{aligned} Hom(\mathcal{F}, i_*\mathcal{G}')(T) &= \operatorname{Hom}_{\mathcal{O}_{X_T}}(h^*\mathcal{F}, h^*i_*\mathcal{G}') \\ &= \operatorname{Hom}_{\mathcal{O}_{X_T}}(h^*\mathcal{F}, i_{T,*}(h')^*\mathcal{G}) \\ &= \operatorname{Hom}_{\mathcal{O}_{X'_T}}(i_T^*h^*\mathcal{F}, (h')^*\mathcal{G}) \\ &= \operatorname{Hom}_{\mathcal{O}_{X'_T}}((h')^*i^*\mathcal{F}, (h')^*\mathcal{G}) \\ &= Hom(i^*\mathcal{F}, \mathcal{G}')(T) \end{aligned}$$

as desired. The middle equality follows from the adjointness of the functors $i_{T,*}$ and i_T^*. □

Lemma 72.3.7. *Let S be a scheme. Let B be an algebraic space over S. Let K be a pseudo-coherent object of $D(\mathcal{O}_B)$.*

(1) *If for all $g : T \to B$ in (Sch/B) the cohomology sheaf $H^{-1}(Lg^*K)$ is zero, then the functor*

$$(Sch/B)^{opp} \longrightarrow Sets, \quad (g : T \to B) \longmapsto H^0(T, H^0(Lg^*K))$$

is an algebraic space affine and of finite presentation over B.

(2) *If for all $g : T \to B$ in (Sch/B) the cohomology sheaves $H^i(Lg^*K)$ are zero for $i < 0$, then K is perfect with tor amplitude in $[0, b]$ for some $b \geq 0$ and the functor*

$$(Sch/B)^{opp} \longrightarrow Sets, \quad (g : T \to B) \longmapsto H^0(T, Lg^*K)$$

is an algebraic space affine and of finite presentation over B.

Proof. Under the assumptions of (2) we have $H^0(T, Lg^*K) = H^0(T, H^0(Lg^*K))$. Let us prove that the rule $T \mapsto H^0(T, H^0(Lg^*K))$ satisfies the sheaf property for the fppf topology. To do this assume we have an fppf covering $\{h_i : T_i \to T\}$ of a scheme $g : T \to B$ over B. Set $g_i = g \circ h_i$. Note that since h_i is flat, we have $Lh_i^* = h_i^*$ and h_i^* commutes with taking cohomology. Hence

$$H^0(T_i, H^0(Lg_i^*K)) = H^0(T_i, H^0(h_i^*Lg^*K)) = H^0(T, h_i^*H^0(Lg^*K))$$

Similarly for the pullback to $T_i \times_T T_j$. Since Lg^*K is a pseudo-coherent complex on T (Cohomology on Sites, Lemma 21.33.3) the cohomology sheaf $\mathcal{F} = H^0(Lg^*K)$ is quasi-coherent (Derived Categories of Spaces, Lemma 56.12.5). Hence by Descent on Spaces, Proposition 55.4.1 we see that

$$H^0(T, \mathcal{F}) = \mathrm{Ker}(\prod H^0(T_i, h_i^*\mathcal{F}) \to \prod H^0(T_i, h_i^*\mathcal{F}))$$

In this way we see that the rules in (1) and (2) satisfy the sheaf property for fppf coverings. This mean we may apply Bootstrap, Lemma 61.11.2 it suffices to prove the representability étale locally on B. Moreover, we may check whether the end result is affine and of finite presentation étale locally on B, see Morphisms of Spaces, Lemmas 49.19.3 and 49.26.4. Hence we may assume that B is an affine scheme.

Assume $B = \mathrm{Spec}(A)$ is an affine scheme. By the results of Derived Categories of Spaces, Lemmas 56.12.5, 56.4.2, and 56.12.2 we deduce that in the rest of the proof we may think of K as a perfect object of the derived category of complexes of modules on B in the Zariski topology. By Derived Categories of Schemes, Lemmas 35.9.1, 35.3.4, and 35.9.3 we can find a pseudo-coherent complex $M^\bullet$ of A-modules such that K is the corresponding object of $D(\mathcal{O}_B)$. Our assumption on pullbacks implies that $M^\bullet \otimes^{\mathbf{L}}_A \kappa(\mathfrak{p})$ has vanishing H^{-1} for all primes $\mathfrak{p} \subset A$. By More on Algebra, Lemma 15.50.14 we can write

$$M^\bullet = \tau_{\geq 0}M^\bullet \oplus \tau_{\leq -1}M^\bullet$$

with $\tau_{\geq 0}M^\bullet$ perfect with Tor amplitude in $[0, b]$ for some $b \geq 0$ (here we also have used More on Algebra, Lemmas 15.50.11 and 15.49.13). Note that in case (2) we also see that $\tau_{\leq -1}M^\bullet = 0$ in $D(A)$ whence $M^\bullet$ and K are perfect with tor amplitude in $[0, b]$. For any B-scheme $g : T \to B$ we have

$$H^0(T, H^0(Lg^*K)) = H^0(T, H^0(Lg^*\tau_{\geq 0}K))$$

(by the dual of Derived Categories, Lemma 13.16.1) hence we may replace K by $\tau_{\geq 0}K$ and correspondingly $M^\bullet$ by $\tau_{\geq 0}M^\bullet$. In other words, we may assume $M^\bullet$ has tor amplitude in $[0, b]$.

Assume $M^\bullet$ has tor amplitude in $[0, b]$. We may assume $M^\bullet$ is a bounded above complex of finite free A-modules (by our definition of pseudo-coherent complexes, see More on Algebra, Definition 15.48.1 and the discussion following the definition). By More on Algebra, Lemma 15.49.2 we see that $M = \mathrm{Coker}(M^{-1} \to M^0)$ is flat. By Algebra, Lemma 10.73.2 we see that M is finite locally free. Hence $M^\bullet$ is quasi-isomorphic to

$$M \to M^1 \to M^2 \to \ldots \to M^d \to 0 \ldots$$

Note that this is a K-flat complex (Cohomology, Lemma 20.25.8), hence derived pullback of K via a morphism $T \to B$ is computed by the complex

$$g^*\widetilde{M} \to g^*\widetilde{M^1} \to \ldots$$

Thus it suffices to show that the functor

$$(g : T \to B) \longmapsto \mathrm{Ker}(\Gamma(T, g^*\widetilde{M}) \to \Gamma(T, g^*(\widetilde{M^1}))$$

is representable by an affine scheme of finite presentation over B.

We may still replace B by the members of an affine open covering in order to prove this last statement. Hence we may assume that M is finite free (recall that M^1 is finite free to begin with). Write $M = A^{\oplus n}$ and $M^1 = A^{\oplus m}$. Let the map $M \to M^1$ be given by the $m \times n$ matrix (a_{ij}) with coefficients in A. Then $\widetilde{M} = \mathcal{O}_B^{\oplus n}$ and $\widetilde{M^1} = \mathcal{O}_B^{\oplus m}$. Thus the functor above is equal to the functor

$$(g : T \to B) \longmapsto \{(f_1, \ldots, f_n) \in \Gamma(T, \mathcal{O}_T) \mid \sum g^\sharp(a_{ij} f_i = 0, j = 1, \ldots, m\}$$

Clearly this is representable by the affine scheme

$$\mathrm{Spec}\left(A[x_1, \ldots, x_n]/(\sum a_{ij}x_i; j = 1, \ldots, m)\right)$$

and the lemma has been proved. □

The functor $Hom(\mathcal{F}, \mathcal{G})$ is representable in a number of situations. All of our results will be based on the following basic case. The proof of this lemma as given below is in some sense the natural generalization to the proof of [**DG67**, III, Cor 7.7.8].

Lemma 72.3.8. *In Situation 72.3.1 assume that*

(1) *B is a Noetherian algebraic space,*
(2) *f is locally of finite type and quasi-separated,*
(3) *$\mathcal{F}$ is a finite type $\mathcal{O}_X$-module, and*
(4) *$\mathcal{G}$ is a finite type $\mathcal{O}_X$-module, flat over B, with scheme theoretic support proper over B.*

Then the functor $Hom(\mathcal{F}, \mathcal{G})$ is representable by an algebraic space affine and of finite presentation over B.

Proof. We may replace X by a quasi-compact open neighbourhood of the support of $\mathcal{G}$, hence we may assume X is Noetherian. In this case X and f are quasi-compact and quasi-separated. Choose an approximation $P \to \mathcal{F}$ by a perfect complex P

of the triple $(X, \mathcal{F}, 0)$, see Derived Categories of Spaces, Definition 56.13.1 and Theorem 56.13.7). Then the induced map

$$\mathrm{Hom}_{\mathcal{O}_X}(\mathcal{F}, \mathcal{G}) \longrightarrow \mathrm{Hom}_{D(\mathcal{O}_X)}(P, \mathcal{G})$$

is an isomorphism because $P \to \mathcal{F}$ induces an isomorphism $H^0(P) \to \mathcal{F}$ and $H^i(P) = 0$ for $i > 0$. Moreover, for any morphism $g : T \to B$ denote $h : X_T = T \times_B X \to X$ the projection and set $P_T = Lh^*P$. Then it is equally true that

$$\mathrm{Hom}_{\mathcal{O}_{X_T}}(\mathcal{F}_T, \mathcal{G}_T) \longrightarrow \mathrm{Hom}_{D(\mathcal{O}_{X_T})}(P_T, \mathcal{G}_T)$$

is an isomorphism, as $P_T = Lh^*P \to Lh^*\mathcal{F} \to \mathcal{F}_T$ induces an isomorphism $H^0(P_T) \to \mathcal{F}_T$ (because h^* is right exact and $H^i(P) = 0$ for $i > 0$). Thus it suffices to prove the result for the functor

$$T \longmapsto \mathrm{Hom}_{D(\mathcal{O}_{X_T})}(P_T, \mathcal{G}_T).$$

By the Leray spectral sequence (see Cohomology on Sites, Remark 21.14.4) we have

$$\mathrm{Hom}_{D(\mathcal{O}_{X_T})}(P_T, \mathcal{G}_T) = H^0(X_T, R\mathcal{H}om(P_T, \mathcal{G}_T)) = H^0(T, Rf_{T,*}R\mathcal{H}om(P_T, \mathcal{G}_T))$$

where $f_T : X_T \to T$ is the base change of f. By Derived Categories of Spaces, Lemma 56.18.3 we have

$$Rf_{T,*}R\mathcal{H}om(P_T, \mathcal{G}_T) = Lg^*Rf_*R\mathcal{H}om(P, \mathcal{G}).$$

By Derived Categories of Spaces, Lemma 56.18.1 the object $K = Rf_*R\mathcal{H}om(P, \mathcal{G})$ of $D(\mathcal{O}_B)$ is perfect. This means we can apply Lemma 72.3.7 as long as we can prove that the cohomology sheaf $H^i(Lg^*K)$ is 0 for all $i < 0$ and $g : T \to B$ as above. This is clear from the last displayed formula as the cohomology sheaves of $Rf_{T,*}R\mathcal{H}om(P_T, \mathcal{G}_T)$ are zero in negative degrees due to the fact that $R\mathcal{H}om(P_T, \mathcal{G}_T)$ has vanishing cohomology sheaves in negative degrees as P_T is perfect with vanishing cohomology sheaves in positive degrees. □

Here is a cheap consequence of Lemma 72.3.8.

Proposition 72.3.9. *In Situation 72.3.1 assume that*

(1) *f is of finite presentation, and*
(2) *$\mathcal{G}$ is a finitely presented $\mathcal{O}_X$-module, flat over B, with scheme theoretic support proper over B.*

Then the functor $Hom(\mathcal{F}, \mathcal{G})$ is representable by an algebraic space affine over B. If $\mathcal{F}$ is of finite presentation, then $Hom(\mathcal{F}, \mathcal{G})$ is of finite presentation over B.

Proof. By Lemma 72.3.2 the functor $Hom(\mathcal{F}, \mathcal{G})$ satisfies the sheaf property for fppf coverings. This mean we may[1] apply Bootstrap, Lemma 61.11.1 to check the representability étale locally on B. Moreover, we may check whether the end result is affine or of finite presentation étale locally on B, see Morphisms of Spaces, Lemmas 49.19.3 and 49.26.4. Hence we may assume that B is an affine scheme.

Assume B is an affine scheme. As f is of finite presentation, it follows X is quasi-compact and quasi-separated. Thus we can write $\mathcal{F} = \mathrm{colim}\, \mathcal{F}_i$ as a filtered colimit of $\mathcal{O}_X$-modules of finite presentation (Limits of Spaces, Lemma 52.9.1). It is clear that

$$Hom(\mathcal{F}, \mathcal{G}) = \lim Hom(\mathcal{F}_i, \mathcal{G})$$

[1] We omit the verification of the set theoretical condition (3) of the referenced lemma.

Hence if we can show that each $Hom(\mathcal{F}_i, \mathcal{G})$ is representable by an affine scheme, then we see that the same thing holds for $Hom(\mathcal{F}, \mathcal{G})$. Use the material in Limits, Section 31.2 and Limits of Spaces, Section 52.4. Thus we may assume that $\mathcal{F}$ is of finite presentation.

Say $B = \mathrm{Spec}(R)$. Write $R = \mathrm{colim}\, R_i$ with each R_i a finite type $\mathbf{Z}$-algebra. Set $B_i = \mathrm{Spec}(R_i)$. By the results of Limits of Spaces, Lemmas 52.7.1 and 52.7.2 we can find an i, a morphism of algebraic spaces $X_i \to B_i$, and finitely presented $\mathcal{O}_{X_i}$-modules $\mathcal{F}_i$ and $\mathcal{G}_i$ such that the base change of $(X_i, \mathcal{F}_i, \mathcal{G}_i)$ to B recovers $(X, \mathcal{F}, \mathcal{G})$. By Limits of Spaces, Lemma 52.6.11 we may, after increasing i, assume that $\mathcal{G}_i$ is flat over B_i. By Limits of Spaces, Lemma 52.6.13 we may similarly assume the scheme theoretic support of $\mathcal{G}_i$ is proper over B_i. At this point we can apply Lemma 72.3.8 to see that $H_i = Hom(\mathcal{F}_i, \mathcal{G}_i)$ is an algebraic space affine of finite presentation over B_i. Pulling back to B (using Remark 72.3.3) we see that $H_i \times_{B_i} B = Hom(\mathcal{F}, \mathcal{G})$ and we win. □

72.4. The Isom functor

In Situation 72.3.1 we can consider the subfunctor

$$Isom(\mathcal{F}, \mathcal{G}) \subset Hom(\mathcal{F}, \mathcal{G})$$

whose value on a scheme T over B is the set of *invertible* $\mathcal{O}_{X_T}$-homomorphisms $u : \mathcal{F}_T \to \mathcal{G}_T$. In this brief section we quickly point out some properties of this functor.

Lemma 72.4.1. *In Situation 72.3.1 the functor $Isom(\mathcal{F}, \mathcal{G})$ satisfies the sheaf property for the fpqc topology.*

Proof. We have already seen that $Hom(\mathcal{F}, \mathcal{G})$ satisfies the sheaf property. Hence it remains to show the following: Given an fpqc covering $\{T_i \to T\}_{i \in I}$ of schemes over B and an $\mathcal{O}_{X_T}$-linear map $u : \mathcal{F}_T \to \mathcal{G}_T$ such that u_{T_i} is an isomorphism for all i, then u is an isomorphism. Since $\{X_i \to X_T\}_{i \in I}$ is an fpqc covering of X_T, see Topologies on Spaces, Lemma 54.3.2, this follows from Descent on Spaces, Proposition 55.4.1. □

Proposition 72.4.2. *In Situation 72.3.1 assume that*

1. *f is of finite presentation, and*
2. *$\mathcal{F}$ and $\mathcal{G}$ are finitely presented $\mathcal{O}_X$-modules, flat over B, with scheme theoretic support proper over B.*

Then the functor $Isom(\mathcal{F}, \mathcal{G})$ is representable by an algebraic space affine of finite presentation over B.

Proof. We will use the abbreviations $H = Hom(\mathcal{F}, \mathcal{G})$, $I = Hom(\mathcal{F}, \mathcal{F})$, $H' = Hom(\mathcal{G}, \mathcal{F})$, and $I' = Hom(\mathcal{G}, \mathcal{G})$. By Proposition 72.3.9 the functors H, I, H', I' are algebraic spaces and the morphisms $H \to B$, $I \to B$, $H' \to B$, and $I' \to B$ are affine and of finite presentation. The composition of maps gives a morphism

$$c : H' \times_B H \longrightarrow I \times_B I', \quad (u', u) \longmapsto (u \circ u', u' \circ u)$$

of algebraic spaces over B. Since $I \times_B I' \to B$ is separated, the section $\sigma : B \to I \times_B I'$ corresponding to $(\mathrm{id}_{\mathcal{F}}, \mathrm{id}_{\mathcal{G}})$ is a closed immersion (Morphisms of Spaces,

Lemma 49.5.7). Moreover, σ is of finite presentation (Morphisms of Spaces, Lemma 49.26.9). Hence

$$Isom(\mathcal{F}, \mathcal{G}) = (H' \times_B H) \times_{c, I \times_B I', \sigma} B$$

is an algebraic space affine of finite presentation over B as well. Some details omitted. □

72.5. The stack of coherent sheaves

In this section we prove that the stack of coherent sheaves on X/B is algebraic under suitable hypotheses. This is a special case of [**Lie06b**, Theorem 2.1.1] which treats the case of the stack of coherent sheaves on an Artin stack over a base.

Situation 72.5.1. Let S be a scheme. Let $f : X \to B$ be a morphism of algebraic spaces over S. Assume that f is of finite presentation. We denote $Coh_{X/B}$ the category whose objects are triples $(T, g, \mathcal{F})$ where

(1) T is a scheme over S,
(2) $g : T \to B$ is a morphism over S, and setting $X_T = T \times_{g,B} X$
(3) $\mathcal{F}$ is a quasi-coherent $\mathcal{O}_{X_T}$-module of finite presentation, flat over T, with scheme theoretic support proper over B.

A morphism $(T, g, \mathcal{F}) \to (T', g', \mathcal{F}')$ is given by a pair (h, φ) where

(1) $h : T \to T'$ is a morphism of schemes over B (i.e., $g' \circ h = g$), and
(2) $\varphi : (h')^* \mathcal{F}' \to \mathcal{F}$ is an isomorphism of $\mathcal{O}_{X_T}$-modules where $h' : X_T \to X_{T'}$ is the base change of h.

Thus $Coh_{X/B}$ is a category and the rule

$$p : Coh_{X/B} \longrightarrow (Sch/S)_{fppf}, \quad (T, g, \mathcal{F}) \longmapsto T$$

is a functor. For a scheme T over S we denote $Coh_{X/B,T}$ the fibre category of p over T. These fibre categories are groupoids.

Lemma 72.5.2. *In Situation 72.5.1 the functor $p : Coh_{X/B} \longrightarrow (Sch/S)_{fppf}$ is fibred in groupoids.*

Proof. We show that p is fibred in groupoids by checking conditions (1) and (2) of Categories, Definition 4.33.1. Given an object $(T', g', \mathcal{F}')$ of $Coh_{X/B}$ and a morphism $h : T \to T'$ of schemes over S we can set $g = h \circ g'$ and $\mathcal{F} = (h')^* \mathcal{F}'$ where $h' : X_T \to X_{T'}$ is the base change of h. Then it is clear that we obtain a morphism $(T, g, \mathcal{F}) \to (T', g', \mathcal{F}')$ of $Coh_{X/B}$ lying over h. This proves (1). For (2) suppose we are given morphisms

$$(h_1, \varphi_1) : (T_1, g_1, \mathcal{F}_1) \to (T, g, \mathcal{F}) \quad \text{and} \quad (h_2, \varphi_2) : (T_2, g_2, \mathcal{F}_2) \to (T, g, \mathcal{F})$$

of $Coh_{X/B}$ and a morphism $h : T_1 \to T_2$ such that $h_2 \circ h = h_1$. Then we can let φ be the composition

$$(h')^* \mathcal{F}_2 \xrightarrow{(h')^* \varphi_2^{-1}} (h')^* (h_2)^* \mathcal{F} = (h_1)^* \mathcal{F} \xrightarrow{\varphi_1} \mathcal{F}_1$$

to obtain the morphism $(h, \varphi) : (T_1, g_1, \mathcal{F}_1) \to (T_2, g_2, \mathcal{F}_2)$ that witnesses the truth of condition (2). □

Lemma 72.5.3. *In Situation 72.5.1. Denote $\mathcal{X} = Coh_{X/B}$. Then $\Delta : \mathcal{X} \to \mathcal{X} \times \mathcal{X}$ is representable by algebraic spaces.*

Proof. Consider two objects $x = (T, g, \mathcal{F})$ and $y = (T, h, \mathcal{G})$ of $\mathcal{X}$ over a scheme T. We have to show that $Isom_{\mathcal{X}}(x, y)$ is representable by an algebraic space over T, see Algebraic Stacks, Lemma 67.10.11. If for $a : T' \to T$ the restrictions $x|_{T'}$ and $y|_{T'}$ are isomorphic in the fibre category $\mathcal{X}_{T'}$, then $g \circ a = h \circ a$. Hence there is a transformation of presheaves

$$Isom_{\mathcal{X}}(x, y) \longrightarrow \text{Equalizer}(g, h)$$

Since the diagonal of B is representable by schemes this equalizer is a scheme. Thus we may replace T by this equalizer and the sheaves $\mathcal{F}$ and $\mathcal{G}$ by their pullbacks. Thus we may assume $g = h$. In this case we have $Isom_{\mathcal{X}}(x, y) = Isom(\mathcal{F}, \mathcal{G})$ and the result follows from Proposition 72.4.2. □

Lemma 72.5.4. *In Situation 72.5.1 the functor $p : Coh_{X/B} \longrightarrow (Sch/S)_{fppf}$ is a stack in groupoids.*

Proof. To prove that $Coh_{X/B}$ is a stack in groupoids, we have to show that the presheaves *Isom* are sheaves and that descent data are effective. The statement on *Isom* follows from Lemma 72.5.3, see Algebraic Stacks, Lemma 67.10.11. Let us prove the statement on descent data. Suppose that $\{a_i : T_i \to T\}$ is an fppf covering of schemes over S. Let (ξ_i, φ_{ij}) be a descent datum for $\{T_i \to T\}$ with values in $Coh_{X/B}$. For each i we can write $\xi_i = (T_i, g_i, \mathcal{F}_i)$. Denote $\text{pr}_0 : T_i \times_T T_j \to T_i$ and $\text{pr}_1 : T_i \times_T T_j \to T_j$ the projections. The condition that $\xi_i|_{T_i \times_T T_j} = \xi_j|_{T_i \times_T T_j}$ implies in particular that $g_i \circ \text{pr}_0 = g_j \circ \text{pr}_1$. Thus there exists a unique morphism $g : T \to B$ such that $g_i = g \circ a_i$, see Descent on Spaces, Lemma 55.6.2. Denote $X_T = T \times_{g,B} X$. Set $X_i = X_{T_i} = T_i \times_{g_i,B} X = T_i \times_{a_i,T} X_T$ and

$$X_{ij} = X_{T_i} \times_{X_T} X_{T_j} = X_i \times_{X_T} X_j$$

with projections pr_i and pr_j to X_i and X_j. Observe that the pullback of $(T_i, g_i, \mathcal{F}_i)$ by $\text{pr}_0 : T_i \times_T T_j \to T_i$ is given by $(T_i \times_T T_j, g_i \circ \text{pr}_0, \text{pr}_i^* \mathcal{F}_i)$. Hence a descent datum for $\{T_i \to T\}$ in $Coh_{X/B}$ is given by the objects $(T_i, g \circ a_i, \mathcal{F}_i)$ and for each pair i, j an isomorphism of $\mathcal{O}_{X_{ij}}$-modules

$$\varphi_{ij} : \text{pr}_i^* \mathcal{F}_i \longrightarrow \text{pr}_j^* \mathcal{F}_j$$

satisfying the cocycle condition over (the pullback of X to) $T_i \times_T T_j \times_T T_k$. Ok, and now we simply use that $\{X_i \to X_T\}$ is an fppf covering so that we can view $(\mathcal{F}_i, \varphi_{ij})$ as a descent datum for this covering. By Descent on Spaces, Proposition 55.4.1 this descent datum is effective and we obtain a quasi-coherent sheaf $\mathcal{F}$ over X_T restricting to $\mathcal{F}_i$ on X_i. By Morphisms of Spaces, Lemma 49.28.5 we see that $\mathcal{F}$ is flat over T and Descent on Spaces, Lemma 55.5.2 guarantees that $\mathcal{Q}$ is of finite presentation as an $\mathcal{O}_{X_T}$-module. Finally, by Descent on Spaces, Lemma 55.10.17 we see that the scheme theoretic support of $\mathcal{F}$ is proper over T as we've assume the scheme theoretic support of $\mathcal{F}_i$ is proper over T_i (note that taking scheme theoretic support commutes with flat base change by Morphisms of Spaces, Lemma 49.27.10). In this way and we obtain our desired object over T. □

Remark 72.5.5. In Situation 72.5.1 the rule $(T, g, \mathcal{F}) \mapsto (T, g)$ defines a 1-morphism

$$Coh_{X/B} \longrightarrow \mathcal{S}_B$$

of categories fibred in groupoids (see Lemma 72.5.4, Algebraic Stacks, Section 67.7, and Examples of Stacks, Section 68.9). Let $B' \to B$ be a morphism of algebraic spaces over S. Let $\mathcal{S}_{B'} \to \mathcal{S}_B$ be the associated 1-morphism of stacks fibred in

sets. Set $X' = X \times_B B'$. We obtain a stack in groupoids $Coh_{X'/B'} \to (Sch/S)_{fppf}$ associated to the base change $f' : X' \to B'$. In this situation the diagram

$$\begin{array}{ccc} Coh_{X'/B'} & \longrightarrow & Coh_{X/B} \\ \downarrow & & \downarrow \\ \mathcal{S}_{B'} & \longrightarrow & \mathcal{S}_B \end{array}$$

is 2-fibre product square. This trivial remark will occasionally be useful to change the base algebraic space.

Lemma 72.5.6. *In Situation 72.5.1 assume that $B \to S$ is locally of finite presentation. Then $p : Coh_{X/B} \to (Sch/S)_{fppf}$ is limit preserving (Artin's Axioms, Definition 71.13.1).*

Proof. Write $B(T)$ for the discrete category whose objects are the S-morphisms $T \to B$. Let $T = \lim T_i$ be a filtered limit of affine schemes over S. Assigning to an object $(T, h, \mathcal{F})$ of $Coh_{X/B,T}$ the object h of $B(T)$ gives us a commutative diagram of fibre categories

$$\begin{array}{ccc} \operatorname{colim} Coh_{X/B,T_i} & \longrightarrow & Coh_{X/B,T} \\ \downarrow & & \downarrow \\ \operatorname{colim} B(T_i) & \longrightarrow & B(T) \end{array}$$

We have to show the top horizontal arrow is an equivalence. Since we have assume that B is locally of finite presentation over S we see from Limits of Spaces, Remark 52.3.10 that the bottom horizontal arrow is an equivalence. This means that we may assume $T = \lim T_i$ be a filtered limit of affine schemes over B. Denote $g_i : T_i \to B$ and $g : T \to B$ the corresponding morphisms. Set $X_i = T_i \times_{g_i,B} X$ and $X_T = T \times_{g,B} X$. Observe that $X_T = \operatorname{colim} X_i$ and that the algebraic spaces X_i and X_T are quasi-separated and quasi-compact (as they are of finite presentation over the affines T_i and T). By Limits of Spaces, Lemma 52.7.2 we see that

$$\operatorname{colim} FP(X_i) = FP(X_T).$$

where $FP(W)$ is short hand for the category of finitely presented $\mathcal{O}_W$-modules. The results of Limits of Spaces, Lemmas 52.6.11 and 52.6.13 tell us the same thing is true if we replace $FP(X_i)$ and $FP(X_T)$ by the full subcategory of objects flat over T_i and T with scheme theoretic support proper over T_i and T. This proves the lemma. □

Lemma 72.5.7. *In Situation 72.5.1. Let*

$$\begin{array}{ccc} Z & \longrightarrow & Z' \\ \downarrow & & \downarrow \\ Y & \longrightarrow & Y' \end{array}$$

be a pushout in the category of schemes over S where $Z \to Z'$ is a thickening and $Z \to Y$ is affine, see More on Morphisms, Lemma 36.11.1. Then the functor on fibre categories

$$Coh_{X/B,Y'} \longrightarrow Coh_{X/B,Y} \times_{Coh_{X/B,Z}} Coh_{X/B,Z'}$$

is an equivalence.

Proof. Observe that the corresponding map

$$B(Y') \longrightarrow B(Y) \times_{B(Z)} B(Z')$$

is a bijection, see More on Morphisms of Spaces, Lemma 57.17.2. Thus using the commutative diagram

$$\begin{array}{ccc} Coh_{X/B,Y'} & \longrightarrow & Coh_{X/B,Y} \times_{Coh_{X/B,Z}} Coh_{X/B,Z'} \\ \downarrow & & \downarrow \\ B(Y') & \longrightarrow & B(Y) \times_{B(Z)} B(Z') \end{array}$$

we see that we may assume that Y' is a scheme over B'. By Remark 72.5.5 we may replace B by Y' and X by $X \times_B Y'$. Thus we may assume $B = Y'$. In this case the statement follows from More on Morphisms of Spaces, Lemma 57.17.7. □

Lemma 72.5.8. *Let*

$$\begin{array}{ccc} X & \xrightarrow[i]{} & X' \\ \downarrow & & \downarrow \\ T & \longrightarrow & T' \end{array}$$

be a cartesian square of algebraic spaces where $T \to T'$ is a first order thickening. Let $\mathcal{F}'$ be an $\mathcal{O}_{X'}$-module flat over T'. Set $\mathcal{F} = i^*\mathcal{F}'$. The following are equivalent

(1) *$\mathcal{F}$ is a quasi-coherent $\mathcal{O}_{X'}$-module of finite presentation,*
(2) *$\mathcal{F}$ is an $\mathcal{O}_{X'}$-module of finite presentation,*
(3) *$\mathcal{F}$ is a quasi-coherent $\mathcal{O}_X$-module of finite presentation,*
(4) *$\mathcal{F}$ is an $\mathcal{O}_X$-module of finite presentation,*

Proof. Recall that a finitely presented module is quasi-coherent hence the equivalence of (1) and (2) and (3) and (4). The equivalence of (2) and (4) is a special case of Deformation Theory, Lemma 65.10.3. □

Lemma 72.5.9. *In Situation 72.5.1 assume that S is a locally Noetherian scheme and $B \to S$ is locally of finite presentation. Let k be a finite type field over S and let $x_0 = (\mathrm{Spec}(k), g_0, \mathcal{G}_0)$ be an object of $\mathcal{X} = Coh_{X/B}$ over k. Then the spaces $T\mathcal{F}_{\mathcal{X},k,x_0}$ and $Inf_{x_0}(\mathcal{F}_{\mathcal{X},k,x_0})$ (Artin's Axioms, Section 71.8) are finite dimensional.*

Proof. Observe that by Lemma 72.5.7 our stack in groupoids $\mathcal{X}$ satisfies property (RS*) defined in Artin's Axioms, Section 71.18. In particular $\mathcal{X}$ satisfies (RS). Hence all associated predeformation categories are deformation categories (Artin's Axioms, Lemma 71.6.1) and the statement makes sense.

In this paragraph we show that we can reduce to the case $B = \mathrm{Spec}(k)$. Set $X_0 = \mathrm{Spec}(k) \times_{g_0,B} X$ and denote $\mathcal{X}_0 = Coh_{X_0/k}$. In Remark 72.5.5 we have seen that $\mathcal{X}_0$ is the 2-fibre product of $\mathcal{X}$ with $\mathrm{Spec}(k)$ over B as categories fibred in groupoids over $(Sch/S)_{fppf}$. Thus by Artin's Axioms, Lemma 71.8.2 we reduce to proving that B, $\mathrm{Spec}(k)$, and $\mathcal{X}_0$ have finite dimensional tangent spaces and infinitesimal automorphism spaces. The tangent space of B and $\mathrm{Spec}(k)$ are finite dimensional by Artin's Axioms, Lemma 71.8.1 and of course these have vanishing Inf. Thus it suffices to deal with $\mathcal{X}_0$.

Let $k[\epsilon]$ be the dual numbers over k. Let $\mathrm{Spec}(k[\epsilon]) \to B$ be the composition of $g_0 : \mathrm{Spec}(k) \to B$ and the morphism $\mathrm{Spec}(k[\epsilon]) \to \mathrm{Spec}(k)$ coming from the inclusion

$k \to k[\epsilon]$. Set $X_0 = \mathrm{Spec}(k) \times_B X$ and $X_\epsilon = \mathrm{Spec}(k[\epsilon]) \times_B X$. Observe that X_ϵ is a first order thickening of X_0 flat over the first order thickening $\mathrm{Spec}(k) \to \mathrm{Spec}(k[\epsilon])$. Unwinding the definitions and using Lemma 72.5.8 we see that $T\mathcal{F}_{\mathcal{X}_0,k,x_0}$ is the set of lifts of $\mathcal{G}_0$ to a flat module on X_ϵ. By Deformation Theory, Lemma 65.11.1 we conclude that

$$T\mathcal{F}_{\mathcal{X}_0,k,x_0} = \mathrm{Ext}^1_{\mathcal{O}_{X_0}}(\mathcal{G}_0, \mathcal{G}_0)$$

Here we have used the identification $\epsilon k[\epsilon] \cong k$ of $k[\epsilon]$-modules. Using Deformation Theory, Lemma 65.11.1 once more we see that

$$\mathrm{Inf}_{x_0}(\mathcal{F}_{\mathcal{X},k,x_0}) = \mathrm{Ext}^0_{\mathcal{O}_{X_0}}(\mathcal{G}_0, \mathcal{G}_0)$$

These spaces are finite dimensional over k as $\mathcal{G}_0$ has support proper over $\mathrm{Spec}(k)$. Namely, X_0 is of finite presentation over $\mathrm{Spec}(k)$, hence Noetherian. Since $\mathcal{G}_0$ is of finite presentation it is a coherent $\mathcal{O}_{X_0}$-module. Thus we may apply Derived Categories of Spaces, Lemma 56.18.4 to conclude the desired finiteness. □

Lemma 72.5.10. *In Situation 72.5.1 assume that S is a locally Noetherian scheme. Let $\mathcal{X} = Coh_{X/B}$. Then the functor Artin's Axioms, Equation (71.9.2.1) is an equivalence.*

Proof. Let A be an S-algebra which is a complete local Noetherian ring with maximal ideal $\mathfrak{m}$ whose residue field k is of finite type over S. We have to show that the category of objects over A is equivalent to the category of formal objects over A. Since we know this holds for the category $\mathcal{S}_B$ fibred in sets associated to B by Artin's Axioms, Lemma 71.9.4, it suffices to prove this for those objects lying over a given morphism $\mathrm{Spec}(A) \to B$.

Set $X_A = \mathrm{Spec}(A) \times_B X$ and $X_n = \mathrm{Spec}(A/\mathfrak{m}^n) \times_B X$. By Grothendieck's existence theorem (More on Morphisms of Spaces, Theorem 57.30.11) we see that the category of coherent modules $\mathcal{F}$ on X_A with support proper over $\mathrm{Spec}(A)$ is equivalent to the category of systems $(\mathcal{F}_n)$ of coherent modules $\mathcal{F}_n$ on X_n with support proper over $\mathrm{Spec}(A/\mathfrak{m}^n)$. The equivalence sends $\mathcal{F}$ to the system $(\mathcal{F} \otimes_A A/\mathfrak{m}^n)$. See discussion in More on Morphisms of Spaces, Remark 57.30.12. To finish the proof of the lemma, it suffices to show that $\mathcal{F}$ is flat over A if and only if all $\mathcal{F} \otimes_A A/\mathfrak{m}^n$ are flat over $A/\mathfrak{m}^n$. This follows from More on Morphisms of Spaces, Lemma 57.20.1. □

Lemma 72.5.11. *In Situation 72.5.1 assume that S is a locally Noetherian scheme, $S = B$, and $f : X \to B$ is flat. Let $\mathcal{X} = Coh_{X/B}$. Then we have openness of versality for $\mathcal{X}$ (see Artin's Axioms, Definition 71.14.1).*

Proof. Let $U \to S$ be of finite type morphism of schemes, x an object of $\mathcal{X}$ over U and $u_0 \in U$ a finite type point such that x is versal at u_0. After shrinking U we may assume that u_0 is a closed point (Morphisms, Lemma 28.17.1) and $U = \mathrm{Spec}(A)$ with $U \to S$ mapping into an affine open $\mathrm{Spec}(\Lambda)$ of S. We will use Artin's Axioms, Lemma 71.21.4 to prove the lemma. Let $\mathcal{F}$ be the coherent module on $X_A = \mathrm{Spec}(A) \times_S X$ flat over A corresponding to the given object x.

According to Deformation Theory, Lemma 65.11.1 we have an isomorphism of functors

$$T_x(M) = \mathrm{Ext}^1_{X_A}(\mathcal{F}, \mathcal{F} \otimes_A M)$$

and given any surjection $A' \to A$ of Λ-algebras with square zero kernel I we have an obstruction class

$$\xi_{A'} \in \mathrm{Ext}^2_{X_A}(\mathcal{F}, \mathcal{F} \otimes_A I)$$

This uses that for any $A' \to A$ as above the base change $X_{A'} = \mathrm{Spec}(A') \times_B X$ is flat over A'. Apply Derived Categories of Spaces, Lemma 56.18.4 to the computation of the Ext groups $\mathrm{Ext}^i_{X_A}(\mathcal{F}, \mathcal{F} \otimes_A M)$ for $i \leq m$ with $m = 2$. We find a perfect object $K \in D(A)$ and functorial isomorphisms

$$H^i(K \otimes^{\mathbf{L}}_A M) \longrightarrow \mathrm{Ext}^i_{X_A}(\mathcal{F}, \mathcal{F} \otimes_A M)$$

for $i \leq m$ compatible with boundary maps. This object K, together with the displayed identifications above gives us a datum as in Artin's Axioms, Situation 71.21.2. Finally, condition (iv) of Artin's Axioms, Lemma 71.21.3 holds by Deformation Theory, Lemma 65.11.3. Thus Artin's Axioms, Lemma 71.21.4 does indeed apply and the lemma is proved. □

Theorem 72.5.12 (Algebraicity of stack coherent sheaves)**.** *Let S be a scheme. Let $f : X \to B$ be morphism of algebraic spaces over S. Assume that f is of finite presentation and flat*[2]*. Then $Coh_{X/B}$ is an algebraic stack over S.*

Proof. Set $\mathcal{X} = Coh_{X/B}$. We have seen that $\mathcal{X}$ is a stack in groupoids over $(Sch/S)_{fppf}$ with diagonal representable by algebraic spaces (Lemmas 72.5.4 and 72.5.3). Hence it suffices to find a scheme W and a surjective and smooth morphism $W \to \mathcal{X}$.

Let B' be a scheme and let $B' \to B$ be a surjective étale morphism. Set $X' = B' \times_B X$ and denote $f' : X' \to B'$ the projection. Then $\mathcal{X}' = Coh_{X'/B'}$ is equal to the 2-fibre product of $\mathcal{X}$ with the category fibred in sets associated to B' over the category fibred in sets associated to B (Remark 72.5.5). By the material in Algebraic Stacks, Section 67.10 the morphism $\mathcal{X}' \to \mathcal{X}$ is surjective and étale. Hence it suffices to prove the result for $\mathcal{X}'$. In other words, we may assume B is a scheme.

Assume B is a scheme. In this case we may replace S by B, see Algebraic Stacks, Section 67.19. Thus we may assume $S = B$.

Assume $S = B$. Choose an affine open covering $S = \bigcup U_i$. Denote $\mathcal{X}_i$ the restriction of $\mathcal{X}$ to $(Sch/U_i)_{fppf}$. If we can find schemes W_i over U_i and surjective smooth morphisms $W_i \to \mathcal{X}_i$, then we set $W = \coprod W_i$ and we obtain a surjective smooth morphism $W \to \mathcal{X}$. Thus we may assume $S = B$ is affine.

Assume $S = B$ is affine, say $S = \mathrm{Spec}(\Lambda)$. Write $\Lambda = \mathrm{colim}\, \Lambda_i$ as a filtered colimit with each Λ_i of finite type over $\mathbf{Z}$. For some i we can find a morphism of algebraic spaces $X_i \to \mathrm{Spec}(\Lambda_i)$ which is of finite presentation and flat and whose base change to Λ is X. See Limits of Spaces, Lemmas 52.7.1 and 52.6.11. If we show that $Coh_{X_i/\mathrm{Spec}(\Lambda_i)}$ is an algebraic stack, then it follows by base change (Remark 72.5.5 and Algebraic Stacks, Section 67.19) that $\mathcal{X}$ is an algebraic stack. Thus we may assume that Λ is a finite type $\mathbf{Z}$-algebra.

Assume $S = B = \mathrm{Spec}(\Lambda)$ is affine of finite type over $\mathbf{Z}$. In this case we will verify conditions (1), (2), (3), and (4) of Artin's Axioms, Lemma 71.17.1 to conclude that $\mathcal{X}$ is an algebraic stack. Note that Λ is a G-ring, see More on Algebra, Proposition 15.37.12. Hence all local rings of S are G-rings. Thus (4) holds. By Lemma 72.5.11 we have that $\mathcal{X}$ satisfies openness of versality, hence (3) holds. To check (2) we have to verify axioms [-1], [0], [1], [2], [3], and [4] of Artin's Axioms, Section 71.12. We

[2]This assumption is not necessary. See discussion in Section 72.6.

omit the verification of [-1] and axioms [0], [1], [2], [3], [4] correspond respectively to Lemmas 72.5.4, 72.5.6, 72.5.7, 72.5.9, and 72.5.10. Finally, condition (1) is Lemma 72.5.3. This finishes the proof of the theorem. □

72.6. The stack of coherent sheaves in the non-flat case

In Theorem 72.5.12 the assumption that $f : X \to B$ is flat is not necessary. In this section we explain where this assumption is used in the proof and one way to get around it.

For a different approach to this problem the reader may wish to consult [**Art69b**] and follow the method discussed in the papers [**OS03**], [**Lie06b**], [**Ols05**], [**HR13**], [**HR10**], [**Ryd11**]. Some of these papers deal with the more general case of the stack of coherent sheaves on an algebraic stack over an algebraic stack and others deal with similar problems in the case of Hilbert stacks or Quot functors. Our strategy will be to show algebraicity of some cases of Hilbert stacks and Quot functors as a consequence of the algebraicity of the stack of coherent sheaves.

The only step in the proof of Theorem 72.5.12 which uses flatness is in the application of Lemma 72.5.11. The lemma is used to construct an obstruction theory as in Artin's Axioms, Section 71.21. The proof of the lemma relies on Deformation Theory, Lemmas 65.11.1 and 65.11.3 from Deformation Theory, Section 65.11. This is how the assumption that f is flat comes about. Before we go on, note that results (2) and (3) of Deformation Theory, Lemmas 65.11.1 do hold without the assumption that f is flat as they rely on Deformation Theory, Lemmas 65.10.7. and 65.10.4 which do not have any flatness assumptions.

Before we give the details we give some motivation for the construction from derived algebraic geometry, since we think it will clarify what follows. Let A be a finite type algebra over the locally Noetherian base S. Denote $X \otimes^{\mathbf{L}} A$ a "derived base change" of X to A and denote $i : X_A \to X \otimes^{\mathbf{L}} A$ the canonical inclusion morphism. The object $X \otimes^{\mathbf{L}} A$ does not (yet) have a definition in the Stacks project; we may think of it as the algebraic space X_A endowed with a simplicial sheaf of rings $\mathcal{O}_{X \otimes^{\mathbf{L}} A}$ whose homology sheaves are

$$H_i(\mathcal{O}_{X \otimes^{\mathbf{L}} A}) = \mathrm{Tor}_i^{\mathcal{O}_S}(\mathcal{O}_X, A).$$

The morphism $X \otimes^{\mathbf{L}} A \to \mathrm{Spec}(A)$ is flat (the terms of the simplicial sheaf of rings being A-flat), so the usual material for deformations of flat modules applies to it. Thus we see that we get an obstruction theory using the groups

$$\mathrm{Ext}^i_{X \otimes^{\mathbf{L}} A}(i_*\mathcal{F}, i_*\mathcal{F} \otimes_A M)$$

where $i = 0, 1, 2$ for inf auts, inf defs, obstructions. Note that a flat deformation of $i_*\mathcal{F}$ to $X \otimes^{\mathbf{L}} A'$ is automatically of the form $i'_*\mathcal{F}'$ where $\mathcal{F}'$ is a flat deformation of $\mathcal{F}$. By adjunction of the functors Li^* and $i_* = Ri_*$ these ext groups are equal to

$$\mathrm{Ext}^i_{X_A}(Li^*(i_*\mathcal{F}), \mathcal{F} \otimes_A M)$$

Thus we obtain obstruction groups of exactly the same form as in the proof of Lemma 72.5.11 with the only change being that one replaces the first occurrence of $\mathcal{F}$ by the complex $Li^*(i_*\mathcal{F})$.

Below we prove the non-flat version of the lemma by a "direct" construction of $E(\mathcal{F}) = Li^*(i_*\mathcal{F})$ and direct proof of its relationship to the deformation theory of $\mathcal{F}$. In fact, it suffices to construct $\tau_{\geq -2}E(\mathcal{F})$, as we are only interested in the

ext groups $\mathrm{Ext}^i_{X_A}(Li^*(i_*\mathcal{F}), \mathcal{F} \otimes_A M)$ for $i = 0, 1, 2$. We can even identify the cohomology sheaves

$$H^i(E(\mathcal{F})) = \begin{cases} 0 & \text{if } i > 0 \\ \mathcal{F} & \text{if } i = 0 \\ 0 & \text{if } i = -1 \\ \mathrm{Tor}_1^{\mathcal{O}_S}(\mathcal{O}_X, A) \otimes_{\mathcal{O}_X} \mathcal{F} & \text{if } i = -2 \end{cases}$$

This observation will guide our construction of $E(\mathcal{F})$ in the remarks below.

Remark 72.6.1 (Direct construction)**.** Let S be a scheme. Let $f : X \to B$ be a morphism of algebraic spaces over S. Let U be another algebraic space over B. Denote $q : X \times_B U \to U$ the second projection. Consider the distinguished triangle

$$Lq^*L_{U/B} \to L_{X \times_B U/B} \to E \to Lq^*L_{U/B}[1]$$

of Cotangent, Section 66.27. For any sheaf $\mathcal{F}$ of $\mathcal{O}_{X \times_B U}$-modules we have the Atiyah class

$$\mathcal{F} \to L_{X \times_B U/B} \otimes^{\mathbf{L}}_{\mathcal{O}_{X \times_B U}} \mathcal{F}[1]$$

see Cotangent, Section 66.18. We can compose this with the map to E and choose a distinguished triangle

$$E(\mathcal{F}) \to \mathcal{F} \to \mathcal{F} \otimes^{\mathbf{L}}_{\mathcal{O}_{X \times_B U}} E[1] \to E(\mathcal{F})[1]$$

in $D(\mathcal{O}_{X \times_B U})$. By construction the Atiyah class lifts to a map

$$e_{\mathcal{F}} : E(\mathcal{F}) \longrightarrow Lq^*L_{U/B} \otimes^{\mathbf{L}}_{\mathcal{O}_{X \times_B U}} \mathcal{F}[1]$$

fitting into a morphism of distinguished triangles

$$\begin{array}{ccccc} \mathcal{F} \otimes^{\mathbf{L}} Lq^*L_{U/B}[1] & \longrightarrow & \mathcal{F} \otimes^{\mathbf{L}} L_{X \times_B U/B}[1] & \longrightarrow & \mathcal{F} \otimes^{\mathbf{L}} E[1] \\ \uparrow{\scriptstyle e_{\mathcal{F}}} & & \uparrow{\scriptstyle Atiyah} & & \uparrow{\scriptstyle =} \\ E(\mathcal{F}) & \longrightarrow & \mathcal{F} & \longrightarrow & \mathcal{F} \otimes^{\mathbf{L}} E[1] \end{array}$$

Given $S, B, X, f, U, \mathcal{F}$ we fix a choice of $E(\mathcal{F})$ and $e_{\mathcal{F}}$.

Remark 72.6.2 (Construction of obstruction class)**.** With notation as in Remark 72.6.1 let $i : U \to U'$ be a first order thickening of U over B. Let $\mathcal{I} \subset \mathcal{O}_{U'}$ be the quasi-coherent sheaf of ideals cutting out B in B'. The fundamental triangle

$$Li^*L_{U'/B} \to L_{U/B} \to L_{U/U'} \to Li^*L_{U'/B}[1]$$

together with the map $L_{U/U'} \to \mathcal{I}[1]$ determine a map $e_{U'} : L_{U/B} \to \mathcal{I}[1]$. Combined with the map $e_{\mathcal{F}}$ of the previous remark we obtain

$$(\mathrm{id}_{\mathcal{F}} \otimes Lq^*e_{U'}) \cup e_{\mathcal{F}} : E(\mathcal{F}) \longrightarrow \mathcal{F} \otimes_{\mathcal{O}_{X \times_B U}} q^*\mathcal{I}[2]$$

(we have also composed with the map from the derived tensor product to the usual tensor product). In other words, we obtain an element

$$\xi_{U'} \in \mathrm{Ext}^2_{\mathcal{O}_{X \times_B U}}(E(\mathcal{F}), \mathcal{F} \otimes_{\mathcal{O}_{X \times_B U}} q^*\mathcal{I})$$

Lemma 72.6.3. *In the situation of Remark 72.6.2 assume that $\mathcal{F}$ is flat over U. Then the vanishing of the class $\xi_{U'}$ is a necessary and sufficient condition for the existence of a $\mathcal{O}_{X \times_B U'}$-module $\mathcal{F}'$ flat over U' with $i^*\mathcal{F}' \cong \mathcal{F}$.*

Proof (sketch). We will use the criterion of Deformation Theory, Lemma 65.10.8. We will abbreviate $\mathcal{O} = \mathcal{O}_{X \times_B U}$ and $\mathcal{O}' = \mathcal{O}_{X \times_B U'}$. Consider the short exact sequence

$$0 \to \mathcal{I} \to \mathcal{O}_{U'} \to \mathcal{O}_U \to 0.$$

Let $\mathcal{J} \subset \mathcal{O}'$ be the quasi-coherent sheaf of ideals cutting out $X \times_B U$. By the above we obtain an exact sequence

$$\mathrm{Tor}_1^{\mathcal{O}_B}(\mathcal{O}_X, \mathcal{O}_U) \to q^*\mathcal{I} \to \mathcal{J} \to 0$$

where the $\mathrm{Tor}_1^{\mathcal{O}_B}(\mathcal{O}_X, \mathcal{O}_U)$ is an abbreviation for

$$\mathrm{Tor}_1^{h^{-1}\mathcal{O}_B}(p^{-1}\mathcal{O}_X, q^{-1}\mathcal{O}_U) \otimes_{(p^{-1}\mathcal{O}_X \otimes_{h^{-1}\mathcal{O}_B} q^{-1}\mathcal{O}_U)} \mathcal{O}.$$

Tensoring with $\mathcal{F}$ we obtain the exact sequence

$$\mathcal{F} \otimes_{\mathcal{O}} \mathrm{Tor}_1^{\mathcal{O}_B}(\mathcal{O}_X, \mathcal{O}_U) \to \mathcal{F} \otimes_{\mathcal{O}} q^*\mathcal{I} \to \mathcal{F} \otimes_{\mathcal{O}} \mathcal{J} \to 0$$

(Note that the roles of the letters $\mathcal{I}$ and $\mathcal{J}$ are reversed relative to the notation in Deformation Theory, Lemma 65.10.8.) Condition (1) of the lemma is that the last map above is an isomorphism, i.e., that the first map is zero. The vanishing of this map may be checked on stalks at geometric points $\overline{z} = (\overline{x}, \overline{u}) : \mathrm{Spec}(k) \to X \times_B U$. Set $R = \mathcal{O}_{B,\overline{b}}$, $A = \mathcal{O}_{X,\overline{x}}$, $B = \mathcal{O}_{U,\overline{u}}$, and $C = \mathcal{O}_{\overline{z}}$. By Cotangent, Lemma 66.27.2 and the defining triangle for $E(\mathcal{F})$ we see that

$$H^{-2}(E(\mathcal{F}))_{\overline{z}} = \mathcal{F}_{\overline{z}} \otimes \mathrm{Tor}_1^R(A, B)$$

The map $\xi_{U'}$ therefore induces a map

$$\mathcal{F}_{\overline{z}} \otimes \mathrm{Tor}_1^R(A, B) \longrightarrow \mathcal{F}_{\overline{z}} \otimes_B \mathcal{I}_{\overline{u}}$$

We claim this map is the same as the stalk of the map described above (proof omitted; this is a purely ring theoretic statement). Thus we see that condition (1) of Deformation Theory, Lemma 65.10.8 is equivalent to the vanishing $H^{-2}(\xi_{U'}) : H^{-2}(E(\mathcal{F})) \to \mathcal{F} \otimes \mathcal{I}$.

To finish the proof we show that, assuming that condition (1) is satisfied, condition (2) is equivalent to the vanising of $\xi_{U'}$. In the rest of the proof we write $\mathcal{F} \otimes \mathcal{I}$ to denote $\mathcal{F} \otimes_{\mathcal{O}} q^*\mathcal{I} = \mathcal{F} \otimes_{\mathcal{O}} \mathcal{J}$. A consideration of the spectral sequence

$$\mathrm{Ext}^i(H^{-j}(E(\mathcal{F})), \mathcal{F} \otimes \mathcal{I}) \Rightarrow \mathrm{Ext}^{i+j}(E(\mathcal{F}), \mathcal{F} \otimes \mathcal{I})$$

using that $H^0(E(\mathcal{F})) = \mathcal{F}$ and $H^{-1}(E(\mathcal{F})) = 0$ shows that there is an exact sequence

$$0 \to \mathrm{Ext}^2(\mathcal{F}, \mathcal{F} \otimes \mathcal{I}) \to \mathrm{Ext}^2(E(\mathcal{F}), \mathcal{F} \otimes \mathcal{I}) \to \mathrm{Hom}(H^{-2}(E(\mathcal{F})), \mathcal{F} \otimes \mathcal{I})$$

Thus our element $\xi_{U'}$ is an element of $\mathrm{Ext}^2(\mathcal{F}, \mathcal{F} \otimes \mathcal{I})$. The proof is finished by showing this element agrees with the element of Deformation Theory, Lemma 65.10.8 a verification we omit. □

Lemma 72.6.4. *In Situation 72.5.1 assume that S is a locally Noetherian scheme and $S = B$. Let $\mathcal{X} = Coh_{X/B}$. Then we have openness of versality for $\mathcal{X}$ (see Artin's Axioms, Definition 71.14.1).*

Proof (sketch). Let $U \to S$ be of finite type morphism of schemes, x an object of $\mathcal{X}$ over U and $u_0 \in U$ a finite type point such that x is versal at u_0. After shrinking U we may assume that u_0 is a closed point (Morphisms, Lemma 28.17.1) and $U = \mathrm{Spec}(A)$ with $U \to S$ mapping into an affine open $\mathrm{Spec}(\Lambda)$ of S. We will use Artin's Axioms, Lemma 71.21.4 to prove the lemma. Let $\mathcal{F}$ be the coherent module on $X_A = \mathrm{Spec}(A) \times_S X$ flat over A corresponding to the given object x.

Choose $E(\mathcal{F})$ and $e_{\mathcal{F}}$ as in Remark 72.6.1. The description of the cohomology sheaves of $E(\mathcal{F})$ shows that

$$\mathrm{Ext}^1(E(\mathcal{F}), \mathcal{F} \otimes_A M) = \mathrm{Ext}^1(\mathcal{F}, \mathcal{F} \otimes_A M)$$

for any A-module M. Using this and using Deformation Theory, Lemma 65.10.7 we have an isomorphism of functors

$$T_x(M) = \mathrm{Ext}^1_{X_A}(E(\mathcal{F}), \mathcal{F} \otimes_A M)$$

By Lemma 72.6.3 given any surjection $A' \to A$ of Λ-algebras with square zero kernel I we have an obstruction class

$$\xi_{A'} \in \mathrm{Ext}^2_{X_A}(E(\mathcal{F}), \mathcal{F} \otimes_A I)$$

Apply Derived Categories of Spaces, Lemma 56.18.4 to the computation of the Ext groups $\mathrm{Ext}^i_{X_A}(E(\mathcal{F}), \mathcal{F} \otimes_A M)$ for $i \leq m$ with $m = 2$. We omit the verification that $E(\mathcal{F})$ is in D^-_{Coh}; hint: use Cotangent, Lemma 66.5.4. We find a perfect object $K \in D(A)$ and functorial isomorphisms

$$H^i(K \otimes^{\mathbf{L}}_A M) \longrightarrow \mathrm{Ext}^i_{X_A}(E(\mathcal{F}), \mathcal{F} \otimes_A M)$$

for $i \leq m$ compatible with boundary maps. This object K, together with the displayed identifications above gives us a datum as in Artin's Axioms, Situation 71.21.2. Finally, condition (iv) of Artin's Axioms, Lemma 71.21.3 holds by a variant of Deformation Theory, Lemma 65.11.3 whose formulation and proof we omit. Thus Artin's Axioms, Lemma 71.21.4 applies and the lemma is proved. □

Theorem 72.6.5 (Algebraicity of stack coherent sheaves; general case)**.** *Let S be a scheme. Let $f : X \to B$ be morphism of algebraic spaces over S. Assume that f is of finite presentation. Then $Coh_{X/B}$ is an algebraic stack over S.*

Proof. Identical to the proof of Theorem 72.5.12 except that we substitute Lemma 72.6.4 for Lemma 72.5.11. □

72.7. Flattening functors

This section is the analogue of More on Flatness, Section 37.19. We urge the reader to skip this section on a first reading.

Situation 72.7.1. Let S be a scheme. Let $f : X \to B$ be a morphism of algebraic spaces over S. Let $u : \mathcal{F} \to \mathcal{G}$ be a homomorphism of quasi-coherent $\mathcal{O}_X$-modules. For any scheme T over B we will denote $u_T : \mathcal{F}_T \to \mathcal{G}_T$ the base change of u to T, in other words, u_T is the pullback of u via the projection morphism $X_T = X \times_B T \to X$. In this situation we can consider the functor

$$F_{iso} : (Sch/B)^{opp} \longrightarrow Sets, \quad T \longrightarrow \begin{cases} \{*\} & \text{if } u_T \text{ is an isomorphism,} \\ \emptyset & \text{else.} \end{cases} \tag{72.7.1.1}$$

There are variants F_{inj}, F_{surj}, F_{zero} where we ask that u_T is injective, surjective, or zero.

In Situation 72.7.1 we sometimes think of the functors F_{iso}, F_{inj}, F_{surj}, and F_{zero} as functors $(Sch/S)^{opp} \to Sets$ endowed with a morphism $F_{iso} \to B$, $F_{inj} \to B$, $F_{surj} \to B$, and $F_{zero} \to B$. Namely, if T is a scheme over S, then an element $h \in F_{iso}(T)$ is just a morphism $h : T \to B$, i.e., an element $h \in B(T)$, such that the base change of u via h is an isomorphism. In particular, when we say that F_{iso} is an algebraic space, we mean that the corresponding functor $(Sch/S)^{opp} \to Sets$ is an algebraic space.

Lemma 72.7.2. *In Situation 72.7.1. Each of the functors F_{iso}, F_{inj}, F_{surj}, F_{zero} satisfies the sheaf property for the fpqc topology.*

Proof. Let $\{T_i \to T\}_{i\in I}$ be an fpqc covering of schemes over B. Set $X_i = X_{T_i} = X \times_S T_i$ and $u_i = u_{T_i}$. Note that $\{X_i \to X_T\}_{i\in I}$ is an fpqc covering of X_T, see Topologies on Spaces, Lemma 54.3.2. In particular, for every $x \in |X_T|$ there exists an $i \in I$ and an $x_i \in |X_i|$ mapping to x. Since $\mathcal{O}_{X_T,\overline{x}} \to \mathcal{O}_{X_i,\overline{x_i}}$ is flat, hence faithfully flat (see Morphisms of Spaces, Section 49.27). we conclude that $(u_i)_{x_i}$ is injective, surjective, bijective, or zero if and only if $(u_T)_x$ is injective, surjective, bijective, or zero. The lemma follows. □

Lemma 72.7.3. *In Situation 72.7.1 let $X' \to X$ be a flat morphism of algebraic spaces. Denote $u' : \mathcal{F}' \to \mathcal{G}'$ the pullback of u to X'. Denote F'_{iso}, F'_{inj}, F'_{surj}, F'_{zero} the functors on Sch/B associated to u'.*

1. *If $\mathcal{G}$ is of finite type and the image of $|X'| \to |X|$ contains the support of $\mathcal{G}$, then $F_{surj} = F'_{surj}$ and $F_{zero} = F'_{zero}$.*
2. *If $\mathcal{F}$ is of finite type and the image of $|X'| \to |X|$ contains the support of $\mathcal{F}$, then $F_{inj} = F'_{inj}$ and $F_{zero} = F'_{zero}$.*
3. *If $\mathcal{F}$ and $\mathcal{G}$ are of finite type and the image of $|X'| \to |X|$ contains the supports of $\mathcal{F}$ and $\mathcal{G}$, then $F_{iso} = F'_{iso}$.*

Proof. let $v : \mathcal{H} \to \mathcal{E}$ be a map of quasi-coherent modules on an algebraic space Y and let $\varphi : Y' \to Y$ be a surjective flat morphism of algebraic spaces, then v is an isomorphism, injective, surjective, or zero if and only if $\varphi^* v$ is an isomorphism, injective, surjective, or zero. Namely, for every $y \in |Y|$ there exists a $y' \in |Y'|$ and the map of local rings $\mathcal{O}_{Y,\overline{y}} \to \mathcal{O}_{Y',\overline{y'}}$ is faithfully flat (see Morphisms of Spaces, Section 49.27). Of course, to check for injectivity or being zero it suffices to look at the points in the support of $\mathcal{H}$, and to check for surjectivity it suffices to look at points in the support of $\mathcal{E}$. Moreover, under the finite type assumptions as in the statement of the lemma, taking the supports commutes with base change, see Morphisms of Spaces, Lemma 49.15.2. Thus the lemma is clear. □

Recall that we've defined the scheme theoretic support of a finite type quasi-coherent module in Morphisms of Spaces, Definition 49.15.4.

Lemma 72.7.4. *In Situation 72.7.1.*

1. *If $\mathcal{G}$ is of finite type and the scheme theoretic support of $\mathcal{G}$ is quasi-compact over B, then F_{surj} is limit preserving.*
2. *If $\mathcal{F}$ of finite type and the scheme theoretic support of $\mathcal{F}$ is quasi-compact over B, then F_{zero} is limit preserving.*
3. *If $\mathcal{F}$ is of finite type, $\mathcal{G}$ is of finite presentation, and the scheme theoretic supports of $\mathcal{F}$ and $\mathcal{G}$ are quasi-compact over B, then F_{iso} is limit preserving.*

Proof. Proof of (1). Let $i : Z \to X$ be the scheme theoretic support of $\mathcal{G}$ and think of $\mathcal{G}$ as a finite type quasi-coherent module on Z. We may replace X by Z and u by the map $i^*\mathcal{F} \to \mathcal{G}$ (details omitted). Hence we may assume f is quasi-compact and $\mathcal{G}$ of finite type. Let $T = \lim_{i \in I} T_i$ be a directed limit of affine B-schemes and assume that u_T is surjective. Set $X_i = X_{T_i} = X \times_S T_i$ and $u_i = u_{T_i} : \mathcal{F}_i = \mathcal{F}_{T_i} \to \mathcal{G}_i = \mathcal{G}_{T_i}$. To prove (1) we have to show that u_i is surjective for some i. Pick $0 \in I$ and replace I by $\{i \mid i \geq 0\}$. Since f is quasi-compact we see X_0 is quasi-compact. Hence we may choose a surjective étale morphism $\varphi_0 : W_0 \to X_0$ where W_0 is an affine scheme. Set $W = W_0 \times_{T_0} T$ and $W_i = W_0 \times_{T_0} T_i$ for $i \geq 0$. These are affine schemes endowed with a surjective étale morphisms $\varphi : W \to X_T$ and $\varphi_i : W_i \to X_i$. Note that $W = \lim W_i$. Hence $\varphi^* u_T$ is surjective and it suffices to prove that $\varphi_i^* u_i$ is surjective for some i. Thus we have reduced the problem to the affine case which is Algebra, Lemma 10.121.3 part (2).

Proof of (2). Assume $\mathcal{F}$ is of finite type with scheme theoretic support $Z \subset B$ quasi-compact over B. Let $T = \lim_{i \in I} T_i$ be a directed limit of affine B-schemes and assume that u_T is zero. Set $X_i = T_i \times_B X$ and denote $u_i : \mathcal{F}_i \to \mathcal{G}_i$ the pullback. Choose $0 \in I$ and replace I by $\{i \mid i \geq 0\}$. Set $Z_0 = Z \times_X X_0$. By Morphisms of Spaces, Lemma 49.15.2 the support of $\mathcal{F}_i$ is $|Z_0|$. Since $|Z_0|$ is quasi-compact we can find an affine scheme W_0 and an étale morphism $W_0 \to X_0$ such that $|Z_0| \subset \mathrm{Im}(|W_0| \to |X_0|)$. Set $W = W_0 \times_{T_0} T$ and $W_i = W_0 \times_{T_0} T_i$ for $i \geq 0$. These are affine schemes endowed with étale morphisms $\varphi : W \to X_T$ and $\varphi_i : W_i \to X_i$. Note that $W = \lim W_i$ and that the support of $\mathcal{F}_T$ and $\mathcal{F}_i$ is contained in the image of $|W| \to |X_T|$ and $|W_i| \to |X_i|$. Now $\varphi^* u_T$ is injective and it suffices to prove that $\varphi_i^* u_i$ is injective for some i. Thus we have reduced the problem to the affine case which is Algebra, Lemma 10.121.3 part (1).

Proof of (3). This can be proven in exactly the same manner as in the previous two paragraphs using Algebra, Lemma 10.121.3 part (3). We can also deduce it from (1) and (2) as follows. Let $T = \lim_{i \in I} T_i$ be a directed limit of affine B-schemes and assume that u_T is an isomorphism. By part (1) there exists an $0 \in I$ such that u_{T_0} is surjective. Set $\mathcal{K} = \mathrm{Ker}(u_{T_0})$ and consider the map of quasi-coherent modules $v : \mathcal{K} \to \mathcal{F}_{T_0}$. For $i \geq 0$ the base change v_{T_i} is zero if and only if u_i is an isomorphism. Moreover, v_T is zero. Since $\mathcal{G}_{T_0}$ is of finite presentation, $\mathcal{F}_{T_0}$ is of finite type, and u_{T_0} is surjective we conclude that $\mathcal{K}$ is of finite type (Modules on Sites, Lemma 18.24.1). It is clear that the support of $\mathcal{K}$ is contained in the support of $\mathcal{F}_{T_0}$ which is quasi-compact over T_0. Hence we can apply part (2) to see that v_{T_i} is zero for some i. □

Lemma 72.7.5. *Let $S = \mathrm{Spec}(R)$ be an affine scheme. Let X be an algebraic space over S. Let $u : \mathcal{F} \to \mathcal{G}$ be a map of quasi-coherent $\mathcal{O}_X$-modules. Assume $\mathcal{G}$ flat over S. Let $T \to S$ be a quasi-compact morphism of schemes such that the base change u_T is zero. Then exists a closed subscheme $Z \subset S$ such that (a) $T \to S$ factors through Z and (b) the base change u_Z is zero. If $\mathcal{F}$ is a finite type $\mathcal{O}_X$-module and the scheme theoretic support of $\mathcal{F}$ is quasi-compact, then we can take $Z \to S$ of finite presentation.*

Proof. Let $U \to X$ be a surjective étale morphism of algebraic spaces where $U = \coprod U_i$ is a disjoint union of affine schemes (see Properties of Spaces, Lemma 48.6.1). By Lemma 72.7.3 we see that we may replace X by U. In other words, we may assume that $X = \coprod X_i$ is a disjoint union of affine schemes X_i. Suppose that we can

prove the lemma for $u_i = u|_{X_i}$. Then we find a closed subscheme $Z_i \subset S$ such that $T \to S$ factors through Z_i and u_{i,Z_i} is zero. If $Z_i = \mathrm{Spec}(R/I_i) \subset \mathrm{Spec}(R) = S$, then taking $Z = \mathrm{Spec}(R/\sum I_i)$ works. Thus we may assume that $X = \mathrm{Spec}(A)$ is affine.

Choose a finite affine open covering $T = T_1 \cup \ldots \cup T_m$. It is clear that we may replace T by $\coprod_{j=1,\ldots,m} T_j$. Hence we may assume T is affine. Say $T = \mathrm{Spec}(R')$. Let $u : M \to N$ be the homomorphisms of A-modules corresponding to $u : \mathcal{F} \to \mathcal{G}$. Then N is a flat R-module as $\mathcal{G}$ is flat over S. The assumption of the lemma means that the composition

$$M \otimes_R R' \to N \otimes_R R'$$

is zero. Let $z \in M$. By Lazard's theorem (Algebra, Theorem 10.76.4) and the fact that $\otimes$ commutes with colimits we can find free R-module F_z, an element $\tilde{z} \in F_z$, and a map $F_z \to N$ such that $u(z)$ is the image of $\tilde{z}$ and $\tilde{z}$ maps to zero in $F_z \otimes_R R'$. Choose a basis $\{e_{z,\alpha}\}$ of F_z and write $\tilde{z} = \sum f_{z,\alpha} e_{z,\alpha}$ with $f_{z,\alpha} \in R$. Let $I \subset R$ be the ideal generated by the elements $f_{z,\alpha}$ with z ranging over all elements of M. By construction I maps to zero in R' and the elements $\tilde{z}$ map to zero in F_z/IF_z whence in N/IN. Thus $Z = \mathrm{Spec}(R/I)$ is a solution to the problem in this case.

Assume $\mathcal{F}$ is of finite type with quasi-compact scheme theoretic support. Write $Z = \mathrm{Spec}(R/I)$. Write $I = \bigcup I_\lambda$ as a filtered union of finitely generated ideals. Set $Z_\lambda = \mathrm{Spec}(R/I_\lambda)$, so $Z = \operatorname{colim} Z_\lambda$. Since u_Z is zero, we see that u_{Z_λ} is zero for some λ by Lemma 72.7.4. This finishes the proof of the lemma. □

Lemma 72.7.6. *Let A be a ring. Let $u : M \to N$ be a map of A-modules. If N is projective as an A-module, then there exists an ideal $I \subset A$ such that for any ring map $\varphi : A \to B$ the following are equivalent*

(1) *$u \otimes 1 : M \otimes_A B \to N \otimes_A B$ is zero, and*
(2) *$\varphi(I) = 0$.*

Proof. As M is projective we can find a projective A-module C such that $F = N \oplus C$ is a free R-module. By replacing u by $u \oplus 1 : F = M \oplus C \to N \oplus C$ we see that we may assume N is free. In this case let I be the ideal of A generated by coefficients of all the elements of $\mathrm{Im}(u)$ with respect to some (fixed) basis of N. □

It would be interesting to find a simple direct proof of the following lemma using the result of Lemma 72.7.5. A "classical" proof of this lemma when $f : X \to B$ is a projective morphism and B a Noetherian scheme would be: (a) choose a relatively ample invertible sheaf $\mathcal{O}_X(1)$, (b) set $u_n : f_*\mathcal{F}(n) \to f_*\mathcal{G}(n)$, (c) observe that $f_*\mathcal{G}(n)$ is a finite locally free sheaf for all $n \gg 0$, and (d) F_{zero} is represented by the vanishing locus of u_n for some $n \gg 0$.

Lemma 72.7.7. *In Situation 72.7.1. Assume*

(1) *f is locally of finite presentation,*
(2) *$\mathcal{G}$ is an $\mathcal{O}_X$-module of finite presentation flat over B,*
(3) *the scheme theoretic support of $\mathcal{G}$ is proper over B.*

Then the functor F_{zero} is an algebraic space and $F_{zero} \to B$ is a closed immersion. If $\mathcal{F}$ is of finite type, then $F_{zero} \to B$ is of finite presentation.

Proof. In order to prove that F_{zero} is an algebraic space, it suffices to show that $F_{zero} \to B$ is representable, see Spaces, Lemma 47.11.1. Let $B' \to B$ be a morphism where B' is a scheme and let $u' : \mathcal{F}' \to \mathcal{G}'$ be the pullback of u to $X' = X_{B'}$. Then

the associated functor F'_{zero} equals $F_{zero} \times_B B'$. This reduces us to the case that B is a scheme.

Assume B is a scheme. We will show that F_{zero} is representable by a closed subscheme of B. By Lemma 72.7.2 and Descent, Lemmas 34.33.2 and 34.35.1 the question is local for the étale topology on B. Let $b \in B$. We first replace B by an affine neighbourhood of b. Denote $Z \subset X$ the scheme theoretic support of $\mathcal{G}$. Denote $Z_b \subset X_b$ the fibre of $Z \subset X \to B$ over b. The space $|Z_b|$ is quasi-compact by the last assumption of the lemma. Choose an affine scheme U and an étale morphism $\varphi : U \to X$ such that $|Z_b| \subset \mathrm{Im}(|U| \to |X|)$. After replacing B by an affine elementary étale neighbourhood of b and replacing U by some affine U' étale over U with $U'_b \to U_b$ surjective, we may assume that $\Gamma(U, \varphi^*\mathcal{G})$ is a projective $\Gamma(B, \mathcal{O}_B)$-module, see More on Flatness, Lemma 37.11.5. Since $Z \to B$ is proper the image of

$$|Z| \setminus \mathrm{Im}(|U| \to |X|)$$

in $|B|$ is a closed subset not containing b. Hence, after replacing B by an affine open containing b, we may assume that $|Z| \subset \mathrm{Im}(|U| \to |X|)$. (To be sure, after this replacement it is still true that $\Gamma(U, \varphi^*\mathcal{G})$ is a projective $\Gamma(B, \mathcal{O}_B)$-module.) By Lemma 72.7.3 we see that F_{zero} is the same as the corresponding functor for the map $\varphi^*\mathcal{F} \to \varphi^*\mathcal{G}$. This case follows immediately from Lemma 72.7.6.

We still have to show that $F_{zero} \to B$ is of finite presentation if $\mathcal{F}$ is of finite type. Let $\mathcal{F}' \subset \mathcal{G}$ be the image of u and denote F'_{zero} the functor corresponding to $\mathcal{F}' \to \mathcal{G}$. Then $F_{zero} = F'_{zero}$ and the scheme theoretic support of $\mathcal{F}'$ is a closed subspace of the scheme theoretic support of $\mathcal{G}$, hence proper over B. Thus Lemma 72.7.4 implies that $F_{zero} = F'_{zero}$ is limit preserving over B. We conclude by Limits of Spaces, Proposition 52.3.9. □

The following result is a variant of More on Flatness, Theorem 37.22.3.

Lemma 72.7.8. *In Situation 72.7.1. Assume*

(1) *f is locally of finite presentation,*
(2) *$\mathcal{F}$ is locally of finite presentation and flat over B,*
(3) *the scheme theoretic support of $\mathcal{F}$ is proper over B, and*
(4) *u is surjective.*

Then the functor F_{iso} is an algebraic space and $F_{iso} \to B$ is a closed immersion. If $\mathcal{G}$ is of finite presentation, then $F_{iso} \to B$ is of finite presentation.

Proof. Let $\mathcal{K} = \mathrm{Ker}(u)$ and apply Lemma 72.7.7 to $\mathcal{K} \to \mathcal{F}$. Note that $\mathcal{K}$ is of finite type if $\mathcal{G}$ is of finite presentation, see Modules on Sites, Lemma 18.24.1. □

We will use the following (easy) result when discussing the Quot functor.

Lemma 72.7.9. *In Situation 72.7.1. Assume*

(1) *f is locally of finite presentation,*
(2) *$\mathcal{G}$ is of finite type,*
(3) *the scheme theoretic support of $\mathcal{G}$ is proper over B.*

Then F_{surj} is an algebraic space and $F_{surj} \to B$ is an open immersion.

Proof. Consider $\mathrm{Coker}(u)$. Observe that $\mathrm{Coker}(u_T) = \mathrm{Coker}(u)_T$ for any T/B. Note that formation of the support of a finite type quasi-coherent module commutes

with pullback (Morphisms of Spaces, Lemma 49.15.1). Hence F_{surj} is representable by the open subspace of B corresponding to the open set

$$|B| \setminus |f|(\mathrm{Supp}(\mathrm{Coker}(u)))$$

see Properties of Spaces, Lemma 48.4.8. This is an open because $|f|$ is closed on $\mathrm{Supp}(\mathcal{G})$ and $\mathrm{Supp}(\mathrm{Coker}(u))$ is a closed subset of $\mathrm{Supp}(\mathcal{G})$. □

72.8. The functor of quotients

In this section we discuss some generalities regarding the functor $Q_{\mathcal{F}/X/B}$ defined below. The notation $\mathrm{Quot}_{\mathcal{F}/X/B}$ is reserved for a subfunctor of $\mathrm{Q}_{\mathcal{F}/X/B}$. We urge the reader to skip this section on a first reading.

Situation 72.8.1. Let S be a scheme. Let $f : X \to B$ be a morphism of algebraic spaces over S. Let $\mathcal{F}$ be a quasi-coherent $\mathcal{O}_X$-module. For any scheme T over B we will denote X_T the base change of X to T and $\mathcal{F}_T$ the pullback of $\mathcal{F}$ via the projection morphism $X_T = X \times_S T \to X$. Given such a T we set

$$\mathrm{Q}_{\mathcal{F}/X/B}(T) = \left\{ \begin{array}{c} \text{quotients } \mathcal{F}_T \to \mathcal{Q} \text{ where } \mathcal{Q} \text{ is a quasi-coherent} \\ \mathcal{O}_{X_T}\text{-module of finite presentation, flat over } T \end{array} \right\}$$

We identify quotients if they have the same kernel. Suppose that $T' \to T$ is a morphism of schemes over B and $\mathcal{F}_T \to \mathcal{Q}$ is an element of $\mathrm{Q}_{\mathcal{F}/X/B}(T)$. Then the pullback $\mathcal{Q}' = (X_{T'} \to X_T)^*\mathcal{Q}$ is a quasi-coherent $\mathcal{O}_{X_{T'}}$-module of finite presentation flat over T' (see Properties of Spaces, Section 48.28 and Morphisms of Spaces, Lemma 49.28.3). Thus we obtain a functor

$$\mathrm{Q}_{\mathcal{F}/X/B} : (Sch/B)^{opp} \longrightarrow Sets \tag{72.8.1.1}$$

This is the functor of quotients of $\mathcal{F}/X/B$.

In Situation 72.8.1 we sometimes think of $\mathrm{Q}_{\mathcal{F}/X/B}$ as a functor $(Sch/S)^{opp} \to Sets$ endowed with a morphism $\mathrm{Q}_{\mathcal{F}/X/S} \to B$. Namely, if T is a scheme over S, then we can think of an element of $\mathrm{Q}_{\mathcal{F}/X/B}$ as a pair $(h, \mathcal{Q})$ where h a morphism $h : T \to B$, i.e., an element $h \in B(T)$, and $\mathcal{Q}$ is a T-flat quotient $\mathcal{F}_T \to \mathcal{Q}$ of finite presentation on $X_T = X \times_{B,h} T$. In particular, when we say that $\mathrm{Q}_{\mathcal{F}/X/S}$ is an algebraic space, we mean that the corresponding functor $(Sch/S)^{opp} \to Sets$ is an algebraic space.

Remark 72.8.2. In Situation 72.8.1 let $B' \to B$ be a morphism of algebraic spaces over S. Set $X' = X \times_B B'$ and denote $\mathcal{F}'$ the pullback of $\mathcal{F}$ to X'. Thus we have the functor $Q_{\mathcal{F}'/X'/B'}$ on the category of schemes over B'. For a scheme T over B' it is clear that we have

$$Q_{\mathcal{F}'/X'/B'}(T) = Q_{\mathcal{F}/X/B}(T)$$

where on the right hand side we think of T as a scheme over B via the composition $T \to B' \to B$. This trivial remark will occasionally be useful to change the base algebraic space.

Remark 72.8.3. Let S be a scheme, X an algebraic space over S, and $\mathcal{F}$ a quasi-coherent $\mathcal{O}_X$-module. Suppose that $\{f_i : X_i \to X\}_{i \in I}$ is an fpqc covering and for each $i, j \in I$ we are given an fpqc covering $\{X_{ijk} \to X_i \times_X X_j\}$. In this situation we have a bijection

$$\left\{ \begin{array}{c} \text{quotients } \mathcal{F} \to \mathcal{Q} \text{ where} \\ \mathcal{Q} \text{ is a quasi-coherent} \end{array} \right\} \longrightarrow \left\{ \begin{array}{c} \text{families of quotients } f_i^*\mathcal{F} \to \mathcal{Q}_i \text{ where} \\ \mathcal{Q}_i \text{ is quasi-coherent and } \mathcal{Q}_i \text{ and } \mathcal{Q}_j \\ \text{restrict to the same quotient on } X_{ijk} \end{array} \right\}$$

Namely, let $(f_i^*\mathcal{F} \to \mathcal{Q}_i)_{i\in I}$ be an element of the right hand side. Then since $\{X_{ijk} \to X_i \times_X X_j\}$ is an fpqc covering we see that the pullbacks of $\mathcal{Q}_i$ and $\mathcal{Q}_j$ restrict to the same quotient of the pullback of $\mathcal{F}$ to $X_i \times_X X_j$ (by fully faithfulness in Descent on Spaces, Proposition 55.4.1). Hence we obtain a descent datum for quasi-coherent modules with respect to $\{X_i \to X\}_{i\in I}$. By Descent on Spaces, Proposition 55.4.1 we find a map of quasi-coherent $\mathcal{O}_X$-modules $\mathcal{F} \to \mathcal{Q}$ whose restriction to X_i recovers the given maps $f_i^*\mathcal{F} \to \mathcal{Q}_i$. Since the family of morphisms $\{X_i \to X\}$ is jointly surjective and flat, for every point $x \in |X|$ there exists an i and a point $x_i \in |X_i|$ mapping to x. Note that the induced map on local rings $\mathcal{O}_{X,\overline{x}} \to \mathcal{O}_{X_i,\overline{x_i}}$ is faithfully flat, see Morphisms of Spaces, Section 49.27. Thus we see that $\mathcal{F} \to \mathcal{Q}$ is surjective.

Lemma 72.8.4. *In Situation 72.8.1. The functor $Q_{\mathcal{F}/X/B}$ satisfies the sheaf property for the fpqc topology.*

Proof. Let $\{T_i \to T\}_{i\in I}$ be an fpqc covering of schemes over S. Set $X_i = X_{T_i} = X \times_S T_i$ and $\mathcal{F}_i = \mathcal{F}_{T_i}$. Note that $\{X_i \to X_T\}_{i\in I}$ is an fpqc covering of X_T (Topologies on Spaces, Lemma 54.3.2) and that $X_{T_i\times_T T_{i'}} = X_i \times_{X_T} X_{i'}$. Suppose that $\mathcal{F}_i \to \mathcal{Q}_i$ is a collection of elements of $\mathrm{Q}_{\mathcal{F}/X/B}(T_i)$ such that $\mathcal{Q}_i$ and $\mathcal{Q}_{i'}$ restrict to the same element of $\mathrm{Q}_{\mathcal{F}/X/B}(T_i \times_T T_{i'})$. By Remark 72.8.3 we obtain a surjective map of quasi-coherent $\mathcal{O}_{X_T}$-modules $\mathcal{F}_T \to \mathcal{Q}$ whose restriction to X_i recovers the given quotients. By Morphisms of Spaces, Lemma 49.28.5 we see that $\mathcal{Q}$ is flat over T. Finally, Descent on Spaces, Lemma 55.5.2 guarantees that $\mathcal{Q}$ is of finite presentation as an $\mathcal{O}_{X_T}$-module. □

Lemma 72.8.5. *In Situation 72.8.1 let $\{X_i \to X\}_{i\in I}$ be an fppf covering and for each $i, j \in I$ let $\{X_{ijk} \to X_i \times_X X_j\}$ be an fppf covering. Denote $\mathcal{F}_i$, resp. $\mathcal{F}_{ijk}$ the pullback of $\mathcal{F}$ to X_i, resp. X_{ijk}. For every scheme T over B the diagram*

$$Q_{\mathcal{F}/X/B}(T) \longrightarrow \prod_i Q_{\mathcal{F}_i/X_i/B}(T) \underset{pr_1^*}{\overset{pr_0^*}{\rightrightarrows}} \prod_{i,j,k} Q_{\mathcal{F}_{ijk}/X_{ijk}/B}(T)$$

presents the first arrow as the equalizer of the other two.

Proof. Let $\mathcal{F}_{i,T} \to \mathcal{Q}_i$ be an element in the equalizer of pr_0^* and pr_1^*. By Remark 72.8.3 we obtain a surjection $\mathcal{F}_T \to \mathcal{Q}$ of quasi-coherent $\mathcal{O}_{X_T}$-modules whose restriction to $X_{i,T}$ recovers $\mathcal{F}_i \to \mathcal{Q}_i$. By Morphisms of Spaces, Lemma 49.28.5 we see that $\mathcal{Q}$ is flat over T as desired. □

Lemma 72.8.6. *In Situation 72.8.1 assume also that (a) f is quasi-compact and quasi-separated and (b) $\mathcal{F}$ is of finite presentation. Then the functor $Q_{\mathcal{F}/X/B}$ is limit preserving in the following sense: If $T = \lim T_i$ is a directed limit of affine schemes over B, then $Q_{\mathcal{F}/X/B}(T) = \operatorname{colim} Q_{\mathcal{F}/X/B}(T_i)$.*

Proof. Let $T = \lim T_i$ be as in the statement of the lemma. Choose $i_0 \in I$ and replace I by $\{i \in I \mid i \geq i_0\}$. We may set $B = S = T_{i_0}$ and we may replace X by X_{T_0} and $\mathcal{F}$ by the pullback to X_{T_0}. Then $X_T = \lim X_{T_i}$, see Limits of Spaces, Lemma 52.4.1. Let $\mathcal{F}_T \to \mathcal{Q}$ be an element of $\mathrm{Q}_{\mathcal{F}/X/B}(T)$. By Limits of Spaces, Lemma 52.7.2 there exists an i and a map $\mathcal{F}_{T_i} \to \mathcal{Q}_i$ of $\mathcal{O}_{X_{T_i}}$-modules of finite presentation whose pullback to X_T is the given quotient map.

We still have to check that, after possibly increasing i, the map $\mathcal{F}_{T_i} \to \mathcal{Q}_i$ is surjective and $\mathcal{Q}_i$ is flat over T_i. To do this, choose an affine scheme U and a

surjective étale morphism $U \to X$ (see Properties of Spaces, Lemma 48.6.3). We may check surjectivity and flatness over T_i after pulling back to the étale cover $U_{T_i} \to X_{T_i}$ (by definition). This reduces us to the case where $X = \mathrm{Spec}(B_0)$ is an affine scheme of finite presentation over $B = S = T_0 = \mathrm{Spec}(A_0)$. Writing $T_i = \mathrm{Spec}(A_i)$, then $T = \mathrm{Spec}(A)$ with $A = \mathrm{colim}\, A_i$ we have reached the following algebra problem. Let $M_i \to N_i$ be a map of finitely presented $B_0 \otimes_{A_0} A_i$-modules such that $M_i \otimes_{A_i} A \to N_i \otimes_{A_i} A$ is surjective and $N_i \otimes_{A_i} A$ is flat over A. Show that for some $i' \geq i$ $M_i \otimes_{A_i} A_{i'} \to N_i \otimes_{A_i} A_{i'}$ is surjective and $N_i \otimes_{A_i} A_{i'}$ is flat over A. The first follows from Algebra, Lemma 10.121.3 and the second from Algebra, Lemma 10.154.1. □

Lemma 72.8.7. *In Situation 72.8.1 assume $X \to B$ locally of finite presentation. Let*

$$\begin{array}{ccc} Z & \longrightarrow & Z' \\ \downarrow & & \downarrow \\ Y & \longrightarrow & Y' \end{array}$$

be a pushout in the category of schemes over B where $Z \to Z'$ is a thickening and $Z \to Y$ is affine, see More on Morphisms, Lemma 36.11.1. Then the natural map

$$Q_{\mathcal{F}/X/B}(Y') \longrightarrow Q_{\mathcal{F}/X/B}(Y) \times_{Q_{\mathcal{F}/X/B}(Z)} Q_{\mathcal{F}/X/B}(Z')$$

is bijective.

Proof. We first argue that it suffices to prove this when all the schemes and algebraic spaces in sight are affine schemes. Let $Y' = \bigcup Y'_i$ be an affine open covering and let Y_i, Z'_i, and Z_i be the corresponding (affine) opens of Y, Z', and Z. Since $Q_{\mathcal{F}/X/B}$ satisfies the sheaf property for the fpqc topology (Lemma 72.8.4), it suffices to prove the result of the lemma for the diagrams

$$\begin{array}{ccc} Z_i & \longrightarrow & Z'_i \\ \downarrow & & \downarrow \\ Y_i & \longrightarrow & Y'_i \end{array} \quad \text{and} \quad \begin{array}{ccc} Z_i \cap Z_j & \longrightarrow & Z'_i \cap Z'_j \\ \downarrow & & \downarrow \\ Y_i \cap Y_j & \longrightarrow & Y'_i \cap Y'_j \end{array}$$

This reduces us to the case where the schemes Y', Y, Z', Z are separated and a second application of this argument to the case where Y', Y, Z', Z are affine.

Assume Y' (and hence also Y, Z', and Z) is affine. By Remark 72.8.2 we may replace B by Y' and X by $X \times_B Y'$, and $\mathcal{F}$ by the pullback. Thus we may assume $B = Y'$.

Assume $B = Y'$ (and hence also Y, Z', and Z) is affine. Choose an étale covering $\{X_i \to X\}_{i \in I}$ with each X_i affine and similarly choose étale coverings $\{X_{ijk} \to X_i \times_X X_j\}$ with each X_{ijk} affine (Properties of Spaces, Lemma 48.6.1). By Lemma 72.8.5 it suffices to prove the lemma for each of the functors associated to X_i and X_{ijk}. Hence we may assume X is affine as well. This reduces the lemma to More on Algebra, Remark 15.4.15. □

72.9. The quot functor

In this section we prove the Quot functor is representable by an algebraic space.

Situation 72.9.1. Let S be a scheme. Let $f : X \to B$ be a morphism of algebraic spaces over S. Let $\mathcal{F}$ be a quasi-coherent $\mathcal{O}_X$-module. For any scheme T over B we will denote X_T the base change of X to T and $\mathcal{F}_T$ the pullback of $\mathcal{F}$ via the projection morphism $X_T = X \times_S T \to X$. Given such a T we set

$$\mathrm{Quot}_{\mathcal{F}/X/B}(T) = \left\{ \begin{array}{c} \text{quotients } \mathcal{F}_T \to \mathcal{Q} \text{ where } \mathcal{Q} \text{ is a quasi-coherent} \\ \mathcal{O}_{X_T}\text{-module of finite presentation, flat over } T \\ \text{with scheme theoretic support proper over } T \end{array} \right\}$$

This is a subfunctor of $Q_{\mathcal{F}/X/T}$ discussed in Section 72.8. Thus we obtain a functor

$$\mathrm{Quot}_{\mathcal{F}/X/B} : (Sch/B)^{opp} \longrightarrow Sets \tag{72.9.1.1}$$

This is the *quot functor* associated to $\mathcal{F}/X/B$.

In Situation 72.9.1 we may think of $\mathrm{Quot}_{\mathcal{F}/X/B}$ as a functor $(Sch/S)^{opp} \to Sets$ endowed with a morphism $\mathrm{Quot}_{\mathcal{F}/X/S} \to B$. Namely, if T is a scheme over S, then we can think of an element of $\mathrm{Quot}_{\mathcal{F}/X/B}$ as a pair $(h, \mathcal{Q})$ where h a morphism $h : T \to B$, i.e., an element $h \in B(T)$, and Q is a finitely presented, T-flat quotient $\mathcal{F}_T \to \mathcal{Q}$ on $X_T = X \times_{B,h} T$ with support proper over T. In particular, when we say that $\mathrm{Quot}_{\mathcal{F}/X/S}$ is an algebraic space, we mean that the corresponding functor $(Sch/S)^{opp} \to Sets$ is an algebraic space.

Lemma 72.9.2. *In Situation 72.9.1. The functor $Quot_{\mathcal{F}/X/B}$ satisfies the sheaf property for the fpqc topology.*

Proof. In Lemma 72.8.4 we have seen that the functor $\mathrm{Q}_{\mathcal{F}/X/S}$ is a sheaf. Recall that for a scheme T over S the subset $\mathrm{Quot}_{\mathcal{F}/X/S}(T) \subset \mathrm{Q}_{\mathcal{F}/X/S}(T)$ picks out those quotients whose support is proper over T. This defines a subsheaf by the result of Descent on Spaces, Lemma 55.10.17 (combined with Morphisms of Spaces, Lemma 49.27.10) which shows that taking scheme theoretic support commutes with flat base change). □

Proposition 72.9.3. *Let S be a scheme. Let $f : X \to B$ be a morphism of algebraic spaces over S. Let $\mathcal{F}$ be a quasi-coherent sheaf on X. If f is of finite presentation, then $Quot_{\mathcal{F}/X/B}$ is an algebraic space. If $\mathcal{F}$ is of finite presentation, then $Quot_{\mathcal{F}/X/B} \to B$ is locally of finite presentation.*

Proof. Note that $\mathrm{Quot}_{\mathcal{F}/X/B}$ is a sheaf in the fppf topology. Let $Quot_{\mathcal{F}/X/B}$ be the stack in groupoids corresponding to $\mathrm{Quot}_{\mathcal{F}/X/S}$, see Algebraic Stacks, Section 67.7. By Algebraic Stacks, Proposition 67.13.3 it suffices to show that $Quot_{\mathcal{F}/X/B}$ is an algebraic stack. Consider the 1-morphism of stacks in groupoids

$$Quot_{\mathcal{F}/X/S} \longrightarrow Coh_{X/B}$$

on $(Sch/S)_{fppf}$ which associates to the quotient $\mathcal{F}_T \to \mathcal{Q}$ the coherent sheaf $\mathcal{Q}$. By Theorem 72.6.5 we know that $Coh_{X/B}$ is an algebraic stack. By Algebraic Stacks, Lemma 67.15.4 it suffices to show that this 1-morphism is representable by algebraic spaces.

Let T be a scheme over S and let the object $(h, \mathcal{G})$ of $Coh_{X/B}$ over T correspond to a 1-morphism $\xi : (Sch/T)_{fppf} \to Coh_{X/B}$. The 2-fibre product

$$\mathcal{Z} = (Sch/T)_{fppf} \times_{\xi, Coh_{X/B}} Quot_{\mathcal{F}/X/S}$$

is a stack in setoids, see Stacks, Lemma 8.6.7. The corresponding sheaf of sets (i.e., functor, see Stacks, Lemmas 8.6.7 and 8.6.2) assigns to a scheme T'/T the set of surjections $u : \mathcal{F}_{T'} \to \mathcal{G}_{T'}$ of quasi-coherent modules on $X_{T'}$. Thus we see that $\mathcal{Z}$ is representable by an open subspace (by Lemma 72.7.9) of the algebraic space $Hom(\mathcal{F}_T, \mathcal{G})$ from Proposition 72.3.9. □

72.10. Other chapters

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CHAPTER 73

Properties of Algebraic Stacks

73.1. Introduction

Please see Algebraic Stacks, Section 67.1 for a brief introduction to algebraic stacks, and please read some of that chapter for our foundations of algebraic stacks. The intent is that in that chapter we are careful to distinguish between schemes, algebraic spaces, algebraic stacks, and starting with this chapter we employ the customary abuse of language where all of these concepts are used interchangeably.

The goal of this chapter is to introduce some basic notions and properties of algebraic stacks. A fundamental reference for the case of quasi-separated algebraic stacks with representable diagonal is [**LMB00**].

73.2. Conventions and abuse of language

We choose a big fppf site Sch_{fppf}. All schemes are contained in Sch_{fppf}. And all rings A considered have the property that $\mathrm{Spec}(A)$ is (isomorphic) to an object of this big site.

We also fix a base scheme S, by the conventions above an element of Sch_{fppf}. The reader who is only interested in the absolute case can take $S = \mathrm{Spec}(\mathbf{Z})$.

Here are our conventions regarding algebraic stacks:

(1) When we say *algebraic stack* we will mean an algebraic stacks over S, i.e., a category fibred in groupoids $p : \mathcal{X} \to (Sch/S)_{fppf}$ which satisfies the conditions of Algebraic Stacks, Definition 67.12.1.
(2) We will say $f : \mathcal{X} \to \mathcal{Y}$ is a *morphism of algebraic stacks* to indicate a 1-morphism of algebraic stacks over S, i.e., a 1-morphism of categories fibred in groupoids over $(Sch/S)_{fppf}$, see Algebraic Stacks, Definition 67.12.3.
(3) A *2-morphism* $\alpha : f \to g$ will indicate a 2-morphism in the 2-category of algebraic stacks over S, see Algebraic Stacks, Definition 67.12.3.
(4) Given morphisms $\mathcal{X} \to \mathcal{Z}$ and $\mathcal{Y} \to \mathcal{Z}$ of algebraic stacks we abusively call the 2-fibre product $\mathcal{X} \times_{\mathcal{Z}} \mathcal{Y}$ the *fibre product*.
(5) We will write $\mathcal{X} \times_S \mathcal{Y}$ for the product of the algebraic stacks $\mathcal{X}$, $\mathcal{Y}$.
(6) We will often abuse notation and say two algebraic stacks $\mathcal{X}$ and $\mathcal{Y}$ are *isomorphic* if they are equivalent in this 2-category.

Here are our conventions regarding algebraic spaces.

(1) If we say X is an *algebraic space* then we mean that X is an algebraic space over S, i.e., X is a presheaf on $(Sch/S)_{fppf}$ which satisfies the conditions of Spaces, Definition 47.6.1.
(2) A *morphism of algebraic spaces* $f : X \to Y$ is a morphism of algebraic spaces over S as defined in Spaces, Definition 47.6.3.

(3) We will **not** distinguish between an algebraic space X and the algebraic stack $\mathcal{S}_X \to (Sch/S)_{fppf}$ it gives rise to, see Algebraic Stacks, Lemma 67.13.1.
(4) In particular, a *morphism* $f : X \to \mathcal{Y}$ from X to an algebraic stack $\mathcal{Y}$ means a morphism $f : \mathcal{S}_X \to \mathcal{Y}$ of algebraic stacks. Similarly for morphisms $\mathcal{Y} \to X$.
(5) Moreover, given an algebraic stack $\mathcal{X}$ we say $\mathcal{X}$ *is an algebraic space* to indicate that $\mathcal{X}$ is representable by an algebraic space, see Algebraic Stacks, Definition 67.8.1.
(6) We will use the following notational convention: If we indicate an algebraic stack by a roman capital (such as $X, Y, Z, A, B, \ldots$) then it will be the case that its inertia stack is trivial, and hence it is an algebraic space, see Algebraic Stacks, Proposition 67.13.3.

Here are our conventions regarding schemes.

(1) If we say X is a *scheme* then we mean that X is a scheme over S, i.e., X is an object of $(Sch/S)_{fppf}$.
(2) By a *morphism of schemes* we mean a morphism of schemes over S.
(3) We will **not** distinguish between a scheme X and the algebraic stack $\mathcal{S}_X \to (Sch/S)_{fppf}$ it gives rise to, see Algebraic Stacks, Lemma 67.13.1.
(4) In particular, a *morphism* $f : X \to \mathcal{Y}$ from a scheme X to an algebraic stack $\mathcal{Y}$ means a morphism $f : \mathcal{S}_X \to \mathcal{Y}$ of algebraic stacks. Similarly for morphisms $\mathcal{Y} \to X$.
(5) Moreover, given an algebraic stack $\mathcal{X}$ we say $\mathcal{X}$ *is a scheme* to indicate that $\mathcal{X}$ is representable, see Algebraic Stacks, Section 67.4.

Here are our conventions regarding morphism of algebraic stacks:

(1) A morphism $f : \mathcal{X} \to \mathcal{Y}$ of algebraic stacks is *representable*, or *representable by schemes* if for every scheme T and morphism $T \to \mathcal{Y}$ the fibre product $T \times_{\mathcal{Y}} \mathcal{X}$ is a scheme. See Algebraic Stacks, Section 67.6.
(2) A morphism $f : \mathcal{X} \to \mathcal{Y}$ of algebraic stacks is *representable by algebraic spaces* if for every scheme T and morphism $T \to \mathcal{Y}$ the fibre product $T \times_{\mathcal{Y}} \mathcal{X}$ is an algebraic space. See Algebraic Stacks, Definition 67.9.1. In this case $Z \times_{\mathcal{Y}} \mathcal{X}$ is an algebraic space whenever $Z \to \mathcal{Y}$ is a morphism whose source is an algebraic space, see Algebraic Stacks, Lemma 67.9.8.

Note that every morphism $X \to \mathcal{Y}$ from an algebraic space to an algebraic stack is representable by algebraic spaces, see Algebraic Stacks, Lemma 67.10.11. We will use this basic result without further mention.

73.3. Properties of morphisms representable by algebraic spaces

We will study properties of (arbitrary) morphisms of algebraic stacks in its own chapter. For morphisms representable by algebraic spaces we know what it means to be surjective, smooth, or étale, etc. This applies in particular to morphisms $X \to \mathcal{Y}$ from algebraic spaces to algebraic stacks. In this section, we recall how this works, we list the properties to which this applies, and we prove a few easy lemmas.

Our first lemma says a morphism is representable by algebraic spaces if it is so after a base change by a flat, locally finitely presented, surjective morphism.

Lemma 73.3.1. *Let $f : \mathcal{X} \to \mathcal{Y}$ be a morphism of algebraic stacks. Let W be an algebraic space and let $W \to \mathcal{Y}$ be surjective, locally of finite presentation, and flat. The following are equivalent*

(1) *f is representable by algebraic spaces, and*
(2) *$W \times_{\mathcal{Y}} \mathcal{X}$ is an algebraic space.*

Proof. The implication (1) $\Rightarrow$ (2) is Algebraic Stacks, Lemma 67.9.8. Conversely, let $W \to \mathcal{Y}$ be as in (2). To prove (1) it suffices to show that f is faithful on fibre categories, see Algebraic Stacks, Lemma 67.15.2. Assumption (2) implies in particular that $W \times_{\mathcal{Y}} \mathcal{X} \to W$ is faithful. Hence the faithfulness of f follows from Stacks, Lemma 8.6.9. □

Let P be a property of morphisms of algebraic spaces which is fppf local on the target and preserved by arbitrary base change. Let $f : \mathcal{X} \to \mathcal{Y}$ be a morphism of algebraic stacks representable by algebraic spaces. Then we say *f has property P* if and only if for every scheme T and morphism $T \to \mathcal{Y}$ the morphism of algebraic spaces $T \times_{\mathcal{Y}} \mathcal{X} \to T$ has property P, see Algebraic Stacks, Definition 67.10.1.

It turns out that if $f : \mathcal{X} \to \mathcal{Y}$ is representable by algebraic spaces and has property P, then for any morphism of algebraic stacks $\mathcal{Y}' \to \mathcal{Y}$ the base change $\mathcal{Y}' \times_{\mathcal{Y}} \mathcal{X} \to \mathcal{Y}'$ has property P, see Algebraic Stacks, Lemmas 67.9.7 and 67.10.6. If the property P is preserved under compositions, then this holds also in the setting of morphisms of algebraic stacks representable by algebraic spaces, see Algebraic Stacks, Lemmas 67.9.9 and 67.10.5. Moreover, in this case products $\mathcal{X}_1 \times \mathcal{X}_2 \to \mathcal{Y}_1 \times \mathcal{Y}_2$ of morphisms representable by algebraic spaces having property $\mathcal{P}$ have property $\mathcal{P}$, see Algebraic Stacks, Lemma 67.10.8.

Finally, if we have two properties P, P' of morphisms of algebraic spaces which are fppf local on the target and preserved by arbitrary base change and if $P(f) \Rightarrow P'(f)$ for every morphism f, then the same implication holds for the corresponding property of morphisms of algebraic stacks representable by algebraic spaces, see Algebraic Stacks, Lemma 67.10.9. We will use this *without further mention* in the following and in the following chapters.

The discussion above applies to each of the following properties of morphisms of algebraic spaces

(1) quasi-compact, see Morphisms of Spaces, Lemma 49.9.3 and Descent on Spaces, Lemma 55.10.1,
(2) quasi-separated, see Morphisms of Spaces, Lemma 49.5.4 and Descent on Spaces, Lemma 55.10.2,
(3) universally closed, see Morphisms of Spaces, Lemma 49.10.3 and Descent on Spaces, Lemma 55.10.3,
(4) universally open, see Morphisms of Spaces, Lemma 49.7.3 and Descent on Spaces, Lemma 55.10.4,
(5) surjective, see Morphisms of Spaces, Lemma 49.6.5 and Descent on Spaces, Lemma 55.10.5,
(6) universally injective, see Morphisms of Spaces, Lemma 49.18.5 and Descent on Spaces, Lemma 55.10.6,
(7) locally of finite type, see Morphisms of Spaces, Lemma 49.22.3 and Descent on Spaces, Lemma 55.10.7,

(8) locally of finite presentation, see Morphisms of Spaces, Lemma 49.26.3 and Descent on Spaces, Lemma 55.10.8,
(9) finite type, see Morphisms of Spaces, Lemma 49.22.3 and Descent on Spaces, Lemma 55.10.9,
(10) finite presentation, see Morphisms of Spaces, Lemma 49.26.3 and Descent on Spaces, Lemma 55.10.10,
(11) flat, see Morphisms of Spaces, Lemma 49.27.4 and Descent on Spaces, Lemma 55.10.11,
(12) open immersion, see Morphisms of Spaces, Section 49.4 and Descent on Spaces, Lemma 55.10.12,
(13) isomorphism, see Descent on Spaces, Lemma 55.10.13,
(14) affine, see Morphisms of Spaces, Lemma 49.19.5 and Descent on Spaces, Lemma 55.10.14,
(15) closed immersion, see Morphisms of Spaces, Section 49.4 and Descent on Spaces, Lemma 55.10.15,
(16) separated, see Morphisms of Spaces, Lemma 49.5.4 and Descent on Spaces, Lemma 55.10.16,
(17) proper, see Morphisms of Spaces, Lemma 49.36.3 and Descent on Spaces, Lemma 55.10.17,
(18) quasi-affine, see Morphisms of Spaces, Lemma 49.20.5 and Descent on Spaces, Lemma 55.10.18,
(19) integral, see Morphisms of Spaces, Lemma 49.40.5 and Descent on Spaces, Lemma 55.10.20,
(20) finite, see Morphisms of Spaces, Lemma 49.40.5 and Descent on Spaces, Lemma 55.10.21,
(21) (locally) quasi-finite, see Morphisms of Spaces, Lemma 49.25.3 and Descent on Spaces, Lemma 55.10.22,
(22) syntomic, see Morphisms of Spaces, Lemma 49.32.3 and Descent on Spaces, Lemma 55.10.23,
(23) smooth, see Morphisms of Spaces, Lemma 49.33.3 and Descent on Spaces, Lemma 55.10.24,
(24) unramified, see Morphisms of Spaces, Lemma 49.34.4 and Descent on Spaces, Lemma 55.10.25,
(25) étale, see Morphisms of Spaces, Lemma 49.35.4 and Descent on Spaces, Lemma 55.10.26,
(26) finite locally free, see Morphisms of Spaces, Lemma 49.41.5 and Descent on Spaces, Lemma 55.10.27,
(27) monomorphism, see Morphisms of Spaces, Lemma 49.11.5 and Descent on Spaces, Lemma 55.10.28,
(28) immersion, see Morphisms of Spaces, Section 49.4 and Descent on Spaces, Lemma 55.11.1,
(29) locally separated, see Morphisms of Spaces, Lemma 49.5.4 and Descent on Spaces, Lemma 55.11.2,

Lemma 73.3.2. *Let P be a property of morphisms of algebraic spaces as above. Let $f : \mathcal{X} \to \mathcal{Y}$ be a morphism of algebraic stacks representable by algebraic spaces. The following are equivalent:*

(1) *f has P,*

(2) *for every algebraic space Z and morphism $Z \to \mathcal{Y}$ the morphism $Z \times_{\mathcal{Y}} \mathcal{X} \to Z$ has P.*

Proof. The implication (2) $\Rightarrow$ (1) is immediate. Assume (1). Let $Z \to \mathcal{Y}$ be as in (2). Choose a scheme U and a surjective étale morphism $U \to Z$. By assumption the morphism $U \times_{\mathcal{Y}} \mathcal{X} \to U$ has P. But the diagram

$$\begin{array}{ccc} U \times_{\mathcal{Y}} \mathcal{X} & \longrightarrow & Z \times_{\mathcal{Y}} \mathcal{X} \\ \downarrow & & \downarrow \\ U & \longrightarrow & Z \end{array}$$

is cartesian, hence the right vertical arrow has P as $\{U \to Z\}$ is an fppf covering. □

The following lemma tells us it suffices to check P after a base change by a surjective, flat, locally finitely presented morphism.

Lemma 73.3.3. *Let P be a property of morphisms of algebraic spaces as above. Let $f : \mathcal{X} \to \mathcal{Y}$ be a morphism of algebraic stacks representable by algebraic spaces. Let W be an algebraic space and let $W \to \mathcal{Y}$ be surjective, locally of finite presentation, and flat. Set $V = W \times_{\mathcal{Y}} \mathcal{X}$. Then*

$$(f \text{ has } P) \Leftrightarrow (\text{the projection } V \to W \text{ has } P).$$

Proof. The implication from left to right follows from Lemma 73.3.2. Assume $V \to W$ has P. Let T be a scheme, and let $T \to \mathcal{Y}$ be a morphism. Consider the commutative diagram

$$\begin{array}{ccccc} T \times_{\mathcal{Y}} \mathcal{X} & \longleftarrow & T \times_{\mathcal{Y}} W & \longrightarrow & W \\ \downarrow & & \downarrow & & \downarrow \\ T & \longleftarrow & T \times_{\mathcal{Y}} V & \longrightarrow & V \end{array}$$

of algebraic spaces. The squares are cartesian. The bottom left morphism is a surjective, flat morphism which is locally of finite presentation, hence $\{T \times_{\mathcal{Y}} V \to T\}$ is an fppf covering. Hence the fact that the right vertical arrow has property P implies that the left vertical arrow has property P. □

Lemma 73.3.4. *Let P be a property of morphisms of algebraic spaces as above. Let $f : \mathcal{X} \to \mathcal{Y}$ be a morphism of algebraic stacks representable by algebraic spaces. Let $\mathcal{Z} \to \mathcal{Y}$ be a morphism of algebraic stacks which is representable by algebraic spaces, surjective, flat, and locally of finite presentation. Set $\mathcal{W} = \mathcal{Z} \times_{\mathcal{Y}} \mathcal{X}$. Then*

$$(f \text{ has } P) \Leftrightarrow (\text{the projection } \mathcal{W} \to \mathcal{Z} \text{ has } P).$$

Proof. Choose an algebraic space W and a morphism $W \to \mathcal{Z}$ which is surjective, flat, and locally of finite presentation. By the discussion above the composition $W \to \mathcal{Y}$ is also surjective, flat, and locally of finite presentation. Denote $V = W \times_{\mathcal{Z}} \mathcal{W} = V \times_{\mathcal{Y}} \mathcal{X}$. By Lemma 73.3.3 we see that f has $\mathcal{P}$ if and only if $V \to W$ does and that $\mathcal{W} \to \mathcal{Z}$ has $\mathcal{P}$ if and only if $V \to W$ does. The lemma follows. □

Lemma 73.3.5. *Let P be a property of morphisms of algebraic spaces as above. Let $\mathcal{X} \to \mathcal{Y}$ and $\mathcal{Y} \to \mathcal{Z}$ be morphisms of algebraic stacks representable by algebraic spaces. Assume*

(1) *$\mathcal{X} \to \mathcal{Y}$ is surjective, flat, and locally of finite presentation,*

(2) *the composition has P, and*
(3) *P is local on the source in the fppf topology.*

Then $\mathcal{Y} \to \mathcal{Z}$ has property P.

Proof. Let Z be a scheme and let $Z \to \mathcal{Z}$ be a morphism. Set $X = \mathcal{X} \times_{\mathcal{Z}} Z$, $Y = \mathcal{Y} \times_{\mathcal{Z}} Z$. By (1) $\{X \to Y\}$ is an fppf covering of algebraic spaces and by (2) $X \to Z$ has property P. By (3) this implies that $Y \to Z$ has property P and we win. □

Lemma 73.3.6. *Let $g : \mathcal{X}' \to \mathcal{X}$ be a morphism of algebraic stacks which is representable by algebraic spaces. Let $[U/R] \to \mathcal{X}$ be a presentation. Set $U' = U \times_{\mathcal{X}} \mathcal{X}'$, and $R' = R \times_{\mathcal{X}} \mathcal{X}'$. Then there exists a groupoid in algebraic spaces of the form (U', R', s', t', c'), a presentation $[U'/R'] \to \mathcal{X}'$, and the diagram*

$$\begin{array}{ccc} [U'/R'] & \longrightarrow & \mathcal{X}' \\ {\scriptstyle [pr]}\downarrow & & \downarrow{\scriptstyle g} \\ [U/R] & \longrightarrow & \mathcal{X} \end{array}$$

is 2-commutative where the morphism $[pr]$ comes from a morphism of groupoids $pr : (U', R', s', t', c') \to (U, R, s, t, c)$.

Proof. Since $U \to \mathcal{Y}$ is surjective and smooth, see Algebraic Stacks, Lemma 67.17.2 the base change $U' \to \mathcal{X}'$ is also surjective and smooth. Hence, by Algebraic Stacks, Lemma 67.16.2 it suffices to show that $R' = U' \times_{\mathcal{X}'} U'$ in order to get a smooth groupoid (U', R', s', t', c') and a presentation $[U'/R'] \to \mathcal{X}'$. Using that $R = V \times_{\mathcal{Y}} V$ (see Groupoids in Spaces, Lemma 59.21.2) this follows from

$$R' = U \times_{\mathcal{X}} U \times_{\mathcal{X}} \mathcal{X}' = (U \times_{\mathcal{X}} \mathcal{X}') \times_{\mathcal{X}'} (U \times_{\mathcal{X}} \mathcal{X}')$$

see Categories, Lemmas 4.29.8 and 4.29.10. Clearly the projection morphisms $U' \to U$ and $R' \to R$ give the desired morphism of groupoids $\mathrm{pr} : (U', R', s', t', c') \to (U, R, s, t, c)$. Hence the morphism [pr] of quotient stacks by Groupoids in Spaces, Lemma 59.20.1.

We still have to show that the diagram 2-commutes. It is clear that the diagram

$$\begin{array}{ccc} U' & \xrightarrow[f']{} & \mathcal{X}' \\ {\scriptstyle \mathrm{pr}_U}\downarrow & & \downarrow{\scriptstyle g} \\ U & \xrightarrow{f} & \mathcal{X} \end{array}$$

2-commutes where $\mathrm{pr}_U : U' \to U$ is the projection. There is a canonical 2-arrow $\tau : f \circ t \to f \circ s$ in $\mathrm{Mor}(R, \mathcal{X})$ coming from $R = U \times_{\mathcal{X}} U$, $t = \mathrm{pr}_0$, and $s = \mathrm{pr}_1$. Using the isomorphism $R' \to U' \times_{\mathcal{X}'} U'$ we get similarly an isomorphism $\tau' : f' \circ t' \to f' \circ s'$. Note that $g \circ f' \circ t' = f \circ t \circ \mathrm{pr}_R$ and $g \circ f' \circ s' = f \circ s \circ \mathrm{pr}_R$, where $\mathrm{pr}_R : R' \to R$ is the projection. Thus it makes sense to ask if

$$\tau \star \mathrm{id}_{\mathrm{pr}_R} = \mathrm{id}_g \star \tau'. \tag{73.3.6.1}$$

Now we make two claims: (1) if Equation (73.3.6.1) holds, then the diagram 2-commutes, and (2) Equation (73.3.6.1) holds. We omit the proof of both claims. Hints: part (1) follows from the construction of $f = f_{can}$ and $f' = f'_{can}$ in Algebraic Stacks, Lemma 67.16.1. Part (2) follows by carefuly working through the definitions. □

Remark 73.3.7. Let $\mathcal{Y}$ be an algebraic stack. Consider the following 2-category:

(1) An object is a morphism $f : \mathcal{X} \to \mathcal{Y}$ which is representable by algebraic spaces,
(2) a 1-morphism $(g, \beta) : (f_1 : \mathcal{X}_1 \to \mathcal{Y}) \to (f_2 : \mathcal{X}_2 \to \mathcal{Y})$ consists of a morphism $g : \mathcal{X}_1 \to \mathcal{X}_2$ and a 2-morphism $\beta : f_1 \to f_2 \circ g$, and
(3) a 2-morphism between $(g, \beta), (g', \beta') : (f_1 : \mathcal{X}_1 \to \mathcal{Y}) \to (f_2 : \mathcal{X}_2 \to \mathcal{Y})$ is a 2-morphism $\alpha : g \to g'$ such that $(\mathrm{id}_{f_2} \star \alpha) \circ \beta = \beta'$.

Let us denote this 2-category *Spaces*/$\mathcal{Y}$ by analogy with the notation of Topologies on Spaces, Section 54.2. Now we claim that in this 2-category the morphism categories

$$\mathrm{Mor}_{\mathit{Spaces}/\mathcal{Y}}((f_1 : \mathcal{X}_1 \to \mathcal{Y}), (f_2 : \mathcal{X}_2 \to \mathcal{Y}))$$

are all setoids. Namely, a 2-morphism α is a rule which to each object x_1 of $\mathcal{X}_1$ assigns an isomorphism $\alpha_{x_1} : g(x_1) \longrightarrow g'(x_1)$ in the relevant fibre category of $\mathcal{X}_2$ such that the diagram

$$\begin{array}{ccc} & f_2(x_1) & \\ {\scriptstyle \beta_{x_1}} \swarrow & & \searrow {\scriptstyle \beta'_{x_1}} \\ f_2(g(x_1)) & \xrightarrow{f_2(\alpha_{x_1})} & f_2(g'(x_1)) \end{array}$$

commutes. But since f_2 is faithful (see Algebraic Stacks, Lemma 67.15.2) this means that if α_{x_1} exists, then it is unique! In other words the 2-category *Spaces*/$\mathcal{Y}$ is very close to being a category. Namely, if we replace 1-morphisms by isomorphism classes of 1-morphisms we obtain a category. We will often perform this replacement without further mention.

73.4. Points of algebraic stacks

Let $\mathcal{X}$ be an algebraic stack. Let K, L be two fields and let $p : \mathrm{Spec}(K) \to \mathcal{X}$ and $q : \mathrm{Spec}(L) \to \mathcal{X}$ be morphisms. We say that p and q are *equivalent* if there exists a field Ω and a 2-commutative diagram

$$\begin{array}{ccc} \mathrm{Spec}(\Omega) & \longrightarrow & \mathrm{Spec}(L) \\ \downarrow & & \downarrow {\scriptstyle q} \\ \mathrm{Spec}(K) & \xrightarrow{p} & \mathcal{X}. \end{array}$$

Lemma 73.4.1. *The notion above does indeed define an equivalence relation on morphisms from spectra of fields into the algebraic stack $\mathcal{X}$.*

Proof. It is clear that the relation is reflexive and symmetric. Hence we have to prove that it is transitive. This comes down to the following: Given a diagram

$$\begin{array}{ccccc} \mathrm{Spec}(\Omega) & \xrightarrow[b]{} & \mathrm{Spec}(L) & \xleftarrow[b']{} & \mathrm{Spec}(\Omega') \\ {\scriptstyle a}\downarrow & & \downarrow {\scriptstyle q} & & \downarrow {\scriptstyle a'} \\ \mathrm{Spec}(K) & \xrightarrow{p} & \mathcal{X} & \xleftarrow{p'} & \mathrm{Spec}(K') \end{array}$$

with both squares 2-commutative we have to show that p is equivalent to p'. By the 2-Yoneda lemma (see Algebraic Stacks, Section 67.5) the morphisms p, p', and q are given by objects x, x', and y in the fibre categories of $\mathcal{X}$ over $\mathrm{Spec}(K)$,

$\mathrm{Spec}(K')$, and $\mathrm{Spec}(L)$. The 2-commutativity of the squares means that there are isomorphisms $\alpha : a^*x \to b^*y$ and $\alpha' : (a')^*x' \to (b')^*y$ in the fibre categories of $\mathcal{X}$ over $\mathrm{Spec}(\Omega)$ and $\mathrm{Spec}(\Omega')$. Choose any field Ω'' and embeddings $\Omega \to \Omega''$ and $\Omega' \to \Omega''$ agreeing on L. Then we can extend the diagram above to

$$\begin{array}{ccccc}
 & & \mathrm{Spec}(\Omega'') & & \\
 & \swarrow^{c} & \downarrow{\scriptstyle q'} & \searrow^{c'} & \\
\mathrm{Spec}(\Omega) & \xrightarrow[b]{} & \mathrm{Spec}(L) & \xleftarrow[b']{} & \mathrm{Spec}(\Omega') \\
\downarrow{\scriptstyle a} & & \downarrow{\scriptstyle q} & & \downarrow{\scriptstyle a'} \\
\mathrm{Spec}(K) & \xrightarrow{p} & \mathcal{X} & \xleftarrow{p'} & \mathrm{Spec}(K')
\end{array}$$

with commutative triangles and

$$(q')^*(\alpha')^{-1} \circ (q')^*\alpha : (a \circ c)^*x \longrightarrow (a' \circ c')^*x'$$

is an isomorphism in the fibre category over $\mathrm{Spec}(\Omega'')$. Hence p is equivalent to p' as desired. □

Definition 73.4.2. Let $\mathcal{X}$ be an algebraic stack. A *point* of $\mathcal{X}$ is an equivalence class of morphisms from spectra of fields into $\mathcal{X}$. The set of points of $\mathcal{X}$ is denoted $|\mathcal{X}|$.

This agrees with our definition of points of algebraic spaces, see Properties of Spaces, Definition 48.4.1. Moreover, for a scheme we recover the usual notion of points, see Properties of Spaces, Lemma 48.4.2. If $f : \mathcal{X} \to \mathcal{Y}$ is a morphism of algebraic stacks then there is an induced map $|f| : |\mathcal{X}| \to |\mathcal{Y}|$ which maps a representative $x : \mathrm{Spec}(K) \to \mathcal{X}$ to the representative $f \circ x : \mathrm{Spec}(K) \to \mathcal{Y}$. This is well defined: namely 2-isomorphic 1-morphisms remain 2-isomorphic after pre- or post-composing by a 1-morphism because you can horizontally pre- or post-compose by the identity of the given 1-morphism. This holds in any (strict) $(2,1)$-category. If

$$\begin{array}{ccc}
\mathcal{X} & \longrightarrow & \mathcal{Y} \\
\downarrow & & \downarrow \\
\mathcal{W} & \longrightarrow & \mathcal{Z}
\end{array}$$

is a 2-commutative diagram of algebraic stacks, then the diagram of sets

$$\begin{array}{ccc}
|\mathcal{X}| & \longrightarrow & |\mathcal{Y}| \\
\downarrow & & \downarrow \\
|\mathcal{W}| & \longrightarrow & |\mathcal{Z}|
\end{array}$$

is commutative. In particular, if $\mathcal{X} \to \mathcal{Y}$ is an equivalence then $|\mathcal{X}| \to |\mathcal{Y}|$ is a bijection.

Lemma 73.4.3. *Let*

$$\begin{array}{ccc}
\mathcal{Z} \times_{\mathcal{Y}} \mathcal{X} & \longrightarrow & \mathcal{X} \\
\downarrow & & \downarrow \\
\mathcal{Z} & \longrightarrow & \mathcal{Y}
\end{array}$$

be a fibre product of algebraic stacks. Then the map of sets of points

$$|\mathcal{Z} \times_{\mathcal{Y}} \mathcal{X}| \longrightarrow |\mathcal{Z}| \times_{|\mathcal{Y}|} |\mathcal{X}|$$

is surjective.

Proof. Namely, suppose given fields K, L and morphisms $\mathrm{Spec}(K) \to \mathcal{X}$, $\mathrm{Spec}(L) \to \mathcal{Z}$, then the assumption that they agree as elements of $|\mathcal{Y}|$ means that there is a common extension $K \subset M$ and $L \subset M$ such that $\mathrm{Spec}(M) \to \mathrm{Spec}(K) \to \mathcal{X} \to \mathcal{Y}$ and $\mathrm{Spec}(M) \to \mathrm{Spec}(L) \to \mathcal{Z} \to \mathcal{Y}$ are 2-isomorphic. And this is exactly the condition which says you get a morphism $\mathrm{Spec}(M) \to \mathcal{Z} \times_{\mathcal{Y}} \mathcal{X}$. □

Lemma 73.4.4. *Let $f : \mathcal{X} \to \mathcal{Y}$ be a morphism of algebraic stacks which is representable by algebraic spaces. The following are equivalent:*

(1) *$|f| : |\mathcal{X}| \to |\mathcal{Y}|$ is surjective, and*
(2) *f is surjective.*

Proof. Assume (1). Let $T \to \mathcal{Y}$ be a morphism whose source is a scheme. To prove (2) we have to show that the morphism of algebraic spaces $T \times_{\mathcal{Y}} \mathcal{X} \to T$ is surjective. By Morphisms of Spaces, Definition 49.6.2 this means we have to show that $|T \times_{\mathcal{Y}} \mathcal{X}| \to |T|$ is surjective. Applying Lemma 73.4.3 we see that this follows from (1).

Conversely, assume (2). Let $y : \mathrm{Spec}(K) \to \mathcal{Y}$ be a morphism from the spectrum of a field into $\mathcal{Y}$. By assumption the morphism $\mathrm{Spec}(K) \times_{y,\mathcal{Y}} \mathcal{X} \to \mathrm{Spec}(K)$ of algebraic spaces is surjective. By Morphisms of Spaces, Definition 49.6.2 this means there exists a field extension $K \subset K'$ and a morphism $\mathrm{Spec}(K') \to \mathrm{Spec}(K) \times_{y,\mathcal{Y}} \mathcal{X}$ such that the left square of the diagram

$$\begin{array}{ccccc} \mathrm{Spec}(K') & \longrightarrow & \mathrm{Spec}(K) \times_{y,\mathcal{Y}} \mathcal{X} & \longrightarrow & \mathcal{X} \\ \downarrow & & \downarrow & & \downarrow \\ \mathrm{Spec}(K) & = & \mathrm{Spec}(K) & \xrightarrow{y} & \mathcal{Y} \end{array}$$

is commutative. This shows that $|X| \to |\mathcal{Y}|$ is surjective. □

Here is a lemma explaining how to compute the set of points in terms of a presentation.

Lemma 73.4.5. *Let $\mathcal{X}$ be an algebraic stack. Let $\mathcal{X} = [U/R]$ be a presentation of $\mathcal{X}$, see Algebraic Stacks, Definition 67.16.5. Then the image of $|R| \to |U| \times |U|$ is an equivalence relation and $|\mathcal{X}|$ is the quotient of $|U|$ by this equivalence relation.*

Proof. The assumption means that we have a smooth groupoid (U, R, s, t, c) in algebraic spaces, and an equivalence $f : [U/R] \to \mathcal{X}$. We may assume $\mathcal{X} = [U/R]$. The induced morphism $p : U \to \mathcal{X}$ is smooth and surjective, see Algebraic Stacks, Lemma 67.17.2. Hence $|U| \to |\mathcal{X}|$ is surjective by Lemma 73.4.4. Note that $R = U \times_{\mathcal{X}} U$, see Groupoids in Spaces, Lemma 59.21.2. Hence Lemma 73.4.3 implies the map

$$|R| \longrightarrow |U| \times_{|\mathcal{X}|} |U|$$

is surjective. Hence the image of $|R| \to |U| \times |U|$ is exactly the set of pairs $(u_1, u_2) \in |U| \times |U|$ such that u_1 and u_2 have the same image in $|\mathcal{X}|$. Combining these two statements we get the result of the lemma. □

Remark 73.4.6. The result of Lemma 73.4.5 can be generalized as follows. Let $\mathcal{X}$ be an algebraic stack. Let U be an algebraic space and let $f : U \to \mathcal{X}$ be a surjective morphism (which makes sense by Section 73.3). Let $R = U \times_{\mathcal{X}} U$, let (U, R, s, t, c) be the groupoid in algebraic spaces, and let $f_{can} : [U/R] \to \mathcal{X}$ be the canonical morphism as constructed in Algebraic Stacks, Lemma 67.16.1. Then the image of $|R| \to |U| \times |U|$ is an equivalence relation and $|\mathcal{X}| = |U|/|R|$. The proof of Lemma 73.4.5 works without change. (Of course in general $[U/R]$ is not an algebraic stack, and in general f_{can} is not an isomorphism.)

Lemma 73.4.7. *There exists a unique topology on the sets of points of algebraic stacks with the following properties:*

(1) *for every morphism of algebraic stacks $\mathcal{X} \to \mathcal{Y}$ the map $|\mathcal{X}| \to |\mathcal{Y}|$ is continuous, and*
(2) *for every morphism $U \to \mathcal{X}$ which is flat and locally of finite presentation with U an algebraic space the map of topological spaces $|U| \to |\mathcal{X}|$ is continuous and open.*

Proof. Choose a morphism $p : U \to \mathcal{X}$ which is surjective, flat, and locally of finite presentation with U an algebraic space. Such exist by the definition of an algebraic stack, as a smooth morphism is flat and locally of finite presentation (see Morphisms of Spaces, Lemmas 49.33.5 and 49.33.7). We define a topology on $|\mathcal{X}|$ by the rule: $W \subset |\mathcal{X}|$ is open if and only if $|p|^{-1}(W)$ is open in $|U|$. To show that this is independent of the choice of p, let $p' : U' \to \mathcal{X}$ be another morphism which is surjective, flat, locally of finite presentation from an algebraic space to $\mathcal{X}$. Set $U'' = U \times_{\mathcal{X}} U'$ so that we have a 2-commutative diagram

$$\begin{array}{ccc} U'' & \longrightarrow & U' \\ \downarrow & & \downarrow \\ U & \longrightarrow & \mathcal{X} \end{array}$$

As $U \to \mathcal{X}$ and $U' \to \mathcal{X}$ are surjective, flat, locally of finite presentation we see that $U'' \to U'$ and $U'' \to U$ are surjective, flat and locally of finite presentation, see Lemma 73.3.2. Hence the maps $|U''| \to |U'|$ and $|U''| \to |U|$ are continuous, open and surjective, see Morphisms of Spaces, Definition 49.6.2 and Lemma 49.27.6. This clearly implies that our definition is independent of the choice of $p : U \to \mathcal{X}$.

Let $f : \mathcal{X} \to \mathcal{Y}$ be a morphism of algebraic stacks. By Algebraic Stacks, Lemma 67.15.1 we can find a 2-commutative diagram

$$\begin{array}{ccc} U & \xrightarrow{a} & V \\ {\scriptstyle x}\downarrow & & \downarrow{\scriptstyle y} \\ \mathcal{X} & \xrightarrow{f} & \mathcal{Y} \end{array}$$

with surjective smooth vertical arrows. Consider the associated commutative diagram

$$\begin{array}{ccc} |U| & \xrightarrow{|a|} & |V| \\ {\scriptstyle |x|}\downarrow & & \downarrow{\scriptstyle |y|} \\ |\mathcal{X}| & \xrightarrow{|f|} & |\mathcal{Y}| \end{array}$$

of sets. If $W \subset |\mathcal{Y}|$ is open, then by the definition above this means exactly that $|y|^{-1}(W)$ is open in $|V|$. Since $|a|$ is continuous we conclude that $|a|^{-1}|y|^{-1}(W) = |x|^{-1}|f|^{-1}(W)$ is open in $|W|$ which means by definition that $|f|^{-1}(W)$ is open in $|\mathcal{X}|$. Thus $|f|$ is continuous.

Finally, we have to show that if U is an algebraic space, and $U \to \mathcal{X}$ is flat and locally of finite presentation, then $|U| \to |\mathcal{X}|$ is open. Let $V \to \mathcal{X}$ be surjective, flat, and locally of finite presentation with V an algebraic space. Consider the commutative diagram

$$\begin{array}{ccccc} |U \times_{\mathcal{X}} V| & \xrightarrow{e} & |U| \times_{|\mathcal{X}|} |V| & \xrightarrow{d} & |V| \\ & \searrow^{f} & \downarrow{\scriptstyle c} & & \downarrow{\scriptstyle b} \\ & & |U| & \xrightarrow{a} & |\mathcal{X}| \end{array}$$

Now the morphism $U \times_{\mathcal{X}} V \to U$ is surjective, i.e, $f : |U \times_{\mathcal{X}} V| \to |U|$ is surjective. The left top horizontal arrow is surjective, see Lemma 73.4.3. The morphism $U \times_{\mathcal{X}} V \to V$ is flat and locally of finite presentation, hence $d \circ e : |U \times_{\mathcal{X}} V| \to |V|$ is open, see Morphisms of Spaces, Lemma 49.27.6. Pick $W \subset |U|$ open. The properties above imply that $b^{-1}(a(W)) = (d \circ e)(f^{-1}(W))$ is open, which by construction means that $a(W)$ is open as desired. □

Definition 73.4.8. Let $\mathcal{X}$ be an algebraic stack. The underlying *topological space* of $\mathcal{X}$ is the set of points $|\mathcal{X}|$ endowed with the topology constructed in Lemma 73.4.7.

This definition does not conflict with the already existing topology on $|\mathcal{X}|$ if $\mathcal{X}$ is an algebraic space.

Lemma 73.4.9. *Let $\mathcal{X}$ be an algebraic stack. Every point of $|\mathcal{X}|$ has a fundamental system of quasi-compact open neighbourhoods. In particular $|\mathcal{X}|$ is locally quasi-compact in the sense of Topology, Definition 5.12.1.*

Proof. This follows formally from the fact that there exists a scheme U and a surjective, open, continuous map $U \to |\mathcal{X}|$ of topological spaces. Namely, if $U \to \mathcal{X}$ is surjective and smooth, then Lemma 73.4.7 guarantees that $|U| \to |\mathcal{X}|$ is continuous, surjective, and open. □

73.5. Surjective morphisms

Let $f : \mathcal{X} \to \mathcal{Y}$ be a morphism of algebraic stacks which is representable by algebraic spaces. In Section 73.3 we have already defined what it means for f to be surjective. In Lemma 73.4.4 we have seen that this is equivalent to requiring $|f| : |\mathcal{X}| \to |\mathcal{Y}|$ to be surjective. This clears the way for the following definition.

Definition 73.5.1. Let $f : \mathcal{X} \to \mathcal{Y}$ be a morphism of algebraic stacks. We say f is *surjective* if the map $|f| : |\mathcal{X}| \to |\mathcal{Y}|$ of associated topological spaces is surjective.

Here are some lemmas.

Lemma 73.5.2. *The composition of surjective morphisms is surjective.*

Proof. Omitted. □

Lemma 73.5.3. *The base change of a surjective morphism is surjective.*

Proof. Omitted. Hint: Use Lemma 73.4.3. □

Lemma 73.5.4. *Let $f : \mathcal{X} \to \mathcal{Y}$ be a morphism of algebraic stacks. Let $\mathcal{Y}' \to \mathcal{Y}$ be a surjective morphism of algebraic stacks. If the base change $f' : \mathcal{Y}' \times_{\mathcal{Y}} \mathcal{X} \to \mathcal{Y}'$ of f is surjective, then f is surjective.*

Proof. Immediate from Lemma 73.4.3. □

Lemma 73.5.5. *Let $\mathcal{X} \to \mathcal{Y} \to \mathcal{Z}$ be morphisms of algebraic stacks. If $\mathcal{X} \to \mathcal{Z}$ is surjective so is $\mathcal{Y} \to \mathcal{Z}$.*

Proof. Immediate. □

73.6. Quasi-compact algebraic stacks

The following definition is equivalent with the definition for algebraic spaces by Properties of Spaces, Lemma 48.5.2.

Definition 73.6.1. Let $\mathcal{X}$ be an algebraic stack. We say $\mathcal{X}$ is *quasi-compact* if and only if $|\mathcal{X}|$ is quasi-compact.

Lemma 73.6.2. *Let $\mathcal{X}$ be an algebraic stack. The following are equivalent:*

(1) *$\mathcal{X}$ is quasi-compact,*
(2) *there exists a surjective smooth morphism $U \to \mathcal{X}$ with U a quasi-compact scheme,*
(3) *there exists a surjective smooth morphism $U \to \mathcal{X}$ with U a quasi-compact algebraic space, and*
(4) *there exists a surjective morphism $\mathcal{U} \to \mathcal{X}$ of algebraic stacks such that $\mathcal{U}$ is quasi-compact.*

Proof. We will use Lemma 73.4.4. Suppose $\mathcal{U}$ and $\mathcal{U} \to \mathcal{X}$ are as in (4). Then since $|\mathcal{U}| \to |\mathcal{X}|$ is surjective and continuous we conclude that $|\mathcal{X}|$ is quasi-compact. Thus (4) implies (1). The implications (2) $\Rightarrow$ (3) $\Rightarrow$ (4) are immediate. Assume (1), i.e., $\mathcal{X}$ is quasi-compact, i.e., that $|\mathcal{X}|$ is quasi-compact. Choose a scheme U and a surjective smooth morphism $U \to \mathcal{X}$. Then since $|U| \to |\mathcal{X}|$ is open we see that there exists a quasi-compact open $U' \subset U$ such that $|U'| \to |X|$ is surjective (and still smooth). Hence (2) holds. □

Lemma 73.6.3. *A finite disjoint union of quasi-compact algebraic stacks is a quasi-compact algebraic stack.*

Proof. This is clear from the corresponding topological fact. □

73.7. Properties of algebraic stacks defined by properties of schemes

Any smooth local property of schemes gives rise to a corresponding property of algebraic stacks via the following lemma. Note that a property of schemes which is smooth local is also étale local as any étale covering is also a smooth covering. Hence for a smooth local property P of schemes we know what it means to say that an algebraic space has P, see Properties of Spaces, Section 73.7.

Lemma 73.7.1. *Let $\mathcal{P}$ be a property of schemes which is local in the smooth topology, see Descent, Definition 34.11.1. Let $\mathcal{X}$ be an algebraic stack. The following are equivalent*

(1) *for some scheme U and some surjective smooth morphism $U \to \mathcal{X}$ the scheme U has property $\mathcal{P}$,*
(2) *for every scheme U and every smooth morphism $U \to \mathcal{X}$ the scheme U has property $\mathcal{P}$,*
(3) *for some algebraic space U and some surjective smooth morphism $U \to \mathcal{X}$ the algebraic space U has property $\mathcal{P}$, and*
(4) *for every algebraic space U and every smooth morphism $U \to \mathcal{X}$ the algebraic space U has property $\mathcal{P}$.*

If $\mathcal{X}$ is a scheme this is equivalent to $\mathcal{P}(U)$. If $\mathcal{X}$ is an algebraic space this is equivalent to X having property $\mathcal{P}$.

Proof. Let $U \to \mathcal{X}$ surjective and smooth with U an algebraic space. Let $V \to \mathcal{X}$ be a smooth morphism with V an algebraic space. Choose schemes U' and V' and surjective étale morphisms $U' \to U$ and $V' \to V$. Finally, choose a scheme W and a surjective étale morphism $W \to V' \times_{\mathcal{X}} U'$. Then $W \to V'$ and $W \to U'$ are smooth morphisms of schemes as compositions of étale and smooth morphisms of algebraic spaces, see Morphisms of Spaces, Lemmas 49.35.6 and 49.33.2. Moreover, $W \to V'$ is surjective as $U' \to \mathcal{X}$ is surjective. Hence, we have

$$\mathcal{P}(U) \Leftrightarrow \mathcal{P}(U') \Rightarrow \mathcal{P}(W) \Rightarrow \mathcal{P}(V') \Leftrightarrow \mathcal{P}(V)$$

where the equivalences are by definition of property $\mathcal{P}$ for algebraic spaces, and the two implications come from Descent, Definition 34.11.1. This proves (3) $\Rightarrow$ (4).

The implications (2) $\Rightarrow$ (1), (1) $\Rightarrow$ (3), and (4) $\Rightarrow$ (2) are immediate. □

Definition 73.7.2. Let $\mathcal{X}$ be an algebraic stack. Let $\mathcal{P}$ be a property of schemes which is local in the smooth topology. We say $\mathcal{X}$ *has property* $\mathcal{P}$ if any of the equivalent conditions of Lemma 73.7.1 hold.

Remark 73.7.3. Here is a list of properties which are local for the smooth topology (keep in mind that the fpqc, fppf, and syntomic topologies are stronger than the smooth topology):

(1) locally Noetherian, see Descent, Lemma 34.12.1,
(2) Jacobson, see Descent, Lemma 34.12.2,
(3) locally Noetherian and (S_k), see Descent, Lemma 34.13.1,
(4) Cohen-Macaulay, see Descent, Lemma 34.13.2,
(5) reduced, see Descent, Lemma 34.14.1,
(6) normal, see Descent, Lemma 34.14.2,
(7) locally Noetherian and (R_k), see Descent, Lemma 34.14.3,
(8) regular, see Descent, Lemma 34.14.4,
(9) Nagata, see Descent, Lemma 34.14.5.

Any smooth local property of germs of schemes gives rise to a corresponding property of algebraic stacks. Note that a property of germs which is smooth local is also étale local. Hence for a smooth local property of germs of schemes P we know what it means to say that an algebraic space X has property P at $x \in |X|$, see Properties of Spaces, Section 73.7.

Lemma 73.7.4. *Let $\mathcal{X}$ be an algebraic stack. Let $x \in |\mathcal{X}|$ be a point of $\mathcal{X}$. Let $\mathcal{P}$ be a property of germs of schemes which is smooth local, see Descent, Definition 34.17.1. The following are equivalent*

(1) *for any smooth morphism $U \to \mathcal{X}$ with U a scheme and $u \in U$ with $a(u) = x$ we have $\mathcal{P}(U, u)$,*
(2) *for some smooth morphism $U \to \mathcal{X}$ with U a scheme and some $u \in U$ with $a(u) = x$ we have $\mathcal{P}(U, u)$,*
(3) *for any smooth morphism $U \to \mathcal{X}$ with U an algebraic space and $u \in |U|$ with $a(u) = x$ the algebraic space U has property $\mathcal{P}$ at u, and*
(4) *for some smooth morphism $U \to \mathcal{X}$ with U a an algebraic space and some $u \in |U|$ with $a(u) = x$ the algebraic space U has property $\mathcal{P}$ at u.*

If $\mathcal{X}$ is representable, then this is equivalent to $\mathcal{P}(\mathcal{X}, x)$. If $\mathcal{X}$ is an algebraic space then this is equivalent to $\mathcal{X}$ having property $\mathcal{P}$ at x.

Proof. Let $a : U \to \mathcal{X}$ and $u \in |U|$ as in (3). Let $b : V \to \mathcal{X}$ be another smooth morphism with V an algebraic space and $v \in |V|$ with $b(v) = x$ also. Choose a scheme U', an étale morphism $U' \to U$ and $u' \in U'$ mapping to u. Choose a scheme V', an étale morphism $V' \to V$ and $v' \in V'$ mapping to v. By Lemma 73.4.3 there exists a point $\overline{w} \in |V' \times_{\mathcal{X}} U'|$ mapping to u' and v'. Choose a scheme W and a surjective étale morphism $W \to V' \times_{\mathcal{X}} U'$. We may choose a $w \in |W|$ mapping to $\overline{w}$ (see Properties of Spaces, Lemma 48.4.4). Then $W \to V'$ and $W \to U'$ are smooth morphisms of schemes as compositions of étale and smooth morphisms of algebraic spaces, see Morphisms of Spaces, Lemmas 49.35.6 and 49.33.2. Hence

$$\mathcal{P}(U, u) \Leftrightarrow \mathcal{P}(U', u') \Leftrightarrow \mathcal{P}(W, w) \Leftrightarrow \mathcal{P}(V', v') \Leftrightarrow \mathcal{P}(V, v)$$

The outer two equivalences by Properties of Spaces, Definition 48.7.5 and the other two by what it means to be a smooth local property of germs of schemes. This proves (4) $\Rightarrow$ (3).

The implications (1) $\Rightarrow$ (2), (2) $\Rightarrow$ (4), and (3) $\Rightarrow$ (1) are immediate. □

Definition 73.7.5. Let $\mathcal{P}$ be a property of germs of schemes which is smooth local. Let $\mathcal{X}$ be an algebraic stack. Let $x \in |\mathcal{X}|$. We say $\mathcal{X}$ *has property $\mathcal{P}$ at x* if any of the equivalent conditions of Lemma 73.7.4 holds.

73.8. Monomorphisms of algebraic stacks

We define a monomorphism of algebraic stacks in the following way. We will see in Lemma 73.8.4 that this is compatible with the corresponding 2-category theoretic notion.

Definition 73.8.1. Let $f : \mathcal{X} \to \mathcal{Y}$ be a morphism of algebraic stacks. We say f is a *monomorphism* if it is representable by algebraic spaces and a monomorphism in the sense of Section 73.3.

First some basic lemmas.

Lemma 73.8.2. *Let $\mathcal{X} \to \mathcal{Y}$ be a morphism of algebraic stacks. Let $\mathcal{Z} \to \mathcal{Y}$ be a monomorphism. Then $\mathcal{Z} \times_{\mathcal{Y}} \mathcal{X} \to \mathcal{X}$ is a monomorphism.*

Proof. This follows from the general discussion in Section 73.3. □

Lemma 73.8.3. *Compositions of monomorphisms of algebraic stacks are monomorphisms.*

Proof. This follows from the general discussion in Section 73.3 and Morphisms of Spaces, Lemma 49.11.4. □

Lemma 73.8.4. *Let $f : \mathcal{X} \to \mathcal{Y}$ be a morphism of algebraic stacks. The following are equivalent:*

(1) *f is a monomorphism,*
(2) *f is fully faithful,*
(3) *the diagonal $\Delta_f : \mathcal{X} \to \mathcal{X} \times_{\mathcal{Y}} \mathcal{X}$ is an equivalence, and*
(4) *there exists an algebraic space W and a surjective, flat morphism $W \to \mathcal{Y}$ which is locally of finite presentation such that $V = \mathcal{X} \times_{\mathcal{Y}} W$ is an algebraic space, and the morphism $V \to W$ is a monomorphism of algebraic spaces.*

Proof. The equivalence of (1) and (4) follows from the general discussion in Section 73.3 and in particular Lemmas 73.3.1 and 73.3.3.

The equivalence of (2) and (3) is Categories, Lemma 4.33.9.

Assume the equivalent conditions (2) and (3). Then f is representable by algebraic spaces according to Algebraic Stacks, Lemma 67.15.2. Moreover, the 2-Yoneda lemma combined with the fully faithfulness implies that for every scheme T the functor

$$\mathrm{Mor}(T, \mathcal{X}) \longrightarrow \mathrm{Mor}(T, \mathcal{Y})$$

is fully faithful. Hence given a morphism $y : T \to \mathcal{Y}$ there exists up to unique 2-isomorphism at most one morphism $x : T \to \mathcal{X}$ such that $y \cong f \circ x$. In particular, given a morphism of schemes $h : T' \to T$ there exists at most one lift $\tilde{h} : T' \to T \times_{\mathcal{Y}} \mathcal{X}$ of h. Thus $T \times_{\mathcal{Y}} \mathcal{X} \to T$ is a monomorphism of algebraic spaces, which proves that (1) holds.

Finally, assume that (1) holds. Then for any scheme T and morphism $y : T \to \mathcal{Y}$ the fibre product $T \times_{\mathcal{Y}} \mathcal{X}$ is an algebraic space, and $T \times_{\mathcal{Y}} \mathcal{X} \to T$ is a monomorphism. Hence there exists up to unique isomorphism exactly one pair (x, α) where $x : T \to \mathcal{X}$ is a morphism and $\alpha : f \circ x \to y$ is a 2-morphism. Applying the 2-Yoneda lemma this says exactly that f is fully faithful, i.e., that (2) holds. □

Lemma 73.8.5. *A monomorphism of algebraic stacks induces an injective map of sets of points.*

Proof. Let $f : \mathcal{X} \to \mathcal{Y}$ be a monomorphism of algebraic stacks. Suppose that $x_i : \mathrm{Spec}(K_i) \to \mathcal{X}$ be morphisms such that $f \circ x_1$ and $f \circ x_2$ define the same element of $|\mathcal{Y}|$. Applying the definition we find a common extension Ω with corresponding morphisms $c_i : \mathrm{Spec}(\Omega) \to \mathrm{Spec}(K_i)$ and a 2-isomorphism $\beta : f \circ x_1 \circ c_1 \to f \circ x_1 \circ c_2$. As f is fully faithful, see Lemma 73.8.4, we can lift β to an isomorphism $\alpha : f \circ x_1 \circ c_1 \to f \circ x_1 \circ c_2$. Hence x_1 and x_2 define the same point of $|\mathcal{X}|$ as desired. □

73.9. Immersions of algebraic stacks

Immersions of algebraic stacks are defined as follows.

Definition 73.9.1. Immersions.

(1) A morphism of algebraic stacks is called an *open immersion* if it is representable, and an open immersion in the sense of Section 73.3.
(2) A morphism of algebraic stacks is called a *closed immersion* if it is representable, and a closed immersion in the sense of Section 73.3.
(3) A morphism of algebraic stacks is called an *immersion* if it is representable, and an immersion in the sense of Section 73.3.

This is not the most convenient way to think about immersions for us. For us it is a little bit more convenient to think of an immersion as a morphism of algebraic stacks which is representable by algebraic spaces and is an immersion in the sense of Section 73.3. Similarly for closed and open immersions. Since this is clearly equivalent to the notion just defined we shall use this characterization without further mention. We prove a few simple lemmas about this notion.

Lemma 73.9.2. *Let $\mathcal{X} \to \mathcal{Y}$ be a morphism of algebraic stacks. Let $\mathcal{Z} \to \mathcal{Y}$ be a (closed, resp. open) immersion. Then $\mathcal{Z} \times_{\mathcal{Y}} \mathcal{X} \to \mathcal{X}$ is a (closed, resp. open) immersion.*

Proof. This follows from the general discussion in Section 73.3. □

Lemma 73.9.3. *Compositions of immersions of algebraic stacks are immersions. Similarly for closed immersions and open immersions.*

Proof. This follows from the general discussion in Section 73.3 and Spaces, Lemma 47.12.2. □

Lemma 73.9.4. *Let $f : \mathcal{X} \to \mathcal{Y}$ be a morphism of algebraic stacks. let W be an algebraic space and let $W \to \mathcal{Y}$ be a surjective, flat morphism which is locally of finite presentation. The following are equivalent:*

1. *f is an (open, resp. closed) immersion, and*
2. *$V = W \times_{\mathcal{Y}} \mathcal{X}$ is an algebraic space, and $V \to W$ is an (open, resp. closed) immersion.*

Proof. This follows from the general discussion in Section 73.3 and in particular Lemmas 73.3.1 and 73.3.3. □

Lemma 73.9.5. *An immersion is a monomorphism.*

Proof. See Morphisms of Spaces, Lemma 49.11.7. □

The following two lemmas explain how to think about immersions in terms of presentations.

Lemma 73.9.6. *Let (U, R, s, t, c) be a smooth groupoid in algebraic spaces. Let $i : \mathcal{Z} \to [U/R]$ be an immersion. Then there exists an R-invariant locally closed subspace $Z \subset U$ and a presentation $[Z/R_Z] \to \mathcal{Z}$ where R_Z is the restriction of R to Z such that*

$$\begin{array}{ccc} [Z/R_Z] & \longrightarrow & \mathcal{Z} \\ & \searrow \quad \swarrow_{i} & \\ & [U/R] & \end{array}$$

is 2-commutative. If i is a closed (resp. open) immersion then Z is a closed (resp. open) subspace of U.

Proof. By Lemma 73.3.6 we get a commutative diagram

$$\begin{array}{ccc} [U'/R'] & \longrightarrow & \mathcal{Z} \\ & \searrow \quad \swarrow & \\ & [U/R] & \end{array}$$

where $U' = \mathcal{Z} \times_{[U/R]} U$ and $R' = \mathcal{Z} \times_{[U/R]} R$. Since $\mathcal{Z} \to [U/R]$ is an immersion we see that $U' \to U$ is an immersion of algebraic spaces. Let $Z \subset U$ be the locally closed subspace such that $U' \to U$ factors through Z and induces an isomorphism $U' \to Z$. It is clear from the construction of R' that $R' = U' \times_{U,t} R = R \times_{s,U} U'$. This implies that $Z \cong U'$ is R-invariant and that the image of $R' \to R$ identifies R' with the restriction $R_Z = s^{-1}(Z) = t^{-1}(Z)$ of R to Z. Hence the lemma holds. □

Lemma 73.9.7. *Let (U, R, s, t, c) be a smooth groupoid in algebraic spaces. Let $\mathcal{X} = [U/R]$ be the associated algebraic stack, see Algebraic Stacks, Theorem 67.17.3. Let $Z \subset U$ be an R-invariant locally closed subspace. Then*

$$[Z/R_Z] \longrightarrow [U/R]$$

is an immersion of algebraic stacks, where R_Z is the restriction of R to Z. If $Z \subset U$ is open (resp. closed) then the morphism is an open (resp. closed) immersion of algebraic stacks.

Proof. Recall that by Groupoids in Spaces, Definition 59.17.1 (see also discussion following the definition) we have $R_Z = s^{-1}(Z) = t^{-1}(Z)$ as locally closed subspaces of R. Hence the two morphisms $R_Z \to Z$ are smooth as base changes of s and t. Hence $(Z, R_Z, s|_{R_Z}, t|_{R_Z}, c|_{R_Z \times_{s,Z,t} R_Z})$ is a smooth groupoid in algebraic spaces, and we see that $[Z/R_Z]$ is an algebraic stack, see Algebraic Stacks, Theorem 67.17.3. The assumptions of Groupoids in Spaces, Lemma 59.24.3 are all satisfied and it follows that we have a 2-fibre square

$$\begin{array}{ccc} Z & \longrightarrow & [Z/R_Z] \\ \downarrow & & \downarrow \\ U & \longrightarrow & [U/R] \end{array}$$

It follows from this and Lemma 73.3.1 that $[Z/R_Z] \to [U/R]$ is representable by algebraic spaces, whereupon it follows from Lemma 73.3.3 that the right vertical arrow is an immersion (resp. closed immersion, resp. open immersion) if and only if the left vertical arrow is. □

We can define open, closed, and locally closed substacks as follows.

Definition 73.9.8. Let $\mathcal{X}$ be an algebraic stack.

(1) An *open substack* of $\mathcal{X}$ is a strictly full subcategory $\mathcal{X}' \subset \mathcal{X}$ such that $\mathcal{X}'$ is an algebraic stack and $\mathcal{X}' \to \mathcal{X}$ is an open immersion.
(2) A *closed substack* of $\mathcal{X}$ is a strictly full subcategory $\mathcal{X}' \subset \mathcal{X}$ such that $\mathcal{X}'$ is an algebraic stack and $\mathcal{X}' \to \mathcal{X}$ is a closed immersion.
(3) A *locally closed substack* of $\mathcal{X}$ is a strictly full subcategory $\mathcal{X}' \subset \mathcal{X}$ such that $\mathcal{X}'$ is an algebraic stack and $\mathcal{X}' \to \mathcal{X}$ is an immersion.

This definition should be used with caution. Namely, if $f : \mathcal{X} \to \mathcal{Y}$ is an equivalence of algebraic stacks and $\mathcal{X}' \subset \mathcal{X}$ is an open substack, then it is not necessarily the case that the subcategory $f(\mathcal{X}')$ is an open substack of $\mathcal{Y}$. The problem is that it may not be a *strictly* full subcategory; but this is also the only problem. Here is a formal statement.

Lemma 73.9.9. *For any immersion $i : \mathcal{Z} \to \mathcal{X}$ there exists a unique locally closed substack $\mathcal{X}' \subset \mathcal{X}$ such that i factors as the composition of an equivalence $i' : \mathcal{Z} \to \mathcal{X}'$*

followed by the inclusion morphism $\mathcal{X}' \to \mathcal{X}$. If i is a closed (resp. open) immersion, then $\mathcal{X}'$ is a closed (resp. open) substack of $\mathcal{X}$.

Proof. Omitted. □

Lemma 73.9.10. *Let $[U/R] \to \mathcal{X}$ be a presentation of an algebraic stack. There is a canonical bijection*

$$R\text{-invariant locally closed subspaces } Z \text{ of } U \leftrightarrow \text{locally closed substacks } \mathcal{Z} \text{ of } \mathcal{X}$$

where if Z corresponds to $\mathcal{Z}$, then $[Z/R_Z] \to \mathcal{Z}$ is a presentation fitting into a 2-commutative diagram with the given presentation of $\mathcal{X}$. Similarly for closed substacks and open substacks.

Proof. Omitted. Hints: Use Lemma 73.9.6 to go from right to left and Lemma 73.9.7 from left to right. □

Lemma 73.9.11. *Let $\mathcal{X}$ be an algebraic stack. The rule $\mathcal{U} \mapsto |\mathcal{U}|$ defines an inclusion preserving bijection between open substacks of $\mathcal{X}$ and open subsets of $|\mathcal{X}|$.*

Proof. Choose a presentation $[U/R] \to \mathcal{X}$, see Algebraic Stacks, Lemma 67.16.2. By Lemma 73.9.10 we see that open substacks correspond to R-invariant open subschemes of U. On the other hand Lemmas 73.4.5 and 73.4.7 guarantee these correspond bijectively to open subsets of $|\mathcal{X}|$. □

Lemma 73.9.12. *Let $\mathcal{X}$ be an algebraic stack. Let U be an algebraic space and $U \to \mathcal{X}$ a surjective smooth morphism. For an open immersion $V \hookrightarrow U$, there exists an algebraic stack $\mathcal{Y}$, an open immersion $\mathcal{Y} \to \mathcal{X}$, and a surjective smooth morphism $V \to \mathcal{Y}$.*

Proof. We define a category fibred in groupoids $\mathcal{Y}$ by letting the fiber category $\mathcal{Y}_T$ over an object T of $(Sch/S)_{fppf}$ be the full subcategory of $\mathcal{X}_T$ consisting of all $y \in \mathrm{Ob}(\mathcal{X}_T)$ such that the projection morphism $V \times_{\mathcal{X},y} T \to T$ surjective. Now for any morphism $x : T \to \mathcal{X}$, the 2-fibred product $T \times_{x,\mathcal{X}} \mathcal{Y}$ has fiber category over T' consisting of triples $(f : T' \to T, y \in \mathcal{X}_{T'}, f^*x \simeq y)$ such that $V \times_{\mathcal{X},y} T' \to T'$ is surjective. Note that $T \times_{x,\mathcal{X}} \mathcal{Y}$ is fibered in setoids since $\mathcal{Y} \to \mathcal{X}$ is faithful (see Stacks, Lemma 8.6.7). Now the isomorphism $f^*x \simeq y$ gives the diagram

$$\begin{array}{ccccc} V \times_{\mathcal{X},y} T' & \longrightarrow & V \times_{\mathcal{X},x} T & \longrightarrow & V \\ \downarrow & & \downarrow & & \downarrow \\ T' & \xrightarrow{f} & T & \xrightarrow{x} & \mathcal{X} \end{array}$$

where both squares are cartesian. The morphism $V \times_{\mathcal{X},x} T \to T$ is smooth by base change, and hence open. Let $T_0 \subset T$ be its image. From the cartesian squares we deduce that $V \times_{\mathcal{X},y} T' \to T'$ is surjective if and only if f lands in T_0. Therefore $T \times_{x,\mathcal{X}} \mathcal{Y}$ is representable by T_0, so the inclusion $\mathcal{Y} \to \mathcal{X}$ is an open immersion. By Algebraic Stacks, Lemma 67.15.5 we conclude that $\mathcal{Y}$ is an algebraic stack. Lastly if we denote the morphism $V \to \mathcal{X}$ by g, we have $V \times_{\mathcal{X}} V \to V$ is surjective (the diagonal gives a section). Hence g is in the image of $\mathcal{Y}_V \to \mathcal{X}_V$, i.e., we obtain a

morphism $g' : V \to \mathcal{Y}$ fitting into the commutative diagram

$$\begin{array}{ccc} V & \longrightarrow & U \\ \downarrow{\scriptstyle g'} & & \downarrow \\ \mathcal{Y} & \longrightarrow & \mathcal{X} \end{array}$$

Since $V \times_{g,\mathcal{X}} \mathcal{Y} \to V$ is a monomorphism, it is in fact an isomorphism since $(1, g')$ defines a section. Therefore $g' : V \to \mathcal{Y}$ is a smooth morphism, as it is the base change of the smooth morphism $g : V \to \mathcal{X}$. It is surjective by our construction of $\mathcal{Y}$ which finishes the proof of the lemma. □

Lemma 73.9.13. *Let $\mathcal{X}$ be an algebraic stack and $\mathcal{X}_i \subset \mathcal{X}$ a collection of open substacks indexed by $i \in I$. Then there exists an open substack, which we denote $\bigcup_{i\in I} \mathcal{X}_i \subset \mathcal{X}$, such that the $\mathcal{X}_i$ are open substacks covering it.*

Proof. We define a fibred subcategory $\mathcal{X}' = \bigcup_{i\in I} \mathcal{X}_i$ by letting the fiber category over an object T of $(Sch/S)_{fppf}$ be the full subcategory of $\mathcal{X}_T$ consisting of all $x \in \mathrm{Ob}(\mathcal{X}_T)$ such that the morphism $\bigsqcup_{i\in I}(\mathcal{X}_i \times_{\mathcal{X}} T) \to T$ is surjective. Let $x_i \in \mathrm{Ob}((\mathcal{X}_i)_T)$. Then $(x_i, 1)$ gives a section of $\mathcal{X}_i \times_{\mathcal{X}} T \to T$, so we have an isomorphism. Thus $\mathcal{X}_i \subset \mathcal{X}'$ is a full subcategory. Now let $x \in \mathrm{Ob}(\mathcal{X}_T)$. Then $\mathcal{X}_i \times_{\mathcal{X}} T$ is representable by an open subscheme $T_i \subset T$. The 2-fibred product $\mathcal{X}' \times_{\mathcal{X}} T$ has fiber over T' consisting of $(y \in \mathcal{X}_{T'}, f : T' \to T, f^*x \simeq y)$ such that $\bigsqcup(\mathcal{X}_i \times_{\mathcal{X},y} T') \to T'$ is surjective. The isomorphism $f^*x \simeq y$ induces an isomorphism $\mathcal{X}_i \times_{\mathcal{X},y} T' \simeq T_i \times_T T'$. Then the $T_i \times_T T'$ cover T' if and only if f lands in $\bigcup T_i$. Therefore we have a diagram

$$\begin{array}{ccccc} T_i & \longrightarrow & \bigcup T_i & \longrightarrow & T \\ \downarrow & & \downarrow & & \downarrow \\ \mathcal{X}_i & \longrightarrow & \mathcal{X}' & \longrightarrow & \mathcal{X} \end{array}$$

with both squares cartesian. By Algebraic Stacks, Lemma 67.15.5 we conclude that $\mathcal{X}' \subset \mathcal{X}$ is algebraic and an open substack. It is also clear from the cartesian squares above that the morphism $\bigsqcup_{i\in I} \mathcal{X}_i \to \mathcal{X}'$ which finishes the proof of the lemma. □

Lemma 73.9.14. *Let $\mathcal{X}$ be an algebraic stack and $\mathcal{X}' \subset \mathcal{X}$ a quasi-compact open substack. Suppose that we have a collection of open substacks $\mathcal{X}_i \subset \mathcal{X}$ indexed by $i \in I$ such that $\mathcal{X}' \subset \bigcup_{i\in I} \mathcal{X}_i$, where we define the union as in Lemma 73.9.13. Then there exists a finite subset $I' \subset I$ such that $\mathcal{X}' \subset \bigcup_{i\in I'} \mathcal{X}_i$.*

Proof. Since $\mathcal{X}$ is algebraic, there exists a scheme U with a surjective smooth morphism $U \to \mathcal{X}$. Let $U_i \subset U$ be the open subscheme representing $\mathcal{X}_i \times_{\mathcal{X}} U$ and $U' \subset U$ the open subscheme representing $\mathcal{X}' \times_{\mathcal{X}} U$. By hypothesis, $U' \subset \bigcup_{i\in I} U_i$. From the proof of Lemma 73.6.2, there is a quasi-compact open $V \subset U'$ such that $V \to \mathcal{X}'$ is a surjective smooth morphism. Therefore there exists a finite subset $I' \subset I$ such that $V \subset \bigcup_{i\in I'} U_i$. We claim that $\mathcal{X}' \subset \bigcup_{i\in I'} \mathcal{X}_i$. Take $x \in \mathrm{Ob}(\mathcal{X}'_T)$ for $T \in \mathrm{Ob}((Sch/S)_{fppf})$. Since $\mathcal{X}' \to \mathcal{X}$ is a monomorphism, we have cartesian squares

$$\begin{array}{ccccc} V \times_{\mathcal{X}} T & \longrightarrow & T & = & T \\ \downarrow & & \downarrow{\scriptstyle x} & & \downarrow{\scriptstyle x} \\ V & \longrightarrow & \mathcal{X}' & \longrightarrow & \mathcal{X} \end{array}$$

By base change, $V \times_{\mathcal{X}} T \to T$ is surjective. Therefore $\bigcup_{i \in I'} U_i \times_{\mathcal{X}} T \to T$ is also surjective. Let $T_i \subset T$ be the open subscheme representing $\mathcal{X}_i \times_{\mathcal{X}} T$. By a formal argument, we have a Cartesian square

$$\begin{array}{ccc} U_i \times_{\mathcal{X}_i} T_i & \longrightarrow & U \times_{\mathcal{X}} T \\ \downarrow & & \downarrow \\ T_i & \longrightarrow & T \end{array}$$

where the vertical arrows are surjective by base change. Since $U_i \times_{\mathcal{X}_i} T_i \simeq U_i \times_{\mathcal{X}} T$, we find that $\bigcup_{i \in I'} T_i = T$. Hence x is an object of $(\bigcup_{i \in I'} \mathcal{X}_i)_T$ by definition of the union. Observe that the inclusion $\mathcal{X}' \subset \bigcup_{i \in I'} \mathcal{X}_i$ is automatically an open substack. □

Lemma 73.9.15. *Let $\mathcal{X}$ be an algebraic stack. Let $\mathcal{X}_i$, $i \in I$ be a set of open substacks of $\mathcal{X}$. Assume*

(1) *$\mathcal{X} = \bigcup_{i \in I} \mathcal{X}_i$, and*
(2) *each $\mathcal{X}_i$ is an algebraic space.*

Then $\mathcal{X}$ is an algebraic space.

Proof. Apply Stacks, Lemma 8.6.10 to the morphism $\coprod_{i \in I} \mathcal{X}_i \to \mathcal{X}$ and the morphism $\mathrm{id} : \mathcal{X} \to \mathcal{X}$ to see that $\mathcal{X}$ is a stack in setoids. Hence $\mathcal{X}$ is an algebraic space, see Algebraic Stacks, Proposition 67.13.3. □

Lemma 73.9.16. *Let $\mathcal{X}$ be an algebraic stack. Let $\mathcal{X}_i$, $i \in I$ be a set of open substacks of $\mathcal{X}$. Assume*

(1) *$\mathcal{X} = \bigcup_{i \in I} \mathcal{X}_i$, and*
(2) *each $\mathcal{X}_i$ is a scheme*

Then $\mathcal{X}$ is a scheme.

Proof. By Lemma 73.9.15 we see that $\mathcal{X}$ is an algebraic space. Since any algebraic space has a largest open subspace which is a scheme, see Properties of Spaces, Lemma 48.10.1 we see that $\mathcal{X}$ is a scheme. □

The following lemma is the analogue of More on Groupoids, Lemma 39.5.1.

Lemma 73.9.17. *Let $\mathcal{P}, \mathcal{Q}, \mathcal{R}$ be properties of morphisms of algebraic spaces. Assume*

(1) *$\mathcal{P}, \mathcal{Q}, \mathcal{R}$ are fppf local on the target and stable under arbitrary base change,*
(2) *smooth $\Rightarrow \mathcal{R}$,*
(3) *for any morphism $f : X \to Y$ which has $\mathcal{Q}$ there exists a largest open subspace $W(\mathcal{P}, f) \subset X$ such that $f|_{W(\mathcal{P},f)}$ has $\mathcal{P}$, and*
(4) *for any morphism $f : X \to Y$ which has $\mathcal{Q}$, and any morphism $Y' \to Y$ which has $\mathcal{R}$ we have $Y' \times_Y W(\mathcal{P}, f) = W(\mathcal{P}, f')$, where $f' : X_{Y'} \to Y'$ is the base change of f.*

Let $f : \mathcal{X} \to \mathcal{Y}$ be a morphisms of algebraic stacks representable by algebraic spaces. Assume f has $\mathcal{Q}$. Then

(A) *there exists a largest open substack $\mathcal{X}' \subset \mathcal{X}$ such that $f|_{\mathcal{X}'}$ has $\mathcal{P}$, and*
(B) *if $\mathcal{Z} \to \mathcal{Y}$ is a morphism of algebraic stacks representable by algebraic spaces which has $\mathcal{R}$ then $\mathcal{Z} \times_{\mathcal{Y}} \mathcal{X}'$ is the largest open substack of $\mathcal{Z} \times_{\mathcal{Y}} \mathcal{X}$ over which the base change $\mathrm{id}_{\mathcal{Z}} \times f$ has property $\mathcal{P}$.*

Proof. Choose a scheme V and a surjective smooth morphism $V \to \mathcal{Y}$. Set $U = V \times_{\mathcal{Y}} \mathcal{X}$ and let $f' : U \to V$ be the base change of f. The morphism of algebraic spaces $f' : U \to V$ has property $\mathcal{Q}$. Thus we obtain the open $W(\mathcal{P}, f') \subset U$ by assumption (3). Note that $U \times_{\mathcal{X}} U = (V \times_{\mathcal{Y}} V) \times_{\mathcal{Y}} \mathcal{X}$ hence the morphism $f'' : U \times_{\mathcal{X}} U \to V \times_{\mathcal{Y}} V$ is the base change of f via either projection $V \times_{\mathcal{Y}} V \to V$. By our choice of V these projections are smooth, hence have property $\mathcal{R}$ by (2). Thus by (4) we see that the inverse images of $W(\mathcal{P}, f')$ under the two projections $\mathrm{pr}_i : U \times_{\mathcal{X}} U \to U$ agree. In other words, $W(\mathcal{P}, f')$ is an R-invariant subspace of U (where $R = U \times_{\mathcal{X}} U$). Let $\mathcal{X}'$ be the open substack of $\mathcal{X}$ corresponding to $W(\mathcal{P}, f)$ via Lemma 73.9.6. By construction $W(\mathcal{P}, f') = \mathcal{X}' \times_{\mathcal{Y}} V$ hence $f|_{\mathcal{X}'}$ has property $\mathcal{P}$ by Lemma 73.3.3. Also, $\mathcal{X}'$ is the largest open substack such that $f|_{\mathcal{X}'}$ has $\mathcal{P}$ as the same maximality holds for $W(\mathcal{P}, f)$. This proves (A).

Finally, if $\mathcal{Z} \to \mathcal{Y}$ is a morphism of algebraic stacks representable by algebraic spaces which has $\mathcal{R}$ then we set $T = V \times_{\mathcal{Y}} \mathcal{Z}$ and we see that $T \to V$ is a morphism of algebraic spaces having property $\mathcal{R}$. Set $f'_T : T \times_V U \to T$ the base change of f'. By (4) again we see that $W(\mathcal{P}, f'_T)$ is the inverse image of $W(\mathcal{P}, f)$ in $T \times_V U$. This implies (B); some details omitted. □

Remark 73.9.18. Warning: Lemma 73.9.17 should be used with care. For example, it applies to $\mathcal{P}$ ="flat", $\mathcal{Q}$ ="empty", and $\mathcal{R}$ ="flat and locally of finite presentation". But given a morphism of algebraic spaces $f : X \to Y$ the largest open subspace $W \subset X$ such that $f|_W$ is flat is *not* the set of points where f is flat!

Remark 73.9.19. Notwithstanding the warning in Remark 73.9.18 there are some cases where Lemma 73.9.17 can be used without causing ambiguity. We give a list. In each case we omit the verification of assumptions (1) and (2) and we give references which imply (3) and (4). Here is the list:

(1) $\mathcal{Q}$ ="locally of finite type", $\mathcal{R} = \emptyset$, and $\mathcal{P}$ ="relative dimension $\leq d$". See Morphisms of Spaces, Definition 49.30.2 and Morphisms of Spaces, Lemmas 49.31.4 and 49.31.3.
(2) $\mathcal{Q}$ ="locally of finite type", $\mathcal{R} = \emptyset$, and $\mathcal{P}$ ="locally quasi-finite". This is the case $d = 0$ of the previous item, see Morphisms of Spaces, Lemma 49.31.6. On the other hand, properties (3) and (4) are spelled out in Morphisms of Spaces, Lemma 49.31.7.
(3) $\mathcal{Q}$ ="locally of finite type", $\mathcal{R} = \emptyset$, and $\mathcal{P}$ ="unramified". This is Morphisms of Spaces, Lemma 49.34.10.
(4) $\mathcal{Q}$ ="locally of finite presentation", $\mathcal{R}$ ="flat and locally of finite presentation", and $\mathcal{P}$ ="flat". See More on Morphisms of Spaces, Theorem 57.18.1 and Lemma 57.18.2. Note that here $W(\mathcal{P}, f)$ is always exactly the set of points where the morphism f is flat because we only consider this open when f has $\mathcal{Q}$ (see loc.cit.).
(5) $\mathcal{Q}$ ="locally of finite presentation", $\mathcal{R}$ ="flat and locally of finite presentation", and $\mathcal{P}$ ="étale". This follows on combining (3) and (4) because an unramified morphism which is flat and locally of finite presentation is étale, see Morphisms of Spaces, Lemma 49.35.12.
(6) Add more here as needed (compare with the longer list at More on Groupoids, Remark 39.5.3).

73.10. Reduced algebraic stacks

We have already defined reduced algebraic stacks in Section 73.7.

Lemma 73.10.1. *Let $\mathcal{X}$ be an algebraic stack. Let $T \subset |\mathcal{X}|$ be a closed subset. There exists a unique closed substack $\mathcal{Z} \subset \mathcal{X}$ with the following properties: (a) we have $|\mathcal{Z}| = T$, and (b) $\mathcal{Z}$ is reduced.*

Proof. Let $U \to \mathcal{X}$ be a surjective smooth morphism, where U is an algebraic space. Set $R = U \times_{\mathcal{X}} U$, so that there is a presentation $[U/R] \to \mathcal{X}$, see Algebraic Stacks, Lemma 67.16.2. As usual we denote $s, t : R \to U$ the two smooth projection morphisms. By Lemma 73.4.5 we see that T corresponds to a closed subset $T' \subset |U|$ such that $|s|^{-1}(T') = |t|^{-1}(T')$. Let $Z \subset U$ be the reduced induced algebraic space structure on T', see Properties of Spaces, Definition 48.9.4. The fibre products $Z \times_{U,t} R$ and $R \times_{s,U} Z$ are closed subspaces of R (Spaces, Lemma 47.12.3). The projections $Z \times_{U,t} R \to Z$ and $R \times_{s,U} Z \to Z$ are smooth by Morphisms of Spaces, Lemma 49.33.3. Thus as Z is reduced, it follows that $Z \times_{U,t} R$ and $R \times_{s,U} Z$ are reduced, see Remark 73.7.3. Since

$$|Z \times_{U,t} R| = |t|^{-1}(T') = |s|^{-1}(T') = R \times_{s,U} Z$$

we conclude from the uniqueness in Properties of Spaces, Lemma 48.9.2 that $Z \times_{U,t} R = R \times_{s,U} Z$. Hence Z is an R-invariant closed subspace of U. By the correspondence of Lemma 73.9.10 (and its proof) we obtain a closed substack $\mathcal{Z} \subset \mathcal{Z}$ with a presentation $[Z/R_Z] \to \mathcal{Z}$. Then $|\mathcal{Z}| = |Z|/|R_Z| = |T'|/\sim$ is the given closed subset T. We omit the proof of unicity. □

Lemma 73.10.2. *Let $\mathcal{X}$ be an algebraic stack. If $\mathcal{X}' \subset \mathcal{X}$ is a closed substack, $\mathcal{X}$ is reduced and $|\mathcal{X}'| = |\mathcal{X}|$, then $\mathcal{X}' = \mathcal{X}$.*

Proof. Choose a presentation $[U/R] \to \mathcal{X}$ with U a scheme. As $\mathcal{X}$ is reduced, we see that U is reduced (by definition of reduced algebraic stacks). By Lemma 73.9.10 $\mathcal{X}'$ corresponds to an R-invariant closed subscheme $Z \subset U$. But now $|Z| \subset |U|$ is the inverse image of $|\mathcal{X}'|$, and hence $|Z| = |U|$. Hence Z is a closed subscheme of U whose underlying sets of points agree. By Schemes, Lemma 25.12.6 the map $\mathrm{id}_U : U \to U$ factors through $Z \to U$, and hence $Z = U$, i.e., $\mathcal{X}' = \mathcal{X}$. □

Lemma 73.10.3. *Let $\mathcal{X}$, $\mathcal{Y}$ be algebraic stacks. Let $\mathcal{Z} \subset \mathcal{X}$ be a closed substack Assume $\mathcal{Y}$ is reduced. A morphism $f : \mathcal{Y} \to \mathcal{X}$ factors through $\mathcal{Z}$ if and only if $f(|\mathcal{Y}|) \subset |\mathcal{Z}|$.*

Proof. Assume $f(|\mathcal{Y}|) \subset |\mathcal{Z}|$. Consider $\mathcal{Y} \times_{\mathcal{X}} \mathcal{Z} \to \mathcal{Y}$. There is an equivalence $\mathcal{Y} \times_{\mathcal{X}} \mathcal{Z} \to \mathcal{Y}'$ where $\mathcal{Y}'$ is a closed substack of $\mathcal{Y}$, see Lemmas 73.9.2 and 73.9.9. Using Lemmas 73.4.3, 73.8.5, and 73.9.5 we see that $|\mathcal{Y}'| = |\mathcal{Y}|$. Hence we have reduced the lemma to Lemma 73.10.2. □

Definition 73.10.4. Let $\mathcal{X}$ be an algebraic stack. Let $Z \subset |\mathcal{X}|$ be a closed subset. An *algebraic stack structure on Z* is given by a closed substack $\mathcal{Z}$ of $\mathcal{X}$ with $|\mathcal{Z}|$ equal to Z. The *reduced induced algebraic stack structure* on Z is the one constructed in Lemma 73.10.1. The *reduction $\mathcal{X}_{red}$ of $\mathcal{X}$* is the reduced induced algebraic stack structure on $|\mathcal{X}|$.

In fact we can use this to define the reduced induced algebraic stack structure on a locally closed subset.

Remark 73.10.5. Let X be an algebraic stack. Let $T \subset |\mathcal{X}|$ be a locally closed subset. Let ∂T be the boundary of T in the topological space $|\mathcal{X}|$. In a formula

$$\partial T = \overline{T} \setminus T.$$

Let $\mathcal{U} \subset \mathcal{X}$ be the open substack of X with $|\mathcal{U}| = |\mathcal{X}| \setminus \partial T$, see Lemma 73.9.11. Let $\mathcal{Z}$ be the reduced closed substack of $\mathcal{U}$ with $|\mathcal{Z}| = T$ obtained by taking the reduced induced closed subspace structure, see Definition 73.10.4. By construction $\mathcal{Z} \to \mathcal{U}$ is a closed immersion of algebraic stacks and $\mathcal{U} \to \mathcal{X}$ is an open immersion, hence $\mathcal{Z} \to \mathcal{X}$ is an immersion of algebraic stacks by Lemma 73.9.3. Note that $\mathcal{Z}$ is a reduced algebraic stack and that $|\mathcal{Z}| = T$ as subsets of $|X|$. We sometimes say $\mathcal{Z}$ is the *reduced induced substack structure* on T.

73.11. Residual gerbes

In the stacks project we would like to define the *residual gerbe* of an algebraic stack $\mathcal{X}$ at a point $x \in |\mathcal{X}|$ to be a monomorphism of algebraic stacks $m_x : \mathcal{Z}_x \to \mathcal{X}$ where $\mathcal{Z}_x$ is a reduced algebraic stack having a unique point which is mapped by m_x to x. It turns out that there are many issues with this notion; existence is not clear in general and neither is uniqueness. We resolve the uniqueness issue by imposing a slightly stronger condition on the algebraic stacks $\mathcal{Z}_x$. We discuss this in more detail by working through a few simple lemmas regarding reduced algebraic stacks having a unique point.

Lemma 73.11.1. *Let $\mathcal{Z}$ be an algebraic stack. Let k be a field and let $\mathrm{Spec}(k) \to \mathcal{Z}$ be surjective and flat. Then any morphism $\mathrm{Spec}(k') \to \mathcal{Z}$ where k' is a field is surjective and flat.*

Proof. Consider the fibre square

$$\begin{array}{ccc} T & \longrightarrow & \mathrm{Spec}(k) \\ \downarrow & & \downarrow \\ \mathrm{Spec}(k') & \longrightarrow & \mathcal{Z} \end{array}$$

Note that $T \to \mathrm{Spec}(k')$ is flat and surjective hence T is not empty. On the other hand $T \to \mathrm{Spec}(k)$ is flat as k is a field. Hence $T \to \mathcal{Z}$ is flat and surjective. It follows from Morphisms of Spaces, Lemma 49.28.5 (via the discussion in Section 73.3) that $\mathrm{Spec}(k') \to \mathcal{Z}$ is flat. It is clear that it is surjective as by assumption $|\mathcal{Z}|$ is a singleton. □

Lemma 73.11.2. *Let $\mathcal{Z}$ be an algebraic stack. The following are equivalent*

1. *$\mathcal{Z}$ is reduced and $|\mathcal{Z}|$ is a singleton,*
2. *there exists a surjective flat morphism $\mathrm{Spec}(k) \to \mathcal{Z}$ where k is a field, and*
3. *there exists a locally of finite type, surjective, flat morphism $\mathrm{Spec}(k) \to \mathcal{Z}$ where k is a field.*

Proof. Assume (1). Let W be a scheme and let $W \to \mathcal{Z}$ be a surjective smooth morphism. Then W is a reduced scheme. Let $\eta \in W$ be a generic point of an irreducible component of W. Since W is reduced we have $\mathcal{O}_{W,\eta} = \kappa(\eta)$. It follows that the canonical morphism $\eta = \mathrm{Spec}(\kappa(\eta)) \to W$ is flat. We see that the composition

$\eta \to \mathcal{Z}$ is flat (see Morphisms of Spaces, Lemma 49.27.3). It is also surjective as $|\mathcal{Z}|$ is a singleton. In other words (2) holds.

Assume (2). Let W be a scheme and let $W \to \mathcal{Z}$ be a surjective smooth morphism. Choose a field k and a surjective flat morphism $\mathrm{Spec}(k) \to \mathcal{Z}$. Then $W \times_{\mathcal{Z}} \mathrm{Spec}(k)$ is an algebraic space smooth over k, hence regular (see Spaces over Fields, Lemma 58.5.1) and in particular reduced. Since $W \times_{\mathcal{Z}} \mathrm{Spec}(k) \to W$ is surjective and flat we conclude that W is reduced (Descent on Spaces, Lemma 55.8.2). In other words (1) holds.

It is clear that (3) implies (2). Finally, assume (2). Pick a nonempty affine scheme W and a smooth morphism $W \to \mathcal{Z}$. Pick a closed point $w \in W$ and set $k = \kappa(w)$. The composition

$$\mathrm{Spec}(k) \xrightarrow{w} W \longrightarrow \mathcal{Z}$$

is locally of finite type by Morphisms of Spaces, Lemmas 49.22.2 and 49.33.6. It is also flat and surjective by Lemma 73.11.1. Hence (3) holds. □

The following lemma singles out a slightly better class of singleton algebraic stacks than the preceding lemma.

Lemma 73.11.3. *Let $\mathcal{Z}$ be an algebraic stack. The following are equivalent*

(1) *$\mathcal{Z}$ is reduced, locally Noetherian, and $|\mathcal{Z}|$ is a singleton, and*
(2) *there exists a locally finitely presented, surjective, flat morphism $\mathrm{Spec}(k) \to \mathcal{Z}$ where k is a field.*

Proof. Assume (2) holds. By Lemma 73.11.2 we see that $\mathcal{Z}$ is reduced and $|\mathcal{Z}|$ is a singleton. Let W be a scheme and let $W \to \mathcal{Z}$ be a surjective smooth morphism. Choose a field k and a locally finitely presented, surjective, flat morphism $\mathrm{Spec}(k) \to \mathcal{Z}$. Then $W \times_{\mathcal{Z}} \mathrm{Spec}(k)$ is an algebraic space smooth over k, hence locally Noetherian (see Morphisms of Spaces, Lemma 49.22.5). Since $W \times_{\mathcal{Z}} \mathrm{Spec}(k) \to W$ is flat, surjective, and locally of finite presentation, we see that $\{W \times_{\mathcal{Z}} \mathrm{Spec}(k) \to W\}$ is an fppf covering and we conclude that W is locally Noetherian (Descent on Spaces, Lemma 55.8.3). In other words (1) holds.

Assume (1). Pick a nonempty affine scheme W and a smooth morphism $W \to \mathcal{Z}$. Pick a closed point $w \in W$ and set $k = \kappa(w)$. Because W is locally Noetherian the morphism $w : \mathrm{Spec}(k) \to W$ is of finite presentation, see Morphisms, Lemma 28.22.7. Hence the composition

$$\mathrm{Spec}(k) \xrightarrow{w} W \longrightarrow \mathcal{Z}$$

is locally of finite presentation by Morphisms of Spaces, Lemmas 49.26.2 and 49.33.5. It is also flat and surjective by Lemma 73.11.1. Hence (2) holds. □

Lemma 73.11.4. *Let $\mathcal{Z}' \to \mathcal{Z}$ be a monomorphism of algebraic stacks. Assume there exists a field k and a locally finitely presented, surjective, flat morphism $\mathrm{Spec}(k) \to \mathcal{Z}$. Then either $\mathcal{Z}'$ is empty or $\mathcal{Z}' \to \mathcal{Z}$ is an equivalence.*

Proof. We may assume that $\mathcal{Z}'$ is nonempty. In this case the fibre product $T = \mathcal{Z}' \times_{\mathcal{Z}} \mathrm{Spec}(k)$ is nonempty, see Lemma 73.4.3. Now T is an algebraic space and the projection $T \to \mathrm{Spec}(k)$ is a monomorphism. Hence $T = \mathrm{Spec}(k)$, see Morphisms of Spaces, Lemma 49.11.8. We conclude that $\mathrm{Spec}(k) \to \mathcal{Z}$ factors through $\mathcal{Z}'$. Suppose the morphism $z : \mathrm{Spec}(k) \to \mathcal{Z}$ is given by the object ξ over $\mathrm{Spec}(k)$. We have just seen that ξ is isomorphic to an object ξ' of $\mathcal{Z}'$ over $\mathrm{Spec}(k)$. Since

z is is surjective, flat, and locally of finite presentation we see that every object of $\mathcal{Z}$ over any scheme is fppf locally isomorphic to a pullback of ξ, hence also to a pullback of ξ'. By descent of objects for stacks in groupoids this implies that $\mathcal{Z}' \to \mathcal{Z}$ is essentially surjective (as well as fully faithful, see Lemma 73.8.4). Hence we win. □

Lemma 73.11.5. *Let $\mathcal{Z}$ be an algebraic stack. Assume $\mathcal{Z}$ satisfies the equivalent conditions of Lemma 73.11.2. Then there exists a unique strictly full subcategory $\mathcal{Z}' \subset \mathcal{Z}$ such that $\mathcal{Z}'$ is an algebraic stack which satisfies the equivalent conditions of Lemma 73.11.3. The inclusion morphism $\mathcal{Z}' \to \mathcal{Z}$ is a monomorphism of algebraic stacks.*

Proof. The last part is immediate from the first part and Lemma 73.8.4. Pick a field k and a morphism $\mathrm{Spec}(k) \to \mathcal{Z}$ which is surjective, flat, and locally of finite type. Set $U = \mathrm{Spec}(k)$ and $R = U \times_{\mathcal{Z}} U$. The projections $s, t : R \to U$ are locally of finite type. Since U is the spectrum of a field, it follows that s, t are flat and locally of finite presentation (by Morphisms of Spaces, Lemma 49.26.7). We see that $\mathcal{Z}' = [U/R]$ is an algebraic stack by Criteria for Representability, Theorem 70.17.2. By Algebraic Stacks, Lemma 67.16.1 we obtain a canonical morphism

$$f : \mathcal{Z}' \longrightarrow \mathcal{Z}$$

which is fully faithful. Hence this morphism is representable by algebraic spaces, see Algebraic Stacks, Lemma 67.15.2 and a monomorphism, see Lemma 73.8.4. By Criteria for Representability, Lemma 70.17.1 the morphism $U \to \mathcal{Z}'$ is surjective, flat, and locally of finite presentation. Hence $\mathcal{Z}'$ is an algebraic stack which satisfies the equivalent conditions of Lemma 73.11.3. By Algebraic Stacks, Lemma 67.12.4 we may replace $\mathcal{Z}'$ by its essential image in $\mathcal{Z}$. Hence we have proved all the assertions of the lemma except for the uniqueness of $\mathcal{Z}' \subset \mathcal{Z}$. Suppose that $\mathcal{Z}'' \subset \mathcal{Z}$ is a second such algebraic stack. Then the projections

$$\mathcal{Z}' \longleftarrow \mathcal{Z}' \times_{\mathcal{Z}} \mathcal{Z}'' \longrightarrow \mathcal{Z}''$$

are monomorphisms. The algebraic stack in the middle is nonempty by Lemma 73.4.3. Hence the two projections are isomorphisms by Lemma 73.11.4 and we win. □

Example 73.11.6. Here is an example where the morphism constructed in Lemma 73.11.5 isn't an isomorphism. This example shows that imposing that residual gerbes are locally Noetherian is necessary in Definition 73.11.8. In fact, the example is even an algebraic space! Let $\mathrm{Gal}(\overline{\mathbf{Q}}/\mathbf{Q})$ be the absolute Galois group of $\mathbf{Q}$ with the pro-finite topology. Let

$$U = \mathrm{Spec}(\overline{\mathbf{Q}}) \times_{\mathrm{Spec}(\mathbf{Q})} \mathrm{Spec}(\overline{\mathbf{Q}}) = \mathrm{Gal}(\overline{\mathbf{Q}}/\mathbf{Q}) \times \mathrm{Spec}(\overline{\mathbf{Q}})$$

(we omit a precise explanation of the meaning of the last equal sign). Let G denote the absolute Galois group $\mathrm{Gal}(\overline{\mathbf{Q}}/\mathbf{Q})$ with the discrete topology viewed as a constant group scheme over $\mathrm{Spec}(\overline{\mathbf{Q}})$, see Groupoids, Example 38.5.6. Then G acts freely and transitively on U. Let $X = U/G$, see Spaces, Definition 47.14.4. Then X is a non-noetherian reduced algebraic space with exactly one point. Furthermore, X has a (locally) finite type point:

$$x : \mathrm{Spec}(\overline{\mathbf{Q}}) \longrightarrow U \longrightarrow X$$

Indeed, every point of U is actually closed! As X is an algebraic space over $\overline{\mathbf{Q}}$ it follows that x is a monomorphism. So x is the morphism constructed in Lemma 73.11.5 but x is not an isomorphism. In fact $\mathrm{Spec}(\overline{\mathbf{Q}}) \to X$ is the residual gerbe of X at x.

It will turn out later that under mild assumptions on the algebraic stack $\mathcal{X}$ the equivalent conditions of the following lemma are satisfied for every point $x \in |\mathcal{X}|$ (insert future reference here).

Lemma 73.11.7. *Let $\mathcal{X}$ be an algebraic stack. Let $x \in |\mathcal{X}|$ be a point. The following are equivalent*

1. *there exists an algebraic stack $\mathcal{Z}$ and a monomorphism $\mathcal{Z} \to \mathcal{X}$ such that $|\mathcal{Z}|$ is a singleton and such that the image of $|\mathcal{Z}|$ in $|\mathcal{X}|$ is x,*
2. *there exists a reduced algebraic stack $\mathcal{Z}$ and a monomorphism $\mathcal{Z} \to \mathcal{X}$ such that $|\mathcal{Z}|$ is a singleton and such that the image of $|\mathcal{Z}|$ in $|\mathcal{X}|$ is x,*
3. *there exists an algebraic stack $\mathcal{Z}$, a monomorphism $f : \mathcal{Z} \to \mathcal{X}$, and a surjective flat morphism $z : \mathrm{Spec}(k) \to \mathcal{X}$ where k is a field such that $x = f(z)$.*

Moreover, if these conditions hold, then there exists a unique strictly full subcategory $\mathcal{Z}_x \subset \mathcal{X}$ such that $\mathcal{Z}_x$ is a reduced, locally Noetherian algebraic stack and $|\mathcal{Z}_x|$ is a singleton which maps to x via the map $|\mathcal{Z}_x| \to |\mathcal{X}|$.

Proof. If $\mathcal{Z} \to \mathcal{X}$ is as in (1), then $\mathcal{Z}_{red} \to \mathcal{X}$ is as in (2). (See Section 73.10 for the notion of the reduction of an algebraic stack.) Hence (1) implies (2). It is immediate that (2) implies (1). The equivalence of (2) and (3) is immediate from Lemma 73.11.2.

At this point we've seen the equivalence of (1) – (3). Pick a monomorphism $f : \mathcal{Z} \to \mathcal{X}$ as in (2). Note that this implies that f is fully faithful, see Lemma 73.8.4. Denote $\mathcal{Z}' \subset \mathcal{X}$ the essential image of the functor f. Then $f : \mathcal{Z} \to \mathcal{Z}'$ is an equivalence and hence $\mathcal{Z}'$ is an algebraic stack, see Algebraic Stacks, Lemma 67.12.4. Apply Lemma 73.11.5 to get a strictly full subcategory $\mathcal{Z}_x \subset \mathcal{Z}'$ as in the statement of the lemma. This proves all the statements of the lemma except for uniqueness.

In order to prove the uniqueness suppose that $\mathcal{Z}_x \subset \mathcal{X}$ and $\mathcal{Z}'_x \subset \mathcal{X}$ are two strictly full subcategories as in the statement of the lemma. Then the projections

$$\mathcal{Z}'_x \longleftarrow \mathcal{Z}'_x \times_{\mathcal{X}} \mathcal{Z}_x \longrightarrow \mathcal{Z}_x$$

are monomorphisms. The algebraic stack in the middle is nonempty by Lemma 73.4.3. Hence the two projections are isomorphisms by Lemma 73.11.4 and we win. □

Having explained the above we can now make the following definition.

Definition 73.11.8. Let $\mathcal{X}$ be an algebraic stack. Let $x \in |\mathcal{X}|$.

1. We say the *residual gerbe of $\mathcal{X}$ at x exists* if the equivalent conditions (1), (2), and (3) of Lemma 73.11.7 hold.

(2) If the residual gerbe of $\mathcal{X}$ at x exists, then the *residual gerbe of* $\mathcal{X}$ *at* x[1] is the strictly full subcategory $\mathcal{Z}_x \subset \mathcal{X}$ constructed in Lemma 73.11.7.

In particular we know that $\mathcal{Z}_x$ (if it exists) is a locally Noetherian, reduced algebraic stack and that there exists a field and a surjective, flat, locally finitely presented morphism

$$\mathrm{Spec}(k) \longrightarrow \mathcal{Z}_x.$$

We will see in Morphisms of Stacks, Lemma 74.19.10 that $\mathcal{Z}_x$ is a gerbe. It turns out that $\mathcal{Z}_x$ is a regular algebraic stack as follows from the following lemma.

Lemma 73.11.9. *A reduced, locally Noetherian algebraic stack $\mathcal{Z}$ such that $|\mathcal{Z}|$ is a singleton is regular.*

Proof. Let $W \to \mathcal{Z}$ be a surjective smooth morphism where W is a scheme. Let k be a field and let $\mathrm{Spec}(k) \to \mathcal{Z}$ be surjective, flat, and locally of finite presentation (see Lemma 73.11.3). The algebraic space $T = W \times_{\mathcal{Z}} \mathrm{Spec}(k)$ is smooth over k in particular regular, see Spaces over Fields, Lemma 58.5.1. Since $T \to W$ is locally of finite presentation, flat, and surjective it follows that W is regular, see Descent on Spaces, Lemma 55.8.4. By definition this means that $\mathcal{Z}$ is regular. □

Lemma 73.11.10. *Let $\mathcal{X}$ be an algebraic stack. Let $x \in |\mathcal{X}|$. Assume that the residual gerbe $\mathcal{Z}_x$ of $\mathcal{X}$ exists. Let $f : \mathrm{Spec}(K) \to \mathcal{X}$ be a morphism where K is a field in the equivalence class of x. Then f factors through the inclusion morphism $\mathcal{Z}_x \to \mathcal{X}$.*

Proof. Choose a field k and a surjective flat locally finite presentation morphism $\mathrm{Spec}(k) \to \mathcal{Z}_x$. Set $T = \mathrm{Spec}(K) \times_{\mathcal{X}} \mathcal{Z}_x$. By Lemma 73.4.3 we see that T is nonempty. As $\mathcal{Z}_x \to \mathcal{X}$ is a monomorphism we see that $T \to \mathrm{Spec}(K)$ is a monomorphism. Hence by Morphisms of Spaces, Lemma 49.11.8 we see that $T = \mathrm{Spec}(K)$ which proves the lemma. □

Lemma 73.11.11. *Let $\mathcal{X}$ be an algebraic stack. Let $x \in |\mathcal{X}|$. Let $\mathcal{Z}$ be an algebraic stack satisfying the equivalent conditions of Lemma 73.11.3 and let $\mathcal{Z} \to \mathcal{X}$ be a monomorphism such that the image of $|\mathcal{Z}| \to |\mathcal{X}|$ is x. Then the residual gerbe $\mathcal{Z}_x$ of $\mathcal{X}$ at x exists and $\mathcal{Z} \to \mathcal{X}$ factors as $\mathcal{Z} \to \mathcal{Z}_x \to \mathcal{X}$ where the first arrow is an equivalence.*

Proof. Let $\mathcal{Z}_x \subset \mathcal{X}$ be the full subcategory corresponding to the essential image of the functor $\mathcal{Z} \to \mathcal{X}$. Then $\mathcal{Z} \to \mathcal{Z}_x$ is an equivalence, hence $\mathcal{Z}_x$ is an algebraic stack, see Algebraic Stacks, Lemma 67.12.4. Since $\mathcal{Z}_x$ inherits all the properties of $\mathcal{Z}$ from this equivalence it is clear from the uniqueness in Lemma 73.11.7 that $\mathcal{Z}_x$ is the residual gerbe of $\mathcal{X}$ at x. □

73.12. Other chapters

Preliminaries

(1) Introduction
(2) Conventions
(3) Set Theory

[1] This clashes with [**LMB00**] in spirit, but not in fact. Namely, in Chapter 11 they associate to any point on any quasi-separated algebraic stack a gerbe (not necessarily algebraic) which they call the residual gerbe. We will see in Morphisms of Stacks, Lemma 74.21.1 that on a quasi-separated algebraic stack every point has a residual gerbe in our sense which is then equivalent to theirs. For more information on this topic see [**Ryd10**, Appendix B].

(4) Categories
(5) Topology
(6) Sheaves on Spaces
(7) Sites and Sheaves
(8) Stacks
(9) Fields
(10) Commutative Algebra
(11) Brauer Groups
(12) Homological Algebra
(13) Derived Categories
(14) Simplicial Methods
(15) More on Algebra
(16) Smoothing Ring Maps
(17) Sheaves of Modules
(18) Modules on Sites
(19) Injectives
(20) Cohomology of Sheaves
(21) Cohomology on Sites
(22) Differential Graded Algebra
(23) Divided Power Algebra
(24) Hypercoverings

Schemes

(25) Schemes
(26) Constructions of Schemes
(27) Properties of Schemes
(28) Morphisms of Schemes
(29) Cohomology of Schemes
(30) Divisors
(31) Limits of Schemes
(32) Varieties
(33) Topologies on Schemes
(34) Descent
(35) Derived Categories of Schemes
(36) More on Morphisms
(37) More on Flatness
(38) Groupoid Schemes
(39) More on Groupoid Schemes
(40) Étale Morphisms of Schemes

Topics in Scheme Theory

(41) Chow Homology
(42) Adequate Modules
(43) Dualizing Complexes
(44) Étale Cohomology
(45) Crystalline Cohomology
(46) Pro-étale Cohomology

Algebraic Spaces

(47) Algebraic Spaces
(48) Properties of Algebraic Spaces
(49) Morphisms of Algebraic Spaces
(50) Decent Algebraic Spaces
(51) Cohomology of Algebraic Spaces
(52) Limits of Algebraic Spaces
(53) Divisors on Algebraic Spaces
(54) Topologies on Algebraic Spaces
(55) Descent and Algebraic Spaces
(56) Derived Categories of Spaces
(57) More on Morphisms of Spaces
(58) Algebraic Spaces over Fields
(59) Groupoids in Algebraic Spaces
(60) More on Groupoids in Spaces
(61) Bootstrap
(62) Quotients of Groupoids
(63) Simplicial Spaces

Deformation Theory

(64) Formal Deformation Theory
(65) Deformation Theory
(66) The Cotangent Complex

Algebraic Stacks

(67) Algebraic Stacks
(68) Examples of Stacks
(69) Sheaves on Algebraic Stacks
(70) Criteria for Representability
(71) Artin's Axioms
(72) Quot and Hilbert Spaces
(73) Properties of Algebraic Stacks
(74) Morphisms of Algebraic Stacks
(75) Cohomology of Algebraic Stacks
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Miscellany

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CHAPTER 74

Morphisms of Algebraic Stacks

74.1. Introduction

In this chapter we introduce some types of morphisms of algebraic stacks. A reference in the case of quasi-separated algebraic stacks with representable diagonal is [**LMB00**].

The goal is to extend the definition of each of the types of morphisms of algebraic spaces to morphisms of algebraic stacks. Each case is slightly different and it seems best to treat them all separately.

For morphisms of algebraic stacks which are representable by algebraic spaces we have already defined a large number of types of morphisms, see Properties of Stacks, Section 73.3. For each corresponding case in this chapter we have to make sure the definition in the general case is compatible with the definition given there.

74.2. Conventions and abuse of language

We continue to use the conventions and the abuse of language introduced in Properties of Stacks, Section 73.2.

74.3. Properties of diagonals

The diagonal of an algebraic stack is closely related to the *Isom*-sheaves, see Algebraic Stacks, Lemma 67.10.11. By the second defining property of an algebraic stack these *Isom*-sheaves are always algebraic spaces.

Lemma 74.3.1. *Let $\mathcal{X}$ be an algebraic stack. Let T be a scheme and let x, y be objects of the fibre category of $\mathcal{X}$ over T. Then the morphism $Isom_{\mathcal{X}}(x, y) \to T$ is locally of finite type.*

Proof. By Algebraic Stacks, Lemma 67.16.2 we may assume that $\mathcal{X} = [U/R]$ for some smooth groupoid in algebraic spaces. By Descent on Spaces, Lemma 55.10.7 it suffices to check the property fppf locally on T. Thus we may assume that x, y come from morphisms $x', y' : T \to U$. By Groupoids in Spaces, Lemma 59.21.1 we see that in this case $Isom_{\mathcal{X}}(x, y) = T \times_{(y', x'), U \times_S U} R$. Hence it suffices to prove that $R \to U \times_S U$ is locally of finite type. This follows from the fact that the composition $s : R \to U \times_S U \to U$ is smooth (hence locally of finite type, see Morphisms of Spaces, Lemmas 49.33.5 and 49.26.5) and Morphisms of Spaces, Lemma 49.22.6. □

Lemma 74.3.2. *Let $\mathcal{X}$ be an algebraic stack. Let T be a scheme and let x, y be objects of the fibre category of $\mathcal{X}$ over T. Then*

(1) *$Isom_{\mathcal{X}}(y, y)$ is a group algebraic space over T, and*
(2) *$Isom_{\mathcal{X}}(x, y)$ is a pseudo torsor for $Isom_{\mathcal{X}}(y, y)$ over T.*

Proof. See Groupoids in Spaces, Definitions 59.5.1 and 59.9.1. The lemma follows immediately from the fact that $\mathcal{X}$ is a stack in groupoids. □

Let $f : \mathcal{X} \to \mathcal{Y}$ be a morphism of algebraic stacks. The *diagonal of* f is the morphism

$$\Delta_f : \mathcal{X} \longrightarrow \mathcal{X} \times_{\mathcal{Y}} \mathcal{X}$$

Here are two properties that every diagonal morphism has.

Lemma 74.3.3. *Let $f : \mathcal{X} \to \mathcal{Y}$ be a morphism of algebraic stacks. Then*

(1) *Δ_f is representable by algebraic spaces, and*
(2) *Δ_f is locally of finite type.*

Proof. Let T be a scheme and let $a : T \to \mathcal{X} \times_{\mathcal{Y}} \mathcal{X}$ be a morphism. By definition of the fibre product and the 2-Yoneda lemma the morphism a is given by a triple $a = (x, x', \alpha)$ where x, x' are objects of $\mathcal{X}$ over T, and $\alpha : f(x) \to f(x')$ is a morphism in the fibre category of $\mathcal{Y}$ over T. By definition of an algebraic stack the sheaves $Isom_{\mathcal{X}}(x, x')$ and $Isom_{\mathcal{Y}}(f(x), f(x'))$ are algebraic spaces over T. In this language α defines a section of the morphism $Isom_{\mathcal{X}}(x, x') \to T$. A T'-valued point of $\mathcal{X} \times_{\mathcal{X} \times_{\mathcal{Y}} \mathcal{X}, a} T$ for $T' \to T$ a scheme over T is the same thing as an isomorphism $x|_{T'} \to x'|_{T'}$ whose image under f is $\alpha|_{T'}$. Thus we see that

$$\begin{array}{ccc} \mathcal{X} \times_{\mathcal{X} \times_{\mathcal{Y}} \mathcal{X}, a} T & \longrightarrow & Isom_{\mathcal{X}}(x, x') \\ \downarrow & & \downarrow \\ T & \xrightarrow{\alpha} & Isom_{\mathcal{Y}}(f(x), f(x')) \end{array} \tag{74.3.3.1}$$

is a fibre square of sheaves over T. In particular we see that $\mathcal{X} \times_{\mathcal{X} \times_{\mathcal{Y}} \mathcal{X}, a} T$ is an algebraic space which proves part (1) of the lemma.

To prove the second statement we have to show that the left vertical arrow of Diagram (74.3.3.1) is locally of finite type. By Lemma 74.3.1 the algebraic space $Isom_{\mathcal{X}}(x, x')$ and is locally of finite type over T. Hence the right vertical arrow of Diagram (74.3.3.1) is locally of finite type, see Morphisms of Spaces, Lemma 49.22.6. We conclude by Morphisms of Spaces, Lemma 49.22.3. □

Lemma 74.3.4. *Let $f : \mathcal{X} \to \mathcal{Y}$ be a morphism of algebraic stacks which is representable by algebraic spaces. Then*

(1) *Δ_f is representable (by schemes),*
(2) *Δ_f is locally of finite type,*
(3) *Δ_f is a monomorphism,*
(4) *Δ_f is separated, and*
(5) *Δ_f is locally quasi-finite.*

Proof. We have already seen in Lemma 74.3.3 that Δ_f is representable by algebraic spaces. Hence the statements (2) – (5) make sense, see Properties of Stacks, Section 73.3. Also Lemma 74.3.3 guarantees (2) holds. Let $T \to \mathcal{X} \times_{\mathcal{Y}} \mathcal{X}$ be a morphism and contemplate Diagram (74.3.3.1). By Algebraic Stacks, Lemma 67.9.2 the right vertical arrow is injective as a map of sheaves, i.e., a monomorphism of algebraic spaces. Hence also the morphism $T \times_{\mathcal{X} \times_{\mathcal{Y}} \mathcal{X}} \mathcal{X} \to T$ is a monomorphism. Thus (3) holds. We already know that $T \times_{\mathcal{X} \times_{\mathcal{Y}} \mathcal{X}} \mathcal{X} \to T$ is locally of finite type. Thus Morphisms of Spaces, Lemma 49.25.8 allows us to conclude that $T \times_{\mathcal{X} \times_{\mathcal{Y}} \mathcal{X}} \mathcal{X} \to T$ is locally quasi-finite and separated. This proves (4) and (5). Finally, Morphisms

of Spaces, Proposition 49.43.2 implies that $T \times_{\mathcal{X} \times_{\mathcal{Y}} \mathcal{X}} \mathcal{X}$ is a scheme which proves (1). □

Lemma 74.3.5. *Let $f : \mathcal{X} \to \mathcal{Y}$ be a morphism of algebraic stacks representable by algebraic spaces. Then the following are equivalent*

(1) *f is separated,*
(2) *Δ_f is a closed immersion,*
(3) *Δ_f is proper, or*
(4) *Δ_f is universally closed.*

Proof. The statements "f is separated", "Δ_f is a closed immersion", "Δ_f is universally closed", and "Δ_f is proper" refer to the notions defined in Properties of Stacks, Section 73.3. Choose a scheme V and a surjective smooth morphism $V \to \mathcal{Y}$. Set $U = \mathcal{X} \times_{\mathcal{Y}} V$ which is an algebraic space by assumption, and the morphism $U \to \mathcal{X}$ is surjective and smooth. By Categories, Lemma 4.29.14 and Properties of Stacks, Lemma 73.3.3 we see that for any property P (as in that lemma) we have: Δ_f has P if and only if $\Delta_{U/V} : U \to U \times_V U$ has P. Hence the equivalence of (2), (3) and (4) follows from Morphisms of Spaces, Lemma 49.36.8 applied to $U \to V$. Moreover, if (1) holds, then $U \to V$ is separated and we see that $\Delta_{U/V}$ is a closed immersion, i.e., (2) holds. Finally, assume (2) holds. Let T be a scheme, and $a : T \to \mathcal{Y}$ a morphism. Set $T' = \mathcal{X} \times_{\mathcal{Y}} T$. To prove (1) we have to show that the morphism of algebraic spaces $T' \to T$ is separated. Using Categories, Lemma 4.29.14 once more we see that $\Delta_{T'/T}$ is the base change of Δ_f. Hence our assumption (2) implies that $\Delta_{T'/T}$ is a closed immersion, hence $T' \to T$ is separated as desired. □

Lemma 74.3.6. *Let $f : \mathcal{X} \to \mathcal{Y}$ be a morphism of algebraic stacks representable by algebraic spaces. Then the following are equivalent*

(1) *f is quasi-separated,*
(2) *Δ_f is quasi-compact, or*
(3) *Δ_f is finite type.*

Proof. The statements "f is quasi-separated", "Δ_f is quasi-compact", and "Δ_f is finite type" refer to the notions defined in Properties of Stacks, Section 73.3. Note that (2) and (3) are equivalent in view of the fact that Δ_f is locally of finite type by Lemma 74.3.4 (and Algebraic Stacks, Lemma 67.10.9). Choose a scheme V and a surjective smooth morphism $V \to \mathcal{Y}$. Set $U = \mathcal{X} \times_{\mathcal{Y}} V$ which is an algebraic space by assumption, and the morphism $U \to \mathcal{X}$ is surjective and smooth. By Categories, Lemma 4.29.14 and Properties of Stacks, Lemma 73.3.3 we see that we have: Δ_f is quasi-compact if and only if $\Delta_{U/V} : U \to U \times_V U$ is quasi-compact. If (1) holds, then $U \to V$ is quasi-separated and we see that $\Delta_{U/V}$ is quasi-compact, i.e., (2) holds. Assume (2) holds. Let T be a scheme, and $a : T \to \mathcal{Y}$ a morphism. Set $T' = \mathcal{X} \times_{\mathcal{Y}} T$. To prove (1) we have to show that the morphism of algebraic spaces $T' \to T$ is quasi-separated. Using Categories, Lemma 4.29.14 once more we see that $\Delta_{T'/T}$ is the base change of Δ_f. Hence our assumption (2) implies that $\Delta_{T'/T}$ is quasi-compact, hence $T' \to T$ is quasi-separated as desired. □

Lemma 74.3.7. *Let $f : \mathcal{X} \to \mathcal{Y}$ be a morphism of algebraic stacks representable by algebraic spaces. Then the following are equivalent*

(1) *f is locally separated, and*

(2) Δ_f *is an immersion.*

Proof. The statements "f is quasi-separated", and "Δ_f is an immersion" refer to the notions defined in Properties of Stacks, Section 73.3. Proof omitted. Hint: Argue as in the proofs of Lemmas 74.3.5 and 74.3.6. □

74.4. Separation axioms

Let $\mathcal{X} = [U/R]$ be a presentation of an algebraic stack. Then the properties of the diagonal of $\mathcal{X}$ over S, are the properties of the morphism $j : R \to U \times_S U$. For example, if $\mathcal{X} = [S/G]$ for some smooth group G in algebraic spaces over S then j is the structure morphism $G \to U$. Hence the diagonal is not automatically separated itself (contrary to what happens in the case of schemes and algebraic spaces). To say that $[S/G]$ is quasi-separated over S should certainly imply that $G \to S$ is quasi-compact, but we hesitate to say that $[S/G]$ is quasi-separated over S without also requiring the morphism $G \to S$ to be quasi-separated. In other words, requiring the diagonal morphism to be quasi-compact does not really agree with our intuition for a "quasi-separated algebraic stack", and we should also require the diagonal itself to be quasi-separated.

What about "separated algebraic stacks"? We have seen in Morphisms of Spaces, Lemma 49.36.8 that an algebraic space is separated if and only if the diagonal is proper. This is the condition that is usually used to define separated algebraic stacks too. In the example $[S/G] \to S$ above this means that $G \to S$ is a proper group scheme. This means algebraic stacks of the form $[\mathrm{Spec}(k)/E]$ are proper over k where E is an elliptic curve over k (insert future reference here). In certain situations it may be more natural to assume the diagonal is finite.

Definition 74.4.1. Let $f : \mathcal{X} \to \mathcal{Y}$ be a morphism of algebraic stacks.

(1) We say f is *DM* if Δ_f is unramified[1].
(2) We say f is *quasi-DM* if Δ_f is locally quasi-finite[2].
(3) We say f is *separated* if Δ_f is proper.
(4) We say f is *quasi-separated* if Δ_f is quasi-compact and quasi-separated.

In this definition we are using that Δ_f is representable by algebraic spaces and we are using Properties of Stacks, Section 73.3 to make sense out of imposing conditions on Δ_f. We note that these definitions do not conflict with the already existing notions if f is representable by algebraic spaces, see Lemmas 74.3.6 and 74.3.5. There is an interesting way to characterize these conditions by looking at higher diagonals, see Lemma 74.6.3.

Definition 74.4.2. Let $\mathcal{X}$ be an algebraic stack over the base scheme S. Denote $p : \mathcal{X} \to S$ the structure morphism.

[1]The letters DM stand for Deligne-Mumford. If f is DM then given any scheme T and any morphism $T \to \mathcal{Y}$ the fibre product $\mathcal{X}_T = \mathcal{X} \times_{\mathcal{Y}} T$ is an algebraic stack over T whose diagonal is unramified, i.e., $\mathcal{X}_T$ is DM. This implies $\mathcal{X}_T$ is a Deligne-Mumford stack, see Theorem 74.15.6. In other words a DM morphism is one whose "fibres" are Deligne-Mumford stacks. This hopefully at least motivates the terminology.

[2]If f is quasi-DM, then the "fibres" $\mathcal{X}_T$ of $\mathcal{X} \to \mathcal{Y}$ are quasi-DM. An algebraic stack $\mathcal{X}$ is quasi-DM exactly if there exists a scheme U and a surjective flat morphism $U \to \mathcal{X}$ of finite presentation which is locally quasi-finite, see Theorem 74.15.3. Note the similarity to being Deligne-Mumford, which is defined in terms of having an étale covering by a scheme.

(1) We say $\mathcal{X}$ is *DM over* S if $p : \mathcal{X} \to S$ is DM.
(2) We say $\mathcal{X}$ is *quasi-DM over* S if $p : \mathcal{X} \to S$ is quasi-DM.
(3) We say $\mathcal{X}$ is *separated over* S if $p : \mathcal{X} \to S$ is separated.
(4) We say $\mathcal{X}$ is *quasi-separated over* S if $p : \mathcal{X} \to S$ is quasi-separated.
(5) We say $\mathcal{X}$ is *DM* if $\mathcal{X}$ is DM[3] over $\mathrm{Spec}(\mathbf{Z})$.
(6) We say $\mathcal{X}$ is *quasi-DM* if $\mathcal{X}$ is quasi-DM over $\mathrm{Spec}(\mathbf{Z})$.
(7) We say $\mathcal{X}$ is *separated* if $\mathcal{X}$ is separated over $\mathrm{Spec}(\mathbf{Z})$.
(8) We say $\mathcal{X}$ is *quasi-separated* if $\mathcal{X}$ is quasi-separated over $\mathrm{Spec}(\mathbf{Z})$.

In the last 4 definitions we view $\mathcal{X}$ as an algebraic stack over $\mathrm{Spec}(\mathbf{Z})$ via Algebraic Stacks, Definition 67.19.2.

Thus in each case we have an absolute notion and a notion relative to our given base scheme (mention of which is usually suppressed by our abuse of notation introduced in Properties of Stacks, Section 73.2). We will see that (1) $\Leftrightarrow$ (5) and (2) $\Leftrightarrow$ (6) in Lemma 74.4.13. We spend some time proving some standard results on these notions.

Lemma 74.4.3. *Let $f : \mathcal{X} \to \mathcal{Y}$ be a morphism of algebraic stacks.*

(1) *If f is separated, then f is quasi-separated.*
(2) *If f is DM, then f is quasi-DM.*
(3) *If f is representable by algebraic spaces, then f is DM.*

Proof. To see (1) note that a proper morphism of algebraic spaces is quasi-compact and quasi-separated, see Morphisms of Spaces, Definition 49.36.1. To see (2) note that an unramified morphism of algebraic spaces is locally quasi-finite, see Morphisms of Spaces, Lemma 49.34.7. Finally (3) follows from Lemma 74.3.4. □

Lemma 74.4.4. *All of the separation axioms listed in Definition 74.4.1 are stable under base change.*

Proof. Let $f : \mathcal{X} \to \mathcal{Y}$ and $\mathcal{Y}' \to \mathcal{Y}$ be morphisms of algebraic stacks. Let $f' : \mathcal{Y}' \times_{\mathcal{Y}} \mathcal{X} \to \mathcal{Y}'$ be the base change of f by $\mathcal{Y}' \to \mathcal{Y}$. Then $\Delta_{f'}$ is the base change of Δ_f by the morphism $\mathcal{X}' \times_{\mathcal{Y}'} \mathcal{X}' \to \mathcal{X} \times_{\mathcal{Y}} \mathcal{X}$, see Categories, Lemma 4.29.14. By the results of Properties of Stacks, Section 73.3 each of the properties of the diagonal used in Definition 74.4.1 is stable under base change. Hence the lemma is true. □

Lemma 74.4.5. *Let $f : \mathcal{X} \to \mathcal{Y}$ be a morphism of algebraic stacks. Let $W \to \mathcal{Y}$ be a surjective, flat, and locally of finite presentation where W is an algebraic space. If the base change $W \times_{\mathcal{Y}} \mathcal{X} \to W$ has one of the separation properties of Definition 74.4.1 then so does f.*

Proof. Denote $g : W \times_{\mathcal{Y}} \mathcal{X} \to W$ the base change. Then Δ_g is the base change of Δ_f by the morphism $q : W \times_{\mathcal{Y}} (\mathcal{X} \times_{\mathcal{Y}} \mathcal{X}) \to \mathcal{X} \times_{\mathcal{Y}} \mathcal{X}$. Since q is the base change of $W \to \mathcal{Y}$ we see that q is representable by algebraic spaces, surjective, flat, and locally of finite presentation. Hence the result follows from Properties of Stacks, Lemma 73.3.4. □

Lemma 74.4.6. *Let S be a scheme. The property of being quasi-DM over S, quasi-separated over S, or separated over S (see Definition 74.4.2) is stable under change of base scheme, see Algebraic Stacks, Definition 67.19.3.*

[3] Theorem 74.15.6 shows that this is equivalent to $\mathcal{X}$ being a Deligne-Mumford stack.

Proof. Follows immediately from Lemma 74.4.4. □

Lemma 74.4.7. *Let $f : \mathcal{X} \to \mathcal{Z}$, $g : \mathcal{Y} \to \mathcal{Z}$ and $\mathcal{Z} \to \mathcal{T}$ be morphisms of algebraic stacks. Consider the induced morphism $i : \mathcal{X} \times_{\mathcal{Z}} \mathcal{Y} \to \mathcal{X} \times_{\mathcal{T}} \mathcal{Y}$. Then*

(1) *i is representable by algebraic spaces and locally of finite type,*
(2) *if $\Delta_{\mathcal{Z}/\mathcal{T}}$ is quasi-separated, then i is quasi-separated,*
(3) *if $\Delta_{\mathcal{Z}/\mathcal{T}}$ is separated, then i is separated,*
(4) *if $\mathcal{Z} \to \mathcal{T}$ is DM, then i is unramified,*
(5) *if $\mathcal{Z} \to \mathcal{T}$ is quasi-DM, then i is locally quasi-finite,*
(6) *if $\mathcal{Z} \to \mathcal{T}$ is separated, then i is proper, and*
(7) *if $\mathcal{Z} \to \mathcal{T}$ is quasi-separated, then i is quasi-compact and quasi-separated.*

Proof. The following diagram

$$\begin{array}{ccc} \mathcal{X} \times_{\mathcal{Z}} \mathcal{Y} & \xrightarrow[i]{} & \mathcal{X} \times_{\mathcal{T}} \mathcal{Y} \\ \downarrow & & \downarrow \\ \mathcal{Z} & \xrightarrow{\Delta_{\mathcal{Z}/\mathcal{T}}} & \mathcal{Z} \times_{\mathcal{T}} \mathcal{Z} \end{array}$$

is a 2-fibre product diagram, see Categories, Lemma 4.29.13. Hence i is the base change of the diagonal morphism $\Delta_{\mathcal{Z}/\mathcal{T}}$. Thus the lemma follows from Lemma 74.3.3, and the material in Properties of Stacks, Section 73.3. □

Lemma 74.4.8. *Let $\mathcal{T}$ be an algebraic stack. Let $g : \mathcal{X} \to \mathcal{Y}$ be a morphism of algebraic stacks over $\mathcal{T}$. Consider the graph $i : \mathcal{X} \to \mathcal{X} \times_{\mathcal{T}} \mathcal{Y}$ of g. Then*

(1) *i is representable by algebraic spaces and locally of finite type,*
(2) *if $\mathcal{Y} \to \mathcal{T}$ is DM, then i is unramified,*
(3) *if $\mathcal{Y} \to \mathcal{T}$ is quasi-DM, then i is locally quasi-finite,*
(4) *if $\mathcal{Y} \to \mathcal{T}$ is separated, then i is proper, and*
(5) *if $\mathcal{Y} \to \mathcal{T}$ is quasi-separated, then i is quasi-compact and quasi-separated.*

Proof. This is a special case of Lemma 74.4.7 applied to the morphism $\mathcal{X} = \mathcal{X} \times_{\mathcal{Y}} \mathcal{Y} \to \mathcal{X} \times_{\mathcal{T}} \mathcal{Y}$. □

Lemma 74.4.9. *Let $f : \mathcal{X} \to \mathcal{T}$ be a morphism of algebraic stacks. Let $s : \mathcal{T} \to \mathcal{X}$ be a morphism such that $f \circ s$ is 2-isomorphic to $id_{\mathcal{T}}$. Then*

(1) *s is representable by algebraic spaces and locally of finite type,*
(2) *if f is DM, then s is unramified,*
(3) *if f is quasi-DM, then s is locally quasi-finite,*
(4) *if f is separated, then s is proper, and*
(5) *if f is quasi-separated, then s is quasi-compact and quasi-separated.*

Proof. This is a special case of Lemma 74.4.8 applied to $g = s$ and $\mathcal{Y} = \mathcal{T}$ in which case $i : \mathcal{T} \to \mathcal{T} \times_{\mathcal{T}} \mathcal{X}$ is 2-isomorphic to s. □

Lemma 74.4.10. *All of the separation axioms listed in Definition 74.4.1 are stable under composition of morphisms.*

Proof. Let $f : \mathcal{X} \to \mathcal{Y}$ and $g : \mathcal{Y} \to \mathcal{Z}$ be morphisms of algebraic stacks to which the axiom in question applies. The diagonal $\Delta_{\mathcal{X}/\mathcal{Z}}$ is the composition

$$\mathcal{X} \longrightarrow \mathcal{X} \times_{\mathcal{Y}} \mathcal{X} \longrightarrow \mathcal{X} \times_{\mathcal{Z}} \mathcal{X}.$$

Our separation axiom is defined by requiring the diagonal to have some property $\mathcal{P}$. By Lemma 74.4.7 above we see that the second arrow also has this property. Hence the lemma follows since the composition of morphisms which are representable by algebraic spaces with property $\mathcal{P}$ also is a morphism with property $\mathcal{P}$, see our general discussion in Properties of Stacks, Section 73.3 and Morphisms of Spaces, Lemmas 49.34.3, 49.25.2, 49.36.4, 49.9.4, and 49.5.8. □

Lemma 74.4.11. *Let $f : \mathcal{X} \to \mathcal{Y}$ be a morphism of algebraic stacks over the base scheme S.*

(1) *If $\mathcal{Y}$ is DM over S and f is DM, then $\mathcal{X}$ is DM over S.*
(2) *If $\mathcal{Y}$ is quasi-DM over S and f is quasi-DM, then $\mathcal{X}$ is quasi-DM over S.*
(3) *If $\mathcal{Y}$ is separated over S and f is separated, then $\mathcal{X}$ is separated over S.*
(4) *If $\mathcal{Y}$ is quasi-separated over S and f is quasi-separated, then $\mathcal{X}$ is quasi-separated over S.*
(5) *If $\mathcal{Y}$ is DM and f is DM, then $\mathcal{X}$ is DM.*
(6) *If $\mathcal{Y}$ is quasi-DM and f is quasi-DM, then $\mathcal{X}$ is quasi-DM.*
(7) *If $\mathcal{Y}$ is separated and f is separated, then $\mathcal{X}$ is separated.*
(8) *If $\mathcal{Y}$ is quasi-separated and f is quasi-separated, then $\mathcal{X}$ is quasi-separated.*

Proof. Parts (1), (2), (3), and (4) follow immediately from Lemma 74.4.10 and Definition 74.4.2. For (5), (6), (7), and (8) think of $\mathcal{X}$ and $\mathcal{Y}$ as algebraic stacks over $\mathrm{Spec}(\mathbf{Z})$ and apply Lemma 74.4.10. Details omitted. □

The following lemma is a bit different to the analogue for algebraic spaces. To compare take a look at Morphisms of Spaces, Lemma 49.5.10.

Lemma 74.4.12. *Let $f : \mathcal{X} \to \mathcal{Y}$ and $g : \mathcal{Y} \to \mathcal{Z}$ be morphisms of algebraic stacks.*

(1) *If $g \circ f$ is DM then so is f.*
(2) *If $g \circ f$ is quasi-DM then so is f.*
(3) *If $g \circ f$ is separated and Δ_g is separated, then f is separated.*
(4) *If $g \circ f$ is quasi-separated and Δ_g is quasi-separated, then f is quasi-separated.*

Proof. Consider the factorization

$$\mathcal{X} \to \mathcal{X} \times_{\mathcal{Y}} \mathcal{X} \to \mathcal{X} \times_{\mathcal{Z}} \mathcal{X}$$

of the diagonal morphism of $g \circ f$. Both morphisms are representable by algebraic spaces, see Lemmas 74.3.3 and 74.4.7. Hence for any scheme T and morphism $T \to \mathcal{X} \times_{\mathcal{Y}} \mathcal{X}$ we get morphisms of algebraic spaces

$$A = \mathcal{X} \times_{(\mathcal{X} \times_{\mathcal{Z}} \mathcal{X})} T \longrightarrow B = (\mathcal{X} \times_{\mathcal{Y}} \mathcal{X}) \times_{(\mathcal{X} \times_{\mathcal{Z}} \mathcal{X})} T \longrightarrow T.$$

If $g \circ f$ is DM (resp. quasi-DM), then the composition $A \to T$ is unramified (resp. locally quasi-finite). Hence (1) (resp. (2)) follows on applying Morphisms of Spaces, Lemma 49.34.11 (resp. Morphisms of Spaces, Lemma 49.25.7). This proves (1) and (2).

Proof of (3). Assume $g \circ f$ is quasi-separated and Δ_g is quasi-separated. Consider the factorization

$$\mathcal{X} \to \mathcal{X} \times_{\mathcal{Y}} \mathcal{X} \to \mathcal{X} \times_{\mathcal{Z}} \mathcal{X}$$

of the diagonal morphism of $g \circ f$. Both morphisms are representable by algebraic spaces and the second one is quasi-separated, see Lemmas 74.3.3 and 74.4.7. Hence for any scheme T and morphism $T \to \mathcal{X} \times_{\mathcal{Y}} \mathcal{X}$ we get morphisms of algebraic spaces

$$A = \mathcal{X} \times_{(\mathcal{X} \times_{\mathcal{Z}} \mathcal{X})} T \longrightarrow B = (\mathcal{X} \times_{\mathcal{Y}} \mathcal{X}) \times_{(\mathcal{X} \times_{\mathcal{Z}} \mathcal{X})} T \longrightarrow T$$

such that $B \to T$ is quasi-separated. The composition $A \to T$ is quasi-compact and quasi-separated as we have assumed that $g \circ f$ is quasi-separated. Hence $A \to B$ is quasi-separated by Morphisms of Spaces, Lemma 49.5.10. And $A \to B$ is quasi-compact by Morphisms of Spaces, Lemma 49.9.8. Thus f is quasi-separated.

Proof of (4). Assume $g \circ f$ is separated and Δ_g is separated. Consider the factorization

$$\mathcal{X} \to \mathcal{X} \times_{\mathcal{Y}} \mathcal{X} \to \mathcal{X} \times_{\mathcal{Z}} \mathcal{X}$$

of the diagonal morphism of $g \circ f$. Both morphisms are representable by algebraic spaces and the second one is separated, see Lemmas 74.3.3 and 74.4.7. Hence for any scheme T and morphism $T \to \mathcal{X} \times_{\mathcal{Y}} \mathcal{X}$ we get morphisms of algebraic spaces

$$A = \mathcal{X} \times_{(\mathcal{X} \times_{\mathcal{Z}} \mathcal{X})} T \longrightarrow B = (\mathcal{X} \times_{\mathcal{Y}} \mathcal{X}) \times_{(\mathcal{X} \times_{\mathcal{Z}} \mathcal{X})} T \longrightarrow T$$

such that $B \to T$ is separated. The composition $A \to T$ is proper as we have assumed that $g \circ f$ is quasi-separated. Hence $A \to B$ is proper by Morphisms of Spaces, Lemma 49.36.6 which means that f is separated. □

Lemma 74.4.13. *Let $\mathcal{X}$ be an algebraic stack over the base scheme S.*

1. *$\mathcal{X}$ is DM $\Leftrightarrow$ $\mathcal{X}$ is DM over S.*
2. *$\mathcal{X}$ is quasi-DM $\Leftrightarrow$ $\mathcal{X}$ is quasi-DM over S.*
3. *If $\mathcal{X}$ is separated, then $\mathcal{X}$ is separated over S.*
4. *If $\mathcal{X}$ is quasi-separated, then $\mathcal{X}$ is quasi-separated over S.*

Let $f : \mathcal{X} \to \mathcal{Y}$ be a morphism of algebraic stacks over the base scheme S.

5. *If $\mathcal{X}$ is DM over S, then f is DM.*
6. *If $\mathcal{X}$ is quasi-DM over S, then f is quasi-DM.*
7. *If $\mathcal{X}$ is separated over S and $\Delta_{\mathcal{Y}/S}$ is separated, then f is separated.*
8. *If $\mathcal{X}$ is quasi-separated over S and $\Delta_{\mathcal{Y}/S}$ is quasi-separated, then f is quasi-separated.*

Proof. Parts (5), (6), (7), and (8) follow immediately from Lemma 74.4.12 and Spaces, Definition 47.13.2. To prove (3) and (4) think of X and Y as algebraic stacks over $\mathrm{Spec}(\mathbf{Z})$ and apply Lemma 74.4.12. Similarly, to prove (1) and (2), think of $\mathcal{X}$ as an algebraic stack over $\mathrm{Spec}(\mathbf{Z})$ consider the morphisms

$$\mathcal{X} \longrightarrow \mathcal{X} \times_S \mathcal{X} \longrightarrow \mathcal{X} \times_{\mathrm{Spec}(\mathbf{Z})} \mathcal{X}$$

Both arrows are representable by algebraic spaces. The second arrow is unramified and locally quasi-finite as the base change of the immersion $\Delta_{S/\mathbf{Z}}$. Hence the composition is unramified (resp. locally quasi-finite) if and only if the first arrow is unramified (resp. locally quasi-finite), see Morphisms of Spaces, Lemmas 49.34.3 and 49.34.11 (resp. Morphisms of Spaces, Lemmas 49.25.2 and 49.25.7). □

Lemma 74.4.14. *Let $\mathcal{X}$ be an algebraic stack. Let W be an algebraic space, and let $f : W \to \mathcal{X}$ be a surjective, flat, locally finitely presented morphism.*

1. *If f is unramified (i.e., étale, i.e., $\mathcal{X}$ is Deligne-Mumford), then $\mathcal{X}$ is DM.*
2. *If f is locally quasi-finite, then $\mathcal{X}$ is quasi-DM.*

Proof. Note that if f is unramified, then it is étale by Morphisms of Spaces, Lemma 49.35.12. This explains the parenthetical remark in (1). Assume f is unramified (resp. locally quasi-finite). We have to show that $\Delta_{\mathcal{X}} : \mathcal{X} \to \mathcal{X} \times \mathcal{X}$ is unramified (resp. locally quasi-finite). Note that $W \times W \to \mathcal{X} \times \mathcal{X}$ is also surjective, flat, and locally of finite presentation. Hence it suffices to show that

$$W \times_{\mathcal{X} \times \mathcal{X}, \Delta_{\mathcal{X}}} \mathcal{X} = W \times_{\mathcal{X}} W \longrightarrow W \times W$$

is unramified (resp. locally quasi-finite), see Properties of Stacks, Lemma 73.3.3. By assumption the morphism $\mathrm{pr}_i : W \times_{\mathcal{X}} W \to W$ is unramified (resp. locally quasi-finite). Hence the displayed arrow is unramified (resp. locally quasi-finite) by Morphisms of Spaces, Lemma 49.34.11 (resp. Morphisms of Spaces, Lemma 49.25.7). □

Lemma 74.4.15. *A monomorphism of algebraic stacks is separated and DM. The same is true for immersions of algebraic stacks.*

Proof. If $f : \mathcal{X} \to \mathcal{Y}$ is a monomorphism of algebraic stacks, then Δ_f is an isomorphism, see Properties of Stacks, Lemma 73.8.4. Since an isomorphism of algebraic spaces is proper and unramified we see that f is separated and DM. The second assertion follows from the first as an immersion is a monomorphism, see Properties of Stacks, Lemma 73.9.5. □

Lemma 74.4.16. *Let $\mathcal{X}$ be an algebraic stack. Let $x \in |\mathcal{X}|$. Assume the residual gerbe $\mathcal{Z}_x$ of $\mathcal{X}$ at x exists. If $\mathcal{X}$ is DM, resp. quasi-DM, resp. separated, resp. quasi-separated, then so is $\mathcal{Z}_x$.*

Proof. This is true because $\mathcal{Z}_x \to \mathcal{X}$ is a monomorphism hence DM and separated by Lemma 74.4.15. Apply Lemma 74.4.11 to conclude. □

74.5. Inertia stacks

The (relative) inertia stack of a stack in groupoids is defined in Stacks, Section 8.7. The actual construction, in the setting of fibred categories, and some of its properties is in Categories, Section 4.32.

Lemma 74.5.1. *Let $\mathcal{X}$ be an algebraic stack. Then the inertia stack $\mathcal{I}_{\mathcal{X}}$ is an algebraic stack as well. The morphism*

$$\mathcal{I}_{\mathcal{X}} \longrightarrow \mathcal{X}$$

is representable by algebraic spaces and locally of finite type. More generally, let $f : \mathcal{X} \to \mathcal{Y}$ be a morphism of algebraic stacks. Then the morphism

$$\mathcal{I}_{\mathcal{X}/\mathcal{Y}} \longrightarrow \mathcal{X}$$

is representable by algebraic spaces and locally of finite type.

Proof. By Categories, Lemma 4.32.1 there are equivalences

$$\mathcal{I}_{\mathcal{X}} \to \mathcal{X} \times_{\Delta, \mathcal{X} \times_S \mathcal{X}, \Delta} \mathcal{X} \quad \text{and} \quad \mathcal{I}_{\mathcal{X}/\mathcal{Y}} \to \mathcal{X} \times_{\Delta, \mathcal{X} \times_{\mathcal{Y}} \mathcal{X}, \Delta} \mathcal{X}$$

which shows that the inertia stacks are algebraic stacks. Let $T \to \mathcal{X}$ be a morphism given by the object x of the fibre category of $\mathcal{X}$ over T. Then we get a 2-fibre product

square

$$\begin{array}{ccc} Isom_{\mathcal{X}}(x,x) & \longrightarrow & \mathcal{I}_{\mathcal{X}} \\ \downarrow & & \downarrow \\ T & \xrightarrow{x} & \mathcal{X} \end{array}$$

This follows immediately from the definition of $\mathcal{I}_{\mathcal{X}}$. Since $Isom_{\mathcal{X}}(x,x)$ is always an algebraic space locally of finite type over T (see Lemma 74.3.1) we conclude that $\mathcal{I}_{\mathcal{X}} \to \mathcal{X}$ is representable by algebraic spaces and locally of finite type. Finally, for the relative inertia we get

$$\begin{array}{ccccc} Isom_{\mathcal{X}}(x,x) & \longleftarrow & K & \longrightarrow & \mathcal{I}_{\mathcal{X}/\mathcal{Y}} \\ \downarrow & & \downarrow & & \downarrow \\ Isom_{\mathcal{Y}}(f(x),f(x)) & \xleftarrow{e} & T & \xrightarrow{x} & \mathcal{X} \end{array}$$

with both squares 2-fibre products. This follows from Categories, Lemma 4.32.3. The left vertical arrow is a morphism of algebraic spaces locally of finite type over T, and hence is locally of finite type, see Morphisms of Spaces, Lemma 49.22.6. Thus K is an algebraic space and $K \to T$ is locally of finite type. This proves the assertion on the relative inertia. □

Remark 74.5.2. Let $\mathcal{X}$ be an algebraic stack. In Properties of Stacks, Remark 73.3.7 we have seen that the 2-category of morphisms $\mathcal{X}' \to \mathcal{X}$ representable by algebraic spaces with target $\mathcal{X}$ forms a category. In this category the inertia stack of $\mathcal{X}$ is a *group object*. Recall that an object of $\mathcal{I}_{\mathcal{X}}$ is just a pair (x, α) where x is an object of $\mathcal{X}$ and α is an automorphism of x in the fibre category of $\mathcal{X}$ that x lives in. The composition

$$c : \mathcal{I}_{\mathcal{X}} \times_{\mathcal{X}} \mathcal{I}_{\mathcal{X}} \longrightarrow \mathcal{I}_{\mathcal{X}}$$

is given by the rule on objects

$$((x,\alpha),(x',\alpha'),\beta) \mapsto (x, \alpha \circ \beta^{-1} \circ \alpha' \circ \beta)$$

which makes sense as $\beta : x \to x'$ is an isomorphism in the fibre category by our definition of fibre products. The neutral element $e : \mathcal{X} \to \mathcal{I}_{\mathcal{X}}$ is given by the functor $x \mapsto (x, \mathrm{id}_x)$. We omit the proof that the axioms of a group object hold. There is a variant of this remark for relative inertia stacks.

Let $\mathcal{X}$ be an algebraic stack and let $\mathcal{I}_{\mathcal{X}}$ be its inertia stack. We have seen in the proof of Lemma 74.5.1 that for any scheme T and object x of $\mathcal{X}$ over T there is a canonical cartesian square

$$\begin{array}{ccc} Isom_{\mathcal{X}}(x,x) & \longrightarrow & \mathcal{I}_{\mathcal{X}} \\ \downarrow & & \downarrow \\ T & \xrightarrow{x} & \mathcal{X} \end{array}$$

The group structure on $\mathcal{I}_{\mathcal{X}}$ discussed in Remark 74.5.2 induces the group structure on $Isom_{\mathcal{X}}(x,x)$ of Lemma 74.3.2. This allows us to define the sheaf $Isom_{\mathcal{X}}$ also for morphisms from algebraic spaces to $\mathcal{X}$. We formalize this in the following definition.

Definition 74.5.3. Let $\mathcal{X}$ be an algebraic stack and let X be an algebraic space. Let $x : X \to \mathcal{X}$ be a morphism. We set

$$Isom_{\mathcal{X}}(x, x) = X \times_{x,\mathcal{X}} \mathcal{I}_{\mathcal{X}}$$

We endow it with the structure of a group algebraic space over X by pulling back the composition law discussed in Remark 74.5.2. We will sometimes refer to $Isom_{\mathcal{X}}(x, x)$ as the *sheaf of automorphisms of* x.

As a variant we may occasionally use the notation $Isom_{\mathcal{X}}(x, y)$ when given two morphisms $x, y : X \to \mathcal{X}$. This will mean simply the algebraic space

$$(X \times_{x,\mathcal{X},y} X) \times_{X \times X, \Delta_X} X.$$

Then it is true, as in Lemma 74.3.2, that $Isom_{\mathcal{X}}(x, y)$ is a pseudo torsor for $Isom_{\mathcal{X}}(x, x)$ over X. We omit the verification.

Lemma 74.5.4. *Let $\pi : \mathcal{X} \to X$ be a morphism from an algebraic stack to an algebraic space. Let $f : X' \to X$ be a morphism of algebraic spaces. Set $\mathcal{X}' = X' \times_X \mathcal{X}$. Then both squares in the diagram*

$$\begin{array}{ccccc} \mathcal{I}_{\mathcal{X}'} & \longrightarrow & \mathcal{X}' & \longrightarrow & X' \\ \downarrow{\scriptstyle \textit{Categories, Equation (4.32.2.1)}} & & \downarrow & & \downarrow \\ \mathcal{I}_{\mathcal{X}} & \longrightarrow & \mathcal{X} & \longrightarrow & X \end{array}$$

are fibre product squares.

Proof. The inertia stack $\mathcal{I}_{\mathcal{X}'}$ is the defined as the category of pairs (x', α') where x' is an object of $\mathcal{X}'$ and α' is an automorphism of x' in its fibre category over $(Sch/S)_{fppf}$, see Categories, Section 4.32. Suppose that x' lies over the scheme U and maps to the object x of $\mathcal{X}$. By the construction of the 2-fibre product in Categories, Lemma 4.30.3 we see that $x' = (U, a', x, 1)$ where $a' : U \to X'$ is a morphism and 1 indicates that $f \circ a' = \pi \circ x$ as morphisms $U \to X$. Moreover we have $Isom_{\mathcal{X}'}(x', x') = Isom_{\mathcal{X}}(x, x)$ as sheaves on U (by the very construction of the 2-fibre product). This implies that the left square is a fibre product square (details omitted). □

Lemma 74.5.5. *Let $f : \mathcal{X} \to \mathcal{Y}$ be a monomorphism of algebraic stacks. Then the diagram*

$$\begin{array}{ccc} \mathcal{I}_{\mathcal{X}} & \longrightarrow & \mathcal{X} \\ \downarrow & & \downarrow \\ \mathcal{I}_{\mathcal{Y}} & \longrightarrow & \mathcal{Y} \end{array}$$

is a fibre product square.

Proof. This follows immediately from the fact that f is fully faithful (see Properties of Stacks, Lemma 73.8.4) and the definition of the inertia in Categories, Section 4.32. Namely, an object of $\mathcal{I}_{\mathcal{X}}$ over a scheme T is the same thing as a pair (x, α) consisting of an object x of $\mathcal{X}$ over T and a morphism $\alpha : x \to x$ in the fibre category of $\mathcal{X}$ over T. As f is fully faithful we see that α is the same thing as a morphism $\beta : f(x) \to f(x)$ in the fibre category of $\mathcal{Y}$ over T. Hence we can think of objects of $\mathcal{I}_{\mathcal{X}}$ over T as triples $((y, \beta), x, \gamma)$ where y is an object of $\mathcal{Y}$ over T,

$\beta : y \to y$ in $\mathcal{Y}_T$ and $\gamma : y \to f(x)$ is an isomorhism over T, i.e., an object of $\mathcal{I}_\mathcal{Y} \times_\mathcal{Y} \mathcal{X}$ over T. □

Lemma 74.5.6. *Let $\mathcal{X}$ be an algebraic stack. Let $[U/R] \to \mathcal{X}$ be a presentation. Let G/U be the stabilizer group algebraic space associated to the groupoid (U, R, s, t, c). Then*

$$\begin{array}{ccc} G & \longrightarrow & U \\ \downarrow & & \downarrow \\ \mathcal{I}_\mathcal{X} & \longrightarrow & \mathcal{X} \end{array}$$

is a fibre product diagram.

Proof. Immediate from Groupoids in Spaces, Lemma 59.25.2. □

74.6. Higher diagonals

Let $f : \mathcal{X} \to \mathcal{Y}$ be a morphism of algebraic stacks. In this situation it makes sense to consider not only the diagonal

$$\Delta_f : \mathcal{X} \to \mathcal{X} \times_\mathcal{Y} \mathcal{X}$$

but also the diagonal of the diagonal, i.e., the morphism

$$\Delta_{\Delta_f} : \mathcal{X} \longrightarrow \mathcal{X} \times_{(\mathcal{X} \times_\mathcal{Y} \mathcal{X})} \mathcal{X}$$

Because of this we sometimes use the following terminology. We denote $\Delta_{f,0} = f$ the *zeroth diagonal*, we denote $\Delta_{f,1} = \Delta_f$ the *first diagonal*, and we denote $\Delta_{f,2} = \Delta_{\Delta_f}$ the *second diagonal*. Note that $\Delta_{f,1}$ is representable by algebraic spaces and locally of finite type, see Lemma 74.3.3. Hence $\Delta_{f,2}$ is representable, a monomorphism, locally of finite type, separated, and locally quasi-finite, see Lemma 74.3.4.

We can describe the second diagonal using the relative inertia stack. Namely, the fibre product $\mathcal{X} \times_{(\mathcal{X} \times_\mathcal{Y} \mathcal{X})} \mathcal{X}$ is equivalent to the relative inertia stack $\mathcal{I}_{\mathcal{X}/\mathcal{Y}}$ by Categories, Lemma 4.32.1. Moreover, via this identification the second diagonal becomes the *neutral section*

$$e : \mathcal{X} \to \mathcal{I}_{\mathcal{X}/\mathcal{Y}}$$

of the relative inertia stack. Moreover, recall from the proof of Lemma 74.5.1 that given a morphism $x : T \to \mathcal{X}$ the fibre product $T \times_{x,\mathcal{X}} \mathcal{I}_{\mathcal{X}/\mathcal{Y}}$ is given as the *kernel* K of the homomorphism of group algebraic spaces

$$Isom_\mathcal{X}(x, x) \longrightarrow Isom_\mathcal{Y}(f(x), f(x))$$

over T. The morphism e corresponds to the neutral section $e : T \to K$ in this situation.

Lemma 74.6.1. *Let $f : \mathcal{X} \to \mathcal{Y}$ be a morphism of algebraic stacks. Then f is representable by algebraic spaces if and only if the second diagonal is an isomorphism.*

Proof. Namely, f is representable by algebraic spaces if and only if f is faithful, see Algebraic Stacks, Lemma 67.15.2. On the other hand, f is faithful if and only if for every object x of $\mathcal{X}$ over a scheme T the functor f induces an injection $Isom_\mathcal{X}(x, x) \to Isom_\mathcal{Y}(f(x), f(x))$, which happens if and only if the kernel K is trivial, which happens if and only if $e : T \to K$ is an isomorphism for every $x : T \to \mathcal{X}$. Since $K = T \times_{x,\mathcal{X}} \mathcal{I}_{\mathcal{X}/\mathcal{Y}}$ as discussed above, this proves the lemma. □

Lemma 74.6.2. *Let $f : \mathcal{X} \to \mathcal{Y}$ be a morphism of algebraic stacks. Then*

(1) *$\Delta_{f,1}$ separated $\Leftrightarrow$ $\Delta_{f,2}$ closed immersion $\Leftrightarrow$ $\Delta_{f,2}$ proper $\Leftrightarrow$ $\Delta_{f,2}$ universally closed,*
(2) *$\Delta_{f,1}$ quasi-separated $\Leftrightarrow$ $\Delta_{f,2}$ finite type $\Leftrightarrow$ $\Delta_{f,2}$ quasi-compact, and*
(3) *$\Delta_{f,1}$ locally separated $\Leftrightarrow$ $\Delta_{f,2}$ immersion.*

Proof. Follows from Lemmas 74.3.5, 74.3.6, and 74.3.7 applied to $\Delta_{f,1}$. □

The following lemma is kind of cute and it may suggest a generalization of these conditions to higher algebraic stacks.

Lemma 74.6.3. *Let $f : \mathcal{X} \to \mathcal{Y}$ be a morphism of algebraic stacks. Then*

(1) *f is separated if and only if $\Delta_{f,1}$ and $\Delta_{f,2}$ are universally closed, and*
(2) *f is quasi-separated if and only if $\Delta_{f,1}$ and $\Delta_{f,2}$ are quasi-compact.*
(3) *f is quasi-DM if and only if $\Delta_{f,1}$ and $\Delta_{f,2}$ are locally quasi-finite.*
(4) *f is DM if and only if $\Delta_{f,1}$ and $\Delta_{f,2}$ are unramified.*

Proof. Proof of (1). Assume that $\Delta_{f,2}$ and $\Delta_{f,1}$ are universally closed. Then $\Delta_{f,1}$ is separated and universally closed by Lemma 74.6.2. By Morphisms of Spaces, Lemma 49.10.7 and Algebraic Stacks, Lemma 67.10.9 we see that $\Delta_{f,1}$ is quasi-compact. Hence it is quasi-compact, separated, universally closed and locally of finite type (by Lemma 74.3.3) so proper. This proves "$\Leftarrow$" of (1). The proof of the implication in the other direction is omitted.

Proof of (2). This follows immediately from Lemma 74.6.2.

Proof of (3). This follows from the fact that $\Delta_{f,2}$ is always locally quasi-finite by Lemma 74.3.4 applied to $\Delta_f = \Delta_{f,1}$.

Proof of (4). This follows from the fact that $\Delta_{f,2}$ is always unramified as Lemma 74.3.4 applied to $\Delta_f = \Delta_{f,1}$ shows that $\Delta_{f,2}$ is locally of finite type and a monomorphism. See More on Morphisms of Spaces, Lemma 57.11.8. □

74.7. Quasi-compact morphisms

Let f be a morphism of algebraic stacks which is representable by algebraic spaces. In Properties of Stacks, Section 73.3 we have defined what it means for f to be quasi-compact. Here is another characterization.

Lemma 74.7.1. *Let $f : \mathcal{X} \to \mathcal{Y}$ be a morphism of algebraic stacks which is representable by algebraic spaces. The following are equivalent:*

(1) *f is quasi-compact, and*
(2) *for every quasi-compact algebraic stack $\mathcal{Z}$ and any morphism $\mathcal{Z} \to \mathcal{Y}$ the algebraic stack $\mathcal{Z} \times_{\mathcal{Y}} \mathcal{X}$ is quasi-compact.*

Proof. Assume (1), and let $\mathcal{Z} \to \mathcal{Y}$ be a morphism of algebraic stacks with $\mathcal{Z}$ quasi-compact. By Properties of Stacks, Lemma 73.6.2 there exists a quasi-compact scheme U and a surjective smooth morphism $U \to \mathcal{Z}$. Since f is representable by algebraic spaces and quasi-compact we see by definition that $U \times_{\mathcal{Y}} \mathcal{X}$ is an algebraic space, and that $U \times_{\mathcal{Y}} \mathcal{X} \to U$ is quasi-compact. Hence $U \times_Y X$ is a quasi-compact algebraic space. The morphism $U \times_{\mathcal{Y}} \mathcal{X} \to \mathcal{Z} \times_{\mathcal{Y}} \mathcal{X}$ is smooth and surjective (as the base change of the smooth and surjective morphism $U \to \mathcal{Z}$). Hence $\mathcal{Z} \times_{\mathcal{Y}} \mathcal{X}$ is quasi-compact by another application of Properties of Stacks, Lemma 73.6.2

Assume (2). Let $Z \to \mathcal{Y}$ be a morphism, where Z is a scheme. We have to show that the morphism of algebraic spaces $p : Z \times_{\mathcal{Y}} \mathcal{X} \to Z$ is quasi-compact. Let $U \subset Z$ be affine open. Then $p^{-1}(U) = U \times_{\mathcal{Y}} \mathcal{Z}$ and the algebraic space $U \times_{\mathcal{Y}} \mathcal{Z}$ is quasi-compact by assumption (2). Hence p is quasi-compact, see Morphisms of Spaces, Lemma 49.9.7. □

This motivates the following definition.

Definition 74.7.2. Let $f : \mathcal{X} \to \mathcal{Y}$ be a morphism of algebraic stacks. We say f is *quasi-compact* if for every quasi-compact algebraic stack $\mathcal{Z}$ and morphism $\mathcal{Z} \to \mathcal{Y}$ the fibre product $\mathcal{Z} \times_{\mathcal{Y}} \mathcal{X}$ is quasi-compact.

By Lemma 74.7.1 above this agrees with the already existing notion for morphisms of algebraic stacks representable by algebraic spaces. In particular this notion agrees with the notions already defined for morphisms between algebraic stacks and schemes.

Lemma 74.7.3. *The base change of a quasi-compact morphism of algebraic stacks by any morphism of algebraic stacks is quasi-compact.*

Proof. Omitted. □

Lemma 74.7.4. *The composition of a pair of quasi-compact morphisms of algebraic stacks is quasi-compact.*

Proof. Omitted. □

Lemma 74.7.5. *Let*

$$\begin{array}{ccc} \mathcal{X} & \xrightarrow{f} & \mathcal{Y} \\ & {\scriptstyle p} \searrow \quad \swarrow {\scriptstyle q} & \\ & \mathcal{Z} & \end{array}$$

be a 2-commutative diagram of morphisms of algebraic stacks. If f is surjective and p is quasi-compact, then q is quasi-compact.

Proof. Let $\mathcal{T}$ be a quasi-compact algebraic stack, and let $\mathcal{T} \to \mathcal{Z}$ be a morphism. By Properties of Stacks, Lemma 73.5.3 the morphism $\mathcal{T} \times_{\mathcal{Z}} \mathcal{X} \to \mathcal{T} \times_{\mathcal{Z}} \mathcal{Y}$ is surjective and by assumption $\mathcal{T} \times_{\mathcal{Z}} \mathcal{X}$ is quasi-compact. Hence $\mathcal{T} \times_{\mathcal{Z}} \mathcal{Y}$ is quasi-compact by Properties of Stacks, Lemma 73.6.2. □

Lemma 74.7.6. *Let $f : \mathcal{X} \to \mathcal{Y}$ and $g : \mathcal{Y} \to \mathcal{Z}$ be morphisms of algebraic stacks. If $g \circ f$ is quasi-compact and g is quasi-separated then f is quasi-compact.*

Proof. This is true because f equals the composition $(1, f) : \mathcal{X} \to \mathcal{X} \times_{\mathcal{Z}} \mathcal{Y} \to \mathcal{Y}$. The first map is quasi-compact by Lemma 74.4.9 because it is a section of the quasi-separated morphism $\mathcal{X} \times_{\mathcal{Z}} \mathcal{Y} \to \mathcal{X}$ (a base change of g, see Lemma 74.4.4). The second map is quasi-compact as it is the base change of f, see Lemma 74.7.3. And compositions of quasi-compact morphisms are quasi-compact, see Lemma 74.7.4. □

Lemma 74.7.7. *Let $f : \mathcal{X} \to \mathcal{Y}$ be a morphism of algebraic stacks.*

(1) *If $\mathcal{X}$ is quasi-compact and $\mathcal{Y}$ is quasi-separated, then f is quasi-compact.*
(2) *If $\mathcal{X}$ is quasi-compact and quasi-separated and $\mathcal{Y}$ is quasi-separated, then f is quasi-compact and quasi-separated.*

(3) *A fibre product of quasi-compact and quasi-separated algebraic stacks is quasi-compact and quasi-separated.*

Proof. Part (1) follows from Lemma 74.7.6. Part (2) follows from (1) and Lemma 74.4.12. For (3) let $\mathcal{X} \to \mathcal{Y}$ and $\mathcal{Z} \to \mathcal{Y}$ be morphisms of quasi-compact and quasi-separated algebraic stacks. Then $\mathcal{X} \times_{\mathcal{Y}} \mathcal{Z} \to \mathcal{Z}$ is quasi-compact and quasi-separated as a base change of $\mathcal{X} \to \mathcal{Y}$ using (2) and Lemmas 74.7.3 and 74.4.4. Hence $\mathcal{X} \times_{\mathcal{Y}} \mathcal{Z}$ is quasi-compact and quasi-separated as an algebraic stack quasi-compact and quasi-separated over $\mathcal{Z}$, see Lemmas 74.4.11 and 74.7.4. □

74.8. Noetherian algebraic stacks

We have already defined locally Noetherian algebraic stacks in Properties of Stacks, Section 73.7.

Definition 74.8.1. Let $\mathcal{X}$ be an algebraic stack. We say $\mathcal{X}$ is *Noetherian* if $\mathcal{X}$ is quasi-compact, quasi-separated and locally Noetherian.

Note that a Noetherian algebraic stack $\mathcal{X}$ is not just quasi-compact and locally Noetherian, but also quasi-separated. In the language of Section 74.6 if we denote $p : \mathcal{X} \to \mathrm{Spec}(\mathbf{Z})$ the "absolute" structure morphism (i.e., the structure morphism of $\mathcal{X}$ viewed as an algebraic stack over $\mathbf{Z}$), then

$$\mathcal{X} \text{ Noetherian} \Leftrightarrow \mathcal{X} \text{ locally Noetherian and } \Delta_{p,0}, \Delta_{p,1}, \Delta_{p,2} \text{ quasi-compact.}$$

This will later mean that an algebraic stack of finite type over a Noetherian algebraic stack is not automatically Noetherian.

74.9. Open morphisms

Let f be a morphism of algebraic stacks which is representable by algebraic spaces. In Properties of Stacks, Section 73.3 we have defined what it means for f to be universally open. Here is another characterization.

Lemma 74.9.1. *Let $f : \mathcal{X} \to \mathcal{Y}$ be a morphism of algebraic stacks which is representable by algebraic spaces. The following are equivalent*

(1) *f is universally open, and*
(2) *for every morphism of algebraic stacks $\mathcal{Z} \to \mathcal{Y}$ the morphism of topological spaces $|\mathcal{Z} \times_{\mathcal{Y}} \mathcal{X}| \to |\mathcal{Z}|$ is open.*

Proof. Assume (1), and let $\mathcal{Z} \to \mathcal{Y}$ be as in (2). Choose a scheme V and a surjective smooth morphism $V \to \mathcal{Z}$. By assumption the morphism $V \times_{\mathcal{Y}} \mathcal{X} \to V$ of algebraic spaces is universally open, in particular the map $|V \times_{\mathcal{Y}} \mathcal{X}| \to |V|$ is open. By Properties of Stacks, Section 73.4 in the commutative diagram

$$\begin{array}{ccc} |V \times_{\mathcal{Y}} \mathcal{X}| & \longrightarrow & |\mathcal{Z} \times_{\mathcal{Y}} \mathcal{X}| \\ \downarrow & & \downarrow \\ |V| & \longrightarrow & |\mathcal{Z}| \end{array}$$

the horizontal arrows are open and surjective, and moreover

$$|V \times_{\mathcal{Y}} \mathcal{X}| \longrightarrow |V| \times_{|\mathcal{Z}|} |\mathcal{Z} \times_{\mathcal{Y}} \mathcal{X}|$$

is surjective. Hence as the left vertical arrow is open it follows that the right vertical arrow is open. This proves (2). The implication (2) $\Rightarrow$ (1) follows from the definitions. □

Thus we may use the following natural definition.

Definition 74.9.2. Let $f : \mathcal{X} \to \mathcal{Y}$ be a morphism of algebraic stacks.

(1) We say f is *open* if the map of topological spaces $|\mathcal{X}| \to |\mathcal{Y}|$ is open.
(2) We say f is *universally open* if for every morphism of algebraic stacks $\mathcal{Z} \to \mathcal{Y}$ the morphism of topological spaces

$$|\mathcal{Z} \times_{\mathcal{Y}} \mathcal{X}| \to |\mathcal{Z}|$$

is open, i.e., the base change $\mathcal{Z} \times_{\mathcal{Y}} \mathcal{X} \to \mathcal{Z}$ is open.

Lemma 74.9.3. *The base change of a universally open morphism of algebraic stacks by any morphism of algebraic stacks is universally open.*

Proof. This is immediate from the definition. □

Lemma 74.9.4. *The composition of a pair of (universally) open morphisms of algebraic stacks is (universally) open.*

Proof. Omitted. □

74.10. Submersive morphisms

Definition 74.10.1. Let $f : \mathcal{X} \to \mathcal{Y}$ be a morphism of algebraic stacks.

(1) We say f is *submersive*[4] if the continuous map $|\mathcal{X}| \to |\mathcal{Y}|$ is submersive, see Topology, Definition 5.5.3.
(2) We say f is *universally submersive* if for every morphism of algebraic stacks $\mathcal{Y}' \to \mathcal{Y}$ the base change $\mathcal{Y}' \times_{\mathcal{Y}} \mathcal{X} \to \mathcal{Y}'$ is submersive.

We note that a submersive morphism is in particular surjective.

74.11. Universally closed morphisms

Let f be a morphism of algebraic stacks which is representable by algebraic spaces. In Properties of Stacks, Section 73.3 we have defined what it means for f to be universally closed. Here is another characterization.

Lemma 74.11.1. *Let $f : \mathcal{X} \to \mathcal{Y}$ be a morphism of algebraic stacks which is representable by algebraic spaces. The following are equivalent*

(1) *f is universally closed, and*
(2) *for every morphism of algebraic stacks $\mathcal{Z} \to \mathcal{Y}$ the morphism of topological spaces $|\mathcal{Z} \times_{\mathcal{Y}} \mathcal{X}| \to |\mathcal{Z}|$ is closed.*

Proof. Assume (1), and let $\mathcal{Z} \to \mathcal{Y}$ be as in (2). Choose a scheme V and a surjective smooth morphism $V \to \mathcal{Z}$. By assumption the morphism $V \times_{\mathcal{Y}} \mathcal{X} \to V$

[4] This is very different from the notion of a submersion of differential manifolds.

of algebraic spaces is universally closed, in particular the map $|V \times_{\mathcal{Y}} \mathcal{X}| \to |V|$ is closed. By Properties of Stacks, Section 73.4 in the commutative diagram

$$\begin{array}{ccc} |V \times_{\mathcal{Y}} \mathcal{X}| & \longrightarrow & |\mathcal{Z} \times_{\mathcal{Y}} \mathcal{X}| \\ \downarrow & & \downarrow \\ |V| & \longrightarrow & |\mathcal{Z}| \end{array}$$

the horizontal arrows are open and surjective, and moreover

$$|V \times_{\mathcal{Y}} \mathcal{X}| \longrightarrow |V| \times_{|\mathcal{Z}|} |\mathcal{Z} \times_{\mathcal{Y}} \mathcal{X}|$$

is surjective. Hence as the left vertical arrow is closed it follows that the right vertical arrow is closed. This proves (2). The implication (2) $\Rightarrow$ (1) follows from the definitions. □

Thus we may use the following natural definition.

Definition 74.11.2. Let $f : \mathcal{X} \to \mathcal{Y}$ be a morphism of algebraic stacks.

(1) We say f is *closed* if the map of topological spaces $|\mathcal{X}| \to |\mathcal{Y}|$ is closed.
(2) We say f is *universally closed* if for every morphism of algebraic stacks $\mathcal{Z} \to \mathcal{Y}$ the morphism of topological spaces
$$|\mathcal{Z} \times_{\mathcal{Y}} \mathcal{X}| \to |\mathcal{Z}|$$
is closed, i.e., the base change $\mathcal{Z} \times_{\mathcal{Y}} \mathcal{X} \to \mathcal{Z}$ is closed.

Lemma 74.11.3. *The base change of a universally closed morphism of algebraic stacks by any morphism of algebraic stacks is universally closed.*

Proof. This is immediate from the definition. □

Lemma 74.11.4. *The composition of a pair of (universally) closed morphisms of algebraic stacks is (universally) closed.*

Proof. Omitted. □

74.12. Types of morphisms smooth local on source-and-target

Given a property of morphisms of algebraic spaces which is *smooth local on the source-and-target*, see Descent on Spaces, Definition 55.18.1 we may use it to define a corresponding property of morphisms of algebraic stacks, namely by imposing either of the equivalent conditions of the lemma below.

Lemma 74.12.1. *Let $\mathcal{P}$ be a property of morphisms of algebraic spaces which is smooth local on the source-and-target. Let $f : \mathcal{X} \to \mathcal{Y}$ be a morphism of algebraic stacks. Consider commutative diagrams*

$$\begin{array}{ccc} U & \xrightarrow{h} & V \\ {\scriptstyle a}\downarrow & & \downarrow{\scriptstyle b} \\ \mathcal{X} & \xrightarrow{f} & \mathcal{Y} \end{array}$$

where U and V are algebraic spaces and the vertical arrows are smooth. The following are equivalent

(1) *for any diagram as above such that in addition $U \to \mathcal{X} \times_{\mathcal{Y}} V$ is smooth the morphism h has property $\mathcal{P}$, and*

(2) *for some diagram as above with $a : U \to \mathcal{X}$ surjective the morphism h has property $\mathcal{P}$.*

If $\mathcal{X}$ and $\mathcal{Y}$ are representable by algebraic spaces, then this is also equivalent to f (as a morphism of algebraic spaces) having property $\mathcal{P}$. If $\mathcal{P}$ is also preserved under any base change, and fppf local on the base, then for morphisms f which are representable by algebraic spaces this is also equivalent to f having property $\mathcal{P}$ in the sense of Properties of Stacks, Section 73.3.

Proof. Let us prove the implication (1) $\Rightarrow$ (2). Pick an algebraic space V and a surjective and smooth morphism $V \to \mathcal{Y}$. Pick an algebraic space U and a surjective and smooth morphism $U \to \mathcal{X} \times_{\mathcal{Y}} V$. Note that $U \to \mathcal{X}$ is surjective and smooth as well, as a composition of the base change $\mathcal{X} \times_{\mathcal{Y}} V \to \mathcal{X}$ and the chosen map $U \to \mathcal{X} \times_{\mathcal{Y}} V$. Hence we obtain a diagram as in (1). Thus if (1) holds, then $h : U \to V$ has property $\mathcal{P}$, which means that (2) holds as $U \to \mathcal{X}$ is surjective.

Conversely, assume (2) holds and let U, V, a, b, h be as in (2). Next, let U', V', a', b', h' be any diagram as in (1). Picture

$$\begin{array}{ccc} U & \xrightarrow{h} & V \\ \downarrow & & \downarrow \\ \mathcal{X} & \xrightarrow{f} & \mathcal{Y} \end{array} \qquad \begin{array}{ccc} U' & \xrightarrow{h'} & V' \\ \downarrow & & \downarrow \\ \mathcal{X} & \xrightarrow{f} & \mathcal{Y} \end{array}$$

To show that (2) implies (1) we have to prove that h' has $\mathcal{P}$. To do this consider the commutative diagram

$$\begin{array}{ccccc} U & \longleftarrow & U \times_{\mathcal{X}} U' & \longrightarrow & U' \\ \downarrow{\scriptstyle h} & \swarrow & \downarrow & \searrow^{(h,h')} & \downarrow{\scriptstyle h'} \\ V & \longleftarrow & V \times_{\mathcal{Y}} V' & \longrightarrow & V' \end{array}$$

of algebraic spaces. Note that the horizontal arrows are smooth as base changes of the smooth morphisms $V \to \mathcal{Y}$, $V' \to \mathcal{Y}$, $U \to \mathcal{X}$, and $U' \to \mathcal{X}$. Note that

$$\begin{array}{ccc} U \times_{\mathcal{X}} U' & \longrightarrow & U' \\ \downarrow & & \downarrow \\ U \times_{\mathcal{Y}} V' & \longrightarrow & \mathcal{X} \times_{\mathcal{Y}} V' \end{array}$$

is cartesian, hence the left vertical arrow is smooth as U', V', a', b', h' is as in (1). Since $\mathcal{P}$ is local on the target we see that the base change $U \times_{\mathcal{Y}} V' \to V \times_{\mathcal{Y}} V'$ has $\mathcal{P}$ and hence after precomposing by the smooth morphism $U \times_{\mathcal{X}} U' \to U \times_{\mathcal{Y}} V'$ the morphism we conclude (h, h') has $\mathcal{P}$. Finally, since $U \times_X U' \to U'$ is surjective this implies that h' has $\mathcal{P}$ as $\mathcal{P}$ is local on the source-and-target. This finishes the proof of the equivalence of (1) and (2).

If $\mathcal{X}$ and $\mathcal{Y}$ are representable, then Descent on Spaces, Lemma 55.18.3 applies which shows that (1) and (2) are equivalent to f having $\mathcal{P}$.

Finally, suppose f is representable, and U, V, a, b, h are as in part (2) of the lemma, and that $\mathcal{P}$ is preserved under arbitrary base change. We have to show that for any scheme Z and morphism $Z \to \mathcal{X}$ the base change $Z \times_{\mathcal{Y}} \mathcal{X} \to Z$ has property $\mathcal{P}$. Consider the diagram

$$\begin{array}{ccc} Z \times_{\mathcal{Y}} U & \longrightarrow & Z \times_{\mathcal{Y}} V \\ \downarrow & & \downarrow \\ Z \times_{\mathcal{Y}} \mathcal{X} & \longrightarrow & Z \end{array}$$

Note that the top horizontal arrow is a base change of h and hence has property $\mathcal{P}$. The left vertical arrow is smooth and surjective and the right vertical arrow is smooth. Thus Descent on Spaces, Lemma 55.18.3 kicks in and shows that $Z \times_{\mathcal{Y}} \mathcal{X} \to Z$ has property $\mathcal{P}$. □

Definition 74.12.2. Let $\mathcal{P}$ be a property of morphisms of algebraic spaces which is smooth local on the source-and-target. We say a morphism $f : \mathcal{X} \to \mathcal{Y}$ of algebraic stacks *has property* $\mathcal{P}$ if the equivalent conditions of Lemma 74.12.1 hold.

Remark 74.12.3. Let $\mathcal{P}$ be a property of morphisms of algebraic spaces which is smooth local on the source-and-target and stable under composition. Then the property of morphisms of algebraic stacks defined in Definition 74.12.2 is stable under composition. Namely, let $f : \mathcal{X} \to \mathcal{Y}$ and $g : \mathcal{Y} \to \mathcal{Z}$ be morphisms of algebraic stacks having property $\mathcal{P}$. Choose an algebraic space W and a surjective smooth morphism $W \to \mathcal{Z}$. Choose an algebraic space V and a surjective smooth morphism $V \to \mathcal{Y} \times_{\mathcal{Z}} W$. Finally, choose an algebraic space U and a surjective and smooth morphism $U \to \mathcal{X} \times_{\mathcal{Y}} V$. Then the morphisms $V \to W$ and $U \to V$ have property $\mathcal{P}$ by definition. Whence $U \to W$ has property $\mathcal{P}$ as we assumed that $\mathcal{P}$ is stable under composition. Thus, by definition again, we see that $g \circ f : \mathcal{X} \to \mathcal{Z}$ has property $\mathcal{P}$.

Remark 74.12.4. Let $\mathcal{P}$ be a property of morphisms of algebraic spaces which is smooth local on the source-and-target and stable under base change. Then the property of morphisms of algebraic stacks defined in Definition 74.12.2 is stable under base change. Namely, let $f : \mathcal{X} \to \mathcal{Y}$ and $g : \mathcal{Y}' \to \mathcal{Y}$ be morphisms of algebraic stacks and assume f has property $\mathcal{P}$. Choose an algebraic space V and a surjective smooth morphism $V \to \mathcal{Y}$. Choose an algebraic space U and a surjective smooth morphism $U \to \mathcal{X} \times_{\mathcal{Y}} V$. Finally, choose an algebraic space V' and a surjective and smooth morphism $V' \to \mathcal{Y}' \times_{\mathcal{Y}} V$. Then the morphism $U \to V$ has property $\mathcal{P}$ by definition. Whence $V' \times_V U \to V'$ has property $\mathcal{P}$ as we assumed that $\mathcal{P}$ is stable under base change. Considering the diagram

$$\begin{array}{ccccc} V' \times_V U & \longrightarrow & \mathcal{Y}' \times_{\mathcal{Y}} \mathcal{X} & \longrightarrow & \mathcal{X} \\ \downarrow & & \downarrow & & \downarrow \\ V' & \longrightarrow & \mathcal{Y}' & \longrightarrow & \mathcal{Y} \end{array}$$

we see that the left top horizontal arrow is smooth and surjective, whence by definition we see that the projection $\mathcal{Y}' \times_{\mathcal{Y}} \mathcal{X} \to \mathcal{Y}'$ has property $\mathcal{P}$.

Remark 74.12.5. Let $\mathcal{P}, \mathcal{P}'$ be properties of morphisms of algebraic spaces which are smooth local on the source-and-target and stable under base change. Suppose

that we have $\mathcal{P} \Rightarrow \mathcal{P}'$ for morphisms of algebraic spaces. Then we also have $\mathcal{P} \Rightarrow \mathcal{P}'$ for the properties of morphisms of algebraic stacks defined in Definition 74.12.2 using $\mathcal{P}$ and $\mathcal{P}'$. This is clear from the definition.

74.13. Morphisms of finite type

The property "locally of finite type" of morphisms of algebraic spaces is smooth local on the source-and-target, see Descent on Spaces, Remark 55.18.5. It is also stable under base change and fpqc local on the target, see Morphisms of Spaces, Lemma 49.22.3 and Descent on Spaces, Lemma 55.10.7. Hence, by Lemma 74.12.1 above, we may define what it means for a morphism of algebraic spaces to be locally of finite type as follows and it agrees with the already existing notion defined in Properties of Stacks, Section 73.3 when the morphism is representable by algebraic spaces.

Definition 74.13.1. Let $f : \mathcal{X} \to \mathcal{Y}$ be a morphism of algebraic stacks.

(1) We say f *locally of finite type* if the equivalent conditions of Lemma 74.12.1 hold with $\mathcal{P} =$ locally of finite type.
(2) We say f is *of finite type* if it is locally of finite type and quasi-compact.

Lemma 74.13.2. *The composition of finite type morphisms is of finite type. The same holds for locally of finite type.*

Proof. Combine Remark 74.12.3 with Morphisms of Spaces, Lemma 49.22.2. □

Lemma 74.13.3. *A base change of a finite type morphism is finite type. The same holds for locally of finite type.*

Proof. Combine Remark 74.12.4 with Morphisms of Spaces, Lemma 49.22.3. □

Lemma 74.13.4. *An immersion is locally of finite type.*

Proof. Follows from Morphisms of Spaces, Lemma 49.22.7. □

Lemma 74.13.5. *Let $f : \mathcal{X} \to \mathcal{Y}$ be a morphism of algebraic stacks. If f is locally of finite type and $\mathcal{Y}$ is locally Noetherian, then $\mathcal{X}$ is locally Noetherian.*

Proof. Let

$$\begin{array}{ccc} U & \longrightarrow & V \\ \downarrow & & \downarrow \\ \mathcal{X} & \longrightarrow & \mathcal{Y} \end{array}$$

be a commutative diagram where U, V are schemes, $V \to \mathcal{Y}$ is surjective and smooth, and $U \to V \times_{\mathcal{Y}} \mathcal{X}$ is surjective and smooth. Then $U \to V$ is locally of finite type. If $\mathcal{Y}$ is locally Noetherian, then V is locally Noetherian. By Morphisms, Lemma 28.16.6 we see that U is locally Noetherian, which means that $\mathcal{X}$ is locally Noetherian. □

The following two lemmas will be improved on later (after we have discussed morphisms of algebraic stacks which are locally of finite presentation).

Lemma 74.13.6. *Let $f : \mathcal{X} \to \mathcal{Y}$ be a morphism of algebraic stacks. Let $W \to \mathcal{Y}$ be a surjective, flat, and locally of finite presentation where W is an algebraic space. If the base change $W \times_{\mathcal{Y}} \mathcal{X} \to W$ is locally of finite type, then f is locally of finite type.*

Proof. Choose an algebraic space V and a surjective smooth morphism $V \to \mathcal{Y}$. Choose an algebraic space U and a surjective smooth morphism $U \to V \times_{\mathcal{Y}} \mathcal{X}$. We have to show that $U \to V$ is locally of finite presentation. Now we base change everything by $W \to \mathcal{Y}$: Set $U' = W \times_{\mathcal{Y}} U$, $V' = W \times_{\mathcal{Y}} V$, $\mathcal{X}' = W \times_{\mathcal{Y}} \mathcal{X}$, and $\mathcal{Y}' = W \times_{\mathcal{Y}} \mathcal{Y} = W$. Then it is still true that $U' \to V' \times_{\mathcal{Y}'} \mathcal{X}'$ is smooth by base change. Hence by our definition of locally finite type morphisms of algebraic stacks and the assumption that $\mathcal{X}' \to \mathcal{Y}'$ is locally of finite type, we see that $U' \to V'$ is locally of finite type. Then, since $V' \to V$ is surjective, flat, and locally of finite presentation as a base change of $W \to \mathcal{Y}$ we see that $U \to V$ is locally of finite type by Descent on Spaces, Lemma 55.10.7 and we win. □

Lemma 74.13.7. *Let $\mathcal{X} \to \mathcal{Y} \to \mathcal{Z}$ be morphisms of algebraic stacks. Assume $\mathcal{X} \to \mathcal{Z}$ is locally of finite type and that $\mathcal{X} \to \mathcal{Y}$ is representable by algebraic spaces, surjective, flat, and locally of finite presentation. Then $\mathcal{Y} \to \mathcal{Z}$ is locally of finite type.*

Proof. Choose an algebraic space W and a surjective smooth morphism $W \to \mathcal{Z}$. Choose an algebraic space V and a surjective smooth morphism $V \to W \times_{\mathcal{Z}} \mathcal{Y}$. Set $U = V \times_{\mathcal{Y}} \mathcal{X}$ which is an algebraic space. We know that $U \to V$ is surjective, flat, and locally of finite presentation and that $U \to W$ is locally of finite type. Hence the lemma reduces to the case of morphisms of algebraic spaces. The case of morphisms of algebraic spaces is Descent on Spaces, Lemma 55.14.2. □

Lemma 74.13.8. *Let $f : \mathcal{X} \to \mathcal{Y}$, $g : \mathcal{Y} \to \mathcal{Z}$ be morphisms of algebraic stacks. If $g \circ f : \mathcal{X} \to \mathcal{Z}$ is locally of finite type, then $f : \mathcal{X} \to \mathcal{Y}$ is locally of finite type.*

Proof. We can find a diagram

$$\begin{array}{ccccc} U & \longrightarrow & V & \longrightarrow & W \\ \downarrow & & \downarrow & & \downarrow \\ \mathcal{X} & \longrightarrow & \mathcal{Y} & \longrightarrow & \mathcal{Z} \end{array}$$

where U, V, W are schemes, the vertical arrow $W \to \mathcal{Z}$ is surjective and smooth, the arrow $V \to \mathcal{Y} \times_{\mathcal{Z}} W$ is surjective and smooth, and the arrow $U \to \mathcal{X} \times_{\mathcal{Y}} V$ is surjective and smooth. Then also $U \to \mathcal{X} \times_{\mathcal{Z}} V$ is surjective and smooth (as a composition of a surjective and smooth morphism with a base change of such). By definition we see that $U \to W$ is locally of finite type. Hence $U \to V$ is locally of finite type by Morphisms, Lemma 28.16.8 which in turn means (by definition) that $\mathcal{X} \to \mathcal{Y}$ is locally of finite type. □

74.14. Points of finite type

Let $\mathcal{X}$ be an algebraic stack. A finite type point $x \in |\mathcal{X}|$ is a point which can be represented by a morphism $\mathrm{Spec}(k) \to \mathcal{X}$ which is locally of finite type. Finite type points are a suitable replacement of closed points for algebraic spaces and algebraic stacks. There are always "enough of them" for example.

Lemma 74.14.1. *Let $\mathcal{X}$ be an algebraic stack. Let $x \in |\mathcal{X}|$. The following are equivalent:*

(1) *There exists a morphism $\mathrm{Spec}(k) \to \mathcal{X}$ which is locally of finite type and represents x.*

(2) *There exists a scheme U, a closed point $u \in U$, and a smooth morphism $\varphi : U \to \mathcal{X}$ such that $\varphi(u) = x$.*

Proof. Let $u \in U$ and $U \to \mathcal{X}$ be as in (2). Then $\mathrm{Spec}(\kappa(u)) \to U$ is of finite type, and $U \to \mathcal{X}$ is representable and locally of finite type (by Morphisms of Spaces, Lemmas 49.35.8 and 49.26.5). Hence we see (1) holds by Lemma 74.13.2.

Conversely, assume $\mathrm{Spec}(k) \to \mathcal{X}$ is locally of finite type and represents x. Let $U \to \mathcal{X}$ be a surjective smooth morphism where U is a scheme. By assumption $U \times_{\mathcal{X}} \mathrm{Spec}(k) \to U$ is a morphism of algebraic spaces which is locally of finite type. Pick a finite type point v of $U \times_{\mathcal{X}} \mathrm{Spec}(k)$ (there exists at least one, see Morphisms of Spaces, Lemma 49.24.3). By Morphisms of Spaces, Lemma 49.24.4 the image $u \in U$ of v is a finite type point of U. Hence by Morphisms, Lemma 28.17.4 after shrinking U we may assume that u is a closed point of U, i.e., (2) holds. □

Definition 74.14.2. Let $\mathcal{X}$ be an algebraic stack. We say a point $x \in |\mathcal{X}|$ is a *finite type point*[5] if the equivalent conditions of Lemma 74.14.1 are satisfied. We denote $\mathcal{X}_{\text{ft-pts}}$ the set of finite type points of $\mathcal{X}$.

We can describe the set of finite type points as follows.

Lemma 74.14.3. *Let $\mathcal{X}$ be an algebraic stack. We have*

$$\mathcal{X}_{ft\text{-}pts} = \bigcup\nolimits_{\varphi : U \to X \text{ smooth}} |\varphi|(U_0)$$

where U_0 is the set of closed points of U. Here we may let U range over all schemes smooth over $\mathcal{X}$ or over all affine schemes smooth over $\mathcal{X}$.

Proof. Immediate from Lemma 74.14.1. □

Lemma 74.14.4. *Let $f : \mathcal{X} \to \mathcal{Y}$ be a morphism of algebraic stacks. If f is locally of finite type, then $f(\mathcal{X}_{ft\text{-}pts}) \subset \mathcal{Y}_{ft\text{-}pts}$.*

Proof. Take $x \in \mathcal{X}_{\text{ft-pts}}$. Represent x by a locally finite type morphism $x : \mathrm{Spec}(k) \to \mathcal{X}$. Then $f \circ x$ is locally of finite type by Lemma 74.13.2. Hence $f(x) \in \mathcal{Y}_{\text{ft-pts}}$. □

Lemma 74.14.5. *Let $f : \mathcal{X} \to \mathcal{Y}$ be a morphism of algebraic stacks. If f is locally of finite type and surjective, then $f(\mathcal{X}_{ft\text{-}pts}) = \mathcal{Y}_{ft\text{-}pts}$.*

Proof. We have $f(\mathcal{X}_{\text{ft-pts}}) \subset \mathcal{Y}_{\text{ft-pts}}$ by Lemma 74.14.4. Let $y \in |\mathcal{Y}|$ be a finite type point. Represent y by a morphism $\mathrm{Spec}(k) \to \mathcal{Y}$ which is locally of finite type. As f is surjective the algebraic stack $\mathcal{X}_k = \mathrm{Spec}(k) \times_{\mathcal{Y}} \mathcal{X}$ is nonempty, therefore has a finite type point $x \in |\mathcal{X}_k|$ by Lemma 74.14.3. Now $\mathcal{X}_k \to \mathcal{X}$ is a morphism which is locally of finite type as a base change of $\mathrm{Spec}(k) \to \mathcal{Y}$ (Lemma 74.13.3). Hence the image of x in $\mathcal{X}$ is a finite type point by Lemma 74.14.4 which maps to y by construction. □

Lemma 74.14.6. *Let $\mathcal{X}$ be an algebraic stack. For any locally closed subset $T \subset |\mathcal{X}|$ we have*

$$T \neq \emptyset \Rightarrow T \cap \mathcal{X}_{ft\text{-}pts} \neq \emptyset.$$

In particular, for any closed subset $T \subset |\mathcal{X}|$ we see that $T \cap \mathcal{X}_{ft\text{-}pts}$ is dense in T.

[5] This is a slight abuse of language as it would perhaps be more correct to say "locally finite type point".

Proof. Let $i : \mathcal{Z} \to \mathcal{X}$ be the reduced induced substack structure on T, see Properties of Stacks, Remark 73.10.5. An immersion is locally of finite type, see Lemma 74.13.4. Hence by Lemma 74.14.4 we see $\mathcal{Z}_{\text{ft-pts}} \subset \mathcal{X}_{\text{ft-pts}} \cap T$. Finally, any nonempty affine scheme U with a smooth morphism towards $\mathcal{Z}$ has at least one closed point, hence $\mathcal{Z}$ has at least one finite type point by Lemma 74.14.3. The lemma follows. □

Here is another, more technical, characterization of a finite type point on an algebraic stack. It tells us in particular that the residual gerbe of $\mathcal{X}$ at x exists whenever x is a finite type point!

Lemma 74.14.7. *Let $\mathcal{X}$ be an algebraic stack. Let $x \in |\mathcal{X}|$. The following are equivalent:*

1. *x is a finite type point,*
2. *there exists an algebraic stack $\mathcal{Z}$ whose underlying topological space $|\mathcal{Z}|$ is a singleton, and a morphism $f : \mathcal{Z} \to \mathcal{X}$ which is locally of finite type such that $\{x\} = |f|(|\mathcal{Z}|)$, and*
3. *the residual gerbe $\mathcal{Z}_x$ of $\mathcal{X}$ at x exists and the inclusion morphism $\mathcal{Z}_x \to \mathcal{X}$ is locally of finite type.*

Proof. (All of the morphisms occurring in this paragraph are representable by algebraic spaces, hence the conventions and results of Properties of Stacks, Section 73.3 are applicable.) Assume x is a finite type point. Choose an affine scheme U, a closed point $u \in U$, and a smooth morphism $\varphi : U \to \mathcal{X}$ with $\varphi(u) = x$, see Lemma 74.14.3. Set $u = \text{Spec}(\kappa(u))$ as usual. Set $R = u \times_{\mathcal{X}} u$ so that we obtain a groupoid in algebraic spaces (u, R, s, t, c), see Algebraic Stacks, Lemma 67.16.1. The projection morphisms $R \to u$ are the compositions

$$R = u \times_{\mathcal{X}} u \to u \times_{\mathcal{X}} U \to u \times_{\mathcal{X}} X = u$$

where the first arrow is of finite type (a base change of the closed immersion of schemes $u \to U$) and the second arrow is smooth (a base change of the smooth morphism $U \to \mathcal{X}$). Hence $s, t : R \to u$ are locally of finite type (as compositions, see Morphisms of Spaces, Lemma 49.22.2). Since u is the spectrum of a field, it follows that s, t are flat and locally of finite presentation (by Morphisms of Spaces, Lemma 49.26.7). We see that $\mathcal{Z} = [u/R]$ is an algebraic stack by Criteria for Representability, Theorem 70.17.2. By Algebraic Stacks, Lemma 67.16.1 we obtain a canonical morphism

$$f : \mathcal{Z} \longrightarrow \mathcal{X}$$

which is fully faithful. Hence this morphism is representable by algebraic spaces, see Algebraic Stacks, Lemma 67.15.2 and a monomorphism, see Properties of Stacks, Lemma 73.8.4. It follows that the residual gerbe $\mathcal{Z}_x \subset \mathcal{X}$ of $\mathcal{X}$ at x exists and that f factors through an equivalence $\mathcal{Z} \to \mathcal{Z}_x$, see Properties of Stacks, Lemma 73.11.11. By construction the diagram

$$\begin{array}{ccc} u & \longrightarrow & U \\ \downarrow & & \downarrow \\ \mathcal{Z} & \xrightarrow{f} & \mathcal{X} \end{array}$$

is commutative. By Criteria for Representability, Lemma 70.17.1 the left vertical arrow is surjective, flat, and locally of finite presentation. Consider

$$\begin{array}{ccccc} u \times_{\mathcal{X}} U & \longrightarrow & \mathcal{Z} \times_{\mathcal{X}} U & \longrightarrow & U \\ \downarrow & & \downarrow & & \downarrow \\ u & \longrightarrow & \mathcal{Z} & \xrightarrow{f} & \mathcal{X} \end{array}$$

As $u \to \mathcal{X}$ is locally of finite type, we see that the base change $u \times_{\mathcal{X}} U \to U$ is locally of finite type. Moreover, $u \times_{\mathcal{X}} U \to \mathcal{Z} \times_{\mathcal{X}} U$ is surjective, flat, and locally of finite presentation as a base change of $u \to \mathcal{Z}$. Thus $\{u \times_{\mathcal{X}} U \to \mathcal{Z} \times_{\mathcal{X}} U\}$ is an fppf covering of algebraic spaces, and we conclude that $\mathcal{Z} \times_{\mathcal{X}} U \to U$ is locally of finite type by Descent on Spaces, Lemma 55.14.1. By definition this means that f is locally of finite type (because the vertical arrow $\mathcal{Z} \times_{\mathcal{X}} U \to \mathcal{Z}$ is smooth as a base change of $U \to \mathcal{X}$ and surjective as $\mathcal{Z}$ has only one point). Since $\mathcal{Z} = \mathcal{Z}_x$ we see that (3) holds.

It is clear that (3) implies (2). If (2) holds then x is a finite type point of $\mathcal{X}$ by Lemma 74.14.4 and Lemma 74.14.6 to see that $\mathcal{Z}_{\text{ft-pts}}$ is nonempty, i.e., the unique point of $\mathcal{Z}$ is a finite type point of $\mathcal{Z}$. □

74.15. Special presentations of algebraic stacks

The following lemma gives a criterion for when a "slice" of a presentation is still flat over the algebraic stack.

Lemma 74.15.1. *Let $\mathcal{X}$ be an algebraic stack. Consider a cartesian diagram*

$$\begin{array}{ccc} U & \xleftarrow{p} & F \\ \downarrow & & \downarrow \\ \mathcal{X} & \longleftarrow & \operatorname{Spec}(k) \end{array}$$

where U is an algebraic space, k is a field, and $U \to \mathcal{X}$ is flat and locally of finite presentation. Let $f_1, \ldots, f_r \in \Gamma(U, \mathcal{O}_U)$ and $z \in |F|$ such that $f_1, \ldots, f_r$ map to a regular sequence in the local ring $\mathcal{O}_{F,\overline{z}}$. Then, after replacing U by an open subspace containing $p(z)$, the morphism

$$V(f_1, \ldots, f_r) \longrightarrow \mathcal{X}$$

is flat and locally of finite presentation.

Proof. Choose a scheme W and a surjective smooth morphism $W \to \mathcal{X}$. Choose an extension of fields $k \subset k'$ and a morphism $w : \operatorname{Spec}(k') \to W$ such that $\operatorname{Spec}(k') \to W \to \mathcal{X}$ is 2-isomorphic to $\operatorname{Spec}(k') \to \operatorname{Spec}(k) \to \mathcal{X}$. This is possible as $W \to \mathcal{X}$ is surjective. Consider the commutative diagram

$$\begin{array}{ccccc} U & \xleftarrow{\text{pr}_0} & U \times_{\mathcal{X}} W & \xleftarrow{p'} & F' \\ \downarrow & & \downarrow & & \downarrow \\ \mathcal{X} & \longleftarrow & W & \longleftarrow & \operatorname{Spec}(k') \end{array}$$

both of whose squares are cartesian. By our choice of w we see that $F' = F \times_{\operatorname{Spec}(k)} \operatorname{Spec}(k')$. Thus $F' \to F$ is surjective and we can choose a point $z' \in |F'|$ mapping to z. Since $F' \to F$ is flat we see that $\mathcal{O}_{F,\overline{z}} \to \mathcal{O}_{F',\overline{z}'}$ is flat, see Morphisms of Spaces,

Lemma 49.27.8. Hence $f_1, \ldots, f_r$ map to a regular sequence in $\mathcal{O}_{F',\overline{z}'}$, see Algebra, Lemma 10.65.7. Note that $U \times_{\mathcal{X}} W \to W$ is a morphism of algebraic spaces which is flat and locally of finite presentation. Hence by More on Morphisms of Spaces, Lemma 57.22.1 we see that there exists an open subspace U' of $U \times_{\mathcal{X}} W$ containing $p(z')$ such that the intersection $U' \cap (V(f_1, \ldots, f_r) \times_{\mathcal{X}} W)$ is flat and locally of finite presentation over W. Note that $\mathrm{pr}_0(U')$ is an open subspace of U containing $p(z)$ as pr_0 is smooth hence open. Now we see that $U' \cap (V(f_1, \ldots, f_r) \times_{\mathcal{X}} W) \to \mathcal{X}$ is flat and locally of finite presentation as the composition

$$U' \cap (V(f_1, \ldots, f_r) \times_{\mathcal{X}} W) \to W \to \mathcal{X}.$$

Hence Properties of Stacks, Lemma 73.3.5 implies $\mathrm{pr}_0(U') \cap V(f_1, \ldots, f_r) \to \mathcal{X}$ is flat and locally of finite presentation as desired. □

Lemma 74.15.2. *Let $\mathcal{X}$ be an algebraic stack. Consider a cartesian diagram*

$$\begin{array}{ccc} U & \xleftarrow{p} & F \\ \downarrow & & \downarrow \\ \mathcal{X} & \longleftarrow & \mathrm{Spec}(k) \end{array}$$

where U is an algebraic space, k is a field, and $U \to \mathcal{X}$ is locally of finite type. Let $z \in |F|$ be such that $\dim_z(F) = 0$. Then, after replacing U by an open subspace containing $p(z)$, the morphism

$$U \longrightarrow \mathcal{X}$$

is locally quasi-finite.

Proof. Since $f : U \to \mathcal{X}$ is locally of finite type there exists a maximal open $W(f) \subset U$ such that the restriction $f|_{W(f)} : W(f) \to \mathcal{X}$ is locally quasi-finite, see Properties of Stacks, Remark 73.9.19 (2). Hence all we need to do is prove that $p(z)$ is a point of $W(f)$. Moreover, the remark referenced above also shows the formation of $W(f)$ commutes with arbitrary base change by a morphism which is representable by algebraic spaces. Hence it suffices to show that the morphism $F \to \mathrm{Spec}(k)$ is locally quasi-finite at z. This follows immediately from Morphisms of Spaces, Lemma 49.31.6. □

A quasi-DM stack has a locally quasi-finite "covering" by a scheme.

Theorem 74.15.3. *Let $\mathcal{X}$ be an algebraic stack. The following are equivalent*

(1) *$\mathcal{X}$ is quasi-DM, and*
(2) *there exists a scheme W and a surjective, flat, locally finitely presented, locally quasi-finite morphism $W \to \mathcal{X}$.*

Proof. The implication (2) $\Rightarrow$ (1) is Lemma 74.4.14. Assume (1). Let $x \in |\mathcal{X}|$ be a finite type point. We will produce a scheme over $\mathcal{X}$ which "works" in a neighbourhood of x. At the end of the proof we will take the disjoint union of all of these to conclude.

Let U be an affine scheme, $U \to \mathcal{X}$ a smooth morphism, and $u \in U$ a closed point which maps to x, see Lemma 74.14.1. Denote $u = \mathrm{Spec}(\kappa(u))$ as usual. Consider

the following commutative diagram

$$\begin{array}{ccc} u & \longleftarrow & R \\ \downarrow & & \downarrow \\ U & \xleftarrow[p]{} & F \\ \downarrow & & \downarrow \\ \mathcal{X} & \longleftarrow & u \end{array}$$

with both squares fibre product squares, in particular $R = u \times_{\mathcal{X}} u$. In the proof of Lemma 74.14.7 we have seen that (u, R, s, t, c) is a groupoid in algebraic spaces with s, t locally of finite type. Let $G \to u$ be the stabilizer group algebraic space (see Groupoids in Spaces, Definition 59.15.2). Note that

$$G = R \times_{(u \times u)} u = (u \times_{\mathcal{X}} u) \times_{(u \times u)} u = \mathcal{X} \times_{\mathcal{X} \times \mathcal{X}} u.$$

As $\mathcal{X}$ is quasi-DM we see that G is locally quasi-finite over u. By More on Groupoids in Spaces, Lemma 60.7.11 we have $\dim(R) = 0$.

Let $e : u \to R$ be the identity of the groupoid. Thus both compositions $u \to R \to u$ are equal to the identity morphism of u. Note that $R \subset F$ is a closed subspace as $u \subset U$ is a closed subscheme. Hence we can also think of e as a point of F. Consider the maps of étale local rings

$$\mathcal{O}_{U,u} \xrightarrow{p^\sharp} \mathcal{O}_{F,\overline{e}} \longrightarrow \mathcal{O}_{R,\overline{e}}$$

Note that $\mathcal{O}_{R,\overline{e}}$ has dimension 0 by the result of the first paragraph. On the other hand, the kernel of the second arrow is $p^\sharp(\mathfrak{m}_u)\mathcal{O}_{F,\overline{e}}$ as R is cut out in F by $\mathfrak{m}_u$. Thus we see that

$$\mathfrak{m}_{\overline{z}} = \sqrt{p^\sharp(\mathfrak{m}_u)\mathcal{O}_{F,\overline{e}}}$$

On the other hand, as the morphism $U \to \mathcal{X}$ is smooth we see that $F \to u$ is a smooth morphism of algebraic spaces. This means that F is a regular algebraic space (Spaces over Fields, Lemma 58.5.1). Hence $\mathcal{O}_{F,\overline{e}}$ is a regular local ring (Properties of Spaces, Lemma 48.23.1). Note that a regular local ring is Cohen-Macaulay (Algebra, Lemma 10.101.3). Let $d = \dim(\mathcal{O}_{F,\overline{e}})$. By Algebra, Lemma 10.99.11 we can find $f_1, \ldots, f_d \in \mathcal{O}_{U,u}$ whose images $\varphi(f_1), \ldots, \varphi(f_d)$ form a regular sequence in $\mathcal{O}_{F,\overline{z}}$. By Lemma 74.15.1 after shrinking U we may assume that $Z = V(f_1, \ldots, f_d) \to \mathcal{X}$ is flat and locally of finite presentation. Note that by construction $F_Z = Z \times_{\mathcal{X}} u$ is a closed subspace of $F = U \times_{\mathcal{X}} u$, that e is a point of this closed subspace, and that

$$\dim(\mathcal{O}_{F_Z,\overline{e}}) = 0.$$

By Morphisms of Spaces, Lemma 49.31.1 it follows that $\dim_e(F_Z) = 0$ because the transcendence degree of e relative to u is zero. Hence it follows from Lemma 74.15.2 that after possibly shrinking U the morphism $Z \to \mathcal{X}$ is locally quasi-finite.

We conclude that for every finite type point x of $\mathcal{X}$ there exists a locally quasi-finite, flat, locally finitely presented morphism $f_x : Z_x \to \mathcal{X}$ with x in the image of $|f_x|$. Set $W = \coprod_x Z_x$ and $f = \coprod f_x$. Then f is flat, locally of finite presentation, and locally quasi-finite. In particular the image of $|f|$ is open, see Properties of Stacks, Lemma 73.4.7. By construction the image contains all finite type points

of $\mathcal{X}$, hence f is surjective by Lemma 74.14.6 (and Properties of Stacks, Lemma 73.4.4). □

Lemma 74.15.4. *Let $\mathcal{Z}$ be a DM, locally Noetherian, reduced algebraic stack with $|\mathcal{Z}|$ a singleton. Then there exists a field k and a surjective étale morphism $\mathrm{Spec}(k) \to \mathcal{Z}$.*

Proof. By Properties of Stacks, Lemma 73.11.3 there exists a field k and a surjective, flat, locally finitely presented morphism $\mathrm{Spec}(k) \to \mathcal{Z}$. Set $U = \mathrm{Spec}(k)$ and $R = U \times_{\mathcal{Z}} U$ so we obtain a groupoid in algebraic spaces (U, R, s, t, c), see Algebraic Stacks, Lemma 67.9.2. Note that by Algebraic Stacks, Remark 67.16.3 we have an equivalence

$$f_{can} : [U/R] \longrightarrow \mathcal{Z}$$

The projections $s, t : R \to U$ are locally of finite presentation. As $\mathcal{Z}$ is DM we see that the stabilizer group algebraic space

$$G = U \times_{U \times U} R = U \times_{U \times U} (U \times_{\mathcal{Z}} U) = U \times_{\mathcal{X} \times \mathcal{X}, \Delta_{\mathcal{X}}} \mathcal{X}$$

is unramified over U. In particular $\dim(G) = 0$ and by More on Groupoids in Spaces, Lemma 60.7.11 we have $\dim(R) = 0$. This implies that R is a scheme, see Spaces over Fields, Lemma 58.4.1. By Varieties, Lemma 32.13.2 we see that R (and also G) is the disjoint union of spectra of Artinian local rings finite over k via either s or t. Let $P = \mathrm{Spec}(A) \subset R$ be the open and closed subscheme whose underlying point is the identity e of the groupoid scheme (U, R, s, t, c). As $s \circ e = t \circ e = \mathrm{id}_{\mathrm{Spec}(k)}$ we see that A is an Artinian local ring whose residue field is identified with k via either $s^\sharp : k \to A$ or $t^\sharp : k \to A$. Note that $s, t : \mathrm{Spec}(A) \to \mathrm{Spec}(k)$ are finite (by the lemma referenced above). Since $G \to \mathrm{Spec}(k)$ is unramified we see that

$$G \cap P = P \times_{U \times U} U = \mathrm{Spec}(A \otimes_{k \otimes k} k)$$

is unramified over k. On the other hand $A \otimes_{k \otimes k} k$ is local as a quotient of A and surjects onto k. We conclude that $A \otimes_{k \otimes k} k = k$. It follows that $P \to U \times U$ is universally injective (as P has only one point with residue field k, unramified (by the computation of the fibre over the unique image point above), and of finite type (because s, t are) hence a monomorphism (see Étale Morphisms, Lemma 40.7.1). Thus $s|_P, t|_P : P \to U$ define a finite flat equivalence relation. Thus we may apply Groupoids, Proposition 38.20.8 to conclude that U/P exists and is a scheme $\overline{U}$. Moreover, $U \to \overline{U}$ is finite locally free and $P = U \times_{\overline{U}} U$. In fact $\overline{U} = \mathrm{Spec}(k_0)$ where $k_0 \subset k$ is the ring of R-invariant functions. As k is a field it follows from the definition Groupoids, Equation (38.20.0.1) that k_0 is a field.

We claim that

$$\mathrm{Spec}(k_0) = \overline{U} = U/P \to [U/R] = \mathcal{Z} \tag{74.15.4.1}$$

is the desired surjective étale morphism. It follows from Properties of Stacks, Lemma 73.11.1 that this morphism is surjective. Thus it suffices to show that (74.15.4.1) is étale[6]. Instead of proving the étaleness directly we first apply Bootstrap, Lemma 61.9.1 to see that there exists a groupoid scheme $(\overline{U}, \overline{R}, \overline{s}, \overline{t}, \overline{c})$ such that (U, R, s, t, c) is the restriction of $(\overline{U}, \overline{R}, \overline{s}, \overline{t}, \overline{c})$ via the quotient morphism $U \to$

[6]We urge the reader to find his/her own proof of this fact. In fact the argument has a lot in common with the final argument of the proof of Bootstrap, Theorem 61.10.1 hence probably should be isolated into its own lemma somewhere.

$\overline{U}$. (We verified all the hypothesis of the lemma above except for the assertion that $j : R \to U \times U$ is separated and locally quasi-finite which follows from the fact that R is a separated scheme locally quasi-finite over k.) Since $U \to \overline{U}$ is finite locally free we see that $[U/R] \to [\overline{U}/\overline{R}]$ is an equivalence, see Groupoids in Spaces, Lemma 59.24.2.

Note that s, t are the base changes of the morphisms $\overline{s}, \overline{t}$ by $U \to \overline{U}$. As $\{U \to \overline{U}\}$ is an fppf covering we conclude $\overline{s}, \overline{t}$ are flat, locally of finite presentation, and locally quasi-finite, see Descent, Lemmas 34.19.13, 34.19.9, and 34.19.22. Consider the commutative diagram

$$\begin{array}{ccccc} U \times_{\overline{U}} U & = & P & \longrightarrow & R \\ & \searrow & \downarrow & & \downarrow \\ & & \overline{U} & \xrightarrow{\overline{e}} & \overline{R} \end{array}$$

It is a general fact about restrictions that the outer four corners form a cartesian diagram. By the equality we see the inner square is cartesian. Since P is open in R we conclude that $\overline{e}$ is an open immersion by Descent, Lemma 34.19.14.

But of course, if $\overline{e}$ is an open immersion and $\overline{s}, \overline{t}$ are flat and locally of finite presentation then the morphisms $\overline{t}, \overline{s}$ are étale. For example you can see this by applying More on Groupoids, Lemma 39.4.1 which shows that $\Omega_{\overline{R}/\overline{U}} = 0$ implies that $\overline{s}, \overline{t} : \overline{R} \to \overline{U}$ is unramified (see Morphisms, Lemma 28.36.2), which in turn implies that $\overline{s}, \overline{t}$ are étale (see Morphisms, Lemma 28.37.16). Hence $\mathcal{Z} = [\overline{U}/\overline{R}]$ is an étale presentation of the algebraic stack $\mathcal{Z}$ and we conclude that $\overline{U} \to \mathcal{Z}$ is étale by Properties of Stacks, Lemma 73.3.3. □

Lemma 74.15.5. *Let $\mathcal{X}$ be an algebraic stack. Consider a cartesian diagram*

$$\begin{array}{ccc} U & \xleftarrow[p]{} & F \\ \downarrow & & \downarrow \\ \mathcal{X} & \longleftarrow & \mathrm{Spec}(k) \end{array}$$

where U is an algebraic space, k is a field, and $U \to \mathcal{X}$ is flat and locally of finite presentation. Let $z \in |F|$ be such that $F \to \mathrm{Spec}(k)$ is unramified at z. Then, after replacing U by an open subspace containing $p(z)$, the morphism

$$U \longrightarrow \mathcal{X}$$

is étale.

Proof. Since $f : U \to \mathcal{X}$ is flat and locally of finite presentation there exists a maximal open $W(f) \subset U$ such that the restriction $f|_{W(f)} : W(f) \to \mathcal{X}$ is étale, see Properties of Stacks, Remark 73.9.19 (5). Hence all we need to do is prove that $p(z)$ is a point of $W(f)$. Moreover, the remark referenced above also shows the formation of $W(f)$ commutes with arbitrary base change by a morphism which is representable by algebraic spaces. Hence it suffices to show that the morphism $F \to \mathrm{Spec}(k)$ is étale at z. Since it is flat and locally of finite presentation as a base change of $U \to \mathcal{X}$ and since $F \to \mathrm{Spec}(k)$ is unramified at z by assumption, this follows from Morphisms of Spaces, Lemma 49.35.12. □

A DM stack is a Deligne-Mumford stack.

Theorem 74.15.6. *Let $\mathcal{X}$ be an algebraic stack. The following are equivalent*

(1) *$\mathcal{X}$ is DM,*
(2) *$\mathcal{X}$ is Deligne-Mumford, and*
(3) *there exists a scheme W and a surjective étale morphism $W \to \mathcal{X}$.*

Proof. Recall that (3) is the definition of (2), see Algebraic Stacks, Definition 67.12.2. The implication (3) $\Rightarrow$ (1) is Lemma 74.4.14. Assume (1). Let $x \in |\mathcal{X}|$ be a finite type point. We will produce a scheme over $\mathcal{X}$ which "works" in a neighbourhood of x. At the end of the proof we will take the disjoint union of all of these to conclude.

By Lemma 74.14.7 the residual gerbe $\mathcal{Z}_x$ of $\mathcal{X}$ at x exists and $\mathcal{Z}_x \to \mathcal{X}$ is locally of finite type. By Lemma 74.4.16 the algebraic stack $\mathcal{Z}_x$ is DM. By Lemma 74.15.4 there exists a field k and a surjective étale morphism $z : \mathrm{Spec}(k) \to \mathcal{Z}_x$. In particular the composition $x : \mathrm{Spec}(k) \to \mathcal{X}$ is locally of finite type (by Morphisms of Spaces, Lemmas 49.22.2 and 49.35.9).

Pick a scheme U and a smooth morphism $U \to \mathcal{X}$ such that x is in the image of $|U| \to |\mathcal{X}|$. Consider the following fibre square

$$\begin{array}{ccc} U & \longleftarrow & F \\ \downarrow & & \downarrow \\ \mathcal{X} & \xleftarrow{x} & \mathrm{Spec}(k) \end{array}$$

in other words $F = U \times_{\mathcal{X},x} \mathrm{Spec}(k)$. By Properties of Stacks, Lemma 73.4.3 we see that F is nonempty. As $\mathcal{Z}_x \to \mathcal{X}$ is a monomorphism we have

$$\mathrm{Spec}(k) \times_{z,\mathcal{Z}_x,z} \mathrm{Spec}(k) = \mathrm{Spec}(k) \times_{x,\mathcal{X},x} \mathrm{Spec}(k)$$

with étale projection maps to $\mathrm{Spec}(k)$ by construction of z. Since

$$F \times_U F = (\mathrm{Spec}(k) \times_{\mathcal{X}} \mathrm{Spec}(k)) \times_{\mathrm{Spec}(k)} F$$

we see that the projections maps $F \times_U F \to F$ are étale as well. It follows that $\Delta_{F/U} : F \to F \times_U F$ is étale (see Morphisms of Spaces, Lemma 49.35.11). By Morphisms of Spaces, Lemma 49.44.2 this implies that $\Delta_{F/U}$ is an open immersion, which finally implies by Morphisms of Spaces, Lemma 49.34.9 that $F \to U$ is unramified.

Pick a nonempty affine scheme V and an étale morphism $V \to F$. (This could be avoided by working directly with F, but it seems easier to explain what's going on by doing so.) Picture

$$\begin{array}{ccccc} U & \longleftarrow & F & \longleftarrow & V \\ \downarrow & & \downarrow & \swarrow & \\ \mathcal{X} & \xleftarrow{x} & \mathrm{Spec}(k) & & \end{array}$$

Then $V \to \mathrm{Spec}(k)$ is a smooth morphism of schemes and $V \to U$ is an unramified morphism of schemes (see Morphisms of Spaces, Lemmas 49.33.2 and 49.34.3). Pick a closed point $v \in V$ with $k \subset \kappa(v)$ finite separable, see Varieties, Lemma 32.15.6. Let $u \in U$ be the image point. The local ring $\mathcal{O}_{V,v}$ is regular (see Varieties, Lemma 32.15.3) and the local ring homomorphism

$$\varphi : \mathcal{O}_{U,u} \longrightarrow \mathcal{O}_{V,v}$$

coming from the morphism $V \to U$ is such that $\varphi(\mathfrak{m}_u)\mathcal{O}_{V,v} = \mathfrak{m}_v$, see Morphisms, Lemma 28.36.14. Hence we can find $f_1, \ldots, f_d \in \mathcal{O}_{U,u}$ such that the images $\varphi(f_1), \ldots, \varphi(f_d)$ form a basis for $\mathfrak{m}_v/\mathfrak{m}_v^2$ over $\kappa(v)$. Since $\mathcal{O}_{V,v}$ is a regular local ring this implies that $\varphi(f_1), \ldots, \varphi(f_d)$ form a regular sequence in $\mathcal{O}_{V,v}$ (see Algebra, Lemma 10.101.3). After replacing U by an open neighbourhood of u we may assume $f_1, \ldots, f_d \in \Gamma(U, \mathcal{O}_U)$. After replacing U by a possibly even smaller open neighbourhood of u we may assume that $V(f_1, \ldots, f_d) \to \mathcal{X}$ is flat and locally of finite presentation, see Lemma 74.15.1. By construction

$$V(f_1, \ldots, f_d) \times_{\mathcal{X}} \mathrm{Spec}(k) \longleftarrow V(f_1, \ldots, f_d) \times_{\mathcal{X}} V$$

is étale and $V(f_1, \ldots, f_d) \times_{\mathcal{X}} V$ is the closed subscheme $T \subset V$ cut out by $f_1|_V, \ldots, f_d|_V$. Hence by construction $v \in T$ and

$$\mathcal{O}_{T,v} = \mathcal{O}_{V,v}/(\varphi(f_1), \ldots, \varphi(f_d)) = \kappa(v)$$

a finite separable extension of k. It follows that $T \to \mathrm{Spec}(k)$ is unramified at v, see Morphisms, Lemma 28.36.14. By definition of an unramified morphism of algebraic spaces this means that $V(f_1, \ldots, f_d) \times_{\mathcal{X}} \mathrm{Spec}(k) \to \mathrm{Spec}(k)$ is unramified at the image of v in $V(f_1, \ldots, f_d) \times_{\mathcal{X}} \mathrm{Spec}(k)$. Applying Lemma 74.15.5 we see that on shrinking U to yet another open neighbourhood of u the morphism $V(f_1, \ldots, f_d) \to \mathcal{X}$ is étale.

We conclude that for every finite type point x of $\mathcal{X}$ there exists an étale morphism $f_x : W_x \to \mathcal{X}$ with x in the image of $|f_x|$. Set $W = \coprod_x W_x$ and $f = \coprod f_x$. Then f is étale. In particular the image of $|f|$ is open, see Properties of Stacks, Lemma 73.4.7. By construction the image contains all finite type points of $\mathcal{X}$, hence f is surjective by Lemma 74.14.6 (and Properties of Stacks, Lemma 73.4.4). □

74.16. Quasi-finite morphisms

The property "locally quasi-finite" of morphisms of algebraic spaces is not smooth local on the source-and-target so we cannot use the material in Section 74.12 to define locally quasi-finite morphisms of algebraic stacks. We do already know what it means for a morphism of algebraic stacks representable by algebraic spaces to be locally quasi-finite, see Properties of Stacks, Section 73.3. To find a condition suitable for general morphisms we make the following observation.

Lemma 74.16.1. *Let $f : \mathcal{X} \to \mathcal{Y}$ be a morphism of algebraic stacks. Assume f is representable by algebraic spaces. The following are equivalent*

1. *f is locally quasi-finite, and*
2. *f is locally of finite type and for every morphism $\mathrm{Spec}(k) \to \mathcal{Y}$ where k is a field the space $|\mathrm{Spec}(k) \times_{\mathcal{Y}} \mathcal{X}|$ is discrete.*

Proof. Assume (1). In this case the morphism of algebraic spaces $\mathcal{X}_k \to \mathrm{Spec}(k)$ is locally quasi-finite as a base change of f. Hence $|\mathcal{X}_k|$ is discrete by Morphisms of Spaces, Lemma 49.25.4. Conversely, assume (2). Pick a surjective smooth morphism $V \to \mathcal{Y}$ where V is a scheme. It suffices to show that the morphism of algebraic spaces $V \times_{\mathcal{Y}} \mathcal{X} \to V$ is locally quasi-finite, see Properties of Stacks, Lemma 73.3.3. The morphism $V \times_{\mathcal{Y}} \mathcal{X} \to V$ is locally of finite type by assumption. For any morphism $\mathrm{Spec}(k) \to V$ where k is a field

$$\mathrm{Spec}(k) \times_V (V \times_{\mathcal{Y}} \mathcal{X}) = \mathrm{Spec}(k) \times_{\mathcal{Y}} \mathcal{X}$$

has a discrete space of points by assumption. Hence we conclude that $V \times_{\mathcal{Y}} \mathcal{X} \to V$ is locally quasi-finite by Morphisms of Spaces, Lemma 49.25.4. □

A morphism of algebraic stacks which is representable by algebraic spaces is quasi-DM, see Lemma 74.4.3. Combined with the lemma above we see that the following definition does not conflict with all of the already existing notion in the case of morphisms representable by algebraic spaces.

Definition 74.16.2. Let $f : \mathcal{X} \to \mathcal{Y}$ be a morphism of algebraic stacks. We say f is *locally quasi-finite* if f is quasi-DM, locally of finite type, and for every morphism $\mathrm{Spec}(k) \to \mathcal{Y}$ where k is a field the space $|\mathcal{X}_k|$ is discrete.

The condition that f be quasi-DM is natural. For example, let k be a field and consider the morphism $\pi : [\mathrm{Spec}(k)/\mathbf{G}_m] \to \mathrm{Spec}(k)$ which has singleton fibres and is locally of finite type. As we will see later this morphism is smooth of relative dimension -1, and we'd like our locally quasi-finite morphisms to have relative dimension 0. Also, note that the section $\mathrm{Spec}(k) \to [\mathrm{Spec}(k)/\mathbf{G}_m]$ does not have discrete fibres, hence is not locally quasi-finite, and we'd like to have the following permanence property for locally quasi-finite morphisms: If $f : \mathcal{X} \to \mathcal{X}'$ is a morphism of algebraic stacks locally quasi-finite over the algebraic stack $\mathcal{Y}$, then f is locally quasi-finite (in fact something a bit stronger holds, see Lemma 74.16.8).

Another justification for the definition above is Lemma 74.16.7 below which characterizes being locally quasi-finite in terms of the existence of suitable "presentations" or "coverings" of $\mathcal{X}$ and $\mathcal{Y}$.

Lemma 74.16.3. *A base change of a locally quasi-finite morphism is locally quasi-finite.*

Proof. We have seen this for quasi-DM morphisms in Lemma 74.4.4 and for locally finite type morphisms in Lemma 74.13.3. It is immediate that the condition on fibres is inherited by a base change. □

Lemma 74.16.4. *Let $\mathcal{X} \to \mathrm{Spec}(k)$ be a locally quasi-finite morphism where $\mathcal{X}$ is an algebraic stack and k is a field. Let $f : V \to \mathcal{X}$ be a locally quasi-finite morphism where V is a scheme. Then $V \to \mathrm{Spec}(k)$ is locally quasi-finite.*

Proof. By Lemma 74.13.2 we see that $V \to \mathrm{Spec}(k)$ is locally of finite type. Assume, to get a contradiction, that $V \to \mathrm{Spec}(k)$ is not locally quasi-finite. Then there exists a nontrivial specialization $v \rightsquigarrow v'$ of points of V, see Morphisms, Lemma 28.21.6. In particular $\mathrm{trdeg}_k(\kappa(v)) > \mathrm{trdeg}_k(\kappa(v'))$, see Morphisms, Lemma 28.29.6. Because $|\mathcal{X}|$ is discrete we see that $|f|(v) = |f|(v')$. Consider $R = V \times_{\mathcal{X}} V$. Then R is an algebraic space and the projections $s, t : R \to V$ are locally quasi-finite as base changes of $V \to \mathcal{X}$ (which is representable by algebraic spaces so this follows from the discussion in Properties of Stacks, Section 73.3). By Properties of Stacks, Lemma 73.4.3 we see that there exists an $r \in |R|$ such that $s(r) = v$ and $t(r) = v'$. By Morphisms of Spaces, Lemma 49.30.3 we see that the transcendence degree of v/k is equal to the transcendence degree of r/k is equal to the transcendence degree of v'/k. This contradiction proves the lemma. □

Lemma 74.16.5. *A composition of a locally quasi-finite morphisms is locally quasi-finite.*

Proof. We have seen this for quasi-DM morphisms in Lemma 74.4.10 and for locally finite type morphisms in Lemma 74.13.2. Let $\mathcal{X} \to \mathcal{Y}$ and $\mathcal{Y} \to \mathcal{Z}$ be locally quasi-finite. Let k be a field and let $\mathrm{Spec}(k) \to \mathcal{Z}$ be a morphism. It suffices to show that $|\mathcal{X}_k|$ is discrete. By Lemma 74.16.3 the morphisms $\mathcal{X}_k \to \mathcal{Y}_k$ and $\mathcal{Y}_k \to \mathrm{Spec}(k)$ are locally quasi-finite. In particular we see that $\mathcal{Y}_k$ is a quasi-DM algebraic stack, see Lemma 74.4.13. By Theorem 74.15.3 we can find a scheme V and a surjective, flat, locally finitely presented, locally quasi-finite morphism $V \to \mathcal{Y}_k$. By Lemma 74.16.4 we see that V is locally quasi-finite over k, in particular $|V|$ is discrete. The morphism $V \times_{\mathcal{Y}_k} \mathcal{X}_k \to \mathcal{X}_k$ is surjective, flat, and locally of finite presentation hence $|V \times_{\mathcal{Y}_k} \mathcal{X}_k| \to |\mathcal{X}_k|$ is surjective and open. Thus it suffices to show that $|V \times_{\mathcal{Y}_k} \mathcal{X}_k|$ is discrete. Note that V is a disjoint union of spectra of Artinian local k-algebras A_i with residue fields k_i, see Varieties, Lemma 32.13.2. Thus it suffices to show that each

$$|\mathrm{Spec}(A_i) \times_{\mathcal{Y}_k} \mathcal{X}_k| = |\mathrm{Spec}(k_i) \times_{\mathcal{Y}_k} \mathcal{X}_k| = |\mathrm{Spec}(k_i) \times_{\mathcal{Y}} \mathcal{X}|$$

is discrete, which follows from the assumption that $\mathcal{X} \to \mathcal{Y}$ is locally quasi-finite. □

Before we characterize locally quasi-finite morphisms in terms of coverings we do it for quasi-DM morphisms.

Lemma 74.16.6. *Let $f : \mathcal{X} \to \mathcal{Y}$ be a morphism of algebraic stacks. The following are equivalent*

1. *f is quasi-DM,*
2. *for any morphism $V \to \mathcal{Y}$ with V an algebraic space there exists a surjective, flat, locally finitely presented, locally quasi-finite morphism $U \to \mathcal{X} \times_{\mathcal{Y}} V$ where U is an algebraic space, and*
3. *there exist algebraic spaces U, V and a morphism $V \to \mathcal{Y}$ which is surjective, flat, and locally of finite presentation, and a morphism $U \to \mathcal{X} \times_{\mathcal{Y}} V$ which is surjective, flat, locally of finite presentation, and locally quasi-finite.*

Proof. The implication (2) $\Rightarrow$ (3) is immediate.

Assume (1) and let $V \to \mathcal{Y}$ be as in (2). Then $\mathcal{X} \times_{\mathcal{Y}} V \to V$ is quasi-DM, see Lemma 74.4.4. By Lemma 74.4.3 the algebraic space V is DM, hence quasi-DM. Thus $\mathcal{X} \times_{\mathcal{Y}} V$ is quasi-DM by Lemma 74.4.11. Hence we may apply Theorem 74.15.3 to get the morphism $U \to \mathcal{X} \times_{\mathcal{Y}} V$ as in (2).

Assume (3). Let $V \to \mathcal{Y}$ and $U \to \mathcal{X} \times_{\mathcal{Y}} V$ be as in (3). To prove that f is quasi-DM it suffices to show that $\mathcal{X} \times_{\mathcal{Y}} V \to V$ is quasi-DM, see Lemma 74.4.5. By Lemma 74.4.14 we see that $\mathcal{X} \times_{\mathcal{Y}} V$ is quasi-DM. Hence $\mathcal{X} \times_{\mathcal{Y}} V \to V$ is quasi-DM by Lemma 74.4.13 and (1) holds. This finishes the proof of the lemma. □

Lemma 74.16.7. *Let $f : \mathcal{X} \to \mathcal{Y}$ be a morphism of algebraic stacks. The following are equivalent*

1. *f is locally quasi-finite,*
2. *f is quasi-DM and for any morphism $V \to \mathcal{Y}$ with V an algebraic space and any locally quasi-finite morphism $U \to \mathcal{X} \times_{\mathcal{Y}} V$ where U is an algebraic space the morphism $U \to V$ is locally quasi-finite,*
3. *for any morphism $V \to \mathcal{Y}$ from an algebraic space V there exists a surjective, flat, locally finitely presented, and locally quasi-finite morphism*

$U \to \mathcal{X} \times_{\mathcal{Y}} V$ where U is an algebraic space such that $U \to V$ is locally quasi-finite,

(4) *there exists algebraic spaces U, V, a surjective, flat, and locally of finitely presented morphism $V \to \mathcal{Y}$, and a morphism $U \to \mathcal{X} \times_{\mathcal{Y}} V$ which is surjective, flat, locally of finite presentation, and locally quasi-finite such that $U \to V$ is locally quasi-finite.*

Proof. Assume (1). Then f is quasi-DM by assumption. Let $V \to \mathcal{Y}$ and $U \to \mathcal{X} \times_{\mathcal{Y}} V$ be as in (2). By Lemma 74.16.5 the composition $U \to \mathcal{X} \times_{\mathcal{Y}} V \to V$ is locally quasi-finite. Thus (1) implies (2).

Assume (2). Let $V \to \mathcal{Y}$ be as in (3). By Lemma 74.16.6 we can find an algebraic space U and a surjective, flat, locally finitely presented, locally quasi-finite morphism $U \to \mathcal{X} \times_{\mathcal{Y}} V$. By (2) the composition $U \to V$ is locally quasi-finite. Thus (2) implies (3).

It is immediate that (3) implies (4).

Assume (4). We will prove (1) holds, which finishes the proof. By Lemma 74.16.6 we see that f is quasi-DM. To prove that f is locally of finite type it suffices to prove that $g : \mathcal{X} \times_{\mathcal{Y}} V \to V$ is locally of finite type, see Lemma 74.13.6. Then it suffices to check that g precomposed with $h : U \to \mathcal{X} \times_{\mathcal{Y}} V$ is locally of finite type, see Lemma 74.13.7. Since $g \circ h : U \to V$ was assumed to be locally quasi-finite this holds, hence f is locally of finite type. Finally, let k be a field and let $\mathrm{Spec}(k) \to \mathcal{Y}$ be a morphism. Then $V \times_{\mathcal{Y}} \mathrm{Spec}(k)$ is a nonempty algebraic space which is locally of finite presentation over k. Hence we can find a finite extension $k \subset k'$ and a morphism $\mathrm{Spec}(k') \to V$ such that

$$\begin{array}{ccc} \mathrm{Spec}(k') & \longrightarrow & V \\ \downarrow & & \downarrow \\ \mathrm{Spec}(k) & \longrightarrow & \mathcal{Y} \end{array}$$

commutes (details omitted). Then $\mathcal{X}_{k'} \to \mathcal{X}_k$ is representable (by schemes), surjective, and finite locally free. In particular $|\mathcal{X}_{k'}| \to |\mathcal{X}_k|$ is surjective and open. Thus it suffices to prove that $|\mathcal{X}_{k'}|$ is discrete. Since

$$U \times_V \mathrm{Spec}(k') = U \times_{\mathcal{X} \times_{\mathcal{Y}} V} \mathcal{X}_{k'}$$

we see that $U \times_V \mathrm{Spec}(k') \to \mathcal{X}_{k'}$ is surjective, flat, and locally of finite presentation (as a base change of $U \to \mathcal{X} \times_{\mathcal{Y}} V$). Hence $|U \times_V \mathrm{Spec}(k')| \to |\mathcal{X}_{k'}|$ is surjective and open. Thus it suffices to show that $|U \times_V \mathrm{Spec}(k')|$ is discrete. This follows from the fact that $U \to V$ is locally quasi-finite (either by our definition above or from the original definition for morphisms of algebraic spaces, via Morphisms of Spaces, Lemma 49.25.4). □

Lemma 74.16.8. *Let $\mathcal{X} \to \mathcal{Y} \to \mathcal{Z}$ be morphisms of algebraic stacks. Assume that $\mathcal{X} \to \mathcal{Z}$ is locally quasi-finite and $\mathcal{Y} \to \mathcal{Z}$ is quasi-DM. Then $\mathcal{X} \to \mathcal{Y}$ is locally quasi-finite.*

Proof. Write $\mathcal{X} \to \mathcal{Y}$ as the composition

$$\mathcal{X} \longrightarrow \mathcal{X} \times_{\mathcal{Z}} \mathcal{Y} \longrightarrow \mathcal{Y}$$

The second arrow is locally quasi-finite as a base change of $\mathcal{X} \to \mathcal{Z}$, see Lemma 74.16.3. The first arrow is locally quasi-finite by Lemma 74.4.8 as $\mathcal{Y} \to \mathcal{Z}$ is quasi-DM. Hence $\mathcal{X} \to \mathcal{Y}$ is locally quasi-finite by Lemma 74.16.5. □

74.17. Flat morphisms

The property "being flat" of morphisms of algebraic spaces is smooth local on the source-and-target, see Descent on Spaces, Remark 55.18.5. It is also stable under base change and fpqc local on the target, see Morphisms of Spaces, Lemma 49.27.4 and Descent on Spaces, Lemma 55.10.11. Hence, by Lemma 74.12.1 above, we may define what it means for a morphism of algebraic spaces to be flat as follows and it agrees with the already existing notion defined in Properties of Stacks, Section 73.3 when the morphism is representable by algebraic spaces.

Definition 74.17.1. Let $f : \mathcal{X} \to \mathcal{Y}$ be a morphism of algebraic stacks. We say f is *flat* if the equivalent conditions of Lemma 74.12.1 hold with $\mathcal{P} = \text{flat}$.

Lemma 74.17.2. *The composition of flat morphisms is flat.*

Proof. Combine Remark 74.12.3 with Morphisms of Spaces, Lemma 49.27.3. □

Lemma 74.17.3. *A base change of a flat morphism is flat.*

Proof. Combine Remark 74.12.4 with Morphisms of Spaces, Lemma 49.27.4. □

Lemma 74.17.4. *Let $f : \mathcal{X} \to \mathcal{Y}$ be a morphism of algebraic stacks. Let $\mathcal{Z} \to \mathcal{Y}$ be a surjective flat morphism of algebraic stacks. If the base change $\mathcal{Z} \times_{\mathcal{Y}} \mathcal{X} \to \mathcal{Z}$ is flat, then f is flat.*

Proof. Choose an algebraic space W and a surjective smooth morphism $W \to \mathcal{Z}$. Then $W \to \mathcal{Z}$ is surjective and flat (Morphisms of Spaces, Lemma 49.33.7) hence $W \to \mathcal{Y}$ is surjective and flat (by Properties of Stacks, Lemma 73.5.2 and Lemma 74.17.2). Since the base change of $\mathcal{Z} \times_{\mathcal{Y}} \mathcal{X} \to \mathcal{Z}$ by $W \to \mathcal{Z}$ is a flat morphism (Lemma 74.17.3) we may replace $\mathcal{Z}$ by W.

Choose an algebraic space V and a surjective smooth morphism $V \to \mathcal{Y}$. Choose an algebraic space U and a surjective smooth morphism $U \to V \times_{\mathcal{Y}} \mathcal{X}$. We have to show that $U \to V$ is flat. Now we base change everything by $W \to \mathcal{Y}$: Set $U' = W \times_{\mathcal{Y}} U$, $V' = W \times_{\mathcal{Y}} V$, $\mathcal{X}' = W \times_{\mathcal{Y}} \mathcal{X}$, and $\mathcal{Y}' = W \times_{\mathcal{Y}} \mathcal{Y} = W$. Then it is still true that $U' \to V' \times_{\mathcal{Y}'} \mathcal{X}'$ is smooth by base change. Hence by our definition of flat morphisms of algebraic stacks and the assumption that $\mathcal{X}' \to \mathcal{Y}'$ is flat, we see that $U' \to V'$ is flat. Then, since $V' \to V$ is surjective as a base change of $W \to \mathcal{Y}$ we see that $U \to V$ is flat by Morphisms of Spaces, Lemma 49.28.3 (2) and we win. □

Lemma 74.17.5. *Let $\mathcal{X} \to \mathcal{Y} \to \mathcal{Z}$ be morphisms of algebraic stacks. If $\mathcal{X} \to \mathcal{Z}$ is flat and $\mathcal{X} \to \mathcal{Y}$ is surjective and flat, then $\mathcal{Y} \to \mathcal{Z}$ is flat.*

Proof. Choose an algebraic space W and a surjective smooth morphism $W \to \mathcal{Z}$. Choose an algebraic space V and a surjective smooth morphism $V \to W \times_{\mathcal{Z}} \mathcal{Y}$. Choose an algebraic space U and a surjective smooth morphism $U \to V \times_{\mathcal{Y}} \mathcal{X}$. We know that $U \to V$ is flat and that $U \to W$ is flat. Also, as $\mathcal{X} \to \mathcal{Y}$ is surjective we see that $U \to V$ is surjective (as a composition of surjective morphisms). Hence the lemma reduces to the case of morphisms of algebraic spaces. The case of morphisms of algebraic spaces is Morphisms of Spaces, Lemma 49.28.5. □

74.18. Morphisms of finite presentation

The property "locally of finite presentation" of morphisms of algebraic spaces is smooth local on the source-and-target, see Descent on Spaces, Remark 55.18.5. It is also stable under base change and fpqc local on the target, see Morphisms of Spaces, Lemma 49.26.3 and Descent on Spaces, Lemma 55.10.8. Hence, by Lemma 74.12.1 above, we may define what it means for a morphism of algebraic spaces to be locally of finite presentation as follows and it agrees with the already existing notion defined in Properties of Stacks, Section 73.3 when the morphism is representable by algebraic spaces.

Definition 74.18.1. Let $f : \mathcal{X} \to \mathcal{Y}$ be a morphism of algebraic stacks.

(1) We say f *locally of finite presentation* if the equivalent conditions of Lemma 74.12.1 hold with $\mathcal{P} =$ locally of finite presentation.
(2) We say f is *of finite presentation* if it is locally of finite presentation, quasi-compact, and quasi-separated.

Note that a morphism of finite presentation is **not** just a quasi-compact morphism which is locally of finite presentation.

Lemma 74.18.2. *The composition of finitely presented morphisms is of finite presentation. The same holds for morphisms which are locally of finite presentation.*

Proof. Combine Remark 74.12.3 with Morphisms of Spaces, Lemma 49.26.2. □

Lemma 74.18.3. *A base change of a finitely presented morphism is of finite presentation. The same holds for morphisms which are locally of finite presentation.*

Proof. Combine Remark 74.12.4 with Morphisms of Spaces, Lemma 49.26.3. □

Lemma 74.18.4. *A morphism which is locally of finite presentation is locally of finite type. A morphism of finite presentation is of finite type.*

Proof. Combine Remark 74.12.5 with Morphisms of Spaces, Lemma 49.26.5. □

Lemma 74.18.5. *Let $f : \mathcal{X} \to \mathcal{Y}$ and $g : \mathcal{Y} \to \mathcal{Z}$ be morphisms of algebraic stacks If $g \circ f$ is locally of finite presentation and g is locally of finite type, then f is locally of finite presentation.*

Proof. Choose an algebraic space W and a surjective smooth morphism $W \to \mathcal{Z}$. Choose an algebraic space V and a surjective smooth morphism $V \to \mathcal{Y} \times_{\mathcal{Z}} W$. Choose an algebraic space U and a surjective smooth morphism $U \to \mathcal{X} \times_{\mathcal{Y}} V$. The lemma follows upon applying Morphisms of Spaces, Lemma 49.26.9 to the morphisms $U \to V \to W$. □

Lemma 74.18.6. *An open immersion is locally of finite presentation.*

Proof. Follows from Morphisms of Spaces, Lemma 49.26.11. □

Lemma 74.18.7. *Let $f : \mathcal{X} \to \mathcal{Y}$ be a morphism of algebraic stacks. Let $\mathcal{Z} \to \mathcal{Y}$ be a surjective, flat, locally finitely presented morphism of algebraic stacks. If the base change $\mathcal{Z} \times_{\mathcal{Y}} \mathcal{X} \to \mathcal{Z}$ is locally of finite presentation, then f is locally of finite presentation.*

Proof. Choose an algebraic space W and a surjective smooth morphism $W \to \mathcal{Z}$. Then $W \to \mathcal{Z}$ is surjective, flat, and locally of finite presentation (Morphisms of Spaces, Lemmas 49.33.7 and 49.33.5) hence $W \to \mathcal{Y}$ is surjective, flat, and locally of finite presentation (by Properties of Stacks, Lemma 73.5.2 and Lemmas 74.17.2 and 74.18.2). Since the base change of $\mathcal{Z} \times_{\mathcal{Y}} \mathcal{X} \to \mathcal{Z}$ by $W \to \mathcal{Z}$ is locally of finite presentation (Lemma 74.17.3) we may replace $\mathcal{Z}$ by W.

Choose an algebraic space V and a surjective smooth morphism $V \to \mathcal{Y}$. Choose an algebraic space U and a surjective smooth morphism $U \to V \times_{\mathcal{Y}} \mathcal{X}$. We have to show that $U \to V$ is locally of finite presentation. Now we base change everything by $W \to \mathcal{Y}$: Set $U' = W \times_{\mathcal{Y}} U$, $V' = W \times_{\mathcal{Y}} V$, $\mathcal{X}' = W \times_{\mathcal{Y}} \mathcal{X}$, and $\mathcal{Y}' = W \times_{\mathcal{Y}} \mathcal{Y} = W$. Then it is still true that $U' \to V' \times_{\mathcal{Y}'} \mathcal{X}'$ is smooth by base change. Hence by our definition of locally finitely presented morphisms of algebraic stacks and the assumption that $\mathcal{X}' \to \mathcal{Y}'$ is locally of finite presentation, we see that $U' \to V'$ is locally of finite presentation. Then, since $V' \to V$ is surjective, flat, and locally of finite presentation as a base change of $W \to \mathcal{Y}$ we see that $U \to V$ is locally of finite presentation by Descent on Spaces, Lemma 55.10.8 and we win. □

Lemma 74.18.8. *Let $\mathcal{X} \to \mathcal{Y} \to \mathcal{Z}$ be morphisms of algebraic stacks. If $\mathcal{X} \to \mathcal{Z}$ is locally of finite presentation and $\mathcal{X} \to \mathcal{Y}$ is surjective, flat, and locally of finite presentation, then $\mathcal{Y} \to \mathcal{Z}$ is locally of finite presentation.*

Proof. Choose an algebraic space W and a surjective smooth morphism $W \to \mathcal{Z}$. Choose an algebraic space V and a surjective smooth morphism $V \to W \times_{\mathcal{Z}} \mathcal{Y}$. Choose an algebraic space U and a surjective smooth morphism $U \to V \times_{\mathcal{Y}} \mathcal{X}$. We know that $U \to V$ is flat and locally of finite presentation and that $U \to W$ is locally of finite presentation. Also, as $\mathcal{X} \to \mathcal{Y}$ is surjective we see that $U \to V$ is surjective (as a composition of surjective morphisms). Hence the lemma reduces to the case of morphisms of algebraic spaces. The case of morphisms of algebraic spaces is Descent on Spaces, Lemma 55.14.1. □

Lemma 74.18.9. *Let $f : \mathcal{X} \to \mathcal{Y}$ be a morphism of algebraic stacks which is surjective, flat, and locally of finite presentation. Then for every scheme U and object y of $\mathcal{Y}$ over U there exists an fppf covering $\{U_i \to U\}$ and objects x_i of $\mathcal{X}$ over U_i such that $f(x_i) \cong y|_{U_i}$ in $\mathcal{Y}_{U_i}$.*

Proof. We may think of y as a morphism $U \to \mathcal{Y}$. By Properties of Stacks, Lemma 73.5.3 and Lemmas 74.18.3 and 74.17.3 we see that $\mathcal{X} \times_{\mathcal{Y}} U \to U$ is surjective, flat, and locally of finite presentation. Let V be a scheme and let $V \to \mathcal{X} \times_{\mathcal{Y}} U$ smooth and surjective. Then $V \to \mathcal{X} \times_{\mathcal{Y}} U$ is also surjective, flat, and locally of finite presentation (see Morphisms of Spaces, Lemmas 49.33.7 and 49.33.5). Hence also $V \to U$ is surjective, flat, and locally of finite presentation, see Properties of Stacks, Lemma 73.5.2 and Lemmas 74.18.2, and 74.17.2. Hence $\{V \to U\}$ is the desired fppf covering and $x : V \to \mathcal{X}$ is the desired object. □

Lemma 74.18.10. *Let $f_j : \mathcal{X}_j \to \mathcal{X}$, $j \in J$ be a family of morphisms of algebraic stacks which are each flat and locally of finite presentation and which are jointly surjective, i.e., $|\mathcal{X}| = \bigcup |\mathcal{X}_i|$. Then for every scheme U and object x of $\mathcal{X}$ over U there exists an fppf covering $\{U_i \to U\}_{i \in I}$, a map $a : I \to J$, and objects x_i of $\mathcal{X}_{a(i)}$ over U_i such that $f_{a(i)}(x_i) \cong y|_{U_i}$ in $\mathcal{X}_{U_i}$.*

Proof. Apply Lemma 74.18.9 to the morphism $\coprod_{j\in J}\mathcal{X}_j \to \mathcal{X}$. (There is a slight set theoretic issue here – due to our setup of things – which we ignore.) To finish, note that a morphism $x_i : U_i \to \coprod_{j\in J}\mathcal{X}_j$ is given by a disjoint union decomposition $U_i = \coprod U_{i,j}$ and morphisms $U_{i,j} \to \mathcal{X}_j$. Then the fppf covering $\{U_{i,j} \to U\}$ and the morphisms $U_{i,j} \to \mathcal{X}_j$ do the job. □

Lemma 74.18.11. *Let $f : \mathcal{X} \to \mathcal{Y}$ be flat and locally of finite presentation. Then $|f| : |\mathcal{X}| \to |\mathcal{Y}|$ is open.*

Proof. Choose a scheme V and a smooth surjective morphism $V \to \mathcal{Y}$. Choose a scheme U and a smooth surjective morphism $U \to V \times_{\mathcal{Y}} \mathcal{X}$. By assumption the morphism of schemes $U \to V$ is flat and locally of finite presentation. Hence $U \to V$ is open by Morphisms, Lemma 28.26.9. By construction of the topology on $|\mathcal{Y}|$ the map $|V| \to |\mathcal{Y}|$ is open. The map $|U| \to |\mathcal{X}|$ is surjective. The result follows from these facts by elementary topology. □

74.19. Gerbes

An important type of algebraic stack are the stacks of the form $[B/G]$ where B is an algebraic space and G is a flat and locally finitely presented group algebraic space over B (acting trivially on B), see Criteria for Representability, Lemma 70.18.3. It turns out that an algebraic stack is a gerbe when it locally in the fppf topology is of this form, see Lemma 74.19.8. In this section we briefly discuss this notion and the corresponding relative notion.

Definition 74.19.1. Let $f : \mathcal{X} \to \mathcal{Y}$ be a morphism of algebraic stacks. We say $\mathcal{X}$ is a *gerbe over* $\mathcal{Y}$ if $\mathcal{X}$ is a gerbe over $\mathcal{Y}$ as stacks in groupoids over $(Sch/S)_{fppf}$, see Stacks, Definition 8.11.4. We say an algebraic stack $\mathcal{X}$ is a *gerbe* if there exists a morphism $\mathcal{X} \to X$ where X is an algebraic space which turns $\mathcal{X}$ into a gerbe over X.

The condition that $\mathcal{X}$ be a gerbe over $\mathcal{Y}$ is defined purely in terms of the topology and category theory underlying the given algebraic stacks; but as we will see later this condition has geometric consequences. For example it implies that $\mathcal{X} \to \mathcal{Y}$ is surjective, flat, and locally of finite presentation, see Lemma 74.19.7. The absolute notion is trickier to parse, because it may not be at first clear that X is well determined. Actually, it is.

Lemma 74.19.2. *Let $\mathcal{X}$ be an algebraic stack. If $\mathcal{X}$ is a gerbe, then the sheafification of the presheaf*

$$(Sch/S)^{opp}_{fppf} \to Sets, \quad U \mapsto \mathrm{Ob}(\mathcal{X}_U)/\cong$$

is an algebraic space and $\mathcal{X}$ is a gerbe over it.

Proof. (In this proof the abuse of language introduced in Section 74.2 really pays off.) Choose a morphism $\pi : \mathcal{X} \to X$ where X is an algebraic space which turns $\mathcal{X}$ into a gerbe over X. It suffices to prove that X is the sheafification of the presheaf $\mathcal{F}$ displayed in the lemma. It is clear that there is a map $c : \mathcal{F} \to X$. We will use Stacks, Lemma 8.11.3 properties (2)(a) and (2)(b) to see that the map $c^\# : \mathcal{F}^\# \to X$ is surjective and injective, hence an isomorphism, see Sites, Lemma 7.12.2. Surjective: Let T be a scheme and let $f : T \to X$. By property (2)(a) there exists an fppf covering $\{h_i : T_i \to T\}$ and morphisms $x_i : T_i \to \mathcal{X}$ such that $f \circ h_i$ corresponds to $\pi \circ x_i$. Hence we see that $f|_{T_i}$ is in the image of c. Injective: Let T

be a scheme and let $x, y : T \to \mathcal{X}$ be morphisms such that $c \circ x = c \circ y$. By (2)(b) we can find a covering $\{T_i \to T\}$ and morphisms $x|_{T_i} \to y|_{T_i}$ in the fibre category $\mathcal{X}_{T_i}$. Hence the restrictions $x|_{T_i}, y|_{T_i}$ are equal in $\mathcal{F}(T_i)$. This proves that x, y give the same section of $\mathcal{F}^\#$ over T as desired. □

Lemma 74.19.3. *Let*

$$\begin{array}{ccc} \mathcal{X}' & \longrightarrow & \mathcal{X} \\ \downarrow & & \downarrow \\ \mathcal{Y}' & \longrightarrow & \mathcal{Y} \end{array}$$

be a fibre product of algebraic stacks. If $\mathcal{X}$ is a gerbe over $\mathcal{Y}$, then $\mathcal{X}'$ is a gerbe over $\mathcal{Y}'$.

Proof. Immediate from the definitions and Stacks, Lemma 8.11.5. □

Lemma 74.19.4. *Let $\mathcal{X} \to \mathcal{Y}$ and $\mathcal{Y} \to \mathcal{Z}$ be morphisms of algebraic stacks. If $\mathcal{X}$ is a gerbe over $\mathcal{Y}$ and $\mathcal{Y}$ is a gerbe over $\mathcal{Z}$, then $\mathcal{X}$ is a gerbe over $\mathcal{Z}$.*

Proof. Immediate from Stacks, Lemma 8.11.6. □

Lemma 74.19.5. *Let*

$$\begin{array}{ccc} \mathcal{X}' & \longrightarrow & \mathcal{X} \\ \downarrow & & \downarrow \\ \mathcal{Y}' & \longrightarrow & \mathcal{Y} \end{array}$$

be a fibre product of algebraic stacks. If $\mathcal{Y}' \to \mathcal{Y}$ is surjective, flat, and locally of finite presentation and $\mathcal{X}'$ is a gerbe over $\mathcal{Y}'$, then $\mathcal{X}$ is a gerbe over $\mathcal{Y}$.

Proof. Follows immediately from Lemma 74.18.9 and Stacks, Lemma 8.11.7. □

Lemma 74.19.6. *Let $\pi : \mathcal{X} \to U$ be a morphism from an algebraic stack to an algebraic space and let $x : U \to \mathcal{X}$ be a section of π. Set $G = Isom_{\mathcal{X}}(x, x)$, see Definition 74.5.3. If $\mathcal{X}$ is a gerbe over U, then*

1. *there is a canonical equivalence of stacks in groupoids*
$$x_{can} : [U/G] \longrightarrow \mathcal{X}.$$
where $[U/G]$ is the quotient stack for the trivial action of G on U,
2. *$G \to U$ is flat and locally of finite presentation, and*
3. *$U \to \mathcal{X}$ is surjective, flat, and locally of finite presentation.*

Proof. Set $R = U \times_{x,\mathcal{X},x} U$. The morphism $R \to U \times U$ factors through the diagonal $\Delta_U : U \to U \times U$ as it factors through $U \times_U U = U$. Hence $R = G$ because

$$\begin{aligned} G &= Isom_{\mathcal{X}}(x, x) \\ &= U \times_{x,\mathcal{X}} \mathcal{I}_{\mathcal{X}} \\ &= U \times_{x,\mathcal{X}} (\mathcal{X} \times_{\Delta,\mathcal{X}\times_S\mathcal{X},\Delta} \mathcal{X}) \\ &= (U \times_{x,\mathcal{X},x} U) \times_{U\times U,\Delta_U} U \\ &= R \times_{U\times U,\Delta_U} U \\ &= R \end{aligned}$$

for the fourth equality use Categories, Lemma 4.29.12. Let $t, s : R \to U$ be the projections. The composition law $c : R \times_{s,U,t} R \to R$ constructed on R in Algebraic Stacks, Lemma 67.16.1 agrees with the group law on G (proof omitted). Thus Algebraic Stacks, Lemma 67.16.1 shows we obtain a canonical fully faithful 1-morphism

$$x_{can} : [U/G] \longrightarrow \mathcal{X}$$

of stacks in groupoids over $(Sch/S)_{fppf}$. To see that it is an equivalence it suffices to show that it is essentially surjective. To do this it suffices to show that any object of $\mathcal{X}$ over a scheme T comes fppf locally from x via a morphism $T \to U$, see Stacks, Lemma 8.4.8. However, this follows the condition that π turns $\mathcal{X}$ into a gerbe over X, see property (2)(a) of Stacks, Lemma 8.11.3.

By Criteria for Representability, Lemma 70.18.3 we conclude that $G \to U$ is flat and locally of finite presentation. Finally, $U \to \mathcal{X}$ is surjective, flat, and locally of finite presentation by Criteria for Representability, Lemma 70.17.1. □

Lemma 74.19.7. *Let $\pi : \mathcal{X} \to \mathcal{Y}$ be a morphism of algebraic stacks. The following are equivalent*

(1) *$\mathcal{X}$ is a gerbe over $\mathcal{Y}$, and*
(2) *there exists an algebraic space U, a group algebraic space G flat and locally of finite presentation over U, and a surjective, flat, and locally finitely presented morphism $U \to \mathcal{Y}$ such that $\mathcal{X} \times_{\mathcal{Y}} U \cong [U/G]$ over U.*

Proof. Assume (2). By Lemma 74.19.5 to prove (1) it suffices to show that $[U/G]$ is a gerbe over U. This is immediate from Groupoids in Spaces, Lemma 59.26.2.

Assume (1). Any base change of π is a gerbe, see Lemma 74.19.3. As a first step we choose a scheme V and a surjective smooth morphism $V \to \mathcal{Y}$. Thus we may assume that $\pi : \mathcal{X} \to V$ is a gerbe over a scheme. This means that there exists an fppf covering $\{V_i \to V\}$ such that the fibre category $\mathcal{X}_{V_i}$ is nonempty, see Stacks, Lemma 8.11.3 (2)(a). Note that $U = \coprod V_i \to U$ is surjective, flat, and locally of finite presentation. Hence we may replace V by U and assume that $\pi : \mathcal{X} \to U$ is a gerbe over a scheme U and that there exists an object x of $\mathcal{X}$ over U. By Lemma 74.19.6 we see that $\mathcal{X} = [U/G]$ over U for some flat and locally finitely presented group algebraic space G over U. □

Lemma 74.19.8. *Let $\pi : \mathcal{X} \to \mathcal{Y}$ be a morphism of algebraic stacks. If $\mathcal{X}$ is a gerbe over $\mathcal{Y}$, then π is surjective, flat, and locally of finite presentation.*

Proof. By Properties of Stacks, Lemma 73.5.4 and Lemmas 74.17.4 and 74.18.7 it suffices to prove to the lemma after replacing π by a base change with a surjective, flat, locally finitely presented morphism $\mathcal{Y}' \to \mathcal{Y}$. By Lemma 74.19.7 we may assume $\mathcal{Y} = U$ is an algebraic space and $\mathcal{X} = [U/G]$ over U. Then $U \to [U/G]$ is surjective, flat, and locally of finite presentation, see Lemma 74.19.6. This implies that π is surjective, flat, and locally of finite presentation by Properties of Stacks, Lemma 73.5.5 and Lemmas 74.17.5 and 74.18.8. □

Proposition 74.19.9. *Let $\mathcal{X}$ be an algebraic stack. The following are equivalent*

(1) *$\mathcal{X}$ is a gerbe, and*
(2) *$\mathcal{I}_{\mathcal{X}} \to \mathcal{X}$ is flat and locally of finite presentation.*

Proof. Assume (1). Choose a morphism $\mathcal{X} \to X$ into an algebraic space X which turns $\mathcal{X}$ into a gerbe over X. Let $X' \to X$ is a surjective, flat, locally finitely presented morphism and set $\mathcal{X}' = X' \times_X \mathcal{X}$. Note that $\mathcal{X}'$ is a gerbe over X' by Lemma 74.19.3. Then both squares in

$$\begin{array}{ccccc} \mathcal{I}_{\mathcal{X}'} & \longrightarrow & \mathcal{X}' & \longrightarrow & X' \\ \downarrow & & \downarrow & & \downarrow \\ \mathcal{I}_{\mathcal{X}} & \longrightarrow & \mathcal{X} & \longrightarrow & X \end{array}$$

are fibre product squares, see Lemma 74.5.4. Hence to prove $\mathcal{I}_{\mathcal{X}} \to \mathcal{X}$ is flat and locally of finite presentation it suffices to do so after such a base change by Lemmas 74.17.4 and 74.18.7. Thus we can apply Lemma 74.19.7 to assume that $\mathcal{X} = [U/G]$. By Lemma 74.19.6 we see G is flat and locally of finite presentation over U and that $x : U \to [U/G]$ is surjective, flat, and locally of finite presentation. Moreover, the pullback of $\mathcal{I}_{\mathcal{X}}$ by x is G and we conclude that (2) holds by descent again, i.e., by Lemmas 74.17.4 and 74.18.7.

Conversely, assume (2). Choose a smooth presentation $\mathcal{X} = [U/R]$, see Algebraic Stacks, Section 67.16. Denote $G \to U$ the stabilizer group algebraic space of the groupoid (U, R, s, t, c, e, i), see Groupoids in Spaces, Definition 59.15.2. By Lemma 74.5.6 we see that $G \to U$ is flat and locally of finite presentation as a base change of $\mathcal{I}_{\mathcal{X}} \to \mathcal{X}$, see Lemmas 74.17.3 and 74.18.3. Consider the following action

$$a : G \times_{U,t} R \to R, \quad (g, r) \mapsto c(g, r)$$

of G on R. This action is free on T-valued points for any scheme T as R is a groupoid. Hence $R' = R/G$ is an algebraic space and the quotient morphism $\pi : R \to R'$ is surjective, flat, and locally of finite presentation by Bootstrap, Lemma 61.11.5. The projections $s, t : R \to U$ are G-invariant, hence we obtain morphisms $s', t' : R' \to U$ such that $s = s' \circ \pi$ and $t = t' \circ \pi$. Since $s, t : R \to U$ are flat and locally of finite presentation we conclude that s', t' are flat and locally of finite presentation, see Morphisms of Spaces, Lemmas 49.28.5 and Descent on Spaces, Lemma 55.14.1. Consider the morphism

$$j' = (t', s') : R' \longrightarrow U \times U.$$

We claim this is a monomorphism. Namely, suppose that T is a scheme and that $a, b : T \to R'$ are morphisms which have the same image in $U \times U$. By definition of the quotient $R' = R/G$ there exists an fppf covering $\{h_j : T_j \to T\}$ such that $a \circ h_j = \pi \circ a_j$ and $b \circ h_j = \pi \circ b_j$ for some morphisms $a_j, b_j : T_j \to R$. Since a_j, b_j have the same image in $U \times U$ we see that $g_j = c(a_j, i(b_j))$ is a T_j-valued point of G such that $c(g_j, b_j) = a_j$. In other words, a_j and b_j have the same image in R' and the claim is proved. Since $j : R \to U \times U$ is a pre-equivalence relation (see Groupoids in Spaces, Lemma 59.11.2) and $R \to R'$ is surjective (as a map of sheaves) we see that $j' : R' \to U \times U$ is an equivalence relation. Hence Bootstrap, Theorem 61.10.1 shows that $X = U/R'$ is an algebraic space. Finally, we claim that the morphism

$$\mathcal{X} = [U/R] \longrightarrow X = U/R'$$

turns $\mathcal{X}$ into a gerbe over X. This follows from Groupoids in Spaces, Lemma 59.26.1 as $R \to R'$ is surjective, flat, and locally of finite presentation (if needed use Bootstrap, Lemma 61.4.5 to see this implies the required hypothesis). $\square$

At this point we have developed enough machinery to prove that residual gerbes (when they exist) are gerbes.

Lemma 74.19.10. *Let $\mathcal{Z}$ be a reduced, locally Noetherian algebraic stack such that $|\mathcal{Z}|$ is a singleton. Then $\mathcal{Z}$ is a gerbe over a reduced, locally Noetherian algebraic space Z with $|Z|$ a singleton.*

Proof. By Properties of Stacks, Lemma 73.11.3 there exists a surjective, flat, locally finitely presented morphism $\mathrm{Spec}(k) \to \mathcal{Z}$ where k is a field. Then $\mathcal{I}_Z \times_{\mathcal{Z}} \mathrm{Spec}(k) \to \mathrm{Spec}(k)$ is representable by algebraic spaces and locally of finite type (as a base change of $\mathcal{I}_{\mathcal{Z}} \to \mathcal{Z}$, see Lemmas 74.5.1 and 74.13.3). Therefore it is locally of finite presentation, see Morphisms of Spaces, Lemma 49.26.7. Of course it is also flat as k is a field. Hence we may apply Lemmas 74.17.4 and 74.18.7 to see that $\mathcal{I}_{\mathcal{Z}} \to \mathcal{Z}$ is flat and locally of finite presentation. We conclude that $\mathcal{Z}$ is a gerbe by Proposition 74.19.9. Let $\pi : \mathcal{Z} \to Z$ be a morphism to an algebraic space such that $\mathcal{Z}$ is a gerbe over Z. Then π is surjective, flat, and locally of finite presentation by Lemma 74.19.8. Hence $\mathrm{Spec}(k) \to Z$ is surjective, flat, and locally of finite presentation as a composition, see Properties of Stacks, Lemma 73.5.2 and Lemmas 74.17.2 and 74.18.2. Hence by Properties of Stacks, Lemma 73.11.3 we see that $|Z|$ is a singleton and that Z is locally Noetherian and reduced. □

Lemma 74.19.11. *Let $f : \mathcal{X} \to \mathcal{Y}$ be a morphism of algebraic stacks. If $\mathcal{X}$ is a gerbe over $\mathcal{Y}$ then the map $|\mathcal{X}| \to |\mathcal{Y}|$ is a homeomorphism of topological spaces.*

Proof. Let k be a field and let y be an object of $\mathcal{Y}$ over $\mathrm{Spec}(k)$. By Stacks, Lemma 8.11.3 property (2)(a) there exists an fppf covering $\{T_i \to \mathrm{Spec}(k)\}$ and objects x_i of $\mathcal{X}$ over T_i with $f(x_i) \cong y|_{T_i}$. Choose an i such that $T_i \neq \emptyset$. Choose a morphism $\mathrm{Spec}(K) \to T_i$ for some field K. Then $k \subset K$ and $x_i|_K$ is an object of $\mathcal{X}$ lying over $y|_K$. Thus we see that $|\mathcal{Y}| \to |\mathcal{X}|$. is surjective. The map $|\mathcal{Y}| \to |\mathcal{X}|$ is also injective. Namely, if x, x' are objects of $\mathcal{X}$ over $\mathrm{Spec}(k)$ whose images $f(x), f(x')$ become isomorphic (over an extension) in $\mathcal{Y}$, then Stacks, Lemma 8.11.3 property (2)(b) guarantees the existence of an extension of k over which x and x' become isomorphic (details omitted). Hence $|\mathcal{X}| \to |\mathcal{Y}|$ is continuous and bijective and it suffices to show that it is also open. This follows from Lemmas 74.19.8 and 74.18.11. □

The following lemma tells us that residual gerbes exist for all points on any algebraic stack which is a gerbe.

Lemma 74.19.12. *Let $\mathcal{X}$ be an algebraic stack. If $\mathcal{X}$ is a gerbe then for every $x \in |\mathcal{X}|$ the residual gerbe of $\mathcal{X}$ at x exists.*

Proof. Let $\pi : \mathcal{X} \to X$ be a morphism from $\mathcal{X}$ into an algebraic space X which turns $\mathcal{X}$ into a gerbe over X. Let $Z_x \to X$ be the residual space of X at x, see Decent Spaces, Definition 50.10.6. Let $\mathcal{Z} = \mathcal{X} \times_X Z_x$. By Lemma 74.19.3 the algebraic stack $\mathcal{Z}$ is a gerbe over Z_x. Hence $|\mathcal{Z}| = |Z_x|$ (Lemma 74.19.11) is a singleton. Since $\mathcal{Z} \to Z_x$ is locally of finite presentation as a base change of π (see Lemmas 74.19.8 and 74.18.3) we see that $\mathcal{Z}$ is locally Noetherian, see Lemma 74.13.5. Thus the residual gerbe $\mathcal{Z}_x$ of $\mathcal{X}$ at x exists and is equal to $\mathcal{Z}_x = \mathcal{Z}_{red}$ the reduction of the algebraic stack $\mathcal{Z}$. Namely, we have seen above that $|\mathcal{Z}_{red}|$ is a singleton mapping to $x \in |\mathcal{X}|$, it is reduced by construction, and it is locally Noetherian (as the reduction of a locally Noetherian algebraic stack is locally Noetherian, details omitted). □

74.20. Stratification by gerbes

The goal of this section is to show that many algebraic stacks $\mathcal{X}$ have a "stratification" by locally closed substacks $\mathcal{X}_i \subset \mathcal{X}$ such that each $\mathcal{X}_i$ is a gerbe. This shows that in some sense gerbes are the building blocks out of which any algebraic stack is constructed. Note that by stratification we only mean that

$$|\mathcal{X}| = \bigcup\nolimits_i |\mathcal{X}_i|$$

is a stratification of the topological space associated to $\mathcal{X}$ and nothing more (in this section). Hence it is harmless to replace $\mathcal{X}$ by its reduction (see Properties of Stacks, Section 73.10) in order to study this stratification.

The following proposition tells us there is (almost always) a dense open substack of the reduction of $\mathcal{X}$

Proposition 74.20.1. *Let $\mathcal{X}$ be a reduced algebraic stack such that $\mathcal{I}_\mathcal{X} \to \mathcal{X}$ is quasi-compact. Then there exists a dense open substack $\mathcal{U} \subset \mathcal{X}$ which is a gerbe.*

Proof. According to Proposition 74.19.9 it is enough to find a dense open substack $\mathcal{U}$ such that $\mathcal{I}_\mathcal{U} \to \mathcal{U}$ is flat and locally of finite presentation. Note that $\mathcal{I}_\mathcal{U} = \mathcal{I}_\mathcal{X} \times_\mathcal{X} \mathcal{U}$, see Lemma 74.5.4.

Choose a presentation $\mathcal{X} = [U/R]$. Let $G \to U$ be the stabilizer group algebraic space of the groupoid R. By Lemma 74.5.6 we see that $G \to U$ is the base change of $\mathcal{I}_\mathcal{X} \to \mathcal{X}$ hence quasi-compact (by assumption) and locally of finite type (by Lemma 74.5.1). Let $W \subset U$ be the largest open (possibly empty) subscheme such that the restriction $G_W \to W$ is flat and locally of finite presentation (we omit the proof that W exists; hint: use that the properties are local). By Morphisms of Spaces, Proposition 49.29.1 we see that $W \subset U$ is dense. Note that $W \subset U$ is R-invariant by More on Groupoids in Spaces, Lemma 60.4.2. Hence W corresponds to an open substack $\mathcal{U} \subset \mathcal{X}$ by Properties of Stacks, Lemma 73.9.10. Since $|U| \to |\mathcal{X}|$ is open and $|W| \subset |U|$ is dense we conclude that $\mathcal{U}$ is dense in $\mathcal{X}$. Finally, the morphism $\mathcal{I}_\mathcal{U} \to \mathcal{U}$ is flat and locally of finite presentation because the base change by the surjective smooth morphism $W \to \mathcal{U}$ is the morphism $G_W \to W$ which is flat and locally of finite presentation by construction. See Lemmas 74.17.4 and 74.18.7. □

The above proposition immediately implies that any point has a residual gerbe on an algebraic stack with quasi-compact inertia, as we will show in Lemma 74.21.1. It turns out that there doesn't always exist a finite stratification by gerbes. Here is an example.

Example 74.20.2. Let k be a field. Take $U = \mathrm{Spec}(k[x_0, x_1, x_2, \ldots])$ and let $\mathbf{G}_m$ act by $t(x_0, x_1, x_2, \ldots) = (tx_0, t^p x_1, t^{p^2} x_2, \ldots)$ where p is a prime number. Let $\mathcal{X} = [U/\mathbf{G}_m]$. This is an algebraic stack. There is a stratification of $\mathcal{X}$ by strata

(1) $\mathcal{X}_0$ is where x_0 is not zero,
(2) $\mathcal{X}_1$ is where x_0 is zero but x_1 is not zero,
(3) $\mathcal{X}_2$ is where x_0, x_1 are zero, but x_2 is not zero,
(4) and so on, and
(5) $\mathcal{X}_\infty$ is where all the x_i are zero.

Each stratum is a gerbe over a scheme with group μ_{p^i} for $\mathcal{X}_i$ and $\mathbf{G}_m$ for $\mathcal{X}_\infty$. The strata are reduced locally closed substacks. There is no coarser stratification with the same properties.

Nonetheless, using transfinite induction we can use Proposition 74.20.1 find possibly infinite stratifications by gerbes...!

Lemma 74.20.3. *Let $\mathcal{X}$ be an algebraic stack such that $\mathcal{I}_\mathcal{X} \to \mathcal{X}$ is quasi-compact. Then there exists a well ordered index set I and for every $i \in I$ a reduced locally closed substack $\mathcal{U}_i \subset \mathcal{X}$ such that*

(1) *each $\mathcal{U}_i$ is a gerbe,*
(2) *we have $|\mathcal{X}| = \bigcup_{i \in I} |\mathcal{U}_i|$,*
(3) *$T_i = |\mathcal{X}| \setminus \bigcup_{i' < i} |\mathcal{U}_{i'}|$ is closed in $|\mathcal{X}|$ for all $i \in I$, and*
(4) *$|\mathcal{U}_i|$ is open in T_i.*

We can moreover arrange it so that either (a) $|\mathcal{U}_i| \subset T_i$ is dense, or (b) $\mathcal{U}_i$ is quasi-compact. In case (a), if we choose $\mathcal{U}_i$ as large as possible (see proof for details), then the stratification is canonical.

Proof. Let $T \subset |\mathcal{X}|$ be a nonempty closed subset. We are going to find (resp. choose) for every such T a reduced locally closed substack $\mathcal{U}(T) \subset \mathcal{X}$ with $|\mathcal{U}(T)| \subset T$ open dense (resp. nonempty quasi-compact). Namely, by Properties of Stacks, Lemma 73.10.1 there exists a unique reduced closed substack $\mathcal{X}' \subset \mathcal{X}$ such that $T = |\mathcal{X}'|$. Note that $\mathcal{I}_{\mathcal{X}'} = \mathcal{I}_\mathcal{X} \times_\mathcal{X} \mathcal{X}'$ by Lemma 74.5.5. Hence $\mathcal{I}_{\mathcal{X}'} \to \mathcal{X}'$ is quasi-compact as a base change, see Lemma 74.7.3. Therefore Proposition 74.20.1 implies there exists a dense maximal (see proof proposition) open substack $\mathcal{U} \subset \mathcal{X}'$ which is a gerbe. In case (a) we set $\mathcal{U}(T) = \mathcal{U}$ (this is canonical) and in case (b) we simply choose a nonempty quasi-compact open $\mathcal{U}(T) \subset \mathcal{U}$, see Properties of Stacks, Lemma 73.4.9 (we can do this for all T simultaneously by the axiom of choice).

By transfinite induction we construct for every ordinal α a closed subset $T_\alpha \subset |\mathcal{X}|$. For $\alpha = 0$ we set $T_0 = |\mathcal{X}|$. Given T_α set

$$T_{\alpha+1} = T_\alpha \setminus |\mathcal{U}(T_\alpha)|.$$

If β is a limit ordinal we set

$$T_\beta = \bigcap\nolimits_{\alpha < \beta} T_\alpha.$$

We claim that $T_\alpha = \emptyset$ for all α large enough. Namely, assume that $T_\alpha \neq \emptyset$ for all α. Then we obtain an injective map from the class of ordinals into the set of subsets of $|\mathcal{X}|$ which is a contradiction.

The claim implies the lemma. Namely, let

$$I = \{\alpha \mid \mathcal{U}_\alpha \neq \emptyset\}.$$

This is a well ordered set by the claim. For $i = \alpha \in I$ we set $\mathcal{U}_i = \mathcal{U}_\alpha$. So $\mathcal{U}_i$ is a reduced locally closed substack and a gerbe, i.e., (1) holds. By construction $T_i = T\alpha$ if $i = \alpha \in I$, hence (3) holds. Also, (4) and (a) or (b) hold by our choice of $\mathcal{U}(T)$ as well. Finally, to see (2) let $x \in |\mathcal{X}|$. There exists a smallest ordinal β with $x \notin T_\beta$ (because the ordinals are well-ordered). In this case β has to be a successor ordinal by the definition of T_β for limit ordinals. Hence $\beta = \alpha + 1$ and $x \in |\mathcal{U}(T_\alpha)|$ and we win. □

Remark 74.20.4. We can wonder about the order type of the canonical stratifications which occur as output of the stratifications of type (a) constructed in Lemma 74.20.3. A natural guess is that the well ordered set I has *cardinality* at most $\aleph_0$. We have no idea if this is true or false. If you do please email stacks.project@gmail.com.

74.21. Existence of residual gerbes

In this section we prove that residual gerbes (as defined in Properties of Stacks, Definition 73.11.8) exist on many algebraic stacks. First, here is the promised application of Proposition 74.20.1.

Lemma 74.21.1. *Let $\mathcal{X}$ be an algebraic stack such that $\mathcal{I}_\mathcal{X} \to \mathcal{X}$ is quasi-compact. Then the residual gerbe of $\mathcal{X}$ at x exists for every $x \in |\mathcal{X}|$.*

Proof. Let $T = \overline{\{x\}} \subset |\mathcal{X}|$ be the closure of x. By Properties of Stacks, Lemma 73.10.1 there exists a reduced closed substack $\mathcal{X}' \subset \mathcal{X}$ such that $T = |\mathcal{X}'|$. Note that $\mathcal{I}_{\mathcal{X}'} = \mathcal{I}_\mathcal{X} \times_\mathcal{X} \mathcal{X}'$ by Lemma 74.5.5. Hence $\mathcal{I}_{\mathcal{X}'} \to \mathcal{X}'$ is quasi-compact as a base change, see Lemma 74.7.3. Therefore Proposition 74.20.1 implies there exists a dense open substack $\mathcal{U} \subset \mathcal{X}'$ which is a gerbe. Note that $x \in |\mathcal{U}|$ because $\{x\} \subset T$ is a dense subset too. Hence a residual gerbe $\mathcal{Z}_x \subset \mathcal{U}$ of $\mathcal{U}$ at x exists by Lemma 74.19.12. It is immediate from the definitions that $\mathcal{Z}_x \to \mathcal{X}$ is a residual gerbe of $\mathcal{X}$ at x. □

If the stack is quasi-DM then residual gerbes exist too. In particular, residual gerbes always exist for Delinge-Mumford stacks.

Lemma 74.21.2. *Let $\mathcal{X}$ be a quasi-DM algebraic stack. Then the residual gerbe of $\mathcal{X}$ at x exists for every $x \in |\mathcal{X}|$.*

Proof. Choose a scheme U and a surjective, flat, locally finite presented, and locally quasi-finite morphism $U \to \mathcal{X}$, see Theorem 74.15.3. Set $R = U \times_\mathcal{X} U$. The projections $s, t : R \to U$ are surjective, flat, locally of finite presentation, and locally quasi-finite as base changes of the morphism $U \to \mathcal{X}$. There is a canonical morphism $[U/R] \to \mathcal{X}$ (see Algebraic Stacks, Lemma 67.16.1) which is an equivalence because $U \to \mathcal{X}$ is surjective, flat, and locally of finite presentation, see Algebraic Stacks, Remark 67.16.3. Thus we may assume that $\mathcal{X} = [U/R]$ where (U, R, s, t, c) is a groupoid in algebraic spaces such that $s, t : R \to U$ are surjective, flat, locally of finite presentation, and locally quasi-finite. Set

$$U' = \coprod\nolimits_{u \in U \text{ lying over } x} \operatorname{Spec}(\kappa(u)).$$

The canonical morphism $U' \to U$ is a monomorphism. Let

$$R' = U' \times_\mathcal{X} U' = R \times_{(U \times U)} (U' \times U')$$

Because $U' \to U$ is a monomorphism we see that both projections $s', t' : R' \to U'$ factor as a monomorphism followed by a locally quasi-finite morphism. Hence, as U' is a disjoint union of spectra of fields, using Spaces over Fields, Lemma 58.4.3 we conclude that the morphisms $s', t' : R' \to U'$ are locally quasi-finite. Again since U' is a disjoint union of spectra of fields, the morphisms s', t' are also flat. Finally, s', t' locally quasi-finite implies s', t' locally of finite type, hence s', t' locally of finite presentation (because U' is a disjoint union of spectra of fields in particular locally Noetherian, so that Morphisms of Spaces, Lemma 49.26.7 applies). Hence $\mathcal{Z} = [U'/R']$ is an algebraic stack by Criteria for Representability, Theorem 70.17.2. As R' is the restriction of R by $U' \to U$ we see $\mathcal{Z} \to \mathcal{X}$ is a monomorphism by Groupoids in Spaces, Lemma 59.24.1 and Properties of Stacks, Lemma 73.8.4. Since $\mathcal{Z} \to \mathcal{X}$ is a monomorphism we see that $|\mathcal{Z}| \to |\mathcal{X}|$ is injective, see Properties of Stacks, Lemma 73.8.5. By Properties of Stacks, Lemma 73.4.3 we see that

$$|U'| = |\mathcal{Z} \times_\mathcal{X} U'| \longrightarrow |\mathcal{Z}| \times_{|\mathcal{X}|} |U'|$$

is surjective which implies (by our choice of U') that $|\mathcal{Z}| \to |\mathcal{X}|$ has image $\{x\}$. We conclude that $|\mathcal{Z}|$ is a singleton. Finally, by construction U' is locally Noetherian and reduced, i.e., $\mathcal{Z}$ is reduced and locally Noetherian. This means that the essential image of $\mathcal{Z} \to \mathcal{X}$ is the residual gerbe of $\mathcal{X}$ at x, see Properties of Stacks, Lemma 73.11.11. □

74.22. Smooth morphisms

The property "being smooth" of morphisms of algebraic spaces is smooth local on the source-and-target, see Descent on Spaces, Remark 55.18.5. It is also stable under base change and fpqc local on the target, see Morphisms of Spaces, Lemma 49.33.3 and Descent on Spaces, Lemma 55.10.24. Hence, by Lemma 74.12.1 above, we may define what it means for a morphism of algebraic spaces to be smooth as follows and it agrees with the already existing notion defined in Properties of Stacks, Section 73.3 when the morphism is representable by algebraic spaces.

Definition 74.22.1. Let $f : \mathcal{X} \to \mathcal{Y}$ be a morphism of algebraic stacks. We say f is *smooth* if the equivalent conditions of Lemma 74.12.1 hold with $\mathcal{P} = \text{smooth}$.

Lemma 74.22.2. *The composition of smooth morphisms is smooth.*

Proof. Combine Remark 74.12.3 with Morphisms of Spaces, Lemma 49.33.2. □

Lemma 74.22.3. *A base change of a smooth morphism is smooth.*

Proof. Combine Remark 74.12.4 with Morphisms of Spaces, Lemma 49.33.3. □

74.23. Other chapters

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CHAPTER 75

Cohomology of Algebraic Stacks

75.1. Introduction

In this chapter we write about cohomology of algebraic stacks. This mean in particular cohomology of quasi-coherent sheaves, i.e., we prove analogues of the results in the chapters entitled "Cohomology of Schemes" and "Cohomology of Algebraic Spaces". The results in this chapter are different from those in [**LMB00**] mainly because we consistently use the "big sites". Before reading this chapter please take a quick look at the chapter "Sheaves on Algebraic Stacks" in order to become familiar with the terminology introduced there, see Sheaves on Stacks, Section 69.1.

75.2. Conventions and abuse of language

We continue to use the conventions and the abuse of language introduced in Properties of Stacks, Section 73.2.

75.3. Notation

Different topologies. If we indicate an algebraic stack by a calligraphic letter, such as $\mathcal{X}, \mathcal{Y}, \mathcal{Z}$, then the notation $\mathcal{X}_{Zar}, \mathcal{X}_{étale}, \mathcal{X}_{smooth}, \mathcal{X}_{syntomic}, \mathcal{X}_{fppf}$ indicates the site introduced in Sheaves on Stacks, Definition 69.4.1. (Think "big site".) Correspondingly the structure sheaf of $\mathcal{X}$ is a sheaf on $\mathcal{X}_{fppf}$. On the other hand, algebraic spaces and schemes are usually indicated by roman capitals, such as X, Y, Z, and in this case $X_{étale}$ indicates the small étale site of X (as defined in Topologies, Definition 33.4.8 or Properties of Spaces, Definition 48.15.1). It seems that the distinction should be clear enough.

The default topology is the fppf topology. Hence we will sometimes say "sheaf on $\mathcal{X}$" or "sheaf of $\mathcal{O}_\mathcal{X}$" modules when we mean sheaf on $\mathcal{X}_{fppf}$ or object of $Mod(\mathcal{X}_{fppf}, \mathcal{O}_\mathcal{X})$.

If $f : \mathcal{X} \to \mathcal{Y}$ is a morphism of algebraic stacks, then the functors f_* and f^{-1} defined on presheaves preserves sheaves for any of the topologies mentioned above. In particular when we discuss the pushforward or pullback of a sheaf we don't have to mention which topology we are working with. The same isn't true when we compute cohomology groups and/or higher direct images. In this case we will always mention which topology we are working with.

Suppose that $f : X \to \mathcal{Y}$ is a morphism from an algebraic space X to an algebraic stack $\mathcal{Y}$. Let $\mathcal{G}$ be a sheaf on $\mathcal{Y}_\tau$ for some topology τ. In this case $f^{-1}\mathcal{G}$ is a sheaf for the τ topology on $\mathcal{S}_X$ (the algebraic stack associated to X) because (by our conventions) f really is a 1-morphism $f : \mathcal{S}_X \to \mathcal{Y}$. If $\tau = étale$ or stronger, then we write $f^{-1}\mathcal{G}|_{X_{étale}}$ to denote the restriction to the étale site of X, see Sheaves on

Stacks, Section 69.21. If $\mathcal{G}$ is an $\mathcal{O}_\mathcal{X}$-module we sometimes write $f^*\mathcal{G}$ and $f^*\mathcal{G}|_{X_{étale}}$ instead.

75.4. Pullback of quasi-coherent modules

Let $f : \mathcal{X} \to \mathcal{Y}$ be a morphism of algebraic stacks. It is a very general fact that quasi-coherent modules on ringed topoi are compatible with pullbacks. In particular the pullback f^* preserves quasi-coherent modules and we obtain a functor

$$f^* : QCoh(\mathcal{O}_\mathcal{Y}) \longrightarrow QCoh(\mathcal{O}_\mathcal{X}),$$

see Sheaves on Stacks, Lemma 69.11.2. In general this functor isn't exact, but if f is flat then it is.

Lemma 75.4.1. *If $f : \mathcal{X} \to \mathcal{Y}$ is a flat morphism of algebraic stacks then $f^* : QCoh(\mathcal{O}_\mathcal{Y}) \to QCoh(\mathcal{O}_\mathcal{X})$ is an exact functor.*

Proof. Choose a scheme V and a surjective smooth morphism $V \to \mathcal{Y}$. Choose a scheme U and a surjective smooth morphism $U \to V \times_\mathcal{Y} \mathcal{X}$. Then $U \to \mathcal{X}$ is still smooth and surjective as a composition of two such morphisms. From the commutative diagram

$$\begin{array}{ccc} U & \xrightarrow{f'} & V \\ \downarrow & & \downarrow \\ \mathcal{X} & \xrightarrow{f} & \mathcal{Y} \end{array}$$

we obtain a commutative diagram

$$\begin{array}{ccc} QCoh(\mathcal{O}_U) & \longleftarrow & QCoh(\mathcal{O}_V) \\ \uparrow & & \uparrow \\ QCoh(\mathcal{O}_\mathcal{X}) & \longleftarrow & QCoh(\mathcal{O}_\mathcal{Y}) \end{array}$$

of abelian categories. Our proof that the bottom two categories in this diagram are abelian showed that the vertical functors are faithful exact functors (see proof of Sheaves on Stacks, Lemma 69.14.1). Since f' is a flat morphism of schemes (by our definition of flat morphisms of algebraic stacks) we see that $(f')^*$ is an exact functor on quasi-coherent sheaves on V. Thus we win. □

75.5. The key lemma

The following lemma is the basis for our understanding of higher direct images of certain types of sheaves of modules. There are two versions: one for the étale topology and one for the fppf topology.

Lemma 75.5.1. *Let $\mathcal{M}$ be a rule which associates to every algebraic stack $\mathcal{X}$ a subcategory $\mathcal{M}_\mathcal{X}$ of $Mod(\mathcal{X}_{étale}, \mathcal{O}_\mathcal{X})$ such that*

1. *$\mathcal{M}_\mathcal{X}$ is a weak Serre subcategory of $Mod(\mathcal{X}_{étale}, \mathcal{O}_\mathcal{X})$ (see Homology, Definition 12.9.1) for all algebraic stacks $\mathcal{X}$,*
2. *for a smooth morphism of algebraic stacks $f : \mathcal{Y} \to \mathcal{X}$ the functor f^* maps $\mathcal{M}_\mathcal{X}$ into $\mathcal{M}_\mathcal{Y}$,*
3. *if $f_i : \mathcal{X}_i \to \mathcal{X}$ is a family of smooth morphisms of algebraic stacks with $|\mathcal{X}| = \bigcup |f_i|(|\mathcal{X}_i|)$, then an object $\mathcal{F}$ of $Mod(\mathcal{X}_{étale}, \mathcal{O}_\mathcal{X})$ is in $\mathcal{M}_\mathcal{X}$ if and only if $f_i^*\mathcal{F}$ is in $\mathcal{M}_{\mathcal{X}_i}$ for all i, and*

(4) *if $f : \mathcal{Y} \to \mathcal{X}$ is a morphism of algebraic stacks such that $\mathcal{X}$ and $\mathcal{Y}$ are representable by affine schemes, then $R^i f_*$ maps $\mathcal{M}_\mathcal{Y}$ into $\mathcal{M}_\mathcal{X}$.*

Then for any quasi-compact and quasi-separated morphism $f : \mathcal{Y} \to \mathcal{X}$ of algebraic stacks $R^i f_*$ maps $\mathcal{M}_\mathcal{Y}$ into $\mathcal{M}_\mathcal{X}$. (Higher direct images computed in étale topology.)

Proof. Let $f : \mathcal{Y} \to \mathcal{X}$ be a quasi-compact and quasi-separated morphism of algebraic stacks and let $\mathcal{F}$ be an object of $\mathcal{M}_\mathcal{Y}$. Choose a surjective smooth morphism $\mathcal{U} \to \mathcal{X}$ where $\mathcal{U}$ is representable by a scheme. By Sheaves on Stacks, Lemma 69.20.3 taking higher direct images commutes with base change. Assumption (2) shows that the pullback of $\mathcal{F}$ to $\mathcal{U} \times_\mathcal{X} \mathcal{Y}$ is in $\mathcal{M}_{\mathcal{U}\times_\mathcal{X}\mathcal{Y}}$ because the projection $\mathcal{U} \times_\mathcal{X} \mathcal{Y} \to \mathcal{Y}$ is smooth as a base change of a smooth morphism. Hence (3) shows we may replace $\mathcal{Y} \to \mathcal{X}$ by the projection $\mathcal{U} \times_\mathcal{X} \mathcal{Y} \to \mathcal{U}$. In other words, we may assume that $\mathcal{X}$ is representable by a scheme. Using (3) once more, we see that the question is Zariski local on $\mathcal{X}$, hence we may assume that $\mathcal{X}$ is representable by an affine scheme. Since f is quasi-compact this implies that also $\mathcal{Y}$ is quasi-compact. Thus we may choose a surjective smooth morphism $g : \mathcal{V} \to \mathcal{Y}$ where $\mathcal{V}$ is representable by an affine scheme.

In this situation we have the spectral sequence

$$E_2^{p,q} = R^q(f \circ g_p)_* g_p^* \mathcal{F} \Rightarrow R^{p+q} f_* \mathcal{F}$$

of Sheaves on Stacks, Proposition 69.20.1. Recall that this is the spectral sequence associated to a double complex. By assumption (1) we may use Homology, Remark 12.22.7. Note that the morphisms

$$g_p : \mathcal{V}_p = \mathcal{V} \times_\mathcal{Y} \ldots \times_\mathcal{Y} \mathcal{V} \longrightarrow \mathcal{Y}$$

are smooth as compositions of base changes of the smooth morphism g. Thus the sheaves $g_p^* \mathcal{F}$ are in $\mathcal{M}_{\mathcal{V}_p}$ by (2). Hence it suffices to prove that the higher direct images of objects of $\mathcal{M}_{\mathcal{V}_p}$ under the morphisms

$$\mathcal{V}_p = \mathcal{V} \times_\mathcal{Y} \ldots \times_\mathcal{Y} \mathcal{V} \longrightarrow \mathcal{X}$$

are in $\mathcal{M}_\mathcal{X}$. The algebraic stacks $\mathcal{V}_p$ are quasi-compact and quasi-separated by Morphisms of Stacks, Lemma 74.7.7. Of course each $\mathcal{V}_p$ is representable by an algebraic space (the diagonal of the algebraic stack $\mathcal{Y}$ is representable by algebraic spaces). This reduces us to the case where $\mathcal{Y}$ is representable by an algebraic space and $\mathcal{X}$ is representable by an affine scheme.

In the situation where $\mathcal{Y}$ is representable by an algebraic space and $\mathcal{X}$ is representable by an affine scheme, we choose anew a surjective smooth morphism $\mathcal{V} \to \mathcal{Y}$ where $\mathcal{V}$ is representable by an affine scheme. Going through the argument above once again we once again reduce to the morphisms $\mathcal{V}_p \to \mathcal{X}$. But in the current situation the algebraic stacks $\mathcal{V}_p$ are representable by quasi-compact and quasi-separated schemes (bacause the diagonal of an algebraic space is representable by schemes).

Thus we may assume $\mathcal{Y}$ is representable by a scheme and $\mathcal{X}$ is representable by an affine scheme. Choose (again) a surjective smooth morphism $\mathcal{V} \to \mathcal{Y}$ where $\mathcal{V}$ is representable by an affine scheme. In this case all the algebraic stacks $\mathcal{V}_p$ are representable by separated schemes (because the diagonal of a scheme is separated).

Thus we may assume $\mathcal{Y}$ is representable by a separated scheme and $\mathcal{X}$ is representable by an affine scheme. Choose (yet again) a surjective smooth morphism

$\mathcal{V} \to \mathcal{Y}$ where $\mathcal{V}$ is representable by an affine scheme. In this case all the algebraic stacks $\mathcal{V}_p$ are representable by affine schemes (because the diagonal of a separated scheme is a closed immersion hence affine) and this case is handled by assumption (4). This finishes the proof. □

Here is the version for the fppf topology.

Lemma 75.5.2. *Let $\mathcal{M}$ be a rule which associates to every algebraic stack $\mathcal{X}$ a subcategory $\mathcal{M}_\mathcal{X}$ of $Mod(\mathcal{O}_\mathcal{X})$ such that*

1. *$\mathcal{M}_\mathcal{X}$ is a weak Serre subcategory of $Mod(\mathcal{O}_\mathcal{X})$ for all algebraic stacks $\mathcal{X}$,*
2. *for a smooth morphism of algebraic stacks $f : \mathcal{Y} \to \mathcal{X}$ the functor f^* maps $\mathcal{M}_\mathcal{X}$ into $\mathcal{M}_\mathcal{Y}$,*
3. *if $f_i : \mathcal{X}_i \to \mathcal{X}$ is a family of smooth morphisms of algebraic stacks with $|\mathcal{X}| = \bigcup |f_i|(|\mathcal{X}_i|)$, then an object $\mathcal{F}$ of $Mod(\mathcal{O}_\mathcal{X})$ is in $\mathcal{M}_\mathcal{X}$ if and only if $f_i^*\mathcal{F}$ is in $\mathcal{M}_{\mathcal{X}_i}$ for all i, and*
4. *if $f : \mathcal{Y} \to \mathcal{X}$ is a morphism of algebraic stacks and $\mathcal{X}$ and $\mathcal{Y}$ are representable by affine schemes, then R^if_* maps $\mathcal{M}_\mathcal{Y}$ into $\mathcal{M}_\mathcal{X}$.*

Then for any quasi-compact and quasi-separated morphism $f : \mathcal{Y} \to \mathcal{X}$ of algebraic stacks R^if_* maps $\mathcal{M}_\mathcal{Y}$ into $\mathcal{M}_\mathcal{X}$. (Higher direct images computed in fppf topology.)

Proof. Identical to the proof of Lemma 75.5.1. □

75.6. Locally quasi-coherent modules

Let $\mathcal{X}$ be an algebraic stack. Let $\mathcal{F}$ be a presheaf of $\mathcal{O}_\mathcal{X}$-modules. We can ask whether $\mathcal{F}$ is *locally quasi-coherent*, see Sheaves on Stacks, Definition 69.11.4. Briefly, this means $\mathcal{F}$ is an $\mathcal{O}_\mathcal{X}$-module for the étale topology such that for any morphism $f : U \to \mathcal{X}$ the restriction $f^*\mathcal{F}|_{U_{étale}}$ is quasi-coherent on $U_{étale}$. (The actual definition is slightly different, but equivalent.) A useful fact is that

$$LQCoh(\mathcal{O}_\mathcal{X}) \subset Mod(\mathcal{X}_{étale}, \mathcal{O}_\mathcal{X})$$

is a weak Serre subcategory, see Sheaves on Stacks, Lemma 69.11.7.

Lemma 75.6.1. *Let $\mathcal{X}$ be an algebraic stack. Let $f_j : \mathcal{X}_j \to \mathcal{X}$ be a family of smooth morphisms of algebraic stacks with $|\mathcal{X}| = \bigcup |f_j|(|\mathcal{X}_j|)$. Let $\mathcal{F}$ be a sheaf of $\mathcal{O}_\mathcal{X}$-modules on $\mathcal{X}_{étale}$. If each $f_j^{-1}\mathcal{F}$ is locally quasi-coherent, then so is $\mathcal{F}$.*

Proof. We may replace each of the algebraic stacks $\mathcal{X}_j$ by a scheme U_j (using that any algebraic stack has a smooth covering by a scheme and that compositions of smooth morphisms are smooth, see Morphisms of Stacks, Lemma 74.22.2). The pullback of $\mathcal{F}$ to $(Sch/U_j)_{étale}$ is still locally quasi-coherent, see Sheaves on Stacks, Lemma 69.11.6. Then $f = \coprod f_j : U = \coprod U_j \to \mathcal{X}$ is a surjective smooth morphism. Let x be an object of $\mathcal{X}$. By Sheaves on Stacks, Lemma 69.18.10 there exists an étale covering $\{x_i \to x\}_{i \in I}$ such that each x_i lifts to an object u_i of $(Sch/U)_{étale}$. This just means that x, x_i live over schemes V, V_i, that $\{V_i \to V\}$ is an étale covering, and that x_i comes from a morphism $u_i : V_i \to U$. The restriction $x_i^*\mathcal{F}|_{V_{i,étale}}$ is equal to the restriction of $f^*\mathcal{F}$ to $V_{i,étale}$, see Sheaves on Stacks, Lemma 69.9.3. Hence $x^*\mathcal{F}|_{V_{étale}}$ is a sheaf on the small étale site of V which is quasi-coherent when restricted to $V_{i,étale}$ for each i. This implies that it is quasi-coherent (as desired), for example by Properties of Spaces, Lemma 48.27.6. □

Lemma 75.6.2. *Let $f : \mathcal{X} \to \mathcal{Y}$ be a quasi-compact and quasi-separated morphism of algebraic stacks. Let $\mathcal{F}$ be a locally quasi-coherent $\mathcal{O}_\mathcal{X}$-module on $\mathcal{X}_{étale}$. Then $R^i f_* \mathcal{F}$ (computed in the étale topology) is a locally quasi-coherent on $\mathcal{Y}_{étale}$.*

Proof. We will use Lemma 75.5.1 to prove this. We will check its assumptions (1) – (4). Parts (1) and (2) follows from Sheaves on Stacks, Lemma 69.11.7. Part (3) follows from Lemma 75.6.1. Thus it suffices to show (4).

Suppose $f : \mathcal{X} \to \mathcal{Y}$ is a morphism of algebraic stacks such that $\mathcal{X}$ and $\mathcal{Y}$ are representable by affine schemes X and Y. Choose any object y of $\mathcal{Y}$ lying over a scheme V. For clarity, denote $\mathcal{V} = (Sch/V)_{fppf}$ the algebraic stack corresponding to V. Consider the cartesian diagram

$$\begin{array}{ccc} \mathcal{Z} & \xrightarrow{g} & \mathcal{X} \\ {\scriptstyle f'}\downarrow & & \downarrow{\scriptstyle f} \\ \mathcal{V} & \xrightarrow{y} & \mathcal{Y} \end{array}$$

Thus $\mathcal{Z}$ is representable by the scheme $Z = V \times_Y X$ and f' is quasi-compact and separated (even affine). By Sheaves on Stacks, Lemma 69.21.3 we have

$$R^i f_* \mathcal{F}|_{V_{étale}} = R^i f'_{small,*}\big(g^* \mathcal{F}|_{Z_{étale}}\big)$$

The right hand side is a quasi-coherent sheaf on $V_{étale}$ by Cohomology of Spaces, Lemma 51.3.1. This implies the left hand side is quasi-coherent which is what we had to prove. □

Lemma 75.6.3. *Let $\mathcal{X}$ be an algebraic stack. Let $f_j : \mathcal{X}_j \to \mathcal{X}$ be a family of flat and locally finitely presented morphisms of algebraic stacks with $|\mathcal{X}| = \bigcup |f_j|(|\mathcal{X}_j|)$. Let $\mathcal{F}$ be a sheaf of $\mathcal{O}_\mathcal{X}$-modules on $\mathcal{X}_{fppf}$. If each $f_j^{-1}\mathcal{F}$ is locally quasi-coherent, then so is $\mathcal{F}$.*

Proof. First, suppose there is a morphism $a : \mathcal{U} \to \mathcal{X}$ which is surjective, flat, locally of finite presentation, quasi-compact, and quasi-separated such that $a^*\mathcal{F}$ is locally quasi-coherent. Then there is an exact sequence

$$0 \to \mathcal{F} \to a_* a^* \mathcal{F} \to b_* b^* \mathcal{F}$$

where b is the morphism $b : \mathcal{U} \times_\mathcal{X} \mathcal{U} \to \mathcal{X}$, see Sheaves on Stacks, Proposition 69.18.7 and Lemma 69.18.10. Moreover, the pullback $b^*\mathcal{F}$ is the pullback of $a^*\mathcal{F}$ via one of the projection morphisms, hence is locally quasi-coherent (Sheaves on Stacks, Lemma 69.11.6). The modules $a_* a^* \mathcal{F}$ and $b_* b^* \mathcal{F}$ are locally quasi-coherent by Lemma 75.6.2. (Note that a_* and b_* don't care about which topology is used to calculate them.) We conclude that $\mathcal{F}$ is locally quasi-coherent, see Sheaves on Stacks, Lemma 69.11.7.

We are going to reduce the proof of the general case the the situation in the first paragraph. Let x be an object of $\mathcal{X}$ lying over the scheme U. We have to show that $\mathcal{F}|_{U_{étale}}$ is a quasi-coherent $\mathcal{O}_U$-module. It suffices to do this (Zariski) locally on U, hence we may assume that U is affine. By Morphisms of Stacks, Lemma 74.18.10 there exists an fppf covering $\{a_i : U_i \to U\}$ such that each $x \circ a_i$ factors through some f_j. Hence $a_i^* \mathcal{F}$ is locally quasi-coherent on $(Sch/U_i)_{fppf}$. After refining the covering we may assume $\{U_i \to U\}_{i=1,\ldots,n}$ is a standard fppf covering. Then $x^*\mathcal{F}$ is an fppf module on $(Sch/U)_{fppf}$ whose pullback by the morphism $a : U_1 \amalg \ldots \amalg U_n \to U$

is locally quasi-coherent. Hence by the first paragraph we see that $x^*\mathcal{F}$ is locally quasi-coherent, which certainly implies that $\mathcal{F}|_{U_{étale}}$ is quasi-coherent. □

75.7. Flat comparison maps

Let $\mathcal{X}$ be an algebraic stack and let $\mathcal{F}$ be an object of $Mod(\mathcal{X}_{étale}, \mathcal{O}_\mathcal{X})$. Given an object x of $\mathcal{X}$ lying over the scheme U the restriction $\mathcal{F}|_{U_{étale}}$ is the restriction of $x^{-1}\mathcal{F}$ to the small étale site of U, see Sheaves on Stacks, Definition 69.9.2. Next, let $\varphi : x \to x'$ be a morphism of $\mathcal{X}$ lying over a morphism of schemes $f : U \to U'$. Thus a 2-commutative diagram

$$\begin{array}{ccc} U & \xrightarrow{\quad f \quad} & U' \\ & {\scriptstyle x}\searrow \quad \swarrow{\scriptstyle x'} & \\ & \mathcal{X} & \end{array}$$

Associated to φ we obtain a comparison map between restrictions

$$c_\varphi : f^*_{small}(\mathcal{F}|_{U'_{étale}}) \longrightarrow \mathcal{F}|_{U_{étale}} \tag{75.7.0.1}$$

see Sheaves on Stacks, Equation (69.9.4.1). In this situation we can consider the following property of $\mathcal{F}$.

Definition 75.7.1. Let $\mathcal{X}$ be an algebraic stack and let $\mathcal{F}$ in $Mod(\mathcal{X}_{étale}, \mathcal{O}_\mathcal{X})$. We say $\mathcal{F}$ has the *flat base change property*[1] if and only if c_φ is an isomorphism whenever f is flat.

Here is a lemma with some properties of this notion.

Lemma 75.7.2. *Let $\mathcal{X}$ be an algebraic stack. Let $\mathcal{F}$ be an $\mathcal{O}_\mathcal{X}$-module on $\mathcal{X}_{étale}$.*

1. *If $\mathcal{F}$ has the flat base change property then for any morphism $g : \mathcal{Y} \to \mathcal{X}$ of algebraic stacks, the pullback $g^*\mathcal{F}$ does too.*
2. *The full subcategory of $Mod(\mathcal{X}_{étale}, \mathcal{O}_\mathcal{X})$ consisting of modules with the flat base change property is a weak Serre subcategory.*
3. *Let $f_i : \mathcal{X}_i \to \mathcal{X}$ be a family of smooth morphisms of algebraic stacks such that $|\mathcal{X}| = \bigcup_i |f_i|(|\mathcal{X}_i|)$. If each $f_i^*\mathcal{F}$ has the flat base change property then so does $\mathcal{F}$.*
4. *The category of $\mathcal{O}_\mathcal{X}$-modules on $\mathcal{X}_{étale}$ with the flat base change property has colimits and they agree with colimits in $Mod(\mathcal{X}_{étale}, \mathcal{O}_\mathcal{X})$.*

Proof. Let $g : \mathcal{Y} \to \mathcal{X}$ be as in (1). Let y be an object of $\mathcal{Y}$ lying over a scheme V. By Sheaves on Stacks, Lemma 69.9.3 we have $(g^*\mathcal{F})|_{V_{étale}} = \mathcal{F}|_{V_{étale}}$. Moreover a comparison mapping for the sheaf $g^*\mathcal{F}$ on $\mathcal{Y}$ is a special case of a comparison map for the sheaf $\mathcal{F}$ on $\mathcal{X}$, see Sheaves on Stacks, Lemma 69.9.3. In this way (1) is clear.

Proof of (2). We use the characterization of weak Serre subcategories of Homology, Lemma 12.9.3. Kernels and cokernels of maps between sheaves having the flat base change property also have the flat base change property. This is clear because f^*_{small} is exact for a flat morphism of schemes and since the restriction functors $(-)|_{U_{étale}}$ are exact (because we are working in the étale topology). Finally, if $0 \to \mathcal{F}_1 \to \mathcal{F}_2 \to \mathcal{F}_3 \to 0$ is a short exact sequence of $Mod(\mathcal{X}_{étale}, \mathcal{O}_\mathcal{X})$ and the outer two sheaves have the flat base change property then the middle one does as

[1] This may be nonstandard notation.

well, again because of the exactness of f_{small}^* and the restriction functors (and the 5 lemma).

Proof of (3). Let $f_i : \mathcal{X}_i \to \mathcal{X}$ be a jointly surjective family of smooth morphisms of algebraic stacks and assume each $f_i^*\mathcal{F}$ has the flat base change property. By part (1), the definition of an algebraic stack, and the fact that compositions of smooth morphisms are smooth (see Morphisms of Stacks, Lemma 74.22.2) we may assume that each $\mathcal{X}_i$ is representable by a scheme. Let $\varphi : x \to x'$ be a morphism of $\mathcal{X}$ lying over a flat morphism $a : U \to U'$ of schemes. By Sheaves on Stacks, Lemma 69.18.10 there exists a jointly surjective family of étale morphisms $U_i' \to U'$ such that $U' \to U' \to \mathcal{X}$ factors through $\mathcal{X}_i$. Thus we obtain commutative diagrams

$$\begin{array}{ccccc} U_i = U \times_{U'} U_i' & \xrightarrow{a_i} & U_i' & \xrightarrow{x_i'} & \mathcal{X}_i \\ \downarrow & & \downarrow & & \downarrow f_i \\ U & \xrightarrow{a} & U' & \xrightarrow{x'} & \mathcal{X} \end{array}$$

Note that each a_i is a flat morphism of schemes as a base change of a. Denote $\psi_i : x_i \to x_i'$ the morphism of $\mathcal{X}_i$ lying over a_i with target x_i'. By assumption the comparison maps $c_{\psi_i} : (a_i)_{small}^*\big(f_i^*\mathcal{F}|_{(U_i')_{étale}}\big) \to f_i^*\mathcal{F}|_{(U_i)_{étale}}$ is an isomorphism. Because the vertical arrows $U_i' \to U'$ and $U_i \to U$ are étale, the sheaves $f_i^*\mathcal{F}|_{(U_i')_{étale}}$ and $f_i^*\mathcal{F}|_{(U_i)_{étale}}$ are the restrictions of $\mathcal{F}|_{U'_{étale}}$ and $\mathcal{F}|_{U_{étale}}$ and the map c_{ψ_i} is the restriction of c_φ to $(U_i)_{étale}$, see Sheaves on Stacks, Lemma 69.9.3. Since $\{U_i \to U\}$ is an étale covering, this implies that the comparison map c_φ is an isomorphism which is what we wanted to prove.

Proof of (4). Let $\mathcal{I} \to Mod(\mathcal{X}_{étale}, \mathcal{O}_\mathcal{X})$, $i \mapsto \mathcal{F}_i$ be a diagram and assume each $\mathcal{F}_i$ has the flat base change property. Recall that $\operatorname{colim}_i \mathcal{F}_i$ is the sheafification of the presheaf colimit. As we are using the étale topology, it is clear that

$$(\operatorname{colim}_i \mathcal{F}_i)|_{U_{étale}} = \operatorname{colim}_i \mathcal{F}_i|_{U_{étale}}$$

As f_{small}^* commutes with colimits (as a left adjoint) we see that (4) holds. □

Lemma 75.7.3. *Let $f : \mathcal{X} \to \mathcal{Y}$ be a quasi-compact and quasi-separated morphism of algebraic stacks. Let $\mathcal{F}$ be an object of $Mod(\mathcal{X}_{étale}, \mathcal{O}_\mathcal{X})$ which is locally quasi-coherent and has the flat base change property. Then each $R^i g_*\mathcal{F}$ (computed in the étale topology) has the flat base change property.*

Proof. We will use Lemma 75.5.1 to prove this. For every algebraic stack $\mathcal{X}$ let $\mathcal{M}_\mathcal{X}$ denote the full subcategory of $Mod(\mathcal{X}_{étale}, \mathcal{O}_\mathcal{X})$ consisting of locally quasi-coherent sheaves with the flat base change property. Once we verify conditions (1) – (4) of Lemma 75.5.1 the lemma will follow. Properties (1), (2), and (3) follow from Sheaves on Stacks, Lemmas 69.11.6 and 69.11.7 and Lemmas 75.6.1 and 75.7.2. Thus it suffices to show part (4).

Suppose $f : \mathcal{X} \to \mathcal{Y}$ is a morphism of algebraic stacks such that $\mathcal{X}$ and $\mathcal{Y}$ are representable by affine schemes X and Y. In this case, suppose that $\psi : y \to y'$ is a morphism of $\mathcal{Y}$ lying over a flat morphism $b : V \to V'$ of schemes. For clarity denote $\mathcal{V} = (Sch/V)_{fppf}$ and $\mathcal{V}' = (Sch/V')_{fppf}$ the corresponding algebraic stacks.

Consider the diagram of algebraic stacks

$$\begin{array}{ccccc} \mathcal{Z} & \xrightarrow{a} & \mathcal{Z}' & \xrightarrow{x'} & \mathcal{X} \\ {\scriptstyle f''}\downarrow & & {\scriptstyle f'}\downarrow & & \downarrow{\scriptstyle f} \\ \mathcal{V} & \xrightarrow{b} & \mathcal{V}' & \xrightarrow{y'} & \mathcal{Y} \end{array}$$

with both squares cartesian. As f is representable by schemes (and quasi-compact and separated – even affine) we see that $\mathcal{Z}$ and $\mathcal{Z}'$ are representable by schemes Z and Z' and in fact $Z = V \times_{V'} Z'$. Since $\mathcal{F}$ has the flat base change property we see that

$$a^*_{small}\big(\mathcal{F}|_{Z'_{étale}}\big) \longrightarrow \mathcal{F}|_{Z_{étale}}$$

is an isomorphism. Moreover,

$$R^i f_*\mathcal{F}|_{V'_{étale}} = R^i(f')_{small,*}\big(\mathcal{F}|_{Z'_{étale}}\big)$$

and

$$R^i f_*\mathcal{F}|_{V_{étale}} = R^i(f'')_{small,*}\big(\mathcal{F}|_{Z_{étale}}\big)$$

by Sheaves on Stacks, Lemma 69.21.3. Hence we see that the comparison map

$$c_\psi : b^*_{small}(R^i f_*\mathcal{F}|_{V'_{étale}}) \longrightarrow R^i f_*\mathcal{F}|_{V_{étale}}$$

is an isomorphism by Cohomology of Spaces, Lemma 51.8.1. Thus $R^i f_*\mathcal{F}$ has the flat base change property. Since $R^i f_*\mathcal{F}$ is locally quasi-coherent by Lemma 75.6.2 we win. □

Proposition 75.7.4. *Summary of results on locally quasi-coherent modules having the flat base change property.*

(1) *Let $\mathcal{X}$ be an algebraic stack. If $\mathcal{F}$ is an object of $Mod(\mathcal{X}_{étale}, \mathcal{O}_\mathcal{X})$ which is locally quasi-coherent and has the flat base change property, then $\mathcal{F}$ is a sheaf for the fppf topology, i.e., it is an object of $Mod(\mathcal{O}_\mathcal{X})$.*
(2) *The category of modules which are locally quasi-coherent and have the flat base change property is a weak Serre subcategory $\mathcal{M}_\mathcal{X}$ of both $Mod(\mathcal{O}_\mathcal{X})$ and $Mod(\mathcal{X}_{étale}, \mathcal{O}_\mathcal{X})$.*
(3) *Pullback f^* along any morphism of algebraic stacks $f : \mathcal{X} \to \mathcal{Y}$ induces a functor $f^* : \mathcal{M}_\mathcal{Y} \to \mathcal{M}_\mathcal{X}$.*
(4) *If $f : \mathcal{X} \to \mathcal{Y}$ is a quasi-compact and quasi-separated morphism of algebraic stacks and $\mathcal{F}$ is an object of $\mathcal{M}_\mathcal{X}$, then*
   (a) *the derived direct image $Rf_*\mathcal{F}$ and the higher direct images $R^i f_*\mathcal{F}$ can be computed in either the étale or the fppf topology with the same result, and*
   (b) *each $R^i f_*\mathcal{F}$ is an object of $\mathcal{M}_\mathcal{Y}$.*
(5) *The category $\mathcal{M}_\mathcal{X}$ has colimits and they agree with colimits in $Mod(\mathcal{X}_{étale}, \mathcal{O}_\mathcal{X})$ as well as in $Mod(\mathcal{O}_\mathcal{X})$.*

Proof. Part (1) is Sheaves on Stacks, Lemma 69.22.1.

Part (2) for the embedding $\mathcal{M}_\mathcal{X} \subset Mod(\mathcal{X}_{étale}, \mathcal{O}_\mathcal{X})$ we have seen in the proof of Lemma 75.7.3. Let us prove (2) for the embedding $\mathcal{M}_\mathcal{X} \subset Mod(\mathcal{O}_\mathcal{X})$. Let $\varphi : \mathcal{F} \to \mathcal{G}$ be a morphism between objects of $\mathcal{M}_\mathcal{X}$. Since $\mathrm{Ker}(\varphi)$ is the same whether computed in the étale or the fppf topology, we see that $\mathrm{Ker}(\varphi)$ is in $\mathcal{M}_\mathcal{X}$ by the étale case. On the other hand, the cokernel computed in the fppf topology is the fppf sheafification of the cokernel computed in the étale topology. However,

this étale cokernel is in $\mathcal{M}_\mathcal{X}$ hence an fppf sheaf by (1) and we see that the cokernel is in $\mathcal{M}_\mathcal{X}$. Finally, suppose that

$$0 \to \mathcal{F}_1 \to \mathcal{F}_2 \to \mathcal{F}_3 \to 0$$

is an exact sequence in $Mod(\mathcal{O}_\mathcal{X})$ (i.e., using the fppf topology) with $\mathcal{F}_1$, $\mathcal{F}_2$ in $\mathcal{M}_\mathcal{X}$. In order to show that $\mathcal{F}_2$ is an object of $\mathcal{M}_\mathcal{X}$ it suffices to show that the sequence is also exact in the étale topology. To do this it suffices to show that any element of $H^1_{fppf}(x, \mathcal{F}_1)$ becomes zero on the members of an étale covering of x (for any object x of $\mathcal{X}$). This is true because $H^1_{fppf}(x, \mathcal{F}_1) = H^1_{étale}(x, \mathcal{F}_1)$ by Sheaves on Stacks, Lemma 69.22.2 and because of locality of cohomology, see Cohomology on Sites, Lemma 21.8.3. This proves (2).

Part (3) follows from Lemma 75.7.2 and Sheaves on Stacks, Lemma 69.11.6.

Part (4)(b) for $R^i f_* \mathcal{F}$ computed in the étale cohomology follows from Lemma 75.7.3. Whereupon part (4)(a) follows from Sheaves on Stacks, Lemma 69.22.2 combined with (1) above.

Part (5) for the étale topology follows from Sheaves on Stacks, Lemma 69.11.7 and Lemma 75.7.2. The fppf version then follows as the colimit in the étale topology is already an fppf sheaf by part (1). □

Lemma 75.7.5. *Let $\mathcal{X}$ be an algebraic stack. With $\mathcal{M}_\mathcal{X}$ the category of locally quasi-coherent modules with the flat base change property.*

1. *Let $f_j : \mathcal{X}_j \to \mathcal{X}$ be a family of smooth morphisms of algebraic stacks with $|\mathcal{X}| = \bigcup |f_j|(|\mathcal{X}_j|)$. Let $\mathcal{F}$ be a sheaf of $\mathcal{O}_\mathcal{X}$-modules on $\mathcal{X}_{étale}$. If each $f_j^{-1}\mathcal{F}$ is in $\mathcal{M}_{\mathcal{X}_i}$, then $\mathcal{F}$ is in $\mathcal{M}_\mathcal{X}$.*
2. *Let $f_j : \mathcal{X}_j \to \mathcal{X}$ be a family of flat and locally finitely presented morphisms of algebraic stacks with $|\mathcal{X}| = \bigcup |f_j|(|\mathcal{X}_j|)$. Let $\mathcal{F}$ be a sheaf of $\mathcal{O}_\mathcal{X}$-modules on $\mathcal{X}_{fppf}$. If each $f_j^{-1}\mathcal{F}$ is in $\mathcal{M}_{\mathcal{X}_i}$, then $\mathcal{F}$ is in $\mathcal{M}_\mathcal{X}$.*

Proof. Part (1) follows from a combination of Lemmas 75.6.1 and 75.7.2. The proof of (2) is analogous to the proof of Lemma 75.6.3. Let $\mathcal{F}$ of a sheaf of $\mathcal{O}_\mathcal{X}$-modules on $\mathcal{X}_{fppf}$.

First, suppose there is a morphism $a : \mathcal{U} \to \mathcal{X}$ which is surjective, flat, locally of finite presentation, quasi-compact, and quasi-separated such that $a^*\mathcal{F}$ is locally quasi-coherent and has the flat base change property. Then there is an exact sequence

$$0 \to \mathcal{F} \to a_* a^* \mathcal{F} \to b_* b^* \mathcal{F}$$

where b is the morphism $b : \mathcal{U} \times_\mathcal{X} \mathcal{U} \to \mathcal{X}$, see Sheaves on Stacks, Proposition 69.18.7 and Lemma 69.18.10. Moreover, the pullback $b^*\mathcal{F}$ is the pullback of $a^*\mathcal{F}$ via one of the projection morphisms, hence is locally quasi-coherent and has the flat base change property, see Proposition 75.7.4. The modules $a_*a^*\mathcal{F}$ and $b_*b^*\mathcal{F}$ are locally quasi-coherent and have the flat base change property by Proposition 75.7.4. We conclude that $\mathcal{F}$ is locally quasi-coherent and has the flat base change property by Proposition 75.7.4.

Choose a scheme U and a surjective smooth morphism $x : U \to \mathcal{X}$. By part (1) it suffices to show that $x^*\mathcal{F}$ is locally quasi-coherent and has the flat base change property. Again by part (1) it suffices to do this (Zariski) locally on U, hence we may assume that U is affine. By Morphisms of Stacks, Lemma 74.18.10 there exists

an fppf covering $\{a_i : U_i \to U\}$ such that each $x \circ a_i$ factors through some f_j. Hence the module $a_i^*\mathcal{F}$ on $(Sch/U_i)_{fppf}$ is locally quasi-coherent and has the flat base change property. After refining the covering we may assume $\{U_i \to U\}_{i=1,\ldots,n}$ is a standard fppf covering. Then $x^*\mathcal{F}$ is an fppf module on $(Sch/U)_{fppf}$ whose pullback by the morphism $a : U_1 \amalg \ldots \amalg U_n \to U$ is locally quasi-coherent and has the flat base change property. Hence by the previous paragraph we see that $x^*\mathcal{F}$ is locally quasi-coherent and has the flat base change property as desired. □

75.8. Parasitic modules

The following definition is compatible with Descent, Definition 34.8.1.

Definition 75.8.1. Let $\mathcal{X}$ be an algebraic stack. A presheaf of $\mathcal{O}_\mathcal{X}$-modules $\mathcal{F}$ is *parasitic* if we have $\mathcal{F}(x) = 0$ for any object x of $\mathcal{X}$ which lies over a scheme U such that the corresponding morphism $x : U \to \mathcal{X}$ is flat.

Here is a lemma with some properties of this notion.

Lemma 75.8.2. *Let $\mathcal{X}$ be an algebraic stack. Let $\mathcal{F}$ be a presheaf of $\mathcal{O}_\mathcal{X}$-modules.*

1. *If $\mathcal{F}$ is parasitic and $g : \mathcal{Y} \to \mathcal{X}$ is a flat morphism of algebraic stacks, then $g^*\mathcal{F}$ is parasitic.*
2. *For $\tau \in \{Zariski, étale, smooth, syntomic, fppf\}$ we have*
   (a) *the τ sheafification of a parasitic presheaf of modules is parasitic, and*
   (b) *the full subcategory of $Mod(\mathcal{X}_\tau, \mathcal{O}_\mathcal{X})$ consisting of parasitic modules is a Serre subcategory.*
3. *Suppose $\mathcal{F}$ is a sheaf for the étale topology. Let $f_i : \mathcal{X}_i \to \mathcal{X}$ be a family of smooth morphisms of algebraic stacks such that $|\mathcal{X}| = \bigcup_i |f_i|(|\mathcal{X}_i|)$. If each $f_i^*\mathcal{F}$ is parasitic then so is $\mathcal{F}$.*
4. *Suppose $\mathcal{F}$ is a sheaf for the fppf topology. Let $f_i : \mathcal{X}_i \to \mathcal{X}$ be a family of flat and locally finitely presented morphisms of algebraic stacks such that $|\mathcal{X}| = \bigcup_i |f_i|(|\mathcal{X}_i|)$. If each $f_i^*\mathcal{F}$ is parasitic then so is $\mathcal{F}$.*

Proof. To see part (1) let y be an object of $\mathcal{Y}$ which lies over a scheme V such that the corresponding morphism $y : V \to \mathcal{Y}$ is flat. Then $g(y) : V \to \mathcal{Y} \to \mathcal{X}$ is flat as a composition of flat morphisms (see Morphisms of Stacks, Lemma 74.17.2) hence $\mathcal{F}(g(y))$ is zero by assumption. Since $g^*\mathcal{F} = g^{-1}\mathcal{F}(y) = \mathcal{F}(g(y))$ we conclude $g^*\mathcal{F}$ is parasitic.

To see part (2)(a) note that if $\{x_i \to x\}$ is a τ-covering of $\mathcal{X}$, then each of the morphisms $x_i \to x$ lies over a flat morphism of schemes. Hence if x lies over a scheme U such that $x : U \to \mathcal{X}$ is flat, so do all of the objects x_i. Hence the presheaf $\mathcal{F}^+$ (see Sites, Section 7.10) is parasitic if the presheaf $\mathcal{F}$ is parasitic. This proves (2)(a) as the sheafification of $\mathcal{F}$ is $(\mathcal{F}^+)^+$.

Let $\mathcal{F}$ be a parasitic τ-module. It is immediate from the definitions that any submodule of $\mathcal{F}$ is parasitic. On the other hand, if $\mathcal{F}' \subset \mathcal{F}$ is a submodule, then it is equally clear that the presheaf $x \mapsto \mathcal{F}(x)/\mathcal{F}'(x)$ is parasitic. Hence the quotient $\mathcal{F}/\mathcal{F}'$ is a parasitic module by (2)(a). Finally, we have to show that given a short exact sequence $0 \to \mathcal{F}_1 \to \mathcal{F}_2 \to \mathcal{F}_3 \to 0$ with $\mathcal{F}_1$ and $\mathcal{F}_3$ parasitic, then $\mathcal{F}_2$ is parasitic. This follows immediately on evaluating on x lying over a scheme flat over $\mathcal{X}$. This proves (2)(b), see Homology, Lemma 12.9.2.

Let $f_i : \mathcal{X}_i \to \mathcal{X}$ be a jointly surjective family of smooth morphisms of algebraic stacks and assume each $f_i^*\mathcal{F}$ is parasitic. Let x be an object of $\mathcal{X}$ which lies over a scheme U such that $x : U \to \mathcal{X}$ is flat. Consider a surjective smooth covering $W_i \to U \times_{x,\mathcal{X}} \mathcal{X}_i$. Denote $y_i : W_i \to \mathcal{X}_i$ the projection. It follows that $\{f_i(y_i) \to x\}$ is a covering for the smooth topology on $\mathcal{X}$. Since a composition of flat morphisms is flat we see that $f_i^*\mathcal{F}(y_i) = 0$. On the other hand, as we saw in the proof of (1), we have $f_i^*\mathcal{F}(y_i) = \mathcal{F}(f_i(y_i))$. Hence we see that for some smooth covering $\{x_i \to x\}_{i\in I}$ in $\mathcal{X}$ we have $\mathcal{F}(x_i) = 0$. This implies $\mathcal{F}(x) = 0$ because the smooth topology is the same as as the étale topology, see More on Morphisms, Lemma 36.28.7. Namely, $\{x_i \to x\}_{i\in I}$ lies over a smooth covering $\{U_i \to U\}_{i\in I}$ of schemes. By the lemma just referenced there exists an étale covering $\{V_j \to U\}_{j\in J}$ which refines $\{U_i \to U\}_{i\in I}$. Denote $x'_j = x|_{V_j}$. Then $\{x'_j \to x\}$ is an étale covering in $\mathcal{X}$ refining $\{x_i \to x\}_{i\in I}$. This means the map $\mathcal{F}(x) \to \prod_{j\in J} \mathcal{F}(x'_j)$, which is injective as $\mathcal{F}$ is a sheaf in the étale topology, factors through $\mathcal{F}(x) \to \prod_{i\in I} \mathcal{F}(x_i)$ which is zero. Hence $\mathcal{F}(x) = 0$ as desired.

Proof of (4): omitted. Hint: similar, but simpler, than the proof of (3). □

Parasitic modules are preserved under absolutely any pushforward.

Lemma 75.8.3. *Let $\tau \in \{étale, fppf\}$. Let $\mathcal{X}$ be an algebraic stack. Let $\mathcal{F}$ be a parasitic object of $Mod(\mathcal{X}_\tau, \mathcal{O}_\mathcal{X})$.*

1. *$H^i_\tau(\mathcal{X}, \mathcal{F}) = 0$ for all i.*
2. *Let $f : \mathcal{X} \to \mathcal{Y}$ be a morphism of algebraic stacks. Then $R^i f_*\mathcal{F}$ (computed in τ-topology) is a parasitic object of $Mod(\mathcal{Y}_\tau, \mathcal{O}_\mathcal{Y})$.*

Proof. We first reduce (2) to (1). By Sheaves on Stacks, Lemma 69.20.2 we see that $R^i f_*\mathcal{F}$ is the sheaf associated to the presheaf

$$y \longmapsto H^i_\tau\Big(V \times_{y,\mathcal{Y}} \mathcal{X},\ \mathrm{pr}^{-1}\mathcal{F}\Big)$$

Here y is a typical object of $\mathcal{Y}$ lying over the scheme V. By Lemma 75.8.2 it suffices to show that these cohomology groups are zero when $y : V \to \mathcal{Y}$ is flat. Note that $\mathrm{pr} : V \times_{y,\mathcal{Y}} \mathcal{X} \to \mathcal{X}$ is flat as a base change of y. Hence by Lemma 75.8.2 we see that $\mathrm{pr}^{-1}\mathcal{F}$ is parasitic. Thus it suffices to prove (1).

To see (1) we can use the spectral sequence of Sheaves on Stacks, Proposition 69.19.1 to reduce this to the case where $\mathcal{X}$ is an algebraic stack representable by an algebraic space. Note that in the spectral sequence each $f_p^{-1}\mathcal{F} = f_p^*\mathcal{F}$ is a parasitic module by Lemma 75.8.2 because the morphisms $f_p : \mathcal{U}_p = \mathcal{U} \times_\mathcal{X} \ldots \times_\mathcal{X} \mathcal{U} \to \mathcal{X}$ are flat. Reusing this spectral sequence one more time (as in the proof of the key Lemma 75.5.1) we reduce to the case where the algebraic stack $\mathcal{X}$ is representable by a scheme X. Then $H^i_\tau(\mathcal{X}, \mathcal{F}) = H^i((Sch/X)_\tau, \mathcal{F})$. In this case the vanishing follows easily from an argument with Čech coverings, see Descent, Lemma 34.8.2. □

The following lemma is one of the major reasons we care about parasitic modules. To understand the statement, recall that the functors $QCoh(\mathcal{O}_\mathcal{X}) \to Mod(\mathcal{X}_{étale}, \mathcal{O}_\mathcal{X})$ and $QCoh(\mathcal{O}_\mathcal{X}) \to Mod(\mathcal{O}_\mathcal{X})$ aren't exact in general.

Lemma 75.8.4. *Let $\mathcal{X}$ be an algebraic stack. Let $\mathcal{F}^\bullet$ be an exact complex in $QCoh(\mathcal{O}_\mathcal{X})$. Then the cohomology sheaves of $\mathcal{F}^\bullet$ in either the étale or the fppf topology are parasitic $\mathcal{O}_\mathcal{X}$-modules.*

Proof. Let $x : U \to \mathcal{X}$ be a flat morphism where U is a scheme. Then $x^*\mathcal{F}^\bullet$ is exact by Lemma 75.4.1. Hence the restriction $x^*\mathcal{F}^\bullet|_{U_{étale}}$ is exact which is what we had to prove. □

75.9. Quasi-coherent modules, I

We have seen that the category of quasi-coherent modules on an algebraic stack is equivalent to the category of quasi-coherent modules on a presentation, see Sheaves on Stacks, Section 69.14. This fact is the basis for the following.

Lemma 75.9.1. *Let $\mathcal{X}$ be an algebraic stack. Let $\mathcal{M}_\mathcal{X}$ be the category of locally quasi-coherent modules with the flat base change property, see Proposition 75.7.4. The inclusion functor $i : QCoh(\mathcal{O}_\mathcal{X}) \to \mathcal{M}_\mathcal{X}$ has a right adjoint*

$$Q : \mathcal{M}_\mathcal{X} \to QCoh(\mathcal{O}_\mathcal{X})$$

such that $Q \circ i$ is the identity functor.

Proof. Choose a scheme U and a surjective smooth morphism $f : U \to \mathcal{X}$. Set $R = U \times_\mathcal{X} U$ so that we obtain a smooth groupoid (U, R, s, t, c) in algebraic spaces with the property that $\mathcal{X} = [U/R]$, see Algebraic Stacks, Lemma 67.16.2. We may and do replace $\mathcal{X}$ by $[U/R]$. In the proof of Sheaves on Stacks, Proposition 69.13.1 we constructed a functor

$$q_1 : QCoh(U, R, s, t, c) \longrightarrow QCoh(\mathcal{O}_\mathcal{X}).$$

The construction of the inverse functor in the proof of Sheaves on Stacks, Proposition 69.13.1 works for objects of $\mathcal{M}_\mathcal{X}$ and induces a functor

$$q_2 : \mathcal{M}_\mathcal{X} \longrightarrow QCoh(U, R, s, t, c).$$

Namely, if $\mathcal{F}$ is an object of $\mathcal{M}_\mathcal{X}$ the we set

$$q_2(\mathcal{F}) = (f^*\mathcal{F}|_{U_{étale}}, \alpha)$$

where α is the isomorphism

$$t^*_{small}(f^*\mathcal{F}|_{U_{étale}}) \to t^*f^*\mathcal{F}|_{R_{étale}} \to s^*f^*\mathcal{F}|_{R_{étale}} \to s^*_{small}(f^*\mathcal{F}|_{U_{étale}})$$

where the outer two morphisms are the comparison maps. Note that $q_2(\mathcal{F})$ is quasi-coherent precisely because $\mathcal{F}$ is locally quasi-coherent (and we used the flat base change property in the construction of the descent datum α). We omit the verification that the cocycle condition (see Groupoids in Spaces, Definition 59.12.1) holds. We define $Q = q_1 \circ q_2$. Let $\mathcal{F}$ be an object of $\mathcal{M}_\mathcal{X}$ and let $\mathcal{G}$ be an object of $QCoh(\mathcal{O}_\mathcal{X})$. We have

$$\begin{aligned} \mathrm{Mor}_{\mathcal{M}_\mathcal{X}}(i(\mathcal{G}), \mathcal{F}) &= \mathrm{Mor}_{QCoh(U,R,s,t,c)}(q_2(\mathcal{G}), q_2(\mathcal{F})) \\ &= \mathrm{Mor}_{QCoh(\mathcal{O}_\mathcal{X})}(\mathcal{G}, Q(\mathcal{F})) \end{aligned}$$

where the first equality is Sheaves on Stacks, Lemma 69.13.2 and the second equality holds because q_1 and q_2 are inverse equivalences of categories. The assertion $Q \circ i \cong \mathrm{id}$ is a formal consequence of the fact that i is fully faithful. □

Lemma 75.9.2. *Let $\mathcal{X}$ be an algebraic stack. Let $Q : \mathcal{M}_\mathcal{X} \to QCoh(\mathcal{O}_\mathcal{X})$ be the functor constructed in Lemma 75.9.1.*

1. *The kernel of Q is exactly the collection of parasitic objects of $\mathcal{M}_\mathcal{X}$.*
2. *For any object $\mathcal{F}$ of $\mathcal{M}_\mathcal{X}$ both the kernel and the cokernel of the adjunction map $Q(\mathcal{F}) \to \mathcal{F}$ are parasitic.*

(3) *The functor Q is exact.*

Proof. Write $\mathcal{X} = [U/R]$ as in the proof of Lemma 75.9.1. Let $\mathcal{F}$ be an object of $\mathcal{M}_\mathcal{X}$. It is clear from the proof of Lemma 75.9.1 that $\mathcal{F}$ is in the kernel of Q if and only if $\mathcal{F}|_{U_{étale}} = 0$. In particular, if $\mathcal{F}$ is parasitic then $\mathcal{F}$ is in the kernel. Next, let $x : V \to \mathcal{X}$ be a flat morphism, where V is a scheme. Set $W = V \times_\mathcal{X} U$ and consider the diagram

$$\begin{array}{ccc} W & \xrightarrow{q} & V \\ {\scriptstyle p}\downarrow & & \downarrow \\ U & \longrightarrow & \mathcal{X} \end{array}$$

Note that the projection $p : W \to U$ is flat and the projection $q : W \to V$ is smooth and surjective. This implies that q^*_{small} is a faithful functor on quasi-coherent modules. By assumption $\mathcal{F}$ has the flat base change property so that we obtain $p^*_{small}\mathcal{F}|_{U_{étale}} \cong q^*_{small}\mathcal{F}|_{V_{étale}}$. Thus if $\mathcal{F}$ is in the kernel of Q, then $\mathcal{F}|_{V_{étale}} = 0$ which completes the proof of (1).

Part (2) follows from the discussion above and the fact that the map $Q(\mathcal{F}) \to \mathcal{F}$ becomes an isomorphism after restricting to $U_{étale}$.

To see part (3) note that Q is left exact as a right adjoint. Suppose that $0 \to \mathcal{F} \to \mathcal{G} \to \mathcal{H} \to 0$ is a short exact sequence in $\mathcal{M}_\mathcal{X}$. Let $\mathcal{E} = \mathrm{Coker}(Q(\mathcal{G}) \to Q(\mathcal{H}))$ in $QCoh(\mathcal{O}_\mathcal{X})$. Since $QCoh(\mathcal{O}_\mathcal{X}) \to \mathcal{M}_\mathcal{X}$ is a left adjoint it is right exact. Hence we see that $Q(\mathcal{G}) \to Q(\mathcal{H}) \to \mathcal{E} \to 0$ is exact in $\mathcal{M}_\mathcal{X}$. Using Lemma 75.8.4 we find that the top row of the following commutative diagram has parasitic cohomology sheaves at $Q(\mathcal{F})$ and $Q(\mathcal{G})$:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & Q(\mathcal{F}) & \longrightarrow & Q(\mathcal{G}) & \longrightarrow & Q(\mathcal{H}) & \longrightarrow & \mathcal{E} & \longrightarrow & 0 \\ & & {\scriptstyle a}\downarrow & & {\scriptstyle b}\downarrow & & {\scriptstyle c}\downarrow & & \downarrow & & \\ 0 & \longrightarrow & \mathcal{F} & \longrightarrow & \mathcal{G} & \longrightarrow & \mathcal{H} & \longrightarrow & 0 & & \end{array}$$

The bottom row is exact and the vertical arrows a, b, c have parasitic kernel and cokernels by part (2). It follows that $\mathcal{E}$ is parasitic: in the quotient category of $Mod(\mathcal{O}_\mathcal{X})$/Parasitic (see Homology, Lemma 12.9.6 and Lemma 75.8.2) we see that a, b, c are isomorphisms and that the top row becomes exact. As it is also quasi-coherent, we conclude that $\mathcal{E}$ is zero because $\mathcal{E} = Q(\mathcal{E}) = 0$ by part (1). □

75.10. Pushforward of quasi-coherent modules

Let $f : \mathcal{X} \to \mathcal{Y}$ be a morphism of algebraic stacks. Consider the pushforward

$$f_* : Mod(\mathcal{O}_\mathcal{X}) \longrightarrow Mod(\mathcal{O}_\mathcal{Y})$$

It turns out that this functor almost never preserves the subcategories of quasi-coherent sheaves. For example, consider the morphism of schemes

$$j : X = \mathbf{A}^2_k \setminus \{0\} \longrightarrow \mathbf{A}^2_k = Y.$$

Associated to this we have the corresponding morphism of algebraic stacks

$$f = j_{big} : \mathcal{X} = (Sch/X)_{fppf} \to (Sch/Y)_{fppf} = \mathcal{Y}$$

The pushforward $f_*\mathcal{O}_\mathcal{X}$ of the structure sheaf has global sections $k[x, y]$. Hence if $f_*\mathcal{O}_\mathcal{X}$ is quasi-coherent on $\mathcal{Y}$ then we would have $f_*\mathcal{O}_\mathcal{X} = \mathcal{O}_\mathcal{Y}$. However, consider $T = \mathrm{Spec}(k) \to \mathbf{A}^2_k = Y$ mapping to 0. Then $\Gamma(T, f_*\mathcal{O}_\mathcal{X}) = 0$ because $X \times_Y T =$

$\emptyset$ whereas $\Gamma(T, \mathcal{O}_\mathcal{Y}) = k$. On the positive side, we know from Cohomology of Schemes, Lemma 29.5.2 that for any flat morphism $T \to Y$ we have the equality $\Gamma(T, f_*\mathcal{O}_\mathcal{X}) = \Gamma(T, \mathcal{O}_\mathcal{Y})$ (this uses that j is quasi-compact and quasi-separated).

Let $f : \mathcal{X} \to \mathcal{Y}$ be a quasi-compact and quasi-separated morphism of algebraic stacks. We work around the problem mentioned above using the following three observations:

(1) f_* does preserve locally quasi-coherent modules (Lemma 75.6.2),
(2) f_* transforms a quasi-coherent sheaf into a locally quasi-coherent sheaf whose flat comparison maps are isomorphisms (Lemma 75.7.3), and
(3) locally quasi-coherent $\mathcal{O}_\mathcal{Y}$-modules with the flat base change property give rise to quasi-coherent modules on a presentation of $\mathcal{Y}$ and hence quasi-coherent modules on $\mathcal{Y}$, see Sheaves on Stacks, Section 69.14.

Thus we obtain a functor

$$f_{QCoh,*} : QCoh(\mathcal{O}_\mathcal{X}) \longrightarrow QCoh(\mathcal{O}_\mathcal{Y})$$

which is a right adjoint to $f^* : QCoh(\mathcal{O}_\mathcal{Y}) \to QCoh(\mathcal{O}_\mathcal{X})$ such that moreover

$$\Gamma(y, f_*\mathcal{F}) = \Gamma(y, f_{QCoh,*}\mathcal{F})$$

for any $y \in \mathrm{Ob}(\mathcal{Y})$ such that the associated 1-morphism $y : V \to \mathcal{Y}$ is flat, see (insert future reference here). Moreover, a similar construction will produce functors $R^i f_{QCoh,*}$. However, these results will not be sufficient to produce a total direct image functor (of complexes with quasi-coherent cohomology sheaves).

Proposition 75.10.1. *Let $f : \mathcal{X} \to \mathcal{Y}$ be a quasi-compact and quasi-separated morphism of algebraic stacks. The functor $f^* : QCoh(\mathcal{O}_\mathcal{Y}) \to QCoh(\mathcal{O}_\mathcal{X})$ has a right adjoint*

$$f_{QCoh,*} : QCoh(\mathcal{O}_\mathcal{X}) \longrightarrow QCoh(\mathcal{O}_\mathcal{Y})$$

which can be defined as the composition

$$QCoh(\mathcal{O}_\mathcal{X}) \to \mathcal{M}_\mathcal{X} \xrightarrow{f_*} \mathcal{M}_\mathcal{Y} \xrightarrow{Q} QCoh(\mathcal{O}_\mathcal{Y})$$

where the functors f_* and Q are as in Proposition 75.7.4 and Lemma 75.9.1. Moreover, if we define $R^i f_{QCoh,*}$ as the composition

$$QCoh(\mathcal{O}_\mathcal{X}) \to \mathcal{M}_\mathcal{X} \xrightarrow{R^i f_*} \mathcal{M}_\mathcal{Y} \xrightarrow{Q} QCoh(\mathcal{O}_\mathcal{Y})$$

then the sequence of functors $\{R^i f_{QCoh,*}\}_{i \geq 0}$ forms a cohomological δ-functor.

Proof. This is a combination of the results mentioned in the statement. The adjointness can be shown as follows: Let $\mathcal{F}$ be a quasi-coherent $\mathcal{O}_\mathcal{X}$-module and let $\mathcal{G}$ be a quasi-coherent $\mathcal{O}_\mathcal{Y}$-module. Then we have

$$\begin{aligned} \mathrm{Mor}_{QCoh(\mathcal{O}_\mathcal{X})}(f^*\mathcal{G}, \mathcal{F}) &= \mathrm{Mor}_{\mathcal{M}_\mathcal{Y}}(\mathcal{G}, f_*\mathcal{F}) \\ &= \mathrm{Mor}_{QCoh(\mathcal{O}_\mathcal{Y})}(\mathcal{G}, Q(f_*\mathcal{F})) \\ &= \mathrm{Mor}_{QCoh(\mathcal{O}_\mathcal{Y})}(\mathcal{G}, f_{QCoh,*}\mathcal{F}) \end{aligned}$$

the first equality by adjointness of f_* and f^* (for arbitrary sheaves of modules). By Proposition 75.7.4 we see that $f_*\mathcal{F}$ is an object of $\mathcal{M}_\mathcal{Y}$ (and can be computed in either the fppf or étale topology) and we obtain the second equality by Lemma 75.9.1. The third equality is the definition of $f_{QCoh,*}$.

To see that $\{R^i f_{QCoh,*}\}_{i\geq 0}$ is a cohomological δ-functor as defined in Homology, Definition 12.11.1 let

$$0 \to \mathcal{F}_1 \to \mathcal{F}_2 \to \mathcal{F}_3 \to 0$$

be a short exact sequence of $QCoh(\mathcal{O}_\mathcal{X})$. This sequence may not be an exact sequence in $Mod(\mathcal{O}_\mathcal{X})$ but we know that it is up to parasitic modules, see Lemma 75.8.4. Thus we may break up the sequence into short exact sequences

$$\begin{aligned}
0 \to \mathcal{P}_1 \to \mathcal{F}_1 \to \mathcal{I}_2 \to 0 \\
0 \to \mathcal{I}_2 \to \mathcal{F}_2 \to \mathcal{Q}_2 \to 0 \\
0 \to \mathcal{P}_2 \to \mathcal{Q}_2 \to \mathcal{I}_3 \to 0 \\
0 \to \mathcal{I}_3 \to \mathcal{F}_3 \to \mathcal{P}_3 \to 0
\end{aligned}$$

of $Mod(\mathcal{O}_\mathcal{X})$ with $\mathcal{P}_i$ parasitic. Note that each of the sheaves $\mathcal{P}_j$, $\mathcal{I}_j$, $\mathcal{Q}_j$ is an object of $\mathcal{M}_\mathcal{X}$, see Proposition 75.7.4. Applying $R^i f_*$ we obtain long exact sequences

$$\begin{aligned}
0 \to f_*\mathcal{P}_1 \to f_*\mathcal{F}_1 \to f_*\mathcal{I}_2 \to R^1 f_*\mathcal{P}_1 \to \ldots \\
0 \to f_*\mathcal{I}_2 \to f_*\mathcal{F}_2 \to f_*\mathcal{Q}_2 \to R^1 f_*\mathcal{I}_2 \to \ldots \\
0 \to f_*\mathcal{P}_2 \to f_*\mathcal{Q}_2 \to f_*\mathcal{I}_3 \to R^1 f_*\mathcal{P}_2 \to \ldots \\
0 \to f_*\mathcal{I}_3 \to f_*\mathcal{F}_3 \to f_*\mathcal{P}_3 \to R^1 f_*\mathcal{I}_3 \to \ldots
\end{aligned}$$

where are the terms are objects of $\mathcal{M}_\mathcal{Y}$ by Proposition 75.7.4. By Lemma 75.8.3 the sheaves $R^i f_*\mathcal{P}_j$ are parasitic, hence vanish on applying the functor Q, see Lemma 75.9.2. Since Q is exact the maps

$$Q(R^i f_*\mathcal{F}_3) \cong Q(R^i f_*\mathcal{I}_3) \cong Q(R^i f_*\mathcal{Q}_2) \to Q(R^{i+1} f_*\mathcal{I}_2) \cong Q(R^{i+1} f_*\mathcal{F}_1)$$

can serve as the connecting map which turns the family of functors $\{R^i f_{QCoh,*}\}_{i\geq 0}$ into a cohomological δ-functor. □

Lemma 75.10.2. *Let $f : \mathcal{X} \to \mathcal{Y}$ be a quasi-compact and quasi-separated morphism of algebraic stacks. Let $\mathcal{F}$ be a quasi-coherent sheaf on $\mathcal{X}$. Then there exists a spectral sequence with E_2-page*

$$E_2^{p,q} = H^p(\mathcal{Y}, R^q f_{QCoh,*}\mathcal{F})$$

converging to $H^{p+q}(\mathcal{X}, \mathcal{F})$.

Proof. By Cohomology on Sites, Lemma 21.14.5 the Leray spectral sequence with

$$E_2^{p,q} = H^p(\mathcal{Y}, R^q f_*\mathcal{F})$$

converges to $H^{p+q}(\mathcal{X}, \mathcal{F})$. The kernel and cokernel of the adjunction map

$$R^q f_{QCoh,*}\mathcal{F} \longrightarrow R^q f_*\mathcal{F}$$

are parasitic modules on $\mathcal{Y}$ (Lemma 75.9.2) hence have vanishing cohomology (Lemma 75.8.3). It follows formally that $H^p(\mathcal{Y}, R^q f_{QCoh,*}\mathcal{F}) = H^p(\mathcal{Y}, R^q f_*\mathcal{F})$ and we win. □

Lemma 75.10.3. *Let $f : \mathcal{X} \to \mathcal{Y}$ and $g : \mathcal{Y} \to \mathcal{Z}$ be quasi-compact and quasi-separated morphisms of algebraic stacks. Let $\mathcal{F}$ be a quasi-coherent sheaf on $\mathcal{X}$. Then there exists a spectral sequence with E_2-page*

$$E_2^{p,q} = R^p g_{QCoh,*}(R^q f_{QCoh,*}\mathcal{F})$$

converging to $R^{p+q}(g \circ f)_{QCoh,*}\mathcal{F}$.

Proof. By Cohomology on Sites, Lemma 21.14.7 the Leray spectral sequence with

$$E_2^{p,q} = R^p g_*(R^q f_* \mathcal{F})$$

converges to $R^{p+q}(g \circ f)_* \mathcal{F}$. By the results of Proposition 75.7.4 all the terms of this spectral sequence are objects of $\mathcal{M}_\mathcal{Z}$. Applying the exact functor $Q_\mathcal{Z} : \mathcal{M}_\mathcal{Z} \to QCoh(\mathcal{O}_\mathcal{Z})$ we obtain a spectral sequence in $QCoh(\mathcal{O}_\mathcal{Z})$ covering to $R^{p+q}(g \circ f)_{QCoh,*}\mathcal{F}$. Hence the result follows if we can show that

$$Q_\mathcal{Z}(R^p g_*(R^q f_* \mathcal{F})) = Q_\mathcal{Z}(R^p g_*(Q_\mathcal{X}(R^q f_* \mathcal{F}))$$

This follows from the fact that the kernel and cokernel of the map

$$Q_\mathcal{X}(R^q f_* \mathcal{F}) \longrightarrow R^q f_* \mathcal{F}$$

are parasitic (Lemma 75.9.2) and that $R^p g_*$ transforms parasitic modules into parasitic modules (Lemma 75.8.3). □

To end this section we make explicit the spectral sequences associated to a smooth covering by a scheme. Please compare with Sheaves on Stacks, Sections 69.19 and 69.20.

Proposition 75.10.4. *Let $f : \mathcal{U} \to \mathcal{X}$ be a morphism of algebraic stacks. Assume f is representable by algebraic spaces, surjective, flat, and locally of finite presentation. Let $\mathcal{F}$ be a quasi-coherent $\mathcal{O}_\mathcal{X}$-module. Then there is a spectral sequence*

$$E_2^{p,q} = H^q(\mathcal{U}_p, f_p^* \mathcal{F}) \Rightarrow H^{p+q}(\mathcal{X}, \mathcal{F})$$

where f_p is the morphism $\mathcal{U} \times_\mathcal{X} \ldots \times_\mathcal{X} \mathcal{U} \to \mathcal{X}$ ($p+1$ factors).

Proof. This is a special case of Sheaves on Stacks, Proposition 69.19.1. □

Proposition 75.10.5. *Let $f : \mathcal{U} \to \mathcal{X}$ and $g : \mathcal{X} \to \mathcal{Y}$ be composable morphisms of algebraic stacks. Assume that*

1. *f is representable by algebraic spaces, surjective, flat, locally of finite presentation, quasi-compact, and quasi-separated, and*
2. *g is quasi-compact and quasi-separated.*

If $\mathcal{F}$ is in $QCoh(\mathcal{O}_\mathcal{X})$ then there is a spectral sequence

$$E_2^{p,q} = R^q(g \circ f_p)_{QCoh,*} f_p^* \mathcal{F} \Rightarrow R^{p+q} g_{QCoh,*} \mathcal{F}$$

in $QCoh(\mathcal{O}_\mathcal{Y})$.

Proof. Note that each of the morphisms $f_p : \mathcal{U} \times_\mathcal{X} \ldots \times_\mathcal{X} \mathcal{U} \to \mathcal{X}$ is quasi-compact and quasi-separated, hence $g \circ f_p$ is quasi-compact and quasi-separated, hence the assertion makes sense (i.e., the functors $R^q(g \circ f_p)_{QCoh,*}$ are defined). There is a spectral sequence

$$E_2^{p,q} = R^q(g \circ f_p)_* f_p^{-1} \mathcal{F} \Rightarrow R^{p+q} g_* \mathcal{F}$$

by Sheaves on Stacks, Proposition 69.20.1. Applying the exact functor $Q_\mathcal{Y} : \mathcal{M}_\mathcal{Y} \to QCoh(\mathcal{O}_\mathcal{Y})$ gives the desired spectral sequence in $QCoh(\mathcal{O}_\mathcal{Y})$. □

75.11. The lisse-étale and the flat-fppf sites

In the book [**LMB00**] many of the results above are proved using the lisse-étale site of an algebraic stack. We define this site here. In Examples, Section 78.44 we show that the lisse-étale site isn't functorial. We also define its analogue, the flat-fppf site, which is better suited to the development of algebraic stacks as given in the stacks project (because we use the fppf topology as our base topology). Of course the flat-fppf site isn't functorial either.

Definition 75.11.1. Let $\mathcal{X}$ be an algebraic stack.

(1) The *lisse-étale site* of $\mathcal{X}$ is the full subcategory $\mathcal{X}_{lisse,étale}$[2] of $\mathcal{X}$ whose objects are those $x \in \mathrm{Ob}(\mathcal{X})$ lying over a scheme U such that $x : U \to \mathcal{X}$ is smooth. A covering of $\mathcal{X}_{lisse,étale}$ is a family of morphisms $\{x_i \to x\}_{i\in I}$ of $\mathcal{X}_{lisse,étale}$ which forms a covering of $\mathcal{X}_{étale}$.

(2) The *flat-fppf site* of $\mathcal{X}$ is the full subcategory $\mathcal{X}_{flat,fppf}$ of $\mathcal{X}$ whose objects are those $x \in \mathrm{Ob}(\mathcal{X})$ lying over a scheme U such that $x : U \to \mathcal{X}$ is flat. A covering of $\mathcal{X}_{flat,fppf}$ is a family of morphisms $\{x_i \to x\}_{i\in I}$ of $\mathcal{X}_{flat,fppf}$ which forms a covering of $\mathcal{X}_{fppf}$.

We denote $\mathcal{O}_{\mathcal{X}_{lisse,étale}}$ the restriction of $\mathcal{O}_{\mathcal{X}}$ to the lisse-étale site and similarly for $\mathcal{O}_{\mathcal{X}_{flat,fppf}}$. The relationship between the lisse-étale site and the étale site is as follows (we mainly stick to "topological" properties in this lemma).

Lemma 75.11.2. *Let $\mathcal{X}$ be an algebraic stack.*

(1) *The inclusion functor $\mathcal{X}_{lisse,étale} \to \mathcal{X}_{étale}$ is fully faithful, continuous and cocontinuous. It follows that*
   (a) *there is a morphism of topoi*
   $$g : Sh(\mathcal{X}_{lisse,étale}) \longrightarrow Sh(\mathcal{X}_{étale})$$
   with g^{-1} given by restriction,
   (b) *the functor g^{-1} has a left adjoint $g_!^{Sh}$ on sheaves of sets,*
   (c) *the adjunction maps $g^{-1}g_* \to id$ and $id \to g^{-1}g_!^{Sh}$ are isomorphisms,*
   (d) *the functor g^{-1} has a left adjoint $g_!$ on abelian sheaves,*
   (e) *the adjunction map $id \to g^{-1}g_!$ is an isomorphism, and*
   (f) *we have $g^{-1}\mathcal{O}_{\mathcal{X}} = \mathcal{O}_{\mathcal{X}_{lisse,étale}}$ hence g induces a flat morphism of ringed topoi such that $g^{-1} = g^*$.*

(2) *The inclusion functor $\mathcal{X}_{flat,fppf} \to \mathcal{X}_{fppf}$ is fully faithful, continuous and cocontinuous. It follows that*
   (a) *there is a morphism of topoi*
   $$g : Sh(\mathcal{X}_{flat,fppf}) \longrightarrow Sh(\mathcal{X}_{fppf})$$
   with g^{-1} given by restriction,
   (b) *the functor g^{-1} has a left adjoint $g_!^{Sh}$ on sheaves of sets,*
   (c) *the adjunction maps $g^{-1}g_* \to id$ and $id \to g^{-1}g_!^{Sh}$ are isomorphisms,*
   (d) *the functor g^{-1} has a left adjoint $g_!$ on abelian sheaves,*
   (e) *the adjunction map $id \to g^{-1}g_!$ is an isomorphism, and*
   (f) *we have $g^{-1}\mathcal{O}_{\mathcal{X}} = \mathcal{O}_{\mathcal{X}_{flat,fppf}}$ hence g induces a flat morphism of ringed topoi such that $g^{-1} = g^*$.*

[2] In the literature the site is denoted Lis-ét($\mathcal{X}$) or Lis-Et($\mathcal{X}$) and the associated topos is denoted $\mathcal{X}_{\text{lis-ét}}$ or $\mathcal{X}_{\text{lis-et}}$. In the stacks project our convention is to name the site and denote the corresponding topos by $Sh(\mathcal{C})$.

Proof. In both cases it is immediate that the functor is fully faithful, continuous, and cocontinuous (see Sites, Definitions 7.14.1 and 7.19.1). Hence properties (a), (b), (c) follow from Sites, Lemmas 7.20.5 and 7.20.7. Parts (d), (e) follow from Modules on Sites, Lemmas 18.16.2 and 18.16.4. Part (f) is immediate. □

Lemma 75.11.3. *Let $\mathcal{X}$ be an algebraic stack. Notation as in Lemma 75.11.2.*

(1) *There exists a functor*
$$g_! : Mod(\mathcal{X}_{lisse,étale}, \mathcal{O}_{\mathcal{X}_{lisse,étale}}) \longrightarrow Mod(\mathcal{X}_{étale}, \mathcal{O}_{\mathcal{X}})$$
which is left adjoint to g^*. Moreover it agrees with the functor $g_!$ on abelian sheaves and $g^*g_! = id$.

(2) *There exists a functor*
$$g_! : Mod(\mathcal{X}_{flat,fppf}, \mathcal{O}_{\mathcal{X}_{flat,fppf}}) \longrightarrow Mod(\mathcal{X}_{fppf}, \mathcal{O}_{\mathcal{X}})$$
which is left adjoint to g^*. Moreover it agrees with the functor $g_!$ on abelian sheaves and $g^*g_! = id$.

Proof. In both cases, the existence of the functor $g_!$ follows from Modules on Sites, Lemma 18.40.1. To see that $g_!$ agrees with the functor on abelian sheaves we will show the maps Modules on Sites, Equation (18.40.2.1) are isomorphisms.

Lisse-étale case. Let $x \in \mathrm{Ob}(\mathcal{X}_{lisse,étale})$ lying over a scheme U with $x : U \to \mathcal{X}$ smooth. Consider the induced fully faithful functor
$$g' : \mathcal{X}_{lisse,étale}/x \longrightarrow \mathcal{X}_{étale}/x$$
The right hand side is identified with $(Sch/U)_{étale}$ and the left hand side with the full subcategory of schemes U'/U such that the composition $U' \to U \to \mathcal{X}$ is smooth. Thus Étale Cohomology, Lemma 44.50.2 applies.

Flat-fppf case. Let $x \in \mathrm{Ob}(\mathcal{X}_{flat,fppf})$ lying over a scheme U with $x : U \to \mathcal{X}$ flat. Consider the induced fully faithful functor
$$g' : \mathcal{X}_{flat,fppf}/x \longrightarrow \mathcal{X}_{fppf}/x$$
The right hand side is identified with $(Sch/U)_{fppf}$ and the left hand side with the full subcategory of schemes U'/U such that the composition $U' \to U \to \mathcal{X}$ is flat. Thus Étale Cohomology, Lemma 44.50.2 applies.

In both cases the equality $g^*g_! = \mathrm{id}$ follows from $g^* = g^{-1}$ and the equality for abelian sheaves in Lemma 75.11.2. □

Lemma 75.11.4. *Let $\mathcal{X}$ be an algebraic stack. Notation as in Lemmas 75.11.2 and 75.11.3.*

(1) *We have $g_!\mathcal{O}_{\mathcal{X}_{lisse,étale}} = \mathcal{O}_{\mathcal{X}}$.*
(2) *We have $g_!\mathcal{O}_{\mathcal{X}_{flat,fppf}} = \mathcal{O}_{\mathcal{X}}$.*

Proof. In this proof we write $\mathcal{C} = \mathcal{X}_{étale}$ (resp. $\mathcal{C} = \mathcal{X}_{fppf}$) and we denote $\mathcal{C}' = \mathcal{X}_{lisse,étale}$ (resp. $\mathcal{C}' = \mathcal{X}_{flat,fppf}$). Then $\mathcal{C}'$ is a full subcategory of $\mathcal{C}$. In this proof we will think of objects V of $\mathcal{C}$ as schemes over $\mathcal{X}$ and objects U of $\mathcal{C}'$ as schemes smooth (resp. flat) over $\mathcal{X}$. Finally, we write $\mathcal{O} = \mathcal{O}_{\mathcal{X}}$ and $\mathcal{O}' = \mathcal{O}_{\mathcal{X}_{lisse,étale}}$ (resp. $\mathcal{O}' = \mathcal{O}_{\mathcal{X}_{flat,fppf}}$). In the notation above we have $\mathcal{O}(V) = \Gamma(V, \mathcal{O}_V)$ and $\mathcal{O}'(U) = \Gamma(U, \mathcal{O}_U)$. Consider the $\mathcal{O}$-module homomorphism $g_!\mathcal{O}' \to \mathcal{O}$ adjoint to the identification $\mathcal{O}' = g^{-1}\mathcal{O}$.

Recall that $g_!\mathcal{O}'$ is the sheaf associated to the presheaf $g_{p!}\mathcal{O}'$ given by the rule

$$V \longmapsto \operatorname{colim}_{V \to U} \mathcal{O}'(U)$$

where the colimit is taken in the category of abelian groups (Modules on Sites, Definition 18.16.1). Below we will use frequently that if

$$V \to U \to U'$$

are morphisms and if $f' \in \mathcal{O}'(U')$ restricts to $f \in \mathcal{O}'(U)$, then $(V \to U, f)$ and $(V \to U', f')$ define the same element of the colimit. Also, $g_!\mathcal{O}' \to \mathcal{O}$ maps the element $(V \to U, f)$ simply to the pullback of f to V.

To see that $g_!\mathcal{O}' \to \mathcal{O}$ is surjective it suffices to show that $1 \in \Gamma(\mathcal{C}, \mathcal{O})$ is locally in the image. Choose an object U of $\mathcal{C}'$ corresponding to a surjective smooth morphism $U \to \mathcal{X}$. Then viewing U both as an object of $\mathcal{C}'$ and $\mathcal{C}$ we see that $(U \to U, 1)$ is an element of the colimit above which maps to $1 \in \mathcal{O}(U)$. Since U surjects onto the final object of $Sh(\mathcal{C})$ we conclude $g_!\mathcal{O}' \to \mathcal{O}$ is surjective.

Suppose that $s \in g_!\mathcal{O}'(V)$ is a section mapping to zero in $\mathcal{O}(V)$. To finish the proof we have to show that s is zero. After replacing V by the members of a covering we may assume s is an element of the colimit

$$\operatorname{colim}_{V \to U} \mathcal{O}'(U)$$

Say $s = \sum(\varphi_i, s_i)$ is a finite sum with $\varphi_i : V \to U_i$, U_i smooth (resp. flat) over $\mathcal{X}$, and $s_i \in \Gamma(U_i, \mathcal{O}_{U_i})$. Choose a scheme W surjective étale over the algebraic space $U = U_1 \times_{\mathcal{X}} \ldots \times_{\mathcal{X}} U_n$. Note that W is still smooth (resp. flat) over $\mathcal{X}$, i.e., defines an object of $\mathcal{C}'$. The fibre product

$$V' = V \times_{(\varphi_1, \ldots, \varphi_n), U} W$$

is surjective étale over V, hence it suffices to show that s maps to zero in $g_!\mathcal{O}'(V')$. Note that the restriction $\sum(\varphi_i, s_i)|_{V'}$ corresponds to the sum of the pullbacks of the functions s_i to W. In other words, we have reduced to the case of (φ, s) where $\varphi : V \to U$ is a morphism with U in $\mathcal{C}'$ and $s \in \mathcal{O}'(U)$ restricts to zero in $\mathcal{O}(V)$. By the commutative diagram

$$\begin{array}{ccc} V & \xrightarrow{(\varphi,0)} & U \times \mathbf{A}^1 \\ & \searrow^{\varphi} & \uparrow{\scriptstyle (\mathrm{id},0)} \\ & & U \end{array}$$

we see that $((\varphi, 0) : V \to U \times \mathbf{A}^1, \mathrm{pr}_2^* x)$ represents zero in the colimit above. Hence we may replace U by $U \times \mathbf{A}^1$, φ by $(\varphi, 0)$ and s by $\mathrm{pr}_1^* s + \mathrm{pr}_2^* x$. Thus we may assume that the vanishing locus $Z : s = 0$ in U of s is smooth (resp. flat) over $\mathcal{X}$. Then we see that $(V \to Z, 0)$ and (φ, s) have the same value in the colimit, i.e., we see that the element s is zero as desired. □

The lisse-étale and the flat-fppf sites can be used to characterize parasitic modules as follows.

Lemma 75.11.5. *Let $\mathcal{X}$ be an algebraic stack.*

(1) *Let $\mathcal{F}$ be an $\mathcal{O}_{\mathcal{X}}$-module with the flat base change property on $\mathcal{X}_{étale}$. The following are equivalent*
   (a) *$\mathcal{F}$ is parasitic, and*

(b) $g^*\mathcal{F} = 0$ *where* $g : Sh(\mathcal{X}_{lisse,étale}) \to Sh(\mathcal{X}_{étale})$ *is as in Lemma 75.11.2.*

(2) *Let* $\mathcal{F}$ *be an* $\mathcal{O}_\mathcal{X}$*-module on* $\mathcal{X}_{fppf}$*. The following are equivalent*
   (a) $\mathcal{F}$ *is parasitic, and*
   (b) $g^*\mathcal{F} = 0$ *where* $g : Sh(\mathcal{X}_{flat,fppf}) \to Sh(\mathcal{X}_{fppf})$ *is as in Lemma 75.11.2.*

Proof. Part (2) is immediate from the definitions (this is one of the advantages of the flat-fppf site over the lisse-étale site). The implication (1)(a) $\Rightarrow$ (1)(b) is immediate as well. To see (1)(b) $\Rightarrow$ (1)(a) let U be a scheme and let $x : U \to \mathcal{X}$ be a surjective smooth morphism. Then x is an object of the lisse-étale site of $\mathcal{X}$. Hence we see that (1)(b) implies that $\mathcal{F}|_{U_{étale}} = 0$. Let $V \to \mathcal{X}$ be an flat morphism where V is a scheme. Set $W = U \times_\mathcal{X} V$ and consider the diagram

$$\begin{array}{ccc} W & \xrightarrow[q]{} & V \\ {\scriptstyle p}\downarrow & & \downarrow \\ U & \longrightarrow & \mathcal{X} \end{array}$$

Note that the projection $p : W \to U$ is flat and the projection $q : W \to V$ is smooth and surjective. This implies that q^*_{small} is a faithful functor on quasi-coherent modules. By assumption $\mathcal{F}$ has the flat base change property so that we obtain $p^*_{small}\mathcal{F}|_{U_{étale}} \cong q^*_{small}\mathcal{F}|_{V_{étale}}$. Thus if $\mathcal{F}$ is in the kernel of g^*, then $\mathcal{F}|_{V_{étale}} = 0$ as desired. □

The lisse-étale site is functorial for smooth morphisms of algebraic stacks and the flat-fppf site is functorial for flat morphisms of algebraic stacks.

Lemma 75.11.6. *Let* $f : \mathcal{X} \to \mathcal{Y}$ *be a morphism of algebraic stacks.*

(1) *If* f *is smooth, then* f *restricts to a continuous and cocontinuous functor* $\mathcal{X}_{lisse,étale} \to \mathcal{Y}_{lisse,étale}$ *which gives a morphism of ringed topoi fitting into the following commutative diagram*

$$\begin{array}{ccc} Sh(\mathcal{X}_{lisse,étale}) & \xrightarrow[g']{} & Sh(\mathcal{X}_{étale}) \\ {\scriptstyle f'}\downarrow & & \downarrow{\scriptstyle f} \\ Sh(\mathcal{Y}_{lisse,étale}) & \xrightarrow{g} & Sh(\mathcal{Y}_{étale}) \end{array}$$

We have $f'_*(g')^{-1} = g^{-1}f_*$ *and* $g'_!(f')^{-1} = f^{-1}g_!$.

(2) *If* f *is flat, then* f *restricts to a continuous and cocontinuous functor* $\mathcal{X}_{flat,fppf} \to \mathcal{Y}_{flat,fppf}$ *which gives a morphism of ringed topoi fitting into the following commutative diagram*

$$\begin{array}{ccc} Sh(\mathcal{X}_{flat,fppf}) & \xrightarrow[g']{} & Sh(\mathcal{X}_{fppf}) \\ {\scriptstyle f'}\downarrow & & \downarrow{\scriptstyle f} \\ Sh(\mathcal{Y}_{flat,fppf}) & \xrightarrow{g} & Sh(\mathcal{Y}_{fppf}) \end{array}$$

We have $f'_*(g')^{-1} = g^{-1}f_*$ *and* $g'_!(f')^{-1} = f^{-1}g_!$.

Proof. The initial statement comes from the fact that if $x \in \mathrm{Ob}(\mathcal{X})$ lies over a scheme U such that $x : U \to \mathcal{X}$ is smooth (resp. flat) and if f is smooth (resp. flat) then $f(x) : U \to \mathcal{Y}$ is smooth (resp. flat), see Morphisms of Stacks, Lemmas 74.22.2

and 74.17.2. The induced functor $\mathcal{X}_{lisse,étale} \to \mathcal{Y}_{lisse,étale}$ (resp. $\mathcal{X}_{flat,fppf} \to \mathcal{Y}_{flat,fppf}$) is continuous and cocontinuous by our definition of coverings in these categories. Finally, the commutativity of the diagram is a consequence of the fact that the horizontal morphisms are given by the inclusion functors (see Lemma 75.11.2) and Sites, Lemma 7.20.2.

To show that $f'_*(g')^{-1} = g^{-1}f_*$ let $\mathcal{F}$ be a sheaf on $\mathcal{X}_{étale}$ (resp. $\mathcal{X}_{fppf}$). There is a canonical pullback map

$$g^{-1}f_*\mathcal{F} \longrightarrow f'_*(g')^{-1}\mathcal{F}$$

see Sites, Section 7.44. We claim this map is an isomorphism. To prove this pick an object y of $\mathcal{Y}_{lisse,étale}$ (resp. $\mathcal{Y}_{flat,fppf}$). Say y lies over the scheme V such that $y : V \to \mathcal{Y}$ is smooth (resp. flat). Since g^{-1} is the restriction we find that

$$\left(g^{-1}f_*\mathcal{F}\right)(y) = \Gamma(V \times_{y,\mathcal{Y}} \mathcal{X},\ \mathrm{pr}^{-1}\mathcal{F})$$

by Sheaves on Stacks, Equation (69.5.0.1). Let $(V \times_{y,\mathcal{Y}} \mathcal{X})' \subset V \times_{y,\mathcal{Y}} \mathcal{X}$ be the full subcategory consisting of objects $z : W \to V \times_{y,\mathcal{Y}} \mathcal{X}$ such that the induced morphism $W \to \mathcal{X}$ is smooth (resp. flat). Denote

$$\mathrm{pr}' : (V \times_{y,\mathcal{Y}} \mathcal{X})' \longrightarrow \mathcal{X}_{lisse,étale} \text{ (resp. } \mathcal{X}_{flat,fppf})$$

the restriction of the functor pr used in the formula above. Exactly the same argument that proves Sheaves on Stacks, Equation (69.5.0.1) shows that for any sheaf $\mathcal{H}$ on $\mathcal{X}_{lisse,étale}$ (resp. $\mathcal{X}_{flat,fppf}$) we have

$$f'_*\mathcal{H}(y) = \Gamma((V \times_{y,\mathcal{Y}} \mathcal{X})',\ (\mathrm{pr}')^{-1}\mathcal{H}) \tag{75.11.6.1}$$

Since $(g')^{-1}$ is restriction we see that

$$\left(f'_*(g')^{-1}\mathcal{F}\right)(y) = \Gamma((V \times_{y,\mathcal{Y}} \mathcal{X})',\ \mathrm{pr}^{-1}\mathcal{F}|_{(V\times_{y,\mathcal{Y}}\mathcal{X})'})$$

By Sheaves on Stacks, Lemma 69.22.3 we see that

$$\Gamma((V \times_{y,\mathcal{Y}} \mathcal{X})',\ \mathrm{pr}^{-1}\mathcal{F}|_{(V\times_{y,\mathcal{Y}}\mathcal{X})'}) = \Gamma(V \times_{y,\mathcal{Y}} \mathcal{X},\ \mathrm{pr}^{-1}\mathcal{F})$$

are equal as desired; although we omit the verification of the assumptions of the lemma we note that the fact that $V \to \mathcal{Y}$ is smooth (resp. flat) is used to verify the second condition.

Finally, the equality $g'_!(f')^{-1} = f^{-1}g_!$ follows formally from the equality $f'_*(g')^{-1} = g^{-1}f_*$ by the adjointness of f^{-1} and f_*, the adjointness of $g_!$ and g^{-1}, and their "primed" versions. □

75.12. Quasi-coherent modules, II

In this section we explain how to think of quasi-coherent modules on an algebraic stack in terms of its lisse-étale or flat-fppf site.

Lemma 75.12.1. *Let $\mathcal{X}$ be an algebraic stack.*

1. *Let $f_j : \mathcal{X}_j \to \mathcal{X}$ be a family of smooth morphisms of algebraic stacks with $|\mathcal{X}| = \bigcup |f_j|(|\mathcal{X}_j|)$. Let $\mathcal{F}$ be a sheaf of $\mathcal{O}_\mathcal{X}$-modules on $\mathcal{X}_{étale}$. If each $f_j^{-1}\mathcal{F}$ is quasi-coherent, then so is $\mathcal{F}$.*
2. *Let $f_j : \mathcal{X}_j \to \mathcal{X}$ be a family of flat and locally finitely presented morphisms of algebraic stacks with $|\mathcal{X}| = \bigcup |f_j|(|\mathcal{X}_j|)$. Let $\mathcal{F}$ be a sheaf of $\mathcal{O}_\mathcal{X}$-modules on $\mathcal{X}_{fppf}$. If each $f_j^{-1}\mathcal{F}$ is quasi-coherent, then so is $\mathcal{F}$.*

Proof. Proof of (1). We may replace each of the algebraic stacks $\mathcal{X}_j$ by a scheme U_j (using that any algebraic stack has a smooth covering by a scheme and that compositions of smooth morphisms are smooth, see Morphisms of Stacks, Lemma 74.22.2). The pullback of $\mathcal{F}$ to $(Sch/U_j)_{étale}$ is still locally quasi-coherent, see Sheaves on Stacks, Lemma 69.11.2. Then $f = \coprod f_j : U = \coprod U_j \to \mathcal{X}$ is a smooth surjective morphism. Let $x : V \to \mathcal{X}$ be an object of $\mathcal{X}$. By Sheaves on Stacks, Lemma 69.18.10 there exists an étale covering $\{x_i \to x\}_{i\in I}$ such that each x_i lifts to an object u_i of $(Sch/U)_{étale}$. This just means that x_i lives over a scheme V_i, that $\{V_i \to V\}$ is an étale covering, and that x_i comes from a morphism $u_i : V_i \to U$. Then $x_i^*\mathcal{F} = u_i^* f^*\mathcal{F}$ is quasi-coherent. This implies that $x^*\mathcal{F}$ on $(Sch/V)_{étale}$ is quasi-coherent, for example by Modules on Sites, Lemma 18.23.3. By Sheaves on Stacks, Lemma 69.11.3 we see that $\mathcal{F}$ is quasi-coherent.

Proof of (2). This is proved using exactly the same argument, which we fully write out here. We may replace each of the algebraic stacks $\mathcal{X}_j$ by a scheme U_j (using that any algebraic stack has a smooth covering by a scheme and that flat and locally finite presented morphisms are preserved by composition, see Morphisms of Stacks, Lemmas 74.17.2 and 74.18.2). The pullback of $\mathcal{F}$ to $(Sch/U_j)_{étale}$ is still locally quasi-coherent, see Sheaves on Stacks, Lemma 69.11.2. Then $f = \coprod f_j : U = \coprod U_j \to \mathcal{X}$ is a surjective, flat, and locally finitely presented morphism. Let $x : V \to \mathcal{X}$ be an object of $\mathcal{X}$. By Sheaves on Stacks, Lemma 69.18.10 there exists an fppf covering $\{x_i \to x\}_{i\in I}$ such that each x_i lifts to an object u_i of $(Sch/U)_{étale}$. This just means that x_i lives over a scheme V_i, that $\{V_i \to V\}$ is an fppf covering, and that x_i comes from a morphism $u_i : V_i \to U$. Then $x_i^*\mathcal{F} = u_i^* f^*\mathcal{F}$ is quasi-coherent. This implies that $x^*\mathcal{F}$ on $(Sch/V)_{étale}$ is quasi-coherent, for example by Modules on Sites, Lemma 18.23.3. By Sheaves on Stacks, Lemma 69.11.3 we see that $\mathcal{F}$ is quasi-coherent. □

We recall that we have defined the notion of a quasi-coherent module on any ringed topos in Modules on Sites, Section 18.23.

Lemma 75.12.2. *Let $\mathcal{X}$ be an algebraic stack. Notation as in Lemma 75.11.2.*

1. *Let $\mathcal{H}$ be a quasi-coherent $\mathcal{O}_{\mathcal{X}_{lisse,étale}}$-module on the lisse-étale site of $\mathcal{X}$. Then $g_!\mathcal{H}$ is a quasi-coherent module on $\mathcal{X}$.*
2. *Let $\mathcal{H}$ be a quasi-coherent $\mathcal{O}_{\mathcal{X}_{flat,fppf}}$-module on the flat-fppf site of $\mathcal{X}$. Then $g_!\mathcal{H}$ is a quasi-coherent module on $\mathcal{X}$.*

Proof. Pick a scheme U and a surjective smooth morphism $x : U \to \mathcal{X}$. By Modules on Sites, Definition 18.23.1 there exists an étale (resp. fppf) covering $\{U_i \to U\}_{i\in I}$ such that each pullback $f_i^{-1}\mathcal{H}$ has a global presentation (see Modules on Sites, Definition 18.17.1). Here $f_i : U_i \to \mathcal{X}$ is the composition $U_i \to U \to \mathcal{X}$ which is a morphism of algebraic stacks. (Recall that the pullback "is" the restriction to $\mathcal{X}/f_i$, see Sheaves on Stacks, Definition 69.9.2 and the discussion following.) Since each f_i is smooth (resp. flat) by Lemma 75.11.6 we see that $f_i^{-1}g_!\mathcal{H} = g_{i,!}(f_i')^{-1}\mathcal{H}$. Using Lemma 75.12.1 we reduce the statement of the lemma to the case where $\mathcal{H}$ has a global presentation. Say we have

$$\bigoplus\nolimits_{j\in J} \mathcal{O} \longrightarrow \bigoplus\nolimits_{i\in I} \mathcal{O} \longrightarrow \mathcal{H} \longrightarrow 0$$

of $\mathcal{O}$-modules where $\mathcal{O} = \mathcal{O}_{\mathcal{X}_{lisse,étale}}$ (resp. $\mathcal{O} = \mathcal{O}_{\mathcal{X}_{flat,fppf}}$). Since $g_!$ commutes with arbitrary colimits (as a left adjoint functor, see Lemma 75.11.3 and Categories,

Lemma 4.24.3) we conclude that there exists an exact sequence

$$\bigoplus\nolimits_{j\in J} g_!\mathcal{O} \longrightarrow \bigoplus\nolimits_{i\in I} g_!\mathcal{O} \longrightarrow g_!\mathcal{H} \longrightarrow 0$$

Finally, Lemma 75.11.4 shows that $g_!\mathcal{O} = \mathcal{O}_\mathcal{X}$ and we win. □

Lemma 75.12.3. *Let $\mathcal{X}$ be an algebraic stack. Let $\mathcal{M}_\mathcal{X}$ be the category of locally quasi-coherent $\mathcal{O}_\mathcal{X}$-modules with the flat base change property.*

(1) *With g as in Lemma 75.11.2 for the lisse-étale site we have*
   (a) *the functors g^{-1} and $g_!$ define mutually inverse functors*

$$QCoh(\mathcal{O}_\mathcal{X}) \underset{g_!}{\overset{g^{-1}}{\rightleftarrows}} QCoh(\mathcal{X}_{lisse,étale}, \mathcal{O}_{\mathcal{X}_{lisse,étale}})$$

   (b) *if $\mathcal{F}$ is in $\mathcal{M}_\mathcal{X}$ then $g^{-1}\mathcal{F}$ is in $QCoh(\mathcal{X}_{lisse,étale}, \mathcal{O}_{\mathcal{X}_{lisse,étale}})$ and*
   (c) *$Q(\mathcal{F}) = g_!g^{-1}\mathcal{F}$ where Q is as in Lemma 75.9.1.*

(2) *With g as in Lemma 75.11.2 for the flat-fppf site we have*
   (a) *the functors g^{-1} and $g_!$ define mutually inverse functors*

$$QCoh(\mathcal{O}_\mathcal{X}) \underset{g_!}{\overset{g^{-1}}{\rightleftarrows}} QCoh(\mathcal{X}_{flat,fppf}, \mathcal{O}_{\mathcal{X}_{flat,fppf}})$$

   (b) *if $\mathcal{F}$ is in $\mathcal{M}_\mathcal{X}$ then $g^{-1}\mathcal{F}$ is in $QCoh(\mathcal{X}_{flat,fppf}, \mathcal{O}_{\mathcal{X}_{flat,fppf}})$ and*
   (c) *$Q(\mathcal{F}) = g_!g^{-1}\mathcal{F}$ where Q is as in Lemma 75.9.1.*

Proof. Pullback by any morphism of ringed topoi preserves categories of quasi-coherent modules, see Modules on Sites, Lemma 18.23.4. Hence g^{-1} preserves the categories of quasi-coherent modules. The same is true for $g_!$ by Lemma 75.12.2. We know that $\mathcal{H} \to g^{-1}g_!\mathcal{H}$ is an isomorphism by Lemma 75.11.2. Conversely, if $\mathcal{F}$ is in $QCoh(\mathcal{O}_\mathcal{X})$ then the map $g_!g^{-1}\mathcal{F} \to \mathcal{F}$ is a map of quasi-coherent modules on $\mathcal{X}$ whose restriction to any scheme smooth over $\mathcal{X}$ is an isomorphism. Then the discussion in Sheaves on Stacks, Sections 69.13 and 69.14 (comparing with quasi-coherent modules on presentations) shows it is an isomorphism. This proves (1)(a) and (2)(a).

Let $\mathcal{F}$ be an object of $\mathcal{M}_\mathcal{X}$. By Lemma 75.9.2 the kernel and cokernel of the map $Q(\mathcal{F}) \to \mathcal{F}$ are parasitic. Hence by Lemma 75.11.5 and since $g^* = g^{-1}$ is exact, we conclude $g^*Q(\mathcal{F}) \to g^*\mathcal{F}$ is an isomorphism. Thus $g^*\mathcal{F}$ is quasi-coherent. This proves (1)(b) and (2)(b). Finally, (1)(c) and (2)(c) follow because $g_!g^*Q(\mathcal{F}) \to Q(\mathcal{F})$ is an isomorphism by our arguments above. □

Remark 75.12.4. Let $\mathcal{X}$ be an algebraic stack. The results of Lemmas 75.9.1 and 75.9.2 imply that

$$QCoh(\mathcal{O}_\mathcal{X}) = \mathcal{M}_\mathcal{X}/\text{Parasitic} \cap \mathcal{M}_\mathcal{X}$$

in words: the category of quasi-coherent modules is the category of locally quasi-coherent modules with the flat base change property divided out by the Serre subcategory consisting of parasitic objects. See Homology, Lemma 12.9.6. The existence of the inclusion functor $i : QCoh(\mathcal{O}_\mathcal{X}) \to \mathcal{M}_\mathcal{X}$ which is left adjoint to the quotient functor means that $\mathcal{M}_\mathcal{X} \to QCoh(\mathcal{O}_\mathcal{X})$ is a *Bousfield colocalization* or a *right Bousfield localization* (insert future reference here). Our next goal is to show a similar result holds on the level of derived categories.

Lemma 75.12.5. *Let $\mathcal{X}$ be an algebraic stack.*

(1) *$QCoh(\mathcal{O}_{\mathcal{X}_{lisse,étale}})$ is a weak Serre subcategory of $Mod(\mathcal{O}_{\mathcal{X}_{lisse,étale}})$.*
(2) *$QCoh(\mathcal{O}_{\mathcal{X}_{flat,fppf}})$ is a weak Serre subcategory of $Mod(\mathcal{O}_{\mathcal{X}_{flat,fppf}})$.*

Proof. We will verify conditions (1), (2), (3), (4) of Homology, Lemma 12.9.3. Since 0 is a quasi-coherent module on any ringed site we see that (1) holds. By definition $QCoh(\mathcal{O})$ is a strictly full subcategory $Mod(\mathcal{O})$, so (2) holds. Let $\varphi : \mathcal{G} \to \mathcal{F}$ be a morphism of quasi-coherent modules on $\mathcal{X}_{lisse,étale}$ or $\mathcal{X}_{flat,fppf}$. We have $g^*g_!\mathcal{F} = \mathcal{F}$ and similarly for $\mathcal{G}$ and φ, see Lemma 75.11.3. By Lemma 75.12.2 we see that $g_!\mathcal{F}$ and $g_!\mathcal{G}$ are quasi-coherent $\mathcal{O}_\mathcal{X}$-modules. Hence we see that $\mathrm{Ker}(g_!\varphi)$ and $\mathrm{Coker}(g_!\varphi)$ are quasi-coherent modules on $\mathcal{X}$. Since g^* is exact (see Lemma 75.11.2) we see that $g^*\mathrm{Ker}(g_!\varphi) = \mathrm{Ker}(g^*g_!\varphi) = \mathrm{Ker}(\varphi)$ and $g^*\mathrm{Coker}(g_!\varphi) = \mathrm{Coker}(g^*g_!\varphi) = \mathrm{Coker}(\varphi)$ are quasi-coherent too (see Lemma 75.12.3). This proves (3).

Finally, suppose that

$$0 \to \mathcal{F} \to \mathcal{E} \to \mathcal{G} \to 0$$

is an extension of $\mathcal{O}_{\mathcal{X}_{lisse,étale}}$-modules (resp. $\mathcal{O}_{\mathcal{X}_{flat,fppf}}$-modules) with $\mathcal{F}$ and $\mathcal{G}$ quasi-coherent. We have to show that $\mathcal{E}$ is quasi-coherent on $\mathcal{X}_{lisse,étale}$ (resp. $\mathcal{X}_{flat,fppf}$). We strongly urge the reader to write out what this means on a napkin and prove it him/herself rather than reading the somewhat labyrinthine proof that follows. By Lemma 75.12.3 this is true if and only if $g_!\mathcal{E}$ is quasi-coherent. By Lemmas 75.12.1 and Lemma 75.11.6 we may check this after replacing $\mathcal{X}$ by a smooth (resp. fppf) cover. Choose a scheme U and a smooth surjective morphism $U \to \mathcal{X}$. By definition there exists an étale (resp. fppf) covering $\{U_i \to U\}_i$ such that $\mathcal{G}$ has a global presentation over each U_i. Replacing $\mathcal{X}$ by U_i (which is permissible by the discussion above) we may assume that the site $\mathcal{X}_{lisse,étale}$ (resp. $\mathcal{X}_{flat,fppf}$) has a final object U (in other words $\mathcal{X}$ is representable by the scheme U) and that $\mathcal{G}$ has a global presentation

$$\bigoplus\nolimits_{j\in J} \mathcal{O} \longrightarrow \bigoplus\nolimits_{i\in I} \mathcal{O} \longrightarrow \mathcal{G} \longrightarrow 0$$

of $\mathcal{O}$-modules where $\mathcal{O} = \mathcal{O}_{\mathcal{X}_{lisse,étale}}$ (resp. $\mathcal{O} = \mathcal{O}_{\mathcal{X}_{flat,fppf}}$). Let $\mathcal{E}'$ be the pullback of $\mathcal{E}$ via the map $\bigoplus_{i\in I} \mathcal{O} \to \mathcal{G}$. Then we see that $\mathcal{E}$ is the cokernel of a map $\bigoplus_{j\in J} \mathcal{O} \to \mathcal{E}'$ hence by property (3) which we proved above, it suffices to prove that $\mathcal{E}'$ is quasi-coherent. Consider the exact sequence

$$L_1g_!\left(\bigoplus\nolimits_{i\in I} \mathcal{O}\right) \to g_!\mathcal{F} \to g_!\mathcal{E}' \to g_!\left(\bigoplus\nolimits_{i\in I} \mathcal{O}\right) \to 0$$

where $L_1g_!$ is the first left derived functor of $g_! : Mod(\mathcal{O}_{\mathcal{X}_{lisse,étale}}) \to Mod(\mathcal{X}_{étale}, \mathcal{O}_\mathcal{X})$ (resp. $g_! : Mod(\mathcal{X}_{flat,fppf}, \mathcal{O}_{\mathcal{X}_{flat,fppf}}) \to Mod(\mathcal{X}_{fppf}, \mathcal{O}_\mathcal{X})$). This derived functor exists by Cohomology on Sites, Lemma 21.26.1. Moreover, since $\mathcal{O} = j_{U!}\mathcal{O}_U$ we have $Lg_!\mathcal{O} = g_!\mathcal{O} = \mathcal{O}_\mathcal{X}$ also by Cohomology on Sites, Lemma 21.26.1. Thus the left hand term vanishes and we obtain a short exact sequence

$$0 \to g_!\mathcal{F} \to g_!\mathcal{E}' \to \bigoplus\nolimits_{i\in I} \mathcal{O}_\mathcal{X} \to 0$$

By Proposition 75.7.4 it follows that $g_!\mathcal{E}'$ is locally quasi-coherent with the flat base change property. Finally, Lemma 75.12.3 implies that $\mathcal{E}' = g^{-1}g_!\mathcal{E}'$ is quasi-coherent as desired. □

75.13. Other chapters

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CHAPTER 76

Derived Categories of Stacks

76.1. Introduction

In this chapter we write about derived categories associated to algebraic stacks. This mean in particular derived categories of quasi-coherent sheaves, i.e., we prove analogues of the results in the chapters entitled "Derived Categories of Schemes" and "Derived Categories of Spaces". The results in this chapter are different from those in [**LMB00**] mainly because we consistently use the "big sites". Before reading this chapter please take a quick look at the chapters "Sheaves on Algebraic Stacks" and "Cohomology of Algebraic Stacks" where the terminology we use here is introduced.

76.2. Conventions, notation, and abuse of language

We continue to use the conventions and the abuse of language introduced in Properties of Stacks, Section 73.2. We use notation as explained in Cohomology of Stacks, Section 75.3.

76.3. The lisse-étale and the flat-fppf sites

The section is the analogue of Cohomology of Stacks, Section 75.11 for derived categories.

Lemma 76.3.1. *Let $\mathcal{X}$ be an algebraic stack. Notation as in Cohomology of Stacks, Lemmas 75.11.2 and 75.11.3.*

(1) *The functor $g_! : Ab(\mathcal{X}_{lisse,étale}) \to Ab(\mathcal{X}_{étale})$ has a left derived functor*
$$Lg_! : D(\mathcal{X}_{lisse,étale}) \longrightarrow D(\mathcal{X}_{étale})$$
which is left adjoint to g^{-1} and such that $g^{-1}Lg_! = id$.

(2) *The functor $g_! : Mod(\mathcal{X}_{lisse,étale}, \mathcal{O}_{\mathcal{X}_{lisse,étale}}) \to Mod(\mathcal{X}_{étale}, \mathcal{O}_{\mathcal{X}})$ has a left derived functor*
$$Lg_! : D(\mathcal{O}_{\mathcal{X}_{lisse,étale}}) \longrightarrow D(\mathcal{X}_{étale}, \mathcal{O}_{\mathcal{X}})$$
which is left adjoint to g^* and such that $g^*Lg_! = id$.

(3) *The functor $g_! : Ab(\mathcal{X}_{flat,fppf}) \to Ab(\mathcal{X}_{fppf})$ has a left derived functor*
$$Lg_! : D(\mathcal{X}_{flat,fppf}) \longrightarrow D(\mathcal{X}_{fppf})$$
which is left adjoint to g^{-1} and such that $g^{-1}Lg_! = id$.

(4) *The functor $g_! : Mod(\mathcal{X}_{flat,fppf}, \mathcal{O}_{\mathcal{X}_{flat,fppf}}) \to Mod(\mathcal{X}_{fppf}, \mathcal{O}_{\mathcal{X}})$ has a left derived functor*
$$Lg_! : D(\mathcal{O}_{\mathcal{X}_{flat,fppf}}) \longrightarrow D(\mathcal{O}_{\mathcal{X}})$$
which is left adjoint to g^* and such that $g^*Lg_! = id$.

Warning: It is not clear (a priori) that $Lg_!$ on modules agrees with $Lg_!$ on abelian sheaves, see Cohomology on Sites, Remark 21.26.2.

Proof. The existence of the functor $Lg_!$ and adjointness to g^* is Cohomology on Sites, Lemma 21.26.1. (For the case of abelian sheaves use the constant sheaf $\mathbf{Z}$ as the structure sheaves.) Moreover, it is computed on a complex $\mathcal{H}^\bullet$ by taking a suitable left resolution $\mathcal{K}^\bullet \to \mathcal{H}^\bullet$ and applying the functor $g_!$ to $\mathcal{K}^\bullet$. Since $g^{-1}g_!\mathcal{K}^\bullet = \mathcal{K}^\bullet$ by Cohomology of Stacks, Lemmas 75.11.3 and 75.11.2 we see that the final assertion holds in each case. □

Lemma 76.3.2. *With assumptions and notation as in Cohomology of Stacks, Lemma 75.11.6. We have*

$$g^{-1} \circ Rf_* = Rf'_* \circ (g')^{-1} \quad \text{and} \quad L(g')_! \circ (f')^{-1} = f^{-1} \circ Lg_!$$

on unbounded derived categories (both for the case of modules and for the case of abelian sheaves).

Proof. Let $\mathcal{F}$ be an abelian sheaf on $\mathcal{X}_{étale}$ (resp. $\mathcal{X}_{fppf}$). We first show that the canonical (base change) map

$$g^{-1}Rf_*\mathcal{F} \longrightarrow Rf'_*(g')^{-1}\mathcal{F}$$

is an isomorphism. To do this let y be an object of $\mathcal{Y}_{lisse,étale}$ (resp. $\mathcal{Y}_{flat,fppf}$). Say y lies over the scheme V such that $y : V \to \mathcal{Y}$ is smooth (resp. flat). Since g^{-1} is the restriction we find that

$$\left(g^{-1}R^pf_*\mathcal{F}\right)(y) = H^p_\tau(V \times_{y,\mathcal{Y}} \mathcal{X},\ \mathrm{pr}^{-1}\mathcal{F})$$

where $\tau = étale$ (resp. $\tau = fppf$), see Sheaves on Stacks, Lemma 69.20.2. By Cohomology of Stacks, Equation (75.11.6.1) for any sheaf $\mathcal{H}$ on $\mathcal{X}_{lisse,étale}$ (resp. $\mathcal{X}_{flat,fppf}$)

$$f'_*\mathcal{H}(y) = \Gamma((V \times_{y,\mathcal{Y}} \mathcal{X})',\ (\mathrm{pr}')^{-1}\mathcal{H})$$

An object of $(V \times_{y,\mathcal{Y}} \mathcal{X})'$ can be seen as a pair (x, φ) where x is an object of $\mathcal{X}_{lisse,étale}$ (resp. $\mathcal{X}_{flat,fppf}$) and $\varphi : f(x) \to y$ is a morphism in $\mathcal{Y}$. We can also think of φ as a section of $(f')^{-1}h_y$ over x. Thus $(V \times_{\mathcal{Y}} \mathcal{X})'$ is the localization of the site $\mathcal{X}_{lisse,étale}$ (resp. $\mathcal{X}_{flat,fppf}$) at the sheaf of sets $(f')^{-1}h_y$, see Sites, Lemma 7.29.3. The morphism

$$\mathrm{pr}' : (V \times_{y,\mathcal{Y}} \mathcal{X})' \to \mathcal{X}_{lisse,étale} \text{ (resp. } \mathrm{pr}' : (V \times_{y,\mathcal{Y}} \mathcal{X})' \to \mathcal{X}_{flat,fppf})$$

is the localization morphism. In particular, the pullback $(\mathrm{pr}')^{-1}$ preserves injective abelian sheaves, see Cohomology on Sites, Lemma 21.13.3. At this point exactly the same argument as in Sheaves on Stacks, Lemma 69.20.2 shows that

$$R^pf'_*\mathcal{H}(y) = H^p_\tau((V \times_{y,\mathcal{Y}} \mathcal{X})',\ (\mathrm{pr}')^{-1}\mathcal{H}) \tag{76.3.2.1}$$

where $\tau = étale$ (resp. $\tau = fppf$). Since $(g')^{-1}$ is given by restriction we conclude that

$$(R^pf'_*(g')^*\mathcal{F})(y) = H^p_\tau((V \times_{y,\mathcal{Y}} \mathcal{X})',\ \mathrm{pr}^{-1}\mathcal{F}|_{(V\times_{y,\mathcal{Y}}\mathcal{X})'})$$

Finally, we can apply Sheaves on Stacks, Lemma 69.22.3 to see that

$$H^p_\tau((V \times_{y,\mathcal{Y}} \mathcal{X})',\ \mathrm{pr}^{-1}\mathcal{F}|_{(V\times_{y,\mathcal{Y}}\mathcal{X})'}) = H^p_\tau(V \times_{y,\mathcal{Y}} \mathcal{X},\ \mathrm{pr}^{-1}\mathcal{F})$$

are equal as desired; although we omit the verification of the assumptions of the lemma we note that the fact that $V \to \mathcal{Y}$ is smooth (resp. flat) is used to verify the second condition.

The rest of the proof is formal. Since cohomology of abelian groups and sheaves of modules agree we also conclude that $g^{-1}Rf_*\mathcal{F} = Rf'_*(g')^{-1}\mathcal{F}$ when $\mathcal{F}$ is a sheaf of modules on $\mathcal{X}_{étale}$ (resp. $\mathcal{X}_{fppf}$).

Next we show that for $\mathcal{G}$ (either sheaf of modules or abelian groups) on $\mathcal{Y}_{lisse,étale}$ (resp. $\mathcal{Y}_{flat,fppf}$) the canonical map

$$L(g')_!(f')^{-1}\mathcal{G} \to f^{-1}Lg_!\mathcal{G}$$

is an isomorphism. To see this it is enough to prove for any injective sheaf $\mathcal{I}$ on $\mathcal{X}_{étale}$ (resp. $\mathcal{X}_{fppf}$) that the induced map

$$\mathrm{Hom}(L(g')_!(f')^{-1}\mathcal{G}, \mathcal{I}[n]) \leftarrow \mathrm{Hom}(f^{-1}Lg_!\mathcal{G}, \mathcal{I}[n])$$

is an isomorphism for all $n \in \mathbf{Z}$. (Hom's taken in suitable derived categories.) By the adjointness of f^{-1} and Rf_*, the adjointness of $Lg_!$ and g^{-1}, and their "primed" versions this follows from the isomorphism $g^{-1}Rf_*\mathcal{I} \to Rf'_*(g')^{-1}\mathcal{I}$ proved above.

In the case of a bounded complex $\mathcal{G}^\bullet$ (of modules or abelian groups) on $\mathcal{Y}_{lisse,étale}$ (resp. $\mathcal{Y}_{fppf}$) the canonical map

$$L(g')_!(f')^{-1}\mathcal{G}^\bullet \to f^{-1}Lg_!\mathcal{G}^\bullet \tag{76.3.2.2}$$

is an isomorphism as follows from the case of a sheaf by the usual arguments involving truncations and the fact that the functors $L(g')_!(f')^{-1}$ and $f^{-1}Lg_!$ are exact functors of triangulated categories.

Suppose that $\mathcal{G}^\bullet$ is a bounded above complex (of modules or abelian groups) on $\mathcal{Y}_{lisse,étale}$ (resp. $\mathcal{Y}_{fppf}$). The canonical map (76.3.2.2) is an isomorphism because we can use the stupid truncations $\sigma_{\geq -n}$ (see Homology, Section 12.13) to write $\mathcal{G}^\bullet$ as a colimit $\mathcal{G}^\bullet = \mathrm{colim}\, \mathcal{G}^\bullet_n$ of bounded complexes. This gives a distinguished triangle

$$\bigoplus\nolimits_{n\geq 1} \mathcal{G}^\bullet_n \to \bigoplus\nolimits_{n\geq 1} \mathcal{G}^\bullet_n \to \mathcal{G}^\bullet \to \ldots$$

and each of the functors $L(g')_!$, $(f')^{-1}$, f^{-1}, $Lg_!$ commutes with direct sums (of complexes).

If $\mathcal{G}^\bullet$ is an arbitrary complex (of modules or abelian groups) on $\mathcal{Y}_{lisse,étale}$ (resp. $\mathcal{Y}_{fppf}$) then we use the canonical truncations $\tau_{\leq n}$ (see Homology, Section 12.13) to write $\mathcal{G}^\bullet$ as a colimit of bounded above complexes and we repeat the argument of the paragraph above.

Finally, by the adjointness of f^{-1} and Rf_*, the adjointness of $Lg_!$ and g^{-1}, and their "primed" versions we conclude that the first identity of the lemma follows from the second in full generality. □

Lemma 76.3.3. *Let $\mathcal{X}$ be an algebraic stack. Notation as in Cohomology of Stacks, Lemma 75.11.2.*

1. *Let $\mathcal{H}$ be a quasi-coherent $\mathcal{O}_{\mathcal{X}_{lisse,étale}}$-module on the lisse-étale site of $\mathcal{X}$. For all $p \in \mathbf{Z}$ the sheaf $H^p(Lg_!\mathcal{H})$ is a locally quasi-coherent module with the flat base change property on $\mathcal{X}$.*
2. *Let $\mathcal{H}$ be a quasi-coherent $\mathcal{O}_{\mathcal{X}_{flat,fppf}}$-module on the flat-fppf site of $\mathcal{X}$. For all $p \in \mathbf{Z}$ the sheaf $H^p(Lg_!\mathcal{H})$ is a locally quasi-coherent module with the flat base change property on $\mathcal{X}$.*

Proof. Pick a scheme U and a surjective smooth morphism $x : U \to \mathcal{X}$. By Modules on Sites, Definition 18.23.1 there exists an étale (resp. fppf) covering $\{U_i \to U\}_{i \in I}$ such that each pullback $f_i^{-1}\mathcal{H}$ has a global presentation (see Modules on Sites, Definition 18.17.1). Here $f_i : U_i \to \mathcal{X}$ is the composition $U_i \to U \to \mathcal{X}$ which is a morphism of algebraic stacks. (Recall that the pullback "is" the restriction to $\mathcal{X}/f_i$, see Sheaves on Stacks, Definition 69.9.2 and the discussion following.) After refining the covering we may assume each U_i is an affine scheme. Since each f_i is smooth (resp. flat) by Lemma 76.3.2 we see that $f_i^{-1}Lg_!\mathcal{H} = Lg_{i,!}(f_i')^{-1}\mathcal{H}$. Using Cohomology of Stacks, Lemma 75.7.5 we reduce the statement of the lemma to the case where $\mathcal{H}$ has a global presentation and where $\mathcal{X} = (Sch/X)_{fppf}$ for some affine scheme $X = \mathrm{Spec}(A)$.

Say our presentation looks like

$$\bigoplus\nolimits_{j \in J} \mathcal{O} \longrightarrow \bigoplus\nolimits_{i \in I} \mathcal{O} \longrightarrow \mathcal{H} \longrightarrow 0$$

where $\mathcal{O} = \mathcal{O}_{\mathcal{X}_{lisse,étale}}$ (resp. $\mathcal{O} = \mathcal{O}_{\mathcal{X}_{flat,fppf}}$). Note that the site $\mathcal{X}_{lisse,étale}$ (resp. $\mathcal{X}_{flat,fppf}$) has a final object, namely X/X which is quasi-compact (see Cohomology on Sites, Section 21.16). Hence we have

$$\Gamma(\bigoplus\nolimits_{i \in I} \mathcal{O}) = \bigoplus\nolimits_{i \in I} A$$

by Sites, Lemma 7.11.2. Hence the map in the presentation corresponds to a similar presentation

$$\bigoplus\nolimits_{j \in J} A \longrightarrow \bigoplus\nolimits_{i \in I} A \longrightarrow M \longrightarrow 0$$

of an A-module M. Moreover, $\mathcal{H}$ is equal to the restriction to the lisse-étale (resp. flat-fppf) site of the quasi-coherent sheaf M^a associated to M. Choose a resolution

$$\ldots \to F_2 \to F_1 \to F_0 \to M \to 0$$

by free A-modules. The complex

$$\ldots \mathcal{O} \otimes_A F_2 \to \mathcal{O} \otimes_A F_1 \to \mathcal{O} \otimes_A F_0 \to \mathcal{H} \to 0$$

is a resolution of $\mathcal{H}$ by free $\mathcal{O}$-modules because for each object U/X of $\mathcal{X}_{lisse,étale}$ (resp. $\mathcal{X}_{flat,fppf}$) the structure morphism $U \to X$ is flat. Hence by construction the value of $Lg_!\mathcal{H}$ is

$$\ldots \to \mathcal{O}_\mathcal{X} \otimes_A F_2 \to \mathcal{O}_\mathcal{X} \otimes_A F_1 \to \mathcal{O}_\mathcal{X} \otimes_A F_0 \to 0 \to \ldots$$

Since this is a complex of quasi-coherent modules on $\mathcal{X}_{étale}$ (resp. $\mathcal{X}_{fppf}$) it follows from Cohomology of Stacks, Proposition 75.7.4 that $H^p(Lg_!\mathcal{H})$ is quasi-coherent. □

76.4. Derived categories of quasi-coherent modules

Let $\mathcal{X}$ be an algebraic stack. As the inclusion functor $QCoh(\mathcal{O}_\mathcal{X}) \to Mod(\mathcal{O}_\mathcal{X})$ isn't exact, we cannot define $D_{QCoh}(\mathcal{O}_\mathcal{X})$ as the full subcategory of $D(\mathcal{O}_\mathcal{X})$ consisting of complexes with quasi-coherent cohomology sheaves. In stead we define the category as follows.

Definition 76.4.1. Let $\mathcal{X}$ be an algebraic stack. Let $\mathcal{M}_\mathcal{X} \subset Mod(\mathcal{O}_\mathcal{X})$ denote the category of locally quasi-coherent $\mathcal{O}_\mathcal{X}$-modules with the flat base change property. Let $\mathcal{P}_\mathcal{X} \subset \mathcal{M}_\mathcal{X}$ be the full subcategory consisting of parasitic objects. We define

the *derived category of $\mathcal{O}_\mathcal{X}$-modules with quasi-coherent cohomology sheaves* as the Verdier quotient[1]

$$D_{QCoh}(\mathcal{O}_\mathcal{X}) = D_{\mathcal{M}_\mathcal{X}}(\mathcal{O}_\mathcal{X})/D_{\mathcal{P}_\mathcal{X}}(\mathcal{O}_\mathcal{X})$$

This definition makes sense: By Cohomology of Stacks, Proposition 75.7.4 we see that $\mathcal{M}_\mathcal{X}$ is a weak Serre subcategory of $Mod(\mathcal{O}_\mathcal{X})$ hence $D_{\mathcal{M}_\mathcal{X}}(\mathcal{O}_\mathcal{X})$ is a strictly full, saturated triangulated subcategory of $D(\mathcal{O}_\mathcal{X})$, see Derived Categories, Lemma 13.12.1. Since parasitic modules form a Serre subcategory of $Mod(\mathcal{O}_\mathcal{X})$ (by Cohomology of Stacks, Lemma 75.8.2) we see that $\mathcal{P}_\mathcal{X} = \text{Parasitic} \cap \mathcal{M}_\mathcal{X}$ is a weak Serre subcategory of $Mod(\mathcal{O}_\mathcal{X})$ and hence $D_{\mathcal{P}_\mathcal{X}}(\mathcal{O}_\mathcal{X})$ is a strictly full, saturated triangulated subcategory of $D(\mathcal{O}_\mathcal{X})$. Since clearly

$$D_{\mathcal{P}_\mathcal{X}}(\mathcal{O}_\mathcal{X}) \subset D_{\mathcal{M}_\mathcal{X}}(\mathcal{O}_\mathcal{X})$$

we conclude that the first is a strictly full, saturated triangulated subcategory of the second. Hence the Verdier quotient exists. A morphism $a : E \to E'$ of $D_{\mathcal{M}_\mathcal{X}}(\mathcal{O}_\mathcal{X})$ becomes an isomorphism in $D_{QCoh}(\mathcal{O}_\mathcal{X})$ if and only if the cone $C(a)$ has parasitic cohomology sheaves, see Derived Categories, Section 13.6 and especially Lemma 13.6.10.

Consider the functors

$$D_{\mathcal{M}_\mathcal{X}}(\mathcal{O}_\mathcal{X}) \xrightarrow{H^i} \mathcal{M}_\mathcal{X} \xrightarrow{Q} QCoh(\mathcal{O}_\mathcal{X})$$

Note that Q annihilates the subcategory $\mathcal{P}_\mathcal{X}$, see Cohomology of Stacks, Lemma 75.9.2. By Derived Categories, Lemma 13.6.8 we obtain a cohomological functor

$$H^i : D_{QCoh}(\mathcal{O}_\mathcal{X}) \longrightarrow QCoh(\mathcal{O}_\mathcal{X}) \tag{76.4.1.1}$$

Moreover, note that $E \in D_{QCoh}(\mathcal{O}_\mathcal{X})$ is zero if and only if $H^i(E) = 0$ for all $i \in \mathbf{Z}$.

Note that the categories $\mathcal{P}_\mathcal{X}$ and $\mathcal{M}_\mathcal{X}$ are also weak Serre subcategories of the abelian category $Mod(\mathcal{X}_{étale}, \mathcal{O}_\mathcal{X})$ of modules in the étale topology, see Cohomology of Stacks, Proposition 75.7.4 and Lemma 75.8.2. Hence the statement of the following lemma makes sense.

Lemma 76.4.2. *Let $\mathcal{X}$ be an algebraic stack. The comparison morphism $\epsilon : \mathcal{X}_{fppf} \to \mathcal{X}_{étale}$ induces a commutative diagram*

$$\begin{array}{ccccc} D_{\mathcal{P}_\mathcal{X}}(\mathcal{O}_\mathcal{X}) & \longrightarrow & D_{\mathcal{M}_\mathcal{X}}(\mathcal{O}_\mathcal{X}) & \longrightarrow & D(\mathcal{O}_\mathcal{X}) \\ \uparrow{\scriptstyle \epsilon^*} & & \uparrow{\scriptstyle \epsilon^*} & & \uparrow{\scriptstyle \epsilon^*} \\ D_{\mathcal{P}_\mathcal{X}}(\mathcal{X}_{étale}, \mathcal{O}_\mathcal{X}) & \longrightarrow & D_{\mathcal{M}_\mathcal{X}}(\mathcal{X}_{étale}, \mathcal{O}_\mathcal{X}) & \longrightarrow & D(\mathcal{X}_{étale}, \mathcal{O}_\mathcal{X}) \end{array}$$

Moreover, the left two vertical arrows are equivalences of triangulated categories, hence we also obtain an equivalence

$$D_{\mathcal{M}_\mathcal{X}}(\mathcal{X}_{étale}, \mathcal{O}_\mathcal{X})/D_{\mathcal{P}_\mathcal{X}}(\mathcal{X}_{étale}, \mathcal{O}_\mathcal{X}) \longrightarrow D_{QCoh}(\mathcal{O}_\mathcal{X})$$

Proof. Since ϵ^* is exact it is clear that we obtain a diagram as in the statement of the lemma. We will show the middle vertical arrow is an equivalence by applying Cohomology on Sites, Lemma 21.22.4 to the following situation: $\mathcal{C} = \mathcal{X}$, $\tau = fppf$, $\tau' = étale$, $\mathcal{O} = \mathcal{O}_\mathcal{X}$, $\mathcal{A} = \mathcal{M}_\mathcal{X}$, and $\mathcal{B}$ is the set of objects of $\mathcal{X}$ lying over affine schemes. To see the lemma applies we have to check conditions (1), (2), (3), (4).

[1]This definition is different from the one in the literature, see [**Ols07b**, 6.3], but it agrees with that definition by Lemma 76.4.3.

Conditions (1) and (2) are clear from the discussion above (explicitly this follows from Cohomology of Stacks, Proposition 75.7.4). Condition (3) holds because every scheme has a Zariski open covering by affines. Condition (4) follows from Descent, Lemma 34.8.4.

We omit the verification that the equivalence of categories $\epsilon^* : D_{\mathcal{M}_\mathcal{X}}(\mathcal{X}_{étale}, \mathcal{O}_\mathcal{X}) \to D_{\mathcal{M}_\mathcal{X}}(\mathcal{O}_\mathcal{X})$ induces an equivalence of the subcategories of complexes with parasitic cohomology sheaves. □

It turns out that $D_{QCoh}(\mathcal{O}_\mathcal{X})$ is the same as the derived category of complexes of modules with quasi-coherent cohomology sheaves on the lisse-étale or flat-fppf site.

Lemma 76.4.3. *Let $\mathcal{X}$ be an algebraic stack. Let $\mathcal{F}^\bullet$ be an object of $D_{\mathcal{M}_\mathcal{X}}(\mathcal{O}_\mathcal{X})$.*

(1) *With g as in Cohomology of Stacks, Lemma 75.11.2 for the lisse-étale site we have*
   (a) *$g^{-1}\mathcal{F}^\bullet$ is in $D_{QCoh}(\mathcal{O}_{\mathcal{X}_{lisse,étale}})$,*
   (b) *$g^{-1}\mathcal{F}^\bullet = 0$ if and only if $\mathcal{F}^\bullet$ is in $D_{\mathcal{P}_\mathcal{X}}(\mathcal{O}_\mathcal{X})$,*
   (c) *$Lg_!\mathcal{H}^\bullet$ is in $D_{\mathcal{M}_\mathcal{X}}(\mathcal{X}_{étale}, \mathcal{O}_\mathcal{X})$ for $\mathcal{H}^\bullet$ in $D_{QCoh}(\mathcal{O}_{\mathcal{X}_{lisse,étale}})$, and*
   (d) *the functors g^{-1} and $Lg_!$ define mutually inverse functors*

$$D_{QCoh}(\mathcal{O}_\mathcal{X}) \underset{Lg_!}{\overset{g^{-1}}{\rightleftarrows}} D_{QCoh}(\mathcal{O}_{\mathcal{X}_{lisse,étale}})$$

(2) *With g as in Cohomology of Stacks, Lemma 75.11.2 for the flat-fppf site we have*
   (a) *$g^{-1}\mathcal{F}^\bullet$ is in $D_{QCoh}(\mathcal{O}_{\mathcal{X}_{lisse,étale}})$,*
   (b) *$g^{-1}\mathcal{F}^\bullet = 0$ if and only if $\mathcal{F}^\bullet$ is in $D_{\mathcal{P}_\mathcal{X}}(\mathcal{O}_\mathcal{X})$,*
   (c) *$Lg_!\mathcal{H}^\bullet$ is in $D_{\mathcal{M}_\mathcal{X}}(\mathcal{O}_\mathcal{X})$ for $\mathcal{H}^\bullet$ in $D_{QCoh}(\mathcal{O}_{\mathcal{X}_{flat,fppf}})$, and*
   (d) *the functors g^{-1} and $Lg_!$ define mutually inverse functors*

$$D_{QCoh}(\mathcal{O}_\mathcal{X}) \underset{Lg_!}{\overset{g^{-1}}{\rightleftarrows}} D_{QCoh}(\mathcal{O}_{\mathcal{X}_{flat,fppf}})$$

Proof. The functor g^{-1} is exact, hence (a) and (b) follow from Cohomology of Stacks, Lemmas 75.12.3 and 75.11.5.

The construction of $Lg_!$ in Lemma 76.3.1 (via Cohomology on Sites, Lemma 21.26.1 which in turn uses Derived Categories, Proposition 13.27.2) shows that $Lg_!$ on any object $\mathcal{H}^\bullet$ of $D(\mathcal{O}_{\mathcal{X}_{lisse,étale}})$ is computed as

$$Lg_!\mathcal{H}^\bullet = \operatorname{colim} g_!\mathcal{K}^\bullet_n = g_! \operatorname{colim} \mathcal{K}^\bullet_n$$

(termwise colimits) where the quasi-isomorphism $\operatorname{colim} \mathcal{K}^\bullet_n \to \mathcal{H}^\bullet$ induces quasi-isomorphisms $\mathcal{K}^\bullet_n \to \tau_{\leq n}\mathcal{H}^\bullet$. Since $\mathcal{M}_\mathcal{X} \subset Mod(\mathcal{X}_{étale}, \mathcal{O}_\mathcal{X})$ (resp. $\mathcal{M}_\mathcal{X} \subset Mod(\mathcal{O}_\mathcal{X})$) is preserved under colimits we see that it suffices to prove (c) on bounded above complexes $\mathcal{H}^\bullet$ in $D_{QCoh}(\mathcal{O}_{\mathcal{X}_{lisse,étale}})$ (resp. $D_{QCoh}(\mathcal{O}_{\mathcal{X}_{flat,fppf}})$). In this case to show that $H^n(Lg_!\mathcal{H}^\bullet)$ is in $\mathcal{M}_\mathcal{X}$ we can argue by induction on the integer m such that $\mathcal{H}^i = 0$ for $i > m$. If $m < n$, then $H^n(Lg_!\mathcal{H}^\bullet) = 0$ and the result holds. In general consider the distinguished triangle

$$\tau_{\leq m-1}\mathcal{H}^\bullet \to \mathcal{H}^\bullet \to H^m(\mathcal{H}^\bullet)[-m] \to \ldots$$

(Derived Categories, Remark 13.11.4) and apply the functor $Lg_!$. Since $\mathcal{M}_\mathcal{X}$ is a weak Serre subcategory of the module category it suffices to prove (c) for two out

of three. We have the result for $Lg_!\tau_{\leq m-1}\mathcal{H}^\bullet$ by induction and we have the result for $Lg_!H^m(\mathcal{H}^\bullet)[-m]$ by Lemma 76.3.3. Whence (c) holds.

Let us prove (2)(d). By (a) and (b) the functor $g^{-1} = g^*$ induces a functor

$$c : D_{QCoh}(\mathcal{O}_\mathcal{X}) \longrightarrow D_{QCoh}(\mathcal{O}_{\mathcal{X}_{flat,fppf}})$$

see Derived Categories, Lemma 13.6.8. Thus we have the following diagram of triangulated categories

$$\begin{array}{ccc} D_{\mathcal{M}_\mathcal{X}}(\mathcal{O}_\mathcal{X}) & \xrightarrow{\quad q \quad} & D_{QCoh}(\mathcal{O}_\mathcal{X}) \\ & {\scriptstyle g^{-1}} \searrow \quad \swarrow {\scriptstyle Lg_!} & \quad \swarrow {\scriptstyle c} \\ & D_{QCoh}(\mathcal{O}_{\mathcal{X}_{flat,fppf}}) & \end{array}$$

where q is the quotient functor, the inner triangle is commutative, and $g^{-1}Lg_! = \mathrm{id}$. For any object of E of $D_{\mathcal{M}_\mathcal{X}}(\mathcal{O}_\mathcal{X})$ the map $a : Lg_!g^{-1}E \to E$ maps to a quasi-isomorphism in $D(\mathcal{O}_{\mathcal{X}_{flat,fppf}})$. Hence the cone on a maps to zero under g^{-1} and by (b) we see that $q(a)$ is an isomorphism. Thus $q \circ Lg_!$ is a quasi-inverse to c.

In the case of the lisse-étale site exactly the same argument as above proves that

$$D_{\mathcal{M}_\mathcal{X}}(\mathcal{X}_{étale}, \mathcal{O}_\mathcal{X})/D_{\mathcal{P}_\mathcal{X}}(\mathcal{X}_{étale}, \mathcal{O}_\mathcal{X})$$

is equivalent to $D_{QCoh}(\mathcal{O}_{\mathcal{X}_{lisse,étale}})$. Applying the last equivalence of Lemma 76.4.2 finishes the proof. □

The following lemma tells us that the quotient functor $D_{\mathcal{M}_\mathcal{X}}(\mathcal{O}_\mathcal{X}) \to D_{QCoh}(\mathcal{O}_\mathcal{X})$ is a Bousfield colocalization (insert future reference here).

Lemma 76.4.4. *Let $\mathcal{X}$ be an algebraic stack. Let E be an object of $D_{\mathcal{M}_\mathcal{X}}(\mathcal{O}_\mathcal{X})$. There exists a canonical distinguished triangle*

$$E' \to E \to P \to E'[1]$$

in $D_{\mathcal{M}_\mathcal{X}}(\mathcal{O}_\mathcal{X})$ such that P is in $D_{\mathcal{P}_\mathcal{X}}(\mathcal{O}_\mathcal{X})$ and

$$\mathrm{Hom}_{D(\mathcal{O}_\mathcal{X})}(E', P') = 0$$

for all P' in $D_{\mathcal{P}_\mathcal{X}}(\mathcal{O}_\mathcal{X})$.

Proof. Consider the morphism of ringed topoi $g : Sh(\mathcal{X}_{flat,fppf}) \longrightarrow Sh(\mathcal{X}_{fppf})$. Set $E' = Lg_!g^{-1}E$ and let P be the cone on the adjunction map $E' \to E$. Since $g^{-1}E' \to g^{-1}E$ is an isomorphism we see that P is an object of $D_{\mathcal{P}_\mathcal{X}}(\mathcal{O}_\mathcal{X})$ by Lemma 76.4.3 (2)(b). Finally, $\mathrm{Hom}(E', P') = \mathrm{Hom}(Lg_!g^{-1}E, P') = \mathrm{Hom}(g^{-1}E, g^{-1}P') = 0$ as $g^{-1}P' = 0$.

Uniqueness. Suppose that $E'' \to E \to P'$ is a second distinguished triangle as in the statement of the lemma. Since $\mathrm{Hom}(E', P') = 0$ the morphism $E' \to E$ factors as $E' \to E'' \to E$, see Derived Categories, Lemma 13.4.2. Similarly, the morphism $E'' \to E$ factors as $E'' \to E' \to E$. Consider the composition $\varphi : E' \to E'$ of the maps $E' \to E''$ and $E'' \to E'$. Note that $\varphi - 1 : E' \to E'$ fits into the commutative diagram

$$\begin{array}{ccc} E' & \longrightarrow & E \\ \downarrow{\scriptstyle \varphi - 1} & & \downarrow{\scriptstyle 0} \\ E' & \longrightarrow & E \end{array}$$

hence factors through $P[-1] \to E$. Since $\mathrm{Hom}(E', P[-1]) = 0$ we see that $\varphi = 1$. Whence the maps $E' \to E''$ and $E'' \to E'$ are inverse to each other. □

76.5. Derived pushforward of quasi-coherent modules

As a first application of the material above we construct the derived pushforward. In Examples, Section 78.45 the reader can find an example of a quasi-compact and quasi-separated morphism $f : \mathcal{X} \to \mathcal{Y}$ of algebraic stacks such that the direct image functor Rf_* does not induce a functor $D_{QCoh}(\mathcal{O}_\mathcal{X}) \to D_{QCoh}(\mathcal{O}_\mathcal{Y})$. Thus restricting to bounded below complexes is necessary.

Proposition 76.5.1. *Let $f : \mathcal{X} \to \mathcal{Y}$ be a quasi-compact and quasi-separated morphism of algebraic stacks. The functor Rf_* induces a commutative diagram*

$$\begin{array}{ccccc} D^+_{\mathcal{P}_\mathcal{X}}(\mathcal{O}_\mathcal{X}) & \longrightarrow & D^+_{\mathcal{M}_\mathcal{X}}(\mathcal{O}_\mathcal{X}) & \longrightarrow & D(\mathcal{O}_\mathcal{X}) \\ \downarrow{\scriptstyle Rf_*} & & \downarrow{\scriptstyle Rf_*} & & \downarrow{\scriptstyle Rf_*} \\ D^+_{\mathcal{P}_\mathcal{Y}}(\mathcal{O}_\mathcal{Y}) & \longrightarrow & D^+_{\mathcal{M}_\mathcal{Y}}(\mathcal{O}_\mathcal{Y}) & \longrightarrow & D(\mathcal{O}_\mathcal{Y}) \end{array}$$

and hence induces a functor

$$Rf_{QCoh,*} : D^+_{QCoh}(\mathcal{O}_\mathcal{X}) \longrightarrow D^+_{QCoh}(\mathcal{O}_\mathcal{Y})$$

on quotient categories. Moreover, the functor $R^i f_{QCoh}$ of Cohomology of Stacks, Proposition 75.10.1 are equal to $H^i \circ Rf_{QCoh,*}$ with H^i as in (76.4.1.1).

Proof. We have to show that Rf_*E is an object of $D^+_{\mathcal{M}_\mathcal{Y}}(\mathcal{O}_\mathcal{Y})$ for E in $D^+_{\mathcal{M}_\mathcal{X}}(\mathcal{O}_\mathcal{X})$. This follows from Cohomology of Stacks, Proposition 75.7.4 and the spectral sequence $R^i f_* H^j(E) \Rightarrow R^{i+j} f_* E$. The case of parasitic modules works the same way using Cohomology of Stacks, Lemma 75.8.3. The final statement is clear from the definition of H^i in (76.4.1.1). □

76.6. Derived pullback of quasi-coherent modules

Derived pullback of complexes with quasi-coherent cohomology sheaves exists in general.

Proposition 76.6.1. *Let $f : \mathcal{X} \to \mathcal{Y}$ be a morphism of algebraic stacks. The exact functor f^* induces a commutative diagram*

$$\begin{array}{ccc} D_{\mathcal{M}_\mathcal{X}}(\mathcal{O}_\mathcal{X}) & \longrightarrow & D(\mathcal{O}_\mathcal{X}) \\ \uparrow{\scriptstyle f^*} & & \uparrow{\scriptstyle f^*} \\ D_{\mathcal{M}_\mathcal{Y}}(\mathcal{O}_\mathcal{Y}) & \longrightarrow & D(\mathcal{O}_\mathcal{Y}) \end{array}$$

The composition

$$D_{\mathcal{M}_\mathcal{Y}}(\mathcal{O}_\mathcal{Y}) \xrightarrow{f^*} D_{\mathcal{M}_\mathcal{X}}(\mathcal{O}_\mathcal{X}) \xrightarrow{q_\mathcal{X}} D_{QCoh}(\mathcal{O}_\mathcal{X})$$

is left deriveable with respect to the localization $D_{\mathcal{M}_\mathcal{Y}}(\mathcal{O}_\mathcal{Y}) \to D_{QCoh}(\mathcal{O}_\mathcal{Y})$ and we may define Lf^*_{QCoh} as its left derived functor

$$Lf^*_{QCoh} : D_{QCoh}(\mathcal{O}_\mathcal{Y}) \longrightarrow D_{QCoh}(\mathcal{O}_\mathcal{X})$$

(see Derived Categories, Definitions 13.14.2 and 13.14.9). If f is quasi-compact and quasi-separated, then Lf^*_{QCoh} and $Rf_{QCoh,*}$ satisfy the following adjointness:

$$\mathrm{Hom}_{D_{QCoh}(\mathcal{O}_\mathcal{X})}(Lf^*_{QCoh}A, B) = \mathrm{Hom}_{D_{QCoh}(\mathcal{O}_\mathcal{Y})}(A, Rf_{QCoh,*}B)$$

for $A \in D_{QCoh}(\mathcal{O}_\mathcal{Y})$ and $B \in D^+_{QCoh}(\mathcal{O}_\mathcal{X})$.

Proof. To prove the first statement, we have to show that f^*E is an object of $D_{\mathcal{M}_\mathcal{X}}(\mathcal{O}_\mathcal{X})$ for E in $D_{\mathcal{M}_\mathcal{Y}}(\mathcal{O}_\mathcal{Y})$. Since $f^* = f^{-1}$ is exact this follows immediately from the fact that f^* maps $\mathcal{M}_\mathcal{Y}$ into $\mathcal{M}_\mathcal{X}$.

Set $\mathcal{D} = D_{\mathcal{M}_\mathcal{Y}}(\mathcal{O}_\mathcal{Y})$. Let S be the collection of morphisms in $\mathcal{D}$ whose cone is an object of $D_{\mathcal{P}_\mathcal{Y}}(\mathcal{O}_\mathcal{Y})$. Set $\mathcal{D}' = D_{QCoh}(\mathcal{O}_\mathcal{X})$. Set $F = q_\mathcal{X} \circ f^* : \mathcal{D} \to \mathcal{D}'$. Then $\mathcal{D}, S, \mathcal{D}', F$ are as in Derived Categories, Situation 13.14.1 and Definition 13.14.2. Let us prove that $LF(E)$ is defined for any object E of $\mathcal{D}$. Namely, consider the triangle

$$E' \to E \to P \to E'[1]$$

constructed in Lemma 76.4.4. Note that $s : E' \to E$ is an element of S. We claim that E' computes LF. Namely, suppose that $s' : E'' \to E$ is another element of S, i.e., fits into a triangle $E'' \to E \to P' \to E''[1]$ with P' in $D_{\mathcal{P}_\mathcal{Y}}(\mathcal{O}_\mathcal{Y})$. By Lemma 76.4.4 (and its proof) we see that $E' \to E$ factors through $E'' \to E$. Thus we see that $E' \to E$ is cofinal in the system S/E. Hence it is clear that E' computes LF.

To see the final statement, write $B = q_\mathcal{X}(H)$ and $A = q_\mathcal{Y}(E)$. Choose $E' \to E$ as above. We will use on the one hand that $Rf_{QCoh,*}(B) = q_\mathcal{Y}(Rf_*H)$ and on the other that $Lf^*_{QCoh}(A) = q_\mathcal{X}(f^*E')$.

$$\begin{aligned}
\mathrm{Hom}_{D_{QCoh}(\mathcal{O}_\mathcal{X})}(Lf^*_{QCoh}A, B) &= \mathrm{Hom}_{D_{QCoh}(\mathcal{O}_\mathcal{X})}(q_\mathcal{X}(f^*E'), q_\mathcal{X}(H)) \\
&= \mathrm{colim}_{H \to H'} \mathrm{Hom}_{D(\mathcal{O}_\mathcal{X})}(f^*E', H') \\
&= \mathrm{colim}_{H \to H'} \mathrm{Hom}_{D(\mathcal{O}_\mathcal{Y})}(E', Rf_*H') \\
&= \mathrm{Hom}_{D(\mathcal{O}_\mathcal{Y})}(E', Rf_*H) \\
&= \mathrm{Hom}_{D_{QCoh}(\mathcal{O}_\mathcal{Y})}(A, Rf_{QCoh,*}B)
\end{aligned}$$

Here the colimit is over morphisms $s : H \to H'$ in $D^+_{\mathcal{M}_\mathcal{X}}(\mathcal{O}_\mathcal{X})$ whose cone $P(s)$ is an object of $D^+_{\mathcal{P}_\mathcal{X}}(\mathcal{O}_\mathcal{X})$. The first equality we've seen above. The second equality holds by construction of the Verdier quotient. The third equality holds by Cohomology on Sites, Lemma 21.19.1. Since $Rf_*P(s)$ is an object of $D^+_{\mathcal{P}_\mathcal{Y}}(\mathcal{O}_\mathcal{Y})$ by Proposition 76.5.1 we see that $\mathrm{Hom}_{D(\mathcal{O}_\mathcal{Y})}(E', Rf_*P(s)) = 0$. Thus the fourth equality holds. The final equality holds by construction of E'. □

76.7. Other chapters

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(2) Conventions
(3) Set Theory
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(7) Sites and Sheaves
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(19) Injectives
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Schemes

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Algebraic Spaces

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CHAPTER 77

Introducing Algebraic Stacks

77.1. Why read this?

We give a very informal introduction to algebraic stacks aimed at graduate students and advanced undergraduates. The goal is to quickly introduce a simple language which you can use to think about local and global properties of your favorite moduli problem. Having done this it should be possible to ask yourself well-posed questions about moduli problems and to start solving them, whilst assuming a general theory exists. If you end up with an interesting result, you can go back to the general theory in the other parts of the stacks project and fill in the gaps as needed.

The point of view we take here is very close to the point of view taken in [**KM85**] and [**Mum65**].

77.2. Preliminary

Let S be a scheme. An *elliptic curve* over S is a triple $(E, f, 0)$ where E is a scheme and $f : E \to S$ and $0 : S \to E$ are morphisms of schemes such that

(1) $f : E \to S$ is proper, smooth of relative dimension 1,
(2) for every $s \in S$ the fibre E_s is a connected curve of genus 1, i.e., $H^0(E_s, \mathcal{O})$ and $H^1(E_s, \mathcal{O})$ both are 1-dimensional $\kappa(s)$-vector spaces, and
(3) 0 is a section of f.

Given elliptic curves $(E, f, 0)/S$ and $(E', f', 0')/S'$ a *morphism of elliptic curves over* $a : S \to S'$ is a morphism $\alpha : E \to E'$ such that the diagram

$$\begin{array}{ccc} E & \xrightarrow{\alpha} & E' \\ {\scriptstyle 0}\uparrow \downarrow{\scriptstyle f} & & {\scriptstyle f'}\downarrow \uparrow{\scriptstyle 0'} \\ S & \xrightarrow{a} & S' \end{array}$$

is commutative and the inner square is cartesian, in other words the morphism α induces an isomorphism $E \to S \times_{S'} E'$. We are going to define the stack of elliptic curves $\mathcal{M}_{1,1}$. In the rest of the stacks project we work out the method introduced in Deligne and Mumford's paper [**DM69**] which consists in presenting $\mathcal{M}_{1,1}$ as a category endowed with a functor

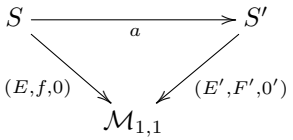
$$p : \mathcal{M}_{1,1} \longrightarrow Sch, \quad (E, f, 0)/S \longmapsto S$$

This means you work with fibred categories over the categories of schemes, topologies, stacks fibred in groupoids, coverings, etc, etc. In this chapter we throw all of that out of the window and we think about it a bit differently – probably closer to how the initiators of the theory started thinking about it themselves.

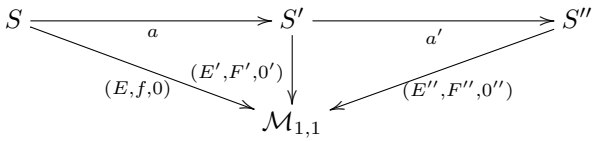
77.3. The moduli stack of elliptic curves

Here is what we are going to do:

(1) Start with your favorite category of schemes *Sch*.
(2) Add a new symbol $\mathcal{M}_{1,1}$.
(3) A morphism $S \to \mathcal{M}_{1,1}$ **is** an elliptic curve $(E, f, 0)$ over S.
(4) A diagram



is commutative if and only if there exists a morphism $\alpha : E \to E'$ of elliptic curves over $a : S \to S'$. We say α *witnesses* the commutativity of the diagram.
(5) Note that commutative diagrams glue as follows



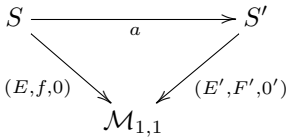
because $\alpha' \circ \alpha$ witnesses the commutativity of the outer triangle if α and α' witness the commutativity of the left and right triangles.
(6) The composition

$$S' \xrightarrow{a} S' \xrightarrow{(E',f',0')} \mathcal{M}_{1,1}$$

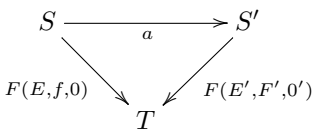
is given by $(E' \times_{S'} S, f' \times_{S'} S, 0' \times_{S'} S)$.

At the end of this procedure we have enlarged the category *Sch* of schemes with exactly one object...

Except that we haven't defined what a morphism from $\mathcal{M}_{1,1}$ to a scheme T is. The answer is that it is the weakest possible notion such that compositions make sense. Thus a morphism $F : \mathcal{M}_{1,1} \to T$ is a rule which to every elliptic curve $(E, f, 0)/S$ associates a morphism $F(E, f, 0) : S \to T$ such that given any commutative diagram



the diagram



is commutative also. An example is the j-invariant

$$j : \mathcal{M}_{1,1} \longrightarrow \mathbf{A}^1_{\mathbf{Z}}$$

which you may have heard of. Aha, so now we're done...

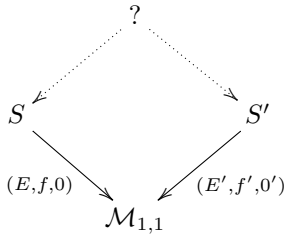
Except, no we're not! We still have to define a notion of morphisms $\mathcal{M}_{1,1} \to \mathcal{M}_{1,1}$. This we do in exactly the same way as before, i.e., a morphism $F : \mathcal{M}_{1,1} \to \mathcal{M}_{1,1}$ is a rule which to every elliptic curve $(E, f, 0)/S$ associates another elliptic curve $F(E, f, 0)$ preserving commutativity of diagrams as above. However, since I don't know of a nontrivial example of such a functor, I'll just define the set of morphisms from $\mathcal{M}_{1,1}$ to itself to consist of the identity for now.

I hope you see how to add other objects to this enlarged category. Somehow it seems intuitively clear that given any "well-behaved" moduli problem we can perform the construction above and add an object to our category. In fact, much of modern day algebraic geometry takes place in such a universe where *Sch* is enlarged with countably many (explicitly constructed) moduli stacks.

You may object that the category we obtain isn't a category because there is a "vagueness" about when diagrams commute and which combinations of diagrams continue to commute as we have to produce a witness to the commutativity. However, it turns out that this, the idea of having witnesses to commutativity, is a valid approach to 2-categories! Thus we stick with it.

77.4. Fibre products

The question we pose here is what should be the fibre product



The answer: A morphism from a scheme T into ? should be a triple (a, a', α) where $a : T \to S$, $a' : T \to S'$ are morphisms of schemes and where $\alpha : E \times_{S,a} T \to E' \times_{S',a'} T$ is an isomorphism of elliptic curves over T. This makes sense because of our definition of composition and commutative diagrams earlier in the discussion.

Lemma 77.4.1 (Key fact)**.** *The functor $Sch^{opp} \to Sets$, $T \mapsto \{(a, a', \alpha)$ as above$\}$ is representable by a scheme $S \times_{\mathcal{M}_{1,1}} S'$.*

Proof. Idea of proof. Relate this functor to

$$Isom_{S \times S'}(E \times S', S \times E')$$

and use Grothendieck's theory of Hilbert schemes. □

Remark 77.4.2. We have the formula $S \times_{\mathcal{M}_{1,1}} S' = (S \times S') \times_{\mathcal{M}_{1,1} \times \mathcal{M}_{1,1}} \mathcal{M}_{1,1}$. Hence the key fact is a property of the diagonal $\Delta_{\mathcal{M}_{1,1}}$ of $\mathcal{M}_{1,1}$.

In any case the key fact allows us to make the following definition.

Definition 77.4.3. We say a morphism $S \to \mathcal{M}_{1,1}$ is *smooth* if for every morphism $S' \to \mathcal{M}_{1,1}$ the projection morphism

$$S \times_{\mathcal{M}_{1,1}} S' \longrightarrow S'$$

is smooth.

Note that this is compatible with the notion of a smooth morphism of schemes as the base change of a smooth morphism is smooth. Moreover, it is clear how to extend this definition to other properties of morphisms into $\mathcal{M}_{1,1}$ (or your own favorite moduli stack). In particular we will use it below for *surjective* morphisms.

77.5. The definition

We'll formulate it as a definition and not as a result since we expect the reader to try out other cases (not just the stack $\mathcal{M}_{1,1}$ and not just *Sch* the category of all schemes).

Definition 77.5.1. We say $\mathcal{M}_{1,1}$ is an *algebraic stack* if and only if

(1) We have descent for objects for the étale topology on *Sch*.
(2) The key fact holds.
(3) there exists a surjective and smooth morphism $S \to \mathcal{M}_{1,1}$.

The first condition is a "sheaf property". We're going to spell it out since there is a technical point we should make. Suppose given a scheme S and an étale covering $\{S_i \to S\}$ and morphisms $e_i : S_i \to \mathcal{M}_{1,1}$ such that the diagrams

$$\begin{array}{ccc} S_i \times_S S_j & \xrightarrow[\text{id}]{} & S_i \times_S S_j \\ & {\scriptstyle e_i \circ \mathrm{pr}_1} \searrow \quad \swarrow {\scriptstyle e_j \circ \mathrm{pr}_2} & \\ & \mathcal{M}_{1,1} & \end{array}$$

commute. The sheaf condition does *not* guarantee the existence of a morphism $e : S \to \mathcal{M}_{1,1}$ in this situation. Namely, we need to pick witnesses α_{ij} for the diagrams above and require that

$$\mathrm{pr}_{02}^* \alpha_{ik} = \mathrm{pr}_{12} \alpha_{jk} \circ \mathrm{pr}_{01}^* \alpha_{ij}$$

as witnesses over $S_i \times_S S_j \times_S S_k$. I think it is clear what this means... If not, then I'm afraid you'll have to read some of the material on categories fibred in groupoids, etc. In any case, the displayed equation is often called the *cocycle condition*. A more precise statement of the "sheaf property" is: given $\{S_i \to S\}$, $e_i : S_i \to \mathcal{M}_{1,1}$ and witnesses α_{ij} satisfying the cocycle condition, there exists a unique (up to unique isomorphism) $e : S \to \mathcal{M}_{1,1}$ with $e_i \cong e|_{S_i}$ recovering the α_{ij}.

As you can see even formulating a precise statement takes a bit of work. The proof of this "sheaf property" relies on a fundamental technique in algebraic geometry, namely descent theory. My suggestion is to initially simply accept the "sheaf property" holds, and see what it implies in practice. In fact, a certain amount of mental agility is required to boil the "sheaf property" down to a manageable statement that you can fit on a napkin. Perhaps the simplest variant which is already a bit interesting is the following: Suppose we have a Galois extension $K \subset L$ of fields with Galois group $G = \mathrm{Gal}(L/K)$. Set $T = \mathrm{Spec}(L)$ and $S = \mathrm{Spec}(K)$. Then $\{T \to S\}$ is an étale covering. Let $(E, f, 0)$ be an elliptic curve over L. (Yes, this just means that $E \subset \mathbf{P}^2_L$ is given by a Weierstrass equation and 0 is the usual point at infinity.) Denote $E_\sigma = E \times_{T, \mathrm{Spec}(\sigma)} T$ the base change. (Yes, this corresponds to applying σ to the coefficients of the Weierstrass equation, or is it σ^{-1}?) Now, suppose moreover that for every $\sigma \in G$ we are given an isomorphism

$$\alpha_\sigma : E \longrightarrow E_\sigma$$

over T. The cocycle condition above means in this situation that

$$(\alpha_\tau)^\sigma \circ \alpha_\sigma = \alpha_{\tau\sigma}$$

for $\sigma, \tau \in G$. If you've ever done any group cohomology then this should be familiar. Anyway, the "glueing" condition on $\mathcal{M}_{1,1}$ says that if you have a solution to this set of equations, then there exists an elliptic curve E' over S such that $E \cong E \times_S T$ (it says a little bit more because it also tells you how to recover the α_σ).

Challenge: Can you prove this entirely using only elliptic curves defined in terms of Weierstrass equations?

77.6. A smooth cover

The last thing we have to do is find a smooth cover of $\mathcal{M}_{1,1}$. In fact, in some sense the existence of a smooth cover *implies*[1] the key fact! In the case of elliptic curves we use the Weierstrass equation to construct one.

Set

$$W = \mathrm{Spec}(\mathbf{Z}[a_1, a_2, a_3, a_4, a_6, 1/\Delta])$$

where $\Delta \in \mathbf{Z}[a_1, a_2, a_3, a_4, a_6]$ is a certain polynomial (see below). Set

$$\mathbf{P}^2_W \supset E_W : zy^2 + a_1xyz + a_3yz^2 = x^3 + a_2x^2z + a_4xz^3 + a_6z^3.$$

Denote $f_W : E_W \to W$ the projection. Finally, denote $0_W : W \to E_W$ the section of f_W given by $(0 : 1 : 0)$. It turns out that there is a degree 12 homogeneous polynomial Δ in a_i where $\deg(a_i) = i$ such that $E_W \to W$ is smooth. You can find it explicitly by computing partials of the Weierstrass equation – of course you can also look it up. You can also use pari/gp to compute it for you. Here it is

$$\begin{aligned}\Delta = &-a_6a_1^6 + a_4a_3a_1^5 + ((-a_3^2 - 12a_6)a_2 + a_4^2)a_1^4 + \\ &(8a_4a_3a_2 + (a_3^3 + 36a_6a_3))a_1^3 + \\ &((-8a_3^2 - 48a_6)a_2^2 + 8a_4^2a_2 + (-30a_4a_3^2 + 72a_6a_4))a_1^2 + \\ &(16a_4a_3a_2^2 + (36a_3^3 + 144a_6a_3)a_2 - 96a_4^2a_3)a_1 + \\ &(-16a_3^2 - 64a_6)a_2^3 + 16a_4^2a_2^2 + (72a_4a_3^2 + 288a_6a_4)a_2 + \\ &- 27a_3^4 - 216a_6a_3^2 - 64a_4^3 - 432a_6^2\end{aligned}$$

You may recognize the last two terms from the case $y^2 = x^3 + Ax + B$ having discriminant $-64A^3 - 432B^2 = -16(4A^3 + 27B^2)$.

Lemma 77.6.1. *The morphism $W \xrightarrow{(E_W, f_W, 0_W)} \mathcal{M}_{1,1}$ is smooth and surjective.*

Proof. Surjectivity follows from the fact that every elliptic curve over a field has a Weierstrass equation. We give a very rough sketch of one way to prove smoothness. Consider the sub group scheme

$$H = \left\{ \left(\begin{array}{ccc} u^2 & s & 0 \\ 0 & u^3 & 0 \\ r & t & 1 \end{array} \right) \middle| \begin{array}{c} u \text{ unit} \\ s, r, t \text{ arbitrary} \end{array} \right\} \subset \mathrm{GL}_{3,\mathbf{Z}}$$

[1] This is a bit of a cheat because in checking the smoothness you have to prove something very close to the key fact – after all smoothness is defined in terms of fibre products. The advantage is that you only have to prove the existence of these fibre products in the case that on one side you have the morphism that you are trying to show provides the smooth cover.

There is an action $H \times W \to W$ of H on the Weierstrass scheme W. To find the equations for this action write out what a coordinate change given by a matrix in H does to the general Weierstrass equation. Then it turns out the following statements hold

(1) any elliptic curve $(E, f, 0)/S$ has Zariski locally on S a Weierstrass equation,
(2) any two Weierstrass equations for $(E, f, 0)$ differ (Zariski locally) by an element of H.

Considering the fibre product $S \times_{\mathcal{M}_{1,1}} W = Isom_{S \times W}(E \times W, S \times E_W)$ we conclude that this means that the morphism $W \to \mathcal{M}_{1,1}$ is an H-torsor. Since $H \to \mathrm{Spec}(\mathbf{Z})$ is smooth, and since torsors over smooth group schemes are smooth we win. □

Remark 77.6.2. The argument sketched above actually shows that $\mathcal{M}_{1,1} = [W/H]$ is a global quotient stack. It is true about 50% of the time that an argument proving a moduli stack is algebraic will show that it is a global quotient stack.

77.7. Properties of algebraic stacks

Ok, so now we know that $\mathcal{M}_{1,1}$ is an algebraic stack. What can we do with this? Well, it isn't so much the fact that it is an algebraic stack that helps us here, but more the point of view that properties of $\mathcal{M}_{1,1}$ should be encoded in the properties of morphisms $S \to \mathcal{M}_{1,1}$, i.e., in families of elliptic curves. We list some examples

Local properties:

$$\mathcal{M}_{1,1} \to \mathrm{Spec}(\mathbf{Z}) \text{ is smooth} \Leftrightarrow W \to \mathrm{Spec}(\mathbf{Z}) \text{ is smooth}$$

Idea. Local properties of an algebraic stack are encoded in the local properties of its smooth cover.

Global properties:

$$\mathcal{M}_{1,1} \text{ is quasi-compact} \Leftarrow W \text{ is quasi-compact}$$
$$\mathcal{M}_{1,1} \text{ is irreducible} \Leftarrow W \text{ is irreducible}$$

Idea. Some global properties of an algebraic stack can be read off from the corresponding property of a *suitable*[2] smooth cover.

Quasi-coherent sheaves:

$$QCoh(\mathcal{O}_{\mathcal{M}_{1,1}}) = H\text{-equivariant quasi-coherent modules on } W$$

Idea. On the one hand a quasi-coherent module on $\mathcal{M}_{1,1}$ should correspond to a quasi-coherent sheaf $\mathcal{F}_{S,e}$ on S for each morphism $e : S \to \mathcal{M}_{1,1}$. In particular for the morphism $(E_W, f_W, 0_W) : W \to \mathcal{M}_{1,1}$. Since this morphism is H-equivariant we see the quasi-coherent module $\mathcal{F}_W$ we obtain is H-equivariant. Conversely, given an H-equivariant module we can recover the sheaves $\mathcal{F}_{S,e}$ by descent theory starting with the observation that $S \times_{e,\mathcal{M}_{1,1}} W$ is an H-torsor.

Picard group:

$$\mathrm{Pic}(\mathcal{M}_{1,1}) = \mathrm{Pic}_H(W) = \mathbf{Z}/12\mathbf{Z}$$

[2] I suppose that it is possible an irreducible algebraic stack exists which doesn't have an irreducible smooth cover – but if so it is going to be quite nasty!

Idea. We have seen the first equality above. Note that $\mathrm{Pic}(W) = 0$ because the ring $\mathbf{Z}[a_1, a_2, a_3, a_4, a_6, 1/\Delta]$ has trivial class group. There is an exact sequence

$$\mathbf{Z}\Delta \to \mathrm{Pic}_H(\mathbf{A}^5_{\mathbf{Z}}) \to \mathrm{Pic}_H(W) \to 0$$

The middle group equals $\mathrm{Hom}(H, \mathbf{G}_m) = \mathbf{Z}$. The image Δ is 12 because Δ has degree 12. This argument is roughly correct, see [**FO10**].

Étale cohomology: Let Λ be a ring. There is a first quadrant spectral sequence converging to $H^{p+q}_{étale}(\mathcal{M}_{1,1}, \Lambda)$ with E_2-page

$$E_2^{p,q} = H^q_{étale}(W \times H \times \ldots \times H, \Lambda) \quad (p \text{ factors } H)$$

Idea. Note that

$$W \times_{\mathcal{M}_{1,1}} W \times_{\mathcal{M}_{1,1}} \ldots \times_{\mathcal{M}_{1,1}} W = W \times H \times \ldots \times H$$

because $W \to \mathcal{M}_{1,1}$ is a H-torsor. The spectral sequence is the Čech-to-cohomology spectral sequence for the smooth cover $\{W \to \mathcal{M}_{1,1}\}$. For example we see that $H^0_{étale}(\mathcal{M}_{1,1}, \Lambda) = \Lambda$ because W is connected, and $H^1_{étale}(\mathcal{M}_{1,1}, \Lambda) = 0$ because $H^1_{étale}(W, \Lambda) = 0$ (of course this requires a proof). Of course, the smooth covering $W \to \mathcal{M}_{1,1}$ may not be "optimal" for the computation of étale cohomology.

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CHAPTER 78

Examples

78.1. Introduction

This chapter will contain examples which illuminate the theory.

78.2. Non-quasi-compact inverse limit of quasi-compact spaces

Let $\mathbf{N}$ denote the set of natural numbers. For every integer n, let I_n denote the set of all natural numbers $> n$. Define T_n to be the unique topology on $\mathbf{N}$ with basis $\{1\}, \ldots, \{n\}, I_n$. Denote by X_n the topological space $(\mathbf{N}, T_n)$. For each $m < n$, the identity map,

$$f_{n,m} : X_n \longrightarrow X_m$$

is continuous. Obviously for $m < n < p$, the composition $f_{p,n} \circ f_{n,m}$ equals $f_{p,m}$. So $((X_n), (f_{n,m}))$ is a directed inverse system of quasi-compact topological spaces.

Let T be the discrete topology on $\mathbf{N}$, and let X be $(\mathbf{N}, T)$. Then for every integer n, the identity map,

$$f_n : X \longrightarrow X_n$$

is continuous. We claim that this is the inverse limit of the directed system above. Let (Y, S) be any topological space. For every integer n, let

$$g_n : (Y, S) \longrightarrow (\mathbf{N}, T_n)$$

be a continuous map. Assume that for every $m < n$ we have $f_{n,m} \circ g_n = g_m$, i.e., the system (g_n) is compatible with the directed system above. In particular, all of the set maps g_n are equal to a common set map

$$g : Y \longrightarrow \mathbf{N}.$$

Moreover, for every integer n, since $\{n\}$ is open in X_n, also $g^{-1}(\{n\}) = g_n^{-1}(\{n\})$ is open in Y. Therefore the set map g is continuous for the topology S on Y and the topology T on $\mathbf{N}$. Thus $(X, (f_n))$ is the inverse limit of the directed system above.

However, clearly X is not quasi-compact, since the infinite open covering by singleton sets has no inverse limit.

Lemma 78.2.1. *There exists an inverse system of quasi-compact topological spaces over* $\mathbf{N}$ *whose limit is not quasi-compact.*

Proof. See discussion above. □

78.3. A nonintegral connected scheme whose local rings are domains

We give an example of an affine scheme $X = \mathrm{Spec}(A)$ which is connected, all of whose local rings are domains, but which is not integral. Connectedness of X means A has no nontrivial idempotents, see Algebra, Lemma 10.19.3. The local rings of X are domains if, whenever $fg = 0$ in A, every point of X has a neighborhood where either f or g vanishes. As long as A is not a domain, then X is not integral (Properties, Definition 27.3.1).

Roughly speaking, the construction is as follows: let X_0 be the cross (the union of coordinate axes) on the affine plane. Then let X_1 be the (reduced) full preimage of X_0 on the blow-up of the plane (X_1 has three rational components forming a chain). Then blow up the resulting surface at the two singularities of X_1, and let X_2 be the reduced preimage of X_1 (which has five rational components), etc. Take X to be the inverse limit. The only problem with this construction is that blow-ups glue in a projective line, so X_1 is not affine. Let us correct this by glueing in an affine line instead (so our scheme will be an open subset in what was described above).

Here is a completely algebraic construction: For every $k \geq 0$, let A_k be the following ring: its elements are collections of polynomials $p_i \in \mathbf{C}[x]$ where $i = 0, \ldots, 2^k$ such that $p_i(1) = p_{i+1}(0)$. Set $X_k = \mathrm{Spec}(A_k)$. Observe that X_k is a union of 2^k+1 affine lines that meet transversally in a chain. Define a ring homomorphism $A_k \to A_{k+1}$ by

$$(p_0, \ldots, p_{2^k}) \longmapsto (p_0, p_0(1), p_1, p_1(1), \ldots, p_{2^k}),$$

in other words, every other polynomial is constant. This identifies A_k with a subring of A_{k+1}. Let A be the direct limit of A_k (basically, their union). Set $X = \mathrm{Spec}(A)$. For every k, we have a natural embedding $A_k \to A$, that is, a map $X \to X_k$. Each A_k is connected but not integral; this implies that A is connected but not integral. It remains to show that the local rings of A are domains.

Take $f, g \in A$ with $fg = 0$ and $x \in X$. Let us construct a neighborhood of x on which one of f and g vanishes. Choose k such that $f, g \in A_{k-1}$ (note the $k-1$ index). Let y be the image of x in X_k. It suffices to prove that y has a neighborhood on which either f or g viewed as sections of $\mathcal{O}_{X_k}$ vanishes. If y is a smooth point of X_k, that is, it lies on only one of the $2^k + 1$ lines, this is obvious. We can therefore assume that y is one of the 2^k singular points, so two components of X_k pass through y. However, on one of these two components (the one with odd index), both f and g are constant, since they are pullbacks of functions on X_{k-1}. Since $fg = 0$ everywhere, either f or g (say, f) vanishes on the other component. This implies that f vanishes on both components, as required.

78.4. Noncomplete completion

Let R be a ring and let $\mathfrak{m}$ be a maximal ideal. Consider the completion

$$R^\wedge = \lim R/\mathfrak{m}^n.$$

Note that $R^\wedge$ is a local ring with maximal ideal $\mathfrak{m}' = \mathrm{Ker}(R^\wedge \to R/\mathfrak{m})$. Namely, if $x = (x_n) \in R^\wedge$ is not in $\mathfrak{m}'$, then $y = (x_n^{-1}) \in R^\wedge$ satisfies $xy = 1$, whence $R^\wedge$ is local by Algebra, Lemma 10.17.2. Now it is always true that $R^\wedge$ complete in its limit topology (see the discussion in More on Algebra, Section 15.25). But beyond that, we have the following questions:

(1) Is it true that $\mathfrak{m}R^\wedge = \mathfrak{m}'$?
(2) Is $R^\wedge$ viewed as an $R^\wedge$-module $\mathfrak{m}'$-adically complete?
(3) Is $R^\wedge$ viewed as an R-module $\mathfrak{m}$-adically complete?

It turns out that of these questions all have a negative answer. The example below was taken from an unpublished note of Bart de Smit and Hendrik Lenstra. See also [**Bou61**, Exercise III.2.12] and [**Yek11**, Example 1.8]

Let k be a field, $R = k[x_1, x_2, x_3, \ldots]$, and $\mathfrak{m} = (x_1, x_2, x_3, \ldots)$. We will think of an element f of $R^\wedge$ as a (possibly) infinite sum

$$f = \sum a_I x^I$$

(using multi-index notation) such that for each $d \geq 0$ there are only finitely many nonzero a_I for $|I| = d$. The maximal ideal $\mathfrak{m}' \subset R^\wedge$ is the collection of f with zero constant term. In particular, the element

$$f = x_1 + x_2^2 + x_3^3 + \ldots$$

is in $\mathfrak{m}'$ but not in $\mathfrak{m}R^\wedge$ which shows that (1) is false in this example. Note that we do have $\mathfrak{m}R^\wedge \subset \mathfrak{m}'$. Hence, $R^\wedge$ is not $\mathfrak{m}$-adically complete as an R-module, then it is also not $\mathfrak{m}'$-adically complete. To show that $R^\wedge$ is not $\mathfrak{m}$-adically complete (as an R-module) it suffices to show that $K_2 = \mathrm{Ker}(R^\wedge \to R/\mathfrak{m}^2)$ is not equal to $\mathfrak{m}^2R^\wedge$, see Algebra, Lemma 10.91.6. Note that an element of $\mathfrak{m}^2R^\wedge \subset (\mathfrak{m}')^2$ can be written as a finite sum

$$\sum\nolimits_{i=1,\ldots,t} f_i g_i \tag{78.4.0.1}$$

with $f_i, g_i \in R^\wedge$ having vanishing constant terms. To get an example we are going to choose an $z \in K_2$ of the form

$$z = z_1 + z_2 + z_3 + \ldots$$

with the following properties

(1) there exist sequences $1 < d_1 < d_2 < d_3 < \ldots$ and $0 < n_1 < n_2 < n_3 < \ldots$ such that $z_i \in k[x_{n_i}, x_{n_i+1}, \ldots, x_{n_{i+1}-1}]$ homogeneous of degree d_i, and
(2) in the ring $k[[x_{n_i}, x_{n_i+1}, \ldots, x_{n_{i+1}-1}]]$ the element z_i cannot be written as a sum (78.4.0.1) with $t \leq i$.

Clearly this implies that z is not in $(\mathfrak{m}')^2$ because the image of the relation (78.4.0.1) in the ring $k[[x_{n_i}, x_{n_i+1}, \ldots, x_{n_{i+1}-1}]]$ for i large enough would produce a contradiction. Hence it suffices to prove that for all $t > 0$ there exists a $d \gg 0$ and an integer n such that we can find an homogeneous element $z \in k[x_1, \ldots, x_n]$ of degree d which cannot be written as a sum (78.4.0.1) for the given t in $k[[x_1, \ldots, x_n]]$. Take $n > 2t$ and any $d > 1$ prime to the characteristic of p and set $z = \sum_{i=1,\ldots,n} x_i^d$. Then the vanishing locus of the ideal

$$(\frac{\partial z}{\partial x_1}, \ldots, \frac{\partial z}{\partial x_n}) = (dx_1^{d-1}, \ldots, dx_n^{d-1})$$

consists of one point. On the other hand,

$$\frac{\partial(\sum_{i=1,\ldots,t} f_i g_i)}{\partial x_j} \in (f_1, \ldots, f_t, g_1, \ldots, g_t)$$

by the Leibniz rule and hence the vanishing locus of these derivatives contains at least

$$V(f_1, \ldots, f_t, g_1, \ldots, g_t) \subset \mathrm{Spec}(k[[x_1, \ldots, x_n]]).$$

Hence this is a contradiction as the dimension of $V(f_1, \ldots, f_t, g_1, \ldots, g_t)$ is at least $n - 2t \geq 1$.

Lemma 78.4.1. *There exists a local ring R and a maximal ideal $\mathfrak{m}$ such that the completion $R^\wedge$ of R with respect to $\mathfrak{m}$ has the following properties*

(1) *$R^\wedge$ is local, but its maximal ideal is not equal to $\mathfrak{m}R^\wedge$,*
(2) *$R^\wedge$ is not a complete local ring, and*
(3) *$R^\wedge$ is not $\mathfrak{m}$-adically complete as an R-module.*

Proof. This follows from the discussion above as (with $R = k[x_1, x_2, x_3, \ldots]$) the completion of the localization $R_\mathfrak{m}$ is equal to the completion of R. □

78.5. Noncomplete quotient

Let k be a field. Let

$$R = k[t, z_1, z_2, z_3, \ldots, w_1, w_2, w_3, \ldots, x]/(z_i t - x^i w_i, z_i w_j)$$

Note that in particular $z_i z_j t = 0$ in this ring. Any element f of R can be uniquely written as a finite sum

$$f = \sum\nolimits_{i=0,\ldots,d} f_i x^i$$

where each $f_i \in k[t, z_i, w_j]$ has no terms involving the products $z_i t$ or $z_i w_j$. Moreover, if f is written in this way, then $f \in (x^n)$ if and only if $f_i = 0$ for $i < n$. So x is a nonzerodivisor and $\bigcap (x^n) = 0$. Let $R^\wedge$ be the completion of R with respect to the ideal (x). Note that $R^\wedge$ is (x)-adically complete, see Algebra, Lemma 10.91.7. By the above we see that an element of $R^\wedge$ can be uniquely written as an infinite sum

$$f = \sum\nolimits_{i=0}^{\infty} f_i x^i$$

where each $f_i \in k[t, z_i, w_j]$ has no terms involving the products $z_i t$ or $z_i w_j$. Consider the element

$$f = \sum\nolimits_{i=1}^{\infty} x^{i-1} w_i = x w_1 + x^2 w_2 + x^3 w_3 + \ldots$$

i.e., we have $f_n = w_n$. Note that $f \in (t, x^n)$ for every n because $x^m w_m \in (t)$ for all m. We claim that $f \notin (t)$. To prove this assume that $tg = f$ where $g = \sum g_l x^l$ in canonical form as above. Since $t z_i z_j = 0$ we may as well assume that none of the g_l have terms involving the products $z_i z_j$. Examining the process to get tg in canonical form we see the following: Given any term cm of g_l where $c \in k$ and m is a monomial in t, z_i, w_j and we make the following replacement

(1) if the monomial m does not involve any z_i, then ctm is a term of f_l, and
(2) if the monomial m does involve a z_i then it is equal to $m = z_i$ and we see that cw_i is term of f_{l+i}.

Since g_0 is a polynomial only finitely many of the variables z_i occur in it. Pick n such that z_n does not occur in g_0. Then the rules above show that w_n does not occur in f_n which is a contradiction. It follows that $R^\wedge/(t)$ is not complete, see Algebra, Lemma 10.91.15.

Lemma 78.5.1. *There exists a ring R complete with respect to a principal ideal I and a principal ideal J such that R/J is not I-adically complete.*

Proof. See discussion above. □

78.6. Completion is not exact

A quick example is the following. Suppose that $R = k[t]$. Let $P = K = \bigoplus_{n \in \mathbf{N}} R$ and $M = \bigoplus_{n \in \mathbf{N}} R/(t^n)$. Then there is a short exact sequence $0 \to K \to P \to M \to 0$ where the first map is given by multiplication by t^n on the nth summand. We claim that $0 \to K^\wedge \to P^\wedge \to M^\wedge \to 0$ is not exact in the middle. Namely, $\xi = (t^2, t^3, t^4, \ldots) \in P^\wedge$ maps to zero in $M^\wedge$ but is not in the image of $K^\wedge \to P^\wedge$, because it would be the image of $(t, t, t, \ldots)$ which is not an element of $K^\wedge$.

A "smaller" example is the following. In the situation of Lemma 78.5.1 the short exact sequence $0 \to J \to R \to R/J \to 0$ does not remain exact after completion. Namely, if $f \in J$ is a generator, then $f : R \to J$ is surjective, hence $R \to J^\wedge$ is surjective, hence the image of $J^\wedge \to R$ is $(f) = J$ but the fact that R/J is noncomplete means that the kernel of the surjection $R \to (R/J)^\wedge$ is strictly bigger than J, see Algebra, Lemmas 10.91.1 and 10.91.15. By the same token the sequence $R \to R \to R/(f) \to 0$ does not remain exact on completion.

Lemma 78.6.1. *Completion is not an exact functor in general; it is not even right exact in general. This holds even when I is finitely generated on the category of finitely presented modules.*

Proof. See discussion above. □

78.7. The category of complete modules is not abelian

Let R be a ring and let $I \subset R$ be a finitely generated ideal. Consider the category $\mathcal{A}$ of I-adically complete R-modules, see Algebra, Definition 10.91.5. Let $\varphi : M \to N$ be a morphism of $\mathcal{A}$. The cokernel of φ in $\mathcal{A}$ is the completion $(\mathrm{Coker}(\varphi))^\wedge$ of the usual cokernel (as I is finitely generated this completion is complete, see Algebra, Lemma 10.91.7). Let $K = \mathrm{Ker}(\varphi)$. We claim that K is complete and hence is the kernel of φ in $\mathcal{A}$. Namely, let $K^\wedge$ be the completion. As M is complete we obtain a factorization

$$K \to K^\wedge \to M \xrightarrow{\varphi} N$$

Since φ is continuous for the I-adic topology, $K \to K^\wedge$ has dense image, and $K = \mathrm{Ker}(\varphi)$ we conclude that $K^\wedge$ maps into K. Thus $K^\wedge = K \oplus C$ and K is a direct sum of a complete module, hence complete.

We will give an example that shows that $\mathrm{Im} \neq \mathrm{Coim}$ in general. We take $R = \mathbf{Z}_p = \lim_n \mathbf{Z}/p^n\mathbf{Z}$ to be the ring of p-adic integers and we take $I = (p)$. Consider the map

$$\mathrm{diag}(1, p, p^2, \ldots) : \Big(\bigoplus\nolimits_{n \geq 1} \mathbf{Z}_p\Big)^\wedge \longrightarrow \prod\nolimits_{n \geq 1} \mathbf{Z}_p$$

where the left hand side is the p-adic completion of the direct sum. Hence an element of the left hand side is a vector $(x_1, x_2, x_3, \ldots)$ with $x_i \in \mathbf{Z}_p$ with p-adic valuation $v_p(x_i) \to \infty$ as $i \to \infty$. This maps to $(x_1, px_2, p^2x_3, \ldots)$. Hence we see that $(1, p, p^2, \ldots)$ is in the closure of the image but not in the image. By our description of kernels and cokernels above it is clear that $\mathrm{Im} \neq \mathrm{Coim}$ for this map.

Lemma 78.7.1. *Let R be a ring and let $I \subset R$ be a finitely generated ideal. The category of I-adically complete R-modules has kernels and cokernels but is not abelian in general.*

Proof. See above. □

78.8. Regular sequences and base change

We are going to construct a ring R with a regular sequence (x, y, z) such that there exists a nonzero element $\delta \in R/zR$ with $x\delta = y\delta = 0$.

To construct our example we first construct a peculiar module E over the ring $k[x, y, z]$ where k is any field. Namely, E will be a push-out as in the following diagram

$$\begin{array}{ccccc} \frac{xk[x,y,z,y^{-1}]}{xyk[x,y,z]} & \longrightarrow & \frac{k[x,y,z,x^{-1},y^{-1}]}{yk[x,y,z,x^{-1}]} & \longrightarrow & \frac{k[x,y,z,x^{-1},y^{-1}]}{yk[x,y,z,x^{-1}]+xk[x,y,z,y^{-1}]} \\ \downarrow z/x & & \downarrow & & \downarrow \\ \frac{k[x,y,z,y^{-1}]}{yzk[x,y,z]} & \longrightarrow & E & \longrightarrow & \frac{k[x,y,z,x^{-1},y^{-1}]}{yk[x,y,z,x^{-1}]+xk[x,y,z,y^{-1}]} \end{array}$$

where the rows are short exact sequences (we dropped the outer zeros due to typesetting problems). Another way to describe E is as

$$E = \{(f, g) \mid f \in k[x, y, z, x^{-1}, y^{-1}], g \in k[x, y, z, y^{-1}]\}/\sim$$

where $(f, g) \sim (f', g')$ if and only if there exists a $h \in k[x, y, z, y^{-1}]$ such that

$$f = f' + xh \bmod yk[x, y, z, x^{-1}], \quad g = g' - zh \bmod yzk[x, y, z]$$

We claim: (a) $x : E \to E$ is injective, (b) $y : E/xE \to E/xE$ is injective, (c) $E/(x, y)E = 0$, (d) there exists a nonzero element $\delta \in E/zE$ such that $x\delta = y\delta = 0$.

To prove (a) suppose that (f, g) is a pair that gives rise to an element of E and that $(xf, xg) \sim 0$. Then there exists a $h \in k[x, y, z, y^{-1}]$ such that $xf + xh \in yk[x, y, z, x^{-1}]$ and $xg - zh \in yzk[x, y, z]$. We may assume that $h = \sum a_{i,j,k}x^i y^j z^k$ is a sum of monomials where only $j \leq 0$ occurs. Then $xg - zh \in yzk[x, y, z]$ implies that only $i > 0$ occurs, i.e., $h = xh'$ for some $h' \in k[x, y, z, y^{-1}]$. Then $(f, g) \sim (f + xh', g - zh')$ and we see that we may assume that $g = 0$ and $h = 0$. In this case $xf \in yk[x, y, z, x^{-1}]$ implies $f \in yk[x, y, z, x^{-1}]$ and we see that $(f, g) \sim 0$. Thus $x : E \to E$ is injective.

Since multiplication by x is an isomorphism on $\frac{k[x,y,z,x^{-1},y^{-1}]}{yk[x,y,z,x^{-1}]}$ we see that E/xE is isomorphic to

$$\frac{k[x, y, z, y^{-1}]}{yzk[x, y, z] + xk[x, y, z, y^{-1}] + zk[x, y, z, y^{-1}]} = \frac{k[x, y, z, y^{-1}]}{xk[x, y, z, y^{-1}] + zk[x, y, z, y^{-1}]}$$

and hence multiplication by y is an isomorphism on E/xE. This clearly implies (b) and (c).

Let $e \in E$ be the equivalence class of $(1, 0)$. Suppose that $e \in zE$. Then there exist $f \in k[x, y, z, x^{-1}, y^{-1}]$, $g \in k[x, y, z, y^{-1}]$, and $h \in k[x, y, z, y^{-1}]$ such that

$$1 + zf + xh \in yk[x, y, z, x^{-1}], \quad 0 + zg - zh \in yzk[x, y, z].$$

This is impossible: the monomial 1 cannot occur in zf, nor in xh. On the other hand, we have $ye = 0$ and $xe = (x, 0) \sim (0, -z) = z(0, -1)$. Hence setting δ equal to the congruence class of e in E/zE we obtain (d).

Lemma 78.8.1. *There exists a local ring R and a regular sequence x, y, z (in the maximal ideal) such that there exists a nonzero element $\delta \in R/zR$ with $x\delta = y\delta = 0$.*

Proof. Let $R = k[x, y, z] \oplus E$ where E is the module above considered as a square zero ideal. Then it is clear that x, y, z is a regular sequence in R, and that the element $\delta \in E/zE \subset R/zR$ gives an element with the desired properties. To get a local example we may localize R at the maximal ideal $\mathfrak{m} = (x, y, z, E)$. The sequence x, y, z remains a regular sequence (as localization is exact), and the element δ remains nonzero as it is supported at $\mathfrak{m}$. □

Lemma 78.8.2. *There exists a local homomorphism of local rings $A \to B$ and a regular sequence x, y in the maximal ideal of B such that $B/(x, y)$ is flat over A, but such that the images $\overline{x}, \overline{y}$ of x, y in $B/\mathfrak{m}_A B$ do not form a regular sequence, nor even a Koszul-regular sequence.*

Proof. Set $A = k[z]_{(z)}$ and let $B = (k[x, y, z] \oplus E)_{(x,y,z,E)}$. Since x, y, z is a regular sequence in B, see proof of Lemma 78.8.1, we see that x, y is a regular sequence in B and that $B/(x, y)$ is a torsion free A-module, hence flat. On the other hand, there exists a nonzero element $\delta \in B/\mathfrak{m}_A B = B/zB$ which is annihilated by $\overline{x}, \overline{y}$. Hence $H_2(K_\bullet(B/\mathfrak{m}_A B, \overline{x}, \overline{y})) \neq 0$. Thus $\overline{x}, \overline{y}$ is not Koszul-regular, in particular it is not a regular sequence, see More on Algebra, Lemma 15.20.2. □

78.9. A Noetherian ring of infinite dimension

A Noetherian local ring has finite dimension as we saw in Algebra, Proposition 10.57.8. But there exist Noetherian rings of infinite dimension. See [**Nag62**, Appendix, Example 1].

Namely, let k be a field, and consider the ring

$$R = k[x_1, x_2, x_3, \ldots].$$

Let $\mathfrak{p}_i = (x_{2^{i-1}}, x_{2^{i-1}+1}, \ldots, x_{2^i-1})$ for $i = 1, 2, \ldots$ which are prime ideals of R. Let S be the multiplicative subset

$$S = \bigcap\nolimits_{i \geq 1} (R \setminus \mathfrak{p}_i).$$

Consider the ring $A = S^{-1}R$. We claim that

(1) The maximal ideals of the ring A are the ideals $\mathfrak{m}_i = \mathfrak{p}_i A$.
(2) We have $A_{\mathfrak{m}_i} = R_{\mathfrak{p}_i}$ which is a Noetherian local ring of dimension 2^i.
(3) The ring A is Noetherian.

Hence it is clear that this is the example we are looking for. Details omitted.

78.10. Local rings with nonreduced completion

In Algebra, Example 10.113.4 we gave an example of a characteristic p Noetherian local domain R of dimension 1 whose completion is nonreduced. In this section we present the example of [**FR70**, Proposition 3.1] which gives a similar ring in characteristic zero.

Let $\mathbf{C}\{x\}$ be the ring of convergent power series over the field $\mathbf{C}$ of complex numbers. The ring of all power series $\mathbf{C}[[x]]$ is its completion. Let $K = \mathbf{C}\{x\}[1/x] = f.f.(B)$ be the field of convergent Laurent series. The K-module $\Omega_{K/\mathbf{C}}$ of algebraic differentials of K over $\mathbf{C}$ is an infinite dimensional K-vector space (proof omitted). We may choose $f_n \in x\mathbf{C}\{x\}$, $n \geq 1$ such that $\mathrm{d}x, \mathrm{d}f_1, \mathrm{d}f_2, \ldots$ are part of a basis of $\Omega_{K/\mathbf{C}}$. Thus we can find a $\mathbf{C}$-derivation

$$D : \mathbf{C}\{x\} \longrightarrow \mathbf{C}((x))$$

such that $D(x) = 0$ and $D(f_i) = x^{-n}$. Let

$$A = \{f \in \mathbf{C}\{x\} \mid D(f) \in \mathbf{C}[[x]]\}$$

We claim that

(1) $\mathbf{C}\{x\}$ is integral over A,
(2) A is a local domain,
(3) $\dim(A) = 1$,
(4) the maximal ideal of A is generated by x and xf_1,
(5) A is Noetherian, and
(6) the completion of A is equal to the ring of dual numbers over $\mathbf{C}[[x]]$.

Since the dual numbers are nonreduced the ring A gives the example.

Note that if $0 \neq f \in x\mathbf{C}\{x\}$ then we may write $D(f) = h/f^n$ for some $n \geq 0$ and $h \in \mathbf{C}[[x]]$. Hence $D(f^{n+1}/(n+1)) \in \mathbf{C}[[x]]$ and $D(f^{n+2}/(n+2)) \in \mathbf{C}[[x]]$. Thus we see $f^{n+1}, f^{n+2} \in A$! In particular we see (1) holds. We also conclude that the fraction field of A is equal to the fraction field of $\mathbf{C}\{x\}$. It also follows immediately that $A \cap x\mathbf{C}\{x\}$ is the set of nonunits of A, hence A is a local domain of dimension 1. If we can show (4) then it will follow that A is Noetherian (proof omitted). Suppose that $f \in A \cap x\mathbf{C}\{x\}$. Write $D(f) = h$, $h \in \mathbf{C}[[x]]$. Write $h = c + xh'$ with $c \in \mathbf{C}$, $h' \in \mathbf{C}[[x]]$. Then $D(f - cxf_1) = c + xh' - c = xh'$. On the other hand $f - cxf_1 = xg$ with $g \in \mathbf{C}\{x\}$, but by the computation above we have $D(g) = h' \in \mathbf{C}[[x]]$ and hence $g \in A$. Thus $f = cxf_1 + xg \in (x, xf_1)$ as desired.

Finally, why is the completion of A nonreduced? Denote $\hat{A}$ the completion of A. Of course this maps surjectively to the completion $\mathbf{C}[[x]]$ of $\mathbf{C}\{x\}$ because $x \in A$. Denote this map $\psi : \hat{A} \to \mathbf{C}[[x]]$. Above we saw that $\mathfrak{m}_A = (x, xf_1)$ and hence $D(\mathfrak{m}_A^n) \subset (x^{n-1})$ by an easy computation. Thus $D : A \to \mathbf{C}[[x]]$ is continuous and gives rise to a continuous derivation $\hat{D} : \hat{A} \to \mathbf{C}[[x]]$ over ψ. Hence we get a ring map

$$\psi + \epsilon\hat{D} : \hat{A} \longrightarrow \mathbf{C}[[x]][\epsilon].$$

Since $\hat{A}$ is a one dimensional Noetherian complete local ring, if we can show this arrow is surjective then it will follow that $\hat{A}$ is nonreduced. Actually the map is an isomorphism but we omit the verification of this. The subring $\mathbf{C}[x]_{(x)} \subset A$ gives rise to a map $i : \mathbf{C}[[x]] \to \hat{A}$ on completions such that $i \circ \psi = \mathrm{id}$ and such that $D \circ i = 0$ (as $D(x) = 0$ by construction). Consider the elements $x^n f_n \in A$. We have

$$(\psi + \epsilon D)(x^n f_n) = x^n f_n + \epsilon$$

for all $n \geq 1$. Surjectivity easily follows from these remarks.

78.11. A non catenary Noetherian local ring

Even though there is a succesful dimension theory of Noetherian local rings there are non-catenary Noetherian local rings. An example may be found in [**Nag62**, Appendix, Example 2]. In fact, we will present this example in the simplest case. Namely, we will construct a local Noetherian domain A of dimension 2 which is not universally catenary. (Note that A is automatically catenary, see Exercises, Exercise 79.12.2.) The existence of a Noetherian local ring which is not universally catenary implies the existence of a Noetherian local ring which is not catenary – and we spell this out at the end of this section in the particular example at hand.

Let k be a field, and consider the formal power series ring $k[[x]]$ in one variable over k. Let

$$z = \sum\nolimits_{i=1}^{\infty} a_i x^i$$

be a formal power series. We assume z as an element of the Laurent series field $k((x)) = f.f.(k[[x]])$ is transcendental over $k(x)$. Put

$$z_j = x^{-j}(z - \sum\nolimits_{i=1,\ldots,j-1} a_i x^i) = \sum\nolimits_{i=j}^{\infty} a_i x^{i-j} \in k[[x]].$$

Note that $Z = z_1$. Let R be the subring of $k[[x]]$ generated by x, z and all of the z_j, in other words

$$R = k[x, z_1, z_2, z_3, \ldots] \subset k[[x]].$$

Consider the ideals $\mathfrak{m} = (x)$ and $\mathfrak{n} = (x - 1, z_1, z_2, \ldots)$ of R.

We have $x(z_{j+1} + a_j) = z_j$. Hence $R/\mathfrak{m} = k$ and $\mathfrak{m}$ is a maximal ideal. Moreover, any element of R not in $\mathfrak{m}$ maps to a unit in $k[[x]]$ and hence $R_{\mathfrak{m}} \subset k[[x]]$. In fact it is easy to deduce that $R_{\mathfrak{m}}$ is a discrete valuation ring and residue field k.

We claim that

$$R/(x-1) = k[x, z_1, z_2, z_3, \ldots]/(x-1) \cong k[z].$$

Namely, the relation above implies that $(x-1)(z_{j+1} + a_j) = -z_{j+1} - a_j + z_j$, and hence we may express the class of z_{j+1} in terms of z_j in the quotient $R/(x-1)$. Since the fraction field of R has transcendence degree 2 over k by construction we see that z is transcendental over k in $R/(x-1)$, whence the desired isomorphism. Hence $\mathfrak{n} = (x-1, z)$ and is a maximal ideal. In fact the map

$$k[x, x^{-1}, z]_{(x-1,z)} \longrightarrow R_{\mathfrak{n}}$$

is an isomorphism (since x^{-1} is invertible in $R_{\mathfrak{n}}$ and since $z_{j+1} = x^{-1}z_j - a_j = \ldots = f_j(x, x^{-1}, z)$). This shows that $R_{\mathfrak{n}}$ is a regular local ring of dimension 2 and residue field k.

Let S be the multiplicative subset

$$S = (R \setminus \mathfrak{m}) \cap (R \setminus \mathfrak{n}) = R \setminus (\mathfrak{m} \cup \mathfrak{n})$$

and set $B = S^{-1}R$. We claim that

(1) The ring B is a k-algebra.
(2) The maximal ideals of the ring B are the two ideals $\mathfrak{m}B$ and $\mathfrak{n}B$.
(3) The residue fields at these maximal ideals is k.
(4) We have $B_{\mathfrak{m}B} = R_{\mathfrak{m}}$ and $B_{\mathfrak{n}B} = R_{\mathfrak{n}}$ which are Noetherian regular local rings of dimensions 1 and 2.
(5) The ring B is Noetherian.

We omit the details of the verifications.

Whenever given a k-algebra B with the properties listed above we get an example as follows. Take $A = k + \mathrm{rad}(B) \subset B$, in our case $\mathrm{rad}(B) = \mathfrak{m}B + \mathfrak{n}B$. It is easy to see that B is finite over A and hence A is Noetherian by Eakin's theorem (see [**Eak68**], or [**Nag62**, Appendix A1], or insert future reference here). Also A is a local domain with the same fraction field as B and residue field k. Since the dimension of B is 2 we see that A has dimension 2 as well, by Algebra, Lemma 10.106.4.

If A were universally catenary then the dimension formula, Algebra, Lemma 10.107.1 would give $\dim(B_{\mathfrak{m}B}) = 2$ contradiction.

Note that B is generated by one element over A. Hence $B = A[x]/\mathfrak{p}$ for some prime $\mathfrak{p}$ of $A[x]$. Let $\mathfrak{m}' \subset A[x]$ be the maximal ideal corresponding to $\mathfrak{m}B$. Then on the one hand $\dim(A[x]_{\mathfrak{m}'}) = 3$ and on the other hand

$$(0) \subset \mathfrak{p}A[x]_{\mathfrak{m}'} \subset \mathfrak{m}'A[x]_{\mathfrak{m}'}$$

is a maximal chain of primes. Hence $A[x]_{\mathfrak{m}'}$ is an example of a non catenary Noetherian local ring.

78.12. Non-quasi-affine variety with quasi-affine normalization

The existence of an example of this kind is mentioned in [**DG67**, II Remark 6.6.13]. They refer to the fifth volume of EGA for such an example, but the fifth volume did not appear.

Let k be a field. Let $Y = \mathbf{A}^2_k \setminus \{(0,0)\}$. We are going to construct a finite surjective birational morphism $\pi : Y \longrightarrow X$ with X a variety over k such that X is not quasi-affine. Namely, consider the following curves in Y:

$$\begin{array}{lcl} C_1 & : & x = 0 \\ C_2 & : & y = 0 \end{array}$$

Note that $C_1 \cap C_2 = \emptyset$. We choose the isomorphism $\varphi : C_1 \to C_2$, $(0, y) \mapsto (y^{-1}, 0)$. We claim there is a unique morphism $\pi : Y \to X$ as above such that

$$C_1 \xrightarrow[\varphi]{\text{id}} Y \xrightarrow{\pi} X$$

is a coequalizer diagram in the category of varieties (and even in the category of schemes). Accepting this for the moment let us show that such an X cannot be quasi-affine. Namely, it is clear that we would get

$$\Gamma(X, \mathcal{O}_X) = \{f \in k[x,y] \mid f(0,y) = f(y^{-1}, 0)\} = k \oplus (xy) \subset k[x,y].$$

In particular these functions do not separate the points $(1,0)$ and $(-1,0)$ whose images in X (we will see below) are distinct (if the characteristic of k is not 2).

To show that X exists consider the Zariski open $D(x+y) \subset Y$ of Y. This is the spectrum of the ring $k[x, y, 1/(x+y)]$ and the curves C_1, C_2 are completely contained in $D(x+y)$. Moreover the morphism

$$C_1 \coprod C_2 \longrightarrow D(x+y) \cap Y = \text{Spec}(k[x, y, 1/(x+y)])$$

is a closed immersion. It follows from More on Algebra, Lemma 15.4.1 that the ring

$$A = \{f \in k[x, y, 1/(x+y)] \mid f(0,y) = f(y^{-1}, 0)\}$$

is of finite type over k. On the other hand we have the open $D(xy) \subset Y$ of Y which is disjoint from the curves C_1 and C_2. It is the spectrum of the ring

$$B = k[x, y, 1/xy].$$

Note that we have $A_{xy} \cong B_{x+y}$ (since A clearly contains the elements $xyP(x,y)$ any polynomial P and the element $xy/(x+y)$). The scheme X is obtained by glueing the affine schemes $\text{Spec}(A)$ and $\text{Spec}(B)$ using the isomorphism $A_{xy} \cong B_{x+y}$ and hence is clearly of finite type over k. To see that it is separated one has to show

that the ring map $A \otimes_k B \to B_{x+y}$ is surjective. To see this use that $A \otimes_k B$ contains the element $xy/(x+y) \otimes 1/xy$ which maps to $1/(x+y)$. The morphism $X \to Y$ is given by the natural maps $D(x+y) \to \mathrm{Spec}(A)$ and $D(xy) \to \mathrm{Spec}(B)$. Since these are both finite we deduce that $X \to Y$ is finite as desired. We omit the verification that X is indeed the coequalizer of the displayed diagram above, however, see (insert future reference for pushouts in the category of schemes here). Note that the morphism $\pi : Y \to X$ does map the points $(1, 0)$ and $(-1, 0)$ to distinct points in X because the function $(x+y^3)/(x+y)^2 \in A$ has value $1/1$, resp. $-1/(-1)^2 = -1$ which are always distinct (unless the characteristic is 2 – please find your own points for characteristic 2). We summarize this discussion in the form of a lemma.

Lemma 78.12.1. *Let k be a field. There exists a variety X whose normalization is quasi-affine but which is itself not quasi-affine.*

Proof. See discussion above and (insert future reference on normalization here). □

78.13. A locally closed subscheme which is not open in closed

This is a copy of Morphisms, Example 28.3.4. Here is an example of an immersion which is not a composition of an open immersion followed by a closed immersion. Let k be a field. Let $X = \mathrm{Spec}(k[x_1, x_2, x_3, \ldots])$. Let $U = \bigcup_{n=1}^{\infty} D(x_n)$. Then $U \to X$ is an open immersion. Consider the ideals

$$I_n = (x_1^n, x_2^n, \ldots, x_{n-1}^n, x_n - 1, x_{n+1}, x_{n+2}, \ldots) \subset k[x_1, x_2, x_3, \ldots][1/x_n].$$

Note that $I_n k[x_1, x_2, x_3, \ldots][1/x_n x_m] = (1)$ for any $m \neq n$. Hence the quasi-coherent ideals $\widetilde{I_n}$ on $D(x_n)$ agree on $D(x_n x_m)$, namely $\widetilde{I_n}|_{D(x_n x_m)} = \mathcal{O}_{D(x_n x_m)}$ if $n \neq m$. Hence these ideals glue to a quasi-coherent sheaf of ideals $\mathcal{I} \subset \mathcal{O}_U$. Let $Z \subset U$ be the closed subscheme corresponding to $\mathcal{I}$. Thus $Z \to X$ is an immersion.

We claim that we cannot factor $Z \to X$ as $Z \to \overline{Z} \to X$, where $\overline{Z} \to X$ is closed and $Z \to \overline{Z}$ is open. Namely, $\overline{Z}$ would have to be defined by an ideal $I \subset k[x_1, x_2, x_3, \ldots]$ such that $I_n = Ik[x_1, x_2, x_3, \ldots][1/x_n]$. But the only element $f \in k[x_1, x_2, x_3, \ldots]$ which ends up in all I_n is 0! Hence I does not exist.

78.14. Nonexistence of suitable opens

This section complements the results of Properties, Section 27.27.

Let k be a field and let $A = k[z_1, z_2, z_3, \ldots]/I$ where I is the ideal generated by all pairwise products $z_i z_j$, $i \neq j$, $i, j \in \mathbf{N}$. Set $S = \mathrm{Spec}(A)$. Let $s \in S$ be the closed point corresponding to the maximal ideal (z_i). We claim there is no quasi-compact open $V \subset S \setminus \{s\}$ which is dense in $S \setminus \{s\}$. Note that $S \setminus \{s\} = \bigcup D(z_i)$. Each $D(z_i)$ is open and irreducible with generic point η_i. We conclude that $\eta_i \in V$ for all i. However, a principal affine open of $S \setminus \{s\}$ is of the form $D(f)$ where $f \in (z_1, z_2, \ldots)$. Then $f \in (z_1, \ldots, z_n)$ for some n and we see that $D(f)$ contains only finitely many of the points η_i. Thus V cannot be quasi-compact.

Let k be a field and let $B = k[x, z_1, z_2, z_3, \ldots]/J$ where J is the ideal generated by the products xz_i, $i \in \mathbf{N}$ and by all pairwise products $z_i z_j$, $i \neq j$, $i, j \in \mathbf{N}$. Set $T = \mathrm{Spec}(B)$. Consider the principal open $U = D(x)$. We claim there is no quasi-compact open $V \subset S$ such that $V \cap U = \emptyset$ and $V \cup U$ is dense in S. Let $t \in T$

be the closed point corresponding to the maximal ideal (x, z_i). The closure of U in T is $\overline{U} = U \cup \{t\}$. Hence $V \subset \bigcup_i D(z_i)$ is a quasi-compact open. By the arguments of the previous paragraph we see that V cannot be dense in $\bigcup D(z_i)$.

Lemma 78.14.1. *Nonexistence quasi-compact opens of affines:*

(1) *There exist an affine scheme S and affine open $U \subset S$ such that there is no quasi-compact open $V \subset S$ with $U \cap V = \emptyset$ and $U \cup V$ dense in S.*
(2) *There exists an affine scheme S and a closed point $s \in S$ such that $S \setminus \{s\}$ does not contain a quasi-compact dense open.*

Proof. See discussion above. □

Let X be the glueing of two copies of the affine scheme T (see above) along the affine open U. Thus there is a morphism $\pi : X \to T$ and $X = U_1 \cup U_2$ such that π maps U_i isomorphically to T and $U_1 \cap U_2$ isomorphically to U. Note that X is quasi-separated (by Schemes, Lemma 25.21.7) and quasi-compact. We claim there does not exist a separated, dense, quasi-compact open $W \subset X$. Namely, consider the two closed points $x_1 \in U_1$, $x_2 \in U_2$ mapping to the closed point $t \in T$ introduced above. Let $\tilde{\eta} \in U_1 \cap U_2$ be the generic point mapping to the (unique) generic point η of U. Note that $\tilde{\eta} \rightsquigarrow x_1$ and $\tilde{\eta} \rightsquigarrow x_2$ lying over the specialization $\eta \rightsquigarrow s$. Since $\pi|_W : W \to T$ is separated we conclude that we cannot have both x_1 and $x_2 \in W$ (by the valuative criterion of separatedness Schemes, Lemma 25.22.2). Say $x_1 \notin W$. Then $W \cap U_1$ is a quasi-compact (as X is quasi-separated) dense open of U_1 which does not contain x_1. Now observe that there exists an isomorphism $(T, t) \cong (S, s)$ of schemes (by sending x to z_1 and z_i to z_{i+1}). Hence by the first paragraph of this section we arrive at a contradiction.

Lemma 78.14.2. *There exists a quasi-compact and quasi-separated scheme X which does not contain a separated quasi-compact dense open.*

Proof. See discussion above. □

78.15. Nonexistence of quasi-compact dense open subscheme

Let X be a quasi-compact and quasi-separated algebraic space over a field k. We know that the schematic locus $X' \subset X$ is a dense open subspace, see Properties of Spaces, Proposition 48.10.3. In fact, this result holds when X is reasonable, see Decent Spaces, Proposition 50.8.3. A natural question is whether one can find a quasi-compact dense open subscheme of X. It turns out this is not possible in general.

Assume the characteristic of k is not 2. Let $B = k[x, z_1, z_2, z_3, \ldots]/J$ where J is the ideal generated by the products xz_i, $i \in \mathbf{N}$ and by all pairwise products $z_i z_j$, $i \neq j$, $i, j \in \mathbf{N}$. Set $U = \operatorname{Spec}(B)$. Denote $0 \in U$ the closed point all of whose coordinates are zero. Set

$$j : R = \Delta \coprod \Gamma \longrightarrow U \times_k U$$

where Δ is the image of the diagonal morphism of U over k and

$$\Gamma = \{((x, 0, 0, 0, \ldots), (-x, 0, 0, 0, \ldots)) \mid x \in \mathbf{A}^1_k, x \neq 0\}.$$

It is clear that $s, t : R \to U$ are étale, and hence j is an étale equivalence relation. The quotient $X = U/R$ is an algebraic space (Spaces, Theorem 47.10.5). Note that j is not an immersion because $(0, 0) \in \Delta$ is in the closure of Γ. Hence X is not a

scheme. On the other hand, X is quasi-separated as R is quasi-compact. Denote 0_X the image of the point $0 \in U$. We claim that $X \setminus \{0_X\}$ is a scheme, namely

$$X \setminus \{0_X\} = \operatorname{Spec}\left(k[x^2, x^{-2}]\right) \coprod \operatorname{Spec}\left(k[z_1, z_2, z_3, \ldots]/(z_i z_j)\right) \setminus \{0\}$$

(details omitted). On the other hand, we have seen in Section 78.14 that the scheme on the right hand side does not contain a quasi-compact dense open.

Lemma 78.15.1. *There exists a quasi-compact and quasi-separated algebraic space which does not contain a quasi-compact dense open subscheme.*

Proof. See discussion above. □

Using the construction of Spaces, Example 47.14.2 in the same manner as we used the construction of Spaces, Example 47.14.1 above, one obtains an example of a quasi-compact, quasi-separated, and locally separated algebraic space which does not contain a quasi-compact dense open subscheme.

78.16. Affines over algebraic spaces

Suppose that $f : Y \to X$ is a morphism of schemes with f locally of finite type and Y affine. Then there exists an immersion $Y \to \mathbf{A}^n_X$ of Y into affine n-space over X. See the slightly more general Morphisms, Lemma 28.40.2.

Now suppose that $f : Y \to X$ is a morphism of algebraic spaces with f locally of finite type and Y an affine scheme. Then it is not true in general that we can find an immersion of Y into affine n-space over X.

A first (nasty) counter example is $Y = \operatorname{Spec}(k)$ and $X = [\mathbf{A}^1_k/\mathbf{Z}]$ where k is a field of characteristic zero and $\mathbf{Z}$ acts on $\mathbf{A}^1_k$ by translation $(n, t) \mapsto t + n$. Namely, for any morphism $Y \to \mathbf{A}^n_X$ over X we can pullback to the covering $\mathbf{A}^1_k$ of X and we get an infinite disjoint union of $\mathbf{A}^1_k$'s mapping into $\mathbf{A}^{n+1}_k$ which is not an immersion.

A second counter example is $Y = \mathbf{A}^1_k \to X = \mathbf{A}^1_k/R$ with $R = \{(t, t)\} \coprod \{(t, -t), t \neq 0\}$. Namely, in this case the morphism $Y \to \mathbf{A}^n_X$ would be given by some regular functions $f_1, \ldots, f_n$ on Y and hence the fibre product of Y with the covering $\mathbf{A}^{n+1}_k \to \mathbf{A}^n_X$ would be the scheme

$$\{(f_1(t), \ldots, f_n(t), t)\} \coprod \{(f_1(t), \ldots, f_n(t), -t), t \neq 0\}$$

with obvious morphism to $\mathbf{A}^{n+1}_k$ which is not an immersion. Note that this gives a counter example with X quasi-separated.

Lemma 78.16.1. *There exists a finite type morphism of algebraic spaces $Y \to X$ with Y affine and X quasi-separated, such that there does not exist an immersion $Y \to \mathbf{A}^n_X$ over X.*

Proof. See discussion above. □

78.17. Pushforward of quasi-coherent modules

In Schemes, Lemma 25.24.1 we proved that f_* transforms quasi-coherent modules into quasi-coherent modules when f is quasi-compact and quasi-separated. Here are some examples to show that these conditions are both necessary.

Suppose that $Y = \mathrm{Spec}(A)$ is an affine scheme and that $X = \coprod_{n \in \mathbf{N}} Y$. We claim that $f_*\mathcal{O}_X$ is not quasi-coherent where $f : X \to Y$ is the obvious morphism. Namely, for $a \in A$ we have

$$f_*\mathcal{O}_X(D(a)) = \prod_{n \in \mathbf{N}} A_a$$

Hence, in order for $f_*\mathcal{O}_X$ to be quasi-coherent we would need

$$\prod_{n \in \mathbf{N}} A_a = \left(\prod_{n \in \mathbf{N}} A\right)_a$$

for all $a \in A$. This isn't true in general, for example if $A = \mathbf{Z}$ and $a = 2$, then $(1, 1/2, 1/4, 1/8, \ldots)$ is an element of the left hand side which is not in the right hand side. Note that f is a non-quasi-compact separated morphism.

Let k be a field. Set

$$A = k[t, z, x_1, x_2, x_3, \ldots]/(tx_1z, t^2x_2^2z, t^3x_3^3z, \ldots)$$

Let $Y = \mathrm{Spec}(A)$. Let $V \subset Y$ be the open subscheme $V = D(x_1) \cup D(x_2) \cup \ldots$. Let X be two copies of Y glued along V. Let $f : X \to Y$ be the obvious morphism. Then we have an exact sequence

$$0 \to f_*\mathcal{O}_X \to \mathcal{O}_Y \oplus \mathcal{O}_Y \xrightarrow{(1,-1)} j_*\mathcal{O}_V$$

where $j : V \to Y$ is the inclusion morphism. Since

$$A \longrightarrow \prod A_{x_n}$$

is injective (details omitted) we see that $\Gamma(Y, f_*\mathcal{O}_X) = A$. On the other hand, the kernel of the map

$$A_t \longrightarrow \prod A_{tx_n}$$

is nonzero because it contains the element z. Hence $\Gamma(D(t), f_*\mathcal{O}_X)$ is strictly bigger than A_t because it contains $(z, 0)$. Thus we see that $f_*\mathcal{O}_X$ is not quasi-coherent. Note that f is quasi-compact but non-quasi-separated.

Lemma 78.17.1. *Schemes, Lemma 25.24.1 is sharp in the sense that one can neither drop the assumption of quasi-compactness nor the assumption of quasi-separatedness.*

Proof. See discussion above. □

78.18. A nonfinite module with finite free rank 1 stalks

Let $R = \mathbf{Q}[x]$. Set $M = \sum_{n \in \mathbf{N}} \frac{1}{x-n} R$ as a submodule of the fraction field of R. Then M is not finitely generated, but for every prime $\mathfrak{p}$ of R we have $M_\mathfrak{p} \cong R_\mathfrak{p}$ as an $R_\mathfrak{p}$-module.

78.19. A finite flat module which is not projective

This is a copy of Algebra, Remark 10.73.3. It is not true that a finite R-module which is R-flat is automatically projective. A counter example is where $R = \mathcal{C}^\infty(\mathbf{R})$ is the ring of infinitely differentiable functions on $\mathbf{R}$, and $M = R_\mathfrak{m} = R/I$ where $\mathfrak{m} = \{f \in R \mid f(0) = 0\}$ and $I = \{f \in R \mid \exists \epsilon, \epsilon > 0 : f(x) = 0 \ \forall x, |x| < \epsilon\}$.

The morphism $\mathrm{Spec}(R/I) \to \mathrm{Spec}(R)$ is also an example of a flat closed immersion which is not open.

Lemma 78.19.1. *Strange flat modules.*

(1) *There exists a ring R and a finite flat R-module M which is not projective.*
(2) *There exists a closed immersion which is flat but not open.*

Proof. See discussion above. □

78.20. A projective module which is not locally free

We give two examples. One where the rank is between 0 and 1 and one where the rank is $\aleph_0$.

Lemma 78.20.1. *Let R be a ring. Let $I \subset R$ be an ideal generated by a countable collection of idempotents. Then I is projective as an R-module.*

Proof. Say $I = (e_1, e_2, e_3, \ldots)$ with e_n an idempotent of R. After inductively replacing e_{n+1} by $e_n + (1 - e_n)e_{n+1}$ we may assume that $(e_1) \subset (e_2) \subset (e_3) \subset \ldots$ and hence $I = \bigcup_{n \geq 1}(e_n) = \operatorname{colim}_n e_n R$. In this case

$$\operatorname{Hom}_R(I, M) = \operatorname{Hom}_R(\operatorname{colim}_n e_n R, M) = \lim_n \operatorname{Hom}_R(e_n R, M) = \lim_n e_n M$$

Note that the transition maps $e_{n+1}M \to e_n M$ are given by multiplication by e_n are are surjective. Hence by Algebra, Lemma 10.81.4 the functor $\operatorname{Hom}_R(I, M)$ is exact, i.e., I is a projective R-module. □

Lemma 78.20.2. *Let R be a ring. Let $n \geq 1$. Let M be an R-module generated by $< n$ elements. Then any R-module map $f : R^{\oplus n} \to M$ has a nonzero kernel.*

Proof. Choose a surjection $R^{\oplus n-1} \to M$. We may lift the map f to a map $f' : R^{\oplus n} \to R^{\oplus n-1}$. It suffices to prove f' has a nonzero kernel. The map $f' : R^{\oplus n} \to R^{\oplus n-1}$ is given by a matrix $A = (a_{ij})$. If one of the a_{ij} is not nilpotent, say $a = a_{ij}$ is not, then we can replace A by the localization A_a and we may assume a_{ij} is a unit. Since if we find a nonzero kernel after localization then there was a nonzero kernel to start with as localization is exact, see Algebra, Proposition 10.9.12. In this case we can do a base change on both $R^{\oplus n}$ and $R^{\oplus n-1}$ and reduce to the case where

$$A = \begin{pmatrix} 1 & 0 & 0 \ldots & \\ 0 & a_{22} & a_{23} & \ldots \\ 0 & a_{32} & \ldots & \\ \ldots & \ldots & & \end{pmatrix}$$

Hence in this case we win by induction on n. If not then each a_{ij} is nilpotent. Set $I = (a_{ij}) \subset R$. Note that $I^{m+1} = 0$ for some $m \geq 0$. Let m be the largest integer such that $I^m \neq 0$. Then we see that $(I^m)^{\oplus n}$ is contained in the kernel of the map and we win. □

Suppose that $P \subset Q$ is an inclusion of R-modules with Q a finite R-module and P locally free, see Algebra, Definition 10.73.1. Suppose that Q can be generated by N elements as an R-module. Then it follows from Lemma 78.20.2 that P is finite locally free (with the free parts having rank at most N). And in this case P is a finite R-module, see Algebra, Lemma 10.73.2.

Combining this with the above we see that a non-finitely-generated ideal which is generated by a countable collection of idempotents is projective but not locally free. An explicit example is $R = \prod_{n \in \mathbf{N}} \mathbf{F}_2$ and I the ideal generated by the idempotents

$$e_n = (1, 1, \ldots, 1, 0, \ldots)$$

where the sequence of 1's has length n.

Lemma 78.20.3. *There exists a ring R and an ideal I such that I is projective as an R-module but not locally free as an R-module.*

Proof. See above. □

Lemma 78.20.4. *Let K be a field. Let C_i, $i = 1, \ldots, n$ be smooth, projective, geometrically irreducible curves over K. Let $P_i \in C_i(K)$ be a rational point and let $Q_i \in C_i$ be a point such that $[\kappa(Q_i) : K] = 2$. Then $[P_1 \times \ldots \times P_n]$ is nonzero in $A_0(U_1 \times_K \ldots \times_K U_n)$ where $U_i = C_i \setminus \{Q_i\}$.*

Proof. There is a degree map $\deg : A_0(C_1 \times_K \ldots \times_K C_n) \to \mathbf{Z}$ Because each Q_i has degree 2 over K we see that any zero cycle supported on the "boundary"

$$C_1 \times_K \ldots \times_K C_n \setminus U_1 \times_K \ldots \times_K U_n$$

has degree divisible by 2. □

We can construct another example of a projective but not locally free module using the lemma above as follows. Let C_n, $n = 1, 2, 3, \ldots$ be smooth, projective, geometrically irreducible curves over $\mathbf{Q}$ each with a pair of points $P_n, Q_n \in C_n$ such that $\kappa(P_n) = \mathbf{Q}$ and $\kappa(Q_n)$ is a quadratic extension of $\mathbf{Q}$. Set $U_n = C_n \setminus \{Q_n\}$; this is an affine curve. Let $\mathcal{L}_n$ be the inverse of the ideal sheaf of P_n on U_n. Note that $c_1(\mathcal{L}_n) = [P_n]$ in the group of zero cycles $A_0(U_n)$. Set $A_n = \Gamma(U_n, \mathcal{O}_{U_n})$. Let $L_n = \Gamma(U_n, \mathcal{L}_n)$ which is a locally free module of rank 1 over A_n. Set

$$B_n = A_1 \otimes_{\mathbf{Q}} A_2 \otimes_{\mathbf{Q}} \ldots \otimes_{\mathbf{Q}} A_n$$

so that $\mathrm{Spec}(B_n) = U_1 \times \ldots \times U_n$ all products over $\mathrm{Spec}(\mathbf{Q})$. For $i \leq n$ we set

$$L_{n,i} = A_1 \otimes_{\mathbf{Q}} \ldots \otimes_{\mathbf{Q}} M_i \otimes_{\mathbf{Q}} \ldots \otimes_{\mathbf{Q}} A_n$$

which is a locally free B_n-module of rank 1. Note that this is also the global sections of $\mathrm{pr}_i^* \mathcal{L}_n$. Set

$$B_\infty = \mathrm{colim}_n B_n \quad \text{and} \quad L_{\infty,i} = \mathrm{colim}_n L_{n,i}$$

Finally, set

$$M = \bigoplus\nolimits_{i \geq 1} L_{\infty,i}.$$

This is a direct sum of finite locally free modules, hence projective. We claim that M is not locally free. Namely, suppose that $f \in B_\infty$ is a nonzero function such that M_f is free over $(B_\infty)_f$. Let $e_1, e_2, \ldots$ be a basis. Choose $n \geq 1$ such that $f \in B_n$. Choose $m \geq n + 1$ such that $e_1, \ldots, e_{n+1}$ are in

$$\bigoplus\nolimits_{1 \leq i \leq m} L_{m,i}.$$

Because the elements $e_1, \ldots, e_{n+1}$ are part of a basis after a faithfully flat base change we conclude that the chern classes

$$c_i(\mathrm{pr}_1^* \mathcal{L}_1 \oplus \ldots \oplus \mathrm{pr}_m^* \mathcal{L}_m), \quad i = m, m-1, \ldots, m-n$$

are zero in the chow group of

$$D(f) \subset U_1 \times \ldots \times U_m$$

Since f is the pullback of a function on $U_1 \times \ldots \times U_n$ this implies in particular that

$$c_{m-n}(\mathcal{O}_W^{\oplus n} \oplus \mathrm{pr}_1^* \mathcal{L}_{n+1} \oplus \ldots \oplus \mathrm{pr}_{m-n}^* \mathcal{L}_m) = 0.$$

on the variety

$$W = (C_{n+1} \times \ldots \times C_m)_K$$

over the field $K = \mathbf{Q}(C_1 \times \ldots \times C_n)$. In other words the cycle

$$[(P_{n+1} \times \ldots \times P_m)_K]$$

is zero in the chow group of zero cycles on W. This contradicts Lemma 78.20.4 above because the points Q_i, $n+1 \leq i \leq m$ induce corresponding points Q_i' on $(C_n)_K$ and as $K/\mathbf{Q}$ is geometrically irreducible we have $[\kappa(Q_i') : K] = 2$.

Lemma 78.20.5. *There exists a countable ring R and a projective module M which is a direct sum of countably many locally free rank 1 modules such that M is not locally free.*

Proof. See above. □

78.21. Zero dimensional local ring with nonzero flat ideal

In [**Laz67**] there is an example of a zero dimensional local ring with a nonzero flat ideal. Here is the construction. Let k be a field. Let X_i, Y_i, $i \geq 1$ be variables. Take $R = k[X_i, Y_i]/(X_i - Y_i X_{i+1}, Y_i^2)$. Denote x_i, resp. y_i the image of X_i, resp. Y_i in this ring. Note that

$$x_i = y_i x_{i+1} = y_i y_{i+1} x_{i+2} = y_i y_{i+1} y_{i+2} x_{i+3} = \ldots$$

in this ring. The ring R has only one prime ideal, namely $\mathfrak{m} = (x_i, y_i)$. We claim that the ideal $I = (x_i)$ is flat as an R-module.

Note that the annihilator of x_i in R is the ideal $(x_1, x_2, x_3, \ldots, y_i, y_{i+1}, y_{i+2}, \ldots)$. Consider the R-module M generated by elements e_i, $i \geq 1$ and relations $e_i = y_i e_{i+1}$. Then M is flat as it is the colimit $\operatorname{colim}_i R$ of copies of R with transition maps

$$R \xrightarrow{y_1} R \xrightarrow{y_2} R \xrightarrow{y_3} \ldots$$

Note that the annihilator of e_i in M is the ideal $(x_1, x_2, x_3, \ldots, y_i, y_{i+1}, y_{i+2}, \ldots)$. Since every element of M, resp. I can be written as fe_i, resp. hx_i for some $f, h \in R$ we see that the map $M \to I$, $e_i \to x_i$ is an isomorphism and I is flat.

Lemma 78.21.1. *There exists a local ring R with a unique prime ideal and a nonzero ideal $I \subset R$ which is a flat R-module*

Proof. See discussion above. □

78.22. An epimorphism of zero-dimensional rings which is not surjective

In [**Laz69**] one can find the following example. Let k be a field. Consider the ring homomorphism

$$k[x_1, x_2, \ldots, z_1, z_2, \ldots]/(x_i^{4^i}, z_i^{4^i}) \longrightarrow k[x_1, x_2, \ldots, y_1, y_2, \ldots]/(x_i^{4^i}, y_i - x_{i+1} y_{i+1}^2)$$

which maps x_i to x_i and z_i to $x_i y_i$. Note that $y_i^{4^{i+1}}$ is zero in the right hand side but that y_1 is not zero (details omitted). This map is not surjective: we can think of the above as a map of $\mathbf{Z}$-graded algebras by setting $\deg(x_i) = -1$, $\deg(z_i) = 0$, and $\deg(y_i) = 1$ and then it is clear that y_1 is not in the image. Finally, the map is an epimorphism because

$$y_{i-1} \otimes 1 = x_i y_i^2 \otimes 1 = y_i \otimes x_i y_i = x_i y_i \otimes y_i = 1 \otimes x_i y_i^2.$$

hence the tensor product of the target over the source is isomorphic to the target.

Lemma 78.22.1. *There exists an epimorphism of local rings of dimension* 0 *which is not a surjection.*

Proof. See discussion above. □

78.23. Finite type, not finitely presented, flat at prime

Let k be a field. Consider the local ring $A_0 = k[x,y]_{(x,y)}$. Denote $\mathfrak{p}_{0,n} = (y + x^n + x^{n+1})$. This is a prime ideal. Set

$$A = A_0[z_1, z_2, z_3, \ldots]/(z_n z_m, z_n(y + x^n + x^{2n+1}))$$

Note that $A \to A_0$ is a surjection whose kernel is an ideal of square zero. Hence A is also a local ring and the prime ideals of A are in one-to-one correspondence with the prime ideals of A_0. Denote $\mathfrak{p}_n$ the prime ideal of A corresponding to $\mathfrak{p}_{0,n}$. Observe that $\mathfrak{p}_n$ is the annihilator of z_n in A. Let

$$C = A[z]/(xz^2 + z + y)[\frac{1}{2zx+1}].$$

Note that $A \to C$ is an étale ring map, see Algebra, Example 10.130.8. Let $\mathfrak{q} \subset C$ be the maximal ideal generated by x, y, z and all z_n. As $A \to C$ is flat we see that the annihilator of z_n in C is $\mathfrak{p}_n C$. We compute

$$\begin{aligned} C/\mathfrak{p}_n C &= A_0/(y + x^n + x^{2n+1}) \\ &= k[x]_{(x)}[z]/(xz^2 + z - x^n - x^{2n+1}) \\ &= k[x]_{(x)}[z]/(z - x^n) \times k[x]_{(x)}[z]/(xz + x^{n+1} + 1) \\ &= k[x]_{(x)} \times k(x) \end{aligned}$$

because $(z - x^n)(xz + x^{n+1} + 1) = xz^2 + z - x^n - x^{2n+1}$. Hence we see that $\mathfrak{p}_n C = \mathfrak{r}_n \cap \mathfrak{q}_n$ with $\mathfrak{r}_n = \mathfrak{p}_n C + (z - x^n)C$ and $\mathfrak{q}_n = \mathfrak{p}_n C + (xz + x^{n+1} + 1)C$. Since $\mathfrak{q}_n + \mathfrak{r}_n = C$ we also get $\mathfrak{p}_n C = \mathfrak{r}_n \mathfrak{q}_n$, It follows that $\mathfrak{q}_n$ is the annihilator of $\xi_n = (z - x^n)z_n$. Observe that on the one hand $\mathfrak{r}_n \subset \mathfrak{q}$, and on the other hand $\mathfrak{q}_n + \mathfrak{q} = C$. This follows for example because $\mathfrak{q}_n$ is a maximal ideal of C distinct from $\mathfrak{q}$. Similarly we have $\mathfrak{q}_n + \mathfrak{q}_m = C$. At this point we let

$$B = \mathrm{Im}(C \longrightarrow C_{\mathfrak{q}})$$

We observe that the elements ξ_n map to zero in B as $xz + x^{n+1} + 1$ is not in $\mathfrak{q}$. Denote $\mathfrak{q}' \subset B$ the image of $\mathfrak{q}$. By construction B is a finite type A-algebra, with $B_{\mathfrak{q}'} \cong C_{\mathfrak{q}}$. In particular we see that $B_{\mathfrak{q}'}$ is flat over A.

We claim there does not exist an element $g' \in B$, $g' \notin \mathfrak{q}'$ such that $B_{g'}$ is of finite presentation over A. We sketch a proof of this claim. Choose an element $g \in C$ which maps to $g' \in B$. Consider the map $C_g \to B_{g'}$. By Algebra, Lemma 10.6.3 we see that B_g is finitely presented over A if and only if the kernel of $C_g \to B_{g'}$ is finitely generated. But the element $g \in C$ is not contained in $\mathfrak{q}$, hence maps to a nonzero element of $A_0[z]/(xz^2 + z + y)$. Hence g can only be contained in finitely many of the prime ideals $\mathfrak{q}_n$, because the primes $(y + x^n + x^{2n+1}, xz + x^{n+1} + 1)$ are an infinite collection of codimension 1 points of the 2-dimensional irreducible Noetherian space $\mathrm{Spec}(k[x,y,z]/(xz^2 + z + y))$. The map

$$\bigoplus\nolimits_{g \notin \mathfrak{q}_n} C/\mathfrak{q}_n \longrightarrow C_g, \quad (c_n) \longrightarrow \sum c_n \xi_n$$

is injective and its image is the kernel of $C_g \to B_{g'}$. We omit the proof of this statement. (Hint: Write $A = A_0 \oplus I$ as an A_0-module where I is the kernel of

$A \to A_0$. Similarly, write $C = C_0 \oplus IC$. Write $IC = \bigoplus Cz_n \cong \bigoplus(C/\mathfrak{r}_n \oplus C/\mathfrak{q}_n)$ and study the effect of multiplication by g on the summands.) This concludes the sketch of the proof of the claim. This also proves that $B_{g'}$ is not flat over A for any g' as above. Namely, if it were flat, then the annihilator of the image of z_n in $B_{g'}$ would be $\mathfrak{p}_n B_{g'}$, and would not contain $z - x^n$.

As a consequence we can answer (negatively) a question posed in [**GR71**, Part I, Remarques (3.4.7) (v)]. Here is a precise statement.

Lemma 78.23.1. *There exists a local ring A, a finite type ring map $A \to B$ and a prime $\mathfrak{q}$ lying over $\mathfrak{m}_A$ such that $B_\mathfrak{q}$ is flat over A, and for any element $g \in B$, $g \notin \mathfrak{q}$ the ring B_g is neither finitely presented over A nor flat over A.*

Proof. See discussion above. □

78.24. Finite type, flat and not of finite presentation

In this section we give some examples of ring maps and morphisms which are of finite type and flat but not of finite presentation.

Let R be a ring which has an ideal I such that R/I is a finite flat module but not projective, see Section 78.19 for an explicit example. Note that this means that I is not finitely generated, see Algebra, Lemma 10.103.5. Note that $I = I^2$, see Algebra, Lemma 10.103.2. The base ring in our examples will be R and correspondingly the base scheme $S = \mathrm{Spec}(R)$.

Consider the ring map $R \to R \oplus R/I\epsilon$ where $\epsilon^2 = 0$ by convention. This is a finite, flat ring map which is not of finite presentation. All the fibre rings are complete intersections and geometrically irreducible.

Let $A = R[x, y]/(xy, ay; a \in I)$. Note that as an R-module we have $A = \bigoplus_{i \geq 0} Ry^i \oplus \bigoplus_{j>0} R/Ix^j$. Hence $R \to A$ is a flat finite type ring map which is not of finite presentation. Each fibre ring is isomorphic to either $\kappa(\mathfrak{p})[x, y]/(xy)$ or $\kappa(\mathfrak{p})[x]$.

We can turn the previous example into a projective morphism by taking $B = R[X_0, X_1, X_2]/(X_1X_2, aX_2; a \in I)$. In this case $X = \mathrm{Proj}(B) \to S$ is a proper flat morphism which is not of finite presentation such that for each $s \in S$ the fibre X_s is isomorphic either to $\mathbf{P}^1_s$ or to the closed subscheme of $\mathbf{P}^2_s$ defined by the vanishing of X_1X_2 (this is a projective nodal curve of arithmetic genus 0).

Let $M = R \oplus R \oplus R/I$. Set $B = \mathrm{Sym}_R(M)$ the symmetric algebra on M. Set $X = \mathrm{Proj}(B)$. Then $X \to S$ is a proper flat morphism, not of finite presentation such that for $s \in S$ the geometric fibre is isomorphic to either $\mathbf{P}^1_s$ or $\mathbf{P}^2_s$. In particular these fibres are smooth and geometrically irreducible.

Lemma 78.24.1. *There exist examples of*

1. *a flat finite type ring map with geometrically irreducible complete intersection fibre rings which is not of finite presentation,*
2. *a flat finite type ring map with geometrically connected, geometrically reduced, dimension 1, complete intersection fibre rings which is not of finite presentation,*
3. *a proper flat morphism of schemes $X \to S$ each of whose fibres is isomorphic to either $\mathbf{P}^1_s$ or to the vanishing locus of X_1X_2 in $\mathbf{P}^2_s$ which is not of finite presentation, and*

(4) *a proper flat morphism of schemes $X \to S$ each of whose fibres is isomorphic to either $\mathbf{P}^1_s$ or $\mathbf{P}^2_s$ which is not of finite presentation.*

Proof. See discussion above. □

78.25. Topology of a finite type ring map

Let $A \to B$ be a local map of local domains. If A is Noetherian, $A \to B$ is essentially of finite type, and $A/\mathfrak{m}_A \subset B/\mathfrak{m}_B$ is finite then there exists a prime $\mathfrak{q} \subset B$, $\mathfrak{q} \neq \mathfrak{m}_B$ such that $A \to B/\mathfrak{q}$ is the localization of a quasi-finite ring map. See More on Morphisms, Lemma 36.35.6.

In this section we give an example that shows this result is false A is no longer Noetherian. Namely, let k be a field and set

$$A = \{a_0 + a_1 x + a_2 x^2 + \ldots \mid a_0 \in k, a_i \in k((y)) \text{ for } i \geq 1\}$$

and

$$C = \{a_0 + a_1 x + a_2 x^2 + \ldots \mid a_0 \in k[y], a_i \in k((y)) \text{ for } i \geq 1\}.$$

The inclusion $A \to C$ is of finite type as C is generated by y over A. We claim that A is a local ring with maximal ideal $\mathfrak{m} = \{a_1 x + a_2 x^2 + \ldots \in A\}$ and no prime ideals besides (0) and $\mathfrak{m}$. Namely, an element $f = a_0 + a_1 x + a_2 x^2 + \ldots$ of A is invertible as soon as $a_0 \neq 0$. If $\mathfrak{q} \subset A$ is a nonzero prime ideal, and $f = a_i x^i + \ldots \in \mathfrak{q}$, then using properties of power series one sees that for any $g \in k((y))$ the element $g^{i+1} x^{i+1} \in \mathfrak{q}$, i.e., $gx \in \mathfrak{q}$. This proves that $\mathfrak{q} = \mathfrak{m}$.

As to the spectrum of the ring C, arguing in the same way as above we see that any nonzero prime ideal contains the prime $\mathfrak{p} = \{a_1 x + a_2 x^2 + \ldots \in C\}$ which lies over $\mathfrak{m}$. Thus the only prime of C which lies over (0) is (0). Set $\mathfrak{m}_C = yC + \mathfrak{p}$ and $B = C_{\mathfrak{m}_C}$. Then $A \to B$ is the desired example.

Lemma 78.25.1. *There exists a local homomorphism $A \to B$ of local domains which is essentially of finite type and such that $A/\mathfrak{m}_A \to B/\mathfrak{m}_B$ is finite such that for every prime $\mathfrak{q} \neq \mathfrak{m}_B$ of B the ring map $A \to B/\mathfrak{q}$ is not the localization of a quasi-finite ring map.*

Proof. See the discussion above. □

78.26. Pure not universally pure

Let k be a field. Let

$$R = k[[x, xy, xy^2, \ldots]] \subset k[[x, y]].$$

In other words, a power series $f \in k[[x, y]]$ is in R if and only if $f(0, y)$ is a constant. In particular $R[1/x] = k[[x, y]][1/x]$ and R/xR is a local ring with a maximal ideal whose square is zero. Denote $R[y] \subset k[[x, y]]$ the set of power series $f \in k[[x, y]]$ such that $f(0, y)$ is a polynomial in y. Then $R \to R[y]$ is a finite type but not finitely presented ring map which induces an isomorphism after inverting x. Also there is a surjection $R[y]/xR[y] \to k[y]$ whose kernel has square zero. Consider the finitely presented ring map $R \to S = R[t]/(xt - xy)$. Again $R[1/x] \to S[1/x]$ is an isomorphism and in this case $S/xS \cong (R/xR)[t]/(xy)$ maps onto $k[t]$ with nilpotent kernel. There is a surjection $S \to R[y]$, $t \longmapsto y$ which induces an isomorphism on inverting x and a surjection with nilpotent kernel modulo x. Hence

the kernel of $S \to R[y]$ is locally nilpotent. In particular $S \to R[y]$ is a universal homeomorphism.

First we claim that S is an S-module which is relatively pure over R. Since on inverting x we obtain an isomorphism we only need to check this at the maximal ideal $\mathfrak{m} \subset R$. Since R is complete with respect to its maximal ideal it is henselian hence we need only check that every prime $\mathfrak{p} \subset R$, $\mathfrak{p} \neq \mathfrak{m}$, the unique prime $\mathfrak{q}$ of S lying over $\mathfrak{p}$ satisfies $\mathfrak{m}S + \mathfrak{q} \neq S$. Since $\mathfrak{p} \neq \mathfrak{m}$ it corresponds to a unique prime ideal of $k[[x, y]][1/x]$. Hence either $\mathfrak{p} = (0)$ or $\mathfrak{p} = (f)$ for some irreducible element $f \in k[[x, y]]$ which is not associated to x (here we use that $k[[x, y]]$ is a UFD – insert future reference here). In the first case $\mathfrak{q} = (0)$ and the result is clear. In the second case we may multiply f by a unit so that $f \in R[y]$ (Weierstrass preparation; details omitted). Then it is easy to see that $R[y]/fR[y] \cong k[[x, y]]/(f)$ hence f defines a prime ideal of $R[y]$ and $\mathfrak{m}R[y] + fR[y] \neq R[y]$. Since $S \to R[y]$ is a universal homeomorphism we deduce the desired result for S also.

Second we claim that S is not universally relatively pure over R. Namely, to see this it sufffices to find a valuation ring $\mathcal{O}$ and a local ring map $R \to \mathcal{O}$ such that $\mathrm{Spec}(R[y] \otimes_R \mathcal{O}) \to \mathrm{Spec}(\mathcal{O})$ does not hit the closed point of $\mathrm{Spec}(\mathcal{O})$. Equivalently, we have to find $\varphi : R \to \mathcal{O}$ such that $\varphi(x) \neq 0$ and $v(\varphi(x)) > v(\varphi(xy))$ where v is the valuation of $\mathcal{O}$. (Because this means that the valuation of y is negative.) To do this consider the ring map

$$R \longrightarrow \{a_0 + a_1 x + a_2 x^2 + \ldots \mid a_0 \in k[y^{-1}], a_i \in k((y))\}$$

defined in the obvious way. We can find a valuation ring $\mathcal{O}$ dominating the localization of the right hand side at the maximal ideal (y^{-1}, x) and we win.

Lemma 78.26.1. *There exists a morphism of affine schemes of finite presentation $X \to S$ and an $\mathcal{O}_X$-module $\mathcal{F}$ of finite presentation such that $\mathcal{F}$ is pure relative to S, but not universally pure relative to S.*

Proof. See discussion above. □

78.27. A formally smooth non-flat ring map

Let k be a field. Consider the k-algebra $k[\mathbf{Q}]$. This is the k-algebra with basis $x_\alpha, \alpha \in \mathbf{Q}$ and multiplication determined by $x_\alpha x_\beta = x_{\alpha+\beta}$. (In particular $x_0 = 1$.) Consider the k-algebra homomorphism

$$k[\mathbf{Q}] \longrightarrow k, \quad x_\alpha \longmapsto 1.$$

It is surjective with kernel J generated by the elements $x_\alpha - 1$. Let us compute J/J^2. Note that multiplication by x_α on J/J^2 is the identity map. Denote z_α the class of $x_\alpha - 1$ modulo J^2. These classes generate J/J^2. Since

$$(x_\alpha - 1)(x_\beta - 1) = x_{\alpha+\beta} - x_\alpha - x_\beta + 1 = (x_{\alpha+\beta} - 1) - (x_\alpha - 1) - (x_\beta - 1)$$

we see that $z_{\alpha+\beta} = z_\alpha + z_\beta$ in J/J^2. A general element of J/J^2 is of the form $\sum \lambda_\alpha z_\alpha$ with $\lambda_\alpha \in k$ (only finitely many nonzero). Note that if the characteristic of k is $p > 0$ then

$$0 = p z_{\alpha/p} = z_{\alpha/p} + \ldots + z_{\alpha/p} = z_\alpha$$

and we see that $J/J^2 = 0$. If the characteristic of k is zero, then

$$J/J^2 = \mathbf{Q} \otimes_{\mathbf{Z}} k \cong k$$

(details omitted) is not zero.

We claim that $k[\mathbf{Q}] \to k$ is a formally smooth ring map if the characteristic of k is positive. Namely, suppose given a solid commutative diagram

$$\begin{array}{ccc} k & \longrightarrow & A \\ \uparrow & \nearrow & \uparrow \\ k[\mathbf{Q}] & \xrightarrow{\varphi} & A' \end{array}$$

with $A' \to A$ a surjection whose kernel I has square zero. To show that $k[\mathbf{Q}] \to k$ is formally smooth we have to prove that φ factors through k. Since $\varphi(x_\alpha - 1)$ maps to zero in A we see that φ induces a map $\overline{\varphi} : J/J^2 \to I$ whose vanishing is the obstruction to the desired factorization. Since $J/J^2 = 0$ if the characteristic is $p > 0$ we get the result we want, i.e., $k[\mathbf{Q}] \to k$ is formally smooth in this case. Finally, this ring map is not flat, for example as the nonzerodivisor $x_2 - 1$ is mapped to zero.

Lemma 78.27.1. *There exists a formally smooth ring map which is not flat.*

Proof. See discussion above. □

78.28. A formally étale non-flat ring map

In this section we give a counterexample to the final sentence in [**DG67**, 0_{IV}, Example 19.10.3(i)] (this was not one of the items caught in their later errata lists). Consider $A \to A/J$ for a local ring A and a nonzero proper ideal J such that $J^2 = J$ (so J isn't finitely generated); the valuation ring of an algebraically closed non-archimedean field with J its maximal ideal is a source of such (A, J). These non-flat quotient maps are formally étale. Namely, suppose given a commutative diagram

$$\begin{array}{ccc} A/J & \longrightarrow & R/I \\ \uparrow & & \uparrow \\ A & \xrightarrow{\varphi} & R \end{array}$$

where I is an ideal of the ring R with $I^2 = 0$. Then $A \to R$ factors uniquely through A/J because

$$\varphi(J) = \varphi(J^2) \subset (\varphi(J)A)^2 \subset I^2 = 0.$$

Hence this also provides a counterexample to the formally étale case of the "structure theorem" for locally finite type and formally étale morphisms in [**DG67**, IV, Theorem 18.4.6(i)] (but not a counterexample to part (ii), which is what people actually use in practice). The error in the proof of the latter is that the very last step of the proof is to invoke the incorrect [**DG67**, 0_{IV}, Example 19.3.10(i)], which is how the counterexample just mentioned creeps in.

Lemma 78.28.1. *There exist formally étale nonflat ring maps.*

Proof. See discussion above. □

78.29. A formally étale ring map with nontrivial cotangent complex

Let k be a field. Consider the ring

$$R = k[\{x_n\}_{n\geq 1}, \{y_n\}_{n\geq 1}]/(x_1y_1, x_{nm}^m - x_n, y_{nm}^m - y_n)$$

Let A be the localization at the maximal ideal generated by all x_n, y_n and denote $J \subset A$ the maximal ideal. Set $B = A/J$. By construction $J^2 = J$ and hence $A \to B$ is formally étale (see Section 78.28). We claim that the element $x_1 \otimes y_1$ is a nonzero element in the kernel of

$$J \otimes_A J \longrightarrow J.$$

Namely, (A, J) is the colimit of the localizations (A_n, J_n) of the rings

$$R_n = k[x_n, y_n]/(x_n^n y_n^n)$$

at their corresponding maximal ideals. Then $x_1 \otimes y_1$ corresponds to the element $x_n^n \otimes y_n^n \in J_n \otimes_{A_n} J_n$ and is nonzero (by an explicit computation which we omit). Since $\otimes$ commutes with colimits we conclude. By [**Ill72**, III Section 3.3] we see that J is not weakly regular. Hence by [**Ill72**, III Proposition 3.3.3] we see that the cotangent complex $L_{B/A}$ is not zero. In fact, we can be more precise. We have $H_0(L_{B/A}) = \Omega_{B/A}$ and $H_1(L_{B/A}) = 0$ because $J/J^2 = 0$. But from the five-term exact sequence of Quillen's fundamental spectral sequence (see Cotangent, Remark 66.11.5 or [**Rei**, Corollary 8.2.6]) and the nonvanishing of $\mathrm{Tor}_2^A(B, B) = \mathrm{Ker}(J \otimes_A J \to J)$ we conclude that $H_2(L_{B/A})$ is nonzero.

Lemma 78.29.1. *There exists a formally étale surjective ring map $A \to B$ with $L_{B/A}$ not equal to zero.*

Proof. See discussion above. □

78.30. Ideals generated by sets of idempotents and localization

Let R be a ring. Consider the ring

$$B(R) = R[x_n; n \in \mathbf{Z}]/(x_n(x_n - 1), x_n x_m; n \neq m)$$

It is easy to show that every prime $\mathfrak{q} \subset B(R)$ is either of the form

$$\mathfrak{q} = \mathfrak{p}B(R) + (x_n; n \in \mathbf{Z})$$

or of the form

$$\mathfrak{q} = \mathfrak{p}B(R) + (x_n - 1) + (x_m; n \neq m, m \in \mathbf{Z}).$$

Hence we see that

$$\mathrm{Spec}(B(R)) = \mathrm{Spec}(R) \amalg \coprod\nolimits_{n\in\mathbf{Z}} \mathrm{Spec}(R)$$

where the topology is not just the disjoint union topology. It has the following properties: Each of the copies indexed by $n \in \mathbf{Z}$ is an open subscheme, namely it is the standard open $D(x_n)$. The "central" copy of $\mathrm{Spec}(R)$ is in the closure of the union of any infinitely many of the other copies of $\mathrm{Spec}(R)$. Note that this last copy of $\mathrm{Spec}(R)$ is cut out by the ideal $(x_n, n \in \mathbf{Z})$ which is generated by the idempotents x_n. Hence we see that if $\mathrm{Spec}(R)$ is connected, then the decomposition above is exactly the decomposition of $\mathrm{Spec}(B(R))$ into connected components.

Next, let $A = \mathbf{C}[x, y]/((y - x^2 + 1)(y + x^2 - 1))$. The spectrum of A consists of two irreducible components $C_1 = \mathrm{Spec}(A_1)$, $C_2 = \mathrm{Spec}(A_2)$ with $A_1 = \mathbf{C}[x, y]/(y - x^2 + 1)$ and $A_2 = \mathbf{C}[x, y]/(y + x^2 - 1)$. Note that these are parametrized by

$(x, y) = (t, t^2-1)$ and $(x, y) = (t, -t^2+1)$ which meet in $P = (-1, 0)$ and $Q = (1, 0)$. We can make a twisted version of $B(A)$ where we glue $B(A_1)$ to $B(A_2)$ in the following way: Above P we let $x_n \in B(A_1) \otimes \kappa(P)$ correspond to $x_n \in B(A_2) \otimes \kappa(P)$, but above Q we let $x_n \in B(A_1) \otimes \kappa(Q)$ correspond to $x_{n+1} \in B(A_2) \otimes \kappa(Q)$. Let $B^{twist}(A)$ denote the resulting A-algebra. Details omitted. By construction $B^{twist}(A)$ is Zariski locally over A isomorphic to the untwisted version. Namely, this happens over both the principal open $\mathrm{Spec}(A) \setminus \{P\}$ and the principal open $\mathrm{Spec}(A) \setminus \{Q\}$. However, our choice of glueing produces enough "monodromy" such that $\mathrm{Spec}(B^{twist}(A))$ is connected (details omitted). Finally, there is a central copy of $\mathrm{Spec}(A) \to \mathrm{Spec}(B^{twist}(A))$ which gives a closed subscheme whose ideal is Zariski locally on $B^{twist}(A)$ cut out by ideals generated by idempotents, but not globally (as $B^{twist}(A)$ has no nontrivial idempotents).

Lemma 78.30.1. *There exists an affine scheme $X = \mathrm{Spec}(A)$ and a closed subscheme $T \subset X$ such that T is Zariski locally on X cut out by ideals generated by idempotents, but T is not cut out by an ideal generated by idempotents.*

Proof. See above. □

78.31. A ring map which identifies local rings which is not ind-étale

Note that the ring map $R \to B(R)$ constructed in Section 78.30 is a colimit of finite products of copies of R. Hence $R \to B(R)$ is ind-Zariski, see Pro-étale Cohomology, Definition 46.4.1. Next, consider the ring map $A \to B^{twist}(A)$ constructed in Section 78.30. Since this ring map is Zariski locally on $\mathrm{Spec}(A)$ isomorphic to an ind-Zariski ring map $R \to B(R)$ we conclude that it identifies local rings (see Pro-étale Cohomology, Lemma 46.4.6). The discussion in Section 78.30 shows there is a section $B^{twist}(A) \to A$ whose kernel is not generated by idempotents. Now, if $A \to B^{twist}(A)$ were ind-étale, i.e., $B^{twist}(A) = \mathrm{colim}\, A_i$ with $A \to A_i$ étale, then the kernel of $A_i \to A$ would be generated by an idempotent (Algebra, Lemmas 10.136.9 and 10.136.10). This would contradict the result mentioned above.

Lemma 78.31.1. *There is a ring map $A \to B$ which identifies local rings but which is not ind-étale. A fortiori it is not ind-Zariski.*

Proof. See discussion above. □

78.32. Non flasque quasi-coherent sheaf associated to injective module

For more examples of this type see [**BGI71**, Exposé II, Appendix I] where Illusie explains some examples due to Verdier.

Consider the affine scheme $X = \mathrm{Spec}(A)$ where

$$A = k[f, g, x, y, \{a_n, b_n\}_{n\geq 1}]/(fy - gx, \{a_n f^n + b_n g^n\}_{n\geq 1})$$

is the ring from Properties, Example 27.23.2. Set $I = (f, g) \subset A$. Consider the quasi-compact open $U = D(f) \cup D(g)$ of X. We have seen in loc. cit. that there is a section $s \in \mathcal{O}_X(U)$ which does not come from an A-module map $I^n \to A$ for any $n \geq 0$.

Let $\alpha : A \to J$ be the embedding of A into an injective A-module. Let $Q = J/\alpha(A)$ and denote $\beta : J \to Q$ the quotient map. We claim that the map

$$\Gamma(X, \widetilde{J}) \longrightarrow \Gamma(U, \widetilde{J})$$

is not surjective. Namely, we claim that $\alpha(s)$ is not in the image. To see this, we argue by contradiction. So assume that $x \in J$ is an element which restricts to $\alpha(s)$ over U. Then $\beta(x) \in Q$ is an element which restricts to 0 over U. Hence we know that $I^n\beta(x) = 0$ for some n, see Properties, Lemma 27.23.1. This implies that we get a morphism $\varphi : I^n \to A$, $h \mapsto \alpha^{-1}(hx)$. It is easy to see that this morphism φ gives rise to the section s via the map of Properties, Lemma 27.23.1 which is a contradiction.

Lemma 78.32.1. *There exists an affine scheme $X = \mathrm{Spec}(A)$ and an injective A-module J such that $\widetilde{J}$ is not a flasque sheaf on X. Even the restriction $\Gamma(X, \widetilde{J}) \to \Gamma(U, \widetilde{J})$ with U quasi-compact open need not be surjective.*

Proof. See above. □

78.33. A non-separated flat group scheme

Every group scheme over a field is separated, see Groupoids, Lemma 38.7.2. This is not true for group schemes over a base.

Let k be a field. Let $S = \mathrm{Spec}(k[x]) = \mathbf{A}^1_k$. Let G be the affine line with 0 doubled (see Schemes, Example 25.14.3) seen as a scheme over S. Thus a fibre of $G \to S$ is either a singleton or a set with two elements (one in U and one in V). Thus we can endow these fibres with the structure of a group (by letting the element in U be the zero of the group structure). More precisely, G has two opens U, V which map isomorphically to S such that $U \cap V$ is mapped isomorphically to $S \setminus \{0\}$. Then

$$G \times_S G = U \times_S U \cup V \times_S U \cup U \times_S V \cup V \times_S V$$

where each piece is isomorphic to S. Hence we can define a multiplication $m : G \times_S G \to G$ as the unique S-morphism which maps the first and the last piece into U and the two middle pieces into V. This matches the pointwise description given above. We omit the verification that this defines a group scheme structure.

Lemma 78.33.1. *There exists a flat group scheme of finite type over the affine line which is not separated.*

Proof. See the discussion above. □

Lemma 78.33.2. *There exists a flat group scheme of finite type over the infinite dimensional affine space which is not quasi-separated.*

Proof. The same construction as above can be carried out with the infinite dimensional affine space $S = \mathbf{A}^\infty_k = \mathrm{Spec}\, k[x_1, x_2, \ldots]$ as the base and the origin $0 \in S$ corresponding to the maximal ideal $(x_1, x_2, \ldots)$ as the closed point which is doubled in G. The resulting group scheme $G \to S$ is not quasi-separated as explained in Schemes, Example 25.21.4. □

78.34. A non-flat group scheme with flat identity component

Let $X \to S$ be a monomorphism of schemes. Let $G = S \amalg X$. Let $m : G \times_S G \to G$ be the S-morphism

$$G \times_S G = X \times_S X \amalg X \amalg X \amalg S \longrightarrow G = X \amalg S$$

which maps the summands $X \times_S X$ and S into S and maps the summands X into X by the identity morphism. This defines a group law. To see this we have to show

that $m \circ (m \times \mathrm{id}_G) = m \circ (\mathrm{id}_G \times m)$ as maps $G \times_S G \times_S G \to G$. Decomposing $G \times_S G \times_S G$ into components as above, we see that we need to verify this for the restriction to each of the 8-pieces. Each piece is isomorphic to either S, X, $X \times_S X$, or $X \times_S X \times_S X$. Moreover, both maps map these pieces to S, X, S, X respectively. Having said this, the fact that $X \to S$ is a monomorphism implies that $X \times_S X \cong X$ and $X \times_S X \times_S X \cong X$ and that there is in each case exactly one S-morphism $S \to S$ or $X \to X$. Thus we see that $m \circ (m \times \mathrm{id}_G) = m \circ (\mathrm{id}_G \times m)$. Thus taking $X \to S$ to be any nonflat monomorphism of schemes (e.g., a closed immersion) we get an example of a group scheme over a base S whose identity component is S (hence flat) but which is not flat.

Lemma 78.34.1. *There exists a group scheme G over a base S whose identity component is flat over S but which is not flat over S.*

Proof. See discussion above. □

78.35. A non-separated group algebraic space over a field

Every group scheme over a field is separated, see Groupoids, Lemma 38.7.2. This is not true for group algebraic spaces over a field (but see end of this section for positive results).

Let k be a field of characteristic zero. Consider the algebraic space $G = \mathbf{A}^1_k/\mathbf{Z}$ from Spaces, Example 47.14.8. By construction G is the fppf sheaf associated to the presheaf

$$T \longmapsto \Gamma(T, \mathcal{O}_T)/\mathbf{Z}$$

on the category of schemes over k. The obvious addition rule on the presheaf induces an addition $m : G \times G \to G$ which turns G into a group algebraic space over $\mathrm{Spec}(k)$. Note that G is not separated (and not even quasi-separated or locally separated). On the other hand $G \to \mathrm{Spec}(k)$ is of finite type!

Lemma 78.35.1. *There exists a group algebraic space of finite type over a field which is not separated (and not even quasi-separated or locally separated).*

Proof. See discussion above. □

Positive results: If the group algebraic space G is either quasi-separated, or locally separated, or more generally a decent algebraic space, then G is in fact separated, see More on Groupoids in Spaces, Lemma 60.7.4. Moreover, a finite type, separated group algebraic space over a field is in fact a scheme (insert future reference here). The idea of the proof is that the schematic locus is open dense, see Properties of Spaces, Proposition 48.10.3 or Decent Spaces, Theorem 50.8.4. By translating this open we see that every point of G has an open neighbourhood which is a scheme.

78.36. Specializations between points in fibre étale morphism

If $f : X \to Y$ is an étale, or more generally a locally quasi-finite morphism of schemes, then there are no specializations between points of fibres, see Morphisms, Lemma 28.21.8. However, for morphisms of algebraic spaces this doesn't hold in general.

To give an example, let k be a field. Set

$$P = k[u, u^{-1}, y, \{x_n\}_{n \in \mathbf{Z}}].$$

Consider the action of $\mathbf{Z}$ on P by k-algebra maps generated by the automorphism τ given by the rules $\tau(u) = u$, $\tau(y) = uy$, and $\tau(x_n) = x_{n+1}$. For $d \geq 1$ set $I_d = ((1-u^d)y, x_n - x_{n+d}, n \in \mathbf{Z})$. Then $V(I_d) \subset \mathrm{Spec}(P)$ is the fix point locus of τ^d. Let $S \subset P$ be the multiplicative subset generated by y and all $1-u^d$, $d \in \mathbf{N}$. Then we see that $\mathbf{Z}$ acts freely on $U = \mathrm{Spec}(S^{-1}P)$. Let $X = U/\mathbf{Z}$ be the quotient algebraic space, see Spaces, Definition 47.14.4.

Consider the prime ideals $\mathfrak{p}_n = (x_n, x_{n+1}, \ldots)$ in $S^{-1}P$. Note that $\tau(\mathfrak{p}_n) = \mathfrak{p}_{n+1}$. Hence each of these define point $\xi_n \in U$ whose image in X is the same point x of X. Moreover we have the specializations

$$\ldots \rightsquigarrow \xi_n \rightsquigarrow \xi_{n-1} \rightsquigarrow \ldots$$

We conclude that $U \to X$ is an example of the promised type.

Lemma 78.36.1. *There exists an étale morphism of algebraic spaces $f : X \to Y$ and a nontrivial specialization of points $x \rightsquigarrow x'$ in $|X|$ with $f(x) = f(x')$ in $|Y|$.*

Proof. See discussion above. □

78.37. A torsor which is not an fppf torsor

In Groupoids, Remark 38.9.5 we raise the question whether any G-torsor is a G-torsor for the fppf topology. In this section we show that this is not always the case.

Let k be a field. All schemes and stacks are over k in what follows. Let $G \to \mathrm{Spec}(k)$ be the group scheme

$$G = (\mu_{2,k})^\infty = \mu_{2,k} \times_k \mu_{2,k} \times_k \mu_{2,k} \times_k \ldots = \lim_n (\mu_{2,k})^n$$

where $\mu_{2,k}$ is the group scheme of second roots of unity over $\mathrm{Spec}(k)$, see Groupoids, Example 38.5.2. As an inverse limit of affine schemes we see that G is an affine group scheme. In fact it is the spectrum of the ring $k[t_1, t_2, t_3, \ldots]/(t_i^2 - 1)$. The multiplication map $m : G \times_k G \to G$ is on the algebra level given by $t_i \mapsto t_i \otimes t_i$.

We claim that any G-torsor over k is of the form

$$P = \mathrm{Spec}(k[x_1, x_2, x_3, \ldots]/(x_i^2 - a_i))$$

for certain $a_i \in k^*$ and with G-action $G \times_k P \to P$ given by $x_i \to t_i \otimes x_i$ on the algebra level. We omit the proof. Actually for the example we only need that P is a G-torsor which is clear since over $k' = k(\sqrt{a_1}, \sqrt{a_2}, \ldots)$ the scheme P becomes isomorphic to G in a G-equivariant manner. Note that P is trivial if and only if $k' = k$ since if P has a k-rational point then all of the a_i are squares.

We claim that P is an fppf torsor if and only if the field extension $k \subset k' = k(\sqrt{a_1}, \sqrt{a_2}, \ldots)$ is finite. If k' is finite over k, then $\{\mathrm{Spec}(k') \to \mathrm{Spec}(k)\}$ is an fppf covering which trivializes P and we see that P is indeed an fppf torsor. Conversely, suppose that P is a G-torsor for the fppf topology. This means that there exists an fppf covering $\{S_i \to \mathrm{Spec}(k)\}$ such that each P_{S_i} is trivial. Pick an i such that S_i is not empty. Let $s \in S_i$ be a closed point. By Varieties, Lemma 32.12.1 the field extension $k \subset \kappa(s)$ is finite, and by construction $P_{\kappa(s)}$ has a $\kappa(s)$-rational point. Thus we see that $k \subset k' \subset \kappa(s)$ and k' is finite over k.

To get an explicit example take $k = \mathbf{Q}$ and $a_i = i$ for example (or a_i is the ith prime if you like).

Lemma 78.37.1. *Let S be a scheme. Let G be a group scheme over S. The stack G-Principal classifying principal homogeneous G-spaces (see Examples of Stacks, Subsection 68.13.5) and the stack G-Torsors classifying fppf G-torsors (see Examples of Stacks, Subsection 68.13.8) are not equivalent in general.*

Proof. The discussion above shows that the functor G-*Torsors* $\to$ G-*Principal* isn't essentially surjective in general. □

78.38. Stack with quasi-compact flat covering which is not algebraic

In this section we briefly describe an example due to Brian Conrad. You can find the example online at this location. Our example is slightly different.

Let k be an algebraically closed field. All schemes and stacks are over k in what follows. Let $G \to \mathrm{Spec}(k)$ be an affine group scheme. In Examples of Stacks, Proposition 68.14.4 we have seen that $\mathcal{X} = [\mathrm{Spec}(k)/G]$ is a stack in groupoids over $(Sch/\mathrm{Spec}(k))_{fppf}$ which can be described as follows. A 1-morphism $T \to \mathcal{X}$ corresponds by definition to an fppf G_T-torsor P over T. The diagonal 1-morphism

$$\Delta : \mathcal{X} \longrightarrow \mathcal{X} \times_{\mathrm{Spec}(k)} \mathcal{X}$$

is representable and affine. The reason for this is that given any pair of G_T-torsors P_1, P_2 in the fppf topology over a scheme S/k the scheme $Isom(P_1, P_2)$ is affine over T. The trivial G-torsor over $\mathrm{Spec}(k)$ defines a 1-morphism

$$f : \mathrm{Spec}(k) \longrightarrow \mathcal{X}.$$

We claim that this is a surjective 1-morphism. The reason is simply that by definition for any 1-morphism $T \to \mathcal{X}$ there exists a fppf covering $\{T_i \to T\}$ such that P_{T_i} is isomorphic to the trivial G_{T_i}-torsor. Hence the compositions $T_i \to T \to \mathcal{X}$ factor through f. Thus it is clear that the projection $T \times_{\mathcal{X}} \mathrm{Spec}(k) \to \mathcal{X}$ is surjective (which is how we define the property that f is surjective, see Algebraic Stacks, Definition 67.10.1). In a similar way you show that f is quasi-compact and flat (details omitted). We also record here the observation that

$$\mathrm{Spec}(k) \times_{\mathcal{X}} \mathrm{Spec}(k) \cong G$$

as schemes over k.

Suppose there exists a surjective smooth morphism $p : U \to \mathcal{X}$ where U is a scheme. Consider the fibre product

$$\begin{array}{ccc} W & \longrightarrow & U \\ \downarrow & & \downarrow \\ \mathrm{Spec}(k) & \longrightarrow & \mathcal{X} \end{array}$$

Then we see that W is a nonempty smooth scheme over k which hence has a k-point. This means that we can factor f through U. Hence we obtain

$$G \cong \mathrm{Spec}(k) \times_{\mathcal{X}} \mathrm{Spec}(k) \cong (\mathrm{Spec}(k) \times_k \mathrm{Spec}(k)) \times_{(U \times_k U)} (U \times_{\mathcal{X}} U)$$

and since the projections $U \times_{\mathcal{X}} U \to U$ were assumed smooth we conclude that $U \times_{\mathcal{X}} U \to U \times_k U$ is locally of finite type, see Morphisms, Lemma 28.16.8. It follows that in this case G is locally of finite type over k. Altogether we have proved the following lemma (which can be significantly generalized).

Lemma 78.38.1. *Let k be a field. Let G be an affine group scheme over k. If the stack $[\mathrm{Spec}(k)/G]$ has a smooth covering by a scheme, then G is of finite type over k.*

Proof. See discussion above. □

To get an explicit example as in the title of this section, take for example $G = (\mu_{2,k})^\infty$ the group scheme of Section 78.37, which is not locally of finite type over k. By the discussion above we see that $\mathcal{X} = [\mathrm{Spec}(k)/G]$ has properties (1) and (2) of Algebraic Stacks, Definition 67.12.1, but not property (3). Hence $\mathcal{X}$ is not an algebraic stack. On the other hand, there does exists a scheme U an a surjective, flat, quasi-compact morphism $U \to \mathcal{X}$, namely the morphism $f : \mathrm{Spec}(k) \to \mathcal{X}$ we studied above.

78.39. Limit preserving on objects, not limit preserving

Let S be a nonempty scheme. Let $\mathcal{G}$ be an injective abelian sheaf on $(Sch/S)_{fppf}$. We obtain a stack in groupoids

$$\mathcal{G}\text{-}Torsors \longrightarrow (Sch/S)_{fppf}$$

over S, see Examples of Stacks, Lemma 68.13.2. This stack is limit preserving on objects over $(Sch/S)_{fppf}$ (see Criteria for Representability, Section 70.5) because every $\mathcal{G}$-torsor is trivial. On the other hand, $\mathcal{G}\text{-}Torsors$ is in general not limit preserving (see Artin's Axioms, Definition 71.13.1) as $\mathcal{G}$ need not be limit preserving as a sheaf. For example, take any nonzero injective sheaf $\mathcal{I}$ and set $\mathcal{G} = \prod_{n \in \mathbf{Z}} \mathcal{I}$ to get an example.

Lemma 78.39.1. *Let S be a nonempty scheme. There exists a stack in groupoids $p : \mathcal{X} \to (Sch/S)_{fppf}$ such that p is limit preserving on objects, but $\mathcal{X}$ is not limit preserving.*

Proof. See discussion above. □

78.40. A non-algebraic classifying stack

Let $S = \mathrm{Spec}(\mathbf{F}_p)$ and let μ_p denote the group scheme of pth roots of unity over S. In Groupoids in Spaces, Section 59.19 we have introduced the quotient stack $[S/\mu_p]$ and in Examples of Stacks, Section 68.14 we have shown $[S/\mu_p]$ is the classifying stack for fppf μ_p-torsors: Given a scheme T over S the category $\mathrm{Mor}_S(T, [S/\mu_p])$ is canonically equivalent to the category of fppf μ_p-torsors over T. Finally, in Criteria for Representability, Theorem 70.17.2 we have seen that $[S/\mu_p]$ is an algebraic stack.

Now we can ask the question: "How about the category fibred in groupoids $\mathcal{S}$ classifying étale μ_p-torsors?" (In other words $\mathcal{S}$ is a category over Sch/S whose fibre category over a scheme T is the category of étale μ_p-torsors over T.)

The first objection is that this isn't a stack for the fppf topology, because descent for objects isn't going to hold. For example the μ_p-torsor $\mathrm{Spec}(\mathbf{F}_p(t)[x]/(x^p - t))$ over $T = \mathrm{Spec}(\mathbf{F}_p(T))$ is fppf locally trivial, but not étale locally trivial.

A fix for this first problem is to work with the étale topology and in this case descent for objects does work. Indeed it is true that $\mathcal{S}$ is a stack in groupoids over $(Sch/S)_{étale}$. Moreover, it is also the case that the diagonal $\Delta : \mathcal{S} \to \mathcal{S} \times \mathcal{S}$ is representable (by schemes). This is true because given two μ_p-torsors (whether

they be étale locally trivial or not) the sheaf of isomorphisms between them is representable by a scheme.

Thus we can finally ask if there exists a scheme U and a smooth and surjective 1-morphism $U \to \mathcal{S}$. We will show in two ways that this is impossible: by a direct argument (which we advise the reader to skip) and by an argument using a general result.

Direct argument (sketch): Note that the 1-morphism $\mathcal{S} \to \mathrm{Spec}(\mathbf{F}_p)$ satisfies the infinitesimal lifting criterion for formal smoothness. This is true because given a first order infinitesimal thickening of schemes $T \to T'$ the kernel of $\mu_p(T') \to \mu_p(T)$ is isomorphic to the sections of the ideal sheaf of T in T', and hence $H^1_{étale}(T, \mu_p) = H^1_{étale}(T', \mu_p)$. Moreover, $\mathcal{S}$ is a limit preserving stack. Hence if $U \to \mathcal{S}$ is smooth, then $U \to \mathrm{Spec}(\mathbf{F}_p)$ is limit preserving and satisfies the infinitesimal lifting criterion for formal smoothness. This implies that U is smooth over $\mathbf{F}_p$. In particular U is reduced, hence $H^1_{étale}(U, \mu_p) = 0$. Thus $U \to \mathcal{S}$ factors as $U \to \mathrm{Spec}(\mathbf{F}_p) \to \mathcal{S}$ and the first arrow is smooth. By descent of smoothness, we see that $U \to \mathcal{S}$ being smooth would imply $\mathrm{Spec}(\mathbf{F}_p) \to \mathcal{S}$ is smooth. However, this is not the case as $\mathrm{Spec}(\mathbf{F}_p) \times_{\mathcal{S}} \mathrm{Spec}(\mathbf{F}_p)$ is μ_p which is not smooth over $\mathrm{Spec}(\mathbf{F}_p)$.

Structural argument: In Criteria for Representability, Section 70.19 we have seen that we can think of algebraic stacks as those stacks in groupoids for the étale topology with diagonal representable by algebraic spaces having a smooth covering. Hence if a smooth surjective $U \to \mathcal{S}$ exists then $\mathcal{S}$ is an algebraic stack, and in particular satisfies descent in the fppf topology. But we've seen above that $\mathcal{S}$ does not satisfies descent in the fppf topology.

Loosely speaking the arguments above show that the classifying stack in the étale topology for étale locally trivial torsors for a group scheme G over a base B is algebraic if and only if G is smooth over B. One of the advantages of working with the fppf topology is that it suffices to assume that $G \to B$ is flat and locally of finite presentation. In fact the quotient stack (for the fppf topology) $[B/G]$ is algebraic if and only if $G \to B$ is flat and locally of finite presentation, see Criteria for Representability, Lemma 70.18.3.

78.41. Sheaf with quasi-compact flat covering which is not algebraic

Consider the functor $F = (\mathbf{P}^1)^\infty$, i.e., for a scheme T the value $F(T)$ is the set of $f = (f_1, f_2, f_3, \ldots)$ where each $f_i : T \to \mathbf{P}^1$ is a morphism of schemes. Note that $\mathbf{P}^1$ satisfies the sheaf property for fpqc coverings, see Descent, Lemma 34.9.3. A product of sheaves is a sheaf, so F also satisfies the sheaf property for the fpqc topology. The diagonal of F is representable: if $f : T \to F$ and $g : S \to F$ are morphisms, then $T \times_F S$ is the scheme theoretic intersection of the closed subschemes $T \times_{f_i, \mathbf{P}^1, g_i} S$ inside the scheme $T \times S$. Consider the group scheme SL_2 which comes with a surjective smooth affine morphism $\mathrm{SL}_2 \to \mathbf{P}^1$. Next, consider $U = (\mathrm{SL}_2)^\infty$ with its canonical (product) morphism $U \to F$. Note that U is an affine scheme. We claim the morphism $U \to F$ is flat, surjective, and universally open. Namely, suppose $f : T \to F$ is a morphism. Then $Z = T \times_F U$ is the infinite fibre product of the schemes $Z_i = T \times_{f_i, \mathbf{P}^1} \mathrm{SL}_2$ over T. Each of the morphisms $Z_i \to T$ is surjective smooth and affine which implies that

$$Z = Z_1 \times_T Z_2 \times_T Z_3 \times_T \ldots$$

is a scheme flat and affine over Z. A simple limit argument shows that $Z \to T$ is open as well.

On the other hand, we claim that F isn't an algebraic space. Namely, if F where an algebraic space it would be a quasi-compact and separated (by our description of fibre products over F) algebraic space. Hence cohomology of quasi-coherent sheaves would vanish above a certain cutoff (see Cohomology of Spaces, Proposition 51.6.2 and remarks preceding it). But clearly by taking the pullback of $\mathcal{O}(-2, -2, \ldots, -2)$ under the projection

$$(\mathbf{P}^1)^\infty \longrightarrow (\mathbf{P}^1)^n$$

(which has a section) we can obtain a quasi-coherent sheaf whose cohomology is nonzero in degree n. Altogether we obtain an answer to a question asked by Anton Geraschenko on mathoverflow.

Lemma 78.41.1. *There exists a functor $F : Sch^{opp} \to Sets$ which satisfies the sheaf condition for the fpqc topology, has representable diagonal $\Delta : F \to F \times F$, and such that there exists a surjective, flat, universally open, quasi-compact morphism $U \to F$ where U is a scheme, but such that F is not an algebraic space.*

Proof. See discussion above. □

78.42. Sheaves and specializations

In the following we fix a big étale site $Sch_{étale}$ as constructed in Topologies, Definition 33.4.6. Moreover, a scheme will be an object of this site. Recall that if x, x' are points of a scheme X we say x is a *specialization* of x' or we write $x' \rightsquigarrow x$ if $x \in \overline{\{x'\}}$. This is true in particular if $x = x'$.

Consider the functor $F : Sch_{étale} \to Ab$ defined by the following rules:

$$F(X) = \prod\nolimits_{x \in X} \prod\nolimits_{x' \in X, x' \rightsquigarrow x, x' \neq x} \mathbf{Z}/2\mathbf{Z}$$

Given a scheme X we denote $|X|$ the underlying set of points. An element $a \in F(X)$ will be viewed as a map of sets $|X| \times |X| \to \mathbf{Z}/2\mathbf{Z}$, $(x, x') \mapsto a(x, x')$ which is zero if $x = x'$ or if x is not a specialization of x'. Given a morphism of schemes $f : X \to Y$ we define

$$F(f) : F(Y) \longrightarrow F(X)$$

by the rule that for $b \in F(Y)$ we set

$$F(f)(b)(x, x') = \begin{cases} 0 & \text{if } x \text{ is not a specialization of } x' \\ b(f(x), f(x')) & \text{else.} \end{cases}$$

Note that this really does define an element of $F(X)$. We claim that if $f : X \to Y$ and $g : Y \to Z$ are composable morphisms then $F(f) \circ F(g) = F(g \circ f)$. Namely, let $c \in F(Z)$ and let $x' \rightsquigarrow x$ be a specialization of points in X, then

$$F(g \circ f)(x, x') = c(g(f(x)), g(f(x'))) = F(g)(F(f)(c))(x, x')$$

because $f(x') \rightsquigarrow f(x)$. (This also works if $f(x) = f(x')$.)

Let G be the sheafification of F in the étale topology.

I claim that if X is a scheme and $x' \rightsquigarrow x$ is a specialization and $x' \neq x$, then $G(X) \neq 0$. Namely, let $a \in F(X)$ be an element such that when we think of a as a function $|X| \times |X| \to \mathbf{Z}/2\mathbf{Z}$ it is nonzero at (x, x'). Let $\{f_i : U_i \to X\}$ be an étale covering of X. Then we can pick an i and a point $u_i \in U_i$ with $f_i(u_i) = x$. Since

generalizations lift along flat morphisms (see Morphisms, Lemma 28.26.8) we can find a specialization $u_i' \rightsquigarrow u_i$ with $f_i(u_i') = x'$. By our construction above we see that $F(f_i)(a) \neq 0$. Hence a determines a nonzero element of $G(X)$.

Note that if $X = \mathrm{Spec}(k)$ where k is a field (or more generally a ring all of whose prime ideals are maximal), then $F(X) = 0$ and for every étale morphism $U \to X$ we have $F(U) = 0$ because there are no specializations between distinct points in fibres of an étale morphism. Hence $G(X) = 0$.

Suppose that $X \subset X'$ is a thickening, see More on Morphisms, Definition 36.2.1. Then the category of schemes étale over X' is equivalent to the category of schemes étale over X by the base change functor $U' \mapsto U = U' \times_{X'} X$, see Étale Cohomology, Theorem 44.46.1. Since it is always the case that $F(U) = F(U')$ in this situation we see that also $G(X) = G(X')$.

As a variant we can consider the presheaf F_n which associates to a scheme X the collection of maps $a : |X|^{n+1} \to \mathbf{Z}/2\mathbf{Z}$ where $a(x_0, \ldots, x_n)$ is nonzero only if $x_n \rightsquigarrow \ldots \rightsquigarrow x_0$ is a sequence of specializations and $x_n \neq x_{n-1} \neq \ldots \neq x_0$. Let G_n be the sheaf associated to F_n. In exactly the same way as above one shows that G_n is nonzero if $\dim(X) \geq n$ and is zero if $\dim(X) < n$.

Lemma 78.42.1. *There exists a sheaf of abelian groups G on $Sch_{étale}$ with the following properties*

(1) $G(X) = 0$ *whenever* $\dim(X) < n$,
(2) $G(X)$ *is not zero if* $\dim(X) \geq n$, *and*
(3) *if $X \subset X'$ is a thickening, then $G(X) = G(X')$.*

Proof. See the discussion above. □

Remark 78.42.2. Here are some remarks:

(1) The presheaves F and F_n are separated presheaves.
(2) It turns out that F, F_n are not sheaves.
(3) One can show that G, G_n is actually a sheaf for the fppf topology.

We will prove these results if we need them.

78.43. Sheaves and constructible functions

In the following we fix a big étale site $Sch_{étale}$ as constructed in Topologies, Definition 33.4.6. Moreover, a scheme will be an object of this site. In this section we say that a *constructible partition* of a scheme X is a locally finite disjoint union decomposition $X = \coprod_{i \in I} X_i$ such that each $X_i \subset X$ is a locally constructible subset of X. Locally finite means that for any quasi-compact open $U \subset X$ there are only finitely many $i \in I$ such that $X_i \cap U$ is not empty. Note that if $f : X \to Y$ is a morphism of schemes and $Y = \coprod Y_j$ is a constructible partition, then $X = \coprod f^{-1}(Y_j)$ is a constructible partition of X. Given a set S and a scheme X a *constructible function* $f : |X| \to S$ is a map such that $X = \coprod_{s \in S} f^{-1}(s)$ is a constructible partition of X. If G is an (abstract group) and $a, b : |X| \to G$ are constructible functions, then $ab : |X| \to G, x \mapsto a(x)b(x)$ is a constructible function too. The reason is that given any two constructible partitions there is a third one refining both.

Let A be any abelian group. For any scheme X we define

$$F(X) = \frac{\{a : |X| \to A \mid a \text{ is a constructible function}\}}{\{\text{locally constant functions } |X| \to A\}}$$

We think of an element a of $F(X)$ simply as a function well defined up to adding a locally constant one. Given a morphism of schemes $f : X \to Y$ and an element $b \in F(Y)$, then we define $F(f)(b) = b \circ f$. Thus F is a presheaf on $Sch_{étale}$.

Note that if $\{f_i : U_i \to X\}$ is an fppf covering, and $a \in F(X)$ is such that $F(f_i)(a) = 0$ in $F(U_i)$, then $a \circ f_i$ is a locally constant function for each i. This means in turn that a is a locally constant function as the morphisms f_i are open. Hence $a = 0$ in $F(X)$. Thus we see that F is a separated presheaf (in the fppf topology hence a fortiori in the étale topology).

Let G be the sheafification of F in the étale topology. Since F is separated, and since $F(X) \neq 0$ for example when X is the spectrum of a discrete valuation ring, we see that G is not zero.

Let $X = \mathrm{Spec}(k)$ where k is a field. Then any étale covering of X can be dominated by a covering $\{\mathrm{Spec}(k') \to \mathrm{Spec}(k)\}$ with $k \subset k'$ a finite separable extension of fields. Since $F(\mathrm{Spec}(k')) = 0$ we see that $G(X) = 0$.

Suppose that $X \subset X'$ is a thickening, see More on Morphisms, Definition 36.2.1. Then the category of schemes étale over X' is equivalent to the category of schemes étale over X by the base change functor $U' \mapsto U = U' \times_{X'} X$, see Étale Cohomology, Theorem 44.46.1. Since $F(U) = F(U')$ in this situation we see that also $G(X) = G(X')$.

The sheaf G is limit preserving, see Limits of Spaces, Definition 52.3.1. Namely, let R be a ring which is written as a directed colimit $R = \mathrm{colim}_i R_i$ of rings. Set $X = \mathrm{Spec}(R)$ and $X_i = \mathrm{Spec}(R_i)$, so that $X = \lim_i X_i$. Then $G(X) = \mathrm{colim}_i G(X_i)$. To prove this one first proves that a constructible partition of $\mathrm{Spec}(R)$ comes from a constructible partitions of some $\mathrm{Spec}(R_i)$. Hence the result for F. To get the result for the sheafification, use that any étale ring map $R \to R'$ comes from an étale ring map $R_i \to R'_i$ for some i. Details omitted.

Lemma 78.43.1. *There exists a sheaf of abelian groups G on $Sch_{étale}$ with the following properties*

(1) *$G(\mathrm{Spec}(k)) = 0$ whenever k is a field,*
(2) *G is limit preserving,*
(3) *if $X \subset X'$ is a thickening, then $G(X) = G(X')$, and*
(4) *G is not zero.*

Proof. See discussion above. □

78.44. The lisse-étale site is not functorial

The *lisse-étale* site $X_{lisse,étale}$ of X is the category of schemes smooth over X endowed with (usual) étale coverings, see Cohomology of Stacks, Section 75.11. Let $f : X \to Y$ be a morphism of schemes. There is a functor

$$u : Y_{lisse,étale} \longrightarrow X_{lisse,étale}, \quad V/Y \longmapsto V \times_Y X$$

which is continuous. Hence we obtain an adjoint pair of functors

$$u^s : Sh(X_{lisse,étale}) \longrightarrow Sh(Y_{lisse,étale}), \quad u_s : Sh(Y_{lisse,étale}) \longrightarrow Sh(X_{lisse,étale}),$$

see Sites, Section 7.14. We claim that, in general, u does **not** define a morphism of sites, see Sites, Definition 7.15.1. In other words, we claim that u_s is not left exact in general. Note that representable presheaves are sheaves on lisse-étale sites.

Hence, by Sites, Lemma 7.14.5 we see that $u_s h_V = h_{V \times_Y X}$. Now consider two morphisms

$$\begin{array}{ccc} V_1 & \overset{a}{\underset{b}{\rightrightarrows}} & V_2 \\ & \searrow \quad \swarrow & \\ & Y & \end{array}$$

of schemes V_1, V_2 smooth over Y. Now if u_s is left exact, then we would have

$$u_s \text{Equalizer}(h_a, h_b : h_{V_1} \to h_{V_2}) = \text{Equalizer}(h_{a \times 1}, h_{b \times 1} : h_{V_1 \times_Y X} \to h_{V_2 \times_Y X})$$

We will take the morphisms $a, b : V_1 \to V_2$ such that there exists no morphism from a scheme smooth over Y into $(a = b) \subset V_1$, i.e., such that the left hand side is the empty sheaf, but such that after base change to X the equalizer is nonempty and smooth over X. A silly example is to take $X = \text{Spec}(\mathbf{F}_p)$, $Y = \text{Spec}(\mathbf{Z})$ and $V_1 = V_2 = \mathbf{A}^1_{\mathbf{Z}}$ with morphisms $a(x) = x$ and $b(x) = x + p$. Note that the equalizer of a and b is the fibre of $\mathbf{A}^1_{\mathbf{Z}}$ over (p).

Lemma 78.44.1. *The lisse-étale site is not functorial, even for morphisms of schemes.*

Proof. See discussion above. □

78.45. Derived pushforward of quasi-coherent modules

Let k be a field of characteristic $p > 0$. Let $S = \text{Spec}(k[x])$. Let $G = \mathbf{Z}/p\mathbf{Z}$ viewed either as an abstract group or as a constant group scheme over S. Consider the algebraic stack $\mathcal{X} = [S/G]$ where G acts trivially on S, see Examples of Stacks, Remark 68.14.3 and Criteria for Representability, Lemma 70.18.3. Consider the structure morphism

$$f : \mathcal{X} \longrightarrow S$$

This morphism is quasi-compact and quasi-separated. Hence we get a functor

$$Rf_{QCoh,*} : D^+_{QCoh}(\mathcal{O}_\mathcal{X}) \longrightarrow D^+_{QCoh}(\mathcal{O}_S),$$

see Derived Categories of Stacks, Proposition 76.5.1. Let's compute $Rf_{QCoh,*}\mathcal{O}_\mathcal{X}$. Since $D_{QCoh}(\mathcal{O}_S)$ is equivalent to the derived category of $k[x]$-modules (see Derived Categories of Schemes, Lemma 35.3.4) this is equivalent to computing $R\Gamma(\mathcal{X}, \mathcal{O}_\mathcal{X})$. For this we can use the covering $S \to \mathcal{X}$ and the spectral sequence

$$H^q(S \times_\mathcal{X} \ldots \times_\mathcal{X} S, O) \Rightarrow H^{p+q}(\mathcal{X}, \mathcal{O}_\mathcal{X})$$

see Cohomology of Stacks, Proposition 75.10.4. Note that

$$S \times_\mathcal{X} \ldots \times_\mathcal{X} S = S \times G^p$$

which is affine. Thus the complex

$$k[x] \to \text{Map}(G, k[x]) \to \text{Map}(G^2, k[x]) \to \ldots$$

computes $R\Gamma(\mathcal{X}, \mathcal{O}_\mathcal{X})$. Here for $\varphi \in \text{Map}(G^{p-1}, k[x])$ its differential is the map which sends $(g_1, \ldots, g_p)$ to

$$\varphi(g_2, \ldots, g_p) + \sum\nolimits_{i=1}^{p-1} (-1)^i \varphi(g_1, \ldots, g_i + g_{i+1}, \ldots, g_p) + (-1)^p \varphi(g_1, \ldots, g_{p-1}).$$

This is just the complex computing the group cohomology of G acting trivially on $k[x]$ (insert future reference here). The cohomology of the cyclic group G on $k[x]$ is

exactly one copy of $k[x]$ in each cohomological degree ≥ 0 (insert future reference here). We conclude that

$$Rf_*\mathcal{O}_\mathcal{X} = \bigoplus\nolimits_{n\geq 0} \mathcal{O}_S[-n]$$

Now, consider the complex

$$E = \bigoplus\nolimits_{m\geq 0} \mathcal{O}_\mathcal{X}[m]$$

This is an object of $D_{QCoh}(\mathcal{O}_\mathcal{X})$. We interrupt the discussion for a general result.

Lemma 78.45.1. *Let $\mathcal{X}$ be an algebraic stack. Let K be an object of $D(\mathcal{O}_\mathcal{X})$ whose cohomology sheaves are locally quasi-coherent (Sheaves on Stacks, Definition 69.11.4) and satisfy the flat base change property (Cohomology of Stacks, Definition 75.7.1). Then there exists a distinguished triangle*

$$K \to \prod\nolimits_{n\geq 0} \tau_{\geq -n}K \to \prod\nolimits_{n\geq 0} \tau_{\geq -n}K \to K[1]$$

in $D(\mathcal{O}_\mathcal{X})$. In other words, K is the derived limit of its canonical truncations.

Proof. Recall that we work on the "big fppf site" $\mathcal{X}_{fppf}$ of $\mathcal{X}$ (by our conventions for sheaves of $\mathcal{O}_\mathcal{X}$-modules in the chapters Sheaves on Stacks and Cohomology on Stacks). Let $\mathcal{B}$ be the set of objects x of $\mathcal{X}_{fppf}$ which lie over an affine scheme U. Combining Sheaves on Stacks, Lemmas 69.22.2, 69.15.1, Descent, Lemma 34.8.4, and Cohomology of Schemes, Lemma 29.2.2 we see that $H^p(x, \mathcal{F}) = 0$ if $\mathcal{F}$ is locally quasi-coherent and $x \in \mathcal{B}$. Now the claim follows from Cohomology on Sites, Lemma 21.22.3. □

Lemma 78.45.2. *Let $\mathcal{X}$ be an algebraic stack. If $\mathcal{F}_n$ is a collection of locally quasi-coherent sheaves with the flat base change property on $\mathcal{X}$, then $\oplus_n \mathcal{F}_n[n] \to \prod_n \mathcal{F}_n[n]$ is an isomorphism in $D(\mathcal{O}_\mathcal{X})$.*

Proof. This is true because by Lemma 78.45.1 we see that the direct sum is isomorphic to the product. □

We continue our discussion. Since a quasi-coherent module is locally quasi-coherent and satisfies the flat base change property (Sheaves on Stacks, Lemma 69.11.5) we get

$$E = \prod\nolimits_{m\geq 0} \mathcal{O}_\mathcal{X}[m]$$

Since cohomology commutes with limits we see that

$$Rf_*E = \prod\nolimits_{m\geq 0} \Big(\bigoplus\nolimits_{n\geq 0} \mathcal{O}_S[m-n]\Big)$$

Note that this complex is not an object of $D_{QCoh}(\mathcal{O}_S)$ because the cohomology sheaf in degree 0 is an infinite product of copies of $\mathcal{O}_S$ which is not even a locally quasi-coherent $\mathcal{O}_S$-module.

Lemma 78.45.3. *A quasi-compact and quasi-separated morphism $f : \mathcal{X} \to \mathcal{Y}$ of algebraic stacks need not induce a functor $Rf_* : D_{QCoh}(\mathcal{O}_\mathcal{X}) \to D_{QCoh}(\mathcal{O}_\mathcal{Y})$.*

Proof. See discussion above. □

78.46. A big abelian category

The purpose of this section is to give an example of a "big" abelian category $\mathcal{A}$ and objects M, N such that the collection of isomorphism classes of extensions $\mathrm{Ext}_{\mathcal{A}}(M, N)$ is not a set. The example is due to Freyd, see [**Fre64**, page 131, Exercise A].

We define $\mathcal{A}$ as follows. An object of $\mathcal{A}$ consists of a triple (M, α, f) where M is an abelian group and α is an ordinal and $f : \alpha \to \mathrm{End}(M)$ is a map. A morphism $(M, \alpha, f) \to (M', \alpha', f')$ is given by a homomorphism of abelian groups $\varphi : M \to M'$ such that for *any* ordinal β we have

$$\varphi \circ f(\beta) = f'(\beta) \circ \varphi$$

Here the rule is that we set $f(\beta) = 0$ if β is not in α and similarly we set $f'(\beta)$ equal to zero if β is not an element of α'. We omit the verification that the category so defined is abelian.

Consider the object $Z = (\mathbf{Z}, \emptyset, f)$, i.e., all the operators are zero. The observation is that computed in $\mathcal{A}$ the group $\mathrm{Ext}^1_{\mathcal{A}}(Z, Z)$ is a proper class and not a set. Namely, for each ordinal α we can find an extension $(M, \alpha+1, f)$ of Z by Z whose underlying group is $M = \mathbf{Z} \oplus \mathbf{Z}$ and where the value of f is always zero except for

$$f(\alpha) = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}.$$

This clearly produces a proper class of isomorphism classes of extensions. In particular, the derived category of $\mathcal{A}$ has proper classes for its collections of morphism, see Derived Categories, Lemma 13.26.6. This means that some care has to be exercised when defining Verdier quotients of triangulated categories.

Lemma 78.46.1. *There exists a "big" abelian category $\mathcal{A}$ whose Ext-groups are proper classes.*

Proof. See discussion above. □

78.47. Weakly associated points and scheme theoretic density

Let k be a field. Let $R = k[z, x_i, y_i]/(z^2, zx_iy_i)$ where i runs over the elements of $\mathbf{N}$. Note that $R = R_0 \oplus M_0$ where $R_0 = k[x_i, y_i]$ is a subring and M_0 is an ideal of square zero with $M_0 \cong R_0/(x_iy_i)$ as R_0-module. The prime $\mathfrak{p} = (z, x_i)$ is weakly associated to R as an R-module (Algebra, Definition 10.63.1). Indeed, the element z in $R_{\mathfrak{p}}$ is nonzero but annihilated by $\mathfrak{p}R_{\mathfrak{p}}$. On the other hand, consider the open subscheme

$$U = \bigcup D(x_i) \subset \mathrm{Spec}(R) = S$$

We claim that $U \subset S$ is scheme theoretically dense (Morphisms, Definition 28.7.1). To prove this it suffices to show that $\mathcal{O}_S \to j_*\mathcal{O}_U$ is injective where $j : U \to S$ is the inclusion morphism, see Morphisms, Lemma 28.7.5. Translated back into algebra, we have to show that for all $g \in R$ the map

$$R_g \longrightarrow \prod R_{x_i g}$$

is injective. Write $g = g_0 + m_0$ with $g_0 \in R_0$ and $m_0 \in M_0$. Then $R_g = R_{g_0}$ (details omitted). Hence we may assume $g \in R_0$. We may also assume g is not zero. Now $R_g = (R_0)_g \oplus (M_0)_g$. Since R_0 is a domain, the map $(R_0)_g \to \prod (R_0)_{x_i g}$ is injective.

If $g \in (x_i y_i)$ then $(M_0)_g = 0$ and there is nothing to prove. If $g \notin (x_i y_i)$ then, since $(x_i y_i)$ is a radical ideal of R_0, we have to show that $M_0 \to \prod (M_0)_{x_i g}$ is injective. The kernel of $R_0 \to M_0 \to (M_0)_{x_n}$ is $(x_i y_i, y_n)$. Since $(x_i y_i, y_n)$ is a radical ideal, if $g \notin (x_i y_i, y_n)$ then the kernel of $R_0 \to M_0 \to (M_0)_{x_n g}$ is $(x_i y_i, y_n)$. As $g \notin (x_i y_i, y_n)$ for all $n \gg 0$ we conclude that the kernel is contained in $\bigcap_{n \gg 0} (x_i y_i, y_n) = (x_i y_i)$ as desired.

Lemma 78.47.1. *There exists a scheme X and a schematically dense open $U \subset X$ such that some weakly associated point $x \in X$ is not in U.*

Proof. In the situation above we have $\mathfrak{p} \notin U$. □

78.48. Example of non-additivity of traces

Let k be a field and let $R = k[\epsilon]$ be the ring of dual numbers over k. In other words, $R = k[x]/(x^2)$ and ϵ is the congruence class of x in R. Consider the short exact sequence of complexes

$$\begin{array}{ccccc} 0 & \longrightarrow & R & \xrightarrow{1} & R \\ \downarrow & & \downarrow{\scriptstyle \epsilon} & & \downarrow \\ R & \xrightarrow{1} & R & \longrightarrow & 0 \end{array}$$

Here the columns are the complexes, the first row is placed in degree 0, and the second row in degree 1. Denote the first complex (i.e., the left column) by $A^\bullet$, the second by $B^\bullet$ and the third $C^\bullet$. We claim that the diagram

$$\begin{array}{ccccc} A^\bullet & \longrightarrow & B^\bullet & \longrightarrow & C^\bullet \\ {\scriptstyle 1+\epsilon}\downarrow & & \downarrow{\scriptstyle 1} & & \downarrow{\scriptstyle 1} \\ A^\bullet & \longrightarrow & B^\bullet & \longrightarrow & C^\bullet \end{array} \tag{78.48.0.1}$$

commutes in $K(R)$, i.e., is a diagram of complexes commuting up to homotopy. Namely, the square on the right commutes and the one on the left is off by the homotopy $1 : A^1 \to B^0$. On the other hand,

$$\mathrm{Tr}_{A^\bullet}(1 + \epsilon) + \mathrm{Tr}_{C^\bullet}(1) \neq \mathrm{Tr}_{B^\bullet}(1).$$

Lemma 78.48.1. *There exists a ring R, a distringuished triangle $(K, L, M, \alpha, \beta, \gamma)$ in the homotopy category $K(R)$, and an endomorphism (a, b, c) of this distinguished triangle, such that K, L, M are perfect complexes and $Tr_K(a) + Tr_M(c) \neq Tr_L(b)$.*

Proof. Consider the example above. The map $\gamma : C^\bullet \to A^\bullet[1]$ is given by multiplication by ϵ in degree 0, see Derived Categories, Definition 13.9.1. Hence it is also true that

$$\begin{array}{ccc} C^\bullet & \xrightarrow{\gamma} & A^\bullet[1] \\ \downarrow & & \downarrow \\ C^\bullet & \xrightarrow{\gamma} & A^\bullet[1] \end{array}$$

commutes in $K(R)$ as $\epsilon(1+\epsilon) = \epsilon$. Thus we indeed have a morphism of distinguished triangles. □

78.49. Being projective is not local on the base

In the chapter on descent we have seen that many properties of morphisms are local on the base, even in the fpqc topology. See Descent, Sections 34.18, 34.19, and 34.20. This is not true for projectivity of morphisms.

Lemma 78.49.1. *The properties*

$$\mathcal{P}(f) = \text{“}f \text{ is projective”, and}$$
$$\mathcal{P}(f) = \text{“}f \text{ is quasi-projective”}$$

are not Zariski local on the base. A fortiori, they are not fpqc local on the base.

Proof. Following Hironaka [**Har77**, Example B.3.4.1], we define a proper morphism of smooth complex 3-folds $f : V_Y \to Y$ which is Zariski-locally projective, but not projective. Since f is proper and not projective, it is also not quasi-projective.

Let Y be projective 3-space over the complex numbers $\mathbf{C}$. Let C and D be smooth conics in Y such that the closed subscheme $C \cap D$ is reduced and consists of two complex points P and Q. (For example, let $C = \{[x, y, z, w] : xy = z^2, w = 0\}$, $D = \{[x, y, z, w] : xy = w^2, z = 0\}$, $P = [1, 0, 0, 0]$, and $Q = [0, 1, 0, 0]$.) On $Y - Q$, first blow up the curve C, and then blow up the strict transform of the curve D (Divisors, Definition 30.19.1). On $Y - P$, first blow up the curve D, and then blow up the strict transform of the curve C. Over $Y - P - Q$, the two varieties we have constructed are canonically isomorphic, and so we can glue them over $Y - P - Q$. The result is a smooth proper 3-fold V_Y over $\mathbf{C}$. The morphism $f : V_Y \to Y$ is proper and Zariski-locally projective (since it is a blow-up over $Y - P$ and over $Y - Q$), by Divisors, Lemma 30.18.11. We will show that V_Y is not projective over $\mathbf{C}$. That will imply that f is not projective.

To do this, let L be the inverse image in V_Y of a complex point of $C - P - Q$, and M the inverse image of a complex point of $D - P - Q$. Then L and M are isomorphic to the projective line $\mathbf{P}^1_{\mathbf{C}}$. Next, let E be the inverse image in V_Y of $C \cup D \subset Y$ in V_Y; thus $E \to C \cup D$ is a proper morphism, with fibers isomorphic to $\mathbf{P}^1$ over $(C \cup D) - \{P, Q\}$. The inverse image of P in E is a union of two lines L_0 and M_0, and we have rational equivalences of cycles $L \sim L_0 + M_0$ and $M \sim M_0$ on E (using that C and D are isomorphic to $\mathbf{P}^1$). Note the asymmetry resulting from the order in which we blew up the two curves. Near Q, the opposite happens. So the inverse image of Q is the union of two lines L'_0 and M'_0, and we have rational equivalences $L \sim L'_0$ and $M \sim L'_0 + M'_0$ on E. Combining these equivalences, we find that $L_0 + M'_0 \sim 0$ on E and hence on V_Y. If V_Y were projective over $\mathbf{C}$, it would have an ample line bundle H, which would have degree > 0 on all curves in V_Y. In particular H would have positive degree on $L_0 + M'_0$, contradicting that the degree of a line bundle is well-defined on 1-cycles modulo rational equivalence on a proper scheme over a field (Chow Homology, Lemma 41.20.2 and Lemma 41.29.2). So V_Y is not projective over $\mathbf{C}$. □

In different terminology, Hironaka's 3-fold V_Y is a small resolution of the blow-up Y' of Y along the reduced subscheme $C \cup D$; here Y' has two node singularities. If we define Z by blowing up Y along C and then along the strict transform of D, then Z is a smooth projective 3-fold, and the non-projective 3-fold V_Y differs from Z by a “flop” over $Y - P$.

78.50. Descent data for schemes need not be effective, even for a projective morphism

In the chapter on descent we have seen that descent data for schemes relative to an fpqc morphism are effective for several classes of morphisms. In particular, affine morphisms and more generally quasi-affine morphisms satisfy descent for fpqc coverings (Descent, Lemma 34.34.1). This is not true for projective morphisms.

Lemma 78.50.1. *There is an etale covering $X \to S$ of schemes and a descent datum $(V/X, \varphi)$ relative to $X \to S$ such that $V \to X$ is projective, but the descent datum is not effective in the category of schemes.*

Proof. We imitate Hironaka's example of a smooth separated complex algebraic space of dimension 3 which is not a scheme [**Har77**, Example B.3.4.2].

Consider the action of the group $G = \mathbf{Z}/2 = \{1, g\}$ on projective 3-space $\mathbf{P}^3$ over the complex numbers by

$$g[x, y, z, w] = [y, x, w, z].$$

The action is free outside the two disjoint lines $L_1 = \{[x, x, z, z]\}$ and $L_2 = \{[x, -x, z, -z]\}$ in $\mathbf{P}^3$. Let $Y = \mathbf{P}^3 - (L_1 \cup L_2)$. There is a smooth quasi-projective scheme $S = Y/G$ over $\mathbf{C}$ such that $Y \to S$ is a G-torsor (Groupoids, Definition 38.9.3). Explicitly, we can define S as the image of the open subset Y in $\mathbf{P}^3$ under the morphism

$$\begin{aligned} \mathbf{P}^3 &\to \text{Proj } \mathbf{C}[x, y, z, w]^G \\ &= \text{Proj } \mathbf{C}[u_0, u_1, v_0, v_1, v_2]/(v_0 v_1 = v_2^2), \end{aligned}$$

where $u_0 = x + y$, $u_1 = z + w$, $v_0 = (x - y)^2$, $v_1 = (z - w)^2$, and $v_2 = (x - y)(z - w)$, and the ring is graded with u_0, u_1 in degree 1 and v_0, v_1, v_2 in degree 2.

Let $C = \{[x, y, z, w] : xy = z^2, w = 0\}$ and $D = \{[x, y, z, w] : xy = w^2, z = 0\}$. These are smooth conic curves in $\mathbf{P}^3$, contained in the G-invariant open subset Y, with $g(C) = D$. Also, $C \cap D$ consists of the two points $P := [1, 0, 0, 0]$ and $Q := [0, 1, 0, 0]$, and these two points are switched by the action of G.

Let $V_Y \to Y$ be the scheme which over $Y - P$ is defined by blowing up D and then the strict transform of C, and over $Y - Q$ is defined by blowing up C and then the strict transform of D. (This is the same construction as in the proof of Lemma 78.49.1, except that Y here denotes an open subset of $\mathbf{P}^3$ rather than all of $\mathbf{P}^3$.) Then the action of G on Y lifts to an action of G on V_Y, which switches the inverse images of $Y - P$ and $Y - Q$. This action of G on V_Y gives a descent datum $(V_Y/Y, \varphi_Y)$ on V_Y relative to the G-torsor $Y \to S$. The morphism $V_Y \to Y$ is proper but not projective, as shown in the proof of Lemma 78.49.1.

Let X be the disjoint union of the open subsets $Y - P$ and $Y - Q$; then we have surjective etale morphisms $X \to Y \to S$. Let V be the pullback of $V_Y \to Y$ to X; then the morphism $V \to X$ is projective, since $V_Y \to Y$ is a blow-up over each of the open subsets $Y - P$ and $Y - Q$. Moreover, the descent datum $(V_Y/Y, \varphi_Y)$ pulls back to a descent datum $(V/X, \varphi)$ relative to the etale covering $X \to S$.

Suppose that this descent datum is effective in the category of schemes. That is, there is a scheme $U \to S$ which pulls back to the morphism $V \to X$ together with

its descent datum. Then U would be the quotient of V_Y by its G-action.

$$\begin{array}{ccc} V & \longrightarrow & X \\ \downarrow & & \downarrow \\ V_Y & \longrightarrow & Y \\ \downarrow & & \downarrow \\ U & \longrightarrow & S \end{array}$$

Let E be the inverse image of $C \cup D \subset Y$ in V_Y; thus $E \to C \cup D$ is a proper morphism, with fibers isomorphic to $\mathbf{P}^1$ over $(C \cup D) - \{P, Q\}$. The inverse image of P in E is a union of two lines L_0 and M_0. It follows that the inverse image of $Q = g(P)$ in E is the union of two lines $L_0' = g(M_0)$ and $M_0' = g(L_0)$. As shown in the proof of Lemma 78.49.1, we have a rational equivalence $L_0 + M_0' = L_0 + g(L_0) \sim 0$ on E.

By descent of closed subschemes, there is a curve $L_1 \subset U$ (isomorphic to $\mathbf{P}^1$) whose inverse image in V_Y is $L_0 \cup g(L_0)$. (Use Descent, Lemma 34.33.1, noting that a closed immersion is an affine morphism.) Let R be a complex point of L_1. Since we assumed that U is a scheme, we can choose a function f in the local ring $O_{U,R}$ that vanishes at R but not on the whole curve L_1. Let D_{loc} be an irreducible component of the closed subset $\{f = 0\}$ in $\operatorname{Spec} O_{U,R}$; then D_{loc} has codimension 1. The closure of D_{loc} in U is an irreducible divisor D_U in U which contains the point R but not the whole curve L_1. The inverse image of D_U in V_Y is an effective divisor D which intersects $L_0 \cup g(L_0)$ but does not contain either curve L_0 or $g(L_0)$.

Since the complex 3-fold V_Y is smooth, $O(D)$ is a line bundle on V_Y. (We use here that a regular local ring is factorial, or in other words is a UFD.) The restriction of $O(D)$ to the proper surface $E \subset V_Y$ is a line bundle which has positive degree on the 1-cycle $L_0 + g(L_0)$, by our information on D. Since $L_0 + g(L_0) \sim 0$ on E, this contradicts that the degree of a line bundle is well-defined on 1-cycles modulo rational equivalence on a proper scheme over a field (Chow Homology, Lemma 41.20.2 and Lemma 41.29.2). Therefore the descent datum $(V/X, \varphi)$ is in fact not effective; that is, U does not exist as a scheme. □

In this example, the descent datum *is* effective in the category of algebraic spaces. More precisely, U exists as a smooth separated algebraic space of dimension 3 over $\mathbf{C}$, for example by Algebraic Spaces, Lemma 47.14.3. Hironaka's 3-fold U is a small resolution of the blow-up S' of the smooth quasi-projective 3-fold S along the irreducible nodal curve $(C \cup D)/G$; the 3-fold S' has a node singularity. The other small resolution of S' (differing from U by a "flop") is again an algebraic space which is not a scheme.

78.51. Derived base change

Let $R \to A$ be a ring map. In More on Algebra, Section 15.44 we construct a derived base change functor $- \otimes^{\mathbf{L}}_R A : D(R) \to D(A)$. Next, let $R \to B$ be a second

ring map. Picture

$$\begin{array}{ccc} B & \longrightarrow & B \otimes_R A \\ \uparrow & & \uparrow \\ R & \longrightarrow & A \end{array}$$

Given a B-module M the tensor product $M \otimes_R A$ is a $B \otimes_R A$-module. In this section we show there does not exist a "derived base change functor" $D(B) \to D(B \otimes_R A)$.

Let k be a field. Set $R = k[x, y]$. Set $A = R/(xy)$ and $B = R/(x^2)$. The object $B \otimes^{\mathbf{L}}_R A$ in $D(A)$ is represented by

$$x^2 : A \longrightarrow A$$

and we have $H^0(B \otimes^{\mathbf{L}}_R A) = B \otimes_R A$. We claim that there does not exist an object E of $D(B \otimes_R A)$ mapping to $B \otimes^{\mathbf{L}}_R A$ in $D(A)$. Namely, for such an E the module $H^0(E)$ would be free, hence E would decompose as $H^0(E)[0] \oplus H^{-1}(E)[1]$. But it is easy to see that $B \otimes^{\mathbf{L}}_R A$ is not isomorphic to the sum of its cohomology groups in $D(A)$.

Lemma 78.51.1. *Let $R \to A$ and $R \to B$ be ring maps. In general there does not exist a functor $T : D(B) \to D(B \otimes_R A)$ of triangulated categories such that a B-module M gives an object $T(M)$ of $D(B \otimes_R A)$ which maps to $M \otimes^{\mathbf{L}}_R A$ under the map $D(B \otimes_R A) \to D(A)$.*

Proof. See discussion above. □

78.52. An interesting compact object

Let R be a ring. Let (A, d) be a differential graded R-algebra. If $A = R$, then we know that every perfect object of $D(A, \mathrm{d}) = D(R)$ is represented by a finite complex of finite projective modules. In other words, compact objects are perfect, see More on Algebra, Proposition 15.51.2. The analogue in the language of differential graded modules would be the question: "Is every compact object of $D(A, \mathrm{d})$ represented by a differential graded A-module P which is finite and graded projective?"

For general differential graded algebras, this is not true. Namely, let k be a field of characteristic 2 (so we don't have to worry about signs). Let $A = k[x, y]/(y^2)$ with

(1) x of degree 0
(2) y of degree -1,
(3) $\mathrm{d}(x) = 0$, and
(4) $\mathrm{d}(y) = x^2 + x$.

Then $x : A \to A$ is a projector in $K(A, \mathrm{d})$. Hence we see that

$$A = \mathrm{Ker}(x) \oplus \mathrm{Im}(1 - x)$$

in $K(A, \mathrm{d})$, see Differential Graded Algebra, Lemma 22.5.4 and Derived Categories, Lemma 13.4.12. It is clear that A is a compact object of $D(A, \mathrm{d})$ (see Differential Graded Algebra, Lemma 22.26.2 for a more general statement) Then $\mathrm{Ker}(x)$ is a compact object of $D(A, \mathrm{d})$ as follows from Derived Categories, Lemma 13.32.2.

Next, suppose that M is a differential graded (right) A-module representing $\mathrm{Ker}(x)$ and suppose that M is finite and projective as a graded A-module. Because every finite graded projective module over $k[x, y]/(y^2)$ is graded free, we see that M is

finite free as a graded $k[x,y]/(y^2)$-module (i.e., when we forget the differential). We set $N = M/M(x^2+x)$. Consider the exact sequence

$$0 \to M \xrightarrow{x^2+x} M \to N \to 0$$

Since x^2+x is of degree 0, in the center of A, and $\mathrm{d}(x^2+x) = 0$ we see that this is a short exact sequence of differential graded A-modules. Moreover, as $\mathrm{d}(y) = x^2+x$ we see that the differential on N is linear. Finally, we conclude that

$$H^{-1}(N) \to H^0(M) \quad \text{and} \quad H^0(M) \to H^0(N)$$

are isomorphisms. A computation of the boundary map shows that $H^*(N) = k[x,y]/(x,y^2)$ as a graded module; we omit the details. Since N is a free $k[x,]/(y^2, x^2+x)$-module we have a resolution

$$\ldots \to N[2] \xrightarrow{y} N[1] \xrightarrow{y} N \to N/Ny \to 0$$

compatible with differentials. Since N is bounded and since $H^0(N) = k[x,y]/(x,y^2)$ it follows from Homology, Lemma 12.22.5 that $H^0(N/Ny) = k[x]/(x)$. But as N/Ny is a finite complex of free $k[x]/(x^2+x) = k \times k$-modules, we see that its cohomology has to have even dimension, a contradiction.

Lemma 78.52.1. *There exists a differential graded algebra (A, d) and a compact object E of $D(A, d)$ such that E cannot be represented by a finite and graded projective differential graded A-module.*

Proof. See discussion above. □

78.53. Two differential graded categories

In this section we construct two differential graded categories satisfying axioms (A), (B), and (C) as in Differential Graded Algebra, Situation 22.20.2 whose objects do not come with a $\mathbf{Z}$-grading.

Example I. Let X be a topological space. Denote $\underline{\mathbf{Z}}$ the constant sheaf with value $\mathbf{Z}$. Let A be an $\underline{\mathbf{Z}}$-torsor. In this setting we say a sheaf of abelian groups $\mathcal{F}$ is *A-graded* if given a local section $a \in A(U)$ there is a projector $p_a : \mathcal{F}|_U \to \mathcal{F}|_U$ such that whenever we have a local isomorphism $\underline{\mathbf{Z}}|_U \to A|_U$ then $\mathcal{F}|_U = \bigoplus_{n \in \mathbf{Z}} p_n(\mathcal{F})$. Another way to say this is that locally on X the abelian sheaf $\mathcal{F}$ has a $\mathbf{Z}$-grading, but on overlaps the different choices of gradings differ by a shift in degree given by the transition functions for the torsor A. We say that a pair $(\mathcal{F}, \mathrm{d})$ is an *A-graded complex of abelian sheaves*, if $\mathcal{F}$ is an A-graded abelian sheaf and $\mathrm{d} : \mathcal{F} \to \mathcal{F}$ is a differential, i.e., $\mathrm{d}^2 = 0$ such that $p_{a+1} \circ \mathrm{d} = \mathrm{d} \circ p_a$ for every local section a of A. In other words, $\mathrm{d}(p_a(\mathcal{F}))$ is contained in $p_{a+1}(\mathcal{F})$.

Next, consider the category $\mathcal{A}$ with

(1) objects are A-graded complexes of abelian sheaves, and
(2) for objects $(\mathcal{F}, \mathrm{d})$, $(\mathcal{G}, \mathrm{d})$ we set

$$\mathrm{Hom}_{\mathcal{A}}((\mathcal{F}, \mathrm{d}), (\mathcal{G}, \mathrm{d})) = \bigoplus \mathrm{Hom}^n(\mathcal{F}, \mathcal{G})$$

where $\mathrm{Hom}^n(\mathcal{F}, \mathcal{G})$ is the group of maps of abelian sheaves f such that $f(p_a(\mathcal{F})) \subset p_{a+n}(\mathcal{G})$ for all local sections a of A. As differential we take $\mathrm{d}(f) = \mathrm{d} \circ f - (-1)^n f \circ \mathrm{d}$, see Differential Graded Algebra, Example 22.19.6.

We omit the verification that this is indeed a differential graded category satisfying (A), (B), and (C). All the properties may be verified locall on X where one just recovers the differential graded category of complexes of abelian sheaves. Thus we obtain a triangulated category $K(\mathcal{A})$.

Twisted derived category of X. Observe that given an object $(\mathcal{F}, \mathrm{d})$ of $\mathcal{A}$, there is a well defined A-graded cohomology sheaf $H(\mathcal{F}, \mathrm{d})$. Hence it is clear what is meant by a quasi-isomorphism in $K(\mathcal{A})$. We can invert quasi-isomorphisms to obtain the *derived category $D(\mathcal{A})$ of complexes of A-graded sheaves*. If A is the trivial torsor, then $D(\mathcal{A})$ is equal to $D(X)$, but for nonzero torsors, one obtains a kind of *twisted* derived category of X.

Example II. Let C be a smooth curve over a perfect field k of characteristic 2. Then $\Omega_{C/k}$ comes endowed with a canonical square root. Namely, we can write $\Omega_{C/k} = \mathcal{L}^{\otimes 2}$ such that for every local function f on C the section $\mathrm{d}(f)$ is equal to $s^{\otimes 2}$ for some local section s of $\mathcal{L}$. The "reason" is that

$$\mathrm{d}(a_0 + a_1 t + \ldots + a_d t^d) = (\sum\nolimits_{i \text{ odd}} a_i^{1/2} t^{(i-1)/2})^2 \mathrm{d}t$$

(insert future reference here). This in particular determines a canonical connection

$$\nabla_{can} : \Omega_{C/k} \longrightarrow \Omega_{C/k} \otimes_{\mathcal{O}_C} \Omega_{C/k}$$

whose 2-curvature is zero (namely, the unique connection such that the squares have derivative equal to zero). Observe that the category of vector bundles with connections is a tensor category, hence we also obtain canonical connections ∇_{can} on the invertible sheaves $\Omega_{C/k}^{\otimes n}$ for all $n \in \mathbf{Z}$.

Let $\mathcal{A}$ be the category with

(1) objects are pairs $(\mathcal{F}, \nabla)$ consisting of a finite locally free sheaf $\mathcal{F}$ endowed with a connection

$$\nabla : \mathcal{F} \longrightarrow \mathcal{F} \otimes_{\mathcal{O}_C} \Omega_{C/k}$$

whose 2-curvature is zero, and

(2) morphisms between $(\mathcal{F}, \nabla_{\mathcal{F}})$ and $(\mathcal{G}, \nabla_{\mathcal{G}})$ are given by

$$\mathrm{Hom}_{\mathcal{A}}((\mathcal{F}, \nabla_{\mathcal{F}}), (\mathcal{G}, \nabla_{\mathcal{G}})) = \bigoplus \mathrm{Hom}_{\mathcal{O}_C}(\mathcal{F}, \mathcal{G} \otimes_{\mathcal{O}_C} \Omega_{C/k}^{\otimes n})$$

For an element $f : \mathcal{F} \to \mathcal{G} \otimes \Omega_{C/k}^{\otimes n}$ of degree n we set

$$\mathrm{d}(f) = \nabla_{\mathcal{G} \otimes \Omega_{C/k}^{\otimes n}} \circ f + f \circ \nabla_{\mathcal{F}}$$

with suitable identifications.

We omit the verification that this forms a differential graded category with properties (A), (B), (C). Thus we obtain a triangulated homotopy category $K(\mathcal{A})$.

If $C = \mathbf{P}^1_k$, then $K(\mathcal{A})$ is the zero category. However, if C is a smooth proper curve of genus > 1, then $K(\mathcal{A})$ is not zero. Namely, suppose that $\mathcal{N}$ is an invertible sheaf of degree $0 \leq d < g - 1$ with a nonzero section σ. Then set $(\mathcal{F}, \nabla_{\mathcal{F}}) = (\mathcal{O}_C, \mathrm{d})$ and $(\mathcal{G}, \nabla_{\mathcal{G}}) = (\mathcal{N}^{\otimes 2}, \nabla_{can})$. We see that

$$\mathrm{Hom}^n_{\mathcal{A}}((\mathcal{F}, \nabla_{\mathcal{F}}), (\mathcal{G}, \nabla_{\mathcal{G}})) = \begin{cases} 0 & \text{if} \quad n < 0 \\ \Gamma(C, \mathcal{N}^{\otimes 2}) & \text{if} \quad n = 0 \\ \Gamma(C, \mathcal{N}^{\otimes 2} \otimes \Omega_{C/k}) & \text{if} \quad n = 1 \end{cases}$$

The first 0 because the degree of $\mathcal{N}^{\otimes 2} \otimes \Omega_{C/k}^{\otimes -1}$ is negative by the condition $d < g-1$. Now, the section $\sigma^{\otimes 2}$ has derivative equal zero, hence the homomorphism group

$$\mathrm{Hom}_{K(\mathcal{A})}((\mathcal{F}, \nabla_{\mathcal{F}}), (\mathcal{G}, \nabla_{\mathcal{G}}))$$

is nonzero.

78.54. Other chapters

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CHAPTER 79

Exercises

79.1. Algebra

This first section just contains some assorted questions.

Exercise 79.1.1. Let A be a ring, and $\mathfrak{m}$ a maximal ideal. In $A[X]$ let $\tilde{\mathfrak{m}}_1 = (\mathfrak{m}, X)$ and $\tilde{\mathfrak{m}}_2 = (\mathfrak{m}, X - 1)$. Show that

$$A[X]_{\tilde{\mathfrak{m}}_1} \cong A[X]_{\tilde{\mathfrak{m}}_2}.$$

Exercise 79.1.2. Find an example of a non Noetherian ring R such that every finitely generated ideal of R is finitely presented as an R-module. (A ring is said to be *coherent* if the last property holds.)

Exercise 79.1.3. Suppose that $(A, \mathfrak{m}, k)$ is a Noetherian local ring. For any finite A-module M define $r(M)$ to be the minimum number of generators of M as an A-module. This number equals $\dim_k M/\mathfrak{m}M = \dim_k M \otimes_A k$ by NAK.

(1) Show that $r(M \otimes_A N) = r(M)r(N)$.
(2) Let $I \subset A$ be an ideal with $r(I) > 1$. Show that $r(I^2) < r(I)^2$.
(3) Conclude that if every ideal in A is a flat module, then A is a PID (or a field).

Exercise 79.1.4. Let k be a field. Show that the following pairs of k-algebras are not isomorphic:

(1) $k[x_1, \ldots, x_n]$ and $k[x_1, \ldots, x_{n+1}]$ for any $n \geq 1$.
(2) $k[a, b, c, d, e, f]/(ab + cd + ef)$ and $k[x_1, \ldots, x_n]$ for $n = 5$.
(3) $k[a, b, c, d, e, f]/(ab + cd + ef)$ and $k[x_1, \ldots, x_n]$ for $n = 6$.

Remark 79.1.5. Of course the idea of this exercise is to find a simple argument in each case rather than applying a "big" theorem. Nonetheless it is good to be guided by general principles.

Exercise 79.1.6. Algebra. (Silly and should be easy.)

(1) Give an example of a ring A and a nonsplit short exact sequence of A-modules
$$0 \to M_1 \to M_2 \to M_3 \to 0.$$
(2) Give an example of a nonsplit sequence of A-modules as above and a faithfully flat $A \to B$ such that
$$0 \to M_1 \otimes_A B \to M_2 \otimes_A B \to M_3 \otimes_A B \to 0.$$
is split as a sequence of B-modules.

Exercise 79.1.7. Suppose that k is a field having a primitive nth root of unity ζ. This means that $\zeta^n = 1$, but $\zeta^m \neq 1$ for $0 < m < n$.

(1) Show that the characteristic of k is prime to n.
(2) Suppose that $a \in k$ is an element of k which is not an dth power in k for any divisor d of n, *in* $\geq d > 1$. Show that $k[x]/(x^n - a)$ is a field. (Hint: Consider a splitting field for $x^n - a$ and use Galois theory.)

Exercise 79.1.8. Let $\nu : k[x] \setminus \{0\} \to \mathbf{Z}$ be a map with the following properties: $\nu(fg) = \nu(f) + \nu(g)$ whenever f, g not zero, and $\nu(f+g) \geq min(\nu(f), \nu(g))$ whenever f, g, $f+g$ are not zero, and $\nu(c) = 0$ for all $c \in k^*$.

(1) Show that if f, g, and $f+g$ are nonzero and $\nu(f) \neq \nu(g)$ then we have equality $\nu(f+g) = min(\nu(f), \nu(g))$.
(2) Show that if $f = \sum a_i x^i$, $f \neq 0$, then $\nu(f) \geq min(\{i\nu(x)\}_{a_i \neq 0})$. When does equality hold?
(3) Show that if ν attains a negative value then $\nu(f) = -n \deg(f)$ for some $n \in \mathbf{N}$.
(4) Suppose $\nu(x) \geq 0$. Show that $\{f \mid f = 0, \text{ or } \nu(f) > 0\}$ is a prime ideal of $k[x]$.
(5) Describe all possible ν.

Let A be a ring. An *idempotent* is an element $e \in A$ such that $e^2 = e$. The elements 1 and 0 are always idempotent. A *nontrivial idempotent* is an idempotent which is not equal to zero. Two idempotents $e, e' \in A$ are called *orthogonal* if $ee' = 0$.

Exercise 79.1.9. Let A be a ring. Show that A is a product of two nonzero rings if and only if A has a nontrivial idempotent.

Exercise 79.1.10. Let A be a ring and let $I \subset A$ be a locally nilpotent ideal. Show that the map $A \to A/I$ induces a bijection on idempotents. (Hint: It may be easier to prove this when I is nilpotent. Do this first. Then use "absolute Noetherian reduction" to reduce to the nilpotent case.)

79.2. Colimits

Definition 79.2.1. A *directed partially ordered set* is a nonempty set I endowed with a partial ordering $\leq$ such that given any pair $i, j \in I$ there exists a $k \in I$ such that $i \leq k$ and $j \leq k$. A *system of rings* over I is given by a ring A_i for each $i \in I$ and a map of rings $\varphi_{ij} : A_i \to A_j$ whenever $i \leq j$ such that the composition $A_i \to A_j \to A_k$ is equal to $A_i \to A_k$ whenever $i \leq j \leq k$.

One similarly defines systems of groups, modules over a fixed ring, vector spaces over a field, etc.

Exercise 79.2.2. Let I be a directed partially ordered set and let (A_i, φ_{ij}) be a system of rings over I. Show that there exists a ring A and maps $\varphi_i : A_i \to A$ such that $\varphi_j \circ \varphi_{ij} = \varphi_i$ for all $i \leq j$ with the following universal property: Given any ring B and maps $\psi_i : A_i \to B$ such that $\psi_j \circ \varphi_{ij} = \psi_i$ for all $i \leq j$, then there exists a unique ring map $\psi : A \to B$ such that $\psi_i = \psi \circ \varphi_i$.

Definition 79.2.3. The ring A constructed in Exercise 79.2.2 is called the *colimit* of the system. Notation $\operatorname{colim} A_i$.

Exercise 79.2.4. Let $(I, \geq)$ be a directed partially ordered set and let (A_i, φ_{ij}) be a system of rings over I with colimit A. Prove that there is a bijection

$$\operatorname{Spec}(A) = \{(\mathfrak{p}_i)_{i \in I} \mid \ \mathfrak{p}_i \subset A_i \text{ and } \mathfrak{p}_i = \varphi_{ij}^{-1}(\mathfrak{p}_j) \ \forall i \leq j\} \subset \prod\nolimits_{i \in I} \operatorname{Spec}(A_i)$$

The set on the right hand side is the limit of the sets $\mathrm{Spec}(A_i)$. Notation $\lim \mathrm{Spec}(A_i)$.

Exercise 79.2.5. Let $(I, \geq)$ be a directed partially ordered set and let (A_i, φ_{ij}) be a system of rings over I with colimit A. Suppose that $\mathrm{Spec}(A_j) \to \mathrm{Spec}(A_i)$ is surjective for all $i \leq j$. Show that $\mathrm{Spec}(A) \to \mathrm{Spec}(A_i)$ is surjective for all i. (Hint: You can try to use Tychonoff, but there is also a basically trivial direct algebraic proof based on Algebra, Lemma 10.16.9.)

Exercise 79.2.6. Let $A \subset B$ be an integral ring extension. Prove that $\mathrm{Spec}(B) \to \mathrm{Spec}(A)$ is surjective. Use the exercises above, the fact that this holds for a finite ring extension (proved in the lectures), and by proving that $B = \mathrm{colim}\, B_i$ is a directed colimit of finite extensions $A \subset B_i$.

Exercise 79.2.7. Let $(I, \geq)$ be a partially ordered set which is directed. Let A be a ring and let $(N_i, \varphi_{i,i'})$ be a directed system of A-modules indexed by I. Suppose that M is another A-module. Prove that

$$\mathrm{colim}_{i \in I}\, M \otimes_A N_i \cong M \otimes_A \Big(\mathrm{colim}_{i \in I}\, N_i \Big).$$

Definition 79.2.8. A module M over R is said to be of *finite presentation* over R if it is isomorphic to the cokernel of a map of finite free modules $R^{\oplus n} \to R^{\oplus m}$.

Exercise 79.2.9. Prove that any module over any ring is

(1) the colimit of its finitely generated submodules, and
(2) in some way a colimit of finitely presented modules.

79.3. Additive and abelian categories

Exercise 79.3.1. Let k be a field. Let $\mathcal{C}$ be the category of filtered vector spaces over k, see Homology, Definition 12.16.1 for the definition of a filtered object of any category.

(1) Show that this is an additive category (explain carefuly what the direct sum of two objects is).
(2) Let $f : (V, F) \to (W, F)$ be a morphism of $\mathcal{C}$. Show that f has a kernel and cokernel (explain precisely what the kernel and cokernel of f are).
(3) Give an example of a map of $\mathcal{C}$ such that the canonical map $\mathrm{Coim}(f) \to \mathrm{Im}(f)$ is not an isomorphism.

Exercise 79.3.2. Let R be a Noetherian domain. Let $\mathcal{C}$ be the category of finitely generated torsion free R-modules.

(1) Show that this is an additive category.
(2) Let $f : N \to M$ be a morphism of $\mathcal{C}$. Show that f has a kernel and cokernel (make sure you define precisely what the kernel and cokernel of f are).
(3) Give an example of a Noetherian domain R and a map of $\mathcal{C}$ such that the canonical map $\mathrm{Coim}(f) \to \mathrm{Im}(f)$ is not an isomorphism.

Exercise 79.3.3. Give an example of a category which is additive and has kernels and cokernels but which is not as in Exercises 79.3.1 and 79.3.2.

79.4. Flat ring maps

Exercise 79.4.1. Let S be a multiplicative subset of the ring A.

(1) For an A-module M show that $S^{-1}M = S^{-1}A \otimes_A M$.
(2) Show that $S^{-1}A$ is flat over A.

Exercise 79.4.2. Find an injection $M_1 \to M_2$ of A-modules such that $M_1 \otimes N \to M_2 \otimes N$ is not injective in the following cases:

(1) $A = k[x, y]$ and $N = (x, y) \subset A$. (Here and below k is a field.)
(2) $A = k[x, y]$ and $N = A/(x, y)$.

Exercise 79.4.3. Give an example of a ring A and a finite A-module M which is a flat but not a projective A-module.

Remark 79.4.4. If M is of finite presentation and flat over A, then M is projective over A. Thus your example will have to involve a ring A which is not Noetherian. I know of an example where A is the ring of $\mathcal{C}^\infty$-functions on $\mathbf{R}$.

Exercise 79.4.5. Find a flat but not free module over $\mathbf{Z}_{(2)}$.

Exercise 79.4.6. Flat deformations.

(1) Suppose that k is a field and $k[\epsilon]$ is the ring of dual numbers $k[\epsilon] = k[x]/(x^2)$ and $\epsilon = \bar{x}$. Show that for any k-algebra A there is a flat $k[\epsilon]$-algebra B such that A is isomorphic to $B/\epsilon B$.
(2) Suppose that $k = \mathbf{F}_p = \mathbf{Z}/p\mathbf{Z}$ and
$$A = k[x_1, x_2, x_3, x_4, x_5, x_6]/(x_1^p, x_2^p, x_3^p, x_4^p, x_5^p, x_6^p).$$
Show that there exists a flat $\mathbf{Z}/p^2\mathbf{Z}$-algebra B such that B/pB is isomorphic to A. (So here p plays the role of ϵ.)
(3) Now let $p = 2$ and consider the same question for $k = \mathbf{F}_2 = \mathbf{Z}/2\mathbf{Z}$ and
$$A = k[x_1, x_2, x_3, x_4, x_5, x_6]/(x_1^2, x_2^2, x_3^2, x_4^2, x_5^2, x_6^2, x_1x_2 + x_3x_4 + x_5x_6).$$
However, in this case show that there does *not* exist a flat $\mathbf{Z}/4\mathbf{Z}$-algebra B such that $B/2B$ is isomorphic to A. (Find the trick! The same example works in arbitrary characteristic $p > 0$, except that the computation is more difficult.)

Exercise 79.4.7. Let $(A, \mathfrak{m}, k)$ be a local ring and let $k \subset k'$ be a finite field extension. Show there exists a flat, local map of local rings $A \to B$ such that $\mathfrak{m}_B = \mathfrak{m}B$ and $B/\mathfrak{m}B$ is isomorphic to k' as k-algebra. (Hint: first do the case where $k \subset k'$ is generated by a single element.)

Remark 79.4.8. The same result holds for arbitrary field extensions $k \subset K$.

79.5. The Spectrum of a ring

Exercise 79.5.1. Compute $\mathrm{Spec}(\mathbf{Z})$ as a set and describe its topology.

Exercise 79.5.2. Let A be any ring. For $f \in A$ we define $D(f) := \{\mathfrak{p} \subset A \mid f \notin \mathfrak{p}\}$. Prove that the open subsets $D(f)$ form a basis of the topology of $\mathrm{Spec}(A)$.

Exercise 79.5.3. Prove that the map $I \mapsto V(I)$ defines a natural bijection
$$\{I \subset A \text{ with } I = \sqrt{I}\} \longrightarrow \{T \subset \mathrm{Spec}(A) \text{ closed}\}$$

Definition 79.5.4. A topological space X is called *quasi-compact* if for any open covering $X = \bigcup_{i \in I} U_i$ there is a finite subset $\{i_1, \ldots, i_n\} \subset I$ such that $X = U_{i_1} \cup \ldots U_{i_n}$.

Exercise 79.5.5. Prove that $\mathrm{Spec}(A)$ is quasi-compact for any ring A.

Definition 79.5.6. A topological space X is said to verify the separation axiom T_0 if for any pair of points $x, y \in X$, $x \neq y$ there is an open subset of X containing one but not the other. We say that X is *Hausdorff* if for any pair $x, y \in X$, $x \neq y$ there are disjoint open subsets U, V such that $x \in U$ and $y \in V$.

Exercise 79.5.7. Show that $\mathrm{Spec}(A)$ is **not** Hausdorff in general. Prove that $\mathrm{Spec}(A)$ is T_0. Give an example of a topological space X that is not T_0.

Remark 79.5.8. Usually the word compact is reserved for quasi-compact and Hausdorff spaces.

Definition 79.5.9. A topological space X is called *irreducible* if X is not empty and if $X = Z_1 \cup Z_2$ with $Z_1, Z_2 \subset X$ closed, then either $Z_1 = X$ or $Z_2 = X$. A subset $T \subset X$ of a topological space is called *irreducible* if it is an irreducible topological space with the topology induced from X. This definition implies T is irreducible if and only if the closure $\bar{T}$ of T in X is irreducible.

Exercise 79.5.10. Prove that $\mathrm{Spec}(A)$ is irreducible if and only if $Nil(A)$ is a prime ideal and that in this case it is the unique minimal prime ideal of A.

Exercise 79.5.11. Prove that a closed subset $T \subset \mathrm{Spec}(A)$ is irreducible if and only if it is of the form $T = V(\mathfrak{p})$ for some prime ideal $\mathfrak{p} \subset A$.

Definition 79.5.12. A point x of an irreducible topological space X is called a *generic point* of X if X is equal to the closure of the subset $\{x\}$.

Exercise 79.5.13. Show that in a T_0 space X every irreducible closed subset has at most one generic point.

Exercise 79.5.14. Prove that in $\mathrm{Spec}(A)$ every irreducible closed subset *does* have a generic point. In fact show that the map $\mathfrak{p} \mapsto \overline{\{\mathfrak{p}\}}$ is a bijection of $\mathrm{Spec}(A)$ with the set of irreducible closed subsets of X.

Exercise 79.5.15. Give an example to show that an irreducible subset of $\mathrm{Spec}(\mathbf{Z})$ does not neccesarily have a generic point.

Definition 79.5.16. A topological space X is called *Noetherian* if any decreasing sequence $Z_1 \supset Z_2 \supset Z_3 \supset \ldots$ of closed subsets of X stabilizes. (It is called *Artinian* if any increasing sequence of closed subsets stabilizes.)

Exercise 79.5.17. Show that if the ring A is Noetherian then the topological space $\mathrm{Spec}(A)$ is Noetherian. Give an example to show that the converse is false. (The same for Artinian if you like.)

Definition 79.5.18. A maximal irreducible subset $T \subset X$ is called an *irreducible component* of the space X. Such an irreducible component of X is automatically a closed subset of X.

Exercise 79.5.19. Prove that any irreducible subset of X is contained in an irreducible component of X.

Exercise 79.5.20. Prove that a Noetherian topological space X has only finitely many irreducible components, say $X_1, \ldots, X_n$, and that $X = X_1 \cup X_2 \cup \ldots \cup X_n$. (Note that any X is always the union of its irreducible components, but that if $X = \mathbf{R}$ with its usual topology for instance then the irreducible components of X are the one point subsets. This is not terribly interesting.)

Exercise 79.5.21. Show that irreducible components of $\mathrm{Spec}(A)$ correspond to minimal primes of A.

Definition 79.5.22. A point $x \in X$ is called *closed* if $\overline{\{x\}} = \{x\}$. Let x, y be points of X. We say that x is a *specialization* of y, or that y is a *generalization* of x if $x \in \overline{\{y\}}$.

Exercise 79.5.23. Show that closed points of $\mathrm{Spec}(A)$ correspond to maximal ideals of A.

Exercise 79.5.24. Show that $\mathfrak{p}$ is a generalization of $\mathfrak{q}$ in $\mathrm{Spec}(A)$ if and only if $\mathfrak{p} \subset \mathfrak{q}$. Characterize closed points, maximal ideals, generic points and minimal prime ideals in terms of generalization and specialization. (Here we use the terminology that a point of a possibly reducible topological space X is called a generic point if it is a generic points of one of the irreducible components of X.)

Exercise 79.5.25. Let I and J be ideals of A. What is the condition for $V(I)$ and $V(J)$ to be disjoint?

Definition 79.5.26. A topological space X is called *connected* if it is nonempty and not the union of two nonempty disjoint open subsets. A *connected component* of X is a maximal connected subset. Any point of X is contained in a connected component of X and any connected component of X is closed in X. (But in general a connected component need not be open in X.)

Exercise 79.5.27. Let A be a nonzero ring. Show that $\mathrm{Spec}(A)$ is disconnected iff $A \cong B \times C$ for certain nonzero rings B, C.

Exercise 79.5.28. Let T be a connected component of $\mathrm{Spec}(A)$. Prove that T is stable under generalization. Prove that T is an open subset of $\mathrm{Spec}(A)$ if A is Noetherian. (Remark: This is wrong when A is an infinite product of copies of $\mathbf{F}_2$ for example. The spectrum of this ring consists of infinitely many closed points.)

Exercise 79.5.29. Compute $\mathrm{Spec}(k[x])$, i.e., describe the prime ideals in this ring, describe the possible specializations, and describe the topology. (Work this out when k is algebraically closed but also when k is not.)

Exercise 79.5.30. Compute $\mathrm{Spec}(k[x, y])$, where k is algebraically closed. [Hint: use the morphism $\varphi : \mathrm{Spec}(k[x, y]) \to \mathrm{Spec}(k[x])$; if $\varphi(\mathfrak{p}) = (0)$ then localize with respect to $S = \{f \in k[x] \mid f \neq 0\}$ and use result of lecture on localization and Spec.] (Why do you think algebraic geometers call this affine 2-space?)

Exercise 79.5.31. Compute $\mathrm{Spec}(\mathbf{Z}[y])$. [Hint: as above.] (Affine 1-space over $\mathbf{Z}$.)

79.6. Localization

Exercise 79.6.1. Let A be a ring. Let $S \subset A$ be a multiplicative subset. Let M be an A-module. Let $N \subset S^{-1}M$ be an $S^{-1}A$-submodule. Show that there exists

an A-submodule $N' \subset M$ such that $N = S^{-1}N'$. (This useful result applies in particular to ideals of $S^{-1}A$.)

Exercise 79.6.2. Let A be a ring. Let M be an A-module. Let $m \in M$.

(1) Show that $I = \{a \in A \mid am = 0\}$ is an ideal of A.
(2) For a prime $\mathfrak{p}$ of A show that the image of m in $M_\mathfrak{p}$ is zero if and only if $I \not\subset \mathfrak{p}$.
(3) Show that m is zero if and only if the image of m is zero in $M_\mathfrak{p}$ for all primes $\mathfrak{p}$ of A.
(4) Show that m is zero if and only if the image of m is zero in $M_\mathfrak{m}$ for all maximal ideals $\mathfrak{m}$ of A.
(5) Show that $M = 0$ if and only if $M_\mathfrak{m}$ is zero for all maximal ideals $\mathfrak{m}$.

Exercise 79.6.3. Find a pair (A, f) where A is a domain with three or more pairwise distinct primes and $f \in A$ is an element such that the principal localization $A_f = \{1, f, f^2, \ldots\}^{-1}A$ is a field.

Exercise 79.6.4. Let A be a ring. Let M be a finite A-module. Let $S \subset A$ be a multiplicative set. Assume that $S^{-1}M = 0$. Show that there exists an $f \in S$ such that the principal localization $M_f = \{1, f, f^2, \ldots\}^{-1}M$ is zero.

Exercise 79.6.5. Give an example of a triple (A, I, S) where A is a ring, $0 \neq I \neq A$ is a proper nonzero ideal, and $S \subset A$ is a multiplicative subset such that $A/I \cong S^{-1}A$ as A-algebras.

79.7. Nakayama's Lemma

Exercise 79.7.1. Let A be a ring. Let I be an ideal of A. Let M be an A-module. Let $x_1, \ldots, x_n \in M$. Assume that

(1) M/IM is generated by $x_1, \ldots, x_n$,
(2) M is a finite A-module,
(3) I is contained in every maximal ideal of A.

Show that $x_1, \ldots, x_n$ generate M. (Suggested solution: Reduce to a localization at a maximal ideal of A using Exercise 79.6.2 and exactness of localization. Then reduce to the statement of Nakayama's lemma in the lectures by looking at the quotient of M by the submodule generated by $x_1, \ldots, x_n$.)

79.8. Length

Definition 79.8.1. Let A be a ring. Let M be an A-module. The *length* of M as an R-module is

$$\text{length}_A(M) = \sup\{n \mid \exists\ 0 = M_0 \subset M_1 \subset \ldots \subset M_n = M,\ M_i \neq M_{i+1}\}.$$

In other words, the supremum of the lengths of chains of submodules.

Exercise 79.8.2. Show that a module M over a ring A has length 1 if and only if it is isomorphic to $A/\mathfrak{m}$ for some maximal ideal $\mathfrak{m}$ in A.

Exercise 79.8.3. Compute the length of the following modules over the following rings. Briefly(!) explain your answer. (Please feel free to use additivity of the length function in short exact sequences, see Algebra, Lemma 10.48.3).

(1) The length of $\mathbf{Z}/120\mathbf{Z}$ over $\mathbf{Z}$.
(2) The length of $\mathbf{C}[x]/(x^{100} + x + 1)$ over $\mathbf{C}[x]$.

(3) The length of $\mathbf{R}[x]/(x^4 + 2x^2 + 1)$ over $\mathbf{R}[x]$.

Exercise 79.8.4. Let $A = k[x, y]_{(x,y)}$ be the local ring of the affine plane at the origin. Make any assumption you like about the field k. Suppose that $f = x^3 + x^2y^2 + y^{100}$ and $g = y^3 - x^{999}$. What is the length of $A/(f, g)$ as an A-module? (Possible way to proceed: think about the ideal that f and g generate in quotients of the form $A/\mathfrak{m}_A^n = k[x, y]/(x, y)^n$ for varying n. Try to find n such that $A/(f, g) + \mathfrak{m}_A^n \cong A/(f, g) + \mathfrak{m}_A^{n+1}$ and use NAK.)

79.9. Singularities

Exercise 79.9.1. Let k be any field. Suppose that $A = k[[x, y]]/(f)$ and $B = k[[u, v]]/(g)$, where $f = xy$ and $g = uv + \delta$ with $\delta \in (u, v)^3$. Show that A and B are isomorphic rings.

Remark 79.9.2. A singularity on a curve over a field k is called an ordinary double point if the complete local ring of the curve at the point is of the form $k'[[x, y]]/(f)$, where (a) k' is a finite separable extension of k, (b) the initial term of f has degree two, i.e., it looks like $q = ax^2 + bxy + cy^2$ for some $a, b, c \in k'$ not all zero, and (c) q is a nondegenerate quadratic form over k' (in char 2 this means that b is not zero). In general there is one isomorphism class of such rings for each isomorphism class of pairs (k', q).

79.10. Hilbert Nullstellensatz

Exercise 79.10.1. *A silly argument using the complex numbers!* Let $\mathbf{C}$ be the complex number field. Let V be a vector space over $\mathbf{C}$. The spectrum of a linear operator $T : V \to V$ is the set of complex numbers $\lambda \in \mathbf{C}$ such that the operator $T - \lambda \mathrm{id}_V$ is not invertible.

(1) Show that $\mathbf{C}(X) = f.f.(\mathbf{C}[X])$ has uncountable dimension over $\mathbf{C}$.
(2) Show that any linear operator on V has a nonempty spectrum if the dimension of V is finite or countable.
(3) Show that if a finitely generated $\mathbf{C}$-algebra R is a field, then the map $\mathbf{C} \to R$ is an isomorphism.
(4) Show that any maximal ideal $\mathfrak{m}$ of $\mathbf{C}[x_1, \ldots, x_n]$ is of the form $(x_1 - \alpha_1, \ldots, x_n - \alpha_n)$ for some $\alpha_i \in \mathbf{C}$.

Remark 79.10.2. Let k be a field. Then for every integer $n \in \mathbf{N}$ and every maximal ideal $\mathfrak{m} \subset k[x_1, \ldots, x_n]$ the quotient $k[x_1, \ldots, x_n]/\mathfrak{m}$ is a finite field extension of k. This will be shown later in the course. Of course (please check this) it implies a similar statement for maximal ideals of finitely generated k-algebras. The exercise above proves it in the case $k = \mathbf{C}$.

Exercise 79.10.3. Let k be a field. Please use Remark 79.10.2.

(1) Let R be a k-algebra. Suppose that $\dim_k R < \infty$ and that R is a domain. Show that R is a field.
(2) Suppose that R is a finitely generated k-algebra, and $f \in R$ not nilpotent. Show that there exists a maximal ideal $\mathfrak{m} \subset R$ with $f \notin \mathfrak{m}$.
(3) Show by an example that this statement fails when R is not of finite type over a field.
(4) Show that any radical ideal $I \subset \mathbf{C}[x_1, \ldots, x_n]$ is the intersection of the maximal ideals containing it.

Remark 79.10.4. This is the Hilbert Nullstellensatz. Namely it says that the closed subsets of $\mathrm{Spec}(k[x_1, \ldots, x_n])$ (which correspond to radical ideals by a previous exercise) are determined by the closed points contained in them.

Exercise 79.10.5. Let $A = \mathbf{C}[x_{11}, x_{12}, x_{21}, x_{22}, y_{11}, y_{12}, y_{21}, y_{22}]$. Let I be the ideal of A generated by the entries of the matrix XY, with

$$X = \begin{pmatrix} x_{11} & x_{12} \\ x_{21} & x_{22} \end{pmatrix} \quad \text{and} \quad Y = \begin{pmatrix} y_{11} & y_{12} \\ y_{21} & y_{22} \end{pmatrix}.$$

Find the irreducible components of the closed subset $V(I)$ of $\mathrm{Spec}(A)$. (I mean describe them and give equations for each of them. You do not have to prove that the equations you write down define prime ideals.) Hints:

(1) You may use the Hilbert Nullstellensatz, and it suffices to find irreducible locally closed subsets which cover the set of closed points of $V(I)$.
(2) There are two easy components.
(3) An image of an irreducible set under a continuous map is irreducible.

79.11. Dimension

Exercise 79.11.1. Construct a ring A with finitely many prime ideals having dimension > 1.

Exercise 79.11.2. Let $f \in \mathbf{C}[x, y]$ be a nonconstant polynomial. Show that $\mathbf{C}[x, y]/(f)$ has dimension 1.

Exercise 79.11.3. Let $(R, \mathfrak{m})$ be a Noetherian local ring. Let $n \geq 1$. Let $\mathfrak{m}' = (\mathfrak{m}, x_1, \ldots, x_n)$ in the polynomial ring $R[x_1, \ldots, x_n]$. Show that

$$\dim(R[x_1, \ldots, x_n]_{\mathfrak{m}'}) = \dim(R) + n.$$

79.12. Catenary rings

Definition 79.12.1. A Noetherian ring A is said to be *catenary* if for any triple of prime ideals $\mathfrak{p}_1 \subset \mathfrak{p}_2 \subset \mathfrak{p}_3$ we have

$$ht(\mathfrak{p}_3/\mathfrak{p}_1) = ht(\mathfrak{p}_3/\mathfrak{p}_2) + ht(\mathfrak{p}_2/\mathfrak{p}_1).$$

Here $ht(\mathfrak{p}/\mathfrak{q})$ means the height of $\mathfrak{p}/\mathfrak{q}$ in the ring $A/\mathfrak{q}$.

Exercise 79.12.2. Show that a Noetherian local domain of dimension 2 is catenary.

Exercise 79.12.3. Let k be a field. Show that a finite type k-algebra is catenary.

79.13. Fraction fields

Exercise 79.13.1. Consider the domain

$$\mathbf{Q}[r, s, t]/(s^2 - (r-1)(r-2)(r-3), t^2 - (r+1)(r+2)(r+3)).$$

Find a domain of the form $\mathbf{Q}[x, y]/(f)$ with isomorphic field of fractions.

79.14. Transcendence degree

Exercise 79.14.1. Let $k \subset K \subset K'$ be field extensions with K' algebraic over K. Prove that $\mathrm{trdeg}_k(K) = \mathrm{trdeg}_k(K')$. (Hint: Show that if $x_1, \ldots, x_d \in K$ are algebraically independent over k and $d < \mathrm{trdeg}_k(K')$ then $k(x_1, \ldots, x_d) \subset K$ cannot be algebraic.)

79.15. Finite locally free modules

Definition 79.15.1. Let A be a ring. Recall that a *finite locally free* A-module M is a module such that for every $\mathfrak{p} \in \mathrm{Spec}(A)$ there exists an $f \in A$, $f \notin \mathfrak{p}$ such that M_f is a finite free A_f-module. We say M is an *invertible module* if M is finite locally free of rank 1, i.e., for every $\mathfrak{p} \in \mathrm{Spec}(A)$ there exists an $f \in A$, $f \notin \mathfrak{p}$ such that $M_f \cong A_f$ as an A_f-module.

Exercise 79.15.2. Prove that the tensor product of finite locally free modules is finite locally free. Prove that the tensor product of two invertible modules is invertible.

Definition 79.15.3. Let A be a ring. The *class group of* A, sometimes called the *Picard group of* A is the set $\mathrm{Pic}(A)$ of isomorphism classes of invertible A-modules endowed with a group operation defined by tensor product (see Exercise 79.15.2).

Note that the class group of A is trivial exactly when every invertible module is isomorphic to a free module of rank 1.

Exercise 79.15.4. Show that the class groups of the following rings are trivial

(1) a polynomial ring $A = k[x]$ where k is a field,
(2) the integers $A = \mathbf{Z}$,
(3) a polynomial ring $A = k[x, y]$ where k is a field, and
(4) the quotient $k[x, y]/(xy)$ where k is a field.

Exercise 79.15.5. Show that the class group of the ring $A = k[x, y]/(y^2 - f(x))$ where k is a field of characteristic not 2 and where $f(x) = (x - t_1) \ldots (x - t_n)$ with $t_1, \ldots, t_n \in k$ distinct and $n \geq 3$ an odd integer is not trivial. (Hint: Show that the ideal $(y, x - t_1)$ defines a nontrivial element of $\mathrm{Pic}(A)$.)

Exercise 79.15.6. Let A be a ring.

(1) Suppose that M is a finite locally free A-module, and suppose that $\varphi : M \to M$ is an endomorphism. Define/construct the *trace* and *determinant* of φ and prove that your construction is "functorial in the triple (A, M, φ)".
(2) Show that if M, N are finite locally free A-modules, and if $\varphi : M \to N$ and $\psi : N \to M$ then $Trace(\varphi \circ \psi) = Trace(\psi \circ \varphi)$ and $Det(\varphi \circ \psi) = Det(\psi \circ \varphi)$.
(3) In case M is finite locally free show that Det defines a multiplicative map $End_A(M) \to A$.

Exercise 79.15.7. Now suppose that B is an A-algebra which is finite locally free as an A-module, in other words B is a finite locally free A-algebra.

(1) Define $Trace_{B/A}$ and $Norm_{B/A}$ using $Trace$ and Det as defined above.
(2) Let $b \in B$ and let $\pi : \mathrm{Spec}(B) \to \mathrm{Spec}(A)$ be the induced morphism. Show that $\pi(V(b)) = V(Norm_{B/A}(b))$. (Recall that $V(f) = \{\mathfrak{p} \mid f \in \mathfrak{p}\}$.)
(3) (Base change.) Suppose that $i : A \to A'$ is a ring map. Set $B' = B \otimes_A A'$. Indicate why $i(Norm_{B/A}(b))$ equals $Norm_{B'/A'}(b \otimes 1)$.
(4) Compute $Norm_{B/A}(b)$ when $B = A \times A \times A \times \ldots \times A$ and $b = (a_1, \ldots, a_n)$.
(5) Compute the norm of $y - y^3$ under the finite flat map $\mathbf{Q}[x] \to \mathbf{Q}[y]$, $x \to y^n$. (Hint: use the "base change" $A = \mathbf{Q}[x] \subset A' = \mathbf{Q}(\zeta_n)(x^{1/n})$.)

79.16. Glueing

Exercise 79.16.1. Suppose that A is a ring and M is an A-module. Let f_i, $i \in I$ be a collection of elements of A such that

$$\mathrm{Spec}(A) = \bigcup D(f_i).$$

(1) Show that if M_{f_i} is a finite A_{f_i}-module, then M is a finite A-module.
(2) Show that if M_{f_i} is a flat A_{f_i}-module, then M is a flat A-module. (This is kind of silly if you think about it right.)

Remark 79.16.2. In algebraic geometric language this means that the property of "being finitely generated" or "being flat" is local for the Zariski topology (in a suitable sense). You can also show this for the property "being of finite presentation".

Exercise 79.16.3. Suppose that $A \to B$ is a ring map. Let $f_i \in A$, $i \in I$ and $g_j \in B$, $j \in J$ be collections of elements such that

$$\mathrm{Spec}(A) = \bigcup D(f_i) \quad \text{and} \quad \mathrm{Spec}(B) = \bigcup D(g_j).$$

Show that if $A_{f_i} \to B_{f_i g_j}$ is of finite type for all i, j then $A \to B$ is of finite type.

79.17. Going up and going down

Definition 79.17.1. Let $\phi : A \to B$ be a homomorphism of rings. We say that the *going-up theorem* holds for ϕ if the following condition is satisfied:

(GU) for any $\mathfrak{p}, \mathfrak{p}' \in \mathrm{Spec}(A)$ such that $\mathfrak{p} \subset \mathfrak{p}'$, and for any $P \in \mathrm{Spec}(B)$ lying over $\mathfrak{p}$, there exists $P' \in \mathrm{Spec}(B)$ lying over $\mathfrak{p}'$ such that $P \subset P'$.

Similarly, we say that the *going-down theorem* holds for ϕ if the following condition is satisfied:

(GD) for any $\mathfrak{p}, \mathfrak{p}' \in \mathrm{Spec}(A)$ such that $\mathfrak{p} \subset \mathfrak{p}'$, and for any $P' \in \mathrm{Spec}(B)$ lying over $\mathfrak{p}'$, there exists $P \in \mathrm{Spec}(B)$ lying over $\mathfrak{p}$ such that $P \subset P'$.

Exercise 79.17.2. In each of the following cases determine whether (GU), (GD) holds, and explain why. (Use any Prop/Thm/Lemma you can find, but check the hypotheses in each case.)

(1) k is a field, $A = k$, $B = k[x]$.
(2) k is a field, $A = k[x]$, $B = k[x, y]$.
(3) $A = \mathbf{Z}$, $B = \mathbf{Z}[1/11]$.
(4) k is an algebraically closed field, $A = k[x, y]$, $B = k[x, y, z]/(x^2 - y, z^2 - x)$.
(5) $A = \mathbf{Z}$, $B = \mathbf{Z}[i, 1/(2 + i)]$.
(6) $A = \mathbf{Z}$, $B = \mathbf{Z}[i, 1/(14 + 7i)]$.
(7) k is an algebraically closed field, $A = k[x]$, $B = k[x, y, 1/(xy - 1)]/(y^2 - y)$.

Exercise 79.17.3. Let k be an algebraically closed field. Compute the image in $\mathrm{Spec}(k[x, y])$ of the following maps:

(1) $\mathrm{Spec}(k[x, yx^{-1}]) \to \mathrm{Spec}(k[x, y])$, where $k[x, y] \subset k[x, yx^{-1}] \subset k[x, y, x^{-1}]$. (Hint: To avoid confusion, give the element yx^{-1} another name.)
(2) $\mathrm{Spec}(k[x, y, a, b]/(ax - by - 1)) \to \mathrm{Spec}(k[x, y])$.
(3) $\mathrm{Spec}(k[t, 1/(t - 1)]) \to \mathrm{Spec}(k[x, y])$, induced by $x \mapsto t^2$, and $y \mapsto t^3$.
(4) $k = \mathbf{C}$ (complex numbers), $\mathrm{Spec}(k[s, t]/(s^3 + t^3 - 1)) \to \mathrm{Spec}(k[x, y])$, where $x \mapsto s^2$, $y \mapsto t^2$.

Remark 79.17.4. Finding the image as above usually is done by using elimination theory.

79.18. Fitting ideals

Exercise 79.18.1. Let R be a ring and let M be a finite R-module. Choose a presentation

$$\bigoplus\nolimits_{j\in J} R \longrightarrow R^{\oplus n} \longrightarrow M \longrightarrow 0.$$

of M. Note that the map $R^{\oplus n} \to M$ is given by a sequence of elements $x_1, \ldots, x_n$ of M. The elements x_i are *generators* of M. The map $\bigoplus_{j\in J} R \to R^{\oplus n}$ is given by a $n \times J$ matrix A with coefficients in R. In other words, $A = (a_{ij})_{i=1,\ldots,n, j\in J}$. The columns $(a_{1j}, \ldots, a_{nj})$, $j \in J$ of A are said to be the *relations*. Any vector $(r_i) \in R^{\oplus n}$ such that $\sum r_i x_i = 0$ is a linear combination of the columns of A. Of course any finite R-module has a lot of different presentations.

(1) Show that the ideal generated by the $(n-k) \times (n-k)$ minors of A is independent of the choice of the presentation. This ideal is the *kth fitting ideal of M*. Notation $Fit_k(M)$.
(2) Show that $Fit_0(M) \subset Fit_1(M) \subset Fit_2(M) \subset \ldots$. (Hint: Use that a determinant can be computed by expanding along a column.)
(3) Show that the following are equivalent:
   (a) $Fit_{r-1}(M) = (0)$ and $Fit_r(M) = R$, and
   (b) M is locally free of rank r.

79.19. Hilbert functions

Definition 79.19.1. A *numerical polynomial* is a polynomial $f(x) \in \mathbf{Q}[x]$ such that $f(n) \in \mathbf{Z}$ for every integer n.

Definition 79.19.2. A *graded module* M over a ring A is an A-module M endowed with a direct sum decomposition $\bigoplus_{n\in\mathbf{Z}} M_n$ into A-submodules. We will say that M is *locally finite* if all of the M_n are finite A-modules. Suppose that A is a Noetherian ring and that φ is a *Euler-Poincaré function* on finite A-modules. This means that for every finitely generated A-module M we are given an integer $\varphi(M) \in \mathbf{Z}$ and for every short exact sequence

$$0 \longrightarrow M' \longrightarrow M \longrightarrow M'' \longrightarrow 0$$

we have $\varphi(M) = \varphi(M') + \varphi(M')$. The *Hilbert function* of a locally finite graded module M (with respect to φ) is the function $\chi_\varphi(M, n) = \varphi(M_n)$. We say that M has a *Hilbert polynomial* if there is some numerical polynomial P_φ such that $\chi_\varphi(M, n) = P_\varphi(n)$ for all sufficiently large integers n.

Definition 79.19.3. A *graded A-algebra* is a graded A-module $B = \bigoplus_{n\geq 0} B_n$ together with an A-bilinear map

$$B \times B \longrightarrow B,\ (b, b') \longmapsto bb'$$

that turns B into an A-algebra so that $B_n \cdot B_m \subset B_{n+m}$. Finally, a *graded module M over a graded A-algebra B* is given by a graded A-module M together with a (compatible) B-module structure such that $B_n \cdot M_d \subset M_{n+d}$. Now you can define *homomorphisms of graded modules/rings*, *graded submodules*, *graded ideals*, *exact sequences of graded modules*, etc, etc.

Exercise 79.19.4. Let $A = k$ a field. What are all possible Euler-Poincaré functions on finite A-modules in this case?

Exercise 79.19.5. Let $A = \mathbf{Z}$. What are all possible Euler-Poincaré functions on finite A-modules in this case?

Exercise 79.19.6. Let $A = k[x, y]/(xy)$ with k algebraically closed. What are all possible Euler-Poincaré functions on finite A-modules in this case?

Exercise 79.19.7. Suppose that A is Noetherian. Show that the kernel of a map of locally finite graded A-modules is locally finite.

Exercise 79.19.8. Let k be a field and let $A = k$ and $B = k[x, y]$ with grading determined by $\deg(x) = 2$ and $\deg(y) = 3$. Let $\varphi(M) = \dim_k(M)$. Compute the Hilbert function of B as a graded k-module. Is there a Hilbert polynomial in this case?

Exercise 79.19.9. Let k be a field and let $A = k$ and $B = k[x, y]/(x^2, xy)$ with grading determined by $\deg(x) = 2$ and $\deg(y) = 3$. Let $\varphi(M) = \dim_k(M)$. Compute the Hilbert function of B as a graded k-module. Is there a Hilbert polynomial in this case?

Exercise 79.19.10. Let k be a field and let $A = k$. Let $\varphi(M) = \dim_k(M)$. Fix $d \in \mathbf{N}$. Consider the graded A-algebra $B = k[x, y, z]/(x^d + y^d + z^d)$, where x, y, z each have degree 1. Compute the Hilbert function of B. Is there a Hilbert polynomial in this case?

79.20. Proj of a ring

Definition 79.20.1. Let R be a graded ring. A *homogeneous* ideal is simply an ideal $I \subset R$ which is also a graded submodule of R. Equivalently, it is an ideal generated by homogeneous elements. Equivalently, if $f \in I$ and

$$f = f_0 + f_1 + \ldots + f_n$$

is the decomposition of f into homogeneous pieces in R then $f_i \in I$ for each i.

Definition 79.20.2. We define the *homogeneous spectrum* $Proj(R)$ of the graded ring R to be the set of homogeneous, prime ideals $\mathfrak{p}$ of R such that $R_+ \not\subset \mathfrak{p}$. Note that $\mathrm{Proj}(R)$ is a subset of $\mathrm{Spec}(R)$ and hence has a natural induced topology.

Definition 79.20.3. Let $R = \oplus_{d \geq 0} R_d$ be a graded ring, let $f \in R_d$ and assume that $d \geq 1$. We define $R_{(f)}$ to be the subring of R_f consisting of elements of the form r/f^n with r homogeneous and $\deg(r) = nd$. Furthermore, we define

$$D_+(f) = \{\mathfrak{p} \in \mathrm{Proj}(R) | f \notin \mathfrak{p}\}.$$

Finally, for a homogeneous ideal $I \subset R$ we define $V_+(I) = V(I) \cap \mathrm{Proj}(R)$.

Exercise 79.20.4. On the topology on $\mathrm{Proj}(R)$. With definitions and notation as above prove the following statements.

(1) Show that $D_+(f)$ is open in $\mathrm{Proj}(R)$.
(2) Show that $D_+(ff') = D_+(f) \cap D_+(f')$.
(3) Let $g = g_0 + \ldots + g_m$ be an element of R with $g_i \in R_i$. Express $D(g) \cap \mathrm{Proj}(R)$ in terms of $D_+(g_i)$, $i \geq 1$ and $D(g_0) \cap \mathrm{Proj}(R)$. No proof necessary.

(4) Let $g \in R_0$ be a homogeneous element of degree 0. Express $D(g) \cap \text{Proj}(R)$ in terms of $D_+(f_\alpha)$ for a suitable family $f_\alpha \in R$ of homogeneous elements of positive degree.
(5) Show that the collection $\{D_+(f)\}$ of opens forms a basis for the topology of $\text{Proj}(R)$.
(6) Show that there is a canonical bijection $D_+(f) \to \text{Spec}(R_{(f)})$. (Hint: Imitate the proof for Spec but at some point thrown in the radical of an ideal.)
(7) Show that the map from (6) is a homeomorphism.
(8) Give an example of an R such that $\text{Proj}(R)$ is not quasi-compact. No proof necessary.
(9) Show that any closed subset $T \subset \text{Proj}(R)$ is of the form $V_+(I)$ for some homogeneous ideal $I \subset R$.

Remark 79.20.5. There is a continuous map $\text{Proj}(R) \longrightarrow \text{Spec}(R_0)$.

Exercise 79.20.6. If $R = A[X]$ with $\deg(X) = 1$, show that the natural map $\text{Proj}(R) \to \text{Spec}(A)$ is a bijection and in fact a homeomorphism.

Exercise 79.20.7. Blowing up: part I. In this exercise $R = Bl_I(A) = A \oplus I \oplus I^2 \oplus \ldots$. Consider the natural map $b : \text{Proj}(R) \to \text{Spec}(A)$. Set $U = \text{Spec}(A) - V(I)$. Show that

$$b : b^{-1}(U) \longrightarrow U$$

is a homeomorphism. Thus we may think of U as an open subset of $\text{Proj}(R)$. Let $Z \subset \text{Spec}(A)$ be an irreducible closed subscheme with generic point $\xi \in Z$. Assume that $\xi \notin V(I)$, in other words $Z \not\subset V(I)$, in other words $\xi \in U$, in other words $Z \cap U \neq \emptyset$. We define the *strict transform* Z' of Z to be the closure of the unique point ξ' lying above ξ. Another way to say this is that Z' is the closure in $\text{Proj}(R)$ of the locally closed subset $Z \cap U \subset U \subset \text{Proj}(R)$.

Exercise 79.20.8. Blowing up: Part II. Let $A = k[x, y]$ where k is a field, and let $I = (x, y)$. Let R be the blow up algebra for A and I.

(1) Show that the strict transforms of $Z_1 = V(\{x\})$ and $Z_2 = V(\{y\})$ are disjoint.
(2) Show that the strict transforms of $Z_1 = V(\{x\})$ and $Z_2 = V(\{x - y^2\})$ are not disjoint.
(3) Find an ideal $J \subset A$ such that $V(J) = V(I)$ and such that the strict transforms of $Z_1 = V(\{x\})$ and $Z_2 = V(\{x - y^2\})$ are disjoint.

Exercise 79.20.9. Let R be a graded ring.

(1) Show that $\text{Proj}(R)$ is empty if $R_n = (0)$ for all $n >> 0$.
(2) Show that $\text{Proj}(R)$ is an irreducible topological space if R is a domain and R_+ is not zero. (Recall that the empty topological space is not irreducible.)

Exercise 79.20.10. Blowing up: Part III. Consider A, I and U, Z as in the definition of strict transform. Let $Z = V(\mathfrak{p})$ for some prime ideal $\mathfrak{p}$. Let $\bar{A} = A/\mathfrak{p}$ and let $\bar{I}$ be the image of I in $\bar{A}$.

(1) Show that there exists a surjective ring map $R := Bl_I(A) \to \bar{R} := Bl_{\bar{I}}(\bar{A})$.
(2) Show that the ring map above induces a bijective map from $\text{Proj}(\bar{R})$ onto the strict transform Z' of Z. (This is not so easy. Hint: Use 5(b) above.)

(3) Conclude that the strict transform $Z' = V_+(P)$ where $P \subset R$ is the homogeneous ideal defined by $P_d = I^d \cap \mathfrak{p}$.
(4) Suppose that $Z_1 = V(\mathfrak{p})$ and $Z_2 = V(\mathfrak{q})$ are irreducible closed subsets defined by prime ideals such that $Z_1 \not\subset Z_2$, and $Z_2 \not\subset Z_1$. Show that blowing up the ideal $I = \mathfrak{p} + \mathfrak{q}$ separates the strict transforms of Z_1 and Z_2, i.e., $Z_1' \cap Z_2' = \emptyset$. (Hint: Consider the homogeneous ideal P and Q from part (c) and consider $V(P + Q)$.)

79.21. Cohen-Macaulay rings of dimension 1

Definition 79.21.1. A Noetherian local ring A is said to be *Cohen-Macaulay* of dimension d if it has dimension d and there exists a system of parameters $x_1, \ldots, x_d$ for A such that x_i is a nonzerodivisor in $A/(x_1, \ldots, x_{i-1})$ for $i = 1, \ldots, d$.

Exercise 79.21.2. Cohen-Macaulay rings of dimension 1. Part I: Theory.

(1) Let $(A, \mathfrak{m})$ be a local Noetherian with $\dim A = 1$. Show that if $x \in \mathfrak{m}$ is not a zerodivisor then
   (a) $\dim A/xA = 0$, in other words A/xA is Artinian, in other words $\{x\}$ is a system of parameters for A.
   (b) A is has no embedded prime.
(2) Conversely, let $(A, \mathfrak{m})$ be a local Noetherian ring of dimension 1. Show that if A has no embedded prime then there exists a nonzerodivisor in $\mathfrak{m}$.

Exercise 79.21.3. Cohen-Macaulay rings of dimension 1. Part II: Examples.

(1) Let A be the local ring at (x, y) of $k[x, y]/(x^2, xy)$.
   (a) Show that A has dimension 1.
   (b) Prove that every element of $\mathfrak{m} \subset A$ is a zerodivisor.
   (c) Find $z \in \mathfrak{m}$ such that $\dim A/zA = 0$ (no proof required).
(2) Let A be the local ring at (x, y) of $k[x, y]/(x^2)$. Find a nonzerodivisor in $\mathfrak{m}$ (no proof required).

Exercise 79.21.4. Local rings of embedding dimension 1. Suppose that $(A, \mathfrak{m}, k)$ is a Noetherian local ring of embedding dimension 1, i.e.,

$$\dim_k \mathfrak{m}/\mathfrak{m}^2 = 1.$$

Show that the function $f(n) = \dim_k \mathfrak{m}^n/\mathfrak{m}^{n+1}$ is either constant with value 1, or its values are

$$1, 1, \ldots, 1, 0, 0, 0, 0, 0, \ldots$$

Exercise 79.21.5. Regular local rings of dimension 1. Suppose that $(A, \mathfrak{m}, k)$ is a regular Noetherian local ring of dimension 1. Recall that this means that A has dimension 1 and embedding dimension 1, i.e.,

$$\dim_k \mathfrak{m}/\mathfrak{m}^2 = 1.$$

Let $x \in \mathfrak{m}$ be any element whose class in $\mathfrak{m}/\mathfrak{m}^2$ is not zero.

(1) Show that for every element y of $\mathfrak{m}$ there exists an integer n such that y can be written as $y = ux^n$ with $u \in A^*$ a unit.
(2) Show that x is a nonzerodivisor in A.
(3) Conclude that A is a domain.

Exercise 79.21.6. Let $(A, \mathfrak{m}, k)$ be a Noetherian local ring with associated graded $Gr_{\mathfrak{m}}(A)$.

(1) Suppose that $x \in \mathfrak{m}^d$ maps to a nonzero divisor $\bar{x} \in \mathfrak{m}^d/\mathfrak{m}^{d+1}$ in degree d of $Gr_{\mathfrak{m}}(A)$. Show that x is a nonzerodivisor.
(2) Suppose the depth of A is at least 1. Namely, suppose that there exists a nonzerodivisor $y \in \mathfrak{m}$. In this case we can do better: assume just that $x \in \mathfrak{m}^d$ maps to the element $\bar{x} \in \mathfrak{m}^d/\mathfrak{m}^{d+1}$ in degree d of $Gr_{\mathfrak{m}}(A)$ which is a nonzerodivisor on sufficiently high degrees: $\exists N$ such that for all $n \geq N$ the map of multiplication by $\bar{x}$

$$\mathfrak{m}^n/\mathfrak{m}^{n+1} \longrightarrow \mathfrak{m}^{n+d}/\mathfrak{m}^{n+d+1}$$

is injective. Then show that x is a nonzerodivisor.

Exercise 79.21.7. Suppose that $(A, \mathfrak{m}, k)$ is a Noetherian local ring of dimension 1. Assume also that the embedding dimension of A is 2, i.e., assume that

$$\dim_k \mathfrak{m}/\mathfrak{m}^2 = 2.$$

Notation: $f(n) = \dim_k \mathfrak{m}^n/\mathfrak{m}^{n+1}$. Pick generators $x, y \in \mathfrak{m}$ and write $Gr_{\mathfrak{m}}(A) = k[\bar{x}, \bar{y}]/I$ for some homogeneous ideal I.

(1) Show that there exists a homogeneous element $F \in k[\bar{x}, \bar{y}]$ such that $I \subset (F)$ with equality in all sufficiently high degrees.
(2) Show that $f(n) \leq n + 1$.
(3) Show that if $f(n) < n + 1$ then $n \geq \deg(F)$.
(4) Show that if $f(n) < n + 1$, then $f(n + 1) \leq f(n)$.
(5) Show that $f(n) = \deg(F)$ for all $n >> 0$.

Exercise 79.21.8. Cohen-Macaulay rings of dimension 1 and embedding dimension 2. Suppose that $(A, \mathfrak{m}, k)$ is a Noetherian local ring which is Cohen-Macaulay of dimension 1. Assume also that the embedding dimension of A is 2, i.e., assume that

$$\dim_k \mathfrak{m}/\mathfrak{m}^2 = 2.$$

Notations: f, F, $x, y \in \mathfrak{m}$, I as in Ex. 6 above. Please use any results from the problems above.

(1) Suppose that $z \in \mathfrak{m}$ is an element whose class in $\mathfrak{m}/\mathfrak{m}^2$ is a linear form $\alpha\bar{x} + \beta\bar{y} \in k[\bar{x}, \bar{y}]$ which is coprime with f.
   (a) Show that z is a nonzerodivisor on A.
   (b) Let $d = \deg(F)$. Show that $\mathfrak{m}^n = z^{n+1-d}\mathfrak{m}^{d-1}$ for all sufficiently large n. (Hint: First show $z^{n+1-d}\mathfrak{m}^{d-1} \to \mathfrak{m}^n/\mathfrak{m}^{n+1}$ is surjective by what you know about $Gr_{\mathfrak{m}}(A)$. Then use NAK.)
(2) What condition on k guarantees the existence of such a z? (No proof required; it's too easy.)
   Now we are going to assume there exists a z as above. This turns out to be a harmless assumption (in the sense that you can reduce to the situation where it holds in order to obtain the results in parts (d) and (e) below).
(3) Now show that $\mathfrak{m}^\ell = z^{\ell-d+1}\mathfrak{m}^{d-1}$ for all $\ell \geq d$.
(4) Conclude that $I = (F)$.
(5) Conclude that the function f has values

$$2, 3, 4, \ldots, d-1, d, d, d, d, d, d, d, \ldots$$

Remark 79.21.9. This suggests that a local Noetherian Cohen-Macaulay ring of dimension 1 and embedding dimension 2 is of the form B/FB, where B is a 2-dimensional regular local ring. This is more or less true (under suitable "niceness" properties of the ring).

79.22. Infinitely many primes

A section with a collection of strange questions on rings where infinitely many primes are not invertible.

Exercise 79.22.1. Give an example of a finite type $\mathbf{Z}$-algebra R with the following two properties:
(a) There is no ring map $R \to \mathbf{Q}$.
(b) For every prime p there exists a maximal ideal $\mathfrak{m} \subset R$ such that $R/\mathfrak{m} \cong \mathbf{F}_p$.

Exercise 79.22.2. For $f \in \mathbf{Z}[x, u]$ we define $f_p(x) = f(x, x^p) \bmod p \in \mathbf{F}_p[x]$. Give an example of an $f \in \mathbf{Z}[x, u]$ such that the following two properties hold:
(a) There exist infinitely many p such that f_p does not have a zero in $\mathbf{F}_p$.
(b) For all $p >> 0$ the polynomial f_p either has a linear or a quadratic factor.

Exercise 79.22.3. For $f \in \mathbf{Z}[x, y, u, v]$ we define $f_p(x, y) = f(x, y, x^p, y^p) \bmod p \in \mathbf{F}_p[x, y]$. Give an "interesting" example of an f such that f_p is reducible for all $p >> 0$. For example, $f = xv - yu$ with $f_p = xy^p - x^p y = xy(x^{p-1} - y^{p-1})$ is "uninteresting"; any f depending only on x, u is "uninteresting", etc.

Remark 79.22.4. Let $h \in \mathbf{Z}[y]$ be a monic polynomial of degree d. Then:

(1) The map $A = \mathbf{Z}[x] \to B = \mathbf{Z}[y]$, $x \mapsto h$ is finite locally free of rank d.
(2) For all primes p the map $A_p = \mathbf{F}_p[x] \to B_p = \mathbf{F}_p[y]$, $y \mapsto h(y) \bmod p$ is finite locally free of rank d.

Exercise 79.22.5. Let h, A, B, A_p, B_p be as in the remark. For $f \in \mathbf{Z}[x, u]$ we define $f_p(x) = f(x, x^p) \bmod p \in \mathbf{F}_p[x]$. For $g \in \mathbf{Z}[y, v]$ we define $g_p(y) = g(y, y^p) \bmod p \in \mathbf{F}_p[y]$.

(1) Give an example of a h and g such that there does not exist a f with the property
$$f_p = Norm_{B_p/A_p}(g_p).$$
(2) Show that for any choice of h and g as above there exists a nonzero f such that for all p we have
$$Norm_{B_p/A_p}(g_p) \text{ divides } f_p.$$
If you want you can restrict to the case $h = y^n$, even with $n = 2$, but it is true in general.
(3) Discuss the relevance of this to Exercises 6 & 7 of the previous set.

Exercise 79.22.6. Unsolved problems. They may be really hard or they may be easy. I don't know.

(1) Is there any $f \in \mathbf{Z}[x, u]$ such that f_p is irreducible for an infinite number of p? (Hint: Yes, this happens for $f(x, u) = u - x - 1$ and also for $f(x, u) = u^2 - x^2 + 1$.)

(2) Let $f \in \mathbf{Z}[x, u]$ nonzero, and suppose $\deg_x(f_p) = dp + d'$ for all large p. (In other words $\deg_u(f) = d$ and the coefficient c of u^d in f has $\deg_x(c) = d'$.) Suppose we can write $d = d_1 + d_2$ and $d' = d'_1 + d'_2$ with $d_1, d_2 > 0$ and $d'_1, d'_2 \geq 0$ such that for all sufficiently large p there exists a factorization

$$f_p = f_{1,p} f_{2,p}$$

with $\deg_x(f_{1,p}) = d_1 p + d'_1$. Is it true that f comes about via a norm construction as in Exercise 4? (More precisely, are there a h and g such that $Norm_{B_p/A_p}(g_p)$ divides f_p for all $p >> 0$.)

(3) Analogous question to the one in (b) but now with $f \in \mathbf{Z}[x_1, x_2, u_1, u_2]$ irreducible and just assuming that $f_p(x_1, x_2) = f(x_1, x_2, x_1^p, x_2^p) \bmod p$ factors for all $p >> 0$.

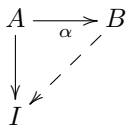
79.23. Filtered derived category

In order to do the exercises in this section, please read the material in Homology, Section 12.16. We will say A is a filtered object of $\mathcal{A}$, to mean that A comes endowed with a filtration F which we omit from the notation.

Exercise 79.23.1. Let $\mathcal{A}$ be an abelian category. Let I be a filtered object of $\mathcal{A}$. Assume that the filtration on I is finite and that each $\mathrm{gr}^p(I)$ is an injective object of $\mathcal{A}$. Show that there exists an isomorphism $I \cong \bigoplus \mathrm{gr}^p(I)$ with filtration $F^p(I)$ corresponding to $\bigoplus_{p' \geq p} \mathrm{gr}^p(I)$.

Exercise 79.23.2. Let $\mathcal{A}$ be an abelian category. Let I be a filtered object of $\mathcal{A}$. Assume that the filtration on I is finite. Show the following are equivalent:

(1) For any solid diagram



of filtered objects with (i) the filtrations on A and B are finite, and (ii) $\mathrm{gr}(\alpha)$ injective the dotted arrow exists making the diagram commute.

(2) Each $\mathrm{gr}^p I$ is injective.

Note that given a morphism $\alpha : A \to B$ of filtered objects with finite filtrations to say that $\mathrm{gr}(\alpha)$ injective is the same thing as saying that α is a *strict monomorphism* in the category $\mathrm{Fil}(\mathcal{A})$. Namely, being a monomorphism means $\mathrm{Ker}(\alpha) = 0$ and strict means that this also implies $\mathrm{Ker}(\mathrm{gr}(\alpha)) = 0$. See Homology, Lemma 12.16.13. (We only use the term "injective" for a morphism in an abelian category, although it makes sense in any additive category having kernels.) The exercises above justifies the following definition.

Definition 79.23.3. Let $\mathcal{A}$ be an abelian category. Let I be a filtered object of $\mathcal{A}$. Assume the filtration on I is finite. We say I is *filtered injective* if each $\mathrm{gr}^p(I)$ is an injective object of $\mathcal{A}$.

We make the following definition to avoid having to keep saying "with a finite filtration" everywhere.

Definition 79.23.4. Let $\mathcal{A}$ be an abelian category. We denote $Fil^f(\mathcal{A})$ the full subcategory of $\mathrm{Fil}(\mathcal{A})$ whose objects consist of those $A \in \mathrm{Ob}(\mathrm{Fil}(\mathcal{A}))$ whose filtration is finite.

Exercise 79.23.5. Let $\mathcal{A}$ be an abelian category. Assume $\mathcal{A}$ has enough injectives. Let A be an object of $\mathrm{Fil}^f(\mathcal{A})$. Show that there exists a strict monomorphism $\alpha : A \to I$ of A into a filtered injective object I of $\mathrm{Fil}^f(\mathcal{A})$.

Definition 79.23.6. Let $\mathcal{A}$ be an abelian category. Let $\alpha : K^\bullet \to L^\bullet$ be a morphism of complexes of $\mathrm{Fil}(\mathcal{A})$. We say that α is a *filtered quasi-isomorphism* if for each $p \in \mathbf{Z}$ the morphism $\mathrm{gr}^p(K^\bullet) \to \mathrm{gr}^p(L^\bullet)$ is a quasi-isomorphism.

Definition 79.23.7. Let $\mathcal{A}$ be an abelian category. Let $K^\bullet$ be a complex of $\mathrm{Fil}^f(\mathcal{A})$. We say that $K^\bullet$ is *filtered acyclic* if for each $p \in \mathbf{Z}$ the complex $\mathrm{gr}^p(K^\bullet)$ is acyclic.

Exercise 79.23.8. Let $\mathcal{A}$ be an abelian category. Let $\alpha : K^\bullet \to L^\bullet$ be a morphism of bounded below complexes of $\mathrm{Fil}^f(\mathcal{A})$. (Note the superscript f.) Show that the following are equivalent:

(1) α is a filtered quasi-isomorphism,
(2) for each $p \in \mathbf{Z}$ the map $\alpha : F^pK^\bullet \to F^pL^\bullet$ is a quasi-isomorphism,
(3) for each $p \in \mathbf{Z}$ the map $\alpha : K^\bullet/F^pK^\bullet \to L^\bullet/F^pL^\bullet$ is a quasi-isomorphism, and
(4) the cone of α (see Derived Categories, Definition 13.8.1) is a filtered acyclic complex.

Moreover, show that if α is a filtered quasi-isomorphism then α is also a usual quasi-isomorphism.

Exercise 79.23.9. Let $\mathcal{A}$ be an abelian category. Assume $\mathcal{A}$ has enough injectives. Let A be an object of $\mathrm{Fil}^f(\mathcal{A})$. Show there exists a complex $I^\bullet$ of $\mathrm{Fil}^f(\mathcal{A})$, and a morphism $A[0] \to I^\bullet$ such that

(1) each I^p is filtered injective,
(2) $I^p = 0$ for $p < 0$, and
(3) $A[0] \to I^\bullet$ is a filtered quasi-isomorphism.

Exercise 79.23.10. Let $\mathcal{A}$ be an abelian category. Assume $\mathcal{A}$ has enough injectives. Let $K^\bullet$ be a bounded below complex of objects of $\mathrm{Fil}^f(\mathcal{A})$. Show there exists a filtered quasi-isomorphism $\alpha : K^\bullet \to I^\bullet$ with $I^\bullet$ a complex of $\mathrm{Fil}^f(\mathcal{A})$ having filtered injective terms I^n, and bounded below. In fact, we may choose α such that each α^n is a strict monomorphism.

Exercise 79.23.11. Let $\mathcal{A}$ be an abelian category. Consider a solid diagram

$$\begin{array}{ccc} K^\bullet & \xrightarrow{\alpha} & L^\bullet \\ {\scriptstyle\gamma}\downarrow & \swarrow_{\beta} & \\ I^\bullet & & \end{array}$$

of complexes of $\mathrm{Fil}^f(\mathcal{A})$. Assume $K^\bullet$, $L^\bullet$ and $I^\bullet$ are bounded below and assume each I^n is a filtered injective object. Also assume that α is a filtered quasi-isomorphism.

(1) There exists a map of complexes β making the diagram commute up to homotopy.

(2) If α is a strict monomorphism in every degree then we can find a β which makes the diagram commute.

Exercise 79.23.12. Let $\mathcal{A}$ be an abelian category. Let $K^\bullet$, $K^\bullet$ be complexes of $\mathrm{Fil}^f(\mathcal{A})$. Assume

(1) $K^\bullet$ bounded below and filtered acyclic, and
(2) $I^\bullet$ bounded below and consisting of filtered injective objects.

Then any morphism $K^\bullet \to I^\bullet$ is homotopic to zero.

Exercise 79.23.13. Let $\mathcal{A}$ be an abelian category. Consider a solid diagram

$$\begin{array}{ccc} K^\bullet & \xrightarrow{\alpha} & L^\bullet \\ {\scriptstyle\gamma}\downarrow & \swarrow_{\beta_i} & \\ I^\bullet & & \end{array}$$

of complexes of $\mathrm{Fil}^f(\mathcal{A})$. Assume $K^\bullet$, $L^\bullet$ and $I^\bullet$ bounded below and each I^n a filtered injective object. Also assume α a filtered quasi-isomorphism. Any two morphisms β_1, β_2 making the diagram commute up to homotopy are homotopic.

79.24. Regular functions

Exercise 79.24.1. In this exercise we try to see what happens with regular functions over non-algebraically closed fields. Let k be a field. Let $Z \subset k^n$ be a Zariski locally closed subset, i.e., there exist ideals $I \subset J \subset k[x_1, \ldots, x_n]$ such that

$$Z = \{a \in k^n \mid f(a) = 0 \ \forall\ f \in I,\ \exists\ g \in J,\ g(a) \neq 0\}.$$

A function $\varphi : Z \to k$ is said to be *regular* if for every $z \in Z$ there exists a Zariski open neighbourhood $z \in U \subset Z$ and polynomials $f, g \in k[x_1, \ldots, x_n]$ such that $g(u) \neq 0$ for all $u \in U$ and such that $\varphi(u) = f(u)/g(u)$ for all $u \in U$.

(1) If $k = \bar{k}$ and $Z = k^n$ show that regular functions are given by polynomials. (Only do this if you haven't seen this argument before.)
(2) If k is finite show that (a) every function φ is regular, (b) the ring of regular functions is finite dimensional over k. (If you like you can take $Z = k^n$ and even $n = 1$.)
(3) If $k = \mathbf{R}$ give an example of a regular function on $Z = \mathbf{R}$ which is not given by a polynomial.
(4) If $k = \mathbf{Q}_p$ give an example of a regular function on $Z = \mathbf{Q}_p$ which is not given by a polynomial.

79.25. Sheaves

A morphism $f : X \to Y$ of a category $\mathcal{C}$ is an *monomorphism* if for every pair of morphisms $a, b : W \to X$ we have $f \circ a = f \circ b \Rightarrow a = b$. A monomorphism in the category of sets is an injective map of sets.

Exercise 79.25.1. Carefully prove that a map of sheaves of sets is an monomorphism (in the category of sheaves of sets) if and only if the induced maps on all the stalks are injective.

A morphism $f : X \to Y$ of a category $\mathcal{C}$ is an *isomorphism* if there exists a morphism $g : Y \to X$ such that $f \circ g = \mathrm{id}_Y$ and $g \circ f = \mathrm{id}_X$. An isomorphism in the category of sets is a bijective map of sets.

Exercise 79.25.2. Carefully prove that a map of sheaves of sets is an isomorphism (in the category of sheaves of sets) if and only if the induced maps on all the stalks are bijective.

A morphism $f : X \to Y$ of a category $\mathcal{C}$ is an *epimorphism* if for every pair of morphisms $a, b : Y \to Z$ we have $a \circ f = b \circ f \Rightarrow a = b$. An epimorphism in the category of sets is a surjective map of sets.

Exercise 79.25.3. Carefully prove that a map of sheaves of sets is an epimorphism (in the category of sheaves of sets) if and only if the induced maps on all the stalks are surjective.

Exercise 79.25.4. Let $f : X \to Y$ be a map of topological spaces. Prove pushforward f_* and pullback f^{-1} for sheaves of **sets** form an adjoint pair of functors.

Exercise 79.25.5. Let $j : U \to X$ be an open immersion. Show that j^{-1} has a left adjoint $j_!$ on the category of sheaves of sets. Characterize the stalks of $j_!(\mathcal{G})$. (Hint: $j_!$ is called extension by zero when you do this for abelian sheaves...)

Exercise 79.25.6. Let $X = \mathbf{R}$ with the usual topology. Let $\mathcal{O}_X = \underline{\mathbf{Z}/2\mathbf{Z}}_X$. Let $i : Z = \{0\} \to X$ be the inclusion and let $\mathcal{O}_Z = \underline{\mathbf{Z}/2\mathbf{Z}}_Z$. Prove the following (the first three follow from the definitions but if you are not clear on the definitions you should elucidate them):

(1) $i_*\mathcal{O}_Z$ is a skyscraper sheaf.
(2) There is a canonical surjective map from $\underline{\mathbf{Z}/2\mathbf{Z}}_X \to i_*\underline{\mathbf{Z}/2\mathbf{Z}}_Z$. Denote the kernel $\mathcal{I} \subset \mathcal{O}_X$.
(3) $\mathcal{I}$ is an ideal sheaf of $\mathcal{O}_X$.
(4) The sheaf $\mathcal{I}$ on X cannot be locally generated by sections (as in Modules, Definition 17.8.1.)

Exercise 79.25.7. Let X be a topological space. Let $\mathcal{F}$ be an abelian sheaf on X. Show that $\mathcal{F}$ is the quotient of a (possibly very large) direct sum of sheaves all of whose terms are of the form

$$j_!(\underline{\mathbf{Z}}_U)$$

where $U \subset X$ is open and $\underline{\mathbf{Z}}_U$ denotes the constant sheaf with value $\mathbf{Z}$ on U.

Remark 79.25.8. Let X be a topological space. In the category of abelian sheaves the direct sum of a family of sheaves $\{\mathcal{F}_i\}_{i\in I}$ is the sheaf associated to the presheaf $U \mapsto \oplus\mathcal{F}_i(U)$. Consequently the stalk of the direct sum at a point x is the direct sum of the stalks of the $\mathcal{F}_i$ at x.

Exercise 79.25.9. Let X be a topological space. Suppose we are given a collection of abelian groups A_x indexed by $x \in X$. Show that the rule $U \mapsto \prod_{x\in U} A_x$ with obvious restriction mappings defines a sheaf $\mathcal{G}$ of abelian groups. Show, by an example, that usually it is not the case that $\mathcal{G}_x = A_x$ for $x \in X$.

Exercise 79.25.10. Let X, A_x, $\mathcal{G}$ be as in Exercise 79.25.9. Let $\mathcal{B}$ be a basis for the topology of X, see Topology, Definition 5.4.1. For $U \in \mathcal{B}$ let A_U be a subgroup $A_U \subset \mathcal{G}(U) = \prod_{x\in U} A_x$. Assume that for $U \subset V$ with $U, V \in \mathcal{B}$ the restriction maps A_V into A_U. For $U \subset X$ open set

$$\mathcal{F}(U) = \left\{ (s_x)_{x\in U} \;\middle|\; \begin{array}{l} \text{for every } x \text{ in } U \text{ there exists } V \in \mathcal{B} \\ x \in V \subset U \text{ such that } (s_y)_{y\in V} \in A_V \end{array} \right\}$$

Show that $\mathcal{F}$ defines a sheaf of abelian groups on X. Show, by an example, that it is usually not the case that $\mathcal{F}(U) = A_U$ for $U \in \mathcal{B}$.

79.26. Schemes

Let LRS be the category of locally ringed spaces. An affine scheme is an object in LRS isomorphic in LRS to a pair of the form $(\mathrm{Spec}(A), \mathcal{O}_{\mathrm{Spec}(A)})$. A scheme is an object $(X, \mathcal{O}_X)$ of LRS such that every point $x \in X$ has an open neighbourhood $U \subset X$ such that the pair $(U, \mathcal{O}_X|_U)$ is an affine scheme.

Exercise 79.26.1. Find a 1-point locally ringed space which is not a scheme.

Exercise 79.26.2. Suppose that X is a scheme whose underlying topological space has 2 points. Show that X is an affine scheme.

Exercise 79.26.3. Suppose that X is a scheme whose underlying topological space is a finite discrete set. Show that X is an affine scheme.

Exercise 79.26.4. Show that there exists a non-affine scheme having three points.

Exercise 79.26.5. Suppose that X is a quasi-compact scheme. Show that X has a closed point.

Remark 79.26.6. When $(X, \mathcal{O}_X)$ is a ringed space and $U \subset X$ is an open subset then $(U, \mathcal{O}_X|_U)$ is a ringed space. Notation: $\mathcal{O}_U = \mathcal{O}_X|_U$. There is a canonical morphisms of ringed spaces

$$j : (U, \mathcal{O}_U) \longrightarrow (X, \mathcal{O}_X).$$

If $(X, \mathcal{O}_X)$ is a locally ringed space, so is $(U, \mathcal{O}_U)$ and j is a morphism of locally ringed spaces. If $(X, \mathcal{O}_X)$ is a scheme so is $(U, \mathcal{O}_U)$ and j is a morphism of schemes. We say that $(U, \mathcal{O}_U)$ is an *open subscheme* of $(X, \mathcal{O}_X)$ and that j is an *open immersion*. More generally, any morphism $j' : (V, \mathcal{O}_V) \to (X, \mathcal{O}_X)$ that is *isomorphic* to a morphism $j : (U, \mathcal{O}_U) \to (X, \mathcal{O}_X)$ as above is called an open immersion.

Exercise 79.26.7. Give an example of an affine scheme $(X, \mathcal{O}_X)$ and an open $U \subset X$ such that $(U, \mathcal{O}_X|U)$ is not an affine scheme.

Exercise 79.26.8. Given an example of a pair of affine schemes $(X, \mathcal{O}_X)$, $(Y, \mathcal{O}_Y)$, an open subscheme $(U, \mathcal{O}_X|_U)$ of X and a morphism of schemes $(U, \mathcal{O}_X|_U) \to (Y, \mathcal{O}_Y)$ that does not extend to a morphism of schemes $(X, \mathcal{O}_X) \to (Y, \mathcal{O}_Y)$.

Exercise 79.26.9. (This is pretty hard.) Given an example of a scheme X, and open subscheme $U \subset X$ and a closed subscheme $Z \subset U$ such that Z does not extend to a closed subscheme of X.

Exercise 79.26.10. Give an example of a scheme X, a field K, and a morphism of ringed spaces $\mathrm{Spec}(K) \to X$ which is NOT a morphism of schemes.

Exercise 79.26.11. Do all the exercises in [**Har77**, Chapter II], Sections 1 and 2... Just kidding!

Definition 79.26.12. A scheme X is called *integral* if for every nonempty affine open $U \subset X$ the ring $\Gamma(U, \mathcal{O}_X) = \mathcal{O}_X(U)$ is a domain.

Exercise 79.26.13. Give an example of a morphism of *integral* schemes $f : X \to Y$ such that the induced maps $\mathcal{O}_{Y,f(x)} \to \mathcal{O}_{X,x}$ are surjective for all $x \in X$, but f is not a closed immersion.

Exercise 79.26.14. Give an example of a fibre product $X \times_S Y$ such that X and Y are affine but $X \times_S Y$ is not.

Remark 79.26.15. It turns out this cannot happen with S separated. Do you know why?

Exercise 79.26.16. Give an example of a scheme V which is integral 1-dimensional scheme of finite type over $\mathbf{Q}$ such that $\mathrm{Spec}(\mathbf{C}) \times_{\mathrm{Spec}(\mathbf{Q})} V$ is not integral.

Exercise 79.26.17. Give an example of a scheme V which is integral 1-dimensional scheme of finite type over a field k such that $\mathrm{Spec}(k') \times_{\mathrm{Spec}(k)} V$ is not reduced for some finite field extension $k \subset k'$.

Remark 79.26.18. If your scheme is affine then dimension is the same as the Krull dimension of the underlying ring. So you can use last semesters results to compute dimension.

79.27. Morphisms

An important question is, given a morphism $\pi : X \to S$, whether the morphism has a section or a rational section. Here are some example exercises.

Exercise 79.27.1. Consider the morphism of schemes

$$\pi : X = \mathrm{Spec}(\mathbf{C}[x, t, 1/xt]) \longrightarrow S = \mathrm{Spec}(\mathbf{C}[t]).$$

(1) Show there does not exist a morphism $\sigma : S \to X$ such that $\pi \circ \sigma = \mathrm{id}_U$.
(2) Show there does exist a nonempty open $U \subset S$ and a morphism $\sigma : U \to X$ such that $\pi \circ \sigma = \mathrm{id}_U$.

Exercise 79.27.2. Consider the morphism of schemes

$$\pi : X = \mathrm{Spec}(\mathbf{C}[x, t]/(x^2 + t)) \longrightarrow S = \mathrm{Spec}(\mathbf{C}[t]).$$

Show there does not exist a nonempty open $U \subset S$ and a morphism $\sigma : U \to X$ such that $\pi \circ \sigma = \mathrm{id}_U$.

Exercise 79.27.3. Let $A, B, C \in \mathbf{C}[t]$ be nonzero polynomials. Consider the morphism of schemes

$$\pi : X = \mathrm{Spec}(\mathbf{C}[x, y, t]/(A + Bx^2 + Cy^2)) \longrightarrow S = \mathrm{Spec}(\mathbf{C}[t]).$$

Show there does exist a nonempty open $U \subset S$ and a morphism $\sigma : U \to X$ such that $\pi \circ \sigma = \mathrm{id}_U$. (Hint: Symbolically, write $x = X/Z$, $y = Y/Z$ for some $X, Y, Z \in \mathbf{C}[t]$ of degree $\leq d$ for some d, and work out the condition that this solves the equation. Then show, using dimension theory, that if $d >> 0$ you can find nonzero X, Y, Z solving the equation.)

Remark 79.27.4. Exercise 79.27.3 is a special case of "Tsen's theorem". Exercise 79.27.5 shows that the method is limited to low degree equations (conics when the base and fibre have dimension 1).

Exercise 79.27.5. Consider the morphism of schemes

$$\pi : X = \mathrm{Spec}(\mathbf{C}[x, y, t]/(1 + tx^3 + t^2y^3)) \longrightarrow S = \mathrm{Spec}(\mathbf{C}[t])$$

Show there does not exist a nonempty open $U \subset S$ and a morphism $\sigma : U \to X$ such that $\pi \circ \sigma = \mathrm{id}_U$.

Exercise 79.27.6. Consider the schemes

$$X = \mathrm{Spec}(\mathbf{C}[\{x_i\}_{i=1}^8, s, t]/(1+sx_1^3+s^2x_2^3+tx_3^3+stx_4^3+s^2tx_5^3+t^2x_6^3+st^2x_7^3+s^2t^2x_8^3))$$

and

$$S = \mathrm{Spec}(\mathbf{C}[s, t])$$

and the morphism of schemes

$$\pi : X \longrightarrow S$$

Show there does not exist a nonempty open $U \subset S$ and a morphism $\sigma : U \to X$ such that $\pi \circ \sigma = \mathrm{id}_U$.

Exercise 79.27.7. (For the number theorists.) Give an example of a closed subscheme

$$Z \subset \mathrm{Spec}\left(\mathbf{Z}[x, \frac{1}{x(x-1)(2x-1)}]\right)$$

such that the morphism $Z \to \mathrm{Spec}(\mathbf{Z})$ is finite and surjective.

Exercise 79.27.8. If you do not like number theory, you can try the variant where you look at

$$\mathrm{Spec}\left(\mathbf{F}_p[t, x, \frac{1}{x(x-t)(tx-1)}]\right) \longrightarrow \mathrm{Spec}(\mathbf{F}_p[t])$$

and you try to find a closed subscheme of the top scheme which maps finite surjectively to the bottom one. (There is a theoretical reason for having a finite ground field here; although it may not be necessary in this particular case.)

Remark 79.27.9. The interpretation of the results of Exercise 79.27.7 and 79.27.8 is that given the morphism $X \to S$ all of whose fibres are nonempty, there exists a finite surjective morphism $S' \to S$ such that the base change $X_{S'} \to S'$ does have a section. This is not a general fact, but it holds if the base is the spectrum of a dedekind ring with finite residue fields at closed points, and the morphism $X \to S$ is flat with geometrically irreducible generic fibre. See Exercise 79.27.10 below for an example where it doesn't work.

Exercise 79.27.10. Prove there exist a $f \in \mathbf{C}[x, t]$ which is not divisible by $t - \alpha$ for any $\alpha \in \mathbf{C}$ such that there does not exist any $Z \subset \mathrm{Spec}(\mathbf{C}[x, t, 1/f])$ which maps finite surjectively to $\mathrm{Spec}(\mathbf{C}[t])$. (I think that $f(x, t) = (xt - 2)(x - t + 3)$ works. To show any candidate has the required property is not so easy I think.)

79.28. Tangent Spaces

Definition 79.28.1. For any ring R we denote $R[\epsilon]$ the ring of *dual numbers*. As an R-module it is free with basis $1, \epsilon$. The ring structure comes from setting $\epsilon^2 = 0$.

Exercise 79.28.2. Let $f : X \to S$ be a morphism of schemes. Let $x \in X$ be a point, let $s = f(x)$. Consider the solid commutative diagram

$$\begin{array}{ccccc} \mathrm{Spec}(\kappa(x)) & \longrightarrow & \mathrm{Spec}(\kappa(x)[\epsilon]) & \dashrightarrow & X \\ & \searrow & \downarrow & & \downarrow \\ & & \mathrm{Spec}(\kappa(s)) & \longrightarrow & S \end{array}$$

with the curved arrow being the canonical morphism of $\mathrm{Spec}(\kappa(x))$ into X. If $\kappa(x) = \kappa(s)$ show that the set of dotted arrows which make the diagram commute are in one to one correspondence with the set of linear maps

$$\mathrm{Hom}_{\kappa(x)}(\frac{\mathfrak{m}_x}{\mathfrak{m}_x^2 + \mathfrak{m}_s\mathcal{O}_{X,x}}, \kappa(x))$$

In other words: describe such a bijection. (This works more generally if $\kappa(x) \supset \kappa(s)$ is a separable algebraic extension.)

Definition 79.28.3. Let $f : X \to S$ be a morphism of schemes. Let $x \in X$. We dub the set of dotted arrows of Exercise 79.28.2 the *tangent space of X over S* and we denote it $T_{X/S,x}$. An element of this space is called a *tangent vector* of X/S at x.

Exercise 79.28.4. For any field K prove that the diagram

$$\begin{array}{ccc} \mathrm{Spec}(K) & \longrightarrow & \mathrm{Spec}(K[\epsilon_1]) \\ \downarrow & & \downarrow \\ \mathrm{Spec}(K[\epsilon_2]) & \longrightarrow & \mathrm{Spec}(K[\epsilon_1, \epsilon_2]/(\epsilon_1\epsilon_2)) \end{array}$$

is a pushout diagram in the category of schemes. (Here $\epsilon_i^2 = 0$ as before.)

Exercise 79.28.5. Let $f : X \to S$ be a morphism of schemes. Let $x \in X$. Define addition of tangent vectors, using Exercise 79.28.4 and a suitable morphism

$$\mathrm{Spec}(K[\epsilon]) \longrightarrow \mathrm{Spec}(K[\epsilon_1, \epsilon_2]/(\epsilon_1\epsilon_2)).$$

Similarly, define scalar multiplication of tangent vectors (this is easier). Show that $T_{X/S,x}$ becomes a $\kappa(x)$-vector space with your constructions.

Exercise 79.28.6. Let k be a field. Consider the structure morphism $f : X = \mathbf{A}^1_k \to \mathrm{Spec}(k) = S$.

(1) Let $x \in X$ be a closed point. What is the dimension of $T_{X/S,x}$?
(2) Let $\eta \in X$ be the generic point. What is the dimension of $T_{X/S,\eta}$?
(3) Consider now X as a scheme over $\mathrm{Spec}(\mathbf{Z})$. What are the dimensions of $T_{X/\mathbf{Z},x}$ and $T_{X/\mathbf{Z},\eta}$?

Remark 79.28.7. Exercise 79.28.6 explains why it is necessary to consider the tangent space of X over S to get a good notion.

Exercise 79.28.8. Consider the morphism of schemes

$$f : X = \mathrm{Spec}(\mathbf{F}_p(t)) \longrightarrow \mathrm{Spec}(\mathbf{F}_p(t^p)) = S$$

Compute the tangent space of X/S at the unique point of X. Isn't that weird? What do you think happens if you take the morphism of schemes corresponding to $\mathbf{F}_p[t^p] \to \mathbf{F}_p[t]$?

Exercise 79.28.9. Let k be a field. Compute the tangent space of X/k at the point $x = (0, 0)$ where $X = \mathrm{Spec}(k[x, y]/(x^2 - y^3))$.

Exercise 79.28.10. Let $f : X \to Y$ be a morphism of schemes over S. Let $x \in X$ be a point. Set $y = f(x)$. Assume that the natural map $\kappa(y) \to \kappa(x)$ is bijective. Show, using the definition, that f induces a natural linear map

$$\mathrm{d}f : T_{X/S,x} \longrightarrow T_{Y/S,y}.$$

Match it with what happens on local rings via Exercise 79.28.2 in case $\kappa(x) = \kappa(s)$.

Exercise 79.28.11. Let k be an algebraically closed field. Let

$$\begin{aligned} f : \mathbf{A}_k^n &\longrightarrow \mathbf{A}_k^m \\ (x_1, \ldots, x_n) &\longmapsto (f_1(x_i), \ldots, f_m(x_i)) \end{aligned}$$

be a morphism of schemes over k. This is given by m polynomials $f_1, \ldots, f_m$ in n variables. Consider the matrix

$$A = \left(\frac{\partial f_j}{\partial x_i} \right)$$

Let $x \in \mathbf{A}_k^n$ be a closed point. Set $y = f(x)$. Show that the map on tangent spaces $T_{\mathbf{A}_k^n/k,x} \to T_{\mathbf{A}_k^m/k,y}$ is given by the value of the matrix A at the point x.

79.29. Quasi-coherent Sheaves

Definition 79.29.1. Let X be a scheme. A sheaf $\mathcal{F}$ of $\mathcal{O}_X$-modules is *quasi-coherent* if for every affine open $\mathrm{Spec}(R) = U \subset X$ the restriction $\mathcal{F}|_U$ is of the form $\widetilde{M}$ for some R-module M.

It is enough to check this conditions on the members of an affine open covering of X. See Schemes, Section 25.24 for more results.

Definition 79.29.2. Let X be a topological space. Let $x, x' \in X$. We say x is a *specialization* of x' if and only if $x \in \overline{\{x'\}}$.

Exercise 79.29.3. Let X be a scheme. Let $x, x' \in X$. Let $\mathcal{F}$ be a quasi-coherent sheaf of $\mathcal{O}_X$-modules. Suppose that (a) x is a specialization of x' and (b) $\mathcal{F}_{x'} \neq 0$. Show that $\mathcal{F}_x \neq 0$.

Exercise 79.29.4. Find an example of a scheme X, points $x, x' \in X$, a sheaf of $\mathcal{O}_X$-modules $\mathcal{F}$ such that (a) x is a specialization of x' and (b) $\mathcal{F}_{x'} \neq 0$ and $\mathcal{F}_x = 0$.

Definition 79.29.5. A scheme X is called *locally Noetherian* if and only if for every point $x \in X$ there exists an affine open $\mathrm{Spec}(R) = U \subset X$ such that R is Noetherian. A scheme is *Noetherian* if it is locally Noetherian and quasi-compact.

If X is locally Noetherian then any affine open of X is the spectrum of a Noetherian ring, see Properties, Lemma 27.5.2.

Definition 79.29.6. Let X be a locally Noetherian scheme. Let $\mathcal{F}$ be a quasi-coherent sheaf of $\mathcal{O}_X$-modules. We say $\mathcal{F}$ is *coherent* if for every point $x \in X$ there exists an affine open $\mathrm{Spec}(R) = U \subset X$ such that $\mathcal{F}|_U$ is isomorphic to $\widetilde{M}$ for some finite R-module M.

Exercise 79.29.7. Let $X = \mathrm{Spec}(R)$ be an affine scheme.

(1) Let $f \in R$. Let $\mathcal{G}$ be a quasi-coherent sheaf of $\mathcal{O}_{D(f)}$-modules on the open subscheme $D(f)$. Show that $\mathcal{G} = \mathcal{F}|_U$ for some quasi-coherent sheaf of $\mathcal{O}_X$-modules $\mathcal{F}$.
(2) Let $I \subset R$ be an ideal. Let $i : Z \to X$ be the closed subscheme of X corresponding to I. Let $\mathcal{G}$ be a quasi-coherent sheaf of $\mathcal{O}_Z$-modules on the closed subscheme Z. Show that $\mathcal{G} = i^*\mathcal{F}$ for some quasi-coherent sheaf of $\mathcal{O}_X$-modules $\mathcal{F}$. (Why is this silly?)
(3) Assume that R is Noetherian. Let $f \in R$. Let $\mathcal{G}$ be a coherent sheaf of $\mathcal{O}_{D(f)}$-modules on the open subscheme $D(f)$. Show that $\mathcal{G} = \mathcal{F}|_U$ for some coherent sheaf of $\mathcal{O}_X$-modules $\mathcal{F}$.

Remark 79.29.8. If $U \to X$ is a quasi-compact immersion then any quasi-coherent sheaf on U is the restriction of a quasi-coherent sheaf on X. If X is a Noetherian scheme, and $U \subset X$ is open, then any coherent sheaf on U is the restriction of a coherent sheaf on X. Of course the exercise above is easier, and shouldn't use these general facts.

79.30. Proj and projective schemes

Exercise 79.30.1. Give examples of graded rings S such that

(1) $\text{Proj}(S)$ is affine and nonempty, and
(2) $\text{Proj}(S)$ is integral, nonempty but not isomorphic to $\mathbf{P}^n_A$ for any $n \geq 0$, any ring A.

Exercise 79.30.2. Give an example of a nonconstant morphism of schemes $\mathbf{P}^1_{\mathbf{C}} \to \mathbf{P}^5_{\mathbf{C}}$ over $\text{Spec}(\mathbf{C})$.

Exercise 79.30.3. Give an example of an isomorphism of schemes

$$\mathbf{P}^1_{\mathbf{C}} \to \text{Proj}(\mathbf{C}[X_0, X_1, X_2]/(X_0^2 + X_1^2 + X_2^2))$$

Exercise 79.30.4. Give an example of a morphism of schemes $f : X \to \mathbf{A}^1_{\mathbf{C}} = \text{Spec}(\mathbf{C}[T])$ such that the (scheme theoretic) fibre X_t of f over $t \in \mathbf{A}^1_{\mathbf{C}}$ is (a) isomorphic to $\mathbf{P}^1_{\mathbf{C}}$ when t is a closed point not equal to 0, and (b) not isomorphic to $\mathbf{P}^1_{\mathbf{C}}$ when $t = 0$. We will call X_0 the *special fibre* of the morphism. This can be done in many, many ways. Try to give examples that satisfy (each of) the following additional restraints (unless it isn't possible):

(1) Can you do it with special fibre projective?
(2) Can you do it with special fibre irreducible and projective?
(3) Can you do it with special fibre integral and projective?
(4) Can you do it with special fibre smooth and projective?
(5) Can you do it with f a flat morphism? This just means that for every affine open $\text{Spec}(A) \subset X$ the induced ring map $\mathbf{C}[t] \to A$ is flat, which in this case means that any nonzero polynomial in t is a nonzerodivisor on A.
(6) Can you do it with f a flat and projective morphism?
(7) Can you do it with f flat, projective and special fibre reduced?
(8) Can you do it with f flat, projective and special fibre irreducible?
(9) Can you do it with f flat, projective and special fibre integral?

What do you think happens when you replace $\mathbf{P}^1_{\mathbf{C}}$ with another variety over $\mathbf{C}$? (This can get very hard depending on which of the variants above you ask for.)

Exercise 79.30.5. Let $n \geq 1$ be any positive integer. Give an example of a surjective morphism $X \to \mathbf{P}^n_{\mathbf{C}}$ with X affine.

Exercise 79.30.6. Maps of Proj. Let R and S be graded rings. Suppose we have a ring map

$$\psi : R \to S$$

and an integer $e \geq 1$ such that $\psi(R_d) \subset S_{de}$ for all $d \geq 0$. (By our conventions this is not a homomorphism of graded rings, unless $e = 1$.)

(1) For which elements $\mathfrak{p} \in \text{Proj}(S)$ is there a well-defined corresponding point in $\text{Proj}(R)$? In other words, find a suitable open $U \subset \text{Proj}(S)$ such that ψ defines a continuous map $r_\psi : U \to \text{Proj}(R)$.

(2) Give an example where $U \neq \mathrm{Proj}(S)$.
(3) Give an example where $U = \mathrm{Proj}(S)$.
(4) (Do not write this down.) Convince yourself that the continuous map $U \to \mathrm{Proj}(R)$ comes canonically with a map on sheaves so that r_ψ is a morphism of schemes:
$$\mathrm{Proj}(S) \supset U \longrightarrow \mathrm{Proj}(R).$$
(5) What can you say about this map if $R = \bigoplus_{d \geq 0} S_{de}$ (as a graded ring with S_e, S_{2e}, etc in degree 1, 2, etc) and ψ is the inclusion mapping?

Notation. Let R be a graded ring as above and let $n \geq 0$ be an integer. Let $X = \mathrm{Proj}(R)$. Then there is a unique quasi-coherent $\mathcal{O}_X$-module $\mathcal{O}_X(n)$ on X such that for every homogeneous element $f \in R$ of positive degree we have $\mathcal{O}_X|_{D_+(f)}$ is the quasi-coherent sheaf associated to the $R_{(f)} = (R_f)_0$-module $(R_f)_n$ (=elements homogeneous of degree n in $R_f = R[1/f]$). See [**Har77**, page 116+]. Note that there are natural maps
$$\mathcal{O}_X(n_1) \otimes_{\mathcal{O}_X} \mathcal{O}_X(n_2) \longrightarrow \mathcal{O}_X(n_1 + n_2)$$

Exercise 79.30.7. Pathologies in Proj. Give examples of R as above such that
(1) $\mathcal{O}_X(1)$ is not an invertible $\mathcal{O}_X$-module.
(2) $\mathcal{O}_X(1)$ is invertible, but the natural map $\mathcal{O}_X(1) \otimes_{\mathcal{O}_X} \mathcal{O}_X(1) \to \mathcal{O}_X(2)$ is NOT an isomorphism.

Exercise 79.30.8. Let S be a graded ring. Let $X = \mathrm{Proj}(S)$. Show that any finite set of points of X is contained in a standard affine open.

Exercise 79.30.9. Let S be a graded ring. Let $X = \mathrm{Proj}(S)$. Let $Z, Z' \subset X$ be two closed subschemes. Let $\varphi : Z \to Z'$ be an isomorphism. Assume $Z \cap Z' = \emptyset$. Show that for any $z \in Z$ there exists an affine open $U \subset X$ such that $z \in U$, $\varphi(z) \in U$ and $\varphi(Z \cap U) = Z' \cap U$. (Hint: Use Exercise 79.30.8 and something akin to Schemes, Lemma 25.11.5.)

79.31. Morphisms from surfaces to curves

Exercise 79.31.1. Let R be a ring. Let $R \to k$ be a map from R to a field. Let $n \geq 0$. Show that
$$\mathrm{Mor}_{\mathrm{Spec}(R)}(\mathrm{Spec}(k), \mathbf{P}^n_R) = (k^{n+1} \setminus \{0\})/k^*$$
where k^* acts via scalar multiplication on k^{n+1}. From now on we denote $(x_0 : \ldots : x_n)$ the morphism $\mathrm{Spec}(k) \to \mathbf{P}^n_k$ corresponding to the equivalence class of the element $(x_0, \ldots, x_n) \in k^{n+1} \setminus \{0\}$.

Exercise 79.31.2. Let k be a field. Let $Z \subset \mathbf{P}^2_k$ be a irreducible closed subscheme. Show that either (a) Z is a closed point, or (b) there exists an homogeneous irreducible $F \in k[X_0, X_1, X_2]$ of degree > 0 such that $Z = V_+(F)$, or (c) $Z = \mathbf{P}^2_k$. (Hint: Look on a standard affine open.)

Exercise 79.31.3. Let k be a field. Let $Z_1, Z_2 \subset \mathbf{P}^2_k$ be irreducible closed subschemes of the form $V_+(F)$ for some homogeneous irreducible $F_i \in k[X_0, X_1, X_2]$ of degree > 0. Show that $Z_1 \cap Z_2$ is not empty. (Hint: Use dimension theory to estimate the dimension of the local ring of $k[X_0, X_1, X_2]/(F_1, F_2)$ at 0.)

Exercise 79.31.4. Show there does not exist a nonconstant morphism of schemes $\mathbf{P}^2_{\mathbf{C}} \to \mathbf{P}^1_{\mathbf{C}}$ over $\mathrm{Spec}(\mathbf{C})$. Here a *constant morphism* is one whose image is a single point. (Hint: If the morphism is not constant consider the fibres over 0 and ∞ and argue that they have to meet to get a contradiction.)

Exercise 79.31.5. Let k be a field. Suppose that $X \subset \mathbf{P}^3_k$ is a closed subscheme given by a single homogeneous equation $F \in k[X_0, X_1, X_2, X_3]$. In other words,

$$X = \mathrm{Proj}(k[X_0, X_1, X_2, X_3]/(F)) \subset \mathbf{P}^3_k$$

as explained in the course. Assume that

$$F = X_0 G + X_1 H$$

for some homogeneous polynomials $G, H \in k[X_0, X_1, X_2, X_3]$ of positive degree. Show that if X_0, X_1, G, H have no common zeros then there exists a nonconstant morphism

$$X \longrightarrow \mathbf{P}^1_k$$

of schemes over $\mathrm{Spec}(k)$ which on field points (see Exercise 79.31.1) looks like $(x_0 : x_1 : x_2 : x_3) \mapsto (x_0 : x_1)$ whenever x_0 or x_1 is not zero.

79.32. Invertible sheaves

Definition 79.32.1. Let X be a locally ringed space. An *invertible $\mathcal{O}_X$-module* on X is a sheaf of $\mathcal{O}_X$-modules $\mathcal{L}$ such that every point has an open neighbourhood $U \subset X$ such that $\mathcal{L}|_U$ is isomorphic to $\mathcal{O}_U$ as $\mathcal{O}_U$-module. We say that $\mathcal{L}$ is trivial if it is isomorphic to $\mathcal{O}_X$ as a $\mathcal{O}_X$-module.

Exercise 79.32.2. General facts.

(1) Show that an invertible $\mathcal{O}_X$-module on a scheme X is quasi-coherent.
(2) Suppose $X \to Y$ is a morphism of ringed spaces, and $\mathcal{L}$ an invertible $\mathcal{O}_Y$-module. Show that $f^*\mathcal{L}$ is an invertible $\mathcal{O}_X$ module.

Exercise 79.32.3. Algebra.

(1) Show that an invertible $\mathcal{O}_X$-module on an affine scheme $\mathrm{Spec}(A)$ corresponds to an A-module M which is (i) finite, (ii) projective, (iii) locally free of rank 1, and hence (iv) flat, and (v) finitely presented. (Feel free to quote things from last semesters course; or from algebra books.)
(2) Suppose that A is a domain and that M is a module as in (a). Show that M is isomorphic as an A-module to an ideal $I \subset A$ such that $IA_{\mathfrak{p}}$ is principal for every prime $\mathfrak{p}$.

Definition 79.32.4. Let R be a ring. An *invertible module* M is an R-module M such that $\widetilde{M}$ is an invertible sheaf on the spectrum of R. We say M is *trivial* if $M \cong R$ as an R-module.

In other words, M is invertible if and only if it satisfies all of the following conditions: it is flat, of finite presentation, projective, and locally free of rank 1. (Of course it suffices for it to be locally free of rank 1).

Exercise 79.32.5. Simple examples.

(1) Let k be a field. Let $A = k[x]$. Show that $X = \mathrm{Spec}(A)$ has only trivial invertible $\mathcal{O}_X$-modules. In other words, show that every invertible A-module is free of rank 1.

(2) Let A be the ring
$$A = \{f \in k[x] \mid f(0) = f(1)\}.$$
Show there exists a nontrivial invertible A-module, unless $k = \mathbf{F}_2$. (Hint: Think about $\mathrm{Spec}(A)$ as identifying 0 and 1 in $\mathbf{A}^1_k = \mathrm{Spec}(k[x])$.)
(3) Same question as in (2) for the ring $A = k[x^2, x^3] \subset k[x]$ (except now $k = \mathbf{F}_2$ works as well).

Exercise 79.32.6. Higher dimensions.

(1) Prove that every invertible sheaf on two dimensional affine space is trivial. More precisely, let $\mathbf{A}^2_k = \mathrm{Spec}(k[x, y])$ where k is a field. Show that every invertible sheaf on $\mathbf{A}^2_k$ is trivial. (Hint: One way to do this is to consider the corresponding module M, to look at $M \otimes_{k[x,y]} k(x)[y]$, and then use Exercise 79.32.5 (1) to find a generator for this; then you still have to think. Another way to is to use Exercise 79.32.3 and use what we know about ideals of the polynomial ring: primes of height one are generated by an irreducible polynomial; then you still have to think.)
(2) Prove that every invertible sheaf on any open subscheme of two dimensional affine space is trivial. More precisely, let $U \subset \mathbf{A}^2_k$ be an open subscheme where k is a field. Show that every invertible sheaf on U is trivial. Hint: Show that every invertible sheaf on U extends to one on $\mathbf{A}^2_k$. Not easy; but you can find it in [**Har77**].
(3) Find an example of a nontrivial invertible sheaf on a punctured cone over a field. More precisely, let k be a field and let $C = \mathrm{Spec}(k[x, y, z]/(xy - z^2))$. Let $U = C \setminus \{(x, y, z)\}$. Find a nontrivial invertible sheaf on U. Hint: It may be easier to compute the group of isomorphism classes of invertible sheaves on U than to just find one. Note that U is covered by the opens $\mathrm{Spec}(k[x, y, z, 1/x]/(xy - z^2))$ and $\mathrm{Spec}(k[x, y, z, 1/y]/(xy - z^2))$ which are "easy" to deal with.

Definition 79.32.7. Let X be a locally ringed space. The *Picard group of* X is the set $\mathrm{Pic}(X)$ of isomorphism classes of invertible $\mathcal{O}_X$-modules with addition given by tensor product. See Modules, Definition 17.21.6. For a ring R we set $\mathrm{Pic}(R) = \mathrm{Pic}(\mathrm{Spec}(R))$.

Exercise 79.32.8. Let R be a ring.

(1) Show that if R is a Noetherian normal domain, then $\mathrm{Pic}(R) = \mathrm{Pic}(R[t])$. [Hint: There is a map $R[t] \to R$, $t \mapsto 0$ which is a left inverse to the map $R \to R[t]$. Hence it suffices to show that any invertible $R[t]$-module M such that $M/tM \cong R$ is free of rank 1. Let $K = f.f.(R)$. Pick a trivialization $K[t] \to M \otimes_{R[t]} K[t]$ which is possible by Exercise 79.32.5 (1). Adjust it so it agrees with the trivialization of M/tM above. Show that it is in fact a trivialization of M over $R[t]$ (this is where normality comes in).]
(2) Let k be a field. Show that $\mathrm{Pic}(k[x^2, x^3, t]) \neq \mathrm{Pic}(k[x^2, x^3])$.

79.33. Čech Cohomology

Exercise 79.33.1. Čech cohomology. Here k is a field.

(1) Let X be a scheme with an open covering $\mathcal{U} : X = U_1 \cup U_2$, with $U_1 = \mathrm{Spec}(k[x])$, $U_2 = \mathrm{Spec}(k[y])$ with $U_1 \cap U_2 = \mathrm{Spec}(k[z, 1/z])$ and with open immersions $U_1 \cap U_2 \to U_1$ resp. $U_1 \cap U_2 \to U_2$ determined by $x \mapsto z$ resp. $y \mapsto z$ (and I really mean this). (We've seen in the lectures that such an X exists; it is the affine line zith zero doubled.) Compute $\check{H}^1(\mathcal{U}, \mathcal{O})$; eg. give a basis for it as a k-vectorspace.
(2) For each element in $\check{H}^1(\mathcal{U}, \mathcal{O})$ construct an exact sequence of sheaves of $\mathcal{O}_X$-modules
$$0 \to \mathcal{O}_X \to E \to \mathcal{O}_X \to 0$$
such that the boundary $\delta(1) \in \check{H}^1(\mathcal{U}, \mathcal{O})$ equals the given element. (Part of the problem is to make sense of this. See also below. It is also OK to show abstractly such a thing has to exist.)

Definition 79.33.2. (Definition of delta.) Suppose that
$$0 \to \mathcal{F}_1 \to \mathcal{F}_2 \to \mathcal{F}_3 \to 0$$
is a short exact sequence of abelian sheaves on any topological space X. The boundary map $\delta : H^0(X, \mathcal{F}_3) \to \check{H}^1(X, \mathcal{F}_1)$ is defined as follows. Take an element $\tau \in H^0(X, \mathcal{F}_3)$. Choose an open covering $\mathcal{U} : X = \bigcup_{i \in I} U_i$ such that for each i there exists a section $\tilde{\tau}_i \in \mathcal{F}_2$ lifting the restriction of τ to U_i. Then consider the assignment
$$(i_0, i_1) \longmapsto \tilde{\tau}_{i_0}|_{U_{i_0 i_1}} - \tilde{\tau}_{i_1}|_{U_{i_0 i_1}}.$$
This is clearly a 1-coboundary in the Čech complex $\check{C}^*(\mathcal{U}, \mathcal{F}_2)$. But we observe that (thinking of $\mathcal{F}_1$ as a subsheaf of $\mathcal{F}_2$) the RHS always is a section of $\mathcal{F}_1$ over $U_{i_0 i_1}$. Hence we see that the assignment defines a 1-cochain in the complex $\check{C}^*(\mathcal{U}, \mathcal{F}_2)$. The cohomology class of this 1-cochain is by definition $\delta(\tau)$.

79.34. Divisors

We collect all relevant definitions here in one spot for convenience.

Definition 79.34.1. Throughout, let S be any scheme and let X be a Noetherian, integral scheme.

(1) A *Weil divisor* on X is a formal linear combination $\Sigma n_i [Z_i]$ of prime divisors Z_i with integer coefficients.
(2) A *prime divisor* is a closed subscheme $Z \subset X$, which is integral with generic point $\xi \in Z$ such that $\mathcal{O}_{X,\xi}$ has dimension 1. We will use the notation $\mathcal{O}_{X,Z} = \mathcal{O}_{X,\xi}$ when $\xi \in Z \subset X$ is as above. Note that $\mathcal{O}_{X,Z} \subset K(X)$ is a subring of the function field of X.
(3) The *Weil divisor associated to a rational function* $f \in K(X)^*$ is the sum $\Sigma v_Z(f)[Z]$. Here $v_Z(f)$ is defined as follows
   (a) If $f \in \mathcal{O}_{X,Z}^*$ then $v_Z(f) = 0$.
   (b) If $f \in \mathcal{O}_{X,Z}$ then
   $$v_Z(f) = \mathrm{length}_{\mathcal{O}_{X,Z}}(\mathcal{O}_{X,Z}/(f)).$$
   (c) If $f = \frac{a}{b}$ with $a, b \in \mathcal{O}_{X,Z}$ then
   $$v_Z(f) = \mathrm{length}_{\mathcal{O}_{X,Z}}(\mathcal{O}_{X,Z}/(a)) - \mathrm{length}_{\mathcal{O}_{X,Z}}(\mathcal{O}_{X,Z}/(b)).$$

(4) An *effective Cartier divisor* on a scheme S is a closed subscheme $D \subset S$ such that every point $d \in D$ has an affine open neighbourhood $\mathrm{Spec}(A) = U \subset S$ in S so that $D \cap U = \mathrm{Spec}(A/(f))$ with $f \in A$ a nonzerodivisor.
(5) The *Weil divisor* $[D]$ *associated to an effective Cartier divisor* $D \subset X$ of our Noetherian integral scheme X is defined as the sum $\Sigma v_Z(D)[Z]$ where $v_Z(D)$ is defined as follows
   (a) If the generic point ξ of Z is not in D then $v_Z(D) = 0$.
   (b) If the generic point ξ of Z is in D then
   $$v_Z(D) = \mathrm{length}_{\mathcal{O}_{X,Z}}(\mathcal{O}_{X,Z}/(f))$$
   where $f \in \mathcal{O}_{X,Z} = \mathcal{O}_{X,\xi}$ is the nonzerodivisor which defines D in an affine neighbourhood of ξ (as in (4) above).
(6) Let S be a scheme. The *sheaf of total quotient rings* $\mathcal{K}_S$ is the sheaf of $\mathcal{O}_S$-algebras which is the sheafification of the pre-sheaf $\mathcal{K}'$ defined as follows. For $U \subset S$ open we set $\mathcal{K}'(U) = S_U^{-1}\mathcal{O}_S(U)$ where $S_U \subset \mathcal{O}_S(U)$ is the multiplicative subset consisting of sections $f \in \mathcal{O}_S(U)$ such that the germ of f in $\mathcal{O}_{S,u}$ is a nonzerodivisor for every $u \in U$. In particular the elements of S_U are all nonzerodivisors. Thus $\mathcal{O}_S$ is a subsheaf of $\mathcal{K}_S$, and we get a short exact sequence
$$0 \to \mathcal{O}_S^* \to \mathcal{K}_S^* \to \mathcal{K}_S^*/\mathcal{O}_S^* \to 0.$$
(7) A *Cartier divisor* on a scheme S is a global section of the quotient sheaf $\mathcal{K}_S^*/\mathcal{O}_S^*$.
(8) The *Weil divisor associated to a Cartier divisor* $\tau \in \Gamma(X, \mathcal{K}_X^*/\mathcal{O}_X^*)$ over our Noetherian integral scheme X is the sum $\Sigma v_Z(\tau)[Z]$ where $v_Z(\tau)$ is defined as by the following recipe
   (a) If the germ of τ at the generic point ξ of Z is zero – in other words the image of τ in the stalk $(\mathcal{K}^*/\mathcal{O}^*)_\xi$ is "zero" – then $v_Z(\tau) = 0$.
   (b) Find an affine open neighbourhood $\mathrm{Spec}(A) = U \subset X$ so that $\tau|_U$ is the image of a section $f \in \mathcal{K}(U)$ and moreover $f = a/b$ with $a, b \in A$. Then we set
   $$v_Z(f) = \mathrm{length}_{\mathcal{O}_{X,Z}}(\mathcal{O}_{X,Z}/(a)) - \mathrm{length}_{\mathcal{O}_{X,Z}}(\mathcal{O}_{X,Z}/(b)).$$

Remarks 79.34.2. Here are some trivial remarks.

(1) On a Noetherian integral scheme X the sheaf $\mathcal{K}_X$ is constant with value the function field $K(X)$.
(2) To make sense out of the definitions above one needs to show that
   $$\mathrm{length}_{\mathcal{O}}(\mathcal{O}/(ab)) = \mathrm{length}_{\mathcal{O}}(\mathcal{O}/(a)) + \mathrm{length}_{\mathcal{O}}(\mathcal{O}/(b))$$
   for any pair (a, b) of nonzero elements of a Noetherian 1-dimensional local domain $\mathcal{O}$. This will be done in the lectures.

Exercise 79.34.3. (On any scheme.) Describe how to assign a Cartier divisor to an effective Cartier divisor.

Exercise 79.34.4. (On an integral scheme.) Describe how to assign a Cartier divisor D to a rational function f such that the Weil divisor associated to D and to f agree. (This is silly.)

Exercise 79.34.5. Give an example of a Weil divisor on a variety which is not the Weil divisor associated to any Cartier divisor.

Exercise 79.34.6. Give an example of a Weil divisor D on a variety which is not the Weil divisor associated to any Cartier divisor but such that nD is the Weil divisor associated to a Cartier divisor for some $n > 1$.

Exercise 79.34.7. Give an example of a Weil divisor D on a variety which is not the Weil divisor associated to any Cartier divisor and such that nD is NOT the Weil divisor associated to a Cartier divisor for any $n > 1$. (Hint: Consider a cone, for example $X : xy - zw = 0$ in $\mathbf{A}^4_k$. Try to show that $D = [x = 0, z = 0]$ works.)

Exercise 79.34.8. On a separated scheme X of finite type over a field: Give an example of a Cartier divisor which is not the difference of two effective Cartier divisors. Hint: Find some X which does not have any nonempty effective Cartier Cartier divisors for example the scheme constructed in [**Har77**, III Exercise 5.9]. There is even an example with X a variety – namely the variety of Exercise 79.34.9.

Exercise 79.34.9. Example of a nonprojective proper variety. Let k be a field. Let $L \subset \mathbf{P}^3_k$ be a line and let $C \subset \mathbf{P}^3_k$ be a nonsingular conic. Assume that $C \cap L = \emptyset$. Choose an isomorphism $\varphi : L \to C$. Let X be the k-variety obtained by glueing C to L via φ. In other words there is a surjective proper birational morphism

$$\pi : \mathbf{P}^3_k \longrightarrow X$$

and an open $U \subset X$ such that $\pi : \pi^{-1}(U) \to U$ is an isomorphism, $\pi^{-1}(U) = \mathbf{P}^3_k \setminus (L \cup C)$ and such that $\pi|_L = \pi|_C \circ \varphi$. (These conditions do not yet uniquely define X. In order to do this you need to specify the structure sheaf of X along points of $Z = X \setminus U$.) Show X exists, is a proper variety, but is not projective. (Hint: For existence use the result of Exercise 79.30.9. For non-projectivity use that $\mathrm{Pic}(\mathbf{P}^3_k) = \mathbf{Z}$ to show that X cannot have an ample invertible sheaf.)

79.35. Differentials

Definitions and results. Kähler differentials.

(1) Let $R \to A$ be a ring map. The *module of Kähler differentials of A over R* is

$$\Omega_{A/R} = \bigoplus\nolimits_{a \in A} A \cdot \mathrm{d}a \Big/ \langle \mathrm{d}(a_1 + a_2) - \mathrm{d}a_1 - \mathrm{d}a_2, \mathrm{d}(a_1 a_2) - a_1 \mathrm{d}a_2 - a_2 \mathrm{d}a_1, \mathrm{d}r \rangle.$$

The canonical universal R-derivation $\mathrm{d} : A \to \Omega_{A/R}$ maps $a \mapsto \mathrm{d}a$.

(2) Consider the short exact sequence

$$0 \to I \to A \otimes_R A \to A \to 0$$

which defines the ideal I. There is a canonical derivation $\mathrm{d} : A \to I/I^2$ which maps a to the class of $a \otimes 1 - 1 \otimes a$. This is another presentation of the module of derivations of A over R, in other words

$$(I/I^2, \mathrm{d}) \cong (\Omega_{A/R}, \mathrm{d}).$$

(3) For multiplicative subsets $S_R \subset R$ and $S_A \subset A$ such that S_R maps into S_A we have

$$\Omega_{S_A^{-1}A/S_R^{-1}R} = S_A^{-1}\Omega_{A/R}.$$

(4) If A is a finitely presented R-algebra then $\Omega_{A/R}$ is a finitely presented A-module. Hence in this case the *fitting* ideals of $\Omega_{A/R}$ are defined. (See exercise set 6 of last semester.)

(5) Let $f : X \to S$ be a morphism of schemes. There is a quasi-coherent sheaf of $\mathcal{O}_X$-modules $\Omega_{X/S}$ and a $\mathcal{O}_S$-linear derivation
$$\mathrm{d} : \mathcal{O}_X \longrightarrow \Omega_{X/S}$$
such that for any affine opens $\mathrm{Spec}(A) = U \subset X$, $\mathrm{Spec}(R) = V \subset S$ with $f(U) \subset V$ we have
$$\Gamma(\mathrm{Spec}(A), \Omega_{X/S}) = \Omega_{A/R}$$
compatibly with d.

Exercise 79.35.1. Let $k[\epsilon]$ be the ring of dual numbers over the field k, i.e., $\epsilon^2 = 0$.

(1) Consider the ring map
$$R = k[\epsilon] \to A = k[x, \epsilon]/(\epsilon x)$$
Show that the fitting ideals of $\Omega_{A/R}$ are (starting with the zeroth fitting ideal)
$$(\epsilon), A, A, \ldots$$
(2) Consider the map $R = k[t] \to A = k[x, y, t]/(x(y - t)(y - 1), x(x - t))$. Show that the fitting ideals of of $\Omega_{A/R}$ in A are (assume characteristic k is zero for simplicity)
$$x(2x - t)(2y - t - 1)A,\ (x, y, t) \cap (x, y - 1, t),\ A,\ A, \ldots$$
So the 0-the fitting ideal is cut out by a single element of A, the 1st fitting ideal defines two closed points of $\mathrm{Spec}(A)$, and the others are all trivial.
(3) Consider the map $R = k[t] \to A = k[x, y, t]/(xy - t^n)$. Compute the fitting ideals of $\Omega_{A/R}$.

Remark 79.35.2. The kth fitting ideal of $\Omega_{X/S}$ is commonly used to define the singular scheme of the morphism $X \to S$ when X has relative dimension k over S. But as part (a) shows, you have to be careful doing this when your family does not have "constant" fibre dimension, e.g., when it is not flat. As part (b) shows, flatness doesn't guarantee it works either (and yes this is a flat family). In "good cases" – such as in (c) – for families of curves you expect the 0-th fitting ideal to be zero and the 1st fitting ideal to define (scheme-theoretically) the singular locus.

Exercise 79.35.3. Suppose that R is a ring and
$$A = k[x_1, \ldots, x_n]/(f_1, \ldots, f_n).$$
Note that we are assuming that A is presented by the same number of equations as variables. Thus the matrix of partial derivatives
$$(\partial f_i/\partial x_j)$$
is $n \times n$, i.e., a square matrix. Assume that its determinant is invertible as an element in A. Note that this is exactly the condition that says that $\Omega_{A/R} = (0)$ in this case of n-generators and n relations. Let $\pi : B' \to B$ be a surjection of R-algebras whose kernel J has square zero (as an ideal in B'). Let $\varphi : A \to B$ be a homomorphism of R-algebras. Show there exists a unique homomorphism of R-algebras $\varphi' : A \to B'$ such that $\varphi = \pi \circ \varphi'$.

Exercise 79.35.4. Find a generalization of the result of the previous exercise to the case where $A = R[x, y]/(f)$.

79.36. Schemes, Final Exam, Fall 2007

These were the questions in the final exam of a course on Schemes, in the Spring of 2007 at Columbia University.

Exercise 79.36.1 (Definitions)**.** Provide definitions of the following concepts.

(1) X is a *scheme*
(2) the morphism of schemes $f : X \to Y$ is *finite*
(3) the morphisms of schemes $f : X \to Y$ is *of finite type*
(4) the scheme X is *Noetherian*
(5) the $\mathcal{O}_X$-module $\mathcal{L}$ on the scheme X is *invertible*
(6) the *genus* of a nonsingular projective curve over an algebraically closed field

Exercise 79.36.2. Let $X = \mathrm{Spec}(\mathbf{Z}[x, y])$, and let $\mathcal{F}$ be a quasi-coherent $\mathcal{O}_X$-module. Suppose that $\mathcal{F}$ is zero when restricted to the standard affine open $D(x)$.

(1) Show that every global section s of $\mathcal{F}$ is killed by some power of x, i.e., $x^n s = 0$ for some $n \in \mathbf{N}$.
(2) Do you think the same is true if we do not assume that $\mathcal{F}$ is quasi-coherent?

Exercise 79.36.3. Suppose that $X \to \mathrm{Spec}(R)$ is a proper morphism and that R is a discrete valuation ring with residue field k. Suppose that $X \times_{\mathrm{Spec}(R)} \mathrm{Spec}(k)$ is the empty scheme. Show that X is the empty scheme.

Exercise 79.36.4. Consider the projective[1] variety

$$\mathbf{P}^1 \times \mathbf{P}^1 = \mathbf{P}^1_{\mathbf{C}} \times_{\mathrm{Spec}(\mathbf{C})} \mathbf{P}^1_{\mathbf{C}}$$

over the field of complex numbers $\mathbf{C}$. It is covered by four affine pieces, corresponding to pairs of standard affine pieces of $\mathbf{P}^1_{\mathbf{C}}$. For example, suppose we use homogeneous coordinates X_0, X_1 on the first factor and Y_0, Y_1 on the second. Set $x = X_1/X_0$, and $y = Y_1/Y_0$. Then the 4 affine open pieces are the spectra of the rings

$$\mathbf{C}[x, y], \quad \mathbf{C}[x^{-1}, y], \quad \mathbf{C}[x, y^{-1}], \quad \mathbf{C}[x^{-1}, y^{-1}].$$

Let $X \subset \mathbf{P}^1 \times \mathbf{P}^1$ be the closed subscheme which is the closure of the closed subset of the first affine piece given by the equation

$$y^3(x^4 + 1) = x^4 - 1.$$

(1) Show that X is contained in the union of the first and the last of the 4 affine open pieces.
(2) Show that X is a nonsingular projective curve.
(3) Consider the morphism $pr_2 : X \to \mathbf{P}^1$ (projection onto the first factor). On the first affine piece it is the map $(x, y) \mapsto x$. Briefly explain why it has degree 3.
(4) Compute the ramification points and ramification indices for the map $pr_2 : X \to \mathbf{P}^1$.
(5) Compute the genus of X.

Exercise 79.36.5. Let $X \to \mathrm{Spec}(\mathbf{Z})$ be a morphism of finite type. Suppose that there is an infinite number of primes p such that $X \times_{\mathrm{Spec}(\mathbf{Z})} \mathrm{Spec}(\mathbf{F}_p)$ is not empty.

[1] The projective embedding is $((X_0, X_1), (Y_0, Y_1)) \mapsto (X_0Y_0, X_0Y_1, X_1Y_0, X_1Y_1)$ in other words $(x, y) \mapsto (1, y, x, xy)$.

(1) Show that $X \times_{\mathrm{Spec}(\mathbf{Z})} \mathrm{Spec}(\mathbf{Q})$ is not empty.
(2) Do you think the same is true if we replace the condition "finite type" by the condition "locally of finite type"?

79.37. Schemes, Final Exam, Spring 2009

These were the questions in the final exam of a course on Schemes, in the Spring of 2009 at Columbia University.

Exercise 79.37.1. Let X be a Noetherian scheme. Let $\mathcal{F}$ be a coherent sheaf on X. Let $x \in X$ be a point. Assume that $\mathrm{Supp}(\mathcal{F}) = \{x\}$.

(1) Show that x is a closed point of X.
(2) Show that $H^0(X, \mathcal{F})$ is not zero.
(3) Show that $\mathcal{F}$ is generated by global sections.
(4) Show that $H^p(X, \mathcal{F}) = 0$ for $p > 0$.

Remark 79.37.2. Let k be a field. Let $\mathbf{P}^2_k = \mathrm{Proj}(k[X_0, X_1, X_2])$. Any invertible sheaf on $\mathbf{P}^2_k$ is isomorphic to $\mathcal{O}_{\mathbf{P}^2_k}(n)$ for some $n \in \mathbf{Z}$. Recall that

$$\Gamma(\mathbf{P}^2_k, \mathcal{O}_{\mathbf{P}^2_k}(n)) = k[X_0, X_1, X_2]_n$$

is the degree n part of the polynomial ring. For a quasi-coherent sheaf $\mathcal{F}$ on $\mathbf{P}^2_k$ set $\mathcal{F}(n) = \mathcal{F} \otimes_{\mathcal{O}_{\mathbf{P}^2_k}} \mathcal{O}_{\mathbf{P}^2_k}(n)$ as usual.

Exercise 79.37.3. Let k be a field. Let $\mathcal{E}$ be a vector bundle on $\mathbf{P}^2_k$, i.e., a finite locally free $\mathcal{O}_{\mathbf{P}^2_k}$-module. We say $\mathcal{E}$ is *split* if $\mathcal{E}$ is isomorphic to a direct sum invertible $\mathcal{O}_{\mathbf{P}^2_k}$-modules.

(1) Show that $\mathcal{E}$ is split if and only if $\mathcal{E}(n)$ is split.
(2) Show that if $\mathcal{E}$ is split then $H^1(\mathbf{P}^2_k, \mathcal{E}(n)) = 0$ for all $n \in \mathbf{Z}$.
(3) Let
$$\varphi : \mathcal{O}_{\mathbf{P}^2_k} \longrightarrow \mathcal{O}_{\mathbf{P}^2_k}(1) \oplus \mathcal{O}_{\mathbf{P}^2_k}(1) \oplus \mathcal{O}_{\mathbf{P}^2_k}(1)$$
be given by linear forms $L_0, L_1, L_2 \in \Gamma(\mathbf{P}^2_k, \mathcal{O}_{\mathbf{P}^2_k}(1))$. Assume $L_i \neq 0$ for some i. What is the condition on L_0, L_1, L_2 such that the cokernel of φ is a vector bundle? Why?
(4) Given an example of such a φ.
(5) Show that $\mathrm{Coker}(\varphi)$ is not split (if it is a vector bundle).

Remark 79.37.4. Freely use the following facts on dimension theory (and add more if you need more).

(1) The dimension of a scheme is the supremum of the length of chains of irreducible closed subsets.
(2) The dimension of a finite type scheme over a field is the maximum of the dimensions of its affine opens.
(3) The dimension of a Noetherian scheme is the maximum of the dimensions of its irreducible components.
(4) The dimension of an affine scheme coincides with the dimension of the corresponding ring.
(5) Let k be a field and let A be a finite type k-algebra. If A is a domain, and $x \neq 0$, then $\dim(A) = \dim(A/xA) + 1$.

Exercise 79.37.5. Let k be a field. Let X be a projective, reduced scheme over k. Let $f : X \to \mathbf{P}^1_k$ be a morphism of schemes over k. Assume there exists an integer $d \geq 0$ such that for every point $t \in \mathbf{P}^1_k$ the fibre $X_t = f^{-1}(t)$ is irreducible of dimension d. (Recall that an irreducible space is not empty.)

(1) Show that $\dim(X) = d + 1$.
(2) Let $X_0 \subset X$ be an irreducible component of X of dimension $d + 1$. Prove that for every $t \in \mathbf{P}^1_k$ the fibre $X_{0,t}$ has dimension d.
(3) What can you conclude about X_t and $X_{0,t}$ from the above?
(4) Show that X is irreducible.

Remark 79.37.6. Given a projective scheme X over a field k and a coherent sheaf $\mathcal{F}$ on X we set

$$\chi(X, \mathcal{F}) = \sum\nolimits_{i \geq 0} (-1)^i \dim_k H^i(X, \mathcal{F}).$$

Exercise 79.37.7. Let k be a field. Write $\mathbf{P}^3_k = \text{Proj}(k[X_0, X_1, X_2, X_3])$. Let $C \subset \mathbf{P}^3_k$ be a *type* $(5, 6)$ *complete intersection curve*. This means that there exist $F \in k[X_0, X_1, X_2, X_3]_5$ and $G \in k[X_0, X_1, X_2, X_3]_6$ such that

$$C = \text{Proj}(k[X_0, X_1, X_2, X_3]/(F, G))$$

is a variety of dimension 1. (Variety implies reduced and irreducible, but feel free to assume C is nonsingular if you like.) Let $i : C \to \mathbf{P}^3_k$ be the corresponding closed immersion. Being a complete intersection also implies that

$$0 \longrightarrow \mathcal{O}_{\mathbf{P}^3_k}(-11) \xrightarrow{\begin{pmatrix} -G \\ F \end{pmatrix}} \mathcal{O}_{\mathbf{P}^3_k}(-5) \oplus \mathcal{O}_{\mathbf{P}^3_k}(-6) \xrightarrow{(F,G)} \mathcal{O}_{\mathbf{P}^3_k} \longrightarrow i_*\mathcal{O}_C \longrightarrow 0$$

is an exact sequence of sheaves. Please use these facts to:

(1) compute $\chi(C, i^*\mathcal{O}_{\mathbf{P}^3_k}(n))$ for any $n \in \mathbf{Z}$, and
(2) compute the dimension of $H^1(C, \mathcal{O}_C)$.

Exercise 79.37.8. Let k be a field. Consider the rings

$$A = k[x, y]/(xy)$$
$$B = k[u, v]/(uv)$$
$$C = k[t, t^{-1}] \times k[s, s^{-1}]$$

and the k-algebra maps

$$\begin{array}{llll} A \longrightarrow C, & x \mapsto (t, 0), & y \mapsto (0, s) \\ B \longrightarrow C, & u \mapsto (t^{-1}, 0), & v \mapsto (0, s^{-1}) \end{array}$$

It is a true fact that these maps induce isomorphisms $A_{x+y} \to C$ and $B_{u+v} \to C$. Hence the maps $A \to C$ and $B \to C$ identify $\text{Spec}(C)$ with open subsets of $\text{Spec}(A)$ and $\text{Spec}(B)$. Let X be the scheme obtained by glueing $\text{Spec}(A)$ and $\text{Spec}(B)$ along $\text{Spec}(C)$:

$$X = \text{Spec}(A) \coprod\nolimits_{\text{Spec}(C)} \text{Spec}(B).$$

As we saw in the course such a scheme exists and there are affine opens $\text{Spec}(A) \subset X$ and $\text{Spec}(B) \subset X$ whose overlap is exactly $\text{Spec}(C)$ identified with an open of each of these using the maps above.

(1) Why is X separated?

(2) Why is X of finite type over k?
(3) Compute $H^1(X, \mathcal{O}_X)$, or what is its dimension?
(4) What is a more geometric way to describe X?

79.38. Schemes, Final Exam, Fall 2010

These were the questions in the final exam of a course on Schemes, in the Fall of 2010 at Columbia University.

Exercise 79.38.1 (Definitions)**.** Provide definitions of the following concepts.

(1) a separated scheme,
(2) a quasi-compact morphism of schemes,
(3) an affine morphism of schemes,
(4) a multiplicative subset of a ring,
(5) a Noetherian scheme,
(6) a variety.

Exercise 79.38.2. Prime avoidance.

(1) Let A be a ring. Let $I \subset A$ be an ideal and let $\mathfrak{q}_1$, $\mathfrak{q}_2$ be prime ideals such that $I \not\subset \mathfrak{q}_i$. Show that $I \not\subset \mathfrak{q}_1 \cup \mathfrak{q}_2$.
(2) What is a geometric interpretation of (1)?
(3) Let $X = \text{Proj}(S)$ for some graded ring S. Let $x_1, x_2 \in X$. Show that there exists a standard open $D_+(F)$ which contains both x_1 and x_2.

Exercise 79.38.3. Why is a composition of affine morphisms affine?

Exercise 79.38.4 (Examples)**.** Give examples of the following:

(1) A reducible projective scheme over a field k.
(2) A scheme with 100 points.
(3) A non-affine morphism of schemes.

Exercise 79.38.5. Chevalley's theorem and the Hilbert Nullstellensatz.

(1) Let $\mathfrak{p} \subset \mathbf{Z}[x_1, \ldots, x_n]$ be a maximal ideal. What does Chevalley's theorem imply about $\mathfrak{p} \cap \mathbf{Z}$?
(2) In turn, what does the Hilbert Nullstellensatz imply about $\kappa(\mathfrak{p})$?

Exercise 79.38.6. Let A be a ring. Let $S = A[X]$ as a graded A-algebra where X has degree 1. Show that $\text{Proj}(S) \cong \text{Spec}(A)$ as schemes over A.

Exercise 79.38.7. Let $A \to B$ be a finite ring map. Show that $\text{Spec}(B)$ is a H-projective scheme over $\text{Spec}(A)$.

Exercise 79.38.8. Give an example of a scheme X over a field k such that X is irreducible and such that for some finite extension $k \subset k$ the base change $X_{k'} = X \times_{\text{Spec}(k)} \text{Spec}(k')$ is connected but reducible.

79.39. Schemes, Final Exam, Spring 2011

These were the questions in the final exam of a course on Schemes, in the Spring of 2011 at Columbia University.

Exercise 79.39.1 (Definitions)**.** Provide definitions of the italicized concepts.

(1) a *separated* scheme,
(2) a *universally closed* morphism of schemes,

(3) *A dominates B* for local rings A, B contained in a common field,
(4) the *dimension* of a scheme X,
(5) the *codimension* of an irreducible closed subscheme Y of a scheme X,

Exercise 79.39.2 (Results)**.** State something formally equivalent to the fact discussed in the course.

(1) The valuative criterion of properness for a morphism $X \to Y$ of varieties for example.
(2) The relationship between $\dim(X)$ and the function field $k(X)$ of X for a variety X over a field k.
(3) Fill in the blank: The category of nonsingular projective curves over k and nonconstant morphisms is anti-equivalent to .....
(4) Noether normalization.
(5) Jacobian criterion.

Exercise 79.39.3. Let k be a field. Let $F \in k[X_0, X_1, X_2]$ be a homogeneous form of degree d. Assume that $C = V_+(F) \subset \mathbf{P}^2_k$ is a smooth curve over k. Denote $i : C \to \mathbf{P}^2_k$ the corresponding closed immersion.

(1) Show that there is a short exact sequence
$$0 \to \mathcal{O}_{\mathbf{P}^2_k}(-d) \to \mathcal{O}_{\mathbf{P}^2_k} \to i_*\mathcal{O}_C \to 0$$
of coherent sheaves on $\mathbf{P}^2_k$: tell me what the maps are and briefly why it is exact.
(2) Conclude that $H^0(C, \mathcal{O}_C) = k$.
(3) Compute the genus of C.
(4) Assume now that $P = (0 : 0 : 1)$ is not on C. Prove that $\pi : C \to \mathbf{P}^1_k$ given by $(a_0 : a_1 : a_2) \mapsto (a_0 : a_1)$ has degree d.
(5) Assume k is algebraically closed, assume all ramification indices (the "e_i") are 1 or 2, and assume the characteristic of k is not equal to 2. How many ramification points does $\pi : C \to \mathbf{P}^1_k$ have?
(6) In terms of F, what do you think is a set of equations of the set of ramification points of π?
(7) Can you guess K_C?

Exercise 79.39.4. Let k be a field. Let X be a "triangle" over k, i.e., you get X by glueing three copies of $\mathbf{A}^1_k$ to each other by identifying 0 on the first copy to 1 on the second copy, 0 on the second copy to 1 on the first copy, and 0 on the third copy to 1 on the first copy. It turns out that X is isomorphic to $\mathrm{Spec}(k[x, y]/(xy(x + y + 1)))$; feel free to use this. Compute the Picard group of X.

Exercise 79.39.5. Let k be a field. Let $\pi : X \to Y$ be a finite birational morphism of curves with X a projective nonsingular curve over k. It follows from the material in the course that Y is a proper curve and that π is the normalization morphism of Y. We have also seen in the course that there exists a dense open $V \subset Y$ such that $U = \pi^{-1}(V)$ is a dense open in X and $\pi : U \to V$ is an isomorphism.

(1) Show that there exists an effective Cartier divisor $D \subset X$ such that $D \subset U$ and such that $\mathcal{O}_X(D)$ is ample on X.
(2) Let D be as in (1). Show that $E = \pi(D)$ is an effective Cartier divisor on Y.
(3) Briefly indicate why

(a) the map $\mathcal{O}_Y \to \pi_*\mathcal{O}_X$ has a coherent cokernel Q which is supported in $Y \setminus V$, and
(b) for every n there is a corresponding map $\mathcal{O}_Y(nE) \to \pi_*\mathcal{O}_X(nD)$ whose cokernel is isomorphic to Q.

(4) Show that $\dim_k H^0(X, \mathcal{O}_X(nD)) - \dim_k H^0(Y, \mathcal{O}_Y(nE))$ is bounded (by what?) and conclude that the invertible sheaf $\mathcal{O}_Y(nE)$ has lots of sections for large n (why?).

79.40. Schemes, Final Exam, Fall 2011

These were the questions in the final exam of a course on Commutative Algebra, in the Fall of 2011 at Columbia University.

Exercise 79.40.1 (Definitions)**.** Provide definitions of the italicized concepts.

(1) a *Noetherian* ring,
(2) a *Noetherian* scheme,
(3) a *finite* ring homomorphism,
(4) a *finite* morphism of schemes,
(5) the *dimension* of a ring.

Exercise 79.40.2 (Results)**.** State something formally equivalent to the fact discussed in the course.

(1) Zariski's Main Theorem.
(2) Noether normalization.
(3) Chinese remainder theorem.
(4) Going up for finite ring maps.

Exercise 79.40.3. Let $(A, \mathfrak{m}, \kappa)$ be a Noetherian local ring whose residue field has characteristic not 2. Suppose that $\mathfrak{m}$ is generated by three elements x, y, z and that $x^2 + y^2 + z^2 = 0$ in A.

(1) What are the possible values of $\dim(A)$?
(2) Give an example to show that each value is possible.
(3) Show that A is a domain if $\dim(A) = 2$. (Hint: look at $\bigoplus_{n\geq 0} \mathfrak{m}^n/\mathfrak{m}^{n+1}$.)

Exercise 79.40.4. Let A be a ring. Let $S \subset T \subset A$ be multiplicative subsets. Assume that

$$\{\mathfrak{q} \mid \mathfrak{q} \cap S = \emptyset\} = \{\mathfrak{q} \mid \mathfrak{q} \cap T = \emptyset\}.$$

Show that $S^{-1}A \to T^{-1}A$ is an isomorphism.

Exercise 79.40.5. Let k be an algebraically closed field. Let

$$V_0 = \{A \in \mathrm{Mat}(3 \times 3, k) \mid \mathrm{rank}(A) = 1\} \subset \mathrm{Mat}(3 \times 3, k) = k^9.$$

(1) Show that V_0 is the set of closed points of a (Zariski) locally closed subset $V \subset \mathbf{A}^9_k$.
(2) Is V irreducible?
(3) What is $\dim(V)$?

Exercise 79.40.6. Prove that the ideal (x^2, xy, y^2) in $\mathbf{C}[x, y]$ cannot be generated by 2 elements.

Exercise 79.40.7. Let $f \in \mathbf{C}[x, y]$ be a nonconstant polynomial. Show that for some $\alpha, \beta \in \mathbf{C}$ the $\mathbf{C}$-algebra map

$$\mathbf{C}[t] \longrightarrow \mathbf{C}[x, y]/(f), \quad t \longmapsto \alpha x + \beta y$$

is finite.

Exercise 79.40.8. Show that given finitely many points $p_1, \ldots, p_n \in \mathbf{C}^2$ the scheme $\mathbf{A}^2_{\mathbf{C}} \setminus \{p_1, \ldots, p_n\}$ is a union of two affine opens.

Exercise 79.40.9. Show that there exists a surjective morphism of schemes $\mathbf{A}^1_{\mathbf{C}} \to \mathbf{P}^1_{\mathbf{C}}$. (Surjective just means surjective on underlying sets of points.)

Exercise 79.40.10. Let k be an algebraically closed field. Let $A \subset B$ be an extension of domains which are both finite type k-algebras. Prove that the image of $\mathrm{Spec}(B) \to \mathrm{Spec}(A)$ contains a nonempty open subset of $\mathrm{Spec}(A)$ using the following steps:

(1) Prove it if $A \to B$ is also finite.
(2) Prove it in case the fraction field of B is a finite extension of the fraction field of A.
(3) Reduce the statement to the previous case.

79.41. Schemes, Final Exam, Fall 2013

These were the questions in the final exam of a course on Commutative Algebra, in the Fall of 2013 at Columbia University.

Exercise 79.41.1 (Definitions)**.** Provide definitions of the italicized concepts.

(1) a *radical ideal* of a ring,
(2) a *finite type* ring homomorphism,
(3) a *differential a la Weil*,
(4) a *scheme*.

Exercise 79.41.2 (Results)**.** State something formally equivalent to the fact discussed in the course.

(1) result on hilbert polynomials of graded modules.
(2) dimension of a Noetherian local ring $(R, \mathfrak{m})$ and $\bigoplus_{n \geq 0} \mathfrak{m}^n/\mathfrak{m}^{n+1}$.
(3) Riemann-Roch.
(4) Clifford's theorem.
(5) Chevalley's theorem.

Exercise 79.41.3. Let $A \to B$ be a ring map. Let $S \subset A$ be a multiplicative subset. Assume that $A \to B$ is of finite type and $S^{-1}A \to S^{-1}B$ is surjective. Show that there exists an $f \in S$ such that $A_f \to B_f$ is surjective.

Exercise 79.41.4. Give an example of an injective local homomorphism $A \to B$ of local rings, such that $\mathrm{Spec}(B) \to \mathrm{Spec}(A)$ is not surjective.

Situation 79.41.5 (Notation plane curve)**.** Let k be an algebraically closed field. Let $F(X_0, X_1, X_2) \in k[X_0, X_1, X_2]$ be an irreducible polynomial homogenenous of degree d. We let

$$D = V(F) \subset \mathbf{P}^2$$

be the projective plane curve given by the vanishing of F. Set $x = X_1/X_0$ and $y = X_2/X_0$ and $f(x, y) = X_0^{-d} F(X_0, X_1, X_2) = F(1, x, y)$. We denote K the fraction

field of the domain $k[x, y]/(f)$. We let C be the abstract curve corresponding to K. Recall (from the lectures) that there is a surjective map $C \to D$ which is bijective over the nonsingular locus of D and an isomorphism if D is nonsingular. Set $f_x = \partial f/\partial x$ and $f_y = \partial f/\partial y$. Finally, we denote $\omega = \mathrm{d}x/f_y = -\mathrm{d}y/f_x$ the element of $\Omega_{K/k}$ discussed in the lectures. Denote K_C the divisor of zeros and poles of ω.

Exercise 79.41.6. In Situation 79.41.5 assume $d \geq 3$ and that the curve D has exactly one singular point, namely $P = (1 : 0 : 0)$. Assume further that we have the expansion

$$f(x, y) = xy + h.o.t$$

around $P = (0, 0)$. Then C has two points v and w lying over over P characterized by

$$v(x) = 1, v(y) > 1 \quad \text{and} \quad w(x) > 1, w(y) = 1$$

(1) Show that the element $\omega = \mathrm{d}x/f_y = -\mathrm{d}y/f_x$ of $\Omega_{K/k}$ has a first order pole at both v and w. (The behaviour of ω at nonsingular points is as discussed in the lectures.)
(2) In the lectures we have shown that ω vanishes to order $d-3$ at the divisor $X_0 = 0$ pulled back to C under the map $C \to D$. Combined with the information of (1) what is the degree of the divisor of zeros and poles of ω on C?
(3) What is the genus of the curve C?

Exercise 79.41.7. In Situation 79.41.5 assume $d = 5$ and that the curve $C = D$ is nonsingular. In the lectures we have shown that the genus of C is 6 and that the linear system K_C is given by

$$L(K_C) = \{h\omega \mid h \in k[x, y],\ \deg(h) \leq 2\}$$

where deg indicates total degree[2]. Let $P_1, P_2, P_3, P_4, P_5 \in D$ be pairwise distinct points lying in the affine open $X_0 \neq 0$. We denote $\sum P_i = P_1 + P_2 + P_3 + P_4 + P_5$ the corresponding divisor of C.

(1) Describe $L(K_C - \sum P_i)$ in terms of polynomials.
(2) What are the possibilities for $l(\sum P_i)$?

Exercise 79.41.8. Write down an F as in Situation 79.41.5 with $d = 100$ such that the genus of C is 0.

Exercise 79.41.9. Let k be an algebraically closed field. Let K/k be finitely generated field extension of transcendence degree 1. Let C be the abstract curve corresponding to K. Let $V \subset K$ be a g^r_d and let $\Phi : C \to \mathbf{P}^r$ be the corresponding morphism. Show that the image of C is contained in a quadric[3] if d is V is a complete linear system and d is large enough relative to the genus of C. (Extra credit: good bound on the degree needed.)

Exercise 79.41.10. Notation as in Situation 79.41.5. Let $U \subset \mathbf{P}^2_k$ be the open subscheme whose complement is D. Describe the k-algebra $A = \mathcal{O}_{\mathbf{P}^2_k}(U)$. Give an upper bound for the number of generators of A as a k-algebra.

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[2] We get ≤ 2 because $d - 3 = 5 - 3 = 2$.

[3] A quadric is a degree 2 hypersurface, i.e., the zero set in $\mathbf{P}^r$ of a degree 2 homogeneous polynomial.

79.42. Other chapters

Preliminaries

(1) Introduction
(2) Conventions
(3) Set Theory
(4) Categories
(5) Topology
(6) Sheaves on Spaces
(7) Sites and Sheaves
(8) Stacks
(9) Fields
(10) Commutative Algebra
(11) Brauer Groups
(12) Homological Algebra
(13) Derived Categories
(14) Simplicial Methods
(15) More on Algebra
(16) Smoothing Ring Maps
(17) Sheaves of Modules
(18) Modules on Sites
(19) Injectives
(20) Cohomology of Sheaves
(21) Cohomology on Sites
(22) Differential Graded Algebra
(23) Divided Power Algebra
(24) Hypercoverings

Schemes

(25) Schemes
(26) Constructions of Schemes
(27) Properties of Schemes
(28) Morphisms of Schemes
(29) Cohomology of Schemes
(30) Divisors
(31) Limits of Schemes
(32) Varieties
(33) Topologies on Schemes
(34) Descent
(35) Derived Categories of Schemes
(36) More on Morphisms
(37) More on Flatness
(38) Groupoid Schemes
(39) More on Groupoid Schemes
(40) Étale Morphisms of Schemes

Topics in Scheme Theory

(41) Chow Homology
(42) Adequate Modules
(43) Dualizing Complexes
(44) Étale Cohomology
(45) Crystalline Cohomology
(46) Pro-étale Cohomology

Algebraic Spaces

(47) Algebraic Spaces
(48) Properties of Algebraic Spaces
(49) Morphisms of Algebraic Spaces
(50) Decent Algebraic Spaces
(51) Cohomology of Algebraic Spaces
(52) Limits of Algebraic Spaces
(53) Divisors on Algebraic Spaces
(54) Topologies on Algebraic Spaces
(55) Descent and Algebraic Spaces
(56) Derived Categories of Spaces
(57) More on Morphisms of Spaces
(58) Algebraic Spaces over Fields
(59) Groupoids in Algebraic Spaces
(60) More on Groupoids in Spaces
(61) Bootstrap
(62) Quotients of Groupoids
(63) Simplicial Spaces

Deformation Theory

(64) Formal Deformation Theory
(65) Deformation Theory
(66) The Cotangent Complex

Algebraic Stacks

(67) Algebraic Stacks
(68) Examples of Stacks
(69) Sheaves on Algebraic Stacks
(70) Criteria for Representability
(71) Artin's Axioms
(72) Quot and Hilbert Spaces
(73) Properties of Algebraic Stacks
(74) Morphisms of Algebraic Stacks
(75) Cohomology of Algebraic Stacks
(76) Derived Categories of Stacks
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Miscellany

(78) Examples
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(82) Coding Style
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CHAPTER 80

A Guide to the Literature

80.1. Short introductory articles

- Barbara Fantechi: *Stacks for Everybody* [**Fan01**]
- Dan Edidin: *What is a stack?* [**Edi03**]
- Dan Edidin: *Notes on the construction of the moduli space of curves* [**Edi00**]
- Angelo Vistoli: *Intersection theory on algebraic stacks and on their moduli spaces*, and especially the appendix. [**Vis89**]

80.2. Classic references

- Mumford: *Picard groups of moduli problems* [**Mum65**]
  Mumford never uses the term "stack" here but the concept is implicit in the paper; he computes the picard group of the moduli stack of elliptic curves.
- Deligne, Mumford: *The irreducibility of the space of curves of given genus* [**DM69**]
  This influential paper introduces "algebraic stacks" in the sense which are now universally called Deligne-Mumford stacks (stacks with representable diagonal which admit étale presentations by schemes). There are many foundational results *without proof*. The paper uses stacks to give two proofs of the irreducibility of the moduli space of curves of genus g.
- Artin: *Versal deformations and algebraic stacks* [**Art74**]
  This paper introduces "algebraic stacks" which generalize Deligne-Mumford stacks and are now commonly referred to as *Artin stacks*, stacks with representable diagonal which admit smooth presentations by schemes. This paper gives deformation-theoretic criterion known as Artin's criterion which allows one to prove that a given moduli stack is an Artin stack without explicitly exhibiting a presentation.

80.3. Books and online notes

- Laumon, Moret-Bailly: *Champs Algébriques* [**LMB00**]
  This book is currently the most exhaustive reference on stacks containing many foundational results. It assumes the reader is familiar with algebraic spaces and frequently references Knutson's book [**Knu71**]. There is an error in chapter 12 concerning the functoriality of the lisse-étale site of an algebraic stack. One doesn't need to worry about this as the error has been patched

by Martin Olsson (see [**Ols07b**]) and the results in the remaining chapters (after perhaps slight modification) are correct.

- The Stacks Project Authors: *Stacks Project* [**Sta**].
  You are reading it!
- Anton Geraschenko: *Lecture notes for Martin Olsson's class on stacks* [**Ols07a**]
  This course systematically develops the theory of algebraic spaces before introducing algebraic stacks (first defined in Lecture 27!). In addition to basic properties, the course covers the equivalence between being Deligne-Mumford and having unramified diagonal, the lisse-étale site on an Artin stack, the theory of quasi-coherent sheaves, the Keel-Mori theorem, cohomological descent, and gerbes (and their relation to the Brauer group). There are also some exercises.
- Behrend, Conrad, Edidin, Fantechi, Fulton, Göttsche, and Kresch: *Algebraic stacks*, online notes for a book being currently written [**BCE$^+$07**]
  The aim of this book is to give a friendly introduction to stacks without assuming a sophisticated background with a focus on examples and applications. Unlike [**LMB00**], it is not assumed that the reader has digested the theory of algebraic spaces. Instead, Deligne-Mumford stacks are introduced with algebraic spaces being a special case with part of the goal being to develop enough theory to prove the assertions in [**DM69**]. The general theory of Artin stacks is to be developed in the second part. Only a fraction of the book is now available on Kresch's website.

80.4. Related references on foundations of stacks

- Vistoli: *Notes on Grothendieck topologies, fibered categories and descent theory* [**Vis05**]
  Contains useful facts on fibered categories, stacks and descent theory in the fpqc topology as well as rigorous proofs.
- Knutson: *Algebraic Spaces* [**Knu71**]
  This book, which evolved from his PhD thesis under Michael Artin, contains the foundations of the theory of algebraic spaces. The book [**LMB00**] frequently references this text. See also Artin's papers on algebraic spaces: [**Art69a**], [**Art69b**], [**Art69c**], [**Art70**], [**Art71b**], [**Art71a**], [**Art73**], and [**Art74**]
- Grothendieck et al, *Théorie des Topos et Cohomologie Étale des Schémas I, II, III* also known as SGA4 [**AGV71**]
  Volume 1 contains many general facts on universes, sites and fibered categories. The word "champ" (French for "stack") appears in Deligne's Exposé XVIII.
- Jean Giraud: *Cohomologie non abélienne* [**Gir65**]
  The book discusses fibered categories, stacks, torsors and gerbes over general sites but does not discuss algebraic stacks. For instance, if G is a sheaf of abelian groups on X, then in the same way $H^1(X, G)$ can be identified with G-torsors, $H^2(X, G)$

can be identified with an appropriately defined set of G-gerbes. When G is not abelian, then $H^2(X, G)$ is defined as the set of G-gerbes.

- Kelly and Street: *Review of the elements of 2-categories* [**KS74**]
  The category of stacks form a 2-category although a simple type of 2-category where are 2-morphisms are invertible. This is a reference on general 2-categories. I have never used this so I cannot say how useful it is. Also note that [**Sta**] contains some basics on 2-categories.

80.5. Papers in the literature

Below is a list of research papers which contain fundamental results on stacks and algebraic spaces. The intention of the summaries is to indicate only the results of the paper which contribute toward stack theory; in many cases these results are subsidiary to the main goals of the paper. We divide the papers into categories with some papers falling into multiple categories.

80.5.1. Deformation theory and algebraic stacks. The first three papers by Artin do not contain anything on stacks but they contain powerful results with the first two papers being essential for [**Art74**].

- Artin: *Algebraic approximation of structures over complete local rings* [**Art69a**]
  It is proved that under mild hypotheses any effective formal deformation can be approximated: if $F : (Sch/S) \to (Sets)$ is a contravariant functor locally of finite presentation with S finite type over a field or excellent DVR, $s \in S$, and $\hat{\xi} \in F(\hat{\mathcal{O}}_{S,s})$ is an effective formal deformation, then for any $n > 0$, there exists an residually trivial étale neighborhood $(S', s') \to (S, s)$ and $\xi' \in F(S')$ such that ξ' and $\hat{\xi}$ agree up to order n (ie. have the same restriction in $F(\mathcal{O}_{S,s}/\mathfrak{m}^n)$).
- Artin: *Algebraization of formal moduli I* [**Art69b**]
  It is proved that under mild hypotheses any effective formal versal deformation is algebraizable. Let $F : (Sch/S) \to (Sets)$ be a contravariant functor locally of finite presentation with S finite type over a field or excellent DVR, $s \in S$ be a locally closed point, $\hat{A}$ be a complete noetherian local $\mathcal{O}_S$-algebra with residue field k' a finite extension of $k(s)$, and $\hat{\xi} \in F(\hat{A})$ be an effective formal versal deformation of an element $\xi_0 \in F(k')$. Then there is a scheme X finite type over S and a closed point $x \in X$ with residue field $k(x) = k'$ and an element $\xi \in F(X)$ such that there is an isomorphism $\hat{\mathcal{O}}_{X,x} \cong \hat{A}$ identifying the restrictions of ξ and $\hat{\xi}$ in each $F(\hat{A}/\mathfrak{m}^n)$. The algebraization is unique if $\hat{\xi}$ is a universal deformation. Applications are given to the representability of the Hilbert and Picard schemes.
- Artin: *Algebraization of formal moduli. II* [**Art70**]
  Vaguely, it is shown that if one can contract a closed subset $Y' \subseteq X'$ formally locally around Y', then exists a global morphism $X' \to X$ contracting Y with X an algebraic space.

- Artin: *Versal deformations and algebraic stacks* [**Art74**]
  This momentous paper builds on his work in [**Art69a**] and [**Art69b**]. This paper introduces Artin's criterion which allows one to prove algebraicity of a stack by verifying deformation-theoretic properties. More precisely (but not very precisely), Artin constructs a presentation of a limit preserving stack $\mathcal{X}$ locally around a point $x \in \mathcal{X}(k)$ as follows: assuming the stack $\mathcal{X}$ satisfies Schlessinger's criterion([**Sch68**]), there exists a formal versal deformation $\hat{\xi} \in \lim \mathcal{X}(\hat{A}/\mathfrak{m}^n)$ of x. Assuming that formal deformations are effective (i.e., $\mathcal{X}(\hat{A}) \to \lim \mathcal{X}(\hat{A}/\mathfrak{m}^n)$ is bijective), then one obtains an effective formal versal deformation $\xi \in \mathcal{X}(\hat{A})$. Using results in [**Art69b**], one produces a finite type scheme U and an element $\xi_U : U \to \mathcal{X}$ which is formally versal at a point $u \in U$ over x. Then if we assume $\mathcal{X}$ admits a deformation and obstruction theory satisfying certain conditions (ie. compatibility with étale localization and completion as well as constructibility condition), then it is shown in section 4 that formal versality is an open condition so that after shrinking U, $U \to \mathcal{X}$ is smooth. Artin also presents a proof that any stack admitting an fppf presentation by a scheme admits a smooth presentation by a scheme so that in particular one can form quotient stacks by flat, separated, finitely presented group schemes.
- Conrad, de Jong: *Approximation of Versal Deformations* [**CdJ02**]
  This paper offers an approach to Artin's algebraization result by applying Popescu's powerful result: if A is a noetherian ring and B a noetherian A-algebra, then the map $A \to B$ is a regular morphism if and only if B is a direct limit of smooth A-algebras. It is not hard to see that Popescu's result implies Artin's approximation over an arbitrary excellent scheme (the excellence hypothesis implies that for a local ring A, the map $A^{\mathrm{h}} \to \hat{A}$ from the henselization to the completion is regular). The paper uses Popescu's result to give a "groupoid" generalization of the main theorem in [**Art69b**] which is valid over arbitrary excellent base schemes and for arbitrary points $s \in S$. In particular, the results in [**Art74**] hold under an arbitrary excellent base. They discuss the étale-local uniqueness of the algebraization and whether the automorphism group of the object acts naturally on the henselization of the algebraization.
- Jason Starr: *Artin's axioms, composition, and moduli spaces* [**Sta06**]
  The paper establishes that Artin's axioms for algebraization are compatible with the composition of 1-morphisms.
- Martin Olsson: *Deformation theory of representable morphism of algebraic stacks* [**Ols06a**]
  This generalizes standard deformation theory results for morphisms of schemes to representable morphisms of algebraic stacks in terms of the cotangent complex. These results cannot be

viewed as consequences of Illusie's general theory as the cotangent complex of a representable morphism $X \to \mathcal{X}$ is not defined in terms of cotangent complex of a morphism of ringed topoi (because the lisse-étale site is not functorial).

80.5.2. Coarse moduli spaces.

- Keel, Mori: *Quotients in Groupoids* [**KM97**]
  It had apparently long been "folklore" that separated Deligne-Mumford stacks admitted coarse moduli spaces. A rigorous (although terse) proof of the following theorem is presented here: if $\mathcal{X}$ is an Artin stack locally of finite type over a noetherian base scheme such that the inertia stack $I_{\mathcal{X}} \to \mathcal{X}$ is finite, then there exists a coarse moduli space $\phi : \mathcal{X} \to Y$ with ϕ separated and Y an algebraic space locally of finite type over S. The hypothesis that the inertia is finite is precisely the right condition: there exists a coarse moduli space $\phi : \mathcal{X} \to Y$ with ϕ separated if and only if the inertia is finite.
- Conrad: *The Keel-Mori Theorem via Stacks* [**Con05**]
  Keel and Mori's paper [**KM97**] is written in the groupoid language and some find it challenging to grasp. Brian Conrad presents a stack-theoretic version of the proof which is quite transparent although it uses the sophisticated language of stacks. Conrad also removes the noetherian hypothesis.
- Rydh: *Existence of quotients by finite groups and coarse moduli spaces* [**Ryd07a**]
  Rydh removes the hypothesis from [**KM97**] and [**Con05**] that $\mathcal{X}$ be finitely presented over some base.
- Abramovich, Olsson, Vistoli: *Tame stacks in positive characteristic* [**AOV08**]
  They define a *tame Artin stack* as an Artin stack with finite inertia such that if $\phi : \mathcal{X} \to Y$ is the coarse moduli space, ϕ_* is exact on quasi-coherent sheaves. They prove that for an Artin stack with finite inertia, the following are equivalent: $\mathcal{X}$ is tame $\iff$ the stabilizers of $\mathcal{X}$ are linearly reductive $\iff$ $\mathcal{X}$ is étale locally on the coarse moduli space a quotient of an affine scheme by a linearly reductive group scheme. For a tame Artin stack, the coarse moduli space is particularly nice. For instance, the coarse moduli space commutes with arbitrary base change while a general coarse moduli space for an Artin stack with finite inertia will only commute with flat base change.
- Alper: *Good moduli spaces for Artin stacks* [**Alp08**]
  For general Artin stacks with infinite affine stabilizer groups (which are necessarily non-separated), coarse moduli spaces often do not exist. The simplest example is $[\mathbf{A}^1/\mathbf{G}_m]$. It is defined here that a quasi-compact morphism $\phi : \mathcal{X} \to Y$ is a *good moduli space* if $\mathcal{O}_Y \to \phi_*\mathcal{O}_{\mathcal{X}}$ is an isomorphism and ϕ_* is exact on quasi-coherent sheaves. This notion generalizes a tame Artin stack in [**AOV08**] as well as encapsulates Mumford's geometric invariant theory: if G is a reductive group acting linearly on $X \subseteq \mathbf{P}^n$, then the morphism from the quotient stack of the

semi-stable locus to the GIT quotient $[X^{ss}/G] \to X//G$ is a good moduli space. The notion of a good moduli space has many nice geometric properties: (1) ϕ is surjective, universally closed, and universally submersive, (2) ϕ identifies points in Y with points in $\mathcal{X}$ up to closure equivalence, (3) ϕ is universal for maps to algebraic spaces, (4) good moduli spaces are stable under arbitrary base change, and (5) a vector bundle on an Artin stack descends to the good moduli space if and only if the representations are trivial at closed points.

80.5.3. Intersection theory.

- Vistoli: *Intersection theory on algebraic stacks and on their moduli spaces* [**Vis89**]

  This paper develops the foundations for intersection theory with rational coefficients for Deligne-Mumford stacks. If $\mathcal{X}$ is a separated Deligne-Mumford stack, the chow group $A_*(\mathcal{X})$ with rational coefficients is defined as the free abelian group of integral closed substacks of dimension k up to rational equivalence. There is a flat pullback, a proper push-forward and a generalized Gysin homomorphism for regular local embeddings. If $\phi : \mathcal{X} \to Y$ is a moduli space (ie. a proper morphism with is bijective on geometric points), there is an induced push-forward $A_*(\mathcal{X}) \to A_k(Y)$ which is an isomorphism.
- Edidin, Graham: *Equivariant Intersection Theory* [**EG98**]

  The purpose of this article is to develop intersection theory with integral coefficients for a quotient stack $[X/G]$ of an action of an algebraic group G on an algebraic space X or, in other words, to develop a G-equivariant intersection theory of X. Equivariant chow groups defined using only invariant cycles does not produce a theory with nice properties. Instead, generalizing Totaro's definition in the case of BG and motivated by the fact that if $V \to X$ is a vector bundle then $A_i(X) \cong A_i(V)$ naturally, the authors define $A_i^G(X)$ as follows: Let $\dim(X) = n$ and $\dim(G) = g$. For each i, choose a l-dimensional G-representation V where G acts freely on an open subset $U \subseteq V$ whose complement as codimension $d > n - i$. So $X_G = [X \times U/G]$ is an algebraic space (it can even be chosen to be a scheme). Then they define $A_i^G(X) = A_{i+l-g}(X_G)$. For the quotient stack, one defines $A_i([X/G]) := A_{i+g}^G(X) = A_{i+l}(X_G)$. In particular, $A_i([X/G]) = 0$ for $i > \dim[X/G] = n-g$ but can be non-zero for $i << 0$ (eg. $A_i(B\mathbf{G}_m) = \mathbf{Z}$ for $i \leq 0$). They establish that these equivariant Chow groups enjoy the same functorial properties as ordinary Chow groups. Furthermore, they establish that if $[X/G] \cong [Y/H]$ that $A_i([X/G]) = A_i([Y/H])$ so that the definition is independent on how the stack is presented as a quotient stack.
- Kresch: *Cycle Groups for Artin Stacks* [**Kre99**]

  Kresch defines Chow groups for arbitrary Artin stacks agreeing with Edidin and Graham's definition in [**EG98**] in the case of

quotient stack. For algebraic stacks with affine stabilizer groups, the theory satisfies the usual properties.

- Behrend and Fantechi: *The intrinsic normal cone* [**BF97**]
  Generalizing a construction due to Li and Tian, Behrend and Fantechi construct a virtual fundamental class for a Deligne-Mumford stack.

80.5.4. Quotient stacks. Quotient stacks[1] form a very important subclass of Artin stacks which include almost all moduli stacks studied by algebraic geometers. The geometry of a quotient stack $[X/G]$ is the G-equivariant geometry of X. It is often easier to show properties are true for quotient stacks and some results are only known to be true for quotient stacks. The following papers address: When is an algebraic stack a global quotient stack? Is an algebraic stack "locally" a quotient stack?

- Laumon, Moret-Bailly: [**LMB00**, Chapter 6]
  Chapter 6 contains several facts about the local and global structure of algebraic stacks. It is proved that an algebraic stack $\mathcal{X}$ over S is a quotient stack $[Y/G]$ with Y an algebraic space (resp. scheme, resp. affine scheme) and G a finite group if and only if there exists an algebraic space (resp. scheme, resp. affine scheme) Y' and an finite étale morphism $Y' \to \mathcal{X}$. It is shown that any Deligne-Mumford stack over S and $x : \mathrm{Spec}(K) \to \mathcal{X}$ admits an representable, étale and separated morphism $\phi : [X/G] \to \mathcal{X}$ where G is a finite group acting on an affine scheme over S such that $\mathrm{Spec}(K) = [X/G] \times_{\mathcal{X}} \mathrm{Spec}(K)$. The existence of presentations with geometrically connected fibers is also discussed in detail.
- Edidin, Hassett, Kresch, Vistoli: *Brauer Groups and Quotient stacks* [**EHKV01**]
  First, they establish some fundamental (although not very difficult) facts concerning when a given algebraic stack (always assumed finite type over a noetherian scheme in this paper) is a quotient stack. For an algebraic stack $\mathcal{X}$: $\mathcal{X}$ is a quotient stack $\iff$ there exists a vector bundle $V \to \mathcal{X}$ such that for every geometric point, the stabilizer acts faithfully on the fiber $\iff$ there exists a vector bundle $V \to \mathcal{X}$ and a locally closed substack $V^0 \subseteq V$ such that V^0 is representable and surjects onto F. They establish that an algebraic stack is a quotient stack if there exists finite flat cover by an algebraic space. Any smooth Deligne-Mumford stack with generically trivial stabilizer is a quotient stack. They show that a $\mathbf{G}_m$-gerbe over a noetherian scheme X corresponding to $\beta \in H^2(X, \mathbf{G}_m)$ is a quotient stack if and only if β is in the image of the Brauer map $\mathrm{Br}(X) \to \mathrm{Br}'(X)$. They use this to produce a non-separated Deligne-Mumford stack that is not a quotient stack.
- Totaro: *The resolution property for schemes and stacks* [**Tot04**]

[1] In the literature, *quotient stack* often means a stack of the form $[X/G]$ with X an algebraic space and G a subgroup scheme of GL_n rather than an arbitrary flat group scheme.

A stack has the resolution property if every coherent sheaf is the quotient of a vector bundle. The first main theorem is that if $\mathcal{X}$ is a normal noetherian algebraic stack with affine stabilizer groups at closed points, then the following are equivalent: (1) $\mathcal{X}$ has the resolution property and (2) $\mathcal{X} = [Y/\mathrm{GL}_n]$ with Y quasi-affine. In the case $\mathcal{X}$ is finite type over a field, then (1) and (2) are equivalent to: (3) $\mathcal{X} = [\mathrm{Spec}(A)/G]$ with G an affine group scheme finite type over k. The implication that quotient stacks have the resolution property was proven by Thomason. The second main theorem is that if $\mathcal{X}$ is a smooth Deligne-Mumford stack over a field which has a finite and generically trivial stabilizer group $I_{\mathcal{X}} \to \mathcal{X}$ and whose coarse moduli space is a scheme with affine diagonal, then $\mathcal{X}$ has the resolution property. Another cool result states that if $\mathcal{X}$ is a noetherian algebraic stack satisfying the resolution property, then $\mathcal{X}$ has affine diagonal if and only if the closed points have affine stabilizer.

- Kresch: *On the Geometry of Deligne-Mumford Stacks* [**Kre09**]
  This article summarizes general structure results of Deligne-Mumford stacks (of finite type over a field) and contains some interesting results concerning quotient stacks. It is shown that any smooth, separated, generically tame Deligne-Mumford stack with quasi-projective coarse moduli space is a quotient stack $[Y/G]$ with Y quasi-projective and G an algebraic group. If $\mathcal{X}$ is a Deligne-Mumford stack whose coarse moduli space is a scheme, then $\mathcal{X}$ is Zariski-locally a quotient stack if and only if it admits a Zariski-open covering by stack quotients of schemes by finite groups. If $\mathcal{X}$ is a Deligne-Mumford stack proper over a field of characteristic 0 with coarse moduli space Y, then: Y is projective and $\mathcal{X}$ is a quotient stack $\iff$ Y is projective and $\mathcal{X}$ possesses a generating sheaf $\iff$ $\mathcal{X}$ admits a closed embedding into a smooth proper DM stack with projective coarse moduli space. This motivates a definition that a Deligne-Mumford stack is *projective* if there exists a closed embedding into a smooth, proper Deligne-Mumford stack with projective coarse moduli space.
- Kresch, Vistoli *On coverings of Deligne-Mumford stacks and surjectivity of the Brauer map* [**KV04**]
  It is shown that in characteristic 0 and for a fixed n, the following two statements are equivalent: (1) every smooth Deligne-Mumford stack of dimension n is a quotient stack and (2) the Azumaya Brauer group coincides with the cohomological Brauer group for smooth schemes of dimension n.
- Kresch: *Cycle Groups for Artin Stacks* [**Kre99**]
  It is shown that a reduced Artin stack finite type over a field with affine stabilizer groups admits a stratification by quotient stacks.
- Abramovich-Vistoli: *Compactifying the space of stable maps* [**AV02**]

Lemma 2.2.3 establishes that for any separated Deligne-Mumford stack is étale-locally on the coarse moduli space a quotient stack $[U/G]$ where U affine and G a finite group. [**Ols06b**, Theorem 2.12] shows in this argument G is even the stabilizer group.

- Abramovich, Olsson, Vistoli: *Tame stacks in positive characteristic* [**AOV08**]
  This paper shows that a tame Artin stack is étale locally on the coarse moduli space a quotient stack of an affine by the stabilizer group.
- Alper: *On the local quotient structure of Artin stacks* [**Alp10**]
  It is conjectured that for an Artin stack $\mathcal{X}$ and a closed point $x \in \mathcal{X}$ with linearly reductive stabilizer, then there is an étale morphism $[V/G_x] \to \mathcal{X}$ with V an algebraic space. Some evidence for this conjecture is given. A simple deformation theory argument (based on ideas in [**AOV08**]) shows that it is true formally locally. A stack-theoretic proof of Luna's étale slice theorem is presented proving that for stacks $\mathcal{X} = [\mathrm{Spec}(A)/G]$ with G linearly reductive, then étale locally on the GIT quotient $\mathrm{Spec}(A^G)$, $\mathcal{X}$ is a quotient stack by the stabilizer.

80.5.5. Cohomology.

- Olsson: *Sheaves on Artin stacks* [**Ols07b**]
  This paper develops the theory of quasi-coherent and constructible sheaves proving basic cohomological properties. This paper corrects a mistake in [**LMB00**] in the functoriality of the lisse-étale site. The cotangent complex is constructed. In addition, the following theorems are proved: Grothendieck's Fundamental Theorem for proper morphisms, Grothendieck's Existence Theorem, Zariski's Connectness Theorem and finiteness theorem for proper pushforwards of coherent and constructible sheaves.
- Behrend: *Derived l-adic categories for algebraic stacks* [**Beh03**]
  Proves the Lefschetz trace formula for algebraic stacks.
- Behrend: *Cohomology of stacks* [**Beh04**]
  Defines the de Rham cohomology for differentiable stacks and singular cohomology for topological stacks.
- Faltings: *Finiteness of coherent cohomology for proper fppf stacks* [**Fal03**]
  Proves coherence for direct images of coherent sheaves for proper morphisms.
- Abramovich, Corti, Vistoli: *Twisted bundles and admissible covers* [**ACV03**]
  The appendix contains the proper base change theorem for étale cohomology for tame Deligne-Mumford stacks.

80.5.6. Existence of finite covers by schemes. The existence of finite covers of Deligne-Mumford stacks by schemes is an important result. In intersection theory on Deligne-Mumford stacks, it is an essential ingredient in defining proper push-forward for non-representable morphisms. There are several results about $\overline{\mathcal{M}}_g$ relying on the existence of a finite cover by a *smooth* scheme which was proven by Looijenga. Perhaps the first result in this direction is [**Ses72**, Theorem 6.1] which treats the equivariant setting.

- Vistoli: *Intersection theory on algebraic stacks and on their moduli spaces* [**Vis89**]
  If $\mathcal{X}$ is a Deligne-Mumford stack with a moduli space (ie. a proper morphism which is bijective on geometric points), then there exists a finite morphism $X \to \mathcal{X}$ from a scheme X.
- Laumon, Moret-Bailly: [**LMB00**, Chapter 16]
  As an application of Zariski's main theorem, Theorem 16.6 establishes: if $\mathcal{X}$ is a Deligne-Mumford stack finite type over a noetherian scheme, then there exists a finite, surjective, generically étale morphism $Z \to \mathcal{X}$ with Z a scheme. It is also shown in Corollary 16.6.2 that any noetherian normal algebraic space is isomorphic to the algebraic space quotient X'/G for a finite group G acting a normal scheme X.
- Edidin, Hassett, Kresch, Vistoli: *Brauer Groups and Quotient stacks* [**EHKV01**]
  Theorem 2.7 states: if $\mathcal{X}$ is an algebraic stack of finite type over a noetherian ground scheme S, then the diagonal $\mathcal{X} \to \mathcal{X} \times_S \mathcal{X}$ is quasi-finite if and only if there exists a finite surjective morphism $X \to F$ from a scheme X.
- Kresch, Vistoli: *On coverings of Deligne-Mumford stacks and surjectivity of the Brauer map* [**KV04**]
  It is proved here that any smooth, separated Deligne-Mumford stack finite type over a field with quasi-projective coarse moduli space admits a finite, flat cover by a smooth quasi-projective scheme.
- Olsson: *On proper coverings of Artin stacks* [**Ols05**]
  Proves that if $\mathcal{X}$ is an Artin stack separated and finite type over S, then there exists a proper surjective morphism $X \to \mathcal{X}$ from a scheme X quasi-projective over S. As an application, Olsson proves coherence and constructibility of direct image sheaves under proper morphisms. As an application, he proves Grothendieck's existence theorem for proper Artin stacks.

80.5.7. Rigidification. Rigidification is a process for removing a flat subgroup from the inertia. For example, if X is a projective variety, the morphism from the Picard stack to the Picard scheme is a rigidification of the group of automorphism $\mathbf{G}_m$.

- Abramovich, Corti, Vistoli: *Twisted bundles and admissible covers* [**ACV03**]
  Let $\mathcal{X}$ be an algebraic stack over S and H be a flat, finitely presented separated group scheme over S. Assume that for every object $\xi \in \mathcal{X}(T)$ there is an embedding $H(S) \hookrightarrow \mathrm{Aut}_{\mathcal{X}(T)}(\xi)$ which is compatible under pullbacks in the sense that for every arrow $\phi : \xi \to \xi'$ over $f : T \to T'$ and $g \in H(T)$, $g \circ \phi = \phi \circ f^* g$. Then there exists an algebraic stack $\mathcal{X}/H$ and a morphism $\rho : \mathcal{X} \to \mathcal{X}/H$ which is an fppf gerbe such that for every $\xi \in \mathcal{X}(T)$, the morphism $\mathrm{Aut}_{\mathcal{X}(T)}(\xi) \to \mathrm{Aut}_{\mathcal{X}/H(T)}(\xi)$ is surjective with kernel $H(T)$.
- Romagny: *Group actions on stacks and applications* [**Rom05**]

Discusses how group actions behave with respect to rigidifications.

- Abramovich, Graber, Vistoli: *Gromov-Witten theory for Deligne-Mumford stacks* [**AGV08**]
  The appendix gives a summary of rigidification as in [**ACV03**] with two alternative interpretations. This paper also contains constructions for gluing algebraic stacks along closed substacks and for taking roots of line bundles.
- Abramovich, Olsson, Vistoli: *Tame stacks in positive characteristic* ([**AOV08**])
  The appendix handles the more complicated situation where the flat subgroup stack of the inertia $H \subseteq I_{\mathcal{X}}$ is normal but not necessarily central.

80.5.8. Stacky curves.

- Abramovich, Vistoli: *Compactifying the space of stable maps* [**AV02**]
  This paper introduces *twisted curves*. The moduli space of stable maps from stable curves into an algebraic stack is typically not compact. By using maps from twisted curves, the authors construct a moduli stack which is proper when the target is a tame Deligne-Mumford stack whose coarse moduli space is projective.
- Behrend, Noohi: *Uniformization of Deligne-Mumford curves* [**BN06**]
  Proves a uniformization theorem of Deligne-Mumford analytic curves.

80.5.9. Hilbert, Quot, Hom and branchvariety stacks.

- Vistoli: *The Hilbert stack and the theory of moduli of families* [**Vis91**]
  If $\mathcal{X}$ is a algebraic stack separated and locally of finite type over a locally noetherian and locally separated algebraic space S, Vistoli defines the Hilbert stack $\mathcal{H}\mathrm{ilb}(\mathcal{F}/S)$ parameterizing finite and unramified morphisms from proper schemes. It is claimed without proof that $\mathcal{H}\mathrm{ilb}(\mathcal{F}/S)$ is an algebraic stack. As a consequence, it is proved that with $\mathcal{X}$ as above, the Hom stack $\mathcal{H}\mathrm{om}_S(T, \mathcal{X})$ is an algebraic stack if T is proper and flat over S.
- Olsson, Starr: *Quot functors for Deligne-Mumford stacks* [**OS03**]
  If $\mathcal{X}$ is a Deligne-Mumford stack separated and locally of finite presentation over an algebraic space S and $\mathcal{F}$ is a locally finitely-presented $\mathcal{O}_{\mathcal{X}}$-module, the quot functor $\mathrm{Quot}(\mathcal{F}/\mathcal{X}/S)$ is represented by an algebraic space separated and locally of finite presentation over S. This paper also defines generating sheaves and proves existence of a generating sheaf for tame, separated Deligne-Mumford stacks which are global quotient stacks of a scheme by a finite group.
- Olsson: *Hom-stacks and Restrictions of Scalars* [**Ols06b**]
  Suppose $\mathcal{X}$ and $\mathcal{Y}$ are Artin stacks locally of finite presentation over an algebraic space S with finite diagonal with $\mathcal{X}$ proper and flat over S such that fppf-locally on S, $\mathcal{X}$ admits a finite finitely presented flat cover by an algebraic space (eg. $\mathcal{X}$ is Deligne-Mumford or a tame Artin stack). Then $\mathrm{Hom}_S(\mathcal{X}, \mathcal{Y})$ is an Artin stack locally of finite presentation over S.

- Alexeev and Knutson: *Complete moduli spaces of branchvarieties* ([**AK10**])
  They define a branchvariety of $\mathbf{P}^n$ as a finite morphism $X \to \mathbf{P}^n$ from a *reduced* scheme X. They prove that the moduli stack of branchvarieties with fixed Hilbert polynomial and total degrees of i-dimensional components is a proper Artin stack with finite stabilizer. They compare the stack of branchvarieties with the Hilbert scheme, Chow scheme and moduli space of stable maps.
- Lieblich: *Remarks on the stack of coherent algebras* [**Lie06b**]
  This paper constructs a generalization of Alexeev and Knutson's stack of branch-varieties over a scheme Y by building the stack as a stack of algebras over the structure sheaf of Y. Existence proofs of Quot and Hom spaces are given.
- Starr: *Artin's axioms, composition, and moduli spaces* [**Sta06**]
  As an application of the main result, a common generalization of Vistoli's Hilbert stack [**Vis91**] and Alexeev and Knutson's stack of branchvarieties [**AK10**] is provided. If $\mathcal{X}$ is an algebraic stack locally of finite type over an an excellent scheme S with finite diagonal, then the stack $\mathcal{H}$ parameterizing morphisms $g : T \to \mathcal{X}$ from a proper algebraic space T with a G-ample line bundle L is an Artin stack locally of finite type over S.
- Lundkvist and Skjelnes: *Non-effective deformations of Grothendieck's Hilbert functor* [**LS08**]
  Shows that the Hilbert functor of a non-separated scheme is not represented since there are non-effective deformations.

80.5.10. Toric stacks. Toric stacks provide a great class of examples and a natural testing ground for conjectures due to the dictionary between the geometry of a toric stack and the combinatorics of its stacky fan in a similar way that toric varieties provide examples and counterexamples in scheme theory.

- Borisov, Chen and Smith: *The orbifold Chow ring of toric Deligne-Mumford stacks* [**BCS05**]
  Inspired by Cox's construction for toric varieties, this paper defines smooth toric DM stacks as explicit quotient stacks associated to a combinatorial object called a *stacky fan*.
- Iwanari: *The category of toric stacks* [**Iwa09**]
  This paper defines a *toric triple* as a smooth Deligne-Mumford stack $\mathcal{X}$ with an open immersion $\mathbf{G}_m \hookrightarrow \mathcal{X}$ with dense image (and therefore $\mathcal{X}$ is an orbifold) and an action $\mathcal{X} \times \mathbf{G}_m \to \mathcal{X}$. It is shown that there is an equivalence between the 2-category of toric triples and the 1-category of stacky fans. The relationship between toric triples and the definition of smooth toric DM stacks in [**BCS05**] is discussed.
- Iwanari: *Integral Chow rings for toric stacks* [**Iwa07**]
  Generalizes Cox's Δ-collections for toric varieties to toric orbifolds.
- Perroni: *A note on toric Deligne-Mumford stacks* [**Per08**]
  Generalizes Cox's Δ-collections and Iwanari's paper [**Iwa07**] to general smooth toric DM stacks.
- Fantechi, Mann, and Nironi: *Smooth toric DM stacks* [**FMN07**]

This paper defines a smooth toric DM stack as a smooth DM stack $\mathcal{X}$ with the action of a DM torus $\mathcal{T}$ (ie. a Picard stack isomorphic to $T \times BG$ with G finite) having an open dense orbit isomorphic to $\mathcal{T}$. They give a "bottom-up description" and prove an equivalence between smooth toric DM stacks and stacky fans.

- Geraschenko and Satriano: *Toric Stacks I and II* [**GS11a, GS11b**]
  These papers define a toric stack as the stack quotient of a toric variety by a subgroup of its torus. A generically stacky toric stack is defined as a torus invariant substack of a toric stack. This definition encompasses and extends previous definitions of toric stacks. The first paper develops a dictionary between the combinatorics of stacky fans and the geometry of the corresponding stacks. It also gives a moduli interpretation of smooth toric stacks, generalizing the one in [**Per08**]. The second paper proves an intrinsic characterization of toric stacks.

80.5.11. Theorem on formal functions and Grothendieck's Existence Theorem. These papers give generalizations of the theorem on formal functions [**DG67**, III.4.1.5] (sometimes referred to Grothendieck's Fundamental Theorem for proper morphisms) and Grothendieck's Existence Theorem [**DG67**, III.5.1.4].

- Knutson: *Algebraic spaces* [**Knu71**, Chapter V]
  Generalizes these theorems to algebraic spaces.
- Abramovich-Vistoli: *Compactifying the space of stable maps* [**AV02**, A.1.1]
  Generalizes these theorems to tame Deligne-Mumford stacks
- Olsson and Starr: *Quot functors for Deligne-Mumford stacks* [**OS03**]
  Generalizes these theorems to separated Deligne-Mumford stacks.
- Olsson: *On proper coverings of Artin stacks* [**Ols05**]
  Provides a generalization to proper Artin stacks.
- Conrad: *Formal GAGA on Artin stacks* [**Con**]
  Provides a generalization to proper Artin stacks and proves a formal GAGA theorem.
- Olsson: *Sheaves on Artin stacks* [**Ols07b**]
  Provides another proof for the generalization to proper Artin stacks.

80.5.12. Group actions on stacks. Actions of groups on algebraic stacks naturally appear. For instance, symmetric group S_n acts on $\overline{\mathcal{M}}_{g,n}$ and for an action of a group G on a scheme X, the normalizer of G in $\mathrm{Aut}(X)$ acts on $[X/G]$. Furthermore, torus actions on stacks often appear in Gromov-Witten theory.

- Romagny: *Group actions on stacks and applications* [**Rom05**]
  This paper makes precise what it means for a group to act on an algebraic stack and proves existence of fixed points as well as existence of quotients for actions of group schemes on algebraic stacks. See also Romagny's earlier note [**Rom03**].

80.5.13. Taking roots of line bundles. This useful construction was discovered independently by Cadman and by Abramovich, Graber and Vistoli. Given a scheme X with an effective Cartier divisor D, the rth root stack is an Artin stack branched over X at D with a μ_r stabilizer over D and scheme-like away from D.

- Charles Cadman *Using Stacks to Impose Tangency Conditions on Curves* [**Cad07**]
- Abramovich, Graber, Vistoli: *Gromov-Witten theory for Deligne-Mumford stacks* [**AGV08**]

80.5.14. Other papers.

- Lieblich: *Moduli of twisted sheaves* [**Lie07**]
  This paper contains a summary of gerbes and twisted sheaves. If $\mathcal{X} \to X$ is a μ_n-gerbe with X a projective relative surface with smooth connected geometric fibers, it is shown that the stack of semistable $\mathcal{X}$-twisted sheaves is an Artin stack locally of finite presentation over S. This paper also develops the theory of associated points and purity of sheaves on Artin stacks.
- Lieblich, Osserman: *Functorial reconstruction theorem for stacks* [**LO08**]
  Proves some surprising and interesting results on when an algebraic stack can be reconstructed from its associated functor.
- David Rydh: *Noetherian approximation of algebraic spaces and stacks* [**Ryd08**]
  This paper shows that every quasi-compact algebraic stack with quasi-finite diagonal can be approximated by a noetherian stack. There are applications to removing the noetherian hypothesis in results of Chevalley, Serre, Zariski and Chow.

80.6. Stacks in other fields

- Behrend and Noohi: *Uniformization of Deligne-Mumford curves* [**BN06**]
  Gives an overview and comparison of topological, analytic and algebraic stacks.
- Behrang Noohi: *Foundations of topological stacks I* [**Noo05**]
- David Metzler: *Topological and smooth stacks* [**Met05**]

80.7. Higher stacks

- Lurie: *Higher topos theory* [**Lur09f**]
- Lurie: *Derived Algebraic Geometry I - V* [**Lur09a**], [**Lur09b**], [**Lur09c**], [**Lur09d**], [**Lur09e**]
- Toën: *Higher and derived stacks: a global overview* [**Toë09**]
- Toën and Vezzosi: *Homotopical algebraic geometry I, II* [**TV05**], [**TV08**]

80.8. Other chapters

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CHAPTER 81

Desirables

81.1. Introduction

This is basically just a list of things that we want to put in the stacks project. As we add material to the project continuously this is always somewhat behind the current state of the project. In fact, it may have been a mistake to try and list things we should add, because it seems impossible to keep it up to date.

Last updated: Thursday, May 9, 2013.

81.2. Conventions

We should have a chapter with a short list of conventions used in the document. This chapter already exists, see Conventions, Section 2.1, but a lot more could be added there. Especially useful would be to find "hidden" conventions and tacit assumptions and put those there.

81.3. Sites and Topoi

We have a chapter on sites and sheaves, see Sites, Section 7.1. We have a chapter on ringed sites (and topoi) and modules on them, see Modules on Sites, Section 18.1. We have a chapter on cohomology in this setting, see Cohomology on Sites, Section 21.1. But a lot more could be added, especially in the chapter on cohomology.

81.4. Stacks

We have a chapter on (abstract) stacks, see Stacks, Section 8.1. It would be nice if

(1) improve the discussion on "stackyfication",
(2) give examples of stackyfication,
(3) more examples in general,
(4) improve the discussion of gerbes.

Example result which has not been added yet: Given a sheaf of abelian groups $\mathcal{F}$ over $\mathcal{C}$ the set of equivalence classes of gerbes with "group" $\mathcal{F}$ is bijective to $H^2(\mathcal{C}, \mathcal{F})$.

81.5. Simplicial methods

We have a chapter on simplicial methods, see Simplicial, Section 14.1. This has to be reviewed and improved. The discussion of the relationship between simplicial homotopy (also known as combinatorial homotopy) and Kan complexes should be improved upon. Moreover, there should be a chapter on "simplicial algebraic geometry", where we discuss simplicial schemes, and how to think of their geometry,

cohomology, etc. Then this should be tied into the chapter on hypercoverings to "explain" the results of this chapter in the new language.

81.6. Cohomology of schemes

There is already a chapter on cohomology of quasi-coherent sheaves, see Cohomology of Schemes, Section 29.1. We also have chapters on étale cohomology of schemes, crystalline cohomology of schemes, derived categories of schemes. But most of the material is very basic and a lot more could be added here.

81.7. Deformation theory à la Schlessinger

We have a chapter on this material, see Formal Deformation Theory, Section 64.1. What is needed is worked out examples of the general theory, for example the case of representations of a fixed abstract group.

81.8. Definition of algebraic stacks

An algebraic stack is a stack in groupoids over the category of schemes with the fppf topology that has a diagonal representable by algebraic spaces and is the target of a surjective smooth morphism from a scheme. The notion "Deligne-Mumford stack" will be reserved for a stack as in [**DM69**]. We will reserve the term "Artin stack" for a stack such as in the papers by Artin [**Art69b**], and [**Art74**]. (See also [**CdJ02**].) In other words, and Artin stack will be an algebraic stack with some reasonable finiteness and separatedness conditions.

81.9. Examples of schemes, algebraic spaces, algebraic stacks

It really is not that hard to show that $\mathcal{M}_g$ is an algebraic stack for $g \geq 2$. We should really have a long list of moduli problems here and prove they are all algebraic stacks. Some of them we can prove are algebraic using Artin approximation. For example the Kontsevich moduli space in characteristic $p > 0$.

Here are some items for the list of moduli problems mentioned above.

(1) $\mathcal{M}_g$, i.e., moduli of smooth projective curves of genus g,
(2) $\overline{\mathcal{M}}_g$, i.e., moduli of stable genus g curves,
(3) $\mathcal{A}_g$, i.e., principally polarized abelian schemes of genus g,
(4) $\mathcal{M}_{1,1}$, i.e., 1-pointed smooth projective genus 1 curves,
(5) $\mathcal{M}_{g,n}$, i.e., smooth projective genus g-curves with n pairwise distinct labeled points,
(6) $\overline{\mathcal{M}}_{g,n}$, i.e., stable n-pointed nodal projective genus g-curves,
(7) $\mathcal{H}om_S(\mathcal{X}, \mathcal{Y})$, moduli of morphisms (with suitable conditions on the stacks $\mathcal{X}$, $\mathcal{Y}$ and the base scheme S),
(8) $Bun_G(X) = \mathcal{H}om_S(X, BG)$, the stack of G-bundles of the geometric Langlands programme (with suitable conditions on the scheme X, the group scheme G, and the base scheme S),
(9) $Pic_{\mathcal{X}/S}$, i.e., the Picard stack associated to an algebraic stack over a base scheme (or space).

How about the algebraic space you get from the deformation theory of a general surface in $\mathbf{P}^3$ with a node? (I mean where you deform it to a general smooth surface in $\mathbf{P}^3$.) Perhaps we can talk about some small dimensional examples here too. For

example the stack where you have $\mathbf{A}^1$ with a $B(\mathbf{Z}/2)$ sitting at 0. Bugeyed covers. And so on.

81.10. Properties of algebraic stacks

This is perhaps one of the easier projects to work on, as most of the basic theory is there now. An interesting project is discussing the various ways of defining what a proper algebraic stack is. Of course these things are really properties of morphisms of stacks. We can define singularities (up to smooth factors) etc. Prove that a connected normal stack is irreducible, etc.

81.11. Lisse étale site of an algebraic stack

This has been introduced in Cohomology of Stacks, Section 75.11. An example to show that it is not functorial with respect to 1-morphisms of algebraic stacks is discussed in Examples, Section 78.44. Of course a lot more could be said about this, but it turns out to be very useful to prove things using the "big" étale site as much as possible.

81.12. Things you always wanted to know but were afraid to ask

There are going to be lots of lemmas that you use over and over again that are useful but aren't really mentioned specifically in the literature, or it isn't easy to find references for. Bag of tricks.

Example: Given two groupoids in schemes $R \Rightarrow U$ and $R' \Rightarrow U'$ what does it mean to have a 1-morphism $[U/R] \to [U'/R']$ purely in terms of groupoids in schemes.

81.13. Quasi-coherent sheaves on stacks

These are defined and discussed in the chapter Cohomology of Stacks, Section 75.1. Derived categories of modules are discussed in the chapter Derived Categories of Stacks, Section 76.1. A lot more could be added to these chapters.

81.14. Flat and smooth

Artin's theorem that having a flat surjection from a scheme is a replacement for the smooth surjective condition. This is now available as Criteria for Representability, Theorem 70.16.1.

81.15. Artin's representability theorem

This is discussed in the chapter Artin's Axioms, Section 71.1. We also have an application, see Quot, Theorem 72.5.12. There should be a lot more applications and the chapter itself has to be cleaned up as well.

81.16. DM stacks are finitely covered by schemes

This all begins with Gabber's lemma I think. Somewhere in Asterisque about Faltings proof of Mordell?

81.17. Martin Olson's paper on properness

This proves two notions of proper are the same. We can also discuss Faltings result that it suffices to use DVR's in certain cases.

81.18. Proper pushforward of coherent sheaves

No comments yet.

81.19. Keel and Mori

See [**KM97**]. This material has been incorporated throughout the Stacks project. See for example More on Groupoids, Section 39.12 and More on Groupoids in Spaces, Section 60.11.

81.20. Add more here

Please.

81.21. Other chapters

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CHAPTER 82

Coding Style

82.1. List of style comments

These will be changed over time, but having some here now will hopefully encourage a consistent LaTeX style. We will call "code[1]" the contents of the source files.

(1) Keep all lines in all tex files to at most 80 characters.

(2) Do not use indentation in the tex file. Use syntax highlighting in your editor, instead of indentation, to visualize environments, etc.

(3) Use

```
\medskip\noindent
```

to start a new paragraph, and use

```
\noindent
```

to start a new paragraph just after an environment.

(4) Do not break the code for mathematical formulas across lines if possible. If the complete code complete with enclosing dollar signs does not fit on the line, then start with the first dollar sign on the first character of the next line. If it still does not fit, find a mathematically reasonable spot to break the code.

(5) Displayed math equations should be coded as follows

```
$$
...
...
$$
```

In other words, start with a double dollar sign on a line by itself and end similarly.

(6) *Do not use any macros.* Rationale: This makes it easier to read the tex file, and start editing an arbitrary part without having to learn innumerable macros. And it doesn't make it harder or more timeconsuming to write. Of course the disadvantage is that the same mathematical object may be TeXed differently in different places in the text, but this should be easy to spot.

(7) The theorem environments we use are: "theorem", "proposition", "lemma" (plain), "definition", "example", "exercise", "situation" (definition), "remark", "remarks" (remark). Of course there is also a "proof" environment.

(8) An environment "foo" should be coded as follows

```
\begin{foo}
```

[1]It is all Knuth's fault. See [**Knu79**].

```
...
...
\end{foo}
```

similarly to the way displayed equations are coded.

(9) Instead of a "corollary", just use "lemma" environment since likely the result will be used to prove the next bigger theorem anyway.

(10) Directly following each lemma, proposition, or theorem is the proof of said lemma, proposition, or theorem. No nested proofs please.

(11) The files preamble.tex, chapters.tex and fdl.tex are special tex files. Apart from these, each tex file has the following structure

```
\input{preamble}
\begin{document}
\title{Title}
\maketitle
\tableofcontents
...
...
\input{chapters}
\bibliography{my}
\bibliographystyle{amsalpha}
\end{document}
```

(12) Try to add labels to lemmas, propositions, theorems, and even remarks, exercise, and other environments. If labelling a lemma use something like

```
\begin{lemma}
\label{lemma-bar}
...
\end{lemma}
```

Similarly for all other environments. In other words, the label of a environment named "foo" starts with "foo-". In addition to this please make all labels consist only of lower case letters, digits, and the symbol "-".

(13) Never refer to "the lemma above" (or proposition, etc). Instead use:

```
Lemma \ref{lemma-bar} above
```

This means that later moving lemmas around is basically harmless.

(14) Cross-file referencing. To reference a lemma labeled "lemma-bar" in the file foo.tex which has title "Foo", please use the following code

```
Foo, Lemma \ref{foo-lemma-bar}
```

If this does not work, then take a look at the file preamble.tex to find the correct expression to use. This will produce the "Foo, Lemma <link>" in the output file so it will be clear that the link points out of the file.

(15) If at all possible avoid forward references in proof environments. (It should be possible to write an automated test for this.)

(16) Do not start any sentence with a mathematical symbol.

(17) Do not have a sentence of the type "This follows from the following" just before a lemma, proposition, or theorem. Every sentence ends with a period.

(18) State all hypotheses in each lemma, proposition, theorem. This makes it easier for readers to see if a given lemma, proposition, or theorem applies to their particular problem.
(19) Keep proofs short; less than 1 page in pdf or dvi. You can always achieve this by splitting out the proof in lemmas etc.
(20) In a defining property foobar use

```
{\it foobar}
```

in the code inside the definition environment. Similarly if the definition occurs in the text of the document. This will make it easier for the reader to see what it is that is being defined.
(21) Put any definition that will be used outside the section it is in, in its own definition environment. Temporary definitions may be made in the text. A tricky case is that of mathematical constructions (which are often definitions involving interrelated lemmas). Maybe a good solution is to have them in their own short section so users can refer to the section instead of a definition.
(22) Do not number equations unless they are actually being referenced somewhere in the text. We can always add labels later.
(23) In statements of lemmas, propositions and theorems and in proofs keep the sentences short. For example, instead of "Let R be a ring and let M be an R-module." write "Let R be a ring. Let M be an R-module.". Rationale: This makes it easier to parse the trickier parts of proofs and statements.
(24) Use the

```
\section
```

command to make sections, but try to avoid using subsections and subsubsections.
(25) Avoid using complicated latex constructions.

82.2. Other chapters

Preliminaries

(1) Introduction
(2) Conventions
(3) Set Theory
(4) Categories
(5) Topology
(6) Sheaves on Spaces
(7) Sites and Sheaves
(8) Stacks
(9) Fields
(10) Commutative Algebra
(11) Brauer Groups
(12) Homological Algebra
(13) Derived Categories
(14) Simplicial Methods
(15) More on Algebra
(16) Smoothing Ring Maps
(17) Sheaves of Modules
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(19) Injectives
(20) Cohomology of Sheaves
(21) Cohomology on Sites
(22) Differential Graded Algebra
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Schemes

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(26) Constructions of Schemes
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(33) Topologies on Schemes
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(39) More on Groupoid Schemes
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(44) Étale Cohomology
(45) Crystalline Cohomology
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(59) Groupoids in Algebraic Spaces
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(85) Auto Generated Index

CHAPTER 83

Obsolete

83.1. Introduction

In this chapter we put some lemmas that have become "obsolete" (see [**Mil17**]).

83.2. Obsolete algebra lemmas

Lemma 83.2.1. *Let M be an R-module of finite presentation. For any surjection $\alpha : R^{\oplus n} \to M$ the kernel of α is a finite R-module.*

Proof. This is a special case of Algebra, Lemma 10.5.3. □

The following technical lemma says that you can lift any sequence of relations from a fibre to the whole space of a ring map which is essentially of finite type, in a suitable sense.

Lemma 83.2.2. *Let $R \to S$ be a ring map. Let $\mathfrak{p} \subset R$ be a prime. Let $\mathfrak{q} \subset S$ be a prime lying over $\mathfrak{p}$. Assume $S_\mathfrak{q}$ is essentially of finite type over $R_\mathfrak{p}$. Assume given*

1. *an integer $n \geq 0$,*
2. *a prime $\mathfrak{a} \subset \kappa(\mathfrak{p})[x_1, \ldots, x_n]$,*
3. *a surjective $\kappa(\mathfrak{p})$-homomorphism*
$$\psi : (\kappa(\mathfrak{p})[x_1, \ldots, x_n])_\mathfrak{a} \longrightarrow S_\mathfrak{q}/\mathfrak{p}S_\mathfrak{q},$$
and
4. *elements $\overline{f}_1, \ldots, \overline{f}_e$ in $Ker(\psi)$.*

Then there exist

1. *an integer $m \geq 0$,*
2. *and element $g \in S$, $g \notin \mathfrak{q}$,*
3. *a map*
$$\Psi : R[x_1, \ldots, x_n, x_{n+1}, \ldots, x_{n+m}] \longrightarrow S_g,$$
and
4. *elements $f_1, \ldots, f_e, f_{e+1}, \ldots, f_{e+m}$ of $Ker(\Psi)$*

such that

1. *the following diagram commutes*
$$\begin{array}{ccc} R[x_1, \ldots, x_{n+m}] & \xrightarrow[x_{n+j} \mapsto 0]{} & (\kappa(\mathfrak{p})[x_1, \ldots, x_n])_\mathfrak{a} \\ \downarrow{\scriptstyle \Psi} & & \downarrow{\scriptstyle \psi} \\ S_g & \longrightarrow & S_\mathfrak{q}/\mathfrak{p}S_\mathfrak{q} \end{array},$$
2. *the element f_i, $i \leq n$ maps to a unit times $\overline{f}_i$ in the local ring*
$$(\kappa(\mathfrak{p})[x_1, \ldots, x_{n+m}])_{(\mathfrak{a}, x_{n+1}, \ldots, x_{n+m})},$$

(3) *the element f_{e+j} maps to a unit times x_{n+j} in the same local ring, and*
(4) *the induced map $R[x_1, \ldots, x_{n+m}]_{\mathfrak{b}} \to S_{\mathfrak{q}}$ is surjective, where $\mathfrak{b} = \Psi^{-1}(\mathfrak{q}S_g)$.*

Proof. We claim that it suffices to prove the lemma in case R and S are local with maximal ideals $\mathfrak{p}$ and $\mathfrak{q}$. Namely, suppose we have constructed

$$\Psi' : R_{\mathfrak{p}}[x_1, \ldots, x_{n+m}] \longrightarrow S_{\mathfrak{q}}$$

and $f'_1, \ldots, f'_{e+m} \in R_{\mathfrak{p}}[x_1, \ldots, x_{n+m}]$ with all the required properties. Then there exists an element $f \in R$, $f \notin \mathfrak{p}$ such that each ff'_k comes from an element $f_k \in R[x_1, \ldots, x_{n+m}]$. Moreover, for a suitable $g \in S$, $g \notin \mathfrak{q}$ the elements $\Psi'(x_i)$ are the image of elements $y_i \in S_g$. Let Ψ be the R-algebra map defined by the rule $\Psi(x_i) = y_i$. Since $\Psi(f_i)$ is zero in the localization $S_{\mathfrak{q}}$ we may after possibly replacing g assume that $\Psi(f_i) = 0$. This proves the claim.

Thus we may assume R and S are local with maximal ideals $\mathfrak{p}$ and $\mathfrak{q}$. Pick $y_1, \ldots, y_n \in S$ such that $y_i \bmod \mathfrak{p}S = \psi(x_i)$. Let $y_{n+1}, \ldots, y_{n+m} \in S$ be elements which generate an R-subalgebra of which S is the localization. These exist by the assumption that S is essentially of finite type over R. Since ψ is surjective we may write $y_{n+j} \bmod \mathfrak{p}S = \psi(h_j)$ for some $h_j \in \kappa(\mathfrak{p})[x_1, \ldots, x_n]_{\mathfrak{a}}$. Write $h_j = g_j/d$, $g_j \in \kappa(\mathfrak{p})[x_1, \ldots, x_n]$ for some common denominator $d \in \kappa(\mathfrak{p})[x_1, \ldots, x_n]$, $d \notin \mathfrak{a}$. Choose lifts $G_j, D \in R[x_1, \ldots, x_n]$ of g_j and d. Set $y'_{n+j} = D(y_1, \ldots, y_n)y_{n+j} - G_j(y_1, \ldots, y_n)$. By construction $y'_{n+j} \in \mathfrak{p}S$. It is clear that $y_1, \ldots, y_n, y'_n, \ldots, y'_{n+m}$ generate an R-subalgebra of S whose localization is S. We define

$$\Psi : R[x_1, \ldots, x_{n+m}] \to S$$

to be the map that sends x_i to y_i for $i = 1, \ldots, n$ and x_{n+j} to y'_{n+j} for $j = 1, \ldots, m$. Properties (1) and (4) are clear by construction. Moreover the ideal $\mathfrak{b}$ maps onto the ideal $(\mathfrak{a}, x_{n+1}, \ldots, x_{n+m})$ in the polynomial ring $\kappa(\mathfrak{p})[x_1, \ldots, x_{n+m}]$.

Denote $J = \mathrm{Ker}(\Psi)$. We have a short exact sequence

$$0 \to J_{\mathfrak{b}} \to R[x_1, \ldots, x_{n+m}]_{\mathfrak{b}} \to S_{\mathfrak{q}} \to 0.$$

The surjectivity comes from our choice of $y_1, \ldots, y_n, y'_n, \ldots, y'_{n+m}$ above. This implies that

$$J_{\mathfrak{b}}/\mathfrak{p}J_{\mathfrak{b}} \to \kappa(\mathfrak{p})[x_1, \ldots, x_{n+m}]_{(\mathfrak{a}, x_{n+1}, \ldots, x_{n+m})} \to S_{\mathfrak{q}}/\mathfrak{p}S_{\mathfrak{q}} \to 0$$

is exact. By construction x_i maps to $\psi(x_i)$ and x_{n+j} maps to zero under the last map. Thus it is easy to choose f_i as in (2) and (3) of the lemma. □

Remark 83.2.3 (Projective resolutions)**.** Let R be a ring. For any set S we let $F(S)$ denote the free R-module on S. Then any left R-module has the following two step resolution

$$F(M \times M) \oplus F(R \times M) \to F(M) \to M \to 0.$$

The first map is given by the rule

$$[m_1, m_2] \oplus [r, m] \mapsto [m_1 + m_2] - [m_1] - [m_2] + [rm] - r[m].$$

83.3. Lemmas related to ZMT

The lemmas in this section were originally used in the proof of the (algebraic version of) Zariski's Main Theorem, Algebra, Theorem 10.117.13.

Lemma 83.3.1. *Let $\varphi : R \to S$ be a ring map. Suppose $t \in S$ satisfies the relation $\varphi(a_0) + \varphi(a_1)t + \ldots + \varphi(a_n)t^n = 0$. Set $u_n = \varphi(a_n)$, $u_{n-1} = u_n t + \varphi(a_{n-1})$, and so on till $u_1 = u_2 t + \varphi(a_1)$. Then all of $u_n, u_{n-1}, \ldots, u_1$ and $u_n t, u_{n-1}t, \ldots, u_1 t$ are integral over R, and the ideals $(\varphi(a_0), \ldots, \varphi(a_n))$ and $(u_n, \ldots, u_1)$ of S are equal.*

Proof. We prove this by induction on n. As $u_n = \varphi(a_n)$ we conclude from Algebra, Lemma 10.117.1 that $u_n t$ is integral over R. Of course $u_n = \varphi(a_n)$ is integral over R. Then $u_{n-1} = u_n t + \varphi(a_{n-1})$ is integral over R (see Algebra, Lemma 10.33.7) and we have

$$\varphi(a_0) + \varphi(a_1)t + \ldots + \varphi(a_{n-1})t^{n-1} + u_{n-1}t^{n-1} = 0.$$

Hence by the induction hypothesis applied to the map $S' \to S$ where S' is the integral closure of R in S and the displayed equation we see that $u_{n-1}, \ldots, u_1$ and $u_{n-1}t, \ldots, u_1 t$ are all in S' too. The statement on the ideals is immediate from the shape of the elements and the fact that $u_1 t + \varphi(a_0) = 0$. □

Lemma 83.3.2. *Let $\varphi : R \to S$ be a ring map. Suppose $t \in S$ satisfies the relation $\varphi(a_0) + \varphi(a_1)t + \ldots + \varphi(a_n)t^n = 0$. Let $J \subset S$ be an ideal such that for at least one i we have $\varphi(a_i) \notin J$. Then there exists a $u \in S$, $u \notin J$ such that both u and ut are integral over R.*

Proof. This is immediate from Lemma 83.3.1 since one of the elements u_i will not be in J. □

The following two lemmas are a way of describing closed subschemes of $\mathbf{P}^1_R$ cut out by one (nondegenerate) equation.

Lemma 83.3.3. *Let R be a ring. Let $F(X, Y) \in R[X, Y]$ be homogeneous of degree d. Assume that for every prime $\mathfrak{p}$ of R at least one coefficient of F is not in $\mathfrak{p}$. Let $S = R[X, Y]/(F)$ as a graded ring. Then for all $n \geq d$ the R-module S_n is finite locally free of rank d.*

Proof. The R-module S_n has a presentation

$$R[X, Y]_{n-d} \to R[X, Y]_n \to S_n \to 0.$$

Thus by Algebra, Lemma 10.74.3 it is enough to show that multiplication by F induces an injective map $\kappa(\mathfrak{p})[X, Y] \to \kappa(\mathfrak{p})[X, Y]$ for all primes $\mathfrak{p}$. This is clear from the assumption that F does not map to the zero polynomial mod $\mathfrak{p}$. The assertion on ranks is clear from this as well. □

Lemma 83.3.4. *Let k be a field. Let $F, G \in k[X, Y]$ be homogeneous of degrees d, e. Assume F, G relatively prime. Then multiplication by G is injective on $S = k[X, Y]/(F)$.*

Proof. This is one way to define "relatively prime". If you have another definition, then you can show it is equivalent to this one. □

Lemma 83.3.5. *Let R be a ring. Let $F(X,Y) \in R[X,Y]$ be homogeneous of degree d. Let $S = R[X,Y]/(F)$ as a graded ring. Let $\mathfrak{p} \subset R$ be a prime such that some coefficient of F is not in $\mathfrak{p}$. There exists an $f \in R$ $f \notin \mathfrak{p}$, an integer e, and a $G \in R[X,Y]_e$ such that multiplication by G induces isomorphisms $(S_n)_f \to (S_{n+e})_f$ for all $n \geq d$.*

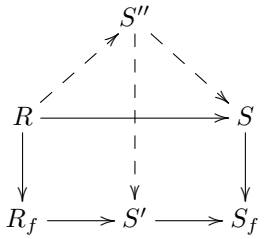
Proof. During the course of the proof we may replace R by R_f for $f \in R$, $f \notin \mathfrak{p}$ (finitely often). As a first step we do such a replacement such that some coefficient of F is invertible in R. In particular the modules S_n are now locally free of rank d for $n \geq d$ by Lemma 83.3.3. Pick any $G \in R[X,Y]_e$ such that the image of G in $\kappa(\mathfrak{p})[X,Y]$ is relatively prime to the image of $F(X,Y)$ (this is possible for some e). Apply Algebra, Lemma 10.74.3 to the map induced by multiplication by G from $S_d \to S_{d+e}$. By our choice of G and Lemma 83.3.4 we see $S_d \otimes \kappa(\mathfrak{p}) \to S_{d+e} \otimes \kappa(\mathfrak{p})$ is bijective. Thus, after replacing R by R_f for a suitable f we may assume that $G : S_d \to S_{d+e}$ is bijective. This in turn implies that the image of G in $\kappa(\mathfrak{p}')[X,Y]$ is relatively prime to the image of F for all primes $\mathfrak{p}'$ of R. And then by Algebra, Lemma 10.74.3 again we see that all the maps $G : S_d \to S_{d+e}$, $n \geq d$ are isomorphisms. □

Remark 83.3.6. Let R be a ring. Suppose that we have $F \in R[X,Y]_d$ and $G \in R[X,Y]_e$ such that, setting $S = R[X,Y]/(F)$ we have (1) S_n is finite locally free of rank d for all $n \geq d$, and (2) multiplication by G defines isomorphisms $S_n \to S_{n+e}$ for all $n \geq d$. In this case we may define a finite, locally free R-algebra A as follows:

(1) as an R-module $A = S_{ed}$, and
(2) multiplication $A \times A \to A$ is given by the rule that $H_1 H_2 = H_3$ if and only if $G^d H_3 = H_1 H_2$ in S_{2ed}.

This makes sense because multiplication by G^d induces a bijective map $S_{de} \to S_{2de}$. It is easy to see that this defines a ring structure. Note the confusing fact that the element G^d defines the unit element of the ring A.

Lemma 83.3.7. *Let R be a ring, let $f \in R$. Suppose we have S, S' and the solid arrows forming the following commutative diagram of rings*



Assume that $R_f \to S'$ is finite. Then we can find a finite ring map $R \to S''$ and dotted arrows as in the diagram such that $S' = (S'')_f$.

Proof. Namely, suppose that S' is generated by x_i over R_f, $i = 1, \ldots, w$. Let $P_i(t) \in R_f[t]$ be a monic polynomial such that $P_i(x_i) = 0$. Say P_i has degree $d_i > 0$. Write $P_i(t) = t^{d_i} + \sum_{j<d_i} (a_{ij}/f^n) t^j$ for some uniform n. Also write the image of x_i in S_f as g_i/f^n for suitable $g_i \in S$. Then we know that the element $\xi_i = f^{nd_i} g_i^{d_i} + \sum_{j<d_i} f^{n(d_i-j)} a_{ij} g_i^j$ of S is killed by a power of f. Hence upon increasing

n to n', which replaces g_i by $f^{n'-n}g_i$ we may assume $\xi_i = 0$. Then S' is generated by the elements $f^n x_i$, each of which is a zero of the monic polynomial $Q_i(t) = t^{d_i} + \sum_{j<d_i} f^{n(d_i-j)} a_{ij} t^j$ with coefficients in R. Also, by construction $Q_i(f^n g_i) = 0$ in S. Thus we get a finite R-algebra $S'' = R[z_1, \ldots, z_w]/(Q_1(z_1), \ldots, Q_w(z_w))$ which fits into a commutative diagram as above. The map $\alpha : S'' \to S$ maps z_i to $f^n g_i$ and the map $\beta : S'' \to S'$ maps z_i to $f^n x_i$. It may not yet be the case that β induces an isomorphism $(S'')_f \cong S'$. For the moment we only know that this map is surjective. The problem is that there could be elements $h/f^n \in (S'')_f$ which map to zero in S' but are not zero. In this case $\beta(h)$ is an element of S such that $f^N \beta(h) = 0$ for some N. Thus $f^N h$ is an element ot the ideal $J = \{h \in S'' \mid \alpha(h) = 0 \text{ and } \beta(h) = 0\}$ of S''. OK, and it is easy to see that S''/J does the job. □

83.4. Formally smooth ring maps

Lemma 83.4.1. *Let R be a ring. Let S be a R-algebra. If S is of finite presentation and formally smooth over R then S is smooth over R.*

Proof. See Algebra, Proposition 10.131.13. □

83.5. Simplicial methods

Lemma 83.5.1. *Assumptions and notation as in Simplicial, Lemma 14.30.1. There exists a section $g : U \to V$ to the morphism f and the composition $g \circ f$ is homotopy equivalent to the identity on V. In particular, the morphism f is a homotopy equivalence.*

Proof. Immediate from Simplicial, Lemmas 14.30.1 and 14.28.8. □

83.6. Obsolete lemmas on schemes

Lemmas that seem superfluous.

Lemma 83.6.1. *Let $(R, \mathfrak{m}, \kappa)$ be a local ring. Let $X \subset \mathbf{P}^n_R$ be a closed subscheme. Assume that $R = \Gamma(X, \mathcal{O}_X)$. Then the special fibre X_k is geometrically connected.*

Proof. This is a special case of More on Morphisms, Theorem 36.36.4. □

Lemma 83.6.2. *Let X be a Noetherian scheme. Let $Z_0 \subset X$ be an irreducible closed subset with generic point ξ. Let $\mathcal{P}$ be a property of coherent sheaves on X such that*

1. *For any short exact sequence of coherent sheaves if two out of three of them have property $\mathcal{P}$ then so does the third.*
2. *If $\mathcal{P}$ holds for a direct sum of coherent sheaves then it holds for both.*
3. *For every integral closed subscheme $Z \subset Z_0 \subset X$, $Z \neq Z_0$ and every quasi-coherent sheaf of ideals $\mathcal{I} \subset \mathcal{O}_Z$ we have $\mathcal{P}$ for $(Z \to X)_*\mathcal{I}$.*
4. *There exists some coherent sheaf $\mathcal{G}$ on X such that*
   (a) *$\text{Supp}(\mathcal{G}) = Z_0$,*
   (b) *$\mathcal{G}_\xi$ is annihilated by $\mathfrak{m}_\xi$, and*
   (c) *property $\mathcal{P}$ holds for $\mathcal{G}$.*

Then property $\mathcal{P}$ holds for every coherent sheaf $\mathcal{F}$ on X whose support is contained in Z_0.

Proof. The proof is a variant on the proof of Cohomology of Schemes, Lemma 29.12.5. In exactly the same manner as in that proof we see that any coherent sheaf whose support is strictly contained in Z_0 has property $\mathcal{P}$.

Consider a coherent sheaf $\mathcal{G}$ as in (3). By Cohomology of Schemes, Lemma 29.12.2 there exists a sheaf of ideals $\mathcal{I}$ on Z_0 and a short exact sequence

$$0 \to ((Z_0 \to X)_*\mathcal{I})^{\oplus r} \to \mathcal{G} \to \mathcal{Q} \to 0$$

where the support of $\mathcal{Q}$ is strictly contained in Z_0. In particular $r > 0$ and $\mathcal{I}$ is nonzero because the support of $\mathcal{G}$ is equal to Z. Since $\mathcal{Q}$ has property $\mathcal{P}$ we conclude that also $((Z_0 \to X)_*\mathcal{I})^{\oplus r}$ has property $\mathcal{P}$. By (2) we deduce property $\mathcal{P}$ for $(Z_0 \to X)_*\mathcal{I}$. Slotting this into the proof of Cohomology of Schemes, Lemma 29.12.5 at the appropriate point gives the lemma. Some details omitted. □

Lemma 83.6.3. *Let X be a Noetherian scheme. Let $\mathcal{P}$ be a property of coherent sheaves on X such that*

1. *For any short exact sequence of coherent sheaves if two out of three of them have property $\mathcal{P}$ then so does the third.*
2. *If $\mathcal{P}$ holds for a direct sum of coherent sheaves then it holds for both.*
3. *For every integral closed subscheme $Z \subset X$ with generic point ξ there exists some coherent sheaf $\mathcal{G}$ such that*
   (a) *$\mathrm{Supp}(\mathcal{G}) = Z$,*
   (b) *$\mathcal{G}_\xi$ is annihilated by $\mathfrak{m}_\xi$, and*
   (c) *property $\mathcal{P}$ holds for $\mathcal{G}$.*

Then property $\mathcal{P}$ holds for every coherent sheaf on X.

Proof. This follows from Lemma 83.6.2 in exactly the same way that Cohomology of Schemes, Lemma 29.12.6 follows from Cohomology of Schemes, Lemma 29.12.5. □

83.7. Functor of quotients

Lemma 83.7.1. *Let $S = \mathrm{Spec}(R)$ be an affine scheme. Let X be an algebraic space over S. Let $q_i : \mathcal{F} \to \mathcal{Q}_i$, $i = 1, 2$ be surjective maps of quasi-coherent $\mathcal{O}_X$-modules. Assume $\mathcal{Q}_1$ flat over S. Let $T \to S$ be a quasi-compact morphism of schemes such that there exists a factorization*

$$\begin{array}{ccc} & \mathcal{F}_T & \\ {\scriptstyle q_{1,T}}\swarrow & & \searrow{\scriptstyle q_{2,T}} \\ \mathcal{Q}_{1,T} & \dashleftarrow & \mathcal{Q}_{2,T} \end{array}$$

Then exists a closed subscheme $Z \subset S$ such that (a) $T \to S$ factors through Z and (b) $q_{1,Z}$ factors through $q_{2,Z}$. If $\mathrm{Ker}(q_2)$ is a finite type $\mathcal{O}_X$-module and X quasi-compact, then we can take $Z \to S$ of finite presentation.

Proof. Apply Quot, Lemma 72.7.5 to the map $\mathrm{Ker}(q_2) \to \mathcal{Q}_1$. □

83.8. Very reasonable algebraic spaces

Material that is somewhat obsolete.

Lemma 83.8.1. *Let S be a scheme. Let X be a reasonable algebraic space over S. Then $|X|$ is Kolmogorov (see Topology, Definition 5.7.4).*

Proof. Follows from the definitions and Decent Spaces, Lemma 50.9.5. □

In the rest of this section we make some remarks about very reasonable algebraic spaces. If there exists a scheme U and a surjective, étale, quasi-compact morphism $U \to X$, then X is very reasonable, see Decent Spaces, Lemma 50.4.7.

Lemma 83.8.2. *A scheme is very reasonable.*

Proof. This is true because the identity map is a quasi-compact, surjective étale morphism. □

Lemma 83.8.3. *Let S be a scheme. Let X be an algebraic space over S. If there exists a Zariski open covering $X = \bigcup X_i$ such that each X_i is very reasonable, then X is very reasonable.*

Proof. This is case (ϵ) of Decent Spaces, Lemma 50.5.2. □

Lemma 83.8.4. *An algebraic space which is Zariski locally quasi-separated is very reasonable. In particular any quasi-separated algebraic space is very reasonable.*

Proof. This is one of the implications of Decent Spaces, Lemma 50.5.1. □

Lemma 83.8.5. *Let S be a scheme. Let X, Y be algebraic spaces over S. Let $Y \to X$ be a representable morphism. If X is very reasonable, so is Y.*

Proof. This is case (ϵ) of Decent Spaces, Lemma 50.5.3. □

Remark 83.8.6. Very reasonable algebraic spaces form a strictly larger collection than Zariski locally quasi-separated algebraic spaces. Consider an algebraic space of the form $X = [U/G]$ (see Spaces, Definition 47.14.4) where G is a finite group acting without fixed points on a non-quasi-separated scheme U. Namely, in this case $U \times_X U = U \times G$ and clearly both projections to U are quasi-compact, hence X is very reasonable. On the other hand, the diagonal $U \times_X U \to U \times U$ is not quasi-compact, hence this algebraic space is not quasi-separated. Now, take U the infinite affine space over a field k of characteristic $\neq 2$ with zero doubled, see Schemes, Example 25.21.4. Let $0_1, 0_2$ be the two zeros of U. Let $G = \{+1, -1\}$, and let -1 act by -1 on all coordinates, and by switching 0_1 and 0_2. Then $[U/G]$ is very reasonable but not Zariski locally quasi-separated (details omitted).

Warning: The following lemma should be used with caution, as the schemes U_i in it are not necessarily separated or even quasi-separated.

Lemma 83.8.7. *Let S be a scheme. Let X be a very reasonable algebraic space over S. There exists a set of schemes U_i and morphisms $U_i \to X$ such that*

1. *each U_i is a quasi-compact scheme,*
2. *each $U_i \to X$ is étale,*
3. *both projections $U_i \times_X U_i \to U_i$ are quasi-compact, and*
4. *the morphism $\coprod U_i \to X$ is surjective (and étale).*

Proof. Decent Spaces, Definition 50.6.1 says that there exist $U_i \to X$ such that (2), (3) and (4) hold. Fix i, and set $R_i = U_i \times_X U_i$, and denote $s, t : R_i \to U_i$ the projections. For any affine open $W \subset U_i$ the open $W' = t(s^{-1}(W)) \subset U_i$ is a quasi-compact R_i-invariant open (see Groupoids, Lemma 38.17.2). Hence W' is a quasi-compact scheme, $W' \to X$ is étale, and $W' \times_X W' = s^{-1}(W') = t^{-1}(W')$ so both projections $W' \times_X W' \to W'$ are quasi-compact. This means the family of $W' \to X$, where $W \subset U_i$ runs through the members of affine open coverings of the U_i gives what we want. □

83.9. Variants of cotangent complexes for schemes

This section gives an alternative construction of the cotangent complex of a morphism of schemes. This section is currently in the obsolete chapter as we can get by with the easier version discussed in Cotangent, Section 66.24 for applications.

Let $f : X \to Y$ be a morphism of schemes. Let $\mathcal{C}_{X/Y}$ be the category whose objects are commutative diagrams

$$\begin{array}{ccccc} X & \longleftarrow & U & \xrightarrow{i} & A \\ \downarrow & & \downarrow & \swarrow & \\ Y & \longleftarrow & V & & \end{array} \tag{83.9.0.1}$$

of schemes where

(1) U is an open subscheme of X,
(2) V is an open subscheme of Y, and
(3) there exists an isomorphism $A = V \times \mathrm{Spec}(P)$ over V where P is a polynomial algebra over $\mathbf{Z}$ (on some set of variables).

In other words, A is an (infinite dimensional) affine space over V. Morphisms are given by commutative diagrams.

Notation. An object of $\mathcal{C}_{X/Y}$, i.e., a diagram (83.9.0.1), is often denoted $U \to A$ where it is understood that (a) U is an open subscheme of X, (b) $U \to A$ is a morphism over Y, (c) the image of the structure morphism $A \to Y$ is an open $V \subset Y$, and (d) $A \to V$ is an affine space. We'll write $U \to A/V$ to indicate $V \subset Y$ is the image of $A \to Y$. Recall that X_{Zar} denotes the small Zariski site X. There are forgetful functors

$$\mathcal{C}_{X/Y} \to X_{Zar},\ (U \to A) \mapsto U \quad \text{and} \quad \mathcal{C}_{X/Y} \mapsto Y_{Zar},\ (U \to A/V) \mapsto V.$$

Lemma 83.9.1. *Let $X \to Y$ be a morphism of schemes.*

(1) *The category $\mathcal{C}_{X/Y}$ is fibred over X_{Zar}.*
(2) *The category $\mathcal{C}_{X/Y}$ is fibred over Y_{Zar}.*
(3) *The category $\mathcal{C}_{X/Y}$ is fibred over the category of pairs (U, V) where $U \subset X$, $V \subset Y$ are open and $f(U) \subset V$.*

Proof. Ad (1). Given an object $U \to A$ of $\mathcal{C}_{X/Y}$ and a morphism $U' \to U$ of X_{Zar} consider the object $i' : U' \to A$ of $\mathcal{C}_{X/Y}$ where i' is the composition of i and $U' \to U$. The morphism $(U' \to A) \to (U \to A)$ of $\mathcal{C}_{X/Y}$ is strongly cartesian over X_{Zar}.

Ad (2). Given an object $U \to A/V$ and $V' \to V$ we can set $U' = U \cap f^{-1}(V')$ and $A' = V' \times_V A$ to obtain a strongly cartesian morphism $(U' \to A') \to (U \to A)$ over $V' \to V$.

Ad (3). Denote $(X/Y)_{Zar}$ the category in (3). Given $U \to A/V$ and a morphism $(U', V') \to (U, V)$ in $(X/Y)_{Zar}$ we can consider $A' = V' \times_V A$. Then the morphism $(U' \to A'/V') \to (U \to A/V)$ is strongly cartesian in $\mathcal{C}_{X/Y}$ over $(X/Y)_{Zar}$. $\square$

We obtain a topology τ_X on $\mathcal{C}_{X/Y}$ by using the topology inherited from X_{Zar} (see Stacks, Section 8.10). If not otherwise stated this is the topology on $\mathcal{C}_{X/Y}$ we will consider. To be precise, a family of morphisms $\{(U_i \to A_i) \to (U \to A)\}$ is a covering of $\mathcal{C}_{X/Y}$ if and only if

(1) $U = \bigcup U_i$, and
(2) $A_i = A$ for all i.

We obtain the same collection of sheaves if we allow $A_i \cong A$ in (2). The functor u defines a morphism of topoi $\pi : Sh(\mathcal{C}_{X/Y}) \to Sh(X_{Zar})$.

The site $\mathcal{C}_{X/Y}$ comes with several sheaves of rings.

(1) The sheaf $\mathcal{O}$ given by the rule $(U \to A) \mapsto \mathcal{O}(A)$.
(2) The sheaf $\underline{\mathcal{O}}_X = \pi^{-1}\mathcal{O}_X$ given by the rule $(U \to A) \mapsto \mathcal{O}(U)$.
(3) The sheaf $\underline{\mathcal{O}}_Y$ given by the rule $(U \to A/V) \mapsto \mathcal{O}(V)$.

We obtain morphisms of ringed topoi

$$\begin{array}{ccc} (Sh(\mathcal{C}_{X/Y}), \underline{\mathcal{O}}_X) & \xrightarrow[i]{} & (Sh(\mathcal{C}_{X/Y}), \mathcal{O}) \\ \downarrow{\scriptstyle \pi} & & \\ (Sh(X_{Zar}), \mathcal{O}_X) & & \end{array} \tag{83.9.1.1}$$

The morphism i is the identity on underlying topoi and $i^\sharp : \mathcal{O} \to \underline{\mathcal{O}}_X$ is the obvious map. The map π is a special case of Cohomology on Sites, Situation 21.27.1. An important role will be played in the following by the derived functors $Li^* : D(\mathcal{O}) \longrightarrow D(\underline{\mathcal{O}}_X)$ left adjoint to $Ri_* = i_* : D(\underline{\mathcal{O}}_X) \to D(\mathcal{O})$ and $L\pi_! : D(\underline{\mathcal{O}}_X) \longrightarrow D(\mathcal{O}_X)$ left adjoint to $\pi^* = \pi^{-1} : D(\mathcal{O}_X) \to D(\underline{\mathcal{O}}_X)$.

Remark 83.9.2. We obtain a second topology τ_Y on $\mathcal{C}_{X/Y}$ by taking the topology inherited from Y_{Zar}. There is a third topology $\tau_{X\to Y}$ where a family of morphisms $\{(U_i \to A_i) \to (U \to A)\}$ is a covering if and only if $U = \bigcup U_i$, $V = \bigcup V_i$ and $A_i \cong V_i \times_V A$. This is the topology inherited from the topology on the site $(X/Y)_{Zar}$ whose underlying category is the category of pairs (U, V) as in Lemma 83.9.1 part (3). The coverings of $(X/Y)_{Zar}$ are families $\{(U_i, V_i) \to (U, V)\}$ such that $U = \bigcup U_i$ and $V = \bigcup V_i$. There are morphisms of topoi

$$Sh(\mathcal{C}_{X/Y}) = Sh(\mathcal{C}_{X/Y}, \tau_X) \longleftarrow Sh(\mathcal{C}_{X/Y}, \tau_{X\to Y}) \longrightarrow Sh(\mathcal{C}_{X/Y}, \tau_Y)$$

(recall that τ_X is our "default" topology). The pullback functors for these arrows are sheafification and pushforward is the identity on underlying presheaves. The diagram of topoi

$$\begin{array}{ccccc} Sh(X_{Zar}) & \xleftarrow[\pi]{} & Sh(\mathcal{C}_{X/Y}) & \longleftarrow & Sh(\mathcal{C}_{X/Y}, \tau_{X\to Y}) \\ \downarrow{\scriptstyle f} & & & & \downarrow \\ Sh(Y_{Zar}) & & \longleftarrow & & Sh(\mathcal{C}_{X/Y}, \tau_Y) \end{array}$$

is **not** commutative. Namely, the pullback of a nonzero abelian sheaf on Y is a nonzero abelian sheaf on $(\mathcal{C}_{X/Y}, \tau_{X\to Y})$, but we can certainly find examples where

such a sheaf pulls back to zero on X. Note that any presheaf $\mathcal{F}$ on Y_{Zar} gives a sheaf $\underline{\mathcal{F}}$ on $\mathcal{C}_{Y/X}$ by the rule which assigns to $(U \to A/V)$ the set $\mathcal{F}(V)$. Even if $\mathcal{F}$ happens to be a sheaf it isn't true in general that $\underline{\mathcal{F}} = \pi^{-1}f^{-1}\mathcal{F}$. This is related to the noncommutativity of the diagram above, as we can describe $\underline{\mathcal{F}}$ as the pushforward of the pullback of $\mathcal{F}$ to $Sh(\mathcal{C}_{X/Y}, \tau_{X\to Y})$ via the lower horizontal and right vertical arrows. An example is the sheaf $\underline{\mathcal{O}_Y}$. But what is true is that there is a map $\underline{\mathcal{F}} \to \pi^{-1}f^{-1}\mathcal{F}$ which is transformed (as we shall see later) into an isomorphism after applying $\pi_!$.

83.10. Deformations and obstructions of flat modules

In this section we sketch a construction of a deformation theory for the stack of coherent sheaves for any algebraic space X over a ring Λ. This material is obsolete due to the improved discussion in Quot, Section 72.6.

Our setup will be the following. We assume given

(1) a ring Λ,
(2) an algebraic space X over Λ,
(3) a Λ-algebra A, set $X_A = X \times_{\mathrm{Spec}(\Lambda)} \mathrm{Spec}(A)$, and
(4) a finitely presented $\mathcal{O}_{X_A}$-module $\mathcal{F}$ flat over A.

In this situation we will consider all possible surjections

$$0 \to I \to A' \to A \to 0$$

where A' is a Λ-algebra whose kernel I is an ideal of square zero in A'. Given A' we obtain a first order thickening $X_A \to X_{A'}$ of algebraic spaces over $\mathrm{Spec}(\Lambda)$. For each of these we consider the problem of lifting $\mathcal{F}$ to a finitely presented module $\mathcal{F}'$ on $X_{A'}$ flat over A'. We would like to replicate the results of Deformation Theory, Lemma 65.11.1 in this setting.

To be more precise let $Lift(\mathcal{F}, A')$ denote the category of pairs $(\mathcal{F}', \alpha)$ where $\mathcal{F}'$ is a finitely presented module on $X_{A'}$ flat over A' and $\alpha : \mathcal{F}'|_{X_A} \to \mathcal{F}$ is an isomorphism. Morphisms $(\mathcal{F}'_1, \alpha_1) \to (\mathcal{F}'_2, \alpha_2)$ are isomorphisms $\mathcal{F}'_1 \to \mathcal{F}'_2$ which are compatible with α_1 and α_2. The set of isomorphism classes of $Lift(\mathcal{F}, A')$ is denoted $\mathrm{Lift}(\mathcal{F}, A')$.

Let $\mathcal{G}$ be a sheaf of $\mathcal{O}_X \otimes_\Lambda A$-modules on $X_{étale}$ flat over A. We introduce the category $Lift(\mathcal{G}, A')$ of pairs $(\mathcal{G}', \beta)$ where $\mathcal{G}'$ is a sheaf of $\mathcal{O}_X \otimes_\Lambda A'$-modules flat over A' and β is an isomorphism $\mathcal{G}' \otimes_{A'} A \to \mathcal{G}$.

Lemma 83.10.1. *Notation and assumptions as above. Let $p : X_A \to X$ denote the projection. Given A' denote $p' : X_{A'} \to X$ the projection. The functor p'_* induces an equivalence of categories between*

(1) *the category $Lift(\mathcal{F}, A')$, and*
(2) *the category $Lift(p_*\mathcal{F}, A')$.*

Proof. FIXME. □

Let $\mathcal{H}$ be a sheaf of $\mathcal{O}\otimes_\Lambda A$-modules on $\mathcal{C}_{X/\Lambda}$ flat over A. We introduce the category $Lift_{\mathcal{O}}(\mathcal{H}, A')$ whose objects are pairs $(\mathcal{H}', \gamma)$ where $\mathcal{H}'$ is a sheaf of $\mathcal{O}\otimes_\Lambda A'$-modules flat over A' and $\gamma : \mathcal{H}' \otimes_A A' \to \mathcal{H}$ is an isomorphism of $\mathcal{O} \otimes_\Lambda A$-modules.

Let $\mathcal{G}$ be a sheaf of $\mathcal{O}_X \otimes_\Lambda A$-modules on $X_{étale}$ flat over A. Consider the morphisms i and π of Cotangent, Equation (66.26.1.1). Denote $\underline{\mathcal{G}} = \pi^{-1}(\mathcal{G})$. It is simply given

by the rule $(U \to \mathbf{A}) \mapsto \mathcal{G}(U)$ hence it is a sheaf of $\underline{\mathcal{O}}_X \otimes_\Lambda A$-modules. Denote $i_*\underline{\mathcal{G}}$ the same sheaf but viewed as a sheaf of $\mathcal{O} \otimes_\Lambda A$-modules.

Lemma 83.10.2. *Notation and assumptions as above. The functor $\pi_!$ induces an equivalence of categories between*

(1) *the category $Lift_{\mathcal{O}}(i_*\underline{\mathcal{G}}, A')$, and*
(2) *the category $Lift(\mathcal{G}, A')$.*

Proof. FIXME. □

Lemma 83.10.3. *Notation and assumptions as in Lemma 83.10.2. Consider the object*

$$L = L(\Lambda, X, A, \mathcal{G}) = L\pi_!(Li^*(i_*(\underline{\mathcal{G}})))$$

of $D(\mathcal{O}_X \otimes_\Lambda A)$. Given a surjection $A' \to A$ of Λ-algebras with square zero kernel I we have

(1) *The category $Lift(\mathcal{G}, A')$ is nonempty if and only if a certain class $\xi \in Ext^2_{\mathcal{O}_X \otimes A}(L, \mathcal{G} \otimes_A I)$ is zero.*
(2) *If $Lift(\mathcal{G}, A')$ is nonempty, then $Lift(\mathcal{G}, A')$ is principal homogeneous under $Ext^1_{\mathcal{O}_X \otimes A}(L, \mathcal{G} \otimes_A I)$.*
(3) *Given a lift $\mathcal{G}'$, the set of automorphisms of $\mathcal{G}'$ which pull back to $id_{\mathcal{G}}$ is canonically isomorphic to $Ext^0_{\mathcal{O}_X \otimes A}(L, \mathcal{G} \otimes_A I)$.*

Proof. FIXME. □

Finally, we put everything together as follows.

Proposition 83.10.4. *With Λ, X, A, $\mathcal{F}$ as above. There exists a canonical object $L = L(\Lambda, X, A, \mathcal{F})$ of $D(X_A)$ such that given a surjection $A' \to A$ of Λ-algebras with square zero kernel I we have*

(1) *The category $Lift(\mathcal{F}, A')$ is nonempty if and only if a certain class $\xi \in Ext^2_{X_A}(L, \mathcal{F} \otimes_A I)$ is zero.*
(2) *If $Lift(\mathcal{F}, A')$ is nonempty, then $Lift(\mathcal{F}, A')$ is principal homogeneous under $Ext^1_{X_A}(L, \mathcal{F} \otimes_A I)$.*
(3) *Given a lift $\mathcal{F}'$, the set of automorphisms of $\mathcal{F}'$ which pull back to $id_{\mathcal{F}}$ is canonically isomorphic to $Ext^0_{X_A}(L, \mathcal{F} \otimes_A I)$.*

Proof. FIXME. □

Lemma 83.10.5. *In the situation of Proposition 83.10.4, if $X \to \mathrm{Spec}(\Lambda)$ is locally of finite type and Λ is Noetherian, then L is pseudo-coherent.*

Proof. FIXME. □

83.11. Duplicate references

This section is a place where we collect duplicates.

Lemma 83.11.1. *Let S be a scheme. Let X be an algebraic space over S. The map $\{\mathrm{Spec}(k) \to X \text{ monomorphism}\} \to |X|$ is injective.*

Proof. This is a duplicate of Properties of Spaces, Lemma 48.4.11. □

Theorem 83.11.2. *Let $S = \mathrm{Spec}(K)$ with K a field. Let $\overline{s}$ be a geometric point of S. Let $G = Gal_{\kappa(s)}$ denote the absolute Galois group. Then there is an equivalence of categories $Sh(S_{étale}) \to G$-Sets, $\mathcal{F} \mapsto \mathcal{F}_{\overline{s}}$.*

Proof. This is a duplicate of Étale Cohomology, Theorem 44.57.3. □

Remark 83.11.3. You got here because of a duplicate tag. Please see Formal Deformation Theory, Section 64.11 for the actual content.

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CHAPTER 84

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85.2. Definitions listed per chapter

Introduction

Conventions

Set Theory

Categories

In 4.2.1: *category*
In 4.2.4: *isomorphism*
In 4.2.5: *groupoid*
In 4.2.8: *functor*
In 4.2.9: *faithful*, *fully faithful*, *essentially surjective*
In 4.2.10: *subcategory*, *full subcategory*, *strictly full*
In 4.2.15: *natural transformation*, *morphism of functors*
In 4.2.17: *equivalence of categories*, *quasi-inverse*
In 4.2.20: *product category*
In 4.3.1: *opposite category*
In 4.3.2: *contravariant*
In 4.3.3: *presheaf of sets on* $\mathcal{C}$, *presheaf*
In 4.3.6: *representable*
In 4.4.1: *product*
In 4.4.2: *has products of pairs of objects*
In 4.5.1: *coproduct*, *amalgamated sum*
In 4.5.2: *has coproducts of pairs of objects*
In 4.6.1: *fibre product*
In 4.6.2: *cartesian*
In 4.6.3: *has fibre products*
In 4.6.4: *representable*
In 4.8.2: *representable*, *F is relatively representable over G*
In 4.9.1: *pushout*
In 4.9.2: *cocartesian*
In 4.10.1: *equalizer*
In 4.11.1: *coequalizer*
In 4.12.1: *initial*, *final*
In 4.13.1: *monomorphism*, *epimorphism*
In 4.14.1: *limit*
In 4.14.2: *colimit*
In 4.14.5: *product*
In 4.14.6: *coproduct*
In 4.16.1: *connected*
In 4.17.1: *$\mathcal{I}$ is cofinal in $\mathcal{J}$*, *cofinal*
In 4.17.3: *$\mathcal{I}$ is initial in $\mathcal{J}$*, *initial*
In 4.19.1: *directed*, *filtered*, *directed*, *filtered*
In 4.20.1: *codirected*, *cofiltered*, *codirected*, *cofiltered*
In 4.21.1: *system over I in $\mathcal{C}$*, *inductive system over I in $\mathcal{C}$*, *inverse system over I*

in $\mathcal{C}$, *projective system over I in $\mathcal{C}$*, *transition maps*
In 4.21.2: *directed system*, *directed inverse system*, *directed*
In 4.22.1: *is essentially constant*, *value*, *essentially constant*, *value*
In 4.22.2: *essentially constant system*, *essentially constant inverse system*
In 4.23.1: *left exact*, *right exact*, *exact*
In 4.24.1: *left adjoint*, *right adjoint*
In 4.25.1: *left multiplicative system*, *right multiplicative system*, *multiplicative system*
In 4.25.17: *saturated*
In 4.26.1: *horizontal*
In 4.27.1: 2-*category*, 1-*morphisms*, 2-*morphisms*, *vertical*, *composition*, *horizontal*
In 4.27.2: *sub* 2-*category*
In 4.27.4: *equivalent*
In 4.27.5: *functor*, *weak functor*, *pseudo functor*
In 4.28.1: $(2,1)$-*category*
In 4.29.1: *final object*
In 4.29.2: *2-fibre product of f and g*
In 4.30.1: 2-*category of categories over $\mathcal{C}$*
In 4.30.2: *fibre category*, *lift*, *x lies over U*, *lift*, *ϕ lies over f*
In 4.31.1: *strongly cartesian morphism*, *strongly $\mathcal{C}$-cartesian morphism*
In 4.31.5: *fibred category over $\mathcal{C}$*
In 4.31.6: *choice of pullbacks*, *pullback functor*
In 4.31.9: 2-*category of fibred categories over $\mathcal{C}$*
In 4.32.2: *relative inertia of $\mathcal{S}$ over $\mathcal{S}'$*, *inertia fibred category $\mathcal{I}_{\mathcal{S}}$ of $\mathcal{S}$*
In 4.33.1: *fibred in groupoids*
In 4.33.6: 2-*category of categories fibred in groupoids over $\mathcal{C}$*
In 4.34.2: *split fibred category*, *$\mathcal{S}_F$*
In 4.35.2: *split category fibred in groupoids*, *$\mathcal{S}_F$*
In 4.36.1: *discrete*
In 4.36.2: *category fibred in sets*, *category fibred in discrete categories*
In 4.36.3: 2-*category of categories fibred in sets over $\mathcal{C}$*
In 4.37.1: *setoid*
In 4.37.2: *category fibred in setoids*
In 4.37.3: 2-*category of categories fibred in setoids over $\mathcal{C}$*
In 4.38.1: *representable*
In 4.39.5: *representable*, *$\mathcal{X}$ is relatively representable over $\mathcal{Y}$*

Topology

In 5.4.1: *base for the topology on X*, *basis for the topology on X*
In 5.4.3: *subbase for the topology on X*, *subbasis for the topology on X*
In 5.5.3: *strict map of topological spaces*, *submersive*
In 5.6.1: *connected*, *connected component*
In 5.6.7: *totally disconnected*
In 5.6.9: *locally connected*
In 5.7.1: *irreducible*, *irreducible component*
In 5.7.4: *generic point*, *Kolmogorov*, *sober*
In 5.8.1: *Noetherian*, *locally Noetherian*
In 5.9.1: *chain of irreducible closed subsets*, *length*, *dimension*, *Krull dimension*, *Krull dimension of X at x*
In 5.9.4: *equidimensional*
In 5.10.1: *codimension*
In 5.10.4: *catenary*
In 5.11.1: *quasi-compact*, *quasi-compact*, *retrocompact*
In 5.12.1: *locally quasi-compact*
In 5.14.1: *constructible*, *locally constructible*
In 5.16.2: *closed*, *proper*, *quasi-proper*, *universally closed*
In 5.17.1: *Jacobson*
In 5.18.1: *specialization*, *generalization*, *stable under specialization*, *stable under generalization*
In 5.18.3: *specializations lift along f*, *specializing*, *generalizations lift along f*, *generalizing*
In 5.19.1: *immediate specialization*, *dimension function*
In 5.20.1: *interior*, *nowhere dense*
In 5.21.1: *profinite*
In 5.22.1: *spectral*, *spectral*
In 5.24.1: *extremally disconnected*

In 5.25.2: *isolated point*
In 5.26.1: *partition*, *parts*, *refines*
In 5.26.2: *good stratification*
In 5.26.3: *stratification*, *strata*

Sheaves on Spaces

In 6.3.1: *presheaf $\mathcal{F}$ of sets on X*, *morphism $\varphi : \mathcal{F} \to \mathcal{G}$ of presheaves of sets on X*
In 6.3.2: *constant presheaf with value A*
In 6.4.4: *presheaf of abelian groups on X*, *abelian presheaf over X*, *morphism of abelian presheaves over X*
In 6.5.1: *presheaf $\mathcal{F}$ on X with values in $\mathcal{C}$*, *morphism $\varphi : \mathcal{F} \to \mathcal{G}$ of presheaves with value in $\mathcal{C}$*
In 6.5.2: *underlying presheaf of sets of $\mathcal{F}$*
In 6.6.1: *presheaf of $\mathcal{O}$-modules*, *morphism $\varphi : \mathcal{F} \to \mathcal{G}$ of presheaves of $\mathcal{O}$-modules*
In 6.7.1: *sheaf $\mathcal{F}$ of sets on X*, *morphism of sheaves of sets*
In 6.7.4: *constant sheaf with value A*
In 6.8.1: *abelian sheaf on X*, *sheaf of abelian groups on X*
In 6.9.1: *sheaf*
In 6.10.1: *sheaf of $\mathcal{O}$-modules*, *morphism of sheaves of $\mathcal{O}$-modules*
In 6.11.2: *separated*
In 6.15.1: *type of algebraic structure*
In 6.16.2: *subpresheaf*, *subsheaf*, *injective*, *surjective*, *injective*, *surjective*
In 6.21.7: *f-map $\xi : \mathcal{G} \to \mathcal{F}$*
In 6.21.9: *composition of φ and ψ*
In 6.25.1: *ringed space*, *morphism of ringed spaces*
In 6.25.3: *composition of morphisms of ringed spaces*
In 6.26.1: *pushforward*, *pullback*
In 6.27.1: *skyscraper sheaf at x with value A*, *skyscraper sheaf*, *skyscraper sheaf*, *skyscraper sheaf*, *skyscraper sheaf*
In 6.30.1: *presheaf $\mathcal{F}$ of sets on $\mathcal{B}$*, *morphism $\varphi : \mathcal{F} \to \mathcal{G}$ of presheaves of sets on $\mathcal{B}$*
In 6.30.2: *sheaf $\mathcal{F}$ of sets on $\mathcal{B}$*, *morphism of sheaves of sets on $\mathcal{B}$*
In 6.30.8: *presheaf $\mathcal{F}$ with values in $\mathcal{C}$ on $\mathcal{B}$*, *morphism $\varphi : \mathcal{F} \to \mathcal{G}$ of presheaves with values in $\mathcal{C}$ on $\mathcal{B}$*, *sheaf $\mathcal{F}$ with values in $\mathcal{C}$ on $\mathcal{B}$*
In 6.30.11: *presheaf of $\mathcal{O}$-modules $\mathcal{F}$ on $\mathcal{B}$*, *morphism $\varphi : \mathcal{F} \to \mathcal{G}$ of presheaves of $\mathcal{O}$-modules on $\mathcal{B}$*, *sheaf $\mathcal{F}$ of $\mathcal{O}$-modules on $\mathcal{B}$*
In 6.31.2: *restriction of $\mathcal{G}$ to U*, *restriction of $\mathcal{G}$ to U*, *open subspace of $(X, \mathcal{O})$ associated to U*, *restriction of $\mathcal{G}$ to U*
In 6.31.3: *extension of $\mathcal{F}$ by the empty set $j_{p!}\mathcal{F}$*, *extension of $\mathcal{F}$ by the empty set $j_!\mathcal{F}$*
In 6.31.5: *extension $j_{p!}\mathcal{F}$ of $\mathcal{F}$ by 0*, *extension $j_!\mathcal{F}$ of $\mathcal{F}$ by 0*, *extension $j_{p!}\mathcal{F}$ of $\mathcal{F}$ by e*, *extension $j_!\mathcal{F}$ of $\mathcal{F}$ by e*, *extension by 0*, *extension by 0*

Sites and Sheaves

In 7.2.1: *presheaf of sets*, *Morphisms of presheaves*
In 7.2.2: *presheaf*, *morphism*
In 7.3.1: *injective*, *surjective*
In 7.3.3: *subpresheaf*
In 7.3.5: *image of φ*
In 7.6.1: *family of morphisms with fixed target*
In 7.6.2: *site*, *coverings of $\mathcal{C}$*
In 7.7.1: *sheaf*
In 7.7.5: *$Sh(\mathcal{C})$*
In 7.7.6: *sheaf*
In 7.8.1: *morphism of families of maps with fixed target of $\mathcal{C}$ from $\mathcal{U}$ to $\mathcal{V}$*, *morphism from $\mathcal{U}$ to $\mathcal{V}$*, *refinement*
In 7.8.2: *combinatorially equivalent*, *tautologically equivalent*
In 7.10.9: *separated*
In 7.10.11: *sheaf associated to $\mathcal{F}$*
In 7.11.1: *quasi-compact*
In 7.12.1: *injective*, *surjective*
In 7.13.1: *effective epimorphism*, *universal effective epimorphism*
In 7.13.2: *weaker than the canonical topology*, *subcanonical*
In 7.13.3: *representable sheaves*, $\underline{U}$
In 7.14.1: *continuous*
In 7.15.1: *morphism of sites*
In 7.15.4: *composition*
In 7.16.1: *topos*, *morphism of topoi*, *composition $f \circ g$*

In 7.19.1: *cocontinuous*
In 7.24.1: *localization of the site $\mathcal{C}$ at the object U*, *localization morphism*, *direct image functor*, *restriction of $\mathcal{F}$ to $\mathcal{C}/U$*, *extension of $\mathcal{G}$ by the empty set*
In 7.28.2: *special cocontinuous functor u from $\mathcal{C}$ to $\mathcal{D}$*
In 7.29.4: *localization of the topos $Sh(\mathcal{C})$ at $\mathcal{F}$*, *localization morphism*
In 7.31.1: *point of the topos $Sh(\mathcal{C})$*
In 7.31.2: *point p of the site $\mathcal{C}$*
In 7.31.6: *skyscraper sheaf*
In 7.35.1: *2-morphism from f to g*
In 7.36.2: *morphism $f : p \to p'$*
In 7.37.1: *conservative*, *has enough points*
In 7.39.2: *weakly contractible*, *enough weakly contractible objects*, *enough P objects*
In 7.41.1: *sheaf theoretically empty*
In 7.41.3: *almost cocontinuous*
In 7.42.1: *embedding*
In 7.42.2: *subtopos*
In 7.42.4: *open subtopos*
In 7.42.6: *closed subtopos*
In 7.42.7: *open immersion*, *closed immersion*
In 7.43.1: *pushforward*
In 7.44.1: *global sections*
In 7.45.1: *sieve S on U*
In 7.45.3: *sieve on U generated by the morphisms f_i*
In 7.45.4: *pullback of S by f*
In 7.45.6: *topology on $\mathcal{C}$*
In 7.45.8: *finer*
In 7.45.10: *sheaf*
In 7.45.12: *canonical topology*
In 7.46.2: *topology associated to $\mathcal{C}$*
In 7.47.2: *separated*
In 7.47.4: *sheaf associated to $\mathcal{F}$*
In 7.50.1: *point p*

Stacks

In 8.2.2: *presheaf of morphisms from x to y*, *presheaf of isomorphisms from x to y*
In 8.3.1: *descent datum (X_i, φ_{ij}) in $\mathcal{S}$ relative to the family $\{f_i : U_i \to U\}$*, *cocycle condition*, *morphism $\psi : (X_i, \varphi_{ij}) \to (X'_i, \varphi'_{ij})$ of descent data*
In 8.3.4: *pullback functor*
In 8.3.5: *trivial descent datum*, *canonical descent datum*, *effective*
In 8.4.1: *stack*
In 8.4.5: *2-category of stacks over $\mathcal{C}$*
In 8.5.1: *stack in groupoids*
In 8.5.5: *2-category of stacks in groupoids over $\mathcal{C}$*
In 8.6.1: *stack in setoids*, *stack in sets*, *stack in discrete categories*
In 8.6.5: *2-category of stacks in setoids over $\mathcal{C}$*
In 8.10.2: *structure of site on $\mathcal{S}$ inherited from $\mathcal{C}$*, *$\mathcal{S}$ is endowed with the topology inherited from $\mathcal{C}$*
In 8.11.1: *gerbe*
In 8.11.4: *gerbe over*
In 8.12.4: *$f_*\mathcal{S}$, pushforward of $\mathcal{S}$ along f*
In 8.12.9: *$f^{-1}\mathcal{S}$, pullback of $\mathcal{S}$ along f*

Fields

In 9.2.1: *field*, *subfield*
In 9.2.2: *domain*, *integral domain*
In 9.5.1: *characteristic*, *prime subfield of F*
In 9.6.2: *field extension*
In 9.6.3: *tower*
In 9.6.6: *generates*
In 9.7.1: *degree*, *finite*
In 9.7.7: *number field*
In 9.8.1: *algebraic*, *algebraic extension*
In 9.9.1: *minimal polynomial*
In 9.10.1: *algebraically closed*
In 9.10.3: *algebraically closed*, *algebraic closure*
In 9.11.1: *relatively prime*
In 9.12.2: *separable*, *separable*, *separable*
In 9.12.6: *separable degree*
In 9.13.1: *purely inseparable*, *purely inseparable*
In 9.13.7: *separable degree*, *inseparable degree*, *degree of inseparability*
In 9.14.1: *normal*
In 9.14.5: *automorphisms of E over F*, *automorphisms of E/F*
In 9.15.2: *splitting field of P over F*
In 9.19.1: *Galois*

In 9.19.3: *Galois group*
In 9.23.1: *algebraically independent*, *purely transcendental extension*, *transcendence basis*
In 9.23.4: *transcendence degree*
In 9.23.9: *algebraic closure of k in K*, *algebraically closed in K*
In 9.24.1: *compositum of K and L in Ω*
In 9.24.2: *linearly disjoint over k in Ω*
In 9.25.1: *algebraic*, *separable*, *purely inseparable*, *normal*, *Galois*

Commutative Algebra

In 10.5.1: *finite R-module*, *finitely generated R-module*, *finitely presented R-module*, *R-module of finite presentation*
In 10.6.1: *finite type*, *S is a finite type R-algebra*, *finite presentation*
In 10.7.1: *finite*
In 10.8.1: *partially ordered set*, *directed set*
In 10.8.2: *system (M_i, μ_{ij}) of R-modules over I*, *directed system*
In 10.8.7: *homomorphism of systems*
In 10.8.12: *relation*
In 10.9.1: *multiplicative subset of R*
In 10.9.2: *localization of A with respect to S*
In 10.9.6: *localization*
In 10.11.1: *R-bilinear*
In 10.11.6: *(A, B)-bimodule*
In 10.13.1: *base change*, *base change*
In 10.16.1: *spectrum*
In 10.16.3: *Zariski*, *standard opens*
In 10.17.1: *local ring*, *local homomorphism of local rings*, *local ring map $\varphi : R \to S$*
In 10.26.2: *Oka family*
In 10.32.1: *Jacobson ring*
In 10.33.1: *integral over R*, *integral*
In 10.33.8: *integral closure*, *integrally closed*
In 10.34.1: *normal*
In 10.34.3: *almost integral over R*, *completely normal*
In 10.34.10: *normal*
In 10.35.1: *integral over I*
In 10.36.1: *flat*, *faithfully flat*, *flat*, *faithfully flat*
In 10.37.1: *support of M*
In 10.37.3: *annihilator of m*, *annihilator of M*
In 10.38.1: *going up*, *going down*
In 10.39.1: *separably generated over k*, *separable over k*
In 10.40.1: *geometrically reduced over k*
In 10.42.1: *perfect*
In 10.42.5: *perfect closure*
In 10.43.6: *geometrically irreducible over k*
In 10.44.3: *geometrically connected over k*
In 10.45.1: *geometrically integral over k*
In 10.46.1: *dominates*, *valuation ring*, *centered*
In 10.46.11: *value group*, *valuation*, *discrete valuation ring*
In 10.47.2: *locally nilpotent*, *nilpotent*
In 10.48.1: *length*
In 10.48.9: *simple*
In 10.49.1: *Artinian*
In 10.50.1: *essentially of finite type*, *essentially of finite presentation*
In 10.53.1: *homogeneous spectrum*
In 10.54.1: *blowup algebra*, *Rees algebra*, *affine blowup algebra*
In 10.55.3: *numerical polynomial*
In 10.56.1: *an ideal of definition of R*
In 10.56.6: *Hilbert polynomial*
In 10.56.8: *$d(M)$*
In 10.57.1: *Krull dimension*
In 10.57.2: *height*
In 10.57.9: *system of parameters of R*, *regular local ring*, *regular system of parameters*
In 10.60.1: *associated*
In 10.61.1: *symbolic power*
In 10.62.2: *relative assassin of N over S/R*
In 10.63.1: *weakly associated*
In 10.64.1: *embedded associated primes*, *embedded primes of R*
In 10.65.1: *M-regular*, *M-regular sequence in I*, *regular sequence*
In 10.65.4: *I-depth*, *depth*
In 10.66.1: *M-quasi-regular*, *quasi-regular sequence*

In 10.67.2: *resolution*, *resolution of M by free R-modules*, *resolution of M by finite free R-modules*
In 10.72.1: *projective*
In 10.73.1: *locally free*, *finite locally free*
In 10.77.1: *universally injective*, *universally exact*
In 10.79.1: *direct sum dévissage*, *Kaplansky dévissage*
In 10.81.1: *Mittag-Leffler inverse system*
In 10.83.1: *Mittag-Leffler directed system of modules*
In 10.83.2: *dominates*
In 10.83.6: *Mittag-Leffler*
In 10.85.1: *coherent module*, *coherent ring*
In 10.91.5: *I-adically complete*, *I-adically complete*
In 10.96.4: *rank*
In 10.98.1: *Cohen-Macaulay*
In 10.99.1: *Cohen-Macaulay*
In 10.99.6: *Cohen-Macaulay*
In 10.99.9: *maximal Cohen-Macaulay*
In 10.100.1: *catenary*
In 10.100.5: *universally catenary*
In 10.103.1: *pure*
In 10.104.2: *finite projective dimension*, *projective dimension*
In 10.104.6: *finite global dimension*, *global dimension*
In 10.105.6: *regular*
In 10.106.5: *local ring of the fibre at $\mathfrak{q}$*
In 10.113.7: *uniformizer*
In 10.114.1: *associates*, *irreducible*, *prime*
In 10.114.4: *unique factorization domain*, *UFD*
In 10.114.6: *principal ideal domain*, *PID*
In 10.114.8: *Dedekind domain*
In 10.115.2: *order of vanishing along R*
In 10.115.3: *lattice in V*
In 10.115.5: *distance between M and M'*
In 10.116.3: *quasi-finite at $\mathfrak{q}$*, *quasi-finite*
In 10.117.8: *strongly transcendental over R*
In 10.119.1: *relative dimension of S/R at $\mathfrak{q}$*, *relative dimension of*
In 10.125.1: *derivation*, *R-derivation*, *Leibniz rule*
In 10.125.2: *module of Kähler differentials*, *module of differentials*
In 10.126.1: *differential operator $D : M \to N$ of order k*
In 10.126.4: *module of principal parts of order k*
In 10.127.1: *naive cotangent complex*
In 10.128.1: *global complete intersection over k*, *local complete intersection over k*
In 10.128.5: *complete intersection (over k)*
In 10.129.1: *syntomic*, *flat local complete intersection over R*
In 10.129.5: *relative global complete intersection*
In 10.130.1: *smooth*
In 10.130.6: *standard smooth algebra over R*
In 10.130.11: *smooth at $\mathfrak{q}$*
In 10.131.1: *formally smooth over R*
In 10.134.1: *small extension*
In 10.136.1: *étale*, *étale at $\mathfrak{q}$*
In 10.136.14: *standard étale*
In 10.139.1: *formally unramified over R*
In 10.140.2: *universal first order thickening*, *conormal module*, *$C_{S/R}$*
In 10.141.1: *formally étale over R*
In 10.142.1: *unramified*, *G-unramified*, *unramified at $\mathfrak{q}$*, *G-unramified at $\mathfrak{q}$*
In 10.143.1: *henselian*, *strictly henselian*
In 10.143.18: *henselization*, *strict henselization of R with respect to $\kappa \subset \kappa^{sep}$*, *strict henselization*
In 10.144.1: *(R_k)*, *regular in codimension $\leq k$*, *(S_k)*
In 10.147.1: *complete local ring*
In 10.147.4: *coefficient ring*
In 10.147.5: *Cohen ring*
In 10.148.1: *N-1*, *N-2*, *Japanese*
In 10.148.15: *universally Japanese*, *Nagata ring*
In 10.148.23: *analytically unramified*, *analytically unramified*
In 10.151.2: *geometrically normal*
In 10.152.2: *geometrically regular*

Brauer groups

In 11.2.1: *finite*

In 11.2.2: *skew field*
In 11.2.3: *simple*, *simple*
In 11.2.4: *central*
In 11.2.5: *opposite algebra*
In 11.5.2: *Brauer group*
In 11.8.1: *splits*, *splitting field*

Homological Algebra

In 12.3.1: *preadditive*, *additive*
In 12.3.3: *zero object*
In 12.3.5: *direct sum*
In 12.3.8: *additive*
In 12.3.9: *kernel*, *cokernel*, *coimage of* f, *image of* f
In 12.4.1: *Karoubian*
In 12.5.1: *abelian*
In 12.5.3: *injective*, *surjective*, *subobject*, *quotient*
In 12.5.7: *complex*, *exact at* y, *exact*, *short exact sequence*
In 12.5.9: *split*
In 12.6.1: *extension* E *of* B *by* A
In 12.6.2: *Ext-group*
In 12.9.1: *Serre subcategory*, *weak Serre subcategory*
In 12.9.5: *kernel of the functor* F
In 12.10.1: *zeroth* K*-group of* $\mathcal{A}$
In 12.11.1: *cohomological* δ*-functor*, δ*-functor*
In 12.11.2: *morphism of* δ*-functors from* F *to* G
In 12.11.3: *universal* δ*-functor*
In 12.12.2: *homotopy equivalence*, *homotopy equivalent*
In 12.12.4: *quasi-isomorphism*, *acyclic*
In 12.12.8: *homotopy equivalence*, *homotopy equivalent*
In 12.12.10: *quasi-isomorphism*, *acyclic*
In 12.14.1: k*-shifted chain complex* $A[k]_\bullet$
In 12.14.2: $H_{i+k}(A_\bullet) \to H_i(A[k]_\bullet)$
In 12.14.7: k*-shifted cochain complex* $A[k]^\bullet$
In 12.14.8: $H^{i+k}(A^\bullet) \longrightarrow H^i(A[k]^\bullet)$
In 12.15.1: *category of graded objects of* $\mathcal{A}$
In 12.15.3: *shift*
In 12.16.1: *decreasing filtration*, *filtered object of* $\mathcal{A}$, *morphism* $(A, F) \to (B, F)$ *of filtered objects*, *induced filtration*, *quotient filtration*, *finite*, *separated*, *exhaustive*
In 12.16.3: *strict*
In 12.17.1: *spectral sequence in* $\mathcal{A}$, *morphism of spectral sequences*
In 12.17.2: *limit*, *collapses at* E_r, *degenerates at* E_r
In 12.18.1: *exact couple*, *morphism of exact couples*
In 12.18.3: *spectral sequence associated to the exact couple*
In 12.19.1: *differential object*, *morphism of differential objects*
In 12.19.3: *homology*
In 12.19.5: *spectral sequence associated to* (A, d, α)
In 12.20.1: *filtered differential object*
In 12.20.4: *induced filtration*
In 12.20.6: *converges*, *abuts to*, *converges to*
In 12.21.1: *filtered complex* $K^\bullet$ *of* $\mathcal{A}$
In 12.21.5: *induced filtration*
In 12.21.7: *converges*
In 12.22.1: *double complex*
In 12.22.2: *associated simple complex* $sA^\bullet$, *associated total complex*
In 12.22.4: *converges*, *converges*
In 12.23.1: *injective*
In 12.23.4: *enough injectives*
In 12.23.5: *functorial injective embeddings*
In 12.24.1: *projective*
In 12.24.4: *enough projectives*
In 12.24.5: *functorial projective surjections*
In 12.26.2: *Mittag-Leffler condition*, *ML*

Derived Categories

In 13.3.1: *triangle*, *morphism of triangles*
In 13.3.2: *triangulated category*, *distinguished triangles*, *pre-triangulated category*
In 13.3.3: *exact functor*, *triangulated functor*
In 13.3.4: *pre-triangulated subcategory*, *triangulated subcategory*
In 13.3.5: *homological*, *cohomological*

In 13.3.6: *δ-functor from $\mathcal{A}$ to $\mathcal{D}$, image of the short exact sequence under the given δ-functor*
In 13.5.1: *compatible with the triangulated structure*
In 13.6.1: *saturated*
In 13.6.5: *kernel of F*, *kernel of H*
In 13.6.7: *quotient category $\mathcal{D}/\mathcal{B}$*, *quotient functor*
In 13.7.1: *category of (cochain) complexes*, *bounded below*, *bounded above*, *bounded*
In 13.8.1: *cone*
In 13.8.4: *termwise split injection $\alpha : A^\bullet \to B^\bullet$*, *termwise split surjection $\beta : B^\bullet \to C^\bullet$*
In 13.8.9: *termwise split sequence of complexes of $\mathcal{A}$*, *triangle associated to the termwise split sequence of complexes*
In 13.9.1: *distinguished triangle of $K(\mathcal{A})$*
In 13.10.3: *derived category of $\mathcal{A}$*, *bounded derived category*
In 13.13.1: *category of finite filtered objects of $\mathcal{A}$*
In 13.13.2: *filtered quasi-isomorphism*, *filtered acyclic*
In 13.13.5: *filtered derived category of $\mathcal{A}$*
In 13.13.7: *bounded filtered derived category*
In 13.14.2: *right derived functor RF is defined at*, *value of RF at X*, *left derived functor LF is defined at*, *value of LF at X*
In 13.14.9: *right deriveable*, *everywhere defined*, *left deriveable*, *everywhere defined*
In 13.14.10: *computes*, *computes*
In 13.15.3: *right derived functors of F*, *left derived functors of F*, *right acyclic for F*, *acyclic for RF*, *left acyclic for F*, *acyclic for LF*
In 13.16.2: *ith right derived functor R^iF of F*
In 13.17.1: *injective resolution of A*, *injective resolution of $K^\bullet$*
In 13.18.1: *projective resolution of A*, *projective resolution of $K^\bullet$*
In 13.20.1: *Cartan-Eilenberg resolution*
In 13.22.2: *resolution functor*
In 13.25.1: *filtered injective*
In 13.26.1: *ith extension group*
In 13.26.4: *Yoneda extension*, *equivalent*
In 13.28.1: *K-injective*
In 13.30.1: *derived colimit*, *homotopy colimit*
In 13.30.2: *derived limit*, *homotopy limit*
In 13.31.1: *classical generator*, *strong generator*, *weak generator*, *generator*
In 13.32.1: *compact object*
In 13.32.5: *compactly generated*

Simplicial Methods

In 14.2.1: $\delta^n_j : [n-1] \to [n]$, $\sigma^n_j : [n+1] \to [n]$
In 14.3.1: *simplicial object U of $\mathcal{C}$*, *simplicial set*, *simplicial abelian group*, *morphism of simplicial objects $U \to U'$*, *category of simplicial objects of $\mathcal{C}$*
In 14.5.1: *cosimplicial object U of $\mathcal{C}$*, *cosimplicial set*, *cosimplicial abelian group*, *morphism of cosimplicial objects $U \to U'$*, *category of cosimplicial objects of $\mathcal{C}$*
In 14.6.1: *product of U and V*
In 14.7.1: *fibre product of V and W over U*
In 14.8.1: *pushout of V and W over U*
In 14.9.1: *product of U and V*
In 14.10.1: *fibre product of V and W over U*
In 14.11.1: *n-simplex of U*, *face of x*, *degeneracy of x*, *degenerate*
In 14.12.1: *product $U \times V$ of U and V*, *product $U \times V$ exists*
In 14.13.1: $\mathrm{Hom}(U, V)$
In 14.15.1: $\mathrm{Hom}(U, V)$
In 14.16.1: *split*
In 14.17.1: *n-truncated simplicial object of $\mathcal{C}$*, *morphism of n-truncated simplicial objects*
In 14.18.1: *augmentation $\epsilon : U \to X$ of U towards an object X of $\mathcal{C}$*
In 14.20.3: *Eilenberg-Maclane object $K(A, k)$*
In 14.24.1: *homotopy connecting a to b*, *homotopic*

In 14.24.6: *homotopy equivalence*, *homotopy equivalent*
In 14.26.1: *homotopic*, *homotopy connecting a and b*
In 14.28.1: *trivial Kan fibration*
In 14.29.1: *Kan fibration*, *Kan complex*

More on Algebra

In 15.5.3: *kth fitting ideal*
In 15.7.1: *henselian pair*
In 15.8.1: *auto-associated*
In 15.15.1: *torsion*, *torsion free*
In 15.17.1: *strict transform of M along $R \to R'$*
In 15.19.1: *Koszul complex*
In 15.19.2: *Koszul complex on $f_1, \ldots, f_r$*
In 15.20.1: *M-Koszul-regular*, *M-H_1-regular*, *Koszul-regular*, *H_1-Koszul-regular*
In 15.21.1: *regular ideal*, *Koszul-regular ideal*, *H_1-regular ideal*, *quasi-regular ideal*
In 15.22.2: *local complete intersection*
In 15.25.1: *topological ring*, *topological module*, *homomorphism of topological modules*, *homomorphism of topological rings*, *linearly topologized*, *linearly topologized*, *ideal of definition*, *pre-admissible*, *admissible*, *pre-adic*, *adic*
In 15.26.1: *formally smooth over R*
In 15.26.3: *formally smooth for the $\mathfrak{n}$-adic topology*
In 15.29.1: *regular*
In 15.33.1: *p-independent over k*, *p-basis of K over k*
In 15.34.1: *J-0*, *J-1*, *J-2*
In 15.37.1: *G-ring*
In 15.38.1: *quasi-excellent*, *excellent*
In 15.40.1: *$M \mapsto M^\vee$*, *free module*
In 15.43.3: *K-flat*
In 15.43.13: *derived tensor product*
In 15.45.1: *Tor independent over R*
In 15.48.1: *m-pseudo-coherent*, *pseudo-coherent*, *m-pseudo-coherent*, *pseudo-coherent*
In 15.49.1: *tor-amplitude in $[a, b]$*, *finite tor dimension*, *tor dimension $\leq d$*, *finite tor dimension*
In 15.50.1: *perfect*, *perfect*
In 15.52.2: *finitely presented relative to R*
In 15.53.4: *m-pseudo-coherent relative to R*, *pseudo-coherent relative to R*, *m-pseudo-coherent relative to R*, *pseudo-coherent relative to R*
In 15.54.1: *pseudo-coherent ring map*, *perfect ring map*
In 15.56.1: *I-power torsion module*, *an f-power torsion module*
In 15.57.4: *derived complete with respect to I*, *derived complete with respect to I*
In 15.60.1: *absolutely flat*, *weakly étale*, *absolutely flat*
In 15.60.3: *weak dimension $\leq d$*
In 15.61.1: *extension of discrete valuation rings*, *ramification index*, *weakly unramified*, *residual degree*, *residue degree*
In 15.61.6: *unramified*, *totally ramified*, *tamely ramified*
In 15.61.9: *decomposition group*, *inertia group*
In 15.61.17: *mixed characteristic*, *absolute ramification index*
In 15.62.2: *weak solution for $A \subset B$*, *solution for $A \subset B$*, *separable solution*

Smoothing Ring Maps

In 16.3.1: *singular ideal of A over R*
In 16.3.3: *elementary standard in A over R*, *strictly standard in A over R*

Sheaves of Modules

In 17.4.1: *generated by global sections*, *generate*
In 17.4.5: *subsheaf generated by the s_i*
In 17.5.1: *support of $\mathcal{F}$*, *support of s*
In 17.8.1: *locally generated by sections*
In 17.9.1: *finite type*
In 17.10.1: *quasi-coherent sheaf of $\mathcal{O}_X$-modules*
In 17.10.6: *sheaf associated to the module M and the ring map α*, *sheaf associated to the module M*
In 17.11.1: *finite presentation*
In 17.12.1: *coherent $\mathcal{O}_X$-module*
In 17.13.1: *closed immersion of ringed spaces*

In 17.14.1: *locally free*, *finite locally free*, *finite locally free of rank r*
In 17.16.1: *flat*
In 17.16.3: *flat at x*
In 17.17.1: *flat at x*, *flat*
In 17.17.3: *flat over Y at a point $x \in X$*, *flat over Y*
In 17.20.1: *Koszul complex*
In 17.20.2: *Koszul complex on $f_1, \ldots, f_r$*
In 17.21.1: *invertible $\mathcal{O}_X$-module*, *trivial*
In 17.21.3: *tensor power of $\mathcal{L}$*
In 17.21.4: *associated graded ring*
In 17.21.6: *Picard group*
In 17.23.1: *$\mathcal{O}_1$-derivation*, *φ-derivation*, *Leibniz rule*
In 17.23.3: *module of differentials*, *universal φ-derivation*
In 17.23.10: *S-derivation*, *sheaf of differentials $\Omega_{X/S}$ of X over S*
In 17.24.1: *naive cotangent complex*
In 17.24.4: *naive cotangent complex*

Modules on Sites

In 18.4.1: *free abelian presheaf*
In 18.5.1: *free abelian sheaf*
In 18.6.1: *ringed site*, *structure sheaf*, *morphism of ringed sites*, *composition of morphisms of ringed sites*
In 18.7.1: *ringed topos*, *structure sheaf*, *morphism of ringed topoi*, *composition of morphisms of ringed topoi*
In 18.8.1: *2-morphism from f to g*
In 18.9.1: *presheaf of $\mathcal{O}$-modules*, *morphism $\varphi : \mathcal{F} \to \mathcal{G}$ of presheaves of $\mathcal{O}$-modules*
In 18.10.1: *sheaf of $\mathcal{O}$-modules*, *morphism of sheaves of $\mathcal{O}$-modules*
In 18.13.1: *pushforward*, *pullback*
In 18.16.1: *$g_{p!}\mathcal{F}$*, *$g_!\mathcal{F} = (g_{p!}\mathcal{F})^\#$*
In 18.17.1: *free $\mathcal{O}$-module*, *finite free*, *generated by global sections*, *generated by r global sections*, *generated by finitely many global sections*, *global presentation*, *global finite presentation*
In 18.19.1: *localization of the ringed site $(\mathcal{C}, \mathcal{O})$ at the object U*, *localization morphism*, *direct image functor*, *restriction of $\mathcal{F}$ to $\mathcal{C}/U$*, *extension by zero*
In 18.21.2: *localization of the ringed topos $(Sh(\mathcal{C}), \mathcal{O})$ at $\mathcal{F}$*, *localization morphism*
In 18.23.1: *locally free*, *finite locally free*, *locally generated by sections*, *locally generated by r sections*, *of finite type*, *quasi-coherent*, *of finite presentation*, *coherent*
In 18.28.1: *flat*, *flat*, *flat*, *flat*
In 18.30.1: *flat*, *flat*
In 18.30.3: *flat over $(Sh(\mathcal{D}), \mathcal{O}')$*
In 18.31.1: *rank r*, *invertible $\mathcal{O}$-module*, *$\mathcal{O}^*$*
In 18.31.4: *Picard group*
In 18.32.1: *$\mathcal{O}_1$-derivation*, *φ-derivation*, *Leibniz rule*
In 18.32.3: *module of differentials*, *universal φ-derivation*
In 18.32.10: *Y-derivation*, *sheaf of differentials $\Omega_{X/Y}$ of X over Y*, *universal Y-derivation*
In 18.33.1: *differential operator $D : \mathcal{F} \to \mathcal{G}$ of order k*
In 18.33.4: *module of principal parts of order k*
In 18.34.1: *naive cotangent complex*
In 18.34.4: *naive cotangent complex*
In 18.39.4: *locally ringed site*
In 18.39.6: *locally ringed*
In 18.39.8: *morphism of locally ringed topoi*, *morphism of locally ringed sites*
In 18.42.1: *constant sheaf*, *locally constant*, *finite locally constant*

Injectives

In 19.2.4: *α-small with respect to I*
In 19.10.1: *generator*, *Grothendieck abelian category*
In 19.11.1: *size*

Cohomology of Sheaves

In 20.5.1: *torsor*, *$\mathcal{G}$-torsor*, *morphism of $\mathcal{G}$-torsors*, *trivial $\mathcal{G}$-torsor*
In 20.10.1: *Čech complex*, *Čech cohomology groups*
In 20.13.1: *flasque*, *flabby*
In 20.22.1: *alternating Čech complex*
In 20.22.2: *ordered Čech complex*
In 20.23.1: *locally finite*
In 20.25.2: *K-flat*

In 20.25.13: *derived tensor product*
In 20.25.14: *Tor*
In 20.32.1: *strictly perfect*
In 20.33.1: *m-pseudo-coherent*, *pseudo-coherent*, *m-pseudo-coherent*, *pseudo-coherent*
In 20.34.1: *tor-amplitude in* $[a, b]$, *finite tor dimension*, *locally has finite tor dimension*
In 20.35.1: *perfect*, *perfect*

Cohomology on Sites

In 21.5.1: *pseudo torsor*, *pseudo* $\mathcal{G}$*-torsor*, *morphism of pseudo* $\mathcal{G}$*-torsors*, *torsor*, $\mathcal{G}$*-torsor*, *morphism of* G*-torsors*, *trivial* $\mathcal{G}$*-torsor*
In 21.9.1: *Cech complex*, *Cech cohomology groups*
In 21.13.4: *limp*
In 21.17.2: *K-flat*
In 21.17.11: *derived tensor product*
In 21.17.12: *Tor*
In 21.23.2: *qc covering*
In 21.30.1: *simplicial* $\mathcal{A}_\bullet$*-module*, *simplicial sheaf of* $\mathcal{A}_\bullet$*-modules*
In 21.32.1: *strictly perfect*
In 21.33.1: *m-pseudo-coherent*, *pseudo-coherent*, *m-pseudo-coherent*, *pseudo-coherent*
In 21.34.1: *tor-amplitude in* $[a, b]$, *finite tor dimension*, *locally has finite tor dimension*
In 21.35.1: *perfect*, *perfect*

Differential Graded Algebra

In 22.3.1: *differential graded algebra over* R
In 22.3.2: *homomorphism of differential graded algebras*
In 22.3.3: *opposite differential graded algebra*
In 22.3.4: *commutative*, *strictly commutative*
In 22.3.5: *tensor product differential graded algebra*
In 22.4.1: *differential graded module*, *homomorphism of differential graded modules*
In 22.4.3: *k-shifted module*
In 22.5.1: *homotopy between* f *and* g, *homotopic*
In 22.5.3: *homotopy category*
In 22.6.1: *cone*
In 22.7.1: *admissible monomorphism*, *admissible epimorphism*, *admissible short exact sequence*
In 22.8.2: *triangle associated to* $0 \to K \to L \to M \to 0$, *distinguished triangle*
In 22.15.2: *derived category of* (A, d)
In 22.17.1: R*-linear category* $\mathcal{A}$
In 22.17.2: *functor of* R*-linear categories*, R*-linear*
In 22.18.1: *graded category* $\mathcal{A}$ *over* R
In 22.18.2: *functor of graded categories over* R, *graded functor*
In 22.18.3: $\mathcal{A}^0$
In 22.18.4: *graded direct sum*
In 22.19.1: *differential graded category* $\mathcal{A}$ *over* R
In 22.19.2: *functor of differential graded categories over* R
In 22.19.3: *category of complexes of* $\mathcal{A}$, *homotopy category of* $\mathcal{A}$
In 22.19.4: *differential graded direct sum*

Divided Power Algebra

In 23.2.1: *divided power structure*
In 23.3.1: *divided power ring*, *homomorphism of divided power rings*
In 23.4.1: *extends*
In 23.6.1: *divided power structure*
In 23.6.4: *compatible with the differential graded structure*
In 23.8.5: *complete intersection*, *local complete intersection*

Hypercoverings

In 24.2.1: *semi-representable objects*, *semi-representable objects over* X
In 24.2.2: *which associates a sheaf to a semi-representable object*
In 24.2.4: *covering*, *covering*
In 24.2.6: *hypercovering of* X
In 24.3.1: *homology of* K
In 24.5.1: *hypercovering of* $\mathcal{G}$, *hypercovering*

Schemes

In 25.2.1: *locally ringed space* $(X, \mathcal{O}_X)$, *local ring of* X *at* x, *residue field of* X *at* x, *morphism of locally ringed spaces*
In 25.3.1: *open immersion*
In 25.3.3: *open subspace of* X *associated to* U
In 25.4.1: *closed immersion*
In 25.4.4: *closed subspace of* X *associated to the sheaf of ideals* $\mathcal{I}$
In 25.5.2: *standard open covering*, *standard open covering*
In 25.5.3: *structure sheaf* $\mathcal{O}_{\mathrm{Spec}(R)}$ *of the spectrum of* R, *spectrum*
In 25.5.5: *affine scheme*, *morphism of affine schemes*
In 25.9.1: *scheme*, *morphism of schemes*
In 25.10.2: *open immersion*, *open subscheme*, *closed immersion*, *closed subscheme*, *immersion*, *locally closed immersion*
In 25.12.1: *reduced*
In 25.12.5: *scheme structure on* Z, *reduced induced scheme structure*, *reduction* X_{red} *of* X
In 25.15.1: *representable by a scheme*, *representable*
In 25.15.3: *satisfies the sheaf property for the Zariski topology*, *subfunctor* $H \subset F$, *representable by open immersions*, *covers* F
In 25.17.1: *fibre product*
In 25.17.7: *inverse image* $f^{-1}(Z)$ *of the closed subscheme* Z
In 25.18.1: *scheme over* S, *structure morphism*, *scheme over* R, *morphism* $f : X \to Y$ *of schemes over* S, *base change*, *base change*, *base change*
In 25.18.3: *preserved under arbitrary base change*, *preserved under base change*, *preserved under arbitrary base change*, *preserved under base change*
In 25.18.4: *scheme theoretic fibre* X_s *of* f *over* s, *fibre of* f *over* s
In 25.19.1: *quasi-compact*
In 25.20.1: *universally closed*
In 25.20.3: *satisfies the existence part of the valuative criterion*, *satisfies the uniqueness part of the valuative criterion*
In 25.21.3: *separated*, *quasi-separated*, *separated*, *quasi-separated*
In 25.23.1: *monomorphism*

Constructions of Schemes

In 26.4.5: *relative spectrum of* $\mathcal{A}$ *over* S, *spectrum of* $\mathcal{A}$ *over* S
In 26.5.1: *affine* n*-space over* S, *affine* n*-space over* R
In 26.6.1: *vector bundle associated to* $\mathcal{E}$
In 26.6.2: *vector bundle* $\pi : V \to S$ *over* S, *morphism of vector bundles over* S
In 26.7.1: *cone associated to* $\mathcal{A}$, *affine cone associated to* $\mathcal{A}$
In 26.7.2: *cone* $\pi : C \to S$ *over* S, *morphism of cones*
In 26.8.2: *standard open covering*
In 26.8.3: *structure sheaf* $\mathcal{O}_{Proj(S)}$ *of the homogeneous spectrum of* S, *homogeneous spectrum*
In 26.10.1: *twist of the structure sheaf of* $Proj(S)$
In 26.13.2: *projective* n*-space over* $\mathbf{Z}$, *projective* n*-space over* S, *projective* n*-space over* R
In 26.16.7: *relative homogeneous spectrum of* $\mathcal{A}$ *over* S, *homogeneous spectrum of* $\mathcal{A}$ *over* S, *relative Proj of* $\mathcal{A}$ *over* S
In 26.21.1: *projective bundle associated to* $\mathcal{E}$, *twist of the structure sheaf*
In 26.22.2: *Grassmannian over* $\mathbf{Z}$, *Grassmannian over* S, *Grassmannian over* R

Properties of Schemes

In 27.3.1: *integral*
In 27.4.1: *local*
In 27.4.2: *locally* P
In 27.5.1: *locally Noetherian*, *Noetherian*
In 27.6.1: *Jacobson*
In 27.7.1: *normal*
In 27.8.1: *Cohen-Macaulay*
In 27.9.1: *regular*, *nonsingular*
In 27.10.1: *dimension*, *dimension of* X *at* x
In 27.11.1: *catenary*
In 27.12.1: *regular in codimension* k, (R_k), (S_k)

In 27.13.1: *Japanese*, *universally Japanese*, *Nagata*
In 27.14.1: *regular locus*, *singular locus*
In 27.15.1: *quasi-affine*
In 27.19.1: *locally projective*
In 27.21.1: *κ-generated*
In 27.22.3: *subsheaf of sections annihilated by $\mathcal{I}$*
In 27.22.6: *subsheaf of sections supported on T*
In 27.24.1: *ample*

Morphisms of Schemes

In 28.5.5: *scheme theoretic support of $\mathcal{F}$*
In 28.6.2: *scheme theoretic image*
In 28.7.1: *scheme theoretic closure of U in X*, *scheme theoretically dense in X*
In 28.8.1: *dominant*
In 28.9.1: *birational*
In 28.10.1: *equivalent*, *rational map from X to Y*, *S-rational map from X to Y*
In 28.10.2: *rational function on X*
In 28.10.3: *ring of rational functions on X*
In 28.10.5: *function field*, *field of rational functions*
In 28.11.1: *surjective*
In 28.12.1: *universally injective*, *radicial*
In 28.13.1: *affine*
In 28.14.1: *quasi-affine*
In 28.15.1: *local*, *stable under base change*, *stable under composition*
In 28.15.2: *locally of type P*
In 28.16.1: *finite type at $x \in X$*, *locally of finite type*, *finite type*
In 28.17.3: *finite type point*
In 28.18.1: *universally catenary*
In 28.20.1: *J-2*
In 28.21.1: *quasi-finite at a point $x \in X$*, *locally quasi-finite*, *quasi-finite*
In 28.22.1: *finite presentation at $x \in X$*, *locally of finite presentation*, *finite presentation*
In 28.24.1: *open*, *universally open*
In 28.25.1: *submersive*, *universally submersive*
In 28.26.1: *flat at a point $x \in X$*, *flat over S at a point $x \in X$*, *flat*, *flat over S*
In 28.27.3: *canonical scheme structure on T*
In 28.30.1: *relative dimension $\leq d$ at x*, *relative dimension $\leq d$*, *relative dimension d*
In 28.32.1: *syntomic at $x \in X$*, *syntomic*, *local complete intersection over k*, *standard syntomic*
In 28.32.15: *syntomic of relative dimension d*
In 28.33.1: *conormal sheaf $\mathcal{C}_{Z/X}$ of Z in X*, *conormal sheaf of i*
In 28.34.1: *sheaf of differentials $\Omega_{X/S}$ of X over S*, *universal S-derivation*
In 28.35.1: *smooth at $x \in X$*, *smooth*, *standard smooth*
In 28.35.13: *smooth of relative dimension d*
In 28.36.1: *unramified at $x \in X$*, *G-unramified at $x \in X$*, *unramified*, *G-unramified*
In 28.37.1: *étale at $x \in X$*, *étale*, *standard étale*
In 28.38.1: *relatively ample*, *f-relatively ample*, *ample on X/S*, *f-ample*
In 28.39.1: *relatively very ample*, *f-relatively very ample*, *very ample on X/S*, *f-very ample*
In 28.41.1: *quasi-projective*, *H-quasi-projective*, *locally quasi-projective*
In 28.42.1: *proper*
In 28.43.1: *projective*, *H-projective*, *locally projective*
In 28.44.1: *integral*, *finite*
In 28.45.1: *universal homeomorphism*
In 28.46.1: *finite locally free*, *rank*, *degree*
In 28.47.5: *degree of X over Y*
In 28.48.2: *integral closure of $\mathcal{O}_X$ in $\mathcal{A}$*
In 28.48.3: *normalization of X in Y*
In 28.48.12: *normalization*
In 28.50.1: *bounds the degrees of the fibres of f*, *fibres of f are universally bounded*

Cohomology of Schemes

In 29.11.1: *depth k at a point*, *depth k at a point*, *(S_k)*, *(S_k)*
In 29.11.2: *Cohen-Macaulay*

Divisors

In 30.2.1: *associated*, *associated points of X*
In 30.4.1: *embedded associated point*, *embedded point*, *embedded component*
In 30.5.1: *weakly associated*, *weakly associated points of X*
In 30.7.1: *relative assassin of $\mathcal{F}$ in X over S*
In 30.8.1: *relative weak assassin of $\mathcal{F}$ in X over S*
In 30.9.1: *locally principal closed subscheme*, *effective Cartier divisor*
In 30.9.6: *sum of the effective Cartier divisors D_1 and D_2*
In 30.9.11: *pullback of D by f is defined*, *pullback of the effective Cartier divisor*
In 30.9.14: *invertible sheaf $\mathcal{O}_S(D)$ associated to D*
In 30.9.16: *regular section*
In 30.9.18: *zero scheme*
In 30.10.2: *relative effective Cartier divisor*
In 30.11.1: *conormal algebra $\mathcal{C}_{Z/X,*}$ of Z in X*, *conormal algebra of f*
In 30.11.5: *normal cone $C_Z X$*, *normal bundle*
In 30.12.2: *regular*, *Koszul-regular*, *H_1-regular*, *quasi-regular*
In 30.13.1: *regular immersion*, *Koszul-regular immersion*, *H_1-regular immersion*, *quasi-regular immersion*
In 30.14.2: *relative quasi-regular immersion*, *relative H_1-regular immersion*
In 30.15.1: *sheaf of meromorphic functions on X*, *$\mathcal{K}_X$*, *meromorphic function*
In 30.15.3: *pullbacks of meromorphic functions are defined for f*
In 30.15.5: *meromorphic section of $\mathcal{F}$*
In 30.15.11: *regular*
In 30.15.15: *ideal sheaf of denominators of s*
In 30.18.1: *blowing up of X along Z*, *blowing up of X in the ideal sheaf $\mathcal{I}$*, *exceptional divisor*, *center*
In 30.19.1: *strict transform*, *strict transform*
In 30.20.1: *U-admissible blowup*

Limits of Schemes

Varieties

In 32.3.1: *variety*
In 32.4.1: *geometrically reduced at x*, *geometrically reduced*
In 32.5.1: *geometrically connected*
In 32.6.1: *geometrically irreducible*
In 32.7.1: *geometrically pointwise integral at x*, *geometrically pointwise integral*, *geometrically integral*
In 32.8.1: *geometrically normal at x*, *geometrically normal*
In 32.10.1: *geometrically regular at x*, *geometrically regular over k*
In 32.13.1: *algebraic k-scheme*, *locally algebraic k-scheme*
In 32.16.1: *affine variety*, *projective variety*, *quasi-projective variety*, *proper variety*
In 32.19.6: *m-regular*
In 32.19.16: *Hilbert polynomial*

Topologies on Schemes

In 33.3.1: *Zariski covering of T*
In 33.3.4: *standard Zariski covering*
In 33.3.5: *big Zariski site*
In 33.3.7: *big Zariski site of S*, *small Zariski site of S*, *big affine Zariski site of S*
In 33.3.14: *restriction to the small Zariski site*
In 33.4.1: *étale covering of T*
In 33.4.5: *standard étale covering*
In 33.4.6: *big étale site*
In 33.4.8: *big étale site of S*, *small étale site of S*, *big affine étale site of S*
In 33.4.14: *restriction to the small étale site*
In 33.5.1: *smooth covering of T*
In 33.5.5: *standard smooth covering*
In 33.5.6: *big smooth site*
In 33.5.8: *big smooth site of S*, *big affine smooth site of S*
In 33.6.1: *syntomic covering of T*
In 33.6.5: *standard syntomic covering*
In 33.6.6: *big syntomic site*
In 33.6.8: *big syntomic site of S*, *big affine syntomic site of S*

In 33.7.1: *fppf covering of T*
In 33.7.5: *standard fppf covering*
In 33.7.6: *big fppf site*
In 33.7.8: *big fppf site of S*, *big affine fppf site of S*
In 33.8.1: *fpqc covering of T*
In 33.8.9: *standard fpqc covering*
In 33.8.12: *satisfies the sheaf property for the given family*, *satisfies the sheaf property for the fpqc topology*

Descent

In 34.2.1: *descent datum $(\mathcal{F}_i, \varphi_{ij})$ for quasi-coherent sheaves*, *cocycle condition*, *morphism $\psi : (\mathcal{F}_i, \varphi_{ij}) \to (\mathcal{F}'_i, \varphi'_{ij})$ of descent data*
In 34.2.3: *trivial descent datum*, *canonical descent datum*, *effective*
In 34.3.1: *descent datum (N, φ) for modules with respect to $R \to A$*, *cocycle condition*, *morphism $(N, \varphi) \to (N', \varphi')$ of descent data*
In 34.3.4: *effective*
In 34.4.2: *split equalizer*
In 34.4.5: *universally injective*
In 34.4.9: *C*
In 34.4.15: *base extension along f*, *descent morphism for modules*, *effective descent morphism for modules*
In 34.4.19: *f_**
In 34.7.2: *structure sheaf of the big site $(Sch/S)_\tau$*, *sheaf of $\mathcal{O}$-modules associated to $\mathcal{F}$*, *sheaf of $\mathcal{O}$-modules associated to $\mathcal{F}$*
In 34.8.1: *parasitic*, *parasitic for the τ-topology*
In 34.11.1: *local in the τ-topology*
In 34.16.1: *germ of X at x*, *morphism of germs*, *composition of morphisms of germs*
In 34.16.2: *étale*, *smooth*
In 34.17.1: *étale local*, *smooth local*
In 34.18.1: *τ local on the base*, *τ local on the target*, *local on the base for the τ-topology*
In 34.22.1: *τ local on the source*, *local on the source for the τ-topology*
In 34.28.3: *étale local on source-and-target*
In 34.29.1: *étale local on the source-and-target*
In 34.30.1: *descent datum for $V/X/S$*, *cocycle condition*, *descent datum relative to $X \to S$*, *morphism $f : (V/X, \varphi) \to (V'/X, \varphi')$ of descent data relative to $X \to S$*
In 34.30.3: *descent datum (V_i, φ_{ij}) relative to the family $\{X_i \to S\}$*, *morphism $\psi : (V_i, \varphi_{ij}) \to (V'_i, \varphi'_{ij})$ of descent data*
In 34.30.7: *pullback functor*
In 34.30.9: *pullback functor*
In 34.30.10: *trivial descent datum*, *canonical descent datum*, *effective*
In 34.30.11: *canonical descent datum*, *effective*
In 34.32.1: *morphisms of type $\mathcal{P}$ satisfy descent for τ-coverings*

Derived Categories of Schemes

In 35.6.4: *supported on T*
In 35.12.1: *approximation holds for the triple*
In 35.12.2: *approximation by perfect complexes holds*
In 35.16.2: *Tor independent over S*

More on Morphisms

In 36.2.1: *thickening*, *first order thickening*, *morphism of thickenings*, *thickenings over S*, *morphisms of thickenings over S*
In 36.3.1: *first order infinitesimal neighbourhood*
In 36.4.1: *formally unramified*
In 36.5.2: *universal first order thickening*, *conormal sheaf of Z over X*
In 36.6.1: *formally étale*
In 36.9.1: *formally smooth*
In 36.15.1: *normal at x*, *normal morphism*
In 36.16.1: *regular at x*, *regular morphism*
In 36.17.1: *Cohen-Macaulay at x*, *Cohen-Macaulay morphism*
In 36.27.1: *étale neighbourhood of (S, s)*, *morphism of étale neighbourhoods*, *elementary étale neighbourhood*

In 36.38.1: *finitely presented relative to S*, *of finite presentation relative to S*
In 36.39.2: *m-pseudo-coherent relative to S*, *pseudo-coherent relative to S*, *m-pseudo-coherent relative to S*, *pseudo-coherent relative to S*
In 36.40.2: *pseudo-coherent*
In 36.41.2: *perfect*
In 36.42.2: *Koszul at x*, *Koszul morphism*, *local complete intersection morphism*
In 36.44.1: *weakly étale*, *absolutely flat*

More on Flatness

In 37.4.1: *one step dévissage of $\mathcal{F}/X/S$ over s*
In 37.4.2: *one step dévissage of $\mathcal{F}/X/S$ at x*
In 37.4.6: *standard shrinking*
In 37.5.1: *complete dévissage of $\mathcal{F}/X/S$ over s*
In 37.5.2: *complete dévissage of $\mathcal{F}/X/S$ at x*
In 37.5.5: *standard shrinking*
In 37.6.1: *elementary étale localization of the ring map $R \to S$ at $\mathfrak{q}$*
In 37.6.2: *complete dévissage of $N/S/R$ over $\mathfrak{r}$*
In 37.6.4: *complete dévissage of $N/S/R$ at $\mathfrak{q}$*
In 37.14.2: *impurity of $\mathcal{F}$ above s*
In 37.15.1: *pure along X_s*, *universally pure along X_s*, *pure along X_s*, *universally S-pure*, *universally pure relative to S*, *S-pure*, *pure relative to S*, *S-pure*, *pure relative to S*
In 37.20.1: *universal flattening of $\mathcal{F}$ exists*, *universal flattening of X exists*
In 37.20.2: *flattening stratification*, *flattening stratification*

Groupoid Schemes

In 38.3.1: *pre-relation*, *relation*, *pre-equivalence relation*, *equivalence relation on U over S*
In 38.3.3: *restriction*, *pullback*
In 38.4.1: *group scheme over S*, *morphism $\psi : (G, m) \to (G', m')$ of group schemes over S*
In 38.4.3: *closed subgroup scheme*, *open subgroup scheme*
In 38.4.4: *smooth group scheme*, *flat group scheme*, *separated group scheme*
In 38.8.1: *action of G on the scheme X/S*, *equivariant*, *G-equivariant*
In 38.8.2: *free*
In 38.9.1: *pseudo G-torsor*, *formally principally homogeneous under G*, *trivial*
In 38.9.3: *principal homogeneous space*, *G-torsor*, *G-torsor in the τ topology*, *τ G-torsor*, *τ torsor*, *quasi-isotrivial*, *locally trivial*
In 38.10.1: *G-equivariant quasi-coherent $\mathcal{O}_X$-module*, *equivariant quasi-coherent $\mathcal{O}_X$-module*
In 38.11.1: *groupoid scheme over S*, *groupoid over S*, *morphism $f : (U, R, s, t, c) \to (U', R', s', t', c')$ of groupoid schemes over S*
In 38.12.1: *quasi-coherent module on (U, R, s, t, c)*
In 38.15.2: *stabilizer of the groupoid scheme (U, R, s, t, c)*
In 38.16.2: *restriction of (U, R, s, t, c) to U'*
In 38.17.1: *R-invariant*, *R-invariant*, *R-invariant*
In 38.18.1: *quotient sheaf U/R*
In 38.18.2: *representable quotient*, *representable quotient*

More on Groupoid Schemes

Étale Morphisms of Schemes

In 40.3.1: *unramified homomorphism of local rings*
In 40.3.5: *unramified at x*, *unramified*
In 40.9.1: *flat*, *faithfully flat*, *flat (resp. faithfully flat)*
In 40.9.3: *flat over Y at $x \in X$*, *flat at $x \in X$*, *flat*, *faithfully flat*
In 40.11.1: *étale homomorphism of local rings*
In 40.11.4: *étale at $x \in X$*, *étale*

Chow Homology and Chern Classes

In 41.2.1: *admissible*, *symbol*, *admissible relation*, *determinant of the finite length R-module M*
In 41.3.1: *2-periodic complex*, *cohomology modules*, *exact*, *$(2,1)$-periodic complex*, *cohomology modules*
In 41.3.2: *multiplicity*, *Herbrand quotient*
In 41.3.4: *determinant of (M, φ, ψ)*
In 41.4.3: *symbol associated to M, a, b*
In 41.4.5: *tame symbol*
In 41.7.5: *δ-dimension of Z*
In 41.8.1: *locally finite*, *cycle on X*, *k-cycle*
In 41.9.2: *multiplicity of Z' in Z*, *k-cycle associated to Z*
In 41.10.2: *multiplicity of Z' in $\mathcal{F}$*, *k-cycle associated to $\mathcal{F}$*
In 41.12.1: *pushforward*
In 41.14.1: *flat pullback of α by f*
In 41.16.1: *order of vanishing of f along Z*
In 41.17.1: *principal divisor associated to f*
In 41.19.1: *rationally equivalent to zero*, *rationally equivalent*, *Chow group of k-cycles on X*, *Chow group of k-cycles module rational equivalence on X*
In 41.23.1: *order of vanishing of s along Z*
In 41.24.1: *Weil divisor associated to s*, *Weil divisor associated to $\mathcal{L}$*
In 41.25.1: *intersection with the first chern class of $\mathcal{L}$*
In 41.27.3: *ϵ-invariant*
In 41.27.5: *sum of the effective Cartier divisors*
In 41.28.1: *Gysin homomorphism*
In 41.34.1: *chern classes of $\mathcal{E}$ on X*, *total chern class of $\mathcal{E}$ on X*
In 41.35.1: *intersection with the jth chern class of $\mathcal{E}$*
In 41.36.1: *polynomial relation among the chern classes*

Adequate Modules

In 42.3.1: *module-valued functor*, *morphism of module-valued functors*
In 42.3.2: *adequate*, *linearly adequate*
In 42.5.1: *adequate*
In 42.5.7: *$Adeq(\mathcal{O})$*, *$Adeq((Sch/S)_\tau, \mathcal{O})$*, *$Adeq(S)$*
In 42.8.1: *pure projective*, *pure injective*
In 42.8.5: *pure projective resolution*, *pure injective resolution*
In 42.8.8: *pure extension module*

Dualizing Complexes

In 43.2.1: *essential*, *essential extension of*, *essential*
In 43.4.1: *projective cover*, *projective envelope*
In 43.5.1: *injective hull*
In 43.5.5: *indecomposable*

Étale Cohomology

In 44.4.1: *étale covering*
In 44.9.1: *presheaf of sets*, *abelian presheaf*
In 44.10.1: *family of morphisms with fixed target*
In 44.10.2: *site*, *coverings*
In 44.11.1: *separated presheaf*, *sheaf*
In 44.11.4: *category of sheaves of sets*, *abelian sheaves*
In 44.13.1: *zeroth Čech cohomology group*
In 44.15.1: *fpqc covering*
In 44.15.5: *satisfies the sheaf property for the fpqc topology*
In 44.16.1: *descent datum*, *effective*
In 44.16.5: *descent datum*
In 44.16.6: *effective*
In 44.17.2: *ringed site*, *quasi-coherent*
In 44.18.1: *Čech complex*, *Čech cohomology groups*
In 44.18.4: *free abelian presheaf on $\mathcal{G}$*
In 44.20.1: *τ-covering*
In 44.20.2: *standard τ-covering*
In 44.20.4: *big τ-site of S*, *small τ-site of S*
In 44.21.1: *étale topos*, *small étale topos*, *Zariski topos*, *small Zariski topos*, *big τ-topos*
In 44.23.1: *constant sheaf*
In 44.23.3: *structure sheaf*
In 44.26.1: *étale*
In 44.26.3: *standard étale*
In 44.27.1: *étale covering*

In 44.27.3: *big étale site over S*, *small étale site over S*, *big*, *small Zariski sites*
In 44.29.1: *geometric point*, *lies over*, *étale neighborhood*, *morphism of étale neighborhoods*
In 44.29.6: *stalk*
In 44.31.3: *support of $\mathcal{F}$*, *support of σ*
In 44.32.2: *henselian*
In 44.32.6: *strictly henselian*
In 44.33.2: *étale local ring of S at $\overline{s}$*, *strict henselization of $\mathcal{O}_{S,s}$*, *henselization of $\mathcal{O}_{S,s}$*, *strict henselization of S at $\overline{s}$*, *henselization of S at s*
In 44.35.1: *direct image*, *pushforward*
In 44.35.3: *direct image*, *pushforward*
In 44.35.4: *higher direct images*
In 44.36.1: *inverse image*, *pullback*
In 44.56.1: *G-set*, *discrete G-set*, *morphism of G-sets*, *G-Sets*
In 44.57.1: *absolute Galois group*, *algebraic*
In 44.58.1: *G-module*, *discrete G-module*, *morphism of G-modules*, *Mod_G*
In 44.58.3: *continuous group cohomology groups*, *group cohomology groups*, *Galois cohomology groups*, *Galois cohomology groups of K with coefficients in M*
In 44.60.3: *similar*, *equivalent*
In 44.60.4: *Brauer group*
In 44.60.8: *C_r*, *nontrivial solution*
In 44.60.13: *variety*, *curve*
In 44.62.1: *abelian variety*
In 44.63.1: *extension by zero*, *extension by zero*
In 44.64.1: *constant sheaf with value E*, *constant sheaf*, *locally constant*, *finite locally constant*, *constant sheaf with value A*, *constant sheaf*, *locally constant*, *finite locally constant*, *constant sheaf with value M*, *constant sheaf*, *locally constant*
In 44.65.1: *constructible*, *constructible*, *constructible*
In 44.71.3: *restriction*, *trace*
In 44.74.3: *absolute frobenius*
In 44.74.5: *geometric frobenius*
In 44.74.9: *arithmetic frobenius*
In 44.74.11: *geometric frobenius*
In 44.75.1: *trace*
In 44.77.4: *total right derived functor of F*, *total right derived functor of G*
In 44.78.1: *filtered injective*, *projective*, *filtered quasi-isomorphism*
In 44.79.1: *filtered derived functor*
In 44.81.1: *perfect*
In 44.83.1: *finite Tor-dimension*
In 44.84.1: *$D_c(X, \Lambda)$*
In 44.84.7: *$D_{ctf}(X, \Lambda)$*
In 44.86.1: *global Lefschetz number*
In 44.86.2: *local Lefschetz number*
In 44.87.2: *G-trace of f on P*
In 44.90.1: *$\mathbf{Z}_\ell$-sheaf*, *lisse*, *morphism*
In 44.90.6: *torsion*, *stalk*
In 44.90.8: *ℓ-adic cohomology*
In 44.91.1: *L-function of $\mathcal{F}$*
In 44.91.3: *L-function of $\mathcal{F}$*
In 44.99.1: *open*
In 44.103.1: *unramified cusp form on $GL_2(\mathbf{A})$ with values in Λ*

Crystalline Cohomology

In 45.2.2: *divided power envelope of J in B relative to (A, I, γ)*
In 45.4.1: *δ is compatible with γ*
In 45.5.2: *divided power thickening*, *homomorphism of divided power thickenings*
In 45.6.1: *divided power A-derivation*
In 45.7.1: *divided power structure γ*
In 45.7.2: *divided power scheme*, *morphism of divided power schemes*
In 45.7.3: *divided power thickening*
In 45.8.1: *divided power thickening of X relative to $(S, \mathcal{I}, \gamma)$*, *morphism of divided power thickenings of X relative to $(S, \mathcal{I}, \gamma)$*
In 45.8.4: *Zariski, étale, smooth, syntomic, or fppf covering*, *big crystalline site*
In 45.9.1: *crystalline site*
In 45.11.1: *locally quasi-coherent*, *quasi-coherent*, *crystal in $\mathcal{O}_{X/S}$-modules*
In 45.11.3: *crystal in quasi-coherent modules*, *crystal in finite locally free modules*
In 45.12.1: *S-derivation $D : \mathcal{O}_{X/S} \to \mathcal{F}$*
In 45.26.2: *F-crystal on X/S (relative to σ)*, *nondegenerate*

Pro-étale Cohomology

In 46.2.3: *w-local*, *w-local*
In 46.3.1: *local isomorphism*, *identifies local rings*
In 46.4.1: *ind-Zariski*
In 46.7.1: *ind-étale*
In 46.10.1: *w-contractible*
In 46.11.1: *pro-étale covering of T*
In 46.11.6: *standard pro-étale covering*
In 46.11.8: *big pro-étale site*
In 46.11.12: *big pro-étale site of S*, *small pro-étale site of S*, *big affine pro-étale site of S*
In 46.11.18: *restriction to the small pro-étale site*
In 46.14.4: *derived complete with respect to $\mathcal{I}$*
In 46.23.1: *extension by zero*, *extension by zero*
In 46.24.1: *constructible*
In 46.25.1: *constructible Λ-sheaf*, *lisse*, *adic lisse*, *adic constructible*
In 46.26.1: *constructible*
In 46.26.4: *adic lisse*, *adic constructible*

Algebraic Spaces

In 47.5.1: *property $\mathcal{P}$*
In 47.6.1: *algebraic space over S*
In 47.6.3: *morphism $f : F \to F'$ of algebraic spaces over S*
In 47.9.2: *étale equivalence relation*
In 47.9.3: *presentation*
In 47.12.1: *open immersion*, *open subspace*, *closed immersion*, *closed subspace*, *immersion*, *locally closed subspace*
In 47.12.5: *Zariski covering*
In 47.12.6: *small Zariski site F_{Zar}*
In 47.13.2: *separated over S*, *locally separated over S*, *quasi-separated over S*, *Zariski locally quasi-separated over S*
In 47.14.4: *acts freely*, *quotient of U by G*
In 47.16.2: *base change of F' to S*, *viewed as an algebraic space over S'*

Properties of Algebraic Spaces

In 48.3.1: *separated*, *locally separated*, *quasi-separated*, *Zariski locally quasi-separated*, *separated*, *locally separated*, *quasi-separated*, *Zariski locally quasi-separated*
In 48.4.1: *point*
In 48.4.7: *topological space*
In 48.5.1: *quasi-compact*
In 48.7.2: *has property $\mathcal{P}$*
In 48.7.5: *has property $\mathcal{P}$ at x*
In 48.8.1: *dimension of X at x*
In 48.8.2: *dimension*
In 48.9.4: *algebraic space structure on Z*, *reduced induced algebraic space structure*, *reduction X_{red} of X*
In 48.13.2: *étale*
In 48.15.1: *small étale site $X_{étale}$*
In 48.15.2: *$X_{spaces,étale}$*
In 48.15.6: *étale topos*, *small étale topos*
In 48.15.8: *f-map $\varphi : \mathcal{G} \to \mathcal{F}$*
In 48.16.1: *geometric point*, *geometric point lying over x*
In 48.16.2: *étale neighborhood*, *morphism of étale neighborhoods*
In 48.16.6: *stalk*
In 48.17.3: *support of $\mathcal{F}$*, *support of σ*
In 48.18.2: *structure sheaf*
In 48.19.2: *étale local ring of X at $\overline{x}$*, *strict henselization of X at $\overline{x}$*
In 48.20.2: *dimension of the local ring of X at x*
In 48.21.2: *geometrically unibranch at x*, *geometrically unibranch*
In 48.22.1: *Noetherian*
In 48.27.1: *quasi-coherent*
In 48.29.2: *locally projective*

Morphisms of Algebraic Spaces

In 49.5.2: *separated*, *locally separated*, *quasi-separated*
In 49.6.2: *surjective*
In 49.7.2: *open*, *universally open*
In 49.8.1: *submersive*, *universally submersive*
In 49.9.2: *quasi-compact*
In 49.10.2: *closed*, *universally closed*
In 49.11.1: *monomorphism*
In 49.13.2: *inverse image $f^{-1}(Z)$ of the closed subspace Z*
In 49.15.4: *scheme theoretic support of $\mathcal{F}$*
In 49.16.2: *scheme theoretic image*

In 49.17.3: *scheme theoretic closure of U in X*, *scheme theoretically dense in X*
In 49.18.3: *universally injective*
In 49.19.2: *affine*
In 49.19.8: *relative spectrum of $\mathcal{A}$ over X*, *spectrum of $\mathcal{A}$ over X*
In 49.20.2: *quasi-affine*
In 49.21.2: *has property $\mathcal{P}$*
In 49.21.4: *has property $\mathcal{Q}$ at x*
In 49.22.1: *locally of finite type*, *finite type at x*, *of finite type*
In 49.24.2: *finite type point*
In 49.25.1: *locally quasi-finite*, *quasi-finite at x*, *quasi-finite*
In 49.26.1: *locally of finite presentation*, *finite presentation at x*, *of finite presentation*
In 49.27.1: *flat*, *flat at x*
In 49.28.2: *flat at x over Y*, *flat over Y*
In 49.30.1: *dimension of the local ring of the fibre of f at x*, *transcendence degree of $x/f(x)$*, *f has relative dimension d at x*
In 49.30.2: *relative dimension $\leq d$*, *relative dimension d*
In 49.32.1: *syntomic*, *syntomic at x*
In 49.33.1: *smooth*, *smooth at x*
In 49.34.1: *unramified*, *unramified at x*, *G-unramified*, *G-unramified at x*
In 49.35.1: *étale at x*
In 49.36.1: *proper*
In 49.37.1: *satisfies the uniqueness part of the valuative criterion*, *satisfies the existence part of the valuative criterion*, *satisfies the valuative criterion*
In 49.40.2: *integral*, *finite*
In 49.41.2: *finite locally free*, *rank*, *degree*
In 49.42.2: *integral closure of $\mathcal{O}_X$ in $\mathcal{A}$*
In 49.42.3: *normalization of X in Y*
In 49.45.2: *universal homeomorphism*

Decent Algebraic Spaces

In 50.3.1: *fibres of f are universally bounded*
In 50.6.1: *decent*, *reasonable*, *very reasonable*
In 50.10.6: *residual space of X at x*
In 50.14.1: *has property (β)*, *has property (β)*, *decent*, *reasonable*, *very reasonable*

Cohomology of Algebraic Spaces

In 51.5.2: *alternating Čech complex*
In 51.9.1: *coherent*

Limits of Algebraic Spaces

In 52.3.1: *locally of finite presentation*, *limit preserving*, *locally of finite presentation over S*, *locally of finite presentation*, *relatively limit preserving*
In 52.13.3: *subsheaf of sections annihilated by $\mathcal{I}$*
In 52.13.6: *subsheaf of sections supported on T*

Divisors on Algebraic Spaces

In 53.2.1: *locally principal closed subspace*, *effective Cartier divisor*
In 53.2.6: *sum of the effective Cartier divisors D_1 and D_2*
In 53.2.10: *pullback of D by f is defined*, *pullback of the effective Cartier divisor*
In 53.2.13: *invertible sheaf $\mathcal{O}_X(D)$ associated to D*
In 53.2.15: *regular section*
In 53.2.17: *zero scheme*
In 53.3.3: *relative homogeneous spectrum of $\mathcal{A}$ over X*, *homogeneous spectrum of $\mathcal{A}$ over X*, *relative Proj of $\mathcal{A}$ over X*
In 53.6.1: *blowing up of X along Z*, *blowing up of X in the ideal sheaf $\mathcal{I}$*, *exceptional divisor*, *center*
In 53.7.1: *strict transform*, *strict transform*
In 53.8.1: *U-admissible blowup*

Topologies on Algebraic Spaces

In 54.3.1: *fpqc covering of X*
In 54.4.1: *fppf covering of X*
In 54.5.1: *syntomic covering of X*
In 54.6.1: *smooth covering of X*
In 54.7.1: *étale covering of X*
In 54.8.1: *Zariski covering of X*

Descent and Algebraic Spaces

In 55.3.1: *descent datum $(\mathcal{F}_i, \varphi_{ij})$ for quasi-coherent sheaves*, *cocycle condition*, *morphism $\psi : (\mathcal{F}_i, \varphi_{ij}) \to (\mathcal{F}'_i, \varphi'_{ij})$ of descent data*

In 55.3.3: *trivial descent datum*, *canonical descent datum*, *effective*
In 55.9.1: *τ local on the base*, *τ local on the target*, *local on the base for the τ-topology*
In 55.12.1: *τ local on the source*, *local on the source for the τ-topology*
In 55.18.1: *smooth local on source-and-target*

Derived Categories of Spaces

In 56.3.2: *supported on T*
In 56.5.1: *derived category of $\mathcal{O}_X$-modules with quasi-coherent cohomology sheaves*
In 56.8.1: *elementary distinguished square*
In 56.13.1: *approximation holds for the triple*
In 56.13.2: *approximation by perfect complexes holds*
In 56.17.2: *Tor independent over B*

More on Morphisms of Spaces

In 57.3.1: *radicial*
In 57.4.1: *conormal sheaf $\mathcal{C}_{Z/X}$ of Z in X*, *conormal sheaf of i*
In 57.5.1: *conormal algebra $\mathcal{C}_{Z/X,*}$ of Z in X*, *conormal algebra of i*
In 57.5.5: *normal cone C_ZX*, *normal bundle*
In 57.6.2: *sheaf of differentials $\Omega_{X/Y}$ of X over Y*, *universal Y-derivation*
In 57.8.1: *thickening*, *first order thickening*, *morphism of thickenings*, *thickenings over B*, *morphisms of thickenings over B*
In 57.9.1: *first order infinitesimal neighbourhood*
In 57.10.1: *formally smooth*, *formally étale*, *formally unramified*
In 57.11.1: *formally unramified*
In 57.12.5: *universal first order thickening*, *conormal sheaf of Z over X*
In 57.13.1: *formally étale*
In 57.16.1: *formally smooth*
In 57.19.2: *the restriction of $\mathcal{F}$ to its fibre over z is flat at x over the fibre of Y over z*, *the fibre of X over z is flat at x over the fibre of Y over z*, *the fibre of X over z is flat over the fibre of Y over z*
In 57.32.2: *Koszul-regular immersion*, *H_1-regular immersion*, *quasi-regular immersion*
In 57.33.1: *pseudo-coherent*, *pseudo-coherent at x*
In 57.34.1: *perfect*, *perfect at x*
In 57.35.1: *Koszul morphism*, *local complete intersection morphism*, *Koszul at x*

Algebraic Spaces over Fields

Groupoids in Algebraic Spaces

In 59.4.1: *pre-relation*, *relation*, *pre-equivalence relation*, *equivalence relation on U over B*
In 59.4.3: *restriction*, *pullback*
In 59.5.1: *group algebraic space over B*, *morphism $\psi : (G, m) \to (G', m')$ of group algebraic spaces over B*
In 59.8.1: *action of G on the algebraic space X/B*, *equivariant*, *G-equivariant*
In 59.8.2: *free*
In 59.9.1: *pseudo G-torsor*, *formally principally homogeneous under G*, *trivial*
In 59.9.3: *principal homogeneous space*, *principal homogeneous G-space over B*, *G-torsor in the τ topology*, *τ G-torsor*, *τ torsor*, *quasi-isotrivial*, *locally trivial*
In 59.10.1: *G-equivariant quasi-coherent $\mathcal{O}_X$-module*, *equivariant quasi-coherent $\mathcal{O}_X$-module*
In 59.11.1: *groupoid in algebraic spaces over B*, *morphism $f : (U, R, s, t, c) \to (U', R', s', t', c')$ of groupoids in algebraic spaces over B*
In 59.12.1: *quasi-coherent module on (U, R, s, t, c)*
In 59.15.2: *stabilizer of the groupoid in algebraic spaces (U, R, s, t, c)*
In 59.16.2: *restriction of (U, R, s, t, c) to U'*
In 59.17.1: *R-invariant*, *R-invariant*, *R-invariant*
In 59.18.1: *quotient sheaf U/R*
In 59.18.3: *quotient representable by an algebraic space*, *representable quotient*,

representable quotient, *quotient representable by an algebraic space*
In 59.19.1: *quotient stack*, *quotient stack*

More on Groupoids in Spaces

In 60.11.1: *split over u*, *splitting of R over u*, *quasi-split over u*, *quasi-splitting of R over u*

Bootstrap

In 61.3.1: *representable by algebraic spaces*
In 61.4.1: *property $\mathcal{P}$*

Quotients of Groupoids

In 62.3.1: *R-invariant*, *G-invariant*
In 62.3.4: *pullback*, *flat pullback*
In 62.4.1: *categorical quotient*, *categorical quotient in $\mathcal{C}$*, *categorical quotient in the category of schemes*, *categorical quotient in schemes*
In 62.4.4: *universal categorical quotient*, *uniform categorical quotient*
In 62.5.1: *orbit*, *R-orbit*
In 62.5.4: *weakly R-equivalent*, *R-equivalent*, *weak orbit*, *weak R-orbit*, *orbit*, *R-orbit*
In 62.5.8: *set-theoretically R-invariant*, *separates orbits*, *separates R-orbits*
In 62.5.13: *set-theoretic pre-equivalence relation*, *set-theoretic equivalence relation*
In 62.5.18: *orbit space for R*
In 62.6.1: *coarse quotient*, *coarse quotient in schemes*
In 62.7.1: *uniformly*, *universally*
In 62.8.1: *sheaf of R-invariant functions on X*, *the functions on X are the R-invariant functions on U*
In 62.9.1: *good quotient*
In 62.10.1: *geometric quotient*

Simplicial Spaces

In 63.8.1: *cartesian*, *Y is cartesian over X*
In 63.8.3: *simplicial scheme associated to f*
In 63.9.1: *quasi-coherent sheaf*, *cartesian*

Formal Deformation Theory

In 64.3.1: *$\mathcal{C}_\Lambda$*, *classical case*
In 64.3.2: *small extension*
In 64.3.6: *relative cotangent space*
In 64.3.9: *essential surjection*
In 64.4.1: *$\widehat{\mathcal{C}}_\Lambda$*
In 64.5.1: *category cofibered in groupoids over $\mathcal{C}$*
In 64.6.1: *prorepresentable*
In 64.6.2: *predeformation category*, *morphism of predeformation categories*
In 64.7.1: *category $\widehat{\mathcal{F}}$ of formal objects of $\mathcal{F}$*, *formal object $\xi = (R, \xi_n, f_n)$ of $\mathcal{F}$*, *morphism $a : \xi \to \eta$ of formal objects*
In 64.7.3: *completion of $\mathcal{F}$*
In 64.8.1: *smooth*
In 64.8.13: *versal*
In 64.9.1: *conditions (S1) and (S2)*
In 64.10.1: *R-linear*
In 64.10.9: *tangent space TF of F*
In 64.11.1: *tangent space $T\mathcal{F}$ of $\mathcal{F}$*
In 64.11.3: *differential $d\varphi : T\mathcal{F} \to T\mathcal{G}$ of φ*
In 64.13.4: *minimal*, *miniversal*
In 64.15.1: *condition (RS)*
In 64.15.8: *deformation category*
In 64.16.1: *lift of x along f*, *morphism of lifts*
In 64.18.1: *group of infinitesimal automorphisms of x' over x*
In 64.18.2: *group of infinitesimal automorphisms of x_0*
In 64.18.5: *automorphism functor of x*
In 64.19.1: *category of groupoids in functors on $\mathcal{C}$*, *groupoid in functors on $\mathcal{C}$*, *morphism $(U, R, s, t, c) \to (U', R', s', t', c')$ of groupoids in functors on $\mathcal{C}$*
In 64.19.4: *representable*
In 64.19.7: *restriction $(U, R, s, t, c)|_{\mathcal{C}'}$ of (U, R, s, t, c) to $\mathcal{C}'$*
In 64.19.9: *quotient category cofibered in groupoids $[U/R] \to \mathcal{C}$*, *quotient morphism $U \to [U/R]$*
In 64.20.1: *prorepresentable*
In 64.20.2: *completion $(U, R, s, t, c)^\wedge$ of (U, R, s, t, c)*
In 64.21.1: *smooth*

In 64.23.1: *presentation of $\mathcal{F}$ by (U, R, s, t, c)*
In 64.25.1: *normalized*, *minimal*

Deformation Theory

In 65.3.2: *strict morphism of thickenings*
In 65.8.2: *strict morphism of thickenings*

The Cotangent Complex

In 66.3.1: *standard resolution of B over A*
In 66.3.2: *cotangent complex*
In 66.12.1: *A-biderivation*
In 66.16.1: *Atiyah class*
In 66.17.1: *standard resolution of $\mathcal{B}$ over $\mathcal{A}$*
In 66.17.2: *cotangent complex*
In 66.18.1: *Atiyah class*
In 66.19.1: *cotangent complex*
In 66.21.1: *cotangent complex*
In 66.23.1: *cotangent complex $L_{X/Y}$ of X over Y*
In 66.25.1: *cotangent complex $L_{X/Y}$ of X over Y*

Algebraic Stacks

In 67.8.1: *representable by an algebraic space over S*
In 67.9.1: *representable by algebraic spaces*
In 67.10.1: *property $\mathcal{P}$*
In 67.12.1: *algebraic stack over S*
In 67.12.2: *Deligne-Mumford stack*
In 67.12.3: *2-category of algebraic stacks over S*
In 67.16.4: *smooth groupoid*
In 67.16.5: *presentation*
In 67.19.2: *viewed as an algebraic stack over S'*
In 67.19.3: *change of base of $\mathcal{X}'$*

Examples of Stacks

In 68.17.2: *degree d finite Hilbert stack of $\mathcal{X}$ over $\mathcal{Y}$*

Sheaves on Algebraic Stacks

In 69.3.1: *presheaf on $\mathcal{X}$*, *morphism of presheaves on $\mathcal{X}$*
In 69.4.1: *associated Zariski site*, *associated étale site*, *associated smooth site*, *associated syntomic site*, *associated fppf site*
In 69.4.3: *Zariski sheaf*, *sheaf for the Zariski topology*, *étale sheaf*, *sheaf for the étale topology*, *smooth sheaf*, *sheaf for the smooth topology*, *syntomic sheaf*, *sheaf for the syntomic topology*, *fppf sheaf*, *sheaf*, *sheaf for the fppf topology*
In 69.4.5: *associated morphism of fppf topoi*
In 69.6.1: *structure sheaf of $\mathcal{X}$*
In 69.7.1: *presheaf of modules on $\mathcal{X}$*, *$\mathcal{O}_\mathcal{X}$-module*, *sheaf of $\mathcal{O}_\mathcal{X}$-modules*
In 69.9.2: *pullback $x^{-1}\mathcal{F}$ of $\mathcal{F}$*, *restriction of $\mathcal{F}$ to $U_{étale}$*
In 69.11.1: *quasi-coherent module on $\mathcal{X}$*, *quasi-coherent $\mathcal{O}_\mathcal{X}$-module*
In 69.11.4: *locally quasi-coherent*

Criteria for Representability

In 70.8.1: *algebraic*

Artin's axioms

In 71.5.1: *condition (RS)*
In 71.9.1: *formal object*, *morphism of formal objects*, *lies over*
In 71.9.3: *effective*
In 71.11.1: *versal*
In 71.13.1: *limit preserving*
In 71.14.1: *openness of versality*, *openness of versality*
In 71.18.2: *condition (RS*)*
In 71.19.1: *obstruction theory*, *obstruction modules*, *obstruction*
In 71.20.5: *naive obstruction theory*

Quot and Hilbert Spaces

Properties of Algebraic Stacks

In 73.4.2: *point*
In 73.4.8: *topological space*
In 73.5.1: *surjective*
In 73.6.1: *quasi-compact*
In 73.7.2: *has property $\mathcal{P}$*
In 73.7.5: *has property $\mathcal{P}$ at x*
In 73.8.1: *monomorphism*
In 73.9.1: *open immersion*, *closed immersion*, *immersion*
In 73.9.8: *open substack*, *closed substack*, *locally closed substack*

In 73.10.4: *algebraic stack structure on Z*, *reduced induced algebraic stack structure*, *reduction $\mathcal{X}_{red}$ of $\mathcal{X}$*
In 73.11.8: *residual gerbe of $\mathcal{X}$ at x exists*, *residual gerbe of $\mathcal{X}$ at x*

Morphisms of Algebraic Stacks

In 74.4.1: *DM*, *quasi-DM*, *separated*, *quasi-separated*
In 74.4.2: *DM over S*, *quasi-DM over S*, *separated over S*, *quasi-separated over S*, *DM*, *quasi-DM*, *separated*, *quasi-separated*
In 74.5.3: *sheaf of automorphisms of x*
In 74.7.2: *quasi-compact*
In 74.8.1: *Noetherian*
In 74.9.2: *open*, *universally open*
In 74.10.1: *submersive*, *universally submersive*
In 74.11.2: *closed*, *universally closed*
In 74.12.2: *has property $\mathcal{P}$*
In 74.13.1: *locally of finite type*, *of finite type*
In 74.14.2: *finite type point*
In 74.16.2: *locally quasi-finite*
In 74.17.1: *flat*
In 74.18.1: *locally of finite presentation*, *of finite presentation*
In 74.19.1: *gerbe over*, *gerbe*
In 74.22.1: *smooth*

Cohomology of Algebraic Stacks

In 75.7.1: *flat base change property*
In 75.8.1: *parasitic*
In 75.11.1: *lisse-étale site*, *flat-fppf site*

Derived Categories of Stacks

In 76.4.1: *derived category of $\mathcal{O}_\mathcal{X}$-modules with quasi-coherent cohomology sheaves*

Introducing Algebraic Stacks

In 77.4.3: *smooth*
In 77.5.1: *algebraic stack*

Examples

Exercises

In 79.2.1: *directed partially ordered set*, *system of rings*
In 79.2.3: *colimit*
In 79.2.8: *finite presentation*
In 79.5.4: *quasi-compact*
In 79.5.6: *Hausdorff*
In 79.5.9: *irreducible*, *irreducible*
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